Derived Differential Geometry

Lecture 1 of 3: Introduction

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For references, see
http://people.maths.ox.ac.uk/~joyce/dmanifolds.html,
http://people.maths.ox.ac.uk/~joyce/Kuranishi.html.
The survey papers arXiv:1104.4951, arXiv:1206.4207,
and arXiv:1510.07444 are a good start.

These slides available at
http://people.maths.ox.ac.uk/~joyce/.

Plan of talk:

1. Introduction

Derived Differential Geometry (DDG) is the study of *derived smooth manifolds* and *derived smooth orbifolds*, where 'derived' is in the sense of the Derived Algebraic Geometry (DAG) of Jacob Lurie and Toën–Vezzosi. Derived manifolds include ordinary smooth manifolds, but also many singular objects.

Derived manifolds and orbifolds form higher categories – 2-categories $\mathbf{dMan}, \mathbf{dOrb}$ or $\mathbf{mKur}, \mathbf{Kur}$ in my set-up, and ∞-categories in the set-ups of Spivak–Borisov–Noel.

Many interesting moduli spaces over $\mathbb{R}$ or $\mathbb{C}$ in both algebraic and differential geometry are naturally derived manifolds or derived orbifolds, including those used to define Donaldson, Donaldson–Thomas, Gromov–Witten and Seiberg–Witten invariants, Floer theories, and Fukaya categories.

A compact, oriented derived manifold or orbifold $\boldsymbol{X}$ has a *virtual class* in homology (or a *virtual chain* if $\partial\boldsymbol{X} \neq \emptyset$), which can be used to define these enumerative invariants, Floer theories, ...

Different definitions of derived manifolds and orbifolds

There are several versions of 'derived manifolds' and 'derived orbifolds' in the literature, in order of increasing simplicity:

- Spivak's ∞-category $\mathbf{DerMan_{Spi}}$ of derived manifolds (2008).
- Borisov–Noël's ∞-category $\mathbf{DerMan_{BN}}$ (2011,2012).
- My d-manifolds and d-orbifolds (2010–2016), which form strict 2-categories $\mathbf{dMan}, \mathbf{dOrb}$.
- My μ-Kuranishi spaces, m-Kuranishi spaces and Kuranishi spaces (2014), which form a category $\boldsymbol{\mu}\mathbf{Kur}$ and weak 2-categories $\mathbf{mKur}, \mathbf{Kur}$.

  Here μ-, m-Kuranishi spaces are types of derived manifold, and Kuranishi spaces a type of derived orbifold.

In fact the Kuranishi space approach is motivated by earlier work by Fukaya, Oh, Ohta and Ono in symplectic geometry (1999,2009–) whose 'Kuranishi spaces' are really a prototype kind of derived orbifold, from before the invention of DAG.

Relation between these definitions

- Borisov–Noel (2011) prove an equivalence of ∞-categories $\mathbf{DerMan_{Spi}} \simeq \mathbf{DerMan_{BN}}$.
- Borisov (2012) gives a 2-functor $\pi_2(\mathbf{DerMan_{BN}}) \to \mathbf{dMan}$ which is nearly an equivalence of 2-categories (e.g. it is a 1-1 correspondence on equivalence classes of objects), where $\pi_2(\mathbf{DerMan_{BN}})$ is the 2-category truncation of $\mathbf{DerMan_{BN}}$.
- I prove (2017) equivalences of 2-categories $\mathbf{dMan} \simeq \mathbf{mKur}$, $\mathbf{dOrb} \simeq \mathbf{Kur}$ and of categories $\mathrm{Ho}(\mathbf{dMan}) \simeq \mathrm{Ho}(\mathbf{mKur}) \simeq \boldsymbol{\mu}\mathbf{Kur}$, where $\mathrm{Ho}(\cdots)$ is the homotopy category.

Thus all these notions of derived manifold are more-or-less equivalent. Kuranishi spaces are easiest. There is a philosophical difference between $\mathbf{DerMan_{Spi}}, \mathbf{DerMan_{BN}}$ (locally modelled on $X \times_Z Y$ for smooth maps of manifolds $g : X \to Z$, $h : Y \to Z$) and $\mathbf{dMan}, \boldsymbol{\mu}\mathbf{Kur}, \mathbf{mKur}$ (locally modelled on $s^{-1}(0)$ for E a vector bundle over a manifold V with $s : V \to E$ a smooth section).

Restriction to the quasi-smooth case

All these definitions of derived manifolds $\boldsymbol{X}$ include the condition that $\boldsymbol{X}$ be *quasi-smooth* in DAG terminology, that is, that the cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ lies in the interval $[-1, 0]$, not in $(-\infty, 0]$. There are several ways to say this:

- A derived manifold $\boldsymbol{X}$ is locally a (homotopy) fibre product $U \times_W V$ of classical manifolds U, V, W.
- A derived manifold is locally $s^{-1}(0)$ of a smooth section s of a vector bundle $E \to V$ on a classical manifold V.

There are more general definitions allowing $\boldsymbol{X}$ not quasi-smooth; I would call these *derived* C^∞*-schemes*.

The quasi-smooth condition is very important in applications: compact, oriented, quasi-smooth derived manifolds have *virtual cycles* in homology, needed for counting invariants, Floer theories, etc. This does not work for non-quasi-smooth derived manifolds. Many moduli spaces are automatically quasi-smooth, e.g. moduli spaces of solutions of nonlinear elliptic p.d.e.s.

Two ways to define ordinary manifolds

Here are two equivalent definitions of classical manifolds:

Definition 1.1

A *manifold* of dimension n is a Hausdorff, second countable topological space X with a sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras (or C^∞-rings) locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$, where $\mathcal{O}_{\mathbb{R}^n}$ is the sheaf of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 1.2

A *manifold* of dimension n is a Hausdorff, second countable topological space X equipped with an atlas of charts $\{(V_i, \psi_i) : i \in I\}$, where $V_i \subseteq \mathbb{R}^n$ is open, and $\psi_i : V_i \to X$ is a homeomorphism with an open subset $\mathrm{Im}\,\psi_i$ of X for all $i \in I$, and $\psi_j^{-1} \circ \psi_i : \psi_i^{-1}(\mathrm{Im}\,\psi_j) \to \psi_j^{-1}(\mathrm{Im}\,\psi_i)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ for all $i, j \in I$.

Two ways to define derived manifolds

The approaches to DDG of Lurie, Spivak, Borisov–Noel, and my d-manifolds, work by generalizing Definition 1.1, and taking a derived manifold $\boldsymbol{X}$ to be a topological space X with a sheaf of derived C^∞-rings $\mathcal{O}_X$. The differences are in the notions of 'derived C^∞-ring' (simplicial/dg), and 'sheaf' (homotopy/strict).
My (μ- and m-)Kuranishi spaces generalize Definition 1.2, giving an 'atlas of charts' definition of derived manifolds/orbifolds. They are equivalent to my d-manifold and d-orbifolds, so we have 2-category equivalences **dMan** $\simeq$ **mKur** and **dOrb** $\simeq$ **Kur**.
Fukaya–Oh–Ohta–Ono have their own definition of Kuranishi space (1999), predating DAG. With hindsight, it is a prototype 'atlas of charts' notion of derived orbifold. It does not work that well, e.g. there is no notion of morphism between FOOO Kuranishi spaces.
My (m-)Kuranishi spaces are a variant of the FOOO definition engineered to be equivalent to d-manifolds and d-orbifolds.

Derived manifolds with boundary, and with corners

As well as classical manifolds **Man**, locally modelled on $\mathbb{R}^n$, in differential geometry we also consider manifolds with boundary $\mathbf{Man^b}$, locally modelled on $[0,\infty)\times\mathbb{R}^{n-1}$, and manifolds with corners $\mathbf{Man^c}$, locally modelled on $[0,\infty)^k\times\mathbb{R}^{n-k}$. Actually there are lots of variations on categories of manifolds with corners.
So we should also consider *derived manifolds/orbifolds with boundary*, and *derived manifolds/orbifolds with corners*. These are very important in applications such as Lagrangian Floer theory, Symplectic Field Theory, and Fukaya categories, as moduli spaces of J-holomorphic curves are derived orbifolds with corners.
For 'things with corners', the Kuranishi space (atlas of charts) approach is much easier than the derived C^∞-scheme approach. This is because the Kuranishi space inputs a category of 'manifolds' satisfying assumptions, which could be $\mathbf{Man^c}$, etc., and outputs 2-categories of 'derived manifolds' and 'derived orbifolds'.
For derived C^∞-schemes, we should modify C^∞-rings to 'C^∞-rings with corners', which changes the theory from the beginning.

2. 2-categories

There are two kinds of higher categories which are well behaved: 2-categories and ∞-categories. Most DAG is written using ∞-categories, but my version of DDG uses 2-categories. There are two main reasons why 2-categories are sufficient in DDG:

- The existence of partitions of unity in differential geometry means that structure sheaves are soft, and means we do not need the extra flexibility of an ∞-category (e.g. we can use strict sheaves, not homotopy sheaves).
- We are only interested in quasi-smooth objects, which are naturally 2-categorical, as they involve complexes of length 2.

There are two kinds of 2-category, strict 2-categories and weak 2-categories. Every weak 2-category $\mathcal{C}$ is equivalent as a weak 2-category to a strict 2-category $\mathcal{C}'$ (weak 2-categories can be 'strictified'), so there is no fundamental difference, but weak 2-categories have more notation.

A 2-category $\boldsymbol{\mathcal{C}}$ has *objects* $X, Y, \ldots$, 1-*morphisms* $f, g : X \to Y$ (morphisms), and 2-*morphisms* $\eta : f \Rightarrow g$ (morphisms between morphisms). Here are some examples to bear in mind:

Example

(a) The strict 2-category $\mathfrak{Cat}$ has objects categories $\mathcal{C}, \mathscr{D}, \ldots$, 1-morphisms functors $F, G : \mathcal{C} \to \mathscr{D}$, and 2-morphisms natural transformations $\eta : F \Rightarrow G$.

(b) The strict 2-category $\mathbf{Top^{ho}}$ of *topological spaces up to homotopy* has objects topological spaces $X, Y, \ldots$, 1-morphisms continuous maps $f, g : X \to Y$, and 2-morphisms isotopy classes $[H] : f \Rightarrow g$ of homotopies H from f to g. That is, $H : X \times [0,1] \to Y$ is continuous with $H(x,0) = f(x)$, $H(x,1) = g(x)$, and $H, H' : X \times [0,1] \to Y$ are isotopic if there exists continuous $I : X \times [0,1]^2 \to Y$ with $I(x,s,0) = H(x,s)$, $I(s,x,1) = H'(x,s)$, $I(x,0,t) = f(x)$, $I(x,1,t) = g(x)$.

Definition

A (*strict*) 2-*category* $\boldsymbol{\mathcal{C}}$ consists of a class of *objects* $\mathrm{Obj}(\boldsymbol{\mathcal{C}})$, for all $X, Y \in \mathrm{Obj}(\boldsymbol{\mathcal{C}})$ an (essentially small) category $\mathrm{Hom}(X,Y)$, for all X in $\mathrm{Obj}(\boldsymbol{\mathcal{C}})$ an object id_X in $\mathrm{Hom}(X,X)$ called the *identity 1-morphism*, and for all X, Y, Z in $\mathrm{Obj}(\boldsymbol{\mathcal{C}})$ a functor $\mu_{X,Y,Z} : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$. These must satisfy the *identity property*, that

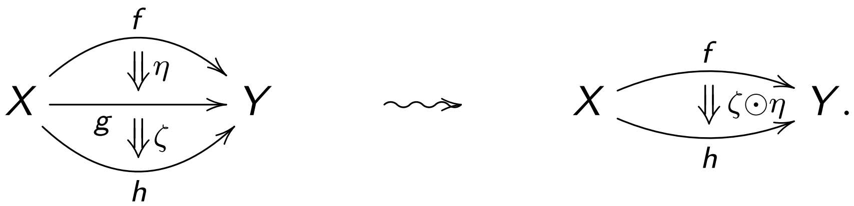
$$\mu_{X,X,Y}(\mathrm{id}_X, -) = \mu_{X,Y,Y}(-, \mathrm{id}_Y) = \mathrm{id}_{\mathrm{Hom}(X,Y)} \tag{2.1}$$

as functors $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Y)$, and the *associativity property*, that

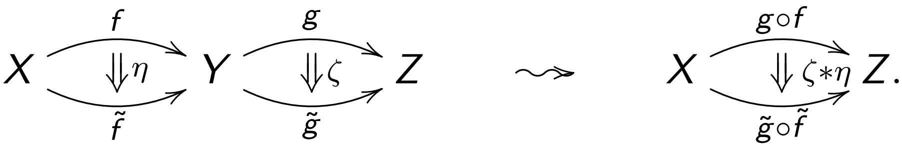
$$\mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}) = \mu_{W,X,Z} \circ (\mathrm{id} \times \mu_{X,Y,Z}) \tag{2.2}$$

as functors $\mathrm{Hom}(W,X) \times \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(W,X)$. Objects f of $\mathrm{Hom}(X,Y)$ are called 1-*morphisms*, written $f : X \to Y$. For 1-morphisms $f, g : X \to Y$, morphisms $\eta \in \mathrm{Hom}_{\mathrm{Hom}(X,Y)}(f,g)$ are called 2-*morphisms*, written $\eta : f \Rightarrow g$.

There are three kinds of composition in a 2-category, satisfying various associativity relations. If $f : X \to Y$ and $g : Y \to Z$ are 1-morphisms then $\mu_{X,Y,Z}(f,g)$ is the *horizontal composition of 1-morphisms*, written $g \circ f : X \to Z$. If $f, g, h : X \to Y$ are 1-morphisms and $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms then composition of η, ζ in $\mathrm{Hom}(X,Y)$ gives the *vertical composition of 2-morphisms* of η, ζ, written $\zeta \odot \eta : f \Rightarrow h$, as a diagram



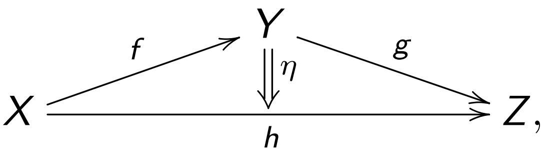
And if $f, \tilde f : X \to Y$ and $g, \tilde g : Y \to Z$ are 1-morphisms and $\eta : f \Rightarrow \tilde f$, $\zeta : g \Rightarrow \tilde g$ are 2-morphisms then $\mu_{X,Y,Z}(\eta,\zeta)$ is the *horizontal composition of 2-morphisms*, written $\zeta * \eta : g \circ f \Rightarrow \tilde g \circ \tilde f$, as a diagram



There are also two kinds of identity: *identity 1-morphisms* $\mathrm{id}_X : X \to X$ and *identity 2-morphisms* $\mathrm{id}_f : f \Rightarrow f$.
A 2-morphism is a 2-*isomorphism* if it is invertible under vertical composition. A 2-category is called a (2,1)-*category* if all 2-morphisms are 2-isomorphisms. For example, stacks in algebraic geometry form a (2,1)-category.

In a 2-category $\mathfrak{C}$, there are three notions of when objects X, Y in $\mathfrak{C}$ are 'the same': *equality* $X = Y$, and *isomorphism*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, and *equivalence*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$. Usually equivalence is the correct notion.

Commutative diagrams in 2-categories should in general only commute *up to (specified) 2-isomorphisms*, rather than strictly. A simple example of a commutative diagram in a 2-category $\mathfrak{C}$ is



which means that X, Y, Z are objects of $\mathfrak{C}$, $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ are 1-morphisms in $\mathfrak{C}$, and $\eta : g \circ f \Rightarrow h$ is a 2-isomorphism.

Definition (Fibre products in 2-categories.)

Let $\mathcal{C}$ be a strict 2-category and $g : X \to Z$, $h : Y \to Z$ be 1-morphisms in $\mathcal{C}$. A *fibre product* $X \times_Z Y$ in $\mathcal{C}$ is an object W, 1-morphisms $\pi_X : W \to X$ and $\pi_Y : W \to Y$ and a 2-isomorphism $\eta : g \circ \pi_X \Rightarrow h \circ \pi_Y$ in $\mathcal{C}$ with the following universal property: suppose $\pi'_X : W' \to X$ and $\pi'_Y : W' \to Y$ are 1-morphisms and $\eta' : g \circ \pi'_X \Rightarrow h \circ \pi'_Y$ is a 2-isomorphism in $\mathcal{C}$. Then there exists a 1-morphism $b : W' \to W$ and 2-isomorphisms $\zeta_X : \pi_X \circ b \Rightarrow \pi'_X$, $\zeta_Y : \pi_Y \circ b \Rightarrow \pi'_Y$ such that the following diagram commutes:

$$\begin{array}{ccc} g \circ \pi_X \circ b & \xRightarrow{\eta * \mathrm{id}_b} & h \circ \pi_Y \circ b \\ {\scriptstyle \mathrm{id}_g * \zeta_X} \Downarrow & & \Downarrow {\scriptstyle \mathrm{id}_h * \zeta_Y} \\ g \circ \pi'_X & \xRightarrow{\eta'} & h \circ \pi'_Y. \end{array}$$

Furthermore, if $\tilde{b}, \tilde{\zeta}_X, \tilde{\zeta}_Y$ are alternative choices of b, ζ_X, ζ_Y then there should exist a unique 2-isomorphism $\theta : \tilde{b} \Rightarrow b$ with

$$\tilde{\zeta}_X = \zeta_X \odot (\mathrm{id}_{\pi_X} * \theta) \quad \text{and} \quad \tilde{\zeta}_Y = \zeta_Y \odot (\mathrm{id}_{\pi_Y} * \theta).$$

If a fibre product $X \times_Z Y$ exists, it is unique up to equivalence.

Weak 2-categories

A *weak 2-category*, or *bicategory*, is like a strict 2-category, except that the equations of functors (2.1), (2.2) are required to hold not up to equality, but up to specified natural isomorphisms. That is, a weak 2-category $\mathcal{C}$ consists of data $\mathrm{Obj}(\mathcal{C}), \mathrm{Hom}(X,Y), \mu_{X,Y,Z}$, id_X as above, but in place of (2.1), a natural isomorphism

$$\alpha : \mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}) \Longrightarrow \mu_{W,X,Z} \circ (\mathrm{id} \times \mu_{X,Y,Z}),$$

and in place of (2.2), natural isomorphisms

$$\beta : \mu_{X,X,Y}(\mathrm{id}_X, -) \Longrightarrow \mathrm{id}, \quad \gamma : \mu_{X,Y,Y}(-, \mathrm{id}_Y) \Longrightarrow \mathrm{id},$$

satisfying some identities. That is, composition of 1-morphisms is associative *only up to specified 2-isomorphisms*, so for 1-morphisms $e : W \to X$, $f : X \to Y$, $g : Y \to Z$ we have a 2-isomorphism

$$\alpha_{g,f,e} : (g \circ f) \circ e \Longrightarrow g \circ (f \circ e).$$

Similarly identities $\mathrm{id}_X, \mathrm{id}_Y$ work up to 2-isomorphism, so for each $f : X \to Y$ we have 2-isomorphisms

$$\beta_f : f \circ \mathrm{id}_X \Longrightarrow f, \qquad \gamma_f : \mathrm{id}_Y \circ f \Longrightarrow f.$$

The homotopy category $\mathrm{Ho}(\mathcal{C})$ of a 2-category $\mathcal{C}$

Let $\mathcal{C}$ be a (strict or weak) 2-category. The *homotopy category* $\mathrm{Ho}(\mathcal{C})$ is the ordinary category whose objects $X, Y, \ldots$ are objects of $\mathcal{C}$, and whose morphisms $[f] : X \to Y$ are 2-isomorphism classes of 1-morphisms $f : X \to Y$ in $\mathcal{C}$.

Thus, we can always reduce a 2-category to an ordinary category. But important information may be lost by doing so. For example:

- A 2-category fibre product $X \times_Z Y$ in $\mathcal{C}$ is generally not a fibre product in $\mathrm{Ho}(\mathcal{C})$, as it is characterized by a universal property involving 2-morphisms in $\mathcal{C}$, which makes no sense in $\mathrm{Ho}(\mathcal{C})$.
- In the 2-category of orbifolds **Orb**, for fixed objects X, Y, the 1-morphisms $f, g : X \to Y$ and 2-morphisms $\eta : f \Rightarrow g$ form a stack (2-sheaf) on X. That is, we can glue 1- and 2-morphisms on an open cover of X. However morphisms $[f] : X \to Y$ in $\mathrm{Ho}(\mathbf{Orb})$ do not form a sheaf on X. The same holds for stacks in algebraic geometry.

3. C^∞-rings and C^∞-schemes

Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes. In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces.

There is a little-known theory of schemes in differential geometry, C^∞-*schemes*, going back to Lawvere, Dubuc, Moerdijk and Reyes, ... in synthetic differential geometry in the 1960s-1980s.

C^∞-schemes are essentially *algebraic* objects, on which smooth real functions and calculus make sense.

The theory works by replacing commutative rings or $\mathbb{K}$-algebras in algebraic geometry by algebraic objects called C^∞-*rings*.

Definition 3.1 (First definition of C^∞-ring)

A C^∞-*ring* is a set $\mathfrak{C}$ together with n-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying:

Let $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathfrak{C}$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have $\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j$.

A *morphism* of C^∞-rings is a map of sets $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Definition 3.2 (Second definition of C^∞-ring)

Write **Euc** for the full subcategory of manifolds **Man** with objects $\mathbb{R}^n$ for $n = 0, 1, \ldots$. That is, **Euc** is the category with objects Euclidean spaces $\mathbb{R}^n$, and morphisms smooth maps $f : \mathbb{R}^m \to \mathbb{R}^n$.
A C^∞-*ring* is a product-preserving functor F : **Euc** $\to$ **Sets**. That is, F is a functor with functorial identifications
$F(\mathbb{R}^{m+n}) = F(\mathbb{R}^m \times \mathbb{R}^n) \cong F(\mathbb{R}^m) \times F(\mathbb{R}^n)$ for all $m, n \geqslant 0$.
A morphism $\phi : F \to G$ of C^∞-rings F, G is a natural transformation of functors $\phi : F \Rightarrow G$.

Definitions 3.1 and 3.2 are equivalent as follows. Given F : **Euc** $\to$ **Sets** as above, define a set $\mathfrak{C} = F(\mathbb{R})$. As F is product-preserving, $F(\mathbb{R}^n) \cong F(\mathbb{R})^n = \mathfrak{C}^n$ for all $n \geqslant 0$. If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth then $F(f) : F(\mathbb{R}^n) \to F(\mathbb{R})$ is identified with a map $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$. Then $\big(\mathfrak{C}, \Phi_{f,\, f:\mathbb{R}^n \to \mathbb{R}\; C^\infty}\big)$ is a C^∞-ring as in Definition 3.1. Conversely, given $\mathfrak{C}$ we define F with $F(\mathbb{R}^n) = \mathfrak{C}^n$.

Manifolds as C^∞-rings

Let X be a manifold, and write $\mathfrak{C} = C^\infty(X)$ for the set of smooth functions $c : X \to \mathbb{R}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by $\Phi_f(c_1, \ldots, c_n)(x) = f\big(c_1(x), \ldots, c_n(x)\big)$ for $x \in X$. These make $C^\infty(X)$ into a C^∞-ring as in Definition 3.1.
Define F : **Euc** $\to$ **Sets** by $F(\mathbb{R}^n) = \mathrm{Hom}_{\mathbf{Man}}(X, \mathbb{R}^n)$ and $F(f) = f\circ : \mathrm{Hom}_{\mathbf{Man}}(X, \mathbb{R}^m) \to \mathrm{Hom}_{\mathbf{Man}}(X, \mathbb{R}^n)$ for $f : \mathbb{R}^m \to \mathbb{R}^n$ smooth. Then F is a C^∞-ring as in Definition 3.2.
If $f : X \to Y$ is smooth map of manifolds then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings; conversely, if $\phi : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings then $\phi = f^*$ for some unique smooth $f : X \to Y$. This gives a *full and faithful functor* $F : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ by $F : X \mapsto C^\infty(X)$, $F : f \mapsto f^*$.
Thus, we can think of manifolds as examples of C^∞-rings. But there are many more C^∞-rings than manifolds. For example, $C^0(X)$ is a C^∞-ring for any topological space X.

C^∞-schemes

We can now develop the whole machinery of scheme theory in algebraic geometry, replacing rings by C^∞-rings throughout.
A C^∞-*ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf of C^∞-rings $\mathcal{O}_X$. It is *local* if the stalks $\mathcal{O}_{X,x}$ for $x \in X$ are local $\mathbb{R}$-algebras with residue field $\mathbb{R}$. Write $\mathbf{LC^\infty RS}$ for the category of local C^∞-ringed spaces.
The *global sections functor* $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings^{op}}$ maps $\Gamma : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$. It has a right adjoint, the *spectrum functor* $\mathrm{Spec} : \mathbf{C^\infty Rings^{op}} \to \mathbf{LC^\infty RS}$. That is, for each C^∞-ring $\mathfrak{C}$ we construct a local C^∞-ringed space $\mathrm{Spec}\,\mathfrak{C}$. Points $x \in \mathrm{Spec}\,\mathfrak{C}$ are $\mathbb{R}$-algebra morphisms $x : \mathfrak{C} \to \mathbb{R}$. We don't use prime ideals.
On the subcategory of *complete* C^∞-rings, Spec is full and faithful.

A local C^∞-ringed space $\underline{X}$ is called an *affine* C^∞-*scheme* if $\underline{X} \cong \mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. It is a C^∞-*scheme* if X can be covered by open $U \subseteq X$ with $(U, \mathcal{O}_X|_U)$ an affine C^∞-scheme.
Write $\mathbf{C^\infty Sch}$ for the full subcategory of C^∞-schemes in $\mathbf{LC^\infty RS}$.
If X is a manifold, define a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ by $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. Then $\underline{X} \cong \mathrm{Spec}\, C^\infty(X)$.
This defines a full and faithful embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$. So we can regard manifolds as examples of C^∞-schemes.
All fibre products exist in $\mathbf{C^\infty Sch}$. In manifolds $\mathbf{Man}$, fibre products $X \times_{g,Z,h} Y$ need exist only if $g : X \to Z$ and $h : Y \to Z$ are transverse. When g, h are not transverse, the fibre product $X \times_{g,Z,h} Y$ exists in $\mathbf{C^\infty Sch}$, but is not a manifold.
We also define *quasicoherent sheaves* on a C^∞-scheme $\underline{X}$, and write $\mathrm{qcoh}(\underline{X})$ for the abelian category of quasicoherent sheaves.
A C^∞-scheme $\underline{X}$ has a well-behaved *cotangent sheaf* $T^*\underline{X}$.