Derived Differential Geometry

Lecture 2 of 3: Two ways to define derived manifolds

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These slides available at
`http://people.maths.ox.ac.uk/~joyce/`.

Plan of talk:

4. Derived C^∞-rings and derived C^∞-schemes
Differential graded C^∞-rings

We can define derived $\mathbb{C}$-schemes by replacing $\mathbb{C}$-algebras A by *dg $\mathbb{C}$-algebras* $A^\bullet$ in the definition of $\mathbb{C}$-scheme — commutative differential graded $\mathbb{C}$-algebras in degrees $\leqslant 0$, of the form $\cdots \to A^{-2} \xrightarrow{\mathrm{d}} A^{-1} \xrightarrow{\mathrm{d}} A^0$, where A^0 is an ordinary $\mathbb{C}$-algebra.
The corresponding 'classical' $\mathbb{C}$-algebra is $H^0(A^\bullet) = A^0/\mathrm{d}[A^{-1}]$.
There is a parallel notion of *dg C^∞-ring* $\mathfrak{C}^\bullet$, of the form $\cdots \to \mathfrak{C}^{-2} \xrightarrow{\mathrm{d}} \mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$, where $\mathfrak{C}^0$ is an ordinary C^∞-ring, and $\mathfrak{C}^{-1}, \mathfrak{C}^{-2}, \ldots$ are modules over $\mathfrak{C}^0$ as an $\mathbb{R}$-algebra. The corresponding 'classical' C^∞-ring is $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.
One could use dg C^∞-rings to define 'derived C^∞-schemes'. It is necessary to regard dg C^∞-rings as in ∞-category. An alternative is to use *simplicial C^∞-rings*, see Lurie DAG V, Spivak arXiv:0810.5175, Borisov–Noel arXiv:1112.0033, and Borisov arXiv:1212.1153.

Square zero dg C^∞-rings

My d-spaces are a 2-category truncation of derived C^∞-schemes.
To define them, I use a special class of dg C^∞-rings called *square zero dg C^∞-rings*, which form a 2-category **SZC$^\infty$Rings**.
A dg C^∞-ring $\mathfrak{C}^\bullet$ is *square zero* if $\mathfrak{C}^i = 0$ for $i < -1$ and $\mathfrak{C}^{-1} \cdot \mathrm{d}[\mathfrak{C}^{-1}] = 0$. Then $\mathfrak{C}$ is $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$, and $\mathrm{d}[\mathfrak{C}^{-1}]$ is a square zero ideal in the (ordinary) C^∞-ring $\mathfrak{C}^0$, and $\mathfrak{C}^{-1}$ is a module over the 'classical' C^∞-ring $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.
A 1-*morphism* $\alpha^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ in **SZC$^\infty$Rings** is maps $\alpha^0 : \mathfrak{C}^0 \to \mathfrak{D}^0$, $\alpha^{-1} : \mathfrak{C}^{-1} \to \mathfrak{D}^{-1}$ preserving all the structure.
Then $H^0(\alpha^\bullet) : H^0(\mathfrak{C}^\bullet) \to H^0(\mathfrak{D}^\bullet)$ is a morphism of C^∞-rings.
For 1-morphisms $\alpha^\bullet, \beta^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ a 2-*morphism* $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ is a C^∞-derivation $\eta : \mathfrak{C}^0 \to \mathfrak{D}^{-1}$ with $\beta^0 = \alpha^0 + \mathrm{d} \circ \eta$, $\beta^{-1} = \alpha^{-1} + \eta \circ \mathrm{d}$.
There is an embedding of (2-)categories **C$^\infty$Rings** $\subset$ **SZC$^\infty$Rings** as the (2-)subcategory of $\mathfrak{C}^\bullet$ with $\mathfrak{C}^{-1} = 0$.

Examples of square zero dg C^∞-rings

Example 4.1

Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section. Then we call (V, E, s) an *m-Kuranishi neighbourhood* (compare m-Kuranishi spaces later). Associate a square zero dg C^∞-ring $\mathfrak{C}^\bullet = (\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0)$ to (V, E, s) by

$$\mathfrak{C}^0 = C^\infty(V)/I_s^2, \qquad \mathfrak{C}^{-1} = C^\infty(E^*)/I_s \cdot C^\infty(E^*),$$
$$\mathrm{d}(\epsilon + I_s \cdot C^\infty(E^*)) = \epsilon(s) + I_s^2,$$

where $I_s = C^\infty(E^*) \cdot s \subset C^\infty(V)$ is the ideal generated by s.
The d-manifold $\boldsymbol{X}$ associated to (V, E, s) is $\operatorname{Spec} \mathfrak{C}^\bullet$. It only knows about functions on V up to $O(s^2)$, and sections of E up to $O(s)$.

5. D-spaces and d-manifolds

A *d-space* $\boldsymbol{X}$ is a topological space X with a sheaf of square zero dg-C^∞-rings $\mathcal{O}_{\boldsymbol{X}}^\bullet = \mathcal{O}_X^{-1} \xrightarrow{\mathrm{d}} \mathcal{O}_{\boldsymbol{X}}^0$, such that $\underline{X} = (X, H^0(\mathcal{O}_{\boldsymbol{X}}^\bullet))$ and $(X, \mathcal{O}_{\boldsymbol{X}}^0)$ are C^∞-schemes, and $\mathcal{O}_X^{-1}$ is quasicoherent over $\underline{X}$. We call $\underline{X}$ the *underlying classical C^∞-scheme*.
D-spaces are a kind of derived C^∞-scheme. They form a strict 2-category **dSpa**, with 1-morphisms and 2-morphisms defined using sheaves of 1-morphisms and 2-morphisms in **SZC$^\infty$Rings** in the obvious way. All (2-category) fibre products exist in **dSpa**.
C^∞-schemes include into d-spaces as those $\boldsymbol{X}$ with $\mathcal{O}_X^{-1} = 0$.
Thus we have inclusions of (2-)categories **Man** $\subset$ **C$^\infty$Sch** $\subset$ **dSpa**, so manifolds are examples of d-spaces.
A d-space $\boldsymbol{X}$ has a *cotangent complex* $\mathbb{L}_{\boldsymbol{X}}^\bullet$, a 2-term complex $\mathbb{L}_{\boldsymbol{X}}^{-1} \xrightarrow{\mathrm{d}_{\boldsymbol{X}}} \mathbb{L}_{\boldsymbol{X}}^0$ of quasicoherent sheaves on $\underline{X}$. Such complexes form a 2-category $\mathrm{qcoh}^{[-1,0]}(\underline{X})$.

D-manifolds

A *d-manifold* $\boldsymbol{X}$ *of virtual dimension* $n \in \mathbb{Z}$ is a d-space $\boldsymbol{X}$ whose topological space X is Hausdorff and second countable, and such that $\boldsymbol{X}$ is covered by open d-subspaces $\boldsymbol{Y} \subset \boldsymbol{X}$ with equivalences $\boldsymbol{Y} \simeq U \times_{g,W,h} V$ in **dSpa**, where U, V, W are manifolds with $\dim U + \dim V - \dim W = n$, and $g : U \to W$, $h : V \to W$ are smooth maps, and $U \times_{g,W,h} V$ is the fibre product in the 2-category **dSpa**. (The 2-category structure is essential here.)
Write **dMan** for the full 2-subcategory of d-manifolds in **dSpa**.
Alternatively, we can write the local models as $\boldsymbol{Y} \simeq V \times_{0,E,s} V$, where V is a manifold, $E \to V$ a vector bundle, $s : V \to E$ a smooth section, and $n = \dim V - \operatorname{rank} E$.
We call a d-manifold $\boldsymbol{X}$ *affine* if it is equivalent in **dMan** to some $V \times_{0,E,s} V$. So any d-manifold has an open cover by affine d-manifolds.

Tangent and obstruction spaces of d-manifolds

Let $\boldsymbol{X}$ be a d-manifold, and $x \in \boldsymbol{X}$. Then we have the *tangent space* $T_x\boldsymbol{X}$ and *obstruction space* $O_x\boldsymbol{X}$, finite-dimensional real vector spaces with $\dim T_x\boldsymbol{X} - \dim O_x\boldsymbol{X} = \operatorname{vdim}\boldsymbol{X}$. The dual vector spaces are the *cotangent space* $T_x^*\boldsymbol{X}$ and *coobstruction space* $O_x^*\boldsymbol{X}$. If $\mathbb{L}_{\boldsymbol{X}} = [\mathbb{L}_{\boldsymbol{X}}^{-1} \xrightarrow{\mathrm{d}} \mathbb{L}_{\boldsymbol{X}}^0]$ is the cotangent complex of $\boldsymbol{X}$ as a d-space, we may define these by the exact sequence

$$0 \longrightarrow O_x^*\boldsymbol{X} \longrightarrow \mathbb{L}_{\boldsymbol{X}}^{-1}|_x \xrightarrow{\mathrm{d}|_x} \mathbb{L}_{\boldsymbol{X}}^0|_x \longrightarrow T_x^*\boldsymbol{X} \longrightarrow 0. \qquad (5.1)$$

If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in **dMan** and $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$, we have natural, functorial linear maps $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$ and $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$. If $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism in **dMan** then $T_x\boldsymbol{f} = T_x\boldsymbol{g}$ and $O_x\boldsymbol{f} = O_x\boldsymbol{g}$.
In contrast to schemes, most d-manifolds are affine:

Theorem

A d-manifold $\boldsymbol{X}$ is affine if and only if $\dim T_x\boldsymbol{X}$ is bounded for $x \in \boldsymbol{X}$. This holds automatically if $\boldsymbol{X}$ is compact.

D-manifolds with boundary, and with corners

I also define 2-categories $\mathbf{dMan^b}, \mathbf{dMan^c}$ of *d-manifolds with boundary* and *corners*, and orbifold versions $\mathbf{dOrb}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$, *d-orbifolds*, using Deligne–Mumford C^∞-stacks.
The currently available definition of d-manifolds with boundary and corners is not that pretty. In joint work with Kelli Francis-Staite, I am planning a different definition based on a new notion of C^∞*-ring with corners*, which is quite like log schemes in algebraic geometry. Including boundaries and corners in the (m-)Kuranishi picture is a lot easier.

6. The 2-category of m-Kuranishi neighbourhoods

We will define m-Kuranishi spaces, a weak 2-category of derived manifolds **mKur**, by an 'atlas of charts'. The charts themselves, *m-Kuranishi neighbourhoods*, form a strict 2-category, with objects, 1- and 2-morphisms.

Definition

Let X be a topological space. An *m-Kuranishi neighbourhood* on X is a quadruple (V, E, s, ψ) such that:

(a) V is a smooth manifold.
(b) $\pi : E \to V$ is a vector bundle over V, the *obstruction bundle*.
(c) $s \in C^\infty(E)$ is a smooth section of E, the *m-Kuranishi section*.
(d) ψ is a homeomorphism from $s^{-1}(0)$ to an open subset $\mathrm{Im}\,\psi$ in X, where $\mathrm{Im}\,\psi$ is called the *footprint* of (V, E, s, ψ).

If $S \subseteq X$ is open, we call (V, E, s, ψ) an *m-Kuranishi neighbourhood over* S if $S \subseteq \mathrm{Im}\,\psi \subseteq X$. We call (V, E, s, ψ) *global* if $S = X$.

Definition

Let $f : X \to Y$ be a continuous map of topological spaces, (V_i, E_i, s_i, ψ_i), (V_j, E_j, s_j, ψ_j) be m-Kuranishi neighbourhoods on X, Y, and $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\psi_j) \subseteq X$ be an open set. A *1-morphism* $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ *over* (S, f) is a triple $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat\phi_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.

(b) $\phi_{ij} : V_{ij} \to V_j$ is smooth.

(c) $\hat\phi_{ij} : E_i|_{V_{ij}} \to \phi_{ij}^*(E_j)$ is a morphism of vector bundles on V_{ij}, with $\hat\phi_{ij}(s_i|_{V_{ij}}) = \phi_{ij}^*(s_j) + O(s_i^2)$.

(d) Parts (b),(c) imply that ϕ_{ij} maps $s_i^{-1}(0) \cap V_{ij} \to s_j^{-1}(0)$. We require that $f \circ \psi_i = \psi_j \circ \phi_{ij}$ on $s_i^{-1}(0) \cap V_{ij}$.

Here the '$O(s_i^2)$' means that $\hat\phi_{ij}(s_i|_{V_{ij}}) - \phi_{ij}^*(s_j)$ lies in the square of the ideal of smooth functions generated by s_i.

We often take $f : X \to Y$ to be $\mathrm{id}_X : X \to X$, and then we call Φ_{ij} a 1-*morphism over* S.
The *identity* 1-*morphism* of (V_i, E_i, s_i, ψ_i) over id_X is

$$\mathrm{id}_{(V_i,E_i,s_i,\psi_i)} = (V_i, \mathrm{id}_{V_i}, \mathrm{id}_{E_i}).$$

Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps, and $T \subseteq Y$, $S \subseteq f^{-1}(T) \subseteq X$ be open, and $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat\phi_{ij}) : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ be a 1-morphism over (S, f), and $\Phi_{jk} = (V_{jk}, \phi_{jk}, \hat\phi_{jk}) : (V_j, E_j, s_j, \psi_j) \to (V_k, E_k, s_k, \psi_k)$ a 1-morphism over (T, g). The *composition* is

$$\Phi_{jk} \circ \Phi_{ij} = \bigl(V_{ij} \cap \phi_{ij}^{-1}(V_{jk}), \phi_{jk} \circ \phi_{ij}|_{\cdots}, \phi_{ij}^{-1}(\hat\phi_{jk}) \circ \hat\phi_{ij}|_{\cdots}\bigr) : (V_i, E_i, s_i, \psi_i) \longrightarrow (V_k, E_k, s_k, \psi_k).$$

Composition is strictly associative, and identities behave as expected, $\Phi_{ij} \circ \mathrm{id}_{(V_i,E_i,s_i,\psi_i)} = \mathrm{id}_{(V_j,E_j,s_j,\psi_j)} \circ \Phi_{ij} = \Phi_{ij}$.

Definition

Let $\Phi_{ij},\Phi'_{ij}:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ be 1-morphisms over (S,f) with $\Phi_{ij}=(V_{ij},\phi_{ij},\hat\phi_{ij}),\Phi'_{ij}=(V'_{ij},\phi'_{ij},\hat\phi'_{ij})$. Consider pairs $(\dot V_{ij},\hat\lambda_{ij})$ with open $\psi_i^{-1}(S)\subseteq\dot V_{ij}\subseteq V_{ij}\cap V'_{ij}$ open, and $\hat\lambda:E_i|_{\dot V_{ij}}\to\phi_{ij}^*(TV_j)|_{\dot V_{ij}}$ a morphism of vector bundles on $\dot V_{ij}$ satisfying $\phi'_{ij}=\phi_{ij}+\Lambda\circ s_i+O(s_i^2)$ and $\hat\phi'_{ij}=\hat\phi_{ij}+\phi_{ij}^*(\mathrm{d}s_j)\circ\Lambda+O(s_i)$ on $\dot V_{ij}$. Define an equivalence relation $\sim$ on such pairs by $(\dot V_{ij},\hat\lambda_{ij})\sim(\dot V'_{ij},\hat\lambda'_{ij})$ if there exists an open $\psi_i^{-1}(S)\subseteq\ddot V_{ij}\subseteq\dot V_{ij}\cap\dot V'_{ij}$ with $\hat\lambda_{ij}|_{\ddot V_{ij}}=\hat\lambda'_{ij}|_{\ddot V_{ij}}+O(s_i)$. Write $\Lambda_{ij}=[\dot V_{ij},\hat\lambda_{ij}]$ for the $\sim$-equivalence class of $(\dot V_{ij},\hat\lambda_{ij})$, and call $\Lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}$ a *2-morphism of m-Kuranishi neighbourhoods over* (S,f).

There are natural notions of vertical and horizontal composition of 2-morphisms, and identity 2-morphisms, which make (global) m-Kuranishi neighbourhoods into a *strict 2-category* **GmKN**. All 2-morphisms are invertible, so **GmKN** is a (2,1)-category.

Global m-Kuranishi neighbourhoods and affine d-manifolds

We can motivate the definitions above using d-manifolds. There is a natural strict 2-functor $F_{\mathbf{GmKN}}^{\mathbf{dMan}}:\mathbf{GmKN}\to\mathbf{dMan}$, which on objects takes a global m-Kuranishi neighbourhood (V,E,s,ψ) on X to the affine d-manifold $\mathrm{Spec}\,\mathfrak{C}^\bullet$, for $\mathfrak{C}^\bullet$ the square zero dg C^∞-ring defined from (V,E,s) in Example 4.1. For fixed objects, $F_{\mathbf{GmKN}}^{\mathbf{dMan}}$ is surjective on 1-morphisms, and a 1-1 correspondence on 2-morphisms. In fact, the correct definition of 2-morphism of m-Kuranishi neighbourhood was deduced from the definition of 2-morphisms of d-manifolds.
This 2-functor $F_{\mathbf{GmKN}}^{\mathbf{dMan}}$ is an equivalence from **GmKN** to the full 2-subcategory $\mathbf{dMan}^{\mathbf{aff}}\subset\mathbf{dMan}$ of affine d-manifolds.

Coordinate changes

For X a topological space and $S \subseteq X$ open, write $\mathbf{mKur}_S(X)$ for the strict 2-category of m-Kuranishi neighbourhoods and 1- and 2-morphisms over (S, id_X). We call a 1-morphism $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ in $\mathbf{mKur}_S(X)$ a *coordinate change* over S if it is an equivalence in $\mathbf{mKur}_S(X)$. That is, there should exist a 1-morphism $\Phi_{ji} : (V_j, E_j, s_j, \psi_j) \to (V_i, E_i, s_i, \psi_i)$ and 2-morphisms $\Lambda_{ii} : \Phi_{ji} \circ \Phi_{ij} \Rightarrow \mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$ and $\Lambda_{jj} : \Phi_{ij} \circ \Phi_{ji} \Rightarrow \mathrm{id}_{(V_j,E_j,s_j,\psi_j)}$. Then we have:

Theorem 6.1

A 1-morphism $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat\phi_{ij}) : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a coordinate change over S if and only if for all $x \in S$ with $v_i = \psi_i^{-1}(x)$ and $v_j = \psi_j^{-1}(x)$, the following sequence is exact:

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{\mathrm{d}s_i|_{v_i} \oplus T_{v_i}\phi_{ij}} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat\phi_{ij}|_{v_i} \oplus -\mathrm{d}s_j|_{v_j}} E_j|_{v_j} \longrightarrow 0.$$

The stack property of 1- and 2-morphisms

Theorem 6.2

Let $f : X \to Y$ be a continuous map, and (V_i, E_i, s_i, ψ_i), (V_j, E_j, s_j, ψ_j) be m-Kuranishi neighbourhoods on X, Y. For each open $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\psi_j)$, write $\mathcal{H}om_f\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)(S)$ for the groupoid of 1-morphisms $\Phi_{ij}, \Phi'_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ and 2-morphisms $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ over (S, f). For all open $T \subseteq S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\psi_j)$ define a functor

$$\rho_{ST} : \mathcal{H}om_f\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)(S) \longrightarrow \mathcal{H}om_f\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)(T)$$ *by restriction to T.*

Then $\mathcal{H}om_f\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)$ is a stack (the 2-category version of sheaf) on $\mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\psi_j)$.

This is not obvious. It means we can glue 1- and 2-morphisms of m-Kuranishi neighbourhoods over the sets of an open cover.

7. The 2-category of m-Kuranishi spaces

Definition 7.1

Let X be a Hausdorff, second countable topological space. An *m-Kuranishi structure* $\mathcal{K}$ on X of *virtual dimension* $n \in \mathbb{Z}$ is data $\mathcal{K} = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i \in I}, \Phi_{ij,\ i,j \in I}, \Lambda_{ijk,\ i,j,k \in I}\bigr)$, where:

(a) I is an indexing set.

(b) (V_i, E_i, s_i, ψ_i) is an m-Kuranishi neighbourhood on X for $i \in I$, with $\dim V_i - \operatorname{rank} E_i = n$. Write $S_{ij} = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, etc.

(c) $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a coordinate change over S_{ij} for $i, j \in I$.

(d) $\Lambda_{ijk} : \Phi_{jk} \circ \Phi_{ij} \Rightarrow \Phi_{ik}$ is a 2-morphism over S_{ijk} for $i, j, k \in I$.

(e) $\bigcup_{i \in I} \operatorname{Im}\psi_i = X$. (f) $\Phi_{ii} = \mathrm{id}_{(V_i, E_i, s_i, \psi_i)}$ for $i \in I$.

(g) $\Lambda_{iij} = \Lambda_{ijj} = \mathrm{id}_{\Phi_{ij}}$ for $i, j \in I$.

(h) $\Lambda_{ikl} \odot (\mathrm{id}_{\Phi_{kl}} * \Lambda_{ijk})|_{S_{ijkl}} = \Lambda_{ijl} \odot (\Lambda_{jkl} * \mathrm{id}_{\Phi_{ij}})|_{S_{ijkl}}$: $\Phi_{kl} \circ \Phi_{jk} \circ \Phi_{ij}|_{S_{ijkl}} \Longrightarrow \Phi_{il}|_{S_{ijkl}}$ for $i, j, k, l \in I$.

We call $\boldsymbol{X} = (X, \mathcal{K})$ an *m-Kuranishi space*, with $\operatorname{vdim} \boldsymbol{X} = n$.

This defines m-Kuranishi spaces using an 'atlas of charts', where the charts are m-Kuranishi neighbourhoods (V_i, E_i, s_i, ψ_i), with coordinate changes Φ_{ij} on double overlaps $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, and 2-isomorphisms $\Lambda_{ijk} : \Phi_{jk} \circ \Phi_{ij}|_{S_{ijk}} \Rightarrow \Phi_{ik}|_{S_{ijk}}$ on triple overlaps $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$, with associativity $\Lambda_{ikl} \odot (\mathrm{id}_{\Phi_{kl}} * \Lambda_{ijk})|_{S_{ijkl}} = \Lambda_{ijl} \odot (\Lambda_{jkl} * \mathrm{id}_{\Phi_{ij}})|_{S_{ijkl}}$ on quadruple overlaps $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k \cap \operatorname{Im}\psi_l$.

Once you have grasped the idea that m-Kuranishi neighbourhoods over $S \subseteq X$ form a 2-category, with restriction $|_T$ to open subsets $T \subseteq S \subseteq X$, Definition 7.1, although complicated, is obvious, and necessary: it is the only sensible way to make a global space by gluing local charts in the world of 2-categories.

Fukaya–Oh–Ohta–Ono define their Kuranishi spaces by a similar atlas of charts approach, but their coordinate changes only go in one direction, and they have no 2-morphisms.

Definition 7.2

Let $\boldsymbol{X}=(X,\mathcal{K})$ and $\boldsymbol{Y}=(Y,\mathcal{L})$ be m-Kuranishi spaces, with $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},\ \Phi_{ii',\,i,i'\in I},\ \Lambda_{ii'i'',\,i,i',i''\in I}\bigr)$ and $\mathcal{L}=\bigl(J,(W_j,F_j,t_j,\chi_j)_{j\in J},\ \Psi_{jj',\,j,j'\in J},\ \mathrm{M}_{jj'j'',\,j,j',j''\in J}\bigr)$. A *1-morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is $\boldsymbol{f}=\bigl(f,\boldsymbol{f}_{ij,\,i\in I,\,j\in J},\ F^{j,\,j\in J}_{ii',\,i,i'\in I},\ F^{jj',\,j,j'\in J}_{i,\,i\in I}\bigr)$, with: (a) $f:X\to Y$ is a continuous map.

(b) $\boldsymbol{f}_{ij}:(V_i,E_i,s_i,\psi_i)\to(W_j,F_j,t_j,\chi_j)$ is a 1-morphism of m-Kuranishi neighbourhoods over $S=\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f for $i\in I$, $j\in J$.

(c) $F^j_{ii'}:\boldsymbol{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\boldsymbol{f}_{ij}$ is a 2-morphism over f for $i,i'\in I$, $j\in J$.

(d) $F^{jj'}_i:\Psi_{jj'}\circ\boldsymbol{f}_{ij}\Rightarrow\boldsymbol{f}_{ij'}$ is a 2-morphism over f for $i\in I$, $j,j'\in J$.

(e) $F^j_{ii}=F^{jj}_i=\mathrm{id}_{\boldsymbol{f}_{ij}}$.

(f) $F^j_{ii''}\odot(\mathrm{id}_{\boldsymbol{f}_{i''j}}*\Lambda_{ii'i''})=F^j_{ii'}\odot(F^j_{i'i''}*\mathrm{id}_{\Phi_{ii'}}):\boldsymbol{f}_{i''j}\circ\Phi_{i'i''}\circ\Phi_{ii'}\Rightarrow\boldsymbol{f}_{i''j}$.

(g) $F^{jj'}_i\odot(\mathrm{id}_{\Psi_{jj'}}*F^j_{ii'})=F^{j'}_{ii'}\odot(F^{jj'}_{i'}*\mathrm{id}_{\Phi_{ii'}}):\Psi_{jj'}\circ\boldsymbol{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\boldsymbol{f}_{ij'}$.

(h) $F^{j'j''}_i\odot(\mathrm{id}_{\Psi_{j'j''}}*F^{jj'}_i)=F^{jj''}_i\odot(\mathrm{M}_{jj'j''}*\mathrm{id}_{\boldsymbol{f}_{ij}}):\Psi_{j'j''}\circ\Psi_{jj'}\circ\boldsymbol{f}_{ij}\Rightarrow\boldsymbol{f}_{ij''}$.

Here (c)–(h) hold for all $i,j,\ldots$, restricted to appropriate domains.

Definition 7.3

Let $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$ be 1-morphisms of m-Kuranishi spaces, with $\boldsymbol{f}=\bigl(f,\boldsymbol{f}_{ij,\,i\in I,\,j\in J},\ F^{j,\,j\in J}_{ii',\,i,i'\in I},\ F^{jj',\,j,j'\in J}_{i,\,i\in I}\bigr)$, $\boldsymbol{g}=\bigl(g,\boldsymbol{g}_{ij,\,i\in I,\,j\in J},\ G^{j,\,j\in J}_{ii',\,i,i'\in I},\ G^{jj',\,j,j'\in J}_{i,\,i\in I}\bigr)$. Suppose the continuous maps $f,g:X\to Y$ satisfy $f=g$. A *2-morphism* $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is data $\boldsymbol{\eta}=(\eta_{ij,\,i\in I,\,j\in J})$, where $\eta_{ij}:\boldsymbol{f}_{ij}\Rightarrow\boldsymbol{g}_{ij}$ is a 2-morphism of m-Kuranishi neighbourhoods over $f=g$, satisfying:

(a) $G^j_{ii'}\odot(\eta_{i'j}*\mathrm{id}_{\Phi_{ii'}})=\eta_{ij}\odot F^j_{ii'}:\boldsymbol{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\boldsymbol{g}_{ij}$ for $i,i'\in I$, $j\in J$.

(b) $G^{jj'}_i\odot(\mathrm{id}_{\Psi_{jj'}}*\eta_{ij})=\eta_{ij'}\odot F^{jj'}_i:\Psi_{jj'}\circ\boldsymbol{f}_{ij}\Rightarrow\boldsymbol{g}_{ij'}$ for $i\in I$, $j,j'\in J$.

We can then define composition of 1- and 2-morphisms, identity 1- and 2-morphisms, and so on, making m-Kuranishi spaces into a weak 2-category **mKur**. Composition of 1-morphisms needs the stack property of 1-morphisms, as in Theorem 6.2.

Composition of 1-morphisms in **mKur**

Suppose $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$, $\boldsymbol{g}:\boldsymbol{Y}\to\boldsymbol{Z}$ are 1-morphisms of m-Kuranishi spaces, where $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ have m-Kuranishi neighbourhoods $(V_i,E_i,s_i,\psi_i)_{i\in I}$, $(W_j,F_j,t_j,\chi_j)_{j\in J}$, $(W_k,F_k,t_k,\xi_k)_{k\in K}$. Consider how to define the composition $\boldsymbol{g}\circ\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Z}$ in **mKur**.

Now $\boldsymbol{g}\circ\boldsymbol{f}$ should contain 1-morphisms of m-Kuranishi neighbourhoods $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}:(V_i,E_i,s_i,\psi_i)\to(W_k,F_k,t_k,\xi_k)$ for all $i\in I$ and $k\in K$, defined over $S_{ik}=\mathrm{Im}\,\psi_i\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$. We do not have this. Instead, for each $j\in J$ we have 1-morphisms $\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}:(V_i,E_i,s_i,\psi_i)\to(W_k,F_k,t_k,\xi_k)$ defined over $S_{ijk}=\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j)\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$.

As the S_{ijk} for $j\in J$ cover S_{ik}, we can use Theorem 6.2 to show that there exists $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}$ defined over S_{ik} with 2-isomorphisms $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}|_{S_{ijk}}\cong\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}$ for all $j\in J$. This $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}$ is only natural up to 2-isomorphism. Thus $\boldsymbol{g}\circ\boldsymbol{f}$ is only natural up to 2-isomorphism in **mKur**, so **mKur** is a weak 2-category. FOOO can't compose morphisms, as they have no stack property.

- The 2-category equivalence $F_{\mathbf{GmKN}}^{\mathbf{dMan}}:\mathbf{GmKN}\to\mathbf{dMan}^{\mathbf{aff}}$ can be extended to an equivalence $F_{\mathbf{mKur}}^{\mathbf{dMan}}:\mathbf{mKur}\to\mathbf{dMan}$. Here **GmKN** is basically the full 2-subcategory of m-Kuranishi spaces $\boldsymbol{X}$ in **mKur** with only one m-Kuranishi neighbourhood.
- There is a full and faithful embedding $\mathbf{Man}\hookrightarrow\mathbf{mKur}$ taking a manifold X to the m-Kuranishi space $\boldsymbol{X}$ with topological space X and one m-Kuranishi neighbourhood $(X,0,0,\mathrm{id}_X)$, where $E\to X$ is the zero vector bundle.
- There is an orbifold version of this construction, *Kuranishi spaces*, a kind of derived orbifold. *Kuranishi neighbourhoods* (V,E,Γ,s,ψ) on X are a manifold V, a vector bundle $E\to V$, a finite group Γ acting on $V.E$, a Γ-equivariant section $s:V\to E$, and a homeomorphism $\psi:s^{-1}(0)/\Gamma\to X$ with an open subset $\mathrm{Im}\,\psi\subseteq X$. They form a 2-category **Kur**.
- We can generalize all this to (m-)Kuranishi spaces with boundary, or with corners, by taking V_i to be manifolds with boundary, or corners. (Not quite as simple as this.)

Tangent and obstruction spaces of m-Kuranishi spaces

As for d-manifolds, for an m-Kuranishi space $\boldsymbol{X}$ we can define the *tangent space* $T_x\boldsymbol{X}$ and *obstruction space* $O_x\boldsymbol{X}$ for any $x \in \boldsymbol{X}$, where if $\boldsymbol{X} = (X, \mathcal{K})$ with $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ii'},\, _{i,i'\in I},$ $\Lambda_{ii'i''},\, _{i,i',i''\in I}\big)$ and $x \in \operatorname{Im}\psi_i$ with $\psi_i^{-1}(x) = v_i \in s_i^{-1}(0) \subseteq V_i$ then as for (5.1) we have an exact sequence

$$0 \longrightarrow T_x\boldsymbol{X} \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}s_i|_{v_i}} E_i|_{v_i} \longrightarrow O_x\boldsymbol{X} \longrightarrow 0.$$

If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism of m-Kuranishi spaces we get functorial linear maps $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$ and $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$. If $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism then $T_x\boldsymbol{f} = T_x\boldsymbol{g}$ and $O_x\boldsymbol{f} = O_x\boldsymbol{g}$.

Theorem 7.4

(a) *An m-Kuranishi space $\boldsymbol{X}$ is a manifold iff $O_x\boldsymbol{X} = 0$ for all $x \in \boldsymbol{X}$.*
(b) *A 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of m-Kuranishi spaces is étale (a local equivalence) iff $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$ and $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$ are isomorphisms for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$. And $\boldsymbol{f}$ is an equivalence in* **mKur** *if also $f : X \to Y$ is a bijection.*

Although tangent and obstruction spaces $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ are only a small part of the data in a derived manifold $\boldsymbol{X}$, they tell you a lot, as Theorem 7.4 illustrates. One can often write (necessary and) sufficient conditions for things in terms of $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ and $T_x\boldsymbol{f}, O_x\boldsymbol{f}$. The same thing does *not* hold for general d-spaces.