Lagrangian Mean Curvature Flow in Calabi–Yau manifolds and the Thomas–Yau Conjecture. I

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These slides available at `http://people.maths.ox.ac.uk/~joyce/`.

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Plan of today's talk:

1. Introduction

A *Calabi–Yau m-fold* (M,J,g,Ω) is a compact Ricci-flat Kähler manifold of complex dimension m, with trivial canonical bundle. The Kähler form ω of g makes (M,ω) into a symplectic manifold. String Theorists conjectured that Calabi–Yau m-folds should come in mirror pairs (M,J,g,Ω), $(\check M,\check J,\check g,\check\Omega)$, where the complex geometry of (M,J,g,Ω) is somehow equivalent to the symplectic geometry of $(\check M,\check J,\check g,\check\Omega)$, and vice versa. In 1994, Kontsevich expressed this in the *Homological Mirror Symmetry Conjecture* as equivalences of triangulated categories:

$$D^b\,\mathrm{coh}(M,J)\simeq D^bF(\check M,\check\omega),\quad D^bF(M,\omega)\simeq D^b\,\mathrm{coh}(\check M,\check J). \quad (1)$$

Here $\mathrm{coh}(M,J)$ is the abelian category of coherent sheaves on (M,J), and $D^b\,\mathrm{coh}(M,J)$ its derived category, and $F(M,\omega)$ is the Fukaya category of Lagrangians in (M,ω), an A_∞-category, and $D^bF(M,\omega)$ is its derived category.

In 2002, motivated by ideas of String Theorists, Tom Bridgeland invented *Bridgeland stability conditions* on triangulated categories. This gives an extension of the HMS Conjecture (folklore):

- There should be a Bridgeland stability condition $\mathcal{S}_{B+i\omega}$ on $D^b\,\mathrm{coh}(M,J)$, depending on the 'complexified Kähler form' $B+i\omega$.
- There should be a Bridgeland stability condition $\mathcal{S}_\Omega$ on $D^bF(M,\omega)$, depending on the 'holomorphic volume form' Ω.
- The HMS equivalences (1) should identify $\mathcal{S}_{B+i\omega}\simeq\mathcal{S}_{\check\Omega}$ and $\mathcal{S}_\Omega\simeq\mathcal{S}_{\check B+i\check\omega}$.

These are not known — it is difficult to construct Bridgeland stability conditions on CY categories, particularly in high dimensions. Bridgeland stability conditions on $D^b\,\mathrm{coh}(M,J)$ are known to exist in dimensions 1,2, and in some special cases in dimension 3. So far as I understand, Bridgeland stability conditions on $D^bF(M,\omega)$ are not known to exist, except via mirror symmetry.

Bridgeland stability conditions

If $\mathcal{T}$ is a triangulated category, a *Bridgeland stability condition* $\mathcal{S} = (Z, P)$ on $\mathcal{T}$ assigns a 'central charge map' $Z : K^{\mathrm{num}}(\mathcal{T}) \to \mathbb{C}$, and for each $\phi \in \mathbb{R}$, a subcategory $P(\phi) \subset \mathcal{T}$ of 'semistable objects with phase ϕ', where if $0 \neq E \in P(\phi)$ then $Z([E]) \in e^{i\pi\phi} \cdot \mathbb{R}_{>0}$, such that every object in $\mathcal{T}$ is built uniquely out of a chain of semistable objects $E_1, \ldots, E_n$ via a kind of Harder–Narasimhan filtration.
Usually it is easy to write down Z, but difficult to construct the subcategories $P(\phi) \subset \mathcal{T}$.
For the conjectural Bridgeland stability condition $\mathcal{S}_\Omega$ on $D^bF(M, \omega)$, it is expected that the subcategories $P(\phi) \subset D^bF(M, \omega)$ should consist of '(graded) special Lagrangian m-folds with phase ϕ'.

(Lagrangian) Mean Curvature Flow

Let (M, g) be a Riemannian manifold, and $L_0 \subset M$ a compact submanifold. Then one can consider the *Mean Curvature Flow* (*MCF*) $L_t : t \in [0, \epsilon)$ of L_0, moving it in the direction of its mean curvature, decreasing its volume. Stationary points of the flow are minimal submanifolds. Finite time singularities can occur in the flow.
If (M, J, g, Ω) is a (Ricci-flat) Calabi–Yau m-fold and L_0 is a Lagrangian, then the L_t for $t \in [0, \epsilon)$ remain Lagrangian (Smoczyk). This is *Lagrangian Mean Curvature Flow* (*LMCF*). If L_0 is *graded*, or *Maslov zero*, then the flow stays in a fixed Hamiltonian isotopy class of Lagrangians. Stationary points of the flow are *special Lagrangian m-folds* (*SL m-folds*).
Lagrangian MCF also works in Kähler–Einstein manifolds.

The Thomas–Yau Conjecture, first attempt

In 2001, motivated by Mirror Symmetry, Thomas and Yau proposed:

Conjecture (Thomas–Yau Conjecture, informal version)

Let (M, J, g, Ω) be a Calabi–Yau m-fold, and L_0 a compact graded Lagrangian in (M, ω). There should be a notion of when L_0 is **stable**, *which Thomas and Yau attempt to define explicitly.*
If L_0 is stable then the LMCF $L_t : t \in [0, \infty)$ of L_0 exists for all time, and $L_t \to L_\infty$ as $t \to \infty$ for an SL m-fold L_∞, which is the unique SL m-fold in the Hamiltonian isotopy class of L_0.

This cannot be true in the precise form they stated it (which doesn't really make sense, because of mistakes inserted by Yau), but that is not the point. Their conjecture was prescient, as it pre-dates both Bridgeland stability (2002), and the definition of $D^bF(M, \omega)$ (2030?). They well knew their conjecture was only a first approximation. Our mission, should we choose to accept it, is to work out the correct conjecture, and then prove that (!).

Revising the Thomas–Yau Conjecture

The aim of these talks is to explain a revised version of the Thomas–Yau Conjecture, updated to include developments since 2001 in Bridgeland stability, Fukaya categories, and LMCF, that I believe has a chance of being true. The complete revised conjecture is fiendishly difficult, with difficulty increasing with dimension – I estimate the 3-dimensional version is about as hard as the Poincaré Conjecture, recently solved by Perelman. But the big picture suggests many smaller, more accessible problems.
Here are the main changes we make to the T–Y Conjecture:

- We should work in the derived Fukaya category $D^bF(M, \omega)$. Objects of $D^bF(M, \omega)$ are pairs (L, b), where L is a (graded) Lagrangian with *unobstructed Lagrangian Floer cohomology*, and b is a *bounding cochain* for L, in the sense of Fukaya–Oh–Ohta–Ono 2009. We should restrict our attention to Lagrangians with unobstructed HF^*.

Revising the Thomas–Yau Conjecture

- 'Stability of Lagrangians' is a Bridgeland stability condition $\mathcal{S}_\Omega$ on $D^bF(M,\omega)$, as in the extended HMS Conjecture. We should not define $\mathcal{S}_\Omega$ explicitly, as Thomas–Yau tried to do; the existence of $\mathcal{S}_\Omega$ is difficult, part of the conjecture.
- Finite time singularities of LMCF are unavoidable, as examples of Neves 2010 show. So our conjecture should concern long-time unique existence of LMCF $L_t : t \in [0,\infty)$ *with surgeries* at times $0 < T_1 < T_2 < \cdots$, in a similar way to Perelman's proof of the Poincaré Conjecture. That is, L_{T_n} is singular, and L_t for $t \in (T_n - \epsilon, T_n)$ and $t \in (T_n, T_n + \epsilon)$ may not be in the same Hamiltonian isotopy class, or even be diffeomorphic. However, L_t must remain in a fixed isomorphism class in $D^bF(M,\omega)$ for all t in $[0,\infty) \setminus \{T_1, T_2, \ldots\}$.

People should talk to each other . . .

In my experience, people that work in Lagrangian MCF often don't know much about the more complicated parts of symplectic topology — Lagrangian Floer cohomology, Fukaya categories, and so on — and people that work in symplectic topology often don't care much about Lagrangian MCF.
I suggest it might be a good idea if this changed.
In particular, I expect that a good understanding of Lagrangian Floer cohomology and Fukaya categories may be very useful to make the next advances in the field of LMCF.
As some evidence for this, see Imagi, Joyce and Oliveira dos Santos, Duke Math. J. 165 (2016), arXiv:1404.0271, which uses Fukaya category techniques to prove uniqueness for certain SL m-folds and LMCF expanders in $\mathbb{C}^m$, motivated by this Thomas–Yau picture.

2. Lagrangians in Calabi–Yau manifolds

Let (M, J, g, Ω) be a Calabi–Yau m-fold, with Kähler form ω. This means that (M, J) is a compact complex manifold of complex dimension m, and g is a Kähler metric on (M, J), and Ω is a holomorphic $(m, 0)$-form on M, which is compatible with ω by

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}. \qquad (2)$$

This implies that $|\Omega|_g = 2^{m/2}$, so Ω has constant length. The Ricci form of g is $\rho = \mathrm{dd}^c(\log |\Omega|_g) = 0$, so g is Ricci-flat. Also Ω is a nonvanishing holomorphic section of the canonical bundle $K_M = \Lambda^{m,0}M$, so $K_M \cong \mathcal{O}_M$ is trivial.

Yau's solution of the Calabi Conjecture shows that if (M, J) is a compact complex manifold admitting Kähler metrics with $K_M \cong \mathcal{O}_M$, then every Kähler class contains a unique Ricci-flat Kähler metric g. If Ω is a nonvanishing holomorphic $(m, 0)$-form on M then $|\Omega|_g$ is constant. Rescaling Ω to get $|\Omega|_g = 2^{m/2}$ gives a Calabi–Yau m-fold. This yields many examples of C–Y m-folds.

Now let L be a Lagrangian in (M, ω), that is, $L \hookrightarrow M$ is a submanifold (embedded or immersed) with $\dim_{\mathbb{R}} L = m$, such that $\omega|_L = 0$. If L is oriented it has a *volume form* vol_L, the unique m-form on L which is positive with respect to the orientation with length $|\mathrm{vol}_L|_g = 1$ w.r.t. the metric g.

As Ω is a complex m-form on M, the restriction $\Omega|_L$ is a complex m-form on L. When L is Lagrangian this has length 1, $|\Omega|_L|_g = 1$, so $\Omega|_L = \Phi \cdot \mathrm{vol}_L$ for some *phase function*

$$\Phi : L \longrightarrow \mathrm{U}(1) = \{z \in \mathbb{C} : |z| = 1\}.$$

This induces a map on cohomology $\Phi^* : \mathbb{Z} \cong H^1(\mathrm{U}(1); \mathbb{Z}) \to H^1(L; \mathbb{Z})$. The *Maslov class* of L is $\mu_L = \Phi^*(1) \in H^1(L; \mathbb{Z})$, and L is *Maslov zero* if $\mu_L = 0$. A *grading* of L is a smooth map $\phi : L \to \mathbb{R}$ with $\Phi = e^{i\pi\phi}$. Gradings exist if and only if L is Maslov zero, and are unique up to addition of $2\mathbb{Z}$.

The Maslov class and gradings are special features of Lagrangians in Calabi–Yau m-folds, which don't work in general symplectic manifolds. They will be important to us for three reasons:

- Special Lagrangians are Maslov zero/graded.
- Lagrangian MCF of a Maslov zero/graded Lagrangian L_0 stays in a fixed Hamiltonian isotopy class / isomorphism class in $D^bF(M,\omega)$. This fails for non-Maslov zero L_0.
- We define the Calabi–Yau Fukaya category $D^bF(M,\omega)$ to have objects *graded* Lagrangians (L,ϕ), plus bounding cochains b. This enables us to make $D^bF(M,\omega)$ into a $\mathbb{Z}$*-graded* triangulated category. For general symplectic manifolds (M,ω), $D^bF(M,\omega)$ is only $\mathbb{Z}_2$-graded. As $D^b\operatorname{coh}(\check M,\check J)$ is $\mathbb{Z}$-graded, we want $D^bF(M,\omega)$ to be $\mathbb{Z}$-graded for the HMS Conjecture to hold.
  Also we need $D^bF(M,\omega)$ to be $\mathbb{Z}$-graded for Bridgeland stability conditions on $D^bF(M,\omega)$ to make sense.

Special Lagrangian m-folds in Calabi–Yau m-folds

An oriented Lagrangian L in (M,J,g,Ω) is *special Lagrangian*, of *phase* $e^{i\pi\phi_0}$ for $\phi_0\in\mathbb{R}$, if $\Omega|_L=e^{i\pi\phi_0}\cdot\mathrm{vol}_L$, that is, the phase function $\Phi:L\to\mathrm{U}(1)$ is constant, $\Phi\equiv e^{i\pi\phi_0}$. Then L is Maslov zero, and (after choosing ϕ_0) has a natural grading $\phi:L\to\mathbb{R}$, the constant function $\phi\equiv\phi_0$. We call L an *SL m-fold*.
In Harvey–Lawson's calibrated geometry, special Lagrangians of phase $e^{i\pi\phi_0}$ are calibrated w.r.t. $\mathrm{Re}(e^{-i\pi\phi_0}\Omega)$ on (M,g), so they are minimal submanifolds in (M,g), and compact SL m-folds are volume-minimizing in their homology class.
Many examples of SL m-folds in $\mathbb{C}^m$ are known, including singular examples. Geometric Measure Theory tells us something about them – there are compactness results for special Lagrangian integral currents. They are important in String Theory, e.g. the SYZ Conjecture explains Mirror Symmetry in terms of fibrations of Calabi–Yau m-folds by SL m-folds, including singular fibres.

3. Lagrangian Floer cohomology and Fukaya categories

The next bit is complicated. For simplicity, I will start by telling some **lies**, and explain how to correct them later. Let (M,ω) be a symplectic manifold, and fix an almost complex structure J on M compatible with ω. We want to define the *Fukaya category* $F(M,\omega)$, an 'A_∞-category', whose objects are embedded Lagrangians L in (M,ω) (**a lie**), and whose morphisms $\mathrm{Hom}^\bullet(L,L')$ for L,L' transversely intersecting in M are complexes over a field $\mathbb{F}$ with basis points $p\in L\cap L'$, and differential $\mathrm{d}(p)=\sum_{q\in L\cap L'}N_{p,q}\cdot q$ (**a lie**), where $N_{p,q}$ in $\mathbb{Z}$ or $\mathbb{Q}$ is the 'number' of J-holomorphic discs Σ in M with boundary in $L\cup L'$ including p,q, of this kind:

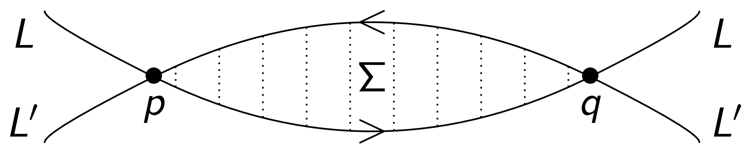

Figure 1: J-holomorphic disc Σ with boundary in $L\cup L'$.

(**Note: defining the 'number' $N_{p,q}$ is tricky.**)

Then for $p,r\in L\cap L'$ we have $\sum_{q\in L\cap L'}N_{p,q}\cdot N_{q,r}=0$ (**a lie**), so that $\mathrm{d}^2=0$ and $\mathrm{Hom}^\bullet(L,L')$ is a complex. The *Lagrangian Floer cohomology* is $HF^*(L,L')=H^*\bigl(\mathrm{Hom}^\bullet(L,L')\bigr)$.

As with all Floer theories, the motivation for $HF^*(L,L')$ is an infinite-dimensional Morse homology. Write $\mathrm{Path}(L,L')$ for the infinite-dimensional manifold of smooth maps $\gamma:[0,1]\to M$ with $\gamma(0)\in L$ and $\gamma(1)\in L'$. Then one defines a functional $F:\mathrm{Path}(L,L')\to\mathbb{R}$ (**a lie**) whose critical points are constant maps γ – hence, intersection points $p\in L\cap L'$ – and whose gradient flow lines from a critical point p to a critical point q correspond to J-holomorphic curves Σ as in Figure 1.

The usual Morse homology argument suggests that the complex $\mathrm{Hom}^\bullet(L,L')$ defined in this way should have $\mathrm{d}^2=0$ – see later.

The *derived Fukaya category* $D^bF(M,\omega)$ is defined from the A_∞-category $F(M,\omega)$ by an algebraic process; objects of $D^bF(M,\omega)$ are 'twisted complexes' of Lagrangians (objects in $F(M,\omega)$), but include single Lagrangians L, which are the objects we are interested in. Then $D^bF(M,\omega)$ is a triangulated category with $\mathrm{Hom}_{D^bF(M,\omega)}(L,L'[k])\cong HF^k(L,L')$.
It is independent of the choice of J up to equivalence.
The idea of the HMS Conjecture is to construct a category $D^bF(M,\omega)$ which could plausibly be equivalent to $D^b\,\mathrm{coh}(\check M,\check J)$, so we want it to be a $\mathbb{Z}$-graded triangulated category over $\mathbb{C}$ with finite-dimensional Hom spaces and Serre duality in dimension m. It is surprising that this is possible.
Note: as far as I know, there isn't yet a full write up of $D^bF(M,\omega)$ available in the literature in the generality we need. Fukaya–Oh–Ohta–Ono's 2009 HF^* book is a good start.

Next we explain (some of) the lies, and how to correct them:
• **Use oriented, graded Lagrangians** (L,ϕ).
We take the Lagrangians L,L' to be oriented, with gradings ϕ,ϕ'. Then each transverse intersection point $p\in L\cap L'$ has a *Maslov index* $\mu(p)\in\mathbb{Z}$, and $\mathrm{Hom}^k(L,L')$ is the $\mathbb{F}$-vector space spanned by $p\in L\cap L'$ with $\mu(p)=k$. This is essential to make $D^bF(M,\omega)$ a $\mathbb{Z}$-graded triangulated category. Without gradings, $\mathrm{Hom}^\bullet(L,L')$ and $D^bF(M,\omega)$ would only be $\mathbb{Z}_2$-graded.
• **Form moduli spaces $\bar{\mathcal{M}}(p,q)$ of J-holomorphic discs.**
To 'count' J-holomorphic discs Σ in Figure 1, we need to construct a moduli space $\bar{\mathcal{M}}(p,q)$ of them, which is a 'Kuranishi space with corners' – a kind of 'derived orbifold with corners' – and compact (**a lie**), and then define a 'virtual chain' for $\bar{\mathcal{M}}(p,q)$ in homology. The virtual dimension is $\mathrm{vdim}\,\bar{\mathcal{M}}(p,q)=\mu(q)-\mu(p)-1$. We can only 'count' those with virtual dimension 0 (so generically we expect $\bar{\mathcal{M}}(p,q)$ to be finitely many points). So $\mu(q)=\mu(p)+1$, which is why d maps $\mathrm{Hom}^k(L,L')\to\mathrm{Hom}^{k+1}(L,L')$.

• **Orienting moduli spaces $\bar{\mathcal{M}}(p,q)$.**
To count J-holomorphic curves Σ with signs, we need orientations on the $\bar{\mathcal{M}}(p,q)$. To define these, we need to choose orientations and spin structures on all Lagrangians L.

• **Why we hope (wrongly) that $\mathrm{d}^2=0$ in $\mathrm{Hom}^\bullet(L,L')$.**
The moduli spaces $\bar{\mathcal{M}}(p,q)$ have boundaries. In good cases we have

$$\partial\bar{\mathcal{M}}(p,r)\cong\textstyle\coprod_{q\in L\cap L'}\bar{\mathcal{M}}(p,q)\times\bar{\mathcal{M}}(q,r). \qquad (3)$$

Let us pretend that all $\bar{\mathcal{M}}(p,q)$ are compact oriented manifolds of dimension $\mu(q)-\mu(p)-1$, and empty if $\mu(q)-\mu(p)-1<0$. Take p,r with $\mu(r)=\mu(p)+2$, so $\dim\bar{\mathcal{M}}(p,r)=1$. Then the number of boundary points of $\bar{\mathcal{M}}(p,r)$, counted with signs, is 0, so $\sum_{q\in L\cap L'}\#(\bar{\mathcal{M}}(p,q)\times\bar{\mathcal{M}}(q,r))=\sum_{q\in L\cap L'}N_{p,q}\cdot N_{q,r}=0$.
This is what we need to show that $\mathrm{d}^2=0$ in $\mathrm{Hom}^\bullet(L,L')$.
It is what we would expect from the infinite-dimensional Morse homology picture.

• **Why $\mathrm{d}^2\neq 0$ in $\mathrm{Hom}^\bullet(L,L')$.**
In general the boundary formula (3) for $\bar{\mathcal{M}}(p,r)$ is incomplete. There are two extra terms, the first from curves like this:

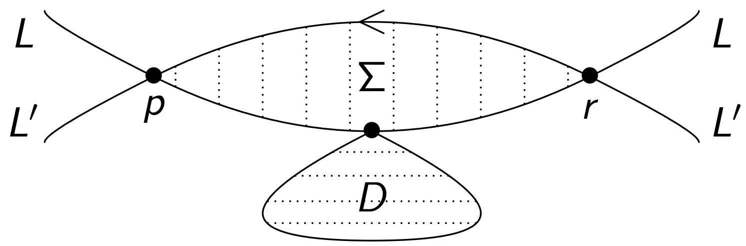

Figure 2: bubbling off a disc D with boundary in L.

in which a J-holomorphic disc D in M with $\partial D\subset L$ 'bubbles off' from Σ, and the second from discs D' with $\partial D'\subset L'$ bubbling off in the same way. These extra terms cause $\mathrm{d}^2\neq 0$, as a component of $\bar{\mathcal{M}}(p,r)$ can be an interval with a point in $\bar{\mathcal{M}}(p,q)\times\bar{\mathcal{M}}(q,r)$ at one end and a bubbled disc at the other, and then $\mathrm{d}^2p=\pm r+\cdots$.
This is only a problem if there exist J-holomorphic discs with boundary in L or L'. If we work with exact Lagrangians in noncompact (M,ω) there are none, hence Seidel's simpler theory.

• **How to fix** $\mathrm{d}^2 \neq 0$ **in** $\mathrm{Hom}^\bullet(L, L')$**.**
Fukaya–Oh–Ohta–Ono define a notion of 'bounding cochains' b, b' for L, L', which are chains $b \in C_{m-1}(L; \mathbb{F})$ in homology, satisfying

$$\partial b = \bigcup_{\Sigma\ J\text{-holomorphic disc},\ \partial\Sigma \subset L} \partial\Sigma + \text{higher order terms} \qquad (4)$$

(**a lie**). They then modify $\mathrm{d} : \mathrm{Hom}^k(L, L') \to \mathrm{Hom}^{k+1}(L, L')$ using b, b' to get $\mathrm{d}^{b,b'}$ with $(\mathrm{d}^{b,b'})^2 = 0$.
Such b need not exist. We say that L has *unobstructed* HF^* if some such b exists, and *obstructed* HF^* otherwise. I claim that this will be an important condition in LMCF.
Thus Lagrangian Floer cohomology is $HF^*\big((L, b), (L', b')\big)$, and objects of $D^bF(M, \omega)$ which are single Lagrangians should be pairs (L, b) where L has unobstructed HF^*.
Lagrangians L with obstructed HF^* do not appear as objects in $D^bF(M, \omega)$, and String Theory does not know about them.

This business of HF^* unobstructed/obstructed and bounding cochains is important, but is not widely known I think – most people working with Fukaya categories use Seidel's simpler exact version, in which it does not arise.
It is necessary for HMS $D^bF(M, \omega) \simeq D^b \operatorname{coh}(\check{M}, \check{J})$ to work.
Moduli of objects in $D^b \operatorname{coh}(\check{M}, \check{J})$ are generally *singular* schemes or Artin stacks. If objects in $D^bF(M, \omega)$ were Lagrangians L up to Hamiltonian isotopy, then moduli of objects in $D^bF(M, \omega)$ would be smooth manifolds of dimension $b^1(L)$, and so could not be isomorphic to singular moduli in $D^b \operatorname{coh}(\check{M}, \check{J})$. But moduli of pairs (L, b) can be singular schemes, as the moduli of solutions to (4) are singular.

• Formal power series and the Novikov ring.

In general there may be *infinitely many* J-holomorphic discs $\Sigma_1, \Sigma_2, \ldots$ as in Figure 1, where $\mathrm{area}(\Sigma_i) \to \infty$ as $i \to \infty$, that is, the moduli space $\bar{\mathcal{M}}(p,q)$ is noncompact. So the 'number' $N_{p,q}$ does not make sense. However, if we fix the area of Σ then the number is finite. So we consider moduli spaces $\bar{\mathcal{M}}(p,q)^A$ of curves Σ with area $A \geqslant 0$, which are compact. We take the field $\mathbb{F}$ to be a 'Novikov ring' of formal Laurent power series $\sum_i c_i t^{A_i}$, where $c_i \in \mathbb{Q}$ or $\mathbb{C}$, and $A_i \in \mathbb{R}$ with $A_i \to +\infty$ as $i \to \infty$. We define $\mathrm{d} : \mathrm{Hom}^k(L,L') \to \mathrm{Hom}^{k+1}(L,L')$ by

$$\mathrm{d}(p) = \sum_{q \in L \cap L',\ \mu(q)=\mu(p)+1,\ A \geqslant 0} N_{p,q}^A t^A \cdot q,$$

where $N_{p,q}^A$ is the 'number' of curves Σ in Figure 1 with area A. This is an infinite but convergent sum in the topology on $\mathbb{F}$.

Hamiltonian isotopy of Lagrangians and HF^*, $D^bF(M,\omega)$

In a symplectic manifold (M,ω), *Hamiltonian isotopy* is a kind of deformation of compact, embedded Lagrangians L in M, moving them by the Hamiltonian flow of a function $H : M \to \mathbb{R}$. It gives an important equivalence relation on Lagrangians L, such that moduli of Lagrangians L up to Hamiltonian isotopy is locally a smooth manifold of dimension $b^1(L)$.

Lagrangian Floer cohomology has invariance properties under Hamiltonian isotopy. If L_1, L_1' are Hamiltonian isotopic, then for any bounding cochain b_1 for L_1 there is a corresponding bounding cochain b_1' for L_1' such that

$$HF^*\big((L_1,b_1),(L_2,b_2)\big) \cong HF^*\big((L_1',b_1'),(L_2,b_2)\big)$$

for any (L_2,b_2). Also $(L_1,b_1),(L_1',b_1')$ are isomorphic in $D^bF(M,\omega)$. This is used to prove many theorems in symplectic topology. We will see tomorrow that Lagrangian MCF L_t, $t \in [0,T)$ moves graded Lagrangians L_t by Hamiltonian isotopy, so stays within a fixed isomorphism class in $D^bF(M,\omega)$.