Lecture 5: More constructions of vertex algebras

Lecture 5 of 8: More constructions of vertex algebras

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References for this lecture: Frenkel and Ben-Zvi, 2nd ed. (2004), §5, R. Borcherds, *Quantum vertex algebras*, math.QA/9903038.

These slides available at
`http://people.maths.ox.ac.uk/~joyce/`

Plan of talk:

5.1. The coset and BRST constructions
The coset construction

Definition 5.1 (Frenkel–Zhu 1992.)

Let $(V, \mathbb{1}, e^{zD}, Y)$ be a vertex algebra over R, and $W \subset V$ be an R-submodule. Write $C(V, W)$ for the R-submodule of V spanned by $v \in V$ such that $Y(w, z)v \in V[[z]]$ for all $w \in W$.

This implies that if $v \in C(V, W)$ and $w \in W$ then

$$Y(v, z_1) \circ Y(w, z_2) = Y(w, z_2) \circ Y(v, z_1).$$

Hence if $v_1, v_2 \in C(V, W)$ then

$$\begin{aligned}\bigl(Y(v_1, z_1)\circ Y(v_2, z_2)\bigr) \circ Y(w, z_3) &= Y(v_1, z_1)\circ Y(w, z_3)\circ Y(v_2, z_2)\\ &= Y(w, z_3) \circ \bigl(Y(v_1, z_1) \circ Y(v_2, z_2)\bigr).\end{aligned}$$

Using this we deduce that $Y(v_1, z_1) \circ Y(v_2, z_2)\mathbb{1} \in C(V, W)[[z_1^{\pm 1}, z_2^{\pm 1}]]$, so $C(V, W)$ is a vertex subalgebra of V, called a *coset vertex algebra*.

Example 5.2

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$, with non-degenerate inner product $\langle\,,\,\rangle$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a reductive Lie subalgebra, with $\langle\,,\,\rangle|_{\mathfrak{h}\times\mathfrak{h}}$ nondegenerate. Generalizing Example 3.10, we have a nontrivial central extension of Lie algebras defined using $\langle\,,\,\rangle$

$$0 \longrightarrow \langle c\rangle_{\mathbb{C}} \longrightarrow \hat{\mathfrak{g}} \longrightarrow \mathfrak{g}[t,t^{-1}] \longrightarrow 0,$$

and for $k \in \mathbb{C}$ we can define an induced representation $V_k^{\hat{\mathfrak{g}}}$ of $\hat{\mathfrak{g}}$ and make it into a vertex algebra using the Reconstruction Theorem. For non-critical k this is a VOA with conformal vector $\omega_{\mathfrak{g}}$.

There is a natural inclusion of vertex algebras $V_k^{\hat{\mathfrak{h}}} \hookrightarrow V_k^{\hat{\mathfrak{g}}}$. Then $C(V_k^{\hat{\mathfrak{g}}}, V_k^{\hat{\mathfrak{h}}})$ is a coset vertex algebra. For non-critical k it is a VOA with conformal vector $\omega_{\mathfrak{g}/\mathfrak{h}} = \omega_{\mathfrak{g}} - \omega_{\mathfrak{h}}$.

The BRST construction

Lemma 5.3

Let $(V, \mathbb{1}, e^{zD}, Y)$ be a vertex algebra over R. Then for all $u, v \in V$ we have

$$u_0(\mathbb{1}) = 0, \quad u_0(D^{(k)}(v)) = D^{(k)}(u_0(v)), \quad [u_0, Y(v,z)] = Y(u_0(v), z).$$

Hence $u_0 : V \to V$ is an infinitesimal automorphism of the vertex algebra structure.

Definition 5.4

Let $(V, \mathbb{1}, e^{zD}, Y)$ be a vertex algebra over R with an additional compatible $\mathbb{Z}$-grading $V = \bigoplus_{n \in \mathbb{Z}} V^n$. Suppose $Q \in V^1$ with $Q_0 \circ Q_0 = 0$. Then we can use Lemma 5.3 to show that $\mathrm{Ker}\, Q_0$ is a vertex subalgebra of V, and $\mathrm{Im}\, Q_0$ is an ideal in $\mathrm{Ker}\, Q_0$. Thus the cohomology $\mathrm{Ker}\, Q_0 / \mathrm{Im}\, Q_0$ of Q_0 is a vertex algebra. Also $(V^0 \cap \mathrm{Ker}\, Q_0)/(V^0 \cap \mathrm{Im}\, Q_0)$ is a vertex subalgebra of this.

The BRST construction comes up in several places. In Physics, it is related to quantizing field theories with gauge symmetries. Lian–Zuckerman define *topological vertex operator algebras*, which are graded VOAs $(V^*, \mathbb{1}, e^{zD}, Y)$ with elements $Q, F, G \in V^*$ satisfying some identities. The BRST VOA $H^*(Q_0)$ has the additional structure of a Batalin–Vilkovisky algebra. The BRST construction is used to define $\mathcal{W}$-algebras, an important class of vertex algebras.

5.2. The Borcherds bicharacter construction

Definition 5.5

Let $\mathbb{K}$ be a field of characteristic zero. Let $(V,\cdot,\Delta,\mathbb{1},\epsilon)$ be a commutative, cocommutative bialgebra over $\mathbb{K}$, with product $\cdot : V\times V\to V$, coproduct $\Delta : V\to V\otimes V$, unit $\mathbb{1}$ and counit $\epsilon : V\to\mathbb{K}$. We use the notation $\Delta v=\sum_v v'\otimes v''$, which is short for $\Delta v=\sum_{i\in I} v_i'\otimes v_i''$ for some finite set I and $v_i',v_i''\in V$.
Let $D : V\to V$ be a *derivation* of $(V,\cdot,\Delta,\mathbb{1},\epsilon)$. That is, D is $\mathbb{K}$-linear with $D(u\cdot v)=(Du)\cdot v+u\cdot(Dv)$ and $D\circ\Delta v=\sum_v(D(v')\otimes v''+v'\otimes D(v''))$.
A *bicharacter* for $(V,\cdot,\Delta,\mathbb{1},\epsilon),D$ is a $\mathbb{K}$-bilinear map $r_z : V\times V\to\mathbb{K}[z,z^{-1}]$ satisfying:

(i) $r_z(u,v)=r_{-z}(v,u)$;
(ii) $r_z(u,\mathbb{1})=\epsilon(u)$;
(iii) $r_z(u\cdot v,w)=\sum_w r_z(u,w')r_z(v,w'')$; and
(iv) $r_z(u,Dv)=-r_z(Du,v)=\frac{\mathrm{d}}{\mathrm{d}z}r_z(u,v)$.

Definition 5.5 (Continued.)

Define $Y : V \otimes V \to V[[z]][z^{-1}]$ by

$$Y(u,z)v = \sum_{n\geqslant 0} \frac{z^n}{n!} \sum_u \sum_v r_z(u',v')(D^n u'') \cdot v''. \tag{5.1}$$

Theorem 5.6 (Borcherds 1999.)

In Definition 5.5, $(V, \mathbb{1}, e^{zD}, Y)$ is a vertex algebra over $\mathbb{K}$.

There are also super- and graded versions: we take $(V_*, \cdot, \Delta, \mathbb{1}, \epsilon)$ to be super- or graded (co)commutative, and D graded of degree 2, and $r : V \times V \to \mathbb{K}[z, z^{-1}]$ to be graded with $\deg z = -2$.

Sketch proof of Theorem 5.6

Let $u, v, w \in V$. Write $(\Delta \times \mathrm{id}_V) \circ \Delta u = \sum_u u' \otimes u'' \otimes u'''$, and similarly for v, w. Consider the element of the meromorphic function space $V[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}]$:

$$\sum_{m,n \geqslant 0} \frac{z_1^m z_2^n}{m!n!} \sum_u \sum_v \sum_w r_{z_1}(u', w') r_{z_2}(v', w'') r_{z_1 - z_2}(u'', v'') \quad (5.2)$$
$$(D^m u''') \cdot (D^n v''') \cdot w''.$$

(This is $X_3(z_1, z_2, 0)(u \otimes v \otimes w)$ in the notation of §2.3.) We can prove that all of

$$Y(u, z_1) \circ Y(v, z_2) w, \quad Y(v, z_2) \circ Y(u, z_1) w,$$
$$Y(Y(u, z_1 - z_2) v, z_2) w$$

are power series expansions of (5.2) using $i_{z_1, z_2}, i_{z_2, z_1}, i_{z_1 - z_2, z_2}$. Thus we can deduce that Y satisfies weak commutativity and weak associativity, the main vertex algebra axioms.

The bicharacter construction is a good place to start for thinking about *generalizations* of vertex algebras. For example, if we omit the condition that $r_z(u,v)=r_{-z}(v,u)$ we get a noncommutative generalization of vertex algebras called *nonlocal vertex algebras* or *field algebras*. There are also 'deformed' notions of vertex algebra in which r_{z_1,z_2} depends on two variables z_1,z_2.

Example 5.7

For any $(V,\cdot,\Delta,\mathbb{1},\epsilon),D$, define $r_z:V\times V\to\mathbb{K}\subset\mathbb{K}[z,z^{-1}]$ by $r_z(u,v)=\epsilon(u)\epsilon(v)$. This is a bicharacter (the *trivial bicharacter*), and $(V,\mathbb{1},e^{zD},Y)$ is the commutative vertex algebra defined from $(V,\cdot,\mathbb{1}),D$ in §1.3, and is independent of the coproduct Δ.

In general we can think of $(V,\mathbb{1},e^{zD},Y)$ as obtained from the commutative vertex algebra from $(V,\cdot,\mathbb{1}),D$ by 'twisting' the vertex algebra operation Y by r_z, making it noncommutative.

Example 5.8

The Heisenberg vertex algebra $V=\mathbb{C}[x_1,x_2,\ldots]$ comes from the bicharacter construction by giving V the obvious algebra structure, coproduct generated by $\Delta(x_n)=x_n\otimes\mathbb{1}+\mathbb{1}\otimes x_n$, and derivation generated by $D(x_n)=nx_{n+1}$. One can show there is a unique bicharacter $r_z:V\times V\to\mathbb{C}[z^{\pm1}]$ with $r_z(x_1,x_1)=z^{-2}$, and that the bicharacter construction yields the Heisenberg vertex algebra. I believe r_z has the form, for $c_{m,n}\in\mathbb{Q}$ I didn't bother to compute

$$r_z(p,q)=\left\{\exp\Bigl(\sum_{m,n\geqslant 0}c_{m,n}z^{-m-n}\frac{\partial^2}{\partial x_m\partial x'_n}\Bigr)\bigl(p(\boldsymbol{x})q(\boldsymbol{x}')\bigr)\right\}\Big|_{\substack{x_i=x'_j=0,\\ \text{all } i,j.}}$$

In general, bicharacters tend to be complicated to write down explicitly in examples.

Example 5.9

As in Example 3.11, let (Λ, χ) be an even lattice, and choose signs $\epsilon_{\lambda,\mu}$ satisfying (3.19). Then we defined a lattice vertex algebra $V_\Lambda = \mathbb{C}[\Lambda] \otimes_{\mathbb{C}} \mathrm{Sym}(t\Lambda_{\mathbb{C}}[t])$, spanned over $\mathbb{C}$ by elements

$$e^{\lambda_0} \otimes (t^{a_1}\lambda_1) \otimes \cdots \otimes (t^{a_n}\lambda_n).$$

It comes from the bicharacter construction. Here V_Λ is a commutative $\mathbb{C}$-algebra in an obvious way. The coproduct Δ and derivation D are generated by

$$\Delta(e^\lambda \otimes 1) = (e^\lambda \otimes 1) \otimes (e^\lambda \otimes 1),$$

$$\Delta(e^0 \otimes (t^a\lambda)) = (e^0 \otimes (t^a\lambda)) \otimes (e^0 \otimes 1) + (e^0 \otimes 1) \otimes (e^0 \otimes (t^a\lambda)),$$

$$D(e^\lambda \otimes 1) = e^\lambda \otimes (t\lambda), \quad D(e^0 \otimes (t^a\lambda)) = a\, e^0 \otimes (t^{a+1}\lambda).$$

There is a unique bicharacter r_z with

$$r_z(e^\lambda \otimes 1, e^\mu \otimes 1) = \epsilon_{\lambda,\mu} z^{\chi(\lambda,\mu)}.$$

Then the bicharacter construction recovers the vertex algebra V_Λ.

Example 5.10 (Joyce 2018–2030?)

Let $\mathcal{A}$ be an abelian or derived category coming from algebraic geometry, e.g. $\mathrm{coh}(X), D^b\,\mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme, or mod-$\mathbb{C}Q$, D^bmod-$\mathbb{C}Q$ for Q a quiver. Let $\mathcal{M}$ be the moduli stack of objects in $\mathcal{A}$, and R a commutative ring. After choosing a bit of extra data, there is a method to make the homology $H_*(\mathcal{M}, R)$ into a graded vertex algebra over R (with shifted grading). The construction can be interpreted as a variant of the bicharacter construction, where $H_*(\mathcal{M}, R)$ is a bialgebra with product from pushforward along the direct sum morphism $\bigoplus : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, and coproduct coming from pushforward along the diagonal map $\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$. The Borcherds Lie algebra constructed from the vertex algebra $H_*(\mathcal{M}, R)$ is important in wall-crossing formulae for enumerative invariants counting τ-semistable objects in $\mathcal{A}$ (e.g. algebraic Donaldson invariants of complex surfaces, Donaldson–Thomas invariants) when these exist (only for special $\mathcal{A}$).

5.3. The orbifold construction

Let $(V, \mathbb{1}, e^{zD}, Y)$ be a vertex algebra over $\mathbb{C}$, and G a finite group acting on V by isomorphisms. Write $R_0, \ldots, R_k$ for the simple G-representations, with R_0 trivial. Then we may decompose

$$V = \bigoplus_{i=0}^{k} V_i \otimes R_i, \tag{5.3}$$

where $V_0 = V^G$ is the G-invariant subspace. Clearly V^G is a vertex subalgebra of V. The rough idea of the *orbifold construction* is to build another vertex algebra

$$V^{\mathrm{orb}} = \bigoplus_{[g] \in G/G} V_{[g]}, \tag{5.4}$$

where the sum is over conjugacy classes $[g]$ in G, with $V_{[1]} = V^G$ as a vertex subalgebra. The $V_{[g]}$ for $[g] \neq [1]$ are *twisted sectors*. As far as I understand, the construction is conjectural in general, but can be proved to work in some examples. Usually one takes $G = \mathbb{Z}_n$, e.g. $G = \mathbb{Z}_2$. The theory of characters gives a kind of duality between representations R_i and conjugacy classes $[g]$.

The motivation comes from Physics. Consider the lattice VOA V_Λ from an even positive definite lattice (Λ, χ), with $\Lambda \cong \mathbb{Z}^d$. Then we may define a torus $\Lambda_{\mathbb{R}}/\Lambda \cong T^d$, where $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, and χ induces a flat Riemannian metric g on T^d. The VOA should be physically interpreted as 'quantizing' maps $\Sigma \to T^d$ for Riemann surfaces Σ. We can pretend V_Λ is like a 'homology group' of a loop space $V_\Lambda \approx H_*(\mathrm{Map}(\mathcal{S}^1, T^d), \mathbb{C})$, where $\mathcal{S}^1$ appears as the boundary of Riemann surfaces Σ. Now $H_1(T^d, \mathbb{Z}) \cong \Lambda$, and we have a decomposition $\mathrm{Map}(\mathcal{S}^1, T^d) = \coprod_{\lambda\in\Lambda} \mathrm{Map}(\mathcal{S}^1, T^d)_\lambda$, where $\mathrm{Map}(\mathcal{S}^1, T^d)_\lambda$ is the loops with homology class λ. Thus we expect

$$V_\Lambda \approx \textstyle\bigoplus_{\lambda\in\Lambda} H_*(\mathrm{Map}(\mathcal{S}^1, T^d)_\lambda, \mathbb{C}),$$

corresponding to the decomposition $V_\Lambda = \bigoplus_{\lambda\in\Lambda} e^\lambda \otimes \mathrm{Sym}(t\Lambda_{\mathbb{C}}[t])$. Now let G be a finite group acting by automorphisms of (Λ, χ) and preserving the $\epsilon_{\lambda,\mu}$. Then G acts by automorphisms of V_Λ. We want to make an 'orbifold vertex algebra' V_Λ^{orb} 'quantizing' maps $\Sigma \to T^d/\Gamma$ to the quotient orbifold T^d/Γ.

Heuristically we expect that $V_\Lambda^{\rm orb}\approx H_*(\mathrm{Map}(\mathcal{S}^1,T^d/G),\mathbb{C})$. Roughly we have

$$\mathrm{Map}(\mathcal{S}^1,T^d/G)=\coprod_{[g]\in G/G}\mathrm{Map}(\mathcal{S}^1,T^d/G)_{[g]},$$

where $\mathrm{Map}(\mathcal{S}^1,T^d/G)_{[g]}$ is the set of loops $\gamma:\mathcal{S}^1\to T^d/G$ that lift to $\tilde\gamma:[0,2\pi]\to T^d$ with $\tilde\gamma(2\pi)=g\cdot\tilde\gamma(0)$ for $g\in G$. Then

$$\mathrm{Map}(\mathcal{S}^1,T^d/G)_{[1]}=\mathrm{Map}(\mathcal{S}^1,T^d)/G.$$

So we expect $V_\Lambda^{\rm orb}=\bigoplus_{[g]\in G/G}V_{[g]}$ as in (5.4) with

$$V_{[g]}\approx H_*(\mathrm{Map}(\mathcal{S}^1,T^d/G)_{[g]},\mathbb{C}),$$

where

$$V_{[1]}\approx H_*(\mathrm{Map}(\mathcal{S}^1,T^d)/G,\mathbb{C})=H_*(\mathrm{Map}(\mathcal{S}^1,T^d),\mathbb{C})^G\approx(V_\Lambda)^G.$$

How can we make all this rigorous?
Suppose the initial vertex algebra $(V, \mathbb{1}, e^{zD}, Y)$ is a holomorphic VOA, that is, it is rational with one simple V-module, which is V itself. Then modulo some assumptions, we can use a G-twisted version of the representation theory of Lecture 4 to construct natural candidates for the twisted sectors $V_{[g]}$ for $[g] \in G/G$, with $V_{[1]} = V^G$. Here each $V_{[g]}$ is a V^G-module. Thus, we can construct the vector space $V^{\mathrm{orb}} = \bigoplus_{[g]\in G/G} V_{[g]}$, and the vertex operators $Y(v,z) : V^{\mathrm{orb}} \to V^{\mathrm{orb}}[[z]][z^{-1}]$ for $v \in V^G$. Also $\mathbb{1} \in V^G$, and the conformal element $\omega \in V^G$ gives us $D = \omega_0 : V^{\mathrm{orb}} \to V^{\mathrm{orb}}$. The missing structure is the vertex operators $Y(v,z)$ for $v \in V^{\mathrm{orb}} \setminus V^G$, which I believe have to be defined by hand for now.

Here is more on how to construct the twisted sectors $V_{[g]}$ when $G=\mathbb{Z}_n$.

Definition 5.11

Let $(V,\mathbb{1},e^{zD},Y)$ be a vertex algebra over $\mathbb{C}$, and $\sigma:V\rightarrow V$ be an automorphism of V with $\sigma^n=\mathrm{id}$. A *σ-twisted V-module* (W,Y^σ) is a $\mathbb{C}$-vector space W and a linear map $Y^\sigma:V\otimes W\rightarrow W[[z^{1/n}]][z^{-1/n}]$ satisfying:

(i) $Y^\sigma(\mathbb{1},z)=\mathrm{id}_W$.

(ii) For all $u,v\in V$ and $w\in W$, in $W[[z_0^{\pm 1},z_1^{\pm 1},z_2^{\pm 1}]]$ we have

$$\frac{1}{n}z_2^{-1}\sum_{m=0}^{n-1}\delta\Big(e^{2\pi i m/n}\Big(\frac{z_1-z_0}{z_2}\Big)^{1/n}\Big)Y^\sigma(Y(u,z_0)v,z_2)w= \tag{5.5}$$

$$z_0^{-1}\delta\Big(\frac{z_1-z_2}{z_0}\Big)Y^\sigma(u,z_1)Y^\sigma(v,z_2)w-z_0^{-1}\delta\Big(\frac{z_2-z_1}{-z_0}\Big)Y^\sigma(v,z_2)Y^\sigma(u,z_1)w.$$

(iii) If $v\in V$ with $\sigma(v)=e^{2\pi i m/n}v$ then $Y^\sigma(v,z)$ maps $W\mapsto z^{-m/n}W[[z]][z^{-1}]$.

This notion of σ-twisted V-module is special to vertex algebras – there is no analogue for representations of groups, algebras, ... If v lies in the σ-fixed vertex subalgebra $V^{\langle\sigma\rangle}$ then (iii) with $m=0$ shows that $Y^\sigma(v,z)$ maps $W \mapsto W[[z]][z^{-1}]$, so restricting to $V^{\langle\sigma\rangle}$ we see that W is an ordinary $V^{\langle\sigma\rangle}$-module.

Conjecture 5.12 (Frenkel–Ben-Zvi §5.7.)

Let V be a holomorphic VOA with an action of $G=\mathbb{Z}_n$. Then:

(a) *For each $\sigma\in G$ there is a unique simple σ-twisted V-module V_σ, with $V_1=V$.*

(b) *V_σ has a natural G-action, so $V_\sigma=\bigoplus_{m=0}^{n-1} V_\sigma^m\otimes R_m$, where R_m is the simple G-representation with eigenvalue $e^{2\pi i m/n}$.*

(c) *$V^{\mathrm{orb}}=\bigoplus_{\sigma\in G} V_\sigma^0$ has a canonical holomorphic VOA structure, including $V_1^0=V^{\langle\sigma\rangle}$ as a vertex subalgebra.*

An important example of this construction is the *Monster vertex algebra* constructed by Frenkel–Lepowsky–Meurmann 1988. Let (Λ, χ) be the *Leech lattice*, a positive definite, even, unimodular lattice of rank 24, and let $G = \mathbb{Z}_2$ act on Λ by multiplication by ± 1. Then FLM construct an orbifold vertex operator algebra V^{orb}, whose automorphism group $\mathrm{Aut}(V^{\mathrm{orb}})$ is the Monster simple group. I'll tell you more about this next time.

Vertex Algebras

Lecture 6 of 8: Monstrous Moonshine

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References for this lecture: T. Gannon, *Moonshine beyond the Monster*, 2006, plus surveys by Goddard (ICM 1998) and Borcherds. V. Kac, *Infinite-dimensional Lie algebras*, 1983.

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`

Plan of talk:

Introduction

The classification of simple Lie algebras has four infinite series A_n, B_n, C_n, D_n plus five exceptional cases G_2, F_4, E_6, E_7, E_8, the largest being E_8 with dimension 248.

Similarly, the (much more difficult) classification of finite simple groups has about 20 infinite families plus 26 exceptional cases, called the sporadic simple groups. The largest of these is the *Monster group* M, of order circa 8×10^{53}, discovered by Fischer and Griess in the 1970's. The smallest irreducible representations of M have dimensions $1, 196883, 21296876, \ldots,$ (there are 194 irreducible representations in all) and Griess constructed M as the automorphism group of a nonassociative product on $\mathbb{C}^{196883}$.

The elliptic modular function $j(\tau)$ and numerology

The *elliptic modular function* $j(\tau)$, beloved by number theorists, is a holomorphic map $j : \mathcal{H} \to \mathbb{C}$, where $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$, that satisfies modular invariance $j(\tau) = j(\tau + 1) = j(-1/\tau)$. It descends to $\mathcal{H}/\mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}$, where $\mathcal{H}/\mathrm{SL}(2,\mathbb{Z})$ is the moduli space of elliptic curves. In terms of $q = e^{2\pi i\tau}$ it has expansion

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots ,$$

and apart from the constant term 744 this determines j uniquely. John McKay noticed that

$$1 = 1, \qquad 196884 = 196883 + 1,$$

$$21493760 = 21296876 + 196883 + 1, \qquad \ldots,$$

where the numbers on the left are coefficients of $j(\tau)$ and the numbers on the right are dimensions of irreducible representations of M. Conway's term *Monstrous Moonshine* refers to various extensions of McKay's observation, and to connections between the Monster (and other sporadic simple groups) and modular functions.

McKay and Thompson suggested that there should be a 'natural' graded representation $V_* = \bigoplus_{n\in\mathbb{N}} V_n$ of the Monster such that $\sum_{n\geqslant 0} q^{n-1} \dim V_n = j(\tau) - 744$. Conway–Norton also conjectured number-theoretically significant values ('Hauptmoduls') for the McKay–Thompson series $T_{[g]}(\tau) = \sum_{n\geqslant 0} q^{n-1}\,\mathrm{Tr}(g|_{V_n})$ for the 194 conjugacy classes $[g]$ in M, where $T_{[1]}(\tau) = j(\tau) - 744$.

Frenkel–Lepowsky–Meurmann constructed a graded representation V_* of M with $\sum_{n\geqslant 0} q^{n-1} \dim V_n = j(\tau) - 744$. This V_* carried a complicated algebraic structure preserved by M.

Borcherds invented vertex algebras to describe this structure, and proved the Conway–Norton Conjectures. So, we have the *Monster VOA* $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$, a holomorphic VOA whose automorphism group is M. In modern language, FLM constructed $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ as an orbifold VOA from the lattice VOA of the Leech lattice (Λ, χ) by $G = \mathbb{Z}_2$ acting by multiplication by ± 1 on Λ.

Today I want to tell you a bit about this story, because I think it is cool, but to start by educating you (and me) a little bit about the theory of infinite-dimensional algebras and their representations, as it is central to Moonshine, and hopefully will be a useful thing for you to take away from the lecture. In particular, I want to ask:

- There are unimaginably huge numbers of infinite-dimensional Lie algebras. What makes certain special infinite-dimensional Lie algebras (e.g. the affine Lie algebras) 'interesting' in the eyes of representation theorists, that they get so much attention?
- Why do modular functions (and generalizations) occur as characters in the theory of (special) infinite-dimensional Lie algebras? (Actually I don't think I can answer this one.)

6.1. Infinite-dimensional Lie algebras and representations

As I understand it (e.g. in the work of Kac), the starting point for thinking about infinite-dimensional representation theory is the well-known theory of finite-dimensional semisimple Lie algebras and their representations (Dynkin diagrams, Cartan matrices, Weyl groups, root systems, highest weight representations, etc.). We choose some package of properties in the finite-dimensional case that we want to extend to infinite dimensions, including some big theorems (e.g. the Weyl character formula), and look for some large class of infinite-dimensional Lie algebras for which this package holds. Sometimes we find new phenomena which didn't occur in the finite-dimensional case (e.g. the existence of 'imaginary roots'), and then we have to adjust the theory to include it.

Properties of finite-dimensional semisimple Lie algebras

Here are properties of finite-dimensional semisimple Lie algebras $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$ that we want to extend to the infinite-dimensional case:
(i) $\mathfrak{g}$ admits a *Cartan decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, where $\mathfrak{h}$ is a finite-dimensional Cartan subalgebra, a maximal abelian Lie subalgebra, and $\Delta \subset \mathfrak{h}^* \setminus \{0\}$ is a set of *roots*, and $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is a finite-dimensional *root space*, the α-eigenspace of $[\mathfrak{h}, -]$ on $\mathfrak{g}$. For finite-dimensional $\mathfrak{g}$ we have $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in \Delta$.
(ii) We can split $\Delta = \Delta_+ \amalg \Delta_-$ such that $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Delta_\pm} \mathfrak{g}_\alpha$ are Lie subalgebras with $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Also $\Delta_- = \{-\alpha : \alpha \in \Delta_+\}$.
(iii) There is a *Cartan involution* $\omega : \mathfrak{g} \to \mathfrak{g}$, a Lie algebra isomorphism with $\omega^2 = 1$, $\omega|_{\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}$, $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, $\omega(\mathfrak{n}_\pm) = \mathfrak{n}_\mp$.
(iv) There is subset of *simple roots* $\{\alpha_1, \ldots, \alpha_r\} = \Delta^{\mathrm{si}} \subset \Delta_+$, a basis for $\mathfrak{h}^*$, where $r = \dim \mathfrak{h} = \operatorname{rank} \mathfrak{g}$, such that if $\beta \in \Delta_+$ then $\beta = \sum_{i=1}^r b_i\alpha_i$ with $b_i \in \mathbb{N}$.
(v) There is a nonsingular invariant bilinear form $(\, , \,)$ on $\mathfrak{g}$ (the Killing form) which restricts to a nonsingular form $(\, , \,)|_{\mathfrak{h}\times\mathfrak{h}}$ on $\mathfrak{h}$.

(vi) The inverse of $(\,,\,)|_{\mathfrak{h}\times\mathfrak{h}}$ gives an inner product on $\mathfrak{h}^*$, and so on the simple roots $\alpha_1,\ldots,\alpha_r$. Writing $a_{ij}=(\alpha_i,\alpha_j)$, we have

$$a_{ii}>0,\;\; a_{ij}=a_{ji},\;\; a_{ij}\leqslant 0 \text{ if } i\neq j, \text{ and } 2a_{ij}/a_{ii}\in\mathbb{Z}. \tag{6.1}$$

We call $A=(a_{ij})_{i,j=1}^r$ the *Cartan matrix* of $\mathfrak{g}$. When $\mathfrak{g}$ is finite-dimensional, A is positive definite.

(vii) We can reconstruct $\mathfrak{g}$ from A by generators e_i,f_i,h_i for $i=1,\ldots,r$, subject to the relations

$$\begin{aligned}&[e_i,f_i]=h_i,\;\; [e_i,f_i]=0,\;\; i\neq j,\;\; [h_i,e_j]=a_{ij}e_j,\;\; [h_i,f_j]=-a_{ij}f_j,\\ &\mathrm{Ad}(e_i)^{n_{ij}}(e_j)=\mathrm{Ad}(f_i)^{n_{ij}}(f_j)=0,\;\; n_{ij}=1-2a_{ij}/a_{ii}.\end{aligned} \tag{6.2}$$

(viii) The *Weyl group* W of $\mathfrak{g}$ is the group of automorphisms of $\mathfrak{h},(\,,\,)|_{\mathfrak{h}\times\mathfrak{h}}$ generated by reflections in roots $\alpha\in\Delta$. It preserves Δ.

(ix) The *fundamental weights* $\omega_1,\ldots,\omega_r\in\mathfrak{h}^*$ are characterized by $2(\omega_i,\alpha_j)=\delta_{ij}(\alpha_j,\alpha_j)$. The *weight lattice* is $\Lambda=\langle\omega_1,\ldots,\omega_r\rangle_{\mathbb{Z}}\subset\mathfrak{h}^*$.

(x) For each $\lambda \in \Lambda$, let $\mathfrak{h} \oplus \mathfrak{n}_-$ act on $\mathbb{C} = \mathbb{C}_\lambda$ such that $\mathfrak{h}$ has eigenvalue λ and $\mathfrak{n}_-$ acts as 0. Write $W_\lambda = \mathbb{C}_\lambda \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_-)} U(\mathfrak{g})$ for the induced (infinite-dimensional) $\mathfrak{g}$-representation, a *Verma module*. Define V_λ to be the quotient of W_λ by its maximal proper submodule. If λ is *dominant* then V_λ is finite-dimensional, a *highest weight representation*. Any irreducible finite-dimensional $\mathfrak{g}$-representation is isomorphic to V_λ for some dominant λ.

(xi) Let V be a finite-dimensional representation of $\mathfrak{g}$. For $h \in \mathfrak{h}$, the *character* of ρ is $\mathrm{ch}_V : \mathfrak{h} \to \mathbb{C}$, $\mathrm{ch}_V(h) = \mathrm{Trace}_V(e^h)$.
If $V = V_\lambda$ is a highest weight representation, the *Weyl character formula* says that

$$\mathrm{ch}_{V_\lambda}(h) = \frac{\sum_{w\in W} \epsilon(w) e^{(\rho+\lambda)(w(h))}}{e^{\rho(h)} \prod_{\alpha\in\Delta_+}(1-e^{-\alpha(h)})}, \tag{6.3}$$

where $\epsilon(w) = \det w : \mathfrak{h} \to \mathfrak{h}$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha = \sum_{i=1}^r \omega_i$ is the *Weyl vector*. When $\lambda = 0$ we have $V_0 = \mathbb{C}$ and $\mathrm{ch}_{V_0} \equiv 1$, so (6.3) reduces to the *Weyl denominator formula*

$$\sum_{w\in W} \epsilon(w) e^{\rho(w(h))} = e^{\rho(h)} \prod_{\alpha\in\Delta_+} (1-e^{-\alpha(h)}). \tag{6.4}$$

6.2. (Generalized) Kac–Moody algebras

It turns out that there is a large class of infinite-dimensional Lie algebras called (*symmetrizable*) *Kac–Moody algebras* for which slightly rewritten versions of (i)–(xi) hold. Some differences are:

- The set of roots Δ and Weyl group W are generally infinite.
- The Cartan matrix A need not be positive definite. If A is positive definite then $\mathfrak{g}$ is finite-dimensional semisimple. If A is positive semi-definite then $\mathfrak{g}$ is a sum of semisimple and infinite-dimensional affine Lie algebras.
- Root spaces $\mathfrak{g}_\alpha$ have $\dim \mathfrak{g}_\alpha = m_\alpha \geqslant 1$, not just $m_\alpha = 1$.
- Simple roots $\alpha_1, \ldots, \alpha_r$ have positive norm (real roots), but other $\alpha \in \Delta_+$ can have $(\alpha, \alpha) = 0$ (null) or $(\alpha, \alpha) < 0$ (imaginary).
- We define a 'category $\mathcal{O}$' of $\mathfrak{g}$-representations, generally infinite-dimensional, which have splittings $V = \bigoplus_{\mu \in \Lambda} V_\mu$, where V_μ is the μ-eigenspace of the action of $\mathfrak{h}$, which must be finite-dimensional. Irreducible representations in category $\mathcal{O}$ are highest weight representations V_λ. The character $\mathrm{ch}_V(h) = \mathrm{Trace}_V(e^h)$ makes sense as a formal sum.

- The *Weyl–Kac character formula* generalizing (6.3) says that

$$\mathrm{ch}_{V_\lambda}(h)=\frac{\sum_{w\in W}\epsilon(w)e^{(\rho+\lambda)(w(h))}}{e^{\rho(h)}\prod_{\alpha\in\Delta_+}(1-e^{-\alpha(h)})^{m_\alpha}}. \tag{6.5}$$

- The *Weyl–Kac denominator formula* generalizing (6.4) says that

$$\sum_{w\in W}\epsilon(w)e^{\rho(w(h))}=e^{\rho(h)}\prod_{\alpha\in\Delta_+}(1-e^{-\alpha(h)})^{m_\alpha}. \tag{6.6}$$

Here (6.6) says an infinite product is an infinite sum, and leads to nontrivial interesting identities. I believe (6.5)–(6.6) are the source of modular functions appearing as characters of representations. For example, an affine Lie algebra $\hat{\mathfrak{g}}$ has Weyl group $W=W_0\ltimes\Lambda^\vee$, where W_0 is the (finite) Weyl group and Λ the weight lattice of the semisimple Lie algebra $\mathfrak{g}$. The roots Δ of $\hat{\mathfrak{g}}$ are $\Delta=((\Delta_0\amalg\{0\})\times\mathbb{Z})\setminus\{(0,0)\}$ for Δ_0 the roots of $\mathfrak{g}$. So both sides of (6.6) can be written explicitly.

Generalized Kac–Moody algebras

Borcherds' proof of the Moonshine Conjectures involved two infinite-dimensional Lie algebras, the 'Fake Monster Lie algebra' and the 'Monster Lie algebra', constructed from vertex algebras. The theory of Kac–Moody algebras turned out not to be general enough to include these, so Borcherds extended it, to 'generalized Kac–Moody algebras' (see Borcherds, J. Algebra 1988).

These satisfy analogues of (i)–(xi) above, with the important difference that we no longer require the simple roots $\alpha_1, \ldots, \alpha_r$ to have positive square norm, that is, we drop the condition $a_{ii} > 0$ in (6.1), and the condition $2a_{ij}/a_{ii} \in \mathbb{Z}$ in (6.1) is only required to hold if $a_{ii} > 0$. We allow $r = \infty$, and do not require the α_i to be a basis of $\mathfrak{h}^*$. In (6.2) the condition $\mathrm{Ad}(e_i)^{n_{ij}}(e_j) = \mathrm{Ad}(f_i)^{n_{ij}}(f_j) = 0$ need hold only if $a_{ii} > 0$, and we add the condition that $[e_i, e_j] = [f_i, f_j] = 0$ if $a_{ij} = 0$. There is a generalization of (6.5)–(6.6) to the GKM case, with corrections from imaginary simple roots.

Borcherds also gave useful sufficient conditions for recognizing when a Lie algebra (e.g. a Lie algebra constructed from a vertex algebra) is a generalized Kac–Moody algebra.

Theorem 6.1 (Borcherds 1988.)

Suppose $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is a Lie algebra graded over $\mathbb{Z}$ such that

(i) $\dim \mathfrak{g}_n < \infty$ *if* $n \neq 0$.
(ii) $\mathfrak{g}$ *has an invariant bilinear form* $(\,,\,)$ *with* $\mathfrak{g}_m \perp \mathfrak{g}_n$ *if* $m \neq -n$.
(iii) $\mathfrak{g}$ *has an involution* $\omega : \mathfrak{g} \to \mathfrak{g}$ *which maps* $\mathfrak{g}_n \mapsto \mathfrak{g}_{-n}$ *and is* -1 *on* $\mathfrak{g}_0$.
(iv) *The bilinear form* $\langle g, h\rangle = -(g, \omega(h))$ *is positive definite on* $\mathfrak{g}_n$ *for* $n \neq 0$.

Let $\mathfrak{k}$ be the kernel of $(\,,\,)$. Then $\mathfrak{k}$ is in the centre of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{k}$ is a generalized Kac–Moody algebra.

Basically this says that a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ with an 'almost positive definite' form is a GKM algebra.

6.3. GKMs from vertex operator algebras

As far as I understand it, Borcherds' proof of the Moonshine Conjectures had the following rough form:

- Start with a suitable vertex operator algebra $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$, e.g. the Monster VOA.
- Construct a Lie algebra $\mathcal{L}(V_*)$ acting on V_*.
- Use the 'no-ghost theorem' from String Theory to show that $\mathcal{L}(V_*)$ has an 'almost positive definite' invariant form.
- Use Theorem 6.1 to recognize that $\mathcal{L}(V_*)$ is a generalized Kac–Moody algebra.
- Use the Borcherds–Weyl–Kac character and denominator formulae for $\mathcal{L}(V_*)$ to deduce stuff, e.g. determine sets of roots Δ and their multiplicities, show characters $T_{[g]}(\tau) = \sum_{n\geqslant 0} q^{n-1}\,\mathrm{Tr}(g|_{V_n})$ of V_* are the conjectured modular-type functions.

Recall from §1.4 that if V_* is a graded VA then $V_2/D(V_0)$ is a Lie algebra, with Lie bracket $\bigl[u+D(V_0), v+D(V_0)\bigr] = u_0(v)+D(V)$. Here we want a smaller Lie algebra, defined using the VOA structure.

Definition 6.2 (Borcherds 1986.)

Let $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ be a VOA with central charge c_{V_*}. Then $L_n = \omega_{n+1} : V_* \to V_*$ defines an action of the Virasoro algebra on V_*. Define $P^i(V_*) = \bigl\{v \in V_i : L_n(v) = 0 \text{ if } n \geqslant 1\bigr\}$, the space of *physical states*. Borcherds shows that $P^2(V_*)/D(P^0(V_*))$ is also a Lie algebra, with Lie bracket

$$\bigl[u + D(P^0(V_*)), v + D(P^0(V_*))\bigr] = u_0(v) + D(P^0(V_*)),$$

with the obvious Lie algebra morphism to $V_2/D(V_0)$.

Now suppose $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ is a lattice VOA from a Lorentizian lattice (Λ, χ) of signature $(N, 1)$ for $N < 26$. Then $P^2(V_*)/D(P^0(V_*))$ has a natural invariant form $(\,,\,)$ as V_* is a lattice VOA, and the 'no-ghost theorem' (proved using the representation theory of the Virasoro algebra with central charge $c_{V_*} = N + 1 \leqslant 26$) says that $(\,,\,)$ is positive semi-definite on $P^2(V_*)$, with lots of null states. So we can quotient $\mathfrak{g} = P^2(V_*)/D(P^0(V_*))$ by the subspace $\mathfrak{k}$ of null states to obtain a smaller Lie algebra $\mathcal{L}(V_*) = \mathfrak{g}/\mathfrak{k}$. We can apply Theorem 6.1 to show that $\mathfrak{k}$ is in the centre of $\mathfrak{g}$ and $\mathcal{L}(V_*)$ is a generalized Kac–Moody algebra, so $P^2(V_*)/D(P^0(V_*))$ is a central extension of a GKM algebra.
When $(\Lambda, \chi) = \mathrm{II}_{25,1}$ is the unique even unimodular lattice of signature $(25, 1)$, $\mathcal{L}(V_*)$ is called the 'Fake Monster Lie algebra' $\mathcal{L}'_M$. Its set of simple roots and root multiplicities are known.

One may write $\mathrm{II}_{25,1} = \Lambda_{\mathrm{Leech}} \oplus \mathrm{II}_{1,1}$, where Λ_{Leech} is the rank 24 Leech lattice, which is even positive definite and unimodular, and $\mathrm{II}_{1,1} \cong \mathbb{Z}^2$ is the hyperbolic lattice. Then the real simple roots of $\mathcal{L}'_M$ are of the form $(\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ for $\lambda \in \Lambda_{\mathrm{Leech}}$, the null simple roots are $(0,0,n)$ for $n > 0$, and there are no imaginary simple roots. The denominator identity for $(0,\sigma,\tau) \in \mathrm{II}_{25,1} \otimes \mathbb{C}$ becomes

$$p^{-1} \prod_{m>0} (1-p^m)^{24} \prod_{m>0,\ n\in\mathbb{Z}} (1-p^m q^n)^{c(mn)} = \Delta(\sigma)\Delta(\tau)(j(\sigma)-j(\tau)), \tag{6.7}$$

where $p = e^{2\pi i\sigma}$, $q = e^{2\pi i\tau}$ and $\Delta(\tau) = q\prod_{n>1}(1-q^n)^{24}$, and $c(n) = \dim(V_{\mathrm{Mon}})_{2n+2}$.

The presence of $j(\sigma)$ in (6.7) suggests a connection to the Moonshine Conjectures. This motivated Borcherds to construct the 'real' Monster Lie algebra $\mathcal{L}_M$, a GKM, whose denominator identity for $(0,\sigma,\tau)$ multiplies (6.7) by $\Delta(\sigma)^{-1}\Delta(\tau)^{-1}$ to get

$$p^{-1} \prod_{m>0,\ n\in\mathbb{Z}} (1-p^m q^n)^{c(mn)} = j(\sigma)-j(\tau). \tag{6.8}$$

Borcherds defined $\mathcal{L}_M$ to be $\mathcal{L}(V_{\mathrm{Mon}} \otimes V_{\mathrm{II}_{1,1}})$, where V_{Mon} is Frenkel–Lepowsky–Meurmann's Monster vertex algebra constructed as a $\mathbb{Z}_2$-orbifold of $V_{\Lambda_{\mathrm{Leech}}}$ as in §5.3. For comparison, $\mathcal{L}'_M$ is $\mathcal{L}(V_{\Lambda_{\mathrm{Leech}}} \otimes V_{\mathrm{II}_{1,1}})$ since $\mathrm{II}_{25,1} = \Lambda_{\mathrm{Leech}} \oplus \mathrm{II}_{1,1}$.

We regard $\mathcal{L}_M$ as graded over $\mathrm{II}_{1,1} = \mathbb{Z}^2$. Then the 'no-ghost theorem' implies that $(\mathcal{L}_M)_{m,n} \cong (V_{\mathrm{Mon}})_{2mn+2}$ as representations of the Monster group M. Borcherds then used twisted versions of the Borcherds–Weyl–Kac denominator identity for $\mathcal{L}_M$ to prove the Moonshine Conjectures.