D-manifolds and derived differential geometry

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September 2014

Based on survey paper:
arXiv:1206.4207, 44 pages
and preliminary version of book which may be downloaded from
people.maths.ox.ac.uk/∼joyce/dmanifolds.html.
These slides available at
people.maths.ox.ac.uk/∼joyce/talks.html.

Plan of talk:

1. Introduction

I will tell you about new classes of geometric objects I call *d-manifolds* and *d-orbifolds* — 'derived' smooth manifolds, in the sense of Derived Algebraic Geometry. Some properties:

- D-manifolds form a *strict 2-category* **dMan**. That is, we have objects $\mathbf{X}$, the d-manifolds, 1-morphisms $\mathbf{f}, \mathbf{g} : \mathbf{X} \to \mathbf{Y}$, the smooth maps, and also 2-morphisms $\eta : \mathbf{f} \Rightarrow \mathbf{g}$.
- Smooth manifolds **Man** embed into d-manifolds as a full (2)-subcategory. So, d-manifolds generalize manifolds.
- There are also 2-categories $\mathbf{dMan^b}$, $\mathbf{dMan^c}$ of d-manifolds *with boundary* and *with corners*, and orbifold versions $\mathbf{dOrb}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$ of these, *d-orbifolds*.
- Much of differential geometry extends nicely to d-manifolds: submersions, immersions, orientations, submanifolds, transverse fibre products, cotangent bundles, ...

- Almost any moduli space used in any enumerative invariant problem over $\mathbb{R}$ or $\mathbb{C}$ has a d-manifold or d-orbifold structure, natural up to equivalence. There are truncation functors to d-manifolds and d-orbifolds from structures currently used — Kuranishi spaces, polyfolds, $\mathbb{C}$-schemes or Deligne–Mumford $\mathbb{C}$-stacks with obstruction theories.
  Combining these truncation functors with known results gives d-manifold/d-orbifold structures on many moduli spaces.
- Virtual classes/cycles/chains can be constructed for compact oriented d-manifolds and d-orbifolds.

So, d-manifolds and d-orbifolds provide a unified framework for studying enumerative invariants and moduli spaces. They also have other applications, and are interesting for their own sake.

Origins in derived algebraic geometry

D-manifolds are based on ideas from *derived algebraic geometry*. First consider an algebro-geometric version of what we want to do. A good algebraic analogue of smooth manifolds are *complex algebraic manifolds*, that is, separated smooth $\mathbb{C}$-schemes S of pure dimension. These form a full subcategory $\mathbf{AlgMan}_{\mathbb{C}}$ in the category $\mathrm{Sch}_{\mathbb{C}}$ of $\mathbb{C}$-schemes, and can roughly be characterized as the (sufficiently nice) objects S in $\mathrm{Sch}_{\mathbb{C}}$ whose cotangent complex $\mathbb{L}_S$ is a vector bundle (i.e. perfect in the interval $[0,0]$).

To make a derived version of this, we first define an ∞-category $\mathbf{DerSch}_{\mathbb{C}}$ of *derived* $\mathbb{C}$*-schemes*, and then define the ∞-category $\mathbf{DerAlgMan}_{\mathbb{C}}$ of *derived complex algebraic manifolds* to be the full ∞-subcategory of objects $\mathbf{S}$ in $\mathbf{DerSch}_{\mathbb{C}}$ which are *quasi-smooth* (have cotangent complex $\mathbb{L}_{\mathbf{S}}$ perfect in the interval $[-1,0]$), and satisfy some other niceness conditions (separated, etc.).

Derived algebraic geometry in the C^∞ world

Thus, we have 'classical' categories $\mathbf{AlgMan}_{\mathbb{C}} \subset \mathrm{Sch}_{\mathbb{C}}$, and related 'derived' ∞-categories $\mathbf{DerAlgMan}_{\mathbb{C}} \subset \mathbf{DerSch}_{\mathbb{C}}$.
David Spivak (arXiv:0810.5175, Duke Math. J.), a student of Jacob Lurie, defined an ∞-category **DerMan** of 'derived smooth manifolds' using a similar structure: he considered 'classical' categories $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ and related 'derived' ∞-categories $\mathbf{DerMan} \subset \mathbf{DerC^\infty Sch}$. Here $\mathbf{C^\infty Sch}$ is C^∞*-schemes*, and $\mathbf{DerC^\infty Sch}$ *derived* C^∞*-schemes*. That is, before we can 'derive', we must first embed **Man** into a larger category of C^∞-schemes, singular generalizations of manifolds.
My set-up is a simplification of Spivak's. I consider 'classical' categories $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ and related 'derived' 2-categories $\mathbf{dMan} \subset \mathbf{dSpa}$, where **dMan** is *d-manifolds*, and **dSpa** *d-spaces*. Here $\mathbf{dMan}, \mathbf{dSpa}$ are roughly 2-category truncations of Spivak's $\mathbf{DerMan}, \mathbf{DerC^\infty Sch}$ — see Borisov arXiv:1212.1153.

2. D-manifolds without boundary

I will concentrate today on d-manifolds *without boundary*.
We begin by discussing C^∞-algebraic geometry, C^∞-rings, and C^∞-schemes. Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes.
In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces.
There is a little-known theory of schemes in differential geometry, C^∞-*schemes*, going back to Lawvere, Dubuc, Moerdijk and Reyes, ... in synthetic differential geometry in the 1960s-1980s.
C^∞-schemes are essentially *algebraic* objects, on which smooth real functions and calculus make sense.

2.1. C^∞-rings

Let X be a manifold, and write $C^\infty(X)$ for the smooth functions $c : X \to \mathbb{R}$. Then $C^\infty(X)$ is an $\mathbb{R}$-*algebra*: we can add smooth functions $(c, d) \mapsto c + d$, and multiply them $(c, d) \mapsto cd$, and multiply by $\lambda \in \mathbb{R}$.
But there are many more operations on $C^\infty(X)$ than this, e.g. if $c : X \to \mathbb{R}$ is smooth then $\exp(c) : X \to \mathbb{R}$ is smooth, giving $\exp : C^\infty(X) \to C^\infty(X)$, which is algebraically independent of addition and multiplication.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by $\Phi_f(c_1, \ldots, c_n)(x) = f\bigl(c_1(x), \ldots, c_n(x)\bigr)$ for all $x \in X$. Then addition comes from $f : \mathbb{R}^2 \to \mathbb{R}$, $f : (x, y) \mapsto x + y$, multiplication from $(x, y) \mapsto xy$, etc. This huge collection of algebraic operations Φ_f make $C^\infty(X)$ into an algebraic object called a C^∞-*ring*.

Definition

A C^∞*-ring* is a set $\mathfrak{C}$ together with n-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying:
Let $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathfrak{C}$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have $\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j$.
A *morphism* of C^∞-rings is $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Examples of C^∞-rings

Then $C^\infty(X)$ is a C^∞-ring for any manifold X, and from $C^\infty(X)$ we can recover X up to canonical isomorphism.
If $f : X \to Y$ is smooth then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings; conversely, if $\phi : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings then $\phi = f^*$ for some unique smooth $f : X \to Y$. This gives a *full and faithful functor* $F : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ by $F : X \mapsto C^\infty(X)$, $F : f \mapsto f^*$.
Thus, we can think of manifolds as examples of C^∞-rings, and C^∞-rings as generalizations of manifolds. But there are many more C^∞-rings than manifolds. For example, $C^0(X)$ is a C^∞-ring for any topological space X.
Any C^∞-ring $\mathfrak{C}$ has a *cotangent module* $\Omega_{\mathfrak{C}}$. If $\mathfrak{C} = C^\infty(X)$ for X a manifold, then $\Omega_{\mathfrak{C}} = C^\infty(T^*X)$.

2.2. C^∞-schemes

We can now develop the whole machinery of scheme theory in algebraic geometry, replacing rings or algebras by C^∞-rings throughout — see my arXiv:1104.4951, arXiv:1001.0023.

A *C^∞-ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf of C^∞-rings $\mathcal{O}_X$. Write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces.

The *global sections functor* $\Gamma : \mathbf{C^\infty RS} \to \mathbf{C^\infty Rings}^{\rm op}$ maps $\Gamma : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$. It has a right adjoint, the *spectrum functor* $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\rm op} \to \mathbf{C^\infty RS}$. That is, for each C^∞-ring $\mathfrak{C}$ we construct a C^∞-ringed space $\mathrm{Spec}\,\mathfrak{C}$. Points $x \in \mathrm{Spec}\,\mathfrak{C}$ are $\mathbb{R}$-algebra morphisms $x : \mathfrak{C} \to \mathbb{R}$ (this implies x is a C^∞-ring morphism). We don't use prime ideals.

On the subcategory of *fair* C^∞-rings, Spec is full and faithful.

A C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme* if $\underline{X} \cong \mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. We call $\underline{X}$ a *C^∞-scheme* if X can be covered by open subsets U with $(U, \mathcal{O}_X|_U)$ an affine C^∞-scheme. Write $\mathbf{C^\infty Sch}$ for the full subcategory of C^∞-schemes in $\mathbf{C^\infty RS}$.

If X is a manifold, define a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ by $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. Then $\underline{X} \cong \mathrm{Spec}\,C^\infty(X)$. This defines a full and faithful embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$. So we can regard manifolds as examples of C^∞-schemes.

All *fibre products* exist in $\mathbf{C^\infty Sch}$. In manifolds $\mathbf{Man}$, fibre products $X \times_{g,Z,h} Y$ need exist only if $g : X \to Z$ and $h : Y \to Z$ are transverse. When g, h are not transverse, the fibre product $X \times_{g,Z,h} Y$ exists in $\mathbf{C^\infty Sch}$, but may not be a manifold.

We also define *vector bundles* and *quasicoherent sheaves* on a C^∞-scheme $\underline{X}$, and write $\mathrm{qcoh}(\underline{X})$ for the abelian category of quasicoherent sheaves. A C^∞-scheme $\underline{X}$ has a well-behaved *cotangent sheaf* $T^*\underline{X}$.

Differences with ordinary Algebraic Geometry

- The topology on C^∞-schemes is finer than the Zariski topology on schemes – affine schemes are always Hausdorff. No need to introduce the étale topology.
- Can find smooth functions supported on (almost) any open set.
- (Almost) any open cover has a subordinate partition of unity.
- Our C^∞-rings $\mathfrak{C}$ are generally *not noetherian* as $\mathbb{R}$-algebras. So ideals I in $\mathfrak{C}$ may not be finitely generated, even in $C^\infty(\mathbb{R}^n)$.

2.3. Differential graded C^∞-rings

We can define derived $\mathbb{C}$-schemes by replacing $\mathbb{C}$-algebras A by *dg $\mathbb{C}$-algebras* $A^\bullet$ in the definition of $\mathbb{C}$-scheme — commutative differential graded $\mathbb{C}$-algebras in degrees $\leqslant 0$, of the form $\cdots \rightarrow A^{-2} \xrightarrow{\mathrm{d}} A^{-1} \xrightarrow{\mathrm{d}} A^0$, where A^0 is an ordinary $\mathbb{C}$-algebra. The corresponding 'classical' $\mathbb{C}$-algebra is $H^0(A^\bullet) = A^0/\mathrm{d}[A^{-1}]$. There is a parallel notion of *dg C^∞-ring* $\mathfrak{C}^\bullet$, of the form $\cdots \rightarrow \mathfrak{C}^{-2} \xrightarrow{\mathrm{d}} \mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$, where $\mathfrak{C}^0$ is an ordinary C^∞-ring, and $\mathfrak{C}^{-1}, \mathfrak{C}^{-2}, \ldots$ are modules over $\mathfrak{C}^0$. The corresponding 'classical' C^∞-ring is $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.
One could use dg C^∞-rings to define 'derived C^∞-schemes'; an alternative is to use *simplicial C^∞-rings*, see Spivak arXiv:0810.5175, Borisov–Noel arXiv:1112.0033, and Borisov arXiv:1212.1153.

Square zero dg C^∞-rings

My d-spaces are a 2-category truncation of derived C^∞-schemes. To define them, I use a special class of dg C^∞-rings called *square zero dg C^∞-rings*, which form a 2-category $\mathbf{SZC^\infty Rings}$.

A dg C^∞-ring $\mathfrak{C}^\bullet$ is *square zero* if $\mathfrak{C}^i = 0$ for $i < -1$ and $\mathfrak{C}^{-1} \cdot \mathrm{d}[\mathfrak{C}^{-1}] = 0$. Then $\mathfrak{C}$ is $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$, and $\mathrm{d}[\mathfrak{C}^{-1}]$ is a square zero ideal in the (ordinary) C^∞-ring $\mathfrak{C}^0$, and $\mathfrak{C}^{-1}$ is a module over the 'classical' C^∞-ring $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.

A 1-*morphism* $\alpha^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ in $\mathbf{SZC^\infty Rings}$ is maps $\alpha^0 : \mathfrak{C}^0 \to \mathfrak{D}^0$, $\alpha^{-1} : \mathfrak{C}^{-1} \to \mathfrak{D}^{-1}$ preserving all the structure. Then $H^0(\alpha^\bullet) : H^0(\mathfrak{C}) \to H^0(\mathfrak{D})$ is a morphism of C^∞-rings.

For 1-morphisms $\alpha^\bullet, \beta^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ a 2-*morphism* $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ is a linear $\eta : \mathfrak{C}^0 \to \mathfrak{D}^{-1}$ with $\beta^0 = \alpha^0 + \mathrm{d} \circ \eta$ and $\beta^{-1} = \alpha^{-1} + \eta \circ \mathrm{d}$.

There is an embedding of (2-)categories $\mathbf{C^\infty Rings} \subset \mathbf{SZC^\infty Rings}$ as the (2-)subcategory of $\mathfrak{C}^\bullet$ with $\mathfrak{C}^{-1} = 0$.

Cotangent complexes in the 2-category setting

Let $\mathfrak{C}^\bullet$ be a square zero dg C^∞-ring. Define the *cotangent complex* $\mathbb{L}_{\mathfrak{C}}^{-1} \xrightarrow{\mathrm{d}_{\mathfrak{C}}} \mathbb{L}_{\mathfrak{C}}^0$ to be the 2-term complex of $H^0(\mathfrak{C}^\bullet)$-modules

$$\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}_{\mathrm{DR}} \circ \mathrm{d}} \Omega_{\mathfrak{C}^0} \otimes_{\mathfrak{C}^0} H^0(\mathfrak{C}^\bullet),$$

regarded as an element of the 2-category of 2-term complexes of $H^0(\mathfrak{C}^\bullet)$-modules. Let $\alpha^\bullet, \beta^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ be 1-morphisms and $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ a 2-morphism in $\mathbf{SZC^\infty Rings}$. Then $H^0(\alpha^\bullet) = H^0(\beta^\bullet)$, so we may regard $\mathfrak{D}^{-1}$ as an $H^0(\mathfrak{C}^\bullet)$-module. And $\eta : \mathfrak{C}^0 \to \mathfrak{D}^{-1}$ is a derivation, so it factors through an $H^0(\mathfrak{C}^\bullet)$-linear map $\hat\eta : \Omega_{\mathfrak{C}^0} \otimes_{\mathfrak{C}^0} H^0(\mathfrak{C}^\bullet) \to \mathfrak{D}^{-1}$. We have a diagram

$$\begin{array}{ccc} \mathbb{L}_{\mathfrak{C}}^{-1} & \xrightarrow{\quad \mathrm{d}_{\mathfrak{C}} \quad} & \mathbb{L}_{\mathfrak{C}}^{0} \\ \mathbb{L}_{\alpha}^{-1} \Big\downarrow \Big\downarrow \mathbb{L}_{\beta}^{-1} & \hat\eta \swarrow & \mathbb{L}_{\alpha}^{0} \Big\downarrow \Big\downarrow \mathbb{L}_{\beta}^{0} \\ \mathbb{L}_{\mathfrak{D}}^{-1} & \xrightarrow{\quad \mathrm{d}_{\mathfrak{D}} \quad} & \mathbb{L}_{\mathfrak{D}}^{0}. \end{array}$$

So 1-morphisms induce morphisms, and 2-morphisms homotopies, of virtual cotangent modules.

Examples of square zero dg C^∞-rings

Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section. Then we call (V, E, s) a *Kuranishi neighbourhood* (compare Kuranishi spaces); for d-orbifolds, we take V an orbifold. Associate a square zero dg C^∞-ring $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$ to (V, E, s) by

$$\mathfrak{C}^0 = C^\infty(V)/I_s^2, \qquad \mathfrak{C}^{-1} = C^\infty(E^*)/I_s \cdot C^\infty(E^*),$$
$$\mathrm{d}(\epsilon + I_s \cdot C^\infty(E^*)) = \epsilon(s) + I_s^2,$$

where $I_s = C^\infty(E^*) \cdot s \subset C^\infty(V)$ is the ideal generated by s.
The d-manifold $\mathbf{X}$ associated to (V, E, s) is $\operatorname{Spec} \mathfrak{C}^\bullet$. It only knows about functions on V up to $O(s^2)$, and sections of E up to $O(s)$.

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2.4. D-spaces

A *d-space* $\mathbf{X}$ is a topological space X with a sheaf of square zero dg-C^∞-rings $\mathcal{O}_{\mathbf{X}}^\bullet = \mathcal{O}_X^{-1} \xrightarrow{\mathrm{d}} \mathcal{O}_{\mathbf{X}}^0$, such that $\underline{X} = (X, H^0(\mathcal{O}_{\mathbf{X}}^\bullet))$ and $(X, \mathcal{O}_{\mathbf{X}}^0)$ are C^∞-schemes, and $\mathcal{O}_X^{-1}$ is quasicoherent over $\underline{X}$. We call $\underline{X}$ the *underlying classical C^∞-scheme*.
D-spaces form a strict 2-category **dSpa**, with 1-morphisms and 2-morphisms defined using sheaves of 1-morphisms and 2-morphisms in **SZC$^\infty$Rings** in the obvious way.
All fibre products exist in **dSpa**.
C^∞-schemes include into d-spaces as those $\mathbf{X}$ with $\mathcal{O}_X^{-1} = 0$.
Thus we have inclusions of (2-)categories **Man** $\subset$ **C$^\infty$Sch** $\subset$ **dSpa**, so manifolds are examples of d-spaces.
The *cotangent complex* $\mathbb{L}_{\mathbf{X}}^\bullet$ of $\mathbf{X}$ is the sheaf of cotangent complexes of $\mathcal{O}_{\mathbf{X}}^\bullet$, a 2-term complex $\mathbb{L}_{\mathbf{X}}^{-1} \xrightarrow{\mathrm{d}_{\mathbf{X}}} \mathbb{L}_{\mathbf{X}}^0$ of quasicoherent sheaves on $\underline{X}$. Such complexes form a 2-category $\mathrm{qcoh}^{[-1,0]}(\underline{X})$.

2.5. D-manifolds

A *d-manifold* **X** of *virtual dimension* $n \in \mathbb{Z}$ is a d-space **X** whose topological space X is Hausdorff and second countable, and such that **X** is covered by open d-subspaces **Y** $\subset$ **X** with equivalences **Y** $\simeq U \times_{g,W,h} V$, where U, V, W are manifolds with $\dim U + \dim V - \dim W = n$, and $g : U \to W$, $h : V \to W$ are smooth maps, and $U \times_{g,W,h} V$ is the fibre product in the 2-category **dSpa**. (The 2-category structure is *essential* to define the fibre product here.)

Write **dMan** for the full 2-subcategory of d-manifolds in **dSpa**.

Alternatively, we can write the local models as **Y** $\simeq V \times_{0,E,s} V$, where V is a manifold, $E \to V$ a vector bundle, $s : V \to E$ a smooth section, and $n = \dim V - \operatorname{rank} E$. Then (V, E, s) is a *Kuranishi neighbourhood* on **X**.

We call such $V \times_{0,E,s} V$ *affine d-manifolds*.

2.6. D-orbifolds, d-manifolds with corners

In a similar way, I define 2-categories of *d-stacks* **dSta**, which are a Deligne–Mumford stack version of d-spaces locally modelled on quotients [**X**/G] for **X** a d-space and G a finite group, and *d-orbifolds* **dOrb** $\subset$ **dSta**. D-orbifolds **X** are locally modelled by Kuranishi neighbourhoods (V, E, s) with V an orbifold, $E \to V$ a vector bundle and $s : V \to E$ a smooth section (that is, **X** is locally equivalent to a fibre product $V \times_{0,E,s} V$ in **dSta**).

I also define 2-categories $\mathbf{dSpa^b}, \mathbf{dSpa^c}, \mathbf{dMan^b}, \mathbf{dMan^c}, \mathbf{dSta^b}, \mathbf{dSta^c}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$ of d-spaces, d-manifolds, d-stacks and d-orbifolds *with boundary*, and *with corners*.

Many moduli spaces of J-holomorphic curves in symplectic geometry will be d-orbifolds, possibly with corners. Doing 'things with corners' properly, especially in the derived context, is more complicated than you would expect.

2.7. Why should **dMan** be a 2-category?

Here is one reason why any class of 'derived manifolds' should be (at least) a 2-category. One property we want of **dMan** (or of Kuranishi spaces, etc.) is that it contains manifolds **Man** as a subcategory, and if X, Y, Z are manifolds and $g : X \to Z$, $h : Y \to Z$ are smooth then a fibre product $\mathbf{W} = X \times_{g,Z,h} Y$ should exist in **dMan**, characterized by a universal property in **dMan**, and should be a d-manifold of 'virtual dimension'

$$\mathrm{vdim}\, \mathbf{W} = \dim X + \dim Y - \dim Z.$$

Note that g, h need not be transverse, and $\mathrm{vdim}\, \mathbf{W}$ may be negative. Consider the case $X = Y = *$, the point, $Z = \mathbb{R}$, and $g, h : * \mapsto 0$. If **dMan** were an ordinary category then as $*$ is a terminal object, the unique fibre product $* \times_{0,\mathbb{R},0} *$ would be $*$. But this has virtual dimension 0, not -1. So **dMan** must be some kind of higher category.

Why is a 2-category enough?

Usually in derived algebraic geometry, one considers an ∞-category of objects (derived stacks, etc.). But we work in a 2-category, a truncation of Spivak's ∞-category of derived manifolds.
Here are two reasons why this truncation does not lose important information. Firstly, d-manifolds correspond to *quasi-smooth* derived schemes $\mathbf{X}$, whose cotangent complexes $\mathbb{L}_{\mathbf{X}}$ lie in degrees $[-1, 0]$. So $\mathbb{L}_{\mathbf{X}}$ lies in a 2-category of complexes, not an ∞-category. Note that $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is étale in **dMan** iff $\Omega_{\mathbf{f}} : \mathbf{f}^*(\mathbb{L}_{\mathbf{Y}}) \to \mathbb{L}_{\mathbf{X}}$ is an equivalence.
Secondly, the existence of *partitions of unity* in differential geometry means that our structure sheaves $\mathcal{O}_{\mathbf{X}}$ are 'fine' or 'soft', which simplifies behaviour. Partitions of unity are also essential in gluing by equivalences in **dMan**. Our '2-category style derived geometry' would not work well in a conventional algebro-geometric context, rather than a differential-geometric one.

3. Differential geometry of d-manifolds

3.1. Cotangent complexes of d-manifolds

If $\mathbf{X}$ is a d-manifold, its cotangent complex $\mathbb{L}_{\mathbf{X}}^{\bullet}$ is *perfect*, that is, $\mathbb{L}_{\mathbf{X}}^{\bullet}$ is equivalent locally on $\underline{X}$ in the 2-category $\mathrm{qcoh}^{[-1,0]}(\underline{X})$ of 2-term complexes of quasicoherent sheaves on $\underline{X}$ to a complex of vector bundles $\mathcal{E}^{-1} \to \mathcal{E}^0$, and $\mathrm{rank}\,\mathcal{E}^0 - \mathrm{rank}\,\mathcal{E}^{-1} = \mathrm{vdim}\,\mathbf{X}$. For $x \in \mathbf{X}$, define the *cotangent space* $T_x^*\mathbf{X} = H^0(\mathbb{L}_{\mathbf{X}}|_x)$ and the *obstruction space* $O_x\mathbf{X} = H^{-1}(\mathbb{L}_{\mathbf{X}}|_x)$, with $\dim T_x^*\mathbf{X} - \dim O_x\mathbf{X} = \mathrm{vdim}\,\mathbf{X}$. A 1-morphism of d-manifolds $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ induces a 1-morphism $\mathrm{d}\mathbf{f} : \underline{f}^*(\mathbb{L}_{\mathbf{Y}}^{\bullet}) \to \mathbb{L}_{\mathbf{X}}^{\bullet}$ in $\mathrm{qcoh}^{[-1,0]}(\underline{X})$.

Theorem

A 1-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in **dMan** *is étale if and only if $\mathrm{d}\mathbf{f} : \underline{f}^*(\mathbb{L}_{\mathbf{Y}}^{\bullet}) \to \mathbb{L}_{\mathbf{X}}^{\bullet}$ is an equivalence in $\mathrm{qcoh}^{[-1,0]}(\underline{X})$, if and only if $H^0(\mathrm{d}\mathbf{f}|_x) : T_{f(x)}^*Y \to T_x^*X$ and $H^{-1}(\mathrm{d}\mathbf{f}|_x) : O_{f(x)}^*Y \to O_x^*X$ are isomorphisms for all $x \in \mathbf{X}$.*

3.2. D-transversality and fibre products

Let $g : X \to Z$, $h : Y \to Z$ be smooth maps of manifolds. Then g, h are *transverse* if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z$ in Z, the map $\mathrm{d}g|_x \oplus \mathrm{d}h|_y : T_z^*Z \to T_x^*X \oplus T_x^*Y$ is injective. If g, h are transverse then a fibre product $X \times_{g,Z,h} Y$ exists in **Man**. Similarly, we call 1-morphisms of d-manifolds $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ *d-transverse* if for all $x \in \mathbf{X}$, $y \in \mathbf{Y}$ with $\mathbf{g}(x) = \mathbf{h}(y) = z$ in $\mathbf{Z}$, the map $H^{-1}(\mathrm{d}\mathbf{g}|_x) \oplus H^{-1}(\mathrm{d}\mathbf{h}|_y) : O_z^*\mathbf{Z} \to O_x^*\mathbf{X} \oplus O_y^*\mathbf{Y}$ is injective. Note that d-transversality is *much weaker* than transversality of manifolds, and often holds automatically.

Theorem

Let $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ *and* $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ *be d-transverse* 1*-morphisms in* **dMan**. *Then a fibre product* $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ *exists in* **dMan**, *with* $\mathrm{vdim}\,\mathbf{W}=\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}$.

If $\mathbf{Z}$ is a manifold, $O_z^*\mathbf{Z}=0$ and d-transversality is trivial, giving:

Corollary

All fibre products of the form $\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y}$ *with* $\mathbf{X},\mathbf{Y}$ *d-manifolds and* $\mathbf{Z}$ *a manifold exist in the* 2*-category* **dMan**.

The same holds in **dOrb**. This is a very useful property of d-manifolds and d-orbifolds.

3.3. Gluing by equivalences

A 1-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in **dMan** is an *equivalence* if there exist $\mathbf{g}:\mathbf{Y}\to\mathbf{X}$ and 2-morphisms $\eta:\mathbf{g}\circ\mathbf{f}\Rightarrow\mathbf{id}_{\mathbf{X}}$ and $\zeta:\mathbf{f}\circ\mathbf{g}\Rightarrow\mathbf{id}_{\mathbf{Y}}$.

Theorem

Let $\mathbf{X},\mathbf{Y}$ *be d-manifolds,* $\emptyset\neq\mathbf{U}\subseteq\mathbf{X}$, $\emptyset\neq\mathbf{V}\subseteq\mathbf{Y}$ *open d-submanifolds, and* $\mathbf{f}:\mathbf{U}\to\mathbf{V}$ *an equivalence. Suppose the topological space* $Z=X\cup_{U=V}Y$ *made by gluing* X,Y *using* f *is Hausdorff. Then there exists a d-manifold* $\mathbf{Z}$, *unique up to equivalence, open* $\hat{\mathbf{X}},\hat{\mathbf{Y}}\subseteq\mathbf{Z}$ *with* $\mathbf{Z}=\hat{\mathbf{X}}\cup\hat{\mathbf{Y}}$, *equivalences* $\mathbf{g}:\mathbf{X}\to\hat{\mathbf{X}}$ *and* $\mathbf{h}:\mathbf{Y}\to\hat{\mathbf{Y}}$, *and a* 2*-morphism* $\eta:\mathbf{g}|_{\mathbf{U}}\Rightarrow\mathbf{h}\circ\mathbf{f}$.

Equivalence is the natural notion of when two objects in **dMan** are 'the same'. In the theorem $\mathbf{Z}$ is a *pushout* $\mathbf{X} \amalg_{\mathbf{id_U},\mathbf{U},\mathbf{f}} \mathbf{Y}$ in **dMan**.
The theorem generalizes to gluing families of d-manifolds $\mathbf{X}_i : i \in I$ by equivalences on double overlaps $\mathbf{X}_i \cap \mathbf{X}_j$, with (weak) conditions on triple overlaps $\mathbf{X}_i \cap \mathbf{X}_j \cap \mathbf{X}_k$.
We can take the $\mathbf{X}_i$ to be 'standard model' d-manifolds $\mathbf{S}_{V_i,E_i,s_i}$, and the equivalences on overlaps $\mathbf{X}_i \cap \mathbf{X}_j$ to be 1-morphisms $\mathbf{S}_{e_{ij},\hat{e}_{ij}}$.
This is very useful for proving existence of d-manifold or d-orbifold structures on moduli spaces. Essentially, from a 'good coordinate system' on a topological space X we can build a d-manifold or d-orbifold $\mathbf{X}$ with the same topological space X.

3.4. D-manifold bordism

Let Y be a manifold. Define the *bordism group* $B_k(Y)$ to have elements $\sim$-equivalence classes $[X, f]$ of pairs (X, f), where X is a compact oriented k-manifold and $f : X \to Y$ is smooth, and $(X, f) \sim (X', f')$ if there exists a $(k+1)$-manifold with boundary W and a smooth map $e : W \to Y$ with $\partial W \cong X \amalg -X'$ and $e|_{\partial W} \cong f \amalg f'$. It is an abelian group, with $[X, f] + [X', f'] = [X \amalg X, f \amalg f']$.

Similarly, define the *derived bordism group* $dB_k(Y)$ with elements $\approx$-equivalence classes $[\mathbf{X},\mathbf{f}]$ of pairs (X,f), where $\mathbf{X}$ is a compact oriented d-manifold with $\mathrm{vdim}\,\mathbf{X}=k$ and $\mathbf{f}:\mathbf{X}\to\mathbf{Y}=F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$ is a 1-morphism in **dMan**, and $(\mathbf{X},\mathbf{f})\approx(\mathbf{X}',\mathbf{f}')$ if there exists a d-manifold with boundary $\mathbf{W}$ with $\mathrm{vdim}\,\mathbf{W}=k+1$ and a 1-morphism $\mathbf{e}:\mathbf{W}\to\mathbf{Y}$ in $\mathbf{dMan^b}$ with $\partial\mathbf{W}\simeq\mathbf{X}\amalg-\mathbf{X}'$ and $\mathbf{e}|_{\partial\mathbf{W}}\cong\mathbf{f}\amalg\mathbf{f}'$. It is an abelian group, with $[\mathbf{X},\mathbf{f}]+[\mathbf{X}',\mathbf{f}']=[\mathbf{X}\amalg\mathbf{X},\mathbf{f}\amalg\mathbf{f}']$.

There is a morphism $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}:B_k(Y)\to dB_k(Y)$ mapping $[X,f]\mapsto[F_{\mathbf{Man}}^{\mathbf{dMan}}(X),F_{\mathbf{Man}}^{\mathbf{dMan}}(f)]$.

Theorem

$\Pi_{\mathrm{bo}}^{\mathrm{dbo}}:B_k(Y)\to dB_k(Y)$ is an isomorphism, with $dB_k(Y)=0$ for $k<0$.

This holds because every d-manifold can be perturbed to a manifold. Composing $(\Pi_{\mathrm{bo}}^{\mathrm{dbo}})^{-1}$ with the projection $B_k(Y)\to H_k(Y,\mathbb{Z})$ gives a morphism $\Pi_{\mathrm{dbo}}^{\mathrm{hom}}:dB_k(Y)\to H_k(Y,\mathbb{Z})$. We can interpret this as a *virtual class map* for compact oriented d-manifolds. Virtual classes (in homology over $\mathbb{Q}$) also exist for compact oriented d-orbifolds.