A Darboux theorem for shifted symplectic derived schemes; d-critical loci

Dominic Joyce, Oxford University

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Plan of talk:

1. PTVV's shifted symplectic geometry

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K} = \mathbb{C}$. Work in the context of Toën and Vezzosi's theory of *derived algebraic geometry*. This gives ∞-categories of *derived $\mathbb{K}$-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived stacks* $\mathbf{dSt}_{\mathbb{K}}$, including *derived Artin stacks* $\mathbf{dArt}_{\mathbb{K}}$. Think of a derived $\mathbb{K}$-scheme $\mathbf{X}$ as a geometric space which can be covered by Zariski open sets $\mathbf{Y} \subseteq \mathbf{X}$ with $\mathbf{Y} \simeq \operatorname{Spec} A$ for $A = (A, \mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{K}$.

Cotangent complexes of derived schemes and stacks

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a notion of *k-shifted symplectic structure* on a derived $\mathbb{K}$-scheme or derived $\mathbb{K}$-stack $\mathbf{X}$, for $k \in \mathbb{Z}$. This is complicated, but here is the basic idea. The *cotangent complex* $\mathbb{L}_{\mathbf{X}}$ of $\mathbf{X}$ is an element of a derived category $L_{\mathrm{qcoh}}(\mathbf{X})$ of quasicoherent sheaves on $\mathbf{X}$. It has exterior powers $\Lambda^p \mathbb{L}_{\mathbf{X}}$ for $p = 0, 1, \ldots$. The *de Rham differential* $\mathrm{d}_{dR} : \Lambda^p \mathbb{L}_{\mathbf{X}} \to \Lambda^{p+1} \mathbb{L}_{\mathbf{X}}$ is a morphism of complexes, though not of $\mathcal{O}_{\mathbf{X}}$-modules. Each $\Lambda^p \mathbb{L}_{\mathbf{X}}$ is a complex, so has an internal differential $\mathrm{d} : (\Lambda^p \mathbb{L}_{\mathbf{X}})^k \to (\Lambda^p \mathbb{L}_{\mathbf{X}})^{k+1}$. We have $\mathrm{d}^2 = \mathrm{d}_{dR}^2 = \mathrm{d} \circ \mathrm{d}_{dR} + \mathrm{d}_{dR} \circ \mathrm{d} = 0$.

p-forms and closed p-forms

A *p-form of degree k* on $\mathbf{X}$ for $k \in \mathbb{Z}$ is an element $[\omega^0]$ of $H^k(\Lambda^p \mathbb{L}_{\mathbf{X}}, \mathrm{d})$. A *closed p-form of degree k* on $\mathbf{X}$ is an element

$$[(\omega^0, \omega^1, \ldots)] \in H^k\big(\textstyle\bigoplus_{i=0}^{\infty} \Lambda^{p+i} \mathbb{L}_{\mathbf{X}}[i], \mathrm{d} + \mathrm{d}_{dR}\big).$$

There is a projection $\pi : [(\omega^0, \omega^1, \ldots)] \mapsto [\omega^0]$ from closed p-forms $[(\omega^0, \omega^1, \ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.
Note that a closed p-form *is not a special example of a p-form*, but a p-form with an extra structure. The map π from closed p-forms to p-forms can be neither injective nor surjective.

PTVV's shifted symplectic geometry
A Darboux theorem for shifted symplectic schemes
D-critical loci
A 'Darboux Theorem' for shifted symplectic stacks
D-critical stacks

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\mathbf{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$, where $\mathbb{T}_{\mathbf{X}} = \mathbb{L}_{\mathbf{X}}^{\vee}$ is the tangent complex of $\mathbf{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ is a quasi-isomorphism.
If $\mathbf{X}$ is a derived scheme then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\mathbf{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k = -1$ then $\mathbf{X}$ is quasi-smooth.
A closed 2-form $\omega = [(\omega^0, \omega^1, \ldots)]$ of degree k on $\mathbf{X}$ is called a *k-shifted symplectic structure* if $[\omega^0] = \pi(\omega)$ is nondegenerate.

Calabi–Yau moduli schemes and moduli stacks

Pantev et al. prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a natural $(2-m)$-shifted symplectic structure ω. So Calabi–Yau 3-folds give -1-shifted derived schemes or stacks.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E]\in\boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{1-i}(E,E)^*$.

The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E)\cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0:\mathbb{T}_{\boldsymbol{\mathcal{M}}}\to\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

Lagrangians and Lagrangian intersections

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\mathbf{L}$ in $(\mathbf{X},\omega)$, which is a morphism $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ of derived schemes or stacks together with a homotopy $i^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\mathbf{L}}\simeq\mathbb{L}_{\mathbf{L}/\mathbf{X}}[k-1]$.

If $\mathbf{L},\mathbf{M}$ are Lagrangians in $(\mathbf{X},\omega)$, then the fibre product $\mathbf{L}\times_{\mathbf{X}}\mathbf{M}$ has a natural $(k-1)$-shifted symplectic structure.

If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

2. A Darboux theorem for shifted symplectic schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\mathbf{X},\omega)$ *is a* k*-shifted symplectic derived* $\mathbb{K}$*-scheme for* $k<0$*. If* $k\not\equiv 2 \mod 4$*, then each* $x\in\mathbf{X}$ *admits a Zariski open neighbourhood* $\mathbf{Y}\subseteq\mathbf{X}$ *with* $\mathbf{Y}\simeq \operatorname{Spec} A$ *for* (A,d) *an explicit cdga over* $\mathbb{K}$ *generated by graded variables* x_j^{-i},y_j^{k+i} *for* $0\leqslant i\leqslant -k/2$*, and* $\omega|_{\mathbf{Y}}=[(\omega^0,0,0,\ldots)]$ *where* x_j^l,y_j^l *have degree* l*, and*

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d *in* (A,d) *is given by Poisson bracket with a Hamiltonian* H *in* A *of degree* $k+1$*.*
If $k\equiv 2 \mod 4$*, we have two statements, one étale local with* ω^0 *standard, and one Zariski local with the components of* ω^0 *in the degree* $k/2$ *variables depending on some invertible functions.*

Sketch of the proof of the theorem

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $x\in\mathbf{X}$. Then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k,0]$. We first show that we can build Zariski open $x\in\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec} A$, for $A=\bigoplus_{i\leqslant 0}A^i$ a cdga over $\mathbb{K}$ with A^0 a smooth $\mathbb{K}$-algebra, and such that A is freely generated over A^0 by graded variables x_j^{-i},y_j^{k+i} in degrees $-1,-2,\ldots,k$. We take $\dim A^0$ and the number of x_j^{-i},y_j^{k+i} to be minimal at x.
Using theorems about periodic cyclic cohomology, we show that on $Y\simeq\operatorname{Spec} A$ we can write $\omega|_Y=[(\omega^0,0,0,\ldots)]$, for ω^0 a 2-form of degree k with $\mathrm{d}\omega^0=\mathrm{d}_{dR}\omega^0=0$. Minimality at x implies ω^0 is strictly nondegenerate near x, so we can change variables to write $\omega^0=\sum_{i,j}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}$. Finally, we show d in (A,d) is a symplectic vector field, which integrates to a Hamiltonian H.

The case of -1-shifted symplectic derived schemes

When $k = -1$ the Hamiltonian H in the theorem has degree 0. Then the theorem reduces to:

Corollary

Suppose $(\mathbf{X}, \omega)$ is a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\mathbf{X}, \omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H : U \to \mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H : U \to \mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X = t_0(\mathbf{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$.

Combining this with results of Pantev et al. from §1 gives interesting consequences in classical algebraic geometry:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

Here we note that $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli scheme, which is -1-shifted symplectic by PTVV.
A complex analytic analogue of this for moduli of coherent sheaves was proved using gauge theory by Joyce and Song arXiv:0810.5645, and for moduli of complexes was claimed by Behrend and Getzler.
Note that the proof of the corollary is wholly algebro-geometric.

Corollary

Let (S, ω) be a classical smooth symplectic $\mathbb{K}$-scheme, and $L, M \subseteq S$ be smooth algebraic Lagrangians. Then the intersection $L \cap M$, as a $\mathbb{K}$-subscheme of S, is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

In real or complex symplectic geometry, where Darboux Theorem holds, the analogue of the corollary is easy to prove, but in classical algebraic symplectic geometry we do not have a Darboux Theorem, so the corollary is not obvious.

3. D-critical loci

Theorem (Joyce arXiv:1304.4508)

Let X be a classical $\mathbb{K}$-scheme. Then there exists a canonical sheaf $\mathcal{S}_X$ of $\mathbb{K}$-vector spaces on X, such that if $R \subseteq X$ is Zariski open and $i : R \hookrightarrow U$ is a closed embedding of R into a smooth $\mathbb{K}$-scheme U, and $I_{R,U} \subseteq \mathcal{O}_U$ is the ideal vanishing on $i(R)$, then

$$\mathcal{S}_X|_R \cong \mathrm{Ker}\left(\frac{\mathcal{O}_U}{I_{R,U}^2} \xrightarrow{\mathrm{d}} \frac{T^*U}{I_{R,U} \cdot T^*U}\right).$$

Also $\mathcal{S}_X$ splits naturally as $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{K}_X$, where $\mathbb{K}_X$ is the sheaf of locally constant functions $X \to \mathbb{K}$.

The meaning of the sheaves $\mathcal{S}_X, \mathcal{S}_X^0$

If $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then taking $R = X$, $i =$inclusion, we see that $f + I_{X,U}^2$ is a section of $\mathcal{S}_X$. Also $f|_{X^{\rm red}} : X^{\rm red} \to \mathbb{K}$ is locally constant, and if $f|_{X^{\rm red}} = 0$ then $f + I_{X,U}^2$ is a section of $\mathcal{S}_X^0$. Note that $f + I_{X,U} = f|_X$ in $\mathcal{O}_X = \mathcal{O}_U/I_{X,U}$. The theorem means that $f + I_{X,U}^2$ makes sense *intrinsically on* X, without reference to the embedding of X into U.

That is, if $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then we can remember f up to second order in the ideal $I_{X,U}$ as a piece of data on X, not on U.

Suppose $X = \mathrm{Crit}(f : U \to \mathbb{A}^1) = \mathrm{Crit}(g : V \to \mathbb{A}^1)$ is written as a critical locus in two different ways. Then $f + I_{X,U}^2$, $g + I_{X,V}^2$ are sections of $\mathcal{S}_X$, so we can ask whether $f + I_{X,U}^2 = g + I_{X,V}^2$. This gives a way to compare isomorphic critical loci in different smooth classical schemes.

The definition of d-critical loci

Definition (Joyce arXiv:1304.4508)

An (*algebraic*) *d-critical locus* (X, s) is a classical $\mathbb{K}$-scheme X and a global section $s \in H^0(\mathcal{S}_X^0)$ such that X may be covered by Zariski open $R \subseteq X$ with an isomorphism $i : R \to \mathrm{Crit}(f : U \to \mathbb{A}^1)$ identifying $s|_R$ with $f + I_{R,U}^2$, for f a regular function on a smooth $\mathbb{K}$-scheme U.

That is, a d-critical locus (X, s) is a $\mathbb{K}$-scheme X which may Zariski locally be written as a critical locus $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, and the section s remembers f up to second order in the ideal $I_{X,U}$.

We also define *complex analytic d-critical loci*, with X a complex analytic space locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ for U a complex manifold and f holomorphic.

Orientations on d-critical loci

Theorem (Joyce arXiv:1304.4508)

Let (X, s) be an algebraic d-critical locus and X^{red} the reduced $\mathbb{K}$-subscheme of X. Then there is a natural line bundle $K_{X,s}$ on X^{red} called the **canonical bundle**, *such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$ then $K_{X,s}$ is locally modelled on $K_U^{\otimes^2}|_{\mathrm{Crit}(f)^{\mathrm{red}}}$, for K_U the usual canonical bundle of U.*

Definition

Let (X, s) be a d-critical locus. An *orientation* on (X, s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on X^{red}.

This is related to *orientation data* in Kontsevich–Soibelman 2008.

A truncation functor from -1-symplectic derived schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\mathbf{X}, \omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then the classical $\mathbb{K}$-scheme $X = t_0(\mathbf{X})$ extends naturally to an algebraic d-critical locus (X, s). The canonical bundle of (X, s) satisfies $K_{X,s} \cong \det \mathbb{L}_{\mathbf{X}}|_{X^{\mathrm{red}}}$.

That is, we define a *truncation functor* from -1-shifted symplectic derived $\mathbb{K}$-schemes to algebraic d-critical loci. Examples show this functor is not full. Think of d-critical loci as *classical truncations* of -1-shifted symplectic derived $\mathbb{K}$-schemes.
An alternative semi-classical truncation, used in D–T theory, is *schemes with symmetric obstruction theory*. D-critical loci appear to be better, for both categorified and motivic D–T theory.

The corollaries in §2 imply:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M}, s)$. The canonical bundle satisfies $K_{\mathcal{M},s} \cong \det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$, where $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is the (symmetric) obstruction theory on $\mathcal{M}$ defined by Thomas or Huybrechts and Thomas.

Corollary

Let (S, ω) be a classical smooth symplectic $\mathbb{K}$-scheme, and $L, M \subseteq S$ be smooth algebraic Lagrangians. Then $X = L \cap M$ extends naturally to a d-critical locus (X, s). The canonical bundle satisfies $K_{X,s} \cong K_L|_{X^{\rm red}} \otimes K_M|_{X^{\rm red}}$. Hence, choices of square roots $K_L^{1/2}, K_M^{1/2}$ give an orientation for (X, s).

4. A 'Darboux Theorem' for shifted symplectic stacks

In Ben-Bassat, Bussi, Brav and Joyce (in progress) we extend the material of §2 from (derived) schemes to (derived) Artin stacks. We define a *derived Artin stack* $\mathbf{X}$ to be 'strongly 1-geometric' in the sense of Toën and Vezzosi. Then the cotangent complex $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty, 1]$, and $X = t_0(\mathbf{X})$ is a classical Artin stack (in particular, not a higher stack). A derived Artin stack $\mathbf{X}$ admits a smooth atlas $\varphi : \mathbf{U} \to \mathbf{X}$ with $\mathbf{U}$ a derived scheme. If Y is a smooth projective scheme and $\boldsymbol{\mathcal{M}}$ is a derived moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b \mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, then $\boldsymbol{\mathcal{M}}$ is a derived Artin stack.

A 'Darboux Theorem' for atlases of derived stacks

Theorem (Ben-Bassat, Bussi, Brav, Joyce)

Let $(\mathbf{X},\omega_{\mathbf{X}})$ be a k-shifted symplectic derived Artin stack for $k<0$, and $p\in\mathbf{X}$. Then there exist 'standard form' affine derived schemes $\mathbf{U}=\operatorname{Spec}A$, $\mathbf{V}=\operatorname{Spec}B$, points $u\in\mathbf{U}$, $v\in\mathbf{V}$ with A,B minimal at u,v, morphisms $\varphi:\mathbf{U}\to\mathbf{X}$ and $\mathbf{i}:\mathbf{U}\to\mathbf{V}$ with $\varphi(u)=p$, $\mathbf{i}(u)=v$, such that φ is smooth of relative dimension $\dim H^1(\mathbb{L}_{\mathbf{X}}|_p)$, and $t_0(\mathbf{i}):t_0(\mathbf{U})\to t_0(\mathbf{V})$ is an isomorphism on classical schemes, and $\mathbb{L}_{\mathbf{U}/\mathbf{V}}\simeq\mathbb{T}_{\mathbf{U}/\mathbf{X}}[1-k]$, and a 'Darboux form' k-shifted symplectic form ω_B on $\mathbf{V}=\operatorname{Spec}B$ such that $\mathbf{i}^*(\omega_B)\sim\varphi^*(\omega_{\mathbf{X}})$ in k-shifted closed 2-forms on $\mathbf{U}$.

Discussion of the 'Darboux Theorem' for stacks

Let $(\mathbf{X},\omega_{\mathbf{X}})$ be a k-shifted symplectic derived Artin stack for $k<0$, and $p\in\mathbf{X}$. Although we do not know how to give a complete, explicit 'standard model' for $(\mathbf{X},\omega_{\mathbf{X}})$ near p, we can give standard models for a smooth atlas $\varphi:\mathbf{U}\to\mathbf{X}$ for $\mathbf{X}$ near p with $\mathbf{U}=\operatorname{Spec}A$ a derived scheme, and for the pullback 2-form $\varphi^*(\omega_{\mathbf{X}})$. We may think of $\varphi:\mathbf{U}\to\mathbf{X}$ as an open neighbourhood of p in the smooth topology, rather than the Zariski topology. Now $(\mathbf{U},\varphi^*(\omega_{\mathbf{X}}))$ is not k-shifted symplectic, as $\varphi^*(\omega_{\mathbf{X}})$ is closed, but not nondegenerate. However, there is a way to modify $\mathbf{U},A$ to get another derived scheme $\mathbf{V}=\operatorname{Spec}B$, where A has generators in degrees $0,-1,\ldots,-k-1$, and $B\subseteq A$ is the dg-subalgebra generated by the generators in degrees $0,-1,\ldots,-k$ only.

Then $\mathbf{V}$ has a natural k-shifted symplectic form ω_B, which we may take to be in 'Darboux form' as in §2, with $\mathbf{i}^*(\omega_B) \sim \varphi^*(\omega_{\mathbf{X}})$. In terms of cotangent complexes, $\mathbb{L}_{\mathbf{U}}$ is obtained from $\varphi^*(\mathbb{L}_{\mathbf{X}})$ by deleting a vector bundle $\mathbb{L}_{\mathbf{U}/\mathbf{X}}$ in degree 1. Also $\mathbb{L}_{\mathbf{V}}$ is obtained from $\mathbb{L}_{\mathbf{U}}$ by deleting the dual vector bundle $\mathbb{T}_{\mathbf{U}/\mathbf{X}}$ in degree $k-1$. As these two deletions are dual under $\varphi^*(\omega_{\mathbf{X}})$, the symplectic form descends to $\mathbf{V}$.

An example in which we have this picture $(\mathbf{V}, \omega_B) \xleftarrow{\mathbf{i}} \mathbf{U} \xrightarrow{\varphi} (\mathbf{X}, \omega_{\mathbf{X}})$ is a 'k-shifted symplectic quotient', when an algebraic group G acts on a k-shifted symplectic derived scheme $(\mathbf{V}, \omega_B)$ with 'moment map' $\mu \in H^k(\mathbf{V}, \mathfrak{g}^* \otimes \mathcal{O}_{\mathbf{V}})$, and $\mathbf{U} = \mu^{-1}(0)$, and $X = [\mathbf{U}/G]$.

-1-shifted symplectic derived stacks

When $k=-1$, $(\mathbf{V}, \omega_B)$ is a derived critical locus $\mathbf{Crit}(f : S \to \mathbb{A}^1)$ for S a smooth scheme. Then $t_0(\mathbf{V}) \cong t_0(\mathbf{U})$ is the classical critical locus $\mathrm{Crit}(f : S \to \mathbb{A}^1)$, and $U = t_0(\mathbf{U})$ is a smooth atlas for the classical Artin stack $X = t_0(\mathbf{X})$. Thus we deduce:

Corollary

Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a -1-shifted symplectic derived stack. Then the classical Artin stack $X = t_0(\mathbf{X})$ locally admits smooth atlases $\varphi : U \to X$ with $U = \mathrm{Crit}(f : S \to \mathbb{A}^1)$, for S a smooth scheme and f a regular function.

Calabi–Yau 3-fold moduli stacks

If Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a moduli stack of coherent sheaves F on Y, or complexes $F^\bullet$ in $D^b\,\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, then by PTVV the corresponding derived moduli stack $\boldsymbol{\mathcal{M}}$ with $t_0(\boldsymbol{\mathcal{M}}) = \mathcal{M}$ has a -1-shifted symplectic structure $\omega_{\boldsymbol{\mathcal{M}}}$. So the previous corollary gives:

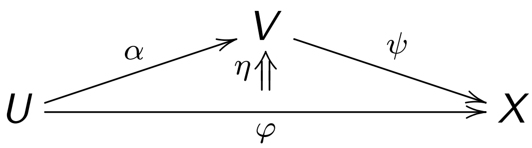
Corollary

Suppose Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b\,\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$. Then $\mathcal{M}$ locally admits smooth atlases $\varphi : U \to X$ with $U = \mathrm{Crit}(f : S \to \mathbb{A}^1)$, for S a smooth scheme.

A holomorphic version of this was proved by Joyce and Song using gauge theory, and is important in D-T theory.

5. D-critical stacks

To generalize the d-critical loci in §3 to Artin stacks, we need a good notion of sheaves on Artin stacks. This is already well understood. Roughly, a sheaf $\mathcal{S}$ on an Artin stack X assigns a sheaf $\mathcal{S}(U,\varphi)$ on U (in the usual sense for schemes) for each smooth morphism $\varphi : U \to X$ with U a scheme, and a morphism $\mathcal{S}(\alpha,\eta) : \alpha^*(\mathcal{S}(V,\psi)) \to \mathcal{S}(U,\varphi)$ (often an isomorphism) for each 2-commutative diagram



(1)

with U, V schemes and φ, ψ smooth, such that $\mathcal{S}(\alpha,\eta)$ have the obvious associativity properties. So, we pass from stacks X to schemes U by working with smooth atlases $\varphi : U \to X$.

The definition of d-critical stacks

Generalizing d-critical loci to stacks is now straightforward. As in §3, on each scheme U we have a canonical sheaf $\mathcal{S}_U^0$. If $\alpha : U \to V$ is a morphism of schemes we have pullback morphisms $\alpha^\star : \alpha^{-1}(\mathcal{S}_V^0) \to \mathcal{S}_U^0$ with associativity properties.
So, for any classical Artin stack X, we define a sheaf $\mathcal{S}_X^0$ on X by $\mathcal{S}_X(U, \varphi) = \mathcal{S}_U^0$ for all smooth $\varphi : U \to X$ with U a scheme, and $\mathcal{S}(\alpha, \eta) = \alpha^\star$ for all diagrams (1).
A global section $s \in H^0(\mathcal{S}_X^0)$ assigns $s(U, \varphi) \in H^0(\mathcal{S}_U^0)$ for all smooth $\varphi : U \to X$ with $\alpha^\star[\alpha^{-1}(s(V, \psi))] = s(U, \varphi)$ for all diagrams (1). We call (X, s) a *d-critical stack* if $\big(U, s(U, \varphi)\big)$ is a d-critical locus for all smooth $\varphi : U \to X$.
That is, if X is a d-critical stack then any smooth atlas $\varphi : U \to X$ for X is a d-critical locus.

A truncation functor from −1-symplectic derived stacks

As for the scheme case in §3, we prove:

Theorem (Ben-Bassat, Brav, Bussi, Joyce)

Let $(\mathbf{X}, \omega)$ be a -1-shifted symplectic derived Artin stack. Then the classical Artin stack $X = t_0(\mathbf{X})$ extends naturally to a d-critical stack (X, s), with canonical bundle $K_{X,s} \cong \det \mathbb{L}_{\mathbf{X}}|_{X^{\mathrm{red}}}$.

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or complexes $F^\bullet$ in $D^b \operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M}, s)$ with canonical bundle $K_{\mathcal{M},s} \cong \det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\mathrm{red}}}$, where $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is the natural obstruction theory on $\mathcal{M}$.

Canonical bundles and orientations

For schemes, a d-critical locus (U, s) has a canonical bundle $K_{U,s} \to U^{\mathrm{red}}$, and an orientation on (U, s) is a square root $K_{U,s}^{1/2}$. Similarly, a d-critical stack (X, s) has a *canonical bundle* $K_{X,s} \to X^{\mathrm{red}}$. For any smooth $\varphi : U \to X$ with U a scheme we have $K_{X,s}(U^{\mathrm{red}}, \varphi^{\mathrm{red}}) = K_{U,s(U,\varphi)} \otimes (\det \mathbb{L}_{U/X})^{\otimes^{-2}}$. An *orientation* on (X, s) is a choice of square root $K_{X,s}^{1/2}$ for $K_{X,s}$. Note that as $(\det \mathbb{L}_{U/X})^{\otimes^{-2}}$ has a natural square root, an orientation for (X, s) gives an orientation for $\big(U, s(U, \varphi)\big)$ for any smooth atlas $\varphi : U \to X$.