Categorification of shifted symplectic geometry using perverse sheaves

Dominic Joyce, Oxford University, April 2016

Based on: arXiv:1304.4508, arXiv:1305.6302, arXiv:1211.3259, arXiv:1305.6428, arXiv:1312.0090, arXiv:1403.2403, arXiv:1404.1329, arXiv:1504.00690, arXiv:1506.04024, arXiv:1509.05672, arXiv:1601.01536 and work in progress.

Joint with Lino Amorim, Oren Ben-Bassat, Chris Brav, Vittoria Bussi, Dennis Borisov, Delphine Dupont, Sven Meinhardt, Pavel Safronov, and Balázs Szendrői. Funded by the EPSRC.

These slides available at
`http://people.maths.ox.ac.uk/~joyce/talks.html`

Plan of talk:

1. Shifted symplectic geometry

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K}=\mathbb{C}$. Work in the context of Toën and Vezzosi's theory of Derived Algebraic Geometry. This gives ∞-categories of *derived $\mathbb{K}$-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived $\mathbb{K}$-stacks* $\mathbf{dSt}_{\mathbb{K}}$, including *derived Artin $\mathbb{K}$-stacks*.

Think of a derived $\mathbb{K}$-scheme $\mathbf{X}$ as a geometric space which can be covered by Zariski open sets $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A^\bullet$ for $A^\bullet=(A^*,\mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{K}$, in degrees $\leqslant 0$.

We require $\mathbf{X}$ to be *locally finitely presented*, that is, we can take the $A^\bullet$ to be finitely presented, a strong condition.

A derived $\mathbb{K}$-scheme or $\mathbb{K}$-stack $\mathbf{X}$ has a *tangent complex* $\mathbb{T}_{\mathbf{X}}$ and a dual *cotangent complex* $\mathbb{L}_{\mathbf{X}}$, which are perfect complexes of coherent sheaves on $\mathbf{X}$, of rank the virtual dimension $\operatorname{vdim}\mathbf{X}\in\mathbb{Z}$.

PTVV's shifted symplectic geometry

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a version of symplectic geometry in the derived world.

Let $\mathbf{X}$ be a derived $\mathbb{K}$-scheme or $\mathbb{K}$-stack. The cotangent complex $\mathbb{L}_{\mathbf{X}}$ has exterior powers $\Lambda^p\mathbb{L}_{\mathbf{X}}$. The *de Rham differential* $\mathrm{d}_{dR}:\Lambda^p\mathbb{L}_{\mathbf{X}}\rightarrow\Lambda^{p+1}\mathbb{L}_{\mathbf{X}}$ is a morphism of complexes. Each $\Lambda^p\mathbb{L}_{\mathbf{X}}$ is a complex, so has an internal differential $\mathrm{d}:(\Lambda^p\mathbb{L}_{\mathbf{X}})^k\rightarrow(\Lambda^p\mathbb{L}_{\mathbf{X}})^{k+1}$. We have $\mathrm{d}^2=\mathrm{d}_{dR}^2=\mathrm{d}\circ\mathrm{d}_{dR}+\mathrm{d}_{dR}\circ\mathrm{d}=0$.

A *p-form of degree k* on $\mathbf{X}$ for $k\in\mathbb{Z}$ is an element $[\omega^0]$ of $H^k\bigl(\Lambda^p\mathbb{L}_{\mathbf{X}},\mathrm{d}\bigr)$. A *closed p-form of degree k* on $\mathbf{X}$ is an element

$$[(\omega^0,\omega^1,\ldots)]\in H^k\bigl(\textstyle\bigoplus_{i=0}^\infty\Lambda^{p+i}\mathbb{L}_{\mathbf{X}}[i],\mathrm{d}+\mathrm{d}_{dR}\bigr).$$

There is a projection $\pi:[(\omega^0,\omega^1,\ldots)]\mapsto[\omega^0]$ from closed p-forms $[(\omega^0,\omega^1,\ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\mathbf{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$, where $\mathbb{T}_{\mathbf{X}} = \mathbb{L}_{\mathbf{X}}^\vee$ is the tangent complex of $\mathbf{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ is a quasi-isomorphism.

If $\mathbf{X}$ is a derived scheme then the complex $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\mathbf{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k = -1$ then $\mathbf{X}$ is quasi-smooth.

A closed 2-form $\omega = [(\omega^0, \omega^1, \ldots)]$ of degree k on $\mathbf{X}$ is called a *k-shifted symplectic structure* if $[\omega^0] = \pi(\omega)$ is nondegenerate.

Calabi–Yau moduli schemes and moduli stacks

PTVV prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a $(2-m)$-shifted symplectic structure ω.

This suggests applications — lots of interesting geometry concerns Calabi–Yau moduli schemes, e.g. Donaldson–Thomas theory.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E] \in \boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{1-i}(E,E)^*$.

The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E) \cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^{i+1}(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong h^{i+1}(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0 : \mathbb{T}_{\boldsymbol{\mathcal{M}}} \to \mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

Lagrangians and Lagrangian intersections

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\mathbf{L}$ in $(\mathbf{X},\omega)$, which is a morphism $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ of derived schemes or stacks together with a homotopy $\mathbf{i}^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\mathbf{L}}\simeq\mathbb{L}_{\mathbf{L}/\mathbf{X}}[k-1]$.
If $\mathbf{L},\mathbf{M}$ are Lagrangians in $(\mathbf{X},\omega)$, then the fibre product $\mathbf{L}\times_{\mathbf{X}}\mathbf{M}$ has a natural $(k-1)$-shifted symplectic structure.
If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

2. A Darboux theorem for shifted symplectic schemes

Theorem 1 (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$. If $k\not\equiv 2 \mod 4$, then each $x\in\mathbf{X}$ admits a Zariski open neighbourhood $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A^\bullet$ for $A^\bullet=(A^*,\mathrm{d})$ an explicit cdga generated by graded variables x_j^{-i},y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\mathbf{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l,y_j^l have degree l, and

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in $A^\bullet$ is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$.
If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 in the degree $k/2$ variables depending on some invertible functions.

Ben-Bassat–Brav–Bussi–Joyce extend this to derived Artin $\mathbb{K}$-stacks.

Sketch of the proof of Theorem 1

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $x\in\mathbf{X}$. Then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k,0]$. We first show that we can build Zariski open $x\in\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A^\bullet$, for $A^\bullet=(\bigoplus_{i\leqslant 0}A^i,\mathrm{d})$ a cdga over $\mathbb{K}$ with A^0 a smooth $\mathbb{K}$-algebra, and such that A^* is freely generated over A^0 by graded variables x_j^{-i},y_j^{k+i} in degrees $-1,-2,\ldots,k$. We take $\dim A^0$ and the number of x_j^{-i},y_j^{k+i} to be minimal at x.
Using theorems about periodic cyclic cohomology, we show that on $Y\simeq\operatorname{Spec}A^\bullet$ we can write $\omega|_Y=[(\omega^0,0,0,\ldots)]$, for ω^0 a 2-form of degree k with $\mathrm{d}\omega^0=\mathrm{d}_{dR}\omega^0=0$. Minimality at x implies ω^0 is strictly nondegenerate near x, so we can change variables to write $\omega^0=\sum_{i,j}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}$. Finally, we show d in $A^\bullet$ is a symplectic vector field, which integrates to a Hamiltonian H.

The case of -1-shifted symplectic derived schemes

When $k=-1$ the Hamiltonian H in Theorem 1 has degree 0. Then Theorem 1 reduces to:

Corollary

Suppose $(\mathbf{X},\omega)$ is a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\mathbf{X},\omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H:U\to\mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H:U\to\mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X=t_0(\mathbf{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$.

This implies that classical Calabi–Yau 3-fold moduli schemes are, Zariski locally, critical loci of regular functions on smooth schemes.

D-critical loci: classical truncations of -1-shifted symplectic schemes

Theorem (Joyce arXiv:1304.4508)

Let X be a classical $\mathbb{K}$-scheme. Then there exists a canonical sheaf $\mathcal{S}_X$ of $\mathbb{K}$-vector spaces on X, such that if $R \subseteq X$ is Zariski open and $i : R \hookrightarrow U$ is a closed embedding of R into a smooth $\mathbb{K}$-scheme U, and $I_{R,U} \subseteq \mathcal{O}_U$ is the ideal vanishing on $i(R)$, then

$$\mathcal{S}_X|_R \cong \mathrm{Ker}\left(\frac{\mathcal{O}_U}{I_{R,U}^2} \xrightarrow{\mathrm{d}} \frac{T^*U}{I_{R,U} \cdot T^*U}\right).$$

Also $\mathcal{S}_X$ splits naturally as $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{K}_X$, where $\mathbb{K}_X$ is the sheaf of locally constant functions $X \to \mathbb{K}$.

The meaning of the sheaves $\mathcal{S}_X, \mathcal{S}_X^0$

If $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then taking $R = X$, $i =$inclusion, we see that $f + I_{X,U}^2$ is a section of $\mathcal{S}_X$. Also $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{K}$ is locally constant, and if $f|_{X^{\mathrm{red}}} = 0$ then $f + I_{X,U}^2$ is a section of $\mathcal{S}_X^0$. Note that $f + I_{X,U} = f|_X$ in $\mathcal{O}_X = \mathcal{O}_U / I_{X,U}$. The theorem means that $f + I_{X,U}^2$ makes sense *intrinsically on* X, without reference to the embedding of X into U.

That is, if $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then we can remember f up to second order in the ideal $I_{X,U}$ as a piece of data on X, not on U.

Suppose $X = \mathrm{Crit}(f : U \to \mathbb{A}^1) = \mathrm{Crit}(g : V \to \mathbb{A}^1)$ is written as a critical locus in two different ways. Then $f + I_{X,U}^2$, $g + I_{X,V}^2$ are sections of $\mathcal{S}_X$, so we can ask whether $f + I_{X,U}^2 = g + I_{X,V}^2$. This gives a way to compare isomorphic critical loci in different smooth classical schemes.

Definition (Joyce arXiv:1304.4508)

An (*algebraic*) *d-critical locus* (X, s) is a classical $\mathbb{K}$-scheme X and a global section $s \in H^0(\mathcal{S}_X^0)$ such that X may be covered by Zariski open $R \subseteq X$ with an isomorphism $i : R \to \mathrm{Crit}(f : U \to \mathbb{A}^1)$ identifying $s|_R$ with $f + I_{R,U}^2$, for f a regular function on a smooth $\mathbb{K}$-scheme U.

That is, a d-critical locus (X, s) is a $\mathbb{K}$-scheme X which may Zariski locally be written as a critical locus $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, and the section s remembers f up to second order in the ideal $I_{X,U}$. We also define *complex analytic d-critical loci*.

Theorem 2 (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\mathbf{X}, \omega)$ *be a* -1*-shifted symplectic derived* $\mathbb{K}$*-scheme. Then the classical* $\mathbb{K}$*-scheme* $X = t_0(\mathbf{X})$ *extends naturally to an algebraic d-critical locus* (X, s)*. The 'canonical bundle' of* (X, s) *satisfies* $K_{X,s} \cong \det \mathbb{L}_{\mathbf{X}}|_{X^{\mathrm{red}}}$*.*

3. Categorification using perverse sheaves: objects

Theorem 3 (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)

Let $(\mathbf{X}, \omega)$ *be a* -1*-shifted symplectic derived* $\mathbb{K}$*-scheme. Then the 'canonical bundle'* $\det(\mathbb{L}_{\mathbf{X}})$ *is a line bundle over the classical scheme* $X = t_0(\mathbf{X})$*. Suppose we are given an* **orientation** *of* $(\mathbf{X}, \omega)$*, i.e. a square root line bundle* $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$*. Then we can construct a canonical perverse sheaf* $P^\bullet_{\mathbf{X},\omega}$ *on* X*, such that if* $(\mathbf{X}, \omega)$ *is Zariski locally modelled on* $\mathbf{Crit}(f : U \to \mathbb{A}^1)$*, then* $P^\bullet_{\mathbf{X},\omega}$ *is locally modelled on the perverse sheaf of vanishing cycles* $\mathcal{PV}^\bullet_{U,f}$ *of* (U, f)*. Similarly, we can construct a natural* $\mathscr{D}$*-module* $D^\bullet_{\mathbf{X},\omega}$ *on* X*, and when* $\mathbb{K} = \mathbb{C}$ *a natural mixed Hodge module* $M^\bullet_{\mathbf{X},\omega}$ *on* X*.*

In fact we actually construct the perverse sheaf on the oriented d-critical locus (X, s) associated to $(\mathbf{X}, \omega)$ in Theorem 2. We also define perverse sheaves on oriented complex analytic d-critical loci.

Sketch of the proof of Theorem 3

Roughly, we prove Theorem 3 by taking a Zariski open cover $\{\mathbf{R}_i : i \in I\}$ of $\mathbf{X}$ with $\mathbf{R}_i \cong \mathbf{Crit}(f_i : U_i \to \mathbb{A}^1)$, and showing that $\mathcal{PV}^\bullet_{U_i,f_i}$ and $\mathcal{PV}^\bullet_{U_j,f_j}$ are canonically isomorphic on $R_i \cap R_j$, so we can glue the $\mathcal{PV}^\bullet_{U_i,f_i}$ to get a global perverse sheaf $P^\bullet_{\mathbf{X},\omega}$ on X.
In fact things are more complicated: the (local) isomorphisms $\mathcal{PV}^\bullet_{U_i,f_i} \cong \mathcal{PV}^\bullet_{U_j,f_j}$ are only canonical *up to sign*. To make them canonical, we use the square root $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$ to define natural principal $\mathbb{Z}_2$-bundles Q_i on R_i, such that $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i \cong \mathcal{PV}^\bullet_{U_j,f_j} \otimes_{\mathbb{Z}_2} Q_j$ is canonical, and then we glue the $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i$ to get $P^\bullet_{\mathbf{X},\omega}$.

Shifted symplectic geometry
A Darboux theorem for shifted symplectic schemes
Categorification using perverse sheaves: objects
Categorification using perverse sheaves: morphisms

Categorifying Calabi–Yau 3-fold moduli spaces

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. ***orientation data****, K–S). Then we have a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a finite-dimensional graded vector space. The pointwise Euler characteristic $\chi(P^\bullet_{\mathcal{M},s})$ is the *Behrend function* $\nu_{\mathcal{M}}$ of $\mathcal{M}$. Thus

$$\textstyle\sum_{i\in\mathbb{Z}}(-1)^i \dim \mathbb{H}^i(P^\bullet_{\mathcal{M},s}) = \chi(\mathcal{M},\nu_{\mathcal{M}}).$$

Now by Behrend 2005, the Donaldson–Thomas invariant of $\mathcal{M}$ is $DT(\mathcal{M}) = \chi(\mathcal{M},\nu_{\mathcal{M}})$. So, $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a graded vector space with dimension $DT(\mathcal{M})$, that is, a *categorification* of $DT(\mathcal{M})$.

Categorifying Lagrangian intersections

Corollary

Let (S,ω) be a classical smooth symplectic $\mathbb{K}$-scheme of dimension $2n$, and $L,M\subseteq S$ be smooth algebraic Lagrangians, with square roots $K_L^{1/2},K_M^{1/2}$ of their canonical bundles. Then we have a natural perverse sheaf $P^\bullet_{L,M}$ on $X=L\cap M$.

We also prove an analogue for complex Lagrangians in holomorphic symplectic manifolds, using complex analytic d-critical loci.
This is related to Kashiwara and Schapira 2008, and Behrend and Fantechi 2009. We think of the hypercohomology $\mathbb{H}^*(P^\bullet_{L,M})$ as being morally related to the (undefined) *Lagrangian Floer cohomology* $HF^*(L,M)$ by $\mathbb{H}^i(P^\bullet_{L,M})\approx HF^{i+n}(L,M)$.
We are working on defining 'Fukaya categories' for algebraic/complex symplectic manifolds using these ideas.

4. Categorification using perverse sheaves: morphisms

We have seen that oriented -1-shifted symplectic derived $\mathbb{K}$-schemes/stacks $(\mathbf{X},\omega)$ carry perverse sheaves $P^\bullet_{\mathbf{X},\omega}$. We also expect that proper, oriented Lagrangians $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ should have associated hypercohomology elements $\mu_{\mathbf{L}}\in\mathbb{H}^*(P^\bullet_{\mathbf{X},\omega})$ with interesting properties, which can be interpreted as the morphisms in a categorification of -1-shifted symplectic geometry.

Definition

Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived scheme, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian. Choose an orientation $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$ for $(\mathbf{X},\omega)$. The Lagrangian structure induces a natural isomorphism $\alpha:\mathcal{O}_L\xrightarrow{\cong}i^*(\det(\mathbb{L}_{\mathbf{X}}))$. An *orientation* for $\mathbf{L}$ is an isomorphism $\beta:\mathcal{O}_L\xrightarrow{\cong}i^*(\det(\mathbb{L}_{\mathbf{X}})^{1/2})$ with $\beta^2=\alpha$.

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian. Then Theorem 1 shows that $\mathbf{X},\omega$ can be put in an explicit local 'Darboux form' $(\mathrm{Spec}\,A^\bullet,\omega_A)$. Joyce and Safronov prove a 'Lagrangian Neighbourhood Theorem' saying that $\mathbf{L},\mathbf{i}$ and the homotopy $h:\mathbf{i}^*(\omega)\sim 0$ can also be put in an explicit local form relative to $A^\bullet,\omega_A$. When $k=-1$ this yields:

Theorem 4 (Joyce and Safronov arXiv:1506.04024)

Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian, and $y\in\mathbf{L}$ with $\mathbf{i}(y)=x\in\mathbf{X}$. Theorem 1 implies that $(\mathbf{X},\omega)$ is equivalent near x to $\mathbf{Crit}(H:U\to\mathbb{A}^1)$, for U a smooth, affine $\mathbb{K}$-scheme. Then $\mathbf{L},\mathbf{i},h$ near y have an explicit local model depending on a smooth, affine $\mathbb{K}$-scheme V, a trivial vector bundle $E\to V$, a nondegenerate quadratic form Q on E, a section $s\in H^0(E)$, and a smooth morphism $\phi:V\to U$ with $Q(s,s)=\phi^*(H)$, where $t_0(\mathbf{L})\cong s^{-1}(0)\subseteq V$ Zariski locally.

Conjecture A

Let $(\mathbf{X},\omega)$ be an oriented -1-shifted symplectic derived $\mathbb{K}$-scheme or $\mathbb{K}$-stack, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ an oriented Lagrangian. Then there is a natural morphism in $D^b_c(\mathbf{L})$

$$\mu_{\mathbf{L}}:\mathbb{Q}_{\mathbf{L}}[\mathrm{vdim}\,\mathbf{L}]\longrightarrow i^!(P^\bullet_{\mathbf{X},\omega}),$$

with given local models in the 'Darboux form' presentations for $\mathbf{X},\omega,\mathbf{L}$ in Theorem 4.

Lino Amorim and I have an outline proof of Conjecture A in the scheme case over $\mathbb{K}=\mathbb{C}$, and also of a complex analytic version. In fact Conjecture A is only the first and simplest in a series of conjectures, which really should be written using ∞-categories, concerning higher coherences of the morphisms $\mu_{\mathbf{L}}$ under products, Verdier duality, composition of Lagrangian correspondences, etc. Our methods also allow us to prove these further conjectures. See Amorim and Ben-Bassat arXiv:1601.01536 for more on this.

Consequences of Conjecture A: perverse COHAs for CY3's

Let Y be a Calabi–Yau 3-fold, and $\boldsymbol{\mathcal{M}}$ the moduli stack of coherent sheaves on Y, so $\boldsymbol{\mathcal{M}}$ is -1-shifted symplectic. Let $\boldsymbol{\mathcal{E}\mathrm{xact}}$ be the derived stack of short exact sequences $0 \to F_1 \to F_2 \to F_3 \to 0$ in $\mathrm{coh}(Y)$, with projections $\pi_1, \pi_2, \pi_3 : \boldsymbol{\mathcal{E}\mathrm{xact}} \to \boldsymbol{\mathcal{M}}$. Ben-Bassat (work in progress) shows $\pi_1 \times \pi_2 \times \pi_3 : \boldsymbol{\mathcal{E}\mathrm{xact}} \to (\boldsymbol{\mathcal{M}}, \omega) \times (\boldsymbol{\mathcal{M}}, -\omega) \times (\boldsymbol{\mathcal{M}}, \omega)$ is Lagrangian.
Suppose we have 'orientation data' for Y, i.e. an orientation for $(\boldsymbol{\mathcal{M}}, \omega)$, with a compatibility condition on exact sequences, which is equivalent to an orientation on $\boldsymbol{\mathcal{E}\mathrm{xact}}$.
Then as in Theorem 3 we have a perverse sheaf $P^\bullet_{\boldsymbol{\mathcal{M}},s}$, with hypercohomology $\mathbb{H}^*(P^\bullet_{\boldsymbol{\mathcal{M}},s})$. Applying Conjecture A to $\boldsymbol{\mathcal{E}\mathrm{xact}}$ and using Verdier duality should (?) give an associative multiplication on $\mathbb{H}^*(P^\bullet_{\boldsymbol{\mathcal{M}},s})$, making it into a *Cohomological Hall Algebra*, as in Kontsevich–Soibelman arXiv:1006.2706, COHAs for CY3 quivers.

Consequences of Conjecture A: 'Fukaya categories' for algebraic / complex symplectic manifolds

Let (S, ω) be a algebraic/complex symplectic manifold, with $\dim_{\mathbb{C}} S = 2n$, and $L, M \subset S$ be algebraic/complex Lagrangians (not supposed compact or closed), with square roots of canonical bundles $K_L^{1/2}, K_M^{1/2}$.
Then the intersection $L \cap M$ is oriented -1-shifted symplectic / an oriented complex analytic d-critical locus, and carries a perverse sheaf $P^\bullet_{L,M}$ by Theorem 3.
We should think of the shifted hypercohomology $\mathbb{H}^{*-n}(P^\bullet_{L,M})$ as a substitute for the Lagrangian Floer cohomology $HF^*(L, M)$ in symplectic geometry. But $HF^*(L, M)$ is the morphisms in the derived Fukaya category $D^b\mathscr{F}(S, \omega)$ in symplectic geometry.

If L, M, N are Lagrangians in (S, ω), then $M \cap L, N \cap M, L \cap N$ are -1-shifted symplectic / d-critical loci, and $L \cap M \cap N$ is Lagrangian in the product $(M \cap L) \times (N \cap M) \times (L \cap N)$ (Ben-Bassat arXiv:1309.0596).
Applying Conjecture A to $L \cap M \cap N$ and rearranging using Verdier duality $P^\bullet_{M,L} \simeq \mathbb{D}(P^\bullet_{M,L})$ gives

$$\mu_{L,M,N} : P^\bullet_{L,M} \overset{L}{\otimes} P^\bullet_{M,N}[n] \longrightarrow P^\bullet_{L,N}.$$

Taking hypercohomology gives the multiplication $HF^*(L, M) \times HF^*(M, N) \to HF^*(L, N)$, which is composition of morphisms in the derived Fukaya category $D^b\mathscr{F}(S, \omega)$.
Higher coherences for such morphisms $\mu_{L,M,N}$ under composition should give the A_∞-structure needed to define a derived 'Fukaya category' $D^b\mathscr{F}(S, \omega)$, which we hope to do.

Comments on a proof of Conjecture A

In Theorem 3 we constructed a perverse sheaf $P^\bullet_{\mathbf{X},\omega}$ on an oriented -1-shifted symplectic $(\mathbf{X}, \omega)$. We did this by constructing a Zariski open cover $\{R_i : i \in I\}$ of $X = t_0(\mathbf{X})$, and perverse sheaves $P^\bullet_i$ on R_i, and isomorphisms $\alpha_{ij} : P^\bullet_i|_{R_i \cap R_j} \to P^\bullet_j|_{R_i \cap R_j}$ on all double overlaps $R_i \cap R_j$, with $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ on triple overlaps $R_i \cap R_j \cap R_k$. Then a unique $P^\bullet_{\mathbf{X},\omega}$ exists with $P^\bullet_{\mathbf{X},\omega}|_{R_i} \cong P^\bullet_i$, as perverse sheaves glue like sheaves.
In Conjecture A, we have explicit local models μ_j for the morphism $\mu_{\mathbf{L}}$ on an open cover $\{S_j : j \in J\}$ of $L = t_0(\mathbf{L})$, constructed using our local models for $\mathbf{L}, \mathbf{X}, \mathbf{i}$ in Theorem 4. However, this is not enough to define $\mu_{\mathbf{L}}$, as such morphisms do not glue like sheaves.
It is an ∞-category gluing problem: we need to construct higher coherences between $\mu_{j_1}, \ldots, \mu_{j_n}$ on n-fold overlaps $S_{j_1} \cap \cdots \cap S_{j_n}$ for all $n = 2, \ldots$. This is difficult, as perverse sheaves of vanishing cycles are not easy to handle on the cochain level.

Actually, to prove Conjecture A we need first to re-prove Theorem 3 in an ∞-categorical way, without using the sheaf property of perverse sheaves, but constructing $P^\bullet_{\mathbf{X},\omega}$ directly as a complex on X. We can define *d-correspondences* $i : L \to (X, s)$ in d-critical loci, which are classical truncations of Lagrangians $\mathbf{i} : \mathbf{L} \to (\mathbf{X}, \omega)$ in -1-shifted symplectic schemes. Our proposed proof of Conjecture A factors through these classical truncations, and also has a complex analytic version.

One of our key ideas is to give a new expression for the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ for a holomorphic function $f : U \to \mathbb{C}$ of a complex manifold, as an explicit complex on $\mathrm{Crit}(f)$, using the theory of 'M-cohomology' in Joyce arXiv:1509.05672. This new expression is easier to glue on overlaps between critical charts $(U_i, f_i), (U_j, f_j)$, and to control the higher coherences on multiple overlaps. This complex is built using differential geometry of manifolds, which is why we need $\mathbb{K} = \mathbb{C}$.