"Fukaya categories" of complex Lagrangians in complex symplectic manifolds

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Based on: arXiv:1304.4508, arXiv:1305.6302, arXiv:1211.3259, arXiv:1403.2403, arXiv:1404.1329, arXiv:1506.04024, arXiv:1509.05672, arXiv:1601.01536 and work in progress.

Joint with Lino Amorim, Oren Ben-Bassat, Chris Brav, Vittoria Bussi, Delphine Dupont, Pavel Safronov, and Balázs Szendrői.

Funded by the EPSRC.

These slides available at
`http://people.maths.ox.ac.uk/~joyce/talks.html`

Plan of talk:

0. Introduction

Let (S,ω) be a real C^∞ symplectic manifold. Then under some assumptions one can define a derived Fukaya category $D^b\mathscr{F}(S,\omega)$, with objects Lagrangians L,M in S, and morphisms $\mathrm{Hom}^*(L,M)=HF^*(L,M)$ the Lagrangian Floer cohomology groups. Here $HF^*(L,M)$ is *not local* on L,M or $L\cap M$, as it is defined by counting 'large' J-holomorphic curves $u:\Sigma\to S$.

Now suppose (S,ω) is a *complex* (holomorphic) symplectic manifold, where S has complex structure I, and we consider complex Lagrangians L,M in S. Then $\mathrm{Re}\,\omega$ is a real C^∞ symplectic structure on the underlying real manifold $S_{\mathbb{R}}$ of S, so we can define $HF^*(L,M)$ for $(S_{\mathbb{R}},\mathrm{Re}\,\omega)$. Note that the almost complex structure J used to do this is not the complex structure I on S, but is orthogonal to it, in a hyperkähler triple I,J,K.

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A simple argument using $\mathrm{Im}\,\omega$ shows that the only J-holomorphic curves in the definition of $HF^*(L,M)$ are constant. This suggests that in the complex case, $HF^*(L,M)$ might be local on $L\cap M$.

Also note that in the real C^∞ case we can always perturb Lagrangians L,M to intersect transversely. But complex Lagrangians are more rigid, we must allow L,M to be non-transverse.

I will outline a programme to define a 'Fukaya category' of complex Lagrangians L,M in a complex symplectic manifold (S,ω), in which the morphisms $\mathrm{Hom}^*(L,M)=$"$HF^*(L,M)$" are defined by constructing a perverse sheaf $P^\bullet_{L,M}$ on $L\cap M$ and taking its hypercohomology $\mathbb{H}^*(P^\bullet_{L,M})$. We do not need S,L,M to be compact or closed. We can also include singular 'derived' Lagrangians in our picture.

This programme also works for algebraic Lagrangians in a symplectic scheme over a field $\mathbb{K}$ of characteristic zero. It originates from the 'shifted symplectic geometry' of Pantev–Toën–Vaquié–Vezzosi in Derived Algebraic Geometry.

1. Shifted symplectic geometry

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K}=\mathbb{C}$. Work in Toën and Vezzosi's theory of Derived Algebraic Geometry. This gives ∞-categories of *derived* $\mathbb{K}$*-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived* $\mathbb{K}$*-stacks* $\mathbf{dSt}_{\mathbb{K}}$. Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a derived version of symplectic geometry.

Let $\mathbf{X}$ be a derived $\mathbb{K}$-scheme or $\mathbb{K}$-stack, supposed locally finitely presented. The cotangent complex $\mathbb{L}_{\mathbf{X}}$ has exterior powers $\Lambda^p\mathbb{L}_{\mathbf{X}}$. The *de Rham differential* is $\mathrm{d}_{dR}:\Lambda^p\mathbb{L}_{\mathbf{X}}\to\Lambda^{p+1}\mathbb{L}_{\mathbf{X}}$. Each $\Lambda^p\mathbb{L}_{\mathbf{X}}$ is a complex, so has an internal differential $\mathrm{d}:(\Lambda^p\mathbb{L}_{\mathbf{X}})^k\to(\Lambda^p\mathbb{L}_{\mathbf{X}})^{k+1}$. We have $\mathrm{d}^2=\mathrm{d}_{dR}^2=\mathrm{d}\circ\mathrm{d}_{dR}+\mathrm{d}_{dR}\circ\mathrm{d}=0$.

A *p-form of degree k* on $\mathbf{X}$ for $k\in\mathbb{Z}$ is an element $[\omega^0]$ of $H^k\bigl(\Lambda^p\mathbb{L}_{\mathbf{X}},\mathrm{d}\bigr)$. A *closed p-form of degree k* on $\mathbf{X}$ is an element

$$[(\omega^0,\omega^1,\ldots)]\in H^k\bigl(\textstyle\bigoplus_{i=0}^{\infty}\Lambda^{p+i}\mathbb{L}_{\mathbf{X}}[i],\mathrm{d}+\mathrm{d}_{dR}\bigr).$$

There is a projection $\pi:[(\omega^0,\omega^1,\ldots)]\mapsto[\omega^0]$ from closed p-forms $[(\omega^0,\omega^1,\ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.

Shifted symplectic structures and Lagrangians

Let $[\omega^0]$ be a 2-form of degree k on $\mathbf{X}$. Then $[\omega^0]$ induces a morphism $\omega^0:\mathbb{T}_{\mathbf{X}}\to\mathbb{L}_{\mathbf{X}}[k]$, where $\mathbb{T}_{\mathbf{X}}=\mathbb{L}_{\mathbf{X}}^{\vee}$ is the tangent complex of $\mathbf{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0:\mathbb{T}_{\mathbf{X}}\to\mathbb{L}_{\mathbf{X}}[k]$ is a quasi-isomorphism.

A closed 2-form $\omega=[(\omega^0,\omega^1,\ldots)]$ of degree k on $\mathbf{X}$ is called a *k-shifted symplectic structure* if $[\omega^0]=\pi(\omega)$ is nondegenerate.

If $\mathbf{X}$ is a derived scheme we must have $k\leqslant 0$, and if $k=0$ then $(\mathbf{X},\omega)$ is a smooth classical $\mathbb{K}$-scheme.

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then PTVV define a notion of *Lagrangian* $\mathbf{L}$ in $(\mathbf{X},\omega)$, which is a morphism $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ of derived schemes or stacks together with a homotopy $\mathbf{i}^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\mathbf{L}}\simeq\mathbb{L}_{\mathbf{L}/\mathbf{X}}[k-1]$.

If $\mathbf{L},\mathbf{M}$ are Lagrangians in $(\mathbf{X},\omega)$, then the fibre product $\mathbf{L}\times_{\mathbf{X}}\mathbf{M}$ has a natural $(k-1)$-shifted symplectic structure.

Derived Lagrangians in classical symplectic schemes

If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L, M \subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV.

However, if $\mathbf{i}:\mathbf{L}\to S$ is a derived Lagrangian in the PTVV sense, it *need not be a classical smooth Lagrangian*. PTVV Lagrangians are more general. This should be of interest even to classical symplectic geometers, we may get an enlarged Fukaya category.

As a typical local model for PTVV derived Lagrangians, suppose $(S_1,\omega_1),(S_2,\omega_2)$ are classical symplectic schemes, and $L_1\to(S_1,\omega_1)$, $L_{12}\to(S_1\times S_2,-\omega_1\boxplus\omega_2)$ are classical Lagrangians. If $L_1\to S_1$, $L_{12}\to S_1$ are transverse, the fibre product $L_1\times_{S_1}L_{12}$ is smooth and a classical Lagrangian in (S_2,ω_2). If they are not transverse, the derived fibre product $L_1\times_{S_1}L_{12}$ is still a derived scheme, and a PTVV derived Lagrangian in (S_2,ω_2).

2. The -1-shifted symplectic case and d-critical loci

Theorem 1 (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$. If $k\not\equiv 2 \mod 4$, then each $x\in\mathbf{X}$ admits a Zariski open neighbourhood $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A^\bullet$ for $A^\bullet=(A^*,\mathrm{d})$ an explicit cdga generated by graded variables x_j^{-i},y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\mathbf{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l,y_j^l have degree l, and

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in $A^\bullet$ is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$.

If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 in the degree $k/2$ variables depending on some invertible functions.

Ben-Bassat–Brav–Bussi–Joyce extend this to derived Artin $\mathbb{K}$-stacks.

The case of −1-shifted symplectic derived schemes

When $k = -1$ the Hamiltonian H in Theorem 1 has degree 0. Then Theorem 1 reduces to:

Corollary

Suppose $(\mathbf{X}, \omega)$ is a −1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\mathbf{X}, \omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H : U \to \mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H : U \to \mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X = t_0(\mathbf{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$.

Note that if $\mathbf{i} : \mathbf{L} \to S$, $\mathbf{j} : \mathbf{M} \to S$ are classical/derived Lagrangians in a classical (0-shifted) symplectic scheme (S, ω), then $\mathbf{X} = \mathbf{L} \times_S \mathbf{M}$ is −1-shifted symplectic. Thus, the corollary tells us that (derived) Lagrangian intersections $L \cap M$ in classical symplectic schemes are locally (derived) critical loci.

Theorem (Joyce arXiv:1304.4508)

Let X be a classical $\mathbb{K}$-scheme. Then there exists a canonical sheaf $\mathcal{S}_X$ of $\mathbb{K}$-vector spaces on X, such that if $R \subseteq X$ is Zariski open and $i : R \hookrightarrow U$ is a closed embedding of R into a smooth $\mathbb{K}$-scheme U, and $I_{R,U} \subseteq \mathcal{O}_U$ is the ideal vanishing on $i(R)$, then

$$\mathcal{S}_X|_R \cong \mathrm{Ker}\left(\frac{\mathcal{O}_U}{I_{R,U}^2} \xrightarrow{\mathrm{d}} \frac{T^*U}{I_{R,U} \cdot T^*U}\right).$$

Also $\mathcal{S}_X$ splits naturally as $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{K}_X$, where $\mathbb{K}_X$ is the sheaf of locally constant functions $X \to \mathbb{K}$.

If $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then taking $R = X$, $i =$inclusion, we see that $f + I_{X,U}^2$ is a section of $\mathcal{S}_X$. Also $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{K}$ is locally constant, and if $f|_{X^{\mathrm{red}}} = 0$ then $f + I_{X,U}^2$ is a section of $\mathcal{S}_X^0$. Note that $f + I_{X,U} = f|_X$ in $\mathcal{O}_X = \mathcal{O}_U / I_{X,U}$. The theorem means that $f + I_{X,U}^2$ makes sense *intrinsically on* X, without reference to the embedding of X into U. This allows us to compare ways of writing a scheme X as a critical locus in different ways.

D-critical loci

Definition (Joyce arXiv:1304.4508)

An (*algebraic*) *d-critical locus* (X, s) is a classical $\mathbb{K}$-scheme X and a global section $s \in H^0(\mathcal{S}_X^0)$ such that X may be covered by Zariski open $R \subseteq X$ with an isomorphism $i : R \to \mathrm{Crit}(f : U \to \mathbb{A}^1)$ identifying $s|_R$ with $f + I_{R,U}^2$, for f a regular function on a smooth $\mathbb{K}$-scheme U.

That is, a d-critical locus (X, s) is a $\mathbb{K}$-scheme X which may Zariski locally be written as a critical locus $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, and the section s remembers f up to second order in the ideal $I_{X,U}$. We also define *complex analytic d-critical loci*, which are complex analytic spaces X with a section of a natural sheaf $\mathcal{S}_X^0$ that are locally modelled on the critical locus of a holomorphic function $f : U \to \mathbb{C}$ for U a complex manifold.

Theorem 2 (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\mathbf{X}, \omega)$ *be a* -1*-shifted symplectic derived* $\mathbb{K}$*-scheme. Then the classical* $\mathbb{K}$*-scheme* $X = t_0(\mathbf{X})$ *extends naturally to an algebraic d-critical locus* (X, s)*. The 'canonical bundle' of* (X, s) *satisfies* $K_{X,s} \cong \det \mathbb{L}_{\mathbf{X}}|_{X^{\mathrm{red}}}$.

This means that d-critical loci are *classical truncations* of -1-shifted symplectic derived schemes. We are working on a similar definition of classical truncation of derived Lagrangians in classical (0-symplectic) symplectic schemes.

Theorem 3 (Bussi arXiv:1404.1329)

Let (S, ω) *be a complex symplectic manifold and* $i : L \to S$, $j : M \to S$ *be smooth complex Lagrangians. Then the fibre product* $X = L \times_{i,S,j} M$ *as a complex analytic space extends naturally to a complex analytic d-critical locus* (X, s).

3. Categorification using perverse sheaves: objects

Theorem 4 (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)

*Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then the 'canonical bundle' $\det(\mathbb{L}_{\mathbf{X}})$ is a line bundle over the classical scheme $X = t_0(\mathbf{X})$. Suppose we are given an **orientation** of $(\mathbf{X},\omega)$, i.e. a square root line bundle $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$. Then we can construct a canonical perverse sheaf $P^\bullet_{\mathbf{X},\omega}$ on X, such that if $(\mathbf{X},\omega)$ is Zariski locally modelled on $\mathbf{Crit}(f : U \to \mathbb{A}^1)$, then $P^\bullet_{\mathbf{X},\omega}$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ of (U,f). Similarly, we can construct a natural $\mathscr{D}$-module $D^\bullet_{\mathbf{X},\omega}$ on X, and when $\mathbb{K}=\mathbb{C}$ a natural mixed Hodge module $M^\bullet_{\mathbf{X},\omega}$ on X.*

In fact we actually construct the perverse sheaf on the oriented d-critical locus (X,s) associated to $(\mathbf{X},\omega)$ in Theorem 2. We also define perverse sheaves on oriented complex analytic d-critical loci.

Sketch of the proof of Theorem 4

Roughly, we prove Theorem 4 by taking a Zariski open cover $\{\mathbf{R}_i : i \in I\}$ of $\mathbf{X}$ with $\mathbf{R}_i \cong \mathbf{Crit}(f_i : U_i \to \mathbb{A}^1)$, and showing that $\mathcal{PV}^\bullet_{U_i,f_i}$ and $\mathcal{PV}^\bullet_{U_j,f_j}$ are canonically isomorphic on $R_i \cap R_j$, so we can glue the $\mathcal{PV}^\bullet_{U_i,f_i}$ to get a global perverse sheaf $P^\bullet_{\mathbf{X},\omega}$ on X.
In fact things are more complicated: the (local) isomorphisms $\mathcal{PV}^\bullet_{U_i,f_i} \cong \mathcal{PV}^\bullet_{U_j,f_j}$ are only canonical *up to sign*. To make them canonical, we use the square root $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$ to define natural principal $\mathbb{Z}_2$-bundles Q_i on R_i, such that $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i \cong \mathcal{PV}^\bullet_{U_j,f_j} \otimes_{\mathbb{Z}_2} Q_j$ is canonical, and then we glue the $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i$ to get $P^\bullet_{\mathbf{X},\omega}$.

Categorifying Lagrangian intersections

Corollary

Let (S,ω) be a classical smooth symplectic $\mathbb{K}$-scheme of dimension $2n$, and $L,M\subseteq S$ be smooth algebraic Lagrangians, with square roots $K_L^{1/2},K_M^{1/2}$ of their canonical bundles. Then we have a natural perverse sheaf $P^\bullet_{L,M}$ on $X=L\cap M$. The analogue holds for complex Lagrangians in complex symplectic manifolds.

This looks similar to results on quantization of symplectic manifolds, e.g. Kashiwara and Schapira's DQ-modules. K-S build a category of modules $\mathcal{V}$ on S supported on Lagrangians. If $\mathcal{V}_L,\mathcal{V}_M$ are supported on L,M, then $\mathcal{H}om(\mathcal{V}_L,\mathcal{V}_M)$ is a perverse sheaf over $\mathbb{C}[[\hbar]]$ supported on $L\cap M$. But our $P^\bullet_{L,M}$ can be defined over *any commutative ring*, not just over $\mathbb{C}[[\hbar]]$.
We think of the hypercohomology $\mathbb{H}^*(P^\bullet_{L,M})$ as related to the (undefined) *Lagrangian Floer cohomology* by $\mathbb{H}^i(P^\bullet_{L,M})\approx HF^{i+n}(L,M)$.

4. Categorification using perverse sheaves: morphisms

We have seen that oriented −1-shifted symplectic derived $\mathbb{K}$-schemes/stacks $(\mathbf{X},\omega)$ carry perverse sheaves $P^\bullet_{\mathbf{X},\omega}$. We also expect that proper, oriented PTVV Lagrangians $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ should have associated hypercohomology elements $\mu_{\mathbf{L}}\in\mathbb{H}^*(P^\bullet_{\mathbf{X},\omega})$ with interesting properties, which can be interpreted as the morphisms in a categorification of −1-shifted symplectic geometry.

Definition

Let $(\mathbf{X},\omega)$ be a −1-shifted symplectic derived scheme, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian. Choose an orientation $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$ for $(\mathbf{X},\omega)$. The Lagrangian structure induces a natural isomorphism $\alpha:\mathcal{O}_L\xrightarrow{\cong} i^*(\det(\mathbb{L}_{\mathbf{X}}))$. An *orientation* for $\mathbf{L}$ is an isomorphism $\beta:\mathcal{O}_L\xrightarrow{\cong} i^*(\det(\mathbb{L}_{\mathbf{X}})^{1/2})$ with $\beta^2=\alpha$.

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian. Then Theorem 1 shows that $\mathbf{X},\omega$ can be put in an explicit local 'Darboux form' $(\operatorname{Spec} A^\bullet,\omega_A)$. Joyce and Safronov prove a 'Lagrangian Neighbourhood Theorem' saying that $\mathbf{L},\mathbf{i}$ and the homotopy $h:\mathbf{i}^*(\omega)\sim 0$ can also be put in an explicit local form relative to $A^\bullet,\omega_A$. When $k=-1$ this yields:

Theorem 5 (Joyce and Safronov arXiv:1506.04024)

Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian, and $y\in\mathbf{L}$ with $\mathbf{i}(y)=x\in\mathbf{X}$. Theorem 1 implies that $(\mathbf{X},\omega)$ is equivalent near x to $\mathbf{Crit}(H:U\to\mathbb{A}^1)$, for U a smooth, affine $\mathbb{K}$-scheme. Then $\mathbf{L},\mathbf{i},h$ near y have an explicit local model depending on a smooth, affine $\mathbb{K}$-scheme V, a trivial vector bundle $E\to V$, a nondegenerate quadratic form Q on E, a section $s\in H^0(E)$, and a smooth morphism $\phi:V\to U$ with $Q(s,s)=\phi^*(H)$, where $t_0(\mathbf{L})\cong s^{-1}(0)\subseteq V$ Zariski locally.

Conjecture A

Let $(\mathbf{X},\omega)$ be an oriented -1-shifted symplectic derived $\mathbb{K}$-scheme or $\mathbb{K}$-stack, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ an oriented Lagrangian. Then there is a natural morphism in $D^b_c(\mathbf{L})$

$$\mu_{\mathbf{L}}:\mathbb{Q}_{\mathbf{L}}[\operatorname{vdim}\mathbf{L}]\longrightarrow i^!(P^\bullet_{\mathbf{X},\omega}),$$

with given local models in the 'Darboux form' presentations for $\mathbf{X},\omega,\mathbf{L}$ in Theorem 5.

Lino Amorim and I have an outline proof of Conjecture A in the scheme case over $\mathbb{K}=\mathbb{C}$, and also of a complex analytic version. In fact Conjecture A is only the first and simplest in a series of conjectures, which really should be written using ∞-categories, concerning higher coherences of the morphisms $\mu_{\mathbf{L}}$ under products, Verdier duality, composition of Lagrangian correspondences, etc. Our methods also allow us to prove these further conjectures. See Amorim and Ben-Bassat arXiv:1601.01536 for more on this.

Consequences of Conjecture A: 'Fukaya categories' for algebraic / complex symplectic manifolds

Let (S,ω) be a algebraic/complex symplectic manifold, with $\dim_{\mathbb{C}} S = 2n$, and $L, M \subset S$ be algebraic/complex Lagrangians (not supposed compact or closed), with square roots of canonical bundles $K_L^{1/2}, K_M^{1/2}$.
Then the intersection $L \cap M$ is oriented -1-shifted symplectic / an oriented complex analytic d-critical locus, and carries a perverse sheaf $P^\bullet_{L,M}$ by Theorem 4.
We should think of the shifted hypercohomology $\mathbb{H}^{*-n}(P^\bullet_{L,M})$ as a substitute for the Lagrangian Floer cohomology $HF^*(L,M)$ in symplectic geometry. But $HF^*(L,M)$ is the morphisms in the derived Fukaya category $D^b\mathscr{F}(S,\omega)$ in symplectic geometry.

If L, M, N are Lagrangians in (S,ω), then $M \cap L, N \cap M, L \cap N$ are -1-shifted symplectic / d-critical loci, and $L \cap M \cap N$ is Lagrangian in the product $(M \cap L) \times (N \cap M) \times (L \cap N)$ (Ben-Bassat arXiv:1309.0596).
Applying Conjecture A to $L \cap M \cap N$ and rearranging using Verdier duality $P^\bullet_{M,L} \simeq \mathbb{D}(P^\bullet_{M,L})$ gives

$$\mu_{L,M,N} : P^\bullet_{L,M} \overset{L}{\otimes} P^\bullet_{M,N}[n] \longrightarrow P^\bullet_{L,N}.$$

Taking hypercohomology gives the multiplication $HF^*(L,M) \times HF^*(M,N) \to HF^*(L,N)$, which is composition of morphisms in the derived Fukaya category $D^b\mathscr{F}(S,\omega)$.
Higher coherences for such morphisms $\mu_{L,M,N}$ under composition should give the A_∞-structure needed to define a derived 'Fukaya category' $D^b\mathscr{F}(S,\omega)$, which we hope to do. [End of talk.]

Comments on a proof of Conjecture A

In Theorem 4 we constructed a perverse sheaf $P^\bullet_{\mathbf{X},\omega}$ on an oriented -1-shifted symplectic $(\mathbf{X},\omega)$. We did this by constructing a Zariski open cover $\{R_i : i \in I\}$ of $X = t_0(\mathbf{X})$, and perverse sheaves $P^\bullet_i$ on R_i, and isomorphisms $\alpha_{ij} : P^\bullet_i|_{R_i\cap R_j} \to P^\bullet_j|_{R_i\cap R_j}$ on all double overlaps $R_i \cap R_j$, with $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ on triple overlaps $R_i \cap R_j \cap R_k$. Then a unique $P^\bullet_{\mathbf{X},\omega}$ exists with $P^\bullet_{\mathbf{X},\omega}|_{R_i} \cong P^\bullet_i$, as perverse sheaves glue like sheaves.

In Conjecture A, we have explicit local models μ_j for the morphism $\mu_{\mathbf{L}}$ on an open cover $\{S_j : j \in J\}$ of $L = t_0(\mathbf{L})$, constructed using our local models for $\mathbf{L}, \mathbf{X}, \mathbf{i}$ in Theorem 5. However, this is not enough to define $\mu_{\mathbf{L}}$, as such morphisms do not glue like sheaves. It is an ∞-category gluing problem: we need to construct higher coherences between $\mu_{j_1}, \ldots, \mu_{j_n}$ on n-fold overlaps $S_{j_1} \cap \cdots \cap S_{j_n}$ for all $n = 2, \ldots$. This is difficult, as perverse sheaves of vanishing cycles are not easy to handle on the cochain level.

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Actually, to prove Conjecture A we need first to re-prove Theorem 4 in an ∞-categorical way, without using the sheaf property of perverse sheaves, but constructing $P^\bullet_{\mathbf{X},\omega}$ directly as a complex on X. We can define *d-correspondences* $i : L \to (X, s)$ in d-critical loci, which are classical truncations of Lagrangians $\mathbf{i} : \mathbf{L} \to (\mathbf{X},\omega)$ in -1-shifted symplectic schemes. Our proposed proof of Conjecture A factors through these classical truncations, and also has a complex analytic version.

One of our key ideas is to give a new expression for the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ for a holomorphic function $f : U \to \mathbb{C}$ of a complex manifold, as an explicit complex on $\mathrm{Crit}(f)$, using the theory of 'M-cohomology' in Joyce arXiv:1509.05672. This new expression is easier to glue on overlaps between critical charts $(U_i, f_i), (U_j, f_j)$, and to control the higher coherences on multiple overlaps. This complex is built using differential geometry of manifolds, which is why we need $\mathbb{K} = \mathbb{C}$.