Categorification of Donaldson–Thomas theory using perverse sheaves

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Plan of talk:

1. PTVV's shifted symplectic geometry
2. A Darboux theorem for shifted symplectic schemes
3. D-critical loci
4. Categorification using perverse sheaves
5. Motivic Milnor fibres

1. PTVV's shifted symplectic geometry

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K} = \mathbb{C}$. Work in the context of Toën and Vezzosi's theory of *derived algebraic geometry*. This gives ∞-categories of *derived $\mathbb{K}$-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived stacks* $\mathbf{dSt}_{\mathbb{K}}$. For this talk we are interested in derived schemes, though we are working on extensions to derived Artin stacks. Think of a derived $\mathbb{K}$-scheme $\mathbf{X}$ as a geometric space which can be covered by Zariski open sets $\mathbf{Y} \subseteq \mathbf{X}$ with $\mathbf{Y} \simeq \operatorname{Spec} A$ for $A = (A, \mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{K}$.

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Cotangent complexes of derived schemes and stacks

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a notion of *k-shifted symplectic structure* on a derived $\mathbb{K}$-scheme or derived $\mathbb{K}$-stack $\mathbf{X}$, for $k \in \mathbb{Z}$. This is complicated, but here is the basic idea. The *cotangent complex* $\mathbb{L}_{\mathbf{X}}$ of $\mathbf{X}$ is an element of a derived category $L_{\mathrm{qcoh}}(\mathbf{X})$ of quasicoherent sheaves on $\mathbf{X}$. It has exterior powers $\Lambda^p \mathbb{L}_{\mathbf{X}}$ for $p = 0, 1, \ldots$. The *de Rham differential* $\mathrm{d}_{dR} : \Lambda^p \mathbb{L}_{\mathbf{X}} \to \Lambda^{p+1} \mathbb{L}_{\mathbf{X}}$ is a morphism of complexes, though not of $\mathcal{O}_{\mathbf{X}}$-modules. Each $\Lambda^p \mathbb{L}_{\mathbf{X}}$ is a complex, so has an internal differential $\mathrm{d} : (\Lambda^p \mathbb{L}_{\mathbf{X}})^k \to (\Lambda^p \mathbb{L}_{\mathbf{X}})^{k+1}$. We have $\mathrm{d}^2 = \mathrm{d}_{dR}^2 = \mathrm{d} \circ \mathrm{d}_{dR} + \mathrm{d}_{dR} \circ \mathrm{d} = 0$.

p-forms and closed p-forms

A *p-form of degree k* on $\mathbf{X}$ for $k\in\mathbb{Z}$ is an element $[\omega^0]$ of $H^k\bigl(\Lambda^p\mathbb{L}_{\mathbf{X}},\mathrm{d}\bigr)$. A *closed p-form of degree k* on $\mathbf{X}$ is an element

$$[(\omega^0,\omega^1,\ldots)]\in H^k\bigl(\textstyle\bigoplus_{i=0}^\infty\Lambda^{p+i}\mathbb{L}_{\mathbf{X}}[i],\mathrm{d}+\mathrm{d}_{dR}\bigr).$$

There is a projection $\pi:[(\omega^0,\omega^1,\ldots)]\mapsto[\omega^0]$ from closed p-forms $[(\omega^0,\omega^1,\ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.
Note that a closed p-form *is not a special example of a p-form*, but a p-form with an extra structure. The map π from closed p-forms to p-forms can be neither injective nor surjective.

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Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\mathbf{X}$. Then $[\omega^0]$ induces a morphism $\omega^0:\mathbb{T}_{\mathbf{X}}\rightarrow\mathbb{L}_{\mathbf{X}}[k]$, where $\mathbb{T}_{\mathbf{X}}=\mathbb{L}_{\mathbf{X}}^\vee$ is the tangent complex of $\mathbf{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0:\mathbb{T}_{\mathbf{X}}\rightarrow\mathbb{L}_{\mathbf{X}}[k]$ is a quasi-isomorphism.
If $\mathbf{X}$ is a derived scheme then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty,0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0,\infty)$. So $\omega^0:\mathbb{T}_{\mathbf{X}}\rightarrow\mathbb{L}_{\mathbf{X}}[k]$ can be a quasi-isomorphism only if $k\leqslant 0$, and then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k,0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0,-k]$. If $k=0$ then $\mathbf{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k=-1$ then $\mathbf{X}$ is quasi-smooth.
A closed 2-form $\omega=[(\omega^0,\omega^1,\ldots)]$ of degree k on $\mathbf{X}$ is called a *k-shifted symplectic structure* if $[\omega^0]=\pi(\omega)$ is nondegenerate.

Dominic Joyce, Oxford University Categorification of D–T theory

Calabi–Yau moduli schemes and moduli stacks

Pantev et al. prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a natural $(2-m)$-shifted symplectic structure ω. So Calabi–Yau 3-folds give -1-shifted derived schemes or stacks.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E]\in\boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{1-i}(E,E)^*$. The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E)\cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0:\mathbb{T}_{\boldsymbol{\mathcal{M}}}\to\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

Lagrangians and Lagrangian intersections

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\mathbf{L}$ in $(\mathbf{X},\omega)$, which is a morphism $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ of derived schemes or stacks together with a homotopy $i^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\mathbf{L}}\simeq\mathbb{L}_{\mathbf{L}/\mathbf{X}}[k-1]$.

If $\mathbf{L},\mathbf{M}$ are Lagrangians in $(\mathbf{X},\omega)$, then the fibre product $\mathbf{L}\times_{\mathbf{X}}\mathbf{M}$ has a natural $(k-1)$-shifted symplectic structure.

If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

2. A Darboux theorem for shifted symplectic schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$. If $k\not\equiv 2 \mod 4$, then each $x\in\mathbf{X}$ admits a Zariski open neighbourhood $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A$ for (A,d) an explicit cdga over $\mathbb{K}$ generated by graded variables x_j^{-i},y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\mathbf{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l,y_j^l have degree l, and

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in (A,d) is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$.
If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 the degree $k/2$ variables depending on some invertible functions.

Sketch of the proof of the theorem

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $x\in\mathbf{X}$. Then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k,0]$. We first show that we can build Zariski open $x\in\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A$, for $A=\bigoplus_{i\leqslant 0}A^i$ a cdga over $\mathbb{K}$ with A^0 a smooth $\mathbb{K}$-algebra, and such that A is freely generated over A^0 by graded variables x_j^{-i},y_j^{k+i} in degrees $-1,-2,\ldots,k$. We take $\dim A^0$ and the number of x_j^{-i},y_j^{k+i} to be minimal at x.
Using theorems about periodic cyclic cohomology, we show that on $Y\simeq\operatorname{Spec}A$ we can write $\omega|_Y=[(\omega^0,0,0,\ldots)]$, for ω^0 a 2-form of degree k with $\mathrm{d}\omega^0=\mathrm{d}_{dR}\omega^0=0$. Minimality at x implies ω^0 is strictly nondegenerate near x, so we can change variables to write $\omega^0=\sum_{i,j}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}$. Finally, we show d in (A,d) is a symplectic vector field, which integrates to a Hamiltonian H.

The case of -1-shifted symplectic derived schemes

When $k = -1$ the Hamiltonian H in the theorem has degree 0. Then the theorem reduces to:

Corollary

Suppose $(\mathbf{X}, \omega)$ is a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\mathbf{X}, \omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H : U \to \mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H : U \to \mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X = t_0(\mathbf{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$.

Combining this with results of Pantev et al. from §1 gives interesting consequences in classical algebraic geometry:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

Here we note that $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli scheme, which is -1-shifted symplectic by PTVV.
A complex analytic analogue of this for moduli of coherent sheaves was proved using gauge theory by Joyce and Song arXiv:0810.5645, and for moduli of complexes was claimed by Behrend and Getzler.
Note that the proof of the corollary is wholly algebro-geometric.

Corollary

Let (S,ω) be a classical smooth symplectic $\mathbb{K}$-scheme, and $L, M \subseteq S$ be smooth algebraic Lagrangians. Then the intersection $L \cap M$, as a $\mathbb{K}$-subscheme of S, is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

In real or complex symplectic geometry, where Darboux Theorem holds, the analogue of the corollary is easy to prove, but in classical algebraic symplectic geometry we do not have a Darboux Theorem, so the corollary is not obvious.

3. D-critical loci

Theorem (Joyce arXiv:1304.4508)

Let X be a classical $\mathbb{K}$-scheme. Then there exists a canonical sheaf $\mathcal{S}_X$ of $\mathbb{K}$-vector spaces on X, such that if $R \subseteq X$ is Zariski open and $i : R \hookrightarrow U$ is a closed embedding of R into a smooth $\mathbb{K}$-scheme U, and $I_{R,U} \subseteq \mathcal{O}_U$ is the ideal vanishing on $i(R)$, then

$$\mathcal{S}_X|_R \cong \mathrm{Ker}\left(\frac{\mathcal{O}_U}{I_{R,U}^2} \xrightarrow{\mathrm{d}} \frac{T^*U}{I_{R,U} \cdot T^*U}\right).$$

Also $\mathcal{S}_X$ splits naturally as $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{K}_X$, where $\mathbb{K}_X$ is the sheaf of locally constant functions $X \to \mathbb{K}$.

The meaning of the sheaves $\mathcal{S}_X, \mathcal{S}_X^0$

If $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then taking $R = X$, $i =$inclusion, we see that $f + I_{X,U}^2$ is a section of $\mathcal{S}_X$. Also $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{K}$ is locally constant, and if $f|_{X^{\mathrm{red}}} = 0$ then $f + I_{X,U}^2$ is a section of $\mathcal{S}_X^0$. Note that $f + I_{X,U} = f|_X$ in $\mathcal{O}_X = \mathcal{O}_U / I_{X,U}$. The theorem means that $f + I_{X,U}^2$ makes sense *intrinsically on* X, without reference to the embedding of X into U.

That is, if $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then we can remember f up to second order in the ideal I_X as a piece of data on X, not on U.

Suppose $X = \mathrm{Crit}(f : U \to \mathbb{A}^1) = \mathrm{Crit}(g : V \to \mathbb{A}^1)$ is written as a critical locus in two different ways. Then $f + I_{X,U}^2$, $g + I_{X,V}^2$ are sections of $\mathcal{S}_X$, so we can ask whether $f + I_{X,U}^2 = g + I_{X,V}^2$. This gives a way to compare isomorphic critical loci in different smooth classical schemes.

The definition of d-critical loci

Definition (Joyce arXiv:1304.4508)

An (*algebraic*) *d-critical locus* (X, s) is a classical $\mathbb{K}$-scheme X and a global section $s \in H^0(\mathcal{S}_X^0)$ such that X may be covered by Zariski open $R \subseteq X$ with an isomorphism $i : R \to \mathrm{Crit}(f : U \to \mathbb{A}^1)$ identifying $s|_R$ with $f + I_{R,U}^2$, for f a regular function on a smooth $\mathbb{K}$-scheme U.

That is, a d-critical locus (X, s) is a $\mathbb{K}$-scheme X which may Zariski locally be written as a critical locus $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, and the section s remembers f up to second order in the ideal $I_{X,U}$.

We also define *complex analytic d-critical loci*, with X a complex analytic space locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ for U a complex manifold and f holomorphic.

Orientations on d-critical loci

Theorem (Joyce arXiv:1304.4508)

Let (X, s) be an algebraic d-critical locus and X^{red} the reduced $\mathbb{K}$-subscheme of X. Then there is a natural line bundle $K_{X,s}$ on X^{red} called the **canonical bundle**, *such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$ then $K_{X,s}$ is locally modelled on $K_U^{\otimes^2}|_{\mathrm{Crit}(f)^{\mathrm{red}}}$, for K_U the usual canonical bundle of U.*

Definition

Let (X, s) be a d-critical locus. An *orientation* on (X, s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on X^{red}.

This is related to *orientation data* in Kontsevich–Soibelman 2008.

A truncation functor from -1-symplectic derived schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\mathbf{X}, \omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then the classical $\mathbb{K}$-scheme $X = t_0(\mathbf{X})$ extends naturally to an algebraic d-critical locus (X, s). The canonical bundle of (X, s) satisfies $K_{X,s} \cong \det \mathbb{L}_{\mathbf{X}}|_{X^{\mathrm{red}}}$.

That is, we define a *truncation functor* from -1-shifted symplectic derived $\mathbb{K}$-schemes to algebraic d-critical loci. Examples show this functor is not full. Think of d-critical loci as *classical truncations* of -1-shifted symplectic derived $\mathbb{K}$-schemes.

An alternative semi-classical truncation, used in D–T theory, is *schemes with symmetric obstruction theory*. D-critical loci appear to be better, for both categorified and motivic D–T theory.

The corollaries in §2 imply:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M}, s)$. The canonical bundle satisfies $K_{\mathcal{M},s} \cong \det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$, where $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is the (symmetric) obstruction theory on $\mathcal{M}$ defined by Thomas or Huybrechts and Thomas.

Corollary

Let (S, ω) be a classical smooth symplectic $\mathbb{K}$-scheme, and $L, M \subseteq S$ be smooth algebraic Lagrangians. Then $X = L \cap M$ extends to naturally to a d-critical locus (X, s). The canonical bundle satisfies $K_{X,s} \cong K_L|_{X^{\rm red}} \otimes K_M|_{X^{\rm red}}$. Hence, choices of square roots $K_L^{1/2}, K_M^{1/2}$ give an orientation for (X, s).

4. Categorification using perverse sheaves

Theorem (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)

Let (X, s) be an algebraic d-critical locus over $\mathbb{K}$, with an orientation $K_{X,s}^{1/2}$. Then we can construct a canonical perverse sheaf $P_{X,s}^\bullet$ on X, such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, then $P_{X,s}^\bullet$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}_{U,f}^\bullet$ of (U, f).
Similarly, we can construct a natural $\mathcal{D}$-module $D_{X,s}^\bullet$ on X, and when $\mathbb{K} = \mathbb{C}$ a natural mixed Hodge module $M_{X,s}^\bullet$ on X.

Sketch of the proof of the theorem

Roughly, we prove the theorem by taking a Zariski open cover $\{R_i : i \in I\}$ of X with $R_i \cong \mathrm{Crit}(f_i : U_i \to \mathbb{A}^1)$, and showing that $\mathcal{PV}^\bullet_{U_i,f_i}$ and $\mathcal{PV}^\bullet_{U_j,f_j}$ are canonically isomorphic on $R_i \cap R_j$, so we can glue the $\mathcal{PV}^\bullet_{U_i,f_i}$ to get a global perverse sheaf $P^\bullet_{X,s}$ on X.
In fact things are more complicated: the (local) isomorphisms $\mathcal{PV}^\bullet_{U_i,f_i} \cong \mathcal{PV}^\bullet_{U_j,f_j}$ are only canonical *up to sign*. To make them canonical, we use the orientation $K^{1/2}_{X,s}$ to define natural principal $\mathbb{Z}_2$-bundles Q_i on R_i, such that $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i \cong \mathcal{PV}^\bullet_{U_j,f_j} \otimes_{\mathbb{Z}_2} Q_j$ is canonical, and then we glue the $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i$ to get $P^\bullet_{X,s}$.

The first corollary in §2 implies:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a finite-dimensional graded vector space. The pointwise Euler characteristic $\chi(P^\bullet_{\mathcal{M},s})$ is the *Behrend function* $\nu_{\mathcal{M}}$ of $\mathcal{M}$. Thus

$$\sum_{i\in\mathbb{Z}}(-1)^i \dim \mathbb{H}^i(P^\bullet_{\mathcal{M},s}) = \chi(\mathcal{M},\nu_{\mathcal{M}}).$$

Now by Behrend 2005, the Donaldson–Thomas invariant of $\mathcal{M}$ is $DT(\mathcal{M}) = \chi(\mathcal{M},\nu_{\mathcal{M}})$. So, $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a graded vector space with dimension $DT(\mathcal{M})$, that is, a *categorification* of $DT(\mathcal{M})$.

Categorifying Lagrangian intersections

The second corollary in §2 implies:

Corollary

Let (S, ω) be a classical smooth symplectic $\mathbb{K}$-scheme of dimension $2n$, and $L, M \subseteq S$ be smooth algebraic Lagrangians, with square roots $K_L^{1/2}, K_M^{1/2}$ of their canonical bundles. Then we have a natural perverse sheaf $P^\bullet_{L,M}$ on $X = L \cap M$.

This is related to Behrend and Fantechi 2009, and Kai's talk. We think of the hypercohomology $\mathbb{H}^*(P^\bullet_{L,M})$ as being morally related to the *Lagrangian Floer cohomology* $HF^*(L, M)$ by

$$\mathbb{H}^i(P^\bullet_{L,M}) \approx HF^{i+n}(L, M).$$

We are working on defining 'Fukaya categories' for algebraic/complex symplectic manifolds using these ideas.

5. Motivic Milnor fibres

By similar arguments to those used to construct the perverse sheaves $P^\bullet_{X,s}$ in §4, we prove:

Theorem (Bussi, Joyce and Meinhardt arXiv:1305.6428)

Let (X, s) be an algebraic d-critical locus over $\mathbb{K}$, with an orientation $K_{X,s}^{1/2}$. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat{\mu}$-equivariant motives $\bar{\mathcal{M}}_X^{\hat{\mu}}$ on X, such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, then $MF_{X,s}$ is locally modelled on $\mathbb{L}^{-\dim U/2}([X] - MF^{\mathrm{mot}}_{U,f})$, where $MF^{\mathrm{mot}}_{U,f}$ is the **motivic Milnor fibre** *of f.*

Vittoria Bussi's talk will give more details.

Relation to motivic D–T invariants

The first corollary in §2 implies:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural motive $MF^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

This motive $MF^\bullet_{\mathcal{M},s}$ is essentially the motivic Donaldson–Thomas invariant of $\mathcal{M}$ defined (partially conjecturally) by Kontsevich and Soibelman 2008. K–S work with motivic Milnor fibres of formal power series at each point of $\mathcal{M}$. Our results show the formal power series can be taken to be a regular function, and clarify how the motivic Milnor fibres vary in families over $\mathcal{M}$.