# Derived differential geometry

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work in progress,

chapters of a book on it

may be downloaded from

`people.ox.ac.uk/~joyce/dmanifolds.html.`

See also arXiv:0910.3518

and arXiv:1001.0023.

These slides available at

`people.ox.ac.uk/~joyce/talks.html.`

## 1. Introduction

Many important areas in both differential and algebraic geometry involve forming 'moduli spaces' $\mathcal{M}$ of some geometric objects, and then 'counting' the points in $\mathcal{M}$ to get an 'invariant' $I(\mathcal{M})$ with interesting properties, for example Donaldson, Seiberg–Witten, Gromov–Witten and Donaldson–Thomas invariants. Taking the 'invariant' to be a vector space, category, . . . , rather than a number, Floer homology theories, contact homology, Symplectic Field Theory, and Fukaya categories also fit in this framework.

All these ‘invariants’ theories have some common features:

- You start with some geometrical space $X$ you want to study.
- You define a moduli space $\mathcal{M}$ of auxiliary geometric objects $E$ on $X$.
- This $\mathcal{M}$ is a topological space, hopefully compact and Hausdorff, but generally not a manifold – it may have bad singularities.
- Nevertheless, $\mathcal{M}$ behaves *as if* it is a compact, oriented manifold of known dimension $k$. One defines a *virtual class* $[\mathcal{M}]_{\mathrm{vir}}$ in $H_k(\mathcal{M};\mathbb{Q})$, which ‘counts’ the points in $\mathcal{M}$.
- This $[\mathcal{M}]_{\mathrm{vir}}$ is then independent of choices in the construction, deformations of $X$ etc.

Methods for defining $[\mathcal{M}]_{\mathrm{vir}}$ vary. In good cases, with generic initial data $\mathcal{M}$ is smooth. Otherwise, we prove $\mathcal{M}$ has some extra geometric structure $\mathcal{G}$, and use $\mathcal{G}$ to define $[\mathcal{M}]_{\mathrm{vir}}$.

- In algebraic geometry problems $\mathcal{M}$ is a scheme or Deligne–Mumford stack with obstruction theory.
- In areas of symplectic geometry based on moduli of $J$-holomorphic curves – Gromov–Witten theory, Lagrangian Floer cohomology, Symplectic Field Theory, Fukaya categories – there are two main geometric structures: *Kuranishi spaces* (Fukaya–Oh–Ohta–Ono) and *polyfolds* (Hofer–Wysocki–Zehnder).

## 2. D-manifolds and d-orbifolds

I will describe a new class of geometric objects I call *d-manifolds* — 'derived' smooth manifolds. Some properties of d-manifolds:

- They form a *strict 2-category* $\mathbf{dMan}$. That is, we have objects $\boldsymbol{X}$, the d-manifolds, 1-morphisms $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$, the smooth maps, and also 2-morphisms $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$.
- Smooth manifolds embed into d-manifolds as a full (2)-subcategory.
- There are also 2-categories $\mathbf{dMan^b}$, $\mathbf{dMan^c}$ of d-manifolds *with boundary* and *with corners*, and orbifold versions $\mathbf{dOrb}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$ of these, *d-orbifolds*.

• Many concepts of differential geometry extend nicely to d-manifolds: submersions, immersions, orientations, submanifolds, transverse fibre products, cotangent bundles, . . . .
• Almost any moduli space used in any enumerative invariant problem over $\mathbb{R}$ or $\mathbb{C}$ has a d-manifold or d-orbifold structure, natural up to equivalence. There are truncation functors to d-manifolds and d-orbifolds from structures currently used – $\mathbb{C}$-schemes with obstruction theories, Kuranishi spaces, polyfolds.
• Virtual classes/cycles/chains can be constructed for compact oriented d-manifolds and d-orbifolds.

So, d-manifolds and d-orbifolds provide a unified framework for studying enumerative invariants and moduli spaces. They also have other applications, and are interesting and beautiful in their own right.

D-manifolds and d-orbifolds are related to other classes of spaces already studied, in particular to the *Kuranishi spaces* of Fukaya–Oh–Ohta–Ono in symplectic geometry, and to David Spivak's *derived manifolds*, from Jacob Lurie's 'derived algebraic geometry' programme.

## 2.1. Kuranishi spaces

*Kuranishi spaces* were defined by Fukaya–Ono 1999 and Fukaya–Oh–Ohta–Ono 2009 as the geometric structure on moduli spaces $\mathcal{M}$ of $J$-holomorphic curves in symplectic geometry. A Kuranishi space is locally modelled on the zeroes $s^{-1}(0)$ of a smooth section $s$ of a vector bundle $E \to V$ over an orbifold $V$. The theory has a lot of problems, and is basically incomplete.

My starting point for this project was to find the 'right' definition of Kuranishi space. I claim that this is: *a Kuranishi space is (should really be) a d-orbifold with corners*.

## 2.2. Derived manifolds

*Derived manifolds* were defined by David Spivak (Duke Math. J. 153, 2010), a student of Jacob Lurie. A lot of my ideas are stolen from Spivak. D-manifolds are much simpler than derived manifolds. D-manifolds are a 2-category, using Hartshorne-level algebraic geometry. Derived manifolds are an $\infty$-category, and use very advanced and scary technology – homotopy sheaves, Bousfeld localization, . . . .

D-manifolds are a 2-category truncation of derived manifolds. I claim that this truncation remembers all the geometric information of importance to symplectic geometers, and other real people.

## 3. The definition of d-manifolds

Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes.

In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces.

There is a little-known theory of schemes in differential geometry, $C^\infty$-*schemes*, going back to Lawvere, Dubuc, Moerdijk and Reyes, . . . in synthetic differential geometry in the 1960s-1980s. This will be the foundation of our d-manifolds.

## 3.1. $C^\infty$-rings

Let $X$ be a manifold, and $C^\infty(X)$ the set of smooth functions $c : X \to \mathbb{R}$. Then $C^\infty(X)$ is an $\mathbb{R}$-*algebra*, by adding and multiplying smooth functions. But there are many more operations on $C^\infty(X)$, e.g. if $c : X \to \mathbb{R}$ is smooth then $\exp(c) : X \to \mathbb{R}$ is smooth, giving $\exp : C^\infty(X) \to C^\infty(X)$, algebraically independent of addition and multiplication.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by

$$\Phi_f(c_1, \ldots, c_n)(x) = f\bigl(c_1(x), \ldots, c_n(x)\bigr)$$

for all $x \in X$. Addition comes from $f : \mathbb{R}^2 \to \mathbb{R}$, $f : (c_1, c_2) \mapsto c_1 + c_2$, multiplication from $(c_1, c_2) \mapsto c_1 c_2$.

**Definition.** A $C^\infty$-*ring* is a set $\mathfrak{C}$ together with $n$-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying the following conditions:
Let $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathfrak{C}$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have $\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j$.
A *morphism* of $C^\infty$-rings is $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of $C^\infty$-rings.

Then $C^\infty(X)$ is a $C^\infty$-ring for any manifold $X$, and from $C^\infty(X)$ we can recover $X$ up to isomorphism. If $f : X \to Y$ is smooth then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of $C^\infty$-rings. This gives a *full and faithful functor* $F : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\mathbf{op}}$ by $F : X \mapsto C^\infty(X)$, $F : f \mapsto f^*$.

Thus, we think of manifolds as examples of $C^\infty$-rings, and $C^\infty$-rings as generalizations of manifolds. But there are many more $C^\infty$-rings than manifolds, e.g. $C^0(X)$ is a $C^\infty$-ring for any topological space $X$.

## 3.2. $C^\infty$-schemes

We can now develop the whole machinery of scheme theory in algebraic geometry, replacing rings or algebras by $C^\infty$-rings throughout — see my arXiv:1001.0023.

We obtain a category $\mathbf{C^\infty Sch}$ of $C^\infty$-*schemes* $\underline{X} = (X, \mathcal{O}_X)$, which are topological space $X$ equipped with a sheaf of $C^\infty$-rings $\mathcal{O}_X$ locally modelled on the spectrum of a $C^\infty$-ring.

If $X$ is a manifold, define a $C^\infty$-scheme $\underline{X} = (X, \mathcal{O}_X)$ by $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. This defines a full and faithful embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$.

We also define *vector bundles*, *coherent sheaves* coh($\underline{X}$) and *quasi-coherent sheaves* qcoh($\underline{X}$), and the *cotangent sheaf* $T^*\underline{X}$ on $\underline{X}$. Then qcoh($\underline{X}$) is an abelian category.

Some differences with conventional algebraic geometry:

• affine schemes are Hausdorff. No need to introduce étale topology.

• partitions of unity exist subordinate to any open cover of a (nice) $C^\infty$-scheme $\underline{X}$.

• $C^\infty$-rings such as $C^\infty(\mathbb{R}^n)$ are not noetherian as $\mathbb{R}$-algebras. Causes problems with coherent sheaves: coh($\underline{X}$) is not closed under kernels, so not an abelian category.

## 3.3. The 2-category of d-spaces

We define d-manifolds as a 2-subcategory of a larger 2-category of *d-spaces*. These are 'derived' versions of $C^\infty$-schemes.

**Definition.** A *d-space* is a is a quintuple $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ where $\underline{X} = (X, \mathcal{O}_X)$ is a separated, second countable, locally fair $C^\infty$-scheme, $\mathcal{O}'_X$ is a second sheaf of $C^\infty$-rings on $X$, and $\mathcal{E}_X$ is a quasicoherent sheaf on $\underline{X}$, and $\imath_X : \mathcal{O}'_X \to \mathcal{O}_X$ is a surjective morphism of sheaves of $C^\infty$-rings whose kernel $\mathcal{I}_X$ is a sheaf of *square zero ideals* in $\mathcal{O}'_X$, and $\jmath_X : \mathcal{E}_X \to \mathcal{I}_X$ is a surjective morphism in qcoh($\underline{X}$), so we have an exact sequence of sheaves on $X$:

$$\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{O}'_X \xrightarrow{\imath_X} \mathcal{O}_X \longrightarrow 0.$$

A 1-*morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a triple $\boldsymbol{f} = (\underline{f}, f', f'')$, where $\underline{f} = (f, f^\sharp) : \underline{X} \to \underline{Y}$ is a morphism of $C^\infty$-schemes and $f' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$, $f'' : \underline{f}^*(\mathcal{E}_Y) \to \mathcal{E}_X$ are sheaf morphisms such that the following commutes:

$$\begin{array}{ccccccc} f^{-1}(\mathcal{E}_Y) & \xrightarrow{f^{-1}(\jmath_Y)} & f^{-1}(\mathcal{O}'_Y) & \xrightarrow{f^{-1}(\imath_Y)} & f^{-1}(\mathcal{O}_X) & \longrightarrow & 0 \\ \downarrow f'' & & \downarrow f' & & \downarrow f^\sharp & & \\ \mathcal{E}_X & \xrightarrow{\jmath_X} & \mathcal{O}'_X & \xrightarrow{\imath_X} & \mathcal{O}_X & \longrightarrow & 0. \end{array}$$

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms with $\boldsymbol{f} = (\underline{f}, f', f'')$, $\boldsymbol{f} = (\underline{g}, g', g'')$. Suppose $\underline{f} = \underline{g}$. A 2-*morphism* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a morphism

$$\eta : f^{-1}(\Omega_{\mathcal{O}'_Y}) \otimes_{f^{-1}(\mathcal{O}'_Y)} \mathcal{O}_X \longrightarrow \mathcal{E}_X$$

in qcoh$(\underline{X})$, where $\Omega_{\mathcal{O}'_Y}$ is the sheaf of cotangent modules of $\mathcal{O}'_Y$, such that $g' = f' + \jmath_X \circ \eta \circ \Pi_{XY}$ and $g'' = f'' + \eta \circ \underline{f}^*(\phi_Y)$, for natural morphisms $\Pi_{XY}, \phi_Y$.

**Theorem 1.** *This defines a strict* 2*-category* $\mathbf{dSpa}$. *All fibre products exist in* $\mathbf{dSpa}$.

We can map $\mathbf{C^\infty Sch}$ into $\mathbf{dSpa}$ by taking a $C^\infty$-scheme $\underline{X} = (X, \mathcal{O}_X)$ to the d-space $\boldsymbol{X} = (\underline{X}, \mathcal{O}_X, 0, \mathrm{id}_{\mathcal{O}_X}, 0)$, with exact sequence

$$0 \xrightarrow{0} \mathcal{O}_X \xrightarrow{\mathrm{id}_{\mathcal{O}_X}} \mathcal{O}_X \longrightarrow 0.$$

This embeds $\mathbf{C^\infty Sch}$, and hence manifolds $\mathbf{Man}$, as discrete 2-subcategories of $\mathbf{dSpa}$. For *transverse* fibre products of manifolds, the fibre products in $\mathbf{Man}$ and $\mathbf{dSpa}$ agree.

## 3.4. The 2-subcategory of d-manifolds

**Definition.** A d-space $\boldsymbol{X}$ is a *d-manifold of dimension* $n\in\mathbb{Z}$ if $\boldsymbol{X}$ may be covered by open d-subspaces $\boldsymbol{Y}$ equivalent in $\mathbf{dSpa}$ to a fibre product $\boldsymbol{U}\times_{\boldsymbol{W}}\boldsymbol{V}$, where $\boldsymbol{U},\boldsymbol{V},\boldsymbol{W}$ are manifolds without boundary and $\dim\boldsymbol{U}+\dim\boldsymbol{V}-\dim\boldsymbol{W}=n$. We allow $n<0$.

Think of a d-manifold $\boldsymbol{X}=(\underline{X},\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X)$ as a 'classical' $C^\infty$-scheme $\underline{X}$, with extra 'derived' data $\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X$.

Write $\mathbf{dMan}$ for the full 2-subcategory of d-manifolds in $\mathbf{dSpa}$. It is not closed under fibre products in $\mathbf{dSpa}$, but we can say:

**Theorem 2.** *All fibre products of the form* $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$ *with* $\boldsymbol{X},\boldsymbol{Y}$ *d-manifolds and* $\boldsymbol{Z}$ *a manifold exist in the 2-category* $\mathbf{dMan}$.

# 4. Properties of d-manifolds

## 4.1. Gluing by equivalences

A 1-morphism $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ in $\mathbf{dMan}$ is an *equivalence* if there exist a 1-morphism $\boldsymbol{g}:\boldsymbol{Y}\rightarrow\boldsymbol{X}$ and 2-morphisms $\eta:\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$ and $\zeta:\boldsymbol{f}\circ\boldsymbol{g}\Rightarrow\mathbf{id}_{\boldsymbol{Y}}$.

**Theorem3.** *Let $\boldsymbol{X},\boldsymbol{Y}$ be d-manifolds, $\emptyset\neq\boldsymbol{U}\subseteq\boldsymbol{X}$, $\emptyset\neq\boldsymbol{V}\subseteq\boldsymbol{Y}$ open d-submanifolds, and $\boldsymbol{f}:\boldsymbol{U}\rightarrow\boldsymbol{V}$ an equivalence. Suppose the topological space $Z=X\cup_{U=V}Y$ made by gluing $X,Y$ using $f$ is Hausdorff. Then there exists a d-manifold $\boldsymbol{Z}$, unique up to equivalence, open $\hat{\boldsymbol{X}},\hat{\boldsymbol{Y}}\subseteq\boldsymbol{Z}$ with $\boldsymbol{Z}=\hat{\boldsymbol{X}}\cup\hat{\boldsymbol{Y}}$, equivalences $\boldsymbol{g}:\boldsymbol{X}\rightarrow\hat{\boldsymbol{X}}$ and $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\hat{\boldsymbol{Y}}$, and a 2-morphism $\eta:\boldsymbol{g}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$.*

Equivalence is the natural notion of when two objects in $\mathbf{dMan}$ are 'the same'. In Theorem 3, $\boldsymbol{Z}$ is a *pushout* $\boldsymbol{X}\amalg_{\mathbf{id}_{\boldsymbol{U}},\boldsymbol{U},\boldsymbol{f}}\boldsymbol{Y}$ in $\mathbf{dMan}$. Theorem 3 generalizes to gluing families of d-manifolds $\boldsymbol{X}_i : i \in I$ by equivalences on double overlaps $\boldsymbol{X}_i \cap \boldsymbol{X}_j$, with (weak) conditions on triple overlaps $\boldsymbol{X}_i \cap \boldsymbol{X}_j \cap \boldsymbol{X}_k$.
This is very useful for proving existence of d-manifold structures on moduli spaces.

## 4.2. Virtual vector bundles

Vector bundle and cotangent bundles have good 2-category generalizations. Let $\underline{X}$ be a $C^\infty$-scheme. Define a 2-category vqcoh($\underline{X}$) of *virtual quasicoherent sheaves* to have objects morphisms $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ in qcoh($\underline{X}$). If $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ and $\psi : \mathcal{F}^1 \to \mathcal{F}^2$ are objects, a 1-morphism $(f^1, f^2) : \phi \to \psi$ is morphisms $f^j : \mathcal{E}^j \to \mathcal{F}^j$ in qcoh($\underline{X}$) for $j = 1, 2$ with $\psi \circ f^1 = f^2 \circ \phi$. If $(f^1, f^2), (g^1, g^2)$ are 1-morphisms $\phi \to \psi$, a 2-morphism $\eta : (f^1, f^2) \Rightarrow (g^1, g^2)$ is a morphism $\eta : \mathcal{E}^2 \to \mathcal{F}^1$ in qcoh($\underline{X}$) with $g^1 = f^1 + \eta \circ \phi$ and $g^2 = f^2 + \psi \circ \eta$.

Call $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ a *virtual vector bundle* on $\underline{X}$ of rank $k \in \mathbb{Z}$ if $X$ may be covered by open $\underline{U} \subseteq \underline{X}$ such that $\phi|_{\underline{U}} : \mathcal{E}^1|_{\underline{U}} \to \mathcal{E}^2|_{\underline{U}}$ is equivalent in the 2-category vqcoh($\underline{U}$) to $\psi : \mathcal{F}^1 \to \mathcal{F}^2$, where $\mathcal{F}^1, \mathcal{F}^2$ are vector bundles on $\underline{U}$ with $\operatorname{rank} \mathcal{F}^2 - \operatorname{rank} \mathcal{F}^1 = k$. Write vvect($\underline{X}$) for the full 2-subcategory of virtual vector bundles on vqcoh($\underline{X}$).

If $\boldsymbol{X}$ is a d-manifold, it has a natural *virtual cotangent bundle* $T^*\boldsymbol{X}$ in vvect($\underline{X}$), of rank $\operatorname{vdim} \boldsymbol{X}$.

If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in $\mathbf{dMan}$, there is a natural 1-morphism $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ in vvect($\underline{X}$).

Then $\boldsymbol{f}$ is *étale* (a local equivalence) if and only if $\Omega_{\boldsymbol{f}}$ is an equivalence in vvect($\underline{X}$). Similarly, $\boldsymbol{f}$ is an *immersion* or *submersion* if $\Omega_{\boldsymbol{f}}$ is surjective or injective in a suitable sense. If $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ lies in vvect($\underline{X}$) we can define a line bundle $\mathcal{L}_\phi$ on $\underline{X}$ analogous to the 'top exterior power' of $\phi : \mathcal{E}^1 \to \mathcal{E}^2$. So for a d-manifold $\boldsymbol{X}$, $\mathcal{L}_{T^*\boldsymbol{X}}$ is a line bundle on $\underline{X}$ which we think of as $\Lambda^{\text{top}} T^*\boldsymbol{X}$. An *orientation* on $\boldsymbol{X}$ is an orientation on the line bundle $\mathcal{L}_{T^*\boldsymbol{X}}$. Orientations have the properties one would expect from the manifold case.