Artin stacks, constructible functions and motivic invariants

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1. Introduction

Let P be a projective $\mathbb{K}$-scheme, coh(P) the abelian category of coherent sheaves on P, and $(\tau, T, \leqslant)$ a *stability condition* on coh(P) – for instance, *Gieseker stability* w.r.t. some ample line bundle L. Then we can form *moduli spaces* $\mathrm{Obj}^{\alpha}_{\mathrm{st}}(\tau), \mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tau)$ of τ-(semi)stable sheaves in coh(P) with fixed Chern character $\alpha \in H^{\mathrm{even}}(P)$.

Basically these moduli spaces are sets of isomorphism (or other equivalence) classes of sheaves, upon which we hope to put some algebraic structure ($\mathbb{K}$-scheme, algebraic $\mathbb{K}$-space,. . .).

The usual approach is to use *S-equivalence* rather than isomorphism, and make $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tau)$ into a *coarse moduli scheme*.

But I'm interested in comparing moduli spaces $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau), \mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tilde\tau)$ for *two different* stability conditions $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant)$ – say, defined using different ample line bundles $L, \tilde L$. To do this we want to regard $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau), \mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tilde\tau)$ as subsets of a larger 'moduli space' $\mathfrak{Obj}_{\mathrm{coh}(P)}(\mathbb{K})$ of *all* coherent sheaves. To define $\mathfrak{Obj}_{\mathrm{coh}(P)}(\mathbb{K})$ we can't use S-equivalence as this depends on τ, so use *isomorphism*.

The right framework is *Artin stacks*. That is, there is a natural moduli stack $\mathfrak{Obj}_{\mathrm{coh}(P)}$ of coherent sheaves on P such that the set of $\mathbb{K}$-points $\mathfrak{Obj}_{\mathrm{coh}(P)}(\mathbb{K})$ is the set of isomorphism classes of coherent sheaves. Then $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau), \mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tilde\tau)$ are *constructible subsets* in $\mathfrak{Obj}_{\mathrm{coh}(P)}$.

My plan for relating τ- and $\tilde{\tau}$-semistability is to construct $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tilde{\tau})$ from the $\mathrm{Obj}^{\beta}_{\mathrm{ss}}(\tau)$ by *adding and subtracting subsets* of $[X] \in \mathfrak{Obj}_{\mathrm{coh}(P)}(\mathbb{K})$ with filtrations of the form $0 = A_0 \subset A_1 \subset \cdots \subset A_n = X$ with $S_i = A_i/A_{i-1}$ τ-semistable.

A convenient way to do this is to represent a constructible set $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tilde{\tau})$ by its *characteristic function*, and work with *constructible functions* on $\mathfrak{Obj}_{\mathrm{coh}(P)}$; then can add and subtract functions rather than sets.

So, we need a theory of *constructible functions on Artin stacks*.

Also, I want to study *invariants* of moduli spaces $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$. For these to be compatible with adding and subtracting subsets, they must:

- be defined for *constructible subsets in Artin* $\mathbb{K}$*-stacks*;
- take values in an *abelian group*;
- be *additive over finite disjoint unions* of constructible subsets.

Call such an invariant *motivic*. For instance, Euler characteristics are motivic, but cohomology is not.

Goals of this talk:

To develop tools to use in the programme.

- Describe the theory of *constructible functions on Artin stacks*, pushforwards and pullbacks along 1-morphisms, and so on.
- Explain *'stack functions'*, a universal generalization of constructible functions.
- Explain how to define natural *motivic invariants* of constructible sets in Artin stacks, such as Euler characteristics and virtual Hodge and Poincaré polynomials. We do this by extending known invariants for $\mathbb{K}$-varieties to stacks.

2. Constructible functions on $\mathbb{C}$-varieties

Let X be a $\mathbb{C}$-variety. A *constructible set* S in X is a finite union of subvarieties in X, regarded as a set of points in X. If $A, B \subseteq X$ are constructible and $\phi : X \to Y$ a morphism then $A \cup B$, $A \cap B$, $A \setminus B$ and $\phi_*(A)$ are constructible.

Call $f : X \to \mathbb{Q}$ *constructible* if $f(X)$ is finite and $f^{-1}(c) \subseteq X$ is constructible for all $c \in \mathbb{Q}$. Write $\mathrm{CF}(X)$ for the $\mathbb{Q}$-algebra of constructible functions on X.

If $\phi : X \to Y$ is a morphism of $\mathbb{C}$-varieties, and $f \in \mathrm{CF}(Y)$, then $f \circ \phi \in \mathrm{CF}(X)$. Define the *pullback* $\phi^* : \mathrm{CF}(Y) \to \mathrm{CF}(X)$ by $\phi^*(f) = f \circ \phi$. It is a contravariant functor, i.e. $(\psi \circ \phi)^* = \phi^* \circ \psi^*$ for morphisms $\phi : X \to Y$, $\psi : Y \to Z$.

Any constructible set $A \subseteq X$ may be written as a *disjoint union* $A = \coprod_{i=1}^{n} Y_i$ for Y_i a (locally closed) subvariety. Define the *Euler characteristic* $\chi(A) = \sum_{i=1}^{n} \chi^{\text{an}}(Y_i)$, where $\chi^{\text{an}}(Y_i)$ is the Euler characteristic of the compactly-supported cohomology of Y_i in the analytic topology.

Write $\chi(X, f) = \sum_{c \in f(X)} c \cdot \chi(f^{-1}(c))$ for $f \in \text{CF}(X)$. Following MacPherson, for $\phi : X \to Y$ a morphism and $f \in \text{CF}(X)$, define the *pushforward* $\phi_*(f) \in \text{CF}(Y)$ by $(\phi_*(f))(y) = \chi(X, f \cdot \delta_{\phi^{-1}(\{y\})})$ for $y \in Y$, where $\delta_{\phi^{-1}(y)}$ is the characteristic function of $\phi^{-1}(\{y\}) \subseteq X$. This is *functorial*, i.e. $(\psi \circ \phi)_* = \psi_* \circ \phi_*$, because of good properties of χ^{an} on fibrations. Pushforwards do not commute with multiplication.

If W, X, Y, Z are $\mathbb{C}$-varieties and

$$\begin{array}{ccc} W & \xrightarrow{\;\;\eta\;\;} & Y \\ \big\downarrow{\scriptstyle\theta} & & {\scriptstyle\psi}\big\downarrow \\ X & \xrightarrow{\;\;\phi\;\;} & Z \end{array}$$

commutes and is a *Cartesian square*,

i.e. W is a fibre product $X \times_{\phi,Z,\psi} Y$, then the following commutes:

$$\begin{array}{ccc} \mathrm{CF}(W) & \xrightarrow{\;\;\eta_*\;\;} & \mathrm{CF}(Y) \\ \big\uparrow{\scriptstyle\theta^*} & & {\scriptstyle\psi^*}\big\uparrow \\ \mathrm{CF}(X) & \xrightarrow{\;\;\phi_*\;\;} & \mathrm{CF}(Z). \end{array}$$

All this extends to $\mathbb{K}$-varieties for $\mathbb{K}$ an algebraically closed field of characteristic zero (Kennedy,. . .), defining ϕ_* using Euler characteristic in compactly-supported l-adic cohomology. But it doesn't work in positive characteristic, as counterexamples show $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ must fail.

3. Extension to Artin stacks

Fix $\mathbb{K}$ algebraically closed of characteristic zero. *Artin $\mathbb{K}$-stacks* $\mathfrak{F}$ are a very general kind of space in algebraic geometry. They include $\mathbb{K}$-*schemes.* Write $\mathfrak{F}(\mathbb{K})$ for the set of *geometric points* of $\mathfrak{F}$. Each $x \in \mathfrak{F}(\mathbb{K})$ has a *stabilizer group* $\mathrm{Iso}_{\mathbb{K}}(x)$, an algebraic $\mathbb{K}$-group, with $\mathrm{Iso}_{\mathbb{K}}(x) = \{1\}$ if $\mathfrak{F}$ is a scheme. Examples are *quotient stacks* $[X/G]$, for X a $\mathbb{K}$-scheme acted on by an algebraic $\mathbb{K}$-group G.

Call $S \subseteq \mathfrak{F}(\mathbb{K})$ *constructible* if $S = \cup_{i=1}^{n} \mathfrak{G}_i(\mathbb{K})$ for finite type $\mathbb{K}$-substacks $\mathfrak{G}_i \subseteq \mathfrak{F}$.

Call $f : \mathfrak{F}(\mathbb{K}) \to \mathbb{Q}$ *constructible* if $f(\mathfrak{F}(\mathbb{K}))$ is finite and $f^{-1}(c)$ is constructible for all $0 \neq c \in \mathbb{Q}$. Write $\mathrm{CF}(\mathfrak{F})$ for the $\mathbb{Q}$-algebra of constructible functions on $\mathfrak{F}$.

For $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$ a *finite type* 1-morphism and $f \in \mathrm{CF}(\mathfrak{G})$ define the *pullback* $\phi^*(g) = g \circ \phi_*$, where $\phi_* : \mathfrak{F}(\mathbb{K}) \rightarrow \mathfrak{G}(\mathbb{K})$. Then $\phi^* : \mathrm{CF}(\mathfrak{G}) \rightarrow \mathrm{CF}(\mathfrak{F})$ is a $\mathbb{Q}$-algebra morphism, and $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

Defining *pushforwards* is more difficult, though: we need a good notion of Euler characteristic of constructible sets in Artin stacks. The next theorem, using results of Kresch and Rosenlicht, is a tool for extending results from varieties to stacks. We say $\mathfrak{F}$ has *affine stabilizers* if every $\mathrm{Iso}_{\mathbb{K}}(x)$ is an *affine* $\mathbb{K}$-group.

Theorem. *Let $\mathfrak{F}$ be a finite type algebraic $\mathbb{K}$-stack with affine stabilizers. Then $\mathfrak{F}(\mathbb{K}) = \coprod_{i=1}^n \mathfrak{F}_i(\mathbb{K})$, where $\mathfrak{F}_i$ is a $\mathbb{K}$-substack of $\mathfrak{F}$ which is 1-isomorphic to a quotient stack $[X_i/G_i]$, for X_i a quasiprojective $\mathbb{K}$-variety and G_i an affine algebraic $\mathbb{K}$-group. Furthermore there exists a geometric quotient $Y_i = X_i//G_i$, a quasiprojective $\mathbb{K}$-variety whose points are G_i-orbits in X_i.*

Then $\mathrm{CF}(\mathfrak{F}) \cong \bigoplus_{i=1}^n \mathrm{CF}(\mathfrak{F}_i)$ and $\mathrm{CF}(\mathfrak{F}_i) \cong \mathrm{CF}(Y_i)$, so we reduce constructible functions on stacks $\mathrm{CF}(\mathfrak{F})$ to those on varieties $\mathrm{CF}(Y_i)$. Define the *naïve Euler characteristic* $\chi^{\mathrm{na}}(\mathfrak{F}) = \sum_{i=1}^n \chi(Y_i)$. It is called *naïve* as it takes no account of stabilizer groups.

Then χ^{na} extends to constructible sets, and for a 1-morphism $\phi : \mathfrak{F} \to \mathfrak{G}$ we can define $\mathsf{CF}^{\mathsf{na}}(\phi) : \mathsf{CF}(\mathfrak{F}) \to \mathsf{CF}(\mathfrak{G})$, with $\mathsf{CF}^{\mathsf{na}}(\psi \circ \phi) = \mathsf{CF}^{\mathsf{na}}(\psi) \circ \mathsf{CF}^{\mathsf{na}}(\phi)$. But because fibre products of stacks involve stabilizer groups, $\mathsf{CF}^{\mathsf{na}}(\cdots)$ do not commute with pullbacks in Cartesian squares.

Instead, for *representable* ϕ define the *stack pushforward* $\mathsf{CF}^{\mathsf{stk}}(\phi)f = \mathsf{CF}^{\mathsf{na}}(\phi)(m_\phi \cdot f)$, where $m_\phi : \mathfrak{F}(\mathbb{K}) \to \mathbb{Z}$ with
$m_\phi(x) = \chi\Big(\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))\Big)$.
Then $\mathsf{CF}^{\mathsf{stk}}(\psi \circ \phi) = \mathsf{CF}^{\mathsf{stk}}(\psi) \circ \mathsf{CF}^{\mathsf{stk}}(\phi)$, and $\mathsf{CF}^{\mathsf{stk}}(\cdots)$ *do* commute with pullbacks in Cartesian squares.

4. Stack functions

Constructible functions on stacks satisfy:

- To each $\mathbb{K}$-stack $\mathfrak{F}$ with affine stabilizers, associate a $\mathbb{Q}$-algebra $\mathrm{CF}(\mathfrak{F})$.
- Constructible $S \subseteq \mathfrak{F}(\mathbb{K})$ have *characteristic functions* $\delta_S \in \mathrm{CF}(\mathfrak{F})$.
- To each finite type 1-morphism $\phi : \mathfrak{F} \to \mathfrak{G}$ associate a *pullback* algebra morphism $\phi^* : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$, with $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
- To each representable 1-morphism $\phi : \mathfrak{F} \to \mathfrak{G}$ associate a linear *pushforward* $\mathrm{CF}^{\mathrm{stk}}(\phi) : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$, with $\mathrm{CF}^{\mathrm{stk}}(\psi \circ \phi) = \mathrm{CF}^{\mathrm{stk}}(\psi) \circ \mathrm{CF}^{\mathrm{stk}}(\phi)$.
- In a *Cartesian square* of Artin $\mathbb{K}$-stacks

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\psi} \\ \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H}, \end{array}$$

the following commutes:

$$\begin{array}{ccc} \mathrm{CF}(\mathfrak{E}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\eta)} & \mathrm{CF}(\mathfrak{G}) \\ \uparrow{\scriptstyle\theta^*} & & \uparrow{\scriptstyle\psi^*} \\ \mathrm{CF}(\mathfrak{F}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\phi)} & \mathrm{CF}(\mathfrak{H}). \end{array}$$

Stack functions are a universal theory with this package of properties. Fix an algebraically closed field $\mathbb{K}$. Let $\mathfrak{F}$ be an Artin $\mathbb{K}$-stack with affine stabilizers. Consider pairs $(\mathfrak{R},\rho)$, where $\mathfrak{R}$ is finite type with affine stabilizers and $\rho:\mathfrak{R}\rightarrow\mathfrak{F}$ a representable 1-morphism. Call $(\mathfrak{R},\rho)$, $(\mathfrak{R}',\rho')$ *equivalent* if there is a 1-isomorphism $\iota:\mathfrak{R}\rightarrow\mathfrak{R}'$ with $\rho'\circ\iota$ and ρ 2-isomorphic 1-morphisms $\mathfrak{R}\rightarrow\mathfrak{F}$.

Write $[(\mathfrak{R},\rho)]$ for the equivalence class of $(\mathfrak{R},\rho)$. Define $\mathrm{SF}(\mathfrak{F})$ to be the $\mathbb{Q}$-vector space generated by such $[(\mathfrak{R},\rho)]$ with for each closed $\mathbb{K}$-substack $\mathfrak{S}$ of $\mathfrak{R}$ a relation $[(\mathfrak{R},\rho)]=[(\mathfrak{S},\rho|_{\mathfrak{S}})]+[(\mathfrak{R}\setminus\mathfrak{S},\rho|_{\mathfrak{R}\setminus\mathfrak{S}})]$.

Define *multiplication* '$\cdot$' on $\mathrm{SF}(\mathfrak{F})$ by

$$[(\mathfrak{R},\rho)]\cdot[(\mathfrak{S},\sigma)]=[(\mathfrak{R}\times_{\rho,\mathfrak{F},\sigma}\mathfrak{S},\rho\circ\pi_{\mathfrak{R}})].$$

For $\mathfrak{G}\subseteq\mathfrak{F}$ a finite type $\mathbb{K}$-substack with inclusion $\iota:\mathfrak{G}\rightarrow\mathfrak{F}$ define the *characteristic function* $\bar\delta_{\mathfrak{G}(\mathbb{K})}=[(\mathfrak{G},\iota)]$.

For $\phi:\mathfrak{F}\rightarrow\mathfrak{G}$ of finite type define the *pullback* $\phi^*:\mathrm{SF}(\mathfrak{G})\rightarrow\mathrm{SF}(\mathfrak{F})$ by

$$\phi^*:[(\mathfrak{R},\rho)]\mapsto[(\mathfrak{R}\times_{\rho,\mathfrak{G},\phi}\mathfrak{F},\pi_{\mathfrak{F}})].$$

For $\phi:\mathfrak{F}\rightarrow\mathfrak{G}$ representable define the *pushforward* $\phi_*:\mathrm{SF}(\mathfrak{F})\rightarrow\mathrm{SF}(\mathfrak{G})$ by

$$\phi_*:[(\mathfrak{R},\rho)]\mapsto[(\mathfrak{R},\phi\circ\rho)].$$

These satisfy the same properties as the constructible functions operations. When $\mathrm{char}\,\mathbb{K}=0$, define $\pi^{\mathrm{stk}}_{\mathfrak{F}}:\mathrm{SF}(\mathfrak{F})\rightarrow\mathrm{CF}(\mathfrak{F})$ by

$$\pi^{\mathrm{stk}}_{\mathfrak{F}}:[(\mathfrak{R},\rho)]\mapsto\mathrm{CF}^{\mathrm{stk}}(\rho)1,$$

Then $\pi^{\mathrm{stk}}_{\mathfrak{F}}$ takes '$\cdot$',$\bar\delta_S$,ϕ^*,ϕ_* on $\mathrm{SF}(\cdots)$ to '$\cdot$',δ_S,ϕ^*,ϕ_* on $\mathrm{CF}(\cdots)$.

5. Motivic invariants

Let Υ be an invariant of quasiprojective $\mathbb{K}$-varieties X up to isomorphism. Suppose:

(i) Υ takes values in a commutative $\mathbb{Q}$-algebra Λ;

(ii) If $Y \subseteq X$ is a closed subvariety then $\Upsilon(X) = \Upsilon(X \setminus Y) + \Upsilon(Y)$;

(iii) $\Upsilon(X \times Y) = \Upsilon(X)\Upsilon(Y)$;

(iv) Write $\ell = \Upsilon(\mathbb{K})$. Then ℓ and $\ell^k - 1$ for $k = 1, 2, \ldots$ are invertible in Λ.

For example, *virtual Hodge polynomials* $H_X(s,t)$ for $\mathbb{K} = \mathbb{C}$ and *virtual Poincaré polynomials* $P_X(z)$ for all $\mathbb{K}$ satisfy these, with Λ an algebra of rational functions of (s,t) or z. *Euler characteristics* satisfy (i)-(iii) but not (iv), since $\ell = \chi(\mathbb{K}) = 1$ and $\ell^k - 1 = 0$ is not invertible.

We want to extend Υ to Υ' on Artin stacks. Using the Theorem in §3 we define a *naïve version* $\Upsilon^{\rm na}(\mathfrak{F}) = \sum_{i=1}^{n} \Upsilon(Y_i)$, ignoring stabilizer groups. But here is a better way.

For $\mathfrak{F}$ 1-isomorphic to a quotient stack $[X/G]$ we would like $\Upsilon'(\mathfrak{F}) = \Upsilon(X)\Upsilon(G)^{-1}$. In general this depends on the choice of X, G, and $\Upsilon(G)$ may not be invertible. But for *special* $\mathbb{K}$-groups G it is independent.

Call a $\mathbb{K}$-group G *special* if every principal G-bundle is Zariski locally trivial. $\mathrm{GL}(m,\mathbb{K})$ is special. Given $[X/G]$ with G affine, have $[X/G] \cong [X \times_G \mathrm{GL}(m,\mathbb{K})/\mathrm{GL}(m,\mathbb{K})]$ for some m. If $\pi : X \to Y$ is a Zariski locally trivial fibration of $\mathbb{K}$-varieties with fibre F then $\Upsilon(X) = \Upsilon(Y)\Upsilon(F)$. Part (iv) implies $\Upsilon(G)$ is invertible in Λ for special G.

Suppose $\mathfrak{F} \cong [X/G] \cong [Y/H]$ for special G, H. Define $Z = X \times_{\mathfrak{F}} Y$, an algebraic $\mathbb{K}$-space, but treat it as a variety. Then Z has a $G \times H$ action with $Z/H \cong X$ and $Z/G \cong Y$, and projections $Z \to X$, $Z \to Y$ are principal H, G-bundles. As G, H are special these are Zariski locally trivial, so $\Upsilon(Z) = \Upsilon(X)\Upsilon(H) = \Upsilon(Y)\Upsilon(G)$. Thus $\Upsilon(X)\Upsilon(G)^{-1} = \Upsilon(Y)\Upsilon(H)^{-1}$.

Using this we define $\Upsilon'(\mathfrak{F})$ for finite type $\mathfrak{F}$ with affine stabilizers, with $\Upsilon'([X/G]) = \Upsilon(X)\Upsilon(G)^{-1}$ for G special, and $\Upsilon'(\mathfrak{F}) = \Upsilon'(\mathfrak{G}) + \Upsilon'(\mathfrak{F} \setminus \mathfrak{G})$ for closed $\mathfrak{G} \subseteq \mathfrak{F}$, and $\Upsilon'(\mathfrak{F} \times \mathfrak{G}) = \Upsilon'(\mathfrak{F})\Upsilon'(\mathfrak{G})$. Also define $\Upsilon'(S)$ for constructible sets $S \subseteq \mathfrak{F}(\mathbb{K})$. Then $\Upsilon'(\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau))$ will be invariants 'counting' τ-semistables with nice properties.

6. Twisting stack functions by motivic invariants

For $\mathfrak{F}$ a $\mathbb{K}$-stack with affine stabilizers and Υ, Λ as in §5, define $\underline{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda)$ to be the Λ-module generated by $[(\mathfrak{R}, \rho)]$ as in §4 with *relations*:

(i) for $\mathfrak{G} \subseteq \mathfrak{R}$ a closed $\mathbb{K}$-substack

$$[(\mathfrak{R}, \rho)] = [(\mathfrak{G}, \rho|_{\mathfrak{G}})] + [(\mathfrak{R} \setminus \mathfrak{G}, \rho|_{\mathfrak{R} \setminus \mathfrak{G}})],$$

(ii) if U is a quasiprojective $\mathbb{K}$-variety and $\pi_{\mathfrak{R}} : \mathfrak{R} \times U \to \mathfrak{R}$ the projection then

$$[(\mathfrak{R} \times U, \rho \circ \pi_{\mathfrak{R}})] = \Upsilon([U])[(\mathfrak{R}, \rho)].$$

(iii) If $\mathfrak{R} \cong [X/G]$ with G special and $\pi : X \to G$ the projection then

$$[(\mathfrak{R}, \rho)] = \Upsilon([G])^{-1}[(X, \rho \circ \pi)].$$

These relations are compatible with the definitions of '$\cdot$',$\bar{\delta}_S$,ϕ^*, ϕ_*, so all the operations on $\mathrm{SF}(\cdots)$ extend to $\underline{\mathrm{SF}}(\cdots, \Upsilon, \Lambda)$.

Using Υ' we prove $\underline{\mathrm{SF}}(\mathrm{Spec}\,\mathbb{K},\Upsilon,\Lambda)\cong\Lambda$. The space $\underline{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda)$ is like Λ-valued constructible functions $\mathrm{CF}(\mathfrak{F})\otimes_{\mathbb{Q}}\Lambda$, with pushforwards ϕ_* defined by 'integration' along the fibres of ϕ using Υ rather than χ. But $\underline{\mathrm{SF}}(\cdots,\Upsilon,\Lambda)$ pushforwards satisfy $(\psi\circ\phi)_*=\psi_*\circ\phi_*$, which does not hold for such $\mathrm{CF}(\mathfrak{F})\otimes_{\mathbb{Q}}\Lambda$ pushforwards.

There are other ways to make stack function spaces twisted by motivic invariants, some of which 'abelianize' stabilizer groups, interesting projections on stack function spaces, and so on. These will be important tools in my programme to define algebras and invariants from an abelian category with a stability condition.