Introduction to calibrated geometry

Dominic Joyce
Oxford University
England

1. Calibrations

Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is an oriented vector subspace V of some tangent space T_xM to M with $\dim V = k$. Each has a *volume form* vol_V defined using g.

A *calibration* on M is a closed k-form φ with $\varphi|_V \leqslant \mathrm{vol}_V$ for every oriented tangent k-plane V on M.

Let N be an oriented k-fold in M with $\dim N = k$. We call N *calibrated* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

If N is compact then $\mathrm{vol}(N) \geqslant [\varphi] \cdot [N]$, and if N is compact and calibrated then $\mathrm{vol}(N) = [\varphi] \cdot [N]$, where $[\varphi] \in H^k(M, \mathbb{R})$ and $[N] \in H_k(M, \mathbb{Z})$.

Thus calibrated submanifolds are volume-minimizing in their homology class, and are *minimal submanifolds*.

1.1 Calibrations on $\mathbb{R}^n$

Let $(\mathbb{R}^n, g)$ be Euclidean, and φ be a constant k-form on $\mathbb{R}^n$ with $\varphi|_V \leqslant \mathrm{vol}_V$ for all oriented k-planes V in $\mathbb{R}^n$.

Let $\mathcal{F}_\varphi$ be the set of oriented k-planes V in $\mathbb{R}^n$ with $\varphi|_V = \mathrm{vol}_V$. Then an oriented k-fold N in $\mathbb{R}^n$ is a φ-submanifold iff $T_xN \in \mathcal{F}_\varphi$ for all $x \in N$.

For φ to be interesting, $\mathcal{F}_\varphi$ must be fairly large, or there will be few φ-submanifolds.

1.2 Calibrations and special holonomy metrics

Let $G \subset \mathrm{O}(n)$ be the holonomy group of a Riemannian metric. Then G acts on $\Lambda^k(\mathbb{R}^n)^*$. Suppose $\varphi_0 \in \Lambda^k(\mathbb{R}^n)^*$ is nonzero and G-invariant. Rescale φ_0 so that $\varphi_0|_V \leqslant \mathrm{vol}_V$ for all oriented k-planes $V \subset \mathbb{R}^n$, and $\varphi_0|_U = \mathrm{vol}_U$ for some U. Then $U \in \mathcal{F}_{\varphi_0}$, so by G-invariance $\mathcal{F}_{\varphi_0}$ contains the G-orbit of U. Usually $\mathcal{F}_{\varphi_0}$ is 'fairly big'.

Let (M, g) be have holonomy G. Then there is constant k-form φ on M corresponding to the G-invariant k-form φ_0. It is a *calibration* on M.

At each $x \in M$ the family of oriented tangent k-planes V with $\varphi|_V = \mathrm{vol}_V$ is $\mathcal{F}_{\varphi_0}$, which is 'fairly big'. So we expect many φ-submanifolds N in M. Thus manifolds with special holonomy often have interesting calibrations.

1.3. Examples

• The group $\mathrm{U}(m) \subset \mathrm{O}(2m)$ preserves a symplectic 2-form ω_0 on $\mathbb{R}^{2m}$. A manifold (M, g) with holonomy $\mathrm{U}(m)$ is a *Kähler* m*-fold*, with *Kähler form* ω and *complex structure* J. For $1 \leqslant k \leqslant m$, the $2k$-form $\omega^k/k!$ is a calibration on M, and its calibrated submanifolds *complex* k*-submanifolds* of (M, J).

• The group SU(m) $\subset$ O($2m$) preserves a complex m-form Ω_0 on $\mathbb{R}^{2m}$. A manifold (M,g) with holonomy SU(m) is a *Calabi–Yau m-fold*, with *complex volume form* Ω. Re Ω is a calibration on M, and its calibrated submanifolds are called *special Lagrangian m-folds*.
An m-fold N in M is special Lagrangian iff $\omega|_N \equiv \mathrm{Im}\,\Omega|_N \equiv 0$.

• The group $G_2 \subset \mathrm{O}(7)$ preserves a 3-form φ_0 and a 4-form $*\varphi_0$ on $\mathbb{R}^7$. A manifold (M, g) with holonomy G_2 carries a constant 3-form φ and 4-form $*\varphi$, which are both calibrations. Their calibrated submanifolds are called *associative 3-folds* and *coassociative 4-folds*.

A 4-fold N in M is coassociative iff $\varphi|_N \equiv 0$.

• The group Spin(7) $\subset$ O(8) preserves a 4-form Ω_0 on $\mathbb{R}^8$. A manifold (M, g) with holonomy Spin(7) carries a constant 4-form Ω, which is a calibration. Its calibrated submanifolds are called *Cayley 4-folds*.

2. Deformation theory

2.1. The local equations

The family of oriented 3-planes in $\mathbb{R}^7$ is $\mathrm{SO}(7)/\mathrm{SO}(3)\times\mathrm{SO}(4)$, dimension 12. The family of associative 3-planes in $\mathbb{R}^7$ is $G_2/\mathrm{SO}(4)$, dimension 8. So the associative 3-planes have *codimension* 4 in all 3-planes. Thus, for a 3-fold L in $\mathbb{R}^7$ or (M,φ,g) to be associative is 4 equations on each tangent plane T_xL.

The freedom to vary L is the sections of its normal bundle, locally 4 real functions on L. So the deformation problem for associative 3-folds is 4 equations on 4 functions, a *determined* problem. The deformation problem for coassociative 4-folds is 4 equations on 3 functions, *overdetermined*, and for Cayley 4-folds is 4 equations on 4 functions, *determined*.

2.2 Deforming compact coassociative 4-folds

Theorem (McLean). *Let (M, φ, g) be a G_2-manifold, and N a compact coassociative 4-fold in M. Then the moduli space $\mathcal{M}_N$ of coassociative deformations of N is smooth of dimension $b^2_+(N)$.*

Roughly, nearby coassociative 4-folds correspond to small closed forms in $\Lambda^2_+ T^*N$, which are $H^2_+(N, \mathbb{R})$ by Hodge theory.

Here is a sketch of the proof. Let $\nu \to N$ be the *normal bundle* of N in M, so that $TM|_N = \nu \oplus TN$ is orthogonal. Then $V \mapsto (V \cdot \varphi)|_{TN}$ defines an isomorphism $\nu \cong \Lambda^2_+ T^*N$. The exponential map $\nu \to M$ identifies a small *tubular neighbourhood* T of N in M with a neighbourhood U of the zero section in $\Lambda^2_+ T^*N$. Let $\pi : T \to N$ be the obvious projection.

Then graphs $\Gamma(\alpha)$ of small self-dual 2-forms α on N are identified with submanifold in $T \subset M$ close to N. Which α correspond to *coassociative* $\Gamma(\alpha)$? Well, $\Gamma(\alpha)$ is coassociative iff $\varphi|_{\Gamma(\alpha)} \equiv 0$. This holds iff $\pi_*\bigl(\varphi|_{\Gamma(\alpha)}\bigr) \equiv 0$, as $\pi : \Gamma(\alpha) \to N$ is a diffeomorphism. Define $P : C^\infty(U) \to C^\infty(\Lambda^3 T^*N)$ by $P(\alpha) = \pi_*\bigl(\varphi|_{\Gamma(\alpha)}\bigr)$. Then $\mathcal{M}_N$ near N is locally isomorphic to $P^{-1}(0)$ near 0.

As a function of $x \in N$
$P(\alpha)(x) = F(x, \alpha(x), \nabla\alpha(x))$,
for F smooth and nonlinear,
so $P(\alpha) = 0$ is a *nonlinear first-order elliptic p.d.e.*
Also $P(\alpha)$ is *exact*, as φ is closed and $[\varphi|_{\Gamma(\alpha)}] = [\varphi|_N] = 0$ in $H^3(N, \mathbb{R})$. For small α, $P(\alpha) \approx \mathrm{d}\alpha$. Thus $\mathcal{M}_N$ locally approximates the set of self-dual 2-forms α with $\mathrm{d}\alpha = 0$.
By Hodge theory this is $H^2_+(N, \mathbb{R})$, of dimension $b^2_+(N)$.

2.3 Deforming the G_2-manifold

Let (M, φ, g) be a G_2-manifold. Then a 4-fold L in M is coassociative iff $\varphi|_L \equiv 0$. This holds only if $[\varphi|_L] = 0$ in $H^3(L, \mathbb{R})$. So we have:

Lemma. *Let* (M, φ, g) *be a* G_2*-manifold, and* L *a compact 4-fold in* M*. Then* L *is isotopic to a coassociative 4-fold* N *in* M *only if* $[\varphi|_L] = 0$ *in* $H^3(L, \mathbb{R})$*.*

This is necessary and *locally sufficient* for (M, φ, g) to have a coassociative 4-fold in a given deformation class.

Theorem. *Let* (M, φ_t, g_t) : $t \in (-\epsilon, \epsilon)$ *be a smooth family of* G_2*-manifolds, and* N_0 *a compact coassociative 4-fold in* (M, φ_0, g_0)*. If* $[\varphi_t|_{N_0}] = 0$ *in* $H^3(N_0, \mathbb{R})$ *for all* t*, then* N_0 *extends to a smooth family of coassociative* N_t *in* (M, φ_t, g_t) *for* $t \in (-\delta, \delta)$*,* $0 < \delta \leqslant \epsilon$*.*

2.4 Associative 3-folds and Cayley 4-folds

Associative 3-folds in G_2-manifolds and *Cayley 4-folds* in Spin(7)-manifolds cannot be defined by the vanishing of closed forms. This gives their deformation theory a different character. Here is how the theories work.

Let N be a compact associative 3-fold or Cayley 4-fold in M. Then there are vector bundles $E, F \to N$ and a first order elliptic operator
$D_N : C^\infty(E) \to C^\infty(F)$.
The *kernel* $\operatorname{Ker} D_N$ is the set of *infinitesimal deformations* of N. The *cokernel* $\operatorname{Coker} D_N$ is the *obstruction space*. The *index* of D_N is $\operatorname{ind}(D_N) = \dim \operatorname{Ker} D_N - \dim \operatorname{Coker} D_N$.

In the associative case $\mathrm{ind}(D_N) = 0$, and in the Cayley case $\mathrm{ind}(D_N) = \tau(N) - \frac{1}{2}\chi(N) - \frac{1}{2}[N] \cdot [N]$, where τ is the signature and χ the Euler characteristic. Generically $\mathrm{Coker}\, D_N = 0$, and then $\mathcal{M}_N$ is locally a manifold with dimension $\mathrm{ind}(D_N)$. If $\mathrm{Coker}\, D_N \neq 0$, then $\mathcal{M}_N$ may be singular, or have a different dimension.

Note that the coassociative and special Lagrangian cases are unusual: there are *no* obstructions, and the moduli space is *always* a manifold of given dimension, without genericity assumptions.
This is a minor mathematical miracle.