Compact manifolds with exceptional holonomy

Dominic Joyce, Lincoln College, Oxford

published as pages 245–252 in J.E. Andersen, J. Dupont, H. Pedersen and A. Swann, editors, 'Geometry and Physics', Lecture notes in pure and applied math. vol. 184, Marcel Dekker, New York, 1997.

In the theory of riemannian holonomy groups, perhaps the most mysterious are the two exceptional cases, the holonomy group G_2 in 7 dimensions and the holonomy group Spin(7) in 8 dimensions. In a series of three recent papers [5], [6], [7] the author constructed the first known examples of compact 7- and 8-manifolds with metrics of holonomy G_2 and Spin(7). This article will give a brief and informal description of the construction.

Our aim is to introduce G_2 and Spin(7), give a little background material, and help the reader understand the main results of [5]-[7] and a taste of their proofs. We will not be able to state the results in full, to give any rigorous proofs, or to give due credit to the many authors on whose work these results rely. For these we refer the reader to the papers [5]-[7], and to Salamon's book [9], which is a good introduction to holonomy groups.

The article is in three sections. Section 1 gives Berger's classification of holonomy groups, defines G_2 and Spin(7), and explains a little about them. Section 2 describes the construction of [5]-[7], and Section 3 discusses the results.

1. Riemannian holonomy groups

Let M be a connected n-dimensional manifold, let g be a riemannian metric on M, and let ∇ be the Levi-Civita connection of g. Let x, y be points in M joined by a smooth path γ. Then *parallel transport* along γ using ∇ defines an isometry between the tangent spaces T_xM, T_yM at x and y.

Definition 1.1 The *holonomy group* $\mathrm{Hol}(g)$ of g is the group of isometries of T_xM generated by parallel transport around closed loops based at x in M. We consider $\mathrm{Hol}(g)$ to be a subgroup of $O(n)$, defined up to conjugation by elements of $O(n)$. Then $\mathrm{Hol}(g)$ is independent of the base point x in M.

The classification of possible holonomy groups was achieved by Berger [1] in 1955, although Alekseevskii later eliminated the group $\mathrm{Spin}(9)$ from Berger's list.

Theorem 1.2 [1] *Let M be a simply-connected, n-dimensional manifold, and g an irreducible, nonsymmetric riemannian metric on M. Then either*

(*i*) $\mathrm{Hol}(g) = SO(n)$,
(*ii*) $n = 2m$ *and* $\mathrm{Hol}(g) = SU(m)$ *or* $U(m)$,
(*iii*) $n = 4m$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *or* $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$,
(*iv*) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$, *or*
(*v*) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$.

Now G_2 and $\mathrm{Spin}(7)$ are the exceptional cases in this classification, so they are called the exceptional holonomy groups. Here is a brief introduction to G_2 and $\mathrm{Spin}(7)$. All unattributed results can be found in [9, §§11 & 12].

Facts about the holonomy group G_2

Let $\mathbb{R}^7$ have coordinates $(x_1, \ldots, x_7)$. Define a 3-form φ_0 on $\mathbb{R}^7$ by

$$\begin{aligned}\varphi_0 = dx_1 \wedge dx_2 \wedge dx_7 + dx_1 \wedge dx_3 \wedge dx_6 + dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_3 \wedge dx_5 \\ - dx_2 \wedge dx_4 \wedge dx_6 + dx_3 \wedge dx_4 \wedge dx_7 + dx_5 \wedge dx_6 \wedge dx_7. \quad (1)\end{aligned}$$

The subgroup of $GL(7, \mathbb{R})$ preserving φ_0 is the *exceptional Lie group* G_2. This group also preserves the euclidean metric $g_0 = dx_1^2 + \cdots + dx_7^2$. It is a compact, semisimple, 14-dimensional Lie group, a subgroup of $SO(7)$.

A G_2-structure on a 7-manifold M gives rise to a 3-form φ and a metric g on M, such that each tangent space of M admits an isomorphism with $\mathbb{R}^7$ identifying φ and g with φ_0 and g_0 respectively.

Proposition 1.3 *Let M be a 7-manifold and (φ, g) a G_2-structure on M. Then the following are equivalent:*

(*i*) $\mathrm{Hol}(g) \subseteq G_2$, *and* φ *is the induced 3-form,*
(*ii*) $\nabla\varphi = 0$ *on* M*, where* ∇ *is the Levi-Civita connection of* g*, and*
(*iii*) $d\varphi = d^*\varphi = 0$ *on* M*.*

The quantity $\nabla\varphi$ is called the *torsion* of the G_2-structure (φ, g). If $\nabla\varphi = 0$ then the G_2-structure is called *torsion-free.* If g has holonomy $\mathrm{Hol}(g) \subseteq G_2$, then g is ricci-flat. Here is a result about compact 7-manifolds with holonomy G_2, taken from [6, Prop. 1.1.1] and [5, Thm. C].

Proposition 1.4 *Let* M *be a compact 7-manifold, and suppose that* (φ, g) *is a torsion-free* G_2*-structure on* M*. Then* $\mathrm{Hol}(g) = G_2$ *if and only if* $\pi_1(M)$ *is finite. In this case the moduli space of metrics with holonomy* G_2 *on* M*, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension* $b^3(M)$*.*

Facts about the holonomy group Spin(7)

Let $\mathbb{R}^8$ have coordinates $(x_1, \ldots, x_8)$. Define a 4-form Ω_0 on $\mathbb{R}^8$ by

$$\Omega_0 = \varphi_0 \wedge dx_8 + *\varphi_0, \tag{2}$$

where φ_0 is the 3-form defined in (1), and $*\varphi_0$ is the Hodge star of φ_0 using the euclidean metric on $\mathbb{R}^7$. The subgroup of $GL(8, \mathbb{R})$ preserving Ω_0 is the holonomy group Spin(7). This group also preserves the orientation on $\mathbb{R}^8$ and the euclidean metric $g_0 = dx_1^2 + \cdots + dx_8^2$. It is a compact, semisimple, 21-dimensional Lie group, a subgroup of $SO(8)$.

A Spin(7)- structure on an 8-manifold M gives rise to a 4-form Ω and a metric g on M, such that each tangent space of M admits an isomorphism with $\mathbb{R}^8$ identifying Ω and g with Ω_0 and g_0 respectively.

Proposition 1.5 *Let* M *be a compact 8-manifold and* (Ω, g) *a* Spin(7)*-structure on* M*. Then the following are equivalent:*

(*i*) $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$, *and* Ω *is the induced 4-form,*
(*ii*) $\nabla\Omega = 0$ *on* M*, where* ∇ *is the Levi-Civita connection of* g*, and*
(*iii*) $d\Omega = 0$ *on* M*.*

Again, $\nabla\Omega$ is the *torsion* of the Spin(7)-structure (Ω, g), and (Ω, g) is *torsion-free* if $\nabla\Omega = 0$. If g has holonomy $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$, then g is ricci-flat. Here is a result on compact 8-manifolds with holonomy Spin(7), taken from [7, Thms C & D].

Proposition 1.6 *Let M be a compact 8-manifold, and suppose that (Ω, g) is a torsion-free* Spin(7)*-structure on M. Then* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *if and only if M is simply-connected, and $b^3(M) + b^4_+(M) = b^2(M) + 2b^4_-(M) + 25$. In this case the moduli space of metrics with holonomy* Spin(7) *on M, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $1 + b^4_-(M)$.*

For some time after Berger's classification, the holonomy groups G_2 and Spin(7) remained a mystery. In 1987, Bryant [2] used the theory of exterior differential systems to show that locally there exist many metrics with these holonomy groups, and gave some explicit, incomplete examples. Then in 1989, Bryant and Salamon [3] found explicit, *complete* metrics with holonomy G_2 and Spin(7) on noncompact manifolds. Recently, the author constructed examples of metrics with holonomy G_2 and Spin(7) on *compact* manifolds [5], [6], [7]. We will now describe this construction.

2. A 'Kummer construction' for 7- and 8-manifolds

It is well known that metrics with holonomy $SU(2)$ on the $K3$ surface can be obtained by resolving the 16 singularities of the orbifold $T^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on T^4 with 16 fixed points. This is called the Kummer construction, and is described in [8]. Our construction is motivated by and modelled on this. It can be divided into four steps. Here is a summary of each. For simplicity we will describe the G_2 case only, but the Spin(7) case is very similar.

Step 1. Let T^7 be the 7-torus. Let (φ_0, g_0) be a flat G_2-structure on T^7. Choose a finite group Γ of isometries of T^7 preserving (φ_0, g_0). Then the quotient T^7/Γ is a singular, compact 7-manifold.

Step 2. For certain special groups Γ there is a method to resolve the singularities of T^7/Γ in a natural way, using complex geometry. We get a nonsingular, compact 7-manifold M, together with a map $\pi : M \to T^7/\Gamma$, the resolving map.

Step 3. On M, we explicitly write down a 1-parameter family of G_2-structures (φ_t, g_t) depending on a real variable $t \in (0, \epsilon)$. These G_2-structures are not torsion-free, but when t is small, they have small torsion. As $t \to 0$, the G_2-structure (φ_t, g_t) converges to the singular G_2-structure $\pi^*(\varphi_0, g_0)$.

Step 4. We prove using analysis that for all sufficiently small t, the G_2-structure (φ_t, g_t) on M, with small torsion, can be deformed to a G_2-structure $(\tilde{\varphi}_t, \tilde{g}_t)$, with zero torsion. Finally, we show that $\tilde{g}_t$ is a metric with holonomy G_2 on the compact 7-manifold M.

We will now explain the steps in greater detail.

Steps 1-2

We begin with an example of a suitable group Γ. Let $(x_1, \ldots, x_7)$ be coordinates on $T^7 = \mathbb{R}^7/\mathbb{Z}^7$, where $x_i \in \mathbb{R}/\mathbb{Z}$. Let (φ_0, g_0) be the flat G_2-structure on T^7 defined by (1). Let α, β and γ be the involutions of T^7 defined by

$$\alpha((x_1, \ldots, x_7)) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7), \tag{3}$$

$$\beta((x_1, \ldots, x_7)) = (-x_1, \tfrac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7), \tag{4}$$

$$\gamma((x_1, \ldots, x_7)) = (\tfrac{1}{2} - x_1, x_2, \tfrac{1}{2} - x_3, x_4, -x_5, x_6, -x_7). \tag{5}$$

By inspection, α, β and γ preserve (φ_0, g_0), because of the careful choice of exactly which signs to change. Also, $\alpha^2 = \beta^2 = \gamma^2 = 1$, and α, β and γ commute. Thus they generate a group $\Gamma = \langle \alpha, \beta, \gamma \rangle \cong \mathbb{Z}_2^3$ of isometries of T^7 preserving the flat G_2-structure (φ_0, g_0). The following Lemma is proved in [5, §2.1].

Lemma 2.1 *The elements $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ and $\alpha\beta\gamma$ of Γ have no fixed points on T^7. The fixed points of α, β, γ are each 16 copies of T^3. The singular set S of T^7/Γ is a disjoint union of 12 copies of T^3, 4 copies from each of α, β, γ. Each component of S is a singularity modelled on that of $T^3 \times \mathbb{C}^2/\{\pm 1\}$.*

Our goal is to resolve the singular set S of T^7/Γ to get a compact 7-manifold M with holonomy G_2. How can we do this? In general we cannot, because we have no idea of how to resolve general orbifold singularities with holonomy G_2. However, there is a good theory of how to resolve complex singularities like $\mathbb{C}^2/G$, with holonomy $SU(2)$. Suppose that we find Γ such that S is a disjoint union of connected components, each component is a copy of T^3, and the singularities are each modelled on the product of T^3 with a *complex* singularity $\mathbb{C}^2/G$, for G a finite subgroup of $SU(2)$. (This happens in the example above, with $G = \{\pm 1\}$.) Then there is a natural method for resolving T^7/Γ.

Let X be the resolution of $\mathbb{C}^2/G$ coming from algebraic geometry, so that X has holonomy $SU(2)$ and $\pi : X \to \mathbb{C}^2/G$ is the resolving map. For $G = \{\pm 1\}$, X is called the *Eguchi-Hanson space*, with metric given explicitly in [4]. Each component of S is modelled on the singularity of $T^3 \times \mathbb{C}^2/G$, and we resolve by replacing this with $T^3 \times X$, using the resolving map $\pi : T^3 \times X \to T^3 \times \mathbb{C}^2/G$ to glue in the patches $T^3 \times X$. In this way we get a compact, nonsingular 7-manifold M.

Step 3

It turns out that X admits not just one, but a 1-parameter family $\{h_t : t > 0\}$ of metrics with holonomy $SU(2)$. Let h be the euclidean metric on $\mathbb{C}^2/G$, and r the distance from the origin. Then h_t satisfies

$$h_t = \pi^*(h) + O(t^4 \pi^*(r)^{-4}), \tag{6}$$

so that h_t is *asymptotically locally euclidean*, and h_t is isometric to $t^2 h_1$ after an automorphism of X. Let h_{T^3} be a flat metric on T^3. Since $SU(2)$ is a subgroup of G_2 and $\mathrm{Hol}(h_t) = SU(2)$, $\hat{g}_t = h_{T^3} + h_t$ is a metric on $T^3 \times X$ with holonomy contained in G_2. Let $\hat{\varphi}_t$ be the associated G_2-structure. Then $(\hat{\varphi}_t, \hat{g}_t)$ is a torsion-free G_2-structure on $T^3 \times X$.

The idea is to make a G_2-structure (φ_t, g_t) on M by gluing together the torsion-free G_2-structures $(\hat{\varphi}_t, \hat{g}_t)$ on the patches $T^3 \times X$, and (φ_0, g_0) on T^7/Γ. The gluing is done using a partition of unity. Naturally, the first derivative of the partition of unity introduces 'errors', so that (φ_t, g_t) is not torsion-free.

The size of the torsion $\nabla\varphi_t$ depends on the difference $\hat{\varphi}_t - \varphi_0$ in the region where the partition of unity changes. Using (6) to estimate this, we find that $\nabla\varphi_t = O(t^4)$. However, as h_t is isometric to $t^2 h_1$, the injectivity radius of h_t is proportional to t, and $\|R(h_t)\|_{C^0}$ is proportional to t^{-2}. Similar estimates therefore apply to g_t. The following result gives the estimates on (φ_t, g_t) that we need.

Theorem A *There is a mathematical construction, which can be written entirely explicitly in coordinates, that gives a compact 7-manifold M and a 1-parameter family of G_2-structures (φ_t, g_t) on M depending on $t \in (0, \epsilon)$.*

These G_2-structures satisfy the following bounds:

- *the torsion $\nabla\varphi_t$ satisfies $\|\nabla\varphi_t\|_{C^1} \le C_1 t^4$,*
- *the injectivity radius $\delta(g_t)$ satisfies $\delta(g_t) \ge C_2 t$,*
- *the riemann curvature $R(g_t)$ satisfies $\|R(g_t)\|_{C^0} \le C_3 t^{-2}$,*
- *the volume $\mathrm{vol}(M)$ satisfies $C_4 \le \mathrm{vol}(M) \le 2C_4$, and*
- *the diameter $\mathrm{diam}(M)$ satisfies $\mathrm{diam}(M) \le C_5$.*

Here $C_1, \ldots, C_5$ are positive constants independent of t.

Step 4

We prove the following analysis result.

Theorem B *Let $C_1, \ldots, C_5$ be positive constants. Then there exist positive constants λ, K depending only on $C_1, \ldots, C_5$, such that for every $t \in (0, \lambda]$, the following is true.*

Let M be a compact 7-manifold, and (φ, g) a G_2-structure on M. Suppose that $d\varphi = 0$, that ψ is a 3-form on M with $d^*\psi = d^*\varphi$, and that these five conditions hold:

(*i*) $\|\psi\|_{C^2} \le C_1 t^4$,
(*ii*) *the injectivity radius $\delta(g)$ satisfies $\delta(g) \ge C_2 t$,*
(*iii*) *the riemann curvature $R(g)$ of g satisfies $\|R(g)\|_{C^0} \le C_3 t^{-2}$,*
(*iv*) *the volume $\mathrm{vol}(M)$ satisfies $C_4 \le \mathrm{vol}(M) \le 2C_4$, and*
(*v*) *the diameter $\mathrm{diam}(M)$ satisfies $\mathrm{diam}(M) \le C_5$.*

Then there exists a smooth, torsion-free G_2-structure $(\tilde\varphi, \tilde g)$ on M with $\|\tilde\varphi - \varphi\|_{C^0} \le K t^{1/2}$.

Basically, this result says that if (φ, g) is a G_2-structure on M, and the torsion $\nabla\varphi$ is sufficiently small, then we can deform to a nearby G_2-structure

$(\tilde{\varphi}, \tilde{g})$ that is torsion-free. Regard ψ as a *first integral* of $\nabla\varphi$. Thus the norm $\|\psi\|_{C^2}$ is a measure of the torsion $\nabla\varphi$, approximately equal to $\|\nabla\varphi\|_{C^1}$. Then parts (i)-(v) say that the torsion $\nabla\varphi$ must be small compared to the injectivity radius, riemann curvature, volume and diameter of M.

Now the hypotheses of Theorem B are, intentionally, nearly the same as the conclusions of Theorem A. With a little work we can close the gap between them. Therefore, applying Theorems A and B together for sufficiently small t, we see that the compact 7-manifold M constructed in Step 2 admits torsion-free G_2-structures $(\tilde{\varphi}, \tilde{g})$. Applying Proposition 1.4, $\mathrm{Hol}(\tilde{g}) = G_2$ if and only if $\pi_1(M)$ is finite. In this way we construct compact riemannian 7-manifolds with holonomy G_2.

The proof of Theorem B is not easy, and it represents most of the hard work in [5]. Here is a sketch, ignoring several technical points. We have a 3-form φ with $d\varphi = 0$ and $d^*\varphi = d^*\psi$, ψ small, and we wish to construct a nearby 3-form $\tilde{\varphi}$ with $d\tilde{\varphi} = 0$ and $\tilde{d}^*\tilde{\varphi} = 0$. Set $\tilde{\varphi} = \varphi + d\eta$, where η is a small 2-form. Then η must satisfy a nonlinear p.d.e., which we write as

$$d^*d\eta = -d^*\psi + d^*F(d\eta), \tag{7}$$

where F is nonlinear, satisfying $F(d\eta) = O(|d\eta|^2)$.

We solve (7) by iteration, introducing a sequence $\{\eta_j\}_{j=0}^{\infty}$ with $\eta_0 = 0$, satisfying the inductive equations

$$d^*d\eta_{j+1} = -d^*\psi + d^*F(d\eta_j), \qquad d^*\eta_{j+1} = 0. \tag{8}$$

If such a sequence exists and converges to η, then taking the limit in (8) shows that η satisfies (7), giving us the solution we want. The key to proving this is an inductive estimate: one must show that if η_j, η_{j+1} satisfy (8) and $\|\eta_j\|_B \leq C$ then $\|\eta_{j+1}\|_B \leq C$, where $\|\,.\,\|_B$ is the norm in some suitable Banach space B, and C is a positive constant. Then by induction the sequence $\{\eta_j\}_{j=0}^{\infty}$ is bounded in B, and convergence follows by standard techniques.

The inductive estimate we use has two ingredients. The first is

$$\|d\eta_{j+1}\|_{L^2} \leq \|\psi\|_{L^2} + \mathrm{cst.}\|d\eta_j\|_{L^2}\|d\eta_j\|_{C^0}, \tag{9}$$

obtained from (8) by taking the L^2-inner product with η_{j+1} and integrating by parts. (The 3-form ψ was introduced solely to achieve this inequality.) The second is the inequality

$$\|d\eta_{j+1}\|_{C^{1,1/2}} \leq \text{cst.}\|\psi\|_{C^{1,1/2}} + \text{cst.}\|d\eta_j\|_{C^{1,1/2}}\|d\eta_j\|_{C^0} + \text{cst.}\|d\eta_{j+1}\|_{L^2}, \quad (10)$$

which is an elliptic regularity estimate. Here the terms 'cst.' are positive constants depending on $C_2, \ldots, C_5$ and t. We show that if $t \leq \lambda$, for $\lambda > 0$ depending on $C_1, \ldots, C_5$, then (9) and (10) combine to give the necessary inductive estimate, and the remainder of the proof is straightforward. For the Spin(7) case, the proof is different, and more difficult.

3. Presentation of results

By considering different groups Γ acting on T^7 and T^8, we are able to find metrics with holonomy G_2 and Spin(7) on many topologically distinct 7- and 8-manifolds. It also happens that the same orbifold T^k/Γ can admit several topologically distinct resolutions, and this increases the number of examples.

Table 1 is a graph of known examples of compact 7-manifolds M with holonomy G_2, from [5] and [6]. We have plotted $b^2(M)$ against $b^3(M)$, as these are the important betti numbers. The symbols '$\bullet$', '$*$' and '$+$' denote the betti numbers of one or more 7-manifolds with holonomy G_2. Here '$\bullet$' means a simply-connected manifold, '$*$' means a non-simply-connected manifold, and '$+$' means both a simply-connected and a non-simply-connected manifold.

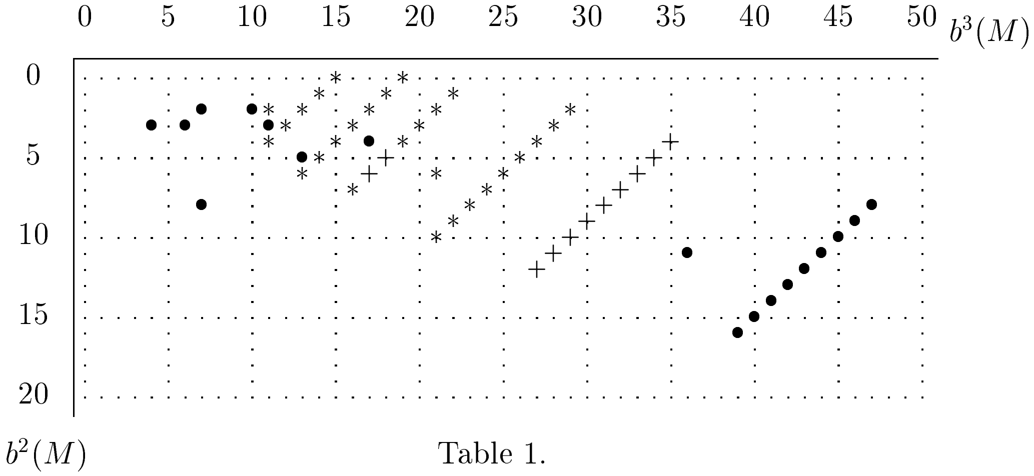

Table 1.

The total number of distinct 7-manifolds known is at least 68.

Similarly, we shall plot a graph of betti numbers $b^2(M)$ against $b^3(M)$ of compact 8-manifolds M with holonomy Spin(7), and this appears in Table 2. Such manifolds are always simply-connected. However, we can use the cup product in cohomology to distinguish 8-manifolds with the same betti numbers. Therefore, in the graph below, the data points are figures, each figure being the number of different 8-manifolds with these betti numbers. All known examples satisfy $b^4_+ = 103 - b^2 + b^3$ and $b^4_- = 39 - b^2 + b^3$, but as far as we know, these are not necessary topological conditions.

$b^3(M)$ \ $b^2(M)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0																					
4					1	2	4	5	7	7	7	5	4	2	1						
5																					
8									1	1	1	1	1								
10																					
15																					
16							1	2	3	4	5	5	5	5	5	4	3	2	1		
20																					

Table 2.

The number of distinct 8-manifolds known is at least 95. Although this is greater than the G_2 case, the variety is less, as all these examples are derived from only 3 groups Γ, giving the 3 horizontal lines on the graph. It seems to be harder to find suitable groups Γ in the Spin(7) case.

Finally, we consider the connections with physics. After many discussions with C. Vafa, S. Shatashvili, E. Witten and M. Roček, it appears that much of the 'mirror symmetry' story for holonomy $SU(3)$ also applies to the holonomy groups G_2 and Spin(7). In particular, Vafa and Shatashvili claim that the reason that the examples are clustered in families with $b^2 + b^3$ constant in Table 1, and with b^3 and $2b^2 + b^4$ constant in Table 2, is because the examples in each family are mirrors of each other. Also, compact riemannian 7-manifolds with holonomy G_2 may become important in physics if 11-dimensional theories come back into fashion, because they are needed to reduce the number of observable dimensions to 4.

References

[1] M. Berger, 'Sur les groupes d'holonomie homogène des variétés à connexion affines et des variétés riemanniennes', Bull. Soc. Math. France 83 (1955) 279-330.

[2] R.L. Bryant, 'Metrics with exceptional holonomy', Ann. Math. 126 (1987) 525-576.

[3] R.L. Bryant & S.M. Salamon, 'On the construction of some complete metrics with exceptional holonomy', Duke Math. J. 58 (1989) 829-850.

[4] T. Eguchi & A.J. Hanson, 'Asymptotically flat solutions to Euclidean gravity', Phys. Lett. 74B (1978) 249-251.

[5] D.D. Joyce, 'Compact riemannian 7-manifolds with holonomy G_2. I', to appear in Journal of Differential Geometry.

[6] D.D. Joyce, 'Compact riemannian 7-manifolds with holonomy G_2. II', to appear in Journal of Differential Geometry.

[7] D.D. Joyce, 'Compact riemannian 8-manifolds with holonomy Spin(7)', preprint (1995).

[8] D.N. Page, 'A physical picture of the $K3$ gravitational instanton', Phys. Lett. 80B (1978), 55-57.

[9] S.M. Salamon, 'Riemannian geometry and holonomy groups', Pitman Res. Notes in Math. 201, Longman, Harlow, 1989.