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Compact Riemannian manifolds with exceptional holonomy

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Suppose that M is an orientable n-dimensional manifold, and g a Riemannian metric on M. Then the *holonomy group* $\mathrm{Hol}(g)$ of g is an important invariant of g. It is a subgroup of $SO(n)$. For generic metrics g on M the holonomy group $\mathrm{Hol}(g)$ is $SO(n)$, but for some special g the holonomy group may be a proper Lie subgroup of $SO(n)$. When this happens the metric g is compatible with some extra geometric structure on M, such as a complex structure.

The possibilities for $\mathrm{Hol}(g)$ were classified in 1955 by Berger. Under conditions on M and g given in §1, Berger found that $\mathrm{Hol}(g)$ must be one of $SO(n)$, $U(m)$, $SU(m)$, $Sp(m)$, $Sp(m)Sp(1)$, G_2 or $Spin(7)$. His methods showed that $\mathrm{Hol}(g)$ is intimately related to the Riemann curvature R of g. One consequence of this is that metrics with holonomy $Sp(m)Sp(1)$ for $m > 1$ are automatically *Einstein*, and metrics with holonomy $SU(m)$, $Sp(m)$, G_2 or $Spin(7)$ are *Ricci-flat*.

Now, people have found many different ways of producing examples of metrics with these holonomy groups, by exploiting the extra geometric structure – for example, quotient constructions, twistor geometry, homogeneous and cohomogeneity one examples, and analytic approaches such as Yau's solution of the Calabi conjecture. Naturally, these methods yield examples of Einstein and Ricci-flat manifolds. In fact, metrics with special holonomy groups provide the *only* examples of compact, Ricci-flat Riemannian manifolds that are known (or known to the author).

The holonomy groups G_2 and $Spin(7)$ are known as the *exceptional holonomy groups*, since they are the exceptional cases in Berger's classification. Here G_2 is a holonomy group in dimension 7, and $Spin(7)$ is a holonomy group in dimension 8. Thus, metrics with holonomy G_2 and $Spin(7)$ are examples of Ricci-flat metrics on 7- and 8-manifolds. The exceptional holonomy groups are the most mysterious of the groups on Berger's list, and have taken longest to reveal their secrets – it was not even known until 1985 that metrics with these holonomy groups existed.

The purpose of this chapter is to describe the construction of compact Riemannian manifolds with holonomy G_2 and $Spin(7)$. These constructions were found in 1994-5 by the present author, and appear in [**16**], [**17**] for the case of G_2, and in

[**15**] for the case of $Spin(7)$. They are also summarized in a short survey paper [**18**], and will be discussed at much greater length in the author's forthcoming book [**19**].

These constructions are interesting because they provide new examples of Ricci-flat 7- and 8-manifolds – in fact, the G_2 case provides the only known examples of compact, simply-connected Ricci-flat manifolds of odd dimension. They are also important to physicists working in String Theory, who need compact 7-manifolds with holonomy G_2 to explain why the universe apparently has only 4 dimensions, rather than the 11 dimensions it really ought to have.

We begin in §1 with an introduction to holonomy groups of Riemannian metrics, and Berger's classification. Sections 2 and 3 define the holonomy groups G_2 and $Spin(7)$, and give the background material we shall need. Section 4 is an aside on metrics with holonomy $SU(2)$, and the Kummer construction for such metrics on the $K3$ surface. The G_2 and $Spin(7)$ constructions are in fact motivated by and modelled on the Kummer construction, so we describe this first as a simple model.

Sections 5-9 explain the construction of metrics of holonomy G_2 and $Spin(7)$ on compact 7- and 8-manifolds. The ideas are first summarized in §5, which divides the proof into four steps. These steps are then covered in more detail in §6-§9 respectively. The most difficult part is Step 3, which uses analysis to construct a solution of a nonlinear elliptic partial differential equation. Finally, in §10 we suggest some areas for future research.

1. Riemannian holonomy groups

Section 1.1 introduces the theory of Riemannian holonomy groups, §1.2 discusses G-structures and their torsion, and §1.3 describes the classification of possible holonomy groups of Riemannian metrics. Some good references on this material are Salamon [**30**], Joyce [**19**], and Kobayashi and Nomizu [**20**, Chapters 2-4].

1.1. Introduction to Riemannian holonomy groups. Throughout this section, let M be a connected manifold of dimension n and g a Riemannian metric on M, and let ∇ be the Levi-Civita connection of g, regarded as a connection on the tangent bundle TM of M. Suppose that $\gamma : [0,1] \to M$ is a smooth path, with $\gamma(0) = p$ and $\gamma(1) = q$. Let s be a smooth section of $\gamma^*(TM)$, so that $s : [0,1] \to TM$ with $s(t) \in T_{\gamma(t)}M$ for each $t \in [0,1]$. Then we say that s is *parallel* if $\nabla_{\dot\gamma(t)}s(t) = 0$ for all $t \in [0,1]$, where $\dot\gamma(t)$ is $\frac{d}{dt}\gamma(t) \in T_{\gamma(t)}M$.

It turns out that for each $v \in T_pM$, there is a unique parallel section s of $\gamma^*(TM)$ with $s(0) = v$. Define a map $P_\gamma : T_pM \to T_qM$ by $P_\gamma(v) = s(1)$. Then P_γ is well-defined and linear, and is called the *parallel transport map* along γ. This definition easily generalizes to piecewise-smooth paths γ. Since ∇ is the Levi-Civita connection of g, we have $\nabla g = 0$. Using this one can show that $P_\gamma : T_pM \to T_qM$ is *orthogonal* with respect to the metric g on T_pM and T_qM.

Here is the definition of holonomy group.

DEFINITION 1.1. Fix a point $p \in M$. We say that γ is a *loop based at* p if $\gamma : [0,1] \to M$ is a piecewise-smooth path with $\gamma(0) = \gamma(1) = p$. If γ is a loop based at p, then the parallel transport map P_γ lies in $O(T_pM)$, the group of orthogonal linear transformations of T_pM. Define the *(Riemannian) holonomy group* $\mathrm{Hol}_p(g)$ *of* g *based at* p to be

$$\mathrm{Hol}_p(g) = \big\{P_\gamma : \gamma \text{ is a loop based at } p\big\} \subset O(T_pM). \tag{1.1}$$

A loop γ based at p is called *null-homotopic* if it can be deformed to the constant loop at p. Define the *restricted (Riemannian) holonomy group* $\mathrm{Hol}^0_p(g)$ *of* g to be

$$\mathrm{Hol}^0_p(g) = \bigl\{P_\gamma : \gamma \text{ is a null-homotopic loop based at } p\bigr\}. \tag{1.2}$$

The following properties are elementary, and easy to prove.

PROPOSITION 1.2. *Both* $\mathrm{Hol}_p(g)$ *and* $\mathrm{Hol}^0_p(g)$ *are subgroups of* $O(T_pM)$. *Suppose that* $p, q \in M$. *Since* M *is connected, we can find a piecewise-smooth path* $\gamma : [0,1] \to M$ *with* $\gamma(0) = p$ *and* $\gamma(1) = q$, *so that* $P_\gamma : T_pM \to T_qM$. *Then*

$$P_\gamma \,\mathrm{Hol}_p(g)\, P_\gamma^{-1} = \mathrm{Hol}_q(g) \qquad \textit{and} \qquad P_\gamma \,\mathrm{Hol}^0_p(g)\, P_\gamma^{-1} = \mathrm{Hol}^0_q(g). \tag{1.3}$$

By choosing an orthonormal basis for T_pM we can identify $O(T_pM)$ with the Lie group $O(n)$. Thus we may regard $\mathrm{Hol}_p(g)$ and $\mathrm{Hol}^0_p(g)$ as subgroups of $O(n)$. Changing the choice of basis changes the subgroups by conjugation by an element of $O(n)$. Thus, $\mathrm{Hol}_p(g)$ and $\mathrm{Hol}^0_p(g)$ may be regarded as subgroups of $O(n)$ defined up to conjugation. Moreover, equation (1.3) shows that in this sense, $\mathrm{Hol}_p(g)$ and $\mathrm{Hol}^0_p(g)$ are *independent of the base point* p. Therefore, we omit the subscript p, and write $\mathrm{Hol}(g)$ for the holonomy group of g and $\mathrm{Hol}^0(g)$ for the reduced holonomy group of g, both of which are subgroups of $O(n)$ defined up to conjugation.

Our next result, taken from [**20**, p. 73, p. 186], is rather more difficult.

THEOREM 1.3. *The reduced holonomy group* $\mathrm{Hol}^0(g)$ *is a closed, connected, Lie subgroup of* $SO(n)$. *It is the connected component of* $\mathrm{Hol}(g)$ *containing the identity, and is normal in* $\mathrm{Hol}(g)$. *There is a surjective group homomorphism* $\phi : \pi_1(M) \to \mathrm{Hol}(g)/\mathrm{Hol}^0(g)$. *Thus* $\mathrm{Hol}(g)/\mathrm{Hol}^0(g)$ *is countable, and if* M *is simply-connected then* $\mathrm{Hol}(g) = \mathrm{Hol}^0(g)$.

Because $\mathrm{Hol}^0(g)$ is a Lie group, it has a Lie algebra. We define the *holonomy algebra* $\mathfrak{hol}(g)$ of g to be the Lie algebra of $\mathrm{Hol}^0(g)$. Then $\mathfrak{hol}(g)$ is a Lie subalgebra of $\mathfrak{o}(n)$, defined up to the adjoint action of $O(n)$. Similarly, define $\mathfrak{hol}_p(g)$ to be the Lie algebra of $\mathrm{Hol}^0_p(g)$, which is a Lie subalgebra of $\mathfrak{o}(T_pM)$. Using g we may identify $\mathfrak{o}(T_pM)$ with $\Lambda^2T^*_pM$, so that $\mathfrak{hol}_p(g)$ becomes a vector subspace of $\Lambda^2T^*_pM$.

Now the holonomy algebra $\mathfrak{hol}(g)$ is intimately connected with the Riemann curvature tensor $R^a{}_{bcd}$ of g. Actually, we find it more convenient to lower the index a and work with the tensor $R_{abcd} = g_{ae}R^e{}_{bcd}$, which we also call the Riemann curvature. Here are two results relating R_{abcd} and $\mathfrak{hol}_p(g)$.

THEOREM 1.4. *The Riemann curvature tensor* R_{abcd} *lies in* $S^2\,\mathfrak{hol}_p(g)$ *at* p, *where* $\mathfrak{hol}_p(g)$ *is regarded as a subspace of* $\Lambda^2T^*_pM$. *It also satisfies the first and second Bianchi identities*

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{1.4}$$

$$\textit{and} \qquad \nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0. \tag{1.5}$$

The second result is the *Ambrose-Singer Holonomy Theorem* [**2**].

THEOREM 1.5. *Let* p *and* q *be points in* M, *let* $\gamma : [0,1] \to M$ *be piecewise-smooth with* $\gamma(0) = p$ *and* $\gamma(1) = q$, *and let* $P_\gamma : T_pM \to T_qM$ *be the parallel* *transport map. Write tensors at p using the tensor indices a, b, and tensors at q using the tensor indices i, j, k, l. Let $v, w \in T_q M$, so that*

$$(P_\gamma)^i_a (P_\gamma)^j_b R_{ijkl} v^k w^l \quad \textit{lies in} \quad \Lambda^2 T^*_p M, \tag{1.6}$$

where R_{ijkl} is the Riemann curvature at q. Then $\mathfrak{hol}_p(g)$ is the vector subspace of $\Lambda^2 T^*_p M$ spanned by all elements of the form (1.6), *for all $q \in M$.*

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, of positive dimension. Then the product $M_1 \times M_2$ has tangent spaces $T_{(p,q)}(M_1 \times M_2) \cong T_p M_1 \oplus T_q M_2$. Thus we may define the *product metric* $g_1 \times g_2$ to be $g_1|_p + g_2|_q$ at each (p, q) in $M_1 \times M_2$. This makes $M_1 \times M_2$ into a Riemannian manifold $(M_1 \times M_2, g_1 \times g_2)$, called a *Riemannian product*. The holonomy group of a product metric is the product of the corresponding holonomy groups.

Proposition 1.6. *Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, with Riemannian product $(M_1 \times M_2, g_1 \times g_2)$. Then* $\mathrm{Hol}(g_1 \times g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$.

We call a Riemannian manifold (M, g) *reducible* if every point has an open neighbourhood isometric to a Riemannian product, and *irreducible* if it is not reducible. Here is a kind of converse to Proposition 1.6.

Theorem 1.7. *Let (M, g) be an irreducible Riemannian manifold of dimension n. Then the natural representation of* $\mathrm{Hol}(g)$ *on* $\mathbb{R}^n$ *is irreducible.*

There is a class of Riemannian manifolds called the *Riemannian symmetric spaces* which are important in the theory of Riemannian holonomy groups. A Riemannian symmetric space is a special kind of Riemannian manifold with a transitive isometry group. The theory of symmetric spaces was worked out by Élie Cartan in the 1920's, who classified them completely, using his own classification of Lie groups and their representations.

A Riemannian metric g is called *locally symmetric* if $\nabla R = 0$, and *nonsymmetric* if it is not locally symmetric. It turns out that every locally symmetric metric is locally isometric to a Riemannian symmetric space. The relevance of symmetric spaces to holonomy groups is that many possible holonomy groups are the holonomy group of a Riemannian symmetric space, but are not realized by any nonsymmetric metric. Therefore, by restricting attention to nonsymmetric metrics one considerably reduces the number of possible Riemannian holonomy groups. For more information about symmetric spaces, see Kobayashi and Nomizu [**21**, §XI] or Helgason [**14**].

1.2. Holonomy groups and torsion-free G-structures. Now we explain a useful mathematical tool for studying holonomy groups.

Definition 1.8. Let M be a manifold of dimension n, and F the frame bundle of M. Then F is a principal bundle over M with fibre $GL(n, \mathbb{R})$. Let G be a Lie subgroup of $GL(n, \mathbb{R})$. Then a *G-structure* on M is a principal subbundle P of F, with fibre G.

Let (M, g) be a riemannian n-manifold, with frame bundle F. Then each point of F is a basis $(e_1, \ldots, e_n)$ for one of the tangent spaces $T_p M$ of M. Define P to be the subset of F of bases $(e_1, \ldots, e_n)$ which are *orthonormal* with respect to g. Then P is a principal subbundle of F with fibre $O(n)$, and so P is an *$O(n)$-structure* on M. This gives a 1-1 correspondence between $O(n)$-structures and Riemannian metrics on M.

Now let M be an n-manifold, G a Lie subgroup of $O(n)$, and Q a G-structure on M. Then $P = O(n)\cdot Q$ is an $O(n)$-structure on M containing Q, which corresponds to a Riemannian metric g on M. Let ∇ be the Levi-Civita connection of g. Then ∇ is a connection on P. We say that Q is *torsion-free* if ∇ preserves the subbundle Q of P. To each G-structure Q on M we can associate a tensor $T(Q)$ called the *torsion* of Q, which measures the failure of ∇ to preserve Q, and Q is torsion-free if and only if $T(Q) = 0$.

The relationship between torsion-free G-structures and holonomy groups is given by the following Proposition.

Proposition 1.9. *Let (M, g) be a manifold of dimension n, and let P be the $O(n)$-structure on M associated to g. Suppose G is a Lie subgroup of $O(n)$. Then there exists a torsion-free G-structure Q contained in P if and only if* $\mathrm{Hol}(g) \subseteq G$. *More generally, there is a 1-1 correspondence between the set of torsion-free G-structures on M contained in P, and the homogeneous space*

$$\big\{a \in O(n) : a^{-1}\mathrm{Hol}(g)a \subseteq G\big\}/G. \tag{1.7}$$

Let Q be a G-structure on M. Then G is a subgroup of $GL(n, \mathbb{R})$, and so it has a natural representation on $\mathbb{R}^n$, and therefore also on $V_{k,l} = \bigotimes^k \mathbb{R}^n \otimes \bigotimes^l (\mathbb{R}^n)^*$. Now the tensor bundle $\bigotimes^k TM \otimes \bigotimes^l T^*M$ over M is canonically isomorphic to $Q \times_G V_{k,l}$. This has two important consequences. Firstly, if the representation $V_{k,l}$ of G splits into a direct sum of subrepresentations, then the tensor bundle $\bigotimes^k TM \otimes \bigotimes^l T^*M$ on M has a corresponding splitting into subbundles. Thus, a G-structure decomposes tensors into components.

Secondly, if an element $t \in V_{k,l}$ is fixed by the action of G, there is a corresponding tensor T in $C^\infty(\bigotimes^k TM \otimes \bigotimes^l T^*M)$. Moreover, if Q is torsion-free, then we have $\nabla T = 0$, where ∇ is the Levi-Civita connection of the metric g associated to Q. Thus, to each torsion-free G-structure we can associate a number of *constant tensors*, that is, tensors T on M with $\nabla T = 0$.

Proposition 1.10. *Let M be a manifold, g a Riemannian metric on M, and ∇ the Levi-Civita connection of g. Then there is a 1-1 correspondence between elements t of $V_{k,l}$ fixed by* $\mathrm{Hol}(g)$, *and constant tensors $T \in C^\infty(\bigotimes^k TM \otimes \bigotimes^l T^*M)$.*

1.3. The classification of Riemannian holonomy groups. The following very important result was proved by Berger [**5**, Thm. 3, p. 318] in 1955.

Theorem 1.11 (Berger). *Suppose that M is a simply-connected manifold of dimension n, and that g is a Riemannian metric on M, that is irreducible and nonsymmetric. Then exactly one of the following seven cases holds.*

(*i*) $\mathrm{Hol}(g) = SO(n)$,
(*ii*) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = U(m)$ *in* $SO(2m)$,
(*iii*) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = SU(m)$ *in* $SO(2m)$,
(*iv*) $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = Sp(m)$ *in* $SO(4m)$,
(*v*) $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = Sp(m)Sp(1)$ *in* $SO(4m)$,

(*vi*) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $SO(7)$, *or*
(*vii*) $n = 8$ *and* $\mathrm{Hol}(g) = Spin(7)$ *in* $SO(8)$.

In fact, Berger also included the possibility $n = 16$ and $\mathrm{Hol}(g) = Spin(9)$, but this was shown not to occur by Alekseevskii [**1**] and Brown and Gray [**8**]. To simplify the classification, Berger made three assumptions: that M is *simply-connected*, that g is *irreducible*, and that g is *nonsymmetric*. If we work with $\mathrm{Hol}^0(g)$ instead of $\mathrm{Hol}(g)$, then we need not suppose M is simply-connected. The holonomy group of a reducible metric is a product of holonomy groups of irreducible metrics, and the holonomy groups of locally symmetric metrics follow from Cartan's classification of Riemannian symmetric spaces. Thus, these three assumptions can easily be removed.

Here is a sketch of Berger's proof of Theorem 1.11. As M is simply-connected, Theorem 1.3 shows that $\mathrm{Hol}(g)$ is a closed, connected Lie subgroup of $SO(n)$, and since g is irreducible, Theorem 1.7 shows that the representation of $\mathrm{Hol}(g)$ on $\mathbb{R}^n$ is irreducible. So, suppose that H is a closed, connected subgroup of $SO(n)$ with irreducible representation on $\mathbb{R}^n$, and Lie algebra $\mathfrak{h}$. The classification of all such groups H follows from the classification of Lie groups (and is of considerable complexity). Berger's method was to take the list of all such groups H, and to apply two tests to each possibility to find out if it could be a holonomy group. The only groups H which passed both tests are those in the Theorem.

Berger's tests are algebraic and involve the curvature tensor. Suppose that R_{abcd} is the Riemann curvature of a metric g with $\mathrm{Hol}(g) = H$. Then Theorem 1.4 shows that $R_{abcd} \in S^2\mathfrak{h}$, and the first Bianchi identity (1.4) applies. But if $\mathfrak{h}$ has large codimension in $\mathfrak{o}(n)$, then the vector space $\mathfrak{R}^H$ of elements of $S^2\mathfrak{h}$ satisfying (1.4) will be small, or even zero. However, Theorem 1.5 shows that $\mathfrak{R}^H$ must be big enough to generate $\mathfrak{h}$. For many of the candidate groups H this does not hold, and so H cannot be a holonomy group. This is the first test.

Now $\nabla_e R_{abcd}$ lies in $(\mathbb{R}^n)^* \otimes \mathfrak{R}^H$, and also satisfies the second Bianchi identity (1.5). Frequently these imply that $\nabla R = 0$, so that g is locally symmetric. Therefore we may exclude such H, and this is Berger's second test. Later, Simons [**31**] found a rather shorter proof of Theorem 1.11 based on showing that $\mathrm{Hol}(g)$ must act transitively on the unit sphere in $\mathbb{R}^n$.

The holonomy groups G_2 and $Spin(7)$, cases (*vi*) and (*vii*) of Theorem 1.11, are known as the *exceptional holonomy groups* because they are the exceptional cases in the classification. The existence of metrics with holonomy G_2 and $Spin(7)$ was first shown in 1985 by Bryant [**9**], using the theory of exterior differential systems. Explicit examples of *complete* metrics with holonomy G_2 and $Spin(7)$ on noncompact manifolds were found in 1989 by Bryant and Salamon [**10**]. Then in 1994-5, the present author constructed examples of metrics with holonomy G_2 on compact 7-manifolds [**16**], [**17**], and of metrics with holonomy $Spin(7)$ on compact 8-manifolds [**15**]. In the rest of this chapter we will explain these examples and the methods used to contruct them.

2. The holonomy group G_2

In this section we give a brief introduction to the geometry of metrics with holonomy G_2, beginning with their local properties, and then moving on to discuss the topology of compact Riemannian manifolds with holonomy G_2. All the results below can be found in Salamon [**30**, Chapter 11], or Joyce [**16, 17**]. Here is a definition of G_2 as a subgroup of $GL(7,\mathbb{R})$.

DEFINITION 2.1. Let $(x_1,\ldots,x_7)$ be coordinates on $\mathbb{R}^7$. Write ω_{ijk} for the 3-form $dx_i\wedge dx_j\wedge dx_k$ on $\mathbb{R}^7$, and ω_{ijkl} for the 4-form $dx_i\wedge dx_j\wedge dx_k\wedge dx_l$. Define a 3-form φ_0 on $\mathbb{R}^7$ by

$$\varphi_0=\omega_{127}+\omega_{136}+\omega_{145}+\omega_{235}-\omega_{246}+\omega_{347}+\omega_{567}. \tag{2.1}$$

The subgroup of $GL(7,\mathbb{R})$ preserving φ_0 is called the *exceptional Lie group* G_2. It is a compact, connected, simply-connected, semisimple, 14-dimensional Lie group. It also preserves the Euclidean metric

$$g_0=dx_1^2+\cdots+dx_7^2 \tag{2.2}$$

on $\mathbb{R}^7$, the orientation on $\mathbb{R}^7$, and the 4-form

$$*\varphi_0=\omega_{1234}+\omega_{1256}-\omega_{1357}+\omega_{1467}+\omega_{2367}+\omega_{2457}+\omega_{3456}. \tag{2.3}$$

Since G_2 is a subgroup of $SO(7)$, a G_2 structure on a 7-manifold M induces a metric g and an orientation on M. Combining g and the orientation gives the *Hodge star*, a linear map $*:\Lambda^kT^*M\rightarrow\Lambda^{7-k}T^*M$. The forms φ_0 and $*\varphi_0$ of (2.1) and (2.3) are related by the Hodge star on $\mathbb{R}^7$, which is why we use this notation.

Let M be a 7-manifold, and φ a 3-form on M. We call φ a *positive 3-form* if for every $p\in M$, there exists an isomorphism between T_pM and $\mathbb{R}^7$ that identifies $\varphi|_p$ and the 3-form φ_0 of (2.1). Since G_2 is the subgroup of $GL(7,\mathbb{R})$ preserving φ_0, it follows that there is a 1-1 correspondence between positive 3-forms φ on M, and G_2-structures Q on M. Moreover, as in §1.2, to each G_2-structure Q on M we may associate a 3-form φ, a metric g, and a 4-form $*\varphi$, corresponding to the tensors (2.1), (2.2) and (2.3) on $\mathbb{R}^7$.

Thus, to each G_2-structure Q on M there corresponds a unique pair (φ,g), where φ is a positive 3-form and g a compatible Riemannian metric. For the rest of this chapter, we will adopt the following abuse of notation: we shall refer to the pair (φ,g) as a *G_2-structure*. Of course it is not, exactly, a G_2-structure, but it does at least define a unique G_2-structure.

PROPOSITION 2.2. *Let M be a 7-manifold and (φ,g) a G_2-structure on M. Then the following are equivalent:*

(*i*) $\mathrm{Hol}(g)\subseteq G_2$, *and φ is the induced 3-form,*
(*ii*) $\nabla\varphi=0$ *on M, where ∇ is the Levi-Civita connection of g, and*
(*iii*) $d\varphi=d^*\varphi=0$ *on M.*

The tensor $\nabla\varphi$ is called the *torsion* of the G_2-structure (φ,g). If $\nabla\varphi=0$ then the G_2-structure is called *torsion-free.* In §1.2 we explained that a G-structure on M induces a splitting of the bundles of tensors on M into irreducible components. Here is the decomposition of the exterior forms on a 7-manifold with a G_2-structure.

PROPOSITION 2.3. *Let M be a 7-manifold and (φ,g) a G_2-structure on M. Then Λ^kT^*M splits orthogonally into components as follows, where Λ^k_l is a vector subbundle of dimension l corresponding to an irreducible representation of G_2:*

(*i*) $\Lambda^1T^*M=\Lambda^1_7$, (*ii*) $\Lambda^2T^*M=\Lambda^2_7\oplus\Lambda^2_{14}$,
(*iii*) $\Lambda^3T^*M=\Lambda^3_1\oplus\Lambda^3_7\oplus\Lambda^3_{27}$, (*iv*) $\Lambda^4T^*M=\Lambda^4_1\oplus\Lambda^4_7\oplus\Lambda^4_{27}$,
(*v*) $\Lambda^5T^*M=\Lambda^5_7\oplus\Lambda^5_{14}$, *and* (*vi*) $\Lambda^6T^*M=\Lambda^6_7$.

The Hodge star $*$ of g gives an isometry between Λ^k_l and Λ^{7-k}_l.

Let the orthogonal projection from $\Lambda^k T^*M$ to Λ^k_l be denoted π_l. So, for instance, if $\xi \in C^\infty(\Lambda^2 T^*M)$, then $\xi = \pi_7(\xi) + \pi_{14}(\xi)$. We saw from Theorem 1.4 that the holonomy group of a Riemannian metric g constrains its Riemann curvature. Using this one can show:

PROPOSITION 2.4. *Let g be a Riemannian metric on a 7-manifold. If $\mathrm{Hol}(g) \subseteq G_2$, then g is Ricci-flat.*

Now suppose that M is a *compact* manifold, and that g is a Riemannian metric on M with $\mathrm{Hol}(g) = G_2$. Then, from the Proposition, g is Ricci-flat. Consider the de Rham cohomology group $H^1(M, \mathbb{R})$. By Hodge theory, each class in $H^1(M, \mathbb{R})$ is represented by a unique 1-form α with $d\alpha = d^*\alpha = 0$. However, since M is compact and g is Ricci-flat, one can prove by a well-known argument of Bochner [**7**] that any such 1-form satisfies $\nabla\alpha = 0$.

But by Proposition 1.10, since $\mathrm{Hol}(g) = G_2$ fixes no nonzero vectors in $(\mathbb{R}^7)^*$, there are no nonzero constant 1-forms on M. Thus $H^1(M, \mathbb{R}) = 0$. One can then use the Cheeger-Gromoll splitting Theorem [**6**, Cor. 6.67] to show that the fundamental group $\pi_1(M)$ of M is *finite*. Thus we prove the following result, which is [**17**, Prop. 1.1.1].

PROPOSITION 2.5. *Let M be a compact 7-manifold, and suppose that (φ, g) is a torsion-free G_2-structure on M. Then $\mathrm{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite.*

The next result is deduced from [**16**, Theorem C].

THEOREM 2.6. *Let M be a compact 7-manifold, let $\mathcal{X}$ be the set of torsion-free G_2-structures on M, and let $\mathcal{D}$ be the group of diffeomorphisms of M isotopic to the identity. Then $\mathcal{X}/\mathcal{D}$ is a smooth manifold of dimension $b^3(M)$. Moreover, the map from $\mathcal{X}/\mathcal{D}$ to $H^3(M, \mathbb{R})$ taking (φ, g) to the cohomology class $[\varphi]$ of φ is a local diffeomorphism.*

This Theorem is proved by studying small deformations of a fixed torsion-free G_2-structure, and is in that sense a purely local result. In fact, at present we know very little about the global geometry of the moduli space of metrics with holonomy G_2 on any 7-manifold.

3. The holonomy group $Spin(7)$

We shall now give a very similar treatment of the holonomy group $Spin(7)$. The material in this section can be found in Salamon [**30**, Chap. 12] or Joyce [**15**]. First we define $Spin(7)$ as a subgroup of $GL(8, \mathbb{R})$.

DEFINITION 3.1. Let $\mathbb{R}^8$ have coordinates $(x_1, \ldots, x_8)$. Write ω_{ijkl} for the 4-form $dx_i \wedge dx_j \wedge dx_k \wedge dx_l$ on $\mathbb{R}^8$. Define a 4-form Ω_0 on $\mathbb{R}^8$ by

$$\begin{aligned} \Omega_0 = {} & \omega_{1256} + \omega_{1278} + \omega_{3456} + \omega_{3478} + \omega_{1357} - \omega_{1368} - \omega_{2457} \\ & + \omega_{2468} - \omega_{1458} - \omega_{1467} - \omega_{2358} - \omega_{2367} + \omega_{1234} + \omega_{5678}. \end{aligned} \tag{3.1}$$

The subgroup of $GL(8, \mathbb{R})$ preserving Ω_0 is the holonomy group $Spin(7)$. It is a compact, connected, simply-connected, semisimple, 21-dimensional Lie group, which is isomorphic as a Lie group to the double cover of $SO(7)$. This group also preserves the orientation on $\mathbb{R}^8$ and the Euclidean metric $g_0 = dx_1^2 + \cdots + dx_8^2$ on $\mathbb{R}^8$.

Let M be an 8-manifold and Ω a 4-form on M. We call Ω an *admissible 4-form* if for every $p \in M$, there is an isomorphism between T_pM and $\mathbb{R}^8$ that identifies $\Omega|_p$ and the 4-form Ω_0 of (3.1). Then there is a 1-1 correspondence between $Spin(7)$-structures Q and admissible 4-forms Ω on M. Each $Spin(7)$-structure Q induces a 4-form Ω on M and a metric g on M, corresponding to Ω_0 and g_0 on $\mathbb{R}^8$. As with G_2, we shall abuse notation by referring to the pair (Ω, g) as a *$Spin(7)$-structure*. The next three results correspond to Propositions 2.2-2.4.

Proposition 3.2. *Let M be a compact 8-manifold and (Ω, g) a $Spin(7)$-structure on M. Then the following are equivalent:*

(i) $\mathrm{Hol}(g) \subseteq Spin(7)$, *and Ω is the induced 4-form,*
(ii) $\nabla\Omega = 0$ *on M, where ∇ is the Levi-Civita connection of g, and*
(iii) $d\Omega = 0$ *on M.*

Again, $\nabla\Omega$ is the *torsion* of the $Spin(7)$-structure (Ω, g), and (Ω, g) is *torsion-free* if $\nabla\Omega = 0$. Since $Spin(7)$ lies in $SO(8)$, a $Spin(7)$-structure on an 8-manifold M induces a natural orientation on M, and so we have the *Hodge star* $* : \Lambda^k T^*M \to \Lambda^{8-k} T^*M$.

Proposition 3.3. *Let M be an 8-manifold and (Ω, g) a $Spin(7)$-structure on M. Then $\Lambda^k T^*M$ splits orthogonally into components as follows, where Λ^k_l is a vector subbundle of dimension l corresponding to an irreducible representation of $Spin(7)$:*

$$(i)\ \Lambda^1 T^*M = \Lambda^1_8, \qquad (ii)\ \Lambda^2 T^*M = \Lambda^2_7 \oplus \Lambda^2_{21}, \qquad (iii)\ \Lambda^3 T^*M = \Lambda^3_8 \oplus \Lambda^3_{48},$$
$$(iv)\ \Lambda^4 T^*M = \Lambda^4_+ T^*M \oplus \Lambda^4_- T^*M,\ \ \Lambda^4_+ T^*M = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27},\ \ \Lambda^4_- T^*M = \Lambda^4_{35},$$
$$(v)\ \Lambda^5 T^*M = \Lambda^5_8 \oplus \Lambda^5_{48}, \qquad (vi)\ \Lambda^6 T^*M = \Lambda^6_7 \oplus \Lambda^6_{21}, \qquad (vii)\ \Lambda^7 T^*M = \Lambda^7_8.$$

The Hodge star $*$ gives an isometry between Λ^k_l and Λ^{8-k}_l. In part (iv), $\Lambda^4_+ T^*M$ and $\Lambda^4_- T^*M$ are the $+1$- and -1- eigenspaces of $*$ on $\Lambda^4 T^*M$ respectively.

The orthogonal projection from $\Lambda^k T^*M$ to Λ^k_l will be written π_l.

Proposition 3.4. *Let g be a Riemannian metric on a 8-manifold. If* $\mathrm{Hol}(g) \subseteq Spin(7)$, *then g is Ricci-flat.*

Now suppose that M is a *compact* 8-manifold, and (Ω, g) a torsion-free $Spin(7)$-structure on M. Since $Spin(7)$ is simply-connected, we deduce that M is spin. Therefore there are positive and negative spin bundles S_+, S_- over M, with fibre $\mathbb{R}^8$, and the Dirac operator D acts by $D : C^\infty(S_+) \to C^\infty(S_-)$. As M is Ricci-flat, it has zero scalar curvature. Thus, by a well-known argument of Lichnerowicz [**26**], all the spinors in $\operatorname{Ker} D$ and $\operatorname{Ker} D^*$ are constant.

However, as for the case of tensors in Proposition 1.10, the constant spinors on a spin Riemannian manifold (M, g) are determined entirely by the holonomy group $\mathrm{Hol}(g)$. Therefore, $\mathrm{Hol}(g)$ determines $\operatorname{Ker} D$ and $\operatorname{Ker} D^*$, and thus it determines the index $\operatorname{ind} D = \dim \operatorname{Ker} D - \dim \operatorname{Ker} D^*$. But D is an elliptic operator, so the Atiyah-Singer Index Theorem shows [**3**, Thm. 5.3] that $\operatorname{ind} D$ is a topological invariant of M known as the *$\hat{A}$-genus $\hat{A}(M)$ of M.*

In this way, the holonomy group $\mathrm{Hol}(g)$ determines a topological invariant $\hat{A}(M)$ of M, which can be written as a linear combination of Betti numbers of M. Conversely, if we know the Betti numbers of M, we can determine $\hat{A}(M)$ and hence $\mathrm{Hol}(g)$. Thus we prove the following result, which is [**15**, Thm. C].

THEOREM 3.5. *Suppose that M is a compact, simply-connected 8-manifold and that (Ω, g) is a torsion-free $Spin(7)$-structure on M. Then M is spin, and the volume form $\Omega\wedge\Omega$ gives a natural orientation on M. Define the $\hat{A}$-genus $\hat{A}(M)$ of M by*

$$24\hat{A}(M) = -1 + b^1 - b^2 + b^3 + b^4_+ - 2b^4_-, \tag{3.2}$$

where b^i are the Betti numbers of M, and $b^4_\pm$ are the dimensions of the spaces of self-dual and anti-self-dual 4-forms in $H^4(M,\mathbb{R})$. Then $\hat{A}(M)$ is equal to $1, 2, 3$ or 4, and the holonomy group $\mathrm{Hol}(g)$ of g is determined by $\hat{A}(M)$ as follows:

(i) $\mathrm{Hol}(g) = Spin(7)$ *if and only if* $\hat{A}(M) = 1$,

(ii) $\mathrm{Hol}(g) = SU(4)$ *if and only if* $\hat{A}(M) = 2$,

(iii) $\mathrm{Hol}(g) = Sp(2)$ *if and only if* $\hat{A}(M) = 3$, *and*

(iv) $\mathrm{Hol}(g) = SU(2)\times SU(2)$ *if and only if* $\hat{A}(M) = 4$.

Every compact, Riemannian 8-manifold with holonomy group $Spin(7)$ is simply-connected.

Finally, here is [**15**, Thm. D], which describes the moduli space of torsion-free $Spin(7)$-structures on a compact 8-manifold.

THEOREM 3.6. *Let M be a simply-connected, compact 8-manifold admitting torsion-free $Spin(7)$-structures, let $\mathcal{X}$ be the set of torsion-free $Spin(7)$-structures on M, and let $\mathcal{D}$ be the group of diffeomorphisms of M isotopic to the identity. Then $\mathcal{X}/\mathcal{D}$ is a smooth manifold of dimension $\hat{A}(M) + b^4_-(M)$.*

4. Metrics with holonomy $SU(2)$ on the $K3$ surface

Before discussing the construction of metrics with holonomy G_2 and $Spin(7)$ on compact 7- and 8-manifolds, we will first explain a simpler construction of the same type: that of metrics of holonomy $SU(2)$ on a particular compact 4-manifold, the $K3$ surface. This is a good illustration of the general plan used for G_2 and $Spin(7)$, but the details are easier because the dimension is lower. Section §4.1 describes the *Eguchi-Hanson space*, an explicit Riemannian manifold with holonony $SU(2)$. Then §4.2 covers the *Kummer construction* for metrics with holonomy $SU(2)$ on the $K3$ surface, in which the Eguchi-Hanson space appears as an important ingredient.

Let $(x_1,\ldots,x_4)$ be the usual coordinates on $\mathbb{R}^4$. Then $SU(2)$ acts on $\mathbb{R}^4$ preserving the Euclidean metric $g_0 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ and the three 2-forms

$$\begin{aligned} \omega_0^I = dx_1\wedge dx_2 + dx_3\wedge dx_4, \qquad \omega_0^J &= dx_1\wedge dx_3 - dx_2\wedge dx_4 \\ \text{and} \qquad \omega_0^K &= dx_1\wedge dx_4 + dx_2\wedge dx_3. \end{aligned} \tag{4.1}$$

Moreover, the subgroup of $GL(4,\mathbb{R})$ that preserves ω_0^I, ω_0^J and ω_0^K is exactly $SU(2)$. Therefore, if X is a 4-manifold, there is a 1-1 correspondence between $SU(2)$-structures on X and triples $(\omega^I, \omega^J, \omega^K)$ of 2-forms on X, such that for each point $p\in X$ there exists an isomorphism $T_pX \cong \mathbb{R}^4$ that identifies ω^I, ω^J and ω^K with ω_0^I, ω_0^J and ω_0^K respectively.

By an abuse of notation, we shall refer to a triple $(\omega^I, \omega^J, \omega^K)$ of 2-forms on X with this property as an $SU(2)$-*structure* on X. To each $SU(2)$-structure $(\omega^I, \omega^J, \omega^K)$ on X, we may associate a Riemannian metric g and almost complex structures I, J, K on X in a natural way, which satisfy the quaternion relations $IJ = -JI = K$.

It can be shown that an $SU(2)$-structure $(\omega^I, \omega^J, \omega^K)$ is *torsion-free* if and only if ω^I, ω^J and ω^K are *closed* 2-forms. If $(\omega^I, \omega^J, \omega^K)$ is torsion-free, then $\mathrm{Hol}(g) \subseteq SU(2)$, the almost complex structures I, J and K are integrable, the metric g is Kähler with respect to each of I, J and K, and the Kähler forms of the Kähler structures (I, g), (J, g) and (K, g) are ω^I, ω^J and ω^K respectively.

As all of this structure shows, Riemannian manifolds with holonomy $SU(2)$ have a very rich and interesting geometry. In particular, metrics with holonomy $SU(2)$ are *Kähler metrics*, in three different ways. Given a complex manifold M, it is often easy to write down a large number of explicit Kähler metrics on M using Kähler potentials, and other devices from Kähler geometry. If g is a Kähler metric on a simply-connected complex surface, then $\mathrm{Hol}(g) \subseteq SU(2)$ if and only if g is Ricci-flat. Thus, one method of constructing metrics with holonomy $SU(2)$ is to find solutions to the equation $\mathrm{Ric}(g) = 0$ for a Kähler metric g. This can be done explicitly in examples with symmetry, or using analysis as in Yau's solution to the Calabi conjecture [**34**].

4.1. The Eguchi-Hanson space. The simplest nontrivial example of a Riemannian manifold with holonomy $SU(2)$ is the Eguchi-Hanson space [**11**], which is a family of complete metrics on the noncompact 4-manifold $T^*\mathbb{CP}^1$. We will write down this metric explicitly in coordinates. Let $\mathbb{C}^2$ be equipped with complex coordinates (z_1, z_2), and the standard flat Kähler metric $g_0 = |dz_1|^2 + |dz_2|^2$. The involution $-1 : (z_1, z_2) \mapsto (-z_1, -z_2)$ acts on $\mathbb{C}^2$, preserving g_0 and fixing 0. Thus $\mathbb{C}^2/\{\pm 1\}$ is a singular complex manifold with one singular point at 0, and the metric g_0 pushes down to $\mathbb{C}^2/\{\pm 1\}$.

Let X be the blow-up of $\mathbb{C}^2/\{\pm 1\}$ at 0, and let $\pi : X \to \mathbb{C}^2/\{\pm 1\}$ be the blow-down map. Then X is a nonsingular complex manifold biholomorphic to $T^*\mathbb{CP}^1$, with $\pi_1(X) = \{1\}$ and $H^2(X, \mathbb{R}) = \mathbb{R}$. The *radius function* r given by $r^2 = |z_1|^2 + |z_2|^2$ on $\mathbb{C}^2$ pushes down to $\mathbb{C}^2/\{\pm 1\}$, and so lifts to X. Let $t \geq 0$, and define a function f_t on X by

$$f_t = \sqrt{r^4 + t^4} + 2t^2 \log r - t^2 \log\left(\sqrt{r^4 + t^4} + t^2\right). \tag{4.2}$$

This is the Kähler potential for the Eguchi-Hanson metric, and is taken from [**24**, p. 593]. For each $t > 0$, define 2-forms ω_t^I, ω_t^J and ω_t^K on X by

$$\omega_t^I = \tfrac{1}{2} i \partial\overline{\partial} f_t, \quad \omega_t^J = \mathrm{Re}(dz_1 \wedge dz_2), \quad \text{and} \quad \omega_t^K = \mathrm{Im}(dz_1 \wedge dz_2). \tag{4.3}$$

Then $(\omega_t^I, \omega_t^J, \omega_t^K)$ is a torsion-free $SU(2)$-structure on X. The associated metric g_t is the *Eguchi-Hanson metric*, and has holonomy $SU(2)$.

From (4.2) we find that the asymptotic behaviour of g_t near infinity in X is

$$g_t = \pi^*(g_0) + O(t^4 r^{-4}). \tag{4.4}$$

Thus, the Eguchi-Hanson metric is asymptotic to the flat metric g_0 on $\mathbb{C}^2/\{\pm 1\}$ at infinity. Metrics with this property are called *Asymptotically Locally Euclidean*, or

ALE for short. Similarly, the 2-forms ω_t^I, ω_t^J and ω_t^K have the asymptotic behaviour

$$\omega_t^I = \pi^*(\omega_0^I) + O(t^4 r^{-4}), \qquad \omega_t^J = \pi^*(\omega_0^J) + O(t^4 r^{-4}) \\ \text{and} \qquad \omega_t^K = \pi^*(\omega_0^K) + O(t^4 r^{-4}). \tag{4.5}$$

4.2. The Kummer construction. The $K3$ *surface* is a compact 4-manifold which has a family of complex structures, each making the $K3$ into a complex surface. These complex surfaces are of particular interest to algebraic geometers. From Yau's proof of the Calabi conjecture [**34**], it is known that the $K3$ surface possesses a 58-parameter family of metrics of holonomy $SU(2)$. An approximate description of some of these metrics was given by Page [**28**], which employs an idea known as the *Kummer construction*.

Proofs of the existence of metrics of holonomy $SU(2)$ on $K3$ using Page's idea have been given by Topiwala [**33**] and LeBrun and Singer [**25**] using twistor theory, and by the author [**17**, Ex. 1] using analysis. Here is a brief sketch of the Kummer construction for the $K3$ surface, and the metrics of holonomy $SU(2)$ upon it.

Let $\mathbb{C}^2$ have complex coordinates (z_1, z_2) and metric g_0 as above. Define a subset Λ of $\mathbb{C}^2$ by $\Lambda = \{(a+ib, c+id) : a,b,c,d \in \mathbb{Z}\}$. Then Λ is a *lattice* in $\mathbb{C}^2$, and $\mathbb{C}^2/\Lambda$ is the 4-torus T^4. It is a complex manifold, with flat Kähler metric g_0. Define an involution $\sigma : T^4 \to T^4$ by

$$\sigma : (z_1, z_2) + \Lambda \longmapsto (-z_1, -z_2) + \Lambda. \tag{4.6}$$

Then σ has 16 fixed points, the points $(z_1, z_2)+\Lambda$ with $z_j \in \{0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2}+\frac{1}{2}i\}$. Thus $T^4/\langle\sigma\rangle$ is a singular complex manifold with a flat Kähler metric g_0, and 16 singular points, $s_1, \ldots, s_{16}$ say, modelled on 0 in $\mathbb{C}^2/\{\pm 1\}$.

Let Y be the blow-up of $T^4/\langle\sigma\rangle$ at each singular point s_j. Then Y is a compact, nonsingular complex surface called a $K3$ surface. The blow-ups replace each singular point with a complex curve $\mathbb{CP}^1$. These blow-ups are modelled on the construction of the Eguchi-Hanson space X. Hence, we may regard Y as the result of gluing 16 copies of X into the orbifold $T^4/\langle\sigma\rangle$.

Define subsets $A_j = \{y \in Y : d(\pi(y), s_j) < \frac{1}{5}\}$ in Y for $j = 1, \ldots, 16$, and let $B = \{y \in Y : d(\pi(y), s_j) > \frac{1}{6} \text{ for } j = 1, \ldots, 16\}$, where $d(\,,\,)$ is the metric on $T^4/\langle\sigma\rangle$ induced by g_0. Then Y is the union of the A_j and B. The intersections $A_j \cap A_k$ are empty for $j \neq k$, but $A_j \cap B$ is an 'annulus' diffeomorphic to $(\frac{1}{6}, \frac{1}{5}) \times \mathcal{S}^3/\langle\sigma\rangle$.

Now A_j is naturally isomorphic to an open subset of the Eguchi-Hanson space X. Therefore on each A_j we have a family $(\omega_t^I, \omega_t^J, \omega_t^K)$ of torsion-free $SU(2)$-structures, depending on $t > 0$. Similarly, B is naturally isomorphic to an open subset of $T^4/\langle\sigma\rangle$, so on B we have the flat $SU(2)$-structure $(\omega_0^I, \omega_0^J, \omega_0^K)$. On the overlap $A_j \cap B$, we can compare these two $SU(2)$-structures using (4.5). Since $r \in (\frac{1}{6}, \frac{1}{5})$ on $A_j \cap B$, we have $r^{-4} < 6^4$, and so

$$\omega_t^I = \omega_0^I + O(t^4), \;\; \omega_t^J = \omega_0^J + O(t^4), \;\; \text{and} \;\; \omega_t^K = \omega_0^K + O(t^4) \quad \text{on } A_j \cap B. \tag{4.7}$$

This equation shows that when $t > 0$ is small, the $SU(2)$-structures $(\omega_t^I, \omega_t^J, \omega_t^K)$ defined on A_j, and $(\omega_0^I, \omega_0^J, \omega_0^K)$ defined on B, are close to each other on $A_j \cap B$. We can exploit this fact to construct torsion-free $SU(2)$-structures on Y. The argument runs as follows. First one uses a partition of unity to write down an $SU(2)$-structure $(\hat\omega_t^I, \hat\omega_t^J, \hat\omega_t^K)$ on Y, that equals $(\omega_t^I, \omega_t^J, \omega_t^K)$ on $A_j \setminus B$ and $(\omega_0^I, \omega_0^J, \omega_0^K)$ on $B \setminus \bigcup_j A_j$, and interpolates smoothly between the two on the overlaps $A_j \cap B$.

Then $(\hat{\omega}_t^I, \hat{\omega}_t^J, \hat{\omega}_t^K)$ is torsion-free on $A_j \setminus B$ and $B \setminus \bigcup_j A_j$, but on $A_j \cap B$ it has nonzero torsion. However, using (4.7) one can ensure that the torsion is $O(t^4)$, and so when t is small, the torsion is also small. Finally, using analysis one proves that an $SU(2)$-structure on Y with sufficiently small torsion, can be deformed to a nearby, torsion-free $SU(2)$-structure. Therefore, for all small enough $t > 0$ we construct a new, torsion-free $SU(2)$-structure $(\tilde{\omega}_t^I, \tilde{\omega}_t^J, \tilde{\omega}_t^K)$ on Y that is close to $(\omega_t^I, \omega_t^J, \omega_t^K)$ on A_j, and close to $(\omega_0^I, \omega_0^J, \omega_0^K)$ on B. The metric $\tilde{g}_t$ associated to this $SU(2)$-structure is a metric on the $K3$ surface with holonomy $SU(2)$.

5. Compact manifolds with holonomy G_2 and $Spin(7)$

In the rest of this chapter we will describe a construction for compact Riemannian 7-manifolds with holonomy G_2, and a very similar construction for compact Riemannian 8-manifolds with holonomy $Spin(7)$. Both constructions are motivated by and modelled on the Kummer construction of §4 for metrics of holonomy $SU(2)$ on the $K3$ surface. They can be divided into four steps. Here is a summary of each. For simplicity we will describe the G_2 case only, but the $Spin(7)$ case is very similar.

Step 1. Let T^7 be the 7-torus. Let (φ_0, g_0) be a flat G_2-structure on T^7. Choose a finite group Γ of isometries of T^7 preserving (φ_0, g_0). Then the quotient T^7/Γ is a singular, compact 7-manifold.

For certain special groups Γ there is a method to resolve the singularities of T^7/Γ in a natural way, using complex geometry. We get a nonsingular, compact 7-manifold M, together with a map $\pi : M \to T^7/\Gamma$, the resolving map.

Step 2. On M, we explicitly write down a 1-parameter family of G_2-structures (φ_t, g_t) depending on a real variable $t \in (0, \epsilon)$. These G_2-structures are not torsion-free, but when t is small, they have small torsion. As $t \to 0$, the G_2-structure (φ_t, g_t) converges to the singular G_2-structure $\pi^*(\varphi_0, g_0)$.

Step 3. We prove using analysis that for all sufficiently small t, the G_2-structure (φ_t, g_t) on M, with small torsion, can be deformed to a G_2-structure $(\tilde{\varphi}_t, \tilde{g}_t)$, with zero torsion.

Step 4. Finally, we show that $\tilde{g}_t$ is a metric with holonomy G_2 on the compact 7-manifold M, using topological invariants of M.

We shall explain Steps 1-4 at much greater length in sections 6-9 respectively. By considering different groups Γ acting on T^7 and T^8, we are able to find metrics with holonomy G_2 and $Spin(7)$ on many topologically distinct 7- and 8-manifolds. It also happens that the same orbifold T^k/Γ can admit several topologically distinct resolutions, and this increases the number of examples.

In [**17**], the author gave examples of 68 distinct, compact 7-manifolds with holonomy G_2, and in [**15**], examples of 95 distinct, compact 8-manifolds with holonomy $Spin(7)$. The forthcoming book [**19**] will provide many more examples, by using more powerful mathematical tools to resolve singularities, and by studying the possibilities for the finite group Γ in a systematic way.

6. Orbifolds and resolutions

This section explains Step 1 of §5 in greater detail. For simplicity, we will mostly confine our attention to the case of 7-manifolds and holonomy G_2, but the case of 8-manifolds with holonomy $Spin(7)$ is very similar. Section 6.1 introduces orbifolds T^7/Γ with flat G_2-structures, and their singular points. Then §6.2 discusses resolutions of T^7/Γ, and describes a special way of resolving orbifolds T^7/Γ using complex geometry. We shall see later that a resolution constructed in this way admits a family of torsion-free G_2-structures. Section 6.3 gives a simple example of an orbifold T^7/Γ, and how to resolve it, and §6.4 gives a similar example in 8 dimensions.

6.1. Orbifolds of T^7 with flat G_2-structures. Let (φ_0, g_0) be the standard, flat G_2-structure on $\mathbb{R}^7$, given in Definition 2.1. Let Λ be a *lattice* in $\mathbb{R}^7$, so that $\Lambda \cong \mathbb{Z}^7$. Then Λ acts as a group of translations on $\mathbb{R}^7$, and the quotient $\mathbb{R}^7/\Lambda$ is the 7-torus T^7. Moreover, this action of Λ on $\mathbb{R}^7$ preserves φ_0 and g_0, and thus there is a flat G_2-structure (φ_0, g_0) on T^7.

Now the group of linear transformations of $\mathbb{R}^7$ preserving (φ_0, g_0) is $G_2 \subset GL(7,\mathbb{R})$, and the group of linear transformations of $\mathbb{R}^7$ preserving Λ is $GL(7,\mathbb{Z}) \subset GL(7,\mathbb{R})$. Thus, the subgroup of $GL(7,\mathbb{R})$ preserving both (φ_0, g_0) and Λ is $F = G_2 \cap GL(7,\mathbb{Z})$. Here F is a finite group, as it is both discrete and compact. Note that the embedding of $GL(7,\mathbb{Z})$ in $GL(7,\mathbb{R})$, and thus the finite group F, is not fixed but depends on the choice of lattice Λ in $\mathbb{R}^7$. The group F acts on T^7 preserving the G_2-structure (φ_0, g_0). But T^7 acts on itself by translations, and this action also preserves (φ_0, g_0). Together, these actions of F and T^7 on T^7 generate a group $F \ltimes T^7$, which turns out to be the full group of automorphisms of T^7 that preserve (φ_0, g_0).

Let Γ be a finite subgroup of $F \ltimes T^7$. Then Γ acts on T^7 preserving (φ_0, g_0), and thus T^7/Γ is an *orbifold*, equipped with a flat G_2-structure (φ_0, g_0). The singular points of T^7/Γ are easy to describe. Let $x \in T^7$, so that $x\Gamma \in T^7/\Gamma$. Let $\Gamma_x = \{\gamma \in \Gamma : \gamma(x) = x\}$ be the *stabilizer* of x. If $\Gamma_x = \{1\}$ then $x\Gamma$ is a *nonsingular point* of T^7/Γ. If $\Gamma_x \neq \{1\}$ then $x\Gamma$ is a *singular point* of T^7/Γ. Moreover, Γ_x acts naturally on $\mathbb{R}^7$, and the singularity at $x\Gamma$ is modelled on the singularity at 0 of $\mathbb{R}^7/\Gamma_x$. This action of Γ_x on $\mathbb{R}^7$ makes Γ_x into a subgroup of G_2.

Thus, $x\Gamma$ is a singular point of T^7/Γ if and only if x is fixed by some nonidentity element of Γ. It is convenient to adopt the following notation: for each nonidentity element $\gamma \in \Gamma$, define S_γ to be the set of points $x\Gamma \in T^7/\Gamma$ for $x \in T^7$ with $\gamma(x) = x$. Define S to be the set of singular points of T^7/Γ. Then S is the union (not necessarily a disjoint union) of the S_γ for $\gamma \neq 1$ in Γ. Suppose that $\alpha \neq 1$ is an element of G_2. Then the subset of $\mathbb{R}^7$ fixed by α is either $\mathbb{R}$ or $\mathbb{R}^3$. Therefore, if $\gamma \neq 1$ lies in Γ then S_γ is either empty, or has dimension 1 or 3. Hence the singular set S is a finite union of (singular) 1-manifolds and 3-manifolds in T^7/Γ.

6.2. Resolutions of T^7/Γ with torsion-free G_2-structures. Our goal is to resolve the singularities of the orbifold T^7/Γ to get a compact 7-manifold M, and to construct a family of torsion-free G_2-structures on M that are in some sense close to the singular G_2-structure (φ_0, g_0) on T^7/Γ. By a *resolution* of T^7/Γ, we mean a pair (M, π), where M is a compact 7-manifold and $\pi : M \to T^7/\Gamma$ a continuous map, such that the restriction $\pi : M \setminus \pi^{-1}(S) \to (T^7/\Gamma) \setminus S$ is a diffeomorphism, and for each $s \in S$, the subset $\pi^{-1}(s)$ is a finite union of compact submanifolds of M.

In general T^7/Γ has not just one, but infinitely many resolutions (M, π). However, nearly all of these resolutions are unsuitable for our purposes, and most of them do not even admit G_2-structures. To be able to construct torsion-free G_2-structures on M, we must restrict our attention to orbifolds T^7/Γ with a particular kind of singularity, and then resolve these singularities in a special way.

Let $x\Gamma$ be a *generic* point of S. Then the singularity of T^7/Γ at $x\Gamma$ is modelled on that of $\mathbb{R}^7/\Gamma_x$ at 0. Now we saw above that the singular set S of T^7/Γ is a finite union of (singular) 1-manifolds and 3-manifolds. As $x\Gamma$ is generic, near $x\Gamma$ we see that S is nonsingular and of dimension 1 or 3. Suppose first that S is of dimension 1 near $x\Gamma$. Then there is a natural splitting $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{R}^6$ preserved by Γ_x, and Γ_x acts trivially on $\mathbb{R}$ and freely on $\mathbb{R}^6 \setminus \{0\}$. As Γ_x preserves the G_2-structure (φ_0, g_0) on $\mathbb{R}^7$, it follows that Γ_x is a subgroup of G_2. But the subgroup of G_2 fixing a subspace $\mathbb{R} \subset \mathbb{R}^7$ is $SU(3)$, so that Γ_x is a subgroup of $SU(3)$.

Similarly, if S has dimension 3 near $x\Gamma$, we can show that Γ_x lies in a subgroup $SU(2)$ of G_2. Thus, if $x\Gamma$ is a generic singular point of T^7/Γ, then one of two possibilities holds:

(i) There is a natural splitting $\mathbb{R}^7 \cong \mathbb{R}^3 \oplus \mathbb{C}^2$, and $SU(2)$ acts trivially on $\mathbb{R}^3$ and in the usual way on $\mathbb{C}^2$. The group Γ_x is a finite subgroup of $SU(2)$ which acts freely on $\mathbb{C}^2 \setminus \{0\}$, and $\mathbb{R}^7/\Gamma_x \cong \mathbb{R}^3 \times (\mathbb{C}^2/\Gamma_x)$.

(ii) There is a natural splitting $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$, and $SU(3)$ acts trivially on $\mathbb{R}$ and in the usual way on $\mathbb{C}^3$. The group Γ_x is a finite subgroup of $SU(3)$ which acts freely on $\mathbb{C}^3 \setminus \{0\}$, and $\mathbb{R}^7/\Gamma_x \cong \mathbb{R} \times (\mathbb{C}^3/\Gamma_x)$.

The key observation is that the singularities $\mathbb{C}^2/\Gamma_x$ and $\mathbb{C}^3/\Gamma_x$ occurring here are *complex singularities*. Now in the field of complex algebraic geometry, the problem of resolving singularities of complex manifolds has been studied for many years, and is very well understood in complex dimensions 2 and 3. In particular, if F is a nontrivial finite subgroup of $SU(2)$, it is known that the quotient singularity $\mathbb{C}^2/F$ can be resolved in a unique way to give a complex manifold X, which has a family of ALE metrics with holonomy $SU(2)$. These metrics were explicitly constructed and classified by Kronheimer [**22**], [**23**]. In the simplest case $F = \{\pm 1\}$, we get the Eguchi-Hanson space of §4.1.

Similar results are known for the case of $SU(3)$. If F is any finite subgroup of $SU(3)$, Roan [**29**] has shown that $\mathbb{C}^3/\Gamma$ admits a suitable complex resolution Y. The family of ALE metrics with holonomy $SU(3)$ on Y are not known explicitly except in special cases, but the author [**19**] has proved that such metrics exist in every case, by following Yau's proof of the Calabi conjecture. Tian and Yau [**32**] have also proved some related results.

Suppose X is a resolution of $\mathbb{C}^2/F$, and g_X is an ALE metric on X with holonomy $SU(2)$. Let $g_{\mathbb{R}^3}$ be a flat metric on $\mathbb{R}^3$. Then $g = g_{\mathbb{R}^3} \times g_X$ is a metric on $\mathbb{R}^3 \times X$ which has holonomy $\mathrm{Hol}(g) = \{1\} \times SU(2)$. Since $\{1\} \times SU(2) \subset G_2$, the metric g extends to a torsion-free G_2-structure (φ, g) on $\mathbb{R}^3 \times X$, which is asymptotic to the flat G_2-structure (φ_0, g_0) on $\mathbb{R}^7/F$ as one approaches infinity in X. In the same way, if Y is a resolution of $\mathbb{C}^3/F$ and g_Y an ALE metric on Y with holonomy $SU(3)$, we may construct torsion-free G_2-structures (φ, g) on $\mathbb{R} \times Y$ that are asymptotic to (φ_0, g_0) on $\mathbb{R}^7/F$.

Thus, if $x\Gamma$ is a singular point of T^7/Γ and Γ_x lies in some subgroup of G_2 conjugate to $SU(2)$ or $SU(3)$, then one can resolve the quotient singularity $\mathbb{R}^7/\Gamma_x$ using complex geometry, in such a way that the resolution carries a family of torsion-free G_2-structures (φ, g) asymptotic to (φ_0, g_0).

For our construction, we first choose an orbifold T^7/Γ for which Γ_x lies in $SU(2)$ or $SU(3)$ for every singular point $x\Gamma$ of T^7/Γ. Next, we build a resolution (M, π) of T^7/Γ, which is modelled at each singular point on the resolution from complex geometry described above. Then, by gluing together torsion-free G_2-structures on the different regions of M, we can write down a family of G_2-structures on M with small torsion. This will be explained in §7, but first we will give examples of orbifolds T^7/Γ and T^8/Γ, and how to resolve them.

6.3. An example of an orbifold T^7/Γ and its resolution. We begin with an example of a suitable group Γ. Let $(x_1, \ldots, x_7)$ be coordinates on $T^7 = \mathbb{R}^7/\mathbb{Z}^7$, where $x_i \in \mathbb{R}/\mathbb{Z}$. Let (φ_0, g_0) be the flat G_2-structure on T^7 defined by (2.1) and (2.2). Let α, β and γ be the involutions of T^7 defined by

$$\alpha\big((x_1, \ldots, x_7)\big) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7), \tag{6.1}$$

$$\beta\big((x_1, \ldots, x_7)\big) = (-x_1, \tfrac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7), \tag{6.2}$$

$$\gamma\big((x_1, \ldots, x_7)\big) = (\tfrac{1}{2} - x_1, x_2, \tfrac{1}{2} - x_3, x_4, -x_5, x_6, -x_7). \tag{6.3}$$

By inspection, α, β and γ preserve (φ_0, g_0), because of the careful choice of exactly which signs to change. Also, $\alpha^2 = \beta^2 = \gamma^2 = 1$, and α, β and γ commute. Thus they generate a group $\Gamma = \langle \alpha, \beta, \gamma \rangle \cong \mathbb{Z}_2^3$ of isometries of T^7 preserving the flat G_2-structure (φ_0, g_0). The following Lemma is proved in [**16**, §2.1].

LEMMA 6.1. *The only nonidentity elements of Γ with fixed points are α, β and γ. Each of S_α, S_β and S_γ are 4 copies of T^3, so that the singular set S of T^7/Γ is a disjoint union of 12 copies of T^3. Every component of S is a singularity modelled on that of $T^3 \times \mathbb{C}^2/\{\pm 1\}$.*

Now the natural resolution of the complex singularity $\mathbb{C}^2/\{\pm 1\}$ is the Eguchi-Hanson space X of §4.1, with its resolving map $\pi : X \to \mathbb{C}^2/\{\pm 1\}$. Therefore, $T^3 \times X$ is a resolution of the singularity $T^3 \times \mathbb{C}^2/\{\pm 1\}$, with resolving map $\pi : T^3 \times X \to T^3 \times \mathbb{C}^2/\{\pm 1\}$. Each component of S is modelled on $T^3 \times \mathbb{C}^2/\{\pm 1\}$, and we resolve T^7/Γ by replacing this with $T^3 \times X$, using the resolving map π. In this way we construct a compact, nonsingular 7-manifold M with a map $\pi : M \to T^7/\Gamma$, making (M, π) into a resolution of T^7/Γ. Later we will construct a family of metrics with holonomy G_2 on M.

6.4. An example of an orbifold T^8/Γ and its resolution. Let $(x_1, \ldots, x_8)$ be coordinates on $T^8 = \mathbb{R}^8/\mathbb{Z}^8$, where $x_i \in \mathbb{R}/\mathbb{Z}$. Define a flat $Spin(7)$-structure (Ω_0, g_0) on T^8 as in Definition 3.1. Let α, β, γ and δ be the involutions of T^8 defined by

$$\alpha\big((x_1, \ldots, x_8)\big) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8), \tag{6.4}$$

$$\beta\big((x_1, \ldots, x_8)\big) = (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8), \tag{6.5}$$

$$\gamma\big((x_1, \ldots, x_8)\big) = (\tfrac{1}{2} - x_1, \tfrac{1}{2} - x_2, x_3, x_4, \tfrac{1}{2} - x_5, \tfrac{1}{2} - x_6, x_7, x_8), \tag{6.6}$$

$$\delta\big((x_1, \ldots, x_8)\big) = (-x_1, x_2, \tfrac{1}{2} - x_3, x_4, \tfrac{1}{2} - x_5, x_6, \tfrac{1}{2} - x_7, x_8). \tag{6.7}$$

By inspection, α, β, γ and δ preserve Ω_0 and g_0. It is easy to see that $\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 1$, and that $\alpha, \beta, \gamma, \delta$ all commute. Define Γ to be the group $\langle \alpha, \beta, \gamma, \delta \rangle$. Then $\Gamma \cong (\mathbb{Z}_2)^4$ is a group of automorphisms of T^8 preserving (Ω_0, g_0).

The following Lemma is proved in [**15**, §3.2].

LEMMA 6.2. *The only nonidentity elements of Γ with fixed points in T^8 are $\alpha, \beta, \gamma, \delta$ and $\alpha\beta$. The corresponding singular sets are as follows:*

(*i*) *S_α is 4 copies of $T^4/\{\pm 1\}$,*

(*ii*) *S_β is 4 copies of $T^4/\{\pm 1\}$,*

(*iii*) *S_γ is 2 copies of T^4,*

(*iv*) *S_δ is 2 copies of T^4, and*

(*v*) *$S_{\alpha\beta}$ is 64 points.*

Here S_α and S_β intersect in $S_{\alpha\beta}$. Each point $x\Gamma \in S_{\alpha\beta}$ has stabilizer $\Gamma_x = \{1, \alpha, \beta, \alpha\beta\}$.

Thus, the singular set of T^8/Γ is rather more complex than that of the previous example. However, we can still resolve T^8/Γ to get a compact 8-manifold M, using only the Eguchi-Hanson space X. First consider S_γ, which is two disjoint copies of T^4, each modelled on the singularity of $T^4 \times \mathbb{C}^2/\{\pm 1\}$. The resolution of this singularity is $T^4 \times X$, and we may use this to resolve both S_γ and S_δ.

Now the singular sets S_α and S_β are not disjoint, but rather, each component $T^4/\{\pm 1\}$ in S_α meets each component $T^4/\{\pm 1\}$ in S_β in 4 points, which lie in $S_{\alpha\beta}$. The singularity at each point in $S_{\alpha\beta}$ is modelled on $\mathbb{C}^2/\{\pm 1\} \times \mathbb{C}^2/\{\pm 1\}$. Here S_α corresponds locally to the subset $\mathbb{C}^2/\{\pm 1\} \times \{0\}$, and S_β corresponds locally to the subset $\{0\} \times \mathbb{C}^2/\{\pm 1\}$. Now the natural resolution of $\mathbb{C}^2/\{\pm 1\} \times \mathbb{C}^2/\{\pm 1\}$ is $X \times X$, and this is how we resolve near each point in $S_{\alpha\beta}$.

Each component $T^4/\{\pm 1\}$ of S_α and S_β is modelled locally on $T^4/\{\pm 1\} \times \mathbb{C}^2/\{\pm 1\}$, and the resolution of this is $K3 \times X$, where $T^4/\{\pm 1\}$ is resolved to give the $K3$ surface using the Eguchi-Hanson space X, as in §4.2. Combining these resolutions gives a compact 8-manifold M with a map $\pi : M \to T^8/\Gamma$, making (M, π) a resolution of T^8/Γ. In [**15**], the author constructs a family of metrics with holonomy $Spin(7)$ on M.

7. G_2- and $Spin(7)$-structures with small torsion

Now we will explain Step 2 of §5, concentrating on the G_2 case. Suppose that we are given an orbifold T^7/Γ equipped with a flat G_2-structure (φ_0, g_0), and a resolution (M, π) of T^7/Γ, constructed in the way explained in §6.2. Let S be the singular set of T^7/Γ, and let $S_1, \ldots, S_l$ be the connected components of S. Now, for simplicity, and in order to be as explicit as possible, we shall suppose that each component S_j is a copy of T^3, and the corresponding singularity is modelled on $T^3 \times \mathbb{C}^2/\Gamma_j$, where Γ_j is a finite subgroup of $SU(2)$. Then each S_j is desingularized using $T^3 \times X_j$, where X_j is a complex resolution of $\mathbb{C}^2/\Gamma_j$. This is the case in the example of §6.3. However, the methods we use also work with more complicated singularities, as explained in [**17**, §2.2].

Let ζ be a positive constant, and define $A_j = \{m \in M : d(\pi(m), S_j) < 2\zeta\}$ for $j = 1, \ldots, l$, and $B = \{m \in M : d(\pi(m), S_j) > \zeta \text{ for } j = 1, \ldots, l\}$, where $d(\,,\,)$ is the metric on T^7/Γ induced by g_0. Let ζ be chosen sufficiently small that the A_j are all disjoint, and each A_j is of the form $A_j \cong T^3 \times Y_j$, where Y_j is an open subset in X_j with boundary $\mathcal{S}^3/\Gamma_j$. Then, as in §4.2, our manifold M is the union of open sets A_j and B, where the A_j are disjoint, but $A_j \cap B$ is an 'annulus' diffeomorphic to the product $T^3 \times (\zeta, 2\zeta) \times \mathcal{S}^3/\Gamma_j$.

The subset B is naturally isomorphic to an open subset of T^7/Γ, and therefore it carries a flat G_2-structure (φ_0, g_0). Consider the restriction of (φ_0, g_0) to $A_j \cap B$. We may identify $A_j \cap B$ with an open subset of $T^3 \times \mathbb{C}^2/\Gamma_j$, and under this identification, φ_0 has the form

$$\varphi_0 = \nu^I \wedge \omega_0^I + \nu^J \wedge \omega_0^J + \nu^K \wedge \omega_0^K + \nu^I \wedge \nu^J \wedge \nu^K, \tag{7.1}$$

where ν^I, ν^J and ν^K are constant, linearly independent 1-forms on T^3, and $(\omega_0^I, \omega_0^J, \omega_0^K)$ is the flat $SU(2)$-structure on $\mathbb{C}^2/\Gamma_j$, as in §4.

Now, as we explained in §6.2, it is known that the manifold X_j admits a family of ALE metrics with holonomy $SU(2)$. Therefore, as in §4.1, for each $t > 0$ we can find a torsion-free $SU(2)$-structure $(\omega_t^I, \omega_t^J, \omega_t^K)$ on X_j, satisfying the asymptotic conditions

$$\omega_t^I = \omega_0^I + O(t^4 r^{-4}), \quad \omega_t^J = \omega_0^J + O(t^4 r^{-4}) \quad \text{and} \quad \omega_t^K = \omega_0^K + O(t^4 r^{-4}) \tag{7.2}$$

near infinity in X_j. Motivated by (7.1), define a 3-form φ_t' on A_j by

$$\varphi_t' = \nu^I \wedge \omega_t^I + \nu^J \wedge \omega_t^J + \nu^K \wedge \omega_t^K + \nu^I \wedge \nu^J \wedge \nu^K. \tag{7.3}$$

Then it turns out that φ_t' induces a *torsion-free G_2-structure* (φ_t', g_t') on A_j. Moreover, since $r \in (\zeta, 2\zeta)$ on $A_j \cap B$, equations (7.1) and (7.2) imply that

$$\varphi_t' = \varphi_0 + O(t^4) \qquad \text{on } A_j \cap B. \tag{7.4}$$

Thus, on the subset A_j of M we have a torsion-free G_2-structure (φ_t', g_t') for each $t > 0$, and on the subset B of M we have a torsion-free G_2-structure (φ_0, g_0). On the overlaps $A_j \cap B$, the difference between the G_2-structures is $O(t^4)$ by (7.4), and so when t is small, the two G_2-structures are close together. It is easy to use a partition of unity to write down a G_2-structure (φ_t, g_t) on M, which equals (φ_t', g_t') on $A_j \setminus B$ and equals (φ_0, g_0) on $B \setminus \bigcup_j A_j$, and interpolates smoothly between the two on the intersections $A_j \cap B$. This G_2-structure will be torsion-free on $A_j \setminus B$ and $B \setminus \bigcup_j A_j$, but will have nonzero torsion on $A_j \cap B$.

Now, for the purposes of the analysis in the next section, we need to estimate three geometric invariants of this G_2-structure (φ_t, g_t). These are the *torsion* $\nabla\varphi_t$, the *injectivity radius* $\delta(g_t)$ of g_t, and the *Riemann curvature* $R(g_t)$ of g_t. Using (7.4), one can ensure that $\|\nabla\varphi_t\|_{C^0} = O(t^4)$, and this is an estimate for the torsion. Let $g_{j,t}$ be the ALE metric with holonomy $SU(2)$ on X_j used in the construction. One can choose the metrics $g_{j,t}$ to be *homothetic* for all t, so that after applying an automorphism of X_j depending on t, we may choose $g_{j,t}$ to be isometric to $t^2 g_{j,1}$ for each $t > 0$.

It immediately follows that $\delta(g_{j,t}) = t\,\delta(g_{j,1})$ and $\|R(g_{j,t})\|_{C^0} = t^{-2}\|R(g_{j,1})\|_{C^0}$. When t is small, it is easy to see that the injectivity radius and curvature of the ALE metrics $g_{j,t}$ make the dominant contribution to the injectivity radius and curvature of the g_t on M. Therefore, we expect $\delta(g_t)$ to be $O(t)$, and $\|R(g_t)\|_{C^0}$ to be $O(t^{-2})$ for small t. Arguing in this way, one may estimate $\delta(g_t)$ and $\|R(g_t)\|_{C^0}$.

We state the existence and some important estimates for the G_2-structures (φ_t, g_t) in the following Theorem, which summarizes the results of this stage of the construction.

THEOREM 7.1. *On the compact 7-manifold M described in §6.3, and on many other compact 7-manifolds constructed in a similar fashion, one can write down the following data explicitly in coordinates:*

- *Positive constants A_1, A_2, A_3 and ϵ,*
- *A G_2-structure (φ_t, g_t) on M with $d\varphi_t = 0$ for each $t \in (0, \epsilon)$, and*
- *A 3-form ψ_t on M with $d^*\psi_t = d^*\varphi_t$ for each $t \in (0, \epsilon)$.*

These satisfy the three conditions:

(i) $\|\psi_t\|_{L^2} \le A_1 t^4$ *and* $\|d^*\psi_t\|_{L^{14}} \le A_1 t^4$,

(ii) the injectivity radius $\delta(g_t)$ satisfies $\delta(g_t) \ge A_2 t$, and

(iii) the Riemann curvature $R(g_t)$ satisfies $\|R(g_t)\|_{C^0} \le A_3 t^{-2}$.

Here the operator d^* and the norms $\|\,.\,\|_{L^2}$, $\|\,.\,\|_{L^{14}}$ and $\|\,.\,\|_{C^0}$ depend on g_t.

For a proof of this result, see [**16**, §2.2] and [**17**, §2.2]. Here is a brief explanation. From Proposition 2.2 we see that $\nabla\varphi_t = 0$ if and only if $d\varphi_t = d^*\varphi_t = 0$. It turns out to be more convenient to work with $d\varphi_t$ and $d^*\varphi_t$, rather than $\nabla\varphi_t$. Also, it is possible to choose the G_2-structure (φ_t, g_t) on M to satisfy $d\varphi_t = 0$, which means that the only nonzero component of the torsion left is $d^*\varphi_t$.

The 3-form ψ_t appearing in Theorem 7.1 should be interpreted as a *first integral* of $d^*\varphi_t$. Since $d^*\varphi_t = d^*\psi_t$, part (i) of the Theorem implies that $\|d^*\varphi_t\|_{L^{14}} \le A_1 t^4$. Thus, part (i) gives an $O(t^4)$ estimate on the torsion $\nabla\varphi_t$ of (φ_t, g_t). The reasons for choosing φ_t closed, and introducing ψ_t in the way we have, will become clear in the next section.

For the $Spin(7)$ construction, the result corresponding to Theorem 7.1 is the following, which is proved in [**15**, §4].

THEOREM 7.2. *On the compact 8-manifold M given in §6.4, and on many other compact 8-manifolds, one can write down the following data explicitly in coordinates:*

- *Positive constants A_1, A_2, A_3 and ϵ,*
- *A $Spin(7)$-structure (Ω_t, g_t) on M for each $t \in (0, \epsilon)$, and*
- *A 4-form ϕ_t on M for each $t \in (0, \epsilon)$ satisfying $d\Omega_t + d\phi_t = 0$.*

These satisfy the three conditions:

(i) $\|\phi_t\|_{L^2} \le A_1 t^{9/2}$ *and* $\|d\phi_t\|_{L^{10}} \le A_1 t$,

(ii) the injectivity radius $\delta(g_t)$ of g_t satisfies $\delta(g_t) \ge A_2 t$, and

(iii) the Riemann curvature $R(g_t)$ of g_t satisfies $\|R(g_t)\|_{C^0} \le A_3 t^{-2}$.

Here all norms are taken w.r.t. the metric g_t on M.

8. Deforming to torsion-free G_2- and $Spin(7)$-structures

Now we explain Step 3 of §5, which in the G_2 case is accomplished by the following Theorem.

THEOREM 8.1. *In the situation of Theorem 7.1, there are positive constants κ, K depending only on A_1, A_2, A_3 and ϵ, such that for every t with $0 < t \le \kappa$, there exists a smooth, torsion-free G_2-structure $(\tilde\varphi_t, \tilde g_t)$ on M with $\|\tilde\varphi_t - \varphi_t\|_{C^0} \le K t^{1/2}$.*

This result is proved in [**16**, §3]. The proof is not easy, and it represents most of the hard work in [**16**]. Note that the proof given there also involves estimates on the volume and diameter of the Riemannian manifold (M, g_t), but it turns out that these are unnecessary and can be removed. An improved proof will be given in [**19**].

The rest of this section gives a sketch of the proof of Theorem 8.1, ignoring several technical points. The treatment follows that given in [**19**], which differs a little from that of [**16**]. We begin in §8.1 with an aside on G_2-structures and exterior forms, which leads in §8.2 to a way to write the deformation problem as a nonlinear elliptic partial differential equation. Section 8.3 then explains how to construct a smooth solution using analytic methods.

8.1. G_2-structures and forms on 7-manifolds. Let φ_0 and $*\varphi_0$ be the 3- and 4-forms defined on $\mathbb{R}^7$ in (2.1) and (2.3). Now $GL(7,\mathbb{R})$ acts linearly on $\mathbb{R}^7$, and this induces an action of $GL(7,\mathbb{R})$ on $\Lambda^3(\mathbb{R}^7)^*$. Let $\mathscr{P}^3$ be the orbit of φ_0 in $\Lambda^3(\mathbb{R}^7)^*$ under this action. From §2, the stabilizer of φ_0 in $GL(7,\mathbb{R})$ is G_2, and therefore $\mathscr{P}^3$ is isomorphic to $GL(7,\mathbb{R})/G_2$. Since $\dim GL(7,\mathbb{R}) = 49$ and $\dim G_2 = 14$, it follows that $\dim \mathscr{P}^3 = 49 - 14 = 35$. But $\dim \Lambda^3(\mathbb{R}^7)^*$ is also $\binom{7}{3} = 35$. Thus, $\mathscr{P}^3$ is an *open set* in $\Lambda^3(\mathbb{R}^7)^*$. This means that the 3-form φ_0 used to define G_2-structures is a generic 3-form on $\mathbb{R}^7$.

In the same way, let $\mathscr{P}^4$ be the orbit of the 4-form $*\varphi_0$ under the action of $GL(7,\mathbb{R})$ on $\Lambda^4(\mathbb{R}^7)^*$. The stabilizer of $*\varphi_0$ is $G_2 \times \{\pm 1\}$, giving $\dim \mathscr{P}^4 = 35 = \dim \Lambda^4(\mathbb{R}^7)^*$, and so $\mathscr{P}^4$ is an open set in $\Lambda^4(\mathbb{R}^7)^*$. Now each element φ of $\mathscr{P}^3$ determines a unique G_2-structure (φ, g) on $\mathbb{R}^7$, and this in turn determines a 4-form $*\varphi$ in $\mathscr{P}^4$. Define a map $\Theta : \mathscr{P}^3 \to \mathscr{P}^4$ by $\Theta(\varphi) = *\varphi$. It is important to note that Θ is a *nonlinear* map. This is because the Hodge star $*$ used to define $*\varphi$ depends on the metric g, but g itself depends on φ.

Next, we extend these ideas from $\mathbb{R}^7$ to a general 7-manifold M. Let $\mathscr{F}$ be the *frame bundle* of M, which is a principal bundle with fibre $GL(7,\mathbb{R})$. Then we may write $\Lambda^3 T^*M = \mathscr{F} \times_{GL(7,\mathbb{R})} \Lambda^3(\mathbb{R}^7)^*$. Now $\mathscr{P}^3$ is a subset of $\Lambda^3(\mathbb{R}^7)^*$ invariant under $GL(7,\mathbb{R})$. Therefore we may define $\mathscr{P}^3 M = \mathscr{F} \times_{GL(7,\mathbb{R})} \mathscr{P}^3$, which is a subbundle of $\Lambda^3 T^*M$ with fibre $\mathscr{P}^3$. Similarly, define $\mathscr{P}^4 M = \mathscr{F} \times_{GL(7,\mathbb{R})} \mathscr{P}^4$, which is a subbundle of $\Lambda^4 T^*M$ with fibre $\mathscr{P}^4$. And as $\Theta : \mathscr{P}^3 \to \mathscr{P}^4$ commutes with the $GL(7,\mathbb{R})$-actions, it induces a map of bundles $\Theta : \mathscr{P}^3 M \to \mathscr{P}^4 M$.

A 3-form φ on M which lies in $\mathscr{P}^3 M$ at every point is called *positive*. Clearly, there is a 1-1 correspondence between G_2-structures on M, and positive 3-forms. By Proposition 2.2, if φ is a positive 3-form and (φ, g) the corresponding G_2-structure, then (φ, g) is *torsion-free* if and only if $d\varphi = d^*\varphi = 0$. But $d^*\varphi = 0$ if and only if $d(*\varphi) = 0$, and $*\varphi = \Theta(\varphi)$. Thus, (φ, g) is torsion-free if and only if $d\varphi = 0$ and $d\,\Theta(\varphi) = 0$.

Therefore, to construct torsion-free G_2-structures on a 7-manifold, we must look for sections φ of $\mathscr{P}^3 M$ satisfying the equations $d\varphi = d\,\Theta(\varphi) = 0$. Because Θ is nonlinear, we have to solve a *nonlinear partial differential equation* on the 3-form φ. The reason it helps to write the equations in this form is that the nonlinearity of the equations is obvious: it is packaged up in the nonlinear function Θ. If, instead, we wrote the condition as $\nabla\varphi = 0$ or $d\varphi = d^*\varphi = 0$, it would still be nonlinear, because the operators ∇ and d^* depend on φ in a nontrivial way, but the nonlinearity would be hidden.

8.2. Reformulating the problem as a nonlinear elliptic p.d.e. Now consider the situation of Theorem 7.1. We are given a 7-manifold M, a family of positive 3-forms φ_t and a family of 3-forms ψ_t for $t \in (0,\epsilon)$, which satisfy $d\varphi_t = 0$ and $d\,\Theta(\varphi_t) = d(*\psi_t)$. We regard t as being small and fixed. Our goal is to deform φ_t to a positive 3-form $\tilde\varphi_t$ satisfying $d\tilde\varphi_t = 0$ and $d\,\Theta(\tilde\varphi_t) = 0$. Let η be a 2-form on M, and put $\tilde\varphi_t = \varphi_t + d\eta$. Since $\mathscr{P}^3 M$ is an open subset of $\Lambda^3 T^*M$, it follows that if $d\eta$ is small in C^0, then $\tilde\varphi_t$ is a positive 3-form, and defines a G_2-structure on M. Also, $d\tilde\varphi_t = 0$ holds automatically.

Thus, the condition for $\tilde\varphi_t$ to define a torsion-free G_2-structure is that $d\eta$ should be small in C^0, and $d\,\Theta(\varphi_t + d\eta) = 0$. The function Θ can be expanded about φ_t to give

$$\Theta(\varphi_t + d\eta) = *\varphi_t + \tfrac{7}{3} * \pi_1(d\eta) + 2 * \pi_7(d\eta) - *d\eta - F(d\eta). \tag{8.1}$$

Here $F : \Lambda^3 T^*M \to \Lambda^4 T^*M$ is a smooth function, such that $F(\chi)$ is defined when $|\chi|$ is small, and satisfies $F(\chi) = O\big(|\chi|^2\big)$. Also, in (8.1) and in the rest of the section, the Hodge star $*$, the projections π_k and the operator d^* depend on the G_2-structure (φ_t, g_t).

Equation (8.1) expresses $\Theta(\varphi_t + d\eta)$ as the sum of a constant term $*\varphi_t$, a term $\frac{7}{3} * \pi_1(d\eta) + 2 * \pi_7(d\eta) - *d\eta$ that is linear in $d\eta$, and a remainder $F(d\eta)$ that is at least quadratic in $d\eta$. We shall use (8.1) to rewrite the equation $d\,\Theta(\varphi_t + d\eta) = 0$ in a form that we are able to solve. Here is the first stage in this.

Lemma 8.2. *Let η be a 2-form on M, and define a real function f on M by*

$$\pi_1(d\eta) = f\varphi_t. \tag{8.2}$$

Then the following two equations are equivalent:

$$d\,\Theta(\varphi_t + d\eta) = \tfrac{7}{3} df \wedge (*\varphi_t - *\psi_t) + 2d(*\pi_7(d\eta)), \tag{8.3}$$

and

$$d^* d\eta = d^*\psi_t + \tfrac{7}{3} d^*(f\psi_t) + *dF(d\eta). \tag{8.4}$$

Proof. Substituting (8.2) into (8.1) and applying d, we find that

$$d\,\Theta(\varphi_t + d\eta) = d * \varphi_t + \tfrac{7}{3} d(f * \varphi_t) + 2d * \pi_7(d\eta) - d * d\eta - dF(d\eta). \tag{8.5}$$

Now $d * \varphi_t = d * \psi_t$, and thus $df \wedge (*\varphi_t - *\psi_t) = d(f * \varphi_t) - d(f * \psi_t)$. Putting these into (8.5) and rearranging gives

$$\begin{aligned} d\,\Theta(\varphi_t + d\eta) = &\tfrac{7}{3} df \wedge (*\varphi_t - *\psi_t) + 2d * \pi_7(d\eta) \\ &- d * d\eta + d * \psi_t + \tfrac{7}{3} d(f * \psi_t) - dF(d\eta). \end{aligned} \tag{8.6}$$

Therefore (8.3) holds if and only if

$$d * d\eta = d * \psi_t + \tfrac{7}{3} d(f * \psi_t) - dF(d\eta). \tag{8.7}$$

Applying the Hodge star to this equation and using the fact that $*d* = -d^*$ on 3-forms gives (8.4), as we have to prove. □

Next, one proves the following Proposition.

Proposition 8.3. *Suppose that η is a smooth 2-form on M with $|d\eta|$ small, that f is a real function on M, and that* (8.3) *holds. Then $\tilde\varphi_t = \varphi_t + d\eta$ is a closed, positive 3-form and*

$$d\,\Theta(\tilde\varphi_t) = 0, \qquad df = 0 \qquad \text{and} \qquad d(*\pi_7(d\eta)) = 0. \tag{8.8}$$

Thus, $\tilde\varphi_t$ defines a torsion-free G_2-structure on M.

This rather curious result is proved using the special geometry of G_2-structures, and it shows that if (8.3) holds, then all three terms in the equation must actually be zero. Combining the Proposition and the previous Lemma, we arrive at the following formulation of the problem.

PROPOSITION 8.4. *Suppose η is a smooth 2-form on M with $|d\eta|$ small, and f a real function on M, that satisfy*

$$(dd^* + d^*d)\eta = d^*\psi_t + \tfrac{7}{3}d^*(f\psi_t) + *dF(d\eta) \tag{8.9}$$

$$\text{and} \qquad f\varphi_t = \pi_1(d\eta). \tag{8.10}$$

Then $\tilde{\varphi}_t = \varphi_t + d\eta$ defines a torsion-free G_2-structure on M.

PROOF. Since M is compact, the subsets $\operatorname{Im} d$ and $\operatorname{Im} d^*$ in $C^\infty(\Lambda^2 T^*M)$ are L^2-orthogonal, and so have zero intersection. Now the l.h.s. of (8.9) is the sum of $dd^*\eta \in \operatorname{Im} d$ and $d^*d\eta \in \operatorname{Im} d^*$, but the r.h.s. lies wholly in $\operatorname{Im} d^*$. Therefore $dd^*\eta = 0$, implying that $d^*\eta = 0$ by integration by parts. Thus, (8.9) implies (8.4), and any solutions η, f of (8.9) and (8.10) also satisfy (8.2) and (8.4). But Lemma 8.2 then shows that (8.3) holds, and finally Proposition 8.3 proves that if in addition $|d\eta|$ is small, then $\tilde{\varphi}_t$ defines a torsion-free G_2-structure, as we want. □

The operator $dd^* + d^*d$ appearing on the left hand side of (8.9) is a *second order linear elliptic operator*. Of course, the terms $d^*(f\psi_t)$ and $*dF(d\eta)$ on the right hand side are also of second order in η, since f is a component of $d\eta$. But when t is small, ψ_t is small, and thus $d^*(f\psi_t)$ is small compared to f. Also, since $F(\chi) = O(|\chi|^2)$, if $d\eta$ is small, then $F(d\eta)$ is even smaller.

Therefore, when ψ_t and $d\eta$ are both small, the second and third terms on the right hand side of (8.9) are small compared to the left hand side. In this case, since ellipticity is an open condition, (8.9) is a *nonlinear elliptic partial differential equation* for η. Now, a great deal is known about the properties of linear and nonlinear elliptic equations, and there is a body of well understood techniques for studying their solutions. For an introduction to this area, see Aubin [**4**], or Gilbarg and Trüdinger [**12**]. We shall use these techniques to show that (8.9) and (8.10) have a smooth solution.

8.3. Constructing a solution to the equation. Theorem 8.1 now follows from Proposition 8.4 and the next Theorem.

THEOREM 8.5. *There exist positive constants κ, K depending on the constants A_1, A_2, A_3 and ϵ of Theorem 7.1, such that for each t with $0 < t \leq \kappa$, there exists a smooth 2-form η on M with $\|d\eta\|_{C^0} \leq Kt^{1/2}$ satisfying* (8.9) *and* (8.10).

Here is a brief sketch of the proof of this result. We solve (8.9) by iteration, introducing sequences $\{\eta_j\}_{j=0}^\infty$ and $\{f_j\}_{j=0}^\infty$ with $\eta_0 = f_0 = 0$, satisfying the inductive relations

$$(dd^* + d^*d)\eta_{j+1} = d^*\psi_t + \tfrac{7}{3}d^*(f_j\psi_t) + *dF(d\eta_j) \tag{8.11}$$

$$\text{and} \qquad f_{j+1}\varphi_t = \pi_1(d\eta_{j+1}). \tag{8.12}$$

Suppose by induction that smooth $\eta_0, \dots, \eta_k$ and $f_0, \dots, f_k$ exist and satisfy (8.11) and (8.12) for $j < k$. Now, provided $F(d\eta_k)$ is well-defined, which happens if $|d\eta_k|$ is small, the r.h.s. of (8.11) for $j = k$ is well-defined and lies in $\operatorname{Im} d^*$. Therefore, by Hodge theory, there exists a smooth 2-form η_{k+1} satisfying (8.11). If in addition we ask that η_{k+1} be L^2-orthogonal to the Hodge forms representing $H^2(M,\mathbb{R})$, then η_{k+1} is unique. The function f_{k+1} is then defined uniquely by (8.12) for $j=k$.

Thus by induction, provided $|d\eta_j|$ remains sufficiently small for $F(d\eta_j)$ to be well-defined, the sequences $\{\eta_j\}_{j=0}^\infty$ and $\{f_j\}_{j=0}^\infty$ exist and can even be chosen uniquely. If these sequences converge to limits η and f, then taking limits in (8.11) and (8.12) shows that η and f satisfy (8.9) and (8.10), giving us the solution we want. The key to proving this is an inductive estimate: one shows that there are positive constants C_1, C_2, K and κ depending only on the constants A_1, A_2, A_3 and ϵ of Theorem 7.1, such that if η_j, η_{j+1}, f_j and f_{j+1} satisfy (8.11) and (8.12) and the inequalities

$$\|d\eta_j\|_{L^2} \le C_1 t^4, \qquad \|\nabla d\eta_j\|_{L^{14}} \le C_2 \quad \text{and} \quad \|d\eta_j\|_{C^0} \le K t^{1/2}, \tag{8.13}$$

and if $t \le \kappa$, then

$$\|d\eta_{j+1}\|_{L^2} \le C_1 t^4, \quad \|\nabla d\eta_{j+1}\|_{L^{14}} \le C_2 \quad \text{and} \quad \|d\eta_{j+1}\|_{C^0} \le K t^{1/2}. \tag{8.14}$$

Here is how the inductive estimates (8.14) are proved. For the first one, we take the L^2-inner product of (8.11) with η_{j+1} and integrate by parts. A short calculation shows that

$$\|d\eta_{j+1}\|_{L^2} \le \|\psi_t\|_{L^2} + \tfrac{7}{3}\|\psi_t\|_{L^2}\|d\eta_j\|_{C^0} + C_3\|d\eta_j\|_{L^2}\|d\eta_j\|_{C^0}, \tag{8.15}$$

where C_3 is a constant such that $|F(\chi)| \le C_3|\chi|^2$, for small 3-forms χ.

Equation (8.15) gives an *a priori estimate* for $\|d\eta_{j+1}\|_{L^2}$ in terms of $\|d\eta_j\|_{L^2}$ and $\|d\eta_j\|_{C^0}$, which are bounded by (8.13), and $\|\psi_t\|_{L^2}$, which is bounded by $A_1 t^4$. When t is small enough (depending on C_1, K, C_3 and A_1), we can show that $\|d\eta_{j+1}\|_{L^2} \le C_1 t^4$, as we have to prove. The 3-form ψ_t was introduced solely to achieve this inequality.

Next we prove the second inequality of (8.14). Using parts (ii) and (iii) of Theorem 7.1 we may show that if χ is a closed 3-form on M then

$$\|\nabla\chi\|_{L^{14}} \le C_4\big(\|d^*\chi\|_{L^{14}} + t^{-4}\|\chi\|_{L^2}\big), \tag{8.16}$$

where C_4 is a positive constant depending on A_2 and A_3. This is an *elliptic regularity result* for the elliptic operator $d+d^*$ acting on exterior forms on M. We substitute $\chi = d\eta_{j+1}$ in (8.16). The term $\|d^*\chi\|_{L^{14}}$ can be estimated in terms of norms of ψ_t and $d\eta_j$ using (8.11), and the term $\|\chi\|_{L^2}$ is $\|d\eta_{j+1}\|_{L^2}$, which we have already bounded. Again, when t is small enough we can show that $\|\nabla d\eta_{j+1}\|_{L^{14}} \le C_2$, as we want.

Lastly, we prove the third inequality of (8.14). If χ is a 3-form on M, then one can use parts (ii) and (iii) of Theorem 7.1 to show that

$$\|\chi\|_{C^0} \le C_5\big(t^{1/2}\|\nabla\chi\|_{L^{14}} + t^{-7/2}\|\chi\|_{L^2}\big), \tag{8.17}$$

where C_5 depends on A_2 and A_3. This is a *Sobolev embedding result*. The third inequality of (8.14) follows from the first two and (8.17), provided we take $K = C_5(C_2 + C_1)$.

The remainder of the proof is comparatively straightforward. By induction on j, the estimates (8.13) hold for all j. It soon follows that the sequences $\{\eta_j\}_{j=0}^\infty$ and $\{f_j\}_{j=0}^\infty$ exist, and converge in the appropriate Sobolev spaces to limits η, f, for which (8.9) and (8.10) hold. Taking the limit in (8.13) shows that $\|d\eta\|_{C^0} \le Kt^{1/2}$. Since (8.9) is elliptic for small t and $|d\eta|$, one can then show that η is smooth using standard analytic techniques, and the proof of Theorem 8.1 is complete.

For the $Spin(7)$ case, the result corresponding to Theorem 8.1 is the following.

THEOREM 8.6. *In the situation of Theorem 7.2, there are positive constants* κ, K *depending only on* A_1, A_2, A_3 *and* ϵ, *such that for every* t *with* $0 < t < \kappa$, *there exists a smooth, torsion-free* $Spin(7)$*-structure* $(\tilde{\Omega}_t, \tilde{g}_t)$ *on* M *with* $\|\tilde{\Omega}_t - \Omega_t\|_{C^0} \leq Kt^{1/2}$.

The proof of this Theorem is given in [**15**, §5]. It is somewhat different to the G_2 case above.

9. Finishing the proof

Having proved Theorem 8.1, we have found examples of compact 7-manifolds M admitting torsion-free G_2-structures. Now if (φ, g) is a torsion-free G_2-structure, then the holonomy group $\mathrm{Hol}(g)$ must be a subgroup of G_2. The final part of the construction, Step 4 of §5, is to show that $\mathrm{Hol}(g) = G_2$, rather than some proper subgroup. From Proposition 2.5, we see that $\mathrm{Hol}(g) = G_2$ if and only if the fundamental group $\pi_1(M)$ of M is finite. Therefore, to show that there exist metrics with holonomy G_2 on M, we just have to compute $\pi_1(M)$ and verify that it is finite.

EXAMPLE 9.1. We shall calculate the Betti numbers $b^k(M)$ and the fundamental group of the compact 7-manifold M of §6.3. Since M is compact and connected, we have $b^0(M) = 1$ and $b^k(M) = b^{7-k}(M)$ by Poincaré duality. Thus, it is enough to work out $b^1(M)$, $b^2(M)$ and $b^3(M)$. Now, the cohomology $H^*(T^7/\Gamma)$ is simply the Γ-invariant part of $H^*(T^7)$. Thus we can easily work out the Betti numbers of T^7/Γ, which are

$$b^1(T^7/\Gamma) = 0, \quad b^2(T^7/\Gamma) = 0 \quad \text{and} \quad b^3(T^7/\Gamma) = 7. \tag{9.1}$$

To make M we glue in 12 patches of the form $T^3 \times X$, where X is the Eguchi-Hanson space. Each of these patches increases b^k by $b^k(T^3 \times X) - b^k(T^3 \times \mathbb{C}^2/\{\pm 1\})$. Using the Künneth theorem and the Betti numbers $b^1(T^3) = 3$, $b^1(X) = 0$ and $b^2(X) = 1$, one can show that each patch adds 0 to b^1, 1 to b^2 and 3 to b^3. Together with (9.1) this gives

$$b^1(M) = 0, \quad b^2(M) = 12 \quad \text{and} \quad b^3(M) = 43. \tag{9.2}$$

The fundamental group of M is also easy to work out: it turns out that T^7/Γ is simply-connected, and the process of resolving does not change the fundamental group, and so M is also simply-connected. Now, Theorem 8.1 constructs torsion-free G_2-structures on M. But the fundamental group of M is finite, so by Proposition 2.5, these torsion-free G_2-structures come from metrics on M with holonomy G_2. Thus, M admits metrics with holonomy G_2. By (9.2) and Theorem 2.6, the moduli space of metrics with holonomy G_2 on M is a smooth, 43-dimensional manifold.

In the case of compact 8-manifolds M with torsion-free $Spin(7)$-structures (Ω, g), the topological test is more complicated: by Theorem 3.5, we have $\mathrm{Hol}(g) = Spin(7)$ if and only if M is simply-connected and $\hat{A}(M) = 1$, where $\hat{A}(M)$ is a linear combination of the Betti numbers $b^k(M)$ and $b^4_\pm(M)$. Thus, to prove that $\mathrm{Hol}(g) = Spin(7)$, we evaluate the fundamental group and Betti numbers of M and verify they satisfy the right conditions. Here is an example.

EXAMPLE 9.2. We shall work out the Betti numbers and fundamental group of the compact 8-manifold M of §6.4. Since M is compact and connected we have $b^0(M) = 1$ and $b^k(M) = b^{8-k}(M)$, so it is enough to find $b^1(M), b^2(M), b^3(M)$ and $b^4(M)$. But M is oriented, and thus $b^4(M)$ splits into the sum of $b^4_+(M)$ and $b^4_-(M)$. Working out the Γ-invariant part of $H^*(T^8)$ shows that

$$(9.3) \quad b^1(T^8/\Gamma) = b^2(T^8/\Gamma) = b^3(T^8/\Gamma) = 0 \quad \text{and} \quad b^4_+(T^8/\Gamma) = b^4_-(T^8/\Gamma) = 7.$$

To find the Betti numbers of the resolution M, we must add contributions from the resolution of each component of the singular set. These are more difficult to work out than in the previous example. In brief, each copy of $T^4/\{\pm 1\}$ in S_α and S_β fixes b^1 and b^3, adds 1 to b^2, and adds 3 to each of b^4_+ and b^4_-. Each copy of T^4 in S_γ and S_δ fixes b^1 and adds 1 to b^2, 4 to b^3 and 3 to each of b^4_+ and b^4_-. Each point in $S_{\alpha\beta}$ fixes b^1, b^2, b^3 and b^4_-, and adds 1 to b^4_+. Combining these with (9.3), we find that M has Betti numbers

$$(9.4) \quad b^1(M) = 0, \quad b^2(M) = 12, \quad b^3(M) = 16, \quad b^4_+(M) = 107, \quad b^4_-(M) = 43.$$

As in Example 9.1, it turns out that T^8/Γ is simply-connected, and the resolution does not change the fundamental group, and so M is simply-connected. From (9.4) and the definition (3.2) of $\hat{A}(M)$, we see that $\hat{A}(M) = 1$. Therefore Theorem 3.5 applies, so M admits metrics with holonomy $Spin(7)$. By (9.4) and Theorem 3.6, the moduli space of metrics of holonomy $Spin(7)$ on M is a smooth, 44-dimensional manifold.

10. Interesting questions

We finish by suggesting some questions and open problems for future research. Here is our first question.

- Which compact 7- and 8-manifolds M admit metrics with holonomy G_2 and $Spin(7)$?

The construction described above yields many examples of compact manifolds with holonomy G_2 and $Spin(7)$ metrics. The author and his research students (Coilin Nunan, Christine Taylor and Christopher Lewis) are making reasonable progress towards describing all compact manifolds with G_2 or $Spin(7)$ holonomy that can be made by resolving orbifolds T^7/Γ or T^8/Γ, and we hope to have a fairly complete picture of these soon. Certainly, only a finite number of manifolds arise in this way.

However, there may be many compact manifolds with holonomy G_2 and $Spin(7)$ that cannot be constructed by this method, but we know almost nothing about them. It is not even sure whether there are finitely or infinitely many compact manifolds admitting metrics with holonomy G_2 and $Spin(7)$. The author guesses that there are only finitely many.

Our second group of questions concerns the moduli space of metrics.

- On a given 7- or 8-manifold M, what does the moduli space of metrics with holonomy G_2 or $Spin(7)$ look like?
- Can this moduli space be compactified by adding extra 'ideal' points, corresponding to singular metrics?
- What kinds of singularities occur in these 'ideal' singular manifolds, and how are they resolved?

These seem to be difficult problems, but ones on which some progress can be made. The simplest sort of singularities to consider are orbifold singularities. When the orbifold group locally lies in some $SU(m)$ in G_2 or $Spin(7)$, we have a good theory of when and how the singularities can be resolved, which will be explained in [**19**]. Some things can also be proved for more general orbifold groups.

Thirdly, we discuss special submanifolds of G_2 and $Spin(7)$ manifolds. There is a beautiful theory called *calibrated geometry*, which was introduced by Harvey and Lawson [**13**]. The idea is that in a Riemannian manifold with an extra geometric structure such as a Kähler structure, there is a special type of minimal submanifold called a *calibrated submanifold*. For example, complex submanifolds of Kähler manifolds are calibrated submanifolds.

Riemannian 7-manifolds with holonomy G_2 have two types of calibrated submanifold, *associative 3-manifolds*, and *coassociative 4-manifolds*. Riemannian 8-manifolds with holonomy $Spin(7)$ have just one type, *Cayley 4-manifolds*. The deformation theory of all three was worked out by McLean [**27**]. Examples of compact associative and coassociative submanifolds in compact 7-manifolds with holonomy G_2 are given in [**17**, §4.2], and similar methods yield examples of Cayley 4-manifolds in $Spin(7)$-manifolds. Calibrated submanifolds seem to play a similar rôle in exceptional geometry to holomorphic curves in complex manifolds.

- Describe the calibrated submanifolds N in a given compact Riemannian manifold M with exceptional holonomy.
- Both coassociative 4-manifolds and Cayley 4-manifolds can occur in families of positive dimension. What do the singular elements of these families look like?
- What happens to the calibrated submanifolds in M as we deform the metric on M?
- One can define invariants of compact manifolds with exceptional holonomy, by counting calibrated submanifolds with a fixed homology class, as the Gromov invariant counts pseudo-holomorphic curves in a symplectic manifold. What is the theory of these invariants? Are there any connections with physics?

Finally, we note that compact manifolds with holonomy G_2 (and to a lesser extent $Spin(7)$) are arousing interest in the branch of theoretical physics known as 'string theory'. String theory is a very complex physical theory that at present has no adequate mathematical desciption. Its practitioners have a track record of making weird mathematical conjectures, which (usually) later turn out to be true. String theorists have already caused controversy in algebraic geometry with the idea of 'mirror manifolds', and formulae for counting holomorphic curves in complex 3-manifolds. It seems probable that string theory will also throw some light on the geometry of compact manifolds with holonomy G_2 and $Spin(7)$.

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