Counting coherent sheaves on Calabi–Yau 3-folds

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1. Motivation

Let $X, \check{X}$ be *mirror Calabi–Yau 3-folds*, coh(X) the abelian category of *coherent sheaves* on X, and $\mathsf{F}(\check{X})$ the *Fukaya category* of *Lagrangians* in $\check{X}$. *Homological Mirror Symmetry* predicts correspondences

$$D^b(\mathrm{coh}(X)) \leftrightarrow D^b(\mathsf{F}(\check{X}))$$

τ-(*semi*)*stable* objects of $D^b(\mathrm{coh}(X))$ $\leftrightarrow$ *special Lagrangian 3-folds* in $\check{X}$.

Here τ is a *stability condition* on coh(X) or $D^b(\mathrm{coh}(X))$, e.g. *Gieseker stability* w.r.t. a *polarization* on X.

We hope to define *invariants* of X which count τ-(semi)stable coherent sheaves on X, and of $\check{X}$ which count SL 3-folds in $\check{X}$, which should be equal under Mirror Symmetry, and of use in String Theory. For the moment, we work with coh(X), as we can use algebraic geometry.

Donaldson–Thomas invariants of X 'count' stable coherent sheaves in class α when stable=semistable. They are unchanged under *deformations* of X.

Issues for this talk:

1. How to define invariants that 'count' τ-(semi)stable sheaves in coh(X), X complex projective variety, that transform nicely under *change of stability condition* τ.

2. *Transformation laws* of these.

3. Special features when X is a Calabi–Yau 3-fold, with *simpler transformation laws*.

4. How best to 'count' *strictly τ-semistable* sheaves on C–Y 3-folds.

2. A research programme

Fix X a *complex projective variety*, $\mathcal{A} = \mathrm{coh}(X)$, $K(\mathcal{A}) \subset H^{\mathrm{even}}(X, \mathbb{Q})$ the *numerical Grothendieck group*, $\tau, \tilde{\tau}$ *stability conditions* on $\mathcal{A}$.

Step 1: Use Artin stacks.

Artin $\mathbb{C}$*-stacks* $\mathfrak{F}$ are a very general kind of space in complex algebraic geometry. They include $\mathbb{C}$-*schemes.* Write $\mathfrak{F}(\mathbb{C})$ for the set of *geometric points* of $\mathfrak{F}$. Each $x \in \mathfrak{F}(\mathbb{C})$ has a *stabilizer group* $\mathrm{Iso}_{\mathbb{C}}(x)$, with $\mathrm{Iso}_{\mathbb{C}}(x) = \{1\}$ if $\mathfrak{F}$ is a scheme.

There is a natural $\mathbb{C}$-stack $\mathfrak{Obj}_{\mathcal{A}}$ of *objects* in $\mathcal{A} = \mathrm{coh}(X)$, with $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{C})$ the set of *isomorphism classes of sheaves* in $\mathrm{coh}(X)$. It has *substacks* $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ of sheaves in class $\alpha \in K(\mathcal{A})$.

Configurations (σ, ι, π) in $\mathcal{A}$ are finite collections of objects and morphisms in $\mathcal{A}$ attached to a *finite poset* $(I, \preceq)$, satisfying axioms.

There are $\mathbb{C}$-stacks $\mathfrak{M}(I, \preceq)_{\mathcal{A}}$ of $(I, \preceq)$-configurations in $\mathcal{A}$, with many natural 1-*morphisms* between them.

Step 2: Regard τ-(semi)stable sheaves as constructible subsets. A *constructible set* in a $\mathbb{C}$-stack $\mathfrak{F}$ is a subset of $\mathfrak{F}(\mathbb{C})$ that is a finite union of $\mathfrak{G}(\mathbb{C})$ for $\mathfrak{G}$ a *finite type* substack of $\mathfrak{F}$. For τ a suitable *stability condition* on $\mathcal{A}$, e.g. *Gieseker stability*, write $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau), \mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$ for the set of $[E] \in \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{C})$ for E a τ-(semi)stable sheaf. Then $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$, $\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$ are *constructible sets* in $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ and $\mathfrak{Obj}_{\mathcal{A}}$.

Advantages of this point of view:

• $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau), \mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tilde{\tau})$ are *subsets of the same space* $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{C})$. This makes comparing them easier than comparing two different moduli schemes.

• $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ parametrizes τ-semistable sheaves up to *isomorphism*, not *S-equivalence*.

• the $\mathbb{C}$-stack $\mathfrak{Obj}_{\mathcal{A}}$ remembers the *stabilizer groups* $\mathrm{Aut}(E)$ of sheaves E, which $\mathbb{C}$-schemes do not. We need this information.

Step 3: Use the algebra of constructible functions.

A *constructible function* on $\mathfrak{F}$ is a map $f:\mathfrak{F}(\mathbb{C})\to\mathbb{Q}$ with finite image and $f^{-1}(c)$ a constructible set for $0\neq c\in f(\mathfrak{F}(\mathbb{C}))$. Write $\mathrm{CF}(\mathfrak{F})$ for the $\mathbb{Q}$-vector space of constructible f. Write $\delta_{\mathrm{ss}}^{\alpha}(\tau),\delta_{\mathrm{st}}^{\alpha}(\tau)\in\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ for the *characteristic functions* of $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau),\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$. Can *pullback* ϕ^*, *pushforward* $\mathrm{CF}^{\mathrm{stk}}(\phi)$ constructible functions along 1-morphisms ϕ.

Write $\mathfrak{Exact}_{\mathcal{A}}$ for the $\mathbb{C}$-stack of *exact sequences*

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

in $\mathcal{A}$, and $\pi_1, \pi_2, \pi_3 : \mathfrak{Exact}_{\mathcal{A}} \to \mathfrak{Obj}_{\mathcal{A}}$ for the 1-morphisms projecting to E_1, E_2, E_3.

Define a *bilinear multiplication* $*$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$f * g = \mathrm{CF}^{\mathrm{stk}}(\pi_2)\big[\pi_1^*(f) \cdot \pi_3^*(g)\big].$$

This makes $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ a $\mathbb{Q}$-*algebra* with *identity* $\delta_{[0]}$. The idea comes from *Ringel–Hall algebras*.

Stack functions $\mathrm{SF}(\mathfrak{F})$ are a universal generalization of $\mathrm{CF}(\mathfrak{F})$ containing more information, and behaving the same way under pushforwards and pullbacks. Then $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ is a *stack algebra*, with surjective algebra morphism $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$.

Using stack functions we generalize results from Euler characteristics χ to other motivic invariants, such as virtual Poincaré polynomials.

Step 4: Transformation law.

We characterize when $E \in \mathcal{A}$ is $\tilde{\tau}$-semistable using an *inclusion-exclusion process* on filtrations

$$0 = A_0 \subset \cdots \subset A_n = E \qquad (1)$$

for $S_i = A_i / A_{i-1}$ τ-*semistable*, with criteria on the orderings of $\tau(\alpha_i)$, $\tilde{\tau}(\alpha_i)$, where $\alpha_i = [S_i] \in K(\mathcal{A})$.

Thus we make $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tilde{\tau})$ by adding and subtracting subsets of $[E] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{C})$ with filtrations (1). The *characteristic function* of this subset is essentially $\delta_{\mathrm{ss}}^{\alpha_1}(\tau) * \cdots * \delta_{\mathrm{ss}}^{\alpha_n}(\tau)$.

Thus we prove a *universal formula*

$$\delta_{\rm ss}^\alpha(\tilde\tau) = \sum_{\substack{\alpha_1,\ldots,\alpha_n\in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha}} \begin{array}{l} S(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\\ \delta_{\rm ss}^{\alpha_1}(\tau)*\cdots*\delta_{\rm ss}^{\alpha_n}(\tau), \end{array} \quad (2)$$

where $C(\mathcal{A}) = \bigl\{[E]\in K(\mathcal{A}) : 0\not\cong E\in\mathcal{A}\bigr\}$. Here $S(\cdots)$ are *explicit combinatorial coefficients* equal to $1,0$ or -1, depending on the orderings of $\tau(\alpha_i)$ and $\tilde\tau(\alpha_i)$.

At each $[E]\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{C})$ there are only *finitely many nonzero terms* on r.h.s. of (2).

If $\dim X\leqslant 2$ there are only *finitely many nonzero functions* on r.h.s. of (2), For $\dim X\geqslant 3$ I can prove this for $\tau,\tilde\tau$ 'close'.

There is a *stack function* analogue of (2).

There are more complicated laws for transformations of $\delta_{\rm st}(\tilde\tau)$.

Step 5: Define 'motivic' invariants.

Let G be an abelian group and

$$F : \{\text{constructible sets in } \mathbb{C}\text{-stacks}\} \to G$$

a map which is *motivic*, that is, $F(S \cup T) = F(S) + F(T)$ if $S \cap T = \emptyset$.

Examples: Euler characteristics, virtual Poincaré polynomials, and virtual Hodge polynomials (can all be *weighted*).

Then define $I_{\mathrm{ss}}(\alpha;\tau) = F(\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau))$, and

$$I_{\mathrm{ss}}(\alpha_1,\ldots,\alpha_n;\tau) = F(\mathcal{M}_{\mathrm{ss}}(\alpha_1,\ldots,\alpha_n;\tau)),$$

where $\mathcal{M}_{\mathrm{ss}}(\alpha_1,\ldots,\alpha_n;\tau)$ is the constructible set of filtrations

$$0 = A_0 \subset \cdots \subset A_n = E \qquad (3)$$

for $S_i = A_i/A_{i-1}$ τ-*semistable* and $[S_i] = \alpha_i$, in the $\mathbb{C}$-stack of all filtrations (3).

Similar reasoning as for (2) shows that

$$I_{\mathrm{ss}}(\alpha,\tilde\tau)=\sum_{\substack{\alpha_1,\ldots,\alpha_n\in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha}} \begin{array}{l} S(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\\ I_{\mathrm{ss}}(\alpha_1,\ldots,\alpha_n;\tau). \end{array} \tag{4}$$

We can also express $I_{\mathrm{ss}}(\alpha_1,\ldots,\alpha_n;\tilde\tau)$ in terms of $I_{\mathrm{ss}}(\beta_1,\ldots,\beta_m;\tau)$ for $m\geqslant n$.

Here F must be *motivic* as the proof involves adding and subtracting sets.

The laws for *stable* invariants $I_{\mathrm{st}}(\cdots)$ are more complicated, involving posets $(I,\preceq)$.

Problem: I can prove there are only finitely many nonzero terms in (4) when $\dim X\leqslant 2$. For $\dim X\geqslant 3$ I can construct a chain of stability conditions $\tau=\tau_0,\tau_1,\ldots,\tau_n=\tilde\tau$ with finitely many terms transforming from τ_{i-1} to τ_i for $i=1,\ldots,n$.

Step 6: Special weights on semistables

For $\alpha \in C(\mathcal{A})$, define

$$\epsilon^\alpha(\tau) = \sum_{\substack{\alpha_1,\ldots,\alpha_n \in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha,\\ \tau(\alpha_i)=\tau(\alpha),\ \forall i}} \frac{(-1)^{n-1}}{n} \cdot \delta_{\mathrm{ss}}^{\alpha_1}(\tau) * \cdots * \delta_{\mathrm{ss}}^{\alpha_n}(\tau). \tag{5}$$

This is *invertible* combinatorially: we have

$$\delta_{\mathrm{ss}}^\alpha(\tau) = \sum_{\substack{\alpha_1,\ldots,\alpha_n \in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha,\\ \tau(\alpha_i)=\tau(\alpha),\ \forall i}} \frac{1}{n!} \cdot \epsilon^{\alpha_1}(\tau) * \cdots * \epsilon^{\alpha_n}(\tau). \tag{6}$$

For $[E] \in \mathfrak{Obj}_{\mathcal{A}}^\alpha(\mathbb{C})$ we have

- $\epsilon^\alpha(\tau)([E]) = 1$ if E is τ-*stable*,
- $\epsilon^\alpha(\tau)([E]) = 0$ is E is τ-*unstable* or *decomposable*,
- $\epsilon^\alpha(\tau)([E]) \in \mathbb{Q}$ if E is *strictly* τ-*semistable* and *indecomposable*.

Combining (2), (5) and (6) gives

$$\epsilon^\alpha(\tilde\tau)=\sum_{\substack{\alpha_1,\ldots,\alpha_n\in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha}} \begin{array}{l} T(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\\ \epsilon^{\alpha_1}(\tau)*\cdots*\epsilon^{\alpha_n}(\tau),\end{array} \quad (7)$$

where $T(\cdots)$ are *explicit combinatorial coefficients* in $\mathbb{Q}$. We can rewrite (7) as

$$\begin{array}{l}\epsilon^\alpha(\tilde\tau)=\epsilon^\alpha(\tau)+\mathbb{Q}\text{-linear combination}\\ \text{of commutators of }\epsilon^{\alpha_1}(\tau),\ldots,\epsilon^{\alpha_n}(\tau),\end{array} \quad (8)$$

where a *commutator* is

$[\epsilon^\alpha(\tau),\epsilon^\beta(\tau)]=\epsilon^\alpha(\tau)*\epsilon^\beta(\tau)-\epsilon^\beta(\tau)*\epsilon^\alpha(\tau)$,

$[\epsilon^\alpha(\tau),[\epsilon^\beta(\tau),\epsilon^\gamma(\tau)]]$, and so on.

Corollary. The $\mathbb{Q}$-Lie subalgebra of $\mathrm{CF}(\mathfrak{Obj}_\mathcal{A})$ generated by the $\epsilon^\alpha(\tau)$ for $\alpha\in C(\mathcal{A})$ is *independent of* τ.

There are *stack function* versions $\bar\epsilon^\alpha(\tau)$.

The moral of this:

What is the best way to define invariants that 'count' τ-(semi)stable sheaves?

- count only τ-*stables*?
- count all τ-*semistables*?

I suggest the 'best' way may be to count τ-semistables E with *weight* $\epsilon^\alpha(\tau)([E])$. Then (8) may give the invariants extra nice properties.

In the case stable=semistable, as for *Donaldson–Thomas invariants*, $\epsilon^\alpha(\tau) \equiv 1$ on τ-semistables, and the issue does not arise.

Step 7: Calabi–Yau 3-folds

Let X be a *Calabi–Yau* 3-*fold*. *Serre duality* in $\mathcal{A} = \mathrm{coh}(X)$ gives

$$\begin{aligned}&\dim \mathrm{Hom}(E,F) - \dim \mathrm{Ext}^1(E,F)\\ &- \dim \mathrm{Hom}(F,E) + \dim \mathrm{Ext}^1(F,E) = \chi([E],[F]),\end{aligned} \tag{9}$$

where $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ is the antisymmetric, biadditive *Euler form*.

Plan: find a way to 'count' sheaves so that for fixed, indecomposable E, F the 'number' of extensions $0 \to E \to G \to F \to 0$ is $\dim \mathrm{Ext}^1(F,E) - \dim \mathrm{Hom}(F,E)$.

Then (9) gives that *the 'number' of extensions of* E *by* F *minus the 'number' of extensions of* F *by* E *is* $\chi([E],[F])$.

This is possible, but very complex. The family of extensions $0 \to E \to G \to F \to 0$ is parametrized by $\mathrm{Ext}^1(F,E)$, and $\mathrm{Hom}(F,E)$ is part of the $\mathbb{C}$-algebra $\mathrm{End}(E \to G \to F)$. For A a finite-dimensional $\mathbb{C}$-algebra, we define a special *weight* $W(A) \in \mathbb{Q}$. Interpret $W(\mathrm{End}(E))$ as the number of 'virtual indecomposables' in E. Then

- $W(\mathrm{End}(E)) = 1$ for E indecomposable,
- $W(A \oplus B) = 0$ for A, B $\mathbb{C}$-algebras.

For $f \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, define $\chi^W(f) \in \mathbb{Q}$ to be $\chi^{\mathrm{na}}\bigl([E] \mapsto f([E]) \cdot W(\mathrm{End}(E))\bigr)$.

For E,F indecomposable, χ^{na} of the family of extensions $0\to E\to G\to F\to 0$ weighted by $W(\mathrm{End}(E\to G\to F))$ is $\dim\mathrm{Ext}^1(F,E)-\dim\mathrm{Hom}(F,E)$. This works as

$$W\Big(\mathrm{End}(E\to E\oplus F\to F)\Big)= \\ -\dim\mathrm{Hom}(F,E).$$

It is *nearly true* that for f,g in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ supported on indecomposables of class α,β we have
$\chi^W([f,g])=\chi^W(f)\cdot\chi^W(g)\cdot\chi(\alpha,\beta)$.
Not quite true when
$W(\mathrm{End}(E\to G\to F))\neq W(\mathrm{End}(G))$.

Define invariants $J(\alpha;\tau)$ of the C-Y 3-fold X for $\alpha \in C(\mathcal{A})$, roughly by $J(\alpha;\tau) = \chi^W(\epsilon^\alpha(\tau))$. Actually, must define $J(\alpha;\tau)$ by applying a linear map to the *stack function* $\bar{\epsilon}^\alpha(\tau)$. They transform by

$$J(\alpha,\tilde\tau) = \sum_{\substack{\text{iso. classes}\\ \text{of } \Gamma, I, \kappa}} \pm\, U(\Gamma,I,\kappa;\tau,\tilde\tau)\cdot \prod_{i\in I} J(\kappa(i),\tau)\cdot \prod_{\substack{\text{edges}\\ i - j \text{ in } \Gamma}} \chi(\kappa(i),\kappa(j)). \qquad (10)$$

Here Γ is a *connected, simply-connected undirected graph* with *vertices* I, $\kappa : I \to C(\mathcal{A})$ has $\sum_{i\in I}\kappa(i) = \alpha$, and $U(\Gamma,I,\kappa;\tau,\tilde\tau)$ in $\mathbb{Q}$ are *explicit combinatorial coefficients*.

Remarks: • I haven't proved (10) has *only finitely many nonzero terms*. But can find $\tau = \tau_0, \tau_1, ..., \tau_n = \tilde\tau$ with finitely many terms going from τ_{i-1} to τ_i, $i = 1, \ldots, n$.

• (10) expresses $J(\alpha; \tilde\tau)$ in terms of invariants $J(\alpha; \tau)$ of the *same type*. This is a special feature of the C–Y 3-fold case.

In contrast, (4) gives $I_{\rm ss}(\alpha, \tilde\tau)$ in terms of *more complex invariants* $I_{\rm ss}(\alpha_1, \ldots, \alpha_n; \tau)$.

• The $J(\alpha; \tau)$ are *not* expected to be *unchanged by deformations of* X, as Donaldson–Thomas invariants are.

Conjecture: there exists an *extension* of D–T invariants to the stable≠semistable case, which are deformation-invariant, and transform according to (10).

• The form of (10) as a sum over graphs Γ emerges combinatorially in a bizarre way. But it is natural in the *mirror picture* of counting SL 3-folds, when one SL 3-fold decays into a tree of intersecting SL 3-folds as the complex structure deforms.

• Can explain the multiplicative identity (10) in terms of a *Lie algebra morphism*

$$\Psi : \mathrm{SF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}) \to C(\mathcal{A}, \mathbb{Q}, \chi)$$

from a Lie algebra of stack functions supported on 'virtual indecomposables' to an explicit algebra.

• The sum over Γ in (10) looks like a sum of *Feynman diagrams*. I think there is some *new physics* behind this, to do with Π-*stability*.