D-manifolds and d-orbifolds: a theory of derived differential geometry

Dominic Joyce
The Mathematical Institute, Oxford University

Semi-final version, December 2012.
Updates will be posted on
`http://people.maths.ox.ac.uk/~joyce/dmanifolds.html`.

Contents

1 Introduction and survey **6**
- 1.1 Introduction . . . 6
- 1.2 C^∞-rings and C^∞-schemes . . . 9
- 1.3 The 2-category of d-spaces . . . 17
- 1.4 The 2-category of d-manifolds . . . 23
- 1.5 Manifolds with boundary and manifolds with corners . . . 40
- 1.6 D-spaces with corners . . . 48
- 1.7 D-manifolds with corners . . . 58
- 1.8 Deligne–Mumford C^∞-stacks . . . 69
- 1.9 Orbifolds . . . 81
- 1.10 The 2-category of d-stacks . . . 87
- 1.11 The 2-category of d-orbifolds . . . 96
- 1.12 Orbifolds with corners . . . 112
- 1.13 D-stacks with corners . . . 122
- 1.14 D-orbifolds with corners . . . 130
- 1.15 D-manifold and d-orbifold bordism, virtual cycles . . . 141
- 1.16 Relation to other classes of spaces in mathematics . . . 146

2 The 2-category of d-spaces **150**
- 2.1 Square zero extensions of C^∞-rings and C^∞-schemes . . . 150
- 2.2 The definition of d-spaces . . . 158
- 2.3 Equivalences in **dSpa** . . . 164
- 2.4 Gluing d-spaces by equivalences . . . 171
- 2.5 Fibre products of d-spaces . . . 186
- 2.6 Fibre products of manifolds in **Man** and **dSpa** . . . 198
- 2.7 Fixed point loci of finite groups in d-spaces . . . 200

3 The 2-category of d-manifolds 205
3.1 The 2-category of virtual quasicoherent sheaves 205
3.2 The definition of d-manifolds 210
3.3 Local properties of d-manifolds 214
3.4 Differential-geometric picture of 1- and 2-morphisms 222
3.5 Equivalences in **dMan** 229
3.6 Gluing d-manifolds by equivalences 233

4 Differential geometry of d-manifolds 236
4.1 Submersions, immersions, and embeddings 236
4.2 Local picture of (w-)submersions and (w-)immersions 242
4.3 D-transversality and fibre products 247
4.4 Embedding d-manifolds into manifolds 259
4.5 Orientation line bundles of virtual vector bundles 266
4.6 Orientations on d-manifolds 272
4.7 The homotopy category of d-manifolds 278

5 Manifolds with corners 281
5.1 Manifolds with corners, and boundaries 281
5.2 Smooth maps of manifolds with corners 282
5.3 Describing how smooth maps act on corners 283
5.4 Simple, semisimple and flat maps, and submersions 284
5.5 Corners $C_k(X)$ and the corner functors 286
5.6 Transversality and fibre products 289
5.7 Immersions, embeddings, and submanifolds 292
5.8 Orientations 294

6 D-spaces with corners 298
6.1 The definition of d-spaces with corners 298
6.2 Boundaries of d-spaces with corners 314
6.3 Simple, semisimple and flat 1-morphisms 321
6.4 Manifolds with corners as d-spaces with corners 329
6.5 Equivalences and étale 1-morphisms in $\mathbf{dSpa^c}$ 332
6.6 Gluing d-spaces with corners by equivalences 337
6.7 Corners $C_k(\mathbf{X})$, and the corner functors $C, \hat{C}$ 339
6.8 Fibre products in $\mathbf{dSpa^c}$ 345
6.9 Boundary and corners of c-transverse fibre products 371
6.10 Fixed points of finite groups in d-spaces with corners 378

7 D-manifolds with corners 385
7.1 Defining d-manifolds with corners 385
7.2 Local properties of d-manifolds with corners 389
7.3 Differential-geometric picture of 1-morphisms 394
7.4 Equivalences of d-manifolds with corners, and gluing 398
7.5 Submersions, immersions, and embeddings 401
7.6 Bd-transversality and fibre products 409

7.7 Embedding d-manifolds with corners into manifolds 414
7.8 Orientations 419

8 Orbifolds and orbifolds with corners 425
8.1 Review of definitions of orbifolds in the literature 425
8.2 Orbifolds as C^∞-stacks 427
8.3 Vector bundles on orbifolds 428
8.4 Orbifold strata of orbifolds, and effective orbifolds 430
8.5 Orbifolds with boundary and orbifolds with corners 439
8.6 Boundaries of orbifolds with corners, and simple, semisimple and flat 1-morphisms 446
8.7 Corners $C_k(\mathcal{X})$ and the corner functors $C, \hat{C}$ 448
8.8 Transversality and fibre products 454
8.9 Orbifold strata of orbifolds with corners 455

9 The 2-category of d-stacks 463
9.1 Square zero extensions of C^∞-stacks 463
9.2 The definition of d-stacks 467
9.3 D-stacks as quotients of d-spaces 472
9.4 Gluing d-stacks by equivalences 476
9.5 Fibre products of d-stacks 479
9.6 Orbifold strata of d-stacks 480

10 The 2-category of d-orbifolds 488
10.1 Definition and local properties of d-orbifolds 488
10.2 Equivalences and gluing 495
10.3 Submersions, immersions and embeddings 500
10.4 D-transversality and fibre products 501
10.5 Embedding d-orbifolds into orbifolds 504
10.6 Orientations on d-orbifolds 506
10.7 Orbifold strata of d-orbifolds 508
10.8 Kuranishi neighbourhoods and good coordinate systems 513
10.9 Semieffective and effective d-orbifolds 520

11 D-stacks with corners 525
11.1 The definition of d-stacks with corners 525
11.2 D-stacks with corners as quotients of d-spaces with corners . . . 533
11.3 Boundaries of d-stacks with corners, and simple, semisimple and flat 1-morphisms 535
11.4 Equivalences of d-stacks with corners, and gluing 537
11.5 Corners $C_k(\boldsymbol{\mathcal{X}})$, and the corner functors $C, \hat{C}$ 540
11.6 Fibre products in $\mathbf{dSta^c}$ 541
11.7 Orbifold strata of d-stacks with corners 546

12 D-orbifolds with corners **550**
12.1 The definition of d-orbifolds with corners 550
12.2 Local properties of d-orbifolds with corners 553
12.3 Equivalences and gluing 555
12.4 Submersions, immersions and embeddings 558
12.5 Bd-transversality and fibre products 561
12.6 Embedding d-orbifolds with corners into orbifolds 563
12.7 Orientations on d-orbifolds with corners 564
12.8 Orbifold strata of d-orbifolds with corners 566
12.9 Kuranishi neighbourhoods and good coordinate systems 568
12.10 Semieffective and effective d-orbifolds with corners 569

13 Bordism for d-manifolds and d-orbifolds **571**
13.1 Classical bordism groups for manifolds 571
13.2 D-manifold bordism groups 575
13.3 Classical bordism for orbifolds 580
13.4 Bordism for d-orbifolds 587
13.5 The proof of Theorem 13.23 591

14 Relating d-manifolds and d-orbifolds to other classes of spaces in mathematics **598**
14.1 Fredholm sections on Banach manifolds and solution spaces of nonlinear elliptic equations 598
14.2 Hofer–Wysocki–Zehnder's polyfolds 610
14.3 Fukaya–Oh–Ohta–Ono's Kuranishi spaces 620
14.4 Derived algebraic geometry and derived schemes 626
14.5 $\mathbb{C}$-schemes and $\mathbb{C}$-stacks with obstruction theories, and quasi-smooth derived $\mathbb{C}$-schemes and $\mathbb{C}$-stacks 630
14.6 The derived manifolds of Spivak and Borisov–Noel 647

A Categories and 2-categories **654**
A.1 Basics of category theory 654
A.2 Limits, colimits and fibre products in categories 655
A.3 2-categories 656
A.4 Fibre products in 2-categories 657

B Algebraic Geometry over C^∞-rings **659**
B.1 C^∞-rings 659
B.2 Special classes of C^∞-ring 661
B.3 Sheaves on topological spaces 662
B.4 C^∞-schemes 665
B.5 Manifolds as C^∞-rings and C^∞-schemes 667
B.6 Modules over C^∞-rings, and cotangent modules 669
B.7 Quasicoherent sheaves on C^∞-schemes 671

C Deligne–Mumford C^∞-stacks **675**
C.1 C^∞-stacks 675
C.2 Gluing C^∞-stacks by equivalences 678
C.3 Strongly representable 1-morphisms of C^∞-stacks 679
C.4 Quotient C^∞-stacks 681
C.5 Deligne–Mumford C^∞-stacks 683
C.6 Quasicoherent sheaves on C^∞-stacks 686
C.7 Sheaves of abelian groups and C^∞-rings on C^∞-stacks 693
C.8 Orbifold strata of C^∞-stacks 695
C.9 Sheaves on orbifold strata 703

D Existence of good coordinate systems **708**
D.1 Outline of the proof of Theorem 12.48 708
D.2 Step 1: Choose an open cover of $\mathcal{X}$ by $\mathcal{X}^a \simeq [\mathbf{Z}^a/\Gamma^a]$ 716
D.3 Step 2: Modify to a better open cover 721
D.4 Step 3: Choose sf-embeddings $\boldsymbol{f}_A : \mathbf{Z}_A \to \mathbf{U}_A$, $A \in I$ 728
D.5 Step 4: Construct the good coordinate system 732

References **738**

Glossary of Notation **745**

Index **753**

1 Introduction and survey

1.1 Introduction

This book develops a new theory of 'derived differential geometry'. The objects in this theory are *d-manifolds*, 'derived' versions of smooth manifolds, which form a (strict) 2-category **dMan**. There are also 2-categories of *d-manifolds with boundary* $\mathbf{dMan^b}$ and *d-manifolds with corners* $\mathbf{dMan^c}$, and orbifold versions of all these, *d-orbifolds* $\mathbf{dOrb}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$.

Here 'derived' is intended in the sense of *derived algebraic geometry.* The original motivating idea for derived algebraic geometry, as in Kontsevich [63] for instance, was that certain moduli schemes $\mathcal{M}$ appearing in enumerative invariant problems may be very singular as schemes. However, it may be natural to realize $\mathcal{M}$ as a categorical truncation of some 'derived' moduli space $\boldsymbol{\mathcal{M}}$, a new kind of geometric object living in a higher category. The geometric structure on $\boldsymbol{\mathcal{M}}$ should encode the full deformation theory of the moduli problem, the obstructions as well as the deformations. It was hoped that $\boldsymbol{\mathcal{M}}$ would be 'smooth', and so in some sense simpler than its truncation $\mathcal{M}$.

Early work in derived algebraic geometry focussed on *dg-schemes*, as in Ciocan-Fontanine and Kapranov [23]. These have largely been replaced by the *derived stacks* of Toën and Vezzosi [100–102], and the *structured spaces* of Lurie [70–72]. *Derived differential geometry* aims to generalize these ideas to differential geometry and smooth manifolds. A brief note about it can be found in Lurie [72, §4.5]; the ideas are worked out in detail by Lurie's student David Spivak [95], who defines an ∞-category of *derived manifolds.*

The author came to these questions from a different direction, symplectic geometry. Many important areas in symplectic geometry involve forming moduli spaces $\overline{\mathcal{M}}_{g,m}(X,J,\beta)$ of J-holomorphic curves in some symplectic manifold (X,ω), possibly with boundary in a Lagrangian Y, and then 'counting' these moduli spaces to get 'invariants' with interesting properties. Such areas include Gromov–Witten invariants (open and closed), Lagrangian Floer cohomology, Symplectic Field Theory, contact homology, and Fukaya categories.

To do this 'counting', one needs to put a suitable geometric structure on $\overline{\mathcal{M}}_{g,m}(X,J,\beta)$ — something like the 'derived' moduli spaces $\boldsymbol{\mathcal{M}}$ above — and use this to define a 'virtual class' or 'virtual chain' in $\mathbb{Z},\mathbb{Q}$ or some homology theory. Two alternative theories for geometric structures to put on moduli spaces $\overline{\mathcal{M}}_{g,m}(X,J,\beta)$ are the *Kuranishi spaces* of Fukaya, Oh, Ohta and Ono [32, 34] and the *polyfolds* of Hofer, Wysocki and Zehnder [43–48].

The philosophies of Kuranishi spaces and of polyfolds are in a sense opposite: Kuranishi spaces remember only the minimal information needed to form virtual chains, but polyfolds remember a huge amount more information, essentially a complete description of the functional-analytic problem which gives rise to the moduli space. There is a truncation functor from polyfolds to Kuranishi spaces.

The theory of Kuranishi spaces in [32, 34] does not go far – they define Kuranishi spaces, and construct virtual cycles upon them, but they do not define morphisms between Kuranishi spaces, for instance. The author tried to study and work with Kuranishi spaces as geometric spaces in their own right, but ran into problems, and became convinced that a new definition was needed. Upon reading Spivak's theory of derived manifolds [95], it became clear that some form of 'derived differential geometry' was required: *Kuranishi spaces in the sense of* [32, §A] *ought to be defined to be 'derived orbifolds with corners'.*

The purpose of this book is to build a comprehensive, rigorous theory of derived differential geometry designed for applications in symplectic geometry, and other areas of mathematics such as String Topology.

As the moduli spaces of interest in the symplectic geometry of Lagrangian submanifolds should be 'derived orbifolds with corners', it was necessary that this theory should cover not just derived manifolds without boundary, but also derived manifolds and derived orbifolds with boundary and with corners. This has much increased the length of the book: the parts dealing with d-manifolds without boundary (Chapters 2–4 and Appendices A–B) are only roughly a quarter of the whole. It turns out that doing 'things with corners' properly is a complex, fascinating, and hitherto almost unexplored area.

The author wants the theory to be easily usable by symplectic geometers, and others who are not specialists in derived algebraic geometry. In applications, much of the theory can be treated as a 'black box', as they do not require a detailed understanding of what a d-manifold or d-orbifold really is, but only a general idea, plus a list of useful properties of the 2-categories $\mathbf{dMan}, \mathbf{dOrb}$.

With this in mind, the rest of Chapter 1 provides a long, detailed, and more-or-less self-contained summary of the rest of the book. The intention is that many readers should be able to find what they need in Chapter 1, only dipping into later chapters for more detail, examples, or proofs.

Our theory of derived differential geometry has a major simplification compared to the derived algebraic geometry of Toën and Vezzosi [100–102] and Lurie [70–72], and the derived manifolds of Spivak [95]. All of the 'derived' spaces in [70–72, 95, 100–102] form some kind of ∞-category (simplicial category, model category, Segal category, quasicategory, ...). In contrast, our d-manifolds and d-orbifolds form (strict) 2-categories $\mathbf{dMan}, \ldots, \mathbf{dOrb^c}$, which are the simplest and most friendly kind of higher category.

Furthermore, the ∞-categories in [70–72, 95, 100–102] are usually formed by localization (inversion of some class of morphisms), so the (higher) morphisms in the resulting ∞-category are difficult to describe and work with. But the 1- and 2-morphisms in $\mathbf{dMan}, \ldots, \mathbf{dOrb^c}$ are defined explicitly, without localization.

The essence of our simplification is this. Consider a 'derived' moduli space $\boldsymbol{\mathcal{M}}$ of some objects E, e.g. vector bundles on some $\mathbb{C}$-scheme X. One expects $\boldsymbol{\mathcal{M}}$ to have a 'cotangent complex' $\mathbb{L}_{\boldsymbol{\mathcal{M}}}$, a complex in some derived category with cohomology $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_E \cong \mathrm{Ext}^{1-i}(E,E)^*$ for $i \in \mathbb{Z}$. In general, $\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ can have nontrivial cohomology in many negative degrees, and because of this such objects $\boldsymbol{\mathcal{M}}$ must form an ∞-category to properly describe their geometry.

However, the moduli spaces relevant to enumerative invariant problems are of a restricted kind: one considers only $\boldsymbol{\mathcal{M}}$ such that $\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ has nontrivial cohomology only in degrees $-1, 0$, where $h^0(\mathbb{L}_{\boldsymbol{\mathcal{M}}})$ encodes the (dual of the) deformations $\mathrm{Ext}^1(E,E)^*$, and $h^{-1}(\mathbb{L}_{\boldsymbol{\mathcal{M}}})$ the (dual of the) obstructions $\mathrm{Ext}^2(E,E)^*$. As in

Toën [100, §4.4.3], such derived spaces are called *quasi-smooth*, and this is a necessary condition on $\boldsymbol{\mathcal{M}}$ for the construction of a virtual fundamental class.

Our construction of d-manifolds replaces complexes in a derived category $D^b \operatorname{coh}(\mathcal{M})$ with a 2-category of complexes in degrees $-1, 0$ only. For general $\boldsymbol{\mathcal{M}}$ this loses a lot of information, but for quasi-smooth $\boldsymbol{\mathcal{M}}$, since $\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ is concentrated in degrees $-1, 0$, the important information is retained. In the language of dg-schemes, this corresponds to working with a subclass of derived schemes whose dg-algebras are of a special kind: they are 2-step supercommutative dg-algebras $A^{-1} \xrightarrow{\mathrm{d}} A^0$ such that $\mathrm{d}(A^{-1}) \cdot A^{-1} = 0$. Then $\mathrm{d}(A^{-1})$ is a square zero ideal in A^0, and A^{-1} is a module over $H^0\bigl(A^{-1} \xrightarrow{\mathrm{d}} A^0\bigr)$.

An important reason why this 2-category style derived geometry works successfully in our differential-geometric context is the existence of *partitions of unity* on smooth manifolds, and on nice C^∞-schemes. This means that (derived) structure sheaves are 'fine' or 'soft', which simplifies their behaviour. Partitions of unity are also essential for constructions such as gluing d-manifolds by equivalences on open d-subspaces in **dMan**. In conventional derived algebraic geometry, where partitions of unity do not exist, one needs the extra freedom of an ∞-category to glue by equivalences.

Apart from Chapter 5, which summarizes [55], and Appendices A–C, the material in this book is new research, being published for the first time. A survey paper on this book, focussing on d-manifolds without boundary, is [58].

Sections 1.2–1.16 summarize the rest of the book, following the order of chapters, except that §1.2 on C^∞-schemes and §1.8 on C^∞-stacks correspond to Appendices B and C, and §1.9 on orbifolds and §1.12 on orbifolds with corners correspond to the first and second halves of Chapter 8.

Throughout the book we will consistently use different typefaces to indicate different classes of geometrical objects. In particular:

- $W, X, Y, \ldots$ will denote manifolds (of any kind), or topological spaces.
- $\underline{W}, \underline{X}, \underline{Y}, \ldots$ will denote C^∞-schemes.
- $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \ldots$ will denote d-spaces, including d-manifolds.
- $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \ldots$ will denote Deligne–Mumford C^∞-stacks, including orbifolds.
- $\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \ldots$ will denote d-stacks, including d-orbifolds.
- $\mathbf{W}, \mathbf{X}, \mathbf{Y}, \ldots$ will denote d-spaces with corners, including d-manifolds with corners.
- $\mathfrak{W}, \mathfrak{X}, \mathfrak{Y}, \ldots$ will denote orbifolds with corners.
- $\boldsymbol{\mathfrak{W}}, \boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Y}}, \ldots$ will denote d-stacks with corners, including d-orbifolds with corners.

Acknowledgements. My particular thanks to Dennis Borisov, Jacob Lurie and Bertrand Toën for help with derived manifolds. I would also like to thank Manabu Akaho, Tom Bridgeland, James Cranch, Oliver Fabert, Kenji Fukaya, Ieke Moerdijk, Hiroshi Ohta, Kauru Ono, and Timo Schürg for useful conversations.

1.2 C^∞-rings and C^∞-schemes

If X is a manifold then the $\mathbb{R}$-algebra $C^\infty(X)$ of smooth functions $c : X \to \mathbb{R}$ is a C^∞-*ring*. That is, for each smooth function $f : \mathbb{R}^n \to \mathbb{R}$ there is an n-fold operation $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ acting by $\Phi_f : c_1, \ldots, c_n \mapsto f(c_1, \ldots, c_n)$, and these operations Φ_f satisfy many natural identities. Thus, $C^\infty(X)$ actually has a far richer algebraic structure than the obvious $\mathbb{R}$-algebra structure.

C^∞-*algebraic geometry* is a version of algebraic geometry in which rings or algebras are replaced by C^∞-rings. The basic objects are C^∞-*schemes*, a category of differential-geometric spaces including smooth manifolds, and also many singular spaces. They were introduced in synthetic differential geometry (see for instance Dubuc [30] and Moerdijk and Reyes [86]), and developed further by the author in [56] (surveyed in [57]).

This section briefly discusses C^∞-rings, C^∞-schemes, and quasicoherent sheaves on C^∞-schemes, with the aim of enabling the reader to understand the definitions of d-spaces and d-manifolds. Appendix B provides a more complete treatment, giving full definitions and results, and going into technical details.

1.2.1 C^∞-rings

Definition 1.2.1. A C^∞-*ring* is a set $\mathfrak{C}$ together with operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all $n \geqslant 0$ and smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, where by convention when $n = 0$ we define $\mathfrak{C}^0$ to be the single point $\{\emptyset\}$. These operations must satisfy the following relations: suppose $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ are smooth functions. Define a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g\bigl(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)\bigr),$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g\bigl(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)\bigr).$$

We also require that for all $1 \leqslant j \leqslant n$, defining $\pi_j : \mathbb{R}^n \to \mathbb{R}$ by $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$, we have $\Phi_{\pi_j}(c_1, \ldots, c_n) = c_j$ for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$.

Usually we refer to $\mathfrak{C}$ as the C^∞-ring, leaving the operations Φ_f implicit.

A *morphism* between C^∞-rings $\bigl(\mathfrak{C}, (\Phi_f)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\bigr)$, $\bigl(\mathfrak{D}, (\Psi_f)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\bigr)$ is a map $\phi : \mathfrak{C} \to \mathfrak{D}$ such that $\Psi_f\bigl(\phi(c_1), \ldots, \phi(c_n)\bigr) = \phi \circ \Phi_f(c_1, \ldots, c_n)$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$ and $c_1, \ldots, c_n \in \mathfrak{C}$. We will write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Here is the motivating example:

Example 1.2.2. Let X be a manifold. Write $C^\infty(X)$ for the set of smooth functions $c : X \to \mathbb{R}$. For $n \geqslant 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ smooth, define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by

$$\bigl(\Phi_f(c_1, \ldots, c_n)\bigr)(x) = f\bigl(c_1(x), \ldots, c_n(x)\bigr), \tag{1.1}$$

for all $c_1, \ldots, c_n \in C^\infty(X)$ and $x \in X$. It is easy to see that $C^\infty(X)$ and the operations Φ_f form a C^∞-ring.

Now let $f : X \to Y$ be a smooth map of manifolds. Then pullback $f^* : C^\infty(Y) \to C^\infty(X)$ mapping $f^* : c \mapsto c \circ f$ is a morphism of C^∞-rings. Furthermore (at least for Y without boundary), every C^∞-ring morphism $\phi : C^\infty(Y) \to C^\infty(X)$ is of the form $\phi = f^*$ for a unique smooth map $f : X \to Y$.

Write $\mathbf{C^\infty Rings}^{\rm op}$ for the opposite category of $\mathbf{C^\infty Rings}$, with directions of morphisms reversed, and **Man** for the category of manifolds without boundary. Then we have a full and faithful functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}} : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\rm op}$ acting by $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}(X) = C^\infty(X)$ on objects and $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}(f) = f^*$ on morphisms. This embeds **Man** as a full subcategory of $\mathbf{C^\infty Rings}^{\rm op}$.

Note that C^∞-rings are far more general than those coming from manifolds. For example, if X is any topological space we could define a C^∞-ring $C^0(X)$ to be the set of *continuous* $c : X \to \mathbb{R}$, with operations Φ_f defined as in (1.1). For X a manifold with $\dim X > 0$, the C^∞-rings $C^\infty(X)$ and $C^0(X)$ are different.

Definition 1.2.3. Let $\mathfrak{C}$ be a C^∞-ring. Then we may give $\mathfrak{C}$ the structure of a *commutative* $\mathbb{R}$*-algebra*. Define addition '+' on $\mathfrak{C}$ by $c + c' = \Phi_f(c, c')$ for $c, c' \in \mathfrak{C}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is $f(x, y) = x + y$. Define multiplication '$\cdot$' on $\mathfrak{C}$ by $c \cdot c' = \Phi_g(c, c')$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is $g(x, y) = xy$. Define scalar multiplication by $\lambda \in \mathbb{R}$ by $\lambda c = \Phi_{\lambda'}(c)$, where $\lambda' : \mathbb{R} \to \mathbb{R}$ is $\lambda'(x) = \lambda x$. Define elements $0, 1 \in \mathfrak{C}$ by $0 = \Phi_{0'}(\emptyset)$ and $1 = \Phi_{1'}(\emptyset)$, where $0' : \mathbb{R}^0 \to \mathbb{R}$ and $1' : \mathbb{R}^0 \to \mathbb{R}$ are the maps $0' : \emptyset \mapsto 0$ and $1' : \emptyset \mapsto 1$. One can show using the relations on the Φ_f that the axioms of a commutative $\mathbb{R}$-algebra are satisfied. In Example 1.2.2, this yields the obvious $\mathbb{R}$-algebra structure on the smooth functions $c : X \to \mathbb{R}$.

An *ideal* I in $\mathfrak{C}$ is an ideal $I \subset \mathfrak{C}$ in $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra. Then we make the quotient $\mathfrak{C}/I$ into a C^∞-ring as follows. If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, define $\Phi_f^I : (\mathfrak{C}/I)^n \to \mathfrak{C}/I$ by

$$\bigl(\Phi_f^I(c_1 + I, \ldots, c_n + I)\bigr)(x) = f\bigl(c_1(x), \ldots, c_n(x)\bigr) + I.$$

Using Hadamard's Lemma, one can show that this is independent of the choice of representatives $c_1, \ldots, c_n$. Then $\bigl(\mathfrak{C}/I, (\Phi_f^I)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\bigr)$ is a C^∞-ring.

A C^∞-ring $\mathfrak{C}$ is called *finitely generated* if there exist $c_1, \ldots, c_n$ in $\mathfrak{C}$ which generate $\mathfrak{C}$ over all C^∞-operations. That is, for each $c \in \mathfrak{C}$ there exists smooth $f : \mathbb{R}^n \to \mathbb{R}$ with $c = \Phi_f(c_1, \ldots, c_n)$. Given such $\mathfrak{C}, c_1, \ldots, c_n$, define $\phi : C^\infty(\mathbb{R}^n) \to \mathfrak{C}$ by $\phi(f) = \Phi_f(c_1, \ldots, c_n)$ for smooth $f : \mathbb{R}^n \to \mathbb{R}$, where $C^\infty(\mathbb{R}^n)$ is as in Example 1.2.2 with $X = \mathbb{R}^n$. Then ϕ is a surjective morphism of C^∞-rings, so $I = \operatorname{Ker} \phi$ is an ideal in $C^\infty(\mathbb{R}^n)$, and $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ as a C^∞-ring. Thus, $\mathfrak{C}$ is finitely generated if and only if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some $n \geqslant 0$ and some ideal I in $C^\infty(\mathbb{R}^n)$.

1.2.2 C^∞-schemes

Next we summarize material in [56, §4] on C^∞-schemes.

Definition 1.2.4. A C^∞*-ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf $\mathcal{O}_X$ of C^∞-rings on X.

A *morphism* $\underline{f} = (f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of C^∞ ringed spaces is a continuous map $f : X \to Y$ and a morphism $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves of C^∞-rings on X, where $f^{-1}(\mathcal{O}_Y)$ is the inverse image sheaf. There is another way to write the data $f^\sharp$: since direct image of sheaves f_* is right adjoint to inverse image f^{-1}, there is a natural bijection

$$\mathrm{Hom}_X\bigl(f^{-1}(\mathcal{O}_Y), \mathcal{O}_X\bigr) \cong \mathrm{Hom}_Y\bigl(\mathcal{O}_Y, f_*(\mathcal{O}_X)\bigr). \tag{1.2}$$

Write $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ for the morphism of sheaves of C^∞-rings on Y corresponding to $f^\sharp$ under (1.2), so that

$$f^\sharp : f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X \quad \rightsquigarrow \quad f_\sharp : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X). \tag{1.3}$$

Depending on the application, either $f^\sharp$ or $f_\sharp$ may be more useful. We choose to regard $f^\sharp$ as primary and write morphisms as $\underline{f} = (f, f^\sharp)$ rather than $(f, f_\sharp)$, because we find it convenient to work uniformly using pullbacks, rather than mixing pullbacks and pushforwards.

Write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces. As in [30, Th. 8] there is a *spectrum functor* $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{C^\infty RS}$, defined explicitly in [56, Def. 4.12]. A C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme* if it is isomorphic in $\mathbf{C^\infty RS}$ to $\mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. A C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$ is called a *C^∞-scheme* if X can be covered by open sets $U \subseteq X$ such that $(U, \mathcal{O}_X|_U)$ is an affine C^∞-scheme. Write $\mathbf{C^\infty Sch}$ for the full subcategory of C^∞-schemes in $\mathbf{C^\infty RS}$.

A C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ is called *locally fair* if X can be covered by open $U \subseteq X$ with $(U, \mathcal{O}_X|_U) \cong \mathrm{Spec}\,\mathfrak{C}$ for some finitely generated C^∞-ring $\mathfrak{C}$. Roughly speaking this means that $\underline{X}$ is locally finite-dimensional. Write $\mathbf{C^\infty Sch^{lf}}$ for the full subcategory of locally fair C^∞-schemes in $\mathbf{C^\infty Sch}$.

We call a C^∞-scheme $\underline{X}$ *separated, second countable, compact, locally compact*, or *paracompact*, if the underlying topological space X is Hausdorff, second countable, compact, locally compact, or paracompact, respectively.

We define a C^∞-scheme $\underline{X}$ for each manifold X.

Example 1.2.5. Let X be a manifold. Define a C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$ to have topological space X and $\mathcal{O}_X(U) = C^\infty(U)$ for each open $U \subseteq X$, where $C^\infty(U)$ is the C^∞-ring of smooth maps $c : U \to \mathbb{R}$, and if $V \subseteq U \subseteq X$ are open define $\rho_{UV} : C^\infty(U) \to C^\infty(V)$ by $\rho_{UV} : c \mapsto c|_V$. Then $\underline{X} = (X, \mathcal{O}_X)$ is a local C^∞-ringed space. It is canonically isomorphic to $\mathrm{Spec}\, C^\infty(X)$, and so is an affine C^∞-scheme. It is locally fair.

Define a functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}} : \mathbf{Man} \to \mathbf{C^\infty Sch^{lf}} \subset \mathbf{C^\infty Sch}$ by $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}} = \mathrm{Spec} \circ F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}$. Then $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ is full and faithful, and embeds $\mathbf{Man}$ as a full subcategory of $\mathbf{C^\infty Sch}$.

By [56, Cor. 4.21 & Th. 4.33] we have:

Theorem 1.2.6. *Fibre products and all finite limits exist in* $\mathbf{C^\infty Sch}$. *The subcategory* $\mathbf{C^\infty Sch^{lf}}$ *is closed under fibre products and finite limits. The functor* $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ *takes transverse fibre products in* $\mathbf{Man}$ *to fibre products in* $\mathbf{C^\infty Sch}$.

The proof of the existence of fibre products in $\mathbf{C^\infty Sch}$ follows that for fibre products of schemes in Hartshorne [38, Th. II.3.3], together with the existence of C^∞-scheme products $\underline{X} \times \underline{Y}$ of affine C^∞-schemes $\underline{X}, \underline{Y}$. The latter follows from the existence of coproducts $\mathfrak{C}\hat{\otimes}\mathfrak{D}$ in $\mathbf{C^\infty Rings}$ of C^∞-rings $\mathfrak{C}, \mathfrak{D}$. Here $\mathfrak{C}\hat{\otimes}\mathfrak{D}$ may be thought of as a 'completed tensor product' of $\mathfrak{C}, \mathfrak{D}$. The actual tensor product $\mathfrak{C} \otimes_{\mathbb{R}} \mathfrak{D}$ is naturally an $\mathbb{R}$-algebra but not a C^∞-ring, with an inclusion of $\mathbb{R}$-algebras $\mathfrak{C} \otimes_{\mathbb{R}} \mathfrak{D} \hookrightarrow \mathfrak{C}\hat{\otimes}\mathfrak{D}$, but $\mathfrak{C}\hat{\otimes}\mathfrak{D}$ is often much larger than $\mathfrak{C} \otimes_{\mathbb{R}} \mathfrak{D}$. For free C^∞-rings we have $C^\infty(\mathbb{R}^m)\hat{\otimes}C^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^{m+n})$.

In [56, Def. 4.34 & Prop. 4.35] we discuss *partitions of unity* on C^∞-schemes.

Definition 1.2.7. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-scheme. Consider a formal sum $\sum_{a\in A} c_a$, where A is an indexing set and $c_a \in \mathcal{O}_X(X)$ for $a \in A$. We say $\sum_{a\in A} c_a$ is a *locally finite sum on* $\underline{X}$ if X can be covered by open $U \subseteq X$ such that for all but finitely many $a \in A$ we have $\rho_{XU}(c_a) = 0$ in $\mathcal{O}_X(U)$.

By the sheaf axioms for $\mathcal{O}_X$, if $\sum_{a\in A} c_a$ is a locally finite sum there exists a unique $c \in \mathcal{O}_X(X)$ such that for all open $U \subseteq X$ with $\rho_{XU}(c_a) = 0$ in $\mathcal{O}_X(U)$ for all but finitely many $a \in A$, we have $\rho_{XU}(c) = \sum_{a\in A} \rho_{XU}(c_a)$ in $\mathcal{O}_X(U)$, where the sum makes sense as there are only finitely many nonzero terms. We call c the *limit* of $\sum_{a\in A} c_a$, written $\sum_{a\in A} c_a = c$.

Let $c \in \mathcal{O}_X(X)$. Then there is a unique maximal open set $V \subseteq X$ with $\rho_{XV}(c) = 0$ in $\mathcal{O}_X(V)$. Define the *support* $\operatorname{supp} c$ to be $X \setminus V$, so that $\operatorname{supp} c$ is closed in X. If $U \subseteq X$ is open, we say that c *is supported in* U if $\operatorname{supp} c \subseteq U$.

Let $\{U_a : a \in A\}$ be an open cover of X. A *partition of unity on* $\underline{X}$ *subordinate to* $\{U_a : a \in A\}$ is $\{\eta_a : a \in A\}$ with $\eta_a \in \mathcal{O}_X(X)$ supported on U_a for $a \in A$, such that $\sum_{a\in A} \eta_a$ is a locally finite sum on $\underline{X}$ with $\sum_{a\in A} \eta_a = 1$.

Proposition 1.2.8. *Suppose $\underline{X}$ is a separated, paracompact, locally fair C^∞-scheme, and $\{\underline{U}_a : a \in A\}$ an open cover of $\underline{X}$. Then there exists a partition of unity $\{\eta_a : a \in A\}$ on $\underline{X}$ subordinate to $\{\underline{U}_a : a \in A\}$.*

Here are some differences between ordinary schemes and C^∞-schemes:

Remark 1.2.9. (i) If A is a ring or algebra, then points of the corresponding scheme $\operatorname{Spec} A$ are prime ideals in A. However, if $\mathfrak{C}$ is a C^∞-ring then (by definition) points of $\operatorname{Spec} \mathfrak{C}$ are maximal ideals in $\mathfrak{C}$ with residue field $\mathbb{R}$, or equivalently, $\mathbb{R}$-algebra morphisms $x : \mathfrak{C} \to \mathbb{R}$. This has the effect that if X is a manifold then points of $\operatorname{Spec} C^\infty(X)$ are just points of X.

(ii) In conventional algebraic geometry, affine schemes are a restrictive class. Central examples such as $\mathbb{CP}^n$ are not affine, and affine schemes are not closed under open subsets, so that $\mathbb{C}^2$ is affine but $\mathbb{C}^2 \setminus \{0\}$ is not. In contrast, affine C^∞-schemes are already general enough for many purposes. For example:

- All manifolds are fair affine C^∞-schemes.
- Open C^∞-subschemes of fair affine C^∞-schemes are fair and affine.
- If $\underline{X}$ is a separated, paracompact, locally fair C^∞-scheme then $\underline{X}$ is affine.

Affine C^∞-schemes are always separated (Hausdorff), so we need general C^∞-schemes to include non-Hausdorff behaviour.

(iii) In conventional algebraic geometry the Zariski topology is too coarse for many purposes, so one has to introduce the étale topology. In C^∞-algebraic geometry there is no need for this, as affine C^∞-schemes are Hausdorff.

(iv) Even very basic C^∞-rings such as $C^\infty(\mathbb{R}^n)$ for $n > 0$ are not noetherian as $\mathbb{R}$-algebras. So C^∞-schemes should be compared to non-noetherian schemes in conventional algebraic geometry.

(v) The existence of partitions of unity, as in Proposition 1.2.8, makes some things easier in C^∞-algebraic geometry than in conventional algebraic geometry. For example, geometric objects can often be 'glued together' over the subsets of an open cover using partitions of unity, and if $\mathcal{E}$ is a quasicoherent sheaf on a separated, paracompact, locally fair C^∞-scheme $\underline{X}$ then $H^i(\mathcal{E}) = 0$ for $i > 0$.

1.2.3 Modules over C^∞-rings, and cotangent modules

In [56, §5] we discuss modules over C^∞-rings.

Definition 1.2.10. Let $\mathfrak{C}$ be a C^∞-ring. A $\mathfrak{C}$-*module* M is a module over $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra as in Definition 1.2.3. $\mathfrak{C}$-modules form an abelian category, which we write as $\mathfrak{C}$-mod. For example, $\mathfrak{C}$ is a $\mathfrak{C}$-module, and more generally $\mathfrak{C} \otimes_{\mathbb{R}} V$ is a $\mathfrak{C}$-module for any real vector space V. Let $\phi : \mathfrak{C} \to \mathfrak{D}$ be a morphism of C^∞-rings. If M is a $\mathfrak{C}$-module then $\phi_*(M) = M \otimes_{\mathfrak{C}} \mathfrak{D}$ is a $\mathfrak{D}$-module. This induces a functor $\phi_* : \mathfrak{C}\text{-mod} \to \mathfrak{D}\text{-mod}$.

Example 1.2.11. Let X be a manifold, and $E \to X$ a vector bundle. Write $C^\infty(E)$ for the vector space of smooth sections e of E. Then $C^\infty(X)$ acts on $C^\infty(E)$ by multiplication, so $C^\infty(E)$ is a $C^\infty(X)$-module.

In [56, §5.3] we define the *cotangent module* $\Omega_{\mathfrak{C}}$ of a C^∞-ring $\mathfrak{C}$.

Definition 1.2.12. Let $\mathfrak{C}$ be a C^∞-ring, and M a $\mathfrak{C}$-module. A C^∞-*derivation* is an $\mathbb{R}$-linear map $\mathrm{d} : \mathfrak{C} \to M$ such that whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and $c_1, \ldots, c_n \in \mathfrak{C}$, we have

$$\mathrm{d}\Phi_f(c_1, \ldots, c_n) = \textstyle\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i.$$

We call such a pair M, d a *cotangent module* for $\mathfrak{C}$ if it has the universal property that for any $\mathfrak{C}$-module M' and C^∞-derivation $\mathrm{d}' : \mathfrak{C} \to M'$, there exists a unique morphism of $\mathfrak{C}$-modules $\phi : M \to M'$ with $\mathrm{d}' = \phi \circ \mathrm{d}$.

Define $\Omega_{\mathfrak{C}}$ to be the quotient of the free $\mathfrak{C}$-module with basis of symbols $\mathrm{d}c$ for $c \in \mathfrak{C}$ by the $\mathfrak{C}$-submodule spanned by all expressions of the form $\mathrm{d}\big(\Phi_f(c_1, \ldots, c_n)\big) - \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i$ for $f : \mathbb{R}^n \to \mathbb{R}$ smooth and $c_1, \ldots, c_n \in \mathfrak{C}$, and define $\mathrm{d}_{\mathfrak{C}} : \mathfrak{C} \to \Omega_{\mathfrak{C}}$ by $\mathrm{d}_{\mathfrak{C}} : c \mapsto \mathrm{d}c$. Then $\Omega_{\mathfrak{C}}, \mathrm{d}_{\mathfrak{C}}$ is a cotangent module for $\mathfrak{C}$. Thus cotangent modules always exist, and are unique up to unique isomorphism.

Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings with cotangent modules $\Omega_{\mathfrak{C}}, \mathrm{d}_{\mathfrak{C}}$, $\Omega_{\mathfrak{D}}, \mathrm{d}_{\mathfrak{D}}$, and $\phi : \mathfrak{C} \to \mathfrak{D}$ be a morphism of C^∞-rings. Then ϕ makes $\Omega_{\mathfrak{D}}$ into a $\mathfrak{C}$-module, and there is a unique morphism $\Omega_\phi : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{D}}$ in $\mathfrak{C}$-mod with $\mathrm{d}_{\mathfrak{D}} \circ \phi = \Omega_\phi \circ \mathrm{d}_{\mathfrak{C}}$. This induces a morphism $(\Omega_\phi)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{D} \to \Omega_{\mathfrak{D}}$ in $\mathfrak{D}$-mod with $(\Omega_\phi)_* \circ (\mathrm{d}_{\mathfrak{C}} \otimes \mathrm{id}_{\mathfrak{D}}) = \mathrm{d}_{\mathfrak{D}}$.

Example 1.2.13. Let X be a manifold. Then the cotangent bundle T^*X is a vector bundle over X, so as in Example 1.2.11 it yields a $C^\infty(X)$-module $C^\infty(T^*X)$. The exterior derivative $\mathrm{d}: C^\infty(X) \to C^\infty(T^*X)$ is a C^∞-derivation. These $C^\infty(T^*X), \mathrm{d}$ have the universal property in Definition 1.2.12, and so form a *cotangent module* for $C^\infty(X)$.

Now let X, Y be manifolds, and $f: X \to Y$ be smooth. Then $f^*(TY), TX$ are vector bundles over X, and the derivative of f is a vector bundle morphism $\mathrm{d}f: TX \to f^*(TY)$. The dual of this morphism is $\mathrm{d}f^*: f^*(T^*Y) \to T^*X$. This induces a morphism of $C^\infty(X)$-modules $(\mathrm{d}f^*)_*: C^\infty\bigl(f^*(T^*Y)\bigr) \to C^\infty(T^*X)$. This $(\mathrm{d}f^*)_*$ is identified with $(\Omega_{f^*})_*$ in Definition 1.2.12 under the natural isomorphism $C^\infty\bigl(f^*(T^*Y)\bigr) \cong C^\infty(T^*Y) \otimes_{C^\infty(Y)} C^\infty(X)$.

Definition 1.2.12 abstracts the notion of cotangent bundle of a manifold in a way that makes sense for any C^∞-ring.

1.2.4 Quasicoherent sheaves on C^∞-schemes

In [56, §6] we discuss sheaves of modules on C^∞-schemes.

Definition 1.2.14. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-scheme. An $\mathcal{O}_X$*-module* $\mathcal{E}$ on $\underline{X}$ assigns a module $\mathcal{E}(U)$ over $\mathcal{O}_X(U)$ for each open set $U \subseteq X$, with $\mathcal{O}_X(U)$-action $\mu_U: \mathcal{O}_X(U) \times \mathcal{E}(U) \to \mathcal{E}(U)$, and a linear map $\mathcal{E}_{UV}: \mathcal{E}(U) \to \mathcal{E}(V)$ for each inclusion of open sets $V \subseteq U \subseteq X$, such that the following commutes:

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{E}(U) \arrow[r, "\mu_U"'] \arrow[d, "\rho_{UV} \times \mathcal{E}_{UV}"] & \mathcal{E}(U) \arrow[d, "\mathcal{E}_{UV}"'] \\ \mathcal{O}_X(V) \times \mathcal{E}(V) \arrow[r, "\mu_V"] & \mathcal{E}(V), \end{tikzcd}$$

and all this data $\mathcal{E}(U), \mathcal{E}_{UV}$ satisfies the usual sheaf axioms [38, §II.1] .

A *morphism of* $\mathcal{O}_X$*-modules* $\phi: \mathcal{E} \to \mathcal{F}$ assigns a morphism of $\mathcal{O}_X(U)$-modules $\phi(U): \mathcal{E}(U) \to \mathcal{F}(U)$ for each open set $U \subseteq X$, such that $\phi(V) \circ \mathcal{E}_{UV} = \mathcal{F}_{UV} \circ \phi(U)$ for each inclusion of open sets $V \subseteq U \subseteq X$. Then $\mathcal{O}_X$-modules form an abelian category, which we write as $\mathcal{O}_X$-mod.

As in [56, §6.2], the spectrum functor $\mathrm{Spec}: \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{C^\infty Sch}$ has a counterpart for modules: if $\mathfrak{C}$ is a C^∞-ring and $(X, \mathcal{O}_X) = \mathrm{Spec}\,\mathfrak{C}$ we can define a functor $\mathrm{MSpec}: \mathfrak{C}\text{-mod} \to \mathcal{O}_X\text{-mod}$. If $\mathfrak{C}$ is a *fair* C^∞-ring, there is a full abelian subcategory $\mathfrak{C}\text{-mod}^{\mathrm{co}}$ of *complete* $\mathfrak{C}$-modules in $\mathfrak{C}$-mod, such that $\mathrm{MSpec}\,|_{\mathfrak{C}\text{-mod}^{\mathrm{co}}}: \mathfrak{C}\text{-mod}^{\mathrm{co}} \to \mathcal{O}_X\text{-mod}$ is an equivalence of categories. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-scheme, and $\mathcal{E}$ an $\mathcal{O}_X$-module. We call $\mathcal{E}$ *quasicoherent* if $\underline{X}$ can be covered by open $\underline{U}$ with $\underline{U} \cong \mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$, and under this identification $\mathcal{E}|_U \cong \mathrm{MSpec}\,M$ for some $\mathfrak{C}$-module M. We call $\mathcal{E}$ a *vector bundle of rank* $n \geqslant 0$ if $\underline{X}$ may be covered by open $\underline{U}$ such that $\mathcal{E}|_{\underline{U}} \cong \mathcal{O}_U \otimes_{\mathbb{R}} \mathbb{R}^n$.

Write $\mathrm{qcoh}(\underline{X}), \mathrm{vect}(\underline{X})$ for the full subcategories of quasicoherent sheaves and vector bundles in $\mathcal{O}_X$-mod. Then $\mathrm{qcoh}(\underline{X})$ is an abelian category. Since $\mathrm{MSpec}: \mathfrak{C}\text{-mod}^{\mathrm{co}} \to \mathcal{O}_X\text{-mod}$ is an equivalence for $\mathfrak{C}$ fair and $(X, \mathcal{O}_X) = \mathrm{Spec}\,\mathfrak{C}$, as in [56, Cor. 6.11] we see that if $\underline{X}$ is a locally fair C^∞-scheme then every $\mathcal{O}_X$-module $\mathcal{E}$ on $\underline{X}$ is quasicoherent, that is, $\mathrm{qcoh}(\underline{X}) = \mathcal{O}_X$-mod.

Remark 1.2.15. (a) If $\underline{X}$ is a separated, paracompact, locally fair C^∞-scheme then vector bundles on $\underline{X}$ are projective objects in the abelian category $\mathrm{qcoh}(\underline{X})$.

(b) In §B.7 we also define a subcategory $\mathrm{coh}(\underline{X})$ of *coherent sheaves* in $\mathrm{qcoh}(\underline{X})$. But we will not really use them in this book, as they do not have all the good properties we want. In conventional algebraic geometry, one usually restricts to noetherian schemes, where coherent sheaves are well behaved, and form an abelian category. However, as in Remark 1.2.9(iv), even very basic C^∞-schemes $\underline{X}$ such as $\underline{\mathbb{R}}^n$ for $n > 0$ are non-noetherian. Because of this, $\mathrm{coh}(\underline{X})$ is not closed under kernels in $\mathrm{qcoh}(\underline{X})$, and is not an abelian category.

Definition 1.2.16. Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of C^∞-schemes, and let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Define the *pullback* $\underline{f}^*(\mathcal{E})$, an $\mathcal{O}_X$-module, by $\underline{f}^*(\mathcal{E}) = f^{-1}(\mathcal{E}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, where $f^{-1}(\mathcal{E}), f^{-1}(\mathcal{O}_Y)$ are inverse image sheaves. If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism in $\mathcal{O}_Y$-mod we have an induced morphism $\underline{f}^*(\phi) = f^{-1}(\phi) \otimes \mathrm{id}_{\mathcal{O}_X} : \underline{f}^*(\mathcal{E}) \to \underline{f}^*(\mathcal{F})$ in $\mathcal{O}_X$-mod. Then $\underline{f}^* : \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod}$ is a *right exact functor* between abelian categories, which restricts to a right exact functor $\underline{f}^* : \mathrm{qcoh}(\underline{Y}) \to \mathrm{qcoh}(\underline{X})$.

Remark 1.2.17. Pullbacks $\underline{f}^*(\mathcal{E})$ are characterized by a universal property, and so are *unique up to canonical isomorphism*, rather than unique. Our definition of $\underline{f}^*(\mathcal{E})$ is not functorial in $\underline{f}$. That is, if $\underline{f} : \underline{X} \to \underline{Y}$, $\underline{g} : \underline{Y} \to \underline{Z}$ are morphisms and $\mathcal{E} \in \mathcal{O}_Z$-mod then $(\underline{g} \circ \underline{f})^*(\mathcal{E})$ and $\underline{f}^*(\underline{g}^*(\mathcal{E}))$ are canonically isomorphic in $\mathcal{O}_X$-mod, but may not be equal. We will write $I_{\underline{f},\underline{g}}(\mathcal{E}) : (\underline{g} \circ \underline{f})^*(\mathcal{E}) \to \underline{f}^*(\underline{g}^*(\mathcal{E}))$ for these canonical isomorphisms. Then $I_{\underline{f},\underline{g}} : (\underline{g} \circ \underline{f})^* \Rightarrow \underline{f}^* \circ \underline{g}^*$ is a natural isomorphism of functors.

Similarly, when $\underline{f}$ is the identity $\underline{\mathrm{id}}_{\underline{X}} : \underline{X} \to \underline{X}$ and $\mathcal{E} \in \mathcal{O}_X$-mod we may not have $\underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E}) = \mathcal{E}$, but there is a canonical isomorphism $\delta_{\underline{X}}(\mathcal{E}) : \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E}) \to \mathcal{E}$, and $\delta_{\underline{X}} : \underline{\mathrm{id}}_{\underline{X}}^* \Rightarrow \mathrm{id}_{\mathcal{O}_X\text{-mod}}$ is a natural isomorphism of functors.

In fact it is a common abuse of notation in algebraic geometry to omit these isomorphisms $I_{\underline{f},\underline{g}}(\mathcal{E}), \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E})$, and just assume that $(\underline{g} \circ \underline{f})^*(\mathcal{E}) = \underline{f}^*(\underline{g}^*(\mathcal{E}))$ and $\underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E}) = \mathcal{E}$. An author who treats them rigorously is Vistoli [103], see in particular [103, Introduction & §3.2.1]. One reason we decided to include them is to be sure that $\mathbf{dSpa}, \mathbf{dMan}, \ldots$ defined below are strict 2-categories, rather than weak 2-categories or some other structure.

Example 1.2.18. Let X be a manifold, and $\underline{X}$ the associated C^∞-scheme from Example 1.2.5, so that $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. Let $E \to X$ be a vector bundle. Define an $\mathcal{O}_X$-module $\mathcal{E}$ on $\underline{X}$ by $\mathcal{E}(U) = C^\infty(E|_U)$, the smooth sections of the vector bundle $E|_U \to U$, and for open $V \subseteq U \subseteq X$ define $\mathcal{E}_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ by $\mathcal{E}_{UV} : e_U \mapsto e_U|_V$. Then $\mathcal{E} \in \mathrm{vect}(\underline{X})$ is a vector bundle on $\underline{X}$, which we think of as a lift of E from manifolds to C^∞-schemes.

Let $f : X \to Y$ be a smooth map of manifolds, and $\underline{f} : \underline{X} \to \underline{Y}$ the corresponding morphism of C^∞-schemes. Let $F \to Y$ be a vector bundle over Y, so that $f^*(F) \to X$ is a vector bundle over X. Let $\mathcal{F} \in \mathrm{vect}(\underline{Y})$ be the vector bundle over $\underline{Y}$ lifting F. Then $\underline{f}^*(\mathcal{F})$ is canonically isomorphic to the vector bundle over $\underline{X}$ lifting $f^*(F)$.

We define *cotangent sheaves*, the sheaf version of cotangent modules in §1.2.3.

Definition 1.2.19. Let $\underline{X}$ be a C^∞-scheme. Define $\mathcal{P}T^*\underline{X}$ to associate to each open $U \subseteq X$ the cotangent module $\Omega_{\mathcal{O}_X(U)}$, and to each inclusion of open sets $V \subseteq U \subseteq X$ the morphism of $\mathcal{O}_X(U)$-modules $\Omega_{\rho_{UV}} : \Omega_{\mathcal{O}_X(U)} \to \Omega_{\mathcal{O}_X(V)}$ associated to the morphism of C^∞-rings $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. Then $\mathcal{P}T^*\underline{X}$ is a *presheaf of $\mathcal{O}_X$-modules on $\underline{X}$*. Define the *cotangent sheaf $T^*\underline{X}$ of $\underline{X}$* to be the sheafification of $\mathcal{P}T^*\underline{X}$, as an $\mathcal{O}_X$-module.

Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of C^∞-schemes. Then by Definition 1.2.16, $\underline{f}^*\bigl(T^*\underline{Y}\bigr) = f^{-1}(T^*\underline{Y}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, where $T^*\underline{Y}$ is the sheafification of the presheaf $V \mapsto \Omega_{\mathcal{O}_Y(V)}$, and $f^{-1}(T^*\underline{Y})$ the sheafification of the presheaf $U \mapsto \lim_{V \supseteq f(U)}(T^*\underline{Y})(V)$, and $f^{-1}(\mathcal{O}_Y)$ the sheafification of the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathcal{O}_Y(V)$. The three sheafifications combine into one, so that $\underline{f}^*\bigl(T^*\underline{Y}\bigr)$ is the sheafification of the presheaf $\mathcal{P}(\underline{f}^*(T^*\underline{Y}))$ acting by

$$U \longmapsto \mathcal{P}(\underline{f}^*(T^*\underline{Y}))(U) = \lim_{V \supseteq f(U)} \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U).$$

Define a morphism of presheaves $\mathcal{P}\Omega_{\underline{f}} : \mathcal{P}(\underline{f}^*(T^*\underline{Y})) \to \mathcal{P}T^*\underline{X}$ on X by

$$(\mathcal{P}\Omega_{\underline{f}})(U) = \lim_{V \supseteq f(U)}(\Omega_{\rho_{f^{-1}(V)\,U} \circ f_\sharp(V)})_*,$$

where $(\Omega_{\rho_{f^{-1}(V)\,U} \circ f_\sharp(V)})_* : \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) \to \Omega_{\mathcal{O}_X(U)} = (\mathcal{P}T^*\underline{X})(U)$ is constructed as in Definition 1.2.12 from the C^∞-ring morphisms $f_\sharp(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ from $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ corresponding to $f^\sharp$ in $\underline{f}$ as in (1.3), and $\rho_{f^{-1}(V)\,U} : \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_X(U)$ in $\mathcal{O}_X$. Define $\Omega_{\underline{f}} : \underline{f}^*\bigl(T^*\underline{Y}\bigr) \to T^*\underline{X}$ to be the induced morphism of the associated sheaves.

Example 1.2.20. Let X be a manifold, and $\underline{X}$ the associated C^∞-scheme. Then $T^*\underline{X}$ is a vector bundle on $\underline{X}$, and is canonically isomorphic to the lift to C^∞-schemes from Example 1.2.18 of the cotangent vector bundle T^*X of X.

Here [56, Th. 6.17] are some properties of cotangent sheaves.

Theorem 1.2.21. (a) *Let $\underline{f} : \underline{X} \to \underline{Y}$ and $\underline{g} : \underline{Y} \to \underline{Z}$ be morphisms of C^∞-schemes. Then*

$$\Omega_{\underline{g}\circ\underline{f}} = \Omega_{\underline{f}} \circ \underline{f}^*(\Omega_{\underline{g}}) \circ I_{\underline{f},\underline{g}}(T^*\underline{Z})$$

as morphisms $(\underline{g}\circ\underline{f})^*(T^*\underline{Z}) \to T^*\underline{X}$. Here $\Omega_{\underline{g}} : \underline{g}^*(T^*\underline{Z}) \to T^*\underline{Y}$ is a morphism in $\mathcal{O}_Y$-mod, *so applying $\underline{f}^*$ gives $\underline{f}^*(\Omega_{\underline{g}}) : \underline{f}^*(\underline{g}^*(T^*\underline{Z})) \to \underline{f}^*(T^*\underline{Y})$ in* $\mathcal{O}_X$-mod, *and $I_{\underline{f},\underline{g}}(T^*\underline{Z}) : (\underline{g}\circ\underline{f})^*(T^*\underline{Z}) \to \underline{f}^*(\underline{g}^*(T^*\underline{Z}))$ is as in Remark* 1.2.17.

(b) *Suppose $\underline{W}, \underline{X}, \underline{Y}, \underline{Z}$ are locally fair C^∞-schemes with a Cartesian square*

$$\begin{tikzcd} \underline{W} \arrow[r, "\underline{f}"] \arrow[d, "\underline{e}"'] & \underline{Y} \arrow[d, "\underline{h}"] \\ \underline{X} \arrow[r, "\underline{g}"] & \underline{Z} \end{tikzcd}$$

in $\mathbf{C^\infty Sch^{lf}}$, *so that* $\underline{W} = \underline{X} \times_{\underline{Z}} \underline{Y}$. *Then the following is exact in* $\mathrm{qcoh}(\underline{W})$:

$$(\underline{g} \circ \underline{e})^*(T^*\underline{Z}) \xrightarrow[{-\underline{f}^*(\Omega_{\underline{h}}) \circ I_{\underline{f},\underline{h}}(T^*\underline{Z})}]{\underline{e}^*(\Omega_{\underline{g}}) \circ I_{\underline{e},\underline{g}}(T^*\underline{Z}) \oplus} \underline{e}^*(T^*\underline{X}) \oplus \underline{f}^*(T^*\underline{Y}) \xrightarrow{\Omega_{\underline{e}} \oplus \Omega_{\underline{f}}} T^*\underline{W} \longrightarrow 0.$$

1.3 The 2-category of d-spaces

We will now define the 2-category of *d-spaces* $\mathbf{dSpa}$, following Chapter 2. D-spaces are 'derived' versions of C^∞-schemes. In §1.4 we will define the 2-category of d-manifolds $\mathbf{dMan}$ as a 2-subcategory of $\mathbf{dSpa}$. For an introduction to 2-categories, see Appendix A.

1.3.1 The definition of d-spaces

Definition 1.3.1. A *d-space* $\boldsymbol{X}$ is a quintuple $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ such that $\underline{X} = (X, \mathcal{O}_X)$ is a separated, second countable, locally fair C^∞-scheme, and $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$ fit into an exact sequence of sheaves on X

$$\mathcal{E}_X \xrightarrow{\quad \jmath_X \quad} \mathcal{O}'_X \xrightarrow{\quad \imath_X \quad} \mathcal{O}_X \longrightarrow 0,$$

satisfying the conditions:

(a) $\mathcal{O}'_X$ is a sheaf of C^∞-rings on X, with $\underline{X}' = (X, \mathcal{O}'_X)$ a C^∞-scheme.

(b) $\imath_X : \mathcal{O}'_X \to \mathcal{O}_X$ is a surjective morphism of sheaves of C^∞-rings on X. Its kernel $\kappa_X : \mathcal{I}_X \to \mathcal{O}'_X$ is a sheaf of ideals $\mathcal{I}_X$ in $\mathcal{O}'_X$, which should be a sheaf of square zero ideals. Here a *square zero ideal* in a commutative $\mathbb{R}$-algebra A is an ideal I with $i \cdot j = 0$ for all $i, j \in I$. Then $\mathcal{I}_X$ is an $\mathcal{O}'_X$-module, but as $\mathcal{I}_X$ consists of square zero ideals and $\imath_X$ is surjective, the $\mathcal{O}'_X$-action factors through an $\mathcal{O}_X$-action. Hence $\mathcal{I}_X$ is an $\mathcal{O}_X$-module, and thus a quasicoherent sheaf on $\underline{X}$, as $\underline{X}$ is locally fair.

(c) $\mathcal{E}_X$ is a quasicoherent sheaf on $\underline{X}$, and $\jmath_X : \mathcal{E}_X \to \mathcal{I}_X$ is a surjective morphism in $\mathrm{qcoh}(\underline{X})$.

As $\underline{X}$ is locally fair, the underlying topological space X is locally homeomorphic to a closed subset of $\mathbb{R}^n$, so it is *locally compact.* But Hausdorff, second countable and locally compact imply paracompact, and thus $\underline{X}$ is *paracompact.*

The sheaf of C^∞-rings $\mathcal{O}'_X$ has a sheaf of cotangent modules $\Omega_{\mathcal{O}'_X}$, which is an $\mathcal{O}'_X$-module with exterior derivative $\mathrm{d} : \mathcal{O}'_X \to \Omega_{\mathcal{O}'_X}$. Define $\mathcal{F}_X = \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}_X$ to be the associated $\mathcal{O}_X$-module, a quasicoherent sheaf on $\underline{X}$, and set $\psi_X = \Omega_{\imath_X} \otimes \mathrm{id} : \mathcal{F}_X \to T^*\underline{X}$, a morphism in $\mathrm{qcoh}(\underline{X})$. Define $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ to be the composition of morphisms of sheaves of abelian groups on X:

$$\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{I}_X \xrightarrow{\mathrm{d}|_{\mathcal{I}_X}} \Omega_{\mathcal{O}'_X} \cong \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}'_X \xrightarrow{\mathrm{id} \otimes \imath_X} \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}_X = \mathcal{F}_X.$$

It turns out that ϕ_X is actually a morphism of $\mathcal{O}_X$-modules, and the following sequence is exact in $\mathrm{qcoh}(\underline{X})$:

$$\mathcal{E}_X \xrightarrow{\phi_X} \mathcal{F}_X \xrightarrow{\psi_X} T^*\underline{X} \longrightarrow 0.$$

The morphism $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ will be called the *virtual cotangent sheaf* of $\boldsymbol{X}$, for reasons we explain in §1.4.3.

Let $\boldsymbol{X}, \boldsymbol{Y}$ be d-spaces. A 1-*morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a triple $\boldsymbol{f} = (\underline{f}, f', f'')$, where $\underline{f} = (f, f^\sharp) : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes, $f' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$ a morphism of sheaves of C^∞-rings on X, and $f'' : \underline{f}^*(\mathcal{E}_Y) \to \mathcal{E}_X$ a morphism in $\mathrm{qcoh}(\underline{X})$, such that the following diagram of sheaves on X commutes:

$$\begin{tikzcd}
f^{-1}(\mathcal{E}_Y) \otimes^{\mathrm{id}}_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{O}_Y) \arrow[r, equal] \arrow[d, "\mathrm{id}\otimes f^\sharp"] & f^{-1}(\mathcal{E}_Y) \arrow[r, "f^{-1}(\jmath_Y)"'] & f^{-1}(\mathcal{O}'_Y) \arrow[r, "f^{-1}(\imath_Y)"'] \arrow[dd, "f'"] & f^{-1}(\mathcal{O}_Y) \arrow[r] \arrow[dd, "f^\sharp"] & 0 \\
\begin{array}{l}\underline{f}^*(\mathcal{E}_Y) = \\ f^{-1}(\mathcal{E}_Y) \otimes^{f^\sharp}_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X\end{array} \arrow[dr, "f''"] & & & & \\
& \mathcal{E}_X \arrow[r, "\jmath_X"] & \mathcal{O}'_X \arrow[r, "\imath_X"] & \mathcal{O}_X \arrow[r] & 0.
\end{tikzcd}$$

Define morphisms $f^2 = \Omega_{f'} \otimes \mathrm{id} : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{F}_X$ and $f^3 = \Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$ in $\mathrm{qcoh}(\underline{X})$. Then the following commutes in $\mathrm{qcoh}(\underline{X})$, with exact rows:

$$\begin{tikzcd}
\underline{f}^*(\mathcal{E}_Y) \arrow[r, "\underline{f}^*(\phi_Y)"'] \arrow[d, "f''"] & \underline{f}^*(\mathcal{F}_Y) \arrow[r, "\underline{f}^*(\psi_Y)"'] \arrow[d, "f^2"] & \underline{f}^*(T^*\underline{Y}) \arrow[r] \arrow[d, "f^3"] & 0 \\
\mathcal{E}_X \arrow[r, "\phi_X"] & \mathcal{F}_X \arrow[r, "\psi_X"] & T^*\underline{X} \arrow[r] & 0.
\end{tikzcd} \tag{1.4}$$

If $\boldsymbol{X}$ is a d-space, the *identity* 1-*morphism* $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ is $\mathbf{id}_{\boldsymbol{X}} = \big(\underline{\mathrm{id}}_{\underline{X}}, \delta_X(\mathcal{O}'_X), \delta_{\underline{X}}(\mathcal{E}_X)\big)$, where $\delta_X(*)$ are the canonical isomorphisms of Remark 1.2.17. Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be d-spaces, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms. Define the *composition of* 1-*morphisms* $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ to be

$$\boldsymbol{g} \circ \boldsymbol{f} = \big(\underline{g} \circ \underline{f}, f' \circ f^{-1}(g') \circ I_{f,g}(\mathcal{O}'_Z), f'' \circ \underline{f}^*(g'') \circ I_{\underline{f},\underline{g}}(\mathcal{E}_Z)\big), \tag{1.5}$$

where $I_{*,*}(*)$ are the canonical isomorphisms of Remark 1.2.17.

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms of d-spaces, where $\boldsymbol{f} = (\underline{f}, f', f'')$ and $\boldsymbol{g} = (\underline{g}, g', g'')$. Suppose $\underline{f} = \underline{g}$. A 2-*morphism* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a morphism $\eta : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{E}_X$ in $\mathrm{qcoh}(\underline{X})$, such that

$$\begin{aligned} g' &= f' + \jmath_X \circ \eta \circ \big(\mathrm{id} \otimes (f^\sharp \circ f^{-1}(\imath_Y))\big) \circ \big(f^{-1}(\mathrm{d})\big) \\ \text{and} \qquad g'' &= f'' + \eta \circ \underline{f}^*(\phi_Y). \end{aligned}$$

Then $g^2 = f^2 + \phi_X \circ \eta$ and $g^3 = f^3$, so (1.4) for $\boldsymbol{f}, \boldsymbol{g}$ combine to give a diagram

$$\begin{tikzcd}
\underline{f}^*(\mathcal{E}_Y) \arrow[r, "\underline{f}^*(\phi_Y)"] \arrow[d, shift right, "f''"'] \arrow[d, shift left, "g''=f''+\eta\circ\underline{f}^*(\phi_Y)"] & \underline{f}^*(\mathcal{F}_Y) \arrow[r, "\underline{f}^*(\psi_Y)"] \arrow[d, shift right, "f^2"'] \arrow[d, shift left, "g^2=f^2+\phi_X\circ\eta"] \arrow[dl, dotted, "\eta"] & \underline{f}^*(T^*\underline{Y}) \arrow[r] \arrow[d, "f^3=g^3"] & 0 \\
\mathcal{E}_X \arrow[r, "\phi_X"] & \mathcal{F}_X \arrow[r, "\psi_X"] & T^*\underline{X} \arrow[r] & 0.
\end{tikzcd} \tag{1.6}$$

That is, η is a homotopy between the morphisms of complexes (1.4) from $\boldsymbol{f}, \boldsymbol{g}$.

If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism, the *identity* 2*-morphism* $\mathrm{id}_{\boldsymbol{f}} : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ is the zero morphism $0 : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{E}_X$. Suppose $\boldsymbol{X}, \boldsymbol{Y}$ are d-spaces, $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Y}$ are 1-morphisms and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$, $\zeta : \boldsymbol{g} \Rightarrow \boldsymbol{h}$ are 2-morphisms. The *vertical composition of* 2*-morphisms* $\zeta \odot \eta : \boldsymbol{f} \Rightarrow \boldsymbol{h}$ as in (A.1) is $\zeta \odot \eta = \zeta + \eta$.

Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be d-spaces, $\boldsymbol{f}, \tilde{\boldsymbol{f}} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}, \tilde{\boldsymbol{g}} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms, and $\eta : \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{f}}$, $\zeta : \boldsymbol{g} \Rightarrow \tilde{\boldsymbol{g}}$ be 2-morphisms. The *horizontal composition of* 2*-morphisms* $\zeta * \eta : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{g}} \circ \tilde{\boldsymbol{f}}$ as in (A.2) is

$$\zeta * \eta = \bigl(\eta \circ \underline{f}^*(g^2) + f'' \circ \underline{f}^*(\zeta) + \eta \circ \underline{f}^*(\phi_Y) \circ \underline{f}^*(\zeta)\bigr) \circ I_{\underline{f},\underline{g}}(\mathcal{F}_Z).$$

This completes the definition of the 2-category of d-spaces **dSpa**.

Regard the category $\mathbf{C^\infty Sch^{lf}_{ssc}}$ of separated, second countable, locally fair C^∞-schemes as a 2-category with only identity 2-morphisms $\mathrm{id}_{\underline{f}}$ for (1-)morphisms $\underline{f} : \underline{X} \to \underline{Y}$. Define a 2-functor $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}} : \mathbf{C^\infty Sch^{lf}_{ssc}} \to \mathbf{dSpa}$ to map $\underline{X}$ to $\boldsymbol{X} = (\underline{X}, \mathcal{O}_X, 0, \mathrm{id}_{\mathcal{O}_X}, 0)$ on objects $\underline{X}$, to map $\underline{f}$ to $\boldsymbol{f} = (\underline{f}, f^\sharp, 0)$ on (1-)morphisms $\underline{f} : \underline{X} \to \underline{Y}$, and to map identity 2-morphisms $\mathrm{id}_{\underline{f}} : \underline{f} \Rightarrow \underline{f}$ to identity 2-morphisms $\mathrm{id}_{\boldsymbol{f}} : \boldsymbol{f} \Rightarrow \boldsymbol{f}$. Define a 2-functor $F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \to \mathbf{dSpa}$ by $F_{\mathbf{Man}}^{\mathbf{dSpa}} = F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}} \circ F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$.

Write $\hat{\mathbf{C}}^{\boldsymbol{\infty}}\mathbf{Sch^{lf}_{ssc}}$ for the full 2-subcategory of objects $\boldsymbol{X}$ in **dSpa** equivalent to $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{X})$ for some $\underline{X}$ in $\mathbf{C^\infty Sch^{lf}_{ssc}}$, and $\hat{\mathbf{M}}\mathbf{an}$ for the full 2-subcategory of objects $\boldsymbol{X}$ in **dSpa** equivalent to $F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$ for some manifold X. When we say that a d-space $\boldsymbol{X}$ *is a C^∞-scheme*, or *is a manifold*, we mean that $\boldsymbol{X} \in \hat{\mathbf{C}}^{\boldsymbol{\infty}}\mathbf{Sch^{lf}_{ssc}}$, or $\boldsymbol{X} \in \hat{\mathbf{M}}\mathbf{an}$, respectively.

In §2.2 we prove:

Theorem 1.3.2. **(a)** *Definition* 1.3.1 *defines a strict* 2*-category* **dSpa**, *in which all* 2*-morphisms are* 2*-isomorphisms.*

(b) *For any* 1*-morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *in* **dSpa** *the* 2*-morphisms* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ *form an abelian group under vertical composition, and in fact a real vector space.*

(c) $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}$ *and* $F_{\mathbf{Man}}^{\mathbf{dSpa}}$ *in Definition* 1.3.1 *are full and faithful strict* 2*-functors. Hence* $\mathbf{C^\infty Sch^{lf}_{ssc}}, \mathbf{Man}$ *and* $\hat{\mathbf{C}}^{\boldsymbol{\infty}}\mathbf{Sch^{lf}_{ssc}}, \hat{\mathbf{M}}\mathbf{an}$ *are equivalent* 2*-categories.*

Remark 1.3.3. **(i)** One should think of a d-space $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ as being a C^∞-scheme $\underline{X}$, which is the 'classical' part of $\boldsymbol{X}$ and lives in a category rather than a 2-category, together with some extra 'derived' information $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$. 2-morphisms in **dSpa** are wholly to do with this derived part. The sheaf $\mathcal{E}_X$ may be thought of as a (dual) 'obstruction sheaf' on $\underline{X}$.

(ii) Readers familiar with derived algebraic geometry may find the following (oversimplified) explanation of d-spaces helpful; more details are given in §14.4.

In conventional algebraic geometry, a $\mathbb{K}$-scheme $(X, \mathcal{O}_X)$ is a topological space X equipped with a sheaf of $\mathbb{K}$-algebras $\mathcal{O}_X$. In derived algebraic geometry, as in Toën and Vezzosi [101,102] and Lurie [70–72], a *derived* $\mathbb{K}$*-scheme* $(X, \mathcal{O}_X)$ is (roughly) a topological space X with a (homotopy) sheaf of (commutative) dg-algebras over $\mathbb{K}$. Here a (*commutative*) *dg-algebra* (A_*, d) is a nonpositively graded $\mathbb{K}$-algebra $\bigoplus_{k\leqslant 0} A_k$, with differentials $\mathrm{d} : A_k \to A_{k+1}$ satisfying $\mathrm{d}^2 = 0$ and $ab = (-1)^{kl}ba$, $\mathrm{d}(ab) = (\mathrm{d}a)b + (-1)^k a(\mathrm{d}b)$ for all $a \in A_k$ and $b \in A_l$.

We call a dg-algebra (A_*, d) *square zero* if $A_k = 0$ for $k \neq 0, -1$ and $A_{-1} \cdot \mathrm{d}(A_{-1}) = 0$. This implies that $\mathrm{d}(A_{-1})$ is a square zero ideal in A_0. General dg-algebras form an ∞-category, but square zero dg-algebras form a 2-category. Ignoring C^∞-rings for the moment, we can think of the data $\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{O}'_X$ in a d-space $\boldsymbol{X}$ as a sheaf of square zero dg-algebras $A_{-1} \xrightarrow{\mathrm{d}} A_0$ on X. The remaining data $\mathcal{O}_X, \imath_X$ can be recovered from this, since $\mathcal{O}'_X \xrightarrow{\imath_X} \mathcal{O}_X$ is the cokernel of $\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{O}'_X$. Thus, a d-space $\boldsymbol{X}$ is like a special kind of derived $\mathbb{R}$-scheme, in which the dg-algebras are all square zero.

1.3.2 Gluing d-spaces by equivalences

Next we discuss gluing of d-spaces and 1-morphisms on open d-subspaces.

Definition 1.3.4. Let $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ be a d-space. Suppose $\underline{U} \subseteq \underline{X}$ is an open C^∞-subscheme. Then $\boldsymbol{U} = \bigl(\underline{U}, \mathcal{O}'_X|_U, \mathcal{E}_X|_U, \imath_X|_U, \jmath_X|_U\bigr)$ is a d-space. We call $\boldsymbol{U}$ an *open d-subspace* of $\boldsymbol{X}$. An *open cover* of a d-space $\boldsymbol{X}$ is a family $\{\boldsymbol{U}_a : a \in A\}$ of open d-subspaces $\boldsymbol{U}_a$ of $\boldsymbol{X}$ with $\underline{X} = \bigcup_{a\in A} \underline{U}_a$.

As in §2.4, we can glue 1-morphisms on open d-subspaces which are 2-isomorphic on the overlap. The proof uses partitions of unity, as in §1.2.2.

Proposition 1.3.5. *Let $\boldsymbol{X}, \boldsymbol{Y}$ be d-spaces, $\boldsymbol{U}, \boldsymbol{V} \subseteq \boldsymbol{X}$ be open d-subspaces with $\boldsymbol{X} = \boldsymbol{U} \cup \boldsymbol{V}$, $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{V} \to \boldsymbol{Y}$ be 1-morphisms, and $\eta : \boldsymbol{f}|_{\boldsymbol{U}\cap\boldsymbol{V}} \Rightarrow \boldsymbol{g}|_{\boldsymbol{U}\cap\boldsymbol{V}}$ a 2-morphism. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Y}$ and 2-morphisms $\zeta : \boldsymbol{h}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{f}$, $\theta : \boldsymbol{h}|_{\boldsymbol{V}} \Rightarrow \boldsymbol{g}$ such that $\theta|_{\boldsymbol{U}\cap\boldsymbol{V}} = \eta \odot \zeta|_{\boldsymbol{U}\cap\boldsymbol{V}} : \boldsymbol{h}|_{\boldsymbol{U}\cap\boldsymbol{V}} \Rightarrow \boldsymbol{g}|_{\boldsymbol{U}\cap\boldsymbol{V}}$. This $\boldsymbol{h}$ is unique up to 2-isomorphism, and independent up to 2-isomorphism of the choice of η.*

Equivalences $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in a 2-category are defined in §A.3, and are the natural notion of when two objects $\boldsymbol{X}, \boldsymbol{Y}$ are 'the same'. In §2.4 we prove theorems on gluing d-spaces by equivalences. See Spivak [95, Lem. 6.8 & Prop. 6.9] for results similar to Theorem 1.3.6 for his 'local C^∞-ringed spaces', an ∞-categorical analogue of our d-spaces.

Theorem 1.3.6. *Suppose $\boldsymbol{X}, \boldsymbol{Y}$ are d-spaces, $\boldsymbol{U} \subseteq \boldsymbol{X}$, $\boldsymbol{V} \subseteq \boldsymbol{Y}$ are open d-subspaces, and $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{V}$ is an equivalence in* **dSpa**. *At the level of topological spaces, we have open $U \subseteq X$, $V \subseteq Y$ with a homeomorphism $f : U \to V$, so we can form the quotient topological space $Z := X \amalg_f Y = (X \amalg Y)/\sim$, where the equivalence relation $\sim$ on $X \amalg Y$ identifies $u \in U \subseteq X$ with $f(u) \in V \subseteq Y$.*

Suppose Z is Hausdorff. Then there exist a d-space $\boldsymbol{Z}$ with topological space Z, open d-subspaces $\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}$ in $\boldsymbol{Z}$ with $\boldsymbol{Z} = \hat{\boldsymbol{X}} \cup \hat{\boldsymbol{Y}}$, equivalences $\boldsymbol{g} : \boldsymbol{X} \to \hat{\boldsymbol{X}}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \hat{\boldsymbol{Y}}$ in **dSpa** *such that $\boldsymbol{g}|_{\boldsymbol{U}}$ and $\boldsymbol{h}|_{\boldsymbol{V}}$ are both equivalences with $\hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$, and a 2-morphism $\eta : \boldsymbol{g}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f} : \boldsymbol{U} \to \hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$. Furthermore, $\boldsymbol{Z}$ is independent of choices up to equivalence.*

In Theorem 1.3.6, $\boldsymbol{Z}$ is a *pushout* $\boldsymbol{X} \amalg_{\mathbf{id}_{\boldsymbol{U}},\boldsymbol{U},\boldsymbol{f}} \boldsymbol{Y}$ in the 2-category **dSpa**.

Theorem 1.3.7. *Suppose I is an indexing set, and $<$ is a total order on I, and $\boldsymbol{X}_i$ for $i \in I$ are d-spaces, and for all $i < j$ in I we are given open d-subspaces $\boldsymbol{U}_{ij} \subseteq \boldsymbol{X}_i$, $\boldsymbol{U}_{ji} \subseteq \boldsymbol{X}_j$ and an equivalence $\boldsymbol{e}_{ij} : \boldsymbol{U}_{ij} \to \boldsymbol{U}_{ji}$, such that for all $i < j < k$ in I we have a 2-commutative diagram*

$$\begin{tikzcd}[column sep=small] & \boldsymbol{U}_{ji} \cap \boldsymbol{U}_{jk} \arrow[dr, "\boldsymbol{e}_{jk}|_{\boldsymbol{U}_{ji}\cap \boldsymbol{U}_{jk}}"] \arrow[d, phantom, "\Downarrow \eta_{ijk}"] & \\ \boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} \arrow[ur, "\boldsymbol{e}_{ij}|_{\boldsymbol{U}_{ij}\cap \boldsymbol{U}_{ik}}"] \arrow[rr, "\boldsymbol{e}_{ik}|_{\boldsymbol{U}_{ij}\cap \boldsymbol{U}_{ik}}"] & & \boldsymbol{U}_{ki} \cap \boldsymbol{U}_{kj} \end{tikzcd}$$

for some η_{ijk}, where all three 1-morphisms are equivalences.

On the level of topological spaces, define the quotient topological space $Y = (\coprod_{i \in I} X_i)/\sim$, where $\sim$ is the equivalence relation generated by $x_i \sim x_j$ if $i < j$, $x_i \in U_{ij} \subseteq X_i$ and $x_j \in U_{ji} \subseteq X_j$ with $e_{ij}(x_i) = x_j$. Suppose Y is Hausdorff and second countable. Then there exist a d-space $\boldsymbol{Y}$ and a 1-morphism $\boldsymbol{f}_i : \boldsymbol{X}_i \to \boldsymbol{Y}$ which is an equivalence with an open d-subspace $\hat{\boldsymbol{X}}_i \subseteq \boldsymbol{Y}$ for all $i \in I$, where $\boldsymbol{Y} = \bigcup_{i \in I} \hat{\boldsymbol{X}}_i$, such that $\boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$ is an equivalence $\boldsymbol{U}_{ij} \to \hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j$ for all $i < j$ in I, and there exists a 2-morphism $\eta_{ij} : \boldsymbol{f}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$. The d-space $\boldsymbol{Y}$ is unique up to equivalence, and is independent of choice of 2-morphisms η_{ijk}.

Suppose also that $\boldsymbol{Z}$ is a d-space, and $\boldsymbol{g}_i : \boldsymbol{X}_i \to \boldsymbol{Z}$ are 1-morphisms for all $i \in I$, and there exist 2-morphisms $\zeta_{ij} : \boldsymbol{g}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{g}_i|_{\boldsymbol{U}_{ij}}$ for all $i < j$ in I. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ and 2-morphisms $\zeta_i : \boldsymbol{h} \circ \boldsymbol{f}_i \Rightarrow \boldsymbol{g}_i$ for all $i \in I$. The 1-morphism $\boldsymbol{h}$ is unique up to 2-isomorphism, and is independent of the choice of 2-morphisms ζ_{ij}.

Remark 1.3.8. In Proposition 1.3.5, it is surprising that $\boldsymbol{h}$ is independent of η up to 2-isomorphism. It holds because of the existence of partitions of unity on nice C^∞-schemes, as in Proposition 1.2.8. Here is a sketch proof: suppose $\eta, \boldsymbol{h}, \zeta, \theta$ and $\eta', \boldsymbol{h}', \zeta', \theta'$ are alternative choices in Proposition 1.3.5. Then we have 2-morphisms $(\zeta')^{-1} \odot \zeta : \boldsymbol{h}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{h}'|_{\boldsymbol{U}}$ and $(\theta')^{-1} \odot \theta : \boldsymbol{h}|_{\boldsymbol{V}} \Rightarrow \boldsymbol{h}'|_{\boldsymbol{V}}$. Choose a partition of unity $\{\alpha, 1-\alpha\}$ on $\underline{X}$ subordinate to $\{\underline{U}, \underline{V}\}$, so that $\alpha : \underline{X} \to \mathbb{R}$ is smooth with α supported on $\underline{U} \subseteq \underline{X}$ and $1 - \alpha$ supported on $\underline{V} \subseteq \underline{X}$. Then $\alpha \cdot \bigl((\zeta')^{-1} \odot \zeta\bigr) + (1-\alpha) \cdot \bigl((\theta')^{-1} \odot \theta\bigr)$ is a 2-morphism $\boldsymbol{h} \Rightarrow \boldsymbol{h}'$, where $\alpha \cdot \bigl((\zeta')^{-1} \odot \zeta\bigr)$ makes sense on all of $\underline{X}$ (rather than just on $\underline{U}$ where $(\zeta')^{-1} \odot \zeta$ is defined) as α is supported on $\underline{U}$, so we extend by zero on $\underline{X} \setminus \underline{U}$.

Similarly, in Theorem 1.3.7, the compatibility conditions on the gluing data $\boldsymbol{X}_i, \boldsymbol{U}_{ij}, \boldsymbol{e}_{ij}$ are significantly weaker than you might expect, because of the existence of partitions of unity. The 2-morphisms η_{ijk} on overlaps $\hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j \cap \hat{\boldsymbol{X}}_k$ are only required to exist, not to satisfy any further conditions. In particular, one might think that on overlaps $\hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j \cap \hat{\boldsymbol{X}}_k \cap \hat{\boldsymbol{X}}_l$ we should require

$$\eta_{ikl} \odot (\mathrm{id}_{\boldsymbol{f}_{kl}} * \eta_{ijk})|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} \cap \boldsymbol{U}_{il}} = \eta_{ijl} \odot (\eta_{jkl} * \mathrm{id}_{\boldsymbol{f}_{ij}})|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} \cap \boldsymbol{U}_{il}}, \tag{1.7}$$

but we do not. Also, one might expect the ζ_{ij} should satisfy conditions on triple overlaps $\hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j \cap \hat{\boldsymbol{X}}_k$, but they need not.

The moral is that constructing d-spaces by gluing together patches $\boldsymbol{X}_i$ is straightforward, as one only has to verify mild conditions on triple overlaps

$\boldsymbol{X}_i\cap\boldsymbol{X}_j\cap\boldsymbol{X}_k$. Again, this works because of the existence of partitions of unity on nice C^∞-schemes, which are used to construct the glued d-spaces $\boldsymbol{Z}$ and 1- and 2-morphisms in Theorems 1.3.6 and 1.3.7.

In contrast, for gluing d-stacks in §1.10.3, we do need compatibility conditions of the form (1.7). The problem of gluing geometric spaces in an ∞-category $\boldsymbol{\mathcal{C}}$ by equivalences, such as Spivak's derived manifolds [94, 95], is discussed by Toën and Vezzosi [101, §1.3.4] and Lurie [70, §6.1.2]. It requires nontrivial conditions on overlaps $\boldsymbol{\mathcal{X}}_{i_1}\cap\cdots\cap\boldsymbol{\mathcal{X}}_{i_n}$ for all $n=2,3,\ldots$.

1.3.3 Fibre products in dSpa

Fibre products in 2-categories are explained in §A.4. In §2.5–§2.6 we discuss fibre products in **dSpa**, and their relation to transverse fibre products in **Man**.

Theorem 1.3.9. (a) *All fibre products exist in the* 2*-category* **dSpa**.

(b) *Let $g:X\rightarrow Z$ and $h:Y\rightarrow Z$ be smooth maps of manifolds, and write $\boldsymbol{X}=F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$, and similarly for $\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{g},\boldsymbol{h}$. If g,h are transverse, so that a fibre product $X\times_{g,Z,h}Y$ exists in* **Man**, *then the fibre product $\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in* **dSpa** *is equivalent in* **dSpa** *to $F_{\mathbf{Man}}^{\mathbf{dSpa}}(X\times_{g,Z,h}Y)$. If g,h are not transverse then $\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in* **dSpa**, *but is not a manifold.*

To prove (a), given 1-morphisms $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$, we write down an explicit d-space $\boldsymbol{W}=(\underline{W},\mathcal{O}_W',\mathcal{E}_W,\imath_W,\jmath_W)$, 1-morphisms $\boldsymbol{e}=(\underline{e},e',e'')$: $\boldsymbol{W}\rightarrow\boldsymbol{X}$ and $\boldsymbol{f}=(\underline{f},f',f''):\boldsymbol{W}\rightarrow\boldsymbol{Y}$ and a 2-morphism $\eta:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$, and verify the universal property for

$$\begin{tikzcd}
\boldsymbol{W} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"'] \arrow[dr, phantom, "\eta\Uparrow"] & \boldsymbol{Y} \arrow[d, "\boldsymbol{h}"] \\
\boldsymbol{X} \arrow[r, "\boldsymbol{g}"] & \boldsymbol{Z}
\end{tikzcd}$$

to be a 2-Cartesian square in **dSpa**. The underlying C^∞-scheme $\underline{W}$ is the fibre product $\underline{W}=\underline{X}\times_{\underline{g},\underline{Z},\underline{h}}\underline{Y}$ in $\mathbf{C^\infty Sch}$, and $\underline{e}:\underline{W}\rightarrow\underline{X}$, $\underline{f}:\underline{W}\rightarrow\underline{Y}$ are the projections from the fibre product. The definitions of $\mathcal{O}_W',\imath_W,\jmath_W,e',f'$ in §2.5 are complex, and we will not give them here. The remaining data $\mathcal{E}_W,e'',f'',\eta$, as well as the virtual cotangent sheaf $\phi_W:\mathcal{E}_W\rightarrow\mathcal{F}_W$, is characterized by the following commutative diagram in qcoh($\underline{W}$), with exact top row:

$$\begin{tikzcd}[column sep=large]
(\underline{g}\circ\underline{e})^*(\mathcal{E}_Z) \arrow[r, "{\left(\begin{smallmatrix} \underline{e}^*(g'')\circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z) \\ -\underline{f}^*(h'')\circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z) \\ (\underline{g}\circ\underline{e})^*(\phi_Z) \end{smallmatrix}\right)}"] & {\begin{array}{l}\underline{e}^*(\mathcal{E}_X)\oplus \\ \underline{f}^*(\mathcal{E}_Y)\oplus \\ (\underline{g}\circ\underline{e})^*(\mathcal{F}_Z)\end{array}} \arrow[r, dotted, "{\left(\begin{smallmatrix} e'' & f'' & \eta \end{smallmatrix}\right)}"] \arrow[d, "{\left(\begin{smallmatrix} -\underline{e}^*(\phi_X) & 0 & \underline{e}^*(g^2)\circ I_{\underline{e},\underline{g}}(\mathcal{F}_Z) \\ 0 & -\underline{f}^*(\phi_Y) & -\underline{f}^*(h^2)\circ I_{\underline{f},\underline{h}}(\mathcal{F}_Z) \end{smallmatrix}\right)}"'] & \mathcal{E}_W \arrow[r] \arrow[d, dotted, "\phi_W"] & 0 \\
& {\begin{array}{l}\underline{e}^*(\mathcal{F}_X)\oplus \\ \underline{f}^*(\mathcal{F}_Y)\end{array}} \arrow[r, dotted, "{\left(\begin{smallmatrix} e^2 & f^2 \end{smallmatrix}\right)}", "\cong"'] & \mathcal{F}_W.
\end{tikzcd}$$

1.3.4 Fixed point loci of finite groups in d-spaces

If a finite group Γ acts on a manifold X by diffeomorphisms, then the fixed point locus X^Γ is a disjoint union of closed, embedded submanifolds of X. In a similar way, if Γ acts on a d-space $\boldsymbol{X}$ by 1-isomorphisms, in §2.7 we define a d-space $\boldsymbol{X}^\Gamma$ called the *fixed d-subspace of* Γ in $\boldsymbol{X}$, with an inclusion 1-morphism $\boldsymbol{j}_{\boldsymbol{X},\Gamma} : \boldsymbol{X}^\Gamma \hookrightarrow \boldsymbol{X}$, whose topological space X^Γ is the fixed point locus of Γ in X.

Note that by an *action* $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\boldsymbol{X})$ *of* Γ *on* $\boldsymbol{X}$ we shall always mean a *strict* action, that is, $\boldsymbol{r}(\gamma) : \boldsymbol{X} \to \boldsymbol{X}$ is a 1-isomorphism for all $\gamma \in \Gamma$ and $\boldsymbol{r}(\gamma\delta) = \boldsymbol{r}(\gamma)\boldsymbol{r}(\delta)$ for all $\gamma, \delta \in \Gamma$, rather than $\boldsymbol{r}(\gamma\delta)$ only being 2-isomorphic to $\boldsymbol{r}(\gamma)\boldsymbol{r}(\delta)$. The next theorem summarizes our results.

Theorem 1.3.10. *Let $\boldsymbol{X}$ be a d-space, Γ a finite group, and $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\boldsymbol{X})$ an action of Γ on $\boldsymbol{X}$ by 1-isomorphisms. Then we can define a d-space $\boldsymbol{X}^\Gamma$ called the **fixed d-subspace of** Γ **in** $\boldsymbol{X}$, with an inclusion 1-morphism $\boldsymbol{j}_{\boldsymbol{X},\Gamma} : \boldsymbol{X}^\Gamma \to \boldsymbol{X}$. It has the following properties:*

- **(a)** *Let $\boldsymbol{X}, \Gamma, \boldsymbol{r}$ and $\boldsymbol{j}_{\boldsymbol{X},\Gamma} : \boldsymbol{X}^\Gamma \to \boldsymbol{X}$ be as above. Suppose $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{X}$ is a 1-morphism in* **dSpa**. *Then $\boldsymbol{f}$ factorizes as $\boldsymbol{f} = \boldsymbol{j}_{\boldsymbol{X},\Gamma} \circ \boldsymbol{g}$ for some 1-morphism $\boldsymbol{g} : \boldsymbol{W} \to \boldsymbol{X}^\Gamma$ in* **dSpa**, *which must be unique, if and only if $\boldsymbol{r}(\gamma) \circ \boldsymbol{f} = \boldsymbol{f}$ for all $\gamma \in \Gamma$.*
- **(b)** *Suppose $\boldsymbol{X}, \boldsymbol{Y}$ are d-spaces, Γ is a finite group, $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\boldsymbol{X})$, $\boldsymbol{s} : \Gamma \to \mathrm{Aut}(\boldsymbol{Y})$ are actions of Γ on $\boldsymbol{X}, \boldsymbol{Y}$, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a Γ-equivariant 1-morphism in* **dSpa**, *that is, $\boldsymbol{f} \circ \boldsymbol{r}(\gamma) = \boldsymbol{s}(\gamma) \circ \boldsymbol{f}$ for $\gamma \in \Gamma$. Then there exists a unique 1-morphism $\boldsymbol{f}^\Gamma : \boldsymbol{X}^\Gamma \to \boldsymbol{Y}^\Gamma$ such that $\boldsymbol{j}_{\boldsymbol{Y},\Gamma} \circ \boldsymbol{f}^\Gamma = \boldsymbol{f} \circ \boldsymbol{j}_{\boldsymbol{X},\Gamma}$.*
- **(c)** *Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be Γ-equivariant 1-morphisms as in* **(b)**, *and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ be a Γ-equivariant 2-morphism, that is, $\eta * \mathrm{id}_{\boldsymbol{r}(\gamma)} = \mathrm{id}_{\boldsymbol{s}(\gamma)} * \eta$ for $\gamma \in \Gamma$. Then there exists a unique 2-morphism $\eta^\Gamma : \boldsymbol{f}^\Gamma \Rightarrow \boldsymbol{g}^\Gamma$ such that $\mathrm{id}_{\boldsymbol{j}_{\boldsymbol{Y},\Gamma}} * \eta^\Gamma = \eta * \mathrm{id}_{\boldsymbol{j}_{\boldsymbol{X},\Gamma}}$.*

Note that **(a)** *is a universal property that determines $\boldsymbol{X}^\Gamma, \boldsymbol{j}_{\boldsymbol{X},\Gamma}$ up to canonical 1-isomorphism.*

We will use fixed d-subspaces $\boldsymbol{X}^\Gamma$ in Theorem 1.10.14 below to describe orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma$ of quotient d-stacks $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$. If $\boldsymbol{X}$ is a d-manifold, as in §1.4, then in general the fixed d-subspaces $\boldsymbol{X}^\Gamma$ are disjoint unions of d-manifolds of different dimensions.

1.4 The 2-category of d-manifolds

We now survey Chapters 3–4 on d-manifolds (without boundary).

1.4.1 The definition of d-manifolds

Definition 1.4.1. A d-space $\boldsymbol{U}$ is called a *principal d-manifold* if is equivalent in **dSpa** to a fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ with $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \hat{\mathbf{Man}}$. That is,

$$\boldsymbol{U} \simeq F_{\mathbf{Man}}^{\mathbf{dSpa}}(X) \times_{F_{\mathbf{Man}}^{\mathbf{dSpa}}(g), F_{\mathbf{Man}}^{\mathbf{dSpa}}(Z), F_{\mathbf{Man}}^{\mathbf{dSpa}}(h)} F_{\mathbf{Man}}^{\mathbf{dSpa}}(Y)$$

for manifolds X, Y, Z and smooth maps $g : X \to Z$ and $h : Y \to Z$. The *virtual dimension* $\operatorname{vdim} \boldsymbol{U}$ of $\boldsymbol{U}$ is defined to be $\operatorname{vdim} \boldsymbol{U} = \dim X + \dim Y - \dim Z$. Proposition 1.4.11(b) below shows that if $\boldsymbol{U} \neq \boldsymbol{\emptyset}$ then $\operatorname{vdim} \boldsymbol{U}$ depends only on the d-space $\boldsymbol{U}$, and not on the choice of X, Y, Z, g, h, and so is well defined.

A d-space $\boldsymbol{W}$ is called a *d-manifold of virtual dimension* $n \in \mathbb{Z}$, written $\operatorname{vdim} \boldsymbol{W} = n$, if $\boldsymbol{W}$ can be covered by nonempty open d-subspaces $\boldsymbol{U}$ which are principal d-manifolds with $\operatorname{vdim} \boldsymbol{U} = n$.

Write **dMan** for the full 2-subcategory of d-manifolds in **dSpa**. If $\boldsymbol{X} \in \hat{\mathbf{Man}}$ then $\boldsymbol{X} \simeq \boldsymbol{X} \times_{\boldsymbol{*}} \boldsymbol{*}$, so $\boldsymbol{X}$ is a principal d-manifold, and thus a d-manifold. Therefore $\hat{\mathbf{Man}}$ in §1.3.1 is a 2-subcategory of **dMan**. We say that a d-manifold $\boldsymbol{X}$ *is a manifold* if it lies in $\hat{\mathbf{Man}}$. The 2-functor $F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \to \mathbf{dSpa}$ maps into **dMan**, and we will write $F_{\mathbf{Man}}^{\mathbf{dMan}} = F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \to \mathbf{dMan}$.

Here, as in §3.2, are alternative descriptions of principal d-manifolds:

Proposition 1.4.2. *The following are equivalent characterizations of when a d-space $\boldsymbol{W}$ is a principal d-manifold:*

(a) $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *for* $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \hat{\mathbf{Man}}$.

(b) $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{i},\boldsymbol{Z},\boldsymbol{j}} \boldsymbol{Y}$, *where X, Y, Z are manifolds, $i : X \to Z$, $j : Y \to Z$ are embeddings, $\boldsymbol{X} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$, and similarly for Y, Z, i, j. That is, $\boldsymbol{W}$ is an intersection of two submanifolds X, Y in Z, in the sense of d-spaces.*

(c) $\boldsymbol{W} \simeq \boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$, *where V is a manifold, $E \to V$ is a vector bundle, $s : V \to E$ is a smooth section, $0 : V \to E$ is the zero section, $\boldsymbol{V} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(V)$, and similarly for $E, s, 0$. That is, $\boldsymbol{W}$ is the zeroes $s^{-1}(0)$ of a smooth section s of a vector bundle E, in the sense of d-spaces.*

Example 1.4.3. Let $X \subseteq \mathbb{R}^n$ be any closed subset. By a lemma of Whitney's, we can write X as the zero set of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$. Then $\boldsymbol{X} = \boldsymbol{\mathbb{R}^n} \times_{\boldsymbol{f},\boldsymbol{\mathbb{R}},\boldsymbol{0}} \boldsymbol{*}$ is a principal d-manifold, with topological space X.

This example shows that the topological spaces X underlying d-manifolds $\boldsymbol{X}$ can be fairly wild, for example, X could be a fractal such as the Cantor set.

1.4.2 'Standard model' d-manifolds, 1- and 2-morphisms

The next three examples, taken from §3.2 and §3.4, give explicit models for principal d-manifolds in the form $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$ from Proposition 1.4.2(c) and their 1- and 2-morphisms, which we call *standard models*.

Example 1.4.4. Let V be a manifold, $E \to V$ a vector bundle (which we sometimes call the *obstruction bundle*), and $s \in C^\infty(E)$. We will write down an explicit principal d-manifold $\boldsymbol{S} = (\underline{S}, \mathcal{O}_S', \mathcal{E}_S, \imath_S, \jmath_S)$ which is equivalent to $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$ in Proposition 1.4.2(c). We call $\boldsymbol{S}$ the *standard model* of (V, E, s), and also write it $\boldsymbol{S}_{V,E,s}$. Proposition 1.4.2 shows that every principal d-manifold $\boldsymbol{W}$ is equivalent to $\boldsymbol{S}_{V,E,s}$ for some V, E, s.

Write $C^\infty(V)$ for the C^∞-ring of smooth functions $c : V \to \mathbb{R}$, and $C^\infty(E)$, $C^\infty(E^*)$ for the vector spaces of smooth sections of E, E^* over V. Then s lies in $C^\infty(E)$, and $C^\infty(E), C^\infty(E^*)$ are modules over $C^\infty(V)$, and there is a natural bilinear product $\cdot : C^\infty(E^*) \times C^\infty(E) \to C^\infty(V)$. Define $I_s \subseteq C^\infty(V)$ to be the ideal generated by s. That is,

$$I_s = \bigl\{\alpha \cdot s : \alpha \in C^\infty(E^*)\bigr\} \subseteq C^\infty(V). \tag{1.8}$$

Let $I_s^2 = \langle fg : f, g \in I_s\rangle_{\mathbb{R}}$ be the square of I_s. Then I_s^2 is an ideal in $C^\infty(V)$, the ideal generated by $s \otimes s \in C^\infty(E \otimes E)$. That is,

$$I_s^2 = \bigl\{\beta \cdot (s \otimes s) : \beta \in C^\infty(E^* \otimes E^*)\bigr\} \subseteq C^\infty(V).$$

Define C^∞-rings $\mathfrak{C} = C^\infty(V)/I_s$, $\mathfrak{C}' = C^\infty(V)/I_s^2$, and let $\pi : \mathfrak{C}' \to \mathfrak{C}$ be the natural projection from the inclusion $I_s^2 \subseteq I_s$. Define a topological space $S = \{v \in V : s(v) = 0\}$, as a subspace of V. Now $s(v) = 0$ if and only if $(s \otimes s)(v) = 0$. Thus S is the underlying topological space for both $\operatorname{Spec}\mathfrak{C}$ and $\operatorname{Spec}\mathfrak{C}'$. So $\operatorname{Spec}\mathfrak{C} = \underline{S} = (S, \mathcal{O}_S)$, $\operatorname{Spec}\mathfrak{C}' = \underline{S}' = (S, \mathcal{O}_S')$, and $\operatorname{Spec}\pi = \underline{\imath}_S = (\mathrm{id}_S, \imath_S) : \underline{S}' \to \underline{S}$, where $\underline{S}, \underline{S}'$ are fair affine C^∞-schemes, and $\mathcal{O}_S, \mathcal{O}_S'$ are sheaves of C^∞-rings on S, and $\imath_S : \mathcal{O}_S' \to \mathcal{O}_S$ is a morphism of sheaves of C^∞-rings. Since π is surjective with kernel the square zero ideal I_s/I_s^2, $\imath_S$ is surjective, with kernel $\mathcal{I}_S$ a sheaf of square zero ideals in $\mathcal{O}_S'$.

From (1.8) we have a surjective $C^\infty(V)$-module morphism $C^\infty(E^*) \to I_s$ mapping $\alpha \mapsto \alpha \cdot s$. Applying $\otimes_{C^\infty(V)}\mathfrak{C}$ gives a surjective $\mathfrak{C}$-module morphism

$$\sigma : C^\infty(E^*)/(I_s \cdot C^\infty(E^*)) \longrightarrow I_s/I_s^2, \quad \sigma : \alpha + (I_s \cdot C^\infty(E^*)) \longmapsto \alpha \cdot s + I_s^2.$$

Define $\mathcal{E}_S = \operatorname{MSpec}\bigl(C^\infty(E^*)/(I_s \cdot C^\infty(E^*))\bigr)$. Also $\operatorname{MSpec}(I_s/I_s^2) = \mathcal{I}_S$, so $\jmath_S = \operatorname{MSpec}\sigma$ is a surjective morphism $\jmath_S : \mathcal{E}_S \to \mathcal{I}_S$ in $\operatorname{qcoh}(\underline{S})$. Therefore $\boldsymbol{S}_{V,E,s} = \boldsymbol{S} = (\underline{S}, \mathcal{O}_S', \mathcal{E}_S, \imath_S, \jmath_S)$ is a d-space.

In fact $\mathcal{E}_S$ is a vector bundle on $\underline{S}$ naturally isomorphic to $\mathcal{E}^*|_{\underline{S}}$, where $\mathcal{E}$ is the vector bundle on $\underline{V} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(V)$ corresponding to $E \to V$. Also $\mathcal{F}_S \cong T^*\underline{V}|_{\underline{S}}$. The morphism $\phi_S : \mathcal{E}_S \to \mathcal{F}_S$ can be interpreted as follows: choose a connection ∇ on $E \to V$. Then $\nabla s \in C^\infty(E \otimes T^*V)$, so we can regard ∇s as a morphism of vector bundles $E^* \to T^*V$ on V. This lifts to a morphism of vector bundles $\hat\nabla s : \mathcal{E}^* \to T^*\underline{V}$ on the C^∞-scheme $\underline{V}$, and ϕ_S is identified with $\hat\nabla s|_{\underline{S}} : \mathcal{E}^*|_{\underline{S}} \to T^*\underline{V}|_{\underline{S}}$ under the isomorphisms $\mathcal{E}_S \cong \mathcal{E}^*|_{\underline{S}}$, $\mathcal{F}_S \cong T^*\underline{V}|_{\underline{S}}$.

Proposition 1.4.2 implies that every principal d-manifold $\boldsymbol{W}$ is equivalent to $\boldsymbol{S}_{V,E,s}$ for some V, E, s. The notation $O(s)$ and $O(s^2)$ used below should be interpreted as follows. Let V be a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$. If $F \to V$ is another vector bundle and $t \in C^\infty(F)$, then we write $t = O(s)$ if $t = \alpha \cdot s$ for some $\alpha \in C^\infty(F \otimes E^*)$, and $t = O(s^2)$ if $t = \beta \cdot (s \otimes s)$ for some $\beta \in C^\infty(F \otimes E^* \otimes E^*)$. Similarly, if W is a manifold and $f, g : V \to W$ are smooth then we write $f = g + O(s)$ if $c \circ f - c \circ g = O(s)$ for all smooth $c : W \to \mathbb{R}$, and $f = g + O(s^2)$ if $c \circ f - c \circ g = O(s^2)$ for all c.

Example 1.4.5. Let V,W be manifolds, $E\to V$, $F\to W$ be vector bundles, and $s\in C^\infty(E)$, $t\in C^\infty(F)$. Write $\boldsymbol{X}=\boldsymbol{S}_{V,E,s}$, $\boldsymbol{Y}=\boldsymbol{S}_{W,F,t}$ for the 'standard model' principal d-manifolds from Example 1.4.4. Suppose $f:V\to W$ is a smooth map, and $\hat f:E\to f^*(F)$ is a morphism of vector bundles on V satisfying

$$\hat f\circ s=f^*(t)+O(s^2)\quad\text{in }C^\infty\big(f^*(F)\big).\tag{1.9}$$

We will define a 1-morphism $\boldsymbol{g}=(\underline{g},g',g''):\boldsymbol{X}\to\boldsymbol{Y}$ in **dMan** using $f,\hat f$. We will also write $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$ as $\boldsymbol{S}_{f,\hat f}:\boldsymbol{S}_{V,E,s}\to\boldsymbol{S}_{W,F,t}$, and call it a *standard model* 1*-morphism.* If $x\in X$ then $x\in V$ with $s(x)=0$, so (1.9) implies that

$$t\big(f(x)\big)=\big(f^*(t)\big)(x)=\hat f\big(s(x)\big)+O\big(s(x)^2\big)=0,$$

so $f(x)\in Y\subseteq W$. Thus $g:=f|_X$ maps $X\to Y$.

Define morphisms of C^∞-rings

$$\begin{aligned}&\phi:C^\infty(W)/I_t\longrightarrow C^\infty(V)/I_s, &&\phi':C^\infty(W)/I_t^2\longrightarrow C^\infty(V)/I_s^2,\\ &\text{by}\quad \phi:c+I_t\longmapsto c\circ f+I_s, &&\phi':c+I_t^2\longmapsto c\circ f+I_s^2.\end{aligned}$$

Here ϕ is well-defined since if $c\in I_t$ then $c=\gamma\cdot t$ for some $\gamma\in C^\infty(F^*)$, so

$$c\circ f=(\gamma\cdot t)\circ f=f^*(\gamma)\cdot f^*(t)=f^*(\gamma)\cdot\big(\hat f\circ s+O(s^2)\big)=\big(\hat f\circ f^*(\gamma)\big)\cdot s+O(s^2)\in I_s.$$

Similarly if $c\in I_t^2$ then $c\circ f\in I_s^2$, so ϕ' is well-defined. Thus we have C^∞-scheme morphisms $\underline{g}=(g,g^\sharp)=\operatorname{Spec}\phi:\underline{X}\to\underline{Y}$, and $(g,g')=\operatorname{Spec}\phi':(X,\mathcal{O}_X')\to(Y,\mathcal{O}_Y')$, both with underlying map g. Hence $g^\sharp:g^{-1}(\mathcal{O}_Y)\to\mathcal{O}_X$ and $g':g^{-1}(\mathcal{O}_Y')\to\mathcal{O}_X'$ are morphisms of sheaves of C^∞-rings on X.

Since $\underline{g}^*(\mathcal{E}_Y)=\operatorname{MSpec}\big(C^\infty(f^*(F^*))/(I_s\cdot C^\infty(f^*(F^*))\big)$, we may define $g'':\underline{g}^*(\mathcal{E}_Y)\to\mathcal{E}_X$ by $g''=\operatorname{MSpec}(G'')$, where

$$\begin{aligned}&G'':C^\infty(f^*(F^*))/(I_s\cdot C^\infty(f^*(F^*))\longrightarrow C^\infty(E^*)/(I_s\cdot C^\infty(E^*))\\ &\text{is defined by}\quad G'':\gamma+I_s\cdot C^\infty(f^*(F^*))\longmapsto\gamma\circ\hat f+I_s\cdot C^\infty(E^*).\end{aligned}$$

This defines $\boldsymbol{g}=(\underline{g},g',g'')$. One can show it is a 1-morphism $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$ in **dMan**, which we also write as $\boldsymbol{S}_{f,\hat f}:\boldsymbol{S}_{V,E,s}\to\boldsymbol{S}_{W,F,t}$.

Suppose $\tilde V\subseteq V$ is open, with inclusion $i_{\tilde V}:\tilde V\to V$. Write $\tilde E=E|_{\tilde V}=i_{\tilde V}^*(E)$ and $\tilde s=s|_{\tilde V}$. Define $\boldsymbol{i}_{\tilde V,V}=\boldsymbol{S}_{i_{\tilde V},\mathrm{id}_{\tilde E}}:\boldsymbol{S}_{\tilde V,\tilde E,\tilde s}\to\boldsymbol{S}_{V,E,s}$. If $s^{-1}(0)\subseteq\tilde V$ then $\boldsymbol{i}_{\tilde V,V}$ is a 1-isomorphism, with inverse $\boldsymbol{i}_{\tilde V,V}^{-1}$. That is, making V smaller without making $s^{-1}(0)$ smaller does not really change $\boldsymbol{S}_{V,E,s}$; the d-manifold $\boldsymbol{S}_{V,E,s}$ depends only on E,s in an arbitrarily small open neighbourhood of $s^{-1}(0)$ in V.

Example 1.4.6. Let V,W be manifolds, $E\to V$, $F\to W$ be vector bundles, and $s\in C^\infty(E)$, $t\in C^\infty(F)$. Suppose $f,g:V\to W$ are smooth and $\hat f:E\to f^*(F)$, $\hat g:E\to g^*(F)$ are vector bundle morphisms with $\hat f\circ s=f^*(t)+O(s^2)$ and $\hat g\circ s=g^*(t)+O(s^2)$, so we have 1-morphisms $\boldsymbol{S}_{f,\hat f},\boldsymbol{S}_{g,\hat g}:\boldsymbol{S}_{V,E,s}\to\boldsymbol{S}_{W,F,t}$. It is easy to show that $\boldsymbol{S}_{f,\hat f}=\boldsymbol{S}_{g,\hat g}$ if and only if $g=f+O(s^2)$ and $\hat g=\hat f+O(s)$.

Now suppose $\Lambda : E \to f^*(TW)$ is a morphism of vector bundles on V. Taking the dual of Λ and lifting to $\underline{V}$ gives $\Lambda^* : \underline{f}^*(T^*\underline{W}) \to \mathcal{E}^*$. Restricting to the C^∞-subscheme $\underline{X} = s^{-1}(0)$ in $\underline{V}$ gives $\lambda = \Lambda^*|_{\underline{X}} : \underline{f}^*(\mathcal{F}_Y) \cong \underline{f}^*(T^*\underline{W})|_{\underline{X}} \to \mathcal{E}^*|_{\underline{X}} = \mathcal{E}_X$. One can show that λ is a 2-morphism $\boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ if and only if

$$g = f + \Lambda \circ s + O(s^2) \quad \text{and} \quad \hat{g} = \hat{f} + f^*(\mathrm{d}t) \circ \Lambda + O(s).$$

Then we write λ as $\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$, and call it a *standard model* 2*-morphism*. Every 2-morphism $\eta : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ is $\boldsymbol{S}_\Lambda$ for some Λ. Two vector bundle morphisms $\Lambda, \Lambda' : E \to f^*(TW)$ have $\boldsymbol{S}_\Lambda = \boldsymbol{S}_{\Lambda'}$ if and only if $\Lambda = \Lambda' + O(s)$.

If $\boldsymbol{X}$ is a d-manifold and $x \in \boldsymbol{X}$ then x has an open neighbourhood $\boldsymbol{U}$ in $\boldsymbol{X}$ equivalent in **dSpa** to $\boldsymbol{S}_{V,E,s}$ for some manifold V, vector bundle $E \to V$ and $s \in C^\infty(E)$. In §3.3 we investigate the extent to which $\boldsymbol{X}$ determines V, E, s near a point in $\boldsymbol{X}$ and V, and prove:

Theorem 1.4.7. *Let $\boldsymbol{X}$ be a d-manifold, and $x \in \boldsymbol{X}$. Then there exists an open neighbourhood $\boldsymbol{U}$ of x in $\boldsymbol{X}$ and an equivalence $\boldsymbol{U} \simeq \boldsymbol{S}_{V,E,s}$ in* **dMan** *for some manifold V, vector bundle $E \to V$ and $s \in C^\infty(E)$ which identifies $x \in \boldsymbol{U}$ with a point $v \in V$ such that $s(v) = \mathrm{d}s(v) = 0$, where $\boldsymbol{S}_{V,E,s}$ is as in Example* 1.4.4*. These V, E, s are determined up to non-canonical isomorphism near v by $\boldsymbol{X}$ near x, and in fact they depend only on the underlying C^∞-scheme $\underline{X}$ and the integer* $\operatorname{vdim} \boldsymbol{X}$.

Thus, if we impose the extra condition $\mathrm{d}s(v) = 0$, which is in fact equivalent to choosing V, E, s with $\dim V$ as small as possible, then V, E, s are determined uniquely near v by $\boldsymbol{X}$ near x (that is, V, E, s are determined locally up to isomorphism, but not up to canonical isomorphism). If we drop the condition $\mathrm{d}s(v) = 0$ then V, E, s are determined uniquely near v by $\boldsymbol{X}$ near x and $\dim V$.

Theorem 1.4.7 shows that any d-manifold $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ is determined up to equivalence in **dSpa** near any point $x \in \boldsymbol{X}$ by the 'classical' underlying C^∞-scheme $\underline{X}$ and the integer $\operatorname{vdim} \boldsymbol{X}$. So we can ask: what extra information about $\boldsymbol{X}$ is contained in the 'derived' data $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$? One can think of this extra information as like a vector bundle $\mathcal{E}$ over $\underline{X}$. The only local information in a vector bundle $\mathcal{E}$ is $\operatorname{rank} \mathcal{E} \in \mathbb{Z}$, but globally it also contains nontrivial algebraic-topological information.

Suppose now that $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in **dMan**, and $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$. Then by Theorem 1.4.7 we have $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ near x and $\boldsymbol{Y} \simeq \boldsymbol{S}_{W,F,t}$ near y. So up to composition with equivalences, we can identify $\boldsymbol{f}$ near x with a 1-morphism $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. Thus, to understand arbitrary 1-morphisms $\boldsymbol{f}$ in **dMan** near a point, it is enough to study 1-morphisms $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. Our next theorem, proved in §3.4, shows that after making V smaller, every 1-morphism $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ is of the form $\boldsymbol{S}_{f,\hat{f}}$.

Theorem 1.4.8. *Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, and $s \in C^\infty(E)$, $t \in C^\infty(F)$. Define principal d-manifolds $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$, $\boldsymbol{Y} = \boldsymbol{S}_{W,F,t}$, with topological spaces $X = \{v \in V : s(v) = 0\}$ and $Y = \{w \in W : t(w) = 0\}$. Suppose $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ is a* 1*-morphism. Then there exist an open neighbourhood*

$\tilde V$ of X in V, a smooth map $f:\tilde V\rightarrow W$, and a morphism of vector bundles $\hat f:\tilde E\rightarrow f^*(F)$ with $\hat f\circ\tilde s=f^*(t)$, where $\tilde E=E|_{\tilde V}$, $\tilde s=s|_{\tilde V}$, such that $\boldsymbol g=\boldsymbol S_{f,\hat f}\circ\boldsymbol i_{\tilde V,V}^{-1}$, where $\boldsymbol i_{\tilde V,V}=\boldsymbol S_{\mathrm{id}_{\tilde V},\mathrm{id}_{\tilde E}}:\boldsymbol S_{\tilde V,\tilde E,\tilde s}\rightarrow\boldsymbol S_{V,E,s}$ is a 1-isomorphism, and $\boldsymbol S_{f,\hat f}:\boldsymbol S_{\tilde V,\tilde E,\tilde s}\rightarrow\boldsymbol S_{W,F,t}$ is as in Example 1.4.5.

These results give a good differential-geometric picture of d-manifolds and their 1- and 2-morphisms near a point. The $O(s)$ and $O(s^2)$ notation helps keep track of what information from V,E,s and $f,\hat f$ and Λ is remembered and what forgotten by the d-manifolds $\boldsymbol S_{V,E,s}$, 1-morphisms $\boldsymbol S_{f,\hat f}$ and 2-morphisms $\boldsymbol S_\Lambda$.

1.4.3 The 2-category of virtual vector bundles

In our theory of derived differential geometry, it is a general principle that categories in classical differential geometry should often be replaced by 2-categories, and classical concepts be replaced by 2-categorical analogues.

In classical differential geometry, if X is a manifold, the vector bundles $E\rightarrow X$ and their morphisms form a category $\mathrm{vect}(X)$. The cotangent bundle T^*X is an important example of a vector bundle. If $f:X\rightarrow Y$ is smooth then pullback $f^*:\mathrm{vect}(Y)\rightarrow\mathrm{vect}(X)$ is a functor. There is a natural morphism $\mathrm{d}f^*:f^*(T^*Y)\rightarrow T^*X$. We now explain 2-categorical analogues of all this for d-manifolds, following §3.1–§3.2.

Definition 1.4.9. Let $\underline{X}$ be a C^∞-scheme, which will usually be the C^∞-scheme underlying a d-manifold $\boldsymbol X$. We will define a 2-category $\mathrm{vqcoh}(\underline{X})$ of *virtual quasicoherent sheaves* on $\underline{X}$. *Objects* of $\mathrm{vqcoh}(\underline{X})$ are morphisms $\phi:\mathcal E^1\rightarrow\mathcal E^2$ in $\mathrm{qcoh}(\underline{X})$, which we also may write as $(\mathcal E^1,\mathcal E^2,\phi)$ or $(\mathcal E^\bullet,\phi)$. Given objects $\phi:\mathcal E^1\rightarrow\mathcal E^2$ and $\psi:\mathcal F^1\rightarrow\mathcal F^2$, a 1-*morphism* $(f^1,f^2):(\mathcal E^\bullet,\phi)\rightarrow(\mathcal F^\bullet,\psi)$ is a pair of morphisms $f^1:\mathcal E^1\rightarrow\mathcal F^1$, $f^2:\mathcal E^2\rightarrow\mathcal F^2$ in $\mathrm{qcoh}(\underline{X})$ such that $\psi\circ f^1=f^2\circ\phi$. We write $f^\bullet$ for (f^1,f^2).

The *identity* 1-*morphism* of $(\mathcal E^\bullet,\phi)$ is $(\mathrm{id}_{\mathcal E^1},\mathrm{id}_{\mathcal E^2})$. The *composition* of 1-morphisms $f^\bullet:(\mathcal E^\bullet,\phi)\rightarrow(\mathcal F^\bullet,\psi)$ and $g^\bullet:(\mathcal F^\bullet,\psi)\rightarrow(\mathcal G^\bullet,\xi)$ is $g^\bullet\circ f^\bullet=(g^1\circ f^1,g^2\circ f^2):(\mathcal E^\bullet,\phi)\rightarrow(\mathcal G^\bullet,\xi)$.

Given $f^\bullet,g^\bullet:(\mathcal E^\bullet,\phi)\rightarrow(\mathcal F^\bullet,\psi)$, a 2-*morphism* $\eta:f^\bullet\Rightarrow g^\bullet$ is a morphism $\eta:\mathcal E^2\rightarrow\mathcal F^1$ in $\mathrm{qcoh}(\underline{X})$ such that $g^1=f^1+\eta\circ\phi$ and $g^2=f^2+\psi\circ\eta$. The *identity* 2-*morphism* for $f^\bullet$ is $\mathrm{id}_{f^\bullet}=0$. If $f^\bullet,g^\bullet,h^\bullet:(\mathcal E^\bullet,\phi)\rightarrow(\mathcal F^\bullet,\psi)$ are 1-morphisms and $\eta:f^\bullet\Rightarrow g^\bullet$, $\zeta:g^\bullet\Rightarrow h^\bullet$ are 2-morphisms, the *vertical composition of* 2-*morphisms* $\zeta\odot\eta:f^\bullet\Rightarrow h^\bullet$ is $\zeta\odot\eta=\zeta+\eta$. If $f^\bullet,\tilde f^\bullet:(\mathcal E^\bullet,\phi)\rightarrow(\mathcal F^\bullet,\psi)$ and $g^\bullet,\tilde g^\bullet:(\mathcal F^\bullet,\psi)\rightarrow(\mathcal G^\bullet,\xi)$ are 1-morphisms and $\eta:f^\bullet\Rightarrow\tilde f^\bullet$, $\zeta:g^\bullet\Rightarrow\tilde g^\bullet$ are 2-morphisms, the *horizontal composition of* 2-*morphisms* $\zeta*\eta:g^\bullet\circ f^\bullet\Rightarrow\tilde g^\bullet\circ\tilde f^\bullet$ is $\zeta*\eta=g^1\circ\eta+\zeta\circ f^2+\zeta\circ\psi\circ\eta$. This defines a strict 2-category $\mathrm{vqcoh}(\underline{X})$, the obvious 2-category of 2-term complexes in $\mathrm{qcoh}(\underline{X})$.

If $\underline{U}\subseteq\underline{X}$ is an open C^∞-subscheme then restriction from $\underline{X}$ to $\underline{U}$ defines a strict 2-functor $|_{\underline{U}}:\mathrm{vqcoh}(\underline{X})\rightarrow\mathrm{vqcoh}(\underline{U})$. An object $(\mathcal E^\bullet,\phi)$ in $\mathrm{vqcoh}(\underline{X})$ is called a *virtual vector bundle of rank* $d\in\mathbb Z$ if $\underline{X}$ may be covered by open $\underline{U}\subseteq\underline{X}$ such that $(\mathcal E^\bullet,\phi)|_{\underline{U}}$ is equivalent in $\mathrm{vqcoh}(\underline{U})$ to some $(\mathcal F^\bullet,\psi)$ for $\mathcal F^1,\mathcal F^2$ vector bundles on $\underline{U}$ with $\mathrm{rank}\,\mathcal F^2-\mathrm{rank}\,\mathcal F^1=d$. We write $\mathrm{rank}(\mathcal E^\bullet,\phi)=d$. If $\underline{X}\neq\emptyset$

then $\operatorname{rank}(\mathcal{E}^\bullet,\phi)$ depends only on $\mathcal{E}^1,\mathcal{E}^2,\phi$, so it is well-defined. Write $\mathrm{vvect}(\underline{X})$ for the full 2-subcategory of virtual vector bundles in $\mathrm{vqcoh}(\underline{X})$.

If $\underline{f}:\underline{X}\to\underline{Y}$ is a C^∞-scheme morphism then pullback gives a strict 2-functor $\underline{f}^*:\mathrm{vqcoh}(\underline{Y})\to\mathrm{vqcoh}(\underline{X})$, which maps $\mathrm{vvect}(\underline{Y})\to\mathrm{vvect}(\underline{X})$.

We apply these ideas to d-spaces.

Definition 1.4.10. Let $\boldsymbol{X}=(\underline{X},\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X)$ be a d-space. Define the *virtual cotangent sheaf* $T^*\boldsymbol{X}$ of $\boldsymbol{X}$ to be the morphism $\phi_X:\mathcal{E}_X\to\mathcal{F}_X$ in $\mathrm{qcoh}(\underline{X})$ from Definition 1.3.1, regarded as a virtual quasicoherent sheaf on $\underline{X}$.

Let $\boldsymbol{f}=(\underline{f},f',f''):\boldsymbol{X}\to\boldsymbol{Y}$ be a 1-morphism in **dSpa**. Then $T^*\boldsymbol{X}=(\mathcal{E}_X,\mathcal{F}_X,\phi_X)$ and $\underline{f}^*(T^*\boldsymbol{Y})=\bigl(\underline{f}^*(\mathcal{E}_Y),\underline{f}^*(\mathcal{F}_Y),\underline{f}^*(\phi_Y)\bigr)$ are virtual quasicoherent sheaves on $\underline{X}$, and $\Omega_{\boldsymbol{f}}:=(f'',f^2)$ is a 1-morphism $\underline{f}^*(T^*\boldsymbol{Y})\to T^*\boldsymbol{X}$ in $\mathrm{vqcoh}(\underline{X})$, as (1.4) commutes.

Let $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$ be 1-morphisms in **dSpa**, and $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a 2-morphism. Then $\eta:\underline{f}^*(\mathcal{F}_Y)\to\mathcal{E}_X$ with $g''=f''+\eta\circ\underline{f}^*(\phi_Y)$ and $g^2=f^2+\phi_X\circ\eta$, as in (1.6). It follows that η is a 2-morphism $\Omega_{\boldsymbol{f}}\Rightarrow\Omega_{\boldsymbol{g}}$ in $\mathrm{vqcoh}(\underline{X})$. Thus, objects, 1-morphisms and 2-morphisms in **dSpa** lift to objects, 1-morphisms and 2-morphisms in $\mathrm{vqcoh}(\underline{X})$.

The next proposition justifies the definition of virtual vector bundle. Because of part (b), if $\boldsymbol{X}$ is a d-manifold we call $T^*\boldsymbol{X}$ the *virtual cotangent bundle* of $\boldsymbol{X}$, rather than the virtual cotangent sheaf.

Proposition 1.4.11. **(a)** *Let V be a manifold, $E\to V$ a vector bundle, and $s\in C^\infty(E)$. Then Example* 1.4.4 *defines a d-manifold $\boldsymbol{S}_{V,E,s}$. Its cotangent bundle $T^*\boldsymbol{S}_{V,E,s}$ is a virtual vector bundle on $\underline{S}_{V,E,s}$ of rank* $\dim V-\operatorname{rank}E$.

(b) *Let $\boldsymbol{X}$ be a d-manifold. Then $T^*\boldsymbol{X}$ is a virtual vector bundle on $\underline{X}$ of rank* $\operatorname{vdim}\boldsymbol{X}$. *Hence if $\boldsymbol{X}\neq\boldsymbol{\emptyset}$ then* $\operatorname{vdim}\boldsymbol{X}$ *is well-defined.*

The virtual cotangent bundle $T^*\boldsymbol{X}$ of a d-manifold $\boldsymbol{X}$ is a d-space analogue of the *cotangent complex* in algebraic geometry, as in Illusie [50, 51]. It contains only a fraction of the information in $\boldsymbol{X}=(\underline{X},\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X)$, but many interesting properties of d-manifolds $\boldsymbol{X}$ and 1-morphisms $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ can be expressed solely in terms of virtual cotangent bundles $T^*\boldsymbol{X},T^*\boldsymbol{Y}$ and 1-morphisms $\Omega_{\boldsymbol{f}}:\underline{f}^*(T^*\boldsymbol{Y})\to T^*\boldsymbol{X}$. Here is an example of this.

Definition 1.4.12. Let $\underline{X}$ be a C^∞-scheme. We say that a virtual vector bundle $(\mathcal{E}^1,\mathcal{E}^2,\phi)$ on $\underline{X}$ *is a vector bundle* if it is equivalent in $\mathrm{vvect}(\underline{X})$ to $(0,\mathcal{E},0)$ for some vector bundle $\mathcal{E}$ on $\underline{X}$. One can show $(\mathcal{E}^1,\mathcal{E}^2,\phi)$ is a vector bundle if and only if ϕ has a left inverse in $\mathrm{qcoh}(\underline{X})$.

Proposition 1.4.13. *Let $\boldsymbol{X}$ be a d-manifold. Then $\boldsymbol{X}$ is a manifold (that is, $\boldsymbol{X}\in\hat{\mathbf{Man}}$) if and only if $T^*\boldsymbol{X}$ is a vector bundle, or equivalently, if $\phi_X:\mathcal{E}_X\to\mathcal{F}_X$ has a left inverse in* $\mathrm{qcoh}(\underline{X})$.

1.4.4 Equivalences in dMan, and gluing by equivalences

Equivalences in a 2-category are defined in §A.3. Equivalences in **dMan** are the best derived analogue of isomorphisms in **Man**, that is, of diffeomorphisms of manifolds. A smooth map of manifolds $f : X \to Y$ is called *étale* if it is a local diffeomorphism. Here is the derived analogue.

Definition 1.4.14. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in **dMan**. We call $\boldsymbol{f}$ *étale* if it is a *local equivalence*, that is, if for each $x \in \boldsymbol{X}$ there exist open $x \in \boldsymbol{U} \subseteq \boldsymbol{X}$ and $\boldsymbol{f}(x) \in \boldsymbol{V} \subseteq \boldsymbol{Y}$ such that $\boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{V}$ and $\boldsymbol{f}|_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ is an equivalence.

If $f : X \to Y$ is a smooth map of manifolds, then f is étale if and only if $\mathrm{d}f^* : f^*(T^*Y) \to T^*X$ is an isomorphism of vector bundles. (The analogue is false for schemes.) In §3.5 we prove a version of this for d-manifolds:

Theorem 1.4.15. *Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism of d-manifolds. Then the following are equivalent:*

(i) *$\boldsymbol{f}$ is étale;*

(ii) *$\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ is an equivalence in* $\mathrm{vqcoh}(\underline{X})$*; and*

(iii) *the following is a split short exact sequence in* $\mathrm{qcoh}(\underline{X})$*:*

$$0 \longrightarrow \underline{f}^*(\mathcal{E}_Y) \xrightarrow{f''\oplus -\underline{f}^*(\phi_Y)} \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y) \xrightarrow{\phi_X \oplus f^2} \mathcal{F}_X \longrightarrow 0. \quad (1.10)$$

If in addition $f : X \to Y$ is a bijection, then $\boldsymbol{f}$ is an equivalence in **dMan***.*

Here a complex $0 \to E \to F \to G \to 0$ in an abelian category $\mathcal{A}$ is called a *split short exact sequence* if there exists an isomorphism $F \cong E \oplus G$ in $\mathcal{A}$ identifying the complex with $0 \to E \xrightarrow{\mathrm{id}\oplus 0} E \oplus G \xrightarrow{0\oplus\mathrm{id}} G \to 0$.

The analogue of Theorem 1.4.15 for d-spaces is false. When $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 'standard model' 1-morphism $\boldsymbol{S}_{f,\hat f} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$, as in §1.4.2, we can express the conditions for $\boldsymbol{S}_{f,\hat f}$ to be étale or an equivalence in terms of $f, \hat f$.

Theorem 1.4.16. *Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, $s \in C^\infty(E)$, $t \in C^\infty(F)$, $f : V \to W$ be smooth, and $\hat f : E \to f^*(F)$ be a morphism of vector bundles on V with $\hat f \circ s = f^*(t) + O(s^2)$. Then Example 1.4.5 defines a 1-morphism $\boldsymbol{S}_{f,\hat f} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ in* **dMan***. This $\boldsymbol{S}_{f,\hat f}$ is étale if and only if for each $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_vV \xrightarrow{\mathrm{d}s(v)\oplus \mathrm{d}f(v)} E_v \oplus T_wW \xrightarrow{\hat f(v)\oplus -\mathrm{d}t(w)} F_w \longrightarrow 0. \quad (1.11)$$

Also $\boldsymbol{S}_{f,\hat f}$ is an equivalence if and only if in addition $f|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection, where $s^{-1}(0) = \{v \in V : s(v)=0\}$, $t^{-1}(0) = \{w \in W : t(w)=0\}$.

Section 1.3.2 discussed gluing d-spaces by equivalences on open d-subspaces. It generalizes immediately to d-manifolds: if in Theorem 1.3.7 we fix $n \in \mathbb{Z}$ and take the initial d-spaces $\boldsymbol{X}_i$ to be d-manifolds with $\mathrm{vdim}\,\boldsymbol{X}_i = n$, then the glued d-space $\boldsymbol{Y}$ is also a d-manifold with $\mathrm{vdim}\,\boldsymbol{Y} = n$.

Here is an analogue of Theorem 1.3.7, taken from §3.6, in which we take the d-spaces $\boldsymbol{X}_i$ to be 'standard model' d-manifolds $\boldsymbol{S}_{V_i,E_i,s_i}$, and the 1-morphisms $\boldsymbol{e}_{ij}$ to be 'standard model' 1-morphisms $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}$. We also use Theorem 1.4.16 in (ii) to characterize when $\boldsymbol{e}_{ij}$ is an equivalence.

Theorem 1.4.17. *Suppose we are given the following data:*

(a) *an integer* n;

(b) *a Hausdorff, second countable topological space* X;

(c) *an indexing set* I, *and a total order* $<$ *on* I;

(d) *for each* i *in* I, *a manifold* V_i, *a vector bundle* $E_i \to V_i$ *with* $\dim V_i - \operatorname{rank} E_i = n$, *a smooth section* $s_i : V_i \to E_i$, *and a homeomorphism* $\psi_i : X_i \to \hat{X}_i$, *where* $X_i = \{v_i \in V_i : s_i(v_i) = 0\}$ *and* $\hat{X}_i \subseteq X$ *is open; and*

(e) *for all* $i < j$ *in* I, *an open submanifold* $V_{ij} \subseteq V_i$, *a smooth map* $e_{ij} : V_{ij} \to V_j$, *and a morphism of vector bundles* $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$.

Using notation $O(s_i), O(s_i^2)$ *as in* §1.4.2, *let this data satisfy the conditions:*

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if* $i < j$ *in* I *then* $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j) + O(s_i^2)$, $\psi_i(X_i \cap V_{ij}) = \hat{X}_i \cap \hat{X}_j$, *and* $\psi_i|_{X_i \cap V_{ij}} = \psi_j \circ e_{ij}|_{X_i \cap V_{ij}}$, *and if* $v_i \in V_{ij}$ *with* $s_i(v_i) = 0$ *and* $v_j = e_{ij}(v_i)$ *then the following is exact:*

$$0 \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} E_i|_{v_i} \oplus T_{v_j}V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} E_j|_{v_j} \longrightarrow 0;$$

(iii) *if* $i < j < k$ *in* I *then*

$$\begin{aligned} e_{ik}|_{V_{ij} \cap V_{ik}} &= e_{jk} \circ e_{ij}|_{V_{ij} \cap V_{ik}} + O(s_i^2) \qquad \text{and} \\ \hat{e}_{ik}|_{V_{ij} \cap V_{ik}} &= e_{ij}|_{V_{ij} \cap V_{ik}}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{V_{ij} \cap V_{ik}} + O(s_i). \end{aligned}$$

Then there exist a d-manifold $\boldsymbol{X}$ *with* $\operatorname{vdim} \boldsymbol{X} = n$ *and underlying topological space* X, *and a* 1*-morphism* $\boldsymbol{\psi}_i : \boldsymbol{S}_{V_i,E_i,s_i} \to \boldsymbol{X}$ *with underlying continuous map* ψ_i, *which is an equivalence with the open d-submanifold* $\hat{\boldsymbol{X}}_i \subseteq \boldsymbol{X}$ *corresponding to* $\hat{X}_i \subseteq X$ *for all* $i \in I$, *such that for all* $i < j$ *in* I *there exists a* 2*-morphism* $\eta_{ij} : \boldsymbol{\psi}_j \circ \boldsymbol{S}_{e_{ij},\hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i \circ \boldsymbol{i}_{V_{ij},V_i}$, *where* $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}} : \boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}} \to \boldsymbol{S}_{V_j,E_j,s_j}$ *and* $\boldsymbol{i}_{V_{ij},V_i} : \boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}} \to \boldsymbol{S}_{V_i,E_i,s_i}$ *are as in Example* 1.4.4. *This d-manifold* $\boldsymbol{X}$ *is unique up to equivalence in* **dMan**.

Suppose also that Y *is a manifold, and* $g_i : V_i \to Y$ *are smooth maps for all* $i \in I$, *and* $g_j \circ e_{ij} = g_i|_{V_{ij}} + O(s_i)$ *for all* $i < j$ *in* I. *Then there exist a* 1*-morphism* $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Y}$ *unique up to* 2*-isomorphism, where* $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y) = \boldsymbol{S}_{Y,0,0}$, *and* 2*-morphisms* $\zeta_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \boldsymbol{S}_{g_i,0}$ *for all* $i \in I$. *Here* $\boldsymbol{S}_{Y,0,0}$ *is from Example* 1.4.4 *with vector bundle* E *and section* s *both zero, and* $\boldsymbol{S}_{g_i,0} : \boldsymbol{S}_{V_i,E_i,s_i} \to \boldsymbol{S}_{Y,0,0} = \boldsymbol{Y}$ *is from Example* 1.4.5 *with* $\hat{g}_i = 0$.

The hypotheses of Theorem 1.4.17 are similar to the notion of *good coordinate system* in the theory of Kuranishi spaces of Fukaya and Ono [34, Def. 6.1], as discussed in §1.11.9. The importance of Theorem 1.4.17 is that all the ingredients are described wholly in differential-geometric or topological terms. So we can use the theorem as a tool to prove the existence of d-manifold structures on spaces coming from other areas of geometry, for instance, on moduli spaces.

1.4.5 Submersions, immersions and embeddings

Let $f : X \to Y$ be a smooth map of manifolds. Then $\mathrm{d}f^* : f^*(T^*Y) \to T^*X$ is a morphism of vector bundles on X, and f is a *submersion* if $\mathrm{d}f^*$ is injective, and f is an *immersion* if $\mathrm{d}f^*$ is surjective. Here the appropriate notions of injective and surjective for morphisms of vector bundles are stronger than the corresponding notions for sheaves: $\mathrm{d}f^*$ is *injective* if it has a left inverse, and *surjective* if it has a right inverse.

In a similar way, if $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism of d-manifolds, we would like to define $\boldsymbol{f}$ to be a submersion or immersion if the 1-morphism $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ in $\mathrm{vvect}(\underline{X})$ is injective or surjective in some suitable sense. It turns out that there are two different notions of injective and surjective 1-morphisms in the 2-category $\mathrm{vvect}(\underline{X})$, a weak and a strong:

Definition 1.4.18. Let $\underline{X}$ be a C^∞-scheme, $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ and $(\mathcal{F}^1, \mathcal{F}^2, \psi)$ be virtual vector bundles on $\underline{X}$, and $(f^1, f^2) : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ be a 1-morphism in $\mathrm{vvect}(\underline{X})$. Then we have a complex in $\mathrm{qcoh}(\underline{X})$:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^1 \arrow[r, shift left, "f^1\oplus -\phi"] & \mathcal{F}^1 \oplus \mathcal{E}^2 \arrow[l, shift left, dotted, "\gamma"] \arrow[r, shift left, "\psi\oplus f^2"] & \mathcal{F}^2 \arrow[l, shift left, dotted, "\delta"] \arrow[r] & 0. \end{tikzcd} \tag{1.12}$$

One can show that $f^\bullet$ is an equivalence in $\mathrm{vvect}(\underline{X})$ if and only if (1.12) is a *split short exact sequence* in $\mathrm{qcoh}(\underline{X})$. That is, $f^\bullet$ is an equivalence if and only if there exist morphisms γ, δ as shown in (1.12) satisfying the conditions:

$$\begin{aligned} \gamma\circ\delta &= 0, & \gamma\circ(f^1\oplus -\phi) &= \mathrm{id}_{\mathcal{E}^1}, \\ (f^1\oplus -\phi)\circ\gamma + \delta\circ(\psi\oplus f^2) &= \mathrm{id}_{\mathcal{F}^1\oplus\mathcal{E}^2}, & (\psi\oplus f^2)\circ\delta &= \mathrm{id}_{\mathcal{F}^2}. \end{aligned} \tag{1.13}$$

Our notions of $f^\bullet$ injective or surjective impose some but not all of (1.13):

(a) We call $f^\bullet$ *weakly injective* if there exists $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ in $\mathrm{qcoh}(\underline{X})$ with $\gamma\circ(f^1\oplus -\phi) = \mathrm{id}_{\mathcal{E}^1}$.

(b) We call $f^\bullet$ *injective* if there exist $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ and $\delta : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ with $\gamma\circ\delta = 0$, $\gamma\circ(f^1\oplus -\phi) = \mathrm{id}_{\mathcal{E}^1}$ and $(f^1\oplus -\phi)\circ\gamma + \delta\circ(\psi\oplus f^2) = \mathrm{id}_{\mathcal{F}^1\oplus\mathcal{E}^2}$.

(c) We call $f^\bullet$ *weakly surjective* if there exists $\delta : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ in $\mathrm{qcoh}(\underline{X})$ with $(\psi\oplus f^2)\circ\delta = \mathrm{id}_{\mathcal{F}^2}$.

(d) We call $f^\bullet$ *surjective* if there exist $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ and $\delta : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ with $\gamma\circ\delta = 0$, $\gamma\circ(f^1\oplus -\phi) = \mathrm{id}_{\mathcal{E}^1}$ and $(\psi\oplus f^2)\circ\delta = \mathrm{id}_{\mathcal{F}^2}$.

If $\underline{X}$ is separated, paracompact, and locally fair, these are local conditions on $\underline{X}$.

Using these we define weak and strong forms of submersions, immersions, and embeddings for d-manifolds.

Definition 1.4.19. Let $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ be a 1-morphism of d-manifolds. Definition 1.4.10 defines a 1-morphism $\Omega_{\boldsymbol{f}}:\underline{f}^*(T^*\boldsymbol{Y})\to T^*\boldsymbol{X}$ in $\mathrm{vvect}(\underline{X})$. Then:

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\Omega_{\boldsymbol{f}}$ is weakly injective.

(b) We call $\boldsymbol{f}$ a *submersion* if $\Omega_{\boldsymbol{f}}$ is injective.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $\Omega_{\boldsymbol{f}}$ is weakly surjective.

(d) We call $\boldsymbol{f}$ an *immersion* if $\Omega_{\boldsymbol{f}}$ is surjective.

(e) We call $\boldsymbol{f}$ a *w-embedding* if it is a w-immersion and $f:X\to f(X)$ is a homeomorphism, so in particular f is injective.

(f) We call $\boldsymbol{f}$ an *embedding* if it is an immersion and f is a homeomorphism with its image.

Here w-submersion is short for *weak submersion*, etc. Conditions (a)–(d) all concern the existence of morphisms γ,δ in the next equation satisfying identities:

$$0 \longrightarrow \underline{f}^*(\mathcal{E}_Y) \underset{\gamma}{\overset{f''\oplus -\underline{f}^*(\phi_Y)}{\rightleftarrows}} \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y) \underset{\delta}{\overset{\phi_X\oplus f^2}{\rightleftarrows}} \mathcal{F}_X \longrightarrow 0.$$

Parts (c)–(f) enable us to define *d-submanifolds* of d-manifolds. *Open d-submanifolds* are open d-subspaces of a d-manifold. More generally, we call $\boldsymbol{i}:\boldsymbol{X}\to\boldsymbol{Y}$ a *w-immersed*, or *immersed*, or *w-embedded*, or *embedded d-submanifold* of $\boldsymbol{Y}$, if $\boldsymbol{X},\boldsymbol{Y}$ are d-manifolds and $\boldsymbol{i}$ is a w-immersion, immersion, w-embedding, or embedding, respectively.

Here are some properties of these, taken from §4.1–§4.2:

Theorem 1.4.20. (i) *Any equivalence of d-manifolds is a w-submersion, submersion, w-immersion, immersion, w-embedding and embedding.*

(ii) *If $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$ are 2-isomorphic 1-morphisms of d-manifolds then $\boldsymbol{f}$ is a w-submersion, submersion, ..., embedding, if and only if $\boldsymbol{g}$ is.*

(iii) *Compositions of w-submersions, submersions, w-immersions, immersions, w-embeddings, and embeddings are 1-morphisms of the same kind.*

(iv) *The conditions that a 1-morphism of d-manifolds $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is a w-submersion, submersion, w-immersion or immersion are local in $\boldsymbol{X}$ and $\boldsymbol{Y}$. That is, for each $x\in\boldsymbol{X}$ with $\boldsymbol{f}(x)=y\in\boldsymbol{Y}$, it suffices to check the conditions for $\boldsymbol{f}|_{\boldsymbol{U}}:\boldsymbol{U}\to\boldsymbol{V}$ with $\boldsymbol{V}$ an open neighbourhood of y in $\boldsymbol{Y}$, and $\boldsymbol{U}$ an open neighbourhood of x in $\boldsymbol{f}^{-1}(\boldsymbol{V})\subseteq\boldsymbol{X}$. The conditions that $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is a w-embedding or embedding are local in $\boldsymbol{Y}$, but not in $\boldsymbol{X}$.*

(v) *Let $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ be a submersion of d-manifolds. Then $\mathrm{vdim}\,\boldsymbol{X}\geqslant\mathrm{vdim}\,\boldsymbol{Y}$, and if $\mathrm{vdim}\,\boldsymbol{X}=\mathrm{vdim}\,\boldsymbol{Y}$ then $\boldsymbol{f}$ is étale.*

(vi) *Let $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ be an immersion of d-manifolds. Then $\mathrm{vdim}\,\boldsymbol{X}\leqslant\mathrm{vdim}\,\boldsymbol{Y}$, and if $\mathrm{vdim}\,\boldsymbol{X}=\mathrm{vdim}\,\boldsymbol{Y}$ then $\boldsymbol{f}$ is étale.*

(vii) *Let $f : X \to Y$ be a smooth map of manifolds, and $\boldsymbol{f} = F_{\mathbf{Man}}^{\mathbf{dMan}}(f)$. Then $\boldsymbol{f}$ is a submersion, immersion, or embedding in* **dMan** *if and only if f is a submersion, immersion, or embedding in* **Man**, *respectively. Also $\boldsymbol{f}$ is a w-immersion or w-embedding if and only if f is an immersion or embedding.*

(viii) *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism of d-manifolds, with $\boldsymbol{Y}$ a manifold. Then $\boldsymbol{f}$ is a w-submersion.*

(ix) *Let $\boldsymbol{X}, \boldsymbol{Y}$ be d-manifolds, with $\boldsymbol{Y}$ a manifold. Then $\boldsymbol{\pi_X} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{X}$ is a submersion.*

(x) *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a submersion of d-manifolds, and $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$. Then there exist open $x \in \boldsymbol{U} \subseteq \boldsymbol{X}$ and $y \in \boldsymbol{V} \subseteq \boldsymbol{Y}$ with $\boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{V}$, a manifold $\boldsymbol{Z}$, and an equivalence $\boldsymbol{i} : \boldsymbol{U} \to \boldsymbol{V} \times \boldsymbol{Z}$, such that $\boldsymbol{f}|_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ is 2-isomorphic to $\boldsymbol{\pi_V} \circ \boldsymbol{i}$, where $\boldsymbol{\pi_V} : \boldsymbol{V} \times \boldsymbol{Z} \to \boldsymbol{V}$ is the projection.*

(xi) *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a submersion of d-manifolds with $\boldsymbol{Y}$ a manifold. Then $\boldsymbol{X}$ is a manifold.*

1.4.6 D-transversality and fibre products

From §1.3.3, if $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are 1-morphisms of d-manifolds then a fibre product $\boldsymbol{W} = \boldsymbol{X}_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in **dSpa**, and is unique up to equivalence. We want to know whether $\boldsymbol{W}$ is a d-manifold. We will define when $\boldsymbol{g}, \boldsymbol{h}$ are *d-transverse*, which is a sufficient condition for $\boldsymbol{W}$ to be a d-manifold.

Recall that if $g : X \to Z$, $h : Y \to Z$ are smooth maps of manifolds, then a fibre product $W = X \times_{g,Z,h} Y$ in **Man** exists if g, h are *transverse*, that is, if $T_z Z = \mathrm{d}g|_x(T_x X) + \mathrm{d}h|_y(T_y Y)$ for all $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$. Equivalently, $\mathrm{d}g|_x^* \oplus \mathrm{d}h|_y^* : T_z Z^* \to T_x^* X \oplus T_y^* Y$ should be injective. Writing $W = X \times_Z Y$ for the topological fibre product and $e : W \to X$, $f : W \to Y$ for the projections, with $g \circ e = h \circ f$, we see that g, h are transverse if and only if

$$e^*(\mathrm{d}g^*) \oplus f^*(\mathrm{d}h^*) : (g \circ e)^*(T^*Z) \to e^*(T^*X) \oplus f^*(T^*Y) \tag{1.14}$$

is an injective morphism of vector bundles on the topological space W, that is, it has a left inverse. The condition that (1.15) has a left inverse is an analogue of this, but on (dual) obstruction rather than cotangent bundles.

Definition 1.4.21. Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be d-manifolds and $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms. Let $\underline{W} = \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y}$ be the C^∞-scheme fibre product, and write $\underline{e} : \underline{W} \to \underline{X}$, $\underline{f} : \underline{W} \to \underline{Y}$ for the projections. Consider the morphism

$$\alpha = \begin{pmatrix} \underline{e}^*(g'') \circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z) \\ -\underline{f}^*(h'') \circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z) \\ (\underline{g} \circ \underline{e})^*(\phi_Z) \end{pmatrix} : \begin{array}{l} (\underline{g} \circ \underline{e})^*(\mathcal{E}_Z) \longrightarrow \\ \qquad \underline{e}^*(\mathcal{E}_X) \oplus \underline{f}^*(\mathcal{E}_Y) \oplus (\underline{g} \circ \underline{e})^*(\mathcal{F}_Z) \end{array} \tag{1.15}$$

in $\mathrm{qcoh}(\underline{W})$. We call $\boldsymbol{g}, \boldsymbol{h}$ *d-transverse* if α has a left inverse. Note that this is a local condition in $\underline{W}$, since local choices of left inverse for α can be combined using a partition of unity on $\underline{W}$ to make a global left inverse.

In the notation of §1.4.3 and §1.4.5, we have 1-morphisms $\Omega_{\boldsymbol{g}} : \underline{g}^*(T^*\boldsymbol{Z}) \to T^*\boldsymbol{X}$ in $\mathrm{vvect}(\underline{X})$ and $\Omega_{\boldsymbol{h}} : \underline{h}^*(T^*\boldsymbol{Z}) \to T^*\boldsymbol{Y}$ in $\mathrm{vvect}(\underline{Y})$. Pulling these back to $\mathrm{vvect}(\underline{W})$ using $\underline{e}^*, \underline{f}^*$ we form the 1-morphism in $\mathrm{vvect}(\underline{W})$:

$$\begin{aligned}\big(\underline{e}^*(\Omega_{\boldsymbol{g}}) \circ I_{\underline{e},\underline{g}}(T^*\boldsymbol{Z})\big) \oplus \big(\underline{f}^*(\Omega_{\boldsymbol{h}}) \circ I_{\underline{f},\underline{h}}(T^*\boldsymbol{Z})\big) : (\underline{g}\circ\underline{e})^*(T^*\boldsymbol{Z})& \\ \longrightarrow \underline{e}^*(T^*\boldsymbol{X}) \oplus \underline{f}^*(T^*\boldsymbol{Y}).&\end{aligned} \tag{1.16}$$

For (1.15) to have a left inverse is equivalent to (1.16) being weakly injective, as in Definition 1.4.18. This is the d-manifold analogue of (1.14) being injective.

Here are the main results of §4.3:

Theorem 1.4.22. *Suppose $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are d-manifolds and $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ are d-transverse 1-morphisms, and let $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ be the d-space fibre product. Then $\boldsymbol{W}$ is a d-manifold, with*

$$\operatorname{vdim}\boldsymbol{W}=\operatorname{vdim}\boldsymbol{X}+\operatorname{vdim}\boldsymbol{Y}-\operatorname{vdim}\boldsymbol{Z}. \tag{1.17}$$

Theorem 1.4.23. *Suppose $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ are 1-morphisms of d-manifolds. The following are sufficient conditions for $\boldsymbol{g},\boldsymbol{h}$ to be d-transverse, so that $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ is a d-manifold of virtual dimension* (1.17):

- **(a)** *$\boldsymbol{Z}$ is a manifold, that is, $\boldsymbol{Z}\in\hat{\mathbf{Man}}$; or*
- **(b)** *$\boldsymbol{g}$ or $\boldsymbol{h}$ is a w-submersion.*

The point here is that roughly speaking, $\boldsymbol{g},\boldsymbol{h}$ are d-transverse if they map the direct sum of the obstruction spaces of $\boldsymbol{X},\boldsymbol{Y}$ surjectively onto the obstruction spaces of $\boldsymbol{Z}$. If $\boldsymbol{Z}$ is a manifold its obstruction spaces are zero. If $\boldsymbol{g}$ is a w-submersion it maps the obstruction spaces of $\boldsymbol{X}$ surjectively onto the obstruction spaces of $\boldsymbol{Z}$. In both cases, d-transversality follows. See [95, Th. 8.15] for the analogue of Theorem 1.4.23(a) for Spivak's derived manifolds.

Theorem 1.4.24. *Let $\boldsymbol{X},\boldsymbol{Z}$ be d-manifolds, $\boldsymbol{Y}$ a manifold, and $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ be 1-morphisms with $\boldsymbol{g}$ a submersion. Then $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ is a manifold, with* $\dim\boldsymbol{W}=\operatorname{vdim}\boldsymbol{X}+\dim\boldsymbol{Y}-\operatorname{vdim}\boldsymbol{Z}$.

Theorem 1.4.24 shows that we may think of submersions as 'representable 1-morphisms' in **dMan**. We can locally characterize embeddings and immersions in **dMan** in terms of fibre products with $\mathbb{R}^{\boldsymbol{n}}$ in **dMan**.

Theorem 1.4.25. **(i)** *Let $\boldsymbol{X}$ be a d-manifold and $\boldsymbol{g}:\boldsymbol{X}\to\mathbb{R}^{\boldsymbol{n}}$ a 1-morphism in* **dMan**. *Then the fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\mathbb{R}^{\boldsymbol{n}},\boldsymbol{0}}\boldsymbol{*}$ exists in* **dMan** *by Theorem* 1.4.23(a), *and the projection $\boldsymbol{\pi_X}:\boldsymbol{W}\to\boldsymbol{X}$ is an embedding.*

(ii) *Suppose $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is an immersion of d-manifolds, and $x\in\boldsymbol{X}$ with $\boldsymbol{f}(x)=y\in\boldsymbol{Y}$. Then there exist open d-submanifolds $x\in\boldsymbol{U}\subseteq\boldsymbol{X}$ and $y\in$*

$\boldsymbol{V} \subseteq \boldsymbol{Y}$ *with* $\boldsymbol{f}(\boldsymbol{U}) \subseteq \boldsymbol{V}$*, and a* 1*-morphism* $\boldsymbol{g} : \boldsymbol{V} \to \mathbb{R}^{\boldsymbol{n}}$ *with* $\boldsymbol{g}(y) = 0$*, where* $n = \operatorname{vdim} \boldsymbol{Y} - \operatorname{vdim} \boldsymbol{X} \geqslant 0$*, fitting into a* 2*-Cartesian square in* **dMan** *:*

$$\begin{tikzcd} \boldsymbol{U} \arrow[rr] \arrow[d, "\boldsymbol{f}|_{\boldsymbol{U}}"'] & \arrow[d, phantom, "\Uparrow"] & * \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{V} \arrow[rr, "\boldsymbol{g}"] & & \mathbb{R}^{\boldsymbol{n}}. \end{tikzcd}$$

If $\boldsymbol{f}$ *is an embedding we may take* $\boldsymbol{U} = \boldsymbol{f}^{-1}(\boldsymbol{V})$*.*

Remark 1.4.26. For the applications the author has in mind, it will be crucial that if $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are 1-morphisms with $\boldsymbol{X}, \boldsymbol{Y}$ d-manifolds and $\boldsymbol{Z}$ a manifold then $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ is a d-manifold, with $\operatorname{vdim} \boldsymbol{W} = \operatorname{vdim} \boldsymbol{X} + \operatorname{vdim} \boldsymbol{Y} - \dim \boldsymbol{Z}$, as in Theorem 1.4.23(a). We will show by example, following Spivak [95, Prop. 1.7], that if d-manifolds **dMan** were an ordinary category containing manifolds as a full subcategory, then this would be false.

Consider the fibre product $* \times_{\boldsymbol{0},\mathbb{R},\boldsymbol{0}} *$ in **dMan**. If **dMan** were a category then as $*$ is a terminal object, the fibre product would be $*$. But then

$$\operatorname{vdim}(* \times_{\boldsymbol{0},\mathbb{R},\boldsymbol{0}} *) = \operatorname{vdim} * = 0 \neq -1 = \operatorname{vdim} * + \operatorname{vdim} * - \operatorname{vdim} \mathbb{R},$$

so equation (1.17) and Theorem 1.4.23(a) would be false. Thus, if we want fibre products of d-manifolds over manifolds to be well behaved, then **dMan** must be at least a 2-category. It could be an ∞-category, as for Spivak's derived manifolds [94, 95], or some other kind of higher category. Making d-manifolds into a 2-category, as we have done, is the simplest of the available options.

1.4.7 Embedding d-manifolds into manifolds

Let V be a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$. Then Example 1.4.4 defines a 'standard model' principal d-manifold $\boldsymbol{S}_{V,E,s}$. When E and s are zero, we have $\boldsymbol{S}_{V,0,0} = \boldsymbol{V} = F_{\mathbf{Man}}^{\mathbf{dMan}}(V)$, so that $\boldsymbol{S}_{V,0,0}$ is a manifold. For general V, E, s, taking $f = \mathrm{id}_V : V \to V$ and $\hat{f} = 0 : E \to 0$ in Example 1.4.5 gives a 'standard model' 1-morphism $\boldsymbol{S}_{\mathrm{id}_V,0} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{V,0,0} = \boldsymbol{V}$. One can show $\boldsymbol{S}_{\mathrm{id}_V,0}$ is an embedding, in the sense of Definition 1.4.19. Any principal d-manifold $\boldsymbol{U}$ is equivalent to some $\boldsymbol{S}_{V,E,s}$. Thus we deduce:

Lemma 1.4.27. *Any principal d-manifold* $\boldsymbol{U}$ *admits an embedding* $\boldsymbol{i} : \boldsymbol{U} \to \boldsymbol{V}$ *into a manifold* $\boldsymbol{V}$*.*

Theorem 1.4.32 below is a converse to this: if a d-manifold $\boldsymbol{X}$ can be embedded into a manifold $\boldsymbol{Y}$, then $\boldsymbol{X}$ is principal. So it will be useful to study embeddings of d-manifolds into manifolds. The following classical facts are due to Whitney [106].

Theorem 1.4.28. **(a)** *Let* X *be an* m*-manifold and* $n \geqslant 2m$*. Then a generic smooth map* $f : X \to \mathbb{R}^n$ *is an immersion.*

(b) *Let* X *be an* m*-manifold and* $n \geqslant 2m+1$*. Then there exists an embedding* $f : X \to \mathbb{R}^n$*, and we can choose such* f *with* $f(X)$ *closed in* $\mathbb{R}^n$*. Generic smooth maps* $f : X \to \mathbb{R}^n$ *are embeddings.*

In §4.4 we generalize Theorem 1.4.28 to d-manifolds.

Theorem 1.4.29. *Let $\boldsymbol{X}$ be a d-manifold. Then there exist immersions and/or embeddings $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{\mathbb{R}^n}$ for some $n\gg 0$ if and only if there is an upper bound for $\dim T^*_x\underline{X}$ for all $x\in\underline{X}$. If there is such an upper bound, then immersions $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{\mathbb{R}^n}$ exist provided $n\geqslant 2\dim T^*_x\underline{X}$ for all $x\in\underline{X}$, and embeddings $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{\mathbb{R}^n}$ exist provided $n\geqslant 2\dim T^*_x\underline{X}+1$ for all $x\in\underline{X}$. For embeddings we may also choose $\boldsymbol{f}$ with $f(X)$ closed in $\mathbb{R}^n$.*

Here is an example in which the condition does not hold.

Example 1.4.30. $\boldsymbol{\mathbb{R}^k}\times_{\boldsymbol{0},\boldsymbol{\mathbb{R}^k},\boldsymbol{0}}\boldsymbol{*}$ is a principal d-manifold of virtual dimension 0, with C^∞-scheme $\underline{\mathbb{R}}^k$, and obstruction bundle $\mathbb{R}^k$. Thus $\boldsymbol{X}=\coprod_{k\geqslant 0}\boldsymbol{\mathbb{R}^k}\times_{\boldsymbol{0},\boldsymbol{\mathbb{R}^k},\boldsymbol{0}}\boldsymbol{*}$ is a d-manifold of virtual dimension 0, with C^∞-scheme $\underline{X}=\coprod_{k\geqslant 0}\underline{\mathbb{R}}^k$. Since $T^*_x\underline{X}\cong\mathbb{R}^n$ for $x\in\mathbb{R}^n\subset\coprod_{k\geqslant 0}\mathbb{R}^k$, $\dim T^*_x\underline{X}$ realizes all values $n\geqslant 0$. Hence there cannot exist immersions or embeddings $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{\mathbb{R}^n}$ for any $n\geqslant 0$.

As $x\mapsto\dim T^*_x\underline{X}$ is an upper semicontinuous map $X\to\mathbb{N}$, if $\boldsymbol{X}$ is compact then $\dim T^*_x\underline{X}$ is bounded above, giving:

Corollary 1.4.31. *Let $\boldsymbol{X}$ be a compact d-manifold. Then there exists an embedding $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{\mathbb{R}^n}$ for some $n\gg 0$.*

If a d-manifold $\boldsymbol{X}$ can be embedded into a manifold Y, we show in §4.4 that we can write $\boldsymbol{X}$ as the zeroes of a section of a vector bundle over Y near its image. See [95, Prop. 9.5] for the analogue for Spivak's derived manifolds.

Theorem 1.4.32. *Suppose $\boldsymbol{X}$ is a d-manifold, Y a manifold, and $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ an embedding, in the sense of Definition 1.4.19. Then there exist an open subset V in Y with $\boldsymbol{f}(\boldsymbol{X})\subseteq\boldsymbol{V}$, a vector bundle $E\to V$, and $s\in C^\infty(E)$ fitting into a 2-Cartesian diagram in* **dSpa**:

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{f}"'] \arrow[dr, phantom, "\eta\Uparrow"] & \boldsymbol{V} \arrow[d, "\boldsymbol{0}"] \\
\boldsymbol{V} \arrow[r, "\boldsymbol{s}"] & \boldsymbol{E}.
\end{tikzcd}$$

Here $\boldsymbol{Y}=F^{\mathbf{dMan}}_{\mathbf{Man}}(Y)$, and similarly for $\boldsymbol{V},\boldsymbol{E},\boldsymbol{s},\boldsymbol{0}$, with $0:V\to E$ the zero section. Hence $\boldsymbol{X}$ is equivalent to the 'standard model' d-manifold $\boldsymbol{S}_{V,E,s}$ of Example 1.4.4, and is a principal d-manifold.

Combining Theorems 1.4.29 and 1.4.32, Lemma 1.4.27, and Corollary 1.4.31 yields:

Corollary 1.4.33. *Let $\boldsymbol{X}$ be a d-manifold. Then $\boldsymbol{X}$ is a principal d-manifold if and only if $\dim T^*_x\underline{X}$ is bounded above for all $x\in\underline{X}$. In particular, if $\boldsymbol{X}$ is compact, then $\boldsymbol{X}$ is principal.*

Corollary 1.4.33 suggests that most interesting d-manifolds are principal, in a similar way to most interesting C^∞-schemes being affine in Remark 1.2.9(ii). Example 1.4.30 gives a d-manifold which is not principal.

1.4.8 Orientations on d-manifolds

Let X be an n-manifold. Then T^*X is a rank n vector bundle on X, so its top exterior power $\Lambda^n T^*X$ is a line bundle (rank 1 vector bundle) on X. In algebraic geometry, $\Lambda^n T^*X$ would be called the canonical bundle of X. We define an *orientation* ω on X to be an *orientation on the fibres of* $\Lambda^n T^*X$. That is, ω is an equivalence class $[\tau]$ of isomorphisms of line bundles $\tau : O_X \to \Lambda^n T^*X$, where O_X is the trivial line bundle $\mathbb{R}\times X \to X$, and τ, τ' are equivalent if $\tau' = \tau \cdot c$ for some smooth $c : X \to (0,\infty)$.

To generalize all this to d-manifolds, we will need a notion of the 'top exterior power' $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ of a virtual vector bundle $(\mathcal{E}^\bullet,\phi)$ in §1.4.3. As the definition in §4.5 is long, we will not give it, but just state its important properties:

Theorem 1.4.34. *Let $\underline{X}$ be a C^∞-scheme, and $(\mathcal{E}^\bullet,\phi)$ a virtual vector bundle on $\underline{X}$. Then in §4.5 we define a line bundle (rank 1 vector bundle) $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ on $\underline{X}$, which we call the* ***orientation line bundle*** *of $(\mathcal{E}^\bullet,\phi)$. This satisfies:*

(a) *Suppose $\mathcal{E}^1,\mathcal{E}^2$ are vector bundles on $\underline{X}$ with ranks k_1,k_2, and $\phi:\mathcal{E}^1\to\mathcal{E}^2$ is a morphism. Then $(\mathcal{E}^\bullet,\phi)$ is a virtual vector bundle of rank k_2-k_1, and there is a canonical isomorphism $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)} \cong \Lambda^{k_1}(\mathcal{E}^1)^*\otimes\Lambda^{k_2}\mathcal{E}^2$.*

(b) *Let $f^\bullet : (\mathcal{E}^\bullet,\phi)\to(\mathcal{F}^\bullet,\psi)$ be an equivalence in* $\mathrm{vvect}(\underline{X})$. *Then there is a canonical isomorphism $\mathcal{L}_{f^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet,\phi)}\to\mathcal{L}_{(\mathcal{F}^\bullet,\psi)}$ in* $\mathrm{qcoh}(\underline{X})$.

(c) *If $(\mathcal{E}^\bullet,\phi)\in\mathrm{vvect}(\underline{X})$ then $\mathcal{L}_{\mathrm{id}_\phi} = \mathrm{id}_{\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}} : \mathcal{L}_{(\mathcal{E}^\bullet,\phi)}\to\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$.*

(d) *If $f^\bullet : (\mathcal{E}^\bullet,\phi)\to(\mathcal{F}^\bullet,\psi)$ and $g^\bullet : (\mathcal{F}^\bullet,\psi)\to(\mathcal{G}^\bullet,\xi)$ are equivalences in* $\mathrm{vvect}(\underline{X})$ *then $\mathcal{L}_{g^\bullet\circ f^\bullet} = \mathcal{L}_{g^\bullet}\circ\mathcal{L}_{f^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet,\phi)}\to\mathcal{L}_{(\mathcal{G}^\bullet,\xi)}$.*

(e) *If $f^\bullet, g^\bullet : (\mathcal{E}^\bullet,\phi)\to(\mathcal{F}^\bullet,\psi)$ are 2-isomorphic equivalences in* $\mathrm{vvect}(\underline{X})$ *then $\mathcal{L}_{f^\bullet} = \mathcal{L}_{g^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet,\phi)}\to\mathcal{L}_{(\mathcal{F}^\bullet,\psi)}$.*

(f) *Let $\underline{f} : \underline{X}\to\underline{Y}$ be a morphism of C^∞-schemes, and $(\mathcal{E}^\bullet,\phi)\in\mathrm{vvect}(\underline{Y})$. Then there is a canonical isomorphism $I_{\underline{f},(\mathcal{E}^\bullet,\phi)} : \underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet,\phi)})\to\mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet,\phi)}$.*

Now we can define orientations on d-manifolds.

Definition 1.4.35. Let $\boldsymbol{X}$ be a d-manifold. Then the virtual cotangent bundle $T^*\boldsymbol{X}$ is a virtual vector bundle on $\underline{X}$ by Proposition 1.4.11(b), so Theorem 1.4.34 gives a line bundle $\mathcal{L}_{T^*\boldsymbol{X}}$ on $\underline{X}$. We call $\mathcal{L}_{T^*\boldsymbol{X}}$ the *orientation line bundle* of $\boldsymbol{X}$.

An *orientation* ω on $\boldsymbol{X}$ is an orientation on $\mathcal{L}_{T^*\boldsymbol{X}}$. That is, ω is an equivalence class $[\tau]$ of isomorphisms $\tau : \mathcal{O}_X\to\mathcal{L}_{T^*\boldsymbol{X}}$ in $\mathrm{qcoh}(\underline{X})$, where τ,τ' are equivalent if they are proportional by a smooth positive function on $\underline{X}$.

If $\omega=[\tau]$ is an orientation on $\boldsymbol{X}$, the *opposite orientation* is $-\omega=[-\tau]$, which changes the sign of the isomorphism $\tau:\mathcal{O}_X\to\mathcal{L}_{T^*\boldsymbol{X}}$. When we refer to $\boldsymbol{X}$ as an oriented d-manifold, $-\boldsymbol{X}$ will mean $\boldsymbol{X}$ with the opposite orientation, that is, $\boldsymbol{X}$ is short for $(\boldsymbol{X},\omega)$ and $-\boldsymbol{X}$ is short for $(\boldsymbol{X},-\omega)$.

Example 1.4.36. (a) Let X be an n-manifold, and $\boldsymbol{X} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X)$ the associated d-manifold. Then $\underline{X} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(X)$, $\mathcal{E}_X = 0$ and $\mathcal{F}_X = T^*\underline{X}$. So $\mathcal{E}_X,\mathcal{F}_X$ are vector bundles of ranks $0,n$. As $\Lambda^0\mathcal{E}_X\cong\mathcal{O}_X$, Theorem 1.4.34(a) gives a canonical isomorphism $\mathcal{L}_{T^*\boldsymbol{X}} \cong \Lambda^n T^*\underline{X}$. That is, $\mathcal{L}_{T^*\boldsymbol{X}}$ is isomorphic to the lift to C^∞-schemes of the line bundle $\Lambda^n T^*X$ on the manifold X.

As above, an orientation on X is an orientation on the line bundle $\Lambda^n T^*X$. Hence orientations on the d-manifold $\boldsymbol{X} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X)$ in the sense of Definition 1.4.35 are equivalent to orientations on the manifold X in the usual sense.

(b) Let V be an n-manifold, $E \to V$ a vector bundle of rank k, and $s \in C^\infty(E)$. Then Example 1.4.4 defines a 'standard model' principal d-manifold $\boldsymbol{S} = \boldsymbol{S}_{V,E,s}$, which has $\mathcal{E}_S \cong \mathcal{E}^*|_{\underline{S}}$, $\mathcal{F}_S \cong T^*\underline{V}|_{\underline{S}}$, where $\mathcal{E}, T^*\underline{V}$ are the lifts of the vector bundles E, T^*V on V to $\underline{V}$. Hence $\mathcal{E}_S, \mathcal{F}_S$ are vector bundles on $\underline{S}_{V,E,s}$ of ranks k, n, so Theorem 1.4.34(a) gives an isomorphism $\mathcal{L}_{T^*\boldsymbol{S}_{V,E,s}} \cong (\Lambda^k\mathcal{E} \otimes \Lambda^n T^*\underline{V})|_{\underline{S}}$.

Thus $\mathcal{L}_{T^*\boldsymbol{S}_{V,E,s}}$ is the lift to $\underline{S}_{V,E,s}$ of the line bundle $\Lambda^k E \otimes \Lambda^n T^*V$ over the manifold V. Therefore we may induce an orientation on the d-manifold $\boldsymbol{S}_{V,E,s}$ from an orientation on the line bundle $\Lambda^k E \otimes \Lambda^n T^*V$ over V. Equivalently, we can induce an orientation on $\boldsymbol{S}_{V,E,s}$ from an orientation on the total space of the vector bundle E^* over V, or from an orientation on the total space of E.

We can construct orientations on d-transverse fibre products of oriented d-manifolds. Note that (1.18) depends on an *orientation convention*: a different choice would change (1.18) by a sign depending on $\operatorname{vdim}\boldsymbol{X}, \operatorname{vdim}\boldsymbol{Y}, \operatorname{vdim}\boldsymbol{Z}$. Our conventions follow those of Fukaya et al. [32, §8.2] for Kuranishi spaces.

Theorem 1.4.37. *Work in the situation of Theorem* 1.4.22, *so that* $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ *are d-manifolds with* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *for* $\boldsymbol{g}, \boldsymbol{h}$ *d-transverse, where* $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ *are the projections. Then we have orientation line bundles* $\mathcal{L}_{T^*\boldsymbol{W}}, \ldots, \mathcal{L}_{T^*\boldsymbol{Z}}$ *on* $\underline{W}, \ldots, \underline{Z}$, *so* $\mathcal{L}_{T^*\boldsymbol{W}}, \underline{e}^*(\mathcal{L}_{T^*\boldsymbol{X}}), \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}), (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\boldsymbol{Z}})$ *are line bundles on* $\underline{W}$. *With a suitable choice of orientation convention, there is a canonical isomorphism*

$$\Phi : \mathcal{L}_{T^*\boldsymbol{W}} \longrightarrow \underline{e}^*(\mathcal{L}_{T^*\boldsymbol{X}}) \otimes_{\mathcal{O}_W} \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}) \otimes_{\mathcal{O}_W} (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\boldsymbol{Z}})^*. \tag{1.18}$$

Hence, if $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ *are oriented d-manifolds, then* $\boldsymbol{W}$ *also has a natural orientation, since trivializations of* $\mathcal{L}_{T^*\boldsymbol{X}}, \mathcal{L}_{T^*\boldsymbol{Y}}, \mathcal{L}_{T^*\boldsymbol{Z}}$ *induce a trivialization of* $\mathcal{L}_{T^*\boldsymbol{W}}$ *by* (1.18).

Fibre products have natural commutativity and associativity properties. When we include orientations, the orientations differ by some sign. Here is an analogue of results of Fukaya et al. [32, Lem. 8.2.3] for Kuranishi spaces.

Proposition 1.4.38. *Suppose* $\boldsymbol{V}, \ldots, \boldsymbol{Z}$ *are oriented d-manifolds,* $\boldsymbol{e}, \ldots, \boldsymbol{h}$ *are* 1*-morphisms, and all fibre products below are d-transverse. Then the following hold, in oriented d-manifolds:*

- **(a)** *For* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *and* $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *we have*

$$\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y} \simeq (-1)^{(\operatorname{vdim}\boldsymbol{X} - \operatorname{vdim}\boldsymbol{Z})(\operatorname{vdim}\boldsymbol{Y} - \operatorname{vdim}\boldsymbol{Z})} \boldsymbol{Y} \times_{\boldsymbol{h},\boldsymbol{Z},\boldsymbol{g}} \boldsymbol{X}.$$

In particular, when $\boldsymbol{Z} = *$ *so that* $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y} = \boldsymbol{X} \times \boldsymbol{Y}$ *we have*

$$\boldsymbol{X} \times \boldsymbol{Y} \simeq (-1)^{\operatorname{vdim}\boldsymbol{X}\operatorname{vdim}\boldsymbol{Y}} \boldsymbol{Y} \times \boldsymbol{X}.$$

(b) *For* $\boldsymbol{e}:\boldsymbol{V}\to\boldsymbol{Y}$, $\boldsymbol{f}:\boldsymbol{W}\to\boldsymbol{Y}$, $\boldsymbol{g}:\boldsymbol{W}\to\boldsymbol{Z}$, *and* $\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Z}$ *we have*

$$\boldsymbol{V}\times_{\boldsymbol{e},\boldsymbol{Y},\boldsymbol{f}\circ\boldsymbol{\pi_W}}\bigl(\boldsymbol{W}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{X}\bigr)\simeq\bigl(\boldsymbol{V}\times_{\boldsymbol{e},\boldsymbol{Y},\boldsymbol{f}}\boldsymbol{W}\bigr)\times_{\boldsymbol{g}\circ\boldsymbol{\pi_W},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{X}.$$

(c) *For* $\boldsymbol{e}:\boldsymbol{V}\to\boldsymbol{Y}$, $\boldsymbol{f}:\boldsymbol{V}\to\boldsymbol{Z}$, $\boldsymbol{g}:\boldsymbol{W}\to\boldsymbol{Y}$, *and* $\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Z}$ *we have*

$$\begin{aligned}&\boldsymbol{V}\times_{(\boldsymbol{e},\boldsymbol{f}),\boldsymbol{Y}\times\boldsymbol{Z},\boldsymbol{g}\times\boldsymbol{h}}(\boldsymbol{W}\times\boldsymbol{X})\simeq\\&\quad(-1)^{\mathrm{vdim}\,\boldsymbol{Z}(\mathrm{vdim}\,\boldsymbol{Y}+\mathrm{vdim}\,\boldsymbol{W})}(\boldsymbol{V}\times_{\boldsymbol{e},\boldsymbol{Y},\boldsymbol{g}}\boldsymbol{W})\times_{\boldsymbol{f}\circ\boldsymbol{\pi_V},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{X}.\end{aligned}$$

1.5 Manifolds with boundary and manifolds with corners

So far we have discussed only manifolds *without boundary* (locally modelled on $\mathbb{R}^n$). One can also consider *manifolds with boundary* (locally modelled on $[0,\infty)\times\mathbb{R}^{n-1}$) and *manifolds with corners* (locally modelled on $[0,\infty)^k\times\mathbb{R}^{n-k}$). In [55], surveyed in Chapter 5, the author studied manifolds with boundary and with corners, giving a new definition of smooth map $f:X\to Y$ between manifolds with corners X,Y, satisfying extra conditions over $\partial^k X,\partial^l Y$. This yields categories $\mathbf{Man^b},\mathbf{Man^c}$ of manifolds with boundary and with corners with good properties *as categories*.

1.5.1 Boundaries and smooth maps

The definition of an n-manifold with corners X in §5.1 involves an atlas of charts (U,ϕ) on X with $U\subseteq[0,\infty)^k\times\mathbb{R}^{n-k}$ open and $\phi:U\hookrightarrow X$ a homeomorphism with an open set in X. Apart from taking $U\subseteq[0,\infty)^k\times\mathbb{R}^{n-k}$ rather than $U\subseteq\mathbb{R}^n$, there is no difference with the usual definition of n-manifold without boundary. The definitions of the *boundary* ∂X of X in §5.1, and of *smooth map* $f:X\to Y$ between manifolds with corners in §5.2, may be surprising for readers who have not thought much about corners, so we give them here.

Definition 1.5.1. Let X be a manifold with corners, of dimension n. Then there is a natural stratification $X=\coprod_{k=0}^n S^k(X)$, where $S^k(X)$ is the *depth k stratum* of X, that is, the set of points $x\in X$ such that X near x is locally modelled on $[0,\infty)^k\times\mathbb{R}^{n-k}$ near 0. Then $S^k(X)$ is an $(n-k)$-manifold without boundary, and $\overline{S^k(X)}=\coprod_{l=k}^n S^l(X)$. The *interior* of X is $X^\circ=S^0(X)$.

A *local boundary component β of X at x* is a local choice of connected component of $S^1(X)$ near x. That is, for each sufficiently small open neighbourhood V of x in X, β gives a choice of connected component W of $V\cap S^1(X)$ with $x\in\overline{W}$, and any two such choices V,W and V',W' must be compatible in the sense that $x\in\overline{(W\cap W')}$. As a set, define the *boundary*

$$\partial X=\bigl\{(x,\beta):x\in X,\ \beta\text{ is a local boundary component for }X\text{ at }x\bigr\}.$$

Then ∂X is an $(n-1)$-manifold with corners if $n>0$, and $\partial X=\emptyset$ if $n=0$.

Define a smooth map $i_X:\partial X\to X$ by $i_X:(x,\beta)\mapsto x$.

Example 1.5.2. The manifold with corners $X = [0,\infty)^2$ has strata $S^0(X) = (0,\infty)^2$, $S^1(X) = \bigl(\{0\}\times(0,\infty)\bigr) \amalg \bigl((0,\infty)\times\{0\}\bigr)$ and $S^2(X) = \{(0,0)\}$. A point (a,b) in X has local boundary components $\{x=0\}$ if $a=0$ and $\{y=0\}$ if $b=0$. Thus

$$\begin{aligned}\partial X &= \bigl\{\bigl((x,0),\{y=0\}\bigr) : x\in[0,\infty)\bigr\} \amalg \bigl\{\bigl((0,y),\{x=0\}\bigr) : y\in[0,\infty)\bigr\}\\ &\cong [0,\infty)\amalg[0,\infty).\end{aligned}$$

Note that $i_X : \partial X \to X$ maps two points $\bigl((0,0),\{x=0\}\bigr), \bigl((0,0),\{y=0\}\bigr)$ to $(0,0)$. In general, if a manifold with corners X has $\partial^2 X \neq \emptyset$ then i_X *is not injective*, so *the boundary* ∂X *is not a subset of* X.

Definition 1.5.3. Let X, Y be manifolds with corners of dimensions m, n. A continuous map $f : X \to Y$ is called *weakly smooth* if whenever $(U,\phi),(V,\psi)$ are charts on X, Y then

$$\psi^{-1}\circ f\circ\phi : (f\circ\phi)^{-1}(\psi(V)) \longrightarrow V$$

is a smooth map from $(f\circ\phi)^{-1}(\psi(V)) \subset \mathbb{R}^m$ to $V\subset\mathbb{R}^n$.

Let $(x,\beta)\in\partial X$. A *boundary defining function for* X *at* (x,β) is a pair (V,b), where V is an open neighbourhood of x in X and $b : V\to[0,\infty)$ is a weakly smooth map, such that $\mathrm{d}b|_v : T_vV \to T_{b(v)}[0,\infty)$ is nonzero for all $v\in V$, and there exists an open neighbourhood U of (x,β) in $i_X^{-1}(V)\subseteq\partial X$, with $b\circ i_X|_U = 0$, and $i_X|_U : U \longrightarrow \bigl\{v\in V : b(v)=0\bigr\}$ is a homeomorphism.

A weakly smooth map of manifolds with corners $f : X\to Y$ is called *smooth* if it satisfies the following additional condition over $\partial X, \partial Y$. Suppose $x\in X$ with $f(x)=y\in Y$, and β is a local boundary component of Y at y. Let (V,b) be a boundary defining function for Y at (y,β). We require that either:

(i) There exists an open $x\in\tilde V\subseteq f^{-1}(V)\subseteq X$ such that $(\tilde V, b\circ f|_{\tilde V})$ is a boundary defining function for X at $(x,\tilde\beta)$, for some unique local boundary component $\tilde\beta$ of X at x; or

(ii) There exists an open $x\in W\subseteq f^{-1}(V)\subseteq X$ with $b\circ f|_W = 0$.

Form the fibre products of topological spaces

$$\begin{aligned}\partial X\times_{f\circ i_X,Y,i_Y}\partial Y &= \bigl\{\bigl((x,\tilde\beta),(y,\beta)\bigr)\in\partial X\times\partial Y : f\circ i_X(x,\tilde\beta)=y=i_Y(y,\beta)\bigr\},\\ X\times_{f,Y,i_Y}\partial Y &= \bigl\{\bigl(x,(y,\beta)\bigr)\in X\times\partial Y : f(x)=y=i_Y(y,\beta)\bigr\}.\end{aligned}$$

Define subsets $S_f\subseteq\partial X\times_Y\partial Y$ and $T_f\subseteq X\times_Y\partial Y$ by $\bigl((x,\tilde\beta),(y,\beta)\bigr)\in S_f$ in case (i) above, and $\bigl(x,(y,\beta)\bigr)\in T_f$ in case (ii) above. Define maps $s_f : S_f\to\partial X$, $t_f : T_f\to X$, $u_f : S_f\to\partial Y$, $v_f : T_f\to\partial Y$ to be the projections from the fibre products. Then S_f, T_f are open and closed in $\partial X\times_Y\partial Y, X\times_Y\partial Y$ and have the structure of manifolds with corners, with $\dim S_f = \dim X - 1$ and $\dim T_f = \dim X$, and s_t, t_f, u_f, v_f are smooth maps with s_f, t_f étale.

We write $\mathbf{Man^c}$ for the category of manifolds with corners, with morphisms smooth maps, and $\mathbf{Man^b}$ for the full subcategory of manifolds with boundary.

1.5.2 (Semi)simple maps, submersions, immersions, embeddings

In §5.4 and §5.7 we define some interesting classes of smooth maps:

Definition 1.5.4. Let $f : X \to Y$ be a smooth map of manifolds with corners.

(a) We call f *simple* if $s_f : S_f \to \partial X$ in Definition 1.5.3 is bijective.

(b) We call f *semisimple* if $s_f : S_f \to \partial X$ is injective.

(c) We call f *flat* if $T_f = \emptyset$ in Definition 1.5.3.

(d) We call f a *diffeomorphism* if it has a smooth inverse $f^{-1} : Y \to X$.

(e) We call f a *submersion* if for all $x \in S^k(X) \subseteq Y$ with $f(x) = y \in S^l(Y) \subseteq Y$, then $\mathrm{d}f|_x : T_xX \to T_{f(x)}Y$ and $\mathrm{d}f|_x : T_x(S^k(X)) \to T_{f(x)}(S^l(Y))$ are surjective. Submersions are automatically semisimple. We call f an *s-submersion* if it is a simple submersion.

(f) We call f an *immersion* if $\mathrm{d}f|_x : T_xX \to T_{f(x)}Y$ is injective for all $x \in X$. We call f an *s-immersion* (or *sf-immersion*) if f is also simple (or simple and flat). We call f an *embedding* (or *s-embedding*, or *sf-embedding*) if f is an immersion (or s-immersion, or sf-immersion), and $f : X \to f(X)$ is a homeomorphism with its image.

For manifolds without boundary, one considers immersed or embedded submanifolds. Part (f) gives six different notions of submanifolds X of manifolds with corners Y: *immersed*, *s-immersed*, *sf-immersed*, *embedded*, *s-embedded* and *sf-embedded submanifolds*.

Example 1.5.5. (i) The inclusion $i : [0,\infty) \hookrightarrow \mathbb{R}$ is an embedding. It is semisimple and flat, but not simple, as $s_i : S_i \to \partial[0,\infty)$ maps $\emptyset \to \{0\}$, and is not surjective, so i is not an s- or sf-embedding. Thus $[0,\infty)$ is an embedded submanifold of $\mathbb{R}$, but not an s- or sf-embedded submanifold.

(ii) The map $f : [0,\infty) \to [0,\infty)^2$ mapping $f : x \mapsto (x,x)$ is an embedding. It is flat, but not semisimple, as $s_f : S_f \to \partial[0,\infty)$ maps two points to one point, and is not injective. Hence f is not an s- or sf-embedding, and $\bigl\{(x,x) : x \in [0,\infty)\bigr\}$ is an embedded submanifold of $[0,\infty)^2$, but not s- or sf-embedded.

(iii) The inclusion $i : \{0\} \hookrightarrow [0,\infty)$ has $\mathrm{d}i|_0$ injective, so it is an embedding. It is simple, but not flat, as $T_i = \bigl\{\bigl(0,(0,\{x=0\})\bigr)\bigr\} \neq \emptyset$. Thus i is an s-embedding, but not an sf-embedding. Hence $\{0\}$ is an s-embedded but not sf-embedded submanifold of $[0,\infty)$.

(iv) Let X be a manifold with corners with $\partial X \neq \emptyset$. Then $i_X : \partial X \to X$ is an immersion. Also $s_{i_X} : S_{i_X} \to \partial^2 X$ is a bijection, so i_X is simple, but $T_{i_X} \cong \partial X \neq \emptyset$, so i_X is not flat. Hence i_X is an s-immersion, but not an sf-immersion. If $\partial^2 X = \emptyset$ then i_X is an s-embedding, but not an sf-embedding.

(v) Let $f : [0,\infty) \to \mathbb{R}$ be smooth. Define $g : [0,\infty) \to [0,\infty) \times \mathbb{R}$ by $g(x) = (x, f(x))$. Then g is an sf-embedding, and $\Gamma_f = \bigl\{(x, f(x) : x \in [0,\infty)\bigr\}$ is an sf-embedded submanifold of $[0,\infty) \times \mathbb{R}$.

Simple and semisimple maps have a property of lifting to boundaries:

Proposition 1.5.6. *Let $f : X \to Y$ be a semisimple map of manifolds with corners. Then there exists a natural decomposition $\partial X = \partial_+^f X \amalg \partial_-^f X$ with $\partial_\pm^f X$ open and closed in ∂X, and semisimple maps $f_+ = f \circ i_X|_{\partial_+^f X} : \partial_+^f X \to Y$ and $f_- : \partial_-^f X \to \partial Y$, such that the following commutes in* $\mathbf{Man^c}$*:*

$$\begin{tikzcd} \partial_-^f X \arrow[rr, "f_-"] \arrow[d, "i_X|_{\partial_-^f X}"] && \partial Y \arrow[d, "i_Y"'] \\ X \arrow[rr, "f"] && Y. \end{tikzcd} \tag{1.19}$$

If f is also flat, then (1.19) *is a Cartesian square, so that $\partial_-^f X \cong X \times_Y \partial Y$. If f is simple then $\partial_+^f X = \emptyset$ and $\partial_-^f X = \partial X$. If f is simple, flat, a submersion, or an s-submersion, then $f_\pm$ are also simple, ..., s-submersions, respectively.*

In fact we define $\partial_-^f X = s_f(S_f)$, so that $s_f : S_f \to \partial_-^f X$ is a bijection since s_f is injective as f is semisimple, and then $f_- = u_f \circ s_f^{-1}$, using the notation of Definition 1.5.3. If $f : X \to Y$ is simple then $f_- : \partial X \to \partial Y$ is also simple, so $f_{-^k} : \partial^k X \to \partial^k Y$ is simple for $k = 1, 2, \ldots$. If f is also flat then f_{-^k} is flat and $\partial^k X \cong X \times_Y \partial^k Y$. A smooth map $f : X \to Y$ is flat if and only if $f(X^\circ) \subseteq Y^\circ$, or equivalently, if $f : X \to Y$ and $i_Y : \partial Y \to Y$ are transverse.

(S-)submersions are locally modelled on projections $\pi_X : X \times Y \to X$:

Proposition 1.5.7. (a) *Let X, Y be manifolds with corners. Then the projection $\pi_X : X \times Y \to X$ is a submersion, and an s-submersion if $\partial Y = \emptyset$.*

(b) *Let $f : X \to Y$ be a submersion of manifolds with corners, and $x \in X$ with $f(x) = y \in Y$. Then there exist open neighbourhoods V of x in X and W of y in Y with $f(V) = W$, a manifold with corners Z, and a diffeomorphism $V \cong W \times Z$ which identifies $f|_V : V \to W$ with $\pi_W : W \times Z \to W$. If f is an s-submersion then $\partial Z = \emptyset$.*

S-immersions and sf-immersions are also locally modelled on products:

Proposition 1.5.8. (a) *Let X be a manifold with corners and $0 \leqslant k \leqslant n$. Then* $\mathrm{id}_X \times 0 : X \to X \times \bigl([0,\infty)^k \times \mathbb{R}^{n-k}\bigr)$ *mapping $x \mapsto (x, 0)$ is an s-embedding, and an sf-embedding if $k = 0$.*

(b) *Let $f : X \to Y$ be an s-immersion of manifolds with corners, and $x \in X$ with $f(x) = y \in Y$. Then there exist open neighbourhoods V of x in X and W of y in Y with $f(V) \subseteq W$, an open neighbourhood Z of 0 in $[0,\infty)^k \times \mathbb{R}^{n-k}$, and a diffeomorphism $W \cong V \times Z$ which identifies $f|_V : V \to W$ with* $\mathrm{id}_V \times 0 : V \to V \times Z$*. If f is an sf-immersion then $k = 0$.*

Example 1.5.5(ii) shows general immersions are not modelled on products.

1.5.3 Corners and the corner functors

As in §5.5, we define the *k-corners* $C_k(X)$ of a manifold with corners X.

Definition 1.5.9. Let X be an n-manifold with corners. Applying ∂ repeatedly gives manifolds with corners $\partial X, \partial^2 X, \ldots$. There is a natural identification

$$\begin{aligned}\partial^k X \cong \bigl\{(x,\beta_1,\ldots,\beta_k) : x \in X,\ \beta_1,\ldots,\beta_k \text{ are distinct}\\ \text{local boundary components for } X \text{ at } x\bigr\}.\end{aligned} \tag{1.20}$$

Using (1.20), we see that the symmetric group S_k of permutations of $\{1,\ldots,k\}$ has a natural, free action on $\partial^k X$ by diffeomorphisms, given by

$$\sigma : (x,\beta_1,\ldots,\beta_k) \longmapsto (x,\beta_{\sigma(1)},\ldots,\beta_{\sigma(k)}).$$

Define the *k-corners* of X, as a set, to be

$$\begin{aligned}C_k(X) = \bigl\{(x,\{\beta_1,\ldots,\beta_k\}) : x \in X,\ \beta_1,\ldots,\beta_k \text{ are distinct}\\ \text{local boundary components for } X \text{ at } x\bigr\}.\end{aligned}$$

Then $C_k(X)$ is naturally a manifold with corners of dimension $n-k$, with $C_k(X) \cong \partial^k X/S_k$. The interior $C_k(X)^\circ$ is naturally diffeomorphic to $S^k(X)$. We have natural diffeomorphisms $C_0(X) \cong X$ and $C_1(X) \cong \partial X$.

A surprising fact about manifolds with corners X is that the disjoint union $C(X) := \coprod_{k=0}^{\dim X} C_k(X)$ has strong functorial properties. Since $C(X)$ is not a manifold with corners, it is helpful to enlarge our category $\mathbf{Man^c}$:

Definition 1.5.10. Write $\check{\mathbf{M}}\mathbf{an^c}$ for the category whose objects are disjoint unions $\coprod_{m=0}^\infty X_m$, where X_m is a manifold with corners of dimension m, and whose morphisms are continuous maps $f : \coprod_{m=0}^\infty X_m \to \coprod_{n=0}^\infty Y_n$, such that $f|_{X_m \cap f^{-1}(Y_n)} : \bigl(X_m \cap f^{-1}(Y_n)\bigr) \to Y_n$ is a smooth map of manifolds with corners for all $m,n \geqslant 0$.

Definition 1.5.11. Define *corner functors* $C, \hat C : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$ by $C(X) = \hat C(X) = \coprod_{k=0}^{\dim X} C_k(X)$ on objects, and on morphisms $f : X \to Y$ in $\mathbf{Man^c}$,

$$\begin{aligned}
&C(f) : \bigl(x,\{\tilde\beta_1,\ldots,\tilde\beta_i\}\bigr) \longmapsto \bigl(y,\{\beta_1,\ldots,\beta_j\}\bigr), \quad \text{where } y = f(x),\\
&\{\beta_1,\ldots,\beta_j\} = \bigl\{\beta : \bigl((x,\tilde\beta_l),(y,\beta)\bigr) \in S_f,\ \text{some } l = 1,\ldots,i\bigr\},\\
&\hat C(f) : \bigl(x,\{\tilde\beta_1,\ldots,\tilde\beta_i\}\bigr) \longmapsto \bigl(y,\{\beta_1,\ldots,\beta_j\}\bigr), \quad \text{where } y = f(x),\\
&\{\beta_1,\ldots,\beta_j\} = \bigl\{\beta : \bigl((x,\tilde\beta_l),(y,\beta)\bigr) \in S_f,\ l = 1,\ldots,i\bigr\} \cup \bigl\{\beta : \bigl(x,(y,\beta)\bigr) \in T_f\bigr\}.
\end{aligned}$$

Write $C_j^{f,k}(X) = C_j(X) \cap C(f)^{-1}(C_k(Y))$ and $C_j^k(f) = C(f)|_{C_j^{f,k}(X)} : C_j^{f,k}(X) \to C_k(Y)$ for all j,k, and similarly for $\hat C_j^{f,k}(X), \hat C_j^k(f)$. Then $C_j^k(f)$ and $\hat C_j^k(f)$ are smooth maps of manifolds with corners. Note that $C_0^{f,0}(X) = C_0(X) \cong X$ and $C_0(Y) \cong Y$, and these isomorphisms identify $C_0^0(f) : C_0(X) \to C_0(Y)$ with $f : X \to Y$.

It turns out that $C, \hat C$ are both functors $\mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$. Furthermore:

(i) For each $X \in \mathbf{Man^c}$ we have a natural diffeomorphism $C(\partial X) \cong \partial C(X)$ identifying $C(i_X) : C(\partial X) \to C(X)$ with $i_{C(X)} : \partial C(X) \to C(X)$

(ii) For all X, Y in $\mathbf{Man^c}$ we have a natural diffeomorphism $C(X \times Y) \cong C(X) \times C(Y)$. These diffeomorphisms commute with product morphisms and direct product morphisms in $\mathbf{Man^c}, \check{\mathbf{M}}\mathbf{an^c}$.

(iii) If $g : X \to Z$ and $h : Y \to Z$ are *strongly transverse* maps in $\mathbf{Man^c}$ then C maps the fibre product $X \times_{g,Z,h} Y$ in $\mathbf{Man^c}$ to the fibre product $C(X) \times_{C(g),C(Z),C(h)} C(Y)$ in $\check{\mathbf{M}}\mathbf{an^c}$.

(iv) If $f : X \to Y$ is semisimple, then $C(f)$ maps $C_k(X) \to \coprod_{l=0}^{k} C_l(Y)$ for all $k \geqslant 0$. The natural diffeomorphisms $C_1(X) \cong \partial X$, $C_0(Y) \cong Y$ and $C_1(Y) \cong \partial Y$ identify $C_1^{f,0}(X) \cong \partial_+^f X$, $C_1^0(f) \cong f_+$, $C_1^{f,1}(X) \cong \partial_-^f X$ and $C_1^1(f) \cong f_-$. If f is simple then $C(f)$ maps $C_k(X) \to C_k(Y)$ for all $k \geqslant 0$.

The analogues hold for $\hat{C}$, except for (iv) and the last part of (i).

1.5.4 (Strong) transversality and fibre products

In §5.6 we discuss conditions for fibre products to exist in $\mathbf{Man^c}$.

Definition 1.5.12. Let $g : X \to Z$, $h : Y \to Z$ be smooth maps of manifolds with corners. We call g, h *transverse* if whenever $x \in S^j(X) \subseteq X$, $y \in S^k(Y) \subseteq Y$ and $z \in S^l(Z) \subseteq Z$ with $g(x) = h(y) = z$, then $T_z Z = \mathrm{d}g|_x(T_x X) + \mathrm{d}h|_y(T_y Y)$ and $T_z(S^l(Z)) = \mathrm{d}g|_x(T_x(S^j(X))) + \mathrm{d}h|_y(T_y(S^k(Y)))$.

We call g, h *strongly transverse* if they are transverse, and whenever there are points in $C_j(X), C_k(Y), C_l(Z)$ with

$$C(g)(x, \{\beta_1, \ldots, \beta_j\}) = C(h)(y, \{\tilde{\beta}_1, \ldots, \tilde{\beta}_k\}) = (z, \{\dot{\beta}_1, \ldots, \dot{\beta}_l\})$$

we have either $j + k > l$ or $j = k = l = 0$.

If one of g, h is a submersion then g, h are strongly transverse. It is well known that transverse fibre products of manifolds without boundary exist. Here is the (more difficult to prove) analogue for manifolds with corners.

Theorem 1.5.13. *Let $g : X \to Z$, $h : Y \to Z$ be transverse smooth maps of manifolds with corners. Then a fibre product $W = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^c}$.*

As a topological space, the fibre product in Theorem 1.5.13 is just the topological fibre product $W = \{(x, y) \in X \times Y : g(x) = h(y)\}$. In general, the boundary ∂W is difficult to describe explicitly: it is the quotient of a subset of $(\partial X \times_Z Y) \amalg (X \times_Z \partial Y)$ by an equivalence relation. Here are some special cases in which we can give an explicit formula for ∂W.

Proposition 1.5.14. *Let $g : X \to Z$, $h : Y \to Z$ be transverse smooth maps in $\mathbf{Man^c}$, so that $X \times_{g,Z,h} Y$ exists by Theorem 1.5.13. Then:*

(a) *If $\partial Z = \emptyset$ then*

$$\partial\big(X \times_{g,Z,h} Y\big) \cong \big(\partial X \times_{g \circ i_X, Z, h} Y\big) \amalg \big(X \times_{g,Z,h \circ i_Y} \partial Y\big). \qquad (1.21)$$

(b) *If g is semisimple then*

$$\partial\bigl(X \times_{g,Z,h} Y\bigr) \cong \bigl(\partial_+^g X \times_{g_+,Z,h} Y\bigr) \amalg \bigl(X \times_{g,Z,h\circ i_Y} \partial Y\bigr). \tag{1.22}$$

(c) *If both g, h are semisimple then*

$$\begin{aligned}&\partial\bigl(X \times_{g,Z,h} Y\bigr) \cong \\ &\quad \bigl(\partial_+^g X \times_{g_+,Z,h} Y\bigr) \amalg \bigl(X \times_{g,Z,h_+} \partial_+^h Y\bigr) \amalg \bigl(\partial_-^g X \times_{g_-,\partial Z,h_-} \partial_-^h Y\bigr).\end{aligned} \tag{1.23}$$

Here all fibre products in (1.21)–(1.23) *are transverse, and so exist.*

For *strongly* transverse smooth maps, fibre products commute with the corner functors $C, \hat{C} : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$. Since $C_1(W) \cong \partial W$, equation (1.24) with $i = 1$ gives another explicit description of ∂W in this case.

Theorem 1.5.15. *Let $g : X \to Z$, $h : Y \to Z$ be strongly transverse smooth maps of manifolds with corners, and write W for the fibre product $X \times_{g,Z,h} Y$ given by Theorem* 1.5.13*. Then there is a canonical diffeomorphism*

$$C_i(W) \cong \coprod_{j,k,l \geqslant 0 : i = j+k-l} C_j^{g,l}(X) \times_{C_j^l(g), C_l(Z), C_k^l(h)} C_k^{h,l}(Y) \tag{1.24}$$

for all $i \geqslant 0$, where the fibre products are all transverse and so exist. Hence

$$C(W) \cong C(X) \times_{C(g),C(Z),C(h)} C(Y) \quad \textit{in } \check{\mathbf{M}}\mathbf{an^c}.$$

1.5.5 Orientations on manifolds with corners

In §5.8 we discuss orientations on manifolds with corners.

Definition 1.5.16. Let X be an n-manifold with corners. An *orientation* ω on X is an orientation on the fibres of the real line bundle $\Lambda^n T^* X$ over X. That is, ω is an equivalence class $[\tau]$ of isomorphisms $\tau : O_X \to \Lambda^n T^* X$, where $O_X = \mathbb{R} \times X \to X$ is the trivial line bundle on X, and τ, τ' are equivalent if $\tau' = \tau \cdot c$ for some smooth $c : X \to (0, \infty)$.

If $\omega = [\tau]$ is an orientation, we write $-\omega$ for the *opposite orientation* $[-\tau]$.

We call the pair (X, ω) an *oriented manifold*. Usually we suppress the orientation ω, and just refer to X as an oriented manifold. When X is an oriented manifold, we write $-X$ for X with the opposite orientation.

If X, Y, Z are oriented manifolds with corners, then we can define orientations on boundaries ∂X, products $X \times Y$, and transverse fibre products $X \times_Z Y$. To do this requires a choice of *orientation convention*. Our orientation conventions are given in Convention 5.35. Having fixed an orientation convention, natural isomorphisms of manifolds with corners such as $X \times_Z Y \cong Y \times_Z X$ lift to isomorphisms of oriented manifolds of corners, modified by signs depending on the dimensions. For example, if $g : X \to Z$ and $h : Y \to Z$ are transverse maps of oriented manifolds with corners then

$$X \times_{g,Z,h} Y \cong (-1)^{(\dim X - \dim Z)(\dim Y - \dim Z)} Y \times_{h,Z,g} X,$$

and with orientations equations (1.21)–(1.23) become

$$\begin{aligned}
\partial\bigl(X \times_{g,Z,h} Y\bigr) &\cong \bigl(\partial X \times_{g\circ i_X,Z,h} Y\bigr) \amalg (-1)^{\dim X + \dim Z}\bigl(X \times_{g,Z,h\circ i_Y} \partial Y\bigr),\\
\partial\bigl(X \times_{g,Z,h} Y\bigr) &\cong \bigl(\partial_+^g X \times_{g_+,Z,h} Y\bigr) \amalg (-1)^{\dim X + \dim Z}\bigl(X \times_{g,Z,h\circ i_Y} \partial Y\bigr),\\
\partial\bigl(X \times_{g,Z,h} Y\bigr) &\cong \bigl(\partial_+^g X \times_{g_+,Z,h} Y\bigr) \amalg (-1)^{\dim X + \dim Z}\bigl(X \times_{h,Z,h_+} \partial_+^h Y\bigr)\\
&\qquad \amalg \bigl(\partial_-^g X \times_{g_-,\partial Z,h_-} \partial_-^h Y\bigr).
\end{aligned}$$

1.5.6 Fixed points of finite groups in manifolds with corners

In §5.5 we study the fixed point locus X^Γ of a group Γ acting on a manifold with corners X. These are related to orbifold strata $\mathfrak{X}^\Gamma$ of orbifolds with corners $\mathfrak{X}$, which we will discuss in §1.12.5. Here is our main result.

Proposition 1.5.17. *Suppose X is a manifold with corners, Γ a finite group, and $r : \Gamma \to \mathrm{Aut}(X)$ an action of Γ on X by diffeomorphisms. Applying the corner functor C of §1.5.3 gives an action $C(r) : \Gamma \to \mathrm{Aut}(C(X))$ of Γ on $C(X)$ by diffeomorphisms. Write $X^\Gamma, C(X)^\Gamma$ for the subsets of $X, C(X)$ fixed by Γ, and $j_{X,\Gamma} : X^\Gamma \to X$ for the inclusion. Then:*

- **(a)** *X^Γ has the structure of an object in $\check{\mathbf{Man}}^{\mathbf{c}}$ (a disjoint union of manifolds with corners of different dimensions, as in §1.5.3) in a unique way, such that $j_{X,\Gamma} : X^\Gamma \to X$ is an embedding. This $j_{X,\Gamma}$ is flat, but need not be (semi)simple.*
- **(b)** *By* **(a)** *we have a smooth map $C(j_{X,\Gamma}) : C(X^\Gamma) \to C(X)$. This $C(j_{X,\Gamma})$ is a diffeomorphism $C(X^\Gamma) \to C(X)^\Gamma$. As $j_{X,\Gamma}$ need not be simple, $C(j_{X,\Gamma})$ need not map $C_k(X^\Gamma) \to C_k(X)$ for $k > 0$.*
- **(c)** *By* **(b)**, *$C(j_{X,\Gamma})$ identifies $C_1(X^\Gamma) \cong \partial(X^\Gamma)$ with a subset of $C(X)^\Gamma \subseteq C(X)$. This gives the following description of $\partial(X^\Gamma)$:*

$$\begin{aligned}
\partial(X^\Gamma) \cong \bigl\{(x, \{\beta_1, \ldots, \beta_k\}) \in C_k(X) : x \in X^\Gamma,\ k \geqslant 1,\ \beta_1, \ldots, \beta_k\\
\text{are distinct local boundary components for } X \text{ at } x,\\
\text{and } \Gamma \text{ acts transitively on } \{\beta_1, \ldots, \beta_k\}\bigr\}.
\end{aligned}$$

- **(d)** *Now suppose Y is a manifold with corners with an action of Γ, and $f : X \to Y$ is a Γ-equivariant smooth map. Then X^Γ, Y^Γ are objects in $\check{\mathbf{Man}}^{\mathbf{c}}$ by* **(a)**, *and $f^\Gamma := f|_{X^\Gamma} : X^\Gamma \to Y^\Gamma$ is a morphism in $\check{\mathbf{Man}}^{\mathbf{c}}$.*

Example 1.5.18. Let $\Gamma = \{1, \sigma\}$ with $\sigma^2 = 1$, so that $\Gamma \cong \mathbb{Z}_2$, and let Γ act on $X = [0,\infty)^2$ by $\sigma : (x_1, x_2) \mapsto (x_2, x_1)$. Then $X^\Gamma = \bigl\{(x,x) : x \in [0,\infty)\bigr\} \cong [0,\infty)$, a manifold with corners, and the inclusion $j_{X,\Gamma} : X^\Gamma \to X$ is

$j_{X,\Gamma}:[0,\infty)\to[0,\infty)^2$, $j_{X,\Gamma}:x\mapsto(x,x)$, a smooth, flat embedding, which is not semisimple. We have $\partial X=\partial\bigl([0,\infty)^2\bigr)\cong[0,\infty)\amalg[0,\infty)$, where Γ acts freely on ∂X by exchanging the two copies of $[0,\infty)$. Hence $(\partial X)^\Gamma=\emptyset$, but $\partial(X^\Gamma)$ is a point $*$, so in this case $(\partial X)^\Gamma\not\cong\partial(X^\Gamma)$. Also $C_2(X)=\bigl\{\bigl(0,\{\{x_1=0\},\{x_2=0\}\}\bigr)\bigr\}$ is a single point, which is Γ-invariant, and $C(j_{X,\Gamma}):C(X^\Gamma)\to C(X)^\Gamma$ identifies $(0,\{\{x=0\}\})\in C_1(X^\Gamma)\cong\partial X$ with this point in $C_2(X)^\Gamma$.

If a finite group Γ acts on a manifold with corners X then as in Proposition 1.5.17(b) we have $C(X)^\Gamma\cong C(X^\Gamma)$, but as in Example 1.5.18 in general we do not have $(\partial X)^\Gamma\cong\partial(X^\Gamma)$, but only $(\partial X)^\Gamma\subseteq\partial(X^\Gamma)$. Thus for fixed point loci, corners have more functorial behaviour than boundaries.

1.6 D-spaces with corners

The goal of Chapters 6 and 7 is to construct and study a well-behaved 2-category $\mathbf{dMan^c}$ of *d-manifolds with corners*, a derived version of $\mathbf{Man^c}$. It is tempting to try and define $\mathbf{dMan^c}$ as a 2-subcategory of d-spaces $\mathbf{dSpa}$, but this turns out not to be a good idea. For example, the natural functor $F_{\mathbf{Man^c}}^{\mathbf{dSpa}}:\mathbf{Man^c}\to\mathbf{dSpa}$ is not full, as 1-morphisms $\boldsymbol{f}:F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(X)\to F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(Y)$ correspond to weakly smooth rather than smooth maps $f:X\to Y$, in the notation of §1.5.1.

Therefore we begin in Chapter 6 by defining a 2-category $\mathbf{dSpa^c}$ of *d-spaces with corners*, and then define $\mathbf{dMan^c}$ in Chapter 7 as a 2-subcategory of $\mathbf{dSpa^c}$. Many properties of manifolds with corners in §1.5 work for d-spaces with corners — for example, boundaries ∂X, simple, semisimple and flat maps $f:X\to Y$, decompositions $\partial X=\partial_+^f X\amalg\partial_-^f X$ and semisimple maps $f_+:\partial_+^f X\to Y$ and $f_-:\partial_-^f X\to\partial Y$ when f is semisimple, and the corner functors $C,\hat C$.

1.6.1 Outline of the definition of the 2-category $\mathbf{dSpa^c}$

The definition of the 2-category of d-spaces with corners $\mathbf{dSpa^c}$ in §6.1 is long, complicated, and not that enlightening. So here we just sketch the main ideas.

Let X be a manifold with corners. Then it has a boundary ∂X with a proper smooth map $i_X:\partial X\to X$. On ∂X we have an exact sequence

$$0\longrightarrow\mathcal{N}_X\longrightarrow i_X^*(T^*X)\xrightarrow{(\mathrm{d}i_X)^*}T^*(\partial X)\longrightarrow 0,\tag{1.25}$$

where $\mathcal{N}_X$ is the conormal bundle of ∂X in X. The line bundle $\mathcal{N}_X$ has a natural orientation ω_X induced by outward-pointing normal vectors to ∂X in X.

Thus, for each manifold with corners X we have a quadruple $(X,\partial X,i_X,\omega_X)$. D-spaces with corners are based on this idea. A *d-space with corners* $\mathbf{X}$ is a quadruple $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i_X},\omega_{\mathbf{X}})$ where $\boldsymbol{X},\boldsymbol{\partial X}$ are d-spaces, and $\boldsymbol{i_X}:\boldsymbol{\partial X}\to\boldsymbol{X}$ is a proper 1-morphism, and we have an exact sequence in $\mathrm{qcoh}(\underline{\partial X})$:

$$0\longrightarrow\mathcal{N}_{\mathbf{X}}\xrightarrow{\nu_{\mathbf{X}}}\underline{i}_{\mathbf{X}}^*(\mathcal{F}_X)\xrightarrow{i_{\mathbf{X}}^2}\mathcal{F}_{\partial X}\longrightarrow 0,\tag{1.26}$$

with $\mathcal{N}_{\mathbf{X}}$ a line bundle, and $\omega_{\mathbf{X}}$ is an orientation on $\mathcal{N}_{\mathbf{X}}$. These $\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i_X},\omega_{\mathbf{X}}$ must satisfy some complicated conditions in §6.1, that we will not give. They require $\boldsymbol{\partial X}$ to be locally equivalent to a fibre product $\boldsymbol{X}\times_{\boldsymbol{[0,\infty)}}\boldsymbol{*}$ in $\mathbf{dSpa}$.

If $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}},\omega_{\mathbf{X}})$ and $\mathbf{Y}=(\boldsymbol{Y},\boldsymbol{\partial Y},\boldsymbol{i}_{\mathbf{Y}},\omega_{\mathbf{Y}})$ are d-spaces with corners, a 1-*morphism* $\boldsymbol{f}:\mathbf{X}\rightarrow\mathbf{Y}$ in $\mathbf{dSpa^c}$ is a 1-morphism $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ in $\mathbf{dSpa}$ satisfying extra conditions over $\boldsymbol{\partial X},\boldsymbol{\partial Y}$, which are analogous to the extra conditions for a weakly smooth map of manifolds with corners $f:X\rightarrow Y$ to be smooth in Definition 1.5.3.

If $\boldsymbol{f}:\mathbf{X}\rightarrow\mathbf{Y}$ is a 1-morphism in $\mathbf{dSpa^c}$, we can form the C^∞-scheme fibre products $\underline{\partial X}\times_{\underline{f}\circ\underline{i}_{\mathbf{X}},\underline{Y},\underline{i}_{\mathbf{Y}}}\underline{\partial Y}$ and $\underline{X}\times_{\underline{f},\underline{Y},\underline{i}_{\mathbf{Y}}}\underline{\partial Y}$. As for S_f,T_f in Definition 1.5.3, we can define open and closed C^∞-subschemes $\underline{S}_{\boldsymbol{f}}\subseteq\underline{\partial X}\times_{\underline{Y}}\underline{\partial Y}$ and $\underline{T}_{\boldsymbol{f}}\subseteq\underline{X}\times_{\underline{Y}}\underline{\partial Y}$, and define C^∞-scheme morphisms $\underline{s}_{\boldsymbol{f}}:\underline{S}_{\boldsymbol{f}}\rightarrow\underline{\partial X}$, $\underline{t}_{\boldsymbol{f}}:\underline{T}_{\boldsymbol{f}}\rightarrow\underline{X}$, $\underline{u}_{\boldsymbol{f}}:\underline{S}_{\boldsymbol{f}}\rightarrow\underline{\partial Y}$ and $\underline{v}_{\boldsymbol{f}}:\underline{T}_{\boldsymbol{f}}\rightarrow\underline{\partial Y}$ to be the projections from the fibre products. Then $\underline{s}_{\boldsymbol{f}},\underline{t}_{\boldsymbol{f}}$ are étale.

If $\boldsymbol{f},\boldsymbol{g}:\mathbf{X}\rightarrow\mathbf{Y}$ are 1-morphisms in $\mathbf{dSpa^c}$, a 2-*morphism* $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ in $\mathbf{dSpa^c}$ is a 2-morphism $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ in $\mathbf{dSpa}$ such that $\underline{S}_{\boldsymbol{f}}=\underline{S}_{\boldsymbol{g}}$, $\underline{T}_{\boldsymbol{f}}=\underline{T}_{\boldsymbol{g}}$ and extra vanishing conditions hold on η over $\underline{S}_{\boldsymbol{f}},\underline{T}_{\boldsymbol{f}}$. Identity 1- and 2-morphisms in $\mathbf{dSpa^c}$, and the compositions of 1- and 2-morphisms in $\mathbf{dSpa^c}$, are all given by identities and compositions in $\mathbf{dSpa}$.

A d-space with corners $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}},\omega_{\mathbf{X}})$ is called a *d-space with boundary* if $\boldsymbol{i}_{\mathbf{X}}:\boldsymbol{\partial X}\rightarrow\boldsymbol{X}$ is injective, and a *d-space without boundary* if $\boldsymbol{\partial X}=\boldsymbol{\emptyset}$. We write $\mathbf{dSpa^b}$ for the full 2-subcategory of d-spaces with boundary, and $\mathbf{d\bar{S}pa}$ for the full 2-subcategory of d-spaces without boundary, in $\mathbf{dSpa^c}$. There is an isomorphism of 2-categories $F_{\mathbf{dSpa}}^{\mathbf{dSpa^c}}:\mathbf{dSpa}\rightarrow\mathbf{d\bar{S}pa}$ mapping $\boldsymbol{X}\mapsto\mathbf{X}=(\boldsymbol{X},\boldsymbol{\emptyset},\boldsymbol{\emptyset},\emptyset)$ on objects, $\boldsymbol{f}\mapsto\boldsymbol{f}$ on 1-morphisms and $\eta\mapsto\eta$ on 2-morphisms. So we can consider d-spaces to be examples of d-spaces with corners.

Remark 1.6.1. If X is a manifold with corners then the orientation ω_X on $\mathcal{N}_X$ is determined uniquely by $X,\partial X,i_X$. But there are examples of d-spaces with corners $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}},\omega_{\mathbf{X}})$ in which $\omega_{\mathbf{X}}$ is not determined by $\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}}$, and really is extra data. We include $\omega_{\mathbf{X}}$ in the definition so that orientations of d-manifolds with corners behave well in relation to boundaries. If we had omitted $\omega_{\mathbf{X}}$ from the definition, then there would exist examples of oriented d-manifolds with corners $\mathbf{X}$ such that $\partial\mathbf{X}$ is not orientable.

For each d-space with corners $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}},\omega_{\mathbf{X}})$, in §6.2 we define a d-space with corners $\partial\mathbf{X}=(\boldsymbol{\partial X},\boldsymbol{\partial^2 X},\boldsymbol{i}_{\partial\mathbf{X}},\omega_{\partial\mathbf{X}})$ called the *boundary* of $\mathbf{X}$, and show that $\boldsymbol{i}_{\mathbf{X}}:\partial\mathbf{X}\rightarrow\mathbf{X}$ is a 1-morphism in $\mathbf{dSpa^c}$. Motivated by (1.20) when $k=2$, the d-space $\boldsymbol{\partial^2 X}$ in $\partial\mathbf{X}$ is given by

$$\boldsymbol{\partial^2 X}\simeq\bigl(\boldsymbol{\partial X}\times_{\boldsymbol{i}_{\mathbf{X}},\boldsymbol{X},\boldsymbol{i}_{\mathbf{X}}}\boldsymbol{\partial X}\bigr)\setminus\boldsymbol{\Delta}_{\boldsymbol{\partial X}}(\boldsymbol{\partial X}),\tag{1.27}$$

where $\Delta_{\boldsymbol{\partial X}}:\boldsymbol{\partial X}\rightarrow\boldsymbol{\partial X}\times_{\boldsymbol{X}}\boldsymbol{\partial X}$ is the diagonal 1-morphism. The 1-morphism $\boldsymbol{i}_{\partial\mathbf{X}}:\boldsymbol{\partial^2 X}\rightarrow\boldsymbol{\partial X}$ is projection to the first factor in the fibre product. There is a natural isomorphism $\mathcal{N}_{\partial\mathbf{X}}\cong\underline{i}_{\mathbf{X}}^*(\mathcal{N}_{\mathbf{X}})$, and the orientation $\omega_{\partial\mathbf{X}}$ on $\mathcal{N}_{\partial\mathbf{X}}$ is defined to correspond to the orientation $\underline{i}_{\mathbf{X}}^*(\omega_{\mathbf{X}})$ on $\underline{i}_{\mathbf{X}}^*(\mathcal{N}_{\mathbf{X}})$.

1.6.2 Simple, semisimple and flat 1-morphisms

In §6.3 we generalize the material on simple, semisimple, and flat maps of manifolds with corners in §1.5.2 to d-spaces with corners. Here are the analogues of

Definition 1.5.4(a)–(c) and Proposition 1.5.6.

Definition 1.6.2. Let $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ be a 1-morphism of d-spaces with corners.

(a) We call $\boldsymbol{f}$ *simple* if $\underline{s}_{\boldsymbol{f}}:\underline{S}_{\boldsymbol{f}}\to\underline{\partial X}$ is bijective.

(b) We call $\boldsymbol{f}$ *semisimple* if $\underline{s}_{\boldsymbol{f}}:\underline{S}_{\boldsymbol{f}}\to\underline{\partial X}$ is injective.

(c) We call $\boldsymbol{f}$ *flat* if $\underline{T}_{\boldsymbol{f}}=\emptyset$.

Theorem 1.6.3. *Let $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ be a semisimple 1-morphism of d-spaces with corners. Then there exists a natural decomposition $\partial\mathbf{X}=\partial_+^{\boldsymbol{f}}\mathbf{X}\amalg\partial_-^{\boldsymbol{f}}\mathbf{X}$ with $\partial_\pm^{\boldsymbol{f}}\mathbf{X}$ open and closed in $\partial\mathbf{X}$, such that:*

(a) *Define $\boldsymbol{f}_+=\boldsymbol{f}\circ\boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}}\mathbf{X}}:\partial_+^{\boldsymbol{f}}\mathbf{X}\to\mathbf{Y}$. Then $\boldsymbol{f}_+$ is semisimple. If $\boldsymbol{f}$ is flat then $\boldsymbol{f}_+$ is also flat.*

(b) *There exists a unique, semisimple 1-morphism $\boldsymbol{f}_-:\partial_-^{\boldsymbol{f}}\mathbf{X}\to\partial\mathbf{Y}$ with $\boldsymbol{f}\circ\boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}}=\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{f}_-$. If $\boldsymbol{f}$ is simple then $\partial_+^{\boldsymbol{f}}\mathbf{X}=\emptyset$, $\partial_-^{\boldsymbol{f}}\mathbf{X}=\partial\mathbf{X}$, and $\boldsymbol{f}_-:\partial\mathbf{X}\to\partial\mathbf{Y}$ is also simple. If $\boldsymbol{f}$ is flat then $\boldsymbol{f}_-$ is flat, and the following diagram is 2-Cartesian in $\mathbf{dSpa^c}$:*

$$\begin{tikzcd}
\partial_-^{\boldsymbol{f}}\mathbf{X} \arrow[r, "\boldsymbol{f}_-"] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}}"'] \arrow[dr, phantom, "\Uparrow \mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{f}_-}"] & \partial\mathbf{Y} \arrow[d, "\boldsymbol{i}_{\mathbf{Y}}"] \\
\mathbf{X} \arrow[r, "\boldsymbol{f}"] & \mathbf{Y}.
\end{tikzcd} \tag{1.28}$$

(c) *Let $\boldsymbol{g}:\mathbf{X}\to\mathbf{Y}$ be another 1-morphism, and $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a 2-morphism in $\mathbf{dSpa^c}$. Then $\boldsymbol{g}$ is also semisimple, with $\partial_-^{\boldsymbol{g}}\mathbf{X}=\partial_-^{\boldsymbol{f}}\mathbf{X}$. If $\boldsymbol{f}$ is simple, or flat, then $\boldsymbol{g}$ is simple, or flat, respectively. Part* **(b)** *defines 1-morphisms $\boldsymbol{f}_-,\boldsymbol{g}_-:\partial_-^{\boldsymbol{f}}\mathbf{X}\to\partial\mathbf{Y}$. There is a unique 2-morphism $\eta_-:\boldsymbol{f}_-\Rightarrow\boldsymbol{g}_-$ in $\mathbf{dSpa^c}$ such that $\mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}}*\eta_-=\eta*\mathrm{id}_{\boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}}}:\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{f}_-\Rightarrow\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{g}_-$.*

We also show that the maps $\boldsymbol{f}\mapsto\boldsymbol{f}_-$, $\eta\mapsto\eta_-$ in Theorem 1.6.3 are functorial, in that they commute with compositions of 1- and 2-morphisms, and take identities to identities. For simple 1-morphisms, this implies:

Corollary 1.6.4. *Write $\mathbf{dSpa^c_{si}}$ for the 2-subcategory of $\mathbf{dSpa^c}$ with arbitrary objects and 2-morphisms, but only simple 1-morphisms. Then there is a strict 2-functor $\partial:\mathbf{dSpa^c_{si}}\to\mathbf{dSpa^c_{si}}$ mapping $\mathbf{X}\mapsto\partial\mathbf{X}$ on objects, $\boldsymbol{f}\mapsto\boldsymbol{f}_-$ on (simple) 1-morphisms, and $\eta\mapsto\eta_-$ on 2-morphisms.*

Thus, boundaries in $\mathbf{dSpa^c}$ have strong functoriality properties.

Remark 1.6.5. According to the general philosophy of working in 2-categories, when one constructs an object with some property in a 2-category, it is usually unique only up to equivalence. When one constructs a 1-morphism with some property in a 2-category, it is usually unique only up to 2-isomorphism. When one considers diagrams of 1-morphisms in a 2-category, they usually commute only up to (specified) 2-isomorphisms.

From this point of view, Theorem 1.6.3(b) looks unnatural, as it gives a 1-morphism $\boldsymbol{f}_-$ which is unique, not just up to 2-isomorphism, and a 1-morphism diagram (1.28) which commutes strictly, not just up to 2-isomorphism.

In fact, this unnaturalness pervades our treatment of boundaries. In our definition of d-space with corners $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$, the conditions on the 1-morphism $\boldsymbol{i}_{\mathbf{X}} : \boldsymbol{\partial X} \rightarrow \boldsymbol{X}$ depend on $\boldsymbol{\partial X}$ up to 1-isomorphism in **dSpa**, rather than up to equivalence, and depend on $\boldsymbol{i}_{\mathbf{X}}$ up to equality, not just up to 2-isomorphism. Boundaries $\partial\mathbf{X}$ are natural up to 1-isomorphism in $\mathbf{dSpa^c}$, not up to equivalence, and 1-morphisms $\boldsymbol{i}_{\mathbf{X}} : \partial\mathbf{X} \rightarrow \mathbf{X}$ natural up to equality.

The author chose this definition of $\mathbf{dSpa^c}$ for its (comparative!) simplicity. In defining objects $\mathbf{X}, \mathbf{Y}$, 1-morphisms $\boldsymbol{f}$, and 2-morphisms η in $\mathbf{dSpa^c}$, we must impose extra conditions, and possibly include extra data, over $\partial\mathbf{X}, \partial\mathbf{Y}$. If these conditions/extra data are imposed weakly, up to equivalence of objects or 2-isomorphism of 1-morphisms, things rapidly become very complicated and unwieldy. For instance, 1-morphisms in $\mathbf{dSpa^c}$ would comprise not just a 1-morphism $\boldsymbol{f} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ in **dSpa**, but also extra 2-morphism data over $\underline{S}_{\boldsymbol{f}}, \underline{T}_{\boldsymbol{f}}$.

So as a matter of policy, we generally do constructions involving boundaries or corners in $\mathbf{dSpa^c}$ strictly, up to 1-isomorphism of objects, and equality of 1-morphisms. One advantage of this is that 1-morphisms $\boldsymbol{f} : \mathbf{X} \rightarrow \mathbf{Y}$ and 2-morphisms $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\mathbf{dSpa^c}$ are special examples of 1- and 2-morphisms in **dSpa** of the underlying d-spaces $\boldsymbol{X}, \boldsymbol{Y}$, rather than also containing further data over $\partial\mathbf{X}, \partial\mathbf{Y}$. Another advantage is that boundaries in $\mathbf{dSpa^c}$ behave in a strictly functorial way, as in Corollary 1.6.4, rather than weakly functorial.

1.6.3 Manifolds with corners as d-spaces with corners

In §6.4 we define a (2-)functor $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \rightarrow \mathbf{dSpa^c}$ from manifolds with corners to d-spaces with corners.

Definition 1.6.6. Let X be a manifold with corners. Then the boundary ∂X is a manifold with corners, with a smooth map $i_{\partial X} : \partial X \rightarrow X$. We will define a d-space with corners $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$. Set $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(X, \partial X, i_X)$. Then the conormal bundle $\mathcal{N}_{\mathbf{X}}$ in (1.26) is the lift to the C^∞-scheme $\underline{\partial X}$ of the conormal line bundle $\mathcal{N}_X$ of ∂X in X, as in (1.25). Let $\omega_{\mathbf{X}}$ be the orientation on $\mathcal{N}_{\mathbf{X}}$ corresponding to that on $\mathcal{N}_X$ induced by outward-pointing normal vectors to ∂X in X. Then $\mathbf{X}$ is a d-space with corners. Set $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X) = \mathbf{X}$.

Let $f : X \rightarrow Y$ be a morphism in $\mathbf{Man^c}$, and set $\mathbf{X}, \mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X, Y)$. Write $\boldsymbol{f} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(f) : \boldsymbol{X} \rightarrow \boldsymbol{Y}$, as a 1-morphism of d-spaces. Then $\boldsymbol{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a 1-morphism of d-spaces with corners. Define $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(f) = \boldsymbol{f}$.

The only 2-morphisms in $\mathbf{Man^c}$, regarded as a 2-category, are identity 2-morphisms $\mathrm{id}_f : f \Rightarrow f$ for smooth $f : X \rightarrow Y$. We define $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\mathrm{id}_f) = \mathrm{id}_{\boldsymbol{f}}$.

Define $F_{\mathbf{Man}}^{\mathbf{d\bar{S}pa}} : \mathbf{Man} \rightarrow \mathbf{d\bar{S}pa}$ and $F_{\mathbf{Man^b}}^{\mathbf{dSpa^b}} : \mathbf{Man^b} \rightarrow \mathbf{dSpa^b}$ to be the restrictions of $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ to the subcategories $\mathbf{Man}, \mathbf{Man^b} \subset \mathbf{Man^c}$.

Write $\bar{\mathbf{Man}}, \bar{\mathbf{Man}}^{\mathbf{b}}, \bar{\mathbf{Man}}^{\mathbf{c}}$ for the full 2-subcategories of objects $\mathbf{X}$ in $\mathbf{dSpa^c}$ equivalent to $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X)$ for some manifold X without boundary, or with boundary, or with corners, respectively. Then $\bar{\mathbf{Man}} \subset \mathbf{d\bar{S}pa}$, $\bar{\mathbf{Man}}^{\mathbf{b}} \subset \mathbf{dSpa^b}$ and $\bar{\mathbf{Man}}^{\mathbf{c}} \subset \mathbf{dSpa^c}$. When we say that a d-space with corners $\mathbf{X}$ *is a manifold*, we mean that $\mathbf{X} \in \bar{\mathbf{Man}}^{\mathbf{c}}$.

In §6.4 we show that $F_{\mathbf{Man}}^{\mathbf{d\bar{S}pa}} : \mathbf{Man} \to \mathbf{d\bar{S}pa}$, $F_{\mathbf{Man^b}}^{\mathbf{dSpa^b}} : \mathbf{Man^b} \to \mathbf{dSpa^b}$ and $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \to \mathbf{dSpa^c}$ are full and faithful strict 2-functors. We also prove that if X is a manifold with corners, then there is a natural 1-isomorphism $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\partial X) \cong \partial F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X)$, and if $f : X \to Y$ is a smooth map of manifolds with corners and $\boldsymbol{f} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(f)$, then f is simple, semisimple or flat in $\mathbf{Man^c}$ if and only if $\boldsymbol{f}$ is simple, semisimple or flat in $\mathbf{dSpa^c}$, respectively.

1.6.4 Equivalences, and gluing d-spaces with corners by equivalences

In §6.5 and §6.6 we discuss *equivalences* in $\mathbf{dSpa^c}$. First we characterize when a 1-morphism $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dSpa^c}$ is an equivalence, in terms of the underlying 1-morphism in $\mathbf{dSpa}$:

Proposition 1.6.7. **(a)** *Suppose* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *is an equivalence in* $\mathbf{dSpa^c}$. *Then* $\boldsymbol{f}$ *is simple and flat, and* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is an equivalence in* $\mathbf{dSpa}$, *where* $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ *and* $\mathbf{Y} = (\boldsymbol{Y}, \boldsymbol{\partial Y}, \boldsymbol{i}_{\mathbf{Y}}, \omega_{\mathbf{Y}})$. *Also* $\boldsymbol{f}_- : \partial\mathbf{X} \to \partial\mathbf{Y}$ *in Theorem* 1.6.3(b) *is an equivalence in* $\mathbf{dSpa^c}$.

(b) *Let* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *be a simple, flat* 1*-morphism in* $\mathbf{dSpa^c}$ *with* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *an equivalence in* $\mathbf{dSpa}$. *Then* $\boldsymbol{f}$ *is an equivalence in* $\mathbf{dSpa^c}$.

Then we consider gluing d-spaces with corners by equivalences, as for d-spaces in §1.3.2. The story is the same. Here is the analogue of Definition 1.3.4:

Definition 1.6.8. Let $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ be a d-space with corners. Suppose $\boldsymbol{U} \subseteq \boldsymbol{X}$ is an open d-subspace in $\mathbf{dSpa}$. Define $\boldsymbol{\partial U} = \boldsymbol{i}_{\mathbf{X}}^{-1}(\boldsymbol{U})$, as an open d-subspace of $\boldsymbol{\partial X}$, and $\boldsymbol{i}_{\mathbf{U}} : \boldsymbol{\partial U} \to \boldsymbol{U}$ by $\boldsymbol{i}_{\mathbf{U}} = \boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{\partial U}}$. Then $\underline{\partial U} \subseteq \underline{\partial X}$ is an open C^∞-subscheme, and the conormal bundle of $\boldsymbol{\partial U}$ in $\boldsymbol{U}$ is $\mathcal{N}_{\mathbf{U}} = \mathcal{N}_{\mathbf{X}}|_{\underline{\partial U}}$ in qcoh$(\underline{\partial U})$. Define an orientation $\omega_{\mathbf{U}}$ on $\mathcal{N}_{\mathbf{U}}$ by $\omega_{\mathbf{U}} = \omega_{\mathbf{X}}|_{\underline{\partial U}}$. Write $\mathbf{U} = (\boldsymbol{U}, \boldsymbol{\partial U}, \boldsymbol{i}_{\mathbf{U}}, \omega_{\mathbf{U}})$. Then $\mathbf{U}$ is a d-space with corners. We call $\mathbf{U}$ an *open d-subspace* of $\mathbf{X}$. An *open cover* of $\mathbf{X}$ is a family $\{\mathbf{U}_a : a \in A\}$ of open d-subspaces $\mathbf{U}_a$ of $\mathbf{X}$ with $\underline{X} = \bigcup_{a \in A} \underline{U}_a$.

Theorem 1.6.9. *Proposition* 1.3.5 *and Theorems* 1.3.6 *and* 1.3.7 *hold without change in the* 2*-category* $\mathbf{dSpa^c}$ *of d-spaces with corners.*

1.6.5 Corners and the corner functors

In §6.7 we extend the material of §1.5.3 on corners and the corner functors from $\mathbf{Man^c}$ to $\mathbf{dSpa^c}$. The next theorem summarizes our results.

Theorem 1.6.10. **(a)** *Let* $\mathbf{X}$ *be a d-space with corners. Then for each* $k = 0, 1, \ldots,$ *we can define a d-space with corners* $C_k(\mathbf{X})$ *called the* k-***corners*** *of* $\mathbf{X}$, *and a* 1*-morphism* $\mathbf{\Pi}_{\mathbf{X}}^k : C_k(\mathbf{X}) \to \mathbf{X}$ *in* $\mathbf{dSpa^c}$. *It has topological space*

$$C_k(X) = \bigl\{(x, \{x'_1, \ldots, x'_k\}) : x \in \boldsymbol{X},\ \ x'_1, \ldots, x'_k \in \boldsymbol{\partial X}, \\ \boldsymbol{i}_{\mathbf{X}}(x'_a) = x,\ \ a = 1, \ldots, k,\ \ x'_1, \ldots, x'_k \text{ are distinct}\bigr\}. \tag{1.29}$$

There is a natural, free action of the symmetric group S_k *on* $\partial^k\mathbf{X}$, *and a* 1*-isomorphism* $C_k(\mathbf{X}) \cong \partial^k\mathbf{X}/S_k$. *We have* 1*-isomorphisms* $C_0(\mathbf{X}) \cong \mathbf{X}$ *and* $C_1(\mathbf{X}) \cong \partial\mathbf{X}$ *in* $\mathbf{dSpa^c}$. *Write* $C(\mathbf{X}) = \coprod_{k=0}^{\infty} C_k(\mathbf{X})$ *and* $\mathbf{\Pi_X} = \coprod_{k=0}^{\infty} \mathbf{\Pi}_{\mathbf{X}}^k$, *so that* $C(\mathbf{X})$ *is a d-space with corners and* $\mathbf{\Pi_X} : C(\mathbf{X}) \to \mathbf{X}$ *is a* 1*-morphism.*

(b) *Let* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *be a* 1*-morphism of d-spaces with corners. Then there is a unique* 1*-morphism* $C(\boldsymbol{f}) : C(\mathbf{X}) \to C(\mathbf{Y})$ *in* $\mathbf{dSpa^c}$ *such that* $\mathbf{\Pi_Y} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \mathbf{\Pi_X} : C(\mathbf{X}) \to \mathbf{Y}$, *and* $C(\boldsymbol{f})$ *acts on points as in* (1.29) *by*

$$C(f) : \bigl(x, \{x'_1, \ldots, x'_k\}\bigr) \longmapsto \bigl(y, \{y'_1, \ldots, y'_l\}\bigr), \quad \text{where} \\ \{y'_1, \ldots, y'_l\} = \bigl\{y' : (x'_i, y') \in \underline{S}_{\boldsymbol{f}},\ \text{some } i = 1, \ldots, k\bigr\}.$$

For all $k, l \geqslant 0$, *write* $C_k^{\boldsymbol{f},l}(\mathbf{X}) = C_k(\mathbf{X}) \cap C(\boldsymbol{f})^{-1}(C_l(\mathbf{Y}))$, *so that* $C_k^{\boldsymbol{f},l}(\mathbf{X})$ *is open and closed in* $C_k(\mathbf{X})$ *with* $C_k(\mathbf{X}) = \coprod_{l=0}^{\infty} C_k^{\boldsymbol{f},l}(\mathbf{X})$, *and write* $C_k^l(\boldsymbol{f}) = C(\boldsymbol{f})|_{C_k^{\boldsymbol{f},l}(\mathbf{X})}$, *so that* $C_k^l(\boldsymbol{f}) : C_k^{\boldsymbol{f},l}(\mathbf{X}) \to C_l(\mathbf{Y})$ *is a* 1*-morphism in* $\mathbf{dSpa^c}$.

(c) *Let* $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ *be* 1*-morphisms and* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ *a* 2*-morphism in* $\mathbf{dSpa^c}$. *Then there exists a unique* 2*-morphism* $C(\eta) : C(\boldsymbol{f}) \Rightarrow C(\boldsymbol{g})$ *in* $\mathbf{dSpa^c}$, *where* $C(\boldsymbol{f}), C(\boldsymbol{g})$ *are as in* **(b)**, *such that*

$$\mathrm{id}_{\mathbf{\Pi_Y}} * C(\eta) = \eta * \mathrm{id}_{\mathbf{\Pi_X}} : \mathbf{\Pi_Y} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \mathbf{\Pi_X} \Longrightarrow \mathbf{\Pi_Y} \circ C(\boldsymbol{g}) = \boldsymbol{g} \circ \mathbf{\Pi_X}.$$

(d) *Define* $C : \mathbf{dSpa^c} \to \mathbf{dSpa^c}$ *by* $C : \mathbf{X} \mapsto C(\mathbf{X})$ *on objects,* $C : \boldsymbol{f} \mapsto C(\boldsymbol{f})$ *on* 1*-morphisms, and* $C : \eta \mapsto C(\eta)$ *on* 2*-morphisms, where* $C(\mathbf{X}), C(\boldsymbol{f}), C(\eta)$ *are as in* **(a)**–**(c)** *above. Then* C *is a strict* 2*-functor, called a* ***corner functor***.

(e) *Let* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *be semisimple. Then* $C(\boldsymbol{f})$ *maps* $C_k(\mathbf{X}) \to \coprod_{l=0}^{k} C_l(\mathbf{Y})$ *for all* $k \geqslant 0$. *The natural* 1*-isomorphisms* $C_1(\mathbf{X}) \cong \partial\mathbf{X}$, $C_0(\mathbf{Y}) \cong \mathbf{Y}$, $C_1(\mathbf{Y}) \cong \partial\mathbf{Y}$ *identify* $C_1^{\boldsymbol{f},0}(\mathbf{X}) \cong \partial_+^{\boldsymbol{f}}\mathbf{X}$, $C_1^{\boldsymbol{f},1}(\mathbf{X}) \cong \partial_-^{\boldsymbol{f}}\mathbf{X}$, $C_1^0(\boldsymbol{f}) \cong \boldsymbol{f}_+$ *and* $C_1^1(\boldsymbol{f}) \cong \boldsymbol{f}_-$.

If $\boldsymbol{f}$ *is simple then* $C(\boldsymbol{f})$ *maps* $C_k(\mathbf{X}) \to C_k(\mathbf{Y})$ *for all* $k \geqslant 0$.

(f) *Analogues of* **(b)**–**(d)** *also hold for a second corner functor* $\hat{C} : \mathbf{dSpa^c} \to \mathbf{dSpa^c}$, *which acts on objects by* $\hat{C} : \mathbf{X} \mapsto C(\mathbf{X})$ *in* **(a)**, *and for* 1*-morphisms* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *in* **(b)**, $\hat{C}(\boldsymbol{f}) : C(\mathbf{X}) \to C(\mathbf{Y})$ *acts on points by*

$$\hat{C}(f) : \bigl(x, \{x'_1, \ldots, x'_k\}\bigr) \longmapsto \bigl(y, \{y'_1, \ldots, y'_l\}\bigr), \quad \text{where} \\ \{y'_1, \ldots, y'_l\} = \bigl\{y' : (x'_i, y') \in \underline{S}_{\boldsymbol{f}},\ \text{some } i = 1, \ldots, k\bigr\} \cup \bigl\{y' : (x, y') \in \underline{T}_{\boldsymbol{f}}\bigr\}.$$

If $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *is flat then* $\hat{C}(\boldsymbol{f}) = C(\boldsymbol{f})$.

The comments of Remark 1.6.5 also apply to Theorem 1.6.10: our construction characterizes $C_k(\mathbf{X})$ up to 1-isomorphism in $\mathbf{dSpa^c}$, not just up to equivalence, the 1-morphisms $C(\boldsymbol{f}), \hat{C}(\boldsymbol{f})$ are characterized up to equality, not just up to 2-isomorphism, and in $\mathbf{\Pi_Y} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \mathbf{\Pi_X}$ we require the 1-morphisms to be equal, not just 2-isomorphic. This may seem unnatural from a 2-category point of view, but it has the advantage that corners are strictly 2-functorial rather than weakly 2-functorial.

1.6.6 Fibre products in $\mathbf{dSpa^c}$

In §6.8–§6.9 we study *fibre products* in $\mathbf{dSpa^c}$. Here the situation is more complex than for d-spaces. As in §1.3.2, all fibre products exist in $\mathbf{dSpa}$, but this fails for $\mathbf{dSpa^c}$. The problem is that in a fibre product $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ in $\mathbf{dSpa^c}$, the boundary $\partial\mathbf{W}$ depends in a complicated way on $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \partial\mathbf{X}, \partial\mathbf{Y}, \partial\mathbf{Z}$, and sometimes there is no good candidate for $\partial\mathbf{W}$. Here is an example.

Example 1.6.11. Let $X = Y = [0,\infty) \times \mathbb{R}$ and $Z = [0,\infty)^2 \times \mathbb{R}$, as manifolds with corners, and define smooth maps $g : X \to Z$ and $h : Y \to Z$ by $g(u,v) = (u,u,v)$ and $h(u,v) = (u, e^v u, v)$. Set $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \boldsymbol{g}, \boldsymbol{h} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X,Y,Z,g,h)$.

In §6.8.6 we show that no fibre product $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ exists in $\mathbf{dSpa^c}$. We do this by showing that $\partial\mathbf{W}$ would have to have exactly one point, lying over $(0,0) \in X$ and $(0,0) \in Y$, which is the only point in $X \times_Z Y$ where normal vectors to $\partial X, \partial Y$ in X, Y project under $\mathrm{d}g, \mathrm{d}h$ to parallel vectors in TZ. But this would contradict other properties of $\partial\mathbf{W}$.

So, we would like to find useful sufficient conditions for existence of fibre products $\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ in $\mathbf{dSpa^c}$; and these conditions should be wholly to do with boundaries, since we already know that fibre products exist in $\mathbf{dSpa}$. In §6.8.1 we define two such sufficient conditions on $\boldsymbol{g}, \boldsymbol{h}$, called *b-transversality* and *c-transversality*.

Definition 1.6.12. Let $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms in $\mathbf{dSpa^c}$. As in §1.6.1 we have line bundles $\mathcal{N}_{\mathbf{X}}, \mathcal{N}_{\mathbf{Z}}$ over the C^∞-schemes $\underline{\partial X}, \underline{\partial Z}$, and a C^∞-subscheme $\underline{S}_{\boldsymbol{g}} \subseteq \underline{\partial X} \times_{\underline{Z}} \underline{\partial Z}$. As in §7.1, there is a natural isomorphism $\lambda_{\boldsymbol{g}} : \underline{u}_{\boldsymbol{g}}^*(\mathcal{N}_{\mathbf{Z}}) \to \underline{s}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}})$ in $\mathrm{qcoh}(\underline{S}_{\boldsymbol{g}})$. The same holds for $\boldsymbol{h}$.

We say that $\boldsymbol{g}, \boldsymbol{h}$ are *b-transverse* if whenever $x \in \underline{X}$ and $y \in \underline{Y}$ with $\underline{g}(x) = \underline{h}(y) = z \in \underline{Z}$, the following morphism in $\mathrm{qcoh}(\underline{*})$ is injective:

$$\begin{aligned}
&\bigoplus_{(x',z') \in \underline{S}_{\boldsymbol{g}} : \underline{i}_{\mathbf{X}}(x') = x} \lambda_{\boldsymbol{g}}|_{(x',z')} \oplus \bigoplus_{(y',z') \in \underline{S}_{\boldsymbol{h}} : \underline{i}_{\mathbf{Y}}(y') = y} \lambda_{\boldsymbol{h}}|_{(y',z')} : \\
&\bigoplus_{z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)} \mathcal{N}_{\mathbf{Z}}|_{z'} \longrightarrow \bigoplus_{x' \in \underline{i}_{\mathbf{X}}^{-1}(x)} \mathcal{N}_{\mathbf{X}}|_{x'} \oplus \bigoplus_{y' \in \underline{i}_{\mathbf{Y}}^{-1}(y)} \mathcal{N}_{\mathbf{Y}}|_{y'}.
\end{aligned}$$

Roughly speaking, this says that the corners of $\mathbf{X}, \mathbf{Y}$ are transverse to the corners of $\mathbf{Z}$. In Example 1.6.11, this condition fails at $x = 0 \in \underline{X}$ and $y = 0 \in \underline{Y}$, so $\boldsymbol{g}, \boldsymbol{h}$ are not b-transverse.

We call $\boldsymbol{g}, \boldsymbol{h}$ *c-transverse* if the following two conditions hold, using the notation of Theorem 1.6.10:

(a) whenever there are points in $C_j(\mathbf{X}), C_k(\mathbf{Y}), C_l(\mathbf{Z})$ with

$$C(\boldsymbol{g})(x,\{x'_1,\ldots,x'_j\}) = C(\boldsymbol{h})(y,\{y'_1,\ldots,y'_k\}) = (z,\{z'_1,\ldots,z'_l\}),$$

we have either $j+k>l$ or $j=k=l=0$; and

(b) whenever there are points in $C_j(\mathbf{X}), C_k(\mathbf{Y}), C_l(\mathbf{Z})$ with

$$\hat C(\boldsymbol{g})(x,\{x'_1,\ldots,x'_j\}) = \hat C(\boldsymbol{h})(y,\{y'_1,\ldots,y'_k\}) = (z,\{z'_1,\ldots,z'_l\}),$$

we have $j+k\geqslant l$.

Here b-transversality is a continuous condition on $\boldsymbol{g},\boldsymbol{h}$, and c-transversality is a discrete condition. Also c-transversality implies b-transversality (though this is not obvious). Part (a) corresponds to the condition in Definition 1.5.12 for transverse g,h in $\mathbf{Man^c}$ to be strongly transverse. We can show:

Lemma 1.6.13. *Let $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$ and $\boldsymbol{h}:\mathbf{Y}\to\mathbf{Z}$ be 1-morphisms in $\mathbf{dSpa^c}$. The following are sufficient conditions for $\boldsymbol{g},\boldsymbol{h}$ to be c-transverse, and hence b-transverse:*

(i) *$\boldsymbol{g}$ or $\boldsymbol{h}$ is semisimple and flat; or*

(ii) *$\mathbf{Z}$ is a d-space without boundary.*

We summarize the main results of §6.8 on fibre products in $\mathbf{dSpa^c}$:

Theorem 1.6.14. (a) *All b-transverse fibre products exist in $\mathbf{dSpa^c}$.*

(b) *The 2-functor $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ of §1.6.3 takes transverse fibre products in $\mathbf{Man^c}$ to b-transverse fibre products in $\mathbf{dSpa^c}$. That is, if*

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

is a Cartesian square in $\mathbf{Man^c}$ with g,h transverse, and $\mathbf{W},\mathbf{X},\mathbf{Y},\mathbf{Z},\boldsymbol{e},\boldsymbol{f},\boldsymbol{g},\boldsymbol{h} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(W,X,Y,Z,e,f,g,h)$, then

$$\begin{tikzcd} \mathbf{W} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"] \arrow[dr, phantom, "\Uparrow \mathrm{id}_{\boldsymbol{g}\circ\boldsymbol{e}}"] & \mathbf{Y} \arrow[d, "\boldsymbol{h}"] \\ \mathbf{X} \arrow[r, "\boldsymbol{g}"] & \mathbf{Z} \end{tikzcd}$$

is 2-Cartesian in $\mathbf{dSpa^c}$, with $\boldsymbol{g},\boldsymbol{h}$ b-transverse. If also g,h are strongly transverse in $\mathbf{Man^c}$, then $\boldsymbol{g},\boldsymbol{h}$ are c-transverse in $\mathbf{dSpa^c}$.

(c) *Suppose we are given a 2-Cartesian diagram in $\mathbf{dSpa^c}$:*

$$\begin{tikzcd} \mathbf{W} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"] \arrow[dr, phantom, "\Uparrow \eta"] & \mathbf{Y} \arrow[d, "\boldsymbol{h}"] \\ \mathbf{X} \arrow[r, "\boldsymbol{g}"] & \mathbf{Z}, \end{tikzcd}$$

with $\boldsymbol{g}, \boldsymbol{h}$ c-transverse. Then the following are also 2*-Cartesian in* $\mathbf{dSpa^c}$*:*

$$\begin{tikzcd} C(\mathbf{W}) \arrow[rr, "C(\boldsymbol{f})"] \arrow[d, "C(\boldsymbol{e})"'] \arrow[drr, phantom, "C(\eta)\Uparrow"] && C(\mathbf{Y}) \arrow[d, "C(\boldsymbol{h})"] \\ C(\mathbf{X}) \arrow[rr, "C(\boldsymbol{g})"] && C(\mathbf{Z}), \end{tikzcd} \tag{1.30}$$

$$\begin{tikzcd} C(\mathbf{W}) \arrow[rr, "\hat{C}(\boldsymbol{f})"] \arrow[d, "\hat{C}(\boldsymbol{e})"'] \arrow[drr, phantom, "\hat{C}(\eta)\Uparrow"] && C(\mathbf{Y}) \arrow[d, "\hat{C}(\boldsymbol{h})"] \\ C(\mathbf{X}) \arrow[rr, "\hat{C}(\boldsymbol{g})"] && C(\mathbf{Z}). \end{tikzcd} \tag{1.31}$$

Also (1.30)–(1.31) *preserve gradings, in that they relate points in* $C_i(\mathbf{W}), C_j(\mathbf{X})$, $C_k(\mathbf{Y}), C_k(\mathbf{Z})$ *with* $i = j + k - l$*. Hence* (1.30) *implies equivalences in* $\mathbf{dSpa^c}$*:*

$$C_i(\mathbf{W}) \simeq \coprod_{j,k,l \geqslant 0: i=j+k-l} C_j^{\boldsymbol{g},l}(\mathbf{X}) \times_{C_j^l(\boldsymbol{g}), C_l(\mathbf{Z}), C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathbf{Y}), \tag{1.32}$$

$$\partial\mathbf{W} \simeq \coprod_{j,k,l \geqslant 0: j+k=l+1} C_j^{\boldsymbol{g},l}(\mathbf{X}) \times_{C_j^l(\boldsymbol{g}), C_l(\mathbf{Z}), C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathbf{Y}). \tag{1.33}$$

Part (a) takes some work to prove. For fibre products in $\mathbf{dSpa}$, as in §1.3.3, we gave an explicit global construction. But for fibre products in $\mathbf{dSpa^c}$, we first prove that local fibre products $\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ exist in $\mathbf{dSpa^c}$ near each $x \in \mathbf{X}$, $y \in \mathbf{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) \in \mathbf{Z}$, and then we use the results of §1.6.4 to glue these local fibre products by equivalences to get a global fibre product.

For general b-transverse fibre products $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ in $\mathbf{dSpa^c}$, the description of $\partial\mathbf{W}$ can be complicated. For c-transverse fibre products, we do at least have a (still complicated) explicit formula (1.33) for $\partial\mathbf{W}$. Here are some cases when this formula simplifies, an analogue of Proposition 1.5.14.

Proposition 1.6.15. *Let* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ *and* $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *be* 1*-morphisms of d-spaces with corners. Then:*

(a) *If* $\partial\mathbf{Z} = \emptyset$ *then there is an equivalence*

$$\partial\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \simeq \big(\partial\mathbf{X} \times_{\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{X}},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \amalg \big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}} \partial\mathbf{Y}\big). \tag{1.34}$$

(b) *If* $\boldsymbol{g}$ *is semisimple and flat then there is an equivalence*

$$\partial\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \simeq \big(\partial_+^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_+,\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \amalg \big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}} \partial\mathbf{Y}\big). \tag{1.35}$$

(c) *If both* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ *are semisimple and flat then there is an equivalence*

$$\begin{aligned} \partial\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \simeq \big(\partial_+^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_+,\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \amalg \big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}_+} \partial_+^{\boldsymbol{h}}\mathbf{Y}\big) \\ \amalg\big(\partial_-^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_-,\partial\mathbf{Z},\boldsymbol{h}_-} \partial_-^{\boldsymbol{h}}\mathbf{Y}\big). \end{aligned} \tag{1.36}$$

Here all fibre products in (1.34)–(1.36) *are c-transverse, and so exist.*

1.6.7 Fixed points of finite groups in d-spaces with corners

In §1.3.4 we discussed the fixed d-subspace $\boldsymbol{X}^\Gamma$ of a finite group Γ acting on a d-space $\boldsymbol{X}$, and in §1.5.6 we considered fixed point loci X^Γ of a finite group Γ acting on a manifold with corners X. Section 6.10 generalizes these to d-spaces with corners. Here is the analogue of Theorem 1.3.10.

Theorem 1.6.16. *Let* $\mathbf{X}$ *be a d-space with corners,* Γ *a finite group, and* $\boldsymbol{r}:\Gamma\rightarrow\mathrm{Aut}(\mathbf{X})$ *an action of* Γ *on* $\mathbf{X}$ *by* 1*-isomorphisms. Then we can define a d-space with corners* $\mathbf{X}^\Gamma$ *called the* ***fixed d-subspace of*** Γ ***in*** $\mathbf{X}$*, with an inclusion* 1*-morphism* $\boldsymbol{j}_{\mathbf{X},\Gamma}:\mathbf{X}^\Gamma\rightarrow\mathbf{X}$*. It has the following properties:*

- **(a)** *Let* $\mathbf{X},\Gamma,\boldsymbol{r}$ *and* $\boldsymbol{j}_{\mathbf{X},\Gamma}:\mathbf{X}^\Gamma\rightarrow\mathbf{X}$ *be as above. Suppose* $\boldsymbol{f}:\mathbf{W}\rightarrow\mathbf{X}$ *is a* 1*-morphism in* $\mathbf{dSpa^c}$*. Then* $\boldsymbol{f}$ *factorizes as* $\boldsymbol{f}=\boldsymbol{j}_{\mathbf{X},\Gamma}\circ\boldsymbol{g}$ *for some* 1*-morphism* $\boldsymbol{g}:\mathbf{W}\rightarrow\mathbf{X}^\Gamma$ *in* $\mathbf{dSpa^c}$*, which must be unique, if and only if* $\boldsymbol{r}(\gamma)\circ\boldsymbol{f}=\boldsymbol{f}$ *for all* $\gamma\in\Gamma$*.*
- **(b)** *Suppose* $\mathbf{X},\mathbf{Y}$ *are d-spaces with corners,* Γ *is a finite group,* $\boldsymbol{r}:\Gamma\rightarrow\mathrm{Aut}(\mathbf{X})$*,* $\boldsymbol{s}:\Gamma\rightarrow\mathrm{Aut}(\mathbf{Y})$ *are actions of* Γ *on* $\mathbf{X},\mathbf{Y}$*, and* $\boldsymbol{f}:\mathbf{X}\rightarrow\mathbf{Y}$ *is a* Γ*-equivariant* 1*-morphism in* $\mathbf{dSpa^c}$*, that is,* $\boldsymbol{f}\circ\boldsymbol{r}(\gamma)=\boldsymbol{s}(\gamma)\circ\boldsymbol{f}$ *for all* $\gamma\in\Gamma$*. Then there exists a unique* 1*-morphism* $\boldsymbol{f}^\Gamma:\mathbf{X}^\Gamma\rightarrow\mathbf{Y}^\Gamma$ *such that* $\boldsymbol{j}_{\mathbf{Y},\Gamma}\circ\boldsymbol{f}^\Gamma=\boldsymbol{f}\circ\boldsymbol{j}_{\mathbf{X},\Gamma}$*.*
- **(c)** *Let* $\boldsymbol{f},\boldsymbol{g}:\mathbf{X}\rightarrow\mathbf{Y}$ *be* Γ*-equivariant* 1*-morphisms as in* **(b)**, *and* $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ *be a* Γ*-equivariant* 2*-morphism, that is,* $\eta*\mathrm{id}_{\boldsymbol{r}(\gamma)}=\mathrm{id}_{\boldsymbol{s}(\gamma)}*\eta$ *for all* $\gamma\in\Gamma$*. Then there exists a unique* 2*-morphism* $\eta^\Gamma:\boldsymbol{f}^\Gamma\Rightarrow\boldsymbol{g}^\Gamma$ *such that* $\mathrm{id}_{\boldsymbol{j}_{\mathbf{Y},\Gamma}}*\eta^\Gamma=\eta*\mathrm{id}_{\boldsymbol{j}_{\mathbf{X},\Gamma}}$*.*

Note that **(a)** *is a universal property that determines* $\boldsymbol{X}^\Gamma,\boldsymbol{j}_{\boldsymbol{X},\Gamma}$ *up to canonical* 1*-isomorphism.*

As for manifolds with corners in §1.5.6, in general $\partial(\mathbf{X}^\Gamma)\not\simeq(\partial\mathbf{X})^\Gamma$, so fixed point loci do not commute with boundaries. But the following analogue of Proposition 1.5.17(b) shows that fixed point loci do commute with corners.

Proposition 1.6.17. *Let* $\mathbf{X}$ *be a d-space with corners,* Γ *a finite group, and* $\boldsymbol{r}:\Gamma\rightarrow\mathrm{Aut}(\mathbf{X})$ *an action of* Γ *on* $\mathbf{X}$*. Applying the corner functor* C *of* §1.6.5 *gives an action* $C(\boldsymbol{r}):\Gamma\rightarrow\mathrm{Aut}(C(\mathbf{X}))$*. Hence Theorem* 1.6.16 *defines fixed d-subspaces* $\mathbf{X}^\Gamma,C(\mathbf{X})^\Gamma$ *and inclusion* 1*-morphisms* $\boldsymbol{j}_{\mathbf{X},\Gamma}:\mathbf{X}^\Gamma\rightarrow\mathbf{X}$*,* $\boldsymbol{j}_{C(\mathbf{X}),\Gamma}:C(\mathbf{X})^\Gamma\rightarrow C(\mathbf{X})$*. Applying* C *to* $\boldsymbol{j}_{\mathbf{X},\Gamma}$ *also gives* $C(\boldsymbol{j}_{\mathbf{X},\Gamma}):C(\mathbf{X}^\Gamma)\rightarrow C(\mathbf{X})$*.*

Then there exists a unique equivalence $\boldsymbol{k}_{\mathbf{X},\Gamma}:C(\mathbf{X}^\Gamma)\rightarrow C(\mathbf{X})^\Gamma$ *in* $\mathbf{dSpa^c}$ *such that* $C(\boldsymbol{j}_{\mathbf{X},\Gamma})=\boldsymbol{j}_{C(\mathbf{X}),\Gamma}\circ\boldsymbol{k}_{\mathbf{X},\Gamma}$*.*

We will use fixed d-subspaces $\mathbf{X}^\Gamma$ in §1.13.7 below to describe orbifold strata $\mathfrak{X}^\Gamma$ of quotient d-stacks with corners $\mathfrak{X}=[\mathbf{X}/G]$. If $\mathbf{X}$ is a d-manifold with corners, as in §1.7, then in general the fixed d-subspaces $\mathbf{X}^\Gamma$ are disjoint unions of d-manifolds with corners of different dimensions, that is, $\mathbf{X}^\Gamma$ lies in $\mathbf{d\check{M}an^c}$.

1.7 D-manifolds with corners

Next we summarize Chapter 7 on d-manifolds with boundary and corners.

1.7.1 The definition of d-manifolds with corners

In §1.4.1 we defined a d-manifold to be a d-space covered by principal open d-submanifolds of fixed dimension, where Proposition 1.4.2 gave three equivalent definitions of principal d-manifolds, the first as a fibre product $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$ in $\mathbf{dSpa}$ with $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\in\hat{\mathbf{Man}}$, and the third as a fibre product $\boldsymbol{V}\times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}}\boldsymbol{V}$ in $\mathbf{dSpa}$, where V is a manifold, $E\rightarrow V$ a vector bundle, and $s\in C^\infty(E)$.

When we pass to d-spaces and d-manifolds with corners in §7.1, the analogues of Proposition 1.4.2(a)–(c) are no longer equivalent. So we have to choose which of them gives the best idea of principal d-manifold with corners. Defining principal d-manifolds with corners to be fibre products $\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y}$ in $\mathbf{dSpa^c}$ with $\mathbf{X},\mathbf{Y},\mathbf{Z}\in\bar{\mathbf{M}}\mathbf{an^c}$ is unsatisfactory, since as in §1.6.6 fibre products $\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y}$ may not exist in $\mathbf{dSpa^c}$. So instead we define principal d-manifolds with corners to be fibre products $\mathbf{V}\times_{\boldsymbol{s},\mathbf{E},\mathbf{0}}\mathbf{V}$ in $\mathbf{dSpa^c}$.

Definition 1.7.1. A d-space with corners $\mathbf{W}$ is called a *principal d-manifold with corners* if is equivalent in $\mathbf{dSpa^c}$ to a fibre product $\mathbf{V}\times_{\boldsymbol{s},\mathbf{E},\mathbf{0}}\mathbf{V}$, where V is a manifold with corners, $E\rightarrow V$ is a vector bundle, $s:V\rightarrow E$ is a smooth section of E, $0:V\rightarrow E$ is the zero section, and $\mathbf{V},\mathbf{E},\boldsymbol{s},\mathbf{0}=F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(V,E,s,0)$. Note that $s,0:V\rightarrow E$ are simple, flat smooth maps in $\mathbf{Man^c}$, so $\boldsymbol{s},\mathbf{0}:\mathbf{V}\rightarrow\mathbf{E}$ are simple, flat 1-morphisms in $\mathbf{dSpa^c}$, and thus $\boldsymbol{s},\mathbf{0}$ are b-transverse by Lemma 1.6.13, and the fibre product $\mathbf{V}\times_{\boldsymbol{s},\mathbf{E},\mathbf{0}}\mathbf{V}$ exists in $\mathbf{dSpa^c}$ by Theorem 1.6.14(a).

If $\mathbf{W}\simeq\mathbf{V}\times_{\boldsymbol{s},\mathbf{E},\mathbf{0}}\mathbf{V}$ then the virtual cotangent sheaf $T^*\boldsymbol{W}$ of the d-space $\boldsymbol{W}$ is a virtual vector bundle with $\operatorname{rank}T^*\boldsymbol{W}=\dim V-\operatorname{rank}E$. Hence, if $\mathbf{W}\neq\emptyset$ then the integer $\dim V-\operatorname{rank}E$ depends only on $\boldsymbol{W}$ up to equivalence in $\mathbf{dSpa}$, and is independent of the choice of V,E,s with $\mathbf{W}\simeq\mathbf{V}\times_{\boldsymbol{s},\mathbf{E},\mathbf{0}}\mathbf{V}$. Define the *virtual dimension* $\operatorname{vdim}\mathbf{W}$ to be $\operatorname{vdim}\mathbf{W}=\operatorname{rank}T^*\boldsymbol{W}=\dim V-\operatorname{rank}E$.

A d-space with corners $\mathbf{X}$ is called a *d-manifold with corners of virtual dimension* $n\in\mathbb{Z}$, written $\operatorname{vdim}\mathbf{X}=n$, if $\mathbf{X}$ can be covered by open d-subspaces $\mathbf{W}$ which are principal d-manifolds with corners with $\operatorname{vdim}\mathbf{W}=n$. A d-manifold with corners $\mathbf{X}$ is called a *d-manifold with boundary* if it is a d-space with boundary, and a *d-manifold without boundary* if it is a d-space without boundary.

Write $\mathbf{d\bar{M}an},\mathbf{dMan^b},\mathbf{dMan^c}$ for the full 2-subcategories of d-manifolds without boundary, and d-manifolds with boundary, and d-manifolds with corners in $\mathbf{dSpa^c}$, respectively. The 2-functor $F_{\mathbf{dSpa}}^{\mathbf{dSpa^c}}:\mathbf{dSpa}\rightarrow\mathbf{dSpa^c}$ in §1.6.1 is an isomorphism of 2-categories $\mathbf{dSpa}\rightarrow\mathbf{d\bar{S}pa}$, and its restriction to $\mathbf{dMan}\subset\mathbf{dSpa}$ gives an isomorphism of 2-categories $F_{\mathbf{dMan}}^{\mathbf{dMan^c}}:\mathbf{dMan}\rightarrow\mathbf{d\bar{M}an}\subset\mathbf{dMan^c}$. So we may as well identify $\mathbf{dMan}$ with its image $\mathbf{d\bar{M}an}$, and consider d-manifolds in §1.4 as examples of d-manifolds with corners.

If $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}},\omega_{\mathbf{X}})$ is a d-manifold with corners, then the virtual cotangent sheaf $T^*\boldsymbol{X}$ of the d-space $\boldsymbol{X}$ from Definition 1.4.10 is a virtual vector bundle on $\underline{X}$, of rank $\operatorname{vdim}\mathbf{X}$. We will call $T^*\boldsymbol{X}\in\operatorname{vvect}(\underline{X})$ the *virtual cotangent bundle* of $\mathbf{X}$, and also write it $T^*\mathbf{X}$.

Much of §1.6 on d-spaces with corners applies immediately to d-manifolds with corners. If $\mathbf{X}$ is a d-manifold with corners with $\operatorname{vdim}\mathbf{X} = n$ then the boundary $\partial\mathbf{X}$ as a d-space with corners from §1.6.1 is a d-manifold with corners, with $\operatorname{vdim}\partial\mathbf{X} = n-1$. The material on simple, semisimple, and flat 1-morphisms in $\mathbf{dSpa^c}$ in §1.6.2 also holds in $\mathbf{dMan^c}$. The functor $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$: $\mathbf{Man^c} \to \mathbf{dSpa^c}$ in §1.6.3 maps to $\mathbf{dMan^c} \subset \mathbf{dSpa^c}$, so we write $F_{\mathbf{Man^c}}^{\mathbf{dMan^c}} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \to \mathbf{dMan^c}$. The 2-categories $\mathbf{\bar{M}an}, \mathbf{\bar{M}an^b}, \mathbf{\bar{M}an^c}$ in Definition 1.6.6 are 2-subcategories of $\mathbf{d\bar{M}an}, \mathbf{dMan^b}, \mathbf{dMan^c}$, respectively. When we say that a d-manifold with corners $\mathbf{X}$ *is a manifold*, we mean that $\mathbf{X} \in \mathbf{\bar{M}an^c}$.

In §1.6.4, if we make a d-space with corners $\mathbf{Y}$ by gluing together d-manifolds with corners $\mathbf{X}_i$ for $i \in I$ by equivalences, then $\mathbf{Y}$ is a d-manifold with corners with $\operatorname{vdim}\mathbf{Y} = n$ provided $\operatorname{vdim}\mathbf{X}_i = n$ for all $i \in I$.

In §1.6.5, if $\mathbf{X}$ is a d-manifold with corners with $\operatorname{vdim}\mathbf{X} = n$ then the k-corners $C_k(\mathbf{X})$ is a d-manifold with corners, with $\operatorname{vdim} C_k(\mathbf{X}) = n-k$. Note however that $C(\mathbf{X}) = \coprod_{k=0}^{\infty} C_k(\mathbf{X})$ in Theorem 1.6.10 is in general *not* a d-manifold with corners, but only a disjoint union of d-manifolds with corners with different dimensions. As for $\mathbf{\check{M}an^c}$ in §1.5.3, define $\mathbf{d\check{M}an^c}$ to be the full 2-subcategory of $\mathbf{X}$ in $\mathbf{dSpa^c}$ which may be written as a disjoint union $\mathbf{X} = \coprod_{n\in\mathbb{Z}} \mathbf{X}_n$ for $\mathbf{X}_n \in \mathbf{dMan^c}$ with $\operatorname{vdim}\mathbf{X}_n = n$, where we allow $\mathbf{X}_n = \boldsymbol{\emptyset}$. We call such $\mathbf{X}$ a *d-manifold with corners of mixed dimension*. Then $C, \hat{C}$ in Theorem 1.6.10 restrict to strict 2-functors $C, \hat{C} : \mathbf{dMan^c} \to \mathbf{d\check{M}an^c}$.

Here are some examples. The fibre products we give all exist in $\mathbf{dMan^c}$ by results in §1.7.5 below.

Example 1.7.2. **(i)** Let $\mathbf{X}$ be the fibre product $\mathbf{[0,\infty)} \times_{\boldsymbol{i},\mathbb{R},\mathbf{0}} \boldsymbol{*}$ in $\mathbf{dMan^c}$, where $i : [0,\infty) \hookrightarrow \mathbb{R}$ is the inclusion. Then $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ is 'a point with point boundary', of virtual dimension 0, and its boundary $\partial\mathbf{X}$ is an 'obstructed point', a point with obstruction space $\mathbb{R}$, of virtual dimension -1.

The conormal bundle $\mathcal{N}_{\mathbf{X}}$ of $\boldsymbol{\partial X}$ in $\boldsymbol{X}$ is the obstruction space $\mathbb{R}$ of $\boldsymbol{\partial X}$. In this case, the orientation $\omega_{\mathbf{X}}$ on $\mathcal{N}_{\mathbf{X}}$ cannot be determined from $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}$, in fact, there is an automorphism of $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}$ which reverses the orientation of $\mathcal{N}_{\mathbf{X}}$. So $\omega_{\mathbf{X}}$ really is extra data. We include $\omega_{\mathbf{X}}$ in the definition of d-manifolds with corners to ensure that orientations of d-manifolds with corners are well-behaved. If we omitted $\omega_{\mathbf{X}}$ from the definition, there would exist oriented d-manifolds with corners $\mathbf{X}$ whose boundaries $\partial\mathbf{X}$ are not orientable.

(ii) The fibre product $\mathbf{[0,\infty)} \times_{\boldsymbol{i},\mathbf{[0,\infty)},\mathbf{0}} \boldsymbol{*}$ is a point $\boldsymbol{*}$ without boundary. The only difference with **(i)** is that we have replaced the target $\mathbb{R}$ with $\mathbf{[0,\infty)}$, adding a boundary. So in a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ in $\mathbf{dMan^c}$, the boundary of $\mathbf{Z}$ affects the boundary of $\mathbf{W}$. This does not happen for fibre products in $\mathbf{Man^c}$.

(iii) Let $\mathbf{X}'$ be the fibre product $\mathbf{[0,\infty)} \times_{\boldsymbol{i},\mathbb{R},\boldsymbol{i}} \mathbf{(-\infty,0]}$ in $\mathbf{dMan^c}$, that is, the derived intersection of submanifolds $[0,\infty), (-\infty,0]$ in $\mathbb{R}$. Topologically, $\mathbf{X}'$ is just the point $\{0\}$, but as a d-manifold with corners $\mathbf{X}'$ has virtual dimension 1. The boundary $\partial\mathbf{X}'$ is the disjoint union of two copies of $\mathbf{X}$ in **(i)**. The C^∞-scheme $\underline{X}'$ in $\mathbf{X}'$ is the spectrum of the C^∞-ring $C^\infty\big([0,\infty)^2\big)/(x+y)$, which is infinite-dimensional, although its topological space is a point.

1.7.2 'Standard model' d-manifolds with corners and 1-morphisms

In Examples 1.4.4 and 1.4.5 of §1.4.2, we defined 'standard model' d-manifolds $\boldsymbol{S}_{V,E,s}$ and 1-morphisms $\boldsymbol{S}_{f,\hat f}:\boldsymbol{S}_{V,E,s}\to\boldsymbol{S}_{W,F,t}$. In §7.1–§7.3 we show that all this extends to d-manifolds with corners in a straightforward way.

Example 1.7.3. Let V be a manifold with corners, $E\to V$ a vector bundle, and $s:V\to E$ a smooth section of E. We will write down an explicit principal d-manifold with corners $\mathbf{S}=(\boldsymbol{S},\boldsymbol{\partial S},\boldsymbol{i}_{\mathbf{S}},\omega_{\mathbf{S}})$.

Define a vector bundle $E_\partial\to\partial V$ by $E_\partial=i_V^*(E)$, and a section $s_\partial:\partial V\to E_\partial$ by $s_\partial=i_V^*(s)$. Define d-spaces $\boldsymbol{S}=\boldsymbol{S}_{V,E,s}$ and $\boldsymbol{\partial S}=\boldsymbol{S}_{\partial V,E_\partial,s_\partial}$ from the triples V,E,s and $\partial V,E_\partial,s_\partial$ exactly as in Example 1.4.4, although now $V,\partial V$ have corners. Define a 1-morphism $\boldsymbol{i}_{\mathbf{S}}:\boldsymbol{\partial S}\to\boldsymbol{S}$ in **dSpa** to be the 'standard model' 1-morphism $\boldsymbol{S}_{i_V,\mathrm{id}_{E_\partial}}:\boldsymbol{S}_{\partial V,E_\partial,s_\partial}\to\boldsymbol{S}_{V,E,s}$ from Example 1.4.5.

Comparing the analogues of (1.25) for $i_V:\partial V\to V$ and (1.26) for $\boldsymbol{i}_{\mathbf{S}}:\boldsymbol{\partial S}\to\boldsymbol{S}$, we see that the conormal bundle $\mathcal{N}_{\mathbf{S}}$ of $\boldsymbol{\partial S}$ in $\boldsymbol{S}$ is canonically isomorphic to the lift to $\underline{\partial S}\subseteq\underline{\partial V}$ of the conormal bundle $\mathcal{N}_V$ of ∂V in V. Define $\omega_{\mathbf{S}}$ to be the orientation on $\mathcal{N}_{\mathbf{S}}$ induced by the orientation on $\mathcal{N}_V$ by outward-pointing normal vectors to ∂V in V. Then $\mathbf{S}=(\boldsymbol{S},\boldsymbol{\partial S},\boldsymbol{i}_{\mathbf{S}},\omega_{\mathbf{S}})$ is a d-space with corners. It is equivalent to $\mathbf{V}\times_{\boldsymbol{s},\mathbf{E},\mathbf{0}}\mathbf{V}$ in Definition 1.7.1, and so is a principal d-manifold with corners. We call $\mathbf{S}$ the *standard model* of (V,E,s), and write it $\mathbf{S}_{V,E,s}$.

There is a natural 1-isomorphism $\partial\mathbf{S}_{V,E,s}\cong\mathbf{S}_{\partial V,E_\partial,s_\partial}$ in $\mathbf{dMan^c}$.

Example 1.7.4. Let V,W be manifolds with corners, $E\to V$, $F\to W$ be vector bundles, and $s:V\to E$, $t:W\to F$ be smooth sections. Then Example 1.7.3 defines 'standard model' principal d-manifolds with corners $\mathbf{S}_{V,E,s},\mathbf{S}_{W,F,t}$, with underlying d-spaces $\boldsymbol{S}_{V,E,s},\boldsymbol{S}_{W,F,t}$. Suppose $f:V\to W$ is a smooth map, and $\hat f:E\to f^*(F)$ is a morphism of vector bundles on V satisfying $\hat f\circ s=f^*(t)+O(s^2)$ in $C^\infty(f^*(F))$, where $f^*(t)=t\circ f$, and $O(s^2)$ is as §1.4.2. Define a 1-morphism $\boldsymbol{S}_{f,\hat f}:\boldsymbol{S}_{V,E,s}\to\boldsymbol{S}_{W,F,t}$ in **dSpa** using $f,\hat f$ exactly as in Example 1.4.5. Then $\mathbf{S}_{f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ is a 1-morphism in $\mathbf{dMan^c}$, which we call a 'standard model' 1-morphism.

Suppose $\tilde V\subseteq V$ is open, with inclusion $i_{\tilde V}:\tilde V\to V$. Write $\tilde E=E|_{\tilde V}=i_{\tilde V}^*(E)$ and $\tilde s=s|_{\tilde V}$. Define $\boldsymbol{i}_{\tilde V,V}=\mathbf{S}_{i_{\tilde V},\mathrm{id}_{\tilde E}}:\mathbf{S}_{\tilde V,\tilde E,\tilde s}\to\mathbf{S}_{V,E,s}$. If $s^{-1}(0)\subseteq\tilde V$ then $\boldsymbol{i}_{\tilde V,V}$ is a 1-isomorphism, with inverse $\boldsymbol{i}_{\tilde V,V}^{-1}$.

In §7.2 and §7.3 we prove analogues of Theorems 1.4.7 and 1.4.8:

Theorem 1.7.5. *Let $\mathbf{X}$ be a d-manifold with corners, and $x\in\mathbf{X}$. Then there exists an open neighbourhood $\mathbf{U}$ of x in $\mathbf{X}$ and an equivalence $\mathbf{U}\simeq\mathbf{S}_{V,E,s}$ in $\mathbf{dMan^c}$ for some manifold with corners V, vector bundle $E\to V$ and smooth section $s:V\to E$ which identifies $x\in\mathbf{U}$ with a point $v\in S^k(V)\subseteq V$, where $S^k(V)$ is as in §1.5.1, such that $s(v)=\mathrm{d}s|_{S^k(V)}(v)=0$. Furthermore, V,E,s and k are determined up to non-canonical isomorphism near v by $\mathbf{X}$ near x.*

Theorem 1.7.6. *Let V,W be manifolds with corners, $E\to V$, $F\to W$ be vector bundles, and $s:V\to E$, $t:W\to F$ be smooth sections. Suppose*

$\boldsymbol{g} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ is a 1-morphism in $\mathbf{dMan^c}$. Then there exist an open neighbourhood $\tilde V$ of $s^{-1}(0)$ in V, a smooth map $f : \tilde V \to W$, and a morphism of vector bundles $\hat f : \tilde E \to f^*(F)$ with $\hat f \circ \tilde s = f^*(t)$, where $\tilde E = E|_{\tilde V}$, $\tilde s = s|_{\tilde V}$, such that $\boldsymbol{g} = \mathbf{S}_{f,\hat f} \circ \boldsymbol{i}^{-1}_{\tilde V,V}$, using the notation of Examples 1.7.3 *and* 1.7.4.

1.7.3 Equivalences of d-manifolds with corners, and gluing

In §7.4 we study equivalences and gluing in $\mathbf{dMan^c}$, as for $\mathbf{dMan}$ in §1.4.4. Here are the analogues of Definition 1.4.14 and Theorems 1.4.15–1.4.17.

Definition 1.7.7. Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism in $\mathbf{dMan^c}$. We call $\boldsymbol{f}$ *étale* if it is a *local equivalence*, that is, if for each $x \in \mathbf{X}$ there exist open $x \in \mathbf{U} \subseteq \mathbf{X}$ and $\boldsymbol{f}(x) \in \mathbf{V} \subseteq \mathbf{Y}$ such that $\boldsymbol{f}(\mathbf{U}) = \mathbf{V}$ and $\boldsymbol{f}|_{\mathbf{U}} : \mathbf{U} \to \mathbf{V}$ is an equivalence.

Theorem 1.7.8. *Suppose $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a* 1*-morphism of d-manifolds with corners. Then the following are equivalent:*

(i) *$\boldsymbol{f}$ is étale;*

(ii) *$\boldsymbol{f}$ is simple and flat, in the sense of* §1.6.2, *and $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\mathbf{Y}) \to T^*\mathbf{X}$ is an equivalence in* $\mathrm{vqcoh}(\underline{X})$; *and*

(iii) *$\boldsymbol{f}$ is simple and flat, and* (1.10) *is a split short exact sequence in* $\mathrm{qcoh}(\underline{X})$.

If in addition $f : X \to Y$ is a bijection, then $\boldsymbol{f}$ is an equivalence in $\mathbf{dMan^c}$.

Theorem 1.7.9. *Let V, W be manifolds with corners, $E \to V$, $F \to W$ be vector bundles, $s : V \to E$, $t : W \to F$ be smooth sections, $f : V \to W$ be smooth, and $\hat f : E \to f^*(F)$ be a morphism of vector bundles on V with $\hat f \circ s = f^*(t) + O(s^2)$. Then Examples* 1.7.3 *and* 1.7.4 *define principal d-manifolds with corners $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$ and a* 1*-morphism $\mathbf{S}_{f,\hat f} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$. This $\mathbf{S}_{f,\hat f}$ is étale if and only if f is simple and flat near $s^{-1}(0) \subseteq V$, in the sense of* §1.5.2, *and for each $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, equation* (1.11) *is exact. Also $\mathbf{S}_{f,\hat f}$ is an equivalence if and only if in addition $f|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection, where $s^{-1}(0) = \{v \in V : s(v) = 0\}$, $t^{-1}(0) = \{w \in W : t(w) = 0\}$.*

Theorem 1.7.10. *Suppose we are given the following data:*

(a) *an integer n;*

(b) *a Hausdorff, second countable topological space X;*

(c) *an indexing set I, and a total order $<$ on I;*

(d) *for each i in I, a manifold with corners V_i, a vector bundle $E_i \to V_i$ with $\dim V_i - \operatorname{rank} E_i = n$, a smooth section $s_i : V_i \to E_i$, and a homeomorphism $\psi_i : X_i \to \hat X_i$, where $X_i = \{v_i \in V_i : s_i(v_i) = 0\}$ and $\hat X_i \subseteq X$ is open; and*

(e) *for all $i < j$ in I, an open submanifold $V_{ij} \subseteq V_i$, a simple, flat map $e_{ij} : V_{ij} \to V_j$, and a morphism of vector bundles $\hat e_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$.*

Let this data satisfy the conditions:

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if* $i < j$ *in* I *then* $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j)$, *and* $\psi_i(X_i \cap V_{ij}) = \hat{X}_i \cap \hat{X}_j$, *and* $\psi_i|_{X_i \cap V_{ij}} = \psi_j \circ e_{ij}|_{X_i \cap V_{ij}}$, *and if* $v_i \in V_i$ *with* $s_i(v_i) = 0$ *and* $v_j = e_{ij}(v_i)$ *then the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} E_j|_{v_j} \longrightarrow 0;$$

(iii) *if* $i < j < k$ *in* I *then* $e_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = e_{jk} \circ e_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} + O(s_i^2)$ *and* $\hat{e}_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = e_{ij}|^*_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} + O(s_i)$.

Then there exist a d-manifold with corners $\mathbf{X}$ *with* $\operatorname{vdim} \mathbf{X} = n$ *and topological space* X, *and a* 1*-morphism* $\boldsymbol{\psi}_i : \mathbf{S}_{V_i, E_i, s_i} \to \mathbf{X}$ *in* $\mathbf{dMan^c}$ *with underlying continuous map* ψ_i *which is an equivalence with the open d-submanifold* $\hat{\mathbf{X}}_i \subseteq \mathbf{X}$ *corresponding to* $\hat{X}_i \subseteq X$ *for all* $i \in I$, *such that for all* $i < j$ *in* I *there exists a* 2*-morphism* $\eta_{ij} : \boldsymbol{\psi}_j \circ \mathbf{S}_{e_{ij}, \hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i \circ \boldsymbol{i}_{V_{ij}, V_i}$, *where* $\mathbf{S}_{e_{ij}, \hat{e}_{ij}} : \mathbf{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}} \to \mathbf{S}_{V_j, E_j, s_j}$ *and* $\boldsymbol{i}_{V_{ij}, V_i} : \mathbf{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}} \to \mathbf{S}_{V_i, E_i, s_i}$ *are as in Example* 1.7.4*. This* $\mathbf{X}$ *is unique up to equivalence in* $\mathbf{dMan^c}$*.*

Suppose also that Y *is a manifold with corners, and* $g_i : V_i \to Y$ *are smooth maps for all* $i \in I$, *and* $g_j \circ e_{ij} = g_i|_{V_{ij}} + O(s_i^2)$ *for all* $i < j$ *in* I*. Then there exist a* 1*-morphism* $\boldsymbol{h} : \mathbf{X} \to \mathbf{Y}$ *unique up to* 2*-isomorphism, where* $\mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(Y) = \mathbf{S}_{Y,0,0}$, *and* 2*-morphisms* $\zeta_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \mathbf{S}_{g_i, 0}$ *for all* $i \in I$*. Here* $\mathbf{S}_{Y,0,0}$ *is from Example* 1.7.3 *with vector bundle* E *and section* s *both zero, and* $\mathbf{S}_{g_i,0} : \mathbf{S}_{V_i, E_i, s_i} \to \mathbf{S}_{Y,0,0} = \mathbf{Y}$ *is from Example* 1.7.4 *with* $\hat{g}_i = 0$*.*

We can use Theorem 1.7.10 as a tool to prove the existence of d-manifold with corner structures on spaces coming from other areas of geometry.

1.7.4 Submersions, immersions and embeddings

In §1.4.5 we defined two kinds of submersions (submersions and w-submersions), immersions, and embeddings for d-manifolds. In §1.5.2 we defined two kinds of submersions (submersions and s-submersions), and three kinds of immersions (immersions, s- and sf-immersions), and embeddings for manifolds with corners. In §7.5, we combine both alternatives for d-manifolds with corners, giving four types of submersions, and six types of immersions and embeddings.

Definition 1.7.11. Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism in $\mathbf{dMan^c}$. As in §1.4.3 and §1.7.1, $T^*\mathbf{X}$ and $\underline{f}^*(T^*\mathbf{Y})$ are virtual vector bundles on $\underline{X}$ of ranks $\operatorname{vdim} \mathbf{X}, \operatorname{vdim} \mathbf{Y}$, and $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\mathbf{Y}) \to T^*\mathbf{X}$ is a 1-morphism in $\mathrm{vvect}(\underline{X})$. Also we have 1-morphisms $C(\boldsymbol{f}), \hat{C}(\boldsymbol{f}) : C(\mathbf{X}) \to C(\mathbf{Y})$ in $\mathbf{d\check{M}an^c} \subset \mathbf{dSpa^c}$ as in §1.6.5 and §1.7.1, so we can form $\Omega_{C(\boldsymbol{f})} : \underline{C(f)}^*(T^*C(\mathbf{Y})) \to T^*C(\mathbf{X})$ and $\Omega_{\hat{C}(\boldsymbol{f})} : \underline{\hat{C}(f)}^*(T^*C(\mathbf{Y})) \to T^*C(\mathbf{X})$. Then:

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{\boldsymbol{f}}$ is weakly injective. We call $\boldsymbol{f}$ an *sw-submersion* if it is also simple.

(b) We call $\boldsymbol{f}$ a *submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{C(\boldsymbol{f})}$ is injective. We call $\boldsymbol{f}$ an *s-submersion* if it is also simple.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $\Omega_{\boldsymbol{f}}$ is weakly surjective. We call $\boldsymbol{f}$ an *sw-immersion*, or *sfw-immersion*, if $\boldsymbol{f}$ is also simple, or simple and flat.

(d) We call $\boldsymbol{f}$ an *immersion* if $\Omega_{\hat{C}(\boldsymbol{f})}$ is surjective. We call $\boldsymbol{f}$ an *s-immersion* if $\boldsymbol{f}$ is also simple, and an *sf-immersion* if $\boldsymbol{f}$ is also simple and flat.

(e) We call $\boldsymbol{f}$ a *w-embedding, sw-embedding, sfw-embedding, embedding, s-embedding*, or *sf-embedding*, if $\boldsymbol{f}$ is a w-immersion, ..., sf-immersion, respectively, and $f : X \to f(X)$ is a homeomorphism, so f is injective.

Here (weakly) injective and (weakly) surjective 1-morphisms in $\mathrm{vvect}(\underline{X})$ are defined in §1.4.5.

Parts (c)–(e) enable us to define *d-submanifolds* $\mathbf{X}$ of a d-manifold with corners $\mathbf{Y}$. *Open d-submanifolds* are open d-subspaces $\mathbf{X}$ in $\mathbf{Y}$. For more general d-submanifolds, we call $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ a *w-immersed, sw-immersed, sfw-immersed, immersed, s-immersed, sf-immersed, w-embedded, sw-embedded, sfw-embedded, embedded, s-embedded*, or *sf-embedded d-submanifold* of $\mathbf{Y}$ if $\mathbf{X}, \mathbf{Y}$ are d-manifolds with corners and $\boldsymbol{f}$ is a w-immersion, ..., sf-embedding, respectively.

Here is the analogue of Theorem 1.4.20, proved in §7.5.

Theorem 1.7.12. (i) *Any equivalence of d-manifolds with corners is a w-submersion, submersion, ..., sf-embedding.*

(ii) *If* $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ *are* 2*-isomorphic* 1*-morphisms of d-manifolds with corners then* $\boldsymbol{f}$ *is a w-submersion, ..., sf-embedding, if and only if* $\boldsymbol{g}$ *is.*

(iii) *Compositions of w-submersions, ..., sf-embeddings are of the same kind.*

(iv) *The conditions that a* 1*-morphism* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *in* $\mathbf{dMan^c}$ *is any kind of submersion or immersion are local in* $\mathbf{X}$ *and* $\mathbf{Y}$*. The conditions that* $\boldsymbol{f}$ *is any kind of embedding are local in* $\mathbf{Y}$*, but not in* $\mathbf{X}$*.*

(v) *Let* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *be a submersion in* $\mathbf{dMan^c}$*. Then* $\mathrm{vdim}\,\mathbf{X} \geqslant \mathrm{vdim}\,\mathbf{Y}$*, and if* $\mathrm{vdim}\,\mathbf{X} = \mathrm{vdim}\,\mathbf{Y}$ *then* $\boldsymbol{f}$ *is étale.*

(vi) *Let* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *be an immersion in* $\mathbf{dMan^c}$*. Then* $\mathrm{vdim}\,\mathbf{X} \leqslant \mathrm{vdim}\,\mathbf{Y}$*. If* $\boldsymbol{f}$ *is an s-immersion and* $\mathrm{vdim}\,\mathbf{X} = \mathrm{vdim}\,\mathbf{Y}$ *then* $\boldsymbol{f}$ *is étale.*

(vii) *Let* $f : X \to Y$ *be a smooth map of manifolds with corners, and* $\boldsymbol{f} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(f)$*. Then* $\boldsymbol{f}$ *is a submersion, s-submersion, immersion, s-immersion, sf-immersion, embedding, s-embedding, or sf-embedding, in* $\mathbf{dMan^c}$ *if and only if* f *is a submersion, ..., an sf-embedding in* $\mathbf{Man^c}$*, respectively. Also* $\boldsymbol{f}$ *is a w-immersion, sw-immersion, sfw-immersion, w-embedding, sw-embedding, or sfw-embedding in* $\mathbf{dMan^c}$ *if and only if* f *is an immersion, ..., sf-embedding in* $\mathbf{Man^c}$*, respectively.*

(viii) *Let* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *be a* 1*-morphism in* $\mathbf{dMan^c}$*, with* $\mathbf{Y}$ *a manifold. Then* $\boldsymbol{f}$ *is a w-submersion if and only if it is semisimple and flat, and* $\boldsymbol{f}$ *is an sw-submersion if and only if it is simple and flat.*

(ix) *Let* $\mathbf{X}, \mathbf{Y}$ *be d-manifolds with corners, with* $\mathbf{Y}$ *a manifold. Then* $\boldsymbol{\pi}_{\mathbf{X}} : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ *is a submersion, and* $\boldsymbol{\pi}_{\mathbf{X}}$ *is an s-submersion if* $\partial \mathbf{Y} = \boldsymbol{\emptyset}$.

(x) *Suppose* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *is a submersion in* $\mathbf{dMan^c}$, *and* $x \in \mathbf{X}$ *with* $\boldsymbol{f}(x) = y \in \mathbf{Y}$. *Then there exist open d-submanifolds* $x \in \mathbf{U} \subseteq \mathbf{X}$ *and* $y \in \mathbf{V} \subseteq \mathbf{Y}$ *with* $\boldsymbol{f}(\mathbf{U}) = \mathbf{V}$, *a manifold with corners* $\mathbf{Z}$, *and an equivalence* $\boldsymbol{i} : \mathbf{U} \to \mathbf{V} \times \mathbf{Z}$, *such that* $\boldsymbol{f}|_{\mathbf{U}} : \mathbf{U} \to \mathbf{V}$ *is* 2*-isomorphic to* $\boldsymbol{\pi}_{\mathbf{V}} \circ \boldsymbol{i}$, *where* $\boldsymbol{\pi}_{\mathbf{V}} : \mathbf{V} \times \mathbf{Z} \to \mathbf{V}$ *is the projection. If* $\boldsymbol{f}$ *is an s-submersion then* $\partial \mathbf{Z} = \boldsymbol{\emptyset}$.

(xi) *Let* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *be a submersion of d-manifolds with corners, with* $\mathbf{Y}$ *a manifold with corners. Then* $\mathbf{X}$ *is a manifold with corners.*

Parts (ix)-(x) are a d-manifold analogue of Proposition 1.5.7.

1.7.5 Bd-transversality and fibre products

In §7.6 we extend §1.4.6 to the corners case. Here are the analogues of Definition 1.4.21 and Theorems 1.4.22–1.4.25:

Definition 1.7.13. Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be d-manifolds with corners and $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$, $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms. We call $\boldsymbol{g}, \boldsymbol{h}$ *bd-transverse* if they are both b-transverse in $\mathbf{dSpa^c}$ in the sense of Definition 1.6.12, and d-transverse in the sense of Definition 1.4.21. We call $\boldsymbol{g}, \boldsymbol{h}$ *cd-transverse* if they are both c-transverse in $\mathbf{dSpa^c}$ in the sense of Definition 1.6.12, and d-transverse. As in §1.6.6, c-transverse implies b-transverse, so cd-transverse implies bd-transverse.

Theorem 1.7.14. *Suppose* $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ *are d-manifolds with corners and* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$, $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *are bd-transverse* 1*-morphisms, and let* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ *be the fibre product in* $\mathbf{dSpa^c}$, *which exists by Theorem* 1.6.14(a) *as* $\boldsymbol{g}, \boldsymbol{h}$ *are b-transverse. Then* $\mathbf{W}$ *is a d-manifold with corners, with*

$$\operatorname{vdim} \mathbf{W} = \operatorname{vdim} \mathbf{X} + \operatorname{vdim} \mathbf{Y} - \operatorname{vdim} \mathbf{Z}. \tag{1.37}$$

Hence, all bd-transverse fibre products exist in $\mathbf{dMan^c}$.

Theorem 1.7.15. *Suppose* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ *and* $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *are* 1*-morphisms in* $\mathbf{dMan^c}$. *The following are sufficient conditions for* $\boldsymbol{g}, \boldsymbol{h}$ *to be cd-transverse, and hence bd-transverse, so that* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ *is a d-manifold with corners of virtual dimension* (1.37):

- **(a)** $\mathbf{Z}$ *is a manifold without boundary, that is,* $\boldsymbol{Z} \in \bar{\mathbf{M}}\mathbf{an}$; *or*
- **(b)** $\boldsymbol{g}$ *or* $\boldsymbol{h}$ *is a w-submersion.*

Theorem 1.7.16. *Let* $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ *be d-manifolds with corners with* $\mathbf{Y}$ *a manifold, and* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$, $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *be* 1*-morphisms with* $\boldsymbol{g}$ *a submersion. Then* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ *is a manifold, with* $\dim \mathbf{W} = \operatorname{vdim} \mathbf{X} + \dim \mathbf{Y} - \operatorname{vdim} \mathbf{Z}$.

Theorem 1.7.17. **(i)** *Let* $\mathbf{X}$ *be a d-manifold with corners and* $\boldsymbol{g} : \mathbf{X} \to [\mathbf{0}, \boldsymbol{\infty})^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}}$ *a semisimple, flat* 1*-morphism in* $\mathbf{dMan^c}$. *Then the fibre product* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},[\mathbf{0},\boldsymbol{\infty})^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}},\mathbf{0}} *$ *exists in* $\mathbf{dMan^c}$, *and* $\boldsymbol{\pi}_{\mathbf{X}} : \mathbf{W} \to \mathbf{X}$ *is an* *s-embedding. When $k=0$, any 1-morphism $\boldsymbol{g}:\mathbf{X}\to\mathbb{R}^{\boldsymbol{n}}$ is semisimple and flat, and $\boldsymbol{\pi}_{\mathbf{X}}:\mathbf{W}\to\mathbf{X}$ is an sf-embedding.*

(ii) *Suppose $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ is an s-immersion of d-manifolds with corners, and $x\in\mathbf{X}$ with $\boldsymbol{f}(x)=y\in\mathbf{Y}$. Then there exist open d-submanifolds $x\in\mathbf{U}\subseteq\mathbf{X}$ and $y\in\mathbf{V}\subseteq\mathbf{Y}$ with $\boldsymbol{f}(\mathbf{U})\subseteq\mathbf{V}$ and a semisimple, flat 1-morphism $\boldsymbol{g}:\mathbf{V}\to[\mathbf{0},\infty)^{\boldsymbol{k}}\times\mathbb{R}^{\boldsymbol{n}-\boldsymbol{k}}$ with $\boldsymbol{g}(y)=0$, where $n=\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{X}\geqslant 0$ and $0\leqslant k\leqslant n$, fitting into a 2-Cartesian square in $\mathbf{dMan^c}$:*

$$\begin{tikzcd} \mathbf{U} \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{f}|_{\mathbf{U}}"'] \arrow[drr, phantom, "\Uparrow"] & & \boldsymbol{*} \arrow[d, "\mathbf{0}"] \\ \mathbf{V} \arrow[rr, "\boldsymbol{g}"] & & {[\mathbf{0},\infty)^{\boldsymbol{k}}\times\mathbb{R}^{\boldsymbol{n}-\boldsymbol{k}}.} \end{tikzcd}$$

If $\boldsymbol{f}$ is an sf-immersion then $k=0$. If $\boldsymbol{f}$ is an s- or sf-embedding then we may take $\mathbf{U}=\boldsymbol{f}^{-1}(\mathbf{V})$.

For ordinary manifolds, a submanifold X in Y may be described locally either as the image of an embedding $X\hookrightarrow Y$, or equivalently as the zeroes of a submersion $Y\to\mathbb{R}^n$, where $n=\dim Y-\dim X$. Theorem 1.7.17 is an analogue of this for d-manifolds with corners. It should be compared with Proposition 1.5.8 for manifolds with corners.

1.7.6 Embedding d-manifolds with corners into manifolds

In §1.4.7 we discussed embeddings of d-manifolds $\boldsymbol{X}$ into manifolds Y. Our two major results were Theorem 1.4.29, which gave necessary and sufficient conditions on $\boldsymbol{X}$ for existence of embeddings $\boldsymbol{f}:\boldsymbol{X}\hookrightarrow\mathbb{R}^{\boldsymbol{n}}$ for $n\gg 0$, and Theorem 1.4.32, which showed that if an embedding $\boldsymbol{f}:\boldsymbol{X}\hookrightarrow\boldsymbol{Y}$ exists with $\boldsymbol{X}$ a d-manifold and $\boldsymbol{Y}=F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$, then $\boldsymbol{X}\simeq\boldsymbol{S}_{V,E,s}$ for open $V\subseteq Y$, so $\boldsymbol{X}$ is a principal d-manifold.

Section 7.7 generalizes these results to d-manifolds with corners. As in §1.7.4, we have three kinds of embeddings in $\mathbf{dMan^c}$, embeddings, s-embeddings and sf-embeddings. The analogue of Theorem 1.4.29 naturally holds for embeddings:

Theorem 1.7.18. *Let $\mathbf{X}$ be a d-manifold with corners. Then there exist immersions and/or embeddings $\boldsymbol{f}:\mathbf{X}\to\mathbb{R}^{\boldsymbol{n}}$ for some $n\gg 0$ if and only if there is an upper bound for $\dim T_x^*\underline{X}$ for all $x\in\underline{X}$. If there is such an upper bound, then immersions $\boldsymbol{f}:\mathbf{X}\to\mathbb{R}^{\boldsymbol{n}}$ exist provided $n\geqslant 2\dim T_x^*\underline{X}$ for all $x\in\underline{X}$, and embeddings $\boldsymbol{f}:\mathbf{X}\to\mathbb{R}^{\boldsymbol{n}}$ exist provided $n\geqslant 2\dim T_x^*\underline{X}+1$ for all $x\in\underline{X}$. For embeddings we may also choose $\boldsymbol{f}$ with $f(X)$ closed in $\mathbb{R}^n$.*

Example 1.4.30 shows the hypotheses of Theorem 1.7.18 need not hold, so there exist d-manifolds with corners $\mathbf{X}$ with no embedding into $\mathbb{R}^n$, or into any manifold with corners. The analogue of Theorem 1.4.32 holds for sf-embeddings:

Theorem 1.7.19. *Let $\mathbf{X}$ be a d-manifold with corners, Y a manifold with corners, and $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ an sf-embedding, in the sense of Definition 1.7.11. Then there exist an open subset V in Y with $\boldsymbol{f}(\mathbf{X})\subseteq\mathbf{V}$, a vector bundle $E\to V$, and* *a smooth section $s : V \to E$ of E fitting into a 2-Cartesian diagram in $\mathbf{dMan^c}$, where $0 : V \to E$ is the zero section and $\mathbf{Y}, \mathbf{V}, \mathbf{E}, \boldsymbol{s}, \mathbf{0} = F^{\mathbf{dMan^c}}_{\mathbf{Man^c}}(Y, V, E, s, 0)$:*

$$\begin{tikzcd} \mathbf{X} \arrow[r, "\boldsymbol{f}"'] \arrow[d, "\boldsymbol{f}"] \arrow[dr, phantom, "\Uparrow"] & \mathbf{V} \arrow[d, "\mathbf{0}"'] \\ \mathbf{V} \arrow[r, "\boldsymbol{s}"] & \mathbf{E}. \end{tikzcd}$$

Hence $\mathbf{X}$ is equivalent to the 'standard model' $\mathbf{S}_{V,E,s}$ of Example 1.7.3, *and is a principal d-manifold with corners.*

Note that, unlike the d-manifolds case in §1.4.7, we cannot immediately combine Theorems 1.7.18 and 1.7.19: we have first to bridge the gap between embeddings and sf-embeddings. For d-manifolds with boundary, we can do this.

Theorem 1.7.20. *Let $\mathbf{X}$ be a d-manifold with boundary. Then there exist sf-immersions and/or sf-embeddings $\boldsymbol{f} : \mathbf{X} \to \boldsymbol{[0,\infty)} \times \boldsymbol{\mathbb{R}^{n-1}}$ for some $n \gg 0$ if and only if $\dim T^*_x \underline{X}$ is bounded above for all $x \in \underline{X}$. Such an upper bound always exists if $\mathbf{X}$ is compact. If there is such an upper bound, then sf-immersions $\boldsymbol{f} : \mathbf{X} \to \boldsymbol{[0,\infty)} \times \boldsymbol{\mathbb{R}^{n-1}}$ exist provided $n \geqslant 2 \dim T^*_x \underline{X} + 1$ for all $x \in \underline{X}$, and sf-embeddings $\boldsymbol{f} : \mathbf{X} \to \boldsymbol{[0,\infty)} \times \boldsymbol{\mathbb{R}^{n-1}}$ exist provided $n \geqslant 2 \dim T^*_x \underline{X} + 2$ for all $x \in \underline{X}$. For sf-embeddings we may also choose $\boldsymbol{f}$ with $f(X)$ closed in $[0,\infty) \times \mathbb{R}^{n-1}$.*

Combining Theorems 1.7.19 and 1.7.20 shows that a d-manifold with boundary $\mathbf{X}$ is principal if and only if $\dim T^*_x \underline{X}$ is bounded above.

Since (nice) d-manifolds with boundary can be embedded into $[0,\infty) \times \mathbb{R}^{n-1}$ for $n \gg 0$, one might guess that (nice) d-manifolds with corners can be embedded into $[0,\infty)^k \times \mathbb{R}^{n-k}$ for $n \gg k \gg 0$. However, this is not true even for manifolds with corners, as the following example from §5.7 shows:

Example 1.7.21. Consider the *teardrop* $T = \{(x,y) \in \mathbb{R}^2 : x \geqslant 0,\ y^2 \leqslant x^2 - x^4\}$, shown in Figure 1.1. It is a compact 2-manifold with corners.

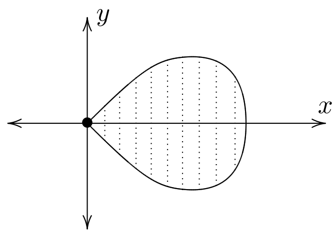

Figure 1.1: The teardrop, a 2-manifold with corners.

Suppose that $f : T \to [0,\infty)^k \times \mathbb{R}^{n-k}$ is an sf-embedding. As f is simple and flat, it maps $S^j(T) \hookrightarrow S^j\big([0,\infty)^k \times \mathbb{R}^{n-k}\big)$ for $j = 0, 1, 2$, in the notation of §1.5.1. The connected components of $S^j\big([0,\infty)^k \times \mathbb{R}^{n-k}\big)$ correspond to subsets $I \subseteq \{1, \ldots, k\}$ with $|I| = j$, with the component corresponding to I given by the equations $x_i = 0$ for $i \in I$ and $x_a > 0$ for $a \in \{1, \ldots, k\} \setminus I$. As

$(0,0) \in S^2(T)$, we see that $f(0,0)$ lies in the component of $S^2\big([0,\infty)^k \times \mathbb{R}^{n-k}\big)$ given by $x_a = x_b = 0$ for $1 \leqslant a < b \leqslant k$.

Considering local models for f near $(0,0) \in T$, we see that f must map the two ends of $S^1(T)$ at $(0,0)$ into different connected components $x_a = 0$ and $x_b = 0$ of $S^1\big([0,\infty)^k \times \mathbb{R}^{n-k}\big)$. However, $S^1(T) \cong (0,1)$ is connected, so f maps $S^1(T)$ into a single connected component of $S^1\big([0,\infty)^k \times \mathbb{R}^{n-k}\big)$, a contradiction. Hence there do not exist sf-embeddings $f : T \to [0,\infty)^k \times \mathbb{R}^{n-k}$ for any n, k.

Here are necessary and sufficient conditions for existence of sf-embeddings from a d-manifold with corners $\mathbf{X}$ into a manifold with corners Y.

Theorem 1.7.22. *Let* $\mathbf{X}$ *be a d-manifold with corners. Then there exist a manifold with corners* Y *and an sf-embedding* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$, *where* $\mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(Y)$, *if and only if* $\dim T_x^*\underline{X} + |\underline{i}_{\mathbf{X}}^{-1}(x)|$ *is bounded above for all* $x \in \underline{X}$. *If such an upper bound exists, then we may take* Y *to be an embedded* n*-dimensional submanifold of* $\mathbb{R}^n$ *for any* n *with* $n \geqslant 2\big(\dim T_x^*\underline{X} + |\underline{i}_{\mathbf{X}}^{-1}(x)|\big) + 1$ *for all* $x \in \underline{X}$.

Such an upper bound always exists if $\mathbf{X}$ *is compact. Thus, every compact d-manifold with corners admits an sf-embedding into a manifold with corners.*

The idea of the proof of Theorem 1.7.22 is that we first choose an embedding $\boldsymbol{g} : \mathbf{X} \to \boldsymbol{\mathbb{R}^n}$ using Theorem 1.7.18, and then show that we can choose a submanifold $Y \subseteq \mathbb{R}^n$ which is the set of points in an open neighbourhood U of $g(X)$ in $\mathbb{R}^n$ satisfying local transverse inequalities of the form $c_i(x) \geqslant 0$ for $i = 1, \ldots, k$, where $c_i : U \to \mathbb{R}$ are local smooth functions which lift under $\boldsymbol{g}$ to local boundary defining functions for $\partial\mathbf{X}$.

Combining Theorems 1.7.19 and 1.7.22 yields:

Corollary 1.7.23. *Let* $\mathbf{X}$ *be a d-manifold with corners. Then* $\mathbf{X}$ *is principal, that is,* $\mathbf{X}$ *is equivalent in* $\mathbf{dMan^c}$ *to some* $\mathbf{S}_{V,E,s}$ *in Example* 1.7.3, *if and only if* $\dim T_x^*\underline{X}$ *and* $|\underline{i}_{\mathbf{X}}^{-1}(x)|$ *are bounded above for all* $x \in \underline{X}$. *This holds automatically if* $\mathbf{X}$ *is compact.*

1.7.7 Orientations

In §7.8 we study orientations on d-manifolds with corners, following the d-manifold case in §1.4.8. Here is the analogue of Definition 1.4.35:

Definition 1.7.24. Let $\mathbf{X}$ be a d-manifold with corners. Then the virtual cotangent bundle $T^*\mathbf{X} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X)$ is a virtual vector bundle on $\underline{X}$, so Theorem 1.4.34 gives a line bundle $\mathcal{L}_{T^*\mathbf{X}}$ on $\underline{X}$. We call $\mathcal{L}_{T^*\mathbf{X}}$ the *orientation line bundle* of $\mathbf{X}$.

An *orientation* ω on $\mathbf{X}$ is an orientation on $\mathcal{L}_{T^*\mathbf{X}}$, in the sense of Definition 1.4.35. An *oriented d-manifold with corners* is a pair $(\mathbf{X}, \omega)$ where $\mathbf{X}$ is a d-manifold with corners and ω an orientation on $\mathbf{X}$. Usually we refer to $\mathbf{X}$ as an oriented d-manifold, leaving ω implicit. We also write $-\mathbf{X}$ for $\mathbf{X}$ with the opposite orientation, that is, $\mathbf{X}$ is short for $(\mathbf{X}, \omega)$ and $-\mathbf{X}$ short for $(\mathbf{X}, -\omega)$.

Example 1.4.36, Theorem 1.4.37 and Proposition 1.4.38 now extend to d-manifolds with corners without change. We can also orient boundaries of oriented d-manifolds with corners. Theorem 1.7.25 is the main reason for including the data $\omega_{\mathbf{X}}$ in a d-manifold with corners $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$.

Theorem 1.7.25. *Let* $\mathbf{X}$ *be a d-manifold with corners. Then* $\partial\mathbf{X}$ *is also a d-manifold with corners, so we have orientation line bundles* $\mathcal{L}_{T^*\mathbf{X}}$ *on* $\underline{X}$ *and* $\mathcal{L}_{T^*(\partial\mathbf{X})}$ *on* $\underline{\partial X}$. *There is a canonical isomorphism of line bundles on* $\underline{\partial X}$:

$$\Psi : \mathcal{L}_{T^*(\partial\mathbf{X})} \longrightarrow \underline{i}_{\mathbf{X}}^*(\mathcal{L}_{T^*\mathbf{X}}) \otimes \mathcal{N}_{\mathbf{X}}^*, \tag{1.38}$$

where $\mathcal{N}_{\mathbf{X}}$ *is the conormal bundle of* $\boldsymbol{\partial X}$ *in* $\boldsymbol{X}$ *from* §1.6.1.

Now $\mathcal{N}_{\mathbf{X}}$ *comes with an orientation* $\omega_{\mathbf{X}}$ *in* $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$. *Hence, if* $\mathbf{X}$ *is an oriented d-manifold with corners, then* $\partial\mathbf{X}$ *also has a natural orientation, by combining the orientations on* $\mathcal{L}_{T^*\mathbf{X}}$ *and* $\mathcal{N}_{\mathbf{X}}^*$ *to get an orientation on* $\mathcal{L}_{T^*(\partial\mathbf{X})}$ *using* (1.38).

As for Proposition 1.4.38, natural equivalences of d-manifolds with corners generally extend to natural equivalences of oriented d-manifolds with corners, with some sign depending on the orientation conventions. Here are two such results, which include signs in Theorem 1.6.3(b) and Proposition 1.6.15.

Proposition 1.7.26. *Suppose* $\mathbf{X}, \mathbf{Y}$ *are oriented d-manifolds with corners, and* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *is a semisimple, flat* 1*-morphism. Then the following holds in oriented d-manifolds with corners, with fibre products cd-transverse:*

$$\partial_-^{\boldsymbol{f}}\mathbf{X} \simeq \partial\mathbf{Y} \times_{\boldsymbol{i}_{\mathbf{Y}},\mathbf{Y},\boldsymbol{f}} \mathbf{X} \simeq (-1)^{\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}}\mathbf{X} \times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\mathbf{Y}}} \partial\mathbf{Y}.$$

If $\boldsymbol{f}$ *is also simple then* $\partial_-^{\boldsymbol{f}}\mathbf{X} = \partial\mathbf{X}$.

Proposition 1.7.27. *Let* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ *and* $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *be* 1*-morphisms of oriented d-manifolds with corners. Then the following hold in oriented d-manifolds with corners, where all the fibre products are cd-transverse, and so exist:*

(a) *If* $\mathbf{Z}$ *is a manifold without boundary then there is an equivalence*

$$\partial\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \simeq \big(\partial\mathbf{X} \times_{\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{X}},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \amalg (-1)^{\mathrm{vdim}\,\mathbf{X}+\dim\mathbf{Z}}\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}} \partial\mathbf{Y}\big).$$

(b) *If* $\boldsymbol{g}$ *is a w-submersion then there is an equivalence*

$$\partial\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \simeq \big(\partial_+^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_+,\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \amalg (-1)^{\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Z}}\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}} \partial\mathbf{Y}\big).$$

(c) *If both* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ *are w-submersions then there is an equivalence*

$$\begin{aligned}\partial\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \simeq{}& \big(\partial_+^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_+,\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big)\\ &\amalg (-1)^{\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Z}}\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}_+} \partial_+^{\boldsymbol{h}}\mathbf{Y}\big) \amalg \big(\partial_-^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_-,\partial\mathbf{Z},\boldsymbol{h}_-} \partial_-^{\boldsymbol{h}}\mathbf{Y}\big).\end{aligned}$$

1.8 Deligne–Mumford C^∞-stacks

Appendix C and [56, §7–§11] discuss C^∞*-stacks*, including *Deligne–Mumford* C^∞*-stacks*, which are related to C^∞-schemes in the same way that Artin stacks and Deligne–Mumford stacks in algebraic geometry are related to schemes.

1.8.1 C^∞-stacks

The next few definitions assume a lot of standard material from stack theory, which is summarized in [56, §7].

Definition 1.8.1. Define a *Grothendieck topology* $\mathcal{J}$ on the category $\mathbf{C^\infty Sch}$ of C^∞-schemes to have coverings $\{\underline{i}_a : \underline{U}_a \to \underline{U}\}_{a\in A}$ where $V_a = i_a(U_a)$ is open in U with $\underline{i}_a : \underline{U}_a \to (V_a, \mathcal{O}_U|_{V_a})$ an isomorphism for all $a \in A$, and $U = \bigcup_{a\in A} V_a$. Up to isomorphisms of the $\underline{U}_a$, the coverings $\{\underline{i}_a : \underline{U}_a \to \underline{U}\}_{a\in A}$ of $\underline{U}$ correspond exactly to open covers $\{V_a : a \in A\}$ of U. Then $(\mathbf{C^\infty Sch}, \mathcal{J})$ is a *site*.

The stacks on $(\mathbf{C^\infty Sch}, \mathcal{J})$ form a 2-category $\mathbf{Sta}_{(\mathbf{C^\infty Sch},\mathcal{J})}$, with all 2-morphisms invertible. As the site $(\mathbf{C^\infty Sch}, \mathcal{J})$ is subcanonical, there is a natural, fully faithful functor $\mathbf{C^\infty Sch} \to \mathbf{Sta}_{(\mathbf{C^\infty Sch},\mathcal{J})}$, defined explicitly below, which we write as $\underline{X} \mapsto \bar{\underline{X}}$ on objects and $\underline{f} \mapsto \bar{\underline{f}}$ on morphisms. A C^∞*-stack* is a stack $\mathcal{X}$ on $(\mathbf{C^\infty Sch}, \mathcal{J})$ such that the diagonal 1-morphism $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}\times\mathcal{X}$ is representable, and there exists a surjective 1-morphism $\Pi : \bar{\underline{U}} \to \mathcal{X}$ called an *atlas* for some C^∞-scheme $\underline{U}$. Write $\mathbf{C^\infty Sta}$ for the full 2-subcategory of C^∞-stacks in $\mathbf{Sta}_{(\mathbf{C^\infty Sch},\mathcal{J})}$. The functor $\mathbf{C^\infty Sch} \to \mathbf{Sta}_{(\mathbf{C^\infty Sch},\mathcal{J})}$ above maps into $\mathbf{C^\infty Sta}$, so we also write it as $F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}} : \mathbf{C^\infty Sch} \to \mathbf{C^\infty Sta}$.

Formally, a C^∞-stack is a category $\mathcal{X}$ with a functor $p_{\mathcal{X}} : \mathcal{X} \to \mathbf{C^\infty Sch}$, where $\mathcal{X}, p_{\mathcal{X}}$ must satisfy many complicated conditions, including sheaf-like conditions for all open covers in $\mathbf{C^\infty Sch}$. A 1*-morphism* $f : \mathcal{X} \to \mathcal{Y}$ of C^∞-stacks is a functor $f : \mathcal{X} \to \mathcal{Y}$ with $p_{\mathcal{Y}} \circ f = p_{\mathcal{X}} : \mathcal{X} \to \mathbf{C^\infty Sch}$. Given 1-morphisms $f, g : \mathcal{X} \to \mathcal{Y}$, a 2*-morphism* $\eta : f \Rightarrow g$ is an isomorphism of functors $\eta : f \Rightarrow g$ with $\mathrm{id}_{p_{\mathcal{Y}}} * \eta = \mathrm{id}_{p_{\mathcal{X}}} : p_{\mathcal{Y}} \circ f \Rightarrow p_{\mathcal{Y}} \circ g$.

If $\underline{X}$ is a C^∞-scheme, the corresponding C^∞-stack $\bar{\underline{X}} = F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}}(\underline{X})$ is the category with objects $(\underline{U}, \underline{u})$ for $\underline{u} : \underline{U} \to \underline{X}$ a morphism in $\mathbf{C^\infty Sch}$, and morphisms $\underline{h} : (\underline{U}, \underline{u}) \to (\underline{V}, \underline{v})$ for $\underline{h} : \underline{U} \to \underline{V}$ a morphism in $\mathbf{C^\infty Sch}$ with $\underline{v}\circ\underline{h} = \underline{u}$. The functor $p_{\bar{\underline{X}}} : \bar{\underline{X}} \to \mathbf{C^\infty Sch}$ maps $p_{\bar{\underline{X}}} : (\underline{U}, \underline{u}) \mapsto \underline{U}$ and $p_{\bar{\underline{X}}} : \underline{h} \mapsto \underline{h}$.

If $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes, the corresponding 1-morphism $\bar{\underline{f}} = F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}}(\underline{f}) : \bar{\underline{X}} \to \bar{\underline{Y}}$ maps $\bar{\underline{f}} : (\underline{U}, \underline{u}) \mapsto (\underline{U}, \underline{f} \circ \underline{u})$ on objects $(\underline{U}, \underline{u})$ and $\bar{\underline{f}} : \underline{h} \mapsto \underline{h}$ on morphisms $\underline{h}$ in $\bar{\underline{X}}$. This defines a functor $\bar{\underline{f}} : \bar{\underline{X}} \to \bar{\underline{Y}}$ with $p_{\bar{\underline{Y}}} \circ \bar{\underline{f}} = p_{\bar{\underline{X}}} : \mathcal{X} \to \mathbf{C^\infty Sch}$, so $\bar{\underline{f}}$ is a 1-morphism $\bar{\underline{f}} : \bar{\underline{X}} \to \bar{\underline{Y}}$ in $\mathbf{C^\infty Sta}$.

We define some classes of morphisms of C^∞-schemes:

Definition 1.8.2. Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism in $\mathbf{C^\infty Sch}$. Then:

- We call $\underline{f}$ an *open embedding* if it is an isomorphism with an open C^∞-subscheme of $\underline{Y}$.
- We call $\underline{f}$ *étale* if it is a local isomorphism (in the Zariski topology).

- We call $\underline{f}$ *proper* if $f : X \to Y$ is a proper map of topological spaces, that is, if $S \subseteq Y$ is compact then $f^{-1}(S) \subseteq X$ is compact.
- We call $\underline{f}$ *universally closed* if whenever $\underline{g} : \underline{W} \to \underline{Y}$ is a morphism then $\pi_W : X \times_{f,Y,g} W \to W$ is a closed map of topological spaces.

Each one is invariant under base change and local in the target in $(\mathbf{C^\infty Sch}, \mathcal{J})$. Thus, they are also defined for representable 1-morphisms of C^∞-stacks.

Definition 1.8.3. Let $\mathcal{X}$ be a C^∞-stack. We say that $\mathcal{X}$ is *separated* if the diagonal 1-morphism $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is universally closed. If $\mathcal{X} \simeq \bar{\underline{X}}$ for some C^∞-scheme $\underline{X}$ then $\mathcal{X}$ is separated if and only if $\underline{X}$ is separated (Hausdorff).

Definition 1.8.4. Let $\mathcal{X}$ be a C^∞-stack. A *C^∞-substack* $\mathcal{Y}$ in $\mathcal{X}$ is a strictly full subcategory $\mathcal{Y}$ in $\mathcal{X}$ such that $p_{\mathcal{Y}} := p_{\mathcal{X}}|_{\mathcal{Y}} : \mathcal{Y} \to \mathbf{C^\infty Sch}$ is also a C^∞-stack. It has a natural inclusion 1-morphism $i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{X}$. We call $\mathcal{Y}$ an *open C^∞-substack* of $\mathcal{X}$ if $i_{\mathcal{Y}}$ is a representable open embedding. An *open cover* $\{\mathcal{Y}_a : a \in A\}$ of $\mathcal{X}$ is a family of open C^∞-substacks $\mathcal{Y}_a$ in $\mathcal{X}$ with $\coprod_{a\in A} i_{\mathcal{Y}_a} : \coprod_{a\in A} \mathcal{Y}_a \to \mathcal{X}$ surjective.

1.8.2 Topological spaces of C^∞-stacks

By [56, §8.4], a C^∞-stack $\mathcal{X}$ has an underlying topological space $\mathcal{X}_{\rm top}$.

Definition 1.8.5. Let $\mathcal{X}$ be a C^∞-stack. Write $\underline{*}$ for the point $\operatorname{Spec}\mathbb{R}$ in $\mathbf{C^\infty Sch}$, and $\bar{\underline{*}}$ for the associated point in $\mathbf{C^\infty Sta}$. Define $\mathcal{X}_{\rm top}$ to be the set of 2-isomorphism classes $[x]$ of 1-morphisms $x : \bar{\underline{*}} \to \mathcal{X}$. If $\mathcal{U} \subseteq \mathcal{X}$ is an open C^∞-substack then any 1-morphism $x : \bar{\underline{*}} \to \mathcal{U}$ is also a 1-morphism $x : \bar{\underline{*}} \to \mathcal{X}$, and $\mathcal{U}_{\rm top}$ is a subset of $\mathcal{X}_{\rm top}$. Define $\mathcal{T}_{\mathcal{X}_{\rm top}} = \big\{\mathcal{U}_{\rm top} : \mathcal{U} \subseteq \mathcal{X}$ is an open C^∞-substack in $\mathcal{X}\big\}$. Then $\mathcal{T}_{\mathcal{X}_{\rm top}}$ is a set of subsets of $\mathcal{X}_{\rm top}$ which is a topology on $\mathcal{X}_{\rm top}$, so $(\mathcal{X}_{\rm top}, \mathcal{T}_{\mathcal{X}_{\rm top}})$ is a topological space, which we call the *underlying topological space* of $\mathcal{X}$, and usually write as $\mathcal{X}_{\rm top}$. If $\underline{X} = (X, \mathcal{O}_X)$ is a C^∞-scheme, so that $\bar{\underline{X}}$ is a C^∞-stack, then $\bar{\underline{X}}_{\rm top}$ is naturally homeomorphic to X.

If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks then there is a natural continuous map $f_{\rm top} : \mathcal{X}_{\rm top} \to \mathcal{Y}_{\rm top}$ defined by $f_{\rm top}([x]) = [f \circ x]$. If $f, g : \mathcal{X} \to \mathcal{Y}$ are 1-morphisms and $\eta : f \Rightarrow g$ is a 2-morphism then $f_{\rm top} = g_{\rm top}$. Mapping $\mathcal{X} \mapsto \mathcal{X}_{\rm top}$, $f \mapsto f_{\rm top}$ and 2-morphisms to identities defines a 2-functor $F_{\mathbf{C^\infty Sta}}^{\mathbf{Top}} : \mathbf{C^\infty Sta} \to \mathbf{Top}$, where the category of topological spaces $\mathbf{Top}$ is regarded as a 2-category with only identity 2-morphisms.

Definition 1.8.6. Let $\mathcal{X}$ be a C^∞-stack, and $[x] \in \mathcal{X}_{\rm top}$. Pick a representative x for $[x]$, so that $x : \bar{\underline{*}} \to \mathcal{X}$ is a 1-morphism. Define the *orbifold group* (or *isotropy group*, or *stabilizer group*) $\operatorname{Iso}_{\mathcal{X}}([x])$ of $[x]$ to be the group of 2-morphisms $\eta : x \Rightarrow x$. It is independent of the choice of $x \in [x]$ up to isomorphism, which is canonical up to conjugation in $\operatorname{Iso}_{\mathcal{X}}([x])$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks and $[x] \in \mathcal{X}_{\rm top}$ with $f_{\rm top}([x]) = [y] \in \mathcal{Y}_{\rm top}$, for $y = f \circ x$, then we define a group morphism $f_* : \operatorname{Iso}_{\mathcal{X}}([x]) \to \operatorname{Iso}_{\mathcal{Y}}([y])$ by $f_*(\eta) = \mathrm{id}_f * \eta$. It is independent of choices of $x \in [x]$, $y \in [y]$ up to conjugation in $\operatorname{Iso}_{\mathcal{X}}([x]), \operatorname{Iso}_{\mathcal{Y}}([y])$.

1.8.3 Strongly representable 1-morphisms

Strongly representable 1-morphisms, discussed in [56, §8.6], will be important in the definitions of orbifolds, d-stacks, and d-orbifolds with corners.

Definition 1.8.7. Let $\mathcal{Y},\mathcal{Z}$ be C^∞-stacks, and $g:\mathcal{Y}\to\mathcal{Z}$ a 1-morphism. Then $\mathcal{Y},\mathcal{Z}$ are categories with functors $p_{\mathcal{Y}}:\mathcal{Y}\to\mathbf{C^\infty Sch}$, $p_{\mathcal{Z}}:\mathcal{Z}\to\mathbf{C^\infty Sch}$, and $g:\mathcal{Y}\to\mathcal{Z}$ is a functor with $p_{\mathcal{Z}}\circ g=p_{\mathcal{Y}}$.

We call g *strongly representable* if whenever $A\in\mathcal{Y}$ with $p_{\mathcal{Y}}(A)=\underline{U}\in\mathbf{C^\infty Sch}$, so that $B=g(A)\in\mathcal{Z}$ with $p_{\mathcal{Z}}(B)=\underline{U}$, and $b:B\to B'$ is an isomorphism in $\mathcal{Z}$ with $p_{\mathcal{Z}}(B')=\underline{U}$ and $p_{\mathcal{Z}}(b)=\underline{\mathrm{id}_U}$, then there exist a unique object A' and isomorphism $a:A\to A'$ in $\mathcal{Y}$ with $g(A')=B'$ and $g(a)=b$.

Here are two important properties of strongly representable 1-morphisms. The first says that we may replace a representable 1-morphism $g:\mathcal{Y}\to\mathcal{Z}$ with a strongly representable 1-morphism $g':\mathcal{Y}'\to\mathcal{Z}$ with $\mathcal{Y}'\simeq\mathcal{Y}$.

Proposition 1.8.8. (a) *Let $g:\mathcal{Y}\to\mathcal{Z}$ be a strongly representable* 1*-morphism of C^∞-stacks. Then g is representable.*

(b) *Suppose $g:\mathcal{Y}\to\mathcal{Z}$ is a representable* 1*-morphism of C^∞-stacks. Then there exist a C^∞-stack $\mathcal{Y}'$, an equivalence $i:\mathcal{Y}\to\mathcal{Y}'$, and a strongly representable* 1*-morphism $g':\mathcal{Y}'\to\mathcal{Z}$ with $g=g'\circ i$. Also $\mathcal{Y}'$ is unique up to canonical* 1*-isomorphism in* $\mathbf{C^\infty Sta}$.

The second says that for some 2-commutative diagrams involving strongly representable morphisms, we can require the diagrams to commute *up to equality*, not just up to 2-isomorphism.

Proposition 1.8.9. *Suppose $\mathcal{X},\mathcal{Y},\mathcal{Z}$ are C^∞-stacks, $f:\mathcal{X}\to\mathcal{Y}$, $g:\mathcal{Y}\to\mathcal{Z}$, $h:\mathcal{X}\to\mathcal{Z}$ are* 1*-morphisms with g strongly representable, and $\eta:g\circ f\Rightarrow h$ is a* 2*-morphism in* $\mathbf{C^\infty Sta}$. *Then as in the diagram below there exist a* 1*-morphism $f':\mathcal{X}\to\mathcal{Y}$ with $g\circ f'=h$, and a* 2*-morphism $\zeta:f\Rightarrow f'$ with $\mathrm{id}_g*\zeta=\eta$, and f',ζ are unique under these conditions.*

$$
\begin{tikzcd}
 & \mathcal{Y} \arrow[dr, "g"] \arrow[d, phantom, "\eta\Downarrow"] & \\
\mathcal{X} \arrow[ur, bend left=40, "f'"] \arrow[ur, bend left=10, "f"'] \arrow[ur, phantom, bend left=25, "\zeta\Uparrow"] \arrow[rr, "h"] & & \mathcal{Z}.
\end{tikzcd}
$$

We will use strongly representable 1-morphisms to define orbifolds, d-stacks, and d-orbifolds with corners so that boundaries behave in a strictly functorial rather than weakly functorial way, as for d-spaces with corners in Remark 1.6.5. Here is an explicit construction of fibre products $\mathcal{X}\times_{g,\mathcal{Z},h}\mathcal{Y}$ in $\mathbf{C^\infty Sta}$ when g is strongly representable, yielding a strictly commutative 2-Cartesian square.

Proposition 1.8.10. *Let $g:\mathcal{X}\to\mathcal{Z}$ and $h:\mathcal{Y}\to\mathcal{Z}$ be* 1*-morphisms of C^∞-stacks with g strongly representable. Define a category $\mathcal{W}$ to have objects pairs (A,B) for $A\in\mathcal{X}$, $B\in\mathcal{Y}$ with $g(A)=h(B)$ in $\mathcal{Z}$, so that $p_{\mathcal{X}}(A)=p_{\mathcal{Y}}(B)$*

in $\mathbf{C^\infty Sch}$, *and morphisms pairs* $(a,b):(A,B)\to(A',B')$ *with* $a:A\to A'$, $b:B\to B'$ *morphisms in* $\mathcal{X},\mathcal{Y}$ *with* $p_{\mathcal{X}}(a)=p_{\mathcal{Y}}(b)$ *in* $\mathbf{C^\infty Sch}$.

Define functors $p_{\mathcal{W}}:\mathcal{W}\to\mathbf{C^\infty Sch}$, $e:\mathcal{W}\to\mathcal{X}$, $f:\mathcal{W}\to\mathcal{Y}$ *by* $p_{\mathcal{W}}:(A,B)\mapsto p_{\mathcal{X}}(A)=p_{\mathcal{Y}}(B)$, $e:(A,B)\mapsto A$, $f:(A,B)\mapsto B$ *on objects and* $p_{\mathcal{W}}:(a,b)\mapsto p_{\mathcal{X}}(a)=p_{\mathcal{Y}}(b)$, $e:(a,b)\mapsto a$, $f:(a,b)\mapsto b$ *on morphisms. Then* $\mathcal{W}$ *is a* C^∞*-stack and* $e:\mathcal{W}\to\mathcal{X}$, $f:\mathcal{W}\to\mathcal{Y}$ *are* 1*-morphisms, with* f *strongly representable, and* $g\circ e=h\circ f$. *Furthermore, the following diagram in* $\mathbf{C^\infty Sta}$ *is* 2*-Cartesian:*

$$\begin{tikzcd}
\mathcal{W} \arrow[rr, "f"] \arrow[d, "e"'] \arrow[drr, phantom, "\mathrm{id}_{g\circ e}\Uparrow"] & & \mathcal{Y} \arrow[d, "h"] \\
\mathcal{X} \arrow[rr, "g"] & & \mathcal{Z}.
\end{tikzcd}$$

If also h *is strongly representable, then* e *is strongly representable.*

1.8.4 Quotient C^∞-stacks

An important class of examples of C^∞-stacks $\mathcal{X}$ are *quotient* C^∞*-stacks* $[\underline{X}/G]$, for $\underline{X}$ a C^∞-scheme acted on by a finite group G. The next three examples define quotient C^∞-stacks $[\underline{X}/G]$, quotient 1-morphisms $[\underline{f},\rho]:[\underline{X}/G]\to[\underline{Y}/H]$, and quotient 2-morphisms $[\delta]:[\underline{f},\rho]\Rightarrow[\underline{g},\sigma]$.

In fact Examples 1.8.11–1.8.13 are simplifications of more complicated definitions given in [56, §9.1]. The construction of [56, §9.1] gives equivalent C^∞-stacks $[\underline{X}/G]$, but has the advantage of being strictly functorial, that is, quotient 1-morphisms compose as $[\underline{g},\sigma]\circ[\underline{f},\rho]=[\underline{g}\circ\underline{f},\sigma\circ\rho]$, whereas in Example 1.8.12 we only have a 2-isomorphism $[\underline{g},\sigma]\circ[\underline{f},\rho]\cong[\underline{g}\circ\underline{f},\sigma\circ\rho]$. We will occasionally assume this strict functoriality below, for instance, in Definition 1.11.26.

Example 1.8.11. Let $\underline{X}$ be a separated C^∞-scheme, G a finite group, and $\underline{r}:G\to\mathrm{Aut}(\underline{X})$ an action of G on $\underline{X}$ by isomorphisms. We will define the *quotient* C^∞*-stack* $\mathcal{X}=[\underline{X}/G]$.

Define a category $\mathcal{X}$ to have objects quintuples $(\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})$, where $\underline{T},\underline{U}$ are C^∞-schemes, $\underline{t}:G\to\mathrm{Aut}(\underline{T})$ is a free action of G on $\underline{T}$ by isomorphisms, $\underline{u}:\underline{T}\to\underline{X}$ is a morphism with $\underline{u}\circ\underline{t}(\gamma)=\underline{r}(\gamma)\circ\underline{u}:\underline{T}\to\underline{X}$ for all $\gamma\in G$, and $\underline{v}:\underline{T}\to\underline{U}$ is a morphism which makes $\underline{T}$ into a principal G-bundle over $\underline{U}$, that is, $\underline{v}$ is proper, étale and surjective, and its fibres are G-orbits in $\underline{T}$ under $\underline{t}$.

A morphism $(\underline{a},\underline{b}):(\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})\to(\underline{T}',\underline{U}',\underline{t}',\underline{u}',\underline{v}')$ in $\mathcal{X}$ is a pair of morphisms $\underline{a}:\underline{U}\to\underline{U}'$ and $\underline{b}:\underline{T}\to\underline{T}'$ such that $\underline{b}\circ\underline{t}(\gamma)=\underline{t}'(\gamma)\circ\underline{b}$ for $\gamma\in G$, and $\underline{u}=\underline{u}'\circ\underline{b}$, and $\underline{a}\circ\underline{v}=\underline{v}'\circ\underline{b}$. Composition is $(\underline{c},\underline{d})\circ(\underline{a},\underline{b})=(\underline{c}\circ\underline{a},\underline{d}\circ\underline{b})$, and identities are $\mathrm{id}_{(\underline{T},\ldots,\underline{v})}=(\underline{\mathrm{id}}_{\underline{U}},\underline{\mathrm{id}}_{\underline{T}})$.

This defines the category $\mathcal{X}$. The functor $p_{\mathcal{X}}:\mathcal{X}\to\mathbf{C^\infty Sch}$ acts by $p_{\mathcal{X}}:(\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})\mapsto\underline{U}$ on objects, and $p_{\mathcal{X}}:(\underline{a},\underline{b})\mapsto\underline{a}$ on morphisms. Then $\mathcal{X}$ is a C^∞-stack, which we write as $[\underline{X}/G]$.

Example 1.8.12. Let $\underline{X},\underline{Y}$ be separated C^∞-schemes acted on by finite groups G,H with actions $\underline{r}:G\to\mathrm{Aut}(\underline{X})$, $\underline{s}:H\to\mathrm{Aut}(\underline{Y})$, so that we have quotient C^∞-stacks $[\underline{X}/G]$ and $[\underline{Y}/H]$ as in Example 1.8.11. Suppose we have morphisms

$\underline{f}:\underline{X}\to\underline{Y}$ of C^∞-schemes and $\rho:G\to H$ of groups, with $\underline{f}\circ\underline{r}(\gamma)=\underline{s}(\rho(\gamma))\circ\underline{f}$ for all $\gamma\in G$. Define a functor $[\underline{f},\rho]:[\underline{X}/G]\to[\underline{Y}/H]$ on objects in $[\underline{X}/G]$ by

$$[\underline{f},\rho]:(\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})\longmapsto\big((\underline{T}\times H)/G,\underline{U},\tilde{\underline{t}},\tilde{\underline{u}},\tilde{\underline{v}}\big).$$

Here for each $\delta\in H$, write $L_\delta,R_\delta:H\to H$ for left and right multiplication by δ. Then to define $(\underline{T}\times H)/G$, each $\gamma\in G$ acts by $\underline{r}(\gamma)\times R_{\rho(\gamma)^{-1}}:\underline{T}\times H\to\underline{T}\times H$. For each $\delta\in H$, the morphism $\tilde{\underline{t}}(\delta):(\underline{T}\times H)/G\to(\underline{T}\times H)/G$ is induced by the morphism $\underline{\mathrm{id}}_{\underline{T}}\times L_\delta:\underline{T}\times H\to\underline{T}\times H$. The morphisms $\tilde{\underline{u}}:(\underline{T}\times H)/G\to\underline{Y}$ and $\tilde{\underline{v}}:(\underline{T}\times H)/G\to\underline{U}$ are induced by $\underline{f}\circ\underline{u}\circ\underline{\pi}_{\underline{T}}:\underline{T}\times H\to\underline{Y}$ and $\underline{v}\circ\underline{\pi}_{\underline{T}}:\underline{T}\times H\to\underline{U}$.

On morphisms $(\underline{a},\underline{b}):(\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})\to(\underline{T}',\underline{U}',\underline{t}',\underline{u}',\underline{v}')$ in $[\underline{X}/G]$, define $[\underline{f},\rho]$ to map $(\underline{a},\underline{b})\mapsto(\underline{a},\tilde{\underline{b}})$, where $\tilde{\underline{b}}:(\underline{T}\times H)/G\to(\underline{T}'\times H)/G$ is induced by $\underline{b}\times\mathrm{id}_H:\underline{T}\times H\to\underline{T}'\times H$. Then $[\underline{f},\rho]:[\underline{X}/G]\to[\underline{Y}/H]$ is a 1-morphism of C^∞-stacks, which we call a *quotient* 1*-morphism.*

If $\rho:G\to H$ is injective, then $[\underline{f},\rho]:[\underline{X}/G]\to[\underline{Y}/H]$ is representable.

Example 1.8.13. Let $[\underline{f},\rho]:[\underline{X}/G]\to[\underline{Y}/H]$ and $[\underline{g},\sigma]:[\underline{X}/G]\to[\underline{Y}/H]$ be quotient 1-morphisms, so that $\underline{f},\underline{g}:\underline{X}\to\underline{Y}$ and $\rho,\sigma:G\to H$ are morphisms. Suppose $\delta\in H$ satisfies $\sigma(\gamma)=\delta\,\rho(\gamma)\,\delta^{-1}$ for all $\gamma\in G$, and $\underline{g}=\underline{s}(\delta)\circ\underline{f}$.

For each object $(\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})$ in $[\underline{X}/G]$, define an isomorphism in $[\underline{Y}/H]$:

$$\begin{aligned}[\delta]\big((\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})\big)=(\underline{\mathrm{id}}_{\underline{U}},\underline{i}_\delta):[\underline{f},\rho]\big((\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})\big)&=\big((\underline{T}\times H)/_{\underline{r}\times R_{\rho^{-1}}}G,\underline{U},\tilde{\underline{t}},\tilde{\underline{u}},\tilde{\underline{v}}\big)\\ \longrightarrow[\underline{g},\sigma]\big((\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})\big)&=\big((\underline{T}\times H)/_{\underline{r}\times R_{\sigma^{-1}}}G,\underline{U},\dot{\underline{t}},\dot{\underline{u}},\dot{\underline{v}}\big),\end{aligned}$$

where $\underline{i}_\delta:(\underline{T}\times H)/_{\underline{r}\times R_{\rho^{-1}}}G\to(\underline{T}\times H)/_{\underline{r}\times R_{\sigma^{-1}}}G$ is induced $\underline{\mathrm{id}}_{\underline{T}}\times R_{\delta^{-1}}:\underline{T}\times H\to\underline{T}\times H$. Then $[\delta]:[\underline{f},\rho]\Rightarrow[\underline{g},\sigma]$ is a natural isomorphism of functors, and a 2-morphism of C^∞-stacks, which we call a *quotient* 2*-morphism.*

1.8.5 Deligne–Mumford C^∞-stacks

Deligne–Mumford C^∞-stacks, studied in [56, §9], are a C^∞ analogue of Deligne–Mumford stacks in algebraic geometry.

Definition 1.8.14. A *Deligne–Mumford C^∞-stack* is a C^∞-stack $\mathcal{X}$ which admits an open cover $\{\mathcal{Y}_a:a\in A\}$ with each $\mathcal{Y}_a$ equivalent to a quotient stack $[\underline{U}_a/G_a]$ in Example 1.8.11 for $\underline{U}_a$ an affine C^∞-scheme and G_a a finite group. We call $\mathcal{X}$ *locally fair* if it has such an open cover with each $\underline{U}_a$ a fair affine C^∞-scheme.

We call a Deligne–Mumford C^∞-stack $\mathcal{X}$ *second countable, compact, locally compact,* or *paracompact*, if the underlying topological space $\mathcal{X}_{\mathrm{top}}$ from §1.8.2 is second countable, compact, locally compact, or paracompact, respectively.

Write $\mathbf{DMC^\infty Sta},\mathbf{DMC^\infty Sta^{lf}},\mathbf{DMC^\infty Sta^{lf}_{ssc}}$ for the full 2-subcategories of Deligne–Mumford C^∞-stacks, and locally fair Deligne–Mumford C^∞-stacks, and separated, second countable, locally fair Deligne–Mumford C^∞-stacks in $\mathbf{C^\infty Sta}$, respectively.

If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack then $\mathrm{Iso}_{\mathcal{X}}([x])$ is finite for all $[x]$ in $\mathcal{X}_{\mathrm{top}}$. If $f:\mathcal{X}\to\mathcal{Y}$ is a 1-morphism of Deligne–Mumford C^∞-stacks then f is representable if and only if the morphisms of orbifold groups $f_*:\mathrm{Iso}_{\mathcal{X}}([x])\to\mathrm{Iso}_{\mathcal{Y}}([y])$ from Definition 1.8.6 are injective for all $[x]\in\mathcal{X}_{\mathrm{top}}$ with $f_{\mathrm{top}}([x])=[x]\in\mathcal{Y}_{\mathrm{top}}$. From [56, §9–§10] we have:

Theorem 1.8.15. (a) *All fibre products exist in the* 2*-category* $\mathbf{C^\infty Sta}$.

(b) $\mathbf{DMC^\infty Sta}, \mathbf{DMC^\infty Sta^{lf}}$ *and* $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$ *are closed under fibre products and under taking open* C^∞*-substacks in* $\mathbf{C^\infty Sta}$.

Proposition 1.8.16. *Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack and $[x]\in\mathcal{X}_{\mathrm{top}}$, so that $\mathrm{Iso}_{\mathcal{X}}([x])\cong H$ for some finite group H. Then there exists an open C^∞-substack $\mathcal{U}$ in $\mathcal{X}$ with $[x]\in\mathcal{U}_{\mathrm{top}}\subseteq\mathcal{X}_{\mathrm{top}}$ and an equivalence $\mathcal{U}\simeq[\underline{Y}/H]$, where $\underline{Y}=(Y,\mathcal{O}_Y)$ is an affine C^∞-scheme with an action of H, and $[x]\in\mathcal{U}_{\mathrm{top}}\cong Y/H$ corresponds to a fixed point y of H in Y.*

Theorem 1.8.17. *Suppose $\mathcal{X}$ is a Deligne–Mumford C^∞-stack with $\mathrm{Iso}_{\mathcal{X}}([x])\cong\{1\}$ for all $[x]\in\mathcal{X}_{\mathrm{top}}$. Then $\mathcal{X}$ is equivalent to $\bar{\underline{X}}$ for some C^∞-scheme $\underline{X}$.*

In conventional algebraic geometry, a stack with all orbifold groups trivial is (equivalent to) an *algebraic space*, but may not be a scheme, so the category of algebraic spaces is larger than the category of schemes. Here algebraic spaces are spaces which are locally isomorphic to schemes in the étale topology, but not necessarily locally isomorphic to schemes in the Zariski topology.

In contrast, as Theorem 1.8.17 shows, in C^∞-algebraic geometry there is no difference between C^∞-schemes and C^∞-algebraic spaces. This is because in C^∞-geometry the Zariski topology is already fine enough, as in Remark 1.2.9(iii), so we gain no extra generality by passing to the étale topology.

1.8.6 Quasicoherent sheaves on C^∞-stacks

In [56, §10] we study sheaves on Deligne–Mumford C^∞-stacks.

Definition 1.8.18. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Define a category $\mathcal{C}_{\mathcal{X}}$ to have objects pairs $(\underline{U},u)$ where $\underline{U}$ is a C^∞-scheme and $u:\bar{\underline{U}}\to\mathcal{X}$ is an étale 1-morphism, and morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ where $\underline{f}:\underline{U}\to\underline{V}$ is an étale morphism of C^∞-schemes, and $\eta:u\Rightarrow v\circ\bar{\underline{f}}$ is a 2-isomorphism. If $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ and $(\underline{g},\zeta):(\underline{V},v)\to(\underline{W},w)$ are morphisms in $\mathcal{C}_{\mathcal{X}}$ then we define the composition $(\underline{g},\zeta)\circ(\underline{f},\eta)$ to be $(\underline{g}\circ\underline{f},\theta):(\underline{U},u)\to(\underline{W},w)$, where θ is the composition of 2-morphisms across the diagram:

$$\begin{tikzcd}
\bar{\underline{U}} \arrow[dr, "\bar{\underline{f}}"] \arrow[drr, bend left, "u"] \arrow[dd, "\overline{\underline{g}\circ\underline{f}}"'] & \arrow[d, phantom, "\Downarrow\eta"] & \\
{} \arrow[r, phantom, "\Leftarrow \mathrm{id}"] & \bar{\underline{V}} \arrow[r, "v"] \arrow[dl, "\bar{\underline{g}}"] \arrow[d, phantom, "\Downarrow\zeta"] & \mathcal{X}. \\
\bar{\underline{W}} \arrow[urr, bend right, "w"'] & {} &
\end{tikzcd}$$

Define an $\mathcal{O}_{\mathcal{X}}$*-module* $\mathcal{E}$ to assign an $\mathcal{O}_U$-module $\mathcal{E}(\underline{U},u)$ on $\underline{U}=(U,\mathcal{O}_U)$ for all objects $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$, and an isomorphism $\mathcal{E}_{(\underline{f},\eta)}:\underline{f}^*(\mathcal{E}(\underline{V},v))\to\mathcal{E}(\underline{U},u)$ for all morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ in $\mathcal{C}_{\mathcal{X}}$, such that for all $(\underline{f},\eta),(\underline{g},\zeta),(\underline{g}\circ\underline{f},\theta)$ as above the following diagram of isomorphisms of $\mathcal{O}_U$-modules commutes:

$$
\begin{tikzcd}
(\underline{g}\circ\underline{f})^*\bigl(\mathcal{E}(\underline{W},w)\bigr) \arrow[rr, "\mathcal{E}_{(\underline{g}\circ\underline{f},\theta)}"'] \arrow[dr, "I_{\underline{f},\underline{g}}(\mathcal{E}(\underline{W},w))"'] & & \mathcal{E}(\underline{U},u), \\
& \underline{f}^*\bigl(\underline{g}^*(\mathcal{E}(\underline{W},w))\bigr) \arrow[r, "\underline{f}^*(\mathcal{E}_{(\underline{g},\zeta)})"] & \underline{f}^*\bigl(\mathcal{E}(\underline{V},v)\bigr) \arrow[u, "\mathcal{E}_{(\underline{f},\eta)}"']
\end{tikzcd}
\tag{1.39}
$$

for $I_{\underline{f},\underline{g}}(\mathcal{E}(\underline{W},w))$ as in Remark 1.2.17.

A *morphism of $\mathcal{O}_{\mathcal{X}}$-modules* $\phi:\mathcal{E}\to\mathcal{F}$ assigns a morphism of $\mathcal{O}_U$-modules $\phi(\underline{U},u):\mathcal{E}(\underline{U},u)\to\mathcal{F}(\underline{U},u)$ for each object $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$, such that for all morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ in $\mathcal{C}_{\mathcal{X}}$ the following commutes:

$$
\begin{tikzcd}
\underline{f}^*\bigl(\mathcal{E}(\underline{V},v)\bigr) \arrow[r, "\mathcal{E}_{(\underline{f},\eta)}"'] \arrow[d, "\underline{f}^*(\phi(\underline{V},v))"'] & \mathcal{E}(\underline{U},u) \arrow[d, "\phi(\underline{U},u)"] \\
\underline{f}^*\bigl(\mathcal{F}(\underline{V},v)\bigr) \arrow[r, "\mathcal{F}_{(\underline{f},\eta)}"] & \mathcal{F}(\underline{U},u).
\end{tikzcd}
$$

We call $\mathcal{E}$ *quasicoherent*, or a *vector bundle of rank n*, if $\mathcal{E}(\underline{U},u)$ is quasicoherent, or a vector bundle of rank n, respectively, for all $(\underline{U},u)\in\mathcal{C}_{\mathcal{X}}$. Write $\mathcal{O}_{\mathcal{X}}$-mod for the category of $\mathcal{O}_{\mathcal{X}}$-modules, and $\mathrm{qcoh}(\mathcal{X})$, $\mathrm{vect}(\mathcal{X})$ for the full subcategories of quasicoherent sheaves and vector bundles, respectively. Then $\mathcal{O}_{\mathcal{X}}$-mod is an abelian category, and $\mathrm{qcoh}(\mathcal{X})$ an abelian subcategory of $\mathcal{O}_{\mathcal{X}}$-mod. If $\mathcal{X}$ is locally fair then $\mathrm{qcoh}(\mathcal{X})=\mathcal{O}_{\mathcal{X}}$-mod.

Note that vector bundles $\mathcal{E}$ on $\mathcal{X}$ are locally trivial in the étale topology, but need not be locally trivial in the Zariski topology. In particular, the orbifold groups $\mathrm{Iso}_{\mathcal{X}}([x])$ of $\mathcal{X}$ can act nontrivially on the fibres $\mathcal{E}|_x$ of $\mathcal{E}$.

As in [56, §10.5], as well as sheaves of $\mathcal{O}_{\mathcal{X}}$-modules, we can define other kinds of sheaves on Deligne–Mumford C^∞-stacks $\mathcal{X}$ by the same method. In particular, to define d-stacks in §1.10, we will need *sheaves of abelian groups* and *sheaves of C^∞-rings* on Deligne–Mumford C^∞-stacks.

Example 1.8.19. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Define a quasicoherent sheaf $\mathcal{O}_{\mathcal{X}}$ on $\mathcal{X}$ called the *structure sheaf* of $\mathcal{X}$ by $\mathcal{O}_{\mathcal{X}}(\underline{U},u)=\mathcal{O}_U$ for all objects $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$, and $(\mathcal{O}_{\mathcal{X}})_{(\underline{f},\eta)}:\underline{f}^*(\mathcal{O}_V)\to\mathcal{O}_U$ is the natural isomorphism for all morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ in $\mathcal{C}_{\mathcal{X}}$.

We may also consider $\mathcal{O}_{\mathcal{X}}$ as a sheaf of C^∞-rings on $\mathcal{X}$.

Example 1.8.20. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Define an $\mathcal{O}_{\mathcal{X}}$-module $T^*\mathcal{X}$ called the *cotangent sheaf* of $\mathcal{X}$ by $(T^*\mathcal{X})(\underline{U},u)=T^*\underline{U}$ for all objects $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$ and $(T^*\mathcal{X})_{(\underline{f},\eta)}=\Omega_{\underline{f}}:\underline{f}^*(T^*\underline{V})\to T^*\underline{U}$ for all morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ in $\mathcal{C}_{\mathcal{X}}$, where $T^*\underline{U}$ and $\Omega_{\underline{f}}$ are as in §1.2.4.

Example 1.8.21. Let $\underline{X}$ be a C^∞-scheme. Then $\mathcal{X}=\bar{\underline{X}}$ is a Deligne–Mumford C^∞-stack. We will define an *inclusion functor* $\mathcal{I}_{\underline{X}}:\mathcal{O}_X\text{-mod}\to\mathcal{O}_{\mathcal{X}}\text{-mod}$. Let $\mathcal{E}$ be an object in $\mathcal{O}_X$-mod. If $(\underline{U},u)$ is an object in $\mathcal{C}_{\mathcal{X}}$ then $u:\bar{\underline{U}}\to\mathcal{X}=\bar{\underline{X}}$ is 2-isomorphic to $\bar{\underline{u}}:\bar{\underline{U}}\to\bar{\underline{X}}$ for some unique morphism $\underline{u}:\underline{U}\to\underline{X}$. Define

$\mathcal{E}'(\underline{U},u)=\underline{u}^*(\mathcal{E})$. If $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ is a morphism in $\mathcal{C}_{\mathcal{X}}$ and $\underline{u},\underline{v}$ are associated to u,v as above, so that $\underline{u}=\underline{v}\circ\underline{f}$, then define

$$\mathcal{E}'_{(\underline{f},\eta)}=I_{\underline{f},\underline{v}}(\mathcal{E})^{-1}:\underline{f}^*(\mathcal{E}'(\underline{V},v))=\underline{f}^*\bigl(\underline{v}^*(\mathcal{E})\bigr)\longrightarrow(\underline{v}\circ\underline{f})^*(\mathcal{E})=\mathcal{E}'(\underline{U},u).$$

Then (1.39) commutes for all $(\underline{f},\eta),(\underline{g},\zeta)$, so $\mathcal{E}'$ is an $\mathcal{O}_{\mathcal{X}}$-module.

If $\phi:\mathcal{E}\to\mathcal{F}$ is a morphism of $\mathcal{O}_X$-modules then we define a morphism $\phi':\mathcal{E}'\to\mathcal{F}'$ in $\mathcal{O}_{\mathcal{X}}$-mod by $\phi'(\underline{U},u)=\underline{u}^*(\phi)$ for $\underline{u}$ associated to u as above. Then defining $\mathcal{I}_{\underline{X}}:\mathcal{E}\mapsto\mathcal{E}'$, $\mathcal{I}_{\underline{X}}:\phi\mapsto\phi'$ gives a functor $\mathcal{O}_X$-mod $\to\mathcal{O}_{\mathcal{X}}$-mod, which induces equivalences between the categories $\mathcal{O}_X$-mod, $\mathrm{qcoh}(\underline{X})$ defined in §1.2.4 and $\mathcal{O}_{\mathcal{X}}$-mod, $\mathrm{qcoh}(\mathcal{X})$ above.

Definition 1.8.22. Let $f:\mathcal{X}\to\mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks, and $\mathcal{F}$ be an $\mathcal{O}_{\mathcal{Y}}$-module. A *pullback* of $\mathcal{F}$ to $\mathcal{X}$ is an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$, together with the following data: if $\underline{U},\underline{V}$ are C^∞-schemes and $u:\bar{U}\to\mathcal{X}$ and $v:\bar{V}\to\mathcal{Y}$ are étale 1-morphisms, then there is a C^∞-scheme $\underline{W}$ and morphisms $\underline{\pi}_{\underline{U}}:\underline{W}\to\underline{U}$, $\underline{\pi}_{\underline{V}}:\underline{W}\to\underline{V}$ giving a 2-Cartesian diagram:

$$\begin{tikzcd}[column sep=huge] \bar{W} \arrow[r, "\bar{\pi}_{\underline{V}}"'] \arrow[d, "\bar{\pi}_{\underline{U}}"'] \arrow[dr, phantom, "\zeta\Uparrow"] & \bar{V} \arrow[d, "v"] \\ \bar{U} \arrow[r, "f\circ u"] & \mathcal{Y}. \end{tikzcd} \tag{1.40}$$

Then an isomorphism $i(\mathcal{F},f,u,v,\zeta):\underline{\pi}_{\underline{U}}^*\bigl(\mathcal{E}(\underline{U},u)\bigr)\to\underline{\pi}_{\underline{V}}^*\bigl(\mathcal{F}(\underline{V},v)\bigr)$ of $\mathcal{O}_W$-modules should be given, which is functorial in $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$ and $(\underline{V},v)$ in $\mathcal{C}_{\mathcal{Y}}$ and the 2-isomorphism ζ in (1.40). We usually write pullbacks $\mathcal{E}$ as $f^*(\mathcal{F})$. Pullbacks $f^*(\mathcal{F})$ exist, and are unique up to unique isomorphism. Using the Axiom of Choice, we choose a pullback $f^*(\mathcal{F})$ for all such $f,\mathcal{F}$.

Let $f:\mathcal{X}\to\mathcal{Y}$ be a 1-morphism, and $\phi:\mathcal{E}\to\mathcal{F}$ be a morphism in $\mathcal{O}_{\mathcal{Y}}$-mod. Then $f^*(\mathcal{E}),f^*(\mathcal{F})\in\mathcal{O}_{\mathcal{X}}$-mod. The *pullback morphism* $f^*(\phi):f^*(\mathcal{E})\to f^*(\mathcal{F})$ is the unique morphism in $\mathcal{O}_{\mathcal{X}}$-mod such that whenever $u:\bar{U}\to\mathcal{X}$, $v:\bar{V}\to\mathcal{Y}$, $\underline{W},\underline{\pi}_{\underline{U}},\underline{\pi}_{\underline{V}}$ are as above, the following diagram in $\mathcal{O}_W$-mod commutes:

$$\begin{tikzcd}[column sep=huge] \underline{\pi}_{\underline{U}}^*\bigl(f^*(\mathcal{E})(\underline{U},u)\bigr) \arrow[r, "{i(\mathcal{E},f,u,v,\zeta)}"'] \arrow[d, "{\underline{\pi}_{\underline{U}}^*(f^*(\phi)(\underline{U},u))}"'] & \underline{\pi}_{\underline{V}}^*\bigl(\mathcal{E}(\underline{V},v)\bigr) \arrow[d, "{\underline{\pi}_{\underline{V}}^*(\phi(\underline{V},v))}"] \\ \underline{\pi}_{\underline{U}}^*\bigl(f^*(\mathcal{F})(\underline{U},u)\bigr) \arrow[r, "{i(\mathcal{F},f,u,v,\zeta)}"] & \underline{\pi}_{\underline{V}}^*\bigl(\mathcal{F}(\underline{V},v)\bigr). \end{tikzcd}$$

This defines a right exact functor $f^*:\mathcal{O}_{\mathcal{Y}}$-mod $\to\mathcal{O}_{\mathcal{X}}$-mod, which also maps $\mathrm{qcoh}(\mathcal{Y})\to\mathrm{qcoh}(\mathcal{X})$.

Let $f,g:\mathcal{X}\to\mathcal{Y}$ be 1-morphisms of Deligne–Mumford C^∞-stacks, $\eta:f\Rightarrow g$ a 2-morphism, and $\mathcal{E}\in\mathcal{O}_{\mathcal{Y}}$-mod. Then we have $\mathcal{O}_{\mathcal{X}}$-modules $f^*(\mathcal{E}),g^*(\mathcal{E})$. Define $\eta^*(\mathcal{E}):f^*(\mathcal{E})\to g^*(\mathcal{E})$ to be the unique isomorphism such that whenever $\underline{U},\underline{V},\underline{W},u,v,\underline{\pi}_{\underline{U}},\underline{\pi}_{\underline{V}}$ are as above, so that we have 2-Cartesian diagrams

$$\begin{tikzcd}[column sep=huge] \bar{W} \arrow[r, "\bar{\pi}_{\underline{V}}"'] \arrow[d, "\bar{\pi}_{\underline{U}}"'] \arrow[dr, phantom, "\zeta\odot(\eta*\mathrm{id}_{u\circ\bar{\pi}_{\underline{U}}})\Uparrow"] & \bar{V} \arrow[d, "v"] \\ \bar{U} \arrow[r, "f\circ u"] & \mathcal{Y}, \end{tikzcd} \qquad \begin{tikzcd}[column sep=huge] \bar{W} \arrow[r, "\bar{\pi}_{\underline{V}}"'] \arrow[d, "\bar{\pi}_{\underline{U}}"'] \arrow[dr, phantom, "\zeta\Uparrow"] & \bar{V} \arrow[d, "v"] \\ \bar{U} \arrow[r, "g\circ u"] & \mathcal{Y}, \end{tikzcd}$$

as in (1.40), then we have commuting isomorphisms of $\mathcal{O}_W$-modules:

$$
\begin{tikzcd}
\underline{\pi}_U^*\bigl(f^*(\mathcal{E})(\underline{U},u)\bigr) \arrow[d, "\underline{\pi}_U^*((\eta^*(\mathcal{E}))(\underline{U},u))"'] \arrow[dr, "i(\mathcal{E},f,u,v,\zeta\odot(\eta*\mathrm{id}_{u\circ\underline{\pi}_U}))"] & \\
\underline{\pi}_U^*\bigl(g^*(\mathcal{E})(\underline{U},u)\bigr) \arrow[r, "i(\mathcal{E},g,u,v,\zeta)"'] & \underline{\pi}_V^*\bigl(\mathcal{E}(\underline{V},v)\bigr).
\end{tikzcd}
$$

This defines a natural isomorphism $\eta^* : f^* \Rightarrow g^*$.

As in Remark 1.2.17, if $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are 1-morphisms of Deligne–Mumford C^∞-stacks and $\mathcal{E} \in \mathcal{O}_{\mathcal{Z}}$-mod, then we have a canonical isomorphism $I_{f,g}(\mathcal{E}) : (g\circ f)^*(\mathcal{E}) \to f^*(g^*(\mathcal{E}))$. If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack and $\mathcal{E} \in \mathcal{O}_{\mathcal{X}}$-mod, we have a canonical isomorphism $\delta_{\mathcal{X}}(\mathcal{E}) : \mathrm{id}_{\mathcal{X}}^*(\mathcal{E}) \to \mathcal{E}$. These $I_{f,g}, \delta_{\mathcal{X}}$ have the same properties as in the C^∞-scheme case.

In a similar way, we can define pullbacks $f^{-1}(\mathcal{E})$ for sheaves of abelian groups and of C^∞-rings $\mathcal{E}$ on $\mathcal{Y}$, and corresponding isomorphisms $I_{f,g}(\mathcal{E}), \delta_{\mathcal{X}}(\mathcal{E})$.

Example 1.8.23. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks. Then Example 1.8.19 defines sheaves of C^∞-rings $\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{Y}}$ on $\mathcal{X}, \mathcal{Y}$, so as in Definition 1.8.22 we have a pullback sheaf $f^{-1}(\mathcal{O}_{\mathcal{Y}})$ of C^∞-rings on $\mathcal{X}$. There is a natural morphism $f^\sharp : f^{-1}(\mathcal{O}_{\mathcal{Y}}) \to \mathcal{O}_{\mathcal{X}}$ of sheaves of C^∞-rings on $\mathcal{X}$, characterized by the following property: for all $(\underline{U},u), (\underline{V},v), \underline{W}, \zeta$ as in Definition 1.8.22, the following diagram of sheaves of C^∞-rings on W commutes:

$$
\begin{tikzcd}
\pi_U^{-1}\bigl(f^{-1}(\mathcal{O}_{\mathcal{Y}})(\underline{U},u)\bigr) \arrow[r, "\pi_U^{-1}(f^\sharp(\underline{U},u))"'] \arrow[d, "\cong"', "i(\mathcal{O}_{\mathcal{Y}},f,u,v,\zeta)"] & \pi_U^{-1}\bigl((\mathcal{O}_{\mathcal{X}})(\underline{U},u)\bigr) = \pi_U^{-1}(\mathcal{O}_U) \arrow[d, "\pi_U^\sharp"', "\cong"] \\
\pi_V^{-1}\bigl(\mathcal{O}_{\mathcal{Y}}(\underline{V},v)\bigr) = \pi_V^{-1}(\mathcal{O}_V) \arrow[r, "\pi_V^\sharp"] & \mathcal{O}_W,
\end{tikzcd}
$$

where $\underline{\pi}_U = (\pi_U, \pi_U^\sharp)$ and $\underline{\pi}_V = (\pi_V, \pi_V^\sharp)$.

Definition 1.8.24. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks. Then $f^*(T^*\mathcal{Y}), T^*\mathcal{X}$ are $\mathcal{O}_{\mathcal{X}}$-modules, by Example 1.8.20 and Definition 1.8.22. Define $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ to be the unique morphism characterized as follows. Let $u : \bar{U} \to \mathcal{X}$, $v : \bar{V} \to \mathcal{Y}$, $\underline{W}, \underline{\pi}_U, \underline{\pi}_V$ be as in Definition 1.8.22, with (1.40) 2-Cartesian. Then the following diagram commutes in $\mathcal{O}_W$-mod:

$$
\begin{tikzcd}
\underline{\pi}_U^*\bigl(f^*(T^*\mathcal{Y})(\underline{U},u)\bigr) \arrow[r, "i(T^*\mathcal{Y},f,u,v,\zeta)"'] \arrow[d, "\underline{\pi}_U^*(\Omega_f(\underline{U},u))"'] & \underline{\pi}_V^*\bigl((T^*\mathcal{Y})(\underline{V},v)\bigr) = \underline{\pi}_V^*(T^*\underline{V}) \arrow[d, "\Omega_{\underline{\pi}_V}"] \\
\underline{\pi}_U^*\bigl((T^*\mathcal{X})(\underline{U},u)\bigr) \arrow[r, "(T^*\mathcal{X})_{(\underline{\pi}_U,\mathrm{id}_{u\circ\underline{\pi}_U})}"] & (T^*\mathcal{X})(\underline{W}, u\circ\underline{\pi}_U) = T^*\underline{W}.
\end{tikzcd}
$$

Here [56, Th. 10.15] is the analogue of Theorem 1.2.21.

Theorem 1.8.25. **(a)** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of Deligne–Mumford C^∞-stacks. Then $\Omega_{g\circ f} = \Omega_f \circ f^*(\Omega_g) \circ I_{f,g}(T^*\mathcal{Z})$.*

(b) *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of Deligne–Mumford C^∞-stacks and $\eta : f \Rightarrow g$ a 2-morphism. Then $\Omega_f = \Omega_g \circ \eta^*(T^*\mathcal{Y}) : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$.*

(c) *Suppose $\mathcal{W},\mathcal{X},\mathcal{Y},\mathcal{Z}$ are locally fair Deligne–Mumford C^∞-stacks with a 2-Cartesian square*

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "f"] \arrow[d, "e"] \arrow[dr, phantom, "\eta \Uparrow"] & \mathcal{Y} \arrow[d, "h"'] \\
\mathcal{X} \arrow[r, "g"] & \mathcal{Z}
\end{tikzcd}$$

in $\mathbf{DMC^\infty Sta^{lf}}$, *so that* $\mathcal{W} \simeq \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. *Then the following is exact in* $\mathrm{qcoh}(\mathcal{W})$:

$$(g\circ e)^*(T^*\mathcal{Z}) \xrightarrow{\substack{e^*(\Omega_g)\circ I_{e,g}(T^*\mathcal{Z})\oplus \\ -f^*(\Omega_h)\circ I_{f,h}(T^*\mathcal{Z})\circ\eta^*(T^*\mathcal{Z})}} \begin{array}{c} e^*(T^*\mathcal{X})\oplus \\ f^*(T^*\mathcal{Y}) \end{array} \xrightarrow{\Omega_e\oplus\Omega_f} T^*\mathcal{W} \longrightarrow 0.$$

1.8.7 Orbifold strata of Deligne–Mumford C^∞-stacks

Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, and Γ a finite group. In [56, §11.1] we define six different notions of *orbifold strata* of $\mathcal{X}$, which are Deligne–Mumford C^∞-stacks written $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$, and open C^∞-substacks $\mathcal{X}^\Gamma_\circ \subseteq \mathcal{X}^\Gamma$, $\tilde{\mathcal{X}}^\Gamma_\circ \subseteq \tilde{\mathcal{X}}^\Gamma$, $\hat{\mathcal{X}}^\Gamma_\circ \subseteq \hat{\mathcal{X}}^\Gamma$. The points and orbifold groups of $\mathcal{X}^\Gamma,\ldots,\hat{\mathcal{X}}^\Gamma_\circ$ are given by:

(i) Points of $\mathcal{X}^\Gamma$ are isomorphism classes $[x,\rho]$, where $[x]\in\mathcal{X}_{\rm top}$ and $\rho:\Gamma\to \mathrm{Iso}_{\mathcal{X}}([x])$ is an injective morphism, and $\mathrm{Iso}_{\mathcal{X}^\Gamma}([x,\rho])$ is the centralizer of $\rho(\Gamma)$ in $\mathrm{Iso}_{\mathcal{X}}([x])$. Points of $\mathcal{X}^\Gamma_\circ\subseteq\mathcal{X}^\Gamma$ are $[x,\rho]$ with ρ an isomorphism, and $\mathrm{Iso}_{\mathcal{X}^\Gamma_\circ}([x,\rho])\cong C(\Gamma)$, the centre of Γ.

(ii) Points of $\tilde{\mathcal{X}}^\Gamma$ are pairs $[x,\Delta]$, where $[x]\in\mathcal{X}_{\rm top}$ and $\Delta\subseteq\mathrm{Iso}_{\mathcal{X}}([x])$ is isomorphic to Γ, and $\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])$ is the normalizer of Δ in $\mathrm{Iso}_{\mathcal{X}}([x])$. Points of $\tilde{\mathcal{X}}^\Gamma_\circ\subseteq\tilde{\mathcal{X}}^\Gamma$ are $[x,\Delta]$ with $\Delta=\mathrm{Iso}_{\mathcal{X}}([x])$, and $\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma_\circ}([x,\Delta])\cong\Gamma$.

(iii) Points $[x,\Delta]$ of $\hat{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma_\circ$ are the same as for $\tilde{\mathcal{X}}^\Gamma,\tilde{\mathcal{X}}^\Gamma_\circ$, but with orbifold groups $\mathrm{Iso}_{\hat{\mathcal{X}}^\Gamma}([x,\Delta])\cong\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])/\Delta$ and $\mathrm{Iso}_{\hat{\mathcal{X}}^\Gamma_\circ}([x,\Delta])\cong\{1\}$.

Since the C^∞-stack $\hat{\mathcal{X}}^\Gamma_\circ$ has trivial orbifold groups, it is (equivalent to) a C^∞-scheme. That is, there is a genuine C^∞-scheme $\underline{\hat{X}}{}^\Gamma_\circ$, unique up to isomorphism in $\mathbf{C^\infty Sch}$, such that $\hat{\mathcal{X}}^\Gamma_\circ\simeq\bar{\underline{\hat{X}}}{}^\Gamma_\circ$ in $\mathbf{C^\infty Sta}$.

There are 1-morphisms $O^\Gamma(\mathcal{X}),\ldots,\hat\Pi^\Gamma_\circ(\mathcal{X})$ forming a strictly commutative diagram, where the columns are inclusions of open C^∞-substacks:

$$\begin{tikzcd}
\mathcal{X}^\Gamma_\circ \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde\Pi^\Gamma_\circ(\mathcal{X})"] \arrow[dd, "\subset"'] \arrow[dr, "O^\Gamma_\circ(\mathcal{X})"'] & & \tilde{\mathcal{X}}^\Gamma_\circ \arrow[r, "\hat\Pi^\Gamma_\circ(\mathcal{X})"] \arrow[dd, "\subset"] \arrow[dl, "\tilde O^\Gamma_\circ(\mathcal{X})"] & \hat{\mathcal{X}}^\Gamma_\circ \simeq \bar{\underline{\hat{X}}}{}^\Gamma_\circ \arrow[dd, "\subset"] \\
& \mathcal{X} & & \\
\mathcal{X}^\Gamma \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde\Pi^\Gamma(\mathcal{X})"'] \arrow[ur, "O^\Gamma(\mathcal{X})"] & & \tilde{\mathcal{X}}^\Gamma \arrow[r, "\hat\Pi^\Gamma(\mathcal{X})"'] \arrow[ul, "\tilde O^\Gamma(\mathcal{X})"'] & \hat{\mathcal{X}}^\Gamma.
\end{tikzcd} \tag{1.41}$$

Also $\mathrm{Aut}(\Gamma)$ acts on $\mathcal{X}^\Gamma,\mathcal{X}^\Gamma_\circ$, with $\tilde{\mathcal{X}}^\Gamma\simeq[\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$, $\tilde{\mathcal{X}}^\Gamma_\circ\simeq[\mathcal{X}^\Gamma_\circ/\mathrm{Aut}(\Gamma)]$. The topological space $\mathcal{X}_{\rm top}$ of $\mathcal{X}$ from §1.8.2 has stratifications

$$\mathcal{X}_{\rm top}\cong\coprod\nolimits_{\substack{\text{iso. classes of}\\ \text{finite groups }\Gamma}}\mathcal{X}^\Gamma_{\circ,\rm top}/\Gamma\cong\coprod\nolimits_\Gamma\tilde{\mathcal{X}}^\Gamma_{\circ,\rm top}\cong\coprod\nolimits_\Gamma\hat{\mathcal{X}}^\Gamma_{\circ,\rm top},$$

which is why $\mathcal{X}^\Gamma,\ldots,\hat{\mathcal{X}}_\circ^\Gamma$ are called orbifold strata. The 1-morphisms $O^\Gamma(\mathcal{X})$, $\tilde{O}^\Gamma(\mathcal{X})$ in (1.41) are proper, and $\hat{\Pi}^\Gamma(\mathcal{X})_{\rm top}:\tilde{\mathcal{X}}^\Gamma_{\rm top}\rightarrow\hat{\mathcal{X}}^\Gamma_{\rm top}$ is a homeomorphism of topological spaces. Hence, if $\mathcal{X}$ is compact then $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma$ are also compact.

Example 1.8.26. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. The *inertia stack* $\mathcal{I}_\mathcal{X}$ of $\mathcal{X}$ is the fibre product $\mathcal{I}_\mathcal{X}=\mathcal{X}\times_{\Delta_\mathcal{X},\mathcal{X}\times\mathcal{X},\Delta_\mathcal{X}}\mathcal{X}$, where $\Delta_\mathcal{X}:\mathcal{X}\rightarrow\mathcal{X}\times\mathcal{X}$ is the diagonal 1-morphism. One can show there is an equivalence

$$\mathcal{I}_\mathcal{X}\simeq\textstyle\coprod_{k\geqslant 1}\mathcal{X}^{\mathbb{Z}_k}.$$

Points of $\mathcal{I}_\mathcal{X}$ are isomorphism classes $[x,\eta]$, where $[x]\in\mathcal{X}_{\rm top}$ and $\eta\in\mathrm{Iso}_\mathcal{X}([x])$. Each such $\eta\in\mathrm{Iso}_\mathcal{X}([x])$ has some finite order $k\geqslant 1$, and generates an injective morphism $\rho:\mathbb{Z}_k\rightarrow\mathrm{Iso}_\mathcal{X}([x])$ mapping $\rho:a\mapsto\eta^a$. We may identify $\mathcal{X}^{\mathbb{Z}_k}$ with the open and closed C^∞-substack of $[x,\eta]$ in $\mathcal{I}_\mathcal{X}$ for which η has order k.

Orbifold strata $\mathcal{X}^\Gamma$ are strongly functorial for representable 1-morphisms and their 2-morphisms. That is, if $f:\mathcal{X}\rightarrow\mathcal{Y}$ is a representable 1-morphism of Deligne–Mumford C^∞-stacks, we define a unique representable 1-morphism $f^\Gamma:\mathcal{X}^\Gamma\rightarrow\mathcal{Y}^\Gamma$ with $O^\Gamma(\mathcal{Y})\circ f^\Gamma=f\circ O^\Gamma(\mathcal{X})$. If $f,g:\mathcal{X}\rightarrow\mathcal{Y}$ are representable and $\eta:f\Rightarrow g$ is a 2-morphism, we define a unique 2-morphism $\eta^\Gamma:f^\Gamma\Rightarrow g^\Gamma$ with $\mathrm{id}_{O^\Gamma(\mathcal{Y})}*\eta^\Gamma=\eta*\mathrm{id}_{O^\Gamma(\mathcal{X})}$. These f^Γ,η^Γ are compatible with compositions of 1- and 2-morphisms, and identities, in the obvious way. Orbifold strata $\tilde{\mathcal{X}}^\Gamma$ have the same kind of functorial behaviour, and $\hat{\mathcal{X}}^\Gamma$ have a weaker functorial behaviour, in that $\hat{f}^\Gamma$ is only natural up to 2-isomorphism.

For $f:\mathcal{X}\rightarrow\mathcal{Y}$ and Γ as above, the restriction $f^\Gamma|_{\mathcal{X}^\Gamma_\circ}$ need not map $\mathcal{X}^\Gamma_\circ\rightarrow\mathcal{Y}^\Gamma_\circ$, but only $\mathcal{X}^\Gamma_\circ\rightarrow\mathcal{Y}^\Gamma$. So we do not define 1-morphisms $f^\Gamma_\circ:\mathcal{X}^\Gamma_\circ\rightarrow\mathcal{Y}^\Gamma_\circ$. The same applies for the actions $\tilde{f}^\Gamma,\hat{f}^\Gamma$ of f on orbifold strata $\tilde{\mathcal{X}}^\Gamma_\circ,\hat{\mathcal{X}}^\Gamma_\circ$.

In [56, §11.3] we describe the orbifold strata of a quotient C^∞-stack $[\underline{X}/G]$.

Theorem 1.8.27. *Suppose $\underline{X}$ is a separated C^∞-scheme and G a finite group acting on $\underline{X}$ by isomorphisms, and write $\mathcal{X}=[\underline{X}/G]$ for the quotient C^∞-stack from Example* 1.8.11, *which is a Deligne–Mumford C^∞-stack. Let Γ be a finite* *group. Then there are equivalences of C^∞-stacks*

$$\mathcal{X}^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective}\\ \text{group morphisms } \rho:\Gamma\to G}} \bigl[\underline{X}^{\rho(\Gamma)}/\bigl\{g\in G: g\rho(\gamma)=\rho(\gamma)g\ \ \forall\gamma\in\Gamma\bigr\}\bigr], \tag{1.42}$$

$$\mathcal{X}^\Gamma_\circ \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective}\\ \text{group morphisms } \rho:\Gamma\to G}} \bigl[\underline{X}^{\rho(\Gamma)}_\circ/\bigl\{g\in G: g\rho(\gamma)=\rho(\gamma)g\ \ \forall\gamma\in\Gamma\bigr\}\bigr], \tag{1.43}$$

$$\tilde{\mathcal{X}}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\underline{X}^{\Delta}/\bigl\{g\in G:\Delta=g\Delta g^{-1}\bigr\}\bigr], \tag{1.44}$$

$$\tilde{\mathcal{X}}^\Gamma_\circ \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\underline{X}^{\Delta}_\circ/\bigl\{g\in G:\Delta=g\Delta g^{-1}\bigr\}\bigr]. \tag{1.45}$$

$$\hat{\mathcal{X}}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\underline{X}^{\Delta}/\bigl(\{g\in G:\Delta=g\Delta g^{-1}\}/\Delta\bigr)\bigr], \tag{1.46}$$

$$\hat{\mathcal{X}}^\Gamma_\circ \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\underline{X}^{\Delta}_\circ/\bigl(\{g\in G:\Delta=g\Delta g^{-1}\}/\Delta\bigr)\bigr]. \tag{1.47}$$

Here for each subgroup $\Delta\subseteq G$, we write $\underline{X}^\Delta$ for the closed C^∞-subscheme in $\underline{X}$ fixed by Δ in G, and $\underline{X}^\Delta_\circ$ for the open C^∞-subscheme in $\underline{X}^\Delta$ of points in $\underline{X}$ whose stabilizer group in G is exactly Δ. In (1.42)–(1.43)*, morphisms $\rho,\rho':\Gamma\to G$ are conjugate if $\rho'=\mathrm{Ad}(g)\circ\rho$ for some $g\in G$, and subgroups $\Delta,\Delta'\subseteq G$ are conjugate if $\Delta=g\Delta'g^{-1}$ for some $g\in G$. In* (1.42)–(1.47) *we sum over one representative ρ or Δ for each conjugacy class.*

Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack and Γ a finite group, so that as above we have an orbifold stratum $\mathcal{X}^\Gamma$ with a 1-morphism $O^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\to\mathcal{X}$. Let $\mathcal{E}$ be a quasicoherent sheaf on $\mathcal{X}$, so that $\mathcal{E}^\Gamma:=O^\Gamma(\mathcal{X})^*(\mathcal{E})$ is a quasicoherent sheaf on $\mathcal{X}^\Gamma$. In [56, §11.4] we show that there is a natural representation of Γ on $\mathcal{E}^\Gamma$ by isomorphisms. Also the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ lifts naturally to $\mathcal{E}^\Gamma$, so that $\mathrm{Aut}(\Gamma)\ltimes\Gamma$ acts equivariantly on $\mathcal{E}^\Gamma$.

Write $R_0,\ldots,R_k$ for the irreducible representations of Γ over $\mathbb{R}$ (that is, we choose one representative R_i in each isomorphism class of irreducible representations), with $R_0=\mathbb{R}$ the trivial representation. Then the Γ-representation on $\mathcal{E}^\Gamma$ induces a splitting

$$\mathcal{E}^\Gamma\cong\textstyle\bigoplus_{i=0}^k\mathcal{E}_i^\Gamma\otimes R_i\quad\text{for } \mathcal{E}_0^\Gamma,\ldots,\mathcal{E}_k^\Gamma\in\mathrm{qcoh}(\mathcal{X}^\Gamma). \tag{1.48}$$

We will be interested in splitting $\mathcal{E}^\Gamma$ into *trivial* and *nontrivial* representations of Γ, denoted by subscripts 'tr' and 'nt'. So we write

$$\mathcal{E}^\Gamma=\mathcal{E}^\Gamma_{\mathrm{tr}}\oplus\mathcal{E}^\Gamma_{\mathrm{nt}}, \tag{1.49}$$

where $\mathcal{E}^\Gamma_{\mathrm{tr}},\mathcal{E}^\Gamma_{\mathrm{nt}}$ are the subsheaves of $\mathcal{E}^\Gamma$ corresponding to the factors $\mathcal{E}_0^\Gamma\otimes R_0$ and $\bigoplus_{i=1}^k\mathcal{E}_i^\Gamma\otimes R_i$ respectively. The same applies for the orbifold stratum $\mathcal{X}^\Gamma_\circ\subseteq\mathcal{X}^\Gamma$.

We also have an orbifold stratum $\tilde{\mathcal{X}}^\Gamma$ with a 1-morphism $\tilde O^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\to\mathcal{X}$, so that $\tilde{\mathcal{E}}^\Gamma:=\tilde O^\Gamma(\mathcal{X})^*(\mathcal{E})$ is a quasicoherent sheaf on $\tilde{\mathcal{X}}^\Gamma$. In general there is no natural Γ-representation on $\tilde{\mathcal{E}}^\Gamma$, as the quotient by $\mathrm{Aut}(\Gamma)$ in $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$ does not preserve the Γ-action. However, we do have a natural splitting

$$\tilde{\mathcal{E}}^\Gamma = \tilde{\mathcal{E}}^\Gamma_{\mathrm{tr}} \oplus \tilde{\mathcal{E}}^\Gamma_{\mathrm{nt}} \tag{1.50}$$

corresponding to (1.49). The same applies for $\tilde{\mathcal{X}}^\Gamma_\circ \subseteq \tilde{\mathcal{X}}^\Gamma$.

As in (1.41), for the orbifold stratum $\hat{\mathcal{X}}^\Gamma$ we do not have a natural 1-morphism $\hat{\mathcal{X}}^\Gamma \to \mathcal{X}$, so we cannot just pull $\mathcal{E}$ back to $\hat{\mathcal{X}}^\Gamma$. Instead, we push $\tilde{\mathcal{E}}^\Gamma$ down to $\hat{\mathcal{X}}^\Gamma$ along the 1-morphism $\hat{\Pi}^\Gamma : \tilde{\mathcal{X}}^\Gamma \to \hat{\mathcal{X}}^\Gamma$. It turns out that in the splitting (1.50), the push down $\hat{\Pi}^\Gamma_*(\tilde{\mathcal{E}}^\Gamma_{\mathrm{nt}})$ is zero, since $\hat{\Pi}^\Gamma$ has fibre $[\underline{*}/\Gamma]$, and $\hat{\Pi}^\Gamma_*$ essentially takes Γ-equivariant parts. So we define $\hat{\mathcal{E}}^\Gamma_{\mathrm{tr}} = \hat{\Pi}^\Gamma_*(\tilde{\mathcal{E}}^\Gamma_{\mathrm{tr}})$, a quasicoherent sheaf on $\hat{\mathcal{X}}^\Gamma$. The same applies for $\hat{\mathcal{X}}^\Gamma_\circ \subseteq \hat{\mathcal{X}}^\Gamma$.

When passing to orbifold strata, it is often natural to restrict to the trivial parts $\mathcal{E}^\Gamma_{\mathrm{tr}}, \tilde{\mathcal{E}}^\Gamma_{\mathrm{tr}}, \hat{\mathcal{E}}^\Gamma_{\mathrm{tr}}$ of the pullbacks of $\mathcal{E}$. The next theorem illustrates this.

Theorem 1.8.28. *Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack and Γ a finite group, so that we have a 1-morphism $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$. As in Example 1.8.20 we have cotangent sheaves $T^*\mathcal{X}, T^*(\mathcal{X}^\Gamma)$ and a morphism $\Omega_{O^\Gamma(\mathcal{X})} : O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) \to T^*(\mathcal{X}^\Gamma)$ in $\mathrm{qcoh}(\mathcal{X}^\Gamma)$. But $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) = (T^*\mathcal{X})^\Gamma$, so by (1.49) we have a splitting $(T^*\mathcal{X})^\Gamma = (T^*\mathcal{X})^\Gamma_{\mathrm{tr}} \oplus (T^*\mathcal{X})^\Gamma_{\mathrm{nt}}$. Then $\Omega_{O^\Gamma(\mathcal{X})}|_{(T^*\mathcal{X})^\Gamma_{\mathrm{tr}}} : (T^*\mathcal{X})^\Gamma_{\mathrm{tr}} \to T^*(\mathcal{X}^\Gamma)$ is an isomorphism, and $\Omega_{O^\Gamma(\mathcal{X})}|_{(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}} = 0$.*

Similarly, using $\tilde{O}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \to \mathcal{X}$ and (1.50) for $(\widetilde{T^*\mathcal{X}})^\Gamma$ we find that $\Omega_{\tilde{O}^\Gamma(\mathcal{X})}|_{(\widetilde{T^*\mathcal{X}})^\Gamma_{\mathrm{tr}}} : (\widetilde{T^*\mathcal{X}})^\Gamma_{\mathrm{tr}} \to T^*(\tilde{\mathcal{X}}^\Gamma)$ is an isomorphism, and $\Omega_{\tilde{O}^\Gamma(\mathcal{X})}|_{(\widetilde{T^*\mathcal{X}})^\Gamma_{\mathrm{nt}}} = 0$. Also, there is a natural isomorphism $(\widehat{T^*\mathcal{X}})^\Gamma_{\mathrm{tr}} \cong T^*(\hat{\mathcal{X}}^\Gamma)$ in $\mathrm{qcoh}(\hat{\mathcal{X}}^\Gamma)$.

1.9 Orbifolds

We now summarize §8.1–§8.4 on orbifolds.

1.9.1 Different ways to define orbifolds

Orbifolds are geometric spaces locally modelled on $\mathbb{R}^n/G$, for $G \subset \mathrm{GL}(n,\mathbb{R})$ a finite group. There are several *nonequivalent* definitions of orbifolds in the literature, which are reviewed in §8.1. They were first defined by Satake [90] (who called them 'V-manifolds') and Thurston [99, §13]. Satake and Thurston defined an orbifold to be a Hausdorff topological space X with an atlas of charts (U_i, Γ_i, ϕ_i) for $i \in I$, where $\Gamma_i \subset \mathrm{GL}(n,\mathbb{R})$ is a finite subgroup, $U_i \subseteq \mathbb{R}^n$ a Γ_i-invariant open subset, and $\phi_i : U_i/\Gamma_i \to X$ a homeomorphism with an open set in X, compatible on overlaps $\phi_i(U_i/\Gamma_i) \cap \phi_j(U_j/\Gamma_j)$ in X. Smooth maps between orbifolds are continuous maps $f : X \to Y$, which lift locally to equivariant smooth maps on the charts.

There is a problem with this notion of smooth maps: some differential-geometric operations, such as pullbacks of vector bundles by smooth maps, may not be well-defined. To fix this problem, new definitions were needed. Moerdijk and Pronk [84, 85] defined orbifolds to be *proper étale Lie groupoids* in **Man**. Chen and Ruan [21, §4] gave an alternative theory more in the spirit of [90, 99]. A book on orbifolds in the sense of [21, 84, 85] is Adem, Leida and Ruan [2].

All of [2, 21, 84, 85, 90, 99] regard orbifolds as an ordinary category. But orbifolds are differential-geometric analogues of Deligne–Mumford stacks, which form a 2-category. So it seems natural to define a 2-category of orbifolds **Orb**. Several important geometric constructions need the extra structure of a 2-category to work properly. For example, transverse fibre products exist in the 2-category **Orb**, where they satisfy a universal property involving 2-morphisms, as in §A.4. In the homotopy category Ho(**Orb**), 'transverse fibre products' can be defined as an *ad hoc* geometric construction, but they are not fibre products in the category-theoretic sense, and do not satisfy a universal property.

There are two main routes in the literature for defining a 2-category of orbifolds **Orb**. The first, as in Pronk [89] and Lerman [67, §3.3], is to define orbifolds to be groupoids in **Man** as in [84,85]. But to define 1- and 2-morphisms in **Orb** one must do more work: one makes proper étale Lie groupoids into a 2-category **Gpoid**, and then **Orb** is defined as a (weak) 2-category localization of **Gpoid** at a suitable class of 1-morphisms.

The second route, as in Behrend and Xu [13, §2], Lerman [67, §4] and Metzler [82, §3.5], is to define orbifolds as a class of Deligne–Mumford stacks on the site $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$ of manifolds with Grothendieck topology $\mathcal{J}_{\mathbf{Man}}$ coming from open covers. The relationship between the two routes is discussed in [13, 67, 89].

In the 'classical' approaches to orbifolds [2, 21, 84, 85, 90, 99], the objects, orbifolds, have a simple definition, but the smooth maps, or 1- and 2-morphisms, are either badly behaved, or very complicated to define. In contrast, in the 'stacky' approaches to orbifolds [13, 56, 67, 82], the objects are very complicated to define, but 1- and 2-morphisms are well-behaved and easy to define — 1-morphisms are just functors, and 2-morphisms are natural isomorphisms.

Our approach in this book, described in §8.2 below, is similar to the second route: we define orbifolds to be special examples of Deligne–Mumford C^∞-stacks, so that they are stacks on the site $(\mathbf{C^\infty Sch}, \mathcal{J})$. This will be convenient for our work on d-stacks and d-orbifolds, which are also based on C^∞-stacks.

Definition 1.9.1. An *orbifold of dimension* n is a separated, second countable Deligne–Mumford C^∞-stack $\mathcal{X}$ such that for every $[x] \in \mathcal{X}_{\mathrm{top}}$ there exist a linear action of $G = \mathrm{Iso}_{\mathcal{X}}([x])$ on $\mathbb{R}^n$, a G-invariant open neighbourhood U of 0 in $\mathbb{R}^n$, and a 1-morphism $i : [\underline{U}/G] \to \mathcal{X}$ which is an equivalence with an open neighbourhood $\mathcal{U} \subseteq \mathcal{X}$ of $[x]$ in $\mathcal{X}$ with $i_{\mathrm{top}}([0]) = [x]$, where $\underline{U} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(U)$.

Write **Orb** for the full 2-subcategory of orbifolds in $\mathbf{DMC^\infty Sta}$. We may refer to 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ in **Orb** as *smooth maps* of orbifolds. Define a full and faithful functor $F_{\mathbf{Man}}^{\mathbf{Orb}} : \mathbf{Man} \to \mathbf{Orb}$ by $F_{\mathbf{Man}}^{\mathbf{Orb}} = F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}} \circ F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$.

Here is [56, Th. 9.26 & Cor. 9.27]. Since equivalent (2-)categories are considered to be 'the same', the moral of Theorem 1.9.2 is that our orbifolds are essentially the same objects as those considered by other recent authors.

Theorem 1.9.2. *The* 2*-category* **Orb** *of orbifolds without boundary defined above is equivalent to the* 2*-categories of orbifolds considered as stacks on* **Man** *defined in Metzler* [82, §3.4] *and Lerman* [67, §4], *and also equivalent as a weak*

2-category to the weak 2-categories of orbifolds regarded as proper étale Lie groupoids defined in Pronk [89] *and Lerman* [67, §3.3].

Furthermore, the homotopy category Ho(**Orb**) *of* **Orb** *(that is, the category whose objects are objects in* **Orb**, *and whose morphisms are 2-isomorphism classes of 1-morphisms in* **Orb***) is equivalent to the category of orbifolds regarded as proper étale Lie groupoids defined in Moerdijk* [84]. *Transverse fibre products in* **Orb** *agree with the corresponding fibre products in* $\mathbf{C^\infty Sta}$.

We define five classes of smooth maps:

Definition 1.9.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth map (1-morphism) of orbifolds.

(i) We call f *representable* if it acts injectively on orbifold groups, that is, $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\bigl(f_{\mathrm{top}}([x])\bigr)$ is an injective morphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$.

Equivalently, f is representable if it is a representable 1-morphism of C^∞-stacks. This means that whenever $\underline{V}$ is a C^∞-scheme and $\Pi : \bar{\underline{V}} \to \mathcal{Y}$ is a 1-morphism then the C^∞-stack fibre product $\mathcal{X} \times_{f,\mathcal{Y},\Pi} \bar{\underline{V}}$ is a C^∞-scheme.

(ii) We call f an *immersion* if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is a surjective morphism of vector bundles, i.e. has a right inverse in $\mathrm{qcoh}(\mathcal{X})$.

(iii) We call f an *embedding* if it is an immersion, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\bigl(f_{\mathrm{top}}([x])\bigr)$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image (so in particular it is injective).

(iv) We call f a *submersion* if $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is an injective morphism of vector bundles, i.e. has a left inverse in $\mathrm{qcoh}(\mathcal{X})$.

(v) We call f *étale* if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is an isomorphism, or equivalently, if f is étale as a 1-morphism of C^∞-stacks.

Note that submersions are not required to be representable.

Definition 1.9.4. An orbifold $\mathcal{X}$ is called *effective* if $\mathcal{X}$ is locally modelled near each $[x] \in \mathcal{X}_{\mathrm{top}}$ on $\mathbb{R}^n/G$, where G acts effectively on $\mathbb{R}^n$, that is, every $1 \neq \gamma \in G$ acts nontrivially on $\mathbb{R}^n$.

In §8.4 we prove a uniqueness property for 2-morphisms of effective orbifolds.

Proposition 1.9.5. *Let $\mathcal{X}, \mathcal{Y}$ be effective orbifolds, and $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms. Suppose that either:*

(i) *f is an embedding, a submersion, étale, or an equivalence;*

(ii) $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\bigl(f_{\mathrm{top}}([x])\bigr)$ *is surjective for all* $[x] \in \mathcal{X}_{\mathrm{top}}$; *or*

(iii) *$\mathcal{Y}$ is a manifold.*

Then there exists at most one 2-morphism $\eta : f \Rightarrow g$.

Some authors include effectiveness in their definition of orbifolds. The Satake–Thurston definitions are not as well-behaved for noneffective orbifolds. One reason is that Proposition 1.9.5 often allows us to treat effective orbifolds as if they were a category rather than a 2-category, that is, one can work in the homotopy category $\mathrm{Ho}(\mathbf{Orb^{eff}})$ of the full 2-subcategory $\mathbf{Orb^{eff}}$ of effective orbifolds, because genuinely 2-categorical behaviour comes from non-uniqueness of 2-morphisms.

In §8.3 we discuss *vector bundles* on orbifolds. Now an orbifold $\mathcal{X}$ is an example of a Deligne–Mumford C^∞-stack, and in §1.8.6 we defined a category $\mathrm{qcoh}(\mathcal{X})$ of quasicoherent sheaves on $\mathcal{X}$, and a full subcategory $\mathrm{vect}(\mathcal{X})$ of vector bundles on $\mathcal{X}$. Unless we say otherwise, a *vector bundle* $\mathcal{E}$ *on an orbifold* $\mathcal{X}$ will just mean an object in $\mathrm{vect}(\mathcal{X})$, a special kind of quasicoherent sheaf on $\mathcal{X}$, and a *smooth section* s *of* $\mathcal{E}$ will mean an element of $C^\infty(\mathcal{E})$, that is, a morphism $s : \mathcal{O}_\mathcal{X} \to \mathcal{E}$ in $\mathrm{vect}(\mathcal{X})$. The cotangent sheaf $T^*\mathcal{X}$ of an n-orbifold $\mathcal{X}$ is a vector bundle on $\mathcal{X}$ of rank n, which we call the *cotangent bundle*.

For some applications below, this point of view on vector bundles is not ideal. If $E \to X$ is a vector bundle on a manifold, then E is itself a manifold (with extra structure), with a submersion $\pi : E \to X$, and a section $s \in C^\infty(E)$ is a smooth map $s : X \to E$ with $\pi \circ s = \mathrm{id}_X$. In §1.4.1–§1.4.2 we considered d-space fibre products $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$ where $\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(V, E, s, 0)$. For the d-orbifold analogue of this, we would like to regard a vector bundle $\mathcal{E}$ over an orbifold $\mathcal{X}$ as being an orbifold in its own right, rather than just a quasicoherent sheaf, and a section $s \in C^\infty(\mathcal{E})$ as being a 1-morphism $s : \mathcal{X} \to \mathcal{E}$ in $\mathbf{Orb}$.

To get round this, in §8.3 we define a *total space functor* Tot, which to each $\mathcal{E}$ in $\mathrm{vect}(\mathcal{X})$ associates an orbifold $\mathrm{Tot}(\mathcal{E})$, called the *total space* of $\mathcal{E}$, and to each section $s \in C^\infty(\mathcal{E})$ associates a 1-morphism $\mathrm{Tot}(s) : \mathcal{X} \to \mathrm{Tot}(\mathcal{E})$ in $\mathbf{Orb}$. Then the d-orbifold analogue of $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$ in Proposition 1.4.2(c) is $\boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}} \boldsymbol{\mathcal{V}}$, where $\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{E}}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Orb}}^{\mathbf{dSta}}\bigl(\mathcal{V}, \mathrm{Tot}(\mathcal{E}), \mathrm{Tot}(s), \mathrm{Tot}(0)\bigr)$.

Many other standard ideas in differential geometry extend simply to orbifolds, such as submanifolds, transverse fibre products, and orientations, and we will generally use these without comment.

1.9.2 Orbifold strata of orbifolds

In §1.8.7 we discussed orbifold strata $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma, \mathcal{X}^\Gamma_\circ, \tilde{\mathcal{X}}^\Gamma_\circ, \hat{\mathcal{X}}^\Gamma_\circ$ of a Deligne–Mumford C^∞-stack $\mathcal{X}$. Section 8.4 works these ideas out for orbifolds. If $\mathcal{X}$ is an orbifold, then $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ need not be orbifolds, as the next example shows, but are disjoint unions of orbifolds of different dimensions.

Example 1.9.6. Let the real projective plane $\mathbb{RP}^2$ have homogeneous coordinates $[x_0, x_1, x_2]$, and let $\mathbb{Z}_2 = \{1, \sigma\}$ act on $\mathbb{RP}^2$ by $\sigma : [x_0, x_1, x_2] \mapsto [x_0, x_1, -x_2]$. The fixed point locus of σ in $\mathbb{RP}^2$ is the disjoint union of the circle $\bigl\{[x_0, x_1, 0] : [x_0, x_1] \in \mathbb{RP}^1\bigr\}$ and the point $\bigl\{[0, 0, 1]\bigr\}$.

Write $\underline{\mathbb{RP}}^2 = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(\mathbb{RP}^2)$, and form the quotient orbifold $\mathcal{X} = [\underline{\mathbb{RP}}^2/\mathbb{Z}_2]$. Then (1.42) shows that the orbifold stratum $\mathcal{X}^{\mathbb{Z}_2}$ is the disjoint union of orbifolds $\underline{\mathbb{RP}}^1 \times [\underline{*}/\mathbb{Z}_2]$ and $[\underline{*}/\mathbb{Z}_2]$ of dimensions 1 and 0, respectively. Note that $\mathcal{X}^{\mathbb{Z}_2}$ is not an orbifold, as it does not have pure dimension, and nor are $\tilde{\mathcal{X}}^{\mathbb{Z}_2}, \ldots, \hat{\mathcal{X}}^{\mathbb{Z}_2}_\circ$.

So that our constructions remain within the world of orbifolds, we will find it useful to define a decomposition $\mathcal{X}^\Gamma = \coprod_{\lambda \in \Lambda^\Gamma_+} \mathcal{X}^{\Gamma,\lambda}$ of $\mathcal{X}^\Gamma$ such that each

$\mathcal{X}^{\Gamma,\lambda}$ is an orbifold of dimension $\dim \mathcal{X} - \dim \lambda$.

Definition 1.9.7. Let Γ be a finite group. Consider representations (V,ρ) of Γ, where V is a finite-dimensional real vector space and $\rho : \Gamma \to \mathrm{Aut}(V)$ a group morphism. We call (V,ρ) *nontrivial* if $V^{\rho(\Gamma)} = \{0\}$. Write $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ for the abelian category of nontrivial (V,ρ), and $K_0(\mathrm{Rep}_{\mathrm{nt}}(\Gamma))$ for its Grothendieck group. Then any (V,ρ) in $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ has a class $\bigl[(V,\rho)\bigr]$ in $K_0(\mathrm{Rep}_{\mathrm{nt}}(\Gamma))$. For brevity, we will use the notation $\Lambda^\Gamma = K_0\bigl(\mathrm{Rep}_{\mathrm{nt}}(\Gamma)\bigr)$ and $\Lambda^\Gamma_+ = \bigl\{\bigl[(V,\rho)\bigr] : (V,\rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)\bigr\} \subseteq \Lambda^\Gamma$. We think of Λ^Γ_+ as the 'positive cone' in Λ^Γ.

By elementary representation theory, up to isomorphism Γ has finitely many irreducible representations. Let $R_0, R_1, \ldots, R_k$ be choices of irreducible representations in these isomorphism classes, with $R_0 = \mathbb{R}$ the trivial irreducible representation, so that $R_1, \ldots, R_k$ are nontrivial. Then Λ^Γ is freely generated over $\mathbb{Z}$ by $[R_1], \ldots, [R_k]$, so that

$$\begin{aligned} \Lambda^\Gamma &= \bigl\{a_1[R_1] + \cdots + a_k[R_k] : a_1, \ldots, a_k \in \mathbb{Z}\bigr\}, \qquad \text{and} \\ \Lambda^\Gamma_+ &= \bigl\{a_1[R_1] + \cdots + a_k[R_k] : a_1, \ldots, a_k \in \mathbb{N}\bigr\} \subseteq \Lambda^\Gamma, \end{aligned}$$

where $\mathbb{N} = \{0,1,2,\ldots\} \subset \mathbb{Z}$. Hence $\Lambda^\Gamma \cong \mathbb{Z}^k$ and $\Lambda^\Gamma_+ \cong \mathbb{N}^k$.

Define a group morphism $\dim : \Lambda^\Gamma \to \mathbb{Z}$ by $\dim : a_1[R_1] + \cdots + a_k[R_k] \mapsto a_1 \dim R_1 + \cdots + a_k \dim R_k$, so that $\dim : [(V,\rho)] \mapsto \dim V$. Then $\dim(\Lambda^\Gamma_+) \subseteq \mathbb{N}$.

Now let $\mathcal{X}$ be an orbifold. As in (1.48)–(1.49) we have decompositions $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) = (T^*\mathcal{X})^\Gamma_{\mathrm{tr}} \oplus (T^*\mathcal{X})^\Gamma_{\mathrm{nt}}$ with $(T^*\mathcal{X})^\Gamma_{\mathrm{tr}} \cong (T^*\mathcal{X})^\Gamma_0 \otimes R_0$ and $(T^*\mathcal{X})^\Gamma_{\mathrm{nt}} \cong \bigoplus_{i=1}^k (T^*\mathcal{X})^\Gamma_i \otimes R_i$, where $(T^*\mathcal{X})^\Gamma_0, \ldots, (T^*\mathcal{X})^\Gamma_k \in \mathrm{qcoh}(\mathcal{X}^\Gamma)$. Since $T^*\mathcal{X}$ is a vector bundle, $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X})$ is a vector bundle, and so the $(T^*\mathcal{X})^\Gamma_i$ are *vector bundles of mixed rank*, that is, locally they are vector bundles, but their ranks may vary on different connected components of $\mathcal{X}^\Gamma$.

For each $\lambda \in \Lambda^\Gamma_+$, define $\mathcal{X}^{\Gamma,\lambda}$ to be the open and closed C^∞-substack in $\mathcal{X}^\Gamma$ with $\mathrm{rank}\bigl((T^*\mathcal{X})^\Gamma_1\bigr)[R_1] + \cdots + \mathrm{rank}\bigl((T^*\mathcal{X})^\Gamma_k\bigr)[R_k] = \lambda$ in Λ^Γ_+. Then $(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}|_{\mathcal{X}^{\Gamma,\lambda}}$ is a vector bundle of rank $\dim \lambda$, so $(T^*\mathcal{X})^\Gamma_{\mathrm{tr}}|_{\mathcal{X}^{\Gamma,\lambda}}$ is a vector bundle of dimension $\dim \mathcal{X} - \dim \lambda$ on $\mathcal{X}^{\Gamma,\lambda}$. But $(T^*\mathcal{X})^\Gamma_{\mathrm{tr}} \cong T^*(\mathcal{X}^\Gamma)$ by Theorem 1.8.28. Hence $T^*(\mathcal{X}^{\Gamma,\lambda})$ is a vector bundle of rank $\dim \mathcal{X} - \dim \lambda$. Since $\mathcal{X}^\Gamma$ is a disjoint union of orbifolds of different dimensions, we see that $\mathcal{X}^{\Gamma,\lambda}$ is an orbifold, with $\dim \mathcal{X}^{\Gamma,\lambda} = \dim \mathcal{X} - \dim \lambda$. Then $\mathcal{X}^\Gamma = \coprod_{\lambda \in \Lambda^\Gamma_+} \mathcal{X}^{\Gamma,\lambda}$.

Write $O^{\Gamma,\lambda}(\mathcal{X}) = O^\Gamma(\mathcal{X})|_{\mathcal{X}^{\Gamma,\lambda}} : \mathcal{X}^{\Gamma,\lambda} \to \mathcal{X}$. It is a proper, representable immersion of orbifolds. We interpret $(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}|_{\mathcal{X}^{\Gamma,\lambda}}$ as the *conormal bundle* of $\mathcal{X}^{\Gamma,\lambda}$ in $\mathcal{X}$. It carries a nontrivial Γ-representation of class $\lambda \in \Lambda^\Gamma_+$, so we refer to λ as the *conormal Γ-representation* of $\mathcal{X}^{\Gamma,\lambda}$.

Define $\mathcal{X}^{\Gamma,\lambda}_\circ = \mathcal{X}^\Gamma_\circ \cap \mathcal{X}^{\Gamma,\lambda}$, and $O^{\Gamma,\lambda}_\circ(\mathcal{X}) = O^\Gamma_\circ(\mathcal{X})|_{\mathcal{X}^{\Gamma,\lambda}_\circ} : \mathcal{X}^{\Gamma,\lambda}_\circ \to \mathcal{X}$. Then $\mathcal{X}^{\Gamma,\lambda}_\circ$ is an orbifold with $\dim \mathcal{X}^{\Gamma,\lambda}_\circ = \dim \mathcal{X} - \dim \lambda$, and $\mathcal{X}^\Gamma_\circ = \coprod_{\lambda \in \Lambda^\Gamma_+} \mathcal{X}^{\Gamma,\lambda}_\circ$.

As in §1.8.7, we have $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma / \mathrm{Aut}(\Gamma)]$. Now $\mathrm{Aut}(\Gamma)$ acts on the right on $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ by $\alpha : (V,\rho) \mapsto (V, \rho \circ \alpha)$ for $\alpha \in \mathrm{Aut}(\Gamma)$, and this induces right actions of $\mathrm{Aut}(\Gamma)$ on $\Lambda^\Gamma = K_0\bigl(\mathrm{Rep}_{\mathrm{nt}}(\Gamma)\bigr)$ and $\Lambda^\Gamma_+ \subseteq \Lambda^\Gamma$. Write these actions as $\alpha : \lambda \mapsto \lambda \cdot \alpha$. Then the action of $\alpha \in \mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ maps $\mathcal{X}^{\Gamma,\lambda} \to \mathcal{X}^{\Gamma,\lambda\cdot\alpha}$. Write $\Lambda^\Gamma_+ / \mathrm{Aut}(\Lambda)$ for the set of $\mathrm{Aut}(\Gamma)$-orbits $\mu = \lambda \cdot \mathrm{Aut}(\Gamma)$ in Λ^Γ_+. The map $\dim : \Lambda^\Gamma \to \mathbb{Z}$ is $\mathrm{Aut}(\Gamma)$-invariant, and so descends to $\dim : \Lambda^\Gamma / \mathrm{Aut}(\Gamma) \to \mathbb{Z}$.

Then $\coprod_{\lambda\in\mu}\mathcal{X}^{\Gamma,\lambda}$ is an open and closed $\mathrm{Aut}(\Gamma)$-invariant C^∞-substack in $\mathcal{X}^\Gamma$ for each $\mu\in\Lambda_+^\Gamma/\mathrm{Aut}(\Lambda)$, so we may define $\tilde{\mathcal{X}}^{\Gamma,\mu}\simeq\bigl[\bigl(\coprod_{\lambda\in\mu}\mathcal{X}^{\Gamma,\lambda}\bigr)/\mathrm{Aut}(\Gamma)\bigr]$, an open and closed C^∞-substack of $\tilde{\mathcal{X}}^\Gamma\simeq[\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$. Write $\tilde{\mathcal{X}}^{\Gamma,\mu}_\circ=\tilde{\mathcal{X}}^\Gamma_\circ\cap\tilde{\mathcal{X}}^{\Gamma,\mu}$. Then $\tilde{\mathcal{X}}^{\Gamma,\mu},\tilde{\mathcal{X}}^{\Gamma,\mu}_\circ$ are orbifolds of dimension $\dim\mathcal{X}-\dim\mu$, with

$$\tilde{\mathcal{X}}^\Gamma=\textstyle\coprod_{\mu\in\Lambda_+^\Gamma/\mathrm{Aut}(\Gamma)}\tilde{\mathcal{X}}^{\Gamma,\mu}\quad\text{and}\quad\tilde{\mathcal{X}}^\Gamma_\circ=\coprod_{\mu\in\Lambda_+^\Gamma/\mathrm{Aut}(\Gamma)}\tilde{\mathcal{X}}^{\Gamma,\mu}_\circ.$$

Set $\tilde{O}^{\Gamma,\mu}(\mathcal{X})=\tilde{O}^\Gamma(\mathcal{X})|_{\tilde{\mathcal{X}}^{\Gamma,\mu}}:\tilde{\mathcal{X}}^{\Gamma,\mu}\to\mathcal{X}$ and $\tilde{O}^{\Gamma,\mu}_\circ(\mathcal{X})=\tilde{O}^\Gamma_\circ(\mathcal{X})|_{\tilde{\mathcal{X}}^{\Gamma,\mu}_\circ}:\tilde{\mathcal{X}}^{\Gamma,\mu}_\circ\to\mathcal{X}$. Then $\tilde{O}^{\Gamma,\mu}(\mathcal{X}),\tilde{O}^{\Gamma,\mu}_\circ(\mathcal{X})$ are representable immersions, with $\tilde{O}^{\Gamma,\mu}(\mathcal{X})$ proper.

The 1-morphism $\hat{\Pi}^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\to\hat{\mathcal{X}}^\Gamma$ maps open and closed C^∞-substacks of $\tilde{\mathcal{X}}^\Gamma$ to open and closed C^∞-substacks of $\hat{\mathcal{X}}^\Gamma$. Let $\hat{\mathcal{X}}^{\Gamma,\mu}=\hat{\Pi}^\Gamma(\mathcal{X})(\tilde{\mathcal{X}}^{\Gamma,\mu})$ for each $\mu\in\Lambda_+^\Gamma/\mathrm{Aut}(\Lambda)$, and write $\hat{\mathcal{X}}^{\Gamma,\mu}_\circ=\hat{\mathcal{X}}^\Gamma_\circ\cap\hat{\mathcal{X}}^{\Gamma,\mu}$. Then $\hat{\mathcal{X}}^{\Gamma,\mu},\hat{\mathcal{X}}^{\Gamma,\mu}_\circ$ are orbifolds of dimension $\dim\mathcal{X}-\dim\mu$, with

$$\hat{\mathcal{X}}^\Gamma=\textstyle\coprod_{\mu\in\Lambda_+^\Gamma/\mathrm{Aut}(\Gamma)}\hat{\mathcal{X}}^{\Gamma,\mu}\quad\text{and}\quad\hat{\mathcal{X}}^\Gamma_\circ=\coprod_{\mu\in\Lambda_+^\Gamma/\mathrm{Aut}(\Gamma)}\hat{\mathcal{X}}^{\Gamma,\mu}_\circ.$$

If $f:\mathcal{X}\to\mathcal{Y}$ is a representable 1-morphism of Deligne–Mumford C^∞-stacks and Γ a finite group, then as in §1.8.7 we have a representable 1-morphism of orbifold strata $f^\Gamma:\mathcal{X}^\Gamma\to\mathcal{Y}^\Gamma$. Note that if $\mathcal{X},\mathcal{Y}$ are orbifolds, then f^Γ need not map $\mathcal{X}^{\Gamma,\lambda}\to\mathcal{Y}^{\Gamma,\lambda}$, or map $\mathcal{X}^\Gamma_\circ\to\mathcal{Y}^\Gamma_\circ$. The analogue applies for $\tilde{f}^\Gamma,\hat{f}^\Gamma$.

Some important properties of orbifolds can be characterized by the vanishing of certain orbifold strata $\mathcal{X}^{\Gamma,\lambda}$. For example:

- An orbifold $\mathcal{X}$ is *locally orientable* if and only if $\mathcal{X}^{\mathbb{Z}_2,\lambda}=\emptyset$ for all odd $\lambda\in\Lambda_+^{\mathbb{Z}_2}\cong\mathbb{N}=\{0,1,2,\ldots\}$.
- An orbifold $\mathcal{X}$ is *effective* in the sense of Definition 1.9.4 if and only if $\mathcal{X}^{\Gamma,0}=\emptyset$ for all nontrivial finite groups Γ.

In §8.4 we consider the question: if $\mathcal{X}$ is an oriented orbifold, can we define orientations on the orbifold strata $\mathcal{X}^{\Gamma,\lambda},\ldots,\hat{\mathcal{X}}^{\Gamma,\mu}_\circ$? Here is an example:

Example 1.9.8. Let $\mathcal{S}^4=\bigl\{(x_1,\ldots,x_5)\in\mathbb{R}^5:x_1^2+\cdots+x_5^2=1\bigr\}$, an oriented 4-manifold. Let $G=\{1,\sigma,\tau,\sigma\tau\}\cong\mathbb{Z}_2^2$ act on $\mathcal{S}^4$ preserving orientations by

$$\begin{aligned}\sigma:(x_1,\ldots,x_5)&\longmapsto(x_1,x_2,x_3,-x_4,-x_5),\\ \tau:(x_1,\ldots,x_5)&\longmapsto(-x_1,-x_2,-x_3,-x_4,x_5),\\ \sigma\tau:(x_1,\ldots,x_5)&\longmapsto(-x_1,-x_2,-x_3,x_4,-x_5).\end{aligned}$$

Then $\mathcal{X}=[\underline{\mathcal{S}}^4/G]$ is an oriented 4-orbifold. The orbifold groups $\mathrm{Iso}_{\mathcal{X}}([x])$ for $[x]\in\mathcal{X}_{\mathrm{top}}$ are all $\{1\}$ or $\mathbb{Z}_2$. The singular locus of $\mathcal{X}$ is the disjoint union of a copy of $\mathbb{RP}^2$ from the fixed points $\pm(x_1,x_2,x_3,0,0)$ of σ, and two isolated points $\{\pm(0,0,0,0,1)\}$ and $\{\pm(0,0,0,1,0)\}$ from the fixed points of τ and $\sigma\tau$.

Identifying $\Lambda_+^{\mathbb{Z}_2}$ and $\Lambda_+^{\mathbb{Z}_2}/\mathrm{Aut}(\mathbb{Z}_2)$ with $\mathbb{N}$, it follows that

$$\begin{aligned}&\mathcal{X}^{\mathbb{Z}_2,2}=\mathcal{X}^{\mathbb{Z}_2,2}_\circ\cong\tilde{\mathcal{X}}^{\mathbb{Z}_2,2}=\tilde{\mathcal{X}}^{\mathbb{Z}_2,2}_\circ\cong\mathbb{RP}^2\times[\underline{*}/\mathbb{Z}_2],\qquad\hat{\mathcal{X}}^{\mathbb{Z}_2,2}=\hat{\mathcal{X}}^{\mathbb{Z}_2,2}_\circ\cong\mathbb{RP}^2,\\ &\mathcal{X}^{\mathbb{Z}_2,4}=\mathcal{X}^{\mathbb{Z}_2,4}_\circ\cong\tilde{\mathcal{X}}^{\mathbb{Z}_2,4}=\tilde{\mathcal{X}}^{\mathbb{Z}_2,4}_\circ\cong[\underline{*}/\mathbb{Z}_2]\amalg[\underline{*}/\mathbb{Z}_2],\quad\hat{\mathcal{X}}^{\mathbb{Z}_2,4}=\hat{\mathcal{X}}^{\mathbb{Z}_2,4}_\circ\cong *\amalg *.\end{aligned}$$

Since $\mathbb{RP}^2$ is not orientable, we see that $\mathcal{X}$ is an oriented orbifold, but none of $\mathcal{X}^{\mathbb{Z}_2,2}, \tilde{\mathcal{X}}^{\mathbb{Z}_2,2}, \hat{\mathcal{X}}^{\mathbb{Z}_2,2}, \mathcal{X}_\circ^{\mathbb{Z}_2,2}, \tilde{\mathcal{X}}_\circ^{\mathbb{Z}_2,2}, \hat{\mathcal{X}}_\circ^{\mathbb{Z}_2,2}$ are orientable.

Thus, we can only orient $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ for all oriented orbifolds $\mathcal{X}$ under some conditions on Γ, λ, μ. The next proposition sets out these conditions:

Proposition 1.9.9. **(a)** *Suppose Γ is a finite group and (V,ρ) a nontrivial Γ-representation which has no odd-dimensional subrepresentations, and write $\lambda = [(V,\rho)] \in \Lambda_+^\Gamma$. Choose an orientation on V. Then for all oriented orbifolds $\mathcal{X}$ we can define natural orientations on the orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}_\circ^{\Gamma,\lambda}$.*

If $|\Gamma|$ is odd then all nontrivial Γ-representations are even-dimensional, so we can orient $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}_\circ^{\Gamma,\lambda}$ for all $\lambda \in \Lambda_+^\Gamma$.

(b) *Let $\Gamma, (V,\rho), \lambda$ be as in* **(a)**, *and set $\mu = \lambda \cdot \mathrm{Aut}(\Gamma) \in \Lambda_+^\Gamma / \mathrm{Aut}(\Gamma)$. Write H for the subgroup of $\mathrm{Aut}(\Gamma)$ fixing λ in Λ_+^Γ. Then for each $\delta \in H$ there exists an isomorphism of Γ-representations $i_\delta : (V, \rho \circ \delta) \to (V,\rho)$. Suppose $i_\delta : V \to V$ is orientation-preserving for all $\delta \in H$. If $\lambda \in 2\Lambda_+^\Gamma$ this holds automatically.*

Then for all oriented orbifolds $\mathcal{X}$ we can define orientations on the orbifold strata $\tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$. For $\tilde{\mathcal{X}}^{\Gamma,\mu}$ this works as $\tilde{\mathcal{X}}^{\Gamma,\mu} \simeq [\mathcal{X}^{\Gamma,\lambda}/H]$, where $\mathcal{X}^{\Gamma,\lambda}$ is oriented by **(a)**, *and the H-action on $\mathcal{X}^{\Gamma,\lambda}$ preserves orientations, so the orientation on $\mathcal{X}^{\Gamma,\lambda}$ descends to an orientation on $\tilde{\mathcal{X}}^{\Gamma,\mu} \simeq [\mathcal{X}^{\Gamma,\lambda}/H]$.*

(c) *Suppose that Γ and $\lambda \in \Lambda_+^\Gamma$ do not satisfy the conditions in* **(a)**, *or Γ and $\mu \in \Lambda_+^\Gamma / \mathrm{Aut}(\Gamma)$ do not satisfy the conditions in* **(b)**. *Then as in Example* 1.9.8 *we can find examples of oriented orbifolds $\mathcal{X}$ such that $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}_\circ^{\Gamma,\lambda}$ are not orientable, or $\tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ are not orientable, respectively. That is, the conditions on Γ, λ, μ in* **(a)**,**(b)** *are necessary as well as sufficient to be able to orient orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ of all oriented orbifolds $\mathcal{X}$.*

Note that Proposition 1.9.9(a),(b) do not apply in Example 1.9.8, since the nontrivial representation of $\mathbb{Z}_2$ on $\mathbb{R}^2$ has an odd-dimensional subrepresentation.

1.10 The 2-category of d-stacks

Chapter 9 studies the 2-category of *d-stacks* **dSta**, which are orbifold versions of d-spaces in §1.3. Broadly, to go from d-spaces $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ to d-stacks we just replace the C^∞-scheme $\underline{X}$ by a Deligne–Mumford C^∞-stack $\mathcal{X}$.

One might expect that combining the 2-categories $\mathbf{DMC^\infty Sta}$ and **dSpa** should result in a 3-category **dSta**, but in fact a 2-category is sufficient. For 1-morphisms $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in **dSta**, a 2-morphism $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in **dSta** is a pair (η, η'), where $\eta : f \Rightarrow g$ is a 2-morphism in $\mathbf{C^\infty Sta}$, and $\eta' : f^*(\mathcal{F}_\mathcal{Y}) \to \mathcal{E}_\mathcal{X}$ is as for 2-morphisms in **dSpa**. These η, η' do not interact very much.

1.10.1 The definition of d-stacks

Definition 1.10.1. A *d-stack* $\boldsymbol{\mathcal{X}}$ is a quintuple $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_\mathcal{X}, \mathcal{E}_\mathcal{X}, \imath_\mathcal{X}, \jmath_\mathcal{X})$, where $\mathcal{X}$ is a separated, second countable, locally fair Deligne–Mumford C^∞-stack in the sense of §1.8, and $\mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}}$ fit into an exact sequence of sheaves of abelian groups on $\mathcal{X}$, in the sense of §1.8.6

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\ \jmath_{\mathcal{X}}\ } \mathcal{O}'_{\mathcal{X}} \xrightarrow{\ \imath_{\mathcal{X}}\ } \mathcal{O}_{\mathcal{X}} \longrightarrow 0,$$

satisfying the conditions:

(a) $\mathcal{O}'_{\mathcal{X}}$ is a sheaf of C^∞-rings on $\mathcal{X}$, and $\imath_{\mathcal{X}} : \mathcal{O}'_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ is a morphism of sheaves of C^∞-rings on $\mathcal{X}$, where $\mathcal{O}_{\mathcal{X}}$ is the structure sheaf of $\mathcal{X}$ as in Example 1.8.19, such that for all $(\underline{U}, u)$ in $\mathcal{C}_{\mathcal{X}}$, $(U, \mathcal{O}'_{\mathcal{X}}(\underline{U}, u))$ is a C^∞-scheme, and $\imath_{\mathcal{X}}(\underline{U}, u) : \mathcal{O}'_{\mathcal{X}}(\underline{U}, u) \to \mathcal{O}_{\mathcal{X}}(\underline{U}, u) = \mathcal{O}_U$ is a surjective morphism of sheaves of C^∞-rings on U, whose kernel is a sheaf of square zero ideals.

   We call $\imath_{\mathcal{X}} : \mathcal{O}'_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ satisfying these conditions a *square zero extension.*

(b) As $\imath_{\mathcal{X}} : \mathcal{O}'_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ is a square zero extension, its kernel $\mathcal{I}_{\mathcal{X}}$ is a quasicoherent sheaf on $\mathcal{X}$. We require that $\mathcal{E}_{\mathcal{X}}$ is also a quasicoherent sheaf on $\mathcal{X}$, and $\jmath_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{I}_{\mathcal{X}}$ is a surjective morphism in $\mathrm{qcoh}(\mathcal{X})$.

The sheaf of C^∞-rings $\mathcal{O}'_{\mathcal{X}}$ has a sheaf of cotangent modules $\Omega_{\mathcal{O}'_{\mathcal{X}}}$, which is an $\mathcal{O}'_{\mathcal{X}}$-module with exterior derivative $\mathrm{d} : \mathcal{O}'_{\mathcal{X}} \to \Omega_{\mathcal{O}'_{\mathcal{X}}}$. Define $\mathcal{F}_{\mathcal{X}} = \Omega_{\mathcal{O}'_{\mathcal{X}}} \otimes_{\mathcal{O}'_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}$ to be the associated $\mathcal{O}_{\mathcal{X}}$-module, a quasicoherent sheaf on $\mathcal{X}$, and set $\psi_{\mathcal{X}} = \Omega_{\imath_{\mathcal{X}}} \otimes \mathrm{id} : \mathcal{F}_{\mathcal{X}} \to T^*\mathcal{X}$, a morphism in $\mathrm{qcoh}(\mathcal{X})$. Define $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ to be the composition of morphisms of sheaves of abelian groups on $\mathcal{X}$:

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\jmath_{\mathcal{X}}} \mathcal{I}_{\mathcal{X}} \xrightarrow{\mathrm{d}|_{\mathcal{I}_{\mathcal{X}}}} \Omega_{\mathcal{O}'_{\mathcal{X}}} \overset{\cong}{=} \Omega_{\mathcal{O}'_{\mathcal{X}}} \otimes_{\mathcal{O}'_{\mathcal{X}}} \mathcal{O}'_{\mathcal{X}} \xrightarrow{\ \mathrm{id}\otimes \imath_{\mathcal{X}}\ } \Omega_{\mathcal{O}'_{\mathcal{X}}} \otimes_{\mathcal{O}'_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}} = \mathcal{F}_{\mathcal{X}}.$$

Then $\phi_{\mathcal{X}}$ is a morphism in $\mathrm{qcoh}(\mathcal{X})$, and the following sequence is exact:

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\ \phi_{\mathcal{X}}\ } \mathcal{F}_{\mathcal{X}} \xrightarrow{\ \psi_{\mathcal{X}}\ } T^*\mathcal{X} \longrightarrow 0. \tag{1.51}$$

The morphism $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ will be called the *virtual cotangent sheaf* of $\boldsymbol{\mathcal{X}}$. It is a d-stack analogue of the cotangent complex in algebraic geometry.

Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ be d-stacks. A 1-*morphism* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a triple $\boldsymbol{f} = (f, f', f'')$, where $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks, $f' : f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \to \mathcal{O}'_{\mathcal{X}}$ a morphism of sheaves of C^∞-rings on $\mathcal{X}$, and $f'' : f^*(\mathcal{E}_{\mathcal{Y}}) \to \mathcal{E}_{\mathcal{X}}$ a morphism in $\mathrm{qcoh}(\mathcal{X})$, such that the following diagram of sheaves on $\mathcal{X}$ commutes:

$$\begin{tikzcd}
f^{-1}(\mathcal{E}_{\mathcal{Y}}) \otimes^{\mathrm{id}}_{f^{-1}(\mathcal{O}_{\mathcal{Y}})} f^{-1}(\mathcal{O}_{\mathcal{Y}}) = f^{-1}(\mathcal{E}_{\mathcal{Y}}) \arrow[r, "f^{-1}(\jmath_{\mathcal{Y}})"'] \arrow[d, "\mathrm{id}\otimes f^\sharp"] & f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \arrow[r, "f^{-1}(\imath_{\mathcal{Y}})"'] \arrow[dd, "f'"] & f^{-1}(\mathcal{O}_{\mathcal{Y}}) \arrow[r] \arrow[dd, "f^\sharp"'] & 0 \\
f^*(\mathcal{E}_{\mathcal{Y}}) = f^{-1}(\mathcal{E}_{\mathcal{Y}}) \otimes^{f^\sharp}_{f^{-1}(\mathcal{O}_{\mathcal{Y}})} \mathcal{O}_{\mathcal{X}} \arrow[dr, "f''"] & & & \\
& \mathcal{E}_{\mathcal{X}} \arrow[r, "\jmath_{\mathcal{X}}"] & \mathcal{O}'_{\mathcal{X}} \arrow[r, "\imath_{\mathcal{X}}"] & \mathcal{O}_{\mathcal{X}} \arrow[r] & 0.
\end{tikzcd}$$

Define morphisms $f^2 = \Omega_{f'} \otimes \mathrm{id} : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{F}_{\mathcal{X}}$ and $f^3 = \Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ in $\mathrm{qcoh}(\mathcal{X})$. Then the following commutes in $\mathrm{qcoh}(\mathcal{X})$, with exact rows:

$$\begin{tikzcd}
f^*(\mathcal{E}_{\mathcal{Y}}) \arrow[r, "f^*(\phi_{\mathcal{Y}})"'] \arrow[d, "f''"] & f^*(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "f^*(\psi_{\mathcal{Y}})"'] \arrow[d, "f^2"] & f^*(T^*\mathcal{Y}) \arrow[r] \arrow[d, "f^3"] & 0 \\
\mathcal{E}_{\mathcal{X}} \arrow[r, "\phi_{\mathcal{X}}"] & \mathcal{F}_{\mathcal{X}} \arrow[r, "\psi_{\mathcal{X}}"] & T^*\mathcal{X} \arrow[r] & 0.
\end{tikzcd} \tag{1.52}$$

If $\boldsymbol{\mathcal{X}}$ is a d-stack, the *identity* 1*-morphism* $\mathbf{id}_{\boldsymbol{\mathcal{X}}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}$ is $\mathbf{id}_{\boldsymbol{\mathcal{X}}} = \big(\mathrm{id}_{\mathcal{X}}, \delta_{\mathcal{X}}(\mathcal{O}'_{\mathcal{X}}), \delta_{\mathcal{X}}(\mathcal{E}_{\mathcal{X}})\big)$, with $\delta_{\mathcal{X}}(*)$ the canonical isomorphisms of Definition 1.8.22.

Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ be d-stacks, and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, $\boldsymbol{g} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms. As in (1.5) define the *composition of* 1*-morphisms* $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$ to be

$$\boldsymbol{g} \circ \boldsymbol{f} = \big(g \circ f, f' \circ f^{-1}(g') \circ I_{f,g}(\mathcal{O}'_{\mathcal{Z}}), f'' \circ f^*(g'') \circ I_{f,g}(\mathcal{E}_{\mathcal{Z}})\big),$$

where $I_{*,*}(*)$ are the canonical isomorphisms of Definition 1.8.22.

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be 1-morphisms of d-stacks, where $\boldsymbol{f} = (f, f', f'')$ and $\boldsymbol{g} = (g, g', g'')$. A 2*-morphism* $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a pair $\boldsymbol{\eta} = (\eta, \eta')$, where $\eta : f \Rightarrow g$ is a 2-morphism in $\mathbf{C^\infty Sta}$ and $\eta' : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{E}_{\mathcal{X}}$ a morphism in $\mathrm{qcoh}(\mathcal{X})$, with

$$\begin{aligned} g' \circ \eta^{-1}(\mathcal{O}'_{\mathcal{Y}}) &= f' + \kappa_{\mathcal{X}} \circ \jmath_{\mathcal{X}} \circ \eta' \circ \big(\mathrm{id} \otimes (f^\sharp \circ f^{-1}(\imath_{\mathcal{Y}}))\big) \circ \big(f^{-1}(\mathrm{d})\big), \\ \text{and} \quad g'' \circ \eta^*(\mathcal{E}_{\mathcal{Y}}) &= f'' + \eta' \circ f^*(\phi_{\mathcal{Y}}). \end{aligned}$$

Then $g^2 \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) = f^2 + \phi_{\mathcal{X}} \circ \eta'$ and $g^3 \circ \eta^*(T^*\mathcal{Y}) = f^3$, so (1.52) for $\boldsymbol{f}, \boldsymbol{g}$ combine to give a commuting diagram (except η') in $\mathrm{qcoh}(\mathcal{X})$, with exact rows:

$$\begin{tikzcd}
f^*(\mathcal{E}_{\mathcal{Y}}) \arrow[rr, "f^*(\phi_{\mathcal{Y}})"] \arrow[dr, "\eta^*(\mathcal{E}_{\mathcal{Y}})"] \arrow[ddr, "f''+"', "\eta' \circ f^*(\phi_{\mathcal{Y}})"'] & & f^*(\mathcal{F}_{\mathcal{Y}}) \arrow[rr, "f^*(\psi_{\mathcal{Y}})"] \arrow[dr, "\eta^*(\mathcal{F}_{\mathcal{Y}})"] \arrow[ddr, "f^2+" near end, "\phi_{\mathcal{X}} \circ \eta'" near end] \arrow[ddl, dotted, "\eta'"'] & & f^*(T^*\mathcal{Y}) \arrow[r] \arrow[dr, "\eta^*(T^*\mathcal{Y})"] \arrow[ddr, "f^3"'] & 0 & \\
& g^*(\mathcal{E}_{\mathcal{Y}}) \arrow[rr, "g^*(\phi_{\mathcal{Y}})" near end] \arrow[d, "g''"] & & g^*(\mathcal{F}_{\mathcal{Y}}) \arrow[rr, "g^*(\psi_{\mathcal{Y}})"] \arrow[d, "g^2"] & & g^*(T^*\mathcal{Y}) \arrow[r] \arrow[d, "g^3"] & 0 \\
& \mathcal{E}_{\mathcal{X}} \arrow[rr, "\phi_{\mathcal{X}}"] & & \mathcal{F}_{\mathcal{X}} \arrow[rr, "\psi_{\mathcal{X}}"] & & T^*\mathcal{X} \arrow[r] & 0.
\end{tikzcd}$$

If $\boldsymbol{f} = (f, f', f'') : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism, the *identity* 2*-morphism* $\mathbf{id}_{\boldsymbol{f}} : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ is $\mathbf{id}_{\boldsymbol{f}} = (\mathrm{id}_f, 0)$.

Let $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be 1-morphisms and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$, $\boldsymbol{\zeta} : \boldsymbol{g} \Rightarrow \boldsymbol{h}$ 2-morphisms. Define the *vertical composition of* 2*-morphisms* $\boldsymbol{\zeta} \odot \boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{h}$ to be

$$\boldsymbol{\zeta} \odot \boldsymbol{\eta} = \big(\zeta \odot \eta, \zeta' \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) + \eta'\big).$$

Suppose $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ are d-stacks, $\boldsymbol{f}, \tilde{\boldsymbol{f}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ and $\boldsymbol{g}, \tilde{\boldsymbol{g}} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ are 1-morphisms, and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{f}}$, $\boldsymbol{\zeta} : \boldsymbol{g} \Rightarrow \tilde{\boldsymbol{g}}$ are 2-morphisms. Define the *horizontal composition of* 2*-morphisms* $\boldsymbol{\zeta} * \boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{g}} \circ \tilde{\boldsymbol{f}}$ to be

$$\boldsymbol{\zeta} * \boldsymbol{\eta} = \big(\zeta * \eta, \big[\eta' \circ f^*(g^2) + f'' \circ f^*(\zeta') + \eta' \circ f^*(\phi_{\mathcal{Y}}) \circ f^*(\zeta')\big] \circ I_{f,g}(\mathcal{F}_{\mathcal{Z}})\big).$$

This completes the definition of the 2-category of d-stacks $\mathbf{dSta}$.

Write $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$ for the 2-category of separated, second countable, locally fair Deligne–Mumford C^∞-stacks. Define a strict 2-functor $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}} : \mathbf{DMC^\infty Sta^{lf}_{ssc}} \to \mathbf{dSta}$ to map objects $\mathcal{X}$ to $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}, 0, \mathrm{id}_{\mathcal{O}_{\mathcal{X}}}, 0)$, to map 1-morphisms f to $\boldsymbol{f} = (f, f^\sharp, 0)$, and to map 2-morphisms η to $\boldsymbol{\eta} = (\eta, 0)$. Write $\mathbf{D\hat{M}C^\infty Sta^{lf}_{ssc}}$ for the full 2-subcategory of $\boldsymbol{\mathcal{X}} \in \mathbf{dSta}$ equivalent to $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\mathcal{X})$ for $\mathcal{X} \in \mathbf{DMC^\infty Sta^{lf}_{ssc}}$. When we say that a d-stack $\boldsymbol{\mathcal{X}}$ *is a* C^∞*-stack*, we mean that $\boldsymbol{\mathcal{X}} \in \mathbf{D\hat{M}C^\infty Sta^{lf}_{ssc}}$.

Define a strict 2-functor $F_{\mathbf{Orb}}^{\mathbf{dSta}} : \mathbf{Orb} \to \mathbf{dSta}$ by $F_{\mathbf{Orb}}^{\mathbf{dSta}} = F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}|_{\mathbf{Orb}}$, noting that $\mathbf{Orb}$ is a full 2-subcategory of $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$. Write $\mathbf{\hat{O}rb}$ for the full 2-subcategory of objects $\boldsymbol{\mathcal{X}}$ in $\mathbf{dSta}$ equivalent to $F_{\mathbf{Orb}}^{\mathbf{dSta}}(\mathcal{X})$ for some orbifold $\mathcal{X}$. When we say that a d-stack $\boldsymbol{\mathcal{X}}$ *is an orbifold*, we mean that $\boldsymbol{\mathcal{X}} \in \hat{\mathbf{Orb}}$.

Recall from §1.8.1 that there is a natural (2-)functor $F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}} : \mathbf{C^\infty Sch} \to \mathbf{C^\infty Sta}$ mapping $\underline{X} \mapsto \bar{\underline{X}}$ on objects and $\underline{f} \mapsto \bar{\underline{f}}$ on morphisms. Also, if $\underline{X}$ is a C^∞-scheme and $\bar{\underline{X}}$ the corresponding C^∞-stack then Example 1.8.21 defines a functor $\mathcal{I}_{\underline{X}} : \mathcal{O}_X\text{-mod} \to \mathcal{O}_{\bar{\underline{X}}}\text{-mod}$. In the same way, we can define functors from the category of sheaves of abelian groups on X to the category of sheaves of abelian groups on $\bar{\underline{X}}$, and from the category of sheaves of C^∞-rings on X to the category of sheaves of C^∞-rings on $\bar{\underline{X}}$, both of which we also denote by $\mathcal{I}_{\underline{X}}$.

With this notation, define a strict 2-functor $F_{\mathbf{dSpa}}^{\mathbf{dSta}} : \mathbf{dSpa} \to \mathbf{dSta}$ to map $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ to $\boldsymbol{\mathcal{X}} = \big(\bar{\underline{X}}, \mathcal{I}_{\underline{X}}(\mathcal{O}'_X), \mathcal{I}_{\underline{X}}(\mathcal{E}_X), \mathcal{I}_{\underline{X}}(\imath_X), \mathcal{I}_{\underline{X}}(\jmath_X)\big)$ on objects, and to map $\boldsymbol{f} = (\underline{f}, f', f'')$ to $\hat{\boldsymbol{f}} = \big(\bar{\underline{f}}, \mathcal{I}_{\underline{X}}(f'), \mathcal{I}_{\underline{X}}(f'')\big)$ on 1-morphisms, and to map η to $\boldsymbol{\eta} = \big(\mathrm{id}_{\bar{\underline{f}}}, \mathcal{I}_{\underline{X}}(\eta)\big)$ on 2-morphisms. Write $\mathbf{d\hat{S}pa}$ for the full 2-subcategory of $\boldsymbol{\mathcal{X}}$ in $\mathbf{dSta}$ equivalent to $F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{X})$ for some $\boldsymbol{X}$ in $\mathbf{dSpa}$.

In §9.2 we prove:

Theorem 1.10.2. **(a)** *Definition* 1.10.1 *defines a strict* 2*-category* $\mathbf{dSta}$, *in which all* 2*-morphisms are* 2*-isomorphisms.*

(b) $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}, F_{\mathbf{Orb}}^{\mathbf{dSta}}$ *and* $F_{\mathbf{dSpa}}^{\mathbf{dSta}}$ *are full and faithful strict* 2*-functors. Hence* $\mathbf{DMC^\infty Sta^{lf}_{ssc}}, \mathbf{Orb}, \mathbf{dSpa}$ *and* $\mathbf{D\hat{M}C^\infty Sta^{lf}_{ssc}}, \hat{\mathbf{Orb}}, \mathbf{d\hat{S}pa}$ *are equivalent* 2*-categories, respectively.*

1.10.2 D-stacks as quotients of d-spaces

In §1.8.4 we defined quotient Deligne–Mumford C^∞-stacks $[\underline{X}/G]$, quotient 1-morphisms $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$, and quotient 2-morphisms $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$. Section 9.3 generalizes all this to d-stacks. The next two theorems summarize our results.

Theorem 1.10.3. **(i)** *Let* $\boldsymbol{X}$ *be a d-space,* G *a finite group, and* $\boldsymbol{r} : G \to \mathrm{Aut}(\boldsymbol{X})$ *a (strict) action of* G *on* $\boldsymbol{X}$ *by* 1*-isomorphisms. Then we can define a* ***quotient d-stack*** $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$, *which is natural up to* 1*-isomorphism in* $\mathbf{dSta}$. *The underlying* C^∞*-stack* $\mathcal{X}$ *is* $[\underline{X}/G]$ *from Example* 1.8.11.

(ii) *Let* $\boldsymbol{X}, \boldsymbol{Y}$ *be d-spaces,* G, H *finite groups, and* $\boldsymbol{r} : G \to \mathrm{Aut}(\boldsymbol{X})$, $\boldsymbol{s} : H \to \mathrm{Aut}(\boldsymbol{Y})$ *be actions of* G, H *on* $\boldsymbol{X}, \boldsymbol{Y}$, *so that by* **(i)** *we have quotient d-stacks* $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$ *and* $\boldsymbol{\mathcal{Y}} = [\boldsymbol{Y}/H]$. *Suppose* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is a* 1*-morphism in* $\mathbf{dSpa}$ *and* $\rho : G \to H$ *is a group morphism, satisfying* $\boldsymbol{f} \circ \boldsymbol{r}(\gamma) = \boldsymbol{s}(\rho(\gamma)) \circ \boldsymbol{f}$ *for all* $\gamma \in G$ *(this is an equality of* 1*-morphisms in* $\mathbf{dSpa}$, *not just a* 2*-isomorphism). Then we can define a* ***quotient*** 1***-morphism*** $\tilde{\boldsymbol{f}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *in* $\mathbf{dSta}$, *which we will also write as* $[\boldsymbol{f}, \rho] : [\boldsymbol{X}/G] \to [\boldsymbol{Y}/H]$.

(iii) *Let* $\tilde{\boldsymbol{f}} = [\boldsymbol{f}, \rho] : [\boldsymbol{X}/G] \to [\boldsymbol{Y}/H]$ *and* $\tilde{\boldsymbol{g}} = [\boldsymbol{g}, \sigma] : [\boldsymbol{X}/G] \to [\boldsymbol{Y}/H]$ *be two quotient* 1*-morphisms as in* **(ii)**. *Suppose* $\delta \in H$ *satisfies* $\delta^{-1}\sigma(\gamma) = \rho(\gamma)\,\delta^{-1}$ *for all* $\gamma \in G$, *and* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{g}$ *is a* 2*-morphism in* $\mathbf{dSpa}$ *such that*

$\eta * \mathrm{id}_{\boldsymbol{r}(\gamma)} = \mathrm{id}_{\boldsymbol{s}(\sigma(\gamma))} * \eta$ *for all* $\gamma \in G$, *using the diagram:*

$$\begin{tikzcd}
\boldsymbol{f} \circ \boldsymbol{r}(\gamma) \arrow[rr, equal] \arrow[d, Rightarrow, "\eta * \mathrm{id}_{\boldsymbol{r}(\gamma)}"] & & \boldsymbol{s}(\rho(\gamma)) \circ \boldsymbol{f} \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{s}(\sigma(\gamma))} * \eta"'] \\
\boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{g} \circ \boldsymbol{r}(\gamma) \arrow[r, equal] & \boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{s}(\sigma(\gamma)) \circ \boldsymbol{g} \arrow[r, equal] & \boldsymbol{s}(\rho(\gamma)) \circ \boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{g}.
\end{tikzcd}$$

Then we can define a ***quotient*** 2***-morphism*** $\boldsymbol{\zeta} : \tilde{\boldsymbol{f}} \Rightarrow \tilde{\boldsymbol{g}}$ *in* **dSta**, *which we also write as* $[\eta, \delta] : [\boldsymbol{f}, \rho] \Rightarrow [\boldsymbol{g}, \sigma]$.

Theorem 1.10.4. **(a)** *Let* $\boldsymbol{\mathcal{X}}$ *be a d-stack and* $[x] \in \mathcal{X}_{\mathrm{top}}$, *and write* $G = \mathrm{Iso}_{\mathcal{X}}([x])$. *Then there exist a quotient d-stack* $[\boldsymbol{U}/G]$, *as in Theorem* 1.10.3(i), *and an equivalence* $\boldsymbol{i} : [\boldsymbol{U}/G] \to \boldsymbol{\mathcal{X}}$ *with an open d-substack* $\boldsymbol{\mathcal{U}}$ *in* $\boldsymbol{\mathcal{X}}$, *with* $i_{\mathrm{top}} : [u] \mapsto [x] \in \mathcal{U}_{\mathrm{top}} \subseteq \mathcal{X}_{\mathrm{top}}$ *for some fixed point* u *of* G *in* U.

(b) *Let* $\tilde{\boldsymbol{f}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *be a* 1*-morphism in* **dSta**, *and* $[x] \in \mathcal{X}_{\mathrm{top}}$ *with* $\tilde{f}_{\mathrm{top}} : [x] \mapsto [y] \in \mathcal{Y}_{\mathrm{top}}$, *and write* $G = \mathrm{Iso}_{\mathcal{X}}([x])$ *and* $H = \mathrm{Iso}_{\mathcal{Y}}([y])$. *Part* **(a)** *gives* 1*-morphisms* $\boldsymbol{i} : [\boldsymbol{U}/G] \to \boldsymbol{\mathcal{X}}$, $\boldsymbol{j} : [\boldsymbol{V}/H] \to \boldsymbol{\mathcal{Y}}$ *which are equivalences with open* $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{Y}}$, *such that* $i_{\mathrm{top}} : [u] \mapsto [x] \in \mathcal{U}_{\mathrm{top}} \subseteq \mathcal{X}_{\mathrm{top}}$, $j_{\mathrm{top}} : [v] \mapsto [y] \in \mathcal{V}_{\mathrm{top}} \subseteq \mathcal{Y}_{\mathrm{top}}$ *for* u, v *fixed points of* G, H *in* U, V.

Then there exist a G*-invariant open d-subspace* $\boldsymbol{U}'$ *of* u *in* $\boldsymbol{U}$ *and a quotient* 1*-morphism* $[\boldsymbol{f}, \rho] : [\boldsymbol{U}'/G] \to [\boldsymbol{V}/H]$, *as in Theorem* 1.10.3(ii), *such that* $\boldsymbol{f}(u) = v$, *and* $\rho : G \to H$ *is* $\tilde{f}_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$, *fitting into a* 2*-commutative diagram:*

$$\begin{tikzcd}
{[\boldsymbol{U}'/G]} \arrow[r, "{[\boldsymbol{f},\rho]}"] \arrow[d, "\boldsymbol{i}|_{[\boldsymbol{U}'/G]}"'] \arrow[dr, phantom, "\boldsymbol{\zeta}\Uparrow"] & {[\boldsymbol{V}/H]} \arrow[d, "\boldsymbol{j}"] \\
\boldsymbol{\mathcal{X}} \arrow[r, "\tilde{\boldsymbol{f}}"] & \boldsymbol{\mathcal{Y}}.
\end{tikzcd}$$

(c) *Let* $\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *be* 1*-morphisms in* **dSta** *and* $\boldsymbol{\eta} : \tilde{\boldsymbol{f}} \Rightarrow \tilde{\boldsymbol{g}}$ *a* 2*-morphism, let* $[x] \in \mathcal{X}_{\mathrm{top}}$ *with* $\tilde{f}_{\mathrm{top}} : [x] \mapsto [y] \in \mathcal{Y}_{\mathrm{top}}$, *and write* $G = \mathrm{Iso}_{\mathcal{X}}([x])$ *and* $H = \mathrm{Iso}_{\mathcal{Y}}([y])$. *Part* **(a)** *gives* $\boldsymbol{i} : [\boldsymbol{U}/G] \to \boldsymbol{\mathcal{X}}$, $\boldsymbol{j} : [\boldsymbol{V}/H] \to \boldsymbol{\mathcal{Y}}$ *which are equivalences with open* $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{Y}}$ *and map* $i_{\mathrm{top}} : [u] \mapsto [x]$, $j_{\mathrm{top}} : [v] \mapsto [y]$ *for* u, v *fixed points of* G, H.

By making $\boldsymbol{U}'$ *smaller, we can take the same* $\boldsymbol{U}'$ *in* **(b)** *for both* $\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}}$. *Thus part* **(b)** *gives a* G*-invariant open* $\boldsymbol{U}' \subseteq \boldsymbol{U}$, *quotient* 1*-morphisms* $[\boldsymbol{f}, \rho] : [\boldsymbol{U}'/G] \to [\boldsymbol{V}/H]$ *and* $[\boldsymbol{g}, \sigma] : [\boldsymbol{U}'/G] \to [\boldsymbol{V}/H]$ *with* $\boldsymbol{f}(u) = \boldsymbol{g}(u) = v$ *and* $\rho = \tilde{f}_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$, $\sigma = \tilde{g}_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$, *and* 2*-morphisms* $\boldsymbol{\zeta} : \tilde{\boldsymbol{f}} \circ \boldsymbol{i}|_{[\boldsymbol{U}'/G]} \Rightarrow \boldsymbol{j} \circ [\boldsymbol{f}, \rho]$, $\boldsymbol{\theta} : \tilde{\boldsymbol{g}} \circ \boldsymbol{i}|_{[\boldsymbol{U}'/G]} \Rightarrow \boldsymbol{j} \circ [\boldsymbol{g}, \sigma]$.

Then there exist a G*-invariant open neighbourhood* $\boldsymbol{U}''$ *of* u *in* $\boldsymbol{U}'$ *and a quotient* 2*-morphism* $[\lambda, \delta] : [\boldsymbol{f}|_{\boldsymbol{U}''}, \rho] \Rightarrow [\boldsymbol{g}|_{\boldsymbol{U}''}, \sigma]$, *as in Theorem* 1.10.3(iii), *such that the following diagram of* 2*-morphisms in* **dSta** *commutes:*

$$\begin{tikzcd}
\tilde{\boldsymbol{f}} \circ \boldsymbol{i}|_{[\boldsymbol{U}''/G]} \arrow[r, Rightarrow, "\boldsymbol{\eta} * \mathrm{id}_{\boldsymbol{i}|_{[\boldsymbol{U}''/G]}}"'] \arrow[d, Rightarrow, "\boldsymbol{\zeta}|_{[\boldsymbol{U}''/G]}"] & \tilde{\boldsymbol{g}} \circ \boldsymbol{i}|_{[\boldsymbol{U}''/G]} \arrow[d, Rightarrow, "\boldsymbol{\theta}|_{[\boldsymbol{U}''/G]}"'] \\
\boldsymbol{j} \circ [\boldsymbol{f}|_{\boldsymbol{U}''}, \rho] \arrow[r, Rightarrow, "\mathrm{id}_{\boldsymbol{j}} * [\lambda, \delta]"] & \boldsymbol{j} \circ [\boldsymbol{g}|_{\boldsymbol{U}''}, \sigma].
\end{tikzcd}$$

Effectively, this says that d-stacks and their 1-morphisms and 2-morphisms are Zariski locally modelled on quotient d-stacks, quotient 1-morphisms, and quotient 2-morphisms, up to equivalence in **dSta**.

In §9.2 we define when a 1-morphism of d-stacks $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ is *étale*. Essentially, $\boldsymbol{f}$ is étale if it is an equivalence locally in the étale topology. It implies that the C^∞-stack 1-morphism $f:\mathcal{X}\to\mathcal{Y}$ in $\boldsymbol{f}$ is étale, and so representable.

We can characterize étale 1-morphisms in **dSta** using Theorem 1.10.4: a 1-morphism $\tilde{\boldsymbol{f}}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ in **dSta** is étale if and only if for all $[\boldsymbol{f},\rho]:[\boldsymbol{U}'/G]\to[\boldsymbol{V}/H]$ in Theorem 1.10.4(b), $\boldsymbol{f}:\boldsymbol{U}'\to\boldsymbol{V}$ is an étale 1-morphism in **dSpa** (that is, a local equivalence in the Zariski topology), and $\rho:G\to H$ is injective.

1.10.3 Gluing d-stacks by equivalences

In §1.3.2 we discussed gluing d-spaces by equivalences in **dSpa**. Section 9.4 generalizes this to **dSta**. Here are the analogues of Definition 1.3.4, Proposition 1.3.5, and Theorems 1.3.6 and 1.3.7.

Definition 1.10.5. Let $\boldsymbol{\mathcal{X}}=(\mathcal{X},\mathcal{O}'_{\mathcal{X}},\mathcal{E}_{\mathcal{X}},\imath_{\mathcal{X}},\jmath_{\mathcal{X}})$ be a d-stack. Suppose $\mathcal{U}\subseteq\mathcal{X}$ is an open C^∞-substack, in the Zariski topology, with inclusion 1-morphism $i_{\mathcal{U}}:\mathcal{U}\to\mathcal{X}$. Then $\boldsymbol{\mathcal{U}}=\bigl(\mathcal{U},i_{\mathcal{U}}^{-1}(\mathcal{O}'_{\mathcal{X}}),i_{\mathcal{U}}^*(\mathcal{E}_{\mathcal{X}}),i_{\mathcal{U}}^\sharp\circ i_{\mathcal{U}}^{-1}(\imath_{\mathcal{X}}),i_{\mathcal{U}}^*(\jmath_{\mathcal{X}})\bigr)$ is a d-stack, where $i_{\mathcal{U}}^\sharp:i_{\mathcal{U}}^{-1}(\mathcal{O}_{\mathcal{X}})\to\mathcal{O}_{\mathcal{U}}$ is as in Example 1.8.23, and is an isomorphism as $i_{\mathcal{U}}$ is étale. We call $\boldsymbol{\mathcal{U}}$ an *open d-substack* of $\boldsymbol{\mathcal{X}}$. An *open cover* of a d-stack $\boldsymbol{\mathcal{X}}$ is a family $\{\boldsymbol{\mathcal{U}}_a:a\in A\}$ of open d-substacks $\boldsymbol{\mathcal{U}}_a$ of $\boldsymbol{\mathcal{X}}$ such that $\{\mathcal{U}_a:a\in A\}$ is an open cover of $\mathcal{X}$, in the Zariski topology.

Proposition 1.10.6. *Let $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}$ be d-stacks, $\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{V}}\subseteq\boldsymbol{\mathcal{X}}$ be open d-substacks with $\boldsymbol{\mathcal{X}}=\boldsymbol{\mathcal{U}}\cup\boldsymbol{\mathcal{V}}$, $\boldsymbol{f}:\boldsymbol{\mathcal{U}}\to\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{g}:\boldsymbol{\mathcal{V}}\to\boldsymbol{\mathcal{Y}}$ be 1-morphisms, and $\boldsymbol{\eta}:\boldsymbol{f}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}\Rightarrow\boldsymbol{g}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}$ a 2-morphism. Then there exist a 1-morphism $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ and 2-morphisms $\boldsymbol{\zeta}:\boldsymbol{h}|_{\boldsymbol{\mathcal{U}}}\Rightarrow\boldsymbol{f}$, $\boldsymbol{\theta}:\boldsymbol{h}|_{\boldsymbol{\mathcal{V}}}\Rightarrow\boldsymbol{g}$ in* **dSta** *such that $\boldsymbol{\theta}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}=\boldsymbol{\eta}\odot\boldsymbol{\zeta}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}:\boldsymbol{h}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}\Rightarrow\boldsymbol{g}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}$. This $\boldsymbol{h}$ is unique up to 2-isomorphism.*

Theorem 1.10.7. *Suppose $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}$ are d-stacks, $\boldsymbol{\mathcal{U}}\subseteq\boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{V}}\subseteq\boldsymbol{\mathcal{Y}}$ are open d-substacks, and $\boldsymbol{f}:\boldsymbol{\mathcal{U}}\to\boldsymbol{\mathcal{V}}$ is an equivalence in* **dSta**. *At the level of topological spaces, we have open $\mathcal{U}_{\rm top}\subseteq\mathcal{X}_{\rm top}$, $\mathcal{V}_{\rm top}\subseteq\mathcal{Y}_{\rm top}$ with a homeomorphism $f_{\rm top}:\mathcal{U}_{\rm top}\to\mathcal{V}_{\rm top}$, so we can form the quotient topological space $\mathcal{Z}_{\rm top}:=\mathcal{X}_{\rm top}\amalg_{f_{\rm top}}\mathcal{Y}_{\rm top}=(\mathcal{X}_{\rm top}\amalg\mathcal{Y}_{\rm top})/\sim$, where the equivalence relation $\sim$ on $\mathcal{X}_{\rm top}\amalg\mathcal{Y}_{\rm top}$ identifies $[u]\in\mathcal{U}_{\rm top}\subseteq\mathcal{X}_{\rm top}$ with $f_{\rm top}([u])\in\mathcal{V}_{\rm top}\subseteq\mathcal{Y}_{\rm top}$.*

Suppose $\mathcal{Z}_{\rm top}$ is Hausdorff. Then there exist a d-stack $\boldsymbol{\mathcal{Z}}$, open d-substacks $\hat{\boldsymbol{\mathcal{X}}},\hat{\boldsymbol{\mathcal{Y}}}$ in $\boldsymbol{\mathcal{Z}}$ with $\boldsymbol{\mathcal{Z}}=\hat{\boldsymbol{\mathcal{X}}}\cup\hat{\boldsymbol{\mathcal{Y}}}$, equivalences $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\hat{\boldsymbol{\mathcal{X}}}$ and $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\to\hat{\boldsymbol{\mathcal{Y}}}$ such that $\boldsymbol{g}|_{\boldsymbol{\mathcal{U}}}$ and $\boldsymbol{h}|_{\boldsymbol{\mathcal{V}}}$ are both equivalences with $\hat{\boldsymbol{\mathcal{X}}}\cap\hat{\boldsymbol{\mathcal{Y}}}$, and a 2-morphism $\boldsymbol{\eta}:\boldsymbol{g}|_{\boldsymbol{\mathcal{U}}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$. Furthermore, $\boldsymbol{\mathcal{Z}}$ is independent of choices up to equivalence.

Theorem 1.10.8. *Suppose I is an indexing set, and $<$ is a total order on I, and $\boldsymbol{\mathcal{X}}_i$ for $i\in I$ are d-stacks, and for all $i<j$ in I we are given open d-substacks $\boldsymbol{\mathcal{U}}_{ij}\subseteq\boldsymbol{\mathcal{X}}_i$, $\boldsymbol{\mathcal{U}}_{ji}\subseteq\boldsymbol{\mathcal{X}}_j$ and an equivalence $\boldsymbol{e}_{ij}:\boldsymbol{\mathcal{U}}_{ij}\to\boldsymbol{\mathcal{U}}_{ji}$, satisfying the following properties:*

(a) *For all $i<j<k$ in I we have a 2-commutative diagram*

$$\begin{tikzcd}[column sep=huge] & \boldsymbol{\mathcal{U}}_{ji}\cap\boldsymbol{\mathcal{U}}_{jk} \arrow[dr, "\boldsymbol{e}_{jk}|_{\boldsymbol{\mathcal{U}}_{ji}\cap\boldsymbol{\mathcal{U}}_{jk}}"] \arrow[d, phantom, "\Downarrow\boldsymbol{\eta}_{ijk}"] & \\ \boldsymbol{\mathcal{U}}_{ij}\cap\boldsymbol{\mathcal{U}}_{ik} \arrow[ur, "\boldsymbol{e}_{ij}|_{\boldsymbol{\mathcal{U}}_{ij}\cap\boldsymbol{\mathcal{U}}_{ik}}"] \arrow[rr, "\boldsymbol{e}_{ik}|_{\boldsymbol{\mathcal{U}}_{ij}\cap\boldsymbol{\mathcal{U}}_{ik}}"] & & \boldsymbol{\mathcal{U}}_{ki}\cap\boldsymbol{\mathcal{U}}_{kj} \end{tikzcd}$$

for some $\boldsymbol{\eta}_{ijk}$, where all three 1*-morphisms are equivalences; and*

(b) *For all $i<j<k<l$ in I the components η_{ijk} in $\boldsymbol{\eta}_{ijk}=(\eta_{ijk},\eta'_{ijk})$ satisfy*

$$\eta_{ikl}\odot(\mathrm{id}_{f_{kl}}*\eta_{ijk})|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}\cap\mathcal{U}_{il}}=\eta_{ijl}\odot(\eta_{jkl}*\mathrm{id}_{f_{ij}})|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}\cap\mathcal{U}_{il}}. \quad (1.53)$$

On the level of topological spaces, define the quotient topological space $\mathcal{Y}_{\mathrm{top}}=(\coprod_{i\in I}\mathcal{X}_{i,\mathrm{top}})/\sim$, where $\sim$ is the equivalence relation generated by $[x_i]\sim[x_j]$ if $[x_i]\in\mathcal{U}_{ij,\mathcal{X}_i,\mathrm{top}}\subseteq\mathcal{X}_{i,\mathrm{top}}$ and $[x_j]\in\mathcal{U}_{ji,\mathrm{top}}\subseteq\mathcal{X}_{j,\mathrm{top}}$ with $e_{ij,\mathrm{top}}([x_i])=[x_j]$. Suppose $\mathcal{Y}_{\mathrm{top}}$ is Hausdorff and second countable. Then there exist a d-stack $\boldsymbol{\mathcal{Y}}$ and a 1*-morphism $\boldsymbol{f}_i:\boldsymbol{\mathcal{X}}_i\rightarrow\boldsymbol{\mathcal{Y}}$ which is an equivalence with an open d-substack $\hat{\boldsymbol{\mathcal{X}}}_i\subseteq\boldsymbol{\mathcal{Y}}$ for all $i\in I$, where $\boldsymbol{\mathcal{Y}}=\bigcup_{i\in I}\hat{\boldsymbol{\mathcal{X}}}_i$, such that $\boldsymbol{f}_i|_{\boldsymbol{\mathcal{U}}_{ij}}$ is an equivalence $\boldsymbol{\mathcal{U}}_{ij}\rightarrow\hat{\boldsymbol{\mathcal{X}}}_i\cap\hat{\boldsymbol{\mathcal{X}}}_j$ for all $i<j$ in I, and there exists a* 2*-morphism $\boldsymbol{\eta}_{ij}:\boldsymbol{f}_j\circ\boldsymbol{e}_{ij}\Rightarrow\boldsymbol{f}_i|_{\boldsymbol{\mathcal{U}}_{ij}}$. The d-stack $\boldsymbol{\mathcal{Y}}$ is unique up to equivalence.*

Suppose also that $\boldsymbol{\mathcal{Z}}$ is a d-stack, and $\boldsymbol{g}_i:\boldsymbol{\mathcal{X}}_i\rightarrow\boldsymbol{\mathcal{Z}}$ are 1*-morphisms for all $i\in I$, and there exist* 2*-morphisms $\boldsymbol{\zeta}_{ij}:\boldsymbol{g}_j\circ\boldsymbol{e}_{ij}\Rightarrow\boldsymbol{g}_i|_{\boldsymbol{\mathcal{U}}_{ij}}$ for all $i<j$ in I, such that for all $i<j<k$ in I the components ζ_{ij},η_{ijk} in $\boldsymbol{\zeta}_{ij},\boldsymbol{\eta}_{ijk}$ satisfy*

$$\bigl(\zeta_{ij}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\bigr)\odot\bigl(\zeta_{jk}*\mathrm{id}_{e_{ij}}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\bigr)=\bigl(\zeta_{ik}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\bigr)\odot\bigl(\mathrm{id}_{g_k}*\eta_{ijk}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\bigr). \quad (1.54)$$

Then there exist a 1*-morphism $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\rightarrow\boldsymbol{\mathcal{Z}}$ and* 2*-morphisms $\boldsymbol{\zeta}_i:\boldsymbol{h}\circ\boldsymbol{f}_i\Rightarrow\boldsymbol{g}_i$ for all $i\in I$. The* 1*-morphism $\boldsymbol{h}$ is unique up to* 2*-isomorphism.*

Remark 1.10.9. Note that in Proposition 1.3.5 for d-spaces, $\boldsymbol{h}$ is independent of η up to 2-isomorphism, but in Proposition 1.10.6 for d-stacks, $\boldsymbol{h}$ may depend on $\boldsymbol{\eta}$. Similarly, in Theorem 1.3.7 for d-spaces, we impose no conditions on 2-morphisms η_{ijk} on quadruple overlaps or ζ_{ij} on triple overlaps, but in Theorem 1.10.8 for d-stacks, we do impose extra conditions (1.53) on the 2-morphisms $\boldsymbol{\eta}_{ijk}$ on quadruple overlaps and (1.54) on the 2-morphisms $\boldsymbol{\zeta}_{ij}$ on triple overlaps. Thus, the d-stack versions of these results are weaker.

The reason for this is that 2-morphisms $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ of d-space 1-morphisms $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ are morphisms $\eta:\underline{f}^*(\mathcal{F}_Y)\rightarrow\mathcal{E}_X$ in $\mathrm{qcoh}(\underline{X})$. We can interpolate between such morphisms using partitions of unity on $\underline{X}$, and in Remark 1.3.8 we explained why this enables us to prove $\boldsymbol{h}$ is independent of η in Proposition 1.3.5, and to do without overlap conditions on η_{ijk},ζ_{ij} in Theorem 1.3.7.

In contrast, for 2-morphisms $\boldsymbol{\eta}=(\eta,\eta'):\boldsymbol{f}\Rightarrow\boldsymbol{g}$ in **dSta**, the C^∞-stack 2-morphisms $\eta:f\Rightarrow g$ are discrete objects, and we cannot join them using partitions of unity. So $\boldsymbol{h}$ may depend on η in Proposition 1.10.6, and we need overlap conditions on the components η_{ijk},ζ_{ij} in $\boldsymbol{\eta}_{ijk},\boldsymbol{\zeta}_{ij}$ in Theorem 1.10.8.

If $f,g:\mathcal{X}\rightarrow\mathcal{Y}$ are 1-morphisms of Deligne–Mumford C^∞-stacks, we can make extra assumptions on $\mathcal{X},\mathcal{Y}$ or f,g which imply that there is at most one

2-morphism $\eta : f \Rightarrow g$, as in Proposition 1.9.5 for orbifolds. Such assumptions can make (1.53) or (1.54) hold automatically, as both sides of (1.53) or (1.54) are 2-morphisms $f \Rightarrow g$. So, for instance, if the C^∞-stacks $\mathcal{X}_i$ are all effective then (1.53) holds, and if the d-stack $\boldsymbol{\mathcal{Z}}$ is a d-space then (1.54) holds.

1.10.4 Fibre products of d-stacks

In §1.3.3 we discussed fibre products of d-spaces. Section 9.5 generalizes this to d-stacks. Here is the analogue of Theorem 1.3.9:

Theorem 1.10.10. **(a)** *All fibre products exist in the* 2*-category* **dSta***.*

(b) *The* 2*-functor* $F_{\mathbf{dSpa}}^{\mathbf{dSta}} : \mathbf{dSpa} \to \mathbf{dSta}$ *preserves fibre products.*

(c) *Let* $g : \mathcal{X} \to \mathcal{Z}$ *and* $h : \mathcal{Y} \to \mathcal{Z}$ *be smooth maps (*1*-morphisms) of orbifolds, and write* $\boldsymbol{\mathcal{X}} = F_{\mathbf{Orb}}^{\mathbf{dSta}}(\mathcal{X})$*, and similarly for* $\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}, \boldsymbol{g}, \boldsymbol{h}$*. If* g, h *are transverse, so that a fibre product* $\mathcal{X} \times_{g,\mathcal{Z},h} \mathcal{Y}$ *exists in* **Orb***, then the fibre product* $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ *in* **dSta** *is equivalent in* **dSta** *to* $F_{\mathbf{Orb}}^{\mathbf{dSta}}(\mathcal{X} \times_{g,\mathcal{Z},h} \mathcal{Y})$*. If* g, h *are not transverse then* $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ *exists in* **dSta***, but is not an orbifold.*

As for d-spaces, we prove (a) by explicitly constructing a d-stack $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ and showing it satisfies the universal property to be a fibre product in the 2-category **dSta**. The proof follows that of Theorem 1.3.9 closely, inserting extra terms for 2-morphisms of C^∞-stacks.

1.10.5 Orbifold strata of d-stacks

In §1.8.7 we discussed orbifold strata of Deligne–Mumford C^∞-stacks. Section 9.6 generalizes this to d-stacks. The next theorems summarize the results.

Theorem 1.10.11. *Let* $\boldsymbol{\mathcal{X}}$ *be a d-stack, and* Γ *a finite group. Then we can define d-stacks* $\boldsymbol{\mathcal{X}}^\Gamma, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \hat{\boldsymbol{\mathcal{X}}}^\Gamma$*, and open d-substacks* $\boldsymbol{\mathcal{X}}^\Gamma_\circ \subseteq \boldsymbol{\mathcal{X}}^\Gamma$, $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ \subseteq \tilde{\boldsymbol{\mathcal{X}}}^\Gamma$, $\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ \subseteq \hat{\boldsymbol{\mathcal{X}}}^\Gamma$*, all natural up to* 1*-isomorphism in* **dSta***, a d-space* $\hat{\boldsymbol{X}}^\Gamma_\circ$ *natural up to* 1*-isomorphism in* **dSpa***, and* 1*-morphisms* $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}), \tilde{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}}), \ldots$ *fitting into a strictly commutative diagram in* **dSta***:*

$$
\begin{tikzcd}
\operatorname{Aut}(\Gamma) \circlearrowright \boldsymbol{\mathcal{X}}^\Gamma_\circ \arrow[rr, "\tilde{\boldsymbol{\Pi}}^\Gamma_\circ(\boldsymbol{\mathcal{X}})"] \arrow[dr, "\boldsymbol{O}^\Gamma_\circ(\boldsymbol{\mathcal{X}})"'] \arrow[dd, "\subset"'] & & \tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ \arrow[r, "\hat{\boldsymbol{\Pi}}^\Gamma_\circ(\boldsymbol{\mathcal{X}})"] \arrow[dl, "\tilde{\boldsymbol{O}}^\Gamma_\circ(\boldsymbol{\mathcal{X}})"] \arrow[dd, "\subset"] & \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ \simeq F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\hat{\boldsymbol{X}}^\Gamma_\circ) \arrow[dd, "\subset"] \\
& \boldsymbol{\mathcal{X}} & & \\
\operatorname{Aut}(\Gamma) \circlearrowright \boldsymbol{\mathcal{X}}^\Gamma \arrow[rr, "\tilde{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}})"'] \arrow[ur, "\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})"] & & \tilde{\boldsymbol{\mathcal{X}}}^\Gamma \arrow[r, "\hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}})"'] \arrow[ul, "\tilde{\boldsymbol{O}}^\Gamma(\boldsymbol{\mathcal{X}})"'] & \hat{\boldsymbol{\mathcal{X}}}^\Gamma.
\end{tikzcd}
\tag{1.55}
$$

We will call $\boldsymbol{\mathcal{X}}^\Gamma, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \hat{\boldsymbol{\mathcal{X}}}^\Gamma, \boldsymbol{\mathcal{X}}^\Gamma_\circ, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ, \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ, \hat{\boldsymbol{X}}^\Gamma_\circ$ *the* ***orbifold strata*** *of* $\boldsymbol{\mathcal{X}}$*.*

The underlying C^∞*-stacks of* $\boldsymbol{\mathcal{X}}^\Gamma, \ldots, \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ *are the orbifold strata* $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ *from* §1.8.7 *of the* C^∞*-stack* $\mathcal{X}$ *in* $\boldsymbol{\mathcal{X}}$*. The* C^∞*-stack* 1*-morphisms underlying the d-stack* 1*-morphisms in* (1.55) *are those given in* (1.41)*.*

Theorem 1.10.12. **(a)** *Let $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}$ be d-stacks, Γ a finite group, and $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ a representable* 1*-morphism in* **dSta**, *that is, the underlying C^∞-stack* 1*-morphism $f:\mathcal{X}\rightarrow\mathcal{Y}$ is representable. Then there is a unique representable* 1*-morphism $\boldsymbol{f}^\Gamma:\boldsymbol{\mathcal{X}}^\Gamma\rightarrow\boldsymbol{\mathcal{Y}}^\Gamma$ in* **dSta** *with $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})\circ\boldsymbol{f}^\Gamma=\boldsymbol{f}\circ\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})$. Here $\boldsymbol{\mathcal{X}}^\Gamma,\boldsymbol{\mathcal{Y}}^\Gamma,\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}),\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})$ are as in Theorem* 1.10.11.

(b) *Let $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ be representable* 1*-morphisms and $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a* 2*-morphism in* **dSta**, *and $\boldsymbol{f}^\Gamma,\boldsymbol{g}^\Gamma:\boldsymbol{\mathcal{X}}^\Gamma\rightarrow\boldsymbol{\mathcal{Y}}^\Gamma$ be as in* **(a)**. *Then there is a unique* 2*-morphism $\boldsymbol{\eta}^\Gamma:\boldsymbol{f}^\Gamma\Rightarrow\boldsymbol{g}^\Gamma$ in* **dSta** *with $\mathbf{id}_{\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})}*\boldsymbol{\eta}^\Gamma=\boldsymbol{\eta}*\mathbf{id}_{\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})}$.*

(c) *Write* $\mathbf{dSta^{re}}$ *for the* 2*-subcategory of* **dSta** *with only representable* 1*-morphisms. Then mapping $\boldsymbol{\mathcal{X}}\mapsto F^\Gamma(\boldsymbol{\mathcal{X}})=\boldsymbol{\mathcal{X}}^\Gamma$ on objects, $\boldsymbol{f}\mapsto F^\Gamma(\boldsymbol{f})=\boldsymbol{f}^\Gamma$ on (representable)* 1*-morphisms, and $\boldsymbol{\eta}\mapsto F^\Gamma(\boldsymbol{\eta})=\boldsymbol{\eta}^\Gamma$ on* 2*-morphisms defines a strict* 2*-functor $F^\Gamma:\mathbf{dSta^{re}}\rightarrow\mathbf{dSta^{re}}$.*

(d) *Analogues of* **(a)**–**(c)** *hold for the orbifold strata $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma$, yielding a strict* 2*-functor $\tilde F^\Gamma:\mathbf{dSta^{re}}\rightarrow\mathbf{dSta^{re}}$. Weaker analogues of* **(a)**–**(c)** *also hold for the orbifold strata $\hat{\boldsymbol{\mathcal{X}}}^\Gamma$. In* **(a)**, *the* 1*-morphism $\hat{\boldsymbol{f}}^\Gamma:\hat{\boldsymbol{\mathcal{X}}}^\Gamma\rightarrow\hat{\boldsymbol{\mathcal{Y}}}^\Gamma$ is natural only up to* 2*-isomorphism, and in* **(c)** *we get a weak* 2*-functor $\hat F^\Gamma:\mathbf{dSta^{re}}\rightarrow\mathbf{dSta^{re}}$.*

Since equivalences in **dSta** are automatically representable, and (strict or weak) 2-functors take equivalences to equivalences, we deduce:

Corollary 1.10.13. *Suppose $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}$ are equivalent d-stacks, and Γ is a finite group. Then $\boldsymbol{\mathcal{X}}^\Gamma$ and $\boldsymbol{\mathcal{Y}}^\Gamma$ are equivalent in* **dSta**, *and similarly for $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\hat{\boldsymbol{\mathcal{X}}}^\Gamma,\boldsymbol{\mathcal{X}}^\Gamma_\circ,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ,\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ and $\tilde{\boldsymbol{\mathcal{Y}}}^\Gamma,\hat{\boldsymbol{\mathcal{Y}}}^\Gamma,\boldsymbol{\mathcal{Y}}^\Gamma_\circ,\tilde{\boldsymbol{\mathcal{Y}}}^\Gamma_\circ,\hat{\boldsymbol{\mathcal{Y}}}^\Gamma_\circ$. Also $\hat{\boldsymbol{X}}^\Gamma_\circ,\hat{\boldsymbol{Y}}^\Gamma_\circ$ are equivalent in* **dSpa**.

Here are the d-stack analogues of Theorems 1.8.27 and 1.8.28:

Theorem 1.10.14. *Let $\boldsymbol{X}$ be a d-space and G a finite group acting on $\boldsymbol{X}$ by* 1*-isomorphisms, and write $\boldsymbol{\mathcal{X}}=[\boldsymbol{X}/G]$ for the quotient d-stack, from Theorem* 1.10.3. *Let Γ be a finite group. Then there are equivalences of d-stacks*

$$\boldsymbol{\mathcal{X}}^\Gamma\simeq\coprod_{\substack{\text{conjugacy classes }[\rho]\text{ of injective}\\ \text{group morphisms }\rho:\Gamma\rightarrow G}}\bigl[\boldsymbol{X}^{\rho(\Gamma)}/\bigl\{g\in G:g\rho(\gamma)=\rho(\gamma)g\ \ \forall\gamma\in\Gamma\bigr\}\bigr],\tag{1.56}$$

$$\boldsymbol{\mathcal{X}}^\Gamma_\circ\simeq\coprod_{\substack{\text{conjugacy classes }[\rho]\text{ of injective}\\ \text{group morphisms }\rho:\Gamma\rightarrow G}}\bigl[\boldsymbol{X}^{\rho(\Gamma)}_\circ/\bigl\{g\in G:g\rho(\gamma)=\rho(\gamma)g\ \ \forall\gamma\in\Gamma\bigr\}\bigr],\tag{1.57}$$

$$\tilde{\boldsymbol{\mathcal{X}}}^\Gamma\simeq\coprod_{\text{conjugacy classes }[\Delta]\text{ of subgroups }\Delta\subseteq G\text{ with }\Delta\cong\Gamma}\bigl[\boldsymbol{X}^\Delta/\bigl\{g\in G:\Delta=g\Delta g^{-1}\bigr\}\bigr],\tag{1.58}$$

$$\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ\simeq\coprod_{\text{conjugacy classes }[\Delta]\text{ of subgroups }\Delta\subseteq G\text{ with }\Delta\cong\Gamma}\bigl[\boldsymbol{X}^\Delta_\circ/\bigl\{g\in G:\Delta=g\Delta g^{-1}\bigr\}\bigr],\tag{1.59}$$

$$\hat{\boldsymbol{\mathcal{X}}}^\Gamma\simeq\coprod_{\text{conjugacy classes }[\Delta]\text{ of subgroups }\Delta\subseteq G\text{ with }\Delta\cong\Gamma}\bigl[\boldsymbol{X}^\Delta/\bigl(\{g\in G:\Delta=g\Delta g^{-1}\}/\Delta\bigr)\bigr],\tag{1.60}$$

$$\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ\simeq\coprod_{\text{conjugacy classes }[\Delta]\text{ of subgroups }\Delta\subseteq G\text{ with }\Delta\cong\Gamma}\bigl[\boldsymbol{X}^\Delta_\circ/\bigl(\{g\in G:\Delta=g\Delta g^{-1}\}/\Delta\bigr)\bigr].\tag{1.61}$$

Here for each subgroup $\Delta \subseteq G$, we write $\boldsymbol{X}^\Delta$ for the closed d-subspace in $\boldsymbol{X}$ fixed by Δ in G, as in §1.3.4, *and $\boldsymbol{X}^\Delta_\circ$ for the open d-subspace in $\boldsymbol{X}^\Delta$ of points in $\boldsymbol{X}$ whose stabilizer group in G is exactly Δ. In* (1.56)–(1.57), *morphisms $\rho, \rho' : \Gamma \to G$ are conjugate if $\rho' = \mathrm{Ad}(g) \circ \rho$ for some $g \in G$, and subgroups $\Delta, \Delta' \subseteq G$ are conjugate if $\Delta = g\Delta' g^{-1}$ for some $g \in G$. In* (1.56)–(1.61) *we sum over one representative ρ or Δ for each conjugacy class.*

Theorem 1.10.15. *Let $\boldsymbol{\mathcal{X}}$ be a d-stack and Γ a finite group, so that Theorem* 1.10.11 *gives a d-stack $\boldsymbol{\mathcal{X}}^\Gamma$ and a* 1*-morphism $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}) : \boldsymbol{\mathcal{X}}^\Gamma \to \boldsymbol{\mathcal{X}}$. Equation* (1.52) *for $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})$ becomes:*

$$\begin{array}{ccccccc}
\begin{array}{c}O^\Gamma(\mathcal{X})^*(\mathcal{E}_\mathcal{X})= \\ (\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{tr}}\oplus(\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{nt}}\end{array} & \xrightarrow{O^\Gamma(\mathcal{X})^*(\phi_\mathcal{X})} & \begin{array}{c}O^\Gamma(\mathcal{X})^*(\mathcal{F}_\mathcal{X})= \\ (\mathcal{F}_\mathcal{X})^\Gamma_{\mathrm{tr}}\oplus(\mathcal{F}_\mathcal{X})^\Gamma_{\mathrm{nt}}\end{array} & \xrightarrow{O^\Gamma(\mathcal{X})^*(\psi_\mathcal{X})} & \begin{array}{c}O^\Gamma(\mathcal{X})^*(T^*\mathcal{X})= \\ (T^*\mathcal{X})^\Gamma_{\mathrm{tr}}\oplus(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}\end{array} & \longrightarrow & 0 \\
\Big\downarrow {\scriptstyle O^\Gamma(\mathcal{X})''} & & \Big\downarrow {\scriptstyle O^\Gamma(\mathcal{X})^2} & & \Big\downarrow {\scriptstyle O^\Gamma(\mathcal{X})^3= \atop \Omega_{O^\Gamma(\mathcal{X})}} & & \\
\mathcal{E}_{\mathcal{X}^\Gamma} & \xrightarrow{\phi_{\mathcal{X}^\Gamma}} & \mathcal{F}_{\mathcal{X}^\Gamma} & \xrightarrow{\psi_{\mathcal{X}^\Gamma}} & T^*(\mathcal{X}^\Gamma) & \longrightarrow & 0.
\end{array} \tag{1.62}$$

Then the columns $O^\Gamma(\mathcal{X})''$, $O^\Gamma(\mathcal{X})^2$, $O^\Gamma(\mathcal{X})^3$ of (1.62) *are isomorphisms when restricted to the 'trivial' summands $(\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{tr}}, (\mathcal{F}_\mathcal{X})^\Gamma_{\mathrm{tr}}, (T^*\mathcal{X})^\Gamma_{\mathrm{tr}}$, and are zero when restricted to the 'nontrivial' summands $(\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{nt}}, (\mathcal{F}_\mathcal{X})^\Gamma_{\mathrm{nt}}, (T^*\mathcal{X})^\Gamma_{\mathrm{nt}}$. In particular, this implies that the virtual cotangent sheaf $\phi_{\mathcal{X}^\Gamma} : \mathcal{E}_{\mathcal{X}^\Gamma} \to \mathcal{F}_{\mathcal{X}^\Gamma}$ of $\boldsymbol{\mathcal{X}}^\Gamma$ is* 1*-isomorphic in* $\mathrm{vqcoh}(\mathcal{X}^\Gamma)$ *to $(\phi_\mathcal{X})^\Gamma_{\mathrm{tr}} : (\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{tr}} \to (\mathcal{F}_\mathcal{X})^\Gamma_{\mathrm{tr}}$, the 'trivial' part of the pullback to $\mathcal{X}^\Gamma$ of the virtual cotangent sheaf $\phi_\mathcal{X} : \mathcal{E}_\mathcal{X} \to \mathcal{F}_\mathcal{X}$ of $\boldsymbol{\mathcal{X}}$.*

The analogous results also hold for $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \hat{\boldsymbol{\mathcal{X}}}^\Gamma, \boldsymbol{\mathcal{X}}^\Gamma_\circ, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ and $\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$.

1.11 The 2-category of d-orbifolds

Chapter 10 discusses *d-orbifolds*, which are orbifold versions of d-manifolds.

1.11.1 Definition of d-orbifolds

In §1.4.3 we discussed *virtual quasicoherent sheaves* and *virtual vector bundles* on C^∞-schemes $\underline{X}$. The next remark, drawn from §10.1.1, explains how these generalize to Deligne–Mumford C^∞-stacks $\mathcal{X}$.

Remark 1.11.1. In the C^∞-stack analogue of Definition 1.4.9, the 2-categories $\mathrm{vqcoh}(\mathcal{X})$ and $\mathrm{vvect}(\mathcal{X})$ for a Deligne–Mumford C^∞-stack $\mathcal{X}$ are defined exactly as for C^∞-schemes. For $\mathcal{X} \neq \emptyset$, virtual vector bundles $(\mathcal{E}^\bullet, \phi)$ have a well-defined *rank* $\mathrm{rank}(\mathcal{E}^\bullet, \phi) \in \mathbb{Z}$. If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of Deligne–Mumford C^∞-stacks then pullback f^* defines strict 2-functors $f^* : \mathrm{vqcoh}(\mathcal{Y}) \to \mathrm{vqcoh}(\mathcal{X})$ and $f^* : \mathrm{vvect}(\mathcal{Y}) \to \mathrm{vvect}(\mathcal{X})$, as for C^∞-schemes. If $f, g : \mathcal{X} \to \mathcal{Y}$ are 1-morphisms and $\eta : f \Rightarrow g$ a 2-morphism then $\eta^* : f^* \Rightarrow g^*$ is a 2-natural transformation.

In the d-stack version of Definition 1.4.10, we define the *virtual cotangent sheaf* $T^*\boldsymbol{\mathcal{X}}$ of a d-stack $\boldsymbol{\mathcal{X}}$ to be the morphism $\phi_\mathcal{X} : \mathcal{E}_\mathcal{X} \to \mathcal{F}_\mathcal{X}$ in $\mathrm{qcoh}(\mathcal{X})$ from Definition 1.10.1. If $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism in $\mathbf{dSta}$ then $\Omega_{\boldsymbol{f}} := (f'', f^2)$ is a 1-morphism $f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ in $\mathrm{vqcoh}(\mathcal{X})$. If $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ are 1-morphisms and $\boldsymbol{\eta} = (\eta, \eta') : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism in $\mathbf{dSta}$, then we have 1-morphisms $\Omega_{\boldsymbol{f}}$:

$f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$, $\Omega_{\boldsymbol{g}} : g^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$, and $\eta^*(T^*\boldsymbol{\mathcal{Y}}) : f^*(T^*\boldsymbol{\mathcal{Y}}) \to g^*(T^*\boldsymbol{\mathcal{Y}})$ in $\mathrm{qcoh}(\mathcal{X})$, and $\eta' : \Omega_{\boldsymbol{f}} \Rightarrow \Omega_{\boldsymbol{g}} \circ \eta^*(T^*\boldsymbol{\mathcal{Y}})$ is a 2-morphism in $\mathrm{vqcoh}(\mathcal{X})$.

We can now define d-orbifolds.

Definition 1.11.2. A d-stack $\boldsymbol{\mathcal{W}}$ is called a *principal d-orbifold* if is equivalent in **dSta** to a fibre product $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ with $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}} \in \hat{\mathbf{Orb}}$. If $\boldsymbol{\mathcal{W}}$ is a nonempty principal d-orbifold then as in Proposition 1.4.11, the virtual cotangent sheaf $T^*\boldsymbol{\mathcal{W}}$ is a virtual vector bundle on $\mathcal{W}$, in the sense of Remark 1.11.1. We define the *virtual dimension* of $\boldsymbol{\mathcal{W}}$ to be $\mathrm{vdim}\,\boldsymbol{\mathcal{W}} = \mathrm{rank}\,T^*\boldsymbol{\mathcal{W}} \in \mathbb{Z}$. If $\boldsymbol{\mathcal{W}} \simeq \boldsymbol{\mathcal{X}} \times_{\boldsymbol{\mathcal{Z}}} \boldsymbol{\mathcal{Y}}$ for orbifolds $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ then $\mathrm{vdim}\,\boldsymbol{\mathcal{W}} = \dim \mathcal{X} + \dim \mathcal{Y} - \dim \mathcal{Z}$.

A d-stack $\boldsymbol{\mathcal{X}}$ is called a *d-orbifold* (*without boundary*) *of virtual dimension* $n \in \mathbb{Z}$, written $\mathrm{vdim}\,\boldsymbol{\mathcal{X}} = n$, if $\boldsymbol{\mathcal{X}}$ can be covered by open d-substacks $\boldsymbol{\mathcal{W}}$ which are principal d-orbifolds with $\mathrm{vdim}\,\boldsymbol{\mathcal{W}} = n$. The virtual cotangent sheaf $T^*\boldsymbol{\mathcal{X}} = (\mathcal{E}_{\mathcal{X}}, \mathcal{F}_{\mathcal{X}}, \phi_{\mathcal{X}})$ of $\boldsymbol{\mathcal{X}}$ is a virtual vector bundle of rank $\mathrm{vdim}\,\boldsymbol{\mathcal{X}} = n$, so we call it the *virtual cotangent bundle* of $\boldsymbol{\mathcal{X}}$.

Let **dOrb** be the full 2-subcategory of d-orbifolds in **dSta**. The 2-functor $F_{\mathbf{Orb}}^{\mathbf{dSta}} : \mathbf{Orb} \to \mathbf{dSta}$ in Definition 1.10.1 maps into **dOrb**, and we will write $F_{\mathbf{Orb}}^{\mathbf{dOrb}} = F_{\mathbf{Orb}}^{\mathbf{dSta}} : \mathbf{Orb} \to \mathbf{dOrb}$. Also $\hat{\mathbf{Orb}}$ is a 2-subcategory of **dOrb**. We say that a d-orbifold $\boldsymbol{\mathcal{X}}$ *is an orbifold* if it lies in $\hat{\mathbf{Orb}}$. The 2-functor $F_{\mathbf{dSpa}}^{\mathbf{dSta}}$ maps $\mathbf{dMan} \to \mathbf{dOrb}$, and we will write $F_{\mathbf{dMan}}^{\mathbf{dOrb}} = F_{\mathbf{dSpa}}^{\mathbf{dSta}}|_{\mathbf{dMan}} : \mathbf{dMan} \to \mathbf{dOrb}$. Then $F_{\mathbf{dMan}}^{\mathbf{dOrb}} \circ F_{\mathbf{Man}}^{\mathbf{dMan}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}} \circ F_{\mathbf{Man}}^{\mathbf{Orb}} : \mathbf{Man} \to \mathbf{dOrb}$.

Write $\hat{\mathbf{dMan}}$ for the full 2-subcategory of objects $\boldsymbol{\mathcal{X}}$ in **dOrb** equivalent to $F_{\mathbf{dMan}}^{\mathbf{dOrb}}(\boldsymbol{X})$ for some d-manifold $\boldsymbol{X}$. When we say that a d-orbifold $\boldsymbol{\mathcal{X}}$ *is a d-manifold*, we mean that $\boldsymbol{\mathcal{X}} \in \hat{\mathbf{dMan}}$.

The orbifold analogue of Proposition 1.4.2 holds. Using Theorem 1.8.17 we can deduce:

Lemma 1.11.3. *Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. Then $\boldsymbol{\mathcal{X}}$ is a d-manifold, that is, $\boldsymbol{\mathcal{X}}$ is equivalent to $F_{\mathbf{dMan}}^{\mathbf{dOrb}}(\boldsymbol{X})$ for some d-manifold $\boldsymbol{X}$, if and only if* $\mathrm{Iso}_{\mathcal{X}}([x]) \cong \{1\}$ *for all $[x]$ in $\mathcal{X}_{\mathrm{top}}$.*

1.11.2 Local properties of d-orbifolds

Following Examples 1.4.4 and 1.4.5, we define 'standard model' d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ and 1-morphisms $\boldsymbol{\mathcal{S}}_{f,\hat{f}}$.

Example 1.11.4. Let $\mathcal{V}$ be an orbifold, $\mathcal{E} \in \mathrm{vect}(\mathcal{V})$ a vector bundle on $\mathcal{V}$ as in §1.8.6, and $s \in C^\infty(\mathcal{E})$ a smooth section, that is, $s : \mathcal{O}_{\mathcal{V}} \to \mathcal{E}$ is a morphism in $\mathrm{vect}(\mathcal{V})$. We will define a principal d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$, which we call a 'standard model' d-orbifold.

Let the Deligne–Mumford C^∞-stack $\mathcal{S}$ be the C^∞-substack in $\mathcal{V}$ defined by the equation $s = 0$, so that informally $\mathcal{S} = s^{-1}(0) \subset \mathcal{V}$. Explicitly, as in §1.8, a C^∞-stack $\mathcal{V}$ consists of a category $\mathcal{V}$ and a functor $p_{\mathcal{V}} : \mathcal{V} \to \mathbf{C^\infty Sch}$, where there is a 1-1 correspondence between objects u in $\mathcal{V}$ with $p_{\mathcal{V}}(u) = \underline{U}$ in $\mathbf{C^\infty Sch}$ and 1-morphisms $\tilde{u} : \bar{\underline{U}} \to \mathcal{V}$ in $\mathbf{C^\infty Sta}$. Define $\mathcal{S}$ to be the full subcategory of objects u in $\mathcal{V}$ such that the morphism $\tilde{u}^*(s) : \tilde{u}^*(\mathcal{O}_{\mathcal{V}}) \to \tilde{u}^*(\mathcal{E})$ in $\mathrm{qcoh}(\underline{\bar{U}})$ is zero, and define $p_{\mathcal{S}} = p_{\mathcal{V}}|_{\mathcal{S}} : \mathcal{S} \to \mathbf{C^\infty Sch}$.

Since $i_{\mathcal{V}} : \mathcal{S} \to \mathcal{V}$ is the inclusion of a C^∞-substack, $i_{\mathcal{V}}^\sharp : i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}}) \to \mathcal{O}_{\mathcal{S}}$ is a surjective morphism of sheaves of C^∞-rings on $\mathcal{S}$. Write $\mathcal{I}_s$ for the kernel of $i_{\mathcal{V}}^\sharp$, as a sheaf of ideals in $i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})$, and $\mathcal{I}_s^2$ for the corresponding sheaf of squared ideals, and $\mathcal{O}'_{\mathcal{S}} = i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2$ for the quotient sheaf of C^∞-rings, and $\imath_{\mathcal{S}} : \mathcal{O}'_{\mathcal{S}} \to \mathcal{O}_{\mathcal{S}}$ for the natural projection $i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2 \twoheadrightarrow i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s \cong \mathcal{O}_{\mathcal{S}}$ induced by the inclusion $\mathcal{I}_s^2 \subseteq \mathcal{I}_s$.

Write $\mathcal{E}^* \in \mathrm{vect}(\mathcal{V})$ for the dual vector bundle of $\mathcal{E}$, and set $\mathcal{E}_{\mathcal{S}} = i_{\mathcal{V}}^*(\mathcal{E}^*)$. There is a natural, surjective morphism $\jmath_{\mathcal{S}} : \mathcal{E}_{\mathcal{S}} \to \mathcal{I}_{\mathcal{S}} = \mathcal{I}_s/\mathcal{I}_s^2$ in $\mathrm{qcoh}(\mathcal{S})$ which locally maps $\alpha + (\mathcal{I}_s \cdot C^\infty(\mathcal{E}^*)) \mapsto \alpha \cdot s + \mathcal{I}_s^2$. Then $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$ is a d-stack. As in the d-manifold case, we can show that $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is equivalent in **dSta** to $\boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}} \boldsymbol{\mathcal{V}}$, where $\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{E}}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Orb}}^{\mathbf{dSta}}\bigl(\mathcal{V}, \mathrm{Tot}(\mathcal{E}), \mathrm{Tot}(s), \mathrm{Tot}(0)\bigr)$, using the notation of §1.9.1. Thus $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is a principal d-orbifold. Every principal d-orbifold $\boldsymbol{\mathcal{W}}$ is equivalent in **dSta** to some $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$.

Sometimes it is useful to take $\mathcal{V}$ to be an *effective* orbifold, as in §1.9.1.

Example 1.11.5. Let $\mathcal{V}, \mathcal{W}$ be orbifolds, $\mathcal{E}, \mathcal{F}$ be vector bundles on $\mathcal{V}, \mathcal{W}$, and $s \in C^\infty(\mathcal{E})$, $t \in C^\infty(\mathcal{F})$ be smooth sections, so that Example 1.11.4 defines 'standard model' principal d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}, \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$. Write $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = \boldsymbol{\mathcal{S}} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$ and $\boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t} = \boldsymbol{\mathcal{T}} = (\mathcal{T}, \mathcal{O}'_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}}, \imath_{\mathcal{T}}, \jmath_{\mathcal{T}})$. Suppose $f : \mathcal{V} \to \mathcal{W}$ is a 1-morphism, and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ is a morphism in $\mathrm{vect}(\mathcal{V})$ satisfying

$$\hat{f} \circ s = f^*(t). \tag{1.63}$$

We will define a 1-morphism $\boldsymbol{g} = (g, g', g'') : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{T}}$ in **dSta**, which we write as $\boldsymbol{\mathcal{S}}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$, and call a 'standard model' 1-morphism.

As in Example 1.11.4, $\mathcal{V}, \mathcal{W}$ are categories, $\mathcal{S} \subseteq \mathcal{V}$, $\mathcal{T} \subseteq \mathcal{W}$ are full subcategories, and $f : \mathcal{V} \to \mathcal{W}$ is a functor. Using (1.63) one can show that $f(\mathcal{S}) \subseteq \mathcal{T} \subseteq \mathcal{W}$. Define $g = f|_{\mathcal{S}} : \mathcal{S} \to \mathcal{T}$. Then $g : \mathcal{S} \to \mathcal{T}$ is a 1-morphism of Deligne–Mumford C^∞-stacks, with $i_{\mathcal{W}} \circ g = f \circ i_{\mathcal{V}} : \mathcal{S} \to \mathcal{W}$.

To define $g' : g^{-1}(\mathcal{O}'_{\mathcal{T}}) \to \mathcal{O}'_{\mathcal{S}}$, consider the commutative diagram:

$$\begin{tikzcd}
g^{-1}(\mathcal{I}_t^2) \arrow[r] \arrow[d, dotted] & g^{-1}(i_{\mathcal{W}}^{-1}(\mathcal{O}_{\mathcal{W}})) \arrow[r] \arrow[d, "{\substack{i_{\mathcal{V}}^{-1}(f^\sharp)\circ I_{i_{\mathcal{V}},f}(\mathcal{O}_{\mathcal{W}})\circ \\ I_{g,i_{\mathcal{W}}}(\mathcal{O}_{\mathcal{W}})^{-1}}}"] & g^{-1}(\mathcal{O}'_{\mathcal{T}}) = g^{-1}(i_{\mathcal{W}}^{-1}(\mathcal{O}_{\mathcal{W}})/\mathcal{I}_t^2) \arrow[r] \arrow[d, dotted, "g'"] & 0 \\
\mathcal{I}_s^2 \arrow[r] & i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}}) \arrow[r] & \mathcal{O}'_{\mathcal{S}} = i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2 \arrow[r] & 0.
\end{tikzcd}$$

The rows are exact. Using (1.63), we see the central column maps $g^{-1}(\mathcal{I}_t) \to \mathcal{I}_s$, and so maps $g^{-1}(\mathcal{I}_t^2) \to \mathcal{I}_s^2$, and the left column exists. Thus by exactness there is a unique morphism g' making the diagram commute.

We have $\mathcal{E}_{\mathcal{S}} = i_{\mathcal{V}}^*(\mathcal{E}^*)$ and $\mathcal{E}_{\mathcal{T}} = i_{\mathcal{W}}^*(\mathcal{F}^*)$, and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ induces $\hat{f}^* : f^*(\mathcal{F}^*) \to \mathcal{E}^*$. Define $g'' = i_{\mathcal{V}}^*(\hat{f}^*) \circ I_{i_{\mathcal{V}},f}(\mathcal{F}^*) \circ I_{g,i_{\mathcal{W}}}(\mathcal{F}^*)^{-1} : g^*(\mathcal{E}_{\mathcal{T}}) \to \mathcal{E}_{\mathcal{S}}$ in $\mathrm{qcoh}(\mathcal{S})$. Then $\boldsymbol{g} = (g, g', g'') : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{T}}$ is a 1-morphism in **dSta**, which we write as $\boldsymbol{\mathcal{S}}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$.

Suppose now that $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ is open, with inclusion 1-morphism $i_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \to \mathcal{V}$. Write $\tilde{\mathcal{E}} = \mathcal{E}|_{\tilde{\mathcal{V}}} = i_{\tilde{\mathcal{V}}}^*(\mathcal{E})$ and $\tilde{s} = s|_{\tilde{\mathcal{V}}}$. Define $\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}} = \boldsymbol{\mathcal{S}}_{i_{\tilde{\mathcal{V}}},\mathrm{id}_{\tilde{\mathcal{E}}}} : \boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}} \to \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$. If $s^{-1}(0) \subseteq \tilde{\mathcal{V}}$ then $\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}} : \boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}} \to \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is a 1-isomorphism.

We do not define 'standard model' 2-morphisms in **dOrb**, as in Example 1.4.6 for d-manifolds, to avoid inconvenience in combining the $O(s), O(s^2)$ notation with 2-morphisms of orbifolds. But see Example 1.11.9 below for a different form of 'standard model' 2-morphism.

Any d-orbifold $\boldsymbol{\mathcal{X}}$ is locally equivalent near a point $[x]$ to a principal d-orbifold, and so to a standard model d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$. The next theorem, the analogue of Theorem 1.4.7, shows that $\mathcal{V},\mathcal{E},s$ are locally determined essentially uniquely if $\dim\mathcal{V}$ is chosen to be minimal (which corresponds to the condition $\mathrm{d}s(v)=0$).

Theorem 1.11.6. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold, and $[x]\in\mathcal{X}_{\mathrm{top}}$. Then there exists an open neighbourhood $\boldsymbol{\mathcal{U}}$ of $[x]$ in $\boldsymbol{\mathcal{X}}$ and an equivalence $\boldsymbol{\mathcal{U}}\simeq\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ in* **dOrb** *for $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ as in Example* 1.11.4, *such that the equivalence identifies $[x]$ with $[v]\in\mathcal{V}_{\mathrm{top}}$ with $s(v)=\mathrm{d}s(v)=0$. Furthermore, $\mathcal{V},\mathcal{E},s$ are determined up to non-canonical equivalence near $[v]$ by $\boldsymbol{\mathcal{X}}$ near $[x]$. In fact, they depend only on the C^∞-stack $\mathcal{X}$, the point $[x]\in\mathcal{X}_{\mathrm{top}}$, and the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on the finite-dimensional vector space $\mathrm{Ker}\big(x^*(\phi_{\mathcal{X}}):x^*(\mathcal{E}_{\mathcal{X}})\rightarrow x^*(\mathcal{F}_{\mathcal{X}})\big)$.*

In a d-orbifold $\boldsymbol{\mathcal{X}}=(\mathcal{X},\mathcal{O}'_{\mathcal{X}},\mathcal{E}_{\mathcal{X}},\imath_{\mathcal{X}},\jmath_{\mathcal{X}})$, we think of $\mathcal{X}$ as 'classical' and $\mathcal{O}'_{\mathcal{X}},\mathcal{E}_{\mathcal{X}},\imath_{\mathcal{X}},\jmath_{\mathcal{X}}$ as 'derived'. The extra information in the 'derived' data is like a vector bundle $\mathcal{E}$ over $\mathcal{X}$. A vector bundle $\mathcal{E}$ on a Deligne–Mumford C^∞-stack $\mathcal{X}$ is determined locally near $[x]\in\mathcal{X}_{\mathrm{top}}$ by the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on the fibre $x^*(\mathcal{E})$ of $\mathcal{E}$ at $[x]$. Thus, it is reasonable that $\boldsymbol{\mathcal{X}}$ should be determined up to equivalence near $[x]$ by $\mathcal{X}$ and a representation of $\mathrm{Iso}_{\mathcal{X}}([x])$.

Here are alternative forms of 'standard model' d-orbifolds, 1-morphisms and 2-morphisms, using the quotient d-stack notation of §1.10.2.

Example 1.11.7. Let V be a manifold, $E\rightarrow V$ a vector bundle, Γ a finite group acting smoothly on V,E preserving the vector bundle structure, and $s:V\rightarrow E$ a smooth, Γ-equivariant section of E. Write the Γ-actions on V,E as $r(\gamma):V\rightarrow V$ and $\hat r(\gamma):E\rightarrow r(\gamma)^*(E)$ for $\gamma\in\Gamma$. Then Examples 1.4.4 and 1.4.5 give an explicit principal d-manifold $\boldsymbol{S}_{V,E,s}$, and 1-morphisms $\boldsymbol{S}_{r(\gamma),\hat r(\gamma)}:\boldsymbol{S}_{V,E,s}\rightarrow\boldsymbol{S}_{V,E,s}$ for $\gamma\in\Gamma$ which are an action of Γ on $\boldsymbol{S}_{V,E,s}$. Hence Theorem 1.10.3(i) gives a quotient d-stack $[\boldsymbol{S}_{V,E,s}/\Gamma]$.

In fact $[\boldsymbol{S}_{V,E,s}/\Gamma]\simeq\boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde s}$ for $\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde s$ defined using V,E,s,Γ, with $\tilde{\mathcal{V}}=[\underline{V}/\Gamma]$. Thus, $[\boldsymbol{S}_{V,E,s}/\Gamma]$ is a principal d-orbifold. But not all principal d-orbifolds $\boldsymbol{\mathcal{W}}$ have $\boldsymbol{\mathcal{W}}\simeq[\boldsymbol{S}_{V,E,s}/\Gamma]$, as not all orbifolds $\mathcal{V}$ have $\mathcal{V}\simeq[\underline{V}/\Gamma]$ for some manifold V and finite group Γ.

Example 1.11.8. Let $[\boldsymbol{S}_{V,E,s}/\Gamma]$, $[\boldsymbol{S}_{W,F,t}/\Delta]$ be quotient d-orbifolds as in Example 1.11.7, where Γ acts on V,E by $q(\gamma):V\rightarrow V$ and $\hat q(\gamma):E\rightarrow q(\gamma)^*(E)$ for $\gamma\in\Gamma$, and Δ acts on W,F by $r(\delta):W\rightarrow W$ and $\hat r(\delta):F\rightarrow r(\delta)^*(F)$ for $\delta\in\Delta$. Suppose $f:V\rightarrow W$ is a smooth map, and $\hat f:E\rightarrow f^*(F)$ is a morphism of vector bundles on V satisfying $\hat f\circ s=f^*(t)+O(s^2)$, as in (1.9), and $\rho:\Gamma\rightarrow\Delta$ is a group morphism satisfying $f\circ q(\gamma)=r(\rho(\gamma))\circ f:V\rightarrow W$ and $q(\gamma)^*(\hat f)\circ\hat q(\gamma)=f^*(\hat r(\rho(\gamma)))\circ\hat f:E\rightarrow(f\circ q(\gamma))^*(F)$ for all $\gamma\in\Gamma$, so that $f,\hat f$ are equivariant under Γ,Δ,ρ. Then Example 1.4.4 defines a 1-morphism

$\boldsymbol{S}_{f,\hat f} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ in **dMan**. The equivariance conditions on $f, \hat f$ imply that $\boldsymbol{S}_{f,\hat f} \circ \boldsymbol{S}_{q(\gamma),\hat q(\gamma)} = \boldsymbol{S}_{r(\rho(\gamma)),\hat r(\rho(\gamma))} \circ \boldsymbol{S}_{f,\hat f}$ for $\gamma \in \Gamma$. Hence Theorem 1.10.3(ii) gives a quotient 1-morphism $[\boldsymbol{S}_{f,\hat f}, \rho] : [\boldsymbol{S}_{V,E,s}/\Gamma] \to [\boldsymbol{S}_{W,F,t}/\Delta]$.

Example 1.11.9. Suppose $[\boldsymbol{S}_{f,\hat f}, \rho], [\boldsymbol{S}_{g,\hat g}, \sigma] : [\boldsymbol{S}_{V,E,s}/\Gamma] \to [\boldsymbol{S}_{W,F,t}/\Delta]$ are two 1-morphisms as in Example 1.11.8, and write $q, \hat q$ for the actions of Γ on V, E and $r, \hat r$ for the actions of Δ on W, F. Then $\rho, \sigma : \Gamma \to \Delta$ are group morphisms. Suppose $\delta \in \Delta$ satisfies $\sigma(\gamma) = \delta\, \rho(\gamma)\, \delta^{-1}$ for all $\gamma \in \Gamma$, and $\Lambda : E \to f^*(TW)$ is a morphism of vector bundles on V which satisfies

$$r(\delta^{-1}) \circ g = f + \Lambda \cdot s + O(s^2) \;\text{ and }\; g^*(\hat r(\delta^{-1})) \circ \hat g = \hat f + \Lambda \cdot \mathrm{d}t + O(s), \tag{1.64}$$

$$f^*(\mathrm{d}r(\rho(\gamma))) \circ \Lambda = q(\gamma)^*(\Lambda) \circ \hat q(\gamma) : E \longrightarrow (f \circ q(\gamma))^*(TW), \quad \forall \gamma \in \Gamma, \tag{1.65}$$

where $\mathrm{d}r(\rho(\gamma)) : TW \to r(\rho(\gamma))^*(TW)$ is the derivative of $r(\rho(\gamma))$. Here (1.64) is the conditions for Example 1.4.6 to define a 'standard model' 2-morphism $\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat f} \Rightarrow \boldsymbol{S}_{r(\delta^{-1})\circ g, g^*(\hat r(\delta^{-1}))\circ \hat g} = \boldsymbol{S}_{r(\delta^{-1}),\hat r(\delta^{-1})} \circ \boldsymbol{S}_{g,\hat g}$ in **dMan**. Then (1.65) implies that $\boldsymbol{S}_\Lambda * \mathrm{id}_{\boldsymbol{S}_{q(\gamma),\hat q(\gamma)}} = \mathrm{id}_{\boldsymbol{S}_{r(\rho(\gamma)),\hat r(\rho(\gamma))}} * \boldsymbol{S}_\Lambda$ for all $\gamma \in \Gamma$. Hence Theorem 1.10.3(iii) gives a quotient 2-morphism $[\boldsymbol{S}_\Lambda, \delta] : [\boldsymbol{S}_{f,\hat f}, \rho] \Rightarrow [\boldsymbol{S}_{g,\hat g}, \sigma]$.

Here is an analogue of Theorem 1.11.6 for the alternative form $[\boldsymbol{S}_{V,E,s}/\Gamma]$.

Proposition 1.11.10. *A d-stack $\boldsymbol{\mathcal{X}}$ is a d-orbifold of virtual dimension $n \in \mathbb{Z}$ if and only if each $[x] \in \mathcal{X}_{\mathrm{top}}$ has an open neighbourhood $\boldsymbol{\mathcal{U}}$ equivalent to some $[\boldsymbol{S}_{V,E,s}/\Gamma]$ in Example 1.11.7 with $\dim V - \operatorname{rank} E = n$, where $\Gamma = \mathrm{Iso}_{\mathcal{X}}([x])$ and $[x] \in \mathcal{X}_{\mathrm{top}}$ is identified with a fixed point v of Γ in V with $s(v) = 0$ and $\mathrm{d}s(v) = 0$. Furthermore, V, E, s, Γ are determined up to non-canonical isomorphism near v by $\boldsymbol{\mathcal{X}}$ near $[x]$.*

1.11.3 Equivalences in dOrb, and gluing by equivalences

Next we summarize the results of §10.2, the analogue of §1.4.4. Section 1.10.2 discussed étale 1-morphisms in **dSta**. We characterize when 1-morphisms $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{S}}_{f,\hat f} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ in **dOrb** are étale, or equivalences.

Theorem 1.11.11. *Suppose $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism of d-orbifolds, and $f : \mathcal{X} \to \mathcal{Y}$ is representable. Then the following are equivalent:*

(i) *$\boldsymbol{f}$ is étale;*

(ii) *$\Omega_{\boldsymbol{f}} : f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ is an equivalence in* $\mathrm{vqcoh}(\mathcal{X})$*; and*

(iii) *The following is a split short exact sequence in* $\mathrm{qcoh}(\mathcal{X})$*:*

$$0 \longrightarrow f^*(\mathcal{E}_{\mathcal{Y}}) \xrightarrow{f'' \oplus -f^*(\phi_{\mathcal{Y}})} \mathcal{E}_{\mathcal{X}} \oplus f^*(\mathcal{F}_{\mathcal{Y}}) \xrightarrow{\phi_{\mathcal{X}} \oplus f^2} \mathcal{F}_{\mathcal{X}} \longrightarrow 0. \tag{1.66}$$

If in addition $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a bijection, then $\boldsymbol{f}$ is an equivalence in **dOrb**.

Theorem 1.11.12. *Suppose $\boldsymbol{\mathcal{S}}_{f,\hat f}:\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}\rightarrow\boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ is a 'standard model' 1-morphism, in the notation of Examples* 1.11.4 *and* 1.11.5, *with $f:\mathcal{V}\rightarrow\mathcal{W}$ representable. Then $\boldsymbol{\mathcal{S}}_{f,\hat f}$ is étale if and only if for each $[v]\in\mathcal{V}_{\rm top}$ with $s(v)=0$ and $[w]=f_{\rm top}([v])\in\mathcal{W}_{\rm top}$, the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_v\mathcal{V} \xrightarrow{{\rm d}s(v)\oplus {\rm d}f(v)} \mathcal{E}_v\oplus T_w\mathcal{W} \xrightarrow{\hat f(v)\oplus -{\rm d}t(w)} \mathcal{F}_w \longrightarrow 0.$$

Also $\boldsymbol{\mathcal{S}}_{f,\hat f}$ is an equivalence if and only if in addition $f_{\rm top}|_{s^{-1}(0)}:s^{-1}(0)\rightarrow t^{-1}(0)$ is a bijection, where $s^{-1}(0)=\{[v]\in\mathcal{V}_{\rm top}:s(v)=0\}$, $t^{-1}(0)=\{[w]\in\mathcal{W}_{\rm top}:t(w)=0\}$, and $f_*:{\rm Iso}_{\mathcal{V}}([v])\rightarrow{\rm Iso}_{\mathcal{W}}(f_{\rm top}([v]))$ is an isomorphism for all $[v]\in s^{-1}(0)\subseteq\mathcal{V}_{\rm top}$.

Here is an analogue of Theorem 1.4.17 for d-orbifolds, taken from §10.2. It is proved by applying Theorem 1.10.8 to glue together the 'standard model' d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$ by equivalences. Now Theorem 1.10.8 includes extra conditions (1.53)–(1.54) on the 2-morphisms η_{ijk},ζ_{jk}. But by taking the $\mathcal{V}_i,\mathcal{Y}$ to be effective orbifolds and the g_i to be submersions, the η_{ijk},ζ_{jk} are unique by Proposition 1.9.5, and so (1.53)–(1.54) hold automatically.

Theorem 1.11.13. *Suppose we are given the following data:*

- (a) *an integer n;*
- (b) *a Hausdorff, second countable topological space X;*
- (c) *an indexing set I, and a total order $<$ on I;*
- (d) *for each i in I, an effective orbifold $\mathcal{V}_i$ in the sense of Definition* 1.9.4, *a vector bundle $\mathcal{E}_i$ on $\mathcal{V}_i$ with $\dim\mathcal{V}_i-\operatorname{rank}\mathcal{E}_i=n$, a section $s_i\in C^\infty(\mathcal{E}_i)$, and a homeomorphism $\psi_i:s_i^{-1}(0)\rightarrow\hat X_i$, where $s_i^{-1}(0)=\{[v_i]\in\mathcal{V}_{i,\rm top}:s_i(v_i)=0\}$ and $\hat X_i\subseteq X$ is open; and*
- (e) *for all $i<j$ in I, an open suborbifold $\mathcal{V}_{ij}\subseteq\mathcal{V}_i$, a 1-morphism $e_{ij}:\mathcal{V}_{ij}\rightarrow\mathcal{V}_j$, and a morphism of vector bundles $\hat e_{ij}:\mathcal{E}_i|_{\mathcal{V}_{ij}}\rightarrow e_{ij}^*(\mathcal{E}_j)$.*

Let this data satisfy the conditions:

- (i) *$X=\bigcup_{i\in I}\hat X_i$;*
- (ii) *if $i<j$ in I then $(e_{ij})_*:{\rm Iso}_{\mathcal{V}_{ij}}([v])\rightarrow{\rm Iso}_{\mathcal{V}_j}(e_{ij,\rm top}([v]))$ is an isomorphism for all $[v]\in\mathcal{V}_{ij,\rm top}$, and $\hat e_{ij}\circ s_i|_{\mathcal{V}_{ij}}=e_{ij}^*(s_j)\circ\iota_{ij}$ where $\iota_{ij}:\mathcal{O}_{\mathcal{V}_{ij}}\rightarrow e_{ij}^*(\mathcal{O}_{\mathcal{V}_j})$ is the natural isomorphism, and $\psi_i(s_i|_{\mathcal{V}_{ij}}^{-1}(0))=\hat X_i\cap\hat X_j$, and $\psi_i|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)}=\psi_j\circ e_{ij,\rm top}|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)}$, and if $[v_i]\in\mathcal{V}_{ij,\rm top}$ with $s_i(v_i)=0$ and $[v_j]=e_{ij,\rm top}([v_i])$ then the following sequence is exact:*
$$0 \longrightarrow T_{v_i}\mathcal{V}_i \xrightarrow{{\rm d}s_i(v_i)\oplus {\rm d}e_{ij}(v_i)} \mathcal{E}_i|_{v_i}\oplus T_{v_j}\mathcal{V}_j \xrightarrow{\hat e_{ij}(v_i)\oplus -{\rm d}s_j(v_j)} \mathcal{E}_j|_{v_j} \longrightarrow 0;$$
- (iii) *if $i<j<k$ in I then there exists a 2-morphism $\eta_{ijk}:e_{jk}\circ e_{ij}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}\Rightarrow e_{ik}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}$ in* **Orb** *with*
$$\hat e_{ik}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}=\eta_{ijk}^*(\mathcal{E}_k)\circ I_{e_{ij},e_{jk}}(\mathcal{E}_k)^{-1}\circ e_{ij}|^*_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}(\hat e_{jk})\circ\hat e_{ij}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}.$$
Note that η_{ijk} is unique by Proposition 1.9.5.

Then there exist a d-orbifold $\boldsymbol{\mathcal{X}}$ with $\operatorname{vdim}\boldsymbol{\mathcal{X}}=n$ *and underlying topological space* $\mathcal{X}_{\rm top}\cong X$, *and a* 1*-morphism* $\boldsymbol{\psi}_i:\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}\rightarrow\boldsymbol{\mathcal{X}}$ *with underlying continuous map* ψ_i *which is an equivalence with the open d-suborbifold* $\hat{\boldsymbol{\mathcal{X}}}_i\subseteq\boldsymbol{\mathcal{X}}$ *corresponding to* $\hat X_i\subseteq X$ *for all* $i\in I$, *such that for all* $i<j$ *in* I *there exists a* 2*-morphism* $\boldsymbol{\eta}_{ij}:\boldsymbol{\psi}_j\circ\boldsymbol{\mathcal{S}}_{e_{ij},\hat e_{ij}}\Rightarrow\boldsymbol{\psi}_i\circ\boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i}$, *where* $\boldsymbol{\mathcal{S}}_{e_{ij},\hat e_{ij}}:\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}}\rightarrow\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j}$ *and* $\boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i}:\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}}\rightarrow\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$, *using the notation of Examples* 1.11.4 *and* 1.11.5. *This d-orbifold* $\boldsymbol{\mathcal{X}}$ *is unique up to equivalence in* **dOrb**.

Suppose also that $\mathcal{Y}$ *is an effective orbifold, and* $g_i:\mathcal{V}_i\rightarrow\mathcal{Y}$ *are submersions for all* $i\in I$, *and there are* 2*-morphisms* $\zeta_{ij}:g_j\circ e_{ij}\Rightarrow g_i|_{\mathcal{V}_{ij}}$ *in* **Orb** *for all* $i<j$ *in* I. *Then there exist a* 1*-morphism* $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ *in* **dOrb** *unique up to* 2*-isomorphism, where* $\boldsymbol{\mathcal{Y}}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})=\boldsymbol{\mathcal{S}}_{\mathcal{Y},0,0}$, *and* 2*-morphisms* $\boldsymbol{\zeta}_i:\boldsymbol{h}\circ\boldsymbol{\psi}_i\Rightarrow\boldsymbol{\mathcal{S}}_{g_i,0}$ *for all* $i\in I$.

Here is another version of the same result using the alternative form of ‘standard model’ d-orbifolds in §1.11.1.

Theorem 1.11.14. *Suppose we are given the following data:*

(a) *an integer* n;

(b) *a Hausdorff, second countable topological space* X;

(c) *an indexing set* I, *and a total order* $<$ *on* I;

(d) *for each* i *in* I, *a manifold* V_i, *a vector bundle* $E_i\rightarrow V_i$ *with* $\dim V_i-\operatorname{rank}E_i=n$, *a finite group* Γ_i, *smooth, locally effective actions* $r_i(\gamma):V_i\rightarrow V_i$, $\hat r_i(\gamma):E_i\rightarrow r(\gamma)^*(E_i)$ *of* Γ_i *on* V_i,E_i *for* $\gamma\in\Gamma_i$, *a smooth,* Γ_i*-equivariant section* $s_i:V_i\rightarrow E_i$, *and a homeomorphism* $\psi_i:X_i\rightarrow\hat X_i$, *where* $X_i=\{v_i\in V_i:s_i(v_i)=0\}/\Gamma_i$ *and* $\hat X_i\subseteq X$ *is an open set; and*

(e) *for all* $i<j$ *in* I, *an open submanifold* $V_{ij}\subseteq V_i$, *invariant under* Γ_i, *a group morphism* $\rho_{ij}:\Gamma_i\rightarrow\Gamma_j$, *a smooth map* $e_{ij}:V_{ij}\rightarrow V_j$, *and a morphism of vector bundles* $\hat e_{ij}:E_i|_{V_{ij}}\rightarrow e_{ij}^*(E_j)$.

Let this data satisfy the conditions:

(i) $X=\bigcup_{i\in I}\hat X_i$;

(ii) *if* $i<j$ *in* I *then* $\hat e_{ij}\circ s_i|_{V_{ij}}=e_{ij}^*(s_j)+O(s_i^2)$, *and for all* $\gamma\in\Gamma$ *we have*

$$\begin{aligned}e_{ij}\circ r_i(\gamma)&=r_j(\rho_{ij}(\gamma))\circ e_{ij}:V_{ij}\longrightarrow V_j,\\ r_i(\gamma)^*(\hat e_{ij})\circ\hat r_i(\gamma)&=e_{ij}^*(\hat r_j(\rho_{ij}(\gamma)))\circ\hat e_{ij}:E_i|_{V_{ij}}\longrightarrow(e_{ij}\circ r_i(\gamma))^*(E_j),\end{aligned}$$

and $\psi_i(X_i\cap(V_{ij}/\Gamma_i))=\hat X_i\cap\hat X_j$, *and* $\psi_i|_{X_i\cap V_{ij}/\Gamma_i}=\psi_j\circ(e_{ij})_*|_{X_i\cap V_{ij}/\Gamma_j}$, *and if* $v_i\in V_{ij}$ *with* $s_i(v_i)=0$ *and* $v_j=e_{ij}(v_i)$ *then* $\rho|_{\operatorname{Stab}_{\Gamma_i}(v_i)}:\operatorname{Stab}_{\Gamma_i}(v_i)\rightarrow\operatorname{Stab}_{\Gamma_j}(v_j)$ *is an isomorphism, and the following sequence of vector spaces is exact:*

$$0\longrightarrow T_{v_i}V_i\xrightarrow{\mathrm{d}s_i(v_i)\oplus\mathrm{d}e_{ij}(v_i)}E_i|_{v_i}\oplus T_{v_j}V_j\xrightarrow{\hat e_{ij}(v_i)\oplus-\mathrm{d}s_j(v_j)}E_j|_{v_j}\longrightarrow 0;$$

(iii) *if $i<j<k$ in I then there exists $\gamma_{ijk}\in\Gamma_k$ satisfying*

$$\begin{aligned}\rho_{ik}(\gamma)&=\gamma_{ijk}\,\rho_{jk}(\rho_{ij}(\gamma))\,\gamma_{ijk}^{-1}\quad \textit{for all } \gamma\in\Gamma_i,\\ e_{ik}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})}&=r_k(\gamma_{ijk})\circ e_{jk}\circ e_{ij}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})},\quad \textit{and}\\ \hat e_{ik}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})}&=\bigl(e_{ij}^*(e_{jk}^*(\hat r_k(\gamma_{ijk})))\circ e_{ij}^*(\hat e_{jk})\circ \hat e_{ij}\bigr)|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})}.\end{aligned}$$

Then there exist a d-orbifold $\boldsymbol{\mathcal{X}}$ with $\operatorname{vdim}\boldsymbol{\mathcal{X}}=n$ *and underlying topological space $\mathcal{X}_{\rm top}\cong X$, and a* 1*-morphism $\boldsymbol{\psi}_i:[\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]\rightarrow\boldsymbol{\mathcal{X}}$ with underlying continuous map ψ_i which is an equivalence with the open d-suborbifold $\hat{\boldsymbol{\mathcal{X}}}_i\subseteq\boldsymbol{\mathcal{X}}$ corresponding to $\hat X_i\subseteq X$ for all $i\in I$, such that for all $i<j$ in I there exists a* 2*-morphism $\boldsymbol{\eta}_{ij}:\boldsymbol{\psi}_j\circ[\boldsymbol{S}_{e_{ij},\hat e_{ij}},\rho_{ij}]\Rightarrow\boldsymbol{\psi}_i\circ[\boldsymbol{i}_{V_{ij},V_i},\mathrm{id}_{\Gamma_i}]$, where $[\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]$ is as in Example* 1.11.7*, and $[\boldsymbol{S}_{e_{ij},\hat e_{ij}},\rho_{ij}]:[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]\rightarrow[\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j]$ and $[\boldsymbol{i}_{V_{ij},V_i},\mathrm{id}_{\Gamma_i}]:[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]\rightarrow[\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_j]$ as in Example* 1.11.8*. This d-orbifold $\boldsymbol{\mathcal{X}}$ is unique up to equivalence in* **dOrb**.

Suppose also that Y is a manifold, and $g_i:V_i\rightarrow Y$ are smooth maps for all $i\in I$ with $g_i\circ r_i(\gamma)=g_i$ for all $\gamma\in\Gamma_i$, and $g_j\circ e_{ij}=g_i|_{V_{ij}}$ for all $i<j$ in I. Then there exist a 1*-morphism $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ unique up to* 2*-isomorphism, where $\boldsymbol{\mathcal{Y}}=F_{\mathbf{Man}}^{\mathbf{dOrb}}(Y)=[\boldsymbol{S}_{Y,0,0}/\{1\}]$, and* 2*-morphisms $\boldsymbol{\zeta}_i:\boldsymbol{h}\circ\boldsymbol{\psi}_i\Rightarrow[\boldsymbol{S}_{g_i,0},\pi_{\{1\}}]$ for all $i\in I$. Here $[\boldsymbol{S}_{Y,0,0}/\{1\}]$ is from Example* 1.11.7 *with E,s both zero and $\Gamma=\{1\}$, and $[\boldsymbol{S}_{g_i,0},\pi_{\{1\}}]:[\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]\rightarrow[\boldsymbol{S}_{Y,0,0}/\{1\}]=\boldsymbol{\mathcal{Y}}$ is from Example* 1.11.8 *with $\hat g_i=0$ and $\rho=\pi_{\{1\}}:\Gamma_i\rightarrow\{1\}$.*

The importance of Theorems 1.11.13 and 1.11.14 is that all the ingredients are described wholly in differential-geometric or topological terms. So we can use these theorems as tools to prove the existence of d-orbifold structures on spaces coming from other areas of geometry, such as moduli spaces of J-holomorphic curves. The theorems are used to define functors to d-orbifolds from other geometric structures, as discussed in §1.16.

1.11.4 Submersions, immersions, and embeddings

Section 1.4.5 discussed (w-)submersions, (w-)immersions, and (w-)embeddings for d-manifolds. Following §10.3, here are the analogues for d-orbifolds.

Definition 1.11.15. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, so that as in Remark 1.11.1 we have a 2-category $\mathrm{vvect}(\mathcal{X})$ of virtual vector bundles on $\mathcal{X}$. We define when a 1-morphism $f^\bullet:(\mathcal{E}^\bullet,\phi)\rightarrow(\mathcal{F}^\bullet,\psi)$ in $\mathrm{vvect}(\mathcal{X})$ is *weakly injective*, *injective*, *weakly surjective* or *surjective* exactly as in Definition 1.4.18.

Let $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ be a 1-morphism of d-orbifolds. Then $\Omega_{\boldsymbol{f}}:f^*(T^*\boldsymbol{\mathcal{Y}})\rightarrow T^*\boldsymbol{\mathcal{X}}$ is a 1-morphism in $\mathrm{vvect}(\mathcal{X})$.

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\Omega_{\boldsymbol{f}}$ is weakly injective.
(b) We call $\boldsymbol{f}$ a *submersion* if $\Omega_{\boldsymbol{f}}$ is injective.
(c) We call $\boldsymbol{f}$ a *w-immersion* if $f:\mathcal{X}\rightarrow\mathcal{Y}$ is representable, i.e. $f_*:\mathrm{Iso}_{\mathcal{X}}([x])\rightarrow\mathrm{Iso}_{\mathcal{Y}}(f_{\rm top}([x]))$ is injective for all $[x]\in\mathcal{X}_{\rm top}$, and $\Omega_{\boldsymbol{f}}$ is weakly surjective.

(d) We call $\boldsymbol{f}$ an *immersion* if $f : \mathcal{X} \to \mathcal{Y}$ is representable and $\Omega_{\boldsymbol{f}}$ is surjective.

(e) We call $\boldsymbol{f}$ a *w-embedding* or *embedding* if it is a w-immersion or immersion, respectively, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image, so in particular f_{top} is injective.

Parts (c)–(e) enable us to define *d-suborbifolds* of d-orbifolds. *Open d-suborbifolds* are (Zariski) open d-substacks of a d-orbifold. For more general d-suborbifolds, we call $\boldsymbol{i} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ a *w-immersed d-suborbifold*, or *immersed d-suborbifold*, or *w-embedded d-suborbifold*, or *embedded d-suborbifold* of $\boldsymbol{\mathcal{Y}}$, if $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ are d-orbifolds and $\boldsymbol{i}$ is a w-immersion, ..., embedding, respectively.

Theorem 1.4.20 in §1.4.5 holds with orbifolds and d-orbifolds in place of manifolds and d-manifolds, except part (v), when we need also to assume $f : \mathcal{X} \to \mathcal{Y}$ representable to deduce $\boldsymbol{f}$ is étale, and part (x), which is false for d-orbifolds (in the Zariski topology, at least).

1.11.5 D-transversality and fibre products

Section 1.4.6 discussed d-transversality and fibre products for d-manifolds. This is extended to d-orbifolds in §10.4, with little essential change. Here are the analogues of Definition 1.4.21 and Theorems 1.4.22–1.4.25.

Definition 1.11.16. Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ be d-orbifolds and $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$, $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms. Let $\mathcal{W} = \mathcal{X} \times_{g,\mathcal{Z},h} \mathcal{Y}$ be the C^∞-stack fibre product, and write $e : \mathcal{W} \to \mathcal{X}$, $f : \mathcal{W} \to \mathcal{Y}$ for the projection 1-morphisms, and $\eta : g \circ e \Rightarrow h \circ f$ for the 2-morphism from the fibre product. Consider the morphism

$$\alpha = \begin{pmatrix} e^*(g'') \circ I_{e,g}(\mathcal{E}_{\mathcal{Z}}) \\ -f^*(h'') \circ I_{f,h}(\mathcal{E}_{\mathcal{Z}}) \circ \eta^*(\mathcal{E}_{\mathcal{Z}}) \\ (g \circ e)^*(\phi_{\mathcal{Z}}) \end{pmatrix} : \begin{array}{l} (g \circ e)^*(\mathcal{E}_{\mathcal{Z}}) \longrightarrow \\ \qquad e^*(\mathcal{E}_{\mathcal{X}}) \oplus f^*(\mathcal{E}_{\mathcal{Y}}) \oplus (g \circ e)^*(\mathcal{F}_{\mathcal{Z}}) \end{array}$$

in $\mathrm{qcoh}(\mathcal{W})$. We call $\boldsymbol{g}, \boldsymbol{h}$ *d-transverse* if α has a left inverse.

Theorem 1.11.17. *Suppose $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ are d-orbifolds and $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$, $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ are d-transverse* 1*-morphisms, and let $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ be the d-stack fibre product, which exists by Theorem* 1.10.10(a)*. Then $\boldsymbol{\mathcal{W}}$ is a d-orbifold, with*

$$\operatorname{vdim} \boldsymbol{\mathcal{W}} = \operatorname{vdim} \boldsymbol{\mathcal{X}} + \operatorname{vdim} \boldsymbol{\mathcal{Y}} - \operatorname{vdim} \boldsymbol{\mathcal{Z}}. \tag{1.67}$$

Theorem 1.11.18. *Suppose $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$, $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ are* 1*-morphisms of d-orbifolds. The following are sufficient conditions for $\boldsymbol{g}, \boldsymbol{h}$ to be d-transverse, so that $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ is a d-orbifold of virtual dimension* (1.67)*:*

(a) *$\boldsymbol{\mathcal{Z}}$ is an orbifold, that is, $\boldsymbol{\mathcal{Z}} \in \hat{\mathbf{Orb}}$; or*

(b) *$\boldsymbol{g}$ or $\boldsymbol{h}$ is a w-submersion.*

Theorem 1.11.19. *Let $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Z}}$ be d-orbifolds, $\boldsymbol{\mathcal{Y}}$ an orbifold, and $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Z}}$, $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{Z}}$ be* 1*-morphisms with $\boldsymbol{g}$ a submersion. Then $\boldsymbol{\mathcal{W}}=\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}$ is an orbifold, with* $\dim\boldsymbol{\mathcal{W}}=\operatorname{vdim}\boldsymbol{\mathcal{X}}+\dim\boldsymbol{\mathcal{Y}}-\operatorname{vdim}\boldsymbol{\mathcal{Z}}$.

Theorem 1.11.20. **(i)** *Let $\rho:G\to H$ be a morphism of finite groups, and H act linearly on $\mathbb{R}^n$. Then as in* §1.10.2 *we have quotient d-orbifolds $[*/G]$, $[\boldsymbol{\mathbb{R}^n}/H]$ and a quotient* 1*-morphism $[\mathbf{0},\rho]:[*/G]\to[\boldsymbol{\mathbb{R}^n}/H]$. Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold and $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to[\boldsymbol{\mathbb{R}^n}/H]$ a* 1*-morphism in* **dOrb**. *Then the fibre product $\boldsymbol{\mathcal{W}}=\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},[\boldsymbol{\mathbb{R}^n}/H],[\mathbf{0},\rho]}[*/G]$ exists in* **dOrb** *by Theorem* 1.11.18(a). *The projection $\boldsymbol{\pi_{\mathcal{X}}}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{X}}$ is an immersion if ρ is injective, and an embedding if ρ is an isomorphism.*

(ii) *Suppose $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ is an immersion of d-orbifolds, and $[x]\in\mathcal{X}_{\rm top}$ with $f_{\rm top}([x])=[y]\in\mathcal{Y}_{\rm top}$. Write $\rho:G\to H$ for $f_*:\mathrm{Iso}_{\mathcal{X}}([x])\to\mathrm{Iso}_{\mathcal{Y}}([y])$. Then ρ is injective, and there exist open neighbourhoods $\boldsymbol{\mathcal{U}}\subseteq\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{V}}\subseteq\boldsymbol{\mathcal{Y}}$ of $[x],[y]$ with $\boldsymbol{f}(\boldsymbol{\mathcal{U}})\subseteq\boldsymbol{\mathcal{V}}$, a linear action of H on $\mathbb{R}^n$ where* $n=\operatorname{vdim}\boldsymbol{\mathcal{Y}}-\operatorname{vdim}\boldsymbol{\mathcal{X}}\geqslant 0$, *and a* 1*-morphism $\boldsymbol{g}:\boldsymbol{\mathcal{V}}\to[\boldsymbol{\mathbb{R}^n}/H]$ with* $g_{\rm top}([y])=[0]$, *fitting into a* 2*-Cartesian square in* **dOrb** :

$$\begin{tikzcd}
\boldsymbol{\mathcal{U}} \arrow[rr] \arrow[d, "\boldsymbol{f}|_{\boldsymbol{\mathcal{U}}}"'] \arrow[drr, phantom, "\Uparrow"] && {[*/G]} \arrow[d, "{[\mathbf{0},\rho]}"] \\
\boldsymbol{\mathcal{V}} \arrow[rr, "\boldsymbol{g}"] && {[\boldsymbol{\mathbb{R}^n}/H].}
\end{tikzcd}$$

If $\boldsymbol{f}$ is an embedding then ρ is an isomorphism, and we may take $\boldsymbol{\mathcal{U}}=\boldsymbol{f}^{-1}(\boldsymbol{\mathcal{V}})$.

1.11.6 Embedding d-orbifolds into orbifolds

Section 1.4.7 discussed embeddings of d-manifolds into manifolds. Theorem 1.4.29 gave necessary and sufficient conditions for the existence of embeddings $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{\mathbb{R}^n}$ for any d-manifold $\boldsymbol{X}$, and Theorem 1.4.32 showed that if a d-manifold $\boldsymbol{X}$ has an embedding $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ for a manifold Y then $\boldsymbol{X}\simeq\boldsymbol{S}_{V,E,s}$ for open $f(X)\subset V\subseteq Y$. Combining these proves that large classes of d-manifolds — all compact d-manifolds, for instance — are principal d-manifolds.

Section 10.5 considers how to generalize all this to d-orbifolds. The proof of Theorem 1.4.32 extends to (d-)orbifolds, giving:

Theorem 1.11.21. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold, $\mathcal{Y}$ an orbifold, and $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ an embedding, in the sense of Definition* 1.11.15. *Then there exist an open suborbifold $\mathcal{V}\subseteq\mathcal{Y}$ with $\boldsymbol{f}(\boldsymbol{\mathcal{X}})\subseteq\boldsymbol{\mathcal{V}}$, a vector bundle $\mathcal{E}$ on $\mathcal{V}$, and a smooth section $s\in C^\infty(\mathcal{E})$ fitting into a* 2*-Cartesian diagram in* **dOrb**, *where* $\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{E}},\boldsymbol{s},\mathbf{0}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}\big(\mathcal{Y},\mathcal{V},\mathrm{Tot}(\mathcal{E}),\mathrm{Tot}(s),\mathrm{Tot}(0)\big)$, *in the notation of* §1.9.1:

$$\begin{tikzcd}
\boldsymbol{\mathcal{X}} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{f}"'] \arrow[dr, phantom, "\Uparrow"] & \boldsymbol{\mathcal{V}} \arrow[d, "\mathbf{0}"] \\
\boldsymbol{\mathcal{V}} \arrow[r, "\boldsymbol{s}"] & \boldsymbol{\mathcal{E}}.
\end{tikzcd}$$

Hence $\boldsymbol{\mathcal{X}}$ is equivalent to the 'standard model' d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ of Example 1.11.4, *and is a principal d-orbifold.*

However, we do not presently have a good analogue of Theorem 1.4.29 for d-orbifolds, so we cannot state useful necessary and sufficient conditions for when a d-orbifold $\boldsymbol{\mathcal{X}}$ can be embedded into an orbifold, or is a principal d-orbifold.

1.11.7 Orientations of d-orbifolds

Section 1.4.8 discusses orientations on d-manifolds. As in §10.6, all this material generalizes easily to d-orbifolds, so we will give few details.

If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack and $(\mathcal{E}^\bullet,\phi)$ a virtual vector bundle on $\mathcal{X}$, then we define a line bundle $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ on $\mathcal{X}$ called the *orientation line bundle* of $(\mathcal{E}^\bullet,\phi)$. It has functorial properties as in Theorem 1.4.34(a)–(f). If $\boldsymbol{\mathcal{X}}$ is a d-orbifold, the virtual cotangent bundle $T^*\boldsymbol{\mathcal{X}}=(\mathcal{E}_{\mathcal{X}},\mathcal{F}_{\mathcal{X}},\phi_{\mathcal{X}})$ is a virtual vector bundle on $\mathcal{X}$. We define an *orientation* ω on $\boldsymbol{\mathcal{X}}$ to be an orientation on the orientation line bundle $\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}$. The analogues of Theorem 1.4.37 and Proposition 1.4.38 hold for d-orbifolds.

One difference between (d-)manifolds and (d-)orbifolds is that line bundles $\mathcal{L}$ on Deligne–Mumford C^∞-stacks $\mathcal{X}$ (such as orientation line bundles) need only be locally trivial in the étale topology, not in the Zariski topology. Because of this, orbifolds and d-orbifolds need not be (Zariski) locally orientable. For example, the orbifold $[\mathbb{R}^{2n+1}/\{\pm 1\}]$ is not locally orientable near 0.

1.11.8 Orbifold strata of d-orbifolds

Section 1.8.7 discussed the *orbifold strata* $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\ldots,\hat{\mathcal{X}}^\Gamma_\circ$ of a Deligne–Mumford C^∞-stack $\mathcal{X}$. When $\mathcal{X}$ is an orbifold, §1.9.2 explained that $\mathcal{X}^\Gamma$ decomposes as $\mathcal{X}^\Gamma=\coprod_{\lambda\in\Lambda^\Gamma_+}\mathcal{X}^{\Gamma,\lambda}$, where each $\mathcal{X}^{\Gamma,\lambda}$ is an orbifold of dimension $\dim\mathcal{X}-\dim\lambda$, and similarly for $\tilde{\mathcal{X}}^\Gamma,\ldots,\hat{\mathcal{X}}^\Gamma_\circ$. Section 1.10.5 discussed the orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\ldots,\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ of a d-stack $\boldsymbol{\mathcal{X}}$. Section 10.7 shows that for a d-orbifold $\boldsymbol{\mathcal{X}}$, the orbifold strata decompose as $\boldsymbol{\mathcal{X}}^\Gamma=\coprod_{\lambda\in\Lambda^\Gamma}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$, where $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ is a d-orbifold of virtual dimension $\operatorname{vdim}\boldsymbol{\mathcal{X}}-\dim\lambda$, and similarly for $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\ldots,\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$.

Definition 1.11.22. Let Γ be a finite group, and use the notation $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$, $\Lambda^\Gamma=K_0\big(\mathrm{Rep}_{\mathrm{nt}}(\Gamma)\big)$, $\Lambda^\Gamma_+\subseteq\Lambda^\Gamma$ and $\dim:\Lambda^\Gamma\to\mathbb{Z}$ of Definition 1.9.7. Let $R_0,R_1,\ldots,R_k$ be the irreducible Γ-representations up to isomorphism, with $R_0=\mathbb{R}$ the trivial representation, so that $\Lambda^\Gamma\cong\mathbb{Z}^k$ and $\Lambda^\Gamma_+\cong\mathbb{N}^k$.

Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold. Theorem 1.10.11 gives a d-stack $\boldsymbol{\mathcal{X}}^\Gamma$ and a 1-morphism $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}):\boldsymbol{\mathcal{X}}^\Gamma\to\boldsymbol{\mathcal{X}}$. The virtual cotangent bundle of $\boldsymbol{\mathcal{X}}$ is $T^*\boldsymbol{\mathcal{X}}=(\mathcal{E}_{\mathcal{X}},\mathcal{F}_{\mathcal{X}},\phi_{\mathcal{X}})$, a virtual vector bundle of rank $\operatorname{vdim}\boldsymbol{\mathcal{X}}$ on $\mathcal{X}$. So $O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})=\big(O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}),O^\Gamma(\mathcal{X})^*(\mathcal{F}_{\mathcal{X}}),O^\Gamma(\mathcal{X})^*(\phi_{\mathcal{X}})\big)$ is a virtual vector bundle on $\mathcal{X}^\Gamma$. As in §1.8.7, $O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}),O^\Gamma(\mathcal{X})^*(\mathcal{F}_{\mathcal{X}})$ have natural Γ-representations inducing decompositions of the form (1.48)–(1.49), and $O^\Gamma(\mathcal{X})^*(\phi_{\mathcal{X}})$ is Γ-equivariant and so preserves these splittings. Hence we have decompositions in $\mathrm{vqcoh}(\mathcal{X}^\Gamma)$:

$$\begin{gathered}O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})\cong\textstyle\bigoplus_{i=0}^k(T^*\boldsymbol{\mathcal{X}})^\Gamma_i\otimes R_i\ \text{ for }\ (T^*\boldsymbol{\mathcal{X}})^\Gamma_i\in\mathrm{vqcoh}(\mathcal{X}^\Gamma),\\ \text{and }\ O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})=(T^*\boldsymbol{\mathcal{X}})^\Gamma_{\mathrm{tr}}\oplus(T^*\boldsymbol{\mathcal{X}})^\Gamma_{\mathrm{nt}},\ \text{ with}\\ (T^*\boldsymbol{\mathcal{X}})^\Gamma_{\mathrm{tr}}\cong(T^*\boldsymbol{\mathcal{X}})^\Gamma_0\otimes R_0\ \text{ and }\ (T^*\boldsymbol{\mathcal{X}})^\Gamma_{\mathrm{nt}}\cong\textstyle\bigoplus_{i=1}^k(T^*\boldsymbol{\mathcal{X}})^\Gamma_i\otimes R_i.\end{gathered}\tag{1.68}$$

Also Theorem 1.10.15 shows that $T^*(\boldsymbol{\mathcal{X}}^\Gamma)\cong(T^*\boldsymbol{\mathcal{X}})^\Gamma_{\rm tr}$.

As $O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})$ is a virtual vector bundle, (1.68) implies the $(T^*\boldsymbol{\mathcal{X}})^\Gamma_i$ are *virtual vector bundles of mixed rank*, whose ranks may vary on different connected components of $\boldsymbol{\mathcal{X}}^\Gamma$. For each $\lambda\in\Lambda^\Gamma$, define $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ to be the open and closed d-substack in $\boldsymbol{\mathcal{X}}^\Gamma$ with $\operatorname{rank}\bigl((T^*\boldsymbol{\mathcal{X}})^\Gamma_1\bigr)[R_1]+\cdots+\operatorname{rank}\bigl((T^*\boldsymbol{\mathcal{X}})^\Gamma_k\bigr)[R_k]=\lambda$ in Λ^Γ. Then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ is a d-orbifold, with $\operatorname{vdim}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}=\operatorname{vdim}\boldsymbol{\mathcal{X}}-\dim\lambda$. Also we have a decomposition $\boldsymbol{\mathcal{X}}^\Gamma=\coprod_{\lambda\in\Lambda^\Gamma}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ in $\mathbf{dSta}$.

Note that in the d-orbifold case $\dim\lambda$ may be negative, so we can have $\operatorname{vdim}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}>\operatorname{vdim}\boldsymbol{\mathcal{X}}$. This is counterintuitive: the (w-immersed) d-suborbifold $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ has larger dimension than the d-orbifold $\boldsymbol{\mathcal{X}}$ that contains it.

Write $\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}})=\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})|_{\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}}:\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}\to\boldsymbol{\mathcal{X}}$. Then $\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}})$ is a proper w-immersion of d-orbifolds, in the sense of §1.11.4. Define $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ=\boldsymbol{\mathcal{X}}^\Gamma_\circ\cap\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$, and $\boldsymbol{O}^{\Gamma,\lambda}_\circ(\boldsymbol{\mathcal{X}})=\boldsymbol{O}^\Gamma_\circ(\boldsymbol{\mathcal{X}})|_{\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ}:\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ\to\boldsymbol{\mathcal{X}}$. Then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ$ is a d-orbifold with $\operatorname{vdim}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ=\operatorname{vdim}\boldsymbol{\mathcal{X}}-\dim\lambda$, and $\boldsymbol{\mathcal{X}}^\Gamma_\circ=\coprod_{\lambda\in\Lambda^\Gamma}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ$.

As for $\tilde{\mathcal{X}}^{\Gamma,\mu},\ldots,\hat{\mathcal{X}}^{\Gamma,\mu}_\circ$ in §1.9.2, for each $\mu\in\Lambda^\Gamma/\operatorname{Aut}(\Gamma)$ we define $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}\simeq\bigl[\bigl(\coprod_{\lambda\in\mu}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}\bigr)/\operatorname{Aut}(\Gamma)\bigr]$ in $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma\simeq[\boldsymbol{\mathcal{X}}^\Gamma/\operatorname{Aut}(\Gamma)]$, and $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ=\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ\cap\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$, and $\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}=\hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}})(\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu})$, and $\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ=\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ\cap\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$. Then $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\ldots,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ are d-orbifolds with $\operatorname{vdim}\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}=\cdots=\operatorname{vdim}\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ=\operatorname{vdim}\boldsymbol{\mathcal{X}}-\dim\mu$, with

$$\tilde{\boldsymbol{\mathcal{X}}}^\Gamma=\coprod_\mu\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\;\;\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ=\coprod_\mu\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ,\;\;\hat{\boldsymbol{\mathcal{X}}}^\Gamma=\coprod_\mu\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\;\;\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ=\coprod_\mu\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ.$$

Also $\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ is a d-manifold, that is, it lies in $\mathbf{d\hat{M}an}$.

Section 10.7 also considers the question: if $\boldsymbol{\mathcal{X}}$ is an oriented d-orbifold, under what conditions on Γ,λ,μ do the orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\ldots,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ have natural orientations? Here is the analogue of Proposition 1.9.9:

Proposition 1.11.23. **(a)** *If Γ is a finite group with $|\Gamma|$ odd and $\boldsymbol{\mathcal{X}}$ an oriented d-orbifold, then we can define orientations on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ$ for all $\lambda\in\Lambda^\Gamma$.*

(b) *Let Γ be a finite group with $|\Gamma|$ odd, $\lambda\in\Lambda^\Gamma$ and $\mu=\lambda\cdot\operatorname{Aut}(\Gamma)$ in $\Lambda^\Gamma/\operatorname{Aut}(\Gamma)$. We may write $\lambda=[(V^+,\rho^+)]-[(V^-,\rho^-)]$ for nontrivial Γ-representations $(V^\pm,\rho^\pm)$ with no common subrepresentation, and then $(V^\pm,\rho^\pm)$ are unique up to isomorphism. Define H to be the subgroup of $\operatorname{Aut}(\Gamma)$ fixing λ in Λ^Γ. Then for each $\delta\in H$ there exist isomorphisms of Γ-representations $i^\pm_\delta:(V^\pm,\rho^\pm\circ\delta)\to(V^\pm,\rho^\pm)$. Suppose $i^+_\delta\oplus i^-_\delta:V^+\oplus V^-\to V^+\oplus V^-$ is orientation-preserving for all $\delta\in H$. If $\lambda\in 2\Lambda^\Gamma$ this holds automatically.*

Then for all oriented d-orbifolds $\boldsymbol{\mathcal{X}}$ we can define orientations on the orbifold strata $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$. For $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$ this works as $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}\simeq[\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}/H]$, where $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ is oriented by **(a)***, and the H-action on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ preserves orientations, so the orientation on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ descends to an orientation on $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}\simeq[\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}/H]$.*

(c) *Suppose that Γ and $\lambda\in\Lambda^\Gamma$ do not satisfy the conditions in* **(a)** *(i.e. $|\Gamma|$ is even), or Γ and $\mu\in\Lambda^\Gamma/\operatorname{Aut}(\Gamma)$ do not satisfy the conditions in* **(b)***. Then we can find examples of oriented d-orbifolds $\boldsymbol{\mathcal{X}}$ such that $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ$ are not orientable, or $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ are not orientable, respectively. That is, the conditions on Γ,λ,μ in* **(a)**,**(b)** *are necessary as well as sufficient to be able to orient orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\ldots,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ of all oriented d-orbifolds $\boldsymbol{\mathcal{X}}$.*

Note that Proposition 1.11.23 for d-orbifolds is weaker than Proposition 1.9.9 for orbifolds. That is, if Γ is a finite group with $|\Gamma|$ even then for some choices of λ, μ we can orient $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ for all oriented orbifolds $\mathcal{X}$, but we cannot orient $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \ldots, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ for all oriented d-orbifolds $\boldsymbol{\mathcal{X}}$.

1.11.9 Kuranishi neighbourhoods and good coordinate systems

We now explain the main ideas of §10.8, which are based on parallel material about Kuranishi spaces due to Fukaya, Oh, Ohta and Ono [32, 34].

Definition 1.11.24. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. A *type A Kuranishi neighbourhood* on $\boldsymbol{\mathcal{X}}$ is a quintuple $(V, E, \Gamma, s, \boldsymbol{\psi})$ where V is a manifold, $E \to V$ a vector bundle, Γ a finite group acting smoothly and locally effectively on V, E preserving the vector bundle structure, and $s : V \to E$ a smooth, Γ-equivariant section of E. Write the Γ-actions on V, E as $r(\gamma) : V \to V$ and $\hat{r}(\gamma) : E \to r(\gamma)^*(E)$ for $\gamma \in \Gamma$. Then Example 1.11.7 defines a principal d-orbifold $[\boldsymbol{S}_{V,E,s}/\Gamma]$. We require that $\boldsymbol{\psi} : [\boldsymbol{S}_{V,E,s}/\Gamma] \to \boldsymbol{\mathcal{X}}$ is a 1-morphism of d-orbifolds which is an equivalence with a nonempty open d-suborbifold $\boldsymbol{\psi}([\boldsymbol{S}_{V,E,s}/\Gamma]) \subseteq \boldsymbol{\mathcal{X}}$.

Definition 1.11.25. Suppose $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ are type A Kuranishi neighbourhoods on a d-orbifold $\boldsymbol{\mathcal{X}}$, with

$$\boldsymbol{\emptyset} \neq \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \boldsymbol{\mathcal{X}}.$$

A *type A coordinate change from* $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ *to* $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ is a quintuple $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$, where:

(a) $\emptyset \neq V_{ij} \subseteq V_i$ is a Γ_i-invariant open submanifold, with

$$\boldsymbol{\psi}_i\big([\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]\big) = \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \boldsymbol{\mathcal{X}}.$$

(b) $\rho_{ij} : \Gamma_i \to \Gamma_j$ is an injective group morphism.

(c) $e_{ij} : V_{ij} \to V_j$ is an embedding of manifolds with $e_{ij} \circ r_i(\gamma) = r_j(\rho_{ij}(\gamma)) \circ e_{ij} : V_{ij} \to V_j$ for all $\gamma \in \Gamma_i$. If $v_i, v_i' \in V_{ij}$ and $\delta \in \Gamma_j$ with $r_j(\delta) \circ e_{ij}(v_i') = e_{ij}(v_i)$, then there exists $\gamma \in \Gamma_i$ with $\rho_{ij}(\gamma) = \delta$ and $r_i(\gamma)(v_i') = v_i$.

(d) $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$ is an embedding of vector bundles (that is, $\hat{e}_{ij}$ has a left inverse), such that $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j)$ and $r_i(\gamma)^*(\hat{e}_{ij}) \circ \hat{r}_i(\gamma) = e_{ij}^*(\hat{r}_j(\rho_{ij}(\gamma))) \circ \hat{e}_{ij} : E_i|_{V_{ij}} \to (e_{ij} \circ r_i(\gamma))^*(E_j)$ for all $\gamma \in \Gamma_i$. Thus Example 1.11.8 defines a quotient 1-morphism

$$[\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}] : [\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i] \longrightarrow [\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j], \tag{1.69}$$

where $[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]$ is an open d-suborbifold in $[\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]$.

(e) If $v_i \in V_{ij}$ with $s_i(v_i) = 0$ and $v_j = e_{ij}(v_i) \in V_j$ then the following linear map is an isomorphism:

$$\big(\mathrm{d}s_j(v_j)\big)_* : \big(T_{v_j}V_j\big)/\big(\mathrm{d}e_{ij}(v_i)[T_{v_i}V_i]\big) \to \big(E_j|_{v_j}\big)/\big(\hat{e}_{ij}(v_i)[E_i|_{v_i}]\big).$$

Theorem 1.11.12 then implies that $[\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}]$ in (1.69) is an equivalence with an open d-suborbifold of $[\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j]$.

(f) $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ [\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}] \Rightarrow \boldsymbol{\psi}_i|_{[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]}$ is a 2-morphism in **dOrb**.

(g) The quotient topological space $V_i \amalg_{V_{ij}} V_j = (V_i \amalg V_j)/\sim$ is Hausdorff, where the equivalence relation $\sim$ identifies $v \in V_{ij} \subseteq V_i$ with $e_{ij}(v) \in V_j$.

Definition 1.11.26. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. A *type A good coordinate system* on $\boldsymbol{\mathcal{X}}$ consists of the following data satisfying conditions (a)–(e):

(a) We are given a countable indexing set I, and a total order $<$ on I making $(I,<)$ into a well-ordered set.

(b) For each $i \in I$ we are given a Kuranishi neighbourhood $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ of type A on $\boldsymbol{\mathcal{X}}$. Write $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i])$, so that $\boldsymbol{\mathcal{X}}_i \subseteq \boldsymbol{\mathcal{X}}$ is an open d-suborbifold, and $\boldsymbol{\psi}_i : [\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i] \to \boldsymbol{\mathcal{X}}_i$ is an equivalence. We require that $\bigcup_{i\in I} \boldsymbol{\mathcal{X}}_i = \boldsymbol{\mathcal{X}}$, so that $\{\boldsymbol{\mathcal{X}}_i : i \in I\}$ is an open cover of $\boldsymbol{\mathcal{X}}$.

(c) For all $i < j$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \neq \boldsymbol{\emptyset}$ we are given a type A coordinate change $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$ from $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ to $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$.

(d) For all $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$, we are given $\gamma_{ijk} \in \Gamma_k$ satisfying $\rho_{ik}(\gamma) = \gamma_{ijk}\, \rho_{jk}(\rho_{ij}(\gamma))\, \gamma_{ijk}^{-1}$ for all $\gamma \in \Gamma_i$, and

$$\begin{aligned} e_{ik}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})} &= r_k(\gamma_{ijk}) \circ e_{jk} \circ e_{ij}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})}, \\ \hat{e}_{ik}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})} &= \bigl(e_{ij}^*(e_{jk}^*(\hat{r}_k(\gamma_{ijk}))) \circ e_{ij}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}\bigr)|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})}. \end{aligned} \tag{1.70}$$

Combining the first equation of (1.70) with Definition 1.11.25(c) for e_{ik} and Γ_i acting effectively on $V_{ik} \cap e_{ij}^{-1}(V_{jk})$ shows that γ_{ijk} is unique. Example 1.11.9 with $\delta = \gamma_{ijk}$ and $\Lambda = 0$ then gives a 2-morphism in **dOrb**:

$$\begin{aligned} \boldsymbol{\eta}_{ijk} = [\boldsymbol{S}_0, \gamma_{ijk}] : [\boldsymbol{S}_{e_{jk},\hat{e}_{jk}}, \rho_{jk}] \circ [\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}]|_{[\boldsymbol{S}_{V_{ik}\cap e_{ij}^{-1}(V_{jk}),E_i,s_i}/\Gamma_i]} \\ \Longrightarrow [\boldsymbol{S}_{e_{ik},\hat{e}_{ik}}, \rho_{ik}]|_{[\boldsymbol{S}_{V_{ik}\cap e_{ij}^{-1}(V_{jk}),E_i,s_i}/\Gamma_i]}. \end{aligned}$$

(e) For all $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$ and $\boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$, we require that if $v_i \in V_{ik}$, $v_j \in V_{jk}$ and $\delta \in \Gamma_k$ with $e_{jk}(v_j) = r_k(\delta) \circ e_{ik}(v_i)$ in V_k, then $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$, and $v_i \in V_{ij}$, and there exists $\gamma \in \Gamma_j$ with $\rho_{jk}(\gamma) = \delta\, \gamma_{ijk}$ and $v_j = r_j(\gamma) \circ e_{ij}(v_i)$.

Suppose now that Y is a manifold, and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism in **dOrb**, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Man}}^{\mathbf{dOrb}}(Y)$. A *type A good coordinate system for* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ consists of a type A good coordinate system $\bigl(I, <, \ldots, \gamma_{ijk}\bigr)$ for $\boldsymbol{\mathcal{X}}$ as in (a)–(e) above, together with the following data satisfying conditions (f)–(g):

(f) For each $i \in I$, we are given a smooth map $g_i : V_i \to Y$ with $g_i \circ r_i(\gamma) = g_i$ for all $\gamma \in \Gamma_i$, so that Example 1.11.8 defines a quotient 1-morphism

$$[\boldsymbol{S}_{g_i,0}, \pi] : [\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i] \longrightarrow [\boldsymbol{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}},$$

where $\pi : \Gamma_i \to \{1\}$ is the projection. We are given a 2-morphism $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow [\boldsymbol{S}_{g_i,0}, \pi]$ in **dOrb**. Sometimes we require g_i to be a submersion.

(g) For all $i<j$ in I with $\boldsymbol{\mathcal{X}}_i\cap\boldsymbol{\mathcal{X}}_j\neq\emptyset$, we require that $g_j\circ e_{ij}=g_i|_{V_{ij}}$. This implies that

$$[\boldsymbol{S}_{g_j,0},\pi]\circ[\boldsymbol{S}_{e_{ij},\hat e_{ij}},\rho_{ij}]=[\boldsymbol{S}_{g_i,0},\pi]|_{[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]}:\\ [\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]\longrightarrow[\boldsymbol{S}_{Y,0,0}/\{1\}]=\boldsymbol{\mathcal{Y}}.$$

Here is the main result of §10.8, which is proved in Appendix D.

Theorem 1.11.27. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold. Then there exists a type A good coordinate system $\bigl(I,<,(V_i,E_i,\Gamma_i,s_i,\boldsymbol{\psi}_i),(V_{ij},e_{ij},\hat e_{ij},\rho_{ij},\boldsymbol{\eta}_{ij}),\gamma_{ijk}\bigr)$ for $\boldsymbol{\mathcal{X}}$. If $\boldsymbol{\mathcal{X}}$ is compact, we may take I to be finite. If $\{\boldsymbol{\mathcal{U}}_j:j\in J\}$ is an open cover of $\boldsymbol{\mathcal{X}}$, we may take $\boldsymbol{\mathcal{X}}_i=\boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i])\subseteq\boldsymbol{\mathcal{U}}_{j_i}$ for each $i\in I$ and some $j_i\in J$.*

Now let Y be a manifold and $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}=F_{\mathbf{Man}}^{\mathbf{dOrb}}(Y)$ a 1-morphism in **dOrb**. *Then all the above extends to type A good coordinate systems for $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$, and we may take the g_i in Definition* 1.11.26(f) *to be submersions.*

Section 10.8 also gives 'type B' versions of Definitions 1.11.24–1.11.26 and Theorem 1.11.27 using the standard model d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ and 1-morphisms $\boldsymbol{\mathcal{S}}_{e_{ij},\hat e_{ij}}$ of Examples 1.11.4 and 1.11.5 in place of $[\boldsymbol{S}_{V,E,s}/\Gamma]$ and $[\boldsymbol{S}_{e_{ij},\hat e_{ij}},\rho_{ij}]$ from Examples 1.11.7 and 1.11.8.

Observe that Definition 1.11.26 is similar to the hypotheses of Theorem 1.11.14. Given a good coordinate system $I,<,(V_i,E_i,\Gamma_i,s_i,\boldsymbol{\psi}_i),\ldots$ on $\boldsymbol{\mathcal{X}}$, Theorem 1.11.14 reconstructs $\boldsymbol{\mathcal{X}}$ up to equivalence in **dOrb** from the data $I,<,V_i,E_i,\Gamma_i,s_i,V_{ij},e_{ij},\hat e_{ij},\rho_{ij},\gamma_{ijk}$. Thus, we can regard Theorem 1.11.27 as a kind of converse to Theorem 1.11.14. Combining the two, we see that every d-orbifold $\boldsymbol{\mathcal{X}}$ can be described up to equivalence by a collection of differential-geometric data $I,<,V_i,\ldots,\gamma_{ijk}$. The 'type B' version of Theorem 1.11.27 is a kind of converse to Theorem 1.11.13.

Fukaya and Ono [34, §5] and Fukaya, Oh, Ohta and Ono [32, §A1] define *Kuranishi spaces*, the geometric structure they put on moduli spaces of J-holomorphic curves in symplectic geometry. We will argue in §14.3 that their definition is not really satisfactory, and that the 'right' way to define Kuranishi spaces is as d-orbifolds, or d-orbifolds with corners.

A *Kuranishi space* in [32, §A1] is a topological space X with a cover by 'Kuranishi neighbourhoods' (V,E,Γ,s,ψ), which are as in Definition 1.11.24 except that ψ is a homeomorphism with an open set in X, rather than an equivalence with an open d-suborbifold. On overlaps between (images of) Kuranishi neighbourhoods in X we are given 'coordinate changes', roughly as in Definition 1.11.25 except for the 2-morphisms $\boldsymbol{\eta}_{ij}$. Fukaya et al. define 'good coordinate systems' for Kuranishi spaces, roughly as in Definition 1.11.26. They state without proof in [32, Lem. A1.11] that good coordinate systems exist for any (compact) Kuranishi space, the analogue of Theorem 1.11.27.

Good coordinate systems are used in [32,34] in some kinds of proof involving Kuranishi spaces, in particular, in the construction of virtual classes and virtual chains. The proofs involve choosing data (such as a multi-valued perturbation of s_i) on each Kuranishi neighbourhood $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, by induction on i in I in the order $<$, where the data must satisfy compatibility conditions with coordinate changes $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij})$.

In fact we have already met the problem good coordinate systems are designed to solve in §1.11.6: in contrast to the d-manifold case, we do not have useful criteria for when a d-orbifold $\boldsymbol{\mathcal{X}}$ is principal. The parallel issue for Kuranishi spaces is that we cannot cover a general Kuranishi space $\boldsymbol{\mathcal{X}}$ with a single Kuranishi neighbourhood (V, E, Γ, s, ψ). So we cover (compact) $\boldsymbol{\mathcal{X}}$ with (finitely) many Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ with particularly well-behaved coordinate changes on overlaps, and then carry out the construction we want on each $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, compatibly with coordinate changes.

The material above will be used in §14.3 to explain the relations between d-orbifolds and Kuranishi spaces. As for Kuranishi spaces, it is also helpful for some proofs involving d-orbifolds, for instance, in constructing virtual classes for compact oriented d-orbifolds, and in studying d-orbifold bordism.

1.11.10 Semieffective and effective d-orbifolds

Section 10.9 defines *semieffective* and *effective* d-orbifolds, which are related to the notion of effective orbifold in Definition 1.9.4.

Definition 1.11.28. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. For $[x] \in \mathcal{X}_{\rm top}$, so that $x : \underline{\bar{*}} \to \mathcal{X}$ is a C^∞-stack 1-morphism, applying pullback x^* to (1.51) gives an exact sequence in $\mathrm{qcoh}(\underline{\bar{*}})$, where $K_{[x]} = \mathrm{Ker}(x^*(\phi_{\mathcal{X}}))$:

$$0 \longrightarrow K_{[x]} \longrightarrow x^*(\mathcal{E}_{\mathcal{X}}) \xrightarrow{x^*(\phi_{\mathcal{X}})} x^*(\mathcal{F}_{\mathcal{X}}) \xrightarrow{x^*(\psi_{\mathcal{X}})} x^*(T^*\mathcal{X}) \cong T_x^*\mathcal{X} \longrightarrow 0.$$

We may think of this as an exact sequence of real vector spaces, where $K_{[x]}, T_x^*\mathcal{X}$ are finite-dimensional with $\dim T_x^*\mathcal{X} - \dim K_{[x]} = \mathrm{vdim}\,\boldsymbol{\mathcal{X}}$.

The orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$ is the group of 2-morphisms $\eta : x \Rightarrow x$. Definition 1.8.22 defines isomorphisms $\eta^*(\mathcal{E}_{\mathcal{X}}) : x^*(\mathcal{E}_{\mathcal{X}}) \to x^*(\mathcal{E}_{\mathcal{X}})$ in $\mathrm{qcoh}(\underline{\bar{*}})$, which make $x^*(\mathcal{E}_{\mathcal{X}})$ into a representation of $\mathrm{Iso}_{\mathcal{X}}([x])$. The same holds for $x^*(\mathcal{F}_{\mathcal{X}}), x^*(T^*\mathcal{X})$, and $x^*(\phi_{\mathcal{X}}), x^*(\psi_{\mathcal{X}})$ are equivariant. Hence $K_{[x]}, T_x^*\mathcal{X}$ are also $\mathrm{Iso}_{\mathcal{X}}([x])$-representations.

We call $\boldsymbol{\mathcal{X}}$ a *semieffective d-orbifold* if $K_{[x]}$ is a trivial representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ for all $[x] \in \mathcal{X}_{\rm top}$. We call $\boldsymbol{\mathcal{X}}$ an *effective d-orbifold* if it is semieffective, and $T_x^*\mathcal{X}$ is an effective representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ for all $[x] \in \mathcal{X}_{\rm top}$.

That is, $\boldsymbol{\mathcal{X}}$ is semieffective if the orbifold groups $\mathrm{Iso}_{\mathcal{X}}([x])$ act trivially on the obstruction spaces of $\boldsymbol{\mathcal{X}}$, and effective if the $\mathrm{Iso}_{\mathcal{X}}([x])$ also act effectively on the tangent spaces of $\boldsymbol{\mathcal{X}}$. One useful property of (semi)effective d-orbifolds is that generic perturbations of semieffective (or effective) d-orbifolds are (effective) orbifolds. We state this for 'standard model' d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$.

Proposition 1.11.29. *Let $\mathcal{V}$ be an orbifold, $\mathcal{E}$ a vector bundle on $\mathcal{V}$, and $s \in C^\infty(\mathcal{E})$, and let $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ be as in Example* 1.11.4. *Suppose $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is a semieffective d-orbifold. Then for any generic perturbation $\tilde{s}$ of s in $C^\infty(\mathcal{E})$ with $\tilde{s} - s$*

sufficiently small in C^1 locally on $\mathcal{V}$, the d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde{s}}$ is an orbifold, that is, it lies in $\hat{\mathbf{Orb}} \subset \mathbf{dOrb}$. If $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is an effective d-orbifold, then $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde{s}}$ is an effective orbifold.

The proof of results on (semi)effective d-orbifold bordism in §1.15 involves an analogue of Proposition 1.11.29 for general d-orbifolds $\boldsymbol{\mathcal{X}}$, proved using good coordinate systems as in §1.11.9. Here are some other good properties of (semi)effective d-orbifolds:

- If $\mathcal{X}$ is an orbifold then $\boldsymbol{\mathcal{X}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{X})$ is a semieffective d-orbifold, and if $\mathcal{X}$ is effective then $\boldsymbol{\mathcal{X}}$ is effective.
- Let $\boldsymbol{\mathcal{X}}$ be a semieffective d-orbifold, Γ a finite group, and $\lambda \in \Lambda^\Gamma$. Then the orbifold stratum $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda} = \boldsymbol{\emptyset}$ unless $\lambda \in \Lambda_+^\Gamma \subset \Lambda^\Gamma$. If $\boldsymbol{\mathcal{X}}$ is effective then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda} = \boldsymbol{\emptyset}$ unless $\lambda = [R]$ for R an effective Γ-representation.
- If $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ are (semi)effective d-orbifolds, then the product $\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}$ is also (semi)effective. More generally, any fibre product $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{\mathcal{Z}}} \boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb}$ with $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ (semi)effective and $\boldsymbol{\mathcal{Z}}$ a manifold is also (semi)effective.
- Proposition 1.11.23 says that if $\boldsymbol{\mathcal{X}}$ is an oriented d-orbifold, then when $|\Gamma|$ is odd we can define orientations on the orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \boldsymbol{\mathcal{X}}_\circ^{\Gamma,\lambda}$, and under extra conditions on μ we can also orient $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$.

  For general d-orbifolds $\boldsymbol{\mathcal{X}}$, this is the best we can do. But for semieffective d-orbifolds $\boldsymbol{\mathcal{X}}$ the analogue of Proposition 1.9.9 for orbifolds holds. This is stronger, as it orients $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \ldots, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ under weaker conditions on Γ, λ, μ, which allow $|\Gamma|$ even for some λ, μ.

1.12 Orbifolds with corners

Sections 8.5–8.9 discuss 2-categories $\mathbf{Orb^b}$ and $\mathbf{Orb^c}$ of *orbifolds with boundary* and *orbifolds with corners*, which are orbifold versions of manifolds with boundary and corners in §1.5. This is new material, and the author knows of no other foundational work on orbifolds with corners.

1.12.1 The definition of orbifolds with corners

Definition 1.12.1. An *orbifold with corners* $\mathfrak{X}$ of dimension $n \geqslant 0$ is a triple $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ with $\mathcal{X}, \partial\mathcal{X}$ separated, second countable Deligne–Mumford C^∞-stacks, and $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ a proper, strongly representable 1-morphism of C^∞-stacks, in the sense of §1.8.3, such that for each $[x] \in \mathcal{X}_{\rm top}$ there exists a 2-Cartesian diagram in $\mathbf{C^\infty Sta}$:

$$\begin{tikzcd}
\underline{\partial \bar{U}} \arrow[rr, "u_\partial"'] \arrow[d, "\bar{\imath}_U"] & \arrow[d, phantom, "\mathrm{id}\Uparrow"] & \partial\mathcal{X} \arrow[d, "i_{\mathfrak{X}}"'] \\
\underline{\bar{U}} \arrow[rr, "u"] & & \mathcal{X}.
\end{tikzcd}$$

Here U is an n-manifold with corners, so that $i_U : \partial U \to U$ is smooth, and $\underline{U}, \underline{\partial U}, \underline{i}_U = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U, \partial U, i_U)$, and u, u_∂ are étale 1-morphisms, and

$u_{\rm top}([p]) = [x]$ for some $p \in U$. We call $\mathfrak{X}$ an *orbifold with boundary*, or an *orbifold without boundary*, if the above condition holds with U a manifold with boundary, or a manifold without boundary, respectively, for each $[x] \in \mathcal{X}_{\rm top}$.

Now suppose $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ and $\mathfrak{Y} = (\mathcal{Y}, \partial\mathcal{Y}, i_{\mathfrak{Y}})$ are orbifolds with corners. A 1-*morphism* $f : \mathfrak{X} \to \mathfrak{Y}$, or *smooth map*, is a 1-morphism of C^∞-stacks $f : \mathcal{X} \to \mathcal{Y}$ such that for each $[x] \in \mathcal{X}_{\rm top}$ with $f_{\rm top}([x]) = [y] \in \mathcal{Y}_{\rm top}$ there exists a 2-commutative diagram in $\mathbf{C^\infty Sta}$:

$$\begin{tikzcd}
\underline{U} \arrow[rr, "u"] \arrow[d, "\underline{h}"'] & \arrow[d, phantom, "\Downarrow \eta"] & \mathcal{X} \arrow[d, "f"] \\
\underline{V} \arrow[rr, "v"] & & \mathcal{Y}.
\end{tikzcd}$$

Here U, V are manifolds with corners, $h : U \to V$ is a smooth map, $\underline{U}, \underline{V}, \underline{h} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U, V, h)$, and u, v are étale, and $u_{\rm top}([p]) = [x]$ for some $p \in U$.

Let $f, g : \mathfrak{X} \to \mathfrak{Y}$ be 1-morphisms of orbifolds with corners. A 2-*morphism* $\eta : f \Rightarrow g$ is a 2-morphism of 1-morphisms $f, g : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{C^\infty Sta}$.

Composition of 1-*morphisms* $g \circ f$, *identity* 1-*morphisms* $\mathrm{id}_{\mathfrak{X}}$, *vertical* and *horizontal composition of* 2-*morphisms* $\zeta \odot \eta$, $\zeta * \eta$, and *identity* 2-*morphisms* for orbifolds with corners, are all given by the corresponding compositions and identities in $\mathbf{C^\infty Sta}$. This defines the 2-category $\mathbf{Orb^c}$ of orbifolds with corners. Write $\mathbf{Orb^b}$ and $\dot{\mathbf{O}}\mathbf{rb}$ for the full 2-subcategories of orbifolds with boundary, and orbifolds without boundary, in $\mathbf{Orb^c}$.

If $\mathcal{X}$ is an orbifold in the sense of Definition 1.9.1, then $\mathfrak{X} = (\mathcal{X}, \emptyset, \emptyset)$ is an orbifold without boundary in this sense, and vice versa. Thus the 2-functor $F_{\mathbf{Orb}}^{\mathbf{Orb^c}} : \mathbf{Orb} \to \mathbf{Orb^c}$ mapping $\mathcal{X} \mapsto \mathfrak{X} = (\mathcal{X}, \emptyset, \emptyset)$ on objects, $f \mapsto f$ on 1-morphisms, and $\eta \mapsto \eta$ on 2-morphisms, is an isomorphism of 2-categories $\mathbf{Orb} \to \dot{\mathbf{O}}\mathbf{rb}$.

Define $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}} : \mathbf{Man^c} \to \mathbf{Orb^c}$ by $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}} : X \mapsto \mathfrak{X} = (\underline{X}, \underline{\partial X}, \underline{i}_X)$ on objects X in $\mathbf{Man^c}$, where $\underline{X}, \underline{\partial X}, \underline{i}_X = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(X, \partial X, i_X)$, and $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}} : f \mapsto \underline{f}$ on morphisms $f : X \to Y$ in $\mathbf{Man^c}$, where $\underline{f} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(f)$. Then $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}}$ is a full and faithful strict 2-functor.

Let $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ be an orbifold with corners, and $\mathcal{V} \subseteq \mathcal{X}$ an open C^∞-substack. Define $\partial\mathcal{V} = i_{\mathfrak{X}}^{-1}(\mathcal{V})$, as an open C^∞-substack of $\partial\mathcal{X}$, and $i_{\mathcal{V}} : \partial\mathcal{V} \to \mathcal{V}$ by $i_{\mathcal{V}} = i_{\mathfrak{X}}|_{\partial\mathcal{V}}$. Then $\mathfrak{V} = (\mathcal{V}, \partial\mathcal{V}, i_{\mathcal{V}})$ is an orbifold with corners. We call $\mathfrak{V}$ an *open suborbifold* of $\mathfrak{X}$. An *open cover* of $\mathfrak{X}$ is a family $\{\mathfrak{V}_a : a \in A\}$ of open suborbifolds $\mathfrak{V}_a$ of $\mathfrak{X}$ with $\mathcal{X} = \bigcup_{a \in A} \mathcal{V}_a$.

Example 1.12.2. Suppose X is a manifold with corners, G a finite group, and $r : G \to \mathrm{Aut}(X)$ an action of G on X by diffeomorphisms. Since $r(\gamma) : X \to X$ is simple for each $\gamma \in G$, as in §1.5.2 we have $r_-(\gamma) : \partial X \to \partial X$, which is also a diffeomorphism. Then $r_- : G \to \mathrm{Aut}(\partial X)$ is an action of G on ∂X, and $i_X : \partial X \to X$ is G-equivariant. Set $\underline{X}, \underline{\partial X}, \underline{i}_X, \underline{r}, \underline{r}_- = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(X, \partial X, i_X, r, r_-)$. Then $\underline{X}, \underline{\partial X}$ are C^∞-schemes with G-actions $\underline{r}, \underline{r}_-$, and $\underline{i}_X : \underline{\partial X} \to \underline{X}$ is G-equivariant, so Examples 1.8.11 and 1.8.12 define Deligne–Mumford C^∞-stacks $[\underline{X}/G], [\underline{\partial X}/G]$ and a 1-morphism $[\underline{i}_X, \mathrm{id}_G] : [\underline{\partial X}/G] \to [\underline{X}/G]$, which turns out to be strongly representable. One can show that $\mathfrak{X} = \bigl([\underline{X}/G], [\underline{\partial X}/G], [\underline{i}_X, \mathrm{id}_G]\bigr)$ is an orbifold with corners, which we will write as $[X/G]$.

Remark 1.12.3. (a) We could have defined $\mathbf{Orb^c}$ equivalently and more simply as a (non-full) 2-subcategory of $\mathbf{DMC^\infty Sta}$, so that an orbifold with corners would be a C^∞-stack $\mathcal{X}$ rather than a triple $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$. We chose the set-up of Definition 1.12.1 partly for its compatibility with the definitions of d-stacks and d-orbifolds with corners $\boldsymbol{\mathfrak{X}} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i_{\mathfrak{X}}}, \omega_{\mathfrak{X}})$ in §1.13–§1.14, and partly because, to make several important constructions more functorial, it is useful to have a particular choice of boundary $\partial\mathcal{X}$ for $\mathcal{X}$ already made.

(b) In Remark 1.6.5 we noted that boundaries in $\mathbf{dSpa^c}$ are strictly functorial. One sign of this is that for a semisimple 1-morphism $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dSpa^c}$, the 1-morphism $\boldsymbol{f}_- : \partial_-^{\boldsymbol{f}}\mathbf{X} \to \partial\mathbf{Y}$ is unique, not just unique up to 2-isomorphism, with an equality of 1-morphisms $\boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}} = \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{f}_-$, not just a 2-isomorphism. By the general philosophy of 2-categories, this may seem unnatural.

We will arrange that boundaries in $\mathbf{Orb^c}$ and also in $\mathbf{dSta^c}, \mathbf{dOrb^c}$ are strictly functorial in the same way. This is our reason for taking $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ in Definition 1.12.1 to be *strongly representable*, in the sense of §1.8.3. Proposition 1.8.8(b) shows that this is no real restriction: $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is naturally representable, and we can make it strongly representable by replacing $\partial\mathcal{X}$ by an equivalent C^∞-stack. Then Proposition 1.8.9 applied to $i_{\mathfrak{Y}} : \partial\mathcal{Y} \to \mathcal{Y}$ is what we need to show that a semisimple 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{dOrb^c}$ lifts to a unique 1-morphism $f_- : \partial_-^f\mathfrak{X} \to \partial\mathfrak{Y}$ with $f \circ i_{\mathfrak{X}}|_{\partial_-^f\mathfrak{X}} = i_{\mathfrak{Y}} \circ f_-$.

(c) An orbifold with corners $\mathfrak{X}$ of dimension n is locally modelled near each point $[x] \in \mathcal{X}_{\rm top}$ on $\bigl([0,\infty)^k \times \mathbb{R}^{n-k}\bigr)/G$ near 0, where G is a finite group acting linearly on $\mathbb{R}^n$ preserving the subset $[0,\infty)^k \times \mathbb{R}^{n-k}$. Note that G is allowed to permute the coordinates $x_1, \ldots, x_k$ in $[0,\infty)^k$. So, for example, we allow 2-dimensional orbifolds with corners modelled on $[0,\infty)^2/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \langle\sigma\rangle$ acts on $[0,\infty)^2$ by $\sigma : (x_1, x_2) \mapsto (x_2, x_1)$.

This implies that the 1-morphism $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ induces morphisms of orbifold groups $(i_{\mathfrak{X}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ which are injective (so that $i_{\mathfrak{X}}$ is representable), but need not be isomorphisms. We will call an orbifold with corners $\mathfrak{X}$ *straight* if the morphisms $(i_{\mathfrak{X}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ are isomorphisms for all $[x'] \in \partial\mathcal{X}_{\rm top}$ with $i_{\mathfrak{X},\rm top}([x']) = [x]$. That is, straight orbifolds with corners are locally modelled on $[0,\infty)^k \times (\mathbb{R}^{n-k}/G)$. Orbifolds with boundary, with $k = 0$ or 1, are automatically straight. Boundaries of orbifold strata behave better for straight orbifolds with corners.

In §1.9.1 we explained that a vector bundle $\mathcal{E}$ on an orbifold $\mathcal{X}$ is a vector bundle on $\mathcal{X}$ as a Deligne–Mumford C^∞-stack, in the sense of §1.8.6. But sometimes it is convenient to regard $\mathcal{E}$ as an orbifold in its own right, so we define a 'total space functor' mapping vector bundles $\mathcal{E}$ to orbifolds $\mathrm{Tot}(\mathcal{E})$.

In the same way, if $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ is an orbifold with corners, in §8.5 we define a *vector bundle* $\mathcal{E}$ *on* $\mathfrak{X}$ to be a vector bundle on $\mathcal{X}$ as a Deligne–Mumford C^∞-stack. To regard $\mathcal{E}$ as an orbifold with corners in its own right, we define a 'total space functor' $\mathrm{Tot^c} : \mathrm{vect}(\mathcal{X}) \to \mathbf{Orb^c}$, which maps a vector bundle $\mathcal{E}$ on $\mathfrak{X}$ to an orbifold with corners $\mathrm{Tot^c}(\mathcal{E})$, and maps a section $s \in C^\infty(\mathcal{E})$ to a simple, flat 1-morphism $\mathrm{Tot^c}(s) : \mathfrak{X} \to \mathrm{Tot^c}(\mathcal{E})$ in $\mathbf{Orb^c}$.

Definition 1.12.4. An orbifold with corners $\mathfrak{X}$ is called *effective* if $\mathfrak{X}$ is locally modelled near each $[x] \in \mathcal{X}_{\mathrm{top}}$ on $([0,\infty)^k \times \mathbb{R}^{n-k})/G$, where G acts effectively on $\mathbb{R}^n$ preserving $[0,\infty)^k \times \mathbb{R}^{n-k}$, that is, every $1 \neq \gamma \in G$ acts nontrivially.

The analogue of Proposition 1.9.5 holds for effective orbifolds with corners.

1.12.2 Boundaries of orbifolds with corners, and simple, semisimple and flat 1-morphisms

Section 8.6 defines boundaries of orbifolds with corners.

Definition 1.12.5. Let $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ be an orbifold with corners. We will define an orbifold with corners $\partial\mathfrak{X} = (\partial\mathcal{X}, \partial^2\mathcal{X}, i_{\partial\mathfrak{X}})$, called the *boundary* of $\mathfrak{X}$, such that $i_{\mathfrak{X}} : \partial\mathfrak{X} \to \mathfrak{X}$ is a 1-morphism in $\mathbf{Orb^c}$. Here $\partial\mathcal{X}$ and $i_{\mathfrak{X}}$ are given in $\mathfrak{X}$, so the new data we have to construct is $\partial^2\mathcal{X}, i_{\partial\mathfrak{X}}$.

As $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is strongly representable by Definition 1.12.1, Proposition 1.8.10 defines an explicit fibre product $\partial\mathcal{X} \times_{i_{\mathfrak{X}},\mathcal{X},i_{\mathfrak{X}}} \partial\mathcal{X}$ with strongly representable projection morphisms $\pi_1, \pi_2 : \partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X} \to \partial\mathcal{X}$ such that $i_{\mathfrak{X}} \circ \pi_1 = i_{\mathfrak{X}} \circ \pi_2$. We will use this explicit fibre product throughout. There is a unique diagonal 1-morphism $\Delta_{\partial\mathcal{X}} : \partial\mathcal{X} \to \partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X}$ with $\pi_1 \circ \Delta_{\partial\mathcal{X}} = \pi_2 \circ \Delta_{\partial\mathcal{X}} = \mathrm{id}_{\partial\mathcal{X}}$. It is an equivalence with an open and closed C^∞-substack $\Delta_{\partial\mathcal{X}}(\partial\mathcal{X}) \subseteq \partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X}$. Define $\partial^2\mathcal{X} = \partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X} \setminus \Delta_{\partial\mathcal{X}}(\partial\mathcal{X})$. Then $\partial^2\mathcal{X}$ is also an open and closed C^∞-substack in $\partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X}$. Define $i_{\partial\mathfrak{X}} = \pi_1|_{\partial^2\mathcal{X}} : \partial^2\mathcal{X} \to \partial\mathcal{X}$. Then $\partial\mathfrak{X} = (\partial\mathcal{X}, \partial^2\mathcal{X}, i_{\partial\mathfrak{X}})$ is an orbifold with corners, with $\dim(\partial\mathfrak{X}) = \dim\mathfrak{X} - 1$. Also $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ in $\mathfrak{X}$ is a 1-morphism $i_{\mathfrak{X}} : \partial\mathfrak{X} \to \mathfrak{X}$ in $\mathbf{Orb^c}$.

Here is the orbifold analogue of parts of §1.5.1–§1.5.2.

Definition 1.12.6. Let $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ and $\mathfrak{Y} = (\mathcal{Y}, \partial\mathcal{Y}, i_{\mathfrak{Y}})$ be orbifolds with corners, and $f : \mathfrak{X} \to \mathfrak{Y}$ a 1-morphism in $\mathbf{Orb^c}$. Consider the C^∞-stack fibre products $\partial\mathcal{X} \times_{f\circ i_{\mathfrak{X}},\mathcal{Y},i_{\mathfrak{Y}}} \partial\mathcal{Y}$ and $\mathcal{X} \times_{f,\mathcal{Y},i_{\mathfrak{Y}}} \partial\mathcal{Y}$. Since $i_{\mathfrak{Y}}$ is strongly representable, we may define these using the explicit construction of Proposition 1.8.10.

The topological space $(\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\mathrm{top}}$ associated to the C^∞-stack $\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$ may be written explicitly as

$$(\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\mathrm{top}} \cong \bigl\{[x', y'] : x' : \underline{\bar{*}} \to \partial\mathcal{X} \text{ and } y' : \underline{\bar{*}} \to \partial\mathcal{Y} \text{ are } \\ \text{1-morphisms with } f \circ i_{\mathfrak{X}} \circ x' = i_{\mathfrak{Y}} \circ y' : \underline{\bar{*}} \to \mathcal{Y}\bigr\}, \tag{1.71}$$

where $[x', y']$ in (1.71) denotes the $\sim$-equivalence class of pairs (x', y'), with $(x', y') \sim (\tilde{x}', \tilde{y}')$ if there exist 2-morphisms $\eta : x' \Rightarrow \tilde{x}'$ and $\zeta : y' \Rightarrow \tilde{y}'$ with $\mathrm{id}_{f\circ i_{\mathfrak{X}}} * \eta = \mathrm{id}_{i_{\mathfrak{Y}}} * \zeta$. There is a natural open and closed C^∞-substack $\mathcal{S}_f \subseteq \partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$, the analogue of S_f in §1.5.1, such that $[x', y']$ in (1.71) lies in $\mathcal{S}_{f,\mathrm{top}}$ if and only if we can complete the following commutative diagram in qcoh($\bar{*}$) with morphisms '$\dashrightarrow$' as shown:

$$
\begin{tikzcd}
0 \arrow[r] & (y')^*(\mathcal{N}_{\mathcal{Y}}) \arrow[r, "(y')^*(\nu_{\mathcal{Y}})"] \arrow[d, dashed, "\cong"'] & (y')^* \circ i_{\mathcal{Y}}^*(T^*\mathcal{Y}) \arrow[r, "(y')^*(\Omega_{i_{\mathcal{Y}}})"] \arrow[d, "\begin{subarray}{l} I_{x',i_{\mathcal{X}}}(T^*\mathcal{X})\circ(i_{\mathcal{X}}\circ x')^*(\Omega_f)\circ \\ I_{i_{\mathcal{X}}\circ x',f}(T^*\mathcal{Y})\circ I_{y',i_{\mathcal{Y}}}(T^*\mathcal{Y})^{-1}\end{subarray}"] & (y')^*(T^*(\partial\mathcal{Y})) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & (x')^*(\mathcal{N}_{\mathcal{X}}) \arrow[r, "(x')^*(\nu_{\mathcal{X}})"'] & (x')^* \circ i_{\mathcal{X}}^*(T^*\mathcal{X}) \arrow[r, "(x')^*(\Omega_{i_{\mathcal{X}}})"'] & (x')^*(T^*(\partial\mathcal{X})) \arrow[r] & 0,
\end{tikzcd}
$$

where $\mathcal{N}_{\mathcal{X}}, \mathcal{N}_{\mathcal{Y}}$ are the *conormal line bundles* of $\partial\mathcal{X}, \partial\mathcal{Y}$ in $\mathcal{X}, \mathcal{Y}$.

Similarly, the topological space $(\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\rm top}$ may be written explicitly as

$$
\begin{aligned}
(\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\rm top} \cong \bigl\{[x,y'] : x : \bar{*} \to \mathcal{X} \text{ and } y' : \bar{*} \to \partial\mathcal{Y} \text{ are}& \\
\text{1-morphisms with } f \circ x = i_{\mathcal{Y}} \circ y' : \bar{*} \to \mathcal{Y}\bigr\}&,
\end{aligned}
\tag{1.72}
$$

where $[x,y']$ in (1.72) denotes the $\approx$-equivalence class of (x,y'), with $(x,y') \approx (\tilde{x},\tilde{y}')$ if there exist $\eta : x \Rightarrow \tilde{x}$ and $\zeta : y' \Rightarrow \tilde{y}'$ with $\mathrm{id}_f * \eta = \mathrm{id}_{i_{\mathcal{Y}}} * \zeta$. There is a natural open and closed C^∞-substack $\mathcal{T}_f \subseteq \mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$, the analogue of T_f in §1.5.1, such that $[x,y']$ in (1.72) lies in $\mathcal{T}_{f,\rm top}$ if and only if we can complete the following commutative diagram in qcoh($\bar{*}$):

$$
\begin{tikzcd}
0 \arrow[r] & (y')^*(\mathcal{N}_{\mathcal{Y}}) \arrow[r, "(y')^*(\nu_{\mathcal{Y}})"'] & (y')^* \circ i_{\mathcal{Y}}^*(T^*\mathcal{Y}) \arrow[r, "(y')^*(\Omega_{i_{\mathcal{Y}}})"'] \arrow[d, "x^*(\Omega_f)\circ I_{x,f}(T^*\mathcal{Y})\circ I_{y',i_{\mathcal{Y}}}(T^*\mathcal{Y})^{-1}"'] & (y')^*(T^*(\partial\mathcal{Y})) \arrow[r] \arrow[dl, dotted] & 0 \\
& & x^*(T^*\mathcal{X}). & &
\end{tikzcd}
$$

Define $s_f = \pi_{\partial\mathcal{X}}|_{\mathcal{S}_f} : \mathcal{S}_f \to \partial\mathcal{X}$, $u_f = \pi_{\partial\mathcal{Y}}|_{\mathcal{S}_f} : \mathcal{S}_f \to \partial\mathcal{Y}$, $t_f = \pi_{\mathcal{X}}|_{\mathcal{T}_f} : \mathcal{T}_f \to \mathcal{X}$, and $v_f = \pi_{\partial\mathcal{Y}}|_{\mathcal{T}_f} : \mathcal{T}_f \to \partial\mathcal{Y}$. Then s_f, t_f are proper, étale 1-morphisms. We call f *simple* if $s_f : \mathcal{S}_f \to \partial\mathcal{X}$ is an equivalence, and we call f *semisimple* if $s_f : \mathcal{S}_f \to \partial\mathcal{X}$ is injective as a 1-morphism of Deligne–Mumford C^∞-stacks, and we call f *flat* if $\mathcal{T}_f = \emptyset$. Simple implies semisimple.

The condition that $i_{\mathfrak{X}}$ is strongly representable in Definition 1.12.1 is essential in constructing f_-, η_- in parts (b),(c) of the next theorem.

Theorem 1.12.7. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a semisimple 1-morphism of orbifolds with corners. Then there is a natural decomposition $\partial\mathfrak{X} = \partial_+^f\mathfrak{X} \amalg \partial_-^f\mathfrak{X}$, where $\partial_\pm^f\mathfrak{X}$ are open and closed suborbifolds in $\partial\mathfrak{X}$, such that:*

(a) *Define $f_+ = f \circ i_{\mathfrak{X}}|_{\partial_+^f\mathfrak{X}} : \partial_+^f\mathfrak{X} \to \mathfrak{Y}$. Then f_+ is semisimple. If f is flat then f_+ is also flat.*

(b) *There exists a unique, semisimple 1-morphism $f_- : \partial_-^f\mathfrak{X} \to \partial\mathfrak{Y}$ in* $\mathbf{Orb^c}$ *with $f \circ i_{\mathfrak{X}}|_{\partial_-^f\mathfrak{X}} = i_{\mathfrak{Y}} \circ f_-$. If f is simple then $\partial_+^f\mathfrak{X} = \emptyset$, $\partial_-^f\mathfrak{X} = \partial\mathfrak{X}$ and $f_- : \partial\mathfrak{X} \to \partial\mathfrak{Y}$ is simple. If f is flat then f_- is flat.*

(c) *Let $g : \mathfrak{X} \to \mathfrak{Y}$ be another 1-morphism and $\eta : f \Rightarrow g$ a 2-morphism in* $\mathbf{Orb^c}$. *Then g is also semisimple, with $\partial_-^g\mathfrak{X} = \partial_-^f\mathfrak{X}$. If f is simple, or flat, then g is too. Part* **(b)** *defines 1-morphisms $f_-, g_- : \partial_-^f\mathfrak{X} \to \partial\mathfrak{Y}$. There is a unique 2-morphism $\eta_- : f_- \Rightarrow g_-$ in* $\mathbf{Orb^c}$ *such that*

$$\mathrm{id}_{i_{\mathcal{Y}}} * \eta_- = \eta * \mathrm{id}_{i_{\mathcal{X}}|_{\partial^f_- \mathcal{X}}} : f \circ i_{\mathcal{X}}|_{\partial^f_- \mathcal{X}} = i_{\mathcal{Y}} \circ f_- \Longrightarrow g \circ i_{\mathcal{X}}|_{\partial^f_- \mathcal{X}} = i_{\mathcal{Y}} \circ g_-.$$

1.12.3 Corners $C_k(\mathfrak{X})$ and the corner functors $C, \hat{C}$

Section 8.7 extends §1.5.3 to orbifolds. Here is the orbifold analogue of the category $\check{\mathbf{Man}}^{\mathbf{c}}$ in Definition 1.5.10.

Definition 1.12.8. We will define a 2-category $\check{\mathbf{Orb}}^{\mathbf{c}}$ whose objects are disjoint unions $\coprod_{m=0}^{\infty} \mathfrak{X}_m$, where $\mathfrak{X}_m$ is a (possibly empty) orbifold with corners of dimension m. In more detail, objects of $\check{\mathbf{Orb}}^{\mathbf{c}}$ are triples $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathcal{X}})$ with $i_{\mathcal{X}} : \partial\mathcal{X} \to \mathcal{X}$ a strongly representable 1-morphism of Deligne–Mumford C^∞-stacks, such that there exists a decomposition $\mathcal{X} = \coprod_{m=0}^{\infty} \mathcal{X}_m$ with each $\mathcal{X}_m \subseteq \mathcal{X}$ an open and closed C^∞-substack, for which $\mathfrak{X}_m := \bigl(\mathcal{X}_m, i_{\mathcal{X}}^{-1}(\mathcal{X}_m), i_{\mathcal{X}}|_{i_{\mathcal{X}}^{-1}(\mathcal{X}_m)}\bigr)$ is an orbifold with corners of dimension m.

A 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in $\check{\mathbf{Orb}}^{\mathbf{c}}$ is a 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{C^\infty Sta}$ such that $f|_{\mathcal{X}_m \cap f^{-1}(\mathcal{Y}_n)} : \bigl(\mathcal{X}_m \cap f^{-1}(\mathcal{Y}_n)\bigr) \to \mathcal{Y}_n$ is a 1-morphism in $\mathbf{Orb^c}$ for all $m, n \geqslant 0$. For 1-morphisms $f, g : \mathfrak{X} \to \mathfrak{Y}$, a 2-morphism $\eta : f \Rightarrow g$ is a 2-morphism $\eta : f \Rightarrow g$ in $\mathbf{C^\infty Sta}$. Then $\mathbf{Orb^c}$ is a full 2-subcategory of $\check{\mathbf{Orb}}^{\mathbf{c}}$.

The next theorem summarizes our results on corners functors in $\mathbf{Orb^c}$.

Theorem 1.12.9. (a) *Let $\mathfrak{X}$ be an orbifold with corners. Then for each $k = 0, 1, \ldots, \dim \mathfrak{X}$ we can define an orbifold with corners $C_k(\mathfrak{X})$ of dimension $\dim \mathfrak{X} - k$ called the k-**corners** of $\mathfrak{X}$, and a 1-morphism $\Pi_{\mathfrak{X}}^k : C_k(\mathfrak{X}) \to \mathfrak{X}$ in $\mathbf{Orb^c}$. It has topological space*

$$\begin{aligned} C_k(\mathcal{X})_{\mathrm{top}} \cong \bigl\{[x, \{x'_1, \ldots, x'_k\}] : x : \bar{\underline{*}} \to \mathcal{X},\ x'_i : \bar{\underline{*}} \to \partial\mathcal{X} \text{ are 1-morphisms} \\ \text{with } x'_1, \ldots, x'_k \text{ distinct and } x = i_{\mathfrak{X}} \circ x'_1 = \cdots = i_{\mathfrak{X}} \circ x'_k\bigr\}. \end{aligned} \tag{1.73}$$

There is a natural action of the symmetric group S_k on $\partial^k \mathfrak{X}$ by 1-isomorphisms, and an equivalence $C_k(\mathfrak{X}) \simeq \partial^k \mathfrak{X}/S_k$. We have 1-isomorphisms $C_0(\mathfrak{X}) \cong \mathfrak{X}$ and $C_1(\mathfrak{X}) \cong \partial\mathfrak{X}$ in $\mathbf{Orb^c}$. Write $C(\mathfrak{X}) = \coprod_{k=0}^{\dim \mathfrak{X}} C_k(\mathfrak{X})$ and $\Pi_{\mathfrak{X}} = \coprod_{k=0}^{\dim \mathfrak{X}} \Pi_{\mathfrak{X}}^k$, so that $C(\mathfrak{X})$ is an object and $\Pi_{\mathfrak{X}} : C(\mathfrak{X}) \to \mathfrak{X}$ a 1-morphism in $\check{\mathbf{Orb}}^{\mathbf{c}}$.

(b) *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a 1-morphism of orbifolds with corners. Then there is a unique 1-morphism $C(f) : C(\mathfrak{X}) \to C(\mathfrak{Y})$ in $\check{\mathbf{Orb}}^{\mathbf{c}}$ such that $\Pi_{\mathfrak{Y}} \circ C(f) = f \circ \Pi_{\mathfrak{X}} : C(\mathfrak{X}) \to \mathfrak{Y}$, and $C(f)$ acts on points as in (1.73) by*

$$\begin{aligned} C(f)_{\mathrm{top}} : \bigl[x, \{x'_1, \ldots, x'_k\}\bigr] \longmapsto \bigl[y, \{y'_1, \ldots, y'_l\}\bigr], \quad \text{where } y = f \circ x, \\ \text{and} \quad \{y'_1, \ldots, y'_l\} = \bigl\{y' : [x'_i, y'] \in \mathcal{S}_{f,\mathrm{top}},\ \text{some } i = 1, \ldots, k\bigr\}, \end{aligned} \tag{1.74}$$

where $\mathcal{S}_f$ is as in Definition 1.12.6.

For all $k, l \geqslant 0$, write $C_k^{f,l}(\mathfrak{X}) = C_k(\mathfrak{X}) \cap C(f)^{-1}(C_l(\mathfrak{Y}))$, so that $C_k^{f,l}(\mathfrak{X})$ is open and closed in $C_k(\mathfrak{X})$ with $C_k(\mathfrak{X}) = \coprod_{l=0}^{\dim \mathfrak{Y}} C_k^{f,l}(\mathfrak{X})$, and write $C_k^l(f) = C(f)|_{C_k^{f,l}(\mathfrak{X})}$, so that $C_k^l(f) : C_k^{f,l}(\mathfrak{X}) \to C_l(\mathfrak{Y})$ is a 1-morphism in $\mathbf{Orb^c}$.

(c) *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms and $\eta : f \Rightarrow g$ a 2-morphism in* $\mathbf{Orb^c}$*. Then there exists a unique 2-morphism $C(\eta) : C(f) \Rightarrow C(g)$ in* $\check{\mathbf{O}}\mathbf{rb^c}$*, where $C(f), C(g)$ are as in* **(b)***, such that*

$$\mathrm{id}_{\Pi_{\mathcal{Y}}} * C(\eta) = \eta * \mathrm{id}_{\Pi_{\mathcal{X}}} : \Pi_{\mathcal{Y}} \circ C(f) = f \circ \Pi_{\mathcal{X}} \Longrightarrow \Pi_{\mathcal{Y}} \circ C(g) = g \circ \Pi_{\mathcal{X}}.$$

(d) *Define $C : \mathbf{Orb^c} \to \check{\mathbf{O}}\mathbf{rb^c}$ by $C : \mathcal{X} \mapsto C(\mathcal{X})$ on objects, $C : f \mapsto C(f)$ on 1-morphisms, and $C : \eta \mapsto C(\eta)$ on 2-morphisms, where $C(\mathcal{X}), C(f), C(\eta)$ are as in* **(a)**–**(c)** *above. Then C is a strict 2-functor, called a* ***corner functor****.*

(e) *Let $f : \mathcal{X} \to \mathcal{Y}$ be semisimple. Then $C(f)$ maps $C_k(\mathcal{X}) \to \coprod_{l=0}^k C_l(\mathcal{Y})$ for all $k \geqslant 0$. The natural 1-isomorphisms $C_1(\mathcal{X}) \cong \partial\mathcal{X}$, $C_0(\mathcal{Y}) \cong \mathcal{Y}$, $C_1(\mathcal{Y}) \cong \partial\mathcal{Y}$ identify $C_1^{f,0}(\mathcal{X}) \cong \partial_+^f \mathcal{X}$, $C_1^{f,1}(\mathcal{X}) \cong \partial_-^f \mathcal{X}$, $C_1^0(f) \cong f_+$ and $C_1^1(f) \cong f_-$.*

If f is simple then $C(f)$ maps $C_k(\mathcal{X}) \to C_k(\mathcal{Y})$ for all $k \geqslant 0$.

(f) *Analogues of* **(b)**–**(d)** *also hold for a second corner functor $\hat{C} : \mathbf{Orb^c} \to \check{\mathbf{O}}\mathbf{rb^c}$, which acts on objects by $\hat{C} : \mathcal{X} \mapsto C(\mathcal{X})$ in* **(a)***, and for 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ in* **(b)***, $\hat{C}(f) : C(\mathcal{X}) \to C(\mathcal{Y})$ acts on points by*

$$\begin{aligned} &\hat{C}(f)_{\mathrm{top}} : \bigl[x, \{x_1', \ldots, x_k'\}\bigr] \longmapsto \bigl[y, \{y_1', \ldots, y_l'\}\bigr], \quad \text{where } y = f \circ x, \\ &\{y_1', \ldots, y_l'\} = \bigl\{y' : [x_i', y'] \in \mathcal{S}_{f,\mathrm{top}},\ i = 1, \ldots, k\bigr\} \cup \bigl\{y' : [x, y'] \in \mathcal{T}_{f,\mathrm{top}}\bigr\}. \end{aligned}$$

If f is flat then $\hat{C}(f) = C(f)$.

Example 1.12.10. Suppose $\mathcal{X}$ is a quotient $[X/G]$ as in Example 1.12.2, where X is a manifold with corners and G is a finite group. Then the action $r : G \to \mathrm{Aut}(X)$ lifts to $C(r) : G \to \mathrm{Aut}(C(X))$, and there is an equivalence $C([X/G]) \simeq [C(X)/G]$ in $\check{\mathbf{O}}\mathbf{rb^c}$, where to define $[C(X)/G]$ we note that Example 1.12.2 also works with X in $\check{\mathbf{M}}\mathbf{an^c}$ rather than $\mathbf{Man^c}$, yielding $[X/G] \in \check{\mathbf{O}}\mathbf{rb^c}$.

Section 1.5.2 defined (s-)submersions, (s- or sf-)immersions and (s- or sf-) embeddings in $\mathbf{Man^c}$. Section 1.9.1 defined submersions, immersions and embeddings in **Orb**. We combine the two definitions.

Definition 1.12.11. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of orbifolds with corners.

(i) We call f a *submersion* if $\Omega_{C(f)} : C(f)^*(T^*C(\mathcal{Y})) \to T^*C(\mathcal{X})$ is an injective morphism of vector bundles, i.e. has a left inverse in $\mathrm{qcoh}(C(\mathcal{X}))$, and f is semisimple and flat. We call f an *s-submersion* if f is also simple.

(ii) We call f an *immersion* if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is a surjective morphism of vector bundles, i.e. has a right inverse in $\mathrm{qcoh}(\mathcal{X})$. We call f an *s-immersion* if f is also simple, and an *sf-immersion* if f is also simple and flat.

(iii) We call f an *embedding, s-embedding*, or *sf-embedding*, if it is an immersion, s-immersion, or sf-immersion, respectively, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\bigl(f_{\mathrm{top}}([x])\bigr)$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image (so in particular it is injective).

Then submersions, ..., sf-embeddings in $\mathbf{Orb^c}$ are étale locally modelled on submersions, ..., sf-embeddings in $\mathbf{Man^c}$.

1.12.4 Transversality and fibre products

Section 1.5.4 discussed transversality and fibre products for manifolds with corners. Section 8.8 generalizes this to orbifolds with corners.

Definition 1.12.12. Let $\mathfrak{X},\mathfrak{Y},\mathfrak{Z}$ be orbifolds with corners and $g:\mathfrak{X}\to\mathfrak{Z}$, $h:\mathfrak{Y}\to\mathfrak{Z}$ be 1-morphisms. Then as in §1.13 we have 1-morphisms $C(g):C(\mathfrak{X})\to C(\mathfrak{Z})$ and $C(h):C(\mathfrak{Y})\to C(\mathfrak{Z})$ in $\check{\mathbf{O}}\mathbf{rb^c}$, and hence 1-morphisms $C(g):C(\mathcal{X})\to C(\mathcal{Z})$ and $C(h):C(\mathcal{Y})\to C(\mathcal{Z})$ in $\mathbf{C^\infty Sta}$. We call g,h *transverse* if the following holds. Suppose $x:\underline{\bar{*}}\to C(\mathcal{X})$ and $y:\underline{\bar{*}}\to C(\mathcal{Y})$ are 1-morphisms in $\mathbf{C^\infty Sta}$, and $\eta:C(g)\circ x\Rightarrow C(h)\circ y$ a 2-morphism. Then the following morphism in qcoh$(\underline{\bar{*}})$ should be injective:

$$\begin{gathered}\bigl(x^*(\Omega_{C(g)})\circ I_{x,C(g)}(T^*C(\mathcal{Z}))\bigr)\oplus\bigl(y^*(\Omega_{C(h)})\circ I_{y,C(h)}(T^*C(\mathcal{Z}))\circ\eta^*(T^*C(\mathcal{Z}))\bigr):\\ (C(g)\circ x)^*(T^*C(\mathcal{Z}))\longrightarrow x^*(T^*C(\mathcal{X}))\oplus y^*(T^*C(\mathcal{Y})).\end{gathered}$$

Now identify $C_k(\mathcal{X})_{\rm top}\subseteq C(\mathcal{X})_{\rm top}$ with the right hand of (1.73), and similarly for $C(\mathcal{Y})_{\rm top},C(\mathcal{Z})_{\rm top}$. Then $C(g)_{\rm top},C(h)_{\rm top}$ act as in (1.74). We call g,h *strongly transverse* if they are transverse, and whenever there are points in $C_j(\mathcal{X})_{\rm top},C_k(\mathcal{Y})_{\rm top},C_l(\mathcal{Z})_{\rm top}$ with

$$C(g)_{\rm top}\bigl([x,\{x_1',\ldots,x_j'\}]\bigr)=C(h)_{\rm top}\bigl([y,\{y_1',\ldots,y_k'\}]\bigr)=[z,\{z_1',\ldots,z_l'\}],$$

we have either $j+k>l$ or $j=k=l=0$.

One can show that g,h are (strongly) transverse if and only if they are étale locally equivalent to (strongly) transverse smooth maps in $\mathbf{Man^c}$.

Here is the analogue of Theorem 1.5.13:

Theorem 1.12.13. *Suppose $g:\mathfrak{X}\to\mathfrak{Z}$ and $h:\mathfrak{Y}\to\mathfrak{Z}$ are transverse* 1-*morphisms in* $\mathbf{Orb^c}$*. Then a fibre product $\mathfrak{W}=\mathfrak{X}\times_{g,\mathfrak{Z},h}\mathfrak{Y}$ exists in the* 2-*category* $\mathbf{Orb^c}$*.*

Proposition 1.5.14 and Theorem 1.5.15 also extend to $\mathbf{Orb^c}$, with equivalences natural up to 2-isomorphism rather than canonical diffeomorphisms.

1.12.5 Orbifold strata of orbifolds with corners

Sections 1.8.7 and 1.9.2 discussed orbifold strata of Deligne–Mumford C^∞-stacks and orbifolds, respectively. Section 8.9 extends this to orbifolds with corners. This is also related to the material on fixed points of finite group actions on manifolds with corners in §1.5.6.

Theorem 1.12.14. *Let $\mathfrak{X}$ be an orbifold with corners, and Γ a finite group. Then we can define objects $\mathfrak{X}^\Gamma,\tilde{\mathfrak{X}}^\Gamma,\hat{\mathfrak{X}}^\Gamma$ in $\check{\mathbf{O}}\mathbf{rb^c}$, and open subobjects $\mathfrak{X}^\Gamma_\circ\subseteq\mathfrak{X}^\Gamma$, $\tilde{\mathfrak{X}}^\Gamma_\circ\subseteq\tilde{\mathfrak{X}}^\Gamma$, $\hat{\mathfrak{X}}^\Gamma_\circ\subseteq\hat{\mathfrak{X}}^\Gamma$, all natural up to* 1-*isomorphism in* $\check{\mathbf{O}}\mathbf{rb^c}$*, and* 1-*morphisms*

$O^\Gamma(\mathcal{X}), \tilde{\Pi}^\Gamma(\mathcal{X}), \ldots$ *fitting into a strictly commutative diagram in* $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$:

$$\begin{tikzcd}
\mathcal{X}^\Gamma_\circ \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde{\Pi}^\Gamma_\circ(\mathcal{X})"] \arrow[dr, "O^\Gamma_\circ(\mathcal{X})"] \arrow[dd, "\subset"'] & & \tilde{\mathcal{X}}^\Gamma_\circ \arrow[r, "\hat{\Pi}^\Gamma_\circ(\mathcal{X})"] \arrow[dl, "\tilde{O}^\Gamma_\circ(\mathcal{X})"'] \arrow[dd, "\subset"] & \hat{\mathcal{X}}^\Gamma_\circ \arrow[dd, "\subset"] \\
& \mathcal{X} & & \\
\mathcal{X}^\Gamma \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[ur, "O^\Gamma(\mathcal{X})"] \arrow[rr, "\tilde{\Pi}^\Gamma(\mathcal{X})"'] & & \tilde{\mathcal{X}}^\Gamma \arrow[ul, "\tilde{O}^\Gamma(\mathcal{X})"] \arrow[r, "\hat{\Pi}^\Gamma(\mathcal{X})"'] & \hat{\mathcal{X}}^\Gamma.
\end{tikzcd} \tag{1.75}$$

The underlying C^∞*-stacks of* $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ *are the orbifold strata* $\mathscr{X}^\Gamma, \ldots, \hat{\mathscr{X}}^\Gamma_\circ$ *from* §1.8.7 *of the* C^∞*-stack* $\mathscr{X}$ *in* $\mathcal{X}$, *and the* 1*-morphisms in* (1.75), *as* C^∞*-stack* 1*-morphisms, are those given in* (1.41).

Use the notation of Definition 1.9.7. *Then there are natural decompositions*

$$\begin{aligned}
\mathcal{X}^\Gamma &= \textstyle\coprod_{\lambda\in\Lambda^\Gamma_+} \mathcal{X}^{\Gamma,\lambda}, & \tilde{\mathcal{X}}^\Gamma &= \textstyle\coprod_{\mu\in\Lambda^\Gamma_+/\,\mathrm{Aut}(\Gamma)} \tilde{\mathcal{X}}^{\Gamma,\mu}, & \hat{\mathcal{X}}^\Gamma &= \textstyle\coprod_{\mu\in\Lambda^\Gamma_+/\,\mathrm{Aut}(\Gamma)} \hat{\mathcal{X}}^{\Gamma,\mu}, \\
\mathcal{X}^\Gamma_\circ &= \textstyle\coprod_{\lambda\in\Lambda^\Gamma_+} \mathcal{X}^{\Gamma,\lambda}_\circ, & \tilde{\mathcal{X}}^\Gamma_\circ &= \textstyle\coprod_{\mu\in\Lambda^\Gamma_+/\,\mathrm{Aut}(\Gamma)} \tilde{\mathcal{X}}^{\Gamma,\mu}_\circ, & \hat{\mathcal{X}}^\Gamma_\circ &= \textstyle\coprod_{\mu\in\Lambda^\Gamma_+/\,\mathrm{Aut}(\Gamma)} \hat{\mathcal{X}}^{\Gamma,\mu}_\circ,
\end{aligned}$$

where $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}^{\Gamma,\mu}_\circ$ *are orbifolds with corners, open and closed in* $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$, *and of dimensions* $\dim\mathcal{X} - \dim\lambda, \dim\mathcal{X} - \dim\mu$. *All of* $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma, \mathcal{X}^\Gamma_\circ, \tilde{\mathcal{X}}^\Gamma_\circ, \hat{\mathcal{X}}^\Gamma_\circ$, $\mathcal{X}^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \mathcal{X}^{\Gamma,\lambda}_\circ, \tilde{\mathcal{X}}^{\Gamma,\mu}_\circ, \hat{\mathcal{X}}^{\Gamma,\mu}_\circ$ *will be called* ***orbifold strata*** *of* $\mathcal{X}$.

The definitions of $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ also make sense if $\mathcal{X}$ lies in $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$ rather than $\mathbf{Orb^c}$. We will not use notation $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}^{\Gamma,\mu}_\circ$ for $\mathcal{X} \in \check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}} \setminus \mathbf{Orb^c}$.

As for Deligne–Mumford C^∞-stacks in §1.8.7, orbifold strata $\mathcal{X}^\Gamma$ are strongly functorial for representable 1-morphisms in $\mathbf{Orb^c}$ and their 2-morphisms. That is, if $f : \mathcal{X} \to \mathcal{Y}$ is a representable 1-morphism in $\mathbf{Orb^c}$, there is a unique representable 1-morphism $f^\Gamma : \mathcal{X}^\Gamma \to \mathcal{Y}^\Gamma$ in $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$ with $O^\Gamma(\mathcal{Y}) \circ f^\Gamma = f \circ O^\Gamma(\mathcal{X})$, which is just the 1-morphism f^Γ from §1.8.7 for the C^∞-stack 1-morphism $f : \mathscr{X} \to \mathscr{Y}$. Note however that f^Γ need not map $\mathcal{X}^{\Gamma,\lambda} \to \mathcal{Y}^{\Gamma,\lambda}$ for $\lambda \in \Lambda^\Gamma_+$.

If $f, g : \mathcal{X} \to \mathcal{Y}$ are representable and $\eta : f \Rightarrow g$ is a 2-morphism in $\mathbf{Orb^c}$, there is a unique 2-morphism $\eta^\Gamma : f^\Gamma \Rightarrow g^\Gamma$ in $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$ with $\mathrm{id}_{O^\Gamma(\mathcal{Y})} * \eta^\Gamma = \eta * \mathrm{id}_{O^\Gamma(\mathcal{X})}$, which is just the C^∞-stack 2-morphism η^Γ from §1.8.7. These f^Γ, η^Γ are compatible with compositions of 1- and 2-morphisms, and identities, in the obvious way. Orbifold strata $\tilde{\mathcal{X}}^\Gamma$ have the same strong functorial behaviour, and orbifold strata $\hat{\mathcal{X}}^\Gamma$ a weaker functorial behaviour.

We also investigate the relationship between orbifold strata and corners.

Theorem 1.12.15. *Let* $\mathcal{X}$ *be an orbifold with corners, and* Γ *a finite group. The corners* $C(\mathcal{X})$ *lie in* $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$ *as in* §1.12.3, *so we have orbifold strata* $\mathcal{X}^\Gamma, C(\mathcal{X})^\Gamma$ *and* 1*-morphisms* $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$, $O^\Gamma(C(\mathcal{X})) : C(\mathcal{X})^\Gamma \to C(\mathcal{X})$. *Applying the corner functor* C *from* §1.12.3 *gives a* 1*-morphism* $C(O^\Gamma(\mathcal{X})) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})$. *Then there exists a unique equivalence* $K^\Gamma(\mathcal{X}) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})^\Gamma$ *such that* $O^\Gamma(C(\mathcal{X})) \circ K^\Gamma(\mathcal{X}) = C(O^\Gamma(\mathcal{X})) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})$. *It restricts to an equivalence* $K^\Gamma_\circ(\mathcal{X}) := K^\Gamma(\mathcal{X})|_{C(\mathcal{X}^\Gamma_\circ)} : C(\mathcal{X}^\Gamma_\circ) \to C(\mathcal{X})^\Gamma_\circ$.

Similarly, there is a unique equivalence $\tilde{K}^\Gamma(\mathcal{X}) : C(\tilde{\mathcal{X}}^\Gamma) \to \widetilde{C(\mathcal{X})}^\Gamma$ *with* $\tilde{O}^\Gamma(C(\mathcal{X})) \circ \tilde{K}^\Gamma(\mathcal{X}) = C(\tilde{O}^\Gamma(\mathcal{X}))$ *and* $\tilde{\Pi}^\Gamma(C(\mathcal{X})) \circ K^\Gamma(\mathcal{X}) = \tilde{K}^\Gamma(\mathcal{X}) \circ C(\tilde{\Pi}^\Gamma(\mathcal{X}))$.

There is an equivalence $\hat{K}^\Gamma(\mathcal{X}) : C(\hat{\mathcal{X}}^\Gamma) \to \widehat{C(\mathcal{X})}^\Gamma$, *unique up to* 2*-isomorphism, with a* 2*-morphism* $\hat{\Pi}^\Gamma(C(\mathcal{X})) \circ \hat{K}^\Gamma(\mathcal{X}) \Rightarrow \hat{K}^\Gamma(\mathcal{X}) \circ C(\hat{\Pi}^\Gamma(\mathcal{X}))$. *They both restrict to equivalences* $\tilde{K}^\Gamma_\circ(\mathcal{X}) : C(\tilde{\mathcal{X}}^\Gamma_\circ) \to \widetilde{C(\mathcal{X})}^\Gamma_\circ$ *and* $\hat{K}^\Gamma_\circ(\mathcal{X}) : C(\hat{\mathcal{X}}^\Gamma_\circ) \to \widehat{C(\mathcal{X})}^\Gamma_\circ$.

Here is an example:

Example 1.12.16. Let $\mathbb{Z}_2 = \{1, \sigma\}$ with $\sigma^2 = 1$ act on $X = [0,\infty)^2$ by $\sigma : (x_1, x_2) \mapsto (x_2, x_1)$. Then $\mathcal{X} = \bigl[[0,\infty)^2/\mathbb{Z}_2\bigr]$ is an orbifold with corners. We have $\partial\mathcal{X} \cong [0,\infty)$ and $\partial^2\mathcal{X} \cong *$, so that $C_2(\mathcal{X}) \simeq [*/S_2] = [*/\mathbb{Z}_2]$. Hence $C(\mathcal{X}) = C_0(\mathcal{X}) \amalg C_1(\mathcal{X}) \amalg C_2(\mathcal{X})$ with $C_0(\mathcal{X}) \simeq \bigl[[0,\infty)^2/\mathbb{Z}_2\bigr]$, $C_1(\mathcal{X}) \simeq [0,\infty)$ and $C_2(\mathcal{X}) \simeq [*/\mathbb{Z}_2]$. The orbifold strata $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ are given by

$$\mathcal{X}^{\mathbb{Z}_2} = \mathcal{X}^{\mathbb{Z}_2}_\circ \simeq \tilde{\mathcal{X}}^{\mathbb{Z}_2} = \tilde{\mathcal{X}}^{\mathbb{Z}_2}_\circ \simeq [0,\infty) \times [*/\mathbb{Z}_2], \qquad \hat{\mathcal{X}}^{\mathbb{Z}_2} = \hat{\mathcal{X}}^{\mathbb{Z}_2}_\circ \simeq [0,\infty).$$

Therefore

$$\begin{aligned} &C_0(\mathcal{X}^{\mathbb{Z}_2}) \simeq [0,\infty) \times [*/\mathbb{Z}_2], && C_1(\mathcal{X}^{\mathbb{Z}_2}) \simeq [*/\mathbb{Z}_2], && C_2(\mathcal{X}^{\mathbb{Z}_2}) = \emptyset, \\ &C_0(\mathcal{X})^{\mathbb{Z}_2} \simeq [0,\infty) \times [*/\mathbb{Z}_2], && C_1(\mathcal{X})^{\mathbb{Z}_2} = \emptyset, && C_2(\mathcal{X})^{\mathbb{Z}_2} \simeq [*/\mathbb{Z}_2]. \end{aligned}$$

We see from this that $K^{\mathbb{Z}_2}(\mathcal{X}) : C(\mathcal{X}^{\mathbb{Z}_2}) \to C(\mathcal{X})^{\mathbb{Z}_2}$ identifies $C_1(\mathcal{X}^{\mathbb{Z}_2})$ with $C_2(\mathcal{X})^{\mathbb{Z}_2}$, so $K^\Gamma(\mathcal{X})$ need not map $C_k(\mathcal{X}^\Gamma)$ to $C_k(\mathcal{X})^\Gamma$ for $k > 0$. The same applies to $\tilde{K}^\Gamma(\mathcal{X}), \hat{K}^\Gamma(\mathcal{X})$.

The construction of $K^\Gamma(\mathcal{X})$ in Theorem 1.12.15 implies that it maps $C_k(\mathcal{X}^\Gamma)$ into $\coprod_{l \geqslant k} C_l(\mathcal{X})^\Gamma$ for $k > 0$. This implies that $C_1(\mathcal{X})^\Gamma \simeq (\partial\mathcal{X})^\Gamma$ is equivalent to an open and closed subobject of $C_1(\mathcal{X}^\Gamma) \simeq \partial(\mathcal{X}^\Gamma)$. Hence we can choose a 1-morphism $J^\Gamma(\mathcal{X}) : (\partial\mathcal{X})^\Gamma \to \partial(\mathcal{X}^\Gamma)$ identified with a quasi-inverse for $K^\Gamma(\mathcal{X})|_{\cdots} : K^\Gamma(\mathcal{X})^{-1}(C_1(\mathcal{X})^\Gamma) \to C_1(\mathcal{X})^\Gamma$ by the equivalences $C_1(\mathcal{X})^\Gamma \simeq (\partial\mathcal{X})^\Gamma$ and $C_1(\mathcal{X}^\Gamma) \simeq \partial(\mathcal{X}^\Gamma)$, and $J^\Gamma(\mathcal{X})$ is an equivalence between $(\partial\mathcal{X})^\Gamma$ and an open and closed subobject of $\partial(\mathcal{X}^\Gamma)$. We then deduce:

Corollary 1.12.17. *Let* $\mathcal{X}$ *be an orbifold with corners, and* Γ *a finite group. Then there exist* 1*-morphisms* $J^\Gamma(\mathcal{X}) : (\partial\mathcal{X})^\Gamma \to \partial(\mathcal{X}^\Gamma)$, $\tilde{J}^\Gamma(\mathcal{X}) : (\widetilde{\partial\mathcal{X}})^\Gamma \to \partial(\tilde{\mathcal{X}}^\Gamma)$, $\hat{J}^\Gamma(\mathcal{X}) : (\widehat{\partial\mathcal{X}})^\Gamma \to \partial(\hat{\mathcal{X}}^\Gamma)$ *in* $\check{\mathbf{O}}\mathbf{rb^c}$, *natural up to* 2*-isomorphism, such that* $J^\Gamma(\mathcal{X})$ *is an equivalence from* $(\partial\mathcal{X})^\Gamma$ *to an open and closed subobject of* $\partial(\mathcal{X}^\Gamma)$, *and similarly for* $\tilde{J}^\Gamma(\mathcal{X}), \hat{J}^\Gamma(\mathcal{X})$.

For $\lambda \in \Lambda^\Gamma_+$, $\mu \in \Lambda^\Gamma_+/\operatorname{Aut}(\Lambda)$ *these restrict to* 1*-morphisms* $J^{\Gamma,\lambda}(\mathcal{X}) : (\partial\mathcal{X})^{\Gamma,\lambda} \to \partial(\mathcal{X}^{\Gamma,\lambda})$, $\tilde{J}^{\Gamma,\mu}(\mathcal{X}) : (\widetilde{\partial\mathcal{X}})^{\Gamma,\mu} \to \partial(\tilde{\mathcal{X}}^{\Gamma,\mu})$, $\hat{J}^{\Gamma,\mu}(\mathcal{X}) : (\widehat{\partial\mathcal{X}})^{\Gamma,\mu} \to \partial(\hat{\mathcal{X}}^{\Gamma,\mu})$ *in* $\mathbf{Orb^c}$, *which are equivalences with open and closed suborbifolds. Hence, if* $\mathcal{X}^{\Gamma,\lambda} = \emptyset$ *then* $(\partial\mathcal{X})^{\Gamma,\lambda} = \emptyset$, *and similarly for* $\tilde{\mathcal{X}}^{\Gamma,\mu}, (\widetilde{\partial\mathcal{X}})^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, (\widehat{\partial\mathcal{X}})^{\Gamma,\mu}$.

As in Remark 1.12.3(c), an orbifold with corners $\mathcal{X}$ is called *straight* if $(i_{\mathcal{X}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ is an isomorphism for all $[x'] \in \partial\mathcal{X}_{\mathrm{top}}$ with $i_{\mathcal{X},\mathrm{top}}([x']) = [x]$. If $\mathcal{X}$ is straight then $K^\Gamma(\mathcal{X})$ in Theorem 1.12.15 is an equivalence $C_k(\mathcal{X}^\Gamma) \to C_k(\mathcal{X})^\Gamma$ for all $k \geqslant 0$, and so $J^\Gamma(\mathcal{X})$ in Corollary 1.12.17 is an equivalence $(\partial\mathcal{X})^\Gamma \to \partial(\mathcal{X}^\Gamma)$. The same applies for $\tilde{J}^\Gamma(\mathcal{X}), \hat{J}^\Gamma(\mathcal{X}), \tilde{K}^\Gamma(\mathcal{X}), \hat{K}^\Gamma(\mathcal{X})$.

Proposition 1.9.9 on orientations of orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}^{\Gamma,\mu}_\circ$ of oriented orbifolds $\mathcal{X}$ also holds without change for orbifolds with corners $\mathcal{X}$.

1.13 D-stacks with corners

Chapter 11 discusses the 2-category $\mathbf{dSta^c}$ of *d-stacks with corners*. There are few new issues here: almost all the material just combines ideas we have seen already on d-spaces with corners from §1.6, and on d-stacks from §1.10, and on orbifolds with corners from §1.12. So we will be brief.

1.13.1 Outline of the definition of the 2-category $\mathbf{dSta^c}$

The definition of the 2-category $\mathbf{dSta^c}$ in §11.1 is long and complicated. So as for $\mathbf{dSpa^c}$ in §1.6.1, we will just sketch the main ideas.

A *d-stack with corners* is a quadruple $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$, where $\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}$ are d-stacks and $\boldsymbol{i}_{\mathfrak{X}} : \boldsymbol{\partial\mathcal{X}} \to \boldsymbol{\mathcal{X}}$ is a 1-morphism of d-stacks with $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ a proper, strongly representable 1-morphism of Deligne–Mumford C^∞-stacks, as in §1.8.3. We should have an exact sequence in qcoh$(\partial\mathcal{X})$:

$$0 \longrightarrow \mathcal{N}_{\mathfrak{X}} \xrightarrow{\nu_{\mathfrak{X}}} i_{\mathfrak{X}}^*(\mathcal{F}_{\mathcal{X}}) \xrightarrow{i_{\mathfrak{X}}^2} \mathcal{F}_{\partial\mathcal{X}} \longrightarrow 0, \tag{1.76}$$

where $\mathcal{N}_{\mathfrak{X}}$ is a line bundle on $\partial\mathcal{X}$, the *conormal bundle* of $\boldsymbol{\partial\mathcal{X}}$ in $\boldsymbol{\mathcal{X}}$, and $\omega_{\mathfrak{X}}$ is an orientation on $\mathcal{N}_{\mathfrak{X}}$. These $\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}}$ must satisfy some complicated conditions in §11.1, that we will not give. They require $\boldsymbol{\partial\mathcal{X}}$ to be locally equivalent to a fibre product $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{[0,\infty)}} \boldsymbol{*}$ in $\mathbf{dSta}$.

If $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ and $\mathfrak{Y} = (\boldsymbol{\mathcal{Y}}, \boldsymbol{\partial\mathcal{Y}}, \boldsymbol{i}_{\mathfrak{Y}}, \omega_{\mathfrak{Y}})$ are d-stacks with corners, a 1-*morphism* $\boldsymbol{f} : \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{dSta^c}$ is a 1-morphism $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dSta}$ satisfying extra conditions over $\boldsymbol{\partial\mathcal{X}}, \boldsymbol{\partial\mathcal{Y}}$. If $\boldsymbol{f}, \boldsymbol{g} : \mathfrak{X} \to \mathfrak{Y}$ be 1-morphisms in $\mathbf{dSta^c}$, so $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ are 1-morphisms in $\mathbf{dSta}$, a 2-morphism $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\mathbf{dSta^c}$ is a 2-morphism $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\mathbf{dSta}$ satisfying extra conditions over $\boldsymbol{\partial\mathcal{X}}, \boldsymbol{\partial\mathcal{Y}}$. In both cases, 1- and 2-morphisms in $\mathbf{dSta^c}$ are étale locally modelled on 1- and 2-morphisms in $\mathbf{dSpa^c}$. Identity 1- and 2-morphisms in $\mathbf{dSta^c}$, and the compositions of 1- and 2-morphisms in $\mathbf{dSta^c}$, are all given by identities and compositions in $\mathbf{dSta}$.

A d-stack with corners $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ is called a *d-stack with boundary* if $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is injective as a representable 1-morphism of C^∞-stacks, and a *d-stack without boundary* if $\boldsymbol{\partial\mathcal{X}} = \boldsymbol{\emptyset}$. We write $\mathbf{dSta^b}$ for the full 2-subcategory of d-stacks with boundary, and $\mathbf{d\bar{S}ta}$ for the full 2-subcategory of d-stacks without boundary, in $\mathbf{dSta^c}$. There is an isomorphism of 2-categories $F_{\mathbf{dSta}}^{\mathbf{dSta^c}} : \mathbf{dSta} \to \mathbf{d\bar{S}ta}$ mapping $\boldsymbol{\mathcal{X}} \mapsto \mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\emptyset}, \boldsymbol{\emptyset}, \emptyset)$ on objects, $\boldsymbol{f} \mapsto \boldsymbol{f}$ on 1-morphisms and $\boldsymbol{\eta} \mapsto \boldsymbol{\eta}$ on 2-morphisms. So we can consider d-stacks to be examples of d-stacks with corners.

Define a strict 2-functor $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}} : \mathbf{dSpa^c} \to \mathbf{dSta^c}$ as follows. If $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ is an object in $\mathbf{dSpa^c}$, set $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\mathbf{X}) = \mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$, where $\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}} = F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}})$. Then comparing equations (1.26) and (1.76), we find there is a natural isomorphism of line bundles $\mathcal{N}_{\mathfrak{X}} \cong \mathcal{I}_{\underline{\partial X}}(\mathcal{N}_{\mathbf{X}})$, where $\mathcal{I}_{\underline{\partial X}} : \text{qcoh}(\underline{\partial X}) \to \text{qcoh}(\bar{\partial X})$ is the equivalence of categories from Example 1.8.21. We define $\omega_{\mathfrak{X}}$ to be the orientation on $\mathcal{N}_{\mathfrak{X}}$ identified with the orientation $\mathcal{I}_{\underline{\partial X}}(\omega_{\mathbf{X}})$ on $\mathcal{I}_{\underline{\partial X}}(\mathcal{N}_{\mathbf{X}})$ by this isomorphism. On 1- and 2-morphisms $\boldsymbol{f},\eta$ in $\mathbf{dSpa^c}$, we define $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\boldsymbol{f})=F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{f})$ and $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\eta)=F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\eta)$.

Write $\mathbf{d\hat{S}pa^c}$ for the full 2-subcategory of objects $\mathfrak{X}$ in $\mathbf{dSta^c}$ equivalent to $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\mathbf{X})$ for some d-space with corners $\mathbf{X}$. When we say that a d-stack with corners $\mathfrak{X}$ *is a d-space*, we mean that $\mathfrak{X}\in\mathbf{d\hat{S}pa^c}$.

Define a strict 2-functor $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}:\mathbf{Orb^c}\to\mathbf{dSta^c}$ as follows. If $\mathfrak{X}=(\mathcal{X},\partial\mathcal{X},i_{\mathcal{X}})$ is an orbifold with corners, as in §1.12.1, define $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(\mathfrak{X})=\boldsymbol{\mathfrak{X}}=(\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i_{\mathcal{X}}},\omega_{\mathfrak{X}})$, where $\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i_{\mathcal{X}}}=F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\mathcal{X},\partial\mathcal{X},i_{\mathcal{X}})$. Then $\mathcal{N}_{\mathfrak{X}}$ in (1.76) is isomorphic to the conormal line bundle of $\partial\mathcal{X}$ in $\mathcal{X}$, and we define $\omega_{\mathfrak{X}}$ to be the orientation on $\mathcal{N}_{\mathfrak{X}}$ induced by 'outward-pointing' normal vectors to $\partial\mathcal{X}$ in $\mathcal{X}$. Then $\boldsymbol{\mathfrak{X}}=(\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i_{\mathcal{X}}},\omega_{\mathfrak{X}})$ is a d-orbifold with corners. On 1- and 2-morphisms f,η in $\mathbf{Orb^c}$, we define $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(f)=F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(f)$ and $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(\eta)=F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\eta)$.

Write $\mathbf{\bar{O}rb},\mathbf{\bar{O}rb^b},\mathbf{\bar{O}rb^c}$ for the full 2-subcategories of objects $\mathfrak{X}$ in $\mathbf{dSta^c}$ equivalent to $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(\mathfrak{X})$ for some orbifold $\mathfrak{X}$ without boundary, or with boundary, or with corners, respectively. Then $\mathbf{\bar{O}rb}\subset\mathbf{d\bar{S}ta}$, $\mathbf{\bar{O}rb^b}\subset\mathbf{dSta^b}$ and $\mathbf{\bar{O}rb^c}\subset\mathbf{dSta^c}$. When we say that a d-stack with corners $\mathfrak{X}$ *is an orbifold*, we mean that $\mathfrak{X}\in\mathbf{\bar{O}rb^c}$.

Remark 1.13.1. As discussed for orbifolds with corners in Remark 1.12.3(b), in a d-stack with corners $\boldsymbol{\mathfrak{X}}=(\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i_{\mathcal{X}}},\omega_{\mathfrak{X}})$ we require $i_{\mathcal{X}}:\partial\mathcal{X}\to\mathcal{X}$ to be *strongly representable*, in the sense of §1.8.3, so that we can make boundaries and corners in $\mathbf{dSta^c}$ strongly functorial, as in Remark 1.6.5 for $\mathbf{dSpa^c}$.

For each d-stack with corners $\boldsymbol{\mathfrak{X}}=(\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i_{\mathcal{X}}},\omega_{\mathfrak{X}})$, in §11.3 we define a d-stack with corners $\partial\boldsymbol{\mathfrak{X}}=(\boldsymbol{\partial\mathcal{X}},\boldsymbol{\partial^2\mathcal{X}},\boldsymbol{i_{\partial\mathcal{X}}},\omega_{\partial\mathfrak{X}})$ called the *boundary* of $\boldsymbol{\mathfrak{X}}$, and show that $\boldsymbol{i_{\mathfrak{X}}}:\partial\boldsymbol{\mathfrak{X}}\to\boldsymbol{\mathfrak{X}}$ is a 1-morphism in $\mathbf{dSta^c}$. As for d-spaces with corners in (1.27), the d-stack $\boldsymbol{\partial^2\mathcal{X}}$ in $\partial\boldsymbol{\mathfrak{X}}$ satisfies

$$\boldsymbol{\partial^2\mathcal{X}}\simeq\big(\boldsymbol{\partial\mathcal{X}}\times_{\boldsymbol{i_{\mathcal{X}}},\boldsymbol{\mathcal{X}},\boldsymbol{i_{\mathcal{X}}}}\boldsymbol{\partial\mathcal{X}}\big)\setminus\boldsymbol{\Delta_{\partial\mathcal{X}}}(\boldsymbol{\partial\mathcal{X}}),$$

where $\Delta_{\boldsymbol{\partial\mathcal{X}}}:\boldsymbol{\partial\mathcal{X}}\to\boldsymbol{\partial\mathcal{X}}\times_{\boldsymbol{\mathcal{X}}}\boldsymbol{\partial\mathcal{X}}$ is the diagonal 1-morphism. The 1-morphism $\boldsymbol{i_{\partial\mathcal{X}}}:\boldsymbol{\partial^2\mathcal{X}}\to\boldsymbol{\partial\mathcal{X}}$ is projection to the first factor in the fibre product. There is a natural isomorphism $\mathcal{N}_{\partial\mathfrak{X}}\cong i_{\mathfrak{X}}^*(\mathcal{N}_{\mathfrak{X}})$, and the orientation $\omega_{\partial\mathfrak{X}}$ on $\mathcal{N}_{\partial\mathfrak{X}}$ corresponds to the orientation $i_{\mathfrak{X}}^*(\omega_{\mathfrak{X}})$ on $i_{\mathfrak{X}}^*(\mathcal{N}_{\mathfrak{X}})$.

1.13.2 D-stacks with corners as quotients of d-spaces with corners

Section 1.10.2 discussed quotient d-stacks $[\boldsymbol{X}/G]$, for $\boldsymbol{X}$ a d-space and $\boldsymbol{r}:G\to\mathrm{Aut}(\boldsymbol{X})$ an action of G on $\boldsymbol{X}$ by 1-isomorphisms. Section 11.2 extends this to d-spaces with corners and d-stacks with corners, and proves:

Theorem 1.13.2. *Theorems* 1.10.3 *and* 1.10.4 *hold unchanged in* $\mathbf{dSta^c}$*.*

Here if $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i_{\mathbf{X}}},\omega_{\mathbf{X}})$ is a d-space with corners and $\boldsymbol{r}:G\to\mathrm{Aut}(\mathbf{X})$ an action of G on $\mathbf{X}$ then each $\boldsymbol{r}(\gamma):\mathbf{X}\to\mathbf{X}$ for $\gamma\in G$ is simple, so Theorem 1.6.3(b) gives a lift $\boldsymbol{r}_-(\gamma):\partial\mathbf{X}\to\partial\mathbf{X}$, defining an action $\boldsymbol{r}_-:G\to\mathrm{Aut}(\partial\mathbf{X})$ of G on $\partial\mathbf{X}$. Then $\boldsymbol{r}:G\to\mathrm{Aut}(\boldsymbol{X})$ and $\boldsymbol{r}_-:G\to\mathrm{Aut}(\boldsymbol{\partial X})$ are actions of G on the d-spaces $\boldsymbol{X},\boldsymbol{\partial X}$, and $\boldsymbol{i_{\mathbf{X}}}:\boldsymbol{\partial X}\to\boldsymbol{X}$ is G-equivariant. So

Theorem 1.10.3(a),(b) give quotient d-stacks $[\boldsymbol{X}/G],[\boldsymbol{\partial X}/G]$ and a quotient 1-morphism $[\boldsymbol{i}_{\mathbf{X}},\mathrm{id}_G]:[\boldsymbol{\partial X}/G]\rightarrow[\boldsymbol{X}/G]$. The quotient d-stack with corners $[\mathbf{X}/G]$ given by the analogue of Theorem 1.10.3 is defined to be $[\mathbf{X}/G]=\bigl([\boldsymbol{X}/G],[\boldsymbol{\partial X}/G],[\boldsymbol{i}_{\mathbf{X}},\mathrm{id}_G],\omega_{[\mathbf{X}/G]}\bigr)$, for a natural orientation $\omega_{[\mathbf{X}/G]}$ on $\mathcal{N}_{[\mathbf{X}/G]}$ constructed from $\omega_{\mathbf{X}}$.

In §11.4 we define when a 1-morphism of d-stacks with corners $\boldsymbol{f}:\boldsymbol{\mathfrak{X}}\rightarrow\boldsymbol{\mathfrak{Y}}$ is *étale*. Essentially, $\boldsymbol{f}$ is étale if it is an equivalence locally in the étale topology. It implies that the C^∞-stack 1-morphism $f:\mathcal{X}\rightarrow\mathcal{Y}$ in $\boldsymbol{f}$ is étale, and so representable. As for d-stacks in §1.10.2, we can characterize étale 1-morphisms in $\mathbf{dSta^c}$ using the corners analogue of Theorem 1.10.4(b) and the definition of étale 1-morphisms in $\mathbf{dSpa^c}$ as (Zariski) local equivalences.

1.13.3 Simple, semisimple and flat 1-morphisms

Section 11.3 generalizes §1.6.2 to d-stacks with corners. Here is the analogue of Definition 1.12.6.

Definition 1.13.3. Let $\boldsymbol{\mathfrak{X}}=(\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i}_{\mathfrak{X}},\omega_{\mathfrak{X}})$ and $\boldsymbol{\mathfrak{Y}}=(\boldsymbol{\mathcal{Y}},\boldsymbol{\partial\mathcal{Y}},\boldsymbol{i}_{\mathfrak{Y}},\omega_{\mathfrak{Y}})$ be d-stacks with corners, and $\boldsymbol{f}:\boldsymbol{\mathfrak{X}}\rightarrow\boldsymbol{\mathfrak{Y}}$ a 1-morphism in $\mathbf{dSta^c}$. Consider the C^∞-stack fibre products $\partial\mathcal{X}\times_{f\circ i_{\mathcal{X}},\mathcal{Y},i_{\mathcal{Y}}}\partial\mathcal{Y}$ and $\mathcal{X}\times_{f,\mathcal{Y},i_{\mathcal{Y}}}\partial\mathcal{Y}$. Since $i_{\mathfrak{Y}}$ is strongly representable, we may define these using the construction of Proposition 1.8.10.

As in (1.71), we may write $(\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y})_{\mathrm{top}}$ explicitly as

$$\begin{aligned}(\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y})_{\mathrm{top}}\cong\bigl\{[x',y']:x':\underline{\bar{*}}\rightarrow\partial\mathcal{X}\text{ and }y':\underline{\bar{*}}\rightarrow\partial\mathcal{Y}\text{ are}\\ \text{1-morphisms with }f\circ i_{\mathfrak{X}}\circ x'=i_{\mathfrak{Y}}\circ y':\underline{\bar{*}}\rightarrow\mathcal{Y}\bigr\},\end{aligned}\tag{1.77}$$

where $[x',y']$ in (1.77) denotes the $\sim$-equivalence class of pairs (x',y'), with $(x',y')\sim(\tilde{x}',\tilde{y}')$ if there exist 2-morphisms $\eta:x'\Rightarrow\tilde{x}'$ and $\zeta:y'\Rightarrow\tilde{y}'$ with $\mathrm{id}_{f\circ i_{\mathcal{X}}}*\eta=\mathrm{id}_{i_{\mathfrak{Y}}}*\zeta$. There is a natural open and closed C^∞-substack $\mathcal{S}_{\boldsymbol{f}}\subseteq\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ such that $[x',y']$ in (1.77) lies in $\mathcal{S}_{\boldsymbol{f},\mathrm{top}}$ if and only if we can complete the following commutative diagram in $\mathrm{qcoh}(\underline{\bar{*}})$ with morphisms '$\dashrightarrow$':

$$\begin{tikzcd}
0 \arrow[r] & (y')^*(\mathcal{N}_{\mathfrak{Y}}) \arrow[r, "(y')^*(\nu_{\mathfrak{Y}})"] \arrow[d, dashed, "\cong"'] & (y')^*\circ i_{\mathfrak{Y}}^*(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "(y')^*(i_{\mathfrak{Y}}^2)"] \arrow[d, "\begin{subarray}{l} I_{x',i_{\mathfrak{X}}}(\mathcal{F}_{\mathcal{X}})\circ(i_{\mathfrak{X}}\circ x')^*(f^2)\circ \\ I_{i_{\mathfrak{X}}\circ x',f}(\mathcal{F}_{\mathcal{Y}})\circ I_{y',i_{\mathfrak{Y}}}(\mathcal{F}_{\mathcal{Y}})^{-1}\end{subarray}"] & (y')^*(\mathcal{F}_{\partial\mathcal{Y}}) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & (x')^*(\mathcal{N}_{\mathfrak{X}}) \arrow[r, "(x')^*(\nu_{\mathfrak{X}})"'] & (x')^*\circ i_{\mathfrak{X}}^*(\mathcal{F}_{\mathcal{X}}) \arrow[r, "(x')^*(i_{\mathfrak{X}}^2)"'] & (x')^*(\mathcal{F}_{\partial\mathcal{X}}) \arrow[r] & 0.
\end{tikzcd}$$

Similarly, as in (1.72) we may write $(\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y})_{\mathrm{top}}$ explicitly as

$$\begin{aligned}(\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y})_{\mathrm{top}}\cong\bigl\{[x,y']:x:\underline{\bar{*}}\rightarrow\mathcal{X}\text{ and }y':\underline{\bar{*}}\rightarrow\partial\mathcal{Y}\text{ are}\\ \text{1-morphisms with }f\circ x=i_{\mathfrak{Y}}\circ y':\underline{\bar{*}}\rightarrow\mathcal{Y}\bigr\},\end{aligned}\tag{1.78}$$

where $[x,y']$ in (1.78) denotes the $\approx$-equivalence class of (x,y'), with $(x,y')\approx(\tilde{x},\tilde{y}')$ if there exist $\eta:x\Rightarrow\tilde{x}$ and $\zeta:y'\Rightarrow\tilde{y}'$ with $\mathrm{id}_f*\eta=\mathrm{id}_{i_{\mathfrak{Y}}}*\zeta$. There is a natural open and closed C^∞-substack $\mathcal{T}_{\boldsymbol{f}} \subseteq \mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$ with $[x, y']$ in (1.78) lies in $\mathcal{T}_{\boldsymbol{f},\mathrm{top}}$ if and only if we can complete the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & (y')^*(\mathcal{N}_{\boldsymbol{\mathcal{Y}}}) \arrow[r, "(y')^*(\nu_{\boldsymbol{\mathcal{Y}}})"'] & (y')^* \circ i_{\boldsymbol{\mathcal{Y}}}^*(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "(y')^*(i_{\boldsymbol{\mathcal{Y}}}^2)"'] \arrow[d, "x^*(f^2)\circ I_{x,f}(\mathcal{F}_{\mathcal{Y}})\circ I_{y',i_{\boldsymbol{\mathcal{Y}}}}(\mathcal{F}_{\mathcal{Y}})^{-1}"'] & (y')^*(\mathcal{F}_{\partial\mathcal{Y}}) \arrow[r] \arrow[dl, dotted] & 0 \\
& & x^*(\mathcal{F}_{\mathcal{X}}). & &
\end{tikzcd}$$

Define $s_{\boldsymbol{f}} = \pi_{\partial\mathcal{X}}|_{\mathcal{S}_{\boldsymbol{f}}} : \mathcal{S}_{\boldsymbol{f}} \to \partial\mathcal{X}$, $u_{\boldsymbol{f}} = \pi_{\partial\mathcal{Y}}|_{\mathcal{S}_{\boldsymbol{f}}} : \mathcal{S}_{\boldsymbol{f}} \to \partial\mathcal{Y}$, $t_{\boldsymbol{f}} = \pi_{\mathcal{X}}|_{\mathcal{T}_{\boldsymbol{f}}} : \mathcal{T}_{\boldsymbol{f}} \to \mathcal{X}$, and $v_{\boldsymbol{f}} = \pi_{\partial\mathcal{Y}}|_{\mathcal{T}_{\boldsymbol{f}}} : \mathcal{T}_{\boldsymbol{f}} \to \partial\mathcal{Y}$. Then $s_{\boldsymbol{f}}, t_{\boldsymbol{f}}$ are proper, étale 1-morphisms. We call $\boldsymbol{f}$ *simple* if $s_{\boldsymbol{f}} : \mathcal{S}_{\boldsymbol{f}} \to \partial\mathcal{X}$ is an equivalence, and we call $\boldsymbol{f}$ *semisimple* if $s_{\boldsymbol{f}} : \mathcal{S}_{\boldsymbol{f}} \to \partial\mathcal{X}$ is injective, as a 1-morphism of Deligne–Mumford C^∞-stacks, and we call $\boldsymbol{f}$ *flat* if $\mathcal{T}_{\boldsymbol{f}} = \emptyset$. Simple implies semisimple.

Theorem 1.13.4. *Let $\boldsymbol{f} : \mathfrak{X} \to \mathfrak{Y}$ be a semisimple 1-morphism of d-stacks with corners. Then there exists a natural decomposition $\partial\mathfrak{X} = \partial_+^{\boldsymbol{f}}\mathfrak{X} \amalg \partial_-^{\boldsymbol{f}}\mathfrak{X}$ with $\partial_\pm^{\boldsymbol{f}}\mathfrak{X}$ open and closed in $\partial\mathfrak{X}$, such that:*

(a) *Define $\boldsymbol{f}_+ = \boldsymbol{f} \circ \boldsymbol{i}_{\mathfrak{X}}|_{\partial_+^{\boldsymbol{f}}\mathfrak{X}} : \partial_+^{\boldsymbol{f}}\mathfrak{X} \to \mathfrak{Y}$. Then $\boldsymbol{f}_+$ is semisimple. If $\boldsymbol{f}$ is flat then $\boldsymbol{f}_+$ is also flat.*

(b) *There exists a unique, semisimple 1-morphism $\boldsymbol{f}_- : \partial_-^{\boldsymbol{f}}\mathfrak{X} \to \partial\mathfrak{Y}$ with $\boldsymbol{f} \circ \boldsymbol{i}_{\mathfrak{X}}|_{\partial_-^{\boldsymbol{f}}\mathfrak{X}} = \boldsymbol{i}_{\mathfrak{Y}} \circ \boldsymbol{f}_-$. If $\boldsymbol{f}$ is simple then $\partial_+^{\boldsymbol{f}}\mathfrak{X} = \emptyset$, $\partial_-^{\boldsymbol{f}}\mathfrak{X} = \partial\mathfrak{X}$, and $\boldsymbol{f}_- : \partial\mathfrak{X} \to \partial\mathfrak{Y}$ is also simple. If $\boldsymbol{f}$ is flat then $\boldsymbol{f}_-$ is flat, and the following diagram is 2-Cartesian in* $\mathbf{dSta^c}$:

$$\begin{tikzcd}
\partial_-^{\boldsymbol{f}}\mathfrak{X} \arrow[rr, "\boldsymbol{f}_-"'] \arrow[d, "\boldsymbol{i}_{\mathfrak{X}}|_{\partial_-^{\boldsymbol{f}}\mathfrak{X}}"'] \arrow[drr, phantom, "\Uparrow \mathbf{id}_{\boldsymbol{i}_{\mathfrak{Y}}\circ \boldsymbol{f}_-}"] & & \partial\mathfrak{Y} \arrow[d, "\boldsymbol{i}_{\mathfrak{Y}}"] \\
\mathfrak{X} \arrow[rr, "\boldsymbol{f}"] & & \mathfrak{Y}.
\end{tikzcd}$$

(c) *Let $\boldsymbol{g} : \mathfrak{X} \to \mathfrak{Y}$ be another 1-morphism and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism in* $\mathbf{dSta^c}$. *Then $\boldsymbol{g}$ is also semisimple, with $\partial_-^{\boldsymbol{g}}\mathfrak{X} = \partial_-^{\boldsymbol{f}}\mathfrak{X}$. If $\boldsymbol{f}$ is simple, or flat, then $\boldsymbol{g}$ is simple, or flat, respectively. Part* **(b)** *defines 1-morphisms $\boldsymbol{f}_-, \boldsymbol{g}_- : \partial_-^{\boldsymbol{f}}\mathfrak{X} \to \partial\mathfrak{Y}$. There is a unique 2-morphism $\boldsymbol{\eta}_- : \boldsymbol{f}_- \Rightarrow \boldsymbol{g}_-$ in* $\mathbf{dSpa^c}$ *such that $\mathbf{id}_{\boldsymbol{i}_{\mathfrak{Y}}} * \boldsymbol{\eta}_- = \boldsymbol{\eta} * \mathbf{id}_{\boldsymbol{i}_{\mathfrak{X}}|_{\partial_-^{\boldsymbol{f}}\mathfrak{X}}} : \boldsymbol{i}_{\mathfrak{Y}} \circ \boldsymbol{f}_- \Rightarrow \boldsymbol{i}_{\mathfrak{Y}} \circ \boldsymbol{g}_-$.*

1.13.4 Equivalences of d-stacks with corners, and gluing

Sections 1.3.2, 1.6.4 and 1.10.3 discussed equivalences and gluing for d-spaces, d-spaces with corners, and d-stacks. Section 11.4 generalizes these to $\mathbf{dSta^c}$.

Proposition 1.13.5. **(a)** *Suppose $\boldsymbol{f} : \mathfrak{X} \to \mathfrak{Y}$ is an equivalence in* $\mathbf{dSta^c}$. *Then $\boldsymbol{f}$ is simple and flat, and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is an equivalence in* $\mathbf{dSta}$, *where $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ and $\mathfrak{Y} = (\boldsymbol{\mathcal{Y}}, \boldsymbol{\partial\mathcal{Y}}, \boldsymbol{i}_{\mathfrak{Y}}, \omega_{\mathfrak{Y}})$. Also $\boldsymbol{f}_- : \partial\mathfrak{X} \to \partial\mathfrak{Y}$ in Theorem* 1.13.4(b) *is an equivalence in* $\mathbf{dSta^c}$.

(b) *Let $\boldsymbol{f} : \mathfrak{X} \to \mathfrak{Y}$ be a simple, flat 1-morphism in $\mathbf{dSta^c}$ with $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ an equivalence in* $\mathbf{dSta}$. *Then $\boldsymbol{f}$ is an equivalence in* $\mathbf{dSta^c}$.

Here is the analogue of Definition 1.10.5:

Definition 1.13.6. Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ be a d-stack with corners. Suppose $\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{X}}$ is an open d-substack in **dSta**. Define $\boldsymbol{\partial\mathcal{V}} = \boldsymbol{i}_{\mathfrak{X}}^{-1}(\boldsymbol{\mathcal{V}})$, as an open d-substack of $\boldsymbol{\partial\mathcal{X}}$, and $\boldsymbol{i}_{\mathcal{V}} : \boldsymbol{\partial\mathcal{V}} \to \boldsymbol{\mathcal{V}}$ by $\boldsymbol{i}_{\mathcal{V}} = \boldsymbol{i}_{\mathfrak{X}}|_{\boldsymbol{\partial\mathcal{V}}}$. Then $\partial\mathcal{V} \subseteq \partial\mathcal{X}$ is open, and the conormal bundle of $\boldsymbol{\partial\mathcal{V}}$ in $\boldsymbol{\mathcal{V}}$ is $\mathcal{N}_{\mathcal{V}} = \mathcal{N}_{\mathfrak{X}}|_{\partial\mathcal{V}}$ in $\mathrm{qcoh}(\partial\mathcal{V})$. Define an orientation $\omega_{\mathcal{V}}$ on $\mathcal{N}_{\mathcal{V}}$ by $\omega_{\mathcal{V}} = \omega_{\mathfrak{X}}|_{\partial\mathcal{V}}$. Write $\boldsymbol{\mathcal{V}} = (\boldsymbol{\mathcal{V}}, \boldsymbol{\partial\mathcal{V}}, \boldsymbol{i}_{\mathcal{V}}, \omega_{\mathcal{V}})$. Then $\boldsymbol{\mathcal{V}}$ is a d-stack with corners. We call $\boldsymbol{\mathcal{V}}$ an *open d-substack* of $\mathfrak{X}$. An *open cover* of $\mathfrak{X}$ is a family $\{\boldsymbol{\mathcal{V}}_a : a \in A\}$ of open d-substacks $\boldsymbol{\mathcal{V}}_a$ of $\mathfrak{X}$ with $\mathcal{X} = \bigcup_{a\in A} \mathcal{V}_a$.

Theorem 1.13.7. *Proposition* 1.10.6 *and Theorems* 1.10.7 *and* 1.10.8 *hold without change in the* 2*-category* $\mathbf{dSta^c}$ *of d-stacks with corners.*

1.13.5 Corners $C_k(\mathfrak{X})$, and the corner functors $C, \hat{C}$

Section 11.5 generalizes the material of §1.5.3, §1.6.5, and §1.12.3 to d-stacks with corners. Here are the main results.

Theorem 1.13.8. (a) *Let $\mathfrak{X}$ be a d-stack with corners. Then for each $k \geqslant 0$ we can define a d-stack with corners $C_k(\mathfrak{X})$ called the k-**corners** of $\mathfrak{X}$, and a* 1*-morphism $\boldsymbol{\Pi}_{\mathfrak{X}}^k : C_k(\mathfrak{X}) \to \mathfrak{X}$, such that $C_k(\mathfrak{X})$ is equivalent to a quotient d-stack $[\partial^k\mathfrak{X}/S_k]$ for a natural action of S_k on $\partial^k\mathfrak{X}$ by* 1*-isomorphisms. The construction of $C_k(\mathfrak{X})$ is unique up to canonical* 1*-isomorphism.*

We can describe the topological space $C_k(\mathcal{X})_{\mathrm{top}}$ as follows. Consider pairs $(x, \{x'_1, \ldots, x'_k\})$, where $x : \underline{\bar{*}} \to \mathcal{X}$ and $x'_i : \underline{\bar{*}} \to \partial\mathcal{X}$ for $i = 1, \ldots, k$ are 1*-morphisms in $\mathbf{C^\infty Sta}$ with $x'_1, \ldots, x'_k$ distinct and $x = i_{\mathfrak{X}} \circ x'_1 = \cdots = i_{\mathfrak{X}} \circ x'_k$. Define an equivalence relation $\approx$ on such pairs by $(x, \{x'_1, \ldots, x'_k\}) \approx (\tilde{x}, \{\tilde{x}'_1, \ldots, \tilde{x}'_k\})$ if there exist $\sigma \in S_k$ and* 2*-morphisms $\eta : x \Rightarrow \tilde{x}$ and $\eta'_i : x'_i \Rightarrow \tilde{x}'_{\sigma(i)}$ for $i = 1, \ldots, k$ with $\eta = \mathrm{id}_{i_{\mathfrak{X}}} * \eta'_1 = \cdots = \mathrm{id}_{i_{\mathfrak{X}}} * \eta'_k$. Write $[x, \{x'_1, \ldots, x'_k\}]$ for the $\approx$-equivalence class of $(x, \{x'_1, \ldots, x'_k\})$. Then*

$$C_k(\mathcal{X})_{\mathrm{top}} \cong \bigl\{[x, \{x'_1, \ldots, x'_k\}] : x : \underline{\bar{*}} \to \mathcal{X},\ x'_i : \underline{\bar{*}} \to \partial\mathcal{X} \text{ 1-morphisms with } x'_1, \ldots, x'_k \text{ distinct and } x = i_{\mathfrak{X}} \circ x'_1 = \cdots = i_{\mathfrak{X}} \circ x'_k\bigr\}. \tag{1.79}$$

We have 1*-isomorphisms $C_0(\mathfrak{X}) \cong \mathfrak{X}$ and $C_1(\mathfrak{X}) \cong \partial\mathfrak{X}$. We write $C(\mathfrak{X}) = \coprod_{k\geqslant 0} C_k(\mathfrak{X})$, so that $C(\mathfrak{X})$ is a d-stack with corners, called the **corners** of $\mathfrak{X}$.*
(b) *Let $\boldsymbol{f} : \mathfrak{X} \to \mathfrak{Y}$ be a* 1*-morphism of d-stacks with corners. Then there are unique* 1*-morphisms $C(\boldsymbol{f}) : C(\mathfrak{X}) \to C(\mathfrak{Y})$ and $\hat{C}(\boldsymbol{f}) : C(\mathfrak{X}) \to C(\mathfrak{Y})$ in $\mathbf{dSta^c}$ such that $\boldsymbol{\Pi}_{\mathfrak{Y}} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \boldsymbol{\Pi}_{\mathfrak{X}} = \boldsymbol{\Pi}_{\mathfrak{Y}} \circ \hat{C}(\boldsymbol{f}) : C(\mathfrak{X}) \to \mathfrak{Y}$, with maps $C(f)_{\mathrm{top}} : C(\mathcal{X})_{\mathrm{top}} \to C(\mathcal{Y})_{\mathrm{top}}$, $\hat{C}(f)_{\mathrm{top}} : C(\mathcal{X})_{\mathrm{top}} \to C(\mathcal{Y})_{\mathrm{top}}$ characterized as follows. Identify $C_k(\mathcal{X})_{\mathrm{top}} \subseteq C(\mathcal{X})_{\mathrm{top}}$ with the right hand side of* (1.79)*, and similarly for $C_l(\mathcal{Y})_{\mathrm{top}}$, and identify $\mathcal{S}_{\boldsymbol{f},\mathrm{top}}, \mathcal{T}_{\boldsymbol{f},\mathrm{top}}$ with subsets of* (1.77)–(1.78)

as in §1.13.3. *Then* $C(f)_{\rm top}$ *and* $\hat C(f)_{\rm top}$ *act by*

$$\begin{aligned}
&C(f)_{\rm top}:\bigl[x,\{x'_1,\ldots,x'_k\}\bigr]\longmapsto\bigl[y,\{y'_1,\ldots,y'_l\}\bigr],\;\; \textit{where } y=f\circ x,\\
&\{y'_1,\ldots,y'_l\}=\bigl\{y':[x'_i,y']\in\mathcal{S}_{\boldsymbol{f},\rm top},\;\textit{some } i=1,\ldots,k\bigr\},\;\textit{and}
\end{aligned}\tag{1.80}$$

$$\begin{aligned}
&\hat C(f)_{\rm top}:\bigl[x,\{x'_1,\ldots,x'_k\}\bigr]\longmapsto\bigl[y,\{y'_1,\ldots,y'_l\}\bigr],\;\; \textit{where } y=f\circ x,\\
&\{y'_1,\ldots,y'_l\}=\bigl\{y':[x'_i,y']\in\mathcal{S}_{\boldsymbol{f},\rm top},\;i=1,\ldots,k\bigr\}\cup\bigl\{y':[x,y']\in\mathcal{T}_{\boldsymbol{f},\rm top}\bigr\}.
\end{aligned}\tag{1.81}$$

For all $k,l\geqslant 0$, *write* $C_k^{\boldsymbol{f},l}(\boldsymbol{\mathcal{X}})=C_k(\boldsymbol{\mathcal{X}})\cap C(\boldsymbol{f})^{-1}(C_l(\boldsymbol{\mathcal{Y}}))$, *so that* $C_k^{\boldsymbol{f},l}(\boldsymbol{\mathcal{X}})$ *is an open and closed d-substack of* $C_k(\boldsymbol{\mathcal{X}})$ *with* $C_k(\boldsymbol{\mathcal{X}})=\coprod_{l=0}^\infty C_k^{\boldsymbol{f},l}(\boldsymbol{\mathcal{X}})$, *and write* $C_k^l(\boldsymbol{f})=C(\boldsymbol{f})|_{C_k^{\boldsymbol{f},l}(\boldsymbol{\mathcal{X}})}:C_k^{\boldsymbol{f},l}(\boldsymbol{\mathcal{X}})\rightarrow C_l(\boldsymbol{\mathcal{Y}})$. *If* $\boldsymbol{f}$ *is semisimple then* $C(\boldsymbol{f})$ *maps* $C_k(\boldsymbol{\mathcal{X}})\rightarrow\coprod_{l=0}^k C_l(\boldsymbol{\mathcal{Y}})$ *for all* $k\geqslant 0$. *If* $\boldsymbol{f}$ *is simple then* $C(\boldsymbol{f})$ *maps* $C_k(\boldsymbol{\mathcal{X}})\rightarrow C_k(\boldsymbol{\mathcal{Y}})$ *for all* $k\geqslant 0$. *If* $\boldsymbol{f}$ *is flat then* $C(\boldsymbol{f})=\hat C(\boldsymbol{f})$.

(c) *Let* $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ *be* 1*-morphisms and* $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ *a* 2*-morphism in* $\mathbf{dSta^c}$. *Then there exist unique* 2*-morphisms* $C(\boldsymbol{\eta}):C(\boldsymbol{f})\Rightarrow C(\boldsymbol{g})$, $\hat C(\boldsymbol{\eta}):\hat C(\boldsymbol{f})\Rightarrow\hat C(\boldsymbol{g})$ *in* $\mathbf{dSta^c}$, *where* $C(\boldsymbol{f}),C(\boldsymbol{g}),\hat C(\boldsymbol{f}),\hat C(\boldsymbol{g})$ *are as in* **(b)**, *such that*

$$\begin{aligned}
&\mathrm{id}_{\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}}*C(\boldsymbol{\eta})=\boldsymbol{\eta}*\mathrm{id}_{\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}}:\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}\circ C(\boldsymbol{f})=\boldsymbol{f}\circ\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}\Longrightarrow\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}\circ C(\boldsymbol{g})=\boldsymbol{g}\circ\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}},\\
&\mathrm{id}_{\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}}*\hat C(\boldsymbol{\eta})=\boldsymbol{\eta}*\mathrm{id}_{\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}}:\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}\circ\hat C(\boldsymbol{f})=\boldsymbol{f}\circ\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}\Longrightarrow\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}\circ\hat C(\boldsymbol{g})=\boldsymbol{g}\circ\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}.
\end{aligned}$$

If $\boldsymbol{f},\boldsymbol{g}$ *are flat then* $C(\boldsymbol{\eta})=\hat C(\boldsymbol{\eta})$.

(d) *Define* $C:\mathbf{dSta^c}\rightarrow\mathbf{dSta^c}$ *by* $C:\boldsymbol{\mathcal{X}}\mapsto C(\boldsymbol{\mathcal{X}})$, $C:\boldsymbol{f}\mapsto C(\boldsymbol{f})$, $C:\boldsymbol{\eta}\mapsto C(\boldsymbol{\eta})$ *on objects,* 1*- and* 2*-morphisms, where* $C(\boldsymbol{\mathcal{X}}),C(\boldsymbol{f}),C(\boldsymbol{\eta})$ *are as in* **(a)**–**(c)** *above. Similarly, define* $\hat C:\mathbf{dSta^c}\rightarrow\mathbf{dSta^c}$ *by* $\hat C:\boldsymbol{\mathcal{X}}\mapsto C(\boldsymbol{\mathcal{X}})$, $\hat C:\boldsymbol{f}\mapsto\hat C(\boldsymbol{f})$, $\hat C:\boldsymbol{\eta}\mapsto\hat C(\boldsymbol{\eta})$. *Then* $C,\hat C$ *are strict* 2*-functors, called* ***corner functors***.

1.13.6 Fibre products in $\mathbf{dSta^c}$

Section 11.6 generalizes §1.6.6 and §1.10.4 to d-stacks with corners. Here are the analogues of Definition 1.6.12, Lemma 1.6.13 and Theorem 1.6.14:

Definition 1.13.9. Let $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Z}}$ and $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\rightarrow\boldsymbol{\mathcal{Z}}$ be 1-morphisms in $\mathbf{dSta^c}$. As in §1.13.1 we have line bundles $\mathcal{N}_{\boldsymbol{\mathcal{X}}},\mathcal{N}_{\boldsymbol{\mathcal{Z}}}$ over the C^∞-stacks $\partial\mathcal{X},\partial\mathcal{Z}$, and §1.13.3 defines a C^∞-substack $\mathcal{S}_{\boldsymbol{g}}\subseteq\partial\mathcal{X}\times_{\mathcal{Z}}\partial\mathcal{Z}$. As in §11.1, there is a natural isomorphism $\lambda_{\boldsymbol{g}}:u_{\boldsymbol{g}}^*(\mathcal{N}_{\boldsymbol{\mathcal{Z}}})\rightarrow s_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathcal{X}}})$ in $\mathrm{qcoh}(\mathcal{S}_{\boldsymbol{g}})$. The same holds for $\boldsymbol{h}$.

We call $\boldsymbol{g},\boldsymbol{h}$ *b-transverse* if the following holds. Suppose $x:\underline{\bar *}\rightarrow\mathcal{X}$ and $y:\underline{\bar *}\rightarrow\mathcal{Y}$ are 1-morphisms in $\mathbf{C^\infty Sta}$, and $\eta:g\circ x\Rightarrow h\circ y$ is a 2-morphism. Since $i_{\boldsymbol{\mathcal{X}}}:\partial\mathcal{X}\rightarrow\mathcal{X}$ is finite and strongly representable, there are finitely many 1-morphisms $x':\underline{\bar *}\rightarrow\partial\mathcal{X}$ with $x=i_{\boldsymbol{\mathcal{X}}}\circ x'$. Write these x' as $x'_1,\ldots,x'_j$. Similarly, write $y'_1,\ldots,y'_k$ for the 1-morphisms $y':\underline{\bar *}\rightarrow\partial\mathcal{Y}$ with $y=i_{\boldsymbol{\mathcal{Y}}}\circ y'$. Write $z=g\circ x$ and $\tilde z=h\circ y$, so that $z,\tilde z:\underline{\bar *}\rightarrow\mathcal{Z}$ and $\eta:z\Rightarrow\tilde z$. Write $z'_1,\ldots,z'_l$ for the 1-morphisms $z':\underline{\bar *}\rightarrow\partial\mathcal{Z}$ with $z=i_{\boldsymbol{\mathcal{Z}}}\circ z'$. Then by Proposition 1.8.9, for each $c=1,\ldots,l$ there are unique $\tilde z'_c:\underline{\bar *}\rightarrow\partial\mathcal{Z}$ and $\eta'_c:z'_c\Rightarrow\tilde z'_c$ with $i_{\boldsymbol{\mathcal{Z}}}\circ\tilde z'_c=\tilde z$ and $\mathrm{id}_{i_{\boldsymbol{\mathcal{Z}}}}*\eta'_c=\eta$.

Definition 1.13.3 defined $\mathcal{S}_{\boldsymbol{g}} \subseteq \partial\mathcal{X} \times_{\mathcal{Z}} \partial\mathcal{Z}$ in terms of points $[x', z']$ in (1.77); write $(x', z') : \bar{*} \to \mathcal{S}_{\boldsymbol{g}}$ for the corresponding 1-morphisms. Then we require that for all such x, y, η, the following morphism in $\mathrm{qcoh}(\bar{*})$ is injective:

$$\begin{aligned}&\textstyle\bigoplus_{a=1,\ldots,j,\ c=1,\ldots,l:\ [x'_a,z'_c]\in\mathcal{S}_{\boldsymbol{g},\mathrm{top}}} I_{(x'_a,z'_c),s_{\boldsymbol{g}}}(\mathcal{N}_{\boldsymbol{\mathcal{X}}})^{-1} \circ (x'_a, z'_c)^*(\lambda_{\boldsymbol{g}}) \circ I_{(x'_a,z'_c),u_{\boldsymbol{g}}}(\mathcal{N}_{\boldsymbol{\mathcal{Z}}}) \oplus \\ &\textstyle\bigoplus_{b=1,\ldots,k,\ c=1,\ldots,l:\ [y'_b,\tilde{z}'_c]\in\mathcal{S}_{\boldsymbol{h},\mathrm{top}}} I_{(y'_b,\tilde{z}'_c),s_{\boldsymbol{h}}}(\mathcal{N}_{\boldsymbol{\mathcal{Y}}})^{-1} \circ (y'_b, \tilde{z}'_c)^*(\lambda_{\boldsymbol{h}}) \circ I_{(y'_b,\tilde{z}'_c),u_{\boldsymbol{h}}}(\mathcal{N}_{\boldsymbol{\mathcal{Z}}}) \circ (\eta'_c)^*(\mathcal{N}_{\boldsymbol{\mathcal{Z}}}) : \\ &\textstyle\bigoplus_{c=1}^{l} (z'_c)^*(\mathcal{N}_{\boldsymbol{\mathcal{Z}}}) \longrightarrow \bigoplus_{a=1}^{j} (x'_a)^*(\mathcal{N}_{\boldsymbol{\mathcal{X}}}) \oplus \bigoplus_{b=1}^{k} (y'_b)^*(\mathcal{N}_{\boldsymbol{\mathcal{Y}}}).\end{aligned}$$

We call $\boldsymbol{g}, \boldsymbol{h}$ *c-transverse* if the following holds. Identify $C_k(\mathcal{X})_{\mathrm{top}} \subseteq C(\mathcal{X})_{\mathrm{top}}$ with the right hand of (1.79), and similarly for $C(\mathcal{Y})_{\mathrm{top}}, C(\mathcal{Z})_{\mathrm{top}}$. Then $C(g)_{\mathrm{top}}$, $C(h)_{\mathrm{top}}, \hat{C}(g)_{\mathrm{top}}, \hat{C}(h)_{\mathrm{top}}$ act as in (1.80)–(1.81). We require that:

(a) whenever there are points in $C_j(\mathcal{X})_{\mathrm{top}}, C_k(\mathcal{Y})_{\mathrm{top}}, C_l(\mathcal{Z})_{\mathrm{top}}$ with

$$C(g)_{\mathrm{top}}\big([x, \{x'_1, \ldots, x'_j\}]\big) = C(h)_{\mathrm{top}}\big([y, \{y'_1, \ldots, y'_k\}]\big) = [z, \{z'_1, \ldots, z'_l\}],$$

we have either $j + k > l$ or $j = k = l = 0$; and

(b) whenever there are points in $C_j(\mathcal{X})_{\mathrm{top}}, C_k(\mathcal{Y})_{\mathrm{top}}, C_l(\mathcal{Z})_{\mathrm{top}}$ with

$$\hat{C}(g)_{\mathrm{top}}\big([x, \{x'_1, \ldots, x'_j\}]\big) = \hat{C}(h)_{\mathrm{top}}\big([y, \{y'_1, \ldots, y'_k\}]\big) = [z, \{z'_1, \ldots, z'_l\}],$$

we have $j + k \geqslant l$.

Then $\boldsymbol{g}, \boldsymbol{h}$ c-transverse implies $\boldsymbol{g}, \boldsymbol{h}$ b-transverse.

Lemma 1.13.10. *Let $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$ and $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms in $\mathbf{dSta^c}$. The following are sufficient conditions for $\boldsymbol{g}, \boldsymbol{h}$ to be c-transverse, and hence b-transverse:*

(i) *$\boldsymbol{g}$ or $\boldsymbol{h}$ is semisimple and flat; or*

(ii) *$\boldsymbol{\mathcal{Z}}$ is a d-stack without boundary.*

Theorem 1.13.11. (a) *All b-transverse fibre products exist in $\mathbf{dSta^c}$.*

(b) *The 2-functor $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}} : \mathbf{dSpa^c} \to \mathbf{dSta^c}$ of §1.13.1 takes b- and c-transverse fibre products in $\mathbf{dSpa^c}$ to b- and c-transverse fibre products in $\mathbf{dSta^c}$.*

(c) *The 2-functor $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}$ of §1.13.1 takes transverse fibre products in $\mathbf{Orb^c}$ to b-transverse fibre products in $\mathbf{dSta^c}$. That is, if*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "f"] \arrow[d, "e"] \arrow[dr, phantom, "\eta \Uparrow"] & \mathcal{Y} \arrow[d, "h"] \\ \mathcal{X} \arrow[r, "g"] & \mathcal{Z} \end{tikzcd}$$

is a 2-Cartesian square in $\mathbf{Orb^c}$ with g, h transverse, and $\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{\eta} = F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}, e, f, g, h, \eta)$, then

$$\begin{tikzcd} \boldsymbol{\mathcal{W}} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"] \arrow[dr, phantom, "\boldsymbol{\eta} \Uparrow"] & \boldsymbol{\mathcal{Y}} \arrow[d, "\boldsymbol{h}"] \\ \boldsymbol{\mathcal{X}} \arrow[r, "\boldsymbol{g}"] & \boldsymbol{\mathcal{Z}} \end{tikzcd}$$

is 2*-Cartesian in* $\mathbf{dSta^c}$*, with* $\boldsymbol{g},\boldsymbol{h}$ *b-transverse. If also* g,h *are strongly transverse in* $\mathbf{Orb^c}$*, then* $\boldsymbol{g},\boldsymbol{h}$ *are c-transverse in* $\mathbf{dSta^c}$*.*

(d) *Suppose we are given a* 2*-Cartesian diagram in* $\mathbf{dSta^c}$*:*

$$\begin{tikzcd}
\mathcal{W} \arrow[rr, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"'] & \arrow[d, phantom, "\boldsymbol{\eta}\Uparrow"] & \mathcal{Y} \arrow[d, "\boldsymbol{h}"] \\
\mathcal{X} \arrow[rr, "\boldsymbol{g}"] & & \mathcal{Z},
\end{tikzcd}$$

with $\boldsymbol{g},\boldsymbol{h}$ *c-transverse. Then the following are also* 2*-Cartesian in* $\mathbf{dSta^c}$*:*

$$\begin{tikzcd}
C(\mathcal{W}) \arrow[rr, "C(\boldsymbol{f})"] \arrow[d, "C(\boldsymbol{e})"'] & \arrow[d, phantom, "C(\boldsymbol{\eta})\Uparrow"] & C(\mathcal{Y}) \arrow[d, "C(\boldsymbol{h})"] \\
C(\mathcal{X}) \arrow[rr, "C(\boldsymbol{g})"] & & C(\mathcal{Z}),
\end{tikzcd} \tag{1.82}$$

$$\begin{tikzcd}
C(\mathcal{W}) \arrow[rr, "\hat C(\boldsymbol{f})"] \arrow[d, "\hat C(\boldsymbol{e})"'] & \arrow[d, phantom, "\hat C(\boldsymbol{\eta})\Uparrow"] & C(\mathcal{Y}) \arrow[d, "\hat C(\boldsymbol{h})"] \\
C(\mathcal{X}) \arrow[rr, "\hat C(\boldsymbol{g})"] & & C(\mathcal{Z}).
\end{tikzcd} \tag{1.83}$$

Also (1.82)–(1.83) *preserve gradings, in that they relate points in* $C_i(\mathcal{W}),C_j(\mathcal{X})$*,* $C_k(\mathcal{Y}),C_k(\mathcal{Z})$ *with* $i=j+k-l$*. Hence* (1.82) *implies equivalences in* $\mathbf{dSta^c}$*:*

$$\begin{aligned}
C_i(\mathcal{W}) &\simeq \coprod_{j,k,l\geqslant 0: i=j+k-l} C_j^{\boldsymbol{g},l}(\mathcal{X}) \times_{C_j^l(\boldsymbol{g}),C_l(\mathcal{Z}),C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathcal{Y}),\\
\partial\mathcal{W} &\simeq \coprod_{j,k,l\geqslant 0: j+k=l+1} C_j^{\boldsymbol{g},l}(\mathcal{X}) \times_{C_j^l(\boldsymbol{g}),C_l(\mathcal{Z}),C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathcal{Y}).
\end{aligned}$$

The analogue of Proposition 1.6.15 also holds in $\mathbf{dSta^c}$.

1.13.7 Orbifold strata of d-stacks with corners

Section 11.7 combines material in §1.10.5 and §1.12.5 on orbifold strata of d-stacks and of orbifolds with corners. It is also related to §1.6.7 on fixed loci in d-spaces with corners. Here is the analogue of Theorem 1.10.11.

Theorem 1.13.12. *Let* $\mathcal{X}$ *be a d-stack with corners, and* Γ *a finite group. Then we can define d-stacks with corners* $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma$*, and open d-substacks* $\mathcal{X}^\Gamma_\circ\subseteq\mathcal{X}^\Gamma$*,* $\tilde{\mathcal{X}}^\Gamma_\circ\subseteq\tilde{\mathcal{X}}^\Gamma$*,* $\hat{\mathcal{X}}^\Gamma_\circ\subseteq\hat{\mathcal{X}}^\Gamma$*, all natural up to* 1*-isomorphism in* $\mathbf{dSta^c}$*, a d-space with corners* $\hat{\mathbf{X}}^\Gamma_\circ$ *natural up to* 1*-isomorphism in* $\mathbf{dSpa^c}$*, and* 1*-morphisms* $\boldsymbol{O}^\Gamma(\mathcal{X}),\tilde{\boldsymbol{\Pi}}^\Gamma(\mathcal{X}),\ldots$ *fitting into a strictly commutative diagram in* $\mathbf{dSta^c}$*:*

$$\begin{tikzcd}
\mathcal{X}^\Gamma_\circ \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde{\boldsymbol{\Pi}}^\Gamma_\circ(\mathcal{X})"] \arrow[d, "\subset"', "\boldsymbol{O}^\Gamma_\circ(\mathcal{X})"] \arrow[dr] & & \tilde{\mathcal{X}}^\Gamma_\circ \arrow[r, "\hat{\boldsymbol{\Pi}}^\Gamma_\circ(\mathcal{X})"] \arrow[d, "\subset", "\tilde{\boldsymbol{O}}^\Gamma_\circ(\mathcal{X})"'] \arrow[dl] & \hat{\mathcal{X}}^\Gamma_\circ \simeq F^{\mathbf{dSta^c}}_{\mathbf{dSpa^c}}(\hat{\mathbf{X}}^\Gamma_\circ) \arrow[d, "\subset"] \\
\mathcal{X}^\Gamma \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde{\boldsymbol{\Pi}}^\Gamma(\mathcal{X})"'] \arrow[r, "\boldsymbol{O}^\Gamma(\mathcal{X})"] & \mathcal{X} & \tilde{\mathcal{X}}^\Gamma \arrow[r, "\hat{\boldsymbol{\Pi}}^\Gamma(\mathcal{X})"'] \arrow[l, "\tilde{\boldsymbol{O}}^\Gamma(\mathcal{X})"'] & \hat{\mathcal{X}}^\Gamma.
\end{tikzcd} \tag{1.84}$$

We will call $\mathfrak{X}^\Gamma, \tilde{\mathfrak{X}}^\Gamma, \hat{\mathfrak{X}}^\Gamma, \mathfrak{X}^\Gamma_\circ, \tilde{\mathfrak{X}}^\Gamma_\circ, \hat{\mathfrak{X}}^\Gamma_\circ, \hat{\mathbf{X}}^\Gamma_\circ$ *the* ***orbifold strata*** *of* $\mathfrak{X}$.

The underlying d-stacks of $\mathfrak{X}^\Gamma, \ldots, \hat{\mathfrak{X}}^\Gamma_\circ$ *are the orbifold strata* $\boldsymbol{\mathcal{X}}^\Gamma, \ldots, \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ *from* §1.10.5 *of the d-stack* $\boldsymbol{\mathcal{X}}$ *in* $\mathfrak{X}$. *The* 1*-morphisms* (1.84), *as* 1*-morphisms in* $\mathbf{dSta}$, *are those given in* (1.55).

The rest of §1.10.5 also extends to $\mathbf{dSta^c}$:

Theorem 1.13.13. *Theorems* 1.10.12, 1.10.14, 1.10.15 *and Corollary* 1.10.13 *hold without change in* $\mathbf{dSta^c}, \mathbf{dSpa^c}$ *rather than* $\mathbf{dSta}, \mathbf{dSpa}$.

Here are analogues of Theorem 1.12.15 and Corollary 1.12.17.

Theorem 1.13.14. *Let* $\mathfrak{X}$ *be a d-stack with corners, and* Γ *a finite group. The corners* $C(\mathfrak{X})$ *from* §1.13.5 *lie in* $\mathbf{dSta^c}$, *so Theorem* 1.13.12 *gives orbifold strata* $\mathfrak{X}^\Gamma, C(\mathfrak{X})^\Gamma$ *and* 1*-morphisms* $\boldsymbol{O}^\Gamma(\mathfrak{X}) : \mathfrak{X}^\Gamma \to \mathfrak{X}$, $\boldsymbol{O}^\Gamma(C(\mathfrak{X})) : C(\mathfrak{X})^\Gamma \to C(\mathfrak{X})$. *Applying the corner functor* C *from* §1.13.5 *gives a* 1*-morphism* $C(\boldsymbol{O}^\Gamma(\mathfrak{X})) : C(\mathfrak{X}^\Gamma) \to C(\mathfrak{X})$. *There exists a unique equivalence* $\boldsymbol{K}^\Gamma(\mathfrak{X}) : C(\mathfrak{X}^\Gamma) \to C(\mathfrak{X})^\Gamma$ *in* $\mathbf{dSta^c}$ *with* $\boldsymbol{O}^\Gamma(C(\mathfrak{X})) \circ \boldsymbol{K}^\Gamma(\mathfrak{X}) = C(\boldsymbol{O}^\Gamma(\mathfrak{X})) : C(\mathfrak{X}^\Gamma) \to C(\mathfrak{X})$. *It restricts to an equivalence* $\boldsymbol{K}^\Gamma_\circ(\mathfrak{X}) := \boldsymbol{K}^\Gamma(\mathfrak{X})|_{C(\mathfrak{X}^\Gamma_\circ)} : C(\mathfrak{X}^\Gamma_\circ) \to C(\mathfrak{X})^\Gamma_\circ$.

Similarly, there is a unique equivalence $\tilde{\boldsymbol{K}}^\Gamma(\mathfrak{X}) : C(\tilde{\mathfrak{X}}^\Gamma) \to \widetilde{C(\mathfrak{X})}^\Gamma$ *with* $\tilde{\boldsymbol{O}}^\Gamma(C(\mathfrak{X})) \circ \tilde{\boldsymbol{K}}^\Gamma(\mathfrak{X}) = C(\tilde{\boldsymbol{O}}^\Gamma(\mathfrak{X}))$ *and* $\tilde{\mathbf{\Pi}}^\Gamma(C(\mathfrak{X})) \circ \boldsymbol{K}^\Gamma(\mathfrak{X}) = \tilde{\boldsymbol{K}}^\Gamma(\mathfrak{X}) \circ C(\tilde{\mathbf{\Pi}}^\Gamma(\mathfrak{X}))$. *There is an equivalence* $\hat{\boldsymbol{K}}^\Gamma(\mathfrak{X}) : C(\hat{\mathfrak{X}}^\Gamma) \to \widehat{C(\mathfrak{X})}^\Gamma$, *unique up to* 2*-isomorphism, with a* 2*-morphism* $\hat{\mathbf{\Pi}}^\Gamma(C(\mathfrak{X})) \circ \tilde{\boldsymbol{K}}^\Gamma(\mathfrak{X}) \Rightarrow \hat{\boldsymbol{K}}^\Gamma(\mathfrak{X}) \circ C(\hat{\mathbf{\Pi}}^\Gamma(\mathfrak{X}))$. *They restrict to equivalences* $\tilde{\boldsymbol{K}}^\Gamma_\circ(\mathfrak{X}) : C(\tilde{\mathfrak{X}}^\Gamma_\circ) \to \widetilde{C(\mathfrak{X})}^\Gamma_\circ$ *and* $\hat{\boldsymbol{K}}^\Gamma_\circ(\mathfrak{X}) : C(\hat{\mathfrak{X}}^\Gamma_\circ) \to \widehat{C(\mathfrak{X})}^\Gamma_\circ$.

Corollary 1.13.15. *Let* $\mathfrak{X}$ *be a d-stack with corners, and* Γ *a finite group. Then there exist* 1*-morphisms* $\boldsymbol{J}^\Gamma(\mathfrak{X}) : (\partial\mathfrak{X})^\Gamma \to \partial(\mathfrak{X}^\Gamma)$, $\tilde{\boldsymbol{J}}^\Gamma(\mathfrak{X}) : (\widetilde{\partial\mathfrak{X}})^\Gamma \to \partial(\tilde{\mathfrak{X}}^\Gamma)$, $\hat{\boldsymbol{J}}^\Gamma(\mathfrak{X}) : (\widehat{\partial\mathfrak{X}})^\Gamma \to \partial(\hat{\mathfrak{X}}^\Gamma)$ *in* $\mathbf{dSta^c}$, *natural up to* 2*-isomorphism, such that* $\boldsymbol{J}^\Gamma(\mathfrak{X})$ *is an equivalence from* $(\partial\mathfrak{X})^\Gamma$ *to an open and closed d-substack of* $\partial(\mathfrak{X}^\Gamma)$, *and similarly for* $\tilde{\boldsymbol{J}}^\Gamma(\mathfrak{X}), \hat{\boldsymbol{J}}^\Gamma(\mathfrak{X})$.

A d-stack with corners $\mathfrak{X}$ is called *straight* if $(i_{\mathfrak{X}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ is an isomorphism for all $[x'] \in \partial\mathcal{X}_{\mathrm{top}}$ with $i_{\mathfrak{X},\mathrm{top}}([x']) = [x]$. D-stacks with boundary are automatically straight. If $\mathfrak{X}$ is straight then $\partial\mathfrak{X}$ is straight, so by induction $\partial^k\mathfrak{X}$ is also straight for all $k \geqslant 0$. If $\mathfrak{X}$ is straight then $\boldsymbol{K}^\Gamma(\mathfrak{X})$ in Theorem 1.13.14 is an equivalence $C_k(\mathfrak{X}^\Gamma) \to C_k(\mathfrak{X})^\Gamma$ for all $k \geqslant 0$, and so $\boldsymbol{J}^\Gamma(\mathfrak{X})$ in Corollary 1.13.15 is an equivalence $(\partial\mathfrak{X})^\Gamma \to \partial(\mathfrak{X}^\Gamma)$. The same applies for $\tilde{\boldsymbol{J}}^\Gamma(\mathfrak{X}), \hat{\boldsymbol{J}}^\Gamma(\mathfrak{X}), \tilde{\boldsymbol{K}}^\Gamma(\mathfrak{X}), \hat{\boldsymbol{K}}^\Gamma(\mathfrak{X})$.

1.14 D-orbifolds with corners

Chapter 12 discusses the 2-category $\mathbf{dOrb^c}$ of *d-orbifolds with corners*. Again, there are few new issues here: almost all the material just combines ideas we have seen already on d-manifolds with corners from §1.7, on orbifolds with corners from §1.12, and on d-stacks with corners from §1.13. So we will be brief.

1.14.1 Definition of d-orbifolds with corners

Section 12.1 defines d-orbifolds with corners, following §1.7.1 and §1.11.1.

Definition 1.14.1. A d-stack with corners $\boldsymbol{\mathcal{W}}$ is called a *principal d-orbifold with corners* if is equivalent in $\mathbf{dSta^c}$ to a fibre product $\boldsymbol{\mathcal{V}}\times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}}\boldsymbol{\mathcal{V}}$, where $\mathcal{V}$ is an orbifold with corners, $\mathcal{E}$ is a vector bundle on $\mathcal{V}$, $s\in C^\infty(\mathcal{E})$, and $\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{E}},\boldsymbol{s},\boldsymbol{0}=F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}\bigl(\mathcal{V},\mathrm{Tot^c}(\mathcal{E}),\mathrm{Tot^c}(s),\mathrm{Tot^c}(0)\bigr)$, for $\mathrm{Tot^c}$ as in §1.12.1. Note that $\mathrm{Tot^c}(s),\mathrm{Tot^c}(0):\mathcal{V}\to\mathrm{Tot^c}(\mathcal{E})$ are simple, flat 1-morphisms in $\mathbf{Orb^c}$, so $\boldsymbol{s},\boldsymbol{0}:\boldsymbol{\mathcal{V}}\to\boldsymbol{\mathcal{E}}$ are simple, flat 1-morphisms in $\mathbf{dSta^c}$. Thus $\boldsymbol{s},\boldsymbol{0}$ are b-transverse by Lemma 1.13.10(i), and $\boldsymbol{\mathcal{V}}\times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}}\boldsymbol{\mathcal{V}}$ exists in $\mathbf{dSta^c}$ by Theorem 1.13.11(a).

If $\boldsymbol{\mathcal{W}}$ is a nonempty principal d-orbifold with corners, then $T^*\boldsymbol{\mathcal{W}}$ is a virtual vector bundle. We define the *virtual dimension* of $\boldsymbol{\mathcal{W}}$ to be $\operatorname{vdim}\boldsymbol{\mathcal{W}}=\operatorname{rank}T^*\boldsymbol{\mathcal{W}}\in\mathbb{Z}$. If $\boldsymbol{\mathcal{W}}\simeq\boldsymbol{\mathcal{V}}\times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}}\boldsymbol{\mathcal{V}}$ then $\operatorname{vdim}\boldsymbol{\mathcal{W}}=\dim\mathcal{V}-\operatorname{rank}\mathcal{E}$.

A d-stack with corners $\boldsymbol{\mathcal{X}}$ is called a *d-orbifold with corners of virtual dimension $n\in\mathbb{Z}$*, written $\operatorname{vdim}\boldsymbol{\mathcal{X}}=n$, if $\boldsymbol{\mathcal{X}}$ can be covered by open d-substacks $\boldsymbol{\mathcal{W}}$ which are principal d-orbifolds with corners with $\operatorname{vdim}\boldsymbol{\mathcal{W}}=n$. A d-orbifold with corners $\boldsymbol{\mathcal{X}}$ is called a *d-orbifold with boundary* if it is a d-stack with boundary, and a *d-orbifold without boundary* if it is a d-stack without boundary.

Write $\mathbf{d\bar{O}rb},\mathbf{dOrb^b},\mathbf{dOrb^c}$ for the full 2-subcategories of d-orbifolds without boundary, and with boundary, and with corners, in $\mathbf{dSta^c}$, respectively. Then $\mathbf{\bar{O}rb},\mathbf{\bar{O}rb^b},\mathbf{\bar{O}rb^c}$ in §1.13.1 are full 2-subcategories of $\mathbf{d\bar{O}rb},\mathbf{dOrb^b}$, $\mathbf{dOrb^c}$. When we say that a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ *is an orbifold*, we mean that $\boldsymbol{\mathcal{X}}$ lies in $\mathbf{\bar{O}rb^c}$. Define full and faithful strict 2-functors

$$
\begin{aligned}
&F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}}:\mathbf{dOrb}\to\mathbf{d\bar{O}rb}\subset\mathbf{dOrb^c}, &&F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}:\mathbf{Orb^c}\to\mathbf{dOrb^c},\\
&F_{\mathbf{Orb^b}}^{\mathbf{dOrb^c}}:\mathbf{Orb^b}\to\mathbf{dOrb^b}\subset\mathbf{dOrb^c}, &&F_{\mathbf{Orb}}^{\mathbf{dOrb^c}}:\mathbf{Orb}\to\mathbf{d\bar{O}rb}\subset\mathbf{dOrb^c},\\
&F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}}:\mathbf{dMan^c}\to\mathbf{dOrb^c}, &&F_{\mathbf{dMan^b}}^{\mathbf{dOrb^c}}:\mathbf{dMan^b}\to\mathbf{dOrb^b}\subset\mathbf{dOrb^c},\\
&\text{and}\quad F_{\mathbf{dMan}}^{\mathbf{dOrb^c}}:\mathbf{dMan}\to\mathbf{d\bar{O}rb}\subset\mathbf{dOrb^c}, &&\text{by}
\end{aligned}
$$

$$
\begin{gathered}
F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}}=F_{\mathbf{dSta}}^{\mathbf{dSta^c}}|_{\mathbf{dOrb}},\quad F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}=F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}},\qquad F_{\mathbf{Orb^b}}^{\mathbf{dOrb^c}}=F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}|_{\mathbf{Orb^b}},\\
F_{\mathbf{Orb}}^{\mathbf{Orb^c}}=F_{\mathbf{dSta}}^{\mathbf{dSta^c}}\circ F_{\mathbf{Orb}}^{\mathbf{dSta}},\ F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}}=F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}|_{\mathbf{dMan^c}},\ F_{\mathbf{dMan^b}}^{\mathbf{dOrb^c}}=F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}|_{\mathbf{dMan^b}},\\
\text{and}\qquad F_{\mathbf{dMan}}^{\mathbf{dOrb^c}}=F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}\circ F_{\mathbf{dMan}}^{\mathbf{dMan^c}}=F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}}\circ F_{\mathbf{dMan}}^{\mathbf{dOrb}},
\end{gathered}
$$

where $F_{\mathbf{Orb}}^{\mathbf{Orb^c}},F_{\mathbf{Orb}}^{\mathbf{dSta}},F_{\mathbf{dSta}}^{\mathbf{dSta^c}},F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}},F_{\mathbf{dMan}}^{\mathbf{dMan^c}},F_{\mathbf{dMan}}^{\mathbf{dOrb}},F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}$ are as in §1.7.1, §1.11.1, §1.12.1, and §1.13.1. Here $F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}}:\mathbf{dOrb}\to\mathbf{d\bar{O}rb}$ is an isomorphism of 2-categories. So we may as well identify $\mathbf{dOrb}$ with its image $\mathbf{d\bar{O}rb}$, and consider d-orbifolds in §1.11 as examples of d-orbifolds with corners.

Write $\mathbf{d\hat{M}an^c}$ for the full 2-subcategory of objects $\boldsymbol{\mathcal{X}}$ in $\mathbf{dOrb^c}$ equivalent to $F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}}(\mathbf{X})$ for some d-manifold with corners $\mathbf{X}$. When we say that a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ *is a d-manifold*, we mean that $\boldsymbol{\mathcal{X}}\in\mathbf{d\hat{M}an^c}$.

These 2-categories lie in a commutative diagram:

$$
\begin{tikzcd}
\mathbf{dSpa} \arrow[ddd, bend left, "F^{\mathbf{dSta}}_{\mathbf{dSpa}}"] & \mathbf{Man} \arrow[r, "\subset"] \arrow[dl, "F^{\mathbf{dMan}}_{\mathbf{Man}}"'] \arrow[d, "F^{\mathbf{dMan^c}}_{\mathbf{Man}}"] \arrow[ddd, bend left, "F^{\mathbf{Orb}}_{\mathbf{Man}}"] & \mathbf{Man^b} \arrow[r, "\subset"] \arrow[d, "F^{\mathbf{dMan^c}}_{\mathbf{Man^b}}"'] \arrow[ddd, bend left, "F^{\mathbf{Orb^c}}_{\mathbf{Man^b}}"] & \mathbf{Man^c} \arrow[d, "F^{\mathbf{dMan^c}}_{\mathbf{Man^c}}"'] \arrow[ddd, bend left, "F^{\mathbf{Orb^c}}_{\mathbf{Man^c}}"] \arrow[dr, "F^{\mathbf{dSpa^c}}_{\mathbf{Man^c}}"] & \\
\mathbf{dMan} \arrow[u, "\subset"] \arrow[r, "F^{\mathbf{dMan^c}}_{\mathbf{dMan}}", "\cong"'] \arrow[d, "F^{\mathbf{dOrb}}_{\mathbf{dMan}}"'] & \mathbf{d\bar{M}an} \arrow[r, "\subset"] \arrow[d] & \mathbf{dMan^b} \arrow[r, "\subset"] \arrow[d, "F^{\mathbf{dOrb^c}}_{\mathbf{dMan^b}}"'] & \mathbf{dMan^c} \arrow[r, "\subset"] \arrow[d, "F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}"'] & \mathbf{dSpa^c} \arrow[d, "F^{\mathbf{dSta^c}}_{\mathbf{dSpa^c}}"'] \\
\mathbf{dOrb} \arrow[r, "\cong", "F^{\mathbf{dOrb^c}}_{\mathbf{dOrb}}"'] \arrow[d, "\subset"'] & \mathbf{d\bar{O}rb} \arrow[r, "\subset"] & \mathbf{dOrb^b} \arrow[r, "\subset"] & \mathbf{dOrb^c} \arrow[r, "\subset"] & \mathbf{dSta^c} \\
\mathbf{dSta} & \mathbf{Orb} \arrow[ul, "F^{\mathbf{dOrb}}_{\mathbf{Orb}}"] \arrow[u, "F^{\mathbf{dOrb^c}}_{\mathbf{Orb}}"] \arrow[r, "F^{\mathbf{Orb^c}}_{\mathbf{Orb}}"] & \mathbf{Orb^b} \arrow[u, "F^{\mathbf{dOrb^c}}_{\mathbf{Orb^b}}"] \arrow[r, "\subset"] & \mathbf{Orb^c}. \arrow[u, "F^{\mathbf{dOrb^c}}_{\mathbf{Orb^c}}"] \arrow[ur, "F^{\mathbf{dSta^c}}_{\mathbf{Orb^c}}"'] &
\end{tikzcd}
$$

If $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ is a d-orbifold with corners, then the virtual cotangent sheaf $T^*\boldsymbol{\mathcal{X}}$ of the d-stack $\boldsymbol{\mathcal{X}}$ from Remark 1.11.1 is a virtual vector bundle on $\mathcal{X}$, of rank $\mathrm{vdim}\,\mathfrak{X}$. We will call $T^*\boldsymbol{\mathcal{X}} \in \mathrm{vvect}(\mathcal{X})$ the *virtual cotangent bundle* of $\mathfrak{X}$, and also write it $T^*\mathfrak{X}$.

Here is the analogue of Lemma 1.11.3:

Lemma 1.14.2. *Let $\mathfrak{X}$ be a d-orbifold with corners. Then $\mathfrak{X}$ is a d-manifold, that is, $\mathfrak{X} \simeq F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\mathbf{X})$ for some d-manifold with corners $\mathbf{X}$, if and only if $\mathrm{Iso}_{\mathcal{X}}([x]) \cong \{1\}$ for all $[x]$ in $\mathcal{X}_{\mathrm{top}}$.*

D-orbifolds with corners are preserved by boundaries and corners.

Proposition 1.14.3. *Suppose $\mathfrak{X}$ is a d-orbifold with corners. Then $\partial\mathfrak{X}$ in* §1.13.1 *and $C_k(\mathfrak{X})$ in* §1.13.5 *are d-orbifolds with corners, with $\mathrm{vdim}\,\partial\mathfrak{X} = \mathrm{vdim}\,\mathfrak{X} - 1$ and $\mathrm{vdim}\,C_k(\mathfrak{X}) = \mathrm{vdim}\,\mathfrak{X} - k$ for all $k \geqslant 0$.*

Definition 1.14.4. As for $\mathbf{d\check{M}an^c}$ in §1.7.1, define $\mathbf{d\check{O}rb^c}$ to be the full 2-subcategory of $\mathfrak{X}$ in $\mathbf{dSta^c}$ which may be written as a disjoint union $\mathfrak{X} = \coprod_{n\in\mathbb{Z}} \mathfrak{X}_n$ for $\mathfrak{X}_n \in \mathbf{dOrb^c}$ with $\mathrm{vdim}\,\mathfrak{X}_n = n$, where we allow $\mathfrak{X}_n = \boldsymbol{\emptyset}$. Then $\mathbf{dOrb^c} \subset \mathbf{d\check{O}rb^c} \subset \mathbf{dSta^c}$, and the corner functors $C, \hat{C} : \mathbf{dSta^c} \to \mathbf{dSta^c}$ in §1.13.5 restrict to strict 2-functors $C, \hat{C} : \mathbf{dOrb^c} \to \mathbf{d\check{O}rb^c}$.

1.14.2 Local properties of d-orbifolds with corners

Section 12.2 combines §1.7.2 and §1.11.2. Here are analogues of Examples 1.7.3, 1.7.4 and Theorem 1.7.5, and of Examples 1.11.4, 1.11.5 and Theorem 1.11.6.

Example 1.14.5. Let $\mathcal{V} = (\mathcal{V}, \partial\mathcal{V}, i_{\mathcal{V}})$ be an orbifold with corners, $\mathcal{E}$ a vector bundle on $\mathcal{V}$ as in §1.12.1, and $s \in C^\infty(\mathcal{E})$. We will define an explicit principal d-orbifold with corners $\mathfrak{S} = (\boldsymbol{\mathcal{S}}, \boldsymbol{\partial\mathcal{S}}, \boldsymbol{i}_{\mathfrak{S}}, \omega_{\mathfrak{S}})$

Define a vector bundle $\mathcal{E}_\partial$ on $\partial\mathcal{V}$ by $\mathcal{E}_\partial = i_{\mathcal{V}}^*(\mathcal{E})$, and a section $s_\partial = i_{\mathcal{V}}^*(s) \in C^\infty(\mathcal{E}_\partial)$. Define d-stacks $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ and $\boldsymbol{\partial\mathcal{S}} = \boldsymbol{S}_{\partial\mathcal{V},\mathcal{E}_\partial,s_\partial}$ from the triples $\mathcal{V}, \mathcal{E}, s$ and $\partial\mathcal{V}, \mathcal{E}_\partial, s_\partial$ exactly as in Example 1.11.4, although now $\mathcal{V}, \partial\mathcal{V}$ have corners. Define a 1-morphism $\boldsymbol{i}_{\boldsymbol{\mathcal{S}}} : \boldsymbol{\partial\mathcal{S}} \to \boldsymbol{S}$ in $\mathbf{dSta}$ to be the 'standard model' 1-morphism $\boldsymbol{\mathcal{S}}_{i_{\mathcal{V}},\mathrm{id}_{\mathcal{E}_\partial}} : \boldsymbol{\mathcal{S}}_{\partial\mathcal{V},\mathcal{E}_\partial,s_\partial} \to \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ from Example 1.11.5.

As in Example 1.7.3, the conormal bundle $\mathcal{N}_{\mathbf{S}}$ of $\boldsymbol{\partial\mathcal{S}}$ in $\boldsymbol{\mathcal{S}}$ is canonically isomorphic to the lift to $\partial\mathcal{S} \subseteq \partial\mathcal{V}$ of the conormal bundle $\mathcal{N}_{\mathcal{V}}$ of $\partial\mathcal{V}$ in $\mathcal{V}$. Define $\omega_{\mathbf{S}}$ to be the orientation on $\mathcal{N}_{\mathbf{S}}$ induced by the orientation on $\mathcal{N}_{\mathcal{V}}$ by outward-pointing normal vectors to $\partial\mathcal{V}$ in $\mathcal{V}$. Then $\mathbf{S} = (\boldsymbol{\mathcal{S}}, \boldsymbol{\partial\mathcal{S}}, \boldsymbol{i}_{\mathbf{S}}, \omega_{\mathbf{S}})$ is a d-stack with corners. It is equivalent in $\mathbf{dSta^c}$ to $\boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\mathbf{0}} \boldsymbol{\mathcal{V}}$ in Definition 1.14.1. We call $\mathbf{S}$ a 'standard model' d-orbifold with corners, and write it $\mathbf{S}_{\mathcal{V},\mathcal{E},s}$.

Every principal d-orbifold with corners $\boldsymbol{\mathcal{W}}$ is equivalent in $\mathbf{dOrb^c}$ to some $\mathbf{S}_{\mathcal{V},\mathcal{E},s}$. Sometimes it is useful to take $\mathcal{V}$ to be an *effective* orbifold with corners, as in §1.12.1. There is a natural 1-isomorphism $\partial\mathbf{S}_{\mathcal{V},\mathcal{E},s} \cong \mathbf{S}_{\partial\mathcal{V},\mathcal{E}_\partial,s_\partial}$ in $\mathbf{dOrb^c}$.

Example 1.14.6. Let $\mathcal{V}, \mathcal{W}$ be orbifolds with corners, $\mathcal{E}, \mathcal{F}$ be vector bundles on $\mathcal{V}, \mathcal{W}$, and $s \in C^\infty(\mathcal{E})$, $t \in C^\infty(\mathcal{F})$, so that Example 1.14.5 defines d-orbifolds with corners $\mathbf{S}_{\mathcal{V},\mathcal{E},s}, \mathbf{S}_{\mathcal{W},\mathcal{F},t}$. Suppose $f : \mathcal{V} \to \mathcal{W}$ is a 1-morphism in $\mathbf{Orb^c}$, and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ is a morphism in $\mathrm{vect}(\mathcal{V})$ satisfying $\hat{f} \circ s = f^*(t)$, as in (1.63).

The d-stacks $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}, \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ in $\mathbf{S}_{\mathcal{V},\mathcal{E},s}, \mathbf{S}_{\mathcal{W},\mathcal{F},t}$ are defined as for 'standard model' d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ in Example 1.11.4. Thus we can follow Example 1.11.5 to define a 1-morphism $\mathbf{S}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ in $\mathbf{dSta}$. Then $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{\mathcal{V},\mathcal{E},s} \to \mathbf{S}_{\mathcal{W},\mathcal{F},t}$ is a 1-morphism in $\mathbf{dOrb^c}$. We call it a 'standard model' 1-morphism.

Suppose now that $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ is open, with inclusion 1-morphism $i_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \to \mathcal{V}$. Write $\tilde{\mathcal{E}} = \mathcal{E}|_{\tilde{\mathcal{V}}} = i_{\tilde{\mathcal{V}}}^*(\mathcal{E})$ and $\tilde{s} = s|_{\tilde{\mathcal{V}}}$. Define $\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}} = \mathbf{S}_{i_{\tilde{\mathcal{V}}},\mathrm{id}_{\tilde{\mathcal{E}}}} : \mathbf{S}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}} \to \mathbf{S}_{\mathcal{V},\mathcal{E},s}$. If $s^{-1}(0) \subseteq \tilde{\mathcal{V}}$ then $\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}} : \mathbf{S}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}} \to \mathbf{S}_{\mathcal{V},\mathcal{E},s}$ is a 1-isomorphism.

Theorem 1.14.7. *Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold with corners, and $[x] \in \mathcal{X}_{\mathrm{top}}$. Then there exists an open neighbourhood $\boldsymbol{\mathcal{U}}$ of $[x]$ in $\boldsymbol{\mathcal{X}}$ and an equivalence $\boldsymbol{\mathcal{U}} \simeq \mathbf{S}_{\mathcal{V},\mathcal{E},s}$ in $\mathbf{dOrb^c}$ for some orbifold with corners $\mathcal{V}$, vector bundle $\mathcal{E}$ over $\mathcal{V}$ and $s \in C^\infty(\mathcal{E})$ which identifies $[x] \in \mathcal{U}_{\mathrm{top}}$ with a point $[v] \in S^k(\mathcal{V})_{\mathrm{top}} \subseteq \mathcal{V}_{\mathrm{top}}$ such that $s(v) = \mathrm{d}s|_{S^k(\mathcal{V})}(v) = 0$, where $S^k(\mathcal{V}) \subseteq \mathcal{V}$ is the locally closed C^∞-substack of $[v] \in \mathcal{V}_{\mathrm{top}}$ such that $\bar{*} \times_{v,\mathcal{V},i_{\mathcal{V}}} \partial\mathcal{V}$ is k points, for $k \geqslant 0$. Furthermore, $\mathcal{V}, \mathcal{E}, s, k$ are determined up to non-canonical equivalence near $[v]$ by $\boldsymbol{\mathcal{X}}$ near $[x]$.*

As in Examples 1.11.7–1.11.8 for d-orbifolds, we can combine the 'standard model' d-manifolds with corners $\mathbf{S}_{V,E,s}$ and 1-morphisms $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ of Examples 1.7.3–1.7.4 with quotient d-stacks with corners of §1.13.2 to define an alternative form of 'standard model' d-orbifolds with corners $[\mathbf{S}_{V,E,s}/\Gamma]$ and 'standard model' 1-morphisms $[\mathbf{S}_{f,\hat{f}}, \rho] : [\mathbf{S}_{V,E,s}/\Gamma] \to [\mathbf{S}_{W,F,t}/\Delta]$.

1.14.3 Equivalences in $\mathbf{dOrb^c}$, and gluing by equivalences

Section 12.3 combines §1.7.3 and §1.11.3. Here are the analogues of Theorems 1.11.11–1.11.14. Étale 1-morphisms in $\mathbf{dSta^c}$ were discussed in §1.13.2.

Theorem 1.14.8. *Suppose $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism of d-orbifolds with corners, and $f : \mathcal{X} \to \mathcal{Y}$ is representable. Then the following are equivalent:*

(i) *$\boldsymbol{f}$ is étale;*

(ii) *$\boldsymbol{f}$ is simple and flat, in the sense of* §1.13.3, *and $\Omega_{\boldsymbol{f}} : f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ is an equivalence in* $\mathrm{vqcoh}(\mathcal{X})$; *and*

(iii) $\boldsymbol{f}$ *is simple and flat, and* (1.66) *is a split short exact sequence in* $\mathrm{qcoh}(\mathcal{X})$.

If in addition $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ *is an isomorphism for all* $[x] \in \mathcal{X}_{\mathrm{top}}$, *and* $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ *is a bijection, then* $\boldsymbol{f}$ *is an equivalence in* $\mathbf{dOrb^c}$.

Theorem 1.14.9. *Suppose* $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{\mathcal{V},\mathcal{E},s} \to \mathbf{S}_{\mathcal{W},\mathcal{F},t}$ *is a 'standard model'* 1-*morphism in* $\mathbf{dOrb^c}$, *in the notation of Examples* 1.14.5 *and* 1.14.6, *with* $f : \mathcal{V} \to \mathcal{W}$ *representable. Then* $\mathbf{S}_{f,\hat{f}}$ *is étale if and only if* f *is simple and flat near* $s^{-1}(0) \subseteq \mathcal{V}$, *in the sense of* §1.12.2, *and for each* $[v] \in \mathcal{V}_{\mathrm{top}}$ *with* $s(v) = 0$ *and* $[w] = f_{\mathrm{top}}([v]) \in \mathcal{W}_{\mathrm{top}}$, *the following sequence is exact:*

$$0 \longrightarrow T_v\mathcal{V} \xrightarrow{\mathrm{d}s(v)\oplus \mathrm{d}f(v)} \mathcal{E}_v \oplus T_w\mathcal{W} \xrightarrow{\hat{f}(v)\oplus -\mathrm{d}t(w)} \mathcal{F}_w \longrightarrow 0.$$

Also $\mathbf{S}_{f,\hat{f}}$ *is an equivalence if and only if in addition* $f_{\mathrm{top}}|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ *is a bijection, where* $s^{-1}(0) = \{[v] \in \mathcal{V}_{\mathrm{top}} : s(v) = 0\}$, $t^{-1}(0) = \{[w] \in \mathcal{W}_{\mathrm{top}} : t(w) = 0\}$, *and* $f_* : \mathrm{Iso}_{\mathcal{V}}([v]) \to \mathrm{Iso}_{\mathcal{W}}(f_{\mathrm{top}}([v]))$ *is an isomorphism for all* $[v] \in s^{-1}(0) \subseteq \mathcal{V}_{\mathrm{top}}$.

Theorem 1.14.10. *Suppose we are given the following data:*

(a) *an integer* n;

(b) *a Hausdorff, second countable topological space* X;

(c) *an indexing set* I, *and a total order* $<$ *on* I;

(d) *for each* i *in* I, *an effective orbifold with corners* $\mathcal{V}_i$, *a vector bundle* $\mathcal{E}_i$ *on* $\mathcal{V}_i$ *with* $\dim \mathcal{V}_i - \mathrm{rank}\, \mathcal{E}_i = n$, *a section* $s_i \in C^\infty(\mathcal{E}_i)$, *and a homeomorphism* $\psi_i : s_i^{-1}(0) \to \hat{X}_i$, *where* $s_i^{-1}(0) = \{[v_i] \in \mathcal{V}_{i,\mathrm{top}} : s_i(v_i) = 0\}$ *and* $\hat{X}_i \subseteq X$ *is open; and*

(e) *for all* $i < j$ *in* I, *an open suborbifold* $\mathcal{V}_{ij} \subseteq \mathcal{V}_i$, *a simple, flat* 1-*morphism* $e_{ij} : \mathcal{V}_{ij} \to \mathcal{V}_j$, *and a morphism of vector bundles* $\hat{e}_{ij} : \mathcal{E}_i|_{\mathcal{V}_{ij}} \to e_{ij}^*(\mathcal{E}_j)$.

Let this data satisfy the conditions:

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if* $i < j$ *in* I *then* $(e_{ij})_* : \mathrm{Iso}_{\mathcal{V}_{ij}}([v]) \to \mathrm{Iso}_{\mathcal{V}_j}(e_{ij,\mathrm{top}}([v]))$ *is an isomorphism for all* $[v] \in \mathcal{V}_{ij,\mathrm{top}}$, *and* $\hat{e}_{ij} \circ s_i|_{\mathcal{V}_{ij}} = e_{ij}^*(s_j) \circ \iota_{ij}$ *where* $\iota_{ij} : \mathcal{O}_{\mathcal{V}_{ij}} \to e_{ij}^*(\mathcal{O}_{\mathcal{V}_j})$ *is the natural isomorphism, and* $\psi_i(s_i|_{\mathcal{V}_{ij}}^{-1}(0)) = \hat{X}_i \cap \hat{X}_j$, *and* $\psi_i|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)} = \psi_j \circ e_{ij,\mathrm{top}}|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)}$, *and if* $[v_i] \in \mathcal{V}_{ij,\mathrm{top}}$ *with* $s_i(v_i) = 0$ *and* $[v_j] = e_{ij,\mathrm{top}}([v_i])$ *then the following sequence is exact:*

$$0 \longrightarrow T_{v_i}\mathcal{V}_i \xrightarrow{\mathrm{d}s_i(v_i)\oplus \mathrm{d}e_{ij}(v_i)} \mathcal{E}_i|_{v_i} \oplus T_{v_j}\mathcal{V}_j \xrightarrow{\hat{e}_{ij}(v_i)\oplus -\mathrm{d}s_j(v_j)} \mathcal{E}_j|_{v_j} \longrightarrow 0;$$

(iii) *if* $i < j < k$ *in* I *then there exists a* 2-*morphism* $\eta_{ijk} : e_{jk} \circ e_{ij}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})} \Rightarrow e_{ik}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})}$ *in* $\mathbf{Orb^c}$ *with*

$$\hat{e}_{ik}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})} = \eta_{ijk}^*(\mathcal{E}_k) \circ I_{e_{ij},e_{jk}}(\mathcal{E}_k)^{-1} \circ e_{ij}|^*_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}.$$

Note that η_{ijk} *is unique by the corners analogue of Proposition* 1.9.5(ii).

Then there exist a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ with $\operatorname{vdim}\boldsymbol{\mathcal{X}}=n$ *and underlying topological space* $\mathcal{X}_{\rm top}\cong X$, *and a* 1*-morphism* $\boldsymbol{\psi}_i:\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}\rightarrow\boldsymbol{\mathcal{X}}$ *in* $\mathbf{dOrb^c}$ *with underlying continuous map ψ_i which is an equivalence with the open d-suborbifold $\hat{\boldsymbol{\mathcal{X}}}_i\subseteq\boldsymbol{\mathcal{X}}$ corresponding to $\hat X_i\subseteq X$ for all $i\in I$, such that for all $i<j$ in I there exists a* 2*-morphism* $\boldsymbol{\eta}_{ij}:\boldsymbol{\psi}_j\circ\boldsymbol{\mathcal{S}}_{e_{ij},\hat e_{ij}}\Rightarrow\boldsymbol{\psi}_i\circ\boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i}$, *where* $\boldsymbol{\mathcal{S}}_{e_{ij},\hat e_{ij}}:\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}}\rightarrow\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j}$ *and* $\boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i}:\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}}\rightarrow\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$ *are as in Examples* 1.14.5–1.14.6*. This $\boldsymbol{\mathcal{X}}$ is unique up to equivalence in* $\mathbf{dOrb^c}$.

Suppose also that $\mathcal{Y}$ is an effective orbifold with corners, and $g_i:\mathcal{V}_i\rightarrow\mathcal{Y}$ are submersions for all $i\in I$, and there are 2*-morphisms* $\zeta_{ij}:g_j\circ e_{ij}\Rightarrow g_i|_{\mathcal{V}_{ij}}$ *in* $\mathbf{Orb^c}$ *for all $i<j$ in I. Then there exist a* 1*-morphism* $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ *in* $\mathbf{dOrb^c}$ *unique up to* 2*-isomorphism, where* $\boldsymbol{\mathcal{Y}}=F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}(\mathcal{Y})=\boldsymbol{\mathcal{S}}_{\mathcal{Y},0,0}$, *and* 2*-morphisms* $\boldsymbol{\zeta}_i:\boldsymbol{h}\circ\boldsymbol{\psi}_i\Rightarrow\boldsymbol{\mathcal{S}}_{g_i,0}$ *for all $i\in I$.*

Theorem 1.14.11. *Suppose we are given the following data:*

(a) *an integer n;*

(b) *a Hausdorff, second countable topological space X;*

(c) *an indexing set I, and a total order $<$ on I;*

(d) *for each i in I, a manifold with corners V_i, a vector bundle $E_i\rightarrow V_i$ with* $\dim V_i-\operatorname{rank}E_i=n$, *a finite group Γ_i, smooth, locally effective actions $r_i(\gamma):V_i\rightarrow V_i$, $\hat r_i(\gamma):E_i\rightarrow r(\gamma)^*(E_i)$ of Γ_i on V_i,E_i for $\gamma\in\Gamma_i$, a smooth, Γ_i-equivariant section $s_i:V_i\rightarrow E_i$, and a homeomorphism $\psi_i:X_i\rightarrow\hat X_i$, where $X_i=\{v_i\in V_i:s_i(v_i)=0\}/\Gamma_i$ and $\hat X_i\subseteq X$ is an open set; and*

(e) *for all $i<j$ in I, an open submanifold $V_{ij}\subseteq V_i$, invariant under Γ_i, a group morphism $\rho_{ij}:\Gamma_i\rightarrow\Gamma_j$, a simple, flat, smooth map $e_{ij}:V_{ij}\rightarrow V_j$, and a morphism of vector bundles $\hat e_{ij}:E_i|_{V_{ij}}\rightarrow e_{ij}^*(E_j)$.*

Let this data satisfy the conditions:

(i) $X=\bigcup_{i\in I}\hat X_i$;

(ii) *if $i<j$ in I then $\hat e_{ij}\circ s_i|_{V_{ij}}=e_{ij}^*(s_j)+O(s_i^2)$, and for all $\gamma\in\Gamma$ we have*

$$e_{ij}\circ r_i(\gamma)=r_j(\rho_{ij}(\gamma))\circ e_{ij}:V_{ij}\longrightarrow V_j,$$
$$r_i(\gamma)^*(\hat e_{ij})\circ\hat r_i(\gamma)=e_{ij}^*(\hat r_j(\rho_{ij}(\gamma)))\circ\hat e_{ij}:E_i|_{V_{ij}}\longrightarrow(e_{ij}\circ r_i(\gamma))^*(E_j),$$

and $\psi_i(X_i\cap(V_{ij}/\Gamma_i))=\hat X_i\cap\hat X_j$, and $\psi_i|_{X_i\cap V_{ij}/\Gamma_i}=\psi_j\circ(e_{ij})_*|_{X_i\cap V_{ij}/\Gamma_j}$, and if $v_i\in V_{ij}$ with $s_i(v_i)=0$ and $v_j=e_{ij}(v_i)$ then $\rho|_{\mathrm{Stab}_{\Gamma_i}(v_i)}:\mathrm{Stab}_{\Gamma_i}(v_i)\rightarrow\mathrm{Stab}_{\Gamma_j}(v_j)$ is an isomorphism, and the following sequence of vector spaces is exact:

$$0\longrightarrow T_{v_i}V_i\xrightarrow{\mathrm{d}s_i(v_i)\oplus\,\mathrm{d}e_{ij}(v_i)}E_i|_{v_i}\oplus T_{v_j}V_j\xrightarrow{\hat e_{ij}(v_i)\oplus\,-\mathrm{d}s_j(v_j)}E_j|_{v_j}\longrightarrow 0;$$

(iii) *if $i<j<k$ in I then there exists $\gamma_{ijk}\in\Gamma_k$ satisfying*

$$\begin{aligned}\rho_{ik}(\gamma)&=\gamma_{ijk}\,\rho_{jk}(\rho_{ij}(\gamma))\,\gamma_{ijk}^{-1}\quad\text{for all }\gamma\in\Gamma_i,\\ e_{ik}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})}&=r_k(\gamma_{ijk})\circ e_{jk}\circ e_{ij}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})},\quad\text{and}\\ \hat e_{ik}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})}&=\bigl(e_{ij}^*(e_{jk}^*(\hat r_k(\gamma_{ijk})))\circ e_{ij}^*(\hat e_{jk})\circ\hat e_{ij}\bigr)|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})}.\end{aligned}$$

Then there exist a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ with $\operatorname{vdim}\boldsymbol{\mathcal{X}}=n$ *and underlying topological space $\mathcal{X}_{\rm top}\cong X$, and a 1-morphism $\boldsymbol{\psi}_i:[\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i]\rightarrow\boldsymbol{\mathcal{X}}$ in $\mathbf{dOrb^c}$ with underlying continuous map ψ_i which is an equivalence with the open d-suborbifold $\hat{\boldsymbol{\mathcal{X}}}_i\subseteq\boldsymbol{\mathcal{X}}$ corresponding to $\hat X_i\subseteq X$ for all $i\in I$, such that for all $i<j$ in I there exists a 2-morphism $\boldsymbol{\eta}_{ij}:\boldsymbol{\psi}_j\circ[\mathbf{S}_{e_{ij},\hat e_{ij}},\rho_{ij}]\Rightarrow\boldsymbol{\psi}_i\circ[\boldsymbol{i}_{V_{ij},V_i},\mathrm{id}_{\Gamma_i}]$, where $[\mathbf{S}_{e_{ij},\hat e_{ij}},\rho_{ij}]:[\mathbf{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]\rightarrow[\mathbf{S}_{V_j,E_j,s_j}/\Gamma_j]$ and $[\boldsymbol{i}_{V_{ij},V_i},\mathrm{id}_{\Gamma_i}]:[\mathbf{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]\rightarrow[\mathbf{S}_{V_i,E_i,s_i}/\Gamma_j]$ combine the notation of Examples 1.7.3–1.7.4 and §1.13.2. This $\boldsymbol{\mathcal{X}}$ is unique up to equivalence in $\mathbf{dOrb^c}$.*

Suppose also that Y is a manifold with corners, and $g_i:V_i\rightarrow Y$ are smooth maps for all $i\in I$ with $g_i\circ r_i(\gamma)=g_i$ for all $\gamma\in\Gamma_i$, and $g_j\circ e_{ij}=g_i|_{V_{ij}}$ for all $i<j$ in I. Then there exist a 1-morphism $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb^c}$ unique up to 2-isomorphism, where $\boldsymbol{\mathcal{Y}}=F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y)=[\mathbf{S}_{Y,0,0}/\{1\}]$, and 2-morphisms $\boldsymbol{\zeta}_i:\boldsymbol{h}\circ\boldsymbol{\psi}_i\Rightarrow[\mathbf{S}_{g_i,0},\pi_{\{1\}}]$ for all $i\in I$. Here $[\mathbf{S}_{g_i,0},\pi_{\{1\}}]:[\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i]\rightarrow[\mathbf{S}_{Y,0,0}/\{1\}]=\boldsymbol{\mathcal{Y}}$ with $\hat g_i=0$ and $\rho=\pi_{\{1\}}:\Gamma_i\rightarrow\{1\}$.

We can use Theorems 1.14.10 and 1.14.11 to prove the existence of d-orbifold with corners structures on spaces coming from other areas of geometry, such as moduli spaces of J-holomorphic curves.

1.14.4 Submersions, immersions and embeddings

Section 12.4 extends §1.7.4 and §1.11.4 to d-orbifolds with corners.

Definition 1.14.12. Let $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ be a 1-morphism in $\mathbf{dOrb^c}$. Then $T^*\boldsymbol{\mathcal{X}}$ and $f^*(T^*\boldsymbol{\mathcal{Y}})$ are virtual vector bundles on $\mathcal{X}$ of ranks $\operatorname{vdim}\boldsymbol{\mathcal{X}},\operatorname{vdim}\boldsymbol{\mathcal{Y}}$, and $\Omega_{\boldsymbol{f}}:f^*(T^*\boldsymbol{\mathcal{Y}})\rightarrow T^*\boldsymbol{\mathcal{X}}$ is a 1-morphism in $\operatorname{vvect}(\mathcal{X})$, as in Remark 1.11.1 and Definition 1.14.1. 'Weakly injective', …, below are as in Definition 1.11.15.

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{\boldsymbol{f}}$ is weakly injective. We call $\boldsymbol{f}$ an *sw-submersion* if it is also simple.

(b) We call $\boldsymbol{f}$ a *submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{C(\boldsymbol{f})}$ is injective, for $C(\boldsymbol{f})$ as in §1.13.5. We call $\boldsymbol{f}$ an *s-submersion* if it is also simple.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $f:\mathcal{X}\rightarrow\mathcal{Y}$ is representable and $\Omega_{\boldsymbol{f}}$ is weakly surjective. We call $\boldsymbol{f}$ an *sw-immersion*, or *sfw-immersion*, if $\boldsymbol{f}$ is also simple, or simple and flat.

(d) We call $\boldsymbol{f}$ an *immersion* if $f:\mathcal{X}\rightarrow\mathcal{Y}$ is representable and $\Omega_{\hat C(\boldsymbol{f})}$ is surjective, for $\hat C(\boldsymbol{f})$ as in §1.13.5. We call $\boldsymbol{f}$ an *s-immersion* if $\boldsymbol{f}$ is also simple, and an *sf-immersion* if $\boldsymbol{f}$ is also simple and flat.

(e) We call $\boldsymbol{f}$ a *w-embedding, sw-embedding, sfw-embedding, embedding, s-embedding*, or *sf-embedding*, if $\boldsymbol{f}$ is a w-immersion, ..., sf-immersion, respectively, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image, so in particular f_{top} is injective.

Parts (c)–(e) enable us to define *d-suborbifolds* $\boldsymbol{\mathcal{X}}$ of a d-orbifold with corners $\boldsymbol{\mathcal{Y}}$. *Open d-suborbifolds* are open d-substacks $\boldsymbol{\mathcal{X}}$ in $\boldsymbol{\mathcal{Y}}$. For more general d-suborbifolds, we call $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ a *w-immersed, sw-immersed, sfw-immersed, immersed, s-immersed, sf-immersed, w-embedded, sw-embedded, sfw-embedded, embedded, s-embedded*, or *sf-embedded suborbifold* of $\boldsymbol{\mathcal{Y}}$ if $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ are d-orbifolds with corners and $\boldsymbol{f}$ is a w-immersion, ..., sf-embedding, respectively.

Theorem 1.7.12 in §1.7.4 holds with orbifolds and d-orbifolds with corners in place of manifolds and d-manifolds with corners, except part (v), when we need also to assume $f : \mathcal{X} \to \mathcal{Y}$ representable to deduce $\boldsymbol{f}$ is étale, and part (x), which is false for d-orbifolds with corners (in the Zariski topology, at least).

1.14.5 Bd-transversality and fibre products

Section 12.5 generalizes §1.7.5 and §1.11.5 to $\mathbf{dOrb^c}$. Here are the analogues of Definition 1.7.13 and Theorems 1.7.14–1.7.17.

Definition 1.14.13. Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ be d-orbifolds with corners and $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$, $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms. We call $\boldsymbol{g}, \boldsymbol{h}$ *bd-transverse* if they are both b-transverse in $\mathbf{dSta^c}$ in the sense of Definition 1.13.9, and d-transverse in the sense of Definition 1.11.16. We call $\boldsymbol{g}, \boldsymbol{h}$ *cd-transverse* if they are both c-transverse in $\mathbf{dSta^c}$ in the sense of Definition 1.13.9, and d-transverse. As in §1.13.6, c-transverse implies b-transverse, so cd-transverse implies bd-transverse.

Theorem 1.14.14. *Suppose $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ are d-orbifolds with corners and $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$, $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ are bd-transverse* 1*-morphisms, and let $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g}, \boldsymbol{\mathcal{Z}}, \boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ be the fibre product in* $\mathbf{dSta^c}$, *which exists by Theorem* 1.13.11(a) *as $\boldsymbol{g}, \boldsymbol{h}$ are b-transverse. Then $\boldsymbol{\mathcal{W}}$ is a d-orbifold with corners, with*

$$\operatorname{vdim} \boldsymbol{\mathcal{W}} = \operatorname{vdim} \boldsymbol{\mathcal{X}} + \operatorname{vdim} \boldsymbol{\mathcal{Y}} - \operatorname{vdim} \boldsymbol{\mathcal{Z}}. \tag{1.85}$$

Hence, all bd-transverse fibre products exist in $\mathbf{dOrb^c}$.

Theorem 1.14.15. *Suppose $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$ and $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ are* 1*-morphisms in* $\mathbf{dOrb^c}$. *The following are sufficient conditions for $\boldsymbol{g}, \boldsymbol{h}$ to be cd-transverse, and hence bd-transverse, so that $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g}, \boldsymbol{\mathcal{Z}}, \boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ is a d-orbifold with corners of virtual dimension* (1.85):

(a) *$\boldsymbol{\mathcal{Z}}$ is an orbifold without boundary, that is, $\boldsymbol{\mathcal{Z}} \in \bar{\mathbf{O}}\mathbf{rb}$; or*

(b) *$\boldsymbol{g}$ or $\boldsymbol{h}$ is a w-submersion.*

Theorem 1.14.16. *Let $\mathcal{X},\mathcal{Y},\mathcal{Z}$ be d-orbifolds with corners with $\mathcal{Y}$ an orbifold, and $\boldsymbol{g}:\mathcal{X}\to\mathcal{Z}$, $\boldsymbol{h}:\mathcal{Y}\to\mathcal{Z}$ be 1-morphisms with $\boldsymbol{g}$ a submersion. Then $\mathcal{W}=\mathcal{X}\times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}}\mathcal{Y}$ is an orbifold, with* $\dim\mathcal{W}=\mathrm{vdim}\,\mathcal{X}+\dim\mathcal{Y}-\mathrm{vdim}\,\mathcal{Z}$.

Theorem 1.14.17. (i) *Let $\rho:G\to H$ be an injective morphism of finite groups, and H act linearly on $\mathbb{R}^n$ preserving $[0,\infty)^k\times\mathbb{R}^{n-k}$. Then* §1.13.2 *gives a quotient 1-morphism $[\boldsymbol{0},\rho]:[*/G]\to\big[[\boldsymbol{0},\boldsymbol{\infty})^{\boldsymbol{k}}\times\mathbb{R}^{\boldsymbol{n}-\boldsymbol{k}}/H\big]$ in* $\mathbf{dOrb^c}$.
Suppose $\mathcal{X}$ is a d-orbifold with corners and $\boldsymbol{g}:\mathcal{X}\to\big[[\boldsymbol{0},\boldsymbol{\infty})^{\boldsymbol{k}}\times\mathbb{R}^{\boldsymbol{n}-\boldsymbol{k}}/H\big]$ is a semisimple, flat 1-morphism in $\mathbf{dOrb^c}$. Then the fibre product $\mathcal{W}=\mathcal{X}\times_{\boldsymbol{g},[[\boldsymbol{0},\boldsymbol{\infty})^{\boldsymbol{k}}\times\mathbb{R}^{\boldsymbol{n}-\boldsymbol{k}}/H],[\boldsymbol{0},\rho]}[*/G]$ exists in $\mathbf{dOrb^c}$. The projection $\boldsymbol{\pi}_{\mathcal{X}}:\mathcal{W}\to\mathcal{X}$ is an s-immersion, and an s-embedding if ρ is an isomorphism.

When $k=0$, any 1-morphism $\boldsymbol{g}:\mathcal{X}\to[\mathbb{R}^{\boldsymbol{n}}/H]$ is semisimple and flat, and $\boldsymbol{\pi}_{\mathcal{X}}:\mathcal{W}\to\mathcal{X}$ is an sf-immersion, and an sf-embedding if ρ is an isomorphism.

(ii) *Suppose $\boldsymbol{f}:\mathcal{X}\to\mathcal{Y}$ is an s-immersion of d-orbifolds with corners, and $[x]\in\mathcal{X}_{\mathrm{top}}$ with $f_{\mathrm{top}}([x])=[y]\in\mathcal{Y}_{\mathrm{top}}$. Write $\rho:G\to H$ for $f_*:\mathrm{Iso}_{\mathcal{X}}([x])\to\mathrm{Iso}_{\mathcal{Y}}([y])$. Then ρ is injective, and there exist open neighbourhoods $\mathcal{U}\subseteq\mathcal{X}$ and $\mathcal{V}\subseteq\mathcal{Y}$ of $[x],[y]$ with $\boldsymbol{f}(\mathcal{U})\subseteq\mathcal{V}$, a linear action of H on $\mathbb{R}^n$ preserving $[0,\infty)^k\times\mathbb{R}^{n-k}\subseteq\mathbb{R}^n$ where $n=\mathrm{vdim}\,\mathcal{Y}-\mathrm{vdim}\,\mathcal{X}\geqslant 0$ and $0\leqslant k\leqslant n$, and a 1-morphism $\boldsymbol{g}:\mathcal{V}\to\big[[\boldsymbol{0},\boldsymbol{\infty})^{\boldsymbol{k}}\times\mathbb{R}^{\boldsymbol{n}-\boldsymbol{k}}/H\big]$ with $g_{\mathrm{top}}([y])=[0]$, fitting into a 2-Cartesian square in $\mathbf{dOrb^c}$:*

$$\begin{tikzcd}
\mathcal{U} \arrow[rr] \arrow[d, "\boldsymbol{f}|_{\mathcal{U}}"'] & \arrow[d, phantom, "\Uparrow"] & {[*/G]} \arrow[d, "{[\boldsymbol{0},\rho]}"] \\
\mathcal{V} \arrow[r, "\boldsymbol{g}"] & \big[[\boldsymbol{0},\boldsymbol{\infty})^{\boldsymbol{k}}\times\mathbb{R}^{\boldsymbol{n}-\boldsymbol{k}}/H\big]. &
\end{tikzcd}$$

If $\boldsymbol{f}$ is an sf-immersion then $k=0$. If $\boldsymbol{f}$ is an s- or sf-embedding then ρ is an isomorphism, and we may take $\mathcal{U}=\boldsymbol{f}^{-1}(\mathcal{V})$.

1.14.6 Embedding d-orbifolds with corners into orbifolds

Section 1.4.7 discussed embeddings of d-manifolds $\boldsymbol{X}$ into manifolds Y. Our two main results were Theorem 1.4.29, which gave necessary and sufficient conditions on $\boldsymbol{X}$ for existence of embeddings $\boldsymbol{f}:\boldsymbol{X}\hookrightarrow\mathbb{R}^{\boldsymbol{n}}$ for $n\gg 0$, and Theorem 1.4.32, which showed that if an embedding $\boldsymbol{f}:\boldsymbol{X}\hookrightarrow\boldsymbol{Y}$ exists with $\boldsymbol{X}$ a d-manifold and $\boldsymbol{Y}=F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$, then $\boldsymbol{X}\simeq\boldsymbol{S}_{V,E,s}$ for open $V\subseteq Y$.

Section 1.7.6 generalized §1.4.7 to d-manifolds with corners, requiring $\boldsymbol{f}:\mathbf{X}\hookrightarrow\mathbf{Y}$ to be an sf-embedding for the analogue of Theorem 1.4.32. Section 1.11.6 explained that while Theorem 1.4.32 generalizes to d-orbifolds, we do not have a good d-orbifold generalization of Theorem 1.4.29. Thus, we do not have a useful necessary and sufficient criterion for when a d-orbifold is principal.

As in §12.6, the situation is the same for d-orbifolds with corners as for d-orbifolds. Here is the analogue of Theorem 1.4.32:

Theorem 1.14.18. *Suppose $\mathcal{X}$ is a d-orbifold with corners, $\mathcal{Y}$ an orbifold with corners, and $\boldsymbol{f}:\mathcal{X}\to\mathcal{Y}$ an sf-embedding, in the sense of* §1.14.4. *Then there* *exist an open suborbifold $\mathcal{V}\subseteq\mathcal{Y}$ with $\boldsymbol{f}(\boldsymbol{\mathcal{X}})\subseteq\boldsymbol{\mathcal{V}}$, a vector bundle $\mathcal{E}$ on $\mathcal{V}$, and a section $s\in C^\infty(\mathcal{E})$ fitting into a 2-Cartesian diagram in* $\mathbf{dOrb^c}$*:*

$$\begin{tikzcd} \boldsymbol{\mathcal{X}} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{f}"'] & \boldsymbol{\mathcal{V}} \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{\mathcal{V}} \arrow[r, "\boldsymbol{s}"'] \arrow[ur, phantom, "\Uparrow"] & \boldsymbol{\mathcal{E}}, \end{tikzcd}$$

where $\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{E}},\boldsymbol{s},\boldsymbol{0}=F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}\bigl(\mathcal{Y},\mathcal{V},\mathrm{Tot^c}(\mathcal{E}),\mathrm{Tot^c}(s),\mathrm{Tot^c}(0)\bigr)$*, in the notation of* §1.12.1*. Thus $\boldsymbol{\mathcal{X}}$ is equivalent to the 'standard model' $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ of Example* 1.14.5*, and is a principal d-orbifold with corners.*

Again, in contrast to d-manifolds with corners, the author does not know useful necessary and sufficient conditions for when a d-orbifold with corners admits an sf-embedding into an orbifold, or is a principal d-orbifold with corners.

1.14.7 Orientations on d-orbifolds with corners

Section 1.4.8 discussed orientations on d-manifolds. This was extended to d-manifolds with corners in §1.7.7, and to d-orbifolds in §1.11.7. As in §12.7, all this generalizes easily to d-orbifolds with corners, so we will give few details.

If $\boldsymbol{\mathcal{X}}$ is a d-orbifold with corners, the virtual cotangent bundle $T^*\boldsymbol{\mathcal{X}}$ is a virtual vector bundle on $\mathcal{X}$. We define an *orientation* ω on $\boldsymbol{\mathcal{X}}$ to be an orientation on the orientation line bundle $\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}$. The analogues of Example 1.4.36, Theorem 1.4.37, Proposition 1.4.38, Theorem 1.7.25, and Propositions 1.7.26 and 1.7.27 all hold for d-orbifolds with corners.

1.14.8 Orbifold strata of d-orbifolds with corners

Sections 1.8.7, 1.9.2, 1.11.8, 1.12.5, and 1.13.7 discussed orbifold strata for Deligne–Mumford C^∞-stacks, orbifolds, d-orbifolds, orbifolds with corners, and d-stacks with corners. Section 12.8 extends this to d-orbifolds with corners.

Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold with corners and Γ a finite group, so that §1.13.7 gives orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\hat{\boldsymbol{\mathcal{X}}}^\Gamma,\boldsymbol{\mathcal{X}}^\Gamma_\circ,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ,\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$, which are d-stacks with corners. Use the notation $\Lambda^\Gamma,\Lambda^\Gamma/\operatorname{Aut}(\Gamma)$ of Definition 1.9.7. Exactly as in the d-orbifold case in §1.11.8, there are natural decompositions

$$\begin{aligned} &\boldsymbol{\mathcal{X}}^\Gamma=\textstyle\coprod_{\lambda\in\Lambda^\Gamma}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, &&\tilde{\boldsymbol{\mathcal{X}}}^\Gamma=\textstyle\coprod_{\mu\in\Lambda^\Gamma/\operatorname{Aut}(\Gamma)}\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, &&\hat{\boldsymbol{\mathcal{X}}}^\Gamma=\textstyle\coprod_{\mu\in\Lambda^\Gamma/\operatorname{Aut}(\Gamma)}\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\\ &\boldsymbol{\mathcal{X}}^\Gamma_\circ=\textstyle\coprod_{\lambda\in\Lambda^\Gamma}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ, &&\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ=\textstyle\coprod_{\mu\in\Lambda^\Gamma/\operatorname{Aut}(\Gamma)}\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ, &&\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ=\textstyle\coprod_{\mu\in\Lambda^\Gamma/\operatorname{Aut}(\Gamma)}\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ, \end{aligned}$$

where $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\ldots,\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ are d-orbifolds with corners with $\operatorname{vdim}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}=\operatorname{vdim}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ=\operatorname{vdim}\boldsymbol{\mathcal{X}}-\dim\lambda$ and $\operatorname{vdim}\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}=\cdots=\operatorname{vdim}\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ=\operatorname{vdim}\boldsymbol{\mathcal{X}}-\dim\mu$.

The analogue of Proposition 1.11.23 on orientations of orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\ldots,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ for oriented d-orbifolds $\boldsymbol{\mathcal{X}}$ also holds for d-orbifolds with corners. In an analogue of Corollary 1.12.17, we can relate boundaries of orbifold strata to orbifold strata of boundaries:

Proposition 1.14.19. *Let $\mathcal{X}$ be a d-orbifold with corners, and Γ a finite group. Then Corollary* 1.13.15 *gives* 1*-morphisms* $\boldsymbol{J}^\Gamma(\mathcal{X}) : (\partial\mathcal{X})^\Gamma \to \partial(\mathcal{X}^\Gamma)$, $\tilde{\boldsymbol{J}}^\Gamma(\mathcal{X}) : (\widetilde{\partial\mathcal{X}})^\Gamma \to \partial(\tilde{\mathcal{X}}^\Gamma)$, $\hat{\boldsymbol{J}}^\Gamma(\mathcal{X}) : (\widehat{\partial\mathcal{X}})^\Gamma \to \partial(\hat{\mathcal{X}}^\Gamma)$ *in* $\mathbf{d\check{O}rb^c}$, *which are equivalences with open and closed subobjects in* $\partial(\mathcal{X}^\Gamma), \partial(\tilde{\mathcal{X}}^\Gamma), \partial(\hat{\mathcal{X}}^\Gamma)$.

These restrict to 1*-morphisms* $\boldsymbol{J}^{\Gamma,\lambda}(\mathcal{X}) : (\partial\mathcal{X})^{\Gamma,\lambda} \to \partial(\mathcal{X}^{\Gamma,\lambda})$ *in* $\mathbf{dOrb^c}$ *for* $\lambda \in \Lambda^\Gamma$ *and* $\tilde{\boldsymbol{J}}^{\Gamma,\mu}(\mathcal{X}) : (\widetilde{\partial\mathcal{X}})^{\Gamma,\mu} \to \partial(\tilde{\mathcal{X}}^{\Gamma,\mu})$, $\hat{\boldsymbol{J}}^{\Gamma,\mu}(\mathcal{X}) : (\widehat{\partial\mathcal{X}})^{\Gamma,\mu} \to \partial(\hat{\mathcal{X}}^{\Gamma,\mu})$ *for* $\mu \in \Lambda^\Gamma / \operatorname{Aut}(\Lambda)$, *which are equivalences with open and closed d-suborbifolds. Hence, if* $\mathcal{X}^{\Gamma,\lambda} = \emptyset$ *then* $(\partial\mathcal{X})^{\Gamma,\lambda} = \emptyset$, *and similarly for* $\tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}$.

Now suppose $\mathcal{X}$ *is straight, in the sense of* §1.13.7, *for instance* $\mathcal{X}$ *could be a d-orbifold with boundary. Then* $\boldsymbol{J}^\Gamma(\mathcal{X}), \ldots, \hat{\boldsymbol{J}}^{\Gamma,\mu}(\mathcal{X})$ *are equivalences, so that* $(\partial\mathcal{X})^\Gamma \simeq \partial(\mathcal{X}^\Gamma)$, $(\partial\mathcal{X})^{\Gamma,\lambda} \simeq \partial(\mathcal{X}^{\Gamma,\lambda})$, *and so on.*

1.14.9 Kuranishi neighbourhoods and good coordinate systems

In §1.11.9 we defined type A Kuranishi neighbourhoods, coordinate changes, and good coordinate systems, on d-orbifolds. Section 12.9 generalizes these to d-orbifolds with corners. The definitions in the corners case are obtained by replacing $\mathbf{Man}, \mathbf{Orb}, \mathbf{dMan}, \mathbf{dOrb}$ by $\mathbf{Man^c}, \mathbf{Orb^c}, \mathbf{dMan^c}, \mathbf{dOrb^c}$ throughout, and making a few other easy changes such as taking the e_{ij} to be sf-embeddings in Definitions 1.11.25(c). For brevity we will not write the definitions out again, but just indicate the differences.

Definition 1.14.20. Let $\mathcal{X}$ be a d-orbifold with corners. Define a *type A Kuranishi neighbourhood* $(V, E, \Gamma, s, \boldsymbol{\psi})$ on $\mathcal{X}$ following Definition 1.11.24, but taking V to be a manifold with corners, and defining the principal d-orbifold with corners $[\mathbf{S}_{V,E,s}/\Gamma]$ by combining Example 1.7.3 and §1.13.2, as in §1.14.2.

If $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ are type A Kuranishi neighbourhoods on $\mathcal{X}$ with $\emptyset \neq \boldsymbol{\psi}_i([\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\mathbf{S}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \mathcal{X}$, define a *type A coordinate change* $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$ *from* $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ *to* $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ following Definition 1.11.25, but taking $e_{ij} : V_{ij} \to V_j$ to be an sf-embedding of manifolds with corners, and defining the quotient 1-morphism $[\mathbf{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}]$ by combining Example 1.7.4 and §1.13.2, as in §1.14.2.

Define a *type A good coordinate system* on $\mathcal{X}$ following Definition 1.11.26, defining quotient 2-morphisms $\boldsymbol{\eta}_{ijk} = [0, \gamma_{ijk}]$ in (d) using §1.13.2. Let Y be a manifold with corners, and $\boldsymbol{h} : \mathcal{X} \to \mathcal{Y}$ a 1-morphism in $\mathbf{dOrb^c}$, where $\mathcal{Y} = F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y)$. Define a *type A good coordinate system for* $\boldsymbol{h} : \mathcal{X} \to \mathcal{Y}$ following Definition 1.11.26.

Here is the analogue of Theorem 1.11.27. It will be proved in Appendix D.

Theorem 1.14.21. *Suppose* $\mathcal{X}$ *is a d-orbifold with corners. Then there exists a type A good coordinate system* $\bigl(I, <, (V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij}), \gamma_{ijk}\bigr)$ *for* $\mathcal{X}$, *in the sense of Definition* 1.14.20. *If* $\mathcal{X}$ *is compact, we may take* I *to be finite. If* $\{\mathcal{U}_j : j \in J\}$ *is an open cover of* $\mathcal{X}$, *we may take* $\mathcal{X}_i = \boldsymbol{\psi}_i([\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i]) \subseteq \mathcal{U}_{j_i}$ *for each* $i \in I$ *and some* $j_i \in J$. *If* $\mathcal{X}$ *is a d-orbifold with boundary, we may take the* V_i *to be manifolds with boundary.*

Now let Y be a manifold with corners and $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}=F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y)$ a semisimple, flat 1-morphism in $\mathbf{dOrb^c}$. Then all the above extends to type A good coordinate systems for $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$, and we may take the $g_i:V_i\rightarrow Y$ to be submersions in $\mathbf{Man^c}$.

We can regard Theorem 1.14.21 as a kind of converse to Theorem 1.14.11. Note that we make the extra assumption that $\boldsymbol{h}$ is semisimple and flat in the last part. This happens automatically if Y is without boundary. Since submersions in $\mathbf{Man^c}$ are automatically semisimple and flat, $\boldsymbol{h}$ being semisimple and flat is a necessary condition for the $g_i:V_i\rightarrow Y$ to be submersions. Section 12.9 also gives 'type B' versions of Definition 1.14.20 and Theorem 1.14.21 using the 'standard model' d-orbifolds with corners $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ and 1-morphisms $\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}$ of Examples 1.14.5–1.14.6 instead of $[\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i]$ and $[\mathbf{S}_{e_{ij},\hat{e}_{ij}},\rho_{ij}]$.

1.14.10 Semieffective and effective d-orbifolds with corners

Section 1.11.10 discussed semieffective and effective d-orbifolds. As in §12.10, all this material extends to d-orbifolds with corners essentially unchanged. We define *semieffective* and *effective* d-orbifolds with corners $\boldsymbol{\mathcal{X}}$ following Definition 1.11.28. The analogues of Proposition 1.11.29 and the rest of §1.11.9 then hold, with (d-)orbifolds replaced by (d-)orbifolds with corners throughout.

Proposition 1.14.22. *Let $\boldsymbol{\mathcal{X}}$ be an effective (or semieffective) d-orbifold with corners. Then $\partial^k\boldsymbol{\mathcal{X}}$ is also effective (or semieffective), for all $k\geqslant 0$.*

However, $\boldsymbol{\mathcal{X}}$ (semi)effective does not imply $C_k(\boldsymbol{\mathcal{X}})$ (semi)effective.

1.15 D-manifold and d-orbifold bordism, virtual cycles

Chapter 13 discusses *bordism groups* of manifolds and orbifolds, defined using manifolds, d-manifolds, orbifolds, and d-orbifolds. We can use these to prove that compact, oriented d-manifolds admit virtual cycles, and so can be used in enumerative invariant problems. The same applies for compact, oriented d-orbifolds, although the direct proof using bordism no longer works.

1.15.1 Classical bordism groups for manifolds

Section 13.1 reviews background material on bordism from the literature. Classical bordism groups $MSO_k(Y)$ were defined by Atiyah [6] for topological spaces Y, using continuous maps $f:X\rightarrow Y$ for X a compact, oriented manifold. Conner [24, §I] gives a good introduction. We define bordism $B_k(Y)$ only for manifolds Y, using smooth $f:X\rightarrow Y$, following Conner's *differential bordism groups* [24, §I.9]. By [24, Th. I.9.1], the natural projection $B_k(Y)\rightarrow MSO_k(Y)$ is an isomorphism, so our notion of bordism agrees with the usual definition.

Definition 1.15.1. Let Y be a manifold without boundary, and $k\in\mathbb{Z}$. Consider pairs (X,f), where X is a compact, oriented manifold without boundary with $\dim X=k$, and $f:X\rightarrow Y$ is a smooth map. Define an equivalence relation $\sim$ on such pairs by $(X,f)\sim(X',f')$ if there exists a compact, oriented $(k+1)$-manifold with boundary W, a smooth map $e:W\to Y$, and a diffeomorphism of oriented manifolds $j:-X\amalg X'\to\partial W$, such that $f\amalg f'=e\circ i_W\circ j$, where $-X$ is X with the opposite orientation.

Write $[X,f]$ for the $\sim$-equivalence class (*bordism class*) of a pair (X,f). For each $k\in\mathbb{Z}$, define the k^{th} *bordism group* $B_k(Y)$ of Y to be the set of all such bordism classes $[X,f]$ with $\dim X=k$. We give $B_k(Y)$ the structure of an abelian group, with zero element $0_Y=[\emptyset,\emptyset]$, and addition given by $[X,f]+[X',f']=[X\amalg X',f\amalg f']$, and additive inverses $-[X,f]=[-X,f]$.

Define $\Pi_{\rm bo}^{\rm hom}:B_k(Y)\to H_k(Y;\mathbb{Z})$ by $\Pi_{\rm bo}^{\rm hom}:[X,f]\mapsto f_*([X])$, where $H_*(-;\mathbb{Z})$ is singular homology, and $[X]\in H_k(X;\mathbb{Z})$ is the fundamental class.

If Y is oriented and of dimension n, there is a biadditive, associative, supercommutative *intersection product* $\bullet:B_k(Y)\times B_l(Y)\to B_{k+l-n}(Y)$, such that if $[X,f],[X',f']$ are classes in $B_*(Y)$, with f,f' transverse, then the fibre product $X\times_{f,Y,f'}X'$ exists as a compact oriented manifold, and

$$[X,f]\bullet[X',f']=[X\times_{f,Y,f'}X',f\circ\pi_X].$$

As in [24, §I.5], bordism is a generalized homology theory. Results of Thom, Wall and others in [24, §I.2] compute the bordism groups $B_k(*)$ of the point $*$. This partially determines the bordism groups of general manifolds Y, as there is a spectral sequence $H_i\big(Y;B_j(*)\big)\Rightarrow B_{i+j}(Y)$.

1.15.2 D-manifold bordism groups

Section 13.2 defines *d-manifold bordism* by replacing manifolds X in pairs $[X,f]$ in §1.15.1 by d-manifolds $\boldsymbol{X}$. For simplicity, we identify the 2-category $\mathbf{dMan}$ of d-manifolds $\boldsymbol{X}$ in §1.4.1, and the 2-subcategory $\mathbf{d\bar{M}an}$ of d-manifolds without boundary $\mathbf{X}=(\boldsymbol{X},\emptyset,\emptyset,\emptyset)$ in $\mathbf{dMan^c}$ in §1.7.1, writing both as $\boldsymbol{X}$.

Definition 1.15.2. Let Y be a manifold without boundary, and $k\in\mathbb{Z}$. Consider pairs $(\boldsymbol{X},\boldsymbol{f})$, where $\boldsymbol{X}\in\mathbf{dMan}$ is a compact, oriented d-manifold without boundary with $\operatorname{vdim}\boldsymbol{X}=k$, and $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is a 1-morphism in $\mathbf{dMan}$, where $\boldsymbol{Y}=F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$.

Define an equivalence relation $\sim$ between such pairs by $(\boldsymbol{X},\boldsymbol{f})\sim(\boldsymbol{X}',\boldsymbol{f}')$ if there exists a compact, oriented d-manifold with boundary $\mathbf{W}$ with $\operatorname{vdim}\mathbf{W}=k+1$, a 1-morphism $\boldsymbol{e}:\mathbf{W}\to\boldsymbol{Y}$ in $\mathbf{dMan^b}$, an equivalence of oriented d-manifolds $\boldsymbol{j}:-\boldsymbol{X}\amalg\boldsymbol{X}'\to\partial\mathbf{W}$, and a 2-morphism $\eta:\boldsymbol{f}\amalg\boldsymbol{f}'\Rightarrow\boldsymbol{e}\circ\boldsymbol{i}_{\mathbf{W}}\circ\boldsymbol{j}$.

Write $[\boldsymbol{X},\boldsymbol{f}]$ for the $\sim$-equivalence class (*d-bordism class*) of a pair $(\boldsymbol{X},\boldsymbol{f})$. For each $k\in\mathbb{Z}$, define the k^{th} *d-manifold bordism group*, or *d-bordism group*, $dB_k(Y)$ of Y to be the set of all such d-bordism classes $[\boldsymbol{X},\boldsymbol{f}]$ with $\operatorname{vdim}\boldsymbol{X}=k$. As for $B_k(Y)$, we give $dB_k(Y)$ the structure of an abelian group, with zero element $0_Y=[\boldsymbol{\emptyset},\boldsymbol{\emptyset}]$, addition $[\boldsymbol{X},\boldsymbol{f}]+[\boldsymbol{X}',\boldsymbol{f}']=[\boldsymbol{X}\amalg\boldsymbol{X}',\boldsymbol{f}\amalg\boldsymbol{f}']$, and additive inverses $-[\boldsymbol{X},\boldsymbol{f}]=[-\boldsymbol{X},\boldsymbol{f}]$.

If Y is oriented and of dimension n, we define a biadditive, associative, supercommutative *intersection product* $\bullet : dB_k(Y) \times dB_l(Y) \to dB_{k+l-n}(Y)$ by

$$[\boldsymbol{X}, \boldsymbol{f}] \bullet [\boldsymbol{X}', \boldsymbol{f}'] = [\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Y},\boldsymbol{f}'} \boldsymbol{X}', \boldsymbol{f} \circ \boldsymbol{\pi_X}]. \tag{1.86}$$

Here $\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Y},\boldsymbol{f}'} \boldsymbol{X}'$ exists as a d-manifold by Theorem 1.4.23(a), and is oriented by Theorem 1.4.37. Note that we do not need to restrict to $[X, f], [X', f']$ with f, f' transverse as in Definition 1.15.1. Define a morphism $\Pi_{\mathrm{bo}}^{\mathrm{dbo}} : B_k(Y) \to dB_k(Y)$ for $k \geqslant 0$ by $\Pi_{\mathrm{bo}}^{\mathrm{dbo}} : [X, f] \mapsto \bigl[F_{\mathbf{Man}}^{\mathbf{dMan}}(X), F_{\mathbf{Man}}^{\mathbf{dMan}}(f)\bigr]$.

In §13.2 we prove that $B_*(Y)$ and $dB_*(Y)$ are isomorphic. See Spivak [95, Th. 2.6] for the analogous result for Spivak's derived manifolds.

Theorem 1.15.3. *For any manifold Y, we have $dB_k(Y) = 0$ for $k < 0$, and $\Pi_{\mathrm{bo}}^{\mathrm{dbo}} : B_k(Y) \to dB_k(Y)$ is an isomorphism for $k \geqslant 0$. When Y is oriented, $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}$ identifies the intersection products $\bullet$ on $B_*(Y)$ and $dB_*(Y)$.*

Here is the main idea in the proof of Theorem 1.15.3. Let $[\boldsymbol{X}, \boldsymbol{f}] \in dB_k(Y)$. By Corollary 1.4.31 there exists an embedding $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{\mathbb{R}^n}$ for $n \gg 0$. Then the direct product $(\boldsymbol{f}, \boldsymbol{g}) : \boldsymbol{X} \to \boldsymbol{Y} \times \boldsymbol{\mathbb{R}^n}$ is also an embedding. Theorem 1.4.32 shows that there exist an open set $V \subseteq Y \times \mathbb{R}^n$, a vector bundle $E \to V$ and $s \in C^\infty(E)$ such that $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$. Let $\tilde{s} \in C^\infty(E)$ be a small, generic perturbation of s. As $\tilde{s}$ is generic, the graph of $\tilde{s}$ in E intersects the zero section transversely. Hence $\tilde{X} = \tilde{s}^{-1}(0)$ is a k-manifold for $k \geqslant 0$, which is compact and oriented for $\tilde{s} - s$ small, and $\tilde{X} = \emptyset$ for $k < 0$. Set $\tilde{f} = \pi_Y|_{\tilde{X}} : \tilde{X} \to Y$. Then $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}\bigl([\tilde{X}, \tilde{f}]\bigr) = [\boldsymbol{X}, \boldsymbol{f}]$, so that $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}$ is surjective. A similar argument for $\mathbf{W}, \boldsymbol{e}$ in Definition 1.15.2 shows that $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}$ is injective.

Theorem 1.15.3 implies that we may define projections

$$\Pi_{\mathrm{dbo}}^{\mathrm{hom}} : dB_k(Y) \longrightarrow H_k(Y; \mathbb{Z}) \text{ by } \Pi_{\mathrm{dbo}}^{\mathrm{hom}} = \Pi_{\mathrm{bo}}^{\mathrm{hom}} \circ (\Pi_{\mathrm{bo}}^{\mathrm{dbo}})^{-1}. \tag{1.87}$$

We think of $\Pi_{\mathrm{dbo}}^{\mathrm{hom}}$ as a *virtual class map.* Virtual classes (or virtual cycles, or virtual chains) are used in several areas of geometry to construct enumerative invariants using moduli spaces. In algebraic geometry, Behrend and Fantechi [12] construct virtual classes for schemes with obstruction theories. In symplectic geometry, there are many versions — see for example Fukaya et al. [34, §6], [32, §A1], Hofer et al. [46], and McDuff [77].

The main message we want to draw from this is that *compact oriented d-manifolds admit virtual classes.* Thus, we can use d-manifolds (and d-orbifolds) as the geometric structure on moduli spaces in enumerative invariant problems such as Gromov–Witten invariants, Lagrangian Floer cohomology, Donaldson–Thomas invariants, ..., as this structure is strong enough to contain all the 'counting' information.

1.15.3 Classical bordism for orbifolds

Section 13.3 generalizes §1.15.1 to orbifolds. Here we will be brief, much more information is given in §13.3. We use the notation of §1.9 on orbifolds **Orb** and §1.12 on orbifolds with boundary $\mathbf{Orb^b}$ freely. For simplicity we do not distinguish between the 2-categories **Orb** in §1.9.1 and $\dot{\mathbf{Orb}}$ in §1.12.1.

Definition 1.15.4. Let $\mathcal{Y}$ be an orbifold, and $k \in \mathbb{Z}$. Consider pairs $(\mathcal{X}, f)$, where $\mathcal{X}$ is a compact, oriented orbifold (without boundary) with $\dim \mathcal{X} = k$, and $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism in **Orb**. Define an equivalence relation $\sim$ between such pairs by $(\mathcal{X}, f) \sim (\mathcal{X}', f')$ if there exists a compact, oriented $(k+1)$-orbifold with boundary $\mathcal{W}$, a 1-morphism $e : \mathcal{W} \to \mathcal{Y}$ in $\mathbf{Orb^b}$, an orientation-preserving equivalence $j : -\mathcal{X} \amalg \mathcal{X}' \to \partial \mathcal{W}$, and a 2-morphism $\eta : f \amalg f' \Rightarrow e \circ i_{\mathcal{W}} \circ j$ in $\mathbf{Orb^b}$.

Write $[\mathcal{X}, f]$ for the $\sim$-equivalence class (*bordism class*) of a pair $(\mathcal{X}, f)$. For each $k \in \mathbb{Z}$, define the k^{th} *orbifold bordism group* $B_k^{\mathrm{orb}}(\mathcal{Y})$ of $\mathcal{Y}$ to be the set of all such bordism classes $[\mathcal{X}, f]$ with $\dim \mathcal{X} = k$. It is an abelian group, with zero $0_{\mathcal{Y}} = [\emptyset, \emptyset]$, addition $[\mathcal{X}, f] + [\mathcal{X}', f'] = [\mathcal{X} \amalg \mathcal{X}', f \amalg f']$, and additive inverses $-[\mathcal{X}, f] = [-\mathcal{X}, f]$. If $k < 0$ then $B_k^{\mathrm{orb}}(\mathcal{Y}) = 0$.

Define *effective orbifold bordism* $B_k^{\mathrm{eff}}(\mathcal{Y})$ in the same way, but requiring both orbifolds $\mathcal{X}$ and orbifolds with boundary $\mathcal{W}$ to be effective (as in §1.9.1 and §1.12.1) in pairs $(\mathcal{X}, f)$ and the definition of $\sim$.

If $\mathcal{Y}$ is an orbifold, define group morphisms

$$\begin{gathered}\Pi_{\mathrm{eff}}^{\mathrm{orb}} : B_k^{\mathrm{eff}}(\mathcal{Y}) \longrightarrow B_k^{\mathrm{orb}}(\mathcal{Y}), \;\; \Pi_{\mathrm{orb}}^{\mathrm{hom}} : B_k^{\mathrm{orb}}(\mathcal{Y}) \longrightarrow H_k(\mathcal{Y}_{\mathrm{top}}; \mathbb{Q}) \\ \text{and} \qquad \Pi_{\mathrm{eff}}^{\mathrm{hom}} : B_k^{\mathrm{eff}}(\mathcal{Y}) \longrightarrow H_k(\mathcal{Y}_{\mathrm{top}}; \mathbb{Z})\end{gathered}$$

by $\Pi_{\mathrm{eff}}^{\mathrm{orb}} : [\mathcal{X}, f] \mapsto [\mathcal{X}, f]$ and $\Pi_{\mathrm{orb}}^{\mathrm{hom}}, \Pi_{\mathrm{eff}}^{\mathrm{hom}} : [\mathcal{X}, f] \mapsto (f_{\mathrm{top}})_*([\mathcal{X}])$, where $[\mathcal{X}]$ is the fundamental class of the compact, oriented k-orbifold $\mathcal{X}$, which lies in $H_k(\mathcal{X}_{\mathrm{top}}; \mathbb{Q})$ for general $\mathcal{X}$, and in $H_k(\mathcal{X}_{\mathrm{top}}; \mathbb{Z})$ for effective $\mathcal{X}$. The morphisms $\Pi_{\mathrm{eff}}^{\mathrm{orb}} : B_k^{\mathrm{eff}}(\mathcal{Y}) \to B_k^{\mathrm{orb}}(\mathcal{Y})$ are injective.

Suppose $\mathcal{Y}$ is an oriented orbifold of dimension n which is a manifold, that is, the orbifold groups $\mathrm{Iso}_{\mathcal{Y}}([y])$ are trivial for all $[y] \in \mathcal{Y}_{\mathrm{top}}$. Define biadditive, associative, supercommutative *intersection products* $\bullet : B_k^{\mathrm{orb}}(\mathcal{Y}) \times B_l^{\mathrm{orb}}(\mathcal{Y}) \to B_{k+l-n}^{\mathrm{orb}}(\mathcal{Y})$ and $\bullet : B_k^{\mathrm{eff}}(\mathcal{Y}) \times B_l^{\mathrm{eff}}(\mathcal{Y}) \to B_{k+l-n}^{\mathrm{eff}}(\mathcal{Y})$ as follows. Given classes $[\mathcal{X}, f], [\mathcal{X}', f']$, we perturb f, f' in their bordism classes to make $f : \mathcal{X} \to \mathcal{Y}$ and $f' : \mathcal{X}' \to \mathcal{Y}$ transverse 1-morphisms, and then as in (1.86) we set

$$[\mathcal{X}, f] \bullet [\mathcal{X}', f'] = [\mathcal{X} \times_{f, \mathcal{Y}, f'} \mathcal{X}', f \circ \pi_{\mathcal{X}}].$$

If $\mathcal{Y}$ is not a manifold then f, f' may not admit transverse perturbations.

Again, orbifold bordism is a generalized homology theory. Results of Druschel [28, 29] and Angel [3–5] give a complete description of the rational effective orbifold bordism ring $B_*^{\mathrm{eff}}(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ when $\mathcal{Y}$ is the point $*$, and some information on the full ring $B_*^{\mathrm{eff}}(*)$. It is much more complicated than bordism $B_*(*)$ for manifolds in §1.15.1, because of contributions from orbifold strata.

As in §1.9.2, if $\mathcal{X}$ is an orbifold and Γ a finite group then we may define orbifold strata $\mathcal{X}^{\Gamma,\lambda}$ for $\lambda \in \Lambda_+^{\Gamma}$ and $\tilde{\mathcal{X}}^{\Gamma,\mu}$ for $\mu \in \Lambda_+^{\Gamma} / \mathrm{Aut}(\Gamma)$, which are orbifolds, with proper 1-morphisms $O^{\Gamma,\lambda}(\mathcal{X}) : \mathcal{X}^{\Gamma,\lambda} \to \mathcal{X}$ and $\tilde{O}^{\Gamma,\mu}(\mathcal{X}) : \tilde{\mathcal{X}}^{\Gamma,\mu} \to \mathcal{X}$. Hence, if $\mathcal{X}$ is compact then $\mathcal{X}^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}$ are compact. If $\mathcal{X}$ is oriented then under extra conditions on Γ, λ, μ, which hold automatically for $\mathcal{X}^{\Gamma,\lambda}$ if $|\Gamma|$ is odd, we can define natural orientations on $\mathcal{X}^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}$. Using these ideas, under the assumptions on Γ, λ, μ needed to orient $\mathcal{X}^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}$ we define morphisms

$$\Pi_{\rm orb}^{\Gamma,\lambda} : B_k^{\rm orb}(\mathcal{Y}) \to B_{k-\dim\lambda}^{\rm orb}(\mathcal{Y}) \text{ by } \Pi_{\rm orb}^{\Gamma,\lambda} : [\mathcal{X}, f] \mapsto [\mathcal{X}^{\Gamma,\lambda}, f \circ O^{\Gamma,\lambda}(\mathcal{X})], \tag{1.88}$$
$$\tilde{\Pi}_{\rm orb}^{\Gamma,\mu} : B_k^{\rm orb}(\mathcal{Y}) \to B_{k-\dim\mu}^{\rm orb}(\mathcal{Y}) \text{ by } \tilde{\Pi}_{\rm orb}^{\Gamma,\mu} : [\mathcal{X}, f] \mapsto [\tilde{\mathcal{X}}^{\Gamma,\mu}, f \circ \tilde{O}^{\Gamma,\mu}(\mathcal{X})]. \tag{1.89}$$

One moral is that orbifold bordism groups $B_*^{\rm orb}(\mathcal{Y}), B_*^{\rm eff}(\mathcal{Y})$ are generally much bigger than manifold bordism groups $B_*(Y)$, because in elements $[\mathcal{X}, f]$ of orbifold bordism groups, extra information is contained in the orbifold strata of $\mathcal{X}$. The morphisms $\Pi_{\rm orb}^{\Gamma,\lambda}, \tilde{\Pi}_{\rm orb}^{\Gamma,\mu}$ recover some of this extra information.

1.15.4 Bordism for d-orbifolds

Section 13.4 combines the ideas of §1.15.2 and §1.15.3 to define bordism for d-orbifolds. As for $\mathbf{Orb}, \dot{\mathbf{O}}\mathbf{rb}$ in §1.15.3, for simplicity we do not distinguish between the 2-categories $\mathbf{dOrb}$ in §1.11.1 and $\mathbf{d\bar{O}rb} \subset \mathbf{dOrb^c}$ in §1.14.1.

Definition 1.15.5. Let $\mathcal{Y}$ be an orbifold, and $k \in \mathbb{Z}$. Consider pairs $(\boldsymbol{\mathcal{X}}, \boldsymbol{f})$, where $\boldsymbol{\mathcal{X}} \in \mathbf{dOrb}$ is a compact, oriented d-orbifold without boundary with $\operatorname{vdim} \boldsymbol{\mathcal{X}} = k$, and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism in $\mathbf{dOrb}$, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$.

Define an equivalence relation $\sim$ between such pairs by $(\boldsymbol{\mathcal{X}}, \boldsymbol{f}) \sim (\boldsymbol{\mathcal{X}}', \boldsymbol{f}')$ if there exists a compact, oriented d-orbifold with boundary $\boldsymbol{\mathcal{W}}$ with $\operatorname{vdim} \boldsymbol{\mathcal{W}} = k+1$, a 1-morphism $\boldsymbol{e} : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb^b}$, an equivalence of oriented d-orbifolds $\boldsymbol{j} : -\boldsymbol{\mathcal{X}} \amalg \boldsymbol{\mathcal{X}}' \to \partial \boldsymbol{\mathcal{W}}$, and a 2-morphism $\eta : \boldsymbol{f} \amalg \boldsymbol{f}' \Rightarrow \boldsymbol{e} \circ \boldsymbol{i_{\mathcal{W}}} \circ \boldsymbol{j}$.

Write $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$ for the $\sim$-equivalence class (*d-bordism class*) of a pair $(\boldsymbol{\mathcal{X}}, \boldsymbol{f})$. For each $k \in \mathbb{Z}$, define the k^{th} *d-orbifold bordism group* $dB_k^{\rm orb}(\mathcal{Y})$ of $\mathcal{Y}$ to be the set of all such d-bordism classes $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$ with $\operatorname{vdim} \boldsymbol{\mathcal{X}} = k$. We give $dB_k^{\rm orb}(\mathcal{Y})$ the structure of an abelian group, with zero element $0_{\mathcal{Y}} = [\boldsymbol{\emptyset}, \boldsymbol{\emptyset}]$, addition $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}] + [\boldsymbol{\mathcal{X}}', \boldsymbol{f}'] = [\boldsymbol{\mathcal{X}} \amalg \boldsymbol{\mathcal{X}}', \boldsymbol{f} \amalg \boldsymbol{f}']$, and additive inverses $-[\boldsymbol{\mathcal{X}}, \boldsymbol{f}] = [-\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$.

Similarly, define the *semieffective d-orbifold bordism group* $dB_k^{\rm sef}(\mathcal{Y})$ and the *effective d-orbifold bordism group* $dB_k^{\rm eff}(\mathcal{Y})$ as above, but taking $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{W}}$ to be semieffective, or effective, respectively, in the sense of §1.11.10 and §1.14.10.

If $\mathcal{Y}$ is oriented and of dimension n, we define a biadditive, associative, supercommutative *intersection product* $\bullet : dB_k^{\rm orb}(\mathcal{Y}) \times dB_l^{\rm orb}(\mathcal{Y}) \to dB_{k+l-n}^{\rm orb}(\mathcal{Y})$ by

$$[\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \bullet [\boldsymbol{\mathcal{X}}', \boldsymbol{f}'] = [\boldsymbol{\mathcal{X}} \times_{\boldsymbol{f},\boldsymbol{\mathcal{Y}},\boldsymbol{f}'} \boldsymbol{\mathcal{X}}', \boldsymbol{f} \circ \boldsymbol{\pi_{\mathcal{X}}}],$$

as in (1.86). Here $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{f},\boldsymbol{\mathcal{Y}},\boldsymbol{f}'} \boldsymbol{\mathcal{X}}'$ exists in $\mathbf{dOrb}$ by Theorem 1.11.18(a), and is oriented by the d-orbifold analogue of Theorem 1.4.37.

If $\mathcal{Y}$ is an orbifold, define group morphisms

$$\begin{aligned} \Pi_{\rm orb}^{\rm sef} &: B_k^{\rm orb}(\mathcal{Y}) \longrightarrow dB_k^{\rm sef}(\mathcal{Y}), \quad \Pi_{\rm eff}^{\rm deff} : B_k^{\rm eff}(\mathcal{Y}) \longrightarrow dB_k^{\rm eff}(\mathcal{Y}), \\ \Pi_{\rm deff}^{\rm sef} &: dB_k^{\rm eff}(\mathcal{Y}) \longrightarrow dB_k^{\rm sef}(\mathcal{Y}), \quad \Pi_{\rm deff}^{\rm dorb} : dB_k^{\rm eff}(\mathcal{Y}) \longrightarrow dB_k^{\rm orb}(\mathcal{Y}), \\ &\text{and} \qquad \Pi_{\rm sef}^{\rm dorb} : dB_k^{\rm sef}(\mathcal{Y}) \longrightarrow dB_k^{\rm orb}(\mathcal{Y}) \end{aligned} \tag{1.90}$$

by $\Pi_{\rm orb}^{\rm sef}, \Pi_{\rm eff}^{\rm deff} : [\mathcal{X}, f] \mapsto [\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$, where $\boldsymbol{\mathcal{X}}, \boldsymbol{f} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{X}, f)$, and $\Pi_{\rm deff}^{\rm sef}, \Pi_{\rm deff}^{\rm dorb}, \Pi_{\rm sef}^{\rm dorb} : [\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \mapsto [\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$.

Here is the main result of §13.4, an orbifold analogue of Theorem 1.15.3. The key idea is that semieffective (or effective) d-orbifolds $\boldsymbol{\mathcal{X}}$ can be perturbed to (effective) orbifolds, as in §1.11.10; to make this rigorous, we use good coordinate systems on $\boldsymbol{\mathcal{X}}$, as in §1.11.9.

Theorem 1.15.6. *For any orbifold $\mathcal{Y}$, the maps $\Pi_{\rm orb}^{\rm sef}: B_k^{\rm orb}(\mathcal{Y}) \to dB_k^{\rm sef}(\mathcal{Y})$ and $\Pi_{\rm eff}^{\rm deff}: B_k^{\rm eff}(\mathcal{Y}) \to dB_k^{\rm eff}(\mathcal{Y})$ in* (1.90) *are isomorphisms for all $k \in \mathbb{Z}$.*

As for (1.87), the theorem implies that we may define projections

$$\Pi_{\rm sef}^{\rm hom}: dB_k^{\rm sef}(\mathcal{Y}) \to H_k(\mathcal{Y}_{\rm top};\mathbb{Q}), \;\; \Pi_{\rm deff}^{\rm hom}: dB_k^{\rm eff}(\mathcal{Y}) \to H_k(\mathcal{Y}_{\rm top};\mathbb{Z})$$
$$\text{by } \Pi_{\rm sef}^{\rm hom} = \Pi_{\rm orb}^{\rm hom} \circ (\Pi_{\rm orb}^{\rm sef})^{-1} \text{ and } \Pi_{\rm deff}^{\rm hom} = \Pi_{\rm eff}^{\rm hom} \circ (\Pi_{\rm eff}^{\rm deff})^{-1}.$$

We think of these $\Pi_{\rm sef}^{\rm hom}, \Pi_{\rm deff}^{\rm hom}$ as *virtual class maps* on $dB_*^{\rm sef}(\mathcal{Y}), dB_*^{\rm eff}(\mathcal{Y})$. In fact, with more work, one can also define virtual class maps on $dB_*^{\rm orb}(\mathcal{Y})$:

$$\Pi_{\rm dorb}^{\rm hom}: dB_k^{\rm orb}(\mathcal{Y}) \longrightarrow H_k(\mathcal{Y}_{\rm top};\mathbb{Q}), \tag{1.91}$$

satisfying $\Pi_{\rm dorb}^{\rm hom} \circ \Pi_{\rm sef}^{\rm dorb} = \Pi_{\rm sef}^{\rm hom}$, for instance following the method of Fukaya et al. [34, §6], [32, §A1] for virtual classes of Kuranishi spaces using 'multisections'.

In future work the author intends to define a virtual chain construction for d-manifolds and d-orbifolds, expressed in terms of new (co)homology theories whose (co)chains are built from d-manifolds or d-orbifolds, as for the 'Kuranishi (co)homology' described in [53, 54].

As in §1.11.8, if $\boldsymbol{\mathcal{X}}$ is a d-orbifold and Γ a finite group then we may define orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ for $\lambda \in \Lambda^\Gamma$ and $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$ for $\mu \in \Lambda^\Gamma / \operatorname{Aut}(\Gamma)$, which are d-orbifolds, with proper 1-morphisms $\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}}): \boldsymbol{\mathcal{X}}^{\Gamma,\lambda} \to \boldsymbol{\mathcal{X}}$ and $\tilde{\boldsymbol{O}}^{\Gamma,\mu}(\boldsymbol{\mathcal{X}}): \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu} \to \boldsymbol{\mathcal{X}}$. Hence, if $\boldsymbol{\mathcal{X}}$ is compact then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$ are compact. If $\boldsymbol{\mathcal{X}}$ is oriented and Γ is odd then we under extra conditions on μ can define natural orientations on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$. As in (1.88)–(1.89), for such Γ, λ, μ we define morphisms

$$\Pi_{\rm dorb}^{\Gamma,\lambda}: dB_k^{\rm orb}(\mathcal{Y}) \to dB_{k-\dim\lambda}^{\rm orb}(\mathcal{Y}) \text{ by } \Pi_{\rm dorb}^{\Gamma,\lambda}: [\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \mapsto [\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \boldsymbol{f} \circ \boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}})],$$
$$\tilde{\Pi}_{\rm dorb}^{\Gamma,\mu}: dB_k^{\rm orb}(\mathcal{Y}) \to dB_{k-\dim\mu}^{\rm orb}(\mathcal{Y}) \text{ by } \tilde{\Pi}_{\rm dorb}^{\Gamma,\mu}: [\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \mapsto [\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \boldsymbol{f} \circ \tilde{\boldsymbol{O}}^{\Gamma,\mu}(\boldsymbol{\mathcal{X}})].$$

We can use these operators $\Pi_{\rm dorb}^{\Gamma,\lambda}$ to study the d-orbifold bordism ring $dB_*^{\rm orb}(*)$. Let Γ be a finite group with $|\Gamma|$ odd, and R be a nontrivial Γ-representation. Define an element $[\boldsymbol{*} \times_{\boldsymbol{0},\boldsymbol{R},\boldsymbol{0}} \boldsymbol{*}/\Gamma, \boldsymbol{\pi}]$ in $dB_{-\dim R}^{\rm orb}(*)$, where $\boldsymbol{R} = F_{\mathbf{Man}}^{\mathbf{dMan}}(R)$, and set $\lambda = [R] \in \Lambda_+^\Gamma$. Then $\Pi_{\rm dorb}^{\Gamma,-\lambda}\bigl([\boldsymbol{*} \times_{\boldsymbol{0},\boldsymbol{R},\boldsymbol{0}} \boldsymbol{*}/\Gamma, \boldsymbol{\pi}]\bigr) \in dB_0^{\rm orb}(*)$, so $\Pi_{\rm dorb}^{\rm hom} \circ \Pi_{\rm dorb}^{\Gamma,-\lambda}\bigl([\boldsymbol{*} \times_{\boldsymbol{0},\boldsymbol{R},\boldsymbol{0}} \boldsymbol{*}/\Gamma, \boldsymbol{\pi}]\bigr)$ lies in $H_0^{\rm orb}(*;\mathbb{Q}) \cong \mathbb{Q}$ by (1.91). Calculation shows that $\Pi_{\rm dorb}^{\rm hom} \circ \Pi_{\rm dorb}^{\Gamma,-\lambda}\bigl([\boldsymbol{*} \times_{\boldsymbol{0},\boldsymbol{R},\boldsymbol{0}} \boldsymbol{*}/\Gamma, \boldsymbol{\pi}]\bigr)$ is either $|\operatorname{Aut}(\Gamma)|/|\Gamma|$ or 0, depending on λ. In the first case, $[\boldsymbol{*} \times_{\boldsymbol{0},\boldsymbol{R},\boldsymbol{0}} \boldsymbol{*}/\Gamma, \boldsymbol{\pi}]$ has infinite order in $dB_{-\dim R}^{\rm orb}(*)$. Extending this argument, we can show that $dB_{4k}^{\rm orb}(*)$ has infinite rank for all $k \leqslant 0$. In contrast, $dB_k^{\rm sef}(*) = dB_k^{\rm eff}(*) = 0$ for all $k < 0$ by Theorem 1.15.6.

1.16 Relation to other classes of spaces in mathematics

In Chapter 14 we study the relationships between d-manifolds and d-orbifolds and other classes of geometric spaces in the literature. The next theorem summarizes our results:

Theorem 1.16.1. *We may construct 'truncation functors' from various classes of geometric spaces to d-manifolds and d-orbifolds, as follows:*

(a) *There is a functor* $\Pi_{\mathbf{BManFS}}^{\mathbf{dMan}} : \mathbf{BManFS} \to \mathrm{Ho}(\mathbf{dMan})$, *where* $\mathbf{BManFS}$ *is a category whose objects are triples* (V, E, s) *of a Banach manifold* V, *Banach vector bundle* $E \to V$, *and smooth section* $s : V \to E$ *whose linearization* $\mathrm{d}s|_x : T_xV \to E|_x$ *is Fredholm with index* $n \in \mathbb{Z}$ *for each* $x \in V$ *with* $s|_x = 0$, *and* $\mathrm{Ho}(\mathbf{dMan})$ *is the homotopy category of the* 2*-category of d-manifolds* $\mathbf{dMan}$.

There is also an orbifold version $\Pi_{\mathbf{BOrbFS}}^{\mathbf{dOrb}} : \mathrm{Ho}(\mathbf{BOrbFS}) \to \mathrm{Ho}(\mathbf{dOrb})$ *of this using Banach orbifolds* $\mathcal{V}$, *and 'corners' versions of both.*

(b) *There is a functor* $\Pi_{\mathbf{MPolFS}}^{\mathbf{dMan}} : \mathbf{MPolFS} \to \mathrm{Ho}(\mathbf{dMan})$, *where* $\mathbf{MPolFS}$ *is a category whose objects are triples* (V, E, s) *of an* ***M-polyfold*** *without boundary* V *as in Hofer, Wysocki and Zehnder* [43, §3.3], *a fillable strong M-polyfold bundle* E *over* V [43, §4.3], *and an sc-smooth Fredholm section* s *of* E [43, §4.4] *whose linearization* $\mathrm{d}s|_x : T_xV \to E|_x$ [43, §4.4] *has Fredholm index* $n \in \mathbb{Z}$ *for all* $x \in V$ *with* $s|_x = 0$.

There is also an orbifold version $\Pi_{\mathbf{PolFS}}^{\mathbf{dOrb}} : \mathrm{Ho}(\mathbf{PolFS}) \to \mathrm{Ho}(\mathbf{dOrb})$ *of this using* ***polyfolds*** $\mathcal{V}$, *and 'corners' versions of both.*

(c) *Given a d-orbifold with corners* $\mathfrak{X}$, *we can construct a* ***Kuranishi space*** (X, κ) *in the sense of Fukaya, Oh, Ohta and Ono* [32, §A], *with the same underlying topological space* X. *Conversely, given a Kuranishi space* (X, κ), *we can construct a d-orbifold with corners* $\mathfrak{X}'$. *Composing the two constructions,* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *are equivalent in* $\mathbf{dOrb^c}$.

Very roughly speaking, this means that the 'categories' of d-orbifolds with corners, and Kuranishi spaces, are equivalent. However, Fukaya et al. [32] *do not define morphisms of Kuranishi spaces, so we have no category of Kuranishi spaces.*

(d) *There is a functor* $\Pi_{\mathbf{SchObs}}^{\mathbf{dMan}} : \mathbf{Sch}_{\mathbb{C}}\mathbf{Obs} \to \mathrm{Ho}(\mathbf{dMan})$, *where* $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$ *is a category whose objects are triples* $(X, E^\bullet, \phi)$, *for* X *a separated, second countable* $\mathbb{C}$*-scheme and* $\phi : E^\bullet \to \tau_{\geqslant -1}(L_X)$ *a perfect obstruction theory on* X *with constant virtual dimension, in the sense of Behrend and Fantechi* [12]. *We may define a natural orientation on* $\Pi_{\mathbf{SchObs}}^{\mathbf{dMan}}(X, E^\bullet, \phi)$ *for each* $(X, E^\bullet, \phi)$.

There is also an orbifold version $\Pi_{\mathbf{StaObs}}^{\mathbf{dOrb}} : \mathrm{Ho}(\mathbf{Sta}_{\mathbb{C}}\mathbf{Obs}) \to \mathrm{Ho}(\mathbf{dOrb})$, *taking* $\mathcal{X}$ *to be a Deligne–Mumford* $\mathbb{C}$*-stack.*

(e) *There is a functor* $\Pi_{\mathbf{QsDSch}}^{\mathbf{dMan}} : \mathrm{Ho}(\mathbf{QsDSch}_{\mathbb{C}}) \longrightarrow \mathrm{Ho}(\mathbf{dMan})$, *where* $\mathbf{QsDSch}_{\mathbb{C}}$ *is the* ∞*-category of separated, second countable, quasi-smooth derived* $\mathbb{C}$*-schemes* X *of constant dimension, as in Toën and Vezzosi* [100–102]. *We may define a natural orientation on* $\Pi_{\mathbf{QsDSch}}^{\mathbf{dMan}}(X)$ *for each* X.

There is also an orbifold version $\Pi_{\mathbf{QsDSta}}^{\mathbf{dOrb}} : \mathrm{Ho}(\mathbf{QsDSta}_{\mathbb{C}}) \to \mathrm{Ho}(\mathbf{dOrb})$, *taking* $\mathcal{X}$ *to be a derived Deligne–Mumford* $\mathbb{C}$*-stack.*

(f) **(Borisov [16])** *There is a natural functor* $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}} : \mathrm{Ho}(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}) \to \mathrm{Ho}(\mathbf{dMan_{pr}})$ *from the homotopy category of the* ∞*-category* $\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}$ *of* ***derived manifolds*** *of finite type with pure dimension, in the sense of Spivak* [95], *to the homotopy category of the full* 2*-subcategory* $\mathbf{dMan_{pr}}$ *of principal d-manifolds in* $\mathbf{dMan}$. *This functor induces a bijection between isomorphism classes of objects in* $\mathrm{Ho}(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}})$ *and* $\mathrm{Ho}(\mathbf{dMan_{pr}})$. *It is full, but not faithful. If* $[\boldsymbol{f}]$ *is a morphism in* $\mathrm{Ho}(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}})$, *then* $[\boldsymbol{f}]$ *is an isomorphism if and only if* $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}([\boldsymbol{f}])$ *is an isomorphism.*

Here, as in §A.3, if $\boldsymbol{\mathcal{C}}$ is a 2-category (or ∞-category), the *homotopy category* $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$ of $\boldsymbol{\mathcal{C}}$ is the category whose objects are objects of $\boldsymbol{\mathcal{C}}$, and whose morphisms are 2-isomorphism classes of 1-morphisms in $\boldsymbol{\mathcal{C}}$. Then equivalences in $\boldsymbol{\mathcal{C}}$ become isomorphisms in $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$, 2-commutative diagrams in $\boldsymbol{\mathcal{C}}$ become commutative diagrams in $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$, and so on.

One moral of Theorem 1.16.1 is that essentially every geometric structure on moduli spaces which is used to define enumerative invariants, either in differential geometry, or in algebraic geometry over $\mathbb{C}$, has a truncation functor to d-manifolds or d-orbifolds. Combining Theorem 1.16.1 with proofs from the literature of the existence on moduli spaces of the geometric structures listed in Theorem 1.16.1, in Chapter 14 we deduce:

Theorem 1.16.2. **(i)** *Any solution set of a smooth nonlinear elliptic equation with fixed topological invariants on a compact manifold naturally has the structure of a d-manifold, uniquely up to equivalence in* $\mathbf{dMan}$.

For example, let $(M,g),(N,h)$ *be Riemannian manifolds, with* M *compact. Then the family of* ***harmonic maps*** $f : M \to N$ *is a d-manifold* $\boldsymbol{\mathcal{H}}_{M,N}$ *with* $\mathrm{vdim}\, \boldsymbol{\mathcal{H}}_{M,N} = 0$. *If* $M = \mathcal{S}^1$, *then* $\boldsymbol{\mathcal{H}}_{M,N}$ *is the moduli space of* ***parametrized closed geodesics*** *in* (N,h).

(ii) *Let* (X,ω) *be a compact symplectic manifold of dimension* $2n$, *and* J *an almost complex structure on* X *compatible with* ω. *For* $\beta \in H_2(X,\mathbb{Z})$ *and* $g,m \geqslant 0$, *write* $\overline{\mathcal{M}}_{g,m}(X,J,\beta)$ *for the moduli space of stable triples* $(\Sigma,\vec{z},u)$ *for* Σ *a genus* g *prestable Riemann surface with* m *marked points* $\vec{z} = (z_1,\ldots,z_m)$ *and* $u : \Sigma \to X$ *a* J*-holomorphic map with* $[u(\Sigma)] = \beta$ *in* $H_2(X,\mathbb{Z})$. *Using results of Hofer, Wysocki and Zehnder* [48] *involving their theory of polyfolds, we can make* $\overline{\mathcal{M}}_{g,m}(X,J,\beta)$ *into a compact, oriented d-orbifold* $\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)$.

(iii) *Let* (X,ω) *be a compact symplectic manifold,* J *an almost complex structure on* X *compatible with* ω, *and* Y *a compact, embedded Lagrangian submanifold in* X. *For* $\beta \in H_2(X,Y;\mathbb{Z})$ *and* $k \geqslant 0$, *write* $\overline{\mathcal{M}}_k(X,Y,J,\beta)$ *for the moduli space of* J*-**holomorphic stable maps*** $(\Sigma,\vec{z},u)$ *to* X *from a prestable holomorphic disc* Σ *with* k *boundary marked points* $\vec{z} = (z_1,\ldots,z_k)$, *with* $u(\partial\Sigma) \subseteq Y$ *and* $[u(\Sigma)] = \beta$ *in* $H_2(X,Y;\mathbb{Z})$. *Using results of Fukaya, Oh, Ohta and Ono* [32, §7–§8] *involving their theory of Kuranishi spaces, we can make* $\overline{\mathcal{M}}_k(X,Y,J,\beta)$ *into a compact d-orbifold with corners* $\overline{\boldsymbol{\mathcal{M}}}_k(X,Y,J,\beta)$. *Given a relative spin structure for* (X,Y), *we may define an orientation on* $\overline{\boldsymbol{\mathcal{M}}}_k(X,Y,J,\beta)$.

(iv) *Let* X *be a complex projective manifold, and* $\overline{\mathcal{M}}_{g,m}(X,\beta)$ *the Deligne–Mumford moduli* $\mathbb{C}$*-stack of stable triples* $(\Sigma,\vec{z},u)$ *for* Σ *a genus* g *prestable*

Riemann surface with m marked points $\vec{z} = (z_1, \ldots, z_m)$ and $u : \Sigma \to X$ a morphism with $u_*([\Sigma]) = \beta \in H_2(X;\mathbb{Z})$. Then Behrend [7] *defines a perfect obstruction theory on $\overline{\mathcal{M}}_{g,m}(X,\beta)$, so we can make $\overline{\mathcal{M}}_{g,m}(X,\beta)$ into a compact, oriented d-orbifold $\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,\beta)$.*

(v) *Let X be a complex algebraic surface, and $\mathcal{M}$ a stable moduli $\mathbb{C}$-scheme of vector bundles or coherent sheaves E on X with fixed Chern character. Then Mochizuki* [83] *defines a perfect obstruction theory on $\mathcal{M}$, so we can make $\mathcal{M}$ into an oriented d-manifold $\boldsymbol{\mathcal{M}}$.*

(vi) *Let X be a complex Calabi–Yau* 3*-fold or smooth Fano* 3*-fold, and $\mathcal{M}$ a stable moduli $\mathbb{C}$-scheme of coherent sheaves E on X with fixed Hilbert polynomial. Then Thomas* [98] *defines a perfect obstruction theory on $\mathcal{M}$, so we can make $\mathcal{M}$ into an oriented d-manifold $\boldsymbol{\mathcal{M}}$.*

(vii) *Let X be a smooth complex projective* 3*-fold, and $\mathcal{M}$ a moduli $\mathbb{C}$-scheme of 'stable PT pairs' (C, D) in X, where $C \subset X$ is a curve and $D \subset C$ is a divisor. Then Pandharipande and Thomas* [88] *define a perfect obstruction theory on $\mathcal{M}$, so we can make $\mathcal{M}$ into a compact, oriented d-manifold $\boldsymbol{\mathcal{M}}$.*

(ix) *Let X be a complex Calabi–Yau* 3*-fold, and $\mathcal{M}$ a separated moduli $\mathbb{C}$-scheme of simple perfect complexes in the derived category $D^b \operatorname{coh}(X)$. Then Huybrechts and Thomas* [49] *define a perfect obstruction theory on $\mathcal{M}$, so we can make $\mathcal{M}$ into an oriented d-manifold $\boldsymbol{\mathcal{M}}$.*

We can use d-manifolds and d-orbifolds to construct *virtual classes* or *virtual chains* for all these moduli spaces.

Remark 1.16.3. D-manifolds should not be confused with *differential graded manifolds*, or *dg-manifolds*. This term is used in two senses, in algebraic geometry to mean a special kind of dg-scheme, as in Ciocan-Fontanine and Kapranov [23, Def. 2.5.1], and in differential geometry to mean a supermanifold with extra structure, as in Cattaneo and Schätz [19, Def. 3.6]. In both cases, a dg-manifold $\mathfrak{E}$ is roughly the total space of a graded vector bundle $E^\bullet$ over a manifold V, with a vector field Q of degree 1 satisfying $[Q, Q] = 0$.

For example, if E is a vector bundle over V and $s \in C^\infty(E)$, we can make E into a dg-manifold $\mathfrak{E}$ by giving E the grading -1, and taking Q to be the vector field on E corresponding to s. To this $\mathfrak{E}$ we can associate the d-manifold $\boldsymbol{S}_{V,E,s}$ from Example 1.4.4. Note that $\boldsymbol{S}_{V,E,s}$ only knows about an infinitesimal neighbourhood of $s^{-1}(0)$ in V, but $\mathfrak{E}$ remembers all of V, E, s.

2 The 2-category of d-spaces

We now define and study the 2-category of *d-spaces* **dSpa**, which are derived versions of C^∞-schemes. In Chapter 3 we will define d-manifolds (without boundary) to be a 2-subcategory of d-spaces, just as §B.5 allows us to regard manifolds as a subcategory of the category of C^∞-schemes. We will assume familiarity with C^∞-schemes, which are explained in §1.2 and Appendix B, and with basic facts about 2-categories, which are summarized in §A.3–§A.4.

2.1 Square zero extensions of C^∞-rings and C^∞-schemes

Square zero extensions are an important tool in deformation theory in algebraic geometry. They appear in the mathematics of cotangent complexes, obstruction theories, and the construction of virtual classes, as in Behrend and Fantechi [12, §4], for instance. They enter our story for similar reasons: we want a d-manifold to be roughly the same as a C^∞-scheme with an obstruction theory. We begin with some basic facts about square zero extensions in C^∞-geometry.

Definition 2.1. A *square zero extension of C^∞-rings* is a surjective morphism of C^∞-rings $\phi : \mathfrak{C}' \to \mathfrak{C}$ such that the kernel I of ϕ in $\mathfrak{C}'$ is a *square zero ideal*, that is, $ij = 0$ in $\mathfrak{C}'$ for all $i, j \in I$. Thus we have an exact sequence

$$0 \longrightarrow I \xrightarrow{\kappa_\phi} \mathfrak{C}' \xrightarrow{\phi} \mathfrak{C} \longrightarrow 0, \tag{2.1}$$

where we write $\kappa_\phi : I \to \mathfrak{C}'$ for the kernel of ϕ.

As I is an ideal in $\mathfrak{C}'$, it is a $\mathfrak{C}'$-module. But it also has the structure of a $\mathfrak{C}$-module: if $i \in I$ and $c \in \mathfrak{C}$, we may choose $c' \in \mathfrak{C}'$ with $\phi(c') = c$, and define $c \cdot i = c' \cdot i$. If $c'' \in \mathfrak{C}'$ is an alternative choice with $\phi(c'') = c$, then $\phi(c'' - c') = 0$, so $c'' - c' \in I$, and $(c'' - c') \cdot i = 0$ as I is square zero. Thus $c' \cdot i = c'' \cdot i$, and the $\mathfrak{C}$-module structure on I is well-defined.

If $\phi : \mathfrak{C}' \to \mathfrak{C}$ and $\psi : \mathfrak{D}' \to \mathfrak{D}$ are square zero extensions, a *morphism of square zero extensions* $(\alpha, \alpha') : \phi \to \psi$ is a pair of morphisms of C^∞-rings $\alpha : \mathfrak{C} \to \mathfrak{D}$ and $\alpha' : \mathfrak{C}' \to \mathfrak{D}'$ with $\alpha \circ \phi = \psi \circ \alpha'$. Then α' takes the kernel I of ϕ to the kernel J of ψ, giving $\alpha'' := \alpha'|_I : I \to J$, in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\kappa_\phi"] \arrow[d, "\alpha''"] & \mathfrak{C}' \arrow[r, "\phi"] \arrow[d, "\alpha'"] & \mathfrak{C} \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & J \arrow[r, "\kappa_\psi"] & \mathfrak{D}' \arrow[r, "\psi"] & \mathfrak{D} \arrow[r] & 0. \end{tikzcd} \tag{2.2}$$

Example 2.2. Suppose $\phi : \mathfrak{C}' \to \mathfrak{C}$ is a square zero extension with $\mathfrak{C}'$ finitely generated. Choosing generators $x_1, \ldots, x_n$ in $\mathfrak{C}$ induces a surjective morphism of C^∞-rings $\pi : C^\infty(\mathbb{R}^n) \to \mathfrak{C}'$. Let J be the kernel of π, and I the kernel of $\phi \circ \pi$. Then $J \subseteq I$, and $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$, $\mathfrak{C}' \cong C^\infty(\mathbb{R}^n)/J$, and (2.1) becomes

$$0 \longrightarrow I/J \longrightarrow C^\infty(\mathbb{R}^n)/J \longrightarrow C^\infty(\mathbb{R}^n)/I \longrightarrow 0.$$

For I/J to be square zero is equivalent to $I^2 \subseteq J$, so $I^2 \subseteq J \subseteq I$. For example, we can take I to be any ideal in $C^\infty(\mathbb{R}^n)$, and $J = I^2$.

Write $\underline{X} = \operatorname{Spec}\mathfrak{C}$, $\underline{X}' = \operatorname{Spec}\mathfrak{C}'$ and $\underline{\imath}_X = \operatorname{Spec}\phi : \underline{X} \to \underline{X}'$. The underlying topological spaces are

$$\begin{aligned} X &\cong \big\{(x_1,\dots,x_n) \in \mathbb{R}^n : i(x_1,\dots,x_n) = 0 \text{ for all } i \in I\big\}, \\ X' &\cong \big\{(x_1,\dots,x_n) \in \mathbb{R}^n : j(x_1,\dots,x_n) = 0 \text{ for all } j \in J\big\}. \end{aligned}$$

Now $i(x_1,\dots,x_n) = 0$ for all $i \in I$ if and only if $k(x_1,\dots,x_n) = 0$ for all $k \in I^2$, so $I^2 \subseteq J \subseteq I$ implies that $X \cong X'$, and the continuous map $\imath_X : X \to X'$ in $\underline{\imath}_X$ is a homeomorphism. The reduced C^∞-schemes $\underline{X}^{\mathrm{red}}, (\underline{X}')^{\mathrm{red}}$ are also the same, but $\underline{X}, \underline{X}'$ can differ in their nonreduced structure.

We associate an exact sequence of $\mathfrak{C}$-modules to a square zero extension.

Definition 2.3. Let $\phi : \mathfrak{C}' \to \mathfrak{C}$ be a square zero extension of C^∞-rings, with kernel $\kappa_\phi : I \to \mathfrak{C}'$. As in §B.6, we have cotangent modules $\Omega_{\mathfrak{C}}, \Omega_{\mathfrak{C}'}$ and a morphism of $\mathfrak{C}$-modules $(\Omega_\phi)_* : \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C} \to \Omega_{\mathfrak{C}}$. Define a linear map $\Xi_\phi : I \to \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}$ to be the composition

$$I \xrightarrow{\kappa_\phi} \mathfrak{C}' \xrightarrow{\mathrm{d}_{\mathfrak{C}'}} \Omega_{\mathfrak{C}'} = \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}' \xrightarrow{\mathrm{id}\otimes\phi} \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}. \tag{2.3}$$

In the next proposition, note that it is not obvious that Ξ_ϕ is a $\mathfrak{C}$-module morphism, since in (2.3) none of $\kappa_\phi, \mathrm{d}_{\mathfrak{C}'}, \mathrm{id}\otimes\phi$ is a $\mathfrak{C}$-module morphism, and $\mathrm{d}_{\mathfrak{C}'}$ is not even a $\mathfrak{C}'$-module morphism.

Proposition 2.4. *In Definition* 2.3, Ξ_ϕ *is a morphism of* $\mathfrak{C}$*-modules, and*

$$I \xrightarrow{\Xi_\phi} \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C} \xrightarrow{(\Omega_\phi)_*} \Omega_{\mathfrak{C}} \longrightarrow 0 \tag{2.4}$$

is an exact sequence of $\mathfrak{C}$*-modules.*

Proof. Let $i \in I$ and $c \in \mathfrak{C}$. Then $c = \phi(c')$ for $c' \in \mathfrak{C}'$, as ϕ is surjective. So

$$\begin{aligned} \Xi_\phi(c\cdot i) &= \Xi_\phi(c'\cdot i) = (\mathrm{id}\otimes\phi)\big(\mathrm{d}_{\mathfrak{C}'}(c'\cdot i)\big) = (\mathrm{id}\otimes\phi)\big(c'\cdot \mathrm{d}_{\mathfrak{C}'}i + i\cdot \mathrm{d}_{\mathfrak{C}'}c'\big) \\ &= \phi(c')\cdot \mathrm{d}_{\mathfrak{C}'}i + \phi(i)\cdot \mathrm{d}_{\mathfrak{C}'}c' = c\cdot \mathrm{d}_{\mathfrak{C}'}i + 0\cdot \mathrm{d}_{\mathfrak{C}'}c' = c\cdot \Xi_\phi(i). \end{aligned}$$

Hence Ξ_ϕ is a $\mathfrak{C}$-module morphism. Now $\Omega_{\mathfrak{C}}$ is generated by $\mathrm{d}_{\mathfrak{C}}c$ for $c \in \mathfrak{C}$. Each c is $\phi(c')$ for some $c' \in \mathfrak{C}$, and then $(\Omega_\phi)_*(\mathrm{d}_{\mathfrak{C}'}c') = \mathrm{d}_{\mathfrak{C}}c$. Thus $(\Omega_\phi)_*$ is surjective on generators of $\Omega_{\mathfrak{C}}$, so it is surjective, and (2.4) is exact at $\Omega_{\mathfrak{C}}$.

To show (2.4) is exact at $\Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}$, suppose $\beta \in \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}$ with $(\Omega_\phi)_*(\beta) = 0$. Write $\beta = \sum_{j=1}^m b_j \,\mathrm{d}_{\mathfrak{C}'}c'_j$ with $b_j \in \mathfrak{C}$ and $c'_j \in \mathfrak{C}'$, and set $c_j = \phi(c'_j)$. Then $(\Omega_\phi)_*(\beta) = \sum_{j=1}^m b_j \,\mathrm{d}_{\mathfrak{C}}c_j = 0$ in $\Omega_{\mathfrak{C}}$. Now $\Omega_{\mathfrak{C}}$ is the quotient of the free $\mathfrak{C}$-module with basis $\mathrm{d}_{\mathfrak{C}}c$ for $c \in \mathfrak{C}$ by the relations $\mathrm{d}_{\mathfrak{C}}\big(\Phi_f(h_1,\dots,h_n)\big) - \sum_{l=1}^n \Phi_{\frac{\partial f}{\partial x_l}}(h_1,\dots,h_n)\cdot \mathrm{d}_{\mathfrak{C}}h_l = 0$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$ and $h_1,\dots,h_n \in \mathfrak{C}$.

Hence for $k = 1, \ldots, p$ there exist smooth functions $f_k : \mathbb{R}^{n_k} \to \mathbb{R}$ and elements $g_k, h_k^1, \ldots, h_k^{n_k}$ in $\mathfrak{C}$ such that

$$\sum_{j=1}^m b_j \cdot \mathrm{d}_{\mathfrak{C}} c_j = \sum_{k=1}^p g_k \cdot \begin{aligned}[t] &\bigl[\mathrm{d}_{\mathfrak{C}}\bigl(\Phi_{f_k}(h_k^1, \ldots, h_k^{n_k})\bigr) \\ &\quad - \textstyle\sum_{l=1}^{n_k} \Phi_{\frac{\partial f_k}{\partial x_l}}(h_k^1, \ldots, h_k^{n_k}) \cdot \mathrm{d}_{\mathfrak{C}} h_k^l\bigr] \end{aligned} \tag{2.5}$$

holds in the free $\mathfrak{C}$-module with basis of symbols $\mathrm{d}_{\mathfrak{C}} c$ for $c \in \mathfrak{C}$.

For smooth $f : \mathbb{R}^n \to \mathbb{R}$ we write $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ and $\Phi'_f : (\mathfrak{C}')^n \to \mathfrak{C}'$ for the operations in the C^∞-ring structures of $\mathfrak{C}, \mathfrak{C}'$. Since (2.1) is exact, we may choose a right inverse $r : \mathfrak{C} \to \mathfrak{C}'$ for ϕ such that $\mathfrak{C}' = r(\mathfrak{C}) \oplus I$ and $\phi \circ r = \mathrm{id}_{\mathfrak{C}}$. Then $\phi(c'_j - r(c_j)) = c_j - c_j = 0$, so $c'_j - r(c_j) \in I$. Also

$$\begin{aligned} \phi \circ \Phi'_{f_k}\bigl(r(h_k^1), \ldots, r(h_k^{n_k})\bigr) &= \Phi_{f_k}\bigl(\phi \circ r(h_k^1), \ldots, \phi \circ r(h_k^{n_k})\bigr) \\ &= \Phi_{f_k}(h_k^1, \ldots, h_k^{n_k}) = \phi \circ r\bigl(\Phi_{f_k}(h_k^1, \ldots, h_k^{n_k})\bigr), \end{aligned}$$

as ϕ is a C^∞-ring morphism. Thus $\Phi'_{f_k}\bigl(r(h_k^1), \ldots, r(h_k^{n_k})\bigr) - r\bigl(\Phi_{f_k}(h_k^1, \ldots, h_k^{n_k})\bigr)$ lies in I. Define

$$\gamma = \sum_{j=1}^m b_j \cdot (c'_j - r(c_j)) - \sum_{k=1}^p g_k \cdot \bigl[\Phi'_{f_k}\bigl(r(h_k^1), \ldots, r(h_k^{n_k})\bigr) - r\bigl(\Phi_{f_k}(h_k^1, \ldots, h_k^{n_k})\bigr)\bigr]$$

in I. Then

$$\begin{aligned} \Xi_\phi(\gamma) &= \sum_{j=1}^m \mathrm{d}_{\mathfrak{C}'}\bigl[b_j \cdot (c'_j - r(c_j))\bigr] - \sum_{k=1}^p \mathrm{d}_{\mathfrak{C}'}\bigl[g_k \cdot \Phi'_{f_k}(r(h_k^1), \ldots, r(h_k^{n_k})) - \\ &\qquad\qquad g_k \cdot r(\Phi_{f_k}(h_k^1, \ldots, h_k^{n_k}))\bigr] \\ &= \sum_{j=1}^m b_j \cdot \mathrm{d}_{\mathfrak{C}'} c'_j + \sum_{j=1}^m \phi\bigl[c'_j - r(c_j)\bigr] \cdot \mathrm{d}_{\mathfrak{C}'} b'_j - \sum_{k=1}^p \phi\bigl[\Phi'_{f_k}(r(h_k^1), \ldots, r(h_k^{n_k})) \\ &\qquad - r(\Phi_{f_k}(h_k^1, \ldots, h_k^{n_k}))\bigr] \cdot \mathrm{d}_{\mathfrak{C}'} g'_k \\ &\quad - \left\{ \begin{aligned} &\sum_{j=1}^m b_j \cdot \mathrm{d}_{\mathfrak{C}'}(r(c_j)) - \sum_{k=1}^p g_k \cdot \bigl[\mathrm{d}_{\mathfrak{C}'}\bigl(r(\Phi_{f_k}(h_k^1, \ldots, h_k^{n_k}))\bigr) \\ &\qquad - \textstyle\sum_{l=1}^{n_k} \Phi_{\frac{\partial f_k}{\partial x_l}}(h_k^1, \ldots, h_k^{n_k}) \cdot \mathrm{d}_{\mathfrak{C}'}(r(h_k^l))\bigr] \end{aligned} \right\} = \beta, \end{aligned}$$

where $b'_j, g'_k \in \mathfrak{C}'$ map to b_j, g_k under ϕ, and in the second line the first term is β, the terms $\phi[\cdots]$ are zero as '$\cdots$' lies in I, and in the third line $\{\cdots\}$ is zero by (2.5), replacing $\mathrm{d}_{\mathfrak{C}} c$ by $\mathrm{d}_{\mathfrak{C}'}(r(c))$. Therefore (2.4) is exact at $\Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}$. $\square$

The next lemma is straightforward:

Lemma 2.5. *Let $\phi : \mathfrak{C}' \to \mathfrak{C}$ and $\psi : \mathfrak{D}' \to \mathfrak{D}$ be square zero extensions, and $(\alpha, \alpha') : \phi \to \psi$ a morphism of square zero extensions. Then we have a commutative diagram*

$$\begin{tikzcd} I \arrow[r, "\Xi_\phi"] \arrow[d, "\alpha''"] & \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C} \arrow[r, "(\Omega_\phi)_*"] \arrow[d, "\Omega_{\alpha'} \otimes \alpha"] & \Omega_{\mathfrak{C}} \arrow[r] \arrow[d, "\Omega_\alpha"] & 0 \\ J \arrow[r, "\Xi_\psi"] & \Omega_{\mathfrak{D}'} \otimes_{\mathfrak{D}'} \mathfrak{D} \arrow[r, "(\Omega_\psi)_*"] & \Omega_{\mathfrak{D}} \arrow[r] & 0, \end{tikzcd} \tag{2.6}$$

where the columns are equivariant under $\alpha : \mathfrak{C} \to \mathfrak{D}$.

Here is an identity satisfied by operations Φ_f, Φ'_f for square zero extensions.

Lemma 2.6. *Let* $\phi : \mathfrak{C}' \to \mathfrak{C}$ *be a square zero extension of* C^∞*-rings with kernel* $I \subseteq \mathfrak{C}$*, and for smooth* $f : \mathbb{R}^n \to \mathbb{R}$ *write* $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ *and* $\Phi'_f : (\mathfrak{C}')^n \to \mathfrak{C}'$ *for the* C^∞*-ring operations in* $\mathfrak{C}, \mathfrak{C}'$*. Suppose* $c'_1, \ldots, c'_n \in \mathfrak{C}'$ *and* $i_1, \ldots, i_n \in I$ *with* $\phi(c'_j) = c_j \in \mathfrak{C}$ *for* $i = 1, \ldots, n$*. Then*

$$\Phi'_f(c'_1 + i_1, \ldots, c'_n + i_n) = \Phi'_f(c'_1, \ldots, c'_n) + \textstyle\sum_{j=1}^n \Phi_{\frac{\partial f}{\partial x_j}}(c_1, \ldots, c_n) \cdot i_j.$$

Proof. By Hadamard's Lemma there exist smooth functions $g_j : \mathbb{R}^{2n} \to \mathbb{R}$ for $j = 1, \ldots, n$ with

$$f(y_1, \ldots, y_n) - f(x_1, \ldots, x_n) = \textstyle\sum_{j=1}^n (y_j - x_j) g_j(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

for all $x_j, y_j \in \mathbb{R}$; also $g_j(x_1, \ldots, x_n, x_1, \ldots, x_n) = \frac{\partial f}{\partial x_j}(x_1, \ldots, x_n)$. Applying the axioms of C^∞-rings in $\mathfrak{C}'$ with $c'_j, c'_j + i_j$ in place of x_j, y_j yields

$$\begin{aligned}
&\Phi'_f(c'_1 + i_1, \ldots, c'_n + i_n) - \Phi'_f(c'_1, \ldots, c'_n) \\
&\quad = \textstyle\sum_{j=1}^n (c'_j + i_j - c'_j) \cdot \Phi'_{g_j}(c'_1, \ldots, c'_n, c'_1 + i_1, \ldots, c'_n + i_n) \\
&\quad = \textstyle\sum_{j=1}^n \phi\big(\Phi'_{g_j}(c'_1, \ldots, c'_n, c'_1 + i_1, \ldots, c'_n + i_n)\big) \cdot i_j \\
&\quad = \textstyle\sum_{j=1}^n \Phi_{g_j}(c_1, \ldots, c_n, c_1, \ldots, c_n) \cdot i_j = \textstyle\sum_{j=1}^n \Phi_{\frac{\partial f}{\partial x_j}}(c_1, \ldots, c_n) \cdot i_j,
\end{aligned}$$

where in the third step we use that the $\mathfrak{C}'$-action on I is induced by a $\mathfrak{C}$-action, in the fourth that ϕ is a morphism of C^∞-rings and $\phi(c'_j) = \phi(c'_j + i_j) = c_j$, and in the last that $g_j(x_1, \ldots, x_n, x_1, \ldots, x_n) = \frac{\partial f}{\partial x_j}(x_1, \ldots, x_n)$. □

The lemma motivates a construction of square zero extensions:

Example 2.7. Let $\mathfrak{C}$ be a C^∞-ring, and G a $\mathfrak{C}$-module. Set $\mathfrak{C}' = \mathfrak{C} \oplus G$, and write elements c' of $\mathfrak{C}'$ as pairs (c, g) for $c \in \mathfrak{C}$ and $g \in G$. Since $\mathfrak{C}$ is a C^∞-ring, it has operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$ satisfying axioms, as in Definition B.1. Define $\Phi'_f : (\mathfrak{C}')^n \to \mathfrak{C}'$ by

$$\Phi'_f\big((c_1, g_1), \ldots, (c_n, g_n)\big) = \big(\Phi_f(c_1, \ldots, c_n), \textstyle\sum_{j=1}^n \Phi_{\frac{\partial f}{\partial x_j}}(c_1, \ldots, c_n) \cdot g_j\big).$$

It is easy to show that these operations make $\mathfrak{C}'$ into a C^∞-ring, which we will write as $\mathfrak{C} \ltimes G$. Define $\phi : \mathfrak{C}' \to \mathfrak{C}$ by $\phi : (c, g) \mapsto c$. Then ϕ is a square zero extension of C^∞-rings. Not all square zero extensions $\phi : \mathfrak{C}' \to \mathfrak{C}$ are of this form: we have $\mathfrak{C}' \cong \mathfrak{C} \ltimes G$ if and only if ϕ admits a right inverse $r : \mathfrak{C} \to \mathfrak{C}'$ which is a C^∞-ring morphism. One can show that $\Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C} \cong \Omega_{\mathfrak{C}} \oplus G$.

More generally, if $\phi : \mathfrak{C}' \to \mathfrak{C}$ is a square zero extension and G is a $\mathfrak{C}$-module we may define $\hat{\mathfrak{C}}' = \mathfrak{C}' \oplus G$ and $\hat{\Phi}'_f : (\hat{\mathfrak{C}}')^n \to \hat{\mathfrak{C}}'$ by

$$\hat{\Phi}'_f\big((c'_1, g_1), \ldots, (c'_n, g_n)\big) = \big(\Phi'_f(c'_1, \ldots, c'_n), \textstyle\sum_{j=1}^n \Phi_{\frac{\partial f}{\partial x_j}}(\phi(c'_1), \ldots, \phi(c'_n)) \cdot g_j\big).$$

Then $\hat{\mathfrak{C}}'$ is a C^∞-ring, which we will write as $\mathfrak{C}' \ltimes G$, and $\hat{\phi} : \hat{\mathfrak{C}}' \to \mathfrak{C}$ given by $\hat{\phi} : (c', g) \mapsto \phi(c)$ is a square zero extension, with $\Omega_{\hat{\mathfrak{C}}'} \otimes_{\hat{\mathfrak{C}}'} \mathfrak{C} \cong (\Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}) \oplus G$.

The next proposition will be important for defining 2-morphisms of d-spaces.

Proposition 2.8. *Suppose $\phi : \mathfrak{C}' \to \mathfrak{C}$ and $\psi : \mathfrak{D}' \to \mathfrak{D}$ are square zero extensions of C^∞-rings with kernels I, J, and $(\alpha, \alpha'_1), (\alpha, \alpha'_2)$ are both morphisms $\phi \to \psi$ which induce morphisms $\alpha''_1, \alpha''_2 : I \to J$, so we have a diagram*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\kappa_\phi"] \arrow[d, shift right, "\alpha''_1"'] \arrow[d, shift left, "\alpha''_2"] & \mathfrak{C}' \arrow[r, "\phi"] \arrow[d, shift right, "\alpha'_1"'] \arrow[d, shift left, "\alpha'_2"] & \mathfrak{C} \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & J \arrow[r, "\kappa_\psi"] & \mathfrak{D}' \arrow[r, "\psi"] & \mathfrak{D} \arrow[r] & 0. \end{tikzcd}$$

Then there exists a unique $\mathfrak{D}$-module morphism $\mu : \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{D} \to J$ such that

$$\alpha'_2 = \alpha'_1 + \kappa_\psi \circ \mu \circ \big(\mathrm{id} \otimes (\alpha \circ \phi)\big) \circ \mathrm{d}_{\mathfrak{C}'}, \tag{2.7}$$

where the morphisms are given in

$$\mathfrak{C}' \xrightarrow{\mathrm{d}_{\mathfrak{C}'}} \Omega_{\mathfrak{C}'} = \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}' \xrightarrow{\mathrm{id} \otimes (\alpha\circ\phi)} \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{D} \xrightarrow{\mu} J \xrightarrow{\kappa_\psi} \mathfrak{D}'.$$

We also have

$$\begin{aligned} \alpha''_2 &= \alpha''_1 + \mu \circ \big(\mathrm{id} \otimes (\alpha \circ \phi)\big) \circ \mathrm{d}_{\mathfrak{C}'} \circ \kappa_\phi \\ \text{and} \quad \Omega_{\alpha'_2} &= \Omega_{\alpha'_1} + \mathrm{d}_{\mathfrak{D}'} \circ \kappa_\psi \circ \mu \circ \big(\mathrm{id} \otimes (\alpha \circ \phi)\big). \end{aligned} \tag{2.8}$$

Conversely, if $(\alpha, \alpha'_1) : \phi \to \psi$ is a morphism, and $\mu : \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{D} \to J$ a $\mathfrak{D}$-module morphism, then defining α'_2 by (2.7) gives a C^∞-ring morphism $\alpha'_2 : \mathfrak{C}' \to \mathfrak{D}'$ with $(\alpha, \alpha'_2) : \phi \to \psi$ a morphism of square zero extensions.

Proof. Define $\beta = \alpha'_2 - \alpha'_1 : \mathfrak{C}' \to J$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, and write $\Phi_f, \Phi'_f, \hat{\Phi}_f, \hat{\Phi}'_f$ for the C^∞-ring operations on $\mathfrak{C}, \mathfrak{C}', \mathfrak{D}, \mathfrak{D}'$ respectively. Let $c'_1, \ldots, c'_n \in \mathfrak{C}'$. Then $\alpha'_j\big(\Phi'_f(c_1, \ldots, c_n)\big) = \hat{\Phi}'_f\big(\alpha'_j(c'_1), \ldots, \alpha'_j(c'_n)\big)$ for $j = 1, 2$ as α'_j is a C^∞-ring morphism. Hence

$$\begin{aligned} \beta\big(\Phi'_f(c'_1, \ldots, c'_n)\big) &= \hat{\Phi}'_f\big(\alpha'_2(c'_1), \ldots, \alpha'_j(c'_n)\big) - \hat{\Phi}'_f\big(\alpha'_1(c'_1), \ldots, \alpha'_1(c'_n)\big) \\ &= \hat{\Phi}'_f\big(\alpha'_1(c'_1) + \beta(c'_1), \ldots, \alpha'_1(c'_n) + \beta(c'_1)\big) - \hat{\Phi}'_f\big(\alpha'_1(c'_1), \ldots, \alpha'_1(c'_n)\big) \\ &= \textstyle\sum_{j=1}^n \hat{\Phi}_{\frac{\partial f}{\partial x_j}}\big(\psi \circ \alpha'_1(c'_1), \ldots, \psi \circ \alpha'_1(c'_n)\big) \cdot \beta(c'_j) \\ &= \textstyle\sum_{j=1}^n \big(\psi \circ \alpha'_1\big(\Phi'_{\frac{\partial f}{\partial x_j}}(c'_1, \ldots, c'_n)\big)\big) \cdot \beta(c'_j). \end{aligned}$$

Therefore β is a C^∞-derivation on $\mathfrak{C}'$ in the sense of Definition B.31, making J a $\mathfrak{C}'$-module via its $\mathfrak{D}$-module structure and $\psi\circ\alpha_1':\mathfrak{C}'\to\mathfrak{D}$. Hence by the universal property of the cotangent module $\Omega_{\mathfrak{C}'}$, there is a unique morphism of $\mathfrak{C}'$-modules $\lambda:\Omega_{\mathfrak{C}'}\to J$ with $\beta=\lambda\circ\mathrm{d}_{\mathfrak{C}'}$. As the $\mathfrak{C}'$-action on J factors through the $\mathfrak{D}$-action, λ factors through $\Omega_{\mathfrak{C}'}\otimes_{\mathfrak{C}'}\mathfrak{D}$, so there is a unique $\mathfrak{D}$-module morphism $\mu:\Omega_{\mathfrak{C}'}\otimes_{\mathfrak{C}'}\mathfrak{D}\to J$ with $\lambda=\mu\circ(\mathrm{id}\otimes(\alpha\circ\phi))$. Putting this all together gives (2.7), and (2.8) easily follows. This proves the first part.

For the converse, given $(\alpha,\alpha_1'),\mu$ and defining α_2' by (2.7), we have $\alpha_1'\bigl(\Phi'_f(c_1,\ldots,c_n)\bigr)=\hat\Phi'_f\bigl(\alpha_1'(c_1'),\ldots,\alpha_1'(c_n')\bigr)$ since α_1' is a C^∞-ring morphism. Running the argument above in reverse then shows that $\alpha_2'\bigl(\Phi'_f(c_1,\ldots,c_n)\bigr)=\hat\Phi'_f\bigl(\alpha_2'(c_1'),\ldots,\alpha_2'(c_n')\bigr)$. Thus α_2' is a C^∞-ring morphism, and $\psi\circ\alpha_2'=\psi\circ\alpha_1'=\alpha\circ\phi$ as $\psi\circ\kappa_\psi=0$, so (α,α_2') is a morphism of square zero extensions. □

All the above material for C^∞-rings quickly translates to square zero extensions of C^∞-schemes, by applying the spectrum functor, and working with sheaves of C^∞-rings or abelian groups and sheaves of $\mathcal{O}_X$-modules, rather than just C^∞-rings and modules. We leave the proofs as an exercise. See §B.3 and §B.7 for the necessary background. Note in particular, as in Definitions B.15 and B.34, that there are two different kinds of pullback $f^{-1},\underline{f}^*$, and we will make use of both: if $f:X\to Y$ is a continuous map and $\mathcal{E}$ is a sheaf of abelian groups or C^∞-rings on Y then $f^{-1}(\mathcal{E})$ is a sheaf of abelian groups or C^∞-rings on Y. Also, if $\underline{f}=(f,f^\sharp):\underline{X}\to\underline{Y}$ is a morphism of C^∞-schemes and $\mathcal{E}\in\mathrm{qcoh}(\underline{Y})$ then $\underline{f}^*(\mathcal{E})=f^{-1}(\mathcal{E})\otimes_{f^{-1}(\mathcal{O}_Y)}\mathcal{O}_X$ in $\mathrm{qcoh}(\underline{X})$.

Here are the C^∞-scheme analogues of Definitions 2.1 and 2.3, Example 2.7, and Proposition 2.8.

Definition 2.9. Let $\underline{X}=(X,\mathcal{O}_X)$ be a locally fair C^∞-scheme. By Theorem B.36(c), this implies all $\mathcal{O}_X$-modules are quasicoherent. A *square zero extension* $(\mathcal{O}_X',\imath_X)$ *of* $\underline{X}$ consists of a sheaf of C^∞-rings $\mathcal{O}_X'$ on X, such that $\underline{X}'=(X,\mathcal{O}_X')$ is a C^∞-scheme, and a morphism $\imath_X:\mathcal{O}_X'\to\mathcal{O}_X$ of sheaves of C^∞-rings on X, which is a sheaf of square zero extensions of C^∞-rings, in the sense of Definition 2.1. Then $\underline{\imath}_X=(\mathrm{id}_X,\imath_X)$ is a morphism of C^∞-schemes $\underline{\imath}_X:\underline{X}\to\underline{X}'$. We also call $(\underline{X},\mathcal{O}_X',\imath_X)$ a *square zero extension of* C^∞*-schemes*.

Write $\kappa_X:\mathcal{I}_X\to\mathcal{O}_X'$ for the kernel of $\imath_X$. Then $\mathcal{I}_X$ is a sheaf of square zero ideals in $\mathcal{O}_X'$, so it is a sheaf of $\mathcal{O}_X$-modules, and thus a quasicoherent sheaf on $\underline{X}$. As for (2.1), we have an exact sequence of sheaves of abelian groups on X:

$$0\longrightarrow\mathcal{I}_X\xrightarrow{\kappa_X}\mathcal{O}_X'\xrightarrow{\imath_X}\mathcal{O}_X\longrightarrow 0.\tag{2.9}$$

The sheaf of C^∞-rings $\mathcal{O}_X'$ has a sheaf of cotangent modules $\Omega_{\mathcal{O}_X'}$, which is a sheaf of $\mathcal{O}_X'$-modules with exterior derivative $\mathrm{d}:\mathcal{O}_X'\to\Omega_{\mathcal{O}_X'}$. Define $\mathcal{F}_X=\Omega_{\mathcal{O}_X'}\otimes_{\mathcal{O}_X'}\mathcal{O}_X$ to be the associated sheaf of $\mathcal{O}_X$-modules, a quasicoherent sheaf on $\underline{X}$, and set $\psi_X=\Omega_{\imath_X}\otimes\mathrm{id}:\mathcal{F}_X\to T^*\underline{X}$, a morphism in $\mathrm{qcoh}(\underline{X})$. Equivalently, $\mathcal{F}_X=\underline{\imath}_X^*(T^*\underline{X}')$ and $\psi_X=\Omega_{\underline{\imath}_X}$. In the analogue of (2.3), define a morphism of sheaves of abelian groups $\xi_X:\mathcal{I}_X\to\mathcal{F}_X$ to be the composition

$$\mathcal{I}_X\xrightarrow{\kappa_X}\mathcal{O}_X'\xrightarrow{\mathrm{d}}\Omega_{\mathcal{O}_X'}=\Omega_{\mathcal{O}_X'}\otimes_{\mathcal{O}_X'}\mathcal{O}_X'\xrightarrow{\mathrm{id}\otimes\imath_X}\Omega_{\mathcal{O}_X'}\otimes_{\mathcal{O}_X'}\mathcal{O}_X=\mathcal{F}_X.\tag{2.10}$$

Proposition 2.4 then implies that ξ_X is a morphism of quasicoherent sheaves on $\underline{X}$, and the following sequence is exact in qcoh($\underline{X}$):

$$\mathcal{I}_X \xrightarrow{\xi_X} \mathcal{F}_X \xrightarrow{\psi_X} T^*\underline{X} \longrightarrow 0. \tag{2.11}$$

Remark 2.10. Of course, we could have defined a square zero extension of C^∞-schemes to be a morphism of C^∞-schemes $\underline{\imath}_X : \underline{X} \to \underline{X}'$ with the properties above. But this would run counter to the philosophy of this chapter. We do not want to regard $\underline{X}'$ as a separate C^∞-scheme in its own right; instead, we try to work always with sheaves (whenever possible quasicoherent sheaves) on $\underline{X}$. So we express $\underline{X}'$ in terms of a second sheaf of C^∞-rings $\mathcal{O}'_X$ on X.

This will become important in the definition of d-stacks in §9.2. Then there is a real difference between working with a morphism of C^∞-stacks $\imath_X : \mathcal{X} \to \mathcal{X}'$, and working with a C^∞-stack $\mathcal{X}$ together with a sheaf of C^∞-rings $\mathcal{O}'_X$ on $\mathcal{X}$. In the first case we would have to worry about 2-morphisms of both $\mathcal{X}$ and $\mathcal{X}'$, but in the second only about 2-morphisms of $\mathcal{X}$, which makes things simpler.

Example 2.11. Let $\underline{X} = (X, \mathcal{O}_X)$ be a locally fair C^∞-scheme, and $\mathcal{G} \in$ qcoh($\underline{X}$). Define a sheaf of C^∞-rings $\mathcal{O}_X \ltimes \mathcal{G}$ on X by $(\mathcal{O}_X \ltimes \mathcal{G})(U) = \mathcal{O}_X(U) \ltimes \mathcal{G}(U) \cong \mathcal{O}_X(U) \oplus \mathcal{G}(U)$ for each open $U \subseteq X$, where the C^∞-ring $\mathcal{O}_X(U) \ltimes \mathcal{G}(U)$ is defined in Example 2.7, with restriction maps $\rho_{UV} : (\mathcal{O}_X \ltimes \mathcal{G})(U) \to (\mathcal{O}_X \ltimes \mathcal{G})(V)$ for open $V \subseteq U \subseteq X$ induced from the restriction maps $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ and $\rho_{UV} : \mathcal{G}(U) \to \mathcal{G}(V)$ in $\mathcal{O}_X, \mathcal{G}$ in the obvious way. Define a sheaf of morphisms of C^∞-rings $\imath_X : \mathcal{O}_X \ltimes \mathcal{G} \to \mathcal{O}_X$ by taking $\imath_X(U) : (\mathcal{O}_X \ltimes \mathcal{G})(U) \cong \mathcal{O}_X(U) \oplus \mathcal{G}(U) \to \mathcal{O}_X(U)$ to be the natural projection. Then $(\mathcal{O}_X \ltimes \mathcal{G}, \imath_X)$ is a square zero extension of $\underline{X}$. It is easy to see that the associated quasicoherent sheaves $\mathcal{I}_X, \mathcal{F}_X$ are $\mathcal{I}_X \cong \mathcal{G}$ and $\mathcal{F}_X \cong T^*\underline{X} \oplus \mathcal{G}$.

More generally, let $(\mathcal{O}'_X, \imath_X)$ be a square zero extension of $\underline{X}$, with sheaves $\mathcal{I}_X, \mathcal{F}_X$, and let $\mathcal{G} \in$ qcoh($\underline{X}$). Define a sheaf of C^∞-rings $\mathcal{O}'_{\hat{X}}$ on X by $\mathcal{O}'_{\hat{X}} = \mathcal{O}'_X \ltimes \mathcal{G}$, where $(\mathcal{O}'_X \ltimes \mathcal{G})(U) = \mathcal{O}'_X(U) \ltimes \mathcal{G}(U) \cong \mathcal{O}'_X(U) \oplus \mathcal{G}(U)$ for all open $U \subseteq X$, and the C^∞-ring $\mathcal{O}'_X(U) \ltimes \mathcal{G}(U)$ is defined in Example 2.7, and with the obvious restriction maps ρ_{UV}. Define a morphism of sheaves of C^∞-rings $\imath_{\hat{X}} : \mathcal{O}'_{\hat{X}} \to \mathcal{O}_X$ by $\imath_{\hat{X}}(U) = \imath_X(U) \circ \pi_{\mathcal{O}'_X(U)}$, where $\pi_{\mathcal{O}'_X(U)} : (\mathcal{O}'_X \ltimes \mathcal{G})(U) \cong \mathcal{O}'_X(U) \oplus \mathcal{G}(U) \to \mathcal{O}'_X(U)$ is the natural projection. Then $(\mathcal{O}'_{\hat{X}}, \imath_{\hat{X}})$ is a square zero extension of $\underline{X}$, with sheaves $\mathcal{I}_{\hat{X}} \cong \mathcal{I}_X \oplus \mathcal{G}$ and $\mathcal{F}_{\hat{X}} \cong \mathcal{F}_X \oplus \mathcal{G}$.

Definition 2.12. Let $(\underline{X}, \mathcal{O}'_X, \imath_X)$ and $(\underline{Y}, \mathcal{O}'_Y, \imath_Y)$ be square zero extensions of C^∞-schemes. A *morphism of square zero extensions* from $(\underline{X}, \mathcal{O}'_X, \imath_X)$ to $(\underline{Y}, \mathcal{O}'_Y, \imath_Y)$ is a pair $(\underline{f}, f')$, where $\underline{f} = (f, f^\sharp)$ is a morphism of C^∞-schemes $\underline{f} : \underline{X} \to \underline{Y}$, so that $f : X \to Y$ is continuous and $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ is a morphism of sheaves of C^∞-rings on X, and $f' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$ is a morphism of sheaves of C^∞-rings on X such that $f^\sharp \circ f^{-1}(\imath_Y) = \imath_X \circ f' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}_X$.

We will define a commutative diagram of sheaves of abelian groups on X:

$$
\begin{tikzcd}
f^{-1}(\mathcal{I}_Y)\otimes^{\mathrm{id}}_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{O}_Y) \arrow[d, "\mathrm{id}\otimes f^\sharp"] \arrow[r, phantom, "="] & f^{-1}(\mathcal{I}_Y) \arrow[r, "f^{-1}(\kappa_Y)"] \arrow[dd, "f'|_{f^{-1}(\mathcal{I}_Y)}"] & f^{-1}(\mathcal{O}'_Y) \arrow[r, "f^{-1}(\imath_Y)"] \arrow[dd, "f'"] & f^{-1}(\mathcal{O}_Y) \arrow[r] \arrow[dd, "f^\sharp"] & 0 \\
\begin{array}{l}\underline{f}^*(\mathcal{I}_Y) = \\ f^{-1}(\mathcal{I}_Y)\otimes^{f^\sharp}_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X\end{array} \arrow[dr, "f^1"] & & & & \\
& \mathcal{I}_X \arrow[r, "\kappa_X"] & \mathcal{O}'_X \arrow[r, "\imath_X"] & \mathcal{O}_X \arrow[r] & 0.
\end{tikzcd}
\tag{2.12}
$$

Here the bottom line is (2.9) for $(\underline{X}, \mathcal{O}'_X, \imath_X)$, and the top line the pullback of (2.9) for $(\underline{Y}, \mathcal{O}'_Y, \imath_Y)$ by $f : X \to Y$, so both are exact as f^{-1} is right exact. We use the notation that if for instance $\imath_X : \mathcal{O}'_X \to \mathcal{O}_X$ is a morphism of sheaves of C^∞-rings and $\mathcal{E}$ a sheaf of $\mathcal{O}'_X$-modules, then $\mathcal{E}\otimes^{\imath_X}_{\mathcal{O}'_X}\mathcal{O}_X$ or just $\mathcal{E}\otimes_{\mathcal{O}'_X}\mathcal{O}_X$ is the associated sheaf of $\mathcal{O}_X$-modules.

As $f^\sharp \circ f^{-1}(\imath_Y) = \imath_X \circ f'$, the right hand square of (2.12) commutes. Hence f' maps the kernel $f^{-1}(\mathcal{I}_Y)$ of $f^{-1}(\imath_Y)$ to the kernel $\mathcal{I}_X$ of $\imath_X$, and $f'|_{f^{-1}(\mathcal{I}_Y)}$ in (2.12) is well-defined. This $f'|_{f^{-1}(\mathcal{I}_Y)}$ is the analogue of α'' in (2.2). Now $f^{-1}(\mathcal{I}_Y)$ is an $f^{-1}(\mathcal{O}_Y)$-module, and $\mathcal{I}_X$ an $\mathcal{O}_X$-module, and $f'|_{f^{-1}(\mathcal{I}_Y)}$ is equivariant under $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. Therefore $f'|_{f^{-1}(\mathcal{I}_Y)}$ factors through $f^{-1}(\mathcal{I}_Y)\otimes_{f^{-1}(\mathcal{O}_Y)}\mathcal{O}_X = \underline{f}^*(\mathcal{I}_Y)$, and there is a unique morphism $f^1 : \underline{f}^*(\mathcal{I}_Y) \to \mathcal{I}_X$ in $\mathrm{qcoh}(\underline{X})$ making (2.12) commute.

Define a morphism $f^3 = \Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$ in $\mathrm{qcoh}(\underline{X})$, and define $f^2 : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{F}_X$ in $\mathrm{qcoh}(\underline{X})$ by the commutative diagram

$$
\begin{tikzcd}
\underline{f}^*(\mathcal{F}_Y) = f^{-1}(\Omega_{\mathcal{O}'_Y}\otimes^{\imath_Y}_{\mathcal{O}'_Y}\mathcal{O}_Y)\otimes^{f^\sharp}_{f^{-1}(\mathcal{O}_Y)}\mathcal{O}_X \arrow[r, "\cong"] \arrow[d, "f^2"] & f^{-1}(\Omega_{\mathcal{O}'_Y})\otimes^{f^\sharp\circ f^{-1}(\imath_Y)}_{f^{-1}(\mathcal{O}'_Y)}\mathcal{O}_X \arrow[d, equal] \\
\mathcal{F}_X = \Omega_{\mathcal{O}'_X}\otimes^{\imath_X}_{\mathcal{O}'_X}\mathcal{O}_X & f^{-1}(\Omega_{\mathcal{O}'_Y})\otimes^{\imath_X\circ f'}_{f^{-1}(\mathcal{O}'_Y)}\mathcal{O}_X. \arrow[l, "\Omega_{f'}\otimes\mathrm{id}"']
\end{tikzcd}
\tag{2.13}
$$

Then the analogue of (2.6) is the following commutative diagram in $\mathrm{qcoh}(\underline{X})$:

$$
\begin{tikzcd}
\underline{f}^*(\mathcal{I}_Y) \arrow[r, "\underline{f}^*(\xi_Y)"'] \arrow[d, "f^1"] & \underline{f}^*(\mathcal{F}_Y) \arrow[r, "\underline{f}^*(\psi_Y)"'] \arrow[d, "f^2"] & \underline{f}^*(T^*\underline{Y}) \arrow[r] \arrow[d, "f^3=\Omega_{\underline{f}}"] & 0 \\
\mathcal{I}_X \arrow[r, "\xi_X"] & \mathcal{F}_X \arrow[r, "\psi_X"] & T^*\underline{X} \arrow[r] & 0.
\end{tikzcd}
\tag{2.14}
$$

Note that both rows are exact, as (2.11) is exact and $\underline{f}^*$ is right exact.

Suppose $(\underline{X}, \mathcal{O}'_X, \imath_X), (\underline{Y}, \mathcal{O}'_Y, \imath_Y), (\underline{Z}, \mathcal{O}'_Z, \imath_Z)$ are square zero extensions of C^∞-schemes, and $(\underline{f}, f') : (\underline{X}, \mathcal{O}'_X, \imath_X) \to (\underline{Y}, \mathcal{O}'_Y, \imath_Y)$, $(\underline{g}, g') : (\underline{Y}, \mathcal{O}'_Y, \imath_Y) \to (\underline{Z}, \mathcal{O}'_Z, \imath_Z)$ are morphisms. Define the composition of $(\underline{f}, f')$ and $(\underline{g}, g')$ to be

$$
(\underline{g}, g') \circ (\underline{f}, f') = \bigl(\underline{g}\circ\underline{f}, f' \circ f^{-1}(g') \circ I_{f,g}(\mathcal{O}'_Z)\bigr). \tag{2.15}
$$

One can check that this is a morphism of square zero extensions, and that composition is associative.

Proposition 2.13. *Let $(\underline{X}, \mathcal{O}'_X, \imath_X)$ and $(\underline{Y}, \mathcal{O}'_Y, \imath_Y)$ be square zero extensions of C^∞-schemes with kernel sheaves $\mathcal{I}_X, \mathcal{I}_Y$, and $(\underline{f}, f'), (\underline{g}, g')$ be morphisms from $(\underline{X}, \mathcal{O}'_X, \imath_X)$ to $(\underline{Y}, \mathcal{O}'_Y, \imath_Y)$. Suppose $\underline{f} = \underline{g}$. Use the notation κ_X, ξ_X, ψ_X, κ_Y, ξ_Y, ψ_Y from Definition 2.9 and $f^1, f^2, f^3, g^1, g^2, g^3$ from Definition 2.12. Then there exists a unique morphism $\mu : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{I}_X$ in $\mathrm{qcoh}(\underline{X})$ such that*

$$g' = f' + \kappa_X \circ \mu \circ \big(\mathrm{id} \otimes (f^\sharp \circ f^{-1}(\imath_Y))\big) \circ \big(f^{-1}(\mathrm{d})\big), \tag{2.16}$$

where the morphisms are given in the diagram

$$\begin{tikzcd}
f^{-1}(\mathcal{O}'_Y) \arrow[rr, "f^{-1}(\mathrm{d})"'] \arrow[d, shift right, "f'"'] \arrow[d, shift left, "g'"] & & f^{-1}(\Omega_{\mathcal{O}'_Y}) \arrow[r, equal] & f^{-1}(\Omega_{\mathcal{O}'_Y}) \otimes^{\mathrm{id}}_{f^{-1}(\mathcal{O}'_Y)} f^{-1}(\mathcal{O}'_Y) \arrow[d, "\mathrm{id}\otimes(f^\sharp\circ f^{-1}(\imath_Y))"'] \\
\mathcal{O}'_X & \mathcal{I}_X \arrow[l, "\kappa_X"'] & \underline{f}^*(\mathcal{F}_Y) \arrow[l, "\mu"'] \arrow[r, equal] & f^{-1}(\Omega_{\mathcal{O}'_Y}) \otimes^{f^\sharp\circ f^{-1}(\imath_Y)}_{f^{-1}(\mathcal{O}'_Y)} \mathcal{O}_X.
\end{tikzcd}$$

We also have

$$g^1 = f^1 + \mu \circ \underline{f}^*(\xi_Y), \quad g^2 = f^2 + \xi_X \circ \mu, \quad \text{and} \quad g^3 = f^3. \tag{2.17}$$

So μ is a homotopy between morphisms of complexes (2.14) *from $(\underline{f}, f'), (\underline{g}, g')$.*

Conversely, if $(\underline{f}, f')$ is a morphism $(\underline{X}, \mathcal{O}'_X, \imath_X) \to (\underline{Y}, \mathcal{O}'_Y, \imath_Y)$ and $\mu : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{I}_X$ is a morphism in $\mathrm{qcoh}(\underline{X})$, then there exists a unique morphism $g' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$ of sheaves of C^∞-rings on X such that $(\underline{g}, g')$ with $\underline{g} = \underline{f}$ is a morphism $(\underline{X}, \mathcal{O}'_X, \imath_X) \to (\underline{Y}, \mathcal{O}'_Y, \imath_Y)$, and (2.16)–(2.17) *hold.*

2.2 The definition of d-spaces

We will now define the 2-category of d-spaces **dSpa**. Theorem 2.15 will show it satisfies the axioms of a 2-category.

Definition 2.14. A *d-space* $\boldsymbol{X}$ is a quintuple $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ such that $\underline{X}$ is a separated, second countable, locally fair C^∞-scheme, and $(\mathcal{O}'_X, \imath_X)$ is a square zero extension of $\underline{X}$ in the sense of Definition 2.9 with kernel $\kappa_X : \mathcal{I}_X \to \mathcal{O}'_X$, so that $\mathcal{I}_X \in \mathrm{qcoh}(\underline{X})$, and $\mathcal{E}_X$ is a quasicoherent sheaf on $\underline{X}$, and $\jmath_X : \mathcal{E}_X \to \mathcal{I}_X$ is a surjective morphism in $\mathrm{qcoh}(\underline{X})$. Thus as for (2.9) we have an exact sequence of sheaves of abelian groups on $\underline{X}$:

$$\mathcal{E}_X \xrightarrow{\kappa_X \circ \jmath_X} \mathcal{O}'_X \xrightarrow{\imath_X} \mathcal{O}_X \longrightarrow 0. \tag{2.18}$$

As $\underline{X}$ is locally fair, the underlying topological space X is locally homeomorphic to a closed subset of $\mathbb{R}^n$, so it is *locally compact*. But Hausdorff, second countable and locally compact imply paracompact, so $\underline{X}$ is *paracompact*.

Let $\mathcal{F}_X, \psi_X, \xi_X$ be as in Definition 2.9, and define $\phi_X = \xi_X \circ \jmath_X : \mathcal{E}_X \to \mathcal{F}_X$. Then from (2.11) and $\jmath_X$ surjective we have an exact sequence in $\mathrm{qcoh}(\underline{X})$:

$$\mathcal{E}_X \xrightarrow{\phi_X} \mathcal{F}_X \xrightarrow{\psi_X} T^*\underline{X} \longrightarrow 0. \tag{2.19}$$

The morphism $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ will be called the *virtual cotangent sheaf* of $\boldsymbol{X}$, for reasons we explain in §3.1.

Write $\lambda_X : \mathcal{C}_X \to \mathcal{E}_X$ for the kernel of $\jmath_X : \mathcal{E}_X \to \mathcal{I}_X$ in $\mathrm{qcoh}(\underline{X})$, and $\mu_X : \mathcal{D}_X \to \mathcal{E}_X$ for the kernel of $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ in $\mathrm{qcoh}(\underline{X})$. Since $\phi_X = \xi_X \circ \jmath_X$, λ_X factors via μ_X, so there exists a unique morphism $\nu_X : \mathcal{C}_X \to \mathcal{D}_X$ with $\lambda_X = \mu_X \circ \nu_X$. Thus we have a commutative diagram with exact diagonals:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_X \arrow[dr, "\lambda_X"] \arrow[dd, "\nu_X"'] & & \mathcal{F}_X \arrow[r, "\psi_X"] & T^*\underline{X} \arrow[r] & 0 \\
& & \mathcal{E}_X \arrow[ur, "\phi_X"] \arrow[dr, "\jmath_X"'] & & & \\
0 \arrow[r] & \mathcal{D}_X \arrow[ur, "\mu_X"'] & & \mathcal{I}_X \arrow[uu, "\xi_X"'] \arrow[rr] & & 0.
\end{tikzcd}
\tag{2.20}
$$

Let $\boldsymbol{X}, \boldsymbol{Y}$ be d-spaces. A 1-*morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a triple $\boldsymbol{f} = (\underline{f}, f', f'')$, where $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes, and $f' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$ is a morphism of sheaves of C^∞-rings on X such that $(\underline{f}, f')$ is a morphism of square zero extensions $(\underline{X}, \mathcal{O}'_X, \imath_X) \to (\underline{Y}, \mathcal{O}'_Y, \imath_Y)$ in the sense of Definition 2.12, and $f'' : \underline{f}^*(\mathcal{E}_Y) \to \mathcal{E}_X$ is a morphism in $\mathrm{qcoh}(\underline{X})$ satisfying

$$\jmath_X \circ f'' = f^1 \circ \underline{f}^*(\jmath_Y) : \underline{f}^*(\mathcal{E}_Y) \longrightarrow \mathcal{I}_X, \tag{2.21}$$

where f^1, f^2, f^3 are as in Definition 2.12. Then from (2.14), (2.19) and (2.21) we have a commutative diagram in $\mathrm{qcoh}(\underline{X})$, with exact rows:

$$
\begin{tikzcd}
\underline{f}^*(\mathcal{E}_Y) \arrow[r, "\underline{f}^*(\phi_Y)"'] \arrow[d, "f''"] & \underline{f}^*(\mathcal{F}_Y) \arrow[r, "\underline{f}^*(\psi_Y)"'] \arrow[d, "f^2"] & \underline{f}^*(T^*\underline{Y}) \arrow[r] \arrow[d, "f^3=\Omega_{\underline{f}}"] & 0 \\
\mathcal{E}_X \arrow[r, "\phi_X"] & \mathcal{F}_X \arrow[r, "\psi_X"] & T^*\underline{X} \arrow[r] & 0.
\end{tikzcd}
\tag{2.22}
$$

Composing (2.21) with $\underline{f}^*(\lambda_Y) : \underline{f}^*(\mathcal{C}_Y) \to \underline{f}^*(\mathcal{E}_Y)$ gives $\jmath_X \circ f'' \circ \underline{f}^*(\lambda_Y) = f^1 \circ \underline{f}^*(\jmath_Y) \circ \underline{f}^*(\lambda_Y) = 0$, since $\jmath_Y \circ \lambda_Y = 0$. Hence $f'' \circ \underline{f}^*(\lambda_Y)$ factors through the kernel of $\jmath_X$, which is λ_X. Thus there exists a unique morphism $f^4 : \underline{f}^*(\mathcal{C}_Y) \to \mathcal{C}_X$ in $\mathrm{qcoh}(\underline{X})$ with $\lambda_X \circ f^4 = f'' \circ \underline{f}^*(\lambda_Y)$. Similarly there exists a unique $f^5 : \underline{f}^*(\mathcal{D}_Y) \to \mathcal{D}_X$ with $\mu_X \circ f^5 = f'' \circ \underline{f}^*(\mu_Y)$.

If $\boldsymbol{X}$ is a d-space, the *identity* 1-*morphism* $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ is $\mathbf{id}_{\boldsymbol{X}} = \bigl(\mathrm{id}_{\underline{X}}, \delta_X(\mathcal{O}'_X), \delta_{\underline{X}}(\mathcal{E}_X)\bigr)$. It is easy to check $\mathbf{id}_{\boldsymbol{X}}$ is a 1-morphism.

Now let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be d-spaces, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms. Generalizing (2.15), define the *composition of* 1-*morphisms* to be

$$\boldsymbol{g} \circ \boldsymbol{f} = \bigl(\underline{g} \circ \underline{f}, f' \circ f^{-1}(g') \circ I_{f,g}(\mathcal{O}'_Z), f'' \circ \underline{f}^*(g'') \circ I_{\underline{f},\underline{g}}(\mathcal{E}_Z)\bigr). \tag{2.23}$$

One can check that $\boldsymbol{g} \circ \boldsymbol{f}$ is a 1-morphism $\boldsymbol{X} \to \boldsymbol{Z}$, with

$$
\begin{aligned}
(g\circ f)^1 &= f^1 \circ \underline{f}^*(g^1) \circ I_{\underline{f},\underline{g}}(\mathcal{I}_Z), \qquad (g\circ f)^2 = f^2 \circ \underline{f}^*(g^2) \circ I_{\underline{f},\underline{g}}(\mathcal{F}_Z),\\
(g\circ f)^3 &= f^3 \circ \underline{f}^*(g^3) \circ I_{\underline{f},\underline{g}}(T^*\underline{Z}), \quad (g\circ f)^4 = f^4 \circ \underline{f}^*(g^4) \circ I_{\underline{f},\underline{g}}(\mathcal{C}_Z),\\
&\text{and} \qquad (g \circ f)^5 = f^5 \circ \underline{f}^*(g^5) \circ I_{\underline{f},\underline{g}}(\mathcal{D}_Z).
\end{aligned}
\tag{2.24}
$$

It is also easy to show that $\boldsymbol{f} \circ \mathbf{id}_{\boldsymbol{X}} = \mathbf{id}_{\boldsymbol{Y}} \circ \boldsymbol{f} = \boldsymbol{f}$.

Let $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ be 1-morphisms of d-spaces, where $\boldsymbol{f}=(\underline{f},f',f'')$ and $\boldsymbol{g}=(\underline{g},g',g'')$, and let $f^1,\ldots,g^5$ be as above. Suppose $\underline{f}=\underline{g}$. A 2-*morphism* $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is a morphism $\eta:\underline{f}^*(\mathcal{F}_Y)\rightarrow\mathcal{E}_X$ in $\operatorname{qcoh}(\underline{X})$, such that

$$
\begin{gathered}
g'=f'+\kappa_X\circ\jmath_X\circ\eta\circ\bigl(\mathrm{id}\otimes(f^\sharp\circ f^{-1}(\imath_Y))\bigr)\circ\bigl(f^{-1}(\mathrm{d})\bigr)\\
\text{and}\qquad g''=f''+\eta\circ\underline{f}^*(\phi_Y).
\end{gathered}
\tag{2.25}
$$

Here the first equation makes sense as Proposition 2.13 shows that (2.16) holds for some $\mu:\underline{f}^*(\mathcal{F}_Y)\rightarrow\mathcal{I}_X$, and we take $\mu=\jmath_X\circ\eta$. Thus (2.17) implies that

$$
\begin{gathered}
g^1=f^1+\jmath_X\circ\eta\circ\underline{f}^*(\xi_Y),\quad g^2=f^2+\phi_X\circ\eta,\\
g^3=f^3,\quad g^4=f^4,\quad\text{and}\quad g^5=f^5.
\end{gathered}
\tag{2.26}
$$

So (2.22) for $\boldsymbol{f},\boldsymbol{g}$ combine to give a diagram

$$
\begin{tikzcd}[column sep=large]
\underline{f}^*(\mathcal{E}_Y) \arrow[r, "\underline{f}^*(\phi_Y)"] \arrow[d, shift right, "f''"'] \arrow[d, shift left, "g''=f''+\eta\circ\underline{f}^*(\phi_Y)"] & \underline{f}^*(\mathcal{F}_Y) \arrow[r, "\underline{f}^*(\psi_Y)"] \arrow[dl, "\eta"] \arrow[d, shift right, "f^2"'] \arrow[d, shift left, "g^2=f^2+\phi_X\circ\eta"] & \underline{f}^*(T^*\underline{Y}) \arrow[r] \arrow[d, "f^3=g^3=\Omega_{\underline{f}}"] & 0 \\
\mathcal{E}_X \arrow[r, "\phi_X"] & \mathcal{F}_X \arrow[r, "\psi_X"] & T^*\underline{X} \arrow[r] & 0.
\end{tikzcd}
$$

That is, η is a homotopy between the morphisms of complexes (2.22) from $\boldsymbol{f},\boldsymbol{g}$.

If $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ is a 1-morphism, the *identity* 2-*morphism* $\mathrm{id}_{\boldsymbol{f}}:\boldsymbol{f}\Rightarrow\boldsymbol{f}$ is the zero morphism $0:\underline{f}^*(\mathcal{F}_Y)\rightarrow\mathcal{E}_X$. Suppose $\boldsymbol{X},\boldsymbol{Y}$ are d-spaces, $\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ are 1-morphisms and $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$, $\zeta:\boldsymbol{g}\Rightarrow\boldsymbol{h}$ are 2-morphisms. Then $\underline{f}=\underline{g}$, $\underline{g}=\underline{h}$, so $\underline{f}=\underline{h}$. Combining (2.25) for η,ζ gives

$$
\begin{gathered}
h'=f'+\kappa_X\circ\jmath_X\circ(\zeta+\eta)\circ\bigl(\mathrm{id}\otimes(f^\sharp\circ f^{-1}(\imath_Y))\bigr)\circ\bigl(f^{-1}(\mathrm{d})\bigr)\\
\text{and}\qquad h''=f''+(\zeta+\eta)\circ\underline{f}^*(\phi_Y).
\end{gathered}
$$

Hence the sum $\zeta+\eta:\underline{f}^*(\mathcal{F}_X)\rightarrow\mathcal{E}_X$ is a 2-morphism $\boldsymbol{f}\Rightarrow\boldsymbol{h}$. Define the *vertical composition of* 2-*morphisms* $\zeta\odot\eta:\boldsymbol{f}\Rightarrow\boldsymbol{h}$ as in (A.1) to be $\zeta\odot\eta=\zeta+\eta$.

Let $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ be d-spaces, $\boldsymbol{f},\tilde{\boldsymbol{f}}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ and $\boldsymbol{g},\tilde{\boldsymbol{g}}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ be 1-morphisms, and $\eta:\boldsymbol{f}\Rightarrow\tilde{\boldsymbol{f}}$, $\zeta:\boldsymbol{g}\Rightarrow\tilde{\boldsymbol{g}}$ be 2-morphisms. Then $\underline{f}=\underline{\tilde{f}}$, $\underline{g}=\underline{\tilde{g}}$, so $\underline{g}\circ\underline{f}=\underline{\tilde{g}}\circ\underline{\tilde{f}}$. Combining (2.24) and (2.25) and using $\underline{f}=\underline{\tilde{f}}$, $\underline{g}=\underline{\tilde{g}}$, we may prove that

$$
\begin{aligned}
&(\tilde{g}\circ\tilde{f})'=\tilde{f}'\circ f^{-1}(\tilde{g}')\circ I_{f,g}(\mathcal{O}_Z')\\
&\quad=\bigl[f'+\kappa_X\circ\jmath_X\circ\eta\circ\bigl(\mathrm{id}\otimes(f^\sharp\circ f^{-1}(\imath_Y))\bigr)\circ\bigl(f^{-1}(\mathrm{d})\bigr)\bigr]\circ\\
&\quad f^{-1}\bigl[g'+\kappa_Y\circ\jmath_Y\circ\zeta\circ\bigl(\mathrm{id}\otimes(g^\sharp\circ g^{-1}(\imath_Z))\bigr)\circ\bigl(g^{-1}(d)\bigr)\bigr]\circ I_{f,g}(\mathcal{O}_Z')\\
&\quad=(g\circ f)'+\kappa_X\circ\jmath_X\circ\theta\circ\bigl(\mathrm{id}\otimes((g\circ f)^\sharp\circ(g\circ f)^{-1}(\imath_Z))\bigr)\circ\bigl((g\circ f)^{-1}(\mathrm{d})\bigr),\\
&(\tilde{g}\circ\tilde{f})''=\tilde{f}''\circ\underline{f}^*(\tilde{g}'')\circ I_{\underline{f},\underline{g}}(\mathcal{E}_Z)\\
&\quad=\bigl[f''+\eta\circ\underline{f}^*(\phi_Y)\bigr]\circ\underline{f}^*\bigl(g''+\zeta\circ\underline{g}^*(\phi_Z)\bigr)\circ I_{\underline{f},\underline{g}}(\mathcal{E}_Z)\\
&\quad=(g\circ f)''+\theta\circ(\underline{g}\circ\underline{f})^*(\phi_Z),\\
&\text{where }\theta=\bigl(\eta\circ\underline{f}^*(g^2)+f''\circ\underline{f}^*(\zeta)+\eta\circ\underline{f}^*(\phi_Y)\circ\underline{f}^*(\zeta)\bigr)\circ I_{\underline{f},\underline{g}}(\mathcal{F}_Z).
\end{aligned}
$$

Here suppressing boring terms such as $\mathrm{id}\otimes(f^\sharp\circ f^{-1}(\imath_Y))$ and $I_{\underline{f},\underline{g}}(\mathcal{E}_Z)$, the essential points for the $(\tilde g\circ\tilde f)'$ equation are that $g^2\circ g^{-1}(\mathrm{d})=\mathrm{d}\circ g'$, $f'\circ f^{-1}(\kappa_Y\circ\jmath_Y)=\kappa_X\circ\jmath_X\circ f''$, and $\mathrm{d}\circ\kappa_Y\circ\jmath_Y=\phi_Y$, and for the $(\tilde g\circ\tilde f)''$ equation is that $\phi_Y\circ g''=g^2\circ\underline{g}^*(\phi_Z)$. Hence $\theta:\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\tilde{\boldsymbol{g}}\circ\tilde{\boldsymbol{f}}$ is a 2-morphism. Define the *horizontal composition of 2-morphisms* $\zeta*\eta:\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\tilde{\boldsymbol{g}}\circ\tilde{\boldsymbol{f}}$ as in (A.2) by

$$\zeta*\eta=\bigl(\eta\circ\underline{f}^*(g^2)+f''\circ\underline{f}^*(\zeta)+\eta\circ\underline{f}^*(\phi_Y)\circ\underline{f}^*(\zeta)\bigr)\circ I_{\underline{f},\underline{g}}(\mathcal{F}_Z). \qquad (2.27)$$

This completes the definition of the 2-category of d-spaces **dSpa**.

Regard the category $\mathbf{C^\infty Sch^{lf}_{ssc}}$ of separated, second countable, locally fair C^∞-schemes as a 2-category with only identity 2-morphisms $\mathrm{id}_{\underline{f}}$ for (1-)morphisms $\underline{f}:\underline{X}\to\underline{Y}$. Define a strict 2-functor $F_{\mathbf{dSpa}}^{\mathbf{C^\infty Sch}}:\mathbf{dSpa}\to\mathbf{C^\infty Sch^{lf}_{ssc}}$ to map $\boldsymbol{X}\mapsto\underline{X}$ on objects $\boldsymbol{X}$, $\boldsymbol{f}\mapsto\underline{f}$ on 1-morphisms $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$, and $\eta\mapsto\mathrm{id}_{\underline{f}}$ on 2-morphisms $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$. That is, $F_{\mathbf{dSpa}}^{\mathbf{C^\infty Sch}}$ forgets the information $\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X$ in $\boldsymbol{X}$, remembering only $\underline{X}$, and forgets the information f',f'' in $\boldsymbol{f}$, remembering only $\underline{f}$.

Define a strict 2-functor $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}:\mathbf{C^\infty Sch^{lf}_{ssc}}\to\mathbf{dSpa}$ to map $\underline{X}$ to $\boldsymbol{X}=(\underline{X},\mathcal{O}_X,0,\mathrm{id}_{\mathcal{O}_X},0)$ on objects $\underline{X}$, to map $\underline{f}$ to $\boldsymbol{f}=(\underline{f},f^\sharp,0)$ on (1-)morphisms $\underline{f}:\underline{X}\to\underline{Y}$, and to map identity 2-morphisms $\mathrm{id}_{\underline{f}}:\underline{f}\Rightarrow\underline{f}$ to identity 2-morphisms $\mathrm{id}_{\boldsymbol{f}}:\boldsymbol{f}\Rightarrow\boldsymbol{f}$. That is, on objects $\boldsymbol{X}$ we have $\mathcal{O}'_X=\mathcal{O}_X$, $\imath_X=\mathrm{id}_{\mathcal{O}_X}$ and $\mathcal{E}_X=\jmath_X=0$, and on 1-morphisms $f'=f^\sharp:f^{-1}(\mathcal{O}_Y)\to\mathcal{O}_X$ and $f''=0$.

Write $\mathbf{Man},\mathbf{Man^b},\mathbf{Man^c}$ for the categories of manifolds, and manifolds with boundary, and manifolds with corners, respectively. We will discuss $\mathbf{Man^b}$, $\mathbf{Man^c}$ in more detail in Chapter 5. Define strict 2-functors $F_{\mathbf{Man}}^{\mathbf{dSpa}}:\mathbf{Man}\to\mathbf{dSpa}$, $F_{\mathbf{Man^b}}^{\mathbf{dSpa}}:\mathbf{Man^b}\to\mathbf{dSpa}$, $F_{\mathbf{Man^c}}^{\mathbf{dSpa}}:\mathbf{Man^c}\to\mathbf{dSpa}$ by $F_{\mathbf{Man^*}}^{\mathbf{dSpa}}=F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}\circ F_{\mathbf{Man^*}}^{\mathbf{C^\infty Sch}}$. Write $\hat{\mathbf{C}}^\infty\mathbf{Sch^{lf}_{ssc}}$ for the full 2-subcategory of objects $\boldsymbol{X}$ in **dSpa** equivalent to $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{X})$ for some $\underline{X}$ in $\mathbf{C^\infty Sch^{lf}_{ssc}}$, and $\hat{\mathbf{M}}\mathbf{an}$ for the full 2-subcategory of objects $\boldsymbol{X}$ in **dSpa** equivalent to $F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$ for some manifold X.

When we say that a d-space $\boldsymbol{X}$ *is a C^∞-scheme*, or *is a manifold*, we mean that $\boldsymbol{X}\in\hat{\mathbf{C}}^\infty\mathbf{Sch^{lf}_{ssc}}$, or $\boldsymbol{X}\in\hat{\mathbf{M}}\mathbf{an}$, respectively.

Theorem 2.15. **(a)** *Definition* 2.14 *defines a (strict)* 2*-category* **dSpa**, *in which all* 2*-morphisms are* 2*-isomorphisms.*

(b) *For any* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ *in* **dSpa** *the* 2*-morphisms* $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{f}$ *form an abelian group under vertical composition, and in fact a real vector space.*

(c) $F_{\mathbf{dSpa}}^{\mathbf{C^\infty Sch}},F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}$ *and* $F_{\mathbf{Man^*}}^{\mathbf{dSpa}}$ *in Definition* 2.14 *are (strict)* 2*-functors.*

(d) $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}$ *and* $F_{\mathbf{Man}}^{\mathbf{dSpa}}$ *are full and faithful in the* 2*-categorical sense, that is, for all objects X,Y in the domain they induce equivalences (in fact isomorphisms) of categories* $F_*^*:\mathbf{Hom}(X,Y)\to\mathbf{Hom}\bigl(F_*^*(X),F_*^*(Y)\bigr)$. *Hence* $\mathbf{C^\infty Sch^{lf}_{ssc}},\mathbf{Man}$ *and* $\hat{\mathbf{C}}^\infty\mathbf{Sch^{lf}_{ssc}},\hat{\mathbf{M}}\mathbf{an}$ *are equivalent* 2*-categories.*

The 2*-functors* $F_{\mathbf{Man^b}}^{\mathbf{dSpa}},F_{\mathbf{Man^c}}^{\mathbf{dSpa}}$ *are faithful, but not full.*

Proof. For (a), let $\boldsymbol{f}:\boldsymbol{W}\to\boldsymbol{X}$, $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ be 1-morphisms in

dSpa. Using (2.23) we see that

$$\begin{aligned}
\boldsymbol{h}\circ(\boldsymbol{g}\circ\boldsymbol{f}) = \big(&\underline{h}\circ(\underline{g}\circ\underline{f}),[f'\circ f^{-1}(g')\circ I_{f,g}(\mathcal{O}'_Y)]\circ(g\circ f)^{-1}(h')\circ I_{g\circ f,h}(\mathcal{O}'_Z),\\
&[f''\circ\underline{f}^*(g'')\circ I_{\underline{f},\underline{g}}(\mathcal{E}_Y)]\circ(\underline{g}\circ\underline{f})^*(h'')\circ I_{\underline{g}\circ\underline{f},\underline{h}}(\mathcal{E}_Z)\big),\\
(\boldsymbol{h}\circ\boldsymbol{g})\circ\boldsymbol{f} = \big(&(\underline{h}\circ\underline{g})\circ\underline{f},f'\circ f^{-1}[g'\circ g^{-1}(h')\circ I_{g,h}(\mathcal{O}'_Z)]\circ I_{f,h\circ g}(\mathcal{O}'_Z),\\
&f''\circ\underline{f}^*[g''\circ\underline{g}^*(h'')\circ I_{\underline{g},\underline{h}}(\mathcal{E}_Z)]\circ I_{\underline{f},\underline{h}\circ\underline{g}}(\mathcal{E}_Z)\big).
\end{aligned}$$

As $\mathbf{C^\infty Sch}$ is a category $\underline{h}\circ(\underline{g}\circ\underline{f})=(\underline{h}\circ\underline{g})\circ\underline{f}$. Also $(h\circ(g\circ f))'=((h\circ g)\circ f)'$ and $(h\circ(g\circ f))''=((h\circ g)\circ f)''$ follow from the identities

$$\begin{aligned}
I_{f,g}(\mathcal{O}'_Y)\circ(g\circ f)^{-1}(h') &= f^{-1}(g^{-1}(h'))\circ I_{f,g}(h^{-1}(\mathcal{O}'_Z)),\\
I_{f,g}(h^{-1}(\mathcal{O}'_Z))\circ I_{g\circ f,h}(\mathcal{O}'_Z) &= f^{-1}(I_{g,h}(\mathcal{O}'_Z))\circ I_{f,h\circ g}(\mathcal{O}'_Z),\\
I_{\underline{f},\underline{g}}(\mathcal{E}_Y)\circ(\underline{g}\circ\underline{f})^*(h'') &= \underline{f}^*(\underline{g}^*(h''))\circ I_{\underline{f},\underline{g}}(\underline{h}^*(\mathcal{E}_Z)),\\
I_{\underline{f},\underline{g}}(\underline{h}^*(\mathcal{E}_Z))\circ I_{\underline{g}\circ\underline{f},\underline{h}}(\mathcal{E}_Z) &= \underline{f}^*(I_{\underline{g},\underline{h}}(\mathcal{E}_Z))\circ I_{\underline{f},\underline{h}\circ\underline{g}}(\mathcal{E}_Z),
\end{aligned}$$

by properties of the $I_{*,*}(*)$. Hence $\boldsymbol{h}\circ(\boldsymbol{g}\circ\boldsymbol{f})=(\boldsymbol{h}\circ\boldsymbol{g})\circ\boldsymbol{f}$, and composition of 1-morphisms in **dSpa** is (strictly) associative.

If $\boldsymbol{e},\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Y}$ are 1-morphisms and $\eta:\boldsymbol{e}\Rightarrow\boldsymbol{f}$, $\zeta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$, $\theta:\boldsymbol{g}\Rightarrow\boldsymbol{h}$ are 2-morphisms in **dSpa** then

$$\theta\odot(\zeta\odot\eta)=\theta+(\zeta+\eta)=(\theta+\zeta)+\eta=(\theta\odot\zeta)\odot\eta.$$

Thus vertical composition of 2-morphisms is associative.

Let $\boldsymbol{f},\tilde{\boldsymbol{f}}:\boldsymbol{W}\to\boldsymbol{X}$, $\boldsymbol{g},\tilde{\boldsymbol{g}}:\boldsymbol{X}\to\boldsymbol{Y}$, $\boldsymbol{h},\tilde{\boldsymbol{h}}:\boldsymbol{Y}\to\boldsymbol{Z}$ be 1-morphisms, and $\eta:\boldsymbol{f}\Rightarrow\tilde{\boldsymbol{f}}$, $\zeta:\boldsymbol{g}\Rightarrow\tilde{\boldsymbol{g}}$, $\theta:\boldsymbol{h}\Rightarrow\tilde{\boldsymbol{h}}$ be 2-morphisms. Since composition of 1-morphisms is associative we may write $\boldsymbol{h}\circ\boldsymbol{g}\circ\boldsymbol{f}$, $\tilde{\boldsymbol{h}}\circ\tilde{\boldsymbol{g}}\circ\tilde{\boldsymbol{f}}$ without ambiguity. Thus $\theta*(\zeta*\eta)$ and $(\theta*\zeta)*\eta$ are both 2-morphisms $\boldsymbol{h}\circ\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\tilde{\boldsymbol{h}}\circ\tilde{\boldsymbol{g}}\circ\tilde{\boldsymbol{f}}$. To prove $\theta*(\zeta*\eta)=(\theta*\zeta)*\eta$, consider the diagram:

$$\begin{tikzcd}
(\underline{h}\circ\underline{g}\circ\underline{f})^*(\mathcal{F}_Z) \arrow[r, "{\substack{(\underline{g}\circ\underline{f})^*(h^2)\circ\\ I_{\underline{g}\circ\underline{f},\underline{h}}(\mathcal{F}_Z)}}"] \arrow[dr, "{\substack{(\underline{g}\circ\underline{f})^*(\theta)\circ\\ I_{\underline{g}\circ\underline{f},\underline{h}}(\mathcal{F}_Z)}}"'] & (\underline{g}\circ\underline{f})^*(\mathcal{F}_Y) \arrow[r, "{\substack{\underline{f}^*(g^2)\circ\\ I_{\underline{f},\underline{g}}(\mathcal{F}_Y)}}"] \arrow[dr, "{\substack{\underline{f}^*(\zeta)\circ\\ I_{\underline{f},\underline{g}}(\mathcal{F}_Y)}}"] & \underline{f}^*(\mathcal{F}_X) \arrow[dr, "\eta"] & \\
& (\underline{g}\circ\underline{f})^*(\mathcal{E}_Y) \arrow[u, "(\underline{g}\circ\underline{f})^*(\phi_Y)"'] \arrow[r, "\underline{f}^*(g'')"] & \underline{f}^*(\mathcal{E}_X) \arrow[u, "\underline{f}^*(\phi_X)"'] \arrow[r, "f''"] & \mathcal{E}_W.
\end{tikzcd} \tag{2.28}$$

Here (2.28) need not be commutative, but the central rectangle commutes by (2.22) for $\boldsymbol{g}$. There are 8 possible routes from the top left to the bottom right corner in (2.28) by composing arrows, but the 2 involving two sides of the central rectangle give the same answer, so these yield 7 possibly different morphisms $(\underline{h}\circ\underline{g}\circ\underline{f})^*(\mathcal{F}_Z)\to\mathcal{E}_W$. Using (2.27) and properties of the $I_{*,*}(*)$ we find that both $\theta*(\zeta*\eta)$ and $(\theta*\zeta)*\eta$ are the sum of these 7 terms, so they are equal, and horizontal composition of 2-morphisms is associative.

Let $\boldsymbol{f},\tilde{\boldsymbol{f}},\hat{\boldsymbol{f}}:\boldsymbol{X}\to\boldsymbol{Y}$ and $\boldsymbol{g},\tilde{\boldsymbol{g}},\hat{\boldsymbol{g}}:\boldsymbol{Y}\to\boldsymbol{Z}$ be 1-morphisms and $\eta:\boldsymbol{f}\Rightarrow\tilde{\boldsymbol{f}}$, $\dot\eta:\tilde{\boldsymbol{f}}\Rightarrow\hat{\boldsymbol{f}}$, $\zeta:\boldsymbol{g}\Rightarrow\tilde{\boldsymbol{g}}$, $\dot\zeta:\tilde{\boldsymbol{g}}\Rightarrow\hat{\boldsymbol{g}}$ be 2-morphisms in **dSpa**. Then $(\dot\zeta\odot\zeta)*(\dot\eta\odot\eta)$

and $(\dot\zeta*\dot\eta)\odot(\zeta*\eta)$ are 2-morphisms $\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\hat{\boldsymbol{g}}\circ\hat{\boldsymbol{f}}$. We find that

$$
\begin{aligned}
(\dot\zeta*\dot\eta)\odot(\zeta*\eta) &= \bigl(\eta\circ\underline{f}^*(g^2)+f''\circ\underline{f}^*(\zeta)+\eta\circ\underline{f}^*(\phi_Y)\circ\underline{f}^*(\zeta)\bigr)\circ I_{\underline{f},\underline{g}}(\mathcal{F}_Z)\\
&\quad+\bigl(\dot\eta\circ\underline{f}^*(\tilde g^2)+\tilde f''\circ\underline{f}^*(\dot\zeta)+\dot\eta\circ\underline{f}^*(\phi_Y)\circ\underline{f}^*(\dot\zeta)\bigr)\circ I_{\underline{f},\underline{g}}(\mathcal{F}_Z)\\
&=\bigl(\eta\circ\underline{f}^*(g^2)+f''\circ\underline{f}^*(\zeta)+\eta\circ\underline{f}^*(\phi_Y)\circ\underline{f}^*(\zeta)\bigr)\circ I_{\underline{f},\underline{g}}(\mathcal{F}_Z)\\
&\quad+\bigl(\dot\eta\circ\underline{f}^*(g^2+\phi_Y\circ\zeta)+(f''+\eta\circ\underline{f}^*(\phi_Y))\circ\underline{f}^*(\dot\zeta)+\dot\eta\circ\underline{f}^*(\phi_Y)\circ\underline{f}^*(\dot\zeta)\bigr)\circ I_{\underline{f},\underline{g}}(\mathcal{F}_Z)\\
&=\bigl((\dot\eta+\eta)\circ\underline{f}^*(g^2)+f''\circ\underline{f}^*(\dot\zeta+\zeta)+(\dot\eta+\eta)\circ\underline{f}^*(\phi_Y)\circ\underline{f}^*(\dot\zeta+\zeta)\bigr)\circ I_{\underline{f},\underline{g}}(\mathcal{F}_Z)\\
&=(\dot\zeta\odot\zeta)*(\dot\eta\odot\eta),
\end{aligned}
$$

using $\zeta\odot\eta=\zeta+\eta$, equation (2.27), $\underline{\tilde f}=\underline{f}$ and $\underline{\tilde g}=\underline{g}$ as $\boldsymbol{f},\boldsymbol{g}$ and $\tilde{\boldsymbol{f}},\tilde{\boldsymbol{g}}$ are 2-isomorphic, and (2.25)–(2.26) to substitute for $\tilde f'',\tilde g^2$ in the second step. Thus horizontal and vertical composition of 2-morphisms are compatible. All the axioms concerning identity 1-morphisms and identity 2-morphisms are basically trivial. Hence **dSpa** is a strict 2-category. If $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is a 2-morphism then from (2.25) we see that $-\eta:\boldsymbol{g}\Rightarrow\boldsymbol{f}$ is a 2-morphism with $(-\eta)\odot\eta=-\eta+\eta=0=\mathrm{id}_{\boldsymbol{f}}$ and $\eta\odot(-\eta)=\eta-\eta=0=\mathrm{id}_{\boldsymbol{g}}$, so $-\eta$ is the inverse of η, and η is a 2-isomorphism. This proves (a).

For (b), from (2.25), since κ_X is injective and the image of $\bigl(\mathrm{id}\otimes(f^\sharp\circ f^{-1}(\imath_Y))\bigr)\circ\bigl(f^{-1}(\mathrm{d})\bigr)$ generates $\underline{f}^*(\mathcal{F}_Y)$ as a sheaf of abelian groups, we see that the group of 2-isomorphisms $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{f}$ is the real vector space of $\eta:\underline{f}^*(\mathcal{F}_Y)\to\mathcal{E}_X$ with $\jmath_X\circ\eta=\eta\circ\underline{f}^*(\phi_Y)=0$, and the group operation is addition. Part (c) is more-or-less immediate from the definitions.

For (d), let $\underline{X},\underline{Y}$ lie in $\mathbf{C^\infty Sch^{lf}_{ssc}}$ and $\boldsymbol{X},\boldsymbol{Y}=F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{X},\underline{Y})$. Then $\mathcal{O}'_X=\mathcal{O}_X$, $\imath_X=\mathrm{id}_{\mathcal{O}_X}$, $\mathcal{E}_X=0$ and $\mathcal{O}'_Y=\mathcal{O}_Y$, $\imath_Y=\mathrm{id}_{\mathcal{O}_Y}$, $\mathcal{E}_Y=0$. Suppose $\boldsymbol{f}=(\underline{f},f',f''):\boldsymbol{X}\to\boldsymbol{Y}$ is a 1-morphism in **dSpa**. Then $\underline{f}:\underline{X}\to\underline{Y}$ is a morphism of C^∞-schemes and $f^\sharp\circ f^{-1}(\mathrm{id}_{\mathcal{O}_Y})=\mathrm{id}_{\mathcal{O}_X}\circ f'$, so $f'=f^\sharp$. Also f'' is a morphism $\underline{f}^*(0)\to 0$, so $f''=0$. Hence $\boldsymbol{f}=(\underline{f},f^\sharp,0)=F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{f})$. For 1-morphisms $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$, a 2-morphism $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is a morphism $\eta:\underline{f}^*(\mathcal{F}_Y)\to\mathcal{E}_X=0$, and so is zero, forcing $\boldsymbol{f}=\boldsymbol{g}$ and $\eta=\mathrm{id}_{\boldsymbol{f}}=F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\mathrm{id}_{\underline{f}})$. Therefore $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}:\mathbf{Hom}(\underline{X},\underline{Y})\to\mathbf{Hom}(\boldsymbol{X},\boldsymbol{Y})$ is an isomorphism of categories, and $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}$ is full and faithful. Corollary B.27 now implies that $F_{\mathbf{Man}}^{\mathbf{dSpa}}=F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}\circ F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ is full and faithful, and that $F_{\mathbf{Man^b}}^{\mathbf{dSpa}},F_{\mathbf{Man^c}}^{\mathbf{dSpa}}$ are faithful, but not full. $\square$

Remark 2.16. (a) We think of a d-space $\boldsymbol{X}=(\underline{X},\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X)$ as being a separated, second countable, locally fair C^∞-scheme $\underline{X}$, which is the 'classical' part of $\boldsymbol{X}$ and lives in a category rather than a 2-category, together with some extra 'derived' information $\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X$. 2-morphisms in **dSpa** are wholly to do with this 'derived' part. The 2-functor $F_{\mathbf{dSpa}}^{\mathbf{C^\infty Sch}}$ forgets the 'derived' information $\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X$.

The equations $g^4=f^4$, $g^5=f^5$ in (2.26) show that the kernels $\mathcal{C}_X,\mathcal{D}_X$ and the morphism $\nu_X:\mathcal{C}_X\to\mathcal{D}_X$ are unaffected by 2-morphisms, so they can also be regarded as 'classical' data.

(b) The 2-functor $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}$ embeds $\mathbf{C^\infty Sch^{lf}_{ssc}}$ as a 2-subcategory of **dSpa**, so we can think of such C^∞-schemes $\underline{X}$ as examples of d-spaces. When we do this we make the 'derived' information $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$ in $\boldsymbol{X} = F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{X})$ as simple as possible, by taking $\mathcal{O}'_X = \mathcal{O}_X$, $\imath_X = \mathrm{id}_{\mathcal{O}_X}$ and $\mathcal{E}_X = \jmath_X = 0$.

Proposition 2.17. *Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism of d-spaces, and $\eta : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{E}_X$ is a morphism in* $\mathrm{qcoh}(\underline{X})$. *Then there exists a unique 1-morphism $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ in* **dSpa** *such that $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism.*

Proof. Applying the final part of Proposition 2.13 to $(\underline{f}, f') : (\underline{X}, \mathcal{O}'_X, \imath_X) \to (\underline{Y}, \mathcal{O}'_Y, \imath_U)$ with morphism $\mu = \jmath_X \circ \eta : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{E}_X$ gives a unique morphism $g' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$ such that $(\underline{g}, g')$ with $\underline{g} = \underline{f}$ is a morphism $(\underline{X}, \mathcal{O}'_X, \imath_X) \to (\underline{Y}, \mathcal{O}'_Y, \imath_U)$, and (2.16)–(2.17) hold with $\mu = \jmath_X \circ \eta$. Thus (2.16) gives the first equation of (2.25). Define $g'' = f'' + \eta \circ \underline{f}^*(\phi_Y)$, as in the second equation of (2.25). Then

$$\jmath_X \circ g'' = \jmath_X \circ f'' + \jmath_X \circ \eta \circ \underline{f}^*(\phi_Y) = f^1 \circ \underline{f}^*(\jmath_Y) + \jmath_X \circ \eta \circ \underline{f}^*(\xi_Y) \circ \underline{f}^*(\jmath_Y) = g^1 \circ \underline{f}^*(\jmath_Y),$$

using (2.21) for $\boldsymbol{f}$, $\phi_Y = \xi_Y \circ \jmath_Y$, and (2.17) with $\mu = \jmath_X \circ \eta$. Hence (2.21) holds for $\boldsymbol{g}$, and $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism. Also $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism as (2.25) holds. Uniqueness of $\boldsymbol{g}$ is clear from uniqueness of g' above. □

2.3 Equivalences in dSpa

Recall as in §A.3 that if $\boldsymbol{\mathcal{C}}$ is a 2-category, an *equivalence* between objects X, Y in $\boldsymbol{\mathcal{C}}$ is a 1-morphism $f : X \to Y$ such that there exists a 1-morphism $g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$. Equivalence is usually the best notion of when objects in $\boldsymbol{\mathcal{C}}$ are 'the same'. In this section we will study equivalences in the 2-category **dSpa**.

The question we are interested in is this: if $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ and $\boldsymbol{Y} = (\underline{Y}, \mathcal{O}'_Y, \mathcal{E}_Y, \imath_Y, \jmath_Y)$ are equivalent in **dSpa**, what is the relationship between the 'derived' data $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$ on $\underline{X}$ and $\mathcal{O}'_Y, \mathcal{E}_Y, \imath_Y, \jmath_Y$ on $\underline{Y}$? We first show that equivalence can add an arbitrary quasicoherent sheaf $\mathcal{G}$ to $\mathcal{O}'_X$ and $\mathcal{E}_X$.

Example 2.18. Let $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ be a d-space and $\mathcal{G} \in \mathrm{qcoh}(\underline{X})$. From $\underline{X}, \mathcal{O}'_X, \imath_X, \mathcal{G}$, Example 2.11 constructs a square zero extension $(\mathcal{O}'_{\hat{X}}, \imath_{\hat{X}})$ of $\underline{X}$, with $\mathcal{O}'_{\hat{X}} = \mathcal{O}'_X \ltimes \mathcal{G}$, and with sheaves $\mathcal{I}_{\hat{X}} \cong \mathcal{I}_X \oplus \mathcal{G}$ and $\mathcal{F}_{\hat{X}} \cong \mathcal{F}_X \oplus \mathcal{G}$. The morphisms $\xi_{\hat{X}} : \mathcal{I}_{\hat{X}} \to \mathcal{F}_{\hat{X}}$, $\psi_{\hat{X}} : \mathcal{F}_{\hat{X}} \to T^*\underline{X}$ are given by

$$\xi_{\hat{X}} = \begin{pmatrix} \xi_X & 0 \\ 0 & \mathrm{id}_{\mathcal{G}} \end{pmatrix} : \mathcal{I}_{\hat{X}} \cong \mathcal{I}_X \oplus \mathcal{G} \longrightarrow \mathcal{F}_X \oplus \mathcal{G} \cong \mathcal{F}_{\hat{X}},$$
$$\psi_{\hat{X}} = \begin{pmatrix} \psi_X & 0 \end{pmatrix} : \mathcal{F}_{\hat{X}} \cong \mathcal{F}_X \oplus \mathcal{G} \longrightarrow T^*\underline{X}.$$

Define $\mathcal{E}_{\hat{X}} = \mathcal{E}_X \oplus \mathcal{G}$ in $\mathrm{qcoh}(\underline{X})$, and define $\jmath_{\hat{X}} : \mathcal{E}_{\hat{X}} \to \mathcal{I}_{\hat{X}}$ by

$$\jmath_{\hat{X}} = \begin{pmatrix} \jmath_X & 0 \\ 0 & \mathrm{id}_{\mathcal{G}} \end{pmatrix} : \mathcal{E}_{\hat{X}} = \mathcal{E}_X \oplus \mathcal{G} \longrightarrow \mathcal{I}_X \oplus \mathcal{G} \cong \mathcal{I}_{\hat{X}}. \tag{2.29}$$

Then $\jmath_{\hat{X}}$ is surjective, as $\jmath_X$ is. Therefore $\hat{\boldsymbol{X}} = (\underline{X}, \mathcal{O}'_{\hat{X}}, \mathcal{E}_{\hat{X}}, \imath_{\hat{X}}, \jmath_{\hat{X}})$ is a d-space. We also have $\mathcal{C}_{\hat{X}} \cong \mathcal{C}_X$ and $\mathcal{D}_{\hat{X}} \cong \mathcal{D}_X$.

Define $\underline{f} = \underline{g} = \underline{\mathrm{id}}_{\underline{X}} : \underline{X} \to \underline{X}$. There are natural morphisms of sheaves of C^∞-rings $f' : \underline{\mathrm{id}}_X^{-1}(\mathcal{O}'_{\hat{X}}) \cong \mathcal{O}'_X \ltimes \mathcal{G} \to \mathcal{O}'_X$ and $g' : \underline{\mathrm{id}}_X^{-1}(\mathcal{O}'_X) \cong \mathcal{O}'_X \to \mathcal{O}'_X \ltimes \mathcal{G} = \mathcal{O}'_{\hat{X}}$; in the C^∞-ring picture of Example 2.7, f' corresponds to the C^∞-ring morphism $\mathfrak{C}' \ltimes G \to \mathfrak{C}'$ mapping $(c', \gamma) \mapsto c'$, and g' to the morphism $\mathfrak{C}' \to \mathfrak{C}' \ltimes G$ mapping $c' \mapsto (c', 0)$. These satisfy $f^\sharp \circ f^{-1}(\imath_{\hat{X}}) = \imath_X \circ f'$ and $g^\sharp \circ g^{-1}(\imath_X) = \imath_{\hat{X}} \circ g'$, so $(\underline{f}, f')$ is a morphism of square zero extensions $(\underline{X}, \mathcal{O}'_X, \imath_X) \to (\underline{X}, \mathcal{O}'_{\hat{X}}, \imath_{\hat{X}})$, and $(\underline{g}, g')$ a morphism $(\underline{X}, \mathcal{O}'_{\hat{X}}, \imath_{\hat{X}}) \to (\underline{X}, \mathcal{O}'_X, \imath_X)$. Therefore Definition 2.12 defines morphisms f^1, f^2, f^3 and g^1, g^2, g^3, given in

$$\begin{tikzcd}[column sep=large]
{\begin{array}{c}\underline{f}^*(\mathcal{I}_{\hat{X}}) \cong \\ \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{I}_X) \oplus \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{G})\end{array}} \arrow[r, "{\underline{f}^*(\xi_{\hat{X}}) = \begin{pmatrix} \underline{\mathrm{id}}_{\underline{X}}^*(\xi_X) & 0 \\ 0 & \underline{\mathrm{id}}_{\underline{X}}^*(\mathrm{id}_{\mathcal{G}}) \end{pmatrix}}"] \arrow[d, "{f^1 = \begin{pmatrix} \delta_{\underline{X}}(\mathcal{I}_X) \\ 0 \end{pmatrix}}"] & {\begin{array}{c}\underline{f}^*(\mathcal{F}_{\hat{X}}) \cong \\ \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{F}_X) \oplus \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{G})\end{array}} \arrow[r, "{\underline{f}^*(\psi_{\hat{X}}) = \begin{pmatrix} \underline{\mathrm{id}}_{\underline{X}}^*(\psi_X) & 0 \end{pmatrix}}"] \arrow[d, "{f^2 = \begin{pmatrix} \delta_{\underline{X}}(\mathcal{F}_X) \\ 0 \end{pmatrix}}"] & {\begin{array}{c}\underline{f}^*(T^*\underline{X}) = \\ \underline{\mathrm{id}}_{\underline{X}}^*(T^*\underline{X})\end{array}} \arrow[r] \arrow[d, "{f^3 = \delta_{\underline{X}}(T^*\underline{X})}"] & 0 \\
\mathcal{I}_X \arrow[r, "\xi_X"] & \mathcal{F}_X \arrow[r, "\psi_X"] & T^*\underline{X} \arrow[r] & 0,
\end{tikzcd} \tag{2.30}$$

$$\begin{tikzcd}[column sep=large]
{\begin{array}{c}\underline{g}^*(\mathcal{I}_X) = \\ \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{I}_X)\end{array}} \arrow[r, "\underline{\mathrm{id}}_{\underline{X}}^*(\xi_X)"] \arrow[d, "{g^1 = \begin{pmatrix} \delta_{\underline{X}}(\mathcal{I}_X) & 0 \end{pmatrix}}"] & {\begin{array}{c}\underline{g}^*(\mathcal{F}_X) = \\ \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{F}_X)\end{array}} \arrow[r, "\underline{\mathrm{id}}_{\underline{X}}^*(\psi_X)"] \arrow[d, "{g^2 = \begin{pmatrix} \delta_{\underline{X}}(\mathcal{F}_X) & 0 \end{pmatrix}}"] & {\begin{array}{c}\underline{g}^*(T^*\underline{X}) = \\ \underline{\mathrm{id}}_{\underline{X}}^*(T^*\underline{X})\end{array}} \arrow[r] \arrow[d, "{g^3 = \delta_{\underline{X}}(T^*\underline{X})}"] & 0 \\
{\begin{array}{c}\mathcal{I}_X \oplus \mathcal{G} \\ \cong \mathcal{I}_{\hat{X}}\end{array}} \arrow[r, swap, "{\xi_{\hat{X}} = \begin{pmatrix} \xi_X & 0 \\ 0 & \mathrm{id}_{\mathcal{G}} \end{pmatrix}}"] & {\begin{array}{c}\mathcal{F}_X \oplus \mathcal{G} \\ \cong \mathcal{F}_{\hat{X}}\end{array}} \arrow[r, swap, "{\psi_{\hat{X}} = \begin{pmatrix} \psi_X & 0 \end{pmatrix}}"] & T^*\underline{X} \arrow[r] & 0.
\end{tikzcd} \tag{2.31}$$

Define $f'' : \underline{f}^*(\mathcal{E}_{\hat{X}}) \to \mathcal{E}_X$ and $g'' : \underline{g}^*(\mathcal{E}_X) \to \mathcal{E}_{\hat{X}}$ by

$$\begin{aligned}
f'' &= \begin{pmatrix} \delta_{\underline{X}}(\mathcal{E}_X) & 0 \end{pmatrix} : \underline{f}^*(\mathcal{E}_{\hat{X}}) = \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E}_X) \oplus \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{G}) \longrightarrow \mathcal{E}_X, \\
g'' &= \begin{pmatrix} \delta_{\underline{X}}(\mathcal{E}_X) \\ 0 \end{pmatrix} : \underline{g}^*(\mathcal{E}_X) = \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E}_X) \longrightarrow \mathcal{E}_X \oplus \mathcal{G} = \mathcal{E}_{\hat{X}}.
\end{aligned} \tag{2.32}$$

Then (2.29)–(2.32) imply that $\jmath_X \circ f'' = f^1 \circ \underline{f}^*(\jmath_{\hat{X}})$ and $\jmath_{\hat{X}} \circ g'' = g^1 \circ \underline{g}^*(\jmath_X)$, as in (2.21). Hence $\boldsymbol{f} = (\underline{f}, f', f'')$ is a 1-morphism $\boldsymbol{X} \to \hat{\boldsymbol{X}}$, and $\boldsymbol{g} = (\underline{g}, g', g'')$ is a 1-morphism $\hat{\boldsymbol{X}} \to \boldsymbol{X}$. Definition 2.14 defines morphisms f^4, f^5, g^4, g^5, which all turn out to be isomorphisms.

One can now show that $\boldsymbol{g} \circ \boldsymbol{f} = \mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$, so $\eta = 0 = \mathrm{id}_{\mathbf{id}_{\boldsymbol{X}}} : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \mathbf{id}_{\boldsymbol{X}}$ is a 2-morphism in **dSpa**. If $\mathcal{G} \neq 0$ it is not true that $\boldsymbol{f} \circ \boldsymbol{g} = \mathbf{id}_{\hat{\boldsymbol{X}}}$, so $\boldsymbol{f}, \boldsymbol{g}$ are not 1-inverse. However, it is easy to show that

$$\zeta = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\underline{X}}(\mathcal{G}) \end{pmatrix} : (\underline{g} \circ \underline{f})^*(\mathcal{F}_{\hat{X}}) \cong \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{F}_X) \oplus \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{G}) \longrightarrow \mathcal{E}_X \oplus \mathcal{G} = \mathcal{E}_{\hat{X}}$$

is a 2-morphism $\zeta : \boldsymbol{f} \circ \boldsymbol{g} \Rightarrow \mathbf{id}_{\hat{\boldsymbol{X}}}$ in **dSpa**. Therefore $\boldsymbol{f} : \boldsymbol{X} \to \hat{\boldsymbol{X}}$ and $\boldsymbol{g} : \hat{\boldsymbol{X}} \to \boldsymbol{X}$ are both *equivalences* in **dSpa**.

Thus if $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ and $\boldsymbol{Y} = (\underline{Y}, \mathcal{O}'_Y, \mathcal{E}_Y, \imath_Y, \jmath_Y)$ are equivalent, the data $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$ and $\mathcal{O}'_Y, \mathcal{E}_Y, \imath_Y, \jmath_Y$ need not be isomorphic.

Definition 2.19. Let $0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$ be a complex in $\mathrm{qcoh}(\underline{X})$ for some C^∞-scheme $\underline{X}$, so that $\beta \circ \alpha = 0$. We call this a *split short exact sequence*, or just *split exact*, if there exist $\gamma : \mathcal{F} \to \mathcal{E}$, $\delta : \mathcal{G} \to \mathcal{F}$ in $\mathrm{qcoh}(\underline{X})$ with

$$\gamma \circ \delta = 0, \;\; \gamma \circ \alpha = \mathrm{id}_{\mathcal{E}}, \;\; \alpha \circ \gamma + \delta \circ \beta = \mathrm{id}_{\mathcal{F}}, \;\; \beta \circ \delta = \mathrm{id}_{\mathcal{G}}. \tag{2.33}$$

Equation (2.33) implies that $0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$ is exact.

Equivalently, $0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$ is split exact if there exists an isomorphism $\mathcal{F} \cong \mathcal{E} \oplus \mathcal{G}$ identifying it with $0 \to \mathcal{E} \xrightarrow{\mathrm{id}_{\mathcal{E}} \oplus 0} \mathcal{E} \oplus \mathcal{G} \xrightarrow{0 \oplus \mathrm{id}_{\mathcal{G}}} \mathcal{G} \to 0$, so that

$$\alpha \cong \begin{pmatrix} \mathrm{id}_{\mathcal{E}} \\ 0 \end{pmatrix}, \;\; \beta \cong \begin{pmatrix} 0 & \mathrm{id}_{\mathcal{G}} \end{pmatrix}, \;\; \gamma \cong \begin{pmatrix} \mathrm{id}_{\mathcal{E}} & 0 \end{pmatrix}, \;\; \delta \cong \begin{pmatrix} 0 \\ \mathrm{id}_{\mathcal{G}} \end{pmatrix}.$$

The next two propositions characterize equivalences in **dSpa**. For an arbitrary 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, equation (2.34) is a complex as (2.22) commutes, but it may not be exact. So our condition is that (2.34) is exact, and also split.

Proposition 2.20. *Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is an equivalence in* **dSpa**. *Then $\underline{f} : \underline{X} \to \underline{Y}$ and $f^4 : \underline{f}^*(\mathcal{C}_Y) \to \mathcal{C}_X$ and $f^5 : \underline{f}^*(\mathcal{D}_Y) \to \mathcal{D}_X$ are all isomorphisms, and the following is a split short exact sequence in* $\mathrm{qcoh}(\underline{X})$:

$$0 \longrightarrow \underline{f}^*(\mathcal{E}_Y) \xrightarrow{f'' \oplus -\underline{f}^*(\phi_Y)} \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y) \xrightarrow{\phi_X \oplus f^2} \mathcal{F}_X \longrightarrow 0. \tag{2.34}$$

Proof. As $\boldsymbol{f}$ is an equivalence, there exist $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\eta : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \mathbf{id}_{\boldsymbol{X}}$ and $\zeta : \boldsymbol{f} \circ \boldsymbol{g} \Rightarrow \mathbf{id}_{\boldsymbol{Y}}$. By Proposition A.6 we may also choose η, ζ to satisfy $\mathrm{id}_{\boldsymbol{f}} * \eta = \zeta * \mathrm{id}_{\boldsymbol{f}}$ and $\mathrm{id}_{\boldsymbol{g}} * \zeta = \eta * \mathrm{id}_{\boldsymbol{g}}$. These 2-morphisms imply that $\underline{g} \circ \underline{f} = \underline{\mathrm{id}}_{\underline{X}}$ and $\underline{f} \circ \underline{g} = \underline{\mathrm{id}}_{\underline{Y}}$, so $\underline{g} = \underline{f}^{-1}$, and $\underline{f}$ is an isomorphism. Using $\eta : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \mathbf{id}_{\boldsymbol{X}}$ and $\zeta : \boldsymbol{f} \circ \boldsymbol{g} \Rightarrow \mathbf{id}_{\boldsymbol{Y}}$ and equations (2.24) and (2.26) shows that

$$\begin{aligned} f^4 \circ \underline{f}^*(g^4) \circ I_{\underline{f},\underline{g}}(\mathcal{C}_X) &= (g \circ f)^4 = (\mathrm{id}_X)^4 = \delta_{\underline{X}}(\mathcal{C}_X), \\ g^4 \circ \underline{g}^*(f^4) \circ I_{\underline{g},\underline{f}}(\mathcal{C}_Y) &= (f \circ g)^4 = (\mathrm{id}_Y)^4 = \delta_{\underline{Y}}(\mathcal{C}_Y). \end{aligned}$$

The first equation implies that $\underline{f}^*(g^4) \circ I_{\underline{f},\underline{g}}(\mathcal{C}_Z) \circ \delta_{\underline{X}}(\mathcal{C}_X)^{-1}$ is a right inverse for f^4. Applying $\underline{f}^*$ to the second equation and using $\underline{g} \circ \underline{f} = \underline{\mathrm{id}}_{\underline{X}}$ and properties of $I_{*,*}(*)$ and $\delta_*(*)$ also shows that $\underline{f}^*(g^4) \circ I_{\underline{f},\underline{g}}(\mathcal{C}_Z) \circ \delta_{\underline{X}}(\mathcal{C}_X)^{-1}$ is a left inverse for f^4. Hence f^4 is an isomorphism, and similarly f^5 is an isomorphism.

Define $\gamma = \gamma_1 \oplus \gamma_2 : \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y) \to \underline{f}^*(\mathcal{E}_Y)$ and $\delta = \delta_1 \oplus \delta_2 : \mathcal{F}_X \to \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y)$ by

$$\begin{gathered} \gamma_1 = \underline{f}^*(g'') \circ I_{\underline{f},\underline{g}}(\mathcal{E}_X) \circ \delta_{\underline{X}}(\mathcal{E}_X)^{-1}, \quad \gamma_2 = -\underline{f}^*(\zeta \circ \delta_{\underline{Y}}(\mathcal{F}_Y)^{-1}), \\ \delta_1 = \eta \circ \delta_{\underline{X}}(\mathcal{F}_X)^{-1}, \quad \delta_2 = \underline{f}^*(g^2) \circ I_{\underline{f},\underline{g}}(\mathcal{F}_X) \circ \delta_{\underline{X}}(\mathcal{F}_X)^{-1}. \end{gathered} \tag{2.35}$$

Then using all four conditions on η,ζ we may show that γ,δ satisfy (2.33) for the complex (2.34). So (2.34) is split exact by Definition 2.19. $\square$

Here is a converse to Proposition 2.20.

Proposition 2.21. *Let $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ be a 1-morphism in* **dSpa**. *Suppose $\underline{f}:\underline{X}\rightarrow\underline{Y}$ and $f^4:\underline{f}^*(\mathcal{C}_Y)\rightarrow\mathcal{C}_X$ are isomorphisms, and* (2.34) *is a split short exact sequence. Then $\boldsymbol{f}$ is an equivalence.*

Proof. As $\underline{f}$ is an isomorphism, we may define $\underline{g}=\underline{f}^{-1}:\underline{Y}\rightarrow\underline{X}$. Since (2.34) is split exact, there exist $\gamma=\gamma_1\oplus\gamma_2$ and $\delta=\delta_1\oplus\delta_2$ satisfying (2.33) for α,β the morphisms in (2.34). Define g'',g^2,η,ζ uniquely such that (2.35) holds. We claim there exists a unique morphism $g':g^{-1}(\mathcal{O}'_X)\rightarrow\mathcal{O}_Y$ of sheaves of C^∞-rings on Y such that $\boldsymbol{g}=(\underline{g},g',g'')$ is a 1-morphism $\boldsymbol{Y}\rightarrow\boldsymbol{X}$, realizing the value of g^2 determined by δ_2, and such that $\eta:\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$ and $\zeta:\boldsymbol{f}\circ\boldsymbol{g}\Rightarrow\mathbf{id}_{\boldsymbol{Y}}$ are 2-morphisms. This then proves $\boldsymbol{f}$ is an equivalence, with quasi-inverse $\boldsymbol{g}$.

It is enough to construct g' on each $\underline{V}_a$ for an open cover of C^∞-subschemes $\{\underline{V}_a:a\in A\}$ of $\underline{Y}$, since by uniqueness the constructed values of g'_a on $\underline{V}_a$ agree on overlaps $\underline{V}_a\cap\underline{V}_b$, and so glue to give a global morphism g'. Fix $y\in Y$, let V be an open neighbourhood of y in Y, and set $U=f^{-1}(V)\subseteq X$. Let $\underline{U},\underline{V}$ be the corresponding open C^∞-subschemes of $\underline{X},\underline{Y}$. As $\underline{X},\underline{Y}$ are locally fair C^∞-schemes, $(X,\mathcal{O}'_X)$, $(Y,\mathcal{O}'_Y)$ are C^∞-schemes, and f is a homeomorphism, making V smaller if necessary, we can suppose that $\underline{U},\underline{V}$ are fair affine C^∞-schemes and $(U,\mathcal{O}'_X|_U),(V,\mathcal{O}'_Y|_Y)$ are affine C^∞-schemes. Also $\underline{U}\cong\underline{V}$ as $\underline{f}$ is an isomorphism.

Hence there exist C^∞-rings $\mathfrak{C},\mathfrak{C}',\mathfrak{D}'$ with $\mathfrak{C}$ fair, and isomorphisms $\underline{U}\cong\operatorname{Spec}\mathfrak{C}\cong\underline{V}$, $(U,\mathcal{O}'_X|_U)\cong\operatorname{Spec}\mathfrak{C}'$, $(V,\mathcal{O}'_Y|_Y)\cong\operatorname{Spec}\mathfrak{D}'$. The morphisms $\imath_X|_U:\mathcal{O}'_X|_U\rightarrow\mathcal{O}_X|_U$, $\imath_Y|_V:\mathcal{O}'_Y|_V\rightarrow\mathcal{O}_Y|_V$, $f'|_U:f^{-1}(\mathcal{O}'_Y)|_U\rightarrow\mathcal{O}'_X|_U$ correspond to morphisms of C^∞-rings $\phi:\mathfrak{C}'\rightarrow\mathfrak{C}$, $\psi:\mathfrak{D}'\rightarrow\mathfrak{C}$, $\xi:\mathfrak{D}'\rightarrow\mathfrak{C}'$ respectively, where $f^\sharp\circ f^{-1}(\imath_Y)=\imath_X\circ f'$ implies that $\phi\circ\xi=\psi$. Here ϕ,ψ are square zero extensions of C^∞-rings, with kernels $I\subset\mathfrak{C}'$ and $J\subset\mathfrak{D}'$, both $\mathfrak{C}$-modules, and inclusions $\kappa:I\rightarrow\mathfrak{C}'$ and $\lambda:J\rightarrow\mathfrak{D}'$.

The isomorphisms $\underline{U}\cong\operatorname{Spec}\mathfrak{C}\cong\underline{V}$ mean that quasicoherent sheaves on $\underline{U},\underline{V}$ are identified under MSpec with $\mathfrak{C}$-modules, and morphisms of quasicoherent sheaves on $\underline{U},\underline{V}$ are identified under MSpec with morphisms of $\mathfrak{C}$-modules. Let the sheaves $\mathcal{E}_X|_{\underline{U}},\mathcal{I}_X|_{\underline{U}},\mathcal{F}_X|_{\underline{U}}$ on $\underline{U}$ and $\mathcal{E}_Y|_{\underline{V}},\mathcal{I}_Y|_{\underline{V}},\mathcal{F}_Y|_{\underline{V}}$ on $\underline{V}$ be identified with $\mathfrak{C}$-modules $E,I,\Omega_{\mathfrak{C}'}\otimes_{\mathfrak{C}'}\mathfrak{C},F,J,\Omega_{\mathfrak{D}'}\otimes_{\mathfrak{D}'}\mathfrak{C}$ respectively.

Let the morphisms $\jmath_X|_{\underline{U}}$, $f''|_{\underline{U}}$, $f^1|_{\underline{U}}$, $f^2|_{\underline{U}}$, $f^3|_{\underline{U}}$, $\eta|_{\underline{U}}$ on $\underline{U}$ and $\jmath_Y|_{\underline{V}}$, $g''|_{\underline{V}}$, $g^2|_{\underline{V}}$, $\zeta|_{\underline{V}}$ on $\underline{V}$ be identified with $\alpha:E\rightarrow I$, $\mu:F\rightarrow E$, $\rho:J\rightarrow I$, $\Omega_\xi\otimes\mathrm{id}_{\mathfrak{C}}:\Omega_{\mathfrak{D}'}\otimes_{\mathfrak{D}'}\mathfrak{C}\rightarrow\Omega_{\mathfrak{C}'}\otimes_{\mathfrak{C}'}\mathfrak{C}$, $\mathrm{id}_{\Omega_{\mathfrak{C}}}:\Omega_{\mathfrak{C}}\rightarrow\Omega_{\mathfrak{C}}$, $\tau:\Omega_{\mathfrak{C}'}\otimes_{\mathfrak{C}'}\mathfrak{C}\rightarrow E$ and $\beta:F\rightarrow J$, $\nu:E\rightarrow F$, $\omega:\Omega_{\mathfrak{C}'}\otimes_{\mathfrak{C}'}\mathfrak{C}\rightarrow\Omega_{\mathfrak{D}'}\otimes_{\mathfrak{D}'}\mathfrak{C}$, $\upsilon:\Omega_{\mathfrak{D}'}\otimes_{\mathfrak{D}'}\mathfrak{C}\rightarrow F$ respectively. We have diagrams

$$
\begin{tikzcd}
F \arrow[r, "\beta"] \arrow[d, shift right, "\mu"'] & J \arrow[r, "\lambda"] \arrow[d, shift right, "\rho"'] & \mathfrak{D}' \arrow[rd, "\psi"] \arrow[d, shift right, "\xi"'] & \\
E \arrow[r, "\alpha"] \arrow[u, shift right, "\nu"'] & I \arrow[r, "\kappa"] \arrow[u, dotted, shift right, "\sigma"'] & \mathfrak{C}' \arrow[r, "\phi"'] \arrow[u, dotted, shift right, "\chi"'] & \mathfrak{C}, \arrow[lu, dotted, bend right, "s"'] \arrow[l, dotted, bend left, "r"]
\end{tikzcd}
\tag{2.36}
$$

$$
\begin{tikzcd}
F \arrow[r, shift left, "\mathrm{d}_{\mathfrak{D}'}\circ\lambda\circ\beta"] \arrow[d, shift right, "\mu"'] & \Omega_{\mathfrak{D}'}\otimes_{\mathfrak{D}'}\mathfrak{C} \arrow[l, shift left, "\upsilon"] \arrow[r, "\Omega_\psi"] \arrow[d, shift right, "\Omega_\xi\otimes\mathrm{id}_{\mathfrak{C}}"'] & \Omega_{\mathfrak{C}} \arrow[r] \arrow[d, leftrightarrow, "\mathrm{id}_{\Omega_{\mathfrak{C}}}"] & 0 \\
E \arrow[r, shift right, "\mathrm{d}_{\mathfrak{C}'}\circ\kappa\circ\alpha"'] \arrow[u, shift right, "\nu"'] & \Omega_{\mathfrak{C}'}\otimes_{\mathfrak{C}'}\mathfrak{C} \arrow[l, shift right, "\tau"'] \arrow[r, "\Omega_\phi"] \arrow[u, shift right, "\omega"'] & \Omega_{\mathfrak{C}} \arrow[r] & 0,
\end{tikzcd}
$$

where σ, χ, r, s remain to be defined. Then (2.33) for γ, δ implies that

$$
\begin{aligned}
&\mu\circ\nu+\tau\circ\mathrm{d}_{\mathfrak{C}'}\circ\kappa\circ\alpha=\mathrm{id}_E, \quad \mathrm{d}_{\mathfrak{C}'}\circ\kappa\circ\alpha\circ\tau+(\Omega_\xi\otimes\mathrm{id}_{\mathfrak{C}})\circ\omega=\mathrm{id}_{\Omega_{\mathfrak{C}'}\otimes_{\mathfrak{C}'}\mathfrak{C}},\\
&\nu\circ\mu+\upsilon\circ\mathrm{d}_{\mathfrak{D}'}\circ\lambda\circ\beta=\mathrm{id}_F, \quad \mathrm{d}_{\mathfrak{D}'}\circ\lambda\circ\beta\circ\upsilon+\omega\circ(\Omega_\xi\otimes\mathrm{id}_{\mathfrak{C}})=\mathrm{id}_{\Omega_{\mathfrak{D}'}\otimes_{\mathfrak{D}'}\mathfrak{C}},\\
&\mathrm{d}_{\mathfrak{C}'}\circ\kappa\circ\alpha\circ\mu=(\Omega_\xi\otimes\mathrm{id}_{\mathfrak{C}})\circ\mathrm{d}_{\mathfrak{D}'}\circ\lambda\circ\beta, \quad \mu\circ\upsilon=\tau\circ(\Omega_\xi\otimes\mathrm{id}_{\mathfrak{C}}),\\
&\mathrm{d}_{\mathfrak{D}'}\circ\lambda\circ\beta\circ\nu=\omega\circ\mathrm{d}_{\mathfrak{C}'}\circ\kappa\circ\alpha, \qquad \nu\circ\tau=\upsilon\circ\omega.
\end{aligned}
\tag{2.37}
$$

Since f^4 is an isomorphism, μ induces an isomorphism from the kernel of β to the kernel of α. The first and third equations of (2.37) imply that μ and ν are inverse on the kernels of α, β. Hence ν induces an isomorphism from the kernel of α to the kernel of β. As α, β are surjective, it follows that there exists a unique $\sigma : I \to J$ with $\sigma\circ\alpha = \beta\circ\nu$. Then (2.37) implies that

$$
\rho\circ\sigma+\alpha\circ\tau\circ\mathrm{d}_{\mathfrak{C}'}\circ\kappa=\mathrm{id}_I, \quad \sigma\circ\rho+\beta\circ\upsilon\circ\mathrm{d}_{\mathfrak{D}'}\circ\lambda=\mathrm{id}_J. \tag{2.38}
$$

As $\psi : \mathfrak{D}' \to \mathfrak{C}$ is surjective with kernel J, we may choose a right inverse $s : \mathfrak{C} \to \mathfrak{D}'$ for ψ. Then $\psi\circ s = \mathrm{id}_{\mathfrak{C}}$, and $\mathfrak{D}' = s(\mathfrak{C}) \oplus J$. Define $r = \xi\circ s : \mathfrak{C} \to \mathfrak{C}'$. As $\phi\circ\xi = \psi$ we have $\phi\circ r = \mathrm{id}_{\mathfrak{C}}$, so r is a right inverse for ϕ, and $\mathfrak{C}' = r(\mathfrak{C}) \oplus I$. Note that r, s are only linear maps, not morphisms of C^∞-rings. Using $\mathfrak{C}' = r(\mathfrak{C}) \oplus I$, $\mathfrak{D}' = s(\mathfrak{C}) \oplus J$ we identify $\mathfrak{C}' = \mathfrak{C} \oplus I$, $\mathfrak{D}' = \mathfrak{C} \oplus J$. Define $\chi = \begin{pmatrix} \mathrm{id}_{\mathfrak{C}} & 0 \\ -\beta\circ\upsilon\circ\mathrm{d}_{\mathfrak{D}'} & \sigma \end{pmatrix} : \mathfrak{C}' = \mathfrak{C} \oplus I \to \mathfrak{C} \oplus J = \mathfrak{D}'$. Then (2.36) becomes

$$
\begin{tikzcd}[column sep=large, row sep=huge]
F \arrow[r, "\beta"] \arrow[d, shift right, "\mu"'] & J \arrow[r, "\lambda=\begin{pmatrix} 0 & \mathrm{id}_J \end{pmatrix}^T"] \arrow[d, shift right, "\rho"'] & \mathfrak{D}'=\mathfrak{C}\oplus J \arrow[rd, shift left, "\psi=\begin{pmatrix} \mathrm{id}_{\mathfrak{C}} & 0 \end{pmatrix}"] \arrow[d, shift right, "\xi=\begin{pmatrix} \mathrm{id}_{\mathfrak{C}} & 0 \\ 0 & \rho \end{pmatrix}"'] & \\
E \arrow[r, "\alpha"] \arrow[u, shift right, "\nu"'] & I \arrow[r, "\kappa=\begin{pmatrix} 0 & \mathrm{id}_I \end{pmatrix}^T"] \arrow[u, shift right, "\sigma"'] & \mathfrak{C}'=\mathfrak{C}\oplus I \arrow[r, shift right, "\phi=\begin{pmatrix} \mathrm{id}_{\mathfrak{C}} & 0 \end{pmatrix}"'] \arrow[u, shift right, "\chi=\begin{pmatrix} \mathrm{id}_{\mathfrak{C}} & 0 \\ -\beta\circ\upsilon\circ\mathrm{d}_{\mathfrak{D}'} & \sigma \end{pmatrix}"'] & \mathfrak{C}. \arrow[lu, shift left, "s"] \arrow[l, shift right, "r"']
\end{tikzcd}
\tag{2.39}
$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, and write $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$, $\Phi'_f : (\mathfrak{C}')^n \to \mathfrak{C}'$, $\tilde{\Phi}_f : (\mathfrak{D}')^n \to \mathfrak{D}'$ for the C^∞-ring operations on $\mathfrak{C}, \mathfrak{C}', \mathfrak{D}'$. Lemma 2.6, equation

(2.39) and ϕ,ψ,ξ morphisms of C^∞-rings imply that we may write

$$\begin{aligned}\Phi'_f\bigl((c_1,i_1),\ldots,(c_n,i_n)\bigr)=\bigl(&\Phi_f(c_1,\ldots,c_n),\rho\circ\Psi_f(c_1,\ldots,c_n)\\&+\textstyle\sum_{a=1}^n\Phi_{\frac{\partial f}{\partial x_a}}(c_1,\ldots,c_n)\cdot i_a\bigr),\end{aligned}\tag{2.40}$$

$$\begin{aligned}\tilde\Phi_f\bigl((c_1,j_1),\ldots,(c_n,j_n)\bigr)=\bigl(&\Phi_f(c_1,\ldots,c_n),\Psi_f(c_1,\ldots,c_n)\\&+\textstyle\sum_{a=1}^n\Phi_{\frac{\partial f}{\partial x_a}}(c_1,\ldots,c_n)\cdot j_a\bigr),\end{aligned}\tag{2.41}$$

for all $c_1,\ldots,c_n\in\mathfrak{C}$, $i_1,\ldots,i_n\in I$ and $j_1,\ldots,j_n\in J$, and some $\Psi_f:\mathfrak{C}^n\to I$.

Using (2.41), the C^∞-derivation property of $\mathrm{d}_{\mathfrak{D}'},\Omega_{\mathfrak{D}'}$, ψ a C^∞-ring morphism, and that the $\mathfrak{D}'$-action on $\Omega_{\mathfrak{D}'}\otimes_{\mathfrak{D}'}\mathfrak{C}$ factors through the $\mathfrak{C}$-action, we see that

$$\begin{aligned}&\mathrm{d}_{\mathfrak{D}'}\circ\Phi_f(c_1,\ldots,c_n)+\mathrm{d}_{\mathfrak{D}'}\circ\lambda\circ\Psi_f(c_1,\ldots,c_n)=\mathrm{d}_{\mathfrak{D}'}\tilde\Phi_f\bigl((c_1,0),\ldots,(c_n,0)\bigr)\\&=\sum_{a=1}^n\tilde\Phi_{\frac{\partial f}{\partial x_a}}\bigl((c_1,0),\ldots,(c_n,0)\bigr)\cdot\mathrm{d}_{\mathfrak{D}'}c_a=\sum_{a=1}^n\Phi_{\frac{\partial f}{\partial x_a}}(c_1,\ldots,c_n)\cdot\mathrm{d}_{\mathfrak{D}'}c_a.\end{aligned}\tag{2.42}$$

Combining equations (2.38)–(2.42) gives

$$\begin{aligned}\chi\circ\Phi'_f&\bigl((c_1,i_1),\ldots,(c_n,i_n)\bigr)=\bigl(\Phi_f(c_1,\ldots,c_n),-\beta\circ\upsilon\circ\mathrm{d}_{\mathfrak{D}'}\circ\Phi_f(c_1,\ldots,c_n)\\&+\sigma\circ\rho\circ\Psi_f(c_1,\ldots,c_n)+\sigma\bigl[\textstyle\sum_{a=1}^n\Phi_{\frac{\partial f}{\partial x_a}}(c_1,\ldots,c_n)\cdot i_a\bigr]\bigr)\\&=\bigl(\Phi_f(c_1,\ldots,c_n),-\beta\circ\upsilon\circ\mathrm{d}_{\mathfrak{D}'}\circ\Phi_f(c_1,\ldots,c_n)\\&\quad+(\mathrm{id}_J-\beta\circ\upsilon\circ\mathrm{d}_{\mathfrak{D}'}\circ\lambda)\circ\Psi_f(c_1,\ldots,c_n)+\textstyle\sum_{a=1}^n\Phi_{\frac{\partial f}{\partial x_a}}(c_1,\ldots,c_n)\cdot\sigma(i_a)\bigr)\\&=\bigl(\Phi_f(c_1,\ldots,c_n),\Psi_f(c_1,\ldots,c_n)\\&\qquad+\textstyle\sum_{a=1}^n\Phi_{\frac{\partial f}{\partial x_a}}(c_1,\ldots,c_n)\cdot(-\beta\circ\upsilon\circ\mathrm{d}_{\mathfrak{D}'}(c_a)+\sigma(i_a))\bigr)\\&\qquad-\bigl(0,\beta\circ\upsilon\bigl[\mathrm{d}_{\mathfrak{D}'}\circ\Phi_f(c_1,\ldots,c_n)+\mathrm{d}_{\mathfrak{D}'}\circ\lambda\circ\Psi_f(c_1,\ldots,c_n)\\&\qquad\qquad-\textstyle\sum_{a=1}^n\Phi_{\frac{\partial f}{\partial x_a}}(c_1,\ldots,c_n)\cdot\mathrm{d}_{\mathfrak{D}'}(c_a)\bigr]\bigr)\\&=\tilde\Phi_f\bigl(\bigl(c_1,-\beta\circ\upsilon\circ\mathrm{d}_{\mathfrak{D}'}(c_1)+\sigma(i_1)\bigr),\ldots,\bigl(c_n,-\beta\circ\upsilon\circ\mathrm{d}_{\mathfrak{D}'}(c_n)+\sigma(i_n)\bigr)\bigr)-0\\&=\tilde\Phi_f\bigl(\chi(c_1,i_1),\ldots,\chi(c_n,i_n)\bigr).\end{aligned}$$

Therefore $\chi:\mathfrak{C}'\to\mathfrak{D}'$ is a C^∞-ring morphism. As $(U,\mathcal{O}'_X|_U)\cong\operatorname{Spec}\mathfrak{C}'$ and $(V,\mathcal{O}'_Y|_Y)\cong\operatorname{Spec}\mathfrak{D}'$, and the identification $V\to U$ is $g|_V:V\to U$, $\operatorname{Spec}\chi$ induces a morphism of sheaves of C^∞-rings $g'|_V:g^{-1}(\mathcal{O}'_X)|_V\to\mathcal{O}'_Y|_V$. One can show this is the unique morphism $g^{-1}(\mathcal{O}'_X)|_V\to\mathcal{O}'_Y|_V$ satisfying the conditions above. Gluing these over an open cover of $\underline{Y}$ defines $g':g^{-1}(\mathcal{O}'_X)\to\mathcal{O}'_Y$, and completes the definition of $\boldsymbol{g}$. Hence $\boldsymbol{f}$ is an equivalence. □

Lemma 2.22. *Suppose $\underline{X}$ is a separated, paracompact, locally fair C^∞-scheme, and $0\to\mathcal{E}\xrightarrow{\alpha}\mathcal{F}\xrightarrow{\beta}\mathcal{G}\to 0$ is a complex in $\mathrm{qcoh}(\underline{X})$. Then being split exact is a local condition in $\underline{X}$. That is, if the restrictions of the complex to the sets of an open cover of $\underline{X}$ are split exact, then the complex is split exact.*

Proof. Let $\{\underline{U}_a : a \in A\}$ be an open cover of $\underline{X}$ such that the restriction of $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ to each $\underline{U}_a$ is split exact. Then there exist $\gamma_a : \mathcal{F}|_{\underline{U}_a} \to \mathcal{E}|_{\underline{U}_a}$ and $\delta_a : \mathcal{G}|_{\underline{U}_a} \to \mathcal{F}|_{\underline{U}_a}$ satisfying (2.33) on $\underline{U}_a$. As $\underline{X}$ is separated, paracompact and locally fair, Proposition B.21 gives a partition of unity $\{\eta_a : a \in A\}$ on $\underline{X}$ subordinate to $\{\underline{U}_a : a \in A\}$. That is, for each $a \in A$ we have $\eta_a \in \mathcal{O}_X(X)$ supported in $\underline{U}_a$, and $\sum_{a\in A} \eta_a = 1$, where the (possibly infinite) sum makes sense as it is locally finite. Then $\eta_a \cdot \gamma_a : \mathcal{F}|_{\underline{U}_a} \to \mathcal{E}|_{\underline{U}_a}$ and $\eta_a \cdot \delta_a : \mathcal{G}|_{\underline{U}_a} \to \mathcal{F}|_{\underline{U}_a}$ are morphisms of sheaves on $\underline{U}_a$, and as η_a is supported on $\underline{U}_a$ we may extend them uniquely over $\underline{X}\backslash\underline{U}_a$ to get $\gamma'_a : \mathcal{F} \to \mathcal{E}$ and $\delta'_a : \mathcal{G} \to \mathcal{E}$, with $\gamma'_a|_{\underline{U}_a} = \eta_a \cdot \gamma_a$, $\delta'_a|_{\underline{U}_a} = \eta_a \cdot \delta_a$ and γ'_a, δ'_a zero outside $\underline{U}_a$.

Define $\gamma' = \sum_{a\in A} \gamma'_a$ and $\delta' = \sum_{a\in A} \delta'_a$. These sums are locally finite, and so well-defined, and give morphisms $\gamma' : \mathcal{F} \to \mathcal{E}$, $\delta' : \mathcal{G} \to \mathcal{F}$ in $\mathrm{qcoh}(\underline{X})$. As γ_a, δ_a satisfy (2.33) for each $a \in A$, using $\sum_{a\in A} \eta_a = 1$ we find that γ', δ' satisfy

$$\gamma' \circ \alpha = \mathrm{id}_{\mathcal{E}}, \quad \alpha \circ \gamma' + \delta' \circ \beta = \mathrm{id}_{\mathcal{F}} \text{ and } \beta \circ \delta' = \mathrm{id}_{\mathcal{G}}, \tag{2.43}$$

as these are linear in γ', δ'. But they may not satisfy $\gamma' \circ \delta' = 0$. However, defining $\gamma = \gamma'$, $\delta = \delta' - \alpha \circ \gamma' \circ \delta'$, equation (2.43) implies that γ, δ satisfy (2.33), so $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is split exact. $\square$

We define open d-subspaces, open covers, and étale 1-morphisms.

Definition 2.23. Let $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ be a d-space. Suppose $\underline{U} \subseteq \underline{X}$ is an open C^∞-subscheme. Then $\boldsymbol{U} = \bigl(\underline{U}, \mathcal{O}'_X|_{\underline{U}}, \mathcal{E}_X|_{\underline{U}}, \imath_X|_{\underline{U}}, \jmath_X|_{\underline{U}}\bigr)$ is a d-space. We call $\boldsymbol{U}$ an *open d-subspace* of $\boldsymbol{X}$. An *open cover* of a d-space $\boldsymbol{X}$ is a family $\{\boldsymbol{U}_a : a \in A\}$ of open d-subspaces $\boldsymbol{U}_a$ of $\boldsymbol{X}$ with $\underline{X} = \bigcup_{a\in A} \underline{U}_a$.

Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in $\mathbf{dSpa}$. We call $\boldsymbol{f}$ *étale* if it is a *local equivalence*, that is, if for each $x \in \boldsymbol{X}$ there exist open $x \in \boldsymbol{U} \subseteq \boldsymbol{X}$ and $\boldsymbol{f}(x) \in \boldsymbol{V} \subseteq \boldsymbol{Y}$ such that $\boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{V}$ and $\boldsymbol{f}|_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ is an equivalence.

As $\boldsymbol{X}$ is a d-space, $\underline{X}$ is separated, paracompact, and locally fair. So by Lemma 2.22, for (2.34) to be split exact is a local condition in $\underline{X}$. Thus Propositions 2.20 and 2.21 imply a characterization of étale 1-morphisms:

Corollary 2.24. *Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in* $\mathbf{dSpa}$. *Then $\boldsymbol{f}$ is étale if and only if $\underline{f} : \underline{X} \to \underline{Y}$ is étale, $f^4 : \underline{f}^*(\mathcal{C}_Y) \to \mathcal{C}_X$ is an isomorphism, and equation* (2.34) *is a split short exact sequence in* $\mathrm{qcoh}(\underline{X})$.

Here is a necessary and sufficient condition for a d-space $\boldsymbol{X}$ to be a C^∞-scheme, that is, for $\boldsymbol{X}$ to lie in $\hat{\mathbf{C}}^{\boldsymbol{\infty}}\mathbf{Sch}^{\mathbf{lf}}_{\mathbf{ssc}}$. A *left inverse* for $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ is $\gamma : \mathcal{F}_X \to \mathcal{E}_X$ in $\mathrm{qcoh}(\underline{X})$ with $\gamma \circ \phi_X = \mathrm{id}_{\mathcal{E}_X}$.

Proposition 2.25. *Let $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ be a d-space. Then $\boldsymbol{X}$ lies in $\hat{\mathbf{C}}^{\boldsymbol{\infty}}\mathbf{Sch}^{\mathbf{lf}}_{\mathbf{ssc}}$, that is, $\boldsymbol{X}$ is equivalent to an object in the image of $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}$, which can be $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{X})$, if and only if $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ has a left inverse.*

Proof. Suppose $\gamma : \mathcal{F}_X \to \mathcal{E}_X$ is a left inverse for ϕ_X. Consider the diagram:

$$\begin{tikzcd} 0 \arrow[r, shift left] & \mathcal{E}_X \arrow[l, shift left] \arrow[r, shift left, "\phi_X"] & \mathcal{F}_X \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\psi_X"] & T^*\underline{X} \arrow[l, shift left, dotted, "\delta"] \arrow[r, shift left] & 0. \arrow[l, shift left] \end{tikzcd} \tag{2.44}$$

Since $\gamma \circ \phi_X = \mathrm{id}_{\mathcal{E}_X}$ we see that ϕ_X is injective, so as (2.19) is exact, the rightwards sequence in (2.44) is exact. Also, as γ exists this exact sequence is split, in the sense of Definition 2.19, so that $\mathcal{F}_X \cong \mathcal{E}_X \oplus T^*\underline{X}$, and there exists a unique $\delta : T^*\underline{X} \to \mathcal{F}_X$ such that the leftwards sequence in (2.44) is exact, and

$$\gamma \circ \delta = 0, \;\; \gamma \circ \phi_X = \mathrm{id}_{\mathcal{E}_X}, \;\; \phi_X \circ \gamma + \delta \circ \psi_X = \mathrm{id}_{\mathcal{F}_X}, \;\; \psi_X \circ \delta = \mathrm{id}_{T^*\underline{X}}.$$

Set $\tilde{\boldsymbol{X}} = F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{X}) = (\underline{X}, \mathcal{O}_X, 0, \mathrm{id}_{\mathcal{O}_X}, 0)$. Define a 1-morphism $\boldsymbol{f} : \tilde{\boldsymbol{X}} \to \boldsymbol{X}$ in $\mathbf{dSpa}$ by $\boldsymbol{f} = (\underline{\mathrm{id}}_{\underline{X}}, \imath_X \circ \delta_X(\mathcal{O}'_X), 0)$. Then $\mathcal{C}_{\tilde{X}} = 0$ as $\mathcal{E}_{\tilde{X}} = 0$. Since $\phi_X = \xi_X \circ \jmath_X$ is injective, $\jmath_X$ is injective, and so $\mathcal{C}_X = \mathrm{Ker}\, \jmath_X = 0$. Hence $f^4 : \underline{f}^*(\mathcal{C}_Y) \to \mathcal{C}_X$ is $0 \to 0$, and so is an isomorphism. Also (2.34) for $\boldsymbol{f}$ is the rightwards sequence in (2.44), and so is split exact from above. Hence $\boldsymbol{f}$ is an equivalence by Proposition 2.21. This proves the 'if' part of the proposition.

For the 'only if' part, suppose $\tilde{\boldsymbol{X}} = F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\tilde{\underline{X}})$ and $\boldsymbol{f} : \boldsymbol{X} \to \tilde{\boldsymbol{X}}$ is an equivalence, so that there exist $\boldsymbol{g} : \tilde{\boldsymbol{X}} \to \boldsymbol{X}$ and $\eta : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \mathbf{id}_{\boldsymbol{X}}$ and $\zeta : \boldsymbol{f} \circ \boldsymbol{g} \Rightarrow \mathbf{id}_{\tilde{\boldsymbol{X}}}$. Then from the definitions one can show that $\gamma = \eta \circ \delta_{\underline{X}}(\mathcal{F}_X)^{-1} : \mathcal{F}_X \to \mathcal{E}_X$ is a left inverse for ϕ_X. $\square$

If ϕ_X has a left inverse then ϕ_X is injective; having a left inverse is a strong form of injectivity. Having a left inverse is invariant under pullbacks, since if γ is a left inverse for $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ then $\underline{g}^*(\gamma)$ is a left inverse for $\underline{g}^*(\phi_X) : \underline{g}^*(\mathcal{E}_X) \to \underline{g}^*(\mathcal{F}_X)$ with $\underline{g} : \underline{Y} \to \underline{X}$. Note that injectivity is not invariant under pullbacks, i.e. ϕ_X injective does not imply $\underline{g}^*(\phi_X)$ injective, since $\underline{g}^*$ may not be left exact. For ϕ_X to have a left inverse is a *local* condition on $\underline{X}$:

Lemma 2.26. *Let $\boldsymbol{X}$ be a d-space. Then $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ has a left inverse if and only if $\underline{X}$ may be covered by open C^∞-subschemes $\underline{U}$ such that $\phi_X|_{\underline{U}} : \mathcal{E}_X|_{\underline{U}} \to \mathcal{F}_X|_{\underline{U}}$ has a left inverse in* $\mathrm{qcoh}(\underline{U})$.

Proof. The 'only if' is trivial, taking $\underline{U} = \underline{X}$. For the 'if' part, we use the method of Lemma 2.22. Let $\{\underline{U}_a : a \in A\}$ be an open cover of $\underline{X}$ such that $\phi_X|_{\underline{U}_a}$ has a left inverse γ'_a on each $\underline{U}_a$. As $\underline{X}$ is separated, paracompact, and locally fair, by Proposition B.21 there exists a partition of unity $\{\eta_a : a \in A\}$ subordinate to $\{\underline{U}_a : a \in A\}$. Hence $\eta_a \cdot \gamma'_a$ satisfies $(\eta_a \cdot \gamma'_a) \circ \phi_X|_{\underline{U}_a} = \eta_a \cdot \mathrm{id}_{\mathcal{E}_X|_{\underline{U}_a}}$. We may extend $\eta_a \cdot \gamma'_a$ uniquely to $\underline{X}$ to get $\gamma_a : \mathcal{F}_X \to \mathcal{E}_X$ which restricts to $\eta_a \cdot \gamma'_a$ on $\underline{U}_a$ and to zero on $\underline{X} \setminus \underline{U}_a$, with $\gamma_a \circ \phi_X = \eta_a \cdot \mathrm{id}_{\mathcal{E}_X}$. Then $\gamma = \sum_{a \in A} \gamma_a$ is a locally finite sum of morphisms in $\mathrm{qcoh}(\underline{X})$, so is a well-defined morphism $\gamma : \mathcal{F}_X \to \mathcal{E}_X$ with $\gamma \circ \phi_X = \sum_{a \in A} \gamma_a \circ \phi_X = \sum_{a \in A} \eta_a \cdot \mathrm{id}_{\mathcal{E}_X} = \mathrm{id}_{\mathcal{E}_X}$, since $\sum_{a \in A} \eta_a = 1$. Therefore γ is a left inverse for ϕ_X. $\square$

2.4 Gluing d-spaces by equivalences

Given a collection of d-spaces $\boldsymbol{X}_i$, $i \in I$, with open d-subspaces $\boldsymbol{U}_{ij} \subset \boldsymbol{X}_i$ for $i, j \in I$ and equivalences $\boldsymbol{e}_{ij} : \boldsymbol{U}_{ij} \xrightarrow{\sim} \boldsymbol{U}_{ji}$ satisfying appropriate conditions on triple overlaps, we study the problem of constructing a d-space $\boldsymbol{Z}$ by gluing the $\boldsymbol{X}_i$, $i \in I$ together on $\boldsymbol{U}_{ij}$, so that $\boldsymbol{Z}$ has open d-subspaces $\hat{\boldsymbol{X}}_i$ equivalent to $\boldsymbol{X}_i$ for $i \in I$, with $\boldsymbol{Z} = \bigcup_{i \in I} \hat{\boldsymbol{X}}_i$, and $\hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j \simeq \boldsymbol{U}_{ij} \simeq \boldsymbol{U}_{ji}$. This will be important in relating d-manifolds to other geometric objects in Chapter 14, and in proving the existence of d-manifold structures on moduli spaces.

We begin by showing that we can glue 1-morphisms on open d-subspaces $\boldsymbol{U},\boldsymbol{V}\subseteq\boldsymbol{X}$ provided they are 2-isomorphic on the overlap $\boldsymbol{U}\cap\boldsymbol{V}$. The last part, in which $\boldsymbol{h}$ is independent of η, will simplify the statements of Theorems 2.29–2.32. As we will see in §9.4, the analogue of $\boldsymbol{h}$ independent of η is not true for d-stacks, and this makes gluing d-stacks by equivalences more complicated.

Proposition 2.27. *Suppose $\boldsymbol{X},\boldsymbol{Y}$ are d-spaces, $\boldsymbol{U},\boldsymbol{V}\subseteq\boldsymbol{X}$ are open d-subspaces with $\boldsymbol{X}=\boldsymbol{U}\cup\boldsymbol{V}$, $\boldsymbol{f}:\boldsymbol{U}\rightarrow\boldsymbol{Y}$ and $\boldsymbol{g}:\boldsymbol{V}\rightarrow\boldsymbol{Y}$ are 1-morphisms, and $\eta:\boldsymbol{f}|_{\boldsymbol{U}\cap\boldsymbol{V}}\Rightarrow\boldsymbol{g}|_{\boldsymbol{U}\cap\boldsymbol{V}}$ is a 2-morphism. Then there exist a 1-morphism $\boldsymbol{h}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ and 2-morphisms $\zeta:\boldsymbol{h}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{f}$, $\theta:\boldsymbol{h}|_{\boldsymbol{V}}\Rightarrow\boldsymbol{g}$ such that $\theta|_{\boldsymbol{U}\cap\boldsymbol{V}}=\eta\odot\zeta|_{\boldsymbol{U}\cap\boldsymbol{V}}:\boldsymbol{h}|_{\boldsymbol{U}\cap\boldsymbol{V}}\Rightarrow\boldsymbol{g}|_{\boldsymbol{U}\cap\boldsymbol{V}}$. This $\boldsymbol{h}$ is unique up to 2-isomorphism.*

Furthermore, $\boldsymbol{h}$ is independent up to 2-isomorphism of the choice of η.

Proof. Since $\{\underline{U},\underline{V}\}$ is an open cover of $\underline{X}$, which is separated, paracompact and locally fair, by Proposition B.21 there exists a partition of unity $\{\epsilon_U,\upsilon_V\}$ on $\underline{X}$ subordinate to $\{\underline{U},\underline{V}\}$. That is, $\epsilon_U,\epsilon_V\in\mathcal{O}_X(X)$ with $\epsilon_U+\epsilon_V=1$, and ϵ_U,ϵ_V are supported on $\underline{U},\underline{V}$. The 2-morphism η is a morphism $\eta:\underline{f}^*(\mathcal{F}_Y)|_{\underline{U}\cap\underline{V}}=\underline{g}^*(\mathcal{F}_Y)|_{\underline{U}\cap\underline{V}}\rightarrow\mathcal{E}_X|_{\underline{U}\cap\underline{V}}$ in qcoh$(\underline{U}\cap\underline{V})$. Hence $-\epsilon_V|_{\underline{U}\cap\underline{V}}\cdot\eta$ is also a morphism $\underline{f}^*(\mathcal{F}_Y)|_{\underline{U}\cap\underline{V}}\rightarrow\mathcal{E}_X|_{\underline{U}\cap\underline{V}}$. Since $\epsilon_V|_{\underline{U}}$ is supported on $\underline{U}\cap\underline{V}$ in $\underline{U}$, we can extend $-\epsilon_V|_{\underline{U}\cap\underline{V}}\cdot\eta$ uniquely by zero over $\underline{U}\setminus\underline{V}$ to get a morphism $\zeta:\underline{f}^*(\mathcal{F}_Y)\rightarrow\mathcal{E}_X|_{\underline{U}}$ in qcoh$(\underline{U})$ such that ζ is supported in $\underline{U}\cap\underline{V}$ and $\zeta|_{\underline{U}\cap\underline{V}}=-\epsilon_V\cdot\eta$. Similarly, we can extend $\epsilon_U\cdot\eta$ uniquely by zero over $\underline{V}\setminus\underline{U}$ to get $\theta:\underline{g}^*(\mathcal{F}_Y)\rightarrow\mathcal{E}_X|_{\underline{V}}$ in qcoh$(\underline{V})$ such that θ is supported in $\underline{U}\cap\underline{V}$ and $\theta|_{\underline{U}\cap\underline{V}}=\epsilon_U\cdot\eta$.

Proposition 2.17 now shows that there are unique 1-morphisms $\boldsymbol{h_U}:\boldsymbol{U}\rightarrow\boldsymbol{Y}$ and $\boldsymbol{h_V}:\boldsymbol{V}\rightarrow\boldsymbol{Y}$ such that $\zeta:\boldsymbol{h_U}\Rightarrow\boldsymbol{f}$ and $\theta:\boldsymbol{h_V}\Rightarrow\boldsymbol{g}$ are 2-morphisms. On $\boldsymbol{U}\cap\boldsymbol{V}$ we have 2-morphisms

$$\boldsymbol{h_U}|_{\boldsymbol{U}\cap\boldsymbol{V}}\xRightarrow{\zeta|_{\boldsymbol{U}\cap\boldsymbol{V}}=-\epsilon_V\cdot\eta}\boldsymbol{f}|_{\boldsymbol{U}\cap\boldsymbol{V}}\xRightarrow{\ \ \eta\ \ }\boldsymbol{g}|_{\boldsymbol{U}\cap\boldsymbol{V}}\xRightarrow{-\theta|_{\boldsymbol{U}\cap\boldsymbol{V}}=-\epsilon_U\cdot\eta}\boldsymbol{h_V}|_{\boldsymbol{U}\cap\boldsymbol{V}}.$$

Composing under $\odot$ and using $\epsilon_U+\epsilon_V=1$ shows $(-\epsilon_U\cdot\eta)+\eta+(-\epsilon_V\cdot\eta)=0$ is a 2-morphism $\boldsymbol{h_U}|_{\boldsymbol{U}\cap\boldsymbol{V}}\Rightarrow\boldsymbol{h_V}|_{\boldsymbol{U}\cap\boldsymbol{V}}$, so $\boldsymbol{h_U}|_{\boldsymbol{U}\cap\boldsymbol{V}}=\boldsymbol{h_V}|_{\boldsymbol{U}\cap\boldsymbol{V}}$. Hence there exists a unique 1-morphism $\boldsymbol{h}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ such that $\boldsymbol{h}|_{\boldsymbol{U}}=\boldsymbol{h_U}$ and $\boldsymbol{h}|_{\boldsymbol{V}}=\boldsymbol{h_V}$. We have 2-morphisms $\zeta:\boldsymbol{h}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{f}$, $\theta:\boldsymbol{h}|_{\boldsymbol{V}}\Rightarrow\boldsymbol{g}$, and $\theta|_{\boldsymbol{U}\cap\boldsymbol{V}}=\epsilon_U\cdot\eta=\eta+(-\epsilon_V\cdot\eta)=\eta\odot\zeta|_{\boldsymbol{U}\cap\boldsymbol{V}}$, as we have to prove.

Suppose $\tilde{\boldsymbol{h}},\tilde\zeta,\tilde\theta$ are alternative choices of $\boldsymbol{h},\zeta,\theta$. Then we have 2-morphisms $\zeta:\boldsymbol{h}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{f}$, $\tilde\zeta:\tilde{\boldsymbol{h}}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{f}$, giving a 2-morphism $\zeta-\tilde\zeta:\boldsymbol{h}|_{\boldsymbol{U}}\Rightarrow\tilde{\boldsymbol{h}}|_{\boldsymbol{U}}$, and similarly $\theta-\tilde\theta:\boldsymbol{h}|_{\boldsymbol{V}}\Rightarrow\tilde{\boldsymbol{h}}|_{\boldsymbol{V}}$. On $\underline{U}\cap\underline{V}$ we have

$$\begin{aligned}(\theta-\tilde\theta)|_{\underline{U}\cap\underline{V}}&=\theta|_{\boldsymbol{U}\cap\boldsymbol{V}}-\tilde\theta|_{\boldsymbol{U}\cap\boldsymbol{V}}=(\eta\odot\zeta|_{\boldsymbol{U}\cap\boldsymbol{V}})-(\eta\odot\tilde\zeta|_{\boldsymbol{U}\cap\boldsymbol{V}})\\&=(\eta+\zeta|_{\underline{U}\cap\underline{V}})-(\eta+\tilde\zeta|_{\underline{U}\cap\underline{V}})=(\zeta-\tilde\zeta)|_{\underline{U}\cap\underline{V}}.\end{aligned}$$

Therefore there exists a unique morphism $\omega:\underline{h}^*(\mathcal{F}_Y)\rightarrow\mathcal{E}_X$ with $\omega|_{\underline{U}}=\zeta-\tilde\zeta$ and $\omega|_{\underline{V}}=\theta-\tilde\theta$. Thus we have 2-morphisms $\omega|_{\underline{U}}:\boldsymbol{h}|_{\boldsymbol{U}}\Rightarrow\tilde{\boldsymbol{h}}|_{\boldsymbol{U}}$ and $\omega|_{\underline{V}}:\boldsymbol{h}|_{\boldsymbol{V}}\Rightarrow\tilde{\boldsymbol{h}}|_{\boldsymbol{V}}$, so as being a 2-morphism is a local condition, $\omega:\boldsymbol{h}\Rightarrow\tilde{\boldsymbol{h}}$ is a 2-morphism, and $\boldsymbol{h}$ is unique up to 2-isomorphism.

Finally, suppose $\hat{\eta}$ is an alternative choice for η, which yields $\hat{\zeta}, \hat{\theta}, \hat{\boldsymbol{h}}_{\boldsymbol{U}}, \hat{\boldsymbol{h}}_{\boldsymbol{V}}, \hat{\boldsymbol{h}}$ in place of $\zeta, \theta, \boldsymbol{h}_{\boldsymbol{U}}, \boldsymbol{h}_{\boldsymbol{V}}, \boldsymbol{h}$. Then $(\hat{\eta}-\eta) : \boldsymbol{f}|_{\boldsymbol{U}\cap\boldsymbol{V}} \Rightarrow \boldsymbol{f}|_{\boldsymbol{U}\cap\boldsymbol{V}}$, so as in the proof of Theorem 2.15(b) we have $(\hat{\eta}-\eta)\circ \underline{f}^*(\phi_Y)|_{\underline{U}\cap\underline{V}} = \jmath_X|_{\underline{U}\cap\underline{V}}\circ(\hat{\eta}-\eta) = 0$. Therefore $(\hat{\zeta}-\zeta)\circ \underline{f}^*(\phi_Y) = \jmath_X|_{\underline{U}}\circ(\hat{\zeta}-\zeta) = 0$, since $\hat{\zeta}-\zeta = -\epsilon_V\cdot(\hat{\eta}-\eta)$ on $\underline{U}\cap\underline{V}$ and $\hat{\zeta}=\zeta=0$ on $\underline{U}\setminus\underline{V}$. Thus $\hat{\zeta}-\zeta : \boldsymbol{f}\Rightarrow\boldsymbol{f}$, so as $\zeta : \boldsymbol{h}_{\boldsymbol{U}}\Rightarrow\boldsymbol{f}$, $\hat{\zeta} : \hat{\boldsymbol{h}}_{\boldsymbol{U}}\Rightarrow\boldsymbol{f}$ we see that $0 : \boldsymbol{h}_{\boldsymbol{U}}\Rightarrow\hat{\boldsymbol{h}}_{\boldsymbol{U}}$, so $\boldsymbol{h}_{\boldsymbol{U}} = \hat{\boldsymbol{h}}_{\boldsymbol{U}}$. Similarly $\boldsymbol{h}_{\boldsymbol{V}} = \hat{\boldsymbol{h}}_{\boldsymbol{V}}$, so $\boldsymbol{h} = \hat{\boldsymbol{h}}$. Hence $\boldsymbol{h}$ is independent of η, though it does depend on the choice of ϵ_U, ϵ_V. The previous part thus shows that up to 2-isomorphism $\boldsymbol{h}$ is independent of η. □

Next we glue two d-spaces $\boldsymbol{X}, \boldsymbol{Y}$ at equivalent open d-subspaces $\boldsymbol{U}\subseteq\boldsymbol{X}$, $\boldsymbol{V}\subseteq\boldsymbol{Y}$ to get a d-space $\boldsymbol{Z}$ which is the union of open d-subspaces $\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}\subseteq\boldsymbol{Z}$ with $\boldsymbol{X}\simeq\hat{\boldsymbol{X}}$, $\boldsymbol{Y}\simeq\hat{\boldsymbol{Y}}$ and $\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}\simeq\boldsymbol{U}\simeq\boldsymbol{V}$. This is not always possible: our definition of d-spaces $\boldsymbol{Z}$ requires that $\underline{Z}$ is separated, that is, the topological space Z is Hausdorff, but $Z = X\amalg_{U=V} Y$ may not be Hausdorff, for instance if $X=Y=\mathbb{R}$ and $U=V=(0,\infty)$.

Thus, assuming $Z = X\amalg_{U=V} Y$ is Hausdorff is a necessary condition for the d-space $\boldsymbol{Z}$ to exist. Our next theorem shows it is also sufficient. See Spivak [95, Lem. 6.8 & Prop. 6.9] for similar results for his derived manifolds.

Theorem 2.28. *Suppose $\boldsymbol{X}, \boldsymbol{Y}$ are d-spaces, $\boldsymbol{U}\subseteq\boldsymbol{X}$, $\boldsymbol{V}\subseteq\boldsymbol{Y}$ are open d-subspaces, and $\boldsymbol{f} : \boldsymbol{U}\to\boldsymbol{V}$ is an equivalence in* **dSpa**. *At the level of topological spaces, we have open $U\subseteq X$, $V\subseteq Y$ with a homeomorphism $f : U\to V$, so we can form the quotient topological space $Z := X\amalg_f Y = (X\amalg Y)/\sim$, where the equivalence relation $\sim$ on $X\amalg Y$ identifies $u\in U\subseteq X$ with $f(u)\in V\subseteq Y$.*

Suppose Z is Hausdorff. This condition may also equivalently be imposed at the level of C^∞-schemes, that is, we may form a quotient C^∞-scheme $\underline{Z} = \underline{X}\amalg_{\underline{f}}\underline{Y}$, and we require $\underline{Z}$ separated. Then there exist a d-space $\boldsymbol{Z}$, open d-subspaces $\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}$ in $\boldsymbol{Z}$ with $\boldsymbol{Z} = \hat{\boldsymbol{X}}\cup\hat{\boldsymbol{Y}}$, equivalences $\boldsymbol{g} : \boldsymbol{X}\to\hat{\boldsymbol{X}}$ and $\boldsymbol{h} : \boldsymbol{Y}\to\hat{\boldsymbol{Y}}$ such that $\boldsymbol{g}|_{\boldsymbol{U}}$ and $\boldsymbol{h}|_{\boldsymbol{V}}$ are both equivalences with $\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}$, and a 2-morphism $\eta : \boldsymbol{g}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f} : \boldsymbol{U}\to\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}$. Furthermore, $\boldsymbol{Z}$ is independent of choices up to equivalence.

Proof. As $\boldsymbol{f}$ is an equivalence there exist $\boldsymbol{e} : \boldsymbol{V}\to\boldsymbol{U}$ and 2-morphisms $\zeta : \boldsymbol{e}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{U}}$, $\theta : \boldsymbol{f}\circ\boldsymbol{e}\Rightarrow\mathbf{id}_{\boldsymbol{V}}$. By Proposition A.6 we may also suppose that $\mathrm{id}_{\boldsymbol{f}} * \zeta = \theta * \mathrm{id}_{\boldsymbol{f}}$ and $\mathrm{id}_{\boldsymbol{e}} * \theta = \zeta * \mathrm{id}_{\boldsymbol{e}}$. Then $\underline{f} : \underline{U}\to\underline{V}$ and $\underline{e} : \underline{V}\to\underline{U}$ are inverse. Therefore there exists a C^∞-scheme $\underline{Z} := (\underline{X}\amalg\underline{Y})/\sim$, where the equivalence relation $\sim$ identifies $\underline{U}\subseteq\underline{X}$ and $\underline{V}\subseteq\underline{Y}$ by the isomorphism $\underline{f}$. The underlying topological space is $Z = (X\amalg Y)/\sim$. Suppose Z is Hausdorff, i.e. $\underline{Z}$ is separated. As $\underline{X}, \underline{Y}$ are second countable and locally fair, so is $\underline{Z}$, and these and $\underline{Z}$ separated imply that $\underline{Z}$ is paracompact.

Write $\hat{\underline{X}}, \hat{\underline{Y}}$ for the open C^∞-subschemes of $\underline{Z}$ corresponding to $\underline{X}, \underline{Y}$, and $\underline{g} : \underline{X}\to\hat{\underline{X}}$, $\underline{h} : \underline{Y}\to\hat{\underline{Y}}$ for the natural isomorphisms. Then $\underline{Z} = \hat{\underline{X}}\cup\hat{\underline{Y}}$ and $\underline{g}|_{\underline{U}} : \underline{U}\to\hat{\underline{X}}\cap\hat{\underline{Y}}$, $\underline{h}|_{\underline{V}} : \underline{V}\to\hat{\underline{X}}\cap\hat{\underline{Y}}$ are isomorphisms with $\underline{g}|_{\underline{U}} = \underline{h}|_{\underline{V}}\circ\underline{f} : \underline{U}\to\hat{\underline{X}}\cap\hat{\underline{Y}}$. As Z is Hausdorff and paracompact, it is a *normal* topological space. Thus, as $\hat{X}\setminus\hat{Y}$ and $\hat{Y}\setminus\hat{X}$ are disjoint closed subsets in Z, there exist open

$\hat{A},\hat{B}\subset Z$ with $(\hat{X}\setminus\hat{Y})\subseteq\hat{A}$, $(\hat{Y}\setminus\hat{X})\subseteq\hat{B}$, and $\overline{\hat{A}}\cap\overline{\hat{B}}=\emptyset$, where $\overline{\hat{A}},\overline{\hat{B}}$ are the closures of $\hat{A},\hat{B}$ in Z.

Define $\hat{C}=Z\setminus\overline{\hat{B}}$ and $\hat{D}=Z\setminus\overline{\hat{A}}$. Then $\hat{A}\subseteq\hat{C}\subseteq\hat{X}$, $\hat{B}\subseteq\hat{D}\subseteq\hat{Y}$, and $Z=\hat{C}\cup\hat{D}$. Set $A=g^{-1}(\hat{A})$, $C=g^{-1}(\hat{C})$, $E=g^{-1}(\hat{C}\cap\hat{D})$, $G=g^{-1}(\hat{D})$, $B=h^{-1}(\hat{B})$, $D=h^{-1}(\hat{D})$, $F=h^{-1}(\hat{C}\cap\hat{D})$, $H=h^{-1}(\hat{C})$, so that $A,C,E,G\subseteq X$ and $B,D,F,H\subseteq Y$ are open. Write $\underline{A},\underline{C},\underline{E},\underline{G}\subseteq\underline{X}$, $\underline{B},\underline{D},\underline{F},\underline{H}\subseteq\underline{Y}$, for the corresponding open C^∞-subschemes, and $\boldsymbol{C},\boldsymbol{E},\boldsymbol{G}\subseteq\boldsymbol{X}$, $\boldsymbol{D},\boldsymbol{F},\boldsymbol{H}\subseteq\boldsymbol{Y}$ for the open d-subspaces. Then $\boldsymbol{X}=\boldsymbol{C}\cup\boldsymbol{G}$, $\boldsymbol{E}=\boldsymbol{C}\cap\boldsymbol{G}$, $\boldsymbol{Y}=\boldsymbol{D}\cup\boldsymbol{H}$ with $\boldsymbol{F}=\boldsymbol{D}\cap\boldsymbol{H}$, $\boldsymbol{E}\subseteq\boldsymbol{G}\subseteq\boldsymbol{U}$, $\boldsymbol{F}\subseteq\boldsymbol{H}\subseteq\boldsymbol{V}$, and $\boldsymbol{f}|_{\boldsymbol{E}}:\boldsymbol{E}\rightarrow\boldsymbol{F}$ is an equivalence.

We will first define d-spaces $\dot{\boldsymbol{C}},\dot{\boldsymbol{D}}$ with open d-subspaces $\dot{\boldsymbol{E}}\subseteq\dot{\boldsymbol{C}}$, $\dot{\boldsymbol{F}}\subseteq\dot{\boldsymbol{F}}$, and equivalences $\boldsymbol{a}:\boldsymbol{C}\rightarrow\dot{\boldsymbol{C}}$, $\boldsymbol{c}:\dot{\boldsymbol{C}}\rightarrow\boldsymbol{C}$, $\boldsymbol{b}:\boldsymbol{D}\rightarrow\dot{\boldsymbol{D}}$, $\boldsymbol{d}:\dot{\boldsymbol{D}}\rightarrow\boldsymbol{D}$ which restrict to equivalences $\boldsymbol{a}|_{\boldsymbol{E}}:\boldsymbol{E}\rightarrow\dot{\boldsymbol{E}}$, $\boldsymbol{c}|_{\dot{\boldsymbol{E}}}:\dot{\boldsymbol{E}}\rightarrow\boldsymbol{E}$, $\boldsymbol{b}|_{\boldsymbol{F}}:\boldsymbol{F}\rightarrow\dot{\boldsymbol{F}}$, $\boldsymbol{d}|_{\dot{\boldsymbol{F}}}:\dot{\boldsymbol{F}}\rightarrow\boldsymbol{F}$, such that $\boldsymbol{c}\circ\boldsymbol{a}=\mathbf{id}_{\boldsymbol{C}}$ and $\boldsymbol{d}\circ\boldsymbol{b}=\mathbf{id}_{\boldsymbol{D}}$, and with 2-morphisms $\eta_C:\boldsymbol{a}\circ\boldsymbol{c}\Rightarrow\mathbf{id}_{\dot{\boldsymbol{C}}}$, $\eta_D:\boldsymbol{b}\circ\boldsymbol{d}\Rightarrow\mathbf{id}_{\dot{\boldsymbol{D}}}$. Then we will define a 1-isomorphism $\boldsymbol{j}:\dot{\boldsymbol{E}}\rightarrow\dot{\boldsymbol{F}}$ with inverse $\boldsymbol{k}:\dot{\boldsymbol{F}}\rightarrow\dot{\boldsymbol{E}}$, and we will define $\boldsymbol{Z}$ by gluing $\dot{\boldsymbol{C}},\dot{\boldsymbol{D}}$ on $\dot{\boldsymbol{E}},\dot{\boldsymbol{F}}$ using $\boldsymbol{j}$.

Using [56, Prop. 4.24] and ideas on partitions of unity in [56, §§4.4, 4.5 & 5.2], as in §B.4, since $\underline{C},\underline{D}$ are separated, paracompact, locally fair C^∞-schemes, we can choose $\gamma\in\mathcal{O}_X(C)$ and $\delta\in\mathcal{O}_Y(D)$ with $\gamma|_{\underline{A}}>0$, $\gamma|_{\underline{E}}=0$, $\delta|_{\underline{B}}>0$, $\delta|_{\underline{F}}=0$. Note that $\gamma|_{\underline{A}}>0$, $\delta|_{\underline{B}}>0$ imply that $\gamma|_{\underline{A}},\delta|_{\underline{B}}$ are invertible. Define $\mathcal{G}_C\in\mathrm{qcoh}(\underline{C})$, $\mathcal{G}_D\in\mathrm{qcoh}(\underline{D})$ by the exact sequences

$$\begin{aligned}&\underline{f}^{-1}(\mathcal{F}_Y)|_{\underline{C}\cap\underline{U}}\xrightarrow[\gamma\cdot\mathrm{id}_{\underline{f}^{-1}(\mathcal{F}_Y)|_{\underline{C}\cap\underline{U}}}]{}\underline{f}^{-1}(\mathcal{F}_Y)|_{\underline{C}\cap\underline{U}}\longrightarrow\mathcal{G}_C|_{\underline{C}\cap\underline{U}}\longrightarrow 0,\\&\underline{e}^{-1}(\mathcal{F}_X)|_{\underline{D}\cap\underline{V}}\xrightarrow{\delta\cdot\mathrm{id}_{\underline{e}^{-1}(\mathcal{F}_X)|_{\underline{D}\cap\underline{V}}}}\underline{e}^{-1}(\mathcal{F}_X)|_{\underline{D}\cap\underline{V}}\longrightarrow\mathcal{G}_D|_{\underline{D}\cap\underline{V}}\longrightarrow 0,\end{aligned}\tag{2.45}$$

and $\mathcal{G}_C|_{\underline{A}}=0$, $\mathcal{G}_D|_{\underline{B}}=0$. For $\mathcal{G}_C$, these definitions are consistent on the overlap $\underline{A}\cap(\underline{C}\cap\underline{U})$ as $\gamma>0$ on $\underline{A}$, so $\gamma\cdot\mathrm{id}_{\underline{f}^{-1}(\mathcal{F}_Y)|_{\underline{C}\cap\underline{U}}}$ is an isomorphism on $\underline{A}\cap(\underline{C}\cap\underline{U})$, and its cokernel is 0 there. Thus $\mathcal{G}_C$ is well-defined up to canonical isomorphism, and similarly so is $\mathcal{G}_D$. In (2.45) on $\underline{E},\underline{F}$ the first morphisms are zero, so the second morphisms are isomorphisms. Thus we have

$$\mathcal{G}_C|_{\underline{A}}=0,\;\;\mathcal{G}_C|_{\underline{E}}\cong\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}},\;\;\mathcal{G}_D|_{\underline{B}}=0,\;\;\mathcal{G}_D|_{\underline{F}}\cong\underline{e}^*(\mathcal{F}_X)|_{\underline{F}}.\tag{2.46}$$

We have a d-space $\boldsymbol{C}=(\underline{C},\mathcal{O}'_X|_C,\mathcal{E}_X|_{\underline{C}},\imath_X|_C,\jmath_X|_{\underline{C}})$ and a quasicoherent sheaf $\mathcal{G}_C$ on $\underline{C}$. Examples 2.11 and 2.18 now construct a d-space by adding $\mathcal{G}_C$ to $\mathcal{O}'_X|_C$ and $\mathcal{E}_X|_{\underline{C}}$, which we write as $\dot{\boldsymbol{C}}=(\dot{\underline{C}},\mathcal{O}'_{\dot{C}},\mathcal{E}_{\dot{C}},\imath_{\dot{C}},\jmath_{\dot{C}})$, and equivalences which we write as $\boldsymbol{a}=(\underline{a},a',a''):\boldsymbol{C}\rightarrow\dot{\boldsymbol{C}}$ and $\boldsymbol{c}=(\underline{c},c',c''):\dot{\boldsymbol{C}}\rightarrow\boldsymbol{C}$, with $\boldsymbol{c}\circ\boldsymbol{a}=\mathbf{id}_{\boldsymbol{C}}$, and a 2-morphism we write as $\eta_C:\boldsymbol{a}\circ\boldsymbol{c}\Rightarrow\mathbf{id}_{\dot{\boldsymbol{C}}}$. Explicitly, these are given by

$$
\begin{aligned}
&\underline{\dot C}=\underline{C},\;\; \mathcal{O}'_{\dot C}=\mathcal{O}'_X|_C\ltimes\mathcal{G}_C,\;\; \mathcal{E}_{\dot C}=\mathcal{E}_X|_{\underline{C}}\oplus\mathcal{G}_C,\;\; \imath_{\dot C}=\imath_X|_C\circ\pi_{\mathcal{O}'_X|_C},\\
&\jmath_{\dot C}=\begin{pmatrix}\jmath_X|_{\underline{C}} & 0\\ 0 & \mathrm{id}_{\mathcal{G}_C}\end{pmatrix}:\mathcal{E}_{\dot C}=\mathcal{E}_X|_{\underline{C}}\oplus\mathcal{G}_C\longrightarrow\mathcal{I}_X|_{\underline{C}}\oplus\mathcal{G}_C\cong\mathcal{I}_{\dot C},\\
&\underline{a}=\underline{c}=\underline{\mathrm{id}}_C,\;\; a':\mathrm{id}_C^{-1}(\mathcal{O}'_{\dot C})\cong\mathcal{O}'_X|_C\oplus\mathcal{G}_C\xrightarrow{(\mathrm{id}\ 0)}\mathcal{O}'_X|_C,\\
&c':\mathrm{id}_C^{-1}(\mathcal{O}'_X|_C)\cong\mathcal{O}'_X|_C\xrightarrow{\binom{\mathrm{id}}{0}}\mathcal{O}'_X|_C\oplus\mathcal{G}_C\cong\mathcal{O}'_{\dot C},\\
&a''=\begin{pmatrix}\delta_{\underline{C}}(\mathcal{E}_X|_{\underline{C}}) & 0\end{pmatrix}:\underline{a}^*(\mathcal{E}_{\dot C})=\underline{\mathrm{id}}_{\underline{C}}^*(\mathcal{E}_X|_{\underline{C}})\oplus\underline{\mathrm{id}}_{\underline{C}}^*(\mathcal{G}_C)\longrightarrow\mathcal{E}_X|_{\underline{C}},\\
&c''=\begin{pmatrix}\delta_{\underline{C}}(\mathcal{E}_X|_{\underline{C}})\\ 0\end{pmatrix}:\underline{c}^*(\mathcal{E}_X|_{\underline{C}})=\underline{\mathrm{id}}_{\underline{C}}^*(\mathcal{E}_X|_{\underline{C}})\longrightarrow\mathcal{E}_X|_{\underline{C}}\oplus\mathcal{G}_C=\mathcal{E}_{\dot C},\\
&\text{and}\quad \eta_C=\begin{pmatrix}0 & 0\\ 0 & \delta_{\underline{C}}(\mathcal{G}_C)\end{pmatrix}:(\underline{a}\circ\underline{c})^*(\mathcal{F}_{\dot C})\cong\underline{\mathrm{id}}_{\underline{C}}^*(\mathcal{F}_X|_{\underline{C}})\oplus\underline{\mathrm{id}}_{\underline{C}}^*(\mathcal{G}_C)\\
&\qquad\qquad\longrightarrow\mathcal{E}_X|_{\underline{C}}\oplus\mathcal{G}_C=\mathcal{E}_{\dot C}.
\end{aligned}
\tag{2.47}
$$

Let $\dot{\boldsymbol{E}}\subseteq\dot{\boldsymbol{C}}$ be the open d-subspace associated to $\underline{E}\subseteq\underline{C}=\underline{\dot C}$. Then $\boldsymbol{a}|_{\boldsymbol{E}}:\boldsymbol{E}\to\dot{\boldsymbol{E}}$, $\boldsymbol{c}|_{\dot{\boldsymbol{E}}}:\dot{\boldsymbol{E}}\to\boldsymbol{E}$ are equivalences. Similarly we define $\dot{\boldsymbol{F}}\subseteq\dot{\boldsymbol{D}}$ and equivalences $\boldsymbol{b}:\boldsymbol{D}\to\dot{\boldsymbol{D}}$, $\boldsymbol{d}:\dot{\boldsymbol{D}}\to\boldsymbol{D}$, $\boldsymbol{b}|_{\boldsymbol{F}}:\boldsymbol{F}\to\dot{\boldsymbol{F}}$, $\boldsymbol{d}|_{\dot{\boldsymbol{F}}}:\dot{\boldsymbol{F}}\to\boldsymbol{F}$ from $\boldsymbol{D},\mathcal{G}_D,\underline{F}$, with the analogous notation.

Define $\tilde{\boldsymbol{\jmath}}:\dot{\boldsymbol{E}}\to\dot{\boldsymbol{F}}$ and $\tilde{\boldsymbol{k}}:\dot{\boldsymbol{F}}\to\dot{\boldsymbol{E}}$ by $\tilde{\boldsymbol{\jmath}}=\boldsymbol{b}\circ\boldsymbol{f}\circ\boldsymbol{c}|_{\dot{\boldsymbol{E}}}$, $\tilde{\boldsymbol{k}}=\boldsymbol{a}\circ\boldsymbol{e}\circ\boldsymbol{d}|_{\dot{\boldsymbol{F}}}$. Then $\underline{\tilde\jmath}=\underline{f}|_{\underline{E}}$, $\underline{\tilde k}=\underline{e}|_{\underline{F}}$. Define $\zeta_{\dot E}:\underline{\tilde\jmath}^*(\mathcal{F}_{\dot D})\to\mathcal{E}_{\dot C}|_{\underline{E}}$ and $\zeta_{\dot F}:\underline{\tilde k}^*(\mathcal{F}_{\dot C})\to\mathcal{E}_{\dot D}|_{\underline{F}}$ by

$$
\begin{aligned}
&\zeta_{\dot E}=\begin{pmatrix}0 & \zeta|_{\underline{E}}\circ I_{\underline{f},\underline{e}}(\mathcal{F}_X|_{\underline{E}})^{-1}\\ \mathrm{id}_{\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}} & \underline{f}^*(e^2)|_{\underline{E}}\end{pmatrix}:\\
&\quad\underline{\tilde\jmath}^*(\mathcal{F}_{\dot D})\cong\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}\oplus\underline{f}^*(\underline{e}^*(\mathcal{F}_X))|_{\underline{E}}\longrightarrow\mathcal{E}_X|_{\underline{E}}\oplus\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}\cong\mathcal{E}_{\dot C}|_{\underline{E}},
\end{aligned}
\tag{2.48}
$$

$$
\begin{aligned}
&\zeta_{\dot F}=\begin{pmatrix}0 & \theta|_{\underline{F}}\circ I_{\underline{e},\underline{f}}(\mathcal{F}_Y|_{\underline{F}})^{-1}\\ \mathrm{id}_{\underline{e}^*(\mathcal{F}_X)|_{\underline{F}}} & \underline{e}^*(f^2)|_{\underline{F}}\end{pmatrix}:\\
&\quad\underline{\tilde k}^*(\mathcal{F}_{\dot C})\cong\underline{e}^*(\mathcal{F}_X)|_{\underline{F}}\oplus\underline{e}^*(\underline{f}^*(\mathcal{F}_Y))|_{\underline{F}}\longrightarrow\mathcal{E}_Y|_{\underline{F}}\oplus\underline{e}^*(\mathcal{F}_X)|_{\underline{F}}\cong\mathcal{E}_{\dot D}|_{\underline{F}}.
\end{aligned}
\tag{2.49}
$$

Then Proposition 2.17 shows that there exist unique 1-morphisms $\boldsymbol{j}:\dot{\boldsymbol{E}}\to\dot{\boldsymbol{F}}$ and $\boldsymbol{k}:\dot{\boldsymbol{F}}\to\dot{\boldsymbol{E}}$ such that $\zeta_{\dot E}:\boldsymbol{j}\Rightarrow\tilde{\boldsymbol{\jmath}}$ and $\zeta_{\dot F}:\boldsymbol{k}\Rightarrow\tilde{\boldsymbol{k}}$ are 2-morphisms.

Define $\theta_{\dot E}:\boldsymbol{k}\circ\boldsymbol{j}\Rightarrow\mathbf{id}_{\dot{\boldsymbol{E}}}$ and $\theta_{\dot F}:\boldsymbol{j}\circ\boldsymbol{k}\Rightarrow\mathbf{id}_{\dot{\boldsymbol{F}}}$ by the commutative diagrams

$$
\begin{tikzcd}
\boldsymbol{k}\circ\boldsymbol{j} \arrow[r, Rightarrow, "\zeta_{\dot F}*\zeta_{\dot E}"'] \arrow[d, Rightarrow, "\theta_{\dot E}"] & \tilde{\boldsymbol{k}}\circ\tilde{\boldsymbol{\jmath}} \arrow[r, equal] & \boldsymbol{a}\circ\boldsymbol{e}\circ\boldsymbol{d}\circ\boldsymbol{b}\circ\boldsymbol{f}\circ\boldsymbol{c}|_{\dot{\boldsymbol{E}}} \arrow[d, equal] \\
\mathbf{id}_{\dot{\boldsymbol{E}}} & \boldsymbol{a}\circ\boldsymbol{c}|_{\dot{\boldsymbol{E}}} \arrow[l, Rightarrow, "\eta_C|_{\dot{\boldsymbol{E}}}"'] & \boldsymbol{a}\circ\boldsymbol{e}\circ\boldsymbol{f}\circ\boldsymbol{c}|_{\dot{\boldsymbol{E}}}, \arrow[l, Rightarrow, "\mathrm{id}_{\boldsymbol{b}}*\zeta*\mathrm{id}_{\boldsymbol{c}|_{\dot{\boldsymbol{E}}}}"']
\end{tikzcd}
\tag{2.50}
$$

$$
\begin{tikzcd}
\boldsymbol{j}\circ\boldsymbol{k} \arrow[r, Rightarrow, "\zeta_{\dot E}*\zeta_{\dot F}"'] \arrow[d, Rightarrow, "\theta_{\dot F}"] & \tilde{\boldsymbol{\jmath}}\circ\tilde{\boldsymbol{k}} \arrow[r, equal] & \boldsymbol{b}\circ\boldsymbol{f}\circ\boldsymbol{c}\circ\boldsymbol{a}\circ\boldsymbol{e}\circ\boldsymbol{d}|_{\dot{\boldsymbol{F}}} \arrow[d, equal] \\
\mathbf{id}_{\dot{\boldsymbol{F}}} & \boldsymbol{b}\circ\boldsymbol{d}|_{\dot{\boldsymbol{F}}} \arrow[l, Rightarrow, "\eta_D|_{\dot{\boldsymbol{F}}}"'] & \boldsymbol{b}\circ\boldsymbol{f}\circ\boldsymbol{e}\circ\boldsymbol{d}|_{\dot{\boldsymbol{F}}}. \arrow[l, Rightarrow, "\mathrm{id}_{\boldsymbol{a}}*\theta*\mathrm{id}_{\boldsymbol{d}|_{\dot{\boldsymbol{F}}}}"']
\end{tikzcd}
\tag{2.51}
$$

Then using $\zeta\odot\eta=\zeta+\eta$ and equations (2.27) and (2.46)–(2.49), we find that

$$\theta_{\dot E}:(\underline{k}\circ\underline{j})^*(\mathcal{F}_{\dot E})\cong\underline{\mathrm{id}}_{\underline{E}}^*(\mathcal{F}_X|_{\underline{E}})\oplus\underline{\mathrm{id}}_{\underline{E}}^*(\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}})\to\mathcal{E}_X|_{\underline{E}}\oplus\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}\cong\mathcal{E}_{\dot E}$$

is given by

$$
\begin{aligned}
\theta_{\dot E} &= \bigl(\eta_C|_{\dot{\boldsymbol{E}}}\bigr) \odot \bigl(\mathrm{id}_{\boldsymbol{a}} * \zeta * \mathrm{id}_{\boldsymbol{c}|_{\dot{\boldsymbol{E}}}}\bigr) \odot \bigl(\zeta_{\dot F} * \zeta_{\dot E}\bigr) \\
&= \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\underline{E}}(\underline{f}^{-1}(\mathcal{F}_Y)|_{\underline{E}}) \end{pmatrix} + \begin{pmatrix} \zeta|_{\underline{E}} & 0 \\ 0 & 0 \end{pmatrix} + \Biggl[\begin{pmatrix} 0 & \zeta|_{\underline{E}} \circ I_{\underline{f},\underline{e}}(\mathcal{F}_X|_{\underline{E}})^{-1} \\ \mathrm{id}_{\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}} & \underline{f}^*(e^2)|_{\underline{E}} \end{pmatrix} \\
&\circ \underline{f}|_{\underline{E}}^* \begin{pmatrix} e^2|_{\underline{F}} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f''|_{\underline{E}} & 0 \\ 0 & 0 \end{pmatrix} \circ \underline{f}|_{\underline{E}}^* \begin{pmatrix} 0 & \theta|_{\underline{F}} \circ I_{\underline{e},\underline{f}}(\mathcal{F}_Y|_{\underline{F}})^{-1} \\ \mathrm{id}_{\underline{e}^*(\mathcal{F}_X)|_{\underline{F}}} & \underline{e}^*(f^2)|_{\underline{F}} \end{pmatrix} \\
&- \begin{pmatrix} 0 & \zeta|_{\underline{E}} \circ I_{\underline{f},\underline{e}}(\mathcal{F}_X|_{\underline{E}})^{-1} \\ \mathrm{id}_{\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}} & \underline{f}^*(e^2)|_{\underline{E}} \end{pmatrix} \circ \underline{f}|_{\underline{E}}^* \begin{pmatrix} \phi_Y|_{\underline{F}} & 0 \\ 0 & \mathrm{id}_{\underline{e}^*(\mathcal{F}_X)|_{\underline{F}}} \end{pmatrix} \\
&\circ \underline{f}|_{\underline{E}}^* \begin{pmatrix} 0 & \theta|_{\underline{F}} \circ I_{\underline{e},\underline{f}}(\mathcal{F}_Y|_{\underline{F}})^{-1} \\ \mathrm{id}_{\underline{e}^*(\mathcal{F}_X)|_{\underline{F}}} & \underline{e}^*(f^2)|_{\underline{F}} \end{pmatrix} \Biggr] \circ \begin{pmatrix} I_{\underline{f},\underline{e}}(\mathcal{F}_X|_{\underline{E}}) & 0 \\ 0 & I_{\underline{f},\underline{e}}(\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}) \end{pmatrix} = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}.
\end{aligned}
\tag{2.52}
$$

Here in the second step the terms $[\cdots]$ are the expansion of $\zeta_{\dot F} * \zeta_{\dot E}$ using

$$
\zeta_{\dot F} * \zeta_{\dot E} = \bigl[\zeta_{\dot E} \circ \underline{f}|_{\underline{E}}^*(\tilde k^2) + \tilde\jmath'' \circ \underline{f}|_{\underline{E}}^*(\zeta_{\dot F}) - \zeta_{\dot E} \circ \underline{f}|_{\underline{E}}^*(\phi_{\dot F}) \circ \underline{f}|_{\underline{E}}^*(\zeta_{\dot F})\bigr] \circ I_{\underline{f},\underline{e}}(\mathcal{F}_{\dot E}),
$$

which follows from (2.27) and $\underline{\tilde\jmath} = \underline{f}|_{\underline{E}}$, $\underline{\tilde k} = \underline{e}|_{\underline{F}}$. Multiplying out (2.52) gives

$$
\begin{aligned}
\Theta_{11} &= \zeta|_{\underline{E}} - \zeta|_{\underline{E}} \circ I_{\underline{f},\underline{e}}(\mathcal{F}_X|_{\underline{E}})^{-1} \circ \underline{f}|_{\underline{E}}^*(\mathrm{id}_{\underline{e}^*(\mathcal{F}_X)|_{\underline{F}}}) \circ I_{\underline{f},\underline{e}}(\mathcal{F}_X|_{\underline{E}}), \\
\Theta_{12} &= \bigl[f''|_{\underline{E}} \circ \underline{f}|_{\underline{E}}^*(\theta|_{\underline{F}} \circ I_{\underline{e},\underline{f}}(\mathcal{F}_Y|_{\underline{F}})^{-1}) \\
&\qquad - \zeta|_{\underline{E}} \circ I_{\underline{f},\underline{e}}(\mathcal{F}_X|_{\underline{E}})^{-1} \circ \underline{f}|_{\underline{E}}^*(\underline{e}^*(f^2)|_{\underline{F}})\bigr] \circ I_{\underline{f},\underline{e}}(\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}), \\
\Theta_{21} &= \bigl[\mathrm{id}_{\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}} \circ \underline{f}|_{\underline{E}}^*(e^2|_{\underline{F}}) - \underline{f}^*(e^2)|_{\underline{E}} \circ \underline{f}|_{\underline{E}}^*(\mathrm{id}_{\underline{e}^*(\mathcal{F}_X)|_{\underline{F}}})\bigr] \circ I_{\underline{f},\underline{e}}(\mathcal{F}_X|_{\underline{E}}), \\
\Theta_{22} &= \delta_{\underline{E}}(\underline{f}^{-1}(\mathcal{F}_Y)|_{\underline{E}}) - \bigl[\mathrm{id}_{\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}} \circ \underline{f}|_{\underline{E}}^*(\phi_Y|_{\underline{F}}) \circ \underline{f}|_{\underline{E}}^*(\theta|_{\underline{F}} \circ I_{\underline{e},\underline{f}}(\mathcal{F}_Y|_{\underline{F}})^{-1}) \\
&\qquad + \underline{f}^*(e^2)|_{\underline{E}} \circ \underline{f}|_{\underline{E}}^*(\mathrm{id}_{\underline{e}^*(\mathcal{F}_X)|_{\underline{F}}}) \circ \underline{f}|_{\underline{E}}^*(\underline{e}^*(f^2)|_{\underline{F}})\bigr] \circ I_{\underline{f},\underline{e}}(\underline{f}^*(\mathcal{F}_Y)|_{\underline{E}}).
\end{aligned}
$$

Here $\Theta_{11} = \Theta_{21} = 0$ are immediate, and $\Theta_{12} = 0$ follows from $\mathrm{id}_{\boldsymbol{f}} * \zeta = \theta * \mathrm{id}_{\boldsymbol{f}}$, and $\Theta_{22} = 0$ follows from $\theta : \boldsymbol{f} \circ \boldsymbol{e} \Rightarrow \mathbf{id}_{\boldsymbol{V}}$ a 2-morphism. Hence $\theta_{\dot E} = 0$, so $\boldsymbol{k} \circ \boldsymbol{j} = \mathbf{id}_{\dot{\boldsymbol{E}}}$ as $\theta_{\dot E} : \boldsymbol{k} \circ \boldsymbol{j} \Rightarrow \mathbf{id}_{\dot{\boldsymbol{E}}}$. Similarly $\theta_{\dot F} = 0$ and $\boldsymbol{j} \circ \boldsymbol{k} = \mathbf{id}_{\dot{\boldsymbol{F}}}$. Therefore $\boldsymbol{j} : \dot{\boldsymbol{E}} \to \dot{\boldsymbol{F}}$ and $\boldsymbol{k} : \dot{\boldsymbol{F}} \to \dot{\boldsymbol{E}}$ are inverse 1-isomorphisms.

We can now define the d-space $\boldsymbol{Z} = (\underline{Z}, \mathcal{O}'_Z, \mathcal{E}_Z, \imath_Z, \jmath_Z)$ by gluing $\dot{\boldsymbol{C}}, \dot{\boldsymbol{D}}$ on the open d-subspaces $\dot{\boldsymbol{E}} \subseteq \dot{\boldsymbol{C}}$, $\dot{\boldsymbol{F}} \subseteq \dot{\boldsymbol{D}}$ using the 1-isomorphism $\boldsymbol{j} : \dot{\boldsymbol{E}} \to \dot{\boldsymbol{F}}$. The C^∞-scheme is $\underline{Z} = \underline{C} \amalg_{\underline{j}} \underline{D} = \underline{X} \amalg_{\underline{f}} \underline{Y}$ as above. Write $\hat{\boldsymbol{C}}, \hat{\boldsymbol{D}} \subseteq \boldsymbol{Z}$ for the open d-subspaces identified with $\dot{\boldsymbol{C}}, \dot{\boldsymbol{D}}$, and $\boldsymbol{l} : \dot{\boldsymbol{C}} \to \hat{\boldsymbol{C}}$, $\boldsymbol{m} : \dot{\boldsymbol{D}} \to \hat{\boldsymbol{D}}$ for the natural 1-isomorphisms. Then $\boldsymbol{l}|_{\dot{\boldsymbol{E}}} : \dot{\boldsymbol{E}} \to \hat{\boldsymbol{C}} \cap \hat{\boldsymbol{D}}$, $\boldsymbol{m}|_{\dot{\boldsymbol{F}}} : \dot{\boldsymbol{F}} \to \hat{\boldsymbol{C}} \cap \hat{\boldsymbol{D}}$ are also 1-isomorphisms, with $\boldsymbol{l}|_{\dot{\boldsymbol{E}}} = \boldsymbol{m}|_{\dot{\boldsymbol{F}}} \circ \boldsymbol{j}$.

Define $\omega_E : \boldsymbol{l} \circ \boldsymbol{a}|_{\boldsymbol{E}} \Rightarrow \boldsymbol{m} \circ \boldsymbol{b} \circ \boldsymbol{f}|_{\boldsymbol{E}}$ and $\omega_F : \boldsymbol{m} \circ \boldsymbol{b}|_{\boldsymbol{F}} \Rightarrow \boldsymbol{l} \circ \boldsymbol{a} \circ \boldsymbol{e}|_{\boldsymbol{F}}$ by

$$
\begin{tikzcd}
\boldsymbol{l} \circ \boldsymbol{a}|_{\boldsymbol{E}} \arrow[r, equal] \arrow[d, Rightarrow, "\omega_E = \mathrm{id}_{\boldsymbol{m}} * \zeta_{\dot E} * \mathrm{id}_{\boldsymbol{a}|_{\boldsymbol{E}}}"] & \boldsymbol{m} \circ \boldsymbol{j} \circ \boldsymbol{a}|_{\boldsymbol{E}} \arrow[r, Rightarrow, "\mathrm{id}_{\boldsymbol{m}} * \zeta_{\dot E} * \mathrm{id}_{\boldsymbol{a}|_{\boldsymbol{E}}}"'] & \boldsymbol{m} \circ \tilde{\boldsymbol{\jmath}} \circ \boldsymbol{a}|_{\boldsymbol{E}} \arrow[d, equal] \\
\boldsymbol{m} \circ \boldsymbol{b} \circ \boldsymbol{f}|_{\boldsymbol{E}} \arrow[r, equal] & \boldsymbol{m} \circ \boldsymbol{b} \circ \boldsymbol{f} \circ \mathbf{id}_{\boldsymbol{E}} \arrow[r, equal] & \boldsymbol{m} \circ \boldsymbol{b} \circ \boldsymbol{f} \circ \boldsymbol{c} \circ \boldsymbol{a}|_{\boldsymbol{E}},
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boldsymbol{m} \circ \boldsymbol{b}|_{\boldsymbol{F}} \arrow[r, equal] \arrow[d, Rightarrow, "\omega_F = \mathrm{id}_{\boldsymbol{l}} * \zeta_{\dot F} * \mathrm{id}_{\boldsymbol{b}|_{\boldsymbol{F}}}"] & \boldsymbol{l} \circ \boldsymbol{k} \circ \boldsymbol{b}|_{\boldsymbol{F}} \arrow[r, Rightarrow, "\mathrm{id}_{\boldsymbol{l}} * \zeta_{\dot F} * \mathrm{id}_{\boldsymbol{b}|_{\boldsymbol{F}}}"'] & \boldsymbol{l} \circ \tilde{\boldsymbol{k}} \circ \boldsymbol{b}|_{\boldsymbol{F}} \arrow[d, equal] \\
\boldsymbol{l} \circ \boldsymbol{a} \circ \boldsymbol{e}|_{\boldsymbol{F}} \arrow[r, equal] & \boldsymbol{l} \circ \boldsymbol{a} \circ \boldsymbol{e} \circ \mathbf{id}_{\boldsymbol{F}} \arrow[r, equal] & \boldsymbol{l} \circ \boldsymbol{a} \circ \boldsymbol{e} \circ \boldsymbol{d} \circ \boldsymbol{b}|_{\boldsymbol{F}}.
\end{tikzcd}
$$

As (2.50) commutes with $\theta_{\dot E}=0$, composing on the left with $\mathrm{id}_{\boldsymbol{l}}$ and on the right with $\mathrm{id}_{\boldsymbol{a}|_{\boldsymbol{E}}}$ shows the following commutes:

$$\begin{tikzcd}[column sep=small]
\boldsymbol{l}\circ\boldsymbol{a}|_{\boldsymbol{E}} \arrow[r, equals] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{l}\circ\boldsymbol{a}}*(-\zeta)|_{\underline{E}}"] & \boldsymbol{m}\circ\boldsymbol{j}\circ\boldsymbol{a}|_{\boldsymbol{E}} \arrow[r, Rightarrow, "\mathrm{id}_{\boldsymbol{m}}*\zeta_{\dot E}*\mathrm{id}_{\boldsymbol{a}|_{\boldsymbol{E}}}"'] & \boldsymbol{m}\circ\tilde{\boldsymbol{j}}\circ\boldsymbol{a}|_{\boldsymbol{E}} \arrow[r, equals] & \boldsymbol{m}\circ\boldsymbol{b}\circ\boldsymbol{f}\circ\boldsymbol{c}\circ\boldsymbol{a}|_{\boldsymbol{E}} \arrow[d, equals] \\
\boldsymbol{l}\circ\boldsymbol{a}\circ\boldsymbol{e}\circ\boldsymbol{f}|_{\boldsymbol{E}} \arrow[r, equals] & \boldsymbol{l}\circ\boldsymbol{a}\circ\boldsymbol{e}\circ\boldsymbol{d}\circ\boldsymbol{b}\circ\boldsymbol{f}|_{\boldsymbol{E}} \arrow[r, equals] & \boldsymbol{l}\circ\tilde{\boldsymbol{k}}\circ\boldsymbol{b}\circ\boldsymbol{f}|_{\boldsymbol{E}} & \boldsymbol{l}\circ\boldsymbol{k}\circ\boldsymbol{b}\circ\boldsymbol{f}|_{\boldsymbol{E}}. \arrow[l, Rightarrow, "\mathrm{id}_{\boldsymbol{l}}*\zeta_{\dot F}*\mathrm{id}_{\boldsymbol{b}|_{\boldsymbol{F}}}*\mathrm{id}_{\boldsymbol{f}|_{\boldsymbol{E}}}"']
\end{tikzcd}$$

Thus the following diagram of 2-morphisms commutes:

$$\begin{tikzcd}
\boldsymbol{l}\circ\boldsymbol{a}|_{\boldsymbol{E}} \arrow[rr, Rightarrow, "\omega_E"'] \arrow[d, equals] && \boldsymbol{m}\circ\boldsymbol{b}\circ\boldsymbol{f}|_{\boldsymbol{E}} \arrow[d, Rightarrow, "\omega_F*\mathrm{id}_{\boldsymbol{f}|_{\boldsymbol{E}}}"'] \\
\boldsymbol{l}\circ\boldsymbol{a}\circ\mathbf{id}_{\boldsymbol{U}}|_{\boldsymbol{E}} \arrow[rr, Rightarrow, "\mathrm{id}_{\boldsymbol{l}\circ\boldsymbol{a}}*(-\zeta)|_{\underline{E}}"] && \boldsymbol{l}\circ\boldsymbol{a}\circ\boldsymbol{e}\circ\boldsymbol{f}|_{\boldsymbol{E}}.
\end{tikzcd} \tag{2.53}$$

Next we construct the equivalences $\boldsymbol{g}:\boldsymbol{X}\to\hat{\boldsymbol{X}}$, $\boldsymbol{h}:\boldsymbol{Y}\to\hat{\boldsymbol{Y}}$. To define $\boldsymbol{g}$, note that we have open d-subspaces $\boldsymbol{C},\boldsymbol{G}\subseteq\boldsymbol{X}$ with $\boldsymbol{X}=\boldsymbol{C}\cup\boldsymbol{G}$ and $\boldsymbol{E}=\boldsymbol{C}\cap\boldsymbol{G}$, and 1-morphisms $\boldsymbol{l}\circ\boldsymbol{a}:\boldsymbol{C}\to\hat{\boldsymbol{X}}$, $\boldsymbol{m}\circ\boldsymbol{b}\circ\boldsymbol{f}|_{\boldsymbol{G}}:\boldsymbol{G}\to\hat{\boldsymbol{X}}$, and a 2-morphism $\omega_E:\boldsymbol{l}\circ\boldsymbol{a}|_{\boldsymbol{E}}\Rightarrow\boldsymbol{m}\circ\boldsymbol{b}\circ\boldsymbol{f}|_{\boldsymbol{E}}$. Proposition 2.27 therefore constructs a 1-morphism $\boldsymbol{g}:\boldsymbol{X}\to\hat{\boldsymbol{X}}$ with 2-morphisms $\zeta_C:\boldsymbol{g}|_{\boldsymbol{C}}\Rightarrow\boldsymbol{l}\circ\boldsymbol{a}$, $\theta_G:\boldsymbol{g}|_{\boldsymbol{G}}\Rightarrow\boldsymbol{m}\circ\boldsymbol{b}\circ\boldsymbol{f}|_{\boldsymbol{G}}$ such that $\theta_G|_{\boldsymbol{C}\cap\boldsymbol{G}}=\omega_E\odot\zeta_C|_{\boldsymbol{C}\cap\boldsymbol{G}}$. In the same way we obtain $\boldsymbol{h}:\boldsymbol{Y}\to\hat{\boldsymbol{Y}}$ with $\zeta_D:\boldsymbol{h}_{\boldsymbol{D}}\Rightarrow\boldsymbol{m}\circ\boldsymbol{b}$ and $\theta_H:\boldsymbol{h}|_{\boldsymbol{H}}\Rightarrow\boldsymbol{l}\circ\boldsymbol{a}\circ\boldsymbol{e}|_{\boldsymbol{H}}$ satisfying $\theta_H|_{\boldsymbol{D}\cap\boldsymbol{H}}=\omega_F\odot\zeta_D|_{\boldsymbol{D}\cap\boldsymbol{H}}$.

To see that $\boldsymbol{g}$ is an equivalence, note that $\boldsymbol{a}:\boldsymbol{C}\to\dot{\boldsymbol{C}}$ is an equivalence and $\boldsymbol{l}:\dot{\boldsymbol{C}}\to\hat{\boldsymbol{C}}$ a 1-isomorphism, so $\boldsymbol{l}\circ\boldsymbol{a}:\boldsymbol{C}\to\hat{\boldsymbol{C}}$ is an equivalence, and thus $\boldsymbol{g}|_{\boldsymbol{C}}:\boldsymbol{C}\to\hat{\boldsymbol{C}}$ is an equivalence as $\boldsymbol{g}|_{\boldsymbol{C}}\cong\boldsymbol{l}\circ\boldsymbol{a}$. Similarly $\boldsymbol{g}|_{\boldsymbol{G}}$ is an equivalence. The morphism $\underline{g}$ in $\boldsymbol{g}=(\underline{g},g',g'')$ is the isomorphism $\underline{g}:\underline{X}\to\underline{\hat{X}}$ from the beginning of the proof. These imply that $\boldsymbol{g}:\boldsymbol{X}\to\hat{\boldsymbol{X}}$ is an equivalence, and similarly $\boldsymbol{h}:\boldsymbol{Y}\to\hat{\boldsymbol{Y}}$ is an equivalence.

To construct $\eta:\boldsymbol{g}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$, define $\eta:\underline{g}^*(\mathcal{F}_Z)|_{\underline{U}}\to\mathcal{E}_X|_{\underline{U}}$ by

$$\begin{aligned}
\eta|_{\underline{U}\cap\underline{C}}&=\bigl[(-\theta_H)*\mathrm{id}_{\boldsymbol{f}|_{\boldsymbol{U}\cap\boldsymbol{C}}}\bigr]+\bigl[\mathrm{id}_{\boldsymbol{l}\circ\boldsymbol{a}}*(-\zeta)|_{\underline{U}\cap\underline{C}}\bigr]+\bigl[\zeta_C|_{\underline{U}\cap\underline{C}}\bigr],\\
\eta|_{\underline{G}}&=\bigl[(-\zeta_D)*\mathrm{id}_{\boldsymbol{f}|_{\boldsymbol{G}}}\bigr]+\bigl[\theta_G\bigr].
\end{aligned}$$

On $(\underline{U}\cap\underline{C})\cap\underline{G}$, the two lines agree as $\theta_G|_{\boldsymbol{C}\cap\boldsymbol{G}}=\omega_E\odot\zeta_C|_{\boldsymbol{C}\cap\boldsymbol{G}}$, $\theta_H|_{\boldsymbol{D}\cap\boldsymbol{H}}=\omega_F\odot\zeta_D|_{\boldsymbol{D}\cap\boldsymbol{H}}$ and (2.53) commutes. Thus we have commutative diagrams

$$\begin{tikzcd}
\boldsymbol{g}|_{\boldsymbol{U}\cap\boldsymbol{C}} \arrow[r, Rightarrow, "\zeta_C|_{\underline{U}\cap\underline{C}}"'] \arrow[d, Rightarrow, "\eta|_{\underline{U}\cap\underline{C}}"] & \boldsymbol{l}\circ\boldsymbol{a}|_{\boldsymbol{U}\cap\boldsymbol{C}} \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{l}\circ\boldsymbol{a}}*(-\zeta)|_{\underline{U}\cap\underline{C}}"'] \\
\boldsymbol{h}\circ\boldsymbol{f}|_{\boldsymbol{U}\cap\boldsymbol{C}} & \boldsymbol{l}\circ\boldsymbol{a}\circ\boldsymbol{e}\circ\boldsymbol{f}|_{\boldsymbol{U}\cap\boldsymbol{C}}, \arrow[l, Rightarrow, "(-\theta_H)*\mathrm{id}_{\boldsymbol{f}|_{\boldsymbol{U}\cap\boldsymbol{C}}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{g}|_{\boldsymbol{G}} \arrow[r, Rightarrow, "\theta_G"'] \arrow[d, Rightarrow, "\eta|_{\underline{G}}"] & \boldsymbol{m}\circ\boldsymbol{b}\circ\boldsymbol{f}|_{\boldsymbol{G}} \arrow[d, equals] \\
\boldsymbol{h}\circ\boldsymbol{f}|_{\boldsymbol{G}} & \boldsymbol{m}\circ\boldsymbol{b}\circ\boldsymbol{f}|_{\boldsymbol{G}}. \arrow[l, Rightarrow, "(-\zeta_D)*\mathrm{id}_{\boldsymbol{f}|_{\boldsymbol{G}}}"']
\end{tikzcd}$$

Hence $\eta|_{\underline{U}\cap\underline{C}}:\boldsymbol{g}|_{\boldsymbol{U}\cap\boldsymbol{C}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}|_{\boldsymbol{U}\cap\boldsymbol{C}}$ and $\eta|_{\underline{G}}:\boldsymbol{g}|_{\boldsymbol{G}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}|_{\boldsymbol{G}}$ are 2-morphisms, so as $\boldsymbol{U}=(\boldsymbol{U}\cap\boldsymbol{C})\cup\boldsymbol{G}$ and being a 2-morphism is local, $\eta:\boldsymbol{g}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ is a 2-morphism. Finally, $\boldsymbol{Z}$ being independent of choices up to equivalence will follow from the characterization of $\boldsymbol{Z}$ as a pushout in **dSpa** in Theorem 2.29 below. This completes the proof of Theorem 2.28. $\square$

We can rewrite Theorem 2.28 in terms of pushouts in the 2-category **dSpa**. As in §A.2, a *pushout* in a (1-)category $\mathcal{C}$ is a colimit of a diagram $X\xleftarrow{e}W\xrightarrow{f}Y$.

It is an object Z in $\mathcal{C}$ with morphisms $g : X \to Z$ and $h : Y \to Z$ with $g \circ e = h \circ f$, satisfying a universal property, and is unique up to isomorphism in $\mathcal{C}$. We write the pushout as $Z = X \amalg_{e,W,f} Y$ or $Z = X \amalg_W Y$. Pushouts are dual to fibre products, that is, a pushout in $\mathcal{C}$ is a fibre product in the opposite category $\mathcal{C}^{\mathrm{op}}$, with directions of morphisms reversed.

There is an explicit construction of pushouts in the category **Top** of topological spaces: given topological spaces W, X, Y and continuous maps $e : W \to X$, $f : W \to Y$, we define $Z = (X \amalg Y)/\sim$, where $\sim$ is the equivalence relation on $X \amalg Y$ generated by $x \sim y$ if $x = e(w) \in X$ and $y = f(w) \in Y$ for $w \in W$, and we give Z the quotient topology from $X \amalg Y \twoheadrightarrow Z$. The continuous maps $g : X \to Z$ and $h : Y \to Z$ are the compositions of inclusions $X, Y \hookrightarrow X \amalg Y$ and the projection $X \amalg Y \twoheadrightarrow Z$. If e, f are homeomorphisms with open sets U, V in X, Y then g, h are homeomorphisms with open sets $\hat{X}, \hat{Y}$ in Z, with $Z = \hat{X} \cup \hat{Y}$.

One can also define pushouts in 2-categories, by the dual definition to that of fibre products in 2-categories in §A.4, with arrows reversed. Given 1-morphisms $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ in a 2-category $\mathfrak{C}$, the pushout $\boldsymbol{Z} = \boldsymbol{X} \amalg_{\boldsymbol{e},\boldsymbol{W},\boldsymbol{f}} \boldsymbol{Y}$ in $\mathfrak{C}$, if it exists, is unique up to equivalence. In the next theorem, $\boldsymbol{b}$ being independent of $\tilde{\eta}$ up to 2-isomorphism is not true of all pushouts in 2-categories, it is due to $\boldsymbol{e}, \boldsymbol{f}$ being equivalences with open d-subspaces, and the existence of suitable partitions of unity which we can use to glue 1- and 2-morphisms on open sets.

Theorem 2.29. *Let $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ be 1-morphisms of d-spaces which are equivalences with open d-subspaces $\boldsymbol{U} \subseteq \boldsymbol{X}$, $\boldsymbol{V} \subseteq \boldsymbol{Y}$. Suppose that the pushout of topological spaces $Z = X \amalg_{e,W,f} Y$ is Hausdorff. Then a pushout $\boldsymbol{Z} = \boldsymbol{X} \amalg_{\boldsymbol{e},\boldsymbol{W},\boldsymbol{f}} \boldsymbol{Y}$ exists in the 2-category* **dSpa**, *as constructed in Theorem 2.28, with 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ and a 2-morphism $\eta : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$.*

By definition, pushouts have the property that if $\tilde{\boldsymbol{g}} : \boldsymbol{X} \to \tilde{\boldsymbol{Z}}$, $\tilde{\boldsymbol{h}} : \boldsymbol{Y} \to \tilde{\boldsymbol{Z}}$ are 1-morphisms and $\tilde{\eta} : \tilde{\boldsymbol{g}} \circ \boldsymbol{e} \Rightarrow \tilde{\boldsymbol{h}} \circ \boldsymbol{f}$ a 2-morphism in **dSpa**, *then there exist a 1-morphism $\boldsymbol{b} : \boldsymbol{Z} \to \tilde{\boldsymbol{Z}}$ and 2-morphisms $\zeta_{\boldsymbol{X}} : \boldsymbol{b} \circ \boldsymbol{g} \Rightarrow \tilde{\boldsymbol{g}}$, $\zeta_{\boldsymbol{Y}} : \boldsymbol{b} \circ \boldsymbol{h} \Rightarrow \tilde{\boldsymbol{h}}$, where $\boldsymbol{b}$ is unique up to 2-isomorphism. In our case, this $\boldsymbol{b}$ has the additional property of being independent up to 2-isomorphism of the choice of 2-morphism $\tilde{\eta}$.*

Proof. As $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{U}$ is an equivalence we can choose a quasi-inverse $\boldsymbol{i} : \boldsymbol{U} \to \boldsymbol{W}$, with 2-morphisms $\zeta_{\boldsymbol{e}} : \boldsymbol{i} \circ \boldsymbol{e} \Rightarrow \mathbf{id}_{\boldsymbol{W}}$, $\theta_{\boldsymbol{e}} : \boldsymbol{e} \circ \boldsymbol{i} \Rightarrow \mathbf{id}_{\boldsymbol{U}}$, which by Proposition A.6 we take to satisfy $\mathrm{id}_{\boldsymbol{e}} * \zeta_{\boldsymbol{e}} = \theta_{\boldsymbol{e}} * \mathrm{id}_{\boldsymbol{e}}$ and $\mathrm{id}_{\boldsymbol{i}} * \theta_{\boldsymbol{e}} = \zeta_{\boldsymbol{e}} * \mathrm{id}_{\boldsymbol{i}}$. Let $\boldsymbol{f'} = \boldsymbol{f} \circ \boldsymbol{i} : \boldsymbol{U} \to \boldsymbol{V}$. Then $\boldsymbol{f'}$ is an equivalence, as $\boldsymbol{f}, \boldsymbol{i}$ are. So Theorem 2.28 applies, and constructs a d-space $\boldsymbol{Z}$, 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$, and a 2-morphism $\eta' : \boldsymbol{g}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f'}$. Define a 2-morphism $\eta : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$ by $\eta = (\mathrm{id}_{\boldsymbol{h} \circ \boldsymbol{f}} * \zeta_{\boldsymbol{e}}) \odot (\eta' * \mathrm{id}_{\boldsymbol{e}})$, as a composition of 2-morphisms $\boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f'} \circ \boldsymbol{e} = \boldsymbol{h} \circ \boldsymbol{f} \circ \boldsymbol{i} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f} \circ \mathbf{id}_{\boldsymbol{W}} = \boldsymbol{h} \circ \boldsymbol{f}$.

We claim that $\boldsymbol{Z}, \boldsymbol{g}, \boldsymbol{h}, \eta$ are a pushout $\boldsymbol{X} \amalg_{\boldsymbol{e},\boldsymbol{W},\boldsymbol{f}} \boldsymbol{Y}$ in the 2-category **dSpa**. That is, given $\tilde{\boldsymbol{g}} : \boldsymbol{X} \to \tilde{\boldsymbol{Z}}$, $\tilde{\boldsymbol{h}} : \boldsymbol{Y} \to \tilde{\boldsymbol{Z}}$ and $\tilde{\eta} : \tilde{\boldsymbol{g}} \circ \boldsymbol{e} \Rightarrow \tilde{\boldsymbol{h}} \circ \boldsymbol{f}$, we should construct a 1-morphism $\boldsymbol{b} : \boldsymbol{Z} \to \tilde{\boldsymbol{Z}}$ and 2-morphisms $\zeta_{\boldsymbol{X}} : \boldsymbol{b} \circ \boldsymbol{g} \Rightarrow \tilde{\boldsymbol{g}}$, $\zeta_{\boldsymbol{Y}} : \boldsymbol{b} \circ \boldsymbol{h} \Rightarrow \tilde{\boldsymbol{h}}$ such that such that the following diagram of 2-isomorphisms commutes:

$$\begin{tikzcd} \boldsymbol{b}\circ\boldsymbol{g}\circ\boldsymbol{e} \arrow[r, Rightarrow, "\mathrm{id}_{\boldsymbol{b}}*\eta"'] \arrow[d, Rightarrow, "\zeta_{\boldsymbol{X}}*\mathrm{id}_{\boldsymbol{e}}"'] & \boldsymbol{b}\circ\boldsymbol{h}\circ\boldsymbol{f} \arrow[d, Rightarrow, "\zeta_{\boldsymbol{Y}}*\mathrm{id}_{\boldsymbol{f}}"] \\ \tilde{\boldsymbol{g}}\circ\boldsymbol{e} \arrow[r, Rightarrow, "\tilde\eta"] & \tilde{\boldsymbol{h}}\circ\boldsymbol{f}. \end{tikzcd} \tag{2.54}$$

Furthermore, if $\dot{\boldsymbol{b}},\dot\zeta_{\boldsymbol{X}},\dot\zeta_{\boldsymbol{Y}}$ are alternative choices of $\boldsymbol{b},\zeta_{\boldsymbol{X}},\zeta_{\boldsymbol{Y}}$ then there should exist a unique 2-morphism $\theta:\dot{\boldsymbol{b}}\Rightarrow\boldsymbol{b}$ with

$$\dot\zeta_{\boldsymbol{X}}=\zeta_{\boldsymbol{X}}\odot(\theta*\mathrm{id}_{\boldsymbol{g}})\quad\text{and}\quad\dot\zeta_{\boldsymbol{Y}}=\zeta_{\boldsymbol{Y}}\odot(\theta*\mathrm{id}_{\boldsymbol{h}}). \tag{2.55}$$

Let $\tilde{\boldsymbol{g}},\tilde{\boldsymbol{h}},\tilde\eta$ be as above. Let $\boldsymbol{p}:\hat{\boldsymbol{X}}\rightarrow\boldsymbol{X}$ and $\boldsymbol{q}:\hat{\boldsymbol{Y}}\rightarrow\boldsymbol{Y}$ be quasi-inverses for $\boldsymbol{g},\boldsymbol{h}$, with 2-morphisms $\zeta_{\boldsymbol{g}}:\boldsymbol{p}\circ\boldsymbol{g}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$, $\theta_{\boldsymbol{g}}:\boldsymbol{g}\circ\boldsymbol{p}\Rightarrow\mathbf{id}_{\hat{\boldsymbol{X}}}$, $\zeta_{\boldsymbol{h}}:\boldsymbol{q}\circ\boldsymbol{h}\Rightarrow\mathbf{id}_{\boldsymbol{Y}}$, $\theta_{\boldsymbol{h}}:\boldsymbol{h}\circ\boldsymbol{q}\Rightarrow\mathbf{id}_{\hat{\boldsymbol{Y}}}$, which by Proposition A.6 we take to satisfy $\mathrm{id}_{\boldsymbol{g}}*\zeta_{\boldsymbol{g}}=\theta_{\boldsymbol{g}}*\mathrm{id}_{\boldsymbol{g}}$, $\mathrm{id}_{\boldsymbol{p}}*\theta_{\boldsymbol{g}}=\zeta_{\boldsymbol{g}}*\mathrm{id}_{\boldsymbol{p}}$, $\mathrm{id}_{\boldsymbol{h}}*\zeta_{\boldsymbol{h}}=\theta_{\boldsymbol{h}}*\mathrm{id}_{\boldsymbol{h}}$, and $\mathrm{id}_{\boldsymbol{q}}*\theta_{\boldsymbol{h}}=\zeta_{\boldsymbol{h}}*\mathrm{id}_{\boldsymbol{q}}$. Then we obtain a 2-morphism $\omega:\tilde{\boldsymbol{g}}\circ\boldsymbol{p}|_{\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}}\Rightarrow\tilde{\boldsymbol{h}}\circ\boldsymbol{q}|_{\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}}$ by composition in the diagram:

$$\begin{tikzcd} \tilde{\boldsymbol{g}}\circ\boldsymbol{p}|_{\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}} \arrow[r, Rightarrow, "\mathrm{id}_{\tilde{\boldsymbol{g}}}*(-\theta_{\boldsymbol{e}})*\mathrm{id}_{\boldsymbol{p}}"'] \arrow[d, Rightarrow, "\omega"] & \tilde{\boldsymbol{g}}\circ\boldsymbol{e}\circ\boldsymbol{i}\circ\boldsymbol{p}|_{\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}} \arrow[r, Rightarrow, "\tilde\eta*\mathrm{id}_{\boldsymbol{i}\circ\boldsymbol{p}}"'] & \tilde{\boldsymbol{h}}\circ\boldsymbol{f}\circ\boldsymbol{i}\circ\boldsymbol{p}|_{\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}} \arrow[d, Rightarrow, "\mathrm{id}_{\tilde{\boldsymbol{h}}}*(-\zeta_{\boldsymbol{h}})*\mathrm{id}_{\boldsymbol{f}'\circ\boldsymbol{p}}"'] \\ \tilde{\boldsymbol{h}}\circ\boldsymbol{q}|_{\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}} & \tilde{\boldsymbol{h}}\circ\boldsymbol{q}\circ\boldsymbol{g}\circ\boldsymbol{p}|_{\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}} \arrow[l, Rightarrow, "\mathrm{id}_{\tilde{\boldsymbol{h}}\circ\boldsymbol{q}}*\theta_{\boldsymbol{g}}"'] & \tilde{\boldsymbol{h}}\circ\boldsymbol{q}\circ\boldsymbol{h}\circ\boldsymbol{f}'\circ\boldsymbol{p}|_{\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}}. \arrow[l, Rightarrow, "\mathrm{id}_{\tilde{\boldsymbol{h}}\circ\boldsymbol{q}}*(-\eta')*\mathrm{id}_{\boldsymbol{p}}"'] \end{tikzcd}$$

Proposition 2.27 thus constructs a 1-morphism $\boldsymbol{b}:\boldsymbol{Z}\rightarrow\tilde{\boldsymbol{Z}}$, unique up to 2-isomorphism, and 2-morphisms $\zeta_{\boldsymbol{b}}:\boldsymbol{b}|_{\hat{\boldsymbol{X}}}\Rightarrow\tilde{\boldsymbol{g}}\circ\boldsymbol{p}$, $\theta_{\boldsymbol{b}}:\boldsymbol{b}|_{\hat{\boldsymbol{Y}}}\Rightarrow\tilde{\boldsymbol{h}}\circ\boldsymbol{q}$ such that $\theta_{\boldsymbol{b}}|_{\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}}=\omega\odot\zeta_{\boldsymbol{b}}|_{\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}}$. This $\boldsymbol{b}$ is independent up to 2-isomorphism of the choice of ω, so in particular is independent of $\tilde\eta$, as we have to prove.

Define $\zeta_{\boldsymbol{X}}:\boldsymbol{b}\circ\boldsymbol{g}\Rightarrow\tilde{\boldsymbol{g}}$ and $\zeta_{\boldsymbol{Y}}:\boldsymbol{b}\circ\boldsymbol{h}\Rightarrow\tilde{\boldsymbol{h}}$ by

$$\zeta_{\boldsymbol{X}}=(\mathrm{id}_{\tilde{\boldsymbol{g}}}*\zeta_{\boldsymbol{g}})\odot(\zeta_{\boldsymbol{b}}*\mathrm{id}_{\boldsymbol{g}})\quad\text{and}\quad\zeta_{\boldsymbol{Y}}=(\mathrm{id}_{\tilde{\boldsymbol{h}}}*\zeta_{\boldsymbol{h}})\odot(\theta_{\boldsymbol{b}}*\mathrm{id}_{\boldsymbol{h}}).$$

The identities on $\zeta_{\boldsymbol{e}},\theta_{\boldsymbol{e}},\zeta_{\boldsymbol{g}},\theta_{\boldsymbol{g}},\zeta_{\boldsymbol{h}},\theta_{\boldsymbol{h}},\omega,\zeta_{\boldsymbol{b}},\theta_{\boldsymbol{b}}$ above then imply that (2.54) commutes. Given alternatives $\dot{\boldsymbol{b}},\dot\zeta_{\boldsymbol{X}},\dot\zeta_{\boldsymbol{Y}}$, we define $\theta|_{\underline{\hat X}},\theta|_{\underline{\hat Y}}$ by the diagrams

$$\begin{tikzcd} \dot{\boldsymbol{b}}|_{\hat{\boldsymbol{X}}} \arrow[r, Rightarrow, "\mathrm{id}_{\dot{\boldsymbol{b}}}*(-\theta_{\boldsymbol{g}})"'] \arrow[d, Rightarrow, "\theta|_{\underline{\hat X}}"] & \dot{\boldsymbol{b}}\circ\boldsymbol{g}\circ\boldsymbol{p} \arrow[r, Rightarrow, "\dot\zeta_{\boldsymbol{X}}*\mathrm{id}_{\boldsymbol{p}}"'] & \tilde{\boldsymbol{g}}\circ\boldsymbol{p} \arrow[d, equal] \\ \boldsymbol{b}|_{\hat{\boldsymbol{X}}} & \boldsymbol{b}\circ\boldsymbol{g}\circ\boldsymbol{p} \arrow[l, Rightarrow, "\mathrm{id}_{\boldsymbol{b}}*\theta_{\boldsymbol{g}}"'] & \tilde{\boldsymbol{g}}\circ\boldsymbol{p}, \arrow[l, Rightarrow, "(-\zeta_{\boldsymbol{X}})*\mathrm{id}_{\boldsymbol{p}}"'] \end{tikzcd} \quad \begin{tikzcd} \dot{\boldsymbol{b}}|_{\hat{\boldsymbol{Y}}} \arrow[r, Rightarrow, "\mathrm{id}_{\dot{\boldsymbol{b}}}*(-\theta_{\boldsymbol{h}})"'] \arrow[d, Rightarrow, "\theta|_{\underline{\hat Y}}"] & \dot{\boldsymbol{b}}\circ\boldsymbol{h}\circ\boldsymbol{q} \arrow[r, Rightarrow, "\dot\zeta_{\boldsymbol{Y}}*\mathrm{id}_{\boldsymbol{q}}"'] & \tilde{\boldsymbol{h}}\circ\boldsymbol{q} \arrow[d, equal] \\ \boldsymbol{b}|_{\hat{\boldsymbol{Y}}} & \boldsymbol{b}\circ\boldsymbol{h}\circ\boldsymbol{q} \arrow[l, Rightarrow, "\mathrm{id}_{\boldsymbol{b}}*\theta_{\boldsymbol{h}}"'] & \tilde{\boldsymbol{h}}\circ\boldsymbol{q}. \arrow[l, Rightarrow, "(-\zeta_{\boldsymbol{Y}})*\mathrm{id}_{\boldsymbol{q}}"'] \end{tikzcd}$$

The identities above imply that these coincide on $\underline{\hat X}\cap\underline{\hat Y}$, and so they glue to define a morphism θ on $\underline{Z}$. One can show it is the unique 2-morphism $\theta:\dot{\boldsymbol{b}}\Rightarrow\boldsymbol{b}$ satisfying (2.55). This completes the proof. $\square$

Next we generalize Theorems 2.28–2.29 to glue together n different d-spaces $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$ by equivalences on open d-subspaces, rather than just two.

Theorem 2.30. *Suppose $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$ are d-spaces, and for all $i,j=1,\ldots,n$ we are given an open d-subspace $\boldsymbol{U}_{ij}\subseteq\boldsymbol{X}_i$ and an equivalence $\boldsymbol{e}_{ij}:\boldsymbol{U}_{ij}\rightarrow\boldsymbol{U}_{ji}$, satisfying the following properties:*

(a) *We have* $\boldsymbol{U}_{ii}=\boldsymbol{X}_i$ *and* $\boldsymbol{e}_{ii}=\mathbf{id}_{\boldsymbol{X}_i}$ *for* $i=1,\ldots,n$; *and*

(b) *For all* $i,j,k=1,\ldots,n$ *we have a* 2*-commutative diagram*

$$\begin{tikzcd}[column sep=small] & \boldsymbol{U}_{ji}\cap\boldsymbol{U}_{jk} \arrow[dr, "\boldsymbol{e}_{jk}|_{\boldsymbol{U}_{ji}\cap\boldsymbol{U}_{jk}}"] \arrow[d, phantom, "\Downarrow\eta_{ijk}"] & \\ \boldsymbol{U}_{ij}\cap\boldsymbol{U}_{ik} \arrow[ur, "\boldsymbol{e}_{ij}|_{\boldsymbol{U}_{ij}\cap\boldsymbol{U}_{ik}}"] \arrow[rr, "\boldsymbol{e}_{ik}|_{\boldsymbol{U}_{ij}\cap\boldsymbol{U}_{ik}}"] & & \boldsymbol{U}_{ki}\cap\boldsymbol{U}_{kj} \end{tikzcd}$$

for some η_{ijk}, *where all three* 1*-morphisms are equivalences.*

On the level of topological spaces, define the quotient topological space $Y=(\coprod_{i=1}^n X_i)/\sim$, *where* $\sim$ *is the equivalence relation generated by* $x_i\sim x_j$ *if* $x_i\in U_{ij}\subseteq X_i$ *and* $x_j\in U_{ji}\subseteq X_j$ *with* $e_{ij}(x_i)=x_j$. *Suppose* Y *is Hausdorff. Then there exist a d-space* $\boldsymbol{Y}$ *and a* 1*-morphism* $\boldsymbol{f}_i:\boldsymbol{X}_i\rightarrow\boldsymbol{Y}$ *which is an equivalence with an open d-subspace* $\hat{\boldsymbol{X}}_i\subseteq\boldsymbol{Y}$ *for* $i=1,\ldots,n$, *where* $\boldsymbol{Y}=\hat{\boldsymbol{X}}_1\cup\cdots\cup\hat{\boldsymbol{X}}_n$, *such that* $\boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$ *is an equivalence* $\boldsymbol{U}_{ij}\rightarrow\hat{\boldsymbol{X}}_i\cap\hat{\boldsymbol{X}}_j$ *for all* $i,j=1,\ldots,n$, *and there exists a* 2*-morphism* $\eta_{ij}:\boldsymbol{f}_j\circ\boldsymbol{e}_{ij}\Rightarrow\boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$. *The d-space* $\boldsymbol{Y}$ *is unique up to equivalence, and is independent of choice of* 2*-morphisms* η_{ijk} *in* (b).

Suppose also that $\boldsymbol{Z}$ *is a d-space, and* $\boldsymbol{g}_i:\boldsymbol{X}_i\rightarrow\boldsymbol{Z}$ *are* 1*-morphisms for* $i=1,\ldots,n$, *and there exist* 2*-morphisms* $\zeta_{ij}:\boldsymbol{g}_j\circ\boldsymbol{e}_{ij}\Rightarrow\boldsymbol{g}_i|_{\boldsymbol{U}_{ij}}$ *for all* $i,j=1,\ldots,n$. *Then there exist a* 1*-morphism* $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ *and* 2*-morphisms* $\zeta_i:\boldsymbol{h}\circ\boldsymbol{f}_i\Rightarrow\boldsymbol{g}_i$ *for* $i=1,\ldots,n$. *The* 1*-morphism* $\boldsymbol{h}$ *is unique up to* 2*-isomorphism, and is independent of the choice of* 2*-morphisms* ζ_{ij}.

Proof. The proof is by induction on n. The case $n=1$ is trivial, with $\boldsymbol{Y}=\boldsymbol{X}_1$ and $\boldsymbol{f}_1=\mathbf{id}_{\boldsymbol{X}_1}$ and $\boldsymbol{h}=\boldsymbol{g}_1$, and the case $n=2$ follows from Theorems 2.28 and 2.29. Suppose by induction that the theorem holds whenever $n\leqslant m$, where $m\geqslant 2$, let $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_{m+1},\boldsymbol{U}_{ij},\boldsymbol{e}_{ij}$ be as in the first part of the theorem for $n=m+1$, and suppose $Y=(\coprod_{i=1}^{m+1}X_i)/\sim$ is Hausdorff.

Apply Theorem 2.28 to the d-spaces $\boldsymbol{X}_m,\boldsymbol{X}_{m+1}$, open d-subspaces $\boldsymbol{U}_{m(m+1)}\subseteq\boldsymbol{X}_m$, $\boldsymbol{U}_{(m+1)m}\subseteq\boldsymbol{X}_{m+1}$, and equivalence $\boldsymbol{e}_{m(m+1)}:\boldsymbol{U}_{m(m+1)}\rightarrow\boldsymbol{U}_{(m+1)m}$. This gives a d-space $\tilde{\boldsymbol{X}}_m$ with open d-subspaces $\dot{\boldsymbol{X}}_m,\dot{\boldsymbol{X}}_{m+1}\subseteq\tilde{\boldsymbol{X}}_m$, and equivalences $\boldsymbol{j}_m:\boldsymbol{X}_m\rightarrow\dot{\boldsymbol{X}}_m$, $\boldsymbol{j}_{m+1}:\boldsymbol{X}_{m+1}\rightarrow\dot{\boldsymbol{X}}_{m+1}$, such that $\boldsymbol{j}_m|_{\boldsymbol{U}_{m(m+1)}}:\boldsymbol{U}_{m(m+1)}\rightarrow\dot{\boldsymbol{X}}_m\cap\dot{\boldsymbol{X}}_{m+1}$ and $\boldsymbol{j}_{m+1}|_{\boldsymbol{U}_{(m+1)m}}:\boldsymbol{U}_{(m+1)m}\rightarrow\dot{\boldsymbol{X}}_m\cap\dot{\boldsymbol{X}}_{m+1}$ are equivalences, and a 2-morphism $\theta_m:\boldsymbol{j}_m|_{\boldsymbol{U}_{m(m+1)}}\Rightarrow\boldsymbol{j}_{m+1}\circ\boldsymbol{e}_{m(m+1)}$.

For $i=1,\ldots,m-1$ define open d-subspaces $\tilde{\boldsymbol{U}}_{im}=\boldsymbol{U}_{im}\cup\boldsymbol{U}_{i(m+1)}\subseteq\boldsymbol{X}_i$ and $\tilde{\boldsymbol{U}}_{mi}=\boldsymbol{j}_m(\boldsymbol{U}_{mi})\cup\boldsymbol{j}_{m+1}(\boldsymbol{U}_{(m+1)i})\subseteq\tilde{\boldsymbol{X}}_m$. Then $\tilde{\boldsymbol{U}}_{mi}$ with 1-morphisms $\boldsymbol{j}_m|_{\boldsymbol{U}_{mi}}:\boldsymbol{U}_{mi}\rightarrow\tilde{\boldsymbol{U}}_{mi}$ and $\boldsymbol{j}_{m+1}|_{\boldsymbol{U}_{(m+1)i}}:\boldsymbol{U}_{(m+1)i}\rightarrow\tilde{\boldsymbol{U}}_{mi}$ is the result of applying Theorem 2.28 to glue the d-spaces $\boldsymbol{U}_{mi},\boldsymbol{U}_{(m+1)i}$ via the equivalence $\boldsymbol{e}_{m(m+1)}|_{\boldsymbol{U}_{mi}\cap\boldsymbol{U}_{m(m+1)}}:\boldsymbol{U}_{mi}\cap\boldsymbol{U}_{m(m+1)}\rightarrow\boldsymbol{U}_{(m+1)i}\cap\boldsymbol{U}_{(m+1)m}$. We have 1-morphisms $\boldsymbol{e}_{mi}:\boldsymbol{U}_{mi}\rightarrow\tilde{\boldsymbol{U}}_{im}$ and $\boldsymbol{e}_{(m+1)i}:\boldsymbol{U}_{(m+1)i}\rightarrow\tilde{\boldsymbol{U}}_{im}$, and (b) with $m,m+1,i$ in place of i,j,k shows that $\boldsymbol{e}_{mi}|_{\boldsymbol{U}_{mi}\cap\boldsymbol{U}_{m(m+1)}}\cong\boldsymbol{e}_{(m+1)i}\circ\boldsymbol{e}_{m(m+1)}|_{\boldsymbol{U}_{mi}\cap\boldsymbol{U}_{m(m+1)}}$, where '$\cong$' denotes 2-isomorphic. Thus the second part of Theorem 2.29 gives a 1-morphism $\tilde{\boldsymbol{e}}_{mi}:\tilde{\boldsymbol{U}}_{mi}\rightarrow\tilde{\boldsymbol{U}}_{im}$ with $\tilde{\boldsymbol{e}}_{mi}\circ\boldsymbol{j}_m|_{\boldsymbol{U}_{mi}}\cong\boldsymbol{e}_{mi}$ and $\tilde{\boldsymbol{e}}_{mi}\circ\boldsymbol{j}_{m+1}|_{\boldsymbol{U}_{(m+1)i}}\cong\boldsymbol{e}_{(m+1)i}$.

We have 1-morphisms $\boldsymbol{j}_m \circ \boldsymbol{e}_{im} : \boldsymbol{U}_{im} \to \tilde{\boldsymbol{U}}_{im}$ and $\boldsymbol{j}_{m+1} \circ \boldsymbol{e}_{i(m+1)} : \boldsymbol{U}_{i(m+1)} \to \tilde{\boldsymbol{U}}_{im}$, and on the overlap $\boldsymbol{U}_{im} \cap \boldsymbol{U}_{i(m+1)}$ we have 2-morphisms

$$\begin{tikzcd}[column sep=large] \boldsymbol{j}_m \circ \boldsymbol{e}_{im}|_{\boldsymbol{U}_{im}\cap \boldsymbol{U}_{i(m+1)}} \arrow[r, Rightarrow, "\theta_m * \mathrm{id}_{\boldsymbol{e}_{im}}"'] \arrow[dr, Rightarrow] & \boldsymbol{j}_{m+1} \circ \boldsymbol{e}_{m(m+1)} \circ \boldsymbol{e}_{im}|_{\boldsymbol{U}_{im}\cap \boldsymbol{U}_{i(m+1)}} \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{j}_{m+1}} * \eta_{im(m+1)}"] \\ & \boldsymbol{j}_{m+1} \circ \boldsymbol{e}_{i(m+1)}|_{\boldsymbol{U}_{im}\cap \boldsymbol{U}_{i(m+1)}}. \end{tikzcd}$$

Proposition 2.27 thus constructs a 1-morphism $\tilde{\boldsymbol{e}}_{im} : \tilde{\boldsymbol{U}}_{im} \to \tilde{\boldsymbol{U}}_{mi}$ such that $\tilde{\boldsymbol{e}}_{im}|_{\boldsymbol{U}_{im}} \cong \boldsymbol{j}_m \circ \boldsymbol{e}_{im}$ and $\tilde{\boldsymbol{e}}_{im}|_{\boldsymbol{U}_{i(m+1)}} \cong \boldsymbol{j}_{m+1} \circ \boldsymbol{e}_{i(m+1)}$.

Since $\boldsymbol{e}_{im}, \boldsymbol{e}_{mi}$ are quasiinverse and $\boldsymbol{e}_{i(m+1)}, \boldsymbol{e}_{(m+1)i}$ are quasiinverse, one can now show that $\tilde{\boldsymbol{e}}_{mi} : \tilde{\boldsymbol{U}}_{mi} \to \tilde{\boldsymbol{U}}_{im}$ and $\tilde{\boldsymbol{e}}_{im} : \tilde{\boldsymbol{U}}_{im} \to \tilde{\boldsymbol{U}}_{mi}$ are quasiinverse, so both are equivalences. Define $\tilde{\boldsymbol{U}}_{mm} = \tilde{\boldsymbol{X}}_m$ and $\tilde{\boldsymbol{e}}_{mm} = \mathbf{id}_{\tilde{\boldsymbol{X}}_m}$ as in (a). We now claim that the data $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{m-1}, \tilde{\boldsymbol{X}}_m$, $\boldsymbol{U}_{ij}, \boldsymbol{e}_{ij}$ for $i,j = 1, \ldots, m-1$, $\tilde{\boldsymbol{U}}_{im}, \tilde{\boldsymbol{U}}_{mi}, \tilde{\boldsymbol{e}}_{im}, \tilde{\boldsymbol{e}}_{mi}$ for $i = 1, \ldots, m-1$ and $\tilde{\boldsymbol{U}}_{mm}, \tilde{\boldsymbol{e}}_{mm}$ satisfies the hypotheses of the theorem with $n = m$. The only nontrivial thing to check is that (b) holds when exactly one of i,j,k are m, and one can prove this by gluing together 2-morphisms on the parts coming from $\boldsymbol{X}_m, \boldsymbol{X}_{m+1}$ using a partition of unity argument, in a similar way to Proposition 2.27.

By induction, we now get a d-space $\boldsymbol{Y}$ and 1-morphisms $\boldsymbol{f}_i : \boldsymbol{X}_i \xrightarrow{\sim} \hat{\boldsymbol{X}}_i \subseteq \boldsymbol{Y}$ for $i = 1, \ldots, m-1$ and $\tilde{\boldsymbol{f}}_m : \tilde{\boldsymbol{X}}_m \to \boldsymbol{Y}$. Set $\boldsymbol{f}_m = \tilde{\boldsymbol{f}}_m \circ \boldsymbol{j}_m$, $\boldsymbol{f}_{m+1} = \tilde{\boldsymbol{f}}_m \circ \boldsymbol{j}_{m+1}$, $\hat{\boldsymbol{X}}_m = \tilde{\boldsymbol{f}}_m(\dot{\boldsymbol{X}}_m) = \boldsymbol{f}_m(\boldsymbol{X}_m)$ and $\hat{\boldsymbol{X}}_{m+1} = \tilde{\boldsymbol{f}}_m(\dot{\boldsymbol{X}}_{m+1}) = \boldsymbol{f}_m(\boldsymbol{X}_{m+1})$. These satisfy the conclusions of the first part of the theorem, proving the first part of the inductive step. For the second part of the inductive step, given $\boldsymbol{Z}$ and $\boldsymbol{g}_i : \boldsymbol{X}_i \to \boldsymbol{Z}$ for $i = 1, \ldots, m+1$, we first apply Proposition 2.27 to obtain $\tilde{\boldsymbol{g}}_m : \tilde{\boldsymbol{X}}_m \to \boldsymbol{Z}$ with $\tilde{\boldsymbol{g}}_m \circ \boldsymbol{j}_m \cong \boldsymbol{g}_m$ and $\tilde{\boldsymbol{g}}_m \circ \boldsymbol{j}_{m+1} \cong \boldsymbol{g}_{m+1}$. Then we apply the second part of the theorem with $n = m$ to $\boldsymbol{Z}$ and $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_{m-1}, \tilde{\boldsymbol{g}}_m$ to get $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$, which satisfies the conditions we need. This completes the inductive step. That $\boldsymbol{Y}, \boldsymbol{h}$ are independent of choices of 2-morphisms η_{ijk}, ζ_{ij} holds by induction and the final parts of Proposition 2.27 and Theorem 2.29. $\square$

We can also generalize Theorem 2.30 to gluing infinitely many d-spaces $\{\boldsymbol{X}_i : i \in I\}$, provided only finitely many $\boldsymbol{X}_i$ intersect near any point in $Y = (\coprod_{i \in I} X_i)/\sim$. The assumption that I be countable is necessary, since otherwise $\underline{Y}$ would not be second countable, so $\boldsymbol{Y}$ would not be a d-space.

Theorem 2.31. *Suppose I is a countable indexing set, and $\boldsymbol{X}_i$ for $i \in I$ are d-spaces, and for all $i,j \in I$ we are given an open d-subspace $\boldsymbol{U}_{ij} \subseteq \boldsymbol{X}_i$ and an equivalence $\boldsymbol{e}_{ij} : \boldsymbol{U}_{ij} \to \boldsymbol{U}_{ji}$, satisfying the following properties:*

(a) *We have $\boldsymbol{U}_{ii} = \boldsymbol{X}_i$ and $\boldsymbol{e}_{ii} = \mathbf{id}_{\boldsymbol{X}_i}$ for all $i \in I$;*

(b) *For all $i,j,k \in I$ we have a 2-commutative diagram*

$$\begin{tikzcd} & \boldsymbol{U}_{ji} \cap \boldsymbol{U}_{jk} \arrow[dr, "\boldsymbol{e}_{jk}|_{\boldsymbol{U}_{ji}\cap \boldsymbol{U}_{jk}}"] \arrow[d, phantom, "\Downarrow \eta_{ijk}"] & \\ \boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} \arrow[ur, "\boldsymbol{e}_{ij}|_{\boldsymbol{U}_{ij}\cap \boldsymbol{U}_{ik}}"] \arrow[rr, "\boldsymbol{e}_{ik}|_{\boldsymbol{U}_{ij}\cap \boldsymbol{U}_{ik}}"] & & \boldsymbol{U}_{ki} \cap \boldsymbol{U}_{kj} \end{tikzcd}$$

for some η_{ijk}, where all three 1-morphisms are equivalences; and

(c) *For all $i \in I$ and $x \in \boldsymbol{X}_i$, there exists an open neighbourhood $\boldsymbol{V}$ of x in $\boldsymbol{X}_i$ such that $\boldsymbol{V} \cap \boldsymbol{U}_{ij} \neq \emptyset$ for only finitely many $j \in I$.*

On the level of topological spaces, define the quotient topological space $Y = (\coprod_{i\in I} X_i)/\sim$, where $\sim$ is the equivalence relation generated by $x_i \sim x_j$ if $x_i \in U_{ij} \subseteq X_i$ and $x_j \in U_{ji} \subseteq X_j$ with $e_{ij}(x_i) = x_j$. Suppose Y is Hausdorff. Then there exist a d-space $\boldsymbol{Y}$ and a 1-morphism $\boldsymbol{f}_i : \boldsymbol{X}_i \to \boldsymbol{Y}$ which is an equivalence with an open d-subspace $\hat{\boldsymbol{X}}_i \subseteq \boldsymbol{Y}$ for all $i \in I$, where $\boldsymbol{Y} = \bigcup_{i\in I} \hat{\boldsymbol{X}}_i$, such that $\boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$ is an equivalence $\boldsymbol{U}_{ij} \to \hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j$ for all $i,j \in I$, and there exists a 2-morphism $\eta_{ij} : \boldsymbol{f}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$. The d-space $\boldsymbol{Y}$ is unique up to equivalence, and is independent of choice of 2-morphisms η_{ijk} in (b).

Suppose also that $\boldsymbol{Z}$ is a d-space, and $\boldsymbol{g}_i : \boldsymbol{X}_i \to \boldsymbol{Z}$ are 1-morphisms for all $i \in I$, and there exist 2-morphisms $\zeta_{ij} : \boldsymbol{g}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{g}_i|_{\boldsymbol{U}_{ij}}$ for all $i,j \in I$. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ and 2-morphisms $\zeta_i : \boldsymbol{h} \circ \boldsymbol{f}_i \Rightarrow \boldsymbol{g}_i$ for all $i \in I$. The 1-morphism $\boldsymbol{h}$ is unique up to 2-isomorphism, and is independent of the choice of 2-morphisms ζ_{ij}.

Proof. When I is finite the theorem reduces to Theorem 2.31 with $n = |I|$, so suppose I is countably infinite. Then we may identify $I = \mathbb{N} = \{1, 2, \ldots\}$. Form the topological space $Y = (\coprod_{i=1}^{\infty} X_i)/\sim$, with projections $f_i : X_i \to Y$ which are homeomorphisms with open subsets $\hat{X}_i \subset Y$ for $i = 1, 2, \ldots$. Then Y is Hausdorff by assumption. Also, $\hat{X}_i \cong X_i$ is second countable (has a countable basis for its topology) as $\boldsymbol{X}_i$ is a d-space. Thus Y is a countable union of second countable open sets $\hat{X}_i$, so Y is second countable.

The gluing of topological spaces also works at the level of C^∞-schemes. That is, we may define a natural C^∞-scheme $\underline{Y} = (\coprod_{i\in I} \underline{X}_i)/\sim$ with underlying topological space Y, and morphisms $\underline{f}_i : \underline{X}_i \to \underline{Y}$ for $i \in I$ which are isomorphisms with open C^∞-subschemes $\hat{\underline{X}}_i$ in $\underline{Y}$, such that $\underline{Y} = \bigcup_{i\in I} \hat{\underline{X}}_i$, and $\underline{f}_j \circ \underline{e}_{ij} = \underline{f}_i|_{\underline{U}_{ij}}$ for all $i,j \in I$. As Y is Hausdorff and second countable, and each $\underline{X}_i$ is locally fair, $\underline{Y}$ is separated, second countable, and locally fair.

The idea of the proof is to take the limit $n \to \infty$ in Theorem 2.30. For the first part, for each $n = 1, 2, \ldots$ we may apply Theorem 2.30 to the data $\boldsymbol{X}_i, \boldsymbol{U}_{ij}, \boldsymbol{e}_{ij}$ for $i,j \leqslant n$ to get a d-space $\boldsymbol{Y}^n$ and 1-morphisms $\boldsymbol{f}_i^n : \boldsymbol{X}_i \to \boldsymbol{Y}^n$ for $i = 1, \ldots, n$, with $\boldsymbol{f}_j^n \circ \boldsymbol{e}_{ij} \cong \boldsymbol{f}_i^n|_{\boldsymbol{U}_{ij}}$. On the level of C^∞-schemes we may take $\underline{Y}^n = \hat{\underline{X}}_1 \cup \cdots \cup \hat{\underline{X}}_n$, as a C^∞-open subscheme of $\underline{Y}$. Also, in the inductive step of Theorem 2.30, $\boldsymbol{Y}^{n+1}$ is in effect constructed from $\boldsymbol{Y}^n$ by gluing in $\boldsymbol{X}_{n+1}$, where the gluing was done using Theorem 2.28. This gluing does not change $\boldsymbol{Y}^n, \boldsymbol{X}_{n+1}$ outside their intersection. Hence $\boldsymbol{Y}^n, \boldsymbol{Y}^{n+1}$ are 1-isomorphic over $(\hat{X}_1 \cup \cdots \cup \hat{X}_n) \setminus \hat{X}_{n+1}$, and we can take them to be actually equal.

Let $y \in \underline{Y}$. As $\underline{Y} = \bigcup_{i=1}^{\infty} \hat{\underline{X}}_i$ we have $y \in \hat{\underline{X}}_i$ for some i, so $y = \underline{f}_i(x)$ for $x \in \underline{X}_i$. Part (c) then gives an open $x \in \underline{V} \subseteq \underline{X}_i$ such that $\underline{V} \cap \underline{U}_{ij} \neq \emptyset$ for only finitely many $j = 1, 2, \ldots$. Let N_y be the maximum of these j, and set $\hat{\underline{V}} = \underline{f}_i(\underline{V})$. Then $\hat{\underline{V}}$ is open in $\underline{Y}$, and $y \in \hat{\underline{V}} \subseteq \hat{\underline{X}}_1 \cup \cdots \cup \hat{\underline{X}}_{N_y}$, and $\hat{\underline{X}}_n \cap \hat{\underline{V}} = \emptyset$ for all $n > N_y$. Thus $\boldsymbol{Y}^n$ is independent of n over $y \in \hat{\underline{V}} \subseteq \underline{Y}$ for all $n > N_y$.

We can now construct a d-space $\boldsymbol{Y} = (\underline{Y}, \mathcal{O}_Y', \mathcal{E}_Y, \imath_Y, \jmath_Y)$ by taking the limit $n \to \infty$ in $\boldsymbol{Y}^n$. Near each $y \in \underline{Y}$ the data $\mathcal{O}_Y'^n, \mathcal{E}_Y^n, \imath_Y^n, \jmath_Y^n$ become independent of n for $n \gg 0$. Formally, we define presheaves $\mathcal{PO}'_Y, \mathcal{PE}_Y$ and morphisms of presheaves $\mathcal{P}\imath_Y, \mathcal{P}\jmath_Y$ over Y using the limiting data over $\underline{\hat{V}} \ni y$ as above for each $y \in \underline{Y}$, and then $\mathcal{O}'_Y, \mathcal{E}_Y, \imath_Y, \jmath_Y$ are the sheafifications of these presheaves and morphisms. By similar limiting arguments as $n \to \infty$ in Theorem 2.30 we obtain $\boldsymbol{f}_i : \boldsymbol{X}_i \to \boldsymbol{Y}$, $\eta_{ij} : \boldsymbol{f}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$ for $i, j = 1, 2, \ldots$ satisfying the first part of the theorem, and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$, $\zeta_i : \boldsymbol{h} \circ \boldsymbol{f}_i \Rightarrow \boldsymbol{g}_i$ for $i = 1, 2, \ldots$ satisfying the second part. This completes the proof. $\square$

We can eliminate the assumptions that I is countable and the $\boldsymbol{X}_i$ have locally finite intersections if we suppose that Y is second countable.

Theorem 2.32. *Suppose $\boldsymbol{X}_i$ for $i \in I$ are d-spaces, and for all $i, j \in I$ we are given an open d-subspace $\boldsymbol{U}_{ij} \subseteq \boldsymbol{X}_i$ and an equivalence $\boldsymbol{e}_{ij} : \boldsymbol{U}_{ij} \to \boldsymbol{U}_{ji}$, satisfying the following properties:*

(a) *We have $\boldsymbol{U}_{ii} = \boldsymbol{X}_i$ and $\boldsymbol{e}_{ii} = \mathbf{id}_{\boldsymbol{X}_i}$ for all $i \in I$; and*

(b) *For all $i, j, k \in I$ we have a 2-commutative diagram*

$$\begin{tikzcd}[column sep=small]
& \boldsymbol{U}_{ji} \cap \boldsymbol{U}_{jk} \arrow[dr, "\boldsymbol{e}_{jk}|_{\boldsymbol{U}_{ji} \cap \boldsymbol{U}_{jk}}"] \arrow[d, phantom, "\Downarrow \eta_{ijk}"] & \\
\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} \arrow[ur, "\boldsymbol{e}_{ij}|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik}}"] \arrow[rr, "\boldsymbol{e}_{ik}|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik}}"] & & \boldsymbol{U}_{ki} \cap \boldsymbol{U}_{kj}
\end{tikzcd}$$

for some η_{ijk}, where all three 1-morphisms are equivalences.

On the level of topological spaces, define the quotient topological space $Y = (\coprod_{i \in I} X_i)/\sim$, where $\sim$ is the equivalence relation generated by $x_i \sim x_j$ if $x_i \in U_{ij} \subseteq X_i$ and $x_j \in U_{ji} \subseteq X_j$ with $e_{ij}(x_i) = x_j$. Suppose Y is Hausdorff and second countable. Then there exist a d-space $\boldsymbol{Y}$ and a 1-morphism $\boldsymbol{f}_i : \boldsymbol{X}_i \to \boldsymbol{Y}$ which is an equivalence with an open d-subspace $\hat{\boldsymbol{X}}_i \subseteq \boldsymbol{Y}$ for all $i \in I$, where $\boldsymbol{Y} = \bigcup_{i \in I} \hat{\boldsymbol{X}}_i$, such that $\boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$ is an equivalence $\boldsymbol{U}_{ij} \to \hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j$ for all $i, j \in I$, and there exists a 2-morphism $\eta_{ij} : \boldsymbol{f}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$. The d-space $\boldsymbol{Y}$ is unique up to equivalence, and is independent of choice of 2-morphisms η_{ijk} in (b).

Suppose also that $\boldsymbol{Z}$ is a d-space, and $\boldsymbol{g}_i : \boldsymbol{X}_i \to \boldsymbol{Z}$ are 1-morphisms for all $i \in I$, and there exist 2-morphisms $\zeta_{ij} : \boldsymbol{g}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{g}_i|_{\boldsymbol{U}_{ij}}$ for all $i, j \in I$. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ and 2-morphisms $\zeta_i : \boldsymbol{h} \circ \boldsymbol{f}_i \Rightarrow \boldsymbol{g}_i$ for all $i \in I$. The 1-morphism $\boldsymbol{h}$ is unique up to 2-isomorphism, and is independent of the choice of 2-morphisms ζ_{ij}.

Proof. As the $\boldsymbol{X}_i$ are d-spaces, the $\underline{X}_i$ are locally fair, so the X_i are locally compact. Hence Y is locally compact, and Hausdorff and second countable by assumption. These imply that Y is paracompact. Write $\hat{X}_i$ for the open subset of Y corresponding to X_i, and $f_i : X_i \to \hat{X}_i$ for the natural homeomorphism. Then $\{\hat{X}_i : i \in I\}$ is an open cover of Y, so as Y is paracompact it has a locally finite refinement. That is, there exists $K \subseteq I$ and an open subset $\emptyset \neq \hat{W}_k \subseteq \hat{X}_k$ for $k \in K$ such that $\{\hat{W}_k : k \in K\}$ is a locally finite open cover of Y. As Y is second countable, K is countable.

Let $W_k = f_k^{-1}(\hat{W}_k) \subset X_k$ for $k \in K$, and $\boldsymbol{W}_k \subseteq \boldsymbol{X}_k$ be the corresponding open d-subspace. For $k, l \in K$, set $\boldsymbol{T}_{kl} = \boldsymbol{U}_{kl} \cap \boldsymbol{W}_k \cap \boldsymbol{e}_{kl}^{-1}(\boldsymbol{W}_l)$ and $\boldsymbol{d}_{kl} = \boldsymbol{e}_{kl}|_{\boldsymbol{T}_{kl}}$, so that $\boldsymbol{T}_{kl} \subseteq \boldsymbol{U}_{kl}$ is an open d-subspace and $\boldsymbol{d}_{kl} : \boldsymbol{T}_{kl} \to \boldsymbol{T}_{lk}$ an equivalence. Then $\boldsymbol{W}_k$, $k \in K$, $\boldsymbol{T}_{kl}, \boldsymbol{d}_{kl}$, $k, l \in K$ satisfy the hypotheses of Theorem 2.31, noting that K is countable, and (c) holds as $\{\hat{W}_k : k \in K\}$ is locally finite. Thus the first part of Theorem 2.31 gives a d-space $\boldsymbol{Y}$ and 1-morphisms $\tilde{\boldsymbol{f}}_k : \boldsymbol{W}_k \to \boldsymbol{Y}$ for $k \in K$ which are equivalences with $\hat{\boldsymbol{W}}_k \subseteq \boldsymbol{Y}$, such that $\boldsymbol{Y} = \bigcup_{k \in K} \hat{\boldsymbol{W}}_k$, and $\tilde{\boldsymbol{f}}_k|_{\boldsymbol{T}_{kl}}$ is an equivalence $\boldsymbol{T}_{kl} \to \hat{\boldsymbol{W}}_k \cap \hat{\boldsymbol{W}}_l$ for all $k, l \in K$, and $\tilde{\boldsymbol{f}}_l \circ \boldsymbol{d}_{kl} \cong \tilde{\boldsymbol{f}}_k|_{\boldsymbol{T}_{kl}}$, where '$\cong$' denotes 2-isomorphic.

Regard these $\boldsymbol{Y}$ and $\tilde{\boldsymbol{f}}_k$, $k \in K$ as fixed for the rest of the proof. Let $i \in I$. To construct $\boldsymbol{f}_i : \boldsymbol{X}_i \to \boldsymbol{Y}$, we apply the first part of Theorem 2.31 to the data $\boldsymbol{W}_k$, $k \in K$, $\boldsymbol{T}_{kl}, \boldsymbol{d}_{kl}$, $k, l \in K$, together with the one extra space $\boldsymbol{X}_i$, and extra overlap data $\tilde{\boldsymbol{U}}_{ik}, \tilde{\boldsymbol{U}}_{ki}, \tilde{\boldsymbol{e}}_{ik}, \tilde{\boldsymbol{e}}_{ki}$ for $k \in K$, where we define $\tilde{\boldsymbol{U}}_{ik} \subseteq \boldsymbol{X}_i$, $\tilde{\boldsymbol{U}}_{ki} \subseteq \boldsymbol{W}_k$, $\tilde{\boldsymbol{e}}_{ik} : \tilde{\boldsymbol{U}}_{ik} \to \tilde{\boldsymbol{U}}_{ki}$, $\tilde{\boldsymbol{e}}_{ki} : \tilde{\boldsymbol{U}}_{ki} \to \tilde{\boldsymbol{U}}_{ik}$ by $\tilde{\boldsymbol{U}}_{ik} = \boldsymbol{U}_{ik} \cap \boldsymbol{e}_{ik}^{-1}(\boldsymbol{W}_k)$, $\tilde{\boldsymbol{U}}_{ki} = \boldsymbol{U}_{ki} \cap \boldsymbol{W}_k$, $\tilde{\boldsymbol{e}}_{ik} = \boldsymbol{e}_{ik}|_{\tilde{\boldsymbol{U}}_{ik}}$ and $\tilde{\boldsymbol{e}}_{ki} = \boldsymbol{e}_{ki}|_{\tilde{\boldsymbol{U}}_{ki}}$. As we are adding only one extra space, the indexing set $K \amalg \{i\}$ is countable, and condition (c) still holds.

Then the first part of Theorem 2.31 yields a d-space $\boldsymbol{Y}^i$, 1-morphisms $\tilde{\boldsymbol{f}}_k^i : \boldsymbol{W}_k \to \boldsymbol{Y}^i$ for $k \in K$, and a 1-morphism $\boldsymbol{f}_i : \boldsymbol{X}_i \to \boldsymbol{Y}^i$. But forgetting $\boldsymbol{X}_i, \tilde{\boldsymbol{U}}_{ik}, \ldots, \boldsymbol{f}_i$, these $\boldsymbol{Y}^i, \tilde{\boldsymbol{f}}_k^i$ satisfy the conclusions of Theorem 2.31 for the data $\boldsymbol{W}_k$, $k \in K$, $\boldsymbol{T}_{kl}, \boldsymbol{d}_{kl}$, $k, l \in K$. Thus by uniqueness of $\boldsymbol{Y}$ up to equivalence, we may take $\boldsymbol{Y}^i = \boldsymbol{Y}$, and $\tilde{\boldsymbol{f}}_k^i = \tilde{\boldsymbol{f}}_k$ for $k \in K$. Hence we have constructed a 1-morphism $\boldsymbol{f}_i : \boldsymbol{X}_i \to \boldsymbol{Y}$ for each $i \in I$, which is an equivalence with an open d-subspace $\hat{\boldsymbol{X}}_i \subseteq \boldsymbol{Y}$. Regard these $\boldsymbol{f}_i, \hat{\boldsymbol{X}}_i$ as fixed for the rest of the proof.

Let $i, j \in I$. To prove pairwise compatibility between $\boldsymbol{f}_i$ and $\boldsymbol{f}_j$, we apply the first part of Theorem 2.31 to the data $\boldsymbol{W}_k$, $k \in K$, $\boldsymbol{T}_{kl}, \boldsymbol{d}_{kl}$, $k, l \in K$, together with the two extra spaces $\boldsymbol{X}_i, \boldsymbol{X}_j$, and corresponding overlap data including $\boldsymbol{U}_{ij}, \boldsymbol{U}_{ji}, \boldsymbol{e}_{ij}$. As above we can take the resulting d-space to be $\boldsymbol{Y}$, and the 1-morphisms to be $\tilde{\boldsymbol{f}}_k : \boldsymbol{W}_k \to \boldsymbol{Y}$ and $\boldsymbol{f}_i : \boldsymbol{X}_i \to \boldsymbol{Y}$, $\boldsymbol{f}_j : \boldsymbol{X}_j \to \boldsymbol{Y}$. Thus Theorem 2.31 shows that $\boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$ is an equivalence $\boldsymbol{U}_{ij} \to \hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j$, and $\boldsymbol{f}_j \circ \boldsymbol{e}_{ij} \cong \boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$. This completes the proof of the first part.

For the second part, suppose $\boldsymbol{Z}$ and $\boldsymbol{g}_i : \boldsymbol{X}_i \to \boldsymbol{Z}$ for $i \in I$ are as in the theorem. Then applying the second part of Theorem 2.31 with data $\boldsymbol{W}_k$, $k \in K$, $\boldsymbol{T}_{kl}, \boldsymbol{d}_{kl}$, $k, l \in K$, and 1-morphisms $\boldsymbol{g}_k|_{\boldsymbol{W}_k} : \boldsymbol{W}_k \to \boldsymbol{Z}$ for $k \in K$, gives a 1-morphism $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$, unique up to 2-isomorphism, such that $\boldsymbol{h} \circ \tilde{\boldsymbol{f}}_k \cong \boldsymbol{g}_k|_{\boldsymbol{W}_k}$ for all $k \in K$. Regard this $\boldsymbol{h}$ as fixed for the rest of the proof.

Fix $i \in I$ and apply the second part of Theorem 2.31 with data $\boldsymbol{W}_k$, $k \in K$, $\boldsymbol{T}_{kl}, \boldsymbol{d}_{kl}$, $k, l \in K$, together with one extra space $\boldsymbol{X}_i$, and extra overlap data $\tilde{\boldsymbol{U}}_{ik}, \tilde{\boldsymbol{U}}_{ki}, \tilde{\boldsymbol{e}}_{ik}, \tilde{\boldsymbol{e}}_{ki}$ for $k \in K$, and 1-morphisms $\boldsymbol{g}_k|_{\boldsymbol{W}_k} : \boldsymbol{W}_k \to \boldsymbol{Z}$ for $k \in K$ and $\boldsymbol{g}_i : \boldsymbol{X}_i \to \boldsymbol{Z}$. This yields a 1-morphism $\boldsymbol{h}^i : \boldsymbol{Y} \to \boldsymbol{Z}$. But forgetting $\boldsymbol{X}_i, \tilde{\boldsymbol{U}}_{ik}, \ldots, \boldsymbol{g}_i$, this $\boldsymbol{h}^i$ satisfies the conclusions of Theorem 2.31 with data $\boldsymbol{W}_k, \boldsymbol{T}_{kl}, \boldsymbol{d}_{kl}, \boldsymbol{g}_k|_{\boldsymbol{W}_k}$. Thus by uniqueness of $\boldsymbol{h}$ up to 2-isomorphism we may take $\boldsymbol{h}^i = \boldsymbol{h}$. Hence $\boldsymbol{h} \circ \boldsymbol{f}_i \cong \boldsymbol{g}_i$. This completes the second part. □

Some of the data in Theorem 2.32 is redundant: as $\boldsymbol{e}_{ij} : \boldsymbol{U}_{ij} \to \boldsymbol{U}_{ji}$ and $\boldsymbol{e}_{ji} : \boldsymbol{U}_{ji} \to \boldsymbol{U}_{ij}$ are quasi-inverse, $\boldsymbol{e}_{ij}$ determines $\boldsymbol{e}_{ij}$ up to 2-isomorphism, so we need remember only one of $\boldsymbol{e}_{ij}$ and $\boldsymbol{e}_{ij}$.

Theorem 2.33. *Suppose I is an indexing set, and $<$ is a total order on I, and*

$\boldsymbol{X}_i$ *for* $i \in I$ *are d-spaces, and for all* $i < j$ *in* I *we are given open d-subspaces* $\boldsymbol{U}_{ij} \subseteq \boldsymbol{X}_i$, $\boldsymbol{U}_{ji} \subseteq \boldsymbol{X}_j$ *and an equivalence* $\boldsymbol{e}_{ij} : \boldsymbol{U}_{ij} \to \boldsymbol{U}_{ji}$, *such that for all* $i < j < k$ *in* I *we have a 2-commutative diagram*

$$\begin{tikzcd} & \boldsymbol{U}_{ji} \cap \boldsymbol{U}_{jk} \arrow[dr, "\boldsymbol{e}_{jk}|_{\boldsymbol{U}_{ji} \cap \boldsymbol{U}_{jk}}"] \arrow[d, phantom, "\Downarrow \eta_{ijk}"] & \\ \boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} \arrow[ur, "\boldsymbol{e}_{ij}|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik}}"] \arrow[rr, "\boldsymbol{e}_{ik}|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik}}"] & & \boldsymbol{U}_{ki} \cap \boldsymbol{U}_{kj} \end{tikzcd}$$

for some η_{ijk}, *where all three* 1*-morphisms are equivalences.*

On the level of topological spaces, define the quotient topological space $Y = (\coprod_{i \in I} X_i)/\sim$, *where* $\sim$ *is the equivalence relation generated by* $x_i \sim x_j$ *if* $i < j$, $x_i \in U_{ij} \subseteq X_i$ *and* $x_j \in U_{ji} \subseteq X_j$ *with* $e_{ij}(x_i) = x_j$. *Suppose* Y *is Hausdorff and second countable. Then there exist a d-space* $\boldsymbol{Y}$ *and a* 1*-morphism* $\boldsymbol{f}_i : \boldsymbol{X}_i \to \boldsymbol{Y}$ *which is an equivalence with an open d-subspace* $\hat{\boldsymbol{X}}_i \subseteq \boldsymbol{Y}$ *for all* $i \in I$, *where* $\boldsymbol{Y} = \bigcup_{i \in I} \hat{\boldsymbol{X}}_i$, *such that* $\boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$ *is an equivalence* $\boldsymbol{U}_{ij} \to \hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j$ *for all* $i < j$ *in* I, *and there exists a* 2*-morphism* $\eta_{ij} : \boldsymbol{f}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$. *The d-space* $\boldsymbol{Y}$ *is unique up to equivalence, and is independent of choice of* 2*-morphisms* η_{ijk}.

Suppose also that $\boldsymbol{Z}$ *is a d-space, and* $\boldsymbol{g}_i : \boldsymbol{X}_i \to \boldsymbol{Z}$ *are* 1*-morphisms for all* $i \in I$, *and there exist* 2*-morphisms* $\zeta_{ij} : \boldsymbol{g}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{g}_i|_{\boldsymbol{U}_{ij}}$ *for all* $i < j$ *in* I. *Then there exist a* 1*-morphism* $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *and* 2*-morphisms* $\zeta_i : \boldsymbol{h} \circ \boldsymbol{f}_i \Rightarrow \boldsymbol{g}_i$ *for all* $i \in I$. *The* 1*-morphism* $\boldsymbol{h}$ *is unique up to* 2*-isomorphism, and is independent of the choice of* 2*-morphisms* ζ_{ij}.

Proof. If $i < j$ in I then $\boldsymbol{e}_{ij} : \boldsymbol{U}_{ij} \to \boldsymbol{U}_{ji}$ is an equivalence, so we can choose a quasi-inverse $\boldsymbol{e}_{ji} : \boldsymbol{U}_{ji} \to \boldsymbol{U}_{ij}$, and 2-morphisms $\theta_{ij} : \boldsymbol{e}_{ji} \circ \boldsymbol{e}_{ij} \Rightarrow \mathbf{id}_{\boldsymbol{U}_{ij}}$, $\theta_{ji} : \boldsymbol{e}_{ij} \circ \boldsymbol{e}_{ji} \Rightarrow \mathbf{id}_{\boldsymbol{U}_{ji}}$. Define $\boldsymbol{U}_{ii} = \boldsymbol{X}_i$ and $\boldsymbol{e}_{ii} = \mathbf{id}_{\boldsymbol{X}_i}$ for all $i \in I$ as in Theorem 2.32(a). We are given 2-morphisms η_{ijk} whenever $i < j < k$ in I. Using these and the θ_{ij}, θ_{ji} we can define 2-morphisms η_{ijk} for all $i, j, k \in I$, such that the hypotheses of the first part of Theorem 2.32 hold. The first part of the theorem then follows from Theorem 2.32. The second part is similar. $\square$

We can simplify Theorem 2.33, making it more restrictive, by taking all the η_{ijk} and ζ_{ij} to be identities. That is, we assume that $\boldsymbol{e}_{ik}|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik}} = \boldsymbol{e}_{jk}|_{\boldsymbol{U}_{ji} \cap \boldsymbol{U}_{jk}} \circ \boldsymbol{e}_{ij}|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik}}$ for all $i < j < k$, and $\boldsymbol{g}_j \circ \boldsymbol{e}_{ij} = \boldsymbol{g}_i|_{\boldsymbol{U}_{ij}}$ for all $i < j$. This would not be a useful modification to Theorem 2.32, as it would force the $\boldsymbol{e}_{ij}$ to be 1-isomorphisms rather than equivalences.

Remark 2.34. In Theorems 2.30–2.33, the compatibility conditions on the gluing data $\boldsymbol{X}_i, \boldsymbol{U}_{ij}, \boldsymbol{e}_{ij}$ are significantly weaker than you might expect. In each theorem, the 2-morphisms η_{ijk} on triple overlaps $\boldsymbol{X}_i \cap \boldsymbol{X}_j \cap \boldsymbol{X}_k$ are only required to exist, not to satisfy any further conditions. In particular, one might think that on quadruple overlaps $\boldsymbol{X}_i \cap \boldsymbol{X}_j \cap \boldsymbol{X}_k \cap \boldsymbol{X}_l$ we should require

$$\eta_{ikl} \odot (\mathrm{id}_{\boldsymbol{f}_{kl}} * \eta_{ijk})|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} \cap \boldsymbol{U}_{il}} = \eta_{ijl} \odot (\eta_{jkl} * \mathrm{id}_{\boldsymbol{f}_{ij}})|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} \cap \boldsymbol{U}_{il}},$$

but we do not. This is a consequence of $\boldsymbol{b}$ being independent of $\tilde{\eta}$ in Theorem 2.29. Similarly, the 2-morphisms ζ_{ij} are only required to exist, though you might expect that on triple overlaps $\boldsymbol{X}_i \cap \boldsymbol{X}_j \cap \boldsymbol{X}_k$ we should require

$$\big(\zeta_{ij}|_{\underline{U}_{ij}\cap\underline{U}_{ik}}\big) \odot \big(\zeta_{jk} * \mathrm{id}_{\boldsymbol{e}_{ij}}|_{\underline{U}_{ij}\cap\underline{U}_{ik}}\big) = \big(\zeta_{ik}|_{\underline{U}_{ij}\cap\underline{U}_{ik}}\big) \odot \big(\mathrm{id}_{\boldsymbol{g}_k} * \eta_{ijk}|_{\underline{U}_{ij}\cap\underline{U}_{ik}}\big).$$

As we explain in §4.7, these weak conditions mean that Theorems 2.30–2.33 can be stated in the homotopy category of d-spaces Ho(**dSpa**).

In contrast, in the analogous result Theorem 9.19 for gluing d-stacks by equivalences, we need compatibility conditions for the η_{ijk} on quadruple overlaps $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \cap \boldsymbol{\mathcal{X}}_l$, and for the ζ_{ij} on triple overlaps $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k$. This is what you would expect for gluing geometric spaces in a 2-category by equivalences. We give some sufficient conditions for these compatibility conditions to hold automatically, using a uniqueness result for 2-morphisms.

The problem of gluing geometric spaces in an ∞-category $\boldsymbol{\mathcal{C}}$ by equivalences, such as Spivak's derived manifolds [95], is discussed by Toën and Vezzosi [101, §1.3.4] and Lurie [70, §6.1.2]. One sets the problem up as finding a homotopy colimit of a Segal groupoid or simplicial object in $\boldsymbol{\mathcal{C}}$. It is complicated, and requires nontrivial conditions on overlaps $\boldsymbol{X}_{i_1} \cap \cdots \cap \boldsymbol{X}_{i_n}$ for all $n = 2, 3, \ldots$.

2.5 Fibre products of d-spaces

In the next definition we write down an explicit construction of fibre products in **dSpa**. See §A.4 for an explanation of fibre products in 2-categories. Theorem 2.36 proves our construction has the universal property required.

Definition 2.35. Let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms of d-spaces. We will define a d-space $\boldsymbol{W} = (\underline{W}, \mathcal{O}'_W, \mathcal{E}_W, \imath_W, \jmath_W)$, and 1-morphisms $\boldsymbol{e} = (\underline{e}, e', e'') : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} = (\underline{f}, f', f'') : \boldsymbol{W} \to \boldsymbol{Y}$, and a 2-morphism $\eta : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$, such that $\boldsymbol{W}, \boldsymbol{e}, \boldsymbol{f}, \eta$ are a fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ in **dSpa**, in the sense of Definition A.7. That is, we have a 2-Cartesian square in **dSpa**:

$$\begin{tikzcd} \boldsymbol{W} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"'] \arrow[dr, phantom, "\eta\Uparrow"] & \boldsymbol{Y} \arrow[d, "\boldsymbol{h}"] \\ \boldsymbol{X} \arrow[r, "\boldsymbol{g}"] & \boldsymbol{Z}. \end{tikzcd}$$

Define $\underline{W} = \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y}$ to be the fibre product in $\mathbf{C^\infty Sch}$. Then $\underline{W}$ is separated, second countable and locally fair by Theorem B.19(b), as $\underline{X}, \underline{Y}, \underline{Z}$ are. Write $\underline{e} : \underline{W} \to \underline{X}$ and $\underline{f} : \underline{W} \to \underline{Y}$ for the projections. Then $\underline{g} \circ \underline{e} = \underline{h} \circ \underline{f}$.

To define $\mathcal{O}'_W, \mathcal{E}_W, \imath_W, \jmath_W, e', e'', f', f'', \eta$ we must first discuss tensor products of sheaves of C^∞-rings. Proposition B.4 and Theorem B.19 imply that products $\underline{X} \times \underline{Y}$ of C^∞-schemes $\underline{X}, \underline{Y}$ exist in $\mathbf{C^\infty Sch}$, and dually, coproducts of C^∞-rings $\mathfrak{C}, \mathfrak{D}$ exist in $\mathbf{C^\infty Rings}$. We will write the coproduct of $\mathfrak{C}, \mathfrak{D}$ in $\mathbf{C^\infty Rings}$ as $\mathfrak{C} \hat{\otimes} \mathfrak{D}$, and think of it as a completion of the tensor product $\mathfrak{C} \otimes_{\mathbb{R}} \mathfrak{D}$ of $\mathfrak{C}$ and $\mathfrak{D}$ as real vector spaces. The tensor product $\mathfrak{C} \otimes_{\mathbb{R}} \mathfrak{D}$ is an $\mathbb{R}$-algebra, but in general not a C^∞-ring. Coproducts of C^∞-rings are unique up to canonical isomorphism; there is also a natural construction for them. By definition, the coproduct $\mathfrak{C} \hat{\otimes} \mathfrak{D}$ comes with C^∞-ring morphisms $i_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \hat{\otimes} \mathfrak{D}$ and $i_{\mathfrak{D}} : \mathfrak{D} \to \mathfrak{C} \hat{\otimes} \mathfrak{D}$, and these induce a morphism of $\mathbb{R}$-algebras $\mathfrak{C} \otimes_{\mathbb{R}} \mathfrak{D} \to \mathfrak{C} \hat{\otimes} \mathfrak{D}$.

These have the universal property that if $\phi : \mathfrak{C} \to \mathfrak{E}$ and $\psi : \mathfrak{D} \to \mathfrak{E}$ are morphisms of C^∞-rings, there is a unique morphism $\phi \hat{\otimes} \psi : \mathfrak{C} \hat{\otimes} \mathfrak{D} \to \mathfrak{E}$ with $(\phi \hat{\otimes} \psi) \circ i_{\mathfrak{C}} = \phi$ and $(\phi \hat{\otimes} \psi) \circ i_{\mathfrak{D}} = \psi$. There is also a second kind of product morphism: if $\phi : \mathfrak{C} \to \mathfrak{E}$ and $\psi : \mathfrak{D} \to \mathfrak{F}$ are morphisms of C^∞-rings, there is a unique morphism $\phi \hat{\boxtimes} \psi : \mathfrak{C} \hat{\otimes} \mathfrak{D} \to \mathfrak{E} \hat{\otimes} \mathfrak{F}$ with $(\phi \hat{\boxtimes} \psi) \circ i_{\mathfrak{C}} = i_{\mathfrak{E}} \circ \phi$ and $(\phi \hat{\boxtimes} \psi) \circ i_{\mathfrak{D}} = i_{\mathfrak{F}} \circ \psi$.

All this also works at the level of sheaves of C^∞-rings. Thus, if $\mathcal{O}, \mathcal{P}$ are sheaves of C^∞-rings on a topological space X, then the (completed) *tensor product* $\mathcal{O} \hat{\otimes} \mathcal{P}$ is a sheaf of C^∞-rings on X equipped with morphisms of sheaves of C^∞-rings $i_{\mathcal{O}} : \mathcal{O} \to \mathcal{O} \hat{\otimes} \mathcal{P}$ and $i_{\mathcal{P}} : \mathcal{P} \to \mathcal{O} \hat{\otimes} \mathcal{P}$, such that if $\phi : \mathcal{O} \to \mathcal{Q}$ and $\psi : \mathcal{P} \to \mathcal{Q}$ are morphisms of sheaves of C^∞-rings on X, then there is a unique morphism $\phi \hat{\otimes} \psi : \mathcal{O} \hat{\otimes} \mathcal{P} \to \mathcal{Q}$ with $(\phi \hat{\otimes} \psi) \circ i_{\mathcal{O}} = \phi$ and $(\phi \hat{\otimes} \psi) \circ i_{\mathcal{P}} = \psi$, and if $\phi : \mathcal{O} \to \mathcal{Q}$ and $\psi : \mathcal{P} \to \mathcal{R}$ are morphisms of sheaves of C^∞-rings on X, then there is a unique morphism $\phi \hat{\boxtimes} \psi : \mathcal{O} \hat{\otimes} \mathcal{P} \to \mathcal{Q} \hat{\otimes} \mathcal{R}$ with $(\phi \hat{\boxtimes} \psi) \circ i_{\mathcal{O}} = i_{\mathcal{Q}} \circ \phi$ and $(\phi \hat{\boxtimes} \psi) \circ i_{\mathcal{P}} = i_{\mathcal{R}} \circ \psi$.

On W we have seven sheaves of C^∞-rings, namely $\mathcal{O}_W, e^{-1}(\mathcal{O}_X), f^{-1}(\mathcal{O}_Y)$, $(g \circ e)^{-1}(\mathcal{O}_Z) = (h \circ f)^{-1}(\mathcal{O}_Z), e^{-1}(\mathcal{O}'_X), e^{-1}(\mathcal{O}'_Y)$ and $(g \circ e)^{-1}(\mathcal{O}'_Z) = (h \circ f)^{-1}(\mathcal{O}'_Z)$. We also have nine morphisms of sheaves of C^∞-rings, that is

$$
\begin{gathered}
e^\sharp : e^{-1}(\mathcal{O}_X) \longrightarrow \mathcal{O}_W, \qquad\qquad f^\sharp : f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_W, \\
e^{-1}(\imath_X) : e^{-1}(\mathcal{O}'_X) \longrightarrow e^{-1}(\mathcal{O}_X), \quad f^{-1}(\imath_Y) : f^{-1}(\mathcal{O}'_Y) \longrightarrow f^{-1}(\mathcal{O}_Y), \\
e^{-1}(g^\sharp) \circ I_{e,g}(\mathcal{O}_Z) : (g \circ e)^{-1}(\mathcal{O}_Z) \longrightarrow e^{-1}(\mathcal{O}_X), \\
f^{-1}(h^\sharp) \circ I_{f,h}(\mathcal{O}_Z) : (g \circ e)^{-1}(\mathcal{O}_Z) \longrightarrow f^{-1}(\mathcal{O}_Y), \\
(g \circ e)^{-1}(\imath_Z) : (g \circ e)^{-1}(\mathcal{O}'_Z) \longrightarrow (g \circ e)^{-1}(\mathcal{O}_Z), \\
e^{-1}(g') \circ I_{e,g}(\mathcal{O}'_Z) : (g \circ e)^{-1}(\mathcal{O}'_Z) \longrightarrow e^{-1}(\mathcal{O}'_X), \\
f^{-1}(h') \circ I_{f,h}(\mathcal{O}'_Z) : (g \circ e)^{-1}(\mathcal{O}'_Z) \longrightarrow f^{-1}(\mathcal{O}'_Y).
\end{gathered}
$$

We will build $\mathcal{O}'_W, \ldots, \eta$ using tensor products of these sheaves and morphisms.

First note that the direct product $(\underline{e}, \underline{f}) : \underline{W} \to \underline{X} \times \underline{Y}$ embeds $\underline{W}$ as a C^∞-subscheme of $\underline{X} \times \underline{Y}$, so it is an injective morphism of C^∞-schemes. The morphism $(e,f)^\sharp : (e,f)^{-1}(\mathcal{O}_{X\times Y}) \to \mathcal{O}_W$ in $(\underline{e}, \underline{f})$ is naturally identified with $e^\sharp \hat{\otimes} f^\sharp : e^{-1}(\mathcal{O}_X) \hat{\otimes} f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_W$. Thus, as $(\underline{e}, \underline{f})$ is injective, $e^\sharp \hat{\otimes} f^\sharp$ is a surjective morphism of sheaves of C^∞-rings. Also $\imath_X, \imath_Y$ are surjective, so $e^{-1}(\imath_X), f^{-1}(\imath_Y)$ are surjective, and therefore $e^{-1}(\imath_X) \hat{\boxtimes} f^{-1}(\imath_Y)$ is too. Hence

$$
\bigl(e^\sharp \hat{\otimes} f^\sharp\bigr) \circ \bigl(e^{-1}(\imath_X) \hat{\boxtimes} f^{-1}(\imath_Y)\bigr) : e^{-1}(\mathcal{O}'_X) \hat{\otimes} f^{-1}(\mathcal{O}'_Y) \longrightarrow \mathcal{O}_W \tag{2.56}
$$

is a surjective morphism of sheaves of C^∞-rings on W.

Write $\mathcal{J}_W$ for the kernel of (2.56), as a sheaf of ideals in $e^{-1}(\mathcal{O}'_X) \hat{\otimes} f^{-1}(\mathcal{O}'_Y)$. Then $\mathcal{J}_W^2$ is another sheaf of ideals in $e^{-1}(\mathcal{O}'_X) \hat{\otimes} f^{-1}(\mathcal{O}'_Y)$. Define $\mathcal{O}'_W = \bigl(e^{-1}(\mathcal{O}'_X) \hat{\otimes} f^{-1}(\mathcal{O}'_Y)\bigr)/\mathcal{J}_W^2$ to be the quotient sheaf of C^∞-rings on W, with projection $\pi_{\mathcal{O}'_W} : e^{-1}(\mathcal{O}'_X) \hat{\otimes} f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_W$. Then $\pi_{\mathcal{O}'_W}$ is surjective, with kernel $\mathcal{J}_W^2$. As $\mathcal{J}_W^2 \subseteq \mathcal{J}_W$, the morphism (2.56) factors through $\pi_{\mathcal{O}'_W}$. Thus, there exists a unique morphism of sheaves of C^∞-rings $\imath_W : \mathcal{O}'_W \to \mathcal{O}_W$ such that

$$
\bigl(e^\sharp \hat{\otimes} f^\sharp\bigr) \circ \bigl(e^{-1}(\imath_X) \hat{\boxtimes} f^{-1}(\imath_Y)\bigr) = \imath_W \circ \pi_{\mathcal{O}'_W}. \tag{2.57}
$$

As (2.56) is surjective, so is $\imath_W$. The kernel of $\imath_W$ is $\mathcal{J}_W/\mathcal{J}_W^2$, which is a sheaf of square zero ideals in $\mathcal{O}'_W$. Thus, $(\mathcal{O}'_W, \imath_W)$ is a square zero extension of $\underline{W}$.

Definition 2.9 now defines quasicoherent sheaves $\mathcal{I}_W = \mathcal{J}_W/\mathcal{J}_W^2$ and $\mathcal{F}_W = \Omega_{\mathcal{O}'_W} \otimes_{\mathcal{O}'_W} \mathcal{O}_W$ on $\underline{W}$, and morphisms $\kappa_W : \mathcal{I}_W \to \mathcal{O}'_W$, $\xi_W : \mathcal{I}_W \to \mathcal{F}_W$ and $\psi_W : \mathcal{F}_W \to T^*\underline{W}$. Define $e' = \pi_{\mathcal{O}'_W} \circ i_{e^{-1}(\mathcal{O}'_X)} : e^{-1}(\mathcal{O}'_X) \to \mathcal{O}'_W$ and $f' = \pi_{\mathcal{O}'_W} \circ i_{f^{-1}(\mathcal{O}'_Y)} : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_W$. Then

$$\begin{aligned}\imath_W \circ e' &= \imath_W \circ \pi_{\mathcal{O}'_W} \circ i_{e^{-1}(\mathcal{O}'_X)} = \bigl(e^\sharp \hat{\otimes} f^\sharp\bigr) \circ \bigl(e^{-1}(\imath_X) \hat{\boxtimes} f^{-1}(\imath_Y)\bigr) \circ i_{e^{-1}(\mathcal{O}'_X)} \\ &= \bigl(e^\sharp \hat{\otimes} f^\sharp\bigr) \circ i_{e^{-1}(\mathcal{O}_X)} \circ e^{-1}(\imath_X) = e^\sharp \circ e^{-1}(\imath_X).\end{aligned}$$

Thus $(\underline{e}, e')$ is a morphism of square zero extensions $(\underline{W}, \mathcal{O}'_W, \imath_W) \to (\underline{X}, \mathcal{O}'_X, \imath_X)$. Similarly $\imath_W \circ f' = f^\sharp \circ f^{-1}(\imath_Y)$, and $(\underline{f}, f')$ is a morphism $(\underline{W}, \mathcal{O}'_W, \imath_W) \to (\underline{Y}, \mathcal{O}'_Y, \imath_Y)$. Thus Definition 2.12 defines morphisms $e^1, e^2, e^3, f^1, f^2, f^3$.

As in (2.15) we have morphisms of square zero extensions $(\underline{g}, g') \circ (\underline{e}, e') = (\underline{g}\circ\underline{e}, (g\circ e)')$ and $(\underline{h}, h')\circ(\underline{f}, h') = (\underline{h}\circ\underline{f}, (h\circ f)')$ from $(\underline{W}, \mathcal{O}'_W, \imath_W)$ to $(\underline{Z}, \mathcal{O}'_Z, \imath_Z)$, where $(g\circ e)' = f' \circ f^{-1}(g') \circ I_{f,g}(\mathcal{O}'_Z)$, and similarly for $(h\circ f)'$. Definition 2.12 defines morphisms $(g\circ e)^j, (h\circ f)^j$ for $j = 1, 2, 3$. We have $\underline{g} \circ \underline{e} = \underline{h} \circ \underline{f}$, but in general $(g \circ e)' \neq (h \circ f)'$. Therefore Proposition 2.13 applies, so there exists a unique morphism $\mu : (\underline{g} \circ \underline{e})^*(\mathcal{F}_Z) \to \mathcal{I}_W$ in qcoh$(\underline{W})$ such that

$$\begin{aligned}&(h \circ f)' = (g\circ e)' + \kappa_W \circ \mu \circ \bigl(\mathrm{id} \otimes ((g\circ e)^\sharp \circ (g\circ e)^{-1}(\imath_Z))\bigr) \circ \bigl((g\circ e)^{-1}(\mathrm{d})\bigr), \\ &(h \circ f)^1 = (g \circ e)^1 + \mu \circ (\underline{g} \circ \underline{e})^*(\xi_Z), \\ &(h \circ f)^2 = (g \circ e)^2 + \xi_W \circ \mu, \quad \text{and} \quad (h \circ f)^3 = (g \circ e)^3.\end{aligned} \tag{2.58}$$

Define morphisms α_1, α_2 in qcoh$(\underline{W})$ by

$$(\underline{g} \circ \underline{e})^*(\mathcal{E}_Z) \xrightarrow{\alpha_1 := \begin{pmatrix} \underline{e}^*(g'')\circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z) \\ -\underline{f}^*(h'')\circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z) \\ (\underline{g}\circ\underline{e})^*(\phi_Z) \end{pmatrix}} \begin{matrix} \underline{e}^*(\mathcal{E}_X)\oplus \\ \underline{f}^*(\mathcal{E}_Y)\oplus \\ (\underline{g} \circ \underline{e})^*(\mathcal{F}_Z) \end{matrix} \xrightarrow{\alpha_2 := \begin{pmatrix} e^1\circ\underline{e}^*(\jmath_X) \\ f^1\circ\underline{f}^*(\jmath_Y) \\ \mu \end{pmatrix}^T} \mathcal{I}_W. \tag{2.59}$$

Then (2.59) is a complex in qcoh$(\underline{W})$, as

$$\begin{aligned}&\alpha_2 \circ \alpha_1 \\ &= e^1 \circ \underline{e}^*(\jmath_X) \circ \underline{e}^*(g'') \circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z) - f^1 \circ \underline{f}^*(\jmath_Y) \circ \underline{f}^*(h'') \circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z) + \mu \circ (\underline{g}\circ\underline{e})^*(\phi_Z) \\ &= e^1 \circ \underline{e}^*(g^1 \circ \underline{g}^*(\jmath_Z)) \circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z) - f^1 \circ \underline{f}^*(h^1 \circ \underline{h}^*(\jmath_Z)) \circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z) + \mu \circ (\underline{g}\circ\underline{e})^*(\xi_Z \circ \jmath_Z) \\ &= \bigl[e^1 \circ \underline{e}^*(g^1) \circ I_{\underline{e},\underline{g}}(\mathcal{I}_Z) - f^1 \circ \underline{f}^*(h^1) \circ I_{\underline{f},\underline{h}}(\mathcal{I}_Z) + \mu \circ (\underline{g}\circ\underline{e})^*(\xi_Z)\bigr] \circ (\underline{g}\circ\underline{e})^*(\jmath_Z) \\ &= \bigl[(g \circ e)^1 - (h \circ f)^1 + \mu \circ (\underline{g} \circ \underline{e})^*(\xi_Z)\bigr] \circ (\underline{g} \circ \underline{e})^*(\jmath_Z) = 0,\end{aligned}$$

where in the second step we use (2.21) for $\boldsymbol{g}, \boldsymbol{h}$ and $\phi_Z = \xi_Z \circ \jmath_Z$, in the third that $I_{\underline{e},\underline{g}}, I_{\underline{f},\underline{h}}$ are natural isomorphisms, in the fourth the third equation of (2.24) for $\boldsymbol{g} \circ \boldsymbol{e}, \boldsymbol{h} \circ \boldsymbol{f}$, and in the fifth the third equation of (2.58).

We will show that α_2 in (2.59) is surjective. Consider the morphisms of sheaves of C^∞-rings on W:

$$\begin{aligned}
e^{-1}(\imath_X)\hat{\boxtimes}\mathrm{id}_{f^{-1}(\mathcal{O}'_Y)} &: e^{-1}(\mathcal{O}'_X)\hat{\otimes}f^{-1}(\mathcal{O}'_Y) \longrightarrow e^{-1}(\mathcal{O}_X)\hat{\otimes}f^{-1}(\mathcal{O}'_Y),\\
\mathrm{id}_{e^{-1}(\mathcal{O}'_X)}\hat{\boxtimes}f^{-1}(\imath_Y) &: e^{-1}(\mathcal{O}'_X)\hat{\otimes}f^{-1}(\mathcal{O}'_Y) \longrightarrow e^{-1}(\mathcal{O}'_X)\hat{\otimes}f^{-1}(\mathcal{O}_Y),\\
e^{-1}(\imath_X)\hat{\boxtimes}f^{-1}(\imath_Y) &: e^{-1}(\mathcal{O}'_X)\hat{\otimes}f^{-1}(\mathcal{O}'_Y) \longrightarrow e^{-1}(\mathcal{O}_X)\hat{\otimes}f^{-1}(\mathcal{O}_Y).
\end{aligned}$$

Write $\mathcal{J}_{X\times Y'}, \mathcal{J}_{X'\times Y}$ and $\mathcal{J}_{X\times Y}$ respectively for the kernels of these, as sheaves of ideals in $e^{-1}(\mathcal{O}'_X)\hat{\otimes}f^{-1}(\mathcal{O}'_Y)$. Then $\mathcal{J}_{X\times Y}=\mathcal{J}_{X\times Y'}\oplus\mathcal{J}_{X'\times Y}$. Equation (2.57) factors through all three morphisms, so $\mathcal{J}_{X\times Y'},\mathcal{J}_{X'\times Y},\mathcal{J}_{X\times Y}\subseteq\mathcal{J}_W$. One can now form a commutative diagram in $\mathrm{qcoh}(\underline{W})$:

$$\begin{tikzcd}[column sep=large]
0 \arrow[d] & & 0 \arrow[d] \\
\begin{array}{l}\underline{e}^*(\mathcal{E}_X)\oplus\\ \underline{f}^*(\mathcal{E}_Y)\end{array} \arrow[r, "\begin{pmatrix} \underline{e}^*(\jmath_X) & 0 \\ 0 & \underline{f}^*(\jmath_Y)\end{pmatrix}"] \arrow[d, "\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \\ 0 & 0\end{pmatrix}"] & \begin{array}{l}\underline{e}^*(\mathcal{I}_X)\oplus\\ \underline{f}^*(\mathcal{I}_Y)\end{array} \arrow[r, "\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2\end{pmatrix}"] & \begin{array}{l}(\mathcal{J}_{X\times Y'}+\mathcal{J}_W^2)/\mathcal{J}_W^2\oplus\\ (\mathcal{J}_{X'\times Y}+\mathcal{J}_W^2)/\mathcal{J}_W^2\end{array} \arrow[d] \\
\begin{array}{l}\underline{e}^*(\mathcal{E}_X)\oplus\underline{f}^*(\mathcal{E}_Y)\\ \oplus(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z)\end{array} \arrow[rr, "\alpha_2"] \arrow[d, "\begin{pmatrix} 0 & 0 & \mathrm{id}\end{pmatrix}"] & & \mathcal{I}_W=\mathcal{J}_W/\mathcal{J}_W^2 \arrow[d] \\
(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z) \arrow[r, "(\underline{g}\circ\underline{e})^*(\psi_Z)"] \arrow[d] & (\underline{g}\circ\underline{e})^*(T^*\underline{Z}) \arrow[r, "\beta_3"] & \mathcal{J}_W/(\mathcal{J}_{X\times Y}+\mathcal{J}_W^2) \arrow[d] \\
0 & & 0,
\end{tikzcd} \tag{2.60}$$

in which the outer columns are exact, and $\beta_1:\underline{e}^*(\mathcal{I}_X)\to(\mathcal{J}_{X\times Y'}+\mathcal{J}_W^2)/\mathcal{J}_W^2$, $\beta_2:\underline{f}^*(\mathcal{I}_Y)\to(\mathcal{J}_{X'\times Y}+\mathcal{J}_W^2)/\mathcal{J}_W^2$, and $\beta_3:(\underline{g}\circ\underline{e})^*(T^*\underline{Z})\to\mathcal{J}_W/(\mathcal{J}_{X\times Y}+\mathcal{J}_W^2)$ are natural, surjective morphisms. Since $\jmath_X,\jmath_Y,\psi_Z$ are surjective, so are $\underline{e}^*(\jmath_X),\underline{f}^*(\jmath_Y),(\underline{g}\circ\underline{e})^*(\psi_Z)$ as pullbacks are right exact, and thus the first and third rows in (2.60) are surjective. Hence α_2 is also surjective, by exactness of the columns.

Define $\mathcal{E}_W$ to be the cokernel of the morphism α_1 in (2.59), with projection $\pi:\underline{e}^*(\mathcal{E}_X)\oplus\underline{f}^*(\mathcal{E}_Y)\oplus(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z)\to\mathcal{E}_W$. Since $\alpha_2\circ\alpha_1=0$ there is a unique morphism $\jmath_W:\mathcal{E}_W\to\mathcal{I}_W$ with $\jmath_W\circ\pi=\alpha_2$, and as α_2 is surjective $\jmath_W$ is also surjective. This completes the definition of the d-space $\boldsymbol{W}=(\underline{W},\mathcal{O}'_W,\mathcal{E}_W,\imath_W,\jmath_W)$. Define morphisms $e'':\underline{e}^*(\mathcal{E}_X)\to\mathcal{E}_W$, $f'':\underline{f}^*(\mathcal{E}_Y)\to\mathcal{E}_W$ and $\eta:(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z)\to\mathcal{E}_W$ in $\mathrm{qcoh}(\underline{W})$ by

$$e''=\pi\circ\begin{pmatrix}\mathrm{id}_{\underline{e}^*(\mathcal{E}_X)}\\0\\0\end{pmatrix},\quad f''=\pi\circ\begin{pmatrix}0\\ \mathrm{id}_{\underline{f}^*(\mathcal{E}_Y)}\\0\end{pmatrix},\quad \eta=\pi\circ\begin{pmatrix}0\\0\\ \mathrm{id}_{(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z)}\end{pmatrix}. \tag{2.61}$$

Then equations (2.59)–(2.61) and $\jmath_W\circ\pi=\alpha_2$ give

$$\jmath_W\circ e''=\jmath_W\circ\pi\circ\begin{pmatrix}\mathrm{id}_{\underline{e}^*(\mathcal{E}_X)}\\0\\0\end{pmatrix}=\alpha_2\circ\begin{pmatrix}\mathrm{id}_{\underline{e}^*(\mathcal{E}_X)}\\0\\0\end{pmatrix}=e^1\circ\underline{e}^*(\jmath_X),$$

so (2.21) holds for $\boldsymbol{e}=(\underline{e},e',e'')$, and thus $\boldsymbol{e}:\boldsymbol{W}\to\boldsymbol{X}$ is a 1-morphism in **dSpa**. Similarly $\jmath_W\circ f''=f^1\circ\underline{f}^*(\jmath_Y)$, so $\boldsymbol{f}=(\underline{f},f',f''):\boldsymbol{W}\to\boldsymbol{Y}$ is a 1-morphism.

Again, equations (2.59)–(2.61) and $\jmath_W\circ\pi=\alpha_2$ give

$$\jmath_W\circ\eta=\jmath_W\circ\pi\circ\begin{pmatrix}0\\0\\\mathrm{id}_{(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z)}\end{pmatrix}=\alpha_2\circ\begin{pmatrix}0\\0\\\mathrm{id}_{(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z)}\end{pmatrix}=\mu.$$

Thus the first equation of (2.58) yields

$$(h\circ f)'=(g\circ e)'+\kappa_W\circ\jmath_W\circ\eta\circ\bigl(\mathrm{id}\otimes((g\circ e)^\sharp\circ(g\circ e)^{-1}(\imath_Z))\circ((g\circ e)^{-1}(\mathrm{d})\bigr).\quad(2.62)$$

Also, we have

$$\begin{aligned}(h\circ f)''&=f''\circ\underline{f}^*(h'')\circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z)=\pi\circ\begin{pmatrix}0\\\mathrm{id}_{\underline{f}^*(\mathcal{E}_Y)}\\0\end{pmatrix}\circ\underline{f}^*(h'')\circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z)\\&=\pi\circ\begin{pmatrix}0\\\underline{f}^*(h'')\circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z)\\0\end{pmatrix}+\pi\circ\begin{pmatrix}\underline{e}^*(g'')\circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z)\\-\underline{f}^*(h'')\circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z)\\(\underline{g}\circ\underline{e})^*(\phi_Z)\end{pmatrix}\\&=\pi\circ\begin{pmatrix}\mathrm{id}_{\underline{e}^*(\mathcal{E}_X)}\\0\\0\end{pmatrix}\circ\underline{e}^*(g'')\circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z)+\pi\circ\begin{pmatrix}0\\0\\\mathrm{id}_{(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z)}\end{pmatrix}\circ(\underline{g}\circ\underline{e})^*(\phi_Z)\\&=e''\circ\underline{e}^*(g'')\circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z)+\eta\circ(\underline{g}\circ\underline{e})^*(\phi_Z)\\&=(g\circ e)''+\eta\circ(\underline{g}\circ\underline{e})^*(\phi_Z),\end{aligned}\quad(2.63)$$

where we use (2.23) in the first and sixth steps, and (2.61) in the second and fifth steps, and in the third step we add on $\pi\circ\alpha_1$, where α_1 is given in (2.59), and $\pi\circ\alpha_1=0$ as π is the cokernel of α_1. Equations (2.25) and (2.62)–(2.63) and $\underline{g}\circ\underline{e}=\underline{h}\circ\underline{f}$ imply that η is a 2-morphism $\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ in **dSpa**.

Definition 2.12 gives $e^2:\underline{e}^*(\mathcal{F}_X)\to\mathcal{F}_W$, $f^2:\underline{f}^*(\mathcal{F}_Y)\to\mathcal{F}_W$, so we form

$$e^2\oplus f^2:\underline{e}^*(\mathcal{F}_X)\oplus\underline{f}^*(\mathcal{F}_Y)\xrightarrow{\cong}\mathcal{F}_W.\quad(2.64)$$

One can show that (2.64) is an isomorphism; essentially this comes down to

$$\begin{aligned}\Omega_{\mathcal{O}'_W}\otimes_{\mathcal{O}'_W}\mathcal{O}_W&=\Omega_{(e^{-1}(\mathcal{O}'_X)\hat\otimes f^{-1}(\mathcal{O}'_Y))/\mathcal{J}_W^2}\otimes_{(e^{-1}(\mathcal{O}'_X)\hat\otimes f^{-1}(\mathcal{O}'_Y))/\mathcal{J}_W^2}\mathcal{O}_W\\&\cong\Omega_{e^{-1}(\mathcal{O}'_X)\hat\otimes f^{-1}(\mathcal{O}'_Y)}\otimes_{e^{-1}(\mathcal{O}'_X)\hat\otimes f^{-1}(\mathcal{O}'_Y)}\mathcal{O}_W\\&\cong\Omega_{e^{-1}(\mathcal{O}'_X)}\otimes_{\mathcal{O}'_X}\mathcal{O}_W\oplus\Omega_{f^{-1}(\mathcal{O}'_Y)}\otimes_{\mathcal{O}'_Y}\mathcal{O}_W,\end{aligned}\quad(2.65)$$

where the first isomorphism in (2.65) holds as the first line may be identified with the quotient of the second line by the submodule generated by derivatives of functions in $\mathcal{J}_W^2$. However, derivatives of functions in $\mathcal{J}_W^2$ lie in $\mathcal{J}_W$, and tensoring by $\mathcal{O}_W$ makes functions in $\mathcal{J}_W$ zero, so the two agree. The second isomorphism in (2.65) holds by a property of cotangent modules of coproducts of C^∞-rings. Then $\phi_W : \mathcal{E}_W \to \mathcal{F}_W$ and $\psi_W : \mathcal{F}_W \to T^*\underline{W}$ are determined by

$$
\begin{aligned}
\phi_W \circ \pi &= (e^2 \oplus f^2) \circ \begin{pmatrix} \underline{e}^*(\phi_X) & 0 & -\underline{e}^*(g^2) \circ I_{\underline{e},\underline{g}}(\mathcal{F}_Z) \\ 0 & \underline{f}^*(\phi_Y) & \underline{f}^*(h^2) \circ I_{\underline{f},\underline{h}}(\mathcal{F}_Z) \end{pmatrix}, \\
\psi_W \circ (e^2 \oplus f^2) &= \bigl(e^3 \circ \underline{e}^*(\psi_X)\bigr) \oplus \bigl(f^3 \circ \underline{f}^*(\psi_Y)\bigr).
\end{aligned} \tag{2.66}
$$

Theorem 2.36. **(a)** *The construction of Definition* 2.35 *gives fibre products in the* 2*-category* **dSpa***. Hence, all fibre products exist in* **dSpa***.*

(b) *The* 2*-functor* $F_{\mathbf{dSpa}}^{\mathbf{C^\infty Sch}}$ *preserves fibre products.*

Proof. For (a), suppose $\tilde{\boldsymbol{W}}$ is a d-space, $\tilde{\boldsymbol{e}} : \tilde{\boldsymbol{W}} \to \boldsymbol{X}$ and $\tilde{\boldsymbol{f}} : \tilde{\boldsymbol{W}} \to \boldsymbol{Y}$ are 1-morphisms, and $\tilde{\eta} : \boldsymbol{g} \circ \tilde{\boldsymbol{e}} \Rightarrow \boldsymbol{h} \circ \tilde{\boldsymbol{f}}$ is a 2-morphism in **dSpa**. Then $\underline{g} \circ \underline{\tilde{e}} = \underline{h} \circ \underline{\tilde{f}}$. Using (2.23) to compute $(g \circ \tilde{e})'', (h \circ \tilde{f})''$, equation (2.25) for $\tilde{\eta}$ becomes

$$
\begin{aligned}
(h \circ \tilde{f})' = (g \circ \tilde{e})' + \kappa_{\tilde{W}} \circ \jmath_{\tilde{W}} \circ \tilde{\eta} \circ \bigl(\mathrm{id} \otimes ((g \circ \tilde{e})^\sharp \circ (g \circ \tilde{e})^{-1}(\imath_Z))\bigr) \\
\circ \bigl((g \circ \tilde{e})^{-1}(\mathrm{d})\bigr),
\end{aligned} \tag{2.67}
$$

$$
\tilde{f}'' \circ \underline{\tilde{f}}^*(h'') \circ I_{\underline{\tilde{f}},\underline{h}}(\mathcal{E}_Z) = \tilde{e}'' \circ \underline{\tilde{e}}^*(g'') \circ I_{\underline{\tilde{e}},\underline{g}}(\mathcal{E}_Z) + \tilde{\eta} \circ (\underline{g} \circ \underline{\tilde{e}})^*(\phi_Z). \tag{2.68}
$$

We will construct a 1-morphism $\boldsymbol{b} = (\underline{b}, b', b'') : \tilde{\boldsymbol{W}} \to \boldsymbol{W}$ and 2-morphisms $\zeta : \boldsymbol{e} \circ \boldsymbol{b} \Rightarrow \tilde{\boldsymbol{e}}$, $\theta : \boldsymbol{f} \circ \boldsymbol{b} \Rightarrow \tilde{\boldsymbol{f}}$, unique up to suitable 2-isomorphisms, such that the following analogue of (A.4) commutes:

$$
\begin{tikzcd}
\boldsymbol{g} \circ \boldsymbol{e} \circ \boldsymbol{b} \arrow[r, Rightarrow, "\eta * \mathrm{id}_{\boldsymbol{b}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}} * \zeta"'] & \boldsymbol{h} \circ \boldsymbol{f} \circ \boldsymbol{b} \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{h}} * \theta"] \\
\boldsymbol{g} \circ \tilde{\boldsymbol{e}} \arrow[r, Rightarrow, "\tilde{\eta}"] & \boldsymbol{h} \circ \tilde{\boldsymbol{f}}.
\end{tikzcd} \tag{2.69}
$$

Since $\underline{W} = \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y}$ is a fibre product in $\mathbf{C^\infty Sch}$, there exists a unique morphism $\underline{b} : \underline{\tilde{W}} \to \underline{W}$ such that $\underline{e} \circ \underline{b} = \underline{\tilde{e}}$ and $\underline{f} \circ \underline{b} = \underline{\tilde{f}}$.

Consider the diagram of morphisms of sheaves of C^∞-rings on $\tilde{W}$:

$$
\begin{tikzcd}
b^{-1}(\mathcal{J}_W) \arrow[r, dotted, "(\tilde{e}' \hat{\otimes} \tilde{f}') \circ (I_{b,e}(\mathcal{O}'_X)^{-1} \hat{\boxtimes} I_{b,f}(\mathcal{O}'_Y)^{-1})|_{b^{-1}(\mathcal{J}_W)}"'] \arrow[d] & \mathcal{I}_{\tilde{W}} \arrow[d, "\kappa_{\tilde{W}}"'] \\
b^{-1}\bigl(e^{-1}(\mathcal{O}'_X) \hat{\otimes} f^{-1}(\mathcal{O}'_Y)\bigr) \arrow[r, "(\tilde{e}' \hat{\otimes} \tilde{f}') \circ (I_{b,e}(\mathcal{O}'_X)^{-1} \hat{\boxtimes} I_{b,f}(\mathcal{O}'_Y)^{-1})"'] \arrow[d, "b^{-1}[(e^\sharp \hat{\otimes} f^\sharp) \circ (e^{-1}(\imath_X) \hat{\boxtimes} f^{-1}(\imath_Y))]"] & \mathcal{O}'_{\tilde{W}} \arrow[d, "\imath_{\tilde{W}}"'] \\
b^{-1}(\mathcal{O}_W) \arrow[r, "b^\sharp"] \arrow[d] & \mathcal{O}_{\tilde{W}} \arrow[d] \\
0 & 0,
\end{tikzcd} \tag{2.70}
$$

where we identify

$$
b^{-1}\bigl(e^{-1}(\mathcal{O}'_X) \hat{\otimes} f^{-1}(\mathcal{O}'_Y)\bigr) = b^{-1}(e^{-1}(\mathcal{O}'_X)) \hat{\otimes} b^{-1}(f^{-1}(\mathcal{O}'_Y)).
$$

The left morphism in (2.70) is the pullback of (2.56) by b. We have

$$
\begin{aligned}
&\imath_{\tilde W}\circ(\tilde e'\hat\otimes\tilde f')\circ(I_{b,e}(\mathcal{O}'_X)^{-1}\hat\boxtimes I_{b,f}(\mathcal{O}'_Y)^{-1})\\
&\quad=\bigl(\imath_{\tilde W}\circ\tilde e'\circ I_{b,e}(\mathcal{O}'_X)^{-1}\bigr)\hat\otimes\bigl(\imath_{\tilde W}\circ\tilde f'\circ I_{b,f}(\mathcal{O}'_Y)^{-1}\bigr)\\
&\quad=\bigl(\tilde e^\sharp\circ\tilde e^{-1}(\imath_X)\circ I_{b,e}(\mathcal{O}'_X)^{-1}\bigr)\hat\otimes\bigl(\tilde f^\sharp\circ\tilde f^{-1}(\imath_Y)\circ I_{b,f}(\mathcal{O}'_Y)^{-1}\bigr)\\
&\quad=\bigl(b^\sharp\circ b^{-1}(e^\sharp)\circ I_{b,e}(\mathcal{O}_X)\circ\tilde e^{-1}(\imath_X)\circ I_{b,e}(\mathcal{O}'_X)^{-1}\bigr)\\
&\qquad\quad\hat\otimes\bigl(b^\sharp\circ b^{-1}(f^\sharp)\circ I_{b,f}(\mathcal{O}_Y)\circ\tilde f^{-1}(\imath_Y)\circ I_{b,f}(\mathcal{O}'_Y)^{-1}\bigr)\\
&\quad=\bigl(b^\sharp\circ b^{-1}(e^\sharp)\circ b^{-1}(e^{-1}(\imath_X))\bigr)\hat\otimes\bigl(b^\sharp\circ b^{-1}(f^\sharp)\circ b^{-1}(f^{-1}(\imath_Y))\bigr)\\
&\quad=b^\sharp\circ b^{-1}\bigl[(e^\sharp\hat\otimes f^\sharp)\circ(e^{-1}(\imath_X)\hat\boxtimes f^{-1}(\imath_Y))\bigr],
\end{aligned}
\tag{2.71}
$$

using $\tilde{\boldsymbol e},\tilde{\boldsymbol f}$ 1-morphisms in the second step, and $\underline{e}\circ\underline{b}=\underline{\tilde e}$, $\underline{f}\circ\underline{b}=\underline{\tilde f}$ and equation (B.5) to rewrite $\tilde e^\sharp,\tilde f^\sharp$ in the third, and properties of $\hat\otimes$, $\hat\otimes$ and $I_{*,*}(*)$.

Thus the central square in(2.70) commutes. As the columns are exact, this implies that $(\tilde e'\hat\otimes\tilde f')\circ(I_{b,e}(\mathcal{O}'_X)^{-1}\hat\boxtimes I_{b,f}(\mathcal{O}'_Y)^{-1})$ restricts to a morphism from $b^{-1}(\mathcal{J}_W)$ to the kernel $\mathcal{I}_{\tilde W}$ of $\imath_{\tilde W}$, as shown. Therefore the morphism of sheaves of C^∞-rings $(\tilde e'\hat\otimes\tilde f')\circ(I_{b,e}(\mathcal{O}'_X)^{-1}\hat\boxtimes I_{b,f}(\mathcal{O}'_Y)^{-1})$ takes $b^{-1}(\mathcal{J}_W)$ to $\mathcal{I}_{\tilde W}$, so it takes $b^{-1}(\mathcal{J}^2_W)$ to $\mathcal{I}^2_{\tilde W}=0$, as $\mathcal{I}_{\tilde W}$ is a sheaf of square zero ideals.

Hence $(\tilde e'\hat\otimes\tilde f')\circ(I_{b,e}(\mathcal{O}'_X)^{-1}\hat\boxtimes I_{b,f}(\mathcal{O}'_Y)^{-1})$ factors through $b^{-1}\bigl(e^{-1}(\mathcal{O}'_X)\hat\otimes f^{-1}(\mathcal{O}'_Y)\bigr)/b^{-1}(\mathcal{J}^2_W)=b^{-1}(\mathcal{O}'_W)$. That is, there exists a unique morphism of sheaves of C^∞-rings $b':b^{-1}(\mathcal{O}'_W)\to\mathcal{O}'_{\tilde W}$ such that

$$
(\tilde e'\hat\otimes\tilde f')\circ(I_{b,e}(\mathcal{O}'_X)^{-1}\hat\boxtimes I_{b,f}(\mathcal{O}'_Y)^{-1})=b'\circ b^{-1}(\pi_{\mathcal{O}'_W}).
\tag{2.72}
$$

Combining equations (2.57) and (2.71)–(2.72) yields

$$
\begin{aligned}
&\imath_{\tilde W}\circ b'\circ b^{-1}(\pi_{\mathcal{O}'_W})=\imath_{\tilde W}\circ(\tilde e'\hat\otimes\tilde f')\circ(I_{b,e}(\mathcal{O}'_X)^{-1}\hat\boxtimes I_{b,f}(\mathcal{O}'_Y)^{-1})\\
&\quad=b^\sharp\circ b^{-1}\bigl[(e^\sharp\hat\otimes f^\sharp)\circ(e^{-1}(\imath_X)\hat\boxtimes f^{-1}(\imath_Y))\bigr]=b^\sharp\circ b^{-1}[\imath_W\circ\pi_{\mathcal{O}'_W}]\\
&\quad=b^\sharp\circ b^{-1}(\imath_W)\circ b^{-1}(\pi_{\mathcal{O}'_W}).
\end{aligned}
$$

Since $b^{-1}(\pi_{\mathcal{O}'_W})$ is surjective, this implies that $\imath_{\tilde W}\circ b'=b^\sharp\circ b^{-1}(\imath_W)$. Hence $(\underline{b},b')$ is a morphism of square zero extensions $(\underline{\tilde W},\mathcal{O}'_{\tilde W},\imath_{\tilde W})\to(\underline{W},\mathcal{O}'_W,\imath_W)$.

As in (2.15), we can form the composition $(\underline{e},e')\circ(\underline{b},b')=(\underline{e}\circ\underline{b},(e\circ b)')$: $(\underline{\tilde W},\mathcal{O}'_{\tilde W},\imath_{\tilde W})\to(\underline{X},\mathcal{O}'_X,\imath_X)$, where

$$
\begin{aligned}
(e\circ b)'&=b'\circ b^{-1}(e')\circ I_{b,e}(\mathcal{O}'_X)=b'\circ b^{-1}\bigl(\pi_{\mathcal{O}'_W}\circ i_{e^{-1}(\mathcal{O}'_X)}\bigr)\circ I_{b,e}(\mathcal{O}'_X)\\
&=(\tilde e'\hat\otimes\tilde f')\circ(I_{b,e}(\mathcal{O}'_X)^{-1}\hat\boxtimes I_{b,f}(\mathcal{O}'_Y)^{-1})\circ b^{-1}(i_{e^{-1}(\mathcal{O}'_X)})\circ I_{b,e}(\mathcal{O}'_X)\\
&=\tilde e'\circ I_{b,e}(\mathcal{O}'_X)^{-1}\circ I_{b,e}(\mathcal{O}'_X)=\tilde e',
\end{aligned}
$$

using the definition of e' and (2.72). Also $\underline{e}\circ\underline{b}=\underline{\tilde e}$. Hence $(\underline{e},e')\circ(\underline{b},b')=(\underline{\tilde e},\tilde e')$, and similarly $(\underline{f},f')\circ(\underline{b},b')=(\underline{\tilde f},\tilde f')$. Thus by (2.24) we have

$$
\tilde e^1=b^1\circ\underline{b}^*(e^1)\circ I_{\underline{b},\underline{e}}(\mathcal{I}_X)\quad\text{and}\quad\tilde f^1=b^1\circ\underline{b}^*(f^1)\circ I_{\underline{b},\underline{f}}(\mathcal{I}_Y).
\tag{2.73}
$$

Combining the first equation of (2.58) and the second of (2.24) gives

$$
\begin{aligned}
(h\circ f\circ b)' &= b'\circ b^{-1}\bigl[(h\circ f)'\bigr]\circ I_{b,h\circ f}(\mathcal{O}'_Z)\\
&= b'\circ b^{-1}\bigl[(g\circ e)'\bigr]\circ I_{b,g\circ e}(\mathcal{O}'_Z) + b'\circ b^{-1}\bigl[\kappa_W\circ\mu\circ\\
&\quad \circ\bigl(\mathrm{id}\otimes((g\circ e)^\sharp\circ(g\circ e)^{-1}(\imath_Z))\bigr)\circ\bigl((g\circ e)^{-1}(\mathrm{d})\bigr)\bigr]\circ I_{b,g\circ e}(\mathcal{O}'_Z)\\
&= b'\circ b^{-1}\bigl[(g\circ e)'\bigr]\circ I_{b,g\circ e}(\mathcal{O}'_Z)+\kappa_{\tilde W}\circ b^1\circ(\mathrm{id}\otimes b^\sharp)\circ b^{-1}(\mu)\\
&\quad\circ b^{-1}\bigl[\bigl(\mathrm{id}\otimes((g\circ e)^\sharp\circ(g\circ e)^{-1}(\imath_Z))\bigr)\circ\bigl((g\circ e)^{-1}(\mathrm{d})\bigr)\bigr]\circ I_{b,g\circ e}(\mathcal{O}'_Z)\\
&= (g\circ e\circ b)'+\kappa_{\tilde W}\circ b^1\circ \underline{b}^*(\mu)\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)\\
&\quad\circ\bigl(\mathrm{id}\otimes((g\circ\tilde e)^\sharp\circ(g\circ\tilde e)^{-1}(\imath_Z))\bigr)\circ\bigl((g\circ\tilde e)^{-1}(\mathrm{d})\bigr),
\end{aligned}
$$

using $b'\circ b^{-1}(\kappa_W)=\kappa_{\tilde W}\circ b^1\circ(\mathrm{id}\otimes b^\sharp)$ in the third step as (2.12) commutes for $(\underline{b},b')$, and $(\mathrm{id}\otimes b^\sharp)\circ b^{-1}(\mu)=\underline{b}^*(\mu)\circ(\mathrm{id}\otimes b^\sharp)$ and properties of $I_{*,*}(*)$ in the fourth step. Comparing this with (2.67) shows that

$$\jmath_{\tilde W}\circ\tilde\eta = b^1\circ\underline{b}^*(\mu)\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z), \tag{2.74}$$

where we have used $(\underline{e},e')\circ(\underline{b},b')=(\underline{\tilde e},\tilde e')$, $(\underline{f},f')\circ(\underline{b},b')=(\underline{\tilde f},\tilde f')$, and that the image of d generates $\mathcal{F}_Z$.

Define $\beta:\underline{b}^*(\underline{e}^*(\mathcal{E}_X))\oplus\underline{b}^*(\underline{f}^*(\mathcal{E}_Y))\oplus\underline{b}^*((\underline{g}\circ\underline{e})^*(\mathcal{F}_Z))\to\mathcal{E}_{\tilde W}$ by

$$\beta=\begin{pmatrix}\tilde e''\circ I_{\underline{b},\underline{e}}(\mathcal{E}_X)^{-1} & \tilde f''\circ I_{\underline{b},\underline{f}}(\mathcal{E}_Y)^{-1} & \tilde\eta\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)^{-1}\end{pmatrix}. \tag{2.75}$$

Then

$$
\begin{aligned}
&\beta\circ\underline{b}^*(\alpha_1)=\tilde e''\circ I_{\underline{b},\underline{e}}(\mathcal{E}_X)^{-1}\circ\underline{b}^*(\underline{e}^*(g'')\circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z))\\
&-\tilde f''\circ I_{\underline{b},\underline{f}}(\mathcal{E}_Y)^{-1}\circ\underline{b}^*(\underline{f}^*(h'')\circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z))+\tilde\eta\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)^{-1}\circ\underline{b}^*((\underline{g}\circ\underline{e})^*(\phi_Z))=0,
\end{aligned}
$$

where in the second step we recognize the whole expression as the composition of (2.68) with $I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{E}_Z)^{-1}$, using properties of the $I_{*,*}(*)$. Since π is the cokernel of α_1 in $\mathrm{qcoh}(\underline{W})$ and $\underline{b}^*:\mathrm{qcoh}(\underline{W})\to\mathrm{qcoh}(\underline{\tilde W})$ is right exact, $\underline{b}^*(\pi)$ is the cokernel of $\underline{b}^*(\alpha_1)$ in $\mathrm{qcoh}(\underline{\tilde W})$. Hence as $\beta\circ\underline{b}^*(\alpha_1)=0$ there exists a unique morphism $b'':\underline{b}^*(\mathcal{E}_W)\to\mathcal{E}_{\tilde W}$ with $b''\circ\underline{b}^*(\pi)=\beta$. We have

$$
\begin{aligned}
\jmath_{\tilde W}\circ b''\circ\underline{b}^*(\pi)&=\jmath_{\tilde W}\circ\begin{pmatrix}\tilde e''\circ I_{\underline{b},\underline{e}}(\mathcal{E}_X)^{-1} & \tilde f''\circ I_{\underline{b},\underline{f}}(\mathcal{E}_Y)^{-1} & \tilde\eta\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)^{-1}\end{pmatrix}\\
&=\begin{pmatrix}\tilde e^1\circ\underline{\tilde e}^*(\jmath_W)\circ I_{\underline{b},\underline{e}}(\mathcal{E}_X)^{-1} & \tilde f^1\circ\underline{\tilde f}^*(\jmath_W)\circ I_{\underline{b},\underline{f}}(\mathcal{E}_Y)^{-1} & \jmath_{\tilde W}\circ\tilde\eta\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)^{-1}\end{pmatrix}\\
&=\begin{pmatrix}\tilde e^1\circ I_{\underline{b},\underline{e}}(\mathcal{I}_X)^{-1}\circ\underline{b}^*\circ\underline{e}^*(\jmath_X) & \tilde f^1\circ I_{\underline{b},\underline{f}}(\mathcal{I}_Y)^{-1}\circ\underline{b}^*\circ\underline{f}^*(\jmath_Y) & b^1\circ\underline{b}^*(\mu)\end{pmatrix}\\
&=b^1\circ\underline{b}^*\begin{pmatrix}e^1\circ\underline{e}^*(\jmath_X) & f^1\circ\underline{f}^*(\jmath_Y) & \mu\end{pmatrix}=b^1\circ\underline{b}^*(\alpha_2)=b^1\circ\underline{b}^*(\jmath_W)\circ\underline{b}^*(\pi),
\end{aligned}
$$

using (2.75) in the first step, (2.21) for $\tilde{\boldsymbol{e}},\tilde{\boldsymbol{f}}$ in the second, (2.74) and properties of $I_{*,*}(*)$ in the third, (2.73) in the fourth, (2.59) in the fifth, and $\jmath_W\circ\pi=\alpha_2$ in the sixth. Since $\underline{b}^*(\pi)$ is surjective, this proves that

$$\jmath_{\tilde W}\circ b''=b^1\circ\underline{b}^*(\jmath_W):\underline{b}^*(\mathcal{E}_W)\longrightarrow\mathcal{I}_{\tilde W},$$

which is (2.21) for $\boldsymbol{b} = (\underline{b}, b', b'')$. Hence $\boldsymbol{b} : \tilde{\boldsymbol{W}} \to \boldsymbol{W}$ is a 1-morphism.

From equations (2.23), (2.61), (2.75) and $b'' \circ \underline{b}^*(\pi) = \beta$ we see that

$$(e \circ b)'' = b'' \circ \underline{b}^*(e'') \circ I_{\underline{b},\underline{e}}(\mathcal{E}_X) = b'' \circ \underline{b}^*(\pi) \circ \underline{b}^* \begin{pmatrix} \mathrm{id}_{\underline{e}^*(\mathcal{E}_X)} \\ 0 \\ 0 \end{pmatrix} \circ I_{\underline{b},\underline{e}}(\mathcal{E}_X)$$
$$= \begin{pmatrix} \tilde{e}'' \circ I_{\underline{b},\underline{e}}(\mathcal{E}_X)^{-1} & \tilde{f}'' \circ I_{\underline{b},\underline{f}}(\mathcal{E}_Y)^{-1} & \tilde{\eta} \circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)^{-1} \end{pmatrix} \begin{pmatrix} I_{\underline{b},\underline{e}}(\mathcal{E}_X) \\ 0 \\ 0 \end{pmatrix} = \tilde{e}''.$$

Together with $\underline{\tilde{e}} = \underline{e} \circ \underline{b}$ and $\tilde{e}' = (e \circ b)'$ this proves that $\tilde{\boldsymbol{e}} = \boldsymbol{e} \circ \boldsymbol{b}$. Therefore $\zeta = 0 = \mathrm{id}_{\tilde{\boldsymbol{e}}}$ is a 2-morphism $\boldsymbol{e} \circ \boldsymbol{b} \Rightarrow \tilde{\boldsymbol{e}}$. Similarly $\tilde{\boldsymbol{f}} = \boldsymbol{f} \circ \boldsymbol{b}$, so $\theta = 0 = \mathrm{id}_{\tilde{\boldsymbol{f}}}$ is a 2-morphism $\boldsymbol{f} \circ \boldsymbol{b} \Rightarrow \tilde{\boldsymbol{f}}$.

Using $\zeta = \theta = \mathrm{id}_{\boldsymbol{b}} = \mathrm{id}_{\boldsymbol{g}} = \mathrm{id}_{\boldsymbol{h}} = 0$, $b'' \circ \underline{b}^*(\pi) = \beta$, and (2.61), (2.75) we have

$$(\mathrm{id}_{\boldsymbol{h}} * \theta) \odot (\eta * \mathrm{id}_{\boldsymbol{b}}) = 0 + b'' \circ \underline{b}^*(\eta) \circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z) = b'' \circ \underline{b}^*(\pi) \circ \underline{b}^* \begin{pmatrix} 0 \\ 0 \\ \mathrm{id} \end{pmatrix} \circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)$$
$$= \begin{pmatrix} \tilde{e}'' \circ I_{\underline{b},\underline{e}}(\mathcal{E}_X)^{-1} & \tilde{f}'' \circ I_{\underline{b},\underline{f}}(\mathcal{E}_Y)^{-1} & \tilde{\eta} \circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z) \end{pmatrix} = \tilde{\eta} + 0 = \tilde{\eta} \odot (\mathrm{id}_{\boldsymbol{g}} * \zeta).$$

Thus (2.69) commutes. This proves $\boldsymbol{W}, \boldsymbol{e}, \boldsymbol{f}, \eta$ satisfy the first universal property of fibre products in 2-categories from Definition A.7.

For the second universal property, suppose that $\tilde{\boldsymbol{b}} = (\underline{\tilde{b}}, \tilde{b}', \tilde{b}'')$, $\tilde{\zeta}$ and $\tilde{\theta}$ are alternative choices for $\boldsymbol{b}, \zeta, \theta$, so that the analogue of (2.69) with $\tilde{\boldsymbol{b}}, \tilde{\zeta}, \tilde{\theta}$ in place of $\boldsymbol{b}, \zeta, \theta$ commutes. Then $\underline{e} \circ \underline{\tilde{b}} = \underline{\tilde{e}} = \underline{e} \circ \underline{b}$ and $\underline{f} \circ \underline{\tilde{b}} = \underline{\tilde{f}} = \underline{f} \circ \underline{b}$, so $\underline{\tilde{b}} = \underline{b}$ by properties of the fibre product $\underline{W} = \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y}$. We must show that there exists a unique 2-morphism $\epsilon : \tilde{\boldsymbol{b}} \Rightarrow \boldsymbol{b}$ with $\tilde{\zeta} = \zeta \odot (\mathrm{id}_{\boldsymbol{e}} * \epsilon)$ and $\tilde{\theta} = \theta \odot (\mathrm{id}_{\boldsymbol{f}} * \epsilon)$.

Using $\zeta = \theta = \mathrm{id}_{\boldsymbol{e}} = \mathrm{id}_{\boldsymbol{f}} = 0$, $\zeta \odot \eta = \zeta + \eta$ and equation (2.27) shows that $\tilde{\zeta} = \zeta \odot (\mathrm{id}_{\boldsymbol{e}} * \epsilon)$ and $\tilde{\theta} = \theta \odot (\mathrm{id}_{\boldsymbol{f}} * \epsilon)$ are equivalent to

$$\tilde{\zeta} = \epsilon \circ \underline{b}^*(e^2) \circ I_{\underline{b},\underline{e}}(\mathcal{F}_X) \quad \text{and} \quad \tilde{\theta} = \epsilon \circ \underline{b}^*(f^2) \circ I_{\underline{b},\underline{f}}(\mathcal{F}_Y).$$

Since $e^2 \oplus f^2$ in (2.64) is an isomorphism, these determine ϵ uniquely, namely

$$\epsilon = \begin{pmatrix} \tilde{\zeta} \circ I_{\underline{b},\underline{e}}(\mathcal{F}_X)^{-1} & \tilde{\theta} \circ I_{\underline{b},\underline{f}}(\mathcal{F}_Y)^{-1} \end{pmatrix} \circ \underline{b}^*\bigl((e^2 \oplus f^2)^{-1}\bigr). \tag{2.76}$$

We will prove that this ϵ is a 2-morphism $\tilde{\boldsymbol{b}} \Rightarrow \boldsymbol{b}$. From above $\underline{\tilde{b}} = \underline{b}$. Proposition 2.13 applied to the pairs $(\underline{\tilde{b}}, \tilde{b}'), (\underline{b}, b')$ shows that there is a unique morphism $\tilde{\mu} : \underline{b}^*(\mathcal{F}_W) \to \mathcal{I}_{\tilde{W}}$ such that

$$b' = \tilde{b}' + \kappa_{\tilde{W}} \circ \tilde{\mu} \circ \bigl(\mathrm{id} \otimes (b^\sharp \circ b^{-1}(\imath_W))\bigr) \circ \bigl(b^{-1}(\mathrm{d})\bigr). \tag{2.77}$$

Proposition 2.13 applied to the pairs $(\underline{e}, e') \circ (\underline{\tilde{b}}, \tilde{b}')$ and $(\underline{\tilde{e}}, \tilde{e}') = (\underline{e}, e') \circ (\underline{b}, b')$ gives a unique morphism $\underline{\tilde{e}}^*(\mathcal{F}_X) \to \mathcal{I}_{\tilde{W}}$, which we can write in two ways: the 2-morphism $\tilde{\zeta} : \boldsymbol{e} \circ \tilde{\boldsymbol{b}} \Rightarrow \tilde{\boldsymbol{e}}$ shows that this morphism is $\jmath_{\tilde{W}} \circ \tilde{\zeta}$, and composing

$\tilde\mu$ relating $(\tilde{\underline{b}},\tilde b'),(\underline{b},b')$ with the identity on $(\underline{e},e')$ shows that the morphism is $\tilde\mu\circ\underline{b}^*(e^2)\circ I_{\underline{b},\underline{e}}(\mathcal{F}_X)$. Hence $\jmath_{\tilde W}\circ\tilde\zeta=\tilde\mu\circ\underline{b}^*(e^2)\circ I_{\underline{b},\underline{e}}(\mathcal{F}_X)$, and similarly $\jmath_{\tilde W}\circ\tilde\theta=\tilde\mu\circ\underline{b}^*(f^2)\circ I_{\underline{b},\underline{f}}(\mathcal{F}_Y)$. Combining these with (2.76) shows that $\tilde\mu=\jmath_{\tilde W}\circ\epsilon$. Substituting this into (2.77) proves the first equation of (2.25) for ϵ.

Using (2.23) for $\boldsymbol{e}\circ\boldsymbol{b},\boldsymbol{e}\circ\tilde{\boldsymbol{b}}$ and (2.25) for $\tilde\zeta:\boldsymbol{e}\circ\tilde{\boldsymbol{b}}\Rightarrow\tilde{\boldsymbol{e}}=\boldsymbol{e}\circ\boldsymbol{b}$ gives

$$\tilde e''=b''\circ\underline{b}^*(e'')\circ I_{\underline{b},\underline{e}}(\mathcal{E}_X)=\tilde b''\circ\underline{b}^*(e'')\circ I_{\underline{b},\underline{e}}(\mathcal{E}_X)+\tilde\zeta\circ(\underline{e}\circ\underline{b})^*(\phi_X).$$

Subtracting gives

$$\begin{aligned}&(b''-\tilde b'')\circ\underline{b}^*(e'')=\tilde\zeta\circ(\underline{e}\circ\underline{b})^*(\phi_X)\circ I_{\underline{b},\underline{e}}(\mathcal{E}_X)^{-1}=\tilde\zeta\circ I_{\underline{b},\underline{e}}(\mathcal{F}_X)^{-1}\circ\underline{b}^*(\underline{e}^*(\phi_X))\\&=\begin{pmatrix}\tilde\zeta\circ I_{\underline{b},\underline{e}}(\mathcal{F}_X)^{-1} & \tilde\theta\circ I_{\underline{b},\underline{f}}(\mathcal{F}_Y)^{-1}\end{pmatrix}\circ\underline{b}^*\bigl((e^2\ \ f^2)^{-1}\bigr)\circ\underline{b}^*\bigl(e^2\ \ f^2\bigr)\circ\begin{pmatrix}\underline{b}^*(\underline{e}^*(\phi_X))\\0\end{pmatrix}\\&=\epsilon\circ\underline{b}^*(e^2)\circ\underline{b}^*(\underline{e}^*(\phi_X))=\epsilon\circ\underline{b}^*(\phi_W)\circ\underline{b}^*(e''),\end{aligned}\tag{2.78}$$

using (2.64) an isomorphism in the second step, (2.76) in the third, and (2.22) for $\boldsymbol{e}$ in the fourth. Similarly

$$(b''-\tilde b'')\circ\underline{b}^*(f'')=\epsilon\circ\underline{b}^*(\phi_W)\circ\underline{b}^*(f'').\tag{2.79}$$

Since the analogue of (2.69) for $\tilde{\boldsymbol{b}},\tilde\zeta,\tilde\theta$ commutes we have

$$\tilde\eta+\tilde\zeta\circ(\underline{e}\circ\underline{b})^*(g^2)\circ I_{\underline{e}\circ\underline{b},\underline{g}}(\mathcal{F}_Z)=\tilde\theta\circ(\underline{f}\circ\underline{b})^*(h^2)\circ I_{\underline{f}\circ\underline{b},\underline{h}}(\mathcal{F}_Z)+\tilde b''\circ\underline{b}^*(\eta)\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z).$$

But $\tilde\eta=b''\circ\underline{b}^*(\eta)\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)$ as (2.69) commutes, so subtracting gives

$$\begin{aligned}&(b''-\tilde b'')\circ\underline{b}^*(\eta)\\&=\bigl(-\tilde\zeta\circ(\underline{e}\circ\underline{b})^*(g^2)\circ I_{\underline{e}\circ\underline{b},\underline{g}}(\mathcal{F}_Z)+\tilde\theta\circ(\underline{f}\circ\underline{b})^*(h^2)\circ I_{\underline{f}\circ\underline{b},\underline{h}}(\mathcal{F}_Z)\bigr)\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)^{-1}\\&=-\tilde\zeta\circ I_{\underline{b},\underline{e}}(\mathcal{F}_X)^{-1}\circ\underline{b}^*(\underline{e}^*(g^2))\circ I_{\underline{b},\underline{e}}(\underline{g}^*(\mathcal{F}_Z))\circ I_{\underline{e}\circ\underline{b},\underline{g}}(\mathcal{F}_Z)\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)^{-1}\\&\qquad+\tilde\theta\circ I_{\underline{b},\underline{f}}(\mathcal{F}_Y)^{-1}\circ\underline{b}^*(\underline{f}^*(h^2))\circ I_{\underline{b},\underline{f}}(\underline{g}^*(\mathcal{F}_Z))\circ I_{\underline{f}\circ\underline{b},\underline{h}}(\mathcal{F}_Z)\circ I_{\underline{b},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)^{-1}\\&=\begin{pmatrix}\tilde\zeta\circ I_{\underline{b},\underline{e}}(\mathcal{F}_X)^{-1} & \tilde\theta\circ I_{\underline{b},\underline{f}}(\mathcal{F}_Y)^{-1}\end{pmatrix}\circ\underline{b}^*\bigl((e^2\ \ f^2)^{-1}\bigr)\\&\qquad\circ\underline{b}^*\bigl(e^2\ \ f^2\bigr)\circ\begin{pmatrix}-\underline{b}^*(\underline{e}^*(g^2))\circ\underline{b}^*(I_{\underline{e},\underline{g}}(\mathcal{F}_Z))\\\underline{b}^*(\underline{f}^*(h^2))\circ\underline{b}^*(I_{\underline{f},\underline{h}}(\mathcal{F}_Z))\end{pmatrix}\\&=\epsilon\circ\underline{b}^*\bigl[f^2\circ\underline{f}^*(h^2)\circ I_{\underline{f},\underline{h}}(\mathcal{F}_Z)-e^2\circ\underline{e}^*(g^2)\circ I_{\underline{e},\underline{g}}(\mathcal{F}_Z)\bigr]\\&=\epsilon\circ\underline{b}^*\bigl[(h\circ f)^2-(g\circ e)^2\bigr]=\epsilon\circ\underline{b}^*(\phi_W\circ\eta)=\epsilon\circ\underline{b}^*(\phi_W)\circ\underline{b}^*(\eta),\end{aligned}\tag{2.80}$$

using properties of the $I_{*,*}(*)$ in the second and third steps, (2.64) an isomorphism in the third, (2.76) in the fourth, (2.24) for $\boldsymbol{g}\circ\boldsymbol{e}$, $\boldsymbol{h}\circ\boldsymbol{f}$ in the fifth, and (2.26) for $\eta:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ in the sixth.

Combining equations (2.61) and (2.78)–(2.80) shows that $(b''-\tilde b'')\circ\underline{b}^*(\pi)=\epsilon\circ\underline{b}^*(\phi_W)\circ\underline{b}^*(\pi)$. Hence $b''=\tilde b''+\epsilon\circ\underline{b}^*(\phi_W)$, as $\underline{b}^*(\pi)$ is surjective. This is the second equation of (2.25) for ϵ. Therefore ϵ is a 2-morphism $\tilde{\boldsymbol{b}}\Rightarrow\boldsymbol{b}$, and is unique with $\tilde\zeta=\zeta\odot(\mathrm{id}_{\boldsymbol{e}}*\epsilon)$ and $\tilde\theta=\theta\odot(\mathrm{id}_{\boldsymbol{f}}*\epsilon)$. This proves the second universal property (A.5) of fibre products in 2-categories from Definition A.7. So $(\boldsymbol{W},\boldsymbol{e},\boldsymbol{f},\eta)$ is a fibre product in **dSpa**, proving part (a). Part (b) is immediate from the construction. This completes the proof of Theorem 2.36. $\square$

Suppose now that $\underline{X}, \underline{Y}, \underline{Z} \in \mathbf{C^\infty Sch^{lf}_{ssc}}$ and $\underline{g} : \underline{X} \to \underline{Z}$, $\underline{h} : \underline{Y} \to \underline{Z}$ are morphisms. Then we can form the fibre product $\underline{W} = \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y}$ in $\mathbf{C^\infty Sch^{lf}_{ssc}}$ by Theorem B.19(b). We can also define d-spaces $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} = F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{X}, \underline{Y}, \underline{Z})$ and 1-morphisms $\boldsymbol{g}, \boldsymbol{h} = F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{g}, \underline{h})$. Thus we may form the explicit 2-category fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ in $\mathbf{dSpa}$ by Definition 2.35. So we can ask whether $\boldsymbol{W}$ is equivalent to $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{W})$, that is, whether $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}$ preserves fibre products. Since the underlying C^∞-scheme of $\boldsymbol{W}$ is $\underline{W}$, Proposition 2.25 gives a necessary and sufficient condition for this to happen.

Corollary 2.37. *Suppose $\underline{g} : \underline{X} \to \underline{Z}$, $\underline{h} : \underline{Y} \to \underline{Z}$ are morphisms in $\mathbf{C^\infty Sch^{lf}_{ssc}}$, and define $\underline{W} = \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y}$ to be the fibre product in $\mathbf{C^\infty Sch^{lf}_{ssc}}$, with projections $\underline{e} : \underline{W} \to \underline{X}$ and $\underline{f} : \underline{W} \to \underline{Y}$. Then $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{X} \times_{\underline{Z}} \underline{Y})$ is equivalent to $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{X}) \times_{F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{Z})} F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{Y})$ in $\mathbf{dSpa}$ if and only if the morphism*

$$\begin{aligned} \underline{e}^*(\Omega_{\underline{g}}) \circ I_{\underline{e},\underline{g}}(T^*\underline{Z}) \oplus \underline{f}^*(\Omega_{\underline{h}}) \circ I_{\underline{f},\underline{h}}(T^*\underline{Z}) &: \\ (\underline{g} \circ \underline{e})^*(T^*\underline{Z}) \longrightarrow \underline{e}^*(T^*\underline{X}) \oplus \underline{f}^*(T^*\underline{Y})& \end{aligned} \tag{2.81}$$

in $\mathrm{qcoh}(\underline{W})$ *has a left inverse.*

As (2.81) need not have a left inverse, the 2-functor $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}$ does not preserve fibre products in general, and the 2-subcategory $\hat{\mathbf{C}}^\infty\mathbf{Sch}^{\mathbf{lf}}_{\mathbf{ssc}}$ in $\mathbf{dSpa}$ is not closed under fibre products.

Example 2.38. Take $\underline{X} = \underline{Y} = \operatorname{Spec}\mathbb{R} = \underline{*}$ and $\underline{Z} = \underline{\mathbb{R}}$, and $\underline{g} : \underline{X} \to \underline{Z}$, $\underline{h} : \underline{Y} \to \underline{Z}$ to be the unique morphisms taking $*$ to the point $0 \in \mathbb{R}$. Then the fibre product $\underline{W} = \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y} = \underline{*} \times_{0,\underline{\mathbb{R}},0} \underline{*}$ in $\mathbf{C^\infty Sch^{lf}_{ssc}}$ is a point $\underline{*}$. Quasicoherent sheaves on $\underline{*}$ are just real vector spaces, and coherent sheaves are finite-dimensional vector spaces. The morphism (2.81) in $\mathrm{qcoh}(\underline{*})$ is $0 : \mathbb{R} \to 0 \oplus 0$, which does not have a left inverse, so $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{X} \times_{\underline{Z}} \underline{Y}) \not\simeq \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ by Corollary 2.37. The fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ in $\mathbf{dSpa}$ has $\underline{W} = (W, \mathcal{O}'_W) = \underline{*}$ and $\mathcal{E}_W = \mathcal{O}_*$. It is a *d-manifold of dimension* -1, in the language of Chapter 3, which we could call the *obstructed point.*

Example 2.39. Take $\underline{X} = \underline{Z} = \underline{\mathbb{R}}$ and $\underline{X} = \operatorname{Spec}\mathbb{R} = \underline{*}$, and $\underline{g} : \underline{X} \to \underline{Z}$ to be the image $F^{\mathbf{C^\infty Sch}}_{\mathbf{Man}}(x^2)$ of the map $x \mapsto x^2$, and $\underline{h} : \underline{Y} \to \underline{Z}$ to be the unique morphism taking $*$ to the point $0 \in \mathbb{R}$. Then the fibre product $\underline{\mathbb{R}} \times_{x^2,\underline{\mathbb{R}},0} \underline{*}$ in $\mathbf{C^\infty Sch^{lf}_{ssc}}$ is the *double point* $\underline{W} = \operatorname{Spec}(\mathbb{R}[x]/(x^2))$. The morphism (2.81) in $\mathrm{qcoh}(\underline{W})$ is $x \oplus 0 : \mathcal{O}_W \to \mathcal{O}_W \oplus 0$, which does not have a left inverse, so $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{X} \times_{\underline{Z}} \underline{Y}) \not\simeq \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ by Corollary 2.37. We could call $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{X} \times_{\underline{Z}} \underline{Y})$ the *classical double point.* It is not a d-manifold. The fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ has $\underline{W} = \operatorname{Spec}(\mathbb{R}[x]/(x^2))$, $(W, \mathcal{O}'_W) = \operatorname{Spec}(\mathbb{R}[x]/(x^4))$, and $\mathcal{E}_W = \mathcal{O}_W$. It is a d-manifold of dimension 0, which we could call the *derived double point.*

Products of d-spaces $\boldsymbol{X} \times \boldsymbol{Y}$ are a useful special case of fibre products $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ with $\boldsymbol{Z} = *$, the point, which is a terminal object in $\mathbf{dSpa}$.

Example 2.40. Let $\boldsymbol{X}, \boldsymbol{Y}$ be d-spaces. The *product* $\boldsymbol{X} \times \boldsymbol{Y}$ is the d-space fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{*},\boldsymbol{h}} \boldsymbol{Y}$, where $\boldsymbol{Z} = \boldsymbol{*} = F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{*})$ is the point, and $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{*}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{*}$ are the unique 1-morphisms. A product comes with projection 1-morphisms $\boldsymbol{\pi_X} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{X}$, $\boldsymbol{\pi_Y} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{Y}$, which satisfy a universal property, so that if $\boldsymbol{V}$ is a d-space and $\boldsymbol{a} : \boldsymbol{V} \to \boldsymbol{X}$, $\boldsymbol{b} : \boldsymbol{V} \to \boldsymbol{Y}$ are 1-morphisms then there exists $\boldsymbol{c} : \boldsymbol{V} \to \boldsymbol{X} \times \boldsymbol{Y}$, unique up to 2-isomorphism, such that $\boldsymbol{\pi_X} \circ \boldsymbol{c} \cong \boldsymbol{a}$ and $\boldsymbol{\pi_Y} \circ \boldsymbol{c} \cong \boldsymbol{b}$.

Definition 2.35 and Theorem 2.36 allow us to write down explicit formulae for the d-space product $\boldsymbol{W} = \boldsymbol{X} \times \boldsymbol{Y}$ and its projections $\boldsymbol{\pi_X}, \boldsymbol{\pi_Y}$. We have $\underline{W} = \underline{X} \times \underline{Y}$, and $\mathcal{O}'_W \cong (\mathcal{O}'_X \hat{\otimes} \mathcal{O}'_Y)/(\mathcal{I}_X \hat{\otimes} \mathcal{I}_Y)$, and $\mathcal{E}_W = \underline{\pi}_X^*(\mathcal{E}_X) \oplus \underline{\pi}_Y^*(\mathcal{E}_Y)$, and $\mathcal{F}_W = \underline{\pi}_X^*(\mathcal{F}_X) \oplus \underline{\pi}_Y^*(\mathcal{F}_Y)$. These have the usual package of properties of products for other classes of geometric spaces. So if $\boldsymbol{a} : \boldsymbol{V} \to \boldsymbol{X}$, $\boldsymbol{b} : \boldsymbol{V} \to \boldsymbol{Y}$ are 1-morphisms there is a *direct product* 1*-morphism* $(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{V} \to \boldsymbol{X} \times \boldsymbol{Y}$, which is just $\boldsymbol{c}$ above, and if $\boldsymbol{e} : \boldsymbol{V} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ are 1-morphisms there is a *product* 1*-morphism* $\boldsymbol{e} \times \boldsymbol{f} : \boldsymbol{V} \times \boldsymbol{W} \to \boldsymbol{X} \times \boldsymbol{Y}$, satisfying some universal properties. Both $(\boldsymbol{a}, \boldsymbol{b})$ and $\boldsymbol{e} \times \boldsymbol{f}$ are unique up to 2-isomorphism.

Here is a d-space analogue of Theorem B.39(c).

Proposition 2.41. *Suppose we are given a* 2*-Cartesian square in* **dSpa***:*

$$\begin{tikzcd} \boldsymbol{W} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"'] \arrow[dr, phantom, "\eta\Uparrow"] & \boldsymbol{Y} \arrow[d, "\boldsymbol{h}"] \\ \boldsymbol{X} \arrow[r, "\boldsymbol{g}"] & \boldsymbol{Z}. \end{tikzcd} \tag{2.82}$$

Then the following sequence in $\mathrm{qcoh}(\underline{W})$ *is exact:*

$$(\underline{g} \circ \underline{e})^*(\mathcal{E}_Z) \xrightarrow{\beta_1} \begin{array}{l} \underline{e}^*(\mathcal{E}_X) \oplus \\ \underline{f}^*(\mathcal{E}_Y) \oplus \\ (\underline{g} \circ \underline{e})^*(\mathcal{F}_Z) \end{array} \xrightarrow{\beta_2} \begin{array}{l} \mathcal{E}_W \oplus \\ \underline{e}^*(\mathcal{F}_X) \oplus \\ \underline{f}^*(\mathcal{F}_Y) \end{array} \xrightarrow{\beta_3} \mathcal{F}_W \to 0, \tag{2.83}$$

where

$$\beta_1 = \begin{pmatrix} \underline{e}^*(g'') \circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z) \\ -\underline{f}^*(h'') \circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z) \\ (\underline{g} \circ \underline{e})^*(\phi_Z) \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} e'' & f'' & \eta \\ -\underline{e}^*(\phi_X) & 0 & \underline{e}^*(g^2) \circ I_{\underline{e},\underline{g}}(\mathcal{F}_Z) \\ 0 & -\underline{f}^*(\phi_Y) & -\underline{f}^*(h^2) \circ I_{\underline{f},\underline{h}}(\mathcal{F}_Z) \end{pmatrix} \tag{2.84}$$
$$\textit{and} \qquad \beta_3 = \begin{pmatrix} \phi_W & e^2 & f^2 \end{pmatrix}.$$

Furthermore, equation (2.83) *is split exact at the third term. That is, if we replace the first two terms of* (2.83) *with* $0 \longrightarrow \mathrm{Coker}\,\beta_1$*, the resulting short exact sequence is split exact.*

Proof. First note that $\beta_2 \circ \beta_1 = 0$ follows from $\eta : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$ and $\boldsymbol{g}, \boldsymbol{h}$ 1-morphisms, and $\beta_3 \circ \beta_2 = 0$ follows from $\boldsymbol{e}, \boldsymbol{f}$ 1-morphisms and $\eta : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$. Hence (2.83) is a complex. Let $\tilde{\boldsymbol{W}}$ be the explicit fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ constructed in Definition 2.35, with projections $\tilde{\boldsymbol{e}} : \tilde{\boldsymbol{W}} \to \boldsymbol{X}$ and $\tilde{\boldsymbol{f}} : \tilde{\boldsymbol{W}} \to \boldsymbol{Y}$. Then as (2.82) 2-commutes, there is a 1-morphism $\boldsymbol{b} : \boldsymbol{W} \to \tilde{\boldsymbol{W}}$ with 2-morphisms $\zeta : \tilde{\boldsymbol{e}} \circ \boldsymbol{b} \Rightarrow \boldsymbol{e}$, $\theta : \tilde{\boldsymbol{f}} \circ \boldsymbol{b} \Rightarrow \boldsymbol{f}$, and as (2.82) is 2-Cartesian, $\boldsymbol{b}$ is an equivalence. Hence Proposition 2.20 shows that the following is a split short exact sequence in $\mathrm{qcoh}(\underline{W})$:

$$0 \longrightarrow \underline{b}^*(\mathcal{E}_{\tilde{W}}) \xrightarrow{b'' \oplus -\underline{b}^*(\phi_{\tilde{W}})} \mathcal{E}_W \oplus \underline{b}^*(\mathcal{F}_{\tilde{W}}) \xrightarrow{\phi_W \oplus b^2} \mathcal{F}_W \longrightarrow 0. \tag{2.85}$$

Note that β_1 in (2.84) is α_1 in (2.59). By definition $\mathcal{E}_{\tilde{W}}$ is the cokernel of $\alpha_1 = \beta_1$, and (2.64) shows that $\mathcal{F}_{\tilde{W}} \cong \underline{\tilde{e}}^*(\mathcal{F}_X) \oplus \underline{\tilde{f}}^*(\mathcal{F}_Y)$, and $\phi_{\tilde{W}}$ is given in (2.66), and b'', b^2 are constructed in the proof of Theorem 2.36. Making these substitutions and using isomorphisms $I_{*,*}(*)$, we see that (2.85) becomes (2.83). Hence (2.83) is exact, and split exact at the third term. $\square$

2.6 Fibre products of manifolds in Man and dSpa

Definition 2.14 defined a functor $F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \to \mathbf{dSpa}$, which is full and faithful by Theorem 2.15(d). We now consider when $F_{\mathbf{Man}}^{\mathbf{dSpa}}$ takes fibre products in **Man** to fibre products in **dSpa**. Recall that if $g : X \to Z$, $h : Y \to Z$ are smooth maps of manifolds then we call g, h *transverse* if whenever $x \in X$, $y \in Y$ and $z \in Z$ with $g(x) = z = h(y)$ we have $T_zZ = \mathrm{d}g|_x(T_xX) + \mathrm{d}h|_y(T_yY)$. It is well known that if g, h are transverse then a fibre product $X \times_{g,Z,h} Y$ exists in **Man**. If g, h are not transverse then a fibre product $X \times_{g,Z,h} Y$ may or may not exist in **Man**.

We now show that fibre products in **Man** and **dSpa** agree if and only if g, h are transverse. Thus, if a non-transverse fibre product $X \times_{g,Z,h} Y$ exists in **Man**, it is the 'wrong' fibre product, from the point of view of derived differential geometry. A similar result for his derived manifolds is Spivak [95, Prop. 8.13].

Theorem 2.42. *Let X, Y, Z be manifolds and $g : X \to Z$, $h : Y \to Z$ be smooth maps. Write $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{g}, \boldsymbol{h} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(X, Y, Z, g, h)$, and let $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ and $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ be the explicit fibre product in* **dSpa** *and projections constructed in §2.5. Then*

(a) *Suppose g, h are transverse. Then a fibre product $\tilde{W} = X \times_Z Y$ exists in* **Man**, *with smooth projections $\tilde{e} = \pi_X : \tilde{W} \to X$ and $\tilde{f} = \pi_Y : \tilde{W} \to Y$ with $g \circ \tilde{e} = h \circ \tilde{f}$. Set $\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{f}} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(\tilde{W}, \tilde{e}, \tilde{f})$, so we have a 2-commutative diagram in* **dSpa** :

$$\begin{tikzcd} \tilde{\boldsymbol{W}} \arrow[r, "\tilde{\boldsymbol{f}}"] \arrow[d, "\tilde{\boldsymbol{e}}"'] \arrow[dr, phantom, "\mathrm{id}_{\boldsymbol{g}\circ\tilde{\boldsymbol{e}}} \Uparrow"] & \boldsymbol{Y} \arrow[d, "\boldsymbol{h}"] \\ \boldsymbol{X} \arrow[r, "\boldsymbol{g}"] & \boldsymbol{Z}. \end{tikzcd} \tag{2.86}$$

As in §2.5, from (2.86) we get a 1-morphism $\boldsymbol{b} : \tilde{\boldsymbol{W}} \to \boldsymbol{W}$ and 2-morphisms $\zeta : \boldsymbol{e} \circ \boldsymbol{b} \Rightarrow \tilde{\boldsymbol{e}}$, $\theta : \boldsymbol{f} \circ \boldsymbol{b} \Rightarrow \tilde{\boldsymbol{f}}$. This $\boldsymbol{b}$ is an equivalence.

(b) *Suppose g, h are not transverse. Then $\boldsymbol{W}$ is not a manifold. Thus, if a fibre product $\tilde{W} = X \times_Z Y$ does exist in* **Man**, *we have $F_{\mathbf{Man}}^{\mathbf{dSpa}}(\tilde{W}) \not\simeq \boldsymbol{W}$.*

Proof. For (a), let g,h be transverse, and $\boldsymbol{W},\boldsymbol{e},\boldsymbol{f},\tilde W,\tilde e,\tilde f,\tilde{\boldsymbol{W}},\tilde{\boldsymbol{e}},\tilde{\boldsymbol{f}},\boldsymbol{b},\zeta,\theta$ be as above. Write $\underline{\tilde W},\underline{X},\underline{Y},\underline{Z},\underline{\tilde e},\underline{\tilde f},\underline{g},\underline{h}=F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(\tilde W,X,Y,Z,\tilde e,\tilde f,g,h)$. Then $\boldsymbol{W}=(\underline{W},\mathcal{O}'_W,\mathcal{E}_W,\imath_W,\jmath_W)$, where $\underline{W}=\underline{X}\times_{\underline{g},\underline{Z},\underline{h}}\underline{Y}$ is the fibre product in $\mathbf{C^\infty Sch}$, and $\boldsymbol{b}=(\underline{b},b',b'')$ where $\underline{b}:\underline{\tilde W}\to\underline{W}$ is the unique morphism with $\underline{e}\circ\underline{b}=\underline{\tilde e}$ and $\underline{f}\circ\underline{b}=\underline{\tilde f}$. We will apply Proposition 2.21, so we must show that $\underline{b}$ and $b^4:\underline{b}^*(\mathcal{C}_W)\to\mathcal{C}_{\tilde W}$ are isomorphisms, and (2.34) is a split exact sequence.

For the first, $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ takes transverse fibre products in $\mathbf{Man}$ to fibre products in $\mathbf{C^\infty Sch}$ by Theorem B.28, so $\underline{b}:\underline{\tilde W}\to\underline{W}$ is an isomorphism. For the second, $\phi_W:\mathcal{E}_W\to\mathcal{F}_W$ is

$$\begin{pmatrix}-\underline{e}^*(\Omega_{\underline{g}})\circ I_{\underline{e},\underline{g}}(T^*\underline{Z})\\ \underline{f}^*(\Omega_{\underline{h}})\circ I_{\underline{f},\underline{h}}(T^*\underline{Z})\end{pmatrix}:(\underline{g}\circ\underline{e})^*(T^*\underline{Z})\longrightarrow\underline{e}^*(T^*\underline{X})\oplus\underline{f}^*(T^*\underline{Y}),\tag{2.87}$$

and this is injective as g,h are transverse. Hence $\mathcal{D}_W=\operatorname{Ker}\phi_W=0$, so $\mathcal{C}_W=0$ as $\nu_W:\mathcal{C}_W\to\mathcal{D}_W$ is injective. Also $\mathcal{C}_{\tilde W}=0$ as by definition $\mathcal{E}_{\tilde W}=0$. Thus $b^4:0\to 0$ is trivially an isomorphism. For the third, the complex

$$0\longrightarrow\underline{b}^*(\mathcal{E}_W)\xrightarrow{b''\oplus-\underline{b}^*(\phi_W)}\mathcal{E}_{\tilde W}\oplus\underline{b}^*(\mathcal{F}_W)\xrightarrow{\phi_{\tilde W}\oplus b^2}\mathcal{F}_{\tilde W}\longrightarrow 0,$$

which is (2.34) for $\boldsymbol{b}$, is

$$0\to\underline{b}^*\circ(\underline{g}\circ\underline{e})^*(T^*\underline{Z})\xrightarrow{\substack{\underline{b}^*\circ\underline{e}^*(\Omega_{\underline{g}})\circ\underline{b}^*(I_{\underline{e},\underline{g}}(T^*\underline{Z}))\oplus\\ -\underline{b}^*\circ\underline{f}^*(\Omega_{\underline{h}})\circ\underline{b}^*(I_{\underline{f},\underline{h}}(T^*\underline{Z}))}}\begin{matrix}\underline{b}^*\circ\underline{e}^*(T^*\underline{X})\oplus\\ \underline{b}^*\circ\underline{f}^*(T^*\underline{Y})\end{matrix}\xrightarrow{\substack{\Omega_{\underline{\tilde e}}\circ I_{\underline{b},\underline{e}}(T^*\underline{X})^{-1}\oplus\\ \Omega_{\underline{\tilde f}}\circ I_{\underline{b},\underline{f}}(T^*\underline{Y})^{-1}}}T^*\underline{\tilde W}\to 0.$$

This is the lift to C^∞-schemes of the sequence of vector bundles on $\tilde W$

$$0\to(g\circ\tilde e)^*(T^*Z)\xrightarrow{\tilde e^*((\mathrm{d}g)^*)\oplus\tilde f^*((\mathrm{d}h)^*)}\tilde e^*(T^*X)\oplus\tilde f^*(T^*Y)\xrightarrow{(\mathrm{d}\tilde e)^*\oplus(\mathrm{d}\tilde f)^*}T^*\tilde W\to 0,$$

which is split exact as $\tilde W=X\times_{g,Z,h}Y$ with g,h transverse. Hence $\boldsymbol{b}$ is an equivalence by Proposition 2.21, proving (a).

For (b), as g,h are not transverse there exist $x\in X$ and $y\in Y$ with $g(x)=h(y)=z\in Z$ and $T_zZ\neq\mathrm{d}g|_x(T_xX)+\mathrm{d}h|_y(T_yY)$. Thus,

$$-(\mathrm{d}g)|_x^*\oplus(\mathrm{d}h)|_y^*:T_z^*Z\longrightarrow T_x^*X\oplus T_y^*Y\tag{2.88}$$

is not injective. Now in $\boldsymbol{W}$, the morphism $\phi_W:\mathcal{E}_W\to\mathcal{F}_W$ is (2.87), and the fibre of (2.87) over $(x,y)\in\underline{W}$ is (2.88). If ϕ_W had a left inverse, then its pullback to (x,y) would be a left inverse for (2.88). But (2.88) is not injective, and so has no left inverse. Thus ϕ_W has no left inverse, and therefore $\boldsymbol{W}$ is not equivalent to an object in the image of $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}$ by Proposition 2.25. As $F_{\mathbf{Man}}^{\mathbf{dSpa}}=F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}\circ F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$, this implies $\boldsymbol{W}$ is not a manifold. $\square$

2.7 Fixed point loci of finite groups in d-spaces

If a finite group Γ acts on a manifold X by diffeomorphisms, then the fixed point locus X^Γ is a disjoint union of closed, embedded submanifolds of X. In a similar way, if Γ acts on a d-space $\boldsymbol{X}$ by 1-isomorphisms, we will define a d-space $\boldsymbol{X}^\Gamma$ called the fixed d-subspace of Γ in $\boldsymbol{X}$, with an inclusion 1-morphism $\boldsymbol{j}_{\boldsymbol{X},\Gamma}:\boldsymbol{X}^\Gamma\hookrightarrow\boldsymbol{X}$, whose topological space X^Γ is the fixed point locus of Γ in X.

When $\boldsymbol{X}$ is a d-manifold, such $\boldsymbol{X}^\Gamma$ are (locally) examples of w-embedded d-submanifolds in the sense of §4.1. Also these ideas will be important in our treatment of d-stacks in Chapter 9. When Γ acts on a d-space $\boldsymbol{X}$, in §9.3 we will define a quotient d-stack $\boldsymbol{\mathcal{X}}=[\boldsymbol{X}/\Gamma]$. Fixed d-subspaces $\boldsymbol{X}^\Gamma$ of finite groups Γ acting on d-spaces $\boldsymbol{X}$ are étale local models for orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma$ of a d-stack $\boldsymbol{\mathcal{X}}$ in §9.6, and Theorem 9.28 will use fixed d-subspaces to describe orbifold strata of quotient d-stacks.

Definition 2.43. Let $\boldsymbol{X}=(\underline{X},\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X)$ be a d-space and Γ a finite group. An *action* $\boldsymbol{r}$ *of* Γ *on* $\boldsymbol{X}$ is an action $\boldsymbol{r}=(\underline{r},r',r''):\Gamma\to\mathrm{Aut}(\boldsymbol{X})$ of Γ on $\boldsymbol{X}$ by 1-isomorphisms. Note that $\boldsymbol{r}(\gamma):\boldsymbol{X}\to\boldsymbol{X}$ must be a 1-isomorphism, not merely an equivalence, for each $\gamma\in\Gamma$, and we require that $\boldsymbol{r}(\gamma)\circ\boldsymbol{r}(\delta)=\boldsymbol{r}(\gamma\delta)$, not just that $\boldsymbol{r}(\gamma)\circ\boldsymbol{r}(\delta)$ is 2-isomorphic to $\boldsymbol{r}(\gamma\delta)$, for all $\gamma,\delta\in\Gamma$. The 2-morphisms in **dSpa** play no rôle in defining Γ-actions on d-spaces.

Given such $\boldsymbol{X},\Gamma,\boldsymbol{r}$, we will define a d-space $\boldsymbol{X}^\Gamma$, which we call the *fixed d-subspace of* Γ *in* $\boldsymbol{X}$, with a natural inclusion 1-morphism $\boldsymbol{j}_{\boldsymbol{X},\Gamma}:\boldsymbol{X}^\Gamma\to\boldsymbol{X}$. Since $\underline{r}$ is an action of Γ on the separated, second countable, locally fair C^∞-scheme $\underline{X}$, it has a fixed C^∞-subscheme $\underline{X}^\Gamma$, a closed C^∞-subscheme in $\underline{X}$, which is also separated, second countable, and locally fair. Write $\underline{j}_{\boldsymbol{X},\Gamma}:\underline{X}^\Gamma\hookrightarrow\underline{X}$ for the inclusion morphism. Then $\underline{r}(\gamma)\circ\underline{j}_{\boldsymbol{X},\Gamma}=\underline{j}_{\boldsymbol{X},\Gamma}$ for all $\gamma\in\Gamma$.

For each $\gamma\in\Gamma$, define isomorphisms $A(\gamma),B(\gamma)$ of sheaves of C^∞-rings and $C(\gamma)$ of quasicoherent sheaves on $\underline{X}^\Gamma$ by

$$\begin{aligned}
A(\gamma)&=j^{-1}_{\boldsymbol{X},\Gamma}(r^\sharp(\gamma^{-1}))\circ I_{j_{\boldsymbol{X},\Gamma},r(\gamma^{-1})}(\mathcal{O}_X):j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)\longrightarrow j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X),\\
B(\gamma)&=j^{-1}_{\boldsymbol{X},\Gamma}(r'(\gamma^{-1}))\circ I_{j_{\boldsymbol{X},\Gamma},r(\gamma^{-1})}(\mathcal{O}'_X):j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)\longrightarrow j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X),\\
C(\gamma)&=\underline{j}^*_{\boldsymbol{X},\Gamma}(r''(\gamma^{-1}))\circ I_{\underline{j}_{\boldsymbol{X},\Gamma},\underline{r}(\gamma^{-1})}(\mathcal{E}_X):\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)\longrightarrow\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X).
\end{aligned}\tag{2.89}$$

If $\gamma,\delta\in\Gamma$ then as $\boldsymbol{r}(\gamma\delta)=\boldsymbol{r}(\gamma)\boldsymbol{r}(\delta)$, equation (2.23) gives

$$\begin{aligned}
r^\sharp(\gamma\delta)&=r^\sharp(\delta)\circ r(\delta)^{-1}(r^\sharp(\gamma))\circ I_{r(\delta),r(\gamma)}(\mathcal{O}_X),\\
r'(\gamma\delta)&=r'(\delta)\circ r(\delta)^{-1}(r'(\gamma))\circ I_{r(\delta),r(\gamma)}(\mathcal{O}'_X)\\
r''(\gamma\delta)&=r''(\delta)\circ\underline{r}(\delta)^*(r''(\gamma))\circ I_{\underline{r}(\delta),\underline{r}(\gamma)}(\mathcal{E}_X),
\end{aligned}$$

so using properties of $I_{*,*}(*)$ and noting that the use of γ^{-1} rather than γ on the r.h.s. of (2.89) deals with the contravariance of r',r'', we find that $A(\gamma\delta)=A(\gamma)A(\delta)$, $B(\gamma\delta)=B(\gamma)B(\delta)$ and $C(\gamma\delta)=C(\gamma)C(\delta)$.

Hence A,B,C are Γ-actions on the sheaves $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X),j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X),\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)$. Note that this is similar to sheaves on orbifold strata in §C.9, where we found that if $\mathcal{X}$ is a Deligne–Mumford C^∞-stack, $\mathcal{E}\in\mathrm{qcoh}(\mathcal{X})$, and $O^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\to\mathcal{X}$

is an orbifold stratum, then the pullback sheaf $\mathcal{E}^\Gamma := O^\Gamma(\mathcal{X})^*(\mathcal{E})$ on $\mathcal{X}^\Gamma$ has a natural Γ-action. Thus as for (C.23) in §C.9, we have canonical splittings

$$j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X) = j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)^\Gamma_{\rm tr} \oplus j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)^\Gamma_{\rm nt}, \;\; j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X) = j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)^\Gamma_{\rm tr} \oplus j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)^\Gamma_{\rm nt}$$
$$\text{and} \quad \underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X) = \underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)^\Gamma_{\rm tr} \oplus \underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)^\Gamma_{\rm nt}$$

into trivial 'tr' and nontrivial 'nt' Γ-representations.

Since $j^{-1}_{\boldsymbol{X},\Gamma}(\imath_X)$ maps $(\mathcal{O}'_X)^\Gamma_{\rm nt} \to (\mathcal{O}_X)^\Gamma_{\rm nt}$ it induces a quotient morphism

$$j^{-1}_{\boldsymbol{X},\Gamma}(\imath_X)_* : j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)/\big(j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)^\Gamma_{\rm nt}\big) \longrightarrow j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)/\big(j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)^\Gamma_{\rm nt}\big), \tag{2.90}$$

where $\big(j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)^\Gamma_{\rm nt}\big), \big(j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)^\Gamma_{\rm nt}\big)$ are the sheaves of ideals in $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)$ and $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)$ generated by $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)^\Gamma_{\rm nt}, j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)^\Gamma_{\rm nt}$. Also $j^\sharp_{\boldsymbol{X},\Gamma} : j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X) \to \mathcal{O}_{X^\Gamma}$ is surjective, with kernel $\big((\mathcal{O}_X)^\Gamma_{\rm nt}\big)$. Hence it induces an isomorphism

$$(j^\sharp_{\boldsymbol{X},\Gamma})_* : j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)/\big(j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)^\Gamma_{\rm nt}\big) \xrightarrow{\cong} \mathcal{O}_{X^\Gamma}. \tag{2.91}$$

Define the sheaf of C^∞-rings $\mathcal{O}'_{X^\Gamma}$ by $\mathcal{O}'_{X^\Gamma} = j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)/\big(j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)^\Gamma_{\rm nt}\big)$, and the morphism $\imath_{X^\Gamma} : \mathcal{O}'_{X^\Gamma} \to \mathcal{O}_{X^\Gamma}$ by $\imath_{X^\Gamma} = (j^\sharp_{\boldsymbol{X},\Gamma})_* \circ j^{-1}_{\boldsymbol{X},\Gamma}(\imath_X)_*$, using the notation of (2.90)–(2.91). We will show that $\big(\mathcal{O}'_{X^\Gamma}, \imath_{X^\Gamma}\big)$ is a square zero extension of $\underline{X}^\Gamma$, in the sense of §2.1. First note that as $(\mathcal{O}'_X, \imath_X)$ is a square zero extension, $\imath_X$ is surjective, so (2.90) is surjective, and (2.91) is an isomorphism, so $\imath_{X^\Gamma}$ is a surjective morphism of sheaves of C^∞-rings.

Next, observe that as $j^{-1}_{\boldsymbol{X},\Gamma}(\imath_X)$ induces a surjective morphism of sheaves of abelian groups $(\mathcal{O}'_X)^\Gamma_{\rm nt} \to (\mathcal{O}_X)^\Gamma_{\rm nt}$, it induces a surjective morphism of sheaves of ideals $\big((\mathcal{O}'_X)^\Gamma_{\rm nt}\big) \to \big((\mathcal{O}_X)^\Gamma_{\rm nt}\big)$. Therefore

$$\begin{aligned}
&\mathrm{Ker}\Big(j^{-1}_{\boldsymbol{X},\Gamma}(\imath_X)_* : j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)/\big((\mathcal{O}'_X)^\Gamma_{\rm nt}\big) \longrightarrow j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)/\big((\mathcal{O}_X)^\Gamma_{\rm nt}\big)\Big)\\
&\cong \big[\big(\mathrm{Ker}\, j^{-1}_{\boldsymbol{X},\Gamma}(\imath_X) : j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X) \to j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)\big) + \big((\mathcal{O}'_X)^\Gamma_{\rm nt}\big)\big]/\big((\mathcal{O}'_X)^\Gamma_{\rm nt}\big)\\
&= \big[j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X) + \big((\mathcal{O}'_X)^\Gamma_{\rm nt}\big)\big]/\big((\mathcal{O}'_X)^\Gamma_{\rm nt}\big),
\end{aligned} \tag{2.92}$$

using $j^{-1}_{\boldsymbol{X},\Gamma}(\imath_X)_* : \big((\mathcal{O}'_X)^\Gamma_{\rm nt}\big) \to \big((\mathcal{O}_X)^\Gamma_{\rm nt}\big)$ surjective in the first step. Since $\mathcal{I}_X$ is a sheaf of square zero ideals in $\mathcal{O}'_X$, it follows that the kernel of (2.90) is a sheaf of square zero ideals in $\mathcal{O}'_{X^\Gamma}$. As (2.91) is an isomorphism, this kernel is the kernel of $\imath_{X^\Gamma}$. Hence $\big(\mathcal{O}'_{X^\Gamma}, \imath_{X^\Gamma}\big)$ is a square zero extension of $\underline{X}^\Gamma$.

As in §2.1, $\mathcal{I}_{X^\Gamma}$ is defined to be the kernel of $\imath_{X^\Gamma}$, and is a quasicoherent sheaf on $\underline{X}^\Gamma$. By (2.90)–(2.92) we have canonical isomorphisms

$$\begin{aligned}
\mathcal{I}_{X^\Gamma} &\cong \big[j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X) + \big((\mathcal{O}'_X)^\Gamma_{\rm nt}\big)\big]/\big((\mathcal{O}'_X)^\Gamma_{\rm nt}\big)\\
&\cong j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)/\big[j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X) \cap \big((\mathcal{O}'_X)^\Gamma_{\rm nt}\big)\big].
\end{aligned} \tag{2.93}$$

The pullback $\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)$ of $\mathcal{I}_X$ to $\underline{X}^\Gamma$ satisfies

$$\begin{aligned}
\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X) &= j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X) \otimes_{j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)} \mathcal{O}_{X^\Gamma}\\
&\cong j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X) \otimes_{j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)} \big[j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)/\big((\mathcal{O}_X)^\Gamma_{\rm nt}\big)\big]\\
&\cong j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)/\big[j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X) \cdot \big((\mathcal{O}_X)^\Gamma_{\rm nt}\big)\big],
\end{aligned} \tag{2.94}$$

using the definition of $\underline{j}^*_{\boldsymbol{X},\Gamma}$ in the first step, and (2.91) in the second. But

$$j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)\cdot\bigl((\mathcal{O}_X)^\Gamma_{\rm nt}\bigr)=j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)\cdot\bigl((\mathcal{O}'_X)^\Gamma_{\rm nt}\bigr)\subseteq j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)\cap\bigl((\mathcal{O}'_X)^\Gamma_{\rm nt}\bigr),$$

where for the first step we use that the $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)$-action on $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)$ factors through $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)$, and for the second that both $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)$ and $\bigl((\mathcal{O}'_X)^\Gamma_{\rm nt}\bigr)$ are sheaves of ideals. Thus by (2.93)–(2.94), there is a natural, surjective morphism

$$\pi_{\mathcal{I}_{X^\Gamma}}:\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)\longrightarrow\mathcal{I}_{X^\Gamma}. \tag{2.95}$$

The Γ-action B on $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)$ above preserves $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)\subset j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)$, so that $j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)=j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)^\Gamma_{\rm tr}\oplus j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)^\Gamma_{\rm nt}$, and $\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)=\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)^\Gamma_{\rm tr}\oplus\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)^\Gamma_{\rm nt}$ as $\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)=j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)\otimes_{j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}_X)}\mathcal{O}_{X^\Gamma}$, where $\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)^\Gamma_{\rm nt}\subseteq\operatorname{Ker}\pi_{\mathcal{I}_{X^\Gamma}}$. As $\jmath_X:\mathcal{E}_X\to\mathcal{I}_X$ is surjective, $\underline{j}^*_{\boldsymbol{X},\Gamma}(\jmath_X):\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)\to\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)$ is, and it is Γ-equivariant, so $\underline{j}^*_{\boldsymbol{X},\Gamma}(\jmath_X)$ induces surjective morphisms $\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)^\Gamma_{\rm tr}\to\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)^\Gamma_{\rm tr}$ and $\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)^\Gamma_{\rm nt}\to\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)^\Gamma_{\rm nt}$. Define $\mathcal{E}_{X^\Gamma}=\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)^\Gamma_{\rm tr}$, as a quasicoherent sheaf on $\underline{X}^\Gamma$, and define $\jmath_{X^\Gamma}=\pi_{\mathcal{I}_{X^\Gamma}}\circ\underline{j}^*_{\boldsymbol{X},\Gamma}(\jmath_X)\vert_{\mathcal{E}_{X^\Gamma}}:\mathcal{E}_{X^\Gamma}\to\mathcal{I}_{X^\Gamma}$. Then $\jmath_{X^\Gamma}$ is surjective as $\pi_{\mathcal{I}_{X^\Gamma}},\underline{j}^*_{\boldsymbol{X},\Gamma}(\jmath_X)$ are, and $\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{I}_X)^\Gamma_{\rm nt}\subseteq\operatorname{Ker}\pi_{\mathcal{I}_{X^\Gamma}}$. Set $\boldsymbol{X}^\Gamma=\bigl(\underline{X}^\Gamma,\mathcal{O}'_{X^\Gamma},\mathcal{E}_{X^\Gamma},\imath_{X^\Gamma},\jmath_{X^\Gamma}\bigr)$. The discussion above shows $\boldsymbol{X}^\Gamma$ is a d-space.

Define $\boldsymbol{j}_{\boldsymbol{X},\Gamma}=(\underline{j}_{\boldsymbol{X},\Gamma},j'_{\boldsymbol{X},\Gamma},j''_{\boldsymbol{X},\Gamma})$, where $j'_{\boldsymbol{X},\Gamma},j''_{\boldsymbol{X},\Gamma}$ are the projections in

$$\begin{aligned}
&j'_{\boldsymbol{X},\Gamma}:j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)\longrightarrow j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)/\bigl(j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)^\Gamma_{\rm nt}\bigr)=\mathcal{O}'_{X^\Gamma},\\
&j''_{\boldsymbol{X},\Gamma}:\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)=\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)^\Gamma_{\rm tr}\oplus\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)^\Gamma_{\rm nt}\longrightarrow\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)^\Gamma_{\rm tr}=\mathcal{E}_{X^\Gamma}.
\end{aligned}$$

It is easy to check that $\boldsymbol{j}_{\boldsymbol{X},\Gamma}:\boldsymbol{X}^\Gamma\to\boldsymbol{X}$ is a 1-morphism in $\mathbf{dSpa}$.

A 1-morphism $\boldsymbol{f}:\boldsymbol{W}\to\boldsymbol{X}$ factorizes via $\boldsymbol{j}_{\boldsymbol{X},\Gamma}:\boldsymbol{X}^\Gamma\to\boldsymbol{X}$ if and only if it is Γ-invariant. This is a universal property of $\boldsymbol{X}^\Gamma,\boldsymbol{j}_{\boldsymbol{X},\Gamma}$, which determines $\boldsymbol{X}^\Gamma$ uniquely up to canonical 1-isomorphism.

Proposition 2.44. *Let $\boldsymbol{X},\Gamma,\boldsymbol{r},\boldsymbol{X}^\Gamma$ and $\boldsymbol{j}_{\boldsymbol{X},\Gamma}:\boldsymbol{X}^\Gamma\to\boldsymbol{X}$ be as in Definition 2.43. Suppose $\boldsymbol{f}:\boldsymbol{W}\to\boldsymbol{X}$ is a 1-morphism in* $\mathbf{dSpa}$. *Then $\boldsymbol{f}$ factorizes as $\boldsymbol{f}=\boldsymbol{j}_{\boldsymbol{X},\Gamma}\circ\boldsymbol{g}$ for some 1-morphism $\boldsymbol{g}:\boldsymbol{W}\to\boldsymbol{X}^\Gamma$ in* $\mathbf{dSpa}$, *which must be unique, if and only if $\boldsymbol{r}(\gamma)\circ\boldsymbol{f}=\boldsymbol{f}$ for all $\gamma\in\Gamma$.*

Proof. The definition of the fixed C^∞-subscheme $\underline{X}^\Gamma$ implies that $\underline{f}:\underline{W}\to\underline{X}$ factorizes as $\underline{f}=\underline{j}_{\boldsymbol{X},\Gamma}\circ\underline{g}$ for some necessarily unique $\underline{g}:\underline{W}\to\underline{X}^\Gamma$ if and only if $\underline{r}(\gamma)\circ\underline{f}=\underline{f}$ for all $\gamma\in\Gamma$. Suppose this holds. Then we have a diagram

$$\begin{tikzcd}
\underline{f}^*(\mathcal{E}_X) \arrow[r, equal] \arrow[d, "f''"] & (\underline{j}_{\boldsymbol{X},\Gamma}\circ\underline{g})^*(\mathcal{E}_X) \arrow[r, "I_{\underline{g},\underline{j}_{\boldsymbol{X},\Gamma}}(\mathcal{E}_X)"'] & \underline{g}^*(\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)) \arrow[d, equal] \\
\mathcal{E}_W & & \underline{g}^*(\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)_{\rm tr})\oplus\underline{g}^*(\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)_{\rm nt}) \arrow[ll, "f''\circ I_{\underline{g},\underline{j}_{\boldsymbol{X},\Gamma}}(\mathcal{E}_X)^{-1}"']
\end{tikzcd}$$

in $\operatorname{qcoh}(\underline{W})$, where the splitting $\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)=\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)_{\rm tr}\oplus\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)_{\rm nt}$ into trivial and nontrivial representations comes from the Γ-action C on $\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)$ from

Definition 2.43. Considering the Γ-actions, we see that $f'' \circ I_{\underline{g},\underline{j}_{\boldsymbol{X},\Gamma}}(\mathcal{E}_X)^{-1}$ is Γ-invariant if and only if the component of $f'' \circ I_{\underline{g},\underline{j}_{\boldsymbol{X},\Gamma}}(\mathcal{E}_X)^{-1}$ mapping $\underline{g}^*(\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)_{\mathrm{nt}}) \to \mathcal{E}_W$ is zero, if and only if $(r(\gamma) \circ f)'' = f''$ for all $\gamma \in \Gamma$.

Thus, $(r(\gamma) \circ f)'' = f''$ for all $\gamma \in \Gamma$ if and only if $f'' \circ I_{\underline{g},\underline{j}_{\boldsymbol{X},\Gamma}}(\mathcal{E}_X)^{-1}$ factorizes via the projection $\underline{g}^*(j''_{\boldsymbol{X},\Gamma}) : \underline{g}^*(\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)) \to \underline{g}^*(\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)_{\mathrm{tr}}) = \underline{g}^*(\mathcal{E}_{X^\Gamma})$. That is, $(r(\gamma) \circ f)'' = f''$ for all $\gamma \in \Gamma$ if and only if there exists a (necessarily unique) morphism $g'' : \underline{g}^*(\mathcal{E}_{X^\Gamma}) \to \mathcal{E}_W$ with $f'' \circ I_{\underline{g},\underline{j}_{\boldsymbol{X},\Gamma}}(\mathcal{E}_X)^{-1} = g'' \circ \underline{g}^*(j''_{\boldsymbol{X},\Gamma})$, so that $f'' = g'' \circ \underline{g}^*(j''_{\boldsymbol{X},\Gamma}) \circ I_{\underline{g},\underline{j}_{\boldsymbol{X},\Gamma}}(\mathcal{E}_X)$, which is the condition on f'' in $\boldsymbol{f} = \boldsymbol{j}_{\boldsymbol{X},\Gamma} \circ \boldsymbol{g}$. A similar proof shows that $(r(\gamma) \circ f)' = f'$ for all $\gamma \in \Gamma$ if and only if there exists a (necessarily unique) morphism $g' : g^{-1}(\mathcal{O}'_{X^\Gamma}) \to \mathcal{O}'_W$ with $f' = g' \circ g^{-1}(j'_{\boldsymbol{X},\Gamma}) \circ I_{g,j_{\boldsymbol{X},\Gamma}}(\mathcal{O}'_X)$. The proposition follows. $\square$

In particular, when $\boldsymbol{W} = \boldsymbol{X}^\Gamma$, $\boldsymbol{f} = \boldsymbol{j}_{\boldsymbol{X},\Gamma}$ and $\boldsymbol{g} = \mathbf{id}_{\boldsymbol{X}^\Gamma}$ this implies that $\boldsymbol{r}(\gamma) \circ \boldsymbol{j}_{\boldsymbol{X},\Gamma} = \boldsymbol{j}_{\boldsymbol{X},\Gamma}$ for all $\gamma \in \Gamma$. We can lift Γ-equivariant 1- and 2-morphisms to fixed d-subspaces.

Proposition 2.45. *Suppose $\boldsymbol{X}, \boldsymbol{Y}$ are d-spaces, Γ is a finite group, $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\boldsymbol{X})$, $\boldsymbol{s} : \Gamma \to \mathrm{Aut}(\boldsymbol{Y})$ are actions of Γ on $\boldsymbol{X}, \boldsymbol{Y}$, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a Γ-equivariant 1-morphism in* **dSpa**, *that is, $\boldsymbol{f} \circ \boldsymbol{r}(\gamma) = \boldsymbol{s}(\gamma) \circ \boldsymbol{f}$ for $\gamma \in \Gamma$. Then there exists a unique 1-morphism $\boldsymbol{f}^\Gamma : \boldsymbol{X}^\Gamma \to \boldsymbol{Y}^\Gamma$ such that $\boldsymbol{j}_{\boldsymbol{Y},\Gamma} \circ \boldsymbol{f}^\Gamma = \boldsymbol{f} \circ \boldsymbol{j}_{\boldsymbol{X},\Gamma}$.*

Now let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be Γ-equivariant 1-morphisms and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a Γ-equivariant 2-morphism, that is, $\eta * \mathrm{id}_{\boldsymbol{r}(\gamma)} = \mathrm{id}_{\boldsymbol{s}(\gamma)} * \eta$ for $\gamma \in \Gamma$. Then there exists a unique 2-morphism $\eta^\Gamma : \boldsymbol{f}^\Gamma \Rightarrow \boldsymbol{g}^\Gamma$ such that $\mathrm{id}_{\boldsymbol{j}_{\boldsymbol{Y},\Gamma}} * \eta^\Gamma = \eta * \mathrm{id}_{\boldsymbol{j}_{\boldsymbol{X},\Gamma}}$.

Proof. The C^∞-scheme morphism $\underline{f} \circ \underline{j}_{\boldsymbol{X},\Gamma} : \underline{X}^\Gamma \to \underline{Y}$ satisfies

$$\underline{s}(\gamma) \circ (\underline{f} \circ \underline{j}_{\boldsymbol{X},\Gamma}) = \underline{f} \circ \underline{r}(\gamma) \circ \underline{j}_{\boldsymbol{X},\Gamma} = \underline{f} \circ \underline{j}_{\boldsymbol{X},\Gamma},$$

by Γ-equivariance of $\underline{f}$ and the defining property of $\underline{j}_{\boldsymbol{X},\Gamma}$. Therefore $\underline{f} \circ \underline{j}_{\boldsymbol{X},\Gamma}$ maps to the C^∞-subscheme of $\underline{Y}$ fixed by Γ, so there exists a unique morphism $\underline{f}^\Gamma : \underline{X}^\Gamma \to \underline{Y}^\Gamma$ with $\underline{j}_{\boldsymbol{Y},\Gamma} \circ \underline{f}^\Gamma = \underline{f} \circ \underline{j}_{\boldsymbol{X},\Gamma}$.

We will construct a 1-morphism $\boldsymbol{f}^\Gamma = (\underline{f}^\Gamma, f^{\Gamma\prime}, f^{\Gamma\prime\prime}) : \boldsymbol{X}^\Gamma \to \boldsymbol{Y}^\Gamma$. To define $f^{\Gamma\prime}, f^{\Gamma\prime\prime}$, consider the commutative diagrams:

$$\begin{tikzcd}
(f^\Gamma)^{-1}\bigl((j^{-1}_{\boldsymbol{Y},\Gamma}(\mathcal{O}'_Y)^\Gamma_{\mathrm{nt}})\bigr) \arrow[r] \arrow[d, dotted] & (f^\Gamma)^{-1}\bigl(j^{-1}_{\boldsymbol{Y},\Gamma}(\mathcal{O}'_Y)\bigr) \arrow[r, "(f^\Gamma)^{-1}(j'_{\boldsymbol{Y},\Gamma})"'] \arrow[d, "{\begin{subarray}{l} j^{-1}_{\boldsymbol{X},\Gamma}(f')\circ I_{j_{\boldsymbol{X},\Gamma},f}(\mathcal{O}'_Y) \\ \circ I_{f^\Gamma,j_{\boldsymbol{Y},\Gamma}}(\mathcal{O}'_Y)^{-1}\end{subarray}}"] & (f^\Gamma)^{-1}\bigl(\mathcal{O}'_{Y^\Gamma}\bigr) \arrow[r] \arrow[d, dotted, "f^{\Gamma\prime}"] & 0 \\
(j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)^\Gamma_{\mathrm{nt}}) \arrow[r] & j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X) \arrow[r, "j'_{\boldsymbol{X},\Gamma}"] & \mathcal{O}'_{X^\Gamma} \arrow[r] & 0,
\end{tikzcd} \tag{2.96}$$

$$\begin{tikzcd}
(\underline{f}^\Gamma)^*(\underline{j}^*_{\boldsymbol{Y},\Gamma}(\mathcal{E}_Y)) \arrow[r, "(\underline{f}^\Gamma)^*(j''_{\boldsymbol{X},\Gamma})"'] \arrow[d, "\underline{j}^*_{\boldsymbol{X},\Gamma}(f'')\circ I_{\underline{j}_{\boldsymbol{X},\Gamma},\underline{f}}(\mathcal{E}_Y)\circ I_{\underline{f}^\Gamma,\underline{j}_{\boldsymbol{Y},\Gamma}}(\mathcal{E}_Y)^{-1}"] & (\underline{f}^\Gamma)^*\bigl(\underline{j}^*_{\boldsymbol{Y},\Gamma}(\mathcal{E}_Y)^\Gamma_{\mathrm{tr}}\bigr) \arrow[r, equal] & (\underline{f}^\Gamma)^*(\mathcal{E}_{Y^\Gamma}) \arrow[d, dotted, "f^{\Gamma\prime\prime}"'] \\
\underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X) \arrow[r, "j''_{\boldsymbol{X},\Gamma}"] & \underline{j}^*_{\boldsymbol{X},\Gamma}(\mathcal{E}_X)^\Gamma_{\mathrm{tr}} \arrow[r, equal] & \mathcal{E}_{X^\Gamma}.
\end{tikzcd} \tag{2.97}$$

In (2.96), the rows are exact by definition of $\mathcal{O}'_{X^\Gamma}, \mathcal{O}'_{Y^\Gamma}$. By Γ-equivariance the central column maps $(f^\Gamma)^{-1}(j^{-1}_{\boldsymbol{Y},\Gamma}(\mathcal{O}'_Y)^\Gamma_{\mathrm{nt}}) \to j^{-1}_{\boldsymbol{X},\Gamma}(\mathcal{O}'_X)^\Gamma_{\mathrm{nt}}$, so the left column

'$\dashrightarrow$' exists, and by exactness there is a unique morphism $f^{\Gamma\prime}$ making (2.96) commute. In (2.97), the left hand column is Γ-equivariant, and so induces a unique morphism $f^{\Gamma\prime\prime}$ on the trivial components to make (2.97) commute. The rest of (a) is straightforward, and (b) follows by similar arguments. $\square$

Uniqueness in Proposition 2.45 implies that these 1- and 2-morphisms $\boldsymbol{f}^\Gamma, \eta^\Gamma$ have the obvious functorial properties under composition, e.g. $(\boldsymbol{g}\circ\boldsymbol{f})^\Gamma = \boldsymbol{g}^\Gamma \circ \boldsymbol{f}^\Gamma$. Thus, we can express the results of this section as a functor: writing $\mathbf{dSpa}^\Gamma$ for the strict 2-category whose objects are pairs $(\boldsymbol{X}, \boldsymbol{r})$ of a d-space $\boldsymbol{X}$ and an action $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\boldsymbol{X})$, and whose 1- and 2-morphisms are Γ-equivariant 1- and 2-morphisms in $\mathbf{dSpa}$, then mapping $(\boldsymbol{X}, \boldsymbol{r}) \mapsto \boldsymbol{X}^\Gamma$ on objects, $\boldsymbol{f} \mapsto \boldsymbol{f}^\Gamma$ on 1-morphisms and $\eta \mapsto \eta^\Gamma$ on 2-morphisms gives a strict 2-functor $\mathbf{dSpa}^\Gamma \to \mathbf{dSpa}$.

3 The 2-category of d-manifolds

We now define and study the 2-category **dMan** of *d-manifolds without boundary*, or just *d-manifolds*, as a 2-subcategory of the 2-category of d-spaces **dSpa** from Chapter 2. We begin in §3.1 by discussing the 2-categories of *virtual quasicoherent sheaves* and *virtual vector bundles* on a C^∞-scheme $\underline{X}$. We introduce these first as we will need ideas from §3.1 to define d-manifolds in §3.2.

Given a manifold V, a vector bundle $E \to V$, and a smooth section $s : V \to E$, in §3.2 we will define a 'standard model' d-manifold $\boldsymbol{S}_{V,E,s}$, with topological space $s^{-1}(0) = \{v \in V : s(v) = 0\}$. These $\boldsymbol{S}_{V,E,s}$ are our preferred local models for d-manifolds, up to equivalence. Section 3.3 studies their local properties, and §3.4 describes 1- and 2-morphisms of $\boldsymbol{S}_{V,E,s}$ in differential-geometric terms. This is applied in §3.5 to characterize equivalences in **dMan**, and in §3.6 to explain how to make a d-manifold by gluing together local charts $\boldsymbol{S}_{V_i,E_i,s_i}$. In this chapter all manifolds will be without boundary.

3.1 The 2-category of virtual quasicoherent sheaves

If X is a manifold of dimension n, the cotangent bundle T^*X is a vector bundle of rank n on X. Our derived analogue of this will be that if $\boldsymbol{X}$ is a d-manifold of virtual dimension $n \in \mathbb{Z}$, the morphism $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ in $\mathrm{qcoh}(\underline{X})$ from §2.2 will be a 'virtual vector bundle of rank n on $\underline{X}$', in the sense below, which we call the 'virtual cotangent bundle' $T^*\boldsymbol{X}$.

Definition 3.1. Let $\underline{X}$ be a C^∞-scheme. We will define a 2-category $\mathrm{vqcoh}(\underline{X})$ of *virtual quasicoherent sheaves* on $\underline{X}$. *Objects* of $\mathrm{vqcoh}(\underline{X})$ will be morphisms $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ in $\mathrm{qcoh}(\underline{X})$, which we also may write as $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ or $(\mathcal{E}^\bullet, \phi)$. Given objects $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ and $\psi : \mathcal{F}^1 \to \mathcal{F}^2$, a 1-*morphism* $(f^1, f^2) : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ is a pair of morphisms $f^1 : \mathcal{E}^1 \to \mathcal{F}^1$, $f^2 : \mathcal{E}^2 \to \mathcal{F}^2$ in $\mathrm{qcoh}(\underline{X})$ such that $\psi \circ f^1 = f^2 \circ \phi$. We may write $f^\bullet$ for (f^1, f^2).

The *identity* 1-*morphism* of $(\mathcal{E}^\bullet, \phi)$ is $(\mathrm{id}_{\mathcal{E}^1}, \mathrm{id}_{\mathcal{E}^2})$. If $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ and $g^\bullet : (\mathcal{F}^\bullet, \psi) \to (\mathcal{G}^\bullet, \xi)$ are 1-morphisms, define the *composition of* 1-*morphisms* to be $g^\bullet \circ f^\bullet = (g^1 \circ f^1, g^2 \circ f^2)$.

Given $f^\bullet, g^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$, a 2-*morphism* $\eta : f^\bullet \Rightarrow g^\bullet$ is a morphism $\eta : \mathcal{E}^2 \to \mathcal{F}^1$ in $\mathrm{qcoh}(\underline{X})$ such that $g^1 = f^1 + \eta \circ \phi$ and $g^2 = f^2 + \psi \circ \eta$. The *identity* 2-*morphism* for $f^\bullet$ is $\mathrm{id}_{f^\bullet} = 0$. If $f^\bullet, g^\bullet, h^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ are 1-morphisms and $\eta : f^\bullet \Rightarrow g^\bullet$, $\zeta : g^\bullet \Rightarrow h^\bullet$ are 2-morphisms, the *vertical composition of* 2-*morphisms* $\zeta \odot \eta : f^\bullet \Rightarrow h^\bullet$ is $\zeta \odot \eta = \zeta + \eta$. If $f^\bullet, \tilde{f}^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ and $g^\bullet, \tilde{g}^\bullet : (\mathcal{F}^\bullet, \psi) \to (\mathcal{G}^\bullet, \xi)$ are 1-morphisms and $\eta : f^\bullet \Rightarrow \tilde{f}^\bullet$, $\zeta : g^\bullet \Rightarrow \tilde{g}^\bullet$ are 2-morphisms, the *horizontal composition of* 2-*morphisms* $\zeta * \eta : g^\bullet \circ f^\bullet \Rightarrow \tilde{g}^\bullet \circ \tilde{f}^\bullet$ is $\zeta * \eta = g^1 \circ \eta + \zeta \circ f^2 + \zeta \circ \psi \circ \eta$. With these definitions, it is not difficult to check that $\mathrm{vqcoh}(\underline{X})$ is a strict 2-category.

The underlying category of $\mathrm{vqcoh}(\underline{X})$ is an abelian category; short exact sequences in vqcoh($\underline{X}$) are commutative diagrams of the form

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^1 \arrow[r, "f^1"] \arrow[d, "\phi"] & \mathcal{F}^1 \arrow[r, "g^1"] \arrow[d, "\psi"] & \mathcal{G}^1 \arrow[r] \arrow[d, "\chi"] & 0 \\ 0 \arrow[r] & \mathcal{E}^2 \arrow[r, "f^2"] & \mathcal{F}^2 \arrow[r, "g^2"] & \mathcal{G}^2 \arrow[r] & 0, \end{tikzcd}$$

with rows exact in qcoh($\underline{X}$).

If $\underline{U} \subseteq \underline{X}$ is an open C^∞-subscheme then restriction from $\underline{X}$ to $\underline{U}$ defines a strict 2-functor $|_{\underline{U}} : \mathrm{vqcoh}(\underline{X}) \to \mathrm{vqcoh}(\underline{U})$.

An object $(\mathcal{E}^\bullet, \phi)$ in vqcoh($\underline{X}$) is called a *virtual vector bundle of rank* $d \in \mathbb{Z}$ if $\underline{X}$ may be covered by open C^∞-subschemes $\underline{U}$ such that $(\mathcal{E}^\bullet, \phi)|_{\underline{U}}$ is equivalent in vqcoh($\underline{U}$) to some $(\mathcal{F}^\bullet, \psi)$ for $\mathcal{F}^1, \mathcal{F}^2$ vector bundles on $\underline{U}$ with $\mathrm{rank}\, \mathcal{F}^2 - \mathrm{rank}\, \mathcal{F}^1 = d$. We write $\mathrm{rank}(\mathcal{E}^\bullet, \phi) = d$. Proposition 3.7 below shows that d does depend only on $\mathcal{E}^1, \mathcal{E}^2, \phi$, so this is well-defined. Define the 2-category vvect($\underline{X}$) to be the full 2-subcategory of virtual vector bundles in vqcoh($\underline{X}$).

If $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes then pullback defines a strict 2-functor $\underline{f}^* : \mathrm{vqcoh}(\underline{Y}) \to \mathrm{vqcoh}(\underline{X})$, which is left exact on the underlying abelian category. Also $\underline{f}^*$ maps $\mathrm{vvect}(\underline{Y}) \to \mathrm{vvect}(\underline{X})$.

The next example explains how this is related to d-spaces in Chapter 2.

Example 3.2. Let $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ be a d-space. Define the *virtual cotangent sheaf* $T^*\boldsymbol{X}$ of $\boldsymbol{X}$ to be the morphism $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ in qcoh($\underline{X}$) from Definition 2.14, regarded as a virtual quasicoherent sheaf on $\underline{X}$. It is a d-space analogue of the *cotangent complex* in algebraic geometry, as in Illusie [50, 51].

Let $\boldsymbol{f} = (\underline{f}, f', f'') : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in **dSpa**. Then $T^*\boldsymbol{X} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X)$ and $\underline{f}^*(T^*\boldsymbol{Y}) = \big(\underline{f}^*(\mathcal{E}_Y), \underline{f}^*(\mathcal{F}_Y), \underline{f}^*(\phi_Y)\big)$ are virtual quasicoherent sheaves on $\underline{X}$, and $\Omega_{\boldsymbol{f}} := (f'', f^2)$ is a 1-morphism $\underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ in vqcoh($\underline{X}$), as (2.22) commutes.

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms in **dSpa**, and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism. Then $\eta : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{E}_X$ with $g'' = f'' + \eta \circ \underline{f}^*(\phi_Y)$ and $g^2 = f^2 + \phi_X \circ \eta$, by (2.25)–(2.26). It follows that η is a 2-morphism $\Omega_{\boldsymbol{f}} \Rightarrow \Omega_{\boldsymbol{g}}$ in vqcoh($\underline{X}$).

Thus, objects, 1-morphisms and 2-morphisms in **dSpa** lift to objects, 1-morphisms and 2-morphisms in vqcoh($\underline{X}$). This is not a functor, as to make vqcoh($\underline{X}$) we have to fix an object $\boldsymbol{X}$. Nonetheless, one can show that composition of 1-morphisms, vertical and horizontal composition of 2-morphisms, and identities in **dSpa** lift to composition of 1-morphisms, vertical and horizontal composition of 2-morphisms, and identities in vqcoh($\underline{X}$) in a meaningful sense, and the proof that vqcoh($\underline{X}$) is a strict 2-category is a simplification of the proof that **dSpa** is a strict 2-category in Theorem 2.15.

Since 1-morphisms and 2-morphisms for $\boldsymbol{X}$ in **dSpa** lift to 1-morphisms and 2-morphisms in vqcoh($\underline{X}$), equivalences lift to equivalences. This gives:

Lemma 3.3. *Suppose* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is an equivalence of d-spaces. Then*

$$\begin{aligned} \Omega_{\boldsymbol{f}} = (f'', f^2) : \underline{f}^*(T^*\boldsymbol{Y}) = \big(\underline{f}^*(\mathcal{E}_Y), \underline{f}^*(\mathcal{F}_Y), \underline{f}^*(\phi_Y)\big) \\ \longrightarrow T^*\boldsymbol{X} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X) \end{aligned} \tag{3.1}$$

is an equivalence in vqcoh($\underline{X}$).

We collect some results on equivalences in the 2-category vqcoh($\underline{X}$). The first is related to f^5 an isomorphism in Proposition 2.20.

Proposition 3.4. *Let $\underline{X}$ be a C^∞-scheme, $\phi^1 : \mathcal{E}^1 \to \mathcal{E}^2$ and $\psi^1 : \mathcal{F}^1 \to \mathcal{F}^2$ be virtual quasicoherent sheaves on $\underline{X}$, and $(f^1, f^2) : (\mathcal{E}^1, \mathcal{E}^2, \phi^1) \to (\mathcal{F}^1, \mathcal{F}^2, \psi^1)$ be a* 1*-morphism in* vqcoh($\underline{X}$). *Write $\phi^0 : \mathcal{E}^0 \to \mathcal{E}^1$, $\phi^2 : \mathcal{E}^2 \to \mathcal{E}^3$ for the kernel and cokernel of ϕ^1, and $\psi^0 : \mathcal{F}^0 \to \mathcal{F}^1$, $\psi^2 : \mathcal{F}^2 \to \mathcal{F}^3$ for the kernel and cokernel of ψ^1. Then there are unique morphisms f^0, f^3 such that the following commutes in* qcoh($\underline{X}$):

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^0 \arrow[r, "\phi^0"'] \arrow[d, dotted, "f^0"] & \mathcal{E}^1 \arrow[r, "\phi^1"'] \arrow[d, "f^1"] & \mathcal{E}^2 \arrow[r, "\phi^2"'] \arrow[d, "f^2"] & \mathcal{E}^3 \arrow[r] \arrow[d, dotted, "f^3"] & 0 \\
0 \arrow[r] & \mathcal{F}^0 \arrow[r, "\psi^0"] & \mathcal{F}^1 \arrow[r, "\psi^1"] & \mathcal{F}^2 \arrow[r, "\psi^2"] & \mathcal{F}^3 \arrow[r] & 0.
\end{tikzcd} \tag{3.2}$$

If (f^1, f^2) is an equivalence in vqcoh($\underline{X}$) *then f^0, f^3 are isomorphisms.*

Proof. The middle square of (3.2) commutes by definition of (f^1, f^2). Hence $\psi^1 \circ (f^1 \circ \phi^0) = f^2 \circ \phi^1 \circ \phi^0 = 0$, so as $\psi^0 = \operatorname{Ker} \psi^1$ there is a unique morphism $f^0 : \mathcal{E}^0 \to \mathcal{F}^0$ with $\psi^0 \circ f^0 = f^1 \circ \phi^0$. Similarly $(\psi^2 \circ f^2) \circ \phi^1 = \psi^2 \circ \psi^1 \circ f^1 = 0$, so as $\phi^2 = \operatorname{Coker} \phi^1$ there is a unique morphism $f^3 : \mathcal{E}^3 \to \mathcal{F}^3$ with $f^3 \circ \phi^2 = \psi^2 \circ f^2$. Therefore (3.2) commutes.

Now suppose (f^1, f^2) is an equivalence in vqcoh($\underline{X}$). Then there exist a 1-morphism $(g^1, g^2) : (\mathcal{F}^1, \mathcal{F}^2, \psi^1) \to (\mathcal{E}^1, \mathcal{E}^2, \phi^1)$ and 2-morphisms $\eta : (g^1, g^2) \circ (f^1, f^2) \Rightarrow \mathrm{id}_{(\mathcal{E}^1, \mathcal{E}^2, \phi^1)}$ and $\zeta : (f^1, f^2) \circ (g^1, g^2) \Rightarrow \mathrm{id}_{(\mathcal{F}^1, \mathcal{F}^2, \psi^1)}$ in vqcoh($\underline{X}$). As above we get a unique $g^0 : \mathcal{F}^0 \to \mathcal{E}^0$, $g^3 : \mathcal{F}^3 \to \mathcal{E}^3$ in qcoh($\underline{X}$) making the analogue of (3.2) commute. Consider the diagram in qcoh($\underline{X}$):

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^0 \arrow[r, "\phi^0"'] \arrow[d, "f^0"] & \mathcal{E}^1 \arrow[r, "\phi^1"'] \arrow[d, "f^1"] & \mathcal{E}^2 \arrow[r, "\phi^2"'] \arrow[d, "f^2"] \arrow[ddl, dotted, "\eta"] & \mathcal{E}^3 \arrow[r] \arrow[d, "f^3"] & 0 \\
0 \arrow[r] & \mathcal{F}^0 \arrow[r, "\psi^0"] \arrow[d, "g^0"] & \mathcal{F}^1 \arrow[r, "\psi^1"] \arrow[d, "g^1"] & \mathcal{F}^2 \arrow[r, "\psi^2"] \arrow[d, "g^2"] & \mathcal{F}^3 \arrow[r] \arrow[d, "g^3"] & 0 \\
0 \arrow[r] & \mathcal{E}^0 \arrow[r, "\phi^0"] & \mathcal{E}^1 \arrow[r, "\phi^1"] & \mathcal{E}^2 \arrow[r, "\phi^2"] & \mathcal{E}^3 \arrow[r] & 0.
\end{tikzcd} \tag{3.3}$$

This commutes, apart from η marked '$\dashrightarrow$'. Since $\eta : (g^1, g^2) \circ (f^1, f^2) \Rightarrow \mathrm{id}_{(\mathcal{E}^1, \mathcal{E}^2, \phi^1)}$ is a 2-morphism, we find the composition of the second column in (3.3) is $g^1 \circ f^1 = \mathrm{id}_{\mathcal{E}^1} - \eta \circ \phi^1 : \mathcal{E}^1 \to \mathcal{E}^1$. Composing this with ϕ^0 gives $\mathrm{id}_{\mathcal{E}^1} \circ \phi^0 = \phi^0 = \phi^0 \circ \mathrm{id}_{\mathcal{E}^0}$, since $\phi^1 \circ \phi^0 = 0$. Thus as (3.3) is commutative, ϕ^0 composed with the first column is $\phi^0 \circ \mathrm{id}_{\mathcal{E}^0}$, so the composition of the first column is $\mathrm{id}_{\mathcal{E}^0}$ since ϕ^0 is injective. Hence $g^0 \circ f^0 = \mathrm{id}_{\mathcal{E}^0}$. Similarly $g^3 \circ f^3 = \mathrm{id}_{\mathcal{E}^3}$. By the analogue of (3.3) with f^j, g^j and η, ζ exchanged we can show that $f^0 \circ g^0 = \mathrm{id}_{\mathcal{F}^0}$ and $f^3 \circ g^3 = \mathrm{id}_{\mathcal{F}^3}$. Therefore f^0, f^3 are isomorphisms, with inverses g^0, g^3. $\square$

The second is related to Proposition 2.20 and 2.21:

Proposition 3.5. *Let $\underline{X}$ be a C^∞-scheme and $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ a 1-morphism in* $\mathrm{vqcoh}(\underline{X})$*. Then $f^\bullet$ is an equivalence if and only if the following complex is a split short exact sequence in* $\mathrm{qcoh}(\underline{X})$*:*

$$0 \longrightarrow \mathcal{E}^1 \xrightarrow{f^1 \oplus -\phi} \mathcal{F}^1 \oplus \mathcal{E}^2 \xrightarrow{\psi \oplus f^2} \mathcal{F}^2 \longrightarrow 0. \tag{3.4}$$

In particular, if $f^\bullet$ is an equivalence then $\mathcal{E}^1 \oplus \mathcal{F}^2 \cong \mathcal{F}^1 \oplus \mathcal{E}^2$ in $\mathrm{qcoh}(\underline{X})$*.*

Proof. Note that (3.4) is a complex since $\psi \circ f^1 = f^2 \circ \phi$ as $f^\bullet$ is a 1-morphism. Suppose $f^\bullet$ is an equivalence. Then there exists a 1-morphism $g^\bullet : (\mathcal{F}^\bullet, \psi) \to (\mathcal{E}^\bullet, \phi)$ and 2-morphisms $\eta : g^\bullet \circ f^\bullet \Rightarrow \mathrm{id}_{(\mathcal{E}^\bullet,\phi)}$ and $\zeta : f^\bullet \circ g^\bullet \Rightarrow \mathrm{id}_{(\mathcal{F}^\bullet,\psi)}$. By Proposition A.6 we can also choose η, ζ to satisfy $\mathrm{id}_{f^\bullet} * \eta = \zeta * \mathrm{id}_{f^\bullet}$ and $\mathrm{id}_{g^\bullet} * \zeta = \eta * \mathrm{id}_{g^\bullet}$. Define morphisms $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ and $\delta : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ by $\gamma = g^1 \oplus -\eta$ and $\delta = \zeta \oplus g^2$. Then using all the relations on $f^1, f^2, g^1, g^2, \phi, \psi, \eta, \zeta$ we find that γ, δ satisfy (2.33) for the complex (3.4), so (3.4) is a split short exact sequence by Definition 2.19. This proves the 'only if' part.

Suppose (3.4) is a split short exact sequence. Then there exist $\gamma = \gamma_1 \oplus \gamma_2 : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ and $\delta = \delta_1 \oplus \delta_2 : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ satisfying (2.33) for (3.4). Define $g^1 = \gamma_1$, $g^2 = \delta_2$, $\eta = -\gamma_2$ and $\zeta = \delta_1$. Then (2.33) implies that (g^1, g^2) is a 1-morphism $(\mathcal{F}^\bullet, \psi) \to (\mathcal{E}^\bullet, \phi)$ in $\mathrm{vqcoh}(\underline{X})$ and $\eta : g^\bullet \circ f^\bullet \Rightarrow \mathrm{id}_{(\mathcal{E}^\bullet,\phi)}$, $\zeta : f^\bullet \circ g^\bullet \Rightarrow \mathrm{id}_{(\mathcal{F}^\bullet,\psi)}$ are 2-morphisms. Hence $f^\bullet$ is an equivalence. This proves the 'if' part. The last part is immediate. □

For the next corollary, the last part of Proposition 3.5 gives $\mathcal{F}^1 \oplus \mathcal{G}^2 \cong \mathcal{G}^1 \oplus \mathcal{F}^2$, so taking ranks gives $\mathrm{rank}\,\mathcal{F}^1 + \mathrm{rank}\,\mathcal{G}^2 = \mathrm{rank}\,\mathcal{G}^1 + \mathrm{rank}\,\mathcal{F}^2$.

Corollary 3.6. *Let $\underline{U}$ be a nonempty C^∞-scheme, and suppose $(\mathcal{F}^\bullet, \psi)$ and $(\mathcal{G}^\bullet, \xi)$ are equivalent in* $\mathrm{vqcoh}(\underline{U})$*, with $\mathcal{F}^1, \mathcal{F}^2, \mathcal{G}^1, \mathcal{G}^2$ vector bundles on $\underline{U}$. Then* $\mathrm{rank}\,\mathcal{F}^2 - \mathrm{rank}\,\mathcal{F}^1 = \mathrm{rank}\,\mathcal{G}^2 - \mathrm{rank}\,\mathcal{G}^1$.

Proposition 3.7. *Suppose $(\mathcal{E}^\bullet, \phi)$ is a virtual vector bundle of rank d on a nonempty C^∞-scheme $\underline{X}$. Then the rank d can be recovered from the restrictions of $\phi, \mathcal{E}^1, \mathcal{E}^2$ to an arbitrarily small neighbourhood of any point $x \in \underline{X}$. Hence* $\mathrm{rank}(\mathcal{E}^\bullet, \phi)$ *is well-defined.*

Proof. Let $x \in \underline{X}$. By Definition 3.1 there exists open $x \in \underline{U} \subseteq \underline{X}$ such that $\phi|_{\underline{U}} : \mathcal{E}^1|_{\underline{U}} \to \mathcal{E}^2|_{\underline{U}}$ is equivalent to $\psi : \mathcal{F}^1 \to \mathcal{F}^2$, where $\mathcal{F}^1, \mathcal{F}^2$ are vector bundles on $\underline{U}$ with $\mathrm{rank}\,\mathcal{F}^2 - \mathrm{rank}\,\mathcal{F}^1 = d$. Suppose also that $x \in \underline{V} \subseteq \underline{X}$ is open and $\phi|_{\underline{V}} : \mathcal{E}^1|_{\underline{V}} \to \mathcal{E}^2|_{\underline{V}}$ is equivalent to $\xi : \mathcal{G}^1 \to \mathcal{G}^2$, where $\mathcal{G}^1, \mathcal{G}^2$ are vector bundles on $\underline{V}$. Then the restrictions of $\psi : \mathcal{F}^1 \to \mathcal{F}^2$ and $\xi : \mathcal{G}^1 \to \mathcal{G}^2$ to $\underline{U} \cap \underline{V}$ are equivalent, so Corollary 3.6 gives $\mathrm{rank}\,\mathcal{G}^2 - \mathrm{rank}\,\mathcal{G}^1 = \mathrm{rank}\,\mathcal{F}^2 - \mathrm{rank}\,\mathcal{F}^1 = d$.

Therefore we can characterize the rank d as follows: if $\phi|_{\underline{V}} : \mathcal{E}^1|_{\underline{V}} \to \mathcal{E}^2|_{\underline{V}}$ is equivalent to $\xi : \mathcal{G}^1 \to \mathcal{G}^2$ on any open $x \in \underline{V} \subseteq \underline{X}$ for $\mathcal{G}^1, \mathcal{G}^2$ vector bundles on $\underline{V}$, then $d = \mathrm{rank}\,\mathcal{G}^2 - \mathrm{rank}\,\mathcal{G}^1$. Such $\underline{V}, \xi, \mathcal{G}^1, \mathcal{G}^2$ exist by definition. By restricting to a smaller open neighbourhood we can make $\underline{V}$ arbitrarily small. Thus we can recover the rank d from the restriction of $\phi, \mathcal{E}^1, \mathcal{E}^2$ to an arbitrarily small neighbourhood of any $x \in \underline{X}$. □

In Definition 2.14 we defined a functor $F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}} : \mathbf{C^\infty Sch^{lf}_{ssc}} \to \mathbf{dSpa}$ and a 2-subcategory $\hat{\mathbf{C}}^\infty\mathbf{Sch^{lf}_{ssc}}$ in $\mathbf{dSpa}$ equivalent to $\mathbf{C^\infty Sch^{lf}_{ssc}}$. Here is an analogue of this at the level of quasicoherent and virtual quasicoherent sheaves on $\underline{X}$.

Definition 3.8. Let $\underline{X}$ be a C^∞-scheme. Regard the category $\mathrm{qcoh}(\underline{X})$ as a 2-category with only identity 2-morphisms. Define a strict 2-functor $F_{\mathrm{qcoh}(\underline{X})}^{\mathrm{vqcoh}(\underline{X})}$: $\mathrm{qcoh}(\underline{X}) \to \mathrm{vqcoh}(\underline{X})$ by $F_{\mathrm{qcoh}(\underline{X})}^{\mathrm{vqcoh}(\underline{X})}(\mathcal{E}) = (0, \mathcal{E}, 0)$ on objects, $F_{\mathrm{qcoh}(\underline{X})}^{\mathrm{vqcoh}(\underline{X})}(\alpha) = (0, \alpha) : (0, \mathcal{E}, 0) \to (0, \mathcal{F}, 0)$ on (1-)morphisms $\alpha : \mathcal{E} \to \mathcal{F}$, and $F_{\mathrm{qcoh}(\underline{X})}^{\mathrm{vqcoh}(\underline{X})}(\mathrm{id}_\alpha) = 0 = \mathrm{id}_{(0,\alpha)} : (0, \alpha) \Rightarrow (0, \alpha)$ on 2-morphisms $\mathrm{id}_\alpha : \alpha \Rightarrow \alpha$. It is easy to check that $F_{\mathrm{qcoh}(\underline{X})}^{\mathrm{vqcoh}(\underline{X})}$ is full and faithful.

The restriction of $F_{\mathrm{qcoh}(\underline{X})}^{\mathrm{vqcoh}(\underline{X})}$ to $\mathrm{vect}(\underline{X})$ maps to $\mathrm{vvect}(\underline{X})$, and so defines a full and faithful strict 2-functor $F_{\mathrm{vect}(\underline{X})}^{\mathrm{vvect}(\underline{X})} : \mathrm{vect}(\underline{X}) \to \mathrm{vvect}(\underline{X})$. Define $\hat{\mathrm{q}}\mathrm{coh}(\underline{X})$ and $\hat{\mathrm{v}}\mathrm{ect}(\underline{X})$ to be the full 2-subcategories of objects in $\mathrm{vqcoh}(\underline{X})$ and $\mathrm{vvect}(\underline{X})$ equivalent to objects in the images of $F_{\mathrm{qcoh}(\underline{X})}^{\mathrm{vqcoh}(\underline{X})}$ and $F_{\mathrm{vect}(\underline{X})}^{\mathrm{vvect}(\underline{X})}$. Then $\hat{\mathrm{q}}\mathrm{coh}(\underline{X}), \hat{\mathrm{v}}\mathrm{ect}(\underline{X})$ are equivalent as 2-categories to $\mathrm{qcoh}(\underline{X}), \mathrm{vect}(\underline{X})$.

By an abuse of notation, we will say that a virtual quasicoherent sheaf $(\mathcal{E}^\bullet, \phi)$ *is a quasicoherent sheaf*, or *is a vector bundle*, if it lies in $\hat{\mathrm{q}}\mathrm{coh}(\underline{X})$ or $\hat{\mathrm{v}}\mathrm{ect}(\underline{X})$.

The next result is related to Proposition 2.25.

Proposition 3.9. *Let $(\mathcal{E}^\bullet, \phi)$ be a virtual quasicoherent sheaf (or a virtual vector bundle) on $\underline{X}$. Then $(\mathcal{E}^\bullet, \phi)$ is a quasicoherent sheaf (respectively a vector bundle) if and only if $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ has a left inverse in* $\mathrm{qcoh}(\underline{X})$.

Proof. Suppose $(\mathcal{E}^\bullet, \phi)$ is a quasicoherent sheaf. Then by definition $(\mathcal{E}^\bullet, \phi)$ is equivalent to $(0, \mathcal{F}, 0)$ for some $\mathcal{F} \in \mathrm{qcoh}(\underline{X})$, so there exist 1-morphisms $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (0, \mathcal{F}, 0)$, $g^\bullet : (0, \mathcal{F}, 0) \to (\mathcal{E}^\bullet, \phi)$ and 2-morphisms $\eta : g^\bullet \circ f^\bullet \Rightarrow \mathrm{id}_{(\mathcal{E}^\bullet,\phi)}$ and $\zeta : f^\bullet \circ g^\bullet \Rightarrow \mathrm{id}_{(0,\mathcal{F},0)}$. We have $\eta\circ\phi + g^1 \circ f^1 = \mathrm{id}_{\mathcal{E}^1}$, but $f^1 = g^1 = 0$, so $\eta \circ \phi = \mathrm{id}_{\mathcal{E}^1}$, and η is a left inverse for ϕ.

Suppose $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ has a left inverse $\eta : \mathcal{E}^2 \to \mathcal{E}^1$. Let $f^2 : \mathcal{E}^2 \to \mathcal{F}$ be the cokernel of ϕ. Since ϕ has a left inverse η we have $\mathcal{E}^2 \cong \mathcal{E}^1 \oplus \mathcal{F}$, and there is a unique morphism $g^2 : \mathcal{F} \to \mathcal{E}^2$ which is the kernel of η and has $g^2 \circ f^2 = \mathrm{id}_{\mathcal{F}}$. Define $f^1 = g^1 = \zeta = 0$. Then we find that $(f^1, f^2) : (\mathcal{E}^\bullet, \phi) \to (0, \mathcal{F}, 0)$ and $(g^1, g^2) : (0, \mathcal{F}, 0) \to (\mathcal{E}^\bullet, \phi)$ are 1-morphisms and $\eta : g^\bullet \circ f^\bullet \Rightarrow \mathrm{id}_{(\mathcal{E}^\bullet,\phi)}$, $\zeta : f^\bullet \circ g^\bullet \Rightarrow \mathrm{id}_{(0,\mathcal{F},0)}$ are 2-morphisms. Hence $f^\bullet$ is an equivalence, and $(\mathcal{E}^\bullet, \phi)$ is a quasicoherent sheaf. For the (virtual) vector bundle parts, if $(\mathcal{E}^\bullet, \phi)$ is a virtual vector bundle, one can show using the last part of Proposition 3.5 that $\mathcal{F}$ above is a vector bundle. $\square$

Propositions 2.25 and 3.9 characterize when a d-space is a C^∞-scheme.

Corollary 3.10. *Let $\boldsymbol{X}$ be a d-space. Then $\boldsymbol{X}$ lies in $\hat{\mathbf{C}}^\infty\mathbf{Sch^{lf}_{ssc}}$ if and only if its virtual cotangent sheaf $T^*\boldsymbol{X}$ is a quasicoherent sheaf.*

This illustrates the fact that although the virtual cotangent sheaf $T^*\boldsymbol{X} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X)$ is only a small part of the data in a d-space $\boldsymbol{X}$, it is useful information, and questions about d-spaces and d-manifolds can often be answered by working with virtual cotangent sheaves.

3.2 The definition of d-manifolds

We will now define the 2-category **dMan** of d-manifolds, as a 2-subcategory of the 2-category of d-spaces **dSpa**. We follow Spivak's treatment of his derived manifolds [94, 95]. He first defines an ∞-category $\mathbf{DLC^\infty RS}$ of 'derived local C^∞-ringed spaces', with manifolds **Man** embedded as a discrete ∞-subcategory, and shows (homotopy) fibre products exist in $\mathbf{DLC^\infty RS}$. Then he defines *principal derived manifolds* as (homotopy) fibre products $\mathbb{R}^m \times_{g,\mathbb{R}^n,0} \{0\}$ in $\mathbf{DLC^\infty RS}$ for $g : \mathbb{R}^m \to \mathbb{R}^n$ smooth. Finally he defines $\boldsymbol{X}$ in $\mathbf{DLC^\infty RS}$ to be a derived manifold if it can be covered by open subspaces $\boldsymbol{U}$ which are principal derived manifolds.

For d-manifolds we replace $\mathbf{DLC^\infty RS}$ by our 2-category of d-spaces **dSpa**. Our notion of principal d-manifold is more general, as we allow fibre products $X \times_{g,Z,h} Y$ for manifolds X, Y, Z, but this is a cosmetic difference and does not change the definition of d-manifold. We also define the virtual dimension of a principal d-manifold, and require all the principal covering subspaces $\boldsymbol{U}$ of a d-manifold $\boldsymbol{X}$ to have the same virtual dimension, which Spivak does not do.

Definition 3.11. A d-space $\boldsymbol{W}$ is called a *principal d-manifold* if is equivalent in **dSpa** to a fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ with $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \hat{\mathbf{Man}}$. That is, $\boldsymbol{W} \simeq F_{\mathbf{Man}}^{\mathbf{dSpa}}(X) \times_{F_{\mathbf{Man}}^{\mathbf{dSpa}}(g), F_{\mathbf{Man}}^{\mathbf{dSpa}}(Z), F_{\mathbf{Man}}^{\mathbf{dSpa}}(h)} F_{\mathbf{Man}}^{\mathbf{dSpa}}(Y)$ for manifolds X, Y, Z and smooth maps $g : X \to Z$, $h : Y \to Z$.

If $\boldsymbol{W} = (\underline{W}, \mathcal{O}'_W, \mathcal{E}_W, \imath_W, \jmath_W)$ then the underlying C^∞-scheme $\underline{W}$ has $\underline{W} \cong \underline{X} \times_{\underline{Z}} \underline{Y}$, where $\underline{X}, \underline{Y}, \underline{Z} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(X, Y, Z)$. From §B.5, $\underline{X}, \underline{Y}, \underline{Z}$ are finitely presented affine C^∞-schemes. But these are closed under fibre products by [56, Th. 4.19], so $\underline{W}$ is also a finitely presented affine C^∞-scheme. If X is a manifold, then taking $Y = Z = *$, the point, and $g = \pi : X \to *$, $h = \mathrm{id}_* : * \to *$, we get $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{*}} \boldsymbol{*} \simeq \boldsymbol{X}$. Thus $F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$ is a principal d-manifold, and any object in $\hat{\mathbf{Man}}$ is a principal d-manifold.

Actually, since anything equivalent to a (2-category) fibre product is also a fibre product, we could just say that principal d-manifolds *are* fibre products $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ with $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ in $\hat{\mathbf{Man}}$. Here are two alternative definitions of principal d-manifolds:

Proposition 3.12. *The following are equivalent characterizations of when a d-space $\boldsymbol{W}$ is a principal d-manifold:*

(a) $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *for* $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \hat{\mathbf{Man}}$.

(b) $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{i},\boldsymbol{Z},\boldsymbol{j}} \boldsymbol{Y}$, *where X, Y, Z are manifolds, $i : X \to Z$, $j : Y \to Z$ are embeddings, and $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{i}, \boldsymbol{j} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(X, Y, Z, i, j)$. That is, $\boldsymbol{W}$ is an intersection of two submanifolds X, Y in Z, in the sense of d-spaces.*

(c) *$\boldsymbol{W} \simeq \boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$, where V is a manifold, $E \to V$ is a vector bundle, $s : V \to E$ is a smooth section of E, $0 : V \to E$ is the zero section, and $\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(V, E, s, 0)$. That is, $\boldsymbol{W}$ is the zeroes $s^{-1}(0)$ of a smooth section s of a vector bundle E, in the sense of d-spaces.*

Proof. Clearly (c) implies (b) implies (a). So it is enough to show (a) implies (c). For $\boldsymbol{W}$ as in (a) we can take $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{g}, \boldsymbol{h} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(X, Y, Z, g, h)$ for $g : X \to Z$, $h : Y \to Z$ smooth maps of manifolds. By the existence of tubular neighbourhoods, we can choose an open neighbourhood U of the zero section in TZ and an étale map $\Phi : U \to Z \times Z$ such that $\Phi(z, 0) = (z, z)$ for all $z \in Z$.

Define a manifold $V = (X \times Y) \times_{g \times h, Z \times Z, \Phi} U$, where the fibre product exists as Φ is étale. Define a vector bundle $E \to V$ by $E = (\pi_Z \circ \pi_U)^*(TZ)$, where $\pi_U : V \to U$ is the projection from the fibre product and $\pi_Z : U \to Z$ is the restriction of the projection $TZ \to Z$. Define a smooth section s of V by $s : \bigl((x, y), (z, v)\bigr) \mapsto \bigl(((x, y), (z, v)), v\bigr)$. Then we have equivalences of fibre products in **dSpa**:

$$
\begin{aligned}
&\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y} \simeq (\boldsymbol{X} \times \boldsymbol{Y}) \times_{\boldsymbol{g}\times\boldsymbol{h},\boldsymbol{Z}\times\boldsymbol{Z},\boldsymbol{\Delta_Z}} \boldsymbol{Z} \\
&\quad \simeq \bigl((\boldsymbol{X}\times\boldsymbol{Y})\times_{\boldsymbol{g}\times\boldsymbol{h},\boldsymbol{Z}\times\boldsymbol{Z},\boldsymbol{\Phi}} \boldsymbol{U}\bigr)\times_{\boldsymbol{\pi_U},\boldsymbol{U},\boldsymbol{0}} \boldsymbol{Z} \simeq \boldsymbol{V}\times_{\boldsymbol{\pi_U},\boldsymbol{TZ},\boldsymbol{0}} \boldsymbol{Z} \simeq \boldsymbol{V}\times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V},
\end{aligned}
$$

where $\Delta_Z : Z \to Z \times Z$ is the diagonal map. So (c) implies (a). $\square$

Of (a)–(c) in Proposition 3.12, we usually prefer to work with form (c). It is related to the notion of *Kuranishi neighbourhood* in the Kuranishi spaces of Fukaya, Oh, Ohta and Ono [32,34], as we will explain in Chapter 14. The fibre product $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$ is only defined up to equivalence in **dSpa**. We will find it convenient to work with a particular explicit choice in this equivalence class, which we call the *standard model* $\boldsymbol{S}_{V,E,s}$.

Definition 3.13. Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section of E. We will write down an explicit principal d-manifold $\boldsymbol{S} = (\underline{S}, \mathcal{O}'_S, \mathcal{E}_S, \imath_S, \jmath_S)$ which is equivalent to $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$ in Proposition 3.12(c). We call $\boldsymbol{S}$ the *standard model* of (V, E, s), and also write it $\boldsymbol{S}_{V,E,s}$. Proposition 3.12 shows that every principal d-manifold $\boldsymbol{W}$ is equivalent to $\boldsymbol{S}_{V,E,s}$ for some V, E, s.

Write $C^\infty(V)$ for the C^∞-ring of smooth functions $c : V \to \mathbb{R}$, and $C^\infty(E)$, $C^\infty(E^*)$ for the vector spaces of smooth sections of E, E^* over V. Then $s \in C^\infty(E)$, and $C^\infty(E), C^\infty(E^*)$ are modules over $C^\infty(V)$, and there is a natural bilinear product $\cdot : C^\infty(E^*) \times C^\infty(E) \to C^\infty(V)$ from the dual pairing $E^* \times E \to \mathbb{R}$. Define $I_s \subseteq C^\infty(V)$ to be the ideal generated by s. That is,

$$I_s = \bigl\{\alpha \cdot s : \alpha \in C^\infty(E^*)\bigr\} \subseteq C^\infty(V). \tag{3.5}$$

Let $I_s^2 = \langle fg : f, g \in I_s \rangle_{\mathbb{R}}$ be the square of I_s. Then I_s^2 is an ideal in $C^\infty(V)$, the ideal generated by $s \otimes s \in C^\infty(E \otimes E)$. That is,

$$I_s^2 = \bigl\{\beta \cdot (s \otimes s) : \beta \in C^\infty(E^* \otimes E^*)\bigr\} \subseteq C^\infty(V). \tag{3.6}$$

Define C^∞-rings $\mathfrak{C} = C^\infty(V)/I_s$, $\mathfrak{C}' = C^\infty(V)/I_s^2$, and let $\pi : \mathfrak{C}' \to \mathfrak{C}$ be the natural projection from the inclusion $I_s^2 \subseteq I_s$. Then π is a square zero extension of C^∞-rings, in an exact sequence as for (2.1):

$$0 \longrightarrow I_s/I_s^2 \xrightarrow{\kappa_\pi} \mathfrak{C}' \xrightarrow{\pi} \mathfrak{C} \longrightarrow 0.$$

Define a topological space $S = \{v \in V : s(v) = 0\}$, as a subspace of V. Now $s(v) = 0$ if and only if $(s \otimes s)(v) = 0$. Thus S is the underlying topological space for both $\operatorname{Spec}\mathfrak{C}$ and $\operatorname{Spec}\mathfrak{C}'$. So applying Spec gives $\operatorname{Spec}\mathfrak{C} = \underline{S} = (S, \mathcal{O}_S)$, $\operatorname{Spec}\mathfrak{C}' = \underline{S}' = (S, \mathcal{O}_S')$, and $\operatorname{Spec}\pi = \underline{\imath}_S = (\mathrm{id}_S, \imath_S) : \underline{S}' \to \underline{S}$, where $\underline{S}, \underline{S}'$ are fair affine C^∞-schemes, and $\mathcal{O}_S, \mathcal{O}_S'$ are sheaves of C^∞-rings on S, and $\imath_S : \mathcal{O}_S' \to \mathcal{O}_S$ is a morphism of sheaves of C^∞-rings. Then $(\underline{S}, \mathcal{O}_S', \imath_S)$ is a square zero extension of C^∞-schemes, as π is a square zero extension of C^∞-rings. As $\underline{S}$ is fair and affine it is separated, second countable and locally fair.

From (3.5) we have a surjective morphism of $C^\infty(V)$-modules $C^\infty(E^*) \to I_s$ mapping $\alpha \mapsto \alpha \cdot s$. Applying $\otimes_{C^\infty(V)}\mathfrak{C}$ gives a morphism of $\mathfrak{C}$-modules

$$\sigma : C^\infty(E^*)/(I_s \cdot C^\infty(E^*)) \longrightarrow I_s/I_s^2, \quad \sigma : \alpha + (I_s \cdot C^\infty(E^*)) \longmapsto \alpha \cdot s + I_s^2.$$

Here σ is surjective as $\otimes_{C^\infty(V)}\mathfrak{C}$ is right exact and so preserves surjectivity. Define a quasicoherent sheaf $\mathcal{E}_S$ on $\underline{S}$ by $\mathcal{E}_S = \operatorname{MSpec}\bigl(C^\infty(E^*)/(I_s \cdot C^\infty(E^*))\bigr)$. Also $\operatorname{MSpec}(I_s/I_s^2)$ is the sheaf $\mathcal{I}_S$ for the square zero extension $(\underline{S}, \mathcal{O}_S', \imath_S)$. Thus $\jmath_S = \operatorname{MSpec}\sigma$ is a morphism $\jmath_S : \mathcal{E}_S \to \mathcal{I}_S$ in $\mathrm{qcoh}(\underline{S})$, which is surjective as σ is surjective and $\operatorname{MSpec} : \mathfrak{C}\text{-mod} \to \mathrm{qcoh}(\underline{S})$ is an exact functor by [56, §6.2]. Therefore $\boldsymbol{S}_{V,E,s} = \boldsymbol{S} = (\underline{S}, \mathcal{O}_S', \mathcal{E}_S, \imath_S, \jmath_S)$ is a d-space.

In fact $\mathcal{E}_S$ is a vector bundle on $\underline{S}$, naturally isomorphic to $\mathcal{E}^*|_{\underline{S}}$, where $\mathcal{E}$ is the vector bundle on $\underline{V} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(V)$ corresponding to $E \to V$, and $\mathcal{E}^*$ the dual vector bundle, and $\mathcal{E}^*|_{\underline{S}}$ is the restriction of $\mathcal{E}^*$ to the C^∞-subscheme $\underline{S}$ in $\underline{V}$. One can also show that $\mathcal{F}_S \cong T^*\underline{V}|_{\underline{S}}$. The morphism $\phi_S : \mathcal{E}_S \to \mathcal{F}_S$ can be interpreted as follows: choose a connection ∇ on $E \to V$. Then $\nabla s \in C^\infty(E \otimes T^*V)$, so we can regard ∇s as a morphism of vector bundles $E^* \to T^*V$ on V. This lifts to a morphism of vector bundles $\hat\nabla s : \mathcal{E}^* \to T^*\underline{V}$ on the C^∞-scheme $\underline{V}$, and ϕ_S is identified with $\hat\nabla s|_{\underline{S}} : \mathcal{E}^*|_{\underline{S}} \to T^*\underline{V}|_{\underline{S}}$ under the isomorphisms $\mathcal{E}^*|_{\underline{S}} \cong \mathcal{E}_S$, $T^*\underline{V}|_{\underline{S}} \cong \mathcal{F}_S$. Although ∇s depends on the choice of ∇, its restriction to $\underline{S} = s^{-1}(0)$ is independent of this choice.

This $\boldsymbol{S}$ is equivalent to the fibre product $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$ in Proposition 3.12(c). One way to prove this is to compare it to the fibre product

$$\tilde{\boldsymbol{S}} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(V) \times_{F_{\mathbf{Man}}^{\mathbf{dSpa}}(s), F_{\mathbf{Man}}^{\mathbf{dSpa}}(E), F_{\mathbf{Man}}^{\mathbf{dSpa}}(0)} F_{\mathbf{Man}}^{\mathbf{dSpa}}(V),$$

where the fibre product is done using the explicit construction of §2.5. One can show that $\tilde{\boldsymbol{S}}$ is 1-isomorphic to the d-space constructed in Example 2.18 starting with the d-space $\boldsymbol{S}$ and the quasicoherent sheaf $\mathcal{F}_S$, so that $\mathcal{E}_{\tilde S} \cong \mathcal{E}_S \oplus \mathcal{F}_S$, $\mathcal{I}_{\tilde S} \cong \mathcal{I}_S \oplus \mathcal{I}_S$, $\mathcal{F}_{\tilde S} \cong \mathcal{F}_S \oplus \mathcal{F}_S$. Thus $\boldsymbol{S}, \tilde{\boldsymbol{S}}$ are equivalent by Example 2.18.

Lemma 3.14. *Let $\boldsymbol{W}$ be a principal d-manifold and $\boldsymbol{U}$ an open d-subspace of $\boldsymbol{W}$. Then $\boldsymbol{U}$ is also a principal d-manifold.*

Proof. By Proposition 3.12(b) we may write $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{i},\boldsymbol{Z},\boldsymbol{j}} \boldsymbol{Y}$, where $i(X), j(Y)$ are embedded submanifolds of Z. The underlying topological space W of $\boldsymbol{W}$ is $i(X) \cap j(Y)$. So any open $U \subseteq W$ is of the form $V \cap i(X) \cap j(Y)$ for $V \subseteq Z$ open. Equivalently, $U = i(\tilde{X}) \cap j(\tilde{Y})$, where $\tilde{X} = i^{-1}(V) \subseteq X$, $\tilde{Y} = j^{-1}(V) \subseteq Y$. Hence $\boldsymbol{U} \simeq \tilde{\boldsymbol{X}} \times_{\boldsymbol{i},\boldsymbol{Z},\boldsymbol{j}} \tilde{\boldsymbol{Y}}$, and so is a principal d-manifold. $\square$

Proposition 3.15. *Suppose $\boldsymbol{W}$ is a principal d-manifold, so that for manifolds X, Y, Z we have $\boldsymbol{W} \simeq F_{\mathbf{Man}}^{\mathbf{dSpa}}(X) \times_{F_{\mathbf{Man}}^{\mathbf{dSpa}}(Z)} F_{\mathbf{Man}}^{\mathbf{dSpa}}(Y)$. Then the virtual cotangent sheaf $T^*\boldsymbol{W} = (\mathcal{E}_W, \mathcal{F}_W, \phi_W)$ is a virtual vector bundle on $\underline{W}$, with*

$$\operatorname{rank} T^*\boldsymbol{W} = \dim X + \dim Y - \dim Z. \tag{3.7}$$

Proof. Define $\tilde{\boldsymbol{W}} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(X) \times_{F_{\mathbf{Man}}^{\mathbf{dSpa}}(Z)} F_{\mathbf{Man}}^{\mathbf{dSpa}}(Y)$, where the fibre product is done using the explicit construction of Definition 2.35. Then there exists an equivalence $\boldsymbol{b} : \boldsymbol{W} \to \tilde{\boldsymbol{W}}$ in $\mathbf{dSpa}$. As in the proof of Theorem 2.42, $\phi_{\tilde{W}} : \mathcal{E}_{\tilde{W}} \to \mathcal{F}_{\tilde{W}}$ is given by (2.87), so $\mathcal{E}_{\tilde{W}} = (\underline{g} \circ \underline{e})^*(T^*\underline{Z})$ and $\mathcal{F}_{\tilde{W}} \cong \underline{e}^*(T^*\underline{X}) \oplus \underline{f}^*(T^*\underline{Y})$. Hence $\mathcal{E}_{\tilde{W}}, \mathcal{F}_{\tilde{W}}$ are vector bundles on $\underline{\tilde{W}}$ with $\operatorname{rank} \mathcal{E}_{\tilde{W}} = \dim Z$ and $\operatorname{rank} \mathcal{F}_{\tilde{W}} = \dim X + \dim Y$. Lemma 3.3 now shows that $\Omega_{\boldsymbol{b}} = (b'', b^2)$ is an equivalence from $\big(\underline{b}^*(\mathcal{E}_{\tilde{W}}), \underline{b}^*(\mathcal{F}_{\tilde{W}}), \underline{b}^*(\phi_{\tilde{W}})\big)$ to $(\mathcal{E}_W, \mathcal{F}_W, \phi_W)$. Therefore $(\mathcal{E}_W, \mathcal{F}_W, \phi_W)$ is a virtual vector bundle on $\underline{W}$, with rank $\operatorname{rank} \underline{b}^*(\mathcal{F}_{\tilde{W}}) - \operatorname{rank} \underline{b}^*(\mathcal{E}_{\tilde{W}}) = \dim X + \dim Y - \dim Z$. $\square$

Definition 3.16. Let $\boldsymbol{W}$ be a nonempty principal d-manifold. Then its virtual cotangent sheaf $T^*\boldsymbol{W} = (\mathcal{E}_W, \mathcal{F}_W, \phi_W)$ is a virtual vector bundle by Proposition 3.15, so we will call it the *virtual cotangent bundle* of $\boldsymbol{W}$. Define the *virtual dimension* $\operatorname{vdim} \boldsymbol{W}$ of $\boldsymbol{W}$ to be the rank of $T^*\boldsymbol{W}$. This is well-defined by Proposition 3.7. If $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ with X, Y, Z manifolds $\operatorname{vdim} \boldsymbol{W} = \dim X + \dim Y - \dim Z$ by (3.7).

Example 3.17. Let V be a manifold and $E \to V$ a vector bundle. Let $\underline{V} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(V)$, and $\mathcal{E} \in \operatorname{qcoh}(\underline{V})$ be the vector bundle on $\underline{V}$ corresponding to E. Define a d-space $\boldsymbol{S} = (\underline{V}, \mathcal{O}_V, \mathcal{E}^*, \mathrm{id}_{\mathcal{O}_V}, 0)$. Then $\boldsymbol{V}$ comes from Definition 3.13 with section $s = 0$, so $\boldsymbol{S}$ is a principal d-manifold, which has virtual dimension $\dim V - \operatorname{rank} E$. Note that this virtual dimension can be negative, and can take all values in $\mathbb{Z}$. We have modified the manifold V by adding an 'obstruction bundle' E^*, which reduces the virtual dimension by $\operatorname{rank} E$.

We can now define d-manifolds.

Definition 3.18. A d-space $\boldsymbol{X}$ is called a *d-manifold of virtual dimension* $n \in \mathbb{Z}$, written $\operatorname{vdim} \boldsymbol{X} = n$, if $\boldsymbol{X}$ can be covered by nonempty open d-subspaces $\boldsymbol{U}$ which are principal d-manifolds with $\operatorname{vdim} \boldsymbol{U} = n$.

Then the underlying C^∞-scheme $\underline{X}$ is covered by finitely presented, affine open $\underline{U} \subseteq \underline{X}$, so $\underline{X}$ is locally finitely presented. It is also separated, second countable, locally compact, paracompact, and locally fair by Definition 2.14.

Proposition 3.15 and Definition 3.16 imply that the virtual cotangent sheaf $T^*\boldsymbol{X} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X)$ of $\boldsymbol{X}$ is a virtual vector bundle of rank $\operatorname{vdim}\boldsymbol{X} = n$, so we call it the *virtual cotangent bundle* of $\boldsymbol{X}$. It is a d-manifold analogue of the *cotangent complex* in algebraic geometry, as in Illusie [50,51]. If $\boldsymbol{X}$ is nonempty then $T^*\boldsymbol{X}$ determines $\operatorname{vdim}\boldsymbol{X}$, so $\operatorname{vdim}\boldsymbol{X}$ depends only on $\boldsymbol{X}$. The empty d-manifold $\boldsymbol{\emptyset}$ can be considered a d-manifold of any virtual dimension $n \in \mathbb{Z}$, just as $\emptyset$ is a manifold of any dimension, so we leave $\operatorname{vdim}\boldsymbol{\emptyset}$ undefined.

Let **dMan** be the full 2-subcategory of d-manifolds in **dSpa**. As $F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$ is a principal d-manifold for any manifold X by Definition 3.11, it is a d-manifold, so the 2-functor $F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \rightarrow \mathbf{dSpa}$ maps into **dMan**, and we will write $F_{\mathbf{Man}}^{\mathbf{dMan}} = F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \rightarrow \mathbf{dMan}$. Also $\hat{\mathbf{Man}}$ is a 2-subcategory of **dMan**. We say that a d-manifold $\boldsymbol{X}$ *is a manifold* if it lies in $\hat{\mathbf{Man}}$.

In §4.4 we will prove that every compact d-manifold is principal, and give other sufficient conditions for d-manifolds to be principal, which suggest that most interesting d-manifolds are principal d-manifolds. Lemma 3.14 implies:

Lemma 3.19. *Let $\boldsymbol{W}$ be a d-manifold, and $\boldsymbol{U}$ an open d-subspace of $\boldsymbol{W}$. Then $\boldsymbol{U}$ is also a d-manifold, with* $\operatorname{vdim}\boldsymbol{U} = \operatorname{vdim}\boldsymbol{W}$.

3.3 Local properties of d-manifolds

We now study the local geometry of a d-manifold $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ near a point $x \in \underline{X}$, focussing on the C^∞-scheme $\underline{X}$. The questions we ask are:

- When can a C^∞-scheme $\underline{X}$ be (locally) extended to a d-manifold $\boldsymbol{X}$?
- How much of the information in $\boldsymbol{X}$, up to equivalence, is contained in the C^∞-scheme $\underline{X}$, and how much in the 'derived' data $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$?

From Definition 3.18, if $\boldsymbol{X}$ is a d-manifold then $\underline{X}$ is locally finitely presented, so we shall restrict to locally finitely presented C^∞-schemes.

Definition 3.20. Let $\underline{X}$ be a locally finitely presented C^∞-scheme, and $x \in \underline{X}$. We will define the *cotangent space* $T_x^*\underline{X}$ and the *obstruction space* $O_x\underline{X}$, both finite-dimensional real vector spaces. Let $\underline{X}_x$ be the localization of $\underline{X}$ at x, as a C^∞-scheme. Then $\underline{X}_x \cong \operatorname{Spec}\mathfrak{C}_x$, where $\mathfrak{C}_x$ is a C^∞-local ring, as in Definition B.7. Since $\underline{X}$ is locally finitely presented, $\mathfrak{C}_x$ is the localization at a point of a finitely presented C^∞-ring. Thus $\mathfrak{C}_x$ fits into an exact sequence

$$\begin{gathered} 0 \longrightarrow I \xrightarrow{\;\;\iota\;\;} C_0^\infty(\mathbb{R}^n) \xrightarrow{\;\;\pi\;\;} \mathfrak{C}_x \longrightarrow 0, \quad \text{where} \\ I = (f_1, \ldots, f_k) \subset C_0^\infty(\mathbb{R}^n) \text{ for } f_1, \ldots, f_k \in C^\infty(\mathbb{R}^n) \text{ with } f_j(0) = 0. \end{gathered} \tag{3.8}$$

Writing $y_1, \ldots, y_n$ for the generators of $C_0^\infty(\mathbb{R}^n)$, $\pi(y_1), \ldots, \pi(y_n)$ generate $\mathfrak{C}_x$.

Suppose $\mathrm{d}f_j(0) \neq 0 \in (\mathbb{R}^n)^*$ for some $j = 1, \ldots, k$. Then $f_j^{-1}(0)$ is an $(n-1)$-submanifold in $\mathbb{R}^n$ near 0, and $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ are coordinates on $f_j^{-1}(0)$ for some $i = 1, \ldots, n$ with $\frac{\partial f_j}{\partial y_i}(0) \neq 0$. We can then find an alternative presentation for $\mathfrak{C}_x$ with $n-1, k-1$ in place of n, k, omitting the generator

y_i and relation f_j, and replacing f_l for $l \neq j$ by $f_l|_{f_j^{-1}(0)}$ near 0, regarded as a function of $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$. Thus, if we choose a presentation (3.8) of $\mathfrak{C}_x$ with n as small as possible, then $\mathrm{d}f_j(0) = 0$ for $j = 1, \ldots, k$.

Write $\mathfrak{m}_x$ for the maximal ideal in $\mathfrak{C}_x$, identified with $\{h + I_x : h \in C^\infty(\mathbb{R}^n), h(0) = 0\}$ under (3.8), and $\Omega_{\mathfrak{C}_x}$ for the cotangent module of $\mathfrak{C}_x$. Define

$$T_x^*\underline{X} = \Omega_{\mathfrak{C}_x}/(\mathfrak{m}_x \cdot \Omega_{\mathfrak{C}_x}). \tag{3.9}$$

From the presentation (3.8) we may write $\mathfrak{C}_x$ as a pushout $C_0^\infty(\mathbb{R}^n) \amalg_{C^\infty(\mathbb{R}^k)} \mathbb{R}$, so by Theorem B.33(b), we get an exact sequence in $\mathfrak{C}_x$-mod

$$\mathbb{R}^k \otimes_{\mathbb{R}} \mathfrak{C}_x \xrightarrow{\alpha} \mathbb{R}^n \otimes_{\mathbb{R}} \mathfrak{C}_x \xrightarrow{\beta} \Omega_{\mathfrak{C}_x} \longrightarrow 0, \tag{3.10}$$

where $\alpha(c_1, \ldots, c_k) = (d_1, \ldots, d_n)$ with $d_i = \sum_{j=1}^k \frac{\partial f_j}{\partial y_i}(y_1, \ldots, y_n) \cdot c_j$. Applying the right exact functor $\otimes_{\mathfrak{C}_x}(\mathfrak{C}_x/\mathfrak{m}_x)$ to (3.10) and using (3.9) shows that

$$T_x^*\underline{X} \cong \frac{\langle \mathrm{d}y_1, \ldots, \mathrm{d}y_n \rangle_{\mathbb{R}}}{\langle \sum_{i=1}^n \frac{\partial f_1}{\partial y_i}(0)\,\mathrm{d}y_i, \ldots, \sum_{i=1}^n \frac{\partial f_k}{\partial y_i}(0)\,\mathrm{d}y_i \rangle_{\mathbb{R}}}. \tag{3.11}$$

When n is least, so that $\mathrm{d}f_j(0) = 0$ for all j, we have $T_x^*\underline{X} \cong \langle \mathrm{d}y_1, \ldots, \mathrm{d}y_n \rangle_{\mathbb{R}}$. Hence $\dim T_x^*\underline{X} = n$ *is the minimal number of generators of* $\mathfrak{C}_x$.

Next, choose a presentation (3.8) for $\mathfrak{C}_x$ with n least, and define

$$O_x\underline{X} = I/(\mathfrak{m}_0 \cdot I), \tag{3.12}$$

where $\mathfrak{m}_0$ is the maximal ideal in $C_0^\infty(\mathbb{R}^n)$. We claim that $O_x\underline{X}$ is independent of choices up to isomorphism. To see this, let $\tilde{I}, \tilde{\iota}, \tilde{\pi}, \tilde{f}_1, \ldots, \tilde{f}_{\tilde{k}}$ be an alternative presentation (3.8) for $\mathfrak{C}_x$ with n least, so the same n. As $\tilde{\pi}$ is surjective, for each $i = 1, \ldots, n$ we can choose $a_i \in C^\infty(\mathbb{R}^n)$ with $a_i(0, \ldots, 0) = 0$ such that $\tilde{\pi}(a_i(y_1, \ldots, y_n)) = \pi(y_i)$. Define a morphism of C^∞-rings $A : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$ by

$$A : g(y_1, \ldots, y_n) \longmapsto g\bigl(a_1(y_1, \ldots, y_n), \ldots, a_n(y_1, \ldots, y_n)\bigr).$$

Then $A(y_i) = a_i(y_1, \ldots, y_n)$, so $\tilde{\pi} \circ A(y_i) = \pi(y_i)$ for $i = 1, \ldots, n$, and thus $\tilde{\pi} \circ A = \pi : C_0^\infty(\mathbb{R}^n) \to \mathfrak{C}_x$ as $y_1, \ldots, y_n$ generate $C_0^\infty(\mathbb{R}^n)$.

Using (3.11) for both presentations, we can make a commutative diagram

$$\begin{tikzcd}
\langle \mathrm{d}y_1, \ldots, \mathrm{d}y_n \rangle_{\mathbb{R}} \arrow[d, "\mathrm{d}y_i \mapsto \sum_{j=1}^n \frac{\partial a_i}{\partial y_j}(0,\ldots,0)\,\mathrm{d}y_j"'] \arrow[dr, "\mathrm{d}\pi", "\cong"'] & \\
& T_x^*\underline{X}. \\
\langle \mathrm{d}y_1, \ldots, \mathrm{d}y_n \rangle_{\mathbb{R}} \arrow[ur, "\cong", "\mathrm{d}\tilde{\pi}"'] &
\end{tikzcd}$$

Then $\mathrm{d}\pi, \mathrm{d}\tilde{\pi}$ are isomorphisms as n is minimal, so the vertical map is an isomorphism, and $\bigl(\frac{\partial a_i}{\partial y_j}(0, \ldots, 0)\bigr)_{i,j=1}^n$ is an invertible $n \times n$ matrix. Hence the map

$\mathbb{R}^n \to \mathbb{R}^n$ taking $(y_1,\ldots,y_n) \mapsto \bigl(a_1(y_1,\ldots,y_n),\ldots,a_n(y_1,\ldots,y_n)\bigr)$ is a diffeomorphism near 0, and $A : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$ is an isomorphism. Therefore we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, "\cong"', "A|_I"] & C_0^\infty(\mathbb{R}^n) \arrow[r, "\pi"] \arrow[d, "\cong"', "A"] & \mathfrak{C}_x \arrow[r] \arrow[d, "\cong"', "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \tilde I \arrow[r, "\tilde\iota"] & C_0^\infty(\mathbb{R}^n) \arrow[r, "\tilde\pi"] & \mathfrak{C}_x \arrow[r] & 0, \end{tikzcd}$$

with columns isomorphisms. This induces an isomorphism $I/(\mathfrak{m}_0\cdot I) \cong \tilde I/(\mathfrak{m}_0\cdot \tilde I)$, so $O_x\underline{X}$ in (3.12) is independent of presentation up to isomorphism.

We have an exact sequence of $C_0^\infty(\mathbb{R}^n)$-modules:

$$0 \to \bigl\{(g_1,\ldots,g_k) : g_i \in C_0^\infty(\mathbb{R}^n),\ f_1g_1+\cdots+f_kg_k=0\bigr\} \xrightarrow{\subseteq} C_0^\infty(\mathbb{R}^n)^k \to I \to 0.$$

Applying $\otimes_{C_0^\infty(\mathbb{R}^n)}(C_0^\infty(\mathbb{R}^n)/\mathfrak{m}_0)$ to this and using (3.12) shows that

$$O_x\underline{X} \cong \frac{\mathbb{R}^k}{\bigl\{\bigl(g_1(0),\ldots,g_k(0)\bigr) : g_i \in C_0^\infty(\mathbb{R}^n),\ f_1g_1+\cdots+f_kg_k=0\bigr\}}. \tag{3.13}$$

Suppose $g_1,\ldots,g_k \in C_0^\infty(\mathbb{R}^n)$ with $f_1g_1+\cdots+f_kg_k = 0 \in C_0^\infty(\mathbb{R}^n)$, and $g_i(0) \neq 0$ for some $i = 1,\ldots,k$. Then g_i is invertible in $C_0^\infty(\mathbb{R}^n)$, so $f_i = \sum_{j=1,\ldots,k,\ i\neq j}(-g_i^{-1}g_j)\cdot f_j$. Hence f_i is dependent on $f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_k$, and $I = (f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_k)$. Thus if we choose the generators $f_1,\ldots,f_k$ of I so that k is minimal, then $f_1g_1+\cdots+f_kg_k = 0 \in C_0^\infty(\mathbb{R}^n)$ implies that $g_i(0) = 0$ for $i = 1,\ldots,k$, so (3.13) gives $O_x\underline{X} \cong \mathbb{R}^k$. Hence $\dim O_x\underline{X} = n$ *is the minimal number of relations for* $\mathfrak{C}_x$ *in* $C_0^\infty(\mathbb{R}^n)$.

The next proposition follows from Definition 3.20. When we lift from the C^∞-local ring $\mathfrak{C}_x$ of $\underline{X}$ at x to the C^∞-ring $\mathfrak{C}$ of a small neighbourhood $\underline{U}$ of x in $\underline{X}$, we replace (3.8) by a presentation $\mathfrak{C} = C^\infty(W)/(f_1|_W,\ldots,f_k|_W)$ for some small open neighbourhood W of 0 in $\mathbb{R}^n$.

Proposition 3.21. *Let* $\underline{X}$ *be a locally finitely presented* C^∞*-scheme, and* $x \in \underline{X}$. *Set* $n = \dim T_x^*\underline{X}$ *and* $k = \dim O_x\underline{X}$. *Then for any small enough open neighbourhood* $\underline{U}$ *of* x *in* $\underline{X}$ *there exists a* C^∞*-ring* $\mathfrak{C} = C^\infty(W)/(f_1|_W,\ldots,f_k|_W)$ *and an isomorphism* $\underline{U} \cong \operatorname{Spec}\mathfrak{C}$ *identifying* $x \in U$ *with* $0 \in W$, *where* W *is an open neighbourhood of* 0 *in* $\mathbb{R}^n$, *and* $f_1,\ldots,f_k \in C^\infty(\mathbb{R}^n)$ *satisfy* $f_j(0) = \mathrm{d}f_j(0) = 0$ *for* $j = 1,\ldots,k$, *and if* $g_1,\ldots,g_k \in C_0^\infty(\mathbb{R}^n)$ *with* $\sum_{j=1}^k f_j\cdot g_j = 0$ *in* $C_0^\infty(\mathbb{R}^n)$ *then* $g_j(0) = 0$ *for all* $j = 1,\ldots,k$.

If $\dim T_x\underline{X} = n$ and $\dim O_x\underline{X} = 0$ then $\underline{X}$ is isomorphic to $\operatorname{Spec} C^\infty(W)$ near x for W open in $\mathbb{R}^n$. That is, $\underline{X}$ is an n-manifold near x. We easily deduce:

Corollary 3.22. *A* C^∞*-scheme* $\underline{X}$ *is an* n*-manifold without boundary if and only if* $\underline{X}$ *is second countable, separated, and locally finitely presented and* $\dim T_x\underline{X} = n$, $\dim O_x\underline{X} = 0$ *for all* $x \in \underline{X}$.

We use the ideas of Definition 3.20 to show that if V is a manifold, $E \to V$ a vector bundle, $s \in C^\infty(E)$ and $x \in V$ with $s(x) = 0$, then V, E, s are completely determined up to isomorphism near x by the induced C^∞-scheme $\underline{X} = \underline{s}^{-1}(0)$ and two nonnegative integers $a, b \geqslant 0$.

Proposition 3.23. *Suppose V is a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section. Write $\underline{X}$ for the locally finitely presented C^∞-scheme $\underline{V} \times_{\underline{s},\underline{E},\underline{0}} \underline{V}$, where $\underline{V}, \underline{E}, \underline{s}, \underline{0} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(V, E, s, 0)$ and $0 : V \to E$ is the zero section. Equivalently, $\underline{X} = \operatorname{Spec}(C^\infty(V)/I_s)$ for I_s as in (3.5). The underlying topological space is $X = \{v \in V : s(v) = 0\}$.*

Fix $x \in X \subseteq V$, and let $\underline{U} \subseteq \underline{X}$, $n = \dim T_x^*\underline{X}$, $k = \dim O_x\underline{X}$, $W \subseteq \mathbb{R}^n$ and $f_1, \ldots, f_k \in C^\infty(\mathbb{R}^n)$ be as in Proposition 3.21. Then for some $a, b \geqslant 0$ we have $\dim V = n + b$ and $\operatorname{rank} E = k + a + b$, and we can choose an open neighbourhood $\tilde{V}$ of x in V with $\tilde{U} = \tilde{V} \cap \tilde{X} \subseteq U$, coordinates $(z_1, \ldots, z_{n+b})$ on $\tilde{V}$ with $x = (0, \ldots, 0)$, and a trivialization $E|_{\tilde{V}} \cong \mathbb{R}^{k+a+b} \times \tilde{V} \to \tilde{V}$, such that using these coordinates and trivialization, we have

$$
\begin{aligned}
&s(z_1, \ldots, z_{n+b}) \\
&\quad = \bigl(f_1(z_1, \ldots, z_n), \ldots, f_k(z_1, \ldots, z_n), \overbrace{0, \ldots, 0}^{a \text{ zeroes}}, z_{n+1}, \ldots, z_{n+b}\bigr)
\end{aligned} \tag{3.14}
$$

for all $(z_1, \ldots, z_{n+b}) \in \tilde{V}$. Furthermore, the coordinates $y_1, \ldots, y_n$ on W and $z_1, \ldots, z_n$ on $\tilde{V}$ map to the same functions on the C^∞-subscheme $\underline{\tilde{U}}$ in $\underline{U}$.

Proof. Set $m = \dim V$ and $l = \operatorname{rank} E$. Let $\tilde{V}$ be an open neighbourhood of x in V with $\tilde{V} \cap \tilde{X} \subseteq U$, and define $\tilde{U} = \tilde{V} \cap \tilde{X}$, and $\underline{\tilde{U}} \subseteq \underline{U}$ for the associated C^∞-subscheme. Now the isomorphism $\underline{U} \cong \operatorname{Spec} \mathfrak{C}$ induces a homeomorphism

$$
U \cong \bigl\{(y_1, \ldots, y_n) \in W : f_1(y_1, \ldots, y_n) = \cdots = f_k(y_1, \ldots, y_n) = 0\bigr\}. \tag{3.15}
$$

Choose an open neighbourhood $\tilde{W}$ of 0 in $W \subseteq \mathbb{R}^n$ such that $\tilde{U}$ is identified with the intersection of the right hand side of (3.15) with $\tilde{W}$. Then we have an isomorphism $\underline{\tilde{U}} \cong \operatorname{Spec} \tilde{\mathfrak{C}}$, where $\tilde{\mathfrak{C}} = C^\infty(\tilde{W})/(f_1|_{\tilde{W}}, \ldots, f_k|_{\tilde{W}})$. At several points in the proof we will need to make $\tilde{V}$ and hence $\tilde{U}, \tilde{W}$ smaller.

Making $\tilde{U}, \tilde{V}, \tilde{W}$ smaller if necessary, we can suppose there exist coordinates $(x_1, \ldots, x_m)$ on $\tilde{V}$ with $x = (0, \ldots, 0)$, and that $E|_{\tilde{V}}$ is trivial, so we can choose an isomorphism $E|_{\tilde{V}} \cong \mathbb{R}^l \times \tilde{V} \to \tilde{V}$. In this trivialization, $s|_{\tilde{V}}$ corresponds to an l-tuple of functions $(s_1, \ldots, s_l)$ on $\tilde{V}$, that is, $s_i \in C^\infty(\tilde{V})$. The coordinates $x_1, \ldots, x_m$ on $\tilde{V}$ identify $\tilde{V}$ with an open subset $\hat{V}$ of $\mathbb{R}^m$, that is, $s_i = s_i(x_1, \ldots, x_m)$ for $(x_1, \ldots, x_m) \in \hat{V}$. Making $\tilde{U}, \tilde{V}, \tilde{W}$ smaller again, we can suppose $s_1, \ldots, s_l$ extend smoothly to $\mathbb{R}^m$, so we regard $s_1, \ldots, s_l$ as lying in $C^\infty(\mathbb{R}^m)$. Then we have isomorphisms

$$
\underline{\tilde{U}} \cong \operatorname{Spec}\bigl(C^\infty(\tilde{W})/(f_1|_{\tilde{W}}, \ldots, f_k|_{\tilde{W}}\bigr) \cong \operatorname{Spec}\bigl(C^\infty(\tilde{V})/(s_1, \ldots, s_l)\bigr),
$$

which induce an isomorphism of C^∞-rings

$$
\Phi : C^\infty(\tilde{W})/(f_1|_{\tilde{W}}, \ldots, f_k|_{\tilde{W}}) \xrightarrow{\cong} C^\infty(\tilde{V})/(s_1, \ldots, s_l).
$$

Localizing Φ at 0 gives an isomorphism of C^∞-local rings

$$\Phi_0 : C_0^\infty(\mathbb{R}^n)/(f_1,\ldots,f_k) \xrightarrow{\cong} C_0^\infty(\mathbb{R}^m)/(s_1,\ldots,s_l). \tag{3.16}$$

Thus equation (3.11) implies that

$$\langle \mathrm{d}y_1,\ldots,\mathrm{d}y_n\rangle_{\mathbb{R}} \cong T_x^*\underline{X} \cong \frac{\langle \mathrm{d}x_1,\ldots,\mathrm{d}x_m\rangle_{\mathbb{R}}}{\langle\sum_{j=1}^m \frac{\partial s_1}{\partial x_j}(0)\,\mathrm{d}x_j,\ldots,\sum_{j=1}^m \frac{\partial s_l}{\partial x_j}(0)\,\mathrm{d}x_j\rangle_{\mathbb{R}}}. \tag{3.17}$$

Hence $m \geqslant n$, so we may write $m = n+b$ for $b = m-n \geqslant 0$, and the matrix $\bigl(\frac{\partial s_i}{\partial x_j}(0,\ldots,0)\bigr)_{i=1,\ldots,l}^{j=1,\ldots,n+b}$ has rank b, so that $l \geqslant b$. Applying a $\mathrm{GL}(m,\mathbb{R})$ transformation to $V \subseteq \mathbb{R}^m$ and $(x_1,\ldots,x_m)$, and a $\mathrm{GL}(l,\mathbb{R})$ transformation to the trivialization $E|_{\tilde V} \cong \mathbb{R}^l \times \tilde V \to \tilde V$ and to $s_1,\ldots,s_l$, we can suppose that

$$\frac{\partial s_i}{\partial x_j}(0,\ldots,0) = \begin{cases} 1, & i = l-b+c,\ j = n+c,\ c = 1,\ldots,b, \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

Then (3.17) becomes

$$\langle \mathrm{d}y_1,\ldots,\mathrm{d}y_n\rangle_{\mathbb{R}} \cong T_x^*\underline{X} \cong \langle \mathrm{d}x_1,\ldots,\mathrm{d}x_n\rangle_{\mathbb{R}}. \tag{3.19}$$

Choose $w_1,\ldots,w_n \in C^\infty(\mathbb{R}^m)$ whose projections to $C_0^\infty(\mathbb{R}^m)/(s_1,\ldots,s_l)$ are $\Phi_0(y_1),\ldots,\Phi_0(y_n)$, so that $w_i = w_i(x_1,\ldots,x_m)$ and $w_i(0,\ldots,0) = 0$. Then the isomorphism (3.19) is given by $\mathrm{d}y_i \longmapsto \sum_{j=1}^n \frac{\partial w_i}{\partial x_j}(0,\ldots,0)\,\mathrm{d}x_j$. Therefore the $n\times n$ matrix $\bigl(\frac{\partial w_i}{\partial x_j}(0,\ldots,0)\bigr)_{i,j=1}^n$ is invertible. Define $z_1,\ldots,z_m \in C^\infty(\tilde V)$ by $z_i = w_i(x_1,\ldots,x_{n+b})$ for $i = 1,\ldots,n$ and $z_{n+c} = s_{l-b+c}(x_1,\ldots,x_{n+b})$ for $c = 1,\ldots,b$. Then (3.18) and $\bigl(\frac{\partial w_i}{\partial x_j}(0,\ldots,0)\bigr)_{i,j=1}^n$ invertible implies that the $m\times m$ matrix $\bigl(\frac{\partial z_i}{\partial x_j}(0,\ldots,0)\bigr)_{i,j=1}^m$ is invertible. Hence $(z_1,\ldots,z_m)$ are coordinates on $\tilde V$ near $x = (0,\ldots,0)$.

Making $\tilde U,\tilde V,\tilde W$ smaller if necessary, we can suppose $(z_1,\ldots,z_{n+b})$ are coordinates on $\tilde V$, and from now on we use these instead of $(x_1,\ldots,x_m)$. So we may regard $s_1,\ldots,s_l \in C^\infty(\tilde V)$ as functions of $(z_1,\ldots,z_{n+b})$; making $\tilde U,\tilde V,\tilde W$ smaller, we can suppose $s_1,\ldots,s_l$ extend to smooth functions of $(z_1,\ldots,z_{n+b})$ on all of $\mathbb{R}^{n+b}$, and so regard $s_i = s_i(z_1,\ldots,z_{n+b})$ as lying in $C^\infty(\mathbb{R}^{n+b})$.

By definition we have $\Phi_0(y_i) = z_i$ for $i = 1,\ldots,n$, and $s_{l-b+c}(z_1,\ldots,z_{n+b}) = z_{n+c}$ for $c = 1,\ldots,b$. Define $t_1,\ldots,t_{l-b} \in C^\infty(\mathbb{R}^n)$ by $t_i(z_1,\ldots,z_n) = s_i(z_1,\ldots,z_n,0,\ldots,0)$. Then by Hadamard's Lemma, there exist functions u_{ic} in $C^\infty(\mathbb{R}^{n+b})$ for $i = 1,\ldots,l-b$ and $c = 1,\ldots,b$ such that

$$s_i(z_1,\ldots,z_{n+a}) = t_i(z_1,\ldots,z_n) + \textstyle\sum_{c=1}^b z_{n+c}\cdot u_{ic}(z_1,\ldots,z_{n+b})$$

for all $(z_1,\ldots,z_{n+b}) \in \mathbb{R}^{n+b}$. Let us now change the trivialization $E|_{\tilde V} \cong \mathbb{R}^l \times \tilde V \to \tilde V$ by applying the automorphism

$$\begin{aligned}(e_1,\ldots,e_l) \longmapsto \bigl(&e_1 - \textstyle\sum_{c=1}^b u_{1c}(z_1,\ldots,z_{n+b})\cdot e_{l-b+c},\ldots,\\ &e_{l-b} - \textstyle\sum_{c=1}^b u_{(l-b)c}(z_1,\ldots,z_{n+b})\cdot e_{l-b+c}, e_{l-b+1},\ldots,e_l\bigr).\end{aligned}$$

Since $s_{l-b+c}(z_1,\ldots,z_{n+b})=z_{n+c}$, this has the effect of mapping $s_i(z_1,\ldots,z_{n+b})$ to $t_i(z_1,\ldots,z_n)$ for $i=1,\ldots,l-b$. Thus, in the new coordinates $(z_1,\ldots,z_{n+b})$ and trivialization $E|_{\tilde V}\cong\mathbb{R}^l\times\tilde V\to\tilde V$, the section $s|_{\tilde V}\in C^\infty(E|_{\tilde V})$ is given by

$$s|_{\tilde V}(z_1,\ldots,z_{n+b})=\bigl(t_1(z_1,\ldots,z_n),\ldots,t_{l-b}(z_1,\ldots,z_n),z_{n+1},\ldots,z_{n+b}\bigr).$$

In the new coordinates $(z_1,\ldots,z_{n+b})$, the isomorphism Φ_0 of (3.16) becomes

$$\begin{aligned}\Phi_0:C_0^\infty(\mathbb{R}^n)/(f_1,\ldots,f_k)&\longrightarrow C_0^\infty(\mathbb{R}^n)/\bigl(t_1(z_1,\ldots,z_n),\ldots,t_{l-b}(z_1,\ldots,z_n)\bigr)\\&\cong C_0^\infty(\mathbb{R}^{n+b})/\bigl(t_1(z_1,\ldots,z_n),\ldots,t_{l-b}(z_1,\ldots,z_n),z_{n+1},\ldots,z_{n+b}\bigr).\end{aligned}$$

Since $\Phi_0(y_i)=z_i$ for $i=1,\ldots,n$, this gives an equality of ideals in $C^\infty(\mathbb{R}^n)$:

$$\bigl(f_1(z_1,\ldots,z_n),\ldots,f_k(z_1,\ldots,z_n)\bigr)=\bigl(t_1(z_1,\ldots,z_n),\ldots,t_{l-b}(z_1,\ldots,z_n)\bigr).$$

But by Definition 3.20, $k=\dim O_x\underline{X}$ is the minimal number of generators for this ideal, so $l-b\geqslant k$, and $l=k+a+b$ for some $a\geqslant 0$. Since the ideals are equal, there exist $A_{ij},B_{ji}\in C^\infty(\mathbb{R}^n)$ for $i=1,\ldots,k$ and $j=1,\ldots,k+a$ with

$$\begin{aligned}f_i(z_1,\ldots,z_n)&=\textstyle\sum_{j=1}^{k+a}A_{ij}(z_1,\ldots,z_n)\cdot t_j(z_1,\ldots,z_n),\\t_j(z_1,\ldots,z_n)&=\textstyle\sum_{i=1}^{k}B_{ji}(z_1,\ldots,z_n)\cdot f_i(z_1,\ldots,z_n),\end{aligned}\quad\text{in } C_0^\infty(\mathbb{R}^n).\tag{3.20}$$

From (3.20) we see that $\sum_{p=1}^k(\sum_{j=1}^{k+a}A_{ij}B_{jp}-\delta_{ip})\cdot f_p=0$ for $i=1,\ldots,k$. Thus the last part of Proposition 3.21 implies that

$$\sum_{j=1}^{k+a}A_{ij}(0,\ldots,0)B_{jp}(0,\ldots,0)=\begin{cases}1, & i=p\\0 & \text{otherwise.}\end{cases}$$

So both matrices $\bigl(A_{ij}(0,\ldots,0)\bigr)_{i=1,\ldots,k}^{j=1,\ldots,k+a}$ and $\bigl(B_{jp}(0,\ldots,0)\bigr)_{j=1,\ldots,k+a}^{p=1,\ldots,k}$ have rank k. By applying a $\mathrm{GL}(k+a,\mathbb{R})$ transformation to the trivialization $E|_{\tilde V}\cong\mathbb{R}^{k+a}\times\mathbb{R}^b\times\tilde V\to\tilde V$ and to $t_1,\ldots,t_{k+a}$, we can suppose that

$$A_{ij}(0,\ldots,0)=B_{ji}(0,\ldots,0)=\begin{cases}1, & i=j=1,\ldots,k,\\0, & \text{otherwise.}\end{cases}$$

Now define more functions $B_{ji}\in C^\infty(\mathbb{R}^n)$ for $i=k+1,\ldots,k+a$ and $j=1,\ldots,k+a$ by $B_{ji}=1$ if $i=j$ and $B_{ji}=0$ otherwise. Then $(B_{ji})_{j=1,\ldots,k+a}^{i=1,\ldots,k+a}$ is a $(k+a)\times(k+a)$ matrix in $C^\infty(\mathbb{R}^n)$ such that $B_{ij}(0,\ldots,0)=\delta_{ij}$. Thus $(B_{ji})_{j=1,\ldots,k+a}^{i=1,\ldots,k+a}$ is invertible at $0\in\mathbb{R}^n$, so it is invertible near 0 in $\mathbb{R}^n$. Making $\tilde U,\tilde V,\tilde W$ smaller if necessary, we can suppose $(B_{ji})_{j=1,\ldots,k+a}^{i=1,\ldots,k+a}$ is invertible at $(z_1,\ldots,z_n)$ for all $(z_1,\ldots,z_{n+b})\in\tilde V$.

Equation (3.20) holds in $C_0^\infty(\mathbb{R}^n)$ rather than in $C^\infty(\mathbb{R}^n)$, which implies that each equation holds for $(z_1,\ldots,z_n)$ close to zero in $\mathbb{R}^n$. Making $\tilde U,\tilde V,\tilde W$

smaller if necessary, we can suppose all equations of (3.20) hold at $(z_1,\ldots,z_n)$ for all $(z_1,\ldots,z_{n+b})\in\tilde V$. Then at each $(z_1,\ldots,z_{n+b})\in\tilde V$ we have

$$\begin{pmatrix} t_1(z_1,\ldots,z_n) \\ \vdots \\ t_{k+a}(z_1,\ldots,z_n)\end{pmatrix}=\bigl(B_{ji}(z_1,\ldots,z_n)\bigr)_{j=1,\ldots,k+a}^{i=1,\ldots,k+a}\begin{pmatrix} f_1(z_1,\ldots,z_n) \\ \vdots \\ f_k(z_1,\ldots,z_n) \\ 0 \\ \vdots \\ 0\end{pmatrix}.$$

Thus, changing the trivialization $E|_{\tilde V}\cong\mathbb{R}^{k+a}\times\mathbb{R}^b\times\tilde V\to\tilde V$ in the first $\mathbb{R}^{k+a}$ by the invertible matrix of functions $\bigl(B_{ji}(z_1,\ldots,z_n)\bigr)_{j=1,\ldots,k+a}^{i=1,\ldots,k+a}$ on $\tilde V$, we can suppose $t_i(z_1,\ldots,z_n)=f_i(z_1,\ldots,z_n)$ for $i=1,\ldots,k$ and $t_i(z_1,\ldots,z_n)=0$ for $i=k+1,\ldots,k+a$. In the new trivialization, s is given by (3.14). Also $y_1,\ldots,y_n$ on W and $z_1,\ldots,z_n$ on $\tilde V$ map to the same functions on $\underline{U}$ by construction. This completes the proof of Proposition 3.23. $\square$

We can make $\underline{U}$ in Proposition 3.21 into a family of d-manifolds $\boldsymbol{U}_{a,b}$.

Example 3.24. Let $\underline{X},x,n,k,\underline{U},\mathfrak{C},W,f_1,\ldots,f_k$ be as in Proposition 3.21, and a,b be nonnegative integers. Define a manifold $V=W\times\mathbb{R}^b$, a vector bundle $E\to V$ to be $\mathbb{R}^{k+a+b}\times V\to V$, and a section $s\in C^\infty(E)$ by (3.14). Let $\boldsymbol{U}_{a,b}$ be the d-manifold $\boldsymbol{S}_{V,E,s}$ of Definition 3.13. Then $\boldsymbol{U}_{a,b}$ is a principal d-manifold with $\mathrm{vdim}\,\boldsymbol{U}_{a,b}=n-k-a$, with underlying C^∞-scheme $\underline{U}$.

This proves that if $\underline{X}$ is a locally finitely presented C^∞-scheme and $x\in\underline{X}$, then $\underline{X}$ near x can be made into a d-manifold $\boldsymbol{X}$, where $\mathrm{vdim}\,\boldsymbol{X}$ can take any value with $\mathrm{vdim}\,\boldsymbol{X}\leqslant\dim T_x^*\underline{X}-\dim O_x\underline{X}$. Therefore a C^∞-scheme $\underline{X}$ locally extends to a d-manifold $\boldsymbol{X}$ if and only if $\underline{X}$ is locally finitely presented.

Using the explicit description of $\boldsymbol{U}_{a,b}$ in Definition 3.13, it is easy to see that $\boldsymbol{U}_{a,b}$ is 1-isomorphic to the d-space constructed in Example 2.18 starting with the d-space $\boldsymbol{U}_{a,0}$ and the quasicoherent sheaf $\mathbb{R}^b\otimes_{\mathbb{R}}\mathcal{O}_U$ on $\underline{U}$. Therefore $\boldsymbol{U}_{a,b}$ is equivalent to $\boldsymbol{U}_{a,0}$, so $\boldsymbol{U}_{a,b}$ is independent of $b\geqslant 0$ up to equivalence.

Proposition 3.25. *Suppose $\boldsymbol{X}$ is a d-manifold with C^∞-scheme $\underline{X}$, and $x\in\underline{X}$. Let $\underline{U}\subseteq\underline{X}$, $n=\dim T_x^*\underline{X}$, $k=\dim O_x\underline{X}$, $W\subseteq\mathbb{R}^n$ and $f_1,\ldots,f_k\in C^\infty(\mathbb{R}^n)$ be as in Proposition 3.21. Write $\boldsymbol{U}\subseteq\boldsymbol{X}$ for the open d-submanifold of $\boldsymbol{X}$ corresponding to $\underline{U}\subseteq\underline{X}$. Then making $\underline{U},\boldsymbol{U},W$ smaller if necessary, $\boldsymbol{U}$ is equivalent in* **dMan** *to the principal d-manifold $\boldsymbol{U}_{a,0}$ of Example 3.24, where*

$$a=\dim T_x^*\underline{X}-\dim O_x\underline{X}-\mathrm{vdim}\,\boldsymbol{X}\geqslant 0. \tag{3.21}$$

Proof. Making $\underline{U},\boldsymbol{U},W$ smaller if necessary, we can suppose $\boldsymbol{U}$ is a principal d-manifold. Thus by Proposition 3.12 and Definition 3.13 we have $\boldsymbol{U}\simeq\boldsymbol{S}_{V,E,s}$ for some V,E,s. Proposition 3.23 shows that V,E,s are given up to isomorphism near x in terms of $x,\underline{U},n,k,W,f_1,\ldots,f_k$ and two integers $a,b\geqslant 0$. So making $\underline{U},\boldsymbol{U},W$ and V smaller, Example 3.24 shows that $\boldsymbol{U}$ is equivalent to $\boldsymbol{U}_{a,b}$, and hence to $\boldsymbol{U}_{a,0}$ as $\boldsymbol{U}_{a,b}\simeq\boldsymbol{U}_{a,0}$. But $\mathrm{vdim}\,\boldsymbol{X}=\mathrm{vdim}\,\boldsymbol{U}=n-k-a$ and $n=\dim T_x^*\underline{X}$, $k=\dim O_x\underline{X}$, so a is given by (3.21). $\square$

The next two corollaries follow from Proposition 3.25.

Corollary 3.26. *Let $\boldsymbol{X}$ be a d-manifold, and $x \in \boldsymbol{X}$. Then there exists an open neighbourhood $\boldsymbol{U}$ of x in $\boldsymbol{X}$ and an equivalence $\boldsymbol{U} \simeq \boldsymbol{S}_{V,E,s}$ in* **dMan** *for some manifold V, vector bundle $E \to V$ and smooth section $s : V \to E$ which identifies $x \in \boldsymbol{U}$ with a point $v \in V$ such that $s(v) = \mathrm{d}s(v) = 0$. Furthermore, V, E, s are determined up to non-canonical isomorphism near v by $\boldsymbol{X}$ near x.*

Corollary 3.27. *Let $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ be a d-manifold. Then $\boldsymbol{X}$ is determined up to non-canonical equivalence near each point $x \in \underline{X}$ by the C^∞-scheme $\underline{X}$ and the integer* $\mathrm{vdim}\,\boldsymbol{X}$.

For a d-manifold $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$, we can now explain how much information in $\boldsymbol{X}$ up to equivalence is stored in the C^∞-scheme $\underline{X}$, and how much in the 'derived' data $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$. By Corollary 3.27, locally the *only* extra information in $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$ up to non-canonical equivalence is the virtual dimension $\mathrm{vdim}\,\boldsymbol{X}$ in $\mathbb{Z}$, and everything else depends only on $\underline{X}$.

Globally, the extra information in $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$ is like a vector bundle $\mathcal{E}$ over $\underline{X}$. The only local information in a vector bundle $\mathcal{E}$ is $\mathrm{rank}\,\mathcal{E} \in \mathbb{Z}$, but globally it also contains nontrivial algebraic-topological information such as the Pontryagin classes $p_i(\mathcal{E}) \in H^{4i}(X;\mathbb{Z})$. This is illustrated by Example 3.17, which builds a d-manifold $\boldsymbol{S} = (\underline{V}, \mathcal{O}_V, \mathcal{E}^*, \mathrm{id}_{\mathcal{O}_V}, 0)$ from a manifold V and a vector bundle $E \to V$. In this case $\mathcal{C}_S \cong \mathcal{D}_S \cong \mathcal{E}^*$, and equivalences of $\boldsymbol{S}$ preserve $\mathcal{C}_S, \mathcal{D}_S$ up to isomorphism by Proposition 2.20, so equivalences of $\boldsymbol{S}$ preserve the manifold V and vector bundle $E \to V$ up to isomorphism.

Here are two criteria for when a d-manifold is a manifold. Proposition 3.9 shows the two are equivalent.

Proposition 3.28. *Let $\boldsymbol{X}$ be a d-manifold. Then $\boldsymbol{X}$ is a manifold (that is, $\boldsymbol{X} \in \hat{\mathbf{Man}}$) if and only if $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ has a left inverse, or equivalently, if and only if its virtual cotangent bundle $T^*\boldsymbol{X}$ is a vector bundle.*

Proof. By Proposition 2.25, $\boldsymbol{X} \simeq F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{X})$ if and only if ϕ_X has a left inverse. The 'only if' part is immediate: if $\boldsymbol{X} \simeq F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$ for a manifold X then $\boldsymbol{X} \simeq F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{X})$ for $\underline{X} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(X)$, so ϕ_X has a left inverse.

For the 'if' part, suppose ϕ_X has a left inverse, so that $\boldsymbol{X} \simeq F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{X})$. Let $x \in \underline{X}$. By Proposition 3.25 there an open $x \in \boldsymbol{U} \subseteq \boldsymbol{X}$ with $\boldsymbol{U} \simeq \boldsymbol{U}_{a,0}$ in Example 3.24. But also $\boldsymbol{U} \simeq F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{U})$ as this holds for $\boldsymbol{X}$, so $\boldsymbol{U}_{a,0} \simeq F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}(\underline{U})$. Thus $\phi_{U_{a,0}} : \mathcal{E}_{U_{a,0}} \to \mathcal{F}_{U_{a,0}}$ has a left inverse by Proposition 2.25. But from the construction of $\boldsymbol{U}_{a,0}$ we can show that the fibre of $\phi_{U_{a,0}}$ over $x \in \underline{U}$ has kernel $\mathbb{R}^{k+a}$. This fibre must be injective as it has a left inverse, so $k = \dim O_x\underline{X} = 0$ and $a = 0$, and (3.21) gives $\dim T_x^*\underline{X} = \mathrm{vdim}\,\boldsymbol{X}$. As this holds for all $x \in \underline{X}$, and $\underline{X}$ is second countable and separated, Corollary 3.22 shows $\underline{X}$ is a manifold. □

3.4 Differential-geometric picture of 1- and 2-morphisms

Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, and $s : V \to E$, $t : W \to F$ be smooth sections. Then Definition 3.13 defines 'standard model' principal d-manifolds $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$, $\boldsymbol{Y} = \boldsymbol{S}_{W,F,t}$. We will now interpret 1-morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in terms of a pair $(f, \hat{f})$ of a smooth map $f : V \to W$ and a vector bundle morphism $\hat{f} : E \to f^*(F)$, and interpret 2-morphisms $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ as a relation between $(f, \hat{f})$ and $(g, \hat{g})$. We will need some notation:

Definition 3.29. Let V be a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$ a smooth section. If $F \to V$ is another vector bundle and $t_1, t_2 \in C^\infty(F)$ are smooth sections, we write $t_1 = t_2 + O(s)$ if there exists $\alpha \in C^\infty(E^* \otimes F)$ such that $t_1 = t_2 + \alpha \cdot s$ in $C^\infty(F)$, where the contraction $\alpha \cdot s$ is formed using the natural pairing of vector bundles $(E^* \otimes F) \times E \to F$ over V. Similarly, we write $t_1 = t_2 + O(s^2)$ if there exists $\alpha \in C^\infty(E^* \otimes E^* \otimes F)$ such that $t_1 = t_2 + \alpha \cdot (s \otimes s)$ in $C^\infty(F)$, where $\alpha \cdot (s \otimes s)$ uses the pairing $(E^* \otimes E^* \otimes F) \times (E \otimes E) \to F$.

Now let W be another manifold, and $f, g : V \to W$ be smooth maps. We write $f = g + O(s)$ if whenever $h : W \to \mathbb{R}$ is a smooth map, there exists $\alpha \in C^\infty(E^*)$ such that $h \circ f = h \circ g + \alpha \cdot s$. Similarly, we write $f = g + O(s^2)$ if whenever $h : W \to \mathbb{R}$ is a smooth map, there exists $\alpha \in C^\infty(E^* \otimes E^*)$ such that $h \circ f = h \circ g + \alpha \cdot (s \otimes s)$.

Now suppose $f, g : V \to W$ with $f = g + O(s^2)$, and $F \to W$ is a vector bundle, and $t_1, t_2 \in C^\infty(W)$. Later we will wish to compare $f^*(t_1)$ and $g^*(t_2)$. Strictly speaking these are sections of different vector bundles $f^*(F), g^*(F)$ over V, so are not comparable. But as $f = g + O(s^2)$, we make the convention that $f^*(t_2) = g^*(t_2) + O(s^2)$ for any t_2. Thus, informally we have

$$f^*(t_1) - g^*(t_2) = \bigl(f^*(t_1) - f^*(t_2)\bigr) + \bigl(f^*(t_2) - g^*(t_2)\bigr) = f^*(t_1) - f^*(t_2) + O(s^2).$$

Therefore, if $f = g + O(s^2)$, we will write $f^*(t_1) = g^*(t_2) + O(s)$ if $f^*(t_1) = f^*(t_2) + O(s)$, and $f^*(t_1) = g^*(t_2) + O(s^2)$ if $f^*(t_1) = f^*(t_2) + O(s^2)$.

This has a simple interpretation using C^∞-subschemes: if $\underline{V} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(V)$, and $\underline{X}, \underline{X}'$ are the C^∞-subschemes in $\underline{V}$ defined by the equations $s = 0$ and $s \otimes s = 0$, then $t_1 = t_2 + O(s)$, $f = g + O(s)$ mean $t_1|_{\underline{X}} = t_2|_{\underline{X}}$, $\underline{f}|_{\underline{X}} = \underline{g}|_{\underline{X}}$, and $t_1 = t_2 + O(s^2)$, $f = g + O(s^2)$ mean $t_1|_{\underline{X}'} = t_2|_{\underline{X}'}$, $\underline{f}|_{\underline{X}'} = \underline{g}|_{\underline{X}'}$. When $f = g + O(s^2)$, $f^*(t_1) = g^*(t_2) + O(s)$ means $(\underline{f}|_{\underline{X}})^*(t_1) = (\underline{g}|_{\underline{X}})^*(t_2)$ and $f^*(t_1) = g^*(t_2) + O(s^2)$ means $(\underline{f}|_{\underline{X}'})^*(t_1) = (\underline{g}|_{\underline{X}'})^*(t_2)$. These make sense as $\underline{f}|_{\underline{X}} = \underline{g}|_{\underline{X}}$ and $\underline{f}|_{\underline{X}'} = \underline{g}|_{\underline{X}'}$.

Definition 3.30. Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, and $s : V \to E$, $t : W \to F$ be smooth sections. Write $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$, $\boldsymbol{Y} = \boldsymbol{S}_{W,F,t}$ for the 'standard model' principal d-manifolds from Definition 3.13. Suppose $f : V \to W$ is a smooth map, and $\hat{f} : E \to f^*(F)$ is a morphism of vector bundles on V satisfying

$$\hat{f} \circ s = f^*(t) + O(s^2) \quad \text{in } C^\infty(f^*(F)), \tag{3.22}$$

where $f^*(t)=t\circ f$, and $O(s^2)$ is as in Definition 3.29. We will define a 1-morphism $\boldsymbol g=(\underline{g},g',g''):\boldsymbol X\to\boldsymbol Y$ in **dMan** using $f,\hat f$. We will also write $\boldsymbol g:\boldsymbol X\to\boldsymbol Y$ as $\boldsymbol S_{f,\hat f}:\boldsymbol S_{V,E,s}\to\boldsymbol S_{W,F,t}$, and call it a *standard model* 1-*morphism.*

With our usual notation for $\boldsymbol X,\boldsymbol Y$, by Definition 3.13 we have

$$
\begin{gathered}
X=\{v\in V:s(v)=0\},\ \underline{X}=\operatorname{Spec}\bigl(C^\infty(V)/I_s\bigr),\ (X,\mathcal O_X')=\operatorname{Spec}\bigl(C^\infty(V)/I_s^2\bigr),\\
Y=\{w\in W:t(w)=0\},\ \underline{Y}=\operatorname{Spec}\bigl(C^\infty(W)/I_t\bigr),\ (Y,\mathcal O_Y')=\operatorname{Spec}\bigl(C^\infty(W)/I_t^2\bigr),\\
\mathcal E_X=\operatorname{MSpec}\bigl(C^\infty(E^*)/(I_s\cdot C^\infty(E^*))\bigr),\ \mathcal E_Y=\operatorname{MSpec}\bigl(C^\infty(F^*)/(I_t\cdot C^\infty(F^*))\bigr),\\
(\mathrm{id}_X,\imath_X)=\operatorname{Spec}\bigl(\pi_X:C^\infty(V)/I_s^2\longrightarrow C^\infty(V)/I_s\bigr),\\
\jmath_X=\operatorname{MSpec}\bigl(s\cdot-:C^\infty(E^*)/(I_s\cdot C^\infty(E^*))\longrightarrow I_s/I_s^2\bigr),\\
(\mathrm{id}_Y,\imath_Y)=\operatorname{Spec}\bigl(\pi_Y:C^\infty(W)/I_t^2\longrightarrow C^\infty(W)/I_t\bigr),\\
\text{and}\quad \jmath_Y=\operatorname{MSpec}\bigl(t\cdot-:C^\infty(F^*)/(I_t\cdot C^\infty(F^*))\longrightarrow I_t/I_t^2\bigr).
\end{gathered}
$$

If $x\in X$ then $x\in V$ with $s(x)=0$, so (3.22) implies that

$$t\bigl(f(x)\bigr)=\bigl(f^*(t)\bigr)(x)=\hat f\bigl(s(x)\bigr)+O\bigl(s(x)^2\bigr)=0,$$

so $f(x)\in Y\subseteq W$. Thus $g:=f|_X$ maps $X\to Y$.

Define morphisms of C^∞-rings

$$
\begin{gathered}
\phi:C^\infty(W)/I_t\longrightarrow C^\infty(V)/I_s,\quad \phi':C^\infty(W)/I_t^2\longrightarrow C^\infty(V)/I_s^2,\\
\text{by}\quad \phi:c+I_t\longmapsto c\circ f+I_s,\quad \phi':c+I_t^2\longmapsto c\circ f+I_s^2.
\end{gathered}
$$

Here ϕ is well-defined since if $c\in I_t$ then $c=\gamma\cdot t$ for some $\gamma\in C^\infty(F^*)$, so

$$c\circ f=(\gamma\cdot t)\circ f=f^*(\gamma)\cdot f^*(t)=f^*(\gamma)\cdot\bigl(\hat f\circ s+O(s^2)\bigr)=\bigl(\hat f\circ f^*(\gamma)\bigr)\cdot s+O(s^2)\in I_s.$$

Similarly if $c\in I_t^2$ then $c\circ f\in I_s^2$, so ϕ' is well-defined. Thus we have C^∞-scheme morphisms $\underline{g}=(g,g^\sharp)=\operatorname{Spec}\phi:\underline{X}\to\underline{Y}$, and $(g,g')=\operatorname{Spec}\phi':(X,\mathcal O_X')\to(Y,\mathcal O_Y')$, which both have underlying continuous map g. Hence $g^\sharp:g^{-1}(\mathcal O_Y)\to\mathcal O_X$ and $g':g^{-1}(\mathcal O_Y')\to\mathcal O_X'$ are morphisms of sheaves of C^∞-rings on X. Also $\pi_X\circ\phi'=\phi\circ\pi_Y$, so applying Spec implies that $g^\sharp\circ g^{-1}(\imath_Y)=\imath_X\circ g'$. Therefore $(\underline{g},g')$ is a morphism of square zero extensions $(\underline{X},\mathcal O_X',\imath_X)\to(\underline{Y},\mathcal O_Y',\imath_Y)$.

Since $\underline{g}^*(\mathcal E_Y)=\operatorname{MSpec}\bigl(C^\infty(f^*(F^*))/(I_s\cdot C^\infty(f^*(F^*)))\bigr)$, we may define $g'':\underline{g}^*(\mathcal E_Y)\to\mathcal E_X$ by $g''=\operatorname{MSpec}(G'')$, where

$$
\begin{gathered}
G'':C^\infty(f^*(F^*))/(I_s\cdot C^\infty(f^*(F^*))\longrightarrow C^\infty(E^*)/(I_s\cdot C^\infty(E^*))\\
\text{is defined by}\quad G'':\gamma+I_s\cdot C^\infty(f^*(F^*))\longmapsto\gamma\circ\hat f+I_s\cdot C^\infty(E^*).
\end{gathered}
$$

Here $\gamma\in C^\infty(f^*(F^*))$, so γ gives a bundle map $f^*(F)\to\mathbb R$. Composing with $\hat f:E\to f^*(F)$ gives a bundle map $E\to\mathbb R$, that is, $\gamma\circ\hat f\in C^\infty(E^*)$. Equation (3.22) implies that $\jmath_X\circ g''=g^1\circ\underline{g}^*(\jmath_Y)$, as in (2.41). Hence $\boldsymbol g=(\underline{g},g',g'')$ is a 1-morphism $\boldsymbol X\to\boldsymbol Y$, which we also write as $\boldsymbol S_{f,\hat f}:\boldsymbol S_{V,E,s}\to\boldsymbol S_{W,F,t}$.

Suppose now that $\tilde V\subseteq V$ is open, with inclusion map $i_{\tilde V}:\tilde V\to V$. Write $\tilde E=E|_{\tilde V}=i_{\tilde V}^*(E)$ and $\tilde s=s|_{\tilde V}$. Define $\boldsymbol i_{\tilde V,V}=\boldsymbol S_{i_{\tilde V},\mathrm{id}_{\tilde E}}:\boldsymbol S_{\tilde V,\tilde E,\tilde s}\to\boldsymbol S_{V,E,s}$.

Remark 3.31. Equation (3.22) means there exists $\alpha \in C^\infty\bigl(E^* \otimes E^* \otimes f^*(F)\bigr)$ with $\hat f \circ s = f^*(t) + \alpha \cdot (s \otimes s)$. Hence $\bigl(\hat f - (\alpha \cdot s)\bigr) \cdot s = f^*(t)$. So (3.22) becomes $\hat f' \circ s = f^*(t)$, where $\hat f' = \hat f - (\alpha \cdot s)$. Also $\boldsymbol{S}_{f,\hat f} = \boldsymbol{S}_{f,\hat f'}$ by Lemma 3.32 below. Thus we can always eliminate $O(s^2)$ in (3.22) by modifying $\hat f$.

Lemma 3.32. *Let V, W be manifolds, $E \to V$, $F \to W$ vector bundles, $s : V \to E$, $t : W \to F$ smooth sections, $f_1, f_2 : V \to W$ smooth maps, and $\hat f_1 : E \to f_1^*(F)$, $\hat f_2 : E \to f_2^*(F)$ vector bundle morphisms with $\hat f_1 \circ s = f_1^*(t) + O(s^2)$ and $\hat f_2 \circ s = f_2^*(t) + O(s^2)$, so we have 1-morphisms $\boldsymbol{S}_{f_1,\hat f_1}, \boldsymbol{S}_{f_2,\hat f_2} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. Then $\boldsymbol{S}_{f_1,\hat f_1} = \boldsymbol{S}_{f_2,\hat f_2}$ if and only if $f_1 = f_2 + O(s^2)$ and $\hat f_1 = \hat f_2 + O(s)$, in the notation of Definition* 3.29.

Proof. $f_1 = f_2 + O(s^2)$ is necessary and sufficient for f_1, f_2 to induce the same morphisms ϕ, ϕ', and then $\hat f_1 = \hat f_2 + O(s)$ is necessary and sufficient for $\hat f_1, \hat f_2$ to induce the same morphisms G. Since ϕ, ϕ', G can be recovered from $\boldsymbol{S}_{f,\hat f}$ by $\phi \cong f^\sharp(X)$, $\phi' \cong f'(X)$ and $G \cong f''(X)$, the lemma follows. $\square$

Lemma 3.33. *Let V be a manifold, $E \to V$ a vector bundle, $s : V \to E$ a smooth section, and $\tilde V \subseteq V$ be open. Then $\boldsymbol{i}_{\tilde V,V} : \boldsymbol{S}_{\tilde V,\tilde E,\tilde s} \to \boldsymbol{S}_{V,E,s}$ is a 1-isomorphism with an open d-submanifold of $\boldsymbol{S}_{V,E,s}$. If $s^{-1}(0) \subseteq \tilde V$ then $\boldsymbol{i}_{\tilde V,V} : \boldsymbol{S}_{\tilde V,\tilde E,\tilde s} \to \boldsymbol{S}_{V,E,s}$ is a 1-isomorphism.*

Proof. This follows from the fact that if $\underline{V} = (V, \mathcal{O}_V) = \operatorname{Spec} C^\infty(V)$ and $\tilde V \subseteq V$ is open, then $\mathcal{O}_V(\tilde V) \cong C^\infty(\tilde V)$. $\square$

Our next result describes 1-morphisms $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ not up to 2-isomorphism, but up to equality. Combined with Lemma 3.32, it gives a differential-geometric classification of 1-morphisms $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$.

Theorem 3.34. *Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, and $s : V \to E$, $t : W \to F$ be smooth sections. Define principal d-manifolds $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$, $\boldsymbol{Y} = \boldsymbol{S}_{W,F,t}$, with topological spaces $X = \{v \in V : s(v) = 0\}$ and $Y = \{w \in W : t(w) = 0\}$. Suppose $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism. Then there exist an open neighbourhood $\tilde V$ of X in V, a smooth map $f : \tilde V \to W$, and a morphism of vector bundles $\hat f : \tilde E \to f^*(F)$ with $\hat f \circ \tilde s = f^*(t)$, where $\tilde E = E|_{\tilde V}$, $\tilde s = s|_{\tilde V}$, such that $\boldsymbol{g} = \boldsymbol{S}_{f,\hat f} \circ \boldsymbol{i}_{\tilde V,V}^{-1}$, where $\boldsymbol{i}_{\tilde V,V} : \boldsymbol{S}_{\tilde V,\tilde E,\tilde s} \to \boldsymbol{S}_{V,E,s}$, $\boldsymbol{S}_{f,\hat f} : \boldsymbol{S}_{\tilde V,\tilde E,\tilde s} \to \boldsymbol{S}_{W,F,t}$, and $\boldsymbol{i}_{\tilde V,V}^{-1}$ exists by Lemma* 3.33.

Proof. We run the construction of Definition 3.30 in reverse. In $\boldsymbol{g} = (\underline{g}, g', g'')$, we have C^∞-scheme morphisms $\underline{g} : \operatorname{Spec}\bigl(C^\infty(V)/I_s\bigr) \to \operatorname{Spec}\bigl(C^\infty(W)/I_t\bigr)$ and $(\underline{g}, g') : \operatorname{Spec}\bigl(C^\infty(V)/I_s^2\bigr) \to \operatorname{Spec}\bigl(C^\infty(W)/I_t^2\bigr)$. Hence there exist unique C^∞-ring morphisms $\phi : C^\infty(W)/I_t \to C^\infty(V)/I_s$ and $\phi' : C^\infty(W)/I_t^2 \to C^\infty(V)/I_s^2$ such that $\underline{g} = \operatorname{Spec}\phi$ and $(\underline{g}, g') = \operatorname{Spec}\phi'$. Since $g^\sharp \circ g^{-1}(\imath_Y) = \imath_X \circ g'$ we have $\pi_X \circ \phi' = \phi \circ \pi_Y$.

We will show that we can choose an open neighbourhood $\tilde{V}$ of X in V and a smooth map $f : \tilde{V} \to W$ such that the following diagram commutes:

$$\begin{tikzcd}
C^\infty(W) \arrow[rr, "f^*"'] \arrow[d, "\pi'_Y"] & & C^\infty(\tilde{V}) \arrow[d, "\pi'_X|_{\tilde{V}}"'] \\
C^\infty(W)/I_t^2 \arrow[r, "\phi'"] \arrow[d, "\pi_Y"] & C^\infty(V)/I_s^2 \arrow[r, "|_{\tilde{V}}", "\cong"'] & C^\infty(\tilde{V})/I_{\tilde{s}}^2 \arrow[d, "\pi_X|_{\tilde{V}}"'] \\
C^\infty(W)/I_t \arrow[r, "\phi"] & C^\infty(V)/I_s \arrow[r, "|_{\tilde{V}}", "\cong"'] \arrow[from=u, "\pi_X"'] & C^\infty(\tilde{V})/I_{\tilde{s}},
\end{tikzcd} \tag{3.23}$$

where $\pi'_X : C^\infty(V) \to C^\infty(V)/I_s^2$ and $\pi'_Y : C^\infty(W) \to C^\infty(W)/I_t^2$ are the projections, and $I_{\tilde{s}}$ is the ideal in $C^\infty(\tilde{V})$ generated by $\tilde{s} = s|_{\tilde{V}}$. Note that if (3.23) commutes then for $x \in X$ and $h \in C^\infty(W)$ we have $h(f(x)) = \bigl(\phi' \circ \pi'_Y(h)\bigr)(x) = h(g(x))$ as $\operatorname{Spec} \phi' = (g, g')$. Since this holds for all h we have $f(x) = g(x)$. Thus

$$f|_X = g : X \longrightarrow Y \subseteq W. \tag{3.24}$$

First we show we can choose such f near any point $x \in X$. Let $x \in X \subseteq V$ and $y = g(x) \in Y \subseteq W$. Write $m = \dim V$, $n = \dim W$ and choose coordinates $(x_1, \ldots, x_m)$ on V near x with $x = (0, \ldots, 0)$ and $(y_1, \ldots, y_n)$ on W near y with $y = (0, \ldots, 0)$. Then $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ induce isomorphisms of C^∞-local rings $C_x^\infty(V) \cong C_0^\infty(\mathbb{R}^m)$, $C_y^\infty(W) \cong C_0^\infty(\mathbb{R}^n)$. Write $I_{s,x}$ for the image of $I_s \subset C^\infty(V)$ in $C_x^\infty(V)$ under the projection $C^\infty(V) \to C_x^\infty(V)$, and $I_{t,y}$ for the image of $I_t \subset C^\infty(W)$ in $C_y^\infty(W)$. Consider the diagram

$$\begin{tikzcd}
C_y^\infty(W) \cong C_0^\infty(\mathbb{R}^n) \arrow[r, "\psi = f_x^*"'] \arrow[d] & C_x^\infty(V) \cong C_0^\infty(\mathbb{R}^m) \arrow[d] \\
C_y^\infty(W)/I_{t,y}^2 \arrow[r, "\phi'_{x,y}"] & C_x^\infty(V)/I_{s,x}^2 \\
C^\infty(W)/I_t^2 \arrow[r, "\phi'"] \arrow[u] & C^\infty(V)/I_s^2. \arrow[u]
\end{tikzcd} \tag{3.25}$$

As $\operatorname{Spec} \phi' = (g, g')$ and $g(x) = y$, there is a unique localized morphism $\phi'_{x,y}$ making the bottom square commute. Since $C_y^\infty(W) \cong C_0^\infty(\mathbb{R}^n)$ is free as a C^∞-local ring, we can choose a morphism $\psi : C_y^\infty(W) \to C_x^\infty(V)$ making the top square of (3.25) commute: choose an element $z_i \in C_x^\infty(V)$ with $\phi'_{x,y}(y_i + I_{t,y}^2) = z_i + I_{s,x}^2$ for each $i = 1, \ldots, n$, and then there is a unique morphism ψ with $\psi(y_i) = z_i$ for $i = 1, \ldots, n$.

This morphism $\psi : C_y^\infty(W) \to C_x^\infty(V)$ may be thought of as Spec of a germ at x of smooth maps $f_x : V \to W$ with $f_x(x) = y$. That is, there exists an open neighbourhood $\tilde{V}_x$ of x in V and a smooth map $f_x : \tilde{V}_x \to W$ with $f_x(x) = y$, such that the induced morphism $f_x^* : C_y^\infty(W) \to C_x^\infty(\tilde{V}_x) \cong C_x^\infty(V)$ is ψ. These $\tilde{V}_x, f_x$ are not unique, but if $\tilde{V}'_x, f'_x$ are alternative choices for $\tilde{V}_x, f_x$ then there is an open neighbourhood U_x of x in $\tilde{V}_x \cap \tilde{V}'_x$ with $f_x|_{U_x} = f'_x|_{U_x}$.

Choose such $\tilde V_x, f_x$, and consider the diagram

$$\begin{tikzcd}[column sep=small]
C^\infty(W) \arrow[rrr, "f_x^*"'] \arrow[d, "\pi'_Y"] & & & C^\infty(\tilde V_x) \arrow[r] \arrow[d, "\pi'_X|_{\tilde V_x}"'] & C^\infty_x(\tilde V_x) \cong C^\infty_x(V) \arrow[d] \\
C^\infty(W)/I_t^2 \arrow[r, "\phi'"] & C^\infty(V)/I_s^2 \arrow[rr, "|_{\tilde V_x}"] & & C^\infty(\tilde V_x)/I^2_{\tilde s_x} \arrow[r] & C^\infty_x(V)/I^2_{s,x}.
\end{tikzcd} \tag{3.26}$$

The outer rectangle commutes as (3.25) does. This does not imply the left hand rectangle commutes. Now $C^\infty(W)$ is a fair C^∞-ring, so we can choose a set of generators $c_1,\ldots,c_N$ for $C^\infty(W)$. Then for $i=1,\ldots,N$, the projection of $\pi'_X|_{\tilde V_x}\circ f_x^*(c_i)-\phi'\circ\pi'_Y(c_i)|_{\tilde V_x}$ to $C^\infty_x(V)/I^2_{s,x}$ is zero as the outer rectangle of (3.26) commutes, so $\pi'_X|_{\tilde V_x}\circ f_x^*(c_i)-\phi'\circ\pi'_Y(c_i)|_{\tilde V_x}$ is zero near x in $C^\infty(\tilde V_x)/I^2_{\tilde s_x}$. Therefore making $\tilde V_x$ smaller we can suppose $\pi'_X|_{\tilde V_x}\circ f_x^*(c_i)-\phi'\circ\pi'_Y(c_i)|_{\tilde V_x}=0$ in $C^\infty(\tilde V_x)/I^2_{\tilde s_x}$ for $i=1,\ldots,N$. As $c_1,\ldots,c_N$ generate $C^\infty(W)$, this means the left hand rectangle of (3.26) commutes.

This proves that for each $x\in X$ we can choose an open neighbourhood $\tilde V_x$ of x in V and a smooth map $f_x:\tilde V_x\to W$ such that the top rectangle of (3.23) commutes with $\tilde V_x$ in place of V. But the bottom left rectangle commutes as $\pi_X\circ\phi'=\phi\circ\pi_Y$, and the bottom right rectangle commutes trivially, so (3.23) commutes. Therefore we can choose V,f to make (3.23) commute locally in X.

To make a global choice of V,f, we will combine local choices V_x,f_x using a partition of unity. Choose such $\tilde V_x,f_x$ for all $x\in X$. Then $\{\tilde V_x:x\in X\}$ is an open cover of X, which is paracompact, so we can choose a locally finite refinement, and then as V is a manifold we can choose a subordinate partition of unity. Thus we may choose a countable subset $S\subseteq X$ and a choice $(\tilde V_x,f_x)$ for each $x\in S$ as above (where we may make $\tilde V_x$ smaller to make the locally finite refinement), such that setting $\tilde V=\bigcup_{x\in S}\tilde V_x$, then $\{\tilde V_x:x\in S\}$ is a locally finite open cover of $\tilde V$, and there is a family $\{\eta_x:x\in S\}$ with $\eta_x:\tilde V\to[0,1]$ smooth and supported in $\tilde V_x$, and $\sum_{x\in S}\eta_x=1$.

Roughly speaking we want to define a smooth map $f:\tilde V\to W$ by $f=\sum_{x\in S}\eta_x f_x$. If W were a vector space this would make sense, but for general manifolds it does not. So we need a way to combine smooth maps $f_x:V\to W$ using a partition of unity. We first explain the case when $S=\{x_1,x_2\}$, so we must combine $f_{x_1}:\tilde V_{x_1}\to W$ and $f_{x_2}:\tilde V_{x_2}\to W$ on the overlap $\tilde V_{x_1}\cap\tilde V_{x_2}$.

Choose a Riemannian metric h on W. Then whenever $w_0,w_1\in W$ are sufficiently close there exists a unique short geodesic on W w.r.t. h from w_0 to w_1. We can write this uniquely as a smooth map $\gamma_{w_0,w_1}:[0,1]\to W$ with $\gamma_{w_0,w_1}(0)=w_0$, $\gamma_{w_0,w_1}(1)=w_1$, and $|\dot\gamma_{w_0,w_1}(t)|=d_h(w_0,w_1)^{-1}$ for $t\in[0,1]$, that is, the parametrization of γ_{w_0,w_1} is proportional to arc-length. These pairs (w_0,w_1) and maps γ_{w_0,w_1} combine to give an open neighbourhood U_W of the diagonal $\Delta_W=\{(w,w):w\in W\}$ in $W\times W$ and a smooth map $\Gamma:U_W\times[0,1]\to W$ such that $\Gamma(w_0,w_1,t)=\gamma_{w_0,w_1}(t)$, so that $\Gamma(w_0,w_1,0)=w_0$ and $\Gamma(w_0,w_1,1)=w_1$ for all $(w_0,w_1)\in U_W$, and $\Gamma(w_1,w_0,1-t)=\Gamma(w_0,w_1,t)$.

Now suppose $S=\{x_1,x_2\}$ and $\tilde V_{x_1},\tilde V_{x_2},f_{x_1}f_{x_2}$ and $\tilde V=\tilde V_{x_1}\cup\tilde V_{x_2}$ are as above. If $v\in\tilde V_{x_1}\cap\tilde V_{x_2}\cap X$ then $f_{x_1}(v)=f_{x_2}(v)=g(v)\in W$, so $\bigl(f_{x_1}(v),f_{x_2}(v)\bigr)\in U_W$ as $\Delta_W\subset U_W$. Hence $\bigl(f_{x_1}(v),f_{x_2}(v)\bigr)\in U_W$ for $v\in$

$\tilde{V}_{x_1} \cap \tilde{V}_{x_2}$ close to X, as U_W is open. Thus, making $\tilde{V}_{x_1}, \tilde{V}_{x_1}, \tilde{V}$ smaller without changing $\tilde{V}_{x_1} \cap X$ and $\tilde{V}_{x_1} \cap X$, we may assume $\bigl(f_{x_1}(v), f_{x_2}(v)\bigr) \in U_W$ for all $v \in \tilde{V}_{x_1} \cap \tilde{V}_{x_2}$. Define $f : \tilde{V} \to W$ by

$$f(v) = \begin{cases} f_{x_1}(v), & v \in \tilde{V}_{x_1} \setminus \tilde{V}_{x_2}, \\ f_{x_2}(v), & v \in \tilde{V}_{x_2} \setminus \tilde{V}_{x_1}, \\ \Gamma\bigl(f_{x_1}(v), f_{x_2}(v), \eta_{x_2}(v)\bigr), & v \in \tilde{V}_{x_1} \cap \tilde{V}_{x_2}. \end{cases} \tag{3.27}$$

Near the boundary between $\tilde{V}_{x_1} \setminus \tilde{V}_{x_2}$ and $\tilde{V}_{x_1} \cap \tilde{V}_{x_2}$ we have $\eta_{x_2} = 0$, as η_{x_2} is supported on $\tilde{V}_{x_2}$, so $\Gamma\bigl(f_{x_1}(v), f_{x_2}(v), \eta_{x_2}(v)\bigr) = f_{x_1}(v)$, so the first and third lines of (3.27) join smoothly. Similarly, near the boundary between $\tilde{V}_{x_2} \setminus \tilde{V}_{x_1}$ and $\tilde{V}_{x_1} \cap \tilde{V}_{x_2}$ we have $\eta_{x_1} = 0$, so $\eta_{x_2} = 1$, and $\Gamma\bigl(f_{x_1}(v), f_{x_2}(v), \eta_{x_2}(v)\bigr) = f_{x_2}(v)$, so the second and third lines of (3.27) join smoothly. Hence f is smooth. As (3.23) commutes for $(\tilde{V}_{x_1}, f_{x_1})$ and $(\tilde{V}_{x_2}, f_{x_2})$ we can show that it commutes for $(\tilde{V}, f)$. This constructs $\tilde{V}, f$ in the case $|S| = 2$.

For the general case, choose a total order $<$ on S. Then any finite subset of S may be numbered $x_1, \ldots, x_k$ with $x_1 < x_2 < \cdots < x_k$ in this total order. For such $x_1, \ldots, x_k$, define a smooth function $f_{x_1,\ldots,x_k} : \tilde{V}_{x_1} \cap \cdots \cap \tilde{V}_{x_k} \to W$ by induction on k by $f_{x_1,\ldots,x_k} = f_{x_1}$ when $k = 1$ and

$$f_{x_1,\ldots,x_k}(v) = \Gamma\Bigl(f_{x_1,\ldots,x_{k-1}}(v), f_{x_k}(v), \frac{\eta_{x_k}(v)}{\eta_{x_1}(v) + \cdots + \eta_{x_k}(v)}\Bigr).$$

We make the $\tilde{V}_x$ for $x \in S$ and $\tilde{V}$ smaller if necessary without changing $\tilde{V}_x \cap X$ to ensure that $\bigl(f_{x_1,\ldots,x_{k-1}}(v), f_{x_k}(v)\bigr) \in U_W$ whenever $v \in \tilde{V}_{x_1} \cap \cdots \cap \tilde{V}_{x_k}$, so that $f_{x_1,\ldots,x_k}$ is well defined. Then we define $f : \tilde{V} \to W$ by $f(v) = f_{x_1,\ldots,x_k}(v)$ whenever $\{x \in S : v \in \tilde{V}_x\} = \{x_1, \ldots, x_k\}$ and $x_1, \ldots, x_k$ are numbered so that $x_1 < x_2 < \cdots < x_k$. One can then show that the transitions between the different regions $\tilde{V}_{x_1} \cap \cdots \cap \tilde{V}_{x_k}$ are smooth, so f is smooth, and (3.23) commutes, as we want.

Next we construct $\hat{f}$. We have $g'' : \underline{g}^*(\mathcal{E}_Y) \to \mathcal{E}_X$ in $\mathrm{qcoh}(\underline{X})$, where

$$\begin{aligned} \mathcal{E}_X &= \mathrm{MSpec}\bigl(C^\infty(E^*)/(I_s \cdot C^\infty(E^*))\bigr) \cong \mathrm{MSpec}\bigl(C^\infty(\tilde{E}^*)/(I_{\tilde{s}} \cdot C^\infty(\tilde{E}^*))\bigr), \\ \underline{g}^*(\mathcal{E}_Y) &\cong \mathrm{MSpec}\bigl(C^\infty(F^*)/(I_t \cdot C^\infty(F^*)) \otimes^{\phi}_{C^\infty(W)/I_t} C^\infty(V)/I_s\bigr) \\ &\cong \mathrm{MSpec}\bigl(C^\infty(F^*)/(I_t \cdot C^\infty(F^*)) \otimes^{|_{\tilde{V}} \circ \phi}_{C^\infty(W)/I_t} C^\infty(\tilde{V})/I_{\tilde{s}}\bigr). \end{aligned}$$

So $g'' \cong \mathrm{MSpec}\, G''$ for some unique $C^\infty(\tilde{V})/I_{\tilde{s}}$-module morphism G'' in (3.28).

We claim that we can choose a morphism of vector bundles $\hat{f} : \tilde{E} \to f^*(F)$ such that the following commutes:

$$\begin{tikzcd}[column sep=small] C^\infty(F^*) \otimes^{\phi}_{C^\infty(W)} C^\infty(\tilde{V}) \arrow[r, "\cong"'] \arrow[d, "\Pi_{F^*}"] & C^\infty(f^*(F^*)) \arrow[r, "\circ \hat{f}"'] & C^\infty(E^*) \arrow[d, "\Pi_{\tilde{E}^*}"'] \\ C^\infty(F^*)/(I_t \cdot C^\infty(F^*)) \otimes^{|_{\tilde{V}} \circ \phi}_{C^\infty(W)/I_t} (C^\infty(\tilde{V})/I_{\tilde{s}}) \arrow[rr, "G''"] & & C^\infty(\tilde{E}^*)/(I_{\tilde{s}} \cdot C^\infty(\tilde{E}^*)). \end{tikzcd} \tag{3.28}$$

To see this, first suppose that $f^*(F^*)$ is a trivial vector bundle $\mathbb{R}^k \times \tilde{V} \to \tilde{V}$, and $C^\infty(f^*(F^*))$ is a free $C^\infty(\tilde{V})$-module $\mathbb{R}^k \otimes_{\mathbb{R}} C^\infty(\tilde{V})$ with basis $\delta_1, \ldots, \delta_k$ over $C^\infty(\tilde{V})$. Choose $\epsilon_1, \ldots, \epsilon_k \in C^\infty(\tilde{E}^*)$ such that $\Pi_{\tilde{E}^*}(\epsilon_i) = G'' \circ \Pi_{F^*}(\delta_i)$ for $i = 1, \ldots, k$. Then as $C^\infty(f^*(F^*))$ is free, there exists a unique $C^\infty(\tilde{V})$-module morphism $\hat{F} : C^\infty(f^*(F^*)) \to C^\infty(\tilde{E}^*)$ such that $\hat{F}(\delta_i) = \epsilon_i$ for $i = 1, \ldots, k$, so there is a unique morphism $\hat{f} : \tilde{E} \to f^*(F)$ with $\hat{F}(\delta) = \delta \circ \hat{f}$ for all $\delta \in C^\infty(f^*(F^*))$. If $f^*(F^*)$ is not trivial, then we can cover $\tilde{V}$ by open subsets $\tilde{V}_a$ on which it is trivial, choose $\hat{f}_a$ on $\tilde{V}_a$ to make (3.28) commute on $\tilde{V}_a$, and then combine these choices with a partition of unity to get $\hat{f}$ on $\tilde{V}$.

Now consider the diagram

$$\begin{tikzcd}
(I_t/I_t^2)\otimes_{C^\infty(W)/I_t}(C^\infty(W)/I_t) \arrow[rr, "\cong"'] \arrow[d, "\mathrm{id}\otimes\phi"] & & I_t/I_t^2 \arrow[dr, "h+I_t^2 \mapsto h\circ f+I_{\tilde{s}}^2"] & \\
(I_t/I_t^2)\otimes^{|_{\tilde{V}}\circ\phi}_{C^\infty(W)/I_t}(C^\infty(\tilde{V})/I_{\tilde{s}}) \arrow[rrr, "G^1"] & & & I_{\tilde{s}}/I_{\tilde{s}}^2 \\
C^\infty(F^*)/(I_t\cdot C^\infty(F^*))\otimes^{|_{\tilde{V}}\circ\phi}_{C^\infty(W)/I_t}(C^\infty(\tilde{V})/I_{\tilde{s}}) \arrow[u, "t\cdot"] \arrow[rrr, "G''"] & & & C^\infty(\tilde{E}^*)/(I_{\tilde{s}}\cdot C^\infty(\tilde{E}^*)). \arrow[u, "\tilde{s}\cdot"']
\end{tikzcd} \tag{3.29}$$

Here the top quadrilateral is mapped under MSpec to the left quadrilateral in (2.12) defining the morphism $g^1 : \underline{g}^*(\mathcal{I}_Y) \to \mathcal{I}_X$ for $\boldsymbol{g}$. Hence there is a unique morphism G^1 with $\mathrm{MSpec}\, G^1 = g^1$ making the top quadrilateral commute. Then applying MSpec to the condition that the bottom rectangle commutes gives $\jmath_X \circ g'' = g^1 \circ \underline{g}^*(\jmath_Y)$, which is (2.21) for $\boldsymbol{g}$. Thus (3.29) commutes.

Combining (3.28) and (3.29) shows that the following commutes:

$$\begin{tikzcd}
C^\infty(f^*(F^*)) \arrow[r, "f^*(t)\cdot"'] \arrow[d, "\circ\hat{f}"] & I_{\tilde{s}} \arrow[dr, "\pi"] & \\
 & & I_{\tilde{s}}/I_{\tilde{s}}^2. \\
C^\infty(E^*) \arrow[r, "\tilde{s}\cdot"] & I_{\tilde{s}} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

Hence $\hat{f} \circ s = f^*(t) + O(s^2)$, as in (3.22). As in Remark 3.31, by replacing $\hat{f}$ by $\hat{f} - (\alpha \cdot s)$ we can arrange that $\hat{f} \circ s = f^*(t)$, and (3.28) still commutes. It now follows from Definition 3.30 and (3.23), (3.24), (3.28) and (3.29) that $\boldsymbol{S}_{f,\hat{f}} = \boldsymbol{g} \circ \boldsymbol{i}_{\tilde{V},V}$, so that $\boldsymbol{g} = \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{i}_{\tilde{V},V}^{-1}$, as we want. $\square$

One can also give a differential-geometric interpretation of 2-morphisms $\lambda : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ of 1-morphisms $\boldsymbol{S}_{f,\hat{f}}, \boldsymbol{S}_{g,\hat{g}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$.

Definition 3.35. Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, and $s : V \to E$, $t : W \to F$ be smooth sections. Suppose $f, g : V \to W$ are smooth, and $\hat{f} : E \to f^*(F)$, $\hat{g} : E \to g^*(F)$ are morphisms of vector bundles satisfying $\hat{f} \circ s = f^*(t) + O(s^2)$ and $\hat{g} \circ s = g^*(t) + O(s^2)$. Then Definition 3.30 defines 1-morphisms $\boldsymbol{S}_{f,\hat{f}}, \boldsymbol{S}_{g,\hat{g}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$.

Choose some extra data: a complete Riemannian metric h on W, and a connection ∇^F on $F \to W$. Then if f, g are sufficiently close in C^0, for each $v \in V$ there is a unique short geodesic for the metric h in W joining $f(v)$ and

$g(v)$. Thus we may write $g(v) = \exp_{f(v)}(\gamma(v))$ for some $\gamma(v) \in T_{f(v)}W$, where $\exp_{f(v)} : T_{f(v)}W \to W$ is the geodesic exponential map. Then γ is a smooth section of $f^*(TW) \to V$, and we write $g = \exp_f(\gamma) \circ f$. We also define an isomorphism $\Theta_{f,g} : f^*(F) \to g^*(F)$ by defining $\Theta_{f,g}(v) : F_{f(v)} \to F_{g(v)}$ to be the parallel transport map in F using ∇^F along the unique short geodesic from $f(v)$ to $g(v)$, for each $v \in V$.

Now suppose $\Lambda : E \to f^*(TW)$ is a morphism of vector bundles on V. Then $\Lambda \circ s \in C^\infty(f^*(TW))$, so we can require that $g = \exp_f(\Lambda \circ s) \circ f$. Also $\nabla^F t$ is a section of $T^*W \otimes F \to W$, so $f^*(\nabla^F t)$ is a section of $f^*(T^*W) \otimes f^*(F) \to V$, or equivalently a morphism $f^*(TW) \to f^*(F)$. Hence $f^*(\nabla^F t) \circ \Lambda$ is a morphism $E \to f^*(F)$. Thus we can require that $\hat g = \Theta_{f,g} \circ (\hat f + f^*(\nabla^F t) \circ \Lambda)$. Taking the dual of Λ and lifting to $\underline{V}$ gives a morphism $\Lambda^* : \underline{f}^*(T^*\underline{W}) \to \mathcal{E}^*$. Restricting to the C^∞-subscheme $\underline{X} = s^{-1}(0)$ in $\underline{V}$ gives $\lambda = \Lambda^*|_{\underline{X}} : \underline{f}^*(\mathcal{F}_Y) \cong \underline{f}^*(T^*\underline{W})|_{\underline{X}} \to \mathcal{E}^*|_{\underline{X}} = \mathcal{E}_X$. One can show that λ is a 2-morphism $\boldsymbol{S}_{f,\hat f} \Rightarrow \boldsymbol{S}_{g,\hat g}$ if and only if

$$g = \exp_f(\Lambda \circ s) \circ f + O(s^2) \;\text{ and }\; \hat g = \Theta_{f,g} \circ (\hat f + f^*(\nabla^F t) \circ \Lambda) + O(s).$$

A more informal way to write these equations is

$$g = f + \Lambda \cdot s + O(s^2) \;\text{ and }\; \hat g = \hat f + \Lambda \cdot f^*(\mathrm{d}t) + O(s). \tag{3.30}$$

We write λ as $\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat f} \Rightarrow \boldsymbol{S}_{g,\hat g}$, and call it a *standard model 2-morphism.*

If $\eta : \boldsymbol{S}_{f,\hat f} \Rightarrow \boldsymbol{S}_{g,\hat g}$ is a 2-morphism in **dSpa** then we can regard η as a morphism of vector bundles $\eta : f^*(T^*W)|_{\underline{X}} \to E^*|_{\underline{X}}$ in qcoh($\underline{X}$), where $\underline{X} \subseteq \underline{X}$ is the C^∞-subscheme $s^{-1}(0)$ in $\underline{V}$. We may extend η to a morphism of vector bundles $\Lambda^* : f^*(T^*W) \to E^*$ on V, or equivalently, a morphism $\Lambda : E \to f^*(TW)$. The fact that η is a 2-morphism implies that (3.30) holds, and then $\eta = \boldsymbol{S}_\Lambda$. So every $\eta : \boldsymbol{S}_{f,\hat f} \Rightarrow \boldsymbol{S}_{g,\hat g}$ is a 'standard model' 2-morphism.

It is also easy to see that $\boldsymbol{S}_{\Lambda'} = \boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat f} \Rightarrow \boldsymbol{S}_{g,\hat g}$ if and only if $\Lambda' = \Lambda + O(s)$.

3.5 Equivalences in dMan

Suppose $f : X \to Y$ is a smooth map of manifolds without boundary. Then f is étale if and only if $\mathrm{d}f^* : f^*(T^*Y) \to T^*X$ is an isomorphism of vector bundles, and f is a diffeomorphism if in addition f is a bijection. This is not true for C^∞-schemes, or even for manifolds with boundary, as Examples 3.37 and 3.38 show. Our next theorem is a 'derived' analogue of this result. It should be compared with Propositions 2.20 and 2.21 and Corollary 2.24 for 1-morphisms of d-spaces $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$; we show that when $\boldsymbol{X}, \boldsymbol{Y}$ are d-manifolds, we can omit the conditions that $\underline{f} : \underline{X} \to \underline{Y}$ is an isomorphism or étale and $f^4 : \underline{f}^*(\mathcal{C}_Y) \to \mathcal{C}_X$ is an isomorphism.

Theorem 3.36. *Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism of d-manifolds. Then the following are equivalent:*

(i) *$\boldsymbol{f}$ is étale;*

(ii) $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ *is an equivalence in* $\mathrm{vqcoh}(\underline{X})$; *and*

(iii) *equation* (2.34) *is a split short exact sequence in* $\mathrm{qcoh}(\underline{X})$.

If in addition $f : X \to Y$ *is a bijection, then* $\boldsymbol{f}$ *is an equivalence in* **dMan**.

Proof. Parts (ii) and (iii) are equivalent by Proposition 3.5 and the definition of $\Omega_{\boldsymbol{f}}$ in Example 3.2, noting that (3.4) for $\Omega_{\boldsymbol{f}}$ is (2.34). If $\boldsymbol{f}$ is étale then Corollary 2.24 implies that (2.34) is split exact, so (i) implies (ii),(iii). We will show that (ii),(iii) imply (i).

Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ satisfies (ii),(iii), and let $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$. By Proposition 3.25 and Example 3.24, there exist open neighbourhoods $\boldsymbol{U} \subseteq \boldsymbol{X}$, $\tilde{\boldsymbol{U}} \subseteq \boldsymbol{Y}$ of x, y and equivalences $\boldsymbol{i} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{U}$, $\boldsymbol{j} : \tilde{\boldsymbol{U}} \to \boldsymbol{S}_{W,F,t}$, where V, W are open neighbourhoods of 0 in $\mathbb{R}^m, \mathbb{R}^n$ for $m = \dim T_x^*\underline{X}$, $n = \dim T_y^*\underline{Y}$, and E, F are trivial vector bundles over V, W, and $s \in C^\infty(E)$, $t \in C^\infty(F)$ with $s(0) = \mathrm{d}s(0) = 0$ and $t(0) = \mathrm{d}t(0) = 0$, and $\boldsymbol{i}(0) = x$, $\boldsymbol{j}(y) = 0$. Making $\boldsymbol{U}, V$ smaller if necessary, we can suppose that $\boldsymbol{f}(\boldsymbol{U}) \subseteq \tilde{\boldsymbol{U}}$.

Define $\boldsymbol{g} = \boldsymbol{j} \circ \boldsymbol{f} \circ \boldsymbol{i} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. Then $\boldsymbol{g}(0) = 0$. Applying Theorem 3.34 to $\boldsymbol{g}$ and replacing V, E, s by $\tilde{V}, \tilde{E}, \tilde{s}$ gives a smooth map $f : V \to W$ with $f(0) = 0$ and a morphism of vector bundles $\hat{f} : E \to f^*(F)$ on V with $\hat{f} \circ s = f^*(t)$, such that $\boldsymbol{g} = \boldsymbol{S}_{f,\hat{f}}$. Now $\Omega_{\boldsymbol{g}} : \underline{g}^*(T^*\boldsymbol{S}_{W,F,t}) \to \boldsymbol{S}_{V,E,s}$ is the composition of $\Omega_{\boldsymbol{i}}$ and pullbacks of $\Omega_{\boldsymbol{f}}$ and $\Omega_{\boldsymbol{j}}$. But $\Omega_{\boldsymbol{i}}, \Omega_{\boldsymbol{j}}$ are equivalences by Lemma 3.3, and $\Omega_{\boldsymbol{f}}$ is an equivalence by assumption. Hence $\Omega_{\boldsymbol{g}}$ is a composition of equivalences in $\mathrm{vqcoh}(\underline{S}_{V,E,s})$, and is an equivalence. Pulling back by the morphism $\underline{0} : \underline{*} \to \underline{S}_{V,E,s}$ taking the point $*$ to $0 \in s^{-1}(0) \subseteq V$, we see that $\underline{0}^*(\Omega_{\boldsymbol{g}})$ is an equivalence in $\mathrm{vqcoh}(\underline{*})$.

Now $T^*\boldsymbol{S}_{V,E,s}$ is the morphism $\mathrm{d}s^* : \mathcal{E}^*|_{\underline{S}_{V,E,s}} \to T^*\underline{V}|_{\underline{S}_{V,E,s}}$, where $\mathcal{E}^*$ is the vector bundle on $\underline{V} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(V)$ corresponding to $E^* \to V$, and $\underline{S}_{V,E,s}$ is the C^∞-subscheme $s = 0$ in $\underline{V}$. Hence $\underline{0}^*(T^*\boldsymbol{S}_{V,E,s})$ is the morphism $\mathrm{d}s(0)^* : E|_0^* \to T_0^*V$ in $\mathrm{qcoh}(\underline{*})$, which we identify with the category of real vector spaces. But $\mathrm{d}s(0) = 0$ from above, so $\underline{0}^*(T^*\boldsymbol{S}_{V,E,s})$ is the morphism $0 : E|_0^* \to T_0^*V$. Similarly $\underline{0}^*(\underline{g}^*(T^*\boldsymbol{S}_{W,F,t}))$ is the morphism $0 : F|_0^* \to T_0^*W$, and $\underline{0}^*(\Omega_{\boldsymbol{g}})$ is given by the columns in the diagram

$$\begin{tikzcd} F|_0^* \arrow[r, "0"'] \arrow[d, "\hat{f}|_0^*"] & T_0^*W \arrow[d, "\mathrm{d}f|_0^*"'] \\ E|_0^* \arrow[r, "0"] & T_0^*V. \end{tikzcd}$$

Thus, $\underline{0}^*(\Omega_{\boldsymbol{g}})$ an equivalence implies that $\mathrm{d}f|_0 : T_0V \to T_0W$ and $\hat{f}|_0 : E|_0 \to F|_0$ are isomorphisms. Therefore $f : V \to W$ is a local diffeomorphism near $0 \in V$, and $\hat{f} : E \to f^*(F)$ is an isomorphism of vector bundles near $0 \in V$. Making $\boldsymbol{U}, \tilde{\boldsymbol{U}}, V, W$ smaller if necessary, we can suppose that $f : V \to W$ is a diffeomorphism and $\hat{f} : E \to f^*(F)$ is an isomorphism. Since $\hat{f} \circ s = f^*(t)$, this implies that $(f, \hat{f})$ is an isomorphism of triples $(V, E, s) \to (W, F, t)$, and $\boldsymbol{g} = \boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ is a 1-isomorphism. As $\boldsymbol{g} = \boldsymbol{j} \circ \boldsymbol{f} \circ \boldsymbol{i}$ with $\boldsymbol{i}, \boldsymbol{j}$ equivalences, we see that $\boldsymbol{f}|_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ is an equivalence. As we can find such

$x \in \boldsymbol{U}$, $y \in \boldsymbol{V}$ for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$, $\boldsymbol{f}$ is a local equivalence, that is, $\boldsymbol{f}$ is étale. Hence (ii)–(iii) imply (i), and (i)–(iii) are equivalent.

For the last part, if $\boldsymbol{f}$ is étale then the underlying continuous map $f : X \to Y$ is a local homeomorphism. As X, Y are Hausdorff, if f is a bijection then f is a homeomorphism, so $\underline{f} : \underline{X} \to \underline{Y}$ is an isomorphism as it is étale, and thus $\boldsymbol{f}$ is an equivalence by Corollary 2.24 and Proposition 2.21. $\square$

Here are examples of morphisms in $\mathbf{Man^b}, \mathbf{C^\infty Sch}$ which induce isomorphisms on cotangent bundles, but are not étale. Applying $F_{\mathbf{Man^b}}^{\mathbf{dSpa}}, F_{\mathbf{C^\infty Sch}}^{\mathbf{dSpa}}$ gives 1-morphisms of d-spaces $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ satisfying Theorem 3.36(ii),(iii), but which are not étale. Thus, Theorem 3.36 is special to d-manifolds.

Example 3.37. The inclusion $i : [0, \infty) \hookrightarrow \mathbb{R}$ is a smooth map of manifolds with boundary and $\mathrm{d}i^* : i^*(T^*\mathbb{R}) \to T^*[0, \infty)$ is an isomorphism, but i is not étale near $0 \in [0, \infty)$. This also holds for C^∞-schemes $\underline{i} : \underline{[0, \infty)} \hookrightarrow \underline{\mathbb{R}}$.

Example 3.38. Let $f \in C^\infty(\mathbb{R}^n)$. Define ideals $J \subseteq I \subseteq C^\infty(\mathbb{R}^n)$ by $I = \bigl(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\bigr)$ and $I = \bigl(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\bigr)$. Then we have C^∞-rings $C^\infty(\mathbb{R}^n)/I$, $C^\infty(\mathbb{R}^n)/J$ with a natural projection $\pi : C^\infty(\mathbb{R}^n)/J \to C^\infty(\mathbb{R}^n)/J$. Set $\underline{X} = \operatorname{Spec}\bigl(C^\infty(\mathbb{R}^n)/I\bigr)$, $\underline{Y} = \operatorname{Spec}\bigl(C^\infty(\mathbb{R}^n)/J\bigr)$ and $\underline{f} = \operatorname{Spec}\pi$. Then $\underline{X}, \underline{Y}$ are finitely presented affine C^∞-schemes and $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism.

We have cotangent modules

$$\begin{aligned}
\Omega_{C^\infty(\mathbb{R}^n)/I} &\cong \frac{\bigl\langle \mathrm{d}x_1, \ldots, \mathrm{d}x_n \bigr\rangle_{\mathbb{R}} \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^n)/I}{\bigl\langle \sum_{i=1}^n \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i,\ \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}\,\mathrm{d}x_i,\ j = 1, \ldots, n \bigr\rangle} \\
&= \frac{\bigl\langle \mathrm{d}x_1, \ldots, \mathrm{d}x_n \bigr\rangle_{\mathbb{R}} \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^n)/I}{\bigl\langle \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}\,\mathrm{d}x_i,\ \ j = 1, \ldots, n \bigr\rangle}, \\
\Omega_{C^\infty(\mathbb{R}^n)/J} &\cong \frac{\bigl\langle \mathrm{d}x_1, \ldots, \mathrm{d}x_n \bigr\rangle_{\mathbb{R}} \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^n)/J}{\bigl\langle \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}\,\mathrm{d}x_i,\ \ j = 1, \ldots, n \bigr\rangle},
\end{aligned}$$

where the second expression for $\Omega_{C^\infty(\mathbb{R}^n)/I}$ holds as $\frac{\partial f}{\partial x_i} = 0$ in $C^\infty(\mathbb{R}^n)/I$. The morphism $\Omega_\pi : \Omega_{C^\infty(\mathbb{R}^n)/J} \to \Omega_{C^\infty(\mathbb{R}^n)/I}$ mapping $\mathrm{d}x_i \mapsto \mathrm{d}x_i$ which induces $(\Omega_\pi)_* : \Omega_{C^\infty(\mathbb{R}^n)/J} \otimes_{C^\infty(\mathbb{R}^n)/J} (C^\infty(\mathbb{R}^n)/I) \to \Omega_{C^\infty(\mathbb{R}^n)/I}$. Comparing the expressions for $\Omega_{C^\infty(\mathbb{R}^n)/I}, \Omega_{C^\infty(\mathbb{R}^n)/J}$ we see that $(\Omega_\pi)_*$ is an isomorphism. But $\Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$ is $\operatorname{MSpec}(\Omega_\pi)_*$, so $\Omega_{\underline{f}}$ is an isomorphism.

We have found a class of morphisms $\underline{f} : \underline{X} \to \underline{Y}$ in $\mathbf{C^\infty Sch}$ such that $\Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$ is an isomorphism, but $\underline{f}$ is in general not étale. On the reduced C^∞-subschemes $\underline{f}^{\mathrm{red}} : \underline{X}^{\mathrm{red}} \to \underline{Y}^{\mathrm{red}}$ is étale, but the nonreduced structures on $\underline{X}, \underline{Y}$ can be different. Note that this holds for $\underline{X}, \underline{Y}$ finitely presented affine C^∞-schemes, which are about as well-behaved as C^∞-schemes can be; in Example 3.37, $\underline{[0, \infty)}$ is not finitely presented.

We can apply Theorem 3.36 to characterize equivalences in the differential-geometric description of 1-morphisms in §3.4. Here is how to understand equation (3.31) below. Given $V, E \to V$ and $s \in C^\infty(E)$, choose a connection ∇^E

on E. Then $\nabla^E s \in C^\infty(E \otimes T^*V)$, so we can regard $\nabla^E s$ as a vector bundle morphism $TV \to E$ on E. Now $\nabla^E s$ depends on the choice of ∇^E: if $\tilde{\nabla}^E$ is another possible choice then $\tilde{\nabla}^E = \nabla^E + \Gamma$, for $\Gamma \in C^\infty(E \otimes E^* \otimes T^*V)$, and $\tilde{\nabla}^E s = \nabla^E s + \Gamma \cdot s$. Hence at a point $v \in V$ with $s(v) = 0$ we have $(\tilde{\nabla}^E s)(v) = (\nabla^E s)(v) + \Gamma(v) \cdot s(v) = (\nabla^E s)(v)$, so $(\nabla^E s)(v)$ is independent of the choice of ∇^E, and we write it $\mathrm{d}s(v)$.

Now choose connections ∇^E, ∇^F on E, F. Differentiating the equation $\hat{f} \circ s = f^*(t) + O(s^2)$ using ∇^E, ∇^F, noting that $\hat{f} \circ s$ should be differentiated using the product rule but $f^*(t) = t \circ f$ should be differentiated using the chain rule, and the derivative of $\hat{f}$ uses both ∇^E, ∇^F so we write it $\nabla^{E,F}\hat{f}$, gives

$$(\nabla^{E,F}\hat{f}) \circ s + \hat{f} \circ \nabla^E s = f^*(\nabla^F t) \circ \mathrm{d}f + O(s \nabla s).$$

At a point $v \in V$ with $s(v) = 0$ and $f(v) = w$, the terms $(\nabla^{E,F}\hat{f}) \circ s$ and $O(s\nabla s)$ vanish, giving $\hat{f}(v) \circ \mathrm{d}s(v) = \mathrm{d}t(w) \circ \mathrm{d}f(v)$. Therefore (3.31) is automatically a complex. It is in fact the dual of the complex (2.34) for $\boldsymbol{S}_{f,\hat{f}}$, specialized at v.

Theorem 3.39. *Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, $s : V \to E$, $t : W \to F$ be smooth sections, $f : V \to W$ be smooth, and $\hat{f} : E \to f^*(F)$ be a morphism of vector bundles on V with $\hat{f} \circ s = f^*(t) + O(s^2)$. Then Definitions* 3.13 *and* 3.30 *define principal d-manifolds $\boldsymbol{S}_{V,E,s}, \boldsymbol{S}_{W,F,t}$ and a* 1*-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. This $\boldsymbol{S}_{f,\hat{f}}$ is étale if and only if for each $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, the following sequence of vector spaces (explained above) is exact:*

$$0 \longrightarrow T_vV \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} E_v \oplus T_wW \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)} F_w \longrightarrow 0. \tag{3.31}$$

Also $\boldsymbol{S}_{f,\hat{f}}$ is an equivalence if and only if in addition $f|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection, where $s^{-1}(0) = \{v \in V : s(v) = 0\}$, $t^{-1}(0) = \{w \in W : t(w) = 0\}$.

Proof. Use the notation of Definition 3.30, so that $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$, $\boldsymbol{g} = \boldsymbol{S}_{f,\hat{f}}$ etc. Then equation (2.34) for $\boldsymbol{g}$ is

$$0 \to \underline{f}^*(\mathcal{F}^*)|_{\underline{X}} \xrightarrow{\underline{\hat{f}}|_{\underline{X}} \oplus -\underline{f}^*(\underline{\nabla^F t})|_{\underline{X}}} \mathcal{E}^*|_{\underline{X}} \oplus \underline{f}^*(T^*\underline{W})|_{\underline{X}} \xrightarrow{\underline{\nabla^E s}|_{\underline{X}} \oplus \Omega_{\underline{f}}|_{\underline{X}}} T^*\underline{V}|_{\underline{X}} \to 0, \tag{3.32}$$

where $\underline{V}, \underline{W}, \underline{f} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(V, W, f)$, and $\underline{X} \subseteq \underline{V}$, $\underline{Y} \subseteq \underline{W}$ the C^∞-subschemes defined by $s = 0$, $t = 0$, and $\mathcal{E}^*, \mathcal{F}^*$ the vector bundles on $\underline{V}, \underline{W}$ corresponding to $E^* \to V$, $F^* \to W$, and $\underline{\hat{f}} : \underline{f}^*(\mathcal{F}^*) \to \mathcal{E}^*$ the morphism in $\mathrm{qcoh}(\underline{V})$ lifting $(\hat{f})^* : f^*(F^*) \to E^*$, and ∇^E, ∇^F are connections on E, F so that $\nabla^E s : E^* \to T^*V$ and $\nabla^F t : F^* \to T^*W$ are vector bundle morphisms, and $\underline{\nabla^E s} : \mathcal{E}^* \to T^*\underline{V}$, $\underline{\nabla^F t} : \mathcal{F}^* \to T^*\underline{W}$ are the corresponding morphisms on $\underline{V}, \underline{W}$.

Then Theorem 3.36 shows that $\boldsymbol{S}_{f,\hat{f}}$ is étale if and only if (3.32) is split exact. The terms in (3.32) are all vector bundles over $\underline{X}$, which is a separated, paracompact, locally fair C^∞-scheme. Exact sequence of vector bundles over such $\underline{X}$ have better properties than exact sequences of quasicoherent sheaves:

(a) If $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is an exact sequence of vector bundles over $\underline{X}$ and $\underline{h} : \underline{Z} \to \underline{X}$ is a morphism of C^∞-schemes then $0 \to \underline{h}^*(\mathcal{E}) \to \underline{h}^*(\mathcal{F}) \to \underline{h}^*(\mathcal{G}) \to 0$ is an exact sequence of vector bundles over $\underline{Z}$. That is, the pullback $\underline{h}^*$ is exact, not just right exact.

(b) Suppose $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is a complex of vector bundles on $\underline{X}$ and $x \in \underline{X}$, with corresponding morphism $\underline{x} : \underline{*} \to \underline{X}$. If $0 \to \underline{x}^*(\mathcal{E}) \to \underline{x}^*(\mathcal{F}) \to \underline{x}^*(\mathcal{G}) \to 0$ is exact in $\mathrm{qcoh}(\underline{*})$, then $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is exact on an open neighbourhood $\underline{U}$ of x in $\underline{X}$.

(c) Any exact sequence of vector bundles $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ over $\underline{X}$ is automatically split exact.

For the 'only if' part, suppose $\boldsymbol{S}_{f,\hat{f}}$ is étale, so that (3.32) is split exact. Let $v \in V$ with $s(v) = 0$. Then $v \in \underline{X}$, so $\underline{v} : \underline{*} \to \underline{X}$ is a morphism, and the pullback of (3.32) by $\underline{v}^*$ is exact by (a). But this pullback is the dual complex of (3.31), so (3.31) is exact. For the 'if' part, suppose is (3.31) is exact for every $v \in s^{-1}(0)$. As this is dual to the pullback of (3.32) by $\underline{v}^*$, this pullback is exact, so (b) shows that (3.32) is exact near v for every $v \in \underline{X}$. Since exactness is local, this proves (3.32) is exact, so it is split exact by (c). Therefore $\boldsymbol{S}_{f,\hat{f}}$ is étale by Theorem 3.36. The last part, with $\boldsymbol{S}_{f,\hat{f}}$ an equivalence, follows from the last part of Theorem 3.36. □

Here is a case in which Theorem 3.39 simplifies. It is related to coordinate changes of Kuranishi neighbourhoods in Fukaya et al. [32, App. A] and §14.3.

Corollary 3.40. *Let V, W be manifolds, $E \to V$, $F \to W$ vector bundles, $s : V \to E$, $t : W \to F$ smooth sections, $f : V \to W$ an embedding, and $\hat{f} : E \to f^*(F)$ an injective morphism of vector bundles on V with $\hat{f} \circ s = f^*(t) + O(s^2)$. For each $v \in s^{-1}(0) \subseteq V$ and $w = f(v) \in W$, we have a linear map*

$$\mathrm{d}t(w)_* : T_wW \big/ \mathrm{d}f(v)[T_vV] \longrightarrow F_w \big/ \hat{f}(v)[E_v]. \tag{3.33}$$

Suppose (3.33) *is an isomorphism for all such v, and $f|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection. Then $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ is an equivalence in* **dMan**.

3.6 Gluing d-manifolds by equivalences

Theorems 2.28–2.33 of §2.4 explain how to glue d-spaces by equivalences on open d-subspaces. All these generalize immediately to d-manifolds: if we fix $n \in \mathbb{Z}$ and take the initial d-spaces $\boldsymbol{X}_i$ to be d-manifolds with $\mathrm{vdim}\, \boldsymbol{X}_i = n$, then the glued d-space $\boldsymbol{Y}$ is also a d-manifold with $\mathrm{vdim}\, \boldsymbol{Y} = n$. Here is the analogue of Theorem 2.28. See Spivak [95, Lem. 6.8 & Prop. 6.9] for similar results for his derived manifolds.

Theorem 3.41. *Suppose $\boldsymbol{X}, \boldsymbol{Y}$ are d-manifolds with* $\mathrm{vdim}\, \boldsymbol{X} = \mathrm{vdim}\, \boldsymbol{Y} = n$ *in $\mathbb{Z}$, and $\boldsymbol{U} \subseteq \boldsymbol{X}$, $\boldsymbol{V} \subseteq \boldsymbol{Y}$ are open d-submanifolds, and $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{V}$ is an equivalence in* **dMan**. *At the level of topological spaces, we have open $U \subseteq X$, $V \subseteq Y$ with a homeomorphism $f : U \to V$, so we can form the quotient* *topological space* $Z := X \amalg_f Y = (X \amalg Y)/\sim$, *where the equivalence relation* $\sim$ *on* $X \amalg Y$ *identifies* $u \in U \subseteq X$ *with* $f(u) \in V \subseteq Y$.

Suppose Z *is Hausdorff. Then there exist a d-manifold* $\boldsymbol{Z}$ *with* $\operatorname{vdim} \boldsymbol{Z} = n$, *open d-submanifolds* $\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}$ *in* $\boldsymbol{Z}$ *with* $\boldsymbol{Z} = \hat{\boldsymbol{X}} \cup \hat{\boldsymbol{Y}}$, *equivalences* $\boldsymbol{g} : \boldsymbol{X} \to \hat{\boldsymbol{X}}$ *and* $\boldsymbol{h} : \boldsymbol{Y} \to \hat{\boldsymbol{Y}}$ *such that* $\boldsymbol{g}|_{\boldsymbol{U}}$ *and* $\boldsymbol{h}|_{\boldsymbol{V}}$ *are both equivalences with* $\hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$, *and a* 2*-morphism* $\eta : \boldsymbol{g}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f} : \boldsymbol{U} \to \hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$. *Furthermore,* $\boldsymbol{Z}$ *is independent of choices up to equivalence.*

Here is an analogue of Theorem 2.33 in which we take the d-spaces $\boldsymbol{X}_i$ to be 'standard model' d-manifolds $\boldsymbol{S}_{V_i,E_i,s_i}$, and the 1-morphisms $\boldsymbol{e}_{ij}$ to be 'standard model' 1-morphisms $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}$, and the 2-morphisms η_{ijk} to be 'standard model' 2-morphisms $\boldsymbol{S}_{\Lambda_{ijk}}$, and replace $\boldsymbol{Y}, \boldsymbol{Z}$ by $\boldsymbol{X}, \boldsymbol{Y}$. We also use Theorem 3.39 in (ii) to characterize when $\boldsymbol{e}_{ij} = \boldsymbol{S}_{e_{ij},\hat{e}_{ij}}$ is an equivalence, we take Y to be a manifold rather than a d-space, and we rewrite by taking X to be given *a priori* rather than constructing it by gluing the X_i together.

In the last part of the theorem, taking $\zeta_{ij} = \boldsymbol{S}_{\Lambda_{ij}}$ for $\Lambda_{ij} : E_i|_{V_{ij}} \to g_i^*(TY)|_{V_{ij}}$, the condition in Theorem 2.33 that $\zeta_{ij} : \boldsymbol{g}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{g}_i|_{\boldsymbol{U}_{ij}}$ is a 2-morphism reduces by (3.30) to $g_j \circ e_{ij} = g_i|_{V_{ij}} + \Lambda_{ij} \cdot s_i + O(s_i^2)$ for some arbitrary Λ_{ij}. We can rewrite this as $g_j \circ e_{ij} = g_i|_{V_{ij}} + O(s_i)$.

Theorem 3.42. *Suppose we are given the following data:*

(a) *an integer* n;

(b) *a Hausdorff, second countable topological space* X;

(c) *an indexing set* I, *and a total order* $<$ *on* I;

(d) *for each* i *in* I, *a manifold* V_i, *a vector bundle* $E_i \to V_i$ *with* $\dim V_i - \operatorname{rank} E_i = n$, *a smooth section* $s_i : V_i \to E_i$, *and a homeomorphism* $\psi_i : X_i \to \hat{X}_i$, *where* $X_i = \{v_i \in V_i : s_i(v_i) = 0\}$ *and* $\hat{X}_i \subseteq X$ *is open; and*

(e) *for all* $i < j$ *in* I, *an open submanifold* $V_{ij} \subseteq V_i$, *a smooth map* $e_{ij} : V_{ij} \to V_j$, *and a morphism of vector bundles* $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$.

Let this data satisfy the conditions:

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if* $i < j$ *in* I *then* $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j) + O(s_i^2)$, $\psi_i(X_i \cap V_{ij}) = \hat{X}_i \cap \hat{X}_j$, *and* $\psi_i|_{X_i \cap V_{ij}} = \psi_j \circ e_{ij}|_{X_i \cap V_{ij}}$, *and if* $v_i \in V_{ij}$ *with* $s_i(v_i) = 0$ *and* $v_j = e_{ij}(v_i)$ *then the following is exact:*
$$0 \longrightarrow T_{v_i} V_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} E_j|_{v_j} \longrightarrow 0;$$

(iii) *if* $i < j < k$ *in* I *then there exists a morphism of vector bundles* $\Lambda_{ijk} : E_i|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} \to e_{ik}^*(TV_k)|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}$ *on* $V_{ik} \cap e_{ij}^{-1}(V_{jk}) \subseteq V_i$, *where* $e_{ij}^{-1}(V_{jk}) \subseteq V_{ij}$, *satisfying, as in* (3.30):
$$e_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = e_{jk} \circ e_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} + \Lambda_{ijk} \cdot s_i + O(s_i^2), \tag{3.34}$$
$$\begin{aligned} \hat{e}_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = {} & e_{ij}|^*_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} \\ & + \Lambda_{ijk} \cdot e_{ik}^*(\mathrm{d}s_k)|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} + O(s_i). \end{aligned} \tag{3.35}$$

Then there exist a d-manifold $\boldsymbol{X}$ *with* $\mathrm{vdim}\,\boldsymbol{X} = n$ *and underlying topological space* X, *and a* 1*-morphism* $\boldsymbol{\psi}_i : \boldsymbol{S}_{V_i,E_i,s_i} \to \boldsymbol{X}$ *with underlying continuous map* ψ_i *which is an equivalence with the open d-submanifold* $\hat{\boldsymbol{X}}_i \subseteq \boldsymbol{X}$ *corresponding to* $\hat{X}_i \subseteq X$ *for all* $i \in I$, *such that for all* $i < j$ *in* I *there exists a* 2*-morphism* $\eta_{ij} : \boldsymbol{\psi}_j \circ \boldsymbol{S}_{e_{ij},\hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i \circ \boldsymbol{i}_{V_{ij},V_i}$, *where* $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}} : \boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}} \to \boldsymbol{S}_{V_j,E_j,s_j}$ *and* $\boldsymbol{i}_{V_{ij},V_i} : \boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}} \to \boldsymbol{S}_{V_i,E_i,s_i}$. *This d-manifold* $\boldsymbol{X}$ *is unique up to equivalence in* **dMan**.

Suppose also that Y *is a manifold, and* $g_i : V_i \to Y$ *are smooth maps for all* $i \in I$, *and* $g_j \circ e_{ij} = g_i|_{V_{ij}} + O(s_i)$ *for all* $i < j$ *in* I. *Then there exist a* 1*-morphism* $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Y}$ *unique up to* 2*-isomorphism, where* $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y) = \boldsymbol{S}_{Y,0,0}$, *and* 2*-morphisms* $\zeta_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \boldsymbol{S}_{g_i,0}$ *for all* $i \in I$. *Here* $\boldsymbol{S}_{Y,0,0}$ *is from Definition* 3.13 *with vector bundle* E *and section* s *both zero, and* $\boldsymbol{S}_{g_i,0} : \boldsymbol{S}_{V_i,E_i,s_i} \to \boldsymbol{S}_{Y,0,0} = \boldsymbol{Y}$ *is from Definition* 3.30 *with* $\hat{g}_i = 0$.

The hypotheses of Theorem 3.42 are similar to *good coordinate systems* in §10.8. The importance of Theorem 3.42 is that all the ingredients are described wholly in differential-geometric or topological terms. So we can use the theorem as a tool to prove the existence of d-manifold structures on spaces coming from other areas of geometry, e.g. on moduli spaces. We return to this in Chapter 14.

4 Differential geometry of d-manifolds

We now develop some of the basic ideas of differential geometry for the d-manifolds of Chapter 3: submersions, immersions, embeddings, submanifolds, fibre products, and (co)orientations. We define two different notions of each of submersions, immersions, embeddings, and submanifolds, a weak and a strong. We define when 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are *d-transverse*, and show that if $\boldsymbol{g}, \boldsymbol{h}$ are d-transverse then the fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists as a d-manifold, with $\operatorname{vdim} \boldsymbol{W} = \operatorname{vdim} \boldsymbol{X} + \operatorname{vdim} \boldsymbol{Y} - \operatorname{vdim} \boldsymbol{Z}$. D-transversality is weaker than transversality for manifolds. For example, if $\boldsymbol{Z}$ is a manifold then any $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are d-transverse.

In §4.4 we will show that any compact d-manifold $\boldsymbol{X}$ can be embedded into $\mathbb{R}^n$ for $n \gg 0$, and that if a d-manifold $\boldsymbol{X}$ can be embedded into a manifold Y then $\boldsymbol{X}$ may be written as the zeroes $s^{-1}(0)$ of a smooth section s of a vector bundle E over an open set in Y, and so is a principal d-manifold. After some preparatory material in §4.5, section 4.6 defines orientations on d-manifolds, and constructs orientations on d-transverse fibre products of oriented d-manifolds. Finally, §4.7 explains that for many purposes one can treat d-manifolds as an ordinary category, rather than a 2-category.

4.1 Submersions, immersions, and embeddings

Submersions, immersions, and embeddings are classes of smooth maps of manifolds. We will generalize these to d-manifolds in Definition 4.4 below. To motivate our definition, first consider submersions and immersions of manifolds. Let $f : X \to Y$ be a smooth map of manifolds. Then $\mathrm{d}f^* : f^*(T^*Y) \to T^*X$ is a morphism of vector bundles on X, and f is a *submersion* if $\mathrm{d}f^*$ is injective, and f is an *immersion* if $\mathrm{d}f^*$ is surjective.

Here the appropriate notions of injective and surjective for morphisms of vector bundles are stronger than the corresponding notions for sheaves: $\mathrm{d}f^*$ is *injective* if it has a left inverse, and *surjective* if it has a right inverse. For example, let $E \to \mathbb{R}$ be the trivial line bundle $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and consider the vector bundle morphism $x : E \to E$ multiplying by the coordinate x on $\mathbb{R}$. Then x is injective as a morphism of quasicoherent sheaves on $\underline{\mathbb{R}}$, but we do not consider it injective as a morphism of vector bundles, as $x|_0 : E|_0 \to E|_0$ is not injective, so x does not have a left inverse.

Now let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism of d-manifolds. Then as in §3.1–§3.2, $T^*\boldsymbol{X}, \underline{f}^*(T^*\boldsymbol{Y})$ are virtual vector bundles on $\underline{X}$, and $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ is a 1-morphism in $\operatorname{vvect}(\underline{X})$ which is the analogue of $\mathrm{d}f^* : f^*(T^*Y) \to T^*X$ for manifolds. So to define submersions and immersions of d-manifolds, we need to find suitable definitions of when a morphism $\Omega_{\boldsymbol{f}}$ in the 2-category $\operatorname{vvect}(\underline{X})$ is injective or surjective, which should be analogues of having a left or right inverse for the category of ordinary vector bundles.

It turns out that there are two different sensible definitions for each of injective and surjective 1-morphisms in $\operatorname{vvect}(\underline{X})$, a weak and a strong, and these will yield weak and strong notions of submersions and immersions in **dMan**.

Definition 4.1. Let $\underline{X}$ be a C^∞-scheme, $(\mathcal{E}^1,\mathcal{E}^2,\phi)$ and $(\mathcal{F}^1,\mathcal{F}^2,\psi)$ be virtual vector bundles on $\underline{X}$, and $(f^1,f^2):(\mathcal{E}^\bullet,\phi)\to(\mathcal{F}^\bullet,\psi)$ be a 1-morphism in $\mathrm{vvect}(\underline{X})$, in the sense of §3.1. Then Proposition 3.5 says that $f^\bullet$ is an equivalence if and only if the complex (3.4) in $\mathrm{qcoh}(\underline{X})$

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^1 \arrow[r, shift left, "f^1\oplus-\phi"] & \mathcal{F}^1\oplus\mathcal{E}^2 \arrow[l, dotted, shift left, "\gamma"] \arrow[r, shift left, "\psi\oplus f^2"] & \mathcal{F}^2 \arrow[l, dotted, shift left, "\delta"] \arrow[r] & 0
\end{tikzcd}
\tag{4.1}
$$

is split exact. So by Definition 2.19, $f^\bullet$ is an equivalence if and only if there exist morphisms γ,δ as in (4.1) satisfying (2.33), which is

$$
\begin{aligned}
\gamma\circ\delta&=0, & \gamma\circ(f^1\oplus-\phi)&=\mathrm{id}_{\mathcal{E}^1},\\
(f^1\oplus-\phi)\circ\gamma+\delta\circ(\psi\oplus f^2)&=\mathrm{id}_{\mathcal{F}^1\oplus\mathcal{E}^2}, & (\psi\oplus f^2)\circ\delta&=\mathrm{id}_{\mathcal{F}^2}.
\end{aligned}
\tag{4.2}
$$

Our notions of $f^\bullet$ injective or surjective impose some but not all of (4.2):

(a) We call $f^\bullet$ *weakly injective* if there exists $\gamma:\mathcal{F}^1\oplus\mathcal{E}^2\to\mathcal{E}^1$ in $\mathrm{qcoh}(\underline{X})$ with $\gamma\circ(f^1\oplus-\phi)=\mathrm{id}_{\mathcal{E}^1}$.

(b) We call $f^\bullet$ *injective* if there exist $\gamma:\mathcal{F}^1\oplus\mathcal{E}^2\to\mathcal{E}^1$ and $\delta:\mathcal{F}^2\to\mathcal{F}^1\oplus\mathcal{E}^2$ with $\gamma\circ\delta=0$, $\gamma\circ(f^1\oplus-\phi)=\mathrm{id}_{\mathcal{E}^1}$ and $(f^1\oplus-\phi)\circ\gamma+\delta\circ(\psi\oplus f^2)=\mathrm{id}_{\mathcal{F}^1\oplus\mathcal{E}^2}$.

(c) We call $f^\bullet$ *weakly surjective* if there exists $\delta:\mathcal{F}^2\to\mathcal{F}^1\oplus\mathcal{E}^2$ in $\mathrm{qcoh}(\underline{X})$ with $(\psi\oplus f^2)\circ\delta=\mathrm{id}_{\mathcal{F}^2}$.

(d) We call $f^\bullet$ *surjective* if there exist $\gamma:\mathcal{F}^1\oplus\mathcal{E}^2\to\mathcal{E}^1$ and $\delta:\mathcal{F}^2\to\mathcal{F}^1\oplus\mathcal{E}^2$ with $\gamma\circ\delta=0$, $\gamma\circ(f^1\oplus-\phi)=\mathrm{id}_{\mathcal{E}^1}$ and $(\psi\oplus f^2)\circ\delta=\mathrm{id}_{\mathcal{F}^2}$.

Here are some properties of these definitions.

Proposition 4.2. **(i)** *If $f^\bullet$ is an equivalence in* $\mathrm{vvect}(\underline{X})$ *then $f^\bullet$ is also weakly injective, injective, weakly surjective, and surjective.*

(ii) *If $f^\bullet,g^\bullet$ are* 2*-isomorphic* 1*-morphisms in* $\mathrm{vvect}(\underline{X})$ *then $f^\bullet$ is weakly injective, injective, weakly surjective, or surjective, if and only if $g^\bullet$ is.*

(iii) *Compositions of weakly injective, injective, weakly surjective, or surjective* 1*-morphisms in* $\mathrm{vvect}(\underline{X})$ *are* 1*-morphisms of the same kind.*

(iv) *Suppose $\underline{X}$ is separated, paracompact, and locally fair. Then the conditions that $f^\bullet$ be weakly injective, injective, weakly surjective, and surjective, are local in $\underline{X}$. That is, it suffices to check them on the sets of an open cover of $\underline{X}$.*

Proof. Part (i) is immediate from Definition 4.1. For (ii), let $\eta:f^\bullet\Rightarrow g^\bullet$ be a 2-morphism. Suppose that one of Definition 4.1(a)–(d) hold for $f^\bullet$, with morphisms $\gamma=\gamma^1\oplus\gamma^2$ and/or $\delta=\delta^1\oplus\delta^2$. One can then check that

$$\tilde\gamma=\gamma^1\oplus(\gamma^2+\gamma^1\circ\eta)\quad\text{and}\quad\tilde\delta=(\delta^1-\eta\circ\delta^2)\oplus\delta^2$$

satisfy the corresponding one of Definition 4.1(a)–(d) for $g^\bullet$. Thus, if $f^\bullet,g^\bullet$ are 2-isomorphic then $f^\bullet$ weakly injective, …, surjective implies $f^\bullet$ weakly injective, …, surjective, and vice versa.

For (iii), let $f^\bullet:(\mathcal{E}^\bullet,\phi)\to(\mathcal{F}^\bullet,\psi)$, $g^\bullet:(\mathcal{F}^\bullet,\psi)\to(\mathcal{G}^\bullet,\xi)$ be 1-morphisms in $\mathrm{vvect}(\underline{X})$. Suppose one of Definition 4.1(a)–(d) hold for $f^\bullet,g^\bullet$, with morphisms $\gamma=\gamma^1\oplus\gamma^2$ and/or $\delta=\delta^1\oplus\delta^2$ for $f^\bullet$ and $\tilde\gamma=\tilde\gamma^1\oplus\tilde\gamma^2$ and/or $\tilde\delta=\tilde\delta^1\oplus\tilde\delta^2$ for $g^\bullet$. By a long but elementary calculation, one can then check that

$$\hat\gamma=(\tilde\gamma^1\circ\gamma^1)\oplus(\tilde\gamma^2+\tilde\gamma^1\circ\gamma^2\circ g^2),\quad \hat\delta=(\delta^1+f^1\circ\tilde\delta^1\circ\delta^2)\oplus(\tilde\delta^2\circ\delta^2)$$

satisfy the corresponding one of Definition 4.1(a)–(d) for $g^\bullet\circ f^\bullet$. Part (iv) follows from the proofs of Lemmas 2.22 and 2.26. $\square$

We study the cohomology of the complex (4.1).

Proposition 4.3. *Suppose $\underline{X}$ is a separated, paracompact, locally fair C^∞-scheme and $f^\bullet:(\mathcal{E}^\bullet,\phi)\to(\mathcal{F}^\bullet,\psi)$ is a 1-morphism in $\mathrm{vvect}(\underline{X})$, so that (4.1) is a complex in $\mathrm{qcoh}(\underline{X})$. Define $\mathcal{G},\mathcal{H},\mathcal{I}\in\mathrm{qcoh}(\underline{X})$ by*

$$\mathcal{G}=\mathrm{Ker}\bigl(f^1\oplus-\phi:\mathcal{E}^1\longrightarrow\mathcal{F}^1\oplus\mathcal{E}^2\bigr),\tag{4.3}$$

$$\mathcal{H}=\frac{\mathrm{Ker}\bigl(\psi\oplus f^2:\mathcal{F}^1\oplus\mathcal{E}^2\longrightarrow\mathcal{F}^2\bigr)}{\mathrm{Im}\bigl(f^1\oplus-\phi:\mathcal{E}^1\longrightarrow\mathcal{F}^1\oplus\mathcal{E}^2\bigr)},\tag{4.4}$$

$$\mathcal{I}=\mathrm{Coker}\bigl(\psi\oplus f^2:\mathcal{F}^1\oplus\mathcal{E}^2\longrightarrow\mathcal{F}^2\bigr),\tag{4.5}$$

the cohomology of (4.1) at the second, third and fourth terms. Then:

- **(i)** *Let $f^\bullet,\tilde f^\bullet:(\mathcal{E}^\bullet,\phi)\to(\mathcal{F}^\bullet,\psi)$ be 1-morphisms, $\eta:f^\bullet\Rightarrow\tilde f^\bullet$ a 2-morphism, and $\mathcal{G},\mathcal{H},\mathcal{I}$ and $\tilde{\mathcal{G}},\tilde{\mathcal{H}},\tilde{\mathcal{I}}$ be as in (4.3)–(4.5) for $f^\bullet$ and $\tilde f^\bullet$. Then there are canonical isomorphisms $\tilde{\mathcal{G}}\cong\mathcal{G}$, $\tilde{\mathcal{H}}\cong\mathcal{H}$ and $\tilde{\mathcal{I}}\cong\mathcal{I}$ in $\mathrm{qcoh}(\underline{X})$.*
- **(ii)** *Let $i^\bullet:(\tilde{\mathcal{E}}^\bullet,\tilde\phi)\to(\mathcal{E}^\bullet,\phi)$, $j^\bullet:(\mathcal{F}^\bullet,\psi)\to(\tilde{\mathcal{F}}^\bullet,\tilde\psi)$ be equivalences in $\mathrm{vvect}(\underline{X})$, and set $\tilde f^\bullet=j^\bullet\circ f^\bullet\circ i^\bullet:(\tilde{\mathcal{E}}^\bullet,\tilde\phi)\to(\tilde{\mathcal{F}}^\bullet,\tilde\psi)$. Let $\mathcal{G},\mathcal{H},\mathcal{I}$ and $\tilde{\mathcal{G}},\tilde{\mathcal{H}},\tilde{\mathcal{I}}$ be as in (4.3)–(4.5) for $f^\bullet$ and $\tilde f^\bullet$. Then there are canonical isomorphisms $\tilde{\mathcal{G}}\cong\mathcal{G}$, $\tilde{\mathcal{H}}\cong\mathcal{H}$ and $\tilde{\mathcal{I}}\cong\mathcal{I}$ in $\mathrm{qcoh}(\underline{X})$.*
- **(iii)** *Suppose $f^\bullet$ is weakly injective. Then $\mathcal{G}=0$.*
- **(iv)** *Suppose $f^\bullet$ is injective. Then $\mathrm{rank}(\mathcal{E}^\bullet,\phi)\leqslant\mathrm{rank}(\mathcal{F}^\bullet,\psi)$, and $\mathcal{G}=\mathcal{H}=0$, and $\mathcal{I}$ is a vector bundle on $\underline{X}$ of rank $\mathrm{rank}(\mathcal{F}^\bullet,\psi)-\mathrm{rank}(\mathcal{E}^\bullet,\phi)$. If also $\mathrm{rank}(\mathcal{E}^\bullet,\phi)=\mathrm{rank}(\mathcal{F}^\bullet,\psi)$ then $\mathcal{I}=0$ and $f^\bullet$ is an equivalence.*
- **(v)** *Suppose $f^\bullet$ is weakly surjective. Then $\mathcal{I}=0$.*
- **(vi)** *Suppose $f^\bullet$ is surjective. Then $\mathrm{rank}(\mathcal{E}^\bullet,\phi)\geqslant\mathrm{rank}(\mathcal{F}^\bullet,\psi)$, and $\mathcal{G}=\mathcal{I}=0$, and $\mathcal{H}$ is a vector bundle on $\underline{X}$ of rank $\mathrm{rank}(\mathcal{E}^\bullet,\phi)-\mathrm{rank}(\mathcal{F}^\bullet,\psi)$. If also $\mathrm{rank}(\mathcal{E}^\bullet,\phi)=\mathrm{rank}(\mathcal{F}^\bullet,\psi)$ then $\mathcal{H}=0$ and $f^\bullet$ is an equivalence.*

Proof. For (i), consider the diagram in $\mathrm{qcoh}(\underline{X})$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^1 \arrow[r, "f^1\oplus-\phi"] \arrow[d, "\mathrm{id}_{\mathcal{E}^1}"'] & \mathcal{F}^1\oplus\mathcal{E}^2 \arrow[r, "\psi\oplus f^2"] \arrow[d, "\begin{pmatrix}\mathrm{id}_{\mathcal{F}^1} & -\eta \\ 0 & \mathrm{id}_{\mathcal{E}^2}\end{pmatrix}"'] & \mathcal{F}^2 \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{F}^2}"] & 0 \\
0 \arrow[r] & \mathcal{E}^1 \arrow[r, "\tilde f^1\oplus-\phi"] & \mathcal{F}^1\oplus\mathcal{E}^2 \arrow[r, "\psi\oplus\tilde f^2"] & \mathcal{F}^2 \arrow[r] & 0.
\end{tikzcd}$$

The rows are complexes, the columns are isomorphisms, and as $\tilde f^1 = f^1 + \eta\circ\phi$ and $\tilde f^2 = f^2 + \psi\circ\eta$ the diagram commutes. Thus the columns induce isomorphisms on cohomology, which is $\mathcal{G},\mathcal{H},\mathcal{I}$ on the top row and $\tilde{\mathcal{G}},\tilde{\mathcal{H}},\tilde{\mathcal{I}}$ on the bottom. Hence $\tilde{\mathcal{G}}\cong\mathcal{G}$, $\tilde{\mathcal{H}}\cong\mathcal{H}$ and $\tilde{\mathcal{I}}\cong\mathcal{I}$.

For (ii), as $i^\bullet:(\mathcal{E}^\bullet,\phi)\to(\tilde{\mathcal{E}}^\bullet,\tilde\phi)$ is an equivalence there exists a 1-morphism $k^\bullet:(\tilde{\mathcal{E}}^\bullet,\tilde\phi)\to(\mathcal{E}^\bullet,\phi)$ and 2-morphisms $\eta: k^\bullet\circ i^\bullet\Rightarrow \mathrm{id}_{(\mathcal{E}^\bullet,\phi)}$, $\zeta: i^\bullet\circ k^\bullet\Rightarrow \mathrm{id}_{(\tilde{\mathcal{E}}^\bullet,\tilde\phi)}$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^1 \arrow[r, "f^1\oplus-\phi"] \arrow[d, "k^1"'] & \mathcal{F}^1\oplus\mathcal{E}^2 \arrow[r, "\psi\oplus f^2"] \arrow[d, "\begin{pmatrix} j^1 & j^1\circ f^1\circ\eta \\ 0 & k^2\end{pmatrix}"'] & \mathcal{F}^2 \arrow[r] \arrow[d, "j^2"'] & 0 \\
0 \arrow[r] & \tilde{\mathcal{E}}^1 \arrow[r, "\tilde f^1\oplus-\tilde\phi"] & \tilde{\mathcal{F}}^1\oplus\tilde{\mathcal{E}}^2 \arrow[r, "\tilde\psi\oplus\tilde f^2"] & \tilde{\mathcal{F}}^2 \arrow[r] & 0.
\end{tikzcd}$$

By a similar but more complicated argument to (i), one can show this commutes and induces isomorphisms $\tilde{\mathcal{G}}\cong\mathcal{G}$, $\tilde{\mathcal{H}}\cong\mathcal{H}$, $\tilde{\mathcal{I}}\cong\mathcal{I}$.

For (iii),(iv),(vi) we have $\gamma\circ(f^1\oplus-\phi)=\mathrm{id}_{\mathcal{E}^1}$, so $f^1\oplus-\phi$ has a left inverse γ and is injective, and $\mathcal{G}=0$. For (v),(vi) we have $(\psi\oplus f^2)\circ\delta=\mathrm{id}_{\mathcal{F}^2}$, so $\psi\oplus f^2$ has a right inverse δ and is surjective, and $\mathcal{I}=0$. This proves (iii),(v).

To show $\mathcal{I}$ is a vector bundle in (iv), let $x\in\underline{X}$. Then as $(\mathcal{E}^\bullet,\phi),(\mathcal{F}^\bullet,\psi)$ are virtual vector bundles, there exists an open neighbourhood $\underline{U}$ of x in $\underline{X}$, objects $(\tilde{\mathcal{E}}^\bullet,\tilde\phi),(\tilde{\mathcal{F}}^\bullet,\tilde\psi)$ in $\mathrm{vvect}(\underline{U})$ with $\tilde{\mathcal{E}}^1,\tilde{\mathcal{E}}^2,\tilde{\mathcal{F}}^1,\tilde{\mathcal{F}}^2$ vector bundles on $\underline{U}$, and equivalences $i^\bullet:(\tilde{\mathcal{E}}^\bullet,\tilde\phi)\to(\mathcal{E}^\bullet,\phi)|_{\underline{U}}$, $j^\bullet:(\mathcal{F}^\bullet,\psi)|_{\underline{U}}\to(\tilde{\mathcal{F}}^\bullet,\tilde\psi)$ in $\mathrm{vvect}(\underline{U})$. Define $\tilde f^\bullet=j^\bullet\circ f^\bullet|_{\underline{U}}\circ i^\bullet:(\tilde{\mathcal{E}}^\bullet,\tilde\phi)\to(\tilde{\mathcal{F}}^\bullet,\tilde\psi)$, and let $\tilde{\mathcal{I}}\in\mathrm{qcoh}(\underline{U})$ be as in (4.5) for $\tilde f^\bullet$. Then (ii) shows that $\tilde{\mathcal{I}}\cong\mathcal{I}|_{\underline{U}}$. Also Proposition 4.2(i),(iii) shows that $\tilde f^\bullet$ is injective, as $f^\bullet$ is and $i^\bullet,j^\bullet$ are equivalences.

Thus we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\mathcal{E}}^1 \arrow[r, shift left, "\tilde f^1\oplus-\tilde\phi"] & \tilde{\mathcal{F}}^1\oplus\tilde{\mathcal{E}}^2 \arrow[l, shift left, "\tilde\gamma"] \arrow[r, shift left, "\tilde\psi\oplus\tilde f^2"] & \tilde{\mathcal{F}}^2 \arrow[l, shift left, "\tilde\delta"] \arrow[r] & 0,
\end{tikzcd} \tag{4.6}$$

with $\tilde\gamma,\tilde\delta$ satisfying three equations. These equations imply that setting $\tilde{\mathcal{J}}=\mathrm{Coker}(\tilde\psi\oplus\tilde f^2)\cong\mathrm{Ker}\,\tilde\delta$, there are isomorphisms $\tilde{\mathcal{F}}^1\oplus\tilde{\mathcal{E}}^2\cong\tilde{\mathcal{E}}^1\oplus\tilde{\mathcal{J}}$, $\tilde{\mathcal{F}}^2\cong\tilde{\mathcal{I}}\oplus\tilde{\mathcal{J}}$ which identify (4.6) with the diagram

$$\begin{tikzcd}[column sep=huge]
0 \arrow[r] & \tilde{\mathcal{E}}^1 \arrow[r, shift left, "\begin{pmatrix}\mathrm{id}_{\tilde{\mathcal{E}}^1}\\ 0\end{pmatrix}"] & \tilde{\mathcal{E}}^1\oplus\tilde{\mathcal{J}} \arrow[l, shift left, "\begin{pmatrix}\mathrm{id}_{\tilde{\mathcal{E}}^1} & 0\end{pmatrix}"] \arrow[r, shift left, "\begin{pmatrix}0 & 0\\ 0 & \mathrm{id}_{\tilde{\mathcal{J}}}\end{pmatrix}"] & \tilde{\mathcal{I}}\oplus\tilde{\mathcal{J}} \arrow[l, shift left, "\begin{pmatrix}0 & 0\\ 0 & \mathrm{id}_{\tilde{\mathcal{J}}}\end{pmatrix}"] \arrow[r] & 0.
\end{tikzcd} \tag{4.7}$$

Since $\tilde{\mathcal{E}}^i,\tilde{\mathcal{F}}^i$ are vector bundles, so are $\tilde{\mathcal{H}},\tilde{\mathcal{I}}$, and we have

$$\begin{aligned}
0\leqslant \operatorname{rank}\tilde{\mathcal{I}} &= \operatorname{rank}\tilde{\mathcal{F}}^2-\operatorname{rank}(\tilde{\mathcal{F}}^1\oplus\tilde{\mathcal{E}}^2)+\operatorname{rank}(\tilde{\mathcal{E}}^1)\\
&=(\operatorname{rank}\tilde{\mathcal{F}}^2-\operatorname{rank}\tilde{\mathcal{F}}^1)-(\operatorname{rank}\tilde{\mathcal{E}}^2-\operatorname{rank}\tilde{\mathcal{E}}^1)=\operatorname{rank}(\mathcal{F}^\bullet,\psi)-\operatorname{rank}(\mathcal{E}^\bullet,\phi).
\end{aligned}$$

Hence $\mathrm{rank}(\mathcal{F}^\bullet,\psi)-\mathrm{rank}(\mathcal{E}^\bullet,\phi)\geqslant 0$, and $\mathcal{I}|_{\underline{U}}\cong\tilde{\mathcal{I}}$ is a vector bundle on $\underline{U}$ of the prescribed rank. As $\underline{X}$ can be covered by such open $\underline{U}$, $\mathcal{I}$ is a vector bundle on $\underline{X}$ of the prescribed rank.

If $\mathrm{rank}(\mathcal{E}^\bullet,\phi)=\mathrm{rank}(\mathcal{F}^\bullet,\psi)$ then $\mathrm{rank}\,\mathcal{I}=0$, so $\mathcal{I}=0$, and it then follows from the equivalence of (4.6)–(4.7) that $(\tilde\psi\oplus\tilde f^2)\circ\tilde\delta=\mathrm{id}_{\tilde{\mathcal{F}}^2}$, so $\tilde f^\bullet$ is an equivalence in $\mathrm{vvect}(\underline{U})$. Therefore $f^\bullet|_{\underline{U}}$ is an equivalence. As $\underline{X}$ is separated, paracompact and locally fair and can be covered by such $\underline{U}$, Lemma 2.22 shows that $f^\bullet$ is an equivalence. This proves (iv).

For (vi), we use a very similar argument, but replacing (4.7) with the diagram

$$\begin{tikzcd} 0 \arrow[r] & \tilde{\mathcal{E}}^1 \arrow[r, shift left, "\mathrm{id}_{\tilde{\mathcal{E}}^1}\oplus 0\oplus 0"] & \tilde{\mathcal{E}}^1\oplus\tilde{\mathcal{F}}^2\oplus\tilde{\mathcal{H}} \arrow[l, shift left, "\mathrm{id}_{\tilde{\mathcal{E}}^1}\oplus 0\oplus 0"] \arrow[r, shift left, "0\oplus\mathrm{id}_{\tilde{\mathcal{F}}^2}\oplus 0"] & \tilde{\mathcal{F}}^2 \arrow[l, shift left, "0\oplus\mathrm{id}_{\tilde{\mathcal{F}}^2}\oplus 0"] \arrow[r] & 0. \end{tikzcd}$$

□

Now we can define weak and strong forms of submersions, immersions, and embeddings for d-manifolds.

Definition 4.4. Let $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ be a 1-morphism of d-manifolds. Example 3.2 defines $T^*\boldsymbol{X}=(\mathcal{E}_X,\mathcal{F}_X,\phi_X)$ and $\underline{f}^*(T^*\boldsymbol{Y})=\bigl(\underline{f}^*(\mathcal{E}_Y),\underline{f}^*(\mathcal{F}_Y),\underline{f}^*(\phi_Y)\bigr)$, which are virtual vector bundles on $\underline{X}$ of ranks $\mathrm{vdim}\,\boldsymbol{X},\mathrm{vdim}\,\boldsymbol{Y}$ as in Definition 3.18, and a 1-morphism $\Omega_{\boldsymbol{f}}=(f'',f^2):\underline{f}^*(T^*\boldsymbol{Y})\to T^*\boldsymbol{X}$ in $\mathrm{vvect}(\underline{X})$. Then:

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\Omega_{\boldsymbol{f}}$ is weakly injective.

(b) We call $\boldsymbol{f}$ a *submersion* if $\Omega_{\boldsymbol{f}}$ is injective.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $\Omega_{\boldsymbol{f}}$ is weakly surjective.

(d) We call $\boldsymbol{f}$ an *immersion* if $\Omega_{\boldsymbol{f}}$ is surjective.

(e) We call $\boldsymbol{f}$ a *w-embedding* if it is a w-immersion and $f:X\to f(X)$ is a homeomorphism, so in particular f is injective.

(f) We call $\boldsymbol{f}$ an *embedding* if it is an immersion and f is a homeomorphism with its image.

Here w-submersion is short for *weak submersion*, etc. These conditions all concern the existence of morphisms γ,δ in the next equation satisfying identities.

$$\begin{tikzcd} 0 \arrow[r] & \underline{f}^*(\mathcal{E}_Y) \arrow[r, shift left, "f''\oplus -\underline{f}^*(\phi_Y)"] & \mathcal{E}_X\oplus\underline{f}^*(\mathcal{F}_Y) \arrow[l, shift left, dotted, "\gamma"] \arrow[r, shift left, "\phi_X\oplus f^2"] & \mathcal{F}_X \arrow[l, shift left, dotted, "\delta"] \arrow[r] & 0. \end{tikzcd} \tag{4.8}$$

Parts (c)–(f) enable us to define *d-submanifolds* of d-manifolds. In classical differential geometry, if X,Y are manifolds and $i:X\to Y$ is an immersion or an embedding, we consider X to be an *immersed* or *embedded submanifold* of Y. In the embedded case we can think of X as a subset $i(X)\subset Y$, but for immersed submanifolds we need to remember X and $i:X\to Y$.

Open d-submanifolds are open d-subspaces of a d-manifold. For more general d-submanifolds, we call $\boldsymbol{i}:\boldsymbol{X}\to\boldsymbol{Y}$ a *w-immersed d-submanifold*, or *immersed* *d-submanifold*, or *w-embedded d-submanifold*, or *embedded d-submanifold*, of $\boldsymbol{Y}$, if $\boldsymbol{X}, \boldsymbol{Y}$ are d-manifolds and $\boldsymbol{i}$ is a w-immersion, immersion, w-embedding, or embedding, respectively. We discuss these in Remark 4.15.

Propositions 4.2 and 4.3 now imply properties of these:

Proposition 4.5. **(i)** *Any equivalence of d-manifolds is a w-submersion, submersion, w-immersion, immersion, w-embedding and embedding.*

(ii) *If* $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ *are* 2*-isomorphic* 1*-morphisms of d-manifolds then* $\boldsymbol{f}$ *is a w-submersion, submersion, ..., embedding, if and only if* $\boldsymbol{g}$ *is.*

(iii) *Compositions of w-submersions, submersions, w-immersions, immersions, w-embeddings, and embeddings are* 1*-morphisms of the same kind.*

(iv) *The conditions that a* 1*-morphism of d-manifolds* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is a w-submersion, submersion, w-immersion or immersion are local in* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*. That is, for each* $x \in \boldsymbol{X}$ *with* $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$*, it suffices to check the conditions for* $\boldsymbol{f}|_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ *with* $\boldsymbol{V}$ *an open neighbourhood of* y *in* $\boldsymbol{Y}$*, and* $\boldsymbol{U}$ *an open neighbourhood of* x *in* $\boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$*.*

For $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ to be a w-embedding or embedding is local in $\boldsymbol{Y}$, but not in $\boldsymbol{X}$, as for f to be a homeomorphism with its image is local in Y but not in X. Our next result follows from Proposition 4.3(iv),(vi) and Theorem 3.36.

Proposition 4.6. **(a)** *Let* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *be a submersion of d-manifolds. Then* $\operatorname{vdim} \boldsymbol{X} \geqslant \operatorname{vdim} \boldsymbol{Y}$*, and if* $\operatorname{vdim} \boldsymbol{X} = \operatorname{vdim} \boldsymbol{Y}$ *then* $\boldsymbol{f}$ *is étale.*

(b) *Let* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *be an immersion of d-manifolds. Then* $\operatorname{vdim} \boldsymbol{X} \leqslant \operatorname{vdim} \boldsymbol{Y}$*, and if* $\operatorname{vdim} \boldsymbol{X} = \operatorname{vdim} \boldsymbol{Y}$ *then* $\boldsymbol{f}$ *is étale.*

Our next result shows that when $\boldsymbol{X}, \boldsymbol{Y}$ are manifolds, submersions, immersions, and embeddings are equivalent to their usual definition in **Man**. Also w-immersions are immersions, and w-submersions are arbitrary smooth maps.

Proposition 4.7. **(a)** *Let* $f : X \to Y$ *be a smooth map of manifolds, and* $\boldsymbol{f} = F_{\mathbf{Man}}^{\mathbf{dMan}}(f)$*. Then* $\boldsymbol{f}$ *is a submersion, immersion, or embedding in* **dMan** *if and only if* f *is a submersion, immersion, or embedding in* **Man***, respectively. Also* $\boldsymbol{f}$ *is a w-immersion or w-embedding if and only if* f *is an immersion or embedding.*

(b) *Let* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *be a* 1*-morphism of d-manifolds, with* $\boldsymbol{Y}$ *a manifold. Then* $\boldsymbol{f}$ *is a w-submersion.*

Proof. For (a), by definition of $F_{\mathbf{Man}}^{\mathbf{dMan}}$ we have $\mathcal{E}_X = \mathcal{E}_Y = 0$, $\mathcal{F}_X = T^*\underline{X}$, $\mathcal{F}_Y = T^*\underline{Y}$ and $f^2 = \Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$. Thus in (4.8) $\gamma = 0$, and δ is a morphism $T^*\underline{X} \to \underline{f}^*(T^*\underline{Y})$, and the conditions reduce to:

- f is a submersion if there exists δ with $\delta \circ \Omega_{\underline{f}} = \mathrm{id}_{\underline{f}^*(T^*\underline{Y})}$; and
- f is a w-immersion or an immersion if there exists δ with $\Omega_{\underline{f}} \circ \delta = \mathrm{id}_{T^*\underline{X}}$.

Now $\Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$ is the lift to C^∞-schemes of $\mathrm{d}f^* : f^*(T^*Y) \to T^*X$. By definition, f is a submersion if $\mathrm{d}f^*$ is injective as a morphism of vector bundles (has a left inverse), and f is an immersion if $\mathrm{d}f^*$ is surjective (has a right inverse). Taking δ to be the lift to C^∞-schemes of this left/right inverse, part (a) follows. For (b), as $\boldsymbol{Y}$ is a manifold $\phi_Y : \mathcal{E}_Y \to \mathcal{F}_Y$ has a left inverse β by Proposition 3.28. Then $\gamma = 0 \oplus -\underline{f}^*(\beta)$ satisfies Definition 4.1(a), so $\Omega_{\boldsymbol{f}}$ is weakly injective, and $\boldsymbol{f}$ is a w-submersion. $\square$

4.2 Local picture of (w-)submersions and (w-)immersions

As in Definitions 4.1 and 4.4, the conditions that $\boldsymbol{f}$ is w-submersion, submersion, w-immersion or immersion are weakenings of the condition in Theorem 3.36 that $\boldsymbol{f}$ be étale. Now Theorem 3.39 gave a differential-geometric criterion for when a 'standard model' 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ is étale. By essentially the same proof, we obtain criteria for when $\boldsymbol{S}_{f,\hat{f}}$ is a w-submersion, submersion, w-immersion or immersion:

Theorem 4.8. *Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, $s : V \to E$, $t : W \to F$ be smooth sections, $f : V \to W$ be smooth, and $\hat{f} : E \to f^*(F)$ be a morphism of vector bundles on V with $\hat{f} \circ s = f^*(t) + O(s^2)$. Then Definitions* 3.13 *and* 3.30 *define principal d-manifolds $\boldsymbol{S}_{V,E,s}, \boldsymbol{S}_{W,F,t}$ and a* 1*-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. As in* (3.31)*, we have a complex*

$$0 \longrightarrow T_vV \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} E_v \oplus T_wW \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)} F_w \longrightarrow 0 \tag{4.9}$$

for each $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$. Then:

- **(a)** *$\boldsymbol{S}_{f,\hat{f}}$ is a w-submersion if and only if for all $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, equation* (4.9) *is exact at the fourth term.*
- **(b)** *$\boldsymbol{S}_{f,\hat{f}}$ is a submersion if and only if for all $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, equation* (4.9) *is exact at the third and fourth terms.*
- **(c)** *$\boldsymbol{S}_{f,\hat{f}}$ is a w-immersion if and only if for all $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, equation* (4.9) *is exact at the second term.*
- **(d)** *$\boldsymbol{S}_{f,\hat{f}}$ is an immersion if and only if for all $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, equation* (4.9) *is exact at the second and fourth terms.*

The conditions in **(a)**–**(d)** *are open conditions on v in $\{v \in V : s(v) = 0\}$.*

In the next theorem we write (w-)submersions and (w-)immersions locally in a canonical form. Parts (a)–(d) show how (w-)submersions and (w-)immersions in **dMan** are related to submersions and immersions in **Man**.

Theorem 4.9. *Suppose $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ is a* 1*-morphism of d-manifolds, and $x \in \boldsymbol{X}$ with $\boldsymbol{g}(x) = y \in \boldsymbol{Y}$. Then there exist open d-submanifolds $\boldsymbol{T} \subseteq \boldsymbol{X}$ and $\boldsymbol{U} \subseteq \boldsymbol{Y}$ with $x \in \boldsymbol{T}$, $y \in \boldsymbol{U}$ and $\boldsymbol{g}(\boldsymbol{T}) \subseteq \boldsymbol{U}$, manifolds V, W, vector bundles $E \to V$, $F \to W$, smooth sections $s : V \to E$, $t : W \to F$, a smooth map $f : V \to W$, a morphism of vector bundles $\hat{f} : E \to f^*(F)$ with $\hat{f} \circ s = f^*(t)$, equivalences $\boldsymbol{i} : \boldsymbol{T} \to \boldsymbol{S}_{V,E,s}$, $\boldsymbol{j} : \boldsymbol{S}_{W,F,t} \to \boldsymbol{U}$, and a* 2*-morphism $\eta : \boldsymbol{j} \circ \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{i} \Rightarrow \boldsymbol{g}|_{\boldsymbol{T}}$, where $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ is as in Definitions* 3.13 *and* 3.30*. Furthermore:*

(a) *If $\boldsymbol{g}$ is a **w-submersion** then we can choose the data $\boldsymbol{T}, \boldsymbol{U}, \ldots, \boldsymbol{j}$ above such that $f : V \to W$ is a submersion in* **Man**, *and $\hat{f} : E \to f^*(F)$ is a surjective morphism of vector bundles.*

(b) *If $\boldsymbol{g}$ is a **submersion** we can choose $\boldsymbol{T}, \ldots, \boldsymbol{j}$ such that $f : V \to W$ is a submersion and $\hat{f} : E \to f^*(F)$ is an isomorphism.*

(c) *If $\boldsymbol{g}$ is a **w-immersion** we can choose $\boldsymbol{T}, \ldots, \boldsymbol{j}$ such that $f : V \to W$ is an immersion in* **Man**, *and $\hat{f} : E \to f^*(F)$ is an injective morphism.*

(d) *If $\boldsymbol{g}$ is an **immersion** we can choose $\boldsymbol{T}, \ldots, \boldsymbol{j}$ such that $f : V \to W$ is an immersion and $\hat{f} : E \to f^*(F)$ is an isomorphism.*

Here are alternative forms for **(a)**–**(d)**:

(a′) *If $\boldsymbol{g}$ is a **w-submersion** we can choose $\boldsymbol{T}, \ldots, \boldsymbol{j}$ such that $V = W \times Z$ for some manifold Z, and $f = \pi_W$, $E = \pi_W^*(F) \oplus G$ for some vector bundle $G \to V$, $\hat{f} = \mathrm{id}_{\pi_W^*(F)} \oplus 0$, and $s = \pi_W^*(t) \oplus u$ for some $u \in C^\infty(G)$.*

(b′) *If $\boldsymbol{g}$ is a **submersion** we can choose $\boldsymbol{T}, \ldots, \boldsymbol{j}$ such that $V = W \times Z$ for some manifold Z, and $f = \pi_W$, $E = \pi_W^*(F)$, $\hat{f} = \mathrm{id}_{\pi_W^*(F)}$, $s = \pi_W^*(t)$.*

(c′) *If $\boldsymbol{g}$ is a **w-immersion** we can choose $\boldsymbol{T}, \ldots, \boldsymbol{j}$ such that $W = V \times Z$ for open $0 \in Z \subseteq \mathbb{R}^n$, and f maps $v \mapsto (v, 0)$, and $f^*(F) = E \oplus G$ for some $G \to V$, and $\hat{f} = \mathrm{id}_E \oplus 0$, $f^*(t) = s \oplus 0$.*

(d′) *If $\boldsymbol{g}$ is an **immersion** we can choose $\boldsymbol{T}, \ldots, \boldsymbol{j}$ such that $W = V \times Z$ for open $0 \in Z \subseteq \mathbb{R}^n$, and $f : v \mapsto (v, 0)$, $f^*(F) = E$, $\hat{f} = \mathrm{id}_E$, $f^*(t) = s$.*

Proof. By Proposition 3.25 and Example 3.24, there exist open neighbourhoods $\boldsymbol{T} \subseteq \boldsymbol{X}$, $\boldsymbol{U} \subseteq \boldsymbol{Y}$ of x, y and quasi-inverse equivalences $\boldsymbol{i} : \boldsymbol{T} \to \boldsymbol{S}_{V,E,s}$, $\boldsymbol{k} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{T}$ and $\boldsymbol{j} : \boldsymbol{S}_{W,F,t} \to \boldsymbol{U}$, $\boldsymbol{l} : \boldsymbol{U} \to \boldsymbol{S}_{W,F,t}$ with 2-morphisms $\zeta_U : \boldsymbol{k} \circ \boldsymbol{i} \Rightarrow \mathbf{id}_{\boldsymbol{T}}$ and $\zeta_V : \boldsymbol{l} \circ \boldsymbol{j} \Rightarrow \mathbf{id}_{\boldsymbol{U}}$, where V, W are open neighbourhoods of 0 in $\mathbb{R}^m, \mathbb{R}^n$ for $m = \dim T_x^* \underline{X}$, $n = \dim T_y^* \underline{Y}$, and $E = \mathbb{R}^a \times V \to V$, $F = \mathbb{R}^b \times W \to W$ are trivial vector bundles over V, W of ranks a, b, and $s = (s_1, \ldots, s_a) \in C^\infty(E)$, $t = (t_1, \ldots, t_b) \in C^\infty(F)$ with $s(0) = \mathrm{d}s(0) = 0$ and $t(0) = \mathrm{d}t(0) = 0$, and $\boldsymbol{k}(0) = x$, $\boldsymbol{l}(y) = 0$. Making $\boldsymbol{T}, V$ smaller if necessary, we can suppose $\boldsymbol{g}(\boldsymbol{T}) \subseteq \boldsymbol{U}$.

Applying Theorem 3.34 to the 1-morphism $\boldsymbol{l} \circ \boldsymbol{g} \circ \boldsymbol{k} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ and replacing V, E, s by $\tilde{V}, \tilde{E}, \tilde{s}$ gives a smooth map $f = (f_1, \ldots, f_n) : V \to W$ with $f(0) = 0$ and a morphism of vector bundles $\hat{f} : E \to f^*(F)$ on V with $\hat{f} \circ s = f^*(t)$, such that $\boldsymbol{l} \circ \boldsymbol{g} \circ \boldsymbol{k} = \boldsymbol{S}_{f,\hat{f}}$. We may write $\hat{f}$ as a matrix of smooth functions $\bigl(A_{ji}(x_1, \ldots, x_m)\bigr)_{i=1,\ldots,a}^{j=1,\ldots,b}$ on V satisfying

$$\begin{aligned} &\textstyle\sum_{i=1}^a A_{ji}(x_1, \ldots, x_m) s_i(x_1, \ldots, x_m) \\ &\quad = t_j\bigl(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)\bigr) \end{aligned} \tag{4.10}$$

for $j = 1, \ldots, b$. Hence $\boldsymbol{j} \circ \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{i} = (\boldsymbol{j} \circ \boldsymbol{l}) \circ \boldsymbol{g} \circ (\boldsymbol{k} \circ \boldsymbol{i})$. Thus $\eta = \zeta_U * \mathrm{id}_{\boldsymbol{g}} * \zeta_V$ is a 2-morphism $\boldsymbol{j} \circ \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{i} \Rightarrow \boldsymbol{g}|_{\boldsymbol{U}}$. This completes the first part.

For (a)–(d), if $\boldsymbol{g}$ is a w-submersion, …, immersion then $\boldsymbol{S}_{f,\hat{f}} = \boldsymbol{l} \circ \boldsymbol{g} \circ \boldsymbol{k}$ is also a w-submersion, …, immersion by Proposition 4.5(i),(iii), as $\boldsymbol{k}, \boldsymbol{l}$ are equivalences. Thus we may apply Theorem 4.8 to $\boldsymbol{S}_{f,\hat{f}}$ at the point $v = 0$. Because $\mathrm{d}s(0) = \mathrm{d}t(0) = 0$, the exactness conditions simplify, and yield:

(a) if $\boldsymbol{g}$ is a w-submersion then $\hat{f}(0) : E_0 \to F_0$ is surjective;

(b) if $\boldsymbol{g}$ is a submersion then $\mathrm{d}f(0) : T_0V \to T_0W$ is surjective and $\hat{f}(0) : E_0 \to F_0$ is an isomorphism;

(c) if $\boldsymbol{g}$ is a w-immersion then $\mathrm{d}f(0) : T_0V \to T_0W$ is injective; and

(d) if $\boldsymbol{g}$ is an immersion then $\mathrm{d}f(0) : T_0V \to T_0W$ is injective and $\hat{f}(0) : E_0 \to F_0$ is surjective.

Each of these conditions is open in $v \in V$ and so hold near $v = 0$. So making $\boldsymbol{T}, V$ smaller if necessary we can suppose that $\hat{f}(v) : E_v \to F_{f(v)}$ is surjective for all $v \in V$ in (a), and similarly for (b)–(d).

To prove (a)–(d), we will modify these choices above. For (a), we leave W, F, t unchanged, but we replace V by

$$V' = \bigl\{(x_1, \ldots, x_m, z_1, \ldots, z_n) \in V \times \mathbb{R}^n : f(x_1, \ldots, x_m) + (z_1, \ldots, z_n) \in W \subseteq \mathbb{R}^n\bigr\}.$$

We replace E by the vector bundle $E' = \pi_V^*(E) \oplus \mathbb{R}^n$ over V', and s by $s' = \pi_V^*(s) \oplus \mathrm{id}_{\mathbb{R}^n}$ in $C^\infty(E')$ so that $s'(x_1, \ldots, x_m, z_1, \ldots, z_n) = s(x_1, \ldots, x_m) \oplus (z_1, \ldots, z_n)$ and f by $f' : V' \to W$ given by $f'(x_1, \ldots, x_m, y_1, \ldots, y_n) = f(x_1, \ldots, x_n) + (y_1, \ldots, y_n)$, where addition is in $\mathbb{R}^n \supseteq W$, and V' is chosen small enough that f' maps $V' \to W \subseteq \mathbb{R}^n$. By Hadamard's Lemma, there exist functions B_{ji} for $j = 1, \ldots, b$ and $i = 1, \ldots, n$ on $\bigl\{(y_1, \ldots, y_n, z_1, \ldots, z_n) \in \mathbb{R}^{2n} : (y_1, \ldots, y_n), (y_1 + z_1, \ldots, y_n + z_n) \in W\bigr\}$ such that

$$\begin{aligned} t_j(y_1 + z_1, \ldots, y_n + z_n) = t_j(y_1, \ldots, y_n) \\ + \textstyle\sum_{i=1}^n B_{ji}(y_1, \ldots, y_n, z_1, \ldots, z_n) \cdot z_i. \end{aligned} \tag{4.11}$$

Define a morphism of vector bundles $\hat{f}' : E' \to f^*(F)$ on V' by

$$\begin{aligned} &\hat{f}'|_{(x_1,\ldots,x_m,z_1,\ldots,z_n)} : (u_1, \ldots, u_a, v_1, \ldots, v_n) \longmapsto (w_1, \ldots, w_b), \\ &\text{where } w_j = \textstyle\sum_{i=1}^a A_{ji}(x_1, \ldots, x_m) \cdot u_i \\ &+ \textstyle\sum_{i=1}^n B_{ji}\bigl(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m), z_1, \ldots, z_n\bigr) \cdot v_i. \end{aligned} \tag{4.12}$$

Combining equations (4.10)–(4.12) shows that $\hat{f}' \circ s' = (f')^*(t)$, so we have $\boldsymbol{S}_{f',\hat{f}'} : \boldsymbol{S}_{V',E',s'} \to \boldsymbol{S}_{W,F,t}$. Since $\hat{f}$ is surjective, the matrix $(A_{ji})_{i=1,\ldots,a}^{j=1,\ldots,b}$ is surjective, so $\hat{f}'$ is surjective by (4.12). Also $\mathrm{d}f' : \mathbb{R}^m \oplus \mathbb{R}^n \to \mathbb{R}^n$ is the identity on the second factor, and so surjective, and thus $f' : V' \to W$ is a submersion.

Define $h : V \to V'$ by $h : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$ and $\hat{h} = \mathrm{id}_E \oplus 0 : E \to h^*(E') \cong E \oplus \mathbb{R}^n$. Then $\hat{h} \circ s = h^*(s')$, so we have a 1-morphism $\boldsymbol{S}_{h,\hat{h}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{V',E',s'}$. Theorem 3.39 implies that $\boldsymbol{S}_{h,\hat{h}}$ is an equivalence. As

$f = f' \circ h$ and $\hat{f} = h^*(\hat{f}') \circ \hat{h}$ we have $\boldsymbol{S}_{f,\hat{f}} = \boldsymbol{S}_{f',\hat{f}'} \circ \boldsymbol{S}_{h,\hat{h}}$. Define $\boldsymbol{i}' = \boldsymbol{S}_{h,\hat{h}} \circ \boldsymbol{i}$. Then $\boldsymbol{i}' : \boldsymbol{T} \to \boldsymbol{S}_{V',E',s'}$ is an equivalence as $\boldsymbol{i}, \boldsymbol{S}_{h,\hat{h}}$ are equivalences, and

$$\boldsymbol{j} \circ \boldsymbol{S}_{f',\hat{f}'} \circ \boldsymbol{i}' = \boldsymbol{j} \circ \boldsymbol{S}_{f',\hat{f}'} \circ \boldsymbol{S}_{h,\hat{h}} \circ \boldsymbol{i} = \boldsymbol{j} \circ \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{i},$$

so we still have a 2-morphism $\eta : \boldsymbol{j} \circ \boldsymbol{S}_{f',\hat{f}'} \circ \boldsymbol{i}' \Rightarrow \boldsymbol{g}|_{\boldsymbol{U}}$. Replacing $V, E, s, f, \hat{f}, \boldsymbol{i}$ by $V', E', s', f', \hat{f}', \boldsymbol{i}'$ proves (a).

For (b), no changes are needed, as f is a submersion and $\hat{f}$ is an isomorphism already. For (c) and (d), let $\hat{f}(0) : E|_0 = \mathbb{R}^a \to F|_0 = \mathbb{R}^b$ have kernel of dimension c, where $c = b - a$ in case (d) as $\hat{f}(0)$ is surjective. Apply a $\mathrm{GL}(a, \mathbb{R})$ transformation to E to make $\mathrm{Ker}\big(\hat{f}(0)\big) = \big\{(u_1, \ldots, u_c, 0, \ldots, 0) : u_i \in \mathbb{R}\big\}$. In a similar way to (a), define $W' = W \times \mathbb{R}^c$ with coordinates $(y_1, \ldots, y_n, z_1, \ldots, z_c)$, and $F' = \pi_W^*(F) \oplus \mathbb{R}^c$, and $t' = \pi_W^*(t) \oplus \mathrm{id}_{\mathbb{R}^c}$ in $C^\infty(F')$, so that $t'(y_1, \ldots, y_n, z_1, \ldots, z_c) = t(y_1, \ldots, y_n) \oplus (z_1, \ldots, z_c)$.

Define $f' : V \to W'$ and $\hat{f}' : E \to (f')^*(F')$ by

$$\begin{aligned} f'(x_1, \ldots, x_m) &= \big(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m), \\ &\qquad s_1(x_1, \ldots, x_m), \ldots, s_c(x_1, \ldots, x_m)\big), \\ \hat{f}'|_{(x_1,\ldots,x_m)}\big((u_1, \ldots, u_a)\big) &= \hat{f}|_{(x_1,\ldots,x_m)}\big((u_1, \ldots, u_a)\big) \oplus (u_1, \ldots, u_c). \end{aligned}$$

Observe that $\hat{f}'(0)$ is injective in case (c), and an isomorphism in case (d), since $\hat{f}'(0) = \hat{f}(0) \oplus \mathrm{id}_{\mathrm{Ker}\,\hat{f}(0)}$. Also $\mathrm{d}f'(0)$ is injective. Making V' smaller we can suppose these hold for all $v' \in V'$, not just at 0. So f' is an immersion, and $\hat{f}'$ is injective in (c) and an isomorphism in (d). Also $\hat{f}' \circ s = (f')^*(t')$, so we have $\boldsymbol{S}_{f',\hat{f}'} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W',F',t'}$.

Define $h = \pi_W : W' \to W$ and $\hat{h} = \mathrm{id}_{h^*(F)} \oplus 0 : F' = h^*(F) \oplus \mathbb{R}^c \to h^*(F)$. Then $\hat{h} \circ t' = h^*(t)$, so we have a 1-morphism $\boldsymbol{S}_{h,\hat{h}} : \boldsymbol{S}_{W',F',t'} \to \boldsymbol{S}_{W,F,t}$. Theorem 3.39 implies that $\boldsymbol{S}_{h,\hat{h}}$ is an equivalence. As $f = h \circ f'$ and $\hat{f} = (f')^*(\hat{h}) \circ \hat{f}'$ we have $\boldsymbol{S}_{f,\hat{f}} = \boldsymbol{S}_{h,\hat{h}} \circ \boldsymbol{S}_{f',\hat{f}'}$. Define $\boldsymbol{j}' = \boldsymbol{j} \circ \boldsymbol{S}_{h,\hat{h}}$. Then $\boldsymbol{j}' : \boldsymbol{S}_{W',F',t'} \to \boldsymbol{U}$ is an equivalence as $\boldsymbol{j}, \boldsymbol{S}_{h,\hat{h}}$ are equivalences, and

$$\boldsymbol{j}' \circ \boldsymbol{S}_{f',\hat{f}'} \circ \boldsymbol{i} = \boldsymbol{j} \circ \boldsymbol{S}_{h,\hat{h}} \circ \boldsymbol{S}_{f',\hat{f}'} \circ \boldsymbol{i} = \boldsymbol{j} \circ \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{i},$$

so we have a 2-morphism $\eta : \boldsymbol{j}' \circ \boldsymbol{S}_{f',\hat{f}'} \circ \boldsymbol{i} \Rightarrow \boldsymbol{g}|_{\boldsymbol{U}}$. Replacing $W, F, t, f, \hat{f}, \boldsymbol{j}$ by $W', F', t', f', \hat{f}', \boldsymbol{j}'$ proves (c) and (d).

For (a′)–(d′) we use the fact that submersions in **Man** are locally modelled on projections $\pi_W : W \times Z \to W$, and immersions are locally modelled on inclusions $\mathrm{id}_V \times 0 : V \to V \times Z$ for open $0 \in Z \subseteq \mathbb{R}^n$. Thus, making V, W smaller, we may replace V, f by $W \times Z, \pi_W$ in (a),(b), and replace W, f by $V \times Z, \mathrm{id} \times 0$ in (c),(d). Also, as $\hat{f}$ is surjective in (a) we may replace E by $\pi_W^*(F) \oplus G$ and $\hat{f}$ by $\mathrm{id}_{\pi_W^*(F)} \oplus 0$, for $G = \mathrm{Ker}\,\hat{f}$ a vector bundle on $V = W \times Z$. This gives (a′), and (b′)–(d′) follow in a similar way. □

The following lemma is easy to prove. Note that $\boldsymbol{\pi_X}$ is not a submersion if $\boldsymbol{X} \neq \emptyset$ and $\boldsymbol{Y}$ is not a manifold.

Lemma 4.10. *Let $\boldsymbol{X}, \boldsymbol{Y}$ be d-manifolds, with $\boldsymbol{Y}$ a manifold. Then $\boldsymbol{\pi_X} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{X}$ is a submersion.*

Theorem 4.9(b′) implies the following two corollaries. The first is a local converse to Lemma 4.10.

Corollary 4.11. *Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a submersion of d-manifolds, and $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$. Then there exist open d-submanifolds $x \in \boldsymbol{U} \subseteq \boldsymbol{X}$ and $y \in \boldsymbol{V} \subseteq \boldsymbol{Y}$ with $\boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{V}$, a manifold $\boldsymbol{Z}$, and an equivalence $\boldsymbol{i} : \boldsymbol{U} \to \boldsymbol{V} \times \boldsymbol{Z}$, such that $\boldsymbol{f}|_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ is 2-isomorphic to $\boldsymbol{\pi_V} \circ \boldsymbol{i}$, where $\boldsymbol{\pi_V} : \boldsymbol{V} \times \boldsymbol{Z} \to \boldsymbol{V}$ is the projection.*

Corollary 4.12. *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a submersion of d-manifolds with $\boldsymbol{Y}$ a manifold. Then $\boldsymbol{X}$ is a manifold.*

Fixed d-subspaces in d-manifolds are w-embedded d-submanifolds.

Example 4.13. Let V be a manifold, $E \to V$ a vector bundle, Γ a finite group acting smoothly on V, E preserving the vector bundle structure, and $s : V \to E$ a smooth, Γ-equivariant section of E. Write the Γ-actions on V, E as $r(\gamma) : V \to V$ and $\hat{r}(\gamma) : E \to r(\gamma)^*(E)$ for $\gamma \in \Gamma$. Then Definitions 3.13 and 3.30 give an explicit principal d-manifold $\boldsymbol{S}_{V,E,s}$, and 1-morphisms $\boldsymbol{S}_{r(\gamma),\hat{r}(\gamma)} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{V,E,s}$ for $\gamma \in \Gamma$ which are an action of Γ on $\boldsymbol{S}_{V,E,s}$. Therefore §2.7 defines a fixed d-subspace $(\boldsymbol{S}_{V,E,s})^\Gamma$, and 1-morphism $\boldsymbol{j}_{\boldsymbol{S}_{V,E,s},\Gamma} : (\boldsymbol{S}_{V,E,s})^\Gamma \to \boldsymbol{S}_{V,E,s}$.

Write V^Γ for the fixed locus of Γ in V. Then V^Γ is a disjoint union of closed, embedded submanifolds of V of different dimensions. The restriction $E|_{V^\Gamma}$ is a vector bundle on V^Γ, and $\hat{r}|_{V^\Gamma}$ is a linear action of Γ on $E|_{V^\Gamma}$. Write E^Γ for the subbundle of $E|_{V^\Gamma}$ fixed by Γ. Then E^Γ is a vector bundle of mixed rank on V^Γ. Since s is Γ-equivariant, $s|_{V^\Gamma}$ is a smooth section of E^Γ.

For $i = 0, \ldots, \dim V$ and $j = 0, \ldots, \operatorname{rank} E$, write V^Γ_{ij} for the open and closed subset of V^Γ where $\dim V - \dim V^\Gamma = i$ has dimension i and $\operatorname{rank} E - \operatorname{rank} E^\Gamma = j$, and write $E^\Gamma_{ij} = E^\Gamma|_{V^\Gamma_{ij}}$, and $s^\Gamma_{ij} = s|_{V^\Gamma_{ij}}$. Then V^Γ_{ij} is a submanifold of V, and E^Γ_{ij} a vector bundle on V^Γ_{ij}, and $s^\Gamma_{ij} \in C^\infty(E^\Gamma_{ij})$. Hence Definition 3.13 gives a d-manifold $\boldsymbol{S}_{V^\Gamma_{ij},E^\Gamma_{ij},s^\Gamma_{ij}}$. From §2.7, we can see there is a natural 1-isomorphism

$$(\boldsymbol{S}_{V,E,s})^\Gamma \cong \textstyle\coprod_{i=0}^{\dim V} \coprod_{j=0}^{\operatorname{rank} E} \boldsymbol{S}_{V^\Gamma_{ij},E^\Gamma_{ij},s^\Gamma_{ij}}. \tag{4.13}$$

Therefore $(\boldsymbol{S}_{V,E,s})^\Gamma$ is a disjoint union of d-manifolds of different dimensions.

Also $\boldsymbol{j}_{\boldsymbol{S}_{V,E,s},\Gamma}$ is identified on $\boldsymbol{S}_{V^\Gamma_{ij},E^\Gamma_{ij},s^\Gamma_{ij}}$ with $\boldsymbol{S}_{f^\Gamma_{ij},\hat{f}^\Gamma_{ij}} : \boldsymbol{S}_{V^\Gamma_{ij},E^\Gamma_{ij},s^\Gamma_{ij}} \to \boldsymbol{S}_{V,E,s}$, where $f^\Gamma_{ij} : V^\Gamma_{ij} \hookrightarrow V$ and $\hat{f}^\Gamma_{ij} : E^\Gamma_{ij} \hookrightarrow E|_{V^\Gamma_{ij}} = (f^\Gamma_{ij})^*(E)$ are the natural inclusions. As f^Γ_{ij} is an embedding, Theorem 4.8(c) shows that $\boldsymbol{S}_{f^\Gamma_{ij},\hat{f}^\Gamma_{ij}}$ is a w-immersion, and in fact a w-embedding. Thus $\boldsymbol{j}_{\boldsymbol{S}_{V,E,s},\Gamma}$ is a w-embedding.

To prove the next result, note that if $x \in \boldsymbol{X}^\Gamma$ then $\boldsymbol{X}$ is locally equivalent near $\boldsymbol{j}_{\boldsymbol{X},\Gamma}(x)$ to some $\boldsymbol{S}_{V,E,s}$. We can choose V, E, s and the equivalence to be Γ-equivariant, and then $\boldsymbol{X}^\Gamma$ is locally equivalent near x to $(\boldsymbol{S}_{V,E,s})^\Gamma$. The proposition then follows from Example 4.13.

Proposition 4.14. *Suppose $\boldsymbol{X}$ is a d-manifold, and Γ is a finite group acting on $\boldsymbol{X}$. Section* 2.7 *defines the fixed d-subspace $\boldsymbol{X}^\Gamma$ of Γ in $\boldsymbol{X}$, and an inclusion* 1*-morphism $\boldsymbol{j}_{\boldsymbol{X},\Gamma} : \boldsymbol{X}^\Gamma \hookrightarrow \boldsymbol{X}$. Then $\boldsymbol{X}^\Gamma = \coprod_{n\in\mathbb{Z}} \boldsymbol{X}^\Gamma_n$, where $\boldsymbol{X}^\Gamma_n$ is a d-manifold with* $\operatorname{vdim} \boldsymbol{X}^\Gamma_n = n$, *and $\boldsymbol{j}_{\boldsymbol{X},\Gamma}|_{\boldsymbol{X}^\Gamma_n} : \boldsymbol{X}^\Gamma_n \to \boldsymbol{X}$ is a w-embedding.*

Remark 4.15. In Definition 4.1 we defined (w-)immersed and (w-)embedded d-submanifolds of a d-manifold. Theorem 4.9(c),(d) give us local models for these, and so help us understand in what sense they are submanifolds.

First let $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{Y}$ be an immersion or an embedding, so that we think of $\boldsymbol{X}$ as an immersed or embedded d-submanifold in $\boldsymbol{Y}$. Then Proposition 2.17(b) shows that $\operatorname{vdim} \boldsymbol{X} \leqslant \operatorname{vdim} \boldsymbol{Y}$, and if $\operatorname{vdim} \boldsymbol{X} = \operatorname{vdim} \boldsymbol{Y}$ then $\boldsymbol{i}$ is étale, so that locally $\boldsymbol{X}$ is (equivalent to) an open d-submanifold in $\boldsymbol{Y}$. Theorem 4.9(d) shows that locally we can write $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ and $\boldsymbol{Y} \simeq \boldsymbol{S}_{W,F,t}$, where V is a submanifold in W and $E = F|_V$, $s = t|_V$. Also, Proposition 4.27 below shows that locally $\boldsymbol{X}$ is (equivalent to) the zeroes of finitely many real equations in $\boldsymbol{Y}$. Immersions and embeddings are the most obvious notions of d-submanifold, with most of the properties one would expect from classical differential geometry.

Now let $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{Y}$ be a w-immersion or a w-embedding. Then Theorem 4.9(c) shows that locally we can write $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ and $\boldsymbol{Y} \simeq \boldsymbol{S}_{W,F,t}$, where V is a submanifold in W and E is a vector subbundle of $F|_V$, with $t|_V \in C^\infty(E) \subseteq C^\infty(F|_V)$, and $s = t|_V$. So V, E are both subobjects of W, F. Note that $\operatorname{vdim} \boldsymbol{Y} - \operatorname{vdim} \boldsymbol{X} = (\dim W - \dim V) - (\operatorname{rank} F - \operatorname{rank} E)$, which can take any value in $\mathbb{Z}$. In particular, if $\boldsymbol{X}$ is a w-immersed or w-embedded d-submanifold in $\boldsymbol{Y}$ we can have $\operatorname{vdim} \boldsymbol{X} > \operatorname{vdim} \boldsymbol{Y}$, which is counterintuitive. So w-immersed and w-embedded d-submanifolds are submanifolds in a weaker sense.

One area these ideas are important is *orbifold strata*, as in §10.7, which are very similar to fixed d-subspaces in Proposition 4.14 above. Orbifold strata of d-orbifolds are w-immersed d-orbifolds, and can have larger dimension than the ambient d-orbifold. So there are some natural problems in which we have to deal with w-immersed and w-embedded d-submanifolds or d-orbifolds.

4.3 D-transversality and fibre products

Next we consider fibre products. From §2.5, a fibre product $\boldsymbol{W} = \boldsymbol{X}_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ of d-manifolds $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ always exists as a d-space. We want to know whether $\boldsymbol{W}$ is a d-manifold. We will define when $\boldsymbol{g}, \boldsymbol{h}$ are *d-transverse*, and show in Theorem 4.21 that if $\boldsymbol{g}, \boldsymbol{h}$ are d-transverse then $\boldsymbol{W}$ is a d-manifold.

To motivate the definition, recall that if $g : X \to Z$, $h : Y \to Z$ are smooth maps of manifolds, then a fibre product $W = X \times_{g,Z,h} Y$ in **Man** exists if g, h are *transverse*. By definition, this means that $T_zZ = \mathrm{d}g|_x(T_xX) + \mathrm{d}h|_y(T_yY)$ for all $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$. That is, $\mathrm{d}g|_x \oplus \mathrm{d}h|_y : T_xX \oplus T_yY \to T_zZ$ is surjective, or dually, $\mathrm{d}g|_x^* \oplus \mathrm{d}h|_y^* : T_zZ^* \to T_x^*X \oplus T_y^*Y$ is injective. Writing $W = X \times_Z Y$ for the topological fibre product and $e : W \to X$, $f : W \to Y$ for the projections, with $g \circ e = h \circ f$, we see that g, h are transverse if and only if the morphism

$$e^*(\mathrm{d}g^*) \oplus f^*(\mathrm{d}h^*) : (g \circ e)^*(T^*Z) \to e^*(T^*X) \oplus f^*(T^*Y) \tag{4.14}$$

of vector bundles on the topological space W is injective, that is, has a left inverse. The condition that (4.15) has a left inverse is an analogue of this, but on obstruction rather than cotangent bundles.

Definition 4.16. Let $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ be d-manifolds and $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ be 1-morphisms, and let $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ be the explicit d-space fibre product from §2.5. Equation (2.59) defines a morphism α_1 in $\mathrm{qcoh}(\underline{W})$:

$$\alpha_1=\begin{pmatrix} \underline{e}^*(g'')\circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z) \\ -\underline{f}^*(h'')\circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z) \\ (\underline{g}\circ\underline{e})^*(\phi_Z)\end{pmatrix}:\begin{aligned}[t]&(\underline{g}\circ\underline{e})^*(\mathcal{E}_Z)\longrightarrow\\ &\qquad\underline{e}^*(\mathcal{E}_X)\oplus\underline{f}^*(\mathcal{E}_Y)\oplus(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z).\end{aligned}\tag{4.15}$$

We call $\boldsymbol{g},\boldsymbol{h}$ *d-transverse* if α_1 has a left inverse

$$\beta=\bigl(\beta_1\,\beta_2\,\beta_3\bigr):\underline{e}^*(\mathcal{E}_X)\oplus\underline{f}^*(\mathcal{E}_Y)\oplus(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z)\longrightarrow(\underline{g}\circ\underline{e})^*(\mathcal{E}_Z)\tag{4.16}$$

with $\beta\circ\alpha_1=\mathrm{id}_{(\underline{g}\circ\underline{e})^*(\mathcal{E}_Z)}$. Note that this is a local condition in $\underline{W}$, since local choices of left inverse for α_1 can be combined using a partition of unity on $\underline{W}$ to make a global left inverse.

Here is a way to interpret this in the notation of §4.1. On $\underline{X}$ we have virtual vector bundles $T^*\boldsymbol{X},\underline{g}^*(T^*\boldsymbol{Z})$ and a 1-morphism $\Omega_{\boldsymbol{g}}:\underline{g}^*(T^*\boldsymbol{Z})\to T^*\boldsymbol{X}$ in $\mathrm{vvect}(\underline{X})$. Pulling back to $\underline{W}$ by $\underline{e}^*$ gives a morphism $\underline{e}^*(\Omega_{\boldsymbol{g}}):\underline{e}^*\circ\underline{g}^*(T^*\boldsymbol{Z})\to\underline{e}^*(T^*\boldsymbol{X})$ in $\mathrm{vvect}(\underline{W})$. Composing with $I_{\underline{e},\underline{g}}(T^*\boldsymbol{Z}):=(I_{\underline{e},\underline{g}}(\mathcal{E}_Z),I_{\underline{e},\underline{g}}(\mathcal{F}_Z))$ gives $\underline{e}^*(\Omega_{\boldsymbol{g}})\circ I_{\underline{e},\underline{g}}(T^*\boldsymbol{Z}):(\underline{g}\circ\underline{e})^*(T^*\boldsymbol{Z})\to\underline{e}^*(T^*\boldsymbol{X})$. Similarly we have $\underline{f}^*(\Omega_{\boldsymbol{h}})\circ I_{\underline{f},\underline{h}}(T^*\boldsymbol{Z}):(\underline{g}\circ\underline{e})^*(T^*\boldsymbol{Z})\to\underline{f}^*(T^*\boldsymbol{Y})$, as $\underline{g}\circ\underline{e}=\underline{g}\circ\underline{f}$. Combining these gives a 1-morphism

$$\begin{aligned}\bigl(\underline{e}^*(\Omega_{\boldsymbol{g}})\circ I_{\underline{e},\underline{g}}(T^*\boldsymbol{Z})\bigr)\oplus\bigl(\underline{f}^*(\Omega_{\boldsymbol{h}})\circ I_{\underline{f},\underline{h}}(T^*\boldsymbol{Z})\bigr)&:(\underline{g}\circ\underline{e})^*(T^*\boldsymbol{Z})\\ &\longrightarrow\underline{e}^*(T^*\boldsymbol{X})\oplus\underline{f}^*(T^*\boldsymbol{Y})\end{aligned}\tag{4.17}$$

in $\mathrm{vvect}(\underline{W})$. For (4.15) to have a left inverse is equivalent to (4.17) being weakly injective, in the sense of Definition 4.1. This is the d-manifold analogue of (4.14) being injective.

We will show in Theorem 4.21 below that if $\boldsymbol{g},\boldsymbol{h}$ are d-transverse then a fibre product $\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in **dMan**. First, here are some elementary properties of d-transversality. They follow from the characterization of $\boldsymbol{g},\boldsymbol{h}$ transverse as (4.17) being weakly injective, Proposition 4.2, and the facts from Example 3.2 that 2-morphisms $\boldsymbol{g}\Rightarrow\tilde{\boldsymbol{g}}$ in **dMan** translate to 2-morphisms $\Omega_{\boldsymbol{g}}\Rightarrow\Omega_{\tilde{\boldsymbol{g}}}$ in $\mathrm{vvect}(\underline{X})$, and $\boldsymbol{a}$ an equivalence in **dMan** implies $\Omega_{\boldsymbol{a}}$ an equivalence in $\mathrm{vvect}(\underline{X})$, and compositions of 1-morphisms in **dMan** lifting to compositions of cotangent 1-morphisms in $\mathrm{vvect}(\underline{X})$.

Proposition 4.17. **(a)** *The condition that $\boldsymbol{g},\boldsymbol{h}$ be d-transverse in Definition 4.16 is local in $\underline{W}$, that is, it is enough to check it on any open cover of $\underline{W}$.*

(b) *Suppose $\boldsymbol{g},\tilde{\boldsymbol{g}}:\boldsymbol{X}\to\boldsymbol{Z}$ and $\boldsymbol{h},\tilde{\boldsymbol{h}}:\boldsymbol{Y}\to\boldsymbol{Z}$ are 1-morphisms of d-manifolds, and $\eta:\boldsymbol{g}\Rightarrow\tilde{\boldsymbol{g}}$, $\zeta:\boldsymbol{h}\Rightarrow\tilde{\boldsymbol{h}}$ are 2-morphisms. Then $\boldsymbol{g},\boldsymbol{h}$ are d-transverse if and only if $\tilde{\boldsymbol{g}},\tilde{\boldsymbol{h}}$ are d-transverse.*

(c) *Suppose* $\boldsymbol{a}:\tilde{\boldsymbol{X}}\rightarrow\boldsymbol{X}$, $\boldsymbol{b}:\tilde{\boldsymbol{Y}}\rightarrow\boldsymbol{Y}$, $\boldsymbol{c}:\boldsymbol{Z}\rightarrow\tilde{\boldsymbol{Z}}$, $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ *are* 1*-morphisms of d-manifolds with* $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ *equivalences. Then* $\boldsymbol{g},\boldsymbol{h}$ *are d-transverse if and only if* $\tilde{\boldsymbol{g}}:=\boldsymbol{c}\circ\boldsymbol{g}\circ\boldsymbol{a}:\tilde{\boldsymbol{X}}\rightarrow\tilde{\boldsymbol{Z}}$ *and* $\tilde{\boldsymbol{h}}:=\boldsymbol{c}\circ\boldsymbol{h}\circ\boldsymbol{b}:\tilde{\boldsymbol{Y}}\rightarrow\tilde{\boldsymbol{Z}}$ *are d-transverse.*

Theorems 3.39 and 4.8 gave differential-geometric characterizations of when a 'standard model' 1-morphism $\boldsymbol{S}_{f,\hat{f}}:\boldsymbol{S}_{V,E,s}\rightarrow\boldsymbol{S}_{W,F,t}$ is étale, a w-submersion, ..., an immersion. By the same method we can prove the following characterization of when two 'standard model' 1-morphisms are d-transverse.

Proposition 4.18. *Suppose* T,U,V *are manifolds,* $F\rightarrow T$, $G\rightarrow U$, $H\rightarrow V$ *are vector bundles,* $t:T\rightarrow F$, $u:U\rightarrow G$, $v:V\rightarrow H$ *are smooth sections,* $p:T\rightarrow V$, $q:U\rightarrow V$ *are smooth maps, and* $\hat{p}:F\rightarrow p^*(H)$, $\hat{q}:G\rightarrow q^*(H)$ *are vector bundle morphisms with* $\hat{p}\circ t=p^*(v)+O(t^2)$ *and* $\hat{q}\circ u=q^*(v)+O(u^2)$. *Then Definitions* 3.13 *and* 3.30 *give* 1*-morphisms* $\boldsymbol{S}_{p,\hat{p}}:\boldsymbol{S}_{T,F,t}\rightarrow\boldsymbol{S}_{V,H,v}$ *and* $\boldsymbol{S}_{q,\hat{q}}:\boldsymbol{S}_{U,G,u}\rightarrow\boldsymbol{S}_{V,H,v}$ *in* **dMan**. *These* $\boldsymbol{S}_{p,\hat{p}},\boldsymbol{S}_{q,\hat{q}}$ *are d-transverse if and only if for all* $x\in T$ *with* $t(x)=0$ *and* $y\in Y$ *with* $u(y)=0$ *and* $p(x)=q(y)=z$ *in* V, *the following morphism of vector spaces is surjective:*

$$\hat{p}(x)\oplus-\hat{q}(y)\oplus\mathrm{d}v(z):F_x\oplus G_y\oplus T_zV\longrightarrow H_z. \tag{4.18}$$

With some simplifying assumptions on $V,H,p,q,\hat{p},\hat{q}$, the next definition and theorem show that a d-transverse fibre product $\boldsymbol{S}_{T,F,t}\times_{\boldsymbol{S}_{p,\hat{p}},\boldsymbol{S}_{V,H,v},\boldsymbol{S}_{q,\hat{q}}}\boldsymbol{S}_{U,G,u}$ in **dSpa** may be written in the form $\boldsymbol{S}_{S,E,s}$ for explicit S,E,s.

Definition 4.19. Let T,U,V be manifolds with $V\subseteq\mathbb{R}^n$ open, and $F\rightarrow T$, $G\rightarrow U$, $H\rightarrow V$ be vector bundles with H the trivial vector bundle $\mathbb{R}^k\times V\rightarrow V$, and $t:T\rightarrow F$, $u:U\rightarrow G$, $v:V\rightarrow H$ be smooth sections with $v=(v_1,\ldots,v_k)$ for $v_i\in C^\infty(V)$, and $p:T\rightarrow V$, $q:U\rightarrow V$ be smooth with $p=(p_1,\ldots,p_n)$, $q=(q_1,\ldots,q_n)$ for $p_i\in C^\infty(T)$, $q_i\in C^\infty(U)$, and $\hat{p}:F\rightarrow p^*(H)=\mathbb{R}^k$, $\hat{q}:G\rightarrow h^*(H)=\mathbb{R}^k$ be vector bundle morphisms with $\hat{p}=(\hat{p}_1,\ldots,\hat{p}_k)$, $\hat{q}=(\hat{q}_1,\ldots,\hat{q}_k)$ for $\hat{p}_i\in C^\infty(F^*)$, $\hat{q}_i\in C^\infty(G^*)$, such that $\hat{p}\circ t=p^*(v)$ and $\hat{q}\circ u=q^*(v)$. Then we have 1-morphisms $\boldsymbol{S}_{p,\hat{p}}:\boldsymbol{S}_{T,F,t}\rightarrow\boldsymbol{S}_{V,H,v}$ and $\boldsymbol{S}_{q,\hat{q}}:\boldsymbol{S}_{U,G,u}\rightarrow\boldsymbol{S}_{V,H,v}$ in **dMan**. Suppose $\boldsymbol{S}_{p,\hat{p}},\boldsymbol{S}_{q,\hat{q}}$ are d-transverse. Then (4.18) is surjective for all $x\in T$, $y\in U$ with $s(x)=0$, $t(y)=0$ and $p(x)=q(y)=z\in V$ by Proposition 4.18. Suppose too that the closure of $\{z\in V:z=p(x)=q(y)$ for $x\in T$, $y\in U$ with $t(x)=u(y)=0\}$ in $\mathbb{R}^n$ is a subset of V.

Applying Hadamard's Lemma to the functions $v_1,\ldots,v_k$ on $V\subseteq\mathbb{R}^n$ shows that there exist $A_{ij}\in C^\infty(V\times V)$ for all $i=1,\ldots,k$, $j=1,\ldots,n$ such that

$$v_i(\tilde{z}_1,\ldots,\tilde{z}_n)-v_i(z_1,\ldots,z_n)=\sum_{j=1}^n A_{ij}(z_1,\ldots,z_n,\tilde{z}_1,\ldots,\tilde{z}_n)\cdot(\tilde{z}_j-z_j), \tag{4.19}$$

for all $(z_1,\ldots,z_n),(\tilde{z}_1,\ldots,\tilde{z}_n)\in V\subseteq\mathbb{R}^n$. Differentiating (4.20) w.r.t. $\tilde{z}_j$ at $(z_1,\ldots,z_n)=(\tilde{z}_1,\ldots,\tilde{z}_n)$ shows that

$$A_{ij}(z_1,\ldots,z_n,z_1,\ldots,z_n)=\frac{\partial v_i}{\partial z_j}(z_1,\ldots,z_n). \tag{4.20}$$

On the manifold $T\times U$, consider the two vector bundles $\pi_T^*(F)\oplus\pi_U^*(G)\oplus\mathbb{R}^n$ and $\mathbb{R}^k$, where $\mathbb{R}^n,\mathbb{R}^k$ stand for the trivial vector bundles $\mathbb{R}^n\times(T\times U)\to T\times U$, $\mathbb{R}^k\times(T\times U)\to T\times U$. Define a morphism of vector bundles

$$\begin{aligned} &B:\pi_T^*(F)\oplus\pi_U^*(G)\oplus\mathbb{R}^n\longrightarrow\mathbb{R}^k \quad\text{over } T\times U \text{ by}\\ &B|_{(x,y)}=\bigl(\hat p_1(x),\ldots,\hat p_k(x)\bigr)\oplus\bigl(-\hat q_1(y),\ldots,-\hat q_k(y)\bigr)\\ &\qquad\qquad\oplus\bigl(A_{ij}\bigl(p_1(x),\ldots,p_n(x),q_1(y),\ldots,q_n(y)\bigr)\bigr)_{i=1,\ldots,k}^{j=1,\ldots,n}. \end{aligned}\tag{4.21}$$

Define a section $s\in C^\infty\bigl(\pi_T^*(F)\oplus\pi_T^*(G)\oplus\mathbb{R}^n\bigr)$ by

$$s(x,y)=t(x)\oplus u(y)\oplus\bigl(q_1(x)-p_1(y),\ldots,q_n(x)-p_n(y)\bigr).\tag{4.22}$$

Then we have

$$\begin{aligned} B(s)|_{x,y}&=\hat p\circ t(x)-\hat q\circ u(y)\\ &+\bigl(\textstyle\sum_{j=1}^n A_{ij}\bigl(p_1(x),\ldots,p_n(x),q_1(y),\ldots,q_n(y)\bigr)\cdot(q_j(x)-p_j(y))\bigr)_{i=1,\ldots,k}\\ &=v(p(x))-v(q(y))+\bigl(v_i(q_1(y),\ldots,q_n(y))-v_i(p_1(x),\ldots,p_n(x))\bigr)_{i=1,\ldots,k}\\ &=v(p(x))-v(q(y))+v(q(y))-v(p(x))=0, \end{aligned}$$

using $\hat p\circ t=p^*(v)$, $\hat q\circ u=q^*(v)$ and (4.19). So $B(s)=0$.

Suppose $x\in T$, $y\in U$ with $t(x)=0$, $u(y)=0$ and $p(x)=q(y)=z=(z_1,\ldots,z_n)\in V$. Then by (4.20) we have

$$B|_{(x,y)}=\hat p(x)\oplus-\hat q(y)\oplus\bigl(\tfrac{\partial v_i}{\partial z_j}(z_1,\ldots,z_n)\bigr)_{i=1,\ldots,k}^{j=1,\ldots,n}.$$

The third term is $\mathrm{d}v(z)$, so $B|_{(x,y)}$ is (4.19), and thus is surjective. This is an open condition in $(x,y)\in T\times U$. Hence we may choose an open neighbourhood S of $W=\bigl\{(x,y)\in T\times U: t(x)=0,\ u(y)=0,\ p(x)=q(y)\bigr\}$ in $T\times U$ such that $B|_{(x,y)}$ is surjective for all $(x,y)\in S$. Then S is a manifold. As $B|_S:\bigl(\pi_T^*(F)\oplus\pi_U^*(G)\oplus\mathbb{R}^n\bigr)|_S\to\mathbb{R}^k|_S$ is surjective, its kernel is a vector bundle on S. Define $E=\mathrm{Ker}(B|_S)$, as a vector subbundle of $\bigl(\pi_T^*(F)\oplus\pi_U^*(G)\oplus\mathbb{R}^n\bigr)|_S$. Then since $B(s)=0$, we have $s\in C^\infty(E)$. Hence Definition 3.13 defines a 'standard model' d-manifold $\boldsymbol{S}_{S,E,s}$.

Theorem 4.20. *In the situation of Definition* 4.19, *with* $\boldsymbol{S}_{p,\hat p}:\boldsymbol{S}_{T,F,t}\to\boldsymbol{S}_{V,H,v}$ *and* $\boldsymbol{S}_{q,\hat q}:\boldsymbol{S}_{U,G,u}\to\boldsymbol{S}_{V,H,v}$ *d-transverse* 1*-morphisms in* **dMan**, *let* $\boldsymbol{W}=\boldsymbol{S}_{T,F,t}\times_{\boldsymbol{S}_{p,\hat p},\boldsymbol{S}_{V,H,v},\boldsymbol{S}_{q,\hat q}}\boldsymbol{S}_{U,G,u}$ *be the explicit fibre product in* **dSpa** *defined in* §2.5. *Then* $\boldsymbol{W}$ *is* 1*-isomorphic to* $\boldsymbol{S}_{S,E,s}$ *in* **dSpa**. *Also*

$$\mathrm{vdim}\,\boldsymbol{S}_{S,E,s}=\mathrm{vdim}\,\boldsymbol{S}_{T,F,t}+\mathrm{vdim}\,\boldsymbol{S}_{U,G,u}-\mathrm{vdim}\,\boldsymbol{S}_{V,H,v}.\tag{4.23}$$

Proof. Write $\boldsymbol{X}=\boldsymbol{S}_{T,F,t}$, $\boldsymbol{Y}=\boldsymbol{S}_{U,G,u}$, $\boldsymbol{Z}=\boldsymbol{S}_{V,H,v}$, $\boldsymbol{g}=\boldsymbol{S}_{p,\hat p}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}=\boldsymbol{S}_{q,\hat q}:\boldsymbol{Y}\to\boldsymbol{Z}$, and $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}=\boldsymbol{S}_{T,F,t}\times_{\boldsymbol{S}_{p,\hat p},\boldsymbol{S}_{V,H,v},\boldsymbol{S}_{q,\hat q}}\boldsymbol{S}_{U,G,u}$ for the explicit fibre product in **dSpa** defined in §2.5, and $\tilde{\boldsymbol{W}}=\boldsymbol{S}_{S,E,s}$. We must show that $\boldsymbol{W},\tilde{\boldsymbol{W}}$ are 1-isomorphic in **dSpa**.

Now $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \tilde{\boldsymbol{W}}$ are defined in Definition 3.13, and $\boldsymbol{g}, \boldsymbol{h}$ in Definition 3.30. Writing $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$, $\boldsymbol{g} = (\underline{g}, g', g'')$, and so on, we have

$$\begin{gathered} \underline{X} = \mathrm{Spec}\bigl(C^\infty(T)/I_t\bigr), \quad (X, \mathcal{O}'_X) = \mathrm{Spec}\bigl(C^\infty(T)/I_t^2\bigr), \\ X = \{x \in T : t(x) = 0\}, \quad \mathcal{E}_X = \mathrm{MSpec}\bigl(C^\infty(F^*)/(I_t \cdot C^\infty(F^*))\bigr), \\ (\mathrm{id}_X, \imath_X) = \mathrm{Spec}\bigl(\pi_X : C^\infty(T)/I_t^2 \longrightarrow C^\infty(T)/I_t\bigr), \\ \jmath_X = \mathrm{MSpec}\bigl(t \cdot - : C^\infty(F^*)/(I_t \cdot C^\infty(F^*)) \longrightarrow I_t/I_t^2\bigr), \end{gathered} \tag{4.24}$$

with the analogous notation for $\boldsymbol{Y}, \boldsymbol{Z}, \tilde{\boldsymbol{W}}$, and

$$\begin{gathered} \underline{g} = \mathrm{Spec}(\phi_p), \ \ (g, g') = \mathrm{Spec}(\phi'_p), \ \ g'' = \mathrm{MSpec}(\alpha_p), \ \text{ where} \\ \phi_p : C^\infty(V)/I_v \longrightarrow C^\infty(T)/I_t, \quad \phi'_p : C^\infty(V)/I_v^2 \longrightarrow C^\infty(T)/I_t^2, \\ \alpha_p : C^\infty(p^*(H^*))/(I_t \cdot C^\infty(p^*(H^*)) \longrightarrow C^\infty(F^*)/(I_t \cdot C^\infty(F^*)) \\ \text{are given by } \ \phi_p : c + I_v \longmapsto c \circ p + I_t, \ \ \phi'_p : c + I_v^2 \longmapsto c \circ p + I_t^2, \\ \text{and} \quad \alpha_p : \gamma + I_t \cdot C^\infty(f^*(H^*)) \longmapsto \gamma \circ \hat{p} + I_t \cdot C^\infty(F^*), \end{gathered}$$

with the analogous notation using $\phi_q, \phi'_q, \alpha_q$ for $\boldsymbol{h} = (\underline{h}, h', h'')$.

Since all the C^∞-schemes involved are fair and affine, the construction of $\boldsymbol{W} = (\underline{W}, \mathcal{O}'_W, \mathcal{E}_W, \imath_W, \jmath_W)$ in Definition 2.35 may be done at the level of C^∞-rings and modules, rather than sheaves. We find that

$$\begin{aligned} &\underline{W} = \mathrm{Spec}\,\mathfrak{C}_W, \ \ (W, \mathcal{O}'_W) = \mathrm{Spec}\,\mathfrak{C}'_W, \ \ (\mathrm{id}_W, \imath_W) = \mathrm{Spec}\bigl(\pi_W : \mathfrak{C}'_W \to \mathfrak{C}_W\bigr), \\ &K_W = \mathrm{Ker}\,\pi_W \subseteq \mathfrak{C}'_W, \ \ \mathcal{I}_W = \mathrm{MSpec}\,K_W, \ \ \mathcal{E}_W = \mathrm{MSpec}\,M_W \ \text{ and} \\ &\jmath_W = \mathrm{MSpec}\bigl(\beta_W : M_W \to K_W\bigr), \ \text{ where} \end{aligned}$$

$$\mathfrak{C}_W = \frac{(C^\infty(T)/I_t)\hat{\otimes}(C^\infty(U)/I_u)}{\bigl(i_{C^\infty(U)/I_u} \circ \phi_q(c) - i_{C^\infty(T)/I_t} \circ \phi_p(c) : \ c \in C^\infty(V)/I_v\bigr)}, \tag{4.25}$$

$$\mathfrak{C}'_W = \frac{(C^\infty(T)/I_t^2)\hat{\otimes}(C^\infty(U)/I_u^2)}{\begin{array}{c}\bigl[\bigl(i_{C^\infty(T)/I_t^2}(c) : c \in I_t/I_t^2\bigr) + \bigl(i_{C^\infty(U)/I_u^2}(c) : c \in I_u/I_u^2\bigr) + \\ \bigl(i_{C^\infty(U)/I_u^2} \circ \phi'_q(c) - i_{C^\infty(T)/I_t^2} \circ \phi'_p(c) : \ c \in C^\infty(V)/I_v^2\bigr)\bigr]^2\end{array}}, \tag{4.26}$$

$$\begin{aligned} \pi_W \ \text{is induced by } \pi_X \hat{\boxtimes} \pi_Y : {}&(C^\infty(T)/I_t^2)\hat{\otimes}(C^\infty(U)/I_u^2) \\ &\longrightarrow (C^\infty(T)/I_t)\hat{\otimes}(C^\infty(U)/I_u), \end{aligned} \tag{4.27}$$

$$M_W = \frac{\begin{array}{l}\bigl(C^\infty(F^*)/I_t \cdot C^\infty(F^*)\bigr) \otimes_{C^\infty(T)/I_t} \mathfrak{C}_W \oplus \\ \bigl(C^\infty(G^*)/I_u \cdot C^\infty(G^*)\bigr) \otimes_{C^\infty(U)/I_u} \mathfrak{C}_W \oplus \\ \bigl(\Omega_{C^\infty(V)/I_v^2} \otimes_{C^\infty(V)/I_v^2} C^\infty(V)/I_v\bigr) \otimes_{C^\infty(V)/I_v} \mathfrak{C}_W\end{array}}{\bigl(C^\infty(H^*)/I_v \cdot C^\infty(H^*)\bigr) \otimes_{C^\infty(V)/I_v} \mathfrak{C}_W}, \tag{4.28}$$

$$
\begin{aligned}
&\beta_W \circ \Pi = \beta_1 \oplus \beta_2 \oplus \beta_3, \quad \text{where for } a, b \in C^\infty(V),\ c \in \mathfrak{C}_W, \\
&\qquad \gamma \in C^\infty(F^*) \text{ and } \delta \in C^\infty(G^*) \text{ we have} \\
&\beta_1 : (\gamma + I_t \cdot C^\infty(F^*)) \otimes c \longmapsto c\bigl(i_{C^\infty(T)/I_t^2}(\gamma \cdot t + I_t^2) + \text{ideal}\bigr), \\
&\beta_2 : (\delta + I_u \cdot C^\infty(G^*)) \otimes c \longmapsto c\bigl(i_{C^\infty(U)/I_u^2}(\delta \cdot u + I_u^2) + \text{ideal}\bigr), \\
&\beta_3 : ((a + I_v^2)\,\mathrm{d}(b + I_v^2)) \otimes c \mapsto c\bigl((i_{C^\infty(T)/I_t^2} \circ \phi'_p(a)) \cdot \\
&\qquad\qquad (i_{C^\infty(U)/I_u^2} \circ \phi'_q(b) - i_{C^\infty(T)/I_t^2} \circ \phi'_p(b)) + \text{ideal}\bigr).
\end{aligned}
\tag{4.29}
$$

for Π the projection from the numerator of (4.28) to M_W.

The projections $\pi_T : C^\infty(T) \to C^\infty(T)/I_t$, $\pi_U : C^\infty(U) \to C^\infty(U)/I_u$ induce a surjective $\pi_T \hat{\boxtimes} \pi_U : C^\infty(T) \hat{\otimes} C^\infty(U) \to (C^\infty(T)/I_t) \hat{\otimes} (C^\infty(U)/I_u)$. But $C^\infty(T) \hat{\otimes} C^\infty(U) \cong C^\infty(T \times U)$. Thus by (4.25) we may write $\mathfrak{C}_W$ as a quotient of $C^\infty(T \times U)$. The same works for $\mathfrak{C}'_W$ in (4.26), giving

$$
\mathfrak{C}_W \cong \frac{C^\infty(T \times U)}{\begin{array}{l}\bigl(\pi_T^*(i) : i \in I_t\bigr) + \bigl(\pi_U^*(j) : j \in I_u\bigr) \\ +\bigl(c \circ q \circ \pi_U - c \circ p \circ \pi_T : c \in C^\infty(V)\bigr)\end{array}}, \tag{4.30}
$$

$$
\mathfrak{C}'_W \cong \frac{C^\infty(T \times U)}{\begin{array}{l}\bigl[\bigl(\pi_T^*(i) : i \in I_t\bigr) + \bigl(\pi_U^*(j) : j \in I_u\bigr) \\ +\bigl(c \circ q \circ \pi_U - c \circ p \circ \pi_T : c \in C^\infty(V)\bigr)\bigr]^2\end{array}}, \tag{4.31}
$$

$$
\pi_W \text{ is induced by } \mathrm{id} : C^\infty(T \times U) \to C^\infty(T \times U). \tag{4.32}
$$

For the first term in the numerator of (4.28) we have

$$
\begin{aligned}
\bigl(C^\infty(F^*)/I_t \cdot C^\infty(F^*)\bigr) \otimes_{C^\infty(T)/I_t} \mathfrak{C}_W &\cong C^\infty(F^*) \otimes_{C^\infty(T)} \mathfrak{C}_W \\
&\cong \bigl(C^\infty(F^*) \otimes_{C^\infty(T)} C^\infty(T \times U)\bigr) \otimes_{C^\infty(T \times U)} \mathfrak{C}_W \\
&\cong C^\infty(\pi_T^*(F^*)) \otimes_{C^\infty(T \times U)} \mathfrak{C}_W.
\end{aligned}
$$

We treat the second term the same way. For the third term, we have

$$
\begin{aligned}
&\bigl(\Omega_{C^\infty(V)/I_v^2} \otimes_{C^\infty(V)/I_v^2} (C^\infty(V)/I_v)\bigr) \otimes_{C^\infty(V)/I_v} \mathfrak{C}_W \\
&\cong \bigl(\Omega_{C^\infty(V)} \otimes_{C^\infty(V)} (C^\infty(V)/I_v)\bigr) \otimes_{C^\infty(V)/I_v} \mathfrak{C}_W \cong C^\infty(T^*V) \otimes_{C^\infty(V)} \mathfrak{C}_W \\
&\cong \bigl(C^\infty(T^*V) \otimes_{C^\infty(V)} C^\infty(T \times U)\bigr) \otimes_{C^\infty(T \times U)} \mathfrak{C}_W \\
&\cong C^\infty\bigl((p \circ \pi_T)^*(T^*V)\bigr) \otimes_{C^\infty(T \times U)} \mathfrak{C}_W.
\end{aligned}
$$

Here the first step is as in (2.65). For the last two steps, the morphism $C^\infty(V) \to \mathfrak{C}_W$ factors through $C^\infty(T \times U)$ in two different ways, via $(p \circ \pi_T)^* : C^\infty(V) \to C^\infty(T \times U)$ or $(q \circ \pi_U)^* : C^\infty(V) \to C^\infty(T \times U)$. The terms $c \circ p \circ \pi_T - c \circ q \circ \pi_U$ in (4.30) imply that the compositions of $(p \circ \pi_T)^*, (q \circ \pi_U)^*$ with the projection $C^\infty(T \times U) \to \mathfrak{C}_W$ are equal. Treating the denominator of (4.28) in the same way as the third term and putting all this together gives

$$M_W \cong \frac{C^\infty\bigl(\pi_T^*(F^*) \oplus \pi_U^*(G^*) \oplus (p\circ\pi_T)^*(T^*V)\bigr) \otimes_{C^\infty(T\times U)} \mathfrak{C}_W}{C^\infty\bigl((p\circ\pi_T)^*(H^*)\bigr) \otimes_{C^\infty(T\times U)} \mathfrak{C}_W}\,, \tag{4.33}$$

$$\begin{aligned}
&\beta_W \circ \tilde\Pi = \tilde\beta_1 \oplus \tilde\beta_2 \oplus \tilde\beta_3, \quad \text{where for } a \in C^\infty(T\times U),\ b\in C^\infty(V),\\
&\qquad c\in\mathfrak{C}_W,\ \gamma\in C^\infty(\pi_T^*(F^*)) \text{ and } \delta\in C^\infty(\pi_U^*(G^*)) \text{ we have}\\
&\tilde\beta_1 : \gamma\otimes c \longmapsto c(\gamma\cdot\pi_T^*(t) + \text{ideal}),\\
&\tilde\beta_2 : \delta\otimes c \longmapsto c(\delta\cdot\pi_U^*(u) + \text{ideal}),\\
&\tilde\beta_3 : \bigl(a\,(p\circ\pi_T)^*(\mathrm{d}b)\bigr)\otimes c \longmapsto c\bigl(a(b\circ q\circ\pi_U - b\circ p\circ\pi_T) + \text{ideal}\bigr),
\end{aligned} \tag{4.34}$$

for $\tilde\Pi$ the projection from the numerator of (4.33) to M_W.

Next we restrict from $T\times U$ to the open set $S\subseteq T\times U$. This transforms (4.30)–(4.34) to (4.35)–(4.39). Restriction gives a morphism $C^\infty(T\times U)\to C^\infty(S)$, which clearly induces morphisms from the right hand sides of (4.30), (4.31), (4.33) to the right hand sides of (4.35), (4.36), (4.38). But it is not immediately obvious that these are isomorphisms, since restriction $C^\infty(T\times U)\to C^\infty(S)$ is neither injective nor surjective – there can be smooth functions on S which do not extend to $T\times U$.

However, S is an open neighbourhood of W in $T\times U$, and the ideals in the denominators of (4.30)–(4.31) include all functions which are zero near W. Hence, any function in $C^\infty(S)$ is equal near W to some other function in $C^\infty(T\times U)$, and so is equal modulo the relevant ideal to something in the image of $C^\infty(T\times U)\to C^\infty(S)$. Hence the natural morphisms from (4.30)–(4.31) to (4.35)–(4.36) are surjective, and the same argument on ideals shows that they are injective. In this way we prove:

$$\mathfrak{C}_W \cong \frac{C^\infty(S)}{\begin{array}{l}\bigl(\pi_T\vert_S^*(i): i\in I_t\bigr) + \bigl(\pi_U\vert_S^*(j): j\in I_u\bigr)\\ +\bigl(c\circ q\circ\pi_U\vert_S - c\circ p\circ\pi_T\vert_S : c\in C^\infty(V)\bigr)\end{array}}\,, \tag{4.35}$$

$$\mathfrak{C}'_W \cong \frac{C^\infty(S)}{\begin{array}{l}\bigl[\bigl(\pi_T\vert_S^*(i): i\in I_t\bigr) + \bigl(\pi_U\vert_S^*(j): j\in I_u\bigr)\\ +\bigl(c\circ q\circ\pi_U\vert_S - c\circ p\circ\pi_T\vert_S : c\in C^\infty(V)\bigr)\bigr]^2\end{array}}\,, \tag{4.36}$$

$$\pi_W \text{ is induced by } \mathrm{id}: C^\infty(S)\to C^\infty(S), \tag{4.37}$$

$$M_W \cong \frac{C^\infty\bigl(\pi_T\vert_S^*(F^*) \oplus \pi_U\vert_S^*(G^*) \oplus (p\circ\pi_T\vert_S)^*(T^*V)\bigr) \otimes_{C^\infty(S)} \mathfrak{C}_W}{C^\infty\bigl((p\circ\pi_T\vert_S)^*(H^*)\bigr) \otimes_{C^\infty(S)} \mathfrak{C}_W}\,, \tag{4.38}$$

$$\begin{aligned}
&\beta_W \circ \dot\Pi = \dot\beta_1 \oplus \dot\beta_2 \oplus \dot\beta_3, \quad \text{where for } a \in C^\infty(S),\ b\in C^\infty(V),\\
&\qquad c\in\mathfrak{C}_W,\ \gamma\in C^\infty(\pi_T\vert_S^*(F^*)) \text{ and } \delta\in C^\infty(\pi_U\vert_S^*(G^*)) \text{ we have}\\
&\dot\beta_1 : \gamma\otimes c \longmapsto c(\gamma\cdot\pi_T\vert_S^*(t) + \text{ideal}),\\
&\dot\beta_2 : \delta\otimes c \longmapsto c(\delta\cdot\pi_U\vert_S^*(u) + \text{ideal}),\\
&\dot\beta_3 : \bigl(a\,(p\circ\pi_T\vert_S)^*(\mathrm{d}b)\bigr)\otimes c \longmapsto c\bigl(a(b\circ q\circ\pi_U\vert_S - b\circ p\circ\pi_T\vert_S) + \text{ideal}\bigr),
\end{aligned} \tag{4.39}$$

for $\dot{\Pi}$ the projection from the numerator of (4.38) to M_W.

Next we use $V \subseteq \mathbb{R}^n$. Consider the term $\bigl(c\circ p\circ\pi_T|_S - c\circ q\circ\pi_U|_S : c \in C^\infty(V)\bigr)$ in (4.35). By assumption in Definition 4.19, the closure of $\bigl\{z \in V : z = p(x) = q(y)$ for $x \in T$, $y \in U$ with $t(x) = u(y) = 0\bigr\}$ in $\mathbb{R}^n$ is a subset of V. Therefore given any $c \in C^\infty(V)$, we can choose $c' \in C^\infty(\mathbb{R}^n)$ such that c and c' agree on a neighbourhood of $\bigl\{z \in V : z = p(x) = q(y)$ for $x \in T$, $y \in U$ with $t(x) = u(y) = 0\bigr\}$ in V. Because of the other two terms in the denominator of (4.35), replacing c by c' has no effect. Hence we may replace $C^\infty(V)$ by $C^\infty(\mathbb{R}^n)$ in this term in (4.35). As $C^\infty(\mathbb{R}^n)$ is generated by the functions $z_1, \ldots, z_n$, it is enough to take $c = z_1,\ldots,c = z_n$. So we see that

$$\mathfrak{C}_W \cong \frac{C^\infty(S)}{\begin{array}{l}\bigl(\pi_T|_S^*(i) : i \in I_t\bigr) + \bigl(\pi_U|_S^*(j) : j \in I_u\bigr)\\ +\bigl(q_i \circ \pi_U|_S - p_i \circ \pi_T|_S : i = 1, \ldots, n\bigr)\end{array}} = C^\infty(S)/I_s, \tag{4.40}$$

$$\mathfrak{C}'_W \cong \frac{C^\infty(S)}{\begin{array}{l}\bigl[\bigl(\pi_T|_S^*(i) : i \in I_t\bigr) + \bigl(\pi_U|_S^*(j) : j \in I_u\bigr)\\ +\bigl(q_i \circ \pi_U|_S - p_i \circ \pi_T|_S : i = 1, \ldots, n\bigr)\bigr]^2\end{array}} = C^\infty(S)/I_s^2, \tag{4.41}$$

by definition of the section s in (4.22). Hence

$$K_W \cong I_s/I_s^2. \tag{4.42}$$

Now T^*V and H are the trivial vector bundles $\mathbb{R}^k \times V \to V$, $\mathbb{R}^k \times V \to V$, so $(p \circ \pi_T|_S)^*(T^*V), (p \circ \pi_T|_S)^*(H^*)$ are the trivial vector bundles $\mathbb{R}^k \times S \to S$ and $\mathbb{R}^k \times S \to S$, and we may rewrite (2.36) as

$$M_W \cong \frac{C^\infty\bigl(\pi_T|_S^*(F^*) \oplus \pi_U|_S^*(G^*) \oplus (\mathbb{R}^n \times S \to S)\bigr) \otimes_{C^\infty(S)} \mathfrak{C}_W}{C^\infty\bigl(\mathbb{R}^k \times S \to S\bigr) \otimes_{C^\infty(S)} \mathfrak{C}_W}. \tag{4.43}$$

We claim that the morphism from the denominator to the numerator of (4.43) which defines the quotient is induced by the morphism $B^*|_S$ of vector bundles on S, for B as in (4.21). To see this, note that our first formula (4.28) for M_W came from Definition 2.35, and MSpec of the morphism from the denominator to the numerator of (2.26) is α_1 in equation (2.59). The first two terms $\underline{e}^*(g'') \circ I_{\underline{e},\underline{g}}(\mathcal{E}_Z)$ and $-\underline{f}^*(h'') \circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z)$ in (2.59) correspond immediately to the first two terms $(\hat{p}_1(x), \ldots, \hat{p}_k(x))$ and $(-\hat{q}_1(y), \ldots, -\hat{q}_k(y))$ in (4.21).

For the third term $(\underline{g} \circ \underline{e})^*(\phi_Z)$ in (2.59), ϕ_Z is the pullback to $\underline{Z} = v^{-1}(0)$ of the vector bundle morphism $\nabla v : H^* \to T^*V$ on V. Since $V \subseteq \mathbb{R}^n$ and $v = (v_1, \ldots, v_k)$, ϕ_Z is induced by the pullback to $\underline{Z}$ of the matrix of functions $\bigl(\frac{\partial v_i}{\partial z_j}(z_1, \ldots, z_n)\bigr)_{i=1,\ldots,k}^{j=1,\ldots,n}$. Hence the morphism from the denominator of (4.43) to the third term in the numerator is induced by the morphism of vector bundles $\tilde{B}_3 : \mathbb{R}^k \to \mathbb{R}^n$ on S given by

$$\tilde{B}_3|_{(x,y)} = \bigl(\tfrac{\partial v_i}{\partial z_j}(p_1(x), \ldots, p_n(x))\bigr)_{i=1,\ldots,k}^{j=1,\ldots,n}. \tag{4.44}$$

The third term of $B^*|_S$ in (4.21) induces the morphism $B_3 : \mathbb{R}^k \to \mathbb{R}^n$ given by

$$B_3|_{(x,y)} = \bigl(A_{ij}(p_1(x), \ldots, p_n(x), q_1(y), \ldots, q_n(y))\bigr)_{i=1,\ldots,k}^{j=1,\ldots,n}. \tag{4.45}$$

Equation (4.20) shows that (4.44) and (4.45) agree at points $(x,y)\in T\times U$ with $q_j(y)=p_j(x)$ for $j=1,\ldots,n$. Thus we may write

$$\tilde{B}_3|_{(x,y)}-B_3|_{(x,y)}=\textstyle\sum_{j=1}^n\bigl(q_j(y)-p_j(x)\bigr)C_j|_{(x,y)}$$

for some vector bundle morphisms $C_1,\ldots,C_n:\mathbb{R}^k\to\mathbb{R}^n$ on S. The terms $q_i\circ\pi_U|_S-p_i\circ\pi_T|_S$ in (4.40) imply that when we apply $\otimes_{C^\infty(S)}\mathfrak{C}_W$ in (4.43), the terms $(q_j(y)-p_j(x))C_j|_{(x,y)}$ become zero, so B_3 and $\tilde{B}_3$ induce the same morphism. So we see from (4.43) that

$$\begin{aligned}M_W&\cong\frac{C^\infty\bigl(\pi_T|_S^*(F^*)\oplus\pi_U|_S^*(G^*)\oplus(\mathbb{R}^n\times S\to S)\bigr)}{B^*\bigl(C^\infty(\mathbb{R}^k\times S\to S)\bigr)}\otimes_{C^\infty(S)}\mathfrak{C}_W\\&\cong C^\infty\left(\frac{\pi_T|_S^*(F^*)\oplus\pi_U|_S^*(G^*)\oplus(\mathbb{R}^n\times S\to S)}{B^*(\mathbb{R}^k\times S\to S)}\right)\otimes_{C^\infty(S)}\mathfrak{C}_W\\&\cong C^\infty(E^*)\otimes_{C^\infty(S)}(C^\infty(S)/I_s)=C^\infty(E^*)/(I_s\cdot C^\infty(E^*)),\end{aligned}\tag{4.46}$$

where as by definition $E\to S$ is the kernel of B, so E^* is the cokernel of B^*.

Since T^*V is the trivial vector bundles $\mathbb{R}^n\times V\to V$, where the basis for $\mathbb{R}^n$ fibres of T^*V is $\mathrm{d}z_1,\ldots,\mathrm{d}z_n$, making the identifications (4.40), (4.42) we can think of $\dot\beta_3$ in (4.39) as mapping $\mathbb{R}^n\otimes(C^\infty(S)/I_s)\to I_s/S_s^2$. Applying (4.39) with $a=a_i\in C^\infty(S)$, $b=z_i$ and $c=1$ for $i=1,\ldots,n$ and using $y_i\circ p=p_i$ and $z_i\circ q=q_i$ shows that in this representation we have

$$\dot\beta_3:(a_1+I_s,\ldots,a_n+I_s)\longmapsto\textstyle\sum_{i=1}^n a_i(q_i\circ\pi_U|_S-p_i\circ\pi_T|_S)+I_s^2.$$

Combining this with the expressions for $\dot\beta_1,\dot\beta_2$ in (4.39) and the definition of s in (4.22) shows that under the identifications (4.42) and (4.46), the morphism $\beta_W:M_W\to K_W$ is identified with

$$s\cdot-:C^\infty(E^*)/(I_s\cdot C^\infty(E^*))\longrightarrow I_s/I_s^2.\tag{4.47}$$

Equations (4.37), (4.40), (4.41), (4.46) and (4.47) now show that $\mathfrak{C}_W,\mathfrak{C}'_W,\pi_W$, M_W,β_W are isomorphic to those defining $\tilde{\boldsymbol{W}}=\boldsymbol{S}_{S,E,s}$ from S,E,s in Definition 3.13, the analogues of (4.24) for $\boldsymbol{X}$. Hence $\boldsymbol{W}$ and $\tilde{\boldsymbol{W}}$ are 1-isomorphic.

Finally, note that

$$\begin{aligned}\operatorname{vdim}\boldsymbol{S}_{S,E,s}&=\dim S-\operatorname{rank}E=(\dim T+\dim U)-(\operatorname{rank}F+\operatorname{rank}G+n-k)\\&=(\dim T-\operatorname{rank}F)+(\dim U-\operatorname{rank}G)-(n-k)\\&=\operatorname{vdim}\boldsymbol{S}_{T,F,t}+\operatorname{vdim}\boldsymbol{S}_{U,G,u}-\operatorname{vdim}\boldsymbol{S}_{V,H,v}.\end{aligned}$$

This proves equation (4.23), and completes the proof of Theorem 4.20. □

As in §2.6, it is well known that if $g:X\to Z$, $h:Y\to Z$ are transverse smooth maps of manifolds then a fibre product $X\times_{g,Z,h}Y$ exists in **Man**. Here is our analogue for d-manifolds.

Theorem 4.21. *Suppose $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are d-manifolds and $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ are d-transverse 1-morphisms, and let $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ be the d-space fibre product. Then $\boldsymbol{W}$ is a d-manifold, with*

$$\operatorname{vdim}\boldsymbol{W}=\operatorname{vdim}\boldsymbol{X}+\operatorname{vdim}\boldsymbol{Y}-\operatorname{vdim}\boldsymbol{Z}. \tag{4.48}$$

Proof. Let $w\in\boldsymbol{W}$. Then $w=(x,y)$ with $x\in\boldsymbol{X}$, $y\in\boldsymbol{Y}$ and $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z\in\boldsymbol{Z}$. Choose a principal d-submanifold $\hat{\boldsymbol{Z}}$ open in $\boldsymbol{Z}$ containing z. Then by Proposition 3.12 there exists an equivalence $\boldsymbol{k}:\hat{\boldsymbol{Z}}\rightarrow\boldsymbol{S}_{V,H,v}$ for some manifold V, vector bundle $H\rightarrow V$ and smooth section $v:V\rightarrow H$, where $\boldsymbol{k}(z)=z_0$ for $z_0\in V$ with $v(z_0)=0$. Choose a neighbourhood V' of z_0 in V such that V' is diffeomorphic to an open set in $\mathbb{R}^n$, and $H|_{V'}$ is isomorphic to a trivial vector bundle $\mathbb{R}^k\times V'\rightarrow V'$. Making $\hat{\boldsymbol{Z}}$ smaller, we may replace V by V', and then take $V\subseteq\mathbb{R}^n$ open, and H to be $\mathbb{R}^k\times V\rightarrow V$.

Now choose principal d-submanifolds $\hat{\boldsymbol{X}}$ open in $\boldsymbol{g}^{-1}(\hat{\boldsymbol{Z}})\subseteq\boldsymbol{X}$ containing x, and $\hat{\boldsymbol{Y}}$ open in $\boldsymbol{h}^{-1}(\hat{\boldsymbol{Z}})\subseteq\boldsymbol{Y}$ containing y. Then there exist equivalences $\boldsymbol{i}:\boldsymbol{S}_{T,F,t}\rightarrow\hat{\boldsymbol{X}}$, $\boldsymbol{j}:\boldsymbol{S}_{U,G,u}\rightarrow\hat{\boldsymbol{Y}}$ for manifolds T,U, vector bundles $F\rightarrow T$, $G\rightarrow U$ and smooth sections $t:T\rightarrow F$, $u:U\rightarrow G$. Apply Theorem 3.34 to the 1-morphisms $\boldsymbol{k}\circ\boldsymbol{g}\circ\boldsymbol{i}:\boldsymbol{S}_{T,F,t}\rightarrow\boldsymbol{S}_{V,H,v}$ and $\boldsymbol{k}\circ\boldsymbol{h}\circ\boldsymbol{j}:\boldsymbol{S}_{U,G,u}\rightarrow\boldsymbol{S}_{V,H,v}$. Replacing T,U by their open subsets $\tilde{T},\tilde{U}$, this gives smooth maps $p:T\rightarrow V$, $q:U\rightarrow V$ and vector bundle morphisms $\hat{p}:F\rightarrow p^*(H)$, $\hat{q}:G\rightarrow q^*(H)$ such that $\boldsymbol{k}\circ\boldsymbol{g}\circ\boldsymbol{i}=\boldsymbol{S}_{p,\hat{p}}$ and $\boldsymbol{k}\circ\boldsymbol{h}\circ\boldsymbol{j}=\boldsymbol{S}_{q,\hat{q}}$. Making $\hat{\boldsymbol{X}},\hat{\boldsymbol{Y}},T,U$ smaller if necessary, we can suppose that the closure of $\bigl\{z\in V: z=p(x)=q(y)$ for $x\in T$, $y\in U$ with $t(x)=u(y)=0\bigr\}$ in $\mathbb{R}^n$ is a subset of V.

Since $\boldsymbol{g}|_{\hat{\boldsymbol{X}}},\boldsymbol{h}|_{\hat{\boldsymbol{Y}}}$ are d-transverse and $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ are equivalences, Proposition 4.17(c) shows that $\boldsymbol{S}_{p,\hat{p}}=\boldsymbol{k}\circ\boldsymbol{g}|_{\hat{\boldsymbol{X}}}\circ\boldsymbol{i}$ and $\boldsymbol{S}_{q,\hat{q}}=\boldsymbol{k}\circ\boldsymbol{h}|_{\hat{\boldsymbol{Y}}}\circ\boldsymbol{j}$ are d-transverse. Hence Theorem 4.21 shows that $\boldsymbol{S}_{T,F,t}\times_{\boldsymbol{S}_{V,H,v}}\boldsymbol{S}_{U,G,u}$ is 1-isomorphic to $\boldsymbol{S}_{S,E,s}$.

Let $\hat{\boldsymbol{W}}$ be the open neighbourhood of w in $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$ equivalent to $\hat{\boldsymbol{X}}\times_{\hat{\boldsymbol{Z}}}\hat{\boldsymbol{Y}}$. Then $\hat{\boldsymbol{W}}$ is equivalent to $\boldsymbol{S}_{S,E,s}$, as $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ are equivalences. So $\hat{\boldsymbol{W}}$ is a principal d-manifold, and (4.23) gives

$$\begin{aligned}\operatorname{vdim}\hat{\boldsymbol{W}}&=\operatorname{vdim}\boldsymbol{S}_{S,E,s}=\operatorname{vdim}\boldsymbol{S}_{T,F,t}+\operatorname{vdim}\boldsymbol{S}_{U,G,u}-\operatorname{vdim}\boldsymbol{S}_{V,H,v}\\&=\operatorname{vdim}\hat{\boldsymbol{X}}+\operatorname{vdim}\hat{\boldsymbol{Y}}-\operatorname{vdim}\hat{\boldsymbol{Z}}=\operatorname{vdim}\boldsymbol{X}+\operatorname{vdim}\boldsymbol{Y}-\operatorname{vdim}\boldsymbol{Z}.\end{aligned}$$

Therefore $\boldsymbol{W}$ can be covered by open principal d-submanifolds $\hat{\boldsymbol{W}}$ of virtual dimension (4.48), and so is a d-manifold of virtual dimension (4.48). □

Part (a) of the next theorem will be central to future applications of d-manifolds. The analogue for derived manifolds is Spivak [95, Th. 8.15], and a partial analogue for Kuranishi spaces is Fukaya et al. [32, §A1.2].

For 1-morphisms of d-manifolds to be d-transverse should be thought of as a significantly weaker condition than for smooth maps of manifolds to be transverse. Thus, fibre products exist more often in **dMan** than they do in **Man**. Theorem 4.22 illustrates this: each of (a),(b) is a fairly weak condition, e.g. for $\boldsymbol{g},\boldsymbol{h}$ to be w-submersions is weaker than for $\boldsymbol{g},\boldsymbol{h}$ to be submersions.

Theorem 4.22. *Suppose $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ are* 1*-morphisms of d-manifolds. The following are sufficient conditions for $\boldsymbol{g},\boldsymbol{h}$ to be d-transverse, so that $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ is a d-manifold of virtual dimension* (4.48):

(a) *$\boldsymbol{Z}$ is a manifold, that is, $\boldsymbol{Z}\in\hat{\mathbf{Man}}$; or*

(b) *$\boldsymbol{g}$ or $\boldsymbol{h}$ is a w-submersion.*

Proof. For (a), since $\boldsymbol{Z}$ lies in $\hat{\mathbf{Man}}$, $\phi_Z:\mathcal{E}_Z\rightarrow\mathcal{F}_Z$ has a left inverse γ by Proposition 3.28. Then $\beta=\bigl(0\,0\,(\underline{g}\circ\underline{e})^*(\gamma)\bigr)$ is a left inverse for α_1 in (4.15), and $\boldsymbol{g},\boldsymbol{h}$ are d-transverse. For (b), if $\boldsymbol{g}$ is a w-submersion then by Definition 4.4(a) there exists $\gamma=(\gamma_1\,\gamma_2):\mathcal{E}_X\oplus\underline{g}^*(\mathcal{F}_Z)\rightarrow\underline{g}^*(\mathcal{E}_Z)$ in $\mathrm{qcoh}(\underline{X})$ with $\gamma\circ(g''\oplus-\underline{g}^*(\phi_Z))=\mathrm{id}_{\underline{g}^*(\mathcal{E}_Z)}$. Define β as in (4.16) by

$$\beta=\bigl(\beta_1\;\beta_2\;\beta_3\bigr)=\bigl(I_{\underline{e},\underline{g}}(\mathcal{E}_Z)^{-1}\circ\underline{e}^*(\gamma_1)\;\;0\;\;-I_{\underline{e},\underline{g}}(\mathcal{E}_Z)^{-1}\circ\underline{e}^*(\gamma_2)\circ I_{\underline{e},\underline{g}}(\mathcal{F}_Z)\bigr).$$

Then β is a left inverse for α_1 in (4.15), and $\boldsymbol{g},\boldsymbol{h}$ are d-transverse. The proof for $\boldsymbol{h}$ a w-submersion is similar. □

If $g:X\rightarrow Z$ and $h:Y\rightarrow Z$ are smooth maps of manifolds and g or h is a submersion then g,h are transverse, so $X\times_{g,Z,h}Y$ exists as a manifold. Theorem 4.22(b), and our next theorem, are different analogues of this for d-manifolds. They justify our definitions of w-submersions and submersions in §4.1. Also, Theorem 4.23 shows that we can think of submersions as *representable* 1*-morphisms* in **dMan**.

Theorem 4.23. *Let $\boldsymbol{X},\boldsymbol{Z}$ be d-manifolds, $\boldsymbol{Y}$ a manifold, and $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ be* 1*-morphisms with $\boldsymbol{g}$ a submersion. Then $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ is a manifold, with* $\dim\boldsymbol{W}=\mathrm{vdim}\,\boldsymbol{X}+\dim\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{Z}$.

Proof. As $\boldsymbol{g}$ is a submersion it is a w-submersion, so $\boldsymbol{W}$ is a d-manifold of virtual dimension (4.48) by Theorem 4.22(b). Write $\boldsymbol{e}:\boldsymbol{W}\rightarrow\boldsymbol{X}$, $\boldsymbol{f}:\boldsymbol{W}\rightarrow\boldsymbol{Y}$ for the projections. Let $w\in\boldsymbol{W}$ with $\boldsymbol{e}(w)=x\in\boldsymbol{X}$ and $\boldsymbol{f}(w)=y$. Since $\boldsymbol{g}$ is a submersion, by Corollary 4.11 we can choose open neighbourhoods $\boldsymbol{T},\boldsymbol{V}$ of x,z in $\boldsymbol{X},\boldsymbol{Z}$ and an equivalence $\boldsymbol{i}:\boldsymbol{T}\rightarrow\boldsymbol{V}\times\boldsymbol{S}$ for some manifold $\boldsymbol{S}$ such that $\boldsymbol{g}|_{\boldsymbol{T}}:\boldsymbol{T}\rightarrow\boldsymbol{V}$ is 2-isomorphic to $\boldsymbol{\pi_V}\circ\boldsymbol{i}$, where $\boldsymbol{\pi}_V:\boldsymbol{V}\times\boldsymbol{S}\rightarrow\boldsymbol{V}$ is the projection. Choose an open neighbourhood $\boldsymbol{U}$ of y in $\boldsymbol{Y}$ with $\boldsymbol{h}(\boldsymbol{U})\subseteq\boldsymbol{V}$. Then $\boldsymbol{T}\times_{\boldsymbol{V}}\boldsymbol{U}$ is an open neighbourhood of w in $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$, with an equivalence $\boldsymbol{T}\times_{\boldsymbol{V}}\boldsymbol{U}\simeq(\boldsymbol{V}\times\boldsymbol{S})\times_{\boldsymbol{\pi}_V,\boldsymbol{V},\boldsymbol{h}|_{\boldsymbol{U}}}\boldsymbol{U}\simeq\boldsymbol{S}\times\boldsymbol{U}$. But $\boldsymbol{S}\times\boldsymbol{U}$ is a manifold as $\boldsymbol{S},\boldsymbol{Y}$ are manifolds, so w has an open neighbourhood $\boldsymbol{T}\times_{\boldsymbol{V}}\boldsymbol{U}$ in $\boldsymbol{W}$ which is a manifold. Since we can cover $\boldsymbol{W}$ by such neighbourhoods, $\boldsymbol{W}$ is a manifold. □

Example 4.24. Let $\boldsymbol{X}=\boldsymbol{Y}=\boldsymbol{*}$, the point, a d-manifold of dimension 0. Let $\boldsymbol{Z}=\boldsymbol{*}\times_{\boldsymbol{0},\mathbb{R},\boldsymbol{0}}\boldsymbol{*}$ be the 'obstructed point' of Example 2.38, with $\mathcal{E}_Z=\mathbb{R}$ and $\mathcal{F}_Z=0$, a d-manifold of dimension -1. There is a unique 1-morphism $\boldsymbol{\pi}:\boldsymbol{*}\rightarrow\boldsymbol{Z}$. Let $\boldsymbol{W}$ be the d-space fibre product $\boldsymbol{X}\times_{\boldsymbol{\pi},\boldsymbol{Z},\boldsymbol{\pi}}\boldsymbol{Y}$. From §2.5 we see that $\boldsymbol{W}=\boldsymbol{*}$, the point, a d-manifold of dimension 0. Hence in this case the d-space fibre product $\boldsymbol{W}$ of d-manifolds $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ is again a d-manifold, but

$$\mathrm{vdim}\,\boldsymbol{W}=0\neq1=0+0-(-1)=\mathrm{vdim}\,\boldsymbol{X}+\mathrm{vdim}\,\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{Z},$$

so that (4.48) does not hold. There is no contradiction, as $\boldsymbol{\pi}, \boldsymbol{\pi}$ are not d-transverse, so Theorem 4.21 does not apply.

Remark 4.25. For the applications the author has in mind, it will be crucial that if $\boldsymbol{g} : \boldsymbol{X} \rightarrow \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \rightarrow \boldsymbol{Z}$ are 1-morphisms with $\boldsymbol{X}, \boldsymbol{Y}$ d-manifolds and $\boldsymbol{Z}$ a manifold then $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ is a d-manifold, with $\operatorname{vdim} \boldsymbol{W} = \operatorname{vdim} \boldsymbol{X} + \operatorname{vdim} \boldsymbol{Y} - \dim \boldsymbol{Z}$, as in Theorem 4.22(a). We will show by example, following Spivak [95, Prop. 1.7], that if d-manifolds **dMan** were an ordinary category containing manifolds as a full subcategory, then this would be false.

Consider the fibre product $* \times_{\boldsymbol{0}, \mathbb{R}, \boldsymbol{0}} *$ in **dMan**, as in Example 2.38. If **dMan** were a category then as $*$ is a terminal object, the category fibre product would be $*$. But then

$$\operatorname{vdim}(* \times_{\boldsymbol{0}, \mathbb{R}, \boldsymbol{0}} *) = \operatorname{vdim} * = 0 \neq -1 = \operatorname{vdim} * + \operatorname{vdim} * - \operatorname{vdim} \mathbb{R},$$

so equation (4.48) and Theorem 4.22(a) would be false.

Thus, if we want fibre products of d-manifolds over manifolds to be well behaved, then **dMan** must be at least a 2-category. It could be an ∞-category, as for Spivak's derived manifolds [94, 95], or some other kind of higher category. Making d-manifolds into a 2-category, as we have done, is the simplest of the available options.

In Theorem 2.42 we showed that a d-space fibre product of manifolds $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g}, \boldsymbol{Z}, \boldsymbol{h}} \boldsymbol{Y}$ is a manifold if and only if g, h are transverse. For some non-transverse $g : X \rightarrow Z$, $h : Y \rightarrow Z$ a category fibre product $W = X \times_{g, Z, h} Y$ may exist in **Man**, but from the point of view of d-manifolds it is the 'wrong' fibre product, and may have the wrong dimension. This happens in our example above, as $* = * \times_{0, \mathbb{R}, 0} *$ is a non-transverse fibre product in **Man**. Example 4.24 illustrates a 2-category analogue of this. If $\boldsymbol{g}, \boldsymbol{h}$ are not d-transverse then a 2-category fibre product $\boldsymbol{X} \times_{\boldsymbol{g}, \boldsymbol{Z}, \boldsymbol{h}} \boldsymbol{Y}$ may or may not exist in **dMan**, and if it does then it may not have the expected dimension, as in Example 4.24.

We can also use fibre products with $\mathbb{R}^{\boldsymbol{n}}$ in **dMan** to locally characterize embeddings and immersions in **dMan**.

Proposition 4.26. *Let $\boldsymbol{X}$ be a d-manifold and $\boldsymbol{g} : \boldsymbol{X} \rightarrow \mathbb{R}^{\boldsymbol{n}}$ a 1-morphism in* **dMan**. *Then the fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g}, \mathbb{R}^{\boldsymbol{n}}, \boldsymbol{0}} *$ exists in* **dMan** *by Theorem* 4.22(a), *and the projection $\boldsymbol{\pi}_{\boldsymbol{X}} : \boldsymbol{W} \rightarrow \boldsymbol{X}$ is an embedding.*

Proof. Taking $\boldsymbol{W}$ to be the explicit fibre product from §2.5, we have $\mathcal{E}_W \cong \underline{\pi}_X^*(\mathcal{E}_X) \oplus (\mathcal{O}_W \otimes_{\mathbb{R}} \mathbb{R}^n)$ and $\mathcal{F}_W \cong \underline{\pi}_X^*(\mathcal{F}_X)$, and these isomorphisms identify $\pi_X'' : \underline{\pi}_X^*(\mathcal{E}_X) \rightarrow \mathcal{E}_W$ with $\mathrm{id}_{\underline{\pi}_X^*(\mathcal{E}_X)} \oplus 0 : \underline{\pi}_X^*(\mathcal{E}_X) \rightarrow \underline{\pi}_X^*(\mathcal{E}_X) \oplus (\mathcal{O}_W \otimes_{\mathbb{R}} \mathbb{R}^n)$, and $\pi_X^2 : \underline{\pi}_X^*(\mathcal{F}_X) \rightarrow \mathcal{F}_W$ with $\mathrm{id}_{\underline{\pi}_X^*(\mathcal{F}_X)}$. It follows easily that $\Omega_{\boldsymbol{\pi}_{\boldsymbol{X}}} = (\pi_X'', \pi_X^2)$ is surjective, so $\boldsymbol{\pi}_{\boldsymbol{X}}$ is an immersion. On topological spaces $\pi_X : W \rightarrow X$ is the inclusion $\{x \in X : g(x) = 0\} \hookrightarrow X$, so π_X is a homeomorphism with its image. Hence $\boldsymbol{\pi}_{\boldsymbol{X}}$ is an embedding. $\square$

Here is a local converse to Proposition 4.26.

Proposition 4.27. *Suppose $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is an immersion of d-manifolds, and $x\in\boldsymbol{X}$ with $\boldsymbol{f}(x)=y\in\boldsymbol{Y}$. Then there exist open d-submanifolds $x\in\boldsymbol{U}\subseteq\boldsymbol{X}$ and $y\in\boldsymbol{V}\subseteq\boldsymbol{Y}$ with $\boldsymbol{f}(\boldsymbol{U})\subseteq\boldsymbol{V}$, and a* 1*-morphism $\boldsymbol{g}:\boldsymbol{V}\to\mathbb{R}^{\boldsymbol{n}}$ with $\boldsymbol{g}(y)=0$, where $n=\mathrm{vdim}\,\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{X}\geqslant 0$, fitting into a* 2*-Cartesian square in* **dMan** :

$$\begin{tikzcd} \boldsymbol{U} \arrow[rr] \arrow[d, "\boldsymbol{f}|_{\boldsymbol{U}}"'] & \arrow[d, phantom, "\Uparrow"] & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{V} \arrow[rr, "\boldsymbol{g}"] & & \mathbb{R}^{\boldsymbol{n}}. \end{tikzcd} \tag{4.49}$$

If $\boldsymbol{f}$ is an embedding we may take $\boldsymbol{U}=\boldsymbol{f}^{-1}(\boldsymbol{V})$.

Proof. Theorem 4.9(d′) shows that we may choose $\boldsymbol{U},\boldsymbol{V}$ with equivalences $\boldsymbol{i}:\boldsymbol{U}\to\boldsymbol{S}_{W,E,s}$, $\boldsymbol{j}:\boldsymbol{S}_{W\times Z,F,t}\to\boldsymbol{V}$ and a 2-morphism $\boldsymbol{f}|_{\boldsymbol{U}}\cong\boldsymbol{j}\circ\boldsymbol{S}_{f,\hat f}\circ\boldsymbol{i}$, for W a manifold, $0\in Z\subseteq\mathbb{R}^n$ open, $f:W\to W\times Z$ mapping $f:w\mapsto(w,0)$, $E\to W$, $F\to W\times Z$ vector bundles with $E=f^*(F)$ and $\hat f=\mathrm{id}_E$, and $s:W\to E$, $t:W\times Z\to F$ smooth sections with $s=f^*(t)$. Using Theorem 4.20 we can show that the following square is 2-Cartesian:

$$\begin{tikzcd} \boldsymbol{S}_{W,E,s} \arrow[rr] \arrow[d, "\boldsymbol{S}_{f,\hat f}"'] & \arrow[d, phantom, "\Uparrow"] & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{S}_{W\times Z,F,t} \arrow[rr, "\boldsymbol{S}_{\pi_Z,0}"] & & \boldsymbol{S}_{\mathbb{R}^n,0,0}=\mathbb{R}^{\boldsymbol{n}}, \end{tikzcd}$$

where $\pi_Z:W\times Z\to Z\subseteq\mathbb{R}^n$ is the projection. Defining $\boldsymbol{g}:\boldsymbol{V}\to\mathbb{R}^{\boldsymbol{n}}$ by $\boldsymbol{g}=\boldsymbol{S}_{\pi_Z,0}\circ\boldsymbol{l}$, where $\boldsymbol{l}:\boldsymbol{V}\to\boldsymbol{S}_{W\times Z,F,t}$ is a quasi-inverse for $\boldsymbol{j}$, implies that (4.49) is 2-Cartesian. The last part follows on making $\boldsymbol{U},\boldsymbol{V}$ smaller. □

4.4 Embedding d-manifolds into manifolds

The following classical facts are due to Whitney [106]. For a detailed discussion see Adachi [1]. Part (b) implies that every m-manifold may be realized as a closed, embedded submanifold of $\mathbb{R}^{2m+1}$.

Theorem 4.28 (Whitney [106])**. (a)** *Let X be an m-manifold and $n\geqslant 2m$. Then a generic smooth map $f:X\to\mathbb{R}^n$ is an immersion.*

(b) *Let X be an m-manifold and $n\geqslant 2m+1$. Then there exists an embedding $f:X\to\mathbb{R}^n$, and we can choose such f with $f(X)$ closed in $\mathbb{R}^n$. Generic smooth maps $f:X\to\mathbb{R}^n$ are embeddings.*

It is also known that there exist embeddings $f:X\to\mathbb{R}^{2m}$ and immersions $f:X\to\mathbb{R}^{2m-\alpha(m)}$, where $\alpha(m)$ is the number of 1's in the binary expansion of m, though just taking f generic is not enough to ensure this, and that these bounds are sharp for some X.

We will generalize Theorem 4.28 to d-manifolds $\boldsymbol{X}$, giving sufficient conditions for existence of immersions and embeddings $\boldsymbol{f}:\boldsymbol{X}\to\mathbb{R}^{\boldsymbol{n}}$. Then we will show that if $\boldsymbol{f}:\boldsymbol{X}\to\mathbb{R}^{\boldsymbol{n}}$ is an embedding then $\boldsymbol{X}$ is equivalent to a 'standard model' d-manifold $\boldsymbol{S}_{V,E,s}$ for V an open neighbourhood of $f(X)$ in $\mathbb{R}^n$. Our results are modelled on Spivak [94, §6.1], [95, Prop. 3.3], who proves analogues of Theorems 4.29 and 4.34 for his derived manifolds.

Theorem 4.29. *Let $\boldsymbol{X}$ be a compact d-manifold. Then there exists an embedding $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{\mathbb{R}^n}$ for some $n\gg 0$.*

Proof. Let $x\in\boldsymbol{X}$. Choose a principal open neighbourhood $\boldsymbol{U}_x$ of x in $\boldsymbol{X}$ and an equivalence $\boldsymbol{i}:\boldsymbol{U}_x\rightarrow\boldsymbol{S}_{V_x,E_x,s_x}$ for some V_x,E_x,s_x. Then $\boldsymbol{i}(x)=v_x\in V_x$ with $s_x(v_x)=0$. As X is paracompact and Hausdorff it is a normal topological space, so we can choose an open neighbourhood U'_x of x in U_x such that the closure $\overline{U'_x}$ of U'_x in X is a subset of U_x. Let $\boldsymbol{U}'_x\subseteq\boldsymbol{U}_x$ be the corresponding open d-submanifold, and choose an open $V'_x\subseteq V_x$ such that $\boldsymbol{i}(\boldsymbol{U}'_x)=\boldsymbol{S}_{V'_x,E'_x,s'_x}\subseteq\boldsymbol{S}_{V_x,E_x,s_x}$, where $E'_x=E_x|_{V'_x}$, $s'_x=s_x|_{V'_x}$.

For some $n_x>\dim V_x$, choose an open neighbourhood V''_x of v_x in V'_x and a smooth map $g_x:V_x\rightarrow\mathbb{R}^{n_x}$ with the properties that $g_x|_{V_x\setminus V'_x}=0$, $g_x|_{V''_x}:V''_x\rightarrow\mathbb{R}^{n_x}$ is an embedding, and $0\notin g_x(V''_x)$, and $g_x(V''_x)\cap g_x(V_x\setminus V''_x)=\emptyset$. This is possible. Set $E''_x=E_x|_{V''_x}$, $s''_x=s_x|_{V''_x}$ and $\boldsymbol{U}''_x=\boldsymbol{i}^{-1}(\boldsymbol{S}_{V''_x,E''_x,s''_x})$, so that $\boldsymbol{U}''_x$ is an open neighbourhood of x in $\boldsymbol{U}'_x\subseteq\boldsymbol{X}$ and $\boldsymbol{i}|_{\boldsymbol{U}''_x}\rightarrow\boldsymbol{S}_{V''_x,E''_x,s''_x}$ is an equivalence.

Consider the 1-morphism $\boldsymbol{S}_{g_x,0}\circ\boldsymbol{i}:\boldsymbol{U}_x\rightarrow\boldsymbol{S}_{\mathbb{R}^{n_x},0,0}=F_{\mathbf{Man}}^{\mathbf{dMan}}(\mathbb{R}^{n_x})=\boldsymbol{\mathbb{R}^{n_x}}$. On $\boldsymbol{U}_x\setminus\overline{\boldsymbol{U}'_x}$ this 1-morphism is identically zero, since $g_x|_{V_x\setminus V'_x}=0$. That is, $\boldsymbol{S}_{g_x,0}\circ\boldsymbol{i}|_{\boldsymbol{U}_x\setminus\overline{\boldsymbol{U}'_x}}=\boldsymbol{0}\circ\boldsymbol{\pi}$, where $\boldsymbol{\pi}:\boldsymbol{U}_x\setminus\overline{\boldsymbol{U}'_x}\rightarrow\boldsymbol{*}$ is the unique morphism and $\boldsymbol{0}:\boldsymbol{*}\rightarrow\boldsymbol{\mathbb{R}^{n_x}}=F_{\mathbf{Man}}^{\mathbf{dSpa}}(0:*\rightarrow\mathbb{R}^{n_x})$. Since $\overline{U'_x}\subseteq U_x$, we may extend $\boldsymbol{S}_{g_x,0}\circ\boldsymbol{i}$ uniquely by zero to all of $\boldsymbol{X}$. That is, there exists a unique 1-morphism $\boldsymbol{f}_x:\boldsymbol{X}\rightarrow\boldsymbol{\mathbb{R}^{n_x}}$ such that $\boldsymbol{f}_x|_{\boldsymbol{U}_x}=\boldsymbol{S}_{g_x,0}\circ\boldsymbol{i}$ and $\boldsymbol{f}_x|_{\boldsymbol{X}\setminus\overline{\boldsymbol{U}'_x}}=\boldsymbol{0}\circ\boldsymbol{\pi}$.

Since $0\notin g_x(V''_x)$ and $g_x(V''_x)\cap g_x(V_x\setminus V''_x)=\emptyset$, we see that $\boldsymbol{f}_x(\boldsymbol{U}''_x)\cap\boldsymbol{f}_x(\boldsymbol{X}\setminus\boldsymbol{U}''_x)=\emptyset$. We claim that $\boldsymbol{f}_x|_{\boldsymbol{U}''_x}:\boldsymbol{U}''_x\rightarrow\boldsymbol{\mathbb{R}^{n_x}}$ is an embedding. To see this, note that $\boldsymbol{f}_x|_{\boldsymbol{U}''_x}=\boldsymbol{S}_{g_x|_{V''_x},0}\circ\boldsymbol{i}|_{\boldsymbol{U}''_x}$, where $\boldsymbol{i}|_{\boldsymbol{U}''_x}:\boldsymbol{U}''_x\rightarrow\boldsymbol{S}_{V''_x,E''_x,s''_x}$ is an equivalence, and $\boldsymbol{S}_{g_x|_{V''_x},0}:\boldsymbol{S}_{V''_x,E''_x,s''_x}\rightarrow\boldsymbol{S}_{\mathbb{R}^{n_x},0,0}=\boldsymbol{\mathbb{R}^{n_x}}$ with $g_x|_{V''_x}:V''_x\rightarrow\mathbb{R}^{n_x}$ an embedding. Equation (4.9) for $\boldsymbol{S}_{g_x|_{V''_x},0}$ is

$$0\longrightarrow T_vV''_x\xrightarrow{\mathrm{d}s_x(v)\oplus\mathrm{d}g_x(v)}E_v\oplus\mathbb{R}^{n_x}\xrightarrow{0\oplus 0}0\longrightarrow 0,$$

which is exact at the second and fourth terms as $\mathrm{d}g_x(v)$ is injective, so $\boldsymbol{S}_{g_x|_{V''_x},0}$ is an immersion by Theorem 4.8(d), and thus an embedding as $g_x|_{V''_x}$ is an embedding and so a homeomorphism with its image. Hence, $\boldsymbol{f}_x|_{\boldsymbol{U}''_x}$ is an embedding by Proposition 4.5(i),(iii).

Choose such $n_x,\boldsymbol{U}''_x,\boldsymbol{f}_x$ for all $x\in X$. Then $\{\boldsymbol{U}''_x:x\in\boldsymbol{X}\}$ is an open cover of $\boldsymbol{X}$, so as $\boldsymbol{X}$ is compact we may choose a finite subcover $\{\boldsymbol{U}''_{x_i}:i=1,\ldots,k\}$. Define $n=n_{x_1}+\cdots+n_{x_k}$, and $\boldsymbol{f}=\boldsymbol{f}_{x_1}\times\cdots\times\boldsymbol{f}_{x_k}:\boldsymbol{X}\rightarrow\boldsymbol{\mathbb{R}^{n_{x_1}}}\times\cdots\boldsymbol{\mathbb{R}^{n_{x_k}}}=\boldsymbol{\mathbb{R}^n}$. We claim $\boldsymbol{f}$ is an embedding. Since $\boldsymbol{f}_{x_i}|_{\boldsymbol{U}''_{x_i}}$ is an embedding, $\boldsymbol{f}|_{\boldsymbol{U}''_{x_i}}$ is an immersion for $i=1,\ldots,k$, so $\boldsymbol{f}$ is an immersion as $\boldsymbol{X}=\boldsymbol{U}''_{x_1}\cup\cdots\cup\boldsymbol{U}''_{x_k}$. Suppose $x\neq y\in X$. Then $x\in U''_{x_i}$ for some $i=1,\ldots,k$. If $y\in U''_{x_i}$ then $f_{x_i}(x)\neq f_{x_i}(y)$ since $f_{x_i}|_{U''_{x_i}}$ is injective as $\boldsymbol{f}_{x_i}|_{\boldsymbol{U}''_{x_i}}$ is an embedding. If $y\notin U''_{x_i}$ then $f_{x_i}(x)\neq f_{x_i}(y)$ as $\boldsymbol{f}_{x_i}(\boldsymbol{U}''_{x_i})\cap\boldsymbol{f}_{x_i}(\boldsymbol{X}\setminus\boldsymbol{U}''_{x_i})=\emptyset$. Hence $f(x)\neq f(y)$, and $f:X\rightarrow\mathbb{R}^n$ is injective. As f is locally an embedding and X is compact, this implies f is a homeomorphism with its image. Hence $\boldsymbol{f}$ is an embedding. $\square$

If $\boldsymbol{X}$ is not compact, there may not exist immersions or embeddings $\boldsymbol{f}: \boldsymbol{X} \to \mathbb{R}^n$, as the next lemma and example show.

Lemma 4.30. *Let $\boldsymbol{X}$ be a d-manifold. A necessary condition for there to exist an immersion or embedding $\boldsymbol{f}: \boldsymbol{X} \to \mathbb{R}^n$ is that $\dim T_x^*\underline{X} \leqslant n$ for all $x \in \underline{X}$.*

Proof. If $\boldsymbol{f}: \boldsymbol{X} \to \mathbb{R}^n$ is an immersion or embedding then from Definitions 4.1 and 4.4, the morphism $\phi_X \oplus f^2 : \mathcal{E}_X \oplus \underline{f}^*(T^*\underline{\mathbb{R}}^n) \to \mathcal{F}_X$ has a right inverse δ. But $\psi_X : \mathcal{F}_X \to T^*\underline{X}$ is the cokernel of ϕ_X by (2.19), and $\psi_X \circ f^2 = \Omega_{\underline{f}} : \underline{f}^*(T^*\underline{\mathbb{R}}^n) \to T^*\underline{X}$. Let $x \in \underline{X}$, and $\underline{x} : \underline{*} \to \underline{X}$ be the corresponding C^∞-scheme morphism. Pulling the above facts back to qcoh($\underline{*}$) by $\underline{x}^*$, which is right exact, and identifying qcoh($\underline{*}$) with the abelian category of real vector spaces, shows that $\phi_X|_x \oplus f^2|_x : \mathcal{E}_X|_x \oplus \underline{\mathbb{R}}^n|_x \longrightarrow \mathcal{F}_X|_x$ has a right inverse $\delta|_x$, and $\psi_X|_x : \mathcal{F}_X|_x \to T_x^*\underline{X}$ is the cokernel of $\phi_X|_x$, and $\psi_X|_x \circ f^2|_x = \Omega_{\underline{f}}|_x$. These imply that $\Omega_{\underline{f}}|_x : T_x^*\underline{X} \to \mathbb{R}^n$ is injective, so $\dim T_x^*\underline{X} \leqslant n$. $\square$

Example 4.31. $\boldsymbol{\mathbb{R}^k} \times_{\boldsymbol{0},\boldsymbol{\mathbb{R}^k},\boldsymbol{0}} \boldsymbol{*}$ is a principal d-manifold of virtual dimension 0, with C^∞-scheme $\underline{\mathbb{R}}^k$, and obstruction bundle $\mathbb{R}^k$. Thus $\boldsymbol{X} = \coprod_{k \geqslant 0} \boldsymbol{\mathbb{R}^k} \times_{\boldsymbol{0},\boldsymbol{\mathbb{R}^k},\boldsymbol{0}} \boldsymbol{*}$ is a d-manifold of virtual dimension 0, with C^∞-scheme $\underline{X} = \coprod_{k \geqslant 0} \underline{\mathbb{R}}^k$. Since $T_x^*\underline{X} \cong \mathbb{R}^n$ for $x \in \mathbb{R}^n \subset \coprod_{k \geqslant 0} \mathbb{R}^k$, $\dim T_x^*\underline{X}$ realizes all values $n \geqslant 0$. Hence there cannot exist immersions or embeddings $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{\mathbb{R}^n}$ for any $n \geqslant 0$.

Here are sufficient lower bounds for n.

Theorem 4.32. *Let $\boldsymbol{X}$ be a compact d-manifold and $n \geqslant 0$. Then immersions $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{\mathbb{R}^n}$ exist provided $n \geqslant 2\dim T_x^*\underline{X}$ for all $x \in \underline{X}$, and embeddings $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{\mathbb{R}^n}$ exist provided $n \geqslant 2\dim T_x^*\underline{X} + 1$ for all $x \in \underline{X}$.*

Proof. By Theorem 4.29 there exists an embedding $\boldsymbol{g}: \boldsymbol{X} \to \boldsymbol{\mathbb{R}^m}$ for some $m \gg 0$. Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map, write $\boldsymbol{L} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(L)$, and set $\boldsymbol{f} = \boldsymbol{L} \circ \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{\mathbb{R}^n}$. We claim that if L is generic in $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ then $\boldsymbol{f}$ is an immersion if $n \geqslant 2\dim T_x^*\underline{X}$ for all $x \in \underline{X}$, and an embedding if $n \geqslant 2\dim T_x^*\underline{X} + 1$ for all $x \in \underline{X}$.

Let $x \in \boldsymbol{X}$. Then by Proposition 3.25 there exists an open neighbourhood $\boldsymbol{U}_x$ of x in $\boldsymbol{X}$ and an equivalence $\boldsymbol{i}: \boldsymbol{S}_{V_x,E_x,s_x} \to \boldsymbol{U}_x$ for some V_x, E_x, s_x with $\boldsymbol{i}(v_x) = x$, where $\dim V_x = \dim T_x^*\underline{X}$ and $v_x \in V_x$ with $s(v_x) = \mathrm{d}s_x(v_x) = 0$. Then $\boldsymbol{g} \circ \boldsymbol{i}$ is an embedding $\boldsymbol{S}_{V_x,E_x,s_x} \to \boldsymbol{S}_{\mathbb{R}^m,0,0} = \boldsymbol{\mathbb{R}^m}$. So by Theorem 3.34, making V_x smaller if necessary there exists a smooth map $p : V_x \to \mathbb{R}^m$ with $\boldsymbol{g} \circ \boldsymbol{i} = \boldsymbol{S}_{p,0}$. As $\boldsymbol{S}_{p,0}$ is an immersion, applying Theorem 4.8(d) at $v = v_x$ and using $\mathrm{d}s_x(v_x) = 0$ shows that $\mathrm{d}p(v_x) : T_{v_x}V_x \to \mathbb{R}^m$ is injective. Thus making $\boldsymbol{U}_x, V_x$ smaller if necessary, we can suppose $p : V_x \to \mathbb{R}^m$ is an embedding.

By dimension counting arguments as in [1, §II.6], [106, §III], as $p : V_x \to \mathbb{R}^m$ is an embedding and $\dim V_x = \dim T_x^*\underline{X}$, for a generic linear $L : \mathbb{R}^m \to \mathbb{R}^n$, if $n \geqslant 2\dim T_x^*\underline{X}$ then $L \circ p : V_x \to \mathbb{R}^n$ is an immersion, and if $n \geqslant 2\dim T_x^*\underline{X} + 1$ then $L \circ p : V_x \to \mathbb{R}^n$ is an embedding. Hence $\boldsymbol{L} \circ \boldsymbol{g} \circ \boldsymbol{i} : \boldsymbol{S}_{V_x,E_x,s_x} \to \boldsymbol{\mathbb{R}^n}$ is an immersion if $n \geqslant 2\dim T_x^*\underline{X}$ and an embedding if $n \geqslant 2\dim T_x^*\underline{X} + 1$ for generic

L, so $\boldsymbol{f}|_{\boldsymbol{U}_x} = \boldsymbol{L} \circ \boldsymbol{g}|_{\boldsymbol{U}_x} : \boldsymbol{U}_x \to \mathbb{R}^{\boldsymbol{n}}$ is an immersion if $n \geqslant 2\dim T^*_x\underline{X}$ and an embedding if $n \geqslant 2\dim T^*_x\underline{X} + 1$ for generic L.

Choose such $\boldsymbol{U}_x$ for each $x \in \boldsymbol{X}$. Then $\{\boldsymbol{U}_x : x \in \boldsymbol{X}\}$ is an open cover of $\boldsymbol{X}$, so as $\boldsymbol{X}$ is compact we may choose a finite subcover $\{\boldsymbol{U}_{x_i} : i = 1, \ldots, k\}$. Suppose $n \geqslant 2\dim T^*_x\underline{X}$ for all $x \in \underline{X}$. Then $\boldsymbol{f}|_{\boldsymbol{U}_{x_i}}$ is an immersion for generic L for each $i = 1, \ldots, k$, so $\boldsymbol{f}$ is an immersion for generic L as $\boldsymbol{X} = \boldsymbol{U}_{x_1} \cup \cdots \cup \boldsymbol{U}_{x_k}$. The embedding case follows in a similar way. $\square$

The assumption that $\boldsymbol{X}$ is compact in Theorems 4.29 and 4.32 can be removed if $\dim T^*_x\underline{X}$ is bounded above for all $x \in \underline{X}$, using the same method as in the classical case [1, §II], [106, §III]. So we may prove:

Theorem 4.33. *Let $\boldsymbol{X}$ be a d-manifold. Then there exist immersions and/or embeddings $\boldsymbol{f} : \boldsymbol{X} \to \mathbb{R}^{\boldsymbol{n}}$ for some $n \gg 0$ if and only if there is an upper bound for $\dim T^*_x\underline{X}$ for all $x \in \underline{X}$. If there is such an upper bound, then immersions $\boldsymbol{f} : \boldsymbol{X} \to \mathbb{R}^{\boldsymbol{n}}$ exist provided $n \geqslant 2\dim T^*_x\underline{X}$ for all $x \in \underline{X}$, and embeddings $\boldsymbol{f} : \boldsymbol{X} \to \mathbb{R}^{\boldsymbol{n}}$ exist provided $n \geqslant 2\dim T^*_x\underline{X} + 1$ for all $x \in \underline{X}$. For embeddings we may also choose $\boldsymbol{f}$ with $f(X)$ closed in $\mathbb{R}^n$.*

If a d-manifold $\boldsymbol{X}$ can be embedded into a manifold Y, we can write $\boldsymbol{X}$ as the zeroes of a section of a vector bundle over Y near its image.

Theorem 4.34. *Suppose $\boldsymbol{X}$ is a d-manifold, Y a manifold, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ an embedding, in the sense of Definition 4.4. Then there exist an open subset V in Y with $\boldsymbol{f}(\boldsymbol{X}) \subseteq \boldsymbol{V}$, a vector bundle $E \to V$, and a smooth section $s : V \to E$ of E fitting into a 2-Cartesian diagram in* **dMan**, *where $0 : V \to E$ is the zero section and $\boldsymbol{Y}, \boldsymbol{V}, \boldsymbol{E}, \boldsymbol{s}, \boldsymbol{0} = F^{\mathbf{dMan}}_{\mathbf{Man}}(Y, V, E, s, 0)$:*

$$\begin{tikzcd} \boldsymbol{X} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{f}"'] \arrow[dr, phantom, "\eta\Uparrow"] & \boldsymbol{V} \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{V} \arrow[r, "\boldsymbol{s}"] & \boldsymbol{E}, \end{tikzcd} \tag{4.50}$$

for some 2-morphism $\eta : \boldsymbol{s} \circ \boldsymbol{f} \Rightarrow \boldsymbol{0} \circ \boldsymbol{f}$. Hence $\boldsymbol{X}$ is equivalent to the 'standard model' d-manifold $\boldsymbol{S}_{V,E,s}$ of Definition 3.13, and is a principal d-manifold.

Proof. As $\boldsymbol{Y} = F^{\mathbf{dMan}}_{\mathbf{Man}}(Y)$ we have $\mathcal{E}_Y = 0$ and $\mathcal{F}_Y = T^*\underline{Y}$. Since $\boldsymbol{f}$ is an embedding, Definitions 4.1(d) and 4.4(d),(f) give morphisms γ, δ in

$$0 \longrightarrow 0 \underset{\gamma=0}{\overset{0}{\rightleftarrows}} \mathcal{E}_X \oplus \underline{f}^*(T^*\underline{Y}) \underset{\delta=\delta^1\oplus\delta^2}{\overset{\phi_X \oplus f^2}{\rightleftarrows}} \mathcal{F}_X \longrightarrow 0. \tag{4.51}$$

Writing $\delta = \delta^1 \oplus \delta^2$, the identities satisfied by γ, δ reduce to $\phi_X \circ \delta^1 + f^2 \circ \delta^2 = \mathrm{id}_{\mathcal{F}_X}$. Proposition 4.3(vi) shows that $\mathcal{H} = \mathrm{Ker}(\phi_X \oplus f^2)$ is a vector bundle on $\underline{X}$ of rank $r := \dim Y - \mathrm{vdim}\,\boldsymbol{X} \geqslant 0$.

On the level of topological spaces, $f(X)$ is a closed subset of Y homeomorphic to X, since $f : X \to Y$ is closed and injective. As $\mathcal{H} \to \underline{X}$ is a vector bundle, $f_*(\mathcal{H})$ is a vector bundle over $f(X)$, so we can choose an open neighbourhood V of $f(X)$ in Y and a vector bundle $E \to V$ with $f^*(E^*) \cong \mathcal{H}$ as vector bundles over X, where E^* is the dual vector bundle to E. This is clear for topological vector bundles over topological spaces, but the argument also works at the level of C^∞-schemes. That is, writing $\underline{V} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(V)$ we have a vector bundle $\mathcal{E}$ over $\underline{V}$ in $\mathrm{qcoh}(\underline{V})$ and an isomorphism $I : \underline{f}^*(\mathcal{E}^*) \to \mathcal{H}$ in $\mathrm{qcoh}(\underline{X})$. As $E \to V$ is a vector bundle over a manifold, (the total space of) E is also a manifold, so $\boldsymbol{E} = F_{\mathbf{Man}}^{\mathbf{dMan}}(E)$ lies in $\hat{\mathbf{Man}} \subset \mathbf{dMan}$.

Let $x \in X$ and $y = f(x) \in V \subseteq Y$. As $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{V}$ is an embedding, Proposition 4.27 gives an open neighbourhood V_x of y in V and a smooth map $g = (g_1, \ldots, g_r) : V_x \to \mathbb{R}^r$, where $r = \dim \boldsymbol{Y} - \mathrm{vdim}\, \boldsymbol{X} = \mathrm{rank}\, E$, such that $\boldsymbol{U}_x = \boldsymbol{f}^{-1}(\boldsymbol{V}_x) \subseteq \boldsymbol{X}$ fits into a 2-Cartesian diagram

$$\begin{tikzcd} \boldsymbol{U}_x \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{f}|_{\boldsymbol{U}_x}"] \arrow[drr, phantom, "\theta_x \Uparrow"] & & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{V}_x \arrow[rr, "\boldsymbol{g}"] & & \boldsymbol{\mathbb{R}}^r. \end{tikzcd} \tag{4.52}$$

As in §2.2, the 1-morphisms $\boldsymbol{g} \circ \boldsymbol{f}|_{\boldsymbol{U}_x}, \boldsymbol{0} \circ \boldsymbol{\pi} : \boldsymbol{U}_x \to \boldsymbol{\mathbb{R}}^r$ induce morphisms

$$\begin{aligned} (g \circ f|_{U_x})^2, (0 \circ \pi)^2 : (\underline{g} \circ \underline{f}|_{\underline{U}_x})^*(T^*\underline{\mathbb{R}}^r) \longrightarrow \mathcal{F}_X|_{\underline{U}_x} \text{ in } \mathrm{qcoh}(\underline{U}_x) \text{ given by} \\ (g \circ f|_{U_x})^2 = f^2|_{\underline{U}_x} \circ \underline{f}|_{\underline{U}_x}^*(\Omega_{\underline{g}}) \circ I_{\underline{f}|_{\underline{U}_x}, \underline{g}}(T^*\underline{\mathbb{R}}^r) \text{ and } (0 \circ \pi)^2 = 0, \end{aligned}$$

using (2.24) and the definition of $F_{\mathbf{Man}}^{\mathbf{dMan}}$. Thus the 2-morphism θ_x in (4.52) is a morphism $\theta_x : (\underline{g} \circ \underline{f}|_{\underline{U}_x})^*(T^*\underline{\mathbb{R}}^r) \to \mathcal{E}_X|_{\underline{U}_x}$ in $\mathrm{qcoh}(\underline{X})$, which by (2.26) satisfies

$$\begin{pmatrix} \phi_X|_{\underline{U}_x} \\ f^2|_{\underline{U}_x} \end{pmatrix} \begin{pmatrix} \underline{f}|_{\underline{U}_x}^*(\Omega_{\underline{g}}) \circ I_{\underline{f}|_{\underline{U}_x}, \underline{g}}(T^*\underline{\mathbb{R}}^r) & \theta_x \end{pmatrix} = 0. \tag{4.53}$$

As $\mathcal{H} = \mathrm{Ker}(\phi_X \oplus f^2)$, equation (4.53) implies there is a unique morphism

$$\begin{aligned} \Phi_x : (\underline{g} \circ \underline{f}|_{\underline{U}_x})^*(T^*\underline{\mathbb{R}}^r) \longrightarrow \mathcal{H}|_{\underline{U}_x} \qquad \text{with} \\ i_{\mathcal{H}}|_{\underline{U}_x} \circ \Phi_x = \begin{pmatrix} \underline{f}|_{\underline{U}_x}^*(\Omega_{\underline{g}}) \circ I_{\underline{f}|_{\underline{U}_x}, \underline{g}}(T^*\underline{\mathbb{R}}^r) & \theta_x \end{pmatrix}, \end{aligned}$$

where $i_{\mathcal{H}} : \mathcal{H} \to \mathcal{E}_X \oplus \underline{f}^*(T^*\underline{Y})$ is the inclusion morphism. Since (4.52) is 2-Cartesian it induces an equivalence $\boldsymbol{b} : \boldsymbol{U}_x \to \boldsymbol{V}_x \times_{\boldsymbol{\mathbb{R}}^r} \boldsymbol{*}$, where $\boldsymbol{V}_x \times_{\boldsymbol{\mathbb{R}}^r} \boldsymbol{*}$ is defined by the explicit construction of §2.5. Equation (2.34) for $\boldsymbol{b}$ is

$$0 \to \underline{f}|_{\underline{U}_x}^*(\underline{g}^*(T^*\underline{\mathbb{R}}^r)) \xrightarrow{-\theta_x \circ I_{\underline{f}|_{\underline{U}_x}, \underline{g}}(T^*\underline{\mathbb{R}}^r)^{-1} \oplus -\underline{f}|_{\underline{U}_x}^*(\Omega_{\underline{g}})} \mathcal{E}_X|_{\underline{U}_x} \oplus \underline{f}|_{\underline{U}_x}^*(T^*\underline{V}_x) \xrightarrow{\phi_X|_{\underline{U}_x} \oplus f^2|_{\underline{U}_x}} \mathcal{F}_X|_{\underline{U}_x} \to 0.$$

Since $\boldsymbol{b}$ is an equivalence, this equation is split exact by Proposition 2.20. This implies that Φ_x is an isomorphism. Making V_x smaller if necessary, we can choose an isomorphism $\Psi_x : \mathbb{R}^r \cong g^*(T^*\mathbb{R}^r) \to E_x^*$ of vector bundles over V_x, where $E_x = E|_{V_x}$, such that $I \circ \underline{f}|_{\underline{U}_x}^*(\underline{\Psi}_x) = \Phi_x$, where $\underline{\Psi}_x$ is the lift of Ψ_x to vector bundles on the C^∞-scheme $\underline{V}_x$, and $I : \underline{f}^*(\mathcal{E}^*) \to \mathcal{H}$ is as above.

Now define $s_x \in C^\infty(E_x)$ by $s_x = (\Psi_x^{-1})^*(g_1, \ldots, g_r)$. The isomorphism Φ_x^* identifies the triples $V_x, \mathbb{R}^r \times V_x \to V_x, g$ and V_x, E_x, s_x. Hence we have a 1-isomorphism $\boldsymbol{S}_{V_x,\mathbb{R}^r,g} \cong \boldsymbol{S}_{V_X,E_x,s_x}$. Since the 2-Cartesian diagram (4.52) implies that $\boldsymbol{U}_x$ is equivalent to $\boldsymbol{S}_{V_x,\mathbb{R}^r,g}$, it is also equivalent to $\boldsymbol{S}_{V_X,E_x,s_x}$. Thus, we have a 2-Cartesian diagram

$$\begin{tikzcd} \boldsymbol{U}_x \arrow[rr, "\boldsymbol{f}|_{\boldsymbol{U}_x}"] \arrow[d, "\boldsymbol{f}|_{\boldsymbol{U}_x}"'] & \arrow[d, phantom, "\eta_x\Uparrow"] & \boldsymbol{V}_x \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{V}_x \arrow[rr, "\boldsymbol{s}_x"] & & \boldsymbol{E}_x. \end{tikzcd} \tag{4.54}$$

Furthermore, this choice of section s_x has an extra property: from the data in (4.54), following the construction of Φ_x above, we can construct a natural isomorphism $\Xi_x : \underline{f}|_{\underline{U}_x}^*(\mathcal{E}^*) \to \mathcal{H}|_{\underline{U}_x}$, and this Ξ_x is exactly $I|_{\underline{U}_x}$.

Here the cotangent bundle T^*E_x at the zero section has a natural isomorphism $T^*E_x|_{0(V_x)} \cong T^*V_x \oplus E_x^*$, so we may identify

$$(\underline{0} \circ \underline{f}|_{\underline{X}_x})^*(T^*\underline{E}_x) \cong \big(\underline{f}^*(T^*\underline{V}) \oplus \underline{f}^*(\mathcal{E})\big)|_{\underline{X}_x}.$$

With this identification, η_x is a morphism

$$\begin{aligned} \eta_x = \eta_x^1 \oplus \eta_x^2 : \big(\underline{f}^*(T^*\underline{V}) \oplus \underline{f}^*(\mathcal{E})\big)|_{\underline{U}_x} \longrightarrow \mathcal{E}_X|_{\underline{U}_x} \quad \text{satisfying} \\ \phi_X|_{\underline{U}_x} \circ \eta_x^1 = 0 \quad \text{and} \quad \phi_X|_{\underline{U}_x} \circ \eta_x^2 = -f^2|_{\underline{U}_x} \circ \underline{f}|_{\underline{U}_x}^*(\mathrm{d}s_x). \end{aligned} \tag{4.55}$$

Now consider the extent to which s_x is unique. If $\tilde{V}_x, \tilde{s}_x$ are alternative choices of V_x, s_x, then we can compare $s_x, \tilde{s}_x$ on $V_x \cap \tilde{V}_x$. The fact that (4.54) is 2-Cartesian for both V_x, s_x and $\tilde{V}_x, \tilde{s}_x$ implies that $s_x = 0$ and $\tilde{s}_x = 0$ define the same C^∞-subscheme of $\underline{V}_x \cap \underline{\tilde{V}}_x$. Therefore locally there exists an automorphism A of the vector bundle $E|_{V_x \cap \tilde{V}_x}$ such that $\tilde{s}_x = A \cdot s_x$. The extra property of s_x above, in which (4.54) induces the isomorphism $\Xi_x = I|_{\underline{U}_x}$, implies that the pullback of A to $\underline{U}_x \cap \underline{U}_x$ is the identity. Since $\underline{U}_x \cap \underline{U}_x$ is isomorphic the C^∞-subscheme $s_x = 0$ in $\underline{V}_x \cap \underline{\tilde{V}}_x$, this implies that $A = \mathrm{id}_E + O(s_x)$. Hence

$$\tilde{s}_x = s_x + O(s_x^2). \tag{4.56}$$

Choose such data V_x, s_x and 2-morphism η_x in (4.54) for each $x \in X$. As each V_x is open in V, replacing V by the open subset $\bigcup_{x\in X} V_x$, the $\{V_x : x \in X\}$ is an open cover of the manifold V. Therefore we can choose a partition of unity $\{\chi_x : x \in X\}$, so that $\chi_x : V \to [0,1]$ is smooth and supported in V_x, and $\sum_{x\in X} \chi_x = 1$, where the sum is locally finite and so makes sense. Define a smooth section s of $E \to V$ by $s = \sum_{x\in S} \chi_x \cdot s_x$, and a morphism $\eta : \underline{f}^*(T^*\underline{V}) \oplus (\underline{0} \circ \underline{f})^*(T^*\underline{E}) \longrightarrow \mathcal{E}_X$ in $\mathrm{qcoh}(\underline{X})$ by $\eta = \sum_{x\in X} \chi_x \cdot \eta_x$.

For $x, \tilde{x} \in X$, equation (4.56) shows that $s_{\tilde{x}} = s_x + O(s_x^2)$ on $V_x \cap V_{\tilde{x}}$. Hence by Definition 3.29 there exists $A_{x,\tilde{x}} \in C^\infty(E \otimes E^* \otimes E^*|_{V_x \cap V_{\tilde{x}}})$ such that $s_{\tilde{x}} = s_x + A_{x,\tilde{x}} \cdot (s_x \otimes s_x)$. Multiplying by $\chi_{\tilde{x}}$ and summing over $\tilde{x} \in X$ gives

$$s|_{V_x} = s_x + \big(\textstyle\sum_{\tilde{x}\in X} \chi_{\tilde{x}} \cdot A_{x,\tilde{x}}\big) \cdot (s_x \otimes s_x) = \big(\mathrm{id}_{E_x} + \big(\textstyle\sum_{\tilde{x}\in X} \chi_{\tilde{x}} \cdot A_{x,\tilde{x}}\big) \cdot s_x\big) \cdot s_x.$$

Making V and each V_x smaller if necessary, we can suppose that the morphism $\mathrm{id}_{E_x} + \big(\sum_{\tilde{x}\in X} \chi_{\tilde{x}} \cdot A_{x,\tilde{x}}\big) \cdot s_x : E_x \to E_x$ is invertible on V_x for each $x \in X$.

We now claim that with these choices of V,E,s,η, equation (4.50) is a 2-Cartesian diagram. To see this, first note that $\underline{s}_x\circ\underline{f}|_{\underline{U}_x}=\underline{0}\circ\underline{f}|_{\underline{U}_x}:\underline{U}_x\to\underline{E}$ for each $s\in S$ and $s=\sum_{x\in X}\chi_x\cdot s_x$ imply that $\underline{s}\circ\underline{f}=\underline{0}\circ\underline{f}:\underline{X}\to\underline{E}$. So (4.50) commutes at the level of C^∞-schemes. For it to 2-commute we need to prove the analogue of (4.55) on $\underline{X}$, with η_x,s_x replaced by η,s. Now

$$\underline{f}|_{\underline{U}_x}^*\bigl(\mathrm{d}(\chi_x\cdot s_x)\bigr)=\underline{f}|_{\underline{U}_x}^*\bigl((\mathrm{d}\chi_x)\otimes s_x+\chi_x\cdot\mathrm{d}s_x\bigr)=\underline{f}|_{\underline{U}_x}^*\bigl(\chi_x\cdot\mathrm{d}s_x\bigr),$$

since $\underline{f}|_{\underline{U}_x}^*(s_x)=0$ as $\underline{U}_x$ is isomorphic to the C^∞-subscheme $s_x=0$ in $\underline{V}_x$, with inclusion morphism $\underline{f}|_{\underline{U}_x}:\underline{U}_x\to\underline{V}_x$. Therefore (4.55) for $\eta,s,\underline{X}$ follows by multiplying (4.55) by χ_x and summing over $x\in X$. Hence (4.50) 2-commutes.

A possible fibre product $\boldsymbol{V}\times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}}\boldsymbol{V}$ is $\boldsymbol{S}_{V,E,s}$, with projection 1-morphism $\boldsymbol{i}=\boldsymbol{S}_{\mathrm{id}_V,0}:\boldsymbol{S}_{V,E,s}\to\boldsymbol{S}_{V,0,0}=\boldsymbol{V}$ to the first factor of $\boldsymbol{V}$. Since (4.50) 2-commutes there exists a 1-morphism $\boldsymbol{j}:\boldsymbol{X}\to\boldsymbol{S}_{V,E,s}$ with $\boldsymbol{f}$ 2-isomorphic to $\boldsymbol{i}\circ\boldsymbol{j}$. We must show $\boldsymbol{j}$ is an equivalence. As for (4.51) we have a complex

$$0\longrightarrow\underline{j}^*\circ\underline{i}^*(\mathcal{E}^*)\xrightarrow{j^1\oplus-\underline{j}^*\circ\underline{i}^*(\mathrm{d}s)}\mathcal{E}_X\oplus\underline{j}^*\circ\underline{i}^*(T^*\underline{V})\xrightarrow{\phi_X\oplus j^2}\mathcal{F}_X\longrightarrow 0\qquad(4.57)$$

in qcoh$(\underline{X})$. Since $\boldsymbol{f}$ is 2-isomorphic to $\boldsymbol{i}\circ\boldsymbol{j}$, so $\underline{f}=\underline{i}\circ\underline{j}$, the morphism $\phi_X\oplus j^2$ in (4.57) is naturally identified with $\phi_X\oplus f^2$ in (4.51). Thus $\phi_X\oplus j^2$ has a right inverse identified with δ, and $\mathrm{Ker}(\phi_X\oplus j^2)$ is naturally isomorphic to $\mathcal{H}$.

Hence (4.57) induces a morphism $\underline{f}^*(\mathcal{E}^*)\cong\underline{j}^*\circ\underline{i}^*(\mathcal{E}^*)\to\mathcal{H}$. Using $s=\sum_{x\in S}\chi_x\cdot s_x$ we can show that this morphism $\underline{f}^*(\mathcal{E}^*)\to\mathcal{H}$ is $\sum_{x\in S}\chi_x\cdot\Xi_x$, where $\Xi_x:\underline{f}^*(\mathcal{E}^*)|_{\underline{U}_x}\to\mathcal{H}|_{\underline{U}_x}$ is the (iso)morphism induced by (4.54). But $\Xi_x=I|_{\underline{U}_x}$ by choice of s_x. Thus the morphism $\underline{f}^*(\mathcal{E}^*)\to\mathcal{H}$ induced by (4.57) is I, and is an isomorphism.

We have now proved that in (4.57), $\phi_X\oplus j^2$ has a right inverse, and $j^1\oplus-\underline{j}^*\circ\underline{i}^*(\mathrm{d}s)$ is an isomorphism with $\mathrm{Ker}(\phi_X\oplus j^2)\cong\mathcal{H}$. Hence (4.57) is a split short exact sequence. Also $f:X\to V$ is injective as f is an embedding, and $s^{-1}(0)=f(X)$ by construction, so $j:X\to s^{-1}(0)=S_{V,E,s}$ is a bijection. Therefore $\boldsymbol{j}:\boldsymbol{X}\to\boldsymbol{S}_{V,E,s}$ is an equivalence by Theorem 3.36, and (4.50) is 2-Cartesian. This completes the proof of Theorem 4.34. □

Combining Theorems 4.33 and 4.34, and noting by Proposition 3.12 that if $\boldsymbol{X}$ is a principal d-manifold then $\boldsymbol{X}\simeq\boldsymbol{V}\times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}}\boldsymbol{V}$ with $\boldsymbol{\pi_V}:\boldsymbol{X}\to\boldsymbol{V}$ an embedding of $\boldsymbol{X}$ in a manifold $\boldsymbol{V}$, yields:

Corollary 4.35. *Let $\boldsymbol{X}$ be a d-manifold. Then $\boldsymbol{X}$ is a principal d-manifold if and only if* $\dim T_x^*\underline{X}$ *is bounded above for all $x\in\underline{X}$.*

Here are some sufficient conditions for a d-manifold to be principal:

Corollary 4.36. *Let $\boldsymbol{W}$ be a d-manifold. Then $\boldsymbol{W}$ is a principal d-manifold if any of the following hold:* **(i)** *$\boldsymbol{W}$ is compact;*

(ii) *$\boldsymbol{W}$ can be covered by finitely many principal open d-submanifolds; and*

(iii) $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$*, where* $\boldsymbol{Z}$ *is a d-manifold and* $\boldsymbol{X}, \boldsymbol{Y}$ *are principal d-manifolds.*

Proof. Part (i) follows from Theorems 4.29 and 4.34. For (ii), if $\boldsymbol{W}$ is covered by principal open $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_n$, then an upper bound C_i exists for $\dim T^*_w \underline{W}$ for $w \in \underline{X}_i$, each $i = 1, \ldots, n$, and $\max(C_1, \ldots, C_n)$ is an upper bound for $\dim T^*_w \underline{W}$ for $w \in \underline{W}$. For (iii), if $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ are the projections and $w \in \boldsymbol{W}$ with $\boldsymbol{e}(w) = x$, $\boldsymbol{f}(w) = y$, then as $\underline{W} \cong \underline{X} \times_{\underline{Z}} \underline{Y}$ we have $\dim T^*_w \underline{W} \leqslant \dim T^*_x \underline{W} + \dim T^*_y \underline{Y}$. But $\dim T^*_x \underline{X}, \dim T^*_y \underline{Y}$ are bounded above as $\boldsymbol{X}, \boldsymbol{Y}$ are principal, so $\dim T^*_w \underline{W}$ is bounded above. $\square$

Corollary 4.36 suggests that most interesting d-manifolds are principal. Example 4.31 gives a d-manifold which is not principal.

4.5 Orientation line bundles of virtual vector bundles

Let $\underline{X}$ be a C^∞-scheme. Definition 3.1 defined the 2-category $\mathrm{vvect}(\underline{X})$ of *virtual vector bundles* $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ on $\underline{X}$. This section will define a line bundle $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ on $\underline{X}$ which we call the *orientation line bundle* of $(\mathcal{E}^\bullet, \phi)$, and explore its basic properties. Section 4.6 will use this to define orientations on d-manifolds $\boldsymbol{X}$, as an orientation (in the sense of Definition B.40) on the orientation line bundle $\mathcal{L}_{T^*\boldsymbol{X}}$ of the virtual cotangent bundle $T^*\boldsymbol{X}$ of $\boldsymbol{X}$.

The classical analogue of our orientation line bundles is that if $E \to X$ is a vector bundle of rank k, then the orientation line bundle of E is the top exterior power $\Lambda^k E$. For a virtual vector bundle $(\mathcal{E}^\bullet, \phi)$ with $\mathcal{E}^1, \mathcal{E}^2$ vector bundles of ranks k_1, k_2, we have $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)} \cong \Lambda^{k_1}(\mathcal{E}^1)^* \otimes \Lambda^{k_2}\mathcal{E}^2$. But when $\mathcal{E}^1, \mathcal{E}^2$ are general quasicoherent sheaves on $\underline{X}$, the definition of $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ is complex, and needs some preparation in the next definition and proposition.

Definition 4.37. Let $\underline{X}$ be a C^∞-scheme, $(\mathcal{E}^\bullet, \phi)$ a virtual vector bundle on $\underline{X}$ of rank $d \in \mathbb{Z}$, and $\underline{U} = (U, \mathcal{O}_U)$ an open C^∞-subscheme in $\underline{X}$. Consider 9-tuples $\boldsymbol{\alpha} = (k_1, k_2, \alpha_1, \ldots, \alpha_7)$, where $k_1, k_2 \geqslant 0$ satisfy $k_2 - k_1 = d$, and $\alpha_1, \ldots, \alpha_7$ are morphisms in $\mathrm{qcoh}(\underline{U})$ as in the following diagram

$$\begin{tikzcd} \mathbb{R}^{k_1} \otimes_{\mathbb{R}} \mathcal{O}_U \arrow[r, shift left, "\alpha_1"] \arrow[d, shift right, "\alpha_3"'] & \mathbb{R}^{k_2} \otimes_{\mathbb{R}} \mathcal{O}_U \arrow[l, shift left, "\alpha_2"] \arrow[d, shift right, "\alpha_5"'] \\ \mathcal{E}^1|_{\underline{U}} \arrow[r, shift left, "\phi|_{\underline{U}}"] \arrow[u, shift right, "\alpha_4"'] & \mathcal{E}^2|_{\underline{U}}, \arrow[l, shift left, "\alpha_7"] \arrow[u, shift right, "\alpha_6"'] \end{tikzcd} \tag{4.58}$$

satisfying the relations

$$\begin{aligned} \alpha_2 \circ \alpha_1 + \alpha_4 \circ \alpha_3 &= \mathrm{id}_{\mathbb{R}^{k_1} \otimes_{\mathbb{R}} \mathcal{O}_U}, & \alpha_1 \circ \alpha_2 + \alpha_6 \circ \alpha_5 &= \mathrm{id}_{\mathbb{R}^{k_2} \otimes_{\mathbb{R}} \mathcal{O}_U}, \\ \alpha_3 \circ \alpha_4 + \alpha_7 \circ \phi|_{\underline{U}} &= \mathrm{id}_{\mathcal{E}^1|_{\underline{U}}}, & \alpha_5 \circ \alpha_6 + \phi|_{\underline{U}} \circ \alpha_7 &= \mathrm{id}_{\mathcal{E}^2|_{\underline{U}}}, \\ \alpha_5 \circ \alpha_1 &= \phi|_{\underline{U}} \circ \alpha_3, & \alpha_1 \circ \alpha_4 &= \alpha_6 \circ \phi|_{\underline{U}}, \\ \alpha_3 \circ \alpha_2 &= \alpha_7 \circ \alpha_5, & \alpha_2 \circ \alpha_6 &= \alpha_4 \circ \alpha_7. \end{aligned} \tag{4.59}$$

We will call such $\boldsymbol{\alpha}$ *orientation generators for* $(\mathcal{E}^\bullet,\phi)$ *on* $\underline{U}$.

Suppose $\boldsymbol{\alpha}=(k_1,k_2,\alpha_1,\ldots,\alpha_7)$ and $\boldsymbol{\beta}=(l_1,l_2,\beta_1,\ldots,\beta_7)$ are orientation generators. Then $k_2-k_1=d=l_2-l_1$, so $k_1+l_2=l_1+k_2$. Define

$$\begin{aligned} M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta}):\mathbb{R}^{k_1}\otimes_{\mathbb{R}}\mathcal{O}_U\oplus\mathbb{R}^{l_2}\otimes_{\mathbb{R}}\mathcal{O}_U &\longrightarrow \mathbb{R}^{k_2}\otimes_{\mathbb{R}}\mathcal{O}_U\oplus\mathbb{R}^{l_1}\otimes_{\mathbb{R}}\mathcal{O}_U \\ \text{by}\quad M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta}) &= \begin{pmatrix} \alpha_1 & -\alpha_6\circ\beta_5 \\ \beta_4\circ\alpha_3 & \beta_2+\beta_4\circ\alpha_7\circ\beta_5\end{pmatrix}. \end{aligned}\tag{4.60}$$

As $k_1+l_2=k_2+l_1$, we can regard $M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ as a $(k_1+l_2)\times(k_1+l_2)$ matrix with entries in $\mathrm{Hom}(\mathcal{O}_U,\mathcal{O}_U)$, which is the C^∞-ring $\mathcal{O}_X(U)$. Thus the determinant $\det M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ makes sense, as an element of $\mathcal{O}_X(U)$. Define

$$F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})=(-1)^{(1+l_1)d}\det M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{O}_X(U).\tag{4.61}$$

Now let $\underline{V}\subseteq\underline{U}$ be an open C^∞-subscheme. Then $\mathcal{O}_U|_V=\mathcal{O}_V$, and we have a restriction map $\rho_{UV}:\mathcal{O}_X(U)\to\mathcal{O}_X(V)$. If $\boldsymbol{\alpha},\boldsymbol{\beta}$ are orientation generators for $(\mathcal{E}^\bullet,\phi)$ on $\underline{U}$, then $\boldsymbol{\alpha}|_{\underline{V}}=(k_1,k_2,\alpha_1|_{\underline{V}},\ldots,\alpha_7|_{\underline{V}})$ and $\boldsymbol{\beta}|_{\underline{V}}=(l_1,l_2,\beta_1|_{\underline{V}},\ldots,\beta_7|_{\underline{V}})$ are orientation generators for $(\mathcal{E}^\bullet,\phi)$ on $\underline{V}$. Equations (4.58)–(4.61) restrict from $\underline{U}$ to $\underline{V}$. It easily follows that

$$\rho_{UV}\bigl(F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})\bigr)=F_{\underline{V}}\bigl(\boldsymbol{\alpha}|_{\underline{V}},\boldsymbol{\beta}|_{\underline{V}}\bigr)\quad\text{in }\mathcal{O}_X(V).\tag{4.62}$$

Here are some properties of these definitions:

Proposition 4.38. *In the situation of Definition* 4.37:

(a) *If* $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ *are orientation generators for* $(\mathcal{E}^\bullet,\phi)$ *on* $\underline{U}$ *then*

$$F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\alpha})=1,\tag{4.63}$$

$$F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})\cdot F_{\underline{U}}(\boldsymbol{\beta},\boldsymbol{\alpha})=1,\quad\textit{and}\tag{4.64}$$

$$F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\gamma})=F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})\cdot F_{\underline{U}}(\boldsymbol{\beta},\boldsymbol{\gamma}).\tag{4.65}$$

(b) *Any* $x\in\underline{X}$ *has an open neighbourhood* $\underline{U}$ *for which there exists an orientation generator* $\boldsymbol{\alpha}$ *for* $(\mathcal{E}^\bullet,\phi)$ *on* $\underline{U}$.

Proof. For (a), write $\boldsymbol{\alpha}=(k_1,k_2,\alpha_1,\ldots,\alpha_7)$, $\boldsymbol{\beta}=(l_1,l_2,\beta_1,\ldots,\beta_7)$ and $\boldsymbol{\gamma}=(m_1,m_2,\gamma_1,\ldots,\gamma_7)$. For (4.63), by equations (4.59) and (4.60) we have

$$\begin{aligned} M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\alpha}) &= \begin{pmatrix} \alpha_1 & -\alpha_6\circ\alpha_5 \\ \alpha_4\circ\alpha_3 & \alpha_2+\alpha_4\circ\alpha_7\circ\alpha_5\end{pmatrix} = \begin{pmatrix} \alpha_1 & -I_{k_2}+\alpha_1\circ\alpha_2 \\ I_{k_1}-\alpha_2\circ\alpha_1 & 2\alpha_2-\alpha_2\circ\alpha_1\circ\alpha_2\end{pmatrix} \\ &= \begin{pmatrix} I_{k_2} & 0 \\ -\alpha_2 & I_{k_1}\end{pmatrix}\begin{pmatrix} \alpha_1 & -I_{k_2} \\ I_{k_1} & 0\end{pmatrix}\begin{pmatrix} I_{k_1} & \alpha_2 \\ 0 & I_{k_2}\end{pmatrix}, \end{aligned}$$

where I_k is the $k\times k$ identity matrix in $\mathcal{O}_X(U)$. The three matrices on the second line have determinants $1,(-1)^{k_2+k_1k_2},1$. Hence

$$F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\alpha})=(-1)^{(1+k_1)d}\det M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\alpha})=(-1)^{(1+k_1)(k_2-k_1)}\cdot(-1)^{k_2+k_1k_2}=1.$$

For (4.65), using (4.59) for $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ we can show by a long calculation that

$$
\begin{gathered}
\begin{pmatrix} \alpha_1 & -\alpha_6\beta_5 & 0 \\ \beta_4\alpha_3 & \beta_2+\beta_4\alpha_7\beta_5 & 0 \\ 0 & 0 & I_{m_1} \end{pmatrix} \begin{pmatrix} I_{k_1} & 0 & 0 \\ 0 & \beta_1 & -\beta_6\gamma_5 \\ 0 & \gamma_4\beta_3 & \gamma_2+\gamma_4\beta_7\gamma_5 \end{pmatrix} \\
\begin{pmatrix} I_{k_1} & \alpha_4\beta_3 & \alpha_4(\alpha_7+\beta_7-\beta_7\phi\alpha_7)\gamma_5 \\ 0 & I_{l_1} & \beta_4\alpha_7\gamma_5 \\ 0 & 0 & I_{m_2} \end{pmatrix} \begin{pmatrix} -I_{k_1} & 0 & 0 \\ \beta_4\alpha_3 & I_{l_1} & 0 \\ 0 & 0 & I_{m_2} \end{pmatrix} = \\
\begin{pmatrix} -I_{k_2} & 0 & 0 \\ 0 & I_{l_1} & 0 \\ -\gamma_4\beta_7\alpha_5 & \gamma_4\beta_3 & I_{m_1} \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 & -\alpha_6\gamma_5 \\ 0 & I_{l_1} & 0 \\ \gamma_4\alpha_3 & 0 & \gamma_2+\gamma_4\alpha_7\gamma_5 \end{pmatrix}.
\end{gathered}
\tag{4.66}
$$

Here the four matrices on the left hand side are morphisms

$$
\begin{matrix}\mathbb{R}^{k_2}\oplus \\ \mathbb{R}^{l_1}\oplus \\ \mathbb{R}^{m_1}\end{matrix} \longleftarrow \begin{matrix}\mathbb{R}^{k_1}\oplus \\ \mathbb{R}^{l_2}\oplus \\ \mathbb{R}^{m_1}\end{matrix} \longleftarrow \begin{matrix}\mathbb{R}^{k_1}\oplus \\ \mathbb{R}^{l_1}\oplus \\ \mathbb{R}^{m_2}\end{matrix} \longleftarrow \begin{matrix}\mathbb{R}^{k_1}\oplus \\ \mathbb{R}^{l_1}\oplus \\ \mathbb{R}^{m_2}\end{matrix} \longleftarrow \begin{matrix}\mathbb{R}^{k_1}\oplus \\ \mathbb{R}^{l_1}\oplus \\ \mathbb{R}^{m_2},\end{matrix}
$$

and the two matrices on the right hand side are morphisms

$$
\begin{matrix}\mathbb{R}^{k_2}\oplus \\ \mathbb{R}^{l_1}\oplus \\ \mathbb{R}^{m_1}\end{matrix} \longleftarrow \begin{matrix}\mathbb{R}^{k_2}\oplus \\ \mathbb{R}^{l_1}\oplus \\ \mathbb{R}^{m_1}\end{matrix} \longleftarrow \begin{matrix}\mathbb{R}^{k_1}\oplus \\ \mathbb{R}^{l_1}\oplus \\ \mathbb{R}^{m_2}.\end{matrix}
$$

As $k_2-k_1=l_2-l_1=m_2-m_1=d$, we have $k_2+l_1+m_1=k_1+l_2+m_1=k_1+l_1+m_2$, so all six matrices in (4.66) are square, and we can take determinants in (4.66). The matrices on the first line of (4.66) are $M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})\oplus I_{m_1}$ and $I_{k_1}\oplus M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})$, with determinants $\det M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and $\det M_{\underline{U}}(\boldsymbol{\beta},\boldsymbol{\gamma})$. The final matrix in (4.66) is $M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\gamma})\oplus I_{l_1}$, with determinant $(-1)^{l_1(k_1+k_2)}\det M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\gamma})$. The remaining three matrices have determinants $1,(-1)^{k_1},(-1)^{k_2}$. Hence taking determinants in (4.66) yields

$$
\det M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})\cdot\det M_{\underline{U}}(\boldsymbol{\beta},\boldsymbol{\gamma})\cdot 1\cdot(-1)^{k_1}=(-1)^{k_2}\cdot(-1)^{l_1(k_1+k_2)}\det M_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\gamma}).
$$

Substituting in (4.61) then gives

$$
\begin{aligned}
(-1)^{(1+l_1)d}F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})\cdot(-1)^{(1+m_1)d}F_{\underline{U}}(\boldsymbol{\beta},\boldsymbol{\gamma})\cdot(-1)^{k_1}& \\
=(-1)^{k_2+l_1d}\cdot(-1)^{(1+m_1)d}F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\gamma}),&
\end{aligned}
$$

and (4.65) follows as $k_2-k_1=d$. Combining equations (4.65) with $\boldsymbol{\gamma}=\boldsymbol{\alpha}$ and (4.63) gives (4.64). This proves (a).

For (b), let $x\in\underline{X}$. As $(\mathcal{E}^\bullet,\phi)$ is a virtual vector bundle of rank d, there exists an open $x\in\underline{U}\subseteq\underline{X}$ such that $(\mathcal{E}^\bullet,\phi)|_{\underline{U}}$ is equivalent in $\mathrm{vqcoh}(\underline{U})$ to $(\mathcal{F}^\bullet,\psi)$ for $\mathcal{F}^1,\mathcal{F}^2$ vector bundles on $\underline{U}$. Making $\underline{U}$ smaller if necessary, we can suppose $\mathcal{F}^1,\mathcal{F}^2$ are trivial vector bundles on $\underline{U}$, and so take $\mathcal{F}^1=\mathbb{R}^{k_1}\otimes_{\mathbb{R}}\mathcal{O}_U$ and $\mathcal{F}^2=\mathbb{R}^{k_2}\otimes_{\mathbb{R}}\mathcal{O}_U$ for $k_1,k_2\geqslant 0$ with $k_2-k_1=d$. So by Proposition A.6 there exist 1-morphisms $(f^1,f^2):(\mathcal{E}^\bullet|_{\underline{U}},\phi|_{\underline{U}})\to(\mathbb{R}^{k_\bullet}\otimes_{\mathbb{R}}\mathcal{O}_U,\psi)$ and $(g^1,g^2):(\mathbb{R}^{k_\bullet}\otimes_{\mathbb{R}}$

$\mathcal{O}_U,\psi) \to (\mathcal{E}^\bullet|_{\underline{U}},\phi|_{\underline{U}})$ in $\mathrm{vqcoh}(\underline{U})$ and 2-isomorphisms $\eta: g^\bullet\circ f^\bullet \Rightarrow \mathrm{id}_{(\mathcal{E}^\bullet|_{\underline{U}},\phi|_{\underline{U}})}$, $\zeta: f^\bullet\circ g^\bullet \Rightarrow \mathrm{id}_{(\mathbb{R}^{k^\bullet}\otimes_{\mathbb{R}}\mathcal{O}_U,\psi)}$ satisfying $\mathrm{id}_{f^\bullet}*\eta=\zeta*\mathrm{id}_{f^\bullet}$ and $\mathrm{id}_{g^\bullet}*\zeta=\eta*\mathrm{id}_{g^\bullet}$.

Then $f^1,f^2,g^1,g^2,\eta,\zeta$ are morphisms in $\mathrm{qcoh}(\underline{U})$:

$$\begin{gathered}f^1:\mathcal{E}^1|_{\underline{U}}\to\mathbb{R}^{k_1}\otimes_{\mathbb{R}}\mathcal{O}_U,\quad f^2:\mathcal{E}^2|_{\underline{U}}\to\mathbb{R}^{k_2}\otimes_{\mathbb{R}}\mathcal{O}_U,\quad g^1:\mathbb{R}^{k_1}\otimes_{\mathbb{R}}\mathcal{O}_U\to\mathcal{E}^1|_{\underline{U}},\\ g^2:\mathbb{R}^{k_2}\otimes_{\mathbb{R}}\mathcal{O}_U\to\mathcal{E}^2|_{\underline{U}},\quad \eta:\mathcal{E}^2|_{\underline{U}}\to\mathcal{E}^1|_{\underline{U}},\qquad \zeta:\mathbb{R}^{k_2}\otimes_{\mathbb{R}}\mathcal{O}_U\to\mathbb{R}^{k_1}\otimes_{\mathbb{R}}\mathcal{O}_U,\\ \text{satisfying}\quad f^2\circ\phi|_{\underline{U}}=\psi\circ f^1,\quad g^2\circ\psi=\phi|_{\underline{U}}\circ g^1,\quad \zeta\circ f^2=f^1\circ\eta,\\ \eta\circ g^2=g^1\circ\zeta,\ \ \mathrm{id}_{\mathcal{E}^1|_{\underline{U}}}=g^1\circ f^1+\eta\circ\phi|_{\underline{U}},\ \ \mathrm{id}_{\mathcal{E}^2|_{\underline{U}}}=g^2\circ f^2+\phi|_{\underline{U}}\circ\eta,\\ \mathrm{id}_{\mathbb{R}^{k_1}\otimes_{\mathbb{R}}\mathcal{O}_U}=f^1\circ g^1+\zeta\circ\psi\ \text{ and }\ \mathrm{id}_{\mathbb{R}^{k_2}\otimes_{\mathbb{R}}\mathcal{O}_U}=f^2\circ g^2+\psi\circ\zeta.\end{gathered}\tag{4.67}$$

Define $\alpha_1=\psi$, $\alpha_2=\zeta$, $\alpha_3=g^1$, $\alpha_4=f^1$, $\alpha_5=g^2$, $\alpha_6=f^2$ and $\alpha_7=\eta$. Then (4.67) is equivalent to (4.59). Hence $\boldsymbol{\alpha}=(k_1,k_2,\alpha_1,\ldots,\alpha_7)$ is an orientation generator for $(\mathcal{E}^\bullet,\phi)$ on $\underline{U}$. □

We can now define orientation line bundles $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ of virtual vector bundles.

Definition 4.39. Let $\underline{X}$ be a C^∞-scheme and $(\mathcal{E}^\bullet,\phi)$ a virtual vector bundle on $\underline{X}$. For each open $\underline{U}\subset\underline{X}$, define an $\mathcal{O}_X(U)$-module $\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)}(U)$ to be the $\mathcal{O}_X(U)$-module generated by all orientation generators $\boldsymbol{\alpha}$ for $(\mathcal{E}^\bullet,\phi)$ on $\underline{U}$, with the relations that

$$\boldsymbol{\beta}=F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})\cdot\boldsymbol{\alpha}\tag{4.68}$$

for all orientation generators $\boldsymbol{\alpha},\boldsymbol{\beta}$. Proposition 4.38(a) implies that these relations are consistent. If follows that if there exists an orientation generator $\boldsymbol{\alpha}$ for $\underline{U}$ then $\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)}(U)$ is the free $\mathcal{O}_X(U)$-module spanned by $\boldsymbol{\alpha}$, so that $\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)}(U)\cong\mathcal{O}_X(U)$, and if there exists no such $\boldsymbol{\alpha}$ then $\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)}(U)=0$.

Let $\underline{V}\subseteq\underline{U}\subseteq\underline{X}$ be open. Define $(\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)})_{UV}:\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)}(U)\to\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)}(V)$ by

$$(\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)})_{UV}:c\cdot\boldsymbol{\alpha}\longmapsto\rho_{UV}(c)\cdot\boldsymbol{\alpha}|_{\underline{V}},$$

for $c\in\mathcal{O}_X(U)$ and $\boldsymbol{\alpha}$ an orientation generator for $(\mathcal{E}^\bullet,\phi)$ on $\underline{U}$. Equation (4.62) implies this is compatible with the relations (4.68), so $(\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)})_{UV}$ is well-defined, and the analogue of (B.9) commutes.

It is now easy to check that $\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)}$ is a presheaf of $\mathcal{O}_X$-modules on $\underline{X}$. Let $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ be the associated sheaf of $\mathcal{O}_X$-modules. If $\underline{U}\subset\underline{X}$ is any open C^∞-subscheme for which there exists an orientation generator $\boldsymbol{\alpha}$ for $(\mathcal{E}^\bullet,\phi)$, then for all open $\underline{V}\subseteq\underline{U}$, $\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)}(V)$ is the free $\mathcal{O}_X(V)$-module spanned by $\boldsymbol{\alpha}|_{\underline{V}}$. Therefore the presheaf $\hat{\mathcal{L}}_{(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}$ is already a sheaf isomorphic to $\mathcal{O}_U$, and the sheafification has no effect, so $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}$ is a sheaf isomorphic to $\mathcal{O}_U$, for which $\boldsymbol{\alpha}$ defines a nonvanishing section. By Proposition 4.38(b), $\underline{X}$ can be covered by such open $\underline{U}$. Hence $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ is a *line bundle* (rank 1 vector bundle) on $\underline{X}$. We call it the *orientation line bundle* of $(\mathcal{E}^\bullet,\phi)$.

When $\mathcal{E}^1,\mathcal{E}^2$ are vector bundles we can identify $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ explicitly:

Proposition 4.40. *Suppose $\mathcal{E}^1,\mathcal{E}^2$ are vector bundles on $\underline{X}$ with ranks k_1,k_2, and $\phi:\mathcal{E}^1\to\mathcal{E}^2$ is a morphism. Then $(\mathcal{E}^\bullet,\phi)$ is a virtual vector bundle of rank k_2-k_1, and there is a canonical isomorphism $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}\cong\Lambda^{k_1}(\mathcal{E}^1)^*\otimes\Lambda^{k_2}\mathcal{E}^2$.*

Proof. The first part is immediate. As $\mathcal{E}^1, \mathcal{E}^2$ are vector bundles on $\underline{X}$ we can cover $\underline{X}$ by open $\underline{U}$ such that $\mathcal{E}^1|_{\underline{U}}, \mathcal{E}^2|_{\underline{U}}$ are trivial. For such $\underline{U}$ we can choose bases of sections $e_1, \ldots, e_{k_1}$ for $\mathcal{E}^1|_{\underline{U}}$ and $f_1, \ldots, f_{k_2}$ for $\mathcal{E}^2|_{\underline{U}}$. Define isomorphisms $\alpha_3 : \mathbb{R}^{k_1} \otimes_{\mathbb{R}} \mathcal{O}_U \to \mathcal{E}^1|_{\underline{U}}$ and $\alpha_5 : \mathbb{R}^{k_2} \otimes_{\mathbb{R}} \mathcal{O}_U \to \mathcal{E}^2|_{\underline{U}}$ by $\alpha_3 : (c_1, \ldots, c_{k_1}) \mapsto c_1 e_1 + \cdots + c_{k_1} e_{k_1}$ and $\alpha_5 : (d_1, \ldots, d_{k_2}) \mapsto d_1 f_1 + \cdots + d_{k_2} f_{k_2}$. Define $\alpha_4 = \alpha_3^{-1}$, $\alpha_6 = \alpha_5^{-1}$, $\alpha_1 = \alpha_6 \circ \phi^1 \circ \alpha_3$ and $\alpha_2 = \alpha_7 = 0$. Then $\boldsymbol{\alpha} = (k_1, k_2, \alpha_1, \ldots, \alpha_7)$ is an orientation generator for $(\mathcal{E}^\bullet, \phi)$ on $\underline{U}$.

Let $\epsilon^1, \ldots, \epsilon^{k_1}$ be the basis of sections of $(\mathcal{E}^1)^*|_{\underline{U}}$ dual to $e_1, \ldots, e_{k_1}$. Then $\epsilon^1 \wedge \cdots \wedge \epsilon^{k_1}$ is a trivializing section of $\Lambda^{k_1}(\mathcal{E}^1)^*|_{\underline{U}}$, and $f_1 \wedge \cdots \wedge f_{k_2}$ a trivializing section of $\Lambda^{k_2}\mathcal{E}^2|_{\underline{U}}$, so $(\epsilon^1 \wedge \cdots \wedge \epsilon^{k_1}) \otimes (f_1 \wedge \cdots \wedge f_{k_2})$ is a trivializing section of $\bigl(\Lambda^{k_1}(\mathcal{E}^1)^* \otimes \Lambda^{k_2}\mathcal{E}^2\bigr)|_{\underline{U}}$. Also $\boldsymbol{\alpha}$ defines a section of $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}$. Hence there is a unique isomorphism $\Phi_{\underline{U}} : \mathcal{L}_{(\mathcal{E}^\bullet,\phi)}|_{\underline{U}} \to \bigl(\Lambda^{k_1}(\mathcal{E}^1)^* \otimes \Lambda^{k_2}\mathcal{E}^2\bigr)|_{\underline{U}}$ with

$$\Phi_{\underline{U}}(\boldsymbol{\alpha}) = (\epsilon^1 \wedge \cdots \wedge \epsilon^{k_1}) \otimes (f_1 \wedge \cdots \wedge f_{k_2}). \tag{4.69}$$

We claim that $\Phi_{\underline{U}}$ is independent of the choices of bases $e_1, \ldots, e_{k_1}$ and $f_1, \ldots, f_{k_2}$. To see this, let $\tilde{e}_1, \ldots, \tilde{e}_{k_2}$ and $\tilde{f}_1, \ldots, \tilde{f}_{k_2}$ be alternative choices, and let $\tilde{\boldsymbol{\alpha}} = (k_1, k_2, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_7)$ be the corresponding orientation generator. We may write

$$e_j = \textstyle\sum_{i=1}^{k_1} A_{ij}\tilde{e}_i \quad \text{and} \quad \tilde{f}_j = \textstyle\sum_{i=1}^{k_2} B_{ij} f_i, \tag{4.70}$$

for $(A_{ij})_{i,j\leqslant k_1}$ and $(B_{ij})_{i,j\leqslant k_2}$ invertible matrices over $\mathcal{O}_X(U)$. We have arranged (4.70) such that (A_{ij}) is the matrix of $\tilde{\alpha}_4 \circ \alpha_3$, and (B_{ij}) is the matrix of $\alpha_6 \circ \tilde{\alpha}_5$. Thus

$$M_{\underline{U}}(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) = \begin{pmatrix} \alpha_1 & -(B_{ij})_{i,j\leqslant k_2} \\ (A_{ij})_{i,j\leqslant k_1} & 0 \end{pmatrix},$$

so that

$$F_{\underline{U}}(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) = (-1)^{(1+k_1)(k_2-k_1)} \det M_{\underline{U}}(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) = \det(A_{ij}) \cdot \det(B_{ij}). \tag{4.71}$$

From (4.70) it follows that

$$e_1 \wedge \cdots \wedge e_{k_1} = \det(A_{ij}) \cdot \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{k_1} \text{ and } \tilde{f}_1 \wedge \cdots \wedge \tilde{f}_{k_2} = \det(B_{ij}) \cdot f_1 \wedge \cdots \wedge f_{k_2}.$$

If $\tilde{\epsilon}^1, \ldots, \tilde{\epsilon}^{k_1}$ is the dual basis of $(\mathcal{E}^1)^*|_{\underline{U}}$ to $\tilde{e}_1, \ldots, \tilde{e}_{k_1}$ then the first equation implies that $\epsilon^1 \wedge \cdots \wedge \epsilon^{k_1} = \det(A_{ij})^{-1} \cdot \tilde{\epsilon}^1 \wedge \cdots \wedge \tilde{\epsilon}^{k_1}$, so combining this with the second equation gives

$$\begin{aligned} &(\tilde{\epsilon}^1 \wedge \cdots \wedge \tilde{\epsilon}^{k_1}) \otimes (\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_{k_2}) \\ &\qquad = \det(A_{ij}) \det(B_{ij}) \cdot (\epsilon^1 \wedge \cdots \wedge \epsilon^{k_1}) \otimes (f_1 \wedge \cdots \wedge f_{k_2}). \end{aligned} \tag{4.72}$$

Comparing equations (4.68), (4.69), (4.71) and (4.72), we see that $\Phi_{\underline{U}}$ is independent of the choices of $e_1, \ldots, e_{k_1}$ and $f_1, \ldots, f_{k_2}$. It is then easy to show that we may glue the local isomorphisms $\Phi_{\underline{U}}$ to get a global canonical isomorphism $\Phi : \mathcal{L}_{(\mathcal{E}^\bullet,\phi)} \to \Lambda^{k_1}(\mathcal{E}^1)^* \otimes \Lambda^{k_2}\mathcal{E}^2$, as we want. $\square$

Definition 4.41. Let $\underline{f}:\underline{X}\to\underline{Y}$ be a morphism of C^∞-schemes, and $(\mathcal{E}^\bullet,\phi)$ a virtual vector bundle on $\underline{Y}$. Then the pullback $\underline{f}^*(\mathcal{E}^\bullet,\phi)=(\underline{f}^*(\mathcal{E}^\bullet),\underline{f}^*(\phi))$ is a virtual vector bundle on $\underline{X}$. Thus we have line bundles $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ on $\underline{Y}$ and $\mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet,\phi)}$ on $\underline{X}$, and the pullback $\underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet,\phi)})$ is also a line bundle on $\underline{X}$. We will construct a canonical isomorphism $I_{\underline{f},(\mathcal{E}^\bullet,\phi)}:\underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet,\phi)})\to\mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet,\phi)}$. Suppose $\underline{V}\subseteq\underline{Y}$ is open and $\boldsymbol{\alpha}=(k_1,k_2,\alpha_1,\ldots,\alpha_7)$ is an orientation generator for $(\mathcal{E}^\bullet,\phi)$ on $\underline{V}$. Then $\underline{U}=\underline{f}^{-1}(\underline{V})\subset\underline{X}$ is open. Define

$$\begin{aligned}\tilde{\boldsymbol{\alpha}}=\bigl(k_1,k_1,\delta_2^{-1}\circ\underline{f}^*(\alpha_1)\circ\delta_1,\delta_1^{-1}\circ\underline{f}^*(\alpha_2)\circ\delta_2,\underline{f}^*(\alpha_3)\circ\delta_1,\delta_1^{-1}\circ\underline{f}^*(\alpha_4),\\ \underline{f}^*(\alpha_5)\circ\delta_2,\delta_2^{-1}\circ\underline{f}^*(\alpha_6),\underline{f}^*(\alpha_7)\bigr),\end{aligned}$$

where $\delta_1:\mathbb{R}^{k_1}\otimes_{\mathbb{R}}\mathcal{O}_U\to\underline{f}^*(\mathbb{R}^{k_1}\otimes_{\mathbb{R}}\mathcal{O}_V)$, $\delta_2:\mathbb{R}^{k_2}\otimes_{\mathbb{R}}\mathcal{O}_U\to\underline{f}^*(\mathbb{R}^{k_2}\otimes_{\mathbb{R}}\mathcal{O}_V)$ are the natural isomorphisms. Then $\tilde{\boldsymbol{\alpha}}$ is an orientation generator for $\underline{f}^*(\mathcal{E}^\bullet,\phi)$ on $\underline{U}$. Now $\tilde{\boldsymbol{\alpha}}$ defines a nonvanishing section of $\mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}$, and $\boldsymbol{\alpha}$ a nonvanishing section of $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}|_{\underline{V}}$, so that $\underline{f}^*(\boldsymbol{\alpha})$ defines a nonvanishing section of $\underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet,\phi)})|_{\underline{U}}$. Define an isomorphism $I_{\underline{f},(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}:\underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet,\phi)})|_{\underline{U}}\to\mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}$ to identify $\underline{f}^*(\boldsymbol{\alpha})$ with $\tilde{\boldsymbol{\alpha}}$. This $I_{\underline{f},(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}$ is independent of the choice of $\boldsymbol{\alpha}$, and the $I_{\underline{f},(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}$ glue to form a global canonical isomorphism $I_{\underline{f},(\mathcal{E}^\bullet,\phi)}:\underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet,\phi)})\to\mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet,\phi)}$ on $\underline{X}$.

Definition 4.42. Let $\underline{X}$ be a C^∞-scheme, $(\mathcal{E}^\bullet,\phi)$ and $(\mathcal{F}^\bullet,\psi)$ be virtual vector bundles on $\underline{X}$, and $f^\bullet:(\mathcal{E}^\bullet,\phi)\to(\mathcal{F}^\bullet,\psi)$ be an equivalence in $\mathrm{vqcoh}(\underline{X})$. We will construct a canonical isomorphism $\mathcal{L}_{f^\bullet}:\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}\to\mathcal{L}_{(\mathcal{F}^\bullet,\psi)}$ on $\underline{X}$.

As $f^\bullet$ is an equivalence, by Proposition A.6 there exist a 1-morphism $g^\bullet:(\mathcal{F}^\bullet,\psi)\to(\mathcal{E}^\bullet,\phi)$ and 2-morphisms $\eta:g^\bullet\circ f^\bullet\Rightarrow\mathrm{id}_{(\mathcal{E}^\bullet,\phi)}$, $\zeta:f^\bullet\circ g^\bullet\Rightarrow\mathrm{id}_{(\mathcal{F}^\bullet,\psi)}$ satisfying $\mathrm{id}_{f^\bullet}*\eta=\zeta*\mathrm{id}_{f^\bullet}$ and $\mathrm{id}_{g^\bullet}*\zeta=\eta*\mathrm{id}_{g^\bullet}$. Then as for (4.67), $f^1,f^2,g^1,g^2,\eta,\zeta$ are morphisms in $\mathrm{qcoh}(\underline{X})$:

$$\begin{gathered}f^j:\mathcal{E}^j\to\mathcal{F}^j,\;\; g^j:\mathcal{F}^j\to\mathcal{E}^j,\;\; g^2:\mathcal{F}^2\to\mathcal{E}^2,\;\; \eta:\mathcal{E}^2\to\mathcal{E}^1,\;\; \zeta:\mathcal{F}^2\to\mathcal{F}^1,\\ \text{satisfying }\; f^2\circ\phi=\psi\circ f^1,\;\; g^2\circ\psi=\phi\circ g^1,\;\; \zeta\circ f^2=f^1\circ\eta,\\ \eta\circ g^2=g^1\circ\zeta,\;\; \mathrm{id}_{\mathcal{E}^1}=g^1\circ f^1+\eta\circ\phi,\;\; \mathrm{id}_{\mathcal{E}^2}=g^2\circ f^2+\phi\circ\eta,\\ \mathrm{id}_{\mathcal{G}^1}=f^1\circ g^1+\zeta\circ\psi\;\text{ and }\;\mathrm{id}_{\mathcal{G}^2}=f^2\circ g^2+\psi\circ\zeta.\end{gathered}\tag{4.73}$$

Suppose $\underline{U}\subseteq\underline{X}$ is open and $\boldsymbol{\alpha}=(k_1,k_2,\alpha_1,\ldots,\alpha_7)$ is an orientation generator for $(\mathcal{E}^\bullet,\phi)$ on $\underline{U}$. Define

$$\begin{gathered}\tilde\alpha_1=\alpha_1,\;\; \tilde\alpha_2=\alpha_2+\alpha_4\circ\eta|_{\underline{U}}\circ\alpha_5,\;\; \tilde\alpha_3=g^1|_{\underline{U}}\circ\alpha_3,\;\; \tilde\alpha_4=\alpha_4\circ f^1|_{\underline{U}},\\ \tilde\alpha_5=g^2|_{\underline{U}}\circ\alpha_5,\;\; \tilde\alpha_6=\alpha_6\circ f^2|_{\underline{U}},\;\; \tilde\alpha_7=\zeta|_{\underline{U}}+g^1|_{\underline{U}}\circ\alpha_7\circ f^2|_{\underline{U}}.\end{gathered}$$

Then using (4.59) and (4.73) one can show that $\tilde\alpha_1,\ldots,\tilde\alpha_7$ satisfy (4.59), so $\tilde{\boldsymbol{\alpha}}=(k_1,k_2,\tilde\alpha_1,\ldots,\tilde\alpha_7)$ is an orientation generator for $(\mathcal{F}^\bullet,\psi)$ on $\underline{U}$.

Thus, $\boldsymbol{\alpha}$ gives a nonvanishing section of $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}$, and $\tilde{\boldsymbol{\alpha}}$ a nonvanishing section of $\mathcal{L}_{(\mathcal{F}^\bullet,\psi)}|_{\underline{U}}$. We define $\mathcal{L}_{f^\bullet}|_{\underline{U}}:\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}\to\mathcal{L}_{(\mathcal{F}^\bullet,\psi)}|_{\underline{U}}$ to be the isomorphism taking $\boldsymbol{\alpha}$ to $\tilde{\boldsymbol{\alpha}}$. If $\boldsymbol{\beta}$ is another orientation generator for $(\mathcal{E}^\bullet,\phi)$ on $\underline{U}$ and $\tilde{\boldsymbol{\beta}}$ the corresponding orientation generator for $(\mathcal{F}^\bullet,\psi)$ on $\underline{U}$, then by a proof similar to that of (4.65) one can show that $F_{\underline{U}}(\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}})=F_{\underline{U}}(\boldsymbol{\alpha},\boldsymbol{\beta})$. So (4.68) implies that $\mathcal{L}_{f^\bullet}|_{\underline{U}}$ is independent of the choice of $\boldsymbol{\alpha}$, and is canonical. Now $\underline{X}$ is covered by open $\underline{U}$ for which there exists an orientation generator $\boldsymbol{\alpha}$ for $(\mathcal{E}^\bullet,\phi)$, and on each such $\underline{U}$ we have defined an isomorphism $\mathcal{L}_{f^\bullet}|_{\underline{U}}:\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}|_{\underline{U}}\rightarrow\mathcal{L}_{(\mathcal{F}^\bullet,\psi)}|_{\underline{U}}$. These glue to give a canonical isomorphism $\mathcal{L}_{f^\bullet}:\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}\rightarrow\mathcal{L}_{(\mathcal{F}^\bullet,\psi)}$.

One can also prove that $\mathcal{L}_{f^\bullet}$ depends only on the equivalence $f^\bullet:(\mathcal{E}^\bullet,\phi)\rightarrow(\mathcal{F}^\bullet,\psi)$, and not on the additional choices of $g^\bullet,\eta,\zeta$ in Proposition A.6. We do this by showing that if $\hat g^\bullet,\hat\eta,\hat\zeta$ are alternative choices, then there exists a 2-morphism $\theta:\hat g^\bullet\rightarrow g^\bullet$ for which $\hat\eta=\eta\odot(\mathrm{id}_{f^\bullet}*\theta)$ and $\hat\zeta=\zeta\odot(\theta*\mathrm{id}_{f^\bullet})$. Then by writing the corresponding orientation generator $\hat{\tilde{\boldsymbol{\alpha}}}$ in terms of θ and $\tilde{\boldsymbol{\alpha}}$, we prove that $F_{\underline{U}}(\hat{\tilde{\boldsymbol{\alpha}}},\tilde{\boldsymbol{\alpha}})=1$, so that $\tilde{\boldsymbol{\alpha}},\hat{\tilde{\boldsymbol{\alpha}}}$ define the same section of $\mathcal{L}_{(\mathcal{F}^\bullet,\psi)}|_{\underline{U}}$, and thus the same isomorphism $\mathcal{L}_{f^\bullet}|_{\underline{U}}$.

Here are some elementary properties of these morphisms $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$. The proofs are straightforward: choose an orientation generator $\boldsymbol{\alpha}$ for $(\mathcal{E}^\bullet,\phi)$ on $\underline{U}\subseteq\underline{X}$, and compare the actions of each side on $\boldsymbol{\alpha}$.

Proposition 4.43. **(a)** *If* $(\mathcal{E}^\bullet,\phi)$ *is a virtual vector bundle on* $\underline{X}$, *so that* $\mathrm{id}_\phi:(\mathcal{E}^\bullet,\phi)\rightarrow(\mathcal{E}^\bullet,\phi)$ *is an equivalence in* $\mathrm{vvect}(\underline{X})$, *then* $\mathcal{L}_{\mathrm{id}_\phi}=\mathrm{id}_{\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}}:\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}\rightarrow\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$.

(b) *If* $f^\bullet:(\mathcal{E}^\bullet,\phi)\rightarrow(\mathcal{F}^\bullet,\psi)$ *and* $g^\bullet:(\mathcal{F}^\bullet,\psi)\rightarrow(\mathcal{G}^\bullet,\xi)$ *are equivalences of virtual vector bundles on* $\underline{X}$ *then* $\mathcal{L}_{g^\bullet\circ f^\bullet}=\mathcal{L}_{g^\bullet}\circ\mathcal{L}_{f^\bullet}:\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}\rightarrow\mathcal{L}_{(\mathcal{G}^\bullet,\xi)}$.

(c) *If* $f^\bullet,g^\bullet:(\mathcal{E}^\bullet,\phi)\rightarrow(\mathcal{F}^\bullet,\psi)$ *are* 2*-isomorphic equivalences of virtual vector bundles on* $\underline{X}$ *then* $\mathcal{L}_{f^\bullet}=\mathcal{L}_{g^\bullet}:\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}\rightarrow\mathcal{L}_{(\mathcal{F}^\bullet,\psi)}$.

4.6 Orientations on d-manifolds

Using §4.5, we can now define orientations on d-manifolds.

Definition 4.44. Let $\boldsymbol{X}$ be a d-manifold. Then the virtual cotangent bundle $T^*\boldsymbol{X}=(\mathcal{E}_X,\mathcal{F}_X,\phi_X)$ is a virtual vector bundle on $\underline{X}$ by Definition 3.18, so Definition 4.39 constructs a line bundle $\mathcal{L}_{T^*\boldsymbol{X}}$ on $\underline{X}$. We call $\mathcal{L}_{T^*\boldsymbol{X}}$ the *orientation line bundle* of $\boldsymbol{X}$.

An *orientation* ω on $\boldsymbol{X}$ is an orientation on $\mathcal{L}_{T^*\boldsymbol{X}}$, in the sense of Definition B.40. That is, ω is an equivalence class $[\tau]$ of isomorphisms $\tau:\mathcal{O}_X\rightarrow\mathcal{L}_{T^*\boldsymbol{X}}$, where τ,τ' are equivalent if they are proportional by a positive function on $\underline{X}$. We call $\boldsymbol{X}$ *orientable* if it admits an orientation, so $\boldsymbol{X}$ is orientable if and only if $\mathcal{L}_{T^*\boldsymbol{X}}$ is trivializable. An *oriented d-manifold* is a pair $(\boldsymbol{X},\omega)$ where $\boldsymbol{X}$ is a d-manifold and ω an orientation on $\boldsymbol{X}$. But we will often refer to $\boldsymbol{X}$ as an oriented d-manifold, leaving the orientation ω implicit.

If $\omega=[\tau]$ is an orientation on $\boldsymbol{X}$, the *opposite orientation* is $-\omega=[-\tau]$, which changes the sign of the isomorphism $\tau:\mathcal{O}_X\rightarrow\mathcal{L}_{T^*\boldsymbol{X}}$. When we refer to $\boldsymbol{X}$ as an oriented d-manifold, $-\boldsymbol{X}$ will mean $\boldsymbol{X}$ with the opposite orientation, that is, $\boldsymbol{X}$ is short for $(\boldsymbol{X},\omega)$ and $-\boldsymbol{X}$ is short for $(\boldsymbol{X},-\omega)$.

Our definitions and conventions on orientations of manifolds will be explained in §5.8 below, together with the analogues for manifolds with corners. We show that when the d-manifold $\boldsymbol{X}$ is a manifold, Definition 4.44 agrees with the definition of orientations in §5.8.

Example 4.45. Let X be an n-manifold, and $\boldsymbol{X} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X)$ the associated d-manifold. Then $\underline{X} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(X)$, $\mathcal{E}_X = 0$ and $\mathcal{F}_X = T^*\underline{X}$. So $\mathcal{E}_X, \mathcal{F}_X$ are vector bundles of ranks $0, n$. As $\Lambda^0\mathcal{E}_X \cong \mathcal{O}_X$, Proposition 4.40 gives a canonical isomorphism $\mathcal{L}_{T^*\boldsymbol{X}} \cong \Lambda^n T^*\underline{X}$. That is, $\mathcal{L}_{T^*\boldsymbol{X}}$ is isomorphic to the lift to C^∞-schemes of the line bundle $\Lambda^n T^*X$ on the manifold X.

By Definition 5.34, an orientation on the manifold X is just an orientation on the line bundle $\Lambda^n T^*X$. Hence orientations on the d-manifold $\boldsymbol{X} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X)$ in the sense of Definition 4.44 are equivalent to orientations on the manifold X in the usual sense.

If $f : X \to Y$ is an étale map of n-manifolds then $(\mathrm{d}f)^* : f^*(T^*Y) \to T^*X$ is an isomorphism of vector bundles, and so induces an isomorphism of line bundles $f^*(\Lambda^n T^*Y) \to \Lambda^n T^*X$. Here is an analogue of this for d-manifolds:

Definition 4.46. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be an étale 1-morphism of d-manifolds, for instance $\boldsymbol{f}$ could be an equivalence. We have orientation line bundles $\mathcal{L}_{T^*\boldsymbol{X}} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X)$ on $\underline{X}$ and $\mathcal{L}_{T^*\boldsymbol{Y}} = (\mathcal{E}_Y, \mathcal{F}_Y, \phi_Y)$ on $\underline{Y}$, so $\underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}})$ is a line bundle on $\underline{X}$. Definition 4.41 gives an isomorphism $I_{\underline{f}, T^*\boldsymbol{Y}} : \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}) \to \mathcal{L}_{\underline{f}^*(T^*\boldsymbol{Y})}$ on $\underline{X}$. But $\Omega_{\boldsymbol{f}} = (f'', f^2) : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ is a 1-morphism in $\mathrm{vvect}(\underline{X})$, which is an equivalence in $\mathrm{vvect}(\underline{X})$ as $\boldsymbol{f}$ is étale. So Definition 4.42 gives an isomorphism $\mathcal{L}_{\Omega_{\boldsymbol{f}}} : \mathcal{L}_{\underline{f}^*(T^*\boldsymbol{Y})} \to \mathcal{L}_{T^*\boldsymbol{X}}$. Define an isomorphism $\mathcal{L}_{\boldsymbol{f}} : \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}) \to \mathcal{L}_{T^*\boldsymbol{X}}$ in $\mathrm{qcoh}(\underline{X})$ by $\mathcal{L}_{\boldsymbol{f}} = \mathcal{L}_{\Omega_{\boldsymbol{f}}} \circ I_{\underline{f}, T^*\boldsymbol{Y}}$.

From Propositions 4.40 and 4.43 we may deduce:

Proposition 4.47. (a) *Let $\boldsymbol{X}$ be a d-manifold, and suppose $\mathcal{E}_X, \mathcal{F}_X$ are vector bundles on $\underline{X}$ with ranks k_1, k_2, so that $k_2 - k_1 = \mathrm{vdim}\,\boldsymbol{X}$. Then there is a canonical isomorphism $\mathcal{L}_{T^*\boldsymbol{X}} \cong \Lambda^{k_1}(\mathcal{E}_X)^* \otimes \Lambda^{k_2}\mathcal{F}_X$.*

(b) *If $\boldsymbol{X}$ is a d-manifold then $\mathcal{L}_{\mathbf{id}_{\boldsymbol{X}}} = \delta_{\underline{X}}(\mathcal{L}_{T^*\boldsymbol{X}}) : \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{L}_{T^*\boldsymbol{X}}) \to \mathcal{L}_{T^*\boldsymbol{X}}$.*

(c) *If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ are étale 1-morphisms of d-manifolds then*

$$\mathcal{L}_{\boldsymbol{g}\circ\boldsymbol{f}} = \mathcal{L}_{\boldsymbol{f}} \circ \underline{f}^*(\mathcal{L}_{\boldsymbol{g}}) \circ I_{\underline{f},\underline{g}}(\mathcal{L}_{T^*\boldsymbol{Z}}) : (\underline{g} \circ \underline{f})^*(\mathcal{L}_{T^*\boldsymbol{Z}}) \longrightarrow \mathcal{L}_{T^*\boldsymbol{X}}.$$

(d) *If $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ are 2-isomorphic étale 1-morphisms of d-manifolds then $\mathcal{L}_{\boldsymbol{f}} = \mathcal{L}_{\boldsymbol{g}} : \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}) = \underline{g}^*(\mathcal{L}_{T^*\boldsymbol{Y}}) \to \mathcal{L}_{T^*\boldsymbol{X}}$.*

We can define orientations on $\boldsymbol{S}_{V,E,s}$ using differential-geometric data.

Definition 4.48. Let V be an n-manifold, $E \to V$ a rank k vector bundle, and $s : V \to E$ a smooth section. Write $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$ for the 'standard model' d-manifold of Definition 3.13. Then we may identify $\underline{X}$ with the C^∞-subscheme $s^{-1}(0)$ in $\underline{V} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(V)$, and $\mathcal{E}_X \cong \mathcal{E}^*|_{\underline{V}}$, $\mathcal{F}_X \cong T^*\underline{V}|_{\underline{X}}$, where $\mathcal{E}$ is the vector bundle on $\underline{V}$ lifting $E \to V$. So Proposition 4.47(a) shows that $\mathcal{L}_{T^*\boldsymbol{X}} \cong \Lambda^k(\mathcal{E}) \otimes \Lambda^n T^*\underline{V}$. That is, $\mathcal{L}_{T^*\boldsymbol{X}}$ is isomorphic to the restriction to $\underline{X}$ of the lift to the C^∞-scheme $\underline{V}$ of the line bundle $\Lambda^k E \otimes \Lambda^n T^*V$ on the manifold V.

Suppose we are given an orientation on the line bundle $\Lambda^k E \otimes \Lambda^n T^*V$ on V, in the sense of Definition 5.34. Then the isomorphism from $\mathcal{L}_{T^*\boldsymbol{X}}$ to the pullback of $\Lambda^k E \otimes \Lambda^n T^*V$ to $\underline{X}$ induces an orientation of $\boldsymbol{S}_{V,E,s}$. We will call this a *standard model orientation*, and $\boldsymbol{S}_{V,E,s}$ with this orientation a *standard model oriented d-manifold*.

Here is another way to write the orientation data on (V,E,s). The total space of E^* is a manifold of dimension $n+k$, and the zero section $0 : V \to E^*$ is a smooth map with $0^*(T^*(E^*)) \cong E^* \otimes T^*V$. Therefore $0^*(\Lambda^{n+k}T^*(E^*)) \cong \Lambda^k E \otimes \Lambda^n T^*V$ is equivalent to an orientation on $0^*(\Lambda^{n+k}T^*(E^*))$ on V, or to an orientation on $\Lambda^{n+k}T^*(E^*)$ on $0(V) \subseteq E^*$, which extends uniquely to an orientation on $\Lambda^{n+k}T^*(E^*)$ on E^* as E^* retracts to $0(V)$. Thus, an orientation on $\Lambda^k E \otimes \Lambda^n T^*V \to V$ is equivalent to an orientation on the total space of E^*, and also to an orientation on the total space of E.

It is easy to prove:

Lemma 4.49. *Let $\boldsymbol{X}$ be an oriented d-manifold. Then every $x \in \boldsymbol{X}$ has an open neighbourhood $\boldsymbol{U}$ equivalent as an oriented d-manifold to a 'standard model' oriented d-manifold $\boldsymbol{S}_{V,E,s}$, as in Definition* 4.48.

Now consider the situation of Theorem 4.21, so that $\boldsymbol{W},\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are d-manifolds with $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ for $\boldsymbol{g},\boldsymbol{h}$ d-transverse, where $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ are the projections. Then we have orientation line bundles $\mathcal{L}_{T^*\boldsymbol{W}},\ldots,\mathcal{L}_{T^*\boldsymbol{Z}}$ on $\underline{W},\ldots,\underline{Z}$. We will construct a canonical isomorphism (4.74) below, which writes $\mathcal{L}_{T^*\boldsymbol{W}}$ in terms of the pullbacks $\underline{e}^*(\mathcal{L}_{T^*\boldsymbol{X}}), \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}), (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\boldsymbol{Z}})$ of $\mathcal{L}_{T^*\boldsymbol{X}}, \mathcal{L}_{T^*\boldsymbol{Y}}, \mathcal{L}_{T^*\boldsymbol{Z}}$. Thus, orientations on $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ induce an orientation on $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$, so we can do oriented fibre products.

Note that (4.74) depends on a choice of *orientation convention*: a different choice would change (4.74) by a sign depending on $\operatorname{vdim}\boldsymbol{X}, \operatorname{vdim}\boldsymbol{Y}, \operatorname{vdim}\boldsymbol{Z}$. Our conventions are chosen to match those of Fukaya et al. [32, §8.2] for fibre products of Kuranishi spaces over orbifolds. Our orientation conventions for manifolds are given in Convention 5.35 below.

Theorem 4.50. *Work in the situation of Theorem* 4.21, *so that $\boldsymbol{W},\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are d-manifolds with $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ for $\boldsymbol{g},\boldsymbol{h}$ d-transverse, where $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ are the projections. Then we have orientation line bundles $\mathcal{L}_{T^*\boldsymbol{W}},\ldots,\mathcal{L}_{T^*\boldsymbol{Z}}$ on $\underline{W},\ldots,\underline{Z}$, so $\mathcal{L}_{T^*\boldsymbol{W}}, \underline{e}^*(\mathcal{L}_{T^*\boldsymbol{X}}), \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}), (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\boldsymbol{Z}})$ are line bundles on $\underline{W}$. With a suitable choice of orientation convention, there is a canonical isomorphism*

$$\Phi : \mathcal{L}_{T^*\boldsymbol{W}} \longrightarrow \underline{e}^*(\mathcal{L}_{T^*\boldsymbol{X}}) \otimes_{\mathcal{O}_W} \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}) \otimes_{\mathcal{O}_W} (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\boldsymbol{Z}})^*. \tag{4.74}$$

Hence, if $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are oriented d-manifolds, then $\boldsymbol{W}$ also has a natural orientation, since trivializations of $\mathcal{L}_{T^*\boldsymbol{X}}, \mathcal{L}_{T^*\boldsymbol{Y}}, \mathcal{L}_{T^*\boldsymbol{Z}}$ induce a trivialization of $\mathcal{L}_{T^*\boldsymbol{W}}$ by (4.74).

Proof. We use the notation of the proof of Theorem 4.21. If $w \in \boldsymbol{W}$ with $w = (x,y)$ with $x \in \boldsymbol{X}$, $y \in \boldsymbol{Y}$ and $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z \in \boldsymbol{Z}$, we choose principal open d-submanifolds $x \in \hat{\boldsymbol{X}} \subseteq \boldsymbol{X}$, $y \in \hat{\boldsymbol{Y}} \subseteq \boldsymbol{Y}$, $z \in \hat{\boldsymbol{Z}} \subseteq \boldsymbol{Z}$ with $\boldsymbol{g}(\hat{\boldsymbol{X}}), \boldsymbol{h}(\hat{\boldsymbol{Y}}) \subseteq \hat{\boldsymbol{Z}}$, manifolds T,U,V, vector bundles $F \to T$, $G \to U$, $H \to V$ sections $t : T \to F$, $u : U \to G$, $v : V \to H$, equivalences $\boldsymbol{i} : \boldsymbol{S}_{T,F,t} \to \hat{\boldsymbol{X}}$, $\boldsymbol{j} : \boldsymbol{S}_{U,G,u} \to \hat{\boldsymbol{Y}}$,

$\boldsymbol{k} : \hat{\boldsymbol{Z}} \to \boldsymbol{S}_{V,H,v}$, smooth maps $p : T \to V$, $q : U \to V$ and vector bundle morphisms $\hat{p} : F \to p^*(H)$, $\hat{q} : G \to q^*(H)$ such that $\boldsymbol{k} \circ \boldsymbol{g} \circ \boldsymbol{i} = \boldsymbol{S}_{p,\hat{p}}$ and $\boldsymbol{k} \circ \boldsymbol{h} \circ \boldsymbol{j} = \boldsymbol{S}_{q,\hat{q}}$. Let $\tilde{\boldsymbol{k}} : \boldsymbol{S}_{V,H,v} \to \hat{\boldsymbol{Z}}$ be a quasi-inverse for $\boldsymbol{k}$.

Write $\hat{\boldsymbol{W}}$ for the open neighbourhood of w in $\boldsymbol{W}$ equivalent to $\hat{\boldsymbol{X}} \times_{\hat{\boldsymbol{Z}}} \hat{\boldsymbol{Y}} \subseteq \boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$. Definition 4.19 defines a manifold S, vector bundle $E \to S$ and section $s : S \to E$ from $T, U, V, F, G, H, t, u, v, p, q, \hat{p}, \hat{q}$, and Theorem 4.20 shows that $\boldsymbol{S}_{S,E,s} \simeq \boldsymbol{S}_{T,F,t} \times_{\boldsymbol{S}_{p,\hat{p}}, \boldsymbol{S}_{V,H,v}, \boldsymbol{S}_{q,\hat{q}}} \boldsymbol{S}_{U,G,u}$. The proof of Theorem 4.21 shows there is an equivalence $\boldsymbol{l} : \boldsymbol{S}_{S,E,s} \to \hat{\boldsymbol{W}}$.

Set $k_1 = \operatorname{rank} E$, $k_2 = \dim S$, $l_1 = \operatorname{rank} F$, $l_2 = \dim T$, $m_1 = \operatorname{rank} G$, $m_2 = \dim U$, $n_1 = \operatorname{rank} H$, and $n_2 = \dim V$. Then $k_2 = l_2 + m_2$ as $S \subseteq T \times U$ is open, and $k_1 = l_1 + m_1 + n_2 - n_1$ by definition of E. Combining Proposition 4.47(a) and Definition 4.46 gives canonical isomorphisms

$$\mathcal{L}_{\boldsymbol{l}} : \underline{i}^*(\mathcal{L}_{T^*\boldsymbol{W}}|_{\underline{\hat{W}}}) \longrightarrow \mathcal{L}_{\boldsymbol{S}_{S,E,s}} \cong (\Lambda^{k_1}\mathcal{E} \otimes \Lambda^{k_2} T^*\underline{S})|_{\underline{S}_{S,E,s}}, \tag{4.75}$$

$$\mathcal{L}_{\boldsymbol{i}} : \underline{i}^*(\mathcal{L}_{T^*\boldsymbol{X}}|_{\underline{\hat{X}}}) \longrightarrow \mathcal{L}_{\boldsymbol{S}_{T,F,t}} \cong (\Lambda^{l_1}\mathcal{F} \otimes \Lambda^{l_2} T^*\underline{T})|_{\underline{S}_{T,F,t}}, \tag{4.76}$$

$$\mathcal{L}_{\boldsymbol{j}} : \underline{j}^*(\mathcal{L}_{T^*\boldsymbol{Y}}|_{\underline{\hat{Y}}}) \longrightarrow \mathcal{L}_{\boldsymbol{S}_{U,G,u}} \cong (\Lambda^{m_1}\mathcal{G} \otimes \Lambda^{m_2} T^*\underline{U})|_{\underline{S}_{U,G,u}}, \tag{4.77}$$

$$\mathcal{L}_{\tilde{\boldsymbol{k}}} : \underline{\tilde{k}}^*(\mathcal{L}_{T^*\boldsymbol{Z}}|_{\underline{\hat{Z}}}) \longrightarrow \mathcal{L}_{\boldsymbol{S}_{V,H,v}} \cong (\Lambda^{n_1}\mathcal{H} \otimes \Lambda^{n_2} T^*\underline{V})|_{\underline{S}_{V,H,v}}. \tag{4.78}$$

By definition of S, E, as $S \subseteq T \times U$ is open we have

$$T^*S \cong \pi_T^*(T^*T) \oplus \pi_U^*(T^*U), \tag{4.79}$$

and there is an exact sequence of vector bundles on S

$$0 \longrightarrow E \longrightarrow \pi_T^*(F) \oplus \pi_U^*(G) \oplus (p \circ \pi_T)^*(TV) \xrightarrow{B} (p \circ \pi_T)^*(H) \longrightarrow 0,$$

noting that $TV \cong \mathbb{R}^n$, $H \cong \mathbb{R}^k$. Splitting this gives an isomorphism

$$E \oplus (p \circ \pi_T)^*(H) \cong \pi_T^*(F) \oplus \pi_U^*(G) \oplus (p \circ \pi_T)^*(TV). \tag{4.80}$$

Now from equations (4.79)–(4.80) we may construct canonical isomorphisms

$$\Lambda^{k_2} T^*S \cong \pi_T^*(\Lambda^{l_2} T^*T) \otimes \pi_U^*(\Lambda^{m_2} T^*U), \tag{4.81}$$

$$\begin{aligned}\Lambda^{k_1} E \otimes (p \circ \pi_T)^*(\Lambda^{n_1} H) \cong \pi_T^*(\Lambda^{l_1} F) \otimes \pi_U^*(\Lambda^{m_1} G) \\ \otimes (p \circ \pi_T)^*(\Lambda^{n_2} TV).\end{aligned} \tag{4.82}$$

Here (4.81) and (4.82) depend, up to sign, on a choice of *sign convention.* We use the most obvious choice: if U, V are vector spaces of dimensions k, l then we define the isomorphism $\Lambda^k U \otimes \Lambda^l V \cong \Lambda^{k+l}(U \oplus V)$ to identify $(u_1 \wedge \cdots \wedge u_k) \otimes (v_1 \wedge \cdots \wedge v_l) \in \Lambda^k U \otimes \Lambda^l V$ with $u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_l \in \Lambda^{k+l}(U \oplus V)$, for $u_1, \ldots, u_k \in U$ and $v_1, \ldots, v_l \in V$, and similarly for triple direct sums $U \oplus V \oplus W$. Note that this depends on the order of U, V: if we swap U, V we multiply the isomorphism by $(-1)^{kl}$. Therefore the signs of the isomorphisms in (4.81)–(4.82) depend on the order we chose to write the factors in (4.79)–(4.80). Combining (4.81)–(4.82) gives a canonical isomorphism of line bundles on S:

$$\begin{aligned}\Lambda^{k_1} E \otimes \Lambda^{k_2} T^*S \cong \pi_T^*(\Lambda^{l_1} F \otimes \Lambda^{l_2} T^*T) \otimes \pi_U^*(\Lambda^{m_1} G \otimes \Lambda^{m_2} T^*U) \\ \otimes \big((p \circ \pi_T)^*(\Lambda^{n_1} H \otimes \Lambda^{n_2} T^*V)\big)^*.\end{aligned} \tag{4.83}$$

Pulling (4.75)–(4.78) and the lift to C^∞-schemes of (4.83) back to $\underline{\hat{W}}$ and combining them using canonical isomorphisms $I_{*,*}(*)$ gives a canonical isomorphism $\mathcal{L}_{T^*\boldsymbol{W}}|_{\underline{\hat{W}}} \to \bigl(\underline{e}^*(\mathcal{L}_{T^*\boldsymbol{X}}) \otimes_{\mathcal{O}_W} \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}) \otimes_{\mathcal{O}_W} (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\boldsymbol{Z}})^*\bigr)|_{\underline{\hat{W}}}$. We define $\Phi_{\underline{\hat{W}}}$ to be this isomorphism, multiplied by a correction factor of

$$(-1)^{l_1+l_2+n_2+l_1m_1+l_1n_1+l_1n_2+l_2m_1+l_2n_1+l_2n_2+n_1n_2}. \tag{4.84}$$

This sign is chosen to correspond to the orientation conventions of Fukaya et al. [32, §8.2] for fibre products of Kuranishi spaces over manifolds. By a long but straightforward calculation, one can show that if $w', x', y', z', \boldsymbol{\hat{W}'}, \boldsymbol{\hat{X}'}, \boldsymbol{\hat{Y}'}, \boldsymbol{\hat{Z}'}, S', T', U', V', E', F', G', H', s', t', u', v', p', q', \hat{p}', \hat{q}', k_i', l_i', m_i', n_i'$ are alternative choices for $w, x, \ldots, n_i$, then $\Phi_{\underline{\hat{W}}}|_{\underline{\hat{W}} \cap \underline{\hat{W}}'} = \Phi_{\underline{\hat{W}}'}|_{\underline{\hat{W}} \cap \underline{\hat{W}}'}$. Clearly this must hold up to sign, and the sign depends only on $l_i, m_i, n_i, l_i', m_i', n_i'$ for $i = 1, 2$. The point of (4.84) is that it yields the sign 1. Since $\underline{W}$ can be covered by such open C^∞-subschemes $\underline{\hat{W}}$, and the $\Phi_{\underline{\hat{W}}}$ are compatible on overlaps $\underline{\hat{W}} \cap \underline{\hat{W}}'$, we can glue the isomorphisms $\Phi_{\underline{\hat{W}}}$ to give a unique isomorphism (4.74) with $\Phi|_{\underline{\hat{W}}} = \Phi_{\underline{\hat{W}}}$, as we have to prove. This proves the isomorphism (4.74). The last part is immediate. This completes the proof of Theorem 4.50. □

In the case in which $W = X \times_{g,Z,h} Y$ is a fibre product of manifolds with h a submersion, we can describe the isomorphism Φ in (4.74) very explicitly.

Example 4.51. Suppose X, Y, Z are manifolds, $g : X \to Z$ is smooth, and $h : Y \to Z$ is a submersion, with $\dim X = l$, $\dim Y = m$, $\dim Z = n$. Then $m \geqslant n$. Let $W = X \times_{g,Z,h} Y$ be the fibre product in **Man**, which exists as h is a submersion, and let $e : W \to X$, $f : W \to Y$ be the projections from the fibre product. Then $\dim W = k = l + m - n$, and e is submersion, as h, g is. As for (B.11) we have an exact sequence of vector bundles on W:

$$0 \to (g\circ e)^*(T^*Z) \xrightarrow{e^*(\mathrm{d}g^*)\oplus -f^*(\mathrm{d}h^*)} e^*(T^*X) \oplus f^*(T^*Y) \xrightarrow{\mathrm{d}e^*\oplus \mathrm{d}f^*} T^*W \to 0, \tag{4.85}$$

where $\mathrm{d}e : TW \to e^*(TX)$ is the derivative of e and $\mathrm{d}e^* : e^*(T^*X) \to T^*W$ the dual map. Let $w \in W$ and set $x = e(w)$, $y = f(w)$ and $z = g(x) = h(y)$.

Choose bases $\alpha_1, \ldots, \alpha_l$ for T_x^*X and $\gamma_1, \ldots, \gamma_n$ for T_z^*Z. As h is a submersion, $\mathrm{d}h^*|_y : T_z^*Z \to T_y^*Y$ is injective. Therefore we can choose $\beta_1, \ldots, \beta_{m-n} \in T_y^*Y$ such that $\mathrm{d}h^*(\gamma_1), \ldots, \mathrm{d}h^*(\gamma_n), \beta_1, \ldots, \beta_{m-n}$ are a basis for T_y^*Y. Then exactness of (4.85) implies that $\mathrm{d}e^*(\alpha_1), \ldots, \mathrm{d}e^*(\alpha_l), \mathrm{d}f^*(\beta_1), \ldots, \mathrm{d}f^*(\beta_{m-n})$ are a basis for T_w^*W. Therefore we have

$$\begin{aligned}
\Lambda^k T_w^*W &= \bigl\langle \mathrm{d}e^*(\alpha_1) \wedge \cdots \wedge \mathrm{d}e^*(\alpha_l) \wedge \mathrm{d}f^*(\beta_1) \wedge \cdots \wedge \mathrm{d}f^*(\beta_{m-n})\bigr\rangle_{\mathbb{R}}, \\
\Lambda^l T_x^*X &= \bigl\langle \alpha_1 \wedge \cdots \wedge \alpha_l \bigr\rangle_{\mathbb{R}}, \\
\Lambda^m T_y^*Y &= \bigl\langle \mathrm{d}h^*(\gamma_1) \wedge \cdots \wedge \mathrm{d}h^*(\gamma_n) \wedge \beta_1 \wedge \cdots \wedge \beta_{m-n} \bigr\rangle_{\mathbb{R}}, \quad \text{and} \\
\Lambda^n T_z^*Z &= \bigl\langle \gamma_1 \wedge \cdots \wedge \gamma_n \bigr\rangle_{\mathbb{R}}, \text{ so that } (\Lambda^n T_z^*Z)^* = \bigl\langle (\gamma_1 \wedge \cdots \wedge \gamma_n)^{-1} \bigr\rangle_{\mathbb{R}}.
\end{aligned} \tag{4.86}$$

By Proposition 4.40 we can rewrite Φ in (4.74) as an isomorphism

$$\Phi : \Lambda^k T^*W \longrightarrow e^*(\Lambda^l T^*X) \otimes f^*(\Lambda^m T^*Y) \otimes (g \circ e)^*(\Lambda^n T^*Z)^*.$$

Using this and (4.86), one can show that Φ acts at w by

$$\Phi|_w : \bigl[\mathrm{d}e^*(\alpha_1)\wedge\cdots\wedge\mathrm{d}e^*(\alpha_l)\wedge\mathrm{d}f^*(\beta_1)\wedge\cdots\wedge\mathrm{d}f^*(\beta_{m-n})\bigr]\longmapsto \tag{4.87}$$
$$\bigl[\alpha_1\wedge\cdots\wedge\alpha_l\bigr]\otimes\bigl[\mathrm{d}h^*(\gamma_1)\wedge\cdots\wedge\mathrm{d}h^*(\gamma_n)\wedge\beta_1\wedge\cdots\wedge\beta_{m-n}\bigr]\otimes\bigl[(\gamma_1\wedge\cdots\wedge\gamma_n)^{-1}\bigr].$$

It is easy to see this is independent of the choice of bases.

Fibre products have natural commutativity and associativity properties. For instance, for any $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ there is a natural equivalence $\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}\simeq\boldsymbol{Y}\times_{\boldsymbol{h},\boldsymbol{Z},\boldsymbol{g}}\boldsymbol{X}$. When we lift these to (multiple) fibre products of oriented d-manifolds, the orientations on each side differ by some sign depending on the virtual dimensions of the factors. The next proposition is the analogue of Proposition 5.37 below for manifolds, and of results by Fukaya et al. [32, Lem. 8.2.3] for Kuranishi spaces, and is proved in the same way.

Proposition 4.52. *Suppose $\boldsymbol{V},\ldots,\boldsymbol{Z}$ are oriented d-manifolds, $\boldsymbol{e},\ldots,\boldsymbol{h}$ are 1-morphisms, and all fibre products below are d-transverse. Then the following hold, in oriented d-manifolds:*

(a) *For $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ we have*

$$\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}\simeq(-1)^{(\mathrm{vdim}\,\boldsymbol{X}-\mathrm{vdim}\,\boldsymbol{Z})(\mathrm{vdim}\,\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{Z})}\boldsymbol{Y}\times_{\boldsymbol{h},\boldsymbol{Z},\boldsymbol{g}}\boldsymbol{X}. \tag{4.88}$$

In particular, when $\boldsymbol{Z}=*$ so that $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}=\boldsymbol{X}\times\boldsymbol{Y}$ we have

$$\boldsymbol{X}\times\boldsymbol{Y}\simeq(-1)^{\mathrm{vdim}\,\boldsymbol{X}\,\mathrm{vdim}\,\boldsymbol{Y}}\boldsymbol{Y}\times\boldsymbol{X}.$$

(b) *For $\boldsymbol{e}:\boldsymbol{V}\to\boldsymbol{Y}$, $\boldsymbol{f}:\boldsymbol{W}\to\boldsymbol{Y}$, $\boldsymbol{g}:\boldsymbol{W}\to\boldsymbol{Z}$, and $\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Z}$ we have*

$$\boldsymbol{V}\times_{\boldsymbol{e},\boldsymbol{Y},\boldsymbol{f}\circ\boldsymbol{\pi_W}}\bigl(\boldsymbol{W}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{X}\bigr)\simeq\bigl(\boldsymbol{V}\times_{\boldsymbol{e},\boldsymbol{Y},\boldsymbol{f}}\boldsymbol{W}\bigr)\times_{\boldsymbol{g}\circ\boldsymbol{\pi_W},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{X}.$$

(c) *For $\boldsymbol{e}:\boldsymbol{V}\to\boldsymbol{Y}$, $\boldsymbol{f}:\boldsymbol{V}\to\boldsymbol{Z}$, $\boldsymbol{g}:\boldsymbol{W}\to\boldsymbol{Y}$, and $\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Z}$ we have*

$$\begin{aligned}&\boldsymbol{V}\times_{(\boldsymbol{e},\boldsymbol{f}),\boldsymbol{Y}\times\boldsymbol{Z},\boldsymbol{g}\times\boldsymbol{h}}(\boldsymbol{W}\times\boldsymbol{X})\simeq\\&\quad(-1)^{\mathrm{vdim}\,\boldsymbol{Z}(\mathrm{vdim}\,\boldsymbol{Y}+\mathrm{vdim}\,\boldsymbol{W})}(\boldsymbol{V}\times_{\boldsymbol{e},\boldsymbol{Y},\boldsymbol{g}}\boldsymbol{W})\times_{\boldsymbol{f}\circ\boldsymbol{\pi_V},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{X}.\end{aligned}$$

To compute the signs in fibre product identities, a simple method is to suppose that all the d-manifolds are manifolds, and all the 1-morphisms are submersions, and then to use (4.87). For example, to prove (4.88), if g,h are both submersions in Example 4.51 we can replace the basis $\alpha_1,\ldots,\alpha_l$ for T_x^*X by $\mathrm{d}g^*(\gamma_1),\ldots,\mathrm{d}g^*(\gamma_n),\alpha_1,\ldots,\alpha_{l-n}$. Then for $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$, equation (4.87) becomes

$$\begin{aligned}\Phi|_w : \bigl[&\mathrm{d}(g\circ e)^*(\gamma_1)\wedge\cdots\wedge\mathrm{d}(g\circ e)^*(\gamma_n)\wedge\mathrm{d}e^*(\alpha_1)\wedge\cdots\wedge\mathrm{d}e^*(\alpha_{l-n})\wedge\mathrm{d}f^*(\beta_1)\\&\wedge\cdots\wedge\mathrm{d}f^*(\beta_{m-n})\bigr]\longmapsto\bigl[\mathrm{d}g^*(\gamma_1)\wedge\cdots\wedge\mathrm{d}g^*(\gamma_n)\wedge\alpha_1\wedge\cdots\wedge\alpha_{l-n}\bigr]\otimes\\&\bigl[\mathrm{d}h^*(\gamma_1)\wedge\cdots\wedge\mathrm{d}h^*(\gamma_n)\wedge\beta_1\wedge\cdots\wedge\beta_{m-n}\bigr]\otimes\bigl[(\gamma_1\wedge\cdots\wedge\gamma_n)^{-1}\bigr],\end{aligned} \tag{4.89}$$

and for $\boldsymbol{Y}\times_{\boldsymbol{Z}}\boldsymbol{X}$, equation (4.87) becomes

$$\begin{aligned}\Phi|_w:\bigl[&\mathrm{d}(g\circ e)^*(\gamma_1)\wedge\cdots\wedge\mathrm{d}(g\circ e)^*(\gamma_n)\wedge\mathrm{d}f^*(\beta_1)\wedge\cdots\wedge\mathrm{d}f^*(\beta_{m-n})\wedge\mathrm{d}e^*(\alpha_1)\\ &\wedge\cdots\wedge\mathrm{d}e^*(\alpha_{l-n})\bigr]\longmapsto\bigl[\mathrm{d}h^*(\gamma_1)\wedge\cdots\wedge\mathrm{d}h^*(\gamma_n)\wedge\beta_1\wedge\cdots\wedge\beta_{m-n}\bigr]\otimes\\ &\bigl[\mathrm{d}g^*(\gamma_1)\wedge\cdots\wedge\mathrm{d}g^*(\gamma_n)\wedge\alpha_1\wedge\cdots\wedge\alpha_{l-n}\bigr]\otimes\bigl[(\gamma_1\wedge\cdots\wedge\gamma_n)^{-1}\bigr].\end{aligned}\tag{4.90}$$

The right hand sides of (4.89)–(4.90) count as the same, as exchanging the order of line bundles $\underline{e}^*(\mathcal{L}_{T^*\boldsymbol{X}}),\underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}})$ does not change signs. The left hand sides of (4.89)–(4.90) differ by a sign $(-1)^{(l-n)(m-n)}$, from exchanging the $(l-n)$ α_i terms and the $(m-n)$ β_j terms. Hence the correct sign in (4.88) is $(-1)^{(l-n)(m-n)}=(-1)^{(\mathrm{vdim}\,\boldsymbol{X}-\mathrm{vdim}\,\boldsymbol{Z})(\mathrm{vdim}\,\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{Z})}$, as we want.

4.7 The homotopy category of d-manifolds

Throughout Chapters 2–4 we have taken pains to stress that **dSpa** and **dMan** are strict 2-categories, to distinguish between 1-isomorphisms and equivalences, to be careful in our use of 2-morphisms and 2-commutative diagrams, and so on. The aim of this final section is to point out that actually **dSpa**, **dMan** can be reduced to ordinary categories, and our results show that these categories are well-behaved. So for many purposes, one can treat d-spaces and d-manifolds as forming categories rather than 2-categories.

Let $\boldsymbol{\mathcal{C}}$ be a 2-category. As in §A.3, the *homotopy category* $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$ of $\boldsymbol{\mathcal{C}}$ is the category whose objects are objects of $\boldsymbol{\mathcal{C}}$, and whose morphisms are 2-isomorphism classes of 1-morphisms in $\boldsymbol{\mathcal{C}}$. Then equivalences in $\boldsymbol{\mathcal{C}}$ become isomorphisms in $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$, 2-commutative diagrams in $\boldsymbol{\mathcal{C}}$ become commutative diagrams in $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$, and so on. Thus we can form the homotopy categories $\mathrm{Ho}(\mathbf{dSpa}),\mathrm{Ho}(\mathbf{dMan})$. If $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is a 1-morphism in **dSpa** or **dMan**, write $[\boldsymbol{f}]:\boldsymbol{X}\to\boldsymbol{Y}$ for the corresponding morphism in $\mathrm{Ho}(\mathbf{dSpa})$ or $\mathrm{Ho}(\mathbf{dMan})$, where $[\boldsymbol{f}]$ is the set of 1-morphisms $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$ which are 2-isomorphic to $\boldsymbol{f}$.

We now interpret parts of Chapters 2–4 in terms of $\mathrm{Ho}(\mathbf{dSpa}),\mathrm{Ho}(\mathbf{dMan})$. Firstly, in §2.4 on gluing d-spaces by equivalences, many of our results yield a d-space unique up to equivalence, or a 1-morphism unique up to 2-isomorphism, and these d-spaces and 1-morphisms are independent of choices of 2-morphisms. These translate in $\mathrm{Ho}(\mathbf{dSpa})$ to:

- Proposition 2.27 implies that if $\boldsymbol{X},\boldsymbol{Y}$ are d-spaces, $\boldsymbol{U},\boldsymbol{V}\subseteq\boldsymbol{X}$ are open d-subspaces with $\boldsymbol{X}=\boldsymbol{U}\cup\boldsymbol{V}$, $[\boldsymbol{f}]:\boldsymbol{U}\to\boldsymbol{Y}$ and $[\boldsymbol{g}]:\boldsymbol{V}\to\boldsymbol{Y}$ are morphisms in $\mathrm{Ho}(\mathbf{dSpa})$ with $[\boldsymbol{f}]|_{\boldsymbol{U}\cap\boldsymbol{V}}=[\boldsymbol{g}]|_{\boldsymbol{U}\cap\boldsymbol{V}}$, then there is a unique morphism $[\boldsymbol{h}]:\boldsymbol{X}\to\boldsymbol{Y}$ in $\mathrm{Ho}(\mathbf{dSpa})$ with $[\boldsymbol{h}]|_{\boldsymbol{U}}=[\boldsymbol{f}]$ and $[\boldsymbol{h}]|_{\boldsymbol{V}}=[\boldsymbol{g}]$.
- Theorem 2.28 implies that if $\boldsymbol{X},\boldsymbol{Y}$ are d-spaces, $\boldsymbol{U}\subseteq\boldsymbol{X}$, $\boldsymbol{V}\subseteq\boldsymbol{Y}$ are open d-subspaces, $[\boldsymbol{f}]:\boldsymbol{U}\to\boldsymbol{V}$ is an isomorphism in $\mathrm{Ho}(\mathbf{dSpa})$, and $X\amalg_f Y$ is Hausdorff, then there exists a d-space $\boldsymbol{Z}$ unique up to canonical isomorphism in $\mathrm{Ho}(\mathbf{dSpa})$, open d-subspaces $\hat{\boldsymbol{X}},\hat{\boldsymbol{Y}}\subseteq\boldsymbol{Z}$ with $\boldsymbol{Z}=\hat{\boldsymbol{X}}\cup\hat{\boldsymbol{Y}}$, and isomorphisms $[\boldsymbol{g}]:\boldsymbol{X}\to\hat{\boldsymbol{X}}$ and $[\boldsymbol{h}]:\boldsymbol{Y}\to\hat{\boldsymbol{Y}}$ in $\mathrm{Ho}(\mathbf{dSpa})$ with $[\boldsymbol{g}]|_{\boldsymbol{U}}=[\boldsymbol{h}]\circ[\boldsymbol{f}]$. Theorem 2.29 implies $\boldsymbol{Z}=\boldsymbol{X}\amalg_{\boldsymbol{U}=\boldsymbol{V}}\boldsymbol{Y}$ is a pushout in the category $\mathrm{Ho}(\mathbf{dSpa})$.

- Theorems 2.30–2.33 yield results on gluing a family of d-spaces $\boldsymbol{X}_i$, $i \in I$ by isomorphisms $[\boldsymbol{e}_{ij}] : \boldsymbol{X}_{ij} \to \boldsymbol{X}_{ji}$ in Ho(**dSpa**) of open d-subspaces $\boldsymbol{X}_{ij} \subseteq \boldsymbol{X}_i$, $\boldsymbol{X}_{ji} \subseteq \boldsymbol{X}_j$.

Sections 2.5 and 4.3 discuss fibre products $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ in **dSpa** and **dMan**. When a fibre product exists, it is unique up to equivalence, and up to equivalence is independent of $\boldsymbol{g}, \boldsymbol{h}$ up to 2-isomorphism. Hence, up to isomorphism in Ho(**dSpa**) or Ho(**dMan**), $\boldsymbol{W}$ depends only on $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ and $[\boldsymbol{g}], [\boldsymbol{h}]$. Thus, given d-spaces $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ and morphisms $[\boldsymbol{g}] : \boldsymbol{X} \to \boldsymbol{Z}$, $[\boldsymbol{h}] : \boldsymbol{Y} \to \boldsymbol{Z}$ in Ho(**dSpa**), we can define a 'fibre product' $\boldsymbol{X} \tilde{\times}_{[\boldsymbol{g}],\boldsymbol{Z},[\boldsymbol{h}]} \boldsymbol{Y}$ to be the fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ in **dSpa** for any representatives $\boldsymbol{g}, \boldsymbol{h}$ for $[\boldsymbol{g}], [\boldsymbol{h}]$. Then $\boldsymbol{W} = \boldsymbol{X} \tilde{\times}_{[\boldsymbol{g}],\boldsymbol{Z},[\boldsymbol{h}]} \boldsymbol{Y}$ is unique up to canonical isomorphism in Ho(**dSpa**), and comes with morphisms $[\boldsymbol{e}] : \boldsymbol{W} \to \boldsymbol{X}$, $[\boldsymbol{f}] : \boldsymbol{W} \to \boldsymbol{Y}$ in Ho(**dSpa**) with $[\boldsymbol{g}] \circ [\boldsymbol{e}] = [\boldsymbol{h}] \circ [\boldsymbol{f}]$.

It is important that this 'fibre product' $\boldsymbol{X} \tilde{\times}_{[\boldsymbol{g}],\boldsymbol{Z},[\boldsymbol{h}]} \boldsymbol{Y}$ is *not* a fibre product in the category Ho(**dSpa**). That is, it is not characterized internally in Ho(**dSpa**) by a universal property. In Remark 4.25 we showed by example that **dMan** must be at least a 2-category for fibre products in **dMan** to have the properties we want, and the category Ho(**dMan**) fails this test. Instead, we can regard the operation $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, [\boldsymbol{g}], [\boldsymbol{h}]) \mapsto \boldsymbol{X} \tilde{\times}_{[\boldsymbol{g}],\boldsymbol{Z},[\boldsymbol{h}]} \boldsymbol{Y}$ as being an extra structure on Ho(**dSpa**), a shadow of the 2-category structure on **dSpa**.

In §3.1, we can pass from the 2-category vvect($\underline{X}$) of virtual vector bundles to its homotopy category Ho(vvect($\underline{X}$)). If $\boldsymbol{X}$ is a d-manifold, we regard $T^*\boldsymbol{X}$ as an object in Ho(vvect($\underline{X}$)). If $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ are 1-morphisms in **dMan** then as in Example 3.2, the corresponding 1-morphisms $\Omega_{\boldsymbol{f}}, \Omega_{\boldsymbol{g}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ in vvect($\underline{X}$) are 2-isomorphic, so $[\Omega_{\boldsymbol{f}}] = [\Omega_{\boldsymbol{g}}]$ in Ho(vvect($\boldsymbol{X}$)). Hence, if $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ is a morphism in Ho(**dMan**) we may define a morphism $\Omega_{[\boldsymbol{f}]} = [\Omega_{\boldsymbol{f}}] : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ in Ho(vvect($\underline{X}$)), where $\boldsymbol{f}$ is any representative for $[\boldsymbol{f}]$. Then:

- Call a morphism $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ in Ho(**dMan**) *étale* if it is a local isomorphism. Then Theorem 3.36 implies $[\boldsymbol{f}]$ is étale if and only if $\Omega_{[\boldsymbol{f}]} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ is an isomorphism in Ho(vvect($\underline{X}$)).
- Call a morphism $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ in Ho(**dMan**) a *w-submersion, submersion, w-immersion, immersion, w-embedding* or *embedding* if any representative $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ for $[\boldsymbol{f}]$ in **dMan** is a w-submersion, ..., embedding, respectively. Proposition 4.5(ii) shows these are independent of the choice of representative $\boldsymbol{f}$, and so are well-defined.
- Similarly, call morphisms $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Z}$, $[\boldsymbol{g}] : \boldsymbol{Y} \to \boldsymbol{Z}$ in Ho(**dMan**) *d-transverse* if any representatives $\boldsymbol{g}, \boldsymbol{h}$ in **dMan** are d-transverse. Proposition 4.17(b) shows this is independent of $\boldsymbol{g}, \boldsymbol{h}$, and so is well-defined.
- Section 4.3 gives sufficient conditions for 'fibre products' $\boldsymbol{X} \tilde{\times}_{[\boldsymbol{g}],\boldsymbol{Z},[\boldsymbol{h}]} \boldsymbol{Y}$ to exist in Ho(**dMan**), for instance, $\boldsymbol{X} \tilde{\times}_{[\boldsymbol{g}],\boldsymbol{Z},[\boldsymbol{h}]} \boldsymbol{Y}$ exists if $[\boldsymbol{g}]$ or $[\boldsymbol{h}]$ is a w-submersion, or if $\boldsymbol{Z}$ is a manifold.

Let $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ be an étale morphism in Ho(**dMan**). Define an isomorphism $\mathcal{L}_{[\boldsymbol{f}]} : \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}) \to \mathcal{L}_{T^*\boldsymbol{X}}$ of orientation line bundles on $\underline{X}$ by $\mathcal{L}_{[\boldsymbol{f}]} = \mathcal{L}_{\boldsymbol{f}}$

for any representative $\boldsymbol{f}$ for $[\boldsymbol{f}]$. Proposition 4.47(d) shows this is well defined. The material on orientations in §4.6 then makes sense in $\mathrm{Ho}(\mathbf{dMan})$.

5 Manifolds with corners

Most of the literature in differential geometry discusses only manifolds without boundary, and a smaller proportion manifolds with boundary. Only a few authors have seriously studied *manifolds with corners* (locally modelled on $[0,\infty)^k \times \mathbb{R}^{n-k}$). Some references are Cerf [20], Douady [27], Jänich [52], Laures [66], Melrose [79, 80], Monthubert [87], and the author [55]. How one sets up manifolds with corners is not universally agreed, but depends on the applications one has in mind. As explained in [55, Rem.s 2.11 & 3.3], there are at least four inequivalent definitions of manifolds with corners, two definitions of boundary, and four definitions of smooth map in use in the literature.

In [55] the author set out a theory of manifolds with corners, including a new notion of smooth map. This theory was designed to be part of the foundations of our theories of d-manifolds and d-orbifolds with corners. We now summarize those parts of [55] we need. Some material and notation is new, particularly in §5.7.

5.1 Manifolds with corners, and boundaries

Here are the principal definitions of [55, §2].

Definition 5.1. For $0 \leqslant k \leqslant n$, write $\mathbb{R}^n_k = [0,\infty)^k \times \mathbb{R}^{n-k}$. Let X be a topological space. An *n-dimensional chart on X* for $n \geqslant 0$ is a pair (U,ϕ), where U is an open subset in $\mathbb{R}^n_k$ for some $0 \leqslant k \leqslant n$, and $\phi : U \to X$ is a homeomorphism with a nonempty open set $\phi(U)$.

Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ and $\alpha : A \to B$ be continuous. We call α *smooth* if it extends to a smooth map between open neighbourhoods of A, B in $\mathbb{R}^m, \mathbb{R}^n$. When $m = n$ we call $\alpha : A \to B$ a *diffeomorphism* if it is a homeomorphism and $\alpha : A \to B$, $\alpha^{-1} : B \to A$ are smooth.

Let $(U,\phi), (V,\psi)$ be n-dimensional charts with corners on X. We call (U,ϕ) and (V,ψ) *compatible* if $\psi^{-1} \circ \phi : \phi^{-1}\big(\phi(U) \cap \psi(V)\big) \to \psi^{-1}\big(\phi(U) \cap \psi(V)\big)$ is a diffeomorphism between subsets of $\mathbb{R}^n$, in the sense above.

An *n-dimensional atlas* for X is a system $\{(U^i,\phi^i) : i \in I\}$ of pairwise compatible n-dimensional charts on X with $X = \bigcup_{i\in I} \phi^i(U^i)$. Any atlas $\{(U^i,\phi^i) : i \in I\}$ is contained in a unique maximal atlas, the set of all charts (U,ϕ) on X which are compatible with (U^i,ϕ^i) for all $i \in I$.

An *n-dimensional manifold with corners* is a second countable Hausdorff topological space X with a maximal n-dimensional atlas. We call X a *manifold without boundary* if it has a compatible atlas $\{(U^i,\phi^i) : i \in I\}$ with all U^i open in $\mathbb{R}^n$, and a *manifold with boundary* if it has a compatible atlas $\{(U^i,\phi^i) : i \in I\}$ with all U^i open in $\mathbb{R}^n$ or $[0,\infty) \times \mathbb{R}^{n-1}$. As a topological space, X is locally compact, and paracompact.

Definition 5.2. Let X be an n-manifold. A map $f : X \to \mathbb{R}$ is called *smooth* if whenever (U,ϕ) is a chart on the manifold X then $f \circ \phi : U \to \mathbb{R}$ is a smooth map between subsets of $\mathbb{R}^n, \mathbb{R}$. Write $C^\infty(X)$ for the $\mathbb{R}$-algebra of smooth functions

$f : X \to \mathbb{R}$. For $x \in X$ define the *tangent space* $T_x X$ by

$$T_x X = \{v : v \text{ is a linear map } C^\infty(X) \to \mathbb{R} \text{ satisfying } v(fg) = v(f)g(x) + f(x)v(g) \text{ for all } f, g \in C^\infty(X)\},$$

and define the *cotangent space* $T_x^* X = (T_x X)^*$. Then $T_x X \cong \mathbb{R}^n \cong T_x^* X$.

Definition 5.3. Let $U \subseteq \mathbb{R}^n_k$ be open. For each $u = (u_1, \ldots, u_n)$ in U, define the *depth* $\mathrm{depth}_U\, u$ of u in U to be the number of $u_1, \ldots, u_k$ which are zero. That is, $\mathrm{depth}_U\, u$ is the number of boundary faces of U containing u.

Let X be an n-manifold with corners. For $x \in X$, choose a chart (U, ϕ) on the manifold X with $\phi(u) = x$ for $u \in U$, and define the *depth* $\mathrm{depth}_X\, x$ of x in X by $\mathrm{depth}_X\, x = \mathrm{depth}_U\, u$. This is independent of the choice of (U, ϕ). For each $k = 0, \ldots, n$, define the *depth k stratum* of X to be

$$S^k(X) = \{x \in X : \mathrm{depth}_X\, x = k\}.$$

A *local boundary component β of X at x* is a local choice of connected component of $S^1(X)$ near x. That is, for each sufficiently small open neighbourhood V of x in X, β gives a choice of connected component W of $V \cap S^1(X)$ with $x \in \overline{W}$, and any two such choices V, W and V', W' must be compatible in the sense that $x \in \overline{(W \cap W')}$. As a set, define the *boundary*

$$\partial X = \{(x, \beta) : x \in X,\ \beta \text{ is a local boundary component for } X \text{ at } x\}.$$

Define a map $i_X : \partial X \to X$ by $i_X : (x, \beta) \mapsto x$. Note that i_X *need not be injective*, as $|i_X^{-1}(x)| = \mathrm{depth}_X\, x$. By [55, Prop. 2.7] ∂X is naturally an $(n-1)$-manifold with corners for $n > 0$, and $\partial X = \emptyset$ if $n = 0$. Thus by induction we may form the $(n-k)$-manifold with corners $\partial^k X$ for $k = 0, \ldots, n$.

Our definition of 'boundary defining function' is a little stronger than that used in [55], but yields the same notion of smooth function.

Definition 5.4. Let X be a manifold with corners, and $(x, \beta) \in \partial X$. A *boundary defining function for X at (x, β)* is a pair (V, b), where V is an open neighbourhood of x in X and $b : V \to [0, \infty)$ is a map, such that $b : V \to \mathbb{R}$ is smooth in the sense of Definition 5.2, and $\mathrm{d}b|_v : T_v V \to T_{b(v)}[0, \infty)$ is nonzero for all $v \in V$, and there exists an open neighbourhood U of (x, β) in $i_X^{-1}(V) \subseteq \partial X$, with $b \circ i_X|_U = 0$, and $i_X|_U : U \longrightarrow \{v \in V : b(v) = 0\}$ is a homeomorphism.

5.2 Smooth maps of manifolds with corners

In [55, Def. 3.1] we define *smooth maps* between manifolds with corners.

Definition 5.5. Let X, Y be manifolds with corners of dimensions m, n. A continuous map $f : X \to Y$ is called *weakly smooth* if whenever $(U, \phi), (V, \psi)$ are charts on the manifolds X, Y then

$$\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}(\psi(V)) \longrightarrow V$$

is a smooth map from $(f \circ \phi)^{-1}(\psi(V)) \subset \mathbb{R}^m$ to $V \subset \mathbb{R}^n$, where smooth maps between subsets of $\mathbb{R}^m, \mathbb{R}^n$ are defined in Definition 5.1.

A weakly smooth map $f : X \to Y$ is called *smooth* if it satisfies the following additional condition over $\partial X, \partial Y$. Suppose $x \in X$ with $f(x) = y \in Y$, and β is a local boundary component of Y at y. Let (V, b) be a boundary defining function for Y at (y, β). We require that either:

(i) There exists an open $x \in \tilde{V} \subseteq f^{-1}(V) \subseteq X$ such that $(\tilde{V}, b \circ f|_{\tilde{V}})$ is a boundary defining function for X at $(x, \tilde{\beta})$, for some unique local boundary component $\tilde{\beta}$ of X at x; or

(ii) There exists an open $x \in W \subseteq f^{-1}(V) \subseteq X$ with $b \circ f|_W = 0$.

We call a smooth map $f : X \to Y$ a *diffeomorphism* if it has a smooth inverse $f^{-1} : Y \to X$. We call f *étale* if it is a local diffeomorphism. That is, for each $x \in X$ there are open $x \in U \subseteq X$ and $f(x) \in V \subseteq Y$ such that $f|_U : U \to V$ is a diffeomorphism.

Most other authors define smooth maps to be weakly smooth maps in the sense above. Our notion of smooth map is related to Monthubert's *morphisms of manifolds with corners* [87, Def. 2.8] and Melrose's *b-maps* [80, §1.12], but is not the same as either. By [55, Lem. 2.8 & Th. 3.4] we have:

Theorem 5.6. *Let W, X, Y, Z be manifolds with corners.*

(a) *If $f : X \to Y$ and $g : Y \to Z$ are smooth then $g \circ f : X \to Z$ is smooth.*

(b) *The identity map $\mathrm{id}_X : X \to X$ is smooth.*

(c) *The map $i_X : \partial X \to X$ in Definition* 5.3 *is a smooth. As a continuous map, it is finite and proper.*

(d) *If $f : W \to Y$ and $g : X \to Z$ are smooth, the* ***product*** *$f \times g : W \times X \to Y \times Z$ given by $(f \times g)(w, x) = \big(f(w), g(x)\big)$ is smooth.*

(e) *If $f : X \to Y$ and $g : X \to Z$ are smooth, the* ***direct product*** *$(f, g) : X \to Y \times Z$ given by $(f, g)(x) = \big(f(x), g(x)\big)$ is smooth.*

Theorem 5.6(a),(b) show that manifolds with corners form a *category*, which we write $\mathbf{Man^c}$, with objects manifolds with corners X and morphisms smooth maps $f : X \to Y$. We write $\mathbf{Man^b}$ for the full subcategory of $\mathbf{Man^c}$ whose objects are manifolds with boundary, and $\mathbf{Man}$ for the full subcategory of $\mathbf{Man^c}$ whose objects are manifolds without boundary, so that $\mathbf{Man} \subset \mathbf{Man^b} \subset \mathbf{Man^c}$.

5.3 Describing how smooth maps act on corners

If $f : X \to Y$ is a smooth map of manifolds with corners, then f may relate $\partial^k X$ to $\partial^l Y$ for $k, l \geqslant 0$ in complicated ways. We now define some notation which is useful for describing this. It is based on [55, §4], but [55] uses the notation $\Xi^f_-, \Xi^f_+, \xi^f_-, \xi^f_+$ in place of $j_f(S_f), T_f, s_f \circ j_f^{-1}, t_f$.

Definition 5.7. Let X,Y be manifolds with corners, and $f:X\rightarrow Y$ a smooth map. Consider $f:X\rightarrow Y$, $i_X:\partial X\rightarrow X$, $i_Y:\partial Y\rightarrow Y$ as continuous maps of topological spaces. Form the fibre products of topological spaces

$$\begin{aligned}\partial X\times_{f\circ i_X,Y,i_Y}\partial Y&=\bigl\{\bigl((x,\tilde\beta),(y,\beta)\bigr)\in\partial X\times\partial Y:f\circ i_X(x,\tilde\beta)=y=i_Y(y,\beta)\bigr\},\\ X\times_{f,Y,i_Y}\partial Y&=\bigl\{\bigl(x,(y,\beta)\bigr)\in X\times\partial Y:f(x)=y=i_Y(y,\beta)\bigr\}.\end{aligned}$$

By Definition 5.5, for each $\bigl(x,(y,\beta)\bigr)\in X\times_Y\partial Y$, if (V,b) is a boundary defining function for Y at (y,β), then either (i) $(\tilde V,b\circ f|_{\tilde V})$ is a boundary defining function for X at some unique $(x,\tilde\beta)$ for open $x\in\tilde V\subseteq f^{-1}(V)$, or (ii) $b\circ f|_W=0$ for open $x\in W\subseteq f^{-1}(V)$. Define subsets $S_f\subseteq\partial X\times_Y\partial Y$ and $T_f\subseteq X\times_Y\partial Y$ by $\bigl((x,\tilde\beta),(y,\beta)\bigr)\in S_f$ in case (i), and $\bigl(x,(y,\beta)\bigr)\in T_f$ in case (ii). These are independent of the choice of (V,b), and so are well-defined. Define maps

$$\begin{gathered}j_f:S_f\rightarrow X\times_Y\partial Y,\ s_f:S_f\rightarrow\partial X,\ t_f:T_f\rightarrow X,\ u_f:S_f\rightarrow\partial Y,\ v_f:T_f\rightarrow\partial Y\\ \text{by}\quad j_f:\bigl((x,\tilde\beta),(y,\beta)\bigr)\mapsto\bigl(x,(y,\beta)\bigr),\quad s_f:\bigl((x,\tilde\beta),(y,\beta)\bigr)\mapsto(x,\tilde\beta),\\ t_f:\bigl(x,(y,\beta)\bigr)\mapsto x,\ u_f:\bigl((x,\tilde\beta),(y,\beta)\bigr)\mapsto(y,\beta),\ v_f:\bigl(x,(y,\beta)\bigr)\mapsto(y,\beta).\end{gathered}$$

Here [55, Prop. 4.2] are some properties of these.

Proposition 5.8. *Let $f:X\rightarrow Y$ be a smooth map of manifolds with corners, and use the notation of Definition* 5.7, *with $n=\dim X$. Then*

(i) *S_f is an open and closed subset of $\partial X\times_Y\partial Y$, and $s_f:S_f\rightarrow\partial X$ is an étale map of topological spaces.*

(ii) *$j_f:S_f\rightarrow X\times_Y\partial Y$ is a homeomorphism with an open and closed subset $j_f(S_f)$ in $X\times_Y\partial Y$.*

(iii) *$T_f=(X\times_Y\partial Y)\setminus j_f(S_f)$, so that T_f is open and closed in $X\times_Y\partial Y$ by* (ii). *Also $t_f:T_f\rightarrow X$ is an étale map of topological spaces.*

(iv) *Parts* (i)-(iii) *imply there is a unique way to make S_f into an $(n-1)$-manifold with corners and T_f into an n-manifold with corners so that s_f,t_f are étale maps of manifolds. Then u_f,v_f are also smooth maps.*

5.4 Simple, semisimple and flat maps, and submersions

In the next definition, submersions are as in [55, §3]. Semisimple, flat maps f below were called semisimple in [55, §3]. The remaining notions are new.

Definition 5.9. Let $f:X\rightarrow Y$ be a smooth map of manifolds with corners.

(i) We call f *simple* if $s_f:S_f\rightarrow\partial X$ is bijective, in the notation of §5.3.

(ii) We call f *semisimple* if $s_f:S_f\rightarrow\partial X$ is injective.

(iii) We call f *flat* if $T_f=\emptyset$.

(iv) Suppose $x \in X$ with $f(x) = y$, and let $x \in S^k(X)$ and $y \in S^l(Y)$. Then the derivative gives a linear map $\mathrm{d}f|_x : T_xX \to T_yY$, which maps $T_x(S^k(X)) \to T_y(S^l(Y))$. We call f a *submersion* if $\mathrm{d}f|_x : T_xX \to T_{f(x)}Y$ and $\mathrm{d}f|_x : T_x(S^k(X)) \to T_{f(x)}(S^l(Y))$ are surjective for all $x \in X$.

(v) We call f an *s-submersion* if f is a simple submersion.

Simple maps are semisimple. Submersions are automatically semisimple and flat. S-submersions are automatically simple and flat. Diffeomorphisms and étale maps are simple and flat. If X is a manifold with corners and $\partial X \neq \emptyset$ then $i_X : \partial X \to X$ is simple, but not flat. For $f : X \to Y$ to be simple, semisimple or flat is an essentially discrete condition on the behaviour of f over $\partial^k X, \partial^l Y$. We will see in §5.6 that if f is simple and flat then $\partial X \cong X \times_Y \partial Y$, and boundaries of fibre products $X \times_{g,Z,h} Y$ are easier to understand when g or h is (semi)simple. Projections from products are examples of (s-)submersions.

Lemma 5.10. *Let X, Y be manifolds with corners. Then the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are submersions. If $\partial Y = \emptyset$ then π_X is an s-submersion, and if $\partial X = \emptyset$ then π_Y is an s-submersion.*

Our next proposition shows that all (s-)submersions are locally modelled on projections from products. It is proved in [55, Prop. 5.1] for submersions. The extension to s-submersions is straightforward.

Proposition 5.11. *Let $f : X \to Y$ be a submersion of manifolds with corners, and $x \in X$ with $f(x) = y \in Y$. Then there exist open neighbourhoods V of x in X and W of y in Y with $f(V) = W$, a manifold with corners Z, and a diffeomorphism $V \cong W \times Z$ which identifies $f|_V : V \to W$ with $\pi_W : W \times Z \to W$. If f is an s-submersion then Z is without boundary.*

Note that we need both $\mathrm{d}f|_x : T_xX \to T_{f(x)}Y$ and $\mathrm{d}f|_x : T_x(S^k(X)) \to T_{f(x)}(S^l(Y))$ to be surjective in Definition 5.9(i) to make Proposition 5.11 true. A smooth map $f : X \to Y$ induces a decomposition $\partial X = \partial_+^f X \amalg \partial_-^f X$:

Definition 5.12. Let $f : X \to Y$ be a smooth map of manifolds with corners. Define subsets $\partial_+^f X, \partial_-^f X$ of ∂X by $\partial_-^f X = s_f(S_f)$ and $\partial_+^f X = \partial X \setminus \partial_-^f X$. Then $\partial X = \partial_+^f X \amalg \partial_-^f X$, and $\partial_+^f X, \partial_-^f X$ are open and closed subsets of ∂X by [55, Prop. 4.3], so they are manifolds with corners.

Now suppose f is semisimple, for instance, f could be a submersion. Then $s_f : S_f \to \partial X$ is injective, with image $\partial_-^f X$, and by Proposition 5.8(iv) s_f is an étale map of manifolds, and $u_f : S_f \to \partial Y$ is smooth. Hence $s_f : S_f \to \partial_-^f X$ is a diffeomorphism. Define smooth maps $f_+ : \partial_+^f X \to Y$ and $f_- : \partial_-^f X \to \partial Y$ by

$$f_+ = f \circ i_X|_{\partial_+^f X} : \partial_+^f X \longrightarrow Y \quad \text{and} \quad f_- = u_f \circ s_f^{-1} : \partial_-^f X \longrightarrow \partial Y. \tag{5.1}$$

If f is simple then $\partial_-^f X = \partial X$, so $\partial_+^f X = \emptyset$, and f_- maps $\partial X \to \partial Y$.

For example, if X is a manifold with corners, then $i_X : \partial X \to X$ is simple, and $(i_X)_- = i_{\partial X} : \partial^2 X \to \partial X$. The next proposition is proved in [55, Prop. 2.28] for submersions; the proof for the other cases is similar.

Proposition 5.13. *Let $f : X \to Y$ be a semisimple map of manifolds with corners, and $\partial_\pm^f X, f_\pm$ be as in Definition* 5.12. *Then f_+, f_- are also semisimple, with $f \circ i_X|_{\partial_-^f X} = i_Y \circ f_-$. If f is simple, flat, a submersion, or an s-submersion, then f_+, f_- are also simple, ..., s-submersions, respectively.*

5.5 Corners $C_k(X)$ and the corner functors

The corners $C_k(X)$ of a manifold with corners X are closely related to $\partial^k X$.

Definition 5.14. Let X be an n-manifold with corners. Define the *k-corners* $C_k(X)$ of X for $k = 0, \ldots, n$ to be

$$\begin{aligned} C_k(X) = \bigl\{(x,\{\beta_1,\ldots,\beta_k\}) : x \in X,\ \beta_1,\ldots,\beta_k \text{ are distinct} \\ \text{local boundary components for } X \text{ at } x\bigr\} \cong \partial^k X/S_k. \end{aligned}$$

As in [55, §2], $C_k(X)$ has the structure of an $(n-k)$-manifold with corners. It is related to the k^{th} boundary $\partial^k X$ of X as follows: by [55, Prop. 2.9] we have a natural isomorphism

$$\begin{aligned} \partial^k X \cong \bigl\{(x,\beta_1,\ldots,\beta_k) : x \in X,\ \beta_1,\ldots,\beta_k \text{ are distinct} \\ \text{local boundary components for } X \text{ at } x\bigr\}. \end{aligned} \tag{5.2}$$

From (5.2) we see that $\partial^k X$ has a natural, free action of the symmetric group S_k of permutations of $\{1,\ldots,k\}$ by diffeomorphisms, given by

$$\sigma : (x,\beta_1,\ldots,\beta_k) \longmapsto (x,\beta_{\sigma(1)},\ldots,\beta_{\sigma(k)}),$$

and $C_k(X) \cong \partial^k X/S_k$. In particular, $C_0(X) \cong X$ and $C_1(X) \cong \partial X$.

Define $\Pi_X : C_k(X) \to X$ by $\Pi_X : (x,\beta_1,\ldots,\beta_k) \mapsto x$. Then Π_X is a smooth map of manifolds with corners. By [55, Prop. 2.13] there are natural identifications, with the first a diffeomorphism:

$$\begin{aligned} \partial\bigl(C_k(X)\bigr) \cong C_k(\partial X) \cong \bigl\{(x,\beta_1,\{\beta_2,\ldots,\beta_{k+1}\}) : x \in X,\ \beta_1,\ldots,\beta_{k+1} \\ \text{are distinct local boundary components for } X \text{ at } x\bigr\}. \end{aligned} \tag{5.3}$$

If X, Y are manifolds with corners then by [55, Prop. 2.12] there is a natural diffeomorphism:

$$C_k(X \times Y) \cong \textstyle\coprod_{i,j\geqslant 0,\ i+j=k} C_i(X) \times C_j(Y). \tag{5.4}$$

Several results in [55] show that the assignment $X \mapsto C(X) = \coprod_{k=0}^{\dim X} C_k(X)$ behaves in a very functorial way. For instance, (5.4) implies that

$$\coprod_{k=0}^{\dim X\times Y} C_k(X\times Y) \cong \bigl[\coprod_{i=0}^{\dim X} C_i(X)\bigr] \times \bigl[\coprod_{j=0}^{\dim Y} C_j(Y)\bigr]. \tag{5.5}$$

Since $\coprod_{k=0}^{\dim X} C_k(X)$ is generally not a manifold, but rather a disjoint union of manifolds of different dimensions, we enlarge our category of manifolds with corners to allow such disjoint unions.

Definition 5.15. Write $\check{\mathbf{Man}}^{\mathbf{c}}$ for the category whose objects are disjoint unions $\coprod_{m=0}^{\infty} X_m$, where X_m is a manifold with corners of dimension m, and whose morphisms are continuous maps $f : \coprod_{m=0}^{\infty} X_m \to \coprod_{n=0}^{\infty} Y_n$, such that $f|_{X_m\cap f^{-1}(Y_n)} : \bigl(X_m \cap f^{-1}(Y_n)\bigr) \to Y_n$ is a smooth map of manifolds with corners for all $m, n \geqslant 0$. Here $X_m \cap f^{-1}(Y_n)$ is open and closed in X_m as f is continuous and Y_n is open and closed in $\coprod_{n=0}^{\infty} Y_n$, so $X_m \cap f^{-1}(Y_n)$ is an m-manifold. A map $f : \coprod_{m=0}^{\infty} X_m \to \coprod_{n=0}^{\infty} Y_n$ satisfying the above conditions will be called *smooth*.

Definition 5.16. We will define a functor $C : \mathbf{Man^c} \to \check{\mathbf{Man}}^{\mathbf{c}}$ called the *corner functor*. On objects, if X is a manifold with corners define $C(X) = \coprod_{k=0}^{\dim X} C_k(X)$. This is a disjoint union of manifolds of dimensions $0, \ldots, \dim X$, and so an object in $\check{\mathbf{Man}}^{\mathbf{c}}$. On morphisms, if $f : X \to Y$ is a smooth map of manifolds with corners, define $C(f) : C(X) \to C(Y)$ by

$$\begin{aligned} &C(f) : \bigl(x, \{\tilde\beta_1, \ldots, \tilde\beta_i\}\bigr) \longmapsto \bigl(y, \{\beta_1, \ldots, \beta_j\}\bigr), \quad \text{where } y = f(x), \\ &\{\beta_1, \ldots, \beta_j\} = \bigl\{\beta : \bigl((x, \tilde\beta_l), (y, \beta)\bigr) \in S_f, \text{ some } l = 1, \ldots, i\bigr\}. \end{aligned} \tag{5.6}$$

Write $C_i^{f,j}(X) = C_i(X) \cap C(f)^{-1}(C_j(Y))$ and $C_i^j(f) = C(f)|_{C_i^{f,j}(X)} : C_i^{f,j}(X) \to C_j(Y)$ for all i, j. Note that $C_0^{f,0}(X) = C_0(X) \cong X$ and $C_0(Y) \cong Y$, and these isomorphisms identify $C_0^0(f) : C_0(X) \to C_0(Y)$ with $f : X \to Y$.

Here is [55, Th. 4.7], except for (vii) which we have expanded. Parts (i)–(iii) show that C is a functor.

Theorem 5.17. *Let W, X, Y, Z be manifolds with corners.*

- **(i)** *If $f : X \to Y$ is smooth then $C(f) : C(X) \to C(Y)$ is smooth in the sense of Definition 5.15. Equivalently, $C_i^{f,j}(X)$ is open and closed in $C_i(X)$ and $C_i^j(f) : C_i^{f,j}(X) \to C_j(Y)$ is a smooth map of manifolds with corners for all $i = 0, \ldots, \dim X$ and $j = 0, \ldots, \dim Y$.*
- **(ii)** *If $f : X \to Y$ and $g : Y \to Z$ are smooth then $C(g \circ f) = C(g) \circ C(f) : C(X) \to C(Z)$.*
- **(iii)** $C(\mathrm{id}_X) = \mathrm{id}_{C(X)} : C(X) \to C(X)$.
- **(iv)** *The diffeomorphisms $C_k(\partial X) \cong \partial C_k(X)$ in (5.3) identify*

$$\begin{aligned} &C(i_X) : \textstyle\coprod_{k\geqslant 0} C_k(\partial X) \longrightarrow \coprod_{k\geqslant 0} C_k(X) \qquad \textit{with} \\ &i_{\coprod_{k\geqslant 0} C_k(X)} := \textstyle\coprod_{k\geqslant 0} i_{C_k(X)} = \coprod_{k\geqslant 0} \partial C_k(X) \longrightarrow \coprod_{k\geqslant 0} C_k(X). \end{aligned}$$

(v) *Let $f : W \to Y$ and $g : X \to Z$ be smooth maps. Then* (5.5) *gives*

$$C(W \times X) \cong C(W) \times C(X) \text{ and } C(Y \times Z) \cong C(Y) \times C(Z). \tag{5.7}$$

These identify $C(f \times g) : C(W \times X) \to C(Y \times Z)$ with $C(f) \times C(g) : C(W) \times C(X) \to C(Y) \times C(Z)$.

(vi) *Let $f : X \to Y$ and $g : X \to Z$ be smooth maps. Then* (5.7) *identifies $C\big((f, g)\big) : C(X) \to C(Y \times Z)$ with $\big(C(f), C(g)\big) : C(X) \to C(Y) \times C(Z)$.*

(vii) *Let $f : X \to Y$ be a semisimple smooth map. Then $C(f)$ maps $C_k(X) \to \coprod_{l=0}^{k} C_l(Y)$ for all $k \geqslant 0$. The natural diffeomorphisms $C_1(X) \cong \partial X$, $C_0(Y) \cong Y$ and $C_1(Y) \cong \partial Y$ identify $C_1^{f,0}(X) \cong \partial_+^f X$, $C_1^0(f) \cong f_+$, $C_1^{f,1}(X) \cong \partial_-^f X$ and $C_1^1(f) \cong f_-$. If f is simple then $C(f)$ maps $C_k(X) \to C_k(Y)$ for all $k \geqslant 0$.*

As in [55, §4], there is also a second way to define a functor $\hat{C} : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$ with almost the same properties. For X a manifold with corners define $\hat{C}(X) = C(X) = \coprod_{k=0}^{\dim X} C_k(X)$, and for $f : X \to Y$ a smooth map of manifolds with corners define

$$\begin{aligned} &\hat{C}(f) : \big(x, \{\tilde{\beta}_1, \ldots, \tilde{\beta}_i\}\big) \longmapsto \big(y, \{\beta_1, \ldots, \beta_j\}\big), \quad \text{where } y = f(x), \\ &\{\beta_1, \ldots, \beta_j\} = \big\{\beta : \big((x, \tilde{\beta}_l), (y, \beta)\big) \in S_f,\ l = 1, \ldots, i\big\} \\ &\qquad\qquad \cup \big\{\beta : \big(x, (y, \beta)\big) \in T_f\big\}. \end{aligned} \tag{5.8}$$

Then the analogues of Theorem 5.17 and Theorem 5.26 below hold for $\hat{C}$, with the exception of Theorem 5.17(iv),(vii). If f is flat then $\hat{C}(f) = C(f)$. One way to understand the relationship between C and $\hat{C}$ is that

$$\begin{aligned} &C(f) : \big(x, \{\tilde{\beta}_1, \ldots, \tilde{\beta}_i\}\big) \longmapsto \big(y, \{\beta_1, \ldots, \beta_j\}\big) \quad \text{if and only if} \\ &\hat{C}(f) : \big(x, \{\tilde{\beta} : (x, \tilde{\beta}) \in \partial X\} \setminus \{\tilde{\beta}_1, \ldots, \tilde{\beta}_i\}\big) \longmapsto \\ &\qquad \big(y, \{\beta : (y, \beta) \in \partial Y\} \setminus \{\beta_1, \ldots, \beta_j\}\big). \end{aligned} \tag{5.9}$$

That is, $C, \hat{C}$ are related by taking complements of subsets in $i_X^{-1}(x), i_Y^{-1}(y)$.

Corners are also useful for understanding the fixed point locus X^Γ of a group Γ acting on a manifold with corners X. The next result, which is new, says that X^Γ lies in $\check{\mathbf{M}}\mathbf{an^c}$ and $C(X^\Gamma) \cong C(X)^\Gamma$, though in general $\partial(X^\Gamma) \ncong (\partial X)^\Gamma$. To prove it, we consider the local form of the Γ-action on X: near a fixed point x of Γ, X looks locally like $[0, \infty)^k \times \mathbb{R}^{m-k}$ near 0, where Γ acts on $[0, \infty)^k \times \mathbb{R}^{m-k}$ by permuting the coordinates $x_1, \ldots, x_k$ in $[0, \infty)^k$, and linearly on $\mathbb{R}^{m-k}$. Then X^Γ near x looks locally like $[0, \infty)^l \times \mathbb{R}^{n-l}$ near 0, where l is the number of Γ-orbits in $\{1, \ldots, k\}$ under the permutation action of Γ on $x_1, \ldots, x_k$, and $n - l = \dim(\mathbb{R}^{m-k})^\Gamma$. We will use Proposition 5.18 to help understand orbifold strata of orbifolds with corners in §8.9.

Proposition 5.18. *Suppose X is a manifold with corners, Γ a finite group, and $r : \Gamma \to \mathrm{Aut}(X)$ an action of Γ on X by diffeomorphisms. Applying the* *corner functor C of Definition 5.16 gives an action $C(r):\Gamma\to\mathrm{Aut}(C(X))$ of Γ on $C(X)$ by diffeomorphisms. Write $X^\Gamma, C(X)^\Gamma$ for the subsets of $X, C(X)$ fixed by Γ, and $j_{X,\Gamma}:X^\Gamma\to X$ for the inclusion. Then:*

(a) *X^Γ has the structure of an object in $\check{\mathbf{M}}\mathbf{an^c}$ (a disjoint union of manifolds with corners of different dimensions, as in Definition 5.15) in a unique way, such that $j_{X,\Gamma}:X^\Gamma\to X$ is an embedding. This $j_{X,\Gamma}$ is flat, but need not be (semi)simple.*

(b) *By **(a)** we have a smooth map $C(j_{X,\Gamma}):C(X^\Gamma)\to C(X)$. This $C(j_{X,\Gamma})$ is a diffeomorphism $C(X^\Gamma)\to C(X)^\Gamma$. As $j_{X,\Gamma}$ need not be simple, $C(j_{X,\Gamma})$ need not map $C_k(X^\Gamma)\to C_k(X)$ for $k>0$.*

(c) *By **(b)**, $C(j_{X,\Gamma})$ identifies $C_1(X^\Gamma)\cong\partial(X^\Gamma)$ with a subset of $C(X)^\Gamma\subseteq C(X)$. This gives the following description of $\partial(X^\Gamma)$:*

$$\begin{aligned}\partial(X^\Gamma)\cong\bigl\{(x,\{\beta_1,\dots,\beta_k\})\in C_k(X):{}&x\in X^\Gamma,\ k\geqslant 1,\ \beta_1,\dots,\beta_k\\ &\textit{are distinct local boundary components for } X \textit{ at } x,\\ &\textit{and } \Gamma \textit{ acts transitively on } \{\beta_1,\dots,\beta_k\}\bigr\}.\end{aligned}\tag{5.10}$$

(d) *Now suppose Y is a manifold with corners with an action of Γ, and $f:X\to Y$ is a Γ-equivariant smooth map. Then X^Γ, Y^Γ are objects in $\check{\mathbf{M}}\mathbf{an^c}$ by **(a)**, and $f^\Gamma:=f|_{X^\Gamma}:X^\Gamma\to Y^\Gamma$ is a morphism in $\check{\mathbf{M}}\mathbf{an^c}$.*

Here is a simple example:

Example 5.19. Let $\Gamma=\{1,\sigma\}$ with $\sigma^2=1$, so that $\Gamma\cong\mathbb{Z}_2$, and let Γ act on $X=[0,\infty)^2$ by $\sigma:(x_1,x_2)\mapsto(x_2,x_1)$. Then $X^\Gamma=\bigl\{(x,x):x\in[0,\infty)\bigr\}\cong[0,\infty)$, a manifold with corners, and the inclusion $j_{X,\Gamma}:X^\Gamma\to X$ is $j_{X,\Gamma}:[0,\infty)\to[0,\infty)^2$, $j_{X,\Gamma}:x\mapsto(x,x)$, a smooth, flat embedding, which is not semisimple. We have $\partial X=\partial\bigl([0,\infty)^2\bigr)\cong[0,\infty)\amalg[0,\infty)$, where Γ acts freely on ∂X by exchanging the two copies of $[0,\infty)$. Hence $(\partial X)^\Gamma=\emptyset$, but $\partial(X^\Gamma)$ is a point $*$, so in this case $(\partial X)^\Gamma\not\cong\partial(X^\Gamma)$. Also $C_2(X)=\bigl\{\bigl(0,\{\{x_1=0\},\{x_2=0\}\}\bigr)\bigr\}$ is a single point, which is Γ-invariant, and $C(j_{X,\Gamma}):C(X^\Gamma)\to C(X)^\Gamma$ identifies $(0,\{\{x=0\}\})\in C_1(X^\Gamma)\cong\partial X$ with this point in $C_2(X)^\Gamma$.

5.6 Transversality and fibre products

In [55, Def. 6.1 & Th. 6.4] we give conditions for fibre products to exist in $\mathbf{Man^c}$.

Definition 5.20. Let X,Y,Z be manifolds with corners and $g:X\to Z$, $h:Y\to Z$ be smooth maps. We call g,h *transverse* if the following holds. Suppose $x\in X$, $y\in Y$ and $z\in Z$ with $g(x)=z=h(y)$, so that there are induced linear maps of tangent spaces $\mathrm{d}g|_x:T_xX\to T_zZ$ and $\mathrm{d}h|_y:T_yY\to T_zZ$. Let $x\in S^j(X)$, $y\in S^k(Y)$ and $z\in S^l(Z)$, so that $\mathrm{d}g|_x$ maps $T_x(S^j(X))\to T_z(S^l(Z))$ and $\mathrm{d}h|_y$ maps $T_y(S^k(Y))\to T_z(S^l(Z))$. Then we require that $T_zZ=\mathrm{d}g|_x(T_xX)+\mathrm{d}h|_y(T_yY)$ and $T_z(S^l(Z))=\mathrm{d}g|_x(T_x(S^j(X)))+\mathrm{d}h|_y(T_y(S^k(Y)))$ for all such x,y,z. If one of g,h is a submersion then g,h are transverse.

Theorem 5.21. *Suppose X, Y, Z are manifolds with corners and $g : X \to Z$, $h : Y \to Z$ are transverse smooth maps. Then there exists a fibre product $W = X \times_{g,Z,h} Y$ in the category* $\mathbf{Man^c}$ *of manifolds with corners, which is given by an explicit construction, as follows.*

As a topological space $W = \{(x, y) \in X \times Y : g(x) = h(y)\}$, with the topology induced by the inclusion $W \subseteq X \times Y$, and the projections $\pi_X : W \to X$ and $\pi_Y : W \to Y$ map $\pi_X : (x, y) \mapsto x$, $\pi_Y : (x, y) \mapsto y$. Let $n = \dim X + \dim Y - \dim Z$, so that $n \geqslant 0$ if $W \neq \emptyset$. The maximal atlas on W is the set of all charts (U, ϕ), where $U \subseteq \mathbb{R}^n_k$ is open and $\phi : U \to W$ is a homeomorphism with a nonempty open set $\phi(U)$ in W, such that $\pi_X \circ \phi : U \to X$ and $\pi_Y \circ \phi : U \to Y$ are smooth maps, and for all $u \in U$ with $\phi(u) = (x, y)$, the following induced linear map of real vector spaces is injective:

$$\mathrm{d}(\pi_X \circ \phi)|_u \oplus \mathrm{d}(\pi_Y \circ \phi)|_u : T_uU = \mathbb{R}^n \longrightarrow T_xX \oplus T_yY.$$

In the general case of Theorem 5.21, the description of ∂W in terms of $\partial X, \partial Y, \partial Z$ is rather complicated. Here [55, Prop.s 6.6–6.8] are three cases in which the expression simplifies. For the first, [55, Prop. 6.6] is proved only for f a submersion, and hence semisimple and flat, but the extension to the other cases is straightforward.

Proposition 5.22. *Let $f : X \to Y$ be a smooth map of manifolds with corners.*

(a) *Suppose f is semisimple and flat, for instance, f could be a submersion. Then f, i_Y are transverse, and there is a canonical diffeomorphism*

$$\partial_-^f X \cong X \times_{f,Y,i_Y} \partial Y, \tag{5.11}$$

which identifies $f_- : \partial_-^f X \to \partial Y$ and $\pi_{\partial Y} : X \times_Y \partial Y \to \partial Y$.

(b) *Suppose f is simple and flat. Then* (5.11) *becomes $\partial X \cong X \times_{f,Y,i_Y} \partial Y$.*

Proposition 5.23. *Let X, Y be manifolds with corners, Z a manifold without boundary, and $g : X \to Z$, $h : Y \to Z$ be transverse smooth maps. Then $g \circ i_X : \partial X \to Z$, $h : Y \to Z$ are transverse, and $g : X \to Z$, $h \circ i_Y : \partial Y \to Z$ are transverse, and there is a canonical diffeomorphism*

$$\partial\bigl(X \times_{g,Z,h} Y\bigr) \cong \bigl(\partial X \times_{g \circ i_X, Z, h} Y\bigr) \amalg \bigl(X \times_{g,Z,h \circ i_Y} \partial Y\bigr). \tag{5.12}$$

Proposition 5.24. *Let X, Y, Z be manifolds with corners, $g : X \to Z$ a submersion and $h : Y \to Z$ smooth. Then there is a canonical diffeomorphism*

$$\partial\bigl(X \times_{g,Z,h} Y\bigr) \cong \bigl(\partial_+^g X \times_{g_+,Z,h} Y\bigr) \amalg \bigl(X \times_{g,Z,h \circ i_Y} \partial Y\bigr). \tag{5.13}$$

If both g, h are submersions there is also a canonical diffeomorphism

$$\begin{aligned}&\partial\bigl(X \times_{g,Z,h} Y\bigr) \cong \\ &\quad\bigl(\partial_+^g X \times_{g_+,Z,h} Y\bigr) \amalg \bigl(X \times_{g,Z,h_+} \partial_+^h Y\bigr) \amalg \bigl(\partial_-^g X \times_{g_-,\partial Z,h_-} \partial_-^h Y\bigr).\end{aligned} \tag{5.14}$$

Equation (5.13) *also holds if* g,h *are transverse and* g *is semisimple, and* (5.14) *also holds if* g,h *are transverse and both are semisimple.*

In [55, Def. 6.10] we also define a stronger notion of transversality.

Definition 5.25. Let X,Y,Z be manifolds with corners and $g:X\rightarrow Z$, $h:Y\rightarrow Z$ be smooth maps. We call g,h *strongly transverse* if they are transverse, and whenever there are points in $C_j(X),C_k(Y),C_l(Z)$ with

$$C(g)(x,\{\beta_1,\ldots,\beta_j\})=C(h)(y,\{\tilde\beta_1,\ldots,\tilde\beta_k\})=(z,\{\dot\beta_1,\ldots,\dot\beta_l\})$$

we have either $j+k>l$ or $j=k=l=0$. Some sufficient conditions for g,h to be strongly transverse are that one of g,h is a submersion, or g,h are transverse and $\partial^2 Z=\emptyset$, or g,h are transverse and one of g,h is semisimple.

In the situation of Theorem 5.21 we have a Cartesian square in $\mathbf{Man^c}$:

$$\begin{tikzcd} W \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z, \end{tikzcd} \quad \begin{array}{l}\text{which induces}\\ \text{a commutative}\\ \text{square in } \mathbf{\check{M}an^c}\end{array} \quad \begin{tikzcd} C(W) \arrow[r, "C(\pi_Y)"] \arrow[d, "C(\pi_X)"] & C(Y) \arrow[d, "C(h)"] \\ C(X) \arrow[r, "C(g)"] & C(Z). \end{tikzcd} \tag{5.15}$$

It is natural to wonder whether the right hand square in (5.15) is Cartesian. By [55, Th. 6.11], the answer is yes if and only if g,h are strongly transverse:

Theorem 5.26. *Let* X,Y,Z *be manifolds with corners, and* $g:X\rightarrow Z$, $h:Y\rightarrow Z$ *be strongly transverse smooth maps, and write* W *for the fibre product* $X\times_{g,Z,h}Y$ *in Theorem* 5.21. *Then there is a canonical diffeomorphism*

$$C_i(W)\cong\coprod_{j,k,l\geqslant 0:i=j+k-l} C_j^{g,l}(X)\times_{C_j^l(g),C_l(Z),C_k^l(h)} C_k^{h,l}(Y) \tag{5.16}$$

for all $i\geqslant 0$, *where the fibre products are all transverse and so exist. Hence*

$$C(W)\cong C(X)\times_{C(g),C(Z),C(h)} C(Y). \tag{5.17}$$

Here the right hand square in (5.15) *induces a map from the left hand side of* (5.17) *to the right hand side, which gives the identification* (5.17).

Since $\partial W\cong C_1(W)$, equation (5.16) when $i=1$ becomes

$$\partial W\cong\coprod_{j,k,l\geqslant 0:j+k=l+1} C_j^{g,l}(X)\times_{C_j^l(g),C_l(Z),C_k^l(h)} C_k^{h,l}(Y). \tag{5.18}$$

Propositions 5.23 and 5.24 may be deduced from this. In [55, Ex. 6.12] we give an example of g,h which are transverse, but not strongly transverse.

Example 5.27. Define smooth maps $g:[0,\infty)\rightarrow[0,\infty)^2$ by $g(x)=(x,2x)$ and $h:[0,\infty)\rightarrow[0,\infty)^2$ by $h(y)=(2y,y)$. Then $f(0)=g(0)=(0,0)$. We have

$$\begin{aligned}\mathrm{d}g|_0\bigl(T_0[0,\infty)\bigr)+\mathrm{d}h|_0\bigl(T_0[0,\infty)\bigr)&=\langle(1,2)\rangle_{\mathbb{R}}+\langle(2,1)\rangle_{\mathbb{R}}=\mathbb{R}^2=T_{(0,0)}[0,\infty)^2,\\ \mathrm{d}g|_0\bigl(T_0\bigl(S^0([0,\infty))\bigr)\bigr)+\mathrm{d}h|_0\bigl(T_0\bigl(S^0([0,\infty))\bigr)\bigr)&=\{0\}=T_{(0,0)}\bigl(S^0([0,\infty)^2)\bigr),\end{aligned}$$

so g, h are transverse. However we have

$$C(g)\big(0, \big\{\{x=0\}\big\}\big) = C(h)\big(0, \big\{\{y=0\}\big\}\big) = \big((0,0), \big\{\{x=0\}, \{y=0\}\big\}\big),$$

with $j = k = 1$ and $l = 2$, so g, h are not strongly transverse. The fibre product $W = [0,\infty)_{f,[0,\infty)^2,g}[0,\infty)$ is a single point $\{0\}$. In (5.16) when $i = 0$ the l.h.s. is one point, and the r.h.s. is two points, one from $j=k=l=0$ and one from $j=k=1$, $l=2$, so (5.16) does not hold. For $i \neq 0$, both sides of (5.16) are empty.

5.7 Immersions, embeddings, and submanifolds

We define three kinds of immersions and embeddings for manifolds with corners, and hence six kinds of submanifolds.

Definition 5.28. Let $f : X \to Y$ be a smooth map of manifolds with corners.

(i) We call f an *immersion* if $\mathrm{d}f|_x : T_xX \to T_{f(x)}Y$ is injective for all $x \in X$.

(ii) We call f an *embedding* if f is an immersion and $f : X \to f(X)$ is a homeomorphism with its image, so in particular f is injective.

(iii) We call f an *s-immersion* or *s-embedding* if f is simple and an immersion or embedding.

(iv) We call f an *sf-immersion* or *sf-embedding* if f is simple and flat and an immersion or embedding.

If $f : X \to Y$ is an immersion, ..., sf-embedding, then we will also call X an *immersed* or *s-immersed* or *sf-immersed* or *embedded* or *s-embedded* or *sf-embedded submanifold* of Y; often one leaves the map f implicit. For (s- or sf-) embedded submanifolds, X, f are determined up to isomorphism by the image $f(X)$ in Y, so we can consider such submanifolds to be subsets of Y. But for (s- or sf-)immersed submanifolds the image $f(X)$ may not determine X, f, so it is better to think of an immersed submanifold as just being an immersion.

Example 5.29. (a) The inclusion $i : [0,\infty) \hookrightarrow \mathbb{R}$ is an embedding. It is semisimple and flat, but not simple, as $s_i : S_i \to \partial[0,\infty)$ maps $\emptyset \to \{0\}$, and is not surjective, so i is not an s- or sf-embedding. Thus $[0,\infty)$ is an embedded submanifold of $\mathbb{R}$, but not an s- or sf-embedded submanifold.

(b) The map $f : [0,\infty) \to [0,\infty)^2$ mapping $f : x \mapsto (x,x)$ is an embedding. It is flat, but not semisimple, as $s_f : S_f \to \partial[0,\infty)$ maps two points to one point, and is not injective. Hence f is not an s- or sf-embedding, and $\big\{(x,x) : x \in [0,\infty)\big\}$ is an embedded submanifold of $[0,\infty)^2$, but not s- or sf-embedded.

(c) The inclusion $i : \{0\} \hookrightarrow [0,\infty)$ has $\mathrm{d}i|_0$ injective, so it is an embedding. It is simple, but not flat, as $T_i = \big\{\big(0,(0,\{x=0\})\big)\big\} \neq \emptyset$. Thus i is an s-embedding, but not an sf-embedding. Hence $\{0\}$ is an s-embedded but not sf-embedded submanifold of $[0,\infty)$.

(d) Let X be a manifold with corners with $\partial X \neq \emptyset$. Then $i_X : \partial X \to X$ is an immersion. Also $s_{i_X} : S_{i_X} \to \partial^2 X$ is a bijection, so i_X is simple, but

$T_{i_X} \cong \partial X \neq \emptyset$, so i_X is not flat. Hence i_X is an s-immersion, but not an sf-immersion. If $\partial^2 X = \emptyset$ then i_X is an s-embedding, but not an sf-embedding.

(e) Let $f : [0, \infty) \to \mathbb{R}$ be smooth. Define $g : [0, \infty) \to [0, \infty) \times \mathbb{R}$ by $g(x) = (x, f(x))$. Then g is an sf-embedding, and $\Gamma_f = \{(x, f(x) : x \in [0, \infty)\}$ is an sf-embedded submanifold of $[0, \infty) \times \mathbb{R}$.

The proof of the next proposition is straightforward.

Proposition 5.30. *Let $f : X \to Y$ be an s-immersion of manifolds with corners, and $x \in X$ with $f(x) = y \in Y$. Then there exist open neighbourhoods V of x in X and W of y in Y with $f(V) \subseteq W$, an open neighbourhood Z of 0 in $\mathbb{R}^n_k$, and a diffeomorphism $W \cong V \times Z$ which identifies $f|_V : V \to W$ with $\mathrm{id}_V \times 0 : V \to V \times Z$ mapping $\mathrm{id}_V \times 0 : v \mapsto (v, 0)$. Hence there is a diffeomorphism $V \cong W \times_{g, \mathbb{R}^n_k, 0} *$, where $g : W \to \mathbb{R}^n_k$ is identified with $\pi_Z : V \times Z \to Z \subseteq \mathbb{R}^n_k$, and the fibre product $W \times_{\mathbb{R}^n_k} *$ in $\mathbf{Man^c}$ is transverse.*

If f is an sf-immersion then $k = 0$, so $V \cong W \times_{g, \mathbb{R}^n, 0} *$.

In Theorem 4.28 we recalled results of Whitney [106] on existence of immersions and embeddings $f : X \to \mathbb{R}^n$ when X is an m-manifold without boundary. Consider how these should be generalized to manifolds with corners. Then we have three kinds of immersions and embeddings. The proof of Theorem 4.28 essentially unchanged yields:

Theorem 5.31. **(a)** *Let X be an m-manifold with corners and $n \geqslant 2m$. Then a generic smooth map $f : X \to \mathbb{R}^n$ is an immersion.*

(b) *Let X be an m-manifold with corners and $n \geqslant 2m + 1$. Then there exists an embedding $f : X \to \mathbb{R}^n$, and we can choose such f with $f(X)$ closed in $\mathbb{R}^n$. Generic smooth maps $f : X \to \mathbb{R}^n$ are embeddings.*

For s- or sf-immersions and s- or sf-embeddings of manifolds with corners, the situation is more complicated. If $\partial X \neq \emptyset$ then no map $f : X \to \mathbb{R}^n$ can be an s-immersion or s-embedding, as $\partial \mathbb{R}^n = \emptyset$ implies $S_f = \emptyset$, so $s_f : S_f \to \partial X$ cannot be surjective. Another obvious possibility is to look for s-immersions and s-embeddings $f : X \to \mathbb{R}^n_k = [0, \infty)^k \times \mathbb{R}^{n-k}$ for $k > 0$. But these need not exist either, as the next example shows.

Example 5.32. Consider the *teardrop* $T = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ y^2 \leqslant x^2 - x^4\}$, shown in Figure 5.1. It is a compact 2-manifold with corners, and we may choose a diffeomorphism $\phi : [0, 1] \to \partial T$.

Suppose for a contradiction that $f : T \to \mathbb{R}^n_k$ is an s-immersion with $f(0, 0) = (0, \ldots, 0)$. At $(0, 0) \in T$ we have $i_T^{-1}(0, 0) = \{\phi(0), \phi(1)\}$. Also $i_{\mathbb{R}^n_k}^{-1}(0, \ldots, 0) = \{x'_1, \ldots, x'_k\}$, where $x'_i \in \partial \mathbb{R}^n_k$ is the local boundary component $\{x_i = 0\}$ at $(0, \ldots, 0)$ for $i = 1, \ldots, k$. As f is an s-immersion, $s_f : S_f \to \partial T$ is surjective, so $(\phi(0), x'_i), (\phi(1), x'_j) \in S_f$ for some $i, j = 1, \ldots, k$, and $i \neq j$ as j_f is injective.

A smooth map $f : X \to Y$ of manifolds with corners maps each connected component of $S^k(X)$ to some connected component of some $S^l(Y)$. Hence $f \circ \phi|_{(0,1)} : (0, 1) \to \mathbb{R}^n_k$ must map into a single connected component of $S^l(\mathbb{R}^n_k)$,

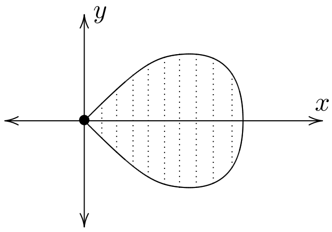

Figure 5.1: The teardrop, a 2-manifold with corners.

as $(0,1)$ is connected. The connected components of $S^l(\mathbb{R}^n_k)$ for $l=0,\ldots,k$ are the 2^k subsets for which either $x_a=0$, or $x_a>0$, for each $a=1,\ldots,k$. Using $(\phi(0),x'_i),(\phi(1),x'_j)\in S_f$ we find that $f\circ\phi|_{(0,1)}$ maps to $x_i=0$, $x_j>0$ near 0 and to $x_i>0$, $x_j=0$ near 1, which lie in different connected components of $S^l(\mathbb{R}^n_k)$, a contradiction. Hence there do not exist s-immersions or s-embeddings $f:T\rightarrow\mathbb{R}^n_k$ for any n,k.

However, for manifolds with boundary, there do exist sf-immersions and sf-embeddings (and hence s-immersions and s-embeddings) into $[0,\infty)\times\mathbb{R}^{n-1}$:

Theorem 5.33. **(a)** *Let X be an m-manifold with boundary and $n\geqslant 2m$. Then there exist sf-immersions $f:X\rightarrow\mathbb{R}^n_1=[0,\infty)\times\mathbb{R}^{n-1}$, and a generic simple flat map $f:X\rightarrow\mathbb{R}^n_1$ is an sf-immersion.*

(b) *Let X be an m-manifold with boundary and $n\geqslant 2m+1$. Then there exist sf-embeddings $f:X\rightarrow\mathbb{R}^n_1$, and we can choose such f with $f(X)$ closed in $\mathbb{R}^n_1$. Generic simple flat maps $f:X\rightarrow\mathbb{R}^n_1$ are sf-embeddings.*

To prove Theorem 5.33, first note that a map $f=(f_1,\ldots,f_n):X\rightarrow\mathbb{R}^n_1$ is smooth, simple and flat if and only if $f_1,\ldots,f_n$ are weakly smooth, and $f_1|_{\partial X}=0$, and $f_1|_{X^\circ}>0$, and the inward normal derivative of f_1 is positive at each point of ∂X. We can construct such a function f_1 by combining boundary defining functions at each point of ∂X using a partition of unity, and extending by an arbitrary smooth positive function away from ∂X. Hence simple flat maps $f:X\rightarrow\mathbb{R}^n_1$ exist for any $n\geqslant 1$. Similar considerations to Theorems 4.28 and 5.31 now imply that a generic simple flat map f is an sf-immersion if $n\geqslant 2m$, and an sf-embedding if $n\geqslant 2m+1$.

5.8 Orientations

Finally we discuss orientations on manifolds with corners, following [55, §7]. There are several equivalent ways to define orientations; our way is chosen to be compatible with our definition of orientations on d-manifolds in §4.5–§4.6.

Definition 5.34. Let X be an n-manifold with corners, and $L\rightarrow X$ a (real) line bundle. Write O_X for the trivial line bundle $\mathbb{R}\times X\rightarrow X$. An *orientation* ω on L is an equivalence class $[\tau]$ of isomorphisms of line bundles $\tau:O_X\rightarrow L$, where τ,τ' are equivalent if $\tau'=\tau\cdot c$ for some smooth $c:X\rightarrow(0,\infty)$. If

$\omega = [\tau]$ is an orientation, we write $-\omega$ for the *opposite orientation* $[-\tau]$, where if $\tau : O_X \to L$ is an isomorphism then $-\tau : O_X \to L$ is also an isomorphism.

Let $E \to X$ be a (real) vector bundle of rank k over X. Then the top exterior power $\Lambda^k E$ is a line bundle. An *orientation ω_E on the fibres of E* is an orientation on $\Lambda^k E$. Then we call (E, ω_E) an *oriented vector bundle.*

Here is an alternative way to say all this. If $L \to X$ is a line bundle, there is a 1-1 correspondence between isomorphisms $\tau : O_X \to L$ and nonvanishing sections $s \in C^\infty(L)$ by $s = \tau(1)$. Given an orientation ω on L, we call a nonvanishing section s *positive* if $\omega = [\tau]$ for τ the isomorphism with $\tau(1) = s$.

If $E \to X$ is an oriented vector bundle of rank k, then locally on X we may choose a basis of sections $(e_1, \ldots, e_k)$ for E, and then $e_1 \wedge \cdots \wedge e_k$ is a nonvanishing local section of $\Lambda^k E$. If $e_1 \wedge \cdots \wedge e_k$ is a positive section, w.r.t. the orientation on $\Lambda^k E$, then we call $(e_1, \ldots, e_k)$ an *oriented basis* for E.

If $E \to X$, $F \to X$ are vector bundles on X of ranks k, l and ω_E, ω_F are orientations on the fibres of E, F, we define the *direct sum orientation* $\omega_{E\oplus F} = \omega_E \oplus \omega_F$ on the fibres of $E \oplus F$ by saying that if $(e_1, \ldots, e_k)$ and $(f_1, \ldots, f_l)$ are oriented bases for E and F locally on X, then $(e_1, \ldots, e_k, f_1, \ldots, f_l)$ is an oriented basis for $E \oplus F$ locally on X. Note that the direct sum orientations $\omega_{E\oplus F}$ on $E \oplus F$ and $\omega_{F\oplus E}$ on $F \oplus E$ differ by a sign $(-1)^{kl}$, under the natural isomorphism $E \oplus F \cong F \oplus E$.

An *orientation ω_X on X* is an orientation on the fibres of the cotangent bundle T^*X, that is, an orientation on the line bundle $\Lambda^n T^*X$. Then we call (X, ω_X) an *oriented manifold.* Usually we suppress the orientation ω_X, and just refer to X as an oriented manifold. When X is an oriented manifold, we write $-X$ for X with the opposite orientation.

We need *orientation conventions* to say how to orient boundaries ∂X and fibre products $X \times_Z Y$ of oriented manifolds X, Y, Z. Our conventions [55, Conv. 7.2] follow those of Fukaya et al. [32, Conv. 45.1].

Convention 5.35. **(a)** Let (X, ω_X) be an oriented manifold with corners. Define $\omega_{\partial X}$ to be the unique orientation on ∂X such that

$$i_X^*(TX) \cong \mathbb{R}_{\mathrm{out}} \oplus T(\partial X) \tag{5.19}$$

is an isomorphism of oriented vector bundles over ∂X, where $i_X^*(TX), T(\partial X)$ are oriented by $\omega_X, \omega_{\partial X}$, and $\mathbb{R}_{\mathrm{out}}$ is oriented by an outward-pointing normal vector to ∂X in X, and the r.h.s. of (5.19) has the direct sum orientation.

(b) Let $(X, \omega_X), (Y, \omega_Y), (Z, \omega_Z)$ be oriented manifolds with corners, and $g : X \to Z$, $h : Y \to Z$ be transverse smooth maps, so that a fibre product $W = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^c}$ by Theorem 5.21. Then we have an exact sequence of vector bundles over W

$$0 \longrightarrow TW \xrightarrow{\mathrm{d}\pi_X \oplus \mathrm{d}\pi_Y} \pi_X^*(TX) \oplus \pi_Y^*(TY) \xrightarrow{\pi_X^*(\mathrm{d}g) - \pi_Y^*(\mathrm{d}h)} (g \circ \pi_X)^*(TZ) \longrightarrow 0. \tag{5.20}$$

Choosing a splitting of (5.20) induces an isomorphism of vector bundles

$$TW \oplus (g \circ \pi_X)^*(TZ) \cong \pi_X^*(TX) \oplus \pi_Y^*(TY). \tag{5.21}$$

Define ω_W to be the unique orientation on W such that the direct sum orientations in (5.21) induced by $\omega_W, \omega_Z, \omega_X, \omega_Y$ differ by a factor $(-1)^{\dim Y \dim Z}$.

Here are two ways to rewrite this convention in special cases. Firstly, suppose g is a submersion. Then $\mathrm{d}g : TX \to g^*(TZ)$ is surjective, so by splitting the exact sequence $0 \to \operatorname{Ker} \mathrm{d}g \to TX \xrightarrow{\mathrm{d}g} g^*(TZ) \to 0$ we obtain an isomorphism

$$TX \cong \operatorname{Ker} \mathrm{d}g \oplus g^*(TZ). \tag{5.22}$$

Give the vector bundle $\operatorname{Ker} \mathrm{d}g \to X$ the unique orientation such that (5.22) is an isomorphism of oriented vector bundles, where $TX, g^*(TZ)$ are oriented using ω_X, ω_Z. As $g : X \to Z$ is a submersion so is $\pi_Y : W \to Y$, and $\mathrm{d}\pi_X$ induces an isomorphism $\operatorname{Ker}(\mathrm{d}\pi_Y) \to \pi_X^*(\operatorname{Ker} \mathrm{d}g)$. Thus we have an exact sequence

$$0 \longrightarrow \pi_X^*(\operatorname{Ker} \mathrm{d}g) \xrightarrow{(\mathrm{d}\pi_X)^{-1}} TW \xrightarrow{\mathrm{d}\pi_Y} \pi_Y^*(TY) \longrightarrow 0.$$

Splitting this gives an isomorphism

$$TW \cong \pi_X^*(\operatorname{Ker} \mathrm{d}g) \oplus \pi_Y^*(TY). \tag{5.23}$$

The orientation on W makes (5.23) into an isomorphism of oriented vector bundles, using ω_Y and the orientation on $\operatorname{Ker} \mathrm{d}g$ to orient the right hand side.

Secondly, let h be a submersion. Then as for (5.22)–(5.23) we have isomorphisms

$$TY \cong h^*(TZ) \oplus \operatorname{Ker} \mathrm{d}h \quad \text{and} \quad TW \cong \pi_X^*(TX) \oplus \pi_Y^*(\operatorname{Ker} \mathrm{d}h). \tag{5.24}$$

We use the first equation of (5.24) to define an orientation on the fibres of $\operatorname{Ker} \mathrm{d}h$, and the second to define an orientation on W.

Given any canonical diffeomorphism between expressions in boundaries and fibre products of oriented manifolds with corners, we can use Convention 5.35 to define orientations on each side. These will be related by some sign ± 1, which we can compute. Here [55, Prop. 7.4] is how to add signs to (5.11)–(5.14).

Proposition 5.36. *In Propositions* 5.22–5.24, *suppose X, Y, Z are oriented. Then in oriented manifolds, equations* (5.11)–(5.14) *respectively become*

$$\partial_-^f X \cong (-1)^{\dim X + \dim Y} X \times_{f,Y,i_Y} \partial Y, \tag{5.25}$$

$$\partial\big(X \times_{g,Z,h} Y\big) \cong \big(\partial X \times_{g\circ i_X,Z,h} Y\big) \amalg (-1)^{\dim X + \dim Z}\big(X \times_{g,Z,h\circ i_Y} \partial Y\big), \tag{5.26}$$

$$\partial\big(X \times_{g,Z,h} Y\big) \cong \big(\partial_+^g X \times_{g_+,Z,h} Y\big) \amalg (-1)^{\dim X + \dim Z}\big(X \times_{g,Z,h\circ i_Y} \partial Y\big), \tag{5.27}$$

$$\begin{aligned} \partial\big(X \times_{g,Z,h} Y\big) \cong{}& \big(\partial_+^g X \times_{g_+,Z,h} Y\big) \amalg (-1)^{\dim X + \dim Z}\big(X \times_{h,Z,h_+} \partial_+^h Y\big) \\ & \amalg \big(\partial_-^g X \times_{g_-,\partial Z,h_-} \partial_-^h Y\big). \end{aligned} \tag{5.28}$$

Here [55, Prop. 7.5] are some more identities involving only fibre products:

Proposition 5.37. **(a)** *If $g : X \to Z$, $h : Y \to Z$ are transverse smooth maps of oriented manifolds with corners then in oriented manifolds we have*

$$X \times_{g,Z,h} Y \cong (-1)^{(\dim X-\dim Z)(\dim Y-\dim Z)} Y \times_{h,Z,g} X. \tag{5.29}$$

(b) *If $e : V \to Y$, $f : W \to Y$, $g : W \to Z$, $h : X \to Z$ are smooth maps of oriented manifolds with corners then in oriented manifolds we have*

$$V \times_{e,Y,f\circ\pi_W} \bigl(W \times_{g,Z,h} X\bigr) \cong \bigl(V \times_{e,Y,f} W\bigr) \times_{g\circ\pi_W,Z,h} X, \tag{5.30}$$

provided all four fibre products are transverse.
(c) *If $e : V \to Y$, $f : V \to Z$, $g : W \to Y$, $h : X \to Z$ are smooth maps of oriented manifolds with corners then in oriented manifolds we have*

$$\begin{aligned} &V \times_{(e,f),Y\times Z,g\times h} (W \times X) \cong \\ &\quad (-1)^{\dim Z(\dim Y+\dim W)}(V \times_{e,Y,g} W) \times_{f\circ\pi_V,Z,h} X, \end{aligned} \tag{5.31}$$

provided all three fibre products are transverse.

Equations (5.26), (5.30), (5.31) can be found in Fukaya et al. [32, Lem. 45.3] for the case of Kuranishi spaces.

6 D-spaces with corners

We now define 2-categories $\mathbf{dSpa^b}, \mathbf{dSpa^c}$ of *d-spaces with boundary*, and *with corners*. In Chapter 7 we will define d-manifolds with boundary or corners as full 2-subcategories of $\mathbf{dSpa^b}, \mathbf{dSpa^c}$.

One might think that the d-spaces of Chapter 2 are already sufficiently general, and that we could just define d-manifolds with corners as a 2-subcategory of $\mathbf{dSpa}$. However, this turns out not to be a good idea. Recall from Chapter 5 that our notion of smooth maps $f : X \to Y$ of manifolds with corners is quite subtle, and includes compatibility conditions over $\partial X, \partial Y$. Because of this, we saw in Corollary B.27 that the functors $F_{\mathbf{Man^b}}^{\mathbf{C^\infty Sch}} : \mathbf{Man^b} \to \mathbf{C^\infty Sch}$ and $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}} : \mathbf{Man^c} \to \mathbf{C^\infty Sch}$ are faithful, but not full.

Thus, if we regard manifolds with corners as C^∞-schemes, then we get the wrong notion of smooth map. Similarly, if we regarded d-manifolds with corners as d-spaces then we would get the wrong notion of 1-morphism. Also, there is no well-behaved notion of boundary of a general d-space, but our d-spaces with corners do have boundaries with all the properties we want.

6.1 The definition of d-spaces with corners

Definitions 6.1, 6.2 and 6.3 will define objects, 1-morphisms and 2-morphisms in the 2-category of *d-spaces with corners* $\mathbf{dSpa^c}$. It is modelled on properties of manifolds with corners in Chapter 5.

Definition 6.1. A *d-space with corners* is a quadruple $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$, where $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ and $\boldsymbol{\partial X} = (\underline{\partial X}, \mathcal{O}'_{\partial X}, \mathcal{E}_{\partial X}, \imath_{\partial X}, \jmath_{\partial X})$ are d-spaces, and $\boldsymbol{i}_{\mathbf{X}} = (\underline{i}_{\mathbf{X}}, i'_{\mathbf{X}}, i''_{\mathbf{X}}) : \boldsymbol{\partial X} \to \boldsymbol{X}$ is a 1-morphism of d-spaces, and $\omega_{\mathbf{X}}$ is defined in (d) below, satisfying the following conditions (a)–(f):

(a) $\underline{i}_{\mathbf{X}} : \underline{\partial X} \to \underline{X}$ is a proper morphism of C^∞-schemes.

(b) $i''_{\mathbf{X}} : \underline{i}_{\mathbf{X}}^*(\mathcal{E}_X) \to \mathcal{E}_{\partial X}$ is an isomorphism in $\mathrm{qcoh}(\underline{\partial X})$.

(c) Regard the point $*$ and $[0,\infty)$ as manifolds with boundary, and $0 : * \to [0,\infty)$ mapping $0 : * \mapsto 0$ as a smooth map. Write $\boldsymbol{[0,\infty)}, \boldsymbol{*}, \boldsymbol{0} = F_{\mathbf{Man^b}}^{\mathbf{dSpa}}\big([0,\infty), *, 0\big)$. Then $\boldsymbol{*}$ is a terminal object in $\mathbf{dSpa}$, and $\boldsymbol{0} : \boldsymbol{*} \to \boldsymbol{[0,\infty)}$ is a 1-morphism in $\mathbf{dSpa}$. Let $x' \in \boldsymbol{\partial X}$ with $\boldsymbol{i}_{\mathbf{X}}(x') = x \in \boldsymbol{X}$. Then there should exist open $x' \in \boldsymbol{U} \subseteq \boldsymbol{\partial X}$ and $x \in \boldsymbol{V} \subseteq \boldsymbol{X}$ with $\boldsymbol{i}_{\mathbf{X}}(\boldsymbol{U}) \subseteq \boldsymbol{V}$ and a 1-morphism $\boldsymbol{b} : \boldsymbol{V} \to \boldsymbol{[0,\infty)}$ in $\mathbf{dSpa}$ such that $\boldsymbol{b} \circ \boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{U}} = \boldsymbol{0} \circ \boldsymbol{\pi}$ as 1-morphisms $\boldsymbol{U} \to \boldsymbol{[0,\infty)}$, where $\boldsymbol{\pi} : \boldsymbol{U} \to \boldsymbol{*}$ is the unique morphism, and the following diagram is 2-Cartesian in $\mathbf{dSpa}$:

$$\begin{tikzcd}
\boldsymbol{U} \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{U}}"'] \arrow[drr, phantom, "\Uparrow \mathrm{id}_{\boldsymbol{0}\circ\boldsymbol{\pi}}"] & & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\
\boldsymbol{V} \arrow[rr, "\boldsymbol{b}"] & & \boldsymbol{[0,\infty)}.
\end{tikzcd} \tag{6.1}$$

We also require that the morphism $b^2 : \underline{b}^*(\mathcal{F}_{[0,\infty)}) \to \mathcal{F}_X|_{\underline{V}}$ in $\mathrm{qcoh}(\underline{V})$ should have a left inverse. That is, there should exist $\beta : \mathcal{F}_X|_{\underline{V}} \to \underline{b}^*(\mathcal{F}_{[0,\infty)})$ with $\beta \circ b^2 = \mathrm{id}_{\underline{b}^*(\mathcal{F}_{[0,\infty)})}$.

Before giving (d)–(f), we pause to examine some implications of (a)–(c). First note that (c) implies that the continuous map $i_{\mathbf{X}}:\partial X\rightarrow X$ locally embeds ∂X as a closed set in X. Hence the fibres $i_{\mathbf{X}}^{-1}(x)$ for $x\in X$ have the discrete topology. But $i_{\mathbf{X}}^{-1}(x)$ is compact, as $i_{\mathbf{X}}$ is proper by (a). Therefore $i_{\mathbf{X}}^{-1}(x)$ is finite for each $x\in X$, that is, $\underline{i}_{\mathbf{X}}:\underline{\partial X}\rightarrow\underline{X}$ is a *finite* morphism.

In (c), Proposition 2.41 shows the following is split exact in $\mathrm{qcoh}(\underline{U})$:

$$0\rightarrow\begin{array}{c}\underline{i}_{\mathbf{X}}|_{\underline{U}}^*(\mathcal{E}_X)\oplus\\ (\underline{0}\circ\underline{\pi})^*(\mathcal{F}_{[0,\infty)})\end{array}\xrightarrow{\begin{pmatrix} i''_{\mathbf{X}}|_{\underline{U}} & 0 \\ -\underline{i}_{\mathbf{X}}|_{\underline{U}}^*(\phi_X) & \begin{array}{c}\underline{i}_{\mathbf{X}}|_{\underline{U}}^*(b^2)\circ\\ I_{\underline{i}_{\mathbf{X}}|_{\underline{U}},\underline{b}}(\mathcal{F}_{[0,\infty)})\end{array}\end{pmatrix}}\begin{array}{c}\mathcal{E}_{\partial X}|_{\underline{U}}\oplus\\ \underline{i}_{\mathbf{X}}|_{\underline{U}}^*(\mathcal{F}_X)\end{array}\xrightarrow{\begin{pmatrix}\phi_{\partial X}|_{\underline{U}}\\ i^2_{\mathbf{X}}|_{\underline{U}}\end{pmatrix}^T}\mathcal{F}_{\partial X}|_{\underline{U}}\rightarrow 0.$$

Since $i''_{\mathbf{X}}|_{\underline{U}}$ is an isomorphism by (b) and $I_{\underline{i}_{\mathbf{X}}|_{\underline{U}},\underline{b}}(\mathcal{F}_{[0,\infty)})$ is an isomorphism, this implies that the following is split exact in $\mathrm{qcoh}(\underline{U})$:

$$0\longrightarrow\underline{i}_{\mathbf{X}}|_{\underline{U}}^*\circ\underline{b}^*(\mathcal{F}_{[0,\infty)})\xrightarrow{\underline{i}_{\mathbf{X}}|_{\underline{U}}^*(b^2)}\underline{i}_{\mathbf{X}}|_{\underline{U}}^*(\mathcal{F}_X)\xrightarrow{i^2_{\mathbf{X}}|_{\underline{U}}}\mathcal{F}_{\partial X}|_{\underline{U}}\longrightarrow 0.\qquad(6.2)$$

Also, as (6.1) is 2-Cartesian $\underline{U}\cong\underline{V}\times_{\underline{b},\underline{[0,\infty)},\underline{0}}\underline{*}$, so Theorem B.39(c) implies the following sequence in $\mathrm{qcoh}(\underline{U})$ is exact:

$$\underline{i}_{\mathbf{X}}|_{\underline{U}}^*\circ\underline{b}^*(T^*\underline{[0,\infty)})\xrightarrow{\underline{i}_{\mathbf{X}}|_{\underline{U}}^*(\Omega_{\underline{b}})}\underline{i}_{\mathbf{X}}|_{\underline{U}}^*(T^*\underline{X})\xrightarrow{\Omega_{\underline{i}_{\mathbf{X}}|_{\underline{U}}}}T^*(\underline{\partial X})|_{\underline{U}}\longrightarrow 0.\qquad(6.3)$$

Define $\mathcal{N}_{\mathbf{X}},\nu_{\mathbf{X}}$ so that $\nu_{\mathbf{X}}:\mathcal{N}_{\mathbf{X}}\rightarrow\underline{i}_{\mathbf{X}}^*(\mathcal{F}_X)$ is the kernel of $i^2_{\mathbf{X}}$. We will call $\mathcal{N}_{\mathbf{X}}$ the *conormal bundle* of $\boldsymbol{\partial X}$ in $\boldsymbol{X}$. Then we have a complex in $\mathrm{qcoh}(\underline{\partial X})$:

$$0\longrightarrow\mathcal{N}_{\mathbf{X}}\xrightarrow{\nu_{\mathbf{X}}}\underline{i}_{\mathbf{X}}^*(\mathcal{F}_X)\xrightarrow{i^2_{\mathbf{X}}}\mathcal{F}_{\partial X}\longrightarrow 0.\qquad(6.4)$$

Since (6.2) is split exact, we see that the restriction of (6.4) to $\underline{U}$ is split exact, and $\mathcal{N}_{\mathbf{X}}|_{\underline{U}}\cong\underline{i}_{\mathbf{X}}|_{\underline{U}}^*\circ\underline{b}^*(\mathcal{F}_{[0,\infty)})$. As $\mathcal{F}_{[0,\infty)}\cong T^*\underline{[0,\infty)}\cong\mathcal{O}_{[0,\infty)}$, this implies that $\mathcal{N}_{\mathbf{X}}|_{\underline{U}}\cong\mathcal{O}_U$, so $\mathcal{N}_{\mathbf{X}}$ is a line bundle on $\underline{U}$. But we can cover $\underline{\partial X}$ by such open $\underline{U}$ by (c). Since being split exact is a local condition on $\underline{\partial X}$ by Lemma 2.22, and being a line bundle is also local, we see that $\mathcal{N}_{\mathbf{X}}$ is a line bundle on $\underline{\partial X}$, and (6.4) is a split exact sequence on $\underline{\partial X}$. In a similar way, from (6.3) we deduce the following sequence in $\mathrm{qcoh}(\underline{\partial X})$ is exact:

$$\mathcal{N}_{\mathbf{X}}\xrightarrow{\underline{i}_{\mathbf{X}}^*(\psi_X)\circ\nu_{\mathbf{X}}}\underline{i}_{\mathbf{X}}^*(T^*\underline{X})\xrightarrow{\Omega_{\underline{i}_{\mathbf{X}}}}T^*(\underline{\partial X})\longrightarrow 0.\qquad(6.5)$$

Let $\underline{x}':\underline{*}\rightarrow\underline{\partial X}$ and $\underline{x}:\underline{*}\rightarrow\underline{X}$ be the C^∞-scheme morphisms corresponding to the points $x'\in\underline{\partial X}$, $x\in\underline{X}$. Then $\underline{x}=\underline{i}_{\mathbf{X}}\circ\underline{x}'$ as $\underline{i}_{\mathbf{X}}(x')=\underline{x}$. As pullbacks take split exact sequences to split exact sequences, applying $(\underline{x}')^*$ to (6.4) and conjugating the middle term by $I_{\underline{x}',\underline{i}_{\mathbf{X}}}(\mathcal{F}_X)$ gives an exact sequence in $\mathrm{qcoh}(\underline{*})$, that is, an exact sequence of vector spaces:

$$0\rightarrow(\underline{x}')^*(\mathcal{N}_{\mathbf{X}})\xrightarrow{\begin{array}{c}I_{\underline{x}',\underline{i}_{\mathbf{X}}}(\mathcal{F}_X)^{-1}\\ \circ(\underline{x}')^*(\nu_{\mathbf{X}})\end{array}}\underline{x}^*(\mathcal{F}_X)\xrightarrow{\begin{array}{c}(\underline{x}')^*(i^2_{\mathbf{X}})\circ\\ I_{\underline{x}',\underline{i}_{\mathbf{X}}}(\mathcal{F}_X)\end{array}}(\underline{x}')^*(\mathcal{F}_{\partial X})\rightarrow 0.\qquad(6.6)$$

Here are the remaining conditions.

(d) As above, the conormal bundle $\mathcal{N}_{\mathbf{X}}$ is a line bundle on $\underline{\partial X}$. We require that $\mathcal{N}_{\mathbf{X}}$ should be trivializable, and that $\omega_{\mathbf{X}}$ is an orientation on $\mathcal{N}_{\mathbf{X}}$, in the sense of Definition B.40.

(e) Let $x', \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{b}$ be as in (c). Then (6.2) is exact, so as $\mathcal{N}_{\mathbf{X}}|_{\underline{U}}$ is the kernel of $i^2_{\mathbf{X}}$, there is a unique isomorphism of line bundles on $\underline{U}$

$$\gamma : \underline{i}_{\mathbf{X}}|^*_{\underline{U}} \circ \underline{b}^*(\mathcal{F}_{[0,\infty)}) \xrightarrow{\cong} \mathcal{N}_{\mathbf{X}}|_{\underline{U}} \tag{6.7}$$

with $\nu_{\mathbf{X}}|_{\underline{U}} \circ \gamma = \underline{i}_{\mathbf{X}}|^*_{\underline{U}}(b^2)$. We require that for all $x' \in \boldsymbol{\partial X}$, there should exist $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{b}$ as in (c) satisfying the extra condition that γ in (6.7) identifies the negative orientation on $\underline{i}_{\mathbf{X}}|^*_{\underline{U}} \circ \underline{b}^*(\mathcal{F}_{[0,\infty)}) \cong \underline{i}_{\mathbf{X}}|^*_{\underline{U}} \circ \underline{b}^*(\mathcal{O}_{[0,\infty)}) \cong \mathcal{O}_U$ with the orientation $\omega_{\mathbf{X}}|_{\underline{U}}$ on $\mathcal{N}_{\mathbf{X}}|_{\underline{U}}$. Then we call $(\boldsymbol{V}, \boldsymbol{b})$ a *boundary defining function* for $\mathbf{X}$ at $x' \in \boldsymbol{\partial X}$.

(f) Let $x \in \boldsymbol{X}$. Since $\underline{i}_{\mathbf{X}} : \underline{\partial X} \to \underline{X}$ is finite from above, we may write $\underline{i}_{\mathbf{X}}^{-1}(x) = \{x'_1, \ldots, x'_k\}$ for $x'_1, \ldots, x'_k \in \underline{\partial X}$. Let $(\boldsymbol{V}_1, \boldsymbol{b}_1), \ldots, (\boldsymbol{V}_k, \boldsymbol{b}_k)$ be boundary defining functions for $\mathbf{X}$ at $x'_1, \ldots, x'_k$ respectively. Write $\underline{x}'_1, \ldots, \underline{x}'_k : \underline{*} \to \underline{\partial X}$, $\underline{x} : \underline{*} \to \underline{X}$ for the morphisms corresponding to $x'_1, \ldots, x'_k, x$. Then as in (6.6) we have a morphism in qcoh($\underline{*}$):

$$\textstyle\bigoplus_{i=1}^k I_{\underline{x}'_i, \underline{i}_{\mathbf{X}}}(\mathcal{F}_X)^{-1} \circ (\underline{x}'_i)^*(\nu_{\mathbf{X}}) : \bigoplus_{i=1}^k (\underline{x}'_i)^*(\mathcal{N}_{\mathbf{X}}) \longrightarrow \underline{x}^*(\mathcal{F}_X). \tag{6.8}$$

We require that (6.8) should be an injective morphism in qcoh($\underline{*}$), that is, an injective linear map of vector spaces. Note that (6.6) exact implies that $I_{\underline{x}'_i, \underline{i}_{\mathbf{X}}}(\mathcal{F}_X)^{-1} \circ (\underline{x}'_i)^*(\nu_{\mathbf{X}})$ is injective for each $i = 1, \ldots, k$, so this is automatic when $k = 1$.

We call $\mathbf{X}$ a *d-space with boundary* if $\underline{i}_{\mathbf{X}} : \underline{\partial X} \to \underline{X}$ is injective (that is, injective on points) and a *d-space without boundary* if $\underline{\partial X} = \emptyset$.

Let $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ be a d-space with corners. Suppose $\boldsymbol{U} \subseteq \boldsymbol{X}$ is an open d-subspace in **dSpa**. Define $\boldsymbol{\partial U} = \boldsymbol{i}_{\mathbf{X}}^{-1}(\boldsymbol{U})$, as an open d-subspace of $\boldsymbol{\partial X}$, and $\boldsymbol{i}_{\mathbf{U}} : \boldsymbol{\partial U} \to \boldsymbol{U}$ by $\boldsymbol{i}_{\mathbf{U}} = \boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{\partial U}}$. Then $\underline{\partial U} \subseteq \underline{\partial X}$ is an open C^∞-subscheme, and the conormal bundle of $\boldsymbol{\partial U}$ in $\boldsymbol{U}$ is $\mathcal{N}_{\mathbf{U}} = \mathcal{N}_{\mathbf{X}}|_{\underline{\partial U}}$ in qcoh($\underline{\partial U}$). Define an orientation $\omega_{\mathbf{U}}$ on $\mathcal{N}_{\mathbf{U}}$ by $\omega_{\mathbf{U}} = \omega_{\mathbf{X}}|_{\underline{\partial U}}$. Write $\mathbf{U} = (\boldsymbol{U}, \boldsymbol{\partial U}, \boldsymbol{i}_{\mathbf{U}}, \omega_{\mathbf{U}})$. Then $\mathbf{U}$ is a d-space with corners. We call $\mathbf{U}$ an *open d-subspace* of $\mathbf{X}$. If $\boldsymbol{U}$ is open and closed in $\boldsymbol{X}$ we call $\mathbf{U}$ an *open and closed d-subspace* of $\mathbf{X}$. An *open cover* of $\mathbf{X}$ is a family $\{\mathbf{U}_a : a \in A\}$ of open d-subspaces $\mathbf{U}_a$ of $\mathbf{X}$ with $\underline{X} = \bigcup_{a \in A} \underline{U}_a$.

Our next definition is an analogue of Definitions 5.5 and 5.7.

Definition 6.2. Let $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ and $\mathbf{Y} = (\boldsymbol{Y}, \boldsymbol{\partial Y}, \boldsymbol{i}_{\mathbf{Y}}, \omega_{\mathbf{Y}})$ be d-spaces with corners. A 1-*morphism* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism of d-spaces $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ satisfying the condition that if $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$ and $y' \in \boldsymbol{\partial Y}$ with $\boldsymbol{i}_{\mathbf{Y}}(y') = y \in \boldsymbol{Y}$, and $(\boldsymbol{V}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{Y}$ at y', then either

(i) there exists $x' \in \boldsymbol{\partial X}$ with $\boldsymbol{i}_{\mathbf{X}}(x') = x$ and an open neighbourhood $\tilde{\boldsymbol{V}}$ of x in $\boldsymbol{X}$ with $\boldsymbol{f}(\tilde{\boldsymbol{V}}) \subseteq \boldsymbol{V}$ such that $\big(\tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}}\big)$ is a boundary defining function for $\mathbf{X}$ at x'; or

(ii) there exists an open neighbourhood $\boldsymbol{W}$ of x in $\boldsymbol{X}$ with $\boldsymbol{f}(\boldsymbol{W}) \subseteq \boldsymbol{V}$ and $\boldsymbol{b} \circ \boldsymbol{f}|_{\boldsymbol{W}} = \boldsymbol{0} \circ \boldsymbol{\pi} : \boldsymbol{W} \to [\boldsymbol{0}, \boldsymbol{\infty})$ in **dSpa**, where $\boldsymbol{\pi} : \boldsymbol{W} \to *$ is the unique 1-morphism and $\boldsymbol{0} : * \to [\boldsymbol{0}, \boldsymbol{\infty})$ is as in Definition 6.1(c).

Note that cases (i),(ii) are exclusive, since $(b \circ f)^2 \neq 0$ at x in (i), but $(b \circ f)^2 = 0$ at x in (ii). In (i), Proposition 6.6(b) below implies that x' is unique. Also, from Proposition 6.6(c),(d) below we see that if (i) or (ii) holds for one boundary defining function $(\boldsymbol{V}, \boldsymbol{b})$ for $\mathbf{Y}$ at y', then (i) or (ii) also holds for every other boundary defining function $(\hat{\boldsymbol{V}}, \hat{\boldsymbol{b}})$ for $\mathbf{Y}$ at y', so the conditions are independent of the choice of $(\boldsymbol{V}, \boldsymbol{b})$.

Let $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ be a d-space with corners. Define the *identity* 1-*morphism* $\mathbf{id}_{\mathbf{X}} : \mathbf{X} \to \mathbf{X}$ to be the identity 1-morphism $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ in **dSpa**. Then (i) above holds with $x' = y'$ and $\tilde{\boldsymbol{V}} = \boldsymbol{V}$ for all $x, y = x, y'$, so $\mathbf{id}_{\mathbf{X}}$ is a 1-morphism of d-spaces with corners.

Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ and $\boldsymbol{g} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms of d-spaces with corners. Define the *composition of* 1-*morphisms* $\boldsymbol{g} \circ \boldsymbol{f} : \mathbf{X} \to \mathbf{Z}$ to be the composition $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ in **dSpa**. We will show in Proposition 6.7(f) below that $\boldsymbol{g} \circ \boldsymbol{f}$ is a 1-morphism $\mathbf{X} \to \mathbf{Z}$.

Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of d-spaces with corners. We have C^∞-scheme morphisms $\underline{i}_{\mathbf{X}} : \underline{\partial X} \to \underline{X}$, $\underline{f} : \underline{X} \to \underline{Y}$, $\underline{i}_{\mathbf{Y}} : \underline{\partial Y} \to \underline{Y}$, so we can form the C^∞-scheme fibre products $\underline{\partial X} \times_{\underline{f} \circ \underline{i}_{\mathbf{X}}, \underline{Y}, \underline{i}_{\mathbf{Y}}} \underline{\partial Y}$ and $\underline{X} \times_{\underline{f}, \underline{Y}, \underline{i}_{\mathbf{Y}}} \underline{\partial Y}$. Define a C^∞-subscheme $\underline{S}_{\boldsymbol{f}} \subseteq \underline{\partial X} \times_{\underline{Y}} \underline{\partial Y}$ by $(x', y') \in \underline{S}_{\boldsymbol{f}}$ if $x' \in \underline{\partial X}$, $y' \in \underline{\partial Y}$ with $\underline{i}_{\mathbf{X}}(x') = x \in \underline{X}$, $\underline{f}(x) = \underline{i}_{\mathbf{Y}}(y') = y \in \underline{Y}$, and if $(\boldsymbol{V}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{Y}$ at y', then there exists open $x \in \tilde{\boldsymbol{V}} \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$ such that $\bigl(\tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}}\bigr)$ is a boundary defining function for $\mathbf{X}$ at x', as in Definition 6.2(i).

Define a C^∞-subscheme $\underline{T}_{\boldsymbol{f}} \subseteq \underline{X} \times_{\underline{Y}} \underline{\partial Y}$ by $(x, y') \in \underline{T}_{\boldsymbol{f}}$ if $x \in \underline{\partial X}$, $y' \in \underline{\partial Y}$ with $\underline{f}(x) = \underline{i}_{\mathbf{Y}}(y') = y \in \underline{Y}$, and if $(\boldsymbol{V}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{Y}$ at y', then there exists open $x \in \boldsymbol{W} \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$ such that $\boldsymbol{b} \circ \boldsymbol{f}|_{\boldsymbol{W}} = \boldsymbol{0} \circ \boldsymbol{\pi}$, as in Definition 6.2(ii). We will show in Proposition 6.7 that $\underline{S}_{\boldsymbol{f}}, \underline{T}_{\boldsymbol{f}}$ are open and closed C^∞-subschemes of $\underline{\partial X} \times_{\underline{Y}} \underline{\partial Y}, \underline{X} \times_{\underline{Y}} \underline{\partial Y}$.

Define $\underline{s}_{\boldsymbol{f}} = \underline{\pi}_{\underline{\partial X}}|_{\underline{S}_{\boldsymbol{f}}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial X}$, $\underline{u}_{\boldsymbol{f}} = \underline{\pi}_{\underline{\partial Y}}|_{\underline{S}_{\boldsymbol{f}}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial Y}$, $\underline{t}_{\boldsymbol{f}} = \underline{\pi}_{\underline{X}}|_{\underline{T}_{\boldsymbol{f}}} : \underline{T}_{\boldsymbol{f}} \to \underline{X}$ and $\underline{v}_{\boldsymbol{f}} = \underline{\pi}_{\underline{\partial Y}}|_{\underline{T}_{\boldsymbol{f}}} : \underline{T}_{\boldsymbol{f}} \to \underline{\partial Y}$. By properties of fibre products there is a unique $\underline{j}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{X} \times_{\underline{f}, \underline{Y}, \underline{i}_{\mathbf{Y}}} \underline{\partial Y}$ with $\underline{\pi}_{\underline{X}} \circ \underline{j}_{\boldsymbol{f}} = \underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}}$ and $\underline{\pi}_{\underline{\partial Y}} \circ \underline{j}_{\boldsymbol{f}} = \underline{u}_{\boldsymbol{f}}$.

Next we define 2-morphisms of d-spaces with corners.

Definition 6.3. Let $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ be 1-morphisms of d-spaces with corners. Then $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ are 1-morphisms of d-spaces. Suppose $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism of d-spaces. Then $\underline{f} = \underline{g}$ by definition, so $\underline{\partial X} \times_{\underline{f} \circ \underline{i}_{\mathbf{X}}, \underline{Y}, \underline{i}_{\mathbf{Y}}} \underline{\partial Y} = \underline{\partial X} \times_{\underline{g} \circ \underline{i}_{\mathbf{X}}, \underline{Y}, \underline{i}_{\mathbf{Y}}} \underline{\partial Y}$, and $\underline{S}_{\boldsymbol{f}}, \underline{S}_{\boldsymbol{g}}$ are open and closed C^∞-subschemes of the same C^∞-scheme, so it makes sense to require $\underline{S}_{\boldsymbol{f}} = \underline{S}_{\boldsymbol{g}}$. Similarly, $\underline{T}_{\boldsymbol{f}}, \underline{T}_{\boldsymbol{g}}$ are open and closed in $\underline{X} \times_{\underline{f}, \underline{Y}, \underline{i}_{\mathbf{Y}}} \underline{\partial Y}$, so $\underline{T}_{\boldsymbol{f}} = \underline{T}_{\boldsymbol{g}}$ makes sense.

A 2-*morphism of d-spaces with corners* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism of d-spaces $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ for which $\underline{S}_{\boldsymbol{f}} = \underline{S}_{\boldsymbol{g}}$, and satisfying

$$(\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}})^*(\eta) \circ I_{\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}}, \underline{f}}(\mathcal{F}_Y) \circ I_{\underline{u}_{\boldsymbol{f}}, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1} \circ \underline{u}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}}) = 0 \text{ in } \mathrm{qcoh}(\underline{S}_{\boldsymbol{f}}), \tag{6.9}$$

$$\underline{t}_{\boldsymbol{f}}^*(\eta) \circ I_{\underline{t}_{\boldsymbol{f}}, \underline{f}}(\mathcal{F}_Y) \circ I_{\underline{v}_{\boldsymbol{f}}, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1} \circ \underline{v}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}}) = 0 \text{ in } \mathrm{qcoh}(\underline{T}_{\boldsymbol{f}}), \tag{6.10}$$

where the morphisms are given in

$$\begin{tikzcd}
\underline{u}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Y}}) \arrow[r, "\underline{u}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})"'] \arrow[d, "0"] & \underline{u}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{Y}}^*(\mathcal{F}_Y) \arrow[r, "I_{\underline{u}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1}"'] & (\underline{i}_{\mathbf{Y}} \circ \underline{u}_{\boldsymbol{f}})^*(\mathcal{F}_Y) \arrow[d, "I_{\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)"'] \\
(\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}})^*(\mathcal{E}_X) & & (\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}})^* \circ \underline{f}^*(\mathcal{F}_Y), \arrow[ll, "(\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}})^*(\eta)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{v}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Y}}) \arrow[r, "\underline{v}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})"'] \arrow[d, "0"] & \underline{v}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{Y}}^*(\mathcal{F}_Y) \arrow[r, "I_{\underline{v}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1}"'] & (\underline{f} \circ \underline{t}_{\boldsymbol{f}})^*(\mathcal{F}_Y) \arrow[d, "I_{\underline{t}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)"'] \\
\underline{t}_{\boldsymbol{f}}^*(\mathcal{E}_X) & & \underline{t}_{\boldsymbol{f}}^* \circ \underline{f}^*(\mathcal{F}_Y). \arrow[ll, "\underline{t}_{\boldsymbol{f}}^*(\eta)"']
\end{tikzcd}$$

If $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism, the *identity* 2*-morphism* $\mathrm{id}_{\boldsymbol{f}} : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ is $\mathrm{id}_{\boldsymbol{f}}$. The condition $\underline{S}_{\boldsymbol{f}} = \underline{S}_{\boldsymbol{f}}$ is trivial.

If $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} : \mathbf{X} \to \mathbf{Y}$ are 1-morphisms and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$, $\zeta : \boldsymbol{g} \Rightarrow \boldsymbol{h}$ are 2-morphisms of d-spaces with corners, define *vertical composition of* 2*-morphisms* $\zeta \odot \eta : \boldsymbol{f} \Rightarrow \boldsymbol{h}$ to be the vertical composition $\zeta \odot \eta$ of 2-morphisms in $\mathbf{dSpa}$. Since $\underline{S}_{\boldsymbol{f}} = \underline{S}_{\boldsymbol{g}}$ and $\underline{S}_{\boldsymbol{g}} = \underline{S}_{\boldsymbol{h}}$ we have $\underline{S}_{\boldsymbol{f}} = \underline{S}_{\boldsymbol{h}}$, and (6.9)–(6.10) for η, ζ imply (6.9)–(6.10) for $\zeta \odot \eta$, so $\zeta \odot \eta$ is a 2-morphism of d-spaces with corners.

If $\boldsymbol{f}, \tilde{\boldsymbol{f}} : \mathbf{X} \to \mathbf{Y}$ and $\boldsymbol{g}, \tilde{\boldsymbol{g}} : \mathbf{Y} \to \mathbf{Z}$ are 1-morphisms of d-spaces with corners, and $\eta : \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{f}}$, $\zeta : \boldsymbol{g} \Rightarrow \tilde{\boldsymbol{g}}$ are 2-morphisms of d-spaces with corners, define *horizontal composition of* 2*-morphisms* $\zeta * \eta : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{g}} \circ \tilde{\boldsymbol{f}}$ to be the horizontal composition $\zeta * \eta$ of 2-morphisms in $\mathbf{dSpa}$. One can easily check that $\zeta * \eta$ is a 2-morphism of d-spaces with corners.

In Definitions 6.1, 6.2 and above we have defined all the structures of a 2-category, which we call the 2*-category of d-spaces with corners*, written $\mathbf{dSpa^c}$. By Theorem 6.4 they satisfy the axioms of a 2-category. Write $\mathbf{dSpa^b}$ for the full 2-subcategory of d-spaces with boundary, and $\mathbf{d\bar{S}pa}$ for the full 2-subcategory of d-spaces without boundary. Define a 2-functor $F_{\mathbf{dSpa}}^{\mathbf{dSpa^c}} : \mathbf{dSpa} \to \mathbf{dSpa^c}$ to map $\boldsymbol{X} \mapsto \mathbf{X} = (\boldsymbol{X}, \emptyset, \emptyset, \emptyset)$ on objects, $\boldsymbol{f} \mapsto \boldsymbol{f}$ on 1-morphisms, and $\eta \mapsto \eta$ on 2-morphisms, where the data $\boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}}$ is trivial as the d-spaces concerned are empty. Then $F_{\mathbf{dSpa}}^{\mathbf{dSpa^c}}$ is a (strict) isomorphism of 2-categories $\mathbf{dSpa} \to \mathbf{d\bar{S}pa}$. So we may as well identify $\mathbf{dSpa}$ with its image $\mathbf{d\bar{S}pa}$, and consider d-spaces in Chapter 2 as examples of d-spaces with corners.

Since 1-morphisms and 2-morphisms in $\mathbf{dSpa^c}$ are just special examples of 1-morphisms and 2-morphisms in $\mathbf{dSpa}$, the proof in Theorem 2.15 that $\mathbf{dSpa}$ is a strict 2-category immediately implies:

Theorem 6.4. *In Definitions* 6.1, 6.2 *and* 6.3, $\mathbf{d\bar{S}pa}, \mathbf{dSpa^b}, \mathbf{dSpa^c}$ *are strict* 2*-categories, in which all* 2*-morphisms are* 2*-isomorphisms, and* $F_{\mathbf{dSpa}}^{\mathbf{dSpa^c}}$ *is a full and faithful strict* 2*-functor.*

We discuss some aspects of the definitions above.

Remark 6.5. (i) Let X be a manifold with corners. Then the boundary ∂X is a manifold with corners, with immersion $i_X : \partial X \to X$. We have an exact sequence of vector bundles on ∂X:

$$0 \longrightarrow T(\partial X) \xrightarrow{\mathrm{d}i_X} i_X^*(TX) \xrightarrow{\pi_\nu} \nu \longrightarrow 0$$

where ν is the *normal bundle* of $i_X(\partial X)$ in X, a line bundle over ∂X, which is canonically oriented by outward-pointing normal vectors. The dual bundle ν^*, the *conormal bundle*, is also an oriented line bundle on ∂X, in the dual sequence

$$0 \longrightarrow \nu^* \xrightarrow{\pi_\nu^*} i_X^*(T^*X) \xrightarrow{(\mathrm{d}i_X)^*} T^*(\partial X) \longrightarrow 0. \tag{6.11}$$

In Definition 6.1, $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \mathcal{N}_{\mathbf{X}}$ and $\omega_{\mathbf{X}}$ are analogues of $X, \partial X, i_X, \nu^*$ and the orientation on ν^* respectively. Both (6.4) and (6.5) are analogues of (6.11). Also boundary defining functions $(\boldsymbol{V}, \boldsymbol{b})$ are the analogue of boundary defining functions (V, b) for X in Definition 5.4.

Our orientation convention will be that (co)normal bundles to boundaries are oriented by *outward-pointing* normal vectors. Thus, in Definition 6.1(e) we choose the negative orientation on $\underline{i}_{\mathbf{X}}|_{\underline{U}}^* \circ \underline{b}^*(\mathcal{F}_{[0,\infty)})$ because outward-pointing normal vectors to $[0,\infty)$ at the boundary 0 point in the negative direction in $\mathbb{R}$.

(ii) For a manifold with corners X, the orientation on ν is determined by $X, \partial X, i_X$. But there exist d-spaces with corners $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ in which the orientation $\omega_{\mathbf{X}}$ is not determined by $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}$, and $(\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, -\omega_{\mathbf{X}})$ is also a d-space with corners. So the orientation $\omega_{\mathbf{X}}$ really is extra data. We include it because in §7.8, if $\mathbf{X}$ is an oriented d-manifold with corners, we will need $\omega_{\mathbf{X}}$ to define an orientation on the boundary $\partial\mathbf{X}$.

(iii) In Definition 6.1, the assumptions that $i''_{\mathbf{X}}$ an isomorphism and (6.1) is 2-Cartesian with 2-morphism $\mathrm{id}_{\boldsymbol{0}\circ\boldsymbol{\pi}}$ ensure that $\boldsymbol{\partial X}$ is immersed in $\boldsymbol{X}$ in a very strong sense, so that $\boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}}$ are locally determined up to canonical (1-)isomorphism by $\boldsymbol{X}$ and a boundary defining function $(\boldsymbol{V}, \boldsymbol{b})$. In Definitions 6.2 and 6.3, we define 1- and 2-morphisms in $\mathbf{dSpa^c}$ to be 1- and 2-morphisms of the underlying d-spaces satisfying extra conditions over the boundaries of $\mathbf{X}, \mathbf{Y}$.

These conditions on the immersion $\boldsymbol{i}_{\mathbf{X}} : \boldsymbol{\partial X} \to \boldsymbol{X}$, and the boundary conditions on 1- and 2-morphisms $\boldsymbol{f}, \eta$, have been carefully chosen to ensure that 1- and 2-morphisms lift uniquely to 1- and 2-morphisms of boundaries, whenever these should exist. For instance, in §6.3 we will discuss the class of *simple* 1-morphisms. These have the property that any simple 1-morphism $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ lifts to a canonical simple 1-morphism $\boldsymbol{f}_- : \partial\mathbf{X} \to \partial\mathbf{Y}$, and if $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ are simple 1-morphisms and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism, then η lifts to a canonical 2-morphism $\eta_- : \boldsymbol{f}_- \Rightarrow \boldsymbol{g}_-$.

(iv) The condition in Definition 6.1(c) that b^2 has a left inverse β will be used infrequently, but is sometimes important for showing that we can construct 2-morphisms η on $\underline{X}$ to satisfy conditions such as (6.9) over $\underline{S}_{\boldsymbol{f}}$ or $\underline{\partial X}$. It will be essential in the proof of Proposition 6.21 and of the existence of b-transverse fibre products in $\mathbf{dSpa^c}$ in §6.8. It will hold automatically in $\mathbf{dMan^c}$. Without assuming b^2 has a left inverse, we showed in Definition 6.1 that (6.2) is split exact, which implies that $\underline{i}_{\mathbf{X}}|_U^*(b^2)$ has a left inverse. So b^2 naturally has a left inverse over $\underline{\partial X}$, and we require this to extend into $\underline{X}$ near $\underline{\partial X}$.

(v) Let $\boldsymbol{f},\boldsymbol{g}:\mathbf{X}\to\mathbf{Y}$ be 1-morphisms in $\mathbf{dSpa^c}$, and $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a 2-morphism. We now motivate the conditions (6.9)–(6.10) on η in Definition 6.3. First note that (6.15) below for $\boldsymbol{f}$ implies an equation in $\mathrm{qcoh}(\underline{S}_{\boldsymbol{f}})$:

$$\underline{s}_{\boldsymbol{f}}^*(i_{\mathbf{X}}^2)\circ I_{\underline{s}_{\boldsymbol{f}},\underline{i}_{\mathbf{X}}}(\mathcal{F}_X)\circ(\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}})^*(f^2)\circ I_{\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1}\circ\underline{u}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})=0.$$

Suppose $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is a 2-morphism in $\mathbf{dSpa}$, but not necessarily in $\mathbf{dSpa^c}$. Then $g^2=f^2+\phi_X\circ\eta$ by (2.26), so subtracting the equation above from its analogue for $\boldsymbol{g}$ yields

$$\begin{aligned}&\underline{s}_{\boldsymbol{f}}^*(i_{\mathbf{X}}^2)\circ I_{\underline{s}_{\boldsymbol{f}},\underline{i}_{\mathbf{X}}}(\mathcal{F}_X)\circ(\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}})^*(\phi_X)\circ\\&(\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}})^*(\eta)\circ I_{\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1}\circ\underline{u}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})=0.\end{aligned}\tag{6.12}$$

The second line of this equation is (6.9), so (6.9) implies (6.12). Thus, (6.9) is a strengthening of an equation (6.12) in $\mathrm{qcoh}(\underline{S}_{\boldsymbol{f}})$ which follows from $\boldsymbol{f},\boldsymbol{g}$ being 1-morphisms in $\mathbf{dSpa^c}$ and $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a 2-morphism in $\mathbf{dSpa}$.

Similarly, from (6.17) below for $\boldsymbol{f},\boldsymbol{g}$ we deduce that

$$\underline{t}_{\boldsymbol{f}}^*(\phi_X)\circ\bigl[\underline{t}_{\boldsymbol{f}}^*(\eta)\circ I_{\underline{t}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{v}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1}\circ\underline{v}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})\bigr]=0,\tag{6.13}$$

where $[\cdots]$ is the left hand side of (6.10). Thus, (6.10) is a strengthening of an equation (6.13) in $\mathrm{qcoh}(\underline{T}_{\boldsymbol{f}})$ which follows from $\boldsymbol{f},\boldsymbol{g}$ being 1-morphisms in $\mathbf{dSpa^c}$ and $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a 2-morphism in $\mathbf{dSpa}$. Equations (6.9)–(6.10) are used in Proposition 6.8, §6.3 and §6.7 to define natural lifts of 2-morphisms η up to the boundary and corners of $\mathbf{X},\mathbf{Y}$, whenever these should exist.

(vi) There may be lots of ways to define a 2-category of d-spaces with corners, and the author experimented with many possibilities before settling on the definitions above. Here we discuss some alternatives.

Let $\boldsymbol{f},\boldsymbol{g}:\mathbf{X}\to\mathbf{Y}$ be 1-morphisms in $\mathbf{dSpa^c}$, and $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ be a 2-morphism in $\mathbf{dSpa}$. We call η a *weak 2-morphism* if $\underline{S}_{\boldsymbol{f}}=\underline{S}_{\boldsymbol{g}}$, but we do not require equations (6.9)–(6.10) to hold. Write $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$ for the 2-category with objects and 1-morphisms as in $\mathbf{dSpa^c}$, but with weak 2-morphisms as 2-morphisms. Then $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$ is also a strict 2-category, which is not equivalent to $\mathbf{dSpa^c}$.

We will show in §6.8 that not all fibre products exist in our 2-category $\mathbf{dSpa^c}$; we will give a sufficient condition on 1-morphisms $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$, $\boldsymbol{h}:\mathbf{Y}\to\mathbf{Z}$, called *b-transversality*, for a fibre product $\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ to exist. In contrast, one can show that in the 2-category $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$, with weak 2-morphisms, all fibre products exist. So $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$ is better behaved than $\mathbf{dSpa^c}$ in at least one respect.

However, in other ways $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$ is badly behaved. The purpose of (6.9)–(6.10) is to ensure that 2-morphisms in $\mathbf{dSpa^c}$ lift up to 2-morphisms of boundaries and corners, and in $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$ this may not happen. For example, if as in (iii) above $\boldsymbol{f},\boldsymbol{g}:\mathbf{X}\to\mathbf{Y}$ are simple 1-morphisms and $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a 2-morphism in $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$, for the 1-morphisms $\boldsymbol{f}_-,\boldsymbol{g}_-:\partial\mathbf{X}\to\partial\mathbf{Y}$ there may exist no 2-morphism $\eta_-:\boldsymbol{f}_-\Rightarrow\boldsymbol{g}_-$ in $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$.

In Chapter 7 we will define the 2-subcategory of d-manifolds with corners $\mathbf{dMan^c}$ in $\mathbf{dSpa^c}$; in a similar way one can define an alternative 2-subcategory of d-manifolds with corners $\widetilde{\mathbf{dMan}}^{\mathbf{c}}$ in $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$, which is not equivalent to $\mathbf{dMan^c}$. The advantage of $\mathbf{dSpa^c}, \mathbf{dMan^c}$ is that boundaries and corners have much better 2-functorial properties, but the advantage of $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}, \widetilde{\mathbf{dMan}}^{\mathbf{c}}$ is that fibre products exist under weaker conditions.

We also get two other non-equivalent strict 2-categories intermediate between $\mathbf{dSpa^c}$ and $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$, if we define 2-morphisms requiring only one of (6.9) and (6.10) to hold. So this yields four different 2-categories of d-spaces with corners.

Here are some properties of boundary defining functions.

Proposition 6.6. *Let* $\mathbf{X}$ *be a d-space with corners,* $x' \in \boldsymbol{\partial X}$ *with* $\boldsymbol{i}_{\mathbf{X}}(x') = x$, *and* $(\boldsymbol{V}, \boldsymbol{b})$ *be a boundary defining function for* $\mathbf{X}$ *at* x'. *Then*

- **(a)** $(\boldsymbol{V}, \boldsymbol{b})$ *is also a boundary defining function for* $\mathbf{X}$ *at* $\tilde{x}'$ *for all* $\tilde{x}'$ *in an open neighbourhood of* x' *in* $\boldsymbol{\partial X}$.
- **(b)** *If* $\tilde{x}' \in \boldsymbol{\partial X}$ *with* $\boldsymbol{i}_{\mathbf{X}}(\tilde{x}') = x$ *and* $\tilde{x}' \neq x'$ *then* $(\boldsymbol{V}, \boldsymbol{b})$ *is not a boundary defining function for* $\mathbf{X}$ *at* $\tilde{x}'$. *Thus,* x *and* $(\boldsymbol{V}, \boldsymbol{b})$ *determine* x' *uniquely.*
- **(c)** *Suppose* $(\tilde{\boldsymbol{V}}, \tilde{\boldsymbol{b}})$ *is a second boundary defining function for* $\mathbf{X}$ *at* x'. *Then there exists an open neighbourhood* $\boldsymbol{W}$ *of* x *in* $\boldsymbol{V} \cap \tilde{\boldsymbol{V}} \subseteq \boldsymbol{X}$ *and a* 1*-morphism* $\boldsymbol{c} : \boldsymbol{W} \to \boldsymbol{(0,\infty)}$ *in* $\mathbf{dSpa}$ *such that* $\tilde{\boldsymbol{b}}|_{\boldsymbol{W}} = \boldsymbol{c} \cdot \boldsymbol{b}|_{\boldsymbol{W}}$, *or equivalently* $\boldsymbol{b}|_{\boldsymbol{W}} = \boldsymbol{c}^{\boldsymbol{-1}} \cdot \tilde{\boldsymbol{b}}|_{\boldsymbol{W}}$.
- **(d)** *Suppose* $\boldsymbol{c} : \boldsymbol{V} \to \boldsymbol{(0,\infty)}$ *is a* 1*-morphism in* $\mathbf{dSpa}$, *and set* $\tilde{\boldsymbol{b}} = \boldsymbol{c} \cdot \boldsymbol{b} : \boldsymbol{V} \to \boldsymbol{[0,\infty)}$. *Then* $(\boldsymbol{V}, \tilde{\boldsymbol{b}})$ *is a boundary defining function for* $\mathbf{X}$ *at* x'.

Here in **(c)**,**(d)** $\boldsymbol{(0,\infty)} = F_{\mathbf{Man}}^{\mathbf{dSpa}}\big((0,\infty)\big)$, *and we use the obvious notions of multiplication* $\boldsymbol{b} \cdot \boldsymbol{c}$ *and multiplicative inverses* $\boldsymbol{c}^{\boldsymbol{-1}}$ *for d-space* 1*-morphisms from* $\boldsymbol{W}$ *to* $\boldsymbol{(0,\infty)}, \boldsymbol{[0,\infty)}$ *or* $\mathbb{R}$. *That is, if* $\boldsymbol{b}, \boldsymbol{c} : \boldsymbol{W} \to \mathbb{R}$ *are* 1*-morphisms we define* $\boldsymbol{b} \cdot \boldsymbol{c} : \boldsymbol{W} \to \mathbb{R}$ *by* $\boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{\mu} \circ (\boldsymbol{b}, \boldsymbol{c})$, *where* $(\boldsymbol{b}, \boldsymbol{c}) : \boldsymbol{W} \to \mathbb{R} \times \mathbb{R}$ *is the direct product and* $\boldsymbol{\mu} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(\mu) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *where* $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *maps* $\mu(x, y) = xy$. *Similarly if* $\boldsymbol{c} : \boldsymbol{W} \to \boldsymbol{(0,\infty)}$ *is a* 1*-morphism we define* $\boldsymbol{c}^{\boldsymbol{-1}} = \boldsymbol{\iota} \circ \boldsymbol{c}$, *where* $\boldsymbol{\iota} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(\iota)$ *and* $\iota : (0,\infty) \to (0,\infty)$ *maps* $\iota : x \mapsto x^{-1}$.

Proof. For (a), Definition 6.1(c),(e) give an open neighbourhood $\boldsymbol{U}$ of x' in $\boldsymbol{\partial X}$, and $(\boldsymbol{V}, \boldsymbol{b})$ is a boundary defining function for all $\tilde{x}' \in \boldsymbol{U}$. For (b), consider the morphisms $I_{\underline{x}', \underline{i}_{\mathbf{X}}}(\mathcal{F}_X)^{-1} \circ (\underline{x}')^*(\nu_{\mathbf{X}}) : (\underline{x}')^*(\mathcal{N}_{\mathbf{X}}) \to \underline{x}^*(\mathcal{F}_X)$ and $I_{\underline{\tilde{x}}', \underline{i}_{\mathbf{X}}}(\mathcal{F}_X)^{-1} \circ (\underline{\tilde{x}}')^*(\nu_{\mathbf{X}}) : (\underline{\tilde{x}}')^*(\mathcal{N}_{\mathbf{X}}) \to \underline{x}^*(\mathcal{F}_X)$ in qcoh$(\underline{*})$. If $(\boldsymbol{V}, \boldsymbol{b})$ were a boundary defining function for $\mathbf{X}$ at $\tilde{x}'$ then (6.6) would be exact for both x' and $\tilde{x}'$, so the morphisms would have the same image in $\underline{x}^*(\mathcal{F}_X)$. But Definition 6.1(f) shows the two morphisms are linearly independent, a contradiction.

For (c), we have sheaves of C^∞-rings $\mathcal{O}_X, \mathcal{O}'_X$ on X, so we can form the stalks $\mathcal{O}_{X,x}, \mathcal{O}'_{X,x}$ at x, which are C^∞-local rings. Similarly, we have stalks $\mathcal{O}_{\partial X',x}, \mathcal{O}'_{\partial X,x'}$ of $\mathcal{O}_{\partial X}, \mathcal{O}'_{\partial X}$ at x', and $\mathcal{O}_{[0,\infty),0} \cong C_0^\infty\big([0,\infty)\big)$ of $\mathcal{O}_{[0,\infty)}$ at 0. The morphisms of sheaves of C^∞-rings $\imath_X : \mathcal{O}'_X \to \mathcal{O}_X$, $\imath_{\partial X} : \mathcal{O}'_{\partial X} \to$

$\mathcal{O}_{\partial X}$, $i_{\mathbf{X}}^\sharp : i_{\mathbf{X}}^{-1}(\mathcal{O}_X) \to \mathcal{O}_{\partial X}$, $i'_{\mathbf{X}} : i_{\mathbf{X}}^{-1}(\mathcal{O}_X{}') \to \mathcal{O}'_{\partial X}$, $b^\sharp : b^{-1}(\mathcal{O}_{[0,\infty)}) \to \mathcal{O}_X$ and $b' : b^{-1}(\mathcal{O}_{[0,\infty)}) \to \mathcal{O}'_X$ (noting that $\mathcal{O}'_{[0,\infty)} = \mathcal{O}_{[0,\infty)}$) have stalks $\imath_{X,x}, \imath_{\partial X,x'}, i^\sharp_{\mathbf{X},x'}, i'_{\mathbf{X},x'}, b^\sharp_x, b'_x$ at x', x, which are morphisms of C^∞-local rings fitting into a commutative diagram:

$$\begin{tikzcd} C_0^\infty\bigl([0,\infty)\bigr) \arrow[r, "b'_x"'] \arrow[d, equal] & \mathcal{O}'_{X,x} \arrow[r, "i'_{\mathbf{X},x'}"'] \arrow[d, "\imath_{X,x}"'] & \mathcal{O}'_{\partial X,x'} \arrow[d, "\imath_{\partial X,x'}"'] \\ C_0^\infty\bigl([0,\infty)\bigr) \arrow[r, "b^\sharp_x"] & \mathcal{O}_{X,x} \arrow[r, "i^\sharp_{\mathbf{X},x'}"] & \mathcal{O}_{\partial X,x'}. \end{tikzcd} \tag{6.14}$$

Using (6.1) 2-Cartesian and $i''_{\mathbf{X}}$ an isomorphism in Definition 6.1(b),(c), and considering properties of fibre products in **dSpa** from §2.5, we can see that $i^\sharp_{\mathbf{X},x'}$ is surjective and the kernel of $i^\sharp_{\mathbf{X},x'}$ is the ideal $\bigl(b^\sharp_x(z)\bigr)$ in $\mathcal{O}_{X,x}$ where z is the coordinate on $[0,\infty)$ (this is immediate from $\underline{\partial X} \cong \underline{X} \times_{\underline{[0,\infty)}} \underline{*}$), and also $i'_{\mathbf{X},x'}$ is surjective and the kernel of $i'_{\mathbf{X},x'}$ is the ideal $\bigl(b'_x(z)\bigr)$ in $\mathcal{O}'_{X,x}$.

Now suppose $(\tilde{\boldsymbol{V}}, \tilde{\boldsymbol{b}})$ is a second boundary defining function for $\mathbf{X}$ at x'. Then we have a diagram (6.14) for $\tilde{\boldsymbol{b}}$, and the kernel of $i^\sharp_{\mathbf{X},x'}$ is $\bigl(\tilde b^\sharp_x(z)\bigr)$, and the kernel of $i'_{\mathbf{X},x'}$ is $\bigl(\tilde b'_x(z)\bigr)$. Therefore $\bigl(b'_x(z)\bigr) = \bigl(\tilde b'_x(z)\bigr)$ as ideals in $\mathcal{O}_{\partial X,x'}$. Thus there exists an invertible element $\alpha \in \mathcal{O}'_{X,x}$ with $\tilde b'_x(z) = \alpha \cdot b'_x(z)$. Since $\mathcal{O}'_{X,x}$ is a C^∞-local ring, α invertible is equivalent to $\alpha(x) \neq 0$ in $\mathbb{R}$.

Let $\gamma, \tilde\gamma$ be the morphisms in (6.7) from $\boldsymbol{b}, \tilde{\boldsymbol{b}}$. Both $\gamma, \tilde\gamma$ are orientation-preserving isomorphisms by Definition 6.1(e), so they are proportional by a positive function on $\underline{U} \cap \underline{\tilde U}$. But $(\underline{x}')^*(\tilde\gamma) = \alpha(x) \cdot (\underline{x}')^*(\gamma)$. Thus $\alpha(x) > 0$. Now α is a germ of sections of $\mathcal{O}_X{}'$ at x. So we may choose a small open neighbourhood W of x in $V \cap \tilde V \subseteq X$ and a section $\beta \in \mathcal{O}'_X(W)$ whose projection to $\mathcal{O}'_{X,x}$ is α. Since $\alpha(x) > 0$, making W smaller we can suppose $\beta > 0$ on W. As $\tilde b'_x(z) = \alpha \cdot b'_x(z)$, we see that $\bigl(\tilde b'(W)\bigr)(z) = \beta \cdot \bigl(b'(W)\bigr)(z)$ holds near x in W, so making W smaller we can suppose that $\bigl(\tilde b'(W)\bigr)(z) = \beta \cdot \bigl(b'(W)\bigr)(z)$.

Using β we can now construct a unique morphism $\boldsymbol{c} = (\underline{c}, c', c'') : \boldsymbol{W} \to \boldsymbol{(0,\infty)}$, where $\underline{c} : \underline{W} \to \underline{(0,\infty)}$ is induced by $\bigl(\imath_X(W)\bigr)(\beta) \in \mathcal{O}_X(W)$, and $c' : c^{-1}(\mathcal{O}_{(0,\infty)}) \to \mathcal{O}_X|_W$ is induced by β, and $c'' : \underline{c}^*(\mathcal{E}_{(0,\infty)}) \to \mathcal{E}_X|_{\underline{W}}$ is zero as $\mathcal{E}_{(0,\infty)} = 0$. This satisfies $\tilde{\boldsymbol{b}}|_{\boldsymbol{W}} = \boldsymbol{c} \cdot \boldsymbol{b}|_{\boldsymbol{W}}$ as $\bigl(\tilde b'(W)\bigr)(z) = \beta \cdot \bigl(b'(W)\bigr)(z)$, so we have proved (c).

For (d), note that if $(\boldsymbol{V}, \boldsymbol{b})$ is a boundary defining function and $\boldsymbol{c} : \boldsymbol{V} \to \boldsymbol{(0,\infty)}$ then (6.1) with $\tilde{\boldsymbol{b}} = \boldsymbol{c} \cdot \boldsymbol{b}$ in place of $\boldsymbol{b}$ is still locally 2-Cartesian, essentially because for any open $W \subseteq V$, $\bigl(c^\sharp(W)\bigr)(z)$ is invertible in $\mathcal{O}_X(W)$ as $\boldsymbol{c} > 0$, so $\bigl(b^\sharp(W)\bigr)(z)$ and $\bigl(c^\sharp(W)\bigr)(z) \cdot \bigl(b^\sharp(W)\bigr)(z)$ generate the same ideal in $\mathcal{O}_X(W)$, and similarly for $\bigl(b'(W)\bigr)(z)$ and $\bigl(c'(W)\bigr)(z) \cdot \bigl(b'(W)\bigr)(z)$ in $\mathcal{O}_X{}'(W)$. So Definition 6.1(c) holds for $\boldsymbol{c} \cdot \boldsymbol{b}$. Also the morphisms $\gamma, \tilde\gamma$ in (6.7) induced by $\boldsymbol{b}$ and $\tilde{\boldsymbol{b}} = \boldsymbol{c} \cdot \boldsymbol{b}$ are related by $\tilde\gamma = (\underline{c} \circ \underline{i}_{\mathbf{X}}|_{\underline{U}}) \cdot \gamma$, regarding $\underline{c} \circ \underline{i}_{\mathbf{X}}|_{\underline{U}} : \underline{U} \to \underline{(0,\infty)} \subset \underline{\mathbb{R}}$ as an element of $\mathcal{O}_X(U)$. So $\tilde\gamma, \gamma$ are proportional by a positive function, as $\boldsymbol{c} > 0$, and γ orientation-preserving implies $\tilde\gamma$ orientation-preserving. Thus Definition 6.1(e) holds for $\boldsymbol{c} \cdot \boldsymbol{b}$, and $\boldsymbol{c} \cdot \boldsymbol{b}$ is a boundary defining function for $\mathbf{X}$ at x'. $\square$

Here are some properties of $\underline{S}_{\boldsymbol{f}}, \underline{T}_{\boldsymbol{f}}, \underline{j}_{\boldsymbol{f}}, \underline{s}_{\boldsymbol{f}}, \underline{t}_{\boldsymbol{f}}, \underline{u}_{\boldsymbol{f}}, \underline{v}_{\boldsymbol{f}}$ in Definition 6.2. Parts (a)–(c) are analogues of Proposition 5.8.

Proposition 6.7. *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a* 1*-morphism of d-spaces with corners, and use the notation of Definition* 6.2*. Then*

(a) *$\underline{S}_{\boldsymbol{f}}, \underline{T}_{\boldsymbol{f}}$ are open and closed C^∞-subschemes in $\underline{\partial X} \times_{\underline{Y}} \underline{\partial Y}$ and $\underline{X} \times_{\underline{Y}} \underline{\partial Y}$.*

(b) *$\underline{j}_{\boldsymbol{f}}$ is an isomorphism of C^∞-schemes $\underline{S}_{\boldsymbol{f}} \to (\underline{X} \times_{\underline{Y}} \underline{\partial Y}) \setminus \underline{T}_{\boldsymbol{f}}$.*

(c) *$\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial X}$ and $\underline{t}_{\boldsymbol{f}} : \underline{T}_{\boldsymbol{f}} \to \underline{X}$ are proper étale morphisms.*

(d) *There exist unique $\lambda_{\boldsymbol{f}} : \underline{u}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Y}}) \to \underline{s}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}})$ and $\mu_{\boldsymbol{f}} : \underline{u}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial Y}) \to \underline{s}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial X})$ in $\mathrm{qcoh}(\underline{S}_{\boldsymbol{f}})$ such that the following diagrams commute:*

$$\begin{tikzcd}
0 \arrow[r] & \underline{u}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Y}}) \arrow[r, "\underline{u}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})"] \arrow[d, "\lambda_{\boldsymbol{f}}"'] & \underline{u}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{Y}}^*(\mathcal{F}_Y) \arrow[r, "\underline{u}_{\boldsymbol{f}}^*(i_{\mathbf{Y}}^2)"] \arrow[d, "{\begin{subarray}{l} I_{\underline{s}_{\boldsymbol{f}},\underline{i}_{\mathbf{X}}}(\mathcal{F}_X)\circ(\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}})^*(f^2)\circ \\ I_{\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1}\end{subarray}}"] & \underline{u}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial Y}) \arrow[r] \arrow[d, "\mu_{\boldsymbol{f}}"] & 0 \\
0 \arrow[r] & \underline{s}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}}) \arrow[r, "\underline{s}_{\boldsymbol{f}}^*(\nu_{\mathbf{X}})"'] & \underline{s}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X) \arrow[r, "\underline{s}_{\boldsymbol{f}}^*(i_{\mathbf{X}}^2)"'] & \underline{s}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial X}) \arrow[r] & 0,
\end{tikzcd} \tag{6.15}$$

$$\begin{tikzcd}
\underline{u}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Y}}) \arrow[r, "\underline{u}_{\boldsymbol{f}}^*(\underline{i}_{\mathbf{Y}}^*(\psi_Y)\circ\nu_{\mathbf{Y}})"] \arrow[d, "\lambda_{\boldsymbol{f}}"'] & \underline{u}_{\boldsymbol{f}}^*\circ\underline{i}_{\mathbf{Y}}^*(T^*\underline{Y}) \arrow[r, "\underline{u}_{\boldsymbol{f}}^*(\Omega_{\underline{i}_{\mathbf{Y}}})"] \arrow[d, "{\begin{subarray}{l} I_{\underline{s}_{\boldsymbol{f}},\underline{i}_{\mathbf{X}}}(T^*\underline{X})\circ(\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}})^*(\Omega_{\underline{f}})\circ \\ I_{\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}},\underline{f}}(T^*\underline{Y})\circ I_{\underline{u}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(T^*\underline{Y})^{-1}\end{subarray}}"] & \underline{u}_{\boldsymbol{f}}^*(T^*(\underline{\partial Y})) \arrow[r] \arrow[d, "{\begin{subarray}{l}\Omega_{\underline{s}_{\boldsymbol{f}}}^{-1}\circ \\ \Omega_{\underline{u}_{\boldsymbol{f}}}\end{subarray}}"] & 0 \\
\underline{s}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}}) \arrow[r, "\underline{s}_{\boldsymbol{f}}^*(\underline{i}_{\mathbf{X}}^*(\psi_X)\circ\nu_{\mathbf{X}})"'] & \underline{s}_{\boldsymbol{f}}^*\circ\underline{i}_{\mathbf{X}}^*(T^*\underline{X}) \arrow[r, "\underline{s}_{\boldsymbol{f}}^*(\Omega_{\underline{i}_{\mathbf{X}}})"'] & \underline{s}_{\boldsymbol{f}}^*(T^*(\underline{\partial X})) \arrow[r] & 0,
\end{tikzcd} \tag{6.16}$$

where the rows are exact, and $\Omega_{\underline{s}_{\boldsymbol{f}}} : \underline{s}_{\boldsymbol{f}}^*(T^*(\underline{\partial X})) \to T^*\underline{S}_{\boldsymbol{f}}$ is an isomorphism as $\underline{s}_{\boldsymbol{f}}$ is étale by **(c)***. Furthermore, $\lambda_{\boldsymbol{f}}$ is an isomorphism, and identifies the orientations $\underline{u}_{\boldsymbol{f}}^*(\omega_{\mathbf{Y}})$ on $\underline{u}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Y}})$ and $\underline{s}_{\boldsymbol{f}}^*(\omega_{\mathbf{X}})$ on $\underline{s}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}})$.*

(e) *The following diagram commutes in $\mathrm{qcoh}(\underline{T}_{\boldsymbol{f}})$, using $\underline{f} \circ \underline{t}_{\boldsymbol{f}} = \underline{i}_{\mathbf{Y}} \circ \underline{v}_{\boldsymbol{f}}$:*

$$\begin{tikzcd}
\underline{v}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Y}}) \arrow[r, "\underline{v}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})"'] \arrow[d, "0"] & \underline{v}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{Y}}^*(\mathcal{F}_Y) \arrow[r, "I_{\underline{v}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1}"'] & (\underline{f} \circ \underline{t}_{\boldsymbol{f}})^*(\mathcal{F}_Y) \arrow[d, "I_{\underline{t}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)"'] \\
\underline{t}_{\boldsymbol{f}}^*(\mathcal{F}_X) & & \underline{t}_{\boldsymbol{f}}^* \circ \underline{f}^*(\mathcal{F}_Y). \arrow[ll, "\underline{t}_{\boldsymbol{f}}^*(f^2)"']
\end{tikzcd} \tag{6.17}$$

(f) *Let $\boldsymbol{g} : \mathbf{Y} \to \mathbf{Z}$ be another* 1*-morphism of d-spaces with corners. Then the composition $\boldsymbol{g} \circ \boldsymbol{f}$ in* **dSpa** *is a* 1*-morphism $\boldsymbol{g} \circ \boldsymbol{f} : \mathbf{X} \to \mathbf{Z}$ in $\mathbf{dSpa^c}$, and $\underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}} \subseteq \underline{\partial X} \times_{\underline{Z}} \underline{\partial Z}$ and $\underline{T}_{\boldsymbol{g}\circ\boldsymbol{f}} \subseteq \underline{X} \times_{\underline{Z}} \underline{\partial Z}$ have underlying sets*

$$\begin{aligned}
S_{\boldsymbol{g}\circ\boldsymbol{f}} &= \bigl\{(x', z') \in \partial X \times_Z \partial Z : \exists\, y' \in \partial Y,\ (x', y') \in S_{\boldsymbol{f}},\ (y', z') \in S_{\boldsymbol{g}}\bigr\}, \\
T_{\boldsymbol{g}\circ\boldsymbol{f}} &= \bigl\{(x, z') \in X \times_Z \partial Z : (f(x), z') \in T_{\boldsymbol{g}}\bigr\} \amalg \\
&\qquad \bigl\{(x, z') \in X \times_Z \partial Z : \exists\, y' \in \partial Y,\ (x, y') \in T_{\boldsymbol{f}},\ (y', z') \in S_{\boldsymbol{g}}\bigr\}.
\end{aligned} \tag{6.18}$$

Compatibly with the first line of (6.18)*, there are natural morphisms $\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}} : \underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}} \to \underline{S}_{\boldsymbol{f}}$ and $\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}} : \underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}} \to \underline{S}_{\boldsymbol{g}}$ satisfying $\underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}} = \underline{s}_{\boldsymbol{f}} \circ \underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}}$, $\underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}} = \underline{u}_{\boldsymbol{g}} \circ \underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}}$ and $\underline{u}_{\boldsymbol{f}} \circ \underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}} = \underline{s}_{\boldsymbol{g}} \circ \underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}}$, and $\lambda_{\boldsymbol{f}}, \lambda_{\boldsymbol{g}}, \lambda_{\boldsymbol{g}\circ\boldsymbol{f}}$ and $\mu_{\boldsymbol{f}}, \mu_{\boldsymbol{g}}, \mu_{\boldsymbol{g}\circ\boldsymbol{f}}$ in part* **(d)** *are related by the commutative diagrams in $\mathrm{qcoh}(\underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}})$:*

$$
\begin{tikzcd}[column sep=large]
\underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Z}}) \arrow[r, "I_{\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}},\underline{u}_{\boldsymbol{g}}}(\mathcal{N}_{\mathbf{Z}})"'] \arrow[d, "\lambda_{\boldsymbol{g}\circ\boldsymbol{f}}"] & (\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^*\circ\underline{u}_{\boldsymbol{g}}^*(\mathcal{N}_{\mathbf{Z}}) \arrow[r, "(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^*(\lambda_{\boldsymbol{g}})"'] & (\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^*\circ\underline{s}_{\boldsymbol{g}}^*(\mathcal{N}_{\mathbf{Y}}) \arrow[d, "I_{\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}},\underline{u}_{\boldsymbol{f}}}(\mathcal{N}_{\mathbf{Y}})\circ I_{\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}},\underline{s}_{\boldsymbol{g}}}(\mathcal{N}_{\mathbf{Y}})^{-1}"'] \\
\underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}}) & (\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^*\circ\underline{s}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}}) \arrow[l, "I_{\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}},\underline{s}_{\boldsymbol{f}}}(\mathcal{N}_{\mathbf{X}})^{-1}"'] & (\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^*\circ\underline{u}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Y}}), \arrow[l, "(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^*(\lambda_{\boldsymbol{f}})"']
\end{tikzcd}
\tag{6.19}
$$

$$
\begin{tikzcd}[column sep=large]
\underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(\mathcal{F}_{\partial Z}) \arrow[r, "I_{\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}},\underline{u}_{\boldsymbol{g}}}(\mathcal{F}_{\partial Z})"'] \arrow[d, "\mu_{\boldsymbol{g}\circ\boldsymbol{f}}"] & (\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^*\circ\underline{u}_{\boldsymbol{g}}^*(\mathcal{F}_{\partial Z}) \arrow[r, "(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^*(\mu_{\boldsymbol{g}})"'] & (\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^*\circ\underline{s}_{\boldsymbol{g}}^*(\mathcal{F}_{\partial Y}) \arrow[d, "I_{\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}},\underline{u}_{\boldsymbol{f}}}(\mathcal{F}_{\partial Y})\circ I_{\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}},\underline{s}_{\boldsymbol{g}}}(\mathcal{F}_{\partial Y})^{-1}"'] \\
\underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(\mathcal{F}_{\partial X}) & (\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^*\circ\underline{s}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial X}) \arrow[l, "I_{\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}},\underline{s}_{\boldsymbol{f}}}(\mathcal{F}_{\partial X})^{-1}"'] & (\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^*\circ\underline{u}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial Y}), \arrow[l, "(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^*(\mu_{\boldsymbol{f}})"']
\end{tikzcd}
\tag{6.20}
$$

Proof. For (a), suppose $(x',y')\in\underline{S}_{\boldsymbol{f}}$. Then $\underline{i}_{\mathbf{X}}(x')=x\in\underline{X}$ and $\underline{f}(x)=y=\underline{i}_{\mathbf{Y}}(y')\in Y$. Let $(\boldsymbol{V},\boldsymbol{b})$ be a boundary defining function for $\mathbf{Y}$ at y'. Then there exists open $x\in\tilde{\boldsymbol{V}}\subseteq\boldsymbol{f}^{-1}(\boldsymbol{V})$ with $\bigl(\tilde{\boldsymbol{V}},\boldsymbol{b}\circ\boldsymbol{f}|_{\tilde{\boldsymbol{V}}}\bigr)$ a boundary defining function for $\mathbf{X}$ at x'. By Proposition 6.6(a) there exist open $x'\in\underline{T}\subseteq\underline{\partial X}$ and $y'\in\underline{U}\subseteq\underline{\partial Y}$ such that $\bigl(\tilde{\boldsymbol{V}},\boldsymbol{b}\circ\boldsymbol{f}|_{\tilde{\boldsymbol{V}}}\bigr)$ is a boundary defining function for $\mathbf{X}$ at all $\tilde{x}'\in\underline{T}$, and $(\boldsymbol{V},\boldsymbol{b})$ is a boundary defining function for $\mathbf{Y}$ at all $\tilde{y}'\in\underline{U}$. Then $\underline{T}\times_{\underline{Y}}\underline{U}$ is an open neighbourhood of (x',y') in $\underline{\partial X}\times_{\underline{Y}}\underline{\partial Y}$ contained in $\underline{S}_{\boldsymbol{f}}$. Hence $\underline{S}_{\boldsymbol{f}}$ is open in $\underline{\partial X}\times_{\underline{Y}}\underline{\partial Y}$, and so is a well-defined C^∞-scheme.

Suppose $(x',y')\in(\underline{\partial X}\times_{\underline{Y}}\underline{\partial Y})\setminus\underline{S}_{\boldsymbol{f}}$. Then $\underline{i}_{\mathbf{X}}(x')=x\in\underline{X}$ and $\underline{f}(x)=y=\underline{i}_{\mathbf{Y}}(y')\in Y$. Let $(\boldsymbol{V},\boldsymbol{b})$ be a boundary defining function for $\mathbf{Y}$ at y', and choose open $y'\in\underline{U}\subseteq\underline{\partial Y}$ such that $(\boldsymbol{V},\boldsymbol{b})$ is also a boundary defining function for $\mathbf{Y}$ at all $\tilde{y}'\in\underline{U}$. By Definition 6.2, either (i) $\bigl(\tilde{\boldsymbol{V}},\boldsymbol{b}\circ\boldsymbol{f}|_{\tilde{\boldsymbol{V}}}\bigr)$ is a boundary defining function for $\mathbf{X}$ at some $x''\in\underline{\partial X}$ with $\underline{i}_{\mathbf{X}}(x'')=x$, for some open $x\in\tilde{\boldsymbol{V}}\subseteq\boldsymbol{f}^{-1}(\boldsymbol{V})$. Then $(x'',y')\in\underline{S}_{\boldsymbol{f}}$, so $x'\neq x''$; or (ii) $\boldsymbol{b}\circ\boldsymbol{f}|_{\boldsymbol{W}}=\boldsymbol{0}\circ\boldsymbol{\pi}$ for some open $x\in\boldsymbol{W}\subseteq\boldsymbol{f}^{-1}(\boldsymbol{V})$.

In case (i), for any $\tilde{x}'\in\underline{\partial X}$ close enough to x', as $\tilde{x}'$ is not close to x'', $\bigl(\tilde{\boldsymbol{V}},\boldsymbol{b}\circ\boldsymbol{f}|_{\tilde{\boldsymbol{V}}}\bigr)$ is not a boundary defining function for $\mathbf{X}$ at $\tilde{x}'$. So we can construct an open neighbourhood of (x',y') in $(\underline{\partial X}\times_{\underline{Y}}\underline{\partial Y})\setminus\underline{S}_{\boldsymbol{f}}$. In case (ii), if $(\tilde{x}',\tilde{y}')\in\underline{i}_{\mathbf{X}}^{-1}(\underline{W})\times_{\underline{Y}}\underline{U}$ with $\underline{i}_{\mathbf{X}}(\tilde{x}')=\tilde{x}$ then Definition 6.2(ii) holds for $(\boldsymbol{V},\boldsymbol{b})$ at $\tilde{x},\tilde{y}'$, so Definition 6.2(i) cannot hold at $\tilde{x}',\tilde{y}'$, and thus $(\tilde{x}',\tilde{y}')\notin\underline{S}_{\boldsymbol{f}}$. Hence $\underline{i}_{\mathbf{X}}^{-1}(\underline{W})\times_{\underline{Y}}\underline{U}$ is an open neighbourhood of (x',y') in $(\underline{\partial X}\times_{\underline{Y}}\underline{\partial Y})\setminus\underline{S}_{\boldsymbol{f}}$. Therefore $(\underline{\partial X}\times_{\underline{Y}}\underline{\partial Y})\setminus\underline{S}_{\boldsymbol{f}}$ is open in $\underline{\partial X}\times_{\underline{Y}}\underline{\partial Y}$, and $\underline{S}_{\boldsymbol{f}}$ is closed. A similar argument shows $\underline{T}_{\boldsymbol{f}}$ is open and closed in $\underline{X}\times_{\underline{Y}}\underline{\partial Y}$. This proves part (a).

For (b) and (c), suppose $(x',y')\in\underline{S}_{\boldsymbol{f}}$. Then there exist a boundary defining function $(\boldsymbol{V},\boldsymbol{b})$ for $\mathbf{Y}$ at y' and open $y'\in\boldsymbol{U}\subseteq\boldsymbol{\partial Y}$ with (6.1) 2-Cartesian. Hence $\underline{U}\cong\underline{V}\times_{\underline{b},\underline{[0,\infty)},\underline{0}}\underline{*}$. There exist open $\underline{i}_{\mathbf{X}}(x')\in\tilde{\boldsymbol{V}}\subseteq\boldsymbol{f}^{-1}(\boldsymbol{V})$ and $x'\in\tilde{\boldsymbol{U}}\subseteq\boldsymbol{\partial X}$ with $\bigl(\tilde{\boldsymbol{V}},\boldsymbol{b}\circ\boldsymbol{f}|_{\tilde{\boldsymbol{V}}}\bigr)$ a boundary defining function for $\mathbf{X}$ at x' and $\tilde{\boldsymbol{U}}\simeq\tilde{\boldsymbol{V}}\times_{\boldsymbol{b}\circ\boldsymbol{f}|_{\tilde{\boldsymbol{V}}},\boldsymbol{[0,\infty)},\boldsymbol{0}}\boldsymbol{*}$, so that $\underline{\tilde{U}}\cong\underline{\tilde{V}}\times_{\underline{b}\circ\underline{f}|_{\underline{\tilde{V}}},\underline{[0,\infty)},\underline{0}},\underline{*}$. We now have

$$
\begin{aligned}
\underline{\tilde{U}}\times_{\underline{f}\circ\underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}},\underline{Y},\underline{i}_{\mathbf{Y}}|_{\underline{U}}}\underline{U}&\cong\underline{\tilde{U}}\times_{\underline{V}}\bigl(\underline{V}\times_{\underline{b},\underline{[0,\infty)},\underline{0}}\underline{*}\bigr)\cong\underline{\tilde{U}}\times_{\underline{b}\circ\underline{f}\circ\underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}},\underline{[0,\infty)},\underline{0}}\underline{*}\cong\underline{\tilde{U}}\\
&\cong\underline{\tilde{V}}\times_{\underline{b}\circ\underline{f}|_{\underline{\tilde{V}}},\underline{[0,\infty)},\underline{0}},\underline{*}\cong\underline{\tilde{V}}\times_{\underline{V}}\bigl(\underline{V}\times_{\underline{b},\underline{[0,\infty)},\underline{0}}\underline{*}\bigr)\cong\underline{\tilde{V}}\times_{\underline{f}|_{\underline{\tilde{V}}},\underline{Y},\underline{i}_{\mathbf{Y}}|_{\underline{U}}}\underline{U},
\end{aligned}
$$

using $\underline{b}\circ\underline{f}\circ\underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}}=\underline{0}\circ\underline{\pi}:\underline{\tilde{U}}\to\underline{[0,\infty)}$.

The first line of this equation implies that $\underline{s}_{\boldsymbol{f}}$ is an isomorphism between the open C^∞-subschemes $\underline{\tilde{U}} \times_{\underline{Y}} \underline{U} \subseteq \underline{S}_{\boldsymbol{f}} \subseteq \underline{\partial X} \times_{\underline{Y}} \underline{\partial Y}$ and $\underline{\tilde{U}} \subseteq \underline{\partial X}$, and the second line that $\underline{j}_{\boldsymbol{f}}$ is an isomorphism between the open C^∞-subschemes $\underline{\tilde{U}} \times_{\underline{Y}} \underline{U} \subseteq \underline{S}_{\boldsymbol{f}} \subseteq \underline{\partial X} \times_{\underline{Y}} \underline{\partial Y}$ and $\underline{\tilde{V}} \times_{\underline{Y}} \underline{U} \subseteq \underline{X} \times_{\underline{Y}} \underline{\partial Y}$. As we can cover $\underline{S}_{\boldsymbol{f}}$ by such open $\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}$, this proves $\underline{s}_{\boldsymbol{f}}$ and $\underline{j}_{\boldsymbol{f}}$ are étale. A similar proof shows $\underline{t}_{\boldsymbol{f}}$ is étale. To show $\underline{s}_{\boldsymbol{f}}, \underline{t}_{\boldsymbol{f}}$ are proper, note that they are the restrictions to closed C^∞-subschemes $\underline{S}_{\boldsymbol{f}}, \underline{T}_{\boldsymbol{f}}$ of $\underline{\pi}_{\partial X} : \underline{\partial X} \times_{\underline{f} \circ \underline{i}_{\mathbf{X}}, \underline{Y}, \underline{i}_{\mathbf{Y}}} \underline{\partial Y} \to \underline{\partial X}$ and $\underline{\pi}_{X} : \underline{X} \times_{\underline{f}, \underline{Y}, \underline{i}_{\mathbf{Y}}} \underline{\partial Y} \to \underline{X}$, and $\underline{\pi}_{\partial X}, \underline{\pi}_{X}$ are proper as $\underline{i}_{\mathbf{Y}}$ is proper. This proves (c).

Suppose $(x, y') \in \underline{X} \times_{\underline{Y}} \underline{\partial Y}$, so that $\underline{f}(x) = \underline{i}_{\mathbf{Y}}(y') = y \in \underline{Y}$. Let $(\boldsymbol{V}, \boldsymbol{b})$ be a boundary defining function for $\mathbf{Y}$ at y'. Then by Definition 6.2, either (i) there is a unique $x' \in \underline{\partial X}$ with $\underline{i}_{\mathbf{X}}(x') = x$ and $(x', y') \in \underline{S}_{\boldsymbol{f}}$, so that $\underline{j}_{\boldsymbol{f}}(x', y') = (x, y')$, or (ii) $(x, y') \in \underline{T}_{\boldsymbol{f}}$, and (i),(ii) are exclusive. Hence $\underline{j}_{\boldsymbol{f}}$ is injective, by uniqueness of x' in (i), and has image $(\underline{X} \times_{\underline{Y}} \underline{\partial Y}) \setminus \underline{T}_{\boldsymbol{f}}$. Therefore $\underline{j}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to (\underline{X} \times_{\underline{Y}} \underline{\partial Y}) \setminus \underline{T}_{\boldsymbol{f}}$ is étale and a bijection on points, which implies it is an isomorphism, proving (b).

For (d), first note that the rows of (6.15) and (6.16) are pullbacks by $\underline{u}_{\boldsymbol{f}}, \underline{s}_{\boldsymbol{f}}$ of (6.4) and (6.5) for $\mathbf{X}, \mathbf{Y}$. Pullbacks are right exact, and also take split exact sequences to split exact sequences. Hence as (6.4) is split exact and (6.5) is exact, the rows of (6.15) and (6.16) are exact. Exactness of the rows in (6.15) implies that $\lambda_{\boldsymbol{f}}, \mu_{\boldsymbol{f}}$ are unique if they exist. Hence it is enough to construct $\lambda_{\boldsymbol{f}}, \mu_{\boldsymbol{f}}$ satisfying the restrictions of (6.15), (6.16) to an open cover of $\underline{S}_{\boldsymbol{f}}$, since by uniqueness the morphisms on the C^∞-subschemes of the open cover can be glued on overlaps to make global morphisms $\lambda_{\boldsymbol{f}}, \mu_{\boldsymbol{f}}$.

Let $(x', y') \in \underline{S}_{\boldsymbol{f}}$, and choose $y' \in \boldsymbol{U}, (\boldsymbol{V}, \boldsymbol{b}), \tilde{\boldsymbol{V}} \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V})$ and $x' \in \tilde{\boldsymbol{U}} \subseteq \boldsymbol{\partial X}$ with $\boldsymbol{U} \simeq \boldsymbol{V} \times_{\boldsymbol{b}, \boldsymbol{[0,\infty)}, \boldsymbol{0}} \boldsymbol{*}$ and $\tilde{\boldsymbol{U}} \simeq \tilde{\boldsymbol{V}} \times_{\boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}}, \boldsymbol{[0,\infty)}, \boldsymbol{0}} \boldsymbol{*}$ as in the proof of (b),(c). Then $\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}$ is an open neighbourhood of (x', y') in $\underline{S}_{\boldsymbol{f}}$. Now consider the diagram in $\mathrm{qcoh}(\underline{\tilde{U}} \times_{\underline{Y}} \underline{U})$:

$$
\begin{tikzcd}[column sep=large]
0 \arrow[r] & \begin{array}{c} \underline{u}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{Y}}|_{\underline{U}}^* \circ \underline{b}^* \\ (\mathcal{F}_{[0,\infty)})|_{\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}} \end{array} \arrow[r, "\underline{u}_{\boldsymbol{f}}^*(\underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}}^*((b \circ f)^2))"] \arrow[d, "\begin{array}{l} I_{\underline{s}_{\boldsymbol{f}}, \underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}}}((\underline{b} \circ \underline{f})^*(\mathcal{F}_{[0,\infty)}) \circ \\ I_{\underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}} \circ \underline{s}_{\boldsymbol{f}}, \underline{b} \circ \underline{f}}(\mathcal{F}_{[0,\infty)}) \circ \\ I_{\underline{i}_{\mathbf{Y}}|_{\underline{U}} \circ \underline{u}_{\boldsymbol{f}}, \underline{b}}(\mathcal{F}_{[0,\infty)})^{-1} \circ \\ I_{\underline{u}_{\boldsymbol{f}}, \underline{i}_{\mathbf{Y}}|_{\underline{U}}} \underline{b}^*(\mathcal{F}_{[0,\infty)})^{-1} \end{array}"] & \underline{u}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{Y}}^*(\mathcal{F}_Y)|_{\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}} \arrow[r, "\underline{u}_{\boldsymbol{f}}^*(i_{\mathbf{Y}}^2)"] \arrow[d, "\begin{array}{l} I_{\underline{s}_{\boldsymbol{f}}, \underline{i}_{\mathbf{X}}}(\mathcal{F}_X) \circ (\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}})^*(f^2) \circ \\ I_{\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}}, \underline{f}}(\mathcal{F}_Y) \circ I_{\underline{u}_{\boldsymbol{f}}, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1} \end{array}"] & \underline{u}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial Y})|_{\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}} \arrow[r] \arrow[d, dotted, "\mu_{\boldsymbol{f}}|_{\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}}"] & 0 \\
0 \arrow[r] & \begin{array}{c} \underline{s}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}}^* \circ (\underline{b} \circ \underline{f})^* \\ (\mathcal{F}_{[0,\infty)})|_{\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}} \end{array} \arrow[r, "\underline{s}_{\boldsymbol{f}}^*(\underline{i}_{\mathbf{Y}}|_{\underline{U}}^*(b^2))"'] & \underline{s}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X)|_{\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}} \arrow[r, "\underline{s}_{\boldsymbol{f}}^*(i_{\mathbf{X}}^2)"'] & \underline{s}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial X})|_{\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}} \arrow[r] & 0.
\end{tikzcd}
\tag{6.21}
$$

Here the rows are the pullbacks of the split exact sequence (6.2) for $(\boldsymbol{V}, \boldsymbol{b})$ on $\underline{U}$ by $\underline{u}_{\boldsymbol{f}}$ and for $(\boldsymbol{f}^{-1}(\boldsymbol{V}), \boldsymbol{b} \circ \boldsymbol{f})$ on $\underline{\tilde{U}}$ by $\underline{s}_{\boldsymbol{f}}$, and so are (split) exact.

There is a natural orientation-preserving isomorphism, shown in the left hand column of (6.21), which makes the left hand square of (6.21) commute. Hence there exists a unique morphism $\mu_{\boldsymbol{f}}|_{\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}}$ making the right hand of (6.21) square commute. In Definition 6.1 we showed there is an orientation-preserving isomorphism $\mathcal{N}_{\mathbf{X}}|_{\underline{U}} \cong \underline{i}_{\mathbf{X}}|_{\underline{U}}^* \circ \underline{b}^*(\mathcal{F}_{[0,\infty)})$ identifying (6.2) with the restriction of (6.4) to $\underline{U}$. These for $\mathbf{X}, \underline{\tilde{U}}$ and $\mathbf{Y}, \underline{U}$ induce an identification of (6.21) with the restriction of (6.15) to $\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}$. Hence there exists a unique orientation-preserving isomorphism $\lambda|_{\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}}$ identified with the left hand column of (6.21), making the restriction of (6.15) to $\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}$ commute. As we can cover $\underline{S}_{\boldsymbol{f}}$ by such open $\underline{\tilde{U}} \times_{\underline{Y}} \underline{U}$, there exist unique global $\lambda_{\boldsymbol{f}}, \mu_{\boldsymbol{f}}$ making (6.15) commute, with $\lambda_{\boldsymbol{f}}$ an orientation-preserving isomorphism. The left hand square of (6.16) commutes by (6.15) and (2.22) for $\boldsymbol{f}$. The right hand square of (6.16) commutes by $\underline{f} \circ \underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}} = \underline{i}_{\mathbf{Y}} \circ \underline{u}_{\boldsymbol{f}}$ and Theorem B.39(b). This proves (d).

For (e), let $(x, y') \in \underline{T}_{\boldsymbol{f}}$, with $\underline{f}(x) = \underline{i}_{\mathbf{Y}}(y') = y \in \boldsymbol{Y}$. Let $(\boldsymbol{V}, \boldsymbol{b})$ be a boundary defining function for $\mathbf{Y}$ at y'. Then as in Definition 6.1 there exists open $y' \in \boldsymbol{U} \subseteq \boldsymbol{\partial Y}$ with (6.1) 2-Cartesian, and as in (6.7) there is an isomorphism $\gamma : \underline{i}_{\mathbf{Y}}|_{\underline{U}}^* \circ \underline{b}^*(\mathcal{F}_{[0,\infty)}) \to \mathcal{N}_{\mathbf{Y}}|_{\underline{U}}$ with $\nu_{\mathbf{Y}}|_{\underline{U}} \circ \gamma = \underline{i}_{\mathbf{Y}}|_{\underline{U}}^*(b^2)$. Also by Definition 6.2 there exists open $x \in \boldsymbol{W} \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$ such that $\boldsymbol{b} \circ \boldsymbol{f}|_{\boldsymbol{W}} = \boldsymbol{0} \circ \boldsymbol{\pi}$. Define $\underline{Z} = \underline{t}_{\boldsymbol{f}}^{-1}(\underline{W}) \cap \underline{v}_{\boldsymbol{f}}^{-1}(\underline{U})$, so that $(x, y') \in \underline{Z} \subseteq \underline{T}_{\boldsymbol{f}}$ is open. We have

$$
\begin{aligned}
&\big(\underline{t}_{\boldsymbol{f}}^*(f^2) \circ I_{\underline{t}_{\boldsymbol{f}}, \underline{f}}(\mathcal{F}_Y) \circ I_{\underline{v}_{\boldsymbol{f}}, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1} \circ \underline{v}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})\big)|_{\underline{Z}} \\
&\quad = \underline{t}_{\boldsymbol{f}}|_{\underline{Z}}^*(f^2) \circ I_{\underline{t}_{\boldsymbol{f}}, \underline{f}}(\mathcal{F}_Y)|_{\underline{Z}} \circ I_{\underline{v}_{\boldsymbol{f}}, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)|_{\underline{Z}}^{-1} \circ \underline{v}_{\boldsymbol{f}}|_{\underline{Z}}^*(\underline{i}_{\mathbf{Y}}|_{\underline{U}}^*(b^2)) \circ \underline{v}_{\boldsymbol{f}}|_{\underline{Z}}^*(\gamma^{-1}) \\
&\quad = \underline{t}_{\boldsymbol{f}}|_{\underline{Z}}^*(f^2 \circ \underline{f}|_{\underline{W}}^*(b^2)) \circ I_{\underline{t}_{\boldsymbol{f}}, \underline{f}}(\mathcal{F}_Y)|_{\underline{Z}} \circ I_{\underline{v}_{\boldsymbol{f}}, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)|_{\underline{Z}}^{-1} \circ \underline{v}_{\boldsymbol{f}}|_{\underline{Z}}^*(\gamma^{-1}) = 0,
\end{aligned}
$$

using $\nu_{\mathbf{Y}}|_{\underline{U}} \circ \gamma = \underline{i}_{\mathbf{Y}}|_{\underline{U}}^*(b^2)$ and γ an isomorphism in the first step, properties of $I_{*,*}(*)$ in the second, and that $(b \circ f|_W)^2 = (0 \circ \pi)^2 = 0$ in the third. This proves the restriction of (6.17) to $\underline{Z}$ commutes. As we can cover $\underline{T}_{\boldsymbol{f}}$ by such open $\underline{Z}$, part (e) follows.

For (f), to prove that $\boldsymbol{g} \circ \boldsymbol{f}$ is a 1-morphism of d-spaces with corners, suppose $x \in \boldsymbol{X}$ with $(\boldsymbol{g} \circ \boldsymbol{f})(x) = z \in \boldsymbol{Z}$, and $z' \in \boldsymbol{\partial Z}$ with $\boldsymbol{i}_{\mathbf{Z}}(z') = z$, and $(\boldsymbol{V}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{Z}$ at z'. Then $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$, and $\boldsymbol{g}(y) = z \in \boldsymbol{Z}$. So as $\boldsymbol{g}$ is a 1-morphism of d-spaces, either (i)$'$ there exists $y' \in \boldsymbol{\partial Y}$ with $\boldsymbol{i}_{\mathbf{Y}}(y') = y$ and open $y \in \tilde{\boldsymbol{V}} \subseteq \boldsymbol{g}^{-1}(\boldsymbol{V})$ such that $(\tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{g}|_{\tilde{\boldsymbol{V}}})$ is a boundary defining function for $\mathbf{Y}$ at y', or (ii)$'$ there exists open $y \in \boldsymbol{W} \subseteq \boldsymbol{g}^{-1}(\boldsymbol{V})$ with $\boldsymbol{b} \circ \boldsymbol{g}|_{\boldsymbol{W}} = \boldsymbol{0} \circ \boldsymbol{\pi}$.

In case (i)$'$, as $\boldsymbol{f}$ is a is a 1-morphism of d-spaces and $(\tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{g}|_{\tilde{\boldsymbol{V}}})$ is a boundary defining function for $\mathbf{Y}$ at y', either (i)$''$ there exists $x' \in \boldsymbol{\partial X}$ with $\boldsymbol{i}_{\mathbf{X}}(x') = x$ and open $x \in \hat{\boldsymbol{V}} \subseteq \boldsymbol{f}^{-1}(\tilde{\boldsymbol{V}})$ such that $(\hat{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{g} \circ \boldsymbol{f}|_{\hat{\boldsymbol{V}}})$ is a boundary defining function for $\mathbf{X}$ at x', so (i) above holds for $\boldsymbol{g} \circ \boldsymbol{f}$, or (ii)$''$ there exists open $x \in \boldsymbol{W} \subseteq \boldsymbol{f}^{-1}(\tilde{\boldsymbol{V}}) \subseteq \boldsymbol{X}$ with $\boldsymbol{b} \circ \boldsymbol{g} \circ \boldsymbol{f}|_{\boldsymbol{W}} = \boldsymbol{0} \circ \boldsymbol{\pi}$, so (ii) holds for $\boldsymbol{g} \circ \boldsymbol{f}$. In case (ii)$'$, setting $\tilde{\boldsymbol{W}} = \boldsymbol{f}^{-1}(\boldsymbol{W})$, we have

$$\boldsymbol{b} \circ (\boldsymbol{g} \circ \boldsymbol{f})|_{\tilde{\boldsymbol{W}}} = (\boldsymbol{b} \circ \boldsymbol{g}|_{\boldsymbol{W}}) \circ \boldsymbol{f}|_{\tilde{\boldsymbol{W}}} = \boldsymbol{0} \circ \boldsymbol{\pi} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{W}}} = \boldsymbol{0} \circ \boldsymbol{\pi},$$

so (ii) holds for $\boldsymbol{g} \circ \boldsymbol{f}$. Hence $\boldsymbol{g} \circ \boldsymbol{f}$ is a 1-morphism of d-spaces with corners.

Equation (6.18) follows from cases (i)$'$,(ii)$'$,(i)$''$,(ii)$''$ above. To construct $\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}}, \underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}}$, note that the morphisms $\underline{f} \circ \underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}} : \underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}} \to \underline{Y}$ and $\underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}} : \underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}} \to \underline{\partial Z}$ satisfy $\underline{g} \circ (\underline{f} \circ \underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}}) = \underline{i}_{\mathbf{Z}} \circ \underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}}$. Hence there is a unique morphism $\underline{\pi} : \underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}} \to \underline{Y} \times_{\underline{Z}} \underline{\partial Z}$ with $\underline{\pi}_Y \circ \underline{\pi} = \underline{f} \circ \underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}}$ and $\underline{\pi}_{\underline{\partial Z}} \circ \underline{\pi} = \underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}}$. Equation (6.18) implies that $\underline{\pi}$ maps $\underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}}$ to $(\underline{Y} \times_{\underline{Z}} \underline{\partial Z}) \setminus \underline{T}_{\boldsymbol{g}}$. Since $\underline{j}_{\boldsymbol{g}} : \underline{S}_{\boldsymbol{g}} \to (\underline{Y} \times_{\underline{Z}} \underline{\partial Z}) \setminus \underline{T}_{\boldsymbol{g}}$ is an isomorphism by part (b), we may define $\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}} : \underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}} \to \underline{S}_{\boldsymbol{g}}$ by $\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}} = \underline{j}_{\boldsymbol{g}}^{-1} \circ \underline{\pi}$. Then $\underline{u}_{\boldsymbol{g}} \circ \underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}} = \underline{u}_{\boldsymbol{g}} \circ \underline{j}_{\boldsymbol{g}}^{-1} \circ \underline{\pi} = \underline{\pi}_{\underline{\partial Z}} \circ \underline{\pi} = \underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}}$, as we want.

Similarly, $\underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}} : \underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}} \to \underline{\partial X}$ and $\underline{s}_{\boldsymbol{g}} \circ \underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}} : \underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}} \to \underline{\partial Y}$ satisfy $\underline{f} \circ \underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}} = \underline{i}_{\mathbf{Y}} \circ (\underline{s}_{\boldsymbol{g}} \circ \underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})$. Hence there is a unique $\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}} : \underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}} \to \underline{\partial X} \times_{\underline{Y}} \underline{\partial Y}$ with $\underline{\pi}_{\underline{\partial X}} \circ \underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}} = \underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}}$ and $\underline{\pi}_{\underline{\partial Y}} \circ \underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}} = \underline{s}_{\boldsymbol{g}} \circ \underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}}$. Equation (6.18) implies that $\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}}$ maps into $\underline{S}_{\boldsymbol{f}} \subseteq \underline{\partial X} \times_{\underline{Y}} \underline{\partial Y}$. Thus $\underline{\pi}_{\underline{\partial X}}, \underline{\pi}_{\underline{\partial Y}}$ become $\underline{s}_{\boldsymbol{f}}, \underline{u}_{\boldsymbol{f}}$, so that $\underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}} = \underline{s}_{\boldsymbol{f}} \circ \underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}}$ and $\underline{u}_{\boldsymbol{f}} \circ \underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}} = \underline{s}_{\boldsymbol{g}} \circ \underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}}$, as we have to prove.

To show (6.19)–(6.20) commute, consider the diagram in $\mathrm{qcoh}(\underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}})$:

$$
\begin{tikzcd}
\underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Z}}) \arrow[r, "\underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(\nu_{\mathbf{Z}})"'] \arrow[d, "I_{\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}},\underline{u}_{\boldsymbol{g}}}(\mathcal{N}_{\mathbf{Z}})"] & \underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{Z}}^*(\mathcal{F}_Z) \arrow[r, "\underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(i_{\mathbf{Z}}^2)"'] \arrow[d, "I_{\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}},\underline{u}_{\boldsymbol{g}}}(\underline{i}_{\mathbf{Z}}^*(\mathcal{F}_Z))"] & \underline{u}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(\mathcal{F}_{\partial Z}) \arrow[d, "I_{\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}},\underline{u}_{\boldsymbol{g}}}(\mathcal{F}_{\partial Z})"'] \\
(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{u}_{\boldsymbol{g}}^*(\mathcal{N}_{\mathbf{Z}}) \arrow[r, "(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{u}_{\boldsymbol{g}}^*(\nu_{\mathbf{Z}})"] \arrow[d, "(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^*(\lambda_{\boldsymbol{g}})"] & (\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{u}_{\boldsymbol{g}}^* \circ \underline{i}_{\mathbf{Z}}^*(\mathcal{F}_Z) \arrow[r, "(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{u}_{\boldsymbol{g}}^*(i_{\mathbf{Z}}^2)"] \arrow[d, "{\begin{array}{l}(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^*(I_{\underline{s}_{\boldsymbol{g}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)\circ(\underline{i}_{\mathbf{Y}}\circ\underline{s}_{\boldsymbol{g}})^*(g^2) \\ \circ I_{\underline{i}_{\mathbf{Y}}\circ\underline{s}_{\boldsymbol{g}},\underline{g}}(\mathcal{F}_Z)\circ I_{\underline{u}_{\boldsymbol{g}},\underline{i}_{\mathbf{Z}}}(\mathcal{F}_Z)^{-1})\end{array}}"] & (\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{u}_{\boldsymbol{g}}^*(\mathcal{F}_{\partial Z}) \arrow[d, "(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^*(\mu_{\boldsymbol{g}})"'] \\
(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{s}_{\boldsymbol{g}}^*(\mathcal{N}_{\mathbf{Y}}) \arrow[r, "(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{s}_{\boldsymbol{g}}^*(\nu_{\mathbf{Y}})"] \arrow[d, "{\begin{array}{l}I_{\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}},\underline{u}_{\boldsymbol{f}}}(\mathcal{N}_{\mathbf{Y}})\circ \\ I_{\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}},\underline{s}_{\boldsymbol{g}}}(\mathcal{N}_{\mathbf{Y}})^{-1}\end{array}}"] & (\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{s}_{\boldsymbol{g}}^* \circ \underline{i}_{\mathbf{Y}}^*(\mathcal{F}_Y) \arrow[r, "(\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{s}_{\boldsymbol{g}}^*(i_{\mathbf{Y}}^2)"] \arrow[d, "{\begin{array}{l}I_{\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}},\underline{u}_{\boldsymbol{f}}}(\underline{i}_{\mathbf{Y}}^*(\mathcal{F}_Y))\circ \\ I_{\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}},\underline{s}_{\boldsymbol{g}}}(\underline{i}_{\mathbf{Y}}^*(\mathcal{F}_Y))^{-1}\end{array}}"] & (\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{s}_{\boldsymbol{g}}^*(\mathcal{F}_{\partial Y}) \arrow[d, "{\begin{array}{l}I_{\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}},\underline{u}_{\boldsymbol{f}}}(\mathcal{F}_{\partial Y})\circ \\ I_{\underline{\Pi}_2^{\boldsymbol{f},\boldsymbol{g}},\underline{s}_{\boldsymbol{g}}}(\mathcal{F}_{\partial Y})^{-1}\end{array}}"'] \\
(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{u}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Y}}) \arrow[r, "(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{u}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})"] \arrow[d, "(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^*(\lambda_{\boldsymbol{f}})"] & (\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{u}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{Y}}^*(\mathcal{F}_Y) \arrow[r, "(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{u}_{\boldsymbol{f}}^*(i_{\mathbf{Y}}^2)"] \arrow[d, "{\begin{array}{l}(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^*(I_{\underline{s}_{\boldsymbol{f}},\underline{i}_{\mathbf{X}}}(\mathcal{F}_X)\circ(\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}})^*(f^2) \\ \circ I_{\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1})\end{array}}"] & (\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{u}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial Y}) \arrow[d, "(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^*(\mu_{\boldsymbol{f}})"'] \\
(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{s}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}}) \arrow[r, "(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{s}_{\boldsymbol{f}}^*(\nu_{\mathbf{X}})"] \arrow[d, "I_{\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}},\underline{s}_{\boldsymbol{f}}}(\mathcal{N}_{\mathbf{X}})^{-1}"] & (\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{s}_{\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X) \arrow[r, "(\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{s}_{\boldsymbol{f}}^*(i_{\mathbf{X}}^2)"] \arrow[d, "I_{\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}},\underline{s}_{\boldsymbol{f}}}(\underline{i}_{\mathbf{X}}^*(\mathcal{F}_X))^{-1}"] & (\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}})^* \circ \underline{s}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial X}) \arrow[d, "I_{\underline{\Pi}_1^{\boldsymbol{f},\boldsymbol{g}},\underline{s}_{\boldsymbol{f}}}(\mathcal{F}_{\partial X})^{-1}"'] \\
\underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}}) \arrow[r, "\underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(\nu_{\mathbf{X}})"] & \underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}}^* \circ \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X) \arrow[r, "\underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(i_{\mathbf{X}}^2)"] & \underline{s}_{\boldsymbol{g}\circ\boldsymbol{f}}^*(\mathcal{F}_{\partial X}).
\end{tikzcd}
$$

This commutes, by (6.15) for $\boldsymbol{f}, \boldsymbol{g}$ and properties of $I_{*,*}(*)$. The composition of the middle column is the middle column of (6.15), by (2.24) and properties of $I_{*,*}(*)$. Hence by uniqueness of $\lambda_{\boldsymbol{g}\circ\boldsymbol{f}}, \mu_{\boldsymbol{g}\circ\boldsymbol{f}}$ in part (d) for $\boldsymbol{g}\circ\boldsymbol{f}$, the compositions of the left and right columns are $\lambda_{\boldsymbol{g}\circ\boldsymbol{f}}, \mu_{\boldsymbol{g}\circ\boldsymbol{f}}$. This proves part (f). $\square$

Here are some properties of 2-morphisms:

Proposition 6.8. *Let $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ be 1-morphisms and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism in* $\mathbf{dSpa^c}$*. Then:*

(a) $\lambda_{\boldsymbol{f}}, \lambda_{\boldsymbol{g}} : \underline{u}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{Y}}) \to \underline{s}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}})$ *in Proposition* 6.7(d) *satisfy* $\lambda_{\boldsymbol{f}} = \lambda_{\boldsymbol{g}}$.

(b) *There is a unique morphism* $\eta_S : \underline{u}_{\boldsymbol f}^*(\mathcal{F}_{\partial Y}) \to \underline{s}_{\boldsymbol f}^*(\mathcal{E}_{\partial X})$ *in* $\mathrm{qcoh}(\underline{S}_{\boldsymbol f})$ *such that the following commutes:*

$$\begin{tikzcd}
(\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f})^*\circ\underline{f}^*(\mathcal{F}_Y) \arrow[r, "{I_{\underline{u}_{\boldsymbol f},\underline{i}_{\mathbf Y}}(\mathcal{F}_Y)\circ}"', "{I_{\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f},\underline{f}}(\mathcal{F}_Y)^{-1}}"' yshift=-12pt] \arrow[d, "(\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f})^*(\eta)"] & \underline{u}_{\boldsymbol f}^*\circ\underline{i}_{\mathbf Y}^*(\mathcal{F}_Y) \arrow[r, "\underline{u}_{\boldsymbol f}^*(i_{\mathbf Y}^2)"'] & \underline{u}_{\boldsymbol f}^*(\mathcal{F}_{\partial Y}) \arrow[d, "\eta_S"] \\
(\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f})^*(\mathcal{E}_X) \arrow[r, "{I_{\underline{s}_{\boldsymbol f},\underline{i}_{\mathbf X}}(\mathcal{E}_X)}"] & \underline{s}_{\boldsymbol f}^*\circ\underline{i}_{\mathbf X}^*(\mathcal{E}_X) \arrow[r, "\underline{s}_{\boldsymbol f}^*(i''_{\mathbf X})"] & \underline{s}_{\boldsymbol f}^*(\mathcal{E}_{\partial X}).
\end{tikzcd} \tag{6.22}$$

(c) *There is a unique morphism* $\eta_T : \underline{v}_{\boldsymbol f}^*(\mathcal{F}_{\partial Y}) \to \underline{t}_{\boldsymbol f}^*(\mathcal{E}_X)$ *in* $\mathrm{qcoh}(\underline{T}_{\boldsymbol f})$ *such that the following commutes:*

$$\begin{tikzcd}
\underline{t}_{\boldsymbol f}^*\circ\underline{f}^*(\mathcal{F}_Y) \arrow[r, "{I_{\underline{t}_{\boldsymbol f},\underline{f}}(\mathcal{F}_Y)^{-1}}"'] \arrow[d, "\underline{t}_{\boldsymbol f}^*(\eta)"] & (\underline{f}\circ\underline{t}_{\boldsymbol f})^*(\mathcal{F}_Y) \arrow[r, "{I_{\underline{v}_{\boldsymbol f},\underline{i}_{\mathbf Y}}(\mathcal{F}_Y)}"'] & \underline{v}_{\boldsymbol f}^*\circ\underline{i}_{\mathbf Y}^*(\mathcal{F}_Y) \arrow[d, "\underline{v}_{\boldsymbol f}^*(i_{\mathbf Y}^2)"'] \\
\underline{t}_{\boldsymbol f}^*(\mathcal{E}_X) & & \underline{v}_{\boldsymbol f}^*(\mathcal{F}_{\partial Y}). \arrow[ll, "\eta_T"']
\end{tikzcd} \tag{6.23}$$

Proof. For (a), we have

$$\begin{aligned}
&\underline{s}_{\boldsymbol f}^*(\nu_{\mathbf X})\circ(\lambda_{\boldsymbol g}-\lambda_{\boldsymbol f})\\
&\quad= I_{\underline{s}_{\boldsymbol f},\underline{i}_{\mathbf X}}(\mathcal{F}_X)\circ(\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f})^*(g^2-f^2)\circ I_{\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol f},\underline{i}_{\mathbf Y}}(\mathcal{F}_Y)^{-1}\circ\underline{u}_{\boldsymbol f}^*(\nu_{\mathbf Y})\\
&\quad= I_{\underline{s}_{\boldsymbol f},\underline{i}_{\mathbf X}}(\mathcal{F}_X)\circ(\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f})^*(\phi_X\circ\eta)\circ I_{\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol f},\underline{i}_{\mathbf Y}}(\mathcal{F}_Y)^{-1}\circ\underline{u}_{\boldsymbol f}^*(\nu_{\mathbf Y})=0,
\end{aligned}$$

using (6.15) for $\boldsymbol f,\boldsymbol g$ in the first step, (2.26) for η in the second, and (6.9) in the third. As $\nu_{\mathbf X}$ is injective by (6.4), and $\underline{s}_{\boldsymbol f}$ is étale by Proposition 6.7(c), $\underline{s}_{\boldsymbol f}^*(\nu_{\mathbf X})$ is injective, so $\lambda_{\boldsymbol g}-\lambda_{\boldsymbol f}=0$.

For (b), first note that as (6.4) for $\mathbf Y$ is split exact, its pullback

$$\begin{tikzcd}
0 \arrow[r] & \underline{u}_{\boldsymbol f}^*(\mathcal{N}_{\mathbf Y}) \arrow[r, "\underline{u}_{\boldsymbol f}^*(\nu_{\mathbf Y})"] & \underline{u}_{\boldsymbol f}^*\circ\underline{i}_{\mathbf Y}^*(\mathcal{F}_Y) \arrow[r, "\underline{u}_{\boldsymbol f}^*(i_{\mathbf Y}^2)"] & \underline{u}_{\boldsymbol f}^*(\mathcal{F}_{\partial Y}) \arrow[r] & 0
\end{tikzcd} \tag{6.24}$$

is also split exact. Now (6.9) implies that

$$\bigl(\underline{s}_{\boldsymbol f}^*(i''_{\mathbf X})\circ I_{\underline{s}_{\boldsymbol f},\underline{i}_{\mathbf X}}(\mathcal{E}_X)\circ(\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f})^*(\eta)\circ I_{\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol f},\underline{i}_{\mathbf Y}}(\mathcal{F}_Y)^{-1}\bigr)\circ\underline{u}_{\boldsymbol f}^*(\nu_{\mathbf Y})=0,$$

so by exactness of (6.24) there is a unique $\eta_S : \underline{u}_{\boldsymbol f}^*(\mathcal{F}_{\partial Y}) \to \underline{s}_{\boldsymbol f}^*(\mathcal{E}_{\partial X})$ with

$$\underline{s}_{\boldsymbol f}^*(i''_{\mathbf X})\circ I_{\underline{s}_{\boldsymbol f},\underline{i}_{\mathbf X}}(\mathcal{E}_X)\circ(\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f})^*(\eta)\circ I_{\underline{i}_{\mathbf X}\circ\underline{s}_{\boldsymbol f},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol f},\underline{i}_{\mathbf Y}}(\mathcal{F}_Y)^{-1}=\eta_S\circ\underline{u}_{\boldsymbol f}^*(i_{\mathbf Y}^2).$$

This is the unique η_S for which (6.22) commutes. The proof for (c) is similar, using equation (6.10). $\square$

Here is the analogue of Proposition 2.17. Note that equations (6.9)–(6.10) are necessary conditions on η for $\eta : \boldsymbol f \Rightarrow \boldsymbol g$ to be a 2-morphism in $\mathbf{dSpa^c}$.

Proposition 6.9. *Suppose* $\boldsymbol f : \mathbf X \to \mathbf Y$ *is a* 1*-morphism of d-spaces with corners, and* $\eta : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{E}_X$ *is a morphism in* $\mathrm{qcoh}(\underline{X})$ *such that* (6.9) *and* (6.10) *hold in* $\mathrm{qcoh}(\underline{S}_{\boldsymbol f})$ *and* $\mathrm{qcoh}(\underline{T}_{\boldsymbol f})$*. Then there exists a unique* 1*-morphism* $\boldsymbol g : \mathbf X \to \mathbf Y$ *in* $\mathbf{dSpa^c}$ *such that* $\eta : \boldsymbol f \Rightarrow \boldsymbol g$ *is a* 2*-morphism in* $\mathbf{dSpa^c}$*.*

Proof. As $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in **dSpa**, Proposition 2.17 gives a unique 1-morphism $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ in **dSpa** such that $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism in **dSpa**. We first show that $\boldsymbol{g}$ is a 1-morphism in $\mathbf{dSpa^c}$. Suppose $x \in \boldsymbol{X}$ with $\boldsymbol{g}(x) = y \in \boldsymbol{Y}$, and $y' \in \boldsymbol{\partial Y}$ with $\boldsymbol{i}_{\mathbf{Y}}(y') = y$, and $(\boldsymbol{V}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{Y}$ at y'. Since $\boldsymbol{f}, \boldsymbol{g}$ are 2-isomorphic in **dSpa** we have $\boldsymbol{f}(x) = \boldsymbol{g}(x) = y$. Thus by Definition 6.2 for $\boldsymbol{f}$, either:

(i) there exists $x' \in \boldsymbol{\partial X}$ with $\boldsymbol{i}_{\mathbf{X}}(x') = x$ and open $x \in \tilde{\boldsymbol{V}} \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$ such that $\bigl(\tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}}\bigr)$ is a boundary defining function for $\mathbf{X}$ at x'; or

(ii) there exists an open $x \in \boldsymbol{W} \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$ with $\boldsymbol{b} \circ \boldsymbol{f}|_{\boldsymbol{W}} = \boldsymbol{0} \circ \boldsymbol{\pi}$.

In case (i), by Definition 6.1 there exist open $y' \in \boldsymbol{U} \subseteq \boldsymbol{i}_{\mathbf{Y}}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{\partial Y}$ and $x' \in \tilde{\boldsymbol{U}} \subseteq \boldsymbol{i}_{\mathbf{X}}^{-1}(\tilde{\boldsymbol{V}}) \subseteq \boldsymbol{\partial X}$ so that $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{b}, \boldsymbol{i}_{\mathbf{Y}}|_{\boldsymbol{U}}$ and $\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}}, \boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{U}}}$ fit into 2-Cartesian diagrams (6.1) in **dSpa**. Since $\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial X}$ is étale by Proposition 6.7(c) and $\underline{s}_{\boldsymbol{f}}(x', y') = x'$, $\underline{u}_{\boldsymbol{f}}(x', y') = y'$, making $\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{V}}$ smaller if necessary, there exists open $(x', y') \in \underline{Z} \subseteq \underline{S}_{\boldsymbol{f}}$ such that $\underline{s}_{\boldsymbol{f}}|_{\underline{Z}} : \underline{Z} \to \underline{\tilde{U}}$ is an isomorphism, and $\underline{u}_{\boldsymbol{f}}(\underline{Z}) \subseteq \underline{U}$. By Definition 6.1(e) there exists an isomorphism $\gamma : \underline{i}_{\mathbf{Y}}|_{\underline{U}}^* \circ \underline{b}^*(\mathcal{F}_{[0,\infty)}) \to \mathcal{N}_{\mathbf{Y}}|_{\underline{U}}$ with $\nu_{\mathbf{Y}}|_{\underline{U}} \circ \gamma = \underline{i}_{\mathbf{Y}}|_{\underline{U}}^*(b^2)$. So by (6.9) we have

$$\begin{aligned}
0 &= \bigl((\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}})^*(\eta) \circ I_{\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}}, \underline{f}}(\mathcal{F}_Y) \circ I_{\underline{u}_{\boldsymbol{f}}, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1} \circ \underline{u}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})\bigr)|_{\underline{Z}} \\
&= (\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}}|_{\underline{Z}})^*(\eta) \circ I_{\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}}|_{\underline{Z}}, \underline{f}}(\mathcal{F}_Y) \circ I_{\underline{u}_{\boldsymbol{f}}|_{\underline{Z}}, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1} \circ \underline{u}_{\boldsymbol{f}}|_{\underline{Z}}^*(\underline{i}_{\mathbf{Y}}|_{\underline{U}}^*(b^2)) \circ \underline{u}_{\boldsymbol{f}}|_{\underline{Z}}^*(\gamma^{-1}) \\
&= (\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}}|_{\underline{Z}})^*(\eta \circ \underline{f}^*(b^2)) \circ I_{\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}}|_{\underline{Z}}, \underline{f}}(\mathcal{F}_{[0,\infty)}) \circ I_{\underline{u}_{\boldsymbol{f}}|_{\underline{Z}}, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_{[0,\infty)})^{-1} \circ \underline{u}_{\boldsymbol{f}}|_{\underline{Z}}^*(\gamma^{-1}).
\end{aligned}$$

Hence $(\underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}}|_{\underline{Z}})^*(\eta \circ \underline{f}^*(b^2)) = 0$ in $\mathrm{qcoh}(\underline{Z})$. As $\underline{s}_{\boldsymbol{f}}|_{\underline{Z}} : \underline{Z} \to \underline{\tilde{U}}$ is an isomorphism, this implies that $\underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}}^*(\eta \circ \underline{f}^*(b^2)) = 0$ in $\mathrm{qcoh}(\underline{\tilde{U}})$.

Since (6.1) for $\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}}, \boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{U}}}$ is 2-Cartesian in **dSpa**, using $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$, we deduce that the following diagram in **dSpa** is also 2-Cartesian:

$$\begin{tikzcd}
\tilde{\boldsymbol{U}} \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{U}}}"'] \arrow[drr, phantom, "\mathrm{id}_{\boldsymbol{b}} * (-\eta|_{\underline{\tilde{V}}}) * \mathrm{id}_{\boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{U}}}} \Uparrow"] & & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\
\tilde{\boldsymbol{V}} \arrow[rr, "\boldsymbol{b} \circ \boldsymbol{g}|_{\tilde{\boldsymbol{V}}}"] & & \boldsymbol{[0,\infty)}.
\end{tikzcd} \tag{6.25}$$

Expanding $\mathrm{id}_{\boldsymbol{b}} * (-\eta|_{\underline{\tilde{V}}}) * \mathrm{id}_{\boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{U}}}}$ using Definition 2.14 gives

$$\begin{aligned}
&\mathrm{id}_{\boldsymbol{b}} * (-\eta|_{\underline{\tilde{V}}}) * \mathrm{id}_{\boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{U}}}} \\
&\quad = -i''_{\mathbf{X}}|_{\underline{\tilde{U}}} \circ \underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}}^*(\eta \circ \underline{f}^*(b^2) \circ I_{\underline{f}|_{\underline{\tilde{V}}}, \underline{b}}(\mathcal{F}_{[0,\infty)})) \circ I_{\underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}}, \underline{b} \circ \underline{f}}(\mathcal{F}_{[0,\infty)})) = 0 = \mathrm{id}_{\boldsymbol{0} \circ \boldsymbol{\pi}},
\end{aligned}$$

since $\underline{i}_{\mathbf{X}}|_{\underline{\tilde{U}}}^*(\eta \circ \underline{f}^*(b^2)) = 0$. Hence (6.25) is of the form (6.1). We now deduce that $\bigl(\tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{g}|_{\tilde{\boldsymbol{V}}}\bigr)$ is a boundary defining function for $\mathbf{X}$ at x', as $\bigl(\tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}}\bigr)$ is. Thus in case (i), Definition 6.2(i) holds for $\boldsymbol{g}$ at $x', y', (\boldsymbol{V}, \boldsymbol{b})$.

In case (ii), by a similar argument using $\underline{t}_{\boldsymbol{f}} : \underline{T}_{\boldsymbol{f}} \to \underline{X}$ étale by Proposition 6.7(c), we can show there exists open $x \in \tilde{\boldsymbol{W}} \subseteq \boldsymbol{W}$ such that the 2-morphism $\mathrm{id}_{\boldsymbol{b}} * \eta|_{\tilde{\boldsymbol{W}}} : \boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{W}}} \Rightarrow \boldsymbol{b} \circ \boldsymbol{g}|_{\tilde{\boldsymbol{W}}}$ is identified with (6.10) and so is zero. Hence

$\boldsymbol{b}\circ\boldsymbol{g}|_{\tilde{\boldsymbol{W}}}=\boldsymbol{b}\circ\boldsymbol{f}|_{\tilde{\boldsymbol{W}}}=\boldsymbol{0}\circ\boldsymbol{\pi}$, so that Definition 6.2(ii) holds for $\boldsymbol{g}$ at $x,y',(\boldsymbol{V},\boldsymbol{b})$. This shows that $\boldsymbol{g}:\mathbf{X}\to\mathbf{Y}$ is a 1-morphism in $\mathbf{dSpa^c}$, and also that $\underline{S}_{\boldsymbol{g}}=\underline{S}_{\boldsymbol{f}}$. Since $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is a 2-morphism in $\mathbf{dSpa}$, and $\underline{S}_{\boldsymbol{g}}=\underline{S}_{\boldsymbol{f}}$, and (6.9)–(6.10) hold, $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is a 2-morphism in $\mathbf{dSpa^c}$. This completes the proof. □

6.2 Boundaries of d-spaces with corners

Let $\mathbf{X}$ be a d-space with corners. In a long definition, we will define a d-space with corners $\partial\mathbf{X}$, the boundary of $\mathbf{X}$, and a 1-morphism $\boldsymbol{i}_{\mathbf{X}}:\partial\mathbf{X}\to\mathbf{X}$ of d-spaces with corners. To understand the definitions of $\boldsymbol{\partial^2 X}$ and $\boldsymbol{i}_{\partial\mathbf{X}}$ below, note that if X is a manifold with corners with boundary ∂X and immersion $i_X:\partial X\to X$, then from (5.2) we see that

$$\partial^2 X\cong\bigl\{(x',x'')\in\partial X\times_{i_X,X,i_X}\partial X:x'\neq x''\bigr\}.$$

That is, as a topological space ∂^2X is the complement of the diagonal in the topological fibre product $\partial X\times_{i_X,X,i_X}\partial X$, and then $i_{\partial X}:\partial^2X\to\partial X$ is the projection to the first factor in the fibre product.

Definition 6.10. Let $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}},\omega_{\mathbf{X}})$ be a d-space with corners, where $\boldsymbol{X}=(\underline{X},\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X)$ and $\boldsymbol{\partial X}=(\underline{\partial X},\mathcal{O}'_{\partial X},\mathcal{E}_{\partial X},\imath_{\partial X},\jmath_{\partial X})$. We will define a d-space with corners $\partial\mathbf{X}=(\boldsymbol{\partial X},\boldsymbol{\partial^2 X},\boldsymbol{i}_{\partial\mathbf{X}},\omega_{\partial\mathbf{X}})$, called the *boundary* of $\mathbf{X}$, and show that $\boldsymbol{i}_{\mathbf{X}}:\partial\mathbf{X}\to\mathbf{X}$ is a 1-morphism in $\mathbf{dSpa^c}$.

Here $\boldsymbol{\partial X}$ and $\boldsymbol{i}_{\mathbf{X}}$ are given in $\mathbf{X}$, so the new data we have to construct is $\boldsymbol{\partial^2 X},\boldsymbol{i}_{\partial\mathbf{X}},\omega_{\partial\mathbf{X}}$, where $\boldsymbol{\partial^2 X}=(\underline{\partial^2X},\mathcal{O}'_{\partial^2X},\mathcal{E}_{\partial^2X},\imath_{\partial^2X},\jmath_{\partial^2X})$ is a d-space, and $\boldsymbol{i}_{\partial\mathbf{X}}=(\underline{i}_{\partial\mathbf{X}},i'_{\partial\mathbf{X}},i''_{\partial\mathbf{X}}):\boldsymbol{\partial^2X}\to\boldsymbol{\partial X}$ is a 1-morphism in $\mathbf{dSpa}$, and $\omega_{\partial\mathbf{X}}$ is an orientation on a line bundle $\mathcal{N}_{\partial\mathbf{X}}$ defined later.

We will also construct a 1-morphism $\boldsymbol{j}_{\partial\mathbf{X}}=(\underline{j}_{\partial\mathbf{X}},j'_{\partial\mathbf{X}},j''_{\partial\mathbf{X}}):\boldsymbol{\partial^2X}\to\boldsymbol{\partial X}$ in $\mathbf{dSpa}$ with $\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{i}_{\partial\mathbf{X}}=\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{j}_{\partial\mathbf{X}}$, such that the 2-commutative diagram

$$\begin{tikzcd}
\boldsymbol{\partial^2X} \arrow[r, "\boldsymbol{j}_{\partial\mathbf{X}}"] \arrow[d, "\boldsymbol{i}_{\partial\mathbf{X}}"'] \arrow[dr, phantom, "\mathrm{id}_{\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{i}_{\partial\mathbf{X}}}\Uparrow"] & \boldsymbol{\partial X} \arrow[d, "\boldsymbol{i}_{\mathbf{X}}"] \\
\boldsymbol{\partial X} \arrow[r, "\boldsymbol{i}_{\mathbf{X}}"] & \boldsymbol{X}
\end{tikzcd} \tag{6.26}$$

is *locally* 2*-Cartesian* in $\mathbf{dSpa}$, in that it induces an equivalence from $\boldsymbol{\partial^2X}$ to an open d-subspace of $\boldsymbol{\partial X}\times_{\boldsymbol{i}_{\mathbf{X}},\boldsymbol{X},\boldsymbol{i}_{\mathbf{X}}}\boldsymbol{\partial X}$, the complement of the diagonal.

Form the C^∞-scheme fibre product $\underline{\partial X}\times_{\underline{i}_{\mathbf{X}},\underline{X},\underline{i}_{\mathbf{X}}}\underline{\partial X}$, and write $\underline{\pi}_1,\underline{\pi}_2:\underline{\partial X}\times_{\underline{X}}\underline{\partial X}\to\underline{\partial X}$ for the projections. The underlying topological space is $\partial X\times_X\partial X=\bigl\{(x',x''):x',x''\in\partial X,\ i_{\mathbf{X}}(x')=i_{\mathbf{X}}(x'')\bigr\}$. We have a subset $\Delta_X\subseteq\partial X\times_X\partial X$, the diagonal, given by $\Delta_X=\bigl\{(x',x'):x'\in\partial X\bigr\}$. It is closed in $\partial X\times_X\partial X$ as ∂X is Hausdorff, and open as $i_{\mathbf{X}}:\partial X\to X$ is an immersion of topological spaces. Hence $\partial^2X=(\partial X\times_X\partial X)\setminus\Delta_X$ is also open and closed in $\partial X\times_X\partial X$. Define $\underline{\partial^2X}$ to be corresponding the open and closed C^∞-subscheme in $\underline{\partial X}\times_{\underline{X}}\underline{\partial X}$. It is separated, second countable and locally fair as $\underline{\partial X},\underline{X}$ are. Define C^∞-scheme morphisms $\underline{i}_{\partial\mathbf{X}},\underline{j}_{\partial\mathbf{X}}:\underline{\partial^2X}\to\underline{\partial X}$ by $\underline{i}_{\partial\mathbf{X}}=\underline{\pi}_1|_{\underline{\partial^2X}}$ and $\underline{j}_{\partial\mathbf{X}}=\underline{\pi}_2|_{\underline{\partial^2X}}$. Since $\underline{i}_{\mathbf{X}}:\underline{\partial X}\to\underline{X}$ is proper, $\underline{\pi}_1,\underline{\pi}_2:\underline{\partial X}\times_{\underline{X}}\underline{\partial X}\to\underline{\partial X}$ are proper, so $\underline{i}_{\partial\mathbf{X}}, \underline{j}_{\partial\mathbf{X}}$ are proper as they are the restrictions of $\underline{\pi}_1, \underline{\pi}_2$ to a closed C^∞-subscheme.

Now fibre products of C^∞-schemes correspond to tensor products of the corresponding sheaves of C^∞-rings. Thus we have

$$\mathcal{O}_{\partial^2 X} = \bigl(\pi_1^{-1}(\mathcal{O}_{\partial X}) \hat{\otimes}_{(i_{\mathbf{X}}\circ\pi_1)^{-1}(\mathcal{O}_X)} \pi_2^{-1}(\mathcal{O}_{\partial X})\bigr)\big|_{\partial^2 X}, \tag{6.27}$$

where the tensor product is taken using the morphisms of sheaves of C^∞-rings

$$\begin{aligned}
\pi_1^{-1}(i_{\mathbf{X}}^\sharp) \circ I_{\pi_1, i_{\mathbf{X}}}(\mathcal{O}_X) &: (i_{\mathbf{X}} \circ \pi_1)^{-1}(\mathcal{O}_X) \longrightarrow \pi_1^{-1}(\mathcal{O}_{\partial X}),\\
\pi_2^{-1}(i_{\mathbf{X}}^\sharp) \circ I_{\pi_2, i_{\mathbf{X}}}(\mathcal{O}_X) &: (i_{\mathbf{X}} \circ \pi_1)^{-1}(\mathcal{O}_X) \longrightarrow \pi_2^{-1}(\mathcal{O}_{\partial X}),
\end{aligned}$$

noting that $i_{\mathbf{X}} \circ \pi_1 = i_{\mathbf{X}} \circ \pi_2 : \partial X \times_X \partial X \to X$. So we have a co-Cartesian square of morphisms of sheaves of C^∞-rings on $\partial^2 X$:

$$\begin{tikzcd}
(i_{\mathbf{X}} \circ \pi_1)^{-1}(\mathcal{O}_X)|_{\partial^2 X} \arrow[r, "\pi_2^{-1}(i_{\mathbf{X}}^\sharp)\circ I_{\pi_2,i_{\mathbf{X}}}(\mathcal{O}_X)|_{\partial^2 X}"'] \arrow[d, "\pi_1^{-1}(i_{\mathbf{X}}^\sharp)\circ I_{\pi_1,i_{\mathbf{X}}}(\mathcal{O}_X)|_{\partial^2 X}"] & \pi_2^{-1}(\mathcal{O}_{\partial X})|_{\partial^2 X} = j_{\partial\mathbf{X}}^{-1}(\mathcal{O}_{\partial X}) \arrow[d, "\pi_2^\sharp|_{\partial^2 X} = j_{\partial\mathbf{X}}^\sharp"'] \\
\pi_1^{-1}(\mathcal{O}_{\partial X})|_{\partial^2 X} = i_{\partial\mathbf{X}}^{-1}(\mathcal{O}_{\partial X}) \arrow[r, "\pi_1^\sharp|_{\partial^2 X} = i_{\partial\mathbf{X}}^\sharp"] & \mathcal{O}_{\partial^2 X}.
\end{tikzcd}$$

In the same way, define a sheaf of C^∞-rings $\mathcal{O}'_{\partial^2 X}$ on $\partial^2 X$ by

$$\begin{aligned}
\mathcal{O}'_{\partial^2 X} &= \bigl(\pi_1^{-1}(\mathcal{O}'_{\partial X}) \hat{\otimes}_{(i_{\mathbf{X}}\circ\pi_1)^{-1}(\mathcal{O}'_X)} \pi_2^{-1}(\mathcal{O}'_{\partial X})\bigr)\big|_{\partial^2 X}\\
&= i_{\partial\mathbf{X}}^{-1}(\mathcal{O}'_{\partial X}) \hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)} j_{\partial\mathbf{X}}^{-1}(\mathcal{O}'_{\partial X}),
\end{aligned} \tag{6.28}$$

to fit in the co-Cartesian square of morphisms of sheaves of C^∞-rings on $\partial^2 X$:

$$\begin{tikzcd}
(i_{\mathbf{X}} \circ \pi_1)^{-1}(\mathcal{O}'_X)|_{\partial^2 X} \arrow[r, "\pi_2^{-1}(i'_{\mathbf{X}})\circ I_{\pi_2,i_{\mathbf{X}}}(\mathcal{O}'_X)|_{\partial^2 X}"'] \arrow[d, "\pi_1^{-1}(i'_{\mathbf{X}})\circ I_{\pi_1,i_{\mathbf{X}}}(\mathcal{O}'_X)|_{\partial^2 X}"] & \pi_2^{-1}(\mathcal{O}'_{\partial X})|_{\partial^2 X} = j_{\partial\mathbf{X}}^{-1}(\mathcal{O}'_{\partial X}) \arrow[d, "\pi'_2|_{\partial^2 X} = j'_{\partial\mathbf{X}}"'] \\
\pi_1^{-1}(\mathcal{O}'_{\partial X})|_{\partial^2 X} = i_{\partial\mathbf{X}}^{-1}(\mathcal{O}'_{\partial X}) \arrow[r, "\pi'_1|_{\partial^2 X} = i'_{\partial\mathbf{X}}"] & \mathcal{O}'_{\partial^2 X}.
\end{tikzcd} \tag{6.29}$$

where $i'_{\partial\mathbf{X}} : i_{\partial\mathbf{X}}^{-1}(\mathcal{O}'_{\partial X}) \to \mathcal{O}'_{\partial^2 X}$ and $j'_{\partial\mathbf{X}} : j_{\partial\mathbf{X}}^{-1}(\mathcal{O}'_{\partial X}) \to \mathcal{O}'_{\partial^2 X}$ are defined by (6.29). We have morphisms of sheaves of C^∞-rings on $\partial^2 X$:

$$\begin{aligned}
\bigl(\pi_1^\sharp \circ \pi_1^{-1}(\imath_{\partial X})\bigr)|_{\partial^2 X} &: \pi_1^{-1}(\mathcal{O}'_{\partial X})|_{\partial^2 X} \longrightarrow \mathcal{O}_{\partial^2 X},\\
\bigl(\pi_2^\sharp \circ \pi_2^{-1}(\imath_{\partial X})\bigr)|_{\partial^2 X} &: \pi_2^{-1}(\mathcal{O}'_{\partial X})|_{\partial^2 X} \longrightarrow \mathcal{O}_{\partial^2 X},
\end{aligned}$$

which satisfy

$$\begin{aligned}
&\bigl(\pi_1^\sharp \circ \pi_1^{-1}(\imath_{\partial X})\bigr)|_{\partial^2 X} \circ \bigl(\pi_1^{-1}(i'_{\mathbf{X}}) \circ I_{\pi_1, i_{\mathbf{X}}}(\mathcal{O}'_X)\bigr)|_{\partial^2 X} =\\
&\bigl(\pi_2^\sharp \circ \pi_2^{-1}(\imath_{\partial X})\bigr)|_{\partial^2 X} \circ \bigl(\pi_2^{-1}(i'_{\mathbf{X}}) \circ I_{\pi_2, i_{\mathbf{X}}}(\mathcal{O}'_X)\bigr)|_{\partial^2 X}.
\end{aligned}$$

Hence by properties of the co-Cartesian square (6.29), there is a unique morphism of sheaves of C^∞-rings $\imath_{\partial^2 X} : \mathcal{O}'_{\partial^2 X} \to \mathcal{O}_{\partial^2 X}$ satisfying

$$\begin{aligned}
\bigl(\pi_1^\sharp \circ \pi_1^{-1}(\imath_{\partial X})\bigr)|_{\partial^2 X} &= \imath_{\partial^2 X} \circ \pi'_1|_{\partial^2 X},\\
\bigl(\pi_2^\sharp \circ \pi_2^{-1}(\imath_{\partial X})\bigr)|_{\partial^2 X} &= \imath_{\partial^2 X} \circ \pi'_2|_{\partial^2 X}.
\end{aligned}$$

These equations may also be written

$$
\begin{aligned}
i_{\partial\mathbf{X}}^{\sharp}\circ i_{\partial\mathbf{X}}^{-1}(\imath_{\partial X})&=\imath_{\partial^2 X}\circ i'_{\partial\mathbf{X}},\\
j_{\partial\mathbf{X}}^{\sharp}\circ i_{\partial\mathbf{X}}^{-1}(\imath_{\partial X})&=\imath_{\partial^2 X}\circ j'_{\partial\mathbf{X}}.
\end{aligned}
\tag{6.30}
$$

Consider the commutative square of sheaves of C^∞-rings on ∂X:

$$
\begin{tikzcd}
i_{\mathbf{X}}^{-1}(\mathcal{O}'_X) \arrow[rr, "i'_{\mathbf{X}}"'] \arrow[d, "i_{\mathbf{X}}^{-1}(\imath_X)"] && \mathcal{O}'_{\partial X} \arrow[d, "\imath_{\partial X}"'] \\
i_{\mathbf{X}}^{-1}(\mathcal{O}_X) \arrow[rr, "i_{\mathbf{X}}^{\sharp}"] && \mathcal{O}_{\partial X}.
\end{tikzcd}
\tag{6.31}
$$

Let $\boldsymbol{U},\boldsymbol{V},\boldsymbol{b}$ be as in Definition 6.1(c), so that $U\subseteq\partial X$ is open. Then (6.1) 2-Cartesian and $i''_{\mathbf{X}}$ an isomorphism implies that

$$
\begin{aligned}
\mathcal{O}_{\partial X}|_U&\cong i_{\mathbf{X}}^{-1}(\mathcal{O}_X)|_U\,\hat{\otimes}_{(b\circ i_{\mathbf{X}}|_U)^{-1}(\mathcal{O}_{[0,\infty)})}\pi^{-1}(\mathcal{O}_*),\\
\mathcal{O}'_{\partial X}|_U&\cong i_{\mathbf{X}}^{-1}(\mathcal{O}_X{}')|_U\,\hat{\otimes}_{(b\circ i_{\mathbf{X}}|_U)^{-1}(\mathcal{O}_{[0,\infty)})}\pi^{-1}(\mathcal{O}_*).
\end{aligned}
$$

It follows that the restriction of (6.31) to U is co-Cartesian. As we can cover ∂X by such U, equation (6.31) is co-Cartesian. That is,

$$
\mathcal{O}_{\partial X}\cong i_{\mathbf{X}}^{-1}(\mathcal{O}_X)\,\hat{\otimes}_{i_{\mathbf{X}}^{-1}(\mathcal{O}'_X)}\mathcal{O}'_{\partial X}.
\tag{6.32}
$$

Applying the exact functor $(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}$ to (2.9) gives an exact sequence of sheaves of $(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)$-modules on $\partial^2 X$:

$$
0\to(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{I}_X)\xrightarrow{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\kappa_X)}(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)\xrightarrow{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\imath_X)}(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}_X)\to 0.
$$

Applying the right exact functor $\hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)}\mathcal{O}'_{\partial^2 X}$ gives an exact sequence of $\mathcal{O}'_{\partial^2 X}$-modules on $\partial^2 X$:

$$
\begin{matrix}(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{I}_X)\\ \hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)}\mathcal{O}'_{\partial^2 X}\end{matrix}\xrightarrow{\substack{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\kappa_X)\\ \otimes\mathrm{id}}}\mathcal{O}'_{\partial^2 X}\xrightarrow{\substack{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\imath_X)\\ \otimes\mathrm{id}}}\begin{matrix}(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}_X)\\ \hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)}\mathcal{O}'_{\partial^2 X}\end{matrix}\to 0.
\tag{6.33}
$$

Now we have natural isomorphisms

$$
\begin{aligned}
&(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}_X)\,\hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)}\mathcal{O}'_{\partial^2 X}\\
&\quad\cong(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}_X)\,\hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)}\big(i_{\partial\mathbf{X}}^{-1}(\mathcal{O}'_{\partial X})\,\hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)}j_{\partial\mathbf{X}}^{-1}(\mathcal{O}'_{\partial X})\big)\\
&\quad\cong i_{\partial\mathbf{X}}^{-1}\big(i_{\mathbf{X}}^{-1}(\mathcal{O}_X)\,\hat{\otimes}_{i_{\mathbf{X}}^{-1}(\mathcal{O}'_X)}\mathcal{O}'_{\partial X}\big)\,\hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)}j_{\partial\mathbf{X}}^{-1}(\mathcal{O}'_{\partial X})\\
&\quad\cong i_{\partial\mathbf{X}}^{-1}(\mathcal{O}_{\partial X})\,\hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)}j_{\partial\mathbf{X}}^{-1}(\mathcal{O}'_{\partial X})\\
&\quad\cong i_{\partial\mathbf{X}}^{-1}(\mathcal{O}_{\partial X})\,\hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}_X)}j_{\partial\mathbf{X}}^{-1}\big(i_{\mathbf{X}}^{-1}(\mathcal{O}_X)\,\hat{\otimes}_{i_{\mathbf{X}}^{-1}(\mathcal{O}'_X)}\mathcal{O}'_{\partial X}\big)\\
&\quad\cong i_{\partial\mathbf{X}}^{-1}(\mathcal{O}_{\partial X})\,\hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}_X)}j_{\partial\mathbf{X}}^{-1}(\mathcal{O}_{\partial X})\cong\mathcal{O}_{\partial^2 X},
\end{aligned}
\tag{6.34}
$$

using (6.28) in the first step, (6.32) in the third and fifth steps, and (6.27) in the last. Also we have natural isomorphisms

$$
\begin{aligned}
&(i_{\mathbf{X}} \circ i_{\partial\mathbf{X}})^{-1}(\mathcal{I}_X)\,\hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)}\mathcal{O}'_{\partial^2 X}\\
&\cong (i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{I}_X)\otimes_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}_X)}\big((i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}_X)\,\hat{\otimes}_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}'_X)}\mathcal{O}'_{\partial^2 X}\big)\\
&\cong (i_{\mathbf{X}} \circ i_{\partial\mathbf{X}})^{-1}(\mathcal{I}_X)\otimes_{(i_{\mathbf{X}}\circ i_{\partial\mathbf{X}})^{-1}(\mathcal{O}_X)}\mathcal{O}_{\partial^2 X} = (\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\mathcal{I}_X),
\end{aligned}
\tag{6.35}
$$

using the fact that the $\mathcal{O}'_X$-action on $\mathcal{I}_X$ factors through an $\mathcal{O}_X$-action in the first step, and (6.34) in the second. Substituting (6.34)–(6.35) into (6.33) gives an exact sequence of $\mathcal{O}'_{\partial^2 X}$-modules on $\partial^2 X$:

$$
(\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\mathcal{I}_X) \longrightarrow \mathcal{O}'_{\partial^2 X} \xrightarrow{\imath_{\partial^2 X}} \mathcal{O}_{\partial^2 X} \longrightarrow 0. \tag{6.36}
$$

Thus $\imath_{\partial^2 X}$ is surjective. Write $\mathcal{I}_{\partial^2 X}$ for the kernel of $\imath_{\partial^2 X}$, a sheaf of ideals in $\mathcal{O}'_{\partial^2 X}$ on $\partial^2 X$. Then (6.36) gives a unique surjective $\mathcal{O}'_{\partial^2 X}$-module morphism

$$
\xi_{\partial\mathbf{X}} : (\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\mathcal{I}_X) \longrightarrow \mathcal{I}_{\partial^2 X}. \tag{6.37}
$$

Since $\mathcal{I}_X$ is a sheaf of square zero ideals and $\xi_{\partial\mathbf{X}}$ is a surjective morphism of ideals, $\mathcal{I}_{\partial^2 X}$ is a sheaf of square zero ideals. Hence $(\underline{\partial^2 X}, \mathcal{O}'_{\partial^2 X}, \imath_{\partial^2 X})$ is a square zero extension of C^∞-schemes, in the sense of Definition 2.9. Also, the $\mathcal{O}'_{\partial^2 X}$-action on both sides of (6.37) factors through an $\mathcal{O}_{\partial^2 X}$-action, so $\xi_{\partial\mathbf{X}}$ is an $\mathcal{O}_{\partial^2 X}$-module morphism, that is, a morphism in $\mathrm{qcoh}(\underline{\partial^2 X})$. By §2.1 we have morphisms $i^1_{\mathbf{X}} : \underline{i}^*_{\mathbf{X}}(\mathcal{I}_X) \to \mathcal{I}_{\partial X}$ in $\mathrm{qcoh}(\underline{\partial X})$, and $i^1_{\partial\mathbf{X}} : \underline{i}^*_{\partial\mathbf{X}}(\mathcal{I}_{\partial X}) \to \mathcal{I}_{\partial^2 X}$ in $\mathrm{qcoh}(\underline{\partial^2 X})$. In terms of these one can show that

$$
\xi_{\partial\mathbf{X}} = i^1_{\partial\mathbf{X}} \circ \underline{i}^*_{\partial\mathbf{X}}(i^1_{\mathbf{X}}) \circ I_{\underline{i}_{\partial\mathbf{X}},\underline{i}_{\mathbf{X}}}(\mathcal{I}_X). \tag{6.38}
$$

From $\boldsymbol{X}$ we have a surjective $\jmath_X : \mathcal{E}_X \to \mathcal{I}_X$. Hence $(\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\jmath_X) : (\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\mathcal{E}_X) \to (\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\mathcal{I}_X)$ is surjective in $\mathrm{qcoh}(\underline{\partial^2 X})$. Define $\mathcal{E}_{\partial^2 X}$ and $\jmath_{\partial^2 X} : \mathcal{E}_{\partial^2 X} \to \mathcal{I}_{\partial^2 X}$ in $\mathrm{qcoh}(\underline{\partial^2 X})$ by $\mathcal{E}_{\partial^2 X} = (\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\mathcal{E}_X)$ and $\jmath_{\partial^2 X} = \xi_{\partial\mathbf{X}} \circ (\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\jmath_X)$. Then $\jmath_{\partial^2 X}$ is surjective, as $\xi_{\partial\mathbf{X}}$ and $(\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\jmath_X)$ are. This completes the definition of $\boldsymbol{\partial^2 X} = (\underline{\partial^2 X}, \mathcal{O}'_{\partial^2 X}, \mathcal{E}_{\partial^2 X}, \imath_{\partial^2 X}, \jmath_{\partial^2 X})$. We have already shown that $\underline{\partial^2 X}$ is separated, second countable and locally fair, and $(\mathcal{O}'_{\partial^2 X}, \imath_{\partial^2 X})$ is a square zero extension of $\underline{\partial^2 X}$, and $\jmath_{\partial^2 X} : \mathcal{E}_{\partial^2 X} \to \mathcal{I}_{\partial^2 X}$ is surjective in $\mathrm{qcoh}(\underline{\partial^2 X})$. Hence $\boldsymbol{\partial^2 X}$ is a d-space.

Define morphisms $i''_{\partial\mathbf{X}} : \underline{i}^*_{\partial\mathbf{X}}(\mathcal{E}_{\partial X}) \to \mathcal{E}_{\partial^2 X}$ and $j''_{\partial\mathbf{X}} : \underline{j}^*_{\partial\mathbf{X}}(\mathcal{E}_{\partial X}) \to \mathcal{E}_{\partial^2 X}$ in $\mathrm{qcoh}(\underline{\partial^2 X})$ by the commutative diagrams

$$
\begin{tikzcd}
\underline{i}^*_{\partial\mathbf{X}}(\mathcal{E}_{\partial X}) \arrow[r, "\underline{i}^*_{\partial\mathbf{X}}((i''_{\mathbf{X}})^{-1})"'] \arrow[d, "i''_{\partial\mathbf{X}}"] & \underline{i}^*_{\partial\mathbf{X}}\circ\underline{i}^*_{\mathbf{X}}(\mathcal{E}_X) \arrow[d, "I_{\underline{i}_{\partial\mathbf{X}},\underline{i}_{\mathbf{X}}}(\mathcal{E}_X)^{-1}"'] \\
\mathcal{E}_{\partial^2 X} \arrow[r, equal] & (\underline{i}_{\mathbf{X}}\circ\underline{i}_{\partial\mathbf{X}})^*(\mathcal{E}_X),
\end{tikzcd}
\quad
\begin{tikzcd}
\underline{j}^*_{\partial\mathbf{X}}(\mathcal{E}_{\partial X}) \arrow[r, "\underline{j}^*_{\partial\mathbf{X}}((i''_{\mathbf{X}})^{-1})"'] \arrow[d, "j''_{\partial\mathbf{X}}"] & \underline{j}^*_{\partial\mathbf{X}}\circ\underline{i}^*_{\mathbf{X}}(\mathcal{E}_X) \arrow[d, "I_{\underline{j}_{\partial\mathbf{X}},\underline{i}_{\mathbf{X}}}(\mathcal{E}_X)^{-1}"'] \\
\mathcal{E}_{\partial^2 X} \arrow[r, equal] & (\underline{j}_{\mathbf{X}}\circ\underline{i}_{\partial\mathbf{X}})^*(\mathcal{E}_X),
\end{tikzcd}
$$

noting that $i''_{\mathbf{X}} : \underline{i}^*_{\mathbf{X}}(\mathcal{E}_X) \to \mathcal{E}_{\partial X}$ is an isomorphism by Definition 6.1(b).

This defines $\boldsymbol{i}_{\partial\mathbf{X}}=(\underline{i}_{\partial\mathbf{X}},i'_{\partial\mathbf{X}},i''_{\partial\mathbf{X}})$ and $\boldsymbol{j}_{\partial\mathbf{X}}=(\underline{j}_{\partial\mathbf{X}},j'_{\partial\mathbf{X}},j''_{\partial\mathbf{X}})$. Equation (6.30) shows that $(\underline{i}_{\partial\mathbf{X}},i'_{\partial\mathbf{X}})$ and $(\underline{j}_{\partial\mathbf{X}},j'_{\partial\mathbf{X}})$ are morphisms of square zero extensions of C^∞-schemes. Equation (2.21) for $\boldsymbol{i}_{\partial\mathbf{X}}$ follows from

$$
\begin{aligned}
\jmath_{\partial^2X}\circ i''_{\partial\mathbf{X}} &= \xi_{\partial\mathbf{X}}\circ(\underline{i}_{\mathbf{X}}\circ\underline{i}_{\partial\mathbf{X}})^*(\jmath_X)\circ I_{\underline{i}_{\partial\mathbf{X}},\underline{i}_{\mathbf{X}}}(\mathcal{E}_X)^{-1}\circ\underline{i}^*_{\partial\mathbf{X}}((i''_{\mathbf{X}})^{-1})\\
&= i^1_{\partial\mathbf{X}}\circ\underline{i}^*_{\partial\mathbf{X}}(i^1_{\mathbf{X}})\circ I_{\underline{i}_{\partial\mathbf{X}},\underline{i}_{\mathbf{X}}}(\mathcal{I}_X)\circ I_{\underline{i}_{\partial\mathbf{X}},\underline{i}_{\mathbf{X}}}(\mathcal{I}_X)^{-1}\circ\underline{i}^*_{\partial\mathbf{X}}\circ\underline{i}_{\mathbf{X}}(\jmath_X)\circ\underline{i}^*_{\partial\mathbf{X}}((i''_{\mathbf{X}})^{-1})\\
&= i^1_{\partial\mathbf{X}}\circ\underline{i}^*_{\partial\mathbf{X}}(i^1_{\mathbf{X}}\circ\underline{i}^*_{\mathbf{X}}(\jmath_X))\circ\underline{i}^*_{\partial\mathbf{X}}((i''_{\mathbf{X}})^{-1})\\
&= i^1_{\partial\mathbf{X}}\circ\underline{i}^*_{\partial\mathbf{X}}(\jmath_{\partial X}\circ i''_{\mathbf{X}})\circ\underline{i}^*_{\partial\mathbf{X}}((i''_{\mathbf{X}})^{-1})=i^1_{\partial\mathbf{X}}\circ\underline{i}^*_{\partial\mathbf{X}}(\jmath_{\partial X}),
\end{aligned}
$$

using (6.38) and properties of $I_{*,*}(*)$ in the second step, and (2.21) for $\boldsymbol{i}_{\mathbf{X}}$ in the fourth. Therefore $\boldsymbol{i}_{\partial\mathbf{X}}:\boldsymbol{\partial^2X}\to\boldsymbol{\partial X}$ and similarly $\boldsymbol{j}_{\partial\mathbf{X}}:\boldsymbol{\partial^2X}\to\boldsymbol{\partial X}$ are 1-morphisms in **dSpa**.

It is now easy to check that $\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{i}_{\partial\mathbf{X}}=\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{j}_{\partial\mathbf{X}}$, so that (6.26) 2-commutes, and induces a 1-morphism $\boldsymbol{h}:\boldsymbol{\partial^2X}\to\boldsymbol{\partial X}\times_{\boldsymbol{i}_{\mathbf{X}},\boldsymbol{X},\boldsymbol{i}_{\mathbf{X}}}\boldsymbol{\partial X}$. One can show $\boldsymbol{h}$ is an equivalence with an open d-subspace by comparing the construction of $\boldsymbol{\partial^2X}$ above with the explicit construction of $\boldsymbol{\partial X}\times_{\boldsymbol{X}}\boldsymbol{\partial X}$ in §2.5; $\boldsymbol{\partial X}\times_{\boldsymbol{X}}\boldsymbol{\partial X}$ is locally obtained from $\boldsymbol{\partial^2X}$ by adding on the sheaf $\mathcal{G}=\mathcal{E}_{\partial^2X}$ using Example 2.18. Hence (6.26) is locally 2-Cartesian.

Next we verify Definition 6.1(a)–(f) for $\partial\mathbf{X}=(\boldsymbol{\partial X},\boldsymbol{\partial^2X},\boldsymbol{i}_{\partial\mathbf{X}},\omega_{\partial\mathbf{X}})$, constructing $\omega_{\partial\mathbf{X}}$ along the way. We have already shown that $\underline{i}_{\partial\mathbf{X}}$ is proper, so (a) holds, and $i''_{\partial\mathbf{X}}=I_{\underline{i}_{\partial\mathbf{X}},\underline{i}_{\mathbf{X}}}(\mathcal{E}_X)^{-1}\circ\underline{i}^*_{\partial\mathbf{X}}((i''_{\mathbf{X}})^{-1})$ is an isomorphism, so (b) holds. For (c), let $(x'_1,x'_2)\in\boldsymbol{\partial^2X}$, so that $x'_1\neq x'_2\in\boldsymbol{\partial X}$ with $\boldsymbol{i}_{\mathbf{X}}(x'_1)=\underline{i}_{\mathbf{X}}(x'_2)=x\in\boldsymbol{X}$. Let $(\boldsymbol{V}_1,\boldsymbol{b}_1),(\boldsymbol{V}_2,\boldsymbol{b}_2)$ be boundary defining functions for $\mathbf{X}$ at x_1,x'_2. Then $x\in\boldsymbol{V}_1,\boldsymbol{V}_2\subseteq\boldsymbol{X}$ are open, and there exist open $x'_1\in\boldsymbol{U}_1\subseteq\boldsymbol{\partial X}$ and $x'_2\in\boldsymbol{U}_2\subseteq\boldsymbol{\partial X}$ with (6.1) 2-Cartesian for $\boldsymbol{U}_1,\boldsymbol{V}_1,\boldsymbol{b}_1$ and $\boldsymbol{U}_2,\boldsymbol{V}_2,\boldsymbol{b}_2$.

Making $\boldsymbol{U}_1,\boldsymbol{V}_1,\boldsymbol{U}_2,\boldsymbol{V}_2$ smaller we can suppose that $\boldsymbol{V}_1=\boldsymbol{V}_2=\boldsymbol{V}$, say. By Definition 6.1(c), $b^2_1:\underline{b}^*_1(\mathcal{F}_{[0,\infty)})\to\mathcal{F}_X|_{\underline{V}}$ and $b^2_2:\underline{b}^*_2(\mathcal{F}_{[0,\infty)})\to\mathcal{F}_X|_{\underline{V}}$ have left inverses β_1,β_2. Hence b^2_1 and b^2_2 embed the line bundles $\underline{b}^*_1(\mathcal{F}_{[0,\infty)})$ and $\underline{b}^*_2(\mathcal{F}_{[0,\infty)})$ as direct summands in $\mathcal{F}_X|_{\underline{V}}$. Definition 6.1(f) shows these embeddings are linearly independent at x, which is an open condition. Hence, making $\boldsymbol{U}_1,\boldsymbol{U}_2,\boldsymbol{V}$ smaller we can suppose that b^2_1,b^2_2 are linearly independent on $\underline{V}$, so that $b^2_1\oplus b^2_2$ embeds $\underline{b}^*_1(\mathcal{F}_{[0,\infty)})\oplus\underline{b}^*_2(\mathcal{F}_{[0,\infty)})$ as a direct summand in $\mathcal{F}_X|_{\underline{V}}$. Thus we can choose the left inverses β_1,β_2 such that $\beta_1\circ b^2_2=\beta_2\circ b^2_1=0$. Suppose $\tilde x'_1\in\boldsymbol{U}_1$ and $\tilde x'_2\in\boldsymbol{U}_2$ with $\boldsymbol{i}_{\mathbf{X}}(\tilde x'_1)=\boldsymbol{i}_{\mathbf{X}}(\tilde x'_2)=\tilde x\in\boldsymbol{V}$. Since b^2_1,b^2_2 are linearly independent at $\tilde x$ we see that $\tilde x'_1\neq\tilde x'_2$. It follows that $\boldsymbol{U}_1\cap\boldsymbol{U}_2=\emptyset$.

Define $\boldsymbol{U}_\partial=\boldsymbol{i}^{-1}_{\partial X}(\boldsymbol{U}_1)\cap\boldsymbol{j}^{-1}_{\partial X}(\boldsymbol{U}_2)$, and $\boldsymbol{V}_\partial=\boldsymbol{U}_1$, and $\boldsymbol{b}_\partial=\boldsymbol{b}_2\circ\boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{V}_\partial}:\boldsymbol{V}_\partial\to\boldsymbol{[0,\infty)}$. Then $(x'_1,x'_2)\in\boldsymbol{U}_\partial\subseteq\boldsymbol{\partial^2X}$ is open, and $x'_1\in\boldsymbol{V}_\partial\subseteq\boldsymbol{\partial X}$ is open, and $\boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{U}_\partial}:\boldsymbol{U}_\partial\to\boldsymbol{V}_\partial$. Consider the 2-commutative diagram

$$
\begin{tikzcd}
\boldsymbol{U}_\partial \arrow[r, "\boldsymbol{j}_{\partial\mathbf{X}}|_{\boldsymbol{U}_\partial}"] \arrow[d, "\boldsymbol{i}_{\partial\mathbf{X}}|_{\boldsymbol{U}_\partial}"'] \arrow[rr, bend left, "\boldsymbol{\pi}_\partial"] \arrow[dr, phantom, "\Uparrow\mathrm{id}_{\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{i}_{\partial\mathbf{X}}|_{\boldsymbol{U}_\partial}}"] & \boldsymbol{U}_2 \arrow[r, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{U}_2}"] \arrow[dr, phantom, "\Uparrow\mathrm{id}_{\boldsymbol{0}\circ\boldsymbol{\pi}}"] & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\
\boldsymbol{V}_\partial \arrow[r, "\boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{V}_\partial}"] \arrow[rr, bend right, "\boldsymbol{b}_\partial"'] & \boldsymbol{V} \arrow[r, "\boldsymbol{b}_2"] & \boldsymbol{[0,\infty)}.
\end{tikzcd}
\tag{6.39}
$$

The right hand square in (6.39) is 2-Cartesian by (6.1) for $\boldsymbol{U}_2,\boldsymbol{V},\boldsymbol{b}_2$. The left hand square is an open subdiagram of (6.16), which is locally 2-Cartesian. As $\boldsymbol{U}_1 \cap \boldsymbol{U}_2 = \emptyset$ the open d-subspace $\boldsymbol{U}_1 \times_{\boldsymbol{V}} \boldsymbol{U}_2$ in $\boldsymbol{\partial X} \times_{\boldsymbol{X}} \boldsymbol{\partial X}$ does not intersect the diagonal, so it lies in $\boldsymbol{\partial^2 X}$, as this is equivalent to the complement of the diagonal in $\boldsymbol{\partial X} \times_{\boldsymbol{X}} \boldsymbol{\partial X}$, and $\boldsymbol{U}_\partial$ is defined so that $\boldsymbol{U}_\partial \simeq \boldsymbol{U}_1 \times_{\boldsymbol{V}} \boldsymbol{U}_2$. Hence the left hand square in (6.39) is 2-Cartesian, so the outer rectangle is 2-Cartesian.

This proves the first part of Definition 6.1(c) for $(x_1', x_2') \in \boldsymbol{\partial^2 X}$ with $\boldsymbol{U}_\partial, (\boldsymbol{V}_\partial, \boldsymbol{b}_\partial)$. For the second part, consider the diagram in qcoh($\underline{V}_\partial$):

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & 0 \arrow[r] \arrow[d, shift left, "0"] & \underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial} \circ \underline{b}_2^*(\mathcal{F}_{[0,\infty)}) \arrow[r] \arrow[d, shift right, "\underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial}(b_2^2)"'] & \underline{b}_\partial^*(\mathcal{F}_{[0,\infty)}) \arrow[r] \arrow[d, shift right, "b_\partial^2"'] & 0 \\
0 \arrow[r] & \underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial} \circ \underline{b}_1^*(\mathcal{F}_{[0,\infty)}) \arrow[r, shift left, "\underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial}(b_1^2)"] \arrow[u, shift left, "0"] & \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X)|_{\underline{V}_\partial} \arrow[r, "i^2_{\mathbf{X}}|_{\underline{V}_\partial}"] \arrow[l, shift left, "\underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial}(\beta_1)"] \arrow[u, shift right, "\underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial}(\beta_2)"'] & \mathcal{F}_{\partial X}|_{\underline{V}_\partial} \arrow[r] \arrow[u, dotted, shift right, "\beta"'] & 0.
\end{tikzcd}$$

The bottom row is (6.2) for $\boldsymbol{U}_1, (\boldsymbol{V}_1, \boldsymbol{b}_1)$, noting that $\underline{V}_\partial = \underline{U}_1$, and so is exact. As $\beta_2 \circ b_1^2 = 0$ we have $\underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial}(\beta_2) \circ \underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial}(b_1^2) = 0$, so by exactness of the bottom row there exists a unique morphism $\beta : \mathcal{F}_{\partial X}|_{\underline{V}_\partial} \to \underline{b}_\partial^*(\mathcal{F}_{[0,\infty)})$ with $\beta \circ i^2_{\mathbf{X}}|_{\underline{V}_\partial} = I_{\underline{i}_{\mathbf{X}}|_{\underline{V}_\partial}, \underline{b}_2}(\mathcal{F}_{[0,\infty)})^{-1} \circ \underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial}(\beta_2)$. Therefore

$$\begin{aligned}
\beta \circ b_\partial^2 &= \beta \circ \big(i^2_{\mathbf{X}}|_{\underline{V}_\partial} \circ \underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial}(b_2^2) \circ I_{\underline{i}_{\mathbf{X}}|_{\underline{V}_\partial}, \underline{b}_2}(\mathcal{F}_{[0,\infty)})\big) \\
&= I_{\underline{i}_{\mathbf{X}}|_{\underline{V}_\partial}, \underline{b}_2}(\mathcal{F}_{[0,\infty)})^{-1} \circ \underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial}(\beta_2) \circ \underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial}(b_2^2) \circ I_{\underline{i}_{\mathbf{X}}|_{\underline{V}_\partial}, \underline{b}_2}(\mathcal{F}_{[0,\infty)}) \\
&= I_{\underline{i}_{\mathbf{X}}|_{\underline{V}_\partial}, \underline{b}_2}(\mathcal{F}_{[0,\infty)})^{-1} \circ \mathrm{id}_{\underline{i}_{\mathbf{X}}|^*_{\underline{V}_\partial} \circ \underline{b}_2^*(\mathcal{F}_{[0,\infty)})} \circ I_{\underline{i}_{\mathbf{X}}|_{\underline{V}_\partial}, \underline{b}_2}(\mathcal{F}_{[0,\infty)}) = \mathrm{id}_{\underline{b}_\partial^*(\mathcal{F}_{[0,\infty)})},
\end{aligned}$$

where in the first line we use $\boldsymbol{b}_\partial = \boldsymbol{b}_2 \circ \boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{V}_\partial}$ and (2.24), and in the third that β_2 is a left inverse for b_2^2. Therefore b_∂^2 has a left inverse β, proving the second part of Definition 6.1(c) for $\boldsymbol{U}_\partial, (\boldsymbol{V}_\partial, \boldsymbol{b}_\partial)$.

Definition 6.1 now defines a conormal line bundle $\mathcal{N}_{\partial \mathbf{X}}$ on $\underline{\partial^2 X}$, in the split exact sequence (6.4). Consider the commutative diagram in qcoh($\underline{\partial^2 X}$):

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & \underline{i}_{\partial \mathbf{X}}^*(\mathcal{N}_{\mathbf{X}}) \arrow[r, "\underline{i}_{\partial \mathbf{X}}^*(\nu_{\mathbf{X}})"'] \arrow[d, dotted, "\chi_{\partial \mathbf{X}}"] & \underline{i}_{\partial \mathbf{X}}^* \circ \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X) \arrow[r, "\underline{i}_{\partial \mathbf{X}}^*(i^2_{\mathbf{X}})"'] \arrow[d, "\underline{i}_{\partial \mathbf{X}}^*(i^2_{\mathbf{X}})"] & \underline{i}_{\partial \mathbf{X}}^*(\mathcal{F}_{\partial X}) \arrow[r] \arrow[d, "i^2_{\partial \mathbf{X}}"] & 0 \\
0 \arrow[r] & \mathcal{N}_{\partial \mathbf{X}} \arrow[r, "\nu_{\partial \mathbf{X}}"] & \underline{i}_{\partial \mathbf{X}}^*(\mathcal{F}_{\partial X}) \arrow[r, "i^2_{\partial \mathbf{X}}"] & \mathcal{F}_{\partial^2 X} \arrow[r] & 0.
\end{tikzcd} \tag{6.40}$$

Here the top line is $\underline{i}_{\partial \mathbf{X}}^*$ applied to (6.4) for $\mathbf{X}$, and the bottom line is (6.4) for $\partial \mathbf{X}$. So both are (split) exact, as pullbacks take split exact sequences to split exact sequences. The right hand square commutes trivially. Therefore there exists a unique morphism $\chi_{\partial \mathbf{X}}$ as shown making (6.40) commute.

In Definition 6.1 we explained that in Definition 6.1(c), we can identify the restriction of (6.4) to $\underline{U}$ with (6.2). In the same way, for $\boldsymbol{U}_2, \boldsymbol{V}, \boldsymbol{b}_2, \boldsymbol{U}_\partial, \boldsymbol{V}_\partial, \boldsymbol{b}_\partial$ as above, we can identify the restriction of (6.40) to $\underline{U}_\partial$ with

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & \underline{i}_{\partial \mathbf{X}}|^*_{\underline{U}_\partial}(\underline{i}_{\mathbf{X}}|^*_{\underline{U}_2} \circ \underline{b}_2^*(\mathcal{F}_{[0,\infty)})) \arrow[r, "\underline{i}_{\partial \mathbf{X}}|^*_{\underline{U}_\partial}(\underline{i}_{\mathbf{X}}|^*_{\underline{U}_2}(b_2^2))"'] \arrow[d, "\underline{i}_{\partial \mathbf{X}}|^*_{\underline{U}_\partial}(I_{\underline{i}_{\mathbf{X}}|_{\underline{U}_2}, \underline{b}_2}(\mathcal{F}_{[0,\infty)}))"] & \underline{i}_{\partial \mathbf{X}}|^*_{\underline{U}_\partial} \circ \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X) \arrow[r, "\underline{i}_{\partial \mathbf{X}}|^*_{\underline{U}_\partial}(i^2_{\mathbf{X}})"'] \arrow[d, "\underline{i}_{\partial \mathbf{X}}|^*_{\underline{U}_\partial}(i^2_{\mathbf{X}})"] & \underline{i}_{\partial \mathbf{X}}|^*_{\underline{U}_\partial}(\mathcal{F}_{\partial X}) \arrow[r] \arrow[d, "i^2_{\partial \mathbf{X}}|_{\underline{U}_\partial}"] & 0 \\
0 \arrow[r] & \underline{i}_{\partial \mathbf{X}}|^*_{\underline{U}_\partial} \circ \underline{b}_\partial^*(\mathcal{F}_{[0,\infty)}) \arrow[r, "\underline{i}_{\partial \mathbf{X}}|^*_{\underline{U}_\partial}(b_\partial^2)"] & \underline{i}_{\partial \mathbf{X}}|^*_{\underline{U}_\partial}(\mathcal{F}_{\partial X}) \arrow[r, "i^2_{\partial \mathbf{X}}|_{\underline{U}_\partial}"] & \mathcal{F}_{\partial^2 X}|_{\underline{U}_\partial} \arrow[r] & 0.
\end{tikzcd} \tag{6.41}$$

This identifies $\chi_{\partial\mathbf{X}}|_{\underline{U}_\partial}$ with an isomorphism. Thus $\chi_{\partial\mathbf{X}} : \underline{i}_{\partial\mathbf{X}}^*(\mathcal{N}_{\mathbf{X}}) \to \mathcal{N}_{\partial\mathbf{X}}$ is an isomorphism of line bundles on $\underline{\partial^2 X}$, as we can cover $\underline{\partial^2 X}$ by such open $\underline{U}_\partial$.

Define $\omega_{\partial\mathbf{X}}$ to be the orientation on $\mathcal{N}_{\partial\mathbf{X}}$ identified with the orientation $\underline{i}_{\partial\mathbf{X}}^*(\omega_{\mathbf{X}})$ on $\underline{i}_{\partial\mathbf{X}}^*(\mathcal{N}_{\mathbf{X}})$ by $\chi_{\partial\mathbf{X}}$. This gives Definition 6.1(d) for $\partial\mathbf{X}$. For (e), as above there is a natural identification between the restriction of (6.40) to $\underline{U}_\partial$ with (6.41). This induces isomorphisms

$$\underline{i}_{\partial\mathbf{X}}|_{\underline{U}_\partial}^*(\gamma) : \underline{i}_{\partial\mathbf{X}}|_{\underline{U}_\partial}^*(\underline{i}_{\mathbf{X}}|_{\underline{U}}^* \circ \underline{b}^*(\mathcal{F}_{[0,\infty)})) \xrightarrow{\cong} \underline{i}_{\partial\mathbf{X}}^*(\mathcal{N}_{\mathbf{X}})|_{\underline{U}_\partial},$$
$$\gamma_\partial : \underline{i}_{\partial\mathbf{X}}|_{\underline{U}_\partial}^* \circ \underline{b}_\partial^*(\mathcal{F}_{[0,\infty)}) \xrightarrow{\cong} \mathcal{N}_{\partial\mathbf{X}}|_{\underline{U}_\partial},$$

for γ as in (6.7) for $\boldsymbol{U}_2, \boldsymbol{V}, \boldsymbol{b}_2$. Now $\underline{i}_{\partial\mathbf{X}}^*(\mathcal{N}_{\mathbf{X}}), \omega_{\partial\mathbf{X}}$ in (6.40) have orientations $\underline{i}_{\partial\mathbf{X}}^*(\omega_{\mathbf{X}}), \omega_{\partial\mathbf{X}}$, where $\chi_{\partial\mathbf{X}}$ is orientation-preserving, and $\underline{i}_{\partial\mathbf{X}}|_{\underline{U}_\partial}^*(\underline{i}_{\mathbf{X}}|_{\underline{U}_2}^* \circ \underline{b}_2^*(\mathcal{F}_{[0,\infty)})), \underline{i}_{\partial\mathbf{X}}|_{\underline{U}_\partial}^* \circ \underline{b}_\partial^*(\mathcal{F}_{[0,\infty)})$ in (6.41) both have orientations coming from the negative orientation on $\mathcal{F}_{[0,\infty)}$, and $\underline{i}_{\partial\mathbf{X}}|_{\underline{U}_\partial}^*(I_{\underline{i}_{\mathbf{X}}|_{\underline{U}_2}, \underline{b}_2}(\mathcal{F}_{[0,\infty)}))$ is orientation-preserving. Since γ identifies orientations by Definition 6.1(e), it follows that γ_∂ also identifies orientations. This proves Definition 6.1(e) for $\boldsymbol{U}_\partial, \boldsymbol{V}_\partial, \boldsymbol{b}_\partial$, so that $(\boldsymbol{V}_\partial, \boldsymbol{b}_\partial)$ is a boundary defining function for $\partial\mathbf{X}$ at (x_1', x_2').

For (f), let $x' \in \boldsymbol{\partial X}$, with $\boldsymbol{i}_{\mathbf{X}}(x') = x \in \boldsymbol{X}$. Write $\boldsymbol{i}_{\mathbf{X}}^{-1}(x) = \{x', x_1', \ldots, x_k'\}$ for distinct $x', x_1', \ldots, x_k' \in \boldsymbol{\partial X}$. Then $\boldsymbol{i}_{\partial\mathbf{X}}^{-1}(x') = \{(x', x_1'), \ldots, (x', x_k')\}$. Consider the diagram in qcoh$(\underline{*})$:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
(\underline{x}')^*(\mathcal{N}_{\mathbf{X}}) \arrow[r, "\mathrm{id}_{(\underline{x}')^*(\mathcal{N}_{\mathbf{X}})}"] \arrow[d, "\mathrm{id} \oplus \bigoplus_{i=1}^k 0"] & (\underline{x}')^*(\mathcal{N}_{\mathbf{X}}) \arrow[d, "(\underline{x}')^*(\nu_{\mathbf{X}})"] \\
{\begin{array}{c}(\underline{x}')^*(\mathcal{N}_{\mathbf{X}}) \oplus \\ \bigoplus_{i=1}^k (\underline{x}_i')^*(\mathcal{N}_{\mathbf{X}})\end{array}} \arrow[r, "{\begin{array}{l} I_{\underline{x}',\underline{i}_{\mathbf{X}}}(\mathcal{F}_X) \circ \\ \bigl(I_{\underline{x}',\underline{i}_{\mathbf{X}}}(\mathcal{F}_X)^{-1} \circ (\underline{x}')^*(\nu_{\mathbf{X}}) \oplus \\ \bigoplus_{i=1}^k I_{\underline{x}_i',\underline{i}_{\mathbf{X}}}(\mathcal{F}_X)^{-1} \circ (\underline{x}_i')^*(\nu_{\mathbf{X}})\bigr) \end{array}}"] \arrow[d, "{\begin{array}{l} 0 \oplus \bigoplus_{i=1}^k (\underline{x}',\underline{x}_i')^*(\chi_{\partial\mathbf{X}}) \\ \circ I_{(\underline{x}',\underline{x}_i'),\underline{i}_{\partial\mathbf{X}}}(\mathcal{N}_{\mathbf{X}}) \end{array}}"] & (\underline{x}')^* \circ \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X) \arrow[d, "(\underline{x}')^*(i_{\mathbf{X}}^2)"] \\
\bigoplus_{i=1}^k (\underline{x}', \underline{x}_i')^*(\mathcal{N}_{\partial\mathbf{X}}) \arrow[r, "{\begin{array}{l} \bigoplus_{i=1}^k I_{(\underline{x}',\underline{x}_i'),\underline{i}_{\partial\mathbf{X}}}(\mathcal{F}_{\partial X})^{-1} \\ \circ (\underline{x}',\underline{x}_i')^*(\nu_{\partial\mathbf{X}}) \end{array}}"] \arrow[d] & (\underline{x}')^*(\mathcal{F}_{\partial X}) \arrow[d] \\
0 & 0.
\end{tikzcd}$$

Here the left hand column is exact as $\chi_{\partial\mathbf{X}}$ in (6.40) is an isomorphism, and the right hand column is pullback of (6.4) by $(\underline{x}')^*$, and so is exact as (6.4) is split exact. The top square commutes trivially, and the bottom square commutes as the right hand column is exact, and the left hand square in (6.40) commutes. The central horizontal morphism is injective by Definition 6.1(f) for $\mathbf{X}$ at x. Therefore the lower horizontal morphism is injective. This proves Definition 6.1(f) for $\partial\mathbf{X}$ at x'. Hence $\partial\mathbf{X}$ is a d-space with corners.

Lastly we show that the d-space morphism $\boldsymbol{i}_{\mathbf{X}} : \boldsymbol{X} \to \boldsymbol{\partial X}$ in $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ is a 1-morphism $\boldsymbol{i}_{\mathbf{X}} : \partial\mathbf{X} \to \mathbf{X}$ in $\mathbf{dSpa^c}$. Suppose $x_1' \in \boldsymbol{\partial X}$ with $\boldsymbol{i}_{\mathbf{X}}(x') = x \in \boldsymbol{X}$, and $x_2' \in \boldsymbol{\partial X}$ with $\boldsymbol{i}_{\mathbf{X}}(x_2') = x \in \boldsymbol{X}$, and let $(\boldsymbol{V}_2, \boldsymbol{b}_2)$ be a boundary defining function for $\mathbf{X}$ at x_2'. Divide into the two cases (i) $x_1' \neq x_2'$, and (ii) $x_1' = x_2'$. In case (i) we may choose a boundary defining function $(\boldsymbol{V}_1, \boldsymbol{b}_1)$ for $\mathbf{X}$ at x_1', and then as above use $(\boldsymbol{V}_1, \boldsymbol{b}_1), (\boldsymbol{V}_2, \boldsymbol{b}_2)$ to construct open $x_1' \in \boldsymbol{V}_\partial \subseteq \underline{i}_{\mathbf{X}}^{-1}(\boldsymbol{V}_2) \subseteq \boldsymbol{\partial X}$ such that $(\boldsymbol{V}_\partial, \boldsymbol{b}_\partial) = (\boldsymbol{V}_\partial, \boldsymbol{b}_2 \circ \boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{V}_\partial})$ is a boundary defining function for $\partial\mathbf{X}$ at (x_1', x_2'), where $\boldsymbol{i}_{\partial\mathbf{X}}(x_1', x_2') = x_1'$. Hence Definition 6.2(i) holds for $\boldsymbol{i}_{\mathbf{X}}$ in this case. In case (ii), let $\boldsymbol{U}_2$ be as in Definition 6.1(c) for $(\boldsymbol{V}_2, \boldsymbol{b}_2)$. Then $x_1' = x_2' \in \boldsymbol{U}_2 \subseteq \boldsymbol{\partial X}$ is open and $\boldsymbol{b}_2 \circ \boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{U}_2} = \boldsymbol{0} \circ \boldsymbol{\pi}$. Thus Definition 6.2(i) holds for $\boldsymbol{i}_{\mathbf{X}}$ in this case, with $\boldsymbol{W} = \boldsymbol{U}_2$. Therefore $\boldsymbol{i}_{\mathbf{X}} : \partial\mathbf{X} \to \mathbf{X}$ is a 1-morphism in $\mathbf{dSpa^c}$. This concludes Definition 6.10.

We will prove properties of boundaries $\partial\mathbf{X}$ in §6.3–§6.5 and §6.8. For now, note that our construction of the boundary $\partial\mathbf{X}$ of $\mathbf{X}$ is canonical: it does not depend on any arbitrary choices (provided we do not regard fibre products in $\mathbf{C^\infty Sch}$ and pullbacks of quasicoherent sheaves as choices). We will show in §6.5 that if $\mathbf{X}$ is equivalent to $\mathbf{Y}$ in $\mathbf{dSpa^c}$, then $\partial\mathbf{X}$ is equivalent to $\partial\mathbf{Y}$. We can iterate the construction to define $\partial^k\mathbf{X}$ for all $k = 1, 2, \ldots$.

6.3 Simple, semisimple and flat 1-morphisms

The next definition and theorem are analogues of Definitions 5.9(i)–(iii) and 5.12 and Proposition 5.13 for d-spaces with corners.

Definition 6.11. Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of d-spaces with corners, and $\partial\mathbf{X}$ the boundary of $\mathbf{X}$ from §6.2. Then $\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial X}$ is proper and étale by Proposition 6.7(c). As $\underline{s}_{\boldsymbol{f}}$ is étale, $\underline{s}_{\boldsymbol{f}}(\underline{S}_{\boldsymbol{f}})$ is open in $\underline{\partial X}$. Also $\underline{\partial X}$ is separated and locally fair, and so locally compact, and this and $\underline{s}_{\boldsymbol{f}}$ proper implies that $\underline{s}_{\boldsymbol{f}}(\underline{S}_{\boldsymbol{f}})$ is closed in $\underline{\partial X}$.

Define $\underline{\partial_-^{\boldsymbol{f}} X} = \underline{s}_{\boldsymbol{f}}(\underline{S}_{\boldsymbol{f}})$, and $\underline{\partial_+^{\boldsymbol{f}} X} = \underline{\partial X} \setminus \underline{\partial_-^{\boldsymbol{f}} X}$. Then $\underline{\partial_\pm^{\boldsymbol{f}} X}$ are open and closed C^∞-subschemes of $\underline{\partial X}$, with $\underline{\partial X} = \underline{\partial_+^{\boldsymbol{f}} X} \amalg \underline{\partial_-^{\boldsymbol{f}} X}$. Write $\partial_+^{\boldsymbol{f}}\mathbf{X}, \partial_-^{\boldsymbol{f}}\mathbf{X}$ for the open and closed d-subspaces of $\partial\mathbf{X}$ corresponding to $\underline{\partial_+^{\boldsymbol{f}} X}, \underline{\partial_-^{\boldsymbol{f}} X}$, as in Definition 6.1. Then $\partial_\pm^{\boldsymbol{f}}\mathbf{X}$ are d-spaces with corners, with $\partial\mathbf{X} = \partial_+^{\boldsymbol{f}}\mathbf{X} \amalg \partial_-^{\boldsymbol{f}}\mathbf{X}$.

We call $\boldsymbol{f}$ *simple* if $\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial X}$ is bijective, and we call $\boldsymbol{f}$ *semisimple* if $\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial X}$ is injective, and we call $\boldsymbol{f}$ *flat* if $\underline{T}_{\boldsymbol{f}} = \emptyset$. Simple implies semisimple. If $\boldsymbol{f}$ is simple then $\partial_-^{\boldsymbol{f}}\mathbf{X} = \partial\mathbf{X}$ and $\partial_+^{\boldsymbol{f}}\mathbf{X} = \boldsymbol{\emptyset}$.

One moral of parts (b),(c) of our next theorem is that when appropriate, 1- and 2-morphisms in $\mathbf{dSpa^c}$ lift uniquely up to 1- and 2-morphisms of boundaries, in a functorial way. In Remark 6.5(vi) we defined an alternative 2-category $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$, with a weaker notion of 2-morphism. Theorem 6.12(c) is false in $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$, as there are examples of semisimple 1-morphisms $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ with a weak 2-morphism $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$, for which there exists no weak 2-morphism $\eta_- : \boldsymbol{f}_- \Rightarrow \boldsymbol{g}_-$.

Theorem 6.12. *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a semisimple 1-morphism of d-spaces with corners. Then:*

- **(a)** *Define $\boldsymbol{f}_+ = \boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}}\mathbf{X}} : \partial_+^{\boldsymbol{f}}\mathbf{X} \to \mathbf{Y}$. Then $\boldsymbol{f}_+$ is semisimple. If $\boldsymbol{f}$ is flat then $\boldsymbol{f}_+$ is also flat.*

(b) *There exists a unique, semisimple* 1*-morphism* $\boldsymbol{f}_- : \partial_-^{\boldsymbol{f}}\mathbf{X} \to \partial\mathbf{Y}$ *with* $\boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}} = \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{f}_-$. *If* $\boldsymbol{f}$ *is simple then* $\boldsymbol{f}_- : \partial\mathbf{X} \to \partial\mathbf{Y}$ *is also simple. If* $\boldsymbol{f}$ *is flat then* $\boldsymbol{f}_-$ *is flat.*

(c) *Let* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ *be another* 1*-morphism and* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ *a* 2*-morphism in* $\mathbf{dSpa^c}$. *Then* $\boldsymbol{g}$ *is also semisimple, with* $\partial_-^{\boldsymbol{g}}\mathbf{X} = \partial_-^{\boldsymbol{f}}\mathbf{X}$. *If* $\boldsymbol{f}$ *is simple, or flat, then* $\boldsymbol{g}$ *is simple, or flat, respectively. Part* **(b)** *defines* 1*-morphisms* $\boldsymbol{f}_-, \boldsymbol{g}_- : \partial_-^{\boldsymbol{f}}\mathbf{X} \to \partial\mathbf{Y}$. *There is a unique* 2*-morphism* $\eta_- : \boldsymbol{f}_- \Rightarrow \boldsymbol{g}_-$ *in* $\mathbf{dSpa^c}$ *such that the following diagram in* $\mathrm{qcoh}(\underline{\partial}_-^{\boldsymbol{f}}\underline{X})$ *commutes:*

$$\begin{tikzcd}[column sep=huge]
\underline{i}_{\mathbf{X}}|^*_{\underline{\partial}_-^{\boldsymbol{f}}\underline{X}} \circ \underline{f}^*(\mathcal{F}_Y) \arrow[r, "{I_{\underline{f}_-,\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)\circ}"', "{I_{\underline{i}_{\mathbf{X}}|_{\underline{\partial}_-^{\boldsymbol{f}}\underline{X}},\underline{f}}(\mathcal{F}_Y)^{-1}}"' yshift=-12pt] \arrow[d, "\underline{i}_{\mathbf{X}}|^*_{\underline{\partial}_-^{\boldsymbol{f}}\underline{X}}(\eta)"] & \underline{f}_-^* \circ \underline{i}_{\mathbf{Y}}^*(\mathcal{F}_Y) \arrow[r, "\underline{f}_-^*(i_{\mathbf{Y}}^2)"'] & \underline{f}_-^*(\mathcal{F}_{\partial Y}) \arrow[d, "\eta_-"] \\
\underline{i}_{\mathbf{X}}|^*_{\underline{\partial}_-^{\boldsymbol{f}}\underline{X}}(\mathcal{E}_X) \arrow[rr, "i''_{\mathbf{X}}|_{\underline{\partial}_-^{\boldsymbol{f}}\underline{X}}"] & & \mathcal{E}_{\partial X}|_{\underline{\partial}_-^{\boldsymbol{f}}\underline{X}}.
\end{tikzcd} \tag{6.42}$$

Equation (6.42) *is equivalent to the* 2*-morphism equation*

$$\mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}} * \eta_- = \eta * \mathrm{id}_{\boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}}} : \boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}} = \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{f}_- \Longrightarrow \boldsymbol{g} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}} = \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{g}_-. \tag{6.43}$$

Proof. For (a), from the proof that $\boldsymbol{i}_{\mathbf{X}} : \partial\mathbf{X} \to \mathbf{X}$ is a 1-morphism in §6.2, one can show the sets $S_{\boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}}\mathbf{X}}}, T_{\boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}}\mathbf{X}}}$ underlying $\underline{S}_{\boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}}\mathbf{X}}}, \underline{T}_{\boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}}\mathbf{X}}}$ are given by

$$S_{\boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}}\mathbf{X}}} = \bigl\{\bigl((x_1', x_2'), x_2'\bigr) : x_1' \in \partial_+^{\boldsymbol{f}}X,\ x_2' \in \partial X,\ x_1' \neq x_2',\ i_{\mathbf{X}}(x_1') = i_{\mathbf{X}}(x_2')\bigr\},$$

$$T_{\boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}}\mathbf{X}}} = \bigl\{(x', x') : x' \in \partial_+^{\boldsymbol{f}}X\bigr\}.$$

Thus from $\boldsymbol{f}_+ = \boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}}\mathbf{X}}$ and Proposition 6.7(f) we see that

$$\begin{aligned} S_{\boldsymbol{f}_+} = \bigl\{\bigl((x_1', x_2'), y'\bigr) \in \partial(\partial_+^{\boldsymbol{f}}X) \times_Y \partial Y : x_1' \in \partial_+^{\boldsymbol{f}}X,\ x_2' \in \partial X,\\ y' \in \partial Y,\ x_1' \neq x_2',\ i_{\mathbf{X}}(x_1') = i_{\mathbf{X}}(x_2'),\ (x_2', y') \in S_{\boldsymbol{f}}\bigr\}, \end{aligned} \tag{6.44}$$

$$\begin{aligned} T_{\boldsymbol{f}_+} = \bigl\{(x', y') \in \partial_+^{\boldsymbol{f}}X \times_Y \partial Y : (i_{\mathbf{X}}(x'), y') \in T_{\boldsymbol{f}}\bigr\} \amalg \\ \bigl\{(x', y') \in \partial_+^{\boldsymbol{f}}X \times_Y \partial Y : (x', y') \in S_{\boldsymbol{f}}\bigr\}. \end{aligned} \tag{6.45}$$

By (6.44), $s_{\boldsymbol{f}_+} : S_{\boldsymbol{f}_+} \to \partial(\partial_+^{\boldsymbol{f}}X)$ maps $s_{\boldsymbol{f}_+} : \bigl((x_1', x_2'), y'\bigr) \mapsto (x_1', x_2')$, where $(x_2', y') \in S_{\boldsymbol{f}}$ and $s_{\boldsymbol{f}} : (x_2', y') \mapsto x_2'$. Since $\boldsymbol{f}$ is semisimple, $s_{\boldsymbol{f}}$ is injective, so $s_{\boldsymbol{f}_+}$ is injective, and $\boldsymbol{f}_+$ is semisimple. Suppose $\boldsymbol{f}$ is flat. Then $T_{\boldsymbol{f}} = \emptyset$, so the first term on the right hand side of (6.45) is empty. If $(x', y') \in S_{\boldsymbol{f}}$ then $s_{\boldsymbol{f}}(x', y') = x'$, so $x' \in \partial_-^{\boldsymbol{f}}X = s_{\boldsymbol{f}}(S_{\boldsymbol{f}})$, and $x' \notin \partial_+^{\boldsymbol{f}}X$. Thus the second term on the right hand side of (6.45) is empty, and $T_{\boldsymbol{f}_+} = \emptyset$. Therefore $\boldsymbol{f}_+$ is flat, proving part (a).

For (b), we construct $\boldsymbol{f}_- = (\underline{f}_-, f_-', f_-'') : \boldsymbol{\partial_-^{f} X} \to \boldsymbol{\partial Y}$ in $\mathbf{dSpa}$ with $\boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}} = \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{f}_-$. Now $\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial X}$ is étale by Proposition 6.7(c), has image $\underline{\partial^{\boldsymbol{f}}_- X}$ by Definition 6.11, and is injective as $\boldsymbol{f}$ is semisimple. Hence $\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial^{\boldsymbol{f}}_- X}$ is an isomorphism. Define $\underline{f}_- = \underline{u}_{\boldsymbol{f}} \circ \underline{s}_{\boldsymbol{f}}^{-1} : \underline{\partial^{\boldsymbol{f}}_- X} \to \underline{\partial Y}$. Then

$$\underline{i}_{\mathbf{Y}} \circ \underline{f}_- = \underline{i}_{\mathbf{Y}} \circ \underline{u}_{\boldsymbol{f}} \circ \underline{s}_{\boldsymbol{f}}^{-1} = \underline{f} \circ \underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}} \circ \underline{s}_{\boldsymbol{f}}^{-1} = \underline{f} \circ \underline{i}_{\mathbf{X}}|_{\underline{\partial^{\boldsymbol{f}}_- X}}, \tag{6.46}$$

since $\underline{i}_{\mathbf{Y}} \circ \underline{u}_{\boldsymbol{f}} = \underline{f} \circ \underline{i}_{\mathbf{X}} \circ \underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{Y}$.

We claim there is a unique morphism $f'_- : f_-^{-1}(\mathcal{O}'_{\partial Y}) \to \mathcal{O}'_{\partial X}|_{\partial^{\boldsymbol{f}}_- X}$ of sheaves of C^∞-rings on $\partial^{\boldsymbol{f}}_- X$ fitting into a commutative diagram:

$$\begin{tikzcd}
(i_{\mathbf{Y}} \circ f_-)^{-1}(\mathcal{O}'_Y) \arrow[rr, "i_{\mathbf{X}}|^{-1}_{\partial^{\boldsymbol{f}}_- X}(f') \circ I_{i_{\mathbf{X}}|_{\partial^{\boldsymbol{f}}_- X}, f}(\mathcal{O}'_Y)"'] \arrow[d, "f_-^{-1}(i'_{\mathbf{Y}}) \circ I_{f_-, i_{\mathbf{Y}}}(\mathcal{O}'_Y)"] & & i_{\mathbf{X}}|^{-1}_{\partial^{\boldsymbol{f}}_- X}(\mathcal{O}'_X) \arrow[d, "i'_{\mathbf{X}}|_{\partial^{\boldsymbol{f}}_- X}"'] \\
f_-^{-1}(\mathcal{O}'_{\partial Y}) \arrow[rr, dotted, "f'_-"] & & \mathcal{O}'_{\partial X}|_{\partial^{\boldsymbol{f}}_- X}.
\end{tikzcd} \tag{6.47}$$

To see f'_- is unique if it exists, note that $i'_{\mathbf{Y}} : i_{\mathbf{Y}}^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_{\partial Y}$ is surjective, so $f_-^{-1}(i'_{\mathbf{Y}}) \circ I_{f_-, i_{\mathbf{Y}}}(\mathcal{O}'_Y)$ in (6.47) is surjective. Thus by uniqueness it is enough to construct f'_- locally on $\partial^{\boldsymbol{f}}_- X$, that is, on the sets of an open cover of $\partial^{\boldsymbol{f}}_- X$.

Let $x' \in \partial^{\boldsymbol{f}}_- X$, and set $f_-(x') = y' \in \partial Y$. Then $(x', y') \in \underline{S}_{\boldsymbol{f}}$. Let $(\boldsymbol{V}, \boldsymbol{b})$ be a boundary defining function for $\mathbf{Y}$ at y'. Then there exists open $x' \in \tilde{\boldsymbol{V}} \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$ such that $(\tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}})$ is a boundary defining function for $\mathbf{X}$ at x'. Hence there exist open $y' \in \boldsymbol{U} \subseteq \boldsymbol{i}_{\mathbf{Y}}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{\partial Y}$ and $x' \in \tilde{\boldsymbol{U}} \subseteq \boldsymbol{i}_{\mathbf{Y}}^{-1}(\tilde{\boldsymbol{V}}) \subseteq \boldsymbol{\partial X}$ such that (6.1) is 2-Cartesian for $y', \mathbf{Y}, \boldsymbol{U}, (\boldsymbol{V}, \boldsymbol{b})$ and for $x', \mathbf{X}, \tilde{\boldsymbol{U}}, (\tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}})$. As $x' \in \partial^{\boldsymbol{f}}_- X$, making $\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{V}}$ smaller if necessary, we can suppose that $\tilde{\boldsymbol{U}} \subseteq \boldsymbol{\partial}^{\boldsymbol{f}}_- \boldsymbol{X}$.

Consider the diagram of sheaves of C^∞-rings on $\tilde{U}$:

$$\begin{tikzcd}
(b \circ i_{\mathbf{Y}} \circ f_-)^{-1}(\mathcal{O}_{[0,\infty)}) \arrow[rr, "\mathrm{id}"'] \arrow[dr, "(i_{\mathbf{Y}} \circ f_-)^{-1}(b') \circ I_{i_{\mathbf{Y}} \circ f_-, b}(\mathcal{O}_{[0,\infty)})"] \arrow[dd, "\pi^{-1}(0') \circ I_{\pi,0}(\mathcal{O}_{[0,\infty)})"] & & (b \circ i_{\mathbf{Y}} \circ f_-)^{-1}(\mathcal{O}_{[0,\infty)}) \arrow[dr, "i_{\mathbf{X}}|^{-1}_{\tilde{U}}((b \circ f|_{\tilde{V}})') \circ I_{i_{\mathbf{X}}|_{\tilde{U}}, b \circ f}(\mathcal{O}_{[0,\infty)})"] \arrow[dd, "\pi^{-1}(0') \circ I_{\pi,0}(\mathcal{O}_{[0,\infty)})"'] & \\
& (i_{\mathbf{Y}} \circ f_-)^{-1}(\mathcal{O}'_Y)|_{\tilde{U}} \arrow[rr, "i_{\mathbf{X}}|^{-1}_{\tilde{U}}(f') \circ I_{i_{\mathbf{X}}|_{\tilde{U}}, f}(\mathcal{O}'_Y)"'] \arrow[dd, "f_-^{-1}(i'_{\mathbf{Y}}) \circ I_{f_-, i_{\mathbf{Y}}}(\mathcal{O}'_Y)|_{\tilde{U}}"] & & i_{\mathbf{X}}|^{-1}_{\tilde{U}}(\mathcal{O}'_X) \arrow[dd, "i'_{\mathbf{X}}|_{\tilde{U}}"'] \\
\pi^{-1}(\mathcal{O}_*) \arrow[rr, "\mathrm{id}"'] \arrow[dr, "f_-^{-1}(\pi') \circ I_{f_-, \pi}(\mathcal{O}_*)"'] & & \pi^{-1}(\mathcal{O}_*) \arrow[dr, "\pi'"] & \\
& f_-^{-1}(\mathcal{O}'_{\partial Y})|_{\tilde{U}} \arrow[rr, dotted, "f'_-|_{\tilde{U}}"] & & \mathcal{O}'_{\partial X}|_{\tilde{U}}.
\end{tikzcd} \tag{6.48}$$

The whole diagram commutes, apart from $f'_-|_{\tilde{U}}$ marked '$\dashrightarrow$' which is not yet constructed. The left hand and right hand quadrilaterals are co-Cartesian because (6.1) is 2-Cartesian for $\boldsymbol{U}, (\boldsymbol{V}, \boldsymbol{b})$ and for $\tilde{\boldsymbol{U}}, (\tilde{\boldsymbol{V}}, \boldsymbol{b} \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}})$, and $i''_{\mathbf{X}}, i''_{\mathbf{Y}}$ are isomorphisms. By the co-Cartesian property of the left hand quadrilateral, there exists a unique morphism $f'_-|_{\tilde{U}}$ making (6.48) commute. As we can cover $\partial^{\boldsymbol{f}}_- X$ by such open $\tilde{U}$, there is a unique morphism f'_- making (6.47) commute.

Consider the cube of morphisms of sheaves of C^∞-rings on $\partial^{\boldsymbol f}_- X$:

$$\begin{tikzcd}[column sep=small]
(i_{\mathbf Y}\circ f_-)^{-1}(\mathcal{O}'_Y) \arrow[rr, "i_{\mathbf X}|^{-1}_{\partial^{\boldsymbol f}_- X}(f')\circ I_{i_{\mathbf X}|_{\partial^{\boldsymbol f}_- X},f}(\mathcal{O}'_Y)"] \arrow[dr, "(i_{\mathbf Y}\circ f_-)^{-1}(\imath_Y)"] \arrow[dd, "{\begin{array}{l} f_-^{-1}(i'_{\mathbf Y})\circ \\ I_{f_-,i_{\mathbf Y}}(\mathcal{O}'_Y)\end{array}}"'] & & i_{\mathbf X}|^{-1}_{\partial^{\boldsymbol f}_- X}(\mathcal{O}'_X) \arrow[dr, "i_{\mathbf X}|^{-1}_{\partial^{\boldsymbol f}_- X}(\imath_X)"] \arrow[dd, "i'_{\mathbf X}|_{\partial^{\boldsymbol f}_- X}" near start] & \\
& (i_{\mathbf Y}\circ f_-)^{-1}(\mathcal{O}_Y) \arrow[rr, "i_{\mathbf X}|^{-1}_{\partial^{\boldsymbol f}_- X}(f^\sharp)\circ I_{i_{\mathbf X}|_{\partial^{\boldsymbol f}_- X},f}(\mathcal{O}_Y)"' near end] \arrow[dd, "f_-^{-1}(i^\sharp_{\mathbf Y})\circ I_{f_-,i_{\mathbf Y}}(\mathcal{O}_Y)" near start] & & i_{\mathbf X}|^{-1}_{\partial^{\boldsymbol f}_- X}(\mathcal{O}_X) \arrow[dd, "i^\sharp_{\mathbf X}|_{\partial^{\boldsymbol f}_- X}"'] \\
f_-^{-1}(\mathcal{O}'_{\partial Y}) \arrow[rr, "f'_-"' near start] \arrow[dr, "f_-^{-1}(\imath_{\partial Y})"'] & & \mathcal{O}'_{\partial X}|_{\partial^{\boldsymbol f}_- X} \arrow[dr, "\imath_{\partial X}|_{\partial^{\boldsymbol f}_- X}"'] & \\
& f_-^{-1}(\mathcal{O}_{\partial Y}) \arrow[rr, "f^\sharp_-"] & & \mathcal{O}_{\partial X}|_{\partial^{\boldsymbol f}_- X}.
\end{tikzcd}$$

Of the six faces, all except possibly the bottom face commute. Using the other five, we see that $\bigl(f_-^\sharp\circ f_-^{-1}(\imath_{\partial Y})-\imath_{\partial X}|_{\partial^{\boldsymbol f}_- X}\circ f'_-\bigr)\circ f_-^{-1}(i'_{\mathbf Y})\circ I_{f_-,i_{\mathbf Y}}(\mathcal{O}'_Y)=0$. As $f_-^{-1}(i'_{\mathbf Y})\circ I_{f_-,i_{\mathbf Y}}(\mathcal{O}'_Y)$ is surjective, this shows the last face commutes, that is,

$$f_-^\sharp\circ f_-^{-1}(\imath_{\partial Y})=\imath_{\partial X}|_{\partial^{\boldsymbol f}_- X}\circ f'_-. \tag{6.49}$$

Consider the diagram in $\mathrm{qcoh}(\underline{\partial^{\boldsymbol f}_- X})$:

$$\begin{tikzcd}
(\underline{i}_{\mathbf Y}\circ \underline{f}_-)^*(\mathcal{E}_Y) \arrow[rr, "\underline{i}_{\mathbf X}|^*_{\underline{\partial^{\boldsymbol f}_- X}}(f'')\circ I_{\underline{i}_{\mathbf X}|_{\underline{\partial^{\boldsymbol f}_- X}},\underline{f}}(\mathcal{E}_Y)"'] \arrow[d, "\cong"', "\underline{f}_-^*(i''_{\mathbf Y})\circ I_{\underline{f}_-,\underline{i}_{\mathbf Y}}(\mathcal{E}_Y)"] & & \underline{i}_{\mathbf X}|^*_{\underline{\partial^{\boldsymbol f}_- X}}(\mathcal{E}_X) \arrow[d, "i''_{\mathbf X}|_{\underline{\partial^{\boldsymbol f}_- X}}"', "\cong"] \\
\underline{f}_-^*(\mathcal{E}'_{\partial Y}) \arrow[rr, dotted, "f''_-"] & & \mathcal{E}_{\partial X}|_{\underline{\partial^{\boldsymbol f}_- X}}.
\end{tikzcd} \tag{6.50}$$

Since $i''_{\mathbf Y}$ is an isomorphism by Definition 6.1(b) for $\mathbf{Y}$, $\underline{f}_-^*(i''_{\mathbf Y})\circ I_{\underline{f}_-,\underline{i}_{\mathbf Y}}(\mathcal{E}_Y)$ is an isomorphism, so there exists a unique morphism f''_- making (6.50) commute. A similar proof to that of (6.49) shows that

$$\jmath_{\partial X}|_{\underline{\partial^{\boldsymbol f}_- X}}\circ f''_-=f_-^1\circ \underline{f}_-^*(\jmath_{\partial Y}):\underline{f}_-^*(\mathcal{E}_{\partial Y})\longrightarrow \mathcal{I}_{\partial X}. \tag{6.51}$$

This completes the definition of $\boldsymbol{f}_-=(\underline{f}_-,f'_-,f''_-)$. Equations (6.49) and (6.51) prove that $\boldsymbol{f}_-:\boldsymbol{\partial}^{\boldsymbol f}_-\boldsymbol{X}\to\boldsymbol{\partial Y}$ is a 1-morphism in **dSpa**. Equations (6.46), (6.47) and (6.50) show that $\boldsymbol{f}\circ\boldsymbol{i}_{\mathbf X}|_{\boldsymbol{\partial}^{\boldsymbol f}_-\boldsymbol{X}}=\boldsymbol{i}_{\mathbf Y}\circ\boldsymbol{f}_-$, as 1-morphisms in **dSpa**. Moreover, (6.46), (6.47) and (6.50) determine $\underline{f}_-,f'_-$ and f''_- uniquely, so $\boldsymbol{f}_-$ is the unique 1-morphism in **dSpa** with $\boldsymbol{f}\circ\boldsymbol{i}_{\mathbf X}|_{\boldsymbol{\partial}^{\boldsymbol f}_-\boldsymbol{X}}=\boldsymbol{i}_{\mathbf Y}\circ\boldsymbol{f}_-$.

Next we show that $\boldsymbol{f}_-$ is a 1-morphism $\partial^{\boldsymbol f}_-\mathbf{X}\to\partial\mathbf{Y}$ in $\mathbf{dSpa^c}$. Let $x'_1\in\boldsymbol{\partial}^{\boldsymbol f}_-\boldsymbol{X}$ with $\boldsymbol{f}_-(x'_1)=y'_1\in\boldsymbol{\partial Y}$ and $\boldsymbol{i}_{\mathbf X}(x'_1)=x\in\boldsymbol{X}$, and let $(y'_1,y'_2)\in\boldsymbol{\partial^2 Y}$ with $\boldsymbol{i}_{\partial\mathbf Y}(y'_1,y'_2)=y'_1$, so that $y'_1\neq y'_2$ and $\boldsymbol{f}(x)=\boldsymbol{i}_{\mathbf Y}(y'_1)=\boldsymbol{i}_{\mathbf Y}(y'_2)=y\in\boldsymbol{Y}$. As in §6.2, we may choose boundary defining functions $(\boldsymbol{V},\boldsymbol{b}_1),(\boldsymbol{V},\boldsymbol{b}_2)$ for $\mathbf{Y}$ at y'_1,y'_2 and open $y'_1\in\boldsymbol{U}_1\subseteq\boldsymbol{i}_{\mathbf Y}^{-1}(\boldsymbol{V})\subseteq\boldsymbol{\partial Y}$, $y'_2\in\boldsymbol{U}_2\subseteq\boldsymbol{i}_{\mathbf Y}^{-1}(\boldsymbol{V})\subseteq\boldsymbol{\partial Y}$ with $\boldsymbol{U}_1\cap\boldsymbol{U}_2=\emptyset$, and define $\boldsymbol{U}_\partial=\boldsymbol{i}_{\partial Y}^{-1}(\boldsymbol{U}_1)\cap\boldsymbol{j}_{\partial Y}^{-1}(\boldsymbol{U}_2)$, $\boldsymbol{V}_\partial=\boldsymbol{U}_1$, and $\boldsymbol{b}_\partial=\boldsymbol{b}_2\circ\boldsymbol{i}_{\mathbf Y}|_{\boldsymbol{V}_\partial}:\boldsymbol{V}_\partial\to\boldsymbol{[0,\infty)}$, and then $(\boldsymbol{V}_\partial,\boldsymbol{b}_\partial)$ is a boundary defining function for $\partial\mathbf{Y}$ at (y'_1,y'_2), and $\boldsymbol{U}_\partial,\boldsymbol{V}_\partial,\boldsymbol{b}_\partial$ form a 2-Cartesian square (6.1).

Since $\boldsymbol{f}_-(x_1') = y_1'$ we have $(x_1', y_1') \in \underline{S}_{\boldsymbol{f}}$. Hence there exist open $x \in \tilde{\boldsymbol{V}}_1 \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$ such that $(\tilde{\boldsymbol{V}}_1, \tilde{\boldsymbol{b}}_1) = (\tilde{\boldsymbol{V}}_1, \boldsymbol{b}_1 \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}_1})$ is a boundary defining function for $\mathbf{X}$ at x_1'. By Definition 6.2 for $\boldsymbol{f}$ at $x, y_2', (\boldsymbol{V}, \boldsymbol{b}_2)$, either

(i) there exist a unique $x_2' \in \boldsymbol{\partial X}$ with $\boldsymbol{i}_{\mathbf{X}}(x_2') = x$ and open $x \in \tilde{\boldsymbol{V}}_2 \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$ such that $(\tilde{\boldsymbol{V}}_2, \tilde{\boldsymbol{b}}_2) = \bigl(\tilde{\boldsymbol{V}}_2, \boldsymbol{b}_2 \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}_2}\bigr)$ is a boundary defining function for $\mathbf{X}$ at x_2'; or

(ii) there exists an open $x \in \boldsymbol{W} \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$ with $\boldsymbol{b}_2 \circ \boldsymbol{f}|_{\boldsymbol{W}} = \boldsymbol{0} \circ \boldsymbol{\pi}$.

In case (i), $(x_2', y_2') \in \underline{S}_{\boldsymbol{f}}$. Since $\underline{s}_{\boldsymbol{f}}$ maps $(x_1', y_1') \mapsto x_1'$, $(x_2', y_2') \mapsto x_2'$ and is injective as $\boldsymbol{f}$ is semisimple, and $y_1' \neq y_2$, we see that $x_1' \neq x_2'$. Hence $(x_1', x_2') \in \boldsymbol{\partial(\partial^{f}_{-} X)}$. As $(\tilde{\boldsymbol{V}}_1, \tilde{\boldsymbol{b}}_1)$ and $(\tilde{\boldsymbol{V}}_2, \tilde{\boldsymbol{b}}_2)$ are boundary defining functions for $\mathbf{X}$ at x_1', x_2', there exist open $x_1' \subseteq \tilde{\boldsymbol{U}}_1 \subseteq \boldsymbol{i}_{\mathbf{X}}^{-1}(\tilde{\boldsymbol{V}}_1) \subseteq \boldsymbol{\partial X}$ and $x_2' \subseteq \tilde{\boldsymbol{U}}_2 \subseteq \boldsymbol{i}_{\mathbf{X}}^{-1}(\tilde{\boldsymbol{V}}_2) \subseteq \boldsymbol{\partial X}$ such that (6.1) is 2-Cartesian for $\tilde{\boldsymbol{U}}_1, \tilde{\boldsymbol{V}}_1, \tilde{\boldsymbol{b}}_1$ and $\tilde{\boldsymbol{U}}_2, \tilde{\boldsymbol{V}}_2, \tilde{\boldsymbol{b}}_2$. As in §6.2, making $\tilde{\boldsymbol{U}}_1, \tilde{\boldsymbol{V}}_1, \tilde{\boldsymbol{U}}_2, \tilde{\boldsymbol{V}}_2$ smaller we can suppose that $\tilde{\boldsymbol{U}}_1 \subseteq \boldsymbol{\partial^{f}_{-} X}$, and $\boldsymbol{f}_-(\tilde{\boldsymbol{U}}_1) \subseteq \boldsymbol{U}_1$, and $\tilde{\boldsymbol{V}}_1 = \tilde{\boldsymbol{V}}_2 = \tilde{\boldsymbol{V}}$, and $\tilde{\boldsymbol{U}}_1 \cap \tilde{\boldsymbol{U}}_2 = \emptyset$. Define $\tilde{\boldsymbol{U}}_\partial = \boldsymbol{i}_{\partial X}^{-1}(\tilde{\boldsymbol{U}}_1) \cap \boldsymbol{j}_{\partial X}^{-1}(\tilde{\boldsymbol{U}}_2)$, $\tilde{\boldsymbol{V}}_\partial = \tilde{\boldsymbol{U}}_1$, and $\tilde{\boldsymbol{b}}_\partial = \tilde{\boldsymbol{b}}_2 \circ \boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{V}}_\partial} : \tilde{\boldsymbol{V}}_\partial \to \boldsymbol{[0,\infty)}$. Then $(\tilde{\boldsymbol{V}}_\partial, \tilde{\boldsymbol{b}}_\partial)$ is a boundary defining function for $\partial^{\boldsymbol{f}}_{-}\mathbf{X}$ at (x_1', x_2'). We now have a boundary defining function $(\tilde{\boldsymbol{V}}_\partial, \tilde{\boldsymbol{b}}_\partial)$ with $x_1' \in \tilde{\boldsymbol{V}}_\partial \subseteq \boldsymbol{f}_-^{-1}(\boldsymbol{V}_\partial) \subseteq \boldsymbol{\partial^{f}_{-} X}$ and

$$\tilde{\boldsymbol{b}}_\partial = \tilde{\boldsymbol{b}}_2 \circ \boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{V}}_\partial} = \boldsymbol{b}_2 \circ \boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{V}}_\partial} = \boldsymbol{b}_2 \circ \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{f}_-|_{\tilde{\boldsymbol{V}}_\partial} = \boldsymbol{b}_\partial \circ \boldsymbol{f}_-|_{\tilde{\boldsymbol{V}}_\partial}.$$

Therefore Definition 6.2(i) holds for $\boldsymbol{f}_-$ in this case.

In case (ii), define $\tilde{\boldsymbol{W}} = \boldsymbol{i}_{\mathbf{X}}^{-1}(\boldsymbol{W}) \cap \boldsymbol{f}_-^{-1}(\boldsymbol{V}_\partial)$, so that $x' \in \tilde{\boldsymbol{W}} \subseteq \boldsymbol{f}_-^{-1}(\boldsymbol{V}_\partial) \subseteq \boldsymbol{\partial^{f}_{-} X}$ is open. Then

$$\boldsymbol{b}_\partial \circ \boldsymbol{f}_-|_{\tilde{\boldsymbol{W}}} = \boldsymbol{b}_2 \circ \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{f}_-|_{\tilde{\boldsymbol{W}}} = \boldsymbol{b}_2 \circ \boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{W}}} = \boldsymbol{0} \circ \boldsymbol{\pi} \circ \boldsymbol{i}_{\mathbf{X}}|_{\tilde{\boldsymbol{W}}} = \boldsymbol{0} \circ \boldsymbol{\pi},$$

so Definition 6.2(ii) holds for $\boldsymbol{f}_-$ in this case. We have shown that for all $x_1' \in \boldsymbol{\partial^{f}_{-} X}$ with $\boldsymbol{f}_-(x_1') = y_1' \in \boldsymbol{\partial Y}$ and $(y_1', y_2') \in \boldsymbol{\partial^2 Y}$ with $\boldsymbol{i}_{\partial\mathbf{Y}}(y_1', y_2') = y_1'$, and for the particular choice of boundary defining function $(\boldsymbol{V}_\partial, \boldsymbol{b}_\partial)$ for $\partial\mathbf{Y}$ at (y_1', y_2'), one of Definition 6.2(i),(ii) holds for $\boldsymbol{f}_-$. As in Definition 6.2, if (i) or (ii) holds for one choice of boundary defining function $(\boldsymbol{V}_\partial, \boldsymbol{b}_\partial)$, then it holds for every choice. Therefore $\boldsymbol{f}_- : \partial^{\boldsymbol{f}}_{-}\mathbf{X} \to \partial\mathbf{Y}$ is a 1-morphism in $\mathbf{dSpa^c}$.

From the proof above it follows that $\underline{S}_{\boldsymbol{f}_-}$ has underlying set

$$\begin{aligned} S_{\boldsymbol{f}_-} = \bigl\{ \bigl((x_1', x_2'), (y_1', y_2')\bigr) : (x_1', y_1'), (x_2', y_2') \in S_{\boldsymbol{f}},\ x_1' \neq x_2',\ y_1' \neq y_2', \\ \boldsymbol{i}_{\mathbf{X}}(x_1') = \boldsymbol{i}_{\mathbf{X}}(x_2') = x \in \boldsymbol{X},\ \boldsymbol{f}(x) = \boldsymbol{i}_{\mathbf{Y}}(y_1') = \boldsymbol{i}_{\mathbf{Y}}(y_1') = y \in \boldsymbol{Y} \bigr\}. \end{aligned} \tag{6.52}$$

Now $\underline{s}_{\boldsymbol{f}}$ maps $(x_1', y_1') \mapsto x_1'$, $(x_2', y_2') \mapsto x_2'$, and is injective as $\boldsymbol{f}$ is semisimple. Hence $\underline{s}_{\boldsymbol{f}_-} : \bigl((x_1', x_2'), (y_1', y_2')\bigr) \mapsto (x_1', x_2')$ is also injective, and $\boldsymbol{f}_-$ is semisimple. Similarly, $\boldsymbol{f}$ simple implies $\underline{s}_{\boldsymbol{f}}, \underline{s}_{\boldsymbol{f}_-}$ bijective, and $\boldsymbol{f}_-$ is simple.

Suppose $\boldsymbol{f}$ is flat, so that $\underline{T}_{\boldsymbol{f}} = \emptyset$. Let $\bigl(x_1', (y_1', y_2')\bigr) \in \underline{\partial^{f}_{-} X} \times_{\underline{\partial Y}} \underline{\partial^2 Y}$. Then $\underline{f}_-(x_1') = y_1' = \underline{i}_{\partial\mathbf{Y}}(y_1', y_2')$, so $(x_1', y_1') \in \underline{S}_{\boldsymbol{f}}$, and $\underline{i}_{\mathbf{X}}(x_1') = x \in \underline{X}$, $\underline{f}(x) =$

$\underline{i}_{\mathbf{Y}}(y_1') = \underline{i}_{\mathbf{Y}}(y_2') = y \in \underline{Y}$. Thus $(x, y_2') \in \underline{X} \times_{\underline{Y}} \underline{\partial Y}$, so $(x, y_2') = \underline{j}_{\boldsymbol{f}}(x_2', y_2')$ for some $(x_2', y_2') \in \underline{S}_{\boldsymbol{f}}$ as $\underline{T}_{\boldsymbol{f}} = \emptyset$. It then follows that $\bigl((x_1', x_2'), (y_1', y_2')\bigr) \in \underline{S}_{\boldsymbol{f}_-}$ and $\underline{j}_{\boldsymbol{f}_-}\bigl((x_1', x_2'), (y_1', y_2')\bigr) = \bigl(x_1', (y_1', y_2')\bigr)$. Thus $\underline{j}_{\boldsymbol{f}_-} : \underline{S}_{\boldsymbol{f}_-} \to \underline{\partial}^{\boldsymbol{f}}_- \underline{X} \times_{\underline{\partial Y}} \underline{\partial^2 Y}$ is surjective, and $\underline{T}_{\boldsymbol{f}_-} = \emptyset$, so that $\boldsymbol{f}_-$ is flat. This proves part (b).

For (c), as $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism in $\mathbf{dSpa^c}$ we have $\underline{f} = \underline{g}$ and $\underline{S}_{\boldsymbol{f}} = \underline{S}_{\boldsymbol{g}}$, which imply that $\underline{T}_{\boldsymbol{f}} = \underline{T}_{\boldsymbol{g}}$ and $\underline{s}_{\boldsymbol{f}} = \underline{s}_{\boldsymbol{g}}$. Since the definitions of $\boldsymbol{f}$ simple, semisimple, or flat and of $\partial^{\boldsymbol{f}}_- \mathbf{X}$ involve only the data $\underline{S}_{\boldsymbol{f}}, \underline{T}_{\boldsymbol{f}}, \underline{s}_{\boldsymbol{f}}$ in $\boldsymbol{f}$, we see that $\boldsymbol{f}$ simple, semisimple, or flat implies $\boldsymbol{g}$ is, and that $\partial^{\boldsymbol{g}}_- \mathbf{X} = \partial^{\boldsymbol{f}}_- \mathbf{X}$.

Since $\boldsymbol{f}$ is semisimple, $\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial}^{\boldsymbol{f}}_- \underline{X}$ is an isomorphism, so $\underline{s}_{\boldsymbol{f}}^* : \mathrm{qcoh}(\underline{\partial}^{\boldsymbol{f}}_- \underline{X}) \to \mathrm{qcoh}(\underline{S}_{\boldsymbol{f}})$ is an equivalence of categories. Now $\underline{s}_{\boldsymbol{f}}^*$ identifies equations (6.22) and (6.42), up to natural isomorphisms $I_{*,*}(*)$. Hence Proposition 6.8(b) shows that there is a unique morphism $\eta_- : \underline{f}_-^*(\mathcal{F}_{\partial Y}) \to \mathcal{E}_{\partial X}|_{\underline{\partial}^{\boldsymbol{f}}_- \underline{X}}$ in $\mathrm{qcoh}(\underline{\partial}^{\boldsymbol{f}}_- \underline{X})$ making (6.42) commute, which corresponds to η_S in (6.22).

We will show $\eta_- : \boldsymbol{f}_- \Rightarrow \boldsymbol{g}_-$ is a 2-morphism in $\mathbf{dSpa^c}$. Consider the diagram in $\mathrm{qcoh}(\underline{\partial}^{\boldsymbol{f}}_- \underline{X})$:

$$
\begin{tikzcd}
\underline{i}_{\mathbf{X}}|^*_{\underline{\partial}^{\boldsymbol{f}}_- \underline{X}} \circ \underline{f}^*(\mathcal{E}_Y) \arrow[rr, "\underline{i}_{\mathbf{X}}^* \circ \underline{f}^*(\phi_Y)"] \arrow[dr, "\underline{f}_-^*(i''_{\mathbf{Y}}) \circ I_{\underline{f}_-, \underline{i}_{\mathbf{Y}}}(\mathcal{E}_Y) \circ I_{\underline{i}_{\mathbf{X}}, \underline{f}}(\mathcal{E}_Y)^{-1}"] \arrow[dd, shift right, "\underline{i}_{\mathbf{X}}^*(f'')"'] \arrow[dd, shift left, "\underline{i}_{\mathbf{X}}^*(g'')"] & & \underline{i}_{\mathbf{X}}|^*_{\underline{\partial}^{\boldsymbol{f}}_- \underline{X}} \circ \underline{f}^*(\mathcal{F}_Y) \arrow[dr, "{\begin{array}{l}\underline{f}_-^*(i^2_{\mathbf{Y}}) \circ \\ I_{\underline{f}_-, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y) \circ \\ I_{\underline{i}_{\mathbf{X}}, \underline{f}}(\mathcal{F}_Y)^{-1}\end{array}}"] \arrow[dd, shift right, "\underline{i}_{\mathbf{X}}^*(f^2)"'] \arrow[dd, shift left, "\underline{i}_{\mathbf{X}}^*(g^2)"] \arrow[dll, dotted, "\underline{i}_{\mathbf{X}}^*(\eta)"'] & \\
& \underline{f}_-^*(\mathcal{E}_{\partial Y}) \arrow[rr, "\underline{f}_-^*(\phi_{\partial Y})"'] \arrow[dd, shift right, "f''_-"'] \arrow[dd, shift left, "g''_-"] & & \underline{f}_-^*(\mathcal{F}_{\partial Y}) \arrow[dd, shift right, "f^2_-"'] \arrow[dd, shift left, "g^2_-"] \arrow[dll, dotted, "\eta_-"'] \\
\underline{i}_{\mathbf{X}}|^*_{\underline{\partial}^{\boldsymbol{f}}_- \underline{X}}(\mathcal{E}_X) \arrow[rr, "\underline{i}_{\mathbf{X}}^*(\phi_X)"'] \arrow[dr, "i''_{\mathbf{X}}"'] & & \underline{i}_{\mathbf{X}}|^*_{\underline{\partial}^{\boldsymbol{f}}_- \underline{X}}(\mathcal{F}_X) \arrow[dr, "i^2_{\mathbf{X}}"'] & \\
& \mathcal{E}_{\partial X}|_{\underline{\partial}^{\boldsymbol{f}}_- \underline{X}} \arrow[rr, "\phi_{\partial X}"] & & \mathcal{F}_{\partial X}|_{\underline{\partial}^{\boldsymbol{f}}_- \underline{X}}.
\end{tikzcd}
$$

We have

$$
\begin{aligned}
&(g''_- - f''_-) \circ \underline{f}_-^*(i''_{\mathbf{Y}}) \circ I_{\underline{f}_-, \underline{i}_{\mathbf{Y}}}(\mathcal{E}_Y) \circ I_{\underline{i}_{\mathbf{X}}, \underline{f}}(\mathcal{E}_Y)^{-1} \\
&\quad = i''_{\mathbf{X}} \circ \bigl(\underline{i}_{\mathbf{X}}^*(g'') - \underline{i}_{\mathbf{X}}^*(f'')\bigr) = i''_{\mathbf{X}} \circ \underline{i}_{\mathbf{X}}^*\bigl(\eta \circ \underline{f}^*(\phi_Y)\bigr) \\
&\quad = \eta_- \circ \underline{f}_-^*(i^2_{\mathbf{Y}}) \circ I_{\underline{f}_-, \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y) \circ I_{\underline{i}_{\mathbf{X}}, \underline{f}}(\mathcal{F}_Y)^{-1} \circ \underline{i}_{\mathbf{X}}^*(\underline{f}^*(\phi_Y)) \\
&\quad = \eta_- \circ \underline{f}_-^*(i^2_{\mathbf{Y}}) \circ \underline{f}_-^*(\underline{i}_{\mathbf{Y}}^*(\phi_Y)) \circ I_{\underline{f}_-, \underline{i}_{\mathbf{Y}}}(\mathcal{E}_Y) \circ I_{\underline{i}_{\mathbf{X}}, \underline{f}}(\mathcal{E}_Y)^{-1} \\
&\quad = \eta_- \circ \underline{f}_-^*(\phi_{\partial Y}) \circ \underline{f}_-^*(i''_{\mathbf{Y}}) \circ I_{\underline{f}_-, \underline{i}_{\mathbf{Y}}}(\mathcal{E}_Y) \circ I_{\underline{i}_{\mathbf{X}}, \underline{f}}(\mathcal{E}_Y)^{-1},
\end{aligned}
$$

using $\boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial^{\boldsymbol{f}}_- \mathbf{X}} = \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{f}_-$ and $\boldsymbol{g} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial^{\boldsymbol{f}}_- \mathbf{X}} = \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{g}_-$ in the first step, and (2.25) for η in the second, and (6.42) in the third, and properties of $I_{*,*}(*)$ in the fourth, and (2.22) for $\boldsymbol{i}_{\mathbf{Y}}$ in the fifth. As $\underline{f}_-^*(i''_{\mathbf{Y}}) \circ I_{\underline{f}_-, \underline{i}_{\mathbf{Y}}}(\mathcal{E}_Y) \circ I_{\underline{i}_{\mathbf{X}}, \underline{f}}(\mathcal{E}_Y)^{-1}$ is an isomorphism by Definition 6.1(b) for $\mathbf{Y}$, this proves that $g''_- - f''_- = \eta_- \circ \underline{f}_-^*(\phi_{\partial Y})$, which is half of the condition (2.25) for $\eta_- : \boldsymbol{f}_- \Rightarrow \boldsymbol{g}_-$ to be a 2-morphism in $\mathbf{dSpa}$. The other half of (2.25) follows by a similar argument involving the diagram of sheaves on $\partial_-^{\boldsymbol{f}} X$:

$$
\begin{tikzcd}[column sep=small]
i_{\mathbf{X}}|^{-1}_{\partial_-^{\boldsymbol{f}} X}\circ \underline{f}^{-1}(\mathcal{E}_Y) \arrow[rr, "i_{\mathbf{X}}^{-1}\circ\underline{f}^{-1}(\kappa_Y\circ\jmath_Y)"] \arrow[dr, "f_-^{-1}(i''_{\mathbf{Y}}\otimes(\mathrm{id}\otimes i^\sharp_{\mathbf{Y}}))\circ I_{f_-,i_{\mathbf{Y}}}(\mathcal{E}_Y)\circ I_{i_{\mathbf{X}},f}(\mathcal{E}_Y)^{-1}"] \arrow[dd, shift right, "{i_{\mathbf{X}}^{-1}(f''\circ(\mathrm{id}\otimes f^\sharp))}"'] \arrow[dd, shift left, "{i_{\mathbf{X}}^{-1}(g''\circ(\mathrm{id}\otimes f^\sharp))}"] & & i_{\mathbf{X}}|^{-1}_{\partial_-^{\boldsymbol{f}} X}\circ \underline{f}^{-1}(\mathcal{O}'_Y) \arrow[dr, "{f_-^{-1}(i'_{\mathbf{Y}})\circ I_{f_-,i_{\mathbf{Y}}}(\mathcal{O}'_Y)\circ I_{i_{\mathbf{X}},f}(\mathcal{O}'_Y)^{-1}}"] \arrow[dd, shift right, "\underline{i}_{\mathbf{X}}^{-1}(f')"'] \arrow[dd, shift left, "\underline{i}_{\mathbf{X}}^{-1}(g')"] \arrow[ddll, dotted, "\underline{i}_{\mathbf{X}}^{-1}(\eta\circ\cdots)"] & \\
& f_-^{-1}(\mathcal{E}_{\partial Y}) \arrow[rr, "f_-^{-1}(\kappa_{\partial Y}\circ\jmath_{\partial Y})"'] \arrow[dd, shift right, "f''_-\circ(\mathrm{id}\otimes f^\sharp_-)"'] \arrow[dd, shift left, "g''_-\circ(\mathrm{id}\otimes f^\sharp_-)"] & & f_-^{-1}(\mathcal{O}'_{\partial Y}) \arrow[dd, shift right, "f'_-"'] \arrow[dd, shift left, "g'_-"] \arrow[ddll, dotted, "\eta_-\circ\cdots"] \\
i_{\mathbf{X}}|^{-1}_{\partial_-^{\boldsymbol{f}} X}(\mathcal{E}_X) \arrow[rr, "i_{\mathbf{X}}^{-1}(\kappa_X\circ\jmath_X)"'] \arrow[dr, "i''_{\mathbf{X}}\circ(\mathrm{id}\otimes i^\sharp_{\mathbf{X}})"'] & & i_{\mathbf{X}}|^{-1}_{\partial_-^{\boldsymbol{f}} X}(\mathcal{O}'_X) \arrow[dr, "i'_{\mathbf{X}}"'] & \\
& \mathcal{E}_{\partial X}|_{\partial_-^{\boldsymbol{f}} X} \arrow[rr, "\kappa_{\partial X}\circ\jmath_{\partial X}|_{\partial_-^{\boldsymbol{f}} X}"] & & \mathcal{O}'_{\partial X}|_{\partial_-^{\boldsymbol{f}} X}.
\end{tikzcd}
$$

Therefore $\eta_- : \boldsymbol{f}_- \Rightarrow \boldsymbol{g}_-$ is a 2-morphism in **dSpa**.

To show that η_- is a 2-morphism in $\mathbf{dSpa^c}$, note that $\underline{S}_{\boldsymbol{f}_-} = \underline{S}_{\boldsymbol{g}_-}$ follows from (6.52) for $\boldsymbol{f},\boldsymbol{g}$ and $\underline{S}_{\boldsymbol{f}} = \underline{S}_{\boldsymbol{g}}$ as $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism. To verify equation (6.9) for η_-, consider the diagram in $\mathrm{qcoh}(\underline{S}_{\boldsymbol{f}_-})$:

$$
\begin{tikzcd}
\underline{u}^*_{\boldsymbol{f}_-}\circ\underline{i}^*_{\partial\mathbf{Y}}(\mathcal{N}_{\mathbf{Y}}) \arrow[r, "\underline{u}^*_{\boldsymbol{f}_-}\circ\underline{i}^*_{\partial\mathbf{Y}}(\nu_{\mathbf{Y}})"'] \arrow[rr, bend left, "0"] \arrow[d, "\cong"', "\underline{u}^*_{\boldsymbol{f}_-}(\chi_{\partial\mathbf{Y}})"] & \underline{u}^*_{\boldsymbol{f}_-}\circ\underline{i}^*_{\partial\mathbf{Y}}\circ\underline{i}^*_{\mathbf{Y}}(\mathcal{F}_Y) \arrow[r, "{\substack{(\underline{i}_{\mathbf{X}}\circ\underline{i}_{\partial\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}_-})^*(\eta)\circ I_{\underline{i}_{\mathbf{X}}\circ\underline{i}_{\partial\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}_-},\underline{f}}(\mathcal{F}_Y)\circ \\ I_{\underline{u}_{\boldsymbol{f}_-},\underline{i}_{\mathbf{Y}}\circ\underline{i}_{\partial\mathbf{Y}}}(\mathcal{F}_Y)^{-1}\circ\underline{u}^*_{\boldsymbol{f}_-}(I_{\underline{i}_{\partial\mathbf{Y}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1})}}"'] \arrow[d, "\underline{u}^*_{\boldsymbol{f}_-}\circ\underline{i}^*_{\partial\mathbf{Y}}(i^2_{\mathbf{Y}})"'] & (\underline{i}_{\mathbf{X}}\circ\underline{i}_{\partial\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}_-})^*(\mathcal{E}_X) \arrow[d, "(\underline{i}_{\partial\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}_-})^*(i''_{\mathbf{Y}})\circ I_{\underline{i}_{\partial\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}_-},\underline{i}_{\mathbf{X}}}(\mathcal{E}_X)"'] \\
\underline{u}^*_{\boldsymbol{f}_-}(\mathcal{N}_{\partial\mathbf{Y}}) \arrow[r, "\underline{u}^*_{\boldsymbol{f}_-}(\nu_{\partial\mathbf{Y}})"] \arrow[rr, bend right, dotted, "0"'] & \underline{u}^*_{\boldsymbol{f}_-}\circ\underline{i}^*_{\partial\mathbf{Y}}(\mathcal{F}_{\partial Y}) \arrow[r, "{\substack{(\underline{i}_{\partial\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}_-})^*(\eta_-)\circ I_{\underline{i}_{\partial\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}_-},\underline{f}_-}(\mathcal{F}_{\partial Y}) \\ \circ I_{\underline{u}_{\boldsymbol{f}_-},\underline{i}_{\partial\mathbf{Y}}}(\mathcal{F}_{\partial Y})^{-1}}}"] & (\underline{i}_{\partial\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}_-})^*(\mathcal{E}_{\partial X}).
\end{tikzcd}
$$

The composition of the top line is zero by (6.9) for η. The left hand square is $\underline{u}^*_{\boldsymbol{f}_-}$ applied to the left hand square of (6.40) for $\mathbf{Y}$, and so commutes, where $\chi_{\partial\mathbf{Y}}$ is an isomorphism. The right hand square is a pullback of (6.42), and so commutes. It follows that the composition of the bottom line is zero, which proves (6.9) for η_-. A similar proof shows (6.10) holds for η_-. Thus $\eta_- : \boldsymbol{f}_- \Rightarrow \boldsymbol{g}_-$ is a 2-morphism in $\mathbf{dSpa^c}$. The equivalence of (6.42) and (6.43) follows from Definition 2.14. This completes the proof of Theorem 6.12. $\square$

Here are some further easy properties of simple, semisimple and flat 1-morphisms. We leave the proof as an exercise.

Proposition 6.13. **(a)** *Suppose $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ and $\boldsymbol{g} : \mathbf{Y} \to \mathbf{Z}$ are 1-morphisms in $\mathbf{dSpa^c}$. If $\boldsymbol{f},\boldsymbol{g}$ are simple, or semisimple, or flat, then $\boldsymbol{g}\circ\boldsymbol{f} : \mathbf{X} \to \mathbf{Z}$ is simple, or semisimple, or flat, respectively. Suppose $\boldsymbol{f},\boldsymbol{g}$ are semisimple. Then $\partial_-^{\boldsymbol{g}\circ\boldsymbol{f}}\mathbf{X} \subseteq \partial_-^{\boldsymbol{f}}\mathbf{X}$, and $\boldsymbol{f}_-(\partial_-^{\boldsymbol{g}\circ\boldsymbol{f}}\mathbf{X}) \subseteq \partial_-^{\boldsymbol{g}}\mathbf{Y}$, and $(\boldsymbol{g}\circ\boldsymbol{f})_- = \boldsymbol{g}_-\circ\boldsymbol{f}_-|_{\partial_-^{\boldsymbol{g}\circ\boldsymbol{f}}\mathbf{X}} : \partial_-^{\boldsymbol{g}\circ\boldsymbol{f}}\mathbf{X} \to \partial\mathbf{Z}$. If $\boldsymbol{f},\boldsymbol{g}$ are simple then $(\boldsymbol{g}\circ\boldsymbol{f})_- = \boldsymbol{g}_-\circ\boldsymbol{f}_- : \partial\mathbf{X} \to \partial\mathbf{Z}$.*

(b) *If* $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ *is a* 1*-morphism and* $\partial^2\mathbf{Y}=\boldsymbol{\emptyset}$*, then* $\boldsymbol{f}$ *is semisimple. If* $\partial\mathbf{Y}=\boldsymbol{\emptyset}$*, then* $\boldsymbol{f}$ *is flat.*

(c) *Identities* $\mathbf{id_X}:\mathbf{X}\to\mathbf{X}$ *in* $\mathbf{dSpa^c}$ *are simple and flat, and* $(\mathbf{id_X})_-=\mathbf{id}_{\partial\mathbf{X}}$.

(d) *If* $\mathbf{X}\in\mathbf{dSpa^c}$ *and* $\partial\mathbf{X}\neq\emptyset$ *then* $\boldsymbol{i}_{\mathbf{X}}:\partial\mathbf{X}\to\mathbf{X}$ *is simple, but not flat.*

(e) *In Theorem* 6.12(c)*, the map* $\eta\mapsto\eta_-$ *is compatible with vertical and horizontal composition of* 2*-morphisms and identity* 2*-morphisms in the obvious ways, so that* $(\zeta\odot\eta)_-=\zeta_-\odot\eta_-$*,* $(\zeta*\eta)_-=\zeta_-*\eta_-$*, and* $(\mathrm{id}_{\boldsymbol{f}})_-=\mathrm{id}_{\boldsymbol{f}_-}$.

Theorem 6.12(b),(c) and Proposition 6.13(a),(c),(e) imply:

Corollary 6.14. *Write* $\mathbf{dSpa^c_{si}}$ *for the* 2*-subcategory of* $\mathbf{dSpa^c}$ *with arbitrary objects and* 2*-morphisms, but only simple* 1*-morphisms. Then there is a strict* 2*-functor* $\partial:\mathbf{dSpa^c_{si}}\to\mathbf{dSpa^c_{si}}$ *mapping* $\mathbf{X}\mapsto\partial\mathbf{X}$ *on objects,* $\boldsymbol{f}\mapsto\boldsymbol{f}_-$ *on (simple)* 1*-morphisms, and* $\eta\mapsto\eta_-$ *on* 2*-morphisms.*

In §6.4 and §6.8 we will prove the following properties of simple, semisimple and flat 1-morphisms $\boldsymbol{f}$ and the induced 1-morphisms $\boldsymbol{f}_-$:

- The functor $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}:\mathbf{Man^c}\to\mathbf{dSpa^c}$ of §6.4 takes simple, semisimple or flat maps in $\mathbf{Man^c}$ to simple, semisimple or flat 1-morphisms in $\mathbf{dSpa^c}$.
- If $\mathbf{X},\mathbf{Y}\in\mathbf{dSpa^c}$ then the projection $\boldsymbol{\pi}_{\mathbf{X}}:\mathbf{X}\times\mathbf{Y}\to\mathbf{X}$ is semisimple and flat, and simple if $\partial\mathbf{Y}=\boldsymbol{\emptyset}$.
- Suppose $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ is a semisimple and flat 1-morphism in $\mathbf{dSpa^c}$. Then the following diagram is 2-Cartesian in $\mathbf{dSpa^c}$:

$$\begin{tikzcd} \partial_-^{\boldsymbol{f}}\mathbf{X} \arrow[rr, "\boldsymbol{f}_-"] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}}"'] \arrow[drr, phantom, "\Uparrow \mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{f}_-}"] & & \partial\mathbf{Y} \arrow[d, "\boldsymbol{i}_{\mathbf{Y}}"] \\ \mathbf{X} \arrow[rr, "\boldsymbol{f}"] & & \mathbf{Y}. \end{tikzcd}$$

  Hence there is an equivalence $\partial_-^{\boldsymbol{f}}\mathbf{X}\simeq\mathbf{X}\times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\mathbf{Y}}}\partial\mathbf{Y}$. If $\boldsymbol{f}$ is simple then $\partial\mathbf{X}\simeq\mathbf{X}\times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\mathbf{Y}}}\partial\mathbf{Y}$. Since $\boldsymbol{f}_-$ is simple, by induction on k we see that

$$\partial^k\mathbf{X}\simeq\mathbf{X}\times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{i}_{\partial\mathbf{Y}}\circ\cdots\circ\boldsymbol{i}_{\partial^{k-1}\mathbf{Y}}}\partial^k\mathbf{Y}.$$

- Let $\boldsymbol{f}:\mathbf{W}\to\mathbf{Y}$, $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$ be 1-morphisms, and $\boldsymbol{f}\times\boldsymbol{g}:\mathbf{W}\times\mathbf{X}\to\mathbf{Y}\times\mathbf{Z}$ the product 1-morphism. If $\boldsymbol{f},\boldsymbol{g}$ are simple, semisimple, or flat, then $\boldsymbol{f}\times\boldsymbol{g}$ is also simple, semisimple, or flat, respectively.
- Let $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$, $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$ be 1-morphisms, and $(\boldsymbol{f},\boldsymbol{g}):\mathbf{X}\to\mathbf{Y}\times\mathbf{Z}$ the direct product 1-morphism. If $\boldsymbol{f},\boldsymbol{g}$ are flat then $(\boldsymbol{f},\boldsymbol{g})$ is flat. However, $\boldsymbol{f},\boldsymbol{g}$ simple or semisimple do not imply $(\boldsymbol{f},\boldsymbol{g})$ simple or semisimple.

To summarize, we have shown that semisimple 1-morphisms $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ are a class of 1-morphisms in $\mathbf{dSpa^c}$ which have a useful extra property — the existence of lifts $\boldsymbol{f}_-:\partial_-^{\boldsymbol{f}}\mathbf{X}\to\partial\mathbf{Y}$ up to the boundaries of $\mathbf{X},\mathbf{Y}$ — and are also closed under most natural operations on 1-morphisms, with the exception of direct products. We can also require $\boldsymbol{f}$ to be simple, or flat. If $\boldsymbol{f}$ is simple and flat then $\boldsymbol{f}_-$ is a 1-morphism $\partial\mathbf{X}\to\partial\mathbf{Y}$, and $\partial\mathbf{X}\simeq\mathbf{X}\times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\mathbf{Y}}}\partial\mathbf{Y}$, so the boundary of $\mathbf{Y}$ determines the boundary of $\mathbf{X}$.

6.4 Manifolds with corners as d-spaces with corners

We now define a (2-)functor $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \to \mathbf{dSpa^c}$ from manifolds with corners to d-spaces with corners, and show that it is full and faithful. Note that in §2.2 we defined $F_{\mathbf{Man^c}}^{\mathbf{dSpa}} : \mathbf{Man^c} \to \mathbf{dSpa}$, and in §6.1 we defined $F_{\mathbf{dSpa}}^{\mathbf{dSpa^c}} : \mathbf{dSpa} \to \mathbf{dSpa^c}$. Our $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ is *not* the composition $F_{\mathbf{dSpa}}^{\mathbf{dSpa^c}} \circ F_{\mathbf{Man^c}}^{\mathbf{dSpa}}$, which is faithful but not full, as for $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}$ in Corollary B.27.

Definition 6.15. Let X be a manifold with corners. Then the boundary ∂X is a manifold with corners, with a smooth map $i_{\partial X} : \partial X \to X$. We will define a d-space with corners $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$. Set $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(X, \partial X, i_X)$. To define $\omega_{\mathbf{X}}$, consider the exact sequence

$$0 \longrightarrow T(\partial X) \xrightarrow{\mathrm{d}i_X} i_X^*(TX) \xrightarrow{\pi_\nu} \nu \longrightarrow 0$$

of vector bundles over ∂X, where ν is the normal bundle of $i_X(\partial X)$ in X, a line bundle over ∂X, which is canonically oriented by outward-pointing normal vectors. The dual ν^* is also an oriented line bundle on ∂X, in the dual sequence

$$0 \longrightarrow \nu^* \xrightarrow{\pi_\nu^*} i_X^*(T^*X) \xrightarrow{(\mathrm{d}i_X)^*} T^*(\partial X) \longrightarrow 0. \tag{6.53}$$

Now as $\boldsymbol{X} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(X)$ we have $\mathcal{F}_X \cong T^*\underline{X}$, and both are isomorphic to the lift to C^∞-schemes of the vector bundle T^*X on X. Similarly $\mathcal{F}_{\partial X} \cong T^*(\underline{\partial X})$ is the lift to C^∞-schemes of the vector bundle $T^*(\partial X)$ on ∂X, and $i_{\mathbf{X}}^2 : \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X) \to \mathcal{F}_{\partial X}$ is the lift to C^∞-schemes of the vector bundle morphism $(\mathrm{d}i_X)^* : i_X^*(T^*X) \to T^*(\partial X)$ on ∂X. Therefore (6.4) for $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}$ is the lift to C^∞-schemes of (6.53), so $\mathcal{N}_{\mathbf{X}} = \operatorname{Ker} i_{\mathbf{X}}^2$ is the C^∞-scheme lift of ν^*. Define $\omega_{\mathbf{X}}$ to be the orientation on $\mathcal{N}_{\mathbf{X}}$ corresponding to the orientation on ν^* induced by outward-pointing normal vectors.

Define $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$, and $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X) = \mathbf{X}$. We will show in Theorem 6.16 that $\mathbf{X}$ is a d-space with corners. Let $f : X \to Y$ be a smooth map of manifolds with corners, and set $\mathbf{X}, \mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X, Y)$. Write $\boldsymbol{f} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(f) : \boldsymbol{X} \to \boldsymbol{Y}$, as a 1-morphism of d-spaces. We will show in Theorem 6.16 that $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism of d-spaces with corners. Define $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(f) = \boldsymbol{f}$.

The only 2-morphisms in $\mathbf{Man^c}$, regarded as a 2-category, are identity 2-morphisms id_f for smooth $f : X \to Y$. We define $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\mathrm{id}_f) = \mathrm{id}_{\boldsymbol{f}}$.

Define $F_{\mathbf{Man}}^{\mathbf{d\bar{S}pa}} : \mathbf{Man} \to \mathbf{d\bar{S}pa}$ and $F_{\mathbf{Man^b}}^{\mathbf{dSpa^b}} : \mathbf{Man^b} \to \mathbf{dSpa^b}$ to be the restrictions of $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ to the subcategories $\mathbf{Man}, \mathbf{Man^b} \subset \mathbf{Man^c}$.

Write $\mathbf{\bar{M}an}, \mathbf{\bar{M}an^b}, \mathbf{\bar{M}an^c}$ for the full 2-subcategories of objects $\mathbf{X}$ in $\mathbf{dSpa^c}$ equivalent to $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X)$ for some manifold X without boundary, or with boundary, or with corners, respectively. Then $\mathbf{\bar{M}an} \subset \mathbf{d\bar{S}pa}$, $\mathbf{\bar{M}an^b} \subset \mathbf{dSpa^b}$ and $\mathbf{\bar{M}an^c} \subset \mathbf{dSpa^c}$. When we say that a d-space with corners $\mathbf{X}$ *is a manifold*, we mean that $\mathbf{X} \in \mathbf{\bar{M}an^c}$.

The next theorem should be contrasted with the fact that $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}} : \mathbf{Man^c} \to \mathbf{C^\infty Sch}$ is faithful but not full, as in Corollary B.27, and therefore $F_{\mathbf{Man^c}}^{\mathbf{dSpa}} :$

$\mathbf{Man^c} \to \mathbf{dSpa}$ is also faithful but not full. So if we regard manifolds with corners either as C^∞-schemes or as d-spaces, then we get the wrong notion of (1-)morphism, but regarding them as d-spaces with corners gives the right notion. This will be essential for our definition of d-manifolds with corners.

Theorem 6.16. $F_{\mathbf{Man}}^{\mathbf{d\bar{S}pa}} : \mathbf{Man} \to \mathbf{d\bar{S}pa}$, $F_{\mathbf{Man^b}}^{\mathbf{dSpa^b}} : \mathbf{Man^b} \to \mathbf{dSpa^b}$ *and* $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \to \mathbf{dSpa^c}$ *are well-defined, full and faithful strict* 2*-functors.*

Proof. First we show that if X is a manifold with corners then $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}}) = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X)$ in Definition 6.15 is a d-space with corners. We must verify Definition 6.1(a)–(f). For (a), $i_X : \partial X \to X$ is proper by Theorem 5.6(c), so $\underline{i}_{\mathbf{X}} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}} : \underline{\partial X} \to \underline{X}$ is proper. For (b), $\mathcal{E}_X = 0 = \mathcal{E}_{\partial X}$, so $i''_{\mathbf{X}} : \underline{i}_{\mathbf{X}}^*(\mathcal{E}_X) \to \mathcal{E}_{\partial X}$ is trivially an isomorphism.

For (c), let $x' \in \boldsymbol{\partial X}$ with $\boldsymbol{i}_{\mathbf{X}}(x') = x \in \underline{b}$. Then $x' \in \partial X$, so there exists a boundary defining function (V, b) for X at x', in the sense of Definition 5.4. Thus $x \in V \subseteq X$ is open, and $b : V \to [0,\infty)$ is smooth with $\mathrm{d}b|_v \neq 0$ for all $v \in V$, and there exists open $x' \in U \subseteq i_X^{-1}(V) \subseteq \partial X$ with $b \circ i_X|_U = 0$, and $i_X|_U : U \longrightarrow \{v \in V : b(v) = 0\}$ is a homeomorphism. Define $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{b} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(U, V, b)$. Then $x' \in \boldsymbol{U} \subseteq \boldsymbol{i}_{\mathbf{X}}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{\partial X}$ and $x \in \boldsymbol{V} \subseteq \boldsymbol{X}$ are open, and $\boldsymbol{b} : \boldsymbol{V} \to \boldsymbol{[0,\infty)}$ is a 1-morphism, and $\boldsymbol{b} \circ \boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{U}} = \boldsymbol{0} \circ \boldsymbol{\pi}$.

Now $U \cong V \times_{b,[0,\infty),0} *$ is a transverse fibre product in $\mathbf{Man^c}$, so the proof of Theorem 2.42 shows that $\boldsymbol{U} \simeq \boldsymbol{V} \times_{\boldsymbol{b},\boldsymbol{[0,\infty)},\boldsymbol{0}} \boldsymbol{*}$ in $\mathbf{dSpa}$. Hence (6.1) is 2-Cartesian. As $\mathrm{d}b|_v \neq 0$ for all $v \in V$, the vector bundle morphism $(\mathrm{d}b)^* : b^*(T^*[0,\infty)) \to T^*V$ over V embeds the line bundle $b^*(T^*[0,\infty))$ as a vector subbundle of T^*V, and there exists a left inverse $\beta : T^*V \to b^*(T^*[0,\infty))$. But $b^2 : \underline{b}^*(\mathcal{F}_{[0,\infty)}) \to \mathcal{F}_X|_{\underline{V}}$ is the lift to C^∞-schemes of $(\mathrm{d}b)^* : b^*(T^*[0,\infty)) \to T^*V$. Hence b^2 also has a left inverse, so Definition 6.1(c) holds for $\mathbf{X}$.

Parts (d),(e) are now immediate. Definition 6.1(f) holds as if $x \in X$ with $i_X^{-1}(x) = \{x'_1, \ldots, x'_k\}$ then (6.8) is identified with

$$\textstyle\bigoplus_{i=1}^k \pi_\nu^*|_{x'_i} : \bigoplus_{i=1}^k \nu^*|_{x'_i} \longrightarrow T_x^* X. \tag{6.54}$$

Locally we may identify $X \cong \mathbb{R}^n_k$, where $\mathbb{R}^n_k$ has coordinates $(y_1, \ldots, y_n)$, such that x'_i corresponds to the local boundary component $\{y_i = 0\}$ at $(0, \ldots, 0)$. Then (6.54) becomes the map $\bigoplus_{i=1}^k \mathbb{R} \to \mathbb{R}^n$ taking $(z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0)$, which is clearly injective. Hence $\mathbf{X}$ is a d-space with corners. This also shows that $(\boldsymbol{V}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{X}$ at x', as in Lemma 6.17(a) below.

Next let $f : X \to Y$ be a smooth map of manifolds with corners, and set $\mathbf{X}, \mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X, Y)$ and $\boldsymbol{f} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(f) : \boldsymbol{X} \to \boldsymbol{Y}$. Suppose $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$ and $y' \in \boldsymbol{\partial Y}$ with $\boldsymbol{i}_{\mathbf{Y}}(y') = y \in \boldsymbol{Y}$. Then $y' \in \partial Y$, so we can choose a boundary defining function (V, b) for Y at y'. Set $(\boldsymbol{V}, \boldsymbol{b}) = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(V, b)$. Then from above $(\boldsymbol{V}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{Y}$ at y'. By Definition 5.5, either (i) there exists open $x \in \tilde{V} \subseteq f^{-1}(V) \subseteq X$ such that $(\tilde{V}, b \circ f|_{\tilde{V}})$ is a boundary defining function for X at x', for some unique $x' \in \partial X$ with $i_X(x') = x$, or (ii) there exists an open $x \in W \subseteq f^{-1}(V) \subseteq X$ with $b \circ f|_W = 0$.

Applying $F_{\mathbf{Man^c}}^{\mathbf{dSpa}}$ we see that these (i),(ii) imply Definition 6.2(i),(ii) for $(\boldsymbol{V}, \boldsymbol{b})$. So Definition 6.2 holds for $\boldsymbol{f}$, at least for the particular choice of boundary defining function $(\boldsymbol{V}, \boldsymbol{b}) = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(V, b)$ for $\mathbf{Y}$ at y'. But as in Definition 6.2, if the condition holds for one choice of $(\boldsymbol{V}, \boldsymbol{b})$ for $\mathbf{Y}$ at y', then it holds for any choice. Hence $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism in $\mathbf{dSpa^c}$. This shows that $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \to \mathbf{dSpa^c}$ is well-defined. Since 1- and 2-morphisms in $\mathbf{dSpa^c}$ are special examples of 1- and 2-morphisms in $\mathbf{dSpa}$, and $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ acts on 1- and 2-morphisms f, id_f as $F_{\mathbf{Man^c}}^{\mathbf{dSpa}}$ does, the proof that $F_{\mathbf{Man^c}}^{\mathbf{dSpa}} : \mathbf{Man^c} \to \mathbf{dSpa}$ is a strict 2-functor in Theorem 2.15(c) now shows that $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \to \mathbf{dSpa^c}$ is a strict 2-functor. Hence $F_{\mathbf{Man}}^{\mathbf{d\bar{S}pa}} : \mathbf{Man} \to \mathbf{d\bar{S}pa}$ and $F_{\mathbf{Man^b}}^{\mathbf{dSpa^b}} : \mathbf{Man^b} \to \mathbf{dSpa^b}$ are also strict 2-functors, as they are restrictions of $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$.

To prove $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ is full and faithful, first note that if $\mathbf{X}, \mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X, Y)$, $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ are 1-morphisms and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism, then as η is a morphism $\underline{f}^*(\mathcal{F}_Y) \to \mathcal{E}_X$ and $\mathcal{E}_X = 0$, the only possibility is $\eta = 0$, so that $\boldsymbol{f} = \boldsymbol{g}$ and $\eta = \mathrm{id}_{\boldsymbol{f}}$. So all relevant 2-morphisms are identities. Under these circumstances, $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ is *full* if it is surjective on 1-morphisms, and *faithful* if it is injective on 1-morphisms. Since f is part of the data of $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(f)$, $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ is clearly injective on 1-morphisms, and so faithful.

To show $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ is full, suppose $\mathbf{X}, \mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X, Y)$ and $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism in $\mathbf{dSpa^c}$, where $\boldsymbol{f} = (\underline{f}, f', f'')$. Then $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes. So by Proposition B.26, there is a unique weakly smooth map $f : X \to Y$ corresponding to $\underline{f}$. Using Definition 6.2 and the fact that boundary defining functions $(\boldsymbol{V}, \boldsymbol{b})$ for $\mathbf{X}, \mathbf{Y}$ in Definition 6.1 correspond to boundary defining functions (V, b) for X, Y in Definition 5.4, we see that f is smooth, so $f : X \to Y$ is a morphism in $\mathbf{Man^c}$. The morphism $f' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$ of sheaves of C^∞-rings on X satisfies $f^\sharp \circ f^{-1}(\imath_Y) = \imath_X \circ f'$. As $\imath_X = \mathrm{id}_{\mathcal{O}_X}$ and $\imath_Y = \mathrm{id}_{\mathcal{O}_Y}$, this forces $f' = f^\sharp$. Also f'' is a morphism $\underline{f}^*(\mathcal{E}_Y) \to \mathcal{E}_X$ and $\mathcal{E}_X = \mathcal{E}_Y = 0$, so $f'' = 0$. Thus $\boldsymbol{f} = (\underline{f}, f^\sharp, 0) = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(f) = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(f)$, and $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ is full. Since $F_{\mathbf{Man}}^{\mathbf{d\bar{S}pa}}, F_{\mathbf{Man^b}}^{\mathbf{dSpa^b}}$ are restrictions of $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ to full subcategories, they are also full and faithful. This completes the proof of Theorem 6.16. $\square$

The proof above also implies:

Lemma 6.17. (a) *If X is a manifold with corners and (V, b) is a boundary defining function for X at $x' \in \partial X$, in the sense of Definition* 5.4, *then $(\boldsymbol{V}, \boldsymbol{b}) = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(V, b)$ is a boundary defining function for $\mathbf{X} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X)$ at $x' \in \boldsymbol{\partial X}$, in the sense of Definition* 6.1.

(b) *Suppose $f : X \to Y$ is a smooth map of manifolds with corners, and let $\boldsymbol{f} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(f)$. Then the C^∞-schemes $\underline{S}_{\boldsymbol{f}}, \underline{T}_{\boldsymbol{f}}$ for $\boldsymbol{f}$ from* §6.1 *have underlying sets S_f, T_f for f from* §5.3.

From Definitions 5.9 and 6.11 and Lemma 6.17(b), we deduce:

Corollary 6.18. *Let $f : X \to Y$ be a smooth map of manifolds with corners, and $\boldsymbol{f} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(f)$. Then $\boldsymbol{f}$ is simple, semisimple, or flat if and only if f is* *simple, semisimple, or flat, respectively.*

We show $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ takes boundaries in $\mathbf{Man^c}$ to boundaries in $\mathbf{dSpa^c}$.

Proposition 6.19. *If X is a manifold with corners and $\mathbf{X}=F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X)$ then $\mathbf{id}_{\boldsymbol{\partial X}}$ is a 1-isomorphism $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\partial X)\rightarrow\partial\mathbf{X}$. Thus, $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ intertwines the constructions of boundaries in $\mathbf{Man^c},\mathbf{dSpa^c}$, up to canonical 1-isomorphism.*

Proof. Write $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}},\omega_{\mathbf{X}})$, $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\partial X)=(\boldsymbol{\partial X},\boldsymbol{\partial^2 X},\boldsymbol{i}_{\boldsymbol{\partial}\mathbf{X}},\omega_{\boldsymbol{\partial}\mathbf{X}})$ and $\partial\mathbf{X}=(\boldsymbol{\partial X},\tilde{\boldsymbol{\partial}}^{\mathbf{2}}\boldsymbol{X},\tilde{\boldsymbol{\imath}}_{\boldsymbol{\partial}\mathbf{X}},\tilde\omega_{\boldsymbol{\partial}\mathbf{X}})$, where $\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{\partial^2 X},\boldsymbol{i}_{\mathbf{X}},\boldsymbol{i}_{\boldsymbol{\partial}\mathbf{X}}=F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(X,\partial X,\partial^2X,i_X,i_{\partial X})$, and $\tilde{\boldsymbol{\partial}}^{\mathbf{2}}\boldsymbol{X},\tilde{\boldsymbol{\imath}}_{\boldsymbol{\partial}\mathbf{X}},\tilde\omega_{\boldsymbol{\partial}\mathbf{X}}$ are constructed in Definition 6.10 from $\boldsymbol{X}$, $\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}},\omega_{\mathbf{X}}$. Comparing the constructions of $\boldsymbol{\partial^2 X}$ and $\tilde{\boldsymbol{\partial}}^{\mathbf{2}}\boldsymbol{X}$, it is easy to show there is a unique 1-isomorphism $\boldsymbol{j}:\boldsymbol{\partial^2 X}\rightarrow\tilde{\boldsymbol{\partial}}^{\mathbf{2}}\boldsymbol{X}$ with $\boldsymbol{i}_{\boldsymbol{\partial}\mathbf{X}}=\tilde{\boldsymbol{\imath}}_{\boldsymbol{\partial}\mathbf{X}}\circ\boldsymbol{j}$. It then follows that $(\boldsymbol{V},\boldsymbol{b})$ is a boundary defining function for $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\partial X)$ at $x''\in\boldsymbol{\partial^2 X}$ if and only if $(\boldsymbol{V},\boldsymbol{b})$ is a boundary defining function for $\partial\mathbf{X}$ at $\boldsymbol{j}(x'')\in\tilde{\boldsymbol{\partial}}^{\mathbf{2}}\boldsymbol{X}$. We deduce that $\mathbf{id}_{\boldsymbol{\partial X}}:F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\partial X)\rightarrow\partial\mathbf{X}$ and $\mathbf{id}_{\boldsymbol{\partial X}}:\partial\mathbf{X}\rightarrow F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\partial X)$ are 1-morphisms in $\mathbf{dSpa^c}$, so $\mathbf{id}_{\boldsymbol{\partial X}}:F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\partial X)\rightarrow\partial\mathbf{X}$ is a 1-isomorphism. □

6.5 Equivalences and étale 1-morphisms in $\mathbf{dSpa^c}$

The next two propositions, analogues of Propositions 2.20 and 2.21, characterize when a 1-morphism in $\mathbf{dSpa^c}$ is an equivalence. The first also shows that if $\mathbf{X}$ is equivalent to $\mathbf{Y}$ in $\mathbf{dSpa^c}$, then $\partial\mathbf{X}$ is equivalent to $\partial\mathbf{Y}$, as one would hope.

Proposition 6.20. *Suppose $\boldsymbol{f}:\mathbf{X}\rightarrow\mathbf{Y}$ is an equivalence in $\mathbf{dSpa^c}$. Then $\boldsymbol{f}$ is simple and flat, and $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ is an equivalence in $\mathbf{dSpa}$. Also $\boldsymbol{f}_-:\partial\mathbf{X}\rightarrow\partial\mathbf{Y}$ is an equivalence in $\mathbf{dSpa^c}$.*

Proof. As $\boldsymbol{f}$ is an equivalence, there exist a 1-morphism $\boldsymbol{g}:\mathbf{Y}\rightarrow\mathbf{X}$ and 2-morphisms $\eta:\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\mathbf{id_X}$ and $\zeta:\boldsymbol{f}\circ\boldsymbol{g}\Rightarrow\mathbf{id_Y}$ in $\mathbf{dSpa^c}$. Regarding $\boldsymbol{f},\boldsymbol{g},\mathbf{id_X},\mathbf{id_Y},\eta,\zeta$ as 1- and 2-morphisms in $\mathbf{dSpa}$, we see that $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ is an equivalence in $\mathbf{dSpa}$. From Definition 6.3 for η,ζ we have $\underline{S}_{\boldsymbol{g}\circ\boldsymbol{f}}=\underline{S}_{\mathbf{id_X}}$ and $\underline{S}_{\boldsymbol{f}\circ\boldsymbol{g}}=\underline{S}_{\mathbf{id_Y}}$. The underlying set $S_{\boldsymbol{g}\circ\boldsymbol{f}}$ is given by (6.18), and $S_{\mathbf{id_X}}=\{(x',x'):x'\in\partial X\}$. Hence we have

$$\begin{aligned}\bigl\{(x',x'')\in\partial X\times_X\partial X:\exists y'\in\partial Y,\ (x',y')\in S_{\boldsymbol{f}},\ (y',x'')\in S_{\boldsymbol{g}}\bigr\}&\\ =\bigl\{(x',x'):x'\in\partial X\bigr\},&\end{aligned}\tag{6.55}$$

$$\begin{aligned}\bigl\{(y',y'')\in\partial Y\times_Y\partial Y:\exists x'\in\partial X,\ (y',x')\in S_{\boldsymbol{g}},\ (x',y'')\in S_{\boldsymbol{f}}\bigr\}&\\ =\bigl\{(y',y'):y'\in\partial Y\bigr\}.&\end{aligned}\tag{6.56}$$

Suppose $x'\in\partial X$. By (6.55) there exists $y'\in\partial Y$ with $(x',y')\in S_{\boldsymbol{f}}$ and $(y',x')\in S_{\boldsymbol{g}}$. Then $\underline{s}_{\boldsymbol{f}}(x',y')=x'$, so $\underline{s}_{\boldsymbol{f}}$ is surjective. Suppose also that $(x',y'')\in S_{\boldsymbol{f}}$. Then (y',y'') lies in the l.h.s. of (6.56), so $y'=y''$. Thus $\underline{s}_{\boldsymbol{f}}$ is injective, so $\underline{s}_{\boldsymbol{f}}:\underline{S}_{\boldsymbol{f}}\rightarrow\underline{\partial X}$ is a bijection, and $\boldsymbol{f}$ is simple. Suppose $(x,y')\in X\times_Y\partial Y$, so that $\underline{f}(x)=y=\underline{i}_{\mathbf{Y}}(y')$. Then by (6.56) there exists $x'\in\partial X$ with $(x',y')\in S_{\boldsymbol{f}}$ and $(y',x')\in S_{\boldsymbol{g}}$, so $\underline{i}_{\mathbf{X}}(x')=\underline{g}(y)=x$ as $\underline{g}=\underline{f}^{-1}$.

Hence $\underline{j}_{\boldsymbol{f}}(x',y')=(x,y')$, so $\underline{j}_{\boldsymbol{f}}:\underline{S}_{\boldsymbol{f}}\to\underline{X}\times_{\underline{Y}}\underline{\partial Y}$ is surjective, and $\underline{T}_{\boldsymbol{f}}=\emptyset$. Therefore $\boldsymbol{f}$ is flat.

Theorem 6.12(b) now gives 1-morphisms $\boldsymbol{f}_-:\partial\mathbf{X}\to\partial\mathbf{Y}$ and $\boldsymbol{g}_-:\partial\mathbf{Y}\to\partial\mathbf{X}$, and Proposition 6.13(a),(c) show that $(\boldsymbol{g}\circ\boldsymbol{f})_-=\boldsymbol{g}_-\circ\boldsymbol{f}_-$, $(\boldsymbol{f}\circ\boldsymbol{g})_-=\boldsymbol{f}_-\circ\boldsymbol{g}_-$, $(\mathbf{id_X})_-=\mathbf{id}_{\partial\mathbf{X}}$ and $(\mathbf{id_Y})_-=\mathbf{id}_{\partial\mathbf{Y}}$. Hence Theorem 6.12(c) gives 2-morphisms $\eta_-:\boldsymbol{g}_-\circ\boldsymbol{f}_-\Rightarrow\mathbf{id}_{\partial\mathbf{X}}$, and $\zeta_-:\boldsymbol{f}_-\circ\boldsymbol{g}_-\Rightarrow\mathbf{id}_{\partial\mathbf{Y}}$ in $\mathbf{dSpa^c}$. These $\boldsymbol{g}_-,\eta_-,\zeta_-$ imply that $\boldsymbol{f}_-$ is an equivalence. □

In the next proposition, note that sufficient conditions for $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ to be an equivalence in $\mathbf{dSpa}$ are given in Proposition 2.21.

Proposition 6.21. *Let $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ be a simple, flat 1-morphism in $\mathbf{dSpa^c}$ with $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ an equivalence in $\mathbf{dSpa}$. Then $\boldsymbol{f}$ is an equivalence in $\mathbf{dSpa^c}$.*

Proof. Since $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is an equivalence in $\mathbf{dSpa}$, there exist a 1-morphism $\boldsymbol{g}:\boldsymbol{Y}\to\boldsymbol{X}$ in $\mathbf{dSpa}$ and 2-morphisms $\eta:\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$ and $\zeta:\boldsymbol{f}\circ\boldsymbol{g}\Rightarrow\mathbf{id}_{\boldsymbol{Y}}$ in $\mathbf{dSpa}$. By Proposition A.6 we may choose $\boldsymbol{g},\eta,\zeta$ to satisfy $\mathrm{id}_{\boldsymbol{f}}*\eta=\zeta*\mathrm{id}_{\boldsymbol{f}}$ and $\mathrm{id}_{\boldsymbol{g}}*\zeta=\eta*\mathrm{id}_{\boldsymbol{g}}$. The issue is that $\boldsymbol{g}$ need not be a 1-morphism $\mathbf{Y}\to\mathbf{X}$ in $\mathbf{dSpa^c}$, and η,ζ need not be 2-morphisms in $\mathbf{dSpa^c}$.

Our solution will run as follows. Suppose $\theta:\underline{g}^*(\mathcal{F}_X)\to\mathcal{E}_Y$ is a morphism in $\mathrm{qcoh}(\underline{Y})$. Then Proposition 2.17 shows that there is a unique 1-morphism $\tilde{\boldsymbol{g}}:\boldsymbol{Y}\to\boldsymbol{X}$ in $\mathbf{dSpa}$ such that $\theta:\boldsymbol{g}\Rightarrow\tilde{\boldsymbol{g}}$ is a 2-morphism in $\mathbf{dSpa}$. Define 2-morphisms $\tilde{\eta}:\tilde{\boldsymbol{g}}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$ and $\tilde{\zeta}:\boldsymbol{f}\circ\tilde{\boldsymbol{g}}\Rightarrow\mathbf{id}_{\boldsymbol{Y}}$ in $\mathbf{dSpa}$ by $\tilde{\eta}=\eta\odot((-\theta)*\mathrm{id}_{\boldsymbol{f}})$ and $\tilde{\zeta}=\zeta\odot(\mathrm{id}_{\boldsymbol{f}}*(-\theta))$. Then $\mathrm{id}_{\boldsymbol{f}}*\eta=\zeta*\mathrm{id}_{\boldsymbol{f}}$ and $\mathrm{id}_{\boldsymbol{g}}*\zeta=\eta*\mathrm{id}_{\boldsymbol{g}}$ imply that $\mathrm{id}_{\boldsymbol{f}}*\tilde{\eta}=\tilde{\zeta}*\mathrm{id}_{\boldsymbol{f}}$ and $\mathrm{id}_{\tilde{\boldsymbol{g}}}*\tilde{\zeta}=\tilde{\eta}*\mathrm{id}_{\tilde{\boldsymbol{g}}}$. We will show that we can choose θ such that $\tilde{\boldsymbol{g}}:\mathbf{Y}\to\mathbf{X}$ is a 1-morphism in $\mathbf{dSpa^c}$, and $\tilde{\eta},\tilde{\zeta}$ are 2-morphisms in $\mathbf{dSpa^c}$. Therefore $\boldsymbol{f}$ is an equivalence in $\mathbf{dSpa^c}$. From (6.55)–(6.56) we can also show that

$$S_{\tilde{\boldsymbol{g}}}=\bigl\{(y',x')\in\partial Y\times_X\partial X:(x',y')\in S_{\boldsymbol{f}}\bigr\}. \tag{6.57}$$

Suppose for the moment that $\tilde{\boldsymbol{g}}$ is a 1-morphism in $\mathbf{dSpa^c}$ and (6.57) holds. As for (6.55)–(6.56), it then follows that $\underline{S}_{\tilde{\boldsymbol{g}}\circ\boldsymbol{f}}=\underline{S}_{\mathbf{id_X}}$ and $\underline{S}_{\boldsymbol{f}\circ\tilde{\boldsymbol{g}}}=\underline{S}_{\mathbf{id_Y}}$. Since $\mathbf{id_X}$ is flat, $\underline{T}_{\mathbf{id_X}}=\emptyset$, so equation (6.10) for $\tilde{\eta}$ is trivial. Hence $\tilde{\eta}$ is a 2-morphism in $\mathbf{dSpa^c}$ if and only if (6.9) holds for $\tilde{\eta}$ in $\mathrm{qcoh}(\underline{S}_{\mathbf{id_X}})$. As $\mathbf{id}_{\boldsymbol{X}}$ is simple, $\underline{s}_{\mathbf{id_X}}:\underline{S}_{\mathbf{id_X}}\to\underline{\partial X}$ is an isomorphism, so we can show (6.9) for $\tilde{\eta}$ is equivalent to the equation in $\mathrm{qcoh}(\underline{\partial X})$:

$$\underline{i}_{\mathbf{X}}^*(\tilde{\eta})\circ I_{\underline{i}_{\mathbf{X}},\underline{\mathrm{id}}_X}(\mathcal{F}_X)\circ\nu_{\mathbf{X}}=0. \tag{6.58}$$

Similarly, $\tilde{\zeta}$ is a 2-morphism in $\mathbf{dSpa^c}$ if and only if the following equivalent of (6.9) for $\tilde{\zeta}$ holds in $\mathrm{qcoh}(\underline{\partial Y})$:

$$\underline{i}_{\mathbf{Y}}^*(\tilde{\zeta})\circ I_{\underline{i}_{\mathbf{Y}},\underline{\mathrm{id}}_Y}(\mathcal{F}_Y)\circ\nu_{\mathbf{Y}}=0. \tag{6.59}$$

Suppose (6.59) holds. Then in $\mathrm{qcoh}(\underline{S}_{\boldsymbol{f}})$ we have

$$
\begin{aligned}
0 &= (\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}})^*(f'')\circ I_{\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}},\underline{f}}(\mathcal{E}_Y)\circ I_{\underline{u}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{E}_Y)^{-1}\circ\underline{u}_{\boldsymbol{f}}^*\big[\underline{i}_{\mathbf{Y}}^*(\tilde\zeta)\circ I_{\underline{i}_{\mathbf{Y}},\underline{\mathrm{id}}_Y}(\mathcal{F}_Y)\circ\nu_{\mathbf{Y}}\big]\\
&= (\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}})^*\big(f''\circ\underline{f}^*(\tilde\zeta)\circ I_{\underline{f},\underline{\mathrm{id}}_Y}(\mathcal{F}_Y)\big)\circ I_{\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1}\circ\underline{u}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})\\
&= (\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}})^*\big(\tilde\eta\circ\underline{\mathrm{id}}_X^*(f^2)\circ I_{\underline{\mathrm{id}}_X,\underline{f}}(\mathcal{F}_Y)\big)\circ I_{\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1}\circ\underline{u}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})\\
&= I_{\underline{s}_{\boldsymbol{f}},\underline{i}_{\mathbf{X}}}(\mathcal{E}_X)\circ\underline{s}_{\boldsymbol{f}}^*(\underline{i}_{\mathbf{X}}^*(\tilde\eta)\circ I_{\underline{i}_{\mathbf{X}},\underline{\mathrm{id}}_X}(\mathcal{F}_X))\circ I_{\underline{s}_{\boldsymbol{f}},\underline{i}_{\mathbf{X}}}(\mathcal{F}_X)\circ(\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}})^*(f^2)\\
&\qquad\circ I_{\underline{i}_{\mathbf{X}}\circ\underline{s}_{\boldsymbol{f}},\underline{f}}(\mathcal{F}_Y)\circ I_{\underline{u}_{\boldsymbol{f}},\underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1}\circ\underline{u}_{\boldsymbol{f}}^*(\nu_{\mathbf{Y}})\\
&= I_{\underline{s}_{\boldsymbol{f}},\underline{i}_{\mathbf{X}}}(\mathcal{E}_X)\circ\underline{s}_{\boldsymbol{f}}^*\big[\underline{i}_{\mathbf{X}}^*(\tilde\eta)\circ I_{\underline{i}_{\mathbf{X}},\underline{\mathrm{id}}_X}(\mathcal{F}_X)\circ\nu_{\mathbf{X}}\big]\circ\lambda_{\boldsymbol{f}},
\end{aligned}
$$

using (6.59) in the first step, properties of $I_{*,*}(*)$ in the second and fourth, $\mathrm{id}_{\boldsymbol{f}}*\tilde\eta=\tilde\zeta*\mathrm{id}_{\boldsymbol{f}}$ in the third, and (6.15) in the fifth, where $\lambda_{\boldsymbol{f}}$ is the isomorphism from Proposition 6.7(d). Since $I_{\underline{s}_{\boldsymbol{f}},\underline{i}_{\mathbf{X}}}(\mathcal{E}_X),\lambda_{\boldsymbol{f}}$ are isomorphisms in $\underline{S}_{\boldsymbol{f}}$ and $\underline{s}_{\boldsymbol{f}}$ an isomorphism in $\mathbf{C^\infty Sch}$, this shows that (6.59) implies (6.58).

Next we show that (6.59) implies that $\tilde{\boldsymbol{g}}$ is a 1-morphism in $\mathbf{dSpa^c}$, and (6.57) holds. Suppose (6.59) holds, let $(x',y')\in\underline{S}_{\boldsymbol{f}}$ with $\underline{i}_{\mathbf{X}}(x')=x$ and $\underline{f}(x)=y=\underline{i}_{\mathbf{X}}(y')$, and choose a boundary defining function $(\boldsymbol{V},\boldsymbol{b})$ for $\mathbf{Y}$ at y'. Then there exists an open $x\in\tilde{\boldsymbol{V}}\subseteq\boldsymbol{f}^{-1}(\boldsymbol{V})$ such that $(\tilde{\boldsymbol{V}},\boldsymbol{b}\circ\boldsymbol{f}|_{\tilde{\boldsymbol{V}}})$ is a boundary defining function for $\mathbf{X}$ at x'.

We claim that there exists open $x\in\hat{\boldsymbol{V}}\subseteq\boldsymbol{g}^{-1}(\tilde{\boldsymbol{V}})\subseteq\boldsymbol{V}$ such that $\big(\hat{\boldsymbol{V}},\boldsymbol{b}\circ\boldsymbol{f}\circ\tilde{\boldsymbol{g}}|_{\hat{\boldsymbol{V}}}\big)$ is a boundary defining function for $\mathbf{Y}$ at y'. To see this, let $y'\in\boldsymbol{U}\subseteq\boldsymbol{i}_{\mathbf{Y}}^{-1}(\boldsymbol{V})\subseteq\boldsymbol{\partial Y}$ be as in Definition 6.1(c) for $(\boldsymbol{V},\boldsymbol{b})$, so that (6.1) is 2-Cartesian for $\boldsymbol{U},\boldsymbol{V},\boldsymbol{b}$. Choose some open $x\in\hat{\boldsymbol{V}}\subseteq\boldsymbol{g}^{-1}(\tilde{\boldsymbol{V}})$, and let $\hat{\boldsymbol{U}}=\boldsymbol{U}\cap\boldsymbol{i}_{\mathbf{Y}}^{-1}(\hat{\boldsymbol{V}})$. Then the analogue of (6.1) for $\hat{\boldsymbol{U}},\hat{\boldsymbol{V}},\boldsymbol{b}|_{\hat{\boldsymbol{V}}}$ is also 2-Cartesian.

We have 1-morphisms $\boldsymbol{b}\circ\boldsymbol{i}_{\mathbf{Y}}|_{\hat{\boldsymbol{U}}},\boldsymbol{b}\circ\boldsymbol{f}\circ\tilde{\boldsymbol{g}}\circ\boldsymbol{i}_{\mathbf{Y}}|_{\hat{\boldsymbol{U}}}:\hat{\boldsymbol{U}}\to\boldsymbol{[0,\infty)}$ and a 2-morphism $\mathrm{id}_{\boldsymbol{b}}*\tilde\zeta*\mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}|_{\hat{\boldsymbol{U}}}}:\boldsymbol{b}\circ\boldsymbol{f}\circ\tilde{\boldsymbol{g}}\circ\boldsymbol{i}_{\mathbf{Y}}|_{\hat{\boldsymbol{U}}}\Rightarrow\boldsymbol{b}\circ\boldsymbol{i}_{\mathbf{Y}}|_{\hat{\boldsymbol{U}}}$. Expanding using Definition 2.14, we have

$$
\mathrm{id}_{\boldsymbol{b}}*\tilde\zeta*\mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}|_{\hat{\boldsymbol{U}}}}=i''_{\mathbf{Y}}|_{\hat{\underline{U}}}\circ\underline{i}_{\mathbf{Y}}|_{\hat{\underline{U}}}^*\big(\tilde\zeta\circ\underline{\mathrm{id}}_Y^*(b^2)\circ I_{\underline{\mathrm{id}}_Y,\underline{b}}(\mathcal{F}_{[0,\infty)})\big)\circ I_{\underline{i}_{\mathbf{Y}}|_{\hat{\boldsymbol{U}}},\underline{b}}(\mathcal{F}_{[0,\infty)}).
$$

Using $i''_{\mathbf{Y}}$ an isomorphism by Definition 6.1(b), and the isomorphism γ of Definition 6.1(e) with $\nu_{\mathbf{Y}}|_{\hat{\underline{U}}}\circ\gamma=\underline{i}_{\mathbf{Y}}|_{\hat{\underline{U}}}^*(b^2)$, we see that $\mathrm{id}_{\boldsymbol{b}}*\tilde\zeta*\mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}|_{\hat{\boldsymbol{U}}}}=0$ is equivalent to the restriction of (6.59) to $\hat{\underline{U}}$. Hence $\boldsymbol{b}\circ\boldsymbol{f}\circ\tilde{\boldsymbol{g}}\circ\boldsymbol{i}_{\mathbf{Y}}|_{\hat{\boldsymbol{U}}}=\boldsymbol{b}\circ\boldsymbol{i}_{\mathbf{Y}}|_{\hat{\boldsymbol{U}}}$. Therefore (6.1) with $\hat{\boldsymbol{U}},\hat{\boldsymbol{V}}$ and $\boldsymbol{b}\circ\boldsymbol{f}\circ\tilde{\boldsymbol{g}}|_{\hat{\boldsymbol{V}}}$ in place of $\boldsymbol{U},\boldsymbol{V},\boldsymbol{b}$ is also 2-Cartesian. This proves the first part of Definition 6.1(c) for $\big(\hat{\boldsymbol{V}},\boldsymbol{b}\circ\boldsymbol{f}\circ\tilde{\boldsymbol{g}}|_{\hat{\boldsymbol{V}}}\big)$.

For the second part, by Definition 6.1(c) for $(\boldsymbol{V},\boldsymbol{b})$ there exists a left inverse $\beta:\mathcal{F}_X|_{\underline{V}}\to\underline{b}^*(\mathcal{F}_{[0,\infty)})$ for $b^2:\underline{b}^*(\mathcal{F}_{[0,\infty)})\to\mathcal{F}_X|_{\underline{V}}$. We have morphisms $b^2|_{\hat{\underline{V}}},(b\circ f\circ\tilde g|_{\hat V})^2:\underline{b}|_{\hat{\underline{V}}}^*(\mathcal{F}_{[0,\infty)})\to\mathcal{F}_X|_{\hat{\underline{V}}}$, where $\beta|_{\hat{\underline{V}}}$ is a left inverse for $b^2|_{\hat{\underline{V}}}$. Using (6.59) we can show that

$$
\underline{i}_{\mathbf{Y}}|_{\hat{\underline{U}}}^*(b^2)=\underline{i}_{\mathbf{Y}}|_{\hat{\underline{U}}}^*\big((b\circ f\circ\tilde g|_{\hat V})^2\big). \tag{6.60}
$$

Hence $b^2|_{\hat{\underline{V}}}$ and $(b\circ f\circ\tilde g|_{\hat V})^2$ coincide on $\underline{i}_{\mathbf{Y}}(\hat{\underline{U}})$, which contains x.

Consider the morphism $\beta|_{\underline{\hat V}}\circ(b\circ f\circ\tilde g|_{\hat V})^2:\underline{b}|_{\underline{\hat V}}^*(\mathcal{F}_{[0,\infty)})\to\underline{b}|_{\underline{\hat V}}^*(\mathcal{F}_{[0,\infty)})$. This is an automorphism of a line bundle on $\underline{\hat V}$, and is the identity on $\underline{i}_{\mathbf{Y}}(\underline{\hat U})$. Thus it is multiplication by a smooth function c on $\underline{\hat V}$, with $c=1$ on $x\in\underline{i}_{\mathbf{Y}}(\underline{\hat U})$. Making $\underline{\hat V},\underline{\hat U}$ smaller we can ensure that $c>0$ on $\underline{\hat V}$, so that $\beta|_{\underline{\hat V}}\circ(b\circ f\circ\tilde g|_{\hat V})^2$ is invertible. Then $\tilde\beta=\bigl(\beta|_{\underline{\hat V}}\circ(b\circ f\circ\tilde g|_{\hat V})^2\bigr)^{-1}\circ\beta|_{\underline{\hat V}}$ is a left inverse for $(b\circ f\circ\tilde g|_{\hat V})^2$. This proves the second part of Definition 6.1(c) for $(\hat{\boldsymbol V},\boldsymbol b\circ\boldsymbol f\circ\tilde{\boldsymbol g}|_{\hat{\boldsymbol V}})$. Equation (6.60) implies that the isomorphism γ in (6.7) for $(\boldsymbol V,\boldsymbol b)$ and for $(\hat{\boldsymbol V},\boldsymbol b\circ\boldsymbol f\circ\tilde{\boldsymbol g}|_{\hat{\boldsymbol V}})$ coincide on $\underline{\hat U}$. Thus Definition 6.1(e) for $(\hat{\boldsymbol V},\boldsymbol b\circ\boldsymbol f\circ\tilde{\boldsymbol g}|_{\hat{\boldsymbol V}})$ follows from that for $(\boldsymbol V,\boldsymbol b)$. Hence $(\hat{\boldsymbol V},\boldsymbol b\circ\boldsymbol f\circ\tilde{\boldsymbol g}|_{\hat{\boldsymbol V}})$ is a boundary defining function for $\mathbf{Y}$ at y'.

Thus, $(\tilde{\boldsymbol V},\boldsymbol b\circ\boldsymbol f|_{\tilde{\boldsymbol V}})$ is a boundary defining function for $\mathbf{X}$ at x', and $(\hat{\boldsymbol V},(\boldsymbol b\circ\boldsymbol f|_{\tilde{\boldsymbol V}})\circ\tilde{\boldsymbol g}|_{\hat{\boldsymbol V}})$ is a boundary defining function for $\mathbf{Y}$ at y'. Therefore Definition 6.2(i) holds for $\tilde{\boldsymbol g}$ at y,x,y',x'. As this holds for all $(x',y')\in\underline{S}_{\boldsymbol f}$, this shows that (6.59) implies that $\tilde{\boldsymbol g}$ is a 1-morphism in $\mathbf{dSpa^c}$, and (6.57) holds.

Let us summarize what we have proved so far. Starting with some choice of $\boldsymbol g,\eta,\zeta$, we choose an arbitrary morphism $\theta:\underline{g}^*(\mathcal{F}_X)\to\mathcal{E}_Y$. Then there exist a unique 1-morphism $\tilde{\boldsymbol g}:\boldsymbol Y\to\boldsymbol X$ and 2-morphisms $\tilde\eta:\tilde{\boldsymbol g}\circ\boldsymbol f\Rightarrow\mathbf{id}_{\boldsymbol X}$ and $\tilde\zeta:\boldsymbol f\circ\tilde{\boldsymbol g}\Rightarrow\mathbf{id}_{\boldsymbol Y}$ in $\mathbf{dSpa}$ with $\theta:\boldsymbol g\Rightarrow\tilde{\boldsymbol g}$, $\tilde\eta=\eta\odot((-\theta)*\mathrm{id}_{\boldsymbol f})$ and $\tilde\zeta=\zeta\odot(\mathrm{id}_{\boldsymbol f}*(-\theta))$. If $\tilde\zeta$ satisfies equation (6.59) then $\tilde{\boldsymbol g}$ is a 1-morphism in $\mathbf{dSpa^c}$ and $\tilde\eta,\tilde\zeta$ are 2-morphisms in $\mathbf{dSpa^c}$, which implies that $\boldsymbol f:\mathbf{X}\to\mathbf{Y}$ is an equivalence in $\mathbf{dSpa^c}$.

It remains to show that we can choose θ so that $\tilde\zeta$ satisfies (6.59). To do this, let $(x',y')\in\underline{S}_{\boldsymbol f}$, $(\boldsymbol V,\boldsymbol b),\boldsymbol U,\tilde{\boldsymbol V},\hat{\boldsymbol V},\hat{\boldsymbol U}$ be as above. Then $\mathrm{id}_{\boldsymbol b}*\tilde\zeta*\mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}|_{\hat{\boldsymbol U}}}=0$ is equivalent to the restriction of (6.59) to $\underline{\hat U}$, and both are implied by $\mathrm{id}_{\boldsymbol b}*\tilde\zeta|_{\underline{\hat V}}=0$. Expanding using Definition 2.14 and $\tilde\zeta=\zeta\odot(\mathrm{id}_{\boldsymbol f}*(-\theta))$, this is equivalent to

$$\begin{aligned}0&=\mathrm{id}_{\boldsymbol b}*\tilde\zeta|_{\hat V}=\tilde\zeta|_{\underline{\hat V}}\circ\underline{\mathrm{id}}_{\underline{\hat V}}^*(b^2)\circ I_{\underline{\mathrm{id}}_{\hat V},\underline{b}}(\mathcal{F}_{[0,\infty)})\\&=\zeta|_{\underline{\hat V}}\circ\underline{\mathrm{id}}_{\underline{\hat V}}^*(b^2)\circ I_{\underline{\mathrm{id}}_{\hat V},\underline{b}}(\mathcal{F}_{[0,\infty)})-\theta|_{\underline{\hat V}}\circ\underline{g}|_{\underline{\hat V}}^*\bigl((b\circ f|_{\tilde V})^2\bigr)\circ I_{\underline{g}|_{\hat V},\underline{b}\circ\underline{f}}(\mathcal{F}_{[0,\infty)}).\end{aligned}\tag{6.61}$$

Since $(\tilde{\boldsymbol V},\boldsymbol b\circ\boldsymbol f|_{\tilde{\boldsymbol V}})$ is a boundary defining function for $\mathbf{X}$ at x', by Definition 6.1(c) there exists a left inverse $\tilde\beta$ for $(b\circ f|_{\tilde V})^2$. Thus we have a canonical isomorphism $\tilde\beta\oplus\tilde\gamma:\mathcal{F}_X|_{\underline{\tilde V}}\to(\underline{b}\circ\underline{f}|_{\underline{\tilde V}})^*(\mathcal{F}_{[0,\infty)})\oplus\mathcal{C}_{\tilde V}$, where $\tilde\gamma:\mathcal{F}_X|_{\underline{\tilde V}}\to\mathcal{C}_{\tilde V}$ is the cokernel of $(b\circ f|_{\tilde V})^2$ in $\mathrm{qcoh}(\underline{\tilde V})$. Hence $\underline{g}|_{\underline{\hat V}}^*(\tilde\beta\oplus\tilde\gamma)$ is also an isomorphism. Thus we may write $\theta|_{\underline{\hat V}}=(\theta^1_{\hat V}\oplus\theta^2_{\hat V})\circ\underline{g}|_{\underline{\hat V}}^*(\tilde\beta\oplus\tilde\gamma)$ for unique $\theta^1_{\hat V}:\underline{g}^*\circ(\underline{b}\circ\underline{f})^*(\mathcal{F}_{[0,\infty)})|_{\underline{\hat V}}\to\mathcal{E}_Y|_{\underline{\hat V}}$ and $\theta^2_{\hat V}:\underline{g}|_{\underline{\hat V}}^*(\mathcal{C}_{\tilde V})\to\mathcal{E}_Y|_{\underline{\hat V}}$. Substituting this into (6.61) and using $\tilde\beta$ a left inverse for $(b\circ f|_{\tilde V})^2$ and $\tilde\gamma\circ(b\circ f|_{\tilde V})^2=0$ shows that (6.61) is equivalent to

$$\theta^1_{\hat V}=\zeta|_{\underline{\hat V}}\circ\underline{\mathrm{id}}_{\underline{\hat V}}^*(b^2)\circ I_{\underline{\mathrm{id}}_{\hat V},\underline{b}}(\mathcal{F}_{[0,\infty)})\circ I_{\underline{g}|_{\hat V},\underline{b}\circ\underline{f}}(\mathcal{F}_{[0,\infty)})^{-1}.\tag{6.62}$$

Hence, a sufficient condition for (6.59) to hold on $\underline{\hat U}$ is that the component $\theta^1_{\hat V}$ of $\theta|_{\underline{\hat V}}$ should be given by (6.62).

Next we show that we can satisfy these conditions near each point $y\in\underline{Y}$. Let $y\in\underline{Y}$ with $\underline{g}(y)=x\in\underline{X}$, and write $\underline{i}_{\mathbf{Y}}^{-1}(y)=\{y'_1,\ldots,y'_k\}$ for $y'_1,\ldots,y'_k$

distinct in $\underline{\partial Y}$. As $\boldsymbol{f}$ is flat there exist unique $x'_1,\ldots,x'_k \in \underline{i}_{\mathbf{X}}^{-1}(x)$ with $(x'_i,y'_i) \in \underline{S}_{\boldsymbol{f}}$ for $i=1,\ldots,k$, and as $\boldsymbol{f}$ is simple $x'_1,\ldots,x'_k$ are distinct, and $\underline{i}_{\mathbf{X}}^{-1}(x) = \{x'_1,\ldots,x'_k\}$. Choose a boundary defining function $(\boldsymbol{V}_i,\boldsymbol{b}_i)$ for $\mathbf{Y}$ at y'_i for $i=1,\ldots,k$. Then as $(x'_i,y'_i) \in \underline{S}_{\boldsymbol{f}}$, there exists an open $x \in \tilde{\boldsymbol{V}}_i \subseteq \boldsymbol{f}^{-1}(\boldsymbol{V}_i)$ such that $(\tilde{\boldsymbol{V}}_i, \boldsymbol{b}_i \circ \boldsymbol{f}|_{\tilde{\boldsymbol{V}}_i})$ is a boundary defining function for $\mathbf{X}$ at x'_i. As above, making $\boldsymbol{V}_1,\ldots,\boldsymbol{V}_k$ and $\tilde{\boldsymbol{V}}_1,\ldots,\tilde{\boldsymbol{V}}_k$ smaller if necessary we can suppose $\boldsymbol{V}_1 = \cdots = \boldsymbol{V}_k = \boldsymbol{V}$ and $\tilde{\boldsymbol{V}}_1 = \cdots = \tilde{\boldsymbol{V}}_k = \tilde{\boldsymbol{V}} = \boldsymbol{f}^{-1}(\boldsymbol{V})$.

Now Definition 6.1(f) for $\mathbf{X}$ at x says that (6.8) is injective. Using the isomorphisms $\mathcal{N}_{\mathbf{X}}|_{\underline{U}_i} \cong \underline{i}_{\mathbf{X}}|_{\underline{U}_i}^* \circ (\underline{b}_i \circ \underline{f})^*(\mathcal{F}_{[0,\infty)})$ from Definition 6.1, we see that

$$\textstyle\bigoplus_{i=1}^k \underline{x}^*\big((b_i \circ f|_{\tilde{V}})^2\big) : \bigoplus_{i=1}^k \underline{x}^* \circ (\underline{b}_i \circ \underline{f}|_{\tilde{\underline{V}}})^*(\mathcal{F}_{[0,\infty)}) \longrightarrow \underline{x}^*(\mathcal{F}_X) \qquad (6.63)$$

is injective. As $(b_i \circ f|_{\tilde{V}})^2$ has a left inverse, $(b_i \circ f|_{\tilde{V}})^2 : (\underline{b}_i \circ \underline{f}|_{\tilde{\underline{V}}})^*(\mathcal{F}_{[0,\infty)}) \to \mathcal{F}_X|_{\tilde{\underline{V}}}$ for $i=1,\ldots,k$ includes $(\underline{b}_i \circ \underline{f}|_{\tilde{\underline{V}}})^*(\mathcal{F}_{[0,\infty)})$ as a direct summand of $\mathcal{F}_X|_{\tilde{\underline{V}}}$. Since (6.63) is injective, these direct summands are linearly independent near x. So making $\boldsymbol{V}, \tilde{\boldsymbol{V}}, \hat{\boldsymbol{V}}$ smaller if necessary, we can suppose the direct summands are linearly independent on $\tilde{\underline{V}}$.

Thus there exists a $\tilde{\beta}_1 \oplus \cdots \oplus \tilde{\beta}_k$ for $(b_1 \circ f|_{\tilde{V}})^2 \oplus \cdots (b_k \circ f|_{\tilde{V}})^2$. That is, by Definition 6.1(c) there exists a left inverse $\tilde{\beta}_i$ for $(b_i \circ f|_{\tilde{V}})^2$ for $i=1,\ldots,k$, and we have shown we can choose $\tilde{\beta}_1,\ldots,\tilde{\beta}_k$ such that $\tilde{\beta}_i \circ (b_j \circ f|_{\tilde{V}})^2 = 0$ for $i \neq j = 1,\ldots,k$. As for $\tilde{\beta}, \tilde{\gamma}$ above we obtain an isomorphism

$$\textstyle\big(\bigoplus_{i=1}^k \tilde{\beta}_i\big) \oplus \tilde{\gamma}_{1,\ldots,k} : \mathcal{F}_X|_{\tilde{\underline{V}}} \xrightarrow{\cong} \big(\bigoplus_{i=1}^k (\underline{b}_i \circ \underline{f}|_{\tilde{\underline{V}}})^*(\mathcal{F}_{[0,\infty)})\big) \oplus \mathcal{C}_{\tilde{V},1,\ldots,k},$$

where $\tilde{\gamma}_{1,\ldots,k} : \mathcal{F}_X|_{\tilde{\underline{V}}} \to \mathcal{C}_{\tilde{V},1,\ldots,k}$ is the cokernel of $\bigoplus_{i=1}^k (b_i \circ f|_{\tilde{V}})^2$. Hence we may write $\theta|_{\hat{\underline{V}}} = (\theta^1_{\hat{V}} \oplus \cdots \oplus \theta^{k+1}_{\hat{V}}) \circ \underline{g}|_{\hat{\underline{V}}}^*(\tilde{\beta}_1 \oplus \cdots \oplus \tilde{\beta}_k \oplus \tilde{\gamma}_{1,\ldots,k})$ for unique $\theta^i_{\hat{V}} : \underline{g}^* \circ (\underline{b}_i \circ \underline{f})^*(\mathcal{F}_{[0,\infty)})|_{\hat{\underline{V}}} \to \mathcal{E}_Y|_{\hat{\underline{V}}}$ for $i=1,\ldots,k$ and $\theta^{k+1}_{\hat{V}} : \underline{g}|_{\hat{\underline{V}}}^*(\mathcal{C}_{\tilde{V},1,\ldots,k}) \to \mathcal{E}_Y|_{\hat{\underline{V}}}$. As for (6.62), we find that (6.59) holds on $\underline{i}_{\mathbf{Y}}^{-1}(\hat{\underline{V}}) = \hat{\underline{U}}_1 \amalg \cdots \amalg \hat{\underline{U}}_k$ if the components $\theta^1_{\hat{V}},\ldots,\theta^k_{\hat{V}}$ of $\theta|_{\hat{\underline{V}}}$ are given by

$$\theta^i_{\hat{V}} = \zeta|_{\hat{\underline{V}}} \circ \underline{\mathrm{id}}_{\hat{\underline{V}}}^*(b_i^2) \circ I_{\underline{\mathrm{id}}_{\hat{V}},\underline{b}_i}(\mathcal{F}_{[0,\infty)}) \circ I_{\underline{g}|_{\hat{V}},\underline{b}_i \circ \underline{f}}(\mathcal{F}_{[0,\infty)})^{-1}, \quad i=1,\ldots,k.$$

This proves that for each $y \in \underline{Y}$ we can choose an open neighbourhood $\hat{\underline{V}}_y$ of y in $\underline{Y}$ and a morphism $\theta_y : \underline{g}^*(\mathcal{F}_X)|_{\hat{\underline{V}}_y} \to \mathcal{E}_Y|_{\hat{\underline{V}}_y}$ such that $\tilde{\zeta}|_{\hat{\underline{V}}_y}$ constructed using θ_y satisfies (6.59) on $\underline{i}_{\mathbf{Y}}^{-1}(\hat{\underline{V}}_y)$. Then $\{\hat{\underline{V}}_y : y \in \underline{Y}\}$ is an open cover of $\underline{Y}$, which is separated, paracompact, and locally fair. Thus by Proposition B.21 we may choose a partition of unity $\{\eta_y : y \in \underline{Y}\}$ subordinate to $\{\hat{\underline{V}}_y : y \in \underline{Y}\}$, and define $\theta = \sum_{y \in \underline{Y}} \eta_y \cdot \theta_y$. Since (6.59) is equivalent to an affine linear equation in θ, combining different local choices of θ_y solving (6.59) using a partition of unity gives a global choice θ solving (6.59). Hence we can choose $\theta : \underline{g}^*(\mathcal{F}_X) \to \mathcal{E}_Y$ in $\mathrm{qcoh}(\underline{Y})$ such that $\tilde{\zeta} = \zeta \odot (\mathrm{id}_{\boldsymbol{f}} * (-\theta))$ satisfies (6.59). This completes the proof of Proposition 6.21. $\square$

Here is the analogue of Definition 2.23 for $\mathbf{dSpa^c}$.

Definition 6.22. Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism in $\mathbf{dSpa^c}$. We call $\boldsymbol{f}$ *étale* if it is a *local equivalence*, that is, if for each $x \in \mathbf{X}$ there exist open $x \in \mathbf{U} \subseteq \mathbf{X}$ and $\boldsymbol{f}(x) \in \mathbf{V} \subseteq \mathbf{Y}$ such that $\boldsymbol{f}(\mathbf{U}) = \mathbf{V}$ and $\boldsymbol{f}|_{\mathbf{U}} : \mathbf{U} \to \mathbf{V}$ is an equivalence.

Propositions 6.20 and 6.21 imply the following characterization of étale 1-morphisms in $\mathbf{dSpa^c}$. Necessary and sufficient conditions for $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ to be étale in $\mathbf{dSpa}$ are given in Corollary 2.24.

Corollary 6.23. *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism in $\mathbf{dSpa^c}$. Then $\boldsymbol{f}$ is étale if and only if $\boldsymbol{f}$ is simple and flat and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is étale in $\mathbf{dSpa}$.*

6.6 Gluing d-spaces with corners by equivalences

All the material of §2.4 on gluing d-spaces by equivalences extends to d-spaces with corners. Here are the analogues of Proposition 2.27 and Theorem 2.28.

Proposition 6.24. *Suppose $\mathbf{X}, \mathbf{Y}$ are d-spaces with corners, $\mathbf{U}, \mathbf{V} \subseteq \mathbf{X}$ are open d-subspaces with $\mathbf{X} = \mathbf{U} \cup \mathbf{V}$, $\boldsymbol{f} : \mathbf{U} \to \mathbf{Y}$ and $\boldsymbol{g} : \mathbf{V} \to \mathbf{Y}$ are 1-morphisms, and $\eta : \boldsymbol{f}|_{\mathbf{U}\cap\mathbf{V}} \Rightarrow \boldsymbol{g}|_{\mathbf{U}\cap\mathbf{V}}$ is a 2-morphism. Then there exists a 1-morphism $\boldsymbol{h} : \mathbf{X} \to \mathbf{Y}$ and 2-morphisms $\zeta : \boldsymbol{h}|_{\mathbf{U}} \Rightarrow \boldsymbol{f}$, $\theta : \boldsymbol{h}|_{\mathbf{V}} \Rightarrow \boldsymbol{g}$ in $\mathbf{dSpa^c}$ such that $\theta|_{\mathbf{U}\cap\mathbf{V}} = \eta \odot \zeta|_{\mathbf{U}\cap\mathbf{V}} : \boldsymbol{h}|_{\mathbf{U}\cap\mathbf{V}} \Rightarrow \boldsymbol{g}|_{\mathbf{U}\cap\mathbf{V}}$. This $\boldsymbol{h}$ is unique up to 2-isomorphism. Furthermore, $\boldsymbol{h}$ is independent of η up to 2-isomorphism.*

Proof. Applying Proposition 2.27 to the d-space data $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{f}, \boldsymbol{g}, \eta$ gives a d-space 1-morphism $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Y}$ and d-space 2-morphisms $\zeta : \boldsymbol{h}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{f}$, $\theta : \boldsymbol{h}|_{\boldsymbol{V}} \Rightarrow \boldsymbol{g}$ with $\theta|_{\boldsymbol{U}\cap\boldsymbol{V}} = \eta \odot \zeta|_{\boldsymbol{U}\cap\boldsymbol{V}}$. Furthermore, by construction we have $\zeta|_{\underline{U}\cap\underline{V}} = -\epsilon \cdot \eta$ and $\theta|_{\underline{U}\cap\underline{V}} = (1-\epsilon)\cdot\eta$, where ϵ is a partition of unity function, and $\zeta|_{\underline{U}\setminus\underline{V}} = 0 = \theta|_{\underline{V}\setminus\underline{U}}$. Since η satisfies (6.9)–(6.10) over $\mathbf{U}\cap\mathbf{V}$, it follows that ζ satisfies (6.9)–(6.10) over $\mathbf{U}$, and θ satisfies (6.9)–(6.10) over $\mathbf{V}$.

As $\boldsymbol{f}, \boldsymbol{g}$ are 1-morphisms in $\mathbf{dSpa^c}$ and $\zeta : \boldsymbol{h}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{f}$, $\theta : \boldsymbol{h}|_{\boldsymbol{V}} \Rightarrow \boldsymbol{g}$ are 2-morphisms in $\mathbf{dSpa}$ with ζ, θ satisfying (6.9)–(6.10), Proposition 6.9 implies that $\boldsymbol{h}|_{\mathbf{U}}$ and $\boldsymbol{h}|_{\mathbf{V}}$ are 1-morphisms in $\mathbf{dSpa^c}$, and ζ, θ are 2-morphisms in $\mathbf{dSpa^c}$. So $\boldsymbol{h}$ is a 1-morphism in $\mathbf{dSpa^c}$ over $\mathbf{U}\cup\mathbf{V} = \mathbf{X}$. The proof in Proposition 2.27 that $\boldsymbol{h}$ is unique and independent of η up to 2-isomorphism is also valid in $\mathbf{dSpa^c}$. □

Theorem 6.25. *Suppose $\mathbf{X}, \mathbf{Y}$ are d-spaces with corners, $\mathbf{U} \subseteq \mathbf{X}$, $\mathbf{V} \subseteq \mathbf{Y}$ are open d-subspaces, and $\boldsymbol{f} : \mathbf{U} \to \mathbf{V}$ is an equivalence in $\mathbf{dSpa^c}$. At the level of topological spaces, we have open $U \subseteq X$, $V \subseteq Y$ with a homeomorphism $f : U \to V$, so we can form the quotient topological space $Z := X \amalg_f Y = (X \amalg Y)/\sim$, where the equivalence relation $\sim$ on $X \amalg Y$ identifies $u \in U \subseteq X$ with $f(u) \in V \subseteq Y$.*

Suppose Z is Hausdorff. Then there exist a d-space with corners $\mathbf{Z}$, open d-subspaces $\hat{\mathbf{X}}, \hat{\mathbf{Y}}$ in $\mathbf{Z}$ with $\mathbf{Z} = \hat{\mathbf{X}} \cup \hat{\mathbf{Y}}$, equivalences $\boldsymbol{g} : \mathbf{X} \to \hat{\mathbf{X}}$ and $\boldsymbol{h} : \mathbf{Y} \to \hat{\mathbf{Y}}$ in $\mathbf{dSpa^c}$ such that $\boldsymbol{g}|_{\mathbf{U}}$ and $\boldsymbol{h}|_{\mathbf{V}}$ are both equivalences with $\hat{\mathbf{X}} \cap \hat{\mathbf{Y}}$, and a 2-morphism $\eta : \boldsymbol{g}|_{\mathbf{U}} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f} : \mathbf{U} \to \hat{\mathbf{X}} \cap \hat{\mathbf{Y}}$. Furthermore, $\mathbf{Z}$ is independent of choices up to equivalence.

Proof. As $\boldsymbol{f}:\mathbf{U}\to\mathbf{V}$ is an equivalence in $\mathbf{dSpa^c}$, we may choose a 1-morphism $\boldsymbol{e}:\mathbf{V}\to\mathbf{U}$ and 2-morphisms $\zeta:\boldsymbol{e}\circ\boldsymbol{f}\Rightarrow\mathbf{id_U}$, $\theta:\boldsymbol{f}\circ\boldsymbol{e}\Rightarrow\mathbf{id_V}$ in $\mathbf{dSpa^c}$. Then $\boldsymbol{f},\boldsymbol{e}$ are simple and flat by Proposition 6.20, so Theorem 6.12(b),(c) gives 1-morphisms $\boldsymbol{f}_-:\partial\mathbf{U}\to\partial\mathbf{V}$, $\boldsymbol{e}_-:\partial\mathbf{V}\to\partial\mathbf{U}$ and 2-morphisms $\zeta_-:\boldsymbol{e}_-\circ\boldsymbol{f}_-\Rightarrow\mathbf{id}_{\partial\mathbf{U}}$, $\theta_-:\boldsymbol{f}_-\circ\boldsymbol{e}_-\Rightarrow\mathbf{id}_{\partial\mathbf{V}}$, so that $\boldsymbol{f}_-,\boldsymbol{e}_-$ are equivalences. By Proposition A.6 we may also suppose that $\mathrm{id}_{\boldsymbol{f}}*\zeta=\theta*\mathrm{id}_{\boldsymbol{f}}$ and $\mathrm{id}_{\boldsymbol{e}}*\theta=\zeta*\mathrm{id}_{\boldsymbol{e}}$, and these imply that $\mathrm{id}_{\boldsymbol{f}_-}*\zeta_-=\theta_-*\mathrm{id}_{\boldsymbol{f}_-}$ and $\mathrm{id}_{\boldsymbol{e}_-}*\theta_-=\zeta_-*\mathrm{id}_{\boldsymbol{e}_-}$.

We now apply Theorem 2.28 twice:

(a) we construct a d-space $\boldsymbol{Z}$ with open $\hat{\boldsymbol{X}},\hat{\boldsymbol{Y}}\subseteq\boldsymbol{Z}$, equivalences $\boldsymbol{g}:\boldsymbol{X}\to\hat{\boldsymbol{X}}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\hat{\boldsymbol{Y}}$ with $\boldsymbol{g}(\boldsymbol{U})=\boldsymbol{h}(\boldsymbol{V})=\hat{\boldsymbol{X}}\cap\hat{\boldsymbol{Y}}$, and a 2-morphism $\eta:\boldsymbol{g}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ in $\mathbf{dSpa}$; and

(b) we construct a d-space $\boldsymbol{\partial Z}$ with open $\boldsymbol{\partial}\hat{\boldsymbol{X}},\boldsymbol{\partial}\hat{\boldsymbol{Y}}\subseteq\boldsymbol{\partial Z}$, equivalences $\boldsymbol{g}_-:\boldsymbol{\partial X}\to\boldsymbol{\partial}\hat{\boldsymbol{X}}$ and $\boldsymbol{h}_-:\boldsymbol{\partial Y}\to\boldsymbol{\partial}\hat{\boldsymbol{Y}}$ with $\boldsymbol{g}_-(\boldsymbol{\partial U})=\boldsymbol{h}_-(\boldsymbol{\partial V})=\boldsymbol{\partial}\hat{\boldsymbol{X}}\cap\boldsymbol{\partial}\hat{\boldsymbol{Y}}$, and a 2-morphism $\eta_-:\boldsymbol{g}_-|_{\boldsymbol{\partial U}}\Rightarrow\boldsymbol{h}_-\circ\boldsymbol{f}_-$ in $\mathbf{dSpa}$.

The proof of Theorem 2.28 involves choices of $\boldsymbol{e},\zeta,\theta$, open $\hat{A},\hat{B}\subseteq Z$, and $\gamma\in\mathcal{O}_X(C)$, $\delta\in\mathcal{O}_Y(D)$. For (a) we choose $\boldsymbol{e},\zeta,\theta$ as above, and $\hat{A},\hat{B},\gamma,\delta$ arbitrary. For (b) we choose $\boldsymbol{e}_-,\zeta_-,\theta_-$ as above, and $\hat{A}_-=i_{\mathbf{X}}^{-1}(\hat{A})$, $\hat{B}_-=i_{\mathbf{Y}}^{-1}(\hat{B})$, $\gamma_-=i_{\mathbf{X}}^\sharp(C)(\gamma)$, and $\delta_-=i_{\mathbf{Y}}^\sharp(D)(\delta)$.

Because the two gluing constructions are done with compatible choices, it is now easy to check that there is a unique, natural d-space 1-morphism $\boldsymbol{i}_{\mathbf{Z}}:\boldsymbol{\partial Z}\to\boldsymbol{Z}$ with $\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{X}}=\boldsymbol{i}_{\mathbf{Z}}\circ\boldsymbol{g}_-:\boldsymbol{\partial X}\to\boldsymbol{Z}$ and $\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}=\boldsymbol{i}_{\mathbf{Z}}\circ\boldsymbol{h}_-:\boldsymbol{\partial Y}\to\boldsymbol{Z}$. This satisfies Definition 6.1(a)–(c), so that we get a conormal bundle $\mathcal{N}_{\mathbf{Z}}$ on $\underline{\partial Z}$. Using $\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{X}}=\boldsymbol{i}_{\mathbf{Z}}\circ\boldsymbol{g}_-$ and $\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}=\boldsymbol{i}_{\mathbf{Z}}\circ\boldsymbol{h}_-$ we construct canonical isomorphisms $\underline{g}_-^*(\mathcal{N}_{\mathbf{Z}})\cong\mathcal{N}_{\mathbf{X}}$, $\underline{h}_-^*(\mathcal{N}_{\mathbf{Z}})\cong\mathcal{N}_{\mathbf{Y}}$. There is then a unique orientation $\omega_{\mathbf{Z}}$ on $\mathcal{N}_{\mathbf{Z}}$ such that these isomorphisms identify $\underline{g}_-^*(\omega_{\mathbf{Z}})\cong\omega_{\mathbf{X}}$ and $\underline{h}_-^*(\omega_{\mathbf{Z}})\cong\omega_{\mathbf{Y}}$.

Write $\mathbf{Z}=(\boldsymbol{Z},\boldsymbol{\partial Z},\boldsymbol{i}_{\mathbf{Z}},\omega_{\mathbf{Z}})$. Then one can show $\mathbf{Z}$ is a d-space with corners, and $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$, $\boldsymbol{h}:\mathbf{Y}\to\mathbf{Z}$ are simple, flat 1-morphisms in $\mathbf{dSpa^c}$, with

$$\underline{S}_{\boldsymbol{g}}=\bigl\{(x',\underline{g}_-(x')):x'\in\underline{\partial X}\bigr\}\quad\text{and}\quad\underline{S}_{\boldsymbol{h}}=\bigl\{(y',\underline{h}_-(y')):y'\in\underline{\partial Y}\bigr\}.\tag{6.64}$$

The identities $\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{X}}=\boldsymbol{i}_{\mathbf{Z}}\circ\boldsymbol{g}_-$ and $\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}=\boldsymbol{i}_{\mathbf{Z}}\circ\boldsymbol{h}_-$ imply that $\boldsymbol{g}_-,\boldsymbol{h}_-$ in (b) above are the same as the 1-morphisms $\boldsymbol{g}_-:\partial\mathbf{X}\to\partial\mathbf{Z}$ and $\boldsymbol{h}_-:\partial\mathbf{Y}\to\partial\mathbf{Z}$ constructed in Theorem 6.12(b). Since $\boldsymbol{g}:\boldsymbol{X}\to\hat{\boldsymbol{X}}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\hat{\boldsymbol{Y}}$ are equivalences in $\mathbf{dSpa}$, and $\boldsymbol{g},\boldsymbol{h}$ are also simple, flat 1-morphisms in $\mathbf{dSpa^c}$, Proposition 6.21 shows that $\boldsymbol{g}:\mathbf{X}\to\hat{\mathbf{X}}$ and $\boldsymbol{h}:\mathbf{Y}\to\hat{\mathbf{Y}}$ are equivalences in $\mathbf{dSpa^c}$. Hence $\boldsymbol{g}|_{\mathbf{U}}$ and $\boldsymbol{h}|_{\mathbf{V}}$ are equivalences with $\hat{\mathbf{X}}\cap\hat{\mathbf{Y}}$.

As $\boldsymbol{f}:\mathbf{U}\to\mathbf{V}$ is simple we have $\underline{S}_{\boldsymbol{f}}=\bigl\{(x',\underline{f}_-(x')):x'\in\underline{\partial U}\bigr\}$. But $\underline{g}_-|_{\underline{U}}=\underline{h}_-\circ\underline{f}_-$, as we have a 2-morphism $\eta_-:\boldsymbol{g}_-|_{\boldsymbol{\partial U}}\Rightarrow\boldsymbol{h}_-\circ\boldsymbol{f}_-$ in $\mathbf{dSpa}$ from (b) above. Using these, (6.64) and Proposition 6.7(f) we see that $\underline{S}_{\boldsymbol{g}|_{\mathbf{U}}}=\underline{S}_{\boldsymbol{h}\circ\boldsymbol{f}}$. The compatibility between (a) and (b) above implies that equation (6.42) in Theorem 6.12(c) commutes. Now (6.42) was proved using (6.9) for η, and reversing the argument using $\boldsymbol{g}|_{\mathbf{U}}$ simple we find that (6.42) implies (6.9) for η. Also $\underline{T}_{\boldsymbol{g}|_{\mathbf{U}}}=\emptyset$ as $\boldsymbol{g}|_{\mathbf{U}}$ is flat, so (6.10) for η is trivial. Hence $\eta:\boldsymbol{g}|_{\mathbf{U}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ is a 2-morphism in $\mathbf{dSpa^c}$. The analogue of Theorem 2.29 implies $\mathbf{Z}$ is independent of choices up to equivalence. $\square$

The proofs of Theorems 2.29–2.33 now work in $\mathbf{dSpa^c}$, using Proposition 6.24 and Theorem 6.25 in place of Proposition 2.27 and Theorem 2.28, and otherwise with only cosmetic changes. Thus we obtain the following analogue of Theorem 2.33:

Theorem 6.26. *Suppose I is an indexing set, and $<$ is a total order on I, and $\mathbf{X}_i$ for $i \in I$ are d-spaces with corners, and for all $i < j$ in I we are given open d-subspaces $\mathbf{U}_{ij} \subseteq \mathbf{X}_i$, $\mathbf{U}_{ji} \subseteq \mathbf{X}_j$ and an equivalence $\boldsymbol{e}_{ij} : \mathbf{U}_{ij} \to \mathbf{U}_{ji}$, such that for all $i < j < k$ in I we have a 2-commutative diagram in $\mathbf{dSpa^c}$:*

$$\begin{tikzcd}
 & \mathbf{U}_{ji} \cap \mathbf{U}_{jk} \arrow[dr, "\boldsymbol{e}_{jk}|_{\mathbf{U}_{ji}\cap\mathbf{U}_{jk}}"] \arrow[d, phantom, "\Downarrow\eta_{ijk}"] & \\
\mathbf{U}_{ij} \cap \mathbf{U}_{ik} \arrow[ur, "\boldsymbol{e}_{ij}|_{\mathbf{U}_{ij}\cap\mathbf{U}_{ik}}"] \arrow[rr, "\boldsymbol{e}_{ik}|_{\mathbf{U}_{ij}\cap\mathbf{U}_{ik}}"] & {} & \mathbf{U}_{ki} \cap \mathbf{U}_{kj}
\end{tikzcd}$$

for some η_{ijk}, where all three 1-morphisms are equivalences.

On the level of topological spaces, define the quotient topological space $Y = (\coprod_{i\in I} X_i)/\sim$, where $\sim$ is the equivalence relation generated by $x_i \sim x_j$ if $i < j$, $x_i \in U_{ij} \subseteq X_i$ and $x_j \in U_{ji} \subseteq X_j$ with $e_{ij}(x_i) = x_j$. Suppose Y is Hausdorff and second countable. Then there exist a d-space with corners $\mathbf{Y}$ and a 1-morphism $\boldsymbol{f}_i : \mathbf{X}_i \to \mathbf{Y}$ which is an equivalence with an open d-subspace $\hat{\mathbf{X}}_i \subseteq \mathbf{Y}$ for all $i \in I$, where $\mathbf{Y} = \bigcup_{i\in I} \hat{\mathbf{X}}_i$, such that $\boldsymbol{f}_i|_{\mathbf{U}_{ij}}$ is an equivalence $\mathbf{U}_{ij} \to \hat{\mathbf{X}}_i \cap \hat{\mathbf{X}}_j$ for all $i < j$ in I, and there exists a 2-morphism $\eta_{ij} : \boldsymbol{f}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{f}_i|_{\mathbf{U}_{ij}}$. The d-space with corners $\mathbf{Y}$ is unique up to equivalence, and is independent of choice of 2-morphisms η_{ijk}.

Suppose also that $\mathbf{Z}$ is a d-space with corners, and $\boldsymbol{g}_i : \mathbf{X}_i \to \mathbf{Z}$ are 1-morphisms in $\mathbf{dSpa^c}$ for all $i \in I$, and there exist 2-morphisms $\zeta_{ij} : \boldsymbol{g}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{g}_i|_{\mathbf{U}_{ij}}$ for all $i < j$ in I. Then there exist a 1-morphism $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ and 2-morphisms $\zeta_i : \boldsymbol{h} \circ \boldsymbol{f}_i \Rightarrow \boldsymbol{g}_i$ for all $i \in I$. The 1-morphism $\boldsymbol{h}$ is unique up to 2-isomorphism, and is independent of the choice of 2-morphisms ζ_{ij}.

6.7 Corners $C_k(\mathbf{X})$, and the corner functors $C, \hat{C}$

We now generalize the material of §5.5 on corners $C_k(X)$ and the corner functors $C, \hat{C}$ from $\mathbf{Man^c}$ to $\mathbf{dSpa^c}$. The construction of $C_k(\mathbf{X})$ in Definition 6.28 essentially follows from that of $\partial^k\mathbf{X}$ in §6.2, and the constructions of 1-morphisms $C(\boldsymbol{f})$ and 2-morphisms $C(\eta)$ in Theorem 6.29 are similar to those of $\boldsymbol{f}_-, \eta_-$ in Theorem 6.12(b),(c) in §6.3. So we will be brief in places. We begin with a remark describing properties of k-fold boundaries $\partial^k\mathbf{X}$ that we will need.

Remark 6.27. Let $\mathbf{X}$ be a d-space with corners. Then §6.2 defines a d-space with corners $\partial^k\mathbf{X}$ for $k = 1, 2, \ldots$, by induction on k. Here $\partial^k\mathbf{X} = (\boldsymbol{\partial^k X}, \boldsymbol{\partial^{k+1} X}, \boldsymbol{i}_{\partial^k\mathbf{X}}, \omega_{\partial^k\mathbf{X}})$, where $\boldsymbol{\partial^k X}, \boldsymbol{\partial^{k+1} X}$ are d-spaces and $\boldsymbol{i}_{\partial^k\mathbf{X}} : \boldsymbol{\partial^k X} \to \boldsymbol{\partial^{k+1} X}$a 1-morphism. From §6.2, the topological space underlying $\boldsymbol{\partial^2 X}$ is

$$\partial^2 X = \bigl\{(x_1', x_2') : x_1', x_2' \in \underline{\partial X},\ x_1' \neq x_2',\ \underline{i}_{\mathbf{X}}(x_1') = \underline{i}_{\mathbf{X}}(x_2')\bigr\}, \tag{6.65}$$

where $\boldsymbol{i}_{\partial\mathbf{X}} : \boldsymbol{\partial^2 X} \to \boldsymbol{\partial X}$ maps $(x_1', x_2') \mapsto x_1'$. In §6.2 we also defined a 1-morphism $\boldsymbol{j}_{\partial\mathbf{X}} : \boldsymbol{\partial^2 X} \to \boldsymbol{\partial X}$ mapping $(x_1', x_2') \mapsto x_2'$, with $\boldsymbol{i}_{\mathbf{X}} \circ \boldsymbol{i}_{\partial\mathbf{X}} = \boldsymbol{i}_{\mathbf{X}} \circ \boldsymbol{j}_{\partial\mathbf{X}}$, such that (6.26) is locally 2-Cartesian. That is, $\boldsymbol{\partial^2 X}$ is equivalent to an open d-subspace of $\boldsymbol{\partial X} \times_{\boldsymbol{i}_{\mathbf{X}},\boldsymbol{X},\boldsymbol{i}_{\mathbf{X}}} \boldsymbol{\partial X}$.

Another way to write (6.65) is

$$\partial^2 X \cong \bigl\{(x,x_1',x_2') : x \in \underline{X},\ \ x_1' \neq x_2' \in \underline{\partial X},\ \ \underline{i}_{\mathbf{X}}(x_1') = \underline{i}_{\mathbf{X}}(x_2') = x\bigr\}. \qquad (6.66)$$

So we regard $\boldsymbol{\partial^2 X}$ as an open d-subspace of $\boldsymbol{X} \times_{\boldsymbol{\Delta}^2_{\boldsymbol{X}},\boldsymbol{X}\times\boldsymbol{X},\boldsymbol{i}_{\mathbf{X}}\times\boldsymbol{i}_{\mathbf{X}}} (\boldsymbol{\partial X}\times\boldsymbol{\partial X})$, where $\boldsymbol{\Delta}^2_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}\times\boldsymbol{X}$ is the diagonal 1-morphism. There is a natural 1-isomorphism $\boldsymbol{\rho}_2^{(12)} : \boldsymbol{\partial^2 X} \to \boldsymbol{\partial^2 X}$ with $\boldsymbol{\rho}_2^{(12)} = \mathbf{id}_{\boldsymbol{\partial^2 X}}$, acting on points in the representation (6.66) by $\boldsymbol{\rho}_2^{(12)} : (x,x_1',x_2') \mapsto (x,x_2',x_1')$, with $\boldsymbol{j}_{\partial\mathbf{X}} = \boldsymbol{i}_{\partial\mathbf{X}} \circ \boldsymbol{\rho}_2^{(12)}$ and $\boldsymbol{i}_{\partial\mathbf{X}} = \boldsymbol{j}_{\partial\mathbf{X}} \circ \boldsymbol{\rho}_2^{(12)}$. We think of $\boldsymbol{\rho}_2^{(12)}$ as defining a free action $\boldsymbol{\rho}_2$ of the symmetric group $S_2 = \bigl\{\mathrm{id},(12)\bigr\} \cong \mathbb{Z}_2$ on $\boldsymbol{\partial^2 X}$ by 1-isomorphisms.

More generally, by induction on k, one can show that the topological space $\partial^k X$ underlying $\boldsymbol{\partial^k X}$ is naturally homeomorphic to

$$\begin{aligned}\partial^k X \cong \bigl\{&(x,x_1',\ldots,x_k') : x \in \underline{X},\ \ x_1',\ldots,x_k' \in \underline{\partial X},\\ &\underline{i}_{\mathbf{X}}(x_1') = \cdots = \underline{i}_{\mathbf{X}}(x_k') = x,\ \ x_1',\ldots,x_k' \text{ are distinct}\bigr\}.\end{aligned} \qquad (6.67)$$

There are natural 1-morphisms $\boldsymbol{\pi}_{\boldsymbol{X}} : \boldsymbol{\partial^k X} \to \boldsymbol{X}$ and $\boldsymbol{\pi}^a_{\boldsymbol{\partial X}} : \boldsymbol{\partial^k X} \to \boldsymbol{\partial X}$ for $a = 1,\ldots,k$, with $\boldsymbol{i}_{\mathbf{X}} \circ \boldsymbol{\pi}^a_{\boldsymbol{\partial X}} = \boldsymbol{\pi}_{\boldsymbol{X}}$ for $a = 1,\ldots,k$, and such that under the isomorphism (6.67), on points we have $\boldsymbol{\pi}_{\boldsymbol{X}} : (x,x_1',\ldots,x_k') \mapsto x$ and $\boldsymbol{\pi}^a_{\boldsymbol{\partial X}} : (x,x_1',\ldots,x_k') \mapsto x_a'$, where $\boldsymbol{\pi}^1_{\boldsymbol{\partial X}} = \boldsymbol{i}_{\partial\mathbf{X}}$ and $\boldsymbol{\pi}^2_{\boldsymbol{\partial X}} = \boldsymbol{j}_{\partial\mathbf{X}}$ when $k = 2$. We have $\boldsymbol{\pi}_{\mathbf{X}} = \boldsymbol{i}_{\mathbf{X}} \circ \boldsymbol{i}_{\partial\mathbf{X}} \circ \cdots \circ \boldsymbol{i}_{\partial^{k-1}\mathbf{X}}$. Definition 6.1(b) for $\boldsymbol{i}_{\mathbf{X}},\ldots,\boldsymbol{i}_{\partial^{k-1}\mathbf{X}}$ therefore implies that $\pi''_{\mathbf{X}} : \underline{\pi}^*_{\mathbf{X}}(\mathcal{E}_X) \to \mathcal{E}_{\partial^k\mathbf{X}}$ is an isomorphism.

There is a natural, free action $\boldsymbol{\rho}_k$ of the symmetric group S_k on $\boldsymbol{\partial^k X}$ by 1-isomorphisms, where for $\sigma \in S_k$, $\boldsymbol{\rho}_k^\sigma : \boldsymbol{\partial^k X} \to \boldsymbol{\partial^k X}$ is a 1-isomorphism acting on points by $\boldsymbol{\rho}_k^\sigma : (x,x_1',\ldots,x_k') \mapsto (x,x_{\sigma(1)}',\ldots,x_{\sigma(k)}')$, and satisfying $\boldsymbol{\pi}_{\boldsymbol{X}} \circ \boldsymbol{\rho}_k^\sigma = \boldsymbol{\pi}_{\boldsymbol{X}}$ and $\boldsymbol{\pi}^a_{\boldsymbol{\partial X}} \circ \boldsymbol{\rho}_k^\sigma = \boldsymbol{\pi}^{\sigma(a)}_{\boldsymbol{\partial X}}$ for $a = 1,\ldots,k$. Under the isomorphisms (6.67), $\boldsymbol{i}_{\partial^k\mathbf{X}}$ maps $(x,x_1',\ldots,x_{k+1}') \mapsto (x,x_1',\ldots,x_k')$, and regarding S_k as the subgroup of S_{k+1} fixing $k+1$, for $\sigma \in S_k$ we have $\boldsymbol{i}_{\partial^k\mathbf{X}} \circ \boldsymbol{\rho}_{k+1}^\sigma = \boldsymbol{\rho}_k^\sigma \circ \boldsymbol{i}_{\partial^k\mathbf{X}}$.

We can relate $\boldsymbol{\partial^k X}$ locally to $\boldsymbol{X}$ in a 2-Cartesian diagram as follows. Let $(x,x_1',\ldots,x_k') \in \boldsymbol{\partial^k X}$, in the representation (6.67). Choose boundary defining functions $(\boldsymbol{V}_1,\boldsymbol{b}_1),\ldots,(\boldsymbol{V}_k,\boldsymbol{b}_k)$ for $\mathbf{X}$ at $x_1',\ldots,x_k'$. Then $x \in \boldsymbol{V}_a \subseteq \boldsymbol{X}$ is open, and there exists open $x_a' \in \boldsymbol{U}_a \subseteq \boldsymbol{\partial X}$ such that (6.1) is 2-Cartesian for $\boldsymbol{U}_a,\boldsymbol{V}_a,\boldsymbol{b}_a$, $a = 1,\ldots,k$. Making $\boldsymbol{U}_a,\boldsymbol{V}_a$ smaller for all a and using $x_1',\ldots,x_k'$ distinct and ∂X Hausdorff, we can suppose $\boldsymbol{V}_1 = \cdots = \boldsymbol{V}_a = \boldsymbol{V}$, say, and $\boldsymbol{U}_1,\ldots,\boldsymbol{U}_k$ are disjoint in $\boldsymbol{\partial X}$.

Define $\boldsymbol{W} = (\boldsymbol{\pi}^1_{\boldsymbol{\partial X}})^{-1}(\boldsymbol{U}_1) \cap \cdots \cap (\boldsymbol{\pi}^k_{\boldsymbol{\partial X}})^{-1}(\boldsymbol{U}_k)$, so that $(x,x_1',\ldots,x_k') \in \boldsymbol{W} \subseteq \boldsymbol{\partial^k X}$ is open. We now claim that the diagram in **dSpa**

$$\begin{tikzcd} \boldsymbol{W} \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{\pi}_{\mathbf{X}}|_{\boldsymbol{W}}"'] \arrow[drr, phantom, "\mathrm{id}_{(\mathbf{0},\ldots,\mathbf{0})\circ\boldsymbol{\pi}}\ \Uparrow"] && \boldsymbol{*} \arrow[d, "(\mathbf{0},\ldots,\mathbf{0})"] \\ \boldsymbol{V} \arrow[rr, "(\boldsymbol{b}_1,\boldsymbol{b}_2,\ldots,\boldsymbol{b}_k)"'] && {[\mathbf{0},\boldsymbol{\infty})^{\boldsymbol{k}}} \end{tikzcd} \qquad (6.68)$$

is 2-commutative and 2-Cartesian. Thus, just as locally $\boldsymbol{\partial X} \simeq \boldsymbol{X} \times_{[\mathbf{0},\boldsymbol{\infty})} \boldsymbol{*}$, so locally $\boldsymbol{\partial^k X} \simeq \boldsymbol{X} \times_{[\mathbf{0},\boldsymbol{\infty})^{\boldsymbol{k}}} \boldsymbol{*}$.

We prove this by induction on k. When $k = 1$ it follows from Definition 6.1(c), as (6.68) reduces to (6.1). For the inductive step, suppose the claim holds for k, let $(x, x'_1, \ldots, x'_{k+1}) \in \boldsymbol{\partial^{k+1} X}$, choose $\boldsymbol{V}, \boldsymbol{U}_1, \ldots, \boldsymbol{U}_{k+1}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{k+1}$ as above, and let $\boldsymbol{W}_k = (\boldsymbol{\pi}^1_{\boldsymbol{\partial X}})^{-1}(\boldsymbol{U}_1) \cap \cdots \cap (\boldsymbol{\pi}^k_{\boldsymbol{\partial X}})^{-1}(\boldsymbol{U}_k) \subseteq \boldsymbol{\partial^k X}$ and $\boldsymbol{W}_{k+1} = (\boldsymbol{\pi}^1_{\boldsymbol{\partial X}})^{-1}(\boldsymbol{U}_1) \cap \cdots \cap (\boldsymbol{\pi}^{k+1}_{\boldsymbol{\partial X}})^{-1}(\boldsymbol{U}_{k+1}) \subseteq \boldsymbol{\partial^{k+1} X}$. Consider the diagram:

$$\begin{tikzcd}
\boldsymbol{W}_{k+1} \arrow[rr, "\boldsymbol{\pi}"'] \arrow[d, "\boldsymbol{i}_{\partial^k \mathbf{X}}|_{\boldsymbol{W}_{k+1}}"] \arrow[dd, bend right, "\boldsymbol{\pi}_{\mathbf{X}}|_{\boldsymbol{W}_{k+1}}"'] & & * \arrow[d, "\boldsymbol{0}"'] \arrow[dd, bend left, "\boldsymbol{0}"] \\
\boldsymbol{W}_k \arrow[rr, "\boldsymbol{b}_{k+1} \circ \boldsymbol{\pi}_{\boldsymbol{X}}|_{\boldsymbol{W}_k}"] \arrow[d, "\boldsymbol{\pi}_{\mathbf{X}}|_{\boldsymbol{W}_k}"] \arrow[urr, phantom, "\mathrm{id}_{\boldsymbol{0} \circ \boldsymbol{\pi}} \Uparrow"] & & {[\boldsymbol{0}, \boldsymbol{\infty}) = * \times [\boldsymbol{0}, \boldsymbol{\infty})} \arrow[d, "(\boldsymbol{0}, \ldots, \boldsymbol{0}) \times \mathbf{id}_{[\boldsymbol{0}, \boldsymbol{\infty})}"'] \\
\boldsymbol{V} \arrow[rr, "(\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{k+1})"] \arrow[urr, phantom, "\mathrm{id}_{(\boldsymbol{0}, \ldots, \boldsymbol{0}) \circ \boldsymbol{\pi}} \Uparrow"] & & {[\boldsymbol{0}, \boldsymbol{\infty})^{\boldsymbol{k+1}} = [\boldsymbol{0}, \boldsymbol{\infty})^{\boldsymbol{k}} \times [\boldsymbol{0}, \boldsymbol{\infty}).}
\end{tikzcd}$$

The top rectangle is (6.1) for the boundary defining function $(\boldsymbol{W}_k, \boldsymbol{b}_{k+1} \circ \boldsymbol{\pi}_{\boldsymbol{X}}|_{\boldsymbol{W}_k})$ for $\partial^k \mathbf{X}$ at $(x, x'_1, \ldots, x'_{k+1})$, and so is 2-commutative and 2-Cartesian. The bottom rectangle is the extension of (6.68) for $\boldsymbol{W}_k$, which is 2-commutative and 2-Cartesian by induction, by a 1-morphism to $[\boldsymbol{0}, \boldsymbol{\infty})$. Thus the bottom rectangle is 2-commutative and 2-Cartesian, and hence the outer rectangle is 2-commutative and 2-Cartesian. This proves the inductive step.

Definition 6.28. Let $\mathbf{X}$ be a d-space with corners. For each $k = 0, 1, 2, \ldots$ we will define a d-space with corners $C_k(\mathbf{X})$ called the *k-corners* of $\mathbf{X}$, and a 1-morphism $\boldsymbol{\Pi}^k_{\mathbf{X}} : C_k(\mathbf{X}) \to \mathbf{X}$ in $\mathbf{dSpa^c}$. These will have the following properties. Write $C_k(\mathbf{X}) = \big(\boldsymbol{C_k(X)}, \boldsymbol{\partial C_k(X)}, \boldsymbol{i}_{C_k(\mathbf{X})}, \omega_{C_k(\mathbf{X})}\big)$, where $\boldsymbol{C_k(X)}, \boldsymbol{\partial C_k(X)}$ are d-spaces with underlying topological spaces $C_k(X), \partial C_k(X)$. Then we have

$$\begin{aligned} C_k(X) = \big\{ &(x, \{x'_1, \ldots, x'_k\}) : x \in \boldsymbol{X},\ \ x'_1, \ldots, x'_k \in \boldsymbol{\partial X}, \\ &\boldsymbol{i}_{\mathbf{X}}(x'_a) = x,\ \ a = 1, \ldots, k,\ \ x'_1, \ldots, x'_k \text{ are distinct} \big\}, \end{aligned} \tag{6.69}$$

$$\begin{aligned} \partial C_k(X) = \big\{ &(x, \{x'_1, \ldots, x'_k\}, x'_{k+1}) : x \in \boldsymbol{X},\ \ x'_1, \ldots, x'_{k+1} \in \boldsymbol{\partial X}, \\ &\boldsymbol{i}_{\mathbf{X}}(x'_a) = x,\ \ a = 1, \ldots, k+1,\ \ x'_1, \ldots, x'_{k+1} \text{ are distinct} \big\}. \end{aligned} \tag{6.70}$$

On points, $\boldsymbol{i}_{C_k(\mathbf{X})}, \boldsymbol{\Pi}^k_{\mathbf{X}}$ act by $\boldsymbol{i}_{C_k(\mathbf{X})} : (x, \{x'_1, \ldots, x'_k\}, x'_{k+1}) \mapsto (x, \{x'_1, \ldots, x'_k\})$ and $\boldsymbol{\Pi}^k_{\mathbf{X}} : (x, \{x'_1, \ldots, x'_k\}) \mapsto x$. There is a unique local 1-isomorphism $\boldsymbol{q}_{\mathbf{X}} : \partial^k \mathbf{X} \to C_k(\mathbf{X})$, acting on points by $\boldsymbol{q}_{\mathbf{X}} : (x, x'_1, \ldots, x'_k) \mapsto (x, \{x'_1, \ldots, x'_k\})$ using the representation (6.67) for $\partial^k \mathbf{X}$, such that

$$\boldsymbol{\Pi}^k_{\mathbf{X}} \circ \boldsymbol{q}_{\mathbf{X}} = \boldsymbol{\pi}_{\mathbf{X}} : \partial^k \mathbf{X} \longrightarrow \mathbf{X}. \tag{6.71}$$

In a similar way to Definition 6.1(b), $(\Pi^k_{\mathbf{X}})'' : (\underline{\Pi}^k_{\mathbf{X}})^*(\mathcal{E}_X) \to \mathcal{E}_{C_k(X)}$ is an isomorphism. Also $\boldsymbol{\Pi}^0_{\mathbf{X}} : C_0(\mathbf{X}) \to \mathbf{X}$ is a 1-isomorphism, and there is a unique 1-isomorphism $\boldsymbol{c}_{\mathbf{X}} : C_1(\mathbf{X}) \to \partial \mathbf{X}$ acting on points by $\boldsymbol{c}_{\mathbf{X}} : \big(\boldsymbol{i}_{\mathbf{X}}(x'), \{x'\}\big) \mapsto x'$, such that $\boldsymbol{\Pi}^1_{\mathbf{X}} = \boldsymbol{i}_{\mathbf{X}} \circ \boldsymbol{c}_{\mathbf{X}} : C_1(\mathbf{X}) \to \mathbf{X}$. Thus, $C_0(\mathbf{X}) \cong \mathbf{X}$ and $C_1(\mathbf{X}) \cong \partial \mathbf{X}$.

In Remark 6.27 we explained that there is a natural, free action $\boldsymbol{\rho}_k$ of the symmetric group S_k on $\partial^k \mathbf{X}$ by 1-isomorphisms. The basic idea is that $C_k(\mathbf{X}) = (\partial^k \mathbf{X}) / \boldsymbol{\rho}_k(S_k)$, and $\boldsymbol{q}_{\mathbf{X}} : \partial^k \mathbf{X} \to \partial^k \mathbf{X} / S_k = C_k(\mathbf{X})$ is the projection to the quotient. It is not difficult to show that quotients of d-spaces by free finite groups of 1-isomorphisms always exist in **dSpa**, and are unique up to canonical 1-isomorphism. The underlying topological space $C_k(X)$ of $\boldsymbol{C_k(X)}$ is therefore homeomorphic to the quotient of (6.67) by S_k, where S_k acts by permuting $x'_1, \ldots, x'_k$.

The effect of quotienting by S_k is to replace the ordered k-tuple $(x'_1, \ldots, x'_k)$ by the unordered set $\{x'_1, \ldots, x'_k\}$. Hence $C_k(X) = \partial^k X / S_k$ is homeomorphic to the right hand side of (6.69). Since the quotient $C_k(\mathbf{X}) = \partial^k \mathbf{X}/S_k$ only specifies $C_k(\mathbf{X})$ up to canonical 1-isomorphism, not equality, we are free to choose $C_k(\mathbf{X})$ so that (6.69) is an equality, not just a canonical homeomorphism. Similarly, the action of S_k on $\boldsymbol{\partial^{k+1} X}$ permutes $x'_1, \ldots, x'_k$ in $(x, x'_1, \ldots, x'_{k+1})$, so the effect of quotienting by S_k is to turn $(x, x'_1, \ldots, x_{k+1})$ into $(x, \{x'_1, \ldots, x'_k\}, x'_{k+1})$, and we may choose $C_k(\mathbf{X})$ so that (6.70) holds.

This explains the construction of $C_k(\mathbf{X})$ and $\boldsymbol{q}_{\mathbf{X}}$. To define $\boldsymbol{\Pi}^k_{\mathbf{X}}$, note that $\boldsymbol{\pi}_{\mathbf{X}} : \partial^k \mathbf{X} \to \mathbf{X}$ satisfies $\boldsymbol{\pi}_{\mathbf{X}} \circ \boldsymbol{\rho}^\sigma_k = \boldsymbol{\pi}_{\mathbf{X}}$ for all $\sigma \in S_k$. Therefore it factors through the quotient $C_k(\mathbf{X}) = (\partial^k \mathbf{X})/\boldsymbol{\rho}_k(S_k)$, and there is a unique 1-morphism $\boldsymbol{\Pi}^k_{\mathbf{X}} : C_k(\mathbf{X}) \to \mathbf{X}$ satisfying (6.71). From (6.71), $\boldsymbol{q}_{\mathbf{X}}$ a local 1-isomorphism, and $\pi''_{\mathbf{X}} : \underline{\pi}^*_{\mathbf{X}}(\mathcal{E}_X) \to \mathcal{E}_{\partial^k \mathbf{X}}$ an isomorphism, we deduce that $(\Pi^k_{\mathbf{X}})'' : (\underline{\Pi}^k_{\mathbf{X}})^*(\mathcal{E}_X) \to \mathcal{E}_{C_k(X)}$ is an isomorphism. The remaining claims are easy to check.

Now let $(x, \{x'_1, \ldots, x'_k\}) \in C_k(\mathbf{X})$. Then $\{x'_1, \ldots, x'_k\}$ is an unordered set, but by numbering the points we have implicitly chosen a preimage $(x, x'_1, \ldots, x'_k)$ of $(x, \{x'_1, \ldots, x'_k\})$ in $\partial^k \mathbf{X}$. Let $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_k, \boldsymbol{V}, \boldsymbol{W}$ and $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ be as in Remark 6.27 for $(x, x'_1, \ldots, x'_k)$, so that $x'_a \in \boldsymbol{U}_a \subseteq \boldsymbol{\partial X}$ is open, and $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_k$ are disjoint in $\boldsymbol{\partial X}$, and $\boldsymbol{W} = (\boldsymbol{\pi}^1_{\boldsymbol{\partial X}})^{-1}(\boldsymbol{U}_1) \cap \cdots \cap (\boldsymbol{\pi}^k_{\boldsymbol{\partial X}})^{-1}(\boldsymbol{U}_k)$ is an open neighbourhood of $(x, x'_1, \ldots, x'_k)$ in $\partial^k \mathbf{X}$ fitting into a 2-Cartesian diagram (6.68) in **dSpa**. Let $\tilde{\boldsymbol{W}} = \boldsymbol{q}_{\mathbf{X}}(\boldsymbol{W})$. Then as $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_k$ are disjoint, $\tilde{\boldsymbol{W}}$ is open in $\boldsymbol{C_k(X)}$, and $\boldsymbol{q}_{\mathbf{X}}|_{\boldsymbol{W}} : \boldsymbol{W} \to \tilde{\boldsymbol{W}}$ is a 1-isomorphism in **dSpa**. So (6.68) and (6.71) imply that we have a 2-Cartesian diagram in **dSpa**:

$$\begin{tikzcd}
\tilde{\boldsymbol{W}} \arrow[rr, "\boldsymbol{\pi}"'] \arrow[d, "\boldsymbol{\Pi}^k_{\mathbf{X}}|_{\tilde{\boldsymbol{W}}}"] & \arrow[d, phantom, "\mathrm{id}_{(\boldsymbol{0},\ldots,\boldsymbol{0})\circ\boldsymbol{\pi}} \Uparrow"] & \boldsymbol{*} \arrow[d, "(\boldsymbol{0},\ldots,\boldsymbol{0})"'] \\
\boldsymbol{V} \arrow[rr, "(\boldsymbol{b}_1,\boldsymbol{b}_2,\ldots,\boldsymbol{b}_k)"] & & {[\boldsymbol{0},\boldsymbol{\infty})^{\boldsymbol{k}}}
\end{tikzcd} \tag{6.72}$$

Thus locally $\boldsymbol{C_k(X)} \simeq \boldsymbol{X} \times_{[\boldsymbol{0},\boldsymbol{\infty})^{\boldsymbol{k}}} \boldsymbol{*}$.

Write $C(\mathbf{X}) = \coprod_{k=0}^\infty C_k(\mathbf{X})$ and $\boldsymbol{\Pi}_{\mathbf{X}} = \coprod_{k=0}^\infty \boldsymbol{\Pi}^k_{\mathbf{X}}$, so that $C(\mathbf{X})$ is a d-space with corners and $\boldsymbol{\Pi}_{\mathbf{X}} : C(\mathbf{X}) \to \mathbf{X}$ is a 1-morphism.

Here is the analogue of Theorem 5.17(i)–(iii). Note the similarity of (a),(b) to Theorem 6.12(b),(c), and of (c) to Corollary 6.14. The proof of Theorem 6.29 is long but uses very similar ideas to those of Theorem 6.12, Proposition 6.13 and Corollary 6.14, so we leave it as an exercise.

Theorem 6.29. **(a)** *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of d-spaces with corners. Then there is a unique 1-morphism $C(\boldsymbol{f}) : C(\mathbf{X}) \to C(\mathbf{Y})$ in $\mathbf{dSpa^c}$ such that $\boldsymbol{\Pi}_{\mathbf{Y}} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \boldsymbol{\Pi}_{\mathbf{X}} : C(\mathbf{X}) \to \mathbf{Y}$, and $C(\boldsymbol{f})$ acts on points as in (5.6) by*

$$\begin{aligned} C(f) : \bigl(x, \{x'_1, \ldots, x'_k\}\bigr) &\longmapsto \bigl(y, \{y'_1, \ldots, y'_l\}\bigr), \quad \textit{where} \\ \{y'_1, \ldots, y'_l\} &= \bigl\{y' : (x'_i, y') \in \underline{S}_{\boldsymbol{f}},\ \textit{some } i = 1, \ldots, k\bigr\}. \end{aligned} \tag{6.73}$$

For all $k,l \geqslant 0$, write $C_k^{\boldsymbol{f},l}(\mathbf{X}) = C_k(\mathbf{X}) \cap C(\boldsymbol{f})^{-1}(C_l(\mathbf{Y}))$, so that $C_k^{\boldsymbol{f},l}(\mathbf{X})$ is an open and closed d-subspace of $C_k(\mathbf{X})$ with $C_k(\mathbf{X}) = \coprod_{l=0}^{\infty} C_k^{\boldsymbol{f},l}(\mathbf{X})$, and write $C_k^l(\boldsymbol{f}) = C(\boldsymbol{f})|_{C_k^{\boldsymbol{f},l}(\mathbf{X})}$, so that $C_k^l(\boldsymbol{f}) : C_k^{\boldsymbol{f},l}(\mathbf{X}) \to C_l(\mathbf{Y})$ is a 1-morphism in $\mathbf{dSpa^c}$. Then $C_k^{\boldsymbol{f},l}(\mathbf{X})$ has underlying topological space

$$
\begin{aligned}
C_k^{\boldsymbol{f},l}(X) = \big\{&(x,\{x'_1,\ldots,x'_k\}) \in C_k(X) : \\
&|\{y' : (x'_i,y') \in \underline{S}_{\boldsymbol{f}},\ \text{some } i=1,\ldots,k\}| = l\big\}.
\end{aligned}
\tag{6.74}
$$

(b) *Let $\boldsymbol{f},\boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ be 1-morphisms and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism in $\mathbf{dSpa^c}$. Then there exists a unique 2-morphism $C(\eta) : C(\boldsymbol{f}) \Rightarrow C(\boldsymbol{g})$ in $\mathbf{dSpa^c}$, where $C(\boldsymbol{f}), C(\boldsymbol{g})$ are as in* **(a)**, *such that*

$$
\mathrm{id}_{\mathbf{\Pi_Y}} * C(\eta) = \eta * \mathrm{id}_{\mathbf{\Pi_X}} : \mathbf{\Pi_Y} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \mathbf{\Pi_X} \Longrightarrow \mathbf{\Pi_Y} \circ C(\boldsymbol{g}) = \boldsymbol{g} \circ \mathbf{\Pi_X}. \tag{6.75}
$$

(c) *Define $C : \mathbf{dSpa^c} \to \mathbf{dSpa^c}$ by $C : \mathbf{X} \mapsto C(\mathbf{X})$ on objects, where $C(\mathbf{X})$ is as in Definition 6.28, and $C : \boldsymbol{f} \mapsto C(\boldsymbol{f})$, $C : \eta \mapsto C(\eta)$ on 1- and 2-morphisms, where $C(\boldsymbol{f}), C(\eta)$ are as in* **(a)**,**(b)** *above. Then C is a strict 2-functor, which we call a* ***corner functor***.

One can also show that in the analogue of Theorem 5.17(iv), there is a natural 1-isomorphism $\partial C_k(\mathbf{X}) \to C_k(\partial \mathbf{X})$, which acts on points by

$$
\big(x,\{x'_1,\ldots,x'_k\},x'_{k+1}\big) \longmapsto \big(x'_{k+1},\big\{(x'_{k+1},x'_1),\ldots,(x'_{k+1},x'_k)\big\}\big),
$$

but we will not use this. The next proposition, the analogue of Theorem 5.17(vii), relates Theorems 6.12 and 6.29. We can also prove that if $\boldsymbol{f},\boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ are semisimple and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism, then η_- and $C(\eta)|_{C_1^{\boldsymbol{f},1}(\mathbf{X})}$ are related in the obvious way.

Proposition 6.30. *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a semisimple 1-morphism in $\mathbf{dSpa^c}$, and use the notation of Theorems 6.12 and 6.29. Then $C(\boldsymbol{f})$ maps $C_k(\mathbf{X}) \to \coprod_{l=0}^{k} C_l(\mathbf{Y})$ for all $k \geqslant 0$, that is, $C_k^{\boldsymbol{f},l}(\mathbf{X}) = \emptyset$ for $l > k$. As in Definition 6.28 there are natural 1-isomorphisms $\boldsymbol{c}_{\mathbf{X}} : C_1(\mathbf{X}) \to \partial\mathbf{X}$, $\boldsymbol{c}_{\mathbf{Y}} : C_1(\mathbf{Y}) \to \partial\mathbf{Y}$ and $\mathbf{\Pi}_{\mathbf{Y}}^0 : C_0(\mathbf{Y}) \to \mathbf{Y}$. We have $\boldsymbol{c}_{\mathbf{X}}\big(C_1^{\boldsymbol{f},0}(\mathbf{X})\big) = \partial_+^{\boldsymbol{f}}\mathbf{X}$ and $\boldsymbol{c}_{\mathbf{X}}\big(C_1^{\boldsymbol{f},1}(\mathbf{X})\big) = \partial_-^{\boldsymbol{f}}\mathbf{X}$, with $\mathbf{\Pi}_{\mathbf{Y}}^0 \circ C_1^0(\boldsymbol{f}) = \boldsymbol{f}_+ \circ \boldsymbol{c}_{\mathbf{X}}|_{C_1^{\boldsymbol{f},0}(\mathbf{X})}$ and $\boldsymbol{c}_{\mathbf{Y}} \circ C_1^1(\boldsymbol{f}) = \boldsymbol{f}_- \circ \boldsymbol{c}_{\mathbf{X}}|_{C_1^{\boldsymbol{f},1}(\mathbf{X})}$.*

If $\boldsymbol{f}$ is simple then $C(\boldsymbol{f})$ maps $C_k(\mathbf{X}) \to C_k(\mathbf{Y})$ for all $k \geqslant 0$.

Proof. Suppose $C(\boldsymbol{f}) : \big(x,\{x'_1,\ldots,x'_k\}\big) \longmapsto \big(y,\{y'_1,\ldots,y'_l\}\big)$. Let $j = 1,\ldots,l$. Then $(x'_i,y'_j) \in \underline{S}_{\boldsymbol{f}}$ for some $i = 1,\ldots,k$. As $\underline{j}_{\boldsymbol{f}} : (x'_i,y'_j) \mapsto (x,y'_j)$ is injective, this i is unique. Since $\boldsymbol{f}$ is semisimple, $\underline{s}_{\boldsymbol{f}} : (x'_i,y'_j) \mapsto x'_i$ is injective, so distinct $j,j' = 1,\ldots,l$ lift to distinct $i,i' = 1,\ldots,k$. Hence the map $j \mapsto i$ is an injective map $\{1,\ldots,l\} \to \{1,\ldots,k\}$, so $k \geqslant l$. This proves $C(\boldsymbol{f})$ maps $C_k(\mathbf{X}) \to \coprod_{l=0}^{k} C_l(\mathbf{Y})$ for all $k \geqslant 0$. If $\boldsymbol{f}$ is simple then $\underline{s}_{\boldsymbol{f}}$ is bijective, so the map $j \mapsto i$ is a bijection $\{1,\ldots,l\} \to \{1,\ldots,k\}$, forcing $k = l$, and $C(\boldsymbol{f})$ maps $C_k(\mathbf{X}) \to C_k(\mathbf{Y})$ for all $k \geqslant 0$, proving the last part.

It now follows from (6.73) that

$$C_1^{\boldsymbol{f},0}(X)=\bigl\{(x,\{x'\})\in C_1(X):\nexists y'\in\partial Y\text{ with }(x',y')\in\underline{S}_{\boldsymbol{f}}\bigr\},$$
$$C_1^{\boldsymbol{f},1}(X)=\bigl\{(x,\{x'\})\in C_1(X):\exists y'\in\partial Y\text{ with }(x',y')\in\underline{S}_{\boldsymbol{f}}\bigr\}.$$

Since $\boldsymbol{c}_{\mathbf{X}}$ maps $(x,\{x'\})\mapsto x'$ on points, we see that $\boldsymbol{c}_{\mathbf{X}}\bigl(C_1^{\boldsymbol{f},0}(\mathbf{X})\bigr)=\partial_+^{\boldsymbol{f}}\mathbf{X}$ and $\boldsymbol{c}_{\mathbf{X}}\bigl(C_1^{\boldsymbol{f},1}(\mathbf{X})\bigr)=\partial_-^{\boldsymbol{f}}\mathbf{X}$. Using $\boldsymbol{\Pi}_{\mathbf{Y}}\circ C(\boldsymbol{f})=\boldsymbol{f}\circ\Pi_{\mathbf{X}}$, $\boldsymbol{\Pi}_{\mathbf{X}}^1=\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{c}_{\mathbf{X}}$ and $\boldsymbol{f}_+=\boldsymbol{f}\circ\boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}}\mathbf{X}}$ we deduce that

$$\boldsymbol{\Pi}_{\mathbf{Y}}^0\circ C_1^0(\boldsymbol{f})=\boldsymbol{f}\circ\boldsymbol{\Pi}_{\mathbf{X}}^1|_{C_1^{\boldsymbol{f},0}(\mathbf{X})}=\boldsymbol{f}\circ\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{c}_{\mathbf{X}}|_{C_1^{\boldsymbol{f},0}(\mathbf{X})}=\boldsymbol{f}_+\circ\boldsymbol{c}_{\mathbf{X}}|_{C_1^{\boldsymbol{f},0}(\mathbf{X})}.$$

Similarly we find that $\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{c}_{\mathbf{Y}}\circ C_1^1(\boldsymbol{f})=\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{f}_-\circ\boldsymbol{c}_{\mathbf{X}}|_{C_1^{\boldsymbol{f},1}(\mathbf{X})}$. Then $\boldsymbol{c}_{\mathbf{Y}}\circ C_1^1(\boldsymbol{f})=\boldsymbol{f}_-\circ\boldsymbol{c}_{\mathbf{X}}|_{C_1^{\boldsymbol{f},1}(\mathbf{X})}$ follows from $\boldsymbol{c}_{\mathbf{X}}$ a 1-isomorphism and uniqueness of $\boldsymbol{f}_-$ in Theorem 6.12(b). $\square$

As in §5.5, in $\mathbf{Man^c}$ there is a second corner functor $\hat C:\mathbf{Man^c}\rightarrow\check{\mathbf{M}}\mathbf{an^c}$ defined by (5.8). The proof of the next theorem is very similar to that of Theorem 6.29, except that Theorem 6.29(b) uses (6.9) for η and arguments similar to Proposition 6.8(b), whereas Theorem 6.31(c) also uses (6.10) for η and arguments similar to Proposition 6.8(c).

Theorem 6.31. **(a)** *Let $\boldsymbol{f}:\mathbf{X}\rightarrow\mathbf{Y}$ be a 1-morphism of d-spaces with corners. Then there is a unique 1-morphism $\hat C(\boldsymbol{f}):C(\mathbf{X})\rightarrow C(\mathbf{Y})$ in $\mathbf{dSpa^c}$ such that $\boldsymbol{\Pi}_{\mathbf{Y}}\circ\hat C(\boldsymbol{f})=\boldsymbol{f}\circ\boldsymbol{\Pi}_{\mathbf{X}}:C(\mathbf{X})\rightarrow\mathbf{Y}$, and $\hat C(\boldsymbol{f})$ acts on points as in* (5.8) *by*

$$\begin{aligned}&\hat C(f):\bigl(x,\{x_1',\ldots,x_k'\}\bigr)\longmapsto\bigl(y,\{y_1',\ldots,y_l'\}\bigr),\quad\textit{where}\\&\{y_1',\ldots,y_l'\}=\bigl\{y':(x_i',y')\in\underline{S}_{\boldsymbol{f}},\ \textit{some } i=1,\ldots,k\bigr\}\cup\bigl\{y':(x,y')\in\underline{T}_{\boldsymbol{f}}\bigr\}.\end{aligned}\tag{6.76}$$

For all $k,l\geqslant 0$, write $\hat C_k^{\boldsymbol{f},l}(\mathbf{X})=C_k(\mathbf{X})\cap\hat C(\boldsymbol{f})^{-1}(C_l(\mathbf{Y}))$, so that $\hat C_k^{\boldsymbol{f},l}(\mathbf{X})$ is an open and closed d-subspace of $C_k(\mathbf{X})$ with $C_k(\mathbf{X})=\coprod_{l=0}^\infty\hat C_k^{\boldsymbol{f},l}(\mathbf{X})$, and write $\hat C_k^l(\boldsymbol{f})=\hat C(\boldsymbol{f})|_{\hat C_k^{\boldsymbol{f},l}(\mathbf{X})}$, so that $\hat C_k^l(\boldsymbol{f}):\hat C_k^{\boldsymbol{f},l}(\mathbf{X})\rightarrow C_l(\mathbf{Y})$ is a 1-morphism in $\mathbf{dSpa^c}$. Then $\hat C_k^{\boldsymbol{f},l}(\mathbf{X})$ has underlying topological space

$$\begin{aligned}&\hat C_k^{\boldsymbol{f},l}(X)=\bigl\{(x,\{x_1',\ldots,x_k'\})\in C_k(X):\\&\bigl|\{y':(x_i',y')\in\underline{S}_{\boldsymbol{f}},\ \textit{some } i=1,\ldots,k\}\cup\{y':(x,y')\in\underline{T}_{\boldsymbol{f}}\}\bigr|=l\bigr\}.\end{aligned}\tag{6.77}$$

If $\boldsymbol{f}$ is flat then $\hat C(\boldsymbol{f})=C(\boldsymbol{f})$, for $C(\boldsymbol{f})$ as in Theorem 6.29(a).

(b) *Let $\boldsymbol{f},\boldsymbol{g}:\mathbf{X}\rightarrow\mathbf{Y}$ be 1-morphisms and $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a 2-morphism in $\mathbf{dSpa^c}$. Then there exists a unique 2-morphism $\hat C(\eta):\hat C(\boldsymbol{f})\Rightarrow\hat C(\boldsymbol{g})$ in $\mathbf{dSpa^c}$, where $\hat C(\boldsymbol{f}),\hat C(\boldsymbol{g})$ are as in* **(a)**, *such that*

$$\mathrm{id}_{\boldsymbol{\Pi}_{\mathbf{Y}}}*\hat C(\eta)=\eta*\mathrm{id}_{\boldsymbol{\Pi}_{\mathbf{X}}}:\boldsymbol{\Pi}_{\mathbf{Y}}\circ\hat C(\boldsymbol{f})=\boldsymbol{f}\circ\boldsymbol{\Pi}_{\mathbf{X}}\Longrightarrow\boldsymbol{\Pi}_{\mathbf{Y}}\circ\hat C(\boldsymbol{g})=\boldsymbol{g}\circ\boldsymbol{\Pi}_{\mathbf{X}}.\tag{6.78}$$

If $\boldsymbol{f},\boldsymbol{g}$ are flat then $\hat C(\eta)=C(\eta)$, for $C(\eta)$ as in Theorem 6.29(b).

(c) *Define* $\hat{C}:\mathbf{dSpa^c}\to\mathbf{dSpa^c}$ *by* $\hat{C}:\mathbf{X}\mapsto C(\mathbf{X})$ *on objects, where* $C(\mathbf{X})$ *is as in Definition* 6.28, *and* $\boldsymbol{f}\mapsto\hat{C}(\boldsymbol{f})$, $\eta\mapsto\hat{C}(\eta)$ *on* 1- *and* 2*-morphisms, where* $\hat{C}(\boldsymbol{f}),\hat{C}(\eta)$ *are as in* **(a)**,**(b)** *above. Then* $\hat{C}$ *is a strict* 2*-functor, which we call a* ***corner functor***.

As for (5.9), on the level of sets the functors $C,\hat{C}$ are related by

$$\begin{aligned}&C(f):\bigl(x,\{x'_1,\ldots,x'_k\}\bigr)\longmapsto\bigl(y,\{y'_1,\ldots,y'_l\}\bigr)\quad\text{if and only if}\\&\hat{C}(f):\bigl(x,\boldsymbol{i}_{\mathbf{X}}^{-1}(x)\setminus\{x'_1,\ldots,x'_k\}\bigr)\longmapsto\bigl(y,\boldsymbol{i}_{\mathbf{Y}}^{-1}(y)\setminus\{y'_1,\ldots,y'_l\}\bigr).\end{aligned}\tag{6.79}$$

That is, $C,\hat{C}$ are related by taking complements of subsets in $\boldsymbol{i}_{\mathbf{X}}^{-1}(x),\boldsymbol{i}_{\mathbf{Y}}^{-1}(y)$.

6.8 Fibre products in $\mathbf{dSpa^c}$

Finally we study fibre products in $\mathbf{dSpa^c}$. In contrast to d-spaces in §2.5, not all fibre products exist in $\mathbf{dSpa^c}$, as Example 6.47 below shows. If $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$ and $\boldsymbol{h}:\mathbf{Y}\to\mathbf{Z}$ are 1-morphisms in $\mathbf{dSpa^c}$, we will define when $\boldsymbol{g},\boldsymbol{h}$ are *b-transverse* and *c-transverse.* These are mild conditions on how $\boldsymbol{g},\boldsymbol{h}$ behave over $\partial^j\mathbf{X},\partial^k\mathbf{Y},\partial^l\mathbf{Z}$, where c-transverse implies b-transverse. B-transverse and c-transverse 1-morphisms are analogous to transverse and strongly transverse maps in $\mathbf{Man^c}$ in §5.6, respectively. If $\boldsymbol{g},\boldsymbol{h}$ are b-transverse, we will prove that a fibre product $\mathbf{W}=\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ exists in $\mathbf{dSpa^c}$. If they are c-transverse, we will show also that

$$C(\mathbf{W})\simeq C(\mathbf{X})\times_{C(\boldsymbol{g}),\mathbf{Z},C(\boldsymbol{h})}C(\mathbf{Y})\simeq C(\mathbf{X})\times_{\hat{C}(\boldsymbol{g}),\mathbf{Z},\hat{C}(\boldsymbol{h})}C(\mathbf{Y}),$$

for $C,\hat{C}$ as in §6.7, where $C(\boldsymbol{g}),C(\boldsymbol{h})$ and $\hat{C}(\boldsymbol{g}),\hat{C}(\boldsymbol{h})$ are also c-transverse.

6.8.1 The definitions of b-transversality and c-transversality

When $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$, $\boldsymbol{h}:\mathbf{Y}\to\mathbf{Z}$ are 1-morphisms in $\mathbf{dSpa^c}$, we will define what it means for $\boldsymbol{g},\boldsymbol{h}$ to be 'b-transverse' and 'c-transverse', which are short for 'boundary-transverse' and 'corners-transverse'. In Corollary 6.39 we will show that c-transverse implies b-transverse. In Theorem 6.45 we will show that if $g:X\to Z$, $h:Y\to Z$ are smooth maps in $\mathbf{Man^c}$ and $\mathbf{X},\mathbf{Y},\mathbf{Z},\boldsymbol{g},\boldsymbol{h}=F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X,Y,Z,g,h)$ then g,h transverse (or strongly transverse) imply that $\boldsymbol{g},\boldsymbol{h}$ are b-transverse (or c-transverse, respectively).

Definition 6.32. Let $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$ and $\boldsymbol{h}:\mathbf{Y}\to\mathbf{Z}$ be 1-morphisms in $\mathbf{dSpa^c}$. We call $\boldsymbol{g},\boldsymbol{h}$ *b-transverse* if whenever $x\in\underline{X}$ and $y\in\underline{Y}$ with $\underline{g}(x)=\underline{h}(y)=z\in\underline{Z}$, the following morphism in qcoh($\underline{*}$) is injective:

$$\begin{aligned}&\bigoplus_{(x',z')\in\underline{S}_{\boldsymbol{g}}:\underline{i}_{\mathbf{X}}(x')=x}I_{\underline{(x',z')},\underline{s}_{\boldsymbol{g}}}(\mathcal{N}_{\mathbf{X}})^{-1}\circ\underline{(x',z')}^*(\lambda_{\boldsymbol{g}})\circ I_{\underline{(x',z')},\underline{u}_{\boldsymbol{g}}}(\mathcal{N}_{\mathbf{Z}})\oplus\\&\bigoplus_{(y',z')\in\underline{S}_{\boldsymbol{h}}:\underline{i}_{\mathbf{Y}}(y')=y}I_{\underline{(y',z')},\underline{s}_{\boldsymbol{h}}}(\mathcal{N}_{\mathbf{Y}})^{-1}\circ\underline{(y',z')}^*(\lambda_{\boldsymbol{h}})\circ I_{\underline{(y',z')},\underline{u}_{\boldsymbol{h}}}(\mathcal{N}_{\mathbf{Z}}):\\&\bigoplus_{z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{z}')^*(\mathcal{N}_{\mathbf{Z}})\longrightarrow\bigoplus_{x'\in\underline{i}_{\mathbf{X}}^{-1}(x)}(\underline{x}')^*(\mathcal{N}_{\mathbf{X}})\oplus\bigoplus_{y'\in\underline{i}_{\mathbf{Y}}^{-1}(y)}(\underline{y}')^*(\mathcal{N}_{\mathbf{Y}}).\end{aligned}\tag{6.80}$$

Here if $x' \in \underline{i}_{\mathbf{X}}^{-1}(x) \subseteq \underline{\partial X}$ we write $\underline{x}' : \underline{*} \to \underline{\partial X}$ for the corresponding morphism in $\mathbf{C^\infty Sch}$, and similarly for $\underline{y}', \underline{z}'$, and if $(x', z') \in \underline{S}_{\boldsymbol{g}}$ we write $\underline{(x', z')} : \underline{*} \to \underline{S}_{\boldsymbol{g}}$ for the corresponding morphism in $\mathbf{C^\infty Sch}$, and similarly for $\underline{(y', z')}$. Also $\underline{s}_{\boldsymbol{g}}, \underline{u}_{\boldsymbol{g}}, \underline{s}_{\boldsymbol{h}}, \underline{u}_{\boldsymbol{h}}$ are as in Definition 6.2, and $\lambda_{\boldsymbol{g}}, \lambda_{\boldsymbol{h}}$ are as in Proposition 6.7(d). Note that if $(x', z') \in \underline{S}_{\boldsymbol{g}}$ then $\underline{x}' = \underline{s}_{\boldsymbol{g}} \circ \underline{(x', z')} : \underline{*} \to \underline{\partial X}$ and then $\underline{z}' = \underline{u}_{\boldsymbol{g}} \circ \underline{(x', z')} : \underline{*} \to \underline{\partial Z}$, and similarly for $(y', z') \in \underline{S}_{\boldsymbol{h}}$.

Informally we may rewrite (6.80) as

$$\begin{aligned} &\bigoplus_{(x',z')\in \underline{S}_{\boldsymbol{g}}:\underline{i}_{\mathbf{X}}(x')=x} \lambda_{\boldsymbol{g}}|_{(x',z')} \oplus \bigoplus_{(y',z')\in \underline{S}_{\boldsymbol{h}}:\underline{i}_{\mathbf{Y}}(y')=y} \lambda_{\boldsymbol{h}}|_{(y',z')} : \\ &\bigoplus_{z'\in \underline{i}_{\mathbf{Z}}^{-1}(z)} \mathcal{N}_{\mathbf{Z}}|_{z'} \longrightarrow \bigoplus_{x'\in \underline{i}_{\mathbf{X}}^{-1}(x)} \mathcal{N}_{\mathbf{X}}|_{x'} \oplus \bigoplus_{y'\in \underline{i}_{\mathbf{Y}}^{-1}(y)} \mathcal{N}_{\mathbf{Y}}|_{y'}, \end{aligned} \tag{6.81}$$

where as $\mathcal{N}_{\mathbf{X}}, \mathcal{N}_{\mathbf{Y}}, \mathcal{N}_{\mathbf{Z}}$ are real line bundles, each of $\mathcal{N}_{\mathbf{X}}|_{x'}, \mathcal{N}_{\mathbf{Y}}|_{y'}, \mathcal{N}_{\mathbf{Z}}|_{z'}$ is a real vector space isomorphic to $\mathbb{R}$, and (6.81) is a linear map between finite-dimensional real vector spaces, whose component mapping $\mathcal{N}_{\mathbf{Z}}|_{z'} \to \mathcal{N}_{\mathbf{X}}|_{x'}$ is $\lambda_{\boldsymbol{g}}|_{(x',z')}$ if $(x', z') \in \underline{S}_{\boldsymbol{g}}$ and zero otherwise, and whose component mapping $\mathcal{N}_{\mathbf{Z}}|_{z'} \to \mathcal{N}_{\mathbf{Y}}|_{y'}$ is $\lambda_{\boldsymbol{h}}|_{(y',z')}$ if $(y', z') \in \underline{S}_{\boldsymbol{h}}$ and zero otherwise. For $\boldsymbol{g}, \boldsymbol{h}$ to be b-transverse the linear maps (6.81) must be injective for all such x, y, z.

Definition 6.33. Let $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms in $\mathbf{dSpa^c}$, and let $C(\boldsymbol{g}), C(\boldsymbol{h}), \hat{C}(\boldsymbol{g}), \hat{C}(\boldsymbol{h})$ be as in §6.7. We call $\boldsymbol{g}, \boldsymbol{h}$ *c-transverse* if the following two conditions hold:

(a) whenever there are points in $C_j(\mathbf{X}), C_k(\mathbf{Y}), C_l(\mathbf{Z})$ with
$$C(\boldsymbol{g})(x, \{x'_1, \ldots, x'_j\}) = C(\boldsymbol{h})(y, \{y'_1, \ldots, y'_k\}) = (z, \{z'_1, \ldots, z'_l\}), \tag{6.82}$$
we have either $j + k > l$ or $j = k = l = 0$; and

(b) whenever there are points in $C_j(\mathbf{X}), C_k(\mathbf{Y}), C_l(\mathbf{Z})$ with
$$\hat{C}(\boldsymbol{g})(x, \{x'_1, \ldots, x'_j\}) = \hat{C}(\boldsymbol{h})(y, \{y'_1, \ldots, y'_k\}) = (z, \{z'_1, \ldots, z'_l\}), \tag{6.83}$$
we have $j + k \geqslant l$.

Note that part (a) corresponds to the condition in Definition 5.25 for transverse g, h in $\mathbf{Man^c}$ to be strongly transverse.

The next example illustrates the fact that b-transversality is a *continuous* condition, as it depends on the value of a real parameter α. In contrast, c-transversality is a discrete condition.

Example 6.34. Define manifolds with corners $X = [0, \infty)$, $Y = [0, \infty)$ and $Z = [0, \infty)^2$. Let $\alpha > 0$, and define smooth maps $g : X \to Z$, $h : Y \to Z$ by $g(x) = (x, x)$, $h(y) = (y, \alpha y)$. In a similar way to Example 5.27, g, h are transverse in $\mathbf{Man^c}$ if and only if $\alpha \neq 1$, and they are not strongly transverse.

Set $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \boldsymbol{g}, \boldsymbol{h} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X, Y, Z, g, h)$. To check whether $\boldsymbol{g}, \boldsymbol{h}$ are b-transverse, note that at $x = 0$, $y = 0$ and $z = (0, 0)$, equation (6.81) is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & \alpha \end{smallmatrix}\right)$. This is injective if and only if $\alpha \neq 1$. When $\alpha \neq 1$ there are no other points $x \in \underline{X}$ and $y \in \underline{Y}$ with $\underline{g}(x) = \underline{h}(y) = z \in \underline{Z}$. Thus $\boldsymbol{g}, \boldsymbol{h}$ are b-transverse if and only if $\alpha \neq 1$.

For c-transversality, note that for any $\alpha > 0$ we have

$$C(\boldsymbol{g})\bigl(0, \bigl\{\{x = 0\}\bigr\}\bigr) = C(\boldsymbol{h})\bigl(0, \bigl\{\{y = 0\}\bigr\}\bigr) = \bigl((0, 0), \bigl\{\{x = 0\}, \{y = 0\}\bigr\}\bigr),$$

so that (6.82) holds with $j = k = 1$ and $l = 2$. Hence Definition 6.33(a) is false, and $\boldsymbol{g}, \boldsymbol{h}$ are not c-transverse.

Here are some sufficient conditions for $\boldsymbol{g}, \boldsymbol{h}$ to be b- and c-transverse. One should think of b- and c-transversality as mild conditions on $\boldsymbol{g}, \boldsymbol{h}$ which are satisfied most of the time — they are much weaker than requiring smooth maps g, h of manifolds with corners to be transverse, for instance.

Lemma 6.35. *Let $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms in $\mathbf{dSpa^c}$. The following are sufficient conditions for $\boldsymbol{g}, \boldsymbol{h}$ to be c-transverse, and hence b-transverse:*

- **(i)** *$\boldsymbol{g}$ or $\boldsymbol{h}$ is semisimple and flat; or*
- **(ii)** *$\mathbf{Z}$ is a d-space without boundary.*

Proof. For (i), suppose $\boldsymbol{g}$ is semisimple and flat. Then $C(\boldsymbol{g}) = \hat{C}(\boldsymbol{g})$ as $\boldsymbol{g}$ is flat. Therefore Proposition 6.30 shows that if (6.82) or (6.83) holds then $j \geqslant l$. Definition 6.33(b) is immediate, as $k \geqslant 0$. For Definition 6.33(a), suppose (6.82) holds, so that $j \geqslant l$. If $k > 0$ then $j + k > l$. If $k = 0$ then $l = 0$, so either $j > 0$ and $j + k > l$, or $j = k = l = 0$. Hence Definition 6.33(a) holds, and $\boldsymbol{g}, \boldsymbol{h}$ are c-transverse. For (ii), as $\partial\mathbf{Z} = \boldsymbol{\emptyset}$ we must have $l = 0$ in (6.82)–(6.83), and c-transversality follows. $\square$

6.8.2 Rewriting b- and c-transversality in terms of graphs $\Gamma_{x,y}$

It will be convenient to think about b- and c-transversality using graphs.

Definition 6.36. Let $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms in $\mathbf{dSpa^c}$, and let $x \in \underline{X}$ and $y \in \underline{Y}$ with $\underline{g}(x) = \underline{h}(y) = z \in \underline{Z}$. Define a finite graph $\Gamma_{x,y}$ to have vertex set $\underline{i}_{\mathbf{X}}^{-1}(x) \amalg \underline{i}_{\mathbf{Y}}^{-1}(y) \amalg \underline{i}_{\mathbf{Z}}^{-1}(z)$, and to have edges $\overset{x'}{\bullet} - \overset{z'}{\bullet}$ if $x' \in \underline{i}_{\mathbf{X}}^{-1}(x)$, $z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)$ and $(x', z') \in \underline{S}_{\boldsymbol{g}}$, and $\overset{y'}{\bullet} - \overset{z'}{\bullet}$ if $y' \in \underline{i}_{\mathbf{Y}}^{-1}(y)$, $z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)$ and $(y', z') \in \underline{S}_{\boldsymbol{h}}$. When we refer to x', y' or z' as vertices of $\Gamma_{x,y}$, we will always mean that $x' \in \underline{i}_{\mathbf{X}}^{-1}(x)$, $y' \in \underline{i}_{\mathbf{Y}}^{-1}(y)$, or $z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)$, respectively.

Definition 6.2 implies that for each vertex z' in $\Gamma_{x,y}$, either there is a unique edge $\overset{x'}{\bullet} - \overset{z'}{\bullet}$, or there is no edge $\overset{x'}{\bullet} - \overset{z'}{\bullet}$ and $(x, z') \in \underline{T}_{\boldsymbol{g}}$. Similarly, either there is a unique edge $\overset{y'}{\bullet} - \overset{z'}{\bullet}$, or there is no edge $\overset{y'}{\bullet} - \overset{z'}{\bullet}$ and $(y, z') \in \underline{T}_{\boldsymbol{h}}$. Hence every vertex z' in $\Gamma_{x,y}$ lies on 0, 1 or 2 edges.

We will be interested in topological properties of $\Gamma_{x,y}$, such as its connected components, and whether it is simply-connected. Let $\hat{\Gamma}$ be a connected component of $\Gamma_{x,y}$, and write j,k,l for the number of vertices x',y',z' in $\hat{\Gamma}$ respectively, and m for the number of edges. Since every edge contains a unique vertex z', and every vertex z' lies on 0, 1 or 2 edges, we have $m \leqslant 2l$. The Euler characteristic of $\hat{\Gamma}$ is $\chi(\hat{\Gamma}) = b^0(\hat{\Gamma}) - b^1(\hat{\Gamma}) = j+k+l-m$. But $b^0(\hat{\Gamma}) = 1$ as $\hat{\Gamma}$ is connected, so we see that

$$b^1(\hat{\Gamma}) = 1 - j - k - l + m \geqslant 0. \tag{6.84}$$

Observe that equation (6.81) splits as a direct sum over connected components $\hat{\Gamma}$ of $\Gamma_{x,y}$, since there are only nonzero morphisms from $\mathcal{N}_{\mathbf{Z}}|_{z'}$ to $\mathcal{N}_{\mathbf{X}}|_{x'}$ or $\mathcal{N}_{\mathbf{X}}|_{y'}$ if z' and x' or y' lie in the same component $\hat{\Gamma}$. Therefore, $\boldsymbol{g},\boldsymbol{h}$ are b-transverse if and only if for all connected components $\hat{\Gamma}$ of graphs $\Gamma_{x,y}$ as above, the following linear map is injective:

$$\begin{aligned}
&\bigoplus_{\text{edges } \overset{x'}{\bullet} - \overset{z'}{\bullet} \text{ in } \hat{\Gamma}} \lambda_{\boldsymbol{g}}|_{(x',z')} \oplus \bigoplus_{\text{edges } \overset{y'}{\bullet} - \overset{z'}{\bullet} \text{ in } \hat{\Gamma}} \lambda_{\boldsymbol{h}}|_{(y',z')} : \\
&\bigoplus_{\text{vertices } z' \text{ in } \hat{\Gamma}} \mathcal{N}_{\mathbf{Z}}|_{z'} \longrightarrow \bigoplus_{\text{vertices } x' \text{ in } \hat{\Gamma}} \mathcal{N}_{\mathbf{X}}|_{x'} \oplus \bigoplus_{\text{vertices } y' \text{ in } \hat{\Gamma}} \mathcal{N}_{\mathbf{Y}}|_{y'}.
\end{aligned} \tag{6.85}$$

Here is a characterization of b-transversality in terms of the graphs $\Gamma_{x,y}$. Note that for types (A),(B) we will prove that the injectivity of (6.85) is automatic, and does not need to be imposed as an extra condition.

Proposition 6.37. *Let $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms in* $\mathbf{dSpa^c}$. *Then $\boldsymbol{g},\boldsymbol{h}$ are b-transverse if and only if whenever $x \in \underline{X}$ and $y \in \underline{Y}$ with $\underline{g}(x) = \underline{h}(y) = z \in \underline{Z}$, and the graph $\Gamma_{x,y}$ is as above, and $\hat{\Gamma}$ is a connected component of $\Gamma_{x,y}$ with j,k,l vertices of the form x',y',z' respectively and m edges, then $\hat{\Gamma}$ satisfies exactly one of the following three conditions:*

- **(A)** *Every vertex z' in $\hat{\Gamma}$ lies on two edges, so $m = 2l$, and $\hat{\Gamma}$ is simply-connected, so $j+k = l+1$ by* (6.84).
- **(B)** *One vertex z'_1 in $\hat{\Gamma}$ lies on one edge, and all other vertices z' in $\hat{\Gamma}$ lie on two edges, so $m = 2l-1$, and $\hat{\Gamma}$ is simply-connected, so $j+k = l$ by* (6.84).
- **(C)** *Every vertex z' in $\hat{\Gamma}$ lies on two edges, so $m = 2l$, and $b^1(\hat{\Gamma}) = 1$, so $j+k = l$ by* (6.84), *and* (6.85) *is an isomorphism of real vector spaces.*

We will call such $\hat{\Gamma}$ ***components of type* (A),(B) *and* (C),** *respectively.*

Proof. For the 'only if' part, suppose $\boldsymbol{g},\boldsymbol{h}$ are b-transverse, and let $x,y,\Gamma_{x,y},\hat{\Gamma}$ and j,k,l,m be as in the proposition. Then $\boldsymbol{g},\boldsymbol{h}$ b-transverse implies that (6.85) is injective. But (6.85) is a linear map $\mathbb{R}^l \to \mathbb{R}^j \oplus \mathbb{R}^k$, so $j+k \geqslant l$. As in Definition 6.36 we also have $m \leqslant 2l$ and $1-j-k-l+m \geqslant 0$, and combining these gives $j+k \leqslant l+1$. Hence either $j+k=l+1$ or $j+k=l$. If $j+k=l+1$ then $1-j-k-l+m \geqslant 0$ gives $m \geqslant 2l$, so as $m \leqslant 2l$ we have $m=2l$. If $j+k=l$ then $1-j-k-l+m \geqslant 0$ gives $m \geqslant 2l-1$, so either $m=2l-1$ or $m=2l$. Therefore we may divide into three cases:

(A) $j+k=l+1$ and $m=2l$.

(B) $j+k=l$ and $m=2l-1$.

(C) $j+k=l$ and $m=2l$.

In case (A), as every vertex z' lies on 0, 1 or 2 edges, $m=2l$ shows every vertex z' in $\hat\Gamma$ lies on two edges, and (6.84) gives $b^1(\hat\Gamma)=0$ so $\hat\Gamma$ is simply-connected, so part (A) of the proposition holds. In case (B) $m=2l-1$ shows one vertex z_1' lies on one edge and all other z' lie on two edges, and (6.84) gives $b^1(\hat\Gamma)=0$ so $\hat\Gamma$ is simply-connected, so part (B) of the proposition holds. In case (C) $m=2l$ shows every vertex z' in $\hat\Gamma$ lies on two edges, and (6.84) gives $b^1(\hat\Gamma)=1$. Also (6.85) is injective as $\boldsymbol{g},\boldsymbol{h}$ are b-transverse, but (6.85) is a linear map $\mathbb{R}^l \to \mathbb{R}^l$ as $j+k=l$, so being injective implies that it is an isomorphism, and part (C) of the proposition holds. This proves the 'only if' part.

For the 'if' part, suppose that for all x,y and components $\hat\Gamma$ of $\Gamma_{x,y}$, one of conditions (A),(B) or (C) holds. To show $\boldsymbol{g},\boldsymbol{h}$ are b-transverse we must show that (6.85) is injective for each such $\hat\Gamma$. In case (C) this is immediate. In case (B), as $\hat\Gamma$ is connected and simply-connected, there is a unique shortest path in $\hat\Gamma$ from the centre of an edge e to the distinguished vertex z_1. Colour an edge e of $\hat\Gamma$ *black* if this path has an even number of edges (including e), and *white* if it has an odd number of edges. It is not difficult to see that every vertex x',y',z' in $\hat\Gamma$ lies on *exactly one white edge*, though it may lie on an arbitrary number of black edges, and this is the unique colouring of $\hat\Gamma$ with this property.

Choose an ordering $z_1',\ldots,z_l'$ of the l vertices z' in $\hat\Gamma$, beginning with the distinguished vertex z_1', with the property that if the unique shortest path in $\hat\Gamma$ from z_1' to z_j' passes through z_i', then $i \leqslant j$. Order the $j+k=l$ vertices x',y' in $\hat\Gamma$ as x_i',y_i' for $i=1,\ldots,l$ such that each white edge $\overset{x'}{\bullet}-\overset{z'}{\bullet}$ or $\overset{y'}{\bullet}-\overset{z'}{\bullet}$ in $\hat\Gamma$ joins x_i' or y_i' to z_i' for $i=1,\ldots,l$. Using these orderings, and choosing isomorphisms $\mathcal{N}_{\mathbf{X}}|_{x'},\mathcal{N}_{\mathbf{Y}}|_{y'},\mathcal{N}_{\mathbf{Z}}|_{z'} \cong \mathbb{R}$, equation (6.85) becomes an $l \times l$ matrix over $\mathbb{R}$, where white edges contribute terms $\lambda_{\boldsymbol{g}}|_{(x_i',z_i')} \neq 0$ or $\lambda_{\boldsymbol{h}}|_{(y_i',z_i')} \neq 0$ in position (i,i) for $i=1,\ldots,l$, and black edges contribute terms in position (i,j) for $i>j$. Hence (6.85) is identified with an upper triangular $l \times l$ matrix with nonzero terms on the diagonal, and so is invertible. Thus in case (B) equation (6.85) is an isomorphism, and so injective.

In case (A), pick an arbitrary vertex x_1' or y_1' in $\hat\Gamma$. Let $\hat\Gamma'$ be the graph obtained from $\hat\Gamma$ by deleting this vertex, and all edges $\overset{x_1'}{\bullet}-\overset{z'}{\bullet}$ or $\overset{y_1'}{\bullet}-\overset{z'}{\bullet}$ containing it. Then each connected component of $\hat\Gamma'$ is simply-connected, as $\hat\Gamma$ is, and has a unique vertex z_1' on one edge (that joined to x_1' or y_1' in $\hat\Gamma$), and other vertices z' lie on two edges. Thus, each connected component of $\hat\Gamma'$ is of type (B), so (6.85) is an isomorphism for $\hat\Gamma'$ from above. But (6.85) for $\hat\Gamma$ is obtained from (6.85) for $\hat\Gamma'$ by adding an extra space $\mathcal{N}_{\mathbf{X}}|_{x_1'}$ or $\mathcal{N}_{\mathbf{X}}|_{y_1'}$ to the right hand side, with extra linear maps $\lambda_{\boldsymbol{g}}|_{(x_1',z')}$ or $\lambda_{\boldsymbol{h}}|_{(y_1',z')}$. Therefore (6.85) for $\hat{\Gamma}$ is injective. So $\boldsymbol{g},\boldsymbol{h}$ are b-transverse, proving the 'if' part. $\square$

Sections 6.8.3–6.8.4 will prove that for b-transverse $\boldsymbol{g},\boldsymbol{h}$ a fibre product $\mathbf{W}=\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ exists in $\mathbf{dSpa^c}$, and there is a 1-1 correspondence between points of $\partial\mathbf{W}$ and type (A) components $\hat{\Gamma}$. Here is a characterization of c-transversality.

Proposition 6.38. *Let $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$ and $\boldsymbol{h}:\mathbf{Y}\to\mathbf{Z}$ be 1-morphisms in $\mathbf{dSpa^c}$. Then $\boldsymbol{g},\boldsymbol{h}$ are c-transverse if and only if whenever $x\in\underline{X}$ and $y\in\underline{Y}$ with $\underline{g}(x)=\underline{h}(y)=z\in\underline{Z}$, and $\hat{\Gamma}$ is a connected component of $\Gamma_{x,y}$ in Definition 6.36, then $\hat{\Gamma}$ satisfies exactly one of conditions* (A),(B) *in Proposition* 6.37*.*

Proof. For the 'if' part, suppose one of Proposition 6.37(A),(B) hold for all $\hat{\Gamma}$ as above. One can show that (6.82) holds for $(x,\{x_1',\ldots,x_j'\})$, $(y,\{y_1',\ldots,y_k'\})$, $(z,\{z_1',\ldots,z_l'\})$ if and only if $\{x_1',\ldots,x_j'\}\amalg\{y_1',\ldots,y_k'\}\amalg\{z_1',\ldots,z_l'\}$ is the disjoint union of the vertex sets of some collection of type (A) components $\hat{\Gamma}$ of $\Gamma_{x,y}$. Summing the equality $j+k=l+1$ in Proposition 6.37(A) over these components $\hat{\Gamma}$ shows that for j,k,l as in (6.82), $j+k-l$ is the number of connected components $\hat{\Gamma}$ involved. Hence either $j+k>l$ or $j=k=l=0$ in (6.82), proving Definition 6.33(a) for $\boldsymbol{g},\boldsymbol{h}$.

Similarly, one can show (6.83) holds for $(x,\{x_1',\ldots,x_j'\})$, $(y,\{y_1',\ldots,y_k'\})$, $(z,\{z_1',\ldots,z_l'\})$ if and only if $\{x_1',\ldots,x_j'\}\amalg\{y_1',\ldots,y_k'\}\amalg\{z_1',\ldots,z_l'\}$ is the disjoint union of the vertex sets of some collection of type (A) components $\hat{\Gamma}$ of $\Gamma_{x,y}$, and all of the type (B) components $\hat{\Gamma}$. Summing the equalities $j+k=l+1$ in (A) and $j+k=l$ in (B) over all such $\hat{\Gamma}$ shows that $j+k\geqslant l$ in (6.83), proving Definition 6.33(a) for $\boldsymbol{g},\boldsymbol{h}$. So $\boldsymbol{g},\boldsymbol{h}$ are c-transverse, and the 'if' part of the proposition holds.

For the 'only if' part, let $\boldsymbol{g},\boldsymbol{h}$ be c-transverse. Suppose $x\in\underline{X}$ and $y\in\underline{Y}$ with $\underline{g}(x)=\underline{h}(y)=z\in\underline{Z}$, and $\hat{\Gamma}$ is a connected component of $\Gamma_{x,y}$. We divide into two cases: (A) every vertex z' in $\hat{\Gamma}$ lies on two edges $\overset{x'}{\bullet}-\overset{z'}{\bullet}$ and $\overset{y'}{\bullet}-\overset{z'}{\bullet}$; and (B) otherwise. In case (A), write the vertices of $\hat{\Gamma}$ as $x_1',\ldots,x_j',y_1',\ldots,y_k',z_1',\ldots,z_l'$. Then (6.82) holds, and we do not have $j=k=l=0$ as $\hat{\Gamma}\neq\emptyset$, so Definition 6.33(a) gives $j+k>l$. But $\hat{\Gamma}$ has $m=2l$ edges as every z' in $\hat{\Gamma}$ lies on two edges, so (6.84) gives $j+k-l=1-b^1(\hat{\Gamma})$. As $j+k>l$ and $b^1(\hat{\Gamma})\geqslant 0$, the only possibility is $j+k=l+1$ and $b^1(\hat{\Gamma})=0$. Thus in case (A), Proposition 6.37(A) holds for $\hat{\Gamma}$.

In case (B), let $\hat{\Gamma}$ have j,k,l vertices of types x',y',z' and m edges. As at least one vertex z' in $\hat{\Gamma}$ lies on 0 or 1 edge, we have $m\leqslant 2l-1$. From (6.84) we have $j+k-l=(m-2l+1)-b^1(\hat{\Gamma})$, so $j+k\leqslant l$ as $m-2l+1\leqslant 0$ and $b^1(\hat{\Gamma})\geqslant 0$. Write the vertices of the union of all the case (B) components $\hat{\Gamma}$ on $\Gamma_{x,y}$ as $x_1',\ldots,x_j',y_1',\ldots,y_k',z_1',\ldots,z_l'$. Then one can show that (6.83) holds, so Definition 6.33(b) gives $j+k\geqslant l$. But for each individual case (B) component we have $j+k\leqslant l$, so the only possibility is that for all $\hat{\Gamma}$ in case (B) we have $j+k=l$. As $m\leqslant 2l-1$, we see from (6.84) that $m=2l-1$, so that one vertex z_1' in $\hat{\Gamma}$ lies on one edge, and all other z' in $\hat{\Gamma}$ lie on two, and $b^1(\hat{\Gamma})=0$.

Thus in case (B), Proposition 6.37(B) holds for $\hat{\Gamma}$, and the 'only if' part of the proposition holds. $\square$

Propositions 6.37 and 6.38 imply:

Corollary 6.39. *Suppose* $\boldsymbol{g}:\mathbf{X}\rightarrow\mathbf{Z}$ *and* $\boldsymbol{h}:\mathbf{Y}\rightarrow\mathbf{Z}$ *are c-transverse* 1*-morphisms in* $\mathbf{dSpa^c}$*. Then* $\boldsymbol{g},\boldsymbol{h}$ *are also b-transverse.*

6.8.3 Local existence of b-transverse fibre products in $\mathbf{dSpa^c}$

We will now prove that if $\boldsymbol{g}:\mathbf{X}\rightarrow\mathbf{Z}$ and $\boldsymbol{h}:\mathbf{Y}\rightarrow\mathbf{Z}$ are b-transverse then a fibre product $\mathbf{W}=\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ exists in $\mathbf{dSpa^c}$. The plan of the proof is that in Definition 6.40 and Theorem 6.41 we first show that a local fibre product $\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y}$ exists near $x\in\mathbf{X}$, $y\in\mathbf{Y}$ with $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z\in\mathbf{Z}$, using the explicit construction of fibre products in $\mathbf{dSpa}$ from §2.5. Then in §6.8.4 we use the results of §6.6 to glue these local fibre products by equivalences to get a global fibre product. The next definition and proof are rather long.

Definition 6.40. Suppose $\boldsymbol{g}:\mathbf{X}\rightarrow\mathbf{Z}$ and $\boldsymbol{h}:\mathbf{Y}\rightarrow\mathbf{Z}$ are b-transverse 1-morphisms in $\mathbf{dSpa^c}$, and $x\in\mathbf{X}$, $y\in\mathbf{Y}$ with $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z\in\mathbf{Z}$. We will construct open d-subspaces $x\in\mathbf{R}\subseteq\mathbf{X}$, $y\in\mathbf{S}\subseteq\mathbf{Y}$ and $z\in\mathbf{T}\subseteq\mathbf{Z}$ with $\boldsymbol{g}(\mathbf{R}),\boldsymbol{h}(\mathbf{S})\subseteq\mathbf{T}$, a d-space with corners $\mathbf{Q}$, 1-morphisms $\boldsymbol{e}:\mathbf{Q}\rightarrow\mathbf{R}$, $\boldsymbol{f}:\mathbf{Q}\rightarrow\mathbf{S}$, and a 2-morphism $\eta:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ in $\mathbf{dSpa^c}$, in the 2-commutative diagram,

$$\begin{tikzcd} \mathbf{Q} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"] \arrow[dr, phantom, "\Uparrow\eta"] & \mathbf{S} \arrow[d, "\boldsymbol{h}|_{\mathbf{S}}"] \\ \mathbf{R} \arrow[r, "\boldsymbol{g}|_{\mathbf{R}}"] & \mathbf{T}. \end{tikzcd} \tag{6.86}$$

Then Theorem 6.41 will show that (6.86) is 2-Cartesian.

Let $\Gamma_{x,y}$ be the graph from Definition 6.36. As $\boldsymbol{g},\boldsymbol{h}$ are b-transverse, Proposition 6.37 shows that every connected component $\hat{\Gamma}$ of $\Gamma_{x,y}$ is of type (A), (B) or (C). As in the proof of Proposition 6.38, every type (A) component $\hat{\Gamma}$ either is a single point x' or y', or has at least two vertices of the form x' or y' lying on only one edge. Choose subsets $I\subseteq\underline{i}_{\mathbf{X}}^{-1}(x)$ and $J\subseteq\underline{i}_{\mathbf{Y}}^{-1}(y)$ such that $I\amalg J$ contains exactly one vertex from each type (A) component $\hat{\Gamma}$, and no vertices from type (B),(C) components.

Consider now the linear map of real vector spaces

$$\begin{aligned}&\bigoplus_{(x',z')\in\underline{S}_{\boldsymbol{g}}:\underline{i}_{\mathbf{X}}(x')=x,\ x'\notin I}\lambda_{\boldsymbol{g}}|_{(x',z')}\oplus\bigoplus_{(y',z')\in\underline{S}_{\boldsymbol{h}}:\underline{i}_{\mathbf{Y}}(y')=y,\ y'\notin J}\lambda_{\boldsymbol{h}}|_{(y',z')}:\\&\bigoplus_{z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}\mathcal{N}_{\mathbf{Z}}|_{z'}\longrightarrow\bigoplus_{x'\in\underline{i}_{\mathbf{X}}^{-1}(x)\setminus I}\mathcal{N}_{\mathbf{X}}|_{x'}\oplus\bigoplus_{y'\in\underline{i}_{\mathbf{Y}}^{-1}(y)\setminus J}\mathcal{N}_{\mathbf{Y}}|_{y'},\end{aligned} \tag{6.87}$$

which is equation (6.81) with spaces $\mathcal{N}_{\mathbf{X}}|_{x'},\mathcal{N}_{\mathbf{Y}}|_{y'}$ and maps $\lambda_{\boldsymbol{g}}|_{(x',z')},\lambda_{\boldsymbol{h}}|_{(y',z')}$ for $x'\in I$ and $y'\in J$ omitted. As in §6.8.2, we may write (6.87) as a direct sum over connected components $\hat{\Gamma}$ of $\Gamma_{x,y}$ of linear maps (6.85), where for $\hat{\Gamma}$ of type (A) we omit $\mathcal{N}_{\mathbf{X}}|_{x'},\lambda_{\boldsymbol{g}}|_{(x',z')}$ for $x'\in I\cap\hat{\Gamma}$, or we omit $\mathcal{N}_{\mathbf{Y}}|_{y'},\lambda_{\boldsymbol{h}}|_{(y',z')}$

for $y' \in J \cap \hat{\Gamma}$. Now the proof of Proposition 6.37 showed that for each $\hat{\Gamma}$, this linear map is an isomorphism. Therefore (6.87) is an isomorphism.

Let $(\boldsymbol{R}_{x'}, \boldsymbol{r}_{x'})$ be a boundary defining function for $\mathbf{X}$ at x' for each $x' \in I$, so that $x \in \boldsymbol{R}_{x'} \subseteq \boldsymbol{X}$ is open. Making the $\boldsymbol{R}_{x'}$ smaller if necessary we can suppose $\boldsymbol{R}_{x'} = \boldsymbol{R}$ for all $x' \in \boldsymbol{i}_{\mathbf{X}}^{-1}(x)$, say. Let $\mathbf{R}$ be the corresponding open d-subspace of $\mathbf{X}$. Similarly, we choose open $y \in \boldsymbol{S} \subseteq \boldsymbol{Y}$ and $\boldsymbol{s}_{y'} : \boldsymbol{S} \to \boldsymbol{[0,\infty)}$ such that $(\boldsymbol{S}, \boldsymbol{s}_{y'})$ is a boundary defining function for $\mathbf{Y}$ at y' for each $y' \in J$, and open $z \in \boldsymbol{T} \subseteq \boldsymbol{Z}$ and $\boldsymbol{t}_{z'} : \boldsymbol{T} \to \boldsymbol{[0,\infty)}$ such that $(\boldsymbol{T}, \boldsymbol{t}_{z'})$ is a boundary defining function for $\mathbf{Z}$ at z' for each $z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)$. Let $\mathbf{S}, \mathbf{T}$ be the corresponding open d-subspaces of $\mathbf{Y}, \mathbf{Z}$. Making $\boldsymbol{R}, \mathbf{R}, \boldsymbol{S}, \mathbf{S}$ smaller if necessary, we can suppose that $\boldsymbol{g}(\mathbf{R}), \boldsymbol{h}(\mathbf{S}) \subseteq \mathbf{T}$.

By Definition 6.2 for $\boldsymbol{g}$, for each $z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)$, either:

(i) there is an open $x \in \tilde{\boldsymbol{V}} \subseteq \boldsymbol{g}^{-1}(\boldsymbol{T})$ such that $\bigl(\tilde{\boldsymbol{V}}, \boldsymbol{t}_{z'} \circ \boldsymbol{g}|_{\tilde{\boldsymbol{V}}}\bigr)$ is a boundary defining function for $\mathbf{X}$ at some $x' \in I$, and $(x', z') \in \underline{S}_{\boldsymbol{g}}$; or

(ii) for some open $x \in \boldsymbol{W} \subseteq \boldsymbol{g}^{-1}(\boldsymbol{T})$ we have $\boldsymbol{t}_{z'} \circ \boldsymbol{g}|_{\boldsymbol{W}} = \boldsymbol{0} \circ \boldsymbol{\pi} : \boldsymbol{W} \to \boldsymbol{[0,\infty)}$ in $\mathbf{dSpa}$, and $(x, z') \in \underline{T}_{\boldsymbol{g}}$.

In case (i), by Proposition 6.6(c) there exists open $x \in \boldsymbol{W} \subseteq \tilde{\boldsymbol{V}}$ and a 1-morphism $\boldsymbol{p}_{x'z'} : \boldsymbol{W} \to \boldsymbol{(0,\infty)}$ in $\mathbf{dSpa}$ such that $\boldsymbol{t}_{z'} \circ \boldsymbol{g}|_{\boldsymbol{W}} = \boldsymbol{p}_{x'z'} \cdot \boldsymbol{r}_{x'}|_{\boldsymbol{W}}$. In both cases, by making $\boldsymbol{R}, \mathbf{R}$ smaller we can take $\boldsymbol{W} = \boldsymbol{R}$ for all $z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)$.

Thus, for each $z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)$, either $(x', z') \in \underline{S}_{\boldsymbol{g}}$ for some unique $x' \in \underline{i}_{\mathbf{X}}^{-1}(x)$, and then $\boldsymbol{t}_{z'} \circ \boldsymbol{g}|_{\boldsymbol{R}} = \boldsymbol{p}_{x'z'} \cdot \boldsymbol{r}_{x'}$ for some 1-morphism $\boldsymbol{p}_{x'z'} : \boldsymbol{R} \to \boldsymbol{(0,\infty)}$ in $\mathbf{dSpa}$, or $(x, z') \in \underline{T}_{\boldsymbol{g}}$ and $\boldsymbol{t}_{z'} \circ \boldsymbol{g}|_{\boldsymbol{R}} = \boldsymbol{0} \circ \boldsymbol{\pi}$. Similarly, using Definition 6.2 for $\boldsymbol{h}$ and making $\boldsymbol{S}, \mathbf{S}$ smaller, we can arrange that for each $z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)$, either $(y', z') \in \underline{S}_{\boldsymbol{h}}$ for some unique $y' \in \underline{i}_{\mathbf{Y}}^{-1}(y)$, and then $\boldsymbol{t}_{z'} \circ \boldsymbol{h}|_{\boldsymbol{S}} = \boldsymbol{q}_{y'z'} \cdot \boldsymbol{s}_{y'}$ for some 1-morphism $\boldsymbol{q}_{y'z'} : \boldsymbol{S} \to \boldsymbol{(0,\infty)}$ in $\mathbf{dSpa}$, or $(y, z') \in \underline{T}_{\boldsymbol{h}}$ and $\boldsymbol{t}_{z'} \circ \boldsymbol{h}|_{\boldsymbol{S}} = \boldsymbol{0} \circ \boldsymbol{\pi}$.

By (6.1) for the boundary defining functions $(\boldsymbol{R}, \boldsymbol{r}_{x'}), (\boldsymbol{S}, \boldsymbol{s}_{y'}), (\boldsymbol{T}, \boldsymbol{t}_{z'})$ for $x' \in \underline{i}_{\mathbf{X}}^{-1}(x)$, $y' \in \underline{i}_{\mathbf{Y}}^{-1}(y)$ and $z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)$, there are open $x' \in \boldsymbol{B}_{x'} \subseteq \boldsymbol{i}_{\mathbf{X}}^{-1}(\boldsymbol{R}) \subseteq \boldsymbol{\partial X}$ and $y' \in \boldsymbol{C}_{y'} \subseteq \boldsymbol{i}_{\mathbf{Y}}^{-1}(\boldsymbol{S}) \subseteq \boldsymbol{\partial Y}$ and $z' \in \boldsymbol{D}_{z'} \subseteq \boldsymbol{i}_{\mathbf{Z}}^{-1}(\boldsymbol{T}) \subseteq \boldsymbol{\partial Z}$ fitting into 2-Cartesian diagrams

$$\begin{tikzcd} \boldsymbol{B}_{x'} \arrow[r, "\boldsymbol{\pi}"'] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{B}_{x'}}"'] \arrow[dr, phantom, "\Nearrow \mathrm{id}_{\boldsymbol{0}\circ\boldsymbol{\pi}}"] & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{R} \arrow[r, "\boldsymbol{r}_{x'}"] & \boldsymbol{[0,\infty)}, \end{tikzcd} \quad \begin{tikzcd} \boldsymbol{C}_{y'} \arrow[r, "\boldsymbol{\pi}"'] \arrow[d, "\boldsymbol{i}_{\mathbf{Y}}|_{\boldsymbol{C}_{y'}}"'] \arrow[dr, phantom, "\Nearrow \mathrm{id}_{\boldsymbol{0}\circ\boldsymbol{\pi}}"] & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{S} \arrow[r, "\boldsymbol{s}_{y'}"] & \boldsymbol{[0,\infty)}, \end{tikzcd} \quad \begin{tikzcd} \boldsymbol{D}_{z'} \arrow[r, "\boldsymbol{\pi}"'] \arrow[d, "\boldsymbol{i}_{\mathbf{Z}}|_{\boldsymbol{D}_{z'}}"'] \arrow[dr, phantom, "\Nearrow \mathrm{id}_{\boldsymbol{0}\circ\boldsymbol{\pi}}"] & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{T} \arrow[r, "\boldsymbol{t}_{z'}"] & \boldsymbol{[0,\infty)}. \end{tikzcd} \tag{6.88}$$

By making $\boldsymbol{R}, \mathbf{R}$ smaller we may suppose that the $\boldsymbol{B}_{x'}$ are disjoint with $\boldsymbol{\partial R} = \boldsymbol{i}_{\mathbf{X}}^{-1}(\boldsymbol{R}) = \coprod_{x' \in \underline{i}_{\mathbf{X}}^{-1}(x)} \boldsymbol{B}_{x'}$. Similarly we can take the $\boldsymbol{C}_{y'}, \boldsymbol{D}_{z'}$ disjoint with $\boldsymbol{\partial S} = \boldsymbol{i}_{\mathbf{Y}}^{-1}(\boldsymbol{S}) = \coprod_{y' \in \underline{i}_{\mathbf{Y}}^{-1}(y)} \boldsymbol{C}_{y'}$ and $\boldsymbol{\partial T} = \boldsymbol{i}_{\mathbf{Z}}^{-1}(\boldsymbol{T}) = \coprod_{z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)} \boldsymbol{D}_{z'}$.

By Definition 6.1(c) for $(\boldsymbol{R}, \boldsymbol{r}_{x'})$, the morphism $r_{x'}^2 : \underline{r}_{x'}^*(\mathcal{F}_{[0,\infty)}) \to \mathcal{F}_X|_{\underline{R}}$ admits a left inverse $\beta_{x'}$ in $\mathrm{qcoh}(\underline{R})$ for $x' \in \underline{i}_{\mathbf{X}}^{-1}(x)$. As in the last part of the proof of Proposition 6.21, using Definition 6.1(f) for $\mathbf{X}$ at x and making $\boldsymbol{R}, \mathbf{R}$ smaller, we can choose these $\beta_{x'}$ such that $\beta_{x'} \circ r_{x''}^2 = 0$ for $x' \neq x''$. Similarly, making $\boldsymbol{S}, \mathbf{S}, \boldsymbol{T}, \mathbf{T}$ smaller, we can choose left inverses $\gamma_{y'}$ for $s_{y'}^2$ in $\mathrm{qcoh}(\underline{S})$ for $y' \in \underline{i}_{\mathbf{Y}}^{-1}(y)$, with $\gamma_{y'} \circ s_{y''}^2 = 0$ for $y' \neq y''$.

We will now construct the d-space with corners $\mathbf{Q}=(\boldsymbol{Q},\boldsymbol{\partial Q},\boldsymbol{i}_{\mathbf{Q}},\omega_{\mathbf{Q}})$. Define $\boldsymbol{Q}=\boldsymbol{R}\times_{\boldsymbol{g}|_{\boldsymbol{R}},\boldsymbol{T},\boldsymbol{h}|_{\boldsymbol{S}}}\boldsymbol{S}$ to be the explicit d-space fibre product from §2.5, and write $\tilde{\boldsymbol{e}}:\boldsymbol{Q}\to\boldsymbol{R}$, $\tilde{\boldsymbol{f}}:\boldsymbol{Q}\to\boldsymbol{S}$ for the explicit 1-morphisms and $\tilde\eta:\boldsymbol{g}\circ\tilde{\boldsymbol{e}}\Rightarrow\boldsymbol{h}\circ\tilde{\boldsymbol{f}}$ for the explicit 2-morphism in $\mathbf{dSpa}$ from §2.5. Define a d-space $\boldsymbol{\partial Q}$ by

$$\boldsymbol{\partial Q}=\textstyle\coprod_{x'\in I}\bigl(\boldsymbol{B}_{x'}\times_{\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{B}_{x'}},\boldsymbol{T},\boldsymbol{h}|_{\boldsymbol{S}}}\boldsymbol{S}\bigr)\amalg\coprod_{y'\in J}\bigl(\boldsymbol{R}\times_{\boldsymbol{g}|_{\boldsymbol{R}},\boldsymbol{T},\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}|_{\boldsymbol{C}_{y'}}}\boldsymbol{C}_{y'}\bigr),\quad(6.89)$$

where again we use the explicit d-space fibre product of §2.5. For $x'\in I$, write $\boldsymbol{a}_{x'}:\boldsymbol{B}_{x'}\times_{\boldsymbol{T}}\boldsymbol{S}\to\boldsymbol{B}_{x'}$ and $\boldsymbol{b}_{x'}:\boldsymbol{B}_{x'}\times_{\boldsymbol{T}}\boldsymbol{S}\to\boldsymbol{S}$ for the projection 1-morphisms and $\kappa_{x'}:\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{a}_{x'}\Rightarrow\boldsymbol{h}\circ\boldsymbol{b}_{x'}$ for the 2-morphism constructed in §2.5. For $y'\in J$, write $\boldsymbol{c}_{y'}:\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{C}_{y'}\to\boldsymbol{R}$ and $\boldsymbol{d}_{y'}:\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{C}_{y'}\to\boldsymbol{C}_{y'}$ for the projection 1-morphisms and $\lambda_{y'}:\boldsymbol{g}\circ\boldsymbol{c}_{y'}\Rightarrow\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{d}_{y'}$ for the 2-morphism from §2.5.

Since $\kappa_{x'}:\boldsymbol{g}\circ(\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{a}_{x'})\Rightarrow\boldsymbol{h}\circ\boldsymbol{b}_{x'}$, the proof in Theorem 2.36 that $\boldsymbol{Q},\tilde{\boldsymbol{e}},\tilde{\boldsymbol{f}},\tilde\eta$ is a fibre product $\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{S}$ in $\mathbf{dSpa}$ constructs an explicit 1-morphism $\boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{B}_{x'}\times_{\boldsymbol{T}}\boldsymbol{S}}:\boldsymbol{B}_{x'}\times_{\boldsymbol{T}}\boldsymbol{S}\to\boldsymbol{Q}$ (called $\boldsymbol{b}$ in that proof) such that $\tilde{\boldsymbol{e}}\circ\boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{B}_{x'}\times_{\boldsymbol{T}}\boldsymbol{S}}=\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{a}_{x'}$ and $\tilde{\boldsymbol{f}}\circ\boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{B}_{x'}\times_{\boldsymbol{T}}\boldsymbol{S}}=\boldsymbol{b}_{x'}$ in $\mathbf{dSpa}$ for $x'\in I$. Note that the construction of §2.5 gives equality of 1-morphisms here, since $\zeta=\eta=0$ in the proof of Theorem 2.36; the universal property of fibre products would only give 2-morphisms $\tilde{\boldsymbol{e}}\circ\boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{B}_{x'}\times_{\boldsymbol{T}}\boldsymbol{S}}\Rightarrow\boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{a}_{x'}$ and $\tilde{\boldsymbol{f}}\circ\boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{B}_{x'}\times_{\boldsymbol{T}}\boldsymbol{S}}\Rightarrow\boldsymbol{b}_{x'}$.

Similarly, since $\lambda_{y'}:\boldsymbol{g}\circ\boldsymbol{c}_{y'}\Rightarrow\boldsymbol{h}\circ(\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{d}_{y'})$, from §2.5 we get explicit $\boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{C}_{y'}}:\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{C}_{y'}\to\boldsymbol{Q}$ with $\tilde{\boldsymbol{e}}\circ\boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{C}_{y'}}=\boldsymbol{c}_{y'}$ and $\tilde{\boldsymbol{f}}\circ\boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{C}_{y'}}=\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{d}_{y'}$ for $y'\in J$. From (6.89), these $\boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{B}_{x'}\times_{\boldsymbol{T}}\boldsymbol{S}}$ and $\boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{C}_{y'}}$ for $x'\in I$ and $y'\in J$ make up a 1-morphism $\boldsymbol{i}_{\mathbf{Q}}:\boldsymbol{\partial Q}\to\boldsymbol{Q}$ in $\mathbf{dSpa}$. We will show in the proof of Theorem 6.41 that there is a unique $\omega_{\mathbf{Q}}$ such that $\mathbf{Q}=(\boldsymbol{Q},\boldsymbol{\partial Q},\boldsymbol{i}_{\mathbf{Q}},\omega_{\mathbf{Q}})$ is a d-space with corners, and $(\boldsymbol{Q},\boldsymbol{r}_{x'}\circ\tilde{\boldsymbol{e}})$ is a boundary defining function for $\mathbf{Q}$ at any $(\tilde x',\tilde y)\in\boldsymbol{B}_{x'}\times_{\boldsymbol{T}}\boldsymbol{S}\subseteq\boldsymbol{\partial Q}$ for $x'\in I$, and $(\boldsymbol{Q},\boldsymbol{s}_{y'}\circ\tilde{\boldsymbol{f}})$ is a boundary defining function for $\mathbf{Q}$ at any $(\tilde x,\tilde y')\in\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{C}_{y'}\subseteq\boldsymbol{\partial Q}$ for $y'\in J$.

Now $\tilde{\boldsymbol{e}},\tilde{\boldsymbol{f}},\tilde\eta$ above are 1- and 2-morphisms in $\mathbf{dSpa}$, but in general are not 1- and 2-morphisms in $\mathbf{dSpa^c}$, so we will construct modified versions $\boldsymbol{e},\boldsymbol{f},\eta$ which are. Consider the morphism in $\mathrm{qcoh}(\underline{Q})$:

$$\begin{aligned}\Phi:{}&\textstyle\bigoplus_{z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{g}\circ\underline{\tilde e})^*\circ\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)})\longrightarrow\\&\textstyle\bigoplus_{x'\in\underline{i}_{\mathbf{X}}^{-1}(x)\setminus I}\underline{\tilde e}^*\circ\underline{r}_{x'}^*(\mathcal{F}_{[0,\infty)})\oplus\bigoplus_{y'\in\underline{i}_{\mathbf{Y}}^{-1}(y)\setminus J}\underline{\tilde f}^*\circ\underline{s}_{y'}^*(\mathcal{F}_{[0,\infty)})\\\Phi={}&\textstyle\bigoplus_{x'\in\underline{i}_{\mathbf{X}}^{-1}(x)\setminus I,\;z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}\underline{\tilde e}^*(\beta_{x'})\circ\underline{\tilde e}^*(g^2)\circ I_{\underline{\tilde e},\underline{g}}(\mathcal{F}_Z)\circ(\underline{g}\circ\underline{\tilde e})^*(t_{z'}^2)\oplus\\&\textstyle\bigoplus_{y'\in\underline{i}_{\mathbf{Y}}^{-1}(y)\setminus J,\;z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}\underline{\tilde f}^*(\gamma_{y'})\circ\underline{\tilde f}^*(h^2)\circ I_{\underline{\tilde f},\underline{h}}(\mathcal{F}_Z)\circ(\underline{g}\circ\underline{\tilde e})^*(t_{z'}^2).\end{aligned}\quad(6.90)$$

Since $\mathcal{F}_{[0,\infty)}$ is a line bundle, Φ is a morphism between vector bundles of rank $|\underline{i}_{\mathbf{Z}}^{-1}(z)|=|\underline{i}_{\mathbf{X}}^{-1}(x)\setminus I|+|\underline{i}_{\mathbf{Y}}^{-1}(y)\setminus J|$. We claim that Φ is invertible near $(x,y)\in\underline{Q}$. To see this, note that for $z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)$ there are natural isomorphisms

$$(\underline{g}\circ\underline{\tilde e})^*\circ\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)})|_{(x,y)}\cong\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)})|_z\cong\underline{i}_{\mathbf{Z}}^*\circ\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)})|_{z'}\cong\mathcal{N}_{\mathbf{Z}}|_{z'},$$

where we use (6.7) for the last step. Thus the domain of $\Phi|_{(x,y)}$ is naturally isomorphic to $\bigoplus_{z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}\mathcal{N}_{\mathbf{Z}}|_{z'}$, which is the domain of (6.87). Similarly the targets of $\Phi|_{(x,y)}$ and (6.87) are naturally isomorphic, and using Definition 6.1(e) and (6.14) commuting we can show that these isomorphisms identify $\Phi|_{(x,y)}$ and (6.87). Therefore $\Phi|_{(x,y)}$ is an isomorphism, as (6.87) is.

As this is an open condition, Φ is invertible near $(x,y)\in \underline{Q}$. Making $\boldsymbol{Q},\boldsymbol{R},\boldsymbol{S}$ smaller if necessary, we can suppose Φ is invertible. Write the inverse of Φ as

$$\rho\oplus\sigma: \bigoplus_{x'\in \underline{i}_{\mathbf{X}}^{-1}(x)\setminus I} \underline{\tilde{e}}^*\circ\underline{r}_{x'}^*(\mathcal{F}_{[0,\infty)})\oplus \bigoplus_{y'\in \underline{i}_{\mathbf{Y}}^{-1}(y)\setminus J} \underline{\tilde{f}}^*\circ\underline{s}_{y'}^*(\mathcal{F}_{[0,\infty)})\to \bigoplus_{z'\in \underline{i}_{\mathbf{Z}}^{-1}(z)} (\underline{g}\circ\underline{\tilde{e}})^*\circ\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)}).$$

Define morphisms $\tau:\underline{\tilde{e}}^*(\mathcal{F}_X)\to\mathcal{E}_Q$, $\upsilon:\underline{\tilde{f}}^*(\mathcal{F}_Y)\to\mathcal{E}_Q$ in $\mathrm{qcoh}(\underline{Q})$ by

$$\begin{aligned}\tau&=\tilde{\eta}\circ\bigl(\textstyle\bigoplus_{z'\in \underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{g}\circ\underline{\tilde{e}})^*(t_{z'}^2)\bigr)\circ\rho\circ\bigl(\textstyle\bigoplus_{x'\in \underline{i}_{\mathbf{X}}^{-1}(x)\setminus I}\underline{\tilde{e}}^*(\beta_{x'})\bigr),\\ \upsilon&=\tilde{\eta}\circ\bigl(\textstyle\bigoplus_{z'\in \underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{g}\circ\underline{\tilde{e}})^*(t_{z'}^2)\bigr)\circ\sigma\circ\bigl(\textstyle\bigoplus_{y'\in \underline{i}_{\mathbf{Y}}^{-1}(y)\setminus J}\underline{\tilde{f}}^*(\gamma_{y'})\bigr).\end{aligned}\tag{6.91}$$

Since $\beta_{x'}\circ r_{x''}^2=0$ for $x'\neq x''$ and $\gamma_{y'}\circ s_{y''}^2=0$ for $y'\neq y''$, (6.91) gives

$$\tau\circ\underline{\tilde{e}}^*(r_{x'}^2)=0,\ \ x'\in I,\ \text{ and }\ \upsilon\circ\underline{\tilde{f}}^*(s_{y'}^2)=0,\ \ y'\in J.\tag{6.92}$$

By Proposition 2.17, there exist unique 1-morphisms $\boldsymbol{e}:\boldsymbol{Q}\to\boldsymbol{R}$, $\boldsymbol{f}:\boldsymbol{Q}\to\boldsymbol{S}$ in **dSpa** such that $\tau:\tilde{\boldsymbol{e}}\Rightarrow\boldsymbol{e}$ and $-\upsilon:\tilde{\boldsymbol{f}}\Rightarrow\boldsymbol{f}$ are 2-morphisms in **dSpa**. For $x'\in I$, we have 1-morphisms $\boldsymbol{r}_{x'}\circ\tilde{\boldsymbol{e}},\boldsymbol{r}_{x'}\circ\boldsymbol{e}:\boldsymbol{Q}\to\boldsymbol{[0,\infty)}$ and a 2-morphism $\mathrm{id}_{\boldsymbol{r}_{x'}}*\tau:\boldsymbol{r}_{x'}\circ\tilde{\boldsymbol{e}}\Rightarrow\boldsymbol{r}_{x'}\circ\boldsymbol{e}$. But $\mathrm{id}_{\boldsymbol{r}_{x'}}*\tau=\tau\circ\underline{\tilde{e}}^*(r_{x'}^2)\circ I_{\underline{\tilde{e}},\underline{r}_{x'}}(\mathcal{F}_{[0,\infty)})$, which is zero by (6.92), so $\boldsymbol{r}_{x'}\circ\tilde{\boldsymbol{e}}=\boldsymbol{r}_{x'}\circ\boldsymbol{e}$. By the same argument for $\upsilon,\boldsymbol{s}_{y'}$ we get

$$\boldsymbol{r}_{x'}\circ\tilde{\boldsymbol{e}}=\boldsymbol{r}_{x'}\circ\boldsymbol{e},\ \ x'\in I,\ \text{ and }\ \boldsymbol{s}_{y'}\circ\tilde{\boldsymbol{f}}=\boldsymbol{s}_{y'}\circ\boldsymbol{f},\ \ y'\in J.\tag{6.93}$$

Define a 2-morphism $\eta:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ in **dSpa** by

$$\eta=(\mathrm{id}_{\boldsymbol{h}}*(-\upsilon))\odot\tilde{\eta}\odot(\mathrm{id}_{\boldsymbol{g}}*(-\tau)).\tag{6.94}$$

Then for $z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)$ we have 1-morphisms $\boldsymbol{t}_{z'}\circ\boldsymbol{g}\circ\boldsymbol{e},\boldsymbol{t}_{z'}\circ\boldsymbol{h}\circ\boldsymbol{f}:\boldsymbol{Q}\to\boldsymbol{[0,\infty)}$ in **dSpa** and a 2-morphism $\mathrm{id}_{\boldsymbol{t}_{z'}}*\eta:\boldsymbol{t}_{z'}\circ\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{t}_{z'}\circ\boldsymbol{h}\circ\boldsymbol{f}$. We find that

$$\begin{aligned}&\mathrm{id}_{\boldsymbol{t}_{z'}}*\eta=\eta\circ(\underline{g}\circ\underline{\tilde{e}})^*(t_{z'}^2)\circ I_{\underline{g}\circ\underline{\tilde{e}},\underline{t}_{z'}}(\mathcal{F}_{[0,\infty)})\\ &=\bigl[\tilde{\eta}-\tau\circ\underline{\tilde{e}}^*(g^2)\circ I_{\underline{\tilde{e}},\underline{g}}(\mathcal{F}_Z)-\upsilon\circ\underline{\tilde{f}}^*(h^2)\circ I_{\underline{\tilde{f}},\underline{h}}(\mathcal{F}_Z)\bigr]\circ(\underline{g}\circ\underline{\tilde{e}})^*(t_{z'}^2)\circ I_{\underline{g}\circ\underline{\tilde{e}},\underline{t}_{z'}}(\mathcal{F}_{[0,\infty)})\\ &=\tilde{\eta}\circ\bigl(\textstyle\bigoplus_{z''\in\underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{g}\circ\underline{\tilde{e}})^*(t_{z''}^2)\bigr)\circ\bigl[\mathrm{id}_{\bigoplus_{z''\in\underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{g}\circ\underline{\tilde{e}})^*\circ\underline{t}_{z''}^*(\mathcal{F}_{[0,\infty)})}-(\rho\oplus\sigma)\circ\Phi\bigr]\\ &\qquad\circ\bigl(\mathrm{id}_{(\underline{g}\circ\underline{\tilde{e}})^*\circ\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)})}\oplus 0\bigr)\circ I_{\underline{g}\circ\underline{\tilde{e}},\underline{t}_{z'}}(\mathcal{F}_{[0,\infty)})=0,\end{aligned}$$

using (6.90)–(6.91), $\rho\oplus\sigma=\Phi^{-1}$ and Definition 2.14, where $\mathrm{id}_{(\underline{g}\circ\underline{\tilde{e}})^*\circ\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)})}\oplus 0$ means the inclusion $(\underline{g}\circ\underline{\tilde{e}})^*\circ\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)})\hookrightarrow\bigoplus_{z''\in\underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{g}\circ\underline{\tilde{e}})^*\circ\underline{t}_{z''}^*(\mathcal{F}_{[0,\infty)})$. So $\mathrm{id}_{\boldsymbol{t}_{z'}}*\eta=\mathrm{id}_{\boldsymbol{t}_{z'}\circ\boldsymbol{g}\circ\boldsymbol{e}}$, which proves that

$$\eta\circ(\underline{g}\circ\underline{\tilde{e}})^*(t_{z'}^2)=0\ \text{ and }\ \boldsymbol{t}_{z'}\circ\boldsymbol{g}\circ\boldsymbol{e}=\boldsymbol{t}_{z'}\circ\boldsymbol{h}\circ\boldsymbol{f}\ \text{ for all }z'\in\underline{i}_{\mathbf{Z}}^{-1}(z).\tag{6.95}$$

We now prove that $\mathbf{Q}, \boldsymbol{e}, \boldsymbol{f}, \eta$ are a fibre product $\mathbf{R} \times_{\boldsymbol{g}|_{\mathbf{R}}, \mathbf{T}, \boldsymbol{h}|_{\mathbf{S}}} \mathbf{S}$ in $\mathbf{dSpa^c}$.

Theorem 6.41. *In Definition* 6.40, $\mathbf{Q}$ *is a d-space with corners, and* $\boldsymbol{e} : \mathbf{Q} \to \mathbf{R}$, $\boldsymbol{f} : \mathbf{Q} \to \mathbf{S}$ *are* 1*-morphisms in* $\mathbf{dSpa^c}$, *and* $\eta : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$ *is a* 2*-morphism in* $\mathbf{dSpa^c}$, *and equation* (6.86) *is* 2*-Cartesian in* $\mathbf{dSpa^c}$.

Proof. To show $\mathbf{Q}$ is a d-space with corners, we will verify Definition 6.1(a)–(f) for $\boldsymbol{Q}, \boldsymbol{\partial Q}, \boldsymbol{i}_{\mathbf{Q}}$, constructing $\omega_{\mathbf{Q}}$ along the way. For (a), for each $x' \in I$ we have a Cartesian square in $\mathbf{C^\infty Sch}$:

$$
\begin{tikzcd}
\underline{B}_{x'} \times_{\underline{T}} \underline{S} \arrow[rr, "\underline{a}_{x'}"'] \arrow[d, "\underline{i}_{\boldsymbol{Q}}|_{\underline{B}_{x'} \times_{\underline{T}} \underline{S}}"] && \underline{B}_{x'} \arrow[d, "\underline{i}_{\mathbf{X}}|_{\underline{B}_{x'}}"'] \\
\underline{Q} = \underline{R} \times_{\underline{T}} \underline{S} \arrow[rr, "\underline{\tilde{e}}"] && \underline{R}.
\end{tikzcd}
$$

Since $\underline{i}_{\mathbf{X}}$ is proper by Definition 6.1(a) for $\mathbf{X}$ and $\underline{B}_{x'}$ is closed in $\underline{i}_{\mathbf{X}}^{-1}(\underline{R})$, we see that $\underline{i}_{\mathbf{X}}|_{\underline{B}_{x'}} : \underline{B}_{x'} \to \underline{R}$ is proper, and therefore $\underline{i}_{\boldsymbol{Q}}|_{\underline{B}_{x'} \times_{\underline{T}} \underline{S}} : \underline{B}_{x'} \times_{\underline{T}} \underline{S} \to \underline{Q}$ is proper by properties of Cartesian squares. Similarly $\underline{i}_{\boldsymbol{Q}}|_{\underline{R} \times_{\underline{T}} \underline{C}_{y'}} : \underline{R} \times_{\underline{T}} \underline{C}_{y'} \to \underline{Q}$ is proper for $y' \in J$. Taking the disjoint union over $x' \in I$ and $y' \in J$ shows that $\underline{i}_{\mathbf{Q}} : \underline{\partial Q} \to \underline{Q}$ is proper, proving Definition 6.1(a) for $\mathbf{Q}$.

For (b), for $x' \in I$, to show that $i''_{\mathbf{Q}}$ is an isomorphism on $\underline{B}_{x'} \times_{\underline{S}} \underline{T} \subseteq \underline{\partial Q}$, consider the diagram in $\mathrm{qcoh}(\underline{B}_{x'} \times_{\underline{S}} \underline{T})$:

$$
\begin{tikzcd}[column sep=small]
\underline{i}_{\boldsymbol{Q}}^* \circ (\underline{g} \circ \underline{\tilde{e}})^*(\mathcal{E}_Z) \arrow[r, "\underline{i}_{\boldsymbol{Q}}^* \circ \begin{pmatrix} \underline{\tilde{e}}^*(g'') \circ I_{\underline{\tilde{e}}, \underline{g}}(\mathcal{E}_Z) \\ -\underline{\tilde{f}}^*(h'') \circ I_{\underline{\tilde{f}}, \underline{h}}(\mathcal{E}_Z) \\ (\underline{g} \circ \underline{\tilde{e}})^*(\phi_Z) \end{pmatrix}"'] \arrow[d, "I_{\underline{i}_{\boldsymbol{Q}}, \underline{g} \circ \underline{\tilde{e}}}(\mathcal{E}_Z)"'] & \begin{array}{c} \underline{i}_{\boldsymbol{Q}}^*(\underline{\tilde{e}}^*(\mathcal{E}_X) \oplus \underline{\tilde{f}}^*(\mathcal{E}_Y) \\ \oplus (\underline{g} \circ \underline{\tilde{e}})^*(\mathcal{F}_Z)) \end{array} \arrow[r] \arrow[d, "\begin{pmatrix} \begin{array}{l} \underline{a}_{x'}^*(i''_{\mathbf{X}}) \circ \\ I_{\underline{a}_{x'}, \underline{i}_{\mathbf{X}}}(\mathcal{E}_X) \circ \\ I_{\underline{i}_{\boldsymbol{Q}}, \underline{\tilde{e}}}(\mathcal{E}_X)^{-1} \end{array} & 0 & 0 \\ 0 & I_{\underline{i}_{\boldsymbol{Q}}, \underline{\tilde{f}}}(\mathcal{E}_Y)^{-1} & 0 \\ 0 & 0 & I_{\underline{i}_{\boldsymbol{Q}}, \underline{g} \circ \underline{\tilde{e}}}(\mathcal{F}_Z)^{-1} \end{pmatrix}"] & \underline{i}_{\boldsymbol{Q}}^*(\mathcal{E}_Q)|_{\underline{B}_{x'} \times_{\underline{T}} \underline{S}} \arrow[r] \arrow[d, "i''_{\mathbf{Q}}|_{\underline{B}_{x'} \times_{\underline{T}} \underline{S}}"] & 0 \\
(\underline{g} \circ \underline{\tilde{e}} \circ \underline{i}_{\boldsymbol{Q}})^*(\mathcal{E}_Z) \arrow[r, "\begin{pmatrix} \underline{a}_{x'}^*((g \circ i_{\mathbf{X}})'') \circ I_{\underline{a}_{x'}, \underline{g} \circ \underline{i}_{\mathbf{X}}}(\mathcal{E}_Z) \\ -\underline{b}_{x'}^*(h'') \circ I_{\underline{b}_{x'}, \underline{h}}(\mathcal{E}_Z) \\ (\underline{g} \circ \underline{\tilde{e}} \circ \underline{i}_{\boldsymbol{Q}})^*(\phi_Z) \end{pmatrix}"] & \begin{array}{c} \underline{a}_{x'}^*(\mathcal{E}_{\partial X}) \oplus \underline{b}_{x'}^*(\mathcal{E}_Y) \oplus \\ (\underline{g} \circ \underline{\tilde{e}} \circ \underline{i}_{\boldsymbol{Q}})^*(\mathcal{F}_Z) \end{array} \arrow[r] & \mathcal{E}_{\partial Q}|_{\underline{B}_{x'} \times_{\underline{T}} \underline{S}} \arrow[r] & 0.
\end{tikzcd}
\tag{6.96}
$$

Here as in §2.5, $\mathcal{E}_Q, \mathcal{E}_{\partial Q}$ are defined to be the cokernels of α_1 in (2.59). The bottom row is the corresponding exact sequence for $\mathcal{E}_{\partial Q}$, and the top row is $\underline{i}_{\boldsymbol{Q}}|_{\underline{B}_{x'} \times_{\underline{T}} \underline{S}}^*$ applied to the exact sequence for $\mathcal{E}_Q$. The definitions imply that (6.96) commutes. Since $i''_{\mathbf{X}}$ is an isomorphism by Definition 6.1(b) for $\mathbf{X}$, the first two columns in (6.96) are isomorphisms. Hence by exactness of the rows, the third column $i''_{\mathbf{Q}}|_{\underline{B}_{x'} \times_{\underline{T}} \underline{S}}$ is an isomorphism for $x' \in I$. Similarly, $i''_{\mathbf{Q}}|_{\underline{R} \times_{\underline{T}} \underline{C}_{y'}}$ is an isomorphism for $y' \in J$. Therefore $i''_{\mathbf{Q}}$ is an isomorphism on $\underline{\partial Q}$, proving Definition 6.1(b) for $\mathbf{Q}$.

For (c), consider the 2-commutative diagram in $\mathbf{dSpa}$ for $x' \in I$:

$$
\begin{tikzcd}
\boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S} \arrow[rr, "\boldsymbol{a}_{x'}"'] \arrow[d, "\boldsymbol{i}_{\boldsymbol{Q}}|_{\boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S}}"] & \arrow[d, phantom, "\mathrm{id}_{\boldsymbol{i}_{\mathbf{X}} \circ \boldsymbol{a}_{x'}} \Nearrow"] & \boldsymbol{B}_{x'} \arrow[rr, "\boldsymbol{\pi}"'] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{B}_{x'}}"] & \arrow[d, phantom, "\mathrm{id}_{\boldsymbol{0} \circ \boldsymbol{\pi}} \Nearrow"] & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"'] \\
\boldsymbol{Q} = \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{S} \arrow[rr, "\tilde{\boldsymbol{e}}"] & & \boldsymbol{R} \arrow[rr, "\boldsymbol{r}_{x'}"] & & {[\boldsymbol{0}, \boldsymbol{\infty})}.
\end{tikzcd}
\tag{6.97}
$$

The right hand square is 2-Cartesian as in (6.88), since $(\boldsymbol{R}, \boldsymbol{r}_{x'})$ is a boundary defining function for $\mathbf{X}$ at x'. The left hand square 2-commutes as $\tilde{\boldsymbol{e}} \circ \boldsymbol{i}_{\mathbf{Q}}|_{\boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S}} = \boldsymbol{i}_{\mathbf{X}} \circ \boldsymbol{a}_{x'}$, and is 2-Cartesian by properties of fibre products. Hence the outer rectangle of (6.97) is 2-Cartesian in **dSpa**. This proves the first part of Definition 6.1(c) for $\mathbf{Q}$ for any $\tilde{x}' \in \boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S} \subseteq \boldsymbol{\partial Q}$, with $\boldsymbol{U} = \boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S}$, $\boldsymbol{V} = \boldsymbol{Q}$ and $\boldsymbol{b} = \boldsymbol{r}_{x'} \circ \tilde{\boldsymbol{e}}$, so that the outer rectangle of (6.97) becomes (6.1).

For the second part of Definition 6.1(c), consider the diagram

$$\begin{array}{c}(\underline{r}_{x'}^* \circ \underline{\tilde{e}})^* \\ (\mathcal{F}_{[0,\infty)})\end{array} \underset{I_{\underline{\tilde{e}},\underline{r}_{x'}}(\mathcal{F}_{[0,\infty)})^{-1}}{\overset{I_{\underline{\tilde{e}},\underline{r}_{x'}}(\mathcal{F}_{[0,\infty)})}{\rightleftarrows}} \begin{array}{c}\underline{\tilde{e}}^* \circ \underline{r}_{x'}^* \\ (\mathcal{F}_{[0,\infty)})\end{array} \underset{\underline{\tilde{e}}^*(\beta_{x'})}{\overset{\underline{\tilde{e}}^*(r_{x'}^2)}{\rightleftarrows}} \underline{\tilde{e}}^*(\mathcal{F}_X) \underset{\mathrm{id}\oplus 0}{\overset{\mathrm{id}\oplus 0}{\rightleftarrows}} \begin{array}{c}\underline{\tilde{e}}^*(\mathcal{F}_X) \oplus \underline{\tilde{f}}^*(\mathcal{F}_Y) \\ \cong \mathcal{F}_Q.\end{array}$$

The composition of the rightwards morphisms is $(r_{x'} \circ \tilde{e})^2$. Each leftwards morphism is a left inverse for the rightwards morphism above it, noting that $\beta_{x'}$ is a left inverse for $r_{x'}^2$. Thus the composition of the leftwards morphisms is a left inverse for $(r_{x'} \circ \tilde{e})^2$. Hence Definition 6.1(c) for $\mathbf{Q}$ holds for any $(\tilde{x}', \tilde{y}) \in \boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S} \subseteq \boldsymbol{\partial Q}$ for $x' \in I$, with $\boldsymbol{U} = \boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S}$, $\boldsymbol{V} = \boldsymbol{Q}$ and $\boldsymbol{b} = \boldsymbol{r}_{x'} \circ \tilde{\boldsymbol{e}}$. Similarly, Definition 6.1(c) for $\mathbf{Q}$ holds for any $(\tilde{x}, \tilde{y}') \in \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{C}_{y'} \subseteq \boldsymbol{\partial Q}$ for $y' \in J$, with $\boldsymbol{U} = \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{C}_{y'}$, $\boldsymbol{V} = \boldsymbol{Q}$ and $\boldsymbol{b} = \boldsymbol{s}_{y'} \circ \tilde{\boldsymbol{f}}$. This proves Definition 6.1(c) for $\mathbf{Q}$.

Definition 6.1 now constructs a conormal line bundle $\mathcal{N}_{\mathbf{Q}}$ for $\mathbf{Q}$ in $\mathrm{qcoh}(\underline{\partial Q})$, in an exact sequence (6.4). For Definition 6.1(d),(e), for $x' \in I$ we define $\omega_{\mathbf{Q}}|_{\underline{B}_{x'} \times_{\underline{T}} \underline{S}}$ to be the unique orientation on $\mathcal{N}_{\mathbf{Q}}|_{\underline{B}_{x'} \times_{\underline{T}} \underline{S}}$ such that (e) holds with $\boldsymbol{U} = \boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S}$, $\boldsymbol{V} = \boldsymbol{Q}$ and $\boldsymbol{b} = \boldsymbol{r}_{x'} \circ \tilde{\boldsymbol{e}}$, and for $y' \in J$ we define $\omega_{\mathbf{Q}}|_{\underline{R} \times_{\underline{T}} \underline{C}_{y'}}$ to be the unique orientation on $\mathcal{N}_{\mathbf{Q}}|_{\underline{R} \times_{\underline{T}} \underline{C}_{y'}}$ such that (e) holds with $\boldsymbol{U} = \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{C}_{y'}$, $\boldsymbol{V} = \boldsymbol{Q}$ and $\boldsymbol{b} = \boldsymbol{s}_{y'} \circ \tilde{\boldsymbol{f}}$. This defines $\omega_{\mathbf{Q}}$, and shows that $(\boldsymbol{Q}, \boldsymbol{r}_{x'} \circ \tilde{\boldsymbol{e}})$ is a boundary defining function for $\mathbf{Q}$ at any $(\tilde{x}', \tilde{y}) \in \boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S} \subseteq \boldsymbol{\partial Q}$ for $x' \in I$, and $(\boldsymbol{Q}, \boldsymbol{s}_{y'} \circ \tilde{\boldsymbol{f}})$ is a boundary defining function for $\mathbf{Q}$ at any $(\tilde{x}, \tilde{y}') \in \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{C}_{y'} \subseteq \boldsymbol{\partial Q}$ for $y' \in J$.

For (f), let $q = (\tilde{x}, \tilde{y}) \in \boldsymbol{Q}$, so that $\tilde{x} \in \boldsymbol{R} \subseteq \boldsymbol{X}$ and $\tilde{y} \in \boldsymbol{S} \subseteq \boldsymbol{Y}$. Then $\underline{q}^*(\mathcal{F}_Q) \cong \underline{\tilde{x}}^*(\mathcal{F}_X) \oplus \underline{\tilde{y}}^*(\mathcal{F}_Y)$. The construction of boundary defining functions for $\mathbf{Q}$ above shows that (6.8) for $\mathbf{Q}$ at q is isomorphic to

$$\begin{pmatrix} \bigoplus_{\substack{\tilde{x}' \in \underline{i}_{\mathbf{X}}^{-1}(\tilde{x}): \\ \tilde{x}' \in \underline{B}_{x'},\, x' \in I}} I_{\underline{\tilde{x}}', \underline{i}_{\mathbf{X}}}(\mathcal{F}_X)^{-1} \circ (\underline{\tilde{x}}')^*(\nu_{\mathbf{X}}) & 0 \\ 0 & \bigoplus_{\substack{\tilde{y}' \in \underline{i}_{\mathbf{Y}}^{-1}(\tilde{y}): \\ \tilde{y}' \in \underline{C}_{y'},\, y' \in J}} I_{\underline{\tilde{y}}', \underline{i}_{\mathbf{Y}}}(\mathcal{F}_Y)^{-1} \circ (\underline{\tilde{y}}')^*(\nu_{\mathbf{Y}}) \end{pmatrix} :$$

$$\bigoplus_{\substack{\tilde{x}' \in \underline{i}_{\mathbf{X}}^{-1}(\tilde{x}): \\ \tilde{x}' \in \underline{B}_{x'},\, x' \in I}} (\underline{\tilde{x}}')^*(\mathcal{N}_{\mathbf{X}}) \oplus \bigoplus_{\substack{\tilde{y}' \in \underline{i}_{\mathbf{Y}}^{-1}(\tilde{y}): \\ \tilde{y}' \in \underline{C}_{y'},\, y' \in J}} (\underline{\tilde{y}}')^*(\mathcal{N}_{\mathbf{Y}}) \longrightarrow \underline{\tilde{x}}^*(\mathcal{F}_X) \oplus \underline{\tilde{y}}^*(\mathcal{F}_Y).$$

Therefore Definition 6.1(f) for $\mathbf{Q}$ at $q = (\tilde{x}, \tilde{y})$ follows from Definition 6.1(f) for $\mathbf{X}, \mathbf{Y}$ at $\tilde{x}, \tilde{y}$. This proves that $\mathbf{Q}$ is a d-space with corners.

Next we prove $\boldsymbol{e} : \mathbf{Q} \to \mathbf{R}$ is a 1-morphism in $\mathbf{dSpa^c}$. Suppose that $(\tilde{x}, \tilde{y}) \in \boldsymbol{Q}$, so that $\boldsymbol{e}(\tilde{x}, \tilde{y}) = \tilde{x} \in \boldsymbol{R}$, and $\tilde{x}' \in \boldsymbol{\partial R}$ with $\boldsymbol{i}_{\mathbf{X}}(\tilde{x}') = \tilde{x}$. Since $\boldsymbol{\partial R} = \coprod_{\underline{i}_{\mathbf{X}}^{-1}(x)} \boldsymbol{B}_{x'}$, we divide into four cases:

(a) $\tilde{x}' \in \boldsymbol{B}_{x'}$ for $x' \in I$;

(b) $\tilde{x}' \in \boldsymbol{B}_{x'}$ for $x' \in i_{\mathbf{X}}^{-1}(x) \setminus I$ a vertex in a type (A) component $\hat{\Gamma}$ of $\Gamma_{x,y}$;

(c) $\tilde{x}' \in \boldsymbol{B}_{x'}$ for $x' \in i_{\mathbf{X}}^{-1}(x) \setminus I$ a vertex in a type (B) component $\hat{\Gamma}$; and

(d) $\tilde{x}' \in \boldsymbol{B}_{x'}$ for $x' \in i_{\mathbf{X}}^{-1}(x) \setminus I$ a vertex in a type (C) component $\hat{\Gamma}$.

In case (a), $(\boldsymbol{R}, \boldsymbol{r}_{x'})$ is a boundary defining function for $\mathbf{R}$ at $\tilde{x}'$, and $(\tilde{x}', \tilde{y}) \in \boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S} \subseteq \boldsymbol{\partial Q}$, and $(\boldsymbol{Q}, \boldsymbol{r}_{x'} \circ \tilde{\boldsymbol{e}})$ is a boundary defining function for $\mathbf{Q}$ at $(\tilde{x}', \tilde{y})$ from above, and $\boldsymbol{r}_{x'} \circ \tilde{\boldsymbol{e}} = \boldsymbol{r}_{x'} \circ \boldsymbol{e}$ by (6.93). Hence $(\boldsymbol{Q}, \boldsymbol{r}_{x'} \circ \boldsymbol{e})$ is a boundary defining function for $\mathbf{Q}$ at $(\tilde{x}', \tilde{y})$. This proves Definition 6.2(i) holds for $\boldsymbol{e}$ at $(\tilde{x}, \tilde{y}), \tilde{x}'$ in case (a), for the particular choice of boundary defining function $(\boldsymbol{R}, \boldsymbol{r}_{x'})$ for $\mathbf{R}$ at $\tilde{x}'$. But as in Definition 6.2, if Definition 6.2(i) for $\boldsymbol{e}$ holds for one choice of boundary defining function, it also holds for any other choice.

In case (b), suppose $\overset{x''}{\bullet} - \overset{z''}{\bullet}$ and $\overset{y''}{\bullet} - \overset{z''}{\bullet}$ are two edges in $\hat{\Gamma}$. Then $\boldsymbol{t}_{z''} \circ \boldsymbol{g}|_{\boldsymbol{R}} = \boldsymbol{p}_{x''z''} \cdot \boldsymbol{r}_{x''}$ for $\boldsymbol{p}_{x''z''} : \boldsymbol{R} \to \boldsymbol{(0, \infty)}$ and $\boldsymbol{t}_{z''} \circ \boldsymbol{h}|_{\boldsymbol{S}} = \boldsymbol{q}_{y''z''} \cdot \boldsymbol{s}_{y''}$ for $\boldsymbol{q}_{y''z''} : \boldsymbol{S} \to \boldsymbol{(0, \infty)}$ from Definition 6.40. So we have equivalences

$$
\begin{aligned}
\boldsymbol{B}_{x''} \times_{\boldsymbol{T}} \boldsymbol{S} &\simeq (\boldsymbol{R} \times_{\boldsymbol{r}_{x''}, \boldsymbol{[0,\infty)}, \boldsymbol{0}} \boldsymbol{*}) \times_{\boldsymbol{T}} \boldsymbol{S} \simeq (\boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{S}) \times_{\boldsymbol{r}_{x''} \circ \boldsymbol{e}, \boldsymbol{[0,\infty)}, \boldsymbol{0}} \boldsymbol{*} \\
&\simeq (\boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{S}) \times_{\boldsymbol{t}_{z''} \circ \boldsymbol{g} \circ \boldsymbol{e}, \boldsymbol{[0,\infty)}, \boldsymbol{0}} \boldsymbol{*} = (\boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{S}) \times_{\boldsymbol{t}_{z''} \circ \boldsymbol{h} \circ \boldsymbol{f}, \boldsymbol{[0,\infty)}, \boldsymbol{0}} \boldsymbol{*} \\
&\simeq (\boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{S}) \times_{\boldsymbol{s}_{y''} \circ \boldsymbol{f}, \boldsymbol{[0,\infty)}, \boldsymbol{0}} \boldsymbol{*} \simeq \boldsymbol{R} \times_{\boldsymbol{T}} (\boldsymbol{S} \times_{\boldsymbol{s}_{y''}, \boldsymbol{[0,\infty)}, \boldsymbol{0}} \boldsymbol{*}) \simeq \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{C}_{y''},
\end{aligned}
$$

where in the third step we note that $\boldsymbol{t}_{z''} \circ \boldsymbol{g}|_{\boldsymbol{R}} = \boldsymbol{p}_{x''z''} \cdot \boldsymbol{r}_{x''}$ and multiplying by $\boldsymbol{p}_{x''z''}$ does not affect the fibre product, in the fifth we use $\boldsymbol{t}_{z''} \circ \boldsymbol{h}|_{\boldsymbol{S}} = \boldsymbol{q}_{y''z''} \cdot \boldsymbol{s}_{y''}$ in the same way, and in the fourth we use (6.95) for z''.

Let $\hat{x}' \in I$ or $\hat{y}' \in J$ be the unique vertex of $\hat{\Gamma}$ in $I \amalg J$. Since $\hat{\Gamma}$ is connected, we can connect the vertex x' from (b) and $\hat{x}'$ or $\hat{y}'$ with a finite sequence of such pairs of edges $\overset{x''}{\bullet} - \overset{z''}{\bullet}$, $\overset{y''}{\bullet} - \overset{z''}{\bullet}$. This gives an equivalence

$$
\boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S} \simeq \boldsymbol{B}_{\hat{x}'} \times_{\boldsymbol{T}} \boldsymbol{S} \quad \text{or} \quad \boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S} \simeq \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{C}_{\hat{y}'}. \tag{6.98}
$$

Note that this shows that $\boldsymbol{\partial Q}$ is independent up to equivalence of the arbitrary choice of $I \amalg J$ representing type (A) components of $\Gamma_{x,y}$ in Definition 6.40.

The equivalences (6.98) identify $(\tilde{x}', \tilde{y}) \in \boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S}$ with a unique point $(\tilde{x}'', \tilde{y}) \in \boldsymbol{B}_{\hat{x}'} \times_{\boldsymbol{T}} \boldsymbol{S}$ or $(\tilde{x}, \tilde{y}'') \in \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{C}_{\hat{y}'}$. As in (a), $(\boldsymbol{Q}, \boldsymbol{r}_{\hat{x}'} \circ \boldsymbol{e})$ is a boundary defining function for $\mathbf{Q}$ at $(\tilde{x}'', \tilde{y})$, or $(\boldsymbol{Q}, \boldsymbol{s}_{\hat{y}'} \circ \boldsymbol{f})$ is a boundary defining function for $\mathbf{Q}$ at $(\tilde{x}, \tilde{y}'')$. Let $\overset{x''}{\bullet} - \overset{z''}{\bullet}$ and $\overset{y''}{\bullet} - \overset{z''}{\bullet}$ be as above. Then

$$
\begin{aligned}
\boldsymbol{r}_{x''} \circ \boldsymbol{e} &= \boldsymbol{e}^*(\boldsymbol{p}_{x''z''}^{-1}) \cdot \big(\boldsymbol{t}_{z''} \circ \boldsymbol{g} \circ \boldsymbol{e}\big) = \boldsymbol{e}^*(\boldsymbol{p}_{x''z''}^{-1}) \cdot \big(\boldsymbol{t}_{z''} \circ \boldsymbol{h} \circ \boldsymbol{f}\big) \\
&= \boldsymbol{e}^*(\boldsymbol{p}_{x''z''}^{-1}) \cdot \boldsymbol{f}^*(\boldsymbol{q}_{y''z''}) \cdot \big(\boldsymbol{s}_{y''} \circ \boldsymbol{f}\big),
\end{aligned} \tag{6.99}
$$

using (6.95) for z''. Thus, Proposition 6.6(c),(d) imply that $(\boldsymbol{Q}, \boldsymbol{r}_{x''} \circ \boldsymbol{e})$ is a boundary defining function for $\mathbf{Q}$ at $(\tilde{x}'', \tilde{y})$ or $(\tilde{x}, \tilde{y}'')$ if and only if $(\boldsymbol{Q}, \boldsymbol{s}_{y''} \circ \boldsymbol{f})$ is one too. Connecting x' to $\hat{x}'$ or $\hat{y}'$ by a finite sequence of such triples x'', y'', z'' we see that $(\boldsymbol{Q}, \boldsymbol{r}_{x'} \circ \boldsymbol{e})$ is a boundary defining function for $\mathbf{Q}$ at $(\tilde{x}'', \tilde{y})$ or $(\tilde{x}, \tilde{y}'')$. This proves Definition 6.2(i) for $\boldsymbol{e}$ holds at $(\tilde{x}, \tilde{y}), \tilde{x}'$ in case (b).

In case (c), by assumption one vertex z_1' in $\hat\Gamma$ lies on one edge, and all other z' in $\hat\Gamma$ lie on two. If z_1' lies on the edge $\overset{x_1'}{\bullet} - \overset{z_1'}{\bullet}$ then $(y,z_1') \in \underline{T}_{\boldsymbol{h}}$, so $\boldsymbol{t}_{z'}\circ\boldsymbol{h}|_{\boldsymbol{S}}=\boldsymbol{0}\circ\boldsymbol{\pi}:\boldsymbol{S}\to[\boldsymbol{0},\boldsymbol{\infty})$. Hence $\boldsymbol{t}_{z'}\circ\boldsymbol{h}\circ\boldsymbol{f}=\boldsymbol{0}\circ\boldsymbol{\pi}\circ\boldsymbol{f}=\boldsymbol{0}\circ\boldsymbol{\pi}:\boldsymbol{Q}\to[\boldsymbol{0},\boldsymbol{\infty})$. But $\boldsymbol{t}_{z_1'}\circ\boldsymbol{g}\circ\boldsymbol{e}=\boldsymbol{t}_{z_1'}\circ\boldsymbol{h}\circ\boldsymbol{f}$ by (6.95), so $\boldsymbol{t}_{z_1'}\circ\boldsymbol{g}\circ\boldsymbol{e}=\boldsymbol{0}\circ\boldsymbol{\pi}$. Similarly, if z_1' lies on the edge $\overset{y_1'}{\bullet} - \overset{z_1'}{\bullet}$ then $(x,z_1')\in\underline{T}_{\boldsymbol{g}}$, so $\boldsymbol{t}_{z'}\circ\boldsymbol{g}|_{\boldsymbol{R}}=\boldsymbol{0}\circ\boldsymbol{\pi}$, and again $\boldsymbol{t}_{z_1'}\circ\boldsymbol{g}\circ\boldsymbol{e}=\boldsymbol{t}_{z_1'}\circ\boldsymbol{h}\circ\boldsymbol{f}=\boldsymbol{0}\circ\boldsymbol{\pi}$.

Since $\hat\Gamma$ is connected, we may go from x' to z_1' by a path of a finite number of edges in $\hat\Gamma$. The argument of (b) then shows that we may write $\boldsymbol{r}_{x'}\circ\boldsymbol{e}$ as the product of $\boldsymbol{t}_{z_1'}\circ\boldsymbol{g}\circ\boldsymbol{e}$ with a term $\boldsymbol{e}^*(\boldsymbol{p}_{x''z''}^{\pm 1})$ or $\boldsymbol{f}^*(\boldsymbol{q}_{y''z''}^{\pm 1})$ for each edge in this path. Since $\boldsymbol{t}_{z_1'}\circ\boldsymbol{g}\circ\boldsymbol{e}=\boldsymbol{0}\circ\boldsymbol{\pi}$, and multiplication by a positive function does not change $\boldsymbol{0}\circ\boldsymbol{\pi}$, we see that $\boldsymbol{r}_{x'}\circ\boldsymbol{e}=\boldsymbol{0}\circ\boldsymbol{\pi}$, and Definition 6.2(ii) for $\boldsymbol{e}$ holds at $(\tilde x,\tilde y),\tilde x'$ in case (c), with $\boldsymbol{W}=\boldsymbol{Q}$.

In case (d), each vertex z'' in $\hat\Gamma$ lies on two edges $\overset{x''}{\bullet} - \overset{z''}{\bullet}$, $\overset{y''}{\bullet} - \overset{z''}{\bullet}$, and (6.99) holds. We may write these equations in matrix form as:

$$\begin{pmatrix} \boldsymbol{r}_{x''}\circ\boldsymbol{e}\ {}_{\substack{\text{vertices}\\ x''\text{ in }\hat\Gamma}} & -\boldsymbol{s}_{y''}\circ\boldsymbol{f}\ {}_{\substack{\text{vertices}\\ y''\text{ in }\hat\Gamma}}\end{pmatrix}\begin{pmatrix} \boldsymbol{e}^*(\boldsymbol{p}_{x''z''})\ {}_{\substack{\text{edges}\\ \overset{x''}{\bullet} - \overset{z''}{\bullet}\text{ in }\hat\Gamma}} \\ \boldsymbol{f}^*(\boldsymbol{q}_{y''z''})\ {}_{\substack{\text{edges}\\ \overset{y''}{\bullet} - \overset{z''}{\bullet}\text{ in }\hat\Gamma}}\end{pmatrix}=\boldsymbol{0}. \tag{6.100}$$

Now the right hand matrix in (6.100) evaluated at $(x,y)\in\boldsymbol{Q}$ is identified with (6.85) under the isomorphisms $\mathcal{N}_{\mathbf{X}}|_{x''}\cong\mathbb{R}$, $\mathcal{N}_{\mathbf{Y}}|_{y''}\cong\mathbb{R}$, $\mathcal{N}_{\mathbf{Z}}|_{z''}\cong\mathbb{R}$ induced by the derivatives of $\boldsymbol{r}_{x''},\boldsymbol{s}_{y''},\boldsymbol{t}_{z''}$ at x'',y'',z''. But (6.85) is an isomorphism by Proposition 6.37(C). Hence the right hand matrix in (6.100) is an isomorphism near $(x,y)\in\boldsymbol{Q}$. Making $\boldsymbol{Q},\boldsymbol{R},\boldsymbol{S},\mathbf{Q},\mathbf{R},\mathbf{S}$ smaller if necessary, we can suppose the right hand matrix in (6.100) is invertible, and hence the left hand matrix is zero. That is, $\boldsymbol{r}_{x''}\circ\boldsymbol{e}=\boldsymbol{0}\circ\boldsymbol{\pi}:\boldsymbol{Q}\to[\boldsymbol{0},\boldsymbol{\infty})$ and $\boldsymbol{s}_{y''}\circ\boldsymbol{f}=\boldsymbol{0}\circ\boldsymbol{\pi}:\boldsymbol{Q}\to[\boldsymbol{0},\boldsymbol{\infty})$ for all vertices x'',y'' in $\hat\Gamma$. When $x''=x'$, this proves Definition 6.2(ii) for $\boldsymbol{e}$ holds at $(\tilde x,\tilde y),\tilde x'$ in case (d), with $\boldsymbol{W}=\boldsymbol{Q}$. So $\boldsymbol{e}:\mathbf{Q}\to\mathbf{R}$ is a 1-morphism in $\mathbf{dSpa^c}$. Similarly $\boldsymbol{f}:\mathbf{Q}\to\mathbf{S}$ is a 1-morphism in $\mathbf{dSpa^c}$.

To prove $\eta:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ is a 2-morphism in $\mathbf{dSpa^c}$, we have to verify (6.9) in $\mathrm{qcoh}(\underline{S}_{\boldsymbol{g}\circ\boldsymbol{e}})$ and (6.10) in $\mathrm{qcoh}(\underline{T}_{\boldsymbol{g}\circ\boldsymbol{e}})$. For (6.9), we have open inclusions $\underline{S}_{\boldsymbol{g}\circ\boldsymbol{e}}\subseteq\coprod_{z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}\underline{\partial Q}\times_{\underline{T}}\underline{D}_{z'}\subseteq\underline{\partial Q}\times_{\underline{T}}\underline{\partial T}$. Fix $z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)$. Then as Definition 6.1(c),(e) hold for $\mathbf{Z}$ with $\boldsymbol{D}_{z'},\boldsymbol{T},\boldsymbol{t}_{z'}$ in place of $\boldsymbol{U},\boldsymbol{V},\boldsymbol{b}$, equation (6.7) gives an isomorphism $\epsilon_{z'}:\underline{i}_{\mathbf{Z}}|_{\underline{D}_{z'}}^*\circ\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)})\to\mathcal{N}_{\mathbf{Z}}|_{\underline{D}_{z'}}$ with $\nu_{\mathbf{Z}}|_{\underline{D}_{z'}}\circ\epsilon_{z'}=\underline{i}_{\mathbf{Z}}|_{\underline{D}_{z'}}^*(t_{z'}^2)$. Thus we have

$$\begin{aligned}
&(\underline{i}_{\mathbf{Z}}\circ\underline{s}_{\boldsymbol{g}\circ\boldsymbol{e}})^*(\eta)\circ I_{\underline{i}_{\mathbf{Q}}\circ\underline{s}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)\circ I_{\underline{u}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{i}_{\mathbf{Z}}}(\mathcal{F}_Z)^{-1}\circ\underline{u}_{\boldsymbol{g}\circ\boldsymbol{e}}^*(\nu_{\mathbf{Z}})|_{\underline{S}_{\underline{g}\circ\underline{e}}\cap(\underline{\partial Q}\times_{\underline{T}}\underline{D}_{z'})}\\
&\ =(\underline{i}_{\mathbf{Z}}\circ\underline{s}_{\boldsymbol{g}\circ\boldsymbol{e}})^*(\eta)\circ I_{\underline{i}_{\mathbf{Q}}\circ\underline{s}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)\circ I_{\underline{u}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{i}_{\mathbf{Z}}}(\mathcal{F}_Z)^{-1}\circ\underline{u}_{\boldsymbol{g}\circ\boldsymbol{e}}^*(\underline{i}_{\mathbf{Z}}|_{\underline{D}_{z'}}^*(t_{z'}^2)\circ\epsilon_{z'}^{-1})|_{\cdots}\\
&\ =(\underline{i}_{\mathbf{Z}}\circ\underline{s}_{\boldsymbol{g}\circ\boldsymbol{e}})^*(\eta\circ(\underline{g}\circ\underline{e})^*(t_{z'}^2))\circ I_{\underline{i}_{\mathbf{Q}}\circ\underline{s}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{g}\circ\underline{e}}(\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)}))\circ\\
&\qquad\quad I_{\underline{u}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{i}_{\mathbf{Z}}}(\underline{t}_{z'}^*(\mathcal{F}_{[0,\infty)}))^{-1}\circ\underline{u}_{\boldsymbol{g}\circ\boldsymbol{e}}^*(\epsilon_{z'}^{-1})|_{\underline{S}_{\underline{g}\circ\underline{e}}\cap(\underline{\partial Q}\times_{\underline{T}}\underline{D}_{z'})}=0
\end{aligned}$$

in $\mathrm{qcoh}\bigl(\underline{S}_{\underline{g}\circ\underline{e}}\cap(\underline{\partial Q}\times_{\underline{T}}\underline{D}_{z'})\bigr)$, using (6.95) and $\underline{u}=\tilde{\underline{e}}$. This proves the restriction of (6.9) for η to $\underline{S}_{\underline{g}\circ\underline{e}}\cap(\underline{\partial Q}\times_{\underline{T}}\underline{D}_{z'})$. So (6.9) holds.

For (6.10), we have open inclusions $\underline{T}_{\boldsymbol{g}\circ\boldsymbol{e}} \subseteq \coprod_{z'\in \underline{i}_{\mathbf{Z}}^{-1}(z)} \underline{Q}\times_{\underline{T}} \underline{D}_{z'} \subseteq \underline{Q}\times_{\underline{T}} \underline{\partial T}$. Fix $z'\in \underline{i}_{\mathbf{Z}}^{-1}(z)$. The same argument then shows that

$$
\begin{aligned}
&\underline{t}^*_{\boldsymbol{g}\circ\boldsymbol{e}}(\eta)\circ I_{\underline{t}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)\circ I_{\underline{v}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{i}_{\mathbf{Z}}}(\mathcal{F}_Z)^{-1}\circ \underline{v}^*_{\boldsymbol{g}\circ\boldsymbol{e}}(\nu_{\mathbf{Z}})|_{\underline{T}_{\underline{g}\circ\underline{e}}\cap(\underline{Q}\times_{\underline{T}}\underline{D}_{z'})}\\
&=\underline{t}^*_{\boldsymbol{g}\circ\boldsymbol{e}}(\eta)\circ I_{\underline{t}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{g}\circ\underline{e}}(\mathcal{F}_Z)\circ I_{\underline{v}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{i}_{\mathbf{Z}}}(\mathcal{F}_Z)^{-1}\circ \underline{v}^*_{\boldsymbol{g}\circ\boldsymbol{e}}(\underline{i}_{\mathbf{Z}}|^*_{\underline{D}_{z'}}(t^2_{z'})\circ\epsilon^{-1}_{z'})|_{\cdots}\\
&=\underline{t}^*_{\boldsymbol{g}\circ\boldsymbol{e}}(\eta\circ(\underline{g}\circ\underline{e})^*(t^2_{z'}))\circ I_{\underline{t}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{g}\circ\underline{e}}(\underline{t}^*_{z'}(\mathcal{F}_{[0,\infty)}))\circ\\
&\qquad I_{\underline{v}_{\boldsymbol{g}\circ\boldsymbol{e}},\underline{i}_{\mathbf{Z}}}(\underline{t}^*_{z'}(\mathcal{F}_{[0,\infty)}))^{-1}\circ \underline{v}^*_{\boldsymbol{g}\circ\boldsymbol{e}}(\underline{i}_{\mathbf{Z}}|^*_{\underline{D}_{z'}}(\epsilon^{-1}_{z'})|_{\underline{T}_{\underline{g}\circ\underline{e}}\cap(\underline{Q}\times_{\underline{T}}\underline{D}_{z'})}=0,
\end{aligned}
$$

which proves the restriction of (6.10) for η to $\underline{T}_{\underline{g}\circ\underline{e}}\cap(\underline{Q}\times_{\underline{T}}\underline{D}_{z'})$. Hence (6.10) holds, and $\eta:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ is a 2-morphism in $\mathbf{dSpa^c}$.

Thus (6.86) is 2-commutative in $\mathbf{dSpa^c}$. We prove it is 2-Cartesian. Suppose $\hat{\mathbf{Q}}$ is a d-space with corners, and $\hat{\boldsymbol{e}}:\hat{\mathbf{Q}}\to\mathbf{R}$, $\hat{\boldsymbol{f}}:\hat{\mathbf{Q}}\to\mathbf{S}$ are 1-morphisms and $\hat\eta:\boldsymbol{g}\circ\hat{\boldsymbol{e}}\Rightarrow\boldsymbol{h}\circ\hat{\boldsymbol{f}}$ a 2-morphism in $\mathbf{dSpa^c}$. Regarding $\hat{\boldsymbol{e}},\hat{\boldsymbol{f}},\hat\eta$ as 1- and 2-morphisms in $\mathbf{dSpa}$, as $\boldsymbol{Q},\tilde{\boldsymbol{e}},\tilde{\boldsymbol{f}},\eta$ are the explicit fibre product $\boldsymbol{R}\times_{\boldsymbol{g}|_{\boldsymbol{R}},\boldsymbol{T},\boldsymbol{h}|_{\boldsymbol{S}}}\boldsymbol{S}$ from Definition 2.36, the proof of Theorem 2.36 constructs a 1-morphism $\hat{\boldsymbol{b}}:\hat{\boldsymbol{Q}}\to\boldsymbol{Q}$ in $\mathbf{dSpa}$ such that $\hat{\boldsymbol{e}}=\tilde{\boldsymbol{e}}\circ\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{f}}=\tilde{\boldsymbol{f}}\circ\hat{\boldsymbol{b}}$. This equality of 1-morphisms is a special feature of the construction of §2.5; the universal property of fibre products would only give 2-morphisms $\tilde{\boldsymbol{e}}\circ\hat{\boldsymbol{b}}\Rightarrow\hat{\boldsymbol{e}}$ and $\tilde{\boldsymbol{f}}\circ\hat{\boldsymbol{b}}\Rightarrow\hat{\boldsymbol{f}}$.

Define a morphism $\omega:\underline{\hat b}^*(\mathcal{F}_Q)\to\mathcal{E}_{\hat Q}$ to be the composition of morphisms

$$
\underline{\hat b}^*(\mathcal{F}_Q)\xrightarrow{\underline{\hat b}^*((\tilde e^2\oplus\tilde f^2)^{-1})}\underline{\hat b}^*\circ\underline{\tilde e}^*(\mathcal{F}_X)\oplus\underline{\hat b}^*\circ\underline{\tilde f}^*(\mathcal{F}_Y)\xrightarrow{\frac12\underline{\hat b}^*(-\tau\oplus\upsilon)}\underline{\hat b}^*(\mathcal{E}_Q)\xrightarrow{\hat b''}\mathcal{E}_{\hat Q}\qquad(6.101)
$$

in $\mathrm{qcoh}(\underline{\hat Q})$, where τ,υ are as in Definition 6.40, and $\tilde e^2\oplus\tilde f^2:\underline{\tilde e}^*(\mathcal{F}_X)\oplus\underline{\tilde f}^*(\mathcal{F}_Y)\to\mathcal{F}_Q$ is an isomorphism as in (2.64), and so has an inverse $(\tilde e^2\oplus\tilde f^2)^{-1}$. By Proposition 2.17 there is a unique morphism $\boldsymbol{b}:\hat{\boldsymbol{Q}}\to\boldsymbol{Q}$ in $\mathbf{dSpa}$ such that $\omega:\hat{\boldsymbol{b}}\Rightarrow\boldsymbol{b}$ is a 2-morphism. The 2-morphism $\tau*\omega:\hat{\boldsymbol{e}}=\tilde{\boldsymbol{e}}\circ\hat{\boldsymbol{b}}\Rightarrow\boldsymbol{e}\circ\boldsymbol{b}$ satisfies

$$
\begin{aligned}
\tau*\omega&=\bigl[\hat b''\circ\underline{\hat b}^*(\tau)+\omega\circ\bigl(\underline{\hat b}^*(\tilde e^2)+\underline{\hat b}^*(\phi_Q)\circ\underline{\hat b}^*(\tau)\bigr)\bigr]\circ I_{\underline{\hat b},\underline{e}}(\mathcal{F}_X)\\
&=\hat b''\circ\underline{\hat b}^*\bigl[\tau+\tfrac12(-\tau\oplus\upsilon)\circ(\tilde e^2\oplus\tilde f^2)^{-1}\circ\bigl(\tilde e^2+\phi_Q\circ\tau\bigr)\bigr]\circ I_{\underline{\hat b},\underline{e}}(\mathcal{F}_X)\\
&=\hat b''\circ\underline{\hat b}^*\bigl[\tau-\tfrac12\tau+\tfrac12\begin{pmatrix}-\tau & \upsilon\end{pmatrix}\begin{pmatrix}\tilde e^2 & \tilde f^2\end{pmatrix}^{-1}\begin{pmatrix}\tilde e^2 & \tilde f^2\end{pmatrix}\begin{pmatrix}-\underline{\tilde e}^*(g^2)\circ I_{\underline{\tilde e},\underline{g}}(\mathcal{F}_Z)\\ \underline{\tilde f}^*(h^2)\circ I_{\underline{\tilde f},\underline{h}}(\mathcal{F}_Z)\end{pmatrix}\\
&\qquad\circ\bigl(\textstyle\bigoplus_{z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{g}\circ\underline{\tilde e})^*(t^2_{z'})\bigr)\circ\rho\circ\bigl(\bigoplus_{x'\in\underline{i}_{\mathbf{X}}^{-1}(x)\setminus I}\underline{\tilde e}^*(\beta_{x'})\bigr)\bigr]\circ I_{\underline{\hat b},\underline{e}}(\mathcal{F}_X)\\
&=\tfrac12\hat b''\circ\underline{\hat b}^*\bigl[\tau-\tilde\eta\circ\bigl(\textstyle\bigoplus_{z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{g}\circ\underline{\tilde e})^*(t^2_{z'})\bigr)\\
&\qquad\circ\begin{pmatrix}\rho & \sigma\end{pmatrix}\begin{pmatrix}(\bigoplus_{x'\in\underline{i}_{\mathbf{X}}^{-1}(x)\setminus I}\underline{\tilde e}^*(\beta_{x'}))\circ\underline{\tilde e}^*(g^2)\circ I_{\underline{\tilde e},\underline{g}}(\mathcal{F}_Z)\circ(\bigoplus_{z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{g}\circ\underline{\tilde e})^*(t^2_{z'}))\\(\bigoplus_{y'\in\underline{i}_{\mathbf{Y}}^{-1}(y)\setminus J}\underline{\tilde f}^*(\gamma_{y'}))\circ\underline{\tilde f}^*(h^2)\circ I_{\underline{\tilde f},\underline{h}}(\mathcal{F}_Z)\circ(\bigoplus_{z'\in\underline{i}_{\mathbf{Z}}^{-1}(z)}(\underline{g}\circ\underline{\tilde e})^*(t^2_{z'}))\end{pmatrix}\\
&\qquad\circ\rho\circ\bigl(\textstyle\bigoplus_{x'\in\underline{i}_{\mathbf{X}}^{-1}(x)\setminus I}\underline{\tilde e}^*(\beta_{x'})\bigr)\bigr]\circ I_{\underline{\hat b},\underline{e}}(\mathcal{F}_X)\\
&=\tfrac12\hat b''\circ\underline{\hat b}^*\bigl[\tau-\tau\bigr]\circ I_{\underline{\hat b},\underline{e}}(\mathcal{F}_X)=0.
\end{aligned}\qquad(6.102)
$$

Here in the first step of (6.102) we use (2.27), in the second (6.101), in the third we switch to matrix notation, substitute in (6.91) for the last τ and use

$$\phi_Q \circ \tilde{\eta} = \begin{pmatrix} \tilde{e}^2 & \tilde{f}^2 \end{pmatrix} \begin{pmatrix} -\underline{\tilde{e}}^*(g^2) \circ I_{\underline{\tilde{e}},\underline{g}}(\mathcal{F}_Z) \\ \underline{\tilde{f}}^*(h^2) \circ I_{\underline{\tilde{f}},\underline{h}}(\mathcal{F}_Z) \end{pmatrix}$$

which follows from expressions (2.61) and (2.66) for $\tilde{\eta}$ and ϕ_Q, and in the fourth we cancel inverse matrices, substitute (6.91) for $(-\tau\ \upsilon)$, and cancel two signs. By (6.90), the column matrix on the sixth line of (6.102) is Φ, so the sixth line is $(\rho \oplus \sigma) \circ \Phi = \mathrm{id}$ by definition of ρ, σ. Cancelling the sixth line, the surrounding terms on the fifth and seventh lines are τ by (6.91), giving the fifth step.

Equation (6.102) implies that $\hat{\boldsymbol{e}} = \boldsymbol{e} \circ \boldsymbol{b}$. Similarly $\hat{\boldsymbol{f}} = \boldsymbol{f} \circ \boldsymbol{b}$. We will show that $\boldsymbol{b} : \hat{\mathbf{Q}} \to \mathbf{Q}$ is a 1-morphism in $\mathbf{dSpa^c}$. Suppose $\hat{q} \in \hat{\boldsymbol{Q}}$ with $\boldsymbol{b}(\hat{q}) = (\tilde{x}, \tilde{y}) \in \boldsymbol{Q}$, and $\tilde{q}' \in \boldsymbol{\partial Q}$ with $\boldsymbol{i}_{\mathbf{Q}}(\tilde{q}') = (\tilde{x}, \tilde{y})$. By (6.89), either

(a) $\tilde{q}' = (\tilde{x}', \tilde{y}) \in \boldsymbol{B}_{x'} \times_{\boldsymbol{T}} \boldsymbol{S}$ for $x' \in I$ and $\tilde{x}' \in \boldsymbol{B}_{x'} \subseteq \boldsymbol{\partial R}$ with $\boldsymbol{i}_{\mathbf{X}}(\tilde{x}') = \tilde{x}$; or

(b) $\tilde{q}' = (\tilde{x}, \tilde{y}') \in \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{C}_{y'}$ for $y' \in J$ and $\tilde{y}' \in \boldsymbol{C}_{y'} \subseteq \boldsymbol{\partial S}$ with $\boldsymbol{i}_{\mathbf{Y}}(\tilde{y}') = \tilde{y}$.

In case (a), from the first part of the proof $(\boldsymbol{Q}, \boldsymbol{r}_{x'} \circ \boldsymbol{e})$ is a boundary defining function for $\mathbf{Q}$ at $\tilde{q}'$, where $(\boldsymbol{R}, \boldsymbol{r}_{x'})$ is a boundary defining function for $\mathbf{R}$ at $\tilde{x}'$. Since $\hat{\boldsymbol{e}} : \hat{\mathbf{Q}} \to \mathbf{R}$ is a 1-morphism in $\mathbf{dSpa^c}$, by Definition 6.2 either

(i) there exists open $\hat{q} \in \boldsymbol{V} \subseteq \hat{\boldsymbol{Q}}$ such that $\big(\boldsymbol{V}, \boldsymbol{r}_{x'} \circ \hat{\boldsymbol{e}}|_{\boldsymbol{V}}\big)$ is a boundary defining function for $\hat{\mathbf{Q}}$ at some $\hat{q}' \in \boldsymbol{i}_{\hat{\mathbf{Q}}}^{-1}(\hat{q})$; or

(ii) there exists open $\hat{q} \in \boldsymbol{W} \subseteq \hat{\boldsymbol{Q}}\ \boldsymbol{X}$ with $\boldsymbol{r}_{x'} \circ \hat{\boldsymbol{e}}|_{\boldsymbol{W}} = \boldsymbol{0} \circ \boldsymbol{\pi} : \boldsymbol{W} \to \boldsymbol{[0,\infty)}$.

Substituting $\hat{\boldsymbol{e}} = \boldsymbol{e} \circ \boldsymbol{b}$, these (i),(ii) imply that Definition 6.2(i),(ii) hold for $\boldsymbol{b}$ at $\hat{q}, \tilde{q}'$, for the particular choice $(\boldsymbol{Q}, \boldsymbol{r}_{x'} \circ \boldsymbol{e})$ of boundary defining function for $\mathbf{Q}$ at $\tilde{q}'$. But as in Definition 6.2, if (i),(ii) hold for one choice they hold for any choice. The proof in case (b) is similar, using $(\boldsymbol{Q}, \boldsymbol{s}_{y'} \circ \boldsymbol{f})$ and $\hat{\boldsymbol{f}} = \boldsymbol{f} \circ \boldsymbol{b}$.

Hence $\boldsymbol{b} : \hat{\mathbf{Q}} \to \mathbf{Q}$ is a 1-morphism in $\mathbf{dSpa^c}$. Also $\zeta = 0 = \mathrm{id}_{\hat{\boldsymbol{e}}} : \boldsymbol{e} \circ \boldsymbol{b} \Rightarrow \hat{\boldsymbol{e}}$ and $\theta = 0 = \mathrm{id}_{\hat{\boldsymbol{f}}} : \boldsymbol{f} \circ \boldsymbol{b} \Rightarrow \hat{\boldsymbol{f}}$ are 2-morphisms in $\mathbf{dSpa^c}$. Consider the diagram of 2-morphisms in $\mathbf{dSpa}$:

$$\begin{tikzcd} \boldsymbol{g} \circ \boldsymbol{e} \circ \boldsymbol{b} \arrow[r, Rightarrow, "\eta * \mathrm{id}_{\boldsymbol{b}}"'] \arrow[d, Rightarrow, "-\mathrm{id}_{\boldsymbol{g}} * \tau * \omega = 0"'] & \boldsymbol{h} \circ \boldsymbol{f} \circ \boldsymbol{b} \arrow[d, Rightarrow, "-\mathrm{id}_{\boldsymbol{h}} * (-\upsilon) * \omega = 0"] \\ \boldsymbol{g} \circ \tilde{\boldsymbol{e}} \circ \hat{\boldsymbol{b}} \arrow[r, Rightarrow, "\tilde{\eta} * \mathrm{id}_{\hat{\boldsymbol{b}}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}} * \tilde{\zeta} = 0"'] & \boldsymbol{h} \circ \tilde{\boldsymbol{f}} \circ \hat{\boldsymbol{b}} \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{h}} * \tilde{\theta} = 0"] \\ \boldsymbol{g} \circ \hat{\boldsymbol{e}} \arrow[r, Rightarrow, "\hat{\eta}"] & \boldsymbol{h} \circ \hat{\boldsymbol{f}}. \end{tikzcd} \tag{6.103}$$

The top square commutes by (6.94). The bottom square commutes by equation (2.69) in the proof that $\boldsymbol{Q} = \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{S}$. The columns are zero as $\tau * \omega = 0$ by (6.102), and $(-\upsilon) * \omega = 0$, and $\tilde{\zeta} = \tilde{\theta} = 0$ as $\zeta = \theta = 0$ in the proof of Theorem 2.36 for $\boldsymbol{Q} = \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{S}$. Thus the outer rectangle of (6.103) commutes. This proves (A.4) commutes for $\zeta = \theta = 0$, as for (2.69) in §2.5, and proves the first universal property for (6.86) to be 2-Cartesian in Definition A.7.

For the second universal property, suppose that $\dot{\boldsymbol{b}} : \hat{\mathbf{Q}} \to \mathbf{Q}$, $\dot{\zeta} : \boldsymbol{e} \circ \dot{\boldsymbol{b}} \Rightarrow \hat{\boldsymbol{e}}$ and $\dot{\theta} : \boldsymbol{f} \circ \dot{\boldsymbol{b}} \Rightarrow \hat{\boldsymbol{f}}$ are alternate choices for $\boldsymbol{b}, \zeta, \theta$. As by construction $\boldsymbol{Q} = \boldsymbol{R} \times_{\boldsymbol{T}} \boldsymbol{S}$

in **dSpa**, equation (6.86) is 2-Cartesian in **dSpa**. Hence there exists a unique 2-morphism $\epsilon : \tilde{\boldsymbol{b}} \Rightarrow \boldsymbol{b}$ in **dSpa** with $\tilde\zeta = \zeta \odot (\mathrm{id}_{\boldsymbol e} * \epsilon)$ and $\tilde\theta = \theta \odot (\mathrm{id}_{\boldsymbol f} * \epsilon)$, as in (A.5). We must prove ϵ is a 2-morphism in $\mathbf{dSpa^c}$. Since $\zeta, \theta, \tilde\zeta, \tilde\theta$ are 2-morphisms in $\mathbf{dSpa^c}$, $\mathrm{id}_{\boldsymbol e} * \epsilon$ and $\mathrm{id}_{\boldsymbol f} * \epsilon$ are too, so (6.9) holds for $\mathrm{id}_{\boldsymbol e} * \epsilon, \mathrm{id}_{\boldsymbol f} * \epsilon$.

One can show that the restriction of (6.9) for ϵ to $\underline{S}_{\boldsymbol b} \cap \big(\partial\underline{\hat Q} \times_{\underline Q} (\underline{B}_{x'} \times_{\underline T} \underline{S})\big)$ for $x' \in I$ is equivalent to the restriction of (6.9) for $\mathrm{id}_{\boldsymbol e} * \epsilon$ to $\underline{S}_{\boldsymbol{e}\circ\boldsymbol{b}} \cap (\partial\underline{\hat Q} \times_{\underline R} \underline{B}_{x'})$, and the restriction of (6.9) for ϵ to $\underline{S}_{\boldsymbol b} \cap \big(\partial\underline{\hat Q} \times_{\underline Q} (\underline{R} \times_{\underline T} \underline{C}_{y'})\big)$ for $y' \in J$ is equivalent to the restriction of (6.9) for $\mathrm{id}_{\boldsymbol f} * \epsilon$ to $\underline{S}_{\boldsymbol{f}\circ\boldsymbol{b}} \cap (\partial\underline{\hat Q} \times_{\underline S} \underline{C}_{y'})$. So (6.9) holds for ϵ. Similarly, the restriction of (6.10) for ϵ to $\underline{T}_{\boldsymbol b} \cap \big(\underline{\hat Q} \times_{\underline Q} (\underline{B}_{x'} \times_{\underline T} \underline{S})\big)$ for $x' \in I$ is equivalent to the restriction of (6.10) for $\mathrm{id}_{\boldsymbol e} * \epsilon$ to $\underline{T}_{\boldsymbol{e}\circ\boldsymbol{b}} \cap (\underline{\hat Q} \times_{\underline R} \underline{B}_{x'})$, and the restriction of (6.10) for ϵ to $\underline{T}_{\boldsymbol b} \cap \big(\underline{\hat Q} \times_{\underline Q} (\underline{R} \times_{\underline T} \underline{C}_{y'})\big)$ for $y' \in J$ is equivalent to the restriction of (6.10) for $\mathrm{id}_{\boldsymbol f} * \epsilon$ to $\underline{T}_{\boldsymbol{f}\circ\boldsymbol{b}} \cap (\underline{\hat Q} \times_{\underline S} \underline{C}_{y'})$. So (6.10) holds for ϵ. Therefore ϵ is a 2-morphism in $\mathbf{dSpa^c}$, proving the second universal property. This completes the proof of Theorem 6.41. □

6.8.4 Global existence of b-transverse fibre products in $\mathbf{dSpa^c}$

We can now prove one of the main results of this chapter, that if $\boldsymbol g : \mathbf X \to \mathbf Z$ and $\boldsymbol h : \mathbf Y \to \mathbf Z$ are b-transverse then a fibre product $\mathbf X \times_{\boldsymbol g, \mathbf Z, \boldsymbol h} \mathbf Y$ exists in $\mathbf{dSpa^c}$.

Theorem 6.42. *All b-transverse fibre products exist in* $\mathbf{dSpa^c}$*.*

Proof. Let $\boldsymbol g : \mathbf X \to \mathbf Z$ and $\boldsymbol h : \mathbf Y \to \mathbf Z$ be b-transverse 1-morphisms in $\mathbf{dSpa^c}$. We will construct a d-space with corners $\mathbf W$, 1-morphisms $\boldsymbol e : \mathbf W \to \mathbf X$ and $\boldsymbol f : \mathbf W \to \mathbf Y$ and a 2-morphism $\eta : \boldsymbol g \circ \boldsymbol e \Rightarrow \boldsymbol h \circ \boldsymbol f$ in $\mathbf{dSpa^c}$ such that

$$\begin{tikzcd} \mathbf W \arrow[r, "\boldsymbol f"] \arrow[d, "\boldsymbol e"'] \arrow[dr, phantom, "\Uparrow\eta"] & \mathbf Y \arrow[d, "\boldsymbol h"] \\ \mathbf X \arrow[r, "\boldsymbol g"] & \mathbf Z \end{tikzcd} \tag{6.104}$$

is 2-Cartesian. Let W be the fibre product of topological spaces

$$W = X \times_{g,Z,h} Y = \big\{(x,y) \in X \times Y : g(x) = h(y)\big\}.$$

Then W is Hausdorff and second countable, as X, Y are.

For each $(x,y) \in W$ with $g(x) = h(y) = z \in Z$, Definition 6.40 constructs open $x \in \mathbf R_x^y \subseteq \mathbf X$, $y \in \mathbf S_x^y \subseteq \mathbf Y$, a d-space with corners $\mathbf Q_x^y$, 1-morphisms $\boldsymbol e_x^y : \mathbf Q_x^y \to \mathbf R_x^y$, $\boldsymbol f_x^y : \mathbf Q_x^y \to \mathbf S_x^y$ and a 2-morphism $\eta_x^y : \boldsymbol g \circ \boldsymbol e_x^y \Rightarrow \boldsymbol h \circ \boldsymbol f_x^y$ in $\mathbf{dSpa^c}$. Theorem 6.41 shows $\mathbf Q_x^y$ is a fibre product $\mathbf R_x^y \times_{\mathbf Z} \mathbf S_x^y$ in $\mathbf{dSpa^c}$. The underlying topological space Q_x^y is the open set $\big\{(x,y) \in W : x \in R_x^y,\ y \in S_x^y\big\}$ in W.

Suppose $(x_1,y_1),(x_2,y_2) \in W$. Then $\mathbf R_{x_1}^{y_1} \cap \mathbf R_{x_2}^{y_2} \subseteq \mathbf X$ and $\boldsymbol S_{x_1}^{y_1} \cap \boldsymbol S_{x_2}^{y_2} \subseteq \mathbf Y$ are open d-subspaces. Define open $\mathbf Q_{x_1x_2}^{y_1y_2} \subseteq \mathbf Q_{x_1}^{y_1}$ and $\mathbf Q_{x_2x_1}^{y_2y_1} \subseteq \mathbf Q_{x_2}^{y_2}$ by

$$\begin{aligned} \mathbf Q_{x_1x_2}^{y_1y_2} &= (\boldsymbol e_{x_1}^{y_1})^{-1}(\mathbf R_{x_1}^{y_1} \cap \mathbf R_{x_2}^{y_2}) \cap (\boldsymbol f_{x_1}^{y_1})^{-1}(\mathbf S_{x_1}^{y_1} \cap \mathbf S_{x_2}^{y_2}), \\ \mathbf Q_{x_2x_1}^{y_2y_1} &= (\boldsymbol e_{x_2}^{y_2})^{-1}(\mathbf R_{x_1}^{y_1} \cap \mathbf R_{x_2}^{y_2}) \cap (\boldsymbol f_{x_2}^{y_2})^{-1}(\mathbf S_{x_1}^{y_1} \cap \mathbf S_{x_2}^{y_2}). \end{aligned}$$

Then $\mathbf{Q}^{y_1y_2}_{x_1x_2},\mathbf{Q}^{y_2y_1}_{x_2x_1}$ are fibre products $(\mathbf{R}^{y_1}_{x_1}\cap\mathbf{R}^{y_2}_{x_2})\times_{\mathbf{Z}}(\mathbf{S}^{y_1}_{x_1}\cap\mathbf{S}^{y_2}_{x_2})$ in $\mathbf{dSpa^c}$, so there is an equivalence $\boldsymbol{i}^{y_1y_2}_{x_1x_2}:\mathbf{Q}^{y_1y_2}_{x_1x_2}\to\mathbf{Q}^{y_2y_1}_{x_2x_1}$, which is natural up to 2-isomorphism, and 2-morphisms $\zeta^{y_1y_2}_{x_1x_2}:\boldsymbol{e}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2}\Rightarrow\boldsymbol{e}^{y_1}_{x_1}|_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}$ and $\theta^{y_1y_2}_{x_1x_2}:\boldsymbol{f}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2}\Rightarrow\boldsymbol{f}^{y_1}_{x_1}|_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}$, such that the following commutes:

$$
\begin{tikzcd}
\boldsymbol{g}\circ\boldsymbol{e}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2} \arrow[r, Rightarrow, "\eta^{y_2}_{x_2}*\mathrm{id}_{\boldsymbol{i}^{y_1y_2}_{x_1x_2}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}}*\zeta^{y_1y_2}_{x_1x_2}"] & \boldsymbol{h}\circ\boldsymbol{f}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2} \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{h}}*\theta^{y_1y_2}_{x_1x_2}"'] \\
\boldsymbol{g}\circ\boldsymbol{e}^{y_1}_{x_1}|_{\mathbf{Q}^{y_1y_2}_{x_1x_2}} \arrow[r, Rightarrow, "\eta^{y_1}_{x_1}|_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}"] & \boldsymbol{h}\circ\boldsymbol{f}^{y_1}_{x_1}|_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}.
\end{tikzcd}
\tag{6.105}
$$

Suppose $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in W$. Then $\mathbf{Q}^{y_1y_2}_{x_1x_2}\cap\mathbf{Q}^{y_1y_3}_{x_1x_3}$, $\mathbf{Q}^{y_2y_1}_{x_2x_1}\cap\mathbf{Q}^{y_2y_3}_{x_2x_3}$, $\mathbf{Q}^{y_3y_1}_{x_3x_1}\cap\mathbf{Q}^{y_3y_2}_{x_3x_2}$ are all fibre products $(\mathbf{R}^{y_1}_{x_1}\cap\mathbf{R}^{y_2}_{x_2}\cap\mathbf{R}^{y_3}_{x_3})\times_{\mathbf{Z}}(\mathbf{S}^{y_1}_{x_1}\cap\mathbf{S}^{y_2}_{x_2}\cap\mathbf{S}^{y_3}_{x_3})$, and we have a triangle of equivalences

$$
\begin{tikzcd}
& \mathbf{Q}^{y_2y_1}_{x_2x_1}\cap\mathbf{Q}^{y_2y_3}_{x_2x_3} \arrow[dr, "\boldsymbol{i}^{y_2y_3}_{x_2x_3}|_{\mathbf{Q}^{y_2y_1}_{x_2x_1}\cap\mathbf{Q}^{y_2y_3}_{x_2x_3}}"] \arrow[d, phantom, "\Downarrow\omega^{y_1y_2y_3}_{x_1x_2x_3}"] & \\
\mathbf{Q}^{y_1y_2}_{x_1x_2}\cap\mathbf{Q}^{y_1y_3}_{x_1x_3} \arrow[ur, "\boldsymbol{i}^{y_1y_2}_{x_1x_2}|_{\mathbf{Q}^{y_1y_2}_{x_1x_2}\cap\mathbf{Q}^{y_1y_3}_{x_1x_3}}"] \arrow[rr, "\boldsymbol{i}^{y_1y_3}_{x_1x_3}|_{\mathbf{Q}^{y_1y_2}_{x_1x_2}\cap\mathbf{Q}^{y_1y_3}_{x_1x_3}}"] & & \mathbf{Q}^{y_3y_1}_{x_3x_1}\cap\mathbf{Q}^{y_3y_2}_{x_3x_2}.
\end{tikzcd}
$$

Since equivalences between different possible fibre products are natural up to 2-isomorphism, there exists a 2-morphism $\omega^{y_1y_2y_3}_{x_1x_2x_3}$ as shown.

As in §6.6, analogues of Theorems 2.28–2.33 hold in $\mathbf{dSpa^c}$, though we only gave analogues of Theorems 2.28 and 2.33. Apply the analogue of Theorem 2.32 with $W,(x,y),\mathbf{Q}^y_x,\mathbf{Q}^{y_1y_2}_{x_1x_2},\boldsymbol{i}^{y_1y_2}_{x_1x_2},\omega^{y_1y_2y_3}_{x_1x_2x_3}$ in place of $I,i,\mathbf{X}_i,\mathbf{U}_{ij},\boldsymbol{e}_{ij},\eta_{ijk}$, respectively. The quotient topological space $Y=(\coprod_{(x,y)\in W}Q^y_x)/\sim$ in Theorem 2.32 is homeomorphic to W, which is Hausdorff. Hence the first part of Theorem 2.32 gives a d-space with corners $\mathbf{W}$, open d-subspaces $\hat{\mathbf{Q}}^y_x\subseteq\mathbf{W}$, equivalences $\boldsymbol{k}^y_x:\mathbf{Q}^y_x\to\hat{\mathbf{Q}}^y_x$ for $(x,y)\in W$ with $\mathbf{W}=\bigcup_{(x,y)\in W}\hat{\mathbf{Q}}^y_x$, and 2-morphisms $\kappa^{y_1y_2}_{x_1x_2}:\boldsymbol{k}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2}\Rightarrow\boldsymbol{k}^{y_1}_{x_1}|_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}$. Then we apply the second part of Theorem 2.32 twice, firstly with $\mathbf{X},\boldsymbol{e}^y_x,\zeta^{y_1y_2}_{x_1x_2}$ and secondly with $\mathbf{Y},\boldsymbol{f}^y_x,\theta^{y_1y_2}_{x_1x_2}$ in place of $\mathbf{Z},\boldsymbol{g}_i,\zeta_{ij}$, respectively. This yields 1-morphisms $\boldsymbol{e}:\mathbf{W}\to\mathbf{X}$ and $\boldsymbol{f}:\mathbf{W}\to\mathbf{Y}$ with 2-morphisms $\alpha^y_x:\boldsymbol{e}\circ\boldsymbol{k}^y_x\Rightarrow\boldsymbol{e}^y_x$ and $\beta^y_x:\boldsymbol{f}\circ\boldsymbol{k}^y_x\Rightarrow\boldsymbol{f}^y_x$.

Using the fact that $\boldsymbol{k}^y_x$ is an equivalence, one can show that there is a unique 2-morphism $\hat{\eta}^y_x:\boldsymbol{g}\circ\boldsymbol{e}|_{\hat{\mathbf{Q}}^y_x}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}|_{\hat{\mathbf{Q}}^y_x}$ for each $(x,y)\in W$ such that the following diagram of 2-morphisms of 1-morphisms $\mathbf{Q}^y_x\to\mathbf{Z}$ commutes:

$$
\begin{tikzcd}
\boldsymbol{g}\circ\boldsymbol{e}\circ\boldsymbol{k}^y_x \arrow[r, Rightarrow, "\hat{\eta}^y_x*\mathrm{id}_{\boldsymbol{k}^y_x}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}}*\alpha^y_x"] & \boldsymbol{h}\circ\boldsymbol{f}\circ\boldsymbol{k}^y_x \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{h}}*\beta^y_x"'] \\
\boldsymbol{g}\circ\boldsymbol{e}^y_x \arrow[r, Rightarrow, "\eta^y_x"] & \boldsymbol{h}\circ\boldsymbol{f}^y_x,
\end{tikzcd}
$$

and then the following diagram is 2-Cartesian in $\mathbf{dSpa^c}$ for all $(x,y)\in W$:

$$
\begin{tikzcd}
\hat{\mathbf{Q}}^y_x \arrow[r, "\boldsymbol{f}|_{\hat{\mathbf{Q}}^y_x}"'] \arrow[d, "\boldsymbol{e}|_{\hat{\mathbf{Q}}^y_x}"] \arrow[dr, phantom, "\Uparrow\hat{\eta}^y_x"] & \mathbf{S}^y_x \arrow[d, "\boldsymbol{h}|_{\mathbf{S}^y_x}"'] \\
\mathbf{R}^y_x \arrow[r, "\boldsymbol{g}|_{\mathbf{R}^y_x}"] & \mathbf{Z}.
\end{tikzcd}
\tag{6.106}
$$

Let $(x_1,y_1),(x_2,y_2)\in W$. Then combining $\kappa^{y_1y_2}_{x_1x_2}:\boldsymbol{k}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2}\Rightarrow\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}$ and (6.105) and (6.106) for $(x_1,y_1),(x_2,y_2)$ gives a commutative diagram:

$$
\begin{tikzcd}[column sep=huge]
\boldsymbol{g}\circ\boldsymbol{e}\circ\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}} \arrow[r, Rightarrow, swap, "\hat\eta^{y_2}_{x_2}*\mathrm{id}_{\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}\circ\boldsymbol{e}}*(-\kappa^{y_1y_2}_{x_1x_2})"] \arrow[dddd, Rightarrow, bend right=60, swap, "\begin{smallmatrix}\mathrm{id}_{\boldsymbol{g}}*\gamma^{y_1y_2}_{x_1x_2}*\\ \mathrm{id}_{\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}}\end{smallmatrix}"] & \boldsymbol{h}\circ\boldsymbol{f}\circ\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}} \arrow[d, Rightarrow, swap, "\mathrm{id}_{\boldsymbol{h}\circ\boldsymbol{h}}*(-\kappa^{y_1y_2}_{x_1x_2})"] \arrow[dddd, Rightarrow, bend left=60, "\begin{smallmatrix}\mathrm{id}_{\boldsymbol{h}}*\delta^{y_1y_2}_{x_1x_2}*\\ \mathrm{id}_{\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}}\end{smallmatrix}"] \\
\boldsymbol{g}\circ\boldsymbol{e}\circ\boldsymbol{k}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2} \arrow[r, Rightarrow, swap, "\hat\eta^{y_2}_{x_2}*\mathrm{id}_{\boldsymbol{k}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}}*\alpha^{y_2}_{x_2}*\mathrm{id}_{\boldsymbol{i}^{y_1y_2}_{x_1x_2}}"] & \boldsymbol{h}\circ\boldsymbol{f}\circ\boldsymbol{k}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2} \arrow[d, Rightarrow, swap, "\mathrm{id}_{\boldsymbol{h}}*\beta^{y_2}_{x_2}*\mathrm{id}_{\boldsymbol{i}^{y_1y_2}_{x_1x_2}}"] \\
\boldsymbol{g}\circ\boldsymbol{e}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2} \arrow[r, Rightarrow, swap, "\eta^{y_2}_{x_2}*\mathrm{id}_{\boldsymbol{i}^{y_1y_2}_{x_1x_2}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}}*\zeta^{y_1y_2}_{x_1x_2}"] & \boldsymbol{h}\circ\boldsymbol{f}^{y_2}_{x_2}\circ\boldsymbol{i}^{y_1y_2}_{x_1x_2} \arrow[d, Rightarrow, swap, "\mathrm{id}_{\boldsymbol{h}}*\theta^{y_1y_2}_{x_1x_2}"] \\
\boldsymbol{g}\circ\boldsymbol{e}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}} \arrow[r, Rightarrow, "\eta^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}}*(-\alpha^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}})"] & \boldsymbol{h}\circ\boldsymbol{f}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}} \arrow[d, Rightarrow, swap, "\mathrm{id}_{\boldsymbol{h}}*(-\beta^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}})"] \\
\boldsymbol{g}\circ\boldsymbol{e}\circ\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}} \arrow[r, Rightarrow, "\hat\eta^{y_1}_{x_1}*\mathrm{id}_{\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}}"] & \boldsymbol{h}\circ\boldsymbol{f}\circ\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}.
\end{tikzcd}
$$

By omitting the '$\boldsymbol{g}\circ$', '$\mathrm{id}_{\boldsymbol{g}}*$' from the left column and using $\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}:\mathbf{Q}^{y_1y_2}_{x_1x_2}\to\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}$ an equivalence, we see that there is a unique 2-morphism $\gamma^{y_1y_2}_{x_1x_2}:\boldsymbol{e}\vert_{\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}}\Rightarrow\boldsymbol{e}\vert_{\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}}$ making the left hand semicircle commute. Similarly there is a unique $\delta^{y_1y_2}_{x_1x_2}:\boldsymbol{f}\vert_{\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}}\Rightarrow\boldsymbol{f}\vert_{\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}}$ making the right semicircle commute. Then considering the outer quadrilateral and using $\boldsymbol{k}^{y_1}_{x_1}\vert_{\mathbf{Q}^{y_1y_2}_{x_1x_2}}$ an equivalence, we see that the following commutes:

$$
\begin{tikzcd}[column sep=huge]
\boldsymbol{g}\circ\boldsymbol{e}\vert_{\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}} \arrow[r, Rightarrow, swap, "\hat\eta^{y_2}_{x_2}\vert_{\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}}*\gamma^{y_1y_2}_{x_1x_2}"] & \boldsymbol{h}\circ\boldsymbol{f}\vert_{\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}} \arrow[d, Rightarrow, swap, "\mathrm{id}_{\boldsymbol{h}}*\delta^{y_1y_2}_{x_1x_2}"] \\
\boldsymbol{g}\circ\boldsymbol{e}\vert_{\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}} \arrow[r, Rightarrow, "\hat\eta^{y_1}_{x_1}\vert_{\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}}"] & \boldsymbol{h}\circ\boldsymbol{f}\vert_{\hat{\mathbf{Q}}^{y_1}_{x_1}\cap\hat{\mathbf{Q}}^{y_2}_{x_2}}.
\end{tikzcd}
\tag{6.107}
$$

Now $\{\underline{\hat{Q}}{}^y_x:(x,y)\in W\}$ is an open cover of $\underline{W}$, which is a separated, paracompact, locally fair C^∞-scheme. Hence by Proposition B.21 there exists a partition of unity $\{\epsilon^y_x:(x,y)\in W\}$ on $\underline{W}$ subordinate to $\{\underline{\hat{Q}}{}^y_x:(x,y)\in W\}$. Define a morphism $\eta:(\underline{g}\circ\underline{e})^*(\mathcal{F}_Z)\to\mathcal{E}_W$ in $\mathrm{qcoh}(\underline{W})$ by $\eta=\sum_{(x,y)\in W}\epsilon^y_x\cdot\hat\eta^y_x$. Since each $\hat\eta^y_x$ is a local choice of 2-morphism $\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ in $\mathbf{dSpa^c}$, and the conditions on 2-morphisms are local, $\eta:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ is a 2-morphism in $\mathbf{dSpa^c}$.

Fix $(x,y)\in W$. Then we have

$$
\begin{aligned}
\eta\vert_{\underline{\hat{Q}}{}^y_x}&=\hat\eta^y_x+\textstyle\sum_{(x,y)\neq(x',y')\in W}\epsilon^{y'}_{x'}\cdot\bigl(\hat\eta^{y'}_{x'}-\hat\eta^y_x\bigr)\\
&=\hat\eta^y_x+\textstyle\sum_{(x,y)\neq(x',y')\in W}\epsilon^{y'}_{x'}\cdot\bigl(\mathrm{id}_{\boldsymbol{g}}*\gamma^{yy'}_{xx'}-\mathrm{id}_{\boldsymbol{h}}*\delta^{yy'}_{xx'}\bigr)\\
&=\hat\eta^y_x+\mathrm{id}_{\boldsymbol{g}}*\gamma^y_x-\mathrm{id}_{\boldsymbol{h}}*\delta^y_x,
\end{aligned}
\tag{6.108}
$$

using (6.107) in the second step, where

$$
\gamma^y_x=\textstyle\sum_{(x,y)\neq(x',y')\in W}\epsilon^{y'}_{x'}\cdot\gamma^{yy'}_{xx'}\quad\text{and}\quad\delta^y_x=\textstyle\sum_{(x,y)\neq(x',y')\in W}\epsilon^{y'}_{x'}\cdot\delta^{yy'}_{xx'}.
$$

As the $\gamma^{yy'}_{xx'},\delta^{yy'}_{xx'}$ are 2-morphisms in $\mathbf{dSpa^c}$ and the conditions on 2-morphisms are linear, we see that $\gamma^y_x:\boldsymbol{e}\vert_{\hat{\mathbf{Q}}^y_x}\Rightarrow\boldsymbol{e}\vert_{\hat{\mathbf{Q}}^y_x}$ and $\delta^y_x:\boldsymbol{f}\vert_{\hat{\mathbf{Q}}^y_x}\Rightarrow\boldsymbol{f}\vert_{\hat{\mathbf{Q}}^y_x}$ are 2-morphisms in $\mathbf{dSpa^c}$. Since (6.106) is 2-Cartesian, and (6.108) shows that by modifying $\boldsymbol{e}|_{\hat{\mathbf{Q}}_x^y}$ and $\boldsymbol{f}|_{\hat{\mathbf{Q}}_x^y}$ by 2-morphisms γ_x^y, δ_x^y we may replace $\hat{\eta}_x^y$ in (6.106) by $\eta|_{\hat{Q}_x^y}$, we see that the following diagram is 2-Cartesian in $\mathbf{dSpa^c}$ for all $(x, y) \in W$:

$$
\begin{tikzcd}
\hat{\mathbf{Q}}_x^y \arrow[rr, "\boldsymbol{f}|_{\hat{\mathbf{Q}}_x^y}"] \arrow[d, "\boldsymbol{e}|_{\hat{\mathbf{Q}}_x^y}"'] \arrow[drr, phantom, "\eta|_{\hat{\mathbf{Q}}_x^y}\Uparrow"] & & \mathbf{S}_x^y \arrow[d, "\boldsymbol{h}|_{\mathbf{S}_x^y}"] \\
\mathbf{R}_x^y \arrow[rr, "\boldsymbol{g}|_{\mathbf{R}_x^y}"] & & \mathbf{Z}.
\end{tikzcd}
\tag{6.109}
$$

We have now constructed $\mathbf{W}, \boldsymbol{e}, \boldsymbol{f}, \eta$ forming a 2-commutative square (6.104), and we have shown that (6.104) is locally 2-Cartesian, that is, we can cover $\mathbf{W}, \mathbf{X}, \mathbf{Y}$ with open d-subspaces $\hat{\mathbf{Q}}_x^y, \mathbf{R}_x^y, \mathbf{S}_x^y$ such that the restriction (6.109) of (6.104) is 2-Cartesian. It remains to show that (6.104) is globally 2-Cartesian. Suppose that $\tilde{\mathbf{W}}$ is a d-space with corners, $\tilde{\boldsymbol{e}} : \tilde{\mathbf{W}} \to \mathbf{X}$ and $\tilde{\boldsymbol{f}} : \tilde{\mathbf{W}} \to \mathbf{Y}$ are 1-morphisms and $\tilde{\eta} : \boldsymbol{g} \circ \tilde{\boldsymbol{e}} \Rightarrow \boldsymbol{h} \circ \tilde{\boldsymbol{f}}$ a 2-morphism in $\mathbf{dSpa^c}$.

For each $(x, y) \in W$, define an open d-subspace $\tilde{\mathbf{Q}}_x^y \subseteq \tilde{\mathbf{W}}$ by $\tilde{\mathbf{Q}}_x^y = \tilde{\boldsymbol{e}}^{-1}(\mathbf{R}_x^y) \cap \tilde{\boldsymbol{f}}^{-1}(\mathbf{S}_x^y)$. Then we have 1-morphisms $\tilde{\boldsymbol{e}}|_{\tilde{\mathbf{Q}}_x^y} : \tilde{\mathbf{Q}}_x^y \to \mathbf{R}_x^y$ and $\tilde{\boldsymbol{f}}|_{\tilde{\mathbf{Q}}_x^y} : \tilde{\mathbf{Q}}_x^y \to \mathbf{S}_x^y$ and a 2-morphism $\tilde{\eta}|_{\tilde{\mathbf{Q}}_x^y} : \boldsymbol{g} \circ \tilde{\boldsymbol{e}}|_{\tilde{\mathbf{Q}}_x^y} \Rightarrow \boldsymbol{h} \circ \tilde{\boldsymbol{f}}|_{\tilde{\mathbf{Q}}_x^y}$. As (6.109) is 2-Cartesian, there exists a 1-morphism $\tilde{\boldsymbol{b}}_x^y : \tilde{\mathbf{Q}}_x^y \to \hat{\mathbf{Q}}_x^y$ and 2-morphisms $\zeta_x^y : \boldsymbol{e} \circ \tilde{\boldsymbol{b}}_x^y \Rightarrow \tilde{\boldsymbol{e}}|_{\tilde{\mathbf{Q}}_x^y}$, $\theta_x^y : \boldsymbol{f} \circ \tilde{\boldsymbol{b}}_x^y \Rightarrow \tilde{\boldsymbol{f}}|_{\tilde{\mathbf{Q}}_x^y}$, such that the following diagram of 2-morphisms of 1-morphisms $\tilde{\mathbf{Q}}_x^y \to \mathbf{Z}$ commutes:

$$
\begin{tikzcd}
\boldsymbol{g} \circ \boldsymbol{e} \circ \tilde{\boldsymbol{b}}_x^y \arrow[rr, Rightarrow, "\eta * \mathrm{id}_{\tilde{\boldsymbol{b}}_x^y}"] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}} * \zeta_x^y"] & & \boldsymbol{h} \circ \boldsymbol{f} \circ \tilde{\boldsymbol{b}}_x^y \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{h}} * \theta_x^y"'] \\
\boldsymbol{g} \circ \tilde{\boldsymbol{e}}|_{\tilde{\mathbf{Q}}_x^y} \arrow[rr, Rightarrow, "\tilde{\eta}|_{\tilde{\mathbf{Q}}_x^y}"] & & \boldsymbol{h} \circ \tilde{\boldsymbol{f}}|_{\tilde{\mathbf{Q}}_x^y}.
\end{tikzcd}
\tag{6.110}
$$

Let $(x_1, y_1), (x_2, y_2) \in W$. Then we have a 2-Cartesian diagram in $\mathbf{dSpa^c}$:

$$
\begin{tikzcd}
\hat{\mathbf{Q}}_{x_1}^{y_1} \cap \hat{\mathbf{Q}}_{x_2}^{y_2} \arrow[rr, "\boldsymbol{f}|_{\hat{\mathbf{Q}}_{x_1}^{y_1} \cap \hat{\mathbf{Q}}_{x_2}^{y_2}}"] \arrow[d, "\boldsymbol{f}|_{\hat{\mathbf{Q}}_{x_1}^{y_1} \cap \hat{\mathbf{Q}}_{x_2}^{y_2}}"'] \arrow[drr, phantom, "\eta|_{\hat{\mathbf{Q}}_{x_1}^{y_1} \cap \hat{\mathbf{Q}}_{x_2}^{y_2}}\Uparrow"] & & \mathbf{S}_{x_1}^{y_1} \cap \mathbf{S}_{x_2}^{y_2} \arrow[d, "\boldsymbol{h}|_{\mathbf{S}_{x_1}^{y_1} \cap \mathbf{S}_{x_2}^{y_2}}"] \\
\mathbf{R}_{x_1}^{y_1} \cap \mathbf{R}_{x_2}^{y_2} \arrow[rr, "\boldsymbol{g}|_{\mathbf{R}_{x_1}^{y_1} \cap \mathbf{R}_{x_2}^{y_2}}"] & & \mathbf{Z}.
\end{tikzcd}
\tag{6.111}
$$

Now both $\tilde{\boldsymbol{b}}_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}}, \zeta_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}}, \theta_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}}$ and $\tilde{\boldsymbol{b}}_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}}, \zeta_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}}, \theta_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}}$ satisfy the first universal property of (6.111) in Definition A.7. Hence by the second universal property, there is a unique $\lambda_{x_1 x_2}^{y_1 y_2} : \tilde{\boldsymbol{b}}_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}} \Rightarrow \tilde{\boldsymbol{b}}_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}}, \zeta_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}}$ in $\mathbf{dSpa^c}$ satisfying the analogue of (A.5):

$$
\begin{aligned}
\zeta_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}} &= \zeta_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}} \odot (\mathrm{id}_{\boldsymbol{e}} * \lambda_{x_1 x_2}^{y_1 y_2}), \\
\theta_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}} &= \theta_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2}} \odot (\mathrm{id}_{\boldsymbol{f}} * \lambda_{x_1 x_2}^{y_1 y_2}).
\end{aligned}
$$

If $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in W$, the uniqueness of $\lambda_{x_i x_j}^{y_i y_j}$ implies that

$$
\lambda_{x_1 x_3}^{y_1 y_3}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2} \cap \tilde{\mathbf{Q}}_{x_3}^{y_3}} = \lambda_{x_1 x_2}^{y_1 y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2} \cap \tilde{\mathbf{Q}}_{x_3}^{y_3}} + \lambda_{x_2 x_3}^{y_2 y_3}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1} \cap \tilde{\mathbf{Q}}_{x_2}^{y_2} \cap \tilde{\mathbf{Q}}_{x_3}^{y_3}}. \tag{6.112}
$$

Using the partition of unity $\{\epsilon_x^y : (x,y) \in W\}$ above, for each $(x,y) \in W$, since the $\lambda_{xx'}^{yy'}$ are 2-morphisms in $\mathbf{dSpa^c}$ and so satisfy (6.9)–(6.10), Proposition 6.9 shows that there is a unique 1-morphism $\boldsymbol{b}_x^y : \tilde{\mathbf{Q}}_x^y \to \hat{\mathbf{Q}}_x^y \subseteq \mathbf{W}$ in $\mathbf{dSpa^c}$ with a 2-morphism $\mu_x^y = \sum_{(x,y)\neq(x',y')\in W} \epsilon_{x'}^{y'} \cdot \lambda_{xx'}^{yy'} : \tilde{\boldsymbol{b}}_x^y \Rightarrow \boldsymbol{b}_x^y$. If $(x_1,y_1),(x_2,y_2) \in W$ we have 2-morphisms

$$\boldsymbol{b}_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}} \xRightarrow{-\mu_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}}} \tilde{\boldsymbol{b}}_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}} \xRightarrow{\lambda_{x_1x_2}^{y_1y_2}} \tilde{\boldsymbol{b}}_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}} \xRightarrow{\mu_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}}} \boldsymbol{b}_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}}.$$

From (6.112) we see that $\mu_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}} + \lambda_{x_1x_2}^{y_1y_2} - \mu_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}} = 0$, so that $\boldsymbol{b}_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}} = \boldsymbol{b}_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}}$. Therefore there is a unique 1-morphism $\boldsymbol{b} : \tilde{\mathbf{W}} \to \mathbf{W}$ in $\mathbf{dSpa^c}$ with $\boldsymbol{b}|_{\tilde{\mathbf{Q}}_x^y} = \boldsymbol{b}_x^y$ for all $(x,y) \in W$.

By a similar argument using (6.112), there exist unique 2-morphisms $\zeta : \boldsymbol{e} \circ \boldsymbol{b} \Rightarrow \tilde{\boldsymbol{e}}$ and $\theta : \boldsymbol{f} \circ \boldsymbol{b} \Rightarrow \tilde{\boldsymbol{f}}$ such that for all $(x,y) \in W$ we have

$$\zeta|_{\tilde{\mathbf{Q}}_x^y} = \zeta_x^y \odot \bigl(\mathrm{id}_{\boldsymbol{e}} * (-\mu_x^y)\bigr), \quad \theta|_{\tilde{\mathbf{Q}}_x^y} = \theta_x^y \odot \bigl(\mathrm{id}_{\boldsymbol{f}} * (-\mu_x^y)\bigr). \tag{6.113}$$

Now consider the diagram

$$\begin{tikzcd}
\boldsymbol{g}\circ\boldsymbol{e}\circ\boldsymbol{b}|_{\tilde{\mathbf{Q}}_x^y} \arrow[rr, Rightarrow, "\eta*\mathrm{id}_{\boldsymbol{b}|_{\tilde{\mathbf{Q}}_x^y}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}\circ\boldsymbol{e}}*(-\mu_x^y)"] \arrow[dd, Rightarrow, bend right=60, "\mathrm{id}_{\boldsymbol{g}}*\zeta|_{\tilde{\mathbf{Q}}_x^y}"'] && \boldsymbol{h}\circ\boldsymbol{f}\circ\boldsymbol{b}|_{\tilde{\mathbf{Q}}_x^y} \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{h}\circ\boldsymbol{f}}*(-\mu_x^y)"'] \arrow[dd, Rightarrow, bend left=60, "\mathrm{id}_{\boldsymbol{h}}*\theta|_{\tilde{\mathbf{Q}}_x^y}"] \\
\boldsymbol{g}\circ\boldsymbol{e}\circ\tilde{\boldsymbol{b}}_x^y \arrow[rr, Rightarrow, "\eta*\mathrm{id}_{\tilde{\boldsymbol{b}}_x^y}"] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}}*\zeta_x^y"] && \boldsymbol{h}\circ\boldsymbol{f}\circ\tilde{\boldsymbol{b}}_x^y \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{h}}*\theta_x^y"'] \\
\boldsymbol{g}\circ\tilde{\boldsymbol{e}}|_{\tilde{\mathbf{Q}}_x^y} \arrow[rr, Rightarrow, "\tilde{\eta}|_{\tilde{\mathbf{Q}}_x^y}"] && \boldsymbol{h}\circ\tilde{\boldsymbol{f}}|_{\tilde{\mathbf{Q}}_x^y}.
\end{tikzcd}$$

The top square commutes trivially, the bottom square commutes by (6.110), and the left and right semicircles commute by (6.113). Hence the outer quadrilateral commutes. But this is the restriction to $\tilde{\mathbf{Q}}_x^y$ of

$$\begin{tikzcd}
\boldsymbol{g}\circ\boldsymbol{e}\circ\boldsymbol{b} \arrow[r, Rightarrow, "\eta*\mathrm{id}_{\boldsymbol{b}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}}*\zeta"] & \boldsymbol{h}\circ\boldsymbol{f}\circ\boldsymbol{b} \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{h}}*\theta"'] \\
\boldsymbol{g}\circ\tilde{\boldsymbol{e}} \arrow[r, Rightarrow, "\tilde{\eta}"] & \boldsymbol{h}\circ\tilde{\boldsymbol{f}}.
\end{tikzcd} \tag{6.114}$$

Since the $\tilde{\mathbf{Q}}_x^y$ cover $\mathbf{W}$, equation (6.114) commutes. This proves the first universal property for (6.104) to be 2-Cartesian in Definition A.7.

For the second universal property, suppose $\tilde{\boldsymbol{b}}, \tilde{\zeta}, \tilde{\theta}$ are alternative choices for $\boldsymbol{b}, \zeta, \theta$. Restricting to $\tilde{\mathbf{Q}}_x^y$, the second universal property for the 2-Cartesian square (6.109) shows there is a unique 2-morphism $\omega_x^y : \tilde{\boldsymbol{b}}|_{\tilde{\mathbf{Q}}_x^y} \Rightarrow \boldsymbol{b}|_{\tilde{\mathbf{Q}}_x^y}$ with

$$\tilde{\zeta}|_{\tilde{\mathbf{Q}}_x^y} = \zeta|_{\tilde{\mathbf{Q}}_x^y} \odot (\mathrm{id}_{\boldsymbol{e}} * \omega_x^y), \quad \tilde{\theta}|_{\tilde{\mathbf{Q}}_x^y} = \theta|_{\tilde{\mathbf{Q}}_x^y} \odot (\mathrm{id}_{\boldsymbol{f}} * \omega_x^y). \tag{6.115}$$

If $(x_1,y_1),(x_2,y_2) \in W$ then as the ω_x^y are unique we see that $\omega_{x_1}^{y_1}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}} = \omega_{x_2}^{y_2}|_{\tilde{\mathbf{Q}}_{x_1}^{y_1}\cap\tilde{\mathbf{Q}}_{x_2}^{y_2}}$. Hence there is a unique 2-morphism $\omega : \tilde{\boldsymbol{b}} \Rightarrow \boldsymbol{b}$ with $\omega|_{\tilde{\mathbf{Q}}_x^y} = \omega_x^y$ for all $(x,y) \in W$, and (6.115) for all (x,y) implies that

$$\tilde{\zeta} = \zeta \odot (\mathrm{id}_{\boldsymbol{e}} * \omega) \quad \text{and} \quad \tilde{\theta} = \theta \odot (\mathrm{id}_{\boldsymbol{f}} * \omega),$$

as we want. This completes the proof of Theorem 6.42. $\square$

For the local b-transverse fibre products $\mathbf{Q} = \mathbf{R} \times_{\boldsymbol{g}|_{\mathbf{R}},\mathbf{T},\boldsymbol{h}|_{\mathbf{S}}} \mathbf{S}$ in $\mathbf{dSpa^c}$ from §6.8.3, the underlying d-space $\boldsymbol{Q}$ is the fibre product $\boldsymbol{R} \times_{\boldsymbol{g}|_{\boldsymbol{R}},\boldsymbol{T},\boldsymbol{h}|_{\boldsymbol{S}}} \boldsymbol{S}$ in **dSpa**. The global b-transverse fibre products of Theorem 6.42 work by gluing together the local fibre products by equivalences, so they are also locally, and thus globally, fibre products in **dSpa**. But from §2.5, fibre products in **dSpa** are also fibre products at the level of C^∞-schemes. We deduce:

Corollary 6.43. *Suppose* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ *is a b-transverse fibre product in* $\mathbf{dSpa^c}$. *Then the d-space* $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *is also a fibre product in* **dSpa**, *and the* C^∞*-scheme* $\underline{W} \cong \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y}$ *is a fibre product in* $\mathbf{C^\infty Sch}$.

As in Example 2.40 for d-spaces, *products of d-spaces with corners* $\mathbf{X} \times \mathbf{Y}$ are a useful special case of fibre products $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ with $\mathbf{Z} = *$, the point.

Example 6.44. Let $\mathbf{X}$ and $\mathbf{Y}$ be d-spaces with corners. The *product* $\mathbf{X} \times \mathbf{Y}$ is the fibre product $\mathbf{X} \times_{\boldsymbol{g},*,\boldsymbol{h}} \mathbf{Y}$ in $\mathbf{dSpa^c}$, where $\mathbf{Z} = *$ is the point, a terminal object in $\mathbf{dSpa^c}$, and $\boldsymbol{g} : \mathbf{X} \to *$, $\boldsymbol{h} : \mathbf{Y} \to *$ are the unique 1-morphisms. As $\partial * = \emptyset$, $\boldsymbol{g}, \boldsymbol{h}$ are always b-transverse by Lemma 6.35(ii), so all products $\mathbf{X} \times \mathbf{Y}$ exist in $\mathbf{dSpa^c}$ by Theorem 6.42. They come with projection 1-morphisms $\boldsymbol{\pi}_{\mathbf{X}} : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$, $\boldsymbol{\pi}_{\mathbf{Y}} : \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$, and satisfy universal properties in $\mathbf{dSpa^c}$.

Defining $\mathbf{X} \times \mathbf{Y}$ as a fibre product only determines it up to equivalence in $\mathbf{dSpa^c}$. There is a canonical representative for this equivalence class, given by

$$\mathbf{X} \times \mathbf{Y} = \big(\boldsymbol{X} \times \boldsymbol{Y}, (\boldsymbol{\partial X} \times \boldsymbol{Y}) \amalg (\boldsymbol{X} \times \boldsymbol{\partial Y}), (\boldsymbol{i}_{\mathbf{X}} \times \mathbf{id}_{\boldsymbol{Y}}) \amalg (\mathbf{id}_{\boldsymbol{X}} \times \boldsymbol{i}_{\mathbf{Y}}), \omega_{\mathbf{X}\times\mathbf{Y}}\big),$$

where $\boldsymbol{X} \times \boldsymbol{Y}, \boldsymbol{\partial X} \times \boldsymbol{Y}, \boldsymbol{X} \times \boldsymbol{\partial Y}$ are explicit d-space products from Example 2.40, and $\boldsymbol{i}_{\mathbf{X}} \times \mathbf{id}_{\boldsymbol{Y}}, \mathbf{id}_{\boldsymbol{X}} \times \boldsymbol{i}_{\mathbf{Y}}$ are the product 1-morphisms. The projection $\boldsymbol{\pi}_{\mathbf{X}} : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ has $\underline{S}_{\boldsymbol{\pi}_{\mathbf{X}}} \cong \underline{\partial X} \times \underline{Y}$ and $\underline{T}_{\boldsymbol{\pi}_{\mathbf{X}}} = \emptyset$, and $\underline{s}_{\boldsymbol{\pi}_{\mathbf{X}}} : \underline{S}_{\boldsymbol{\pi}_{\mathbf{X}}} \to \underline{\partial(X \times Y)}$ is the inclusion $\underline{\partial X} \times \underline{Y} \hookrightarrow (\underline{\partial X} \times \underline{Y}) \amalg (\underline{X} \times \underline{\partial Y})$. Therefore $\boldsymbol{\pi}_{\mathbf{X}} : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ is semisimple and flat, and is simple if $\partial \mathbf{Y} = \boldsymbol{\emptyset}$.

The universal properties of products imply the existence of two kinds of product 1-morphisms. Firstly, if $\boldsymbol{f} : \mathbf{W} \to \mathbf{Y}$ and $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ are 1-morphisms, there is a *product* 1*-morphism* $\boldsymbol{f} \times \boldsymbol{g} : \mathbf{W} \times \mathbf{X} \to \mathbf{Y} \times \mathbf{Z}$. Using expressions for how $\boldsymbol{f} \times \boldsymbol{g}$ acts on boundaries, one can show that if $\boldsymbol{f}, \boldsymbol{g}$ are simple, semisimple, or flat, then $\boldsymbol{f} \times \boldsymbol{g}$ is also simple, semisimple, or flat, respectively.

Secondly, if $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ and $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ are 1-morphisms, there is a *direct product* 1*-morphism* $(\boldsymbol{f}, \boldsymbol{g}) : \mathbf{X} \to \mathbf{Y} \times \mathbf{Z}$. If $\boldsymbol{f}, \boldsymbol{g}$ are flat then $(\boldsymbol{f}, \boldsymbol{g})$ is flat. However, $\boldsymbol{f}, \boldsymbol{g}$ simple or semisimple do not imply $(\boldsymbol{f}, \boldsymbol{g})$ simple or semisimple. For example, if $\boldsymbol{f} = \boldsymbol{g} = \mathbf{id}_{[0,\infty)} : \mathbf{[0,\infty)} \to \mathbf{[0,\infty)}$ then $\boldsymbol{f}, \boldsymbol{g}$ are simple but $(\boldsymbol{f}, \boldsymbol{g})$ is not semisimple.

6.8.5 Transverse fibre products of manifolds with corners

Here is an analogue of Theorem 2.42(a) for manifolds and d-spaces with corners.

Theorem 6.45. *The* 2*-functor* $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ *takes transverse fibre products in* $\mathbf{Man^c}$

to b-transverse fibre products in $\mathbf{dSpa^c}$. *That is, if*

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z \end{tikzcd} \tag{6.116}$$

is a Cartesian square in $\mathbf{Man^c}$ *with* g, h *transverse, and* $\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(W, X, Y, Z, e, f, g, h)$, *then*

$$\begin{tikzcd} \mathbf{W} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"] \arrow[dr, phantom, "\mathrm{id}_{\boldsymbol{g}\circ\boldsymbol{e}} \Uparrow"] & \mathbf{Y} \arrow[d, "\boldsymbol{h}"] \\ \mathbf{X} \arrow[r, "\boldsymbol{g}"] & \mathbf{Z} \end{tikzcd} \tag{6.117}$$

is 2*-Cartesian in* $\mathbf{dSpa^c}$, *with* $\boldsymbol{g}, \boldsymbol{h}$ *b-transverse. If also* g, h *are strongly transverse in* $\mathbf{Man^c}$, *then* $\boldsymbol{g}, \boldsymbol{h}$ *are c-transverse in* $\mathbf{dSpa^c}$.

Proof. Let $g : X \to Z$ and $h : Y \to Z$ be transverse smooth maps in $\mathbf{Man^c}$ in the sense of §5.6, and write $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \boldsymbol{g}, \boldsymbol{h} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X, Y, Z, g, h)$. We first prove $\boldsymbol{g}, \boldsymbol{h}$ are b-transverse. Let $x \in X$ and $y \in Y$ with $g(x) = h(y) = z$. Then as g, h are transverse $\mathrm{d}g|_x \oplus \mathrm{d}h|_y : T_xX \oplus T_yY \to T_zZ$ is surjective, and dually,

$$\mathrm{d}g|_x^* \oplus \mathrm{d}h|_y^* : T_z^*Z \longrightarrow T_x^*X \oplus T_y^*Y \tag{6.118}$$

is an injective linear map.

Write $\nu_Z \to \partial Z$ for the normal line bundle of ∂Z in Z, and ν_Z^* for its dual. Then for each $z' \in \partial Z$ with $i_Z(z') = z$, $\nu_Z^*|_{z'} \subseteq T_z^*Z$, and the restriction of $\mathrm{d}g|_x^*$ to $\nu_Z^*|_{z'}$ is the map $\lambda_g|_{(x',z')} : \nu_Z^*|_{z'} \to \nu_X^*|_{x'}$ if $x' \in i_X^{-1}(x)$ with $(x', z') \in S_g$, where x' is unique, and $\mathrm{d}g|_x^*$ is zero on $\nu_Z^*|_{z'}$ if there is no such x'. Taking the sum over all $z' \in i_Z^{-1}(z)$, and doing the same for h, we see that

$$\begin{aligned} \left(\mathrm{d}g|_x^* \oplus \mathrm{d}h|_y^*\right)\big|_{\bigoplus_{z'\in i_Z^{-1}(z)}\nu_Z^*|_{z'}} &= \bigoplus_{(x',z')\in S_g : i_X(x')=x} \lambda_g|_{(x',z')} \oplus \bigoplus_{(y',z')\in S_h : i_Y(y')=y} \lambda_h|_{(y',z')} : \\ \bigoplus_{z'\in i_Z^{-1}(z)} \nu_Z^*|_{z'} &\longrightarrow \bigoplus_{x'\in i_X^{-1}(x)} \nu_X^*|_{x'} \oplus \bigoplus_{y'\in i_Y^{-1}(y)} \nu_Y^*|_{y'}. \end{aligned} \tag{6.119}$$

As (6.118) is injective, (6.119) is injective. But (6.81) for $\boldsymbol{g}, \boldsymbol{h}$ is the lift to C^∞-schemes of (6.119), so (6.81) is injective, and $\boldsymbol{g}, \boldsymbol{h}$ are b-transverse.

As g, h are transverse a fibre product $W = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^c}$ by Theorem 5.21, with projections $e : W \to X$, $f : W \to Y$. Since $\boldsymbol{g}, \boldsymbol{h}$ are b-transverse a fibre product $\tilde{\mathbf{W}} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ exists in $\mathbf{dSpa^c}$ by Theorem 6.42, with projections $\tilde{\boldsymbol{e}} : \tilde{\mathbf{W}} \to \mathbf{X}$, $\tilde{\boldsymbol{f}} : \tilde{\mathbf{W}} \to \mathbf{Y}$, and a 2-morphism $\tilde{\eta} : \boldsymbol{g} \circ \tilde{\boldsymbol{e}} \Rightarrow \boldsymbol{h} \circ \tilde{\boldsymbol{f}}$. As (6.117) is 2-commutative, properties of fibre products give a 1-morphism $\boldsymbol{b} : \mathbf{W} \to \tilde{\mathbf{W}}$ and 2-morphisms $\zeta : \tilde{\boldsymbol{e}} \circ \boldsymbol{b} \Rightarrow \boldsymbol{e}$, $\theta : \tilde{\boldsymbol{f}} \circ \boldsymbol{b} \Rightarrow \boldsymbol{f}$ in $\mathbf{dSpa^c}$.

Since the d-space $\tilde{\boldsymbol{W}}$ in $\tilde{\mathbf{W}}$ is a fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ in $\mathbf{dSpa}$ by Corollary 6.43, the proof of Theorem 2.42(a) implies that $\boldsymbol{b} : \boldsymbol{W} \to \tilde{\boldsymbol{W}}$ is an equivalence in $\mathbf{dSpa}$. Comparing the constructions of ∂W in the proof of Theorem 5.21 in [55, §8], and of $\partial\tilde{\mathbf{W}}$ in §6.8.3–§6.8.4, we find there is a natural

1-1 correspondence between ∂W and $\partial\tilde{\mathbf{W}}$ on the level of sets, and using this we can show that $\boldsymbol{b}$ is simple and flat. Therefore $\boldsymbol{b}$ is an equivalence in $\mathbf{dSpa^c}$ by Proposition 6.21. Hence (6.117) is 2-Cartesian in $\mathbf{dSpa^c}$.

Finally we show that if g,h are strongly transverse in the sense of §5.6, then $\boldsymbol{g},\boldsymbol{h}$ are c-transverse. Definition 6.33(a) for $\boldsymbol{g},\boldsymbol{h}$ follows immediately from strong transversality in Definition 5.25 for g,h. For (b), suppose equation (6.83) holds. As for (6.119), we find that

$$\left(\mathrm{d}g|_x^*\oplus\mathrm{d}h|_y^*\right)\big|_{\bigoplus_{i=1}^l\nu_Z^*|_{z_i'}}:\bigoplus\nolimits_{i=1}^l\nu_Z^*|_{z_i'}\longrightarrow\bigoplus\nolimits_{i=1}^j\nu_X^*|_{x_i'}\oplus\bigoplus\nolimits_{i=1}^k\nu_Y^*|_{y_i'}. \tag{6.120}$$

As (6.118) is injective, (6.120) is injective. But (6.120) is a linear map $\mathbb{R}^l\rightarrow\mathbb{R}^{j+k}$, so $j+k\geqslant l$, and Definition 6.33(b) holds. Hence $\boldsymbol{g},\boldsymbol{h}$ are c-transverse. □

6.8.6 Examples of non-b-transverse fibre products in $\mathbf{dSpa^c}$

If $\boldsymbol{g}:\mathbf{X}\rightarrow\mathbf{Z}$ and $\boldsymbol{h}:\mathbf{Y}\rightarrow\mathbf{Z}$ are 1-morphisms in $\mathbf{dSpa^c}$ which are not b-transverse, then a fibre product $\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ may or may not exist in $\mathbf{dSpa^c}$, as the following two examples show.

Example 6.46. Let $X=Y=*$, the point in $\mathbf{Man^c}$, and $Z=[0,\infty)$, and $g:X\rightarrow Z$, $h:Y\rightarrow Z$ map $*\mapsto 0$. Then g,h are not transverse in $\mathbf{Man^c}$. Define $W=*$ and $e=\mathrm{id}_*:W\rightarrow X$, $f=\mathrm{id}_*:W\rightarrow Y$. Since X,Y are terminal objects $*$ in $\mathbf{Man^c}$, it easily follows that $W=X\times_{g,Z,h}Y$ is a fibre product in $\mathbf{Man^c}$. Note that $\dim W=0\neq -1=\dim X+\dim Y-\dim Z$, whereas for transverse fibre products $W=X\times_{g,Z,h}Y$ in $\mathbf{Man^c}$ we always have $\dim W=\dim X+\dim Y-\dim Z$.

Set $\mathbf{W},\mathbf{X},\mathbf{Y},\mathbf{Z},\boldsymbol{e},\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}=F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(W,X,Y,Z,e,f,g,h)$. Then $\boldsymbol{g},\boldsymbol{h}$ are not b-transverse in $\mathbf{dSpa^c}$. However, one can show that (6.117) is 2-Cartesian in $\mathbf{dSpa^c}$, so that the fibre product $\mathbf{W}=\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ exists in $\mathbf{dSpa^c}$. To prove this, we verify directly that (6.117) satisfies the universal properties in §A.4.

The important point in the proof is this: suppose $\tilde{\mathbf{W}}$ is a d-space with corners, $\tilde{\boldsymbol{e}}:\tilde{\mathbf{W}}\rightarrow\mathbf{X}$ and $\tilde{\boldsymbol{f}}:\tilde{\mathbf{W}}\rightarrow\mathbf{Y}$ are 1-morphisms and $\tilde{\eta}:\boldsymbol{g}\circ\tilde{\boldsymbol{e}}\Rightarrow\boldsymbol{h}\circ\tilde{\boldsymbol{f}}$ a 2-morphism in $\mathbf{dSpa^c}$. As $\mathbf{X}=\mathbf{Y}=*$, the only possibility for $\tilde{\boldsymbol{e}},\tilde{\boldsymbol{f}}$ is the unique projection $\boldsymbol{\pi}:\tilde{\boldsymbol{e}}:\tilde{\mathbf{W}}\rightarrow *$. Then $\boldsymbol{g}\circ\tilde{\boldsymbol{e}}=\boldsymbol{h}\circ\tilde{\boldsymbol{f}}=\boldsymbol{0}\circ\boldsymbol{\pi}:\tilde{\mathbf{W}}\rightarrow[\boldsymbol{0},\boldsymbol{\infty})$. It follows that $\underline{T}_{\boldsymbol{g}\circ\tilde{\boldsymbol{e}}}=\underline{\tilde{W}}\times_{\underline{[0,\infty)}}\underline{0}\cong\underline{\tilde{W}}$. Equation (6.10) for $\tilde{\eta}$ on $\underline{T}_{\boldsymbol{g}\circ\tilde{\boldsymbol{e}}}$ then implies that $\tilde{\eta}=0=\mathrm{id}_{\boldsymbol{g}\circ\tilde{\boldsymbol{e}}}$. So $\boldsymbol{b}=\boldsymbol{\pi}:\tilde{\mathbf{W}}\rightarrow *=\mathbf{W}$ and $\zeta=0:\boldsymbol{e}\circ\boldsymbol{b}\Rightarrow\tilde{\boldsymbol{e}}$, $\theta=0:\boldsymbol{f}\circ\boldsymbol{b}\Rightarrow\tilde{\boldsymbol{f}}$ make (6.114) commute.

Example 6.46 has several interesting features. Firstly, $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ takes the fibre product in $\mathbf{Man^c}$ to the fibre product in $\mathbf{dSpa^c}$, although they are not (b-)transverse, so Theorem 6.45 does not apply. Thus the analogue of Theorem 2.42(b) is false in the corners case.

Secondly, in the fibre product $\mathbf{W}=\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$, the underlying d-spaces do not satisfy $\boldsymbol{W}\simeq\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$, since the fibre product $*\times_{[\boldsymbol{0},\boldsymbol{\infty})}*$ in $\mathbf{dSpa}$ is the 'obstructed point' of Example 2.38 rather than the point $*$. Hence the analogue of Corollary 6.43 is false for non b-transverse fibre products.

Thirdly, in the language of Chapter 7, $\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are d-manifolds with corners, and $\operatorname{vdim} \mathbf{W} = \operatorname{vdim} \mathbf{X} = \operatorname{vdim} \mathbf{Y} = 0$, $\operatorname{vdim} \mathbf{Z} = 1$, so that $\operatorname{vdim} \mathbf{W} \neq \operatorname{vdim} \mathbf{X} + \operatorname{vdim} \mathbf{Y} - \operatorname{vdim} \mathbf{Z}$. So although the non bd-transverse fibre product $\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ exists in $\mathbf{dMan^c}$, it does not have the expected dimension.

Example 6.47. Let $X = Y = [0, \infty) \times \mathbb{R}$ and $Z = [0, \infty)^2 \times \mathbb{R}$, and define $g : X \to Z$ and $h : Y \to Z$ by $g(u, v) = (u, u, v)$ and $h(u, v) = (u, e^v u, v)$. Then X, Y, Z are manifolds with corners and g, h are smooth, but they are not transverse, since $T_z Z \neq \mathrm{d}g|_x(T_x X) + \mathrm{d}h|_y(T_y Y)$ at $x = (0, 0) \in X$, $y = (0, 0) \in Y$ and $z = (0, 0, 0) \in Z$. No fibre product $X \times_{g,Z,h} Y$ exists in $\mathbf{Man^c}$, as such a fibre product could not be a manifold over $(0, 0) \in X$ and $(0, 0) \in Y$.

Set $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \boldsymbol{g}, \boldsymbol{h} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X, Y, Z, g, h)$. For $v \in \mathbb{R}$, consider the points $x = (0, v) \in \mathbf{X}$, $y = (0, v) \in \mathbf{Y}$ and $z = (0, 0, v) \in \mathbf{Z}$. We have $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$, and $\underline{i}_{\mathbf{X}}^{-1}(x), \underline{i}_{\mathbf{X}}^{-1}(y)$ are each one points, and $\underline{i}_{\mathbf{Z}}^{-1}(z)$ is two points. Equation (6.81) is the map $\mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & e^v \end{smallmatrix}\right)$. This is invertible when $v \neq 0$. Thus (6.81) is injective when $v \neq 0$, but not injective when $v = 0$. Hence $\boldsymbol{g}, \boldsymbol{h}$ are not b-transverse, as (6.81) is not injective at $x = y = (0, 0)$ and $z = (0, 0, 0)$.

We will prove that no fibre product $\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ exists in $\mathbf{dSpa^c}$. Suppose for a contradiction that $\mathbf{W}, \boldsymbol{e}, \boldsymbol{f}, \eta$ are such a fibre product, in a 2-Cartesian diagram (6.104). Define $W_1 = [0, \infty)$ and smooth maps $e_1 : W_1 \to X$, $f_1 : W_1 \to Y$ by $e_1(u) = f_1(u) = (u, 0)$. Then $g \circ e_1 = h \circ f_1$. Set $\mathbf{W}_1, \boldsymbol{e}_1, \boldsymbol{f}_1 = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(W_1, e_1, f_1)$. Then $\boldsymbol{g} \circ \boldsymbol{e}_1 = \boldsymbol{h} \circ \boldsymbol{f}_1$, so $\mathrm{id}_{\boldsymbol{g} \circ \boldsymbol{e}_1} : \boldsymbol{g} \circ \boldsymbol{e}_1 \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}_1$ is a 2-morphism in $\mathbf{dSpa^c}$. The universal property of fibre products gives a 1-morphism $\boldsymbol{b}_1 : \mathbf{W}_1 \to \mathbf{W}$ and 2-morphisms $\zeta_1 : \boldsymbol{e} \circ \boldsymbol{b}_1 \Rightarrow \boldsymbol{e}_1$, $\theta_1 : \boldsymbol{f} \circ \boldsymbol{b}_1 \Rightarrow \boldsymbol{f}_1$.

Identify $\partial X = \mathbb{R}$, with $i_X : \partial X \to X$ mapping $v \mapsto (0, v)$. Then (X, u) is a boundary defining function for X at any $v \in \partial X$. Hence $(\boldsymbol{X}, \boldsymbol{u})$ is a boundary defining function for $\mathbf{X}$ at any $v \in \boldsymbol{\partial X}$ by Lemma 6.17(a). The points $w = \boldsymbol{b}_1(0) \in \boldsymbol{W}$ and $x' = 0 \in \boldsymbol{\partial X}$ satisfy $\boldsymbol{e}(w) = \boldsymbol{e} \circ \boldsymbol{b}_1(0) = \boldsymbol{e}_1(0) = (0, 0) = \boldsymbol{i}_{\mathbf{X}}(0) = \boldsymbol{i}_{\mathbf{X}}(x')$. So as $\boldsymbol{e} : \mathbf{W} \to \mathbf{X}$ is a 1-morphism in $\mathbf{dSpa^c}$, one of Definition 6.2(i),(ii) holds for $\boldsymbol{e}$ at $w \in \boldsymbol{W}$ and $x' \in \boldsymbol{\partial X}$. But if (ii) held, this would force (ii) to hold for $\boldsymbol{e} \circ \boldsymbol{b}_1$ at $0 \in \boldsymbol{W}_1$ and $0 \in \boldsymbol{\partial X}$, and so (ii) would hold for $\boldsymbol{e}_1$ at $0 \in \boldsymbol{W}_1$ and $0 \in \boldsymbol{\partial X}$, a contradiction. Thus Definition 6.2(i) holds for $\boldsymbol{e}$ at $w \in \boldsymbol{W}$ and $x' \in \boldsymbol{\partial X}$. So there exists a unique $w' \in \boldsymbol{i}_{\mathbf{W}}^{-1}(w)$ and an open neighbourhood $\boldsymbol{V}$ of w in $\boldsymbol{W}$ such that $\bigl(\boldsymbol{V}, \boldsymbol{u} \circ \boldsymbol{e}|_{\boldsymbol{V}}\bigr)$ is a boundary defining function for $\mathbf{W}$ at w'. By Definition 6.1, there is an open neighbourhood $\boldsymbol{U}$ of w' in $\boldsymbol{\partial W}$ in a 2-Cartesian square in $\mathbf{dSpa}$:

$$\begin{tikzcd} \boldsymbol{U} \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{W}}|_{\boldsymbol{U}}"'] \arrow[drr, phantom, "\Uparrow \mathrm{id}_{\boldsymbol{0} \circ \boldsymbol{\pi}}"] & & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{V} \arrow[rr, "\boldsymbol{u} \circ \boldsymbol{e}|_{\boldsymbol{V}}"] & & {[\boldsymbol{0}, \boldsymbol{\infty})}. \end{tikzcd} \tag{6.121}$$

The same argument gives a unique $w'' \in \boldsymbol{i}_{\mathbf{W}}^{-1}(w)$ and open $w \in \hat{\boldsymbol{V}} \subseteq \boldsymbol{W}$ such that $\bigl(\hat{\boldsymbol{V}}, \boldsymbol{u} \circ \boldsymbol{f}|_{\hat{\boldsymbol{V}}}\bigr)$ is a boundary defining function for $\mathbf{W}$ at w''. Write (t, u, v) for the coordinates on Z, and let $z' \in \underline{i}_{\mathbf{Z}}^{-1}(0, 0, 0)$ be the boundary point corresponding to the local boundary component $t = 0$ of Z. Since $u = t \circ g$ on X and $u = t \circ h$ on Y, we have $\boldsymbol{u} \circ \boldsymbol{e} = \boldsymbol{t} \circ \boldsymbol{g} \circ \boldsymbol{e}$ and $\boldsymbol{u} \circ \boldsymbol{f} = \boldsymbol{t} \circ \boldsymbol{h} \circ \boldsymbol{f}$. As

$\boldsymbol{g}\circ\boldsymbol{e}$ and $\boldsymbol{h}\circ\boldsymbol{f}$ are 2-isomorphic, $\bigl(\boldsymbol{V},\boldsymbol{t}\circ\boldsymbol{g}\circ\boldsymbol{e}|_{\boldsymbol{V}}\bigr)$ and $\bigl(\hat{\boldsymbol{V}},\boldsymbol{t}\circ\boldsymbol{h}\circ\boldsymbol{f}|_{\hat{\boldsymbol{V}}}\bigr)$ must be boundary defining functions for the same point in $\boldsymbol{i}_{\mathbf{W}}^{-1}(w)$. Therefore $w'=w''$. Making $\boldsymbol{U},\boldsymbol{V},\tilde{\boldsymbol{V}}$ smaller we can take $\hat{\boldsymbol{V}}=\boldsymbol{V}$, and suppose (6.121) with $\boldsymbol{u}\circ\boldsymbol{f}|_{\boldsymbol{V}}$ in place of $\boldsymbol{u}\circ\boldsymbol{e}|_{\boldsymbol{V}}$ is also 2-Cartesian, with the same $\boldsymbol{U}$.

Define $W_2=\mathbb{R}$ and smooth maps $e_2:W_2\rightarrow X$, $f_2:W_2\rightarrow Y$ by $e_2(v)=f_2(v)=(0,v)$. Then $g\circ e_2=h\circ f_2$. Set $\mathbf{W}_2,\boldsymbol{e}_2,\boldsymbol{f}_2=F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(W_2,e_2,f_2)$. Then $\boldsymbol{g}\circ\boldsymbol{e}_2=\boldsymbol{h}\circ\boldsymbol{f}_2$, so as above we get a 1-morphism $\boldsymbol{b}_2:\mathbf{W}_2\rightarrow\mathbf{W}$ and 2-morphisms $\zeta_2:\boldsymbol{e}\circ\boldsymbol{b}_2\Rightarrow\boldsymbol{e}_2$, $\theta_2:\boldsymbol{f}\circ\boldsymbol{b}_2\Rightarrow\boldsymbol{f}_2$. By considering the maps $*\rightarrow W_1$ and $*\rightarrow W_2$ mapping $*\mapsto 0$ we can show that $\boldsymbol{b}_1(0)=\boldsymbol{b}_2(0)=w\in\boldsymbol{W}$. As $\boldsymbol{V}$ is an open neighbourhood of $w=\boldsymbol{b}_2(0)$ in $\boldsymbol{W}$, $\boldsymbol{W}_3=\boldsymbol{b}_2^{-1}(\boldsymbol{V})$ is an open neighbourhood of 0 in $\boldsymbol{W}_2$. Then $\boldsymbol{b}_2|_{\boldsymbol{W}_3}:\boldsymbol{W}_3\rightarrow\boldsymbol{V}$, and we have a 2-morphism $\mathrm{id}_{\boldsymbol{u}}*\theta_2|_{\boldsymbol{W}_3}:\boldsymbol{u}\circ\boldsymbol{e}|_{\boldsymbol{V}}\circ\boldsymbol{b}_2|_{\boldsymbol{W}_3}\Rightarrow\boldsymbol{u}\circ\boldsymbol{e}_2=\boldsymbol{0}\circ\boldsymbol{\pi}$, since $u\circ e_2=0$. Since (6.121) is 2-Cartesian in **dSpa**, there exists a 1-morphism $\boldsymbol{b}_3:\boldsymbol{W}_3\rightarrow\boldsymbol{U}$ and 2-morphism $\boldsymbol{i}_{\mathbf{W}}\circ\boldsymbol{b}_3\Rightarrow\boldsymbol{b}_2|_{\boldsymbol{W}_3}$ in **dSpa**.

Now as $\bigl(\boldsymbol{V},\boldsymbol{u}\circ\boldsymbol{e}|_{\boldsymbol{V}}\bigr)$ and $\bigl(\boldsymbol{V},\boldsymbol{u}\circ\boldsymbol{f}|_{\boldsymbol{V}}\bigr)$ are both boundary defining functions for $\mathbf{W}$ at w', they are also boundary defining functions for $\mathbf{W}$ at any $\tilde{w}'\in\boldsymbol{\partial W}$ sufficiently close to w'. Thus, if $0\neq v\in W_2$ is close to zero then $v\in\boldsymbol{b}_2^{-1}(\boldsymbol{V})$, so $\tilde{w}'=\boldsymbol{b}_3(v)$ lies in $\boldsymbol{\partial W}$, and $\bigl(\boldsymbol{V},\boldsymbol{u}\circ\boldsymbol{e}|_{\boldsymbol{V}}\bigr)$ and $\bigl(\boldsymbol{V},\boldsymbol{u}\circ\boldsymbol{f}|_{\boldsymbol{V}}\bigr)$ are both boundary defining functions for $\mathbf{W}$ at $\tilde{w}'$. Write $\underline{i}_{\mathbf{X}}^{-1}(0,v)=\{\tilde{x}'\}$, $\underline{i}_{\mathbf{Y}}^{-1}(0,v)=\{\tilde{y}'\}$ and $\underline{i}_{\mathbf{Z}}^{-1}(0,0,v)=\{\tilde{z}_1',\tilde{z}_2'\}$, where $\tilde{z}_1',\tilde{z}_2'$ correspond to the boundary components $t=0$ and $u=0$ respectively. Then $\bigl(\boldsymbol{V},\boldsymbol{u}\circ\boldsymbol{e}|_{\boldsymbol{V}}\bigr)$ a boundary defining function for $\mathbf{W}$ at $\tilde{w}'$ implies that $(\tilde{w}',\tilde{x}')\in\underline{S}_{\boldsymbol{e}}$, and similarly $(\tilde{w}',\tilde{y}')\in\underline{S}_{\boldsymbol{f}}$.

As for (6.81), consider the diagram of linear maps

$$\begin{tikzcd}
\mathbb{R}\cong\mathcal{N}_{\mathbf{Z}}|_{\tilde{z}_1'} \arrow[rr, "\lambda_{\boldsymbol{g}}|_{(\tilde{x}',\tilde{z}_1')}=1"'] \arrow[drr, "\lambda_{\boldsymbol{h}}|_{(\tilde{y}',\tilde{z}_1')}=1"' near start] & & \mathbb{R}\cong\mathcal{N}_{\mathbf{X}}|_{\tilde{x}'} \arrow[dr, "\lambda_{\boldsymbol{e}}|_{(\tilde{w}',\tilde{x}')}"] & \\
& & & \mathcal{N}_{\mathbf{W}}|_{\tilde{w}'}. \\
\mathbb{R}\cong\mathcal{N}_{\mathbf{Z}}|_{\tilde{z}_2'} \arrow[rr, "\lambda_{\boldsymbol{h}}|_{(\tilde{y}',\tilde{z}_2')}=e^v"] \arrow[urr, "\lambda_{\boldsymbol{g}}|_{(\tilde{x}',\tilde{z}_2')}=1" near start] & & \mathbb{R}\cong\mathcal{N}_{\mathbf{Y}}|_{\tilde{y}'} \arrow[ur, "\lambda_{\boldsymbol{f}}|_{(\tilde{w}',\tilde{y}')}"'] &
\end{tikzcd} \tag{6.122}$$

Here the identifications $\mathbb{R}\cong\mathcal{N}_{\mathbf{X}}|_{\tilde{x}'}$, $\mathbb{R}\cong\mathcal{N}_{\mathbf{Y}}|_{\tilde{y}'}$, $\mathbb{R}\cong\mathcal{N}_{\mathbf{Z}}|_{\tilde{z}_j'}$ come from $T_0[0,\infty)\cong\mathbb{R}$, and $\lambda_{\boldsymbol{e}},\ldots,\lambda_{\boldsymbol{h}}$ are as in Proposition 6.7(d). The equations $\lambda_{\boldsymbol{g}}|_{(\tilde{x}',\tilde{z}_1')}=1,\ \ldots,\ \lambda_{\boldsymbol{h}}|_{(\tilde{y}',\tilde{z}_2')}=e^v$, mean that $\lambda_{\boldsymbol{g}}|_{(\tilde{x}',\tilde{z}_1')}:\mathbb{R}\rightarrow\mathbb{R}$ and $\lambda_{\boldsymbol{h}}|_{(\tilde{y}',\tilde{z}_2')}:\mathbb{R}\rightarrow\mathbb{R}$ act by multiplication by $1,e^v$, and the values $1,1,1,e^v$ are the appropriate components of $\mathrm{d}g^*|_{(0,v)}$ and $\mathrm{d}h^*|_{(0,v)}$.

Since $\eta:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ is a 2-morphism, Proposition 6.8(a) gives $\lambda_{\boldsymbol{g}\circ\boldsymbol{e}}=\lambda_{\boldsymbol{h}\circ\boldsymbol{f}}$. Hence for $j=1,2$ we have

$$\lambda_{\boldsymbol{e}}|_{(\tilde{w}',\tilde{x}')}\circ\lambda_{\boldsymbol{g}}|_{(\tilde{x}',\tilde{z}_j')}=\lambda_{\boldsymbol{g}\circ\boldsymbol{e}}|_{(\tilde{x}',\tilde{z}_j')}=\lambda_{\boldsymbol{h}\circ\boldsymbol{f}}|_{(\tilde{x}',\tilde{z}_j')}=\lambda_{\boldsymbol{f}}|_{(\tilde{w}',\tilde{y}')}\circ\lambda_{\boldsymbol{h}}|_{(\tilde{y}',\tilde{z}_j')}.$$

Making the substitutions in (6.122), this reduces to the two equations

$$\lambda_{\boldsymbol{e}}|_{(\tilde{w}',\tilde{x}')}=\lambda_{\boldsymbol{f}}|_{(\tilde{w}',\tilde{y}')}\quad\text{and}\quad\lambda_{\boldsymbol{e}}|_{(\tilde{w}',\tilde{x}')}=e^v\cdot\lambda_{\boldsymbol{f}}|_{(\tilde{w}',\tilde{y}')}.$$

As $v\neq 0$ and $\lambda_{\boldsymbol{e}},\lambda_{\boldsymbol{f}}$ are isomorphisms, this is a contradiction. Therefore no fibre product $\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ exists in $\mathbf{dSpa^c}$. This completes Example 6.47.

Remark 6.48. In Example 6.46, the key point in determining the fibre product $\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y}$ was that 2-morphisms in $\mathbf{dSpa^c}$ satisfy equation (6.10). Similarly, in Example 6.47 the key point in showing no fibre product $\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y}$ exists was that $\eta:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ implies that $\lambda_{\boldsymbol{g}\circ\boldsymbol{e}}=\lambda_{\boldsymbol{h}\circ\boldsymbol{f}}$, and this holds as 2-morphisms η in $\mathbf{dSpa^c}$ satisfy equation (6.9).

If we omit (6.9)–(6.10) from the definition of 2-morphisms, as in the alternate 2-category $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$ of Remark 6.5, then Examples 6.46 and 6.47 work out rather differently. In Example 6.46, the fibre product in $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$ is the 'obstructed point' of Example 2.38, rather than the point $*$. In Example 6.47, a fibre product $\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y}$ does exist in $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$. The proof by contradiction fails because $\eta:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ does not imply $\lambda_{\boldsymbol{g}\circ\boldsymbol{e}}=\lambda_{\boldsymbol{h}\circ\boldsymbol{f}}$ in $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$.

If $\boldsymbol{g}:\mathbf{X}\rightarrow\mathbf{Z}$ and $\boldsymbol{h}:\mathbf{Y}\rightarrow\mathbf{Z}$ are c-transverse, then fibre products $\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ in $\mathbf{dSpa^c}$ and $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$ agree. However, if $\boldsymbol{g},\boldsymbol{h}$ are b-transverse but not c-transverse then fibre products $\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ in $\mathbf{dSpa^c}$ and $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$ are different. This is because each type (C) component $\hat\Gamma$ in Proposition 6.37 contributes a point to $\partial(\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y})$ in $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$, but no point in $\mathbf{dSpa^c}$. This implies that Theorem 6.45 fails in $\widetilde{\mathbf{dSpa}}^{\mathbf{c}}$ for $\boldsymbol{g},\boldsymbol{h}$ b-transverse but not c-transverse.

6.9 Boundary and corners of c-transverse fibre products

In §5.6 we saw that if $g:X\rightarrow Z$ and $h:Y\rightarrow Z$ are strongly transverse in $\mathbf{Man^c}$ and $W=X\times_{g,Z,h}Y$ then $C(g):C(X)\rightarrow C(Z)$ and $C(h):C(Y)\rightarrow C(Z)$ are transverse in $\check{\mathbf{M}}\mathbf{an^c}$, and $C(W)\cong C(X)\times_{C(g),C(Z),C(h)}C(Y)$. The same also applies for the second corner functor $\hat C:\mathbf{Man^c}\rightarrow\check{\mathbf{M}}\mathbf{an^c}$ in (5.8). We now prove analogues of all this for c-transverse $\boldsymbol{g},\boldsymbol{h}$ in $\mathbf{dSpa^c}$.

Proposition 6.49. *Let $\boldsymbol{g}:\mathbf{X}\rightarrow\mathbf{Z}$ and $\boldsymbol{h}:\mathbf{Y}\rightarrow\mathbf{Z}$ be c-transverse 1-morphisms in $\mathbf{dSpa^c}$, and use the notation of §6.7. Then $C(\boldsymbol{g}):C(\mathbf{X})\rightarrow C(\mathbf{Z})$ and $C(\boldsymbol{h}):C(\mathbf{Y})\rightarrow C(\mathbf{Z})$ are c-transverse, and $\hat C(\boldsymbol{g}):C(\mathbf{X})\rightarrow C(\mathbf{Z})$ and $\hat C(\boldsymbol{h}):C(\mathbf{Y})\rightarrow C(\mathbf{Z})$ are c-transverse.*

Proof. We first describe the underlying set $C(C(X))$ of $C(C(\mathbf{X}))$. As in §6.7 we have $C_k(\mathbf{X})\cong\partial^k\mathbf{X}/S_k$. Hence

$$C_{k_1}\bigl(C_{k_2}(\mathbf{X})\bigr)\cong\partial^{k_1}\bigl(\partial^{k_2}\mathbf{X}/S_{k_2}\bigr)/S_{k_1}\cong\partial^{k_1+k_2}\mathbf{X}/(S_{k_1}\times S_{k_2}).$$

Here as in (6.67) we may describe points of $\partial^{k_1+k_2}\mathbf{X}$ as $(x,x'_1,\ldots,x'_{k_1+k_2})$ for $x\in\underline{X}$ and distinct $x'_1,\ldots,x'_{k_1+k_2}\in\underline{i}_{\mathbf{X}}^{-1}(x)\subseteq\underline{\partial X}$, and then S_{k_1} acts on $(x,x'_1,\ldots,x'_{k_1+k_2})$ by permuting $x'_1,\ldots,x'_{k_1}$ and fixing $x,x'_{k_1+1},\ldots,x'_{k_1+k_2}$, and S_{k_2} acts on $(x,x'_1,\ldots,x'_{k_1+k_2})$ by permuting $x'_{k_1+1},\ldots,x'_{k_1+k_2}$ and fixing $x,x'_1,\ldots,x'_{k_1}$. Therefore as for (6.69)–(6.70), we have natural homeomorphisms

$$\begin{aligned}&C_{k_1}\bigl(C_{k_2}(X)\bigr)\cong\bigl\{(x,\{x'_1,\ldots,x'_{k_1}\},\{x'_{k_1+1},\ldots,x'_{k_1+k_2}\}):x\in\boldsymbol{X},\\&x'_1,\ldots,x'_{k_1+k_2}\in\boldsymbol{\partial X}\text{ are distinct, }\boldsymbol{i}_{\mathbf{X}}(x'_a)=x,\ \ a=1,\ldots,k_1+k_2\bigr\}.\end{aligned}\tag{6.123}$$

In the representation (6.123) for $\mathbf{X},\mathbf{Z}$, we find $C(C(\boldsymbol{g}))$ acts by $C(C(\boldsymbol{g}))$: $(x,\{x'_1,\ldots,x'_{j_1}\},\{x'_{j_1+1},\ldots,x'_{j_1+j_2}\})\longmapsto(z,\{z'_1,\ldots,z'_{l_1}\},\{z'_{l_1+1},\ldots,z'_{l_1+l_2}\})$

when $C(\boldsymbol{g}) : (x, \{x'_1, \ldots, x'_{j_1}\}) \mapsto (z, \{z'_1, \ldots, z'_{l_1}\})$ and $C(\boldsymbol{g}) : (x, \{x'_{j_1+1}, \ldots, x'_{j_1+j_2}\}) \mapsto (z, \{z'_{l_1+1}, \ldots, z'_{l_1+l_2}\})$. Suppose $C(C(\boldsymbol{g}))\bigl((x, \{x'_1, \ldots, x'_{j_1}\}, \{x'_{j_1+1}, \ldots, x'_{j_1+j_2}\})\bigr) = C(C(\boldsymbol{h}))\bigl((y, \{y'_1, \ldots, y'_{k_1}\}, \{y'_{k_1+1}, \ldots, y'_{k_1+k_2}\})\bigr) = (z, \{z'_1, \ldots, z'_{l_1}\}, \{z'_{l_1+1}, \ldots, z'_{l_1+l_2}\})$. Then $C(\boldsymbol{g})\bigl((x, \{x'_1, \ldots, x'_{j_1}\})\bigr) = C(\boldsymbol{h})\bigl((y, \{y'_1, \ldots, y'_{k_1}\})\bigr) = (z, \{z'_1, \ldots, z'_{l_1}\})$, so as $\boldsymbol{g}, \boldsymbol{h}$ are c-transverse either $j_1 + k_1 > l_1$ or $j_1 = k_1 = l_1 = 0$. This proves $C(\boldsymbol{g}), C(\boldsymbol{h})$ satisfy Definition 6.33(a). Similar proofs show that $C(\boldsymbol{g}), C(\boldsymbol{h})$ satisfy Definition 6.33(b), so that $C(\boldsymbol{g}), C(\boldsymbol{h})$ are c-transverse, and that $\hat{C}(\boldsymbol{g}), \hat{C}(\boldsymbol{h})$ are c-transverse. □

Here is an analogue of Theorem 5.26 for the corner functors $C, \hat{C}$ in $\mathbf{dSpa^c}$.

Theorem 6.50. *Suppose we are given a* 2*-Cartesian diagram in* $\mathbf{dSpa^c}$*:*

$$\begin{tikzcd} \mathbf{W} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"] \arrow[dr, phantom, "\eta\Uparrow"] & \mathbf{Y} \arrow[d, "\boldsymbol{h}"] \\ \mathbf{X} \arrow[r, "\boldsymbol{g}"] & \mathbf{Z}, \end{tikzcd}$$

with $\boldsymbol{g}, \boldsymbol{h}$ *c-transverse. Then the following are also* 2*-Cartesian in* $\mathbf{dSpa^c}$*:*

$$\begin{tikzcd} C(\mathbf{W}) \arrow[r, "C(\boldsymbol{f})"] \arrow[d, "C(\boldsymbol{e})"] \arrow[dr, phantom, "C(\eta)\Uparrow"] & C(\mathbf{Y}) \arrow[d, "C(\boldsymbol{h})"] \\ C(\mathbf{X}) \arrow[r, "C(\boldsymbol{g})"] & C(\mathbf{Z}), \end{tikzcd} \tag{6.124}$$

$$\begin{tikzcd} C(\mathbf{W}) \arrow[r, "\hat{C}(\boldsymbol{f})"] \arrow[d, "\hat{C}(\boldsymbol{e})"] \arrow[dr, phantom, "\hat{C}(\eta)\Uparrow"] & C(\mathbf{Y}) \arrow[d, "\hat{C}(\boldsymbol{h})"] \\ C(\mathbf{X}) \arrow[r, "\hat{C}(\boldsymbol{g})"] & C(\mathbf{Z}). \end{tikzcd} \tag{6.125}$$

Let $(w, P) \in C(\mathbf{W})$ *with* $C(\boldsymbol{e})\bigl((w, P)\bigr) = (x, Q) \in C(\mathbf{X})$, $C(\boldsymbol{f})\bigl((w, P)\bigr) = (y, R) \in C(\mathbf{Y})$, *and* $C(\boldsymbol{g})\bigl((x, Q)\bigr) = C(\boldsymbol{h})\bigl((y, R)\bigr) = (z, S) \in C(\mathbf{Z})$*. Then* $|P| + |S| = |Q| + |R|$*. Therefore* (6.124) *implies equivalences in* $\mathbf{dSpa^c}$*:*

$$C_i(\mathbf{W}) \simeq \coprod_{j,k,l \geqslant 0: i=j+k-l} C_j^{\boldsymbol{g},l}(\mathbf{X}) \times_{C_j^l(\boldsymbol{g}), C_l(\mathbf{Z}), C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathbf{Y}), \tag{6.126}$$

$$\partial \mathbf{W} \simeq \coprod_{j,k,l \geqslant 0: j+k=l+1} C_j^{\boldsymbol{g},l}(\mathbf{X}) \times_{C_j^l(\boldsymbol{g}), C_l(\mathbf{Z}), C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathbf{Y}). \tag{6.127}$$

Similarly, if $(w, \hat{P}) \in C(\mathbf{W})$, $\hat{C}(\boldsymbol{e})\bigl((w, \hat{P})\bigr) = (x, \hat{Q}) \in C(\mathbf{X})$, $\hat{C}(\boldsymbol{f})\bigl((w, \hat{P})\bigr) = (y, \hat{R}) \in C(\mathbf{Y})$, *and* $\hat{C}(\boldsymbol{g})\bigl((x, \hat{Q})\bigr) = \hat{C}(\boldsymbol{h})\bigl((y, \hat{R})\bigr) = (z, \hat{S}) \in C(\mathbf{Z})$*, then* $|\hat{P}| + |\hat{S}| = |\hat{Q}| + |\hat{R}|$*, and we have equivalences in* $\mathbf{dSpa^c}$*:*

$$C_i(\mathbf{W}) \simeq \coprod_{j,k,l \geqslant 0: i=j+k-l} \hat{C}_j^{\boldsymbol{g},l}(\mathbf{X}) \times_{\hat{C}_j^l(\boldsymbol{g}), C_l(\mathbf{Z}), \hat{C}_k^l(\boldsymbol{h})} \hat{C}_k^{\boldsymbol{h},l}(\mathbf{Y}). \tag{6.128}$$

Proof. By Proposition 6.49 $C(\boldsymbol{g}), C(\boldsymbol{h})$ are c-transverse, so by Theorem 6.42 a fibre product $C(\mathbf{X}) \times_{C(\boldsymbol{g}), C(\mathbf{Z}), C(\boldsymbol{h})} C(\mathbf{Y})$ exists in $\mathbf{dSpa^c}$, which we write as

$C'(\mathbf{W})$, with projections $C'(\boldsymbol{e}) : C'(\mathbf{W}) \to C(\mathbf{X})$ and $C'(\boldsymbol{f}) : C'(\mathbf{W}) \to C(\mathbf{Y})$, and a 2-morphism $C'(\eta) : C(\boldsymbol{g}) \circ C'(\boldsymbol{e}) \Rightarrow C(\boldsymbol{h}) \circ C'(\boldsymbol{f})$. As (6.124) is 2-commutative, the universal property of this fibre product gives a 1-morphism $\boldsymbol{b} : C(\mathbf{W}) \to C'(\mathbf{W})$ and 2-morphisms $\zeta : C'(\boldsymbol{e}) \circ \boldsymbol{b} \Rightarrow C(\boldsymbol{e})$, $\theta : C'(\boldsymbol{f}) \circ \boldsymbol{b} \Rightarrow C(\boldsymbol{f})$.

Similarly, $\hat{C}(\boldsymbol{g}), \hat{C}(\boldsymbol{h})$ are c-transverse, so $\hat{C}'(\mathbf{W}) = C(\mathbf{X}) \times_{\hat{C}(\boldsymbol{g}), C(\mathbf{Z}), \hat{C}(\boldsymbol{h})} C(\mathbf{Y})$ exists in $\mathbf{dSpa^c}$ with 1-morphisms $\hat{C}'(\boldsymbol{e}) : \hat{C}'(\mathbf{W}) \to C(\mathbf{X})$, $\hat{C}'(\boldsymbol{f}) : \hat{C}'(\mathbf{W}) \to C(\mathbf{Y})$, $\hat{\boldsymbol{b}} : C(\mathbf{W}) \to \hat{C}'(\mathbf{W})$ and 2-morphisms $\hat{C}'(\eta) : \hat{C}(\boldsymbol{g}) \circ \hat{C}'(\boldsymbol{e}) \Rightarrow \hat{C}(\boldsymbol{h}) \circ \hat{C}'(\boldsymbol{f})$, $\hat{\zeta} : \hat{C}'(\boldsymbol{e}) \circ \hat{\boldsymbol{b}} \Rightarrow \hat{C}(\boldsymbol{e})$, $\hat{\theta} : \hat{C}'(\boldsymbol{f}) \circ \hat{\boldsymbol{b}} \Rightarrow \hat{C}(\boldsymbol{f})$. To show that (6.124)–(6.125) are 2-Cartesian, we must prove that $\boldsymbol{b}, \hat{\boldsymbol{b}}$ are equivalences in $\mathbf{dSpa^c}$.

By Corollary 6.43, as $C(\boldsymbol{g}), C(\boldsymbol{h})$ are c-transverse the d-space $\boldsymbol{C'(W)}$ in $C'(\mathbf{W})$ is a fibre product $\boldsymbol{C(X)} \times_{\boldsymbol{C(Z)}} \boldsymbol{C(Y)}$ in $\mathbf{dSpa}$, and so by §2.5 is also a fibre product on the level of sets, $C'(W) \cong C(X) \times_{C(Z)} C(Y)$. We will prove that on points (6.124) induces a Cartesian square in **Sets**, so that $C(W) \cong C(X) \times_{C(Z)} C(Y)$. This will imply that $b : C(W) \to C'(W)$ is a bijection, so $\boldsymbol{b}$ is a bijection on points. It is enough to check this on the fibres over each $w \in W$, since $W \cong X \times_Z Y$ is Cartesian in **Sets** by the same argument. That is, for each $w \in \underline{W}$ with $\underline{e}(w) = x \in \underline{X}$, $\underline{f}(w) = y \in \underline{Y}$ and $\underline{g}(x) = \underline{h}(y) = z \in \underline{Z}$, we must prove that the commutative diagram

$$\begin{tikzcd}
\bigl\{(w,P) : P \subseteq \underline{i}_{\mathbf{W}}^{-1}(w)\bigr\} \arrow[r, "C(\boldsymbol{f})|_{\Pi_{\mathbf{W}}^{-1}(w)}"'] \arrow[d, "C(\boldsymbol{e})|_{\Pi_{\mathbf{W}}^{-1}(w)}"] & \bigl\{(y,R) : R \subseteq \underline{i}_{\mathbf{Y}}^{-1}(y)\bigr\} \arrow[d, "C(\boldsymbol{h})|_{\Pi_{\mathbf{Y}}^{-1}(y)}"'] \\
\bigl\{(x,Q) : Q \subseteq \underline{i}_{\mathbf{X}}^{-1}(x)\bigr\} \arrow[r, "C(\boldsymbol{g})|_{\Pi_{\mathbf{X}}^{-1}(x)}"] & \bigl\{(z,S) : S \subseteq \underline{i}_{\mathbf{Z}}^{-1}(z)\bigr\}
\end{tikzcd} \tag{6.129}$$

is Cartesian in **Sets**.

Consider the graph $\Gamma_{x,y}$ of Definition 6.36. As $\boldsymbol{g}, \boldsymbol{h}$ are c-transverse, by Proposition 6.38, every connected component $\hat{\Gamma}$ of $\Gamma_{x,y}$ is of type (A) or (B) in Proposition 6.37. The construction of c-transverse fibre products in §6.8.3–§6.8.4 implies that there is a 1-1 correspondence w' between points of $\underline{i}_{\mathbf{W}}^{-1}(w)$ and type (A) components $\hat{\Gamma}$ of $\Gamma_{x,y}$. We can also describe the maps in (6.129) in terms of $\Gamma_{x,y}$. If $P \subseteq \underline{i}_{\mathbf{W}}^{-1}(w)$ then $C(\boldsymbol{e}) : (w,P) \mapsto (x,Q)$ and $C(\boldsymbol{f}) : (w,P) \mapsto (y,R)$, where Q, R are the sets of vertices x', y' respectively in type (A) components $\hat{\Gamma}$ corresponding to $w' \in P$. If $Q \subseteq \underline{i}_{\mathbf{X}}^{-1}(x)$ then $C(\boldsymbol{g}) : (x,Q) \mapsto (z,S)$, where S is the set of z' lying on edges $\overset{x'}{\bullet} - \overset{z'}{\bullet}$ in $\Gamma_{x,y}$ for $x' \in Q$. If $R \subseteq \underline{i}_{\mathbf{Y}}^{-1}(y)$ then $C(\boldsymbol{h}) : (y,R) \mapsto (z,S)$, with S the set of z' lying on $\overset{y'}{\bullet} - \overset{z'}{\bullet}$ for $y' \in R$.

Using this graph description, it is easy to see that (6.129) is Cartesian in sets, and therefore $\boldsymbol{b} : C(\mathbf{W}) \to C'(\mathbf{W})$ is a bijection on points. A similar argument shows that the analogue of (6.129) for $\hat{C}$ is Cartesian, and so $\hat{\boldsymbol{b}} : C(\mathbf{W}) \to \hat{C}'(\mathbf{W})$ is a bijection on points. In this case, if $\hat{P} \subseteq \underline{i}_{\mathbf{W}}^{-1}(w)$ then $\hat{C}(\boldsymbol{e}) : (w,\hat{P}) \mapsto (x,\hat{Q})$ and $\hat{C}(\boldsymbol{f}) : (w,\hat{P}) \mapsto (y,\hat{R})$, where $\hat{Q}, \hat{R}$ are the sets of vertices x', y' in type (A) $\hat{\Gamma}$ corresponding to $w' \in P$, plus those x', y' in any type (B) $\hat{\Gamma}$. If $\hat{Q} \subseteq \underline{i}_{\mathbf{X}}^{-1}(x)$ then $\hat{C}(\boldsymbol{g}) : (x,\hat{Q}) \mapsto (z,\hat{S})$, with $\hat{S}$ the set of z' lying on $\overset{x'}{\bullet} - \overset{z'}{\bullet}$ for $x' \in \hat{Q}$, plus those z' not on any $\overset{x''}{\bullet} - \overset{z'}{\bullet}$, and $\hat{C}(\boldsymbol{h})$ is similar.

By the same kind of method, and using ideas in the proof of Proposition

6.49, we can show that $C(\boldsymbol{b}) : C(C(\mathbf{W})) \to C(C'(\mathbf{W}))$ and $C(\hat{\boldsymbol{b}}) : C(C(\mathbf{W})) \to C(\hat{C}'(\mathbf{W}))$ are bijections on points. It follows that $\boldsymbol{b}, \hat{\boldsymbol{b}}$ are simple and flat.

We can also use these ideas to prove $|P|+|S| = |Q|+|R|$ and $|\hat{P}|+|\hat{S}| = |\hat{Q}|+|\hat{R}|$ in the second and third paragraphs of the theorem. Suppose $C(\boldsymbol{e})\big((w,P)\big) = (x,Q)$, $C(\boldsymbol{f})\big((w,P)\big) = (y,R)$, and $C(\boldsymbol{g})\big((x,Q)\big) = C(\boldsymbol{h})\big((y,R)\big) = (z,S)$. Then Q, R, S are the sets of vertices x', y', z' respectively in type (A) components $\hat{\Gamma}$ in $\Gamma_{x,y}$ corresponding to $w' \in P$. By Proposition 6.37(A), any such $\hat{\Gamma}$ has j, k, l vertices of type x', y', z' respectively, where $1 + l = j + k$. Summing $1 + l = j + k$ over all $\hat{\Gamma}$ corresponding to $w' \in P$ gives $|P| + |S| = |Q| + |R|$.

Similarly, suppose $\hat{C}(\boldsymbol{e})\big((w,\hat{P})\big) = (x,\hat{Q})$, $\hat{C}(\boldsymbol{f})\big((w,\hat{P})\big) = (y,\hat{R})$, and $\hat{C}(\boldsymbol{g})\big((x,\hat{Q})\big) = \hat{C}(\boldsymbol{h})\big((y,\hat{R})\big) = (z,\hat{S})$. Then $\hat{Q}, \hat{R}, \hat{S}$ are the sets of vertices x', y', z' respectively in type (A) components $\hat{\Gamma}$ corresponding to $w' \in \hat{P}$, plus those in all type (B) components $\hat{\Gamma}$. By Proposition 6.37(B), any type (B) component $\hat{\Gamma}$ has j, k, l vertices of type x', y', z', where $0 + l = j + k$. Summing $1 + l = j + k$ over type (A) $\hat{\Gamma}$ corresponding to $w' \in \hat{P}$, and summing $0 + l = j + k$ over all type (B) $\hat{\Gamma}$, gives $|\hat{P}| + |\hat{S}| = |\hat{Q}| + |\hat{R}|$.

Next we prove that $\boldsymbol{b} : \boldsymbol{C(W)} \to \boldsymbol{C'(W)}$ and $\hat{\boldsymbol{b}} : \boldsymbol{C(W)} \to \boldsymbol{\hat{C}'(W)}$ are étale 1-morphisms in **dSpa**. This is a local condition, so it is enough to prove it for the explicit local fibre products of §6.8.3. Work in the situation of §6.8.3, supposing in addition that $\boldsymbol{g}, \boldsymbol{h}$ are c-transverse so that there are no type (C) components $\hat{\Gamma}$ in $\Gamma_{x,y}$. Then $x \in \mathbf{R} \subseteq \mathbf{X}$, $y \in \mathbf{S} \subseteq \mathbf{Y}$, $z \in \mathbf{T} \subseteq \mathbf{Z}$, and we construct an explicit fibre product $\mathbf{Q} = \mathbf{R} \times_{\mathbf{T}} \mathbf{S}$ in $\mathbf{dSpa^c}$. By definition $\boldsymbol{i}_{\mathbf{Q}}^{-1}(w) = I \amalg J$ for subsets $I \subseteq \underline{i}_{\mathbf{X}}^{-1}(x)$ and $J \subseteq \underline{i}_{\mathbf{X}}^{-1}(y)$. Let $A \subseteq I \amalg J$, so that $(w,A) \in C_i(\mathbf{Q})$ for $i = |A|$, and let $C(\boldsymbol{e})\big((w,A)\big) = (x,B) \in C_j(\mathbf{R})$ for $j = |B|$, $C(\boldsymbol{f})\big((w,A)\big) = (y,C) \in C_k(\mathbf{S})$ for $k = |C|$, and $C(\boldsymbol{g})\big((x,B)\big) = C(\boldsymbol{g})\big((y,C)\big) = (z,D) \in C_l(\mathbf{T})$ for $l = |D|$, so that $B \subseteq \underline{i}_{\mathbf{X}}^{-1}(x)$, $C \subseteq \underline{i}_{\mathbf{Y}}^{-1}(y)$ and $D \subseteq \underline{i}_{\mathbf{Z}}^{-1}(z)$. Write $A = A_I \amalg A_J$ where $A_I = A \cap I$ and $A_J = A \cap J$. Then B, C, D are the sets of vertices x', y', z' in type (A) components $\hat{\Gamma}$ of $\Gamma_{x,y}$ containing a vertex x' or y' in A, and $A_I \subseteq B$, $A_J \subseteq C$.

Define open neighbourhoods $(w,A) \in \mathbf{M} \subseteq C_i(\mathbf{Q})$, $(x,B) \in \mathbf{N} \subseteq C_j(\mathbf{R})$, $(y,C) \in \mathbf{O} \subseteq C_k(\mathbf{S})$ and $(z,D) \in \mathbf{P} \subseteq C_l(\mathbf{T})$ to have open sets

$$
\begin{aligned}
M = \big\{ &\big((\tilde{x},\tilde{y}), \{(\tilde{x}'_a,\tilde{y}) : a \in A_I\} \amalg \{(\tilde{x},\tilde{y}'_a) : a \in A_J\}\big) : (\tilde{x},\tilde{y}) \in \mathbf{Q}, \\
&\tilde{x}'_a \in \boldsymbol{i}_{\mathbf{X}}^{-1}(\tilde{x}) \cap \boldsymbol{B}_a,\ \ a \in A_I,\ \ \tilde{y}'_a \in \boldsymbol{i}_{\mathbf{Y}}^{-1}(\tilde{y}) \cap \boldsymbol{C}_a,\ \ a \in A_J \big\}, \\
N = \big\{ &(\tilde{x}, \{\tilde{x}'_b : b \in B\}) : \tilde{x} \in \mathbf{R},\ \ \tilde{x}'_b \in \boldsymbol{i}_{\mathbf{X}}^{-1}(\tilde{x}) \cap \boldsymbol{B}_b,\ \ b \in B \big\}, \\
O = \big\{ &(\tilde{y}, \{\tilde{y}'_b : c \in C\}) : \tilde{y} \in \mathbf{S},\ \ \tilde{y}'_c \in \boldsymbol{i}_{\mathbf{Y}}^{-1}(\tilde{y}) \cap \boldsymbol{C}_c,\ \ c \in C \big\}, \\
P = \big\{ &(\tilde{z}, \{\tilde{z}'_b : d \in D\}) : \tilde{z} \in \mathbf{T},\ \ \tilde{z}'_d \in \boldsymbol{i}_{\mathbf{Z}}^{-1}(\tilde{z}) \cap \boldsymbol{D}_d,\ \ d \in D \big\}.
\end{aligned}
$$

As for (6.72), the d-spaces $\boldsymbol{M},\boldsymbol{N},\boldsymbol{O},\boldsymbol{P}$ fit into 2-Cartesian squares in $\mathbf{dSpa}$:

$$\begin{tikzcd} \boldsymbol{M} \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{\Pi}^i_{\mathbf{W}}|_{\boldsymbol{M}}"'] & \arrow[d, phantom, "\mathrm{id}_{\mathbf{0}\circ\boldsymbol{\pi}}\Uparrow"] & {*} \arrow[d, "\mathbf{0}"] \\ \boldsymbol{Q} \arrow[rr, "\prod_{a\in A_I}\boldsymbol{r}_a\circ\boldsymbol{e}\times\prod_{a\in A_J}\boldsymbol{s}_a\circ\boldsymbol{f}"'] & & {[\mathbf{0},\infty)^{\boldsymbol{i}}}, \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{N} \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{\Pi}^j_{\mathbf{X}}|_{\boldsymbol{N}}"'] & \arrow[d, phantom, "\mathrm{id}_{\mathbf{0}\circ\boldsymbol{\pi}}\Uparrow"] & {*} \arrow[d, "\mathbf{0}"] \\ \boldsymbol{R} \arrow[rr, "\prod_{b\in B}\boldsymbol{r}_b"'] & & {[\mathbf{0},\infty)^{\boldsymbol{j}}}, \end{tikzcd}$$

$$\begin{tikzcd} \boldsymbol{O} \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{\Pi}^k_{\mathbf{Y}}|_{\boldsymbol{O}}"'] & \arrow[d, phantom, "\mathrm{id}_{\mathbf{0}\circ\boldsymbol{\pi}}\Uparrow"] & {*} \arrow[d, "\mathbf{0}"] \\ \boldsymbol{S} \arrow[rr, "\prod_{c\in C}\boldsymbol{s}_c"'] & & {[\mathbf{0},\infty)^{\boldsymbol{k}}}, \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{P} \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{\Pi}^l_{\mathbf{Z}}|_{\boldsymbol{P}}"'] & \arrow[d, phantom, "\mathrm{id}_{\mathbf{0}\circ\boldsymbol{\pi}}\Uparrow"] & {*} \arrow[d, "\mathbf{0}"] \\ \boldsymbol{T} \arrow[rr, "\prod_{d\in D}\boldsymbol{t}_d"'] & & {[\mathbf{0},\infty)^{\boldsymbol{l}}}. \end{tikzcd}$$

Therefore $\boldsymbol{b}|_{\boldsymbol{M}}$ maps $\boldsymbol{M}\to\boldsymbol{N}\times_{\boldsymbol{P}}\boldsymbol{O}$, where

$$\begin{aligned} \boldsymbol{M}&\simeq\bigl(\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{S}\bigr)\times_{\prod_{a\in A_I}\boldsymbol{r}_a\circ\boldsymbol{\pi}_{\boldsymbol{R}}\times\prod_{a\in A_J}\boldsymbol{s}_a\circ\boldsymbol{\pi}_{\boldsymbol{S}},[\mathbf{0},\infty)^{\boldsymbol{i}},\mathbf{0}}*\\ &\simeq\bigl(\boldsymbol{R}\times_{\prod_{a\in A_I}\boldsymbol{r}_a,[\mathbf{0},\infty)^{|\boldsymbol{A_I}|},\mathbf{0}}*\bigr)\times_{\boldsymbol{T}}\bigl(\boldsymbol{S}\times_{\prod_{a\in A_J}\boldsymbol{s}_a,[\mathbf{0},\infty)^{|\boldsymbol{A_J}|},\mathbf{0}}*\bigr), \end{aligned}\tag{6.130}$$

$$\begin{aligned} \boldsymbol{N}\times_{\boldsymbol{P}}\boldsymbol{O}&\simeq\bigl(\boldsymbol{R}\times_{\prod_{b\in B}\boldsymbol{r}_b,[\mathbf{0},\infty)^{\boldsymbol{j}},\mathbf{0}}*\bigr)\times_{\boldsymbol{T}\times_{\prod_{d\in D}\boldsymbol{t}_d,[\mathbf{0},\infty)^{\boldsymbol{l}},\mathbf{0}}*}\bigl(\boldsymbol{S}\times_{\prod_{c\in C}\boldsymbol{s}_c,[\mathbf{0},\infty)^{\boldsymbol{k}},\mathbf{0}}*\bigr)\\ &\simeq\bigl(\boldsymbol{R}\times_{\prod_{a\in A_I}\boldsymbol{r}_a,[\mathbf{0},\infty)^{|\boldsymbol{A_I}|},\mathbf{0}}*\bigr)\times_{\boldsymbol{T}}\bigl(\boldsymbol{S}\times_{\prod_{a\in A_J}\boldsymbol{s}_a,[\mathbf{0},\infty)^{|\boldsymbol{A_J}|},\mathbf{0}}*\bigr). \end{aligned}\tag{6.131}$$

Here in the first step of (6.130) we identify $\boldsymbol{Q},\boldsymbol{e},\boldsymbol{f}$ with $\boldsymbol{R}\times_{\boldsymbol{T}}\boldsymbol{S},\boldsymbol{\pi}_{\boldsymbol{R}},\boldsymbol{\pi}_{\boldsymbol{S}}$, and in the second we transfer the fibre products by $\boldsymbol{r}_a\circ\boldsymbol{\pi}_{\boldsymbol{R}}$ onto $\boldsymbol{R}$, and the fibre products by $\boldsymbol{s}_a\circ\boldsymbol{\pi}_{\boldsymbol{S}}$ onto $\boldsymbol{S}$.

In the second step of (6.131), we argue as follows. Define $\tilde\Gamma$ to be the subgraph of $\Gamma_{x,y}$ with vertices $B\amalg C\amalg D$. Then $\tilde\Gamma$ is a disjoint union of type (A) components $\hat\Gamma$ of $\Gamma_{x,y}$, where each $\hat\Gamma$ has a unique vertex x_1' or y_1' in $A\subseteq I\amalg J$. By choice of $I\amalg J$, each such $\hat\Gamma$ is either a single vertex x_1' or y_1' in A, or the distinguished vertex x_1' or y_1' lies on exactly one edge. Write $\tilde\Gamma'$ for the subgraph of $\tilde\Gamma$ obtained by deleting all vertices x_1',y_1' in A, and all edges $\overset{x_1'}{\bullet}-\overset{z_1'}{\bullet}$ and $\overset{y_1'}{\bullet}-\overset{z_1'}{\bullet}$ for $x_1',y_1'\in A$. Then $\tilde\Gamma'$ has vertices $(B\setminus A_I)\amalg(C\setminus A_J)\amalg D$.

In the proof of Proposition 6.37 we defined a colouring of $\tilde\Gamma'$ into black and white edges, where every vertex x',y',z' in $\tilde\Gamma'$ lies on exactly one white edge. Thus, the white edges in $\hat\Gamma'$ give a 1-1 correspondence between D and $(B\setminus A_I)\amalg(C\setminus A_J)$. Let $\overset{x'}{\bullet}-\overset{z'}{\bullet}$ be a white edge in $\tilde\Gamma'$, so that $(x',z')\in\underline{S}_{\boldsymbol{g}}$, and $\boldsymbol{t}_{z'}\circ\boldsymbol{g}|_{\boldsymbol{R}}=\boldsymbol{p}_{x'z'}\cdot\boldsymbol{r}_{x'}$ for $\boldsymbol{p}_{x'z'}:\boldsymbol{R}\to(\mathbf{0},\infty)$ by Definition 6.40. Therefore, in the first term $\boldsymbol{R}\times_{[\mathbf{0},\infty)^{\boldsymbol{j}}}*$ of the r.h.s. of the first line of (6.131), the fibre product $\times_{\boldsymbol{r}_{x'},[\mathbf{0},\infty)}*$ may be replaced by $\times_{\boldsymbol{t}_{z'}\circ\boldsymbol{g}|_{\boldsymbol{R}},[\mathbf{0},\infty)}*$ without changing it. Then the first term $\boldsymbol{R}\times_{[\mathbf{0},\infty)^{\boldsymbol{j}}}*$ contains $\times_{\boldsymbol{t}_{z'}\circ\boldsymbol{g}|_{\boldsymbol{R}},[\mathbf{0},\infty),\mathbf{0}}*$, and the second term $\boldsymbol{T}\times_{[\mathbf{0},\infty)^{\boldsymbol{l}}}*$ contains $\times_{\boldsymbol{t}_{z'},[\mathbf{0},\infty),\mathbf{0}}*$. We may omit both of these without changing the fibre product up to equivalence.

Similarly, if $\overset{y'}{\bullet}-\overset{z'}{\bullet}$ is a white edge in $\hat\Gamma'$ then we may simultaneously omit $\times_{\boldsymbol{t}_{z'},[\mathbf{0},\infty),\mathbf{0}}*$ from the second term $\boldsymbol{T}\times_{[\mathbf{0},\infty)^{\boldsymbol{l}}}*$, and $\times_{\boldsymbol{s}_{y'},[\mathbf{0},\infty)}*$ from the third term $\boldsymbol{S}\times_{[\mathbf{0},\infty)^{\boldsymbol{k}}}*$, without changing it. Omitting such pairs from all white edges in $\tilde{\Gamma}'$ gives the second line of (6.131). Comparing (6.130) and (6.131) gives an equivalence $\boldsymbol{M} \simeq \boldsymbol{N} \times_{\boldsymbol{P}} \boldsymbol{O}$, which is $\boldsymbol{b}|_{\boldsymbol{M}} : \boldsymbol{M} \to \boldsymbol{N} \times_{\boldsymbol{P}} \boldsymbol{O}$. So $\boldsymbol{b} : \boldsymbol{C(W)} \to \boldsymbol{C'(W)}$ is a local equivalence, that is, it is étale.

We have now proved that $\boldsymbol{b} : C(\mathbf{W}) \to C'(\mathbf{W})$ is a bijection, simple, and flat, and $\boldsymbol{b} : \boldsymbol{C(W)} \to \boldsymbol{C'(W)}$ is étale in $\mathbf{dSpa}$. Proposition 6.21 now shows that $\boldsymbol{b}$ is an equivalence in $\mathbf{dSpa^c}$. Hence (6.124) is 2-Cartesian in $\mathbf{dSpa^c}$. The equality $|P|+|S|=|Q|+|R|$ proved above shows that (6.124) decomposes into equations (6.126) for each $i \geqslant 0$. Since $\partial\mathbf{W} \simeq C_1(\mathbf{W})$, the case $i=1$ in (6.126) gives (6.127). A similar proof shows $\hat{\boldsymbol{b}} : \boldsymbol{C(W)} \to \boldsymbol{\hat{C}'(W)}$ is étale, (6.125) is 2-Cartesian, and (6.128) holds. This completes the proof of Theorem 6.50. □

Remark 6.51. If $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ are b-transverse, but not c-transverse, then $C(\boldsymbol{g}), C(\boldsymbol{h})$ and $\hat{C}(\boldsymbol{g}), \hat{C}(\boldsymbol{h})$ are both b-transverse, so that fibre products $C(\mathbf{X}) \times_{C(\boldsymbol{g}),C(\mathbf{Z}),C(\boldsymbol{h})} C(\mathbf{Y})$ and $C(\mathbf{X}) \times_{\hat{C}(\boldsymbol{g}),C(\mathbf{Z}),\hat{C}(\boldsymbol{h})} C(\mathbf{Y})$ exist in $\mathbf{dSpa^c}$. However, we do not have equivalences $C(\mathbf{W}) \simeq C(\mathbf{X}) \times_{C(\boldsymbol{g}),C(\mathbf{Z}),C(\boldsymbol{h})} C(\mathbf{Y})$ and $C(\mathbf{W}) \simeq C(\mathbf{X}) \times_{\hat{C}(\boldsymbol{g}),C(\mathbf{Z}),\hat{C}(\boldsymbol{h})} C(\mathbf{Y})$, where $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$.

Instead, $C(\mathbf{W})$ is naturally equivalent to open, closed, proper d-subspaces of $C(\mathbf{X}) \times_{C(\boldsymbol{g}),C(\mathbf{Z}),C(\boldsymbol{h})} C(\mathbf{Y})$ and $C(\mathbf{X}) \times_{\hat{C}(\boldsymbol{g}),C(\mathbf{Z}),\hat{C}(\boldsymbol{h})} C(\mathbf{Y})$. For $w \in \mathbf{W}$, the number k of type (C) components $\hat{\Gamma}$ in $\Gamma_{x,y}$ for $x = \boldsymbol{e}(w)$ and $y = \boldsymbol{f}(w)$ is locally constant as a function of w, and locally $C(\mathbf{X}) \times_{C(\boldsymbol{g}),C(\mathbf{Z}),C(\boldsymbol{h})} C(\mathbf{Y})$ and $C(\mathbf{X}) \times_{\hat{C}(\boldsymbol{g}),C(\mathbf{Z}),\hat{C}(\boldsymbol{h})} C(\mathbf{Y})$ may be thought of as 2^k copies of $C(\mathbf{W})$. When $k=0$, so that $2^k=1$, there are no type (C) components $\hat{\Gamma}$, so $\boldsymbol{g}, \boldsymbol{h}$ are c-transverse by Propositions 6.37 and 6.38.

The expression (6.127) for the boundary $\partial(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})$ of a c-transverse fibre product is rather complicated. Also, when we discuss orientations of d-manifolds with corners in §7.8, equation (6.127) will not be helpful, as if $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are oriented d-manifolds with corners, $C^j(\mathbf{X}), C^k(\mathbf{Y}), C^l(\mathbf{Z})$ are generally not oriented for $j,k,l \geqslant 2$. Here are some special cases in which we can give alternative expressions for $\partial(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})$, which are suitable for working with orientations.

Theorem 6.52. *Let $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms of d-spaces with corners. Then the following hold, where all fibre products in* (6.132)–(6.138) *are c-transverse, and so exist:*

(a) *If $\partial\mathbf{Z} = \emptyset$ then there is an equivalence*

$$\partial\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \simeq \big(\partial\mathbf{X} \times_{\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{X}},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \amalg \big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}} \partial\mathbf{Y}\big). \tag{6.132}$$

(b) *If $\boldsymbol{g}$ is semisimple and flat then there is an equivalence*

$$\partial\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \simeq \big(\partial_+^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_+,\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \amalg \big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}} \partial\mathbf{Y}\big). \tag{6.133}$$

(c) *If $\boldsymbol{g}$ is simple and flat then there is an equivalence*

$$\partial\big(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}\big) \simeq \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\mathbf{Y}}} \partial\mathbf{Y}. \tag{6.134}$$

(d) *If $\boldsymbol{h}$ is semisimple and flat then there is an equivalence*

$$\partial\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr)\simeq\bigl(\partial\mathbf{X}\times_{\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{X}},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr)\amalg\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}_+}\partial_+^{\boldsymbol{h}}\mathbf{Y}\bigr). \tag{6.135}$$

(e) *If $\boldsymbol{h}$ is simple and flat then there is an equivalence*

$$\partial\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr)\simeq\partial\mathbf{X}\times_{\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{X}},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}. \tag{6.136}$$

(f) *If both $\boldsymbol{g}$ and $\boldsymbol{h}$ are semisimple and flat then there is an equivalence*

$$\begin{aligned}\partial\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr)\simeq\bigl(\partial_+^{\boldsymbol{g}}\mathbf{X}\times_{\boldsymbol{g}_+,\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr)\amalg\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}_+}\partial_+^{\boldsymbol{h}}\mathbf{Y}\bigr)&\\ \amalg\bigl(\partial_-^{\boldsymbol{g}}\mathbf{X}\times_{\boldsymbol{g}_-,\partial\mathbf{Z},\boldsymbol{h}_-}\partial_-^{\boldsymbol{h}}\mathbf{Y}\bigr).&\end{aligned} \tag{6.137}$$

(g) *If both $\boldsymbol{g}$ and $\boldsymbol{h}$ are simple and flat then there is an equivalence*

$$\partial\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr)\simeq\partial\mathbf{X}\times_{\boldsymbol{g}_-,\partial\mathbf{Z},\boldsymbol{h}_-}\partial\mathbf{Y}. \tag{6.138}$$

Proof. Parts (a),(f),(g) follow immediately from Proposition 6.30, Lemma 6.35, and equation (6.127) of Theorem 6.50. Parts (b)–(e) are not equivalent to (6.127). Instead, we can prove them directly from the local construction of fibre products in §6.8.3.

In case (b), in the situation of §6.8.3, $\boldsymbol{g}$ semisimple and flat implies that every type (A) component $\hat\Gamma$ of $\Gamma_{x,y}$ is either (i) a single vertex x', where $x'\in\partial_+^{\boldsymbol{g}}\mathbf{X}$; or (ii) contains a unique vertex y'. Every $y'\in\underline{i}_{\mathbf{Y}}^{-1}(y)$ lies in a unique type (A) component $\hat\Gamma$ as in (ii). Also, every type (B) component $\hat\Gamma$ is a single edge $\overset{x'}{\bullet}-\overset{z'}{\bullet}$, with $(y,z')\in\underline{T}_{\boldsymbol{h}}$. We choose $I\amalg J$ so that I is all vertices x' of type (i), and $J=\underline{i}_{\mathbf{Y}}^{-1}(y)$ is the set of vertices y' in (ii). Then $\boldsymbol{\partial Q}$ in (6.89) corresponds to the right hand side of (6.133), and a proof similar to Theorem 6.50 gives the result. In case (c) $\partial_+^{\boldsymbol{g}}\mathbf{X}=\boldsymbol{\emptyset}$ as $\boldsymbol{g}$ is simple, so (6.134) follows from (6.133). For (d),(e) we exchange $\boldsymbol{g},\boldsymbol{h}$ in (b),(c). $\square$

Using Theorem 6.52 we show that if $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ is semisimple and flat then $\partial_-^{\boldsymbol{f}}\mathbf{X}\simeq\mathbf{X}\times_{\mathbf{Y}}\partial\mathbf{Y}$. If $\boldsymbol{f}$ is also simple then $\partial^k\mathbf{X}\simeq\mathbf{X}\times_{\mathbf{Y}}\partial^k\mathbf{Y}$ for all $k\geqslant 0$.

Proposition 6.53. *Suppose $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ is semisimple and flat in* $\mathbf{dSpa^c}$*. Then using the notation of* §6.3*, the following diagram is* 2*-Cartesian in* $\mathbf{dSpa^c}$:

$$\begin{tikzcd} \partial_-^{\boldsymbol{f}}\mathbf{X} \arrow[r, "\boldsymbol{f}_-"] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}}"'] \arrow[dr, phantom, "\Uparrow\mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{f}_-}"] & \partial\mathbf{Y} \arrow[d, "\boldsymbol{i}_{\mathbf{Y}}"] \\ \mathbf{X} \arrow[r, "\boldsymbol{f}"] & \mathbf{Y}, \end{tikzcd} \tag{6.139}$$

so that $\partial_-^{\boldsymbol{f}}\mathbf{X}\simeq\mathbf{X}\times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\mathbf{Y}}}\partial\mathbf{Y}$. If $\boldsymbol{f}$ is also simple then (6.139) *becomes*

$$\begin{tikzcd} \partial\mathbf{X} \arrow[r, "\boldsymbol{f}_-"] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}"'] \arrow[dr, phantom, "\Uparrow\mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{f}_-}"] & \partial\mathbf{Y} \arrow[d, "\boldsymbol{i}_{\mathbf{Y}}"] \\ \mathbf{X} \arrow[r, "\boldsymbol{f}"] & \mathbf{Y}, \end{tikzcd} \tag{6.140}$$

so that $\partial\mathbf{X} \simeq \mathbf{X} \times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\mathbf{Y}}} \partial\mathbf{Y}$*. Also for all* $k \geqslant 0$ *we have equivalences*

$$\partial^k\mathbf{X} \simeq \mathbf{X} \times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{i}_{\partial\mathbf{Y}}\circ\cdots\circ\boldsymbol{i}_{\partial^{k-1}\mathbf{Y}}} \partial^k\mathbf{Y}. \tag{6.141}$$

Proof. Apply Theorem 6.52(b) with $\boldsymbol{Z} = \boldsymbol{Y}$, $\boldsymbol{g} = \boldsymbol{f}$ and $\boldsymbol{h} = \mathbf{id}_{\boldsymbol{Y}}$. Then $\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y} = \mathbf{X} \times_{\boldsymbol{f},\mathbf{Y},\mathbf{id}_{\mathbf{Y}}} \mathbf{Y} \simeq \mathbf{X}$, so the left hand side of (6.133) is $\partial\mathbf{X}$. The two terms on the right hand side of (6.133) are equivalent to $\partial_+^{\boldsymbol{f}}\mathbf{X}$ and $\partial_-^{\boldsymbol{f}}\mathbf{X}$ respectively. Equation (6.139) expresses the resulting expression for $\partial_-^{\boldsymbol{f}}\mathbf{X}$ as a 2-Cartesian diagram. The 2-morphism in (6.139) is the identity as $\boldsymbol{f}\circ\boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}} = \boldsymbol{i}_{\mathbf{Y}}\circ\boldsymbol{f}_-$ by Theorem 6.12(b).

When $\boldsymbol{f}$ is simple $\partial_-^{\boldsymbol{f}}\mathbf{X} = \partial\mathbf{X}$, so (6.139) reduces to (6.140). Also in this case $\boldsymbol{f}_-$ is simple and flat by Theorem 6.12(b). Hence

$$\begin{aligned}\partial(\partial\mathbf{X}) &\simeq \partial\mathbf{X} \times_{\boldsymbol{f}_-,\partial\mathbf{Y},\boldsymbol{i}_{\partial\mathbf{Y}}} \partial^2\mathbf{Y} \simeq \bigl(\mathbf{X} \times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\mathbf{Y}}} \partial\mathbf{Y}\bigr) \times_{\boldsymbol{\pi}_{\partial\mathbf{Y}},\partial\mathbf{Y},\boldsymbol{i}_{\partial\mathbf{Y}}} \partial^2\mathbf{Y}\\ &\simeq \mathbf{X} \times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\partial\mathbf{Y}}\circ\boldsymbol{i}_{\mathbf{Y}}} \partial^2\mathbf{Y},\end{aligned}$$

using natural equivalences of fibre products. This proves the case $k = 2$ of (6.141). The general case follows in the same way by induction on k. $\square$

6.10 Fixed points of finite groups in d-spaces with corners

When a finite group Γ acts on a d-space $\boldsymbol{X}$ by 1-isomorphisms, in §2.7 we defined the fixed d-subspace $\boldsymbol{X}^\Gamma$ of Γ in $\boldsymbol{X}$, a d-space with an inclusion 1-morphism $\boldsymbol{j}_{\boldsymbol{X},\Gamma} : \boldsymbol{X}^\Gamma \hookrightarrow \boldsymbol{X}$, whose topological space X^Γ is the fixed point locus of Γ in X. We now generalize this to d-spaces with corners.

The unusual part of the next definition is the choice of boundary d-space $\boldsymbol{\partial}(\boldsymbol{X}^\Gamma)$ in $\mathbf{X}^\Gamma = (\boldsymbol{X}^\Gamma, \boldsymbol{\partial}(\boldsymbol{X}^\Gamma), \boldsymbol{i}_{\mathbf{X}^\Gamma}, \omega_{\mathbf{X}^\Gamma})$. This is modelled on the formula (5.10) in Proposition 5.18(c) for the boundary $\partial(X^\Gamma)$ of the fixed locus X^Γ of a finite group Γ acting on a manifold with corners X.

Definition 6.54. Let $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ be a d-space with corners, and Γ a finite group. An *action* $\boldsymbol{r}$ *of* Γ *on* $\mathbf{X}$ is an action $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\mathbf{X})$ of Γ on $\mathbf{X}$ by 1-isomorphisms in $\mathbf{dSpa^c}$. Sections 6.2 and 6.7 define the boundary $\partial\mathbf{X} = (\boldsymbol{\partial X}, \boldsymbol{\partial^2 X}, \boldsymbol{i}_{\partial\mathbf{X}}, \omega_{\partial\mathbf{X}})$ and corners $C(\mathbf{X}) = \bigl(\boldsymbol{C}(\boldsymbol{X}), \boldsymbol{\partial C}(\boldsymbol{X}), \boldsymbol{i}_{C(\mathbf{X})}, \omega_{C(\mathbf{X})}\bigr)$ of $\mathbf{X}$. As $\boldsymbol{r}(\gamma) : \mathbf{X} \to \mathbf{X}$ is simple for each $\gamma \in \Gamma$, Theorems 6.12(b) and 6.29(a) lift $\boldsymbol{r}$ to actions $\boldsymbol{r}_- : \Gamma \to \mathrm{Aut}(\partial\mathbf{X})$ and $C(\boldsymbol{r}) : \Gamma \to \mathrm{Aut}(C(\mathbf{X}))$ of G on $\partial\mathbf{X}$ and $C(\mathbf{X})$ in $\mathbf{dSpa^c}$.

Hence $\boldsymbol{r}, \boldsymbol{r}_-, C(\boldsymbol{r})$ are also actions $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\boldsymbol{X})$, $\boldsymbol{r}_- : \Gamma \to \mathrm{Aut}(\boldsymbol{\partial X})$, $C(\boldsymbol{r}) : \Gamma \to \mathrm{Aut}(\boldsymbol{C}(\boldsymbol{X}))$ in $\mathbf{dSpa}$ of Γ on the d-spaces $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{C}(\boldsymbol{X})$. Therefore Definition 2.43 defines d-spaces $\boldsymbol{X}^\Gamma, (\boldsymbol{\partial X})^\Gamma, \boldsymbol{C}(\boldsymbol{X})^\Gamma$ which are the fixed loci of Γ in $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{C}(\boldsymbol{X})$, with inclusion 1-morphisms $\boldsymbol{j}_{\boldsymbol{X},\Gamma} : \boldsymbol{X}^\Gamma \to \boldsymbol{X}$, $\boldsymbol{j}_{\boldsymbol{\partial X},\Gamma} : (\boldsymbol{\partial X})^\Gamma \to \boldsymbol{\partial X}$, $\boldsymbol{j}_{\boldsymbol{C}(\boldsymbol{X}),\Gamma} : \boldsymbol{C}(\boldsymbol{X})^\Gamma \to \boldsymbol{C}(\boldsymbol{X})$. Since $\boldsymbol{i}_{\mathbf{X}} : \boldsymbol{\partial X} \to \boldsymbol{X}$ and $\boldsymbol{\Pi}_{\mathbf{X}} : \boldsymbol{C}(\boldsymbol{X}) \to \boldsymbol{X}$ are Γ-equivariant, Proposition 2.45 gives unique 1-morphisms $\boldsymbol{i}_{\mathbf{X}}^\Gamma : (\boldsymbol{\partial X})^\Gamma \to \boldsymbol{X}^\Gamma$ and $\boldsymbol{\Pi}_{\mathbf{X}}^\Gamma : \boldsymbol{C}(\boldsymbol{X})^\Gamma \to \boldsymbol{X}^\Gamma$ such that $\boldsymbol{j}_{\boldsymbol{\partial X},\Gamma}\circ\boldsymbol{i}_{\mathbf{X}}^\Gamma = \boldsymbol{i}_{\mathbf{X}}\circ\boldsymbol{j}_{\boldsymbol{X},\Gamma}$ and $\boldsymbol{j}_{\boldsymbol{C}(\boldsymbol{X}),\Gamma}\circ\boldsymbol{\Pi}_{\mathbf{X}}^\Gamma = \boldsymbol{\Pi}_{\mathbf{X}}\circ\boldsymbol{j}_{\boldsymbol{\partial X},\Gamma}$.

We will define a d-space with corners $\mathbf{X}^\Gamma = (\boldsymbol{X}^\Gamma, \boldsymbol{\partial}(\boldsymbol{X}^\Gamma), \boldsymbol{i}_{\mathbf{X}^\Gamma}, \omega_{\mathbf{X}^\Gamma})$, which we call the *fixed d-subspace of* Γ *in* $\mathbf{X}$, with a natural inclusion 1-morphism $\boldsymbol{j}_{\mathbf{X},\Gamma} : \mathbf{X}^\Gamma \to \mathbf{X}$ in $\mathbf{dSpa^c}$. Here the d-space $\boldsymbol{X}^\Gamma$ and 1-morphism $\boldsymbol{j}_{\mathbf{X},\Gamma} = \boldsymbol{j}_{\boldsymbol{X},\Gamma} : \boldsymbol{X}^\Gamma \to \boldsymbol{X}$ are from Definition 2.43 for the d-space $\boldsymbol{X}$, and $\boldsymbol{\partial}(\boldsymbol{X}^\Gamma), \boldsymbol{i}_{\mathbf{X}^\Gamma}, \omega_{\mathbf{X}^\Gamma}$ remain to be defined. We do not take $\boldsymbol{\partial}(\boldsymbol{X}^\Gamma) = (\boldsymbol{\partial X})^\Gamma$ and $\boldsymbol{i}_{\mathbf{X}^\Gamma} = i^\Gamma_{\mathbf{X}}$, though these would seem to be obvious choices. This is because, as in Proposition 5.18 and Example 5.19, if Γ acts on a manifold with corners X then in general $\partial(X^\Gamma) \not\cong (\partial X)^\Gamma$, but instead $\partial(X^\Gamma)$ is given in terms of $C(X)$ by (5.10).

As in (6.69), points of $\boldsymbol{C(X)}$ are of the form $(x, \{x'_1, \ldots, x'_k\})$ for $x \in \boldsymbol{X}$, $\{x'_1, \ldots, x'_k\} \subseteq \boldsymbol{i}_{\mathbf{X}}^{-1}(x) \subseteq \boldsymbol{\partial X}$ and $k \geqslant 0$, and Γ acts on these by

$$C(\boldsymbol{r})(\gamma) : (x, \{x'_1, \ldots, x'_k\}) \longmapsto \big(\boldsymbol{r}(\gamma)(x), \{\boldsymbol{r}_-(\gamma)(x'_1), \ldots, \boldsymbol{r}_-(\gamma)(x'_k)\}\big).$$

Thus points of $\boldsymbol{C(X)}^\Gamma$ are $(x, \{x'_1, \ldots, x'_k\})$ fixed by this Γ-action. That is, x must be fixed by $\boldsymbol{r}(\Gamma)$, i.e. $x \in \boldsymbol{X}^\Gamma$, and the subset $\{x'_1, \ldots, x'_k\}$ in $\boldsymbol{\partial X}$ must be fixed by $\boldsymbol{r}_-(\Gamma)$, although $\boldsymbol{r}_-(\Gamma)$ may permute $x'_1, \ldots, x'_k$.

Following (5.10), define $\boldsymbol{\partial}(\boldsymbol{X}^\Gamma)$ to be the open and closed d-subspace of $\boldsymbol{C(X)}^\Gamma$ consisting of points $(x, \{x'_1, \ldots, x'_k\})$ such that $k \geqslant 1$ and $\boldsymbol{r}_-(\Gamma)$ acts transitively on $\{x'_1, \ldots, x'_k\}$, and define a 1-morphism $\boldsymbol{i}_{\mathbf{X}^\Gamma} : \boldsymbol{\partial}(\boldsymbol{X}^\Gamma) \to \boldsymbol{X}^\Gamma$ in $\mathbf{dSpa}$ by $\boldsymbol{i}_{\mathbf{X}^\Gamma} = \boldsymbol{\Pi}^\Gamma_{\mathbf{X}}|_{\boldsymbol{\partial}(\boldsymbol{X}^\Gamma)}$. Define $\nu_{\mathbf{X}^\Gamma} : \mathcal{N}_{\mathbf{X}^\Gamma} \to \underline{i}^*_{\mathbf{X}^\Gamma}(\mathcal{F}_{X^\Gamma})$ to be the kernel of $i^2_{\mathbf{X}^\Gamma}$, giving a complex in $\mathrm{qcoh}(\underline{\partial X}^\Gamma)$:

$$0 \longrightarrow \mathcal{N}_{\mathbf{X}^\Gamma} \xrightarrow{\nu_{\mathbf{X}^\Gamma}} \underline{i}^*_{\mathbf{X}^\Gamma}(\mathcal{F}_{X^\Gamma}) \xrightarrow{i^2_{\mathbf{X}^\Gamma}} \mathcal{F}_{\partial X^\Gamma} \longrightarrow 0. \tag{6.142}$$

We claim that $\mathcal{N}_{\mathbf{X}^\Gamma}$ is a line bundle on $\underline{\partial X}^\Gamma$, and that there is a canonical orientation $\omega_{\mathbf{X}^\Gamma}$ on $\mathcal{N}_{\mathbf{X}^\Gamma}$ such that $\mathbf{X}^\Gamma = (\boldsymbol{X}^\Gamma, \boldsymbol{\partial}(\boldsymbol{X}^\Gamma), \boldsymbol{i}_{\mathbf{X}^\Gamma}, \omega_{\mathbf{X}^\Gamma})$ is a d-space with corners. To prove this, we will lift from $C_k(\mathbf{X}) \cong (\partial^k \mathbf{X})/S_k$ to $\partial^k \mathbf{X}$.

From §6.1, equation (6.4) is split exact in $\mathrm{qcoh}(\partial\mathcal{X})$. Hence $\underline{i}^*_{\mathbf{X}}(\mathcal{F}_X) \cong \mathcal{N}_{\mathbf{X}} \oplus \mathcal{F}_{\partial X}$. Pulling back by $\underline{i}_{\partial\mathbf{X}} : \partial^2\mathcal{X} \to \partial\mathcal{X}$ and using $I_{\underline{i}_{\partial\mathbf{X}}, \underline{i}_{\mathbf{X}}}(\mathcal{F}_X)$ gives

$$(\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\mathcal{F}_X) \cong \underline{i}^*_{\partial\mathbf{X}}(\mathcal{N}_{\mathbf{X}}) \oplus \underline{i}^*_{\partial\mathbf{X}}(\mathcal{F}_{\partial X}) \quad \text{in } \mathrm{qcoh}(\partial^2\mathcal{X}).$$

Combining this with (6.4) split exact for $\partial\mathbf{X}$ yields

$$(\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}})^*(\mathcal{F}_X) \cong \underline{i}^*_{\partial\mathbf{X}}(\mathcal{N}_{\mathbf{X}}) \oplus \mathcal{N}_{\partial\mathbf{X}} \oplus \mathcal{F}_{\partial^2 X} \quad \text{in } \mathrm{qcoh}(\partial^2\mathcal{X}).$$

Continuing by induction shows that

$$\begin{aligned}(\underline{i}_{\mathbf{X}} \circ \underline{i}_{\partial\mathbf{X}} \circ \cdots \circ \underline{i}_{\partial^{k-1}\mathbf{X}})^*(\mathcal{F}_X) &\cong (\underline{i}_{\partial\mathbf{X}} \circ \cdots \circ \underline{i}_{\partial^{k-1}\mathbf{X}})^*(\mathcal{N}_{\mathbf{X}}) \oplus \cdots \\ &\oplus \underline{i}^*_{\partial^{k-1}\mathbf{X}}(\mathcal{N}_{\partial^{k-2}\mathbf{X}}) \oplus \mathcal{N}_{\partial^{k-1}\mathbf{X}} \oplus \mathcal{F}_{\partial^k X} \quad \text{in } \mathrm{qcoh}(\underline{\partial^k X}).\end{aligned}$$

Doing the same argument with split exact sequences rather than direct sums shows that we have a split exact sequence in $\mathrm{qcoh}(\underline{\partial^k X})$:

$$0 \to \begin{array}{c}\bigoplus_{i=0}^{k-1}(\underline{i}_{\partial\mathbf{X}} \circ \cdots \\ \circ\, \underline{i}_{\partial^{k-1-i}\mathbf{X}})^*(\mathcal{N}_{\partial^i\mathbf{X}})\end{array} \to \begin{array}{c}(\underline{i}_{\mathbf{X}} \circ \cdots \\ \circ\, \underline{i}_{\partial^{k-1}\mathbf{X}})^*(\mathcal{F}_X)\end{array} \xrightarrow{(i_{\mathbf{X}} \circ \cdots \circ i_{\partial^{k-1}\mathbf{X}})^2} \mathcal{F}_{\partial^k X} \to 0. \tag{6.143}$$

Define K_k, κ_k to be the kernel of $(\Pi^k_{\mathbf{X}})^2$ in $\mathrm{qcoh}(\underline{C_k(X)})$, giving a complex

$$0 \longrightarrow K_k \xrightarrow{\kappa_k} (\underline{\Pi}^k_{\mathbf{X}})^*(\mathcal{F}_X) \xrightarrow{(\Pi^k_{\mathbf{X}})^2} \mathcal{F}_{C_k(X)} \longrightarrow 0. \tag{6.144}$$

Recall from Definition 6.28 that $C_k(\mathbf{X}) \cong (\partial^k\mathbf{X})/S_k$, and we have a local 1-isomorphism $\boldsymbol{q}_{\mathbf{X}} : \partial^k\mathbf{X} \to C_k(\mathbf{X})$ with $\mathbf{\Pi}^k_{\mathbf{X}} \circ \boldsymbol{q}_{\mathbf{X}} = \boldsymbol{i}_{\mathbf{X}} \circ \cdots \circ \boldsymbol{i}_{\partial^{k-1}\mathbf{X}}$. Pulling (6.144) back to $\mathrm{qcoh}(\underline{\partial^k X})$ using $\underline{q}_{\mathbf{X}} : \underline{\partial^k X} \to \underline{C_k(X)}$, and noting that $q^2_{\mathbf{X}} : \underline{q}^*_{\mathbf{X}}(\mathcal{F}_{C_k(X)}) \to \mathcal{F}_{\partial^k X}$ is an isomorphism as $\boldsymbol{q}_{\mathbf{X}}$ is a local 1-isomorphism, we see that the pullback of (6.144) by $\underline{q}_{\mathbf{X}}$ is isomorphic to (6.143). As (6.143) is split exact, which is an étale local condition, and $\underline{q}_{\mathbf{X}}$ is surjective, this implies that (6.144) is split exact. Also we have

$$\underline{q}^*_{\mathbf{X}}(K_k) \cong \textstyle\bigoplus_{i=0}^{k-1} (\underline{i}_{\partial\mathbf{X}} \circ \cdots \circ \underline{i}_{\partial^{k-1-i}\mathbf{X}})^*(\mathcal{N}_{\partial^i\mathbf{X}}). \tag{6.145}$$

As in Definition 2.43, for any Γ-equivariant quasicoherent sheaf on $\underline{C_k(X)}$, its pullback to $\mathrm{qcoh}(\underline{C_k(X)}^\Gamma)$ by $\underline{j}_{\boldsymbol{C_k(X)},\Gamma} : \underline{C_k(X)}^\Gamma \to \underline{C_k(X)}$ has a natural Γ-action, and so splits into trivial and nontrivial Γ-representations. Thus

$$\begin{aligned}
\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}[K_k] &= \underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}[K_k]_{\mathrm{tr}} \oplus \underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}[K_k]_{\mathrm{nt}},\\
\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}\big[\mathcal{F}_{C_k(X)}\big] &= \underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}\big[\mathcal{F}_{C_k(X)}\big]_{\mathrm{tr}} \oplus \underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}\big[\mathcal{F}_{C_k(X)}\big]_{\mathrm{nt}},\\
\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}\big[(\underline{\Pi}^k_{\mathbf{X}})^*(\mathcal{F}_X)\big] &= \underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}\big[(\underline{\Pi}^k_{\mathbf{X}})^*(\mathcal{F}_X)\big]_{\mathrm{tr}} \oplus \underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}\big[(\underline{\Pi}^k_{\mathbf{X}})^*(\mathcal{F}_X)\big]_{\mathrm{nt}},
\end{aligned}$$

and (6.144) splits into 'trivial' and 'nontrivial' split exact sequences. Also by definition of $\boldsymbol{X}^\Gamma, \boldsymbol{C_k(X)}^\Gamma$ in Definition 2.43 we have

$$\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}\big[(\underline{\Pi}^k_{\mathbf{X}})^*(\mathcal{F}_X)\big]_{\mathrm{tr}} \cong ((\underline{\Pi}^k_{\mathbf{X}})^\Gamma)^*(\mathcal{F}_{X^\Gamma}), \quad \underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}\big[\mathcal{F}_{C_k(X)}\big]_{\mathrm{tr}} \cong \mathcal{F}_{C_k(X)^\Gamma}.$$

Taking trivial parts of (6.144) and using the last two equations now shows that we have a split exact sequence in $\mathrm{qcoh}(\underline{C_k(X)}^\Gamma)$:

$$0 \longrightarrow \underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}(K_k)_{\mathrm{tr}} \longrightarrow ((\underline{\Pi}^k_{\mathbf{X}})^\Gamma)^*(\mathcal{F}_{X^\Gamma}) \xrightarrow{((\Pi^k_{\mathbf{X}})^\Gamma)^2} \mathcal{F}_{C_k(X)^\Gamma} \longrightarrow 0. \tag{6.146}$$

As above, $\boldsymbol{\partial(X^\Gamma)}$ is open and closed in $\boldsymbol{C(X)}^\Gamma$, and $\boldsymbol{i}_{\mathbf{X}^\Gamma} = \mathbf{\Pi}^\Gamma_{\mathbf{X}}|_{\boldsymbol{\partial(X^\Gamma)}}$. Thus, the restrictions of (6.142) and (6.146) to $\underline{\partial(X^\Gamma)} \cap \underline{C_k(X)}^\Gamma$ are isomorphic. This proves that (6.142) is split exact, as (6.146) is, and

$$\mathcal{N}_{\mathbf{X}^\Gamma}\big|_{\underline{\partial(X^\Gamma)}\cap\underline{C_k(X)}^\Gamma} \cong \underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}(K_k)_{\mathrm{tr}}\big|_{\underline{\partial(X^\Gamma)}\cap\underline{C_k(X)}^\Gamma}. \tag{6.147}$$

By (6.145), $\underline{q}^*_{\mathbf{X}}(K_k)$ is the direct sum of k line bundles on $\underline{\partial^k X}$, so it is a rank k vector bundle. Hence K_k and $\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}(K_k)$ are also rank k vector bundles on $\underline{C_k(X)}$ and $\underline{C_k(X)}^\Gamma$ respectively, and locally the direct sum of k line bundles.

To describe $\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}(K_k)_{\mathrm{tr}}$ we must understand how Γ acts on these k line local bundles on $\underline{C_k(X)}^\Gamma$. At a point $(x, \{x'_1, \ldots, x'_k\})$ in $\underline{C_k(X)}^\Gamma$, we just have

$$\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}(K_k)|_{(x,\{x'_1,\ldots,x'_k\})} \cong \textstyle\bigoplus_{i=1}^k \mathcal{N}_{\mathbf{X}}|_{x'_i},$$

and Γ acts by permuting the factors of $\mathcal{N}_{\mathbf{X}}|_{x'_1},\ldots,\mathcal{N}_{\mathbf{X}}|_{x'_k}$ in the same way that it permutes $x'_1,\ldots,x'_k$. Therefore the fixed subspace of Γ in $\bigoplus_{i=1}^k \mathcal{N}_{\mathbf{X}}|_{x'_i}$ is the sum of one copy of $\mathbb{R}$ for each orbit of Γ in $\{x'_1,\ldots,x'_k\}$. By definition of $\boldsymbol{\partial}(\boldsymbol{X}^\Gamma)$, for $(x,\{x'_1,\ldots,x'_k\})\in \underline{\partial(X^\Gamma)}\cap\underline{C_k(X)}^\Gamma$ the Γ-action on $\{x'_1,\ldots,x'_k\}$ is transitive. This implies that $\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}(K_k)_{\rm tr}|_{\underline{\partial(X^\Gamma)}\cap\underline{C_k(X)}^\Gamma}$ is a line bundle. So $\mathcal{N}_{\mathbf{X}^\Gamma}$ in (6.142) is a line bundle, as we have to prove.

We can also use these ideas to define the orientation $\omega_{\mathbf{X}^\Gamma}$ on $\mathcal{N}_{\mathbf{X}^\Gamma}$. In (6.145), the $\mathcal{N}_{\partial^i\mathbf{X}}$ have orientations $\omega_{\partial^i\mathbf{X}}$, so $\underline{q}^*_{\mathbf{X}}(K_k)$ is the sum of k oriented line bundles. Thus $\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}(K_k)$ is locally the sum of k oriented line bundles, and Γ acts on $\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}(K_k)$ by permuting these line bundles transitively, preserving the orientations. There is then a unique orientation $\omega_{\mathbf{X}^\Gamma}$ on $\mathcal{N}_{\mathbf{X}^\Gamma}\cong \underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}(K_k)_{\rm tr}$ such that the projection from $\underline{j}^*_{\boldsymbol{C_k(X)},\Gamma}(K_k)_{\rm tr}$ to each oriented local line bundle is orientation-preserving.

This completes the definition of $\mathbf{X}^\Gamma=(\boldsymbol{X}^\Gamma,\boldsymbol{\partial}(\boldsymbol{X}^\Gamma),\boldsymbol{i}_{\mathbf{X}^\Gamma},\omega_{\mathbf{X}^\Gamma})$. To show $\mathbf{X}^\Gamma$ is a d-space with corners, we must verify Definition 6.1(a)–(f). Part (d) is done. For (a), we have $\underline{\Pi}^k_{\mathbf{X}}\circ\underline{q}_{\mathbf{X}}=\underline{i}_{\mathbf{X}}\circ\cdots\circ\underline{i}_{\partial^{k-1}\mathbf{X}}$, where $\underline{i}_{\mathbf{X}},\ldots,\underline{i}_{\partial^{k-1}\mathbf{X}}$ are proper by Definition 6.1(a), and $\underline{q}_{\mathbf{X}}$ is surjective, so $\underline{\Pi}^k_{\mathbf{X}}:\underline{C_k(X)}\to\underline{X}$ is proper. Since $\partial^k\mathbf{X}=\boldsymbol{\emptyset}$ for $k\gg 0$ locally in $\mathbf{X}$, it follows that $\underline{\Pi}_{\mathbf{X}}:\underline{C(X)}\to\underline{X}$ is proper. Hence $\underline{i}_{\mathbf{X}^\Gamma}$ is proper, as it is the restriction of $\underline{\Pi}_{\mathbf{X}}$ to a closed C^∞-subscheme.

For (b), using $\boldsymbol{\Pi}^k_{\mathbf{X}}\circ\boldsymbol{q}_{\mathbf{X}}=\boldsymbol{i}_{\mathbf{X}}\circ\cdots\circ\boldsymbol{i}_{\partial^{k-1}\mathbf{X}}$ and omitting isomorphisms $I_{*,*}(*)$ for simplicity gives an equation in morphisms in $\mathrm{qcoh}(\underline{C_k(X)})$:

$$q''_{\mathbf{X}}\circ\underline{q}^*_{\mathbf{X}}(\Pi^{k\prime\prime}_{\mathbf{X}})=i''_{\partial^{k-1}\mathbf{X}}\circ\cdots\circ(\underline{i}_{\partial\mathbf{X}}\circ\cdots\circ\underline{i}_{\partial^{k-1}\mathbf{X}})^*(i''_{\mathbf{X}}):(\underline{\Pi}^k_{\mathbf{X}}\circ\underline{q}_{\mathbf{X}})^*(\mathcal{E}_X)\to\mathcal{E}_{\partial^kX}.$$

Since $i''_{\partial^j\mathbf{X}}$ is an isomorphism by Definition 6.1(b), and $q''_{\mathbf{X}}$ is an isomorphism as $\boldsymbol{q}_{\mathbf{X}}$ is a 1-isomorphism, this shows $\underline{q}^*_{\mathbf{X}}(\Pi^{k\prime\prime}_{\mathbf{X}})$ is an isomorphism. Because $\underline{q}_{\mathbf{X}}$ is étale and surjective, $\Pi^{k\prime\prime}_{\mathbf{X}}$ is an isomorphism, for each $k\geqslant 0$. Hence $\Pi''_{\mathbf{X}}$ is an isomorphism, and so $i''_{\mathbf{X}^\Gamma}=\Pi''_{\mathbf{X}}|_{\underline{\partial(X^\Gamma)}}$ is.

For (c) and (e), let $(x,\{x'_1,\ldots,x'_k\})\in\boldsymbol{\partial}(\boldsymbol{X}^\Gamma)$. Then x is Γ-invariant, $k\geqslant 1$, and Γ acts transitively on $\{x'_1,\ldots,x'_k\}$. Let Δ be the subgroup of Γ fixing x'_1, so that $|\Gamma|/|\Delta|=k$. Let $(\boldsymbol{V},\boldsymbol{b}_1)$ be a boundary defining function for $\mathbf{X}$ at x'_1. We can choose $\boldsymbol{V},\boldsymbol{b}_1$ so that $\boldsymbol{V}$ is Γ-invariant, and $\boldsymbol{b}_1$ is Δ-invariant, that is, $\boldsymbol{b}_1\circ\boldsymbol{r}(\delta)|_{\boldsymbol{V}}=\boldsymbol{b}_1$ for all $\delta\in\Delta$. For each $j=2,\ldots,k$, let $\gamma_j\in\Gamma$ with $\boldsymbol{r}_-(\gamma_j):x'_1\mapsto x'_j$, and set $\boldsymbol{b}_j=\boldsymbol{b}_1\circ\boldsymbol{r}(\gamma_j^{-1})|_{\boldsymbol{V}}$. Then $(\boldsymbol{V},\boldsymbol{b}_j)$ is a boundary defining function for $\mathbf{X}$ at x'_j, for all $j=1,\ldots,k$.

As for (6.72), making $\boldsymbol{V}$ smaller if necessary, there exists an open neighbourhood $\boldsymbol{U}$ of $(x,\{x'_1,\ldots,x'_k\})$ in $(\boldsymbol{\Pi}^k_{\mathbf{X}})^{-1}(\boldsymbol{V})\subseteq\boldsymbol{C_k(X)}$ in a 2-Cartesian square

$$\begin{tikzcd}
\boldsymbol{U} \arrow[rrr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{\Pi}^k_{\mathbf{X}}|_{\boldsymbol{U}}"'] & \arrow[r, phantom, "\mathrm{id}_{(\boldsymbol{0},\ldots,\boldsymbol{0})\circ\boldsymbol{\pi}}\ \Uparrow"] & & \boldsymbol{*} \arrow[d, "(\boldsymbol{0},\ldots,\boldsymbol{0})"] \\
\boldsymbol{V} \arrow[rrr, "(\boldsymbol{b}_1,\boldsymbol{b}_2,\ldots,\boldsymbol{b}_k)"'] & & & {[\boldsymbol{0},\boldsymbol{\infty})^{\boldsymbol{k}}}
\end{tikzcd} \tag{6.148}$$

in **dSpa**. Making $\boldsymbol{U},\boldsymbol{V}$ smaller if necessary, we can also suppose $\boldsymbol{U},\boldsymbol{V}$ are Γ-invariant, and $\boldsymbol{U}\subseteq\boldsymbol{\partial}(\boldsymbol{X}^\Gamma)$. Now Γ acts on the whole diagram (6.148), so we may restrict to Γ-invariant d-subspaces. Since Γ acts on $[0,\infty)^k$ by permuting the coordinates $x_1, \ldots, x_k$ transitively, the fixed d-subspace $([\mathbf{0},\boldsymbol{\infty})^{\boldsymbol{k}})^\Gamma$ is $[0,\infty)$, diagonally embedded in $[\mathbf{0},\boldsymbol{\infty})^{\boldsymbol{k}}$. Thus the Γ-invariant part of (6.148) is

$$\begin{tikzcd}
\boldsymbol{U}^\Gamma \arrow[rrr, "\boldsymbol{\pi}"] \arrow[d, "(\boldsymbol{\Pi}^k_{\mathbf{X}})^\Gamma|_{\boldsymbol{U}^\Gamma}"'] \arrow[drrr, phantom, "\Uparrow \mathrm{id}_{\mathbf{0}\circ\boldsymbol{\pi}}"] & & & {*} \arrow[d, "\mathbf{0}"] \\
\boldsymbol{V}^\Gamma \arrow[rrr, "\boldsymbol{b}"] & & & {[\mathbf{0},\boldsymbol{\infty}),}
\end{tikzcd} \tag{6.149}$$

where $\boldsymbol{b} = \boldsymbol{b}_1 \circ \boldsymbol{j}_{\boldsymbol{V},\Gamma} = \cdots = \boldsymbol{b}_k \circ \boldsymbol{j}_{\boldsymbol{V},\Gamma}$.

Since (6.149) is the Γ-invariant part of the 2-Cartesian diagram (6.148), we can show (6.149) is 2-Cartesian. As $(\boldsymbol{V},\boldsymbol{b}_i)$ is a boundary defining function for $\mathbf{X}$ at x'_i, by Definition 6.1(c) b_i^2 has a left inverse β_i in qcoh$(\underline{V})$ for $i=1,\ldots,k$, and pulling these back by $\underline{i}_{\boldsymbol{V},\Gamma}$ we can show b^2 has a left inverse β in qcoh$(\underline{V}^\Gamma)$. Hence Definition 6.1(c) holds for $\mathbf{X}^\Gamma$ at $(x,\{x'_1,\ldots,x'_k\})$. The construction of the orientation $\omega_{\mathbf{X}^\Gamma}$ above then implies that Definition 6.1(e) holds, so $(\boldsymbol{V}^\Gamma,\boldsymbol{b})$ is a boundary defining function for $\mathbf{X}^\Gamma$ at $(x,\{x'_1,\ldots,x'_k\})$. For Definition 6.1(f), note that equation (6.8) for $\mathbf{X}^\Gamma$ at $x \in \boldsymbol{X}^\Gamma$ is isomorphic to the Γ-invariant part of (6.8) for $\mathbf{X}$ at $x \in \boldsymbol{X}$. Thus (f) for $\mathbf{X}$ at x implies (f) for $\mathbf{X}^\Gamma$ at x.

Therefore $\mathbf{X}^\Gamma$ is a d-space with corners. Above, $(\boldsymbol{V},\boldsymbol{b}_i)$ was a boundary defining function for $\mathbf{X}$ at x'_i, and $(\boldsymbol{V}^\Gamma,\boldsymbol{b}) = \big(\boldsymbol{j}_{\boldsymbol{X},\Gamma}^{-1}(\boldsymbol{V}),\boldsymbol{b}_i\circ\boldsymbol{j}_{\boldsymbol{X},\Gamma}|_{\boldsymbol{j}_{\boldsymbol{X},\Gamma}^{-1}(\boldsymbol{V})}\big)$ a boundary defining function for $\mathbf{X}^\Gamma$ at $(x,\{x'_1,\ldots,x'_k\})$. From this we see that $\boldsymbol{j}_{\mathbf{X},\Gamma} = \boldsymbol{j}_{\boldsymbol{X},\Gamma} : \mathbf{X}^\Gamma \to \mathbf{X}$ is a flat 1-morphism in $\mathbf{dSpa^c}$.

As for Propositions 2.44 and 2.45, we have:

Proposition 6.55. *Let $\mathbf{X},\Gamma,\boldsymbol{r},\mathbf{X}^\Gamma$ and $\boldsymbol{j}_{\mathbf{X},\Gamma} : \mathbf{X}^\Gamma \to \mathbf{X}$ be as in Definition* 6.54. *Suppose $\boldsymbol{f} : \mathbf{W} \to \mathbf{X}$ is a* 1*-morphism in* $\mathbf{dSpa^c}$. *Then $\boldsymbol{f}$ factorizes as $\boldsymbol{f} = \boldsymbol{j}_{\mathbf{X},\Gamma}\circ\boldsymbol{g}$ for some* 1*-morphism $\boldsymbol{g} : \mathbf{W} \to \mathbf{X}^\Gamma$ in* $\mathbf{dSpa^c}$, *which must be unique, if and only if $\boldsymbol{r}(\gamma)\circ\boldsymbol{f} = \boldsymbol{f}$ for all $\gamma \in \Gamma$.*

Proposition 6.56. *Suppose $\mathbf{X},\mathbf{Y}$ are d-spaces with corners, Γ is a finite group, $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\mathbf{X})$, $\boldsymbol{s} : \Gamma \to \mathrm{Aut}(\mathbf{Y})$ are actions of Γ on $\mathbf{X},\mathbf{Y}$, and $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a Γ-equivariant* 1*-morphism in* $\mathbf{dSpa^c}$, *that is, $\boldsymbol{f}\circ\boldsymbol{r}(\gamma) = \boldsymbol{s}(\gamma)\circ\boldsymbol{f}$ for $\gamma \in \Gamma$. Then there exists a unique* 1*-morphism $\boldsymbol{f}^\Gamma : \mathbf{X}^\Gamma \to \mathbf{Y}^\Gamma$ such that $\boldsymbol{j}_{\mathbf{Y},\Gamma}\circ\boldsymbol{f}^\Gamma = \boldsymbol{f}\circ\boldsymbol{j}_{\mathbf{X},\Gamma}$.*

Now let $\boldsymbol{f},\boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ be Γ-equivariant 1*-morphisms and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a Γ-equivariant* 2*-morphism, that is, $\eta * \mathrm{id}_{\boldsymbol{r}(\gamma)} = \mathrm{id}_{\boldsymbol{s}(\gamma)} * \eta$ for $\gamma \in \Gamma$. Then there exists a unique* 2*-morphism $\eta^\Gamma : \boldsymbol{f}^\Gamma \Rightarrow \boldsymbol{g}^\Gamma$ such that $\mathrm{id}_{\boldsymbol{j}_{\mathbf{Y},\Gamma}} * \eta^\Gamma = \eta * \mathrm{id}_{\boldsymbol{j}_{\mathbf{X},\Gamma}}$.*

To prove Proposition 6.55, note that Proposition 2.44 gives a unique 1-morphism $\boldsymbol{g} : \boldsymbol{W} \to \boldsymbol{X}^\Gamma$ in $\mathbf{dSpa}$ if and only if $\boldsymbol{r}(\gamma)\circ\boldsymbol{f} = \boldsymbol{f}$ for all $\gamma \in \Gamma$, so we have only to show that this $\boldsymbol{g}$ is a 1-morphism in $\mathbf{dSpa^c}$. Similarly, to prove Proposition 6.56, observe that Proposition 2.45 gives unique $\boldsymbol{f}^\Gamma,\eta^\Gamma$ as 1- and 2-morphisms in $\mathbf{dSpa}$, so we have only to show that these $\boldsymbol{f}^\Gamma,\eta^\Gamma$ are 1- and 2-morphisms in $\mathbf{dSpa^c}$, that is, they satisfy the additional conditions of Definitions 6.2 and 6.3. In both cases this can be done by working with boundary defining functions as in Definition 6.54.

In Proposition 5.18 we saw that if a finite group Γ acts on a manifold with corners X then $C(X^\Gamma) \cong C(X)^\Gamma$. The analogue holds for d-spaces with corners.

Proposition 6.57. *Let* $\mathbf{X}$ *be a d-space with corners,* Γ *a finite group, and* $\boldsymbol{r}:\Gamma\to\mathrm{Aut}(\mathbf{X})$ *an action of* Γ *on* $\mathbf{X}$*. Applying the corner functor* C *of* §6.7 *gives an action* $C(\boldsymbol{r}):\Gamma\to\mathrm{Aut}(C(\mathbf{X}))$*. Hence Definition* 6.54 *defines fixed d-subspaces* $\mathbf{X}^\Gamma, C(\mathbf{X})^\Gamma$ *and inclusion* 1*-morphisms* $\boldsymbol{j}_{\mathbf{X},\Gamma}:\mathbf{X}^\Gamma\to\mathbf{X}$*,* $\boldsymbol{j}_{C(\mathbf{X}),\Gamma}: C(\mathbf{X})^\Gamma\to C(\mathbf{X})$*. Applying* C *to* $\boldsymbol{j}_{\mathbf{X},\Gamma}$ *also gives* $C(\boldsymbol{j}_{\mathbf{X},\Gamma}):C(\mathbf{X}^\Gamma)\to C(\mathbf{X})$*.*

Then there exists a unique equivalence $\boldsymbol{k}_{\mathbf{X},\Gamma}:C(\mathbf{X}^\Gamma)\to C(\mathbf{X})^\Gamma$ *in* $\mathbf{dSpa^c}$ *such that* $C(\boldsymbol{j}_{\mathbf{X},\Gamma})=\boldsymbol{j}_{C(\mathbf{X}),\Gamma}\circ\boldsymbol{k}_{\mathbf{X},\Gamma}$*.*

Proof. For each $\gamma\in\Gamma$ we have

$$C(\boldsymbol{r}(\gamma))\circ C(\boldsymbol{j}_{\mathbf{X},\Gamma})=C(\boldsymbol{r}(\gamma)\circ\boldsymbol{j}_{\mathbf{X},\Gamma})=C(\boldsymbol{j}_{\mathbf{X},\Gamma}),$$

as C is a strict 2-functor, and $\boldsymbol{r}(\gamma)\circ\boldsymbol{j}_{\mathbf{X},\Gamma}=\boldsymbol{j}_{\mathbf{X},\Gamma}$ as in §2.7. Hence Proposition 6.55 for $C(\mathbf{X})$ shows there exists a unique 1-morphism $\boldsymbol{k}_{\mathbf{X},\Gamma}:C(\mathbf{X}^\Gamma)\to C(\mathbf{X})^\Gamma$ with $C(\boldsymbol{j}_{\mathbf{X},\Gamma})=\boldsymbol{j}_{C(\mathbf{X}),\Gamma}\circ\boldsymbol{k}_{\mathbf{X},\Gamma}$.

Points of $C_k(\mathbf{X}^\Gamma)$ are $\bigl(x,\bigl\{(x,\{x'_{1,1},\ldots,x'_{1,a_1}\}),\ldots,(x,\{x'_{k,1},\ldots,x'_{k,a_k}\})\bigr\}\bigr)$, where $x\in X^\Gamma$ and $\{x'_{1,1},\ldots,x'_{1,a_1}\},\ldots,\{x'_{k,1},\ldots,x'_{k,a_k}\}$ are distinct (hence disjoint) nonempty Γ-orbits in $i_{\mathbf{X}}^{-1}(x)\subseteq\partial X$. Points of $C_l(\mathbf{X})^\Gamma$ are $(x,\{x'_1,\ldots,x'_l\})$ where $x\in X^\Gamma$ and $\{x'_1,\ldots,x'_l\}$ is a Γ-invariant subset of $i_{\mathbf{X}}^{-1}(x)\subseteq\partial X$. Then

$$\begin{aligned}\boldsymbol{k}_{\mathbf{X},\Gamma}:\bigl(x,\bigl\{(x,&\{x'_{1,1},\ldots,x'_{1,a_1}\}),\ldots,(x,\{x'_{k,1},\ldots,x'_{k,a_k}\})\bigr\}\bigr)\\ &\longmapsto\bigl(x,\{x'_{1,1},\ldots,x'_{1,a_1}\}\cup\cdots\cup\{x'_{k,1},\ldots,x'_{k,a_k}\}\bigr).\end{aligned}$$

This is a bijection, as for any $(x,\{x'_1,\ldots,x'_l\})$ in $C_l(\mathbf{X})^\Gamma$, the Γ-invariant subset $\{x'_1,\ldots,x'_l\}$ is the disjoint union of finitely many Γ-orbits which we write as $\{x'_{1,1},\ldots,x'_{1,a_1}\},\ldots,\{x'_{k,1},\ldots,x'_{k,a_k}\}$ with $a_1+\cdots+a_k=l$, and this gives a unique preimage in $C_k(\mathbf{X}^\Gamma)$. So $\boldsymbol{k}_{\mathbf{X},\Gamma}$ is a bijection on points.

To prove that $\boldsymbol{k}_{\mathbf{X},\Gamma}$ is étale (a local equivalence), we generalize the argument of (6.148)–(6.149). Let $\bigl(x,\bigl\{(x,\{x'_{1,1},\ldots,x'_{1,a_1}\}),\ldots,(x,\{x'_{k,1},\ldots,x'_{k,a_k}\})\bigr\}\bigr)$ be a point in $C_k(\mathbf{X}^\Gamma)$, with $l=a_1+\cdots+a_k$. As in Definition 6.54, we can choose Γ-invariant open neighbourhoods $\boldsymbol{V}$ of x in $\boldsymbol{X}$ and $\boldsymbol{U}$ of $\bigl(x,\{x'_{1,1},\ldots,x'_{1,a_1}\}\cup\cdots\cup\{x'_{k,1},\ldots,x'_{k,a_k}\}\bigr)$ in $\boldsymbol{C_l(X)}$ and 1-morphisms $\boldsymbol{b}_{i,j}:\boldsymbol{V}\to\boldsymbol{[0,\infty)}$ such that $(\boldsymbol{V},\boldsymbol{b}_{i,j})$ is a boundary defining function for $\mathbf{X}$ at $x'_{i,j}$, and the family of $\boldsymbol{b}_{i,j}$ is Γ-invariant, and as in (6.148) we have a 2-Cartesian diagram

$$\begin{tikzcd}[column sep=huge] \boldsymbol{U} \arrow[r, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{\Pi}^l_{\mathbf{X}}|_{\boldsymbol{U}}"'] \arrow[dr, phantom, "\mathrm{id}_{(\boldsymbol{0},\ldots,\boldsymbol{0})\circ\boldsymbol{\pi}}\ \Uparrow"] & * \arrow[d, "(\boldsymbol{0},\ldots,\boldsymbol{0})"] \\ \boldsymbol{V} \arrow[r, "(\boldsymbol{b}_{i,j}:i=1,\ldots,k,\ j=1,\ldots,a_i)"'] & \boldsymbol{[0,\infty)^l.} \end{tikzcd} \tag{6.150}$$

Then Γ acts on (6.150), and the Γ-invariant part is 2-Cartesian, as in (6.149):

$$\begin{tikzcd}[column sep=huge] \boldsymbol{U}^\Gamma \arrow[r, "\boldsymbol{\pi}"] \arrow[d, "(\boldsymbol{\Pi}^l_{\mathbf{X}})^\Gamma|_{\boldsymbol{U}^\Gamma}"'] \arrow[dr, phantom, "\mathrm{id}_{\boldsymbol{0}\circ\boldsymbol{\pi}}\ \Uparrow"] & * \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{V}^\Gamma \arrow[r, "(\boldsymbol{b}_1,\ldots,\boldsymbol{b}_k)"] & \boldsymbol{[0,\infty)^k,} \end{tikzcd} \tag{6.151}$$

where $\boldsymbol{b}_i = \boldsymbol{b}_{i,1} \circ \boldsymbol{j}_{\boldsymbol{V},\Gamma} = \cdots = \boldsymbol{b}_{i,a_i} \circ \boldsymbol{j}_{\boldsymbol{V},\Gamma}$ for $i = 1, \ldots, k$.

From Definition 6.54, $(\boldsymbol{V}^\Gamma, \boldsymbol{b}_i)$ is a boundary defining function for $\mathbf{X}^\Gamma$ at $(x, \{x'_{i,1}, \ldots, x'_{i,a_i}\})$. So by comparing (6.151) with the analogue of (6.150) for $\mathbf{X}^\Gamma$ at $\bigl(x, \bigl\{(x, \{x'_{1,1}, \ldots, x'_{1,a_1}\}), \ldots, (x, \{x'_{k,1}, \ldots, x'_{k,a_k}\})\bigr\}\bigr)$, and noting that both are 2-Cartesian in $\mathbf{dSpa^c}$, we see that neighbourhoods of $\bigl(x, \{x'_{1,1}, \ldots, x'_{1,a_1}\} \cup \cdots \cup \{x'_{k,1}, \ldots, x'_{k,a_k}\}\bigr)$ in $C_k(\mathbf{X}^\Gamma)$ and of $\bigl(x, \{x'_{1,1}, \ldots, x'_{1,a_1}\} \cup \cdots \cup \{x'_{k,1}, \ldots, x'_{k,a_k}\}\bigr)$ in $C_l(\mathbf{X})^\Gamma$ are equivalent. This equivalence is realized by $\boldsymbol{k}_{\mathbf{X},\Gamma}$, so $\boldsymbol{k}_{\mathbf{X},\Gamma}$ is étale. As it is also a bijection, it is an equivalence in $\mathbf{dSpa^c}$. □

7 D-manifolds with corners

We now define 2-categories $\mathbf{dMan^b}, \mathbf{dMan^c}$ of *d-manifolds with boundary*, and *d-manifolds with corners*, as full 2-subcategories of $\mathbf{dSpa^b}, \mathbf{dSpa^c}$ in Chapter 6, and generalize the material on d-manifolds in Chapters 3 and 4 to the boundary and corners case.

7.1 Defining d-manifolds with corners

Proposition 3.12 gave three equivalent characterizations (a)–(c) of principal d-manifolds $\boldsymbol{W}$ in $\mathbf{dMan}$. Our initial definition of principal d-manifold was option (a), a fibre product $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$ in $\mathbf{dSpa}$ with $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\in\hat{\mathbf{M}}\mathbf{an}$, but the most useful notion was option (c), a fibre product $\boldsymbol{V}_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}}\boldsymbol{V}$ for V a manifold, E a vector bundle and $s\in C^\infty(E)$, as we used these in 'standard model' d-manifolds $\boldsymbol{S}_{V,E,s}$.

If we try to generalize Proposition 3.12 to $\mathbf{Man^c}$ and $\mathbf{dSpa^c}$, the given proof no longer works. The problem is that when $\partial Z\neq\emptyset$, we cannot smoothly identify TZ near the zero section with $Z\times Z$ near the diagonal, as their corner structure is different. So it is not obvious that options (a)–(c) of Proposition 3.12 are equivalent in the corners case. Also, in (a),(b) the fibre products may not exist in $\mathbf{dSpa^c}$ unless $\boldsymbol{g},\boldsymbol{h}$ and $\boldsymbol{i},\boldsymbol{j}$ are b-transverse. We model our definition of principal d-manifolds with corners on part (c) of Proposition 3.12, as again this will be the most useful of the three.

Definition 7.1. A d-space with corners $\mathbf{W}$ is called a *principal d-manifold with corners* if is equivalent in $\mathbf{dSpa^c}$ to a fibre product $\mathbf{V}\times_{\boldsymbol{s},\mathbf{E},\mathbf{0}}\mathbf{V}$, where V is a manifold with corners, $E\to V$ is a vector bundle, $s:V\to E$ is a smooth section of E, $0:V\to E$ is the zero section, and $\mathbf{V},\mathbf{E},\boldsymbol{s},\mathbf{0}=F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(V,E,s,0)$. Note that $s,0:V\to E$ are simple, flat smooth maps in $\mathbf{Man^c}$, so $\boldsymbol{s},\mathbf{0}:\mathbf{V}\to\mathbf{E}$ are simple, flat 1-morphisms in $\mathbf{dSpa^c}$. Thus $\boldsymbol{s},\mathbf{0}$ are b-transverse by Lemma 6.35(i), and a fibre product $\mathbf{V}\times_{\boldsymbol{s},\mathbf{E},\mathbf{0}}\mathbf{V}$ exists in $\mathbf{dSpa^c}$ by Theorem 6.42.

By Corollary 6.43, the underlying d-space $\boldsymbol{W}$ and C^∞-scheme $\underline{W}$ are fibre products $\boldsymbol{W}\simeq\boldsymbol{V}\times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}}\boldsymbol{V}$ in $\mathbf{dSpa}$ and $\underline{W}\cong\underline{V}\times_{\underline{s},\underline{E},\underline{0}}\underline{V}$ in $\mathbf{C^\infty Sch}$. We will also see below that $\boldsymbol{\partial W}\simeq\boldsymbol{\partial V}\times_{\boldsymbol{s}\circ\boldsymbol{i}_{\mathbf{V}},\boldsymbol{E},\boldsymbol{0}}\boldsymbol{V}$ and $\underline{\partial W}\cong\underline{\partial V}\times_{\underline{s}\circ\underline{i}_V,\underline{E},\underline{0}}\underline{V}$.

If X is any manifold with corners, taking $V=X$, E the zero vector bundle over X and $s=0$ gives $\mathbf{W}\simeq\mathbf{X}\times_{\mathbf{id_X},\mathbf{X},\mathbf{id_X}}\mathbf{X}\simeq\mathbf{X}$ for $\mathbf{X}=F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X)$. Hence any $\mathbf{X}$ in $\bar{\mathbf{M}}\mathbf{an^c}$ is a principal d-manifold with corners.

In Definition 3.13 we defined a family of principal d-manifolds $\boldsymbol{S}_{V,E,s}$ we called *standard models*. Here is the analogue for d-manifolds with corners.

Definition 7.2. Let V be a manifold with corners, $E\to V$ a vector bundle, and $s:V\to E$ a smooth section of E. We will write down an explicit principal d-manifold with corners $\mathbf{S}=(\boldsymbol{S},\boldsymbol{\partial S},\boldsymbol{i}_{\mathbf{S}},\omega_{\mathbf{S}})$ equivalent to $\mathbf{V}\times_{\boldsymbol{s},\mathbf{E},\mathbf{0}}\mathbf{V}$ in Definition 7.1. We call $\mathbf{S}$ the *standard model* of (V,E,s), and also write it $\mathbf{S}_{V,E,s}$.

Define a vector bundle $E_\partial\to\partial V$ by $E_\partial=E|_{\partial V}=i_V^*(E)$, and a smooth section $s_\partial:\partial V\to E_\partial$ by $s_\partial=s|_{\partial V}=i_V^*(s)$. Define d-spaces $\boldsymbol{S}=\boldsymbol{S}_{V,E,s}$ and $\boldsymbol{\partial S}=\boldsymbol{S}_{\partial V,E_\partial,s_\partial}$ from the triples V,E,s and $\partial V,E_\partial,s_\partial$ exactly as in Definition

3.13, although now $V, \partial V$ have corners. Define a 1-morphism $\boldsymbol{i}_{\mathbf{S}} : \boldsymbol{\partial S} \to \boldsymbol{S}$ in **dSpa** to be the 'standard model' 1-morphism $\boldsymbol{S}_{i_V,\mathrm{id}_{E_\partial}} : \boldsymbol{S}_{\partial V,E_\partial,s_\partial} \to \boldsymbol{S}_{V,E,s}$ from Definition 3.29.

Then from Definitions 3.13 and 3.29, $\mathcal{E}_S, \mathcal{F}_S$ are the restrictions to $\underline{S} \subseteq \underline{V}$ of the vector bundles $\mathcal{E}^*, T^*\underline{V}$ on the C^∞-scheme $\underline{V}$ lifting the vector bundles E^*, T^*V on V, and $\mathcal{E}_{\partial S}, \mathcal{F}_{\partial S}$ are the restrictions to $\underline{\partial S} \subseteq \underline{\partial V}$ of the vector bundles $\mathcal{E}_\partial^*, T^*(\underline{\partial V})$ on $\underline{\partial V}$ lifting the vector bundles $E_\partial^*, T^*(\partial V)$ on ∂V, and $i''_{\mathbf{S}} : \underline{i}_{\mathbf{S}}^*(\mathcal{E}_S) \to \mathcal{E}_{\partial S}$ is the lift to $\underline{\partial S}$ of the identity morphism $\mathrm{id}_{E_\partial^*} : i_V^*(E^*) \to E_\partial^*$ of vector bundles on ∂V, and $i^2_{\mathbf{S}} : \underline{i}_{\mathbf{S}}^*(\mathcal{F}_S) \to \mathcal{F}_{\partial S}$ is the lift to $\underline{\partial S}$ of the vector bundle morphism $\mathrm{d}i_V : i_V^*(T^*V) \to T^*(\partial V)$ on ∂V.

The pullback $E_\partial = i_V^*(E)$ is in effect a transverse fibre product, $E_\partial = E \times_V \partial V$. Theorem 2.42 then implies that $\boldsymbol{E}_\partial \simeq \boldsymbol{E} \times_{\boldsymbol{V}} \boldsymbol{\partial V}$ in **dSpa**. Thus we have equivalences

$$\begin{aligned}\boldsymbol{\partial S} &\simeq \boldsymbol{\partial V} \times_{\boldsymbol{s}_\partial,\boldsymbol{E}_\partial,\boldsymbol{0}} \boldsymbol{\partial V} \simeq \boldsymbol{\partial V} \times_{\boldsymbol{s}_\partial,(\boldsymbol{E}\times_{\boldsymbol{V}}\boldsymbol{\partial V}),\boldsymbol{0}} \boldsymbol{\partial V} \simeq \boldsymbol{\partial V} \times_{\boldsymbol{s}\circ\boldsymbol{i}_{\mathbf{V}},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V} \\ &\simeq \boldsymbol{\partial V} \times_{\boldsymbol{i}_{\mathbf{V}},\boldsymbol{V},\pi_1} (\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}) \simeq \boldsymbol{\partial V} \times_{\boldsymbol{i}_{\mathbf{V}},\boldsymbol{V},\pi_1} \boldsymbol{S}.\end{aligned}$$

Therefore we have a 2-Cartesian diagram in **dSpa**:

$$\begin{tikzcd} \boldsymbol{\partial S} \arrow[rr, "\boldsymbol{\pi_{\partial V}}"] \arrow[d, "\boldsymbol{i}_{\mathbf{S}}"'] \arrow[drr, phantom, "\mathrm{id}_{\boldsymbol{\pi_V}\circ\boldsymbol{i}_{\mathbf{S}}}\Uparrow"] & & \boldsymbol{\partial V} \arrow[d, "\boldsymbol{i}_{\mathbf{V}}"] \\ \boldsymbol{S} \arrow[rr, "\boldsymbol{\pi_V}"] & & \boldsymbol{V}, \end{tikzcd} \tag{7.1}$$

where $\boldsymbol{\pi_V} = \boldsymbol{S}_{\mathrm{id}_V,0} : \boldsymbol{S} = \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{V,0,0} = \boldsymbol{V}$ and $\boldsymbol{\pi_{\partial V}} = \boldsymbol{S}_{\mathrm{id}_{\partial V},0} : \boldsymbol{\partial S} = \boldsymbol{S}_{\partial V,E_\partial,s_\partial} \to \boldsymbol{S}_{\partial V,0,0} = \boldsymbol{\partial V}$. The 2-morphism in (7.1) is the identity as

$$\boldsymbol{\pi_V} \circ \boldsymbol{i}_{\mathbf{S}} = \boldsymbol{S}_{\mathrm{id}_V,0} \circ \boldsymbol{S}_{i_V,\mathrm{id}_{E_\partial}} = \boldsymbol{S}_{i_V,0} = \boldsymbol{S}_{i_V,0} \circ \boldsymbol{S}_{\mathrm{id}_{\partial V},0} = \boldsymbol{i}_{\mathbf{V}} \circ \boldsymbol{\pi_{\partial V}}.$$

We will verify Definition 6.1(a)–(f) for $\mathbf{S} = (\boldsymbol{S}, \boldsymbol{\partial S}, \boldsymbol{i}_{\mathbf{S}}, \omega_{\mathbf{S}})$, constructing $\omega_{\mathbf{S}}$ along the way. For (a), as $i_V : \partial V \to V$ is a proper map of topological spaces, $\boldsymbol{i}_{\mathbf{V}}$ is proper in (7.1), and thus $\boldsymbol{i}_{\mathbf{S}}$ and hence $\underline{i}_{\mathbf{S}}$ are proper by properties of Cartesian squares. For (b), as $i''_{\mathbf{S}}$ is the lift to $\underline{\partial S}$ of $\mathrm{id}_{E_\partial^*}$, it is clearly an isomorphism. For (c), let $x' \in \boldsymbol{\partial S}$ with $\boldsymbol{i}_{\mathbf{S}}(x') = x \in \boldsymbol{S}$. Then $x' \in V'$ and $x \in V$ with $i_V(x') = x$, and $s_\partial(x') = s(x) = 0$. Let (W, b) be a boundary defining function for the manifold with corners V at $x' \in \partial V$, so that by Definition 5.4 $x \in W \subseteq V$ is open and $b : W \to [0,\infty)$ is smooth, and there exists open $x' \in W' \subseteq \partial V$ with $b \circ i_V|_{W'} = 0$ and $i_V|_{W'} : W' \to \{w \in W : b(w) = 0\}$ is a homeomorphism.

Define $\boldsymbol{W}, \boldsymbol{W'}, \boldsymbol{b} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(W, W', b)$, $\tilde{\boldsymbol{W}} = \boldsymbol{\pi_V}^{-1}(\boldsymbol{W})$, $\tilde{\boldsymbol{W}}' = \boldsymbol{\pi_{\partial V}}^{-1}(\boldsymbol{W'})$, and $\tilde{\boldsymbol{b}} = \boldsymbol{b} \circ \boldsymbol{\pi_V}|_{\tilde{\boldsymbol{W}}} : \tilde{\boldsymbol{W}} \to [\boldsymbol{0}, \boldsymbol{\infty})$. Consider the 2-commutative diagram in $\mathbf{dSpa^c}$:

$$\begin{tikzcd} \tilde{\boldsymbol{W}}' \arrow[r, "\boldsymbol{\pi_{\partial V}}|_{\tilde{\boldsymbol{W}}'}"] \arrow[d, "\boldsymbol{i}_{\mathbf{S}}|_{\tilde{\boldsymbol{W}}'}"'] \arrow[dr, phantom, "\mathrm{id}_{\boldsymbol{\pi_V}\circ\boldsymbol{i}_{\mathbf{S}}}\Uparrow"] & \boldsymbol{W'} \arrow[r, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{V}}|_{\boldsymbol{W'}}\ \boldsymbol{i}_{\mathbf{X}}|_U"] \arrow[dr, phantom, "\mathrm{id}_{\boldsymbol{0}\circ\boldsymbol{\pi}}\Uparrow"] & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\ \tilde{\boldsymbol{W}} \arrow[r, "\boldsymbol{\pi_V}|_{\tilde{\boldsymbol{W}}}"] \arrow[rr, bend right, "\tilde{\boldsymbol{b}}"'] & \boldsymbol{W} \arrow[r, "\boldsymbol{b}"] & [\boldsymbol{0}, \boldsymbol{\infty}). \end{tikzcd} \tag{7.2}$$

The left hand square is 2-Cartesian as it is a restriction of (7.1) to open d-subspaces. The right hand square is 2-Cartesian as $W' \cong W \times_{b,[0,\infty),0} *$ is a transverse fibre product in $\mathbf{Man^c}$. Hence the outer square of (7.2) is 2-Cartesian. This is (6.1) for $\mathbf{S}$ at $x' \in \boldsymbol{\partial S}$ with $\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{W}}', \tilde{\boldsymbol{b}}$ in place of $\boldsymbol{V}, \boldsymbol{U}, \boldsymbol{b}$, and proves the first part of Definition 6.1(c) for $\mathbf{S}$.

For the second part, note that $\tilde{b}^2 : \underline{\tilde{b}}^*(\mathcal{F}_{[0,\infty)}) \to \mathcal{F}_S|_{\underline{\tilde{W}}}$ is the lift to $\underline{\tilde{W}}$ of $(\mathrm{d}b)^* : b^*(T^*[0,\infty)) \to T^*W$, which is an embedding of vector bundles as $\mathrm{d}b|_w \neq 0$ for $w \in W$, and so has a right inverse. Therefore $\tilde{b}^2$ has a right inverse β in $\mathrm{qcoh}(\underline{\tilde{W}})$. Hence Definition 6.1(a)–(c) hold for $\mathbf{S}$.

Definition 6.1 now defines a conormal bundle $\mathcal{N}_{\mathbf{S}}$, in an exact sequence (6.4). This is just the lift from ∂V to $\underline{\partial S}$ of the exact sequence (6.53) for V. Hence $\mathcal{N}_{\mathbf{S}}$ is the lift to $\underline{\partial S}$ of the conormal bundle ν^* of ∂V in V. Now ν and ν^* have natural orientations by outward-pointing vectors at the boundary ∂V of V. Define $\omega_{\mathbf{S}}$ to be the orientation on $\mathcal{N}_{\mathbf{S}}$ which lifts this orientation on ν^*. Parts (d),(e) of Definition 6.1 are now immediate, and $(\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{b}})$ above is a boundary defining function for $\mathbf{S}$ at x'. Part (f) follows from the corresponding statement for V at x, since $\underline{x}^*(\mathcal{F}_S) \cong T_x^*V$ and $(\underline{x}_i')^*(\mathcal{N}_{\mathbf{X}}) \cong \nu^*|_{x_i'}$.

Therefore $\mathbf{S}$ is a d-space with corners. One can show using the material of §6.8.3 that $\mathbf{S}$ is equivalent to the fibre product $\mathbf{V} \times_{\boldsymbol{s},\mathbf{E},\mathbf{0}} \mathbf{V}$ in $\mathbf{dSpa^c}$, and so is a principal d-manifold with corners.

It is easy to show that the constructions of boundaries in §6.2 and corners in §6.7 have the obvious compatibilities with 'standard model' d-manifolds.

Lemma 7.3. *Let V be a manifold with corners, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section of E. Define a vector bundle $E_\partial = i_V^*(E)$ on ∂V and a smooth section $s_\partial = i_V^*(s) : \partial V \to E_\partial$. Define $\mathbf{S}_{V,E,s}$ and $\mathbf{S}_{\partial V, E_\partial, s_\partial}$ as in Definition 7.2. Then there is a natural 1-isomorphism $\partial \mathbf{S}_{V,E,s} \cong \mathbf{S}_{\partial V, E_\partial, s_\partial}$ in $\mathbf{dSpa^c}$, where $\partial\mathbf{S}_{V,E,s}$ is as in §6.2.*

Similarly, for $k \geqslant 0$ define a vector bundle $E_k = \Pi_k^*(E)$ on the k-corners $C_k(V)$ and a section $s_k = \Pi_k^*(s) : C_k(V) \to E_k$, where $\Pi_k : C_k(V) \to V$ is the natural projection. Then there is a natural 1-isomorphism $C_k(\mathbf{S}_{V,E,s}) \cong \mathbf{S}_{C_k(V),E_k,s_k}$ in $\mathbf{dSpa^c}$, where $C_k(\mathbf{S}_{V,E,s})$ is as in §6.7.

Here is an analogue of Proposition 3.15.

Lemma 7.4. *Suppose $\mathbf{W} = (\boldsymbol{W}, \boldsymbol{\partial W}, \boldsymbol{i}_{\mathbf{W}}, \omega_{\mathbf{W}})$ is a principal d-manifold with corners, so that $\mathbf{W} \simeq \mathbf{V} \times_{\boldsymbol{s},\mathbf{E},\mathbf{0}} \mathbf{V}$, where V is a manifold with corners, $E \to V$ is a vector bundle, $s \in C^\infty(E)$, and $\mathbf{V}, \mathbf{E}, \boldsymbol{s}, \mathbf{0} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(V, E, s, 0)$. Then the virtual cotangent sheaf $T^*\boldsymbol{W}$ of the d-space $\boldsymbol{W}$ is a virtual vector bundle on $\underline{W}$, with* $\operatorname{rank} T^*\boldsymbol{W} = \dim V - \operatorname{rank} E$.

Proof. Since $\mathbf{W} \simeq \mathbf{V} \times_{\boldsymbol{s},\mathbf{E},\mathbf{0}} \mathbf{V}$ there exists an equivalence $\boldsymbol{i} : \mathbf{W} \to \mathbf{S}_{V,E,s}$. In Definition 7.2, $\mathcal{E}_S, \mathcal{F}_S$ are the lifts of E^*, T^*V to $\underline{S}$, and so are vector bundles of ranks $\operatorname{rank} E, \dim V$ respectively. Therefore $T^*\boldsymbol{S}_{V,E,s} = (\mathcal{E}_S, \mathcal{F}_S, \phi_S)$ is a virtual vector bundle on $\underline{S}$ of rank $\dim V - \operatorname{rank} E$. So $\underline{i}^*(T^*\boldsymbol{S}_{V,E,s})$ is a is a virtual vector bundle on $\underline{W}$ of rank $\dim V - \operatorname{rank} E$. But $\Omega_{\boldsymbol{i}} : \underline{i}^*(T^*\boldsymbol{S}_{V,E,s}) \to T^*\boldsymbol{W}$ is an equivalence in $\mathrm{vqcoh}(\underline{W})$, as $\boldsymbol{i}$ is an equivalence. The lemma follows. □

We can now define the 2-category $\mathbf{dMan^c}$ of d-manifolds with corners.

Definition 7.5. Let $\mathbf{W}$ be a nonempty principal d-manifold with corners. Then the virtual cotangent sheaf $T^*\boldsymbol{W}$ is a virtual vector bundle on $\underline{W}$ by Lemma 7.4. Define the *virtual dimension* $\operatorname{vdim}\mathbf{W}$ of $\mathbf{W}$ to be the rank of $T^*\boldsymbol{W}$.

A d-space with corners $\mathbf{X}$ is called a *d-manifold with corners of virtual dimension* $n \in \mathbb{Z}$, written $\operatorname{vdim}\mathbf{X} = n$, if $\mathbf{X}$ can be covered by open d-subspaces $\mathbf{U}$ which are principal d-manifolds with corners with $\operatorname{vdim}\mathbf{U} = n$. A d-manifold with corners $\mathbf{X}$ is called a *d-manifold with boundary* if it is a d-space with boundary, and a *d-manifold without boundary* if it is a d-space without boundary.

Write $\bar{\mathbf{dMan}}, \mathbf{dMan^b}, \mathbf{dMan^c}$ for the full 2-subcategories of d-manifolds without boundary, and d-manifolds with boundary, and d-manifolds with corners in $\mathbf{dSpa^c}$, respectively. The 2-functor $F_{\mathbf{dSpa}}^{\mathbf{dSpa^c}} : \mathbf{dSpa} \to \mathbf{dSpa^c}$ of Definition 6.3 is an isomorphism of 2-categories $\mathbf{dSpa} \to \bar{\mathbf{dSpa}}$, and its restriction to $\mathbf{dMan} \subset \mathbf{dSpa}$ gives a (strict) isomorphism of 2-categories $F_{\mathbf{dMan}}^{\mathbf{dMan^c}} : \mathbf{dMan} \to \bar{\mathbf{dMan}} \subset \mathbf{dMan^c}$. So we may as well identify $\mathbf{dMan}$ with its image $\bar{\mathbf{dMan}}$, and consider d-manifolds without boundary in Chapters 3–4 as examples of d-manifolds with corners.

If $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ is a d-manifold with corners, then the virtual cotangent sheaf $T^*\boldsymbol{X}$ of the d-space $\boldsymbol{X}$ from Example 3.2 is a virtual vector bundle in $\operatorname{vvect}(\underline{X})$, of rank $\operatorname{vdim}\mathbf{X}$. We will call $T^*\boldsymbol{X}$ the *virtual cotangent bundle* of $\mathbf{X}$, and also write it $T^*\mathbf{X}$.

As $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X)$ is a principal d-manifold with corners for any manifold with corners X by Definition 7.1, it is a d-manifold with corners, so the 2-functor $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \to \mathbf{dSpa^c}$ in Definition 6.15 maps into $\mathbf{dMan^c}$, and we will write $F_{\mathbf{Man^c}}^{\mathbf{dMan^c}} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \to \mathbf{dMan^c}$, and $F_{\mathbf{Man^b}}^{\mathbf{dMan^c}} : \mathbf{Man^b} \to \mathbf{dMan^b} \subset \mathbf{dMan^c}$, $F_{\mathbf{Man}}^{\mathbf{dMan^c}} : \mathbf{Man} \to \bar{\mathbf{dMan}} \subset \mathbf{dMan^c}$ for its restrictions to $\mathbf{Man^b}, \mathbf{Man}$. The 2-categories $\bar{\mathbf{Man}}, \bar{\mathbf{Man}}{}^{\mathbf{b}}, \bar{\mathbf{Man}}{}^{\mathbf{c}}$ in Definition 6.15 are 2-subcategories of $\bar{\mathbf{dMan}}, \mathbf{dMan^b}, \mathbf{dMan^c}$, respectively. When we say that a d-manifold with corners $\mathbf{X}$ *is a manifold*, we mean that $\mathbf{X} \in \bar{\mathbf{Man}}{}^{\mathbf{c}}$.

As for Lemmas 3.14 and 3.19, we prove:

Lemma 7.6. *Let* $\mathbf{W}$ *be a d-manifold with corners, and* $\mathbf{U}$ *an open d-subspace of* $\mathbf{W}$*. Then* $\mathbf{U}$ *is also a d-manifold with corners, with* $\operatorname{vdim}\mathbf{U} = \operatorname{vdim}\mathbf{W}$*. If* $\mathbf{W}$ *is principal, then* $\mathbf{U}$ *is principal.*

Boundaries and corners of d-manifolds with corners are also d-manifolds with corners, with the dimensions one would expect.

Proposition 7.7. *Suppose* $\mathbf{X}$ *is a d-manifold with corners. Then* $\partial\mathbf{X}$ *in* §6.2 *and* $C_k(\mathbf{X})$ *in* §6.7 *are d-manifolds with corners, with* $\operatorname{vdim}\partial\mathbf{X} = \operatorname{vdim}\mathbf{X} - 1$ *and* $\operatorname{vdim} C_k(\mathbf{X}) = \operatorname{vdim}\mathbf{X} - k$ *for all* $k \geqslant 0$*.*

Proof. Let $x' \in \partial\mathbf{X}$ with $\boldsymbol{i}_{\mathbf{X}}(x') = x \in \mathbf{X}$. Then x has a principal open neighbourhood $\mathbf{U}$ in $\mathbf{X}$, so there is an equivalence $\mathbf{U} \simeq \mathbf{S}_{V,E,s}$ for some V, E, s. Thus $\partial\mathbf{U}$ is an open neighbourhood of x' in $\partial\mathbf{X}$. But $\partial\mathbf{U} \simeq \partial\mathbf{S}_{V,E,s} \simeq \mathbf{S}_{\partial V, E_\partial, s_\partial}$ by Proposition 6.20 and Lemma 7.3, and $\mathbf{S}_{\partial V, E_\partial, s_\partial}$ is a principal d-manifold of virtual dimension $\operatorname{vdim}\mathbf{S}_{V,E,s} - 1 = \operatorname{vdim}\mathbf{X} - 1$. Therefore $\partial\mathbf{X}$ can be covered by principal open $\partial\mathbf{U}$ with $\operatorname{vdim}\partial\mathbf{U} = \operatorname{vdim}\mathbf{X} - 1$, so $\partial\mathbf{X}$ is a d-manifold with corners with $\operatorname{vdim}\partial\mathbf{X} = \operatorname{vdim}\mathbf{X} - 1$. The proof for $C_k(\mathbf{X})$ is similar, using $C_k(\mathbf{S}_{V,E,s}) \cong \mathbf{S}_{C_k(V),E_k,s_k}$ from Lemma 7.3. $\square$

In §5.5 we defined a category $\check{\mathbf{M}}\mathbf{an^c}$ generalizing $\mathbf{Man^c}$, whose objects are disjoint unions $\coprod_{n=0}^{\infty} X_n$, where $X_n \in \mathbf{Man^c}$ with $\dim X_n = n$. This was convenient because the corner functor $C : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$ with $C(X) = \coprod_{k=0}^{\dim X} C_k(X)$ maps into $\check{\mathbf{M}}\mathbf{an^c}$, so the category $\check{\mathbf{M}}\mathbf{an^c}$ gives a neat way of packaging the properties of $C_k(X)$ for all $k \geqslant 0$. For the same reason we introduce a 2-category $\mathbf{d}\check{\mathbf{M}}\mathbf{an^c}$ of d-manifolds with corners of mixed dimension.

Definition 7.8. Define $\mathbf{d}\check{\mathbf{M}}\mathbf{an^c}$ to be the full 2-subcategory of $\mathbf{X}$ in $\mathbf{dSpa^c}$ which may be written as a disjoint union $\mathbf{X} = \coprod_{n\in\mathbb{Z}} \mathbf{X}_n$ for $\mathbf{X}_n \in \mathbf{dMan^c}$ with $\operatorname{vdim}\mathbf{X}_n = n$, where we allow $\mathbf{X}_n = \boldsymbol{\emptyset}$. We call such $\mathbf{X}$ a *d-manifold with corners of mixed dimension*. Then $\mathbf{dMan^c} \subset \mathbf{d}\check{\mathbf{M}}\mathbf{an^c} \subset \mathbf{dSpa^c}$, and Proposition 7.7 implies that the corner functors $C, \hat{C} : \mathbf{dSpa^c} \to \mathbf{dSpa^c}$ in §6.7 restrict to strict 2-functors $C, \hat{C} : \mathbf{dMan^c} \to \mathbf{d}\check{\mathbf{M}}\mathbf{an^c}$.

If $\underline{X}$ is a C^∞-scheme, define $\check{\mathrm{v}}\mathrm{vect}(\underline{X})$ to be the full 2-subcategory of objects $(\mathcal{E}^\bullet, \phi)$ in $\mathrm{vqcoh}(\underline{X})$ such that $\underline{X}$ has a decomposition $\underline{X} = \coprod_{n\in\mathbb{Z}} \underline{X}_n$ with $(\mathcal{E}^\bullet, \phi)|_{\underline{X}_n}$ a virtual vector bundle on $\underline{X}$ of rank n. We call $(\mathcal{E}^\bullet, \phi)$ a *virtual vector bundle of mixed rank*. Then $\mathrm{vvect}(\underline{X}) \subset \check{\mathrm{v}}\mathrm{vect}(\underline{X}) \subset \mathrm{vqcoh}(\underline{X})$. If $\mathbf{X}$ is an object in $\mathbf{d}\check{\mathbf{M}}\mathbf{an^c}$ then $T^*\mathbf{X} \in \check{\mathrm{v}}\mathrm{vect}(\underline{X})$. In particular, if $\mathbf{X}$ is a d-manifold with corners, then $T^*(C(\mathbf{X}))$ is a virtual vector bundle of mixed rank on $\underline{C(X)}$. The material of §4.5 on orientation line bundles extends immediately to virtual vector bundles of mixed rank.

We call $(\mathcal{E}^\bullet, \phi)$ a *vector bundle of mixed rank* if $\underline{X} = \coprod_{n\in\mathbb{Z}} \underline{X}_n$ with $(\mathcal{E}^\bullet, \phi)|_{\underline{X}_n}$ a vector bundle on $\underline{X}$ of rank n. By Proposition 3.9, an object $(\mathcal{E}^\bullet, \phi)$ in $\check{\mathrm{v}}\mathrm{vect}(\underline{X})$ is a vector bundle of mixed rank if and only if ϕ has a left inverse.

7.2 Local properties of d-manifolds with corners

In §3.3 we showed that any d-manifold $\boldsymbol{X}$ is determined up to equivalence near $x \in \boldsymbol{X}$ by the 'classical' C^∞-scheme $\underline{X}$ and the integer $\operatorname{vdim}\boldsymbol{X}$. To find an analogue of this for d-manifolds with corners $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$, we must first decide what in $\mathbf{X}$ counts as 'classical' data. Broadly we want to reduce to C^∞-schemes, so from $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}$ we count $\underline{X}, \underline{\partial X}, \underline{i}_{\mathbf{X}}$ as classical. We need the orientation $\omega_{\mathbf{X}}$ on $\mathcal{N}_{\mathbf{X}}$, so we must include the line bundle $\mathcal{N}_{\mathbf{X}}$ on $\underline{\partial X}$. We also need some way to relate $\mathcal{N}_{\mathbf{X}}$ to $\underline{X}, \underline{\partial X}, \underline{i}_{\mathbf{X}}$. Since $\mathcal{N}_{\mathbf{X}}$ is the conormal bundle of $\boldsymbol{\partial X}$ in $\boldsymbol{X}$, the appropriate data is a morphism $\varrho_{\mathbf{X}} : \mathcal{N}_{\mathbf{X}} \to \mathcal{K}_{\mathbf{X}}$, where $\mathcal{K}_{\mathbf{X}}$ is the conormal sheaf of $\underline{\partial X}$ in $\underline{X}$. Here is how to define $\mathcal{K}_{\mathbf{X}}$ and $\varrho_{\mathbf{X}}$.

Definition 7.9. Let $\mathbf{X}$ be a d-manifold with corners. Then $\mathbf{X}$ contains C^∞-schemes $\underline{X} = (X, \mathcal{O}_X)$, $\underline{\partial X} = (\partial X, \mathcal{O}_{\partial X})$ and a morphism $\underline{i}_{\mathbf{X}} = (i_{\mathbf{X}}, i_{\mathbf{X}}^\sharp) : \underline{\partial X} \to \underline{X}$. Then $i_{\mathbf{X}}^\sharp : i_{\mathbf{X}}^{-1}(\mathcal{O}_X) \to \mathcal{O}_{\partial X}$ is a surjective morphism of sheaves of C^∞-rings on ∂X. Write $\mathcal{J}_{\mathbf{X}}$ for the kernel of $i_{\mathbf{X}}^\sharp$, a sheaf of ideals in $i_{\mathbf{X}}^{-1}(\mathcal{O}_X)$ on ∂X. Then $\mathcal{J}_{\mathbf{X}}^2 \subseteq \mathcal{J}_{\mathbf{X}}$ is also a sheaf of ideals in $i_{\mathbf{X}}^{-1}(\mathcal{O}_X)$. Write $\mathcal{K}_{\mathbf{X}} = \mathcal{J}_{\mathbf{X}}/\mathcal{J}_{\mathbf{X}}^2$. As $\mathcal{J}_{\mathbf{X}}$ is an $i_{\mathbf{X}}^{-1}(\mathcal{O}_X)$-module, $\mathcal{K}_{\mathbf{X}}$ is an $i_{\mathbf{X}}^{-1}(\mathcal{O}_X)/\mathcal{J}_{\mathbf{X}}$-module. But $i_{\mathbf{X}}^{-1}(\mathcal{O}_X)/\mathcal{J}_{\mathbf{X}} \cong \mathcal{O}_{\partial X}$ as $i_{\mathbf{X}}^{\sharp}$ is surjective, so $\mathcal{K}_{\mathbf{X}}$ is an $\mathcal{O}_{\partial X}$-module, that is, $\mathcal{K}_{\mathbf{X}} \in \mathrm{qcoh}(\underline{\partial X})$. We think of $\mathcal{K}_{\mathbf{X}}$ as the *conormal sheaf of* $\underline{\partial X}$ *in* $\underline{X}$.

Using similar arguments to §2.1 and §6.1, one can show that there exist unique morphisms $\varrho_{\mathbf{X}} : \mathcal{N}_{\mathbf{X}} \to \mathcal{K}_{\mathbf{X}}$ and $\sigma_{\mathbf{X}} = i_{\mathbf{X}}^{-1}(\mathrm{d}) \otimes \mathrm{id}_{\mathcal{O}_{\partial X}} : \mathcal{K}_{\mathbf{X}} \to \underline{i}_{\mathbf{X}}^*(T^*\underline{X})$ in $\mathrm{qcoh}(\underline{\partial X})$ such that the following diagram commutes, with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathcal{J}'_{\mathbf{X}} = \mathrm{Ker}\, i'_{\mathbf{X}} \arrow[rr] \arrow[dl] \arrow[dd] & & i_{\mathbf{X}}^{-1}(\mathcal{O}'_X) \arrow[rr, "i'_{\mathbf{X}}"'] \arrow[dl, "i_{\mathbf{X}}^{-1}(\imath_X)"'] \arrow[dd, "i_{\mathbf{X}}^{-1}(\mathrm{d}\otimes\imath_X)"] & & \mathcal{O}'_{\partial X} \arrow[r] \arrow[dl, "\imath_{\partial X}"] \arrow[dd, "\mathrm{d}\otimes\imath_{\partial X}"] & 0 \\
0 \arrow[r] & \mathcal{J}_{\mathbf{X}} = \mathrm{Ker}\, i_{\mathbf{X}}^{\sharp} \arrow[rr] \arrow[dd] & & i_{\mathbf{X}}^{-1}(\mathcal{O}_X) \arrow[rr, "i_{\mathbf{X}}^{\sharp}"] \arrow[dd, "i_{\mathbf{X}}^{-1}(\mathrm{d})\otimes i_{\mathbf{X}}^{\sharp}"] & & \mathcal{O}_{\partial X} \arrow[r] \arrow[dd, "\mathrm{d}"] & 0 & \\
 & 0 \arrow[r] & \mathcal{N}_{\mathbf{X}} \arrow[rr, "\nu_{\mathbf{X}}"'] \arrow[dl, "\varrho_{\mathbf{X}}"] & & \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X) \arrow[rr, "i_{\mathbf{X}}^2"'] \arrow[dl, "\underline{i}_{\mathbf{X}}^*(\psi_X)"] & & \mathcal{F}_{\partial X} \arrow[r] \arrow[dl, "\psi_{\partial X}"] & 0 \\
 & \mathcal{K}_{\mathbf{X}} \arrow[rr, "\sigma_{\mathbf{X}}"] & & \underline{i}_{\mathbf{X}}^*(T^*\underline{X}) \arrow[rr, "\Omega_{\underline{i}_{\mathbf{X}}}"] & & T^*(\underline{\partial X}) \arrow[r] & 0. &
\end{tikzcd}$$

Then $\varrho_{\mathbf{X}}$ is surjective, as the projections $\mathcal{J}'_{\mathbf{X}} \to \mathcal{J}_{\mathbf{X}} \to \mathcal{K}_{\mathbf{X}}$ are surjective and the left hand diamond commutes.

We consider the 'classical' data in a d-manifold with corners $\mathbf{X}$ to be $\underline{X}, \underline{\partial X}, \underline{i}_{\mathbf{X}}, \mathcal{N}_{\mathbf{X}}, \varrho_{\mathbf{X}}, \omega_{\mathbf{X}}$. Here $\mathcal{K}_{\mathbf{X}}, \sigma_{\mathbf{X}}$ are constructed from $\underline{X}, \underline{\partial X}, \underline{i}_{\mathbf{X}}$, and so do not count as extra data. All this data is 'classical' in the sense that it is unaffected by 2-morphisms. This means that if $\mathbf{X}, \mathbf{Y}$ are equivalent in $\mathbf{dMan^c}$ then the classical parts $\underline{X}, \underline{\partial X}, \ldots, \omega_{\mathbf{X}}$ and $\underline{Y}, \underline{\partial Y}, \ldots, \omega_{\mathbf{Y}}$ are strictly isomorphic. To see that $\mathcal{N}_{\mathbf{X}}, \mathcal{N}_{\mathbf{Y}}$ are unaffected by 2-morphisms, note that if $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ are 1-morphisms and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism in $\mathbf{dSpa^c}$, then $\lambda_{\boldsymbol{f}} = \lambda_{\boldsymbol{g}}$ by Proposition 6.8(a), where $\lambda_{\boldsymbol{f}}, \lambda_{\boldsymbol{g}}$ are how $\boldsymbol{f}, \boldsymbol{g}$ act on $\mathcal{N}_{\mathbf{X}}, \mathcal{N}_{\mathbf{Y}}$.

We can now prove a partial analogue of Propositions 3.21 and 3.23.

Proposition 7.10. *Suppose V is a manifold with corners, $E \to V$ a vector bundle, $s : V \to E$ a smooth section, and $v \in V$ with $s(v) = 0$. Then for some $k, l, m, a, b \geqslant 0$ we have $\dim V = k + m + b$ and $\operatorname{rank} E = l + a + b$, and we can choose an open neighbourhood $\tilde V$ of v in V and coordinates $(x_1, \ldots, x_{k+m+b})$ on V identifying v with $(0, \ldots, 0)$ and $\tilde V$ with an open neighbourhood W of 0 in $\mathbb{R}_k^{k+m+b}$, so that $x_1, \ldots, x_k \in [0, \infty)$ and $x_{k+1}, \ldots, x_{k+m+b} \in \mathbb{R}$. We can also choose a trivialization $E|_{\tilde V} \cong \mathbb{R}^{l+a+b} \times \tilde V \to \tilde V$, so that $s|_{\tilde V} = (s_1, \ldots, s_{l+a+b})$ for $s_i \in C^\infty(\tilde V)$, where in the coordinates $(x_1, \ldots, x_{k+m+b})$ on $\tilde V$ we have*

$$s_i(x_1, \ldots, x_{k+m+b}) = \begin{cases} f_i(x_1, \ldots, x_{k+m}), & i = 1, \ldots, l, \\ 0, & i = l+1, \ldots, l+a, \\ x_{k+m-l-a+i}, & i = l+a+1, \ldots, l+a+b, \end{cases} \tag{7.3}$$

for functions $f_1, \ldots, f_l \in C^\infty(\mathbb{R}_k^{k+m})$ with $f_i(0, \ldots, 0) = 0$ and $\frac{\partial f_i}{\partial x_j}(0, \ldots, 0) = 0$ for all $i = 1, \ldots, l$ and $j = k+1, \ldots, k+m$, and if $g_1, \ldots, g_l \in C_0^\infty(\mathbb{R}_k^{k+m})$ with $\sum_{j=1}^{l} f_j \cdot g_j = 0$ in $C_0^\infty(\mathbb{R}_k^{k+m})$ then $g_j(0) = 0$ for all $j = 1, \ldots, l$.

Write $\mathbf{S} = \mathbf{S}_{V,E,s}$ for the 'standard model' d-manifold with corners from Definition 7.2. Then k, l, m and possible choices for $f_1, \ldots, f_l \in C^\infty(\mathbb{R}^{k+m})$

may be reconstructed solely from the point $v \in S \subseteq V$ and the 'classical' data $\underline{S}, \underline{\partial S}, \underline{i}_{\mathbf{S}}, \mathcal{N}_{\mathbf{S}}, \varrho_{\mathbf{S}}, \omega_{\mathbf{S}}$ in $\mathbf{S}_{V,E,s}$. Knowing $\dim V, \operatorname{rank} E$ then determines a, b. Hence, V, E, s are determined up to non-canonical isomorphism near v solely by $v, \underline{S}, \underline{\partial S}, \underline{i}_{\mathbf{S}}, \mathcal{N}_{\mathbf{S}}, \varrho_{\mathbf{S}}, \omega_{\mathbf{S}}$ and the integers $\dim V, \operatorname{rank} E$.

Proof. Let $i_V^{-1}(v) = \{v'_1, \ldots, v'_k\}$, and set $n = \dim V$ and $r = \operatorname{rank} E$. Since V is an n-manifold with corners and $v'_1, \ldots, v'_k$ parametrize the local boundary components of V at $v \in V$, we can choose coordinates $(x_1, \ldots, x_n)$ on an open neighbourhood $\tilde{V}$ of v in V which identify v with $(0, \ldots, 0)$ and $\tilde{V}$ with an open neighbourhood W of $(0, \ldots, 0)$ in $\mathbb{R}^n_k = [0, \infty)^k \times \mathbb{R}^{n-k}$, where $k \leqslant n$, and which identify v'_i with $\bigl((0, \ldots, 0), \{x_i = 0\}\bigr)$ in $\partial\mathbb{R}^n_k$, that is, v'_i corresponds to the local boundary component $x_i = 0$ at $(0, \ldots, 0)$ in $\mathbb{R}^n_k$ for $i = 1, \ldots, k$. Making $\tilde{V}, W$ smaller if necessary we can suppose $E|_{\tilde{V}}$ is trivial, and identify it with $\mathbb{R}^r \times \tilde{V} \to \tilde{V}$. So in coordinates $(x_1, \ldots, x_n)$ on $\tilde{V}$ we have $s = \bigl(s_1(x_1, \ldots, x_n), \ldots, s_r(x_1, \ldots, x_n)\bigr)$ for $s_1, \ldots, s_r \in C^\infty(W)$. Making $\tilde{V}, W$ still smaller, we can suppose $s_1, \ldots, s_r$ extend to $\mathbb{R}^n_k$, so that $s_1, \ldots, s_r \in C^\infty(\mathbb{R}^n_k)$.

Let $\underline{S}_v$ be the localization of $\underline{S}$ at v, as a C^∞-scheme, and $\underline{\partial S}_{v'_i}$ the localization of $\underline{\partial S}$ at v'_i for $i = 1, \ldots, k$. As $\underline{i}_{\mathbf{S}}(v'_i) = v$, we have a localized morphism $\underline{i}_{\mathbf{S}, v'_i} : \underline{\partial S}_{v'_i} \to \underline{S}_v$ for $i = 1, \ldots, k$. Then $\underline{S}_v \cong \operatorname{Spec} \mathfrak{C}_v$, $\underline{\partial S}_{v'_i} \cong \operatorname{Spec} \mathfrak{D}_{v'_i}$ and $\underline{i}_{\mathbf{S}, v'_i} \cong \operatorname{Spec} \psi_{v'_i}$ for C^∞-local rings $\mathfrak{C}_v, \mathfrak{D}_{v'_i}$ and morphisms $\psi_{v'_i} : \mathfrak{C}_v \to \mathfrak{D}_{v'_i}$ for $i = 1, \ldots, k$. We now have isomorphisms

$$\begin{gathered}\mathfrak{C}_v \cong C_0^\infty(\mathbb{R}^n_k)/(s_1, \ldots, s_r) \text{ and } \mathfrak{D}_{v'_i} \cong C_0^\infty(\mathbb{R}^n_k)/(x_i, s_1, \ldots, s_r) \\ \text{identifying } \psi_{v'_i} : \mathfrak{C}_v \to \mathfrak{D}_{v'_i} \text{ with } c + (s_1, \ldots, s_r) \mapsto c + (x_i, s_1, \ldots, s_r).\end{gathered} \tag{7.4}$$

Note that $\mathfrak{C}_v, \mathfrak{D}_{v'_1}, \ldots, \mathfrak{D}_{v'_k}$ and $\psi_{v'_1}, \ldots, \psi_{v'_k}$ depend only on $\underline{S}, \underline{\partial S}, \underline{i}_{\mathbf{S}}, v$ up to isomorphisms and reorderings of the $\mathfrak{D}_{v'_i}, \psi_{v'_i}$.

For the first part of the proposition, we must show that we can choose $\tilde{V}, W$, $(x_1, \ldots, x_n)$, and the trivialization $E|_{\tilde{V}} \cong \mathbb{R}^r \times \tilde{V}$ such that $s_1, \ldots, s_r$ assume the special form (7.3). We follow the proof of Proposition 3.23, with the following changes. In the coordinates $(x_1, \ldots, x_n)$, we are only allowed to make coordinate transformations that preserve the conditions $x_j \geqslant 0$ for $j = 1, \ldots, k$, because of the corners of V. So we leave $x_1, \ldots, x_k$ fixed, and only change the choice of $x_{k+1}, \ldots, x_n$. Because of this we can only ensure that $\frac{\partial f_i}{\partial x_j}(0) = 0$ for all i and $j \geqslant k+1$, rather than $\frac{\partial f_i}{\partial x_j}(0) = 0$ for all i, j as in Proposition 3.23.

For the second part of the proposition, we have to explain how to reconstruct k, l, m and possible choices for $f_1, \ldots, f_l \in C^\infty(\mathbb{R}^{k+m})$ solely from $\underline{S}, \underline{\partial S}, \underline{i}_{\mathbf{S}}, \mathcal{N}_{\mathbf{S}}, \varrho_{\mathbf{S}}, \omega_{\mathbf{S}}$ and v. We do this in six steps:

(a) $k = |i_{\mathbf{S}}^{-1}(v)|$, so k depends only on $\underline{S}, \underline{\partial S}, \underline{i}_{\mathbf{S}}, v$.

(b) For $i = 1, \ldots, k$ we choose an image $\tilde{x}_i$ of x_i in $\mathfrak{C}_v$, using $\mathcal{N}_{\mathbf{S}}, \varrho_{\mathbf{S}}, \omega_{\mathbf{S}}$.

(c) m is the minimal number of generators of the C^∞-ring $\mathfrak{C}_v/(\tilde{x}_1, \ldots, \tilde{x}_k)$.

(d) We choose $\tilde{x}_{k+1}, \ldots, \tilde{x}_{k+m} \in \mathfrak{C}_v$ with $\tilde{x}_i(v) = 0$, whose projections to $\mathfrak{C}_v/(\tilde{x}_1, \ldots, \tilde{x}_k)$ are a minimal set of generators. These $\tilde{x}_{k+1}, \ldots, \tilde{x}_{k+m}$ will be the images in $\mathfrak{C}_v$ of the coordinates $x_{k+1}, \ldots, x_{k+m}$.

(e) There is now a unique surjective morphism $\pi_k^{k+m}: C_0^\infty(\mathbb{R}_k^{k+m}) \to \mathfrak{C}_v$ with $\pi_k^{k+m}(x_i) = \tilde{x}_i$ for $i = 1, \ldots, k+m$. So $\operatorname{Ker} \pi_k^{k+m}$ is an ideal in $C_0^\infty(\mathbb{R}_k^{k+m})$. Then l is the minimal number of generators of $\operatorname{Ker} \pi_k^{k+m}$.

(f) Choose $f_1, \ldots, f_l$ in $C^\infty(\mathbb{R}_k^{k+m})$ whose images in $C_0^\infty(\mathbb{R}_k^{k+m})$ are a minimal set of generators for $\operatorname{Ker} \pi_k^{k+m}$.

Part (a) is immediate. For (b), note from (7.4) that $\operatorname{Ker} \psi_{v_i'} = (x_i) \subseteq \mathfrak{C}_v$ for $i = 1, \ldots, k$, so the ideal (x_i) generated by x_i in $\mathfrak{C}_v$ depends only on $\mathfrak{C}_v, \mathfrak{D}_{v_i'}, \psi_{v_i'}$. We will use the data $\mathcal{N}_{\mathbf{S}}, \varrho_{\mathbf{S}}, \omega_{\mathbf{S}}$ to choose an appropriate generator $\tilde{x}_i$ in $\operatorname{Ker} \psi_{v_i'}$. If $\operatorname{Ker} \psi_{v_i'} = \{0\}$ the only choice is $\tilde{x}_i = 0$, so suppose $\operatorname{Ker} \psi_{v_i'} \neq \{0\}$. Pick a representative τ for $\omega_{\mathbf{S}}$, so that $\tau : \mathcal{O}_{\partial S} \to \mathcal{N}_{\mathbf{S}}$ is an isomorphism, and $\varrho_{\mathbf{X}} \circ \tau : \mathcal{O}_{\partial S} \to \mathcal{K}_{\mathbf{S}}$ is surjective in $\operatorname{qcoh}(\underline{\partial S})$. Restricting to the C^∞-subscheme $\underline{\partial S}_{v_i'}$ in $\underline{\partial S}$ gives $\varrho_{\mathbf{X}} \circ \tau|_{\underline{\partial S}_{v_i'}} : \mathcal{O}_{\partial S}|_{\underline{\partial S}_{v_i'}} \to \mathcal{K}_{\mathbf{S}}|_{\underline{\partial S}_{v_i'}}$. We have $\mathcal{O}_{\partial S}|_{\underline{\partial S}_{v_i'}} \cong \operatorname{MSpec} \mathfrak{D}_{v_i'}$ and $\mathcal{K}_{\mathbf{S}}|_{\underline{\partial S}_{v_i'}} \cong \operatorname{MSpec}(x_i)/(x_i^2)$, where $(\tilde{x}_i)/(\tilde{x}_i^2)$ is a $\mathfrak{D}_{v_i'}$-module. Hence $\varrho_{\mathbf{X}} \circ \tau|_{\underline{\partial S}_{v_i'}} \cong \operatorname{MSpec} P_i$ for some surjective $\mathfrak{D}_{v_i'}$-module morphism $P_i : \mathfrak{D}_{v_i'} \to (x_i)/(x_i^2)$. Thus $P_i(-1)$ is a generator of $(x_i)/(x_i^2)$. Hence we can choose a generator $\tilde{x}_i$ of $(x_i) \subseteq \mathfrak{C}_v$ with $\tilde{x}_i + (x_i^2) = P_i(-1)$.

As $\mathfrak{C}_v$ is a local ring, a consequence of the Nakayama Lemma shows that $\tilde{x}_i = c \cdot x_i$ for some invertible element $c \in \mathfrak{C}_v$. Note that $c \in \mathfrak{C}_v$ is invertible if and only if $c(v) \neq 0$ in $\mathbb{R}$. If $\tilde{x}_i = c \cdot x_i = c' \cdot x_i$ for invertible $c, c' \in \mathfrak{C}_v$ then $(c - c') \cdot x_i = 0$. If $c(v) \neq c'(v)$ then $c - c'$ is invertible, so $x_i = 0$ in $\mathfrak{C}_v$, contradicting $\operatorname{Ker} \psi_{v_i'} = (x_i) \neq \{0\}$. Hence $c(v) = c'(v)$, so $c(v) \in \mathbb{R} \setminus \{0\}$ depends only on the choice of τ and not on the choice of c.

Since different $\tau, \tau' \in \omega_{\mathbf{S}}$ are proportional by a positive function, different choices $\tilde{x}_i, \tilde{x}_i'$ for $\tilde{x}_i$ are proportional by a positive function, so the sign of $c(v)$ is independent of choices. As x_i induces a boundary defining function for $\mathbf{S}$ at v_i', we can show that x_i is a possible choice for $\tilde{x}_i$, so that $c(v) > 0$. The sign -1 in $P_i(-1)$ is necessary because in Definition 6.1(e), $\omega_{\mathbf{X}}$ is identified with the negative orientation on $\mathcal{F}_{[0,\infty)}$.

We have now shown that using only the data $\mathfrak{C}_v, \mathfrak{D}_{v_i'}, \psi_{v_i'}$ and $\mathcal{N}_{\mathbf{S}}, \varrho_{\mathbf{S}}, \omega_{\mathbf{S}}$ at v_i', we can choose an element $\tilde{x}_i \in \operatorname{Ker} \psi_{v_i'} = (x_i)$ such that $\tilde{x}_i = c \cdot x_i$ for $c \in \mathfrak{C}_v$ with $c(v) > 0$. The point here is that the inclusion $\underline{i_{\mathbf{S}}} : \underline{\partial S} \to \underline{S}$ at near $v_i' \in \underline{\partial S}$ determines the hypersurface $x_i = 0$ in $\underline{S}$, but it may not be enough to distinguish the two 'sides' $x_i \geqslant 0$ and $x_i \leqslant 0$ of this hypersurface. We use the orientation $\omega_{\mathbf{S}}$ to distinguish the two. This is important because if $x_i, \tilde{x}_i, c$ are functions on $\tilde{V}$ with $\tilde{x}_i = c \cdot x_i$, then $\tilde{x}_i$ is a suitable replacement for the coordinate x_i near v if and only if $c(v) > 0$. This proves (b). Parts (c)–(f) and the rest of the proposition now follow from Definition 3.20 and Propositions 3.21 and 3.23 with only straightforward modifications. □

In Example 3.24 we studied a family of d-manifolds $\boldsymbol{U}_{a,b}$. Proposition 3.23 implies that every 'standard model' d-manifold $\boldsymbol{S}_{V,E,s}$ is locally 1-isomorphic to some $\boldsymbol{U}_{a,b}$. But $\boldsymbol{U}_{a,b}$ is independent of b up to equivalence, so as in Proposition 3.25, every d-manifold $\boldsymbol{X}$ is locally equivalent to some $\boldsymbol{U}_{a,0}$. Now b in Proposition 7.10 plays the same rôle as b in $\boldsymbol{U}_{a,b}$ in Example 3.24. Therefore

$\mathbf{S}_{V,E,s}$ near v in Proposition 7.10 is independent of b up to equivalence. So up to equivalence we can set $b = 0$. Proposition 7.10 then shows that k, l, m and $f_1, \ldots, f_l$ may be reconstructed from $v, \underline{S}, \underline{\partial S}, \underline{i}_{\mathbf{S}}, \mathcal{N}_{\mathbf{S}}, \varrho_{\mathbf{S}}, \omega_{\mathbf{S}}$, and a can be recovered from $\operatorname{vdim} \mathbf{S} = \dim V - \operatorname{rank} E$ by $a = k - l + m - \operatorname{vdim} \mathbf{S}$. Thus we deduce analogues of Corollaries 3.26 and 3.27:

Corollary 7.11. *Let $\mathbf{X}$ be a d-manifold with corners, and $x \in \mathbf{X}$. Then there exists an open neighbourhood $\mathbf{U}$ of x in $\mathbf{X}$ and an equivalence $\mathbf{U} \simeq \mathbf{S}_{V,E,s}$ in $\mathbf{dMan^c}$ for some manifold with corners V, vector bundle $E \to V$ and smooth section $s : V \to E$ which identifies $x \in \mathbf{U}$ with a point $v \in S^k(V) \subseteq V$, where $S^k(V)$ is as in §5.1, such that $s(v) = \mathrm{d}s|_{S^k(V)}(v) = 0$. Furthermore, V, E, s are determined up to non-canonical isomorphism near v by $\mathbf{X}$ near x.*

Here in Proposition 7.10 the depth k stratum $S^k(V)$ of V is locally identified with $0^k \times \mathbb{R}^{m+b} \subseteq [0,\infty)^k \times \mathbb{R}^{m+b} = \mathbb{R}^{k+m+b}_k$, so $\frac{\partial f_i}{\partial x_j}(0, \ldots, 0) = 0$ for $i = 1, \ldots, l$ and $j = k+1, \ldots, k+m$ translates to $\mathrm{d}s|_{S^k(V)}(v) = 0$ by (7.3) and $b = 0$.

Corollary 7.12. *Let $\mathbf{X}$ be a d-manifold with corners. Then $\mathbf{X}$ is determined up to non-canonical equivalence near each point $x \in \mathbf{X}$ by the 'classical' data $\underline{X}, \underline{\partial X}, \underline{i}_{\mathbf{X}}, \mathcal{N}_{\mathbf{X}}, \varrho_{\mathbf{X}}, \omega_{\mathbf{X}}$ in $\mathbf{X}$ and the integer* $\operatorname{vdim} \mathbf{X}$.

Corollary 7.12 shows that locally the *only* extra information in the 'derived' data in $\mathbf{X}$ up to non-canonical equivalence is $\operatorname{vdim} \mathbf{X} \in \mathbb{Z}$. Globally, the extra information is like a vector bundle $\mathcal{E}$ over $\underline{X}$. Here is an analogue of Proposition 3.28 for d-manifolds with corners.

Proposition 7.13. *Let $\mathbf{X}$ be a d-manifold with corners. Then $\mathbf{X}$ is a manifold (that is, $\mathbf{X} \in \bar{\mathbf{Man}}^{\mathbf{c}}$) if and only if $\phi_{C(X)} : \mathcal{E}_{C(X)} \to \mathcal{F}_{C(X)}$ has a left inverse, or equivalently, if the cotangent bundle $T^*(C(\mathbf{X}))$ of the corners $C(\mathbf{X})$ of $\mathbf{X}$ is a vector bundle of mixed rank on $\underline{C(X)}$, in the sense of Definition* 7.8.

Proof. Since $T^*(C(\mathbf{X}))$ is a virtual vector bundle of mixed rank, the two conditions that $\phi_{C(X)}$ has a left inverse, and $T^*(C(\mathbf{X}))$ is a vector bundle of mixed rank, are equivalent by Proposition 3.9. Using Proposition 2.25, Lemma 2.26 and Theorem 6.29 we find the condition that $\phi_{C(X)}$ has a left inverse is local in $\mathbf{X}$, that is, it is enough to check it on any cover of $\mathbf{X}$ by open d-submanifolds $\mathbf{U}$, and unchanged under replacing $\mathbf{X}$ by an equivalent $\tilde{\mathbf{X}}$ in $\mathbf{dMan^c}$.

For the 'only if' part, let X be a manifold with corners and $\mathbf{X} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(X)$, as in Definition 6.15. Then $\mathcal{E}_X = 0$, and $\mathcal{E}_{C(X)} = 0$, so $\phi_{C(X)} : \mathcal{E}_{C(X)} \to \mathcal{F}_{C(X)}$ trivially has a left inverse. More generally, if $\mathbf{X} \in \bar{\mathbf{Man}}^{\mathbf{c}}$ then $\mathbf{X} \simeq F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(X)$ for some $X \in \mathbf{Man^c}$, so by the first part $\phi_{C(X)}$ has a left inverse.

For the 'if' part, suppose $\phi_{C(X)}$ has a left inverse. Fix $x \in \underline{X}$. Then by Corollary 7.11 there exist open $x \in \mathbf{U} \subseteq \mathbf{X}$ and an equivalence $\mathbf{U} \simeq \mathbf{S} = \mathbf{S}_{V,E,s}$ for some V, E, s identifying $x \in \mathbf{U}$ with $v \in S^k(V) \subseteq V$, where $s(v) = \mathrm{d}s|_{S^k(V)}(v) = 0$. Let $i_V^{-1}(v) = \{v_1', \ldots, v_k'\}$, and set $w = \bigl(v, \{v_1', \ldots, v_k'\}\bigr) \in C(\mathbf{S})$. Let $\underline{w} : \underline{*} \to \underline{C(X)}$ be the corresponding morphism in $\mathbf{C^\infty Sch}$. Identifying $\mathrm{qcoh}(\underline{*})$ with the category of real vector spaces, $\underline{w}^*(\phi_{C(S)})$ becomes the map $0 = \mathrm{d}s|_{S^k(V)}(v) : E|_v^* \to T_v^*(S^k(V))$. By the first part, $\phi_{C(S)}$ has a left inverse

$\gamma : \mathcal{F}_{C(S)} \to \mathcal{E}_{C(S)}$, so $\underline{w}^*(\gamma)$ is a left inverse for $0 : E|_v^* \to T_v^*(S^k(V))$, which forces rank $E = 0$. Thus $\mathbf{S}_{V,E,s} = \mathbf{S}_{V,0,0} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(V)$ is a manifold, so $\mathbf{U}$ is a manifold. Since we can cover $\mathbf{X}$ by such $\mathbf{U}$, $\mathbf{X}$ is a manifold. $\square$

In Proposition 3.28, a d-manifold $\boldsymbol{X}$ is a manifold provided $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ has a left inverse. But in the next example, $\mathbf{X}$ is a d-manifold with corners such that $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ has a left inverse, but $\mathbf{X}$ is not a manifold. This shows that the conditions over $C_k(\mathbf{X})$ for $k \geqslant 1$ in Proposition 7.13 are really necessary.

Example 7.14. Let $\mathbf{X}$ be the d-manifold with corners defined informally by the inequality $x^3 \geqslant 0$ in $\mathbb{R}$. We may write $\mathbf{X}$ as a b-transverse fibre product $\boldsymbol{\mathbb{R}} \times_{\boldsymbol{g},\boldsymbol{\mathbb{R}},\boldsymbol{h}} \boldsymbol{[0,\infty)}$ in $\mathbf{dSpa^c}$, where $\boldsymbol{g},\boldsymbol{h} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(g,h)$ for $g : \mathbb{R} \to \mathbb{R}$ and $h : [0,\infty) \to \mathbb{R}$ given by $g(x) = x^3$ and $h(y) = y$. Equivalently, we may write $\mathbf{X} = \mathbf{S}_{V,E,s}$, where $V = \mathbb{R} \times [0,\infty)$ with coordinates (x,y), and E is the trivial vector bundle $\mathbb{R} \times V \to V$, and $s(x,y) = y - x^3$.

Then the d-space $\boldsymbol{X}$ in $\mathbf{X}$ is equivalent to $\boldsymbol{[0,\infty)} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}([0,\infty))$, and so $T^*\boldsymbol{X}$ is a vector bundle, and $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ has a left inverse. But the d-space $\boldsymbol{\partial X} \simeq \boldsymbol{\mathbb{R}} \times_{\boldsymbol{g},\boldsymbol{\mathbb{R}},\boldsymbol{0}} \boldsymbol{*}$ is a non-reduced point, not a manifold, and $\phi_{\partial X}$ does not have a left inverse. So $\mathbf{X}$ is not a manifold, and $\phi_{C(X)}$ has an inverse over $C_0(\mathbf{X}) \simeq \mathbf{X}$, but does not have a left inverse over $C_1(\mathbf{X}) \simeq \partial\mathbf{X}$.

7.3 Differential-geometric picture of 1-morphisms

We now develop versions of the material of §3.4 for d-manifolds with corners. Here is the analogue of Definition 3.30.

Definition 7.15. Let V, W be manifolds with corners, $E \to V$, $F \to W$ be vector bundles, and $s : V \to E$, $t : W \to F$ be smooth sections. Write $\mathbf{X} = \mathbf{S}_{V,E,s}$, $\mathbf{Y} = \mathbf{S}_{W,F,t}$ for the 'standard model' principal d-manifolds with corners from Definition 7.2. Suppose $f : V \to W$ is a smooth map, and $\hat{f} : E \to f^*(F)$ is a morphism of vector bundles on V satisfying $\hat{f} \circ s = f^*(t) + O(s^2)$ in $C^\infty(f^*(F))$, where $f^*(t) = t \circ f$, and $O(s^2)$ is as in Definition 3.29. By Definition 7.2, the d-spaces $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$ and $\boldsymbol{Y} = \boldsymbol{S}_{W,F,t}$ in $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ and $\mathbf{Y} = (\boldsymbol{Y}, \boldsymbol{\partial Y}, \boldsymbol{i}_{\mathbf{Y}}, \omega_{\mathbf{Y}})$ are defined as in Definition 3.13 for the without corners case. Define a 1-morphism $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{dSpa}$ using $f, \hat{f}$ exactly as in Definition 3.30.

To show $\boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism in $\mathbf{dMan^c}$, suppose $x \in \boldsymbol{X}$ with $\boldsymbol{g}(x) = y \in \boldsymbol{Y}$, and $y' \in \boldsymbol{\partial Y}$ with $\boldsymbol{i}_{\mathbf{Y}}(y') = y$. Then $x \in V$ with $s(x) = 0$, and $f(x) = y \in W$ with $t(w) = 0$, and $y' \in \partial W$ with $i_W(y') = y$. Let $(\tilde{W}, b)$ be a boundary defining function for W at y', as in Definition 5.4, so that $y \in \tilde{W} \subseteq W$ is open and $b : \tilde{W} \to [0,\infty)$ is smooth. Then by Definition 5.5, either:

(i) there exists an open $x \in \tilde{V} \subseteq f^{-1}(\tilde{W}) \subseteq V$ such that $(\tilde{V}, b \circ f|_{\tilde{V}})$ is a boundary defining function for V at some $x' \in i_V^{-1}(x) \subset \partial V$; or

(ii) there exists open $x \in \tilde{V} \subseteq f^{-1}(\tilde{W}) \subseteq V$ with $b \circ f|_{\tilde{V}} = 0$.

In case (i), define $\tilde{\boldsymbol{V}}, \tilde{\boldsymbol{W}}, \boldsymbol{b} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(\tilde{V}, \tilde{W}, b)$, and set $\tilde{\boldsymbol{X}} = \boldsymbol{\pi}_{\boldsymbol{V}}^{-1}(\tilde{\boldsymbol{V}})$ and $\tilde{\boldsymbol{Y}} = \boldsymbol{\pi}_{\boldsymbol{W}}{}^{-1}(\tilde{\boldsymbol{W}})$. Then $x \in \tilde{\boldsymbol{X}} \subseteq \boldsymbol{X}$ and $y \in \tilde{\boldsymbol{Y}} \subseteq \boldsymbol{Y}$ are open, with $\boldsymbol{g}(\tilde{\boldsymbol{X}}) \subseteq \tilde{\boldsymbol{Y}}$.

By Definition 7.2, since $(\tilde{V}, b\circ f|_{\tilde{V}})$ and $(\tilde{W}, b)$ are boundary defining functions for V, W at x', y', so $(\tilde{\boldsymbol{X}}, \boldsymbol{b}\circ\boldsymbol{f}\circ\boldsymbol{\pi_V}|_{\tilde{\boldsymbol{X}}})$ and $(\tilde{\boldsymbol{Y}}, \boldsymbol{b}\circ\boldsymbol{\pi_W}|_{\tilde{\boldsymbol{Y}}})$ are boundary defining functions for $\mathbf{X}, \mathbf{Y}$ at x', y', respectively. Since $\boldsymbol{\pi_W}\circ\boldsymbol{g} = \boldsymbol{f}\circ\boldsymbol{\pi_V}$, we see that $\bigl(\tilde{\boldsymbol{X}}, (\boldsymbol{b}\circ\boldsymbol{\pi_W}|_{\tilde{\boldsymbol{Y}}})\circ\boldsymbol{g}|_{\tilde{\boldsymbol{X}}}\bigr)$ is a boundary defining function for $\mathbf{X}$ at x'. Hence Definition 6.2(i) holds for $\boldsymbol{g}$ at x', y'. Similarly, in case (ii) Definition 6.2(ii) holds for $\boldsymbol{g}$ at x, y'. Therefore $\boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism in $\mathbf{dMan^c}$, and

$$\begin{aligned} \underline{S}_{\boldsymbol{g}} &= \bigl\{(x', y') \in \partial\underline{X} \times_{\underline{Y}} \partial\underline{Y} \subseteq \partial\underline{V} \times_{\underline{W}} \partial\underline{W} : (x', y') \in S_f\bigr\}, \\ \underline{T}_{\boldsymbol{g}} &= \bigl\{(x, y') \in \underline{X} \times_{\underline{Y}} \partial\underline{Y} \subseteq \underline{V} \times_{\underline{W}} \partial\underline{W} : (x, y') \in T_f\bigr\}. \end{aligned} \tag{7.5}$$

We will also write $\boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ as $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$, and call it a *standard model* 1*-morphism.*

Suppose now that $\tilde{V} \subseteq V$ is open, with inclusion map $i_{\tilde{V}} : \tilde{V} \to V$. Write $\tilde{E} = E|_{\tilde{E}} = i_{\tilde{V}}^*(E)$ and $\tilde{s} = s|_{\tilde{E}}$. Define $\boldsymbol{i}_{\tilde{V},V} = \mathbf{S}_{i_{\tilde{V}},\mathrm{id}_{\tilde{E}}} : \mathbf{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \mathbf{S}_{V,E,s}$.

Here are the analogues of Lemmas 3.32 and 3.33. The proofs are obvious generalizations of those in §3.4.

Lemma 7.16. *Let V, W be manifolds with corners, $E \to V$, $F \to W$ vector bundles, $s : V \to E$, $t : W \to F$ smooth sections, $f_1, f_2 : V \to W$ smooth maps, and $\hat{f}_1 : E \to f_1^*(F)$, $\hat{f}_2 : E \to f_2^*(F)$ vector bundle morphisms with $\hat{f}_1 \circ s = f_1^*(t) + O(s^2)$ and $\hat{f}_2 \circ s = f_2^*(t) + O(s^2)$, so we have* 1*-morphisms $\mathbf{S}_{f_1,\hat{f}_1}, \mathbf{S}_{f_2,\hat{f}_2} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$. Then $\mathbf{S}_{f_1,\hat{f}_1} = \mathbf{S}_{f_2,\hat{f}_2}$ if and only if $f_1 = f_2 + O(s^2)$ and $\hat{f}_1 = \hat{f}_2 + O(s)$, in the notation of Definition* 3.29*.*

Lemma 7.17. *Let V be a manifold with corners, $E \to V$ a vector bundle, $s : V \to E$ a smooth section, and $\tilde{V} \subseteq V$ be open. Then $\boldsymbol{i}_{\tilde{V},V} : \mathbf{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \mathbf{S}_{V,E,s}$ is a* 1*-isomorphism with an open d-submanifold of $\mathbf{S}_{V,E,s}$. If $\tilde{V}$ is an open neighbourhood of $s^{-1}(0)$ in V then $\boldsymbol{i}_{\tilde{V},V} : \mathbf{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \mathbf{S}_{V,E,s}$ is a* 1*-isomorphism.*

From (7.5) we deduce:

Lemma 7.18. *In Definition* 7.15*, the* 1*-morphism $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$, is simple, semisimple, or flat, if and only if f is simple, semisimple, or flat respectively near $\{v \in V : s(v) = 0\} \subseteq V$.*

Here is the analogue of Theorem 3.34:

Theorem 7.19. *Let V, W be manifolds with corners, $E \to V$, $F \to W$ be vector bundles, and $s : V \to E$, $t : W \to F$ be smooth sections. Define principal d-manifolds with corners $\mathbf{X} = \mathbf{S}_{V,E,s}$, $\mathbf{Y} = \mathbf{S}_{W,F,t}$, with topological spaces $X = \{v \in V : s(v) = 0\}$ and $Y = \{w \in W : t(w) = 0\}$. Suppose $\boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ is a* 1*-morphism. Then there exist an open neighbourhood $\tilde{V}$ of X in V, a smooth map $f : \tilde{V} \to W$, and a morphism of vector bundles $\hat{f} : \tilde{E} \to f^*(F)$ with $\hat{f} \circ \tilde{s} = f^*(t)$, where $\tilde{E} = E|_{\tilde{V}}$, $\tilde{s} = s|_{\tilde{V}}$, such that $\boldsymbol{g} = \mathbf{S}_{f,\hat{f}} \circ \boldsymbol{i}_{\tilde{V},V}^{-1}$, where $\boldsymbol{i}_{\tilde{V},V} : \mathbf{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \mathbf{S}_{V,E,s}$, $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \mathbf{S}_{W,F,t}$, and $\boldsymbol{i}_{\tilde{V},V}^{-1}$ exists by Lemma* 7.17*.*

Proof. We explain how to modify the proof of Theorem 3.34 to include corners. In the first part of the proof we fixed $x \in X \subseteq V$ with $g(x) = y \in Y \subseteq W$, and chose coordinates $(x_1, \ldots, x_m)$ on V near x and $(y_1, \ldots, y_n)$ on W near y which induced isomorphisms of C^∞-local rings $C^\infty_x(V) \cong C^\infty_0(\mathbb{R}^m)$, $C^\infty_y(W) \cong C^\infty_0(\mathbb{R}^n)$, and we showed that we can choose open $x \in \tilde{V}_x \subseteq V$ and smooth $f_x : \tilde{V}_x \to W$ such that (3.23) commutes with $\tilde{V}_x, f_x$ in place of $\tilde{V}, f$.

When V, W have corners, setting $k = |i_V^{-1}(x)|$, $l = |i_W^{-1}(y)|$, we instead have $C^\infty_x(V) \cong C^\infty_0(\mathbb{R}^m_k)$ and $C^\infty_y(W) \cong C^\infty_0(\mathbb{R}^n_l)$, where the coordinates on V near x and on $\mathbb{R}^m_k$ are $(x_1, \ldots, x_m)$ with $x_1, \ldots, x_k \in [0, \infty)$, and the coordinates on W near y and on $\mathbb{R}^n_l$ are $(y_1, \ldots, y_n)$ with $y_1, \ldots, y_l \in [0, \infty)$. Write $i_V^{-1}(x) = \{v'_1, \ldots, v'_k\} \subseteq \underline{\partial X} \subseteq \partial V$ and $i_W^{-1}(y) = \{w'_1, \ldots, w'_l\} \subseteq \underline{\partial Y} \subseteq \partial W$, where v'_i corresponds to the local boundary component $x_i = 0$ of $\mathbb{R}^m_k$ at 0 for $i = 1, \ldots, k$, and w'_j corresponds to $y_j = 0$.

For each $j = 1, \ldots, l$, either $(x, w'_j) \in \underline{T}_{\boldsymbol{g}}$, or $(x, w'_j) = \underline{j}_{\boldsymbol{g}}(v'_i, w'_j)$ for some $i = 1, \ldots, k$ with $(v'_i, w'_j) \in \underline{S}_{\boldsymbol{g}}$, and this v'_i is unique as $\underline{j}_{\boldsymbol{g}}$ is injective. Renumber $w'_1, \ldots, w'_l, y_1, \ldots, y_l$ so that $(v'_{b_j}, w'_j) \in \underline{S}_{\boldsymbol{g}}$ for $j = 1, \ldots, a$ and some $a \leqslant l$ and $b_1, \ldots, b_a \in \{1, \ldots, k\}$, and $(x, w'_j) \in \underline{T}_{\boldsymbol{g}}$ for $j = a + 1, \ldots, l$. We want to construct a morphism $\psi : C^\infty_0(\mathbb{R}^n_l) \to C^\infty_0(\mathbb{R}^m_k)$ such that $\operatorname{Spec}\psi$ is a germ at 0 of smooth maps $\mathbb{R}^m_k \to \mathbb{R}^n_l$, and if f_x is the corresponding germ at x of smooth maps $V \to W$, the following analogue of (3.25) commutes:

$$
\begin{tikzcd}
C^\infty_0(\mathbb{R}^n_l) \arrow[r, "\psi"'] \arrow[d, "\cong"] & C^\infty_0(\mathbb{R}^m_k) \arrow[d, "\cong"'] \\
C^\infty_y(W) \arrow[r, "f_x^*"] \arrow[d] & C^\infty_x(V) \arrow[d] \\
C^\infty_y(W)/I^2_{t,y} \arrow[r, "\phi'_{x,y}"] & C^\infty_x(V)/I^2_{s,x} \\
C^\infty(W)/I^2_t \arrow[r, "\phi'"] \arrow[u] & C^\infty(V)/I^2_s. \arrow[u]
\end{tikzcd}
\tag{7.6}
$$

Suppose we had such a ψ. For $j = 1, \ldots, n$, choose $\psi_j \in C^\infty(\mathbb{R}^m_k)$ such that the image of ψ_j in $C^\infty_0(\mathbb{R}^m_k)$ is $\psi(y_j)$. Then in coordinates $(x_1, \ldots, x_m)$ on $\tilde{V}_x \subseteq V$ and $(y_1, \ldots, y_n)$ on W near y, the smooth map $f_x : \tilde{V}_x \to W$ acts by

$$f_x : (x_1, \ldots, x_m) \longmapsto (y_1, \ldots, y_n) = \bigl(\psi_1(x_1, \ldots, x_m), \ldots, \psi_n(x_1, \ldots, x_n)\bigr).$$

The condition that f_x is a smooth map of manifolds with corners near x turns out to be equivalent to:

(a) for all $j = 1, \ldots, a$ and $(x_1, \ldots, x_m)$ near 0 in $\mathbb{R}^m_k$ with $x_{b_j} = 0$, we have $\psi_j(x_1, \ldots, x_m) = 0$ and $\frac{\partial \psi_j}{\partial x_{b_j}}(x_1, \ldots, x_m) > 0$; and

(b) for all $j = a+1, \ldots, l$ and $(x_1, \ldots, x_m)$ near 0 in $\mathbb{R}^m_k$, $\psi_j(x_1, \ldots, x_m) = 0$.

Here (a) comes from $(v'_{b_j}, w'_j) \in \underline{S}_{\boldsymbol{g}}$ for $j = 1, \ldots, a$, and says that if $(x_1, \ldots, x_m)$ in $\mathbb{R}^m_k$ is near 0 with $x_{b_j} = 0$ then $f_x(x_1, \ldots, x_m) = (y_1, \ldots, y_n)$ has $y_j = 0$, and f_x pulls the boundary defining function $(\mathbb{R}^n_l, y_j)$ for $\mathbb{R}^n_l$ at $(y_1, \ldots, y_n)$ back to the boundary defining function $(\mathbb{R}^n_l, \psi_i)$ for $\mathbb{R}^m_k$ at $(x_1, \ldots, x_m)$, which is equivalent to the boundary defining function $(\mathbb{R}^n_l, x_{b_j})$. Similarly (b) comes from $(x, w'_j) \in \underline{T}_{\boldsymbol{g}}$ for $j = a+1, \ldots, l$, and says that if $(x_1, \ldots, x_m) \in \mathbb{R}^m_k$ is near 0 with $f_x(x_1, \ldots, x_m) = (y_1, \ldots, y_n)$ then $y_i = 0$, and the boundary defining function $(\mathbb{R}^n_l, y_i)$ for $\mathbb{R}^n_l$ at $(y_1, \ldots, y_n)$ pulls back to zero near $(x_1, \ldots, x_m)$.

Let $\tilde{I}_{s,x}, \tilde{I}_{t,y}$ be the ideals in $C^\infty_0(\mathbb{R}^m_k), C^\infty_0(\mathbb{R}^n_l)$ identified with $I_{s,x}, I_{t,y}$ by $C^\infty_x(V) \cong C^\infty_0(\mathbb{R}^m_k)$ and $C^\infty_y(W) \cong C^\infty_0(\mathbb{R}^n_l)$, and let $\tilde{\phi}'_{x,y} : C^\infty_0(\mathbb{R}^n_l)/\tilde{I}^2_{t,y} \to C^\infty_0(\mathbb{R}^m_k)/\tilde{I}^2_{s,x}$ be identified with $\phi'_{x,y}$. Then (7.6) commuting is equivalent to

$$\psi_j(x_1, \ldots, x_m) + \tilde{I}^2_{s,x} = \tilde{\phi}'_{x,y}(y_j + \tilde{I}^2_{t,y}) \quad \text{for } j = 1, \ldots, n. \tag{7.7}$$

We must show that we can choose $\psi_1, \ldots, \psi_n \in C^\infty(\mathbb{R}^m_k)$ such that (7.7) and (a),(b) above hold. For $j = 1, \ldots, a$, using Definition 6.2(i) for $\boldsymbol{g}$ at $(v'_{b_j}, w'_j) \in \underline{S}_{\boldsymbol{g}}$ and noting that $x_{b_j} + \tilde{I}^2_{s,x}$, $y_j + \tilde{I}^2_{t,y}$ are identified with boundary defining functions for $\mathbf{X}$ at v'_{b_j} and $\mathbf{Y}$ at w'_j, we find that $\tilde{\phi}'_{x,y}(y_j + \tilde{I}^2_{t,y})$ lies in the ideal $(x_{b_j} + \tilde{I}^2_{s,x})$ in $C^\infty_0(\mathbb{R}^m_k)/\tilde{I}^2_{s,x}$. Hence we can choose $\psi_j \in C^\infty(\mathbb{R}^m_k)$ of the form $\psi_j(x_1, \ldots, x_m) = x_{b_j} \cdot c_j(x_1, \ldots, x_m)$ for $c_j \in C^\infty(\mathbb{R}^m_k)$ so that (7.7) holds. Furthermore, the compatibility condition of $\boldsymbol{g}$ with $\omega_{\mathbf{X}}, \omega_{\mathbf{Y}}$ imply that we can take $c_j(0, \ldots, 0) > 0$. So (a) holds.

For $j = a+1, \ldots, l$, using Definition 6.2(ii) for $\boldsymbol{g}$ at $(x, w'_j) \in \underline{T}_{\boldsymbol{g}}$ and noting that $y_i + \tilde{I}^2_{t,y}$ is identified with a boundary defining function for $\mathbf{Y}$ at w'_i shows that $\tilde{\phi}'_{x,y}(y_j + \tilde{I}^2_{t,y}) = 0$. So we may take $\psi_j = 0$, and (7.7) and (b) hold. For $j = l+1, \ldots, n$ we choose arbitrary $\psi_j \in C^\infty(\mathbb{R}^m_k)$ such that (7.7) holds. Thus we can choose $\psi_1, \ldots, \psi_n$ to satisfy all the conditions. These define ψ in (7.6), where f_x is a germ at x of smooth functions $V \to W$. This generalizes the choice of $\tilde{V}_x, f_x$ in the proof of Theorem 3.34 to the corners case.

Next we explain how to generalize the proof which joins $\tilde{V}_x, f_x$ for $x \in X$ using a partition of unity on $\tilde{V}$ to get a global choice of open $X \subseteq \tilde{V} \subseteq V$ and smooth $f : \tilde{V} \to W$. The important point here is that the Riemannian metric h on W should be chosen so that the strata $S^k(W)$ of W from §5.1 should be *locally geodesically closed*, that is, a geodesic in $\bigl(S^k(W), h|_{S^k(W)}\bigr)$ should also be a geodesic in (W, h). This implies that the definition of $U_W \subseteq W \times W$ and the smooth map $\Gamma : U_W \times [0,1] \to W$ still works (in particular, U_W is open), and if $(w_0, w_1) \in U_W$ with w_0, w_1 lie in the same local component of $S^k(W)$ then $\Gamma(w_0, w_1, t) \in S^k(W)$ for $t \in [0,1]$, as the geodesic from w_0 to w_1 in $S^k(W)$ is the geodesic from w_0 to w_1 in W.

Let $\tilde{V}, f$ be constructed as in the proof of Theorem 3.34. We claim that making $\tilde{V}$ smaller if necessary, $f : \tilde{V} \to W$ is a smooth map of manifolds with corners. To see this, it is enough to show that in the case $S = \{x_1, x_2\}$, the map f in (3.27) made by combining the smooth maps $f_{x_1} : \tilde{V}_{x_1} \to W$ and $f_{x_2} : \tilde{V}_{x_2} \to W$ is a smooth map of manifolds with corners near $X \subseteq \tilde{V}$. The important point is this: suppose $x \in \tilde{V}_{x_1} \cap \tilde{V}_{x_2} \cap X$. Then $f_{x_1}(x) = f_{x_2}(x) = g(x) = y \in Y$. We have $x \in S^k(V)$ and $y \in S^l(W)$ for some unique k, l. Suppose v lies in the same connected component of $\tilde{V}_{x_1} \cap \tilde{V}_{x_2} \cap S^k(V)$ as x. Then $f_{x_1}(v)$ and $f_{x_2}(v)$ both lie in $S^l(W)$, since it is a property of smooth maps of manifolds with corners

$f : V \to W$ that $\mathrm{depth}_W f(v)$ is locally constant on each stratum $S^k(V)$.

Therefore for $v \in \tilde V_{x_1} \cap \tilde V_{x_2}$ close to x, $f_{x_1}(v)$ and $f_{x_2}(v)$ lie in the same stratum $S^l(W)$ of W. We can also suppose they lie in the same local component, so that $f(v) = \Gamma\bigl(f_{x_1}(v), f_{x_2}(v), \eta_{x_2}(v)\bigr)$ in (3.27) also lies in $S^l(W)$. Thus, the construction of f in the proof of Theorem 3.34 yields a weakly smooth map $f : \tilde V \to W$ such that $\mathrm{depth}_W f(v)$ is locally constant on each stratum $S^k(\tilde V)$ near X. This and the f_{x_i} smooth and coinciding at X imply that f is smooth near X, so making $\tilde V$ smaller if necessary, f is smooth. This generalizes the choice of $\tilde V$ and $f : \tilde V \to W$ in the proof of Theorem 3.34 to the corners case. The rest of the proof works with essentially no changes. □

We can generalize Definition 3.35 to interpret 2-morphisms $\lambda : \mathbf{S}_{f,\hat f} \Rightarrow \mathbf{S}_{g,\hat g}$ in $\mathbf{dMan^c}$ using morphisms $\Lambda : E \to f^*(TW)$ on V, but we will not use this.

7.4 Equivalences of d-manifolds with corners, and gluing

Proposition 2.21 and Corollary 2.24 gave sufficient conditions for a 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{dSpa}$ to be an equivalence or étale. Theorem 3.36 showed that when $\boldsymbol{X}, \boldsymbol{Y} \in \mathbf{dMan}$, these conditions can be weakened. Similarly, Proposition 6.21 and Corollary 6.23 gave sufficient conditions for a 1-morphism $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dSpa^c}$ to be an equivalence or étale. In an analogue of Theorem 3.36, we will show that when $\mathbf{X}, \mathbf{Y} \in \mathbf{dMan^c}$, these conditions can be weakened.

Theorem 7.20. *Suppose $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism of d-manifolds with corners. Then the following are equivalent:*

(i) *$\boldsymbol{f}$ is étale;*

(ii) *$\boldsymbol{f}$ is simple and flat, in the sense of Definition 6.11, and $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\mathbf{Y}) \to T^*\mathbf{X}$ is an equivalence in $\mathrm{vqcoh}(\underline{X})$; and*

(iii) *$\boldsymbol{f}$ is simple and flat, and (2.34) is a split short exact sequence in $\mathrm{qcoh}(\underline{X})$.*

If in addition $f : X \to Y$ is a bijection, then $\boldsymbol{f}$ is an equivalence in $\mathbf{dMan^c}$.

Proof. The proof follows that of Theorem 3.36, using Proposition 6.21, Corollary 6.23, Theorem 7.19 and Corollary 7.11 in place of Proposition 2.21, Corollary 2.24, Theorem 3.34, and Proposition 3.25, respectively. The main difference is this: rather than having open $0 \in V \subseteq \mathbb{R}^m$ and $0 \in W \subseteq \mathbb{R}^n$ and s, t with $s(0) = \mathrm{d}s(0) = 0$ and $t(0) = \mathrm{d}t(0) = 0$, we have open $0 \in V \subseteq \mathbb{R}^m_k$ and $0 \in W \subseteq \mathbb{R}^n_l$ and s, t with $s(0) = t(0) = 0$ and

$$\tfrac{\partial s}{\partial x_i}(0) = 0,\ i = k+1, \ldots, m, \text{ and } \tfrac{\partial t}{\partial y_j}(0) = 0,\ j = l+1, \ldots, n. \tag{7.8}$$

To show (ii),(iii) imply (i), suppose (ii),(iii) hold. Let $\underline{i}_{\mathbf{X}}^{-1}(x) = \{v'_1, \ldots, v'_k\} \subseteq \underline{\partial X} \subseteq \partial X$, where v'_i corresponds to the boundary component $x_i = 0$ in $\mathbb{R}^m_k$, and $\underline{i}_{\mathbf{X}}^{-1}(y) = \{w'_1, \ldots, w'_l\} \subseteq \underline{\partial Y} \subseteq \partial W$, where v'_i corresponds to the boundary component $x_i = 0$ in $\mathbb{R}^m_k$, and w'_j to $y_j = 0$ in $\mathbb{R}^n_l$. For each $j = 1, \ldots, l$ we have $(x, w'_j) \in \underline{X} \times_{\underline{Y}} \underline{\partial Y} = \underline{j}_{\boldsymbol{f}}(\underline{S}_{\boldsymbol{f}})$ since $\underline{T}_{\boldsymbol{f}} = \emptyset$ as $\boldsymbol{f}$ is flat, so $(x, w'_j) =$

$\underline{j}_{\boldsymbol{f}}(v_i', w_j')$ for some $i = 1, \ldots, k$, and i is unique as $\underline{j}_{\boldsymbol{f}}$ is injective. Also, each v_i' is $\underline{s}_{\boldsymbol{f}}(v_i', w_j')$ for some unique j, since $\underline{s}_{\boldsymbol{f}}$ is a bijection as $\boldsymbol{f}$ is simple. Hence there is a 1-1 correspondence between $v_1', \ldots, v_k'$ and $w_1', \ldots, w_l'$ such that $v_i' \leftrightarrow w_j'$ if $(v_i', w_j') \in \underline{S}_{\boldsymbol{f}}$. So $k = l$. Renumber $w_1', \ldots, w_l'$ and $y_1, \ldots, y_l$ so that $(v_i', w_i') \in \underline{S}_{\boldsymbol{f}}$ for $i = 1, \ldots, k$. Then writing $f : V \to W$ as $f = (f_1, \ldots, f_n)$ for $f_j = f_j(x_1, \ldots, x_m)$, we see that

$$\frac{\partial f_j}{\partial x_i}(0, \ldots, 0) = \begin{cases} 0, & i, j \leqslant k \text{ and } i \neq j, \\ > 0, & i = j \leqslant k, \\ \text{in } \mathbb{R}, & i > k \text{ or } j > k. \end{cases} \tag{7.9}$$

The condition that $\underline{0}^*(\Omega_{\boldsymbol{g}})$ is an equivalence in the proof of Theorem 3.36 then becomes that the following is a (split) exact sequence of real vector spaces:

$$0 \longrightarrow F|_0^* \xrightarrow{\hat{f}|_0^* \oplus -\mathrm{d}t(0)^*} E|_0^* \oplus T_0^*\mathbb{R}^n \xrightarrow{\mathrm{d}f|_0^* \oplus \mathrm{d}s(0)^*} T_0^*\mathbb{R}^m \longrightarrow 0. \tag{7.10}$$

But using (7.8)–(7.9) and $k = l$ we can show that (7.10) is exact only if $\mathrm{d}f|_0 : T_0V \to T_0W$ and $\hat{f}|_0 : E|_0 \to F|_0$ are isomorphisms. The rest of the proof is as for Theorem 3.36. □

Here is the analogue of Theorem 3.39 for d-manifolds with corners. To prove it, note that the proof of Theorem 3.39 shows that (2.34) is split exact if and only if (7.11) is exact for all $v \in s^{-1}(0)$. The theorem then follows from Lemma 7.18 and Theorem 7.20.

Theorem 7.21. *Let V, W be manifolds with corners, $E \to V$, $F \to W$ be vector bundles, $s : V \to E$, $t : W \to F$ be smooth sections, $f : V \to W$ be smooth, and $\hat{f} : E \to f^*(F)$ be a morphism of vector bundles on V with $\hat{f} \circ s = f^*(t) + O(s^2)$. Then Definitions* 7.2 *and* 7.15 *define principal d-manifolds with corners $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$ and a 1-morphism $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$. This $\mathbf{S}_{f,\hat{f}}$ is étale if and only if f is simple and flat near $s^{-1}(0) \subseteq V$, in the sense of Definition* 5.9, *and for each $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_vV \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} E_v \oplus T_wW \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)} F_w \longrightarrow 0. \tag{7.11}$$

Also $\mathbf{S}_{f,\hat{f}}$ is an equivalence if and only if in addition $f|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection, where $s^{-1}(0) = \{v \in V : s(v) = 0\}$, $t^{-1}(0) = \{w \in W : t(w) = 0\}$.

We can also prove the following analogues of Theorems 3.41 and 3.42. Theorem 7.22 is immediate from Theorem 6.25. Theorem 7.23 follows from Theorems 6.26 and 7.21 in the same way as Theorem 3.42; we include the condition that e_{ij} is simple and flat in (e) so that, together with (iii), Theorem 7.21 implies that $\mathbf{S}_{e_{ij},\hat{e}_{ij}} : \mathbf{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}} \to \mathbf{S}_{V_j,E_j,s_j}$ is an equivalence with its image in $\mathbf{dMan^c}$.

Theorem 7.22. *Suppose $\mathbf{X}, \mathbf{Y}$ are d-manifolds with corners with* $\operatorname{vdim} \mathbf{X} = \operatorname{vdim} \mathbf{Y} = n$ *in $\mathbb{Z}$, and $\mathbf{U} \subseteq \mathbf{X}$, $\mathbf{V} \subseteq \mathbf{Y}$ are open d-submanifolds, and $\boldsymbol{f} : \mathbf{U} \to \mathbf{V}$*

is an equivalence in $\mathbf{dMan^c}$. *At the level of topological spaces, we have open* $U \subseteq X$, $V \subseteq Y$ *with a homeomorphism* $f : U \to V$, *so we can form the quotient topological space* $Z := X \amalg_f Y = (X \amalg Y)/\sim$, *where the equivalence relation* $\sim$ *on* $X \amalg Y$ *identifies* $u \in U \subseteq X$ *with* $f(u) \in V \subseteq Y$.

Suppose Z *is Hausdorff. Then there exist a d-manifold with corners* $\mathbf{Z}$ *with* $\operatorname{vdim} \mathbf{Z} = n$, *open d-submanifolds* $\hat{\mathbf{X}}, \hat{\mathbf{Y}}$ *in* $\mathbf{Z}$ *with* $\mathbf{Z} = \hat{\mathbf{X}} \cup \hat{\mathbf{Y}}$, *equivalences* $\boldsymbol{g} : \mathbf{X} \to \hat{\mathbf{X}}$ *and* $\boldsymbol{h} : \mathbf{Y} \to \hat{\mathbf{Y}}$ *such that* $\boldsymbol{g}|_{\mathbf{U}}$ *and* $\boldsymbol{h}|_{\mathbf{V}}$ *are both equivalences with* $\hat{\mathbf{X}} \cap \hat{\mathbf{Y}}$, *and a 2-morphism* $\eta : \boldsymbol{g}|_{\mathbf{U}} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f} : \mathbf{U} \to \hat{\mathbf{X}} \cap \hat{\mathbf{Y}}$. *Furthermore,* $\mathbf{Z}$ *is independent of choices up to equivalence.*

Theorem 7.23. *Suppose we are given the following data:*

(a) *an integer* n;

(b) *a Hausdorff, second countable topological space* X;

(c) *an indexing set* I, *and a total order* $<$ *on* I;

(d) *for each* i *in* I, *a manifold with corners* V_i, *a vector bundle* $E_i \to V_i$ *with* $\dim V_i - \operatorname{rank} E_i = n$, *a smooth section* $s_i : V_i \to E_i$, *and a homeomorphism* $\psi_i : X_i \to \hat{X}_i$, *where* $X_i = \{v_i \in V_i : s_i(v_i) = 0\}$ *and* $\hat{X}_i \subseteq X$ *is open; and*

(e) *for all* $i < j$ *in* I, *an open submanifold* $V_{ij} \subseteq V_i$, *a simple, flat map* $e_{ij} : V_{ij} \to V_j$, *and a morphism of vector bundles* $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$.

Let this data satisfy the conditions:

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if* $i < j$ *in* I *then* $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j)$, *and* $\psi_i(X_i \cap V_{ij}) = \hat{X}_i \cap \hat{X}_j$, *and* $\psi_i|_{X_i \cap V_{ij}} = \psi_j \circ e_{ij}|_{X_i \cap V_{ij}}$, *and if* $v_i \in V_i$ *with* $s_i(v_i) = 0$ *and* $v_j = e_{ij}(v_i)$ *then the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} E_j|_{v_j} \longrightarrow 0;$$

(iii) *if* $i < j < k$ *in* I *then* $e_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = e_{jk} \circ e_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} + O(s_i^2)$ *and* $\hat{e}_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = e_{ij}|^*_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} + O(s_i)$.

Then there exist a d-manifold with corners $\mathbf{X}$ *with* $\operatorname{vdim} \mathbf{X} = n$ *and topological space* X, *and a* 1*-morphism* $\boldsymbol{\psi}_i : \mathbf{S}_{V_i, E_i, s_i} \to \mathbf{X}$ *in* $\mathbf{dMan^c}$ *with underlying continuous map* ψ_i *which is an equivalence with the open d-submanifold* $\hat{\mathbf{X}}_i \subseteq \mathbf{X}$ *corresponding to* $\hat{X}_i \subseteq X$ *for all* $i \in I$, *such that for all* $i < j$ *in* I *there exists a* 2*-morphism* $\eta_{ij} : \boldsymbol{\psi}_j \circ \mathbf{S}_{e_{ij}, \hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i \circ \boldsymbol{i}_{V_{ij}, V_i}$, *where* $\mathbf{S}_{e_{ij}, \hat{e}_{ij}} : \mathbf{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}} \to \mathbf{S}_{V_j, E_j, s_j}$ *and* $\boldsymbol{i}_{V_{ij}, V_i} : \mathbf{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}} \to \mathbf{S}_{V_i, E_i, s_i}$ *are as in Definition* 7.15. *This* $\mathbf{X}$ *is unique up to equivalence in* $\mathbf{dMan^c}$.

Suppose also that Y *is a manifold with corners, and* $g_i : V_i \to Y$ *are smooth maps for all* $i \in I$, *and* $g_j \circ e_{ij} = g_i|_{V_{ij}} + O(s_i^2)$ *for all* $i < j$ *in* I. *Then there exist a* 1*-morphism* $\boldsymbol{h} : \mathbf{X} \to \mathbf{Y}$ *unique up to* 2*-isomorphism, where* $\mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(Y) = \mathbf{S}_{Y,0,0}$, *and* 2*-morphisms* $\zeta_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \mathbf{S}_{g_i,0}$ *for all* $i \in I$. *Here* $\mathbf{S}_{Y,0,0}$ *is from Definition* 7.2 *with vector bundle* E *and section* s *both zero, and* $\mathbf{S}_{g_i,0} : \mathbf{S}_{V_i,E_i,s_i} \to \mathbf{S}_{Y,0,0} = \mathbf{Y}$ *is from Definition* 7.15 *with* $\hat{g}_i = 0$.

All the ingredients in Theorem 7.23 are described wholly in differential-geometric or topological terms. So we can use it to construct d-manifold with corner structures on spaces coming from other areas of geometry, for instance, on moduli spaces of J-holomorphic curves with boundary in a Lagrangian.

7.5 Submersions, immersions, and embeddings

In §4.1 we defined two kinds of submersions (submersions and w-submersions), immersions and embeddings for d-manifolds without boundary. In §5.4 we defined two kinds of submersions (submersions and s-submersions), and in §5.7 three kinds of immersions (immersions, s- and sf-immersions), and embeddings for manifolds with corners. For d-manifolds with corners, in an analogue of Definition 4.4, we combine both alternatives, giving four types of submersions, and six types of immersions and embeddings.

Definition 7.24. Let $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ be a 1-morphism in $\mathbf{dMan^c}$. As in Definition 7.5, $T^*\mathbf{X}=(\mathcal{E}_X,\mathcal{F}_X,\phi_X)$ and $\underline{f}^*(T^*\mathbf{Y})=\bigl(\underline{f}^*(\mathcal{E}_Y),\underline{f}^*(\mathcal{F}_Y),\underline{f}^*(\phi_Y)\bigr)$ are virtual vector bundles on $\underline{X}$ of ranks $\operatorname{vdim}\mathbf{X},\operatorname{vdim}\mathbf{Y}$, and $\Omega_{\boldsymbol{f}}=(f'',f^2):\underline{f}^*(T^*\mathbf{Y})\to T^*\mathbf{X}$ is a 1-morphism in $\operatorname{vvect}(\underline{X})$. Also we have 1-morphisms $C(\boldsymbol{f}),\hat C(\boldsymbol{f}):C(\mathbf{X})\to C(\mathbf{Y})$ as in §6.7, so we can form $\Omega_{C(\boldsymbol{f})}:\underline{C(f)}^*(T^*C(\mathbf{Y}))\to T^*C(\mathbf{X})$ and $\Omega_{\hat C(\boldsymbol{f})}:\underline{\hat C(f)}^*(T^*C(\mathbf{Y}))\to T^*C(\mathbf{X})$. Then:

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{\boldsymbol{f}}$ is weakly injective. We call $\boldsymbol{f}$ an *sw-submersion* if it is also simple.

(b) We call $\boldsymbol{f}$ a *submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{C(\boldsymbol{f})}$ is injective. We call $\boldsymbol{f}$ an *s-submersion* if it is also simple.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $\Omega_{\boldsymbol{f}}$ is weakly surjective. We call $\boldsymbol{f}$ an *sw-immersion*, or *sfw-immersion*, if $\boldsymbol{f}$ is also simple, or simple and flat.

(d) We call $\boldsymbol{f}$ an *immersion* if $\Omega_{\hat C(\boldsymbol{f})}$ is surjective. We call $\boldsymbol{f}$ an *s-immersion* if $\boldsymbol{f}$ is also simple, and an *sf-immersion* if $\boldsymbol{f}$ is also simple and flat.

(e) We call $\boldsymbol{f}$ a *w-embedding, sw-embedding, sfw-embedding, embedding, s-embedding*, or *sf-embedding*, if $\boldsymbol{f}$ is a w-immersion, …, sf-immersion, respectively, and $f:X\to f(X)$ is a homeomorphism, so f is injective.

More generally, we make the same definitions for $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ a 1-morphism in $\mathbf{d\check{M}an^c}$ from Definition 7.8.

Parts (c)–(e) enable us to define *d-submanifolds* $\mathbf{X}$ of a d-manifold with corners $\mathbf{Y}$. *Open d-submanifolds* are open d-subspaces $\mathbf{X}$ in $\mathbf{Y}$. For more general d-submanifolds, we call $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ a *w-immersed, sw-immersed, sfw-immersed, immersed, s-immersed, sf-immersed, w-embedded, sw-embedded, sfw-embedded, embedded, s-embedded*, or *sf-embedded d-submanifold* of $\mathbf{Y}$ if $\mathbf{X},\mathbf{Y}$ are d-manifolds with corners and $\boldsymbol{f}$ is a w-immersion, …, sf-embedding, respectively.

Remark 7.25. The conditions in Definition 7.24(a)–(d) are chosen to make the results of Chapter 4 extend to d-manifolds with corners, and in particular so that Theorem 7.31 below holds. For (a),(b), note that submersions in $\mathbf{Man^c}$ are semisimple and flat, so making (w-)submersions in $\mathbf{dMan^c}$ semisimple and flat is natural. As $\boldsymbol{f}$ is flat $C(\boldsymbol{f}) = \hat{C}(\boldsymbol{f})$, so $\Omega_{C(\boldsymbol{f})}$ injective in (b) is equivalent to $\Omega_{\hat{C}(\boldsymbol{f})}$ injective. Also, in (a),(c) one can show that $\Omega_{\boldsymbol{f}}$ weakly injective or weakly surjective is equivalent to $\Omega_{\hat{C}(\boldsymbol{f})}$ weakly injective or weakly surjective. So in (a)–(d) we can rewrite the conditions on $\Omega_{\boldsymbol{f}}, \Omega_{C(\boldsymbol{f})}, \Omega_{\hat{C}(\boldsymbol{f})}$ uniformly to say that $\Omega_{\hat{C}(\boldsymbol{f})}$ is weakly injective/injective/weakly surjective/surjective.

If $\boldsymbol{f}$ is simple then $C(\boldsymbol{f})$ maps $C_k(\mathbf{X}) \to C_k(\mathbf{Y})$ for all k. Using this, one can show that $\boldsymbol{f}$ is an s-submersion if and only if $\boldsymbol{f}$ is simple and flat and $\Omega_{\boldsymbol{f}}$ is injective. Similarly, $\boldsymbol{f}$ is an sf-immersion if and only if $\boldsymbol{f}$ is simple and flat and $\Omega_{\boldsymbol{f}}$ is surjective.

We show that for each kind of submersion and immersion, $C(\boldsymbol{f}), \hat{C}(\boldsymbol{f}), \boldsymbol{f}_+$ and $\boldsymbol{f}_-$ inherit the same properties as $\boldsymbol{f}$. We do not include the various kinds of embeddings, as $\boldsymbol{f}$ injective does not imply $C(\boldsymbol{f}), \hat{C}(\boldsymbol{f}), \boldsymbol{f}_\pm$ injective.

Proposition 7.26. *Suppose* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *is a w-submersion, sw-submersion, …, s-immersion or sf-immersion in* $\mathbf{dMan^c}$. *Then* $C(\boldsymbol{f})$ *and* $\hat{C}(\boldsymbol{f}) : C(\mathbf{X}) \to C(\mathbf{Y})$ *from* §6.7 *are also w-submersions, …, sf-immersions in* $\mathbf{d\check{M}an^c}$. *If* $\boldsymbol{f}$ *is semisimple then* $\boldsymbol{f}_+ : \partial_+^{\boldsymbol{f}}\mathbf{X} \to \mathbf{Y}$ *and* $\boldsymbol{f}_- : \partial_-^{\boldsymbol{f}}\mathbf{X} \to \partial\mathbf{Y}$ *from* §6.3 *are also w-submersions, …, sf-immersions in* $\mathbf{dMan^c}$.

Proof. In the proof of Proposition 6.49 we explained that points of $C_k(C_l(\mathbf{X}))$ may be written $\bigl(x, \{x'_1, \ldots, x'_k\}, \{x'_{k+1}, \ldots, x'_{k+l}\}\bigr)$ for $x \in \boldsymbol{X}$ and distinct $x'_1, \ldots, x'_{k+l}$ in $\boldsymbol{i}_{\mathbf{X}}^{-1}(x)$. Using the ideas of §6.7 one can show there is a natural 1-morphism $\boldsymbol{I}_{\mathbf{X}}^{k,l} : C_k(C_l(\mathbf{X})) \to C_{k+l}(\mathbf{X})$ acting on points by $\bigl(x, \{x'_1, \ldots, x'_k\}, \{x'_{k+1}, \ldots, x'_{k+l}\}\bigr) \mapsto \bigl(x, \{x'_1, \ldots, x'_{k+l}\}\bigr)$. It is a local 1-isomorphism, and so étale. Let $\boldsymbol{I}_{\mathbf{X}} : C(C(\mathbf{X}) \to C(\mathbf{X})$ act by $\boldsymbol{I}_{\mathbf{X}}^{k,l}$ on $C_k(C_l(\mathbf{X}))$ for all $k, l \geqslant 0$.

We now claim that the following diagrams strictly commute in $\mathbf{d\check{M}an^c}$:

$$
\begin{tikzcd}
C(C(\mathbf{X})) \arrow[r, "\hat{C}(C(\boldsymbol{f}))"'] \arrow[d, "\boldsymbol{I}_{\mathbf{X}}"] & C(C(\mathbf{Y})) \arrow[d, "\boldsymbol{I}_{\mathbf{Y}}"'] \\
C(\mathbf{X}) \arrow[r, "\hat{C}(\boldsymbol{f})"] & C(\mathbf{Y}),
\end{tikzcd}
\qquad
\begin{tikzcd}
C(C(\mathbf{X})) \arrow[r, "\hat{C}(\hat{C}(\boldsymbol{f}))"'] \arrow[d, "\boldsymbol{I}_{\mathbf{X}}"] & C(C(\mathbf{Y})) \arrow[d, "\boldsymbol{I}_{\mathbf{Y}}"'] \\
C(\mathbf{X}) \arrow[r, "\hat{C}(\boldsymbol{f})"] & C(\mathbf{Y}).
\end{tikzcd}
\tag{7.12}
$$

To prove this, we verify that (7.12) commutes at the level of points, and use Theorem 6.29(a) for $C(\boldsymbol{f}), C(C(\boldsymbol{f}))$ and $\boldsymbol{\Pi}_{\mathbf{X}} \circ \boldsymbol{I}_{\mathbf{X}} = \boldsymbol{\Pi}_{\mathbf{X}} \circ \boldsymbol{\Pi}_{C(\mathbf{X})}$, $\boldsymbol{\Pi}_{\mathbf{Y}} \circ \boldsymbol{I}_{\mathbf{Y}} = \boldsymbol{\Pi}_{\mathbf{Y}} \circ \boldsymbol{\Pi}_{C(\mathbf{Y})}$. Since (7.13) commutes with $\boldsymbol{I}_{\mathbf{X}}, \boldsymbol{I}_{\mathbf{Y}}$ local 1-isomorphisms, we see that $\Omega_{\hat{C}(C(\boldsymbol{f}))}, \Omega_{\hat{C}(\hat{C}(\boldsymbol{f}))}$ are both locally identified with $\Omega_{\hat{C}(\boldsymbol{f})}$.

Hence $\Omega_{\hat{C}(\boldsymbol{f})}$ weakly injective/injective/weakly surjective/surjective implies that $\Omega_{\hat{C}(C(\boldsymbol{f}))}, \Omega_{\hat{C}(\hat{C}(\boldsymbol{f}))}$ are both weakly injective/…/surjective. But as in Remark 7.25, the conditions on $\Omega_{\boldsymbol{f}}, \Omega_{C(\boldsymbol{f})}, \Omega_{\hat{C}(\boldsymbol{f})}$ in Definition 7.24(a)–(d) equivalently apply to $\Omega_{\hat{C}(\boldsymbol{f})}$. Also, one can show that $\boldsymbol{f}$ simple, semisimple or flat implies $C(\boldsymbol{f}), \hat{C}(\boldsymbol{f})$ are simple, semisimple or flat, respectively. Therefore $\boldsymbol{f}$ a w-submersion, …, sf-immersion implies $C(\boldsymbol{f}), \hat{C}(\boldsymbol{f})$ are w-submersions, …, sf-immersions. The result for $\boldsymbol{f}_+, \boldsymbol{f}_-$ then follows from Proposition 6.30. □

Here is the analogue of Proposition 4.5, proved the same way. By 'w-submersion, …, sf-embedding' we mean all 16 classes in Definition 7.24.

Proposition 7.27. **(i)** *Any equivalence of d-manifolds with corners is a w-submersion, submersion, …, sf-embedding.*

(ii) *If* $\boldsymbol{f},\boldsymbol{g}:\mathbf{X}\to\mathbf{Y}$ *are* 2*-isomorphic* 1*-morphisms of d-manifolds with corners then* $\boldsymbol{f}$ *is a w-submersion, …, sf-embedding, if and only if* $\boldsymbol{g}$ *is.*

(iii) *Compositions of w-submersions, …, sf-embeddings are of the same kind.*

(iv) *The conditions that a* 1*-morphism* $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ *in* $\mathbf{dMan^c}$ *is any kind of submersion or immersion are local in* $\mathbf{X}$ *and* $\mathbf{Y}$*. The conditions that* $\boldsymbol{f}$ *is any kind of embedding are local in* $\mathbf{Y}$*, but not in* $\mathbf{X}$*.*

Here are the analogues of Propositions 4.6 and 4.7 and Theorem 4.8.

Proposition 7.28. **(a)** *Let* $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ *be a submersion in* $\mathbf{dMan^c}$*. Then* $\operatorname{vdim}\mathbf{X}\geqslant\operatorname{vdim}\mathbf{Y}$*, and if* $\operatorname{vdim}\mathbf{X}=\operatorname{vdim}\mathbf{Y}$ *then* $\boldsymbol{f}$ *is étale.*

(b) *Let* $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ *be an immersion in* $\mathbf{dMan^c}$*. Then* $\operatorname{vdim}\mathbf{X}\leqslant\operatorname{vdim}\mathbf{Y}$*. If* $\boldsymbol{f}$ *is an s- or sf-immersion and* $\operatorname{vdim}\mathbf{X}=\operatorname{vdim}\mathbf{Y}$ *then* $\boldsymbol{f}$ *is étale.*

Proof. For (a), as $\Omega_{C(\boldsymbol{f})}$ is injective, and $C_0^0(\boldsymbol{f}):C_0(\mathbf{X})\to C_0(\mathbf{Y})$ is identified with $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$, so $\Omega_{\boldsymbol{f}}:\underline{f}^*(T^*\mathbf{Y})\to T^*\mathbf{X}$ is injective, where $T^*\mathbf{X},\underline{f}^*(T^*\mathbf{Y})\in\operatorname{vvect}(\underline{X})$ have ranks $\operatorname{vdim}\mathbf{X},\operatorname{vdim}\mathbf{Y}$. Therefore Proposition 4.3(iv) shows that $\operatorname{vdim}\mathbf{X}\geqslant\operatorname{vdim}\mathbf{Y}$, and if $\operatorname{vdim}\mathbf{X}=\operatorname{vdim}\mathbf{Y}$ then $\Omega_{\boldsymbol{f}}$ is an equivalence.

Suppose $\operatorname{vdim}\mathbf{X}=\operatorname{vdim}\mathbf{Y}$. By definition $\boldsymbol{f}$ is semisimple and flat. If $\boldsymbol{f}$ is not simple then $\underline{s}_{\boldsymbol{f}}:\underline{S}_{\boldsymbol{f}}\to\underline{\partial X}$ is not surjective, and $\underline{\partial X}\setminus\operatorname{Im}(\underline{s}_{\boldsymbol{f}})\subseteq C_1^{\boldsymbol{f},0}(\underline{X})$, so that $C_1^{\boldsymbol{f},0}(\mathbf{X})\neq\emptyset$. But $\Omega_{C_1^0(\boldsymbol{f})}:C_1^0(\underline{f})^*(T^*C_0(\mathbf{Y}))\to T^*C_1^{\boldsymbol{f},0}(\mathbf{X})$ is injective, and $\operatorname{vdim}C_1^{\boldsymbol{f},0}(\mathbf{X})=\operatorname{vdim}\mathbf{X}-1<\operatorname{vdim}\mathbf{Y}=\operatorname{vdim}C_0(\mathbf{Y})$, contradicting Proposition 4.3(iv). Hence $\boldsymbol{f}$ is simple, so $\boldsymbol{f}$ is étale by Theorem 7.20.

For (b), let $x\in\mathbf{X}$, so that $(x,\emptyset)\in C_0(\mathbf{X})$, and let $\hat C(\boldsymbol{f})=\bigl(y,\{y_1',\ldots,y_k'\}\bigr)\in C_k(\mathbf{Y})$. Then $\Omega_{\hat C(\boldsymbol{f})}$ injective near $(x,\emptyset)$ and Proposition 4.3(vi) imply that $\operatorname{vdim}\mathbf{X}\leqslant\operatorname{vdim}\mathbf{Y}-k$, so $\operatorname{vdim}\mathbf{X}\leqslant\operatorname{vdim}\mathbf{Y}$ as $k\geqslant 0$. If $\operatorname{vdim}\mathbf{X}=\operatorname{vdim}\mathbf{Y}$ then $k=0$ for all $x\in\mathbf{X}$, which implies $\boldsymbol{f}$ is flat, and $\hat C_0^0(\boldsymbol{f}):C_0(\mathbf{X})\to C_0(\mathbf{Y})$ is identified with $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$, so $\Omega_{\boldsymbol{f}}$ is injective, and thus an equivalence by Proposition 4.3(vi). If also $\boldsymbol{f}$ is an s- or sf-embedding then $\boldsymbol{f}$ is simple and flat and $\Omega_{\boldsymbol{f}}$ is an equivalence, so $\boldsymbol{f}$ is étale by Theorem 7.20. $\square$

Proposition 7.29. **(a)** *Let* $f:X\to Y$ *be a smooth map of manifolds with corners, and* $\boldsymbol{f}=F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(f)$*. Then* $\boldsymbol{f}$ *is simple, semisimple, flat, a submersion, s-submersion, immersion, s-immersion, sf-immersion, embedding, s-embedding, or sf-embedding, in* $\mathbf{dMan^c}$ *if and only if* f *is simple, …, an sf-embedding in* $\mathbf{Man^c}$*, respectively. Also* $\boldsymbol{f}$ *is a w-immersion, sw-immersion, sfw-immersion, w-embedding, sw-embedding, or sfw-embedding in* $\mathbf{dMan^c}$ *if and only if* f *is an immersion, …, sf-embedding in* $\mathbf{Man^c}$*, respectively.*

(b) *Let* $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$ *be a* 1*-morphism in* $\mathbf{dMan^c}$*, with* $\mathbf{Y}$ *a manifold. Then* $\boldsymbol{f}$ *is a w-submersion if and only if it is semisimple and flat, and* $\boldsymbol{f}$ *is an sw-submersion if and only if it is simple and flat.*

Proof. For (a), first note that as in §6.4, $F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}$ maps $s_f : S_f \to \partial X$ and T_f from §5.3 to $\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial X}$ and $\underline{T}_{\boldsymbol{f}}$. So comparing Definitions 5.9 and 6.11 shows that $\boldsymbol{f}$ is simple, semisimple or flat if and only if f is. For the rest of (a), note that as in the proof of Proposition 4.7, $\mathrm{d}f : TX \to f^*(TY)$ surjective (or injective) is equivalent to $\Omega_{\boldsymbol{f}}$ injective (or surjective, respectively), and the same holds for $C(f), \hat{C}(f)$. Part (a) then follows by comparing Definitions 5.9, 5.28 and 7.24, and using Theorem 5.17(viii) for submersions. Part (b) is as for Proposition 4.7. $\square$

In the next theorem, note that exactness of (7.13) and (7.14) are not independent: (7.13) exact at the second term implies (7.14) exact at the second term, and (7.14) exact at the fourth term implies (7.13) exact at the fourth term. So in (a) we could replace (7.13)–(7.14) by (7.14), and in (b) we could replace (7.13)–(7.14) by (7.13). When f is simple and flat near v we have $k = l$ in equation (7.14).

Theorem 7.30. *Let V, W be manifolds with corners, $E \to V$, $F \to W$ be vector bundles, $s : V \to E$, $t : W \to F$ be smooth sections, $f : V \to W$ be smooth, and $\hat{f} : E \to f^*(F)$ be a morphism of vector bundles on V with $\hat{f} \circ s = f^*(t) + O(s^2)$. Then Definitions* 7.2 *and* 7.15 *define principal d-manifolds with corners $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$ and a* 1*-morphism $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$. As in* (3.31), *we have a complex*

$$0 \longrightarrow T_v V \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} E_v \oplus T_w W \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)} F_w \longrightarrow 0 \qquad (7.13)$$

for each $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$. If $v \in S^k(V)$ and $w \in S^l(W)$ then we also have a complex

$$0 \to T_v(S^k(V)) \xrightarrow{(\mathrm{d}s(v) \oplus \mathrm{d}f(v))|_{T_v(S^k(V))}} E_v \oplus T_w(S^l(W)) \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)|_{T_w(S^l(W))}} F_w \to 0. \quad (7.14)$$

- **(a)** *$\mathbf{S}_{f,\hat{f}}$ is a w-submersion if and only if $f : V \to W$ is semisimple and flat near $\{v \in V : s(v) = 0\} \subseteq V$, and for all $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, equations* (7.13)–(7.14) *are exact at the fourth term. $\mathbf{S}_{f,\hat{f}}$ is an sw-submersion if and only if also f is simple near $\{v \in V : s(v) = 0\}$.*
- **(b)** *$\mathbf{S}_{f,\hat{f}}$ is a submersion if and only if for all $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, equations* (7.13)–(7.14) *are exact at the third and fourth terms. These imply that f is semisimple and flat near $\{v \in V : s(v) = 0\}$. $\mathbf{S}_{f,\hat{f}}$ is an s-submersion if and only if also f is simple near $\{v \in V : s(v) = 0\}$.*
- **(c)** *$\mathbf{S}_{f,\hat{f}}$ is a w-immersion if and only if for all $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, equations* (7.13)–(7.14) *are exact at the second term. $\mathbf{S}_{f,\hat{f}}$ is an sw-immersion (or sfw-immersion) if and only if also f is simple (or simple and flat) near $\{v \in V : s(v) = 0\}$.*
- **(d)** *$\mathbf{S}_{f,\hat{f}}$ is an immersion if and only if for all $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, equations* (7.13)–(7.14) *are exact at the second and fourth*

terms. $\mathbf{S}_{f,\hat{f}}$ *is an s-immersion (or sf-immersion) if and only if also* f *is simple (or simple and flat) near* $\{v \in V : s(v) = 0\}$.

The conditions in **(a)**–**(d)** *are open conditions on* v *in* $\{v \in V : s(v) = 0\}$.

Proof. In (a),(b) we will need the following facts. Let $f, \hat{f}, v, w, k, l$ be as in the theorem. Consider the induced map

$$(\mathrm{d}f)_* : T_v V / T_v(S^k(V)) \longrightarrow T_w W / T_w(S^l(W)). \tag{7.15}$$

Then f is semisimple and flat (or simple and flat) near v if and only if (7.15) is surjective (or an isomorphism). Now (7.14) is a subcomplex of (7.13), with quotient (7.15) in the second and third places and zeroes elsewhere. Thus we get a long exact sequence of real vector spaces:

$$\begin{tikzcd} H^2(7.13) \arrow[r] & \mathrm{Ker}\,(7.15) \arrow[r] & H^3(7.14) \arrow[r] & H^3(7.13) \arrow[d] \\ 0 & H^4(7.13) \arrow[l] & H^4(7.14) \arrow[l] & \mathrm{Coker}\,(7.15). \arrow[l] \end{tikzcd} \tag{7.16}$$

Using this we will relate surjectivity of (7.15) with exactness of (7.13)–(7.14) at the third and fourth places.

For (a), Definition 7.24(a), Lemma 7.18 and the proof of Theorem 4.8(a), show that $\mathbf{S}_{f,\hat{f}}$ is a w-submersion if and only if f is semisimple and flat near $\{v \in V : s(v) = 0\}$, and (7.13) is exact at the fourth term for all v, w. By the first part, f semisimple and flat imply (7.15) is surjective, so $\mathrm{Coker}\,(7.15) = H^4(7.13) = 0$ in (7.16). Hence $H^4(7.14) = 0$, so (7.14) is exact at the fourth term. This gives the first part of (a), and the second part follows from Lemma 7.18.

For (b), first suppose that $\mathbf{S}_{f,\hat{f}}$ is a submersion. Then f is semisimple and flat near $\{v \in V : s(v) = 0\}$ by Lemma 7.18. Let v, w, k, l be as in the theorem. Then $C(f) : (v, \emptyset) \mapsto (w, \emptyset)$. Write $i_V^{-1}(v) = \{v'_1, \ldots, v'_k\}$ and $i_W^{-1}(w) = \{w'_1, \ldots, w'_l\}$. As f is flat near v, for each $j = 1, \ldots, l$, there exists $i = 1, \ldots, k$ with $(v'_i, w'_j) \in S_f$. Hence $C(f) : \big(v, \{v'_1, \ldots, v'_k\}\big) \mapsto \big(w, \{w'_1, \ldots, w'_k\}\big)$. Since $\Omega_{C(\boldsymbol{f})}$ is injective, the proof of Theorem 4.8(b) at the points $(v, \emptyset)$ and $\big(v, \{v'_1, \ldots, v'_k\}\big)$ in $C(\mathbf{S}_{V,E,s})$ shows that (7.13) and (7.14) are exact at the third and fourth terms. This proves the 'only if' in the first part of (b).

Next, suppose (7.13) and (7.14) are exact at the third and fourth terms for all v, w, k, l as in the theorem. Then (7.16) gives $\mathrm{Coker}\,(7.15) = 0$, so f is semisimple and flat near v, proving the second sentence of (b), and thus $\mathbf{S}_{f,\hat{f}}$ is semisimple and flat by Lemma 7.18. Suppose $I \subseteq \{v'_1, \ldots, v'_k\}$, and let $C(f) : (v, I) \mapsto (w, J)$ for $J \subseteq \{w'_1, \ldots, w'_l\}$. Then the proof of Theorem 4.8(b) shows that $\Omega_{C(\boldsymbol{f})}$ is injective near $(v, I) \in C(\mathbf{S}_{V,E,s})$ if and only if the following complex is exact at the third and fourth terms:

$$0 \to T_{(v,I)}(C_{|I|}(V)) \xrightarrow{(\mathrm{d}s(v) \oplus \mathrm{d}f(v))|_{T_{(v,I)}(C_{|I|}(V))}} E_v \oplus T_{(w,J)}(C_{|J|}(W)) \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)|_{T_{(w,J)}(C_{|J|}(W))}} F_w \to 0. \tag{7.17}$$

Now (7.17) interpolates between (7.13) and (7.14), and one can show that (7.13)–(7.14) exact at third and fourth terms implies (7.17) exact at third and fourth terms. Hence $\Omega_{C(\boldsymbol{f})}$ is injective near $(v,I)\in C(\mathbf{S}_{V,E,s})$ for all (v,I), so $\Omega_{C(\boldsymbol{f})}$ is injective, and $\mathbf{S}_{f,\hat f}$ is a submersion. This proves the 'if' in the first part of (b). The last part of (b) follows from Lemma refdm7lem6. Parts (c),(d) follow using similar arguments, and Theorem 4.8(c),(d). For (c) note that (7.13) exact at the second term implies (7.14) exact at the third term. For the 'only if' part of (d), we deduce (7.13) and (7.14) exact at second and fourth terms from $\Omega_{\hat C(\boldsymbol{f})}$ surjective at the points $(v,\emptyset)$ and $\bigl(v,\{v_1',\ldots,v_k'\}\bigr)$ in $C(\mathbf{S}_{V,E,s})$, respectively. $\square$

Here is the analogue of Theorem 4.9.

Theorem 7.31. *Suppose $\boldsymbol{g}:\mathbf{X}\to\mathbf{Y}$ is a 1-morphism of d-manifolds with corners, and $x\in\mathbf{X}$ with $\boldsymbol{g}(x)=y\in\mathbf{Y}$. Then there exist open d-submanifolds $\mathbf{T}\subseteq\mathbf{X}$ and $\mathbf{U}\subseteq\mathbf{Y}$ with $x\in\mathbf{T}$, $y\in\mathbf{U}$ and $\boldsymbol{g}(\mathbf{T})\subseteq\mathbf{U}$, manifolds with corners V,W, vector bundles $E\to V$, $F\to W$, smooth sections $s:V\to E$, $t:W\to F$, a smooth map $f:V\to W$, a morphism of vector bundles $\hat f:E\to f^*(F)$ with $\hat f\circ s=f^*(t)$, equivalences $\boldsymbol{i}:\mathbf{T}\to\mathbf{S}_{V,E,s}$, $\boldsymbol{j}:\mathbf{S}_{W,F,t}\to\mathbf{U}$, and a 2-morphism $\eta:\boldsymbol{j}\circ\mathbf{S}_{f,\hat f}\circ\boldsymbol{i}\Rightarrow\boldsymbol{g}|_{\mathbf{T}}$, where $\mathbf{S}_{f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ is as in Definitions 7.2 and 7.15. Furthermore:*

- **(a)** *If $\boldsymbol{g}$ is a **w-submersion** then we can choose the data $\boldsymbol{T},\boldsymbol{U},\ldots,\boldsymbol{j}$ above such that $f:V\to W$ is a submersion in $\mathbf{Man^c}$, and $\hat f:E\to f^*(F)$ is a surjective morphism of vector bundles. If $\boldsymbol{g}$ is an **sw-submersion** in $\mathbf{dMan^c}$, then f is an s-submersion in $\mathbf{Man^c}$.*
- **(b)** *If $\boldsymbol{g}$ is a **submersion** we can choose $\boldsymbol{T},\ldots,\boldsymbol{j}$ such that $f:V\to W$ is a submersion in $\mathbf{Man^c}$, and $\hat f:E\to f^*(F)$ is an isomorphism. If $\boldsymbol{g}$ is an **s-submersion** in $\mathbf{dMan^c}$, then f is an s-submersion in $\mathbf{Man^c}$.*
- **(c)** *If $\boldsymbol{g}$ is a **w-immersion** we can choose $\boldsymbol{T},\ldots,\boldsymbol{j}$ such that $f:V\to W$ is an immersion in $\mathbf{Man}$, and $\hat f:E\to f^*(F)$ is an injective morphism. If $\boldsymbol{g}$ is an **sw-immersion** or **sfw-immersion** in $\mathbf{dMan^c}$, then f is an s-immersion or sf-immersion in $\mathbf{Man^c}$.*
- **(d)** *If $\boldsymbol{g}$ is an **immersion** we can choose $\boldsymbol{T},\ldots,\boldsymbol{j}$ such that $f:V\to W$ is an immersion and $\hat f:E\to f^*(F)$ is an isomorphism. If $\boldsymbol{g}$ is an **s-immersion** or **sf-immersion** in $\mathbf{dMan^c}$, then f is an s-immersion or sf-immersion in $\mathbf{Man^c}$.*

Here are alternative forms for **(a)**–**(d)**, *excluding (w-)immersions in* **(c)**,**(d)**:

- **(a′)** *If $\boldsymbol{g}$ is a **w-submersion** we can choose $\boldsymbol{T},\ldots,\boldsymbol{j}$ such that $V=W\times Z$ for some manifold with corners Z, and $f=\pi_W$, $E=\pi_W^*(F)\oplus G$ for some vector bundle $G\to V$, $\hat f=\mathrm{id}_{\pi_W^*(F)}\oplus 0$, and $s=\pi_W^*(t)\oplus u$ for some $u\in C^\infty(G)$. If $\boldsymbol{g}$ is an **sw-submersion** then $\partial Z=\emptyset$.*
- **(b′)** *If $\boldsymbol{g}$ is a **submersion** we can choose $\boldsymbol{T},\ldots,\boldsymbol{j}$ such that $V=W\times Z$ for some manifold with corners Z, and $f=\pi_W$, $E=\pi_W^*(F)$, $\hat f=\mathrm{id}_{\pi_W^*(F)}$, $s=\pi_W^*(t)$. If $\boldsymbol{g}$ is an **s-submersion** then $\partial Z=\emptyset$.*

(c′) *If $\boldsymbol{g}$ is an* ***sw-immersion*** *we can choose $\boldsymbol{T},\ldots,\boldsymbol{j}$ such that $W = V\times Z$ for open $0\in Z\subseteq\mathbb{R}^n_k$, and f maps $v\mapsto(v,0)$, and $f^*(F)=E\oplus G$ for some $G\to V$, and $\hat f=\mathrm{id}_E\oplus 0$, $f^*(t)=s\oplus 0$. If $\boldsymbol{g}$ is an* ***sfw-immersion*** *then $k=0$, so $\partial Z=\emptyset$.*

(d′) *If $\boldsymbol{g}$ is an* ***s-immersion*** *we can choose $\boldsymbol{T},\ldots,\boldsymbol{j}$ such that $W=V\times Z$ for open $0\in Z\subseteq\mathbb{R}^n_k$, and $f: v\mapsto(v,0)$, $f^*(F)=E$, $\hat f=\mathrm{id}_E$, $f^*(t)=s$. If $\boldsymbol{g}$ is an* ***sf-immersion*** *then $k=0$, so $\partial Z=\emptyset$.*

Proof. By Corollary 7.11 there exist open neighbourhoods $\mathbf{T}\subseteq\boldsymbol{X}$, $\mathbf{U}\subseteq\boldsymbol{Y}$ of x,y and quasi-inverse equivalences $\boldsymbol{i}:\mathbf{T}\to\mathbf{S}_{V,E,s}$, $\boldsymbol{k}:\mathbf{S}_{V,E,s}\to\mathbf{T}$ and $\boldsymbol{j}:\mathbf{S}_{W,F,t}\to\mathbf{U}$, $\boldsymbol{l}:\mathbf{U}\to\mathbf{S}_{W,F,t}$ with 2-morphisms $\zeta_U:\boldsymbol{k}\circ\boldsymbol{i}\Rightarrow\mathbf{id_T}$ and $\zeta_V:\boldsymbol{l}\circ\boldsymbol{j}\Rightarrow\mathbf{id_U}$, where we may take V and W to be open neighbourhoods of 0 in $\mathbb{R}^m_k$ and $\mathbb{R}^n_l$, and E,F to be trivial vector bundles $\mathbb{R}^a\times V\to V$, $\mathbb{R}^b\times W\to W$, and $s=(s_1,\ldots,s_a)\in C^\infty(E)$, $t=(t_1,\ldots,t_b)\in C^\infty(F)$ with $s(0)=\mathrm{d}s|_{S^k(V)}(0)=0$ and $t(0)=\mathrm{d}t|_{S^l(W)}(0)=0$, and $\boldsymbol{i}(x)=0$, $\boldsymbol{l}(y)=0$. Note that this is weaker than in the proof of Theorem 4.9, in which we had $\mathrm{d}s(0)=\mathrm{d}t(0)=0$.

Making $\mathbf{T},V$ smaller if necessary, we can suppose $\boldsymbol{g}(\mathbf{T})\subseteq\mathbf{U}$. Applying Theorem 7.19 to the 1-morphism $\boldsymbol{l}\circ\boldsymbol{g}\circ\boldsymbol{k}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ and replacing V,E,s by $\tilde V,\tilde E,\tilde s$ gives a smooth map $f:V\to W$ with $f(0)=0$ and a morphism of vector bundles $\hat f:E\to f^*(F)$ on V with $\hat f\circ s=f^*(t)$, such that $\boldsymbol{l}\circ\boldsymbol{g}\circ\boldsymbol{k}=\mathbf{S}_{f,\hat f}$. Hence $\boldsymbol{j}\circ\mathbf{S}_{f,\hat f}\circ\boldsymbol{i}=(\boldsymbol{j}\circ\boldsymbol{l})\circ\boldsymbol{g}\circ(\boldsymbol{k}\circ\boldsymbol{i})$. Thus $\eta=\zeta_U*\mathrm{id}_{\boldsymbol{g}}*\zeta_V$ is a 2-morphism $\boldsymbol{j}\circ\mathbf{S}_{f,\hat f}\circ\boldsymbol{i}\Rightarrow\boldsymbol{g}|_{\boldsymbol{U}}$. This completes the first part.

For (a)–(d), if $\boldsymbol{g}$ is a w-submersion, …, immersion then $\mathbf{S}_{f,\hat f}=\boldsymbol{l}\circ\boldsymbol{g}\circ\boldsymbol{k}$ is also a w-submersion, …, immersion by Proposition 7.27(i),(iii), as $\boldsymbol{k},\boldsymbol{l}$ are equivalences. Thus we may apply Theorem 7.30 to $\mathbf{S}_{f,\hat f}$. Using $\mathrm{d}s|_{S^k(V)}(0)=\mathrm{d}t|_{S^l(W)}(0)=0$ in equation (7.14), the conditions on (7.13)–(7.14) reduce to:

(a) if $\boldsymbol{g}$ is a w-submersion then $\hat f(0):E_0\to F_0$ is surjective;

(b) if $\boldsymbol{g}$ is a submersion then $\hat f(0):E_0\to F_0$ is an isomorphism and $\mathrm{d}f(0):T_0V\to T_0W$, $\mathrm{d}f(0)|_{T_0(S^k(V))}:T_0(S^k(V))\to T_0(S^l(W))$ are surjective;

(c) if $\boldsymbol{g}$ is a w-immersion then $\mathrm{d}f(0)|_{T_0(S^k(V))}:T_0(S^k(V))\to T_0(S^l(W))$ is injective; and

(d) if $\boldsymbol{g}$ is an immersion then $\mathrm{d}f(0)|_{T_0(S^k(V))}:T_0(S^k(V))\to T_0(S^l(W))$ is injective and $\hat f(0):E_v\to F_v$ is surjective.

The proofs of the first parts of (a)–(d) follow those of Theorem 4.9(a)–(d), with the following changes. For w-submersions in (a), we replace V by

$$V'=\bigl\{(x_1,\ldots,x_m,z_{l+1},\ldots,z_n)\in V\times\mathbb{R}^{n-l}:\\ f(x_1,\ldots,x_m)+(0,\ldots,0,z_{l+1},\ldots,z_n)\in W\subseteq\mathbb{R}^n_l\bigr\},$$

and we replace E by $E'=\pi_V^*(E)\oplus\mathbb{R}^{n-l}$, and s by $s'=\pi_V^*(s)\oplus\mathrm{id}_{\mathbb{R}^{n-l}}$. To prove that f' is a submersion near 0, note that f' maps $S^k(V')\to S^l(W)$ near 0, and $\mathrm{d}f'(0)|_{T_0(S^k(V'))}:T_0(S^k(V'))\to T_0(S^l(W))$ maps $\mathbb{R}^{m-k}\oplus\mathbb{R}^{n-l}\to\mathbb{R}^{n-l}$ and is the identity on the second factor, so $\mathrm{d}f'(0)|_{T_0(S^k(V'))}$ is surjective. Also $\mathbf{S}_{f,\hat{f}}$ is semisimple and flat as $\boldsymbol{g}$ is, so f is semisimple and flat near 0 by Lemma 7.18. Thus $(\mathrm{d}f')_* : T_0V'/T_0(S^k(V')) \longrightarrow T_0W/T_0(S^l(W))$ is surjective. Combining this with $\mathrm{d}f'(0)|_{T_0(S^k(V'))}$ surjective shows $\mathrm{d}f'(0) : T_0V \to T_0W$ is surjective. Hence f' is a submersion near $0 \in V'$ by Definition 5.9(iv). Making V' smaller if necessary, f' is a submersion.

For submersions in (b), f is a submersion and $\hat{f}$ is an isomorphism near $0 \in V$ already, so after making V smaller, no further changes are needed. For w-immersions in (c) and immersions in (d), we define $W', F', t', f', \hat{f}', h, \hat{h}, \boldsymbol{j}'$ as in the proof of Theorem 4.9(c),(d), without change. The analogue of (7.13) with $f', \hat{f}', 0, 0$ in place of $f, \hat{f}, v, w$ is still exact at the second term, and $\hat{f}'(0)$ is injective. From this we see that $\mathrm{d}f'(0)$ is injective, even though $\mathrm{d}f(0)$ need not be injective. So f' is an immersion near $0 \in V'$.

This proves the first parts of (a)–(d). The remaining cases of sw-submersions in (a), …, sf-immersions in (d) follow from the first parts and Lemma 7.18, making V smaller if necessary. Finally, (a′)–(d′) follow from (a)–(d) as for Theorem 4.9(a′)–(d′), but using Proposition 5.11 to give a local form for (s-)submersions in (a′),(b′) and Proposition 5.30 to give a local form for s- and sf-immersions in (c′),(d′). We exclude (w-)immersions in (c′),(d′) as we have no analogue of Proposition 5.30 for immersions in $\mathbf{Man^c}$. □

The following lemma, the analogue of Lemma 4.10, is easy to prove.

Lemma 7.32. *Let $\mathbf{X}, \mathbf{Y}$ be d-manifolds with corners, with $\mathbf{Y}$ a manifold. Then $\boldsymbol{\pi}_{\mathbf{X}} : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ is a submersion, and $\boldsymbol{\pi}_{\mathbf{X}}$ is an s-submersion if $\partial\mathbf{Y} = \boldsymbol{\emptyset}$.*

Theorem 7.31(b′) implies the following analogues of Corollaries 4.11 and 4.12. The first is a local converse for Lemma 7.32, as for Proposition 5.11.

Corollary 7.33. *Suppose $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a submersion in $\mathbf{dMan^c}$, and $x \in \mathbf{X}$ with $\boldsymbol{f}(x) = y \in \mathbf{Y}$. Then there exist open d-submanifolds $x \in \mathbf{U} \subseteq \mathbf{X}$ and $y \in \mathbf{V} \subseteq \mathbf{Y}$ with $\boldsymbol{f}(\mathbf{U}) = \mathbf{V}$, a manifold with corners $\mathbf{Z}$, and an equivalence $\boldsymbol{i} : \mathbf{U} \to \mathbf{V} \times \mathbf{Z}$, such that $\boldsymbol{f}|_{\mathbf{U}} : \mathbf{U} \to \mathbf{V}$ is 2-isomorphic to $\boldsymbol{\pi}_{\mathbf{V}} \circ \boldsymbol{i}$, where $\boldsymbol{\pi}_{\mathbf{V}} : \mathbf{V} \times \mathbf{Z} \to \mathbf{V}$ is the projection. If $\boldsymbol{f}$ is an s-submersion then $\partial\mathbf{Z} = \boldsymbol{\emptyset}$.*

Corollary 7.34. *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a submersion of d-manifolds with corners, with $\mathbf{Y}$ a manifold with corners. Then $\mathbf{X}$ is a manifold with corners.*

Example 4.13 generalizes to d-manifolds with corners: if V is a manifold with corners, $E \to V$ a vector bundle, Γ a finite group acting smoothly on V, E preserving the vector bundle structure, and $s : V \to E$ a smooth, Γ-equivariant section of E, then the standard model d-manifold with corners $\mathbf{S}_{V,E,s}$ in Definition 7.2 has a Γ-action, so §6.10 defines a fixed d-subspace $(\mathbf{S}_{V,E,s})^\Gamma$ in $\mathbf{dSpa^c}$. As for (4.13), we have a 1-isomorphism in $\mathbf{dSpa^c}$

$$(\mathbf{S}_{V,E,s})^\Gamma \cong \textstyle\coprod_{i=0}^{\dim V} \coprod_{j=0}^{\operatorname{rank} E} \mathbf{S}_{V_{ij}^\Gamma, E_{ij}^\Gamma, s_{ij}^\Gamma}.$$

Therefore $(\mathbf{S}_{V,E,s})^\Gamma$ is an object in $\mathbf{d\check{M}an^c}$. As for Proposition 4.14, we prove:

Proposition 7.35. *Let* $\mathbf{X}$ *be a d-manifold with corners, and* Γ *is a finite group acting on* $\mathbf{X}$*. Section* 6.10 *defines the fixed d-subspace* $\mathbf{X}^\Gamma$ *and a* 1*-morphism* $\boldsymbol{j}_{\mathbf{X},\Gamma}:\mathbf{X}^\Gamma\to\mathbf{X}$*. Then* $\mathbf{X}^\Gamma$ *lies in* $\mathbf{d\check{M}an^c}$*, and* $\boldsymbol{j}_{\mathbf{X},\Gamma}$ *is a w-embedding.*

7.6 Bd-transversality and fibre products

We now extend §4.3 to the corners case. Here are the analogues of Definition 4.16 and Theorem 4.21.

Definition 7.36. Let $\mathbf{X},\mathbf{Y},\mathbf{Z}$ be d-manifolds with corners and $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$, $\boldsymbol{h}:\mathbf{Y}\to\mathbf{Z}$ be 1-morphisms. We call $\boldsymbol{g},\boldsymbol{h}$ *bd-transverse* if they are both b-transverse in $\mathbf{dSpa^c}$ in the sense of Definition 6.32, and d-transverse in the sense of Definition 4.16. We call $\boldsymbol{g},\boldsymbol{h}$ *cd-transverse* if they are both c-transverse in $\mathbf{dSpa^c}$ in the sense of Definition 6.33, and d-transverse. As in §6.8.2, c-transverse implies b-transverse, so cd-transverse implies bd-transverse.

Theorem 7.37. *Suppose* $\mathbf{X},\mathbf{Y},\mathbf{Z}$ *are d-manifolds with corners and* $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$*,* $\boldsymbol{h}:\mathbf{Y}\to\mathbf{Z}$ *are bd-transverse* 1*-morphisms, and let* $\mathbf{W}=\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ *be the fibre product in* $\mathbf{dSpa^c}$*, which exists by Theorem* 6.42 *as* $\boldsymbol{g},\boldsymbol{h}$ *are b-transverse. Then* $\mathbf{W}$ *is a d-manifold with corners, with*

$$\operatorname{vdim}\mathbf{W}=\operatorname{vdim}\mathbf{X}+\operatorname{vdim}\mathbf{Y}-\operatorname{vdim}\mathbf{Z}. \tag{7.18}$$

Hence, all bd-transverse fibre products exist in $\mathbf{dMan^c}$*.*

Proof. Since $\boldsymbol{g},\boldsymbol{h}$ are b-transverse, a fibre product $\mathbf{W}=\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ exists in $\mathbf{dSpa^c}$ by Theorem 6.42, with projections $\boldsymbol{e}:\mathbf{W}\to\mathbf{X}$ and $\boldsymbol{f}:\mathbf{W}\to\mathbf{Y}$. We will show that $\mathbf{W}$ is a d-manifold with corners, of virtual dimension (7.18). Thus $\mathbf{W}=\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y}$ is a fibre product in $\mathbf{dMan^c}$, as we want.

Let $w\in\mathbf{W}$, with $\boldsymbol{e}(w)=x\in\mathbf{X}$ and $\boldsymbol{f}(w)=y\in\mathbf{Y}$, so that $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z\in\mathbf{Z}$. In §6.8.3 we constructed open d-subspaces $x\in\mathbf{R}\subseteq\mathbf{X}$, $y\in\mathbf{S}\subseteq\mathbf{Y}$ and $z\in\mathbf{T}\subseteq\mathbf{Z}$ with $\boldsymbol{g}(\mathbf{R}),\boldsymbol{h}(\mathbf{S})\subseteq\mathbf{T}$, and an explicit fibre product $\mathbf{Q}=\mathbf{R}\times_{\boldsymbol{g}|_{\mathbf{R}},\mathbf{T},\boldsymbol{h}|_{\mathbf{S}}}\mathbf{S}$ in $\mathbf{dSpa^c}$. So $\mathbf{Q}$ is equivalent in $\mathbf{dSpa^c}$ to the open neighbourhood $\hat{\mathbf{W}}=\boldsymbol{e}^{-1}(\mathbf{R})\cap\boldsymbol{f}^{-1}(\mathbf{S})$ of w in $\mathbf{W}$.

As $\mathbf{X},\mathbf{Y},\mathbf{Z}$ are d-manifolds with corners, making $\mathbf{R},\mathbf{S},\mathbf{T}$ smaller, we can suppose they are principal d-manifolds with corners. So we have equivalences $\boldsymbol{i}:\mathbf{S}_{T,F,t}\to\mathbf{R}$, $\boldsymbol{j}:\mathbf{S}_{U,G,u}\to\mathbf{S}$, $\boldsymbol{k}:\mathbf{T}\to\mathbf{S}_{V,H,v}$ for manifolds with corners T,U,V, vector bundles $F\to T$, $G\to U$, $H\to V$ with H the trivial bundle $\mathbb{R}^r\times V\to V$, and smooth sections $t:T\to F$, $u:U\to G$, $v:H\to V$. Then $\boldsymbol{k}\circ\boldsymbol{g}\circ\boldsymbol{i}:\mathbf{S}_{T,F,t}\to\mathbf{S}_{V,H,v}$, $\boldsymbol{k}\circ\boldsymbol{h}\circ\boldsymbol{j}:\mathbf{S}_{U,G,u}\to\mathbf{S}_{V,H,v}$ are 1-morphisms, so by Theorem 7.19, making $\mathbf{R},\mathbf{S},T,U$ smaller we may suppose $\boldsymbol{k}\circ\boldsymbol{g}\circ\boldsymbol{i}=\mathbf{S}_{p,\hat{p}}$ and $\boldsymbol{k}\circ\boldsymbol{h}\circ\boldsymbol{j}=\mathbf{S}_{q,\hat{q}}$ for $p:T\to V$, $q:U\to V$ be smooth maps of manifolds with corners, and $\hat{p}:F\to p^*(H)$, $\hat{q}:G\to h^*(H)$ be vector bundle morphisms with $\hat{p}\circ t=p^*(v)$ and $\hat{q}\circ u=q^*(v)$.

Since $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ are equivalences, $\boldsymbol{g},\boldsymbol{h}$ bd-transverse implies that $\mathbf{S}_{p,\hat{p}},\mathbf{S}_{q,\hat{q}}$ are bd-transverse. As they are b-transverse, a fibre product $\mathbf{S}_{T,F,t}\times_{\mathbf{S}_{p,\hat{p}},\mathbf{S}_{V,H,v},\mathbf{S}_{q,\hat{q}}}\mathbf{S}_{U,G,u}$ exists in $\mathbf{dSpa^c}$, with

$$\hat{\mathbf{W}}\simeq\mathbf{R}\times_{\mathbf{T}}\mathbf{S}=\mathbf{Q}\simeq\mathbf{S}_{T,F,t}\times_{\mathbf{S}_{p,\hat{p}},\mathbf{S}_{V,H,v},\mathbf{S}_{q,\hat{q}}}\mathbf{S}_{U,G,u}. \tag{7.19}$$

As they are d-transverse, the analogue of Proposition 4.18 shows that

$$\hat{p}(\tilde{x}) \oplus -\hat{q}(\tilde{y}) \oplus \mathrm{d}v(\tilde{z}) : F_{\tilde{x}} \oplus G_{\tilde{y}} \oplus T_{\tilde{z}}V \longrightarrow H_{\tilde{z}} \tag{7.20}$$

is surjective for all $\tilde{x}\in T$, $\tilde{y}\in U$ with $t(\tilde{x})=0$, $u(\tilde{y})=0$ and $p(\tilde{x})=q(\tilde{y})=\tilde{z}\in V$.

Making $\mathbf{R},\mathbf{S},\mathbf{T},T,U,V$ smaller if necessary, we can suppose that T,U,V are open neighbourhoods of 0 in $\mathbb{R}^l_a,\mathbb{R}^m_b,\mathbb{R}^n_c$ respectively, and $t(0)=u(0)=v(0)=0$ so that $0\in\mathbf{S}_{T,F,t}$, $0\in\mathbf{S}_{u,G,u}$, $0\in\mathbf{S}_{V,H,v}$, and $\boldsymbol{i}(x)=0$, $\boldsymbol{j}(x)=0$, $\boldsymbol{k}(0)=z$, and $p(0)=q(0)=0$. Write $(x_1,\ldots,x_l)$, $(y_1,\ldots,y_m)$, $(z_1,\ldots,z_n)$ for the coordinates on T,U,V, and write $i_T^{-1}(0)=\{t'_1,\ldots,t'_a\}$, $i_U^{-1}(0)=\{u'_1,\ldots,u'_b\}$, and $i_V^{-1}(0)=\{v'_1,\ldots,v'_c\}$, where t'_i,u'_j,v'_k correspond to the local boundary components $x_i=0$, $y_j=0$, $z_k=0$ of T,U,V at 0, respectively.

As $p=(p_1,\ldots,p_n)$ is a smooth map of manifolds with corners, we show that if $(x_1,\ldots,x_l)$ is sufficiently close to 0 in T then for $k=1,\ldots,c$ we have

$$p_k(x_1,\ldots,x_l)=\begin{cases} x_i\cdot g_k(x_1,\ldots,x_l), & (t'_i,v'_k)\in S_p,\ \text{some}\ i=1,\ldots,a, \\ 0, & (0,v'_k)\in T_p, \end{cases} \tag{7.21}$$

where $g_k:T\rightarrow(0,\infty)$ is smooth, and given by $g_k(x_1,\ldots,x_l)=p_k(x_1,\ldots,x_l)/x_i$ if $x_i>0$, and $g_k(x_1,\ldots,x_l)=\frac{\partial p_k}{\partial x_i}(x_1,\ldots,x_l)$ if $x_i=0$. Making $\mathbf{R},T$ smaller we can suppose (7.21) holds for all $(x_1,\ldots,x_l)\in T$. Similarly, we may suppose that for all $k=1,\ldots,c$ and $(y_1,\ldots,y_m)\in U$ we have

$$q_k(y_1,\ldots,y_m)=\begin{cases} y_j\cdot h_k(y_1,\ldots,y_m), & (u'_j,v'_k)\in S_q,\ \text{some}\ j=1,\ldots,b, \\ 0, & (0,v'_k)\in T_q, \end{cases} \tag{7.22}$$

for smooth $h_k:U\rightarrow(0,\infty)$. Equations (7.21)–(7.22) correspond to the conditions on the boundary defining functions $(\boldsymbol{R},\boldsymbol{r}_{x'}),(\boldsymbol{S},\boldsymbol{s}_{y'}),(\boldsymbol{T},\boldsymbol{t}_{z'})$ for $\mathbf{R},\mathbf{S},\mathbf{T}$ in Definition 6.40.

From Definition 5.1, any smooth function on $T\subseteq\mathbb{R}^l_a$ extends to a smooth function on an open neighbourhood $\dot{T}$ of T in $\mathbb{R}^l$. Hence we can choose open sets $\dot{T}\subseteq\mathbb{R}^l$, $\dot{U}\subseteq\mathbb{R}^m$, $\dot{V}\subseteq\mathbb{R}^n$ with $T=\dot{T}\cap\mathbb{R}^l_a$, $U=\dot{U}\cap\mathbb{R}^m_b$, $V=\dot{V}\cap\mathbb{R}^n_c$, and vector bundles $\dot{F}\rightarrow\dot{T}$, $\dot{G}\rightarrow\dot{U}$ and $\dot{H}=\mathbb{R}^r\times\dot{V}\rightarrow\dot{V}$ with $\dot{F}|_T=F$, $\dot{G}|_U=G$, $\dot{H}|_V=H$, so that $p,q,t,u,v,\hat{p},\hat{q}$ extend to smooth maps $\dot{p}:\dot{T}\rightarrow\dot{V}$, $\dot{q}:\dot{U}\rightarrow\dot{V}$, smooth sections $\dot{t}:\dot{T}\rightarrow\dot{F}$, $\dot{u}:\dot{U}\rightarrow\dot{G}$ and $\dot{v}:\dot{V}\rightarrow\dot{H}$, and vector bundle morphisms $\dot{\hat{p}}:\dot{F}\rightarrow\dot{p}^*(\dot{H})$, $\dot{\hat{q}}:\dot{G}\rightarrow\dot{q}^*(\dot{H})$ such that $\dot{\hat{p}}\circ\dot{t}=\dot{p}^*(\dot{v})$ and $\dot{\hat{q}}\circ\dot{u}=\dot{q}^*(\dot{v})$. Making $\dot{T},\dot{U}$ smaller if necessary, we can suppose the analogues of (7.21)–(7.22) hold for smooth $\dot{g}_j:\dot{T}\rightarrow(0,\infty)$ and $\dot{h}_j:\dot{U}\rightarrow(0,\infty)$, and (7.20) is surjective for all $\tilde{x}\in\dot{T}$, $\tilde{y}\in\dot{U}$ with $\dot{s}(\tilde{x})=0$, $\dot{t}(\tilde{y})=0$ and $\dot{p}(\tilde{x})=\dot{q}(\tilde{y})=\tilde{z}\in\dot{V}$.

Now §6.8.3 constructs an explicit fibre product $\mathbf{Q}=\mathbf{R}\times_{\boldsymbol{g}|_{\mathbf{R}},\mathbf{T},\boldsymbol{h}|_{\mathbf{S}}}\mathbf{S}$ in $\mathbf{dSpa^c}$, or equivalently, a fibre product $\mathbf{S}_{T,F,t}\times_{\mathbf{S}_{p,\hat{p}},\mathbf{S}_{V,H,v},\mathbf{S}_{q,\hat{q}}}\mathbf{S}_{U,G,u}$. The construction involves a graph $\Gamma_{x,y}$ from Definition 6.36, where in our case $x=0\in\mathbf{S}_{T,F,t}$ and $y=0\in\mathbf{S}_{U,G,u}$. The vertices of $\Gamma_{0,0}$ are $\{t'_1,\ldots,t'_a\}\amalg\{u'_1,\ldots,u'_b\}\amalg\{v'_1,\ldots,v'_c\}$, and the edges are $\overset{t'_i}{\bullet}-\overset{v'_k}{\bullet}$ if $(t'_i,v'_k)\in S_p$ and $\overset{u'_j}{\bullet}-\overset{v'_k}{\bullet}$ if $(u'_j,v'_k)\in S_q$. Since $\mathbf{S}_{p,\hat{p}},\mathbf{S}_{q,\hat{q}}$ are b-transverse, each connected component $\hat{\Gamma}$ of $\Gamma_{0,0}$ is of type (A),(B) or (C) in Proposition 6.37.

In the construction of $\mathbf{Q}$ in Definition 6.40, we chose subsets $I \subseteq \{t'_1, \ldots, t'_a\}$ and $J \subseteq \{u'_1, \ldots, u'_b\}$ such that $I \amalg J$ contains one vertex from each type (A) component $\hat{\Gamma}$ of $\Gamma_{0,0}$, and each element of $I \amalg J$ lies on at most one edge in $\Gamma_{0,0}$. Define $\tilde{\Gamma}$ to be the subgraph of $\Gamma_{0,0}$ with vertices $\bigl(\{t'_1, \ldots, t'_a\} \setminus I\bigr) \amalg \bigl(\{u'_1, \ldots, u'_b\} \setminus J\bigr) \amalg \{v'_1, \ldots, v'_c\}$. In the proof of Proposition 6.37 we defined a colouring of (each connected component of) $\tilde{\Gamma}$ into black and white edges, where every vertex t'_i, u'_j, v'_k in $\tilde{\Gamma}$ lies on exactly one white edge.

Define manifolds with corners $\acute{T}, \acute{U}$ with $T \subseteq \acute{T} \subseteq \dot{T}$ and $U \subseteq \acute{U} \subseteq \dot{U}$ by

$$\acute{T} = \bigl\{(x_1, \ldots, x_l) \in \dot{T} : x_i \geqslant 0,\ i \in I\bigr\},\ \ \acute{U} = \bigl\{(y_1, \ldots, y_m) \in \dot{U} : y_j \geqslant 0,\ j \in J\bigr\}.$$

We have isomorphisms of transverse fibre products in $\mathbf{Man^c}$:

$$\begin{aligned} T &\cong \acute{T} \times_{\prod_{i=1,\ldots,a:t'_i \notin I} x_i, \mathbb{R}^{a-|I|}, \mathrm{inc}} [0,\infty)^{a-|I|}, \\ U &\cong \acute{U} \times_{\prod_{j=1,\ldots,b:u'_j \notin J} y_j, \mathbb{R}^{b-|J|}, \mathrm{inc}} [0,\infty)^{b-|J|}, \\ V &\cong \dot{V} \times_{\prod_{k=1,\ldots,c} z_k, \mathbb{R}^{c}, \mathrm{inc}} [0,\infty)^{c}, \end{aligned} \tag{7.23}$$

where $\mathrm{inc} : [0,\infty)^d \to \mathbb{R}^d$ is the inclusion.

Set $\acute{F} = \dot{F}|_{\acute{T}}$, $\acute{t} = \dot{t}|_{\acute{T}}$, $\acute{G} = \dot{G}|_{\acute{U}}$, $\acute{u} = \dot{u}|_{\acute{U}}$, $\acute{p} = \dot{p}|_{\acute{T}}$, $\acute{\hat{p}} = \dot{\hat{p}}|_{\acute{T}}$, $\acute{q} = \dot{q}|_{\acute{U}}$, and $\acute{\hat{q}} = \dot{\hat{q}}|_{\acute{U}}$. Then we have d-manifolds with corners $\mathbf{S}_{\acute{T},\acute{F},\acute{t}}, \mathbf{S}_{\acute{U},\acute{G},\acute{u}}$ and 1-morphisms $\mathbf{S}_{\acute{p},\acute{\hat{p}}} : \mathbf{S}_{\acute{T},\acute{F},\acute{t}} \to \mathbf{S}_{\dot{V},\dot{H},\dot{v}}$, $\mathbf{S}_{\acute{q},\acute{\hat{q}}} : \mathbf{S}_{\acute{U},\acute{G},\acute{u}} \to \mathbf{S}_{\dot{V},\dot{H},\dot{v}}$. Following (7.24), we have equivalences of fibre products

$$\mathbf{S}_{T,F,t} \simeq \mathbf{S}_{\acute{T},\acute{F},\acute{t}} \times_{\prod_{i=1,\ldots,a:t'_i \notin I} \boldsymbol{x}_i, \boldsymbol{\mathbb{R}}^{a-|I|}, \mathbf{inc}} \mathbf{[0,\infty)}^{a-|I|}, \tag{7.24}$$

$$\mathbf{S}_{U,G,u} \simeq \mathbf{S}_{\acute{U},\acute{G},\acute{u}} \times_{\prod_{j=1,\ldots,b:u'_j \notin J} \boldsymbol{y}_j, \boldsymbol{\mathbb{R}}^{b-|J|}, \mathbf{inc}} \mathbf{[0,\infty)}^{b-|J|}, \tag{7.25}$$

$$\mathbf{S}_{V,G,v} \simeq \mathbf{S}_{\dot{V},\dot{H},\dot{v}} \times_{\prod_{k=1,\ldots,c} \boldsymbol{z}_k, \boldsymbol{\mathbb{R}}^{c}, \mathbf{inc}} \mathbf{[0,\infty)}^{c}, \tag{7.26}$$

in $\mathbf{dSpa^c}$, where $\boldsymbol{x}_i : \mathbf{S}_{\acute{T},\acute{F},\acute{t}} \to \boldsymbol{\mathbb{R}}$, $\boldsymbol{y}_j : \mathbf{S}_{\acute{U},\acute{G},\acute{u}} \to \boldsymbol{\mathbb{R}}$, $\boldsymbol{z}_k : \mathbf{S}_{\dot{V},\dot{H},\dot{v}} \to \boldsymbol{\mathbb{R}}$ are the 1-morphisms induced by the coordinate functions x_i, y_j, z_k on $\acute{T}, \acute{U}, \dot{V}$ respectively.

We now have equivalences of fibre products in $\mathbf{dSpa^c}$:

$$\begin{aligned} &\mathbf{S}_{T,F,t} \times_{\mathbf{S}_{p,\hat{p}}, \mathbf{S}_{V,H,v}, \mathbf{S}_{q,\hat{q}}} \mathbf{S}_{U,G,u} \simeq \bigl(\mathbf{S}_{\acute{T},\acute{F},\acute{t}} \times_{\prod_{i=1,\ldots,a:t'_i \notin I} \boldsymbol{x}_i, \boldsymbol{\mathbb{R}}^{a-|I|}, \mathbf{inc}} \mathbf{[0,\infty)}^{a-|I|}\bigr) \\ &\times_{\mathbf{S}_{\dot{V},\dot{H},\dot{v}} \times \prod_{k=1,\ldots,c} \boldsymbol{z}_k, \boldsymbol{\mathbb{R}}^c, \mathbf{inc}, \mathbf{[0,\infty)}^c} \bigl(\mathbf{S}_{\acute{U},\acute{G},\acute{u}} \times_{\prod_{j=1,\ldots,b:u'_j \notin J} \boldsymbol{y}_j, \boldsymbol{\mathbb{R}}^{b-|J|}, \mathbf{inc}} \mathbf{[0,\infty)}^{b-|J|}\bigr) \\ &\qquad \simeq \mathbf{S}_{\acute{T},\acute{F},\acute{t}} \times_{\mathbf{S}_{\acute{p},\acute{\hat{p}}}, \mathbf{S}_{\dot{V},\dot{H},\dot{v}}, \mathbf{S}_{\acute{q},\acute{\hat{q}}}} \mathbf{S}_{\acute{U},\acute{G},\acute{u}}. \end{aligned} \tag{7.27}$$

Here in the first step we substitute in (7.24)–(7.26). In the second step of (7.27), as in the proof of (6.131), we argue as follows. Suppose $\overset{t'_i}{\bullet} - \overset{v'_k}{\bullet}$ is a white edge in $\tilde{\Gamma}$. Then $i = 1, \ldots, c$ with $t'_i \notin I$, so that i corresponds to the factor $\times_{\boldsymbol{x}_i, \boldsymbol{\mathbb{R}}, \mathbf{inc}} \mathbf{[0,\infty)}$ in (7.24) and the first line of (7.27), and k corresponds to the factor $\times_{\boldsymbol{z}_k, \boldsymbol{\mathbb{R}}, \mathbf{inc}} \mathbf{[0,\infty)}$ in (7.26) and the second line of (7.27).

As $(t'_i, v'_k) \in \underline{S}_{\mathbf{S}_{\acute{p},\acute{\hat{p}}}}$, by Definition 6.2(i) there exists an open neighbourhood $\boldsymbol{V}$ of 0 in $\boldsymbol{S}_{\acute{T},\acute{F},\acute{t}}$ such that $(\boldsymbol{V}, \boldsymbol{z}_k \circ \mathbf{S}_{\acute{p},\acute{\hat{p}}}|_{\boldsymbol{V}})$ is a boundary defining function for $\mathbf{S}_{\acute{T},\acute{F},\acute{t}}$ at t'_i. But $(\boldsymbol{S}_{\acute{T},\acute{F},\acute{t}}, \boldsymbol{x}_i)$ is also a boundary defining function for $\mathbf{S}_{\acute{T},\acute{F},\acute{t}}$ at t'_i. Hence by Proposition 6.6(c), making $\boldsymbol{V}$ smaller if necessary we have that $\boldsymbol{x}_i|_{\boldsymbol{V}} = \boldsymbol{c} \cdot \boldsymbol{z}_k \circ \mathbf{S}_{\acute{p},\acute{\hat{p}}}|_{\boldsymbol{V}}$, for some $\boldsymbol{c} : \boldsymbol{V} \to \boldsymbol{(0,\infty)}$. Making $\mathbf{R}, T, \dot{T}, \acute{T}$ smaller if necessary we can take $\boldsymbol{V} = \boldsymbol{S}_{\acute{p},\acute{\hat{p}}}$, so that $\boldsymbol{x}_i = \boldsymbol{c} \cdot \boldsymbol{z}_k \circ \mathbf{S}_{\acute{p},\acute{\hat{p}}}$ on $\boldsymbol{S}_{\acute{p},\acute{\hat{p}}}$ for $\boldsymbol{c} > 0$.

Thus, in the term $\mathbf{S}_{\acute{T},\acute{F},\acute{t}} \times_{\prod_i \boldsymbol{x}_i, \mathbb{R}^{a-|I|}, \mathbf{inc}} \boldsymbol{[0,\infty)}^{a-|I|}$ on the first line of (7.27), the fibre product $\times_{\boldsymbol{x}_i, \mathbb{R}, \mathbf{inc}} \boldsymbol{[0,\infty)}$ may be replaced by $\times_{\boldsymbol{z}_k \circ \mathbf{S}_{\acute{p},\acute{\hat{p}}}, \mathbb{R}, \mathbf{inc}} \boldsymbol{[0,\infty)}$ without changing it. We may then simultaneously omit this $\times_{\boldsymbol{z}_k \circ \mathbf{S}_{\acute{p},\acute{\hat{p}}}, \mathbb{R}, \mathbf{inc}} \boldsymbol{[0,\infty)}$, and the term $\times_{\boldsymbol{z}_k, \mathbb{R}, \mathbf{inc}} \boldsymbol{[0,\infty)}$ from the beginning of the second line of (7.27), without changing the fibre product up to equivalence.

Hence, for each white edge $\overset{t'_i}{\bullet} - \overset{v'_k}{\bullet}$ in $\tilde{\Gamma}$, we may omit $\times_{\boldsymbol{x}_i, \mathbb{R}, \mathbf{inc}} \boldsymbol{[0,\infty)}$ and $\times_{\boldsymbol{z}_k, \mathbb{R}, \mathbf{inc}} \boldsymbol{[0,\infty)}$ from the first and second lines of (7.27), without changing the fibre product. Similarly, for each white edge $\overset{u'_j}{\bullet} - \overset{v'_k}{\bullet}$ in $\tilde{\Gamma}$, we may omit $\times_{\boldsymbol{y}_j, \mathbb{R}, \mathbf{inc}} \boldsymbol{[0,\infty)}$ and $\times_{\boldsymbol{z}_k, \mathbb{R}, \mathbf{inc}} \boldsymbol{[0,\infty)}$ from the first and second lines of (7.27), without changing the fibre product. But every vertex t'_i, u'_j, v'_k in $\tilde{\Gamma}$ lies on exactly one white edge, and the vertices of $\tilde{\Gamma}$ correspond to the fibre products $\times_{\mathbb{R}} \boldsymbol{[0,\infty)}$ in (7.27). So making these omissions for all white edges in $\tilde{\Gamma}$ deletes all fibre products $\times_{\mathbb{R}} \boldsymbol{[0,\infty)}$ in the first and second lines of (7.27), leaving the third line of (7.27). This proves (7.27).

Now in the final term $\mathbf{S}_{\acute{T},\acute{F},\acute{t}} \times_{\mathbf{S}_{\dot{V},\dot{H},\dot{v}}} \mathbf{S}_{\acute{U},\acute{G},\acute{u}}$ in (7.27), we have $\partial \mathbf{S}_{\dot{V},\dot{H},\dot{v}} = \emptyset$ as $\partial \dot{V} = \emptyset$. We can apply the construction of Definition 4.19 to the data $\acute{T}, \acute{F}, \acute{t}, \acute{U}, \acute{G}, \acute{u}, \dot{V}, \dot{H}, \dot{v}$ and $\acute{p}, \acute{\hat{p}}, \acute{q}, \acute{\hat{q}}$. This gives a manifold S which is an open neighbourhood of $\acute{W} = \{(x,y) \in \acute{T} \times \acute{U} : \acute{t}(x) = 0,\ \acute{u}(y) = 0,\ \acute{p}(x) = \acute{q}(y)\}$ in $\acute{T} \times \acute{U}$, a vector bundle $E \to S$, and a smooth section $s : S \to E$, satisfying

$$\begin{aligned} \dim S - \operatorname{rank} E &= \dim \acute{T} - \operatorname{rank} \acute{F} + \dim \acute{U} - \operatorname{rank} \acute{G} - \dim \dot{V} + \operatorname{rank} \dot{H} \\ &= \operatorname{vdim} \mathbf{X} + \operatorname{vdim} \mathbf{Y} - \operatorname{vdim} \mathbf{Z}. \end{aligned} \tag{7.28}$$

Since $\acute{T}, \acute{U}$ have corners, S also has corners. The important point for the construction to work is that $\dot{V}$ is without boundary. We can now adapt the proof of Theorem 4.20 to give an equivalence in $\mathbf{dSpa^c}$:

$$\mathbf{S}_{\acute{T},\acute{F},\acute{t}} \times_{\mathbf{S}_{\acute{p},\acute{\hat{p}}}, \mathbf{S}_{\dot{V},\dot{H},\dot{v}}, \mathbf{S}_{\acute{q},\acute{\hat{q}}}} \mathbf{S}_{\acute{U},\acute{G},\acute{u}} \simeq \mathbf{S}_{S,E,s}. \tag{7.29}$$

Equations (7.19), (7.27) and (7.29) give an equivalence $\hat{\mathbf{W}} \simeq \mathbf{S}_{S,E,s}$ in $\mathbf{dSpa^c}$. Hence any point $w \in \mathbf{W}$ has an open neighbourhood $\hat{\mathbf{W}}$ which is a principal d-manifold, and has dimension $\operatorname{vdim} \mathbf{X} + \operatorname{vdim} \mathbf{Y} - \operatorname{vdim} \mathbf{Z}$ by (7.28). Therefore $\mathbf{W}$ is a d-manifold with corners, and (7.18) holds. This completes the proof of Theorem 7.37. $\square$

Here is an analogue of Theorem 4.22.

Theorem 7.38. *Suppose $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ are 1-morphisms in $\mathbf{dMan^c}$. The following are sufficient conditions for $\boldsymbol{g}, \boldsymbol{h}$ to be cd-transverse,* *and hence bd-transverse, so that* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ *is a d-manifold with corners of virtual dimension* (7.18):

(a) $\mathbf{Z}$ *is a manifold without boundary, that is,* $\boldsymbol{Z} \in \bar{\mathbf{M}}\mathbf{an}$; *or*

(b) $\boldsymbol{g}$ *or* $\boldsymbol{h}$ *is a w-submersion.*

Proof. In each case (a),(b), $\boldsymbol{g}, \boldsymbol{h}$ are d-transverse by Theorem 4.22, and c-transverse by Lemma 6.35, as w-submersions are semisimple and flat. Thus they are cd-transverse, and the result follows from Theorem 7.37. $\square$

Here is the analogue of Theorem 4.23. It has the same proof, but using Theorem 7.38(b) and Corollary 7.33 instead of Theorem 4.22(b) and Corollary 4.11.

Theorem 7.39. *Let* $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ *be d-manifolds with corners with* $\mathbf{Y}$ *a manifold, and* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$, $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *be* 1*-morphisms with* $\boldsymbol{g}$ *a submersion. Then* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ *is a manifold, with* $\dim \mathbf{W} = \operatorname{vdim} \mathbf{X} + \dim \mathbf{Y} - \operatorname{vdim} \mathbf{Z}$.

Here are analogues of Propositions 4.26 and 4.27. They are proved by the same method, using the fibre product of §6.8 rather than §2.5, and using Theorems 7.31(d′) and 7.37 instead of Theorems 4.9(d′) and 4.20. Note that Proposition 7.41 discusses s-immersions and sf-immersions, but not immersions, and the analogue of Proposition 7.41 is false for general immersions.

Proposition 7.40. *Let* $\mathbf{X}$ *be a d-manifold with corners and* $\boldsymbol{g} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}_{\boldsymbol{k}}$ *a semisimple, flat* 1*-morphism in* $\mathbf{dMan^c}$. *Then the fibre product* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbb{R}^{\boldsymbol{n}}_{\boldsymbol{k}},\mathbf{0}} *$ *exists in* $\mathbf{dMan^c}$, *and* $\boldsymbol{\pi}_{\mathbf{X}} : \mathbf{W} \to \mathbf{X}$ *is an s-embedding. When* $k = 0$, *any* 1*-morphism* $\boldsymbol{g} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}_{\mathbf{0}} = \mathbb{R}^{\boldsymbol{n}}$ *is semisimple and flat, and* $\boldsymbol{\pi}_{\mathbf{X}} : \mathbf{W} \to \mathbf{X}$ *is an sf-embedding.*

Proposition 7.41. *Suppose* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *is an s-immersion of d-manifolds with corners, and* $x \in \mathbf{X}$ *with* $\boldsymbol{f}(x) = y \in \mathbf{Y}$. *Then there exist open d-submanifolds* $x \in \mathbf{U} \subseteq \mathbf{X}$ *and* $y \in \mathbf{V} \subseteq \mathbf{Y}$ *with* $\boldsymbol{f}(\mathbf{U}) \subseteq \mathbf{V}$ *and a semisimple, flat* 1*-morphism* $\boldsymbol{g} : \mathbf{V} \to \mathbb{R}^{\boldsymbol{n}}_{\boldsymbol{k}}$ *with* $\boldsymbol{g}(y) = 0$, *where* $n = \operatorname{vdim} \mathbf{Y} - \operatorname{vdim} \mathbf{X} \geqslant 0$ *and* $0 \leqslant k \leqslant n$, *fitting into a* 2*-Cartesian square in* $\mathbf{dMan^c}$:

$$\begin{tikzcd} \mathbf{U} \arrow[rr, "\boldsymbol{\pi}"'] \arrow[d, "\boldsymbol{f}|_{\mathbf{U}}"'] \arrow[drr, phantom, "\Uparrow"] & & * \arrow[d, "\mathbf{0}"'] \\ \mathbf{V} \arrow[rr, "\boldsymbol{g}"] & & \mathbb{R}^{\boldsymbol{n}}_{\boldsymbol{k}}. \end{tikzcd}$$

If $\boldsymbol{f}$ *is an sf-immersion then* $k = 0$. *If* $\boldsymbol{f}$ *is an s- or sf-embedding then we may take* $\mathbf{U} = \boldsymbol{f}^{-1}(\mathbf{V})$.

The material of §6.9 extends immediately to cd-transverse fibre products in $\mathbf{dMan^c}$, since these are examples of c-transverse fibre products in $\mathbf{dSpa^c}$. Thus Theorem 6.50 describes $C_i(\mathbf{W})$ and $\partial\mathbf{W}$ for a cd-transverse fibre product $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ in $\mathbf{dMan^c}$. Furthermore, one can show using $\boldsymbol{g}, \boldsymbol{h}$ cd-transverse that the fibre products in (6.126)–(6.127) are also cd-transverse. This gives:

Corollary 7.42. *Suppose* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ *and* $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *are cd-transverse* 1*-morphisms in* $\mathbf{dMan^c}$*, and let* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ *be the fibre product in* $\mathbf{dMan^c}$*, which exists by Theorem 7.37. Then we have equivalences in* $\mathbf{dMan^c}$ *for all* $i \geqslant 0$*, where each fibre product is cd-transverse and so exists in* $\mathbf{dMan^c}$ *:*

$$C_i(\mathbf{W}) \simeq \coprod_{j,k,l\geqslant 0: i=j+k-l} C_j^{\boldsymbol{g},l}(\mathbf{X}) \times_{C_j^l(\boldsymbol{g}),C_l(\mathbf{Z}),C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathbf{Y}), \tag{7.30}$$

$$\partial \mathbf{W} \simeq \coprod_{j,k,l\geqslant 0: j+k=l+1} C_j^{\boldsymbol{g},l}(\mathbf{X}) \times_{C_j^l(\boldsymbol{g}),C_l(\mathbf{Z}),C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathbf{Y}). \tag{7.31}$$

Similarly, Theorems 6.45 and 6.52 and Proposition 6.53 in §6.8–§6.9 extend immediately to fibre products in $\mathbf{dMan^c}$, with all fibre products in equations (6.132)–(6.138) cd-transverse.

7.7 Embedding d-manifolds with corners into manifolds

In §4.4 we showed in Theorem 4.29 that any compact d-manifold $\boldsymbol{X}$ without boundary can be embedded in $\mathbb{R}^n$ for $n \gg 0$, and gave necessary and sufficient conditions for when a noncompact $\boldsymbol{X}$ can be embedded in $\mathbb{R}^n$. We also showed in Theorem 4.34 that if $\boldsymbol{X}$ can be embedded in a manifold Y then $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ for open $V \subseteq Y$. Putting these together shows every compact d-manifold without boundary is principal. We now extend these results to d-manifolds with corners. Some of the issues have already been discussed for $\mathbf{Man^c}$ in §5.7.

For manifolds and d-manifolds with corners, we have three notions of embedding — embeddings, s-embeddings and sf-embeddings — but in this section we will be concerned only with embeddings and sf-embeddings. If $\mathbf{X}$ is a compact d-manifold with corners, the natural analogue of Theorem 4.29 is that $\mathbf{X}$ has an embedding into $\mathbb{R}^n$, and of Theorem 4.34 is that if $\mathbf{X}$ has an sf-embedding into a manifold with corners Y then $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$ for open $V \subseteq Y$. Before we can combine the two to show that $\mathbf{X}$ is principal, we have to bridge the gap between embeddings and sf-embeddings, which requires more work.

Here is the analogue of Theorem 4.29. The proof is the same, using Theorem 7.30(d) and Proposition 7.27 instead of Theorem 4.8(d) and Proposition 4.5.

Theorem 7.43. *Let* $\mathbf{X}$ *be a compact d-manifold with corners. Then there exists an embedding* $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$ *for some* $n \gg 0$*.*

In the same way, the analogues of Lemma 4.30 and Theorems 4.32 and 4.33 for $\mathbf{dMan^c}$ hold, the last being:

Theorem 7.44. *Let* $\mathbf{X}$ *be a d-manifold with corners. Then there exist immersions and/or embeddings* $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$ *for some* $n \gg 0$ *if and only if there is an upper bound for* $\dim T_x^*\underline{X}$ *for all* $x \in \underline{X}$*. If there is such an upper bound, then immersions* $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$ *exist provided* $n \geqslant 2 \dim T_x^*\underline{X}$ *for all* $x \in \underline{X}$*, and embeddings* $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$ *exist provided* $n \geqslant 2 \dim T_x^*\underline{X} + 1$ *for all* $x \in \underline{X}$*. For embeddings we may also choose* $\boldsymbol{f}$ *with* $f(X)$ *closed in* $\mathbb{R}^n$*.*

Next we study existence of sf-embeddings of d-manifolds with corners in manifolds. The next two results deal with d-manifolds with boundary.

Proposition 7.45. *Suppose* $\mathbf{X}$ *is a d-manifold with boundary. Then there exists a simple, flat* 1*-morphism* $\boldsymbol{f}:\mathbf{X}\to\mathbf{[0,\infty)}$, *where* $\mathbf{[0,\infty)}=F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}\big([0,\infty)\big)$, *fitting into a* 2*-Cartesian diagram in* $\mathbf{dMan^c}$*:*

$$
\begin{tikzcd}
\boldsymbol{\partial}\mathbf{X} \arrow[r, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}"'] \arrow[dr, phantom, "\Uparrow \mathrm{id}_{\mathbf{0}\circ\boldsymbol{\pi}}"] & \boldsymbol{*} \arrow[d, "\mathbf{0}"] \\
\mathbf{X} \arrow[r, "\boldsymbol{f}"] & \mathbf{[0,\infty)}.
\end{tikzcd}
\tag{7.32}
$$

Proof. For each $x'\in\boldsymbol{\partial X}$ we may choose a boundary defining function $(\boldsymbol{V}_{x'},\boldsymbol{b}_{x'})$ for $\mathbf{X}$ at x', where $\boldsymbol{V}_{x'}$ is an open neighbourhood of $\boldsymbol{i}_{\mathbf{X}}(x')$ in $\boldsymbol{X}$. Using $\boldsymbol{i}_{\mathbf{X}}$ injective, by making $\boldsymbol{V}_{x'}$ smaller we may suppose that $(\boldsymbol{V}_{x'},\boldsymbol{b}_{x'})$ is also a boundary-defining function for $\mathbf{X}$ at any $x''\in\boldsymbol{i}_{\mathbf{X}}^{-1}(\boldsymbol{V}_{x'})\subseteq\boldsymbol{\partial X}$. As $\boldsymbol{i}_{\mathbf{X}}$ is proper, $\underline{i}_{\mathbf{X}}(\underline{\partial X})$ is closed in $\underline{X}$, so $\underline{X}^\circ=\underline{X}\setminus\underline{i}_{\mathbf{X}}(\underline{\partial X})$ is open in $\underline{X}$. Hence $\{\underline{V}_{x'}:x'\in\underline{\partial X}\}\cup\{\underline{X}^\circ\}$ is an open cover of $\underline{X}$, which is separated, paracompact, and locally fair. Therefore by Proposition B.21 there exists a partition of unity $\{\eta_{x'}:x'\in\underline{\partial X}\}\cup\{\eta_{X^\circ}\}$ on $\underline{X}$ subordinate to $\{\underline{V}_{x'}:x'\in\underline{\partial X}\}\cup\{\underline{X}^\circ\}$, where we take the $\eta_{x'}$ and η_{X° to have values in $[0,1]$.

We will define a 1-morphism $\boldsymbol{f}:\boldsymbol{X}\to\mathbf{[0,\infty)}$ in $\mathbf{dSpa}$ by

$$
\boldsymbol{f}=\textstyle\sum_{x'\in\underline{\partial X}}\boldsymbol{\eta}_{x'}\cdot\boldsymbol{b}_{x'}+\boldsymbol{\eta}_{X^\circ}\cdot\mathbf{1},
\tag{7.33}
$$

Here $\boldsymbol{\eta}_{x'}=(\underline{\eta}_{x'},\eta'_{x'},\eta''_{x'}):\boldsymbol{X}\to\mathbf{[0,\infty)}$ is a 1-morphism in $\mathbf{dSpa}$ with $\underline{\eta}_{x'}:\underline{X}\to\underline{[0,\infty)}$ the unique morphism induced by $\eta_{x'}\in\mathcal{O}_X(X)$, and $\eta''_{x'}=0$ as $\mathcal{E}_{[0,\infty)}=0$, and $\eta'_{x'}$ is chosen arbitrarily such that $\boldsymbol{\eta}_{x'}$ is a 1-morphism which is supported on $\operatorname{supp}\eta_{x'}\subseteq\boldsymbol{V}_{x'}$. One can show this is possible. Although $\boldsymbol{b}_{x'}$ is only defined on $\boldsymbol{V}_{x'}$, since $\boldsymbol{\eta}_{x'}=\mathbf{0}$ outside $\boldsymbol{V}_{x'}$ we can regard $\boldsymbol{\eta}_{x'}\cdot\boldsymbol{b}_{x'}$ as a 1-morphism $\boldsymbol{X}\to\mathbf{[0,\infty)}$ which is zero outside $\boldsymbol{V}_{x'}$. We define $\boldsymbol{\eta}_{X^\circ}$ in the same way as $\boldsymbol{\eta}_{x'}$, and $\mathbf{1}:\boldsymbol{X}\to\mathbf{[0,\infty)}$ is the constant function 1 on $\boldsymbol{X}$. Then (7.33) is a locally finite sum, and $\boldsymbol{f}:\boldsymbol{X}\to\mathbf{[0,\infty)}$ is well-defined.

Now let $x'\in\boldsymbol{\partial X}$. Since $\{\eta_{x'}:x'\in\underline{\partial X}\}\cup\{\eta_{X^\circ}\}$ is locally finite and $\underline{i}_{\mathbf{X}}(x')\notin\underline{X}^\circ$, there exist $x'_1,\ldots,x'_n\in\underline{\partial X}$ and an open neighbourhood $\underline{U}_{x'}$ of $\underline{i}_{\mathbf{X}}(x')$ in $\underline{X}$ such that $x'\in\underline{U}_{x'}\subseteq\underline{V}_{x'_i}$ for $i=1,\ldots,n$, and $\eta_{x''}|_{\underline{U}_{x'}}=0$ for all $x''\in\underline{\partial X}\setminus\{x_1,\ldots,x_n\}$ and $\eta_{X^\circ}|_{\underline{U}_{x'}}=0$. Then $(\boldsymbol{V}_{x'},\boldsymbol{b}_{x'})$ and $(\boldsymbol{V}_{x'_i},\boldsymbol{b}_{x'_i})$ for $i=1,\ldots,n$ are all boundary-defining functions for $\mathbf{X}$ at x'. Thus by Proposition 6.6(c) for $i=1,\ldots,n$ there exists an open neighbourhood of x', which making $\underline{U}_{x'}$ we may take to be $\boldsymbol{U}_{x'}$, and a 1-morphism $\boldsymbol{c}_{x'_i}:\boldsymbol{U}_{x'}\to\mathbf{(0,\infty)}$ such that $\boldsymbol{b}_{x'_i}|_{\boldsymbol{U}_{x'}}=\boldsymbol{c}_{x'_i}\cdot\boldsymbol{b}_{x'}|_{\boldsymbol{U}_{x'}}$. Hence (7.33) gives

$$
\boldsymbol{f}|_{\boldsymbol{U}_{x'}}=\textstyle\sum_{i=1}^n\boldsymbol{\eta}_{x'_i}\cdot\boldsymbol{b}_{x'_i}=\big(\sum_{i=1}^n\boldsymbol{\eta}_{x'_i}\cdot\boldsymbol{c}_{x'_i}\big)\cdot b_{x'}.
$$

Since the $\boldsymbol{\eta}_{x'_i}$ take values in $[0,1]$ with $\sum_{i=1}^n\eta_{x'_i}=1$ on $\underline{U}_{x'}$ and $\boldsymbol{c}_{x'_i}>0$, we see that $\sum_{i=1}^n\boldsymbol{\eta}_{x'_i}\cdot\boldsymbol{c}_{x'_i}:\boldsymbol{U}_{x'}\to\mathbf{(0,\infty)}$. Therefore Proposition 6.6(d) shows that $\big(\boldsymbol{U}_{x'},\boldsymbol{f}|_{\boldsymbol{U}_{x'}}\big)$ is a boundary defining function for $\mathbf{X}$ at $x'\in\boldsymbol{\partial X}$.

Now if $\mathbf{X}$ is a d-manifold with corners and $(\boldsymbol{V}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{X}$ at x' for $\boldsymbol{V} \subseteq \boldsymbol{X}$, then with $\mathbf{V} \subseteq \mathbf{X}$ the d-manifold with corners corresponding to $\boldsymbol{V}$, making $\boldsymbol{V}$ smaller if necessary, $\boldsymbol{b} : \mathbf{V} \to [\mathbf{0}, \boldsymbol{\infty})$ is a semisimple, flat 1-morphism in $\mathbf{dSpa^c}$, which is simple if $\boldsymbol{i}_{\mathbf{X}}$ is injective over $\boldsymbol{V}$, and the analogue of (6.1) in $\mathbf{dSpa^c}$ is 2-Cartesian. Hence $\boldsymbol{f}|_{\mathbf{U}_{x'}} : \mathbf{U}_{x'} \to [\mathbf{0}, \boldsymbol{\infty})$ is a simple, flat 1-morphism in $\mathbf{dSpa^c}$, and the following is 2-Cartesian in $\mathbf{dSpa^c}$:

$$\begin{tikzcd}
\boldsymbol{i}_{\mathbf{X}}^{-1}(\mathbf{U}_{x'}) \arrow[rr, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{X}}|_{\boldsymbol{i}_{\mathbf{X}}^{-1}(\mathbf{U}_{x'})}"'] \arrow[drr, phantom, "\mathrm{id}_{\mathbf{0}\circ\boldsymbol{\pi}}\Uparrow"] & & {*} \arrow[d, "\mathbf{0}"] \\
\mathbf{U}_{x'} \arrow[rr, "\boldsymbol{f}|_{\mathbf{U}_{x'}}"] & & {[\mathbf{0}, \boldsymbol{\infty}).}
\end{tikzcd} \tag{7.34}$$

Therefore $\boldsymbol{f}$ is a simple, flat 1-morphism in $\mathbf{dSpa^c}$ near any point $\boldsymbol{i}_{\mathbf{X}}(x') \in \boldsymbol{X}$ for each $x' \in \boldsymbol{\partial X}$. If $x \in \boldsymbol{X}^\circ = \boldsymbol{X} \setminus \boldsymbol{i}_{\mathbf{X}}(\boldsymbol{\partial X})$ then $\boldsymbol{i}_{\mathbf{X}}^{-1}(x) = \emptyset$, and $\boldsymbol{f}(x) > 0$ so $\boldsymbol{i}_{[0,\infty)}^{-1}(\boldsymbol{f}(x)) = \emptyset$. Thus, as $\boldsymbol{f}$ is a 1-morphism in $\mathbf{dSpa}$, the extra conditions for $\boldsymbol{f}$ to be a 1-morphism in $\mathbf{dSpa^c}$ near $x \in \boldsymbol{X}^\circ$, and to be simple and flat, are trivial. Hence $\boldsymbol{f} : \mathbf{X} \to [\mathbf{0}, \boldsymbol{\infty})$ is a simple, flat 1-morphism in $\mathbf{dSpa^c}$. Also (7.34) 2-Cartesian shows that (7.32) is locally 2-Cartesian near each $x' \in \partial\mathbf{X}$. As $\boldsymbol{i}_{\mathbf{X}}$ is injective, $\boldsymbol{f} = 0$ on $\boldsymbol{i}_{\mathbf{X}}(\boldsymbol{\partial X})$ and $\boldsymbol{f} > 0$ on $\boldsymbol{X} \setminus \boldsymbol{i}_{\mathbf{X}}(\boldsymbol{\partial X})$, we see that (7.32) is globally 2-Cartesian. □

Here is an analogue of Theorem 5.33.

Corollary 7.46. *Let* $\mathbf{X}$ *be a d-manifold with boundary. Then there exist sf-immersions and/or sf-embeddings* $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}_{\mathbf{1}} = [\mathbf{0}, \boldsymbol{\infty}) \times \mathbb{R}^{\boldsymbol{n-1}}$ *for some* $n \gg 0$ *if and only if* $\dim T^*_x\underline{X}$ *is bounded above for all* $x \in \underline{X}$. *Such an upper bound always exists if* $\mathbf{X}$ *is compact. If there is such an upper bound, then sf-immersions* $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}_{\mathbf{1}}$ *exist provided* $n \geqslant 2\dim T^*_x\underline{X} + 1$ *for all* $x \in \underline{X}$, *and sf-embeddings* $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}_{\mathbf{1}}$ *exist provided* $n \geqslant 2\dim T^*_x\underline{X} + 2$ *for all* $x \in \underline{X}$. *For sf-embeddings we may also choose* $\boldsymbol{f}$ *with* $f(X)$ *closed in* $\mathbb{R}^n_1$.

Proof. Write $i : \mathbb{R}^n_1 \hookrightarrow \mathbb{R}^n$ for the inclusion, and $\boldsymbol{i} = F^{\mathbf{dMan^c}}_{\mathbf{Man^c}}(i)$. If $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}_{\mathbf{1}}$ is an sf-immersion or sf-embedding then $\boldsymbol{i} \circ \boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$ is an immersion or embedding. Hence the 'only if' in the first part follows from the 'only if' part of Theorem 7.44. For the 'if' part, Proposition 7.45 gives a simple, flat 1-morphism $\boldsymbol{g} : \mathbf{X} \to [\mathbf{0}, \boldsymbol{\infty})$, and Theorem 7.44 gives an immersion $\boldsymbol{h} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n-1}}$ if $n \geqslant 2\dim T^*_x\underline{X} + 1$ for all x, and an embedding $\boldsymbol{h} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n-1}}$ if $n \geqslant 2\dim T^*_x\underline{X} + 2$ for all x. Then the direct product 1-morphism $\boldsymbol{f} = (\boldsymbol{g}, \boldsymbol{h}) : \mathbf{X} \to [\mathbf{0}, \boldsymbol{\infty}) \times \mathbb{R}^{\boldsymbol{n-1}} = \mathbb{R}^{\boldsymbol{n}}_{\mathbf{1}}$ is simple and flat as $\boldsymbol{g}$ is simple and flat and $\partial\mathbb{R}^{\boldsymbol{n-1}} = \emptyset$, and an immersion or embedding as $\boldsymbol{h}$ is. Hence $\boldsymbol{f}$ is an sf-immersion or sf-embedding. The last part follows from the last part of Theorem 7.44. □

Here are necessary and sufficient conditions for existence of sf-embeddings from a d-manifold with corners $\mathbf{X}$ into a manifold with corners Y.

Theorem 7.47. *Let* $\mathbf{X}$ *be a d-manifold with corners. Then there exist a manifold with corners* Y *and an sf-embedding* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$, *where* $\mathbf{Y} = F^{\mathbf{dMan^c}}_{\mathbf{Man^c}}(Y)$, *if and only if* $\dim T^*_x\underline{X} + |\underline{i}_{\mathbf{X}}^{-1}(x)|$ *is bounded above for all* $x \in \underline{X}$. *If such an upper* *bound exists, then we may take Y to be an embedded n-dimensional submanifold of $\mathbb{R}^n$ for any n with $n \geqslant 2\bigl(\dim T_x^*\underline{X} + |\underline{i}_{\mathbf{X}}^{-1}(x)|\bigr) + 1$ for all $x \in \underline{X}$.*

Such an upper bound always exists if $\mathbf{X}$ is compact. Thus, every compact d-manifold with corners admits an sf-embedding into a manifold with corners.

Proof. For the 'only if' part, if $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is an sf-embedding with Y a manifold then $\dim T_x^*\underline{X} \leqslant \dim T_{\underline{f}(x)}^*\underline{Y} = \dim Y$ and $|\underline{i}_{\mathbf{X}}^{-1}(x)| = |\underline{i}_Y^{-1}(\underline{f}(x))| \leqslant \dim Y$ for all $x \in \underline{X}$, so $\dim T_x^*\underline{X} + |\underline{i}_{\mathbf{X}}^{-1}(x)|$ is bounded above by $2\dim Y$.

For the 'if' part, suppose $n \geqslant 2\bigl(\dim T_x^*\underline{X} + |\underline{i}_{\mathbf{X}}^{-1}(x)|\bigr) + 1$ for all $x \in \underline{X}$. We will first show that we can choose the following data:

(a) a d-manifold with corners $\mathbf{W}$;

(b) a finite local 1-isomorphism $\boldsymbol{p} : \mathbf{W} \to \mathbf{X}$ in $\mathbf{dMan^c}$;

(c) a 1-morphism $\boldsymbol{q} : \partial\mathbf{X} \to \partial\mathbf{W}$ which is a 1-isomorphism with an open and closed d-submanifold $\boldsymbol{q}(\partial\mathbf{X})$ in $\partial\mathbf{W}$, satisfying $\boldsymbol{p} \circ \boldsymbol{i}_{\mathbf{W}} \circ \boldsymbol{q} = \boldsymbol{i}_{\mathbf{X}}$; and

(d) a 1-morphism $\boldsymbol{b} : \boldsymbol{W} \to [\mathbf{0}, \boldsymbol{\infty})$ in $\mathbf{dSpa}$ such that $(\boldsymbol{W}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{W}$ at $\boldsymbol{q}(x')$ for all $x' \in \boldsymbol{\partial X}$.

This part of the proof does not depend on n, and works for all d-manifolds with corners $\mathbf{X}$. We can think of $\mathbf{W}$ as a 'collar' of $\partial\mathbf{X}$ in $\mathbf{X}$, and of $\mathbf{W}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{b}$ as a 'universal boundary defining function' for $\mathbf{X}$ at every point in $\partial\mathbf{X}$. To construct $\mathbf{W}, \boldsymbol{p}, \boldsymbol{q}$, the rough idea is that if $x \in \mathbf{X}$ with $\boldsymbol{i}_{\mathbf{X}}^{-1}(x) = \{x_1', \ldots, x_k'\}$ then $\boldsymbol{p} : \mathbf{W} \to \mathbf{X}$ should be an k-sheeted étale cover of $\mathbf{X}$ near x, with sheets corresponding to $x_1', \ldots, x_k'$, and $\boldsymbol{q}$ should send x_i' to the corresponding point of $\partial\mathbf{W}$ in the sheet corresponding to x_i'.

More formally, we define $\mathbf{W} = \bigl[\coprod_{x' \in \boldsymbol{\partial X}} \mathbf{V}_{x'}\bigr] / \sim$, where $\mathbf{V}_{x'}$ is a small open neighbourhood of $\boldsymbol{i}_{\mathbf{X}}(x')$ in $\mathbf{X}$, and the equivalence relation $\sim$ identifies $\mathbf{V}_{x'}$ and $\mathbf{V}_{x''}$ on their overlap $\mathbf{V}_{x'} \cap \mathbf{V}_{x''}$ in $\mathbf{X}$ provided x' and x'' are 'close' in $\boldsymbol{\partial X}$. If $x' \neq x''$ but $\boldsymbol{i}_{\mathbf{X}}(x') = \boldsymbol{i}_{\mathbf{X}}(x'')$, then x', x'' are not counted as 'close' in $\boldsymbol{\partial X}$, so $\mathbf{V}_{x'}$ and $\mathbf{V}_{x''}$ are not identified on their overlap. It takes some care to show that we can choose such neighbourhoods $\mathbf{V}_{x'}$ and identifications on overlaps so that $\sim$ is an equivalence relation and $\mathbf{W}$ is Hausdorff, but it can be done.

The morphism $\boldsymbol{p} : \mathbf{W} \to \mathbf{X}$ is defined locally to take $\mathbf{V}_{x'} \subset \mathbf{W}$ to $\mathbf{V}_{x'} \subset \mathbf{X}$ by the identity. The morphism $\boldsymbol{q} : \partial\mathbf{X} \to \partial\mathbf{W}$ is defined near $x' \in \boldsymbol{\partial X}$ to take $\partial\mathbf{V}_{x'} \subset \partial\mathbf{X}$ to $\partial\mathbf{V}_{x'} \subset \partial\mathbf{W}$ by the identity. Then $\boldsymbol{p} \circ \boldsymbol{i}_{\mathbf{W}} \circ \boldsymbol{q} = \boldsymbol{i}_{\mathbf{X}}$ is immediate. To construct $\boldsymbol{b}$, making the $\mathbf{V}_{x'}$ smaller if necessary, for each $x' \in \boldsymbol{\partial X}$ we can choose $\boldsymbol{b}_{x'} : \boldsymbol{V}_{x'} \to [\mathbf{0}, \boldsymbol{\infty})$ such that $(\boldsymbol{V}_{x'}, \boldsymbol{b}_{x'})$ is a boundary defining function for $\mathbf{X}$ at x'. Then we combine the $\boldsymbol{b}_{x'}$ using a partition of unity on $\underline{W}$ to get $\boldsymbol{b} : \mathbf{W} \to [\mathbf{0}, \boldsymbol{\infty})$ as in the proof of Proposition 7.45, and making $\mathbf{W}$ smaller if necessary we find $(\boldsymbol{W}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{W}$ at $\boldsymbol{q}(x')$ for all $x' \in \boldsymbol{\partial X}$. Hence we can choose $\mathbf{W}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{b}$ satisfying (a)–(d).

Next, as $n \geqslant 2\bigl(\dim T_x^*\underline{X} + |\underline{i}_{\mathbf{X}}^{-1}(x)|\bigr) + 1 \geqslant 2\dim T_x^*\underline{X} + 1$, by Theorem 7.44 we may choose an embedding $\boldsymbol{g} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$. Since $\boldsymbol{p} : \mathbf{W} \to \mathbf{X}$ is a local 1-isomorphism and $\boldsymbol{g} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$ is an embedding, $\boldsymbol{g} \circ \boldsymbol{p} : \mathbf{W} \to \mathbf{X}$ is an immersion, but may not be an embedding as $\boldsymbol{p}$ may not be injective. By the same argument used to define $\mathbf{W}$ above, making $\mathbf{W}$ smaller if necessary, we can construct an n-manifold without boundary U, a finite étale map $\pi : U \to \mathbb{R}^n$, and an embedding $\boldsymbol{h} : \mathbf{W} \to \mathbf{U}$, such that $\boldsymbol{\pi} \circ \boldsymbol{h} = \boldsymbol{g} \circ \boldsymbol{p}$ for $\mathbf{U}, \boldsymbol{\pi} = F_{\mathbf{Man}}^{\mathbf{dMan^c}}(U, \pi)$.

We have $U = \bigl[\coprod_{x' \in \partial \boldsymbol{X}} U_{x'}\bigr] / \sim$, where $U_{x'}$ is a small open neighbourhood of $\boldsymbol{g} \circ \boldsymbol{i}_{\mathbf{X}}(x')$ in $\mathbb{R}^n$, and $\sim$ identifies $U_{x'}$ and $U_{x''}$ on their overlap $U_{x'} \cap U_{x''}$ in $\mathbb{R}^n$ provided x' and x'' are 'close' in $\partial \boldsymbol{X}$. Making U smaller if necessary, we can also ensure the following property holds: let $x \in \boldsymbol{X}$ with $\boldsymbol{p}^{-1}(x) = \{w_1, \ldots, w_l\} \subseteq \boldsymbol{W}$. Then for sufficiently small $\epsilon > 0$, writing $B_\epsilon(\boldsymbol{g}(x))$ for the open ball about $\boldsymbol{g}(x)$ in $\mathbb{R}^n$, we have a natural diffeomorphism $U \supseteq \pi^{-1}\bigl(B_\epsilon(\boldsymbol{g}(x))\bigr) \cong B_\epsilon(\boldsymbol{g}(x)) \times \{w_1, \ldots, w_l\}$ which identifies w_i with $(\boldsymbol{g}(x), w_i)$ and $\pi|_{\pi^{-1}(B_\epsilon(\boldsymbol{g}(x)))}$ with the projection $B_\epsilon(\boldsymbol{g}(x)) \times \{w_1, \ldots, w_l\} \to B_\epsilon(\boldsymbol{g}(x))$.

We now have an embedding $\boldsymbol{h} : \mathbf{W} \to \mathbf{U}$ and a 1-morphism $\boldsymbol{b} : \mathbf{W} \to \boldsymbol{[0,\infty)} \subset \mathbb{R}$. We claim we can choose a smooth map $c : U \to \mathbb{R}$ such that there exists a 2-morphism $\boldsymbol{b} \Rightarrow \boldsymbol{c} \circ \boldsymbol{h}$, where $\boldsymbol{c} = F_{\mathbf{Man}}^{\mathbf{dMan^c}}(c)$, regarding $\boldsymbol{b}, \boldsymbol{c} \circ \boldsymbol{h}$ as 1-morphisms $\mathbf{W} \to \mathbb{R}$. Since $\boldsymbol{h}$ is an embedding, any 1-morphism $\mathbf{W} \to \mathbb{R}$ can locally be extended to a 1-morphism $\mathbf{U} \to \mathbb{R}$ up to 2-isomorphism, and we combine such local choices with a partition of unity to get $\boldsymbol{c}$.

We need c to satisfy the following extra condition: suppose $x \in \underline{X}$ with $\underline{i}_{\mathbf{X}}^{-1}(x) = \{x_1', \ldots, x_k'\}$. For $i = 1, \ldots, k$ set $u_i = \boldsymbol{h} \circ \boldsymbol{i}_{\mathbf{W}} \circ \boldsymbol{q}(x_i') \in U$, so that $\pi(u_i) = \boldsymbol{g}(x) \in \mathbb{R}^n$. Then $\mathrm{d}c|_{u_i} \in T_{u_i}^* U$. But $\mathrm{d}\pi|_{u_i} : T_{u_i} U \to T_{\boldsymbol{g}(x)} \mathbb{R}^n = (\mathbb{R}^n)^*$ is an isomorphism as π is étale, so $(\mathrm{d}\pi|_{u_i}^{-1})^* : T_{u_i}^* U \to (\mathbb{R}^n)^*$ is an isomorphism, and we write $t_i = (\mathrm{d}\pi|_{u_i}^{-1})^*(\mathrm{d}c|_{u_i}) \in (\mathbb{R}^n)^*$. We require that $t_1, \ldots, t_k$ should be linearly independent in $(\mathbb{R}^n)^*$ for all $x \in \underline{X}$. Using the condition that $n \geqslant 2\bigl(\dim T_x^* \underline{X} + |\underline{i}_{\mathbf{X}}^{-1}(x)|\bigr) + 1$, one can show that this holds for generic $\boldsymbol{g}$ by the same kind of dimension-counting arguments used to prove Theorems 4.32 and 4.33.

Now consider the set

$$T = \bigl\{(y_1, \ldots, y_n) \in \mathbb{R}^n : c(u) \geqslant 0 \text{ for all } u \in U \text{ with } \pi(u) = (y_1, \ldots, y_n)\bigr\}. \quad (7.35)$$

Let $x \in \underline{X}$ with $\boldsymbol{g}(x) = (y_1, \ldots, y_n)$, write $\underline{i}_{\mathbf{X}}^{-1}(x) = \{x_1', \ldots, x_k'\}$, and set $w_i' = \underline{q}(x_i') \in \underline{\partial W}$ and $w_i = \underline{i}_{\mathbf{W}}(w_i') \in \underline{W}$ for $i = 1, \ldots, k$. Then $\underline{p}(w_i) = x$ as $\boldsymbol{p} \circ \boldsymbol{i}_{\mathbf{W}} \circ \boldsymbol{q} = \boldsymbol{i}_{\mathbf{X}}$, so $w_1, \ldots, w_k$ are distinct points of $\underline{p}^{-1}(x)$. As $\underline{p}$ is finite we may write $\underline{p}^{-1}(x) = \{w_1, \ldots, w_k, w_{k+1}, \ldots, w_l\}$ for some $l \geqslant k$. Now $(\boldsymbol{W}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{W}$ at w_i' for $i = 1, \ldots, k$, so $\boldsymbol{b}(w_i) = \boldsymbol{b} \circ \boldsymbol{i}_{\mathbf{W}}(w_i') = 0$ for $i = 1, \ldots, k$. But $w_i \notin \boldsymbol{i}_{\mathbf{W}} \circ \boldsymbol{q}(\partial \boldsymbol{X})$ for $i = k+1, \ldots, l$, and so $\boldsymbol{b}(w_i) > 0$ for $i = k+1, \ldots, l$.

Set $w_i = \boldsymbol{h}(w_i) \in U$ for $i = 1, \ldots, l$. Since $\boldsymbol{b}$ and $\boldsymbol{c} \circ \boldsymbol{h}$ are 2-isomorphic we have $\boldsymbol{b}(w_i) = \boldsymbol{b} \circ \boldsymbol{h}(w_i) = \boldsymbol{c}(u_i) = c(u_i)$. Hence $c(u_i) = 0$ for $i = 1, \ldots, k$ and $c(u_i) > 0$ for $i = k+1, \ldots, l$. The property of U above shows that $\pi^{-1}\bigl(B_\epsilon(y_1, \ldots, y_n)\bigr) \cong \coprod_{i=1}^l B_\epsilon(u_i)$, where $B_\epsilon(u_i)$ is an open ball of radius ϵ about u_i in U. Since $c(u_i) > 0$ for $i = k+1, \ldots, l$, for small ϵ we have $c > 0$ on $B_\epsilon(u_i)$ for $i = k+1, \ldots, l$. Using this and the condition that $t_1, \ldots, t_k$ are linearly independent, we see that for $\epsilon > 0$ small enough, $T \cap B_\epsilon(y_1, \ldots, y_n)$ is a manifold with corners diffeomorphic to a small open ball about 0 in $\mathbb{R}_k^n$.

Thus, close to each point in $\boldsymbol{g}(\boldsymbol{X})$, the set T in (7.35) is an n-manifold with corners. So we may choose an open neighbourhood Y of $\boldsymbol{g}(\boldsymbol{X})$ in T which is an

n-manifold with corners, an embedded submanifold of $\mathbb{R}^n$. Write $r : Y \hookrightarrow \mathbb{R}^n$ for the inclusion, and $\mathbf{Y}, \boldsymbol{r} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(Y, r)$. We claim that there exists an sf-embedding $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ with a 2-morphism $\eta : \boldsymbol{r} \circ \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\mathbf{dMan^c}$. To prove this, we first show that we can choose such $\boldsymbol{f}$ near any point $x \in \underline{X}$. The important point here is that with notation $x, k, l, w_i', w_i, u_i, \epsilon$ as above, Y is defined near $(y_1, \ldots, y_n) = \boldsymbol{g}(x)$ by $c|_{B_\epsilon(u_i)} \circ \pi|_{B_\epsilon(u_i)}^{-1} \geqslant 0$ for $i = 1, \ldots, k$. But as $\boldsymbol{b} \cong \boldsymbol{c} \circ \boldsymbol{h}$ and $(\boldsymbol{W}, \boldsymbol{b})$ is a boundary defining function for $\mathbf{W}$ at w_i', we see that $c|_{B_\epsilon(u_i)} \circ \pi|_{B_\epsilon(u_i)}^{-1}$ is an extension to $B_\epsilon(\boldsymbol{g}(x)) \subset \mathbb{R}^n$ of a boundary defining function for $\mathbf{X}$ at x_i' for $i = 1, \ldots, k$. Finally, we combine the local choices for $\boldsymbol{f}$ near each $x \in \underline{X}$ using a partition of unity to get a global choice for $\boldsymbol{f}$. $\square$

Here is the analogue of Theorem 4.34. The proof is essentially the same, using Proposition 7.41 and Theorem 7.20 instead of Proposition 4.27 and Theorem 3.36. The important thing about requiring $\boldsymbol{f}$ to be an sf-embedding rather than an embedding or s-embedding is that we may apply Proposition 7.41.

Theorem 7.48. *Suppose $\mathbf{X}$ is a d-manifold with corners, Y a manifold with corners, and $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ an sf-embedding, in the sense of Definition 7.24. Then there exist an open subset V in Y with $\boldsymbol{f}(\mathbf{X}) \subseteq \mathbf{V}$, a vector bundle $E \to V$, and a smooth section $s : V \to E$ of E fitting into a 2-Cartesian diagram in $\mathbf{dSpa^c}$, where $0 : V \to E$ is the zero section and $\mathbf{Y}, \mathbf{V}, \mathbf{E}, \boldsymbol{s}, \mathbf{0} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(Y, V, E, s, 0)$:*

$$\begin{tikzcd} \mathbf{X} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{f}"'] \arrow[dr, phantom, "\Uparrow"] & \mathbf{V} \arrow[d, "\mathbf{0}"] \\ \mathbf{V} \arrow[r, "\boldsymbol{s}"] & \mathbf{E}. \end{tikzcd}$$

Hence $\mathbf{X}$ is equivalent to the 'standard model' $\mathbf{S}_{V,E,s}$ of Definition 7.2, and is a principal d-manifold with corners.

Conversely, if $\mathbf{X}$ is a principal d-manifold with corners then $\mathbf{X} \simeq \mathbf{V} \times_{\boldsymbol{s}, \mathbf{E}, \mathbf{0}} \mathbf{V}$ and $\boldsymbol{\pi}_{\mathbf{V}} : \mathbf{X} \to \mathbf{V}$ is an sf-embedding of $\mathbf{X}$ in a manifold $\mathbf{V}$. Thus from Theorems 7.47 and 7.48 we deduce analogues of Corollaries 4.35 and 4.36:

Corollary 7.49. *Let $\mathbf{X}$ be a d-manifold with corners. Then $\mathbf{X}$ is principal if and only if $\dim T_x^* \underline{X}$ and $|\underline{i}_{\mathbf{X}}^{-1}(x)|$ are bounded above for all $x \in \underline{X}$.*

Corollary 7.50. *Let $\mathbf{W}$ be a d-manifold with corners. Then $\mathbf{W}$ is a principal d-manifold with corners if any of the following hold:* **(i)** *$\mathbf{W}$ is compact;*

- **(ii)** *$\mathbf{W}$ can be covered by finitely many principal open d-submanifolds; and*
- **(iii)** *$\mathbf{W} \simeq \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$, where $\mathbf{Z}$ is a d-manifold with corners and $\mathbf{X}, \mathbf{Y}$ are principal d-manifolds with corners.*

7.8 Orientations

We now define orientations on d-manifolds with corners, following §4.6.

Definition 7.51. Let $\mathbf{X}$ be a d-manifold with corners. Then the virtual cotangent bundle $T^*\mathbf{X} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X)$ is a virtual vector bundle on $\underline{X}$ by Definition 7.5, so Definition 4.39 constructs a line bundle $\mathcal{L}_{T^*\mathbf{X}}$ on $\underline{X}$. We call $\mathcal{L}_{T^*\mathbf{X}}$ the *orientation line bundle* of $\mathbf{X}$.

An *orientation* ω on $\mathbf{X}$ is an orientation on $\mathcal{L}_{T^*\mathbf{X}}$, in the sense of Definition B.40. We call $\mathbf{X}$ *orientable* if it admits an orientation, so $\mathbf{X}$ is orientable if and only if $\mathcal{L}_{T^*\mathbf{X}}$ is trivializable. An *oriented d-manifold with corners* is a pair $(\mathbf{X}, \omega)$ where $\mathbf{X}$ is a d-manifold with corners and ω an orientation on $\mathbf{X}$. But we will often refer to $\mathbf{X}$ as an oriented d-manifold, leaving the orientation ω implicit. We will also write $-\mathbf{X}$ for $\mathbf{X}$ with the opposite orientation, that is, $\mathbf{X}$ is short for $(\mathbf{X}, \omega)$ and $-\mathbf{X}$ is short for $(\mathbf{X}, -\omega)$.

All the results of §4.6 now generalize to d-manifolds with corners essentially without change. Thus, as in Example 4.45, if X is a manifold with corners and $\mathbf{X} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(X)$ then orientations on X in the sense of §5.8 are equivalent to orientations on $\mathbf{X}$ in the sense above. The analogues of Theorem 4.50 and Proposition 4.52 are:

Theorem 7.52. *Work in the situation of Theorem* 7.37, *so that* $\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$ *are d-manifolds with corners with* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ *for* $\boldsymbol{g}, \boldsymbol{h}$ *bd-transverse, where* $\boldsymbol{e} : \mathbf{W} \to \mathbf{X}$, $\boldsymbol{f} : \mathbf{W} \to \mathbf{Y}$ *are the projections. Then we have orientation line bundles* $\mathcal{L}_{T^*\mathbf{W}}, \ldots, \mathcal{L}_{T^*\mathbf{Z}}$ *on* $\underline{W}, \ldots, \underline{Z}$, *so* $\mathcal{L}_{T^*\mathbf{W}}, \underline{e}^*(\mathcal{L}_{T^*\mathbf{X}}), \underline{f}^*(\mathcal{L}_{T^*\mathbf{Y}}), (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\mathbf{Z}})$ *are line bundles on* $\underline{W}$. *With a suitable choice of orientation convention, there is a canonical isomorphism*

$$\Phi : \mathcal{L}_{T^*\mathbf{W}} \longrightarrow \underline{e}^*(\mathcal{L}_{T^*\mathbf{X}}) \otimes_{\mathcal{O}_W} \underline{f}^*(\mathcal{L}_{T^*\mathbf{Y}}) \otimes_{\mathcal{O}_W} (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\mathbf{Z}})^*. \tag{7.36}$$

Hence, if $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ *are oriented d-manifolds with corners, then* $\mathbf{W}$ *also has a natural orientation, since trivializations of* $\mathcal{L}_{T^*\mathbf{X}}, \mathcal{L}_{T^*\mathbf{Y}}, \mathcal{L}_{T^*\mathbf{Z}}$ *induce a trivialization of* $\mathcal{L}_{T^*\mathbf{W}}$ *by* (7.36).

Proposition 7.53. *Suppose* $\mathbf{V}, \ldots, \mathbf{Z}$ *are oriented d-manifolds with corners,* $\boldsymbol{e}, \ldots, \boldsymbol{h}$ *are* 1*-morphisms, and all fibre products below are bd-transverse. Then the following hold, in oriented d-manifolds with corners:*

(a) *For* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ *and* $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *we have*

$$\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y} \simeq (-1)^{(\operatorname{vdim}\mathbf{X} - \operatorname{vdim}\mathbf{Z})(\operatorname{vdim}\mathbf{Y} - \operatorname{vdim}\mathbf{Z})} \mathbf{Y} \times_{\boldsymbol{h},\mathbf{Z},\boldsymbol{g}} \mathbf{X}.$$

In particular, when $\mathbf{Z} = *$ *so that* $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y} = \mathbf{X} \times \mathbf{Y}$ *we have*

$$\mathbf{X} \times \mathbf{Y} \simeq (-1)^{\operatorname{vdim}\mathbf{X}\operatorname{vdim}\mathbf{Y}} \mathbf{Y} \times \mathbf{X}.$$

(b) *For* $\boldsymbol{e} : \mathbf{V} \to \mathbf{Y}$, $\boldsymbol{f} : \mathbf{W} \to \mathbf{Y}$, $\boldsymbol{g} : \mathbf{W} \to \mathbf{Z}$, *and* $\boldsymbol{h} : \mathbf{X} \to \mathbf{Z}$ *we have*

$$\mathbf{V} \times_{\boldsymbol{e},\mathbf{Y},\boldsymbol{f}\circ\boldsymbol{\pi}_{\mathbf{W}}} \big(\mathbf{W} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{X}\big) \simeq \big(\mathbf{V} \times_{\boldsymbol{e},\mathbf{Y},\boldsymbol{f}} \mathbf{W}\big) \times_{\boldsymbol{g}\circ\boldsymbol{\pi}_{\mathbf{W}},\mathbf{Z},\boldsymbol{h}} \mathbf{X}.$$

(c) *For* $\boldsymbol{e} : \mathbf{V} \to \mathbf{Y}$, $\boldsymbol{f} : \mathbf{V} \to \mathbf{Z}$, $\boldsymbol{g} : \mathbf{W} \to \mathbf{Y}$, *and* $\boldsymbol{h} : \mathbf{X} \to \mathbf{Z}$ *we have*

$$\begin{aligned}&\mathbf{V} \times_{(\boldsymbol{e},\boldsymbol{f}),\mathbf{Y}\times\mathbf{Z},\boldsymbol{g}\times\boldsymbol{h}} (\mathbf{W} \times \mathbf{X}) \simeq \\ &\quad (-1)^{\operatorname{vdim}\mathbf{Z}(\operatorname{vdim}\mathbf{Y} + \operatorname{vdim}\mathbf{W})} (\mathbf{V} \times_{\boldsymbol{e},\mathbf{Y},\boldsymbol{g}} \mathbf{W}) \times_{\boldsymbol{f}\circ\boldsymbol{\pi}_{\mathbf{V}},\mathbf{Z},\boldsymbol{h}} \mathbf{X}.\end{aligned}$$

One new feature of orientations for d-manifolds with corners is that given an orientation on $\mathbf{X}$, we will define a natural orientation on $\partial\mathbf{X}$. We do this in our next theorem, which is similar to Theorems 4.50 and 7.52. As for (7.36), the isomorphism Ψ in (7.37) depends on a choice of *orientation convention*: a different choice would change (7.37), and the orientation on $\partial\mathbf{X}$, by a sign depending on $\mathrm{vdim}\,\mathbf{X}$.

Our orientation conventions are chosen to match those of Fukaya et al. [32, §8.2] for Kuranishi spaces. For manifolds they are given in Convention 5.35. If X is an n-manifold with corners, they can be expressed as follows: if $x' \in \partial X$ with $i_X(x') = x \in X$, and $v_0 \in T_xX$ is an outward-pointing normal vector to ∂X at x', and $(v_1, \ldots, v_{n-1})$ is an oriented basis for $T_{x'}(\partial X) \subset T_xX$, then $(v_0, v_1, \ldots, v_{n-1})$ is an oriented basis for T_xX.

Theorem 7.54. *Let* $\mathbf{X}$ *be a d-manifold with corners. Then* $\partial\mathbf{X}$ *is also a d-manifold with corners, so we have orientation line bundles* $\mathcal{L}_{T^*\mathbf{X}}$ *on* $\underline{X}$ *and* $\mathcal{L}_{T^*(\partial\mathbf{X})}$ *on* $\underline{\partial X}$*. With a suitable choice of orientation convention, there is a canonical isomorphism*

$$\Psi : \mathcal{L}_{T^*(\partial\mathbf{X})} \longrightarrow \underline{i}_{\mathbf{X}}^*(\mathcal{L}_{T^*\mathbf{X}}) \otimes \mathcal{N}_{\mathbf{X}}^* \tag{7.37}$$

of line bundles on $\underline{\partial X}$*, where* $\mathcal{N}_{\mathbf{X}}$ *is the conormal bundle of* $\boldsymbol{\partial X}$ *in* $\boldsymbol{X}$ *from Definition* 6.1*, and* $\mathcal{N}_{\mathbf{X}}^*$ *its dual line bundle.*

Now $\mathcal{N}_{\mathbf{X}}$ *comes with an orientation* $\omega_{\mathbf{X}}$ *in* $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$*. Hence, if* $\mathbf{X}$ *is an oriented d-manifold with corners, then* $\partial\mathbf{X}$ *also has a natural orientation, by combining the orientations on* $\mathcal{L}_{T^*\mathbf{X}}$ *and* $\mathcal{N}_{\mathbf{X}}^*$ *to get an orientation on* $\mathcal{L}_{T^*(\partial\mathbf{X})}$ *using* (7.37)*.*

Proof. Let $x' \in \underline{\partial X}$ with $\underline{i}_{\mathbf{X}}(x') = x \in \underline{X}$. As $\mathbf{X}$ is a d-manifold with corners, there exists an open neighbourhood $\mathbf{U}$ of x in $\mathbf{X}$ which is a principal d-manifold with corners, and so has an equivalence $\boldsymbol{i} : \mathbf{U} \to \mathbf{S}_{V,E,s}$ for some manifold with corners V, vector bundle $E \to V$ and smooth section $s : V \to E$. Then $\partial\mathbf{U}$ is an open neighbourhood of x' in $\partial\mathbf{X}$, and $\boldsymbol{i}_- : \partial\mathbf{U} \to \partial\mathbf{S}_{V,E,s}$ by Proposition 6.20. By Lemma 7.3 there is a natural 1-isomorphism $\boldsymbol{j} : \partial\mathbf{S}_{V,E,s} \to \mathbf{S}_{\partial V,E_\partial,s_\partial}$, where $E_\partial = i_V^*(E)$ is a vector bundle on ∂V and $s_\partial = i_V^*(s) : \partial V \to E_\partial$. So $\boldsymbol{j} \circ \boldsymbol{i}_- : \partial\mathbf{U} \to \mathbf{S}_{\partial V,E_\partial,s_\partial}$ is an equivalence in $\mathbf{dSpa^c}$.

Write $\dim V = n$ and $\mathrm{rank}\, E = k$, so that $\dim \partial V = n-1$ and $\mathrm{rank}\, E_\partial = k$. Then Definition 4.46 and Proposition 4.47(a) give isomorphisms

$$\begin{aligned} \mathcal{L}_{\boldsymbol{i}} : \underline{i}^*(\mathcal{L}_{T^*\mathbf{S}_{V,E,s}}) &\longrightarrow \mathcal{L}_{T^*\mathbf{U}} = \mathcal{L}_{T^*\mathbf{X}}|_{\underline{U}}, \\ \mathcal{L}_{\boldsymbol{j}\circ\boldsymbol{i}_-} : (\underline{j} \circ \underline{i}_-)^*(\mathcal{L}_{T^*\mathbf{S}_{\partial V,E_\partial,s_\partial}}) &\longrightarrow \mathcal{L}_{T^*\partial\mathbf{U}} = \mathcal{L}_{T^*\partial\mathbf{X}}|_{\underline{\partial U}}, \\ \mathcal{L}_{T^*\mathbf{S}_{V,E,s}} &\cong (\Lambda^k\mathcal{E}^* \otimes \Lambda^n T^*\underline{V})|_{\underline{S}_{V,E,s}}, \\ \mathcal{L}_{T^*\mathbf{S}_{\partial V,E_\partial,s_\partial}} &\cong (\Lambda^k\mathcal{E}_\partial^* \otimes \Lambda^{n-1}(T^*\underline{\partial V}))|_{\underline{S}_{\partial V,E_\partial,s_\partial}}, \end{aligned} \tag{7.38}$$

where $\underline{V}, \underline{\partial V} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(V, \partial V)$, and $\mathcal{E}, \mathcal{E}_\partial$ are the lifts of E, E_∂ to vector bundles on $\underline{V}, \underline{\partial V}$, and $\underline{S}_{V,E,s}, \underline{S}_{\partial V,E_\partial,s_\partial}$ are regarded as C^∞-subschemes of $\underline{V}, \underline{\partial V}$.

As $\boldsymbol{i}$ is an equivalence, $\underline{s}_{\boldsymbol{i}} : \underline{S}_{\boldsymbol{i}} \to \underline{\partial U}$ and $\underline{u}_{\boldsymbol{i}} : \underline{S}_{\boldsymbol{i}} \to \underline{S}_{\partial V, E_\partial, s_\partial}$ are both isomorphisms, with $\underline{u}_{\boldsymbol{i}} \circ \underline{s}_{\boldsymbol{i}}^{-1} = \underline{i}_- : \underline{\partial U} \to \underline{S}_{\partial V, E_\partial, s_\partial}$. So Proposition 6.7(d) gives an isomorphism

$$\begin{aligned}\delta_{\underline{\partial U}}(\mathcal{N}_{\mathbf{U}}) \circ I^{-1}_{\underline{s}_{\boldsymbol{i}}^{-1}, \underline{s}_{\boldsymbol{i}}}(\mathcal{N}_{\mathbf{U}}) \circ (\underline{s}_{\boldsymbol{i}}^{-1})^*(\lambda_{\boldsymbol{i}}) \circ I_{\underline{s}_{\boldsymbol{i}}^{-1}, \underline{u}_{\boldsymbol{i}}}(\mathcal{N}_{\mathbf{S}_{V,E,s}}) :& \\ \underline{i}_-^*(\mathcal{N}_{\mathbf{S}_{V,E,s}}) \longrightarrow \mathcal{N}_{\mathbf{X}}&.\end{aligned} \tag{7.39}$$

We also have natural isomorphisms

$$\mathcal{N}_{\mathbf{S}_{V,E,s}} \cong \nu^*|_{\underline{S}_{\partial V, E_\partial, s_\partial}}, \quad \underline{i}^*_{\mathbf{S}_{V,E,s}}(\mathcal{E}|_{\underline{S}_{V,E,s}}) \cong \mathcal{E}_\partial|_{\underline{S}_{\partial V, E_\partial, s_\partial}}, \tag{7.40}$$

where $\nu = \underline{i}_V^*(T\underline{V})/T\underline{\partial V}$ is the normal line bundle of $\underline{\partial V}$ in $\underline{V}$. Its dual ν^* fits into an exact sequence in $\mathrm{qcoh}(\underline{\partial V})$, as for (6.11):

$$0 \longrightarrow \nu^* \xrightarrow{\pi_\nu^*} \underline{i}_V^*(T^*\underline{V}) \xrightarrow{\Omega_{\underline{i}_V}} T^*(\underline{\partial V}) \longrightarrow 0. \tag{7.41}$$

Combining equations (7.38)–(7.40) gives isomorphisms

$$\begin{aligned}\mathcal{L}_{T^*(\partial\mathbf{X})}|_{\underline{\partial U}} &\cong \underline{i}_-^*\big((\Lambda^k\mathcal{E}_\partial^* \otimes \Lambda^{n-1}(T^*\underline{\partial V}))|_{\underline{S}_{\partial V, E_\partial, s_\partial}}\big), \\ \big(\underline{i}_{\mathbf{X}}^*(\mathcal{L}_{T^*\mathbf{X}}) \otimes \mathcal{N}_{\mathbf{X}}^*\big)|_{\underline{\partial U}} &\cong \underline{i}_-^*\big((\Lambda^k\mathcal{E}_\partial^* \otimes \Lambda^n(\underline{i}_V^*(T^*\underline{V})) \otimes \nu)|_{\underline{S}_{\partial V, E_\partial, s_\partial}}\big).\end{aligned} \tag{7.42}$$

Define an isomorphism $\psi : \Lambda^n(\underline{i}_V^*(T^*\underline{V})) \otimes \nu \to \Lambda^{n-1}(T^*\underline{\partial V}))$ in $\mathrm{qcoh}(\underline{\partial V})$ as follows. At a point $x' \in \underline{\partial V}$, choose a basis element e for $\nu|_{x'}$, and let ϵ be the dual basis element for $\nu^*|_{x'}$. Then $\epsilon_0 = \pi_\nu^*(\epsilon)$ is nonzero in $\underline{i}_V^*(T^*\underline{V})|_{x'}$, so we can choose $\epsilon_1, \ldots, \epsilon_{n-1}$ so that $\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}$ is a basis for $\underline{i}_V^*(T^*\underline{V})|_{x'}$. Then $\Omega_{\underline{i}_V}(\epsilon_1), \ldots, \Omega_{\underline{i}_V}(\epsilon_{n-1})$ is a basis for $T^*(\underline{\partial V})|_{x'}$ by (7.41). We define ψ to satisfy

$$\psi|_{x'} : (\epsilon_0 \wedge \epsilon_1 \wedge \cdots \wedge \epsilon_{n-1}) \otimes e \longmapsto \Omega_{\underline{i}_V}(\epsilon_1) \wedge \cdots \wedge \Omega_{\underline{i}_V}(\epsilon_{n-1}). \tag{7.43}$$

One can show that (7.43) is independent of choices of $e, \epsilon_1, \ldots, \epsilon_{n-1}$, and that this characterizes a unique isomorphism ψ.

Combining ψ with (7.42) gives an isomorphism $\mathcal{L}_{T^*(\partial\mathbf{X})}|_{\underline{\partial U}} \to (\underline{i}_{\mathbf{X}}^*(\mathcal{L}_{T^*\mathbf{X}}) \otimes \mathcal{N}_{\mathbf{X}}^*)|_{\underline{\partial U}}$. We define $\Psi_{\underline{\partial U}}$ to be this, multiplied by a correction factor of $(-1)^k$. By a long but straightforward calculation one can show that if $\tilde x', \tilde{\mathbf{U}}, \tilde{\boldsymbol{\imath}}, \tilde V, \tilde E, \tilde s, \tilde n, \tilde k$ are alternative choices for $x', \mathbf{U}, \boldsymbol{i}, V, E, s, n, k$, then $\Psi_{\underline{\partial U}}|_{\underline{\partial U} \cap \underline{\partial \tilde U}} = \Psi_{\underline{\partial \tilde U}}|_{\underline{\partial U} \cap \underline{\partial \tilde U}}$. Clearly this must hold up to sign, and the sign depends only on $n, k, \tilde n, \tilde k$. The point of the extra factor $(-1)^k$ is that it yields the sign 1. Since $\underline{\partial X}$ can be covered by such open C^∞-subschemes $\underline{\partial U}$, and the $\Psi_{\underline{\partial U}}$ are compatible on overlaps $\underline{\partial U} \cap \underline{\partial \tilde U}$, we can glue the isomorphisms $\Psi_{\underline{\partial U}}$ to give a unique isomorphism (7.37) with $\Psi|_{\underline{\partial U}} = \Psi_{\underline{\partial U}}$. The last part is immediate. □

If $\mathbf{X}$ is an oriented d-manifold with corners then $\partial\mathbf{X}, \partial^2\mathbf{X}, \ldots$ are also oriented d-manifolds with corners, by induction in Theorem 7.54. As in §6.7, the corners $C_k(\mathbf{X})$ satisfy $C_k(\mathbf{X}) \cong \partial^k\mathbf{X}/S_k$. However, for $k \geqslant 2$, the action of the symmetric group S_k on $\partial^k\mathbf{X}$ does not preserve orientations, as $\sigma \in S_k$ multiplies orientations by $\mathrm{sign}(\sigma)$. So $C_k(\mathbf{X})$ has no natural orientation for $k \geqslant 2$, and need not be orientable, as the next example shows.

Example 7.55. Let W be the oriented 4-manifold with corners $\mathcal{S}^2 \times [0,\infty)^2$, and write points of W as $(x_1, x_2, x_3, y_1, y_2)$, where $x_i, y_j \in \mathbb{R}$ with $x_1^2 + x_2^2 + x_3^2 = 1$ and $y_1, y_2 \geqslant 0$. Define $\sigma : W \to W$ by $\sigma : (x_1, x_2, x_3, y_1, y_2) \mapsto (-x_1, -x_2, -x_3, y_2, y_1)$. Then σ is an orientation-preserving free involution, so $X = W/\langle\sigma\rangle$ is also an oriented 4-manifold with corners. We find that $\partial X \cong \mathcal{S}^2 \times [0,\infty)$ is an oriented 3-manifold with boundary, and $\partial^2 X \cong \mathcal{S}^2$ an oriented 2-manifold. However, $C_2(X) \cong \mathcal{S}^2/\mathbb{Z}_2 \cong \mathbb{RP}^2$ is a non-orientable 2-manifold.

In Theorem 6.52, in some special cases we expressed the boundary of a fibre product $\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y}$ in $\mathbf{dSpa^c}$ as a disjoint union of fibre products of $\partial\mathbf{X}, \partial\mathbf{Y}, \partial\mathbf{Z}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$. Here is a version of this for oriented d-manifolds with corners, in which we include signs to compare the orientations on each side. The analogue for manifolds with corners is Proposition 5.36, and Fukaya et al. [32, Lem. 8.2.3(1)] give the analogue of part (a) for Kuranishi spaces.

Theorem 7.56. *Let* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ *and* $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *be* 1*-morphisms of oriented d-manifolds with corners. Then the following hold in oriented d-manifolds with corners, where by Theorem* 7.38 *all fibre products in* (7.44)–(7.50) *are cd-transverse, and so exist, and the orientations on cd-transverse fibre products and boundaries are determined by Theorems* 7.52 *and* 7.54*:*

(a) *If* $\mathbf{Z}$ *is a manifold without boundary then there is an equivalence*

$$\begin{aligned}\partial\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr) \simeq \bigl(&\partial\mathbf{X}\times_{\boldsymbol{g}\circ\boldsymbol{i_{\mathbf{X}}},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr)\\ &\amalg (-1)^{\mathrm{vdim}\,\mathbf{X}+\dim\mathbf{Z}}\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ\boldsymbol{i_{\mathbf{Y}}}}\partial\mathbf{Y}\bigr).\end{aligned} \tag{7.44}$$

(b) *If* $\boldsymbol{g}$ *is a w-submersion then there is an equivalence*

$$\begin{aligned}\partial\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr) \simeq \bigl(&\partial_+^{\boldsymbol{g}}\mathbf{X}\times_{\boldsymbol{g}_+,\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr)\\ &\amalg (-1)^{\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Z}}\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ\boldsymbol{i_{\mathbf{Y}}}}\partial\mathbf{Y}\bigr).\end{aligned} \tag{7.45}$$

(c) *If* $\boldsymbol{g}$ *is an sw-submersion then there is an equivalence*

$$\partial\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr) \simeq (-1)^{\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Z}}\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ\boldsymbol{i_{\mathbf{Y}}}}\partial\mathbf{Y}. \tag{7.46}$$

(d) *If* $\boldsymbol{h}$ *is a w-submersion then there is an equivalence*

$$\begin{aligned}\partial\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr) \simeq \bigl(&\partial\mathbf{X}\times_{\boldsymbol{g}\circ\boldsymbol{i_{\mathbf{X}}},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr)\\ &\amalg (-1)^{\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Z}}\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}_+}\partial_+^{\boldsymbol{h}}\mathbf{Y}\bigr).\end{aligned} \tag{7.47}$$

(e) *If* $\boldsymbol{h}$ *is an sw-submersion then there is an equivalence*

$$\partial\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr) \simeq \partial\mathbf{X}\times_{\boldsymbol{g}\circ\boldsymbol{i_{\mathbf{X}}},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}. \tag{7.48}$$

(f) *If both* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ *are w-submersions then there is an equivalence*

$$\begin{aligned}&\partial\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr) \simeq \bigl(\partial_+^{\boldsymbol{g}}\mathbf{X}\times_{\boldsymbol{g}_+,\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\bigr)\\ &\amalg (-1)^{\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Z}}\bigl(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}_+}\partial_+^{\boldsymbol{h}}\mathbf{Y}\bigr)\amalg\bigl(\partial_-^{\boldsymbol{g}}\mathbf{X}\times_{\boldsymbol{g}_-,\partial\mathbf{Z},\boldsymbol{h}_-}\partial_-^{\boldsymbol{h}}\mathbf{Y}\bigr).\end{aligned} \tag{7.49}$$

(g) *If both* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ *are sw-submersions then there is an equivalence*

$$\partial\big(\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}\big)\simeq\partial\mathbf{X}\times_{\boldsymbol{g}_-,\partial\mathbf{Z},\boldsymbol{h}_-}\partial\mathbf{Y}. \tag{7.50}$$

Proof. Omitting signs, equations (7.44)–(7.50) hold in unoriented d-manifolds with corners by Theorems 6.52 and 7.38. So it remains to determine the signs for each term on the right hand sides of (7.44)–(7.50). On general grounds, the signs can only be functions of $\mathrm{vdim}\,\mathbf{X},\mathrm{vdim}\,\mathbf{Y},\mathrm{vdim}\,\mathbf{Z}$. Also, we only have to compute three signs, as the 12 terms divide into those of type $\partial\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y}$, for which we claim the sign is 1, and those of type $\mathbf{X}\times_{\mathbf{Z}}\partial\mathbf{Y}$, for which we claim the sign is $(-1)^{\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Z}}$, and those of type $\partial\mathbf{X}\times_{\partial\mathbf{Z}}\partial\mathbf{Y}$, for which we claim the sign is 1. To prove these claims it is enough to consider the case in which X,Y,Z are manifolds with corners and g,h are submersions, and these follow from equations (5.26)–(5.28) of Proposition 5.36. □

In a similar way, we can add signs to Proposition 6.53, yielding an analogue of equation (5.25) of Proposition 5.36:

Proposition 7.57. *Suppose* $\mathbf{X},\mathbf{Y}$ *are oriented d-manifolds with corners, and* $\boldsymbol{f}:\mathbf{X}\rightarrow\mathbf{Y}$ *is a semisimple, flat* 1*-morphism. Then the following holds in oriented d-manifolds with corners, with fibre products cd-transverse:*

$$\partial_-^{\boldsymbol{f}}\mathbf{X}\simeq\partial\mathbf{Y}\times_{\boldsymbol{i}_{\mathbf{Y}},\mathbf{Y},\boldsymbol{f}}\mathbf{X}\simeq(-1)^{\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}}\mathbf{X}\times_{\boldsymbol{f},\mathbf{Y},\boldsymbol{i}_{\mathbf{Y}}}\partial\mathbf{Y}. \tag{7.51}$$

If $\boldsymbol{f}$ *is also simple then* $\partial_-^{\boldsymbol{f}}\mathbf{X}=\partial\mathbf{X}$*.*

8 Orbifolds and orbifolds with corners

As a preparation for our work on d-orbifolds and d-orbifolds with corners in Chapters 9–12, we now study *orbifolds* and *orbifolds with corners*. There are several non-equivalent definitions of orbifolds in use in the literature, which are surveyed in §8.1. Our approach, advocated in [56, §9.6] and explained in §8.2, is to regard orbifolds as special examples of *Deligne–Mumford C^∞-stacks*.

Deligne–Mumford C^∞-stacks and sheaves upon them are the foundation of Chapters 8–12, just as C^∞-schemes and their sheaves are the foundation of Chapters 2–7. They are discussed in §1.8 and Appendix C, and readers are advised to peruse §1.8 before proceeding. Parts of the chapter use more detailed knowledge of topics on C^∞-stacks, in particular, §8.4, §8.9 depend on §§C.5, C.8, C.9, and §8.5–§8.7 depend on §C.3.

Sections 8.5–8.9 consider *orbifolds with corners*, the orbifold analogue of Chapter 5. We will define an orbifold with corners to be a triple $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ where $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is a 1-morphism of Deligne–Mumford C^∞-stacks, the analogue of the smooth map $i_X : \partial X \to X$ in §5.1 for a manifold with corners X. Roughly speaking, the orbifold with corners is just the C^∞-stack $\mathcal{X}$, but including $\partial\mathcal{X}, i_{\mathfrak{X}}$ in the definition as extra data enables us to define boundaries and corners in a strictly functorial rather than weakly functorial way, and is more compatible with Chapters 11–12. Sections 8.3–8.9 are new material.

8.1 Review of definitions of orbifolds in the literature

Orbifolds (without boundary) are geometric spaces locally modelled on $\mathbb{R}^n/G$, for G a finite group acting linearly on $\mathbb{R}^n$, just as manifolds without boundary are locally modelled on $\mathbb{R}^n$. Orbifolds were introduced by Satake [90], who called them 'V-manifolds'. Later they were studied by Thurston [99, Ch. 13] in his work on 3-manifolds, who gave them the name 'orbifold', and showed they have well-behaved notions of fundamental group and universal cover.

Satake and Thurston defined n-orbifolds in the spirit of the usual definition of manifolds in §5.1, as a Hausdorff topological space X with extra structure given by an atlas $\{(U_i, \Gamma_i, \phi_i) : i \in I\}$ of orbifold charts (U_i, Γ_i, ϕ_i), where $\Gamma_i \subset \mathrm{GL}(n, \mathbb{R})$ is a finite subgroup, $U_i \subseteq \mathbb{R}^n$ a Γ_i-invariant open subset, $\phi_i : U_i/\Gamma_i \to X$ a homeomorphism with an open set in X, and pairs of charts $(U_i, \Gamma_i, \phi_i), (U_j, \Gamma_j, \phi_j)$ satisfy compatibility conditions on their overlaps in X. Smooth maps between orbifolds are continuous maps $f : X \to Y$ of the underlying spaces, which lift locally to equivariant smooth maps on the charts.

There is a problem with this notion of smooth maps, discussed by Adem et al. [2, p. 23–4, p. 47–50]: some differential-geometric operations, such as the pullback of vector bundles by smooth maps, may not be well-defined. The issue is this: if $f : X \to Y$ is smooth, and $x \in X$ with $f(x) = y \in Y$, and (U, Γ, ϕ), (V, Δ, ψ) are orbifold charts on X, Y near x, y with $f \circ \phi(U/\Gamma) \subseteq \psi(V/\Delta)$, for (U, Γ, ϕ) 'sufficiently small', then there should exist a group morphism $\rho : \Gamma \to \Delta$ and a smooth ρ-equivariant map $g : U \to V$, such that the induced map $g_* : U/\Gamma \to V/\Delta$ satisfies $\psi \circ g_* = f \circ \phi$. These g, ρ may not be unique, but pullbacks of vector bundles, etc., depend on the particular choices of g, ρ.

To fix this problem, new definitions of orbifolds and smooth maps were needed. Moerdijk and Pronk [84, 85] defined orbifolds to be *proper étale Lie groupoids* (U, V, s, t, u, i, m) in **Man**. Roughly speaking, a groupoid-orbifold is a Satake–Thurston orbifold together with particular choices of atlas of charts (U_i, Γ_i, ϕ_i) and transition functions between them. Their definition of smooth map $f : \mathcal{X} \to \mathcal{Y}$, called *strong maps* [85, §5] is complicated: it is an equivalence class of diagrams $\mathcal{X} \xleftarrow{\phi} \mathcal{X}' \xrightarrow{\psi} \mathcal{Y}$, where $\mathcal{X}'$ is a third orbifold, and ϕ, ψ are morphisms of groupoids with ϕ an equivalence (loosely, a diffeomorphism).

Chen and Ruan [21, §4] gave an alternative theory. Their definition of orbifold as a topological space with extra structure is similar to Satake and Thurston, but uses germs of orbifold charts (U, Γ, ϕ). Their notion of smooth map, called *good maps* [21, §4.4], is a continuous map $f : X \to Y$ of the topological spaces together with compatible choices of germs of smooth lifts to the orbifold charts. Lupercio and Uribe [69, Prop. 5.1.7] prove that these notions of strong maps and good maps are equivalent. A book on orbifolds from the point of view of [84, 85] and [21] is Adem, Leida and Ruan [2].

All of [2, 21, 84, 85, 90, 99] regard orbifolds as an ordinary category. But orbifolds are differential-geometric analogues of Deligne–Mumford stacks, which are known to form a 2-category. So it seems natural to try and define a 2-category of orbifolds **Orb**. One reason for doing this is that several important geometric constructions need the extra structure of a 2-category to work properly. For example, transverse fibre products exist and behave nicely in the 2-category **Orb**, where they satisfy a universal property involving 2-morphisms. But fibre products in the homotopy category Ho(**Orb**) are not well behaved, and are not a good generalization of fibre products in **Man**.

There are two main routes in the literature for defining a 2-category of orbifolds **Orb**. The first, as in Pronk [89] and Lerman [67, §3.3], is to define orbifolds to be groupoids (U, V, s, t, u, i, m) in **Man** as in [84, 85]. But to define 1- and 2-morphisms in **Orb** one must do more work: one makes proper étale Lie groupoids into a 2-category **Gpoid**, and then **Orb** is defined as a (weak) 2-category localization of **Gpoid** at a suitable class of 1-morphisms.

The second route, as in Behrend and Xu [13, §2], Lerman [67, §4] and Metzler [82, §3.5], is to define orbifolds as a class of Deligne–Mumford stacks on the site $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$ of manifolds with Grothendieck topology $\mathcal{J}_{\mathbf{Man}}$ coming from open covers. The relationship between the two routes is discussed by Behrend and Xu [13, §2.6], Lerman [67], and Pronk [89], who proves the two approaches give equivalent weak 2-categories.

The author's approach [56, §9.6], described in §8.2 below, is similar to the second route: we define orbifolds to be examples of Deligne–Mumford C^∞-stacks, so that they are stacks on the site $(\mathbf{C^\infty Sch}, \mathcal{J})$. This will be convenient for our work on d-stacks and d-orbifolds, which are also based on C^∞-stacks.

In the 'classical' approaches to orbifolds [2, 21, 84, 85, 90, 99], the objects, orbifolds, have a simple definition, but the smooth maps, or 1- and 2-morphisms, are either badly behaved, or very complicated to define. In contrast, in the 'stacky' approaches to orbifolds [13, 56, 67, 82], the objects are very complicated to define, but 1- and 2-morphisms are well-behaved and easy to define — 1-morphisms are just functors, and 2-morphisms are natural isomorphisms.

8.2 Orbifolds as C^∞-stacks

Here is our definition of orbifolds [56, Def. 9.25], which will be used in the rest of the book. It is based on Metzler [82, §3.5] and Lerman [67, Rem. 4.33], but replacing stacks on the site $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$ by stacks on the site $(\mathbf{C^\infty Sch}, \mathcal{J})$.

Definition 8.1. A C^∞-stack $\mathcal{X}$ is called an *orbifold* (*without boundary*) if it is equivalent to a groupoid stack $[\underline{V} \rightrightarrows \underline{U}]$ for some groupoid $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$ in $\mathbf{C^\infty Sch}$ which is the image under $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ of a groupoid (U, V, s, t, u, i, m) in $\mathbf{Man}$ respectively, where $s : V \to U$ is an étale smooth map, and $s \times t : V \to U \times U$ is a proper smooth map. That is, $\mathcal{X}$ is the C^∞-stack associated to a *proper étale Lie groupoid* in $\mathbf{Man}$, in the sense of [84, 85]. Every orbifold $\mathcal{X}$ is a separated, second countable, locally compact, paracompact, locally finitely presented Deligne–Mumford C^∞-stack.

An equivalent definition, which is closer to Satake and Thurston's definitions [90, 99], is that a separated, second countable Deligne–Mumford C^∞-stack $\mathcal{X}$ is an orbifold of dimension n if for every $[x] \in \mathcal{X}_{\rm top}$ there exist a linear action of $G = \mathrm{Iso}_{\mathcal{X}}([x])$ on $\mathbb{R}^n$, a G-invariant open neighbourhood U of 0 in $\mathbb{R}^n$, and a 1-morphism $i : [\underline{U}/G] \to \mathcal{X}$ which is an equivalence with an open neighbourhood $\mathcal{U} \subseteq \mathcal{X}$ of $[x]$ in $\mathcal{X}$ with $i_{\rm top}([0]) = [x]$, where $\underline{U} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(U)$. That is, $\mathcal{X}$ is a separated, second countable Deligne–Mumford C^∞-stack which is locally modelled on $\mathbb{R}^n/G$ for some finite group G near any point.

Write $\mathbf{Orb}$ for the full 2-subcategory of orbifolds in $\mathbf{DMC^\infty Sta}$. We may refer to 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{Orb}$ as *smooth maps* of orbifolds. Define a full and faithful functor $F_{\mathbf{Man}}^{\mathbf{Orb}} : \mathbf{Man} \to \mathbf{Orb}$ by $F_{\mathbf{Man}}^{\mathbf{Orb}} = F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}} \circ F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$. When we say that an orbifold $\mathcal{X}$ *is a manifold*, we mean that $\mathcal{X} \simeq F_{\mathbf{Man}}^{\mathbf{Orb}}(X)$ for some manifold X.

Here is [56, Th. 9.26 & Cor. 9.27]. Since equivalent (2-)categories are considered to be 'the same', the moral of Theorem 8.2 is that our orbifolds are essentially the same objects as those considered by other recent authors.

Theorem 8.2. *The* 2*-category* $\mathbf{Orb}$ *of orbifolds without boundary defined above is equivalent to the* 2*-categories of orbifolds considered as stacks on* $\mathbf{Man}$ *defined in Metzler* [82, §3.4] *and Lerman* [67, §4], *and also equivalent as a weak* 2*-category to the weak* 2*-categories of orbifolds regarded as proper étale Lie groupoids defined in Pronk* [89] *and Lerman* [67, §3.3].

Furthermore, the homotopy category Ho($\mathbf{Orb}$) *of* $\mathbf{Orb}$ *(that is, the category whose objects are objects in* $\mathbf{Orb}$, *and whose morphisms are* 2*-isomorphism classes of* 1*-morphisms in* $\mathbf{Orb}$*) is equivalent to the category of orbifolds regarded as proper étale Lie groupoids defined in Moerdijk* [84]. *Transverse fibre products in* $\mathbf{Orb}$ *agree with the corresponding fibre products in* $\mathbf{C^\infty Sta}$.

We define five classes of smooth maps:

Definition 8.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth map (1-morphism) of orbifolds.

(i) We call f *representable* if it acts injectively on orbifold groups, that is, $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\bigl(f_{\mathrm{top}}([x])\bigr)$ is an injective morphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$.

By Corollary C.24 this is equivalent to f being a representable 1-morphism of C^∞-stacks. That is, whenever $\underline{V}$ is a C^∞-scheme and $\Pi : \bar{\underline{V}} \to \mathcal{Y}$ is a 1-morphism, the fibre product $\mathcal{X} \times_{f,\mathcal{Y},\Pi} \bar{\underline{V}}$ in $\mathbf{C^\infty Sta}$ is a C^∞-scheme.

(ii) We call f an *immersion* if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is a surjective morphism of vector bundles, i.e. has a right inverse in $\mathrm{qcoh}(\mathcal{X})$.

(iii) We call f an *embedding* if it is an immersion, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\bigl(f_{\mathrm{top}}([x])\bigr)$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image (so in particular it is injective).

(iv) We call f a *submersion* if $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is an injective morphism of vector bundles, i.e. has a left inverse in $\mathrm{qcoh}(\mathcal{X})$.

(v) We call f *étale* if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is an isomorphism, or equivalently, if f is étale as a 1-morphism of C^∞-stacks.

Note that submersions are not required to be representable.

Many other standard ideas in differential geometry extend simply to orbifolds, such as submanifolds, transverse fibre products, and orientations, and we will generally use these without comment.

8.3 Vector bundles on orbifolds

In [56, §10], summarized in [57, §4.3] and §C.6 below, the author defined and studied the category $\mathrm{qcoh}(\mathcal{X})$ of quasicoherent sheaves on a Deligne–Mumford C^∞-stack $\mathcal{X}$, and the subcategory $\mathrm{vect}(\mathcal{X})$ of vector bundles in $\mathrm{qcoh}(\mathcal{X})$. As orbifolds are examples of Deligne–Mumford C^∞-stacks, if $\mathcal{X}$ is an orbifold this defines a category $\mathrm{vect}(\mathcal{X})$ of vector bundles on $\mathcal{X}$. So for us, a vector bundle $\mathcal{E}$ on $\mathcal{X}$ is by definition a special kind of quasicoherent sheaf on $\mathcal{X}$.

A *smooth section* s of a vector bundle $\mathcal{E}$ on $\mathcal{X}$ is a morphism $s : \mathcal{O}_{\mathcal{X}} \to \mathcal{E}$ in $\mathrm{qcoh}(\mathcal{X})$. Smooth sections form a vector space, which we write as $C^\infty(\mathcal{E})$. If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of vector bundles on $\mathcal{X}$ then ϕ induces a linear map $\phi_* : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{F})$ sending $s \mapsto \phi \circ s$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a smooth map (1-morphism) of orbifolds and $\mathcal{F}$ is a vector bundle over $\mathcal{Y}$, the pullback $f^*(\mathcal{F})$ is a vector bundle over $\mathcal{X}$. It induces a linear map $f^* : C^\infty(\mathcal{F}) \to C^\infty(f^*(\mathcal{F}))$ sending $s \mapsto f^*(s) \circ \iota$, where $\iota : \mathcal{O}_{\mathcal{X}} \to f^*(\mathcal{O}_{\mathcal{Y}}) = f^{-1}(\mathcal{O}_{\mathcal{Y}}) \otimes_{f^{-1}(\mathcal{O}_{\mathcal{Y}})} \mathcal{O}_{\mathcal{X}}$ is the natural isomorphism.

As in [56, Prop. 10.14], the cotangent sheaf $T^*\mathcal{X}$ of an n-orbifold $\mathcal{X}$ is a vector bundle on $\mathcal{X}$ of rank n, which we call the *cotangent bundle*. We can then define the *tangent bundle* $T\mathcal{X} = (T^*\mathcal{X})^*$. If $f : \mathcal{X} \to \mathcal{Y}$ is a smooth map of orbifolds we have a natural morphism $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ of vector bundles on $\mathcal{X}$, and a dual morphism $\Omega_f^* : T\mathcal{X} \to f^*(T\mathcal{Y})$. We think of Ω_f^* as $\mathrm{d}f$ and Ω_f as $(\mathrm{d}f)^*$.

Here is a feature of vector bundles on orbifolds which does not appear in the manifold case. Let $\mathcal{X}$ be an n-orbifold, $\mathcal{E} \to \mathcal{X}$ a rank k vector bundle on $\mathcal{X}$, and $[x] \in \mathcal{X}_{\mathrm{top}}$ be a geometric point of $\mathcal{X}$, with orbifold group $G = \mathrm{Iso}_{\mathcal{X}}([x])$. Then $\mathcal{X}$ is locally modelled near $[x]$ on $\mathbb{R}^n/G$ near 0, where G acts linearly on $\mathbb{R}^n$, and $\mathcal{E}$ is locally modelled near $[x]$ on the orbifold vector bundle $(\mathbb{R}^k \times \mathbb{R}^n)/G \to \mathbb{R}^n/G$, where G acts linearly on $\mathbb{R}^k$, and this action need not be trivial. That is, at each geometric point $[x] \in \mathcal{X}_{\mathrm{top}}$ the fibre $\mathcal{E}_x$ is a vector space isomorphic to $\mathbb{R}^k$, *equipped with a representation of* $\mathrm{Iso}_{\mathcal{X}}([x])$, which need not be trivial.

Smooth sections of $\mathcal{E}$ (in the Zariski topology) are locally modelled near $[x]$ on G-equivariant smooth maps $s : \mathbb{R}^n \to \mathbb{R}^k$ near 0. Now $s(0)$ must take values in the G-invariant subspace $(\mathbb{R}^k)^G$ of $\mathbb{R}^k$. Thus a smooth section s of a vector bundle $\mathcal{E}$ over an orbifold $\mathcal{X}$ must take values at each $[x] \in \mathcal{X}_{\mathrm{top}}$ in the subspace of $\mathcal{E}_x$ invariant under $\mathrm{Iso}_{\mathcal{X}}([x])$.

So, for example, there can be nonzero vector bundles $\mathcal{E} \to \mathcal{X}$ which have no nonzero sections. Also, if E is a rank k vector bundle over an n-manifold, then a generic section $s \in C^\infty(E)$ is transverse, so that $s^{-1}(0)$ is a submanifold of X of dimension $n-k$. However, a vector bundle $\mathcal{E}$ over an orbifold $\mathcal{X}$ may have no transverse sections. This will be important in Chapter 13, as it means that every compact d-manifold is bordant to a compact manifold, but compact d-orbifolds may not be bordant to compact orbifolds.

For some applications below, this point of view on vector bundles is not ideal. If $E \to X$ is a vector bundle on a manifold, then E is itself a manifold (with extra structure), with a submersion $\pi : E \to X$, and a section $s \in C^\infty(E)$ is a smooth map $s : X \to E$ with $\pi \circ s = \mathrm{id}_X$. In Proposition 3.12(c) we considered d-space fibre products $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$ where $\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(V, E, s, 0)$. For the d-orbifold analogue of this, we would like to regard a vector bundle $\mathcal{E}$ over an orbifold $\mathcal{X}$ as being an orbifold in its own right, rather than just a quasicoherent sheaf, and a section $s \in C^\infty(\mathcal{E})$ as being a 1-morphism $s : \mathcal{X} \to \mathcal{E}$ in $\mathbf{Orb}$.

To get round this, we will define a *total space functor* Tot, which to each $\mathcal{E}$ in $\mathrm{vect}(\mathcal{X})$ associates an orbifold $\mathrm{Tot}(\mathcal{E})$, called the *total space* of $\mathcal{E}$, and to each section $s \in C^\infty(\mathcal{E})$ associates a 1-morphism $\mathrm{Tot}(s) : \mathcal{X} \to \mathrm{Tot}(\mathcal{E})$ in $\mathbf{Orb}$. Then the d-orbifold analogue of $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$ in Proposition 3.12(c) is $\boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}} \boldsymbol{\mathcal{V}}$, where $\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{E}}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Orb}}^{\mathbf{dSta}}\big(\mathcal{V}, \mathrm{Tot}(\mathcal{E}), \mathrm{Tot}(s), \mathrm{Tot}(0)\big)$.

Definition 8.4. Let $\mathcal{X}$ be an orbifold, and $\mathcal{E} \in \mathrm{vect}(\mathcal{X})$ a vector bundle on $\mathcal{X}$. Then $\mathcal{X}$ is a C^∞-stack, and so as in Definition C.1 consists of a category $\mathcal{X}$ and a functor $p_{\mathcal{X}} : \mathcal{X} \to \mathbf{C^\infty Sch}$ satisfying many complicated conditions. There is a 1-1 correspondence between objects u in $\mathcal{X}$ with $p_{\mathcal{X}}(u) = \underline{U}$ in $\mathbf{C^\infty Sch}$ and 1-morphisms $\tilde{u} : \bar{\underline{U}} \to \mathcal{X}$ in $\mathbf{C^\infty Sta}$. If $u, v \in \mathcal{X}$ with $p_{\mathcal{X}}(u) = \underline{U}$, $p_{\mathcal{X}}(v) = \underline{V}$ correspond to $\tilde{u} : \bar{\underline{U}} \to \mathcal{X}$ and $\tilde{v} : \bar{\underline{V}} \to \mathcal{X}$ in $\mathbf{C^\infty Sta}$, there is a 1-1 correspondence between morphisms $\eta : u \to v$ in $\mathcal{X}$ with $p_{\mathcal{X}}(\eta) = \underline{f} : \underline{U} \to \underline{V}$ and 2-morphisms $\tilde{\eta} : \tilde{u} \Rightarrow \tilde{v} \circ \bar{\underline{f}}$ in $\mathbf{C^\infty Sta}$.

Define a category $\mathrm{Tot}(\mathcal{E})$ to have objects pairs (u, α), where u is an object in $\mathcal{X}$ with $p_{\mathcal{X}}(u) = \underline{U}$ in $\mathbf{C^\infty Sch}$, and $\tilde{u} : \bar{\underline{U}} \to \mathcal{X}$ is the corresponding 1-morphism in $\mathbf{C^\infty Sta}$, and $\alpha : \tilde{u}^*(\mathcal{O}_{\mathcal{X}}) \to \tilde{u}^*(\mathcal{E})$ is a morphism in $\mathrm{vect}(\bar{\underline{U}})$. If (u, α) and (v, β) are objects in $\mathrm{Tot}(\mathcal{E})$ with $p_{\mathcal{X}}(u) = \underline{U}$, $p_{\mathcal{X}}(v) = \underline{V}$ and corresponding

1-morphisms $\tilde{u} : \underline{\bar{U}} \to \mathcal{X}$, $\tilde{v} : \underline{\bar{V}} \to \mathcal{X}$, a morphism $\eta : (u, \alpha) \to (v, \beta)$ in $\mathrm{Tot}(\mathcal{E})$ is a morphism $\eta : u \to v$ in $\mathcal{X}$ with $p_{\mathcal{X}}(\eta) = \underline{f} : \underline{U} \to \underline{V}$ and corresponding 2-morphism $\tilde{\eta} : \tilde{u} \Rightarrow \tilde{v} \circ \underline{\bar{f}}$ in $\mathbf{C^\infty Sta}$, such that the following commutes in $\mathrm{vect}(\underline{\bar{U}})$:

$$
\begin{tikzcd}
\tilde{u}^*(\mathcal{O}_{\mathcal{X}}) \arrow[r, "\tilde{\eta}^*(\mathcal{O}_{\mathcal{X}})"'] \arrow[d, "\alpha"] & (\tilde{v} \circ \underline{\bar{f}})^*(\mathcal{O}_{\mathcal{X}}) \arrow[r, "I_{\underline{\bar{f}}, \tilde{v}}(\mathcal{O}_{\mathcal{X}})"'] & \underline{\bar{f}}^*(\tilde{v}^*(\mathcal{O}_{\mathcal{X}})) \arrow[d, "\underline{\bar{f}}^*(\beta)"'] \\
\tilde{u}^*(\mathcal{E}) \arrow[r, "\tilde{\eta}^*(\mathcal{E})"] & (\tilde{v} \circ \underline{\bar{f}})^*(\mathcal{E}) \arrow[r, "I_{\underline{\bar{f}}, \tilde{v}}(\mathcal{E})"] & \underline{\bar{f}}^*(\tilde{v}^*(\mathcal{E})),
\end{tikzcd}
$$

where $\iota : \mathcal{O}_{\underline{\bar{U}}} \to \underline{\bar{f}}^*(\mathcal{O}_{\underline{\bar{V}}})$ is the natural isomorphism.

Composition of morphisms in $\mathrm{Tot}(\mathcal{E})$ is as in $\mathcal{X}$, and identities are $\mathrm{id}_{(u,\alpha)} = \mathrm{id}_u$. The functor $p_{\mathrm{Tot}(\mathcal{E})} : \mathrm{Tot}(\mathcal{E}) \to \mathbf{C^\infty Sch}$ is given by $p_{\mathrm{Tot}(\mathcal{E})} : (u, \alpha) \mapsto p_{\mathcal{X}}(u)$ on objects and $p_{\mathrm{Tot}(\mathcal{E})} : \eta \mapsto p_{\mathcal{X}}(\eta)$ on morphisms. With these definitions, one can show that $\mathrm{Tot}(\mathcal{E})$ is a category, $p_{\mathrm{Tot}(\mathcal{E})} : \mathrm{Tot}(\mathcal{E}) \to \mathbf{C^\infty Sch}$ is a functor, and $\mathrm{Tot}(\mathcal{E}), p_{\mathrm{Tot}(\mathcal{E})}$ is a C^∞-stack, and in fact an orbifold.

Define $\pi : \mathrm{Tot}(\mathcal{E}) \to \mathcal{X}$ by $\pi : (u, \alpha) \mapsto u$ on objects and $\pi : \eta \to \eta$ on morphisms. Then π is a functor with $p_{\mathcal{X}} \circ \pi = p_{\mathrm{Tot}(\mathcal{E})}$, so $\pi : \mathrm{Tot}(\mathcal{E}) \to \mathcal{X}$ is a 1-morphism in $\mathbf{Orb}$. Let $s \in C^\infty(\mathcal{E})$, so that $s : \mathcal{O}_{\mathcal{X}} \to \mathcal{E}$ is a morphism in $\mathrm{vect}(\mathcal{X})$. Define a functor $\mathrm{Tot}(s) : \mathcal{X} \to \mathrm{Tot}(\mathcal{E})$ to map $\mathrm{Tot}(s) : u \mapsto \big(u, \tilde{u}^*(s)\big)$ on objects, where $\tilde{u} : \underline{\bar{U}} \to \mathcal{X}$ corresponds to u, and to map $\mathrm{Tot}(s) : \eta \mapsto \eta$ on morphisms. Then $p_{\mathrm{Tot}(\mathcal{E})} \circ \mathrm{Tot}(s) = p_{\mathcal{X}}$, so $\mathrm{Tot}(s) : \mathcal{X} \to \mathrm{Tot}(\mathcal{E})$ is a 1-morphism in $\mathbf{Orb}$. It satisfies $\pi \circ \mathrm{Tot}(s) = \mathrm{id}_{\mathcal{X}}$.

These $\mathrm{Tot}(\mathcal{E}), \mathrm{Tot}(s)$ have good functorial properties with respect to morphisms $\phi : \mathcal{E} \to \mathcal{F}$ in $\mathrm{vect}(\mathcal{X})$, and 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ and 2-morphisms $\eta : f \Rightarrow g$ in $\mathbf{Orb}$, but we will not need them, so we leave them as an exercise.

Chen and Ruan [21, §4] give a different treatment of vector bundles on orbifolds. Their definition of orbifold $\mathcal{X}$ involves covering the topological space $\mathcal{X}$ by an atlas of orbifold charts (V, G, π) with V a manifold, G a finite group acting on V and $\pi : V/G \to \mathcal{X}$ a homeomorphism with an open set. Then they define an *orbifold vector bundle* $\mathcal{E}$ to be an orbifold $\mathcal{E}$ whose charts are of the special form $(V \times \mathbb{R}^k, G, \pi)$ for each (V, G, π) in the atlas for $\mathcal{X}$. So their $\mathcal{E}$ is an orbifold in its own right, as for our $\mathrm{Tot}(\mathcal{E})$. Chen and Ruan's orbifold vector bundles form a category (not a 2-category) equivalent to our $\mathrm{vect}(\mathcal{X})$.

8.4 Orbifold strata of orbifolds, and effective orbifolds

8.4.1 Orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \mathcal{X}^{\Gamma,\lambda}_\circ, \tilde{\mathcal{X}}^{\Gamma,\mu}_\circ, \hat{\mathcal{X}}^{\Gamma,\mu}_\circ$ of orbifolds $\mathcal{X}$

In [56, §11] and §C.8 we study *orbifold strata* of a Deligne–Mumford C^∞-stack $\mathcal{X}$. We define six variations of this idea, C^∞-stacks written $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$, and open C^∞-substacks $\mathcal{X}^\Gamma_\circ \subseteq \mathcal{X}^\Gamma$, $\tilde{\mathcal{X}}^\Gamma_\circ \subseteq \tilde{\mathcal{X}}^\Gamma$, $\hat{\mathcal{X}}^\Gamma_\circ \subseteq \hat{\mathcal{X}}^\Gamma$, for each finite group Γ. The C^∞-stack $\hat{\mathcal{X}}^\Gamma_\circ$ is a C^∞-scheme, so that $\hat{\mathcal{X}}^\Gamma_\circ \simeq \bar{\underline{\hat{X}}}{}^\Gamma_\circ$ for a C^∞-scheme $\underline{\hat{X}}{}^\Gamma_\circ$. The geometric points and orbifold groups of $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ are given by:

(i) Points of $\mathcal{X}^\Gamma$ are isomorphism classes $[x, \rho]$, where $[x] \in \mathcal{X}_{\mathrm{top}}$ and $\rho : \Gamma \to \mathrm{Iso}_{\mathcal{X}}([x])$ is an injective morphism, and $\mathrm{Iso}_{\mathcal{X}^\Gamma}([x, \rho])$ is the centralizer of

$\rho(\Gamma)$ in $\mathrm{Iso}_{\mathcal{X}}([x])$. Points of $\mathcal{X}_\circ^\Gamma \subseteq \mathcal{X}^\Gamma$ are $[x,\rho]$ with ρ an isomorphism, and $\mathrm{Iso}_{\mathcal{X}_\circ^\Gamma}([x,\rho]) \cong C(\Gamma)$, the centre of Γ.

(ii) Points of $\tilde{\mathcal{X}}^\Gamma$ are pairs $[x,\Delta]$, where $[x] \in \mathcal{X}_{\mathrm{top}}$ and $\Delta \subseteq \mathrm{Iso}_{\mathcal{X}}([x])$ is isomorphic to Γ, and $\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])$ is the normalizer of Δ in $\mathrm{Iso}_{\mathcal{X}}([x])$. Points of $\tilde{\mathcal{X}}_\circ^\Gamma \subseteq \tilde{\mathcal{X}}^\Gamma$ are $[x,\Delta]$ with $\Delta = \mathrm{Iso}_{\mathcal{X}}([x])$, and $\mathrm{Iso}_{\tilde{\mathcal{X}}_\circ^\Gamma}([x,\Delta]) \cong \Gamma$.

(iii) Points $[x,\Delta]$ of $\hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$ are the same as for $\tilde{\mathcal{X}}^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma$, but with orbifold groups $\mathrm{Iso}_{\hat{\mathcal{X}}^\Gamma}([x,\Delta]) \cong \mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])/\Delta$ and $\mathrm{Iso}_{\hat{\mathcal{X}}_\circ^\Gamma}([x,\Delta]) \cong \{1\}$.

There are 1-morphisms $O^\Gamma(\mathcal{X}),\ldots,\hat\Pi_\circ^\Gamma(\mathcal{X})$ forming a strictly commutative diagram, where the columns are inclusions of open C^∞-substacks:

$$\begin{tikzcd}
\mathcal{X}_\circ^\Gamma \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde\Pi_\circ^\Gamma(\mathcal{X})"] \arrow[dr, "O_\circ^\Gamma(\mathcal{X})"] \arrow[dd, "\subset"'] & & \tilde{\mathcal{X}}_\circ^\Gamma \arrow[r, "\hat\Pi_\circ^\Gamma(\mathcal{X})"] \arrow[dl, "\tilde O_\circ^\Gamma(\mathcal{X})"'] \arrow[dd, "\subset"] & \hat{\mathcal{X}}_\circ^\Gamma \simeq \bar{\underline{X}}_\circ^\Gamma \arrow[dd, "\subset"] \\
& \mathcal{X} & & \\
\mathcal{X}^\Gamma \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde\Pi^\Gamma(\mathcal{X})"'] \arrow[ur, "O^\Gamma(\mathcal{X})"'] & & \tilde{\mathcal{X}}^\Gamma \arrow[r, "\hat\Pi^\Gamma(\mathcal{X})"'] \arrow[ul, "\tilde O^\Gamma(\mathcal{X})"] & \hat{\mathcal{X}}^\Gamma.
\end{tikzcd}$$

Also $\mathrm{Aut}(\Gamma)$ acts on $\mathcal{X}^\Gamma, \mathcal{X}_\circ^\Gamma$, with $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$, $\tilde{\mathcal{X}}_\circ^\Gamma \simeq [\mathcal{X}_\circ^\Gamma/\mathrm{Aut}(\Gamma)]$.

The 1-morphisms $O^\Gamma(\mathcal{X}), \tilde O^\Gamma(\mathcal{X}), \tilde\Pi^\Gamma(\mathcal{X}), \hat\Pi^\Gamma(\mathcal{X})$ are proper, so if $\mathcal{X}$ is compact then $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$ are also compact, although the open C^∞-substacks $\mathcal{X}_\circ^\Gamma$, $\tilde{\mathcal{X}}_\circ^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$ are generally noncompact.

In [56, §11] and §C.9 we discuss sheaves on orbifold strata. We show, for example, that if $\mathcal{E}$ is a quasicoherent sheaf on a Deligne–Mumford C^∞-stack $\mathcal{X}$ and Γ is a finite group, then the pullback $\mathcal{E}^\Gamma := O^\Gamma(\mathcal{X})^*(\mathcal{E})$ of $\mathcal{E}$ to $\mathcal{X}^\Gamma$ carries a natural representation of Γ, and so decomposes as a direct sum of subsheaves corresponding to irreducible representations of Γ.

We now discuss how these ideas work out when $\mathcal{X}$ is an orbifold, where as in §8.2 we regard orbifolds as examples of Deligne–Mumford C^∞-stacks. Let $\mathcal{X}$ be an n-orbifold, and Γ a finite group. Then $\mathcal{X}^\Gamma$ is a C^∞-stack. As $\mathcal{X}$ is an orbifold it can be covered by open $\mathcal{U} \subseteq \mathcal{X}$ with $\mathcal{U} \simeq [\underline{U}/G]$ for G a finite group acting linearly on $\mathbb{R}^n$, and $U \subseteq \mathbb{R}^n$ a G-invariant open subset, and $\underline{U} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(U)$. Equation (C.14) of Theorem C.53 then implies that

$$\mathcal{U}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\rho] \text{ of injective } \rho:\Gamma\rightarrow G} \bigl[\underline{U}^{\rho(\Gamma)}/C_G(\rho(\Gamma))\bigr].$$

Here $\underline{U}^{\rho(\Gamma)} \cong F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}\bigl(U \cap (\mathbb{R}^n)^{\rho(\Gamma)}\bigr)$, where $(\mathbb{R}^n)^{\rho(\Gamma)}$ is the linear subspace of $\mathbb{R}^n$ fixed by the subgroup $\rho(\Gamma) \subseteq G$. Thus, $[\underline{U}^{\rho(\Gamma)}/C_G(\rho(\Gamma))]$ is an orbifold of dimension $\dim(\mathbb{R}^n)^{\rho(\Gamma)}$. But because different choices of ρ may yield different dimensions $\dim(\mathbb{R}^n)^{\rho(\Gamma)}$, in general $\mathcal{U}^\Gamma$ is not an orbifold, but only a disjoint union of orbifolds of different dimensions.

Since $\mathcal{X}^\Gamma$ is covered by such $\mathcal{U}^\Gamma$, it follows that $\mathcal{X}^\Gamma$ is also in general a disjoint union of orbifolds of different dimensions, but may not be an orbifold. The same applies for $\tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma, \mathcal{X}_\circ^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$. As for $\check{\mathbf{M}}\mathbf{an^c}$ in Definition 5.15, write $\check{\mathbf{O}}\mathbf{rb}$ for the full 2-subcategory of $\mathbf{DMC^\infty Sta}$ whose objects are disjoint unions of orbifolds of different dimensions, so that $\mathbf{Orb}\subset\check{\mathbf{O}}\mathbf{rb}\subset\mathbf{DMC^\infty Sta}$. Then $\mathcal{X}^\Gamma,\ldots,\hat{\mathcal{X}}^\Gamma_\circ$ lie in $\check{\mathbf{O}}\mathbf{rb}$ for any orbifold $\mathcal{X}$ and finite group Γ.

So that our constructions remain within the world of orbifolds, we will find it useful to define a decomposition $\mathcal{X}^\Gamma=\coprod_{\lambda\in\Lambda^\Gamma_+}\mathcal{X}^{\Gamma,\lambda}$ of $\mathcal{X}^\Gamma$ such that each $\mathcal{X}^{\Gamma,\lambda}$ is an orbifold of dimension $\dim\mathcal{X}-\dim\lambda$.

Definition 8.5. Let Γ be a finite group. Consider representations (V,ρ) of Γ, where V is a finite-dimensional real vector space and $\rho:\Gamma\to\mathrm{Aut}(V)$ a group morphism. We call (V,ρ) *nontrivial* if $V^{\rho(\Gamma)}=\{0\}$. Write $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ for the abelian category of nontrivial (V,ρ), and $K_0(\mathrm{Rep}_{\mathrm{nt}}(\Gamma))$ for its Grothendieck group. Then any (V,ρ) in $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ has a class $\bigl[(V,\rho)\bigr]$ in $K_0(\mathrm{Rep}_{\mathrm{nt}}(\Gamma))$. For brevity, we will use the notation $\Lambda^\Gamma=K_0\bigl(\mathrm{Rep}_{\mathrm{nt}}(\Gamma)\bigr)$ and $\Lambda^\Gamma_+=\bigl\{\bigl[(V,\rho)\bigr]:(V,\rho)\in\mathrm{Rep}_{\mathrm{nt}}(\Gamma)\bigr\}\subseteq\Lambda^\Gamma$. We think of Λ^Γ_+ as the 'positive cone' in Λ^Γ.

There is a simple description of $\Lambda^\Gamma,\Lambda^\Gamma_+$ in terms of irreducible representations. By elementary representation theory, up to isomorphism Γ has finitely many irreducible representations. Let $R_0,R_1,\ldots,R_k$ be choices of irreducible representations in these isomorphism classes, with $R_0=\mathbb{R}$ the trivial irreducible representation, so that $R_1,\ldots,R_k$ are nontrivial. Then Λ^Γ is freely generated over $\mathbb{Z}$ by $[R_1],\ldots,[R_k]$, so that

$$\begin{aligned}\Lambda^\Gamma&=\bigl\{a_1[R_1]+\cdots+a_k[R_k]:a_1,\ldots,a_k\in\mathbb{Z}\bigr\},\qquad\text{and}\\ \Lambda^\Gamma_+&=\bigl\{a_1[R_1]+\cdots+a_k[R_k]:a_1,\ldots,a_k\in\mathbb{N}\bigr\}\subseteq\Lambda^\Gamma,\end{aligned}\tag{8.1}$$

where $\mathbb{N}=\{0,1,2,\ldots\}\subset\mathbb{Z}$. Hence $\Lambda^\Gamma\cong\mathbb{Z}^k$ and $\Lambda^\Gamma_+\cong\mathbb{N}^k$.

Define a group morphism $\dim:\Lambda^\Gamma\to\mathbb{Z}$ by $\dim:a_1[R_1]+\cdots+a_k[R_k]\mapsto a_1\dim R_1+\cdots+a_k\dim R_k$, so that $\dim:[(V,\rho)]\mapsto\dim V$. Then $\dim(\Lambda^\Gamma_+)\subseteq\mathbb{N}$.

Now let $\mathcal{X}$ be an orbifold. Then $T^*\mathcal{X}$ is a vector bundle, so $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X})$ is a vector bundle on $\mathcal{X}^\Gamma$. Definition C.54 gives an action of Γ on $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X})$ by isomorphisms, so $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X})=(T^*\mathcal{X})^\Gamma_{\mathrm{tr}}\oplus(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}$ as in (C.22)–(C.23), with $(T^*\mathcal{X})^\Gamma_{\mathrm{tr}}\cong(T^*\mathcal{X})^\Gamma_0\otimes R_0$ and $(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}\cong\bigoplus_{i=1}^k(T^*\mathcal{X})^\Gamma_i\otimes R_i$, where $(T^*\mathcal{X})^\Gamma_0,\ldots,(T^*\mathcal{X})^\Gamma_k\in\mathrm{qcoh}(\mathcal{X}^\Gamma)$. As $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X})$ is a vector bundle, the $(T^*\mathcal{X})^\Gamma_i$ are *vector bundles of mixed rank*, that is, locally they are vector bundles, but their ranks may vary on different connected components of $\mathcal{X}^\Gamma$.

For each $\lambda\in\Lambda^\Gamma_+$, define $\mathcal{X}^{\Gamma,\lambda}$ to be the open and closed C^∞-substack in $\mathcal{X}^\Gamma$ with $\mathrm{rank}\bigl((T^*\mathcal{X})^\Gamma_1\bigr)[R_1]+\cdots+\mathrm{rank}\bigl((T^*\mathcal{X})^\Gamma_k\bigr)[R_k]=\lambda$ in Λ^Γ_+. Then $(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}|_{\mathcal{X}^{\Gamma,\lambda}}$ is a vector bundle of rank $\dim\lambda$, so $(T^*\mathcal{X})^\Gamma_{\mathrm{tr}}|_{\mathcal{X}^{\Gamma,\lambda}}$ is a vector bundle of dimension $\dim\mathcal{X}-\dim\lambda$ on $\mathcal{X}^{\Gamma,\lambda}$. But $(T^*\mathcal{X})^\Gamma_{\mathrm{tr}}\cong T^*(\mathcal{X}^\Gamma)$ by Theorem C.58. Hence $T^*(\mathcal{X}^{\Gamma,\lambda})$ is a vector bundle of rank $\dim\mathcal{X}-\dim\lambda$. Since $\mathcal{X}^\Gamma$ is a disjoint union of orbifolds of different dimensions, we see that $\mathcal{X}^{\Gamma,\lambda}$ is an orbifold, with $\dim\mathcal{X}^{\Gamma,\lambda}=\dim\mathcal{X}-\dim\lambda$. (We make the convention that the empty C^∞-stack $\emptyset$ is an orbifold of every dimension $n\in\mathbb{Z}$.) As every point of $\mathcal{X}^\Gamma$ lies in $\mathcal{X}^{\Gamma,\lambda}$ for a unique $\lambda\in\Lambda^\Gamma_+$, we see that

$$\mathcal{X}^\Gamma=\textstyle\coprod_{\lambda\in\Lambda^\Gamma_+}\mathcal{X}^{\Gamma,\lambda}.\tag{8.2}$$

Write $O^{\Gamma,\lambda}(\mathcal{X})=O^\Gamma(\mathcal{X})|_{\mathcal{X}^{\Gamma,\lambda}}:\mathcal{X}^{\Gamma,\lambda}\to\mathcal{X}$. Then $O^{\Gamma,\lambda}(\mathcal{X})$ is a smooth map of orbifolds (i.e. a 1-morphism in $\mathbf{Orb}$). It is a proper, representable immersion, in the sense of Definition 8.3, where proper and representable follow from Theorem C.49(f),(g), as $\mathcal{X}^{\Gamma,\lambda}$ is closed in $\mathcal{X}^{\Gamma}$. We interpret $(T^*\mathcal{X})_{\rm nt}^{\Gamma}|_{\mathcal{X}^{\Gamma,\lambda}}$ as the *conormal bundle* of $\mathcal{X}^{\Gamma,\lambda}$ in $\mathcal{X}$. It carries a nontrivial Γ-representation of class $\lambda\in\Lambda_+^\Gamma$, so we refer to λ as the *conormal* Γ*-representation* of $\mathcal{X}^{\Gamma,\lambda}$.

Define $\mathcal{X}_\circ^{\Gamma,\lambda}=\mathcal{X}_\circ^{\Gamma}\cap\mathcal{X}^{\Gamma,\lambda}$, and $O_\circ^{\Gamma,\lambda}(\mathcal{X})=O_\circ^{\Gamma}(\mathcal{X})|_{\mathcal{X}_\circ^{\Gamma,\lambda}}:\mathcal{X}_\circ^{\Gamma,\lambda}\rightarrow\mathcal{X}$. Then $\mathcal{X}_\circ^{\Gamma,\lambda}$ is an orbifold with $\dim\mathcal{X}_\circ^{\Gamma,\lambda}=\dim\mathcal{X}-\dim\lambda$, and $\mathcal{X}_\circ^{\Gamma}=\coprod_{\lambda\in\Lambda_+^\Gamma}\mathcal{X}_\circ^{\Gamma,\lambda}$, and $O_\circ^{\Gamma,\lambda}(\mathcal{X})$ is a representable immersion, but not necessarily proper.

As in §C.8, we have $\tilde{\mathcal{X}}^\Gamma\simeq[\mathcal{X}^\Gamma/\operatorname{Aut}(\Gamma)]$. Now $\operatorname{Aut}(\Gamma)$ acts on the right on $\operatorname{Rep}_{\rm nt}(\Gamma)$ by $\alpha:(V,\rho)\mapsto(V,\rho\circ\alpha)$ for $\alpha\in\operatorname{Aut}(\Gamma)$, and this induces right actions of $\operatorname{Aut}(\Gamma)$ on $\Lambda^\Gamma=K_0\bigl(\operatorname{Rep}_{\rm nt}(\Gamma)\bigr)$ and $\Lambda_+^\Gamma\subseteq\Lambda^\Gamma$. Write these actions as $\alpha:\lambda\mapsto\lambda\cdot\alpha$. Then the action of $\alpha\in\operatorname{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ maps $\mathcal{X}^{\Gamma,\lambda}\rightarrow\mathcal{X}^{\Gamma,\lambda\cdot\alpha}$. Write $\Lambda_+^\Gamma/\operatorname{Aut}(\Lambda)$ for the set of $\operatorname{Aut}(\Gamma)$-orbits $\mu=\lambda\cdot\operatorname{Aut}(\Gamma)$ in Λ_+^Γ. The map $\dim:\Lambda^\Gamma\rightarrow\mathbb{Z}$ is $\operatorname{Aut}(\Gamma)$-invariant, and so descends to $\dim:\Lambda^\Gamma/\operatorname{Aut}(\Gamma)\rightarrow\mathbb{Z}$.

Then $\coprod_{\lambda\in\mu}\mathcal{X}^{\Gamma,\lambda}$ is an open and closed $\operatorname{Aut}(\Gamma)$-invariant C^∞-substack in $\mathcal{X}^\Gamma$ for each $\mu\in\Lambda_+^\Gamma/\operatorname{Aut}(\Lambda)$, so we may define $\tilde{\mathcal{X}}^{\Gamma,\mu}\simeq\bigl[\bigl(\coprod_{\lambda\in\mu}\mathcal{X}^{\Gamma,\lambda}\bigr)/\operatorname{Aut}(\Gamma)\bigr]$, an open and closed C^∞-substack of $\tilde{\mathcal{X}}^\Gamma\simeq[\mathcal{X}^\Gamma/\operatorname{Aut}(\Gamma)]$. Write $\tilde{\mathcal{X}}_\circ^{\Gamma,\mu}=\tilde{\mathcal{X}}_\circ^{\Gamma}\cap\tilde{\mathcal{X}}^{\Gamma,\mu}$. Then $\tilde{\mathcal{X}}^{\Gamma,\mu},\tilde{\mathcal{X}}_\circ^{\Gamma,\mu}$ are orbifolds of dimension $\dim\mathcal{X}-\dim\mu$, with

$$\tilde{\mathcal{X}}^\Gamma=\textstyle\coprod_{\mu\in\Lambda_+^\Gamma/\operatorname{Aut}(\Gamma)}\tilde{\mathcal{X}}^{\Gamma,\mu}\quad\text{and}\quad\tilde{\mathcal{X}}_\circ^\Gamma=\textstyle\coprod_{\mu\in\Lambda_+^\Gamma/\operatorname{Aut}(\Gamma)}\tilde{\mathcal{X}}_\circ^{\Gamma,\mu}.\tag{8.3}$$

Set $\tilde O^{\Gamma,\mu}(\mathcal{X})=\tilde O^{\Gamma}(\mathcal{X})|_{\tilde{\mathcal{X}}^{\Gamma,\mu}}:\tilde{\mathcal{X}}^{\Gamma,\mu}\rightarrow\mathcal{X}$ and $\tilde O_\circ^{\Gamma,\mu}(\mathcal{X})=\tilde O_\circ^{\Gamma}(\mathcal{X})|_{\tilde{\mathcal{X}}_\circ^{\Gamma,\mu}}:\tilde{\mathcal{X}}_\circ^{\Gamma,\mu}\rightarrow\mathcal{X}$. Then $\tilde O^{\Gamma,\mu}(\mathcal{X}),\tilde O_\circ^{\Gamma,\mu}(\mathcal{X})$ are representable immersions, with $\tilde O^{\Gamma,\mu}(\mathcal{X})$ proper.

The 1-morphism $\hat\Pi^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\rightarrow\hat{\mathcal{X}}^\Gamma$ induces a homeomorphism of topological spaces by Theorem C.49(e), so it maps open and closed C^∞-substacks of $\tilde{\mathcal{X}}^\Gamma$ to open and closed C^∞-substacks of $\hat{\mathcal{X}}^\Gamma$. Let $\hat{\mathcal{X}}^{\Gamma,\mu}=\hat\Pi^\Gamma(\mathcal{X})(\tilde{\mathcal{X}}^{\Gamma,\mu})$ for each $\mu\in\Lambda_+^\Gamma/\operatorname{Aut}(\Lambda)$, and write $\hat{\mathcal{X}}_\circ^{\Gamma,\mu}=\hat{\mathcal{X}}_\circ^{\Gamma}\cap\hat{\mathcal{X}}^{\Gamma,\mu}$. Then $\hat{\mathcal{X}}^{\Gamma,\mu},\hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ are orbifolds of dimension $\dim\mathcal{X}-\dim\mu$, with

$$\hat{\mathcal{X}}^\Gamma=\textstyle\coprod_{\mu\in\Lambda_+^\Gamma/\operatorname{Aut}(\Gamma)}\hat{\mathcal{X}}^{\Gamma,\mu}\quad\text{and}\quad\hat{\mathcal{X}}_\circ^\Gamma=\textstyle\coprod_{\mu\in\Lambda_+^\Gamma/\operatorname{Aut}(\Gamma)}\hat{\mathcal{X}}_\circ^{\Gamma,\mu}.\tag{8.4}$$

Furthermore, $\hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ is a manifold (that is, it is equivalent in **Orb** to something in the image of $F_{\mathbf{Man}}^{\mathbf{Orb}}$), and $\hat{\mathcal{X}}_\circ^{\Gamma,\mu}\simeq\bar{\underline{\hat X}}{}_\circ^{\Gamma,\mu}$ for a C^∞-scheme $\underline{\hat X}{}_\circ^{\Gamma,\mu}$ which is also a manifold. Thus, the decomposition following from (C.7)

$$\mathcal{X}_{\rm top}=\coprod_{\substack{\text{isomorphism classes}\\ \text{of finite groups }\Gamma}}\coprod_{\mu\in\Lambda_+^\Gamma/\operatorname{Aut}(\Gamma)}\hat{\mathcal{X}}_{\circ,\rm top}^{\Gamma,\mu}\tag{8.5}$$

is a stratification of the topological space $\mathcal{X}_{\rm top}$ of an orbifold $\mathcal{X}$ into manifolds $\hat{\mathcal{X}}_{\circ,\rm top}^{\Gamma,\mu}$. This is the usual sense of 'orbifold strata' of $\mathcal{X}$ in the literature.

If $f:\mathcal{X}\rightarrow\mathcal{Y}$ is a representable 1-morphism of Deligne–Mumford C^∞-stacks and Γ a finite group, Definition C.51 defined a representable 1-morphism of orbifold strata $f^\Gamma:\mathcal{X}^\Gamma\rightarrow\mathcal{Y}^\Gamma$. Note that if $\mathcal{X},\mathcal{Y}$ are orbifolds, then f^Γ need not map $\mathcal{X}^{\Gamma,\lambda}\rightarrow\mathcal{Y}^{\Gamma,\lambda}$, or map $\mathcal{X}_\circ^\Gamma\rightarrow\mathcal{Y}_\circ^\Gamma$. The same applies for $\tilde f^\Gamma,\hat f^\Gamma$.

Although the author has not found a treatment of our notions of orbifold strata $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma$ or $\mathcal{X}^{\Gamma,\lambda},\tilde{\mathcal{X}}^{\Gamma,\mu},\hat{\mathcal{X}}^{\Gamma,\mu}$ anywhere in the literature, similar ideas occur in several places. For example:

- Given a Satake–Thurston orbifold $\mathcal{X}$, Kawasaki [60, §1] defines orbifolds of mixed dimension $\tilde{\Sigma}\mathcal{X}$ and $\tilde{\tilde{\Sigma}}\mathcal{X}$, with morphisms $\pi : \tilde{\Sigma}\mathcal{X} \to \mathcal{X}$ and $\pi' : \tilde{\tilde{\Sigma}}\mathcal{X} \to \tilde{\Sigma}\mathcal{X}$, which occur in the orbifold versions of the Signature and Index Theorems. In our notation these are $\tilde{\Sigma}\mathcal{X} \simeq \coprod_{k\geqslant 2} \tilde{\mathcal{X}}^{\mathbb{Z}_k}$ and $\tilde{\tilde{\Sigma}}\mathcal{X} \simeq \coprod_{k\geqslant 2} \mathcal{X}^{\mathbb{Z}_k}$, with $\pi \cong \coprod_{k\geqslant 2} \tilde{O}^{\mathbb{Z}_k}(\mathcal{X})$ and $\pi' \cong \coprod_{k\geqslant 2} \tilde{\Pi}^{\mathbb{Z}_k}(\mathcal{X})$.
- Chen and Ruan [22] define a cohomology theory $H^d_{\rm orb}(\mathcal{X}, J)$ of complex orbifolds $(\mathcal{X}, J)$, graded by $d \in \mathbb{Q}$, which has an associative multiplication, and appears naturally in String Theory. In our notation it is

  $$H^d_{\rm orb}(\mathcal{X}, J) = \bigoplus\nolimits_{k\geqslant 1} \bigoplus\nolimits_{\lambda\in\Lambda^{\mathbb{Z}_k}_+(\mathbb{C})} H^{d-2\iota_k(\lambda)}(\mathcal{X}^{\mathbb{Z}_k,\lambda}_{\rm top}; \mathbb{R}),$$

  where $\Lambda^{\mathbb{Z}_k}_+(\mathbb{C})$ is as for $\Lambda^{\mathbb{Z}_k}_+$ above but using complex representations, and $\iota_k : \Lambda^{\mathbb{Z}_k}_+(\mathbb{C}) \to \mathbb{Q}$ is the unique group morphism mapping $\iota_k : [R_j] \mapsto j/k$ for $j = 0, \ldots, k-1$, where R_j is the representation of $\mathbb{Z}_k$ on $\mathbb{C}$ in which the generator of $\mathbb{Z}_k$ acts by multiplication by $e^{2\pi ij/k}$.
- When $\mathcal{X}$ is an effective Satake–Thurston n-orbifold and $\Gamma \subset \mathrm{SO}(n)$ a finite subgroup, Druschel [28, Def. 2.9] defines an effective Satake–Thurston orbifold $\mathcal{X}_\Gamma$ which is essentially the same as our $\hat{\mathcal{X}}^{\Gamma,\lambda}$ for $\lambda \in \Lambda^\Gamma_+$ the class of the nontrivial part $(\mathbb{R}^n)_{\rm nt}$ of the given representation of Γ on $\mathbb{R}^n$, provided $\hat{\mathcal{X}}^{\Gamma,\lambda}$ is effective, and is the effective truncation of $\hat{\mathcal{X}}^{\Gamma,\lambda}$ otherwise. Here *effective* orbifolds are discussed in §8.4.3.

8.4.2 Orbifold strata and orientations

We now consider issues linking orbifold strata and orientations of orbifolds. First observe that an orbifold $\mathcal{X}$ need not be even locally orientable. For example, the orbifold $[\underline{\mathbb{R}}^n/\{\pm 1\}]$ is not orientable near 0 if n is odd. More generally, if a finite group G acts on $\mathbb{R}^n$ by $\rho : G \to \mathrm{GL}(n, \mathbb{R})$, then $[\underline{\mathbb{R}}^n/G]$ is orientable near 0 if and only if $\det \rho(\gamma) = 1$ for all $\gamma \in G$. This holds if and only if G has no subgroup $\Gamma \cong \mathbb{Z}_2$ whose generator has eigenvalue -1 with odd multiplicity. We can express this in terms of orbifold strata:

Lemma 8.6. *An orbifold $\mathcal{X}$ is locally orientable if and only if $\mathcal{X}^{\mathbb{Z}_2,\lambda} = \emptyset$ for all odd $\lambda \in \Lambda^{\mathbb{Z}_2}_+ \cong \mathbb{N} = \{0, 1, 2, \ldots\}$.*

Next consider the question: if $\mathcal{X}$ is an oriented orbifold, when can we define orientations on the orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}^{\Gamma,\mu}_\circ$? Here is an example:

Example 8.7. Let $\mathcal{S}^4 = \big\{(x_1, \ldots, x_5) \in \mathbb{R}^5 : x_1^2 + \cdots + x_5^2 = 1\big\}$, an oriented 4-manifold. Let $G = \{1, \sigma, \tau, \sigma\tau\} \cong \mathbb{Z}_2^2$ act on $\mathcal{S}^4$ preserving orientations by

$$\begin{aligned} \sigma &: (x_1, \ldots, x_5) \longmapsto (x_1, x_2, x_3, -x_4, -x_5), \\ \tau &: (x_1, \ldots, x_5) \longmapsto (-x_1, -x_2, -x_3, -x_4, x_5), \\ \sigma\tau &: (x_1, \ldots, x_5) \longmapsto (-x_1, -x_2, -x_3, x_4, -x_5). \end{aligned}$$

Then $\mathcal{X} = [\underline{\mathcal{S}}^4/G]$ is an oriented 4-orbifold. The orbifold groups $\mathrm{Iso}_{\mathcal{X}}([x])$ for $[x] \in \mathcal{X}_{\mathrm{top}}$ are all $\{1\}$ or $\mathbb{Z}_2$. The singular locus of $\mathcal{X}$ is the disjoint union of a copy of $\mathbb{RP}^2$ from the fixed points $\pm(x_1, x_2, x_3, 0, 0)$ of σ, and two isolated points $\{\pm(0,0,0,0,1)\}$ and $\{\pm(0,0,0,1,0)\}$ from the fixed points of τ and $\sigma\tau$.

Identifying $\Lambda_+^{\mathbb{Z}_2}$ and $\Lambda_+^{\mathbb{Z}_2}/\mathrm{Aut}(\mathbb{Z}_2)$ with $\mathbb{N}$ as in Lemma 8.6, as $\mathrm{Aut}(\mathbb{Z}_2) = \{\mathrm{id}_{\mathbb{Z}_2}\}$, it follows that

$$\begin{aligned} &\mathcal{X}^{\mathbb{Z}_2,2} = \mathcal{X}_\circ^{\mathbb{Z}_2,2} \cong \tilde{\mathcal{X}}^{\mathbb{Z}_2,2} = \tilde{\mathcal{X}}_\circ^{\mathbb{Z}_2,2} \cong \mathbb{RP}^2 \times [\underline{*}/\mathbb{Z}_2], && \hat{\mathcal{X}}^{\mathbb{Z}_2,2} = \hat{\mathcal{X}}_\circ^{\mathbb{Z}_2,2} \cong \mathbb{RP}^2, \\ &\mathcal{X}^{\mathbb{Z}_2,4} = \mathcal{X}_\circ^{\mathbb{Z}_2,4} \cong \tilde{\mathcal{X}}^{\mathbb{Z}_2,4} = \tilde{\mathcal{X}}_\circ^{\mathbb{Z}_2,4} \cong [\underline{*}/\mathbb{Z}_2] \amalg [\underline{*}/\mathbb{Z}_2], && \hat{\mathcal{X}}^{\mathbb{Z}_2,4} = \hat{\mathcal{X}}_\circ^{\mathbb{Z}_2,4} \cong * \amalg *. \end{aligned}$$

Since $\mathbb{RP}^2$ is not orientable, we see that $\mathcal{X}$ is an oriented orbifold, but none of $\mathcal{X}^{\mathbb{Z}_2,2}, \tilde{\mathcal{X}}^{\mathbb{Z}_2,2}, \hat{\mathcal{X}}^{\mathbb{Z}_2,2}, \mathcal{X}_\circ^{\mathbb{Z}_2,2}, \tilde{\mathcal{X}}_\circ^{\mathbb{Z}_2,2}, \hat{\mathcal{X}}_\circ^{\mathbb{Z}_2,2}$ are orientable.

However, under extra conditions on Γ, λ, μ we can define orientations on $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$. To do this, we will need to choose *coherent orientations* on all Γ-representations in a given class $\lambda \in \Lambda_+^\Gamma$.

Definition 8.8. Let Γ be a finite group. The following facts are easy to prove using Schur's Lemma and elementary representation theory:

(i) Suppose $(V, \rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ has an odd-dimensional subrepresentation W. Then by splitting $V = W \oplus W^\perp$, we can find an orientation-reversing isomorphism $\alpha : (V, \rho) \to (V, \rho)$ in $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ with $\alpha|_W = -1$, $\alpha|_{W^\perp} = 1$.

(ii) Suppose $(V, \rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ has no odd-dimensional subrepresentations. Then every isomorphism $\alpha : (V, \rho) \to (V, \rho)$ is orientation-preserving.

(iii) $\dim V$ is even for all $(V, \rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ if and only if $|\Gamma|$ is odd.

Choose representatives $(R_1, \rho_1), \ldots, (R_k, \rho_k)$ for all the isomorphism classes of nontrivial, irreducible, even-dimensional, real Γ-representations, and choose orientations on the vector spaces $R_1, \ldots, R_k$. Write $\Lambda_{\mathrm{ev}}^\Gamma$ for the sublattice of Λ^Γ spanned by $[(R_1, \rho_1)], \ldots, [(R_k, \rho_k)]$, and $\Lambda_{\mathrm{ev},+}^\Gamma = \Lambda_{\mathrm{ev}}^\Gamma \cap \Lambda_+^\Gamma$. Then $\Lambda_{\mathrm{ev}}^\Gamma \cong \mathbb{Z}^k$ and $\Lambda_{\mathrm{ev},+}^\Gamma \cong \mathbb{N}^k$. Also part (iii) implies that $\Lambda_{\mathrm{ev}}^\Gamma = \Lambda^\Gamma$ and $\Lambda_{\mathrm{ev},+}^\Gamma = \Lambda_+^\Gamma$ if and only if $|\Gamma|$ is odd. If $(V, \rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ then (V, ρ) has no odd-dimensional subrepresentations if and only if $[(V, \rho)] \in \Lambda_{\mathrm{ev},+}^\Gamma$.

Suppose $(V, \rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ with $[(V, \rho)] \in \Lambda_{\mathrm{ev},+}^\Gamma$. Then there exists an isomorphism $\alpha : (V, \rho) \to \bigoplus_{i=1}^k a_i(R_i, \rho_i)$ for unique integers $a_1, \ldots, a_k \geqslant 0$, where $a_i(R_i, \rho_i)$ is the direct sum of a_i copies of (R_i, ρ_i). The chosen orientations on $R_1, \ldots, R_k$ induce an orientation on $\bigoplus_{i=1}^k a_i R_i$, which pulls back by α to an orientation on V. This is independent of the choice of α by (ii), since (V, ρ) has no odd-dimensional subrepresentations.

Thus, we have constructed an orientation on V for every $(V, \rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ with $[(V, \rho)] \in \Lambda_{\mathrm{ev},+}^\Gamma$, such that if $\alpha : (V, \rho) \to (V', \rho')$ is an isomorphism in $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$, then α identifies the orientations on V and V'. We will call these *coherent orientations*. If $[(V, \rho)] \in \Lambda_+^\Gamma \setminus \Lambda_{\mathrm{ev},+}^\Gamma$ it is not possible to choose such coherent orientations, since (V, ρ) admits orientation-reversing automorphisms by (i). If $|\Gamma|$ is odd we have coherent orientations for all $(V, \rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)$.

Next we define a map $\Phi^\Gamma : \mathrm{Aut}(\Gamma) \times \Lambda^\Gamma_{\mathrm{ev}} \to \{\pm 1\}$ which will be important in orienting orbifold strata $\tilde{\mathcal{X}}^{\Gamma,\mu}, \tilde{\mathcal{X}}^{\Gamma,\mu}_\circ, \hat{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}_\circ$. Let $\delta \in \mathrm{Aut}(\Gamma)$, and $i = 1, \ldots, k$. Then $(R_i, \rho_i \circ \delta)$ is an irreducible, nontrivial, even-dimensional Γ-representation, so there exists an isomorphism $\alpha : (R_i, \rho_i \circ \delta) \to (R_j, \rho_j)$ for some unique $j = 1, \ldots, k$. Define $\phi(\delta, i) = 1$ if $\alpha : R_i \to R_j$ identifies the chosen orientations on R_i, R_j, and $\phi(\delta, i) = -1$ otherwise. This is independent of the choice of α, as all automorphisms of (R_j, ρ_j) are orientation-preserving. If $\lambda \in \Lambda^\Gamma_{\mathrm{ev}}$ then $\lambda = \sum_{i=1}^k a_i[(R_i, \rho_i)]$ for unique $a_1, \ldots, a_k \in \mathbb{Z}$. Define $\Phi^\Gamma(\delta, \lambda) = \prod_{i=1}^k \phi(\delta, i)^{a_i}$. These $\Phi^\Gamma(\delta, \lambda)$ have the following properties:

(a) Suppose $(V, \rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ with $[(V, \rho)] = \lambda \in \Lambda^\Gamma_{\mathrm{ev},+}$, and $\delta \in \mathrm{Aut}(\Lambda)$. Then $(V, \rho \circ \delta)$ also lies in $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ with $[(V, \rho \circ \delta)] \in \Lambda^\Gamma_{\mathrm{ev},+}$. As above, we have coherent orientations on (V, ρ) and on $(V, \rho \circ \delta)$. These orientations on V agree if and only if $\Phi^\Gamma(\delta, \lambda) = 1$.

(b) $\Phi^\Gamma(\delta, \lambda)$ depends on the choice of orientations on $R_1, \ldots, R_k$ for general δ, λ. But if $\lambda \cdot \delta = \lambda$, the case of most interest, then $\Phi^\Gamma(\delta, \lambda)$ is independent of the choice of orientations.

(c) Let $\lambda \in \Lambda^\Gamma_{\mathrm{ev}}$ and Δ be the subgroup of $\mathrm{Aut}(\Gamma)$ fixing λ. Then $\delta \mapsto \Phi^\Gamma(\delta, \lambda)$ is a group morphism $\Delta \to \{\pm 1\}$.

(d) $\lambda \mapsto \Phi^\Gamma(\delta, \lambda)$ is a group morphism $\Lambda^\Gamma_{\mathrm{ev}} \to \{\pm 1\}$ for each $\delta \in \mathrm{Aut}(\Lambda)$.

We first study orientations on $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}^{\Gamma,\lambda}_\circ$.

Proposition 8.9. *Let $\mathcal{X}$ be an oriented orbifold, and Γ a finite group. Then we can define orientations on $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}^{\Gamma,\lambda}_\circ$ for all $\lambda \in \Lambda^\Gamma_{\mathrm{ev},+}$. These depend on choices of orientations on $R_1, \ldots, R_k$ for representatives $(R_1, \rho_1), \ldots, (R_k, \rho_k)$ of the nontrivial, irreducible, even-dimensional Γ-representations. If $|\Gamma|$ is odd then $\Lambda^\Gamma_{\mathrm{ev},+} = \Lambda^\Gamma_+$, so all orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}^{\Gamma,\lambda}_\circ$ are oriented.*

Proof. Let $\mathcal{X}$ be an oriented orbifold, Γ a finite group, and $\lambda \in \Lambda^\Gamma_{\mathrm{ev},+}$. Then we have an orbifold stratum $\mathcal{X}^{\Gamma,\lambda}$ and 1-morphism $O^{\Gamma,\lambda}(\mathcal{X}) : \mathcal{X}^{\Gamma,\lambda} \to \mathcal{X}$. As in Definition 8.5, the vector bundle $O^{\Gamma,\lambda}(\mathcal{X})^*(T^*\mathcal{X})$ on $\mathcal{X}^{\Gamma,\lambda}$ has an action of Γ, and splits as $O^{\Gamma,\lambda}(\mathcal{X})^*(T^*\mathcal{X}) = (T^*\mathcal{X})^{\Gamma,\lambda}_{\mathrm{tr}} \oplus (T^*\mathcal{X})^{\Gamma,\lambda}_{\mathrm{nt}}$, where $(T^*\mathcal{X})^{\Gamma,\lambda}_{\mathrm{tr}} \cong T^*(\mathcal{X}^{\Gamma,\lambda})$ by Theorem C.58, and $(T^*\mathcal{X})^{\Gamma,\lambda}_{\mathrm{nt}}$ is a vector bundle on $\mathcal{X}^{\Gamma,\lambda}$, whose fibres $x^*((T^*\mathcal{X})^{\Gamma,\lambda}_{\mathrm{nt}})$ for $[x] \in \mathcal{X}^{\Gamma,\lambda}_{\mathrm{top}}$ are Γ-representations with class λ.

As $\lambda \in \Lambda^\Gamma_{\mathrm{ev},+}$, Definition 8.8 defines coherent orientations on the fibres of $(T^*\mathcal{X})^{\Gamma,\lambda}_{\mathrm{nt}}$. Also the orientation on $\mathcal{X}$ pulls back to an orientation on the fibres of $O^{\Gamma,\lambda}(\mathcal{X})^*(T^*\mathcal{X})$. As $O^{\Gamma,\lambda}(\mathcal{X})^*(T^*\mathcal{X}) \cong T^*(\mathcal{X}^{\Gamma,\lambda}) \oplus (T^*\mathcal{X})^{\Gamma,\lambda}_{\mathrm{nt}}$, combining these two orientations gives an orientation on the fibres of $T^*(\mathcal{X}^{\Gamma,\lambda})$, that is, an orientation on $\mathcal{X}^{\Gamma,\lambda}$, and so on $\mathcal{X}^{\Gamma,\lambda}_\circ \subseteq \mathcal{X}^{\Gamma,\lambda}$. □

In Example 8.7 we have $\Lambda^{\mathbb{Z}_2}_+ \cong \mathbb{N}$ and $\Lambda^{\mathbb{Z}_2}_{\mathrm{ev},+} \cong \{0\}$, and the non-orientable orbifold strata $\mathcal{X}^{\mathbb{Z}_2,\lambda}, \mathcal{X}^{\mathbb{Z}_2,\lambda}_\circ$ with $\lambda \cong 2$ have $\lambda \in \Lambda^{\mathbb{Z}_2}_+ \setminus \Lambda^{\mathbb{Z}_2}_{\mathrm{ev},+}$. If $|\Gamma|$ is even, then $\Lambda^\Gamma_{\mathrm{ev},+} \neq \Lambda^\Gamma_+$, and generalizing Example 8.7 for each $\lambda \in \Lambda^\Gamma_+ \setminus \Lambda^\Gamma_{\mathrm{ev},+}$ we can construct examples of oriented orbifolds $\mathcal{X}$ with $\mathcal{X}^{\Gamma,\lambda},\mathcal{X}^{\Gamma,\lambda}_\circ$ non-orientable. Hence, the conditions on Γ,λ in Proposition 8.9 are both necessary and sufficient to be able to orient $\mathcal{X}^{\Gamma,\lambda},\mathcal{X}^{\Gamma,\lambda}_\circ$ for all oriented orbifolds $\mathcal{X}$.

For the orbifold strata $\tilde{\mathcal{X}}^{\Gamma,\mu},\tilde{\mathcal{X}}^{\Gamma,\mu}_\circ,\hat{\mathcal{X}}^{\Gamma,\mu},\hat{\mathcal{X}}^{\Gamma,\mu}_\circ$, recall from §8.4.1 and §C.8 that $\mathrm{Aut}(\Gamma)$ acts on $\mathcal{X}^\Gamma$ with $\tilde{\mathcal{X}}^\Gamma\simeq[\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$, and $\mathrm{Aut}(\Gamma)$ acts on Λ^Γ_+, and if $\lambda\in\Lambda^\Gamma_+$ with $\mu=\lambda\cdot\mathrm{Aut}(\Gamma)$ and Δ is the subgroup of $\mathrm{Aut}(\Gamma)$ fixing λ in Λ^Γ_+ then $\tilde{\mathcal{X}}^{\Gamma,\mu}\simeq[\mathcal{X}^{\Gamma,\lambda}/\Delta]$. Suppose $\lambda\in\Lambda^\Gamma_{\mathrm{ev},+}$, so that $\mathcal{X}^{\Gamma,\lambda}$ is oriented by Proposition 8.9. Then by considering how $\delta\in\Delta$ acts on the orientation on the vector bundle $(T^*\mathcal{X})^{\Gamma,\lambda}_{\mathrm{nt}}$ in the proof of Proposition 8.9, we see that the action of δ on $\mathcal{X}^{\Gamma,\lambda}$ multiplies the orientation by $\Phi^\Gamma(\delta,\lambda)=\pm1$ in Definition 8.8.

If $\Phi^\Gamma(\delta,\lambda)=1$ for all $\delta\in\Delta$ then Δ acts on $\mathcal{X}^{\Gamma,\lambda}$ preserving orientations, and $\tilde{\mathcal{X}}^{\Gamma,\mu}\simeq[\mathcal{X}^{\Gamma,\lambda}/\Delta]$ is oriented. Also $\hat{\Pi}^{\Gamma,\mu}(\mathcal{X}):\tilde{\mathcal{X}}^{\Gamma,\mu}\to\hat{\mathcal{X}}^{\Gamma,\mu}$ takes orientations on $\tilde{\mathcal{X}}^{\Gamma,\mu}$ to orientations on $\hat{\mathcal{X}}^{\Gamma,\mu}$. Thus we deduce:

Proposition 8.10. *Let Γ be a finite group and $\lambda\in\Lambda^\Gamma_{\mathrm{ev},+}$ with $\Phi^\Gamma(\delta,\lambda)=1$ for all $\delta\in\mathrm{Aut}(\Gamma)$ with $\lambda\cdot\delta=\lambda$. Set $\mu=\lambda\cdot\mathrm{Aut}(\Gamma)\in\Lambda^\Gamma_{\mathrm{ev},+}/\mathrm{Aut}(\Gamma)$. Then for all oriented orbifolds $\mathcal{X}$, the orbifold strata $\tilde{\mathcal{X}}^{\Gamma,\mu},\tilde{\mathcal{X}}^{\Gamma,\mu}_\circ,\hat{\mathcal{X}}^{\Gamma,\mu},\hat{\mathcal{X}}^{\Gamma,\mu}_\circ$ are oriented.*

Note that if $\lambda=2\lambda'$ for $\lambda'\in\Lambda^\Gamma_{\mathrm{ev},+}$ then $\Phi^\Gamma(\delta,\lambda)=1$ for all $\delta\in\mathrm{Aut}(\Gamma)$, so the conditions of Proposition 8.10 are satisfied.

Example 8.11. Let $\Gamma=\mathbb{Z}_3=\{1,\zeta,\zeta^2\}$ where $\zeta^3=1$. Then $\mathrm{Aut}(\mathbb{Z}_3)=\{1,\sigma\}\cong\mathbb{Z}_2$, where σ maps $1\mapsto1$, $\zeta\mapsto\zeta^2$, $\zeta^2\mapsto\zeta$. There is one nontrivial, irreducible, real $\mathbb{Z}_3$-representation (V,ρ) up to isomorphism, with $V\cong\mathbb{R}^2$, so $\Lambda^{\mathbb{Z}_3}=\Lambda^{\mathbb{Z}_3}_{\mathrm{ev}}\cong\mathbb{Z}$ and $\Lambda^{\mathbb{Z}_3}_+=\Lambda^{\mathbb{Z}_3}_{\mathrm{ev},+}\cong\mathbb{N}$. Also $\mathrm{Aut}(\mathbb{Z}_3)$ acts trivially on $\Lambda^{\mathbb{Z}_3}$ and $\Lambda^{\mathbb{Z}_3}_+$. The map $\Phi^{\mathbb{Z}_3}:\mathrm{Aut}(\mathbb{Z}_3)\times\Lambda^{\mathbb{Z}_3}_{\mathrm{ev}}\to\{\pm1\}$ in Definition 8.8 is given by $\Phi^{\mathbb{Z}_3}(1,k)=1$ and $\Phi^{\mathbb{Z}_3}(\sigma,k)=(-1)^k$ for $k\in\mathbb{Z}\cong\Lambda^{\mathbb{Z}_3}_{\mathrm{ev}}$. Thus, Proposition 8.10 shows that if $\mathcal{X}$ is an oriented orbifold and $\mu\in\Lambda^{\mathbb{Z}_3}_+/\mathrm{Aut}(\mathbb{Z}_3)\cong\mathbb{N}$ is even, then $\tilde{\mathcal{X}}^{\mathbb{Z}_3,\mu},\tilde{\mathcal{X}}^{\mathbb{Z}_3,\mu}_\circ,\hat{\mathcal{X}}^{\mathbb{Z}_3,\mu},\hat{\mathcal{X}}^{\mathbb{Z}_3,\mu}_\circ$ are oriented.

Here is an example. Let $X=\mathcal{S}^2\times\mathbb{R}^2$, an oriented 4-manifold, and write points of X as (x_1,x_2,x_3,y_1,y_2) for $x_i,y_j\in\mathbb{R}$ with $x_1^2+x_2^2+x_3^2=1$. Let $G=\mathbb{Z}_2\times\mathbb{Z}_3$ with generators σ,τ with $\sigma^2=\tau^3=1$, and define an orientation-preserving action of G on X by

$$\begin{aligned}\sigma&:(x_1,x_2,x_3,y_1,y_2)\longmapsto(-x_1,-x_2,-x_3,y_1,-y_2),\\ \tau&:(x_1,x_2,x_3,y_1,y_2)\longmapsto(x_1,x_2,x_3,-\tfrac12y_1+\tfrac{\sqrt3}{2}y_2,-\tfrac{\sqrt3}{2}y_1-\tfrac12y_2).\end{aligned}$$

Let $\mathcal{X}=[\underline{X}/G]$ be the corresponding oriented orbifold. Then we find that

$$\mathcal{X}^{\mathbb{Z}_3,\lambda}=\mathcal{X}^{\mathbb{Z}_3,\lambda}_\circ\cong\mathcal{S}^2\times[\underline{*}/\mathbb{Z}_3],\;\tilde{\mathcal{X}}^{\mathbb{Z}_3,\mu}=\tilde{\mathcal{X}}^{\mathbb{Z}_3,\mu}_\circ\cong\mathbb{RP}^2\times[\underline{*}/\mathbb{Z}_3],\;\hat{\mathcal{X}}^{\mathbb{Z}_3,\mu}=\hat{\mathcal{X}}^{\mathbb{Z}_3,\mu}_\circ\cong\mathbb{RP}^2,$$

where $\lambda\in\Lambda^{\mathbb{Z}_3}_+\cong\mathbb{N}$ and $\mu\in\Lambda^{\mathbb{Z}_3}_+/\mathrm{Aut}(\mathbb{Z}_3)\cong\mathbb{N}$ both correspond to $1\in\mathbb{N}$. Thus $\mathcal{X}^{\mathbb{Z}_3,\lambda},\mathcal{X}^{\mathbb{Z}_3,\lambda}_\circ$ are oriented, as in Proposition 8.9, but $\tilde{\mathcal{X}}^{\mathbb{Z}_3,\mu},\tilde{\mathcal{X}}^{\mathbb{Z}_3,\mu}_\circ,\hat{\mathcal{X}}^{\mathbb{Z}_3,\mu},\hat{\mathcal{X}}^{\mathbb{Z}_3,\mu}_\circ$ are not orientable. This is consistent with the first part, as $\mu=1$ is not even.

In a similar way, if Γ and $\mu\in\Lambda^\Gamma_+/\mathrm{Aut}(\Gamma)$ do not satisfy the conditions of Proposition 8.10, we can find examples of oriented orbifolds $\mathcal{X}$ such that

$\tilde{\mathcal{X}}^{\Gamma,\mu}, \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ are not orientable. Hence, the conditions on Γ, μ in Proposition 8.10 are both necessary and sufficient to be able to orient $\tilde{\mathcal{X}}^{\Gamma,\mu}, \ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ for all oriented orbifolds $\mathcal{X}$.

Example 8.12. Suppose Γ is a finite abelian group. Then the inverse map $i : \Gamma \to \Gamma$ mapping $i : \gamma \mapsto \gamma^{-1}$ lies in $\mathrm{Aut}(\Gamma)$, and one can show that $\Phi^\Gamma(i, \lambda) = (-1)^{\dim \lambda/2}$ for all $\lambda \in\in \Lambda^\Gamma_{\mathrm{ev}}$. Thus the conditions of Proposition 8.10 hold for $\lambda \in \Lambda^\Gamma_{\mathrm{ev},+}$ and $\mu = \lambda \cdot \mathrm{Aut}(\Gamma)$ only if $\dim \lambda = \dim \mu = 4k$ for some $k \geqslant 0$.

There do exist examples of nonabelian Γ and $\lambda \in \Lambda^\Gamma_{\mathrm{ev},+}$, $\mu = \lambda \cdot \mathrm{Aut}(\Gamma)$ satisfying the conditions of Proposition 8.10 with $\dim \lambda = \dim \mu = 4k+2$. Here is one way to find some. A finite group Γ is called *complete* if all automorphisms of Γ are inner automorphisms. If Γ is complete and (R, ρ) is a Γ-representation then $\lambda = [(R, \rho)]$ satisfies Proposition 8.10 if and only if (R, ρ) has no odd-dimensional subrepresentations and ρ is orientation-preserving.

It is known that the symmetric groups S_n for $n \neq 2, 6$ are complete. The symmetric group S_8 has an irreducible, faithful, orientation-preserving representation ρ on $\mathbb{R}^{70}$. Hence $\lambda = [(\mathbb{R}^{70}, \rho)]$ satisfies the conditions of Proposition 8.10 with $\dim \lambda = 4k+2$ for $k = 17$. This example is used by Druschel [28, Cor. 3.12] to show that the orbifold bordism group $B^{\mathrm{eff}}_{70}(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ in §13.3 is nonzero.

8.4.3 Effective orbifolds

In §C.5 we defined *effective* Deligne–Mumford C^∞-stacks. Since orbifolds are examples of Deligne–Mumford C^∞-stacks, this gives a notion of *effective orbifold*. Here are four ways to characterize effective orbifolds.

Proposition 8.13. *An orbifold $\mathcal{X}$ is effective if any of the following equivalent conditions hold:*

(i) *$\mathcal{X}$ is locally modelled near each $[x] \in \mathcal{X}_{\mathrm{top}}$ on $\mathbb{R}^n/G$, where G acts effectively on $\mathbb{R}^n$, that is, every $1 \neq \gamma \in G$ acts nontrivially on $\mathbb{R}^n$;*

(ii) *Generic points $[x] \in \mathcal{X}_{\mathrm{top}}$ have $\mathrm{Iso}_{\mathcal{X}}([x]) = \{1\}$;*

(iii) *$\mathcal{X}^{\Gamma,\lambda} = \emptyset$ whenever $\Gamma \neq \{1\}$ is a nontrivial finite group and $\lambda \in \Lambda^\Gamma_+$ with $\lambda \neq [R]$ for R an effective representation of Γ; and*

(iv) *$\mathcal{X}^{\Gamma,0} = \emptyset$ whenever $\Gamma \neq \{1\}$ is a nontrivial finite group.*

Proof. Clearly (i) is equivalent to Definition C.28 for $\mathcal{X}$. If G acts linearly on $\mathbb{R}^n$, then G acts effectively if and only if generic points of $\mathbb{R}^n$ have stabilizer group $\{1\}$ in G. Thus (ii) is equivalent to (i). If $\mathcal{X}$ is locally modelled on $\mathbb{R}^n/G$ then $\mathcal{X}^{\Gamma,\lambda}$ is locally modelled on $[\underline{\mathbb{R}}^n/G]^{\Gamma,\lambda}$. Using the explicit expression for $[\underline{\mathbb{R}}^n/G]^{\Gamma,\lambda}$ from Theorem C.53, we can see that G acts effectively on $\mathbb{R}^n$ if and only if $[\underline{\mathbb{R}}^n/G]^{\Gamma,\lambda} = \emptyset$ whenever $\Gamma \neq \{1\}$ and $\lambda \neq [R]$ for R effective, if and only if $[\underline{\mathbb{R}}^n/G]^{\Gamma,0} = \emptyset$ whenever $\Gamma \neq \{1\}$. Parts (iii),(iv) follow. □

By the method of Proposition C.29, we can also prove:

Proposition 8.14. *Let $\mathcal{X}, \mathcal{Y}$ be effective orbifolds, and $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms. Suppose any one of the following conditions hold:*

(i) *f is an embedding;*

(ii) *f is a submersion;*

(iii) *$f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\big(f_{\mathrm{top}}([x])\big)$ is surjective for all $[x] \in \mathcal{X}_{\mathrm{top}}$;*

(iv) *$\mathrm{Iso}_{\mathcal{Y}}\big(f_{\mathrm{top}}([x])\big) \cong \{1\}$ for generic $[x] \in \mathcal{X}_{\mathrm{top}}$; or*

(v) *$\mathcal{Y}$ is a manifold.*

Then there exists at most one 2-morphism $\eta : f \Rightarrow g$.

Some authors include effectiveness in their definition of orbifolds. The Satake–Thurston definitions are not as well-behaved for noneffective orbifolds. One reason is that Proposition 8.14 often allows us to treat effective orbifolds as if they were a category rather than a 2-category, that is, one can work in the homotopy category $\mathrm{Ho}(\mathbf{Orb^{eff}})$ of the full 2-subcategory $\mathbf{Orb^{eff}}$ of effective orbifolds, because genuinely 2-categorical behaviour comes from non-uniqueness of 2-morphisms. But if $\mathcal{X}, \mathcal{Y}$ are effective orbifolds, and $f, g : \mathcal{X} \to \mathcal{Y}$ are 1-morphisms, there can be many 2-morphisms $\eta : f \Rightarrow g$ if f, g map $\mathcal{X}$ to a nontrivial orbifold stratum $\mathcal{Y}^{\Gamma,\lambda}$ of $\mathcal{Y}$.

If $\mathcal{X}$ is an effective n-orbifold then the frame bundle $F_{\mathcal{X}}$ of $\mathcal{X}$ is a manifold, and we may write $\mathcal{X}$ as a quotient C^∞-stack $[F_{\mathcal{X}}/\,\mathrm{GL}(n,\mathbb{R})]$, or by choosing a Riemannian metric g on $\mathcal{X}$, we can write $\mathcal{X}$ as $[F_{\mathcal{X}}^g/\mathrm{O}(n)]$. An article on non-effective orbifolds $\mathcal{X}$ is Henriques and Metzler [39], who investigate when $\mathcal{X}$ may be written as $[P/K]$ for P a manifold and K a compact Lie group.

Effective orbifolds are also important in questions of integrality in homology and cohomology. Let $\mathcal{X}$ be a compact, oriented n-orbifold. Then the *fundamental class* $[\mathcal{X}]$ is naturally defined in $H_n(\mathcal{X}_{\mathrm{top}};\mathbb{Q})$, as each point $[x] \in \mathcal{X}_{\mathrm{top}}$ contributes to $[\mathcal{X}]$ with the rational weight $1/\big|\mathrm{Iso}_{\mathcal{X}}(x)\big|$. But if $\mathcal{X}$ is effective, then $[\mathcal{X}]$ is defined in $H_n(\mathcal{X}_{\mathrm{top}};\mathbb{Z})$.

8.5 Orbifolds with boundary and orbifolds with corners

We now define 2-categories $\mathbf{Orb^b}, \mathbf{Orb^c}$ of *orbifolds with boundary* and *orbifolds with corners*, the orbifold analogues of $\mathbf{Man^b}, \mathbf{Man^c}$ in Chapter 5. Our definition uses the notion of *strongly representable* 1*-morphisms* from §C.3, and readers may wish to familiarize themselves with these before continuing.

Definition 8.15. An *orbifold with corners* $\mathfrak{X}$ of dimension $n \geqslant 0$ is a triple $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ where $\mathcal{X}, \partial\mathcal{X}$ are separated, second countable Deligne–Mumford C^∞-stacks, and $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is a proper, strongly representable 1-morphism of C^∞-stacks, such that for each $[x] \in \mathcal{X}_{\mathrm{top}}$ there exists a 2-Cartesian diagram in $\mathbf{C^\infty Sta}$ satisfying conditions:

$$\begin{tikzcd} \underline{\partial \bar U} \arrow[rr, "u_\partial"'] \arrow[d, "\bar i_U"] & \arrow[d, phantom, "\mathrm{id}\Uparrow"] & \partial\mathcal{X} \arrow[d, "i_{\mathfrak{X}}"'] \\ \underline{\bar U} \arrow[rr, "u"] & {} & \mathcal{X}. \end{tikzcd} \tag{8.6}$$

Here U is an n-manifold with corners, so that $i_U : \partial U \to U$ is smooth, and $\underline{U}, \underline{\partial U}, \underline{i_U} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U, \partial U, i_U)$, and u, u_∂ are étale 1-morphisms, and $u_{\mathrm{top}}([p]) = [x]$ for some $p \in U$. Note that it is no restriction to take the 2-morphism in (8.6) to be the identity, as if (8.6) held with some other 2-morphism η, then as $i_{\mathfrak{X}}$ is strongly representable, by Proposition C.13 we could replace u_∂ by a unique 2-isomorphic u'_∂ to make $\eta = \mathrm{id}$.

We have an exact sequence of vector bundles on $\partial\mathcal{X}$:

$$0 \longrightarrow \mathcal{N}_{\mathfrak{X}} \xrightarrow{\ \nu_{\mathfrak{X}}\ } i_{\mathfrak{X}}^*(T^*\mathcal{X}) \xrightarrow{\ \Omega_{i_{\mathfrak{X}}}\ } T^*(\partial\mathcal{X}) \longrightarrow 0, \tag{8.7}$$

where $\mathcal{N}_{\mathfrak{X}}, \nu_{\mathfrak{X}}$ are defined to be the kernel of $\Omega_{i_{\mathfrak{X}}}$. Then $\mathcal{N}_{\mathfrak{X}}$ is a line bundle on $\partial\mathcal{X}$, the *conormal line bundle* of $\partial\mathcal{X}$ in $\mathcal{X}$, and has a natural orientation $\omega_{\mathfrak{X}}$ by outward-pointing normal vectors.

We call $\mathfrak{X}$ an *orbifold with boundary*, or an *orbifold without boundary* if the above condition holds with U a manifold with boundary, or a manifold without boundary, respectively, for each $[x] \in \mathcal{X}_{\mathrm{top}}$. Equivalently, $\mathfrak{X}$ is an orbifold with boundary if and only if $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is injective as a representable 1-morphism of C^∞-stacks, that is, if $i_{\mathfrak{X},\mathrm{top}} : \partial\mathcal{X}_{\mathrm{top}} \to \mathcal{X}_{\mathrm{top}}$ is injective and all the induced morphisms $(i_{\mathfrak{X}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ are isomorphisms. And $\mathfrak{X}$ is an orbifold without boundary if and only if $\partial\mathcal{X} = \emptyset$, so that $\mathfrak{X} = (\mathcal{X}, \emptyset, \emptyset)$.

Now suppose $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ and $\mathfrak{Y} = (\mathcal{Y}, \partial\mathcal{Y}, i_{\mathfrak{Y}})$ are orbifolds with corners. A 1-*morphism* $f : \mathfrak{X} \to \mathfrak{Y}$, or *smooth map*, is a 1-morphism of C^∞-stacks $f : \mathcal{X} \to \mathcal{Y}$ such that for each $[x] \in \mathcal{X}_{\mathrm{top}}$ with $f_{\mathrm{top}}([x]) = [y] \in \mathcal{Y}_{\mathrm{top}}$ there exists a 2-commutative diagram in $\mathbf{C^\infty Sta}$ satisfying conditions:

$$\begin{tikzcd} \bar{U} \arrow[rr, "u"] \arrow[d, "\bar{h}"] \arrow[drr, phantom, "\Downarrow\eta"] & & \mathcal{X} \arrow[d, "f"] \\ \bar{V} \arrow[rr, "v"] & & \mathcal{Y}. \end{tikzcd} \tag{8.8}$$

Here U, V are manifolds with corners, $h : U \to V$ is a smooth map, $\underline{U}, \underline{V}, \underline{h} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U, V, h)$, and u, v are étale, and $u_{\mathrm{top}}([p]) = [x]$ for some $p \in U$.

Let $f, g : \mathfrak{X} \to \mathfrak{Y}$ be 1-morphisms of orbifolds with corners. A 2-*morphism* $\eta : f \Rightarrow g$ is a 2-morphism of 1-morphisms $f, g : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{C^\infty Sta}$.

Composition of 1-*morphisms* $g \circ f$, *identity* 1-*morphisms* $\mathrm{id}_{\mathfrak{X}}$, *vertical* and *horizontal composition of* 2-*morphisms* $\zeta \odot \eta$, $\zeta * \eta$, and *identity* 2-*morphisms* for orbifolds with corners, are all given by the corresponding compositions and identities in $\mathbf{C^\infty Sta}$. It is easy to show that 1- and 2-morphisms of orbifolds with corners are closed under these compositions, so these are well-defined.

We have now defined all the structures of the 2-category $\mathbf{Orb^c}$ of orbifolds with corners. Since 1- and 2-morphisms in $\mathbf{Orb^c}$ are just examples of 1- and 2-morphisms in $\mathbf{C^\infty Sta}$, the axioms of a 2-category are satisfied. Write $\mathbf{Orb^b}$ and $\dot{\mathbf{O}}\mathbf{rb}$ for the full 2-subcategories of orbifolds with boundary, and orbifolds without boundary, in $\mathbf{Orb^c}$.

If $\mathcal{X}$ is an orbifold in the sense of §8.2, then $\mathfrak{X} = (\mathcal{X}, \emptyset, \emptyset)$ is an orbifold without boundary in this sense, and vice versa. Thus the 2-functor $F_{\mathbf{Orb}}^{\mathbf{Orb^c}}$:

$\mathbf{Orb} \to \mathbf{Orb^c}$ mapping $\mathcal{X} \mapsto \mathfrak{X} = (\mathcal{X}, \emptyset, \emptyset)$ on objects, $f \mapsto f$ on 1-morphisms, and $\eta \mapsto \eta$ on 2-morphisms, is an isomorphism of 2-categories $\mathbf{Orb} \to \dot{\mathbf{O}}\mathbf{rb}$.

Define a full and faithful strict 2-functor $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}} : \mathbf{Man^c} \to \mathbf{Orb^c}$ by $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}} : X \mapsto \mathfrak{X} = (\bar{\underline{X}}, \overline{\partial \underline{X}}, \bar{\underline{i}}_X)$ on objects X in $\mathbf{Man^c}$, where $\underline{X}, \underline{\partial X}, \underline{i}_X = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(X, \partial X, i_X)$. Proposition C.16 shows $\bar{\underline{i}}_X$ is strongly representable, and for each $[x]$ in $\bar{\underline{X}}_{\rm top}$ we can take (8.6) to have $U = X$ and u, u_∂ identities, so $\mathfrak{X}$ is an orbifold with corners. On (1-)morphisms $f : X \to Y$ in $\mathbf{Man^c}$ define $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}} : f \mapsto \bar{\underline{f}}$, where $\underline{f} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(f)$, and on 2-morphisms $\mathrm{id}_f : f \Rightarrow f$ in $\mathbf{Man^c}$ (regarded as a 2-category with only identity 2-morphisms) define $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}} : \mathrm{id}_f \mapsto \mathrm{id}_{\bar{\underline{f}}}$. When we say an orbifold with corners $\mathfrak{X}$ *is a manifold*, we mean that $\mathfrak{X} \simeq F_{\mathbf{Man^c}}^{\mathbf{Orb^c}}(X)$ for some manifold with corners X.

Let $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ be an orbifold with corners, and $\mathcal{V} \subseteq \mathcal{X}$ an open C^∞-substack. Define $\partial\mathcal{V} = i_{\mathfrak{X}}^{-1}(\mathcal{V})$, as an open C^∞-substack of $\partial\mathcal{X}$, and $i_{\mathcal{V}} : \partial\mathcal{V} \to \mathcal{V}$ by $i_{\mathcal{V}} = i_{\mathfrak{X}}|_{\partial\mathcal{V}}$. Then $\mathfrak{V} = (\mathcal{V}, \partial\mathcal{V}, i_{\mathcal{V}})$ is an orbifold with corners. We call $\mathfrak{V}$ an *open suborbifold* of $\mathfrak{X}$. If $\mathcal{V}$ is open and closed in $\mathcal{X}$ we call $\mathfrak{V}$ an *open and closed suborbifold* of $\mathfrak{X}$. An *open cover* of $\mathfrak{X}$ is a family $\{\mathfrak{V}_a : a \in A\}$ of open suborbifolds $\mathfrak{V}_a$ of $\mathfrak{X}$ with $\mathcal{X} = \bigcup_{a \in A} \mathcal{V}_a$.

Example 8.16. Suppose X is a manifold with corners, G a finite group, and $r : G \to \mathrm{Aut}(X)$ an action of G on X by diffeomorphisms. Since $r(\gamma) : X \to X$ is simple for each $\gamma \in G$, as in §5.4 we have $r_-(\gamma) : \partial X \to \partial X$, which is also a diffeomorphism. Then $r_- : G \to \mathrm{Aut}(\partial X)$ is an action of G on ∂X, and $i_X : \partial X \to X$ is G-equivariant. Set $\underline{X}, \underline{\partial X}, \underline{i}_X, \underline{r}, \underline{r}_- = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(X, \partial X, i_X, r, r_-)$. Then $\underline{X}, \underline{\partial X}$ are C^∞-schemes with G-actions $\underline{r}, \underline{r}_-$, and $\underline{i}_X : \underline{\partial X} \to \underline{X}$ is G-equivariant, so Definitions C.17 and C.18 define Deligne–Mumford C^∞-stacks $[\underline{X}/G], [\underline{\partial X}/G]$ and a 1-morphism $[\underline{i}_X, \mathrm{id}_G] : [\underline{\partial X}/G] \to [\underline{X}/G]$.

We will show $[\underline{i}_X, \mathrm{id}_G]$ is strongly representable. Let $(A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})$ be an object in $[\underline{\partial X}/G]$, and let $\underline{\tilde{p}} : \underline{T} \to \underline{T} \times_A G$ and $\underline{\tilde{u}} : \underline{T} \times_A G \to \underline{\partial X}$ be as in Definition C.17, so that $\underline{u} = \underline{\tilde{u}} \circ \underline{\tilde{p}} : \underline{T} \to \underline{\partial X}$. Then $[\underline{i}_X, \mathrm{id}_G]$ maps $(A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \mapsto (A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{i}_X \circ \underline{u}, \underline{v})$, by Definition C.18. Suppose $(\underline{a}, \underline{\tilde{a}}) : (A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{i}_X \circ \underline{u}, \underline{v}) \to (A', \mu', \underline{T}', \underline{U}', \underline{t}', \underline{u}', \underline{v}')$ is an isomorphism in $[\underline{X}/G]$, with inverse $(\underline{a}^{-1}, \underline{\tilde{a}}^{-1})$. Write $\underline{\tilde{p}}' : \underline{T}' \to \underline{T}' \times_{A'} G$ and $\underline{\tilde{u}}' : \underline{T}' \times_{A'} G \to \underline{X}$ for $\underline{\tilde{p}}, \underline{\tilde{u}}$ for $(A', \mu', \underline{T}', \underline{U}', \underline{t}', \underline{u}', \underline{v}')$. We can now check that

$$(\underline{a}, \underline{\tilde{a}}) : (A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \mapsto (A', \mu', \underline{T}', \underline{U}', \underline{t}', \underline{\tilde{u}} \circ \underline{\tilde{a}}^{-1} \circ \underline{\tilde{p}}', \underline{v}')$$

is the unique isomorphism in $[\underline{\partial X}/G]$ with $[\underline{i}_X, \mathrm{id}_G] : (\underline{a}, \underline{\tilde{a}}) \mapsto (\underline{a}, \underline{\tilde{a}})$. Thus $[\underline{i}_X, \mathrm{id}_G]$ is strongly representable. It is now easy to show that $\mathfrak{X} = \big([\underline{X}/G], [\underline{\partial X}/G], [\underline{i}_X, \mathrm{id}_G]\big)$ is an orbifold with corners, which we may write as $[X/G]$.

Remark 8.17. (a) Our definition of smooth maps $f : X \to Y$ in $\mathbf{Man^c}$ in Chapter 5 has good properties on lifting to boundaries and corners, e.g. from §5.4, if f is simple there is a unique smooth $f_- : \partial X \to \partial Y$ with $i_Y \circ f_- = f \circ i_X$.

When we generalize this to 2-categories, for a simple 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{Orb^c}$, one might expect to define $f_- : \partial\mathfrak{X} \to \partial\mathfrak{Y}$ uniquely only up to 2-isomorphism, with a 2-morphism $\eta : i_{\mathfrak{Y}} \circ f_- \Rightarrow f \circ i_{\mathfrak{X}}$. However, as we can, and it seems more elegant, we have defined $\mathbf{Orb^c}$ so that the lifts $f_- : \partial\mathfrak{X} \to \partial\mathfrak{Y}$ are unique, and satisfy $i_{\mathfrak{Y}} \circ f_- = f \circ i_{\mathfrak{X}}$, and other similar functorial aspects of boundaries and corners also hold strictly, not just weakly up to 2-isomorphism. This works because $i_{\mathcal{Y}} : \partial\mathcal{Y} \to \mathcal{Y}$ is *strongly representable*, in the sense of §C.3, so Proposition C.13 gives uniqueness of f_- with $i_{\mathfrak{Y}} \circ f_- = f \circ i_{\mathfrak{X}}$.

(b) Definition 8.15 is more complicated than it need be. We can define an equivalent 2-category $\tilde{\mathbf{O}}\mathbf{rb^c}$ in a simpler way as a (non-full) 2-subcategory of $\mathbf{DMC^\infty Sta}$, by taking objects $\mathcal{X}$ of $\tilde{\mathbf{O}}\mathbf{rb^c}$ to be separated, second countable Deligne–Mumford C^∞-stacks locally equivalent to $[\underline{U}/G]$ for U an n-manifold with corners, and 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ in $\tilde{\mathbf{O}}\mathbf{rb^c}$ to be 1-morphisms in $\mathbf{DMC^\infty Sta}$ locally 2-isomorphic to $[\underline{f}, \rho] : [\underline{U}/G] \to [\underline{V}/H]$ for $f : U \to V$ a morphism in $\mathbf{Man^c}$, and 2-morphisms $\eta : f \Rightarrow g$ in $\tilde{\mathbf{O}}\mathbf{rb^c}$ to be arbitrary.

We chose the more complicated definition for two reasons. Firstly, it improves compatibility with the definitions of d-stacks and d-orbifolds with corners in Chapters 11 and 12. Secondly, constructions below such as the definitions of boundaries $\partial\mathfrak{X}$, and of $\mathcal{S}_f, \mathcal{T}_f$ and $f_- : \partial_-^f\mathfrak{X} \to \partial\mathfrak{Y}$ for $f : \mathfrak{X} \to \mathfrak{Y}$ a (semisimple) 1-morphism, are more functorial and do not need arbitrary choices if we have a particular choice of boundary $\partial\mathcal{X}$ for $\mathcal{X}$ already made.

(c) Definition 8.15 basically says that objects and 1-morphisms in $\mathbf{Orb^c}$ are étale locally modelled on objects and morphisms in $\mathbf{Man^c}$.

Using Theorem C.25, we can show that an equivalent way to define an orbifold with corners $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ of dimension n is that $\mathcal{X}, \partial\mathcal{X}$ are separated, second countable Deligne–Mumford C^∞-stacks, and $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is a proper, strongly representable 1-morphism of C^∞-stacks, such that for each $[x] \in \mathcal{X}_{\rm top}$ there exists a 2-Cartesian diagram in $\mathbf{C^\infty Sta}$ satisfying conditions:

$$\begin{tikzcd}
{[\underline{\partial U}/G]} \arrow[rr, "u_\partial"] \arrow[d, "{[\underline{i}_U, \mathrm{id}_G]}"'] \arrow[drr, phantom, "\mathrm{id}\Uparrow"] & & \partial\mathcal{X} \arrow[d, "i_{\mathfrak{X}}"] \\
{[\underline{U}/G]} \arrow[rr, "u"] & & \mathcal{X}.
\end{tikzcd} \tag{8.9}$$

Here $G = \mathrm{Iso}_{\mathcal{X}}([x])$ is a finite group acting linearly on $\mathbb{R}^n$ preserving the subset $[0,\infty)^k \times \mathbb{R}^{n-k}$ for some $k = 0, \ldots, n$, and $U \subseteq [0,\infty)^k \times \mathbb{R}^{n-k}$ is a G-invariant open subset, and $\underline{U}, \underline{\partial U}, \underline{i}_U = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U, \partial U, i_U)$, and u, u_∂ are equivalences with open C^∞-substacks of $\mathcal{X}, \partial\mathcal{X}$, and $u_{\rm top}([0]) = [x]$.

(d) In part (c), when G acts linearly on $\mathbb{R}^n$ preserving the subset $[0,\infty)^k \times \mathbb{R}^{n-k}$, note that G is allowed to permute the coordinates $x_1, \ldots, x_k$ in $[0,\infty)^k$. So, for example, we allow 2-dimensional orbifolds with corners modelled on $[0,\infty)^2/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \langle\sigma\rangle$ acts on $[0,\infty)^2$ by $\sigma : (x_1, x_2) \mapsto (x_2, x_1)$.

This means that if $\mathfrak{X}$ is locally modelled on $[U/G]$ for U a manifold with corners, and G fixes some $p \in U$, then G can still act nontrivially on $i_U^{-1}(p) \subseteq \partial U$. If $p' \in \partial U$ with $i_U(p') = p$, then the stabilizer groups satisfy $\mathrm{Stab}_G(p') \subseteq \mathrm{Stab}_G(p)$, but can have $\mathrm{Stab}_G(p') \neq \mathrm{Stab}_G(p)$. So the 1-morphism $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ induces morphisms of orbifold groups $(i_{\mathfrak{X}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ which are injective (so that $i_{\mathfrak{X}}$ is representable), but need not be isomorphisms.

As in Definition 8.37 below, we will call $\mathfrak{X}$ in $\mathbf{Orb^c}$ *straight* if the morphisms $(i_{\mathfrak{X}})_*:\mathrm{Iso}_{\partial\mathcal{X}}([x'])\to\mathrm{Iso}_{\mathcal{X}}([x])$ are isomorphisms for all $[x']\in\partial\mathcal{X}_{\mathrm{top}}$ with $i_{\mathfrak{X},\mathrm{top}}([x'])=[x]$. That is, straight orbifolds with corners are locally modelled on $[0,\infty)^k\times(\mathbb{R}^{n-k}/G)$, where G acts trivially on $[0,\infty)^k$. Orbifolds with boundary, with $k=0$ or 1, are automatically straight. We will see in §8.9 that boundaries of orbifold strata behave better for straight orbifolds with corners.

(e) In §8.1 we explained that one approach to defining orbifolds is as proper étale Lie groupoids in **Man**. In the same way, one approach to defining orbifolds with corners is as *proper étale Lie groupoids* (U,V,s,t,u,i,m) *in* $\mathbf{Man^c}$. That is, U,V are manifolds with corners, and $s,t:V\to U$ are étale smooth maps in $\mathbf{Man^c}$ (so in particular they are simple and flat), and $(s,t):V\to U\times U$ is proper, and U,V,s,t,u,i,m satisfy the groupoid conditions in Definition C.2.

Given such a groupoid $(U,\ldots,m)$, we form a boundary groupoid $(\partial U,\partial V,s_\partial,\ldots,m_\partial)$ where $s_\partial=s_-:\partial V\to\partial U$, well-defined as s is simple, and $t_\partial,\ldots,m_\partial$ also have natural definitions. Thus by Definition C.2 we get C^∞-stacks $[\underline{V}\rightrightarrows\underline{U}]$, $[\underline{\partial V}\rightrightarrows\underline{\partial U}]$, and $i_U:\partial U\to U$, $i_V:\partial V\to V$ induce a 1-morphism of groupoids $(\partial U,\ldots,m_\partial)\to(U,\ldots,m)$, and hence a (representable) 1-morphism of C^∞-stacks $[\underline{i_U},\underline{i_V}]:[\underline{\partial V}\rightrightarrows\underline{\partial U}]\to[\underline{V}\rightrightarrows\underline{U}]$. Define $\mathcal{X}=[\underline{V}\rightrightarrows\underline{U}]$. Apply Proposition C.14(b) to $[\underline{i_U},\underline{i_V}]$ to get a C^∞-stack $\partial\mathcal{X}$, a strongly representable 1-morphism $i_{\mathcal{X}}:\partial\mathcal{X}\to\mathcal{X}$, and an equivalence $j:[\underline{\partial V}\rightrightarrows\underline{\partial U}]\to\partial\mathcal{X}$ with $[\underline{i_U},\underline{i_V}]=i_{\mathfrak{X}}\circ j$. Then $\mathfrak{X}=(\mathcal{X},\partial\mathcal{X},i_{\mathfrak{X}})$ is an orbifold with corners.

Conversely, if $\mathfrak{X}=(\mathcal{X},\partial\mathcal{X},i_{\mathfrak{X}})$ is an orbifold with corners and $u:\bar{\underline{U}}\to\mathcal{X}$ is an atlas for $\mathcal{X}$ with the C^∞-scheme $\underline{U}$ separated and second countable, then $\underline{U}\simeq F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U)$ for some $U\in\mathbf{Man^c}$, and we can extend U naturally to a proper étale Lie groupoid (U,V,s,t,u,i,m) in $\mathbf{Man^c}$ with $\bar{\underline{V}}\simeq\bar{\underline{U}}\times_{u,\mathcal{X},u}\bar{\underline{U}}$.

In §5.3, for $f:X\to Y$ a smooth map in $\mathbf{Man^c}$, we defined open and closed subsets $S_f\subseteq\partial X\times_Y\partial Y$ and $T_f\subseteq X\times_Y\partial Y$. Here is the orbifold analogue.

Definition 8.18. Let $\mathfrak{X}=(\mathcal{X},\partial\mathcal{X},i_{\mathfrak{X}})$ and $\mathfrak{Y}=(\mathcal{Y},\partial\mathcal{Y},i_{\mathfrak{Y}})$ be orbifolds with corners, and $f:\mathfrak{X}\to\mathfrak{Y}$ a 1-morphism. Consider the C^∞-stack fibre products $\partial\mathcal{X}\times_{f\circ i_{\mathfrak{X}},\mathcal{Y},i_{\mathfrak{Y}}}\partial\mathcal{Y}$ and $\mathcal{X}\times_{f,\mathcal{Y},i_{\mathfrak{Y}}}\partial\mathcal{Y}$. Since $i_{\mathfrak{Y}}$ is strongly representable, we may define these using the explicit construction of Proposition C.15, and then we have 2-Cartesian diagrams

$$\begin{tikzcd}
\partial\mathcal{X}\times_{f\circ i_{\mathfrak{X}},\mathcal{Y},i_{\mathfrak{Y}}}\partial\mathcal{Y} \arrow[r, "\pi_{\partial\mathcal{Y}}"'] \arrow[d, "\pi_{\partial\mathcal{X}}"'] \arrow[dr, phantom, "\mathrm{id}\,\Uparrow"] & \partial\mathcal{Y} \arrow[d, "i_{\mathfrak{Y}}"] \\
\partial\mathcal{X} \arrow[r, "f\circ i_{\mathfrak{X}}"] & \mathcal{Y},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{X}\times_{f,\mathcal{Y},i_{\mathfrak{Y}}}\partial\mathcal{Y} \arrow[r, "\pi_{\partial\mathcal{Y}}"'] \arrow[d, "\pi_{\mathcal{X}}"'] \arrow[dr, phantom, "\mathrm{id}\,\Uparrow"] & \partial\mathcal{Y} \arrow[d, "i_{\mathfrak{Y}}"] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y},
\end{tikzcd}
\tag{8.10}$$

with $\pi_{\partial\mathcal{X}},\pi_{\mathcal{X}}$ strongly representable. Define a 1-morphism $\Pi_f:\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}\to\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ by $\Pi_f:(A,B)\mapsto(i_{\mathfrak{X}}(A),B)$ on objects and $\Pi_f:(a,b)\mapsto(i_{\mathfrak{X}}(a),b)$ on morphisms, using the notation of Proposition C.15. Then Π_f is strongly representable, as $i_{\mathfrak{X}}$ is, and $i_{\mathfrak{X}}\circ\pi_{\partial\mathcal{X}}=\pi_{\mathcal{X}}\circ\Pi_f$, $\pi_{\partial\mathcal{Y}}=\pi_{\partial\mathcal{Y}}\circ\Pi_f$.

We will describe the topological space $(\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y})_{\mathrm{top}}$ associated to the C^∞-stack $\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$. Note that this is *not* simply the topological fibre product $\partial\mathcal{X}_{\mathrm{top}}\times_{\mathcal{Y}_{\mathrm{top}}}\partial\mathcal{Y}_{\mathrm{top}}$. Consider pairs (x',y') where $x':\bar{\underline{*}}\to\partial\mathcal{X}$ and $y':\bar{\underline{*}}\to\partial\mathcal{Y}$

are 1-morphisms with $f\circ i_{\mathcal{X}}\circ x'=i_{\mathcal{Y}}\circ y':\bar{\underline{*}}\rightarrow\mathcal{Y}$. Define an equivalence relation $\sim$ on such pairs by $(x',y')\sim(\tilde{x}',\tilde{y}')$ if there exist 2-morphisms $\eta:x'\Rightarrow\tilde{x}'$ and $\zeta:y'\Rightarrow\tilde{y}'$ with $\mathrm{id}_{f\circ i_{\mathcal{X}}}*\eta=\mathrm{id}_{i_{\mathcal{Y}}}*\zeta$. Write $[x',y']$ for the $\sim$-equivalence class of (x',y'). Using the explicit construction of $\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ in Proposition C.15, we can show that such $[x',y']$ correspond to points of $(\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y})_{\mathrm{top}}$. That is,

$$\begin{aligned}(\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y})_{\mathrm{top}}\cong\bigl\{[x',y']:x':\bar{\underline{*}}\rightarrow\partial\mathcal{X}\text{ and }y':\bar{\underline{*}}\rightarrow\partial\mathcal{Y}\text{ are}\\ \text{1-morphisms with }f\circ i_{\mathcal{X}}\circ x'=i_{\mathcal{Y}}\circ y':\bar{\underline{*}}\rightarrow\mathcal{Y}\bigr\}.\end{aligned}\tag{8.11}$$

Similarly, for $\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ consider pairs (x,y') where $x:\bar{\underline{*}}\rightarrow\mathcal{X}$, $y':\bar{\underline{*}}\rightarrow\partial\mathcal{Y}$ with $f\circ x=i_{\mathcal{Y}}\circ y'$. Define an equivalence relation $\approx$ by $(x,y')\approx(\tilde{x},\tilde{y}')$ if there exist $\eta:x\Rightarrow\tilde{x}$ and $\zeta:y'\Rightarrow\tilde{y}'$ with $\mathrm{id}_f*\eta=\mathrm{id}_{i_{\mathcal{Y}}}*\zeta$, and write $[x,y']$ for the $\approx$-equivalence class of (x,y'). Then we have a natural identification

$$\begin{aligned}(\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y})_{\mathrm{top}}\cong\bigl\{[x,y']:x:\bar{\underline{*}}\rightarrow\mathcal{X}\text{ and }y':\bar{\underline{*}}\rightarrow\partial\mathcal{Y}\text{ are}\\ \text{1-morphisms with }f\circ x=i_{\mathcal{Y}}\circ y':\bar{\underline{*}}\rightarrow\mathcal{Y}\bigr\}.\end{aligned}\tag{8.12}$$

Suppose we are given a diagram (8.8) as in Definition 8.15 with 1- and 2-morphisms $\bar{\underline{h}},u,v,\eta$, and corresponding diagrams (8.6) for $\underline{U},\mathfrak{X}$ and $\underline{V},\mathfrak{Y}$ with 1-morphisms u,u_∂ and v,v_∂. Then we have 1-morphisms

$$u_\partial\circ\bar{\underline{\pi}}_{\partial U}:(\overline{\underline{\partial U}\times_{\underline{h}\circ\underline{i}_U,\underline{V},\underline{i}_V}\underline{\partial V}})\rightarrow\partial\mathcal{X},\;\;v_\partial\circ\bar{\underline{\pi}}_{\partial V}:(\overline{\underline{\partial U}\times_{\underline{h}\circ\underline{i}_U,\underline{V},\underline{i}_V}\underline{\partial V}})\rightarrow\partial\mathcal{Y},$$

and a 2-morphism $\eta*\mathrm{id}_{\bar{\underline{i}}_U\circ\bar{\underline{\pi}}_{\partial U}}:(i_{\mathfrak{X}}\circ f)\circ(u_\partial\circ\bar{\underline{\pi}}_{\partial U})\Rightarrow i_{\mathfrak{Y}}\circ(v_\partial\circ\bar{\underline{\pi}}_{\partial V})$. So properties of fibre products give a 1-morphism $a:(\overline{\underline{\partial U}\times_{\underline{V}}\underline{\partial V}})\rightarrow\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$, unique up to 2-isomorphism, with $\pi_{\partial\mathcal{X}}\circ a\cong u_\partial\circ\bar{\underline{\pi}}_{\partial U}$ and $\pi_{\partial\mathcal{Y}}\circ a\cong v_\partial\circ\bar{\underline{\pi}}_{\partial V}$. This a is étale as $u,u_\partial,v,v_\partial$ are. Similarly, we construct étale $b:(\overline{\underline{U}\times_{\underline{f},\underline{V},\underline{i}_V}\underline{\partial V}})\rightarrow\mathcal{X}\times_{f,\mathcal{Y},i_{\mathcal{Y}}}\partial\mathcal{Y}$. Taken over all choices of $\underline{U},u,\underline{V},v,\underline{h}$, such a,b form étale open covers of $\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ and $\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$.

Definition 5.7 defined open $S_h\subseteq\partial U\times_V\partial V$ and $T_h\subseteq U\times_V\partial V$, which lift to open C^∞-subschemes $\underline{S}_h\subseteq\underline{\partial U}\times_{\underline{V}}\underline{\partial V}$ and $\underline{T}_h\subseteq\underline{U}\times_{\underline{V}}\underline{\partial V}$. Since the constructions are étale local, and all choices of a,b give étale open covers of $\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ and $\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$, it follows that there are unique open C^∞-substacks $\mathcal{S}_f\subseteq\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ and $\mathcal{T}_f\subseteq\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ with $a^{-1}(\mathcal{S}_f)=\bar{\underline{S}}_h$ and $b^{-1}(\mathcal{T}_f)=\bar{\underline{T}}_h$ for all such $\underline{U},u,\underline{V},v,\underline{h}$.

Following Definition 5.7, define $s_f=\pi_{\partial\mathcal{X}}|_{\mathcal{S}_f}:\mathcal{S}_f\rightarrow\partial\mathcal{X}$, $u_f=\pi_{\partial\mathcal{Y}}|_{\mathcal{S}_f}:\mathcal{S}_f\rightarrow\partial\mathcal{Y}$, $t_f=\pi_{\mathcal{X}}|_{\mathcal{T}_f}:\mathcal{T}_f\rightarrow\mathcal{X}$, $v_f=\pi_{\partial\mathcal{Y}}|_{\mathcal{T}_f}:\mathcal{T}_f\rightarrow\partial\mathcal{Y}$, and $j_f=\Pi_f|_{\mathcal{S}_f}:\mathcal{S}_f\rightarrow\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$. Then s_f,t_f are strongly representable by Proposition C.14(c), as $\pi_{\partial\mathcal{X}},\pi_{\mathcal{X}}$ are. Also (8.10) and $i_{\mathfrak{X}}\circ\pi_{\partial\mathcal{X}}=\pi_{\mathcal{X}}\circ\Pi_f$, $\pi_{\partial\mathcal{Y}}=\pi_{\partial\mathcal{Y}}\circ\Pi_f$ give identities on $s_f,u_f,\ldots,j_f$, which are equalities rather than 2-isomorphisms:

$$f\circ i_{\mathfrak{X}}\circ s_f=i_{\mathfrak{Y}}\circ u_f,\;\;f\circ t_f=i_{\mathfrak{Y}}\circ v_f,\;\;\pi_{\mathcal{X}}\circ j_f=i_{\mathfrak{X}}\circ s_f,\;\;\pi_{\partial\mathcal{Y}}\circ j_f=u_f.$$

Then $\underline{s}_h,\underline{u}_h,\ldots,\underline{j}_h=F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(s_h,u_h,\ldots,j_h)$ are étale lifts of $s_f,u_f,\ldots,j_f$. Thus, from Proposition 5.8 for h for all $\underline{U},u,\underline{V},v,\underline{h}$ we deduce:

(a) $\mathcal{S}_f,\mathcal{T}_f$ are open and closed C^∞-substacks in $\partial\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ and $\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$.

(b) j_f is an equivalence of C^∞-stacks $\mathcal{S}_f \to (\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}) \setminus \mathcal{T}_f$.

(c) $s_f : \mathcal{S}_f \to \partial\mathcal{X}$ and $t_f : \mathcal{T}_f \to \mathcal{X}$ are proper étale 1-morphisms.

We can also characterize the topological spaces $\mathcal{S}_{f,\mathrm{top}} \subseteq (\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\mathrm{top}}$ and $\mathcal{T}_{f,\mathrm{top}} \subseteq (\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\mathrm{top}}$ of $\mathcal{S}_f, \mathcal{T}_f$ using (8.11)–(8.12). We find that

$$
\begin{aligned}
\mathcal{S}_{f,\mathrm{top}} \cong \bigl\{&[x',y'] : x' : \underline{\bar{*}} \to \partial\mathcal{X},\ y' : \underline{\bar{*}} \to \partial\mathcal{Y} \text{ with } f \circ i_{\mathfrak{X}} \circ x' = i_{\mathfrak{Y}} \circ y', \\
&(i_{\mathfrak{X}} \circ x')^*(\Omega_f) \circ I_{i_{\mathfrak{X}} \circ x',f}(T^*\mathcal{Y}) \circ I_{y',i_{\mathfrak{Y}}}(T^*\mathcal{Y})^{-1} \circ (y')^*(\nu_{\mathfrak{Y}}) \neq 0, \\
&\text{and } (x')^*(\Omega_{i_{\mathfrak{X}}}) \circ I_{x',i_{\mathfrak{X}}}(T^*\mathcal{X}) \circ (i_{\mathfrak{X}} \circ x')^*(\Omega_f) \circ \\
&I_{i_{\mathfrak{X}} \circ x',f}(T^*\mathcal{Y}) \circ I_{y',i_{\mathfrak{Y}}}(T^*\mathcal{Y})^{-1} \circ (y')^*(\nu_{\mathfrak{Y}}) = 0\bigr\}, \qquad (8.13) \\
\mathcal{T}_{f,\mathrm{top}} \cong \bigl\{&[x,y'] : x : \underline{\bar{*}} \to \mathcal{X},\ y' : \underline{\bar{*}} \to \partial\mathcal{Y} \text{ with } f \circ x = i_{\mathfrak{Y}} \circ y', \text{ and} \\
&x^*(\Omega_f) \circ I_{x,f}(T^*\mathcal{Y}) \circ I_{y',i_{\mathfrak{Y}}}(T^*\mathcal{Y})^{-1} \circ (y')^*(\nu_{\mathfrak{Y}}) = 0\bigr\}. \qquad (8.14)
\end{aligned}
$$

These give alternative definitions of $\mathcal{S}_f, \mathcal{T}_f$. Here the morphisms for the final conditions in (8.13)–(8.14) are given in

$$
\begin{tikzcd}
0 \arrow[r] & (y')^*(\mathcal{N}_{\mathfrak{Y}}) \arrow[r, "(y')^*(\nu_{\mathfrak{Y}})"] \arrow[d, dotted, "\cong"'] & (y')^* \circ i_{\mathfrak{Y}}^*(T^*\mathcal{Y}) \arrow[r, "(y')^*(\Omega_{i_{\mathfrak{Y}}})"] \arrow[d, "{\begin{array}{l} I_{x',i_{\mathfrak{X}}}(T^*\mathcal{X}) \circ (i_{\mathfrak{X}} \circ x')^*(\Omega_f) \circ \\ I_{i_{\mathfrak{X}} \circ x',f}(T^*\mathcal{Y}) \circ I_{y',i_{\mathfrak{Y}}}(T^*\mathcal{Y})^{-1} \end{array}}"] & (y')^*(T^*(\partial\mathcal{Y})) \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & (x')^*(\mathcal{N}_{\mathfrak{X}}) \arrow[r, "(x')^*(\nu_{\mathfrak{X}})"'] & (x')^* \circ i_{\mathfrak{X}}^*(T^*\mathcal{X}) \arrow[r, "(x')^*(\Omega_{i_{\mathfrak{X}}})"'] & (x')^*(T^*(\partial\mathcal{X})) \arrow[r] & 0,
\end{tikzcd}
\qquad (8.15)
$$

$$
\begin{tikzcd}
0 \arrow[r] & (y')^*(\mathcal{N}_{\mathfrak{Y}}) \arrow[r, "(y')^*(\nu_{\mathfrak{Y}})"'] & (y')^* \circ i_{\mathfrak{Y}}^*(T^*\mathcal{Y}) \arrow[r, "(y')^*(\Omega_{i_{\mathfrak{Y}}})"'] \arrow[d, "x^*(\Omega_f) \circ I_{x,f}(T^*\mathcal{Y}) \circ I_{y',i_{\mathfrak{Y}}}(T^*\mathcal{Y})^{-1}"'] & (y')^*(T^*(\partial\mathcal{Y})) \arrow[r] \arrow[dl, dotted] & 0 \\
& & x^*(T^*\mathcal{X}). & &
\end{tikzcd}
\qquad (8.16)
$$

The conditions in (8.13),(8.14) are equivalent to the existence of (iso)morphisms '$\dashrightarrow$' in (8.15),(8.16) respectively making (8.15)–(8.16) commute.

The material of §8.3 also extends to orbifolds with corners.

Definition 8.19. Let $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ be an orbifold with corners. By a *vector bundle $\mathcal{E}$ on $\mathfrak{X}$* we mean that $\mathcal{E}$ is an object in the category $\mathrm{vect}(\mathcal{X}) \subset \mathrm{qcoh}(\mathcal{X})$ of vector bundles on the Deligne–Mumford C^∞-stack $\mathcal{X}$, as in §C.6. A *smooth section s* of $\mathcal{E}$ on $\mathcal{X}$ is a morphism $s : \mathcal{O}_{\mathcal{X}} \to \mathcal{E}$ in $\mathrm{qcoh}(\mathcal{X})$. Smooth sections form a vector space, which we write as $C^\infty(\mathcal{E})$.

In §12.1 we will want to regard a vector bundle $\mathcal{E}$ on $\mathfrak{X}$ as being an orbifold with corners in its own right. To do this, we define the analogue $\mathrm{Tot}^{\mathrm{c}}$ of the total space functor Tot of Definition 8.4 for orbifolds with corners.

Define $\mathrm{Tot}^{\mathrm{c}}(\mathcal{E}) = \tilde{\mathfrak{E}} = (\tilde{\mathcal{E}}, \partial\tilde{\mathcal{E}}, i_{\tilde{\mathfrak{E}}})$, where $\tilde{\mathcal{E}} = \mathrm{Tot}(\mathcal{E})$ and $\partial\tilde{\mathcal{E}} = \mathrm{Tot}(i_{\mathfrak{X}}^*(\mathcal{E}))$ are the total spaces of the vector bundles $\mathcal{E}$ and $i_{\mathfrak{X}}^*(\mathcal{E})$ on the Deligne–Mumford C^∞-stacks $\mathcal{X}$ and $\partial\mathcal{X}$, as in Definition 8.4, and $i_{\tilde{\mathfrak{E}}} : \partial\tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ is a 1-morphism acting as a functor by $i_{\tilde{\mathfrak{E}}} : (u, \alpha) \mapsto \bigl(i_{\mathfrak{X}}(u), I_{\tilde{u},i_{\mathfrak{X}}}(\mathcal{E})^{-1} \circ \alpha \circ \iota\bigr)$ on objects, where $\iota : (i_{\mathfrak{X}} \circ \tilde{u})^*(\mathcal{O}_{\mathcal{X}}) \to \tilde{u}^*(\mathcal{O}_{\partial\mathcal{X}})$ is the natural isomorphism, and by $i_{\tilde{\mathfrak{E}}} : \eta \mapsto i_{\mathfrak{X}}(\eta)$

on morphisms. Using $i_{\mathcal{X}}$ strongly representable we can show that $i_{\tilde{\mathcal{E}}}$ is strongly representable. By considering local models, it is easy to see that $\mathrm{Tot}^{\mathrm{c}}(\mathcal{E}) = \tilde{\mathcal{E}}$ is an orbifold with corners, of dimension $\dim \mathfrak{X} + \operatorname{rank} \mathcal{E}$.

Definition 8.4 defines 1-morphisms $\pi : \mathrm{Tot}(\mathcal{E}) \to \mathcal{X}$ and $\mathrm{Tot}(s) : \mathcal{X} \to \mathrm{Tot}(\mathcal{E})$ in **Orb** for $s \in C^\infty(\mathcal{E})$, with $\pi \circ \mathrm{Tot}(s) = \mathrm{id}_{\mathcal{X}}$. The same definitions yield 1-morphisms $\pi : \mathrm{Tot}^{\mathrm{c}}(\mathcal{E}) \to \mathfrak{X}$ and $\mathrm{Tot}^{\mathrm{c}}(s) : \mathfrak{X} \to \mathrm{Tot}^{\mathrm{c}}(\mathcal{E})$ in $\mathbf{Orb^c}$ for $s \in C^\infty(\mathcal{E})$, with $\pi \circ \mathrm{Tot}^{\mathrm{c}}(s) = \mathrm{id}_{\mathfrak{X}}$.

8.6 Boundaries of orbifolds with corners, and simple, semisimple and flat 1-morphisms

Next we define boundaries of orbifolds with corners.

Definition 8.20. Let $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ be an orbifold with corners. We will define an orbifold with corners $\partial\mathfrak{X} = (\partial\mathcal{X}, \partial^2\mathcal{X}, i_{\partial\mathfrak{X}})$, called the *boundary* of $\mathfrak{X}$, such that $i_{\mathfrak{X}} : \partial\mathfrak{X} \to \mathfrak{X}$ is a 1-morphism in $\mathbf{Orb^c}$. Here $\partial\mathcal{X}$ and $i_{\mathfrak{X}}$ are given in $\mathfrak{X}$, so the new data we have to construct is $\partial^2\mathcal{X}, i_{\partial\mathfrak{X}}$.

As $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is strongly representable by Definition 8.15, Proposition C.15 defines an explicit fibre product $\partial\mathcal{X} \times_{i_{\mathfrak{X}},\mathcal{X},i_{\mathfrak{X}}} \partial\mathcal{X}$ with strongly representable projection morphisms $\pi_1, \pi_2 : \partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X} \to \partial\mathcal{X}$ such that $i_{\mathfrak{X}} \circ \pi_1 = i_{\mathfrak{X}} \circ \pi_2$. We will use this explicit fibre product throughout. There is a unique diagonal 1-morphism $\Delta_{\partial\mathcal{X}} : \partial\mathcal{X} \to \partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X}$ with $\pi_1 \circ \Delta_{\partial\mathcal{X}} = \pi_2 \circ \Delta_{\partial\mathcal{X}} = \mathrm{id}_{\partial\mathcal{X}}$. Since $\partial\mathcal{X}$ is separated and $i_{\mathfrak{X}}$ is an immersion, $\Delta_{\partial\mathcal{X}}$ is an equivalence with an open and closed C^∞-substack $\Delta_{\partial\mathcal{X}}(\partial\mathcal{X}) \subseteq \partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X}$. Define $\partial^2\mathcal{X} = \partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X} \setminus \Delta_{\partial\mathcal{X}}(\partial\mathcal{X})$. Then $\partial^2\mathcal{X}$ is also an open and closed C^∞-substack in $\partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X}$. It is separated and second countable as $\partial\mathcal{X}$ is.

Define C^∞-stack 1-morphisms $i_{\partial\mathfrak{X}} = \pi_1|_{\partial^2\mathcal{X}} : \partial^2\mathcal{X} \to \partial\mathcal{X}$ and $j_{\partial\mathfrak{X}} = \pi_2|_{\partial^2\mathcal{X}} : \partial^2\mathcal{X} \to \partial\mathcal{X}$. Since $i_{\mathfrak{X}}$ is proper, π_1, π_2 are proper, so $i_{\partial\mathfrak{X}}, j_{\partial\mathfrak{X}}$ are proper as they are the restrictions of π_1, π_2 to a closed C^∞-substack. Also $i_{\partial\mathfrak{X}}, j_{\partial\mathfrak{X}}$ are strongly representable by Proposition C.14(c), as π_1, π_2 are strongly representable and $\partial^2\mathcal{X}$ is open in $\partial\mathcal{X} \times_{\mathcal{X}} \partial\mathcal{X}$.

To show that $\partial\mathfrak{X} = (\partial\mathcal{X}, \partial^2\mathcal{X}, i_{\partial\mathfrak{X}})$ is an orbifold with corners, note that if $[x'] \in \partial\mathcal{X}_{\mathrm{top}}$ with $i_{\mathfrak{X},\mathrm{top}}([x']) = [x] \in \mathcal{X}_{\mathrm{top}}$, then Definition 8.15 gives a 2-Cartesian diagram (8.6) with $u_{\mathrm{top}}([p]) = [x]$ for $p \in U$. So there is a unique $p' \in \partial U$ with $i_U(p') = p$ and $u_{\partial,\mathrm{top}}([p']) = [x']$. We can then show there is a unique 1-morphism u_{∂^2} such that equation (8.6) for $\partial\mathfrak{X}$ at $[x']$ is

$$\begin{tikzcd}
\underline{\partial^2 \bar{U}} \arrow[rr, "u_{\partial^2}"] \arrow[d, "\bar{i}_{\partial U}"'] \arrow[drr, phantom, "\mathrm{id}\Uparrow"] & & \partial^2\mathcal{X} \arrow[d, "i_{\partial\mathcal{X}}"] \\
\underline{\partial\bar{U}} \arrow[rr, "u_\partial"] & & \partial\mathcal{X}.
\end{tikzcd}$$

The 1-morphism $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is étale locally modelled on the maps $i_U : \partial U \to U$ for U as above, which are smooth maps of manifolds with corners by Theorem 5.6(c). Hence $i_{\mathfrak{X}} : \partial\mathfrak{X} \to \mathfrak{X}$ is a 1-morphism in $\mathbf{Orb^c}$.

Here is the orbifold analogue of parts of §5.4.

Definition 8.21. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a 1-morphism of orbifolds with corners, and $\partial\mathfrak{X}$ the boundary of $\mathfrak{X}$. Then $s_f : \mathcal{S}_f \to \partial\mathcal{X}$ is proper and étale by Definition 8.18(c). This and $\partial\mathcal{X}$ locally compact imply that $s_f(\mathcal{S}_f)$ is open and closed in $\partial\mathcal{X}$. Define $\partial_-^f\mathcal{X} = s_f(\mathcal{S}_f)$ and $\partial_+^f\mathcal{X} = \partial\mathcal{X} \setminus \partial_-^f\mathcal{X}$. Then $\partial_\pm^f\mathcal{X}$ are open and closed C^∞-substacks of $\partial\mathcal{X}$, with $\partial\mathcal{X} = \partial_+^f\mathcal{X} \amalg \partial_-^f\mathcal{X}$. Write $\partial_+^f\mathfrak{X}, \partial_-^f\mathfrak{X}$ for the open and closed suborbifolds of $\partial\mathfrak{X}$ corresponding to $\partial_+^f\mathcal{X}, \partial_-^f\mathcal{X}$, as in Definition 8.15. Then $\partial\mathfrak{X} = \partial_+^f\mathfrak{X} \amalg \partial_-^f\mathfrak{X}$.

We call f *simple* if $s_f : \mathcal{S}_f \to \partial\mathcal{X}$ is an equivalence, so that $\partial_+^f\mathcal{X} = \emptyset$, and we call f *semisimple* if $s_f : \mathcal{S}_f \to \partial_-^f\mathcal{X}$ is an equivalence, and we call f *flat* if $\mathcal{T}_f = \emptyset$. Simple implies semisimple. If f is simple then $\partial_-^f\mathfrak{X} = \partial\mathfrak{X}$ and $\partial_+^f\mathfrak{X} = \emptyset$.

One can show that $f : \mathfrak{X} \to \mathfrak{Y}$ is simple, semisimple or flat if and only if the smooth maps $h : U \to V$ in Definition 8.15 for f are simple, semisimple or flat in the sense of §5.4 for all diagrams (8.8). That is, 1-morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{Orb^c}$ are simple, semisimple or flat if and only if they are étale locally modelled on simple, semisimple or flat morphisms in $\mathbf{Man^c}$.

The condition that $i_\mathfrak{X}$ is *strongly representable* in Definition 8.15 is essential in constructing f_-, η_- in parts (b),(c) of the next theorem, an analogue of Proposition 5.13 and Theorem 6.12, and our main reason for including $i_\mathfrak{X}$ strongly representable in Definition 8.15 was to make the theorem hold.

Theorem 8.22. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a semisimple 1-morphism of orbifolds with corners. Then:*

- **(a)** *Define $f_+ = f \circ i_\mathfrak{X}|_{\partial_+^f\mathfrak{X}} : \partial_+^f\mathfrak{X} \to \mathfrak{Y}$. Then f_+ is semisimple. If f is flat then f_+ is also flat.*
- **(b)** *There exists a unique, semisimple 1-morphism $f_- : \partial_-^f\mathfrak{X} \to \partial\mathfrak{Y}$ in $\mathbf{Orb^c}$ with $f \circ i_\mathfrak{X}|_{\partial_-^f\mathfrak{X}} = i_\mathfrak{Y} \circ f_-$. If f is simple then $f_- : \partial\mathfrak{X} \to \partial\mathfrak{Y}$ is also simple. If f is flat then f_- is flat.*
- **(c)** *Let $g : \mathfrak{X} \to \mathfrak{Y}$ be another 1-morphism and $\eta : f \Rightarrow g$ a 2-morphism in $\mathbf{Orb^c}$. Then g is also semisimple, with $\partial_-^g\mathfrak{X} = \partial_-^f\mathfrak{X}$. If f is simple, or flat, then g is simple, or flat, respectively. Part **(b)** defines 1-morphisms $f_-, g_- : \partial_-^f\mathfrak{X} \to \partial\mathfrak{Y}$. There is a unique 2-morphism $\eta_- : f_- \Rightarrow g_-$ in $\mathbf{Orb^c}$ such that*

$$\mathrm{id}_{i_\mathfrak{Y}} * \eta_- = \eta * \mathrm{id}_{i_\mathfrak{X}|_{\partial_-^f\mathfrak{X}}} : f \circ i_\mathfrak{X}|_{\partial_-^f\mathfrak{X}} = i_\mathfrak{Y} \circ f_- \Longrightarrow g \circ i_\mathfrak{X}|_{\partial_-^f\mathfrak{X}} = i_\mathfrak{Y} \circ g_-. \tag{8.17}$$

Proof. In part (b), we define $f_- : \partial_-^f\mathcal{X} \to \partial\mathcal{Y}$ as follows. As f is semisimple, $s_f : \mathcal{S}_f \to \partial_-^f\mathcal{X}$ is an equivalence, so there exists a quasi-inverse $j : \partial_-^f\mathcal{X} \to \mathcal{S}_f$ and a 2-morphism $\eta : s_f \circ j \Rightarrow \mathrm{id}_{\partial_-^f\mathcal{X}}$. This gives a 2-morphism

$$\mathrm{id}_f * \eta : i_\mathcal{Y} \circ (u_f \circ j) = f \circ i_\mathcal{X} \circ s_f \circ j \Longrightarrow f \circ i_\mathcal{X} \circ \mathrm{id}_{\partial_-^f\mathcal{X}} = f \circ i_\mathcal{X}|_{\partial_-^f\mathcal{X}}.$$

Since $i_\mathcal{Y}$ is strongly representable, Proposition C.13 gives a unique $f_- : \partial_-^f\mathcal{X} \to \partial\mathcal{Y}$ with $i_\mathcal{Y} \circ f_- = f \circ i_\mathcal{X}|_{\partial_-^f\mathcal{X}}$ and $\zeta : u_f \circ j \Rightarrow f_-$ with $\mathrm{id}_{i_\mathcal{Y}} * \zeta = \mathrm{id}_f * \eta$.

For part (c), by considering diagrams combining (8.8) for f and g:

$$\begin{tikzcd}
\underline{\bar U} \arrow[rr, "u"] \arrow[d, "\underline{\bar h}"] \arrow[drr, phantom, "\Downarrow\zeta"] & & \mathcal{X} \arrow[d, bend right, "g"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "\overset{\eta}{\Leftarrow}"] \\
\underline{\bar V} \arrow[rr, "v"] & & \mathcal{Y},
\end{tikzcd}$$

so that f, g are étale locally modelled on the same smooth map $h : U \to V$, we see that f semisimple implies all such h semisimple implies g semisimple. Also, as $\partial_-^f \mathfrak{X}, \partial_-^g \mathfrak{X}$ are both étale locally modelled on $\partial_-^h U$, we see that $\partial_-^g \mathfrak{X} = \partial_-^f \mathfrak{X}$.

From (b) we have unique 1-morphisms $f_-, g_- : \partial_-^f \mathcal{X} \to \partial\mathcal{Y}$ with $i_{\mathcal{Y}} \circ f_- = f \circ i_{\mathcal{X}}|_{\partial_-^f \mathcal{X}}$ and $i_{\mathcal{Y}} \circ g_- = g \circ i_{\mathcal{X}}|_{\partial_-^f \mathcal{X}}$. Thus we have a 2-morphism

$$\eta * \mathrm{id}_{i_{\mathcal{X}}|_{\partial_-^f \mathcal{X}}} : i_{\mathcal{Y}} \circ f_- \Longrightarrow g \circ i_{\mathcal{X}}|_{\partial_-^f \mathcal{X}}.$$

As $i_{\mathcal{Y}}$ is strongly representable, Proposition C.14 gives a unique 1-morphism $\tilde f_- : \partial_-^f \mathcal{X} \to \partial\mathcal{Y}$ with $i_{\mathcal{Y}} \circ \tilde f_- = g \circ i_{\mathcal{X}}|_{\partial_-^f \mathcal{X}}$, and a 2-morphism $\eta_- : f_- \Rightarrow \tilde f_-$ with $\mathrm{id}_{i_{\mathcal{Y}}} * \eta_- = \eta * \mathrm{id}_{i_{\mathcal{X}}|_{\partial_-^f \mathcal{X}}}$. But the uniqueness of g_- with $i_{\mathcal{Y}} \circ g_- = g \circ i_{\mathcal{X}}|_{\partial_-^f \mathcal{X}}$ forces $\tilde f_- = g_-$, so η_- is a 2-morphism $f_- \Rightarrow g_-$ as we want.

The rest of the proof follows from §5.4 by deducing properties in $\mathbf{Orb^c}$ from the corresponding properties on étale open covers in $\mathbf{Man^c}$. □

8.7 Corners $C_k(\mathfrak{X})$ and the corner functors $C, \hat C$

In §5.5 and [55, §2] we explained that if X is a manifold with corners, then the k^{th} boundary $\partial^k X$ has a natural action of the symmetric group S_k by diffeomorphisms. This is not immediately obvious from the definition, but as in (5.2) there is a natural diffeomorphism

$$\begin{aligned}\partial^k X \cong \bigl\{(x,\beta_1,\ldots,\beta_k) : x \in X,\ &\beta_1,\ldots,\beta_k \text{ are distinct}\\ &\text{local boundary components for } X \text{ at } x\bigr\},\end{aligned} \tag{8.18}$$

where $i_{\partial^k X} : \partial^{k+1} X \to \partial^k X$ acts by $i_{\partial^k X} : (x,\beta_1,\ldots,\beta_{k+1}) \mapsto (x,\beta_1,\ldots,\beta_k)$, and in this presentation S_k acts by permuting $\beta_1,\ldots,\beta_k$. The next remark explains the analogue of this for orbifolds with corners.

Remark 8.23. Let $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathcal{X}})$ be an orbifold with corners. Then Definition 8.20 defines orbifolds with corners $\partial\mathfrak{X}, \partial^2\mathfrak{X} = \partial(\partial\mathfrak{X}), \ldots$, where $\partial^k\mathfrak{X} = (\partial^k\mathcal{X}, \partial^{k+1}\mathcal{X}, i_{\partial^k\mathfrak{X}})$ for $i_{\partial^k\mathfrak{X}} : \partial^{k+1}\mathcal{X} \to \partial^k\mathcal{X}$ a 1-morphism of C^∞-stacks. As in the manifold case, there is a natural action r_k of S_k on the orbifold with corners $\partial^k\mathfrak{X}$ by 1-isomorphisms for all $k = 0, 1, \ldots$, which we will write as a group morphism $r_k : S_k \to \mathrm{Aut}(\partial^k\mathcal{X})$. Here is one way to construct this.

Define an explicit C^∞-stack $\tilde\partial^k\mathcal{X}$ as follows. Objects of the category $\tilde\partial^k\mathcal{X}$ are k-tuples $(A_1,\ldots,A_k)$ for $A_1,\ldots,A_k \in \partial\mathcal{X}$ with $i_{\mathfrak{X}}(A_1) = \cdots = i_{\mathfrak{X}}(A_k)$ in $\mathcal{X}$, such that if $1 \leqslant i < j \leqslant k$ then there does not exist an object P and morphisms $a_i : P \to A_i$, $a_j : P \to A_j$ in $\partial\mathcal{X}$ such that $p_{\partial\mathcal{X}}(P) = \underline{*}$ and $p_{\partial\mathcal{X}}(a_i) = p_{\partial\mathcal{X}}(a_j)$ in $\mathbf{C^\infty Sch}$. Morphisms $(a_1,\ldots,a_k) : (A_1,\ldots,A_k) \to (B_1,\ldots,B_k)$ in the category

$\tilde\partial^k\mathcal X$ are k-tuples $(a_1,\ldots,a_k)$ with $a_i:A_i\to B_i$ a morphism in $\partial\mathcal X$ for $i=1,\ldots,k$, with $i_{\mathcal X}(a_1)=\cdots=i_{\mathcal X}(a_k)$. Composition is $(b_1,\ldots,b_k)\circ(a_1,\ldots,a_k)=(b_1\circ a_1,\ldots,b_k\circ a_k)$, and identities $\mathrm{id}_{(A_1,\ldots,A_k)}=(\mathrm{id}_{A_1},\ldots,\mathrm{id}_{A_k})$. Define a functor $p_{\tilde\partial^k\mathcal X}:\tilde\partial^k\mathcal X\to\mathbf{C^\infty Sch}$ by $p_{\tilde\partial^k\mathcal X}:(A_1,\ldots,A_k)\mapsto p_{\partial\mathcal X}(A_1)=\cdots=p_{\partial\mathcal X}(A_k)$ and $p_{\tilde\partial^k\mathcal X}:(a_1,\ldots,a_k)\mapsto p_{\partial\mathcal X}(a_1)=\cdots=p_{\partial\mathcal X}(a_k)$.

Using similar ideas to Proposition C.15, one can now show $\tilde\partial^k\mathcal X$ is a Deligne–Mumford C^∞-stack, with a natural 1-isomorphism $\partial^k\mathcal X\to\tilde\partial^k\mathcal X$. For example, $\partial^2\mathcal X=\tilde\partial^2\mathcal X$, objects of $\partial^3\mathcal X$ are $\bigl((A_1,A_2),(A_1,A_3)\bigr)$ for some object (A_1,A_2,A_3) in $\tilde\partial^3\mathcal X$, objects of $\partial^4\mathcal X$ are $\bigl(\bigl((A_1,A_2),(A_1,A_3)\bigr),\bigl((A_1,A_2),(A_1,A_4)\bigr)\bigr)$ for some (A_1,A_2,A_3,A_4) in $\tilde\partial^4\mathcal X$, and so on. Now S_k acts on $\tilde\partial^k\mathcal X$ by permutation of $A_1,\ldots,A_k$ and $a_1,\ldots,a_k$ in objects $(A_1,\ldots,A_k)$ and morphisms $(a_1,\ldots,a_k)$. Lifting through the 1-isomorphism $\partial^k\mathcal X\to\tilde\partial^k\mathcal X$ gives $r_k:S_k\to\mathrm{Aut}(\partial^k\mathcal X)$.

If we embed S_k into S_{k+1} as the subgroup of permutations of $1,\ldots,k+1$ fixing $k+1$, then $i_{\partial^k\mathcal X}:\partial^{k+1}\mathcal X\to\partial^k\mathcal X$ is S_k-equivariant. Also, the 1-morphism $i_{\mathcal X}\circ i_{\partial\mathcal X}\circ\cdots\circ i_{\partial^{k-1}\mathcal X}:\partial^k\mathcal X\to\mathcal X$ is S_k-invariant, that is,

$$i_{\mathcal X}\circ i_{\partial\mathcal X}\circ\cdots\circ i_{\partial^{k-1}\mathcal X}\circ r_k(\sigma)=i_{\mathcal X}\circ i_{\partial\mathcal X}\circ\cdots\circ i_{\partial^{k-1}\mathcal X}\quad\text{for all }\sigma\in S_k.$$

Here is a description of the topological space $(\partial^k\mathcal X)_{\rm top}$ parallel to (8.18), and similar to (8.11)–(8.12). Consider $(k+1)$-tuples $(x,x_1',\ldots,x_k')$, where $x:\bar{\underline{*}}\to\mathcal X$ and $x_i':\bar{\underline{*}}\to\partial\mathcal X$ for $i=1,\ldots,k$ are 1-morphisms in $\mathbf{C^\infty Sta}$ such that $x_1',\ldots,x_k'$ are distinct and $x=i_{\mathcal X}\circ x_1'=\cdots=i_{\mathcal X}\circ x_k'$. Define an equivalence relation $\sim$ on such $k+1$-tuples by $(x,x_1',\ldots,x_k')\sim(\tilde x,\tilde x_1',\ldots,\tilde x_k')$ if there exists $(\eta,\eta_1',\ldots,\eta_k')$ where $\eta:x\Rightarrow\tilde x$ and $\eta_i':x_i'\Rightarrow\tilde x_i'$ are 2-morphisms with $\eta=\mathrm{id}_{i_{\mathcal X}}*\eta_1'=\cdots=\mathrm{id}_{i_{\mathcal X}}*\eta_k'$. Write $[x,x_1',\ldots,x_k']$ for the $\sim$-equivalence class of $(x,x_1',\ldots,x_k')$. Such $[x,x_1',\ldots,x_k']$ correspond to points of $(\partial^k\mathcal X)_{\rm top}$, that is,

$$\begin{aligned}(\partial^k\mathcal X)_{\rm top}\cong\bigl\{&[x,x_1',\ldots,x_k']:x:\bar{\underline{*}}\to\mathcal X,\ x_i':\bar{\underline{*}}\to\partial\mathcal X\text{ are 1-morphisms}\\&\text{with }x_1',\ldots,x_k'\text{ distinct and }x=i_{\mathcal X}\circ x_1'=\cdots=i_{\mathcal X}\circ x_k'\bigr\}.\end{aligned}\tag{8.19}$$

To prove this, we show that (8.19) is naturally identified with $(\tilde\partial^k\mathcal X)_{\rm top}$. Note that (8.19) relies on $i_{\mathcal X}$ strongly representable, so that equalities of 1-morphisms $x=i_{\mathcal X}\circ x_1'=\cdots=i_{\mathcal X}\circ x_k'$ rather than 2-isomorphisms of 1-morphisms $x\cong i_{\mathcal X}\circ x_1'\cong\cdots\cong i_{\mathcal X}\circ x_k'$ are well-behaved.

From §5.5, for X a manifold with corners, the k-corners $C_k(X)$ of X is

$$\begin{aligned}C_k(X)=\bigl\{&(x,\{\beta_1,\ldots,\beta_k\}):x\in X,\ \beta_1,\ldots,\beta_k\text{ are distinct}\\&\text{local boundary components for }X\text{ at }x\bigr\}\cong\partial^kX/S_k.\end{aligned}\tag{8.20}$$

It is a manifold with corners. The map $\Pi_X^k:C_k(X)\to X$ taking $\Pi_X^k:(x,\{\beta_1,\ldots,\beta_k\})\mapsto x$ is smooth. Here is the orbifold analogue of this.

Definition 8.24. By §C.4, we may now define quotient Deligne–Mumford C^∞-stacks $[\partial^k\mathcal X/S_k]$, $[\partial^{k+1}\mathcal X/S_k]$ and quotient 1-morphisms

$$\begin{aligned}{}[i_{\mathcal X}\circ i_{\partial\mathcal X}\circ\cdots\circ i_{\partial^{k-1}\mathcal X},\pi_{\{1\}}]&:[\partial^k\mathcal X/S_k]\longrightarrow[\mathcal X/\{1\}]=\mathcal X,\\ [i_{\partial^k\mathcal X},\mathrm{id}_{S_k}]&:[\partial^{k+1}\mathcal X/S_k]\longrightarrow[\partial^k\mathcal X/S_k],\end{aligned}$$

where $\pi_{\{1\}} : S_k \to \{1\}$ is the unique projection. If $\mathcal{X}$ is étale locally modelled on $\bar{\underline{U}}$ for U a manifold with corners, then $[\partial^k \mathcal{X}/S_k], [\partial^{k+1}\mathcal{X}/S_k]$ are étale locally modelled on $C_k(U), \partial C_k(U)$, and $[i_{\mathfrak{X}} \circ \cdots \circ i_{\partial^{k-1}\mathfrak{X}}, \pi_{\{1\}}]$ is étale locally modelled (in a 2-Cartesian square) on $\Pi_U^k : C_k(U) \to U$, and $[i_{\partial^k \mathfrak{X}}, \mathrm{id}_{S_k}]$ is étale locally modelled (in a 2-Cartesian square) on $i_{C_k(U)} : \partial C_k(U) \to C_k(U)$.

Since $[i_{\mathfrak{X}} \circ \cdots \circ i_{\partial^{k-1}\mathfrak{X}}, \pi_{\{1\}}]$, $[i_{\partial^k\mathfrak{X}}, \mathrm{id}_{S_k}]$ are étale locally modelled on C^∞-scheme morphisms, they are representable. So by Proposition C.14(b) applied to $[i_{\mathfrak{X}} \circ \cdots \circ i_{\partial^{k-1}\mathfrak{X}}, \pi_{\{1\}}]$, there exists a Deligne–Mumford C^∞-stack $C_k(\mathcal{X})$ unique up to 1-isomorphism, a strongly representable 1-morphism $\Pi_{\mathfrak{X}}^k : C_k(\mathcal{X}) \to \mathcal{X}$, and an equivalence $j : [\partial^k \mathcal{X}/S_k] \to C_k(\mathcal{X})$ with $[i_{\mathfrak{X}} \circ \cdots \circ i_{\partial^{k-1}\mathfrak{X}}, \pi_{\{1\}}] = \Pi_{\mathfrak{X}}^k \circ j$. Then by Proposition C.14(b) applied to $j \circ [i_{\partial^k\mathfrak{X}}, \mathrm{id}_{S_k}]$, there exists a Deligne–Mumford C^∞-stack $\partial C_k(\mathcal{X})$ unique up to 1-isomorphism, a strongly representable 1-morphism $i_{C_k(\mathfrak{X})} : \partial C_k(\mathcal{X}) \to C_k(\mathcal{X})$, and an equivalence $k : [\partial^{k+1}\mathcal{X}/S_k] \to \partial C_k(\mathcal{X})$ with $j \circ [i_{\partial^k\mathfrak{X}}, \mathrm{id}_{S_k}] = i_{C_k(\mathfrak{X})} \circ k$.

Suppose now that U is a manifold with corners, and $u : \bar{\underline{U}} \to \mathcal{X}$ is an étale 1-morphism of C^∞-stacks, as in (8.6). Then $C_k(U)$ and $\partial C_k(U)$ are manifolds with corners, and $\Pi_U^k : C_k(U) \to U$, $i_{C_k(U)} : \partial C_k(U) \to C_k(U)$ are smooth maps, so we have 1-morphisms of C^∞-stacks $\bar{\underline{i}}_{C_k(U)} : \overline{\underline{\partial C_k(U)}} \to \overline{\underline{C_k(U)}}$, $\bar{\underline{\Pi}}_U^k : \overline{\underline{C_k(U)}} \to \bar{\underline{U}}$, where $\underline{U}, \underline{C_k(U)}, \underline{\partial C_k(U)}, \underline{i}_{C_k(U)}, \underline{\Pi}_U^k = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U, C_k(U), \partial C_k(U), i_{C_k(U)}, \Pi_U^k)$. We now claim that there are unique, étale 1-morphisms $u_{C_k}, u_{\partial C_k}$ such that the following diagram 2-commutes, with both squares 2-Cartesian:

$$\begin{tikzcd}
\overline{\underline{\partial C_k(U)}} \arrow[rr, "u_{\partial C_k}"'] \arrow[d, "\bar{\underline{i}}_{C_k(U)}"'] \arrow[drr, phantom, "\mathrm{id}\Uparrow"] & & \partial C_k(\mathcal{X}) \arrow[d, "i_{C_k(\mathfrak{X})}"] \\
\overline{\underline{C_k(U)}} \arrow[rr, "u_{C_k}"'] \arrow[d, "\bar{\underline{\Pi}}_U^k"'] \arrow[drr, phantom, "\mathrm{id}\Uparrow"] & & C_k(\mathcal{X}) \arrow[d, "\Pi_{\mathfrak{X}}^k"] \\
\bar{\underline{U}} \arrow[rr, "u"] & & \mathcal{X}.
\end{tikzcd} \tag{8.21}$$

To see this, note that as $C_k(U) \cong \partial^k U/S_k$, $C_k(\mathcal{X}) \simeq \partial^k \mathcal{X}/S_k$ and $\partial C_k(U) \cong \partial^{k+1}U/S_k$, $\partial C_k(\mathcal{X}) \simeq \partial^{k+1}\mathcal{X}/S_k$, there exists a diagram of the form (8.21) with morphisms $u'_{C_k}, u'_{\partial C_k}$ unique up to 2-isomorphism, and with squares 2-Cartesian, but with 2-morphisms not necessarily identities. Then as $\Pi_{\mathfrak{X}}^k, i_{C_k(\mathfrak{X})}$ are strongly representable, we can apply Proposition C.13 to the bottom and then the top square of (8.21) to show that there exist unique $u_{C_k}, u_{\partial C_k}$ making the squares of (8.21) 2-Cartesian with identity 2-morphisms.

Define $C_k(\mathfrak{X}) = (C_k(\mathcal{X}), \partial C_k(\mathcal{X}), i_{C_k(\mathfrak{X})})$. Using the top square of (8.21) in place of (8.6), and noting that as $u : \bar{\underline{U}} \to \mathcal{X}$ above form an étale open cover of $\mathcal{X}$, the $u_{C_k} : \overline{\underline{C_k(U)}} \to C_k(\mathcal{X})$ form an étale open cover of $C_k(\mathcal{X})$, we see that $C_k(\mathfrak{X})$ is an orbifold with corners, which we call the *k-corners* of $\mathfrak{X}$. Using the bottom square of (8.21) in place of (8.8), we also see that $\Pi_{\mathfrak{X}}^k : C_k(\mathfrak{X}) \to \mathfrak{X}$ is a 1-morphism of orbifolds with corners, which is strongly representable as a 1-morphism of C^∞-stacks $\Pi_{\mathfrak{X}}^k : C_k(\mathcal{X}) \to \mathcal{X}$.

Here is a description of the topological space $C_k(\mathcal{X})_{\mathrm{top}}$ parallel to (8.20). As $C_k(\mathcal{X}) \simeq [\partial^k \mathcal{X}/S_k]$ we have a homeomorphism $C_k(\mathcal{X})_{\mathrm{top}} \cong (\partial^k \mathcal{X})_{\mathrm{top}}/S_k$. In the description (8.19) of $(\partial^k\mathcal{X})_{\rm top}$, S_k acts by permuting $x'_1,\ldots,x'_k$, so dividing by S_k turns an ordered k-tuple $(x'_1,\ldots,x'_k)$ into an unordered set $\{x'_1,\ldots,x'_k\}$. Consider pairs $(x,\{x'_1,\ldots,x'_k\})$, where $x:\bar{\underline{*}}\rightarrow\mathcal{X}$ and $x'_i:\bar{\underline{*}}\rightarrow\partial\mathcal{X}$ for $i=1,\ldots,k$ are 1-morphisms in $\mathbf{C^\infty Sta}$ such that $x'_1,\ldots,x'_k$ are distinct and $x=i_{\mathfrak X}\circ x'_1=\cdots=i_{\mathfrak X}\circ x'_k$. Define an equivalence relation $\approx$ on such pairs by $(x,\{x'_1,\ldots,x'_k\})\approx(\tilde x,\{\tilde x'_1,\ldots,\tilde x'_k\})$ if there exists $(\eta,\eta'_1,\ldots,\eta'_k)$ and $\sigma\in S_k$ (that is, σ is a permutation of $1,\ldots,k$) where $\eta:x\Rightarrow\tilde x$ and $\eta'_i:x'_i\Rightarrow\tilde x'_{\sigma(i)}$ are 2-morphisms in $\mathbf{C^\infty Sta}$ for $i=1,\ldots,k$ with $\eta=\mathrm{id}_{i_{\mathfrak X}}*\eta'_1=\cdots=\mathrm{id}_{i_{\mathfrak X}}*\eta'_k$. Write $[x,\{x'_1,\ldots,x'_k\}]$ for the $\approx$-equivalence class of $(x,\{x'_1,\ldots,x'_k\})$. Such $[x,\{x'_1,\ldots,x'_k\}]$ are naturally identified with points of $C_k(\mathcal{X})_{\rm top}$, that is,

$$\begin{aligned} C_k(\mathcal{X})_{\rm top}\cong\bigl\{[x,\{x'_1,\ldots,x'_k\}]:x:\bar{\underline{*}}\rightarrow\mathcal{X},\ x'_i:\bar{\underline{*}}\rightarrow\partial\mathcal{X}\text{ are 1-morphisms}\\ \text{with }x'_1,\ldots,x'_k\text{ distinct and }x=i_{\mathfrak X}\circ x'_1=\cdots=i_{\mathfrak X}\circ x'_k\bigr\}.\end{aligned}\tag{8.22}$$

Here is the orbifold analogue of the category $\check{\mathbf{Man}}{}^{\mathbf{c}}$ in Definition 5.15.

Definition 8.25. We will define a 2-category $\check{\mathbf{Orb}}{}^{\mathbf{c}}$ whose objects are disjoint unions $\coprod_{m=0}^\infty\mathfrak{X}_m$, where $\mathfrak{X}_m$ is a (possibly empty) orbifold with corners of dimension m. In more detail, objects of $\check{\mathbf{Orb}}{}^{\mathbf{c}}$ are triples $\mathfrak{X}=(\mathcal{X},\partial\mathcal{X},i_{\mathfrak X})$ with $i_{\mathfrak X}:\partial\mathcal{X}\rightarrow\mathcal{X}$ a strongly representable 1-morphism of Deligne–Mumford C^∞-stacks, such that there exists a decomposition $\mathcal{X}=\coprod_{m=0}^\infty\mathcal{X}_m$ with each $\mathcal{X}_m\subseteq\mathcal{X}$ an open and closed C^∞-substack, for which $\mathfrak{X}_m:=\bigl(\mathcal{X}_m,i_{\mathfrak X}^{-1}(\mathcal{X}_m),i_{\mathfrak X}|_{i_{\mathfrak X}^{-1}(\mathcal{X}_m)}\bigr)$ is an orbifold with corners of dimension m. This decomposition is unique, as $\mathcal{X}_m\subseteq\mathcal{X}$ is the open C^∞-substack on which $T^*\mathcal{X}$ has rank m.

A 1-morphism $f:\mathfrak{X}\rightarrow\mathfrak{Y}$ in $\check{\mathbf{Orb}}{}^{\mathbf{c}}$ is a 1-morphism $f:\mathcal{X}\rightarrow\mathcal{Y}$ such that $f|_{\mathcal{X}_m\cap f^{-1}(\mathcal{Y}_n)}:\bigl(\mathcal{X}_m\cap f^{-1}(\mathcal{Y}_n)\bigr)\rightarrow\mathcal{Y}_n$ is a 1-morphism of orbifolds with corners for all $m,n\geqslant 0$. For 1-morphisms $f,g:\mathfrak{X}\rightarrow\mathfrak{Y}$, a 2-morphism $\eta:f\Rightarrow g$ is a 2-morphism $\eta:f\Rightarrow g$ in $\mathbf{C^\infty Sta}$. Then $\mathbf{Orb^c}$ is a full 2-subcategory of $\check{\mathbf{Orb}}{}^{\mathbf{c}}$.

For each orbifold with corners $\mathfrak{X}$, define $C(\mathfrak{X})=\coprod_{k=0}^{\dim\mathfrak{X}}C_k(\mathfrak{X})$, and define $\Pi_{\mathfrak X}:C(\mathfrak{X})\rightarrow\mathfrak{X}$ by $\Pi_{\mathfrak X}|_{C_k(\mathfrak{X})}=\Pi_{\mathfrak X}^k:C_k(\mathfrak{X})\rightarrow\mathfrak{X}$ for $k=0,\ldots,\dim\mathfrak{X}$. Then $C(\mathfrak{X})$ is an object in $\check{\mathbf{Orb}}{}^{\mathbf{c}}$, and $\Pi_{\mathfrak X}:C(\mathfrak{X})\rightarrow\mathfrak{X}$ a 1-morphism in $\check{\mathbf{Orb}}{}^{\mathbf{c}}$.

The definitions of $C(\mathfrak{X}),\Pi_{\mathfrak X}^k,\Pi_{\mathfrak X}$ above also make sense if $\mathfrak{X},\mathfrak{Y}$ are objects in $\check{\mathbf{Orb}}{}^{\mathbf{c}}$ rather than $\mathbf{Orb^c}$.

Example 8.26. Suppose $\mathfrak{X}$ is a quotient $[X/G]$ as in Example 8.16, where X is a manifold with corners and G is a finite group. Then the action $r:G\rightarrow\mathrm{Aut}(X)$ lifts to $C(r):G\rightarrow\mathrm{Aut}(C(X))$, and we find there is an equivalence $C([X/G])\simeq[C(X)/G]$ in $\check{\mathbf{Orb}}{}^{\mathbf{c}}$, where to define $[C(X)/G]$ we note that Example 8.16 also works with X in $\check{\mathbf{Man}}{}^{\mathbf{c}}$ rather than $\mathbf{Man^c}$, yielding $[X/G]\in\check{\mathbf{Orb}}{}^{\mathbf{c}}$.

Here is the analogue of Definition 5.16 and Theorems 5.17 and 6.29.

Theorem 8.27. (a) *Let $f:\mathfrak{X}\rightarrow\mathfrak{Y}$ be a 1-morphism of orbifolds with corners. Then there are unique 1-morphisms $C(f):C(\mathfrak{X})\rightarrow C(\mathfrak{Y})$ and $\hat C(f):C(\mathfrak{X})\rightarrow C(\mathfrak{Y})$ in $\check{\mathbf{Orb}}{}^{\mathbf{c}}$ such that $\Pi_{\mathfrak Y}\circ C(f)=f\circ\Pi_{\mathfrak X}=\Pi_{\mathfrak Y}\circ\hat C(f):C(\mathfrak{X})\rightarrow\mathfrak{Y}$, and* *whenever as in* (8.8) *we are given a* 2*-commutative diagram in* $\mathbf{C^\infty Sta}$

$$\begin{tikzcd} \underline{\bar U} \arrow[rr, "u"] \arrow[d, "\underline{\bar h}"] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{X} \arrow[d, "f"] \\ \underline{\bar V} \arrow[rr, "v"] & & \mathcal{Y}, \end{tikzcd} \tag{8.23}$$

where U,V *are manifolds with corners,* $h:U\rightarrow V$ *is a smooth map,* $\underline{U},\underline{V},\underline{h} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U,V,h)$, *and* u,v *are étale, then there are* 2*-commutative diagrams*

$$\begin{tikzcd} \overline{\underline{C(U)}} \arrow[r, "u_C"] \arrow[d, "\overline{\underline{C(h)}}"] & C(\mathcal{X}) \arrow[d, "C(f)"] \arrow[dl, phantom, "\swArrow\eta_C"] \\ \overline{\underline{C(V)}} \arrow[r, "v_C"] & C(\mathcal{Y}), \end{tikzcd} \qquad \begin{tikzcd} \overline{\underline{C(U)}} \arrow[r, "u_C"] \arrow[d, "\overline{\underline{\hat C(h)}}"] & C(\mathcal{X}) \arrow[d, "\hat C(f)"] \arrow[dl, phantom, "\swArrow\eta_{\hat C}"] \\ \overline{\underline{C(V)}} \arrow[r, "v_C"] & C(\mathcal{Y}), \end{tikzcd} \tag{8.24}$$

where $C(h):C(U)\rightarrow C(V)$ *and* $\hat C(h):C(U)\rightarrow C(V)$ *are defined in* (5.6) *and* (5.8), *and* $\underline{C(U)},\underline{C(V)},\underline{C(h)},\underline{\hat C(h)} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}\bigl(C(U),C(V),C(h),\hat C(h)\bigr)$, *and* $u_C:\overline{\underline{C(U)}}\rightarrow C(\mathcal{X})$ *is defined by* $u_C|_{\overline{\underline{C_k(U)}}} = u_{C_k}:\overline{\underline{C_k(U)}}\rightarrow C_k(\mathcal{X})$ *for* u_{C_k} *as in* (8.21), *and similarly for* v_C.

We can also characterize the maps $C(f)_{\rm top}:C(\mathcal{X})_{\rm top}\rightarrow C(\mathcal{Y})_{\rm top}$, $\hat C(f)_{\rm top}: C(\mathcal{X})_{\rm top}\rightarrow C(\mathcal{Y})_{\rm top}$. *Identify* $C_k(\mathcal{X})_{\rm top}\subseteq C(\mathcal{X})_{\rm top}$ *with the right hand side of* (8.22), *and similarly for* $C(\mathcal{Y})_{\rm top}$, *and identify* $\mathcal{S}_{f,{\rm top}},\mathcal{T}_{f,{\rm top}}$ *with the right hand sides of* (8.13)–(8.14). *Then as in* (5.6) *and* (5.8), $C(f)_{\rm top}$ *and* $\hat C(f)_{\rm top}$ *act by*

$$\begin{aligned} &C(f)_{\rm top}:\bigl[x,\{x'_1,\ldots,x'_k\}\bigr]\longmapsto\bigl[y,\{y'_1,\ldots,y'_l\}\bigr], \quad \text{where } y=f\circ x,\\ &\{y'_1,\ldots,y'_l\}=\bigl\{y':[x'_i,y']\in\mathcal{S}_{f,{\rm top}},\ \text{some } i=1,\ldots,k\bigr\}, \text{ and} \end{aligned} \tag{8.25}$$

$$\begin{aligned} &\hat C(f)_{\rm top}:\bigl[x,\{x'_1,\ldots,x'_k\}\bigr]\longmapsto\bigl[y,\{y'_1,\ldots,y'_l\}\bigr], \quad \text{where } y=f\circ x,\\ &\{y'_1,\ldots,y'_l\}=\bigl\{y':[x'_i,y']\in\mathcal{S}_{f,{\rm top}},\ i=1,\ldots,k\bigr\}\cup\bigl\{y':[x,y']\in\mathcal{T}_{f,{\rm top}}\bigr\}. \end{aligned} \tag{8.26}$$

For $0\leqslant k\leqslant\dim\mathfrak{X}$, $0\leqslant l\leqslant\dim\mathfrak{Y}$ *write* $C_k^{f,l}(\mathfrak{X}) = C_k(\mathfrak{X})\cap C(f)^{-1}(C_l(\mathfrak{Y}))$ *and* $\hat C_k^{f,l}(\mathfrak{X}) = C_k(\mathfrak{X})\cap \hat C(f)^{-1}(C_l(\mathfrak{Y}))$, *so that* $C_k^{f,l}(\mathfrak{X}),\hat C_k^{f,l}(\mathfrak{X})$ *are open and closed suborbifolds of* $C_k(\mathfrak{X})$ *with* $C_k(\mathfrak{X}) = \coprod_{l=0}^{\dim\mathfrak{Y}} C_k^{f,l}(\mathfrak{X}) = \coprod_{l=0}^{\dim\mathfrak{Y}} \hat C_k{}^{f,l}(\mathfrak{X})$, *and write* $C_k^l(f) = C(f)|_{C_k^{f,l}(\mathfrak{X})}$, $\hat C_k^l(f) = \hat C(f)|_{\hat C_k^{f,l}(\mathfrak{X})}$, *so that* $C_k^l(f):C_k^{f,l}(\mathfrak{X})\rightarrow C_l(\mathfrak{Y})$ *and* $\hat C_k^l(f):\hat C_k^{f,l}(\mathfrak{X})\rightarrow C_l(\mathfrak{Y})$ *are* 1*-morphisms in* $\mathbf{Orb^c}$. *If* f *is simple then* $C(f)$ *maps* $C_k(\mathfrak{X})\rightarrow C_k(\mathfrak{Y})$ *for all* $k\geqslant 0$. *If* f *is flat then* $C(f)=\hat C(f)$.

(b) *Let* $f,g:\mathfrak{X}\rightarrow\mathfrak{Y}$ *be* 1*-morphisms and* $\eta:f\Rightarrow g$ *a* 2*-morphism in* $\mathbf{Orb^c}$. *Then there exist unique* 2*-morphisms* $C(\eta):C(f)\Rightarrow C(g)$ *and* $\hat C(\eta):\hat C(f)\Rightarrow \hat C(g)$ *in* $\check{\mathbf{O}}\mathbf{rb^c}$, *where* $C(f),C(g),\hat C(f),\hat C(g)$ *are as in* **(a)**, *such that*

$$\begin{aligned} &\mathrm{id}_{\Pi_\mathfrak{Y}}*C(\eta)=\eta*\mathrm{id}_{\Pi_\mathfrak{X}}:\Pi_\mathfrak{Y}\circ C(f)=f\circ\Pi_\mathfrak{X}\Longrightarrow\Pi_\mathfrak{Y}\circ C(g)=g\circ\Pi_\mathfrak{X},\\ &\mathrm{id}_{\Pi_\mathfrak{Y}}*\hat C(\eta)=\eta*\mathrm{id}_{\Pi_\mathfrak{X}}:\Pi_\mathfrak{Y}\circ\hat C(f)=f\circ\Pi_\mathfrak{X}\Longrightarrow\Pi_\mathfrak{Y}\circ\hat C(g)=g\circ\Pi_\mathfrak{X}. \end{aligned} \tag{8.27}$$

If f,g *are flat then* $C(\eta)=\hat C(\eta)$.

(c) *Define* $C:\mathbf{Orb^c}\rightarrow\check{\mathbf{O}}\mathbf{rb^c}$ *by* $C:\mathfrak{X}\mapsto C(\mathfrak{X})$ *on objects, where* $C(\mathfrak{X})$ *is as in Definition* 8.25, *and* $C:f\mapsto C(f)$, $C:\eta\mapsto C(\eta)$ *on* 1*- and* 2*-morphisms,* *where $C(f), C(\eta)$ are as in* **(a)**,**(b)** *above. Similarly, define $\hat C : \mathbf{Orb^c} \to \check{\mathbf{O}}\mathbf{rb^c}$ by $\hat C : \mathfrak{X} \mapsto C(\mathfrak{X})$, $\hat C : f \mapsto \hat C(f)$, $\hat C : \eta \mapsto \hat C(\eta)$. Then $C, \hat C$ are strict 2-functors, which we call **corner functors**.*

Parts **(a)**–**(c)** *above also hold if $\mathfrak{X}, \mathfrak{Y}$ are objects in $\check{\mathbf{O}}\mathbf{rb^c}$ rather than $\mathbf{Orb^c}$. Thus we get corner functors $C, \hat C : \check{\mathbf{O}}\mathbf{rb^c} \to \check{\mathbf{O}}\mathbf{rb^c}$.*

Proof. For (a), we first show that the 1-morphisms $C(f), \hat C(f)$ exist locally in $C(\mathfrak{X})$. Suppose $[x] \in \mathcal{X}_{\rm top}$ with $f_{\rm top}([x]) = [y] \in \mathcal{Y}_{\rm top}$, and set $G = \mathrm{Iso}_{\mathcal{X}}([x])$, $H = \mathrm{Iso}_{\mathcal{Y}}([y])$, and write $\rho : G \to H$ for $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$. Using Theorem C.25 we can show that there exist manifolds with corners U, V, a smooth map $h : U \to V$, actions of G, H on U, V such that h is equivariant under $\rho : U \to V$, and a 2-commutative diagram

$$\begin{tikzcd} {[\underline{U}/G]} \arrow[rr, "u"'] \arrow[d, "{[\underline{h},\rho]}"] \arrow[drr, phantom, "\Downarrow\eta"] && \mathcal{X} \arrow[d, "f"'] \\ {[\underline{V}/H]} \arrow[rr, "v"] && \mathcal{Y}, \end{tikzcd} \tag{8.28}$$

with $\underline{U}, \underline{V}, \underline{h} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U, V, h)$, in which u, v are equivalences with open neighbourhoods $\mathcal{U} \subseteq \mathcal{X}$ of $[x]$ in $\mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{Y}$ of $[y]$ in $\mathcal{Y}$. Equation (8.28) is a substitute for (8.8), giving an alternative definition of 1-morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{Orb^c}$, just as (8.9) in Remark 8.17(c) is a substitute for (8.6).

Then G, H act on $C(U), C(V)$, and as for u_{C_k} in (8.21), we find there are unique 1-morphisms u_C, v_C in 2-Cartesian diagrams

$$\begin{tikzcd} {[\underline{C(U)}/G]} \arrow[r, "u_C"'] \arrow[d, "{[\underline{\Pi}_U,\mathrm{id}_G]}"] \arrow[dr, phantom, "\overset{\mathrm{id}}{\not\nearrow}"] & C(\mathcal{X}) \arrow[d, "\Pi_{\mathcal{X}}"'] \\ {[\underline{U}/G]} \arrow[r, "u"] & \mathcal{X}, \end{tikzcd} \qquad \begin{tikzcd} {[\underline{C(V)}/H]} \arrow[r, "v_C"'] \arrow[d, "{[\underline{\Pi}_V,\mathrm{id}_H]}"] \arrow[dr, phantom, "\overset{\mathrm{id}}{\not\nearrow}"] & C(\mathcal{Y}) \arrow[d, "\Pi_{\mathcal{Y}}"'] \\ {[\underline{V}/H]} \arrow[r, "v"] & \mathcal{Y}, \end{tikzcd} \tag{8.29}$$

and furthermore u_C, v_C are equivalences with open C^∞-substacks $C(\mathcal{U}) \subseteq C(\mathcal{X})$, $C(\mathcal{V}) \subseteq C(\mathcal{Y})$. By choosing a quasi-inverse for u_C, we see that there exists a 1-morphism $C(f)_{\mathcal{U}}$, unique up to 2-isomorphism, and a 2-morphism ζ to make the following diagram 2-commute:

$$\begin{tikzcd} {[\underline{C(U)}/G]} \arrow[rr, "u_C"'] \arrow[d, "{[\underline{C(h)},\rho]}"] \arrow[drr, phantom, "\Downarrow\zeta"] && C(\mathcal{U}) \subseteq C(\mathcal{X}) \arrow[d, "C(f)_{\mathcal{U}}"'] \\ {[\underline{C(V)}/H]} \arrow[rr, "v_C"] && C(\mathcal{V}) \subseteq C(\mathcal{Y}). \end{tikzcd} \tag{8.30}$$

Combining (8.28)–(8.30) shows that $\Pi_{\mathfrak{Y}} \circ C(f)_{\mathcal{U}} \cong f \circ \Pi_{\mathfrak{X}}|_{C(\mathcal{U})}$. So as $\Pi_{\mathfrak{Y}}$ is strongly representable, Proposition 4.17 shows that we may choose $C(f)_{\mathcal{U}}$ uniquely in its 2-isomorphism class such that $\Pi_{\mathfrak{Y}} \circ C(f)_{\mathcal{U}} = f \circ \Pi_{\mathfrak{X}}|_{C(\mathcal{U})}$. Since $\mathcal{X}$ is covered by such open $\mathcal{U} \subseteq \mathcal{X}$, $C(\mathcal{X})$ is covered by the corresponding $C(\mathcal{U}) \subseteq C(\mathcal{X})$. By uniqueness, these 1-morphisms $C(f)_{\mathcal{U}} : C(\mathcal{U}) \to C(\mathcal{Y})$ agree on overlaps $C(\mathcal{U}) \cap C(\mathcal{U}')$ in $C(\mathcal{X})$. Hence sheafifying gives a 1-morphism $C(f) : C(\mathcal{X}) \to C(\mathcal{Y})$ with $C(f)|_{C(\mathcal{U})} = C(f)_{\mathcal{U}}$ for all such $\mathcal{U}$ above, and applying Proposition 4.17 again shows there is a unique such $C(f)$ with $\Pi_{\mathfrak{Y}} \circ C(f) = f \circ \Pi_{\mathfrak{X}}$. The proof for $\hat C(f)$ is the same.

To prove (8.25)–(8.26), we use (5.6) and (5.8) for the étale local models $h : U \to V$ in $\mathbf{Man^c}$. Part (a) follows. Part (b) is proved by the same argument as for η_- in Theorem 8.22(c). For (c), we may deduce functoriality of $C, \hat{C}$ on $\mathbf{Orb^c}$ from the functoriality of $C, \hat{C}$ on $\mathbf{Man^c}$ by Theorem 5.17(i)–(iii) and its analogue for $\hat{C}$, and uniqueness of $C(f), \hat{C}(f), C(\eta), \hat{C}(\eta)$ in (a),(b). $\square$

Theorem 5.17(iv)–(vii) also extend to orbifolds with corners.

Definitions 5.9 and 5.28 defined (s-)submersions, (s- or sf-)immersions and (s- or sf-)embeddings in $\mathbf{Man^c}$. Definition 8.3 defined submersions, immersions and embeddings in $\mathbf{Orb}$. We combine the two definitions.

Definition 8.28. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a 1-morphism of orbifolds with corners.

- (i) We call f a *submersion* if $\Omega_{C(f)} : C(f)^*(T^*C(\mathcal{Y})) \to T^*C(\mathcal{X})$ is an injective morphism of vector bundles, i.e. has a left inverse in $\mathrm{qcoh}(C(\mathcal{X}))$, and f is semisimple and flat. We call f an *s-submersion* if f is also simple.
- (ii) We call f an *immersion* if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is a surjective morphism of vector bundles, i.e. has a right inverse in $\mathrm{qcoh}(\mathcal{X})$. We call f an *s-immersion* if f is also simple, and an *sf-immersion* if f is also simple and flat.
- (iii) We call f an *embedding, s-embedding*, or *sf-embedding*, if it is an immersion, s-immersion, or sf-immersion, respectively, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\big(f_{\mathrm{top}}([x])\big)$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image (so in particular it is injective).

Then submersions, ..., sf-embeddings in $\mathbf{Orb^c}$ are étale locally modelled on submersions, ..., sf-embeddings in $\mathbf{Man^c}$.

8.8 Transversality and fibre products

Next we extend §5.6 to orbifolds. Here is the analogue of Definitions 5.20 and 5.25.

Definition 8.29. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ be orbifolds with corners and $g : \mathfrak{X} \to \mathfrak{Z}$, $h : \mathfrak{Y} \to \mathfrak{Z}$ be 1-morphisms. Then as in §8.7 we have 1-morphisms $C(g) : C(\mathfrak{X}) \to C(\mathfrak{Z})$ and $C(h) : C(\mathfrak{Y}) \to C(\mathfrak{Z})$ in $\mathbf{\check{O}rb^c}$, and hence 1-morphisms $C(g) : C(\mathcal{X}) \to C(\mathcal{Z})$ and $C(h) : C(\mathcal{Y}) \to C(\mathcal{Z})$ in $\mathbf{C^\infty Sta}$. We call g, h *transverse* if the following holds. Suppose $x : \bar{\underline{*}} \to C(\mathcal{X})$ and $y : \bar{\underline{*}} \to C(\mathcal{Y})$ are 1-morphisms in $\mathbf{C^\infty Sta}$, and $\eta : C(g) \circ x \Rightarrow C(h) \circ y$ a 2-morphism. Then the following morphism in $\mathrm{qcoh}(\bar{\underline{*}})$ should be injective:

$$\begin{aligned}\big(x^*(\Omega_{C(g)}) \circ I_{x,C(g)}(T^*C(\mathcal{Z}))\big) \oplus \big(y^*(\Omega_{C(h)}) \circ I_{y,C(h)}(T^*C(\mathcal{Z})) \circ \eta^*(T^*C(\mathcal{Z}))\big) :\\ (C(g) \circ x)^*(T^*C(\mathcal{Z})) \longrightarrow x^*(T^*C(\mathcal{X})) \oplus y^*(T^*C(\mathcal{Y})). \qquad (8.31)\end{aligned}$$

An equivalent definition is the following: suppose $e : U \to W$ and $f : V \to W$ are smooth maps of manifolds with corners, and $u : \bar{\underline{U}} \to \mathcal{X}$, $v : \bar{\underline{V}} \to \mathcal{Y}$ and

$w : \underline{\bar{W}} \to \mathcal{Z}$ are étale 1-morphisms, fitting into 2-commutative diagrams:

$$\begin{tikzcd} \underline{\bar{U}} \arrow[r, "u"] \arrow[d, "\underline{\bar{e}}"] \arrow[dr, phantom, "\not\Downarrow"] & \mathcal{X} \arrow[d, "g"] \\ \underline{\bar{W}} \arrow[r, "w"] & \mathcal{Z}, \end{tikzcd} \qquad \begin{tikzcd} \underline{\bar{V}} \arrow[r, "v"] \arrow[d, "\underline{\bar{f}}"] \arrow[dr, phantom, "\not\Downarrow"] & \mathcal{Y} \arrow[d, "h"] \\ \underline{\bar{W}} \arrow[r, "w"] & \mathcal{Z}. \end{tikzcd}$$

Then g, h are transverse in $\mathbf{Orb^c}$ if and only if e, f are transverse in $\mathbf{Man^c}$ for all such U, V, W, e, f. That is, g, h are transverse if and only if they are étale locally equivalent to transverse smooth maps in $\mathbf{Man^c}$.

Now identify $C_k(\mathcal{X})_{\rm top} \subseteq C(\mathcal{X})_{\rm top}$ with the right hand of (8.22), and similarly for $C(\mathcal{Y})_{\rm top}, C(\mathcal{Z})_{\rm top}$. Then $C(g)_{\rm top}, C(h)_{\rm top}$ act as in (8.25). We call g, h *strongly transverse* if they are transverse, and whenever there are points in $C_j(\mathcal{X})_{\rm top}, C_k(\mathcal{Y})_{\rm top}, C_l(\mathcal{Z})_{\rm top}$ with

$$C(g)_{\rm top}\bigl([x, \{x'_1, \ldots, x'_j\}]\bigr) = C(h)_{\rm top}\bigl([y, \{y'_1, \ldots, y'_k\}]\bigr) = [z, \{z'_1, \ldots, z'_l\}],$$

we have either $j + k > l$ or $j = k = l = 0$. Again, g, h are strongly transverse if and only if they are étale locally equivalent to strongly transverse smooth maps in $\mathbf{Man^c}$.

Here is the analogue of Theorem 5.21. One can deduce it from Theorem 5.21 using the method of proof of Theorem 11.24 below.

Theorem 8.30. *Suppose $g : \mathfrak{X} \to \mathfrak{Z}$ and $h : \mathfrak{Y} \to \mathfrak{Z}$ are transverse 1-morphisms in $\mathbf{Orb^c}$. Then a fibre product $\mathcal{W} = \mathfrak{X} \times_{g,\mathfrak{Z},h} \mathfrak{Y}$ exists in the 2-category $\mathbf{Orb^c}$.*

Propositions 5.22–5.24 and Theorem 5.26 also extend to $\mathbf{Orb^c}$, with equivalences natural up to 2-isomorphism rather than canonical diffeomorphisms.

8.9 Orbifold strata of orbifolds with corners

In §C.8 we studied *orbifold strata* $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ of a Deligne–Mumford C^∞-stack $\mathcal{X}$ and finite group Γ, and in §8.4.1 we applied these ideas to orbifolds $\mathcal{X}$, showing that the orbifold stratum $\mathcal{X}^\Gamma$ has a decomposition $\mathcal{X}^\Gamma = \coprod_{\lambda \in \Lambda^\Gamma_+} \mathcal{X}^{\Gamma,\lambda}$ with each $\mathcal{X}^{\Gamma,\lambda}$ an orbifold. We now extend these ideas to orbifolds with corners $\mathfrak{X}$, defining orbifold strata $\mathfrak{X}^\Gamma, \tilde{\mathfrak{X}}^\Gamma, \hat{\mathfrak{X}}^\Gamma, \mathfrak{X}^\Gamma_\circ, \tilde{\mathfrak{X}}^\Gamma_\circ, \hat{\mathfrak{X}}^\Gamma_\circ$ in $\mathbf{\check{O}rb^c}$, and refinements $\mathfrak{X}^{\Gamma,\lambda}, \mathfrak{X}^{\Gamma,\lambda}_\circ$ for $\lambda \in \Lambda^\Gamma_+$ and $\tilde{\mathfrak{X}}^{\Gamma,\mu}, \hat{\mathfrak{X}}^{\Gamma,\mu}, \tilde{\mathfrak{X}}^{\Gamma,\mu}_\circ, \hat{\mathfrak{X}}^{\Gamma,\mu}_\circ$ for $\mu \in \Lambda^\Gamma_+/\operatorname{Aut}(\Lambda)$ which are orbifolds with corners of dimensions $\dim \mathfrak{X} - \dim \lambda$, $\dim \mathfrak{X} - \dim \mu$.

In $\mathfrak{X}^\Gamma = \bigl(\mathcal{X}^\Gamma, \partial(\mathcal{X}^\Gamma), i_{\mathfrak{X}^\Gamma}\bigr)$, the underlying C^∞-stack of $\mathfrak{X}^\Gamma$ is the orbifold stratum $\mathcal{X}^\Gamma$ of the C^∞-stack $\mathcal{X}$ in $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$. However (except for *straight* orbifolds with corners $\mathfrak{X}$ below), in general the boundary C^∞-stack $\partial(\mathcal{X}^\Gamma)$ in $\mathfrak{X}^\Gamma$ is not the orbifold stratum $(\partial\mathcal{X})^\Gamma$ of the boundary C^∞-stack $\partial\mathcal{X}$ in $\mathfrak{X}$, but is a certain open and closed C^∞-substack of the orbifold stratum $C(\mathfrak{X})^\Gamma$ of the corners $C(\mathfrak{X})$ from §8.7. The motivation for the definition of $\partial(\mathcal{X}^\Gamma)$ is the description (5.10) in Proposition 5.18(c) of the boundary $\partial(X^\Gamma)$ of the fixed point locus X^Γ of a finite group Γ acting on a manifold with corners X.

Definition 8.31. Let $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ be an orbifold with corners, and Γ a finite group. Then §8.7 defines the corners $C(\mathfrak{X})$ in $\check{\mathbf{O}}\mathbf{rb^c}$, and a projection $\Pi_{\mathfrak{X}} : C(\mathfrak{X}) \to \mathfrak{X}$ in $\check{\mathbf{O}}\mathbf{rb^c}$. As a 1-morphism of the underlying Deligne–Mumford C^∞-stacks, $\Pi_{\mathfrak{X}} : C(\mathcal{X}) \to \mathcal{X}$ is strongly representable. Thus Definitions C.45 and C.51 define orbifold strata $\mathcal{X}^\Gamma, C(\mathcal{X})^\Gamma$ of $\mathcal{X}, C(\mathcal{X})$, which are Deligne–Mumford C^∞-stacks, 1-morphisms $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$, $O^\Gamma(C(\mathcal{X})) : C(\mathcal{X})^\Gamma \to C(\mathcal{X})$ which are strongly representable by Theorem C.49(f), and a 1-morphism $\Pi^\Gamma_{\mathfrak{X}} : C(\mathcal{X})^\Gamma \to \mathcal{X}^\Gamma$ with $O^\Gamma(\mathcal{X}) \circ \Pi^\Gamma_{\mathfrak{X}} = \Pi_{\mathfrak{X}} \circ O^\Gamma(C(\mathcal{X}))$. Since $\Pi_{\mathfrak{X}}, O^\Gamma(\mathcal{X}), O^\Gamma(C(\mathcal{X}))$ are strongly representable, it follows that $\Pi^\Gamma_{\mathfrak{X}}$ is too.

Combining the description (8.22) of points in $C(\mathcal{X})_{\rm top}$ with Theorem C.49(c) shows that we may write the topological space $C(\mathcal{X})^\Gamma_{\rm top}$ of $C(\mathcal{X})^\Gamma$ explicitly as

$$\begin{aligned} C(\mathcal{X})^\Gamma_{\rm top} \cong \bigl\{ & [x, \{x'_1, \ldots, x'_k\}, \rho, \sigma] : k \geqslant 0,\ x : \underline{\bar{*}} \to \mathcal{X} \text{ and } x'_i : \underline{\bar{*}} \to \partial\mathcal{X} \\ & \text{are 1-morphisms, } x'_1, \ldots, x'_k \text{ are distinct, } x = i_{\mathfrak{X}} \circ x'_1 = \cdots = i_{\mathfrak{X}} \circ x'_k, \\ & \sigma : \Gamma \to S_k \text{ is a group morphism, } \rho(\gamma) = (\eta(\gamma), \eta'_1(\gamma), \ldots, \eta'_k(\gamma)) \\ & \text{for } \gamma \in \Gamma \text{ with } \eta(\gamma) : x \Rightarrow x \text{ and } \eta'_i(\gamma) : x'_i \Rightarrow x'_{\sigma(i)} \text{ 2-morphisms,} \\ & \eta(\gamma) * \eta(\delta) = \eta(\gamma\delta),\ \eta'_{\sigma(\delta)(i)}(\gamma) * \eta'_i(\delta) = \eta'_i(\gamma\delta),\ \gamma, \delta \in \Gamma,\ i = 1, \ldots, k \bigr\}. \end{aligned} \tag{8.32}$$

Write $\partial(\mathcal{X}^\Gamma)_{\rm top} \subseteq C(\mathcal{X})^\Gamma_{\rm top}$ for the subset corresponding to $[x, \{x'_1, \ldots, x'_k\}, \rho, \sigma]$ in (8.32) such that $k \geqslant 1$ and $\sigma(\Gamma) \subseteq S_k$ acts transitively on $\{1, \ldots, k\}$ (these conditions are the analogue of (5.10)). We will show $\partial(\mathcal{X}^\Gamma)_{\rm top}$ is open and closed in $C(\mathcal{X})^\Gamma_{\rm top}$, so it induces an open and closed C^∞-substack $\partial(\mathcal{X}^\Gamma)$ in $C(\mathcal{X})^\Gamma$.

Define a 1-morphism $i_{\mathfrak{X}^\Gamma} : \partial(\mathcal{X}^\Gamma) \to \mathcal{X}^\Gamma$ by $i_{\mathfrak{X}^\Gamma} = \Pi^\Gamma_{\mathfrak{X}}|_{\partial(\mathcal{X}^\Gamma)}$. Then $i_{\mathfrak{X}^\Gamma}$ is strongly representable by Proposition C.14(c), as $\Pi^\Gamma_{\mathfrak{X}}$ is strongly representable and $\partial(\mathcal{X}^\Gamma) \subset C(\mathcal{X})^\Gamma$ is open. Define $\mathfrak{X}^\Gamma = \bigl(\mathcal{X}^\Gamma, \partial(\mathcal{X}^\Gamma), i_{\mathfrak{X}^\Gamma}\bigr)$. We claim that $\mathfrak{X}^\Gamma$ is an object in $\check{\mathbf{O}}\mathbf{rb^c}$. To prove this, note that $\mathcal{X}$ is covered by Zariski open C^∞-substacks $\mathcal{V}$ equivalent to $[\underline{U}/G]$ for U a manifold with corners acted on by a finite group G and $\underline{U} = F^{\mathbf{C^\infty Sch}}_{\mathbf{Man^c}}(U)$. Then $C(\mathcal{X})$ is covered by corresponding $C(\mathcal{V}) \simeq [\underline{C(U)}/G]$. Thus by equation (C.10) of Theorem C.53 we have

$$\begin{aligned} \mathcal{U}^\Gamma \simeq [\underline{U}/G]^\Gamma &\simeq \bigl[\bigl(\textstyle\coprod_{\text{injective morphisms } \rho : \Gamma \to G} \underline{U}^{\rho(\Gamma)}\bigr)/G\bigr], \\ C(\mathcal{U})^\Gamma \simeq [\underline{C(U)}/G]^\Gamma &\simeq \bigl[\bigl(\textstyle\coprod_{\text{injective morphisms } \rho : \Gamma \to G} \underline{C(U)}^{\rho(\Gamma)}\bigr)/G\bigr] \\ &\simeq \bigl[\bigl(\textstyle\coprod_{\text{injective morphisms } \rho : \Gamma \to G} \underline{C(U^{\rho(\Gamma)})}\bigr)/G\bigr], \end{aligned}$$

using Proposition 5.18(b) to identify $C(U)^{\rho(\Gamma)}$ and $C(U^{\rho(\Gamma)})$ in the last step.

Comparing (5.10) and (8.32) shows that the subset $\partial(\mathcal{X}^\Gamma)_{\rm top} \cap C(\mathcal{U})^\Gamma_{\rm top}$ in $C(\mathcal{U})^\Gamma_{\rm top}$ is identified with $\bigl[\bigl(\coprod_\rho \underline{C_1(U^{\rho(\Gamma)})}\bigr)/G\bigr]_{\rm top}$ in $\bigl[\bigl(\coprod_\rho \underline{C(U^{\rho(\Gamma)})}\bigr)/G\bigr]_{\rm top}$. Since $C_1(U^{\rho(\Gamma)})$ is open and closed in $C(U^{\rho(\Gamma)})$, this shows that $\partial(\mathcal{X}^\Gamma)_{\rm top} \cap C(\mathcal{U})^\Gamma_{\rm top}$ is open and closed in $C(\mathcal{U})^\Gamma_{\rm top}$, so $\partial(\mathcal{X}^\Gamma)_{\rm top}$ is open and closed in $C(\mathcal{X})^\Gamma_{\rm top}$ as we claimed, since such $C(\mathcal{U})^\Gamma_{\rm top}$ form an open cover of $C(\mathcal{X})^\Gamma_{\rm top}$. Also, as $C_1(U^{\rho(\Gamma)}) \cong \partial(U^{\rho(\Gamma)})$, this shows that

$$\begin{aligned} \mathcal{U}^\Gamma &\simeq \bigl[\bigl(\textstyle\coprod_{\text{injective morphisms } \rho : \Gamma \to G} \underline{U}^{\rho(\Gamma)}\bigr)/G\bigr], \\ \partial(\mathcal{X}^\Gamma) \cap C(\mathcal{U})^\Gamma &\simeq \bigl[\bigl(\textstyle\coprod_{\text{injective morphisms } \rho : \Gamma \to G} \underline{\partial(U^{\rho(\Gamma)})}\bigr)/G\bigr]. \end{aligned}$$

Thus we see that the triple $\big(\mathcal{X}^\Gamma, \partial(\mathcal{X}^\Gamma), i_{\mathfrak{X}^\Gamma}\big)$ is Zariski locally modelled on triples $\big([\underline{U^{\rho(\Gamma)}})/G], [\underline{\partial(U^{\rho(\Gamma)})}/G], [\underline{i}_{U^{\rho(\Gamma)}}, \mathrm{id}_G]\big)$, which is an orbifold with corners as $U^{\rho(\Gamma)}$ is a manifold with corners by Proposition 5.18(a). Since the dimension of $U^{\rho(\Gamma)}$ can vary with connected component of $\mathcal{X}^\Gamma$, this shows that $\mathfrak{X}^\Gamma$ is a disjoint union of orbifolds with corners of different dimensions, that is, an object in $\check{\mathbf{O}}\mathbf{rb^c}$.

Use the notion $\Lambda^\Gamma, \Lambda^\Gamma_+, R_0, R_1, \ldots, R_k$ of Definition 8.5. As $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}})$ is an orbifold with corners, the cotangent bundle $T^*\mathcal{X}$ is a vector bundle on $\mathcal{X}$, so $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X})$ is a vector bundle on $\mathcal{X}^\Gamma$. As in Definition 8.5, we have a splitting $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) \cong \bigoplus_{i=0}^k (T^*\mathcal{X})_i^\Gamma \otimes R_i$, where $(T^*\mathcal{X})_i^\Gamma$ for $i = 0, \ldots, k$ are vector bundles of mixed rank on $\mathcal{X}^\Gamma$. For each $\lambda \in \Lambda^\Gamma_+$, define $\mathcal{X}^{\Gamma,\lambda}$ to be the open and closed C^∞-substack in $\mathcal{X}^\Gamma$ with $\mathrm{rank}\big((T^*\mathcal{X})_1^\Gamma\big)[R_1] + \cdots + \mathrm{rank}\big((T^*\mathcal{X})_k^\Gamma\big)[R_k] = \lambda$ in Λ^Γ_+. Then $\mathcal{X}^\Gamma = \coprod_{\lambda \in \Lambda^\Gamma_+} \mathcal{X}^{\Gamma,\lambda}$. As in §C.8, $\mathcal{X}^\Gamma_\circ \subseteq \mathcal{X}^\Gamma$ is an open C^∞-substack. Set $\mathcal{X}^{\Gamma,\lambda}_\circ = \mathcal{X}^{\Gamma,\lambda} \cap \mathcal{X}^\Gamma_\circ$ for $\lambda \in \Lambda^\Gamma_+$, so that $\mathcal{X}^{\Gamma,\lambda}_\circ$ is an open and closed C^∞-substack of $\mathcal{X}^\Gamma_\circ$ with $\mathcal{X}^\Gamma_\circ = \coprod_{\lambda \in \Lambda^\Gamma_+} \mathcal{X}^{\Gamma,\lambda}_\circ$.

Define $\mathfrak{X}^\Gamma_\circ, \mathfrak{X}^{\Gamma,\lambda}, \mathfrak{X}^{\Gamma,\lambda}_\circ$ to be the open subobjects of $\mathfrak{X}^\Gamma$ with C^∞-stacks $\mathcal{X}^\Gamma_\circ$, $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}^{\Gamma,\lambda}_\circ \subseteq \mathcal{X}^\Gamma$, so that for instance $\mathfrak{X}^\Gamma_\circ = \big(\mathcal{X}^\Gamma_\circ, \partial(\mathcal{X}^\Gamma_\circ), i_{\mathfrak{X}^\Gamma_\circ}\big)$ where $\partial(\mathcal{X}^\Gamma_\circ) = i_{\mathfrak{X}^\Gamma}^{-1}(\mathcal{X}^\Gamma_\circ)$ and $i_{\mathfrak{X}^\Gamma_\circ} = i_{\mathfrak{X}^\Gamma}|_{\partial(\mathcal{X}^\Gamma_\circ)}$. Then $\mathfrak{X}^\Gamma_\circ, \mathfrak{X}^{\Gamma,\lambda}, \mathfrak{X}^{\Gamma,\lambda}_\circ$ are objects in $\check{\mathbf{O}}\mathbf{rb^c}$. As in Definition 8.5, $T^*\mathfrak{X}^{\Gamma,\lambda}$ and $T^*\mathfrak{X}^{\Gamma,\lambda}_\circ$ are vector bundles of rank $\dim \mathfrak{X} - \dim \lambda$, so $\mathfrak{X}^{\Gamma,\lambda}, \mathfrak{X}^{\Gamma,\lambda}_\circ$ are orbifolds with corners of dimension $\dim \mathfrak{X} - \dim \lambda$. We have

$$\mathfrak{X}^\Gamma = \textstyle\coprod_{\lambda \in \Lambda^\Gamma_+} \mathfrak{X}^{\Gamma,\lambda} \quad \text{and} \quad \mathfrak{X}^\Gamma_\circ = \textstyle\coprod_{\lambda \in \Lambda^\Gamma_+} \mathfrak{X}^{\Gamma,\lambda}_\circ$$

In a similar way, using the other classes of orbifold strata $\tilde{\mathcal{X}}^\Gamma, \tilde{\mathcal{X}}^\Gamma_\circ, \hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma_\circ$ in §C.8, we may define objects $\tilde{\mathfrak{X}}^\Gamma, \tilde{\mathfrak{X}}^\Gamma_\circ, \hat{\mathfrak{X}}^\Gamma, \hat{\mathfrak{X}}^\Gamma_\circ$ in $\check{\mathbf{O}}\mathbf{rb^c}$, and following Definition 8.5 for orbifolds without boundary, for each $\mu \in \Lambda^\Gamma_+ / \mathrm{Aut}(\Lambda)$ we may define orbifolds with corners $\tilde{\mathfrak{X}}^{\Gamma,\mu}, \tilde{\mathfrak{X}}^{\Gamma,\mu}_\circ, \hat{\mathfrak{X}}^{\Gamma,\mu}, \hat{\mathfrak{X}}^{\Gamma,\mu}_\circ$ of dimension $\dim \mathfrak{X} - \dim \mu$, such that as in (8.3)–(8.4) we have $\tilde{\mathfrak{X}}^\Gamma = \coprod_{\mu \in \Lambda^\Gamma_+ / \mathrm{Aut}(\Gamma)} \tilde{\mathfrak{X}}^{\Gamma,\mu}$, and similarly for $\tilde{\mathcal{X}}^\Gamma_\circ, \hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma_\circ$.

For $\tilde{\mathfrak{X}}^\Gamma, \tilde{\mathfrak{X}}^\Gamma_\circ$, the definitions are essentially the same as for $\mathfrak{X}^\Gamma, \mathfrak{X}^\Gamma_\circ$. In particular, using $\tilde{O}^\Gamma(\mathcal{X})$ strongly representable by Theorem C.49(f), we show $\tilde{\Pi}^\Gamma_{\mathfrak{X}}$ is strongly representable as for $\Pi^\Gamma_{\mathfrak{X}}$. However, $\hat{\Pi}^\Gamma_{\mathfrak{X}} : \widehat{C(\mathcal{X})}^\Gamma \to \hat{\mathcal{X}}^\Gamma$ need not be strongly representable, though it is representable. So in $\hat{\mathfrak{X}}^\Gamma = \big(\hat{\mathcal{X}}^\Gamma, \partial(\widehat{\mathcal{X}^\Gamma}), i_{\hat{\mathfrak{X}}^\Gamma}\big)$, we should not define $\partial(\widehat{\mathcal{X}^\Gamma})$ as a C^∞-substack of $\widehat{C(\mathcal{X})}^\Gamma$ and $i_{\hat{\mathfrak{X}}^\Gamma}$ as the restriction of $\hat{\Pi}^\Gamma_{\mathfrak{X}}$, since then $i_{\hat{\mathfrak{X}}^\Gamma}$ might not be strongly representable. Instead, as in the definition of $C_k(\mathfrak{X})$ in §8.7, we use Proposition C.14(b) to replace $\hat{\Pi}^\Gamma_{\mathfrak{X}}$ by a strongly representable 1-morphism $\hat{\Pi}'^\Gamma_{\mathfrak{X}} : \widehat{C(\mathcal{X})}'^\Gamma \to \hat{\mathcal{X}}^\Gamma$ with an equivalence $i : \widehat{C(\mathcal{X})}^\Gamma \to \widehat{C(\mathcal{X})}'^\Gamma$ such that $\hat{\Pi}'^\Gamma_{\mathfrak{X}} \circ i = \hat{\Pi}^\Gamma_{\mathfrak{X}}$, and then define $\partial(\widehat{\mathcal{X}^\Gamma})$ as an open and closed C^∞-substack of $\widehat{C(\mathcal{X})}'^\Gamma$.

Since the C^∞-stack $\hat{\mathcal{X}}^\Gamma_\circ$ is a C^∞-scheme (it has trivial orbifold groups), the $\hat{\mathfrak{X}}^{\Gamma,\mu}_\circ$ are manifolds with corners (they are equivalent in $\mathbf{Orb^c}$ to something in the image of $F^{\mathbf{Orb^c}}_{\mathbf{Man^c}}$), so there exists $\hat{X}^{\Gamma,\mu}_\circ$ in $\mathbf{Man^c}$ unique up to isomorphism with $\hat{\mathfrak{X}}^{\Gamma,\mu}_\circ \simeq F^{\mathbf{Orb^c}}_{\mathbf{Man^c}}(\hat{X}^{\Gamma,\mu}_\circ)$, giving a homeomorphism $\hat{\mathfrak{X}}^{\Gamma,\mu}_{\circ,\mathrm{top}} \cong \hat{X}^{\Gamma,\mu}_\circ$. Thus, (C.7) gives a decomposition

$$\mathscr{X}_{\mathrm{top}} \cong \coprod_{\substack{\text{isomorphism classes}\\ \text{of finite groups } \Gamma}} \coprod_{\mu \in \Lambda_+^\Gamma / \operatorname{Aut}(\Gamma)} \hat{X}_\circ^{\Gamma,\mu},$$

a stratification of the topological space $\mathscr{X}_{\mathrm{top}}$ of an orbifold with corners $\mathcal{X}$ into manifolds with corners $\hat{X}_\circ^{\Gamma,\mu}$. All of $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma, \mathcal{X}_\circ^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma, \mathcal{X}^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \mathcal{X}_\circ^{\Gamma,\lambda}, \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}, \hat{X}_\circ^{\Gamma,\mu}$ will be called *orbifold strata* of $\mathcal{X}$.

The definitions of $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \ldots, \hat{\mathcal{X}}_\circ^\Gamma$ also make sense if $\mathcal{X}$ lies in $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$ rather than $\mathbf{Orb}^{\mathbf{c}}$. We will not use notation $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{X}_\circ^{\Gamma,\mu}$ for $\mathcal{X} \in \check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}} \setminus \mathbf{Orb}^{\mathbf{c}}$.

In Definitions C.47, C.48 and C.51 we defined classes of 1- and 2-morphisms between orbifold strata, for instance, if $\mathscr{X}$ is a Deligne–Mumford C^∞-stack we defined 1-morphisms $O^\Gamma(\mathscr{X}) : \mathscr{X}^\Gamma \to \mathscr{X}$ and $\tilde{\Pi}^\Gamma(\mathscr{X}) : \mathscr{X}^\Gamma \to \tilde{\mathscr{X}}^\Gamma$, and if $f : \mathscr{X} \to \mathscr{Y}$ is a representable 1-morphism we defined 1-morphisms $f^\Gamma : \mathscr{X}^\Gamma \to \mathscr{Y}^\Gamma$ and $\tilde{f}^\Gamma : \tilde{\mathscr{X}}^\Gamma \to \tilde{\mathscr{Y}}^\Gamma$. When $\mathscr{X}, \mathscr{Y}$ are the C^∞-stacks of orbifolds with corners $\mathcal{X}, \mathcal{Y}$, and f is a 1-morphism in $\mathbf{Orb}^{\mathbf{c}}$, all these 1- and 2-morphisms are also 1- and 2-morphisms in $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$, and fit into a strictly commutative diagram in $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$:

$$\begin{tikzcd}
\mathcal{X}_\circ^\Gamma \arrow[loop left, "\operatorname{Aut}(\Gamma)"] \arrow[rr, "\tilde{\Pi}_\circ^\Gamma(\mathcal{X})"] \arrow[dr, "O_\circ^\Gamma(\mathcal{X})"] \arrow[dd, "\subset"'] & & \tilde{\mathcal{X}}_\circ^\Gamma \arrow[r, "\hat{\Pi}_\circ^\Gamma(\mathcal{X})"] \arrow[dl, "\tilde{O}_\circ^\Gamma(\mathcal{X})"'] \arrow[dd, "\subset"] & \hat{\mathcal{X}}_\circ^\Gamma \arrow[dd, "\subset"] \\
& \mathcal{X} & & \\
\mathcal{X}^\Gamma \arrow[loop left, "\operatorname{Aut}(\Gamma)"] \arrow[rr, "\tilde{\Pi}^\Gamma(\mathcal{X})"'] \arrow[ur, "O^\Gamma(\mathcal{X})"] & & \tilde{\mathcal{X}}^\Gamma \arrow[r, "\hat{\Pi}^\Gamma(\mathcal{X})"'] \arrow[ul, "\tilde{O}^\Gamma(\mathcal{X})"'] & \hat{\mathcal{X}}^\Gamma.
\end{tikzcd}$$

To prove this, we show that they are étale locally modelled on smooth maps in $\mathbf{Man}^{\mathbf{c}}$, which in most cases follows from Proposition 5.18(a),(d).

Thus, for example, if $f : \mathcal{X} \to \mathcal{Y}$ is a representable 1-morphism in $\mathbf{Orb}^{\mathbf{c}}$ then $f^\Gamma : \mathscr{X}^\Gamma \to \mathscr{Y}^\Gamma$ and $\tilde{f}^\Gamma : \tilde{\mathscr{X}}^\Gamma \to \tilde{\mathscr{Y}}^\Gamma$ in Definition C.51 are also 1-morphisms $f^\Gamma : \mathcal{X}^\Gamma \to \mathcal{Y}^\Gamma$ and $\tilde{f}^\Gamma : \tilde{\mathcal{X}}^\Gamma \to \tilde{\mathcal{Y}}^\Gamma$ in $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$. Note however that f^Γ need not map $\mathcal{X}^{\Gamma,\lambda} \to \mathcal{Y}^{\Gamma,\lambda}$ for $\lambda \in \Lambda_+^\Gamma$, and $\tilde{f}^\Gamma$ need not map $\tilde{\mathcal{X}}^{\Gamma,\mu} \to \tilde{\mathcal{Y}}^{\Gamma,\mu}$ for $\mu \in \Lambda_+^\Gamma / \operatorname{Aut}(\Gamma)$, unless f is an equivalence. As in Definition C.51, it follows that we may define strict 2-functors $F^\Gamma, \tilde{F}^\Gamma : \mathbf{Orb}^{\mathbf{c}}_{\mathbf{re}} \to \check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}_{\mathbf{re}}$ and a weak 2-functor $\hat{F}^\Gamma : \mathbf{Orb}^{\mathbf{c}}_{\mathbf{re}} \to \check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}_{\mathbf{re}}$, where $\mathbf{Orb}^{\mathbf{c}}_{\mathbf{re}}, \check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}_{\mathbf{re}}$ are the 2-subcategories of $\mathbf{Orb}^{\mathbf{c}}, \check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$ with representable 1-morphisms.

Our next theorem says, roughly, that the corner functor C of §8.7 commutes with the orbifold strata functors in $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$. It is related to Proposition 5.18, which roughly says that C commutes with fixed point loci in $\check{\mathbf{M}}\mathbf{an}^{\mathbf{c}}$.

Theorem 8.32. *Let $\mathcal{X}$ be an orbifold with corners, and Γ a finite group. The corners $C(\mathcal{X})$ lie in $\check{\mathbf{O}}\mathbf{rb}^{\mathbf{c}}$ as in §8.7, so we have orbifold strata $\mathcal{X}^\Gamma, C(\mathcal{X})^\Gamma$ and 1-morphisms $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$, $O^\Gamma(C(\mathcal{X})) : C(\mathcal{X})^\Gamma \to C(\mathcal{X})$. Applying the corner functor C from §8.7 gives a 1-morphism $C(O^\Gamma(\mathcal{X})) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})$. Then there exists a unique equivalence $K^\Gamma(\mathcal{X}) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})^\Gamma$ such that $O^\Gamma(C(\mathcal{X})) \circ K^\Gamma(\mathcal{X}) = C(O^\Gamma(\mathcal{X})) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})$. It restricts to an equivalence $K_\circ^\Gamma(\mathcal{X}) := K^\Gamma(\mathcal{X})|_{C(\mathcal{X}_\circ^\Gamma)} : C(\mathcal{X}_\circ^\Gamma) \to C(\mathcal{X})_\circ^\Gamma$.*

Similarly, there is a unique equivalence $\tilde K^\Gamma(\mathfrak{X}) : C(\tilde{\mathfrak{X}}^\Gamma) \to \widetilde{C(\mathfrak{X})}^\Gamma$ *with* $\tilde O^\Gamma(C(\mathfrak{X}))\circ \tilde K^\Gamma(\mathfrak{X}) = C(\tilde O^\Gamma(\mathfrak{X}))$ *and* $\tilde\Pi^\Gamma(C(\mathfrak{X}))\circ K^\Gamma(\mathfrak{X}) = \tilde K^\Gamma(\mathfrak{X})\circ C(\tilde\Pi^\Gamma(\mathfrak{X}))$. *There is an equivalence* $\hat K^\Gamma(\mathfrak{X}) : C(\hat{\mathfrak{X}}^\Gamma)\to \widehat{C(\mathfrak{X})}^\Gamma$, *unique up to* 2*-isomorphism, with a* 2*-morphism* $\hat\Pi^\Gamma(C(\mathfrak{X}))\circ \tilde K^\Gamma(\mathfrak{X}) \Rightarrow \hat K^\Gamma(\mathfrak{X})\circ C(\hat\Pi^\Gamma(\mathfrak{X}))$. *They both restrict to equivalences* $\tilde K^\Gamma_\circ(\mathfrak{X}) : C(\tilde{\mathfrak{X}}^\Gamma_\circ) \to \widetilde{C(\mathfrak{X})}^\Gamma_\circ$ *and* $\hat K^\Gamma_\circ(\mathfrak{X}) : C(\hat{\mathfrak{X}}^\Gamma_\circ)\to \widehat{C(\mathfrak{X})}^\Gamma_\circ$.

Proof. First consider the case in which $\mathfrak{X}$ is a quotient $[X/G]$ as in Example 8.16, for X a manifold with corners and G a finite group acting on X. Then $C(\mathfrak{X}) \simeq [C(X)/G]$ as in Example 8.26. We now have equivalences

$$\begin{aligned} C(\mathfrak{X}^\Gamma) &\simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \\ \text{of injective } \rho:\Gamma\to G}} C\bigl([X^{\rho(\Gamma)}/C_G(\rho(\Gamma))]\bigr) \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \\ \text{of injective } \rho:\Gamma\to G}} \bigl[C(X^{\rho(\Gamma)})/C_G(\rho(\Gamma))\bigr] \\ &\simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \\ \text{of injective } \rho:\Gamma\to G}} \bigl[C(X)^{\rho(\Gamma)}/C_G(\rho(\Gamma))\bigr] \simeq C(\mathfrak{X})^\Gamma, \end{aligned}$$

where $C_G(\rho(\Gamma))$ is the centralizer of $\rho(\Gamma)$ in G. Here in the first and fourth steps we use equation (C.14) of Theorem C.53 transferred to $\check{\mathbf{O}}\mathbf{rb^c}$, in the second $C([X/G]) \simeq [C(X)/G]$ as in Example 8.26, in the third $C(X^\Gamma)\cong C(X)^\Gamma$ by Proposition 5.18(b). Thus there exists an equivalence $K^\Gamma(\mathfrak{X}) : C(\mathfrak{X}^\Gamma)\to C(\mathfrak{X})^\Gamma$ in $\check{\mathbf{O}}\mathbf{rb^c}$, unique up to 2-isomorphism.

Using the formula for $O^\Gamma(\mathfrak{X})$ in the representation (C.14) as in the last part of Theorem C.53, we see that there is a 2-morphism $O^\Gamma(C(\mathfrak{X}))\circ K^\Gamma(\mathfrak{X}) \Rightarrow C(O^\Gamma(\mathfrak{X}))$. But $O^\Gamma(C(\mathfrak{X}))$ is strongly representable by Theorem C.49(f). Hence Proposition C.13 shows that we can choose $K^\Gamma(\mathfrak{X})$ uniquely in its 2-isomorphism class such that $O^\Gamma(C(\mathfrak{X}))\circ K^\Gamma(\mathfrak{X}) = C(O^\Gamma(\mathfrak{X}))$.

In the same way, using equations (C.16), (C.18) we find equivalences $\tilde K^\Gamma(\mathfrak{X}) : C(\tilde{\mathfrak{X}}^\Gamma)\to \widetilde{C(\mathfrak{X})}^\Gamma$ and $\hat K^\Gamma(\mathfrak{X}) : C(\hat{\mathfrak{X}}^\Gamma) \to \widehat{C(\mathfrak{X})}^\Gamma$, unique up to 2-isomorphism, with 2-morphisms $\tilde O^\Gamma(C(\mathfrak{X}))\circ \tilde K^\Gamma(\mathfrak{X}) \Rightarrow C(\tilde O^\Gamma(\mathfrak{X}))$, $\tilde \Pi^\Gamma(C(\mathfrak{X}))\circ K^\Gamma(\mathfrak{X}) \Rightarrow \tilde K^\Gamma(\mathfrak{X})\circ C(\tilde\Pi^\Gamma(\mathfrak{X}))$ and $\hat\Pi^\Gamma(C(\mathfrak{X}))\circ\tilde K^\Gamma(\mathfrak{X}) \Rightarrow \hat K^\Gamma(\mathfrak{X})\circ C(\hat\Pi^\Gamma(\mathfrak{X}))$. As $\tilde O^\Gamma(C(\mathfrak{X}))$ is strongly representable by Theorem C.49(f), we can choose $\tilde K^\Gamma(\mathfrak{X})$ uniquely with $\tilde O^\Gamma(C(\mathfrak{X}))\circ \tilde K^\Gamma(\mathfrak{X}) = C(\tilde O^\Gamma(\mathfrak{X}))$ by Proposition C.13.

We have now chosen $K^\Gamma(\mathfrak{X}), \tilde K^\Gamma(\mathfrak{X})$ uniquely such that $O^\Gamma(C(\mathfrak{X}))\circ K^\Gamma(\mathfrak{X}) = C(O^\Gamma(\mathfrak{X}))$ and $\tilde O^\Gamma(C(\mathfrak{X}))\circ \tilde K^\Gamma(\mathfrak{X}) = C(\tilde O^\Gamma(\mathfrak{X}))$, and there is a 2-morphism $\zeta : \tilde\Pi^\Gamma(C(\mathfrak{X}))\circ K^\Gamma(\mathfrak{X}) \Rightarrow \tilde K^\Gamma(\mathfrak{X})\circ C(\tilde\Pi^\Gamma(\mathfrak{X}))$. Consider the diagram

$$\begin{tikzcd} \tilde O^\Gamma(C(\mathfrak{X}))\circ\tilde\Pi^\Gamma(C(\mathfrak{X}))\circ K^\Gamma(\mathfrak{X}) \arrow[r, equal] \arrow[d, Rightarrow, "\mathrm{id}_{\tilde O^\Gamma(C(\mathfrak{X}))} * \zeta"] & O^\Gamma(C(\mathfrak{X}))\circ K^\Gamma(\mathfrak{X}) \arrow[r, equal] & C(O^\Gamma(\mathfrak{X})) \arrow[d, equal] \\ \tilde O^\Gamma(C(\mathfrak{X}))\circ\tilde K^\Gamma(\mathfrak{X})\circ C(\tilde\Pi^\Gamma(\mathfrak{X})) \arrow[r, equal] & C(\tilde O^\Gamma(\mathfrak{X}))\circ C(\tilde\Pi^\Gamma(\mathfrak{X})) \arrow[r, equal] & C(\tilde O^\Gamma(\mathfrak{X})\circ\tilde\Pi^\Gamma(\mathfrak{X})), \end{tikzcd}$$

using $O^\Gamma(\mathfrak{X}) = \tilde O^\Gamma(\mathfrak{X})\circ\tilde\Pi^\Gamma(\mathfrak{X})$ by Definition C.47 and C a strict 2-functor by Theorem 8.27. Since $\tilde O^\Gamma(C(\mathfrak{X}))$ is strongly representable, Proposition C.13 now implies that $\tilde\Pi^\Gamma(C(\mathfrak{X}))\circ K^\Gamma(\mathfrak{X}) = \tilde K^\Gamma(\mathfrak{X})\circ C(\tilde\Pi^\Gamma(\mathfrak{X}))$. It is also easy to check that $K^\Gamma(\mathfrak{X}), \tilde K^\Gamma(\mathfrak{X}), \hat K^\Gamma(\mathfrak{X})$ map $C(\mathfrak{X}^\Gamma_\circ), C(\tilde{\mathfrak{X}}^\Gamma_\circ), C(\hat{\mathfrak{X}}^\Gamma_\circ)$ to $C(\mathfrak{X})^\Gamma_\circ, \widetilde{C(\mathfrak{X})}^\Gamma_\circ, \widehat{C(\mathfrak{X})}^\Gamma_\circ$, and so restrict to equivalences $K_\circ^\Gamma(\mathcal{X}), \tilde{K}_\circ^\Gamma(\mathcal{X}), \hat{K}_\circ^\Gamma(\mathcal{X})$ as claimed. This proves the theorem in the case that $\mathcal{X} = [X/G]$.

For the general case, if $\mathcal{X}$ is an orbifold with corners, we can cover $\mathcal{X}$ by open suborbifolds $\mathcal{U}$ equivalent to $[U/G]$ for some manifold with corners U and finite group G. Then we have $C(\mathcal{U}^\Gamma) \simeq C([U/G]^\Gamma) \simeq C([U/G])^\Gamma \simeq C(\mathcal{U})^\Gamma$, so there exists an equivalence $K^\Gamma(\mathcal{U}) : C(\mathcal{U}^\Gamma) \to C(\mathcal{U})^\Gamma$. Using $O^\Gamma(C([U/G])) \circ K^\Gamma([U/G]) = C(O^\Gamma([U/G]))$, we see there is a 2-morphism $O^\Gamma(C(\mathcal{U})) \circ K^\Gamma(\mathcal{U}) \Rightarrow C(O^\Gamma(\mathcal{U}))$. Thus as above we can choose $K^\Gamma(\mathcal{U})$ uniquely in its 2-isomorphism class so that $O^\Gamma(C(\mathcal{U})) \circ K^\Gamma(\mathcal{U}) = C(O^\Gamma(\mathcal{U}))$.

Therefore we can cover $\mathcal{X}$ by open $\mathcal{U}$ on which $K^\Gamma(\mathcal{X})|_{C(\mathcal{U}^\Gamma)} = K^\Gamma(\mathcal{U})$ is uniquely defined. By uniqueness these $K^\Gamma(\mathcal{U})$ agree on overlaps $C(\mathcal{U}^\Gamma) \cap C(\tilde{\mathcal{U}}^\Gamma)$, so we can glue them to make a global equivalence $K^\Gamma(\mathcal{X}) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})^\Gamma$ with $O^\Gamma(C(\mathcal{X})) \circ K^\Gamma(\mathcal{X}) = C(O^\Gamma(\mathcal{X}))$. The same argument yields a global equivalence $\tilde{K}^\Gamma(\mathcal{X}) : C(\tilde{\mathcal{X}}^\Gamma) \to \widetilde{C(\mathcal{X})}^\Gamma$ with $\tilde{O}^\Gamma(C(\mathcal{X})) \circ \tilde{K}^\Gamma(\mathcal{X}) = C(\tilde{O}^\Gamma(\mathcal{X}))$ and $\tilde{\Pi}^\Gamma(C(\mathcal{X})) \circ K^\Gamma(\mathcal{X}) = \tilde{K}^\Gamma(\mathcal{X}) \circ C(\tilde{\Pi}^\Gamma(\mathcal{X}))$.

For $\hat{K}^\Gamma(\mathcal{X})$, observe that $C(\hat{\Pi}^\Gamma(\mathcal{X})) : C(\tilde{\mathcal{X}}^\Gamma) \to C(\hat{\mathcal{X}}^\Gamma)$ is a $B\Gamma$-*gerbe*. It follows that any 1-morphism $f : C(\tilde{\mathcal{X}}^\Gamma) \to \mathcal{Y}$ in $\mathbf{DMC^\infty Sta}$ factors via $C(\hat{\Pi}^\Gamma(\mathcal{X}))$ up to 2-isomorphism (that is, $f \cong g \circ C(\hat{\Pi}^\Gamma(\mathcal{X}))$ for some $g : C(\hat{\mathcal{X}}^\Gamma) \to \mathcal{Y}$, which is unique up to 2-isomorphism) if and only if for all $[x] \in C(\tilde{\mathcal{X}}^\Gamma)_{\rm top}$

$$\begin{aligned} &\operatorname{Ker}\bigl[f_* : \operatorname{Iso}_{C(\tilde{\mathcal{X}}^\Gamma)}([x]) \to \operatorname{Iso}_{\mathcal{Y}}(f_{\rm top}([x]))\bigr] \subseteq \\ &\operatorname{Ker}\bigl[C(\hat{\Pi}^\Gamma(\mathcal{X}))_* : \operatorname{Iso}_{C(\tilde{\mathcal{X}}^\Gamma)}([x]) \to \operatorname{Iso}_{C(\hat{\mathcal{X}}^\Gamma)}(C(\hat{\Pi}^\Gamma(\mathcal{X}))_{\rm top}([x]))\bigr]. \end{aligned} \tag{8.33}$$

Applying this to $f = \hat{\Pi}^\Gamma(C(\mathcal{X})) \circ \tilde{K}^\Gamma(\mathcal{X})$, we already know that f factorizes via $C(\hat{\Pi}^\Gamma(\mathcal{X}))$ locally, so (8.33) holds, and thus f factorizes via $C(\hat{\Pi}^\Gamma(\mathcal{X}))$ globally. That is, there exists $\hat{K}^\Gamma(\mathcal{X}) : C(\hat{\mathcal{X}}^\Gamma) \to \widehat{C(\mathcal{X})}^\Gamma$, unique up to 2-isomorphism, with $\hat{\Pi}^\Gamma(C(\mathcal{X})) \circ \tilde{K}^\Gamma(\mathcal{X}) \cong \hat{K}^\Gamma(\mathcal{X}) \circ C(\hat{\Pi}^\Gamma(\mathcal{X}))$. Then for $\mathcal{U} \simeq [U/G]$ as above we have $\hat{K}^\Gamma(\mathcal{X})|_{C(\hat{\mathcal{U}}^\Gamma)} \cong \hat{K}^\Gamma(\mathcal{U})$, where $\hat{K}^\Gamma(\mathcal{U})$ is an equivalence. As such $C(\hat{\mathcal{U}}^\Gamma)$ cover $C(\hat{\mathcal{X}}^\Gamma)$, this implies $\hat{K}^\Gamma(\mathcal{X})$ is an equivalence. □

Here is an example, based on Example 5.19.

Example 8.33. Let $\mathbb{Z}_2 = \{1, \sigma\}$ with $\sigma^2 = 1$ act on $X = [0,\infty)^2$ by $\sigma : (x_1, x_2) \mapsto (x_2, x_1)$. Then $\mathcal{X} = \bigl[[0,\infty)^2/\mathbb{Z}_2\bigr]$ is an orbifold with corners. We have $\partial\mathcal{X} \cong [0,\infty)$ and $\partial^2\mathcal{X} \cong *$, so that $C_2(\mathcal{X}) \simeq [*/S_2] = [*/\mathbb{Z}_2]$. Hence $C(\mathcal{X}) = C_0(\mathcal{X}) \amalg C_1(\mathcal{X}) \amalg C_2(\mathcal{X})$ with $C_0(\mathcal{X}) \simeq \bigl[[0,\infty)^2/\mathbb{Z}_2\bigr]$, $C_1(\mathcal{X}) \simeq [0,\infty)$ and $C_2(\mathcal{X}) \simeq [*/\mathbb{Z}_2]$. The orbifold strata $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}_\circ^\Gamma$ are given by

$$\mathcal{X}^{\mathbb{Z}_2} = \mathcal{X}_\circ^{\mathbb{Z}_2} \simeq \tilde{\mathcal{X}}^{\mathbb{Z}_2} = \tilde{\mathcal{X}}_\circ^{\mathbb{Z}_2} \simeq [0,\infty) \times [*/\mathbb{Z}_2], \qquad \hat{\mathcal{X}}^{\mathbb{Z}_2} = \hat{\mathcal{X}}_\circ^{\mathbb{Z}_2} \simeq [0,\infty).$$

Therefore

$$\begin{aligned} &C_0(\mathcal{X}^{\mathbb{Z}_2}) \simeq [0,\infty) \times [*/\mathbb{Z}_2], && C_1(\mathcal{X}^{\mathbb{Z}_2}) \simeq [*/\mathbb{Z}_2], && C_2(\mathcal{X}^{\mathbb{Z}_2}) = \emptyset, \\ &C_0(\mathcal{X})^{\mathbb{Z}_2} \simeq [0,\infty) \times [*/\mathbb{Z}_2], && C_1(\mathcal{X})^{\mathbb{Z}_2} = \emptyset, && C_2(\mathcal{X})^{\mathbb{Z}_2} \simeq [*/\mathbb{Z}_2]. \end{aligned}$$

We see from this that $K^{\mathbb{Z}_2}(\mathcal{X}) : C(\mathcal{X}^{\mathbb{Z}_2}) \to C(\mathcal{X})^{\mathbb{Z}_2}$ identifies $C_1(\mathcal{X}^{\mathbb{Z}_2})$ with $C_2(\mathcal{X})^{\mathbb{Z}_2}$, so $K^\Gamma(\mathcal{X})$ need not map $C_k(\mathcal{X}^\Gamma)$ to $C_k(\mathcal{X})^\Gamma$ for $k > 0$. The same applies to $\tilde{K}^\Gamma(\mathcal{X}), \hat{K}^\Gamma(\mathcal{X})$.

The 1-morphisms $K^\Gamma(\mathfrak{X}), \tilde{K}^\Gamma(\mathfrak{X}), \hat{K}^\Gamma(\mathfrak{X})$ in Theorem 8.32 are étale locally modelled on the diffeomorphisms $C(j_{X,\Gamma}) : C(X^\Gamma) \to C(X)^\Gamma$ from Proposition 5.18. Now $C(j_{X,\Gamma})$ maps $C_k(X^\Gamma)$ to $\coprod_{l\geqslant k} C_l(X)^\Gamma$ for each $k > 0$. Thus $K^\Gamma(\mathfrak{X})$ maps $C_k(\mathfrak{X}^\Gamma)$ into $\coprod_{l\geqslant k} C_l(\mathfrak{X})^\Gamma$ for $k > 0$, and similarly for $\tilde{K}^\Gamma(\mathfrak{X}), \hat{K}^\Gamma(\mathfrak{X})$. This implies that $K^\Gamma(\mathfrak{X})^{-1}(C_1(\mathfrak{X})^\Gamma)$ is open and closed in $C_1(\mathfrak{X}^\Gamma)$. That is, $C_1(\mathfrak{X})^\Gamma \simeq (\partial\mathfrak{X})^\Gamma$ is equivalent to an open and closed subobject of $C_1(\mathfrak{X}^\Gamma) \simeq \partial(\mathfrak{X}^\Gamma)$.

Hence we can choose a 1-morphism $J^\Gamma(\mathfrak{X}) : (\partial\mathfrak{X})^\Gamma \to \partial(\mathfrak{X}^\Gamma)$ which is identified with a quasi-inverse for $K^\Gamma(\mathfrak{X})|_{\cdots} : K^\Gamma(\mathfrak{X})^{-1}(C_1(\mathfrak{X})^\Gamma) \to C_1(\mathfrak{X})^\Gamma$ by the equivalences $C_1(\mathfrak{X})^\Gamma \simeq (\partial\mathfrak{X})^\Gamma$ and $C_1(\mathfrak{X}^\Gamma) \simeq \partial(\mathfrak{X}^\Gamma)$, and $J^\Gamma(\mathfrak{X})$ is an equivalence between $(\partial\mathfrak{X})^\Gamma$ and an open and closed subobject of $\partial(\mathfrak{X}^\Gamma)$. Considering local models shows that $J^\Gamma(\mathfrak{X})$ maps $(\partial\mathfrak{X})^{\Gamma,\lambda}$ to an open and closed suborbifold of $\partial(\mathfrak{X}^{\Gamma,\lambda})$, for each $\lambda \in \Lambda^\Gamma_+$. The analogues hold for $\tilde{K}^\Gamma(\mathfrak{X}), \hat{K}^\Gamma(\mathfrak{X})$. This proves:

Corollary 8.34. *Let $\mathfrak{X}$ be an orbifold with corners, and Γ a finite group. Then there exist 1-morphisms $J^\Gamma(\mathfrak{X}) : (\partial\mathfrak{X})^\Gamma \to \partial(\mathfrak{X}^\Gamma)$, $\tilde{J}^\Gamma(\mathfrak{X}) : (\widetilde{\partial\mathfrak{X}})^\Gamma \to \partial(\tilde{\mathfrak{X}}^\Gamma)$, $\hat{J}^\Gamma(\mathfrak{X}) : (\widehat{\partial\mathfrak{X}})^\Gamma \to \partial(\hat{\mathfrak{X}}^\Gamma)$ in $\check{\mathbf{O}}\mathbf{rb^c}$, natural up to 2-isomorphism, such that $J^\Gamma(\mathfrak{X})$ is an equivalence from $(\partial\mathfrak{X})^\Gamma$ to an open and closed subobject of $\partial(\mathfrak{X}^\Gamma)$, and similarly for $\tilde{J}^\Gamma(\mathfrak{X}), \hat{J}^\Gamma(\mathfrak{X})$.*

For $\lambda \in \Lambda^\Gamma_+$, $\mu \in \Lambda^\Gamma_+/\operatorname{Aut}(\Lambda)$ these restrict to 1-morphisms $J^{\Gamma,\lambda}(\mathfrak{X}) : (\partial\mathfrak{X})^{\Gamma,\lambda} \to \partial(\mathfrak{X}^{\Gamma,\lambda})$, $\tilde{J}^{\Gamma,\mu}(\mathfrak{X}) : (\widetilde{\partial\mathfrak{X}})^{\Gamma,\mu} \to \partial(\tilde{\mathfrak{X}}^{\Gamma,\mu})$, $\hat{J}^{\Gamma,\mu}(\mathfrak{X}) : (\widehat{\partial\mathfrak{X}})^{\Gamma,\mu} \to \partial(\hat{\mathfrak{X}}^{\Gamma,\mu})$ in $\mathbf{Orb^c}$, which are equivalences with open and closed suborbifolds. Hence, if $\mathfrak{X}^{\Gamma,\lambda} = \emptyset$ then $(\partial\mathfrak{X})^{\Gamma,\lambda} = \emptyset$, and similarly for $\tilde{\mathfrak{X}}^{\Gamma,\mu}, (\widetilde{\partial\mathfrak{X}})^{\Gamma,\mu}, \hat{\mathfrak{X}}^{\Gamma,\mu}, (\widehat{\partial\mathfrak{X}})^{\Gamma,\mu}$.

Actually Corollary 8.34 is clear from Definition 8.31, since $\partial(\mathcal{X}^\Gamma)$ consists of all of $C_1(\mathcal{X})^\Gamma \simeq (\partial\mathcal{X})^\Gamma$ plus pieces of $C_k(\mathcal{X})^\Gamma$ for $k \geqslant 2$.

In general $J^\Gamma(\mathfrak{X}), \tilde{J}^\Gamma(\mathfrak{X}), \hat{J}^\Gamma(\mathfrak{X})$ may not be equivalences, and we can have $(\partial\mathfrak{X})^\Gamma \not\simeq \partial(\mathfrak{X}^\Gamma)$, $(\widetilde{\partial\mathfrak{X}})^\Gamma \not\simeq \partial(\tilde{\mathfrak{X}}^\Gamma)$ and $(\widehat{\partial\mathfrak{X}})^\Gamma \not\simeq \partial(\hat{\mathfrak{X}}^\Gamma)$, as Example 8.33 shows. That is, the 2-functors $F^\Gamma, \tilde{F}^\Gamma, \hat{F}^\Gamma$ do not commute with boundaries ∂. Here is a class of orbifolds with corners for which $\partial(\mathfrak{X}^\Gamma) \simeq (\partial\mathfrak{X})^\Gamma$, etc.

Definition 8.35. An orbifold with corners $\mathfrak{X}$ is called *straight* if the injective morphisms $(i_\mathfrak{X})_* : \operatorname{Iso}_{\partial\mathcal{X}}([x']) \to \operatorname{Iso}_\mathcal{X}([x])$ on orbifold groups are isomorphisms for all $[x'] \in \partial\mathcal{X}_{\rm top}$ with $i_{\mathfrak{X},\rm top}([x']) = [x]$. That is, straight orbifolds with corners are locally modelled on $[0,\infty)^k \times (\mathbb{R}^{n-k}/G)$, where G acts trivially on $[0,\infty)^k$. Orbifolds with boundary, with $k = 0$ or 1, are automatically straight.

If $\mathfrak{X}$ is straight one can show using §8.6 that $\partial\mathfrak{X}$ is straight, so by induction $\partial^k\mathfrak{X}$ is also straight for all $k \geqslant 0$.

If $\mathfrak{X}$ is straight then Definition 8.31 simplifies in the following way. In points $[x, \{x'_1, \ldots, x'_k\}, \rho, \sigma]$ in (8.32), the morphism $\sigma : \Gamma \to S_k$ is automatically trivial, $\sigma = 1$. Thus, when we defined $\partial(\mathcal{X}^\Gamma)_{\rm top} \subseteq C(\mathcal{X})^\Gamma_{\rm top}$ to correspond to $[x, \{x'_1, \ldots, x'_k\}, \rho, \sigma]$ in (8.32) such that $k \geqslant 1$ and $\sigma(\Gamma) \subseteq S_k$ acts transitively on $\{1, \ldots, k\}$, this holds if and only if $k = 1$, so $\partial(\mathcal{X}^\Gamma)_{\rm top} = C_1(\mathcal{X})^\Gamma_{\rm top}$, and $\partial(\mathcal{X}^\Gamma) = C_1(\mathcal{X})^\Gamma$, so $\partial(\mathcal{X}^\Gamma) \cong (\partial\mathcal{X})^\Gamma$ as $C_1(\mathcal{X}) \cong \partial\mathcal{X}$.

Hence for $\mathfrak{X}$ straight we have $\mathfrak{X}^\Gamma = \bigl(\mathcal{X}^\Gamma, C_1(\mathcal{X})^\Gamma, (\Pi^1_\mathfrak{X})^\Gamma\bigr)$. Also $\mathfrak{X}'^\Gamma := \bigl(\mathcal{X}^\Gamma, \partial\mathcal{X}^\Gamma, i^\Gamma_\mathfrak{X}\bigr)$ lies in $\check{\mathbf{O}}\mathbf{rb^c}$, and $\mathrm{id}_\mathcal{X} : \mathfrak{X}^\Gamma \to \mathfrak{X}'^\Gamma$ is a 1-isomorphism in $\check{\mathbf{O}}\mathbf{rb^c}$.

Thus, for straight orbifolds with corners $\mathcal{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathcal{X}})$ we could have adopted the simpler definition $\mathcal{X}^\Gamma = \bigl(\mathcal{X}^\Gamma, \partial\mathcal{X}^\Gamma, i_{\mathcal{X}}^\Gamma\bigr)$. The same holds for $\mathcal{X}^\Gamma_\circ, \tilde{\mathcal{X}}^\Gamma, \tilde{\mathcal{X}}^\Gamma_\circ$.

If $\mathcal{X}$ is straight then $K^\Gamma(\mathcal{X})$ in Theorem 8.32 is an equivalence $C_k(\mathcal{X}^\Gamma) \to C_k(\mathcal{X})^\Gamma$ for all $k \geqslant 0$, and so $J^\Gamma(\mathcal{X})$ in Corollary 8.34 is an equivalence $(\partial\mathcal{X})^\Gamma \to \partial(\mathcal{X}^\Gamma)$. The same applies for $\tilde{J}^\Gamma(\mathcal{X}), \hat{J}^\Gamma(\mathcal{X}), \tilde{K}^\Gamma(\mathcal{X}), \hat{K}^\Gamma(\mathcal{X})$. This gives:

Corollary 8.36. *Let $\mathcal{X}$ be a straight orbifold with corners, and Γ a finite group. Then we have equivalences $(\partial\mathcal{X})^\Gamma \simeq \partial(\mathcal{X}^\Gamma)$ in $\check{\mathbf{O}}\mathbf{rb^c}$ and $(\partial\mathcal{X})^{\Gamma,\lambda} \simeq \partial(\mathcal{X}^{\Gamma,\lambda})$ in $\mathbf{Orb^c}$ for $\lambda \in \Lambda^\Gamma_+$. The analogues hold for the other orbifold strata $\tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma, \mathcal{X}^\Gamma_\circ, \tilde{\mathcal{X}}^\Gamma_\circ, \hat{\mathcal{X}}^\Gamma_\circ$ and $\tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \mathcal{X}^{\Gamma,\lambda}_\circ, \tilde{\mathcal{X}}^{\Gamma,\mu}_\circ, \hat{\mathcal{X}}^{\Gamma,\mu}_\circ$.*

The material of §8.4.2 on orbifold strata and orientations extends to orbifolds with corners essentially without change. As in §8.4.3 and §C.5, we can consider *effective* orbifolds with corners.

Definition 8.37. An orbifold with corners $\mathcal{X}$ is called *effective* if the underlying C^∞-stack $\mathcal{X}$ is effective in the sense of §C.5. Equivalently, $\mathcal{X}$ is effective if it is Zariski locally modelled on quotients $[U/G]$ for U a manifold with corners and G a finite group acting locally effectively on U.

If G acts locally effectively on U, then it acts locally effectively on ∂U. Hence $\mathcal{X}$ effective implies that $\partial\mathcal{X}$ is effective, and $\partial^k\mathcal{X}$ is effective for $k \geqslant 0$. However, $\mathcal{X}$ effective does not imply $C_k(\mathcal{X})$ effective for $k \geqslant 2$. For example, $\mathcal{X} = \bigl[[0,\infty)^2/\mathbb{Z}_2\bigr]$ in Example 8.33 is effective, but $C_2(\mathcal{X}) \simeq [*/\mathbb{Z}_2]$ is not.

The analogues of Propositions 8.13 and 8.14 hold, with the same proofs.

9 The 2-category of d-stacks

Next we define and study the 2-category of *d-stacks* **dSta**, which are derived versions of Deligne–Mumford C^∞-stacks. Sections 9.1–9.5 are a C^∞-stack version of Chapter 2. Broadly, we replace C^∞-schemes by Deligne–Mumford C^∞-stacks throughout, and then deal with the extra issues introduced by 2-morphisms of C^∞-stacks. We will often omit proofs in this part, or just comment on the differences. Section 9.6 discusses orbifold strata of d-stacks.

9.1 Square zero extensions of C^∞-stacks

We extend the material of §2.1 on square zero extensions of C^∞-schemes to Deligne–Mumford C^∞-stacks. See §C.6–§C.7 for the necessary background on quasicoherent sheaves, and sheaves of abelian groups and C^∞-rings, on Deligne–Mumford C^∞-stacks. Here are the analogues of Definitions 2.9 and 2.12:

Definition 9.1. Let $\mathcal{X}$ be a locally fair Deligne–Mumford C^∞-stack. By Proposition C.31, this implies all $\mathcal{O}_{\mathcal{X}}$-modules are quasicoherent. As in §C.6–§C.7, sheaves on $\mathcal{X}$ are defined in terms of the category $\mathcal{C}_{\mathcal{X}}$ from Definition C.30, with objects $(\underline{U},u)$ and morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$.

A *square zero extension* $(\mathcal{O}'_{\mathcal{X}},\imath_{\mathcal{X}})$ *of* $\mathcal{X}$ consists of a sheaf of C^∞-rings $\mathcal{O}'_{\mathcal{X}}$ on $\mathcal{X}$ and a morphism of sheaves of C^∞-rings $\imath_{\mathcal{X}}:\mathcal{O}'_{\mathcal{X}}\to\mathcal{O}_{\mathcal{X}}$ on $\mathcal{X}$, where $\mathcal{O}_{\mathcal{X}}$ is the structure sheaf of $\mathcal{X}$ as in Example C.42, such that for all $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$

$$\imath_{\mathcal{X}}(\underline{U},u):\mathcal{O}'_{\mathcal{X}}(\underline{U},u)\longrightarrow\mathcal{O}_{\mathcal{X}}(\underline{U},u)=\mathcal{O}_U \tag{9.1}$$

is a square zero extension of C^∞-schemes on $\underline{U}$, in the sense of Definition 2.9. We also call $(\mathcal{X},\mathcal{O}'_{\mathcal{X}},\imath_{\mathcal{X}})$ a *square zero extension of C^∞-stacks.*

For each $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$, define quasicoherent sheaves $\mathcal{I}_{\mathcal{X}}(\underline{U},u)$, $\mathcal{F}_{\mathcal{X}}(\underline{U},u)$ on $\underline{U}$, a morphism $\kappa_{\mathcal{X}}(\underline{U},u):\mathcal{I}_{\mathcal{X}}(\underline{U},u)\to\mathcal{O}'_{\mathcal{X}}(\underline{U},u)$ of sheaves of abelian groups on U, and morphisms $\xi_{\mathcal{X}}(\underline{U},u):\mathcal{I}_{\mathcal{X}}(\underline{U},u)\to\mathcal{F}_{\mathcal{X}}(\underline{U},u)$, $\psi_{\mathcal{X}}(\underline{U},u):\mathcal{F}_{\mathcal{X}}(\underline{U},u)\to T^*\underline{U}=(T^*\mathcal{X})(\underline{U},u)$ of quasicoherent sheaves on $\underline{U}$, to be $\mathcal{I}_X,\mathcal{F}_X,\kappa_X,\xi_X,\psi_X$ in Definition 2.9 respectively for the square zero extension of C^∞-schemes (9.1).

If $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ in $\mathcal{C}_{\mathcal{X}}$ then $f':=(\mathcal{O}'_{\mathcal{X}})_{(\underline{f},\eta)}:f^{-1}(\mathcal{O}'_{\mathcal{X}}(\underline{V},v))\to\mathcal{O}'_{\mathcal{X}}(\underline{U},u)$ is a morphism of sheaves of C^∞-rings on U, and $(\underline{f},f')$ is a morphism of square zero extensions of C^∞-schemes $\bigl(\underline{U},\mathcal{O}'_{\mathcal{X}}(\underline{U},u),\imath_{\mathcal{X}}(\underline{U},u)\bigr)\to\bigl(\underline{V},\mathcal{O}'_{\mathcal{X}}(\underline{V},v),\imath_{\mathcal{X}}(\underline{V},v)\bigr)$, in the sense of Definition 2.12. So Definition 2.12 defines morphisms f^1,f^2,f^3 in $\mathrm{qcoh}(\underline{U})$, which are isomorphisms as $\underline{f}$ is étale and f' an isomorphism. Define

$$\begin{aligned}(\mathcal{I}_{\mathcal{X}})_{(\underline{f},\eta)}&=f^1:\underline{f}^*(\mathcal{I}_{\mathcal{X}}(\underline{V},v))\longrightarrow\mathcal{I}_{\mathcal{X}}(\underline{U},u)\\(\mathcal{F}_{\mathcal{X}})_{(\underline{f},\eta)}&=f^2:\underline{f}^*(\mathcal{F}_{\mathcal{X}}(\underline{V},v))\longrightarrow\mathcal{F}_{\mathcal{X}}(\underline{U},u).\end{aligned}$$

It is now easy to check that the data $\mathcal{I}_{\mathcal{X}}(\underline{U},u),\mathcal{F}_{\mathcal{X}}(\underline{U},u),(\mathcal{I}_{\mathcal{X}})_{(\underline{f},\eta)},(\mathcal{F}_{\mathcal{X}})_{(\underline{f},\eta)}$ defines quasicoherent sheaves $\mathcal{I}_{\mathcal{X}},\mathcal{F}_{\mathcal{X}}$ on $\mathcal{X}$, in the sense of Definition C.30, and the data $\xi_{\mathcal{X}}(\underline{U},u),\psi_{\mathcal{X}}(\underline{U},u)$ defines morphisms of quasicoherent sheaves $\xi_{\mathcal{X}}$:

$\mathcal{I}_\mathcal{X} \to \mathcal{F}_\mathcal{X}$, $\psi_\mathcal{X} : \mathcal{F}_\mathcal{X} \to T^*\mathcal{X}$. Also, regarding $\mathcal{I}_\mathcal{X}$ as a sheaf of abelian groups on $\mathcal{X}$ as in Remark C.41, the data $\kappa_\mathcal{X}(\underline{U}, u)$ defines a morphism $\kappa_\mathcal{X} : \mathcal{I}_\mathcal{X} \to \mathcal{O}'_\mathcal{X}$ of sheaves of abelian groups on $\mathcal{X}$, in the sense of Definition C.40.

Equation (2.9) for each $(\underline{U}, u)$ implies that we have an exact sequence of sheaves of abelian groups on $\mathcal{X}$:

$$0 \longrightarrow \mathcal{I}_\mathcal{X} \xrightarrow{\kappa_\mathcal{X}} \mathcal{O}'_\mathcal{X} \xrightarrow{\imath_\mathcal{X}} \mathcal{O}_\mathcal{X} \longrightarrow 0, \tag{9.2}$$

and equation (2.11) for each $(\underline{U}, u)$ implies that we have an exact sequence of sheaves of quasicoherent sheaves on $\mathcal{X}$:

$$\mathcal{I}_\mathcal{X} \xrightarrow{\xi_\mathcal{X}} \mathcal{F}_\mathcal{X} \xrightarrow{\psi_\mathcal{X}} T^*\mathcal{X} \longrightarrow 0. \tag{9.3}$$

Definition 9.2. Let $(\mathcal{X}, \mathcal{O}'_\mathcal{X}, \imath_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{O}'_\mathcal{Y}, \imath_\mathcal{Y})$ be square zero extensions of C^∞-stacks. A *morphism of square zero extensions* from $(\mathcal{X}, \mathcal{O}'_\mathcal{X}, \imath_\mathcal{X})$ to $(\mathcal{Y}, \mathcal{O}'_\mathcal{Y}, \imath_\mathcal{Y})$ is a pair (f, f'), where $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks, and $f' : f^{-1}(\mathcal{O}'_\mathcal{Y}) \to \mathcal{O}'_\mathcal{X}$ a morphism of sheaves of C^∞-rings on $\mathcal{X}$ such that $f^\sharp \circ f^{-1}(\imath_\mathcal{Y}) = \imath_\mathcal{X} \circ f' : f^{-1}(\mathcal{O}'_\mathcal{Y}) \to \mathcal{O}_\mathcal{X}$, for $f^\sharp : f^{-1}(\mathcal{O}_\mathcal{Y}) \to \mathcal{O}_\mathcal{X}$ as in Example C.44. Define morphisms $f^1 : f^*(\mathcal{I}_\mathcal{Y}) \to \mathcal{I}_\mathcal{X}$, $f^2 : f^*(\mathcal{F}_\mathcal{Y}) \to \mathcal{F}_\mathcal{X}$ and $f^3 : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ in $\mathrm{qcoh}(\mathcal{X})$ by $f^3 = \Omega_f$ and the commutative diagrams

$$\begin{tikzcd}
f^{-1}(\mathcal{I}_\mathcal{Y}) \otimes^{\mathrm{id}}_{f^{-1}(\mathcal{O}_\mathcal{Y})} f^{-1}(\mathcal{O}_\mathcal{Y}) = f^{-1}(\mathcal{I}_\mathcal{Y}) \arrow[r, "f^{-1}(\kappa_\mathcal{Y})"'] \arrow[d, "\mathrm{id}\otimes f^\sharp"] \arrow[dr, "f'|_{f^{-1}(\mathcal{I}_\mathcal{Y})}"] & f^{-1}(\mathcal{O}'_\mathcal{Y}) \arrow[r, "f^{-1}(\imath_\mathcal{Y})"'] \arrow[d, "f'"] & f^{-1}(\mathcal{O}_\mathcal{Y}) \arrow[r] \arrow[d, "f^\sharp"'] & 0 \\
f^*(\mathcal{I}_\mathcal{Y}) = f^{-1}(\mathcal{I}_\mathcal{Y}) \otimes^{f^\sharp}_{f^{-1}(\mathcal{O}_\mathcal{Y})} \mathcal{O}_\mathcal{X} \arrow[r, "f^1"] & \mathcal{I}_X \arrow[r, "\kappa_X"] & \mathcal{O}'_\mathcal{X} \arrow[r, "\imath_\mathcal{X}"] & \mathcal{O}_\mathcal{X} \longrightarrow 0,
\end{tikzcd} \tag{9.4}$$

$$\begin{tikzcd}
f^*(\mathcal{F}_\mathcal{Y}) = f^{-1}(\Omega_{\mathcal{O}'_\mathcal{Y}} \otimes^{\imath_\mathcal{Y}}_{\mathcal{O}'_\mathcal{Y}} \mathcal{O}_\mathcal{Y}) \otimes^{f^\sharp}_{f^{-1}(\mathcal{O}_\mathcal{Y})} \mathcal{O}_\mathcal{X} \arrow[r, "\cong"] \arrow[d, "f^2"] & f^{-1}(\Omega_{\mathcal{O}'_\mathcal{Y}}) \otimes^{f^\sharp \circ f^{-1}(\imath_\mathcal{Y})}_{f^{-1}(\mathcal{O}'_\mathcal{Y})} \mathcal{O}_\mathcal{X} \arrow[d, equal] \\
\mathcal{F}_\mathcal{X} = \Omega_{\mathcal{O}'_\mathcal{X}} \otimes^{\imath_\mathcal{X}}_{\mathcal{O}'_\mathcal{X}} \mathcal{O}_\mathcal{X} & f^{-1}(\Omega_{\mathcal{O}'_\mathcal{Y}}) \otimes^{\imath_\mathcal{X} \circ f'}_{f^{-1}(\mathcal{O}'_\mathcal{Y})} \mathcal{O}_\mathcal{X}, \arrow[l, "\Omega_{f'} \otimes \mathrm{id}"']
\end{tikzcd} \tag{9.5}$$

the analogues of (2.12)–(2.13). Here for (9.4) the right hand square commutes as $f^\sharp \circ f^{-1}(\imath_\mathcal{Y}) = \imath_\mathcal{X} \circ f'$, so the morphism $f'|_{f^{-1}(\mathcal{I}_\mathcal{Y})}$ exists by exactness, and it is $f^\sharp$-equivariant, and so factors via a morphism f^1.

Then the analogue of (2.14) is a commutative diagram in $\mathrm{qcoh}(\mathcal{X})$:

$$\begin{tikzcd}
f^*(\mathcal{I}_\mathcal{Y}) \arrow[r, "f^*(\xi_\mathcal{Y})"'] \arrow[d, "f^1"] & f^*(\mathcal{F}_\mathcal{Y}) \arrow[r, "f^*(\psi_\mathcal{Y})"'] \arrow[d, "f^2"] & f^*(T^*\mathcal{Y}) \arrow[r] \arrow[d, "f^3 = \Omega_f"] & 0 \\
\mathcal{I}_\mathcal{X} \arrow[r, "\xi_\mathcal{X}"] & \mathcal{F}_\mathcal{X} \arrow[r, "\psi_\mathcal{X}"] & T^*\mathcal{X} \arrow[r] & 0,
\end{tikzcd} \tag{9.6}$$

with exact rows. We will explain how to deduce this from (2.14), using the definition of pullbacks f^* on C^∞-stacks in Definition C.36.

Suppose $(\underline{U}, u) \in \mathcal{C}_{\mathcal{X}}$ and $(\underline{V}, v) \in \mathcal{C}_{\mathcal{Y}}$. Then $u : \bar{\underline{U}} \to \mathcal{X}$ and $v : \bar{\underline{V}} \to \mathcal{Y}$ are étale 1-morphisms, so there is a C^∞-scheme $\underline{W}$ and morphisms $\pi_{\underline{U}} : \underline{W} \to \underline{U}$, $\pi_{\underline{V}} : \underline{W} \to \underline{V}$ giving a 2-Cartesian diagram in $\mathbf{C^\infty Sta}$:

$$\begin{tikzcd}
\bar{\underline{W}} \arrow[rr, "\bar{\pi}_{\underline{V}}"'] \arrow[d, "\bar{\pi}_{\underline{U}}"'] \arrow[drr, phantom, "\zeta \Uparrow"] & & \bar{\underline{V}} \arrow[d, "v"] \\
\bar{\underline{U}} \arrow[rr, "f \circ u"] & & \mathcal{Y},
\end{tikzcd}$$

as in (C.6). Note that $\pi_{\underline{U}}$ is étale as u is étale, and $\pi_{\underline{V}} : \underline{W} \to \underline{V}$ is an étale lift of $f : \mathcal{X} \to \mathcal{Y}$. Define $\mathcal{O}'_U = \mathcal{O}'_X(\underline{U}, u)$, $\imath_U = \imath_{\mathcal{X}}(\underline{U}, u)$, $\mathcal{O}'_V = \mathcal{O}'_{\mathcal{Y}}(\underline{V}, v)$, $\imath_V = \imath_{\mathcal{Y}}(\underline{V}, v)$, $\mathcal{O}'_W = \pi_U^{-1}(\mathcal{O}'_{\mathcal{X}}(\underline{U}, u))$, and $\imath_W = \pi_U^\sharp \circ \pi_U^{-1}(\imath_{\mathcal{X}}(\underline{U}, u))$.

Then $(\underline{U}, \mathcal{O}'_U, \imath_U)$, $(\underline{V}, \mathcal{O}'_V, \imath_V)$ and $(\underline{W}, \mathcal{O}'_W, \imath_W)$ are square zero extensions of C^∞-schemes, where for $(\underline{W}, \mathcal{O}'_W, \imath_W)$ we use $\pi_U^\sharp : \pi_U^{-1}(\mathcal{O}_U) \to \mathcal{O}_W$ an isomorphism as $\pi_{\underline{U}} : \underline{W} \to \underline{U}$ is étale. Define $\pi'_U = \mathrm{id}_{\mathcal{O}'_W} : \pi_U^{-1}(\mathcal{O}'_U) \to \mathcal{O}'_W$, and define $\pi'_V : \pi_V^{-1}(\mathcal{O}'_V) \to \mathcal{O}'_W$ to be the composition

$$\pi_V^{-1}(\mathcal{O}'_V) = \pi_V^{-1}(\mathcal{O}'_{\mathcal{Y}}(\underline{V}, v)) \xrightarrow{i(\mathcal{O}'_{\mathcal{Y}}, f, u, v, \zeta)^{-1}} \pi_U^{-1}((f^{-1}(\mathcal{O}'_{\mathcal{Y}}))(\underline{U}, u)) \xrightarrow{\pi_U^{-1}(f'(\underline{U}, u))} \pi_U^{-1}(\mathcal{O}'_{\mathcal{X}}(\underline{U}, u)) = \mathcal{O}'_W.$$

Then $(\pi_{\underline{U}}, \pi'_U) : (\underline{W}, \mathcal{O}'_W, \imath_W) \to (\underline{U}, \mathcal{O}'_U, \imath_U)$ and $(\pi_{\underline{V}}, \pi'_V) : (\underline{W}, \mathcal{O}'_W, \imath_W) \to (\underline{V}, \mathcal{O}'_V, \imath_V)$ are morphisms of square zero extensions of C^∞-schemes.

Consider the diagram of morphisms in $\mathrm{qcoh}(\underline{W})$:

$$\begin{tikzcd}[column sep=small]
\underline{\pi}_{\underline{U}}^*\big((f^*(\mathcal{I}_{\mathcal{Y}}))(\underline{U}, u)\big) \arrow[r, "\underline{\pi}_{\underline{U}}^*((f^*(\xi_{\mathcal{Y}}))(\underline{U},u))"'] \arrow[d, "\cong"', "i(\mathcal{I}_{\mathcal{Y}}, f, u, v, \zeta)"] & \underline{\pi}_{\underline{U}}^*\big((f^*(\mathcal{F}_{\mathcal{Y}}))(\underline{U}, u)\big) \arrow[r, "\underline{\pi}_{\underline{U}}^*((f^*(\psi_{\mathcal{Y}}))(\underline{U},u))"'] \arrow[d, "\cong"', "i(\mathcal{F}_{\mathcal{Y}}, f, u, v, \zeta)"] & \underline{\pi}_{\underline{U}}^*\big((f^*(T^*\mathcal{Y}))(\underline{U}, u)\big) \arrow[r] \arrow[d, "\cong"', "i(T^*\mathcal{Y}, f, u, v, \zeta)"] & 0 \\
\underline{\pi}_{\underline{V}}^*\big(\mathcal{I}_{\mathcal{Y}}(\underline{V}, v)\big) \arrow[r, "\underline{\pi}_{\underline{V}}^*(\xi_{\mathcal{Y}}(\underline{V},v))"] \arrow[d, equal] & \underline{\pi}_{\underline{V}}^*\big(\mathcal{F}_{\mathcal{Y}}(\underline{V}, v)\big) \arrow[r, "\underline{\pi}_{\underline{V}}^*(\psi_{\mathcal{Y}}(\underline{V},v))"] \arrow[d, equal] & \underline{\pi}_{\underline{V}}^*\big((T^*\mathcal{Y})(\underline{V}, v)\big) \arrow[r] \arrow[d, equal] & 0 \\
\underline{\pi}_{\underline{V}}^*(\mathcal{I}_V) \arrow[r, "\underline{\pi}_{\underline{V}}^*(\xi_V)"] \arrow[d, "\pi_V^1"] & \underline{\pi}_{\underline{V}}^*(\mathcal{F}_V) \arrow[r, "\underline{\pi}_{\underline{V}}^*(\psi_V)"] \arrow[d, "\pi_V^2"] & \underline{\pi}_{\underline{V}}^*(T^*\underline{V}) \arrow[r] \arrow[d, "\pi_V^3 = \Omega_{\pi_{\underline{V}}}"] & 0 \\
\mathcal{I}_W \arrow[r, "\xi_W"] \arrow[d, "\cong"', "(\pi_U^1)^{-1}"] & \mathcal{F}_W \arrow[r, "\psi_W"] \arrow[d, "\cong"', "(\pi_U^2)^{-1}"] & T^*\underline{W} \arrow[r] \arrow[d, "\cong"', "(\pi_U^3)^{-1} = \Omega_{\pi_{\underline{U}}}^{-1}"] & 0 \\
\underline{\pi}_{\underline{U}}^*(\mathcal{I}_U) \arrow[r, "\underline{\pi}_{\underline{U}}^*(\xi_U)"] \arrow[d, equal] & \underline{\pi}_{\underline{U}}^*(\mathcal{F}_U) \arrow[r, "\underline{\pi}_{\underline{U}}^*(\psi_U)"] \arrow[d, equal] & \underline{\pi}_{\underline{U}}^*(T^*\underline{U}) \arrow[r] \arrow[d, equal] & 0 \\
\underline{\pi}_{\underline{U}}^*\big(\mathcal{I}_{\mathcal{X}}(\underline{U}, u)\big) \arrow[r, "\underline{\pi}_{\underline{U}}^*(\xi_{\mathcal{X}}(\underline{U},u))"] & \underline{\pi}_{\underline{U}}^*\big(\mathcal{F}_{\mathcal{X}}(\underline{U}, u)\big) \arrow[r, "\underline{\pi}_{\underline{U}}^*(\psi_{\mathcal{X}}(\underline{U},u))"] & \underline{\pi}_{\underline{U}}^*\big(T^*\mathcal{X}(\underline{U}, u)\big) \arrow[r] & 0.
\end{tikzcd} \tag{9.7}$$

Here the first two lines of (9.7) commute by definition of f^* in Definition C.36. The second two lines are equal by definition of $(\underline{V}, \mathcal{O}'_V, \imath_V)$. The third two lines are (2.14) for $(\pi_{\underline{V}}, \pi'_V)$. The fourth two lines are the inverse of (2.14) for $(\pi_{\underline{U}}, \pi'_U)$, where $\pi_U^1, \pi_U^2, \pi_U^3$ are isomorphisms as $\pi_{\underline{U}}$ is étale and π'_U an isomorphism. The fifth two lines are equal by definition of $(\underline{U}, \mathcal{O}'_U, \imath_U)$.

Thus (9.7) commutes. The compositions of the columns are $\underline{\pi}_{\underline{U}}^*(f^j(\underline{U},u))$ for $j = 1,2,3$. This proves that (9.6) evaluated on $(\underline{U},u)$, pulled back to $\underline{W}$ by the étale morphism $\underline{\pi}_{\underline{U}}$, commutes. If $v : \bar{\underline{V}} \to \mathcal{Y}$ is an étale atlas for $\mathcal{Y}$ then $\underline{\pi}_{\underline{U}} : \underline{W} \to \underline{U}$ is surjective, so (9.6) commuting implies that (9.6) evaluated on $(\underline{U},u)$ commutes. As this is true for all $(\underline{U},u)$, equation (9.6) commutes.

We have now seen two examples of proofs deducing facts about square zero extensions of C^∞-stacks from corresponding facts about square zero extensions of C^∞-schemes: the easy proof of the exactness of (9.3), and the proof that (9.6) commutes, which is more complicated because the definition of pullbacks f^*, f^{-1} in Definitions C.36 and C.43 is somewhat indirect. Once you have the idea, these proofs are routine, so we will generally omit them.

If $f,g : \mathcal{X} \to \mathcal{Y}$ are 1-morphisms of Deligne–Mumford stacks and $\eta : f \to g$ is a 2-morphism, then as in §C.6–§C.7, we have functors f^*, g^*, f^{-1}, g^{-1} from sheaves on $\mathcal{Y}$ to sheaves on $\mathcal{X}$, and natural isomorphisms $\eta^* : f^* \Rightarrow g^*$, $\eta^{-1} : f^{-1} \Rightarrow g^{-1}$. By considering the commutative diagram:

$$\begin{tikzcd}
g^{-1}(\mathcal{O}'_{\mathcal{Y}}) \arrow[r, "\eta^{-1}(\mathcal{O}'_{\mathcal{Y}})^{-1}"'] \arrow[rr, bend left, "g'"] \arrow[d, "g^{-1}(\imath_{\mathcal{Y}})"] & f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \arrow[r, "f'"'] \arrow[d, "f^{-1}(\imath_{\mathcal{Y}})"] & \mathcal{O}'_{\mathcal{X}} \arrow[d, "\imath_{\mathcal{X}}"'] \\
g^{-1}(\mathcal{O}_{\mathcal{Y}}) \arrow[r, "\eta^{-1}(\mathcal{O}_{\mathcal{Y}})^{-1}"] \arrow[rr, bend right, "g^\sharp"'] & f^{-1}(\mathcal{O}_{\mathcal{Y}}) \arrow[r, "f^\sharp"] & \mathcal{O}_{\mathcal{X}},
\end{tikzcd}$$

we may deduce:

Lemma 9.3. *Let $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$ be square zero extensions of C^∞-stacks, and $(f,f') : (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$ a morphism of square zero extensions. Suppose $g : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks, and $\eta : f \Rightarrow g$ a 2-morphism. Define $g' = f' \circ \eta^{-1}(\mathcal{O}'_{\mathcal{Y}})^{-1} : g^{-1}(\mathcal{O}'_{\mathcal{Y}}) \to \mathcal{O}'_{\mathcal{X}}$, a morphism of sheaves of C^∞-rings on $\mathcal{X}$, so that $g' \circ \eta^{-1}(\mathcal{O}'_{\mathcal{Y}}) = f'$. Then $(g,g') : (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$ is a morphism of square zero extensions. We also have*

$$g^1 \circ \eta^*(\mathcal{I}_{\mathcal{Y}}) = f^1, \;\; g^2 \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) = f^2, \;\; \textit{and} \;\; g^3 \circ \eta^*(T^*\mathcal{Y}) = f^3.$$

Proposition 2.13 lifts unchanged from C^∞-schemes to C^∞-stacks, enabling us to compare $(f,f'), (g,g') : (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$ with $f = g$. Combining this with Lemma 9.3 compares $(f,f'), (g,g')$ with f,g 2-isomorphic:

Proposition 9.4. *Let $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$ be square zero extensions of C^∞-stacks with kernel sheaves $\mathcal{I}_{\mathcal{X}}, \mathcal{I}_{\mathcal{Y}}$, and $(f,f'), (g,g')$ be morphisms from $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}})$ to $(\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$. Suppose $\eta : f \Rightarrow g$ is a 2-morphism of C^∞-stacks. Use the notation $\kappa_{\mathcal{X}}, \xi_{\mathcal{X}}, \psi_{\mathcal{X}}, \kappa_{\mathcal{Y}}, \xi_{\mathcal{Y}}, \psi_{\mathcal{Y}}$ from Definition 9.1 and $f^1, f^2, f^3, g^1, g^2, g^3$ from Definition 9.2. Then there exists a unique morphism $\mu : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{I}_{\mathcal{X}}$ in $\mathrm{qcoh}(\mathcal{X})$ such that*

$$g' \circ \eta^{-1}(\mathcal{O}'_{\mathcal{Y}}) = f' + \kappa_{\mathcal{X}} \circ \mu \circ \bigl(\mathrm{id} \otimes (f^\sharp \circ f^{-1}(\imath_{\mathcal{Y}}))\bigr) \circ \bigl(f^{-1}(\mathrm{d})\bigr), \tag{9.8}$$

where the morphisms are given in the diagram

$$
\begin{tikzcd}
& f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \arrow[dl, "\eta^{-1}(\mathcal{O}'_{\mathcal{Y}})"', "\cong"] \arrow[d, "f'"] \arrow[rr, "f^{-1}(\mathrm{d})"'] & & f^{-1}(\Omega_{\mathcal{O}'_{\mathcal{Y}}}) = f^{-1}(\Omega_{\mathcal{O}'_{\mathcal{Y}}}) \otimes^{\mathrm{id}}_{f^{-1}(\mathcal{O}'_{\mathcal{Y}})} f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \arrow[d, "\mathrm{id}\otimes(f^\sharp\circ f^{-1}(\imath_{\mathcal{Y}}))"'] \\
g^{-1}(\mathcal{O}'_{\mathcal{Y}}) \arrow[r, "g'"] & \mathcal{O}'_{\mathcal{X}} & \mathcal{I}_{\mathcal{X}} \arrow[l, "\kappa_{\mathcal{X}}"'] & f^*(\mathcal{F}_{\mathcal{Y}}) = f^{-1}(\Omega_{\mathcal{O}'_{\mathcal{Y}}}) \otimes^{f^\sharp\circ f^{-1}(\imath_{\mathcal{Y}})}_{f^{-1}(\mathcal{O}'_{\mathcal{Y}})} \mathcal{O}_{\mathcal{X}}. \arrow[l, "\mu"']
\end{tikzcd}
$$

We also have

$$
\begin{gathered}
g^1 \circ \eta^*(\mathcal{I}_{\mathcal{Y}}) = f^1 + \mu \circ f^*(\xi_{\mathcal{Y}}), \qquad g^2 \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) = f^2 + \xi_{\mathcal{X}} \circ \mu, \\
\textit{and} \qquad g^3 \circ \eta^*(T^*\mathcal{Y}) = f^3.
\end{gathered}
\tag{9.9}
$$

Conversely, if $(f, f') : (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$ *is a morphism of square zero extensions,* $g : \mathcal{X} \to \mathcal{Y}$ *is a* 1*-morphism,* $\eta : f \Rightarrow g$ *is a* 2*-morphism, and* $\mu : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{I}_{\mathcal{X}}$ *is a morphism in* $\mathrm{qcoh}(\mathcal{X})$*, then there exists a unique morphism* $g' : g^{-1}(\mathcal{O}'_{\mathcal{Y}}) \to \mathcal{O}'_{\mathcal{X}}$ *of sheaves of* C^∞*-rings on* $\mathcal{X}$ *such that* (g, g') *is a morphism* $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$*, and* (9.8)–(9.9) *hold.*

Remark 9.5. In §2.1 we defined a square zero extension of a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ to be a pair $(\mathcal{O}'_X, \imath_X)$, where in fact $\underline{X}' = (X, \mathcal{O}'_X)$ is a C^∞-scheme, and $\underline{\imath}_X = (\mathrm{id}_X, \imath_X)$ is a morphism of C^∞-schemes $\underline{\imath}_X : \underline{X} \to \underline{X}'$. So we could have written Chapter 2 in terms of two C^∞-schemes $\underline{X}, \underline{X}'$ and morphism $\underline{\imath}_X : \underline{X} \to \underline{X}'$, rather than a single C^∞-scheme $\underline{X}$ with data $\imath_X : \mathcal{O}'_X \to \mathcal{O}_X$. But as in Remark 2.10, we chose not to do this.

For C^∞-stacks, the situation is different. Given a C^∞-stack $\mathcal{X}$ and a square zero extension $(\mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}})$, we do not immediately get a second C^∞-stack $\mathcal{X}'$ and a 1-morphism $\imath_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}'$. We can in fact define such $\mathcal{X}', \imath_{\mathcal{X}}$, but they depend on arbitrary choices, and are only natural up to equivalence and 2-isomorphism.

We could have defined d-stacks using C^∞-stacks $\mathcal{X}, \mathcal{X}'$ and a 1-morphism $\imath_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}'$, rather than a triple $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}})$. This would have made things more complicated, because $\mathcal{X}'$ would also live in a 2-category, and we would have to consider 2-morphisms of 1-morphisms associated to $\mathcal{X}'$, and so on. In contrast, once $\mathcal{X}$ is fixed, $\mathcal{O}'_{\mathcal{X}}$ lives in a category rather than a 2-category. Writing things in terms of $\mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}}$ also makes it easier to generalize proofs from d-spaces to d-stacks.

9.2 The definition of d-stacks

We now define the 2-category **dSta** of *d-stacks*, which are analogues of d-spaces in which C^∞-schemes $\underline{X}, \underline{X}'$ are replaced by Deligne–Mumford C^∞-stacks $\mathcal{X}, \mathcal{X}'$. The main difference with §2.2 is that to define 2-morphisms in **dSta** we have to include 2-morphisms of the Deligne–Mumford C^∞-stacks $\mathcal{X}, \mathcal{Y}$. Here is the analogue of Definition 2.14.

Definition 9.6. A *d-stack* $\boldsymbol{\mathcal{X}}$ is a quintuple $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$, where $\mathcal{X}$ is a separated, second countable, locally fair Deligne–Mumford C^∞-stack in the sense of §C.5, and $(\mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}})$ is a square zero extension of $\mathcal{X}$ in the sense of Definition 9.1 with kernel $\kappa_{\mathcal{X}} : \mathcal{I}_{\mathcal{X}} \to \mathcal{O}'_{\mathcal{X}}$, so that $\mathcal{I}_{\mathcal{X}} \in \mathrm{qcoh}(\mathcal{X})$, and $\mathcal{E}_{\mathcal{X}} \in \mathrm{qcoh}(\mathcal{X})$, and $\jmath_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{I}_{\mathcal{X}}$ is a surjective morphism in $\mathrm{qcoh}(\mathcal{X})$. As for (2.18), by (9.2) we have an exact sequence of sheaves of abelian groups on $\mathcal{X}$:

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\kappa_{\mathcal{X}} \circ \jmath_{\mathcal{X}}} \mathcal{O}'_{\mathcal{X}} \xrightarrow{\imath_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}} \longrightarrow 0.$$

Let $\mathcal{F}_{\mathcal{X}}, \psi_{\mathcal{X}}, \xi_{\mathcal{X}}$ be as in Definition 9.2, and define $\phi_{\mathcal{X}} = \xi_{\mathcal{X}} \circ \jmath_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$. As for (2.19), from (9.3) and $\jmath_{\mathcal{X}}$ surjective we have an exact sequence in $\mathrm{qcoh}(\mathcal{X})$:

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\phi_{\mathcal{X}}} \mathcal{F}_{\mathcal{X}} \xrightarrow{\psi_{\mathcal{X}}} T^*\mathcal{X} \longrightarrow 0. \tag{9.10}$$

The morphism $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ will be called the *virtual cotangent sheaf* of $\boldsymbol{\mathcal{X}}$.

Write $\lambda_{\mathcal{X}} : \mathcal{C}_{\mathcal{X}} \to \mathcal{E}_{\mathcal{X}}$ for the kernel of $\jmath_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{I}_{\mathcal{X}}$ in $\mathrm{qcoh}(\mathcal{X})$, and $\mu_{\mathcal{X}} : \mathcal{D}_{\mathcal{X}} \to \mathcal{E}_{\mathcal{X}}$ for the kernel of $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ in $\mathrm{qcoh}(\mathcal{X})$. Then there exists a unique morphism $\nu_{\mathcal{X}} : \mathcal{C}_{\mathcal{X}} \to \mathcal{D}_{\mathcal{X}}$ with $\lambda_{\mathcal{X}} = \mu_{\mathcal{X}} \circ \nu_{\mathcal{X}}$. Thus we have a commutative diagram with exact diagonals:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_{\mathcal{X}} \arrow[d, "\nu_{\mathcal{X}}"'] \arrow[dr, "\lambda_{\mathcal{X}}"] & & \mathcal{F}_{\mathcal{X}} \arrow[r, "\psi_{\mathcal{X}}"] & T^*\mathcal{X} \arrow[r] & 0 \\
& & \mathcal{E}_{\mathcal{X}} \arrow[ur, "\phi_{\mathcal{X}}"] \arrow[dr, "\jmath_{\mathcal{X}}"'] & & & \\
0 \arrow[r] & \mathcal{D}_{\mathcal{X}} \arrow[ur, "\mu_{\mathcal{X}}"'] & & \mathcal{I}_{\mathcal{X}} \arrow[u, "\xi_{\mathcal{X}}"'] \arrow[r] & 0.
\end{tikzcd}$$

Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ be d-stacks. A 1-*morphism* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a triple $\boldsymbol{f} = (f, f', f'')$, where $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks, $f' : f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \to \mathcal{O}'_{\mathcal{X}}$ a morphism of sheaves of C^∞-rings on $\mathcal{X}$ such that (f, f') is a morphism of square zero extensions $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$ in the sense of Definition 9.2, and $f'' : f^*(\mathcal{E}_{\mathcal{Y}}) \to \mathcal{E}_{\mathcal{X}}$ is a morphism in $\mathrm{qcoh}(\mathcal{X})$ satisfying

$$\jmath_{\mathcal{X}} \circ f'' = f^1 \circ f^*(\jmath_{\mathcal{Y}}) : f^*(\mathcal{E}_{\mathcal{Y}}) \longrightarrow \mathcal{I}_{\mathcal{X}}, \tag{9.11}$$

where f^1, f^2, f^3 are as in Definition 9.2. Then as for (2.22), from (9.6), (9.10) and (9.11) we have a commutative diagram in $\mathrm{qcoh}(\mathcal{X})$, with exact rows:

$$\begin{tikzcd}
f^*(\mathcal{E}_{\mathcal{Y}}) \arrow[r, "f^*(\phi_{\mathcal{Y}})"'] \arrow[d, "f''"] & f^*(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "f^*(\psi_{\mathcal{Y}})"'] \arrow[d, "f^2"] & f^*(T^*\mathcal{Y}) \arrow[r] \arrow[d, "f^3"] & 0 \\
\mathcal{E}_{\mathcal{X}} \arrow[r, "\phi_{\mathcal{X}}"] & \mathcal{F}_{\mathcal{X}} \arrow[r, "\psi_{\mathcal{X}}"] & T^*\mathcal{X} \arrow[r] & 0.
\end{tikzcd} \tag{9.12}$$

There are unique morphisms $f^4 : f^*(\mathcal{C}_{\mathcal{Y}}) \to \mathcal{C}_{\mathcal{X}}$ and $f^5 : f^*(\mathcal{D}_{\mathcal{Y}}) \to \mathcal{D}_{\mathcal{X}}$ in $\mathrm{qcoh}(\mathcal{X})$ with $\lambda_{\mathcal{X}} \circ f^4 = f'' \circ f^*(\lambda_{\mathcal{Y}})$ and $\mu_{\mathcal{X}} \circ f^5 = f'' \circ f^*(\mu_{\mathcal{Y}})$.

If $\boldsymbol{\mathcal{X}}$ is a d-stack, the *identity* 1-*morphism* $\mathbf{id}_{\boldsymbol{\mathcal{X}}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}$ is $\mathbf{id}_{\boldsymbol{\mathcal{X}}} = \bigl(\mathrm{id}_{\mathcal{X}}, \delta_{\mathcal{X}}(\mathcal{O}'_{\mathcal{X}}), \delta_{\mathcal{X}}(\mathcal{E}_{\mathcal{X}})\bigr)$. It is easy to check $\mathbf{id}_{\boldsymbol{\mathcal{X}}}$ is a 1-morphism.

Now let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ be d-stacks, and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, $\boldsymbol{g} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms. As in (2.23) define the *composition of* 1-*morphisms* to be

$$\boldsymbol{g} \circ \boldsymbol{f} = \bigl(g \circ f, f' \circ f^{-1}(g') \circ I_{f,g}(\mathcal{O}'_{\mathcal{Z}}), f'' \circ f^*(g'') \circ I_{f,g}(\mathcal{E}_{\mathcal{Z}})\bigr).$$

Then $\boldsymbol{g} \circ \boldsymbol{f}$ is a 1-morphism $\boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$, and the analogue of (2.24) holds. Also $\boldsymbol{f} \circ \mathbf{id}_{\boldsymbol{\mathcal{X}}} = \mathbf{id}_{\boldsymbol{\mathcal{Y}}} \circ \boldsymbol{f} = \boldsymbol{f}$.

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be 1-morphisms of d-stacks, where $\boldsymbol{f} = (f, f', f'')$ and $\boldsymbol{g} = (g, g', g'')$. A *2-morphism* $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a pair $\boldsymbol{\eta} = (\eta, \eta')$, where $\eta : f \Rightarrow g$ is a 2-morphism in $\mathbf{C^\infty Sta}$ and $\eta' : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{E}_{\mathcal{X}}$ a morphism in $\mathrm{qcoh}(\mathcal{X})$, with

$$\begin{aligned} g' \circ \eta^{-1}(\mathcal{O}'_{\mathcal{Y}}) &= f' + \kappa_{\mathcal{X}} \circ \jmath_{\mathcal{X}} \circ \eta' \circ \big(\mathrm{id} \otimes (f^\sharp \circ f^{-1}(\imath_{\mathcal{Y}}))\big) \circ \big(f^{-1}(\mathrm{d})\big), \\ \text{and} \quad g'' \circ \eta^*(\mathcal{E}_{\mathcal{Y}}) &= f'' + \eta' \circ f^*(\phi_{\mathcal{Y}}). \end{aligned} \tag{9.13}$$

Here the first equation makes sense as Proposition 9.4 shows that (9.8) holds for some $\mu : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{I}_{\mathcal{X}}$, and we take $\mu = \jmath_{\mathcal{X}} \circ \eta'$. Generalizing (2.26) gives

$$\begin{aligned} &g^1 \circ \eta^*(\mathcal{I}_{\mathcal{Y}}) = f^1 + \jmath_{\mathcal{X}} \circ \eta' \circ f^*(\xi_{\mathcal{Y}}), \quad g^2 \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) = f^2 + \phi_{\mathcal{X}} \circ \eta', \\ &g^3 \circ \eta^*(T^*\mathcal{Y}) = f^3, \quad g^4 \circ \eta^*(\mathcal{C}_{\mathcal{Y}}) = f^4, \quad \text{and} \quad g^5 \circ \eta^*(\mathcal{D}_{\mathcal{Y}}) = f^5, \end{aligned}$$

so the following diagram commutes (except η') in $\mathrm{qcoh}(\mathcal{X})$, with exact rows:

$$\begin{tikzcd}
f^*(\mathcal{E}_{\mathcal{Y}}) \arrow[rr, "f^*(\phi_{\mathcal{Y}})"] \arrow[dr, "\eta^*(\mathcal{E}_{\mathcal{Y}})"] \arrow[ddr, "f''+" ', "\eta'\circ f^*(\phi_{\mathcal{Y}})"'] & & f^*(\mathcal{F}_{\mathcal{Y}}) \arrow[rr, "f^*(\psi_{\mathcal{Y}})"] \arrow[dr, "\eta^*(\mathcal{F}_{\mathcal{Y}})"] \arrow[ddr, "f^2+\phi_{\mathcal{X}}\circ\eta'"'] \arrow[ddl, dotted, "\eta'"'] & & f^*(T^*\mathcal{Y}) \arrow[rr] \arrow[dr, "\eta^*(T^*\mathcal{Y})"] \arrow[ddr, "f^3"'] & & 0 \\
& g^*(\mathcal{E}_{\mathcal{Y}}) \arrow[rr, "g^*(\phi_{\mathcal{Y}})"] \arrow[d, "g''"] & & g^*(\mathcal{F}_{\mathcal{Y}}) \arrow[rr, "g^*(\psi_{\mathcal{Y}})"] \arrow[d, "g^2"] & & g^*(T^*\mathcal{Y}) \arrow[r] \arrow[d, "g^3"] & 0 \\
& \mathcal{E}_{\mathcal{X}} \arrow[rr, "\phi_{\mathcal{X}}"] & & \mathcal{F}_{\mathcal{X}} \arrow[rr, "\psi_{\mathcal{X}}"] & & T^*\mathcal{X} \arrow[r] & 0.
\end{tikzcd}$$

If $\boldsymbol{f} = (f, f', f'') : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism, the *identity 2-morphism* $\mathbf{id}_{\boldsymbol{f}} : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ is $\mathbf{id}_{\boldsymbol{f}} = (\mathrm{id}_f, 0)$.

Suppose $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ are 1-morphisms and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$, $\boldsymbol{\zeta} : \boldsymbol{g} \Rightarrow \boldsymbol{h}$ are 2-morphisms. Writing $\boldsymbol{f} = (f, f', f'')$, $\boldsymbol{\eta} = (\eta, \eta')$, and so on, we have 2-morphisms $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ in $\mathbf{C^\infty Sta}$, so that $\zeta \odot \eta : f \Rightarrow h$ is a 2-morphism in $\mathbf{C^\infty Sta}$ by vertical composition of 2-morphisms in $\mathbf{C^\infty Sta}$. Composing the analogue of (9.13) for $\boldsymbol{\zeta}$ with $\eta^{-1}(\mathcal{O}'_{\mathcal{Y}}), \eta^*(\mathcal{E}_{\mathcal{Y}})$ and using properties of the natural isomorphisms η^{-1}, η^* yields

$$\begin{aligned} h' \circ (\zeta \odot \eta)^{-1}(\mathcal{O}'_{\mathcal{Y}}) &= \big[h' \circ \zeta^{-1}(\mathcal{O}'_{\mathcal{Y}})\big] \circ \eta^{-1}(\mathcal{O}'_{\mathcal{Y}}) \\ &= \big[g' + \kappa_{\mathcal{X}} \circ \jmath_{\mathcal{X}} \circ \zeta' \circ (\mathrm{id} \otimes (g^\sharp \circ g^{-1}(\imath_{\mathcal{Y}}))) \circ (g^{-1}(\mathrm{d}))\big] \circ \eta^{-1}(\mathcal{O}'_{\mathcal{Y}}) \\ &= g' \circ \eta^{-1}(\mathcal{O}'_{\mathcal{Y}}) + \kappa_{\mathcal{X}} \circ \jmath_{\mathcal{X}} \circ \zeta' \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) \circ (\mathrm{id} \otimes (f^\sharp \circ f^{-1}(\imath_{\mathcal{Y}}))) \circ (f^{-1}(\mathrm{d})), \\ h'' \circ (\zeta \odot \eta)^*(\mathcal{E}_{\mathcal{Y}}) &= \big[h'' \circ \zeta^*(\mathcal{E}_{\mathcal{Y}})\big] \circ \eta^*(\mathcal{E}_{\mathcal{Y}}) = \big[g'' + \zeta' \circ g^*(\phi_{\mathcal{Y}})\big] \circ \eta^*(\mathcal{E}_{\mathcal{Y}}) \\ &= g'' \circ \eta^*(\mathcal{E}_{\mathcal{Y}}) + \zeta' \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) \circ f^*(\phi_{\mathcal{Y}}). \end{aligned}$$

Combining these with (9.13) yields

$$\begin{aligned} h' \circ (\zeta \odot \eta)^{-1}(\mathcal{O}'_{\mathcal{Y}}) &= f' + \kappa_{\mathcal{X}} \circ \jmath_{\mathcal{X}} \circ \theta' \circ \big(\mathrm{id} \otimes (f^\sharp \circ f^{-1}(\imath_{\mathcal{Y}}))\big) \circ \big(f^{-1}(\mathrm{d})\big), \\ h'' \circ (\zeta \odot \eta)^*(\mathcal{E}_{\mathcal{Y}}) &= f'' + \theta' \circ f^*(\phi_{\mathcal{Y}}), \\ \text{with} \quad \theta' &= \zeta' \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) + \eta'. \end{aligned}$$

Thus we may define the *vertical composition of* 2*-morphisms* $\boldsymbol{\zeta} \odot \boldsymbol{\eta}$ to be

$$\boldsymbol{\zeta} \odot \boldsymbol{\eta} = \big(\zeta \odot \eta, \zeta' \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) + \eta'\big) : \boldsymbol{f} \Rightarrow \boldsymbol{h}.$$

Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ be d-stacks, $\boldsymbol{f}, \tilde{\boldsymbol{f}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ and $\boldsymbol{g}, \tilde{\boldsymbol{g}} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms, and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{f}}$, $\boldsymbol{\zeta} : \boldsymbol{g} \Rightarrow \tilde{\boldsymbol{g}}$ be 2-morphisms. A similar proof to that in Definition 2.14 shows that we have a 2-morphism $\boldsymbol{\zeta} * \boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{g}} \circ \tilde{\boldsymbol{f}}$ given by

$$\boldsymbol{\zeta} * \boldsymbol{\eta} = \big(\zeta * \eta, \big[\eta' \circ f^*(g^2) + f'' \circ f^*(\zeta') + \eta' \circ f^*(\phi_{\mathcal{Y}}) \circ f^*(\zeta')\big] \circ I_{f,g}(\mathcal{F}_{\mathcal{Z}})\big).$$

This is the *horizontal composition of* 2*-morphisms* $\boldsymbol{\eta}, \boldsymbol{\zeta}$. This completes the definition of the 2-category of d-stacks **dSta**.

Write $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$ for the 2-category of separated, second countable, locally fair Deligne–Mumford C^∞-stacks. Define a strict 2-functor $F_{\mathbf{dSta}}^{\mathbf{C^\infty Sta}} : \mathbf{dSta} \to \mathbf{DMC^\infty Sta^{lf}_{ssc}}$ to map $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}}) \mapsto \mathcal{X}$ on objects, $\boldsymbol{f} = (f, f', f'') \mapsto f$ on 1-morphisms, and $\boldsymbol{\eta} = (\eta, \eta') \mapsto \eta$ on 2-morphisms.

Define a strict 2-functor $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}} : \mathbf{DMC^\infty Sta^{lf}_{ssc}} \to \mathbf{dSta}$ to map objects $\mathcal{X}$ to $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}, 0, \mathrm{id}_{\mathcal{O}_{\mathcal{X}}}, 0)$, to map 1-morphisms f to $\boldsymbol{f} = (f, f^\sharp, 0)$, and to map 2-morphisms η to $\boldsymbol{\eta} = (\eta, 0)$. Write $\mathbf{D\hat{M}C^\infty Sta^{lf}_{ssc}}$ for the full 2-subcategory of $\boldsymbol{\mathcal{X}} \in \mathbf{dSta}$ equivalent to $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\mathcal{X})$ for $\mathcal{X} \in \mathbf{DMC^\infty Sta^{lf}_{ssc}}$. When we say that a d-stack $\boldsymbol{\mathcal{X}}$ *is a* C^∞*-stack*, we mean that $\boldsymbol{\mathcal{X}} \in \mathbf{D\hat{M}C^\infty Sta^{lf}_{ssc}}$.

Define a strict 2-functor $F_{\mathbf{Orb}}^{\mathbf{dSta}} : \mathbf{Orb} \to \mathbf{dSta}$ by $F_{\mathbf{Orb}}^{\mathbf{dSta}} = F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}|_{\mathbf{Orb}}$, noting that **Orb** is a full 2-subcategory of $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$. Write $\mathbf{\hat{O}rb}$ for the full 2-subcategory of objects $\boldsymbol{\mathcal{X}}$ in **dSta** equivalent to $F_{\mathbf{Orb}}^{\mathbf{dSta}}(\mathcal{X})$ for some orbifold $\mathcal{X}$. When we say that a d-stack $\boldsymbol{\mathcal{X}}$ *is an orbifold*, we mean that $\boldsymbol{\mathcal{X}} \in \mathbf{\hat{O}rb}$.

Recall from §C.1 that there is a natural (2-)functor $\mathbf{C^\infty Sch} \to \mathbf{C^\infty Sta}$ mapping $\underline{X} \mapsto \bar{\underline{X}}$ on objects and $\underline{f} \mapsto \bar{\underline{f}}$ on morphisms. Also, if $\underline{X}$ is a C^∞-scheme and $\bar{\underline{X}}$ the corresponding C^∞-stack then Example C.32 defines a functor $\mathcal{I}_{\underline{X}} : \mathcal{O}_X\text{-mod} \to \mathcal{O}_{\bar{\underline{X}}}\text{-mod}$. In the same way, we can define functors from the category of sheaves of abelian groups on $\underline{X}$ to the category of sheaves of abelian groups on $\bar{\underline{X}}$, and from the category of sheaves of C^∞-rings on $\underline{X}$ to the category of sheaves of C^∞-rings on $\bar{\underline{X}}$, both of which we also denote by $\mathcal{I}_{\underline{X}}$.

With this notation, define a strict 2-functor $F_{\mathbf{dSpa}}^{\mathbf{dSta}} : \mathbf{dSpa} \to \mathbf{dSta}$ to map $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ to $\boldsymbol{\mathcal{X}} = \big(\bar{\underline{X}}, \mathcal{I}_{\underline{X}}(\mathcal{O}'_X), \mathcal{I}_{\underline{X}}(\mathcal{E}_X), \mathcal{I}_{\underline{X}}(\imath_X), \mathcal{I}_{\underline{X}}(\jmath_X)\big)$ on objects, and to map $\boldsymbol{f} = (\underline{f}, f', f'')$ to $\hat{\boldsymbol{f}} = \big(\bar{\underline{f}}, \mathcal{I}_{\underline{X}}(f'), \mathcal{I}_{\underline{X}}(f'')\big)$ on 1-morphisms, and to map η to $\boldsymbol{\eta} = \big(\mathrm{id}_{\bar{\underline{f}}}, \mathcal{I}_{\underline{X}}(\eta)\big)$ on 2-morphisms. Write $\mathbf{d\hat{S}pa}$ for the full 2-subcategory of $\boldsymbol{\mathcal{X}}$ in **dSta** equivalent to $F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{X})$ for some $\boldsymbol{X}$ in **dSpa**.

Here is the analogue of Theorem 2.15. The proof follows that of Theorem 2.15, but inserting extra canonical isomorphisms like $\eta^{-1}(\mathcal{O}'_{\mathcal{Y}}), \eta^*(\mathcal{E}_{\mathcal{Y}})$ coming from 2-morphisms η in $\mathbf{C^\infty Sta}$. We leave it as an exercise.

Theorem 9.7. (a) *Definition* 9.6 *defines a (strict)* 2*-category* **dSta**, *in which all* 2*-morphisms are* 2*-isomorphisms.*

(b) $F_{\mathbf{dSta}}^{\mathbf{C^\infty Sta}}, F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}, F_{\mathbf{Orb}}^{\mathbf{dSta}}$ *and* $F_{\mathbf{dSpa}}^{\mathbf{dSta}}$ *are (strict)* 2*-functors.*

(c) $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}, F_{\mathbf{Orb}}^{\mathbf{dSta}}$ *and* $F_{\mathbf{dSpa}}^{\mathbf{dSta}}$ *are full and faithful in the* 2*-categorical sense. Hence* $\mathbf{DMC^\infty Sta^{lf}_{ssc}}, \mathbf{Orb}, \mathbf{dSpa}$ *and* $\mathbf{D\hat{M}C^\infty Sta^{lf}_{ssc}}, \mathbf{\hat{O}rb}, \mathbf{d\hat{S}pa}$ *are equivalent* 2*-categories, respectively.*

Remark 9.8. (a) If we replace all Deligne–Mumford C^∞-stacks $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ in Definition 9.6 by C^∞-schemes $\underline{X}, \underline{Y}, \underline{Z}$, and replace 2-morphisms $\eta : f \Rightarrow g$ in $\boldsymbol{\eta}$ above by identity 2-morphisms, so that $f = g$, then Definition 9.6 reduces to Definition 2.14. This is why the embedding $F_{\mathbf{dSpa}}^{\mathbf{dSta}} : \mathbf{dSpa} \hookrightarrow \mathbf{dSta}$ is a 2-functor. Thus, we can regard d-spaces as special examples of d-stacks, just as schemes are regarded as special examples of stacks.

(b) A d-stack $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$ consists of a 'classical' part $\mathcal{X}$, the underlying Deligne–Mumford C^∞-stack $\mathcal{X}$, and a 'derived' part, the data $\mathcal{O}'_{\mathcal{X}}$, $\mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}}$. The truncation functor $F_{\mathbf{dSta}}^{\mathbf{C^\infty Sta}}$ forgets the 'derived' information. The embedding functor $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}$ allows us to regard separated, second countable, locally fair Deligne–Mumford C^∞-stacks $\mathcal{X}$ as special examples of d-stacks, in which the 'derived' information $\mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}}$ is as simple as possible, with $\mathcal{O}'_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}$, $\imath_{\mathcal{X}} = \mathrm{id}_{\mathcal{O}_{\mathcal{X}}}$, and $\mathcal{E}_{\mathcal{X}} = \jmath_{\mathcal{X}} = 0$.

The 2-morphisms $\boldsymbol{\eta} = (\eta, \eta')$ in $\mathbf{dSta}$ combine a 2-morphism η of the 'classical' part $\mathcal{X}$, and a 2-morphism η' of the 'derived' part $\mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}}$, which corresponds to the 2-morphisms η in $\mathbf{dSpa}$ in §2.2. These two components η, η' do not interact very much, and have a rather different flavour.

Here is the d-stack analogue of Proposition 2.17, which we deduce from Proposition 9.4 rather than Proposition 2.13.

Proposition 9.9. *Suppose* $\boldsymbol{f} = (f, f', f'') : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *is a* 1*-morphism of d-stacks,* $g : \mathcal{X} \to \mathcal{Y}$ *a* 1*-morphism of* C^∞*-stacks,* $\eta : f \Rightarrow g$ *a* 2*-morphism, and* $\eta' : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{E}_{\mathcal{X}}$ *a morphism in* $\mathrm{qcoh}(\mathcal{X})$*. Then there exists a unique* 1*-morphism* $\boldsymbol{g} = (g, g', g'') : \boldsymbol{X} \to \boldsymbol{Y}$ *in* $\mathbf{dSta}$ *such that* $\boldsymbol{\eta} = (\eta, \eta') : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ *is a* 2*-morphism.*

We now give d-stack analogues of the material of §2.2. Here are the analogues of Propositions 2.20, 2.21 and 2.25. The proofs are straightforward generalizations, using Proposition 9.9 in the proofs of Propositions 9.11 and 9.12.

Proposition 9.10. *Suppose* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *is an equivalence in* $\mathbf{dSta}$*. Then* $f : \mathcal{X} \to \mathcal{Y}$ *is an equivalence in* $\mathbf{C^\infty Sta}$*, and* $f^4 : f^*(\mathcal{C}_{\mathcal{Y}}) \to \mathcal{C}_{\mathcal{X}}$*,* $f^5 : f^*(\mathcal{D}_{\mathcal{Y}}) \to \mathcal{D}_{\mathcal{X}}$ *are isomorphisms, and the following is a split short exact sequence in* $\mathrm{qcoh}(\mathcal{X})$*:*

$$0 \longrightarrow f^*(\mathcal{E}_{\mathcal{Y}}) \xrightarrow{f'' \oplus -f^*(\phi_{\mathcal{Y}})} \mathcal{E}_{\mathcal{X}} \oplus f^*(\mathcal{F}_{\mathcal{Y}}) \xrightarrow{\phi_{\mathcal{X}} \oplus f^2} \mathcal{F}_{\mathcal{X}} \longrightarrow 0. \tag{9.14}$$

Proposition 9.11. *Let* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *be a* 1*-morphism in* $\mathbf{dSta}$*. Suppose* $f : \mathcal{X} \to \mathcal{Y}$ *is an equivalence,* $f^4 : f^*(\mathcal{C}_{\mathcal{Y}}) \to \mathcal{C}_{\mathcal{X}}$ *is an isomorphism, and* (9.14) *is a split short exact sequence. Then* $\boldsymbol{f}$ *is an equivalence.*

Proposition 9.12. *Let $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$ be a d-stack. Then $\boldsymbol{\mathcal{X}}$ lies in $\mathbf{D\hat{M}C^\infty Sta^{lf}_{ssc}}$, that is, $\boldsymbol{\mathcal{X}}$ is equivalent to an object in the image of $F^{\mathbf{dSta}}_{\mathbf{C^\infty Sta}}$, which can be $F^{\mathbf{dSta}}_{\mathbf{C^\infty Sta}}(\mathcal{X})$, if and only if $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ has a left inverse.*

As in Definition C.4, a representable 1-morphism of C^∞-stacks $f : \mathcal{X} \to \mathcal{Y}$ is called *étale* if it is an equivalence *locally in the étale topology*. This is a weaker condition than being an isomorphism locally in the Zariski topology. For example, if a finite group G acts on a C^∞-scheme $\underline{U}$, then we have a quotient C^∞-stack $[\underline{U}/G]$ with atlas $\Pi : \bar{\underline{U}} \to [\underline{U}/G]$, and Π is étale, though it is not a local equivalence in the Zariski topology wherever the action of G is not free.

For d-stacks, we define étale 1-morphisms by the analogue of Corollary 2.24. By Propositions 9.10 and 9.11, the definition essentially means that $\boldsymbol{f}$ is étale if it is an equivalence locally in the étale topology, though we do not actually define the étale topology for d-stacks.

Definition 9.13. Let $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a 1-morphism of d-stacks. We call $\boldsymbol{f}$ *étale* if $f : \mathcal{X} \to \mathcal{Y}$ is an étale 1-morphism of C^∞-stacks, so in particular f is representable, and $f^4 : f^*(\mathcal{C}_{\mathcal{Y}}) \to \mathcal{C}_{\mathcal{X}}$ is an isomorphism, and (9.14) is a split short exact sequence in $\mathrm{qcoh}(\mathcal{X})$.

Definition 2.23 defined open d-subspaces and open covers for d-spaces. One can generalize this to d-stacks in two different ways, using either the Zariski or the étale topology. We use the Zariski topology.

Definition 9.14. Let $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$ be a d-stack. Suppose $\mathcal{U} \subseteq \mathcal{X}$ is an open C^∞-substack, in the Zariski topology, with inclusion 1-morphism $i_{\mathcal{U}} : \mathcal{U} \to \mathcal{X}$. Then $\boldsymbol{\mathcal{U}} = \bigl(\mathcal{U}, i_{\mathcal{U}}^{-1}(\mathcal{O}'_{\mathcal{X}}), i_{\mathcal{U}}^*(\mathcal{E}_{\mathcal{X}}), i_{\mathcal{U}}^\sharp \circ i_{\mathcal{U}}^{-1}(\imath_{\mathcal{X}}), i_{\mathcal{U}}^*(\jmath_{\mathcal{X}})\bigr)$ is a d-stack, where $i_{\mathcal{U}}^\sharp : i_{\mathcal{U}}^{-1}(\mathcal{O}_{\mathcal{X}}) \to \mathcal{O}_{\mathcal{U}}$ is as in Example C.44, and is an isomorphism as $i_{\mathcal{U}}$ is étale. We call $\boldsymbol{\mathcal{U}}$ an *open d-substack* of $\boldsymbol{\mathcal{X}}$. An *open cover* of a d-stack $\boldsymbol{\mathcal{X}}$ is a family $\{\boldsymbol{\mathcal{U}}_a : a \in A\}$ of open d-substacks $\boldsymbol{\mathcal{U}}_a$ of $\boldsymbol{\mathcal{X}}$ such that $\{\mathcal{U}_a : a \in A\}$ is an open cover of $\mathcal{X}$, in the Zariski topology.

9.3 D-stacks as quotients of d-spaces

In Definitions C.17, C.18 and C.19 we defined quotient C^∞-stacks $[\underline{X}/G]$, 1-morphisms $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$, and 2-morphisms $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$. Thus, we can build Deligne–Mumford C^∞-stacks $\mathcal{X}, \mathcal{Y}$ and their 1- and 2-morphisms out of C^∞-schemes $\underline{X}, \underline{Y}$ with actions of groups G, H, and equivariant morphisms $\underline{f}, \underline{g} : \underline{X} \to \underline{Y}$. All this also works for d-stacks and d-spaces.

Definition 9.15. Let $\boldsymbol{r}$ be an action of G by 1-isomorphisms on a d-space $\boldsymbol{X}$, in the sense of Definition 2.43. We will define a d-stack $\boldsymbol{\mathcal{X}}$, which we call the *quotient d-stack* $[\boldsymbol{X}/G]$. The definition involves some arbitrary choices, but is natural up to canonical 1-isomorphism in $\mathbf{dSta}$. The first component $\underline{r}$ of $\boldsymbol{r} = (\underline{r}, r', r'')$ is an action of G on the separated, second countable, locally fair C^∞-scheme $\underline{X}$ by isomorphisms. Hence the quotient C^∞-stack $\mathcal{X} = [\underline{X}/G]$ from Definition C.17 is a separated, second countable, locally fair Deligne–Mumford C^∞-stack, with étale atlas $\Pi : \bar{\underline{X}} \to \mathcal{X}$.

Now the action of G on $\underline{X}$ defines a *groupoid in C^∞-schemes* $G \times \underline{X} \rightrightarrows \underline{X}$, with $[G\times\underline{X} \rightrightarrows \underline{X}] = [\underline{X}/G] = \mathcal{X}$. A *quasicoherent* $(G\times\underline{X} \rightrightarrows \underline{X})$*-module* $(\mathcal{E},\Phi)$, in the sense of Definition C.34, is just a quasicoherent sheaf $\mathcal{E}$ on $\underline{X}$ together with a lift of the action of G on $\underline{X}$ up to $\mathcal{E}$. That is, $(\mathcal{E},\Phi)$ is a *G-equivariant quasicoherent sheaf* on $\underline{X}$. We write $\mathrm{qcoh}^G(\underline{X}) = \mathrm{qcoh}(G\times\underline{X} \rightrightarrows \underline{X})$. Theorem C.35 shows that $F_\Pi : \mathrm{qcoh}(\mathcal{X}) \to \mathrm{qcoh}^G(\underline{X})$ is an equivalence of categories. The data r'' in $\boldsymbol{r}$ is a lift of the action of G up to $\mathcal{E}_X$, so we can interpret $(\mathcal{E}_X, r'')$ as an object in $\mathrm{qcoh}^G(\underline{X})$. Since F_Π is an equivalence of categories, we can choose $\mathcal{E}_{\mathcal{X}} \in \mathrm{qcoh}(\mathcal{X})$ such that $F_\Pi(\mathcal{E}_{\mathcal{X}}) \cong (\mathcal{E}_X, r'')$. Fix a particular choice of isomorphism $F_\Pi(\mathcal{E}_{\mathcal{X}}) \cong (\mathcal{E}_X, r'')$.

In the same way as Definition C.34 and Theorem C.35, we can define categories of G-equivariant sheaves of abelian groups and C^∞-rings on $\underline{X}$, and equivalences to them from the categories of sheaves of abelian groups and sheaves of C^∞-rings on $\mathcal{X}$, which we will also write as F_Π. With this notation, the data $r^\sharp, r'$ in $\boldsymbol{r}$ are lifts of the action of G up to $\mathcal{O}_X, \mathcal{O}'_X$, so we can interpret $(\mathcal{O}_X, r^\sharp), (\mathcal{O}'_X, r')$ as G-equivariant sheaves of C^∞-rings on X, there is a natural isomorphism $F_\Pi(\mathcal{O}_{\mathcal{X}}) \cong (\mathcal{O}_X, r^\sharp)$, and we can choose a sheaf of C^∞-rings $\mathcal{O}'_{\mathcal{X}}$ on $\mathcal{X}$ with $F_\Pi(\mathcal{O}'_{\mathcal{X}}) \cong (\mathcal{O}'_X, r')$.

The data $\imath_X, \jmath_X$ in $\boldsymbol{X}$ is G-equivariant, and so yields morphisms of $(\mathcal{E}_X, r'')$, $(\mathcal{O}'_X, r'), (\mathcal{O}_X, r^\sharp)$. Hence as the functors F_Π are equivalences of categories, there are unique morphisms $\imath_{\mathcal{X}} : \mathcal{O}'_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ and $\jmath_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{I}_{\mathcal{X}}$ on $\mathcal{X}$ such that $F_\Pi(\imath_{\mathcal{X}}), F_\Pi(\jmath_{\mathcal{X}})$ are identified with $\imath_X, \jmath_X$ by the chosen isomorphisms $F_\Pi(\mathcal{E}_{\mathcal{X}}) \cong (\mathcal{E}_X, r'')$, $F_\Pi(\mathcal{O}'_{\mathcal{X}}) \cong (\mathcal{O}'_X, r')$, $F_\Pi(\mathcal{O}_{\mathcal{X}}) \cong (\mathcal{O}_X, r^\sharp)$. One can now show that $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$ is a d-stack, defined up to canonical 1-isomorphism, which we also write as $[\boldsymbol{X}/G]$.

Next let $\boldsymbol{X}, \boldsymbol{Y}$ be d-spaces, G, H finite groups, and $\boldsymbol{r} : G \to \mathrm{Aut}(\boldsymbol{X})$, $\boldsymbol{s} : H \to \mathrm{Aut}(\boldsymbol{Y})$ be actions of G, H on $\boldsymbol{X}, \boldsymbol{Y}$, so that we have quotient d-stacks $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$ and $\boldsymbol{\mathcal{Y}} = [\boldsymbol{Y}/H]$. Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism and $\rho : G \to H$ is a group morphism, satisfying $\boldsymbol{f} \circ \boldsymbol{r}(\gamma) = \boldsymbol{s}(\rho(\gamma)) \circ \boldsymbol{f}$ for all $\gamma \in G$ — again, this is an equality of 1-morphisms in **dSpa**, not just a 2-isomorphism. We will define a 1-morphism $\tilde{\boldsymbol{f}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in **dSta**, which we will also write as $[\boldsymbol{f}, \rho] : [\boldsymbol{X}/G] \to [\boldsymbol{Y}/H]$, and call a *quotient 1-morphism*.

Write $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ for the C^∞-stack 1-morphism $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$ defined in Definition C.18 using the morphism $\underline{f} : \underline{X} \to \underline{Y}$ in $\boldsymbol{f} = (\underline{f}, f', f'')$. We have categories $\mathrm{qcoh}^G(\underline{X}), \mathrm{qcoh}^H(\underline{Y})$ of equivariant quasicoherent sheaves on $\underline{X}, \underline{Y}$. Since $\underline{f} : \underline{X} \to \underline{Y}$ satisfies $\underline{f} \circ \underline{r}(\gamma) = \underline{s}(\rho(\gamma)) \circ \underline{f}$ for all $\gamma \in G$, we have a pullback functor $(\underline{f}, \rho)^* : \mathrm{qcoh}^H(\underline{Y}) \to \mathrm{qcoh}^G(\underline{X})$ given by $(\underline{f}, \rho)^*(\mathcal{E}, \Phi) = \big(\underline{f}^*(\mathcal{E}), \underline{f}^*(\Phi) \circ \rho\big)$ on objects and $(\underline{f}, \rho)^*(\alpha) = \underline{f}^*(\alpha)$ on morphisms.

As above we have $(\mathcal{E}_X, r'') \in \mathrm{qcoh}^G(\underline{X})$ and $(\mathcal{E}_Y, s'') \in \mathrm{qcoh}^H(\underline{Y})$, so we form $(\underline{f}, \rho)^*(\mathcal{E}_Y, s'') \in \mathrm{qcoh}^G(\underline{X})$. The morphism f'' in $\boldsymbol{f}$ is equivariant, so is a morphism $f'' : (\underline{f}, \rho)^*(\mathcal{E}_Y, s'') \to (\mathcal{E}_X, r'')$ in $\mathrm{qcoh}^G(\underline{X})$. Define $\tilde{f}'' : \tilde{f}^*(\mathcal{E}_{\mathcal{Y}}) \to$

$\mathcal{E}_{\mathcal{X}}$ to be the unique morphism in $\mathrm{qcoh}(\mathcal{X})$ such that the following commutes:

$$\begin{tikzcd} F_\Pi(\tilde f^*(\mathcal{E}_{\mathcal{Y}})) \arrow[r, "F_\Pi(\tilde f'')"'] \arrow[d, "\cong"] & F_\Pi(\mathcal{E}_{\mathcal{X}}) \arrow[d, "\cong"'] \\ (\underline{f},\rho)^*(\mathcal{E}_Y,s'') \arrow[r, "f''"] & (\mathcal{E}_X,r''), \end{tikzcd} \tag{9.15}$$

where the columns come from the isomorphisms $F_\Pi(\mathcal{E}_{\mathcal{X}})\cong(\mathcal{E}_X,r'')$, $F_\Pi(\mathcal{E}_{\mathcal{Y}})\cong(\mathcal{E}_Y,s'')$ chosen while constructing $[\boldsymbol{X}/G],[\boldsymbol{Y}/H]$. Since the functors F_Π are equivalences of categories, $\tilde f''$ is well-defined.

Similarly, we define $\tilde f':\tilde f^{-1}(\mathcal{O}'_{\mathcal{Y}})\to\mathcal{O}'_{\mathcal{X}}$ to be the unique morphism such that the following commutes:

$$\begin{tikzcd} F_\Pi(\tilde f^{-1}(\mathcal{O}'_{\mathcal{Y}})) \arrow[r, "F_\Pi(\tilde f')"'] \arrow[d, "\cong"] & F_\Pi(\mathcal{O}'_{\mathcal{X}}) \arrow[d, "\cong"'] \\ (f,\rho)^{-1}(\mathcal{O}'_Y,s') \arrow[r, "f'"] & (\mathcal{O}'_X,r'). \end{tikzcd}$$

Then $f^\sharp\circ f^{-1}(\imath_Y)=\imath_X\circ f'$ and $\jmath_X\circ f''=f^1\circ\underline{f}^*(\jmath_Y)$ imply $\tilde f^\sharp\circ\tilde f^{-1}(\imath_{\mathcal{Y}})=\imath_{\mathcal{X}}\circ\tilde f'$ and $\jmath_{\mathcal{X}}\circ\tilde f''=\tilde f^1\circ\tilde f^*(\jmath_{\mathcal{Y}})$, so $\tilde{\boldsymbol{f}}=(\tilde f,\tilde f',\tilde f'')$ is a 1-morphism $\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$, which we write as $[\boldsymbol{f},\rho]:[\boldsymbol{X}/G]\to[\boldsymbol{Y}/H]$.

Now let $\tilde{\boldsymbol{f}}=[\boldsymbol{f},\rho]:[\boldsymbol{X}/G]\to[\boldsymbol{Y}/H]$ and $\tilde{\boldsymbol{g}}=[\boldsymbol{g},\sigma]:[\boldsymbol{X}/G]\to[\boldsymbol{Y}/H]$ be two 1-morphisms of the above form, so that $\underline{f},\underline{g}:\underline{X}\to\underline{Y}$ and $\rho,\sigma:G\to H$ are morphisms. Suppose $\delta\in H$ satisfies $\delta^{-1}\,\sigma(\gamma)=\rho(\gamma)\,\delta^{-1}$ for all $\gamma\in G$, and $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{s}(\delta^{-1})\circ\boldsymbol{g}$ is a 2-morphism in **dSpa**, such that $\eta*\mathrm{id}_{\boldsymbol{r}(\gamma)}=\mathrm{id}_{\boldsymbol{s}(\sigma(\gamma))}*\eta$ for all $\gamma\in G$, using the diagram:

$$\begin{tikzcd} \boldsymbol{f}\circ\boldsymbol{r}(\gamma) \arrow[rr, equal] \arrow[d, Rightarrow, "\eta*\mathrm{id}_{\boldsymbol{r}(\gamma)}"] & & \boldsymbol{s}(\rho(\gamma))\circ\boldsymbol{f} \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{s}(\sigma(\gamma))}*\eta"'] \\ \boldsymbol{s}(\delta^{-1})\circ\boldsymbol{g}\circ\boldsymbol{r}(\gamma) \arrow[r, equal] & \boldsymbol{s}(\delta^{-1})\circ\boldsymbol{s}(\sigma(\gamma))\circ\boldsymbol{g} \arrow[r, equal] & \boldsymbol{s}(\rho(\gamma))\circ\boldsymbol{s}(\delta^{-1})\circ\boldsymbol{g}. \end{tikzcd}$$

Then Definition C.19 defines $[\delta]:[\underline{f},\rho]\Rightarrow[\underline{g},\sigma]$, which we write as $\zeta:\tilde f\Rightarrow\tilde g$.

Using $\eta*\mathrm{id}_{\boldsymbol{r}(\gamma)}=\mathrm{id}_{\boldsymbol{s}(\sigma(\gamma))}*\eta$ for $\gamma\in G$ we see that $\eta:\underline{f}^*(\mathcal{F}_Y)\to\mathcal{E}_X$ is G-equivariant, so is also a morphism $\eta:(\underline{f},\rho)^*(\mathcal{F}_Y,s^2)\to(\mathcal{E}_X,r'')$ in $\mathrm{qcoh}^G(\underline{X})$. Define $\zeta':\tilde f^{-1}(\mathcal{F}_{\mathcal{Y}})\to\mathcal{E}_{\mathcal{X}}$ to be the unique morphism in $\mathrm{qcoh}(\mathcal{X})$ such that the following commutes in $\mathrm{qcoh}^G(\underline{X})$:

$$\begin{tikzcd} F_\Pi(\tilde f^{-1}(\mathcal{F}_{\mathcal{Y}})) \arrow[r, "F_\Pi(\zeta')"'] \arrow[d, "\cong"] & F_\Pi(\mathcal{E}_{\mathcal{X}}) \arrow[d, "\cong"'] \\ (\underline{f},\rho)^*(\mathcal{F}_Y,s^2) \arrow[r, "\eta"] & (\mathcal{E}_X,r''), \end{tikzcd} \tag{9.16}$$

where the columns are induced by the isomorphisms $F_\Pi(\mathcal{E}_{\mathcal{X}})\cong(\mathcal{E}_X,r'')$ and $F_\Pi(\mathcal{F}_{\mathcal{Y}})\cong(\mathcal{F}_Y,s^2)$. This is well-defined as F_Π is an equivalence of categories.

We claim that $\boldsymbol{\zeta}=(\zeta,\zeta')$ is a 2-morphism $\boldsymbol{\zeta}:\tilde{\boldsymbol{f}}\Rightarrow\tilde{\boldsymbol{g}}$ in **dSta**, which we also write as $[\eta,\delta]:[\boldsymbol{f},\rho]\Rightarrow[\boldsymbol{g},\sigma]$, and call a *quotient 2-morphism*. We must verify

(9.13) for $\boldsymbol{\zeta}$. The second equation of (9.13) is $\tilde{g}'' \circ \zeta^*(\mathcal{E}_{\mathcal{Y}}) = \tilde{f}'' + \zeta' \circ \tilde{f}^*(\phi_{\mathcal{Y}})$. This follows from

$$g'' \circ \underline{g}^*(s(\delta^{-1})'') \circ I_{\underline{g},\underline{s}(\delta^{-1})}(\mathcal{E}_Y) = \bigl(s(\delta^{-1}) \circ g\bigr)'' = f'' + \eta \circ \underline{f}^*(\phi_Y),$$

which combines the third entry of (2.23) for $\boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{g}$ with the second equation of (2.25) for $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{g}$, and the commutative diagrams (9.15) and

$$\begin{tikzcd} F_\Pi(\tilde{f}^{-1}(\mathcal{E}_{\mathcal{Y}})) \arrow[r, "F_\Pi(\zeta^*(\mathcal{E}_{\mathcal{Y}}))"'] \arrow[d, "\cong"] & F_\Pi(\tilde{g}^*(\mathcal{E}_{\mathcal{Y}})) \arrow[r, "F_\Pi(\tilde{g}'')"'] \arrow[d, "\cong"] & F_\Pi(\mathcal{E}_{\mathcal{X}}) \arrow[d, "\cong"'] \\ (\underline{f},\rho)^*(\mathcal{E}_Y, s'') \arrow[r, "\underline{g}^*(s(\delta^{-1})'')\circ I_{\underline{g},\underline{s}(\delta^{-1})}(\mathcal{E}_Y)"] & (\underline{g},\sigma)^*(\mathcal{E}_Y, s'') \arrow[r, "g''"] & (\mathcal{E}_X, r''), \end{tikzcd} \tag{9.17}$$

$$\begin{tikzcd} F_\Pi(\tilde{f}^{-1}(\mathcal{E}_{\mathcal{Y}})) \arrow[r, "F_\Pi(\tilde{f}^*(\phi_{\mathcal{Y}}))"'] \arrow[d, "\cong"] & F_\Pi(\tilde{f}^{-1}(\mathcal{F}_{\mathcal{Y}})) \arrow[r, "F_\Pi(\zeta')"'] \arrow[d, "\cong"] & F_\Pi(\mathcal{E}_{\mathcal{X}}) \arrow[d, "\cong"'] \\ (\underline{f},\rho)^*(\mathcal{E}_Y, s'') \arrow[r, "\underline{f}^*(\phi_Y)"] & (\underline{f},\rho)^*(\mathcal{F}_Y, s^2) \arrow[r, "\eta"] & (\mathcal{E}_X, r''). \end{tikzcd} \tag{9.18}$$

Here the left hand square of (9.17) follows from the definition of $\zeta = [\delta]$, the left hand square of (9.18) is the pullback by $\underline{f}, \tilde{f}$ of the obvious relation between $\phi_Y, \phi_{\mathcal{Y}}$, and the right hand squares of (9.17)–(9.18) are (9.15) for $\tilde{g}$ and (9.16). The first equation of (9.13) for $\boldsymbol{\zeta}$ follows by a similar but longer argument.

As in Definitions C.17, C.18 and C.19, quotient 1- and 2-morphisms of d-stacks have the obvious, strictly functorial properties under compositions. For instance, if $[\boldsymbol{f}, \rho] : [\boldsymbol{X}/G] \to [\boldsymbol{Y}/H]$, $[\boldsymbol{g}, \sigma] : [\boldsymbol{Y}/H] \to [\boldsymbol{Z}/I]$ are quotient 1-morphisms then $[\boldsymbol{g}, \sigma] \circ [\boldsymbol{f}, \rho] = [\boldsymbol{g} \circ \boldsymbol{f}, \sigma \circ \rho] : [\boldsymbol{X}/G] \to [\boldsymbol{Z}/I]$ (these 1-morphisms are equal, not just 2-isomorphic), and if $[\boldsymbol{f}, \rho], [\boldsymbol{g}, \sigma], [\boldsymbol{h}, \tau] : [\boldsymbol{X}/G] \to [\boldsymbol{Y}/H]$ are 1-morphisms and $[\eta, \delta] : [\boldsymbol{f}, \rho] \Rightarrow [\boldsymbol{g}, \sigma]$, $[\zeta, \epsilon] : [\boldsymbol{g}, \sigma] \Rightarrow [\boldsymbol{h}, \tau]$ are quotient 2-morphisms then $[\zeta, \epsilon] \odot [\eta, \delta] = [(\mathrm{id}_{\boldsymbol{s}(\delta^{-1})} * \zeta) \odot \eta, \epsilon\delta] : [\boldsymbol{f}, \rho] \Rightarrow [\boldsymbol{h}, \tau]$.

Here is an analogue of Theorem C.25 for d-spaces and d-stacks. To prove it we use Theorem C.25 to get the underlying C^∞-scheme and C^∞-stack data $\underline{U}, i, \mathcal{U}, \underline{V}, j, \mathcal{V}, \underline{U}', \underline{f}, \zeta, \underline{g}, \theta, [\delta]$ in $\boldsymbol{U}, \boldsymbol{i}, \boldsymbol{\mathcal{U}}, \boldsymbol{U}, \boldsymbol{j}, \boldsymbol{\mathcal{V}}, \boldsymbol{U}', \boldsymbol{f}, \boldsymbol{\zeta}, \boldsymbol{g}, \boldsymbol{\theta}, [\lambda, \delta]$. Then we essentially run Definition 9.15 in reverse to construct the remaining sheaf data $\mathcal{O}'_U, \mathcal{E}_U, \imath_U, \jmath_U$ in $\boldsymbol{U}$, and similarly for $\boldsymbol{i}, \boldsymbol{\mathcal{U}}, \ldots, [\lambda, \delta]$, frequently using the fact that the F_Π are equivalences of categories. In Definition 9.15, to define $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$ we had to make an arbitrary choice of an object $\mathcal{E}_{\mathcal{X}} \in \mathrm{qcoh}(\mathcal{X})$ and isomorphism $F_\Pi(\mathcal{E}_{\mathcal{X}}) \cong (\mathcal{E}_X, r'')$. But to define $\mathcal{E}_U$ in $\boldsymbol{U}$ in part (a) we can set $(\mathcal{E}_U, r'') = F_\Pi(\mathcal{E}_{\mathcal{U}})$. The rest of the construction of $\boldsymbol{U}, \boldsymbol{i}, \ldots, [\lambda, \delta]$ is also natural, rather than natural up to canonical isomorphism.

Theorem 9.16. (a) *Let $\boldsymbol{\mathcal{X}}$ be a d-stack and $[x] \in \mathcal{X}_{\mathrm{top}}$, and write $G = \mathrm{Iso}_{\mathcal{X}}([x])$. Then there exists a quotient d-stack $[\boldsymbol{U}/G]$ and a 1-morphism $\boldsymbol{i} : [\boldsymbol{U}/G] \to \boldsymbol{\mathcal{X}}$ which is an equivalence with an open d-substack $\boldsymbol{\mathcal{U}}$ in $\boldsymbol{\mathcal{X}}$, and $i_{\mathrm{top}} : [u] \mapsto [x] \in \mathcal{U}_{\mathrm{top}} \subseteq \mathcal{X}_{\mathrm{top}}$ for some fixed point u of G in U.*

(b) *Let $\tilde{\boldsymbol{f}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a 1-morphism in* **dSta**, *and $[x] \in \mathcal{X}_{\mathrm{top}}$ with $\tilde{f}_{\mathrm{top}} : [x] \mapsto [y] \in \mathcal{Y}_{\mathrm{top}}$, and write $G = \mathrm{Iso}_{\mathcal{X}}([x])$ and $H = \mathrm{Iso}_{\mathcal{Y}}([y])$. Part* **(a)** *gives 1-morphisms $\boldsymbol{i} : [\boldsymbol{U}/G] \to \boldsymbol{\mathcal{X}}$, $\boldsymbol{j} : [\boldsymbol{V}/H] \to \boldsymbol{\mathcal{Y}}$ which are equivalences with open*

$\mathcal{U} \subseteq \mathcal{X}$, $\mathcal{V} \subseteq \mathcal{Y}$, *such that* $i_{\rm top} : [u] \mapsto [x] \in \mathcal{U}_{\rm top} \subseteq \mathcal{X}_{\rm top}$, $j_{\rm top} : [v] \mapsto [y] \in \mathcal{V}_{\rm top} \subseteq \mathcal{Y}_{\rm top}$ *for* u, v *fixed points of* G, H *in* U, V.

Then there exists a G*-invariant open d-subspace* $\boldsymbol{U}'$ *of* u *in* $\boldsymbol{U}$ *and a quotient* 1*-morphism* $[\boldsymbol{f}, \rho] : [\boldsymbol{U}'/G] \to [\boldsymbol{V}/H]$ *such that* $\boldsymbol{f}(u) = v$, *and* $\rho : G \to H$ *is* $\tilde{f}_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$, *fitting into a* 2*-commutative diagram:*

$$\begin{tikzcd} {[\boldsymbol{U}'/G]} \arrow[rr, "{[\boldsymbol{f},\rho]}"'] \arrow[d, "\boldsymbol{i}|_{[\boldsymbol{U}'/G]}"] & \arrow[d, phantom, "\boldsymbol{\zeta}\Uparrow"] & {[\boldsymbol{V}/H]} \arrow[d, "\boldsymbol{j}"'] \\ \boldsymbol{\mathcal{X}} \arrow[rr, "\tilde{\boldsymbol{f}}"] & & \boldsymbol{\mathcal{Y}}. \end{tikzcd}$$

(c) *Let* $\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *be* 1*-morphisms in* **dSta** *and* $\boldsymbol{\eta} : \tilde{\boldsymbol{f}} \Rightarrow \tilde{\boldsymbol{g}}$ *a* 2*-morphism, let* $[x] \in \mathcal{X}_{\rm top}$ *with* $\tilde{f}_{\rm top} : [x] \mapsto [y] \in \mathcal{Y}_{\rm top}$, *and write* $G = \mathrm{Iso}_{\mathcal{X}}([x])$ *and* $H = \mathrm{Iso}_{\mathcal{Y}}([y])$. *Part* **(a)** *gives* $\boldsymbol{i} : [\boldsymbol{U}/G] \to \boldsymbol{\mathcal{X}}$, $\boldsymbol{j} : [\boldsymbol{V}/H] \to \boldsymbol{\mathcal{Y}}$ *which are equivalences with open* $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{Y}}$ *and map* $i_{\rm top} : [u] \mapsto [x]$, $j_{\rm top} : [v] \mapsto [y]$ *for* u, v *fixed points of* G, H.

By making $\boldsymbol{U}'$ *smaller, we can take the same* $\boldsymbol{U}'$ *in* **(b)** *for both* $\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}}$. *Thus part* **(b)** *gives a* G*-invariant open* $\boldsymbol{U}' \subseteq \boldsymbol{U}$, *quotient morphisms* $[\boldsymbol{f}, \rho] : [\boldsymbol{U}'/G] \to [\boldsymbol{V}/H]$ *and* $[\boldsymbol{g}, \sigma] : [\boldsymbol{U}'/G] \to [\boldsymbol{V}/H]$ *with* $\boldsymbol{f}(u) = \boldsymbol{g}(u) = v$ *and* $\rho = \tilde{f}_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$, $\sigma = \tilde{g}_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$, *and* 2*-morphisms* $\boldsymbol{\zeta} : \tilde{\boldsymbol{f}} \circ \boldsymbol{i}|_{[\boldsymbol{U}'/G]} \Rightarrow \boldsymbol{j} \circ [\boldsymbol{f}, \rho]$, $\boldsymbol{\theta} : \tilde{\boldsymbol{g}} \circ \boldsymbol{i}|_{[\boldsymbol{U}'/G]} \Rightarrow \boldsymbol{j} \circ [\boldsymbol{g}, \sigma]$.

Then there exists a G*-invariant open neighbourhood* $\boldsymbol{U}''$ *of* u *in* $\boldsymbol{U}'$, *an element* $\delta \in H$ *with* $\sigma(\gamma) = \delta\, \rho(\gamma) \circ \delta^{-1}$ *for all* $\gamma \in G$, *and a* 2*-morphism* $\lambda : \boldsymbol{f}|_{\boldsymbol{U}''} \Rightarrow \boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{g}|_{\boldsymbol{U}''}$ *in* **dSpa** *with* $\lambda * \mathrm{id}_{\boldsymbol{r}(\gamma)|_{\boldsymbol{U}''}} = \mathrm{id}_{\boldsymbol{s}(\sigma(\gamma))|_{\boldsymbol{U}''}} * \lambda$ *for all* $\gamma \in G$, *so that* $[\lambda, \delta] : [\boldsymbol{f}|_{\boldsymbol{U}''}, \rho] \Rightarrow [\boldsymbol{g}|_{\boldsymbol{U}''}, \sigma]$ *is a quotient* 2*-morphism, and the following diagram of* 2*-morphisms in* **dSta** *commutes:*

$$\begin{tikzcd} \tilde{\boldsymbol{f}} \circ \boldsymbol{i}|_{[\boldsymbol{U}''/G]} \arrow[r, Rightarrow, "\boldsymbol{\eta} * \mathrm{id}_{\boldsymbol{i}|_{[\boldsymbol{U}''/G]}}"'] \arrow[d, Rightarrow, "\boldsymbol{\zeta}|_{[\boldsymbol{U}''/G]}"] & \tilde{\boldsymbol{g}} \circ \boldsymbol{i}|_{[\boldsymbol{U}''/G]} \arrow[d, Rightarrow, "\boldsymbol{\theta}|_{[\boldsymbol{U}''/G]}"'] \\ \boldsymbol{j} \circ [\boldsymbol{f}|_{\boldsymbol{U}''}, \rho] \arrow[r, Rightarrow, "\mathrm{id}_{\boldsymbol{j}} * [\lambda, \delta]"] & \boldsymbol{j} \circ [\boldsymbol{g}|_{\boldsymbol{U}''}, \sigma]. \end{tikzcd}$$

9.4 Gluing d-stacks by equivalences

Next we generalize the material of §2.4 to d-stacks. A new issue arises in passing from C^∞-schemes to C^∞-stacks, which is that the claims in §2.4 on independence of choice of 2-morphisms no longer hold for d-stacks, since they are not true for gluing C^∞-stacks and their 1-morphisms. This can be seen from the results of §C.2: Propositions C.9 and C.10 are analogues of Theorem 2.28 and Proposition 2.27 in §2.4 for C^∞-stacks, and Example C.11 shows that the independence of choice of 2-morphism in Proposition 2.27 fails for C^∞-stacks.

For d-spaces, we can join different choices of 2-morphisms using a partition of unity, but for C^∞-stacks 2-morphisms are discrete objects, and we cannot interpolate between them. So, in the analogue of §2.4 for d-stacks, we must impose extra conditions on the 2-morphisms in $\mathbf{C^\infty Sta}$, as in (9.19)–(9.20) below, to ensure that we can glue C^∞-stacks and their 1-morphisms on open C^∞-substacks, using Propositions C.9 and C.10. Apart from this, the generalization to C^∞-stacks is fairly straightforward: the proof of Theorem 2.28 is largely local arguments about sheaves on C^∞-schemes, and these lift to local arguments about sheaves on C^∞-stacks using the techniques of §9.1–§9.2. We explain in Remark C.27 and Example C.33 how to extend partition of unity arguments on C^∞-schemes $\underline{X}$ to Deligne–Mumford C^∞-stacks $\mathcal{X}$, by using partitions of unity on the coarse moduli C^∞-scheme $\underline{\mathcal{X}}_{\rm top}$.

In this way we can prove the following analogues of Proposition 2.27 and Theorems 2.28 and 2.33.

Proposition 9.17. *Suppose $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}$ are d-stacks, $\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{V}}\subseteq\boldsymbol{\mathcal{X}}$ are open d-substacks with $\boldsymbol{\mathcal{X}}=\boldsymbol{\mathcal{U}}\cup\boldsymbol{\mathcal{V}}$, $\boldsymbol{f}:\boldsymbol{\mathcal{U}}\rightarrow\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{g}:\boldsymbol{\mathcal{V}}\rightarrow\boldsymbol{\mathcal{Y}}$ are 1-morphisms, and $\boldsymbol{\eta}:\boldsymbol{f}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}\Rightarrow\boldsymbol{g}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}$ is a 2-morphism. Then there exist a 1-morphism $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ and 2-morphisms $\boldsymbol{\zeta}:\boldsymbol{h}|_{\boldsymbol{\mathcal{U}}}\Rightarrow\boldsymbol{f}$, $\boldsymbol{\theta}:\boldsymbol{h}|_{\boldsymbol{\mathcal{V}}}\Rightarrow\boldsymbol{g}$ in* **dSta** *such that $\boldsymbol{\theta}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}=\boldsymbol{\eta}\odot\boldsymbol{\zeta}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}:\boldsymbol{h}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}\Rightarrow\boldsymbol{g}|_{\boldsymbol{\mathcal{U}}\cap\boldsymbol{\mathcal{V}}}$. This $\boldsymbol{h}$ is unique up to 2-isomorphism.*

Furthermore, $\boldsymbol{h}$ is independent up to 2-isomorphism of the component η' in $\boldsymbol{\eta}=(\eta,\eta')$, but it may depend on η.

Theorem 9.18. *Suppose $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}$ are d-stacks, $\boldsymbol{\mathcal{U}}\subseteq\boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{V}}\subseteq\boldsymbol{\mathcal{Y}}$ are open d-substacks, and $\boldsymbol{f}:\boldsymbol{\mathcal{U}}\rightarrow\boldsymbol{\mathcal{V}}$ is an equivalence in* **dSta**. *At the level of topological spaces, we have open $\mathcal{U}_{\rm top}\subseteq\mathcal{X}_{\rm top}$, $\mathcal{V}_{\rm top}\subseteq\mathcal{Y}_{\rm top}$ with a homeomorphism $f_{\rm top}:\mathcal{U}_{\rm top}\rightarrow\mathcal{V}_{\rm top}$, so we can form the quotient topological space $\mathcal{Z}_{\rm top}:=\mathcal{X}_{\rm top}\amalg_{f_{\rm top}}\mathcal{Y}_{\rm top}=(\mathcal{X}_{\rm top}\amalg\mathcal{Y}_{\rm top})/\sim$, where the equivalence relation $\sim$ on $\mathcal{X}_{\rm top}\amalg\mathcal{Y}_{\rm top}$ identifies $[u]\in\mathcal{U}_{\rm top}\subseteq\mathcal{X}_{\rm top}$ with $f_{\rm top}([u])\in\mathcal{V}_{\rm top}\subseteq\mathcal{Y}_{\rm top}$.*

Suppose $\mathcal{Z}_{\rm top}$ is Hausdorff. This condition may also equivalently be imposed at the level of C^∞-stacks, that is, we may form a pushout C^∞-stack $\mathcal{Z}=\mathcal{X}\amalg_f\mathcal{Y}$ by Proposition C.9, *and we require $\mathcal{Z}$ separated. Then there exist a d-stack $\boldsymbol{\mathcal{Z}}$, open d-substacks $\hat{\boldsymbol{\mathcal{X}}},\hat{\boldsymbol{\mathcal{Y}}}$ in $\boldsymbol{\mathcal{Z}}$ with $\boldsymbol{\mathcal{Z}}=\hat{\boldsymbol{\mathcal{X}}}\cup\hat{\boldsymbol{\mathcal{Y}}}$, equivalences $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\rightarrow\hat{\boldsymbol{\mathcal{X}}}$ and $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\rightarrow\hat{\boldsymbol{\mathcal{Y}}}$ such that $\boldsymbol{g}|_{\boldsymbol{\mathcal{U}}}$ and $\boldsymbol{h}|_{\boldsymbol{\mathcal{V}}}$ are both equivalences with $\hat{\boldsymbol{\mathcal{X}}}\cap\hat{\boldsymbol{\mathcal{Y}}}$, and a 2-morphism $\boldsymbol{\eta}:\boldsymbol{g}|_{\boldsymbol{\mathcal{U}}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}:\boldsymbol{\mathcal{U}}\rightarrow\hat{\boldsymbol{\mathcal{X}}}\cap\hat{\boldsymbol{\mathcal{Y}}}$. Furthermore, $\boldsymbol{\mathcal{Z}}$ is independent of choices up to equivalence.*

Theorem 9.19. *Suppose I is an indexing set, and $<$ is a total order on I, and $\boldsymbol{\mathcal{X}}_i$ for $i\in I$ are d-stacks, and for all $i<j$ in I we are given open d-substacks $\boldsymbol{\mathcal{U}}_{ij}\subseteq\boldsymbol{\mathcal{X}}_i$, $\boldsymbol{\mathcal{U}}_{ji}\subseteq\boldsymbol{\mathcal{X}}_j$ and an equivalence $\boldsymbol{e}_{ij}:\boldsymbol{\mathcal{U}}_{ij}\rightarrow\boldsymbol{\mathcal{U}}_{ji}$, satisfying the following properties:*

(a) *For all $i<j<k$ in I we have a 2-commutative diagram*

$$\begin{tikzcd}
& \boldsymbol{\mathcal{U}}_{ji}\cap\boldsymbol{\mathcal{U}}_{jk} \arrow[dr, "\boldsymbol{e}_{jk}|_{\boldsymbol{\mathcal{U}}_{ji}\cap\boldsymbol{U}_{jk}}"] \arrow[d, phantom, "\Downarrow\boldsymbol{\eta}_{ijk}"] & \\
\boldsymbol{\mathcal{U}}_{ij}\cap\boldsymbol{\mathcal{U}}_{ik} \arrow[ur, "\boldsymbol{e}_{ij}|_{\boldsymbol{\mathcal{U}}_{ij}\cap\boldsymbol{\mathcal{U}}_{ik}}"] \arrow[rr, "\boldsymbol{e}_{ik}|_{\boldsymbol{\mathcal{U}}_{ij}\cap\boldsymbol{\mathcal{U}}_{ik}}"] & {} & \boldsymbol{\mathcal{U}}_{ki}\cap\boldsymbol{\mathcal{U}}_{kj}
\end{tikzcd}$$

for some $\boldsymbol{\eta}_{ijk}$, where all three 1-morphisms are equivalences; and

(b) *For all $i<j<k<l$ in I the components η_{ijk} in $\boldsymbol{\eta}_{ijk}=(\eta_{ijk},\eta'_{ijk})$ satisfy*

$$\eta_{ikl}\odot(\mathrm{id}_{f_{kl}}*\eta_{ijk})|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}\cap\mathcal{U}_{il}}=\eta_{ijl}\odot(\eta_{jkl}*\mathrm{id}_{f_{ij}})|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}\cap\mathcal{U}_{il}}. \quad (9.19)$$

Note that if the C^∞-stacks $\mathcal{X}_i$ in $\boldsymbol{\mathcal{X}}_i$ are effective for all $i \in I$, then Proposition C.29(ii) *implies that* (9.19) *holds automatically, as there is only one* 2*-morphism* $\boldsymbol{e}_{kl} \circ \boldsymbol{e}_{jk} \circ \boldsymbol{e}_{ij}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}\cap\mathcal{U}_{il}} \Rightarrow \boldsymbol{e}_{il}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}\cap\mathcal{U}_{il}}$.

On the level of topological spaces, define the quotient topological space $\mathcal{Y}_{\rm top} = (\coprod_{i\in I} \mathcal{X}_{i,\rm top})/\sim$, *where* $\sim$ *is the equivalence relation generated by* $[x_i] \sim [x_j]$ *if* $[x_i] \in \mathcal{U}_{ij,\mathcal{X}_i,\rm top} \subseteq \mathcal{X}_{i,\rm top}$ *and* $[x_j] \in \mathcal{U}_{ji,\rm top} \subseteq \mathcal{X}_{j,\rm top}$ *with* $e_{ij,\rm top}([x_i]) = [x_j]$. *Suppose* $\mathcal{Y}_{\rm top}$ *is Hausdorff and second countable. Then there exist a d-stack* $\boldsymbol{\mathcal{Y}}$ *and a* 1*-morphism* $\boldsymbol{f}_i : \boldsymbol{\mathcal{X}}_i \to \boldsymbol{\mathcal{Y}}$ *which is an equivalence with an open d-substack* $\hat{\boldsymbol{\mathcal{X}}}_i \subseteq \boldsymbol{\mathcal{Y}}$ *for all* $i \in I$, *where* $\boldsymbol{\mathcal{Y}} = \bigcup_{i\in I} \hat{\boldsymbol{\mathcal{X}}}_i$, *such that* $\boldsymbol{f}_i|_{\boldsymbol{\mathcal{U}}_{ij}}$ *is an equivalence* $\boldsymbol{\mathcal{U}}_{ij} \to \hat{\boldsymbol{\mathcal{X}}}_i \cap \hat{\boldsymbol{\mathcal{X}}}_j$ *for all* $i < j$ *in* I, *and there exists a* 2*-morphism* $\boldsymbol{\eta}_{ij} : \boldsymbol{f}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{f}_i|_{\boldsymbol{\mathcal{U}}_{ij}}$. *The d-stack* $\boldsymbol{\mathcal{Y}}$ *is unique up to equivalence, and is independent of choices of the components* η'_{ijk} *in* $\boldsymbol{\eta}_{ijk} = (\eta_{ijk}, \eta'_{ijk})$ *in* (a). *If the* C^∞*-stacks* $\mathcal{X}_i$ *in* $\boldsymbol{\mathcal{X}}_i$ *are effective for all* $i \in I$, *then the* η_{ijk} *are unique by Proposition* C.29(ii), *so* $\boldsymbol{\mathcal{Y}}$ *is independent of the choices of* $\boldsymbol{\eta}_{ijk}$ *in* (a).

Suppose also that $\boldsymbol{\mathcal{Z}}$ *is a d-stack, and* $\boldsymbol{g}_i : \boldsymbol{\mathcal{X}}_i \to \boldsymbol{\mathcal{Z}}$ *are* 1*-morphisms for all* $i \in I$, *and there exist* 2*-morphisms* $\boldsymbol{\zeta}_{ij} : \boldsymbol{g}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{g}_i|_{\boldsymbol{\mathcal{U}}_{ij}}$ *for all* $i < j$ *in* I, *such that for all* $i < j < k$ *in* I *the components* ζ_{ij}, η_{ijk} *in* $\boldsymbol{\zeta}_{ij}, \boldsymbol{\eta}_{ijk}$ *satisfy*

$$\bigl(\zeta_{ij}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\bigr) \odot \bigl(\zeta_{jk} * \mathrm{id}_{e_{ij}}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\bigr) = \bigl(\zeta_{ik}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\bigr) \odot \bigl(\mathrm{id}_{g_k} * \eta_{ijk}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\bigr). \tag{9.20}$$

Then there exist a 1*-morphism* $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ *and* 2*-morphisms* $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{f}_i \Rightarrow \boldsymbol{g}_i$ *for all* $i \in I$. *The* 1*-morphism* $\boldsymbol{h}$ *is unique up to* 2*-isomorphism, and is independent of the components* ζ'_{ij} *in* $\boldsymbol{\zeta}_{ij} = (\zeta_{ij}, \zeta'_{ij})$.

Note that Proposition C.29 *gives conditions for uniqueness of* C^∞*-stack* 2*-morphisms* $\eta : f \Rightarrow g$, *and if any of these conditions apply to* $g_i : \mathcal{X}_i \to \mathcal{Z}$ *for all* $i \in I$, *then* (9.20) *holds automatically, as there is only one* 2*-morphism* $g_k \circ e_{jk} \circ e_{ij}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}} \Rightarrow g_i|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}$, *and also* ζ_{ij} *is unique, so* $\boldsymbol{h}$ *is independent of the choice of* $\boldsymbol{\zeta}_{ij}$. *In particular, if* $\boldsymbol{\mathcal{Z}}$ *is a d-space, so that* $\mathcal{Z}$ *is a* C^∞*-scheme, then* (9.20) *always holds, and* $\boldsymbol{h}$ *is independent of the choice of* $\boldsymbol{\zeta}_{ij}$.

Remark 9.20. In §4.7 we pointed out that many of the results on d-spaces and d-manifolds in Chapters 2–4 can be stated in terms of the homotopy categories $\mathrm{Ho}(\mathbf{dSpa}), \mathrm{Ho}(\mathbf{dMan})$, and so we can for some purposes treat $\mathbf{dSpa}, \mathbf{dMan}$ as categories rather than 2-categories. Theorem 9.19 is an example in which this fails for d-stacks: the 2-morphism overlap conditions (9.19)–(9.20) do not make sense in $\mathrm{Ho}(\mathbf{dSta})$, they are a genuinely 2-categorical phenomenon.

However, in Theorem 9.19 we give conditions for (9.19)–(9.20) to hold automatically, and under these conditions the theorem makes sense in $\mathrm{Ho}(\mathbf{dSta})$. That is, we can glue d-stacks $\boldsymbol{\mathcal{X}}_i$ whose C^∞-stacks $\mathcal{X}_i$ are effective by isomorphisms $[\boldsymbol{e}_{ij}]$ in $\mathrm{Ho}(\mathbf{dSta})$ on overlaps $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j$, provided $[\boldsymbol{e}_{jk}] \circ [\boldsymbol{e}_{ij}] = [\boldsymbol{e}_{ik}]$ in $\mathrm{Ho}(\mathbf{dSta})$ on triple overlaps $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k$.

Similarly, Proposition C.10 does not descend to $\mathrm{Ho}(\mathbf{DMC^\infty Sta})$, unless we impose conditions as in Proposition C.29 to ensure uniqueness of 2-morphisms.

9.5 Fibre products of d-stacks

We can also modify the material of §2.5–§2.6 to work for d-stacks. This is easier than for §2.4. Given 1-morphisms of d-stacks $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Z}}$, $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{Z}}$, following Definition 2.35 we can define an explicit d-stack $\boldsymbol{\mathcal{W}}$, 1-morphisms $\boldsymbol{e}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{X}}$, $\boldsymbol{f}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{Y}}$ and a 2-morphism $\boldsymbol{\eta}:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ such that $\boldsymbol{\mathcal{W}},\boldsymbol{e},\boldsymbol{f},\boldsymbol{\eta}$ is a fibre product $\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}$ in **dSta**. We will not repeat the long definition, but will comment briefly on the differences.

Define $\mathcal{W}=\mathcal{X}\times_{g,\mathcal{Z},h}\mathcal{Y}$ to be the fibre product in $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$, which exists by Theorem C.22(a), and fits into a 2-Cartesian square in $\mathbf{C^\infty Sta}$:

$$\begin{tikzcd} \mathcal{W} \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\Uparrow\eta"] & \mathcal{Y} \arrow[d, "h"] \\ \mathcal{X} \arrow[r, "g"] & \mathcal{Z}. \end{tikzcd}$$

This defines $\mathcal{W},e,f$ and η. The 2-morphism $\eta:g\circ e\Rightarrow h\circ f$ then induces natural isomorphisms $\eta^{-1}:(g\circ e)^{-1}\Rightarrow(h\circ f)^{-1}$ for sheaves of abelian groups and C^∞-rings, and $\eta^*:(g\circ e)^*\Rightarrow(h\circ f)^*$ for quasicoherent sheaves, as in §C.6–§C.7, and at various points we have to insert terms like $\eta^{-1}(\mathcal{O}'_{\mathcal{Z}})$ and $\eta^*(\mathcal{E}_{\mathcal{Z}})$ in the formulae of Definition 2.35. For instance, (2.59) becomes

$$\begin{tikzcd}[column sep=huge] (g\circ e)^*(\mathcal{E}_{\mathcal{Z}}) \arrow[r, "\alpha_1:=\begin{pmatrix} e^*(g'')\circ I_{e,g}(\mathcal{E}_{\mathcal{Z}}) \\ -f^*(h'')\circ I_{f,h}(\mathcal{E}_{\mathcal{Z}})\circ\eta^*(\mathcal{E}_{\mathcal{Z}}) \\ (g\circ e)^*(\phi_{\mathcal{Z}}) \end{pmatrix}"] & \begin{array}{l} e^*(\mathcal{E}_{\mathcal{X}})\oplus \\ f^*(\mathcal{E}_{\mathcal{Y}})\oplus \\ (g\circ e)^*(\mathcal{F}_{\mathcal{Z}}) \end{array} \arrow[r, "\alpha_2:=\begin{pmatrix} e^1\circ e^*(\jmath_{\mathcal{X}}) \\ f^1\circ f^*(\jmath_{\mathcal{Y}}) \\ \mu \end{pmatrix}^T"] & \mathcal{I}_{\mathcal{W}}. \end{tikzcd} \tag{9.21}$$

Apart from this, there are few differences. The computations with sheaves on $\underline{W}$ in §2.5 lift to computations with sheaves on $\mathcal{W}$ with little change. Thus we prove an analogue of Theorem 2.36. Here $F_{\mathbf{dSpa}}^{\mathbf{dSta}}$ preserves fibre products as it maps the explicit construction of fibre products in **dSpa** in Definition 2.35 to the explicit construction of fibre products in **dSta** described above.

Theorem 9.21. *All fibre products exist in the* 2*-category* **dSta**. *The* 2*-functors* $F_{\mathbf{dSta}}^{\mathbf{C^\infty Sta}}:\mathbf{dSta}\to\mathbf{C^\infty Sta}$ *and* $F_{\mathbf{dSpa}}^{\mathbf{dSta}}:\mathbf{dSpa}\to\mathbf{dSta}$ *preserve fibre products.*

As for Corollary 2.37, from Proposition 9.12 and Theorem 9.21 we deduce:

Corollary 9.22. *Suppose* $g:\mathcal{X}\to\mathcal{Z}$, $h:\mathcal{Y}\to\mathcal{Z}$ *are* 1*-morphisms in* $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$, *and let* $\mathcal{W}=\mathcal{X}\times_{g,\mathcal{Z},h}\mathcal{Y}$ *be the fibre product in* $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$, *with projections* $e:\mathcal{W}\to\mathcal{X}$, $f:\mathcal{W}\to\mathcal{Y}$ *and* 2*-morphism* $\eta:g\circ e\Rightarrow h\circ f$. *Then* $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y})$ *is equivalent to* $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\mathcal{X})\times_{F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\mathcal{Z})}F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\mathcal{Y})$ *in* **dSta** *if and only if the morphism*

$$\begin{gathered} e^*(\Omega_g)\circ I_{e,g}(T^*\mathcal{Z})\oplus f^*(\Omega_h)\circ I_{f,h}(T^*\mathcal{Z})\circ\eta^*(T^*\mathcal{Z}): \\ (g\circ e)^*(T^*\mathcal{Z})\longrightarrow e^*(T^*\mathcal{X})\oplus f^*(T^*\mathcal{Y}) \end{gathered}$$

in $\mathrm{qcoh}(\mathcal{W})$ *has a left inverse.*

Here is the analogue of Theorem 2.42. It has essentially the same proof, using Propositions 9.11 and 9.12 and Theorem 8.2 rather than Propositions 2.21 and 2.25 and Theorem B.28.

Theorem 9.23. *Let $\mathcal{X},\mathcal{Y},\mathcal{Z}$ be orbifolds without boundary, and $g:\mathcal{X}\to\mathcal{Z}$, $h:\mathcal{Y}\to\mathcal{Z}$ be smooth maps (1-morphisms). Write $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}},\boldsymbol{g},\boldsymbol{h}=F_{\mathbf{Orb}}^{\mathbf{dSta}}(\mathcal{X},\mathcal{Y},\mathcal{Z},g,h)$, and let $\boldsymbol{\mathcal{W}}=\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{e}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{X}}$, $\boldsymbol{f}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{Y}}$ be the fibre product and projections from Theorem 9.21. Then*

(a) *Suppose g,h are transverse. Then a fibre product $\tilde{\mathcal{W}}=\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$ exists in* **Orb**, *with smooth projections $\tilde e=\pi_{\mathcal{X}}:\tilde{\mathcal{W}}\to\mathcal{X}$, $\tilde f=\pi_{\mathcal{Y}}:\tilde{\mathcal{W}}\to\mathcal{Y}$ and 2-morphism $\tilde\eta:g\circ\tilde e\Rightarrow h\circ\tilde f$. Set $\tilde{\boldsymbol{\mathcal{W}}},\tilde{\boldsymbol{e}},\tilde{\boldsymbol{f}},\tilde{\boldsymbol{\eta}}=F_{\mathbf{Orb}}^{\mathbf{dSta}}(\tilde W,\tilde e,\tilde f,\tilde\eta)$, so we have a 2-commutative diagram in* **dSta**:

$$\begin{tikzcd} \tilde{\boldsymbol{\mathcal{W}}} \arrow[r, "\tilde{\boldsymbol{f}}"'] \arrow[d, "\tilde{\boldsymbol{e}}"] \arrow[dr, phantom, "\tilde{\boldsymbol{\eta}}\Uparrow"] & \boldsymbol{\mathcal{Y}} \arrow[d, "\boldsymbol{h}"'] \\ \boldsymbol{\mathcal{X}} \arrow[r, "\boldsymbol{g}"] & \boldsymbol{\mathcal{Z}}. \end{tikzcd} \tag{9.22}$$

From (9.22) we get a 1-morphism $\boldsymbol{b}:\tilde{\boldsymbol{\mathcal{W}}}\to\boldsymbol{\mathcal{W}}$ and 2-morphisms $\boldsymbol{\zeta}:\boldsymbol{e}\circ\boldsymbol{b}\Rightarrow\tilde{\boldsymbol{e}}$, $\boldsymbol{\theta}:\boldsymbol{f}\circ\boldsymbol{b}\Rightarrow\tilde{\boldsymbol{f}}$ in **dSta**. *This $\boldsymbol{b}$ is an equivalence.*

(b) *Suppose g,h are not transverse. Then $\boldsymbol{\mathcal{W}}$ is not an orbifold. Thus, if a fibre product $\tilde{\mathcal{W}}=\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$ does exist in* **Orb**, *we have $F_{\mathbf{Orb}}^{\mathbf{dSta}}(\tilde{\mathcal{W}})\not\simeq\boldsymbol{\mathcal{W}}$.*

9.6 Orbifold strata of d-stacks

Finally we generalize the material of §C.8 to d-stacks. If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack and Γ a finite group, then in §C.8 we defined Deligne–Mumford C^∞-stacks $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma$, and open C^∞-substacks $\mathcal{X}^\Gamma_\circ\subseteq\mathcal{X}^\Gamma$, $\tilde{\mathcal{X}}^\Gamma_\circ\subseteq\tilde{\mathcal{X}}^\Gamma$, $\hat{\mathcal{X}}^\Gamma_\circ\subseteq\hat{\mathcal{X}}^\Gamma$, fitting into a strictly commutative diagram (C.8) in $\mathbf{C^\infty Sta}$. In exactly the same way, if $\boldsymbol{\mathcal{X}}$ is a d-stack and Γ a finite group, we will define d-stacks $\boldsymbol{\mathcal{X}}^\Gamma,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\hat{\boldsymbol{\mathcal{X}}}^\Gamma$, and open d-substacks $\boldsymbol{\mathcal{X}}^\Gamma_\circ\subseteq\boldsymbol{\mathcal{X}}^\Gamma$, $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ\subseteq\tilde{\boldsymbol{\mathcal{X}}}^\Gamma$, $\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ\subseteq\hat{\boldsymbol{\mathcal{X}}}^\Gamma$, and a d-space $\hat{\boldsymbol{X}}^\Gamma_\circ$, fitting into a strictly commutative diagram in **dSta**:

$$\begin{tikzcd} \boldsymbol{\mathcal{X}}^\Gamma_\circ \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde{\boldsymbol{\Pi}}^\Gamma_\circ(\boldsymbol{\mathcal{X}})"] \arrow[d, "\subset"', "\boldsymbol{O}^\Gamma_\circ(\boldsymbol{\mathcal{X}})"] \arrow[dr] & & \tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ \arrow[r, "\hat{\boldsymbol{\Pi}}^\Gamma_\circ(\boldsymbol{\mathcal{X}})"] \arrow[dl, "\tilde{\boldsymbol{O}}^\Gamma_\circ(\boldsymbol{\mathcal{X}})"'] \arrow[d, "\subset"] & \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ\simeq F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\hat{\boldsymbol{X}}^\Gamma_\circ) \arrow[d, "\subset"] \\ & \boldsymbol{\mathcal{X}} & & \\ \boldsymbol{\mathcal{X}}^\Gamma \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[ur, "\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})"] \arrow[rr, "\tilde{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}})"'] & & \tilde{\boldsymbol{\mathcal{X}}}^\Gamma \arrow[ul, "\tilde{\boldsymbol{O}}^\Gamma(\boldsymbol{\mathcal{X}})"] \arrow[r, "\hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}})"'] & \hat{\boldsymbol{\mathcal{X}}}^\Gamma. \end{tikzcd} \tag{9.23}$$

We will call $\boldsymbol{\mathcal{X}}^\Gamma,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\hat{\boldsymbol{\mathcal{X}}}^\Gamma,\boldsymbol{\mathcal{X}}^\Gamma_\circ,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ,\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ,\hat{\boldsymbol{X}}^\Gamma_\circ$ *orbifold strata* of $\boldsymbol{\mathcal{X}}$.

Definition 9.24. Let $\boldsymbol{\mathcal{X}}=(\mathcal{X},\mathcal{O}'_{\mathcal{X}},\mathcal{E}_{\mathcal{X}},\imath_{\mathcal{X}},\jmath_{\mathcal{X}})$ be a d-stack, and Γ a finite group. We will define a d-stack $\boldsymbol{\mathcal{X}}^\Gamma=(\mathcal{X}^\Gamma,\mathcal{O}'_{\mathcal{X}^\Gamma},\mathcal{E}_{\mathcal{X}^\Gamma},\imath_{\mathcal{X}^\Gamma},\jmath_{\mathcal{X}^\Gamma})$. The definition is modelled on Definition 2.43. Here $\mathcal{X}^\Gamma$ is defined in Definition C.45. By Theorem C.49(a),(b),(f),(g), $\mathcal{X}^\Gamma$ is a separated, second countable, locally fair

Deligne–Mumford C^∞-stack, as $\mathcal{X}$ is, and has a proper, strongly representable 1-morphism $O^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\to\mathcal{X}$ given in Definition C.47.

Now $\imath_{\mathcal{X}}:\mathcal{O}'_{\mathcal{X}}\to\mathcal{O}_{\mathcal{X}}$ is a morphism of sheaves of C^∞-rings on $\mathcal{X}$, where $\mathcal{O}_{\mathcal{X}}$ is the structure sheaf of $\mathcal{X}$ as in Example C.42. Pulling back to $\mathcal{X}^\Gamma$ gives sheaves of C^∞-rings $O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}}),O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\mathcal{X}})$ on $\mathcal{X}^\Gamma$, and a morphism $O^\Gamma(\mathcal{X})^{-1}(\imath_{\mathcal{X}}):O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})\to O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\mathcal{X}})$. Also, as $O^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\to\mathcal{X}$ is a 1-morphism of Deligne–Mumford C^∞-stacks, we have a natural morphism $O^\Gamma(\mathcal{X})^\sharp:O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\mathcal{X}})\to\mathcal{O}_{\mathcal{X}^\Gamma}$ explained in Example C.44. Since $\mathcal{O}'_{\mathcal{X}},\mathcal{O}_{\mathcal{X}}$ are sheaves of C^∞-rings, they are sheaves of real vector spaces. So, as in Definition C.54, the pullbacks $O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}}),O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\mathcal{X}})$ have natural splittings as sheaves of real vector spaces:

$$O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})=(\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm tr}\oplus(\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt},\;\; O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\mathcal{X}})=(\mathcal{O}_{\mathcal{X}})^\Gamma_{\rm tr}\oplus(\mathcal{O}_{\mathcal{X}})^\Gamma_{\rm nt}. \quad (9.24)$$

As $O^\Gamma(\mathcal{X})^{-1}(\imath_{\mathcal{X}})$ is Γ-equivariant and surjective, it induces surjective morphisms $(\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm tr}\to(\mathcal{O}_{\mathcal{X}})^\Gamma_{\rm tr}$ and $(\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\to(\mathcal{O}_{\mathcal{X}})^\Gamma_{\rm nt}$.

Write $\bigl((\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\bigr)$, $\bigl((\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\bigr)$ for the sheaves of ideals in the sheaves of C^∞-rings $O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}}),O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\mathcal{X}})$ generated by $(\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt},(\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}$. Then as for (2.90)–(2.91) we have morphisms

$$O^\Gamma(\mathcal{X})^{-1}(\imath_{\mathcal{X}})_*:O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})/\bigl((\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\bigr)\to O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\mathcal{X}})/\bigl((\mathcal{O}_{\mathcal{X}})^\Gamma_{\rm nt}\bigr), \quad (9.25)$$
$$\bigl(O^\Gamma(\mathcal{X})^\sharp\bigr)_*:O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\mathcal{X}})/\bigl((\mathcal{O}_{\mathcal{X}})^\Gamma_{\rm nt}\bigr)\xrightarrow{\cong}\mathcal{O}_{\mathcal{X}^\Gamma}. \quad (9.26)$$

Define the sheaf of C^∞-rings $\mathcal{O}'_{\mathcal{X}^\Gamma}$ on $\mathcal{X}^\Gamma$ by $\mathcal{O}'_{\mathcal{X}^\Gamma}=O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})/\bigl((\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\bigr)$, and the morphism $\imath_{\mathcal{X}^\Gamma}:\mathcal{O}'_{\mathcal{X}^\Gamma}\to\mathcal{O}_{\mathcal{X}^\Gamma}$ by $\imath_{\mathcal{X}^\Gamma}=\bigl(O^\Gamma(\mathcal{X})^\sharp\bigr)_*\circ O^\Gamma(\mathcal{X})^{-1}(\imath_{\mathcal{X}})_*$, in the notation of (9.25)–(9.26). Using the argument of (2.92)–(2.95) we may show that $\bigl(\mathcal{O}'_{\mathcal{X}^\Gamma},\imath_{\mathcal{X}^\Gamma}\bigr)$ is a square zero extension of $\mathcal{X}^\Gamma$, in the sense of Definition 9.1, and construct a natural, surjective morphism

$$\pi_{\mathcal{I}_{\mathcal{X}^\Gamma}}:O^\Gamma(\mathcal{X})^*(\mathcal{I}_{\mathcal{X}})\longrightarrow\mathcal{I}_{\mathcal{X}^\Gamma}. \quad (9.27)$$

On $\mathcal{X}$ we have a surjective morphism $\jmath_{\mathcal{X}}:\mathcal{E}_{\mathcal{X}}\to\mathcal{I}_{\mathcal{X}}$ in $\mathrm{qcoh}(\mathcal{X})$. By Definition C.54, the pullbacks $O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}),O^\Gamma(\mathcal{X})^*(\mathcal{I}_{\mathcal{X}})$ to $\mathcal{X}^\Gamma$ have splittings

$$O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}})=(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm tr}\oplus(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm nt},\;\; O^\Gamma(\mathcal{X})^*(\mathcal{I}_{\mathcal{X}})=(\mathcal{I}_{\mathcal{X}})^\Gamma_{\rm tr}\oplus(\mathcal{I}_{\mathcal{X}})^\Gamma_{\rm nt},$$

and $O^\Gamma(\mathcal{X})^*(\jmath_{\mathcal{X}}):O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}})\to O^\Gamma(\mathcal{X})^*(\mathcal{I}_{\mathcal{X}})$ maps $(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm tr}\to(\mathcal{I}_{\mathcal{X}})^\Gamma_{\rm tr}$ and $(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm nt}\to(\mathcal{I}_{\mathcal{X}})^\Gamma_{\rm nt}$. Here $O^\Gamma(\mathcal{X})^*(\jmath_{\mathcal{X}})$ is surjective as $\jmath_{\mathcal{X}}$ is and $O^\Gamma(\mathcal{X})^*$ is right exact, so the morphisms $(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm tr}\to(\mathcal{I}_{\mathcal{X}})^\Gamma_{\rm tr}$ and $(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm nt}\to(\mathcal{I}_{\mathcal{X}})^\Gamma_{\rm nt}$ are also surjective. Also, since $(\mathcal{I}_{\mathcal{X}})^\Gamma_{\rm nt}=O^\Gamma(\mathcal{X})^{-1}(\mathcal{I}_{\mathcal{X}})_{\rm nt}\otimes_{O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\mathcal{X}})}\mathcal{O}_{\mathcal{X}^\Gamma}$ and $O^\Gamma(\mathcal{X})^{-1}(\mathcal{I}_{\mathcal{X}})_{\rm nt}\subseteq\bigl((\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\bigr)$, we see that $(\mathcal{I}_{\mathcal{X}})^\Gamma_{\rm nt}\subseteq\operatorname{Ker}\pi_{\mathcal{I}_{\mathcal{X}^\Gamma}}$.

Define $\mathcal{E}_{\mathcal{X}^\Gamma}=(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm tr}$, as a quasicoherent sheaf on $\mathcal{X}^\Gamma$, and define $\jmath_{\mathcal{X}^\Gamma}:\mathcal{E}_{\mathcal{X}^\Gamma}\to\mathcal{I}_{\mathcal{X}^\Gamma}$ by $\jmath_{\mathcal{X}^\Gamma}=\pi_{\mathcal{I}_{\mathcal{X}^\Gamma}}\circ O^\Gamma(\mathcal{X})^*(\jmath_{\mathcal{X}})|_{(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm tr}}$. Since $O^\Gamma(\mathcal{X})^*(\jmath_{\mathcal{X}})|_{(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm tr}}:(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm tr}\to(\mathcal{I}_{\mathcal{X}})^\Gamma_{\rm tr}$ is surjective, and (9.27) is surjective, and $(\mathcal{I}_{\mathcal{X}})^\Gamma_{\rm nt}\subseteq\operatorname{Ker}\pi_{\mathcal{I}_{\mathcal{X}^\Gamma}}$, we see that $\jmath_{\mathcal{X}^\Gamma}$ is surjective. This shows that $\boldsymbol{\mathcal{X}}^\Gamma=(\mathcal{X}^\Gamma,\mathcal{O}'_{\mathcal{X}^\Gamma},\mathcal{E}_{\mathcal{X}^\Gamma},\imath_{\mathcal{X}^\Gamma},\jmath_{\mathcal{X}^\Gamma})$ is a d-stack, by Definition 9.6. As in §C.8, we have an open C^∞-substack

$\mathcal{X}^\Gamma_\circ \subseteq \mathcal{X}^\Gamma$. Define $\boldsymbol{\mathcal{X}}^\Gamma_\circ = (\mathcal{X}^\Gamma_\circ, \mathcal{O}'_{\mathcal{X}^\Gamma}|_{\mathcal{X}^\Gamma_\circ}, \mathcal{E}_{\mathcal{X}^\Gamma}|_{\mathcal{X}^\Gamma_\circ}, \imath_{\mathcal{X}^\Gamma}|_{\mathcal{X}^\Gamma_\circ}, \jmath_{\mathcal{X}^\Gamma}|_{\mathcal{X}^\Gamma_\circ})$ to be the corresponding open d-substack of $\boldsymbol{\mathcal{X}}^\Gamma$.

By an almost identical definition, but replacing $\mathcal{X}^\Gamma, O^\Gamma(\mathcal{X})$ by $\tilde{\mathcal{X}}^\Gamma, \tilde{O}^\Gamma(\mathcal{X})$ throughout, we can define a d-stack $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma = (\tilde{\mathcal{X}}^\Gamma, \mathcal{O}'_{\tilde{\mathcal{X}}^\Gamma}, \mathcal{E}_{\tilde{\mathcal{X}}^\Gamma}, \imath_{\tilde{\mathcal{X}}^\Gamma}, \jmath_{\tilde{\mathcal{X}}^\Gamma})$ and an open d-substack $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ = (\tilde{\mathcal{X}}^\Gamma_\circ, \mathcal{O}'_{\tilde{\mathcal{X}}^\Gamma}|_{\tilde{\mathcal{X}}^\Gamma_\circ}, \mathcal{E}_{\tilde{\mathcal{X}}^\Gamma}|_{\tilde{\mathcal{X}}^\Gamma_\circ}, \imath_{\tilde{\mathcal{X}}^\Gamma}|_{\tilde{\mathcal{X}}^\Gamma_\circ}, \jmath_{\tilde{\mathcal{X}}^\Gamma}|_{\tilde{\mathcal{X}}^\Gamma_\circ})$ in $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma$. Although the pullbacks $\tilde{O}^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}}), \tilde{O}^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\mathcal{X}}), \tilde{O}^\Gamma(\mathcal{X})^*(\mathcal{I}_{\mathcal{X}}), \tilde{O}^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}})$ of $\mathcal{O}'_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}$, $\mathcal{I}_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}$ to $\tilde{\mathcal{X}}^\Gamma$ do not have natural Γ-actions, as in Definition C.55 they still have decompositions into trivial 'tr' and nontrivial 'nt' parts, and this is what we need to make the construction work.

In §C.8 we defined the orbifold stratum $\hat{\mathcal{X}}^\Gamma$, which has no natural projection $\hat{\mathcal{X}}^\Gamma \to \mathcal{X}$, but does have a natural 1-morphism $\hat{\Pi}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \to \hat{\mathcal{X}}^\Gamma$, a non-representable 1-morphism, with fibre $[\bar{*}/\Gamma]$. If $\mathcal{E}$ is a quasicoherent sheaf on $\mathcal{X}$ then Definition C.55 defined a splitting $\tilde{O}^\Gamma(\mathcal{X})^*(\mathcal{E}) = \tilde{\mathcal{E}}^\Gamma_{\rm tr} \oplus \tilde{\mathcal{E}}^\Gamma_{\rm nt}$, and Definition C.56 defined a quasicoherent sheaf $\hat{\mathcal{E}}^\Gamma_{\rm tr}$ on $\hat{\mathcal{X}}^\Gamma$ by $\hat{\mathcal{E}}^\Gamma_{\rm tr} = \hat{\Pi}^\Gamma(\mathcal{X})_*(\tilde{\mathcal{E}}^\Gamma_{\rm tr})$, using pushforwards along $\hat{\Pi}^\Gamma(\mathcal{X})$. We also have $\hat{\Pi}^\Gamma(\mathcal{X})^*(\hat{\mathcal{E}}^\Gamma_{\rm tr}) \cong \tilde{\mathcal{E}}^\Gamma_{\rm tr}$, so that pushforward $\hat{\Pi}^\Gamma(\mathcal{X})_*$ and pullback $\hat{\Pi}^\Gamma(\mathcal{X})^*$ are inverse in this case, and $\hat{\Pi}^\Gamma(\mathcal{X})_*(\tilde{\mathcal{E}}^\Gamma_{\rm nt}) = 0$, so that nontrivial components $\mathcal{E}^\Gamma_{\rm nt}, \tilde{\mathcal{E}}^\Gamma_{\rm nt}$ on $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma$ have no analogue on $\hat{\mathcal{X}}^\Gamma$.

Motivated by this, we define a d-stack $\hat{\boldsymbol{\mathcal{X}}}^\Gamma = (\hat{\mathcal{X}}^\Gamma, \mathcal{O}'_{\hat{\mathcal{X}}^\Gamma}, \mathcal{E}_{\hat{\mathcal{X}}^\Gamma}, \imath_{\hat{\mathcal{X}}^\Gamma}, \jmath_{\hat{\mathcal{X}}^\Gamma})$ by $\mathcal{O}'_{\hat{\mathcal{X}}^\Gamma} = \hat{\Pi}^\Gamma(\mathcal{X})_*(\mathcal{O}'_{\tilde{\mathcal{X}}^\Gamma})$, $\mathcal{E}_{\hat{\mathcal{X}}^\Gamma} = \hat{\Pi}^\Gamma(\mathcal{X})_*(\mathcal{E}_{\tilde{\mathcal{X}}^\Gamma})$, $\imath_{\hat{\mathcal{X}}^\Gamma} = \iota \circ \hat{\Pi}^\Gamma(\mathcal{X})_*(\imath_{\tilde{\mathcal{X}}^\Gamma})$, and $\jmath_{\hat{\mathcal{X}}^\Gamma} = \hat{\Pi}^\Gamma(\mathcal{X})_*(\jmath_{\tilde{\mathcal{X}}^\Gamma})$, where $\iota : \hat{\Pi}^\Gamma(\mathcal{X})_*(\mathcal{O}_{\tilde{\mathcal{X}}^\Gamma}) \to \mathcal{O}_{\hat{\mathcal{X}}^\Gamma}$ is the natural isomorphism. We also define an open d-substack $\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ = (\hat{\mathcal{X}}^\Gamma_\circ, \mathcal{O}'_{\hat{\mathcal{X}}^\Gamma}|_{\hat{\mathcal{X}}^\Gamma_\circ}, \ldots, \jmath_{\hat{\mathcal{X}}^\Gamma}|_{\hat{\mathcal{X}}^\Gamma_\circ})$ in $\hat{\boldsymbol{\mathcal{X}}}^\Gamma$.

Since $\mathcal{O}'_{\tilde{\mathcal{X}}^\Gamma}, \mathcal{O}_{\tilde{\mathcal{X}}^\Gamma}, \mathcal{E}_{\tilde{\mathcal{X}}^\Gamma}$ all behave like trivial components $\tilde{\mathcal{E}}^\Gamma_{\rm tr}$, pushforwards $\hat{\Pi}^\Gamma(\mathcal{X})_*$ and pullbacks $\hat{\Pi}^\Gamma(\mathcal{X})^{-1}, \hat{\Pi}^\Gamma(\mathcal{X})^*$ are inverse upon them, so that we have canonical isomorphisms $\mathcal{O}'_{\tilde{\mathcal{X}}^\Gamma} \cong \hat{\Pi}^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\hat{\mathcal{X}}^\Gamma})$, $\mathcal{O}_{\tilde{\mathcal{X}}^\Gamma} \cong \hat{\Pi}^\Gamma(\mathcal{X})^{-1}(\mathcal{O}_{\hat{\mathcal{X}}^\Gamma})$ and $\mathcal{E}_{\tilde{\mathcal{X}}^\Gamma} \cong \hat{\Pi}^\Gamma(\mathcal{X})^*(\mathcal{E}_{\hat{\mathcal{X}}^\Gamma})$, which identify $\hat{\Pi}^\Gamma(\mathcal{X})^{-1}(\imath_{\hat{\mathcal{X}}^\Gamma}), \hat{\Pi}^\Gamma(\mathcal{X})^*(\jmath_{\hat{\mathcal{X}}^\Gamma})$ with $\imath_{\tilde{\mathcal{X}}^\Gamma}, \jmath_{\tilde{\mathcal{X}}^\Gamma}$. Thus, the conditions on $\mathcal{O}'_{\hat{\mathcal{X}}^\Gamma}, \ldots, \jmath_{\hat{\mathcal{X}}^\Gamma}$ for $\hat{\boldsymbol{\mathcal{X}}}^\Gamma$ to be a d-stack, when pulled back to $\tilde{\mathcal{X}}^\Gamma$ by $\hat{\Pi}^\Gamma(\mathcal{X})$, follow as $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma$ is a d-stack. But $\hat{\Pi}^\Gamma(\mathcal{X})$ is étale and surjective, and the conditions are local, so they hold on $\hat{\mathcal{X}}^\Gamma$. Therefore $\hat{\boldsymbol{\mathcal{X}}}^\Gamma$ and hence $\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ are d-stacks.

As in §C.8, the C^∞-stack $\hat{\mathcal{X}}^\Gamma_\circ$ has trivial orbifold groups, so there is a C^∞-scheme $\underline{\hat{X}}^\Gamma_\circ$, unique up to isomorphism, such that $\hat{\mathcal{X}}^\Gamma_\circ \simeq \bar{\underline{\hat{X}}}^\Gamma_\circ$. Hence there exists a d-space $\hat{\boldsymbol{X}}^\Gamma_\circ = (\underline{\hat{X}}^\Gamma_\circ, \mathcal{O}'_{\hat{X}^\Gamma_\circ}, \mathcal{E}_{\hat{X}^\Gamma_\circ}, \imath_{\hat{X}^\Gamma_\circ}, \jmath_{\hat{X}^\Gamma_\circ})$ with $\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ \simeq F^{\mathbf{dSta}}_{\mathbf{dSpa}}(\hat{\boldsymbol{X}}^\Gamma_\circ)$, and in fact $\hat{\boldsymbol{X}}^\Gamma_\circ$ is natural up to 1-isomorphism in $\mathbf{dSpa}$, not just up to equivalence.

Here is the analogue of Definitions C.47 and C.48.

Definition 9.25. Let $\boldsymbol{\mathcal{X}}$ be a d-stack, and Γ a finite group. Define a 1-morphism $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}) : \boldsymbol{\mathcal{X}}^\Gamma \to \boldsymbol{\mathcal{X}}$ in $\mathbf{dSta}$ by $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}) = \big(O^\Gamma(\mathcal{X}), O^\Gamma(\mathcal{X})', O^\Gamma(\mathcal{X})''\big)$, where $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$ is given in Definition C.47, and

$$O^\Gamma(\mathcal{X})' : O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}}) \longrightarrow \mathcal{O}'_{\mathcal{X}^\Gamma} := O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})/\big((\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\big)$$

to be the obvious projection, and

$$O^\Gamma(\mathcal{X})'' : O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}) \longrightarrow \mathcal{E}_{\mathcal{X}^\Gamma} := (\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm tr}$$

to project to the first component in $O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}) = (\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm tr} \oplus (\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm nt}$.

Define $\tilde{\boldsymbol{O}}^\Gamma(\boldsymbol{\mathcal{X}}):\tilde{\boldsymbol{\mathcal{X}}}^\Gamma\to\boldsymbol{\mathcal{X}}$ in the same way. Define $\tilde{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}}):\boldsymbol{\mathcal{X}}^\Gamma\to\tilde{\boldsymbol{\mathcal{X}}}^\Gamma$ by $\tilde{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}})=\bigl(\tilde{\Pi}^\Gamma(\mathcal{X}),\tilde{\Pi}^\Gamma(\mathcal{X})',\tilde{\Pi}^\Gamma(\mathcal{X})''\bigr)$, where $\tilde{\Pi}^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\to\tilde{\mathcal{X}}^\Gamma$ is given in Definition C.47, and

$$\begin{aligned}
\tilde{\Pi}^\Gamma(\mathcal{X})':\tilde{\Pi}^\Gamma(\mathcal{X})^{-1}\bigl[\tilde{O}^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})/\bigl(\widetilde{(\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}}\bigr)\bigr]&\longrightarrow O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})/\bigl((\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\bigr),\\
\tilde{\Pi}^\Gamma(\mathcal{X})'':\tilde{\Pi}^\Gamma(\mathcal{X})^*\bigl[\bigl(\tilde{O}^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}})\bigr)_{\rm tr}\bigr]&\longrightarrow\bigl(O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}})\bigr)_{\rm tr}\quad\text{are induced by}\\
I_{\tilde{\Pi}^\Gamma(\mathcal{X}),\tilde{O}^\Gamma(\mathcal{X})}(\mathcal{O}'_{\mathcal{X}})^{-1}:\tilde{\Pi}^\Gamma(\mathcal{X})^{-1}\bigl[\tilde{O}^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})\bigr]&\longrightarrow O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}}),\\
I_{\tilde{\Pi}^\Gamma(\mathcal{X}),\tilde{O}^\Gamma(\mathcal{X})}(\mathcal{E}_{\mathcal{X}})^{-1}:\tilde{\Pi}^\Gamma(\mathcal{X})^*\bigl[\tilde{O}^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}})\bigr]&\longrightarrow O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}),
\end{aligned}$$

using $\tilde{O}^\Gamma(\mathcal{X})\circ\tilde{\Pi}^\Gamma(\mathcal{X})=O^\Gamma(\mathcal{X})$ from Definition C.47.

Define a 1-morphism $\hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}}):\tilde{\boldsymbol{\mathcal{X}}}^\Gamma\to\hat{\boldsymbol{\mathcal{X}}}^\Gamma$ in **dSta** by $\hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}})=\bigl(\hat{\Pi}^\Gamma(\mathcal{X}),\hat{\Pi}^\Gamma(\mathcal{X})',\hat{\Pi}^\Gamma(\mathcal{X})''\bigr)$, where $\hat{\Pi}^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\to\hat{\mathcal{X}}^\Gamma$ is given in Definition C.47, and $\hat{\Pi}^\Gamma(\mathcal{X})':\hat{\Pi}^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\hat{\mathcal{X}}^\Gamma})\to\mathcal{O}'_{\tilde{\mathcal{X}}^\Gamma}$, $\hat{\Pi}^\Gamma(\mathcal{X})'':\hat{\Pi}^\Gamma(\mathcal{X})^*(\mathcal{E}_{\hat{\mathcal{X}}^\Gamma})\to\mathcal{E}_{\tilde{\mathcal{X}}^\Gamma}$ are the natural isomorphisms mentioned in Definition 9.24.

For each $\Lambda\in\mathrm{Aut}(\Gamma)$, Definition C.47 gives a 1-morphism $L^\Gamma(\Lambda,\mathcal{X}):\mathcal{X}^\Gamma\to\mathcal{X}^\Gamma$ with $O^\Gamma(\mathcal{X})\circ L^\Gamma(\Lambda,\mathcal{X})=O^\Gamma(\mathcal{X})$. Define a 1-morphism $\boldsymbol{L}^\Gamma(\Lambda,\boldsymbol{\mathcal{X}}):\boldsymbol{\mathcal{X}}^\Gamma\to\boldsymbol{\mathcal{X}}^\Gamma$ by $\boldsymbol{L}^\Gamma(\Lambda,\boldsymbol{\mathcal{X}})=\bigl(L^\Gamma(\Lambda,\mathcal{X}),L^\Gamma(\Lambda,\mathcal{X})',L^\Gamma(\Lambda,\mathcal{X})''\bigr)$, where

$$\begin{aligned}
L^\Gamma(\Lambda,\mathcal{X})':L^\Gamma(\Lambda,\mathcal{X})^{-1}\bigl[O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})/\bigl((\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\bigr)\bigr]&\to O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})/\bigl((\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\bigr),\\
L^\Gamma(\Lambda,\mathcal{X})'':L^\Gamma(\Lambda,\mathcal{X})^*\bigl[\bigl(\tilde{O}^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}})\bigr)_{\rm tr}\bigr]&\to\bigl(O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}})\bigr)_{\rm tr}\ \text{ are induced by}\\
I_{L^\Gamma(\Lambda,\mathcal{X}),\tilde{O}^\Gamma(\mathcal{X})}(\mathcal{O}'_{\mathcal{X}})^{-1}:L^\Gamma(\Lambda,\mathcal{X})^{-1}\bigl[\tilde{O}^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})\bigr]&\longrightarrow O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}}),\\
I_{L^\Gamma(\Lambda,\mathcal{X}),\tilde{O}^\Gamma(\mathcal{X})}(\mathcal{E}_{\mathcal{X}})^{-1}:L^\Gamma(\Lambda,\mathcal{X})^*\bigl[\tilde{O}^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}})\bigr]&\longrightarrow O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}).
\end{aligned}$$

It is easy to check $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}),\tilde{\boldsymbol{O}}^\Gamma(\boldsymbol{\mathcal{X}}),\tilde{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}}),\hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}}),\boldsymbol{L}^\Gamma(\Lambda,\boldsymbol{\mathcal{X}})$ are 1-morphisms in **dSta**. Define 1-morphisms $\boldsymbol{O}^\Gamma_\circ(\boldsymbol{\mathcal{X}}),\ldots,\boldsymbol{L}^\Gamma_\circ(\Lambda,\boldsymbol{\mathcal{X}})$ to be the restrictions of $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}),\ldots,\boldsymbol{L}^\Gamma(\Lambda,\boldsymbol{\mathcal{X}})$ to the open d-substacks $\boldsymbol{\mathcal{X}}^\Gamma_\circ,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ,\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$. As (C.8) strictly commutes, one can show that (9.23) strictly commutes in **dSta**.

Definition C.48 defined 2-morphisms $E^\Gamma(\gamma,\mathcal{X}):O^\Gamma(\mathcal{X})\Rightarrow O^\Gamma(\mathcal{X})$ in $\mathbf{C^\infty Sta}$ for $\gamma\in\Gamma$, giving an action of Γ on $O^\Gamma(\mathcal{X})$ by 2-morphisms. It is easy to show that $\boldsymbol{E}^\Gamma(\gamma,\boldsymbol{\mathcal{X}})=(E^\Gamma(\gamma,\mathcal{X}),0)$ is a 2-morphism $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})\Rightarrow\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})$ in **dSta**, and gives an action of Γ on $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})$ by 2-morphisms. Similarly $\boldsymbol{E}^\Gamma_\circ(\gamma,\boldsymbol{\mathcal{X}})=(E^\Gamma_\circ(\gamma,\mathcal{X}),0)$ gives an action of Γ on $\boldsymbol{O}^\Gamma_\circ(\boldsymbol{\mathcal{X}})$ by 2-morphisms.

The analogue of Theorem C.49 for d-stacks is now immediate, since everything in the d-stack analogue depends only on the underlying C^∞-stacks anyway. Here is the analogue of Definition C.51.

Definition 9.26. Let $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}$ be d-stacks, Γ a finite group, and $\boldsymbol{f}=(f,f',f''):\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ be a 1-morphism in **dSta** which is representable, that is, $f:\mathcal{X}\to\mathcal{Y}$ is a representable 1-morphism in $\mathbf{C^\infty Sta}$. Then Definition C.51 gives a representable 1-morphism $f^\Gamma:\mathcal{X}^\Gamma\to\mathcal{Y}^\Gamma$, with $O^\Gamma(\mathcal{Y})\circ f^\Gamma=f\circ O^\Gamma(\mathcal{X})$. Define a 1-morphism $\boldsymbol{f}^\Gamma=(f^\Gamma,f^{\Gamma\prime},f^{\Gamma\prime\prime}):\boldsymbol{\mathcal{X}}^\Gamma\rightarrow\boldsymbol{\mathcal{Y}}^\Gamma$ in **dSta**, where

$$
\begin{gathered}
f^{\Gamma\prime}:(f^\Gamma)^{-1}\bigl[O^\Gamma(\mathcal{Y})^{-1}(\mathcal{O}'_{\mathcal{Y}})/\bigl((\mathcal{O}'_{\mathcal{Y}})^\Gamma_{\rm nt}\bigr)\bigr]\longrightarrow O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})/\bigl((\mathcal{O}'_{\mathcal{X}})^\Gamma_{\rm nt}\bigr)\ \text{ and}\\
f^{\Gamma\prime\prime}:(f^\Gamma)^*\bigl[\bigl(O^\Gamma(\mathcal{Y})^*(\mathcal{E}_{\mathcal{Y}})\bigr)_{\rm tr}\bigr]\longrightarrow\bigl(O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}})\bigr)_{\rm tr}\quad\text{are induced by}\\
O^\Gamma(\mathcal{X})^{-1}(f')\circ I_{O^\Gamma(\mathcal{X}),f}(\mathcal{O}'_{\mathcal{Y}})\circ I_{f^\Gamma,O^\Gamma(\mathcal{Y})}(\mathcal{O}'_{\mathcal{Y}})^{-1}:\\
(f^\Gamma)^{-1}\bigl[O^\Gamma(\mathcal{Y})^{-1}(\mathcal{O}'_{\mathcal{Y}})\bigr]\longrightarrow O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})\quad\text{and}\\
O^\Gamma(\mathcal{X})^*(f'')\circ I_{O^\Gamma(\mathcal{X}),f}(\mathcal{E}_{\mathcal{Y}})\circ I_{f^\Gamma,O^\Gamma(\mathcal{Y})}(\mathcal{E}_{\mathcal{Y}})^{-1}:\\
(f^\Gamma)^*\bigl[O^\Gamma(\mathcal{Y})^*(\mathcal{E}_{\mathcal{Y}})\bigr]\longrightarrow O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}).
\end{gathered}
$$

It is straightforward to check that $\boldsymbol{f}^\Gamma$ is a representable 1-morphism in **dSta**, and is the unique 1-morphism with $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})\circ\boldsymbol{f}^\Gamma=\boldsymbol{f}\circ\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})$.

Definition C.51 gives a representable 1-morphism $\tilde f^\Gamma:\tilde{\mathcal{X}}^\Gamma\rightarrow\tilde{\mathcal{Y}}^\Gamma$ with $\tilde O^\Gamma(\mathcal{Y})\circ\tilde f^\Gamma=f\circ\tilde O^\Gamma(\mathcal{X})$, and we define $\tilde{\boldsymbol{f}}^\Gamma:\tilde{\boldsymbol{\mathcal{X}}}^\Gamma\rightarrow\tilde{\boldsymbol{\mathcal{Y}}}^\Gamma$ as for $\boldsymbol{f}^\Gamma$.

Definition C.51 also gives a representable 1-morphism $\hat f^\Gamma:\hat{\mathcal{X}}^\Gamma\rightarrow\hat{\mathcal{Y}}^\Gamma$, unique up to 2-isomorphism, with a 2-morphism $\zeta:\hat\Pi^\Gamma(\mathcal{Y})\circ\tilde f^\Gamma\Rightarrow\hat f^\Gamma\circ\hat\Pi^\Gamma(\mathcal{X})$. Define a 1-morphism $\hat{\boldsymbol{f}}^\Gamma=(\hat f^\Gamma,\hat f^{\Gamma\prime},\hat f^{\Gamma\prime\prime}):\hat{\boldsymbol{\mathcal{X}}}^\Gamma\rightarrow\hat{\boldsymbol{\mathcal{Y}}}^\Gamma$, where $\hat f^{\Gamma\prime}:(\hat f^\Gamma)^{-1}(\mathcal{O}'_{\hat{\mathcal{Y}}^\Gamma})\rightarrow\mathcal{O}'_{\hat{\mathcal{X}}^\Gamma}$ and $\hat f^{\Gamma\prime\prime}:(\hat f^\Gamma)^*(\mathcal{E}_{\hat{\mathcal{Y}}^\Gamma})\rightarrow\mathcal{E}_{\hat{\mathcal{X}}^\Gamma}$ are the unique morphisms such that

$$
\begin{aligned}
&\hat\Pi^\Gamma(\mathcal{X})'\circ\hat\Pi^\Gamma(\mathcal{X})^{-1}(\hat f^{\Gamma\prime})\circ I_{\hat\Pi^\Gamma(\mathcal{X}),\hat f^\Gamma}(\mathcal{O}'_{\hat{\mathcal{Y}}^\Gamma})\circ\zeta^{-1}(\mathcal{O}'_{\hat{\mathcal{Y}}^\Gamma})\\
&\qquad\qquad=\tilde f^{\Gamma\prime}\circ(\tilde f^\Gamma)^{-1}(\hat\Pi^\Gamma(\mathcal{Y})')\circ I_{\tilde f^\Gamma,\hat\Pi^\Gamma(\mathcal{Y})}(\mathcal{O}'_{\hat{\mathcal{Y}}^\Gamma}),\\
&\hat\Pi^\Gamma(\mathcal{X})''\circ\hat\Pi^\Gamma(\mathcal{X})^*(\hat f^{\Gamma\prime\prime})\circ I_{\hat\Pi^\Gamma(\mathcal{X}),\hat f^\Gamma}(\mathcal{E}_{\hat{\mathcal{Y}}^\Gamma})\circ\zeta^*(\mathcal{E}_{\hat{\mathcal{Y}}^\Gamma})\\
&\qquad\qquad=\tilde f^{\Gamma\prime\prime}\circ(\tilde f^\Gamma)^*(\hat\Pi^\Gamma(\mathcal{Y})'')\circ I_{\tilde f^\Gamma,\hat\Pi^\Gamma(\mathcal{Y})}(\mathcal{E}_{\hat{\mathcal{Y}}^\Gamma}).
\end{aligned}
$$

As $\hat\Pi^\Gamma(\mathcal{X})',I_{\hat\Pi^\Gamma(\mathcal{X}),\hat f^\Gamma}(\mathcal{O}'_{\hat{\mathcal{Y}}^\Gamma}),\zeta^{-1}(\mathcal{O}'_{\hat{\mathcal{Y}}^\Gamma})$ are isomorphisms the first equation determines $\hat\Pi^\Gamma(\mathcal{X})^{-1}(\hat f^{\Gamma\prime})$ uniquely, and since as in Definition 9.24 $\hat\Pi^\Gamma(\mathcal{X})^{-1}$ and $\hat\Pi^\Gamma(\mathcal{X})_*$ are inverse in this case, this determines $\hat f^{\Gamma\prime}$. Similarly $\hat f^{\Gamma\prime\prime}$ is uniquely determined. One can now check that $\hat{\boldsymbol{f}}^\Gamma:\hat{\boldsymbol{\mathcal{X}}}^\Gamma\rightarrow\hat{\boldsymbol{\mathcal{Y}}}^\Gamma$ is a 1-morphism in **dSta**, and $\boldsymbol{\zeta}=(\zeta,0):\hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{Y}})\circ\tilde{\boldsymbol{f}}^\Gamma\Rightarrow\hat{\boldsymbol{f}}^\Gamma\circ\hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}})$ is a 2-morphism.

Now let $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ be representable 1-morphisms in **dSta**, and $\boldsymbol{\eta}=(\eta,\eta'):\boldsymbol{f}\Rightarrow\boldsymbol{g}$ be a 2-morphism. Then $\eta:f\Rightarrow g$ is a 2-morphism in $\mathbf{C^\infty Sta}$, so Definition C.51 defines a 2-morphism $\eta^\Gamma:f^\Gamma\Rightarrow g^\Gamma$. Define a 2-morphism $\boldsymbol{\eta}^\Gamma=(\eta^\Gamma,\eta^{\Gamma\prime}):\boldsymbol{f}^\Gamma\Rightarrow\boldsymbol{g}^\Gamma$ in **dSta**, where $\eta^{\Gamma\prime}$ fits in the commutative diagram

$$
\begin{tikzcd}
(f^\Gamma)^*\bigl[(\mathcal{F}_{\mathcal{Y}})^\Gamma_{\rm tr}\bigr] \arrow[d, "(f^\Gamma)^*(\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})^2_{\rm tr})"', "\cong"] \arrow[dr, "{[(O^\Gamma(\mathcal{X})^*(\eta')\circ I_{O^\Gamma(\mathcal{X}),f}(\mathcal{F}_{\mathcal{Y}})\circ I_{f^\Gamma,O^\Gamma(\mathcal{Y})}(\mathcal{F}_{\mathcal{Y}})^{-1})]_{\rm tr}}"] & \\
(f^\Gamma)^*\bigl[\mathcal{F}_{\mathcal{Y}^\Gamma}\bigr] \arrow[r, "\eta^{\Gamma\prime}"] & \mathcal{E}_{\mathcal{X}^\Gamma}=(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm tr}.
\end{tikzcd}
$$

Here $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}}):\boldsymbol{\mathcal{Y}}^\Gamma\rightarrow\boldsymbol{\mathcal{Y}}$ induces $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})^2:O^\Gamma(\mathcal{Y})^*(\mathcal{F}_{\mathcal{Y}})\rightarrow\mathcal{F}_{\mathcal{Y}^\Gamma}$ in $\mathrm{qcoh}(\mathcal{Y}^\Gamma)$, where $O^\Gamma(\mathcal{Y})^*(\mathcal{F}_{\mathcal{Y}})=(\mathcal{F}_{\mathcal{Y}})^\Gamma_{\rm tr}\oplus(\mathcal{F}_{\mathcal{Y}})^\Gamma_{\rm nt}$ as in §C.9. We will show in Theorem 9.29 below that $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})^2$ induces an isomorphism $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})^2_{\rm tr}:(\mathcal{F}_{\mathcal{Y}})^\Gamma_{\rm tr}\rightarrow\mathcal{F}_{\mathcal{Y}^\Gamma}$, and $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})^2=0$ on $(\mathcal{F}_{\mathcal{Y}})^\Gamma_{\rm nt}$. It is straightforward to show $\boldsymbol{\eta}^\Gamma$ is a 2-morphism in **dSta**, and is the unique 2-morphism satisfying $\mathbf{id}_{\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})}*\boldsymbol{\eta}^\Gamma=\boldsymbol{\eta}*\mathbf{id}_{\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})}$.

Similarly, we can define 2-morphisms $\tilde{\boldsymbol{\eta}}^\Gamma : \tilde{\boldsymbol{f}}^\Gamma \Rightarrow \tilde{\boldsymbol{g}}^\Gamma$ and $\hat{\boldsymbol{\eta}}^\Gamma : \hat{\boldsymbol{f}}^\Gamma \Rightarrow \hat{\boldsymbol{g}}^\Gamma$.

As in Definition C.51, we can express all this in terms of (strict or weak) 2-functors. Write $\mathbf{dSta^{re}}$ for the 2-subcategory of $\mathbf{dSta}$ with only representable 1-morphisms. Define strict 2-functors $F^\Gamma, \tilde{F}^\Gamma : \mathbf{dSta^{re}} \to \mathbf{dSta^{re}}$ by $F^\Gamma : \boldsymbol{\mathcal{X}} \mapsto F^\Gamma(\boldsymbol{\mathcal{X}}) = \boldsymbol{\mathcal{X}}^\Gamma$ on objects, $F^\Gamma : \boldsymbol{f} \mapsto F^\Gamma(\boldsymbol{f}) = \boldsymbol{f}^\Gamma$ on representable 1-morphisms, and $F^\Gamma : \boldsymbol{\eta} \mapsto F^\Gamma(\boldsymbol{\eta}) = \boldsymbol{\eta}^\Gamma$ on 2-morphisms, and similarly for $\tilde{F}^\Gamma$. They are strict as they preserve composition of 1-morphisms, that is, $F^\Gamma(\boldsymbol{g} \circ \boldsymbol{f}) = F^\Gamma(\boldsymbol{g}) \circ F^\Gamma(\boldsymbol{f})$ for 1-morphisms $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, $\boldsymbol{g} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$.

For the orbifold strata $\hat{\boldsymbol{\mathcal{X}}}^\Gamma$, we can define a weak 2-functor $\hat{F}^\Gamma : \mathbf{dSta^{re}} \to \mathbf{dSta^{re}}$ mapping $\hat{F}^\Gamma : \boldsymbol{\mathcal{X}} \mapsto \hat{F}^\Gamma(\boldsymbol{\mathcal{X}}) = \hat{\boldsymbol{\mathcal{X}}}^\Gamma$ on objects, $\hat{F}^\Gamma : \boldsymbol{f} \mapsto \hat{F}^\Gamma(\boldsymbol{f}) = \hat{\boldsymbol{f}}^\Gamma$ on representable 1-morphisms, and $\hat{F}^\Gamma : \boldsymbol{\eta} \mapsto \hat{F}^\Gamma(\boldsymbol{\eta}) = \hat{\boldsymbol{\eta}}^\Gamma$ on 2-morphisms. If $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, $\boldsymbol{g} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ are representable then we need not have $\hat{F}^\Gamma(\boldsymbol{g} \circ \boldsymbol{f}) = \hat{F}^\Gamma(\boldsymbol{g}) \circ \hat{F}^\Gamma(\boldsymbol{f})$, since as in Definition C.51 the underlying C^∞-stack 1-morphisms $\hat{f}^\Gamma, \hat{g}^\Gamma, (\widehat{g \circ f})^\Gamma$ are defined by stackification, and involve arbitrary choices. Instead there is a canonical 2-isomorphism $\hat{F}^\Gamma(\boldsymbol{f}, \boldsymbol{g}) = \bigl(\hat{F}^\Gamma(f,g), 0\bigr) : \hat{F}^\Gamma(\boldsymbol{g} \circ \boldsymbol{f}) \Rightarrow \hat{F}^\Gamma(\boldsymbol{g}) \circ \hat{F}^\Gamma(\boldsymbol{f})$, which is the final piece of data in the weak 2-functor $\hat{F}^\Gamma$.

Since equivalences in $\mathbf{dSta}$ are automatically representable, and (strict or weak) 2-functors take equivalences to equivalences, we deduce:

Corollary 9.27. *Suppose $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ are equivalent d-stacks, and Γ is a finite group. Then $\boldsymbol{\mathcal{X}}^\Gamma$ and $\boldsymbol{\mathcal{Y}}^\Gamma$ are equivalent in* $\mathbf{dSta}$, *and similarly for $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \hat{\boldsymbol{\mathcal{X}}}^\Gamma, \boldsymbol{\mathcal{X}}^\Gamma_\circ, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ, \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ and $\tilde{\boldsymbol{\mathcal{Y}}}^\Gamma, \hat{\boldsymbol{\mathcal{Y}}}^\Gamma, \boldsymbol{\mathcal{Y}}^\Gamma_\circ, \tilde{\boldsymbol{\mathcal{Y}}}^\Gamma_\circ, \hat{\boldsymbol{\mathcal{Y}}}^\Gamma_\circ$. Also $\hat{\boldsymbol{X}}^\Gamma_\circ, \hat{\boldsymbol{Y}}^\Gamma_\circ$ are equivalent in* $\mathbf{dSpa}$.

The last part follows as $F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\hat{\boldsymbol{X}}^\Gamma_\circ) \simeq \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ \simeq \hat{\boldsymbol{\mathcal{Y}}}^\Gamma_\circ \simeq F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\hat{\boldsymbol{Y}}^\Gamma_\circ)$ in $\mathbf{d\hat{S}pa}$, and $F_{\mathbf{dSpa}}^{\mathbf{dSta}} : \mathbf{dSpa} \to \mathbf{d\hat{S}pa}$ is an equivalence of categories by Theorem 9.7(c).

Here is a d-stack analogue of Theorem C.53, when the d-stack $\boldsymbol{\mathcal{X}}$ is a quotient $[\boldsymbol{X}/G]$ as in Definition 9.15, written in terms of the fixed d-subspace $\boldsymbol{X}^\Gamma$ of finite group Γ acting on a d-space $\boldsymbol{X}$ defined in §2.7. The analogues of (C.10)–(C.13) also hold, but for brevity we omit them.

To prove Theorem 9.28, we extend the proof of Theorem C.53 to include the sheaf data $\mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}}$ on $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$, and on $\boldsymbol{\mathcal{X}}^\Gamma, \ldots, \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$. We do this using Theorem C.57 (and its analogue for sheaves of C^∞-rings), which is a tool for comparing sheaves on $\mathcal{X}, \mathcal{X}^\Gamma$ and G-equivariant sheaves on $\underline{X}, \coprod_\rho \underline{X}^{\rho(\Gamma)}$. When we make this comparison, we find that Definition 9.24 applied to $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$ is parallel to Definition 2.43 applied to $\boldsymbol{X}$ for subgroups of G isomorphic to Γ. We leave the details to the reader.

Theorem 9.28. *Let $\boldsymbol{X}$ be a d-space and G a finite group acting on $\boldsymbol{X}$ by* 1-*isomorphisms, and write $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$ for the quotient d-stack, from Definition* 9.15. *Let Γ be a finite group. Then there are equivalences of d-stacks*

$$\boldsymbol{\mathcal{X}}^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective} \\ \text{group morphisms } \rho : \Gamma \to G}} \bigl[\boldsymbol{X}^{\rho(\Gamma)} / \bigl\{g \in G : g\rho(\gamma) = \rho(\gamma) g \ \ \forall \gamma \in \Gamma\bigr\}\bigr], \tag{9.28}$$

$$\boldsymbol{\mathcal{X}}_\circ^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective}\\ \text{group morphisms } \rho:\Gamma\to G}} \bigl[\boldsymbol{X}_\circ^{\rho(\Gamma)}/\{g\in G: g\rho(\gamma)=\rho(\gamma)g \;\; \forall\gamma\in\Gamma\}\bigr], \tag{9.29}$$

$$\tilde{\boldsymbol{\mathcal{X}}}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\boldsymbol{X}^\Delta/\{g\in G:\Delta=g\Delta g^{-1}\}\bigr], \tag{9.30}$$

$$\tilde{\boldsymbol{\mathcal{X}}}_\circ^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\boldsymbol{X}_\circ^\Delta/\{g\in G:\Delta=g\Delta g^{-1}\}\bigr], \tag{9.31}$$

$$\hat{\boldsymbol{\mathcal{X}}}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\boldsymbol{X}^\Delta/\bigl(\{g\in G:\Delta=g\Delta g^{-1}\}/\Delta\bigr)\bigr], \tag{9.32}$$

$$\hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\boldsymbol{X}_\circ^\Delta/\bigl(\{g\in G:\Delta=g\Delta g^{-1}\}/\Delta\bigr)\bigr]. \tag{9.33}$$

Here morphisms $\rho,\rho':\Gamma\to G$ are conjugate if $\rho'=\mathrm{Ad}(g)\circ\rho$ for some $g\in G$, and subgroups $\Delta,\Delta'\subseteq G$ are conjugate if $\Delta=g\Delta' g^{-1}$ for some $g\in G$. In (9.28)–(9.33) *we sum over one representative ρ or Δ for each conjugacy class. For each subgroup $\Delta\subseteq G$, allowing $\Delta=\rho(\Gamma)$, we write $\boldsymbol{X}^\Delta$ for the closed d-subspace in $\boldsymbol{X}$ fixed by Δ in G, as in Definition* 2.43*, and $\boldsymbol{X}_\circ^\Delta$ for the open d-subspace in $\boldsymbol{X}^\Delta$ of points in $\boldsymbol{X}$ whose stabilizer group in G is exactly Δ. The groups acting on $\boldsymbol{X}^\Delta$ in* (9.28)–(9.33) *have natural actions induced by the G-action on $\boldsymbol{X}$, such that $\boldsymbol{j}_{\boldsymbol{X},\Delta}:\boldsymbol{X}^\Delta\hookrightarrow\boldsymbol{X}$ is equivariant.*

Under the equivalences (9.28)–(9.33)*, the* 1*-morphisms in* (9.23) *are identified up to* 2*-isomorphism with* 1*-morphisms between quotient d-stacks induced by natural d-space* 1*-morphisms between $\boldsymbol{X}^{\rho(\Gamma)},\boldsymbol{X},\ldots.$*

Here is the analogue of Theorem C.58.

Theorem 9.29. *Let $\boldsymbol{\mathcal{X}}$ be a d-stack and Γ a finite group, so that Definitions* 9.24 *and* 9.25 *define a d-stack $\boldsymbol{\mathcal{X}}^\Gamma$ and a* 1*-morphism $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}):\boldsymbol{\mathcal{X}}^\Gamma\to\boldsymbol{\mathcal{X}}$. Equation* (9.12) *for $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})$ becomes:*

$$\begin{tikzcd}[column sep=large]
{\begin{array}{c}O^\Gamma(\mathcal{X})^*(\mathcal{E}_\mathcal{X})=\\ (\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{tr}}\oplus(\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{nt}}\end{array}} \arrow[r, "O^\Gamma(\mathcal{X})^*(\phi_\mathcal{X})"'] \arrow[d, "O^\Gamma(\mathcal{X})''"] & {\begin{array}{c}O^\Gamma(\mathcal{X})^*(\mathcal{F}_\mathcal{X})=\\ (\mathcal{F}_\mathcal{X})^\Gamma_{\mathrm{tr}}\oplus(\mathcal{F}_\mathcal{X})^\Gamma_{\mathrm{nt}}\end{array}} \arrow[r, "O^\Gamma(\mathcal{X})^*(\psi_\mathcal{X})"'] \arrow[d, "O^\Gamma(\mathcal{X})^2"] & {\begin{array}{c}O^\Gamma(\mathcal{X})^*(T^*\mathcal{X})=\\ (T^*\mathcal{X})^\Gamma_{\mathrm{tr}}\oplus(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}\end{array}} \arrow[r] \arrow[d, "{O^\Gamma(\mathcal{X})^3=\Omega_{O^\Gamma(\mathcal{X})}}"] & 0 \\
\mathcal{E}_{\mathcal{X}^\Gamma} \arrow[r, "\phi_{\mathcal{X}^\Gamma}"] & \mathcal{F}_{\mathcal{X}^\Gamma} \arrow[r, "\psi_{\mathcal{X}^\Gamma}"] & T^*(\mathcal{X}^\Gamma) \arrow[r] & 0.
\end{tikzcd} \tag{9.34}$$

Then the columns $O^\Gamma(\mathcal{X})''$, $O^\Gamma(\mathcal{X})^2$, $O^\Gamma(\mathcal{X})^3$ of (9.34) *are isomorphisms when restricted to the 'trivial' summands $(\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{tr}},(\mathcal{F}_\mathcal{X})^\Gamma_{\mathrm{tr}},(T^*\mathcal{X})^\Gamma_{\mathrm{tr}}$, and are zero when restricted to the 'nontrivial' summands $(\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{nt}},(\mathcal{F}_\mathcal{X})^\Gamma_{\mathrm{nt}},(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}$. In particular, this implies that the virtual cotangent sheaf $\phi_{\mathcal{X}^\Gamma}:\mathcal{E}_{\mathcal{X}^\Gamma}\to\mathcal{F}_{\mathcal{X}^\Gamma}$ of $\boldsymbol{\mathcal{X}}^\Gamma$ is* 1*-isomorphic in* $\mathrm{vqcoh}(\mathcal{X}^\Gamma)$ *to $(\phi_\mathcal{X})^\Gamma_{\mathrm{tr}}:(\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{tr}}\to(\mathcal{F}_\mathcal{X})^\Gamma_{\mathrm{tr}}$, the 'trivial' part of the pullback to $\mathcal{X}^\Gamma$ of the virtual cotangent sheaf $\phi_\mathcal{X}:\mathcal{E}_\mathcal{X}\to\mathcal{F}_\mathcal{X}$ of $\boldsymbol{\mathcal{X}}$.*

The analogous results also hold for $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\hat{\boldsymbol{\mathcal{X}}}^\Gamma,\boldsymbol{\mathcal{X}}^\Gamma_\circ,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ and $\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$.

Proof. For the first column of (9.34), Definitions 9.24 and 9.25 defined $\mathcal{E}_{\mathcal{X}^\Gamma}=(\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{tr}}$ and $O^\Gamma(\mathcal{X})''=\mathrm{id}_{(\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{tr}}}$ on $(\mathcal{E}_\mathcal{X})^\Gamma_{\mathrm{tr}}$, an isomorphism, and $O^\Gamma(\mathcal{X})''=0$ on

$(\mathcal{E}_{\mathcal{X}})^\Gamma_{\rm nt}$. For the third column of (9.34), Theorem C.58 shows that $\Omega_{O^\Gamma(\mathcal{X})}$ is an isomorphism on $(T^*\mathcal{X})^\Gamma_{\rm tr}$ and $\Omega_{O^\Gamma(\mathcal{X})}=0$ on $(T^*\mathcal{X})^\Gamma_{\rm nt}$. For the second column of (9.34), we have a morphism $O^\Gamma(\mathcal{X})':O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})\rightarrow\mathcal{O}'_{\mathcal{X}^\Gamma}$ of sheaves of C^∞-rings on $\mathcal{X}^\Gamma$, so we may consider the diagram

$$\begin{array}{c}
\bigl[(\Omega_{\mathcal{O}'_{\mathcal{X}}})^\Gamma_{\rm tr}\oplus(\Omega_{\mathcal{O}'_{\mathcal{X}}})^\Gamma_{\rm nt}\bigr]\otimes_{O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})}\mathcal{O}'_{\mathcal{X}^\Gamma}\\
\Vert\\
O^\Gamma(\mathcal{X})^{-1}(\Omega_{\mathcal{O}'_{\mathcal{X}}})\otimes_{O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})}\mathcal{O}'_{\mathcal{X}^\Gamma}\\
\downarrow\cong\\
\Omega_{O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})}\otimes_{O^\Gamma(\mathcal{X})^{-1}(\mathcal{O}'_{\mathcal{X}})}\mathcal{O}'_{\mathcal{X}^\Gamma}\\
\downarrow{\scriptstyle\Omega_{O^\Gamma(\mathcal{X})'}}\\
\Omega_{\mathcal{O}'_{\mathcal{X}^\Gamma}}.
\end{array}\tag{9.35}$$

The proof of Theorem C.58 implies that the composition of (9.35) is an isomorphism on the 'tr' summand, and zero on the 'nt' summand. But applying $\otimes_{\mathcal{O}'_{\mathcal{X}^\Gamma}}\mathcal{O}_{\mathcal{X}^\Gamma}$ to (9.35) yields a morphism naturally identified with the second column of (9.34), as $\mathcal{F}_{\mathcal{X}^\Gamma}=\Omega_{\mathcal{O}'_{\mathcal{X}^\Gamma}}\otimes_{\mathcal{O}'_{\mathcal{X}^\Gamma}}\mathcal{O}_{\mathcal{X}^\Gamma}$. The first part of the theorem follows. The proofs for $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\hat{\boldsymbol{\mathcal{X}}}^\Gamma$ are similar, and the result for $\boldsymbol{\mathcal{X}}^\Gamma_\circ,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ,\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ then follows by restriction to $\mathcal{X}^\Gamma_\circ,\tilde{\mathcal{X}}^\Gamma_\circ,\hat{\mathcal{X}}^\Gamma_\circ$. $\square$

10 The 2-category of d-orbifolds

We now define and study the 2-category **dOrb** of *d-orbifolds without boundary*, or just *d-orbifolds*, as a 2-subcategory of the 2-category of d-stacks **dSta** from Chapter 9. Sections 10.1–10.6 are analogues of material on d-manifolds from Chapters 3 and 4, and in these parts we often omit proofs, or just indicate the differences with the d-manifold case. Sections 10.7–10.9 cover material special to (d-)orbifolds, namely orbifold strata, Kuranishi neighbourhoods and good coordinate systems, and (semi)effective d-orbifolds.

10.1 Definition and local properties of d-orbifolds

This section extends §3.1–§3.4 to d-orbifolds.

10.1.1 Virtual quasicoherent sheaves on C^∞-stacks

The material of §3.1 on virtual quasicoherent sheaves and virtual vector bundles on a C^∞-scheme $\underline{X}$ extends easily to Deligne–Mumford C^∞-stacks $\mathcal{X}$, using §C.6. We briefly explain the differences in the C^∞-stack case.

In the C^∞-stack analogue of Definition 3.1, the definition of the 2-category $\mathrm{vqcoh}(\mathcal{X})$ is exactly as for C^∞-schemes. We call $(\mathcal{E}^\bullet,\phi)$ in $\mathrm{vqcoh}(\mathcal{X})$ a *virtual vector bundle of rank* $d\in\mathbb{Z}$ if $\mathcal{X}$ may be covered by *Zariski* open C^∞-substacks $\mathcal{U}$ such that $(\mathcal{E}^\bullet,\phi)|_{\mathcal{U}}$ is equivalent in $\mathrm{vqcoh}(\mathcal{U})$ to some $(\mathcal{F}^\bullet,\psi)$ for $\mathcal{F}^1,\mathcal{F}^2$ vector bundles on $\mathcal{U}$ with $\mathrm{rank}\,\mathcal{F}^2-\mathrm{rank}\,\mathcal{F}^1=d$. But note from Definition C.30 that vector bundles $\mathcal{F}^1,\mathcal{F}^2$ on $\mathcal{U}$ need only be locally trivial *in the étale topology*, so the orbifold groups $\mathrm{Iso}_{\mathcal{U}}([u])$ of $\mathcal{U}$ can act nontrivially on the fibres of $\mathcal{F}^1,\mathcal{F}^2$. Then $\mathrm{vvect}(\mathcal{X})$ is the full 2-subcategory of virtual vector bundles in $\mathrm{vqcoh}(\mathcal{X})$.

If $f:\mathcal{X}\to\mathcal{Y}$ is a 1-morphism of Deligne–Mumford C^∞-stacks then pullback f^* defines strict 2-functors $f^*:\mathrm{vqcoh}(\mathcal{Y})\to\mathrm{vqcoh}(\mathcal{X})$ and $f^*:\mathrm{vvect}(\mathcal{Y})\to\mathrm{vvect}(\mathcal{X})$, as for C^∞-schemes. If $f,g:\mathcal{X}\to\mathcal{Y}$ are 1-morphisms of Deligne–Mumford C^∞-stacks and $\eta:f\Rightarrow g$ is a 2-morphism then $\eta^*:f^*\Rightarrow g^*$ is a strict 2-natural transformation; such η^* do not occur in the C^∞-scheme case.

In the d-stack version of Example 3.2, we define the *virtual cotangent sheaf* $T^*\boldsymbol{\mathcal{X}}$ of a d-stack $\boldsymbol{\mathcal{X}}$ to be the morphism $\phi_{\mathcal{X}}:\mathcal{E}_{\mathcal{X}}\to\mathcal{F}_{\mathcal{X}}$ in $\mathrm{qcoh}(\mathcal{X})$ from Definition 9.6. If $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ is a 1-morphism in **dSta** then $\Omega_{\boldsymbol{f}}:=(f'',f^2)$ is a 1-morphism $f^*(T^*\boldsymbol{\mathcal{Y}})\to T^*\boldsymbol{\mathcal{X}}$ in $\mathrm{vqcoh}(\mathcal{X})$. For 2-morphisms in **dSta**, the picture is more complicated than the d-space case. Suppose $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ are 1-morphisms and $\boldsymbol{\eta}=(\eta,\eta'):\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is a 2-morphism in **dSta**. Then we have 1-morphisms $\Omega_{\boldsymbol{f}}:f^*(T^*\boldsymbol{\mathcal{Y}})\to T^*\boldsymbol{\mathcal{X}}$, $\Omega_{\boldsymbol{g}}:g^*(T^*\boldsymbol{\mathcal{Y}})\to T^*\boldsymbol{\mathcal{X}}$, and $\eta^*(T^*\boldsymbol{\mathcal{Y}}):f^*(T^*\boldsymbol{\mathcal{Y}})\to g^*(T^*\boldsymbol{\mathcal{Y}})$ in $\mathrm{qcoh}(\mathcal{X})$, with $\eta^*(T^*\boldsymbol{\mathcal{Y}})$ a 1-isomorphism, and $\eta':\Omega_{\boldsymbol{f}}\Rightarrow\Omega_{\boldsymbol{g}}\circ\eta^*(T^*\boldsymbol{\mathcal{Y}})$ is a 2-morphism in $\mathrm{vqcoh}(\mathcal{X})$.

The C^∞-stack and d-stack analogues of Lemma 3.3, Propositions 3.4, 3.5 and 3.7, Corollaries 3.6 and 3.10, and Definition 3.8 are immediate.

10.1.2 The definition of d-orbifolds

We will now define the 2-category **dOrb** of d-orbifolds without boundary, following §3.2. Here is the analogue of Definitions 3.11, 3.16 and 3.18.

Definition 10.1. A d-stack $\boldsymbol{\mathcal{W}}$ is called a *principal d-orbifold* (*without boundary*) if is equivalent in **dSta** to a fibre product $\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}$ with $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}\in\hat{\mathbf{Orb}}$. The underlying C^∞-stack $\mathcal{W}\simeq\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$ is locally finitely presented, as $\mathcal{X},\mathcal{Y},\mathcal{Z}$ are. Any object in $\hat{\mathbf{Orb}}$ is a principal d-orbifold.

If $\boldsymbol{\mathcal{W}}$ is a nonempty principal d-orbifold then as in Proposition 3.15, the virtual cotangent sheaf $T^*\boldsymbol{\mathcal{W}}$ is a virtual vector bundle on $\mathcal{W}$. We define the *virtual dimension* of $\boldsymbol{\mathcal{W}}$ to be $\operatorname{vdim}\boldsymbol{\mathcal{W}}=\operatorname{rank}T^*\boldsymbol{\mathcal{W}}\in\mathbb{Z}$. This is well-defined as in Proposition 3.7. If $\boldsymbol{\mathcal{W}}\simeq\boldsymbol{\mathcal{X}}\times_{\boldsymbol{\mathcal{Z}}}\boldsymbol{\mathcal{Y}}$ for orbifolds $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}$ then $\operatorname{vdim}\boldsymbol{\mathcal{W}}=\dim\mathcal{X}+\dim\mathcal{Y}-\dim\mathcal{Z}$.

A d-stack $\boldsymbol{\mathcal{X}}$ is called a *d-orbifold* (*without boundary*) *of virtual dimension* $n\in\mathbb{Z}$, written $\operatorname{vdim}\boldsymbol{\mathcal{X}}=n$, if $\boldsymbol{\mathcal{X}}$ can be covered by open d-substacks $\boldsymbol{\mathcal{W}}$ which are principal d-orbifolds with $\operatorname{vdim}\boldsymbol{\mathcal{W}}=n$. The underlying C^∞-stack $\mathcal{X}$ is separated, second countable, locally compact, paracompact, and locally finitely presented. The virtual cotangent sheaf $T^*\boldsymbol{\mathcal{X}}=(\mathcal{E}_\mathcal{X},\mathcal{F}_\mathcal{X},\phi_\mathcal{X})$ of $\boldsymbol{\mathcal{X}}$ is a virtual vector bundle of rank $\operatorname{vdim}\boldsymbol{\mathcal{X}}=n$, so we call it the *virtual cotangent bundle* of $\boldsymbol{\mathcal{X}}$. We consider the empty d-stack $\boldsymbol{\emptyset}$ to be a d-orbifold of any virtual dimension $n\in\mathbb{Z}$, so we leave $\operatorname{vdim}\boldsymbol{\emptyset}$ undefined.

Let **dOrb** be the full 2-subcategory of d-orbifolds in **dSta**. The 2-functor $F_{\mathbf{Orb}}^{\mathbf{dSta}}:\mathbf{Orb}\to\mathbf{dSta}$ in Definition 9.6 maps into **dOrb**, and we will write $F_{\mathbf{Orb}}^{\mathbf{dOrb}}=F_{\mathbf{Orb}}^{\mathbf{dSta}}:\mathbf{Orb}\to\mathbf{dOrb}$. Also $\hat{\mathbf{Orb}}$ is a 2-subcategory of **dOrb**. We say that a d-orbifold $\boldsymbol{\mathcal{X}}$ *is an orbifold* if it lies in $\hat{\mathbf{Orb}}$. The 2-functor $F_{\mathbf{dSpa}}^{\mathbf{dSta}}$ maps $\mathbf{dMan}\to\mathbf{dOrb}$, and we will write $F_{\mathbf{dMan}}^{\mathbf{dOrb}}=F_{\mathbf{dSpa}}^{\mathbf{dSta}}|_{\mathbf{dMan}}:\mathbf{dMan}\to\mathbf{dOrb}$. Then $F_{\mathbf{dMan}}^{\mathbf{dOrb}}\circ F_{\mathbf{Man}}^{\mathbf{dMan}}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}\circ F_{\mathbf{Man}}^{\mathbf{Orb}}:\mathbf{Man}\to\mathbf{dOrb}$.

Write $\mathbf{d\hat{M}an}$ for the full 2-subcategory of objects $\boldsymbol{\mathcal{X}}$ in **dOrb** equivalent to $F_{\mathbf{dMan}}^{\mathbf{dOrb}}(\boldsymbol{X})$ for some d-manifold $\boldsymbol{X}$. When we say that a d-orbifold $\boldsymbol{\mathcal{X}}$ *is a d-manifold*, we mean that $\boldsymbol{\mathcal{X}}\in\mathbf{d\hat{M}an}$.

Here is the analogue of Proposition 3.12. For part (b), embeddings $i:\mathcal{X}\to\mathcal{Z}$ are defined in Definition 8.3(iii), and include the condition that $i_*:\operatorname{Iso}_\mathcal{X}([x])\to\operatorname{Iso}_\mathcal{Z}\bigl(i_{\rm top}([x])\bigr)$ is an isomorphism for all $[x]\in\mathcal{X}_{\rm top}$.

Proposition 10.2. *The following are equivalent characterizations of when a d-stack $\boldsymbol{\mathcal{W}}$ is a principal d-orbifold:*

- **(a)** $\boldsymbol{\mathcal{W}}\simeq\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}$ *for* $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}\in\hat{\mathbf{Orb}}$.
- **(b)** $\boldsymbol{\mathcal{W}}\simeq\boldsymbol{\mathcal{X}}\times_{\boldsymbol{i},\boldsymbol{\mathcal{Z}},\boldsymbol{j}}\boldsymbol{\mathcal{Y}}$, *where $\mathcal{X},\mathcal{Y},\mathcal{Z}$ are orbifolds, $i:\mathcal{X}\to\mathcal{Z}$, $j:\mathcal{Y}\to\mathcal{Z}$ are embeddings, and $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}},\boldsymbol{i},\boldsymbol{j}=F_{\mathbf{Orb}}^{\mathbf{dSta}}(\mathcal{X},\mathcal{Y},\mathcal{Z},i,j)$. That is, $\boldsymbol{\mathcal{W}}$ is an intersection of two suborbifolds $\mathcal{X},\mathcal{Y}$ in $\mathcal{Z}$, in the sense of d-stacks.*
- **(c)** $\boldsymbol{\mathcal{W}}\simeq\boldsymbol{\mathcal{V}}\times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}}\boldsymbol{\mathcal{V}}$, *where $\mathcal{V}$ is an orbifold, $\mathcal{E}\in\operatorname{vect}(\mathcal{V})$ is a vector bundle on $\mathcal{V}$ in the sense of §8.3, $s\in C^\infty(\mathcal{E})$ is a smooth section of $\mathcal{E}$, $0\in C^\infty(\mathcal{E})$ is the zero section, and $\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{E}},\boldsymbol{s},\boldsymbol{0}=F_{\mathbf{Orb}}^{\mathbf{dSta}}\bigl(\mathcal{V},\operatorname{Tot}(\mathcal{E}),\operatorname{Tot}(s),\operatorname{Tot}(0)\bigr)$, using*

the 'total space functor' of Definition 8.4. *That is, $\boldsymbol{\mathcal{W}}$ is the zeroes $s^{-1}(0)$ of a section s of a vector bundle $\mathcal{E}$, in the sense of d-stacks.*

We have written the proof of Proposition 3.12 so that essentially the same proof works for Proposition 10.2. But note that in the manifold case we can take Φ to be a diffeomorphism with an open neighbourhood U' of the diagonal in $Z \times Z$, and then V is diffeomorphic to the open set $(g \times h)^{-1}(U')$ in $X \times Y$. In the orbifold case, if z is an orbifold point of $\mathcal{Z}$ then Φ is not a diffeomorphism near $(z, 0) \in \mathcal{U}$ and $(z, z) \in \mathcal{Z} \times \mathcal{Z}$, but rather a $|\mathrm{Iso}_{\mathcal{Z}}(z)|$-fold branched cover, and fibre product with Φ modifies orbifold groups.

The analogue of Lemma 3.19 is immediate:

Lemma 10.3. *Let $\boldsymbol{\mathcal{W}}$ be a d-orbifold, and $\boldsymbol{\mathcal{U}}$ an open d-substack of $\boldsymbol{\mathcal{W}}$. Then $\boldsymbol{\mathcal{U}}$ is also a d-orbifold, with* $\mathrm{vdim}\,\boldsymbol{\mathcal{U}} = \mathrm{vdim}\,\boldsymbol{\mathcal{W}}$.

Using Theorem C.23 and Example C.32 we deduce:

Lemma 10.4. *Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. Then $\boldsymbol{\mathcal{X}}$ is a d-manifold, that is, $\boldsymbol{\mathcal{X}}$ is equivalent to $F_{\mathbf{dMan}}^{\mathbf{dOrb}}(\boldsymbol{X})$ for some d-manifold $\boldsymbol{X}$, if and only if* $\mathrm{Iso}_{\mathcal{X}}([x]) \cong \{1\}$ *for all $[x]$ in $\mathcal{X}_{\mathrm{top}}$.*

10.1.3 Local properties of d-orbifolds

In Definition 3.13 we defined 'standard model' d-manifolds $\boldsymbol{S}_{V,E,s}$, a class of explicit principal d-manifolds. We give two d-orbifold analogues of this, starting either with an orbifold $\mathcal{V}$, or with a manifold V acted on by a finite group Γ.

Definition 10.5. Let $\mathcal{V}$ be an orbifold, $\mathcal{E} \in \mathrm{vect}(\mathcal{V})$ a vector bundle on $\mathcal{V}$ as in §8.3, and $s \in C^\infty(\mathcal{E})$ a smooth section, that is, $s : \mathcal{O}_{\mathcal{V}} \to \mathcal{E}$ is a morphism in $\mathrm{vect}(\mathcal{V})$. We will define a principal d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$, which we call a 'standard model' d-orbifold.

Let the Deligne–Mumford C^∞-stack $\mathcal{S}$ be the C^∞-substack in $\mathcal{V}$ defined by the equation $s = 0$, so that informally $\mathcal{S} = s^{-1}(0) \subset \mathcal{V}$. Explicitly, as in Definition C.1, a C^∞-stack $\mathcal{V}$ consists of a category $\mathcal{V}$ and a functor $p_{\mathcal{V}} : \mathcal{V} \to \mathbf{C^\infty Sch}$, where there is a 1-1 correspondence between objects u in $\mathcal{V}$ with $p_{\mathcal{V}}(u) = \underline{U}$ in $\mathbf{C^\infty Sch}$ and 1-morphisms $\tilde{u} : \bar{\underline{U}} \to \mathcal{V}$ in $\mathbf{C^\infty Sta}$. Define $\mathcal{S}$ to be the full subcategory of objects u in $\mathcal{V}$ such that the morphism $\tilde{u}^*(s) : \tilde{u}^*(\mathcal{O}_{\mathcal{V}}) \to \tilde{u}^*(\mathcal{E})$ in $\mathrm{qcoh}(\bar{\underline{U}})$ is zero, and define $p_{\mathcal{S}} = p_{\mathcal{V}}|_{\mathcal{S}} : \mathcal{S} \to \mathbf{C^\infty Sch}$.

The inclusion of categories $i_{\mathcal{V}} : \mathcal{S} \hookrightarrow \mathcal{V}$ is a 1-morphism of C^∞-stacks, which is a closed embedding. It satisfies $0 = i_{\mathcal{V}}^*(s) : i_{\mathcal{V}}^*(\mathcal{O}_{\mathcal{V}}) \to i_{\mathcal{V}}^*(\mathcal{E})$ in $\mathrm{qcoh}(\mathcal{S})$, and $\mathcal{S}$ is the largest C^∞-substack of $\mathcal{V}$ with this property. One can show $\mathcal{S}$ is equivalent to the C^∞-stack fibre product $\mathcal{S} \simeq \mathcal{V} \times_{\mathrm{Tot}(s),\mathrm{Tot}(\mathcal{E}),\mathrm{Tot}(0)} \mathcal{V}$, where the orbifold $\mathrm{Tot}(\mathcal{E})$ and 1-morphisms $\mathrm{Tot}(s), \mathrm{Tot}(0) : \mathcal{V} \to \mathrm{Tot}(\mathcal{E})$ are as in Definition 8.4.

Since $i_{\mathcal{V}} : \mathcal{S} \to \mathcal{V}$ is the inclusion of a C^∞-substack, $i_{\mathcal{V}}^\sharp : i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}}) \to \mathcal{O}_{\mathcal{S}}$ is a surjective morphism of sheaves of C^∞-rings on $\mathcal{S}$. Write $\mathcal{I}_s$ for the kernel of $i_{\mathcal{V}}^\sharp$, as a sheaf of ideals in $i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})$, and $\mathcal{I}_s^2$ for the corresponding sheaf of squared ideals, and $\mathcal{O}'_{\mathcal{S}} = i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2$ for the quotient sheaf of C^∞-rings, and

$\imath_{\mathcal{S}} : \mathcal{O}'_{\mathcal{S}} \to \mathcal{O}_{\mathcal{S}}$ for the natural projection $i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2 \twoheadrightarrow i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s \cong \mathcal{O}_{\mathcal{S}}$ induced by the inclusion $\mathcal{I}_s^2 \subseteq \mathcal{I}_s$. Then $(\mathcal{O}'_{\mathcal{S}}, \imath_{\mathcal{S}})$ is a square zero extension of $\mathcal{S}$, in the sense of §9.1.

Write $\mathcal{E}^* \in \mathrm{vect}(\mathcal{V})$ for the dual vector bundle of $\mathcal{E}$, and set $\mathcal{E}_{\mathcal{S}} = i_{\mathcal{V}}^*(\mathcal{E}^*)$. There is a natural, surjective morphism $\jmath_{\mathcal{S}} : \mathcal{E}_{\mathcal{S}} \to \mathcal{I}_{\mathcal{S}} = \mathcal{I}_s/\mathcal{I}_s^2$ in $\mathrm{qcoh}(\mathcal{S})$ which locally maps $\alpha + (\mathcal{I}_s \cdot C^\infty(\mathcal{E}^*)) \mapsto \alpha \cdot s + \mathcal{I}_s^2$. Then $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$ is a d-stack. As in the d-manifold case, we can show that $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is equivalent in **dSta** to $\boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}} \boldsymbol{\mathcal{V}}$ in Proposition 10.2(c). Thus $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is a principal d-orbifold, and every principal d-orbifold $\boldsymbol{\mathcal{W}}$ is equivalent in **dSta** to some $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$.

Sometimes it is useful to take $\mathcal{V}$ to be an *effective* orbifold, as in §8.4.3.

All the material of §3.3 can now be extended to d-orbifolds. To study a d-orbifold $\boldsymbol{\mathcal{X}}$ near a point $[x] \in \mathcal{X}_{\mathrm{top}}$, we can use two approaches. We could note that $\boldsymbol{\mathcal{X}} \simeq \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ in **dOrb** near $[x]$ for some $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$, and then extend §3.3 from d-manifolds $\boldsymbol{S}_{V,E,s}$ to d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$. Alternatively, we could use Theorem 9.2(a) to show that $\boldsymbol{\mathcal{X}} \simeq [\boldsymbol{U}/G]$ near $[x]$, where $\boldsymbol{U}$ is a d-manifold, $G = \mathrm{Iso}_{\mathcal{X}}([x])$, and the equivalence identifies $[x]$ with a fixed point u of G in $\boldsymbol{U}$. Then $\boldsymbol{U} \simeq \boldsymbol{S}_{V,E,s}$ in **dMan** near u for some $\boldsymbol{S}_{V,E,s}$. We can take G to act on V, E, s and the equivalence to be G-equivariant. Then $\boldsymbol{\mathcal{X}} \simeq [\boldsymbol{S}_{V,E,s}/G]$ near $[x]$. We can then apply the results of §3.3 to $\boldsymbol{S}_{V,E,s}$, showing where necessary that the proofs work equivariantly with respect to G.

In the d-orbifold version of Example 3.24 there are extra choices, namely representations of $G = \mathrm{Iso}_{\mathcal{X}}([x])$ on $\mathbb{R}^a, \mathbb{R}^b$, which help determine the action of G on $\boldsymbol{U}_{a,b}$. Up to equivalence, the d-orbifold $[\boldsymbol{U}_{a,b}/G]$ is independent of b and the representation of G on $\mathbb{R}^b$, but it does remember a and the representation of G on $\mathbb{R}^a$. In this way we may prove a d-orbifold version of Proposition 3.25:

Proposition 10.6. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold, and $[x] \in \mathcal{X}_{\mathrm{top}}$. Let $G, \mathcal{U}, \underline{U}, u$ be as in Theorem* C.25(a), *with $G = \mathrm{Iso}_{\mathcal{X}}([x])$, $\mathcal{U} \subseteq \mathcal{X}$ and $\mathcal{U} \simeq [\underline{U}/G]$ with $[x] \in \mathcal{U}_{\mathrm{top}} \subseteq \mathcal{X}_{\mathrm{top}}$ corresponding to $[u] \in U/G$. Since $\mathcal{X}$ is locally finitely presented, $\underline{U}$ is too, so $T_u^*\underline{U}$ and $O_u\underline{U}$ are defined as in* §3.3.

Then $a := \dim T_u^*\underline{U} - \dim O_u\underline{U} - \mathrm{vdim}\,\boldsymbol{\mathcal{X}} \geqslant 0$, and $\boldsymbol{\mathcal{X}}$ is determined up to non-canonical equivalence near $[x]$ by the C^∞-stack $\mathcal{X}$, the integer $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}$, and a choice of representation of G on $\mathbb{R}^a$, up to automorphisms of $\mathbb{R}^a$.

As for d-manifolds, in a d-orbifold $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$, the extra information in $\mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}}$ is like a vector bundle $\mathcal{E}$ over $\mathcal{X}$. But in the orbifold case, as in §8.3, locally a vector bundle $\mathcal{E}$ is not determined up to isomorphism by $\mathrm{rank}\,\mathcal{E}$: at a point $[x] \in \mathcal{X}_{\mathrm{top}}$, the finite group $\mathrm{Iso}_{\mathcal{X}}([x])$ acts on the fibre $\mathcal{E}|_x$ of $\mathcal{E}$, and $\mathcal{E}$ depends on this representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on $\mathbb{R}^{\mathrm{rank}\,\mathcal{E}}$ locally up to isomorphism. The extra data of the representation of G on $\mathbb{R}^a$ in Proposition 10.6 corresponds to this representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on $\mathbb{R}^{\mathrm{rank}\,\mathcal{E}}$.

Here are d-orbifold analogues of Corollary 3.27 and Proposition 3.28.

Proposition 10.7. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold, and $[x] \in \mathcal{X}_{\mathrm{top}}$. Then there exists an open neighbourhood $\boldsymbol{\mathcal{U}}$ of $[x]$ in $\boldsymbol{\mathcal{X}}$ and an equivalence $\boldsymbol{\mathcal{U}} \simeq \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ in* **dOrb** *for $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ as in Definition* 10.5, *such that the equivalence identifies $[x]$*

with $[v] \in \mathcal{V}_{\mathrm{top}}$ with $s(v) = \mathrm{d}s(v) = 0$. Furthermore, $\mathcal{V}, \mathcal{E}, s$ are determined up to non-canonical equivalence near $[v]$ by $\boldsymbol{\mathcal{X}}$ near $[x]$.

Proposition 10.8. *Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. Then $\boldsymbol{\mathcal{X}}$ is an orbifold (that is, $\boldsymbol{\mathcal{X}} \in \hat{\mathbf{Orb}}$) if and only if $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ has a left inverse, or equivalently, if and only if its virtual cotangent bundle $T^*\boldsymbol{\mathcal{X}}$ is a vector bundle.*

10.1.4 Differential-geometric picture of 1- and 2-morphisms

In §3.4 we defined and studied 'standard model' 1-morphisms $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ in **dSpa**. Here are d-stack analogues for $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ in §10.1.3. There we used notation $f = g + O(s)$, $f = g + O(s^2)$ for $f, g : V \to W$ smooth maps, and $t_1 = t_2 + O(s)$ if $E, F \to V$ are vector bundles and $s \in C^\infty(E)$, $t_1, t_2 \in C^\infty(F)$, and by doing this we weakened the conditions on $f, \hat{f}, g, \hat{g}$ needed to define 1-morphisms $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ and 2-morphisms $\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$.

We could introduce similar notation for orbifolds, but it would be more complicated, as we would need to define 2-morphisms '$\eta : f \Rightarrow g + O(s)$' and '$\eta : f \Rightarrow g + O(s^2)$' for $f, g : \mathcal{V} \to \mathcal{W}$ 1-morphisms of orbifolds. So for simplicity we will omit the $O(s), O(s^2)$ terms and assume the equations hold exactly.

Definition 10.9. Let $\mathcal{V}, \mathcal{W}$ be orbifolds, $\mathcal{E}, \mathcal{F}$ be vector bundles on $\mathcal{V}, \mathcal{W}$, and $s \in C^\infty(\mathcal{E})$, $t \in C^\infty(\mathcal{F})$ be smooth sections, so that Definition 10.5 defines 'standard model' principal d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}, \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$. Write $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = \boldsymbol{\mathcal{S}} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$ and $\boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t} = \boldsymbol{\mathcal{T}} = (\mathcal{T}, \mathcal{O}'_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}}, \imath_{\mathcal{T}}, \jmath_{\mathcal{T}})$. Suppose $f : \mathcal{V} \to \mathcal{W}$ is a 1-morphism, and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ is a morphism in $\mathrm{vect}(\mathcal{V})$ satisfying

$$\hat{f} \circ s = f^*(t). \tag{10.1}$$

We will define a 1-morphism $\boldsymbol{g} = (g, g', g'') : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{T}}$ in **dSta**, which we write as $\boldsymbol{\mathcal{S}}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$, and call a *standard model* 1-morphism.

As in Definition 10.5, $\mathcal{V}, \mathcal{W}$ are categories, and $\mathcal{S} \subseteq \mathcal{V}$, $\mathcal{T} \subseteq \mathcal{W}$ are full subcategories, and $f : \mathcal{V} \to \mathcal{W}$ is a functor. We claim that $f(\mathcal{S}) \subseteq \mathcal{T} \subseteq \mathcal{W}$. To show this, suppose u is an object in $\mathcal{S}$ with $p_{\mathcal{V}}(u) = \underline{U} \in \mathbf{C^\infty Sch}$, and let $\tilde{u} : \bar{\underline{U}} \to \mathcal{V}$ be the corresponding 1-morphism. Then $\tilde{u}^*(s) = 0$. The object $f(u)$ in $\mathcal{V}$ has $p_{\mathcal{W}}(f(u)) = \underline{U}$, and has corresponding 1-morphism $\widetilde{f(u)} = f \circ \tilde{u} : \bar{\underline{U}} \to \mathcal{V}$. Consider the diagram in $\mathrm{qcoh}(\bar{\underline{U}})$:

$$\begin{tikzcd}
\tilde{u}^*(\mathcal{O}_{\mathcal{V}}) \arrow[r, "\cong", "\tilde{u}^*(\iota)"'] \arrow[d, "\tilde{u}^*(s)=0"] & \tilde{u}^*(f^*(\mathcal{O}_{\mathcal{W}})) \arrow[r, "\cong", "I_{\tilde{u},f}(\mathcal{O}_{\mathcal{W}})^{-1}"'] \arrow[d, "\tilde{u}^*(f^*(t))"] & \widetilde{f(u)}^*(\mathcal{O}_{\mathcal{W}}) \arrow[d, "(\widetilde{f(u)})^*(t)"'] \\
\tilde{u}^*(\mathcal{E}) \arrow[r, "\tilde{u}^*(\hat{f})"] & \tilde{u}^*(f^*(\mathcal{F})) \arrow[r, "I_{\tilde{u},f}(\mathcal{F})^{-1}"] & \widetilde{f(u)}^*(\mathcal{F}).
\end{tikzcd}$$

Here the left hand square commutes by applying $\tilde{u}^*$ to (10.1), the right hand square commutes by $\widetilde{f(u)} = f \circ \tilde{u}$ and properties of $I_{*,*}(*)$, and the top row is isomorphisms. So as $\tilde{u}^*(s) = 0$ we see that $(\widetilde{f(u)})^*(t) = 0$, and $f(u)$ is an object in $\mathcal{T}$. Therefore $f|_{\mathcal{S}}$ is a functor $\mathcal{S} \to \mathcal{T}$. Define $g = f|_{\mathcal{S}} : \mathcal{S} \to \mathcal{T}$.

Then $g : \mathcal{S} \to \mathcal{T}$ is a 1-morphism of Deligne–Mumford C^∞-stacks. It satisfies $i_{\mathcal{W}} \circ g = f \circ i_{\mathcal{V}} : \mathcal{S} \to \mathcal{W}$, where the 1-morphisms are equal, not just 2-isomorphic.

To define $g' : g^{-1}(\mathcal{O}'_{\mathcal{T}}) \to \mathcal{O}'_{\mathcal{S}}$, consider the commutative diagram:

$$\begin{tikzcd}
g^{-1}(\mathcal{I}_t^2) \arrow[r] \arrow[d, dotted] & g^{-1}(i_{\mathcal{W}}^{-1}(\mathcal{O}_{\mathcal{W}})) \arrow[r] \arrow[d, "{\substack{i_{\mathcal{V}}^{-1}(f^\sharp)\circ I_{i_{\mathcal{V}},f}(\mathcal{O}_{\mathcal{W}})\circ \\ I_{g,i_{\mathcal{W}}}(\mathcal{O}_{\mathcal{W}})^{-1}}}"] & g^{-1}(\mathcal{O}'_{\mathcal{T}}) = g^{-1}(i_{\mathcal{W}}^{-1}(\mathcal{O}_{\mathcal{W}})/\mathcal{I}_t^2) \arrow[r] \arrow[d, dotted, "g'"] & 0 \\
\mathcal{I}_s^2 \arrow[r] & i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}}) \arrow[r] & \mathcal{O}'_{\mathcal{S}} = i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2 \arrow[r] & 0.
\end{tikzcd}$$

The rows are exact. Using (10.1), one can show the central column maps $g^{-1}(\mathcal{I}_t) \to \mathcal{I}_s$, and so maps $g^{-1}(\mathcal{I}_t^2) \to \mathcal{I}_s^2$, and the left column exists. Thus by exactness there is a unique morphism g' making the diagram commute.

We have $\mathcal{E}_{\mathcal{S}} = i_{\mathcal{V}}^*(\mathcal{E}^*)$ and $\mathcal{E}_{\mathcal{T}} = i_{\mathcal{W}}^*(\mathcal{F}^*)$, where $\mathcal{E}^*, \mathcal{F}^*$ are the duals of $\mathcal{E}, \mathcal{F}$. Then $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ induces $\hat{f}^* : f^*(\mathcal{F}^*) \to \mathcal{E}^*$ in $\mathrm{vect}(\mathcal{V})$. Define $g'' = i_{\mathcal{V}}^*(\hat{f}^*) \circ I_{i_{\mathcal{V}},f}(\mathcal{F}^*) \circ I_{g,i_{\mathcal{W}}}(\mathcal{F}^*)^{-1} : g^*(\mathcal{E}_{\mathcal{T}}) \to \mathcal{E}_{\mathcal{S}}$ in $\mathrm{qcoh}(\mathcal{S})$. One can now show that $\boldsymbol{g} = (g, g', g'') : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{T}}$ is a 1-morphism in **dSta**, which we write as $\boldsymbol{\mathcal{S}}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$, and call a *standard model* 1-morphism.

Suppose now that $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ is open, with inclusion 1-morphism $i_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \to \mathcal{V}$. Write $\tilde{\mathcal{E}} = \mathcal{E}|_{\tilde{\mathcal{V}}} = i_{\tilde{\mathcal{V}}}^*(\mathcal{E})$ and $\tilde{s} = s|_{\tilde{\mathcal{V}}}$. Define $\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}} = \boldsymbol{\mathcal{S}}_{i_{\tilde{\mathcal{V}}},\mathrm{id}_{\tilde{\mathcal{E}}}} : \boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}} \to \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$. As in Lemma 3.33, if $s^{-1}(0) \subseteq \tilde{\mathcal{V}}$ then $\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}}$ is a 1-isomorphism.

We can write down an analogue of Theorem 3.34 for the standard model 1-morphisms of Definition 10.9, but it is weaker. Theorem 3.34 classifies 1-morphisms $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ in **dMan** up to equality, not just up to 2-isomorphism. But if $\boldsymbol{g} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ is a 1-morphism in **dSta**, it is only sensible to classify the C^∞-stack 1-morphism g in $\boldsymbol{g} = (g, g', g'')$ up to 2-isomorphism, not up to equality. Thus, the analogue of Theorem 3.34 should say only that any $\boldsymbol{g} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ is 2-isomorphic in **dOrb** to some $\boldsymbol{\mathcal{S}}_{f,\hat{f}} \circ \boldsymbol{i}_{\tilde{V},V}^{-1}$.

Definition 3.29 also defined 'standard model' 2-morphisms $\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ in **dMan**. We could write down a d-orbifold analogue of this, but will not, to avoid having to combine $O(s), O(s^2)$ and 2-morphisms of orbifolds. But here is a simpler construction of 2-morphisms $\boldsymbol{\mathcal{S}}_{f,\hat{f}} \Rightarrow \boldsymbol{\mathcal{S}}_{g,\hat{g}}$ coming from 2-morphisms $\eta : f \Rightarrow g$. The proof is easy.

Proposition 10.10. *Let $\boldsymbol{\mathcal{S}}_{f,\hat{f}}, \boldsymbol{\mathcal{S}}_{g,\hat{g}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ be 'standard model'* 1-*morphisms of d-orbifolds, in the notation of Definitions* 10.5 *and* 10.9. *Suppose $\eta : f \Rightarrow g$ is a* 2-*morphism in* **Orb** *which satisfies $\hat{g} = \eta^*(\mathcal{F}) \circ \hat{f} : \mathcal{E} \to g^*(\mathcal{F})$. Then $\boldsymbol{\eta} = (\eta|_{\mathcal{S}_{\mathcal{V},\mathcal{E},s}}, 0)$ is a* 2-*morphism $\boldsymbol{\eta} : \boldsymbol{\mathcal{S}}_{f,\hat{f}} \Rightarrow \boldsymbol{\mathcal{S}}_{g,\hat{g}}$ in* **dOrb**.

10.1.5 An alternative form of 'standard model' d-orbifolds

Rather than building 'standard models' $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}, \boldsymbol{\mathcal{S}}_{f,\hat{f}}$ for d-orbifolds and their 1-morphisms using orbifolds $\mathcal{V}$, we can instead use combine the 'standard model' notation $\boldsymbol{S}_{V,E,s}, \boldsymbol{S}_{f,\hat{f}}, \boldsymbol{S}_\Lambda$ for d-manifolds in §3.3–§3.4 with the quotient d-stack notation of §9.3. We explain this for d-orbifolds, 1-morphisms and 2-morphisms in the next three examples.

Example 10.11. Let V be a manifold, $E \to V$ a vector bundle, Γ a finite group acting smoothly on V, E preserving the vector bundle structure, and $s : V \to E$ a smooth, Γ-equivariant section of E. Write the Γ-actions on V, E as $r(\gamma) : V \to V$ and $\hat{r}(\gamma) : E \to r(\gamma)^*(E)$ for $\gamma \in \Gamma$. Then Definitions 3.13 and 3.30 give an explicit principal d-manifold $\boldsymbol{S}_{V,E,s}$, and 1-morphisms $\boldsymbol{S}_{r(\gamma),\hat{r}(\gamma)} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{V,E,s}$ for $\gamma \in \Gamma$ which are an action of Γ on $\boldsymbol{S}_{V,E,s}$. Hence Definition 9.15 defines a quotient d-stack $[\boldsymbol{S}_{V,E,s}/\Gamma]$.

Now $\tilde{\mathcal{V}} = [\underline{V}/\Gamma]$ is an orbifold, and by Definition C.34 and Theorem C.35 E, s induce a vector bundle $\tilde{\mathcal{E}}$ on $\tilde{\mathcal{V}}$ and section $\tilde{s} \in C^\infty(\tilde{\mathcal{E}})$, so that Definition 10.5 gives a 'standard model' principal d-orbifold $\boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}}$. One can show that $[\boldsymbol{S}_{V,E,s}/\Gamma] \simeq \boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}}$, so $[\boldsymbol{S}_{V,E,s}/\Gamma]$ is a principal d-orbifold. But not all principal d-orbifolds $\boldsymbol{\mathcal{W}}$ have $\boldsymbol{\mathcal{W}} \simeq [\boldsymbol{S}_{V,E,s}/\Gamma]$, as not all orbifolds $\mathcal{V}$ have $\mathcal{V} \simeq [\underline{V}/\Gamma]$ for some manifold V and finite group Γ.

Example 10.12. Let $[\boldsymbol{S}_{V,E,s}/\Gamma]$, $[\boldsymbol{S}_{W,F,t}/\Delta]$ be quotient d-orbifolds as in Example 10.11, where Γ acts on V, E by $q(\gamma) : V \to V$ and $\hat{q}(\gamma) : E \to q(\gamma)^*(E)$ for $\gamma \in \Gamma$, and Δ acts on W, F by $r(\delta) : W \to W$ and $\hat{r}(\delta) : F \to r(\delta)^*(F)$ for $\delta \in \Delta$. Suppose $f : V \to W$ is a smooth map, and $\hat{f} : E \to f^*(F)$ is a morphism of vector bundles on V satisfying $\hat{f} \circ s = f^*(t) + O(s^2)$, as in (3.22), and $\rho : \Gamma \to \Delta$ is a group morphism satisfying $f \circ q(\gamma) = r(\rho(\gamma)) \circ f : V \to W$ and $q(\gamma)^*(\hat{f}) \circ \hat{q}(\gamma) = f^*(\hat{r}(\rho(\gamma))) \circ \hat{f} : E \to (f \circ q(\gamma))^*(F)$ for all $\gamma \in \Gamma$, so that $f, \hat{f}$ are equivariant under Γ, Δ, ρ. Then Definition 3.30 defines a 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ in **dSpa**. The equivariance conditions on $f, \hat{f}$ imply that $\boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{S}_{q(\gamma),\hat{q}(\gamma)} = \boldsymbol{S}_{r(\rho(\gamma)),\hat{r}(\rho(\gamma))} \circ \boldsymbol{S}_{f,\hat{f}}$ for $\gamma \in \Gamma$. Hence Definition 9.15 defines a quotient 1-morphism $[\boldsymbol{S}_{f,\hat{f}}, \rho] : [\boldsymbol{S}_{V,E,s}/\Gamma] \to [\boldsymbol{S}_{W,F,t}/\Delta]$.

Example 10.13. Suppose $[\boldsymbol{S}_{f,\hat{f}}, \rho], [\boldsymbol{S}_{g,\hat{g}}, \sigma] : [\boldsymbol{S}_{V,E,s}/\Gamma] \to [\boldsymbol{S}_{W,F,t}/\Delta]$ are two 1-morphisms as in Example 10.12, and write $q, \hat{q}$ for the actions of Γ on V, E and $r, \hat{r}$ for the actions of Δ on W, F. Then $\rho, \sigma : \Gamma \to \Delta$ are group morphisms. Suppose $\delta \in \Delta$ satisfies $\sigma(\gamma) = \delta\, \rho(\gamma)\, \delta^{-1}$ for all $\gamma \in \Gamma$, and $\Lambda : E \to f^*(TW)$ is a morphism of vector bundles on V which satisfies

$$r(\delta^{-1}) \circ g = f + \Lambda \cdot s + O(s^2) \text{ and } g^*(\hat{r}(\delta^{-1})) \circ \hat{g} = \hat{f} + \Lambda \cdot \mathrm{d}t + O(s), \quad (10.2)$$

$$f^*(\mathrm{d}r(\rho(\gamma))) \circ \Lambda = q(\gamma)^*(\Lambda) \circ \hat{q}(\gamma) : E \longrightarrow (f \circ q(\gamma))^*(TW), \quad \forall \gamma \in \Gamma, \quad (10.3)$$

where $\mathrm{d}r(\rho(\gamma)) : TW \to r(\rho(\gamma))^*(TW)$ is the derivative of $r(\rho(\gamma))$. Here (10.2) is the conditions for Definition 3.29 to define a 'standard model' 2-morphism $\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{r(\delta^{-1})\circ g, g^*(\hat{r}(\delta^{-1}))\circ\hat{g}} = \boldsymbol{S}_{r(\delta^{-1}),\hat{r}(\delta^{-1})} \circ \boldsymbol{S}_{g,\hat{g}}$ in **dSpa**. Then (10.3) implies that $\boldsymbol{S}_\Lambda * \mathrm{id}_{\boldsymbol{S}_{q(\gamma),\hat{q}(\gamma)}} = \mathrm{id}_{\boldsymbol{S}_{r(\rho(\gamma)),\hat{r}(\rho(\gamma))}} * \boldsymbol{S}_\Lambda$ for all $\gamma \in \Gamma$. Hence Definition 9.15 defines a quotient 2-morphism $[\boldsymbol{S}_\Lambda, \delta] : [\boldsymbol{S}_{f,\hat{f}}, \rho] \Rightarrow [\boldsymbol{S}_{g,\hat{g}}, \sigma]$.

Proposition 10.14. *A d-stack $\boldsymbol{\mathcal{X}}$ is a d-orbifold of virtual dimension $n \in \mathbb{Z}$ if and only if each $[x] \in \mathcal{X}_{\rm top}$ has an open neighbourhood $\boldsymbol{\mathcal{U}}$ equivalent to some $[\boldsymbol{S}_{V,E,s}/\Gamma]$ in Example 10.11 with $\dim V - \operatorname{rank} E = n$, where $\Gamma = \mathrm{Iso}_{\mathcal{X}}([x])$ and $[x] \in \mathcal{X}_{\rm top}$ is identified with a fixed point v of Γ in V with $s(v) = 0$ and $\mathrm{d}s(v) = 0$. Furthermore, V, E, s, Γ are determined up to non-canonical isomorphism near v by $\boldsymbol{\mathcal{X}}$ near $[x]$.*

Proof. For the 'if' part, note that $[\boldsymbol{S}_{V,E,s}/\Gamma]$ is a principal d-orbifold of virtual dimension n by Example 10.11, so $\boldsymbol{\mathcal{X}}$ is covered by open principal d-orbifolds $\boldsymbol{\mathcal{U}}$ with $\mathrm{vdim}\boldsymbol{\mathcal{U}}=n$, and $\boldsymbol{\mathcal{X}}$ is a d-orbifold with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}=n$.

For the 'only if' part, suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold and $[x]\in\mathcal{X}_{\mathrm{top}}$. Proposition 10.7 gives open $\boldsymbol{\mathcal{U}}\subseteq\boldsymbol{\mathcal{X}}$ and an equivalence $\boldsymbol{\mathcal{U}}\simeq\boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}}$ identifying $[x]$ with $[\tilde v]$ for $[\tilde v]\in\tilde{\mathcal{V}}_{\mathrm{top}}$ with $\tilde s(v)=\mathrm{d}\tilde s(v)=0$. As $\mathcal{V}$ is an orbifold it is equivalent to $[\underline{V}/\Gamma]$ near $[v]$, where $\underline{V}=F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(V)$ and V is a manifold acted on by $\Gamma=\mathrm{Iso}_{\tilde{\mathcal{V}}}([\tilde v])=\mathrm{Iso}_{\mathcal{X}}([x])$. Making $\boldsymbol{\mathcal{U}},\tilde{\mathcal{V}}$ smaller we can take $\tilde{\mathcal{V}}=[\underline{V}/\Gamma]$. Then Definition C.34 and Theorem C.35 show $\mathcal{E},s$ come from a Γ-equivariant vector bundle $E\to V$ and section $s:V\to E$ up to isomorphism. So Example 10.11 defines $[\boldsymbol{S}_{V,E,s}/\Gamma]$ with $[\boldsymbol{S}_{V,E,s}/\Gamma]\simeq\boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}}\simeq\boldsymbol{\mathcal{U}}$. The last part follows from the last part of Proposition 10.7. $\square$

Proposition 10.15. *A quotient d-stack* $[\boldsymbol{U}/G]$ *is a d-orbifold if and only if the d-space* $\boldsymbol{U}$ *is a d-manifold, and then* $\mathrm{vdim}[\boldsymbol{U}/G]=\mathrm{vdim}\,\boldsymbol{U}$.

Proof. Suppose $[\boldsymbol{U}/G]$ is a d-orbifold. Then Proposition 10.14 shows we can cover $[\boldsymbol{U}/G]$ by open $\boldsymbol{\mathcal{W}}$ with equivalences $\boldsymbol{i}:[\boldsymbol{S}_{V,E,s}/\Gamma]\to\boldsymbol{\mathcal{W}}\subseteq[\boldsymbol{U}/G]$. Form the 2-Cartesian square in **dSta**:

$$\begin{tikzcd}
F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{U})\times_{[\boldsymbol{U}/G]}F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{S}_{V,E,s}) \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"] \arrow[dr, phantom, "\Uparrow"] & F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{S}_{V,E,s}) \arrow[d, "\boldsymbol{i}\circ\boldsymbol{h}"] \\
F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{U}) \arrow[r, "\boldsymbol{g}"] & {[\boldsymbol{U}/G],}
\end{tikzcd}$$

where $\boldsymbol{g},\boldsymbol{h}$ are projections of the form $F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{V})\to[\boldsymbol{V}/G]$, and so are étale. Thus $\boldsymbol{i}\circ\boldsymbol{h}$ is étale, so $\boldsymbol{e},\boldsymbol{f}$ are étale by properties of fibre products. Since $F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{S}_{V,E,s})$ is a d-manifold $F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{U})\times_{[\boldsymbol{U}/G]}F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{S}_{V,E,s})$ is a d-manifold as $\boldsymbol{f}$ is étale, so $F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{U})$ and hence $\boldsymbol{U}$ is a d-manifold of dimension $\mathrm{vdim}[\boldsymbol{U}/G]$ on the image of $\boldsymbol{e}$, which is $\boldsymbol{g}^{-1}(\boldsymbol{\mathcal{W}})$. As we can cover $[\boldsymbol{U}/G]$ by such $\boldsymbol{\mathcal{W}}$, $\boldsymbol{U}$ is a d-manifold of dimension $\mathrm{vdim}[\boldsymbol{U}/G]$, proving the 'only if' part.

For the 'if' part, suppose $\boldsymbol{U}$ is a d-manifold, and let $u\in\boldsymbol{U}$ have stabilizer group H in G. Applying Corollary 3.26 to $\boldsymbol{U}$ at u gives an open neighbourhood $\boldsymbol{U}'$ of u in $\boldsymbol{U}$ and an equivalence $\boldsymbol{U}'\simeq\boldsymbol{S}_{V,E,s}$ identifying $u\in\boldsymbol{U}'$ and $v\in V$ with $s(v)=\mathrm{d}s(v)=0$. The proof works equivariantly w.r.t. H, so we can choose $\boldsymbol{U}'$ H-invariant, and V,E with H-actions fixing $v\in V$ and s H-equivariant, so that H acts on $\boldsymbol{S}_{V,E,s}$, and $[\boldsymbol{U}'/H]\simeq[\boldsymbol{S}_{V,E,s}/H]$. Making $\boldsymbol{U}',V$ smaller if necessary we can suppose $\gamma(\boldsymbol{U}')\cap\boldsymbol{U}'=\emptyset$ for $\gamma\in G\setminus H$. Then $[\boldsymbol{U}'/H]$ is equivalent to an open neighbourhood of $[u]$ in $[\boldsymbol{U}/G]$. Hence $[\boldsymbol{U}/G]$ is equivalent to the principal d-orbifold $[\boldsymbol{S}_{V,E,s}/H]$ of virtual dimension $\mathrm{vdim}\,\boldsymbol{U}$ near $[u]$, so $[\boldsymbol{U}/G]$ is a d-orbifold with $\mathrm{vdim}[\boldsymbol{U}/G]=\mathrm{vdim}\,\boldsymbol{U}$. $\square$

10.2 Equivalences and gluing

We now extend §3.5–§3.6 to d-orbifolds, using the results of §9.4. Definition 9.13 defined when a 1-morphism of d-stacks $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ is étale, so this gives a notion of when a 1-morphism of d-orbifolds $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ is étale. Here is the d-orbifold analogue of Theorem 3.36.

Theorem 10.16. *Suppose $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a* 1*-morphism of d-orbifolds, and $f : \mathcal{X} \to \mathcal{Y}$ is representable. Then the following are equivalent:*

(i) *$\boldsymbol{f}$ is étale;*

(ii) *$\Omega_{\boldsymbol{f}} : f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ is an equivalence in* $\mathrm{vqcoh}(\mathcal{X})$*; and*

(iii) *equation* (9.14) *is a split short exact sequence in* $\mathrm{qcoh}(\mathcal{X})$*.*

If in addition $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a bijection, then $\boldsymbol{f}$ is an equivalence in **dOrb**.

Proof. Note that (ii),(iii) are equivalent by Proposition 3.5, and (i) implies (iii) by Definition 9.13. To show (ii),(iii) imply (i), we must prove that (ii),(iii) force $f : \mathcal{X} \to \mathcal{Y}$ to be étale and $f^4 : f^*(\mathcal{C}_{\mathcal{Y}}) \to \mathcal{C}_{\mathcal{X}}$ to be an isomorphism. But we can reduce these to the d-manifold case Theorem 3.36 using Theorem 9.16(b). Hence (i)–(iii) are equivalent. For the final part, f étale, $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ an isomorphism for all $[x]$ and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ a bijection imply that f is an equivalence, so the theorem follows by Proposition 9.11. □

Here is an analogue of Theorem 3.39. The proof follows that of Theorem 3.39, using Theorem 10.16 in place of Theorem 3.36.

Theorem 10.17. *Suppose $\boldsymbol{\mathcal{S}}_{f,\hat f} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ is a 'standard model'* 1*-morphism, in the notation of Definitions* 10.5 *and* 10.9*, with $f : \mathcal{V} \to \mathcal{W}$ representable. Then $\boldsymbol{\mathcal{S}}_{f,\hat f}$ is étale if and only if for each $[v] \in \mathcal{V}_{\mathrm{top}}$ with $s(v) = 0$ and $[w] = f_{\mathrm{top}}([v]) \in \mathcal{W}_{\mathrm{top}}$, the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_v\mathcal{V} \xrightarrow{\mathrm{d}s(v)\oplus \mathrm{d}f(v)} \mathcal{E}_v \oplus T_w\mathcal{W} \xrightarrow{\hat f(v)\oplus -\mathrm{d}t(w)} \mathcal{F}_w \longrightarrow 0.$$

Also $\boldsymbol{\mathcal{S}}_{f,\hat f}$ is an equivalence if and only if in addition $f_{\mathrm{top}}|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection, where $s^{-1}(0) = \{[v] \in \mathcal{V}_{\mathrm{top}} : s(v) = 0\}$, $t^{-1}(0) = \{[w] \in \mathcal{W}_{\mathrm{top}} : t(w) = 0\}$, and $f_* : \mathrm{Iso}_{\mathcal{V}}([v]) \to \mathrm{Iso}_{\mathcal{W}}(f_{\mathrm{top}}([v]))$ is an isomorphism for all $[v] \in s^{-1}(0) \subseteq \mathcal{V}_{\mathrm{top}}$.

Here is the analogue of Corollary 3.40:

Corollary 10.18. *Let $\mathcal{V}, \mathcal{W}$ be orbifolds, $\mathcal{E}, \mathcal{F}$ vector bundles over $\mathcal{V}, \mathcal{W}$, $s \in C^\infty(\mathcal{E})$, $t \in C^\infty(\mathcal{F})$ smooth sections, $f : \mathcal{V} \to \mathcal{W}$ an embedding of orbifolds, and $\hat f : \mathcal{E} \to f^*(\mathcal{F})$ an injective morphism of vector bundles (that is, $\hat f$ has a left inverse) satisfying* (10.1)*. For each $[v] \in \mathcal{V}_{\mathrm{top}}$ with $s(v) = 0$ and $f_{\mathrm{top}}([v]) = [w] \in \mathcal{W}_{\mathrm{top}}$, we have a linear map*

$$\mathrm{d}t(w)_* : T_w\mathcal{W}/\mathrm{d}f(v)[T_v\mathcal{V}] \longrightarrow \mathcal{F}_w/\hat f(v)[\mathcal{E}_v]. \tag{10.4}$$

Suppose (10.4) *is an isomorphism for all $[v] \in s^{-1}(0) \subseteq \mathcal{V}_{\mathrm{top}}$, and $f_{\mathrm{top}}|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection. Then $\boldsymbol{\mathcal{S}}_{f,\hat f} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ is an equivalence.*

Proposition 9.17 and Theorems 9.18 and 9.19 of §9.4 explained how to glue d-stacks and their 1-morphisms by equivalences on open d-substacks. As for d-manifolds in §3.6, these generalize immediately to d-orbifolds: if we fix $n \in \mathbb{Z}$

and take the initial d-stacks $\boldsymbol{\mathcal{X}}_i$ to be d-orbifolds with $\operatorname{vdim}\boldsymbol{\mathcal{X}}_i = n$, then the glued d-stack $\boldsymbol{\mathcal{X}}$ is also a d-orbifold with $\operatorname{vdim}\boldsymbol{\mathcal{X}} = n$. Note that if we apply Theorem 9.19 with the $\boldsymbol{\mathcal{X}}_i$ effective d-orbifolds, then the $\mathcal{X}_i$ are effective by Lemma 10.61, so (9.19) holds automatically. So, gluing effective d-orbifolds by equivalences is simpler than gluing general d-orbifolds by equivalences.

Here is a d-orbifold analogue of Theorem 3.42. To prove it we apply Theorem 9.19 taking the d-stacks $\boldsymbol{\mathcal{X}}_i$ to be 'standard model' d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$ in Definition 10.5, and the 1-morphisms $\boldsymbol{e}_{ij},\boldsymbol{g}_i$ 'standard model' 1-morphisms $\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}},\boldsymbol{\mathcal{S}}_{g_i,0}$ in Definition 10.9, and the 2-morphisms $\boldsymbol{\eta}_{ijk},\boldsymbol{\zeta}_{ij}$ 2-morphisms $(\eta_{ijk}|_{\mathcal{S}_{\cdots}},0)$, $(\zeta_{ij}|_{\mathcal{S}_{\cdots}},0)$ from Proposition 10.10. Part (ii) and Theorem 10.17 imply that $\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}$ is an equivalence. Since $\mathcal{V}_i,\mathcal{Y}$ are effective, Proposition 8.14 implies that (9.19)–(9.20) hold automatically.

Theorem 10.19. *Suppose we are given the following data:*

(a) *an integer n;*

(b) *a Hausdorff, second countable topological space X;*

(c) *an indexing set I, and a total order $<$ on I;*

(d) *for each i in I, an effective orbifold $\mathcal{V}_i$, a vector bundle $\mathcal{E}_i$ on $\mathcal{V}_i$ with $\dim\mathcal{V}_i - \operatorname{rank}\mathcal{E}_i = n$, a section $s_i \in C^\infty(\mathcal{E}_i)$, and a homeomorphism $\psi_i : s_i^{-1}(0) \to \hat{X}_i$, where $s_i^{-1}(0) = \{[v_i] \in \mathcal{V}_{i,\mathrm{top}} : s_i(v_i) = 0\}$ and $\hat{X}_i \subseteq X$ is open; and*

(e) *for all $i < j$ in I, an open suborbifold $\mathcal{V}_{ij} \subseteq \mathcal{V}_i$, a 1-morphism $e_{ij} : \mathcal{V}_{ij} \to \mathcal{V}_j$, and a morphism of vector bundles $\hat{e}_{ij} : \mathcal{E}_i|_{\mathcal{V}_{ij}} \to e_{ij}^*(\mathcal{E}_j)$.*

Let this data satisfy the conditions:

(i) *$X = \bigcup_{i\in I}\hat{X}_i$;*

(ii) *if $i < j$ in I then $(e_{ij})_* : \mathrm{Iso}_{\mathcal{V}_{ij}}([v]) \to \mathrm{Iso}_{\mathcal{V}_j}(e_{ij,\mathrm{top}}([v]))$ is an isomorphism for all $[v] \in \mathcal{V}_{ij,\mathrm{top}}$, and $\hat{e}_{ij}\circ s_i|_{\mathcal{V}_{ij}} = e_{ij}^*(s_j)\circ\iota_{ij}$ where $\iota_{ij} : \mathcal{O}_{\mathcal{V}_{ij}} \to e_{ij}^*(\mathcal{O}_{\mathcal{V}_j})$ is the natural isomorphism, and $\psi_i(s_i|_{\mathcal{V}_{ij}}^{-1}(0)) = \hat{X}_i\cap\hat{X}_j$, and $\psi_i|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)} = \psi_j\circ e_{ij,\mathrm{top}}|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)}$, and if $[v_i] \in \mathcal{V}_{ij,\mathrm{top}}$ with $s_i(v_i) = 0$ and $[v_j] = e_{ij,\mathrm{top}}([v_i])$ then the following sequence is exact:*

$$0 \longrightarrow T_{v_i}\mathcal{V}_i \xrightarrow{\mathrm{d}s_i(v_i)\oplus\,\mathrm{d}e_{ij}(v_i)} \mathcal{E}_i|_{v_i}\oplus T_{v_j}\mathcal{V}_j \xrightarrow{\hat{e}_{ij}(v_i)\oplus\,-\mathrm{d}s_j(v_j)} \mathcal{E}_j|_{v_j} \longrightarrow 0;$$

(iii) *if $i < j < k$ in I then there exists a 2-morphism $\eta_{ijk} : e_{jk}\circ e_{ij}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})} \Rightarrow e_{ik}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}$ in* **Orb** *with*

$$\hat{e}_{ik}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})} = \eta_{ijk}^*(\mathcal{E}_k)\circ I_{e_{ij},e_{jk}}(\mathcal{E}_k)^{-1}\circ e_{ij}|^*_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}(\hat{e}_{jk})\circ\hat{e}_{ij}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}.$$

Note that η_{ijk} is unique by Proposition 8.14.

Then there exist a d-orbifold $\boldsymbol{\mathcal{X}}$ with $\operatorname{vdim}\boldsymbol{\mathcal{X}} = n$ and underlying topological space $\mathcal{X}_{\mathrm{top}} \cong X$, and a 1-morphism $\boldsymbol{\psi}_i : \boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i} \to \boldsymbol{\mathcal{X}}$ with underlying continuous map ψ_i which is an equivalence with the open d-suborbifold $\hat{\boldsymbol{\mathcal{X}}}_i \subseteq \boldsymbol{\mathcal{X}}$

corresponding to $\hat{X}_i \subseteq X$ *for all* $i \in I$*, such that for all* $i < j$ *in* I *there exists a* 2*-morphism* $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ \boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i \circ \boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i}$*, where* $\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}} : \boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}} \to \boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j}$ *and* $\boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i} : \boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}} \to \boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$*. This d-orbifold* $\boldsymbol{\mathcal{X}}$ *is unique up to equivalence in* **dOrb**.

Suppose also that $\mathcal{Y}$ *is an effective orbifold, and* $g_i : \mathcal{V}_i \to \mathcal{Y}$ *are* 1*-morphisms for all* $i \in I$ *satisfying any of Proposition* 8.14(i)–(v)*, and there are* 2*-morphisms* $\zeta_{ij} : g_j \circ e_{ij} \Rightarrow g_i|_{\mathcal{V}_{ij}}$ *in* **Orb** *for all* $i < j$ *in* I*. Then there exist a* 1*-morphism* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *in* **dOrb** *unique up to* 2*-isomorphism, where* $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y}) = \boldsymbol{\mathcal{S}}_{\mathcal{Y},0,0}$*, and* 2*-morphisms* $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \boldsymbol{\mathcal{S}}_{g_i,0}$ *for all* $i \in I$*.*

Remark 10.20. (a) The assumptions in Theorem 10.19 that the $\mathcal{V}_i$ and $\mathcal{Y}$ are *effective* orbifolds, and $(e_{ij})_* : \mathrm{Iso}_{\mathcal{V}_{ij}}([v]) \xrightarrow{\cong} \mathrm{Iso}_{\mathcal{V}_j}(e_{ij,\mathrm{top}}([v]))$, and the g_i satisfy any of Proposition 8.14(i)–(v), are all made so that the conditions (9.19)–(9.20) on 2-morphisms on quadruple and triple overlaps in Theorem 9.19 hold automatically, because of Proposition 8.14, so we can omit them.

(b) Because we have no overlap conditions on 2-morphisms, Theorem 10.19 makes sense in the homotopy category $\mathrm{Ho}(\mathbf{Orb^{eff}})$ of the 2-category of effective orbifolds, as in §4.7 and Remark 9.20. That is, for the purposes of constructing d-orbifolds using Theorem 10.19 we can treat effective orbifolds as forming a category rather than a 2-category.

(c) Taking the orbifolds $\mathcal{V}_i$ effective does not force the d-orbifold $\boldsymbol{\mathcal{X}}$ to be effective. Every d-orbifold $\boldsymbol{\mathcal{X}}$ is locally equivalent to some $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ with $\mathcal{V}$ effective.

(d) We can simplify the theorem by taking the $e_{ij}, \hat{e}_{ij}$ to satisfy the hypotheses of Corollary 10.18 instead of part (iii).

(e) In §10.8 we will prove a kind of converse to Theorem 10.19: we will show that every d-orbifold $\boldsymbol{\mathcal{X}}$ admits a *good coordinate system*, a collection of data $I, <, \mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i, \mathcal{V}_{ij}, e_{ij}, \hat{e}_{ij}, \boldsymbol{\eta}_{ij}, \eta_{ijk}$ satisfying the hypotheses of the first part of Theorem 10.19. This shows these hypotheses are not unrealistically strong.

(f) The importance of Theorem 10.19 is that all the ingredients are described wholly in differential-geometric or topological terms. So we can use the theorem as a tool to prove the existence of d-orbifold structures on spaces coming from other areas of geometry, such as moduli spaces of J-holomorphic curves. We will discuss this in Chapter 14.

Here is a similar result using the quotient d-orbifolds $[\boldsymbol{S}_{V,E,s}/\Gamma]$ and 1-morphisms $[\boldsymbol{S}_{f,\hat{f}}, \rho]$ of §10.1.5, proved in the same way.

Theorem 10.21. *Suppose we are given the following data:*

(a) *an integer* n*;*

(b) *a Hausdorff, second countable topological space* X*;*

(c) *an indexing set* I*, and a total order* $<$ *on* I*;*

(d) *for each* i *in* I*, a manifold* V_i*, a vector bundle* $E_i \to V_i$ *with* $\dim V_i - \mathrm{rank}\, E_i = n$*, a finite group* Γ_i*, smooth, locally effective actions* $r_i(\gamma) : V_i \to V_i$*,* $\hat{r}_i(\gamma) : E_i \to r(\gamma)^*(E_i)$ *of* Γ_i *on* V_i, E_i *for* $\gamma \in \Gamma_i$*, a smooth,*

Γ_i*-equivariant section* $s_i : V_i \to E_i$*, and a homeomorphism* $\psi_i : X_i \to \hat{X}_i$*, where* $X_i = \{v_i \in V_i : s_i(v_i) = 0\}/\Gamma_i$ *and* $\hat{X}_i \subseteq X$ *is an open set; and*

(e) *for all* $i < j$ *in* I*, an open submanifold* $V_{ij} \subseteq V_i$*, invariant under* Γ_i*, a group morphism* $\rho_{ij} : \Gamma_i \to \Gamma_j$*, a smooth map* $e_{ij} : V_{ij} \to V_j$*, and a morphism of vector bundles* $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$*.*

Let this data satisfy the conditions:

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if* $i < j$ *in* I *then* $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j) + O(s_i^2)$*, and for all* $\gamma \in \Gamma$ *we have*

$$\begin{gathered} e_{ij} \circ r_i(\gamma) = r_j(\rho_{ij}(\gamma)) \circ e_{ij} : V_{ij} \longrightarrow V_j, \\ r_i(\gamma)^*(\hat{e}_{ij}) \circ \hat{r}_i(\gamma) = e_{ij}^*(\hat{r}_j(\rho_{ij}(\gamma))) \circ \hat{e}_{ij} : E_i|_{V_{ij}} \longrightarrow (e_{ij} \circ r_i(\gamma))^*(E_j), \end{gathered}$$

and $\psi_i(X_i \cap (V_{ij}/\Gamma_i)) = \hat{X}_i \cap \hat{X}_j$*, and* $\psi_i|_{X_i \cap V_{ij}/\Gamma_i} = \psi_j \circ (e_{ij})_*|_{X_i \cap V_{ij}/\Gamma_j}$*, and if* $v_i \in V_{ij}$ *with* $s_i(v_i) = 0$ *and* $v_j = e_{ij}(v_i)$ *then* $\rho|_{\mathrm{Stab}_{\Gamma_i}(v_i)} : \mathrm{Stab}_{\Gamma_i}(v_i) \to \mathrm{Stab}_{\Gamma_j}(v_j)$ *is an isomorphism, and the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} E_j|_{v_j} \longrightarrow 0;$$

(iii) *if* $i < j < k$ *in* I *then there exists* $\gamma_{ijk} \in \Gamma_k$ *satisfying*

$$\begin{aligned} \rho_{ik}(\gamma) &= \gamma_{ijk}\, \rho_{jk}(\rho_{ij}(\gamma))\, \gamma_{ijk}^{-1} \quad \textit{for all } \gamma \in \Gamma_i, \\ e_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} &= r_k(\gamma_{ijk}) \circ e_{jk} \circ e_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}, \quad \textit{and} \\ \hat{e}_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} &= \big(e_{ij}^*(e_{jk}^*(\hat{r}_k(\gamma_{ijk}))) \circ e_{ij}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}\big)|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}. \end{aligned}$$

Then there exist a d-orbifold $\boldsymbol{\mathcal{X}}$ *with* $\mathrm{vdim}\, \boldsymbol{\mathcal{X}} = n$ *and underlying topological space* $\mathcal{X}_{\mathrm{top}} \cong X$*, and a 1-morphism* $\boldsymbol{\psi}_i : [\boldsymbol{S}_{V_i, E_i, s_i}/\Gamma_i] \to \boldsymbol{\mathcal{X}}$ *with underlying continuous map* ψ_i *which is an equivalence with the open d-suborbifold* $\hat{\boldsymbol{\mathcal{X}}}_i \subseteq \boldsymbol{\mathcal{X}}$ *corresponding to* $\hat{X}_i \subseteq X$ *for all* $i \in I$*, such that for all* $i < j$ *in* I *there exists a 2-morphism* $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ [\boldsymbol{S}_{e_{ij}, \hat{e}_{ij}}, \rho_{ij}] \Rightarrow \boldsymbol{\psi}_i \circ [\boldsymbol{i}_{V_{ij}, V_i}, \mathrm{id}_{\Gamma_i}]$*, where* $[\boldsymbol{S}_{e_{ij}, \hat{e}_{ij}}, \rho_{ij}] : [\boldsymbol{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}}/\Gamma_i] \to [\boldsymbol{S}_{V_j, E_j, s_j}/\Gamma_j]$ *and* $[\boldsymbol{i}_{V_{ij}, V_i}, \mathrm{id}_{\Gamma_i}] : [\boldsymbol{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}}/\Gamma_i] \to [\boldsymbol{S}_{V_i, E_i, s_i}/\Gamma_j]$ *are as in Example* 10.12*. This d-orbifold* $\boldsymbol{\mathcal{X}}$ *is unique up to equivalence in* **dOrb***.*

Suppose also that Y *is a manifold, and* $g_i : V_i \to Y$ *are smooth maps for all* $i \in I$ *with* $g_i \circ r_i(\gamma) = g_i$ *for all* $\gamma \in \Gamma_i$*, and* $g_j \circ e_{ij} = g_i|_{V_{ij}}$ *for all* $i < j$ *in* I*. Then there exist a 1-morphism* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *unique up to 2-isomorphism, where* $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Man}}^{\mathbf{dOrb}}(Y) = [\boldsymbol{S}_{Y,0,0}/\{1\}]$*, and 2-morphisms* $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow [\boldsymbol{S}_{g_i, 0}, \pi_{\{1\}}]$ *for all* $i \in I$*. Here* $[\boldsymbol{S}_{Y,0,0}/\{1\}]$ *is from Example* 10.11 *with* E, s *both zero and* $\Gamma = \{1\}$*, and* $[\boldsymbol{S}_{g_i,0}, \pi_{\{1\}}] : [\boldsymbol{S}_{V_i, E_i, s_i}/\Gamma_i] \to [\boldsymbol{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}}$ *is from Example* 10.12 *with* $\hat{g}_i = 0$ *and* $\rho = \pi_{\{1\}} : \Gamma_i \to \{1\}$*.*

10.3 Submersions, immersions and embeddings

We now extend §4.1–§4.2 to d-orbifolds. If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack and $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ is a 1-morphism in $\mathrm{vvect}(\mathcal{X})$, then we define when $f^\bullet$ is *weakly injective*, *injective*, *weakly surjective* or *surjective* exactly as in Definition 4.1. Propositions 4.2 and 4.3 then hold on $\mathcal{X}$, where Proposition 4.2(iv) is true whether we interpret 'local' in the Zariski or the étale topology. Here is the analogue of Definition 4.4, which uses ideas from Definition 8.3:

Definition 10.22. Let $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a 1-morphism of d-orbifolds. Then $T^*\boldsymbol{\mathcal{X}} = (\mathcal{E}_\mathcal{X}, \mathcal{F}_\mathcal{X}, \phi_\mathcal{X})$ and $f^*(T^*\boldsymbol{\mathcal{Y}}) = \bigl(f^*(\mathcal{E}_\mathcal{Y}), f^*(\mathcal{F}_\mathcal{Y}), f^*(\phi_\mathcal{Y})\bigr)$ are virtual vector bundles on $\mathcal{X}$ of ranks $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}, \mathrm{vdim}\,\boldsymbol{\mathcal{Y}}$, and $\Omega_{\boldsymbol{f}} = (f'', f^2) : f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ is a 1-morphism in $\mathrm{vvect}(\mathcal{X})$.

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\Omega_{\boldsymbol{f}}$ is weakly injective.

(b) We call $\boldsymbol{f}$ a *submersion* if $\Omega_{\boldsymbol{f}}$ is injective.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $f : \mathcal{X} \to \mathcal{Y}$ is representable, i.e. $f_* : \mathrm{Iso}_\mathcal{X}([x]) \to \mathrm{Iso}_\mathcal{Y}(f_{\mathrm{top}}([x]))$ is injective for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $\Omega_{\boldsymbol{f}}$ is weakly surjective.

(d) We call $\boldsymbol{f}$ an *immersion* if $f : \mathcal{X} \to \mathcal{Y}$ is representable and $\Omega_{\boldsymbol{f}}$ is surjective.

(e) We call $\boldsymbol{f}$ a *w-embedding* or *embedding* if it is a w-immersion or immersion, respectively, and $f_* : \mathrm{Iso}_\mathcal{X}([x]) \to \mathrm{Iso}_\mathcal{Y}(f_{\mathrm{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image, so in particular f_{top} is injective.

Parts (c)–(e) enable us to define *d-suborbifolds* of d-orbifolds. *Open d-suborbifolds* are (Zariski) open d-substacks of a d-orbifold. For more general d-suborbifolds, we call $\boldsymbol{i} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ a *w-immersed d-suborbifold*, or *immersed d-suborbifold*, or *w-embedded d-suborbifold*, or *embedded d-suborbifold* of $\boldsymbol{\mathcal{Y}}$, if $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ are d-orbifolds and $\boldsymbol{i}$ is a w-immersion, ..., embedding, respectively.

The 1-morphisms $\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}}) : \boldsymbol{\mathcal{X}}^{\Gamma,\lambda} \to \boldsymbol{\mathcal{X}}$ and $\tilde{\boldsymbol{O}}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}}) : \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\lambda} \to \boldsymbol{\mathcal{X}}$ of §10.7 are examples of w-immersions. Propositions 4.5, 4.6 and 4.7 hold for d-orbifolds and orbifolds, except that in the d-orbifold analogue of Proposition 4.6(a) we also have to assume $f : \mathcal{X} \to \mathcal{Y}$ is representable to deduce $\boldsymbol{f}$ is étale. Here is an analogue of Theorem 4.8. It is proved in the same way, following Theorem 10.17 rather than Theorem 3.39.

Theorem 10.23. *Suppose* $\boldsymbol{S}_{f,\hat f} : \boldsymbol{S}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{S}_{\mathcal{W},\mathcal{F},t}$ *is a 'standard model' 1-morphism in* **dOrb**, *as in Definition* 10.9. *For each* $[v] \in \mathcal{V}_{\mathrm{top}}$ *with* $s(v) = 0$ *and* $[w] = f_{\mathrm{top}}([v]) \in \mathcal{W}_{\mathrm{top}}$, *we have a complex*

$$0 \longrightarrow T_v\mathcal{V} \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} \mathcal{E}_v \oplus T_w\mathcal{W} \xrightarrow{\hat f(v) \oplus -\mathrm{d}t(w)} \mathcal{F}_w \longrightarrow 0. \qquad (10.5)$$

(a) $\boldsymbol{S}_{f,\hat f}$ *is a w-submersion if and only if for all* $[v] \in \mathcal{V}_{\mathrm{top}}$ *with* $s(v) = 0$ *and* $[w] = f_{\mathrm{top}}(v) \in \mathcal{W}_{\mathrm{top}}$, (10.5) *is exact at the fourth term.*

(b) $\boldsymbol{S}_{f,\hat f}$ *is a submersion if and only if for all* $[v] \in \mathcal{V}_{\mathrm{top}}$ *with* $s(v) = 0$ *and* $[w] = f_{\mathrm{top}}(v) \in \mathcal{W}_{\mathrm{top}}$, (10.5) *is exact at the third and fourth terms.*

(c) $\boldsymbol{S}_{f,\hat{f}}$ *is a w-immersion if and only if for all* $[v] \in \mathcal{V}_{\rm top}$ *with* $s(v) = 0$ *and* $[w] = f_{\rm top}(v) \in \mathcal{W}_{\rm top}$, (10.5) *is exact at the second term and* $f_* : \mathrm{Iso}_{\mathcal{V}}([v]) \to \mathrm{Iso}_{\mathcal{W}}([w])$ *is injective.*

(d) $\boldsymbol{S}_{f,\hat{f}}$ *is an immersion if and only if for all* $[v] \in \mathcal{V}_{\rm top}$ *with* $s(v) = 0$ *and* $[w] = f_{\rm top}(v) \in \mathcal{W}_{\rm top}$, (10.5) *is exact at the second and fourth terms and* $f_* : \mathrm{Iso}_{\mathcal{V}}([v]) \to \mathrm{Iso}_{\mathcal{W}}([w])$ *is injective.*

The conditions in **(a)**–**(d)** *are open conditions on* $[v]$ *in* $\{[v] \in \mathcal{V}_{\rm top} : s(v) = 0\}$.

Here is an analogue of the first part of Theorem 4.9, proved in the same way.

Theorem 10.24. *Suppose* $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *is a* 1*-morphism of d-orbifolds, and* $[x] \in \mathcal{X}_{\rm top}$ *with* $g_{\rm top}([x]) = [y] \in \mathcal{Y}_{\rm top}$. *Then there exist open d-suborbifolds* $\boldsymbol{\mathcal{T}} \subseteq \boldsymbol{\mathcal{X}}$ *and* $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{Y}}$ *with* $[x] \in \mathcal{T}_{\rm top}$, $[y] \in \mathcal{U}_{\rm top}$ *and* $\boldsymbol{g}(\boldsymbol{\mathcal{T}}) \subseteq \boldsymbol{\mathcal{U}}$, *a 'standard model'* 1*-morphism* $\boldsymbol{\mathcal{S}}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ *in* **dOrb**, *equivalences* $\boldsymbol{i} : \boldsymbol{\mathcal{T}} \to \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$, $\boldsymbol{j} : \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t} \to \boldsymbol{\mathcal{U}}$, *and a* 2*-morphism* $\boldsymbol{\eta} : \boldsymbol{j} \circ \boldsymbol{\mathcal{S}}_{f,\hat{f}} \circ \boldsymbol{i} \Rightarrow \boldsymbol{g}|_{\boldsymbol{\mathcal{T}}}$. *Furthermore:*

(a) *If* $\boldsymbol{g}$ *is a* ***w-submersion*** *then we can choose the data* $\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{U}}, \ldots, \boldsymbol{j}$ *above such that* $f : \mathcal{V} \to \mathcal{W}$ *is a submersion in* **Orb**, *and* $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ *is a surjective morphism of vector bundles (i.e. has a right inverse).*

(b) *If* $\boldsymbol{g}$ *is a* ***submersion*** *we can choose* $\boldsymbol{\mathcal{T}}, \ldots, \boldsymbol{j}$ *such that* $f : \mathcal{V} \to \mathcal{W}$ *is a submersion and* $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ *is an isomorphism.*

(c) *If* $\boldsymbol{g}$ *is a* ***w-immersion*** *we can choose* $\boldsymbol{\mathcal{T}}, \ldots, \boldsymbol{j}$ *such that* $f : \mathcal{V} \to \mathcal{W}$ *is an immersion in* **Orb**, *and* $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ *is an injective morphism of vector bundles (i.e. has a left inverse).*

(d) *If* $\boldsymbol{g}$ *is an* ***immersion*** *we can choose* $\boldsymbol{\mathcal{T}}, \ldots, \boldsymbol{j}$ *such that* $f : \mathcal{V} \to \mathcal{W}$ *is an immersion and* $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ *is an isomorphism.*

Theorem 4.9(a′)–(d′) do not extend to (d-)orbifolds as stated, since submersions in **Orb** are not (Zariski) locally modelled on projections $\pi_{\mathcal{X}} : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$, and similarly for immersions. As for Corollary 4.12, Theorem 10.24(b) implies:

Corollary 10.25. *Let* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *be a submersion of d-orbifolds with* $\boldsymbol{\mathcal{Y}}$ *an orbifold. Then* $\boldsymbol{\mathcal{X}}$ *is an orbifold.*

10.4 D-transversality and fibre products

Next we generalize §4.3 to d-orbifolds. As for Definition 4.16, we define:

Definition 10.26. Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ be d-manifolds and $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$, $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms, and let $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ be the explicit d-stack fibre product from §9.5. Equation (9.21) defines a morphism α_1 in $\mathrm{qcoh}(\mathcal{W})$:

$$\alpha_1 = \begin{pmatrix} e^*(g'') \circ I_{e,g}(\mathcal{E}_{\mathcal{Z}}) \\ -f^*(h'') \circ I_{f,h}(\mathcal{E}_{\mathcal{Z}}) \circ \eta^*(\mathcal{E}_{\mathcal{Z}}) \\ (g \circ e)^*(\phi_{\mathcal{Z}}) \end{pmatrix} : (g \circ e)^*(\mathcal{E}_{\mathcal{Z}}) \longrightarrow \\ e^*(\mathcal{E}_{\mathcal{X}}) \oplus f^*(\mathcal{E}_{\mathcal{Y}}) \oplus (g \circ e)^*(\mathcal{F}_{\mathcal{Z}}).$$

We call $\boldsymbol{g},\boldsymbol{h}$ *d-transverse* if α_1 has a left inverse

$$\beta = \begin{pmatrix}\beta_1 \, \beta_2 \, \beta_3\end{pmatrix} : e^*(\mathcal{E}_{\mathcal{X}}) \oplus f^*(\mathcal{E}_{\mathcal{Y}}) \oplus (g\circ e)^*(\mathcal{F}_{\mathcal{Z}}) \longrightarrow (g\circ e)^*(\mathcal{F}_{\mathcal{Z}})$$

with $\beta\circ\alpha_1 = \mathrm{id}_{(g\circ e)^*(\mathcal{F}_{\mathcal{Z}})}$.

Propositions 4.17 and 4.18 extend immediately to (d-)orbifolds. Here is the analogue of Theorem 4.21, deduced from Theorem 4.21 using Theorem 9.16.

Theorem 10.27. *Suppose $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}$ are d-orbifolds and $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Z}}$, $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{Z}}$ are d-transverse 1-morphisms, and let $\boldsymbol{\mathcal{W}}=\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}$ be the d-stack fibre product. Then $\boldsymbol{\mathcal{W}}$ is a d-orbifold, with*

$$\mathrm{vdim}\,\boldsymbol{\mathcal{W}} = \mathrm{vdim}\,\boldsymbol{\mathcal{X}} + \mathrm{vdim}\,\boldsymbol{\mathcal{Y}} - \mathrm{vdim}\,\boldsymbol{\mathcal{Z}}. \tag{10.6}$$

Proof. Write $\boldsymbol{e}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{X}}$ and $\boldsymbol{f}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{Y}}$ for the projections from the fibre product. Let $[w]\in\mathcal{W}_{\mathrm{top}}$, and write $e_{\mathrm{top}}([w])=[x]\in\mathcal{X}_{\mathrm{top}}$, $f_{\mathrm{top}}([w])=[y]\in\mathcal{Y}_{\mathrm{top}}$, and $g_{\mathrm{top}}([x])=h_{\mathrm{top}}([y])=[z]\in\mathcal{Z}_{\mathrm{top}}$. Apply Theorem 9.16(a) to $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}$ at $[x],[y],[z]$, and Theorem 9.16(b) to $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Z}}$ at $[x]$ and $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{Z}}$ at $[y]$. Replacing $\boldsymbol{U}$ by $\boldsymbol{U}'$, this yields:

- open neighbourhoods $\boldsymbol{\mathcal{T}}\subseteq\boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{U}}\subseteq\boldsymbol{\mathcal{Y}}$, $\boldsymbol{\mathcal{V}}\subseteq\boldsymbol{\mathcal{Z}}$ of $[x],[y],[z]$ in $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}$ with $\boldsymbol{g}(\boldsymbol{\mathcal{T}}),\boldsymbol{h}(\boldsymbol{\mathcal{U}})\subseteq\boldsymbol{\mathcal{V}}$;
- finite groups $G=\mathrm{Iso}_{\mathcal{X}}([x])$, $H=\mathrm{Iso}_{\mathcal{Y}}([y])$, $K=\mathrm{Iso}_{\mathcal{Z}}([z])$;
- d-spaces $\boldsymbol{T},\boldsymbol{U},\boldsymbol{V}$ with actions $\boldsymbol{p}:G\to\mathrm{Aut}(\boldsymbol{T})$, $\boldsymbol{q}:H\to\mathrm{Aut}(\boldsymbol{U})$, $\boldsymbol{r}:K\to\mathrm{Aut}(\boldsymbol{V})$, so that Definition 9.15 gives quotient d-stacks $[\boldsymbol{T}/G]$, $[\boldsymbol{U}/H]$, $[\boldsymbol{V}/K]$;
- fixed points $t\in\boldsymbol{T}$, $u\in\boldsymbol{U}$, $v\in\boldsymbol{V}$ of G,H,K;
- 1-morphisms $\boldsymbol{i}:[\boldsymbol{T}/G]\to\boldsymbol{\mathcal{X}}$, $\boldsymbol{j}:[\boldsymbol{U}/H]\to\boldsymbol{\mathcal{Y}}$, $\boldsymbol{k}:[\boldsymbol{V}/K]\to\boldsymbol{\mathcal{Z}}$ which are equivalences with $\boldsymbol{\mathcal{T}}\subseteq\boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{U}}\subseteq\boldsymbol{\mathcal{Y}}$, $\boldsymbol{\mathcal{V}}\subseteq\boldsymbol{\mathcal{Z}}$, such that $i_{\mathrm{top}}:[t]\mapsto[x]$, $j_{\mathrm{top}}:[u]\mapsto[y]$ and $k_{\mathrm{top}}:[v]\mapsto[z]$;
- group morphisms $\rho:G\to K$, $\sigma:H\to K$ which are $g_*:\mathrm{Iso}_{\mathcal{X}}([x])\to\mathrm{Iso}_{\mathcal{Z}}([z])$ and $h_*:\mathrm{Iso}_{\mathcal{Y}}([y])\to\mathrm{Iso}_{\mathcal{Z}}([z])$;
- d-space 1-morphisms $\boldsymbol{m}:\boldsymbol{T}\to\boldsymbol{V}$ and $\boldsymbol{n}:\boldsymbol{U}\to\boldsymbol{V}$ which are equivariant w.r.t. $\rho:G\to K$ and $\sigma:H\to K$ and map $\boldsymbol{m}:t\mapsto v$, $\boldsymbol{n}:u\mapsto v$, so that Definition 9.15 gives quotient 1-morphisms $[\boldsymbol{m},\rho]:[\boldsymbol{T}/G]\to[\boldsymbol{V}/K]$ and $[\boldsymbol{n},\sigma]:[\boldsymbol{U}/G]\to[\boldsymbol{V}/K]$; and
- 2-morphisms $\eta:\boldsymbol{g}\circ\boldsymbol{i}\Rightarrow\boldsymbol{k}\circ[\boldsymbol{m},\rho]$ and $\zeta:\boldsymbol{h}\circ\boldsymbol{j}\Rightarrow\boldsymbol{k}\circ[\boldsymbol{n},\sigma]$ in **dSta**.

Let $\boldsymbol{\mathcal{W}}'$ be the open neighbourhood $\boldsymbol{\mathcal{T}}\times_{\boldsymbol{g}|_{\boldsymbol{\mathcal{T}}},\boldsymbol{\mathcal{Z}},\boldsymbol{h}|_{\boldsymbol{\mathcal{U}}}}\boldsymbol{\mathcal{U}}$ of $[w]$ in $\boldsymbol{\mathcal{W}}$. Then we have equivalences of d-stacks

$$\boldsymbol{\mathcal{W}}'\simeq\boldsymbol{\mathcal{T}}\times_{\boldsymbol{g}|_{\boldsymbol{\mathcal{T}}},\boldsymbol{\mathcal{V}},\boldsymbol{h}|_{\boldsymbol{\mathcal{U}}}}\boldsymbol{\mathcal{U}}\simeq[\boldsymbol{T}/G]\times_{[\boldsymbol{m},\rho],[\boldsymbol{V}/K],[\boldsymbol{n},\sigma]}[\boldsymbol{U}/H], \tag{10.7}$$

since $\boldsymbol{i}:[\boldsymbol{T}/G]\to\boldsymbol{\mathcal{X}}$, $\boldsymbol{j}:[\boldsymbol{U}/H]\to\boldsymbol{\mathcal{Y}}$, $\boldsymbol{k}:[\boldsymbol{V}/K]\to\boldsymbol{\mathcal{Z}}$ are equivalences and $\eta:\boldsymbol{g}\circ\boldsymbol{i}\Rightarrow\boldsymbol{k}\circ[\boldsymbol{m},\rho]$ and $\zeta:\boldsymbol{h}\circ\boldsymbol{j}\Rightarrow\boldsymbol{k}\circ[\boldsymbol{n},\sigma]$ are 2-morphisms.

Define a 1-morphism $\boldsymbol{l} : \boldsymbol{T} \times K \to \boldsymbol{V}$ in **dSpa** by $\boldsymbol{l}|_{\boldsymbol{T}\times\{\kappa\}} = \boldsymbol{r}(\kappa) \circ \boldsymbol{m} : \boldsymbol{T} \times \{\kappa\} \cong \boldsymbol{T} \to \boldsymbol{V}$ for $\kappa \in K$. Then we can form the d-space fibre product $(\boldsymbol{T} \times K) \times_{\boldsymbol{l},\boldsymbol{V},\boldsymbol{n}} \boldsymbol{U}$ by the explicit construction of §2.5. Define actions $\boldsymbol{p}' : G \times H \to \mathrm{Aut}(\boldsymbol{T} \times K)$, $\boldsymbol{q}' : G \times H \to \mathrm{Aut}(\boldsymbol{U})$, $\boldsymbol{r}' : G \times H \to \mathrm{Aut}(\boldsymbol{V})$ of $G \times H$ on $\boldsymbol{T} \times K, \boldsymbol{U}, \boldsymbol{V}$ by

$$
\begin{aligned}
\boldsymbol{p}'(\gamma,\delta)|_{\boldsymbol{T}\times\{\kappa\}} &= \boldsymbol{p}(\gamma) \times (\kappa \mapsto \sigma(\delta) \circ \kappa \circ \rho(\gamma)^{-1}) : \\
&\qquad \boldsymbol{T} \times \{\kappa\} \longrightarrow \boldsymbol{T} \times \{\sigma(\delta) \circ \kappa \circ \rho(\gamma)^{-1}\}, \\
\boldsymbol{q}'(\gamma,\delta) &= \boldsymbol{q}(\delta) : \boldsymbol{U} \to \boldsymbol{U} \quad \text{and} \quad \boldsymbol{r}'(\gamma,\delta) = \boldsymbol{r}(\sigma(\delta)) : \boldsymbol{V} \to \boldsymbol{V},
\end{aligned}
\tag{10.8}
$$

for $\gamma \in G$, $\delta \in H$ and $\kappa \in K$. Then we have

$$
\begin{aligned}
\boldsymbol{l} \circ \boldsymbol{p}'(\gamma,\delta)|_{\boldsymbol{T}\times\{\kappa\}} &= \boldsymbol{l}|_{\boldsymbol{T}\times\{\sigma(\delta)\circ\kappa\circ\rho(\gamma)^{-1}\}} \boldsymbol{p}(\gamma) \times (\kappa \mapsto \sigma(\delta) \circ \kappa \circ \rho(\gamma)^{-1}) \\
&= \boldsymbol{r}\big(\sigma(\delta) \circ \kappa \circ \rho(\gamma)^{-1}\big) \circ \boldsymbol{m} \circ \boldsymbol{p}(\gamma) = \boldsymbol{r}\big(\sigma(\delta) \circ \kappa \circ \rho(\gamma)^{-1}\big) \circ \boldsymbol{r}(\rho(\gamma)) \circ \boldsymbol{m} \\
&= \boldsymbol{r}(\sigma(\delta)) \circ \boldsymbol{r}(\kappa) \circ \boldsymbol{m} = \boldsymbol{r}'(\gamma,\delta) \circ \boldsymbol{l}|_{\boldsymbol{T}\times\{\kappa\}},
\end{aligned}
$$

so $\boldsymbol{l} \circ \boldsymbol{p}'(\gamma,\delta) = \boldsymbol{r}'(\gamma,\delta) \circ \boldsymbol{l}$ for all $(\gamma,\delta) \in G \times H$, and $\boldsymbol{n} \circ \boldsymbol{q}'(\gamma,\delta) = \boldsymbol{r}'(\gamma,\delta) \circ \boldsymbol{n}$ follows from $\boldsymbol{n} \circ \boldsymbol{q}(\gamma) = \boldsymbol{r}(\sigma(\gamma)) \circ \boldsymbol{n}$.

Thus $\boldsymbol{l}, \boldsymbol{n}$ are $G \times H$ invariant. As the construction of §2.5 is natural up to 1-isomorphism, the actions of $G \times H$ on $\boldsymbol{T} \times K, \boldsymbol{U}, \boldsymbol{V}$ therefore induce an action of $G \times H$ on $(\boldsymbol{T} \times K) \times_{\boldsymbol{l},\boldsymbol{V},\boldsymbol{n}} \boldsymbol{U}$ by 1-isomorphisms, so we can form the quotient d-stack $\big[(\boldsymbol{T} \times K) \times_{\boldsymbol{l},\boldsymbol{V},\boldsymbol{n}} \boldsymbol{U}/G \times H\big]$. By comparing the definition of quotient d-stacks in §9.3 with the explicit constructions of fibre products of d-spaces and d-stacks in §2.5 and §9.5, one can show there is a natural equivalence

$$
[\boldsymbol{T}/G] \times_{[\boldsymbol{m},\rho],[\boldsymbol{V}/K],[\boldsymbol{n},\sigma]} [\boldsymbol{U}/H] \simeq \big[(\boldsymbol{T} \times K) \times_{\boldsymbol{l},\boldsymbol{V},\boldsymbol{n}} \boldsymbol{U}/G \times H\big]. \tag{10.9}
$$

Since $\boldsymbol{\mathcal{T}} \subseteq \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{Z}}$ are d-orbifolds with $[\boldsymbol{T}/G] \simeq \boldsymbol{\mathcal{T}}$, $[\boldsymbol{U}/H] \simeq \boldsymbol{\mathcal{U}}$, $[\boldsymbol{V}/K] \simeq \boldsymbol{\mathcal{V}}$, Proposition 10.15 shows that $\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{V}$ are d-manifolds with virtual dimensions $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}$, $\mathrm{vdim}\,\boldsymbol{\mathcal{Y}}$, $\mathrm{vdim}\,\boldsymbol{\mathcal{Z}}$. As $\boldsymbol{g}, \boldsymbol{h}$ are d-transverse in **dOrb**, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are equivalences, and $\eta : \boldsymbol{g} \circ \boldsymbol{i} \Rightarrow \boldsymbol{k} \circ [\boldsymbol{m},\rho]$ and $\zeta : \boldsymbol{h} \circ \boldsymbol{j} \Rightarrow \boldsymbol{k} \circ [\boldsymbol{n},\sigma]$ are 2-morphisms, the d-orbifold analogue of Proposition 4.17 shows that $[\boldsymbol{m},\rho]$, $[\boldsymbol{n},\sigma]$ are d-transverse in **dOrb**, and hence $\boldsymbol{m}, \boldsymbol{n}$ are d-transverse in **dMan**, and also $\boldsymbol{l}, \boldsymbol{n}$ are d-transverse in **dMan**.

Theorem 4.21 now proves that $(\boldsymbol{T} \times K) \times_{\boldsymbol{l},\boldsymbol{V},\boldsymbol{n}} \boldsymbol{U}$ is a d-manifold, of virtual dimension $\mathrm{vdim}\,\boldsymbol{\mathcal{X}} + \mathrm{vdim}\,\boldsymbol{\mathcal{Y}} - \mathrm{vdim}\,\boldsymbol{\mathcal{Z}}$. So Proposition 10.15 shows that $\big[(\boldsymbol{T} \times K) \times_{\boldsymbol{l},\boldsymbol{V},\boldsymbol{n}} \boldsymbol{U}/G \times H\big]$ is a d-orbifold of the same dimension. Thus (10.7) and (10.9) imply that $\boldsymbol{\mathcal{W}}'$ is a d-orbifold of virtual dimension $\mathrm{vdim}\,\boldsymbol{\mathcal{X}} + \mathrm{vdim}\,\boldsymbol{\mathcal{Y}} - \mathrm{vdim}\,\boldsymbol{\mathcal{Z}}$. As we can cover $\boldsymbol{\mathcal{W}}$ by such open $\boldsymbol{\mathcal{W}}'$, the theorem follows. $\square$

Here are the analogues of Theorems 4.22 and 4.23, with similar proofs.

Theorem 10.28. *Suppose* $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$, $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ *are* 1*-morphisms of d-orbifolds. The following are sufficient conditions for* $\boldsymbol{g}, \boldsymbol{h}$ *to be d-transverse, so that* $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ *is a d-orbifold of virtual dimension* (10.6):

(a) $\boldsymbol{\mathcal{Z}}$ *is an orbifold, that is,* $\boldsymbol{\mathcal{Z}} \in \hat{\mathbf{O}}\mathbf{rb}$; *or*

(b) *$\boldsymbol{g}$ or $\boldsymbol{h}$ is a w-submersion.*

Theorem 10.29. *Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Z}}$ be d-orbifolds, $\boldsymbol{\mathcal{Y}}$ an orbifold, and $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$, $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms with $\boldsymbol{g}$ a submersion. Then $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ is an orbifold, with $\dim \boldsymbol{\mathcal{W}} = \operatorname{vdim} \boldsymbol{\mathcal{X}} + \dim \boldsymbol{\mathcal{Y}} - \operatorname{vdim} \boldsymbol{\mathcal{Z}}$.*

Next we give d-orbifold analogues of Propositions 4.26 and 4.27, proved in a similar way. In contrast to Proposition 4.27, it is not true that if $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is an immersion of d-orbifolds (or even of orbifolds) then locally $\boldsymbol{\mathcal{X}} \simeq \boldsymbol{\mathcal{Y}} \times_{\boldsymbol{\mathbb{R}^n}} *$; the best we can do is that $\boldsymbol{\mathcal{X}} \simeq \boldsymbol{\mathcal{Y}} \times_{[\boldsymbol{\mathbb{R}^n}/H]} [*/G]$ for finite groups G, H.

Proposition 10.30. *Let $\rho : G \to H$ be a morphism of finite groups, and H act linearly on $\mathbb{R}^n$. Then as in §9.3 we have quotient d-orbifolds $[*/G]$, $[\boldsymbol{\mathbb{R}^n}/H]$ and a quotient 1-morphism $[\boldsymbol{0}, \rho] : [*/G] \to [\boldsymbol{\mathbb{R}^n}/H]$. Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold and $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to [\boldsymbol{\mathbb{R}^n}/H]$ a 1-morphism in* **dOrb***. Then the fibre product $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},[\boldsymbol{\mathbb{R}^n}/H],[\boldsymbol{0},\rho]} [*/G]$ exists in* **dOrb** *by Theorem 10.28(a). The projection $\boldsymbol{\pi_{\mathcal{X}}} : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{X}}$ is an immersion if ρ is injective, and an embedding if ρ is an isomorphism.*

Proposition 10.31. *Suppose $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is an immersion of d-orbifolds, and $[x] \in \mathcal{X}_{\rm top}$ with $f_{\rm top}([x]) = [y] \in \mathcal{Y}_{\rm top}$. Write $\rho : G \to H$ for $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$. Then ρ is injective, and there exist open neighbourhoods $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{Y}}$ of $[x], [y]$ with $\boldsymbol{f}(\boldsymbol{\mathcal{U}}) \subseteq \boldsymbol{\mathcal{V}}$, a linear action of H on $\mathbb{R}^n$ where $n = \operatorname{vdim} \boldsymbol{\mathcal{Y}} - \operatorname{vdim} \boldsymbol{\mathcal{X}} \geqslant 0$, and a 1-morphism $\boldsymbol{g} : \boldsymbol{\mathcal{V}} \to [\boldsymbol{\mathbb{R}^n}/H]$ with $g_{\rm top}([y]) = [0]$, fitting into a 2-Cartesian square in* **dOrb** *:*

$$\begin{tikzcd} \boldsymbol{\mathcal{U}} \arrow[rr] \arrow[d, "\boldsymbol{f}|_{\boldsymbol{\mathcal{U}}}"] & \arrow[d, phantom, "\Uparrow"] & {[*/G]} \arrow[d, "{[\boldsymbol{0},\rho]}"] \\ \boldsymbol{\mathcal{V}} \arrow[rr, "\boldsymbol{g}"] & & {[\boldsymbol{\mathbb{R}^n}/H].} \end{tikzcd}$$

If $\boldsymbol{f}$ is an embedding then ρ is an isomorphism, and we may take $\boldsymbol{\mathcal{U}} = \boldsymbol{f}^{-1}(\boldsymbol{\mathcal{V}})$.

10.5 Embedding d-orbifolds into orbifolds

Section 4.4 proved Theorems 4.29, 4.32 and 4.33 giving necessary and sufficient conditions for the existence of embeddings $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{\mathbb{R}^n}$ for any d-manifold $\boldsymbol{X}$, and Theorem 4.34 showing that if a d-manifold $\boldsymbol{X}$ has an embedding $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ for a manifold Y then $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ for open $f(X) \subset V \subseteq Y$. Combining these theorems in Corollaries 4.35 and 4.36 showed that large classes of d-manifolds — all compact d-manifolds, for instance — are principal d-manifolds.

The proof of Theorem 4.34 extends to (d-)orbifolds, giving:

Theorem 10.32. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold, $\mathcal{Y}$ an orbifold, and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ an embedding, in the sense of Definition 10.22. Then there exist an open suborbifold $\mathcal{V} \subseteq \mathcal{Y}$ with $\boldsymbol{f}(\boldsymbol{\mathcal{X}}) \subseteq \boldsymbol{\mathcal{V}}$, a vector bundle $\mathcal{E}$ on $\mathcal{V}$, and a smooth section $s \in C^\infty(\mathcal{E})$*

fitting into a 2*-Cartesian diagram in* $\mathbf{dOrb}$, *where* $\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{E}},\boldsymbol{s},\mathbf{0}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}\bigl(\mathcal{Y},\mathcal{V},\mathrm{Tot}(\mathcal{E}),\mathrm{Tot}(s),\mathrm{Tot}(0)\bigr)$, *in the notation of Definition* 8.4:

$$\begin{tikzcd} \boldsymbol{\mathcal{X}} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{f}"] \arrow[dr, phantom, "\Uparrow"] & \boldsymbol{\mathcal{V}} \arrow[d, "\mathbf{0}"] \\ \boldsymbol{\mathcal{V}} \arrow[r, "\boldsymbol{s}"] & \boldsymbol{\mathcal{E}}. \end{tikzcd}$$

Hence $\boldsymbol{\mathcal{X}}$ *is equivalent to the 'standard model' d-orbifold* $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ *of Definition* 10.5, *and is a principal d-orbifold.*

However, Theorems 4.29, 4.32 and 4.33 do not seem to extend nicely to d-orbifolds. If $\mathcal{X}$ is an orbifold and $[x]\in\mathcal{X}_{\rm top}$ such that $\mathrm{Iso}_{\mathcal{X}}([x])$ acts nontrivially on $T_x\mathcal{X}$, then it is easy to see that for any 1-morphism $f:\mathcal{X}\to\mathbb{R}^n$ the linear map $\mathrm{d}f|_x:T_x\mathcal{X}\to\mathbb{R}^n$ is not injective, as its kernel contains the nontrivial part of the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on $T_x\mathcal{X}$. Thus f is not an immersion or an embedding. So general d-orbifolds $\boldsymbol{\mathcal{X}}$ do not admit embeddings $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\mathbb{R}^n$.

Another natural thing to try is to look for embeddings $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to[\mathbb{R}^n/G]$ for G a finite group acting linearly on $\mathbb{R}^n$. However, this also does not work. There exist representable 1-morphisms $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to[\mathbb{R}^n/G]$ if and only if $\boldsymbol{\mathcal{X}}\simeq[\boldsymbol{X}/G]$ for some d-manifold $\boldsymbol{X}$. But most orbifolds and d-orbifolds cannot be written as global quotients. For example, the weighted projective space $\mathbb{CP}^1_{1,k}$ for $k>1$ is a 2-orbifold homeomorphic to $\mathcal{S}^2$, with one orbifold point at $[0,1]$ with orbifold group $\mathbb{Z}_k$. As $\mathbb{CP}^1_{1,k}\setminus\{[0,1]\}$ is a simply-connected manifold, one can show $\mathbb{CP}^1_{1,k}\not\simeq[\underline{V}/G]$ for any manifold V and finite group G.

We can ask:

Question 10.33. *Let* $\boldsymbol{\mathcal{X}}$ *be a d-orbifold, and suppose* $\dim T^*_x\mathcal{X}$ *is bounded above for* $[x]\in\mathcal{X}_{\rm top}$. *Does there exist an embedding* $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$, *for* $\boldsymbol{\mathcal{Y}}$ *an orbifold?*

If the answer is yes then Corollaries 4.35 and 4.36 also hold for d-orbifolds. Here is one possibly useful criterion for the existence of embeddings.

Proposition 10.34. *Suppose* $\boldsymbol{\mathcal{X}}$ *is a compact d-orbifold,* $\mathcal{Y}$ *an effective orbifold, and* $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ *a* 1*-morphism with* $f:\mathcal{X}\to\mathcal{Y}$ *representable, where* $\boldsymbol{\mathcal{Y}}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$. *Then there exist a vector bundle* $\mathcal{E}$ *on* $\mathcal{Y}$ *and an embedding* $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{E}}$ *with* $\boldsymbol{\pi}\circ\boldsymbol{g}\cong\boldsymbol{f}$, *where* $\boldsymbol{\mathcal{E}},\boldsymbol{\pi}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}\bigl(\mathrm{Tot}(\mathcal{E}),\pi\bigr)$, *and* $\pi:\mathrm{Tot}(\mathcal{E})\to\mathcal{Y}$ *is as in Definition* 8.4. *Hence* $\boldsymbol{\mathcal{X}}$ *is a principal d-orbifold by Theorem* 10.32.

To prove Proposition 10.34, we set $\mathcal{E}=\bigoplus_{k=0}^N\bigl(\mathbb{R}^{n_k}\otimes\bigotimes^kT^*\mathcal{Y}\bigr)$ for $N,n_0,\ldots,n_N\gg 0$, and build $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{E}}$ from a generic smooth section of $f^*(\mathcal{E})$ on $\mathcal{X}$. If $[x]\in\mathcal{X}_{\rm top}$ with $f_{\rm top}([x])=[y]$ then $f_*:\mathrm{Iso}_{\mathcal{X}}([x])\to\mathrm{Iso}_{\mathcal{Y}}([y])$ is injective as f is representable, and the representation of $\mathrm{Iso}_{\mathcal{Y}}([y])$ on $T^*\mathcal{Y}$ is effective as $\mathcal{Y}$ is effective, so the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on $g^*(\mathcal{E})|_x$ contains all irreducible representations of $\mathrm{Iso}_{\mathcal{X}}([x])$ for $N\gg 0$, and so contains the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on $T_x\mathcal{X}$ for $n_0,\ldots,n_N\gg 0$. Using this, one can show in a similar way to the proof of Theorem 4.29 that for large enough $N,n_0,\ldots,n_N$, a generic section of $f^*(\mathcal{E})$ yields an embedding $\boldsymbol{g}$.

10.6 Orientations on d-orbifolds

Next we generalize the material of §4.5 to d-orbifolds.

Definition 10.35. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Then as in §10.1.1 we have the 2-category $\mathrm{vqcoh}(\mathcal{X})$ of virtual quasicoherent sheaves on $\mathcal{X}$, and the 2-subcategory $\mathrm{vvect}(\mathcal{X})$ of virtual vector bundles on $\mathcal{X}$. Let $\phi:\mathcal{E}^1\to\mathcal{E}^2$ be a virtual vector bundle in $\mathrm{vqcoh}(\mathcal{X})$. We will define a line bundle $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ on $\mathcal{X}$ we call the *orientation line bundle* of $(\mathcal{E}^\bullet,\phi)$, generalizing orientation line bundles for C^∞-schemes in §4.5.

Let $\mathcal{C}_{\mathcal{X}}$ be the category of Definition C.30, with objects pairs $(\underline{U},u)$ for $\underline{U}$ a C^∞-scheme and $u:\bar{\underline{U}}\to\mathcal{X}$ an étale 1-morphism, and morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ for $\underline{f}:\underline{U}\to\underline{V}$ is an étale morphism of C^∞-schemes, and $\eta:u\Rightarrow v\circ\bar{\underline{f}}$. Then since $\phi:\mathcal{E}^1\to\mathcal{E}^2$ is a morphism in $\mathrm{qcoh}(\mathcal{X})$, for each object $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$ we have a morphism $\phi(\underline{U},u):\mathcal{E}^1(\underline{U},u)\to\mathcal{E}^2(\underline{U},u)$ in $\mathrm{qcoh}(\underline{U})$. As $\phi:\mathcal{E}^1\to\mathcal{E}^2$ is a virtual vector bundle on $\mathcal{X}$, $\phi(\underline{U},u):\mathcal{E}^1(\underline{U},u)\to\mathcal{E}^2(\underline{U},u)$ is a virtual vector bundle on the C^∞-scheme $\underline{U}$. Hence §4.5 defines the orientation line bundle $\mathcal{L}_{(\mathcal{E}^\bullet(\underline{U},u),\phi(\underline{U},u))}$ on $\underline{U}$. Define $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}(\underline{U},u)=\mathcal{L}_{(\mathcal{E}^\bullet(\underline{U},u),\phi(\underline{U},u))}$.

Next let $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ be a morphism in $\mathcal{C}_{\mathcal{X}}$. Then by definition of $\mathrm{qcoh}(\mathcal{X})$ we get a commutative diagram in $\mathrm{qcoh}(\underline{U})$, with columns isomorphisms:

$$\begin{tikzcd} \underline{f}^*\bigl(\mathcal{E}^1(\underline{V},v)\bigr) \arrow[r, "\underline{f}^*(\phi^1(\underline{V},v))"'] \arrow[d, "\mathcal{E}^1_{(\underline{f},\eta)}"] & \underline{f}^*\bigl(\mathcal{E}^2(\underline{V},v)\bigr) \arrow[d, "\mathcal{E}^2_{(\underline{f},\eta)}"] \\ \mathcal{E}^1(\underline{U},u) \arrow[r, "\phi(\underline{U},u)"] & \mathcal{E}^2(\underline{U},u). \end{tikzcd}$$

Thus $\mathcal{E}^\bullet_{(\underline{f},\eta)}:\underline{f}^*(\mathcal{E}^\bullet(\underline{V},v),\phi(\underline{V},v))\to(\mathcal{E}^\bullet(\underline{U},u),\phi(\underline{U},u))$ is a 1-isomorphism in $\mathrm{vvect}(\underline{U})$, and hence an equivalence. Definitions 4.41 and 4.42 now give canonical isomorphisms

$$\begin{aligned} I_{\underline{f},(\mathcal{E}^\bullet(\underline{V},v),\phi(\underline{V},v))}&:\underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}(\underline{V},v))=\underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet(\underline{V},v),\phi(\underline{V},v))})\to\mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet(\underline{V},v),\phi(\underline{V},v))},\\ \mathcal{L}_{\mathcal{E}^\bullet_{(\underline{f},\eta)}}&:\mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet(\underline{V},v),\phi(\underline{V},v))}\longrightarrow\mathcal{L}_{(\mathcal{E}^\bullet(\underline{U},u),\phi(\underline{U},u))}=\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}(\underline{U},u).\end{aligned}$$

Define $(\mathcal{L}_{(\mathcal{E}^\bullet,\phi)})_{(\underline{f},\eta)}=\mathcal{L}_{\mathcal{E}^\bullet_{(\underline{f},\eta)}}\circ I_{\underline{f},(\mathcal{E}^\bullet(\underline{V},v),\phi(\underline{V},v))}$. It is now easy to check that this data $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}(\underline{U},u),(\mathcal{L}_{(\mathcal{E}^\bullet,\phi)})_{(\underline{f},\eta)}$ defines a quasicoherent sheaf $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ on $\mathcal{X}$ in the sense of §C.6, which is a line bundle on $\mathcal{X}$ as each $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}(\underline{U},u)$ is a line bundle. We call $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ the *orientation line bundle* of $\phi:\mathcal{E}^1\to\mathcal{E}^2$.

Definitions 4.41 and 4.42 and Propositions 4.40 and 4.43 also generalize immediately to Deligne–Mumford C^∞-stacks. We leave the details as an exercise. Here is the analogue of Definition 4.44.

Definition 10.36. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. Then the virtual cotangent bundle $T^*\boldsymbol{\mathcal{X}}=(\mathcal{E}_{\mathcal{X}},\mathcal{F}_{\mathcal{X}},\phi_{\mathcal{X}})$ is a virtual vector bundle on $\mathcal{X}$, so Definition 10.35 constructs a line bundle $\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}$ on $\mathcal{X}$. We call $\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}$ the *orientation line bundle* of $\boldsymbol{\mathcal{X}}$. An *orientation* ω on $\boldsymbol{\mathcal{X}}$ is an orientation on $\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}$. That is, ω is an equivalence class $[\tau]$ of isomorphisms $\tau:\mathcal{O}_{\mathcal{X}}\to\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}$, where τ,τ' are equivalent if they are proportional by a positive function on $\mathcal{X}$. We call $\boldsymbol{\mathcal{X}}$ *orientable* if it admits an orientation. An *oriented d-orbifold* is a pair $(\boldsymbol{\mathcal{X}}, \omega)$ where $\boldsymbol{\mathcal{X}}$ is a d-orbifold and ω an orientation on $\boldsymbol{\mathcal{X}}$. But we will often refer to $\boldsymbol{\mathcal{X}}$ as an oriented d-orbifold, leaving the orientation ω implicit. If $\omega = [\tau]$ is an orientation on $\boldsymbol{\mathcal{X}}$, the *opposite orientation* is $-\omega = [-\tau]$. When we refer to $\boldsymbol{\mathcal{X}}$ as an oriented d-orbifold, $-\boldsymbol{\mathcal{X}}$ will mean $\boldsymbol{\mathcal{X}}$ with the opposite orientation.

If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack, then line bundles $\mathcal{L}$ on $\mathcal{X}$ are étale locally trivial, but need not be Zariski locally trivial. If $[x] \in \mathcal{X}_{\rm top}$ with $\mathrm{Iso}_{\mathcal{X}}([x]) = G$, then line bundles $\mathcal{L}$ on $\mathcal{X}$ near $[x]$ are classified up to isomorphism by representations of G on $\mathbb{R}$, that is, by group morphisms $\rho : G \to \{\pm 1\}$. If $\rho \not\equiv 1$ then $\mathcal{L}$ is not trivializable, even locally. This means that orbifolds and d-orbifolds need not be orientable even locally near one point. For example, the orbifold $[\underline{\mathbb{R}}^n/\{\pm 1\}]$ is not orientable near $[0]$ for n odd.

All of the results and examples on orientations of d-manifolds in §4.6 now extend to d-orbifolds in the obvious way. Proposition 4.47(d) must be modified to say that if $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ are étale 1-morphisms of d-orbifolds and $\boldsymbol{\eta} = (\eta, \eta') : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism then $\mathcal{L}_{\boldsymbol{f}} = \mathcal{L}_{\boldsymbol{g}} \circ \eta^*(\mathcal{L}_{\boldsymbol{\mathcal{Y}}}) : f^*(\mathcal{L}_{\boldsymbol{\mathcal{Y}}}) \to \mathcal{L}_{\boldsymbol{\mathcal{X}}}$. Here is the analogue of Theorem 4.50:

Theorem 10.37. *Work in the situation of Theorem* 10.27, *so that* $\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ *are d-orbifolds with* $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g}, \boldsymbol{\mathcal{Z}}, \boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ *for* $\boldsymbol{g}, \boldsymbol{h}$ *d-transverse, where* $\boldsymbol{e} : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{X}}$, $\boldsymbol{f} : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{Y}}$ *are the projections. Then we have orientation line bundles* $\mathcal{L}_{T^*\boldsymbol{\mathcal{W}}}, \ldots, \mathcal{L}_{T^*\boldsymbol{\mathcal{Z}}}$ *on* $\mathcal{W}, \ldots, \mathcal{Z}$, *so* $\mathcal{L}_{T^*\boldsymbol{\mathcal{W}}}, e^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}), f^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Y}}}), (g \circ e)^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Z}}})$ *are line bundles on* $\mathcal{W}$. *With a suitable choice of orientation convention, there is a canonical isomorphism*

$$\Phi : \mathcal{L}_{T^*\boldsymbol{\mathcal{W}}} \longrightarrow e^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}) \otimes_{\mathcal{O}_{\mathcal{W}}} f^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Y}}}) \otimes_{\mathcal{O}_{\mathcal{W}}} (g \circ e)^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Z}}})^*. \tag{10.10}$$

Hence, if $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ *are oriented d-orbifolds, then* $\boldsymbol{\mathcal{W}}$ *has a natural orientation.*

Proof. Use the notation of the proof of Theorem 10.27, so that $\boldsymbol{\mathcal{W}}' \subseteq \boldsymbol{\mathcal{W}}$, $\boldsymbol{\mathcal{T}} \subseteq \boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{Y}}$, $\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{Z}}$ are open with equivalences $\boldsymbol{\mathcal{T}} \simeq [\boldsymbol{T}/G]$, $\boldsymbol{\mathcal{U}} \simeq [\boldsymbol{U}/H]$, $\boldsymbol{\mathcal{V}} \simeq [\boldsymbol{V}/K]$, and by (10.7)–(10.9) we have

$$\boldsymbol{\mathcal{W}}' \simeq \boldsymbol{\mathcal{T}} \times_{\boldsymbol{g}|_{\boldsymbol{\mathcal{T}}}, \boldsymbol{\mathcal{V}}, \boldsymbol{h}|_{\boldsymbol{\mathcal{U}}}} \boldsymbol{\mathcal{U}} \simeq \big[\big((\boldsymbol{T} \times K) \times_{\boldsymbol{l}, \boldsymbol{V}, \boldsymbol{n}} \boldsymbol{U}\big)/G \times H\big]. \tag{10.11}$$

The equivalence $\boldsymbol{\mathcal{T}} \simeq [\boldsymbol{T}/G]$ induces an equivalence of categories $F_\Pi : \mathrm{qcoh}(\mathcal{T}) \to \mathrm{qcoh}^G(\underline{T})$, as in Definition C.34 and Theorem C.35, and up to canonical isomorphism F_Π maps $\mathcal{L}_{\boldsymbol{\mathcal{X}}}|_{\mathcal{T}} = \mathcal{L}_{\boldsymbol{\mathcal{T}}} \mapsto (\mathcal{L}_{\boldsymbol{T}}, \rho)$, where $\mathcal{L}_{\boldsymbol{T}} \in \mathrm{qcoh}(\underline{T})$ and ρ is the natural lift of the G-action on $\underline{T}$ to $\mathcal{L}_{\boldsymbol{T}}$. The same applies for $\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{W}}'$. So by (10.11) we have

$$\begin{aligned} F_\Pi[\mathcal{L}_{\boldsymbol{\mathcal{W}}}|_{\mathcal{W}'}] &= F_\Pi[\mathcal{L}_{\boldsymbol{\mathcal{W}}'}] \cong \big(\mathcal{L}_{(\boldsymbol{T} \times K) \times_{\boldsymbol{l}, \boldsymbol{V}, \boldsymbol{n}} \boldsymbol{U}}, \sigma\big) \\ &\cong \big(\underline{e}^*(\mathcal{L}_{T^*(\boldsymbol{T} \times K)}) \otimes \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{U}}) \otimes (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\boldsymbol{V}})^*, \sigma\big) \\ &\cong F_\Pi\big[e^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}})|_{\mathcal{W}'}\big] \otimes F_\Pi\big[f^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Y}}})|_{\mathcal{W}'}\big] \otimes F_\Pi\big[(g \circ e)^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Z}}})^*|_{\mathcal{W}'}\big] \\ &\cong F_\Pi\big[\big(e^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}) \otimes_{\mathcal{O}_{\mathcal{W}}} f^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Y}}}) \otimes_{\mathcal{O}_{\mathcal{W}}} (g \circ e)^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Z}}})^*\big)|_{\mathcal{W}'}\big], \end{aligned} \tag{10.12}$$

where in the second line we use Theorem 4.50 for the d-manifold fibre product $(\boldsymbol{T}\times K)\times_{\boldsymbol{l},\boldsymbol{V},\boldsymbol{n}}\boldsymbol{U}$. As F_Π is an equivalence of categories, (10.12) implies that there is a canonical isomorphism

$$\Phi_{\mathcal{W}'}:\mathcal{L}_{T^*\boldsymbol{\mathcal{W}}}|_{\mathcal{W}'}\longrightarrow\bigl(e^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}})\otimes_{\mathcal{O}_{\mathcal{W}}}f^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Y}}})\otimes_{\mathcal{O}_{\mathcal{W}}}(g\circ e)^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Z}}})^*\bigr)|_{\mathcal{W}'}.$$

We can cover $\mathcal{W}$ by such open $\mathcal{W}'\subseteq\mathcal{W}$. As the isomorphisms $\Phi_{\mathcal{W}'}$ are canonical, they agree on overlaps. Hence they glue to define a unique isomorphism Φ in (10.10) with $\Phi|_{\mathcal{W}'}=\Phi_{\mathcal{W}'}$ for each $\mathcal{W}'\subseteq\mathcal{W}$ as above. $\square$

The analogue of Proposition 4.52 holds for orientations of d-orbifolds, with the same proof.

10.7 Orbifold strata of d-orbifolds

Section C.8 defined orbifold strata $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\ldots,\hat{\mathcal{X}}^\Gamma_\circ$ of a C^∞-stack $\mathcal{X}$. When $\mathcal{X}$ is an orbifold, §8.4.1 showed that $\mathcal{X}^\Gamma$ decomposes as $\mathcal{X}^\Gamma=\coprod_{\lambda\in\Lambda^\Gamma_+}\mathcal{X}^{\Gamma,\lambda}$, where each $\mathcal{X}^{\Gamma,\lambda}$ is an orbifold of dimension $\dim\mathcal{X}-\dim\lambda$, and similarly for $\tilde{\mathcal{X}}^\Gamma,\ldots,\hat{\mathcal{X}}^\Gamma_\circ$. Section 9.6 defined the orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma,\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\ldots,\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ of a d-stack $\boldsymbol{\mathcal{X}}$. We will now show that when $\boldsymbol{\mathcal{X}}$ is a d-orbifold, these decompose naturally as $\boldsymbol{\mathcal{X}}^\Gamma=\coprod_{\lambda\in\Lambda^\Gamma}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$, where $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ is a d-orbifold of virtual dimension $\operatorname{vdim}\boldsymbol{\mathcal{X}}-\dim\lambda$, and similarly for $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\ldots,\hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$. Here is the analogue of Definition 8.5.

Definition 10.38. Let Γ be a finite group, and use the notation $\operatorname{Rep}_{\rm nt}(\Gamma)$, $\Lambda^\Gamma=K_0\bigl(\operatorname{Rep}_{\rm nt}(\Gamma)\bigr)$, $\Lambda^\Gamma_+\subseteq\Lambda^\Gamma$ and $\dim:\Lambda^\Gamma\to\mathbb{Z}$ of Definition 8.5. Let $R_0,R_1,\ldots,R_k$ be representatives for the isomorphism classes of irreducible Γ-representations, with $R_0=\mathbb{R}$ the trivial irreducible representation, so that $R_1,\ldots,R_k$ are nontrivial. Then Λ^Γ is freely generated over $\mathbb{Z}$ by $[R_1],\ldots,[R_k]$, so that (8.1) gives isomorphisms $\Lambda^\Gamma\cong\mathbb{Z}^k$, $\Lambda^\Gamma_+\cong\mathbb{N}^k$.

Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. As $\boldsymbol{\mathcal{X}}$ is a d-stack, Definitions 9.24 and 9.25 define a d-stack $\boldsymbol{\mathcal{X}}^\Gamma$ and a 1-morphism $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}):\boldsymbol{\mathcal{X}}^\Gamma\to\boldsymbol{\mathcal{X}}$. The virtual cotangent bundle of $\boldsymbol{\mathcal{X}}$ is $T^*\boldsymbol{\mathcal{X}}=(\mathcal{E}_{\mathcal{X}},\mathcal{F}_{\mathcal{X}},\phi_{\mathcal{X}})$, a virtual vector bundle of rank $\operatorname{vdim}\boldsymbol{\mathcal{X}}$ on $\mathcal{X}$. So $O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})=\bigl(O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}),O^\Gamma(\mathcal{X})^*(\mathcal{F}_{\mathcal{X}}),O^\Gamma(\mathcal{X})^*(\phi_{\mathcal{X}})\bigr)$ is a virtual vector bundle on $\mathcal{X}^\Gamma$. As in Definition C.54, $O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}),O^\Gamma(\mathcal{X})^*(\mathcal{F}_{\mathcal{X}})$ have natural Γ-representations inducing decompositions of the form (C.22)–(C.23), and $O^\Gamma(\mathcal{X})^*(\phi_{\mathcal{X}})$ is Γ-equivariant and so preserves these splittings. Hence we have decompositions in $\operatorname{vqcoh}(\mathcal{X}^\Gamma)$:

$$\begin{aligned}O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})&\cong\textstyle\bigoplus_{i=0}^k(T^*\boldsymbol{\mathcal{X}})^\Gamma_i\otimes R_i\ \text{ for }\ (T^*\boldsymbol{\mathcal{X}})^\Gamma_i\in\operatorname{vqcoh}(\mathcal{X}^\Gamma),\\ \text{and }\ O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})&=(T^*\boldsymbol{\mathcal{X}})^\Gamma_{\rm tr}\oplus(T^*\boldsymbol{\mathcal{X}})^\Gamma_{\rm nt},\ \text{ with}\\ (T^*\boldsymbol{\mathcal{X}})^\Gamma_{\rm tr}\cong(T^*\boldsymbol{\mathcal{X}})^\Gamma_0\otimes R_0\ &\text{ and }\ (T^*\boldsymbol{\mathcal{X}})^\Gamma_{\rm nt}\cong\textstyle\bigoplus_{i=1}^k(T^*\boldsymbol{\mathcal{X}})^\Gamma_i\otimes R_i.\end{aligned}\tag{10.13}$$

Also Theorem 9.29 shows that $T^*(\boldsymbol{\mathcal{X}}^\Gamma)\cong(T^*\boldsymbol{\mathcal{X}})^\Gamma_{\rm tr}$.

As $O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})$ is a virtual vector bundle, (10.13) implies the $(T^*\boldsymbol{\mathcal{X}})^\Gamma_i$ are *virtual vector bundles of mixed rank*, whose ranks may vary on different connected components of $\boldsymbol{\mathcal{X}}^\Gamma$. For each $\lambda\in\Lambda^\Gamma$, define $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ to be the open and closed d-substack in $\boldsymbol{\mathcal{X}}^\Gamma$ with $\mathrm{rank}\bigl((T^*\boldsymbol{\mathcal{X}})_1^\Gamma\bigr)[R_1]+\cdots+\mathrm{rank}\bigl((T^*\boldsymbol{\mathcal{X}})_k^\Gamma\bigr)[R_k]=\lambda$ in Λ^Γ. Then $(T^*\mathcal{X})_{\mathrm{nt}}^\Gamma|_{\mathcal{X}^{\Gamma,\lambda}}$ is a virtual vector bundle of rank $\dim\lambda$, so $T^*(\boldsymbol{\mathcal{X}}^{\Gamma,\lambda})\cong(T^*\mathcal{X})_{\mathrm{tr}}^\Gamma|_{\mathcal{X}^{\Gamma,\lambda}}$ is a virtual vector bundle of rank $\dim\boldsymbol{\mathcal{X}}-\dim\lambda$ on $\mathcal{X}^{\Gamma,\lambda}$. We will show in Corollary 10.40 that $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ is a d-orbifold, with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}=\mathrm{vdim}\,\boldsymbol{\mathcal{X}}-\dim\lambda$. Note that in the d-orbifold case $\dim\lambda$ may be negative, so we can have $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}>\mathrm{vdim}\,\boldsymbol{\mathcal{X}}$. As for (8.2), we have a decomposition in **dSta**:

$$\boldsymbol{\mathcal{X}}^\Gamma=\textstyle\coprod_{\lambda\in\Lambda^\Gamma}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}. \tag{10.14}$$

Write $\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}})=\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})|_{\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}}:\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}\to\boldsymbol{\mathcal{X}}$. Then $\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}})$ is a proper w-immersion of d-orbifolds, in the sense of §10.3. We interpret $(T^*\boldsymbol{\mathcal{X}})_{\mathrm{nt}}^\Gamma|_{\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}}$ as the *virtual conormal bundle* of $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ in $\boldsymbol{\mathcal{X}}$. It carries a nontrivial (virtual) Γ-representation of class $\lambda\in\Lambda^\Gamma$, so we refer to λ as the (*virtual*) *conormal* Γ-*representation* of $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$.

Define $\boldsymbol{\mathcal{X}}_\circ^{\Gamma,\lambda}=\boldsymbol{\mathcal{X}}_\circ^\Gamma\cap\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$, and $\boldsymbol{O}_\circ^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}})=\boldsymbol{O}_\circ^\Gamma(\boldsymbol{\mathcal{X}})|_{\boldsymbol{\mathcal{X}}_\circ^{\Gamma,\lambda}}:\boldsymbol{\mathcal{X}}_\circ^{\Gamma,\lambda}\to\boldsymbol{\mathcal{X}}$. Then $\boldsymbol{\mathcal{X}}_\circ^{\Gamma,\lambda}$ is a d-orbifold with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}_\circ^{\Gamma,\lambda}=\mathrm{vdim}\,\boldsymbol{\mathcal{X}}-\dim\lambda$, and $\boldsymbol{\mathcal{X}}_\circ^\Gamma=\coprod_{\lambda\in\Lambda^\Gamma}\boldsymbol{\mathcal{X}}_\circ^{\Gamma,\lambda}$, and $\boldsymbol{O}_\circ^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}})$ is a w-immersion, but need not be proper.

As for $\tilde{\mathcal{X}}^{\Gamma,\mu}$ in Definition 8.5, for each $\mu\in\Lambda^\Gamma/\mathrm{Aut}(\Gamma)$ we may define $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}\simeq\bigl[\bigl(\coprod_{\lambda\in\mu}\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}\bigr)/\mathrm{Aut}(\Gamma)\bigr]$, an open and closed d-substack of $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma\simeq[\boldsymbol{\mathcal{X}}^\Gamma/\mathrm{Aut}(\Gamma)]$, and $\tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}=\tilde{\boldsymbol{\mathcal{X}}}_\circ^\Gamma\cap\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$. Then $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ are d-orbifolds with $\mathrm{vdim}\,\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}=\mathrm{vdim}\,\tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}=\mathrm{vdim}\,\boldsymbol{\mathcal{X}}-\dim\mu$, with

$$\tilde{\boldsymbol{\mathcal{X}}}^\Gamma=\textstyle\coprod_{\mu\in\Lambda^\Gamma/\mathrm{Aut}(\Gamma)}\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}\quad\text{and}\quad\tilde{\boldsymbol{\mathcal{X}}}_\circ^\Gamma=\textstyle\coprod_{\mu\in\Lambda^\Gamma/\mathrm{Aut}(\Gamma)}\tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}.$$

Set $\tilde{\boldsymbol{O}}^{\Gamma,\mu}(\boldsymbol{\mathcal{X}})=\tilde{\boldsymbol{O}}^\Gamma(\boldsymbol{\mathcal{X}})|_{\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}}:\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}\to\boldsymbol{\mathcal{X}}$, $\tilde{\boldsymbol{O}}_\circ^{\Gamma,\mu}(\boldsymbol{\mathcal{X}})=\tilde{\boldsymbol{O}}_\circ^\Gamma(\boldsymbol{\mathcal{X}})|_{\tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}}:\tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}\to\boldsymbol{\mathcal{X}}$. Then $\tilde{\boldsymbol{O}}^{\Gamma,\mu}(\boldsymbol{\mathcal{X}}),\tilde{\boldsymbol{O}}_\circ^{\Gamma,\mu}(\boldsymbol{\mathcal{X}})$ are w-immersions, with $\tilde{\boldsymbol{O}}^{\Gamma,\mu}(\boldsymbol{\mathcal{X}})$ proper.

The 1-morphism $\hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}}):\tilde{\boldsymbol{\mathcal{X}}}^\Gamma\to\hat{\boldsymbol{\mathcal{X}}}^\Gamma$ induces a homeomorphism of topological spaces by Theorem C.49(e), so it maps open and closed d-substacks of $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma$ to open and closed d-substacks of $\hat{\boldsymbol{\mathcal{X}}}^\Gamma$. Let $\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}=\hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}})(\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu})$ for each $\mu\in\Lambda^\Gamma/\mathrm{Aut}(\Lambda)$, and write $\hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}=\hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma\cap\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$. Then $\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ are d-orbifolds of virtual dimension $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}-\dim\mu$, with

$$\hat{\boldsymbol{\mathcal{X}}}^\Gamma=\textstyle\coprod_{\mu\in\Lambda^\Gamma/\mathrm{Aut}(\Gamma)}\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}\quad\text{and}\quad\hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma=\textstyle\coprod_{\mu\in\Lambda^\Gamma/\mathrm{Aut}(\Gamma)}\hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}.$$

Also Theorem C.49(e) and Lemma 10.4 imply that $\hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ is a d-manifold, that is, it lies in $\mathbf{d\hat{M}an}$.

Example 10.39. Suppose $\mathcal{V}$ is an orbifold, $\mathcal{E}\to\mathcal{V}$ a vector bundle, $s:\mathcal{V}\to\mathcal{E}$ a smooth section, and $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ the 'standard model' d-orbifold of Definition 10.5. Let Γ be a finite group. We will describe the orbifold strata $(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s})^{\Gamma,\lambda}$ of $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$.

As in §8.4.1 we have a decomposition $\mathcal{V}^\Gamma=\coprod_{\lambda_1\in\Lambda_+^\Gamma}\mathcal{V}^{\Gamma,\lambda_1}$. Also, as in §C.9 the vector bundle $\mathcal{E}^\Gamma=O^\Gamma(\mathcal{V})^*(\mathcal{E})$ on $\mathcal{V}^\Gamma$ has a natural Γ-representation, and has decompositions (C.22)–(C.23). Since $\mathcal{E}^\Gamma$ is a vector bundle, the $\mathcal{E}_i^\Gamma$ for $i=0,\ldots,k$ in (C.22) are vector bundles of mixed rank.

For all $\lambda_1,\lambda_2\in\Lambda_+^\Gamma$, define $\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}$ to be the open and closed suborbifold of $\mathcal{V}^{\Gamma,\lambda_1}$ such that $\mathrm{rank}(\mathcal{E}_1^\Gamma)[R_1]+\cdots+\mathrm{rank}(\mathcal{E}_k^\Gamma)[R_k]=\lambda_2$ in Λ_+^Γ. Then $\mathcal{V}^\Gamma=$

$\coprod_{\lambda_1,\lambda_2\in\Lambda_+^\Gamma}\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}$. For each λ_1,λ_2 we have

$$\begin{aligned}(T^*\mathcal{V})^\Gamma|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}}&=(T^*\mathcal{V})^\Gamma_{\rm tr}|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}}\oplus(T^*\mathcal{V})^\Gamma_{\rm nt}|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}},\\ \mathcal{E}^\Gamma|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}}&=\mathcal{E}^\Gamma_{\rm tr}|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}}\oplus\mathcal{E}^\Gamma_{\rm nt}|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}},\end{aligned}$$

where all the factors are vector bundles with

$$\begin{aligned}\mathrm{rank}(T^*\mathcal{V})^\Gamma_{\rm tr}|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}}&=\dim\mathcal{V}-\dim\lambda_1, & \mathrm{rank}(T^*\mathcal{V})^\Gamma_{\rm nt}|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}}&=\dim\lambda_1,\\ \mathrm{rank}\,\mathcal{E}^\Gamma_{\rm tr}|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}}&=\mathrm{rank}\,\mathcal{E}-\dim\lambda_2, & \mathrm{rank}\,\mathcal{E}^\Gamma_{\rm nt}|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}}&=\dim\lambda_2.\end{aligned}$$

For all $\lambda_1,\lambda_2\in\Lambda_+^\Gamma$, define a vector bundle $\mathcal{E}^{\Gamma,\lambda_1,\lambda_2}$ on $\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}$ by $\mathcal{E}^{\Gamma,\lambda_1,\lambda_2}=\mathcal{E}^\Gamma_{\rm tr}|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}}$. Now $s:\mathcal{V}\to\mathcal{E}$ is a smooth section of $\mathcal{E}\to\mathcal{V}$, so lifts to a smooth section $O^\Gamma(\mathcal{V})^*(s)$ of $\mathcal{E}^\Gamma\to\mathcal{V}^\Gamma$. It is invariant under the Γ-action on $\mathcal{E}^\Gamma$, and so is a section of $\mathcal{E}^\Gamma_{\rm tr}$. Hence $s^{\Gamma,\lambda_1,\lambda_2}:=O^\Gamma(\mathcal{V})^*(s)|_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}}$ is a smooth section of the vector bundle $\mathcal{E}^{\Gamma,\lambda_1,\lambda_2}\to\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}$, so Definition 10.5 gives a 'standard model' d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2},\mathcal{E}^{\Gamma,\lambda_1,\lambda_2},s^{\Gamma,\lambda_1,\lambda_2}}$, with

$$\begin{aligned}\mathrm{vdim}\,\boldsymbol{\mathcal{S}}_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2},\mathcal{E}^{\Gamma,\lambda_1,\lambda_2},s^{\Gamma,\lambda_1,\lambda_2}}&=\dim\mathcal{V}^{\Gamma,\lambda_1,\lambda_2}-\mathrm{rank}\,\mathcal{E}^{\Gamma,\lambda_1,\lambda_2}\\&=\dim\mathcal{V}-\mathrm{rank}\,\mathcal{E}-\lambda_1+\lambda_2.\end{aligned}\tag{10.15}$$

We now claim that for all $\lambda\in\Lambda^\Gamma$ we have an equivalence in **dSta**:

$$(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s})^{\Gamma,\lambda}\simeq\coprod_{\lambda_1,\lambda_2\in\Lambda_+^\Gamma:\lambda_1-\lambda_2=\lambda}\boldsymbol{\mathcal{S}}_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2},\mathcal{E}^{\Gamma,\lambda_1,\lambda_2},s^{\Gamma,\lambda_1,\lambda_2}}.\tag{10.16}$$

To prove this we compare the construction of $(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s})^\Gamma$ in §C.8–§9.6 with the definition of $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ in Definition 10.5. There is a natural equivalence

$$(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s})^\Gamma\simeq\boldsymbol{\mathcal{S}}_{\mathcal{V}^\Gamma,\mathcal{E}^\Gamma,O^\Gamma(\mathcal{V})^*(s)}=\coprod_{\lambda_1,\lambda_2\in\Lambda_+^\Gamma}\boldsymbol{\mathcal{S}}_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2},\mathcal{E}^{\Gamma,\lambda_1,\lambda_2},s^{\Gamma,\lambda_1,\lambda_2}},\tag{10.17}$$

where $\mathcal{V}^\Gamma$ is an orbifold of mixed dimension, and $\mathcal{E}^\Gamma$ a vector bundle of mixed rank. Writing $\boldsymbol{\mathcal{X}}=\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$, we see that $O^\Gamma(\mathcal{X})^*(\mathcal{E}_\mathcal{X})_{\rm nt},O^\Gamma(\mathcal{X})^*(\mathcal{F}_\mathcal{X})_{\rm nt}$ are vector bundles with Γ-representations of class $\lambda_1,\lambda_2\in\Lambda_+^\Gamma$ on $\mathcal{S}_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2},\mathcal{E}^{\Gamma,\lambda_1,\lambda_2},s^{\Gamma,\lambda_1,\lambda_2}}$ in $\mathcal{X}^\Gamma$, so $(T^*\boldsymbol{\mathcal{X}})^\Gamma_{\rm nt}$ is a virtual vector bundle with Γ-representation of class $\lambda_1-\lambda_2$ on $\mathcal{S}_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2},\mathcal{E}^{\Gamma,\lambda_1,\lambda_2},s^{\Gamma,\lambda_1,\lambda_2}}$, and $\boldsymbol{\mathcal{S}}_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2},\mathcal{E}^{\Gamma,\lambda_1,\lambda_2},s^{\Gamma,\lambda_1,\lambda_2}}$ in (10.17) lies in $(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s})^{\Gamma,\lambda_1-\lambda_2}\subseteq(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s})^\Gamma$. Equation (10.16) follows. Note that (10.15)–(10.16) imply that $(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s})^{\Gamma,\lambda}$ is a disjoint union of principal d-orbifolds of virtual dimension $\dim\mathcal{V}-\mathrm{rank}\,\mathcal{E}-\dim\lambda$, so is a d-orbifold of this dimension.

Corollary 10.40. *In Definition* 10.38, $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ,\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ *are d-orbifolds of virtual dimensions* $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}-\dim\lambda$, $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}-\dim\mu$.

Proof. As $\boldsymbol{\mathcal{X}}$ is a d-orbifold it can be covered by principal open d-suborbifolds $\boldsymbol{\mathcal{U}}\subseteq\boldsymbol{\mathcal{X}}$. Then $\boldsymbol{\mathcal{U}}\simeq\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ for some $\mathcal{V},\mathcal{E},s$ as $\boldsymbol{\mathcal{U}}$ is principal. Corollary 9.27 implies that $\boldsymbol{\mathcal{U}}^{\Gamma,\lambda}\simeq(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s})^{\Gamma,\lambda}$ for all Γ,λ. So Example 10.39 shows that $\boldsymbol{\mathcal{U}}^{\Gamma,\lambda}$ is a d-orbifold with $\mathrm{vdim}\boldsymbol{\mathcal{U}}^{\Gamma,\lambda}=\mathrm{vdim}\boldsymbol{\mathcal{U}}-\dim\lambda=\mathrm{vdim}\,\boldsymbol{\mathcal{X}}-\dim\lambda$. But $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ is covered by such open $\boldsymbol{\mathcal{U}}^{\Gamma,\lambda}$, so $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ is also a d-orbifold with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}=\mathrm{vdim}\,\boldsymbol{\mathcal{X}}-\dim\lambda$. Since $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ,\ldots,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ are constructed from $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ by open d-substacks and quotients, they are also d-orbifolds of the same dimension. $\square$

Section 8.4.2 discussed issues involving orbifold strata and orientations. By studying local models $\boldsymbol{S}_{\mathcal{V},\mathcal{E},s}$, the analogue of Lemma 8.6 is easy to prove:

Lemma 10.41. *A d-orbifold $\boldsymbol{\mathcal{X}}$ is locally orientable if and only if $\boldsymbol{\mathcal{X}}^{\mathbb{Z}_2,\lambda}=\emptyset$ for all odd $\lambda\in\Lambda^{\mathbb{Z}_2}\cong\mathbb{Z}$.*

As in §8.4.2, we consider the question: if $\boldsymbol{\mathcal{X}}$ is an oriented orbifold, when can we define orientations on the orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\ldots,\hat{\boldsymbol{\mathcal{X}}}{}^{\Gamma,\mu}_{\circ}$? Use the notation $\Lambda^{\Gamma}_{\rm ev},\Phi^{\Gamma}(\delta,\lambda)$ of Definition 8.8. By analogy with Proposition 8.9, we might expect that if $\boldsymbol{\mathcal{X}}$ is an oriented d-orbifold and $\lambda\in\Lambda^{\Gamma}_{\rm ev}$, then we can define orientations on the orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_{\circ}$. However, this turns out to be false (although we will show in §10.9 that it holds for $\boldsymbol{\mathcal{X}}$ semieffective).

Example 10.42. Let $V=\mathcal{S}^2\times\mathbb{R}$, and write points of V as (x_1,x_2,x_3,y) for $x_i,y\in\mathbb{R}$ with $x_1^2+x_2^2+x_3^2=1$. Let $E\to V$ be the trivial vector bundle $\mathbb{R}\times V\to V$, so that sections of E are functions $V\to\mathbb{R}$, and define $s\in C^\infty(E)$ by $s(x_1,x_2,x_3,y)=x_3y$. Let $G=\mathbb{Z}_2^2=\{1,\sigma,\tau,\sigma\tau\}$, and define actions $r,\hat r$ of G on V and $E=\mathbb{R}\times V$ by

$$\begin{aligned} r(\sigma)&:(x_1,x_2,x_3,y)\longmapsto(-x_1,-x_2,-x_3,y),\\ r(\tau)&:(x_1,x_2,x_3,y)\longmapsto(x_1,x_2,x_3,-y),\\ \hat r(\sigma)&:(e,x_1,x_2,x_3,y)\longmapsto(-e,-x_1,-x_2,-x_3,y),\\ \hat r(\tau)&:(e,x_1,x_2,x_3,y)\longmapsto(-e,x_1,x_2,x_3,-y).\end{aligned}$$

Then $\boldsymbol{S}_{V,E,s}$ is an oriented d-manifold of virtual dimension 2. The G-actions $r,\hat r$ induce a G-action on $\boldsymbol{S}_{V,E,s}$ preserving the orientation, so $\boldsymbol{\mathcal{X}}=[\boldsymbol{S}_{V,E,s}/G]$ is an oriented d-orbifold, as in Example 10.11. We find that

$$\boldsymbol{\mathcal{X}}^{\mathbb{Z}_2,0}=\boldsymbol{\mathcal{X}}^{\mathbb{Z}_2,0}_{\circ}\cong\tilde{\boldsymbol{\mathcal{X}}}{}^{\mathbb{Z}_2,0}=\tilde{\boldsymbol{\mathcal{X}}}{}^{\mathbb{Z}_2,0}_{\circ}\cong\mathbf{RP}^2\times[*/\mathbb{Z}_2],\;\;\hat{\boldsymbol{\mathcal{X}}}{}^{\mathbb{Z}_2,0}=\hat{\boldsymbol{\mathcal{X}}}{}^{\mathbb{Z}_2,0}_{\circ}\cong\mathbf{RP}^2.$$

None of these are orientable d-orbifolds, as $\mathbb{RP}^2$ is not orientable. Observe that if $\mathcal{X}$ is an oriented orbifold, then Propositions 8.9, 8.10 imply that $\mathcal{X}^{\mathbb{Z}_2,0},\ldots,\hat{\mathcal{X}}{}^{\mathbb{Z}_2,0}_{\circ}$ are all oriented, as $0\in\Lambda^{\Gamma}_{{\rm ev},+}$. Thus, this example shows that for d-orbifolds, the direct analogues of Propositions 8.9 and 8.10 fail.

The topological space $\mathcal{X}_{\rm top}$ is the union of two components, an $\mathbb{RP}^2$ of points $[\pm(x_1,x_2,x_3,0)]$ with orbifold group $\mathbb{Z}_2$, and an annulus A of points $[\pm(x_1,x_2,0,\pm y)]$ with orbifold group $\{1\}$ if $y\neq 0$, which intersect in a circle $\mathcal{S}^1$ of points $[\pm(x_1,x_2,0,0)]$. The oriented d-orbifold $\boldsymbol{\mathcal{X}}$ is an oriented orbifold except along this circle $\mathcal{S}^1$. The orientation on $\mathbb{RP}^2$ changes sign across the singular circle $\mathcal{S}^1$.

This example is also interesting in relation to (semi)effective d-orbifolds in §10.9 below. The d-orbifold $\boldsymbol{\mathcal{X}}$ is not (semi)effective. In contrast to Proposition 10.58, for all small G-equivariant perturbations $\tilde s$ of s, $\tilde{\boldsymbol{\mathcal{X}}}=[\boldsymbol{S}_{V,E,\tilde s}/G]$ is not an orbifold, but must have singularities close to $\mathcal{S}^1\subset\mathcal{X}_{\rm top}$. To prove this, note that $\tilde{\boldsymbol{\mathcal{X}}}$ would have to be an oriented orbifold, but also contain $\mathbb{RP}^2\times[\underline{*}/\mathbb{Z}_2]$.

To prove Proposition 8.9, it was enough to coherently orient all $(V,\rho)\in\mathrm{Rep}_{\rm nt}(\Gamma)$ with $[(V,\rho)]=\lambda$. But for the d-orbifold analogue, we need to coherently orient all $(V,\rho)\in\mathrm{Rep}_{\rm nt}(\Gamma)$, since arbitrary nontrivial Γ-representations can occur in the sheaves $(\mathcal{E}_{\mathcal{X}})_{\rm nt}^{\Gamma,\lambda}, (\mathcal{F}_{\mathcal{X}})_{\rm nt}^{\Gamma,\lambda}$ in the virtual normal bundle $(T^*\boldsymbol{\mathcal{X}})_{\rm nt}^{\Gamma,\lambda}$ of $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$. This is possible only if $\Lambda^\Gamma_{\rm ev,+}=\Lambda^\Gamma_+$, which as in Definition 8.8 holds if and only if $|\Gamma|$ is odd. So we must suppose $|\Gamma|$ odd in our d-orbifold version:

Proposition 10.43. *Let Γ be a finite group with $|\Gamma|$ odd, let (R_i,ρ_i) for $i=1,\ldots,k$ be representatives for the nontrivial, irreducible, real Γ-representations up to isomorphism, and choose orientations on $R_1,\ldots,R_k$. Then for all oriented d-orbifolds $\boldsymbol{\mathcal{X}}$ we may define orientations on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ$ for all $\lambda\in\Lambda^\Gamma$.*

Proof. Let $\boldsymbol{\mathcal{X}}$ be an oriented d-orbifold and $\lambda\in\Lambda^\Gamma$, so that we have an orbifold stratum $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ and 1-morphism $\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}}):\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}\to\boldsymbol{\mathcal{X}}$. As in (10.13) and Theorem 9.29 we have decompositions of virtual vector bundles on $\mathcal{X}^{\Gamma,\lambda}$:

$$\begin{aligned} O^{\Gamma,\lambda}(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})&=(T^*\boldsymbol{\mathcal{X}})_{\rm tr}^{\Gamma,\lambda}\oplus(T^*\boldsymbol{\mathcal{X}})_{\rm nt}^{\Gamma,\lambda},\ \text{ where}\\ (T^*\boldsymbol{\mathcal{X}})_{\rm tr}^{\Gamma,\lambda}\cong T^*(\boldsymbol{\mathcal{X}}^{\Gamma,\lambda})\ \text{ and }\ &(T^*\boldsymbol{\mathcal{X}})_{\rm nt}^{\Gamma,\lambda}\cong\textstyle\bigoplus_{i=1}^k(T^*\boldsymbol{\mathcal{X}})_i^{\Gamma,\lambda}\otimes R_i. \end{aligned}\tag{10.18}$$

Also $R_1,\ldots,R_k$ are even-dimensional, as $|\Gamma|$ is odd.

Suppose V,R are finite-dimensional real vector spaces, with R oriented and even-dimensional. Then there is a canonical orientation on $V\otimes R$, without choosing an orientation for V, characterized as follows: let $v_1,\ldots,v_k$ be a basis for V, and $r_1,\ldots,r_{2n}$ be an oriented basis for R. Then $v_1\otimes r_1,\ldots,v_1\otimes r_{2n}$, $v_2\otimes r_1,\ldots,v_2\otimes r_{2n},\ldots,v_k\otimes r_1,\ldots,v_k\otimes r_{2n}$ is an oriented basis for $V\otimes R$.

In the same way, as $(T^*\boldsymbol{\mathcal{X}})_i^{\Gamma,\lambda}$ is a virtual vector bundle on $\mathcal{X}^{\Gamma,\lambda}$, and R_i is an oriented, even-dimensional real vector space, we can define a canonical orientation on the virtual vector bundle $(T^*\boldsymbol{\mathcal{X}})_i^{\Gamma,\lambda}\otimes R_i$, without assuming $(T^*\boldsymbol{\mathcal{X}})_i^{\Gamma,\lambda}$ is oriented or orientable. Combining these orientations for $i=1,\ldots,k$ gives an orientation on the virtual vector bundle $(T^*\boldsymbol{\mathcal{X}})_{\rm nt}^{\Gamma,\lambda}$. Also the orientation on $\boldsymbol{\mathcal{X}}$ gives an orientation on $T^*\boldsymbol{\mathcal{X}}$, which pulls back to an orientation on $O^{\Gamma,\lambda}(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})$. Combining these orientations using (10.18) we can define an orientation on $T^*(\boldsymbol{\mathcal{X}}^{\Gamma,\lambda})$, and hence on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$, and on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ\subseteq\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$. □

For the orbifold strata $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$, we follow the method of Proposition 8.10. Let Γ be a finite group with $|\Gamma|$ odd, and let $\lambda\in\Lambda^\Gamma$ with $\mu=\lambda\cdot\mathrm{Aut}(\Gamma)\in\Lambda^\Gamma/\mathrm{Aut}(\Gamma)$. Write Δ for the subgroup of $\mathrm{Aut}(\Gamma)$ fixing λ in Λ^Γ. Suppose $\boldsymbol{\mathcal{X}}$ is an oriented d-orbifold. Then Δ acts on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$, with $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}\simeq[\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}/\Delta]$. Proposition 10.43 defines an orientation on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$. As in §8.4.2, the action of $\delta\in\Delta$ on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ multiplies the orientation by $\Phi^\Gamma(\delta,\lambda)=\pm1$. If $\Phi^\Gamma(\delta,\lambda)=1$ for all $\delta\in\Delta$ then Δ preserves the orientation on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$, and $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}\simeq[\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}/\Delta]$ is oriented. Also $\hat{\boldsymbol{\Pi}}^{\Gamma,\mu}(\boldsymbol{\mathcal{X}}):\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}\to\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$ takes orientations on $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$ to orientations on $\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$. Thus we deduce:

Proposition 10.44. *Let Γ be a finite group with $|\Gamma|$ odd, and $\lambda\in\Lambda^\Gamma$ with $\Phi^\Gamma(\delta,\lambda)=1$ for all $\delta\in\mathrm{Aut}(\Gamma)$ with $\lambda\cdot\delta=\lambda$. Set $\mu=\lambda\cdot\mathrm{Aut}(\Gamma)\in\Lambda^\Gamma/\mathrm{Aut}(\Gamma)$. Then $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ are oriented for all oriented d-orbifolds $\boldsymbol{\mathcal{X}}$.*

If $|\Gamma|$ is even, then for any $\lambda\in\Lambda^\Gamma$, generalizing Example 10.42 we can construct examples of oriented d-orbifolds $\boldsymbol{\mathcal{X}}$ with $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ$ non-orientable.

Similarly, if Γ, μ do not satisfy the conditions of Proposition 10.44, then we can construct oriented $\boldsymbol{\mathcal{X}}$ with $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ non-orientable. Hence, the conditions on Γ, λ, μ in Propositions 10.43 and 10.44 are both necessary and sufficient to be able to orient $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \ldots, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ for all oriented d-orbifolds $\boldsymbol{\mathcal{X}}$.

10.8 Kuranishi neighbourhoods, good coordinate systems

The material of this section is modelled on the theory of *Kuranishi spaces* of Fukaya, Oh, Ohta and Ono [32, App. A], [34], which will be discussed in Chapter 14. A *Kuranishi structure* [32, §A1.1], [34, §5] on a topological space X comprises a cover of X by *Kuranishi neighbourhoods* $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ or $(\mathcal{V}_i, \mathcal{E}_i, s_i, \psi_i)$, which are like charts in the definition of a manifold, together with *coordinate changes* $(e_{ij}, \hat{e}_{ij}, \rho_{ij})$ or $(e_{ij}, \hat{e}_{ij})$ between the Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, $(V_j, E_j, \Gamma_j, s_j, \psi_j)$ on their overlaps in X, which are like transition functions between charts in the definition of a manifold.

There are two versions of the theory, which we will refer to as *type A* and *type B* respectively: in [32] the (type A) Kuranishi neighbourhoods are $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ with V_i a manifold, E_i a vector bundle on V_i, $s_i \in C^\infty(E_i)$, and Γ_i a finite group acting effectively on V_i, E_i preserving s_i, but in [34] the (type B) Kuranishi neighbourhoods are $(\mathcal{V}_i, \mathcal{E}_i, s_i, \psi_i)$ with $\mathcal{V}_i$ an effective orbifold, $\mathcal{E}_i$ a vector bundle on $\mathcal{V}_i$ and $s_i \in C^\infty(\mathcal{E}_i)$. We can pass from type A to type B by setting $\mathcal{V}_i = [\underline{V}_i/\Gamma_i]$.

Fukaya et al. define *good coordinate systems* on a compact Kuranishi space X [32, Lem. A1.11], [34, Def. 6.1], which are a cover of X by Kuranishi neighbourhoods and coordinate changes with nice properties, and they claim that good coordinate systems exist for any Kuranishi space. They use good coordinate systems to show that any compact Kuranishi space has a (multivalued, $\mathbb{Q}$-weighted) perturbation to a (non-Hausdorff, $\mathbb{Q}$-weighted) compact manifold.

This section will define analogues of these ideas for d-orbifolds, in two versions 'type A' and 'type B', and prove in Theorems 10.48 and 10.54 that every d-orbifold admits good coordinate systems of both types A and B. Theorems 10.48 and 10.54 may be regarded as converses to Theorems 10.19 and 10.21 in §10.2, as they show that given any d-orbifold $\boldsymbol{\mathcal{X}}$, there exist data $I, <, V_i, E_i, \Gamma_i, s_i, \psi_i, \ldots$ or $I, <, \mathcal{V}_i, \mathcal{E}_i, s_i, \psi_i, \ldots$ satisfying the hypotheses of Theorems 10.19 and 10.21, and yielding the d-orbifold $\boldsymbol{\mathcal{X}}$. We will apply these ideas in two ways:

- In §4.4 we showed that every compact d-manifold $\boldsymbol{X}$ is equivalent to some $\boldsymbol{S}_{V,E,s}$. If $\tilde{s} \in C^\infty(E)$ is generic then $\tilde{X} = \tilde{s}^{-1}(0)$ is a manifold, which is compact if $\tilde{s} - s$ is sufficiently small in C^1. Thus, any compact d-manifold $\boldsymbol{X}$ can be deformed to a compact manifold $\tilde{\boldsymbol{X}}$ by a small perturbation. We will use this in §13.2 to show that d-manifold bordism groups $dB_k(Y)$ of a manifold Y are isomorphic to the ordinary bordism groups $B_k(Y)$.

  As in §10.5, the author cannot prove that all compact d-orbifolds $\boldsymbol{\mathcal{X}}$ are equivalent to some $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$. However, by choosing a type B good coordinate system $I, <, \mathcal{V}_i, \mathcal{E}_i, s_i, \psi_i, \ldots$ on $\boldsymbol{\mathcal{X}}$ and choosing small perturbations $\tilde{s}_i$ of s_i by induction on $i \in I$ in a similar way to [32, 34], we will show at the end of §10.9 that every semieffective (or effective) d-orbifold $\boldsymbol{\mathcal{X}}$ can be deformed to an (effective) orbifold $\tilde{\boldsymbol{\mathcal{X}}}$. In §13.4 we will use this argument to prove isomorphisms of orbifold and d-orbifold bordism groups.

Thus, good coordinate systems on d-orbifolds are used as a substitute for results on embedding d-manifolds in manifolds in §4.4.

- In Chapter 14 we will use good coordinate systems on d-orbifolds to show that every d-orbifold can be given a Kuranishi structure. We will conclude that d-orbifolds and Kuranishi spaces are roughly the same thing.

10.8.1 Type A Kuranishi neighbourhoods, good coordinate systems

We now define Kuranishi neighbourhoods, coordinate changes, and good coordinate systems of type A on a d-orbifold $\boldsymbol{\mathcal{X}}$, loosely following Fukaya, Oh, Ohta and Ono [32, Def.s A1.1, A1.3 & Lem. A1.11].

Definition 10.45. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. A *type A Kuranishi neighbourhood* on $\boldsymbol{\mathcal{X}}$ is a quintuple $(V, E, \Gamma, s, \boldsymbol{\psi})$ where V is a manifold, $E \to V$ a vector bundle, Γ a finite group acting smoothly and locally effectively on V, E (in the sense of Definition C.28) preserving the vector bundle structure, and $s : V \to E$ a smooth, Γ-equivariant section of E. Write the Γ-actions on V, E as $r(\gamma) : V \to V$ and $\hat{r}(\gamma) : E \to r(\gamma)^*(E)$ for $\gamma \in \Gamma$. Then Example 10.11 defines a principal d-orbifold $[\boldsymbol{S}_{V,E,s}/\Gamma]$. We require that $\boldsymbol{\psi} : [\boldsymbol{S}_{V,E,s}/\Gamma] \to \boldsymbol{\mathcal{X}}$ is a 1-morphism of d-orbifolds which is an equivalence with a nonempty open d-suborbifold $\boldsymbol{\psi}([\boldsymbol{S}_{V,E,s}/\Gamma]) \subseteq \boldsymbol{\mathcal{X}}$. If $[x] \in \mathcal{X}_{\rm top}$ we call $(V, E, \Gamma, s, \boldsymbol{\psi})$ a *type A Kuranishi neighbourhood of* $[x]$ if $[x] \in \boldsymbol{\psi}([\boldsymbol{S}_{V,E,s}/\Gamma])_{\rm top}$.

Definition 10.46. Suppose $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ are type A Kuranishi neighbourhoods on a d-orbifold $\boldsymbol{\mathcal{X}}$, with

$$\emptyset \neq \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \boldsymbol{\mathcal{X}}.$$

A *type A coordinate change from* $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ *to* $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ is a quintuple $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$, where:

(a) $\emptyset \neq V_{ij} \subseteq V_i$ is a Γ_i-invariant open submanifold, with

$$\boldsymbol{\psi}_i\big([\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]\big) = \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \boldsymbol{\mathcal{X}}.$$

(b) $\rho_{ij} : \Gamma_i \to \Gamma_j$ is an injective group morphism.

(c) $e_{ij} : V_{ij} \to V_j$ is an embedding of manifolds with $e_{ij} \circ r_i(\gamma) = r_j(\rho_{ij}(\gamma)) \circ e_{ij} : V_{ij} \to V_j$ for all $\gamma \in \Gamma_i$. If $v_i, v_i' \in V_{ij}$ and $\delta \in \Gamma_j$ with $r_j(\delta) \circ e_{ij}(v_i') = e_{ij}(v_i)$, then there exists $\gamma \in \Gamma_i$ with $\rho_{ij}(\gamma) = \delta$ and $r_i(\gamma)(v_i') = v_i$.

(d) $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$ is an embedding of vector bundles (that is, $\hat{e}_{ij}$ has a left inverse), such that $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j)$ and $r_i(\gamma)^*(\hat{e}_{ij}) \circ \hat{r}_i(\gamma) = e_{ij}^*(\hat{r}_j(\rho_{ij}(\gamma))) \circ \hat{e}_{ij} : E_i|_{V_{ij}} \to (e_{ij} \circ r_i(\gamma))^*(E_j)$ for all $\gamma \in \Gamma_i$. Thus Example 10.12 defines a quotient 1-morphism

$$[\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}] : [\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i] \longrightarrow [\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j], \tag{10.19}$$

where $[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]$ is an open d-suborbifold in $[\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]$.

(e) If $v_i \in V_{ij}$ with $s_i(v_i) = 0$ and $v_j = e_{ij}(v_i) \in V_j$ then the following linear map is an isomorphism, as in (10.4):

$$\big(\mathrm{d}s_j(v_j)\big)_* : \big(T_{v_j}V_j\big)/\big(\mathrm{d}e_{ij}(v_i)[T_{v_i}V_i]\big) \to \big(E_j|_{v_j}\big)/\big(\hat e_{ij}(v_i)[E_i|_{v_i}]\big).$$

Corollary 10.18 then implies that $[\boldsymbol{S}_{e_{ij},\hat e_{ij}},\rho_{ij}]$ in (10.19) is an equivalence with an open d-suborbifold of $[\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j]$.

(f) $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ [\boldsymbol{S}_{e_{ij},\hat e_{ij}},\rho_{ij}] \Rightarrow \boldsymbol{\psi}_i|_{[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]}$ is a 2-morphism in **dOrb**.

(g) The quotient topological space $V_i \amalg_{V_{ij}} V_j = (V_i \amalg V_j)/\sim$ is Hausdorff, where the equivalence relation $\sim$ identifies $v \in V_{ij} \subseteq V_i$ with $e_{ij}(v) \in V_j$.

Definition 10.47. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. A *type A good coordinate system* on $\boldsymbol{\mathcal{X}}$ consists of the following data satisfying conditions (a)–(e):

(a) We are given a countable indexing set I, and a total order $<$ on I making $(I,<)$ into a well-ordered set.

(b) For each $i \in I$ we are given a Kuranishi neighbourhood $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ of type A on $\boldsymbol{\mathcal{X}}$. Write $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i])$, so that $\boldsymbol{\mathcal{X}}_i \subseteq \boldsymbol{\mathcal{X}}$ is an open d-suborbifold, and $\boldsymbol{\psi}_i : [\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i] \to \boldsymbol{\mathcal{X}}_i$ is an equivalence. We require that $\bigcup_{i\in I} \boldsymbol{\mathcal{X}}_i = \boldsymbol{\mathcal{X}}$, so that $\{\boldsymbol{\mathcal{X}}_i : i \in I\}$ is an open cover of $\boldsymbol{\mathcal{X}}$.

(c) For all $i < j$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \neq \boldsymbol{\emptyset}$ we are given a type A coordinate change $(V_{ij}, e_{ij}, \hat e_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$ from $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ to $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$.

(d) For all $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$, we are given $\gamma_{ijk} \in \Gamma_k$ satisfying $\rho_{ik}(\gamma) = \gamma_{ijk}\,\rho_{jk}(\rho_{ij}(\gamma))\,\gamma_{ijk}^{-1}$ for all $\gamma \in \Gamma_i$, and

$$\begin{aligned} e_{ik}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})} &= r_k(\gamma_{ijk}) \circ e_{jk} \circ e_{ij}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})}, \\ \hat e_{ik}|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})} &= \big(e_{ij}^*(e_{jk}^*(\hat r_k(\gamma_{ijk}))) \circ e_{ij}^*(\hat e_{jk}) \circ \hat e_{ij}\big)|_{V_{ik}\cap e_{ij}^{-1}(V_{jk})}. \end{aligned} \tag{10.20}$$

Combining the first equation of (10.20) with Definition 10.46(c) for e_{ik} and Γ_i acting effectively on $V_{ik} \cap e_{ij}^{-1}(V_{jk})$ shows that γ_{ijk} is unique. Example 10.13 with $\delta = \gamma_{ijk}$ and $\Lambda = 0$ then gives a 2-morphism in **dOrb**:

$$\begin{aligned} \boldsymbol{\eta}_{ijk} = [\boldsymbol{S}_0, \gamma_{ijk}] : [\boldsymbol{S}_{e_{jk},\hat e_{jk}},\rho_{jk}] \circ [\boldsymbol{S}_{e_{ij},\hat e_{ij}},\rho_{ij}]|_{[\boldsymbol{S}_{V_{ik}\cap e_{ij}^{-1}(V_{jk}),E_i,s_i}/\Gamma_i]} \\ \Longrightarrow [\boldsymbol{S}_{e_{ik},\hat e_{ik}},\rho_{ik}]|_{[\boldsymbol{S}_{V_{ik}\cap e_{ij}^{-1}(V_{jk}),E_i,s_i}/\Gamma_i]}. \end{aligned} \tag{10.21}$$

(e) For all $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$ and $\boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$, we require that if $v_i \in V_{ik}$, $v_j \in V_{jk}$ and $\delta \in \Gamma_k$ with $e_{jk}(v_j) = r_k(\delta) \circ e_{ik}(v_i)$ in V_k, then $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$, and $v_i \in V_{ij}$, and there exists $\gamma \in \Gamma_j$ with $\rho_{jk}(\gamma) = \delta\,\gamma_{ijk}$ and $v_j = r_j(\gamma) \circ e_{ij}(v_i)$.

Suppose now that Y is a manifold, and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism in **dOrb**, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Man}}^{\mathbf{dOrb}}(Y)$. A *type A good coordinate system for* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ consists of a type A good coordinate system $\big(I, <, \ldots, \gamma_{ijk}\big)$ for $\boldsymbol{\mathcal{X}}$ as in (a)–(e) above, together with the following data satisfying conditions (f)–(g):

(f) For each $i \in I$, we are given a smooth map $g_i : V_i \to Y$ with $g_i \circ r_i(\gamma) = g_i$ for all $\gamma \in \Gamma_i$, so that Example 10.12 defines a quotient 1-morphism

$$[\boldsymbol{S}_{g_i,0}, \pi] : [\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i] \longrightarrow [\boldsymbol{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}},$$

where $\pi : \Gamma_i \to \{1\}$ is the projection. We are given a 2-morphism $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow [\boldsymbol{S}_{g_i,0}, \pi]$ in **dOrb**. Sometimes we require g_i to be a submersion.

(g) For all $i < j$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \neq \emptyset$, we require that $g_j \circ e_{ij} = g_i|_{V_{ij}}$. This implies that

$$[\boldsymbol{S}_{g_j,0}, \pi] \circ [\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}] = [\boldsymbol{S}_{g_i,0}, \pi]|_{[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]} : \\ [\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i] \longrightarrow [\boldsymbol{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}}.$$

Here is the main result of this section. It will be proved in Appendix D, along with the analogue Theorem 12.48 for d-orbifolds with corners. The analogue for Kuranishi spaces is Fukaya et al. [32, Lem. A1.11], which is stated without proof; the type B version, with some proof, is [34, Lem. 6.3].

Theorem 10.48. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold. Then there exists a type A good coordinate system $\bigl(I, <, (V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij}), \gamma_{ijk}\bigr)$ for $\boldsymbol{\mathcal{X}}$. If $\boldsymbol{\mathcal{X}}$ is compact, we may take I to be finite. If $\{\boldsymbol{\mathcal{U}}_j : j \in J\}$ is an open cover of $\boldsymbol{\mathcal{X}}$, we may take $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]) \subseteq \boldsymbol{\mathcal{U}}_{j_i}$ for each $i \in I$ and some $j_i \in J$.*

Now let Y be a manifold and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Man}}^{\mathbf{dOrb}}(Y)$ a 1*-morphism in* **dOrb**. *Then all the above extends to type A good coordinate systems for $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, and we may take the g_i in Definition* 10.47(f) *to be submersions.*

Remark 10.49. (i) If $\bigl(I, <, (V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij}), \gamma_{ijk}\bigr)$ is a type A good coordinate system on $\boldsymbol{\mathcal{X}}$, then the data $n = \operatorname{vdim} \boldsymbol{\mathcal{X}}$, $X = \mathcal{X}_{\rm top}$, $I, <,$ $V_i, E_i, \Gamma_i, s_i, \psi_i = f_{i,\rm top}$ for $i \in I$ and $V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}$ for $i < j$ in I, γ_{ijk} for $i < j < k$ in I satisfy the hypotheses of Theorem 10.21. The theorem then reconstructs $\boldsymbol{\mathcal{X}}$ up to equivalence, and the 1- and 2-morphisms $\boldsymbol{\psi}_i, \boldsymbol{\eta}_{ij}$. Thus, the data $I, <, V_i, E_i, \Gamma_i, s_i, V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \gamma_{ijk}$ determine $\boldsymbol{\mathcal{X}}$ up to equivalence. So we can regard Theorem 10.48 as a kind of converse to Theorem 10.21.

(ii) The conditions Definition 10.46(g), that $(I, <)$ is *well-ordered* in Definition 10.47(a), and Definition 10.47(f), do not appear in [32, Lem. A1.11], [34, Def. 6.1], but are technical conditions that the author has added in order to make inductive proofs using good coordinate systems work. For example, to perturb a semieffective d-orbifold $\boldsymbol{\mathcal{X}}$ to an orbifold $\tilde{\mathcal{X}}$ as in §13.4, we can choose a type A good coordinate system on $\boldsymbol{\mathcal{X}}$, and then choose small generic Γ_i-equivariant perturbations $\tilde{s}_i$ of s_i on V_i, by transfinite induction on $i \in I$ in the order $<$ on I. We need $(I, <)$ well-ordered so that transfinite induction is valid.

If $i < j$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \neq \boldsymbol{\emptyset}$ then the restriction of $\tilde{s}_j$ to $\Gamma_j \cdot e_{ij}(V_{ij})$ in V_j is determined by $\tilde{s}_i|_{V_{ij}}$. If $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$ and $\boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$, this prescribes $\tilde{s}_k$ on $\Gamma_k \cdot e_{ik}(V_{ik})$ and $\Gamma_k \cdot e_{jk}(V_{jk})$. Part (e) implies that these conditions are consistent on $\bigl(\Gamma_k \cdot e_{ik}(V_{ik})\bigr) \cap \bigl(\Gamma_k \cdot e_{jk}(V_{jk})\bigr)$, so that it is possible to choose $\tilde{s}_k$. Definition 10.46(g) ensures the perturbation $\tilde{\mathcal{X}}$ is Hausdorff (separated).

10.8.2 Type B Kuranishi neighbourhoods, good coordinate systems

Here are the 'type B' analogues of Definitions 10.45–10.47, loosely following Fukaya and Ono [34, Def.s 5.1, 5.3 & 6.1].

Definition 10.50. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. A *type B Kuranishi neighbourhood* on $\boldsymbol{\mathcal{X}}$ is a quadruple $(\mathcal{V}, \mathcal{E}, s, \boldsymbol{\psi})$ where $\mathcal{V}$ is an effective orbifold, $\mathcal{E}$ a vector bundle on $\mathcal{V}$ and $s \in C^\infty(\mathcal{E})$, so that Definition 10.5 defines a 'standard model' d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$, and $\boldsymbol{\psi} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{X}}$ is a 1-morphism of d-orbifolds which is an equivalence with a nonempty open d-suborbifold $\boldsymbol{\psi}(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}) \subseteq \boldsymbol{\mathcal{X}}$. If $[x] \in \mathcal{X}_{\rm top}$ we say that $(\mathcal{V}, \mathcal{E}, s, \boldsymbol{\psi})$ is a *type B Kuranishi neighbourhood of* $[x]$ if also $[x] \in \boldsymbol{\psi}(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s})_{\rm top}$, so that $[x] = f_{\rm top}([v])$ for some $[v] \in \mathcal{V}_{\rm top}$ with $s(v) = 0$.

Definition 10.51. Suppose $(\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i), (\mathcal{V}_j, \mathcal{E}_j, s_j, \boldsymbol{\psi}_j)$ are type B Kuranishi neighbourhoods on a d-orbifold $\boldsymbol{\mathcal{X}}$, with $\boldsymbol{\emptyset} \neq \boldsymbol{\psi}_i(\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}) \cap \boldsymbol{\psi}_j(\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j}) \subseteq \boldsymbol{\mathcal{X}}$. A *type B coordinate change from* $(\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i)$ *to* $(\mathcal{V}_j, \mathcal{E}_j, s_j, \boldsymbol{\psi}_j)$ is a quadruple $(\mathcal{V}_{ij}, e_{ij}, \hat{e}_{ij}, \boldsymbol{\eta}_{ij})$, where:

(a) $\emptyset \neq \mathcal{V}_{ij} \subseteq \mathcal{V}_i$ is an open suborbifold, with

$$\boldsymbol{\psi}_i\bigl(\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}}\bigr) = \boldsymbol{\psi}_i(\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}) \cap \boldsymbol{\psi}_j(\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j}) \subseteq \boldsymbol{\mathcal{X}}.$$

(b) $e_{ij} : \mathcal{V}_{ij} \to \mathcal{V}_j$ is an embedding of orbifolds, in the sense of §8.2.

(c) $\hat{e}_{ij} : \mathcal{E}_i|_{\mathcal{V}_{ij}} \to e_{ij}^*(\mathcal{E}_j)$ is an embedding of vector bundles (that is, $\hat{e}_{ij}$ has a left inverse in $\mathrm{qcoh}(\mathcal{V}_{ij})$), such that $\hat{e}_{ij} \circ s_i|_{\mathcal{V}_{ij}} = e_{ij}^*(s_j) \circ \iota_{ij}$, where $\iota_{ij} : \mathcal{O}_{\mathcal{V}_{ij}} \to e_{ij}^*(\mathcal{O}_{\mathcal{V}_j})$ is the natural isomorphism, so that Definition 10.9 gives a 'standard model' 1-morphism

$$\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}} : \boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}} \longrightarrow \boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j}. \tag{10.22}$$

(d) if $[v_i] \in \mathcal{V}_{ij,\rm top}$ with $s_i(v_i) = 0$ and $[v_j] = e_{ij,\rm top}([v_i])$ then the following linear map is an isomorphism, as in (10.4):

$$\bigl(\mathrm{d}s_j(v_j)\bigr)_*\!: \bigl(T_{v_j}\mathcal{V}_j\bigr)\big/\bigl(\mathrm{d}e_{ij}(v_i)[T_{v_i}\mathcal{V}_i]\bigr) \to \bigl(\mathcal{E}_j|_{v_j}\bigr)\big/\bigl(\hat{e}_{ij}(v_i)[\mathcal{E}_i|_{v_i}]\bigr). \tag{10.23}$$

Corollary 10.18 then implies that $\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}$ in (10.22) is an equivalence with an open d-suborbifold of $\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j}$.

(e) $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ \boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i|_{\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}}}$ is a 2-morphism in $\mathbf{dOrb}$.

(f) The quotient topological space $\mathcal{V}_{i,\rm top} \amalg_{\mathcal{V}_{ij,\rm top}} \mathcal{V}_{j,\rm top} = (\mathcal{V}_{i,\rm top} \amalg \mathcal{V}_{j,\rm top})/\sim$ is Hausdorff, where $\sim$ identifies $[v] \in \mathcal{V}_{ij,\rm top} \subseteq \mathcal{V}_{i,\rm top}$ with $e_{ij,\rm top}([v]) \in \mathcal{V}_{j,\rm top}$.

Note that $e_{ij}, \hat{e}_{ij}$ embeddings imply that $\dim \mathcal{V}_i \leqslant \dim \mathcal{V}_j$ and $\operatorname{rank} \mathcal{E}_i \leqslant \operatorname{rank} \mathcal{E}_j$. Thus, coordinate changes generally exist only in one direction.

Definition 10.52. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. A *type B good coordinate system* on $\boldsymbol{\mathcal{X}}$ consists of the following data satisfying conditions (a)–(e):

(a) We are given a countable indexing set I, and a total order $<$ on I making $(I,<)$ into a well-ordered set.

(b) For each $i \in I$ we are given a Kuranishi neighbourhood $(\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i)$ of type B on $\boldsymbol{\mathcal{X}}$. Write $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i(\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i})$, so that $\boldsymbol{\mathcal{X}}_i \subseteq \boldsymbol{\mathcal{X}}$ is an open d-suborbifold, and $\boldsymbol{\psi}_i : \boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i} \to \boldsymbol{\mathcal{X}}_i$ is an equivalence. We require that $\bigcup_{i\in I} \boldsymbol{\mathcal{X}}_i = \boldsymbol{\mathcal{X}}$, so that $\{\boldsymbol{\mathcal{X}}_i : i \in I\}$ is an open cover of $\boldsymbol{\mathcal{X}}$.

(c) For all $i < j$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \neq \boldsymbol{\emptyset}$ we are given a type B coordinate change $(\mathcal{V}_{ij}, e_{ij}, \hat{e}_{ij}, \boldsymbol{\eta}_{ij})$ from $(\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i)$ to $(\mathcal{V}_j, \mathcal{E}_j, s_j, \boldsymbol{\psi}_j)$.

(d) For all $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$, we are given a 2-morphism $\eta_{ijk} : e_{jk} \circ e_{ij}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})} \Rightarrow e_{ik}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}$ in **Orb** with

$$\hat{e}_{ik}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})} = \eta^*_{ijk}(\mathcal{E}_k) \circ I_{e_{ij},e_{jk}}(\mathcal{E}_k)^{-1} \circ e_{ij}|^*_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}.$$

This η_{ijk} is unique by Proposition 8.14(i). Proposition 10.10 gives a 2-morphism in **dOrb**

$$\boldsymbol{\eta}_{ijk} = \bigl(\eta_{ijk}|_{s_i^{-1}(0)}, 0\bigr) : \boldsymbol{\mathcal{S}}_{e_{jk},\hat{e}_{jk}} \circ \boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}|_{\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk}),\mathcal{E}_i,s_i}} \Longrightarrow \boldsymbol{\mathcal{S}}_{e_{ik},\hat{e}_{ik}}|_{\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk}),\mathcal{E}_i,s_i}}.$$

(e) For all $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$ and $\boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$, we require that if $[v_i] \in \mathcal{V}_{ik,\mathrm{top}} \subseteq \mathcal{V}_{i,\mathrm{top}}$ and $[v_j] \in \mathcal{V}_{jk,\mathrm{top}} \subseteq \mathcal{V}_{j,\mathrm{top}}$ with $e_{ik,\mathrm{top}}([v_i]) = e_{jk,\mathrm{top}}([v_j])$ in $\mathcal{V}_{k,\mathrm{top}}$, then $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \boldsymbol{\emptyset}$, and $[v_i] \in \mathcal{V}_{ij,\mathrm{top}}$ with $[v_j] = e_{ij,\mathrm{top}}([v_i])$.

Suppose now that $\mathcal{Y}$ is an effective orbifold, and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism in **dOrb**, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$. A *type B good coordinate system for* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ consists of a good coordinate system $\bigl(I, <, \ldots, \eta_{ijk}\bigr)$ for $\boldsymbol{\mathcal{X}}$ as in (a)–(e) above, together with the data:

(f) For each $i \in I$, a 1-morphism $g_i : \mathcal{V}_i \to \mathcal{Y}$, and a 2-morphism $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \boldsymbol{\mathcal{S}}_{g_i,0}$ in **dOrb**. Sometimes we require g_i to be a submersion, as in §8.2.

(g) For all $i < j$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \neq \emptyset$, a 2-morphism $\zeta_{ij} : g_j \circ e_{ij} \Rightarrow g_i|_{\mathcal{V}_{ij}}$ in **Orb**. When g_i is a submersion, ζ_{ij} is unique by Proposition 8.14(ii).

Remark 10.53. (i) One reason for requiring $\mathcal{V}_i$ to be effective in Definition 10.50, and e_{ij} an embedding in Definition 10.51(b), and g_i a submersion in Definition 10.52(f), is so that the 2-morphisms η_{ijk}, ζ_{ij} in Definition 10.52(d),(g) are unique by Proposition 8.14(i),(ii). Thus, we can regard the η_{ijk}, ζ_{ij} as not part of the data of a good coordinate system, but just require them to exist.

This means we can express good coordinate systems for $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in terms of the homotopy category $\mathrm{Ho}(\mathbf{Orb^{eff}})$, as in §4.7 and Remarks 9.20 and 10.20(b). That is, when working with good coordinate systems we can treat (effective) orbifolds as forming a category rather than a 2-category.

(ii) There are natural maps from type A Kuranishi neighbourhoods, coordinate changes, and good coordinate systems, to type B Kuranishi neighbourhoods, coordinate changes, and good coordinate systems, defined as follows.

If $(V, E, \Gamma, s, \boldsymbol{\psi})$ is a type A Kuranishi neighbourhood on $\boldsymbol{\mathcal{X}}$ then $\tilde{\mathcal{V}} = [\underline{V}/\Gamma]$ is an orbifold, where $\underline{V} = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(V)$. The Γ-equivariant vector bundle $E \to V$ lifts to a Γ-equivariant vector bundle $(\mathcal{E}, \Phi) \in \mathrm{qcoh}^\Gamma(\underline{V})$, and the equivalence $F_\Pi : \mathrm{qcoh}(\mathcal{V}) \to \mathrm{qcoh}^\Gamma(\underline{V})$ from Definition C.34 and Theorem C.35 implies that we may choose a vector bundle $\tilde{\mathcal{E}}$ on $\mathcal{V}$ with an isomorphism $F_\Pi(\tilde{\mathcal{E}}) \cong (E, \Phi)$. The Γ-equivariant section s of E lifts to $\tilde{s} \in C^\infty(\tilde{\mathcal{E}})$. There is a natural equivalence $\boldsymbol{i} : \boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}} \to [\boldsymbol{S}_{V,E,s}/\Gamma]$, and we set $\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi} \circ \boldsymbol{i} : \boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}} \to \boldsymbol{\mathcal{X}}$. Then $(\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{s}, \tilde{\boldsymbol{\psi}})$ is a type B Kuranishi neighbourhood on $\boldsymbol{\mathcal{X}}$.

Similarly, type A coordinate changes $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$ induce type B coordinate changes $(\tilde{\mathcal{V}}_{ij}, \tilde{e}_{ij}, \hat{\tilde{e}}_{ij}, \tilde{\boldsymbol{\eta}}_{ij})$, where $\tilde{\mathcal{V}}_{ij} = [\underline{V}_{ij}/\Gamma_i]$ and $\tilde{e}_{ij} = [\underline{e}_{ij}, \rho_{ij}]$ in the notation of Definitions C.17 and C.18. Given a type A good coordinate system, map $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i) \mapsto (\tilde{\mathcal{V}}_i, \tilde{\mathcal{E}}_i, \tilde{s}_i, \tilde{\boldsymbol{\psi}}_i)$ and $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij}) \mapsto (\tilde{\mathcal{V}}_{ij}, \tilde{e}_{ij}, \hat{\tilde{e}}_{ij}, \tilde{\boldsymbol{\eta}}_{ij})$ as above, and set

$$\begin{aligned}\tilde{\eta}_{ijk} = [\gamma_{ijk}] : \tilde{e}_{jk} \circ \tilde{e}_{ij}|_{\tilde{\mathcal{V}}_{ik} \cap \tilde{e}_{ij}^{-1}(\tilde{\mathcal{V}}_{jk})} = [\underline{e}_{jk} \circ \underline{e}_{ij}, \rho_{jk} \circ \rho_{ij}]|_{[(\underline{V}_{ik} \cap \underline{e}_{ij}^{-1}(\underline{V}_{jk}))/\Gamma_i]} \\ \Longrightarrow [\underline{e}_{ik}, \rho_{ik}]|_{[(\underline{V}_{ik} \cap \underline{e}_{ij}^{-1}(\underline{V}_{jk}))/\Gamma_i]} = \tilde{e}_{ik}|_{\tilde{\mathcal{V}}_{ik} \cap \tilde{e}_{ij}^{-1}(\tilde{\mathcal{V}}_{jk})},\end{aligned}$$

with $[\gamma_{ijk}]$ as in Definition C.19. Then $\bigl(I,<,(\tilde{\mathcal{V}}_i,\tilde{\mathcal{E}}_i,\tilde{s}_i,\tilde{\boldsymbol{\psi}}_i),(\tilde{\mathcal{V}}_{ij},\tilde{e}_{ij},\hat{\tilde{e}}_{ij},\tilde{\boldsymbol{\eta}}_{ij}),\tilde{\eta}_{ijk}\bigr)$ is a type B good coordinate system.

Here is the type B analogue of Theorem 10.48, a kind of converse to Theorem 10.19. The analogue for Kuranishi spaces is Fukaya and Ono [34, Lem. 6.3]. The first part of Theorem 10.54 follows immediately from Theorem 10.48 by Remark 10.53(ii). For the second part, Theorem 10.48 only covers the case when Y is a manifold, so we need to extend to $\mathcal{Y}$ an effective orbifold. This extension is not difficult, but would take a long time to write out, so we leave it as an exercise.

Theorem 10.54. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold. Then there exists a type B good coordinate system $\bigl(I, <, (\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i), (\mathcal{V}_{ij}, e_{ij}, \hat{e}_{ij}, \boldsymbol{\eta}_{ij}), \eta_{ijk}\bigr)$ for $\boldsymbol{\mathcal{X}}$. If $\boldsymbol{\mathcal{X}}$ is compact, we may take I to be finite. If $\{\boldsymbol{\mathcal{U}}_j : j \in J\}$ is an open cover of $\boldsymbol{\mathcal{X}}$, we may take $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i(\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}) \subseteq \boldsymbol{\mathcal{U}}_{j_i}$ for each $i \in I$ and some $j_i \in J$.*

Now let $\mathcal{Y}$ be an effective orbifold and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ a 1-morphism. Then the above extends to type B good coordinate systems for $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, and we may take the g_i in Definition 10.52(f) *to be submersions.*

We define a special kind of type B good coordinate system.

Definition 10.55. A type B good coordinate system $\bigl(I, <, (\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i), (\mathcal{V}_{ij}, e_{ij}, \hat{e}_{ij}, \boldsymbol{\eta}_{ij}), \eta_{ijk}\bigr)$ on a d-orbifold $\boldsymbol{\mathcal{X}}$, or for a 1-morphism $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ to an orbifold $\mathcal{Y}$, is called a *very good coordinate system* if $I \subset \mathbb{N} = \{0, 1, 2, \ldots\}$, and the order $<$ on I is the restriction of $<$ on $\mathbb{N}$, and $\dim \mathcal{V}^i = i$ for all $i \in I$. That is, there is at most one Kuranishi neighbourhood $(\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i)$ of each dimension $\dim \mathcal{V}_i$, and we use the dimensions $\dim \mathcal{V}_i$ as the indexing set I.

Theorem 10.56. *For each d-orbifold $\boldsymbol{\mathcal{X}}$, there exists a very good coordinate system $\bigl(I,<,(\mathcal{V}_i,\mathcal{E}_i,s_i,\boldsymbol{\psi}_i),(\mathcal{V}_{ij},e_{ij},\hat{e}_{ij},\boldsymbol{\eta}_{ij}),\eta_{ijk}\bigr)$. If also $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ is a 1-morphism to an effective orbifold $\mathcal{Y}$, there exists a very good coordinate system for $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$, and we may take the g_i in Definition* 10.52(f) *to be submersions. If $\boldsymbol{\mathcal{X}}$ is compact, we may take I to be finite in both cases.*

We can deduce this from Theorem 10.54. Let $\bigl(I,<,(\mathcal{V}_i,\mathcal{E}_i,s_i,\boldsymbol{\psi}_i),\ldots,\eta_{ijk}\bigr)$ be as in Theorem 10.54 for $\boldsymbol{\mathcal{X}}$ or $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$. Now the proof of Theorem 10.48 below automatically yields $\bigl(I,<,(V_i,E_i,\Gamma_i,s_i,\boldsymbol{\psi}_i),\ldots,\gamma_{ijk}\bigr)$ with the extra property that $\dim V_i<\dim V_j$ if $i<j$ and $\boldsymbol{\mathcal{X}}_i\cap\boldsymbol{\mathcal{X}}_j\neq\boldsymbol{\emptyset}$, and the same holds for Theorem 10.54 which is deduced from it.

So we put $\dot{I}=\{\dim\tilde{\mathcal{V}}_i:i\in I\}\subseteq\mathbb{N}$, and for each $d\in\dot{I}$ we put $\dot{\mathcal{V}}_d=\coprod_{i\in I:\dim\mathcal{V}_i=d}\mathcal{V}_i$, which is an effective orbifold of dimension d, and we define $\dot{\mathcal{E}}_d,\dot{s}_d,\dot{\boldsymbol{f}}_d$ by $\dot{\mathcal{E}}_d|_{\mathcal{V}_i}=\mathcal{E}_i$, $\dot{s}_d|_{\mathcal{V}_i}=s_i$, $\dot{\boldsymbol{f}}_d|_{\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}}=\dot{\boldsymbol{f}}_i$ for each $i\in I$ with $\dim\mathcal{V}_i=d$. Similarly, for $d<e$ in $\dot{I}$ we put $\dot{\mathcal{V}}_{d,e}=\coprod_{i,j\in I:\dim\mathcal{V}_i=d,\ \dim\mathcal{V}_j=e}\mathcal{V}_{ij}$, and so on. Then $\bigl(\dot{I},<,(\dot{\mathcal{V}}_i,\dot{\mathcal{E}}_i,\dot{s}_i,\dot{\boldsymbol{f}}_i),\ldots,\dot{\eta}_{ijk}\bigr)$ is a very good coordinate system for $\boldsymbol{\mathcal{X}}$ or $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$, and Theorem 10.56 follows.

10.9 Semieffective and effective d-orbifolds

In §C.5 we defined effective C^∞-stacks, and in §8.4.3 we discussed effective orbifolds. We now define *semieffective* and *effective* d-orbifolds. They have the property that a small, generic perturbation of a semieffective d-orbifold is an orbifold, and a small, generic perturbation of an effective d-orbifold is an effective orbifold. This will be important in §13.4, when we show that (semi)effective d-orbifold bordism is isomorphic to (effective) orbifold bordism.

Definition 10.57. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. For $[x]\in\mathcal{X}_{\rm top}$, so that $x:\bar{\ast}\to\mathcal{X}$ is a C^∞-stack 1-morphism, apply the right exact operator x^* to the exact sequence (9.10) to get an exact sequence in $\mathrm{qcoh}(\bar{\ast})$, where $K_{[x]}=\mathrm{Ker}(x^*(\phi_{\mathcal{X}}))$:

$$0\longrightarrow K_{[x]}\longrightarrow x^*(\mathcal{E}_{\mathcal{X}})\xrightarrow{x^*(\phi_{\mathcal{X}})}x^*(\mathcal{F}_{\mathcal{X}})\xrightarrow{x^*(\psi_{\mathcal{X}})}x^*(T^*\mathcal{X})\cong T_x^*\mathcal{X}\longrightarrow 0. \tag{10.24}$$

We may think of this as an exact sequence of real vector spaces, where $K_{[x]},T_x^*\mathcal{X}$ are finite-dimensional with $\dim T_x^*\mathcal{X}-\dim K_{[x]}=\mathrm{vdim}\,\boldsymbol{\mathcal{X}}$.

The orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$ is the group of 2-morphisms $\eta:x\Rightarrow x$. Definition C.36 defines isomorphisms $\eta^*(\mathcal{E}_{\mathcal{X}}):x^*(\mathcal{E}_{\mathcal{X}})\to x^*(\mathcal{E}_{\mathcal{X}})$ in $\mathrm{qcoh}(\bar{\ast})$, which make $x^*(\mathcal{E}_{\mathcal{X}})$ into a representation of $\mathrm{Iso}_{\mathcal{X}}([x])$. The same holds for $x^*(\mathcal{F}_{\mathcal{X}}),x^*(T^*\mathcal{X})$, and $x^*(\phi_{\mathcal{X}}),x^*(\psi_{\mathcal{X}})$ are equivariant. Hence $K_{[x]},T_x^*\mathcal{X}$ are also $\mathrm{Iso}_{\mathcal{X}}([x])$-representations.

We call $\boldsymbol{\mathcal{X}}$ a *semieffective d-orbifold* if $K_{[x]}$ is a trivial representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ for all $[x]\in\mathcal{X}_{\rm top}$. We call $\boldsymbol{\mathcal{X}}$ an *effective d-orbifold* if it is semieffective, and $T_x^*\mathcal{X}$ is an effective representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ for all $[x]\in\mathcal{X}_{\rm top}$.

Here is our result that generic perturbations of (semi)effective d-orbifolds are (effective) orbifolds, stated for 'standard model' d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$. When we say '$\tilde s - s$ is sufficiently small in C^1 locally in $\mathcal{V}$', we mean that $|\tilde s - s|([v]) + |\nabla(\tilde s - s)|([v]) \leqslant C([v])$ for all $[v] \in \mathcal{V}_{\rm top}$, for some choice of connection ∇ on $\mathcal{E}$ and metrics $|\,.\,|$ on $\mathcal{E}, \mathcal{E}\otimes T^*\mathcal{V}$, and some continuous $C : \mathcal{V}_{\rm top} \to (0,\infty)$.

Proposition 10.58. *Let $\mathcal{V}$ be an orbifold, $\mathcal{E}$ a vector bundle on $\mathcal{V}$, and $s \in C^\infty(\mathcal{E})$, and let $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ be as in Definition 10.5. Suppose $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is a semieffective d-orbifold. Then for any generic perturbation $\tilde s$ of s in $C^\infty(\mathcal{E})$ with $\tilde s - s$ sufficiently small in C^1 locally on $\mathcal{V}$, the d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde s}$ is an orbifold, that is, it lies in $\hat{\mathbf{Orb}} \subset \mathbf{dOrb}$. If $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is an effective d-orbifold, then $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde s}$ is an effective orbifold.*

Proof. If $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ then points $[x] \in \mathcal{X}_{\rm top}$ correspond to points $[v] \in \mathcal{V}_{\rm top}$ with $s(v) = 0$, and then (10.24) for $[v]$ becomes

$$0 \longrightarrow K_{[v]} \longrightarrow \mathcal{E}|_v^* \xrightarrow{\mathrm{d}s|_v^*} T_v^*\mathcal{V} \longrightarrow T_v^*\mathcal{X} \longrightarrow 0, \tag{10.25}$$

which is an exact sequence of finite-dimensional representations of $\mathrm{Iso}_{\mathcal{V}}([v])$. As $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is semieffective, the representation of $\mathrm{Iso}_{\mathcal{V}}([v])$ on $K_{[v]}$ is trivial.

Choose an open neighbourhood $\mathcal{U}$ of $[v]$ in $\mathcal{V}$ and a splitting $\mathcal{E}|_{\mathcal{U}} = \mathcal{A}\oplus\mathcal{B}$, where $\mathcal{A},\mathcal{B}$ are vector subbundles of $\mathcal{E}|_{\mathcal{U}}$ with $\mathcal{B}|_v = \mathrm{d}s|_v(T_v\mathcal{V})$, which implies that $\mathcal{A}|_v \cong K_{[v]}^*$ by (10.25). Then $\mathrm{Iso}_{\mathcal{V}}([v])$ acts trivially on $\mathcal{A}|_v$, so making $\mathcal{U}$ smaller if necessary, we can suppose that $\mathrm{Iso}_{\mathcal{V}}([u])$ acts trivially on $\mathcal{A}|_u$ for all $[u] \in \mathcal{U}_{\rm top}$. Choose a connection ∇ on $\mathcal{E}|_{\mathcal{U}}$, and regard $\nabla s|_{\mathcal{U}}$ as a morphism $T\mathcal{U} \to \mathcal{E}|_{\mathcal{U}} = \mathcal{A}\oplus\mathcal{B}$, so we can split $\nabla s|_{\mathcal{U}} = \nabla s^{\mathcal{A}} \oplus \nabla s^{\mathcal{B}}$ for $\nabla s^{\mathcal{A}} : T\mathcal{U} \to \mathcal{A}$ and $\nabla s^{\mathcal{B}} : T\mathcal{U} \to \mathcal{B}$. Then $\nabla s^{\mathcal{B}}|_v : T_v\mathcal{U} \to \mathcal{B}|_v$ is surjective, so making $\mathcal{U}$ smaller if necessary, we can suppose $\nabla s^{\mathcal{B}}$ is surjective on $\mathcal{U}$.

Let $\tilde s$ be a locally C^1-small generic perturbation of s, and write $s|_{\mathcal{U}} = s^{\mathcal{A}} \oplus s^{\mathcal{B}}$ and $\tilde s|_{\mathcal{U}} = \tilde s^{\mathcal{A}} \oplus \tilde s^{\mathcal{B}}$ for $s^{\mathcal{A}}, \tilde s^{\mathcal{A}} \in C^\infty(\mathcal{A})$ and $s^{\mathcal{B}}, \tilde s^{\mathcal{B}} \in C^\infty(\mathcal{B})$. As $\nabla s^{\mathcal{B}} : T\mathcal{U} \to \mathcal{B}$ is surjective and $\nabla(\tilde s - s)$ is small, $\nabla \tilde s^{\mathcal{B}} : T\mathcal{U} \to \mathcal{B}$ is surjective, so $(\tilde s^{\mathcal{B}})^{-1}(0)$ is a suborbifold of $\mathcal{U}$. Since $\tilde s$ is generic, $\tilde s^{\mathcal{A}}|_{(\tilde s^{\mathcal{B}})^{-1}(0)}$ is a transverse section of $\mathcal{A}|_{(\tilde s^{\mathcal{B}})^{-1}(0)}$, so $\tilde s|_{\mathcal{U}}^{-1}(0) = (\tilde s^{\mathcal{A}})^{-1}(0) \cap (\tilde s^{\mathcal{B}})^{-1}(0)$ is a suborbifold of $\mathcal{U}$, and $\boldsymbol{\mathcal{S}}_{\mathcal{U},\mathcal{E}|_{\mathcal{U}},\tilde s|_{\mathcal{U}}} \simeq F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\tilde s|_{\mathcal{U}}^{-1}(0))$. Thus, $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde s}$ is an orbifold near any $[v] \in \mathcal{V}_{\rm top}$ with $s(v) = 0$. But $\tilde s^{-1}(0)$ is empty away from $s^{-1}(0)$, as $\tilde s - s$ is locally C^0-small. So $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde s}$ is an orbifold, proving the first part.

For the second part, if $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is effective then $\mathrm{Iso}_{\mathcal{X}}([v])$ acts effectively on $T_v^*\mathcal{X}$ in (10.25). Making $\mathcal{U}$ smaller if necessary, we can use this to show that $(\tilde s^{\mathcal{B}})^{-1}(0)$ and $\tilde s|_{\mathcal{U}}^{-1}(0)$ above are effective orbifolds, so $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde s}$ is effective near each $[v] \in s^{-1}(0)$, and thus is effective. □

Using good coordinate systems from §10.8, we now sketch an argument that if $\boldsymbol{\mathcal{X}}$ is a general semieffective (or effective) d-orbifold then a small, generic perturbation $\tilde{\boldsymbol{\mathcal{X}}}$ of $\boldsymbol{\mathcal{X}}$ is an (effective) orbifold. Here we are not being precise about what we mean by a 'small, generic perturbation' $\tilde{\boldsymbol{\mathcal{X}}}$ of $\boldsymbol{\mathcal{X}}$, though this will become clearer during the proof. A more rigorous version of the same argument will be used in §13.4 to prove (semi)effective d-orbifold bordism is isomorphic to (effective) orbifold bordism. The argument is modelled on that used by Fukaya et al. [32, Th. A1.23], [34, Th. 6.4] to construct virtual classes and virtual chains for Kuranishi spaces using good coordinate systems and multisections.

Let $\boldsymbol{\mathcal{X}}$ be a (semi)effective d-orbifold, and $\bigl(I,<,(\mathcal{V}_i,\mathcal{E}_i,s_i,\boldsymbol{\psi}_i),(\mathcal{V}_{ij},e_{ij},\hat e_{ij},\boldsymbol{\eta}_{ij}),\eta_{ijk}\bigr)$ be a very good coordinate system for $\boldsymbol{\mathcal{X}}$ in the sense of Definition 10.55, which exists by Theorem 10.56. By induction on $j\in I\subseteq\mathbb{N}$, we choose perturbations $\tilde s_j$ of s_j in $C^\infty(\mathcal{E}_j)$ satisfying the conditions:

(a) if $i<j$ in I with $\boldsymbol{\psi}_i(\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i})\cap\boldsymbol{\psi}_j(\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j})\neq\boldsymbol{\emptyset}$ then $e_{ij}^*(\tilde s_j)=\hat e_{ij}(\tilde s_i)$.

(b) $\tilde s_j$ is generic and locally C^1-small on $\mathcal{V}_j$ away from $\bigcup_{i\,<\,j\text{ in }I}e_{ij}(\mathcal{V}_{ij})$.

Here (a) determines $\tilde s_j$ on the suborbifold $e_{ij}(\mathcal{V}_{ij})$ in $\mathcal{V}_j$, for all such $i<j$. Definition 10.55(d),(e) ensure that the conditions on $\tilde s_j$ in (a) imposed by different $i,i'<j$ are consistent on $e_{ij}(\mathcal{V}_{ij})\cap e_{i'j}(\mathcal{V}_{i'j})$ in $\mathcal{V}_j$. Thus the inductive step is possible, and we can choose $\tilde s_j$ satisfying (a),(b) for all $j\in I$.

We then define a new d-orbifold $\tilde{\boldsymbol{\mathcal{X}}}$ by gluing together the d-orbifolds $\tilde{\boldsymbol{\mathcal{X}}}_i=\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,\tilde s_i}$ for $i\in I$ using Theorem 9.19, with $\boldsymbol{\mathcal{U}}_{ij}=\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},\tilde s_i|_{\mathcal{V}_{ij}}}$, $\boldsymbol{e}_{ij}=\boldsymbol{\mathcal{S}}_{e_{ij},\hat e_{ij}}:\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},\tilde s_i|_{\mathcal{V}_{ij}}}\to\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,\tilde s_j}$, and so on. Since the structures in a type B good coordinate system are nearly the same as the hypotheses of Theorem 9.19, we have exactly the data we need to do this.

Finally we claim that this perturbation $\tilde{\boldsymbol{\mathcal{X}}}$ of $\boldsymbol{\mathcal{X}}$ is an orbifold (and an effective orbifold if $\boldsymbol{\mathcal{X}}$ is effective). To see this, we prove by induction on $j\in I$ that each $\tilde{\boldsymbol{\mathcal{X}}}_j=\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,\tilde s_j}$ is an (effective) orbifold. Proposition 10.58 and (b) above imply that $\tilde{\boldsymbol{\mathcal{X}}}_j=\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,\tilde s_j}$ is an (effective) orbifold away from $\boldsymbol{e}_{ij}(\boldsymbol{\mathcal{U}}_{ij})$ for all $i<j$. But $\tilde{\boldsymbol{\mathcal{X}}}_j$ is equivalent on $\boldsymbol{e}_{ij}(\boldsymbol{\mathcal{U}}_{ij})$ to $\boldsymbol{\mathcal{U}}_{ij}\subset\tilde{\boldsymbol{\mathcal{X}}}_i$, which is an (effective) orbifold by an earlier inductive step. So $\tilde{\boldsymbol{\mathcal{X}}}_j$ is an (effective) orbifold for all j, and thus $\tilde{\boldsymbol{\mathcal{X}}}$ is an (effective) orbifold.

Example 10.42 describes a non-semieffective d-orbifold $\boldsymbol{\mathcal{X}}$ that cannot be perturbed to an orbifold by a small perturbation.

Remark 10.59. We have explained that a d-orbifold $\boldsymbol{\mathcal{X}}$ being semieffective (or effective) is a sufficient condition for a small, generic perturbation $\tilde{\boldsymbol{\mathcal{X}}}$ of $\boldsymbol{\mathcal{X}}$ to be an (effective) orbifold. However, it is not a necessary condition.

The necessary and sufficient condition can be described as follows. Let $[x]\in\mathcal{X}_{\rm top}$, the representations $K_{[x]},T_x^*\mathcal{X}$ of $\mathrm{Iso}_{\mathcal{X}}([x])$, and the splittings $K_{[x]}=K_{[x],\rm tr}\oplus K_{[x],\rm nt}$ and $T_x^*\mathcal{X}=(T_x^*\mathcal{X})_{\rm tr}\oplus(T_x^*\mathcal{X})_{\rm nt}$ be as in Definition 10.57. Write $\mathrm{Hom}(K_{[x],\rm nt},(T_x^*\mathcal{X})_{\rm nt})$ for the finite-dimensional vector space of morphisms of $\mathrm{Iso}_{\mathcal{X}}([x])$-representations $\lambda:K_{[x],\rm nt}\to(T_x^*\mathcal{X})_{\rm nt}$, and $\mathrm{Hom}_0(K_{[x],\rm nt},(T_x^*\mathcal{X})_{\rm nt})$ for the (generally singular) closed subset of such λ which are not injective.

Then small, generic perturbations of $\boldsymbol{\mathcal{X}}$ are orbifolds if and only if for all $[x]\in\mathcal{X}_{\rm top}$, either $K_{[x],\rm nt}=0$, or the codimension of $\mathrm{Hom}_0(K_{[x],\rm nt},(T_x^*\mathcal{X})_{\rm nt})$ in $\mathrm{Hom}(K_{[x],\rm nt},(T_x^*\mathcal{X})_{\rm nt})$ is strictly greater than $\dim(T_x^*\mathcal{X})_{\rm tr}-\dim K_{[x],\rm tr}$. Small, generic perturbations of $\boldsymbol{\mathcal{X}}$ are effective orbifolds if and only if this condition holds, and also for each $[x]\in\mathcal{X}_{\rm top}$, either $[(T_x^*\mathcal{X})_{\rm nt}]-[K_{[x],\rm nt}]=[R]$ in Λ^Γ for some effective representation R of $\mathrm{Iso}_{\mathcal{X}}([x])$, or $\dim(T_x^*\mathcal{X})_{\rm tr}<\dim K_{[x],\rm tr}$.

We could have adopted these more complicated conditions as our definition of (semi)effective d-orbifolds, and then a stronger version of Proposition

10.58 would hold, with $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde{s}}$ an (effective) orbifold if and only if $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde{s}}$ is a semieffective (effective) d-orbifold. However, Lemmas 10.61–10.63 and Propositions 10.64 and 10.65 below would be false for this alternative definition of (semi)effective d-orbifolds.

We now discuss other good properties of (semi)effective d-orbifolds. (Semi)-effectiveness is preserved by equivalences $\boldsymbol{i}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ in **dOrb**, as these induce isomorphisms $K_{[x]}\cong K_{[y]}$, $T_x^*\mathcal{X}\cong T_y^*\mathcal{Y}$ when $i_{\rm top}([x])=[y]$. If $\mathcal{X}$ is an orbifold and $\boldsymbol{\mathcal{X}}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{X})$ then $\mathcal{E}_{\mathcal{X}}=K_{[x]}=0$ in Definition 10.57. So we deduce:

Lemma 10.60. *Let $\mathcal{X}$ be an orbifold, and $\boldsymbol{\mathcal{X}}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{X})$. Then $\boldsymbol{\mathcal{X}}$ is a semieffective d-orbifold, and if $\mathcal{X}$ is effective then $\boldsymbol{\mathcal{X}}$ is effective.*

The last part of Definition 10.57 implies:

Lemma 10.61. *Let $\boldsymbol{\mathcal{X}}$ be an effective d-orbifold. Then the underlying C^∞-stack $\mathcal{X}$ is effective, in the sense of* §C.5.

The converse is false: if $\boldsymbol{\mathcal{X}}$ is a d-orbifold then $\mathcal{X}$ effective as a C^∞-stack does not imply $\boldsymbol{\mathcal{X}}$ is effective as a d-orbifold. We can combine Lemma 10.61 and Proposition C.29 to deduce uniqueness results for the C^∞-stack components of 2-morphisms of effective d-orbifolds.

Being (semi)effective constrains the orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\ldots,\hat{\boldsymbol{\mathcal{X}}}{}^{\Gamma,\mu}_{\circ}$ of $\boldsymbol{\mathcal{X}}$ from §10.7. The next lemma shows that for a semieffective (or effective) d-orbifold $\boldsymbol{\mathcal{X}}$, the same orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ vanish as would automatically vanish if $\boldsymbol{\mathcal{X}}$ were an orbifold (or effective orbifold, respectively). It implies vanishing for the other orbifold strata $\tilde{\boldsymbol{\mathcal{X}}}{}^{\Gamma,\mu}$, $\hat{\boldsymbol{\mathcal{X}}}{}^{\Gamma,\mu}$, $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_{\circ}$, $\tilde{\boldsymbol{\mathcal{X}}}{}^{\Gamma,\mu}_{\circ}$, $\hat{\boldsymbol{\mathcal{X}}}{}^{\Gamma,\mu}_{\circ}$ in the obvious way.

Lemma 10.62. *Let $\boldsymbol{\mathcal{X}}$ be a semieffective d-orbifold, Γ a finite group, and $\lambda\in\Lambda^\Gamma$. Then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}=\boldsymbol{\emptyset}$ unless $\lambda\in\Lambda^\Gamma_+\subset\Lambda^\Gamma$. If $\boldsymbol{\mathcal{X}}$ is effective then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}=\boldsymbol{\emptyset}$ unless $\lambda=[R]$ for R an effective Γ-representation.*

Proof. For any d-orbifold $\boldsymbol{\mathcal{X}}$, points of $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ are isomorphism classes $[x,\rho]$ as in §C.8, where $[x]\in\mathcal{X}_{\rm top}$ and $\rho:\Gamma\rightarrow{\rm Iso}_{\mathcal{X}}([x])$ is injective. Since $K_{[x]},T_x^*\mathcal{X}$ in Definition 10.9 are ${\rm Iso}_{\mathcal{X}}([x])$ representations, ρ makes them into Γ-representations. Thus they split $K_{[x]}=K_{[x],\rm tr}\oplus K_{[x],\rm nt}$ and $T_x^*\mathcal{X}=(T_x^*\mathcal{X})_{\rm tr}\oplus(T_x^*\mathcal{X})_{\rm nt}$ into trivial and nontrivial Γ-representations. It follows from the definition of $\boldsymbol{\mathcal{X}}^\Gamma$ in §10.7 that $\lambda=[(T_x^*\mathcal{X})_{\rm nt}]-[K_{[x],\rm nt}]\in\Lambda^\Gamma$. If $\boldsymbol{\mathcal{X}}$ is semieffective then $K_{[x]}$ is a trivial representation, so $K_{[x],\rm nt}=0$ and $\lambda=[(T_x^*\mathcal{X})_{\rm nt}]\in\Lambda^\Gamma_+$. If $\boldsymbol{\mathcal{X}}$ is effective then $T_x^*\mathcal{X}$ is an effective ${\rm Iso}_{\mathcal{X}}([x])$- and Γ-representation, so $\lambda=[R]$ for $R=(T_x^*\mathcal{X})_{\rm nt}$ an effective Γ-representation. The lemma follows. $\square$

Again, the converse is false: $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}=\boldsymbol{\emptyset}$ for $\lambda\in\Lambda^\Gamma\setminus\Lambda^\Gamma_+$ does not imply a d-orbifold $\boldsymbol{\mathcal{X}}$ is semieffective, and similarly for effective d-orbifolds. Using the explicit construction of fibre products in **dSta** in §9.5 one can prove:

Lemma 10.63. *If $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}$ are (semi)effective d-orbifolds, then the product $\boldsymbol{\mathcal{X}}\times\boldsymbol{\mathcal{Y}}$ is also (semi)effective. More generally, any fibre product $\boldsymbol{\mathcal{X}}\times_{\boldsymbol{\mathcal{Z}}}\boldsymbol{\mathcal{Y}}$ in* **dOrb** *with $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}$ (semi)effective and $\boldsymbol{\mathcal{Z}}$ a manifold is also (semi)effective.*

In §8.4.2 we proved in Propositions 8.9 and 8.10 that if $\mathcal{X}$ is an oriented orbifold, then the orbifold strata $\mathcal{X}^{\Gamma,\lambda},\ldots,\hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ can be oriented under conditions on Γ,λ,μ, which allow $|\Gamma|$ even provided $\lambda\in\Lambda^\Gamma_{\rm ev,+}$. But in §10.7 we showed in Propositions 10.43 and 10.44 that if $\boldsymbol{\mathcal{X}}$ is an oriented d-orbifold, then the orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\ldots,\hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ can in general be oriented under stronger conditions on Γ,λ,μ, which require $|\Gamma|$ odd.

However, if $\boldsymbol{\mathcal{X}}$ is a semieffective d-orbifold, it turns out that $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\ldots,\hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ can be oriented under the weaker conditions on Γ,λ,μ of Propositions 8.9 and 8.10. The reason for this is that in the proof of Proposition 10.43, the virtual vector bundle $(T^*\boldsymbol{\mathcal{X}})^{\Gamma,\lambda}_{\rm nt}$ is a morphism $(\phi_{\mathcal{X}})^{\Gamma,\lambda}_{\rm nt}:(\mathcal{E}_{\mathcal{X}})^{\Gamma,\lambda}_{\rm nt}\rightarrow(\mathcal{F}_{\mathcal{X}})^{\Gamma,\lambda}_{\rm nt}$, and Definition 10.57 implies that the kernel of $(\phi_{\mathcal{X}})^{\Gamma,\lambda}_{\rm nt}$ is zero at each point $[x]\in\mathcal{X}^{\Gamma,\lambda}_{\rm top}$. This is enough to force $(T^*\boldsymbol{\mathcal{X}})^{\Gamma,\lambda}_{\rm nt}$ to be a vector bundle in $\mathrm{vvect}(\mathcal{X}^{\Gamma,\lambda})$, in the sense of Definition 3.8, and $\mathrm{Coker}(\phi_{\mathcal{X}})^{\Gamma,\lambda}_{\rm nt}$ lies in $\mathrm{vect}(\mathcal{X}^{\Gamma,\lambda})$. Then as in Proposition 8.9, $\mathrm{Coker}(\phi_{\mathcal{X}})^{\Gamma,\lambda}_{\rm nt}$ and $(T^*\boldsymbol{\mathcal{X}})^{\Gamma,\lambda}_{\rm nt}$ are oriented provided $\lambda\in\Lambda^\Gamma_{\rm ev,+}$. So we obtain the following analogues of Propositions 8.9 and 8.10:

Proposition 10.64. *Let $\boldsymbol{\mathcal{X}}$ be an oriented, semieffective d-orbifold, and Γ a finite group. Then we can define orientations on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ$ for all $\lambda\in\Lambda^\Gamma_{\rm ev,+}$. These depend on orientations on $R_1,\ldots,R_k$ for representatives $(R_1,\rho_1),\ldots,(R_k,\rho_k)$ of the nontrivial, irreducible, even-dimensional Γ-representations.*

Proposition 10.65. *Let Γ be a finite group and $\lambda\in\Lambda^\Gamma_{\rm ev,+}$ with $\Phi^\Gamma(\delta,\lambda)=1$ for all $\delta\in\mathrm{Aut}(\Gamma)$ with $\lambda\cdot\delta=\lambda$. Set $\mu=\lambda\cdot\mathrm{Aut}(\Gamma)$ in $\Lambda^\Gamma_{\rm ev,+}/\mathrm{Aut}(\Gamma)$. Then $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$, $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ,\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ are oriented for all oriented, semieffective d-orbifolds $\boldsymbol{\mathcal{X}}$.*

11 D-stacks with corners

We now study *d-stacks with corners*. Most of the chapter works by combining ideas on d-spaces with corners from Chapter 6, and on orbifolds with corners from §8.5–§8.9, and on d-stacks from Chapter 9; there are few new issues.

11.1 The definition of d-stacks with corners

Definitions 11.1–11.3 will define objects and 1- and 2-morphisms in the 2-category of *d-stacks with corners* $\mathbf{dSta^c}$. Broadly, we just replace d-spaces and C^∞-schemes in §6.1 by d-stacks and C^∞-stacks, but our definition also includes features of the definition of orbifolds with corners in §8.5.

Definition 11.1. A *d-stack with corners* is a quadruple $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i_{\mathfrak{X}}}, \omega_{\mathfrak{X}})$, where $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$ and $\boldsymbol{\partial\mathcal{X}} = (\partial\mathcal{X}, \mathcal{O}'_{\partial\mathcal{X}}, \mathcal{E}_{\partial\mathcal{X}}, \imath_{\partial\mathcal{X}}, \jmath_{\partial\mathcal{X}})$ are d-stacks, and $\boldsymbol{i_{\mathfrak{X}}} = (i_{\mathfrak{X}}, i'_{\mathfrak{X}}, i''_{\mathfrak{X}}) : \boldsymbol{\partial\mathcal{X}} \to \boldsymbol{\mathcal{X}}$ is a 1-morphism of d-stacks, and $\omega_{\mathfrak{X}}$ is defined in (c) below, satisfying the following conditions (a)–(d):

(a) $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is a proper, strongly representable 1-morphism of Deligne–Mumford C^∞-stacks, as in §C.3.

(b) $i''_{\mathfrak{X}} : i^*_{\mathfrak{X}}(\mathcal{E}_{\mathcal{X}}) \to \mathcal{E}_{\partial\mathcal{X}}$ is an isomorphism in $\mathrm{qcoh}(\partial\mathcal{X})$.

(c) Define $\mathcal{N}_{\mathfrak{X}}, \nu_{\mathfrak{X}}$ so that $\nu_{\mathfrak{X}} : \mathcal{N}_{\mathfrak{X}} \to i^*_{\mathfrak{X}}(\mathcal{F}_{\mathcal{X}})$ is the kernel of $i^2_{\mathfrak{X}}$ in $\mathrm{qcoh}(\partial\mathcal{X})$. We call $\mathcal{N}_{\mathfrak{X}}$ the *conormal bundle* of $\boldsymbol{\partial\mathcal{X}}$ in $\boldsymbol{\mathcal{X}}$. Then we have a complex:

$$0 \longrightarrow \mathcal{N}_{\mathfrak{X}} \xrightarrow{\nu_{\mathfrak{X}}} i^*_{\mathfrak{X}}(\mathcal{F}_{\mathcal{X}}) \xrightarrow{i^2_{\mathfrak{X}}} \mathcal{F}_{\partial\mathcal{X}} \longrightarrow 0. \tag{11.1}$$

We require that $\mathcal{N}_{\mathfrak{X}}$ is a trivializable line bundle on $\partial\mathcal{X}$, and $\omega_{\mathfrak{X}}$ is an orientation on $\mathcal{N}_{\mathfrak{X}}$.

(d) Suppose $\underline{U}$ is a separated, second countable C^∞-scheme, and $u : \bar{\underline{U}} \to \mathcal{X}$ is an étale 1-morphism. For instance, $u : \bar{\underline{U}} \to \mathcal{X}$ could be an atlas for $\mathcal{X}$. Then as $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is representable by part (a) and Proposition C.14(a), we can choose a C^∞-scheme $\underline{\partial U}$ unique up to isomorphism, a morphism $\underline{i}_{\mathbf{U}} : \underline{\partial U} \to \underline{U}$, an étale 1-morphism $u_\partial : \bar{\underline{\partial U}} \to \partial\mathcal{X}$ in a 2-Cartesian diagram in $\mathbf{C^\infty Sta}$, as for (8.6):

$$\begin{tikzcd} \bar{\underline{\partial U}} \arrow[rr, "u_\partial"] \arrow[d, "\bar{\underline{i}}_{\mathbf{U}}"'] & \arrow[d, phantom, "\mathrm{id}\Uparrow"] & \partial\mathcal{X} \arrow[d, "i_{\mathfrak{X}}"] \\ \bar{\underline{U}} \arrow[rr, "u"] & & \mathcal{X}. \end{tikzcd} \tag{11.2}$$

Note that as $i_{\mathfrak{X}}$ is strongly representable, Proposition C.13 allows us to choose the 2-morphism in (11.2) to be the identity.

Recall the definition of sheaves on $\mathcal{X}$ and $\partial\mathcal{X}$ in §C.6. We have $(\underline{U}, u) \in \mathcal{C}_{\mathcal{X}}$ and $(\underline{\partial U}, u_\partial) \in \mathcal{C}_{\partial\mathcal{X}}$, with $\underline{U} = (U, \mathcal{O}_U)$. Hence $\mathcal{O}'_U := \mathcal{O}'_{\mathcal{X}}(\underline{U}, u)$ is a sheaf of C^∞-rings on U, and $\mathcal{E}_U := \mathcal{E}_{\mathcal{X}}(\underline{U}, u)$ is a quasicoherent sheaf on $\underline{U}$, and

$\imath_U : \mathcal{O}'_U = \mathcal{O}'_{\mathcal{X}}(\underline{U}, u) \to \mathcal{O}_{\mathcal{X}}(\underline{U}, u) = \mathcal{O}_U$ is a morphism of sheaves of C^∞-rings on U which is a square zero extension with kernel $\mathcal{I}_U := \mathcal{I}_{\mathcal{X}}(\underline{U}, u)$, and $\jmath_U : \mathcal{E}_U = \mathcal{E}_{\mathcal{X}}(\underline{U}, u) \to \mathcal{I}_{\mathcal{X}}(\underline{U}, u) = \mathcal{I}_U$ is surjective in qcoh$(\underline{U})$.

Comparing the definitions of d-spaces and d-stacks now shows that $\boldsymbol{U} = (\underline{U}, \mathcal{O}'_U, \mathcal{E}_U, \imath_U, \jmath_U)$ is a d-space; we put the assumptions that $\underline{U}$ is separated and second countable in for this reason. Similarly, $\boldsymbol{\partial U} = (\underline{\partial U}, \mathcal{O}'_{\partial U}, \mathcal{E}_{\partial U}, \imath_{\partial U}, \jmath_{\partial U})$ is a d-space, where $\mathcal{O}'_{\partial U} = \mathcal{O}'_{\partial\mathcal{X}}(\underline{\partial U}, u_\partial)$, $\mathcal{E}_{\partial U} = \mathcal{E}_{\partial\mathcal{X}}(\underline{\partial U}, u_\partial)$, $\imath_{\partial U} = \imath_{\partial\mathcal{X}}(\underline{\partial U}, u_\partial)$, $\jmath_{\partial U} = \jmath_{\partial\mathcal{X}}(\underline{\partial U}, u_\partial)$.

Using Definitions C.30 and C.36 we can construct a natural isomorphism $\alpha_{\mathcal{E}_{\mathcal{X}}, \underline{U}} : \underline{i}^*_{\mathbf{U}}(\mathcal{E}_U) = \underline{i}^*_{\mathbf{U}}(\mathcal{E}_{\mathcal{X}}(\underline{U}, u)) \to i^*_{\mathfrak{X}}(\mathcal{E}_{\mathcal{X}})(\underline{\partial U}, u_\partial)$ in qcoh$(\underline{\partial U})$. Also $i''_{\mathfrak{X}} : i^*_{\mathfrak{X}}(\mathcal{E}_{\mathcal{X}}) \to \mathcal{E}_{\partial\mathcal{X}}$ induces $i''_{\mathfrak{X}}(\underline{\partial U}, u_\partial) : i^*_{\mathfrak{X}}(\mathcal{E}_{\mathcal{X}})(\underline{\partial U}, u_\partial) \to \mathcal{E}_{\partial\mathcal{X}}(\underline{\partial U}, u_\partial) = \mathcal{E}_{\partial U}$, so composing gives a morphism $i''_{\mathbf{U}} := i''_{\mathfrak{X}}(\underline{\partial U}, u_\partial) \circ \alpha_{\mathcal{E}_{\mathcal{X}}, \underline{U}} : \underline{i}^*_{\mathbf{U}}(\mathcal{E}_U) \to \mathcal{E}_{\partial U}$. Similarly we set $i'_{\mathbf{U}} := i'_{\mathfrak{X}}(\underline{\partial U}, u_\partial) \circ \alpha_{\mathcal{O}'_{\mathcal{X}}, \underline{U}} : i^{-1}_{\mathbf{U}}(\mathcal{O}'_U) \to \mathcal{O}'_{\partial U}$. Then $\boldsymbol{i}_{\mathbf{U}} = (\underline{i}_{\mathbf{U}}, i'_{\mathbf{U}}, i''_{\mathbf{U}})$ is a d-space 1-morphism $\boldsymbol{i}_{\mathbf{U}} : \boldsymbol{U} \to \boldsymbol{\partial U}$.

We now have a commutative diagram with exact rows in qcoh$(\underline{\partial U})$:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{N}_{\mathfrak{X}}(\underline{\partial U}, u_\partial) \arrow[r, "\nu_{\mathfrak{X}}(\underline{\partial U}, u_\partial)"] \arrow[d, "\cong"] & i^*_{\mathfrak{X}}(\mathcal{F}_{\mathcal{X}})(\underline{\partial U}, u_\partial) \arrow[r, "i^2_{\mathfrak{X}}(\underline{\partial U}, u_\partial)"] \arrow[d, "\cong"] & \mathcal{F}_{\partial\mathcal{X}}(\underline{\partial U}, u_\partial) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathcal{N}_{\mathbf{U}} \arrow[r, "\nu_{\mathbf{U}}"] & i^*_{\mathbf{U}}(\mathcal{F}_U) \arrow[r, "i^2_{\mathbf{U}}"] & \mathcal{F}_{\partial U} \arrow[r] & 0, \end{tikzcd} \tag{11.3}$$

where the top row is (11.1) evaluated at $(\underline{\partial U}, u_\partial) \in \mathcal{C}_{\partial\mathcal{X}}$, and $\mathcal{N}_{\mathbf{U}}, \nu_{\mathbf{U}}$ are defined to be the kernel of $i^2_{\mathbf{U}}$. Part (c) now implies $\mathcal{N}_{\mathbf{U}}$ is a line bundle on $\underline{\partial U}$, with a unique orientation $\omega_{\mathbf{U}}$ identified with $\omega_{\mathfrak{X}}(\underline{\partial U}, u_\partial)$ by (11.3). We require that $\mathbf{U} = (\boldsymbol{U}, \boldsymbol{\partial U}, \boldsymbol{i}_{\mathbf{U}}, \omega_{\mathbf{U}})$ is a d-space with corners.

We call $\mathfrak{X}$ a *d-stack with boundary* if $i_{\mathfrak{X}} : \partial\mathcal{X} \to \mathcal{X}$ is injective as a representable 1-morphism of C^∞-stacks, that is, if $i_{\mathfrak{X},\text{top}} : \partial\mathcal{X}_{\text{top}} \to \mathcal{X}_{\text{top}}$ is injective, and all the induced morphisms on orbifold groups $(i_{\mathfrak{X}})_* : \text{Iso}_{\partial\mathcal{X}}([x']) \to \text{Iso}_{\mathcal{X}}([x])$ are isomorphisms. We call $\mathfrak{X}$ a *d-stack without boundary* if $\boldsymbol{\partial\mathcal{X}} = \boldsymbol{\emptyset}$.

Properties of d-spaces with corners proved in Chapter 6 now hold for each $\mathbf{U}$ in (d), so provided the properties are étale local we can deduce the corresponding facts for $\mathfrak{X}$, noting that the allowed $\underline{U}, \underline{\partial U}$ in (d) form étale open covers of $\mathcal{X}, \partial\mathcal{X}$. For example, as (6.4) for $\mathbf{U}$ is split exact, and as in Lemma 2.22 being split exact is an étale local condition on the separated, paracompact, locally fair Deligne–Mumford C^∞-stack $\partial\mathcal{X}$, so equation (11.1) is split exact. Similarly, (6.5) exact for each $\mathbf{U}$ implies the following sequence in qcoh$(\partial\mathcal{X})$ is exact:

$$\mathcal{N}_{\mathfrak{X}} \xrightarrow{i^*_{\mathfrak{X}}(\psi_X) \circ \nu_{\mathfrak{X}}} i^*_{\mathfrak{X}}(T^*\mathcal{X}) \xrightarrow{\Omega_{i_{\mathfrak{X}}}} T^*(\partial\mathcal{X}) \longrightarrow 0. \tag{11.4}$$

If part (d) holds for a collection $\{(\underline{U}_i, u_i) : i \in I\}$ of pairs $(\underline{U}, u)$ forming an étale open cover of $\mathcal{X}$, then it holds for all such $(\underline{U}, u)$. This is because the $\mathbf{U}$ constructed from $(\underline{U}, u)$ is covered by (Zariski) open subsets 1-isomorphic to open subsets in the d-spaces with corners $\mathbf{U}_i$ for $i \in I$, and apart from the separated, second countable conditions on $\underline{U}$ that we impose by hand, for $\mathbf{U} = (\boldsymbol{U}, \boldsymbol{\partial U}, \boldsymbol{i}_{\mathbf{U}}, \omega_{\mathbf{U}})$ to be a d-space with corners is a (Zariski) local condition.

Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ be a d-stack with corners. Suppose $\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{X}}$ is an open d-substack in **dSta**. Define $\boldsymbol{\partial\mathcal{V}} = \boldsymbol{i}_{\mathfrak{X}}^{-1}(\boldsymbol{\mathcal{V}})$, as an open d-substack of $\boldsymbol{\partial\mathcal{X}}$, and $\boldsymbol{i}_{\mathcal{V}} : \boldsymbol{\partial\mathcal{V}} \to \boldsymbol{\mathcal{V}}$ by $\boldsymbol{i}_{\mathcal{V}} = \boldsymbol{i}_{\mathfrak{X}}|_{\boldsymbol{\partial\mathcal{V}}}$. Then $\partial\mathcal{V} \subseteq \partial\mathcal{X}$ is an open C^∞-substack, and the conormal bundle of $\boldsymbol{\partial\mathcal{V}}$ in $\boldsymbol{\mathcal{V}}$ is $\mathcal{N}_{\mathcal{V}} = \mathcal{N}_{\mathfrak{X}}|_{\partial\mathcal{V}}$ in $\mathrm{qcoh}(\partial\mathcal{V})$. Define an orientation $\omega_{\mathcal{V}}$ on $\mathcal{N}_{\mathcal{V}}$ by $\omega_{\mathcal{V}} = \omega_{\mathfrak{X}}|_{\partial\mathcal{V}}$. Write $\mathfrak{V} = (\boldsymbol{\mathcal{V}}, \boldsymbol{\partial\mathcal{V}}, \boldsymbol{i}_{\mathcal{V}}, \omega_{\mathcal{V}})$. Then $\mathfrak{V}$ is a d-stack with corners. We call $\mathfrak{V}$ an *open d-substack* of $\mathfrak{X}$. If $\boldsymbol{\mathcal{V}}$ is open and closed in $\boldsymbol{\mathcal{X}}$ we call $\mathfrak{V}$ an *open and closed d-substack* of $\mathfrak{X}$. An *open cover* of $\mathfrak{X}$ is a family $\{\mathfrak{V}_a : a \in A\}$ of open d-substacks $\mathfrak{V}_a$ of $\mathfrak{X}$ with $\mathcal{X} = \bigcup_{a\in A} \mathcal{V}_a$.

Definition 11.2. Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ and $\mathfrak{Y} = (\boldsymbol{\mathcal{Y}}, \boldsymbol{\partial\mathcal{Y}}, \boldsymbol{i}_{\mathfrak{Y}}, \omega_{\mathfrak{Y}})$ be d-stacks with corners. A 1-*morphism* $\boldsymbol{f} : \mathfrak{X} \to \mathfrak{Y}$ *of d-stacks with corners* is a 1-morphism $\boldsymbol{f} = (f, f', f'') : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in **dSta** satisfying the following condition:

(∗) Suppose we are given a 2-commutative diagram in $\mathbf{C^\infty Sta}$:

$$\begin{tikzcd} \bar{\underline{U}} \arrow[rr, "u"] \arrow[d, "\bar{\underline{h}}"] & \arrow[d, phantom, "\Downarrow \eta"] & \mathcal{X} \arrow[d, "f"'] \\ \bar{\underline{V}} \arrow[rr, "v"] & & \mathcal{Y}, \end{tikzcd} \tag{11.5}$$

as for (8.8), where $\underline{U}, \underline{V}$ are separated, second countable C^∞-schemes, $\underline{h} : \underline{U} \to \underline{V}$ is a morphism, and $u : \bar{\underline{U}} \to \mathcal{X}$, $v : \bar{\underline{V}} \to \mathcal{Y}$ are étale 1-morphisms. Then Definition 11.1(d) extends $\underline{U}, \underline{V}$ to d-spaces with corners $\mathbf{U}, \mathbf{V}$. As for $\boldsymbol{i}_{\mathbf{U}}$ in Definition 11.1(d), using (11.5) we can extend $\underline{h} : \underline{U} \to \underline{V}$ naturally to 1-morphism $\boldsymbol{h} = (\underline{h}, h', h'') : \boldsymbol{U} \to \boldsymbol{V}$ in **dSpa**, using diagrams (11.2) for $\underline{U}, u$ with 1-morphism u_∂, and for $\underline{V}, v$ with v_∂. We require that $\boldsymbol{h}$ should be a 1-morphism $\boldsymbol{h} : \mathbf{U} \to \mathbf{V}$ in $\mathbf{dSpa^c}$, that is, it should satisfy the conditions of Definition 6.2, for all such diagrams (11.5).

If (∗) holds for each of a collection $\{(\underline{U}_i, u_i, \underline{V}_i, v_i, \underline{h}_i) : i \in I\}$ such that $\{(\underline{U}_i, u_i) : i \in I\}$ are an étale open cover of $\mathcal{X}$, then (∗) holds for all such $(\underline{U}, u, \underline{V}, v, \underline{h})$, by the same argument as for Definition 11.1(d). Properties of 1-morphisms in $\mathbf{dSpa^c}$ proved in Chapter 6 now hold for each such $\boldsymbol{h}$, so provided they are étale local in $\mathcal{X}, \mathcal{Y}$, we can deduce the corresponding properties for $\boldsymbol{f}$.

If $\boldsymbol{f} : \mathfrak{X} \to \mathfrak{Y}$, $\boldsymbol{g} : \mathfrak{Y} \to \mathfrak{Z}$ are 1-morphisms of d-stacks with corners then we define the *composition* $\boldsymbol{g} \circ \boldsymbol{f} : \mathfrak{X} \to \mathfrak{Z}$ to be the composition $\boldsymbol{g} \circ \boldsymbol{f}$ of 1-morphisms in **dSta**. Since $\boldsymbol{f}, \boldsymbol{g}$ satisfy (∗), we see that $\boldsymbol{g} \circ \boldsymbol{f}$ satisfies (∗), since $\boldsymbol{g} \circ \boldsymbol{f}$ is étale locally modelled on compositions of 1-morphisms in $\mathbf{dSpa^c}$ lifting $\boldsymbol{f}, \boldsymbol{g}$, and 1-morphisms in $\mathbf{dSpa^c}$ are closed under composition.

If $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ is a d-stack with corners, we define the *identity* 1-*morphism* $\mathbf{id}_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X}$ to be the identity 1-morphism $\mathrm{id}_{\boldsymbol{\mathcal{X}}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}$ in **dSta**.

Consider the C^∞-stack fibre products $\partial\mathcal{X} \times_{f\circ i_{\mathcal{X}}, \mathcal{Y}, i_{\mathcal{Y}}} \partial\mathcal{Y}$ and $\mathcal{X} \times_{f,\mathcal{Y},i_{\mathcal{Y}}} \partial\mathcal{Y}$. Since $i_{\mathcal{Y}}$ is strongly representable, we may define these using the explicit construction of Proposition C.15, and then we have 2-Cartesian diagrams

$$\begin{tikzcd} \partial\mathcal{X} \times_{f\circ i_{\mathcal{X}}, \mathcal{Y}, i_{\mathcal{Y}}} \partial\mathcal{Y} \arrow[r, "\pi_{\partial\mathcal{Y}}"] \arrow[d, "\pi_{\partial\mathcal{X}}"] \arrow[dr, phantom, "\mathrm{id} \Nearrow"] & \partial\mathcal{Y} \arrow[d, "i_{\mathcal{Y}}"'] \\ \partial\mathcal{X} \arrow[r, "f\circ i_{\mathcal{X}}"] & \mathcal{Y}, \end{tikzcd} \quad \begin{tikzcd} \mathcal{X} \times_{f,\mathcal{Y},i_{\mathcal{Y}}} \partial\mathcal{Y} \arrow[r, "\pi_{\partial\mathcal{Y}}"] \arrow[d, "\pi_{\mathcal{X}}"] \arrow[dr, phantom, "\mathrm{id} \Nearrow"] & \partial\mathcal{Y} \arrow[d, "i_{\mathcal{Y}}"'] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y}, \end{tikzcd} \tag{11.6}$$

as for (8.10), with $\pi_{\partial\mathcal{X}}, \pi_{\mathcal{X}}$ strongly representable.

Following Definition 8.18 we define $\Pi_f : \partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y} \to \mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$ strongly representable with $i_{\boldsymbol{\mathcal{X}}} \circ \pi_{\partial\mathcal{X}} = \pi_{\mathcal{X}} \circ \Pi_f$, $\pi_{\partial\mathcal{Y}} = \pi_{\partial\mathcal{Y}} \circ \Pi_f$, and unique open C^∞-substacks $\mathcal{S}_{\boldsymbol{f}} \subseteq \partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$ and $\mathcal{T}_{\boldsymbol{f}} \subseteq \mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$, which are characterized by the property that in $(*)$ we get étale projections $a : (\underline{\partial U} \times_{\underline{V}} \underline{\partial V}) \to \partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$ and $b : (\underline{U} \times_{\underline{f},\underline{V},\underline{i}_{\boldsymbol{V}}} \underline{\partial V}) \to \mathcal{X} \times_{f,\mathcal{Y},i_{\boldsymbol{\mathcal{Y}}}} \partial\mathcal{Y}$ with $a^{-1}(\mathcal{S}_{\boldsymbol{f}}) = \underline{\bar{S}}_{\boldsymbol{h}}$ and $b^{-1}(\mathcal{T}_{\boldsymbol{f}}) = \underline{\bar{T}}_{\boldsymbol{h}}$, for $\underline{S}_{\boldsymbol{h}} \subseteq \underline{\partial U} \times_{\underline{V}} \underline{\partial V}$ and $\underline{T}_{\boldsymbol{h}} \subseteq \underline{U} \times_{\underline{V}} \underline{\partial V}$ as in Definition 6.2.

As for (8.11)–(8.16) we can describe the sets $(\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\mathrm{top}}, (\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\mathrm{top}}$ and subsets $\mathcal{S}_{\boldsymbol{f},\mathrm{top}} \subseteq (\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\mathrm{top}}$, $\mathcal{T}_{\boldsymbol{f},\mathrm{top}} \subseteq (\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\mathrm{top}}$. We have

$$\begin{aligned}(\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\mathrm{top}} \cong \bigl\{[x', y'] : {} & x' : \underline{\bar{*}} \to \partial\mathcal{X} \text{ and } y' : \underline{\bar{*}} \to \partial\mathcal{Y} \text{ are} \\ & \text{1-morphisms with } f \circ i_{\boldsymbol{\mathcal{X}}} \circ x' = i_{\boldsymbol{\mathcal{Y}}} \circ y' : \underline{\bar{*}} \to \mathcal{Y}\bigr\},\end{aligned} \tag{11.7}$$

$$\begin{aligned}(\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\mathrm{top}} \cong \bigl\{[x, y'] : {} & x : \underline{\bar{*}} \to \mathcal{X} \text{ and } y' : \underline{\bar{*}} \to \partial\mathcal{Y} \text{ are} \\ & \text{1-morphisms with } f \circ x = i_{\boldsymbol{\mathcal{Y}}} \circ y' : \underline{\bar{*}} \to \mathcal{Y}\bigr\},\end{aligned} \tag{11.8}$$

where $[x', y'], [x, y']$ are equivalence classes of $(x', y'), (x, y')$ as in §8.5. Then

$$\begin{aligned}\mathcal{S}_{\boldsymbol{f},\mathrm{top}} \cong \bigl\{[x', y'] : {} & x' : \underline{\bar{*}} \to \partial\mathcal{X},\ y' : \underline{\bar{*}} \to \partial\mathcal{Y} \text{ with } f \circ i_{\boldsymbol{\mathcal{X}}} \circ x' = i_{\boldsymbol{\mathcal{Y}}} \circ y', \\ & (i_{\boldsymbol{\mathcal{X}}} \circ x')^*(f^2) \circ I_{i_{\boldsymbol{\mathcal{X}}} \circ x', f}(\mathcal{F}_{\mathcal{Y}}) \circ I_{y', i_{\boldsymbol{\mathcal{Y}}}}(\mathcal{F}_{\mathcal{Y}})^{-1} \circ (y')^*(\nu_{\boldsymbol{\mathcal{Y}}}) \neq 0, \\ & \text{and } (x')^*(i^2_{\boldsymbol{\mathcal{X}}}) \circ I_{x', i_{\boldsymbol{\mathcal{X}}}}(\mathcal{F}_{\mathcal{X}}) \circ (i_{\boldsymbol{\mathcal{X}}} \circ x')^*(f^2) \circ \\ & I_{i_{\boldsymbol{\mathcal{X}}} \circ x', f}(\mathcal{F}_{\mathcal{Y}}) \circ I_{y', i_{\boldsymbol{\mathcal{Y}}}}(\mathcal{F}_{\mathcal{Y}})^{-1} \circ (y')^*(\nu_{\boldsymbol{\mathcal{Y}}}) = 0\bigr\},\end{aligned} \tag{11.9}$$

$$\begin{aligned}\mathcal{T}_{\boldsymbol{f},\mathrm{top}} \cong \bigl\{[x, y'] : {} & x : \underline{\bar{*}} \to \mathcal{X},\ y' : \underline{\bar{*}} \to \partial\mathcal{Y} \text{ with } f \circ x = i_{\boldsymbol{\mathcal{Y}}} \circ y', \text{ and} \\ & x^*(f^2) \circ I_{x,f}(\mathcal{F}_{\mathcal{Y}}) \circ I_{y', i_{\boldsymbol{\mathcal{Y}}}}(\mathcal{F}_{\mathcal{Y}})^{-1} \circ (y')^*(\nu_{\boldsymbol{\mathcal{Y}}}) = 0\bigr\}.\end{aligned} \tag{11.10}$$

Here the morphisms in (11.9)–(11.10) are given in

$$\begin{tikzcd}
0 \arrow[r] & (y')^*(\mathcal{N}_{\boldsymbol{\mathcal{Y}}}) \arrow[r, "(y')^*(\nu_{\boldsymbol{\mathcal{Y}}})"] \arrow[d, dotted, "\cong"'] & (y')^* \circ i^*_{\boldsymbol{\mathcal{Y}}}(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "(y')^*(i^2_{\boldsymbol{\mathcal{Y}}})"] \arrow[d, "{\begin{array}{l} I_{x', i_{\boldsymbol{\mathcal{X}}}}(\mathcal{F}_{\mathcal{X}}) \circ (i_{\boldsymbol{\mathcal{X}}} \circ x')^*(f^2) \circ \\ I_{i_{\boldsymbol{\mathcal{X}}} \circ x', f}(\mathcal{F}_{\mathcal{Y}}) \circ I_{y', i_{\boldsymbol{\mathcal{Y}}}}(\mathcal{F}_{\mathcal{Y}})^{-1} \end{array}}"] & (y')^*(\mathcal{F}_{\partial\mathcal{Y}}) \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & (x')^*(\mathcal{N}_{\boldsymbol{\mathcal{X}}}) \arrow[r, "(x')^*(\nu_{\boldsymbol{\mathcal{X}}})"'] & (x')^* \circ i^*_{\boldsymbol{\mathcal{X}}}(\mathcal{F}_{\mathcal{X}}) \arrow[r, "(x')^*(i^2_{\boldsymbol{\mathcal{X}}})"'] & (x')^*(\mathcal{F}_{\partial\mathcal{X}}) \arrow[r] & 0,
\end{tikzcd} \tag{11.11}$$

$$\begin{tikzcd}
0 \arrow[r] & (y')^*(\mathcal{N}_{\boldsymbol{\mathcal{Y}}}) \arrow[r, "(y')^*(\nu_{\boldsymbol{\mathcal{Y}}})"'] & (y')^* \circ i^*_{\boldsymbol{\mathcal{Y}}}(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "(y')^*(i^2_{\boldsymbol{\mathcal{Y}}})"'] \arrow[d, "x^*(f^2) \circ I_{x,f}(\mathcal{F}_{\mathcal{Y}}) \circ I_{y', i_{\boldsymbol{\mathcal{Y}}}}(\mathcal{F}_{\mathcal{Y}})^{-1}"'] & (y')^*(\mathcal{F}_{\partial\mathcal{Y}}) \arrow[r] \arrow[dl, dotted] & 0 \\
& & x^*(\mathcal{F}_{\mathcal{X}}). & &
\end{tikzcd} \tag{11.12}$$

The conditions in (11.9),(11.10) are equivalent to the existence of morphisms '$\dashrightarrow$' in (11.11),(11.12) respectively making (11.11)–(11.12) commute.

Define $s_{\boldsymbol{f}} = \pi_{\partial\mathcal{X}}|_{\mathcal{S}_{\boldsymbol{f}}} : \mathcal{S}_{\boldsymbol{f}} \to \partial\mathcal{X}$, $u_{\boldsymbol{f}} = \pi_{\partial\mathcal{Y}}|_{\mathcal{S}_{\boldsymbol{f}}} : \mathcal{S}_{\boldsymbol{f}} \to \partial\mathcal{Y}$, $t_{\boldsymbol{f}} = \pi_{\mathcal{X}}|_{\mathcal{T}_{\boldsymbol{f}}} : \mathcal{T}_{\boldsymbol{f}} \to \mathcal{X}$ and $v_{\boldsymbol{f}} = \pi_{\partial\mathcal{Y}}|_{\mathcal{T}_{\boldsymbol{f}}} : \mathcal{T}_{\boldsymbol{f}} \to \partial\mathcal{Y}$, and $j_{\boldsymbol{f}} = \Pi_f|_{\mathcal{S}_{\boldsymbol{f}}} : \mathcal{S}_{\boldsymbol{f}} \to \mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$. Then $s_{\boldsymbol{f}}, t_{\boldsymbol{f}}$ are strongly representable by Proposition C.14(c), as $\pi_{\partial\mathcal{X}}, \pi_{\mathcal{X}}$ are. Also (11.6) and $i_{\boldsymbol{\mathcal{X}}} \circ \pi_{\partial\mathcal{X}} = \pi_{\mathcal{X}} \circ \Pi_f$, $\pi_{\partial\mathcal{Y}} = \pi_{\partial\mathcal{Y}} \circ \Pi_f$ give identities on $s_{\boldsymbol{f}}, u_{\boldsymbol{f}}, \ldots, j_{\boldsymbol{f}}$, which are equalities rather than 2-isomorphisms:

$$f \circ i_{\boldsymbol{\mathcal{X}}} \circ s_{\boldsymbol{f}} = i_{\boldsymbol{\mathcal{Y}}} \circ u_{\boldsymbol{f}}, \ \ f \circ t_{\boldsymbol{f}} = i_{\boldsymbol{\mathcal{Y}}} \circ v_{\boldsymbol{f}}, \ \ \pi_{\mathcal{X}} \circ j_{\boldsymbol{f}} = i_{\boldsymbol{\mathcal{X}}} \circ s_{\boldsymbol{f}}, \ \ \pi_{\partial\mathcal{Y}} \circ j_{\boldsymbol{f}} = u_{\boldsymbol{f}}.$$

In the situation of $(*)$, $\underline{s}_{\boldsymbol{h}}, \underline{u}_{\boldsymbol{h}}, \ldots, \underline{j}_{\boldsymbol{h}}$ are étale lifts of $s_{\boldsymbol{f}}, u_{\boldsymbol{f}}, \ldots, j_{\boldsymbol{f}}$. Thus, from Proposition 6.7(a)–(e) for $\boldsymbol{h}$ for all $\underline{U}, u, \underline{V}, v, \underline{h}$ we deduce:

(a) $\mathcal{S}_{\boldsymbol{f}}, \mathcal{T}_{\boldsymbol{f}}$ are open and closed C^∞-substacks in $\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$ and $\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$.

(b) $j_{\boldsymbol{f}}$ is an equivalence of C^∞-stacks $\mathcal{S}_{\boldsymbol{f}} \to (\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}) \setminus \mathcal{T}_{\boldsymbol{f}}$.

(c) $s_{\boldsymbol{f}} : \mathcal{S}_{\boldsymbol{f}} \to \partial\mathcal{X}$ and $t_{\boldsymbol{f}} : \mathcal{T}_{\boldsymbol{f}} \to \mathcal{X}$ are proper étale 1-morphisms.

(d) There are unique $\lambda_{\boldsymbol{f}} : u_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathcal{Y}}}) \to s_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathcal{X}}})$ and $\mu_{\boldsymbol{f}} : u_{\boldsymbol{f}}^*(\mathcal{F}_{\partial\mathcal{Y}}) \to s_{\boldsymbol{f}}^*(\mathcal{F}_{\partial\mathcal{X}})$ in $\mathrm{qcoh}(\underline{S}_{\boldsymbol{f}})$ such that the following diagrams commute:

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & u_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathcal{Y}}}) \arrow[r, "u_{\boldsymbol{f}}^*(\nu_{\boldsymbol{\mathcal{Y}}})"] \arrow[d, "\lambda_{\boldsymbol{f}}"'] & u_{\boldsymbol{f}}^* \circ i_{\boldsymbol{\mathcal{Y}}}^*(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "u_{\boldsymbol{f}}^*(i_{\boldsymbol{\mathcal{Y}}}^2)"] \arrow[d, "{\begin{subarray}{l} I_{s_{\boldsymbol{f}}, i_{\boldsymbol{\mathcal{X}}}}(\mathcal{F}_{\mathcal{X}}) \circ (i_{\boldsymbol{\mathcal{X}}} \circ s_{\boldsymbol{f}})^*(f^2) \circ \\ I_{i_{\boldsymbol{\mathcal{X}}} \circ s_{\boldsymbol{f}}, f}(\mathcal{F}_{\mathcal{Y}}) \circ I_{u_{\boldsymbol{f}}, i_{\boldsymbol{\mathcal{Y}}}}(\mathcal{F}_{\mathcal{Y}})^{-1} \end{subarray}}"] & u_{\boldsymbol{f}}^*(\mathcal{F}_{\partial\mathcal{Y}}) \arrow[r] \arrow[d, "\mu_{\boldsymbol{f}}"] & 0 \\
0 \arrow[r] & s_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathcal{X}}}) \arrow[r, "s_{\boldsymbol{f}}^*(\nu_{\boldsymbol{\mathcal{X}}})"'] & s_{\boldsymbol{f}}^* \circ i_{\boldsymbol{\mathcal{X}}}^*(\mathcal{F}_{\mathcal{X}}) \arrow[r, "s_{\boldsymbol{f}}^*(i_{\boldsymbol{\mathcal{X}}}^2)"'] & s_{\boldsymbol{f}}^*(\mathcal{F}_{\partial\mathcal{X}}) \arrow[r] & 0,
\end{tikzcd} \tag{11.13}$$

$$\begin{tikzcd}[column sep=large]
u_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathcal{Y}}}) \arrow[r, "u_{\boldsymbol{f}}^*(i_{\boldsymbol{\mathcal{Y}}}^*(\psi_{\mathcal{Y}}) \circ \nu_{\boldsymbol{\mathcal{Y}}})"] \arrow[d, "\lambda_{\boldsymbol{f}}"'] & u_{\boldsymbol{f}}^* \circ i_{\boldsymbol{\mathcal{Y}}}^*(T^*\mathcal{Y}) \arrow[r, "u_{\boldsymbol{f}}^*(\Omega_{i_{\mathcal{Y}}})"] \arrow[d, "{\begin{subarray}{l} I_{s_{\boldsymbol{f}}, i_{\boldsymbol{\mathcal{X}}}}(T^*\mathcal{X}) \circ (i_{\boldsymbol{\mathcal{X}}} \circ s_{\boldsymbol{f}})^*(\Omega_f) \circ \\ I_{i_{\boldsymbol{\mathcal{X}}} \circ s_{\boldsymbol{f}}, f}(T^*\mathcal{Y}) \circ I_{u_{\boldsymbol{f}}, i_{\boldsymbol{Y}}}(T^*\mathcal{Y})^{-1} \end{subarray}}"] & u_{\boldsymbol{f}}^*(T^*(\partial\mathcal{Y})) \arrow[r] \arrow[d, "{\begin{subarray}{l} \Omega_{s_{\boldsymbol{f}}}^{-1} \circ \\ \Omega_{u_{\boldsymbol{f}}} \end{subarray}}"] & 0 \\
s_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathcal{X}}}) \arrow[r, "s_{\boldsymbol{f}}^*(i_{\boldsymbol{\mathcal{X}}}^*(\psi_{\mathcal{X}}) \circ \nu_{\boldsymbol{\mathcal{X}}})"'] & s_{\boldsymbol{f}}^* \circ i_{\boldsymbol{\mathcal{X}}}^*(T^*\mathcal{X}) \arrow[r, "s_{\boldsymbol{f}}^*(\Omega_{i_{\mathcal{X}}})"'] & s_{\boldsymbol{f}}^*(T^*(\partial\mathcal{X})) \arrow[r] & 0,
\end{tikzcd} \tag{11.14}$$

where the rows are exact, and $\Omega_{s_{\boldsymbol{f}}} : s_{\boldsymbol{f}}^*(T^*(\partial\mathcal{X})) \to T^*\mathcal{S}_{\boldsymbol{f}}$ is an isomorphism as $s_{\boldsymbol{f}}$ is étale by (c). Furthermore, $\lambda_{\boldsymbol{f}}$ is an isomorphism, and identifies the orientations $u_{\boldsymbol{f}}^*(\omega_{\boldsymbol{\mathcal{Y}}})$ on $u_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathcal{Y}}})$ and $s_{\boldsymbol{f}}^*(\omega_{\boldsymbol{\mathcal{X}}})$ on $s_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathcal{X}}})$.

(e) The following diagram commutes in $\mathrm{qcoh}(\mathcal{T}_{\boldsymbol{f}})$, using $f \circ t_{\boldsymbol{f}} = i_{\boldsymbol{\mathcal{Y}}} \circ v_{\boldsymbol{f}}$:

$$\begin{tikzcd}[column sep=large]
v_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathcal{Y}}}) \arrow[r, "v_{\boldsymbol{f}}^*(\nu_{\boldsymbol{\mathcal{Y}}})"'] \arrow[d, "0"] & v_{\boldsymbol{f}}^* \circ i_{\boldsymbol{\mathcal{Y}}}^*(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "I_{v_{\boldsymbol{f}}, i_{\boldsymbol{\mathcal{Y}}}}(\mathcal{F}_{\mathcal{Y}})^{-1}"'] & (f \circ t_{\boldsymbol{f}})^*(\mathcal{F}_{\mathcal{Y}}) \arrow[d, "I_{t_{\boldsymbol{f}}, f}(\mathcal{F}_{\mathcal{Y}})"'] \\
t_{\boldsymbol{f}}^*(\mathcal{F}_{\mathcal{X}}) & & t_{\boldsymbol{f}}^* \circ f^*(\mathcal{F}_{\mathcal{Y}}). \arrow[ll, "t_{\boldsymbol{f}}^*(f^2)"']
\end{tikzcd} \tag{11.15}$$

Definition 11.3. Let $\boldsymbol{f}, \boldsymbol{g} : \mathfrak{X} \to \mathfrak{Y}$ be 1-morphisms of d-stacks with corners. Then $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i_{\mathcal{X}}}, \omega_{\mathcal{X}})$, $\mathfrak{Y} = (\boldsymbol{\mathcal{Y}}, \boldsymbol{\partial\mathcal{Y}}, \boldsymbol{i_{\mathcal{Y}}}, \omega_{\mathcal{Y}})$, and $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ are 1-morphisms in **dSta** satisfying condition $(*)$ above. Suppose $\boldsymbol{\eta} = (\eta, \eta') : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism in **dSta**. Let $\partial\mathcal{X} \times_{f \circ i_{\mathcal{X}}, \mathcal{Y}, i_{\mathcal{Y}}} \partial\mathcal{Y}$ and $\partial\mathcal{X} \times_{g \circ i_{\mathcal{X}}, \mathcal{Y}, i_{\mathcal{Y}}} \partial\mathcal{Y}$ be as in Definition 11.2. Using $\eta : f \Rightarrow g$, properties of fibre products, and $\pi_{\partial\mathcal{X}} : \partial\mathcal{X} \times_{g \circ i_{\mathcal{X}}, \mathcal{Y}, i_{\mathcal{Y}}} \partial\mathcal{Y} \to \partial\mathcal{X}$ strongly representable, we find there is a unique 1-morphism $\Pi_\eta : \partial\mathcal{X} \times_{f \circ i_{\mathcal{X}}, \mathcal{Y}, i_{\mathcal{Y}}} \partial\mathcal{Y} \to \partial\mathcal{X} \times_{g \circ i_{\mathcal{X}}, \mathcal{Y}, i_{\mathcal{Y}}} \partial\mathcal{Y}$ and 2-morphism $\theta_\eta : \pi_{\partial\mathcal{Y}} \Rightarrow \pi_{\partial\mathcal{Y}} \circ \Pi_\eta$ such that $\pi_{\partial\mathcal{X}} \circ \Pi_\eta = \pi_{\partial\mathcal{X}}$ and $\mathrm{id}_{i_{\mathcal{Y}}} * \theta_\eta = \eta * \mathrm{id}_{i_{\mathcal{X}} \circ \pi_{\partial\mathcal{X}}}$. In fact Π_η is a 1-isomorphism, with inverse $\Pi_{\eta^{-1}}$. Similarly, there is a unique 1-isomorphism $\tilde{\Pi}_\eta : \mathcal{X} \times_{f, \mathcal{Y}, i_{\mathcal{Y}}} \partial\mathcal{Y} \to \mathcal{X} \times_{g, \mathcal{Y}, i_{\mathcal{Y}}} \partial\mathcal{Y}$ and 2-morphism $\tilde{\theta}_\eta : \pi_{\partial\mathcal{Y}} \Rightarrow \pi_{\partial\mathcal{Y}} \circ \tilde{\Pi}_\eta$ such that $\pi_{\mathcal{X}} \circ \Pi'_\eta = \pi_{\mathcal{X}}$ and $\mathrm{id}_{i_{\mathcal{Y}}} * \tilde{\theta}_\eta = \eta * \mathrm{id}_{\pi_{\mathcal{X}}}$.

We call $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a *2-morphism of d-stacks with corners* if $\mathcal{S}_{\boldsymbol{g}} = \Pi_\eta(\mathcal{S}_{\boldsymbol{f}})$, and

$$(i_{\mathcal{X}} \circ s_{\boldsymbol{f}})^*(\eta') \circ I_{i_{\mathcal{X}} \circ s_{\boldsymbol{f}}, f}(\mathcal{F}_{\mathcal{Y}}) \circ I_{u_{\boldsymbol{f}}, i_{\mathcal{Y}}}(\mathcal{F}_{\mathcal{Y}})^{-1} \circ u_{\boldsymbol{f}}^*(\nu_{\boldsymbol{\mathcal{Y}}}) = 0 \text{ in } \mathrm{qcoh}(\mathcal{S}_{\boldsymbol{f}}), \tag{11.16}$$

$$t_{\boldsymbol{f}}^*(\eta') \circ I_{t_{\boldsymbol{f}}, f}(\mathcal{F}_{\mathcal{Y}}) \circ I_{v_{\boldsymbol{f}}, i_{\mathcal{Y}}}(\mathcal{F}_{\mathcal{Y}})^{-1} \circ v_{\boldsymbol{f}}^*(\nu_{\boldsymbol{\mathcal{Y}}}) = 0 \text{ in } \mathrm{qcoh}(\mathcal{T}_{\boldsymbol{f}}), \tag{11.17}$$

where the morphisms are given in

$$\begin{tikzcd}
u_{\boldsymbol f}^*(\mathcal{N}_{\boldsymbol{\mathcal Y}}) \arrow[r, "u_{\boldsymbol f}^*(\nu_{\boldsymbol{\mathcal Y}})"'] \arrow[d, "0"] & u_{\boldsymbol f}^* \circ i_{\boldsymbol{\mathcal Y}}^*(\mathcal{F}_{\mathcal Y}) \arrow[r, "I_{u_{\boldsymbol f}, i_{\boldsymbol{\mathcal Y}}}(\mathcal{F}_{\mathcal Y})^{-1}"'] & (i_{\boldsymbol{\mathcal Y}} \circ u_{\boldsymbol f})^*(\mathcal{F}_{\mathcal Y}) \arrow[d, "I_{i_{\boldsymbol{\mathcal X}} \circ s_{\boldsymbol f}, f}(\mathcal{F}_{\mathcal Y})"'] \\
(i_{\boldsymbol{\mathcal X}} \circ s_{\boldsymbol f})^*(\mathcal{E}_{\mathcal X}) & & (i_{\boldsymbol{\mathcal X}} \circ s_{\boldsymbol f})^* \circ f^*(\mathcal{F}_{\mathcal Y}), \arrow[ll, "(i_{\boldsymbol{\mathcal X}} \circ s_{\boldsymbol f})^*(\eta')"']
\end{tikzcd}$$

$$\begin{tikzcd}
v_{\boldsymbol f}^*(\mathcal{N}_{\boldsymbol{\mathcal Y}}) \arrow[r, "v_{\boldsymbol f}^*(\nu_{\boldsymbol{\mathcal Y}})"'] \arrow[d, "0"] & v_{\boldsymbol f}^* \circ i_{\boldsymbol{\mathcal Y}}^*(\mathcal{F}_{\mathcal Y}) \arrow[r, "I_{v_{\boldsymbol f}, i_{\boldsymbol{\mathcal Y}}}(\mathcal{F}_{\mathcal Y})^{-1}"'] & (f \circ t_{\boldsymbol f})^*(\mathcal{F}_{\mathcal Y}) \arrow[d, "I_{t_{\boldsymbol f}, f}(\mathcal{F}_{\mathcal Y})"'] \\
t_{\boldsymbol f}^*(\mathcal{E}_{\mathcal X}) & & t_{\boldsymbol f}^* \circ f^*(\mathcal{F}_{\mathcal Y}), \arrow[ll, "t_{\boldsymbol f}^*(\eta')"']
\end{tikzcd}$$

using $f \circ i_{\boldsymbol{\mathcal X}} \circ s_{\boldsymbol f} = i_{\boldsymbol{\mathcal Y}} \circ u_{\boldsymbol f}$ and $f \circ t_{\boldsymbol f} = i_{\boldsymbol{\mathcal Y}} \circ v_{\boldsymbol f}$. Since $\mathcal{S}_{\boldsymbol f}, \mathcal{S}_{\boldsymbol g}$ determine $\mathcal{T}_{\boldsymbol f}, \mathcal{T}_{\boldsymbol g}$ by Definition 11.2(b), we see that $\mathcal{S}_{\boldsymbol g} = \Pi_\eta(\mathcal{S}_{\boldsymbol f})$ implies that $\mathcal{T}_{\boldsymbol g} = \tilde\Pi_\eta(\mathcal{T}_{\boldsymbol f})$. From $\pi_{\partial\mathcal X} \circ \Pi_\eta = \pi_{\partial\mathcal X}$, $\theta_\eta : \pi_{\partial\mathcal Y} \Rightarrow \pi_{\partial\mathcal Y} \circ \Pi_\eta$, and so on, we see that

$$\begin{aligned}
&s_{\boldsymbol g} \circ \Pi_\eta|_{\mathcal{S}_{\boldsymbol f}} = s_{\boldsymbol f}, && t_{\boldsymbol g} \circ \tilde\Pi_\eta|_{\mathcal{T}_{\boldsymbol f}} = t_{\boldsymbol f}, \\
&\theta_\eta|_{\mathcal{S}_{\boldsymbol f}} : u_{\boldsymbol f} \Rightarrow u_{\boldsymbol g} \circ \Pi_\eta|_{\mathcal{S}_{\boldsymbol f}}, && \tilde\theta_\eta|_{\mathcal{T}_{\boldsymbol f}} : v_{\boldsymbol f} \Rightarrow v_{\boldsymbol g} \circ \tilde\Pi_\eta|_{\mathcal{T}_{\boldsymbol f}}.
\end{aligned}$$

We can also express these conditions on $\boldsymbol\eta$ by lifting étale locally to $\mathbf{dSpa^c}$, as in Definitions 11.1 and 11.2, in the following way. Suppose we are given a 2-commutative diagram in $\mathbf{C^\infty Sta}$:

$$\begin{tikzcd}
\bar{\underline{U}} \arrow[r, "u"'] \arrow[d, "\bar{\underline{h}}"] \arrow[dr, phantom, "\Downarrow\zeta"] & \mathcal{X} \arrow[d, bend right, "g"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "\overset{\eta}{\Leftarrow}"] \\
\bar{\underline{V}} \arrow[r, "v"] & \mathcal{Y},
\end{tikzcd} \tag{11.18}$$

where $\underline{U}, \underline{V}$ are separated, second countable C^∞-schemes, $\underline{h} : \underline{U} \to \underline{V}$ is a morphism, $u : \bar{\underline{U}} \to \mathcal{X}$, $v : \bar{\underline{V}} \to \mathcal{Y}$ are étale 1-morphisms, and $\eta : f \Rightarrow g$ is from $\boldsymbol\eta$. Then condition $(*)$ of Definition 11.2 applies to $\boldsymbol f$ with 2-morphism $\zeta \odot (\eta * \mathrm{id}_u)$, giving a 1-morphism $\boldsymbol h = (\underline{h}, h', h'') : \mathbf{U} \to \mathbf{V}$ in $\mathbf{dSpa^c}$, and also to $\boldsymbol g$ with 2-morphism ζ, giving a 1-morphism $\tilde{\boldsymbol h} = (\underline{h}, \tilde h', \tilde h'') : \mathbf{U} \to \mathbf{V}$. Define a morphism $\theta : \underline{h}^*(\mathcal{F}_V) \to \mathcal{E}_U$ in $\mathrm{qcoh}(\underline{U})$ to be the composition

$$\underline{h}^*(\mathcal{F}_V) \xrightarrow{\cong} \underline{h}^*(\mathcal{F}_{\mathcal Y}(\underline{V}, v)) \xrightarrow{\cong} f^*(\mathcal{F}_{\mathcal Y})(\underline{U}, u) \xrightarrow{\eta'(\underline{U},u)} \mathcal{E}_{\mathcal X}(\underline{U}, u) = \mathcal{E}_U,$$

where the first two terms are natural isomorphisms we can construct using Definitions C.30 and C.36. Then equation (9.13) for η' implies (2.25) for θ, so that $\theta : \boldsymbol h \to \tilde{\boldsymbol h}$ is a 2-morphism in $\mathbf{dSpa}$.

One can show that $\mathcal{S}_{\boldsymbol g} = \Pi_\eta(\mathcal{S}_{\boldsymbol f})$ implies that $\underline{S}_{\tilde{\boldsymbol h}} = \underline{S}_{\boldsymbol h}$, and (11.16)–(11.17) for η' imply (6.9)–(6.10) for θ, so $\theta : \boldsymbol h \Rightarrow \tilde{\boldsymbol h}$ is a 2-morphism in $\mathbf{dSpa^c}$ by Definition 6.3. Conversely, if $\boldsymbol\eta : \boldsymbol f \Rightarrow \boldsymbol g$ is a 2-morphism in $\mathbf{dSta}$, and $\theta : \boldsymbol h \Rightarrow \tilde{\boldsymbol h}$ is a 2-morphism in $\mathbf{dSpa^c}$ for every such diagram (11.18), then $\underline{S}_{\tilde{\boldsymbol h}} = \underline{S}_{\boldsymbol h}$ for all such $\boldsymbol h, \tilde{\boldsymbol h}$ imply that $\mathcal{S}_{\boldsymbol g} = \Pi_\eta(\mathcal{S}_{\boldsymbol f})$, and (6.9)–(6.10) for all such θ imply (11.16)–(11.17) for $\boldsymbol\eta$, so $\boldsymbol\eta : \boldsymbol f \Rightarrow \boldsymbol g$ is a 2-morphism of d-stacks with corners.

As for objects and 1-morphisms in Definitions 11.1 and 11.2, properties of 2-morphisms in $\mathbf{dSpa^c}$ proved in Chapter 6 imply properties of 2-morphisms of d-stacks with corners, provided the properties are étale local. In particular, from Proposition 6.8(a)–(c) we can deduce:

(a) $\lambda_{\boldsymbol{f}}, \lambda_{\boldsymbol{g}}$ in Definition 11.2(d) fit into a commutative diagram in $\mathrm{qcoh}(\mathcal{S}_{\boldsymbol{f}})$:

$$
\begin{tikzcd}
u_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathfrak{Y}}}) \arrow[r, "\theta_\eta|^*_{\mathcal{S}_{\boldsymbol{f}}}(\mathcal{N}_{\boldsymbol{\mathfrak{Y}}})"'] \arrow[d, "\lambda_{\boldsymbol{f}}"] & (u_{\boldsymbol{g}} \circ \Pi_\eta|_{\mathcal{S}_{\boldsymbol{f}}})^*(\mathcal{N}_{\boldsymbol{\mathfrak{Y}}}) \arrow[r, "I_{\Pi_\eta|_{\mathcal{S}_{\boldsymbol{f}}}, u_{\boldsymbol{g}}}(\mathcal{N}_{\boldsymbol{\mathfrak{Y}}})"'] & \Pi_\eta|^*_{\mathcal{S}_{\boldsymbol{f}}}(u_{\boldsymbol{g}}^*(\mathcal{N}_{\boldsymbol{\mathfrak{Y}}})) \arrow[d, "\Pi_\eta|^*_{\mathcal{S}_{\boldsymbol{f}}}(\lambda_{\boldsymbol{g}})"'] \\
s_{\boldsymbol{f}}^*(\mathcal{N}_{\boldsymbol{\mathfrak{X}}}) \arrow[r, equal] & (s_{\boldsymbol{g}} \circ \Pi_\eta|_{\mathcal{S}_{\boldsymbol{f}}})^*(\mathcal{N}_{\boldsymbol{\mathfrak{X}}}) & \Pi_\eta|^*_{\mathcal{S}_{\boldsymbol{f}}}(s_{\boldsymbol{g}}^*(\mathcal{N}_{\boldsymbol{\mathfrak{X}}})). \arrow[l, "I_{\Pi_\eta|_{\mathcal{S}_{\boldsymbol{f}}}, s_{\boldsymbol{g}}}(\mathcal{N}_{\boldsymbol{\mathfrak{X}}})^{-1}"']
\end{tikzcd}
\tag{11.19}
$$

(b) There is a unique morphism $\eta'_S : u_{\boldsymbol{f}}^*(\mathcal{F}_{\partial\mathcal{Y}}) \to s_{\boldsymbol{f}}^*(\mathcal{E}_{\partial\mathcal{X}})$ in $\mathrm{qcoh}(\mathcal{S}_{\boldsymbol{f}})$ such that the following commutes:

$$
\begin{tikzcd}
(i_{\boldsymbol{\mathfrak{X}}} \circ s_{\boldsymbol{f}})^* \circ f^*(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "I_{u_{\boldsymbol{f}}, i_{\boldsymbol{\mathfrak{Y}}}}(\mathcal{F}_{\mathcal{Y}}) \circ \\ I_{i_{\boldsymbol{\mathfrak{X}}} \circ s_{\boldsymbol{f}}, f}(\mathcal{F}_{\mathcal{Y}})^{-1}"'] \arrow[d, "(i_{\boldsymbol{\mathfrak{X}}} \circ s_{\boldsymbol{f}})^*(\eta')"] & u_{\boldsymbol{f}}^* \circ i_{\boldsymbol{\mathfrak{Y}}}^*(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "u_{\boldsymbol{f}}^*(i^2_{\boldsymbol{\mathfrak{Y}}})"'] & u_{\boldsymbol{f}}^*(\mathcal{F}_{\partial\mathcal{Y}}) \arrow[d, "\eta'_S"] \\
(i_{\boldsymbol{\mathfrak{X}}} \circ s_{\boldsymbol{f}})^*(\mathcal{E}_{\mathcal{X}}) \arrow[r, "I_{s_{\boldsymbol{f}}, i_{\boldsymbol{\mathfrak{X}}}}(\mathcal{E}_{\mathcal{X}})"] & s_{\boldsymbol{f}}^* \circ i_{\boldsymbol{\mathfrak{X}}}^*(\mathcal{E}_{\mathcal{X}}) \arrow[r, "s_{\boldsymbol{f}}^*(i''_{\boldsymbol{\mathfrak{X}}})"] & s_{\boldsymbol{f}}^*(\mathcal{E}_{\partial\mathcal{X}}).
\end{tikzcd}
\tag{11.20}
$$

(c) There is a unique morphism $\eta'_T : v_{\boldsymbol{f}}^*(\mathcal{F}_{\partial\mathcal{Y}}) \to t_{\boldsymbol{f}}^*(\mathcal{E}_{\mathcal{X}})$ in $\mathrm{qcoh}(\mathcal{T}_{\boldsymbol{f}})$ such that the following commutes:

$$
\begin{tikzcd}
t_{\boldsymbol{f}}^* \circ f^*(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "I_{t_{\boldsymbol{f}}, f}(\mathcal{F}_{\mathcal{Y}})^{-1}"'] \arrow[d, "t_{\boldsymbol{f}}^*(\eta')"] & (f \circ t_{\boldsymbol{f}})^*(\mathcal{F}_{\mathcal{Y}}) \arrow[r, "I_{v_{\boldsymbol{f}}, i_{\boldsymbol{\mathfrak{Y}}}}(\mathcal{F}_{\mathcal{Y}})"'] & v_{\boldsymbol{f}}^* \circ i_{\boldsymbol{\mathfrak{Y}}}^*(\mathcal{F}_{\mathcal{Y}}) \arrow[d, "v_{\boldsymbol{f}}^*(i^2_{\boldsymbol{\mathfrak{Y}}})"'] \\
t_{\boldsymbol{f}}^*(\mathcal{E}_{\mathcal{X}}) & & v_{\boldsymbol{f}}^*(\mathcal{F}_{\partial\mathcal{Y}}). \arrow[ll, "\eta'_T"']
\end{tikzcd}
\tag{11.21}
$$

We define *vertical composition* $\boldsymbol{\zeta} \odot \boldsymbol{\eta}$ and *horizontal composition* $\boldsymbol{\zeta} * \boldsymbol{\eta}$ of 2-morphisms of d-stacks with corners to be vertical and horizontal composition of 2-morphisms in **dSta**. Using the étale local characterization of 2-morphisms above, as 2-morphisms in $\mathbf{dSpa^c}$ are closed under vertical and horizontal composition, we see that such $\boldsymbol{\zeta} \odot \boldsymbol{\eta}$ and $\boldsymbol{\zeta} * \boldsymbol{\eta}$ are also 2-morphisms of d-stacks with corners. For $\boldsymbol{f} : \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathfrak{Y}}$ a 1-morphism of d-stacks with corners, we define the *identity 2-morphism* to be the identity 2-morphism $\mathbf{id}_{\boldsymbol{f}} : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ in **dSta**.

In Definitions 11.1, 11.2 and above we have defined all the structures of a 2-category, which we call the 2-*category of d-stacks with corners*, written $\mathbf{dSta^c}$. By Theorem 11.4 they satisfy the axioms of a 2-category. Write $\mathbf{dSta^b}$ for the full 2-subcategory of d-stacks with boundary, and $\mathbf{d\bar{S}ta}$ for the full 2-subcategory of d-stacks without boundary.

Define a 2-functor $F_{\mathbf{dSta}}^{\mathbf{dSta^c}} : \mathbf{dSta} \to \mathbf{dSta^c}$ to map $\boldsymbol{\mathcal{X}} \mapsto \boldsymbol{\mathfrak{X}} = (\boldsymbol{\mathcal{X}}, \emptyset, \emptyset, \emptyset)$ on objects, $\boldsymbol{f} \mapsto \boldsymbol{f}$ on 1-morphisms, and $\boldsymbol{\eta} \mapsto \boldsymbol{\eta}$ on 2-morphisms, where the data $\boldsymbol{\partial\mathcal{X}}, \boldsymbol{i_{\mathfrak{X}}}, \omega_{\boldsymbol{\mathfrak{X}}}$ is trivial as the d-stacks concerned are empty. Then $F_{\mathbf{dSta}}^{\mathbf{dSta^c}}$ is a (strict) isomorphism of 2-categories $\mathbf{dSta} \to \mathbf{d\bar{S}ta}$. So we may as well identify **dSta** with its image $\mathbf{d\bar{S}ta}$, and consider d-stacks in Chapter 9 as examples of d-stacks with corners.

Define a 2-functor $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}:\mathbf{dSpa^c}\rightarrow\mathbf{dSta^c}$ as follows. If $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}},\omega_{\mathbf{X}})$ is an object in $\mathbf{dSpa^c}$, we define $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\mathbf{X})=\boldsymbol{\mathfrak{X}}=(\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i}_{\mathfrak{X}},\omega_{\mathfrak{X}})$, where $\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i}_{\mathfrak{X}}=F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i}_{\mathbf{X}})$. Note that then $i_{\mathfrak{X}}=F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}}(\underline{i}_{\mathbf{X}})=\bar{\underline{i}}_{\mathbf{X}}$, which is strongly representable by Proposition C.16, as required by Definition 11.1(a). Roughly speaking, the orientation $\omega_{\mathfrak{X}}$ on the line bundle $\mathcal{N}_{\mathfrak{X}}$ on $\bar{\underline{\partial X}}$ is given by $\omega_{\mathfrak{X}}=\mathcal{I}_{\underline{\partial X}}(\omega_{\mathbf{X}})$, where the equivalence of categories $\mathcal{I}_{\underline{\partial X}}:\mathrm{qcoh}(\underline{\partial X})\rightarrow\mathrm{qcoh}(\bar{\underline{\partial X}})$ is as in Example C.32. More precisely, if $\tau:\mathcal{O}_{\partial X}\rightarrow\mathcal{N}_{\mathbf{X}}$ in $\mathrm{qcoh}(\underline{\partial X})$ represents $\omega_{\mathbf{X}}$, then $\omega_{\mathfrak{X}}$ is the orientation on $\mathcal{N}_{\mathfrak{X}}$ represented by the following composition, with natural isomorphisms in the first and third morphisms:

$$\mathcal{O}_{\partial\mathcal{X}}\xrightarrow{\cong}\mathcal{I}_{\underline{\partial X}}(\mathcal{O}_{\partial X})\xrightarrow{\mathcal{I}_{\underline{\partial X}}(\tau)}\mathcal{I}_{\underline{\partial X}}(\mathcal{N}_{\mathbf{X}})\xrightarrow{\cong}\mathcal{N}_{\mathfrak{X}}.$$

On 1-morphisms $\boldsymbol{f}$ and 2-morphisms η in $\mathbf{dSpa^c}$, define $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\boldsymbol{f})=F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{f})$ and $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\boldsymbol{\eta})=F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{\eta})$. Write $\mathbf{d\hat{S}pa^c}$ for the full 2-subcategory of objects $\boldsymbol{\mathfrak{X}}$ in $\mathbf{dSta^c}$ equivalent to $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\mathbf{X})$ for some d-space with corners $\mathbf{X}$. When we say that a d-stack with corners $\boldsymbol{\mathfrak{X}}$ *is a d-space*, we mean that $\boldsymbol{\mathfrak{X}}\in\mathbf{d\hat{S}pa^c}$.

Define a 2-functor $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}:\mathbf{Orb^c}\rightarrow\mathbf{dSta^c}$ as follows. If $\mathfrak{X}=(\mathcal{X},\partial\mathcal{X},i_{\mathfrak{X}})$ is an orbifold with corners, as in §8.5, define $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(\mathfrak{X})=\boldsymbol{\mathfrak{X}}=(\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i}_{\mathfrak{X}},\omega_{\mathfrak{X}})$, where $\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i}_{\mathfrak{X}}=F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\mathcal{X},\partial\mathcal{X},i_{\mathfrak{X}})$. Then there are canonical isomorphisms $\mathcal{F}_{\mathcal{X}}\cong T^*\mathcal{X}$ and $\mathcal{F}_{\partial\mathcal{X}}\cong T^*\partial\mathcal{X}$, so (11.1) gives an exact sequence

$$0\longrightarrow\mathcal{N}_{\mathfrak{X}}\longrightarrow i_{\mathfrak{X}}^*(T^*\mathcal{X})\xrightarrow{\Omega_{i_{\mathfrak{X}}}}T^*(\partial\mathcal{X})\longrightarrow 0.$$

Thus $\mathcal{N}_{\mathfrak{X}}$ is naturally isomorphic to the conormal line bundle of $\partial\mathcal{X}$ in $\mathcal{X}$. Let $\omega_{\mathfrak{X}}$ be the orientation on $\mathcal{N}_{\mathfrak{X}}$ coming from 'outward-pointing' normal vectors on this conormal line bundle. Then $\boldsymbol{\mathfrak{X}}=(\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i}_{\mathfrak{X}},\omega_{\mathfrak{X}})$ is a d-stack with corners. Define $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}:f\mapsto\boldsymbol{f}=F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(f)$ on 1-morphisms $f:\mathfrak{X}\rightarrow\mathfrak{Y}$ in $\mathbf{Orb^c}$, and $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}:\eta\mapsto\boldsymbol{\eta}=F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\eta)$ on 2-morphisms $\eta:f\Rightarrow g$ in $\mathbf{Orb^c}$.

Write $\mathbf{\bar{O}rb},\mathbf{\bar{O}rb^b},\mathbf{\bar{O}rb^c}$ for the full 2-subcategories of objects $\boldsymbol{\mathfrak{X}}$ in $\mathbf{dSta^c}$ equivalent to $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(\mathfrak{X})$ for some orbifold $\mathfrak{X}$ without boundary, or with boundary, or with corners, respectively. Then $\mathbf{\bar{O}rb}\subset\mathbf{d\bar{S}ta}$, $\mathbf{\bar{O}rb^b}\subset\mathbf{dSta^b}$ and $\mathbf{\bar{O}rb^c}\subset\mathbf{dSta^c}$. When we say that a d-stack with corners $\boldsymbol{\mathfrak{X}}$ *is an orbifold*, we mean that $\boldsymbol{\mathfrak{X}}\in\mathbf{\bar{O}rb^c}$.

Since 1- and 2-morphisms in $\mathbf{dSta^c}$ are just special examples of 1- and 2-morphisms in $\mathbf{dSta}$, the proof in Theorem 9.7 that $\mathbf{dSta}$ is a 2-category and $F_{\mathbf{dSpa}}^{\mathbf{dSta}},F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}$ are full and faithful strict 2-functors implies:

Theorem 11.4. *In Definitions* 11.1–11.3, $\mathbf{d\bar{S}ta},\mathbf{dSta^b},\mathbf{dSta^c}$ *are strict* 2*-categories, in which all* 2*-morphisms are* 2*-isomorphisms, and* $F_{\mathbf{dSta}}^{\mathbf{dSta^c}}$, $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}$ *and* $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}$ *are full and faithful strict* 2*-functors.*

Remark 11.5. (a) Definitions 11.1–11.3 combine ideas from the definitions of $\mathbf{dSpa^c},\mathbf{Orb^c},\mathbf{dSta}$ in §6.1, §8.5 and §9.2. Most of the points about these definitions made in Remarks 6.5, 8.17 and 9.8 are also relevant to $\mathbf{dSta^c}$.

(b) Just as in §8.5 we defined objects and 1-morphisms in $\mathbf{Orb^c}$ to be étale locally modelled on objects and morphisms in $\mathbf{Man^c}$, so above we defined objects, 1- and 2-morphisms in $\mathbf{dSta^c}$ to be étale locally modelled on objects, 1-

and 2-morphisms in $\mathbf{dSpa^c}$. Some advantages of this are that, firstly, writing down more direct definitions is tricky, as to define 'boundary defining functions' for d-stacks with corners one is forced to work étale locally rather than Zariski locally, and secondly, it enables us to easily deduce results in $\mathbf{dSta^c}$ from results in $\mathbf{dSpa^c}$, as in Definitions 11.2(a)–(e) and 11.3(a)–(c).

(c) As for $i_{\mathcal{X}}$ in Definition 8.15, the condition that $i_{\mathfrak{X}}$ is *strongly representable* in Definition 11.1(a) is there to ensure that 1-morphisms $\boldsymbol{f}$ and 2-morphisms $\boldsymbol{\eta}$ lift to boundaries ($\boldsymbol{f}_-, \boldsymbol{\eta}_-$ in §11.3) and corners ($C(\boldsymbol{f}), C(\boldsymbol{\eta}), \hat{C}(\boldsymbol{f}), \hat{C}(\boldsymbol{\eta})$ in §11.5) uniquely, in a strictly functorial way, rather than merely uniquely up to 2-isomorphism and in a weakly functorial way.

Here is the analogue of Propositions 2.17, 6.9 and 9.9. To prove it we first apply Proposition 9.9 to show that there exists a unique 1-morphism $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dSta}$ such that $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism in $\mathbf{dSta}$. Then we show that $\boldsymbol{g}$ satisfies Definition 11.2(∗), so it is a 1-morphism in $\mathbf{dSta^c}$. To do this, we show that $\boldsymbol{g}$ is étale locally modelled on 1-morphisms in $\mathbf{dSpa^c}$ constructed using Proposition 6.9, using the method of étale locally modelling 2-morphisms based on equation (11.18) in Definition 11.3.

Proposition 11.6. *Suppose $\boldsymbol{f} = (f, f', f'') : \mathfrak{X} \to \mathfrak{Y}$ is a 1-morphism of d-stacks with corners, $g : \mathcal{X} \to \mathcal{Y}$ a 1-morphism of C^∞-stacks, $\eta : f \Rightarrow g$ a 2-morphism of C^∞-stacks, and $\eta' : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{E}_{\mathcal{X}}$ a morphism in* $\mathrm{qcoh}(\mathcal{X})$ *such that* (11.16) *and* (11.17) *hold in* $\mathrm{qcoh}(\mathcal{S}_{\boldsymbol{f}})$ *and* $\mathrm{qcoh}(\mathcal{T}_{\boldsymbol{f}})$*. Then there exists a unique 1-morphism $\boldsymbol{g} = (g, g', g'') : \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{dSta^c}$ such that $\boldsymbol{\eta} = (\eta, \eta') : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism in $\mathbf{dSta^c}$.*

11.2 D-stacks with corners as quotients of d-spaces

We now give the analogue of §9.3 for d-spaces and d-stacks with corners.

Definition 11.7. Let $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ be a d-space with corners, G a finite group, and $\boldsymbol{r} : G \to \mathrm{Aut}(\mathbf{X})$ an action of G on $\mathbf{X}$ by 1-isomorphisms, as in Definition 6.54. Since 1-isomorphisms are simple, by §6.3 each $\boldsymbol{r}(\gamma) : \mathbf{X} \to \mathbf{X}$ for $\gamma \in G$ has a unique lift $\boldsymbol{r}_-(\gamma) : \partial\mathbf{X} \to \partial\mathbf{X}$ with $\boldsymbol{i}_{\mathbf{X}} \circ \boldsymbol{r}_-(\gamma) = \boldsymbol{r}(\gamma) \circ \boldsymbol{i}_{\mathbf{X}}$. Then $\boldsymbol{r}_- : G \to \mathrm{Aut}(\partial\mathbf{X})$ is an action of G on $\partial\mathbf{X}$ by 1-isomorphisms. Hence $\boldsymbol{r} : G \to \mathrm{Aut}(\boldsymbol{X})$ and $\boldsymbol{r}_- : G \to \mathrm{Aut}(\boldsymbol{\partial X})$ are actions of G on the d-spaces $\boldsymbol{X}, \boldsymbol{\partial X}$ by 1-isomorphisms, so §9.3 defines quotient d-stacks $[\boldsymbol{X}/G], [\boldsymbol{\partial X}/G]$. Also the d-space 1-morphism $\boldsymbol{i}_{\mathbf{X}} : \boldsymbol{\partial X} \to \boldsymbol{X}$ satisfies $\boldsymbol{i}_{\mathbf{X}} \circ \boldsymbol{r}_-(\gamma) = \boldsymbol{r}(\mathrm{id}_G(\gamma)) \circ \boldsymbol{i}_{\mathbf{X}}$ for all $\gamma \in G$, so §9.3 defines a quotient 1-morphism $[\boldsymbol{i}_{\mathbf{X}}, \mathrm{id}_G] : [\boldsymbol{\partial X}/G] \to [\boldsymbol{X}/G]$. The proof in Example 8.16 shows that the underlying C^∞-stack 1-morphism $[\underline{i}_{\mathbf{X}}, \mathrm{id}_G] : [\underline{\partial X}/G] \to [\underline{X}/G]$ is strongly representable.

Set $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$, $\boldsymbol{\partial\mathcal{X}} = [\boldsymbol{\partial X}/G]$, and $\boldsymbol{i}_{\mathfrak{X}} = [\boldsymbol{i}_{\mathbf{X}}, \mathrm{id}_G]$. We will verify Definition 11.1(a)–(d) for $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathfrak{X}}, \omega_{\mathfrak{X}})$, defining $\omega_{\mathfrak{X}}$ along the way. For (a), $i_{\mathfrak{X}} = [\underline{i}_{\mathbf{X}}, \mathrm{id}_G]$ is strongly representable from above, and $i_{\mathfrak{X}}$ proper in (a), $i''_{\mathfrak{X}}$ an isomorphism in (b), and $\mathcal{N}_{\mathfrak{X}}$ a line bundle in (c), follow from the corresponding properties of $i_{\mathbf{X}}, i''_{\mathbf{X}}, \mathcal{N}_{\mathbf{X}}$ for the d-space with corners $\mathbf{X}$ in Definition 6.1.

For Definition 11.1(d), writing $\iota : \{1\} \to G$ for the inclusion, we now have a 2-Cartesian diagram in $\mathbf{C^\infty Sta}$:

$$\begin{tikzcd}
\underline{\partial\bar{X}} = [\partial\underline{X}/\{1\}] \arrow[rr, "{[\underline{\mathrm{id}}_{\partial X},\iota]}"] \arrow[d, "\underline{\bar{i}}_{\mathbf{X}}"'] & \arrow[d, phantom, "\mathrm{id}\Uparrow"] & {[\partial\underline{X}/G] = \partial\mathcal{X}} \arrow[d, "{[\underline{i}_{\mathbf{X}},\mathrm{id}_G]=i_{\mathcal{X}}}"] \\
\underline{\bar{X}} = [\underline{X}/\{1\}] \arrow[rr, "{[\underline{\mathrm{id}}_{X},\iota]}"] & & {[\underline{X}/G] = \mathcal{X}.}
\end{tikzcd} \tag{11.22}$$

Regard (11.22) as an example of diagram (11.2) for $\boldsymbol{\mathfrak{X}} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i_{\mathcal{X}}}, \omega_{\mathcal{X}})$, where $\omega_{\mathcal{X}}$ remains to be constructed. In Definition 11.1(d) we use (11.2) to construct $\boldsymbol{U}, \boldsymbol{\partial U}, \boldsymbol{i}_{\mathbf{U}}, \omega_{\mathbf{U}}$ from $\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i_{\mathcal{X}}}, \omega_{\mathcal{X}}$. In our case, the constructions of $\boldsymbol{U}, \boldsymbol{\partial U}, \boldsymbol{i}_{\mathbf{U}}$ just recover $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i}_{\mathbf{X}}$ (up to canonical isomorphism). So (11.3) gives a unique isomorphism $\mathcal{N}_{\mathcal{X}}(\underline{\partial X}, [\underline{\mathrm{id}}_{\partial X},\iota]) \cong \mathcal{N}_{\mathbf{X}}$ in $\mathrm{qcoh}(\underline{\partial X})$.

As the orientation $\omega_{\mathbf{X}}$ on $\mathcal{N}_{\mathbf{X}}$ is G-invariant, there is a unique orientation $\omega_{\mathcal{X}}$ on $\mathcal{N}_{\mathcal{X}}$ such that this isomorphism identifies $\omega_{\mathcal{X}}(\underline{\partial X}, [\underline{\mathrm{id}}_{\partial X},\iota])$ with $\omega_{\mathbf{X}}$. Hence Definition 11.1(c) holds, and (d) holds with (11.22) in place of (11.2). But as in Definition 11.1, it is sufficient to verify (d) for an étale open cover $\{(\underline{U}_i, u_i) : i \in I\}$ of $\mathcal{X}$, and $\big\{(\underline{X}, [\underline{\mathrm{id}}_X,\iota])\big\}$ is an étale open cover of $\mathcal{X} = [\underline{X}/G]$. Therefore $\boldsymbol{\mathfrak{X}}$ is a d-stack with corners. We shall also write $\boldsymbol{\mathfrak{X}} = [\mathbf{X}/G]$.

Next let $\mathbf{X}, \mathbf{Y}$ be d-spaces with corners, G, H finite groups, and $\boldsymbol{r} : G \to \mathrm{Aut}(\mathbf{X})$, $\boldsymbol{s} : H \to \mathrm{Aut}(\mathbf{Y})$ be actions of G, H on $\mathbf{X}, \mathbf{Y}$, so that we have quotient d-stacks with corners $\boldsymbol{\mathfrak{X}} = [\mathbf{X}/G]$ and $\boldsymbol{\mathfrak{Y}} = [\mathbf{Y}/H]$. Suppose $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism in $\mathbf{dSpa^c}$ and $\rho : G \to H$ is a group morphism, satisfying $\boldsymbol{f} \circ \boldsymbol{r}(\gamma) = \boldsymbol{s}(\rho(\gamma)) \circ \boldsymbol{f}$ for all $\gamma \in G$. Then Definition 9.15 defines a 1-morphism $[\boldsymbol{f}, \rho] : [\boldsymbol{X}/G] \to [\boldsymbol{Y}/H]$ in $\mathbf{dSta}$. We claim that $[\boldsymbol{f}, \rho] : [\mathbf{X}/G] \to [\mathbf{Y}/H]$ is also a 1-morphism in $\mathbf{dSta^c}$. We must verify Definition 11.2($*$). Consider the 2-commutative diagram in $\mathbf{C^\infty Sta}$:

$$\begin{tikzcd}
\underline{\bar{X}} = [\underline{X}/\{1\}] \arrow[rr, "{[\underline{\mathrm{id}}_{X},\iota]}"] \arrow[d, "\underline{\bar{f}}"'] & \arrow[d, phantom, "\Downarrow\mathrm{id}"] & {[\underline{X}/G] = \mathcal{X}} \arrow[d, "{[\underline{f},\rho]}"] \\
\underline{\bar{Y}} = [\underline{Y}/\{1\}] \arrow[rr, "{[\underline{\mathrm{id}}_{Y},\iota]}"] & & {[\underline{Y}/H] = \mathcal{Y}.}
\end{tikzcd} \tag{11.23}$$

Then $[\underline{\mathrm{id}}_X,\iota], [\underline{\mathrm{id}}_Y,\iota]$ are étale. Applying ($*$) with (11.23) in place of (11.5) and $[\boldsymbol{f}, \rho]$ in place of $\boldsymbol{f}$ reconstructs $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in place of $\boldsymbol{h} : \boldsymbol{U} \to \boldsymbol{V}$. Since by assumption $\boldsymbol{f}$ is a 1-morphism in $\mathbf{dSpa^c}$, condition ($*$) holds in this case. As $\big\{(\underline{X}, [\underline{\mathrm{id}}_X,\iota])\big\}$ is an étale open cover of $\mathcal{X}$, condition ($*$) holds in every case.

Now let $[\boldsymbol{f}, \rho], [\boldsymbol{g}, \sigma] : [\mathbf{X}/G] \to [\mathbf{Y}/H]$ be two quotient 1-morphisms of the above form. Suppose $\delta \in H$ satisfies $\delta^{-1}\,\sigma(\gamma) = \rho(\gamma)\,\delta^{-1}$ for all $\gamma \in G$, and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{g}$ is a 2-morphism in $\mathbf{dSpa^c}$, such that $\eta * \mathrm{id}_{\boldsymbol{r}(\gamma)} = \mathrm{id}_{\boldsymbol{s}(\sigma(\gamma))} * \eta$ for all $\gamma \in G$. Then Definition 9.15 defines a 2-morphism $[\eta, \delta] : [\boldsymbol{f}, \rho] \Rightarrow [\boldsymbol{g}, \sigma]$ in $\mathbf{dSta}$. By a similar argument using the following in place of (11.18):

$$\begin{tikzcd}
\underline{\bar{X}} = [\underline{X}/\{1\}] \arrow[rr, "{[\underline{\mathrm{id}}_{X},\iota]}"] \arrow[d, "\underline{\bar{f}}"'] & \arrow[d, phantom, "\Downarrow\mathrm{id}"] & {[\underline{X}/G] = \mathcal{X}} \arrow[d, bend right, "{[\underline{g},\sigma]}"'] \arrow[d, bend left, "{[\underline{f},\rho]}"] \arrow[d, phantom, "{\overset{[\delta]}{\Leftarrow}}"] \\
\underline{\bar{Y}} = [\underline{Y}/\{1\}] \arrow[rr, "{[\underline{\mathrm{id}}_{Y},\iota]}"] & & {[\underline{Y}/H] = \mathcal{Y},}
\end{tikzcd}$$

for $[\delta]$ as in §C.4, we can prove $[\eta, \delta] : [\boldsymbol{f}, \rho] \Rightarrow [\boldsymbol{g}, \sigma]$ is a 2-morphism in $\mathbf{dSta^c}$.

Generalizing the proof of Theorem 9.16, we can show:

Theorem 11.8. *The analogue of Theorem* 9.16 *holds in* $\mathbf{dSta^c}$.

11.3 Boundaries of d-stacks with corners, and simple, semisimple and flat 1-morphisms

Next we define boundaries of d-stacks with corners, following §6.2 and §8.6.

Definition 11.9. Let $\mathfrak{X}=(\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i}_{\mathfrak{X}},\omega_{\mathfrak{X}})$ be a d-stack with corners. We will define a d-stack with corners $\partial\mathfrak{X}=(\boldsymbol{\partial\mathcal{X}},\boldsymbol{\partial^2\mathcal{X}},\boldsymbol{i}_{\partial\mathfrak{X}},\omega_{\partial\mathfrak{X}})$, called the *boundary* of $\mathfrak{X}$, and show that $\boldsymbol{i}_{\mathfrak{X}}:\partial\mathfrak{X}\rightarrow\mathfrak{X}$ is a 1-morphism in $\mathbf{dSta^c}$. Here $\boldsymbol{\partial\mathcal{X}}$ and $\boldsymbol{i}_{\mathfrak{X}}$ are given in $\mathfrak{X}$, so the new data we have to construct is $\boldsymbol{\partial^2\mathcal{X}},\boldsymbol{i}_{\partial\mathfrak{X}},\omega_{\partial\mathfrak{X}}$.

As $i_{\mathfrak{X}}:\partial\mathcal{X}\rightarrow\mathcal{X}$ is strongly representable by Definition 11.1(a), Proposition C.15 defines an explicit fibre product $\partial\mathcal{X}\times_{i_{\mathfrak{X}},\mathcal{X},i_{\mathfrak{X}}}\partial\mathcal{X}$ with strongly representable projection morphisms $\pi_1,\pi_2:\partial\mathcal{X}\times_{\mathcal{X}}\partial\mathcal{X}\rightarrow\partial\mathcal{X}$ such that $i_{\mathfrak{X}}\circ\pi_1=i_{\mathfrak{X}}\circ\pi_2$. We will use this explicit fibre product throughout. There is a unique diagonal 1-morphism $\Delta_{\partial\mathcal{X}}:\partial\mathcal{X}\rightarrow\partial\mathcal{X}\times_{\mathcal{X}}\partial\mathcal{X}$ with $\pi_1\circ\Delta_{\partial\mathcal{X}}=\pi_2\circ\Delta_{\partial\mathcal{X}}=\mathrm{id}_{\partial\mathcal{X}}$. Since $\partial\mathcal{X}$ is separated and $i_{\mathfrak{X}}$ is an immersion, $\Delta_{\partial\mathcal{X}}$ is an equivalence with an open and closed C^∞-substack $\Delta_{\partial\mathcal{X}}(\partial\mathcal{X})\subseteq\partial\mathcal{X}\times_{\mathcal{X}}\partial\mathcal{X}$. Define $\partial^2\mathcal{X}=\partial\mathcal{X}\times_{\mathcal{X}}\partial\mathcal{X}\setminus\Delta_{\partial\mathcal{X}}(\partial\mathcal{X})$. Then $\partial^2\mathcal{X}$ is also open and closed in $\partial\mathcal{X}\times_{\mathcal{X}}\partial\mathcal{X}$. It is separated and second countable as $\partial\mathcal{X}$ is.

Define C^∞-stack 1-morphisms $i_{\partial\mathfrak{X}}=\pi_1|_{\partial^2\mathcal{X}}:\partial^2\mathcal{X}\rightarrow\partial\mathcal{X}$ and $j_{\partial\mathfrak{X}}=\pi_2|_{\partial^2\mathcal{X}}:\partial^2\mathcal{X}\rightarrow\partial\mathcal{X}$. Since $i_{\mathfrak{X}}$ is proper, π_1,π_2 are proper, so $i_{\partial\mathfrak{X}},j_{\partial\mathfrak{X}}$ are proper as they are the restrictions of π_1,π_2 to a closed C^∞-substack. Also $i_{\partial\mathfrak{X}},j_{\partial\mathfrak{X}}$ are strongly representable by Proposition C.14(c), as π_1,π_2 are strongly representable and $\partial^2\mathcal{X}$ is open in $\partial\mathcal{X}\times_{\mathcal{X}}\partial\mathcal{X}$.

Define the data $\mathcal{O}'_{\partial^2\mathcal{X}},\mathcal{E}_{\partial^2\mathcal{X}},\imath_{\partial^2\mathcal{X}},\jmath_{\partial^2\mathcal{X}}$ and $i'_{\partial\mathfrak{X}},i''_{\partial\mathfrak{X}},j'_{\partial\mathfrak{X}},j''_{\partial\mathfrak{X}}$ in $\boldsymbol{\partial^2\mathcal{X}}=(\partial^2\mathcal{X},\mathcal{O}'_{\partial^2\mathcal{X}},\mathcal{E}_{\partial^2\mathcal{X}},\imath_{\partial^2\mathcal{X}},\jmath_{\partial^2\mathcal{X}})$ and $\boldsymbol{i}_{\partial\mathfrak{X}}=(i_{\partial\mathfrak{X}},i'_{\partial\mathfrak{X}},i''_{\partial\mathfrak{X}})$, $\boldsymbol{j}_{\partial\mathfrak{X}}=(j_{\partial\mathfrak{X}},j'_{\partial\mathfrak{X}},j''_{\partial\mathfrak{X}})$, and the orientation $\omega_{\partial\mathfrak{X}}$ on $\mathcal{N}_{\partial\mathfrak{X}}$, by lifting the definitions of $\mathcal{O}'_{\partial^2X},\mathcal{E}_{\partial^2X},\ldots,$ $j''_{\partial\mathbf{X}},\omega_{\partial\mathbf{X}}$ in Definition 6.10 from sheaves and morphisms on the C^∞-scheme $\underline{\partial^2X}$ to sheaves and morphisms on the Deligne–Mumford C^∞-stack $\partial^2\mathcal{X}$. The same or similar proofs then show that $\boldsymbol{\partial^2\mathcal{X}}$ is a d-stack, and $\boldsymbol{i}_{\partial\mathfrak{X}}:\boldsymbol{\partial^2\mathcal{X}}\rightarrow\boldsymbol{\partial\mathcal{X}}$, $\boldsymbol{j}_{\partial\mathfrak{X}}:\boldsymbol{\partial^2\mathcal{X}}\rightarrow\boldsymbol{\partial\mathcal{X}}$ are 1-morphisms in $\mathbf{dSta}$, and $\mathcal{N}_{\mathfrak{X}}=\mathrm{Ker}(i^2_{\mathfrak{X}})$ is a line bundle on $\partial^2\mathcal{X}$ with orientation $\omega_{\mathfrak{X}}$.

We have already proved Definition 11.1(a),(c) for $\partial\mathfrak{X}$, and part (b) is immediate from the definition of $i''_{\partial\mathfrak{X}}$. For (d), by Definition 11.1(d) for $\mathfrak{X}$, given an étale 1-morphism $u:\bar{\underline{U}}\rightarrow\mathcal{X}$ with $\underline{U}$ separated and second countable, we construct an étale 1-morphism $u_\partial:\bar{\underline{\partial U}}\rightarrow\partial\mathcal{X}$, and then $\underline{U},\underline{\partial U}$ extend to a d-space with corners $\mathbf{U}$ with $\mathfrak{X}$ étale locally modelled on $\mathbf{U}$. Comparing the construction of $\partial\mathbf{U}$ from $\mathbf{U}$ in §6.2 and the construction of $\partial\mathfrak{X}$ from $\mathfrak{X}$ above, we see that applying Definition 11.1(d) with $\partial\mathfrak{X}=(\boldsymbol{\partial\mathcal{X}},\boldsymbol{\partial^2\mathcal{X}},\boldsymbol{i}_{\partial\mathfrak{X}},\omega_{\partial\mathfrak{X}})$ and $u_\partial:\bar{\underline{\partial U}}\rightarrow\partial\mathcal{X}$ in place of $\mathbf{X}$ and $u:\bar{\underline{U}}\rightarrow\mathcal{X}$, reconstructs $\partial\mathbf{U}$ in place of $\mathbf{U}$. Since $\partial\mathbf{U}$ is a d-space with corners, this shows that Definition 11.1(d) holds for $\partial\mathfrak{X}$ and $u_\partial:\bar{\underline{\partial U}}\rightarrow\partial\mathcal{X}$. As all such $u_\partial:\bar{\underline{\partial U}}\rightarrow\partial\mathcal{X}$ form an étale open cover of $\partial\mathcal{X}$, Definition 11.1(d) holds for $\partial\mathfrak{X}$. Hence $\partial\mathfrak{X}$ is a d-stack with corners.

Here are the analogues of Definition 6.11 and Theorem 6.12 in §6.3.

Definition 11.10. Let $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ be a 1-morphism of d-stacks with corners, and $\partial\boldsymbol{\mathcal{X}}$ the boundary of $\boldsymbol{\mathcal{X}}$. Then $s_{\boldsymbol{f}}:\mathcal{S}_{\boldsymbol{f}}\to\partial\mathcal{X}$ is strongly representable, and proper and étale by Definition 11.2(c). This and $\partial\mathcal{X}$ locally compact imply that $s_{\boldsymbol{f}}(\mathcal{S}_{\boldsymbol{f}})$ is open and closed in $\partial\mathcal{X}$. Define $\partial_-^{\boldsymbol{f}}\mathcal{X}=s_{\boldsymbol{f}}(\mathcal{S}_{\boldsymbol{f}})$ and $\partial_+^{\boldsymbol{f}}\mathcal{X}=\partial\mathcal{X}\setminus\partial_-^{\boldsymbol{f}}\mathcal{X}$. Then $\partial_\pm^{\boldsymbol{f}}\mathcal{X}$ are open and closed C^∞-substacks of $\partial\mathcal{X}$, with $\partial\mathcal{X}=\partial_+^{\boldsymbol{f}}\mathcal{X}\amalg\partial_-^{\boldsymbol{f}}\mathcal{X}$. Write $\partial_+^{\boldsymbol{f}}\boldsymbol{\mathcal{X}},\partial_-^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}$ for the open and closed d-substacks of $\partial\boldsymbol{\mathcal{X}}$ corresponding to $\partial_+^{\boldsymbol{f}}\mathcal{X},\partial_-^{\boldsymbol{f}}\mathcal{X}$, as in Definition 11.1. Then $\partial_\pm^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}$ are d-stacks with corners, with $\partial\boldsymbol{\mathcal{X}}=\partial_+^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}\amalg\partial_-^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}$.

We call $\boldsymbol{f}$ *simple* if $s_{\boldsymbol{f}}:\mathcal{S}_{\boldsymbol{f}}\to\partial\mathcal{X}$ is an equivalence, and we call $\boldsymbol{f}$ *semisimple* if $s_{\boldsymbol{f}}:\mathcal{S}_{\boldsymbol{f}}\to\partial_-^{\boldsymbol{f}}\mathcal{X}$ is an equivalence, and we call $\boldsymbol{f}$ *flat* if $\mathcal{T}_{\boldsymbol{f}}=\emptyset$. Simple implies semisimple. If $\boldsymbol{f}$ is simple then $\partial_-^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}=\partial\boldsymbol{\mathcal{X}}$ and $\partial_+^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}=\emptyset$.

One can show that $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ is simple, semisimple or flat if and only if the 1-morphisms $\boldsymbol{h}:\mathbf{U}\to\mathbf{V}$ in Definition 11.2(∗) for $\boldsymbol{f}$ are simple, semisimple or flat in the sense of §6.3 for all $\boldsymbol{h}:\mathbf{U}\to\mathbf{V}$ in (∗), or equivalently, for all $\boldsymbol{h}_i:\mathbf{U}_i\to\mathbf{V}_i$ from a collection $\{(\underline{U}_i,u_i,\underline{V}_i,v_i,\underline{h}_i):i\in I\}$ such that $\{(\underline{U}_i,u_i):i\in I\}$ are an étale open cover of $\mathcal{X}$. That is, 1-morphisms $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ in $\mathbf{dSta^c}$ are simple, semisimple or flat if and only if they are étale locally modelled on simple, semisimple or flat 1-morphisms in $\mathbf{dSpa^c}$.

The condition that $i_{\mathcal{X}}$ is *strongly representable* in Definition 11.1(a) is essential in constructing $\boldsymbol{f}_-,\boldsymbol{\eta}_-$ in parts (b),(c) of the next theorem, and our main reason for including this in Definition 11.1 was to make the theorem hold. The proof of Theorem 11.11 combines those of Theorems 6.12 and 8.22.

Theorem 11.11. *Let $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ be a semisimple 1-morphism of d-stacks with corners. Then:*

(a) *Define $\boldsymbol{f}_+=\boldsymbol{f}\circ\boldsymbol{i}_{\boldsymbol{\mathcal{X}}}|_{\partial_+^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}}:\partial_+^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$. Then $\boldsymbol{f}_+$ is semisimple. If $\boldsymbol{f}$ is flat then $\boldsymbol{f}_+$ is also flat.*

(b) *There exists a unique, semisimple 1-morphism $\boldsymbol{f}_-:\partial_-^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}\to\partial\boldsymbol{\mathcal{Y}}$ with $\boldsymbol{f}\circ\boldsymbol{i}_{\boldsymbol{\mathcal{X}}}|_{\partial_-^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}}=\boldsymbol{i}_{\boldsymbol{\mathcal{Y}}}\circ\boldsymbol{f}_-$. If $\boldsymbol{f}$ is simple then $\boldsymbol{f}_-:\partial\boldsymbol{\mathcal{X}}\to\partial\boldsymbol{\mathcal{Y}}$ is also simple. If $\boldsymbol{f}$ is flat then $\boldsymbol{f}_-$ is flat.*

(c) *Let $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ be another 1-morphism and $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a 2-morphism in $\mathbf{dSta^c}$. Then $\boldsymbol{g}$ is also semisimple, with $\partial_-^{\boldsymbol{g}}\boldsymbol{\mathcal{X}}=\partial_-^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}$. If $\boldsymbol{f}$ is simple, or flat, then $\boldsymbol{g}$ is simple, or flat, respectively. Part* **(b)** *defines 1-morphisms $\boldsymbol{f}_-,\boldsymbol{g}_-:\partial_-^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}\to\partial\boldsymbol{\mathcal{Y}}$. There is a unique 2-morphism $\boldsymbol{\eta}_-:\boldsymbol{f}_-\Rightarrow\boldsymbol{g}_-$ in $\mathbf{dSta^c}$ such that*

$$\mathbf{id}_{\boldsymbol{i}_{\boldsymbol{\mathcal{Y}}}}*\boldsymbol{\eta}_-=\boldsymbol{\eta}*\mathbf{id}_{\boldsymbol{i}_{\boldsymbol{\mathcal{X}}}|_{\partial_-^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}}}:\boldsymbol{f}\circ\boldsymbol{i}_{\boldsymbol{\mathcal{X}}}|_{\partial_-^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}}=\boldsymbol{i}_{\boldsymbol{\mathcal{Y}}}\circ\boldsymbol{f}_-\Longrightarrow\boldsymbol{g}\circ\boldsymbol{i}_{\boldsymbol{\mathcal{X}}}|_{\partial_-^{\boldsymbol{f}}\boldsymbol{\mathcal{X}}}=\boldsymbol{i}_{\boldsymbol{\mathcal{Y}}}\circ\boldsymbol{g}_-. \quad (11.24)$$

The analogue of Proposition 6.13 holds in $\mathbf{dSta^c}$. As for Corollary 6.14, we deduce:

Corollary 11.12. *Write* $\mathbf{dSta^c_{si}}$ *for the* 2*-subcategory of* $\mathbf{dSta^c}$ *with arbitrary objects and* 2*-morphisms, but only simple* 1*-morphisms. Then there is a strict* 2*-functor* $\partial : \mathbf{dSta^c_{si}} \to \mathbf{dSta^c_{si}}$ *mapping* $\mathcal{X} \mapsto \partial\mathcal{X}$ *on objects,* $\boldsymbol{f} \mapsto \boldsymbol{f}_-$ *on (simple)* 1*-morphisms, and* $\boldsymbol{\eta} \mapsto \boldsymbol{\eta}_-$ *on* 2*-morphisms.*

11.4 Equivalences of d-stacks with corners, and gluing

Here are the analogues of Propositions 6.20 and 6.21 for equivalences in $\mathbf{dSta^c}$. The proofs follow those in §6.5 very closely, and we leave them as an exercise. Note that necessary and sufficient conditions for $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ to be an equivalence in $\mathbf{dSta}$ are given in Propositions 9.10 and 9.11.

Proposition 11.13. *Suppose* $\boldsymbol{f} : \mathcal{X} \to \mathcal{Y}$ *is an equivalence in* $\mathbf{dSta^c}$*. Then* $\boldsymbol{f}$ *is simple and flat, and* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *is an equivalence in* $\mathbf{dSta}$*. Also* $\boldsymbol{f}_- : \partial\mathcal{X} \to \partial\mathcal{Y}$ *is an equivalence in* $\mathbf{dSta^c}$*.*

Proposition 11.14. *Let* $\boldsymbol{f} : \mathcal{X} \to \mathcal{Y}$ *be a simple, flat* 1*-morphism in* $\mathbf{dSta^c}$ *with* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *an equivalence in* $\mathbf{dSta}$*. Then* $\boldsymbol{f}$ *is an equivalence in* $\mathbf{dSta^c}$*.*

Here is the analogue of Definitions 2.23, 6.22 and 9.13.

Definition 11.15. Let $\boldsymbol{f} : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism in $\mathbf{dSta^c}$. We call $\boldsymbol{f}$ *étale* if $f : \mathcal{X} \to \mathcal{Y}$ is representable, and every 1-morphism $\boldsymbol{h} : \mathbf{U} \to \mathbf{V}$ in $\mathbf{dSpa^c}$ constructed from $\boldsymbol{f}$ in Definition 11.2(∗) is étale in $\mathbf{dSpa^c}$, as in Definition 6.22. That is, $\boldsymbol{f}$ is étale if it is étale locally an equivalence in $\mathbf{dSpa^c}$.

As for Corollary 6.23, we may deduce:

Corollary 11.16. *Let* $\boldsymbol{f} : \mathcal{X} \to \mathcal{Y}$ *be a* 1*-morphism in* $\mathbf{dSta^c}$*. Then* $\boldsymbol{f}$ *is étale if and only if* $\boldsymbol{f}$ *is simple and flat and* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *is étale in* $\mathbf{dSta}$*.*

In sections 2.4, 6.6 and 9.4 we studied gluing objects and 1-morphisms by equivalences in $\mathbf{dSpa}$, $\mathbf{dSpa^c}$ and $\mathbf{dSta}$. The results of §6.6 were essentially identical to those of §2.4, replacing $\mathbf{dSpa}$ by $\mathbf{dSpa^c}$. The results of §9.4 were more complicated, as we had to impose extra conditions on the C^∞-stack 2-morphism components η in d-stack 2-morphisms $\boldsymbol{\eta} = (\eta, \eta')$.

For d-stacks with corners, our results are essentially identical to those of §9.4, replacing $\mathbf{dSta}$ by $\mathbf{dSta^c}$. The proofs combine those of §6.6 and §9.4. We now give analogues of Proposition 9.17 and Theorems 9.18 and 9.19 for $\mathbf{dSta^c}$, and brief explanations of how to prove them.

Proposition 11.17. *Suppose* $\mathcal{X}, \mathcal{Y}$ *are d-stacks with corners,* $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$ *are open d-substacks with* $\mathcal{X} = \mathcal{U} \cup \mathcal{V}$*,* $\boldsymbol{f} : \mathcal{U} \to \mathcal{Y}$ *and* $\boldsymbol{g} : \mathcal{V} \to \mathcal{Y}$ *are* 1*-morphisms, and* $\boldsymbol{\eta} : \boldsymbol{f}|_{\mathcal{U}\cap\mathcal{V}} \Rightarrow \boldsymbol{g}|_{\mathcal{U}\cap\mathcal{V}}$ *is a* 2*-morphism in* $\mathbf{dSta^c}$*. Then there exist a* 1*-morphism* $\boldsymbol{h} : \mathcal{X} \to \mathcal{Y}$ *and* 2*-morphisms* $\boldsymbol{\zeta} : \boldsymbol{h}|_{\mathcal{U}} \Rightarrow \boldsymbol{f}$*,* $\boldsymbol{\theta} : \boldsymbol{h}|_{\mathcal{V}} \Rightarrow \boldsymbol{g}$ *in* $\mathbf{dSta^c}$ *such that* $\boldsymbol{\theta}|_{\mathcal{U}\cap\mathcal{V}} = \boldsymbol{\eta} \odot \boldsymbol{\zeta}|_{\mathcal{U}\cap\mathcal{V}} : \boldsymbol{h}|_{\mathcal{U}\cap\mathcal{V}} \Rightarrow \boldsymbol{g}|_{\mathcal{U}\cap\mathcal{V}}$*. This* $\boldsymbol{h}$ *is unique up to* 2*-isomorphism.*

Furthermore, $\boldsymbol{h}$ *is independent up to* 2*-isomorphism of the component* η' *in* $\boldsymbol{\eta} = (\eta, \eta')$*, but it may depend on* η*.*

To prove this, first apply Proposition 9.17 to construct $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\zeta},\boldsymbol{\theta}$ as 1- and 2-morphisms in **dSta** rather than $\mathbf{dSta^c}$. Then follow the proof of Proposition 6.24 to show that these particular $\boldsymbol{h},\boldsymbol{\zeta},\boldsymbol{\theta}$ (in which h',h'',ζ',θ' are constructed explicitly from f',f'',g',g'',η' using a partition of unity on $\mathcal{X}$, as in the proof of Proposition 2.27), are actually 1- and 2-morphisms in $\mathbf{dSta^c}$. We explain in Remark C.27 and Example C.33 how to extend partition of unity arguments on C^∞-schemes $\underline{X}$ to Deligne–Mumford C^∞-stacks $\mathcal{X}$, by using partitions of unity on the coarse moduli C^∞-scheme $\underline{\mathcal{X}}_{\rm top}$.

Theorem 11.18. *Suppose $\mathfrak{X},\mathfrak{Y}$ are d-stacks with corners, $\mathfrak{U}\subseteq\mathfrak{X}$, $\mathfrak{V}\subseteq\mathfrak{Y}$ are open d-substacks, and $\boldsymbol{f}:\mathfrak{U}\rightarrow\mathfrak{V}$ is an equivalence in $\mathbf{dSta^c}$. At the level of topological spaces, we have open $\mathcal{U}_{\rm top}\subseteq\mathcal{X}_{\rm top}$, $\mathcal{V}_{\rm top}\subseteq\mathcal{Y}_{\rm top}$ with a homeomorphism $f_{\rm top}:\mathcal{U}_{\rm top}\rightarrow\mathcal{V}_{\rm top}$, so we can form the quotient topological space $\mathcal{Z}_{\rm top}:=\mathcal{X}_{\rm top}\amalg_{f_{\rm top}}\mathcal{Y}_{\rm top}=(\mathcal{X}_{\rm top}\amalg\mathcal{Y}_{\rm top})/\sim$, where the equivalence relation $\sim$ on $\mathcal{X}_{\rm top}\amalg\mathcal{Y}_{\rm top}$ identifies $[u]\in\mathcal{U}_{\rm top}\subseteq\mathcal{X}_{\rm top}$ with $f_{\rm top}([u])\in\mathcal{V}_{\rm top}\subseteq\mathcal{Y}_{\rm top}$.*

Suppose $\mathcal{Z}_{\rm top}$ is Hausdorff. This condition may also equivalently be imposed at the level of C^∞-stacks, that is, we may form a quotient C^∞-stack $\mathcal{Z}=\mathcal{X}\amalg_f\mathcal{Y}$ by Proposition C.9, *and we require $\mathcal{Z}$ separated. Then there exist a d-stack with corners $\mathfrak{Z}$, open d-substacks $\hat{\mathfrak{X}},\hat{\mathfrak{Y}}$ in $\mathfrak{Z}$ with $\mathfrak{Z}=\hat{\mathfrak{X}}\cup\hat{\mathfrak{Y}}$, equivalences $\boldsymbol{g}:\mathfrak{X}\rightarrow\hat{\mathfrak{X}}$ and $\boldsymbol{h}:\mathfrak{Y}\rightarrow\hat{\mathfrak{Y}}$ such that $\boldsymbol{g}|_{\mathfrak{U}}$ and $\boldsymbol{h}|_{\mathfrak{V}}$ are both equivalences with $\hat{\mathfrak{X}}\cap\hat{\mathfrak{Y}}$, and a* 2*-morphism $\boldsymbol{\eta}:\boldsymbol{g}|_{\mathfrak{U}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}:\mathfrak{U}\rightarrow\hat{\mathfrak{X}}\cap\hat{\mathfrak{Y}}$. Furthermore, $\mathfrak{Z}$ is independent of choices up to equivalence.*

The basic idea of the proof of Theorem 11.18 is to follow the proof of Theorem 6.25 for $\mathbf{dSpa^c}$, but using Theorem 9.18 rather than Theorem 2.28. To define $\mathfrak{Z}=(\boldsymbol{\mathcal{Z}},\boldsymbol{\partial\mathcal{Z}},\boldsymbol{i}_{\mathfrak{Z}},\omega_{\mathfrak{Z}})$, we make $\boldsymbol{\mathcal{Z}}$ by gluing the d-stacks $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}$ on the open d-substacks $\boldsymbol{\mathcal{U}}\subseteq\boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{V}}\subseteq\boldsymbol{\mathcal{Y}}$ by the equivalence $\boldsymbol{f}:\boldsymbol{\mathcal{U}}\rightarrow\boldsymbol{\mathcal{V}}$ using Theorem 9.18.

Since $\boldsymbol{f}:\mathbf{U}\rightarrow\mathbf{V}$ is an equivalence in $\mathbf{dSta^c}$, it is simple, so Theorem 11.11(b) gives a 1-morphism $\boldsymbol{f}_-:\partial\mathbf{U}\rightarrow\partial\mathbf{V}$, which is also an equivalence. Then $\boldsymbol{\partial\mathcal{X}},\boldsymbol{\partial\mathcal{Y}}$ are d-stacks, and $\boldsymbol{\partial\mathcal{U}}\subseteq\boldsymbol{\partial\mathcal{X}}$, $\boldsymbol{\partial\mathcal{V}}\subseteq\boldsymbol{\partial\mathcal{V}}$ are open d-substacks, and $\boldsymbol{f}_-:\boldsymbol{\partial\mathcal{U}}\rightarrow\boldsymbol{\partial\mathcal{V}}$ is an equivalence in **dSta**, so Theorem 9.18 applies to glue $\boldsymbol{\partial\mathcal{X}},\boldsymbol{\partial\mathcal{Y}}$ on $\boldsymbol{\partial\mathcal{U}},\boldsymbol{\partial\mathcal{V}}$, yielding a d-stack $\boldsymbol{\partial\tilde{\mathcal{Z}}}$. Doing these two constructions using compatible choices as in the proof of Theorem 6.25, the 1-morphisms $\boldsymbol{i}_{\mathfrak{X}}:\boldsymbol{\partial\mathcal{X}}\rightarrow\boldsymbol{\mathcal{X}}$, $\boldsymbol{i}_{\mathfrak{Y}}:\boldsymbol{\partial\mathcal{Y}}\rightarrow\boldsymbol{\mathcal{Y}}$ also glue to give a 1-morphism $\tilde{\boldsymbol{\imath}}_{\mathfrak{Z}}:\boldsymbol{\partial\tilde{\mathcal{Z}}}\rightarrow\boldsymbol{\mathcal{Z}}$.

The underlying C^∞-stack 1-morphism $\tilde\imath_{\mathfrak{Z}}:\partial\tilde{\mathcal{Z}}\rightarrow\mathcal{Z}$ may not be strongly representable. So we apply Proposition C.14(b) to obtain a C^∞-stack $\partial\mathcal{Z}$, a strongly representable 1-morphism $i_{\mathfrak{Z}}:\partial\mathcal{Z}\rightarrow\mathcal{Z}$ and an equivalence $j:\partial\tilde{\mathcal{Z}}\rightarrow\partial\mathcal{Z}$ with $i_{\mathfrak{Z}}\circ j=\tilde\imath_{\mathfrak{Z}}$. Then we transport the data $\mathcal{O}'_{\partial\tilde{\mathcal{Z}}},\mathcal{E}_{\partial\tilde{\mathcal{Z}}},\imath_{\partial\tilde{\mathcal{Z}}},\jmath_{\partial\tilde{\mathcal{Z}}}$ and $\tilde\imath'_{\mathfrak{Z}},\tilde\imath''_{\mathfrak{Z}}$ in $\boldsymbol{\partial\tilde{\mathcal{Z}}}=(\partial\tilde{\mathcal{Z}},\mathcal{O}'_{\partial\tilde{\mathcal{Z}}},\mathcal{E}_{\partial\tilde{\mathcal{Z}}},\imath_{\partial\tilde{\mathcal{Z}}},\jmath_{\partial\tilde{\mathcal{Z}}})$ and $\tilde{\boldsymbol{\imath}}_{\mathfrak{Z}}=(\tilde\imath_{\mathfrak{Z}},\tilde\imath'_{\mathfrak{Z}},\tilde\imath''_{\mathfrak{Z}})$ along the equivalence j to extend $\partial\mathcal{Z},i_{\mathfrak{X}}$ to a d-stack $\boldsymbol{\partial\mathcal{Z}}=(\partial\mathcal{Z},\mathcal{O}'_{\partial\mathcal{Z}},\mathcal{E}_{\partial\mathcal{Z}},\imath_{\partial\mathcal{Z}},\jmath_{\partial\mathcal{Z}})$ and a 1-morphism $\boldsymbol{i}_{\mathfrak{Z}}=(i_{\mathfrak{Z}},i'_{\mathfrak{Z}},i''_{\mathfrak{Z}}):\boldsymbol{\partial\mathcal{Z}}\rightarrow\boldsymbol{\mathcal{Z}}$, which are unique up to canonical 1-isomorphism.

That is, we make $\boldsymbol{\partial\mathcal{Z}},\boldsymbol{i}_{\mathfrak{Z}}$ from $\boldsymbol{\partial\tilde{\mathcal{Z}}},\tilde{\boldsymbol{\imath}}_{\mathfrak{Z}}$ by replacing $\partial\tilde{\mathcal{Z}}$ by an equivalent C^∞-stack $\partial\mathcal{Z}$ so as to make $i_{\mathfrak{Z}}$ strongly representable. The rest of the proof of Theorem 6.25 extends to $\mathbf{dSta^c}$ by our usual method of deducing results in $\mathbf{dSta^c}$ from corresponding results in $\mathbf{dSpa^c}$ applied to étale open covers.

Theorem 11.19. *Suppose I is an indexing set, and $<$ is a total order on I,* *and* $\boldsymbol{\mathcal{X}}_i$ *for* $i \in I$ *are d-stacks with corners, and for all* $i < j$ *in* I *we are given open d-substacks* $\boldsymbol{\mathcal{U}}_{ij} \subseteq \boldsymbol{\mathcal{X}}_i$, $\boldsymbol{\mathcal{U}}_{ji} \subseteq \boldsymbol{\mathcal{X}}_j$ *and an equivalence* $\boldsymbol{e}_{ij} : \boldsymbol{\mathcal{U}}_{ij} \to \boldsymbol{\mathcal{U}}_{ji}$ *in* $\mathbf{dSta^c}$ *satisfying the following properties:*

(a) *For all* $i < j < k$ *in* I *we have a 2-commutative diagram*

$$\begin{tikzcd} & \boldsymbol{\mathcal{U}}_{ji} \cap \boldsymbol{\mathcal{U}}_{jk} \arrow[dr, "\boldsymbol{e}_{jk}|_{\boldsymbol{\mathcal{U}}_{ji}\cap\boldsymbol{\mathcal{U}}_{jk}}"] \arrow[d, phantom, "\Downarrow \boldsymbol{\eta}_{ijk}"] & \\ \boldsymbol{\mathcal{U}}_{ij} \cap \boldsymbol{\mathcal{U}}_{ik} \arrow[ur, "\boldsymbol{e}_{ij}|_{\boldsymbol{\mathcal{U}}_{ij}\cap\boldsymbol{\mathcal{U}}_{ik}}"] \arrow[rr, "\boldsymbol{e}_{ik}|_{\boldsymbol{\mathcal{U}}_{ij}\cap\boldsymbol{\mathcal{U}}_{ik}}"] & & \boldsymbol{\mathcal{U}}_{ki} \cap \boldsymbol{\mathcal{U}}_{kj} \end{tikzcd}$$

for some $\boldsymbol{\eta}_{ijk}$*, where all three* 1*-morphisms are equivalences; and*

(b) *For all* $i < j < k < l$ *in* I *the components* η_{ijk} *in* $\boldsymbol{\eta}_{ijk} = (\eta_{ijk}, \eta'_{ijk})$ *satisfy*

$$\eta_{ikl} \odot (\mathrm{id}_{f_{kl}} * \eta_{ijk})|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}\cap\mathcal{U}_{il}} = \eta_{ijl} \odot (\eta_{jkl} * \mathrm{id}_{f_{ij}})|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}\cap\mathcal{U}_{il}}. \quad (11.25)$$

Note that if the C^∞*-stacks* $\mathcal{X}_i$ *in* $\boldsymbol{\mathcal{X}}_i$ *are effective for all* $i \in I$*, then Proposition* C.29(ii) *implies that* (11.25) *holds automatically, as there is only one* 2*-morphism* $\boldsymbol{e}_{kl} \circ \boldsymbol{e}_{jk} \circ \boldsymbol{e}_{ij}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}\cap\mathcal{U}_{il}} \Rightarrow \boldsymbol{e}_{il}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}\cap\mathcal{U}_{il}}$.

On the level of topological spaces, define the quotient topological space $\mathcal{Y}_{\mathrm{top}} = (\coprod_{i\in I} \mathcal{X}_{i,\mathrm{top}})/\sim$*, where* $\sim$ *is the equivalence relation generated by* $[x_i] \sim [x_j]$ *if* $[x_i] \in \mathcal{U}_{ij,\mathrm{top}} \subseteq \mathcal{X}_{i,\mathrm{top}}$ *and* $[x_j] \in \mathcal{U}_{ji,\mathrm{top}} \subseteq \mathcal{X}_{j,\mathrm{top}}$ *with* $e_{ij,\mathrm{top}}([x_i]) = [x_j]$*. Suppose* $\mathcal{Y}_{\mathrm{top}}$ *is Hausdorff and second countable. Then there exist a d-stack with corners* $\boldsymbol{\mathcal{Y}}$ *and a* 1*-morphism* $\boldsymbol{f}_i : \boldsymbol{\mathcal{X}}_i \to \boldsymbol{\mathcal{Y}}$ *which is an equivalence with an open d-substack* $\hat{\boldsymbol{\mathcal{X}}}_i \subseteq \boldsymbol{\mathcal{Y}}$ *for all* $i \in I$*, where* $\boldsymbol{\mathcal{Y}} = \bigcup_{i\in I} \hat{\boldsymbol{\mathcal{X}}}_i$*, such that* $\boldsymbol{f}_i|_{\boldsymbol{\mathcal{U}}_{ij}}$ *is an equivalence* $\boldsymbol{\mathcal{U}}_{ij} \to \hat{\boldsymbol{\mathcal{X}}}_i \cap \hat{\boldsymbol{\mathcal{X}}}_j$ *for all* $i < j$ *in* I*, and there exists a* 2*-morphism* $\boldsymbol{\eta}_{ij} : \boldsymbol{f}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{f}_i|_{\boldsymbol{\mathcal{U}}_{ij}}$ *in* $\mathbf{dSta^c}$*. The d-stack with corners* $\boldsymbol{\mathcal{Y}}$ *is unique up to equivalence, and is independent of choices of the components* η'_{ijk} *in* $\boldsymbol{\eta}_{ijk} = (\eta_{ijk}, \eta'_{ijk})$ *in* (a)*. If the* C^∞*-stacks* $\mathcal{X}_i$ *in* $\boldsymbol{\mathcal{X}}_i$ *are effective for all* $i \in I$*, then the* η_{ijk} *are unique by Proposition* C.29(ii)*, so in this case* $\boldsymbol{\mathcal{Y}}$ *is independent of the choices of* $\boldsymbol{\eta}_{ijk}$ *in* (a)*.*

Suppose also that $\boldsymbol{\mathcal{Z}}$ *is a d-stack with corners, and* $\boldsymbol{g}_i : \boldsymbol{\mathcal{X}}_i \to \boldsymbol{\mathcal{Z}}$ *are* 1*-morphisms for all* $i \in I$*, and there exist* 2*-morphisms* $\boldsymbol{\zeta}_{ij} : \boldsymbol{g}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{g}_i|_{\boldsymbol{\mathcal{U}}_{ij}}$ *in* $\mathbf{dSta^c}$ *for all* $i < j$ *in* I*, such that for all* $i < j < k$ *in* I *the components* ζ_{ij}, η_{ijk} *in* $\boldsymbol{\zeta}_{ij}, \boldsymbol{\eta}_{ijk}$ *satisfy*

$$\big(\zeta_{ij}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\big) \odot \big(\zeta_{jk} * \mathrm{id}_{e_{ij}}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\big) = \big(\zeta_{ik}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\big) \odot \big(\mathrm{id}_{g_k} * \eta_{ijk}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}\big). \quad (11.26)$$

Then there exist a 1*-morphism* $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ *and* 2*-morphisms* $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{f}_i \Rightarrow \boldsymbol{g}_i$ *in* $\mathbf{dSta^c}$ *for all* $i \in I$*. The* 1*-morphism* $\boldsymbol{h}$ *is unique up to* 2*-isomorphism, and is independent of the components* ζ'_{ij} *in* $\boldsymbol{\zeta}_{ij} = (\zeta_{ij}, \zeta'_{ij})$*.*

Note that Proposition C.29 *gives conditions for uniqueness of* C^∞*-stack* 2*-morphisms* $\eta : f \Rightarrow g$*, and if any of these apply to* $g_i : \mathcal{X}_i \to \mathcal{Z}$ *for all* $i \in I$*, then* (11.26) *holds automatically, as there is only one* 2*-morphism* $g_k \circ e_{jk} \circ e_{ij}|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}} \Rightarrow g_i|_{\mathcal{U}_{ij}\cap\mathcal{U}_{ik}}$*, and also* ζ_{ij} *is unique, so* $\boldsymbol{h}$ *is independent of the choice of* $\boldsymbol{\zeta}_{ij}$*. In particular, if* $\boldsymbol{\mathcal{Z}}$ *is a d-space with corners, so that* $\mathcal{Z}$ *is a* C^∞*-scheme, then* (11.26) *always holds, and* $\boldsymbol{h}$ *is independent of the choice of* $\boldsymbol{\zeta}_{ij}$*.*

Theorem 11.19 is proved using repeated applications of Theorem 11.18 by an inductive procedure, as for the proofs of Theorem 2.30–2.33 in §2.4, just as Theorem 9.19 is proved using repeated applications of Theorem 9.18.

11.5 Corners $C_k(\boldsymbol{\mathcal{X}})$, and the corner functors $C, \hat{C}$

In §5.5, for a manifold with corners X we defined the k-corners $C_k(X) \cong \partial^k X/S_k$, and the corners $C(X) = \coprod_{k=0}^{\dim X} C_k(X)$, and for $f : X \to Y$ a smooth map of manifolds with corners we defined smooth maps $C(f), \hat{C}(f) : C(X) \to C(Y)$, giving the corner functors $C, \hat{C} : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$. We extended this to d-spaces with corners in §6.7, and to orbifolds with corners in §8.7.

Combining the ideas of §6.7 and §8.7, we can define the k-corners $C_k(\boldsymbol{\mathcal{X}}) \simeq [\partial^k \boldsymbol{\mathcal{X}}/S_k]$ of a d-stack with corners, and the corners $C(\boldsymbol{\mathcal{X}}) = \coprod_{k\geqslant 0} C_k(\boldsymbol{\mathcal{X}})$, and corner functors $C, \hat{C} : \mathbf{dSta^c} \to \mathbf{dSta^c}$. Basically, we follow §8.7 to define the C^∞-stacks $C_k(\mathcal{X}), \partial C_k(\mathcal{X}), \ldots$ and 1-morphisms $\Pi_{\mathcal{X}}, i_{C_k(\mathcal{X})}, C(f), \hat{C}(f), \ldots$ and 2-morphisms $C(\eta), \hat{C}(\eta), \ldots$ in $C_k(\boldsymbol{\mathcal{X}}), \boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}, C(\boldsymbol{f}), \hat{C}(\boldsymbol{f}), C(\boldsymbol{\eta}), \hat{C}(\boldsymbol{\eta})$, and then we follow §6.7 to define the remaining sheaf data $\mathcal{O}'_{C_k(X)}, \mathcal{E}'_{C_k(X)}, \imath_{C_k(X)}, \jmath_{C_k(X)}, \ldots, \Pi'_{\mathcal{X}}, \Pi''_{\mathcal{X}}, \ldots, C(\eta)', \ldots$. The ideas are similar to the definitions of $\partial\boldsymbol{\mathcal{X}}, \boldsymbol{f}_-, \boldsymbol{\eta}_-$ in §11.3. So for brevity we just state the main result:

Theorem 11.20. **(a)** *Let $\boldsymbol{\mathcal{X}}$ be a d-stack with corners. Then for each $k \geqslant 0$ we can define a d-stack with corners $C_k(\boldsymbol{\mathcal{X}})$ called the k-**corners** of $\boldsymbol{\mathcal{X}}$, and a 1-morphism $\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}^k : C_k(\boldsymbol{\mathcal{X}}) \to \boldsymbol{\mathcal{X}}$, such that $C_k(\boldsymbol{\mathcal{X}})$ is equivalent to a quotient d-stack $[\partial^k\boldsymbol{\mathcal{X}}/S_k]$ for a natural action of S_k on $\partial^k\boldsymbol{\mathcal{X}}$ by 1-isomorphisms. The C^∞-stack 1-morphism $\Pi_{\mathcal{X}}^k : C_k(\mathcal{X}) \to \mathcal{X}$ is strongly representable. The construction of $C_k(\boldsymbol{\mathcal{X}})$ is unique up to canonical 1-isomorphism.*

We can describe the topological space $C_k(\mathcal{X})_{\rm top}$ as follows. Consider pairs $(x, \{x'_1, \ldots, x'_k\})$, where $x : \bar{\underline{*}} \to \mathcal{X}$ and $x'_i : \bar{\underline{*}} \to \partial\mathcal{X}$ for $i = 1, \ldots, k$ are 1-morphisms in $\mathbf{C^\infty Sta}$ with $x'_1, \ldots, x'_k$ distinct and $x = i_{\mathcal{X}} \circ x'_1 = \cdots = i_{\mathcal{X}} \circ x'_k$. Define an equivalence relation $\approx$ on such pairs by $(x, \{x'_1, \ldots, x'_k\}) \approx (\tilde{x}, \{\tilde{x}'_1, \ldots, \tilde{x}'_k\})$ if there exist $\sigma \in S_k$ and 2-morphisms $\eta : x \Rightarrow \tilde{x}$ and $\eta'_i : x'_i \Rightarrow \tilde{x}'_{\sigma(i)}$ for $i = 1, \ldots, k$ with $\eta = \mathrm{id}_{i_{\mathcal{X}}} * \eta'_1 = \cdots = \mathrm{id}_{i_{\mathcal{X}}} * \eta'_k$. Write $[x, \{x'_1, \ldots, x'_k\}]$ for the $\approx$-equivalence class of $(x, \{x'_1, \ldots, x'_k\})$. Then

$$\begin{aligned} C_k(\mathcal{X})_{\rm top} \cong \bigl\{[x, \{x'_1, \ldots, x'_k\}] : x : \bar{\underline{*}} \to \mathcal{X},\ x'_i : \bar{\underline{*}} \to \partial\mathcal{X} \ \text{1-morphisms} \\ \text{with } x'_1, \ldots, x'_k \text{ distinct and } x = i_{\mathcal{X}} \circ x'_1 = \cdots = i_{\mathcal{X}} \circ x'_k\bigr\}. \end{aligned} \tag{11.27}$$

*If $\boldsymbol{\mathcal{X}}$ is étale locally modelled on a d-space with corners $\mathbf{U}$, as in Definition 11.1, then $C_k(\boldsymbol{\mathcal{X}})$ is étale locally modelled on $C_k(\mathbf{U})$ from §6.7. We have 1-isomorphisms $C_0(\boldsymbol{\mathcal{X}}) \cong \boldsymbol{\mathcal{X}}$ and $C_1(\boldsymbol{\mathcal{X}}) \cong \partial\boldsymbol{\mathcal{X}}$. We write $C(\boldsymbol{\mathcal{X}}) = \coprod_{k\geqslant 0} C_k(\boldsymbol{\mathcal{X}})$, so that $C(\boldsymbol{\mathcal{X}})$ is also a d-stack with corners, called the **corners** of $\boldsymbol{\mathcal{X}}$.*

(b) *Let $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a 1-morphism of d-stacks with corners. Then there are unique 1-morphisms $C(\boldsymbol{f}) : C(\boldsymbol{\mathcal{X}}) \to C(\boldsymbol{\mathcal{Y}})$ and $\hat{C}(\boldsymbol{f}) : C(\boldsymbol{\mathcal{X}}) \to C(\boldsymbol{\mathcal{Y}})$ in $\mathbf{dSta^c}$ such that $\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}} = \boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}} \circ \hat{C}(\boldsymbol{f}) : C(\boldsymbol{\mathcal{X}}) \to \boldsymbol{\mathcal{Y}}$, and if $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is étale locally modelled on a 1-morphism $\boldsymbol{h} : \mathbf{U} \to \mathbf{V}$ in $\mathbf{dSpa^c}$, as in Definition*

11.2, *then* $C(\boldsymbol{f}),\hat{C}(\boldsymbol{f})$ *are étale locally modelled on* $C(\boldsymbol{h}),\hat{C}(\boldsymbol{h}):C(\mathbf{U})\to C(\mathbf{V})$ *from Theorems* 6.29(a), 6.31(a).

We can also characterize the maps $C(f)_{\rm top}:C(\mathcal{X})_{\rm top}\to C(\mathcal{Y})_{\rm top}$, $\hat{C}(f)_{\rm top}:C(\mathcal{X})_{\rm top}\to C(\mathcal{Y})_{\rm top}$. *Identify* $C_k(\mathcal{X})_{\rm top}\subseteq C(\mathcal{X})_{\rm top}$ *with the right hand side of* (11.27), *and similarly for* $C_l(\mathcal{Y})_{\rm top}$, *and identify* $\mathcal{S}_{\boldsymbol{f},{\rm top}},\mathcal{T}_{\boldsymbol{f},{\rm top}}$ *with the right hand sides of* (11.9)–(11.10). *Then* $C(f)_{\rm top}$ *and* $\hat{C}(f)_{\rm top}$ *act by*

$$\begin{aligned}&C(f)_{\rm top}:\bigl[x,\{x_1',\ldots,x_k'\}\bigr]\longmapsto\bigl[y,\{y_1',\ldots,y_l'\}\bigr],\ \text{where } y=f\circ x,\\&\{y_1',\ldots,y_l'\}=\bigl\{y':[x_i',y']\in\mathcal{S}_{\boldsymbol{f},{\rm top}},\ \text{some } i=1,\ldots,k\bigr\},\ \text{and}\end{aligned}\tag{11.28}$$

$$\begin{aligned}&\hat{C}(f)_{\rm top}:\bigl[x,\{x_1',\ldots,x_k'\}\bigr]\longmapsto\bigl[y,\{y_1',\ldots,y_l'\}\bigr],\ \text{where } y=f\circ x,\\&\{y_1',\ldots,y_l'\}=\bigl\{y':[x_i',y']\in\mathcal{S}_{\boldsymbol{f},{\rm top}},\ i=1,\ldots,k\bigr\}\cup\bigl\{y':[x,y']\in\mathcal{T}_{\boldsymbol{f},{\rm top}}\bigr\}.\end{aligned}\tag{11.29}$$

For all $k,l\geqslant 0$, *write* $C_k^{\boldsymbol{f},l}(\boldsymbol{\mathcal{X}})=C_k(\boldsymbol{\mathcal{X}})\cap C(\boldsymbol{f})^{-1}(C_l(\boldsymbol{\mathcal{Y}}))$, *so that* $C_k^{\boldsymbol{f},l}(\boldsymbol{\mathcal{X}})$ *is an open and closed d-substack of* $C_k(\boldsymbol{\mathcal{X}})$ *with* $C_k(\boldsymbol{\mathcal{X}})=\coprod_{l=0}^{\infty}C_k^{\boldsymbol{f},l}(\boldsymbol{\mathcal{X}})$, *and write* $C_k^l(\boldsymbol{f})=C(\boldsymbol{f})|_{C_k^{\boldsymbol{f},l}(\boldsymbol{\mathcal{X}})}:C_k^{\boldsymbol{f},l}(\boldsymbol{\mathcal{X}})\to C_l(\boldsymbol{\mathcal{Y}})$. *If* $\boldsymbol{f}$ *is semisimple then* $C(\boldsymbol{f})$ *maps* $C_k(\boldsymbol{\mathcal{X}})\to\coprod_{l=0}^{k}C_l(\boldsymbol{\mathcal{Y}})$ *for all* $k\geqslant 0$. *If* $\boldsymbol{f}$ *is simple then* $C(\boldsymbol{f})$ *maps* $C_k(\boldsymbol{\mathcal{X}})\to C_k(\boldsymbol{\mathcal{Y}})$ *for all* $k\geqslant 0$. *If* $\boldsymbol{f}$ *is flat then* $C(\boldsymbol{f})=\hat{C}(\boldsymbol{f})$.

(c) *Let* $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ *be* 1*-morphisms and* $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ *a* 2*-morphism in* $\mathbf{dSta^c}$. *Then there exist unique* 2*-morphisms* $C(\boldsymbol{\eta}):C(\boldsymbol{f})\Rightarrow C(\boldsymbol{g})$, $\hat{C}(\boldsymbol{\eta}):\hat{C}(\boldsymbol{f})\Rightarrow\hat{C}(\boldsymbol{g})$ *in* $\mathbf{dSta^c}$, *where* $C(\boldsymbol{f}),C(\boldsymbol{g}),\hat{C}(\boldsymbol{f}),\hat{C}(\boldsymbol{g})$ *are as in* **(b)**, *such that*

$$\begin{aligned}&{\rm id}_{\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}}*C(\boldsymbol{\eta})=\boldsymbol{\eta}*{\rm id}_{\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}}:\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}\circ C(\boldsymbol{f})=\boldsymbol{f}\circ\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}\Longrightarrow\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}\circ C(\boldsymbol{g})=\boldsymbol{g}\circ\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}},\\&{\rm id}_{\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}}*\hat{C}(\boldsymbol{\eta})=\boldsymbol{\eta}*{\rm id}_{\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}}:\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}\circ\hat{C}(\boldsymbol{f})=\boldsymbol{f}\circ\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}\Longrightarrow\boldsymbol{\Pi}_{\boldsymbol{\mathcal{Y}}}\circ\hat{C}(\boldsymbol{g})=\boldsymbol{g}\circ\boldsymbol{\Pi}_{\boldsymbol{\mathcal{X}}}.\end{aligned}$$

If $\boldsymbol{f},\boldsymbol{g}$ *are flat then* $C(\boldsymbol{\eta})=\hat{C}(\boldsymbol{\eta})$.

(d) *Define* $C:\mathbf{dSta^c}\to\mathbf{dSta^c}$ *by* $C:\boldsymbol{\mathcal{X}}\mapsto C(\boldsymbol{\mathcal{X}})$, $C:\boldsymbol{f}\mapsto C(\boldsymbol{f})$, $C:\boldsymbol{\eta}\mapsto C(\boldsymbol{\eta})$ *on objects,* 1*- and* 2*-morphisms, where* $C(\boldsymbol{\mathcal{X}}),C(\boldsymbol{f}),C(\boldsymbol{\eta})$ *are as in* **(a)**–**(c)** *above. Similarly, define* $\hat{C}:\mathbf{dSta^c}\to\mathbf{dSta^c}$ *by* $\hat{C}:\boldsymbol{\mathcal{X}}\mapsto C(\boldsymbol{\mathcal{X}})$, $\hat{C}:\boldsymbol{f}\mapsto\hat{C}(\boldsymbol{f})$, $\hat{C}:\boldsymbol{\eta}\mapsto\hat{C}(\boldsymbol{\eta})$. *Then* $C,\hat{C}$ *are strict* 2*-functors, called* ***corner functors***.

11.6 Fibre products in dSta$^{\mathbf{c}}$

In §6.8, we defined what it means for 1-morphisms $\boldsymbol{g}:\mathbf{X}\to\mathbf{Z}$ and $\boldsymbol{h}:\mathbf{Y}\to\mathbf{Z}$ in $\mathbf{dSpa^c}$ to be *b-transverse* and *c-transverse*, where c-transverse implies b-transverse. We showed that if $\boldsymbol{g},\boldsymbol{h}$ are b-transverse, then a fibre product $\mathbf{X}\times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}}\mathbf{Y}$ exists in $\mathbf{dSpa^c}$. If $\boldsymbol{g},\boldsymbol{h}$ are c-transverse, we described the boundary $\partial(\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y})$ and corners $C_k(\mathbf{X}\times_{\mathbf{Z}}\mathbf{Y})$ of the fibre product in terms of the boundaries and corners of $\mathbf{X},\mathbf{Y},\mathbf{Z}$. All of this extends to the 2-category $\mathbf{dSta^c}$ of d-stacks with corners. Here are the analogues of Definitions 6.32 and 6.33.

Definition 11.21. Let $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Z}}$ and $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{Z}}$ be 1-morphisms in $\mathbf{dSta^c}$. We call $\boldsymbol{g},\boldsymbol{h}$ *b-transverse* if the following holds. Suppose $x:\bar{\underline{*}}\to\mathcal{X}$ and $y:\bar{\underline{*}}\to\mathcal{Y}$ are 1-morphisms in $\mathbf{C^\infty Sta}$, and $\eta:g\circ x\Rightarrow h\circ y$ is a 2-morphism. Since

$i_{\mathcal{X}} : \partial\mathcal{X} \to \mathcal{X}$ is finite and strongly representable, there are finitely many 1-morphisms $x' : \underline{\bar{*}} \to \partial\mathcal{X}$ with $x = i_{\mathcal{X}} \circ x'$. Write these x' as $x'_1, \ldots, x'_j$; they correspond to the points of $\underline{\bar{*}} \times_{x,\mathcal{X},i_{\mathcal{X}}} \partial\mathcal{X}$. Similarly, write $y'_1, \ldots, y'_k$ for the 1-morphisms $y' : \underline{\bar{*}} \to \partial\mathcal{Y}$ with $y = i_{\mathcal{Y}} \circ y'$. Write $z = g \circ x$ and $\tilde z = h \circ y$, so that $z, \tilde z : \underline{\bar{*}} \to \mathcal{Z}$ and $\eta : z \Rightarrow \tilde z$. Write $z'_1, \ldots, z'_l$ for the 1-morphisms $z' : \underline{\bar{*}} \to \partial\mathcal{Z}$ with $z = i_{\mathcal{Z}} \circ z'$. Then by Proposition C.13, for each $c = 1, \ldots, l$ there are unique $\tilde z'_c : \underline{\bar{*}} \to \partial\mathcal{Z}$ and $\eta'_c : z'_c \Rightarrow \tilde z'_c$ with $i_{\mathcal{Z}} \circ \tilde z'_c = \tilde z$ and $\mathrm{id}_{i_{\mathcal{Z}}} * \eta'_c = \eta$.

Let $a = 1, \ldots, j$ and $c = 1, \ldots, l$. Then $x'_a : \underline{\bar{*}} \to \partial\mathcal{X}$ and $z'_c : \underline{\bar{*}} \to \partial\mathcal{Z}$ with $g \circ i_{\mathcal{X}} \circ x'_a = i_{\mathcal{Z}} \circ z'_c$, so defining $\partial\mathcal{X} \times_{g\circ i_{\mathcal{X}},\mathcal{Z},i_{\mathcal{Z}}} \partial\mathcal{Z}$ by the explicit construction of Proposition C.15, there is a unique 1-morphism $(x'_a, z'_c) : \underline{\bar{*}} \to \partial\mathcal{X} \times_{\mathcal{Z}} \partial\mathcal{Z}$ with $(x'_a, z'_c) \circ \pi_{\partial\mathcal{X}} = x'_a$ and $(x'_a, z'_c) \circ \pi_{\partial\mathcal{Z}} = z'_c$. As in (11.9), we write $[x'_a, z'_c] \in \mathcal{S}_{\boldsymbol{g},\mathrm{top}}$ if (x'_a, z'_c) maps to the open C^∞-substack $\mathcal{S}_{\boldsymbol{g}}$ in $\partial\mathcal{X} \times_{\mathcal{Z}} \partial\mathcal{Z}$. Similarly, for $b = 1, \ldots, k$ and $c = 1, \ldots, l$ we have unique $(y'_b, \tilde z'_c) : \underline{\bar{*}} \to \partial\mathcal{Y} \times_{\mathcal{Z}} \partial\mathcal{Z}$ with $(y'_b, \tilde z'_c) \circ \pi_{\partial\mathcal{Y}} = y'_b$ and $(y'_b, \tilde z'_c) \circ \pi_{\partial\mathcal{Z}} = \tilde z'_c$, and we may have $[y'_b, \tilde z'_c] \in \mathcal{S}_{\boldsymbol{h},\mathrm{top}}$. We require that for all such x, y, η, the following morphism in $\mathrm{qcoh}(\underline{\bar{*}})$ is injective:

$$
\begin{aligned}
&\bigoplus_{a=1,\ldots,j,\ c=1,\ldots,l:\ [x'_a,z'_c]\in\mathcal{S}_{\boldsymbol{g},\mathrm{top}}} I_{(x'_a,z'_c),s_{\boldsymbol{g}}}(\mathcal{N}_{\boldsymbol{\mathcal{X}}})^{-1} \circ (x'_a, z'_c)^*(\lambda_{\boldsymbol{g}}) \circ I_{(x'_a,z'_c),u_{\boldsymbol{g}}}(\mathcal{N}_{\boldsymbol{\mathcal{Z}}}) \oplus \\
&\bigoplus_{b=1,\ldots,k,\ c=1,\ldots,l:\ [y'_b,\tilde z'_c]\in\mathcal{S}_{\boldsymbol{h},\mathrm{top}}} I_{(y'_b,\tilde z'_c),s_{\boldsymbol{h}}}(\mathcal{N}_{\boldsymbol{\mathcal{Y}}})^{-1} \circ (y'_b, \tilde z'_c)^*(\lambda_{\boldsymbol{h}}) \circ I_{(y'_b,\tilde z'_c),u_{\boldsymbol{h}}}(\mathcal{N}_{\boldsymbol{\mathcal{Z}}}) \circ (\eta'_c)^*(\mathcal{N}_{\boldsymbol{\mathcal{Z}}}) : \\
&\bigoplus\nolimits_{c=1}^{l} (z'_c)^*(\mathcal{N}_{\boldsymbol{\mathcal{Z}}}) \longrightarrow \bigoplus\nolimits_{a=1}^{j} (x'_a)^*(\mathcal{N}_{\boldsymbol{\mathcal{X}}}) \oplus \bigoplus\nolimits_{b=1}^{k} (y'_b)^*(\mathcal{N}_{\boldsymbol{\mathcal{Y}}}).
\end{aligned}
\tag{11.30}
$$

Here $\mathcal{N}_{\boldsymbol{\mathcal{X}}}, \mathcal{N}_{\boldsymbol{\mathcal{Y}}}, \mathcal{N}_{\boldsymbol{\mathcal{Z}}}$ and $\lambda_{\boldsymbol{g}}, \lambda_{\boldsymbol{h}}$ are as in Definitions 11.1(c) and 11.2(d).

An equivalent definition is the following: suppose we have 2-commutative diagrams in $\mathbf{C^\infty Sta}$, as for (11.5):

$$
\begin{tikzcd}
\underline{\bar U} \arrow[r, "u"] \arrow[d, "\underline{\bar e}"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{X} \arrow[d, "g"] \\
\underline{\bar W} \arrow[r, "w"] & \mathcal{Z},
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\bar V} \arrow[r, "v"] \arrow[d, "\underline{\bar f}"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{Y} \arrow[d, "h"] \\
\underline{\bar W} \arrow[r, "w"] & \mathcal{Z},
\end{tikzcd}
$$

where $\underline{U}, \underline{V}, \underline{W}$ are separated, second countable C^∞-schemes, $\underline{e} : \underline{U} \to \underline{W}$, $\underline{f} : \underline{V} \to \underline{W}$ are morphisms, and u, v, w are étale 1-morphisms. As in Definitions 11.1 and 11.2 we can extend $\underline{U}, \underline{V}, \underline{W}$ to d-spaces with corners $\mathbf{U}, \mathbf{V}, \mathbf{W}$, and $\underline{e}, \underline{f}$ to 1-morphisms $\boldsymbol{e} : \mathbf{U} \to \mathbf{W}$, $\boldsymbol{f} : \mathbf{V} \to \mathbf{W}$. Then $\boldsymbol{g}, \boldsymbol{h}$ are b-transverse in $\mathbf{dSta^c}$ if and only if $\boldsymbol{e}, \boldsymbol{f}$ are b-transverse in $\mathbf{dSpa^c}$ in the sense of §6.8.1, for all such $\underline{U}, \underline{V}, \underline{W}, \underline{e}, \underline{f}$. That is, $\boldsymbol{g}, \boldsymbol{h}$ are b-transverse if and only if they are étale locally equivalent to b-transverse 1-morphisms in $\mathbf{dSpa^c}$.

Definition 11.22. Let $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$ and $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms in $\mathbf{dSta^c}$, and let $C(\boldsymbol{g}), C(\boldsymbol{h}), \hat C(\boldsymbol{g}), \hat C(\boldsymbol{h})$ be as in §11.5. Identify $C_k(\mathcal{X})_{\mathrm{top}} \subseteq C(\mathcal{X})_{\mathrm{top}}$ with the right hand of (11.27), and similarly for $C(\mathcal{Y})_{\mathrm{top}}, C(\mathcal{Z})_{\mathrm{top}}$. Then $C(g)_{\mathrm{top}}, C(h)_{\mathrm{top}}, \hat C(g)_{\mathrm{top}}, \hat C(h)_{\mathrm{top}}$ act as in (11.28)–(11.29). We call $\boldsymbol{g}, \boldsymbol{h}$ *c-transverse* if the following two conditions hold:

(a) whenever there are points in $C_j(\mathcal{X})_{\mathrm{top}}, C_k(\mathcal{Y})_{\mathrm{top}}, C_l(\mathcal{Z})_{\mathrm{top}}$ with

$$C(g)_{\mathrm{top}}\bigl([x, \{x'_1, \ldots, x'_j\}]\bigr) = C(h)_{\mathrm{top}}\bigl([y, \{y'_1, \ldots, y'_k\}]\bigr) = [z, \{z'_1, \ldots, z'_l\}],$$

we have either $j+k>l$ or $j=k=l=0$; and

(b) whenever there are points in $C_j(\mathcal{X})_{\rm top}, C_k(\mathcal{Y})_{\rm top}, C_l(\mathcal{Z})_{\rm top}$ with

$$\hat C(g)_{\rm top}\bigl([x,\{x'_1,\ldots,x'_j\}]\bigr)=\hat C(h)_{\rm top}\bigl([y,\{y'_1,\ldots,y'_k\}]\bigr)=[z,\{z'_1,\ldots,z'_l\}],$$

we have $j+k\geqslant l$.

As in Definition 11.21, $\boldsymbol g,\boldsymbol h$ are c-transverse if and only if they are étale locally equivalent to c-transverse 1-morphisms in $\mathbf{dSpa^c}$. Since c-transverse implies b-transverse in $\mathbf{dSpa^c}$ by Corollary 6.39, c-transverse also implies b-transverse for 1-morphisms $\boldsymbol g:\boldsymbol{\mathfrak X}\rightarrow\boldsymbol{\mathfrak Z}$ and $\boldsymbol h:\boldsymbol{\mathfrak Y}\rightarrow\boldsymbol{\mathfrak Z}$ in $\mathbf{dSta^c}$.

As for Lemma 6.35, we have:

Lemma 11.23. *Let $\boldsymbol g:\boldsymbol{\mathfrak X}\rightarrow\boldsymbol{\mathfrak Z}$ and $\boldsymbol h:\boldsymbol{\mathfrak Y}\rightarrow\boldsymbol{\mathfrak Z}$ be* 1*-morphisms in* $\mathbf{dSta^c}$*. The following are sufficient conditions for $\boldsymbol g,\boldsymbol h$ to be c-transverse, and hence b-transverse:*

(i) *$\boldsymbol g$ or $\boldsymbol h$ is semisimple and flat; or*

(ii) *$\boldsymbol{\mathfrak Z}$ is a d-stack without boundary.*

Here is the analogue of Theorem 6.42.

Theorem 11.24. *All b-transverse fibre products exist in* $\mathbf{dSta^c}$*.*

Proof. Let $\boldsymbol g:\boldsymbol{\mathfrak X}\rightarrow\boldsymbol{\mathfrak Z}$ and $\boldsymbol h:\boldsymbol{\mathfrak Y}\rightarrow\boldsymbol{\mathfrak Z}$ be b-transverse 1-morphisms in $\mathbf{dSta^c}$. We will show that a fibre product $\boldsymbol{\mathfrak W}=\boldsymbol{\mathfrak X}\times_{\boldsymbol g,\boldsymbol{\mathfrak Z},\boldsymbol h}\boldsymbol{\mathfrak Y}$ exists in $\mathbf{dSta^c}$. The strategy is the same as in §6.8: we first show that b-transverse fibre products exist locally (in the Zariski topology), and then use the results of §11.4 to glue these local fibre products together by equivalences to get a global fibre product.

For the first part, the analogue of §6.8.3, suppose $[x]\in\mathcal{X}_{\rm top}$ and $[y]\in\mathcal{Y}_{\rm top}$ with $g_{\rm top}([x])=h_{\rm top}([y])=[z]\in\mathcal{Z}_{\rm top}$, we use the method of the proof of Theorem 10.27. Set $G=\mathrm{Iso}_{\mathcal X}([x])$, $H=\mathrm{Iso}_{\mathcal Y}([y])$ and $K=\mathrm{Iso}_{\mathcal Z}([z])$, and write $\rho:G\rightarrow K$ and $\sigma:H\rightarrow K$ for $g_*:\mathrm{Iso}_{\mathcal X}([x])\rightarrow\mathrm{Iso}_{\mathcal Z}([z])$ and $h_*:\mathrm{Iso}_{\mathcal Y}([y])\rightarrow\mathrm{Iso}_{\mathcal Z}([z])$. Then by Theorem 11.8 there exist d-spaces with corners $\mathbf{T},\mathbf{U},\mathbf{V}$ with actions $\boldsymbol p:G\rightarrow\mathrm{Aut}(\mathbf{T})$, $\boldsymbol q:H\rightarrow\mathrm{Aut}(\mathbf{U})$, $\boldsymbol r:K\rightarrow\mathrm{Aut}(\mathbf{V})$, 1-morphisms $\boldsymbol i:[\mathbf{T}/G]\rightarrow\boldsymbol{\mathfrak X}$, $\boldsymbol j:[\mathbf{U}/K]\rightarrow\boldsymbol{\mathfrak Y}$, $\boldsymbol k:[\mathbf{V}/K]\rightarrow\boldsymbol{\mathfrak Z}$ which are equivalences with open neighbourhoods $\boldsymbol{\mathfrak T},\boldsymbol{\mathfrak U},\boldsymbol{\mathfrak V}$ of $[x],[y],[z]$ in $\boldsymbol{\mathfrak X},\boldsymbol{\mathfrak Y},\boldsymbol{\mathfrak Z}$ with $\boldsymbol g(\boldsymbol{\mathfrak T}),\boldsymbol h(\boldsymbol{\mathfrak U})\subseteq\boldsymbol{\mathfrak V}\subseteq\boldsymbol{\mathfrak Z}$, 1-morphisms $\boldsymbol m:\mathbf{T}\rightarrow\mathbf{V}$, $\boldsymbol n:\mathbf{U}\rightarrow\mathbf{V}$ which are equivariant under ρ,σ, fitting into 2-commutative diagrams:

$$\begin{tikzcd}[\mathbf{T}/G] \arrow[r, "{[\boldsymbol m,\rho]}"'] \arrow[d, "\boldsymbol i"] \arrow[dr, phantom, "\eta\,\Nearrow"] & {[\boldsymbol V/K]} \arrow[d, "\boldsymbol k"'] \\ \boldsymbol{\mathfrak X} \arrow[r, "\boldsymbol g"] & \boldsymbol{\mathfrak Z},\end{tikzcd}\qquad\begin{tikzcd}[\mathbf{U}/H] \arrow[r, "{[\boldsymbol n,\sigma]}"'] \arrow[d, "\boldsymbol j"] \arrow[dr, phantom, "\zeta\,\Nearrow"] & {[\boldsymbol V/K]} \arrow[d, "\boldsymbol k"'] \\ \boldsymbol{\mathfrak Y} \arrow[r, "\boldsymbol h"] & \boldsymbol{\mathfrak Z}.\end{tikzcd}$$

As in the proof of Theorem 10.27, consider the product $\mathbf{T}\times K$ as a d-space with corners, and define a 1-morphism $\boldsymbol l:\mathbf{T}\times K\rightarrow\mathbf{V}$ in $\mathbf{dSpa^c}$ by

$\boldsymbol{l}|_{\mathbf{T}\times\{\kappa\}} = \boldsymbol{r}(\kappa)\circ\boldsymbol{m} : \mathbf{T}\times\{\kappa\}\cong\mathbf{T}\to\mathbf{V}$. Define actions $\boldsymbol{p}',\boldsymbol{q}',\boldsymbol{r}'$ of G,H,K on $\mathbf{T}\times K,\mathbf{U},\mathbf{V}$ as in (10.8). Then $\boldsymbol{l}:\mathbf{T}\times K\to\mathbf{V}$ and $\boldsymbol{n}:\mathbf{U}\to\mathbf{V}$ are both $G\times H$ equivariant. Since $\boldsymbol{l},\boldsymbol{n}$ are étale locally modelled on $\boldsymbol{g},\boldsymbol{h}$ which are b-transverse, a fibre product $\mathbf{S}=(\mathbf{T}\times K)\times_{\boldsymbol{l},\mathbf{V},\boldsymbol{n}}\mathbf{U}$ exists in $\mathbf{dSpa^c}$ by Theorem 6.42.

We want to choose $\mathbf{S}$ so that the $G\times H$-actions on $\mathbf{T}\times K,\mathbf{U},\mathbf{V}$ lift to a $G\times H$-action on $\mathbf{S}$. To do this, making $\mathbf{T},\mathbf{U},\mathbf{V},\boldsymbol{\mathcal{T}},\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{V}}$ smaller if necessary, we may take $\mathbf{S}$ to be defined by the explicit local construction of fibre products in $\mathbf{dSpa^c}$ in §6.8.3. This involves an arbitrary choice of subsets $I\subseteq \underline{i}_{\mathbf{X}}^{-1}(x)$ and $J\subseteq \underline{i}_{\mathbf{Y}}^{-1}(y)$ such that $I\amalg J$ contains exactly one vertex from each type (A) component $\hat\Gamma$. For the construction of $\mathbf{S}$ to be $G\times H$-equivariant, we need to choose I,J invariant under $G\times H$. This is possible, as if $\hat\Gamma$ is a type (A) component of $\Gamma_{x,y}$ and L is the subgroup of $G\times H$ fixing $\hat\Gamma$, then as $\hat\Gamma$ is simply-connected L must fix at least one vertex of the form x' or y' in $\hat\Gamma$, and we choose I or J to contain the $G\times H$-orbit of such a vertex.

The rest of the construction of $\mathbf{S}$ in (6.8.3) involves no arbitrary choices, and so is $G\times H$-equivariant. Hence $G\times H$ acts on $\mathbf{S}$ by 1-isomorphisms, and as in §11.2 we have a quotient d-stack with corners $[\mathbf{S}/(G\times H)]$. As in (10.8)–(10.9), it then follows on general grounds that

$$[\mathbf{S}/(G\times H)]\simeq[\mathbf{T}/G]\times_{[\boldsymbol{m},\rho],[\mathbf{V}/K],[\boldsymbol{n},\sigma]}[\mathbf{U}/H]\simeq\boldsymbol{\mathcal{T}}\times_{\boldsymbol{g}|_{\boldsymbol{\mathcal{T}}},\boldsymbol{\mathcal{V}},\boldsymbol{h}|_{\boldsymbol{\mathcal{U}}}}\boldsymbol{\mathcal{U}},$$

so a local fibre product $\boldsymbol{\mathcal{T}}\times_{\boldsymbol{g}|_{\boldsymbol{\mathcal{T}}},\boldsymbol{\mathcal{V}},\boldsymbol{h}|_{\boldsymbol{\mathcal{U}}}}\boldsymbol{\mathcal{U}}$ exists in $\mathbf{dSta^c}$.

To prove that a global fibre product $\boldsymbol{\mathcal{W}}=\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}$ exists in $\mathbf{dSta^c}$, we now follow the proof of Theorem 6.42 in §6.8.4, and glue the local fibre products $\boldsymbol{\mathcal{T}}\times_{\boldsymbol{g}|_{\boldsymbol{\mathcal{T}}},\boldsymbol{\mathcal{V}},\boldsymbol{h}|_{\boldsymbol{\mathcal{U}}}}\boldsymbol{\mathcal{U}}$ together by equivalences in $\mathbf{dSta^c}$, using Theorem 11.19 rather than Theorem 6.26. There is something extra to check, since the hypotheses of Theorem 11.19 include conditions (11.25)–(11.26) on 2-morphisms in $\mathbf{C^\infty Sta}$ which have no analogue in Theorem 6.26. In our case, we can choose the $\boldsymbol{\eta}_{ijk}$ so that these conditions are automatically satisfied.

The point of (11.25) is to ensure that it is possible to glue the C^∞-stacks $\mathcal{X}_i$ on overlaps $\mathcal{U}_{ij}$ to make a global C^∞-stack $\mathcal{X}$. But we are gluing a family of local fibre products $\mathcal{T}_i\times_{g|_{\mathcal{T}_i},\mathcal{V}_i,h|_{\mathcal{U}_i}}\mathcal{U}_i$ for $i\in I$ with $\mathcal{T}_i\subseteq\mathcal{X}$, $\mathcal{U}_i\subseteq\mathcal{Y}$, $\mathcal{V}_i\subseteq\mathcal{Z}$. The glued C^∞-stack is just the fibre product $\mathcal{X}\times_{g,\mathcal{Z},h}\mathcal{Y}$, which we know exists. We are free to fix a particular C^∞-stack fibre product $\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$, and then take each local fibre product $\mathcal{T}_i\times_{g|_{\mathcal{T}_i},\mathcal{V}_i,h|_{\mathcal{U}_i}}\mathcal{U}_i$ to be an open C^∞-substack of $\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$, so that the e_{ij} and η_{ijk} in Theorem 11.19 are identities, and (11.25) holds trivially.

Similarly, when the second part of Theorem 11.19 is applied to show that we can glue the projection 1-morphisms $\mathcal{T}_i\times_{\mathcal{V}_i}\mathcal{U}_i\to\mathcal{T}_i$ for $i\in I$, the result is the projection 1-morphism $\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}\to\mathcal{X}$, which we know exists. We are free to take the projections $\mathcal{T}_i\times_{\mathcal{V}_i}\mathcal{U}_i\to\mathcal{T}_i$ to be the restriction of $\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}\to\mathcal{X}$ to an open C^∞-substack, and the ζ_{ij} to be identities, and then (11.26) holds trivially. The rest of the proof is as in §6.8.4. □

As for Corollary 6.43 and Theorem 9.21, comparing the constructions of (b-transverse) fibre products in $\mathbf{dSpa^c},\mathbf{dSta},\mathbf{dSta^c}$ in §6.8, §9.5 and §11.6, we deduce:

Corollary 11.25. *The* 2*-functor* $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}} : \mathbf{dSpa^c} \to \mathbf{dSta^c}$ *takes b- and c-transverse fibre products in* $\mathbf{dSpa^c}$ *to b- and c-transverse fibre products in* $\mathbf{dSta^c}$*. The* 2*-functor* $F_{\mathbf{dSta^c}}^{\mathbf{dSta}} : \mathbf{dSta^c} \to \mathbf{dSta}$ *takes b-transverse fibre products in* $\mathbf{dSta^c}$ *to fibre products in* $\mathbf{dSta}$*.*

The analogue of Example 6.44 holds in $\mathbf{dSta^c}$: *products* $\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}$ exist in $\mathbf{dSta^c}$, and are given explicitly by

$$\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}} = \big(\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}, (\partial\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}) \amalg (\boldsymbol{\mathcal{X}} \times \partial\boldsymbol{\mathcal{Y}}), (\boldsymbol{i_{\mathcal{X}}} \times \mathbf{id}_{\boldsymbol{\mathcal{Y}}}) \amalg (\mathbf{id}_{\boldsymbol{\mathcal{X}}} \times \boldsymbol{i_{\mathcal{Y}}}), \omega_{\boldsymbol{\mathcal{X}}\times\boldsymbol{\mathcal{Y}}}\big).$$

Combining the proofs of Theorems 6.45 and 9.23, we may prove:

Theorem 11.26. *The* 2*-functor* $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}$ *takes transverse fibre products in* $\mathbf{Orb^c}$ *to b-transverse fibre products in* $\mathbf{dSta^c}$*. That is, if*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "f"] \arrow[d, "e"'] & \mathcal{Y} \arrow[d, "h"] \\ \mathcal{X} \arrow[r, "g"'] \arrow[ur, phantom, "\eta \Uparrow"] & \mathcal{Z} \end{tikzcd}$$

is a 2*-Cartesian square in* $\mathbf{Orb^c}$ *with* g, h *transverse, and* $\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{\eta} = F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(W, X, Y, Z, e, f, g, h, \eta)$*, then*

$$\begin{tikzcd} \boldsymbol{\mathcal{W}} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"'] & \boldsymbol{\mathcal{Y}} \arrow[d, "\boldsymbol{h}"] \\ \boldsymbol{\mathcal{X}} \arrow[r, "\boldsymbol{g}"'] \arrow[ur, phantom, "\boldsymbol{\eta} \Uparrow"] & \mathbf{Z} \end{tikzcd}$$

is 2*-Cartesian in* $\mathbf{dSta^c}$*, with* $\boldsymbol{g}, \boldsymbol{h}$ *b-transverse. If also* g, h *are strongly transverse in* $\mathbf{Orb^c}$*, then* $\boldsymbol{g}, \boldsymbol{h}$ *are c-transverse in* $\mathbf{dSta^c}$*.*

We can also extend the material of §6.9 to d-stacks with corners, and show that the analogues of Propositions 6.49 and 6.53, and of Theorems 6.50 and 6.52, hold in $\mathbf{dSta^c}$. The analogue of Theorem 6.50 gives:

Theorem 11.27. *Suppose we are given a* 2*-Cartesian diagram in* $\mathbf{dSta^c}$*:*

$$\begin{tikzcd} \boldsymbol{\mathcal{W}} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{e}"'] & \boldsymbol{\mathcal{Y}} \arrow[d, "\boldsymbol{h}"] \\ \boldsymbol{\mathcal{X}} \arrow[r, "\boldsymbol{g}"'] \arrow[ur, phantom, "\boldsymbol{\eta} \Uparrow"] & \boldsymbol{\mathcal{Z}}, \end{tikzcd}$$

with $\boldsymbol{g}, \boldsymbol{h}$ *c-transverse. Then the following are also* 2*-Cartesian in* $\mathbf{dSta^c}$*:*

$$\begin{tikzcd} C(\boldsymbol{\mathcal{W}}) \arrow[r, "C(\boldsymbol{f})"] \arrow[d, "C(\boldsymbol{e})"'] & C(\boldsymbol{\mathcal{Y}}) \arrow[d, "C(\boldsymbol{h})"] \\ C(\boldsymbol{\mathcal{X}}) \arrow[r, "C(\boldsymbol{g})"'] \arrow[ur, phantom, "C(\boldsymbol{\eta}) \Uparrow"] & C(\boldsymbol{\mathcal{Z}}), \end{tikzcd} \tag{11.31}$$

$$\begin{tikzcd} C(\boldsymbol{\mathcal{W}}) \arrow[r, "\hat{C}(\boldsymbol{f})"] \arrow[d, "\hat{C}(\boldsymbol{e})"'] & C(\boldsymbol{\mathcal{Y}}) \arrow[d, "\hat{C}(\boldsymbol{h})"] \\ C(\boldsymbol{\mathcal{X}}) \arrow[r, "\hat{C}(\boldsymbol{g})"'] \arrow[ur, phantom, "\hat{C}(\boldsymbol{\eta}) \Uparrow"] & C(\boldsymbol{\mathcal{Z}}). \end{tikzcd} \tag{11.32}$$

Also (11.31)–(11.32) *preserve gradings, in that they relate points in* $C_i(\mathcal{W})$, $C_j(\mathcal{X}), C_k(\mathcal{Y}), C_k(\mathcal{Z})$ *with* $i = j + k - l$. *Hence* (11.31) *implies equivalences in* $\mathbf{dSta^c}$:

$$C_i(\mathcal{W}) \simeq \coprod_{j,k,l\geqslant 0: i=j+k-l} C_j^{\boldsymbol{g},l}(\mathcal{X}) \times_{C_j^l(\boldsymbol{g}),C_l(\mathcal{Z}),C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathcal{Y}),$$
$$\partial\mathcal{W} \simeq \coprod_{j,k,l\geqslant 0: j+k=l+1} C_j^{\boldsymbol{g},l}(\mathcal{X}) \times_{C_j^l(\boldsymbol{g}),C_l(\mathcal{Z}),C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathcal{Y}).$$

11.7 Orbifold strata of d-stacks with corners

In [56, §11] and §C.8–§C.9 we studied orbifold strata $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ of a Deligne–Mumford C^∞-stack $\mathcal{X}$, and sheaves upon them. Sections 8.4 and 8.9 applied these ideas to orbifolds and orbifolds with corners, and §9.6 extended them to d-stacks. The closest d-space analogue of orbifold strata is fixed d-subspaces $\boldsymbol{X}^\Gamma$ of a finite group Γ acting on a d-space $\boldsymbol{X}$, and we studied these in §2.7 for d-spaces, and in §6.10 for d-spaces with corners.

We now consider orbifold strata of d-stacks with corners. This is largely a matter of combining the material of §6.10, §8.9 and §9.6.

Definition 11.28. Let $\mathcal{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i_{\mathcal{X}}}, \omega_{\mathcal{X}})$ be a d-stack with corners, and Γ a finite group. We will define a new d-stack with corners $\mathcal{X}^\Gamma = (\boldsymbol{\mathcal{X}}^\Gamma, \boldsymbol{\partial(\mathcal{X}^\Gamma)}, \boldsymbol{i_{\mathcal{X}^\Gamma}}, \omega_{\mathcal{X}^\Gamma})$, called an *orbifold stratum* of $\mathcal{X}$. Here $\boldsymbol{\mathcal{X}}^\Gamma$ is the orbifold stratum of the d-stack $\boldsymbol{\mathcal{X}}$ defined in Definition 9.24, and $\boldsymbol{\partial(\mathcal{X}^\Gamma)}, \boldsymbol{i_{\mathcal{X}^\Gamma}}, \omega_{\mathcal{X}^\Gamma}$ remain to be defined. Note that as for $\mathfrak{X}^\Gamma = \big(\mathcal{X}^\Gamma, \partial(\mathcal{X}^\Gamma), i_{\mathfrak{X}^\Gamma}\big)$ in §8.9, $\boldsymbol{\partial(\mathcal{X}^\Gamma)}$ is not the orbifold stratum $(\boldsymbol{\partial\mathcal{X}})^\Gamma$ of the d-stack $\boldsymbol{\partial\mathcal{X}}$, but something more complicated.

In §11.5 we described the corners $C(\mathcal{X}) = (\boldsymbol{C(\mathcal{X})}, \boldsymbol{\partial C(\mathcal{X})}, \boldsymbol{i_{C(\mathcal{X})}}, \omega_{C(\mathcal{X})})$ of $\mathcal{X}$, a d-stack with corners, with a 1-morphism $\boldsymbol{\Pi_{\mathcal{X}}} : C(\mathcal{X}) \to \mathcal{X}$. Hence $\boldsymbol{C(\mathcal{X})}$ is a d-stack, and $\boldsymbol{\Pi_{\mathcal{X}}} : \boldsymbol{C(\mathcal{X})} \to \boldsymbol{\mathcal{X}}$ is a 1-morphism in $\mathbf{dSta}$, whose underlying C^∞-stack 1-morphism $\Pi_{\mathcal{X}} : C(\mathcal{X}) \to \mathcal{X}$ is strongly representable. Thus, by §9.6 we have a d-stack $\boldsymbol{C(\mathcal{X})}^\Gamma$, an orbifold stratum of $\boldsymbol{C(\mathcal{X})}$, and a d-stack 1-morphism $\boldsymbol{\Pi}^\Gamma_{\mathcal{X}} : \boldsymbol{C(\mathcal{X})}^\Gamma \to \boldsymbol{\mathcal{X}}^\Gamma$, with underlying C^∞-stack 1-morphism $\Pi^\Gamma_{\mathcal{X}} : C(\mathcal{X})^\Gamma \to \mathcal{X}^\Gamma$ defined in §C.8. As $\Pi_{\mathcal{X}}$ is strongly representable, the argument of Definition 8.31 shows that $\Pi^\Gamma_{\mathcal{X}}$ is strongly representable.

Exactly as in Definition 8.31, equation (8.32) describes the topological space $C(\mathcal{X})^\Gamma_{\rm top}$ of $\boldsymbol{C(\mathcal{X})}$ in terms of equivalence classes $[x, \{x'_1, \ldots, x'_k\}, \rho, \sigma]$. We define $\partial(\mathcal{X}^\Gamma)_{\rm top} \subseteq C(\mathcal{X})^\Gamma_{\rm top}$ to be the subset of $[x, \{x'_1, \ldots, x'_k\}, \rho, \sigma]$ such that $k \geqslant 1$ and $\sigma(\Gamma) \subseteq S_k$ acts transitively on $\{1, \ldots, k\}$. Then $\partial(\mathcal{X}^\Gamma)_{\rm top}$ is open and closed in $C(\mathcal{X})^\Gamma_{\rm top}$, so it defines an open and closed d-substack $\boldsymbol{\partial(\mathcal{X}^\Gamma)}$ in $\boldsymbol{C(\mathcal{X})}^\Gamma$. Set $\boldsymbol{i_{\mathcal{X}^\Gamma}} = \boldsymbol{\Pi}^\Gamma_{\mathcal{X}}|_{\boldsymbol{\partial(\mathcal{X}^\Gamma)}}$. Then $\boldsymbol{i_{\mathcal{X}^\Gamma}} : \boldsymbol{\partial(\mathcal{X}^\Gamma)} \to \boldsymbol{\mathcal{X}}^\Gamma$ is a 1-morphism in $\mathbf{dSta}$, whose underlying C^∞-stack 1-morphism $i_{\mathcal{X}^\Gamma} : \partial(\mathcal{X}^\Gamma) \to \mathcal{X}^\Gamma$ is strongly representable by Proposition C.14(c), as $\Pi^\Gamma_{\mathcal{X}}$ is.

Define $\nu_{\mathcal{X}^\Gamma} : \mathcal{N}_{\mathcal{X}^\Gamma} \to i^*_{\mathcal{X}^\Gamma}(\mathcal{F}_{\mathcal{X}^\Gamma})$ to be the kernel of $i^2_{\mathcal{X}^\Gamma}$, giving a complex

$$0 \longrightarrow \mathcal{N}_{\mathcal{X}^\Gamma} \xrightarrow{\nu_{\mathcal{X}^\Gamma}} i^*_{\mathcal{X}^\Gamma}(\mathcal{F}_{\mathcal{X}^\Gamma}) \xrightarrow{i^2_{\mathcal{X}^\Gamma}} \mathcal{F}_{\partial\mathcal{X}^\Gamma} \longrightarrow 0$$

in $\mathrm{qcoh}(\partial\mathcal{X}^\Gamma)$. We claim that $\mathcal{N}_{\mathfrak{X}^\Gamma}$ is an orientable line bundle on $\partial\mathcal{X}^\Gamma$, and that there is a canonical orientation $\omega_{\mathfrak{X}^\Gamma}$ on $\mathcal{N}_{\mathfrak{X}^\Gamma}$ such that $\mathfrak{X}^\Gamma = (\boldsymbol{\mathcal{X}}^\Gamma, \boldsymbol{\partial}(\boldsymbol{\mathcal{X}}^\Gamma), \boldsymbol{i}_{\mathfrak{X}^\Gamma}, \omega_{\mathfrak{X}^\Gamma})$ is a d-stack with corners. To prove this, we note that if $\mathfrak{X}$ is étale locally modelled on a d-space with corners $\mathbf{X}$, then $\mathfrak{X}^\Gamma$ is étale locally modelled on the fixed d-subspace $\mathbf{X}^\Gamma$ of a Γ-action on $\mathbf{X}$, so the proof in Definition 6.54 that $\mathbf{X}^\Gamma$ is a d-space with corners implies $\mathfrak{X}^\Gamma$ is a d-stack with corners. As in §C.8, we have an open C^∞-substack $\mathcal{X}^\Gamma_\circ \subseteq \mathcal{X}^\Gamma$. Define $\mathfrak{X}^\Gamma_\circ \subseteq \mathfrak{X}^\Gamma$ to be the corresponding open d-substack with corners.

In a similar way, using the other classes of orbifold strata $\tilde{\mathcal{X}}^\Gamma, \tilde{\mathcal{X}}^\Gamma_\circ, \hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma_\circ$ in §C.8, combining Definitions 8.31 and 9.24 we may define d-stacks with corners $\tilde{\mathfrak{X}}^\Gamma, \tilde{\mathfrak{X}}^\Gamma_\circ, \hat{\mathfrak{X}}, \hat{\mathfrak{X}}^\Gamma_\circ$. As in Definition 8.31 $\tilde{\Pi}^\Gamma_{\mathfrak{X}} : \widetilde{C(\mathcal{X})}^\Gamma \to \tilde{\mathcal{X}}^\Gamma$ is strongly representable, so for $\tilde{\mathfrak{X}}^\Gamma, \tilde{\mathfrak{X}}^\Gamma_\circ$ we can follow the definitions of $\mathfrak{X}^\Gamma, \mathfrak{X}^\Gamma_\circ$ exactly. However, $\hat{\Pi}^\Gamma_{\mathfrak{X}} : \widehat{C(\mathcal{X})}^\Gamma \to \tilde{\mathcal{X}}^\Gamma$ is representable, but not necessarily strongly representable. Thus, to ensure that $i_{\hat{\mathfrak{X}}^\Gamma}$ is strongly representable in the definition of $\hat{\mathfrak{X}}^\Gamma, \hat{\mathfrak{X}}^\Gamma_\circ$, we have to use Proposition C.14(b) to replace the C^∞-stack in the boundary d-stack $\boldsymbol{\partial}(\hat{\boldsymbol{\mathcal{X}}}^\Gamma)$ in $\hat{\mathfrak{X}}^\Gamma$ by an equivalent C^∞-stack.

The d-stack with corners $\hat{\mathfrak{X}}^\Gamma_\circ$ has trivial orbifold groups, so it is a d-space with corners, that is, there exists $\hat{\mathbf{X}}^\Gamma_\circ$ in $\mathbf{dSpa^c}$ with $\hat{\mathfrak{X}}^\Gamma_\circ \simeq F^{\mathbf{dSta^c}}_{\mathbf{dSpa^c}}(\hat{\mathbf{X}}^\Gamma_\circ)$, and in fact $\hat{\mathbf{X}}^\Gamma_\circ$ is natural up to 1-isomorphism in $\mathbf{dSpa^c}$, not just up to equivalence.

Definition 9.25 defined 1-morphisms $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}), \tilde{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}}), \ldots, \boldsymbol{L}^\Gamma_\circ(\Lambda, \boldsymbol{\mathcal{X}})$ in $\mathbf{dSta}$ between the d-stacks $\boldsymbol{\mathcal{X}}^\Gamma, \ldots, \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ, \boldsymbol{\mathcal{X}}$, in a diagram (9.23). These are all also 1-morphisms in $\mathbf{dSta^c}$ between the d-stacks with corners $\mathfrak{X}^\Gamma, \ldots, \hat{\mathfrak{X}}^\Gamma_\circ, \mathfrak{X}$, so we write them $\boldsymbol{O}^\Gamma(\mathfrak{X}), \tilde{\boldsymbol{\Pi}}^\Gamma(\mathfrak{X}), \ldots, \boldsymbol{L}^\Gamma_\circ(\Lambda, \mathfrak{X})$. To see this, note that $\tilde{\boldsymbol{\Pi}}^\Gamma(\mathfrak{X}), \hat{\boldsymbol{\Pi}}^\Gamma(\mathfrak{X}), \boldsymbol{L}^\Gamma(\Lambda, \mathfrak{X}), \tilde{\boldsymbol{\Pi}}^\Gamma_\circ(\mathfrak{X}), \hat{\boldsymbol{\Pi}}^\Gamma_\circ(\mathfrak{X}), \boldsymbol{L}^\Gamma_\circ(\Lambda, \mathfrak{X})$ are all étale locally modelled on identities $\mathbf{id}_{\mathbf{X}^\Gamma} : \mathbf{X}^\Gamma \to \mathbf{X}^\Gamma$ in $\mathbf{dSpa^c}$ in the sense of Definition 11.2, and $\boldsymbol{O}^\Gamma(\mathfrak{X}), \tilde{\boldsymbol{O}}^\Gamma(\mathfrak{X}), \boldsymbol{O}^\Gamma_\circ(\mathfrak{X}), \tilde{\boldsymbol{O}}^\Gamma_\circ(\mathfrak{X})$ are all étale locally modelled on $\boldsymbol{j}_{\mathbf{X},\Gamma} : \mathbf{X}^\Gamma \to \mathbf{X}$ in $\mathbf{dSpa^c}$ from Definition 6.54. This gives a strictly commutative diagram in $\mathbf{dSta^c}$:

$$\begin{tikzcd}
\mathfrak{X}^\Gamma_\circ \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde{\boldsymbol{\Pi}}^\Gamma_\circ(\mathfrak{X})"] \arrow[dr, "\boldsymbol{O}^\Gamma_\circ(\mathfrak{X})"] \arrow[d, "\subset"'] & & \tilde{\mathfrak{X}}^\Gamma_\circ \arrow[r, "\hat{\boldsymbol{\Pi}}^\Gamma_\circ(\mathfrak{X})"] \arrow[dl, "\tilde{\boldsymbol{O}}^\Gamma_\circ(\mathfrak{X})"'] \arrow[d, "\subset"] & \hat{\mathfrak{X}}^\Gamma_\circ \simeq F^{\mathbf{dSta}}_{\mathbf{dSpa}}(\hat{\mathbf{X}}^\Gamma_\circ) \arrow[d, "\subset"] \\
\mathfrak{X}^\Gamma \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[r, "\boldsymbol{O}^\Gamma(\mathfrak{X})"] \arrow[rr, bend right, "\tilde{\boldsymbol{\Pi}}^\Gamma(\mathfrak{X})"'] & \mathfrak{X} & \tilde{\mathfrak{X}}^\Gamma \arrow[l, "\tilde{\boldsymbol{O}}^\Gamma(\mathfrak{X})"'] \arrow[r, "\hat{\boldsymbol{\Pi}}^\Gamma(\mathfrak{X})"'] & \hat{\mathfrak{X}}^\Gamma.
\end{tikzcd} \tag{11.33}$$

The 2-morphisms $\boldsymbol{E}^\Gamma(\gamma, \boldsymbol{\mathcal{X}}), \boldsymbol{E}^\Gamma_\circ(\gamma, \boldsymbol{\mathcal{X}})$ in Definition 9.25 are also 2-morphisms in $\mathbf{dSta^c}$, since (11.16)–(11.17) hold automatically with $\eta' = 0$.

Now let $\mathfrak{X}, \mathfrak{Y}$ be d-stacks with corners, Γ a finite group, and $\boldsymbol{f} : \mathfrak{X} \to \mathfrak{Y}$ a representable 1-morphism in $\mathbf{dSta^c}$. Then $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a representable 1-morphism in $\mathbf{dSta}$, so Definition 9.26 defines 1-morphisms $\boldsymbol{f}^\Gamma, \tilde{\boldsymbol{f}}^\Gamma, \hat{\boldsymbol{f}}^\Gamma$ in $\mathbf{dSta}$ with $\boldsymbol{f}^\Gamma : \boldsymbol{\mathcal{X}}^\Gamma \to \boldsymbol{\mathcal{Y}}^\Gamma$, and so on. These are all also 1-morphisms $\boldsymbol{f}^\Gamma : \mathfrak{X}^\Gamma \to \mathfrak{Y}^\Gamma$, … in $\mathbf{dSta^c}$. To see this, note that $\boldsymbol{f}^\Gamma, \tilde{\boldsymbol{f}}^\Gamma, \hat{\boldsymbol{f}}^\Gamma$ are étale locally modelled in the sense of Definition 11.2 on the 1-morphisms $\boldsymbol{f}^\Gamma : \mathbf{X}^\Gamma \to \mathbf{Y}^\Gamma$ in $\mathbf{dSpa^c}$ on fixed d-subspaces from Proposition 6.56.

Similarly, if $\boldsymbol{f}, \boldsymbol{g} : \mathfrak{X} \to \mathfrak{Y}$ are representable 1-morphisms in $\mathbf{dSta^c}$ and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism, then Definition 9.26 defines 2-morphisms $\boldsymbol{\eta}^\Gamma, \tilde{\boldsymbol{\eta}}^\Gamma$,

$\hat{\boldsymbol{\eta}}^\Gamma$ in **dSta** with $\boldsymbol{\eta}^\Gamma : \boldsymbol{f}^\Gamma \Rightarrow \boldsymbol{g}^\Gamma$, and so on. These are all also 2-morphisms in $\mathbf{dSta^c}$, as they are étale locally modelled in the sense of Definition 11.3 on the 1-morphisms $\eta^\Gamma : \mathbf{X}^\Gamma \to \mathbf{Y}^\Gamma$ in $\mathbf{dSpa^c}$ from Proposition 6.56.

The analogue of Corollary 9.27 now holds in $\mathbf{dSta^c}$:

Corollary 11.29. *Suppose $\mathfrak{X}, \mathfrak{Y}$ are equivalent d-stacks with corners, and Γ is a finite group. Then $\mathfrak{X}^\Gamma, \mathfrak{Y}^\Gamma$ are equivalent in $\mathbf{dSta^c}$, and similarly for $\tilde{\mathfrak{X}}^\Gamma, \hat{\mathfrak{X}}^\Gamma, \mathfrak{X}^\Gamma_\circ, \tilde{\mathfrak{X}}^\Gamma_\circ, \hat{\mathfrak{X}}^\Gamma_\circ$ and $\tilde{\mathfrak{Y}}^\Gamma, \hat{\mathfrak{Y}}^\Gamma, \mathfrak{Y}^\Gamma_\circ, \tilde{\mathfrak{Y}}^\Gamma_\circ, \hat{\mathfrak{Y}}^\Gamma_\circ$. Also $\hat{\mathbf{X}}^\Gamma_\circ, \hat{\mathbf{Y}}^\Gamma_\circ$ are equivalent in $\mathbf{dSpa^c}$.*

Here is the analogue of Theorems 9.28 and C.53, proved in the same way as Theorem 9.28, but using §6.10 and §11.2 rather than §2.7 and §9.3.

Theorem 11.30. *Let $\mathbf{X}$ be a d-space with corners and G a finite group acting on $\mathbf{X}$ by 1-isomorphisms, and write $\mathfrak{X} = [\mathbf{X}/G]$ for the quotient d-stack with corners, from Definition 11.7. Let Γ be a finite group. Then there are equivalences of d-stacks with corners*

$$\mathfrak{X}^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective}\\ \text{group morphisms } \rho:\Gamma\to G}} \bigl[\mathbf{X}^{\rho(\Gamma)}/\bigl\{g \in G : g\rho(\gamma) = \rho(\gamma)g \ \ \forall \gamma \in \Gamma\bigr\}\bigr], \tag{11.34}$$

$$\mathfrak{X}^\Gamma_\circ \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective}\\ \text{group morphisms } \rho:\Gamma\to G}} \bigl[\mathbf{X}^{\rho(\Gamma)}_\circ/\bigl\{g \in G : g\rho(\gamma) = \rho(\gamma)g \ \ \forall \gamma \in \Gamma\bigr\}\bigr], \tag{11.35}$$

$$\tilde{\mathfrak{X}}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\mathbf{X}^{\Delta}/\bigl\{g \in G : \Delta = g\Delta g^{-1}\bigr\}\bigr], \tag{11.36}$$

$$\tilde{\mathfrak{X}}^\Gamma_\circ \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\mathbf{X}^{\Delta}_\circ/\bigl\{g \in G : \Delta = g\Delta g^{-1}\bigr\}\bigr], \tag{11.37}$$

$$\hat{\mathfrak{X}}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\mathbf{X}^{\Delta}/\bigl(\{g \in G : \Delta = g\Delta g^{-1}\}/\Delta\bigr)\bigr], \tag{11.38}$$

$$\hat{\mathfrak{X}}^\Gamma_\circ \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \bigl[\mathbf{X}^{\Delta}_\circ/\bigl(\{g \in G : \Delta = g\Delta g^{-1}\}/\Delta\bigr)\bigr]. \tag{11.39}$$

Here morphisms $\rho, \rho' : \Gamma \to G$ are conjugate if $\rho' = \mathrm{Ad}(g) \circ \rho$ for some $g \in G$, and subgroups $\Delta, \Delta' \subseteq G$ are conjugate if $\Delta = g\Delta' g^{-1}$ for some $g \in G$. In (11.34)–(11.39) *we sum over one representative ρ or Δ for each conjugacy class. For each subgroup $\Delta \subseteq G$, allowing $\Delta = \rho(\Gamma)$, we write $\mathbf{X}^\Delta$ for the closed d-subspace in $\mathbf{X}$ fixed by Δ in G, as in Definition* 6.54, *and $\mathbf{X}^\Delta_\circ$ for the open d-subspace in $\mathbf{X}^\Delta$ of points in $\mathbf{X}$ whose stabilizer group in G is exactly Δ. The groups acting on $\mathbf{X}^\Delta$ in* (11.34)–(11.39) *have natural actions induced by the G-action on $\mathbf{X}$, such that $\boldsymbol{j}_{\mathbf{X},\Delta} : \mathbf{X}^\Delta \hookrightarrow \mathbf{X}$ is equivariant.*

Under the equivalences (11.34)–(11.39), *the* 1*-morphisms in* (11.33) *are identified up to* 2*-isomorphism with* 1*-morphisms between quotient d-stacks with corners induced by natural* 1*-morphisms in $\mathbf{dSpa^c}$ between $\mathbf{X}^{\rho(\Gamma)}, \mathbf{X}, \ldots$.*

Theorem 9.29 also holds for d-stacks with corners $\mathfrak{X}$ and $\mathfrak{X}^\Gamma, \tilde{\mathfrak{X}}^\Gamma, \ldots, \tilde{\mathfrak{X}}^\Gamma_\circ$, by applying it to the underlying d-stacks $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{X}}^\Gamma, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \ldots, \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$. Here is the analogue of Theorem 8.32. It is proved using essentially the same method, but using Proposition 6.57 and Theorem 11.30 in place of Proposition 5.18 and Theorem C.53, respectively.

Theorem 11.31. *Let $\mathcal{X}$ be a d-stack with corners, and Γ a finite group. The corners $C(\mathcal{X})$ from §11.5 is also a d-stack with corners, so we have orbifold strata $\mathcal{X}^\Gamma, C(\mathcal{X})^\Gamma$ and 1-morphisms $\boldsymbol{O}^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$, $\boldsymbol{O}^\Gamma(C(\mathcal{X})) : C(\mathcal{X})^\Gamma \to C(\mathcal{X})$. Applying the corner functor C from §11.5 gives a 1-morphism $C(\boldsymbol{O}^\Gamma(\mathcal{X})) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})$. There exists a unique equivalence $\boldsymbol{K}^\Gamma(\mathcal{X}) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})^\Gamma$ in $\mathbf{dSta^c}$ with $\boldsymbol{O}^\Gamma(C(\mathcal{X})) \circ \boldsymbol{K}^\Gamma(\mathcal{X}) = C(\boldsymbol{O}^\Gamma(\mathcal{X})) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})$. It restricts to an equivalence $\boldsymbol{K}^\Gamma_\circ(\mathcal{X}) := \boldsymbol{K}^\Gamma(\mathcal{X})|_{C(\mathcal{X}^\Gamma_\circ)} : C(\mathcal{X}^\Gamma_\circ) \to C(\mathcal{X})^\Gamma_\circ$.*

Similarly, there is a unique equivalence $\tilde{\boldsymbol{K}}^\Gamma(\mathcal{X}) : C(\tilde{\mathcal{X}}^\Gamma) \to \widetilde{C(\mathcal{X})}^\Gamma$ with $\tilde{\boldsymbol{O}}^\Gamma(C(\mathcal{X})) \circ \tilde{\boldsymbol{K}}^\Gamma(\mathcal{X}) = C(\tilde{\boldsymbol{O}}^\Gamma(\mathcal{X}))$ and $\tilde{\boldsymbol{\Pi}}^\Gamma(C(\mathcal{X})) \circ \boldsymbol{K}^\Gamma(\mathcal{X}) = \tilde{\boldsymbol{K}}^\Gamma(\mathcal{X}) \circ C(\tilde{\boldsymbol{\Pi}}^\Gamma(\mathcal{X}))$. There is an equivalence $\hat{\boldsymbol{K}}^\Gamma(\mathcal{X}) : C(\hat{\mathcal{X}}^\Gamma) \to \widehat{C(\mathcal{X})}^\Gamma$, unique up to 2-isomorphism, with a 2-morphism $\hat{\boldsymbol{\Pi}}^\Gamma(C(\mathcal{X})) \circ \tilde{\boldsymbol{K}}^\Gamma(\mathcal{X}) \Rightarrow \hat{\boldsymbol{K}}^\Gamma(\mathcal{X}) \circ C(\hat{\boldsymbol{\Pi}}^\Gamma(\mathcal{X}))$. They restrict to equivalences $\tilde{\boldsymbol{K}}^\Gamma_\circ(\mathcal{X}) : C(\tilde{\mathcal{X}}^\Gamma_\circ) \to \widetilde{C(\mathcal{X})}^\Gamma_\circ$ and $\hat{\boldsymbol{K}}^\Gamma_\circ(\mathcal{X}) : C(\hat{\mathcal{X}}^\Gamma_\circ) \to \widehat{C(\mathcal{X})}^\Gamma_\circ$.

Here are the analogues of Corollaries 8.34 and 8.36 and Definition 8.35.

Corollary 11.32. *Let $\mathcal{X}$ be a d-stack with corners, and Γ a finite group. Then there exist 1-morphisms $\boldsymbol{J}^\Gamma(\mathcal{X}) : (\partial\mathcal{X})^\Gamma \to \partial(\mathcal{X}^\Gamma)$, $\tilde{\boldsymbol{J}}^\Gamma(\mathcal{X}) : (\widetilde{\partial\mathcal{X}})^\Gamma \to \partial(\tilde{\mathcal{X}}^\Gamma)$, $\hat{\boldsymbol{J}}^\Gamma(\mathcal{X}) : (\widehat{\partial\mathcal{X}})^\Gamma \to \partial(\hat{\mathcal{X}}^\Gamma)$ in $\mathbf{dSta^c}$, natural up to 2-isomorphism, such that $\boldsymbol{J}^\Gamma(\mathcal{X})$ is an equivalence from $(\partial\mathcal{X})^\Gamma$ to an open and closed d-substack of $\partial(\mathcal{X}^\Gamma)$, and similarly for $\tilde{\boldsymbol{J}}^\Gamma(\mathcal{X}), \hat{\boldsymbol{J}}^\Gamma(\mathcal{X})$.*

Definition 11.33. A d-stack with corners $\mathcal{X}$ is called *straight* if the morphisms $(i_{\mathcal{X}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ on orbifold groups are isomorphisms for all $[x'] \in \partial\mathcal{X}_{\mathrm{top}}$ with $i_{\mathcal{X},\mathrm{top}}([x']) = [x]$. D-stacks with boundary are automatically straight. If $\mathcal{X}$ is straight one can show using §11.3 that $\partial\mathcal{X}$ is straight, so by induction $\partial^k\mathcal{X}$ is also straight for all $k \geqslant 0$.

Corollary 11.34. *Let $\mathcal{X}$ be a straight d-stack with corners, and Γ a finite group. Then we have an equivalence $\partial(\mathcal{X}^\Gamma) \simeq (\partial\mathcal{X})^\Gamma$ in $\mathbf{dSta^c}$. The analogue holds for the other orbifold strata $\tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma, \mathcal{X}^\Gamma_\circ, \tilde{\mathcal{X}}^\Gamma_\circ, \mathcal{X}^\Gamma_\circ$.*

12 D-orbifolds with corners

We now define and study the 2-category $\mathbf{dOrb^c}$ of d-orbifolds with corners, as a full 2-subcategory of the 2-category $\mathbf{dSta^c}$ of d-stacks with corners from Chapter 11. Most of the chapter works by combining ideas on d-manifolds with corners from Chapter 7, and on orbifolds with corners from §8.5–§8.9, and on d-orbifolds from Chapter 10, so we often omit proofs.

12.1 The definition of d-orbifolds with corners

Here are the d-orbifold analogues of Definitions 7.1, 7.2 and 7.5 and Lemmas 7.3 and 7.4, proved in the same way.

Definition 12.1. A d-stack with corners $\boldsymbol{\mathcal{W}}$ is called a *principal d-orbifold with corners* if is equivalent in $\mathbf{dSta^c}$ to a fibre product $\boldsymbol{\mathcal{V}}\times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}}\boldsymbol{\mathcal{V}}$, where $\mathcal{V}$ is an orbifold with corners, $\mathcal{E}$ is a vector bundle on $\mathcal{V}$, $s\in C^\infty(\mathcal{E})$ is a smooth section of $\mathcal{E}$, $0\in C^\infty(\mathcal{E})$ is the zero section, so that Definition 8.19 defines an orbifold with corners $\mathrm{Tot^c}(\mathcal{E})$, the 'total space' of $\mathcal{E}$, and 1-morphisms $\mathrm{Tot^c}(s),\mathrm{Tot^c}(0):\mathcal{X}\rightarrow\mathrm{Tot^c}(\mathcal{E})$, and we set $\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{E}},\boldsymbol{s},\boldsymbol{0}=F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}\bigl(\mathcal{V},\mathrm{Tot^c}(\mathcal{E}),\mathrm{Tot^c}(s),\mathrm{Tot^c}(0)\bigr)$.

Note that $\mathrm{Tot^c}(s),\mathrm{Tot^c}(0):\mathcal{V}\rightarrow\mathrm{Tot^c}(\mathcal{E})$ are simple, flat 1-morphisms in $\mathbf{Orb^c}$, so $\boldsymbol{s},\boldsymbol{0}:\boldsymbol{\mathcal{V}}\rightarrow\boldsymbol{\mathcal{E}}$ are simple, flat 1-morphisms in $\mathbf{dSta^c}$. Thus $\boldsymbol{s},\boldsymbol{0}$ are b-transverse by Lemma 11.23(i), and a fibre product $\boldsymbol{\mathcal{V}}\times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}}\boldsymbol{\mathcal{V}}$ exists in $\mathbf{dSta^c}$ by Theorem 11.24(a).

Definition 12.2. Let $\mathcal{V}=(\mathcal{V},\partial\mathcal{V},i_{\mathcal{V}})$ be an orbifold with corners, $\mathcal{E}\in\mathrm{vect}(\mathcal{V})$ a vector bundle on $\mathcal{V}$ as in Definition 8.19, and $s\in C^\infty(\mathcal{E})$, that is, $s:\mathcal{O}_{\mathcal{V}}\rightarrow\mathcal{E}$ is a morphism in $\mathrm{vect}(\mathcal{V})$. We will define an explicit principal d-orbifold with corners $\mathbf{S}=(\boldsymbol{\mathcal{S}},\boldsymbol{\partial\mathcal{S}},\boldsymbol{i_{\mathbf{S}}},\omega_{\mathbf{S}})$ equivalent in $\mathbf{dSta^c}$ to $\boldsymbol{\mathcal{V}}\times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}}\boldsymbol{\mathcal{V}}$ in Definition 12.1. We call $\mathbf{S}$ a *standard model* d-orbifold with corners, and also write it $\mathbf{S}_{\mathcal{V},\mathcal{E},s}$.

Define the d-stack $\boldsymbol{\mathcal{S}}=(\mathcal{S},\mathcal{O}'_{\mathcal{S}},\mathcal{E}_{\mathcal{S}},\imath_{\mathcal{S}},\jmath_{\mathcal{S}})$ exactly as in Definition 10.5, using the Deligne–Mumford C^∞-stack $\mathcal{V}$, vector bundle $\mathcal{E}$ and section $s\in C^\infty(\mathcal{E})$. This time $\mathcal{V}$ is not an orbifold, but the definition still makes sense. Similarly, define the d-stack $\boldsymbol{\partial\mathcal{S}}=(\partial\mathcal{S},\mathcal{O}'_{\partial\mathcal{S}},\mathcal{E}_{\partial\mathcal{S}},\imath_{\partial\mathcal{S}},\jmath_{\partial\mathcal{S}})$ as in Definition 10.5, but using $\partial\mathcal{V},i_{\mathcal{V}}^*(\mathcal{E}),i_{\mathcal{V}}^*(s)$ in place of $\mathcal{V},\mathcal{E},s$. Define the d-stack 1-morphism $\boldsymbol{i_{\mathbf{S}}}=(i_{\mathbf{S}},i'_{\mathbf{S}},i''_{\mathbf{S}}):\boldsymbol{\partial\mathcal{S}}\rightarrow\boldsymbol{\mathcal{S}}$ as for $\boldsymbol{\mathcal{S}}_{f,\hat f}$ in Definition 10.9, with $\partial\mathcal{V},\mathcal{V},i_{\mathcal{V}}^*(\mathcal{E}),\mathcal{E}$, $i_{\mathcal{V}}^*(s),s,i_{\mathcal{V}},\mathrm{id}_{i_{\mathcal{V}}^*(\mathcal{E})}$ in place of $\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{F},s,t,f,\hat f$, respectively.

Then by definition the C^∞-stack $\mathcal{S}$, as a category, is the full subcategory of objects u in $\mathcal{V}$ with $p_{\mathcal{V}}(u)=\underline{U}$ in $\mathbf{C^\infty Sch}$ such that $0=\tilde u^*(s):\tilde u^*(\mathcal{O}_{\mathcal{V}})\rightarrow\tilde u^*(\mathcal{E})$ in $\mathrm{qcoh}(\bar{\underline{U}})$, and $p_{\mathcal{S}}=p_{\mathcal{V}}|_{\mathcal{S}}:\mathcal{S}\rightarrow\mathbf{C^\infty Sch}$. Similarly $\partial\mathcal{S}$ is the full subcategory of u in $\partial\mathcal{V}$ with $p_{\mathcal{V}}(u)=\underline{U}$ and $0=\tilde u^*\circ i_{\mathcal{V}}^*(s):\tilde u^*(\mathcal{O}_{\partial\mathcal{V}})\rightarrow\tilde u^*\circ i_{\mathcal{V}}^*(\mathcal{E})$ in $\mathrm{qcoh}(\bar{\underline{U}})$, and $i_{\mathbf{S}}:\partial\mathcal{S}\rightarrow\mathcal{S}$ is $i_{\mathbf{S}}=i_{\mathcal{V}}|_{\partial\mathcal{S}}$. As $\partial\mathcal{S},\mathcal{S}$ are full subcategories and $i_{\mathcal{V}}$ is strongly representable, this implies that $i_{\mathbf{S}}$ is strongly representable.

Suppose W is a manifold with corners, $\underline{W}=F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(W)$ and $w:\bar{\underline{W}}\rightarrow\mathcal{V}$ is étale. Then there exist a vector bundle $F\rightarrow W$ and $t\in C^\infty(F)$ whose lifts to $\bar{\underline{W}}$ are isomorphic to $w^*(\mathcal{E})$ and $w^*(s)$. Definition 7.2 defines a d-manifold with corners $\mathbf{S}_{W,F,t}=\tilde{\mathbf{S}}=(\tilde{\boldsymbol{S}},\boldsymbol{\partial}\tilde{\boldsymbol{S}},\boldsymbol{i}_{\tilde{\mathbf{S}}},\omega_{\tilde{\mathbf{S}}})$. Comparing the definitions shows that $\boldsymbol{\mathcal{S}},\boldsymbol{\partial\mathcal{S}},\boldsymbol{i_{\mathbf{S}}}$ are étale locally modelled on $\tilde{\boldsymbol{S}},\boldsymbol{\partial}\tilde{\boldsymbol{S}},\boldsymbol{i}_{\tilde{\mathbf{S}}}$. Hence the conormal bundle $\mathcal{N}_{\mathbf{S}}$ for $\boldsymbol{S},\boldsymbol{\partial S},\boldsymbol{i_S}$ in $\mathrm{qcoh}(\partial\mathcal{S})$ from (11.1) is étale locally modelled on the conormal bundle $\mathcal{N}_{\tilde{\mathbf{S}}}$ for $\tilde{\boldsymbol{S}},\boldsymbol{\partial\tilde{S}},\boldsymbol{i}_{\tilde{\mathbf{S}}}$ from (6.4).

Definition 7.2 shows $\mathcal{N}_{\tilde{\mathbf{S}}}$ is a line bundle isomorphic to the lift to $\underline{\partial\tilde{S}}$ of the conormal bundle $\tilde{\nu}^*$ of ∂W in W. Therefore $\mathcal{N}_{\mathbf{S}}$ is a line bundle isomorphic to the lift to $\partial\mathcal{S}$ of the conormal bundle ν^* of $\partial\mathcal{V}$ in $\mathcal{V}$. Define $\omega_{\mathbf{S}}$ to be the orientation on $\mathcal{N}_{\mathbf{S}}$ induced by the orientation on ν^* by outward-pointing normal vectors. Then $\boldsymbol{S},\boldsymbol{\partial S},\boldsymbol{i_S},\omega_{\mathbf{S}}$ are étale locally modelled on $\tilde{\boldsymbol{S}},\boldsymbol{\partial\tilde{S}},\boldsymbol{i}_{\tilde{\mathbf{S}}},\omega_{\tilde{\mathbf{S}}}$. Hence $\mathbf{S}$ is étale locally modelled on the d-manifold with corners $\mathbf{S}_{W,F,t}=\tilde{\mathbf{S}}$ in the sense of Definition 11.1(d). As such $w:\underline{\bar{W}}\to\mathcal{V}$ form an étale open cover of $\mathcal{V}$, this implies that $\mathbf{S}$ is a d-stack with corners.

As in Definitions 7.2 and 10.5, we can show that $\mathbf{S}_{\mathcal{V},\mathcal{E},s}=\mathbf{S}$ is equivalent in $\mathbf{dSta^c}$ to $\boldsymbol{\mathcal{V}}\times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}}\boldsymbol{\mathcal{V}}$ in Definition 12.1. Thus $\mathbf{S}_{\mathcal{V},\mathcal{E},s}$ is a principal d-orbifold with corners, and every principal d-orbifold with corners $\mathbf{W}$ is equivalent in $\mathbf{dSta^c}$ to some $\mathbf{S}_{\mathcal{V},\mathcal{E},s}$. Sometimes it is useful to take $\mathcal{V}$ to be an *effective* orbifold with corners, as in §8.9.

Lemma 12.3. *Let $\mathcal{V}$ be an orbifold with corners, $\mathcal{E}$ a vector bundle on $\mathcal{V}$, and $s\in C^\infty(\mathcal{E})$. Write $\mathcal{E}_\partial=i_{\mathcal{V}}^*(\mathcal{E})$, a vector bundle on $\partial\mathcal{V}$, and $s_\partial=i_{\mathcal{V}}^*(s)$ in $C^\infty(\mathcal{E}_\partial)$. Define $\mathbf{S}_{\mathcal{V},\mathcal{E},s},\mathbf{S}_{\partial\mathcal{V},\mathcal{E}_\partial,s_\partial}$ as in Definition* 12.2. *Then there is a natural* 1*-isomorphism $\partial\mathbf{S}_{\mathcal{V},\mathcal{E},s}\cong\mathbf{S}_{\partial\mathcal{V},\mathcal{E}_\partial,s_\partial}$ in* $\mathbf{dSta^c}$, *where $\partial\mathbf{S}_{\mathcal{V},\mathcal{E},s}$ is as in* §11.3.

Similarly, for $k\geqslant 0$ define a vector bundle $\mathcal{E}_k=\Pi_k^*(\mathcal{E})$ on the k-corners $C_k(\mathcal{V})$ and a section $s_k=\Pi_k^*(s)\in C^\infty(E_k)$, where $\Pi_k:C_k(\mathcal{V})\to\mathcal{V}$ is the natural projection. Then there is a natural 1*-isomorphism $C_k(\mathbf{S}_{\mathcal{V},\mathcal{E},s})\cong\mathbf{S}_{C_k(\mathcal{V}),\mathcal{E}_k,s_k}$ in* $\mathbf{dSta^c}$, *where $C_k(\mathbf{S}_{\mathcal{V},\mathcal{E},s})$ is as in* §11.5.

Lemma 12.4. *Suppose $\mathbf{W}=(\boldsymbol{\mathcal{W}},\boldsymbol{\partial\mathcal{W}},\boldsymbol{i_W},\omega_{\mathbf{W}})$ is a principal d-orbifold with corners, so that $\mathbf{W}\simeq\boldsymbol{\mathcal{V}}\times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}}\boldsymbol{\mathcal{V}}$ as in Definition* 12.1 *for $\mathcal{V}$ an orbifold with corners, $\mathcal{E}$ a vector bundle on $\mathcal{V}$, and $s\in C^\infty(\mathcal{E})$. Then the virtual cotangent sheaf $T^*\boldsymbol{\mathcal{W}}$ is a virtual vector bundle on $\mathcal{W}$, with* $\operatorname{rank}T^*\boldsymbol{\mathcal{W}}=\dim\mathcal{V}-\operatorname{rank}\mathcal{E}$.

Definition 12.5. If $\mathbf{W}$ is a nonempty principal d-orbifold with corners, then $T^*\boldsymbol{\mathcal{W}}$ is a virtual vector bundle by Lemma 12.2. We define the *virtual dimension* of $\mathbf{W}$ to be $\operatorname{vdim}\mathbf{W}=\operatorname{rank}T^*\mathbf{W}\in\mathbb{Z}$.

A d-stack with corners $\mathbf{X}$ is called a *d-orbifold with corners of virtual dimension* $n\in\mathbb{Z}$, written $\operatorname{vdim}\mathbf{X}=n$, if $\mathbf{X}$ can be covered by open d-substacks $\mathbf{W}$ which are principal d-orbifolds with corners with $\operatorname{vdim}\mathbf{W}=n$. A d-orbifold with corners $\mathbf{X}$ is called a *d-orbifold with boundary* if it is a d-stack with boundary, and a *d-orbifold without boundary* if it is a d-stack without boundary.

If $\mathbf{X}=(\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}},\boldsymbol{i_X},\omega_{\mathbf{X}})$ is a d-orbifold with corners, then the underlying C^∞-stacks $\mathcal{X},\partial\mathcal{X}$ in $\boldsymbol{\mathcal{X}},\boldsymbol{\partial\mathcal{X}}$ are separated, second countable, locally compact, paracompact, and locally fair. The virtual cotangent sheaf $T^*\mathbf{X}=T^*\boldsymbol{\mathcal{X}}=(\mathcal{E}_{\mathcal{X}},\mathcal{F}_{\mathcal{X}},\phi_{\mathcal{X}})$ of $\boldsymbol{\mathcal{X}}$ is a virtual vector bundle of rank $\operatorname{vdim}\mathbf{X}=n$, so we call it the *virtual cotangent bundle* of $\mathbf{X}$. We consider the empty d-stack with corners $\boldsymbol{\emptyset}$ to be a d-orbifold with corners of any virtual dimension $n\in\mathbb{Z}$, so we leave $\operatorname{vdim}\boldsymbol{\emptyset}$ undefined.

Write $\mathbf{d\bar{O}rb},\mathbf{dOrb^b},\mathbf{dOrb^c}$ for the full 2-subcategories of d-orbifolds without boundary, and with boundary, and with corners, in $\mathbf{dSta^c}$, respectively.

Then $\mathbf{\bar{O}rb}, \mathbf{\bar{O}rb^b}, \mathbf{\bar{O}rb^c}$ in Definition 11.3 are full 2-subcategories of $\mathbf{d\bar{O}rb}$, $\mathbf{dOrb^b}, \mathbf{dOrb^c}$. When we say that a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ *is an orbifold*, we mean that $\boldsymbol{\mathcal{X}}$ lies in $\mathbf{\bar{O}rb^c}$. Define full and faithful strict 2-functors

$$
\begin{aligned}
&F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}} : \mathbf{dOrb} \to \mathbf{d\bar{O}rb} \subset \mathbf{dOrb^c}, \quad && F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}} : \mathbf{Orb^c} \to \mathbf{dOrb^c}, \\
&F_{\mathbf{Orb^b}}^{\mathbf{dOrb^c}} : \mathbf{Orb^b} \to \mathbf{dOrb^b} \subset \mathbf{dOrb^c}, \quad && F_{\mathbf{Orb}}^{\mathbf{dOrb^c}} : \mathbf{Orb} \to \mathbf{d\bar{O}rb} \subset \mathbf{dOrb^c}, \\
&F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}} : \mathbf{dMan^c} \to \mathbf{dOrb^c}, \quad && F_{\mathbf{dMan^b}}^{\mathbf{dOrb^c}} : \mathbf{dMan^b} \to \mathbf{dOrb^b} \subset \mathbf{dOrb^c},
\end{aligned}
$$

$$
\text{and} \qquad F_{\mathbf{dMan}}^{\mathbf{dOrb^c}} : \mathbf{dMan} \to \mathbf{d\bar{O}rb} \subset \mathbf{dOrb^c}, \qquad \text{by}
$$

$$
\begin{aligned}
&F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}} = F_{\mathbf{dSta}}^{\mathbf{dSta^c}}|_{\mathbf{dOrb}}, \quad F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}} = F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}, \quad F_{\mathbf{Orb^b}}^{\mathbf{dOrb^c}} = F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}|_{\mathbf{Orb^b}}, \\
&F_{\mathbf{Orb}}^{\mathbf{Orb^c}} = F_{\mathbf{dSta}}^{\mathbf{dSta^c}} \circ F_{\mathbf{Orb}}^{\mathbf{dSta}}, \; F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}} = F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}|_{\mathbf{dMan^c}}, \; F_{\mathbf{dMan^b}}^{\mathbf{dOrb^c}} = F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}|_{\mathbf{dMan^b}}, \\
&\text{and} \qquad F_{\mathbf{dMan}}^{\mathbf{dOrb^c}} = F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}} \circ F_{\mathbf{dMan}}^{\mathbf{dMan^c}} = F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}} \circ F_{\mathbf{dMan}}^{\mathbf{dOrb}},
\end{aligned}
$$

where $F_{\mathbf{Orb}}^{\mathbf{Orb^c}}, F_{\mathbf{Orb}}^{\mathbf{dSta}}, F_{\mathbf{dSta}}^{\mathbf{dSta^c}}, F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}, F_{\mathbf{dMan}}^{\mathbf{dMan^c}}, F_{\mathbf{dMan}}^{\mathbf{dOrb}}, F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}$ are as in Definitions 7.5, 8.15, 10.1 and 11.3. Here $F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}} : \mathbf{dOrb} \to \mathbf{d\bar{O}rb}$ is an isomorphism of 2-categories. So we may as well identify $\mathbf{dOrb}$ with its image $\mathbf{d\bar{O}rb}$, and consider d-orbifolds without boundary in Chapter 10 as examples of d-orbifolds with corners.

Write $\mathbf{d\hat{M}an^c}$ for the full 2-subcategory of objects $\boldsymbol{\mathcal{X}}$ in $\mathbf{dOrb^c}$ equivalent to $F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}}(\mathbf{X})$ for some d-manifold with corners $\mathbf{X}$. When we say that a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ *is a d-manifold*, we mean that $\boldsymbol{\mathcal{X}} \in \mathbf{d\hat{M}an^c}$.

These 2-categories lie in a commutative diagram:

$$
\begin{tikzcd}
\mathbf{dSpa} \arrow[ddd, bend left, "F_{\mathbf{dSpa}}^{\mathbf{dSta}}"] & \mathbf{Man} \arrow[r, "\subset"'] \arrow[dl, "F_{\mathbf{Man}}^{\mathbf{dMan}}"'] \arrow[d, "F_{\mathbf{Man}}^{\mathbf{dMan^c}}"] \arrow[ddd, bend left, "F_{\mathbf{Man}}^{\mathbf{Orb}}"] & \mathbf{Man^b} \arrow[r, "\subset"'] \arrow[d, "F_{\mathbf{Man^b}}^{\mathbf{dMan^c}}"'] \arrow[ddd, bend left, "F_{\mathbf{Man^b}}^{\mathbf{Orb^c}}"] & \mathbf{Man^c} \arrow[d, "F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}"'] \arrow[ddd, bend left, "F_{\mathbf{Man^c}}^{\mathbf{Orb^c}}"] \arrow[dr, "F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}"] & \\
\mathbf{dMan} \arrow[u, "\subset"] \arrow[r, "F_{\mathbf{dMan}}^{\mathbf{dMan^c}}", "\cong"'] \arrow[d, "F_{\mathbf{dMan}}^{\mathbf{dOrb}}"'] & \mathbf{d\bar{M}an} \arrow[r, "\subset"] \arrow[d] & \mathbf{dMan^b} \arrow[r, "\subset"] \arrow[d, "F_{\mathbf{dMan^b}}^{\mathbf{dOrb^c}}"'] & \mathbf{dMan^c} \arrow[r, "\subset"] \arrow[d, "F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}}"'] & \mathbf{dSpa^c} \arrow[d, "F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}"'] \\
\mathbf{dOrb} \arrow[r, "\cong", "F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}}"'] \arrow[d, "\subset"'] & \mathbf{d\bar{O}rb} \arrow[r, "\subset"'] & \mathbf{dOrb^b} \arrow[r, "\subset"'] \arrow[d, leftrightarrow, "F_{\mathbf{Orb^b}}^{\mathbf{dOrb^c}}"'] & \mathbf{dOrb^c} \arrow[r, "\subset"'] & \mathbf{dSta^c} \\
\mathbf{dSta} & \mathbf{Orb} \arrow[ul, "F_{\mathbf{Orb}}^{\mathbf{dOrb}}"] \arrow[u, "F_{\mathbf{Orb}}^{\mathbf{dOrb^c}}"] \arrow[r, "F_{\mathbf{Orb}}^{\mathbf{Orb^c}}"] & \mathbf{Orb^b} \arrow[r, "\subset"] & \mathbf{Orb^c}. \arrow[u, "F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}"] \arrow[ur, "F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}"'] &
\end{tikzcd}
$$

Here are analogues of Lemmas 10.3 and 10.4, with the same proofs, and of Proposition 7.7, which follows from Lemma 12.3.

Lemma 12.6. *Let $\boldsymbol{\mathcal{W}}$ be a d-orbifold with corners, and $\boldsymbol{\mathcal{U}}$ an open d-substack of $\boldsymbol{\mathcal{W}}$. Then $\boldsymbol{\mathcal{U}}$ is also a d-orbifold with corners, with* $\operatorname{vdim} \boldsymbol{\mathcal{U}} = \operatorname{vdim} \boldsymbol{\mathcal{W}}$.

Lemma 12.7. *Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold with corners. Then $\boldsymbol{\mathcal{X}}$ is a d-manifold, that is, $\boldsymbol{\mathcal{X}} \simeq F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}}(\mathbf{X})$ for some d-manifold with corners $\mathbf{X}$, if and only if* $\operatorname{Iso}_{\mathcal{X}}([x]) \cong \{1\}$ *for all $[x]$ in $\mathcal{X}_{\mathrm{top}}$.*

Proposition 12.8. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold with corners. Then $\partial\boldsymbol{\mathcal{X}}$ in §11.3 and $C_k(\boldsymbol{\mathcal{X}})$ in §11.5 are d-orbifolds with corners, with* $\operatorname{vdim} \partial\boldsymbol{\mathcal{X}} = \operatorname{vdim} \boldsymbol{\mathcal{X}} - 1$ *and* $\operatorname{vdim} C_k(\boldsymbol{\mathcal{X}}) = \operatorname{vdim} \boldsymbol{\mathcal{X}} - k$ *for all $k \geqslant 0$.*

Here is the analogue of Definition 7.8.

Definition 12.9. Define $\mathbf{d\check{O}rb^c}$ to be the full 2-subcategory of $\mathcal{X}$ in $\mathbf{dSta^c}$ which may be written as a disjoint union $\mathcal{X} = \coprod_{n\in\mathbb{Z}} \mathcal{X}_n$ for $\mathcal{X}_n \in \mathbf{dOrb^c}$ with $\operatorname{vdim} \mathcal{X}_n = n$, where we allow $\mathcal{X}_n = \emptyset$. We call such $\mathcal{X}$ a *d-orbifold with corners of mixed dimension.* Then $\mathbf{dOrb^c} \subset \mathbf{d\check{O}rb^c} \subset \mathbf{dSta^c}$, and Proposition 12.8 implies that the corner functors $C, \hat{C} : \mathbf{dSta^c} \to \mathbf{dSta^c}$ in §11.5 restrict to strict 2-functors $C, \hat{C} : \mathbf{dOrb^c} \to \mathbf{d\check{O}rb^c}$.

12.2 Local properties of d-orbifolds with corners

Section 10.1.3 explained how to generalize the material of §3.3 from d-manifolds to d-orbifolds. In the same way, we can generalize the material of §7.2 from d-manifolds with corners to d-orbifolds with corners. We will leave the details to the reader, and just state analogues of Corollaries 7.11 and 7.12 and Proposition 7.13 (see also Propositions 10.6–10.8).

Proposition 12.10. *Let $\mathcal{X}$ be a d-orbifold with corners, and $[x] \in \mathcal{X}_{\rm top}$. Then there exists an open neighbourhood $\mathcal{U}$ of $[x]$ in $\mathcal{X}$ and an equivalence $\mathcal{U} \simeq \mathcal{S}_{\mathcal{V},\mathcal{E},s}$ in $\mathbf{dOrb^c}$ for some orbifold with corners $\mathcal{V}$, vector bundle $\mathcal{E}$ over $\mathcal{V}$ and $s \in C^\infty(\mathcal{E})$ which identifies $[x] \in \mathcal{U}_{\rm top}$ with a point $[v] \in S^k(\mathcal{V})_{\rm top} \subseteq \mathcal{V}_{\rm top}$ such that $s(v) = {\rm d}s|_{S^k(\mathcal{V})}(v) = 0$, where $S^k(\mathcal{V}) \subseteq \mathcal{V}$ is the locally closed C^∞-substack of $[v] \in \mathcal{V}_{\rm top}$ such that $\bar{*} \times_{v,\mathcal{V},i_\mathcal{V}} \partial\mathcal{V}$ is k points, for $k \geqslant 0$. Furthermore, $\mathcal{V}, \mathcal{E}, s, k$ are determined up to non-canonical equivalence near $[v]$ by $\mathcal{X}$ near $[x]$.*

Proposition 12.11. *Let $\mathcal{X}$ be a d-orbifold with corners. Then $\mathcal{X}$ is determined up to non-canonical equivalence near each point $[x] \in \mathcal{X}_{\rm top}$ by the 'classical' data $\mathcal{X}, \partial\mathcal{X}, i_\mathcal{X}, \mathcal{N}_\mathcal{X}, \varrho_\mathcal{X}, \omega_\mathcal{X}$ in $\mathcal{X}$, the integer $\operatorname{vdim} \mathcal{X}$, and a choice of representation of $\operatorname{Iso}_\mathcal{X}([x])$ on the obstruction space $x^*(\operatorname{Ker} \phi_\mathcal{X}) \in \operatorname{qcoh}(\bar{*})$ at $[x]$.*

Proposition 12.12. *Let $\mathcal{X}$ be a d-orbifold with corners. Then $\mathcal{X}$ is an orbifold (that is, $\mathcal{X} \in \mathbf{\bar{O}rb^c}$) if and only if $\phi_{C(\mathcal{X})} : \mathcal{E}_{C(\mathcal{X})} \to \mathcal{F}_{C(\mathcal{X})}$ has a left inverse, or equivalently, if the virtual cotangent bundle $T^*(C(\mathcal{X}))$ of the corners $C(\mathcal{X})$ of $\mathcal{X}$ is a vector bundle of mixed rank on $C(\mathcal{X})$.*

We define 'standard model' 1-morphisms in $\mathbf{dOrb^c}$, following Definitions 3.30, 7.15 and 10.9.

Definition 12.13. Let $\mathcal{V}, \mathcal{W}$ be orbifolds with corners, $\mathcal{E}, \mathcal{F}$ vector bundles on $\mathcal{V}, \mathcal{W}$, and $s \in C^\infty(\mathcal{E})$, $t \in C^\infty(\mathcal{F})$, so that Definition 12.2 defines d-orbifolds with corners $\mathcal{S}_{\mathcal{V},\mathcal{E},s}, \mathcal{S}_{\mathcal{W},\mathcal{F},t}$. Suppose $f : \mathcal{V} \to \mathcal{W}$ is a 1-morphism in $\mathbf{Orb^c}$, and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ a morphism in $\operatorname{vect}(\mathcal{V})$ satisfying $\hat{f} \circ s = f^*(t)$, as in (10.1).

The d-stacks $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}, \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ in $\mathcal{S}_{\mathcal{V},\mathcal{E},s}, \mathcal{S}_{\mathcal{W},\mathcal{F},t}$ are defined as for 'standard model' d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ in Definition 10.5. Thus we can follow Definition 10.9 to define a 1-morphism $\mathcal{S}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ in $\mathbf{dSta}$. Now in Definition 7.15 we defined 'standard model' 1-morphisms $\mathbf{S}_{f,\hat{f}}$ in $\mathbf{dMan^c}$ by following Definition 3.30 for 1-morphisms $\boldsymbol{S}_{f,\hat{f}}$ in $\mathbf{dMan}$, and showing that this 1-morphism in $\mathbf{dSpa}$ is also a 1-morphism in $\mathbf{dSpa^c}$. Since $\boldsymbol{\mathcal{S}}_{f,\hat{f}}$ in Definition 10.9 is étale locally modelled on $\boldsymbol{S}_{f,\hat{f}}$ in Definition 3.30, we see that $\mathcal{S}_{f,\hat{f}}$ above is étale locally modelled (in the sense of Definition 11.2(∗)) on the 1-morphism $\mathbf{S}_{f,\hat{f}}$ in $\mathbf{dSpa^c}$ in Definition 7.15. Therefore $\mathcal{S}_{f,\hat{f}}:\mathcal{S}_{\mathcal{V},\mathcal{E},s}\rightarrow\mathcal{S}_{\mathcal{W},\mathcal{F},t}$ is a 1-morphism in $\mathbf{dSta^c}$, and hence in $\mathbf{dOrb^c}$. We call it a *standard model* 1-morphism in $\mathbf{dOrb^c}$.

Suppose now that $\tilde{\mathcal{V}}\subseteq\mathcal{V}$ is open, with inclusion 1-morphism $i_{\tilde{\mathcal{V}}}:\tilde{\mathcal{V}}\rightarrow\mathcal{V}$. Write $\tilde{\mathcal{E}}=\mathcal{E}|_{\tilde{\mathcal{V}}}=i_{\tilde{\mathcal{V}}}^*(\mathcal{E})$ and $\tilde{s}=s|_{\tilde{\mathcal{V}}}=i_{\tilde{\mathcal{V}}}^*(s)$. Then we have a 1-morphism $\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}}=\mathcal{S}_{i_{\tilde{\mathcal{V}}},\mathrm{id}_{\tilde{\mathcal{E}}}}:\mathcal{S}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}}\rightarrow\mathcal{S}_{\mathcal{V},\mathcal{E},s}$. If $s^{-1}(0)\subseteq\tilde{\mathcal{V}}$ then $\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}}$ is a 1-isomorphism.

As for Lemma 7.18 and Proposition 10.10, we have:

Lemma 12.14. *In Definition* 12.13*, the* 1*-morphism* $\mathcal{S}_{f,\hat{f}}:\mathcal{S}_{\mathcal{V},\mathcal{E},s}\rightarrow\mathcal{S}_{\mathcal{W},\mathcal{F},t}$*, is simple, semisimple, or flat, if and only if* f *is simple, semisimple, or flat respectively near* $s^{-1}(0)$ *in* $\mathcal{V}$*.*

Proposition 12.15. *Let* $\mathcal{S}_{f,\hat{f}},\mathcal{S}_{g,\hat{g}}:\mathcal{S}_{\mathcal{V},\mathcal{E},s}\rightarrow\mathcal{S}_{\mathcal{W},\mathcal{F},t}$ *be 'standard model'* 1*-morphisms of d-orbifolds with corners, in the notation of Definitions* 12.2 *and* 12.13*. Suppose* $\eta:f\Rightarrow g$ *is a* 2*-morphism in* $\mathbf{Orb^c}$ *which satisfies* $\hat{g}=\eta^*(\mathcal{F})\circ\hat{f}:\mathcal{E}\rightarrow g^*(\mathcal{F})$*. Then* $\boldsymbol{\eta}=(\eta|_{\mathcal{S}_{\mathcal{V},\mathcal{E},s}},0)$ *is a* 2*-morphism* $\boldsymbol{\eta}:\mathcal{S}_{f,\hat{f}}\Rightarrow\mathcal{S}_{g,\hat{g}}$ *in* $\mathbf{dOrb^c}$*.*

As in §10.1.4 for 1-morphisms in $\mathbf{dOrb}$, we can write down an analogue of Theorems 3.34 and 7.19 for the standard model 1-morphisms of Definition 12.13, but it is weaker, saying only that any 1-morphism $\boldsymbol{g}:\mathcal{S}_{\mathcal{V},\mathcal{E},s}\rightarrow\mathcal{S}_{\mathcal{W},\mathcal{F},t}$ is 2-isomorphic in $\mathbf{dSta^c}$ to some $\mathcal{S}_{f,\hat{f}}\circ\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}}^{-1}$.

The material of §10.1.5 also extends to d-orbifolds with corners. Here are the analogues of Examples 10.11 and 10.12 and Propositions 10.14 and 10.15, with similar proofs.

Example 12.16. Let V be a manifold with corners, $E\rightarrow V$ a vector bundle, Γ a finite group acting smoothly on V,E preserving the vector bundle structure, and $s:V\rightarrow E$ a smooth, Γ-equivariant section of E. Write the Γ-actions on V,E as $r(\gamma):V\rightarrow V$ and $\hat{r}(\gamma):E\rightarrow r(\gamma)^*(E)$ for $\gamma\in\Gamma$. Then Definitions 7.2 and 7.15 give an explicit principal d-manifold with corners $\mathbf{S}_{V,E,s}$, and 1-morphisms $\mathbf{S}_{r(\gamma),\hat{r}(\gamma)}:\mathbf{S}_{V,E,s}\rightarrow\mathbf{S}_{V,E,s}$ for $\gamma\in\Gamma$ which are an action of Γ on $\mathbf{S}_{V,E,s}$. Hence Definition 11.7 defines a quotient d-stack with corners $[\mathbf{S}_{V,E,s}/\Gamma]$.

Example 8.16 defines an orbifold with corners $\tilde{\mathcal{V}}=[V/\Gamma]$, and by Definition C.34 and Theorem C.35 E,s induce a vector bundle $\tilde{\mathcal{E}}$ on $\tilde{\mathcal{V}}=[\underline{V}/\Gamma]$ and section $\tilde{s}\in C^\infty(\tilde{\mathcal{E}})$, so that Definition 12.2 gives a 'standard model' principal d-orbifold with corners $\mathcal{S}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}}$. One can show that $[\mathbf{S}_{V,E,s}/\Gamma]\simeq\mathcal{S}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}}$, so $[\mathbf{S}_{V,E,s}/\Gamma]$ is a principal d-orbifold with corners.

Example 12.17. Let $[\mathbf{S}_{V,E,s}/\Gamma]$, $[\mathbf{S}_{W,F,t}/\Delta]$ be quotient d-orbifolds with corners as in Example 12.16, where Γ acts on V,E by $q(\gamma):V\rightarrow V$ and $\hat{q}(\gamma):E\rightarrow q(\gamma)^*(E)$ for $\gamma\in\Gamma$, and Δ acts on W,F by $r(\delta):W\rightarrow W$ and $\hat{r}(\delta):F\rightarrow r(\delta)^*(F)$ for $\delta\in\Delta$. Suppose $f:V\rightarrow W$ is a smooth map in $\mathbf{Man^c}$, and $\hat{f}:E\rightarrow f^*(F)$ is a morphism of vector bundles on V satisfying $\hat{f}\circ s=f^*(t)+O(s^2)$, and $\rho:\Gamma\rightarrow\Delta$ is a group morphism satisfying $f\circ q(\gamma)=r(\rho(\gamma))\circ f:V\rightarrow W$ and $q(\gamma)^*(\hat{f})\circ\hat{q}(\gamma)=f^*(\hat{r}(\rho(\gamma)))\circ\hat{f}:E\rightarrow(f\circ q(\gamma))^*(F)$ for all $\gamma\in\Gamma$. Then Definition 7.15 defines a 1-morphism

$\mathbf{S}_{f,\hat f} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ in $\mathbf{dMan^c}$. The equivariance conditions on $f, \hat f$ imply that $\mathbf{S}_{f,\hat f} \circ \mathbf{S}_{q(\gamma),\hat q(\gamma)} = \mathbf{S}_{r(\rho(\gamma)),\hat r(\rho(\gamma))} \circ \mathbf{S}_{f,\hat f}$ for $\gamma \in \Gamma$. Hence Definition 11.7 defines a quotient 1-morphism $[\mathbf{S}_{f,\hat f},\rho] : [\mathbf{S}_{V,E,s}/\Gamma] \to [\mathbf{S}_{W,F,t}/\Delta]$ in $\mathbf{dOrb^c}$.

Proposition 12.18. *A d-stack with corners $\mathcal{X}$ is a d-orbifold with corners of virtual dimension $n \in \mathbb{Z}$ if and only if each $[x] \in \mathcal{X}_{\rm top}$ has an open neighbourhood $\mathcal{U}$ equivalent to some $[\mathbf{S}_{V,E,s}/\Gamma]$ in Example 12.16 with $\dim V - \operatorname{rank} E = n$, where $\Gamma = \operatorname{Iso}_{\mathcal{X}}([x])$ and $[x] \in \mathcal{X}_{\rm top}$ is identified with a fixed point $v \in S^k(V) \subseteq V$ of Γ with $s(v) = 0$ and $\mathrm{d}s|_{S^k(V)}(v) = 0$. Furthermore, V, E, s, Γ are determined up to non-canonical isomorphism near v by $\mathcal{X}$ near $[x]$.*

Proposition 12.19. *A quotient d-stack with corners $[\mathbf{U}/G]$ is a d-orbifold with corners if and only if the d-space with corners $\mathbf{U}$ is a d-manifold with corners, and then $\operatorname{vdim}[\mathbf{U}/G] = \operatorname{vdim} \mathbf{U}$.*

12.3 Equivalences and gluing

Sections 7.4 and 10.2 discussed equivalences and gluing for d-manifolds with corners, and for d-orbifolds. Combining the two yields analogous results for d-orbifolds with corners. The proofs in this section combine those of §7.4 and §10.2 in a straightforward way, using results from §11.4, so we leave them as an exercise. Here is the analogue of Theorems 7.20 and 10.16.

Theorem 12.20. *Suppose $\boldsymbol{f} : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of d-orbifolds with corners, and $f : \mathcal{X} \to \mathcal{Y}$ is representable. Then the following are equivalent:*

(i) *$\boldsymbol{f}$ is étale;*

(ii) *$\boldsymbol{f}$ is simple and flat, in the sense of Definition 11.10, and $\Omega_{\boldsymbol{f}} : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is an equivalence in $\operatorname{vqcoh}(\mathcal{X})$; and*

(iii) *$\boldsymbol{f}$ is simple and flat, and (9.14) is a split short exact sequence in $\operatorname{qcoh}(\mathcal{X})$.*

If in addition $f_* : \operatorname{Iso}_{\mathcal{X}}([x]) \to \operatorname{Iso}_{\mathcal{Y}}(f_{\rm top}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\rm top}$, and $f_{\rm top} : \mathcal{X}_{\rm top} \to \mathcal{Y}_{\rm top}$ is a bijection, then $\boldsymbol{f}$ is an equivalence in $\mathbf{dOrb^c}$.

Here is the analogue of Theorems 7.21 and 10.17.

Theorem 12.21. *Suppose $\mathcal{S}_{f,\hat f} : \mathcal{S}_{\mathcal{V},\mathcal{E},s} \to \mathcal{S}_{\mathcal{W},\mathcal{F},t}$ is a 'standard model' 1-morphism in $\mathbf{dOrb^c}$, in the notation of Definitions 12.2 and 12.13, with $f : \mathcal{V} \to \mathcal{W}$ representable. Then $\mathcal{S}_{f,\hat f}$ is étale if and only if f is simple and flat near $s^{-1}(0) \subseteq \mathcal{V}$, in the sense of Definition 8.21, and for each $[v] \in \mathcal{V}_{\rm top}$ with $s(v) = 0$ and $[w] = f_{\rm top}([v]) \in \mathcal{W}_{\rm top}$, the following sequence is exact:*

$$0 \longrightarrow T_v\mathcal{V} \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} \mathcal{E}_v \oplus T_w\mathcal{W} \xrightarrow{\hat f(v) \oplus -\mathrm{d}t(w)} \mathcal{F}_w \longrightarrow 0.$$

Also $\mathcal{S}_{f,\hat f}$ is an equivalence if and only if in addition $f_{\rm top}|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection, where $s^{-1}(0) = \{[v] \in \mathcal{V}_{\rm top} : s(v) = 0\}$, $t^{-1}(0) = \{[w] \in \mathcal{W}_{\rm top} : t(w) = 0\}$, and $f_* : \operatorname{Iso}_{\mathcal{V}}([v]) \to \operatorname{Iso}_{\mathcal{W}}(f_{\rm top}([v]))$ is an isomorphism for all $[v] \in s^{-1}(0) \subseteq \mathcal{V}_{\rm top}$.

As for Corollary 10.18, we deduce:

Corollary 12.22. *Let $\mathcal{V},\mathcal{W}$ be orbifolds with corners, $\mathcal{E},\mathcal{F}$ vector bundles over $\mathcal{V},\mathcal{W}$, $s\in C^\infty(\mathcal{E})$, $t\in C^\infty(\mathcal{F})$ smooth sections, $f:\mathcal{V}\to\mathcal{W}$ an sf-embedding of orbifolds with corners, and $\hat f:\mathcal{E}\to f^*(\mathcal{F})$ an injective morphism of vector bundles (that is, $\hat f$ has a left inverse) satisfying $\hat f\circ s=f^*(t)$. For each $[v]\in\mathcal{V}_{\rm top}$ with $s(v)=0$ and $f_{\rm top}([v])=[w]\in\mathcal{W}_{\rm top}$, we have a linear map*

$$\mathrm{d}t(w)_*:T_w\mathcal{W}/\mathrm{d}f(v)[T_v\mathcal{V}]\longrightarrow\mathcal{F}_w/\hat f(v)[\mathcal{E}_v]. \tag{12.1}$$

Suppose (12.1) *is an isomorphism for all $[v]\in s^{-1}(0)\subseteq\mathcal{V}_{\rm top}$, and $f_{\rm top}|_{s^{-1}(0)}:s^{-1}(0)\to t^{-1}(0)$ is a bijection. Then $\boldsymbol{S}_{f,\hat f}:\boldsymbol{S}_{\mathcal{V},\mathcal{E},s}\to\boldsymbol{S}_{\mathcal{W},\mathcal{F},t}$ is an equivalence.*

Here are analogues of Theorems 10.19 and 10.21, extending Theorem 7.23.

Theorem 12.23. *Suppose we are given the following data:*

- **(a)** *an integer n;*
- **(b)** *a Hausdorff, second countable topological space X;*
- **(c)** *an indexing set I, and a total order $<$ on I;*
- **(d)** *for each i in I, an effective orbifold with corners $\mathcal{V}_i$, a vector bundle $\mathcal{E}_i$ on $\mathcal{V}_i$ with $\dim\mathcal{V}_i-\operatorname{rank}\mathcal{E}_i=n$, a section $s_i\in C^\infty(\mathcal{E}_i)$, and a homeomorphism $\psi_i:s_i^{-1}(0)\to\hat X_i$, where $s_i^{-1}(0)=\{[v_i]\in\mathcal{V}_{i,\rm top}:s_i(v_i)=0\}$ and $\hat X_i\subseteq X$ is open; and*
- **(e)** *for all $i<j$ in I, an open suborbifold $\mathcal{V}_{ij}\subseteq\mathcal{V}_i$, a simple, flat 1-morphism $e_{ij}:\mathcal{V}_{ij}\to\mathcal{V}_j$, and a morphism of vector bundles $\hat e_{ij}:\mathcal{E}_i|_{\mathcal{V}_{ij}}\to e_{ij}^*(\mathcal{E}_j)$.*

Let this data satisfy the conditions:

- **(i)** *$X=\bigcup_{i\in I}\hat X_i$;*
- **(ii)** *if $i<j$ in I then $(e_{ij})_*:\mathrm{Iso}_{\mathcal{V}_{ij}}([v])\to\mathrm{Iso}_{\mathcal{V}_j}(e_{ij,\rm top}([v]))$ is an isomorphism for all $[v]\in\mathcal{V}_{ij,\rm top}$, and $\hat e_{ij}\circ s_i|_{\mathcal{V}_{ij}}=e_{ij}^*(s_j)\circ\iota_{ij}$ where $\iota_{ij}:\mathcal{O}_{\mathcal{V}_{ij}}\to e_{ij}^*(\mathcal{O}_{\mathcal{V}_j})$ is the natural isomorphism, and $\psi_i(s_i|_{\mathcal{V}_{ij}}^{-1}(0))=\hat X_i\cap\hat X_j$, and $\psi_i|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)}=\psi_j\circ e_{ij,\rm top}|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)}$, and if $[v_i]\in\mathcal{V}_{ij,\rm top}$ with $s_i(v_i)=0$ and $[v_j]=e_{ij,\rm top}([v_i])$ then the following sequence is exact:*

$$0\longrightarrow T_{v_i}\mathcal{V}_i\xrightarrow{\mathrm{d}s_i(v_i)\oplus\,\mathrm{d}e_{ij}(v_i)}\mathcal{E}_i|_{v_i}\oplus T_{v_j}\mathcal{V}_j\xrightarrow{\hat e_{ij}(v_i)\oplus-\mathrm{d}s_j(v_j)}\mathcal{E}_j|_{v_j}\longrightarrow 0;$$

- **(iii)** *if $i<j<k$ in I then there exists a 2-morphism $\eta_{ijk}:e_{jk}\circ e_{ij}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}\Rightarrow e_{ik}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}$ in $\mathbf{Orb^c}$ with*

$$\hat e_{ik}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}=\eta_{ijk}^*(\mathcal{E}_k)\circ I_{e_{ij},e_{jk}}(\mathcal{E}_k)^{-1}\circ e_{ij}|^*_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}(\hat e_{jk})\circ\hat e_{ij}|_{\mathcal{V}_{ik}\cap e_{ij}^{-1}(\mathcal{V}_{jk})}.$$

Note that η_{ijk} is unique by Proposition 8.14.

Then there exist a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ *with* $\operatorname{vdim}\boldsymbol{\mathcal{X}}=n$ *and underlying topological space* $\mathcal{X}_{\mathrm{top}}\cong X$, *and a* 1*-morphism* $\boldsymbol{\psi}_i:\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}\to\boldsymbol{\mathcal{X}}$ *in* $\mathbf{dOrb^c}$ *with underlying continuous map* ψ_i *which is an equivalence with the open d-suborbifold* $\hat{\boldsymbol{\mathcal{X}}}_i\subseteq\boldsymbol{\mathcal{X}}$ *corresponding to* $\hat{X}_i\subseteq X$ *for all* $i\in I$, *such that for all* $i<j$ *in* I *there exists a* 2*-morphism* $\boldsymbol{\eta}_{ij}:\boldsymbol{\psi}_j\circ\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}\Rightarrow\boldsymbol{\psi}_i\circ\boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i}$, *where* $\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}:\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}}\to\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j}$ *and* $\boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i}:\boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}}\to\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$. *This* $\boldsymbol{\mathcal{X}}$ *is unique up to equivalence in* $\mathbf{dOrb^c}$.

Suppose also that $\mathcal{Y}$ *is an effective orbifold with corners, and* $g_i:\mathcal{V}_i\to\mathcal{Y}$ *are* 1*-morphisms in* $\mathbf{Orb^c}$ *for all* $i\in I$ *satisfying any of Proposition* 8.14(i)–(v), *and there are* 2*-morphisms* $\zeta_{ij}:g_j\circ e_{ij}\Rightarrow g_i|_{\mathcal{V}_{ij}}$ *in* $\mathbf{Orb^c}$ *for all* $i<j$ *in* I. *Then there exist a* 1*-morphism* $\boldsymbol{h}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ *in* $\mathbf{dOrb^c}$ *unique up to* 2*-isomorphism, where* $\boldsymbol{\mathcal{Y}}=F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}(\mathcal{Y})=\boldsymbol{\mathcal{S}}_{\mathcal{Y},0,0}$, *and* 2*-morphisms* $\boldsymbol{\zeta}_i:\boldsymbol{h}\circ\boldsymbol{\psi}_i\Rightarrow\boldsymbol{\mathcal{S}}_{g_i,0}$ *for all* $i\in I$.

Theorem 12.24. *Suppose we are given the following data:*

(a) *an integer* n;

(b) *a Hausdorff, second countable topological space* X;

(c) *an indexing set* I, *and a total order* $<$ *on* I;

(d) *for each* i *in* I, *a manifold with corners* V_i, *a vector bundle* $E_i\to V_i$ *with* $\dim V_i-\operatorname{rank}E_i=n$, *a finite group* Γ_i, *smooth, locally effective actions* $r_i(\gamma):V_i\to V_i$, $\hat{r}_i(\gamma):E_i\to r(\gamma)^*(E_i)$ *of* Γ_i *on* V_i,E_i *for* $\gamma\in\Gamma_i$, *a smooth,* Γ_i*-equivariant section* $s_i:V_i\to E_i$, *and a homeomorphism* $\psi_i:X_i\to\hat{X}_i$, *where* $X_i=\{v_i\in V_i:s_i(v_i)=0\}/\Gamma_i$ *and* $\hat{X}_i\subseteq X$ *is an open set; and*

(e) *for all* $i<j$ *in* I, *an open submanifold* $V_{ij}\subseteq V_i$, *invariant under* Γ_i, *a group morphism* $\rho_{ij}:\Gamma_i\to\Gamma_j$, *a simple, flat, smooth map* $e_{ij}:V_{ij}\to V_j$, *and a morphism of vector bundles* $\hat{e}_{ij}:E_i|_{V_{ij}}\to e_{ij}^*(E_j)$.

Let this data satisfy the conditions:

(i) $X=\bigcup_{i\in I}\hat{X}_i$;

(ii) *if* $i<j$ *in* I *then* $\hat{e}_{ij}\circ s_i|_{V_{ij}}=e_{ij}^*(s_j)+O(s_i^2)$, *and for all* $\gamma\in\Gamma$ *we have*

$$
\begin{aligned}
e_{ij}\circ r_i(\gamma)&=r_j(\rho_{ij}(\gamma))\circ e_{ij}:V_{ij}\longrightarrow V_j,\\
r_i(\gamma)^*(\hat{e}_{ij})\circ\hat{r}_i(\gamma)&=e_{ij}^*(\hat{r}_j(\rho_{ij}(\gamma)))\circ\hat{e}_{ij}:E_i|_{V_{ij}}\longrightarrow(e_{ij}\circ r_i(\gamma))^*(E_j),
\end{aligned}
$$

and $\psi_i(X_i\cap(V_{ij}/\Gamma_i))=\hat{X}_i\cap\hat{X}_j$, *and* $\psi_i|_{X_i\cap V_{ij}/\Gamma_i}=\psi_j\circ(e_{ij})_*|_{X_i\cap V_{ij}/\Gamma_j}$, *and if* $v_i\in V_{ij}$ *with* $s_i(v_i)=0$ *and* $v_j=e_{ij}(v_i)$ *then* $\rho|_{\mathrm{Stab}_{\Gamma_i}(v_i)}:\mathrm{Stab}_{\Gamma_i}(v_i)\to\mathrm{Stab}_{\Gamma_j}(v_j)$ *is an isomorphism, and the following sequence of vector spaces is exact:*

$$
0\longrightarrow T_{v_i}V_i\xrightarrow{\mathrm{d}s_i(v_i)\oplus\,\mathrm{d}e_{ij}(v_i)}E_i|_{v_i}\oplus T_{v_j}V_j\xrightarrow{\hat{e}_{ij}(v_i)\oplus-\mathrm{d}s_j(v_j)}E_j|_{v_j}\longrightarrow 0;
$$

(iii) *if $i < j < k$ in I then there exists $\gamma_{ijk} \in \Gamma_k$ satisfying*

$$\rho_{ik}(\gamma) = \gamma_{ijk}\,\rho_{jk}(\rho_{ij}(\gamma))\,\gamma_{ijk}^{-1} \quad \textit{for all } \gamma \in \Gamma_i,$$
$$e_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = r_k(\gamma_{ijk}) \circ e_{jk} \circ e_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}, \quad \textit{and}$$
$$\hat e_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = \bigl(e_{ij}^*(e_{jk}^*(\hat r_k(\gamma_{ijk}))) \circ e_{ij}^*(\hat e_{jk}) \circ \hat e_{ij}\bigr)|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}.$$

Then there exist a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ with $\operatorname{vdim} \boldsymbol{\mathcal{X}} = n$ *and underlying topological space $\mathcal{X}_{\rm top} \cong X$, and a 1-morphism $\boldsymbol{\psi}_i : [\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i] \to \boldsymbol{\mathcal{X}}$ in* $\mathbf{dOrb^c}$ *with underlying continuous map ψ_i which is an equivalence with the open d-suborbifold $\hat{\boldsymbol{\mathcal{X}}}_i \subseteq \boldsymbol{\mathcal{X}}$ corresponding to $\hat X_i \subseteq X$ for all $i \in I$, such that for all $i < j$ in I there exists a 2-morphism $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ [\mathbf{S}_{e_{ij},\hat e_{ij}}, \rho_{ij}] \Rightarrow \boldsymbol{\psi}_i \circ [\boldsymbol{i}_{V_{ij},V_i}, \mathrm{id}_{\Gamma_i}]$, where $[\mathbf{S}_{e_{ij},\hat e_{ij}}, \rho_{ij}] : [\mathbf{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i] \to [\mathbf{S}_{V_j,E_j,s_j}/\Gamma_j]$ and $[\boldsymbol{i}_{V_{ij},V_i}, \mathrm{id}_{\Gamma_i}] : [\mathbf{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i] \to [\mathbf{S}_{V_i,E_i,s_i}/\Gamma_j]$ are as in Example* 12.17. *This $\boldsymbol{\mathcal{X}}$ is unique up to equivalence in* $\mathbf{dOrb^c}$.

Suppose also that Y is a manifold with corners, and $g_i : V_i \to Y$ are smooth maps for all $i \in I$ with $g_i \circ r_i(\gamma) = g_i$ for all $\gamma \in \Gamma_i$, and $g_j \circ e_{ij} = g_i|_{V_{ij}}$ for all $i < j$ in I. Then there exist a 1*-morphism $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in* $\mathbf{dOrb^c}$ *unique up to* 2*-isomorphism, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y) = [\mathbf{S}_{Y,0,0}/\{1\}]$, and* 2*-morphisms $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow [\mathbf{S}_{g_i,0}, \pi_{\{1\}}]$ for all $i \in I$. Here $[\mathbf{S}_{Y,0,0}/\{1\}]$ is from Example* 12.16 *with E, s both zero and $\Gamma = \{1\}$, and $[\mathbf{S}_{g_i,0}, \pi_{\{1\}}] : [\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i] \to [\mathbf{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}}$ is from Example* 12.17 *with $\hat g_i = 0$ and $\rho = \pi_{\{1\}} : \Gamma_i \to \{1\}$.*

The effectiveness assumptions in Theorems 12.23 and 12.24 ensure the overlap conditions (11.25)–(11.26) on 2-morphisms in Theorem 11.19 hold automatically. The importance of Theorems 12.23 and 12.24 is that all the ingredients are described wholly in differential-geometric or topological terms, so we can use them to prove the existence of d-orbifold structures on spaces coming from other areas of geometry, such as moduli spaces of J-holomorphic curves.

In §12.9 we will prove that every d-orbifold with corners $\boldsymbol{\mathcal{X}}$ admits *good coordinate systems*, collections of data satisfying the hypotheses of Theorems 12.23 and 12.24. This shows these hypotheses are not unrealistically strong.

12.4 Submersions, immersions and embeddings

Sections 4.1–4.2 studied submersions, immersions and embeddings in **dMan**. This was extended to d-manifolds with corners in §7.5, and to d-orbifolds in §10.3. Also Definition 8.28 defined submersions, s-submersions, ..., sf-embeddings in $\mathbf{Orb^c}$. For d-orbifolds with corners, we combine the modifications of §7.5 and §10.3. Here is the analogue of Definitions 7.24 and 10.22.

Definition 12.25. Let $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a 1-morphism in $\mathbf{dOrb^c}$. Then $T^*\boldsymbol{\mathcal{X}} = (\mathcal{E}_{\mathcal{X}}, \mathcal{F}_{\mathcal{X}}, \phi_{\mathcal{X}})$ and $f^*(T^*\boldsymbol{\mathcal{Y}}) = \bigl(f^*(\mathcal{E}_{\mathcal{Y}}), f^*(\mathcal{F}_{\mathcal{Y}}), f^*(\phi_{\mathcal{Y}})\bigr)$ are virtual vector bundles on $\mathcal{X}$ of ranks $\operatorname{vdim}\boldsymbol{\mathcal{X}}, \operatorname{vdim}\boldsymbol{\mathcal{Y}}$, and $\Omega_{\boldsymbol{f}} = (f'', f^2) : f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ is a 1-morphism in $\mathrm{vvect}(\mathcal{X})$.

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{\boldsymbol{f}}$ is weakly injective. We call $\boldsymbol{f}$ an *sw-submersion* if it is also simple.

(b) We call $\boldsymbol{f}$ a *submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{C(\boldsymbol{f})}$ is injective, for $C(\boldsymbol{f})$ as in §11.5. We call $\boldsymbol{f}$ an *s-submersion* if it is also simple.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $f : \mathcal{X} \to \mathcal{Y}$ is representable and $\Omega_{\boldsymbol{f}}$ is weakly surjective. We call $\boldsymbol{f}$ an *sw-immersion*, or *sfw-immersion*, if $\boldsymbol{f}$ is also simple, or simple and flat.

(d) We call $\boldsymbol{f}$ an *immersion* if $f : \mathcal{X} \to \mathcal{Y}$ is representable and $\Omega_{\hat{C}(\boldsymbol{f})}$ is surjective, for $\hat{C}(\boldsymbol{f})$ as in §11.5. We call $\boldsymbol{f}$ an *s-immersion* if $\boldsymbol{f}$ is also simple, and an *sf-immersion* if $\boldsymbol{f}$ is also simple and flat.

(e) We call $\boldsymbol{f}$ a *w-embedding, sw-embedding, sfw-embedding, embedding, s-embedding*, or *sf-embedding*, if $\boldsymbol{f}$ is a w-immersion, ..., sf-immersion, respectively, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image, so in particular f_{top} is injective.

More generally, we make the same definitions for $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ a 1-morphism in $\mathbf{d\check{O}rb^c}$ from Definition 12.9.

Parts (c)–(e) enable us to define *d-suborbifolds* $\boldsymbol{\mathcal{X}}$ of a d-orbifold with corners $\boldsymbol{\mathcal{Y}}$. *Open d-suborbifolds* are open d-substacks $\boldsymbol{\mathcal{X}}$ in $\boldsymbol{\mathcal{Y}}$. For more general d-suborbifolds, we call $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ a *w-immersed, sw-immersed, sfw-immersed, immersed, s-immersed, sf-immersed, w-embedded, sw-embedded, sfw-embedded, embedded, s-embedded*, or *sf-embedded d-suborbifold* of $\boldsymbol{\mathcal{Y}}$ if $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ are d-orbifolds with corners and $\boldsymbol{f}$ is a w-immersion, ..., sf-embedding, respectively.

The 1-morphisms $\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}}) : \boldsymbol{\mathcal{X}}^{\Gamma,\lambda} \to \boldsymbol{\mathcal{X}}$ and $\tilde{\boldsymbol{O}}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}}) : \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\lambda} \to \boldsymbol{\mathcal{X}}$ of §12.8 are examples of w-immersions. Here is the analogue of Proposition 7.26. To prove it, note by comparing Definitions 7.24 and 12.25 that a 1-morphism $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb^c}$ is a w-submersion, sw-submersion, ..., s-immersion or sf-immersion if and only if it is étale locally modelled in the sense of Definition 11.2(∗) on 1-morphisms $\boldsymbol{h} : \mathbf{U} \to \mathbf{V}$ in $\mathbf{dMan^c}$ which are w-submersions, ..., sf-immersions, plus we must require $f : \mathcal{X} \to \mathcal{Y}$ representable for w-immersions, ..., sf-immersions. But if $\boldsymbol{f}$ is étale locally modelled on $\boldsymbol{h}$ then $C(\boldsymbol{f}), \hat{C}(\boldsymbol{f}), \boldsymbol{f}_\pm$ are étale locally modelled on $C(\boldsymbol{h}), \hat{C}(\boldsymbol{h}), \boldsymbol{h}_\pm$, and if f is representable then $C(f), \hat{C}(f), f_\pm$ are representable. Therefore Proposition 12.26 follows from Proposition 7.26.

Proposition 12.26. *Suppose $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a w-submersion, sw-submersion, ..., s-immersion or sf-immersion in* $\mathbf{dOrb^c}$. *Then $C(\boldsymbol{f})$ and $\hat{C}(\boldsymbol{f}) : C(\boldsymbol{\mathcal{X}}) \to C(\boldsymbol{\mathcal{Y}})$ from* §11.5 *are also w-submersions, ..., sf-immersions in* $\mathbf{d\check{O}rb^c}$. *If $\boldsymbol{f}$ is semisimple then $\boldsymbol{f}_+ : \partial_+^{\boldsymbol{f}} \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ and $\boldsymbol{f}_- : \partial_-^{\boldsymbol{f}} \boldsymbol{\mathcal{X}} \to \partial \boldsymbol{\mathcal{Y}}$ from* §11.3 *are also w-submersions, ..., sf-immersions in* $\mathbf{dOrb^c}$.

Propositions 7.27, 7.28 and 7.29 then hold for d-orbifolds with corners and orbifolds with corners, except that in the d-orbifold analogue of Proposition 7.28(a) we also have to assume $f : \mathcal{X} \to \mathcal{Y}$ is representable to deduce $\boldsymbol{f}$ is étale. The next two theorems are the analogues of Theorems 7.30, 7.31(a)–(d) and 10.23, 10.24, proved by the same methods.

Theorem 12.27. *Suppose $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{\mathcal{V},\mathcal{E},s} \to \mathbf{S}_{\mathcal{W},\mathcal{F},t}$ is a 'standard model' 1-morphism in* $\mathbf{dOrb^c}$*, as in Definition* 12.9*. For each $[v] \in \mathcal{V}_{\rm top}$ with $s(v) = 0$ and $[w] = f_{\rm top}([v]) \in \mathcal{W}_{\rm top}$, we have a complex*

$$0 \longrightarrow T_v\mathcal{V} \xrightarrow{\mathrm{d}s|_v \oplus \mathrm{d}f|_v} \mathcal{E}_v \oplus T_w\mathcal{W} \xrightarrow{\hat{f}|_v \oplus -\mathrm{d}t|_w} \mathcal{F}_w \longrightarrow 0. \qquad (12.2)$$

If also $[v']$ lies in $(\Pi^k_{\mathcal{V},\rm top})^{-1}([v]) \subseteq C_k(\mathcal{V})_{\rm top}$ with $\hat{C}(f)_{\rm top}([v']) = [w']$ in $(\Pi^l_{\mathcal{W},\rm top})^{-1}([w]) \subseteq C_l(\mathcal{W})_{\rm top}$, we have a complex

$$0 \to T_{v'}C_k(\mathcal{V}) \xrightarrow{\mathrm{d}(\Pi^k_{\mathcal{V}})^*(s)|_{v'} \oplus \mathrm{d}\hat{C}(f)|_{v'}} \mathcal{E}_v \oplus T_{w'}C_l(\mathcal{W}) \xrightarrow{\hat{f}|_v \oplus -\mathrm{d}(\Pi^k_{\mathcal{W}})^*(t)|_{w'}} \mathcal{F}_w \to 0. \qquad (12.3)$$

(a) $\mathbf{S}_{f,\hat{f}}$ *is a w-submersion if and only if $f : \mathcal{V} \to \mathcal{W}$ is semisimple and flat near $s^{-1}(0)$ in $\mathcal{V}$, and for all $[v],[w],[v'],[w']$ as above, equations* (12.2)–(12.3) *are exact at the fourth terms.* $\mathbf{S}_{f,\hat{f}}$ *is an sw-submersion if and only if also f is simple near $s^{-1}(0) = \{[v] \in \mathcal{V}_{\rm top} : s(v) = 0\}$.*

(b) $\mathbf{S}_{f,\hat{f}}$ *is a submersion if and only if for all $[v],[w],[v'],[w']$ as above, equations* (12.2)–(12.3) *are exact at the third and fourth terms. These imply that f is semisimple and flat near $s^{-1}(0)$.* $\mathbf{S}_{f,\hat{f}}$ *is an s-submersion if and only if also f is simple near $s^{-1}(0)$.*

(c) $\mathbf{S}_{f,\hat{f}}$ *is a w-immersion if and only if for all $[v],[w],[v'],[w']$ as above, equations* (12.2)–(12.3) *are exact at the second term, and $f_* : \mathrm{Iso}_{\mathcal{V}}([v]) \to \mathrm{Iso}_{\mathcal{W}}([w])$ is injective.* $\mathbf{S}_{f,\hat{f}}$ *is an sw-immersion (or sfw-immersion) if and only if also f is simple (or simple and flat) near $s^{-1}(0)$.*

(d) $\mathbf{S}_{f,\hat{f}}$ *is an immersion if and only if for all $[v],[w],[v'],[w']$ as above, equations* (12.2)–(12.3) *are exact at the second and fourth terms, and $f_* : \mathrm{Iso}_{\mathcal{V}}([v]) \to \mathrm{Iso}_{\mathcal{W}}([w])$ is injective.* $\mathbf{S}_{f,\hat{f}}$ *is an s-immersion (or sf-immersion) if and only if also f is simple (or simple and flat) near $s^{-1}(0)$.*

The conditions in **(a)**–**(d)** *are open conditions on $[v]$ in $s^{-1}(0)$.*

Theorem 12.28. *Suppose $\boldsymbol{g} : \mathcal{\boldsymbol{X}} \to \mathcal{\boldsymbol{Y}}$ is a 1-morphism of d-orbifolds with corners, and $[x] \in \mathcal{X}_{\rm top}$ with $g_{\rm top}([x]) = [y] \in \mathcal{Y}_{\rm top}$. Then there exist open d-suborbifolds $\boldsymbol{\mathcal{T}} \subseteq \boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{Y}}$ with $[x] \in \mathcal{T}_{\rm top}$, $[y] \in \mathcal{U}_{\rm top}$ and $\boldsymbol{g}(\boldsymbol{\mathcal{T}}) \subseteq \boldsymbol{\mathcal{U}}$, a 'standard model' 1-morphism $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{\mathcal{V},\mathcal{E},s} \to \mathbf{S}_{\mathcal{W},\mathcal{F},t}$ in* $\mathbf{dOrb^c}$ *as in Definition* 12.13*, equivalences $\boldsymbol{i} : \boldsymbol{\mathcal{T}} \to \mathbf{S}_{\mathcal{V},\mathcal{E},s}$, $\boldsymbol{j} : \mathbf{S}_{\mathcal{W},\mathcal{F},t} \to \boldsymbol{\mathcal{U}}$, and a 2-morphism $\boldsymbol{\eta} : \boldsymbol{j} \circ \mathbf{S}_{f,\hat{f}} \circ \boldsymbol{i} \Rightarrow \boldsymbol{g}|_{\boldsymbol{\mathcal{T}}}$ in* $\mathbf{dOrb^c}$*. Furthermore:*

(a) *If $\boldsymbol{g}$ is a **w-submersion** then we can choose the data $\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{U}}, \ldots, \boldsymbol{j}$ above such that $f : \mathcal{V} \to \mathcal{W}$ is a submersion in* $\mathbf{Orb^c}$*, and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ is a surjective morphism of vector bundles (i.e. has a right inverse). If $\boldsymbol{g}$ is an **sw-submersion** in* $\mathbf{dOrb^c}$*, then f is an s-submersion in* $\mathbf{Orb^c}$*.*

(b) *If $\boldsymbol{g}$ is a **submersion** we can choose $\boldsymbol{\mathcal{T}}, \ldots, \boldsymbol{j}$ such that $f : \mathcal{V} \to \mathcal{W}$ is a submersion and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ is an isomorphism. If $\boldsymbol{g}$ is an **s-submersion** then f is an s-submersion.*

(c) *If $\boldsymbol{g}$ is a **w-immersion** we can choose $\mathcal{T},\ldots,\boldsymbol{j}$ such that $f:\mathcal{V}\to\mathcal{W}$ is an immersion in* $\mathbf{Orb^c}$*, and $\hat{f}:\mathcal{E}\to f^*(\mathcal{F})$ is an injective morphism of vector bundles (i.e. has a left inverse). If $\boldsymbol{g}$ is an **sw-immersion** or **sfw-immersion** then f is an s-immersion or sf-immersion.*

(d) *If $\boldsymbol{g}$ is an **immersion** we can choose $\mathcal{T},\ldots,\boldsymbol{j}$ such that $f:\mathcal{V}\to\mathcal{W}$ is an immersion and $\hat{f}:\mathcal{E}\to f^*(\mathcal{F})$ is an isomorphism. If $\boldsymbol{g}$ is an **s-immersion** or **sf-immersion** then f is an s-immersion or sf-immersion.*

As for Corollaries 7.34 and 10.25, Theorem 12.28(b) implies:

Corollary 12.29. *Let $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ be a submersion of d-orbifolds with corners, with $\boldsymbol{\mathcal{Y}}$ an orbifold with corners. Then $\boldsymbol{\mathcal{X}}$ is an orbifold with corners.*

12.5 Bd-transversality and fibre products

In §4.3 we studied fibre products in **dMan**, and showed that such fibre products exist under the assumption of d-transversality. This was extended to d-manifolds with corners in §7.6, and to d-orbifolds in §10.4. We now discuss fibre products in $\mathbf{dOrb^c}$. Here is the analogue of Definition 7.36.

Definition 12.30. Let $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}$ be d-orbifolds with corners and $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Z}}$, $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{Z}}$ be 1-morphisms. We call $\boldsymbol{g},\boldsymbol{h}$ *bd-transverse* if they are both b-transverse in $\mathbf{dSta^c}$ in the sense of Definition 11.21, and d-transverse in the sense of Definition 10.26. We call $\boldsymbol{g},\boldsymbol{h}$ *cd-transverse* if they are both c-transverse in $\mathbf{dSta^c}$ in the sense of Definition 11.22, and d-transverse. As in §11.6, c-transverse implies b-transverse, so cd-transverse implies bd-transverse.

Here is the analogue of Theorems 4.21, 7.37 and 10.27. It may be deduced from Theorem 7.37 using the same method by which Theorem 10.27 was deduced from Theorem 4.21.

Theorem 12.31. *Suppose $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}$ are d-orbifolds with corners and $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Z}}$, $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{Z}}$ are bd-transverse 1-morphisms, and let $\boldsymbol{\mathcal{W}}=\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}$ be the fibre product in* $\mathbf{dSta^c}$*, which exists by Theorem 11.24 as $\boldsymbol{g},\boldsymbol{h}$ are b-transverse. Then $\boldsymbol{\mathcal{W}}$ is a d-orbifold with corners, with*

$$\operatorname{vdim}\boldsymbol{\mathcal{W}}=\operatorname{vdim}\boldsymbol{\mathcal{X}}+\operatorname{vdim}\boldsymbol{\mathcal{Y}}-\operatorname{vdim}\boldsymbol{\mathcal{Z}}.\tag{12.4}$$

Hence, all bd-transverse fibre products exist in $\mathbf{dOrb^c}$.

Here are analogues of Theorems 7.38, 7.39 and Propositions 7.40, 7.41, see also Theorems 10.28, 10.29 and Propositions 10.30, 10.31. They are proved using the same methods.

Theorem 12.32. *Suppose $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Z}}$ and $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{Z}}$ are 1-morphisms in* $\mathbf{dOrb^c}$*. The following are sufficient conditions for $\boldsymbol{g},\boldsymbol{h}$ to be cd-transverse, and hence bd-transverse, so that $\boldsymbol{\mathcal{W}}=\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}$ is a d-orbifold with corners of virtual dimension* (12.4):

(a) *$\mathcal{Z}$ is an orbifold without boundary, that is, $\mathcal{Z} \in \bar{\mathbf{O}}\mathbf{rb}$; or*

(b) *$\boldsymbol{g}$ or $\boldsymbol{h}$ is a w-submersion.*

Theorem 12.33. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be d-orbifolds with corners with $\mathcal{Y}$ an orbifold, and $\boldsymbol{g} : \mathcal{X} \to \mathcal{Z}$, $\boldsymbol{h} : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms with $\boldsymbol{g}$ a submersion. Then $\mathcal{W} = \mathcal{X} \times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}} \mathcal{Y}$ is an orbifold, with $\dim \mathcal{W} = \operatorname{vdim} \mathcal{X} + \dim \mathcal{Y} - \operatorname{vdim} \mathcal{Z}$.*

Proposition 12.34. *Let $\rho : G \to H$ be an injective morphism of finite groups, and H act linearly on $\mathbb{R}^n$ preserving the subset $\mathbb{R}^n_k \subseteq \mathbb{R}^n$ for $0 \leqslant k \leqslant n$. Then as in §11.2 we have quotient d-orbifolds with corners $[*/G]$, $[\boldsymbol{\mathbb{R}}^{\boldsymbol{n}}_{\boldsymbol{k}}/H]$ and a quotient 1-morphism $[\mathbf{0}, \rho] : [*/G] \to [\boldsymbol{\mathbb{R}}^{\boldsymbol{n}}_{\boldsymbol{k}}/H]$. Suppose $\mathcal{X}$ is a d-orbifold with corners and $\boldsymbol{g} : \mathcal{X} \to [\boldsymbol{\mathbb{R}}^{\boldsymbol{n}}_{\boldsymbol{k}}/H]$ is a semisimple, flat 1-morphism in $\mathbf{dOrb^c}$. Then the fibre product $\mathcal{W} = \mathcal{X} \times_{\boldsymbol{g},[\boldsymbol{\mathbb{R}}^{\boldsymbol{n}}_{\boldsymbol{k}}/H],[\mathbf{0},\rho]} [*/G]$ exists in $\mathbf{dOrb^c}$. The projection $\boldsymbol{\pi}_{\mathcal{X}} : \mathcal{W} \to \mathcal{X}$ is an s-immersion, and an s-embedding if ρ is an isomorphism.*

When $k = 0$, any 1-morphism $\boldsymbol{g} : \mathcal{X} \to [\boldsymbol{\mathbb{R}}^{\boldsymbol{n}}/H]$ is semisimple and flat, and $\boldsymbol{\pi}_{\mathcal{X}} : \mathcal{W} \to \mathcal{X}$ is an sf-immersion, and an sf-embedding if ρ is an isomorphism.

Proposition 12.35. *Suppose $\boldsymbol{f} : \mathcal{X} \to \mathcal{Y}$ is an s-immersion of d-orbifolds with corners, and $[x] \in \mathcal{X}_{\rm top}$ with $f_{\rm top}([x]) = [y] \in \mathcal{Y}_{\rm top}$. Write $\rho : G \to H$ for $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$. Then ρ is injective, and there exist open neighbourhoods $\mathcal{U} \subseteq \mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{Y}$ of $[x], [y]$ with $\boldsymbol{f}(\mathcal{U}) \subseteq \mathcal{V}$, a linear action of H on $\mathbb{R}^n$ preserving the subset $\mathbb{R}^n_k \subseteq \mathbb{R}^n$ where $n = \operatorname{vdim} \mathcal{Y} - \operatorname{vdim} \mathcal{X} \geqslant 0$ and $0 \leqslant k \leqslant n$, and a 1-morphism $\boldsymbol{g} : \mathcal{V} \to [\boldsymbol{\mathbb{R}}^{\boldsymbol{n}}_{\boldsymbol{k}}/H]$ with $g_{\rm top}([y]) = [0]$, fitting into a 2-Cartesian square in $\mathbf{dOrb^c}$:*

$$\begin{tikzcd} \mathcal{U} \arrow[rr] \arrow[d, "\boldsymbol{f}|_{\mathcal{U}}"'] \arrow[drr, phantom, "\Uparrow"] & & {[*/G]} \arrow[d, "{[\mathbf{0},\rho]}"] \\ \mathcal{V} \arrow[rr, "\boldsymbol{g}"] & & {[\boldsymbol{\mathbb{R}}^{\boldsymbol{n}}_{\boldsymbol{k}}/H].} \end{tikzcd}$$

If $\boldsymbol{f}$ is an sf-immersion then $k = 0$. If $\boldsymbol{f}$ is an s- or sf-embedding then ρ is an isomorphism, and we may take $\mathcal{U} = \boldsymbol{f}^{-1}(\mathcal{V})$.

Since the material of §6.9 extends to d-stacks with corners, as for Corollary 7.42 we may deduce:

Corollary 12.36. *Suppose $\boldsymbol{g} : \mathcal{X} \to \mathcal{Z}$ and $\boldsymbol{h} : \mathcal{Y} \to \mathcal{Z}$ are cd-transverse 1-morphisms in $\mathbf{dOrb^c}$, and let $\mathcal{W} = \mathcal{X} \times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}} \mathcal{Y}$ be the fibre product in $\mathbf{dOrb^c}$, which exists by Theorem 12.31. Then we have equivalences in $\mathbf{dOrb^c}$ for all $i \geqslant 0$, where each fibre product is cd-transverse and so exists in $\mathbf{dOrb^c}$:*

$$C_i(\mathcal{W}) \simeq \coprod_{j,k,l \geqslant 0 : i = j+k-l} C_j^{\boldsymbol{g},l}(\mathcal{X}) \times_{C_j^l(\boldsymbol{g}), C_l(\mathcal{Z}), C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathcal{Y}), \tag{12.5}$$

$$\partial \mathcal{W} \simeq \coprod_{j,k,l \geqslant 0 : j+k = l+1} C_j^{\boldsymbol{g},l}(\mathcal{X}) \times_{C_j^l(\boldsymbol{g}), C_l(\mathcal{Z}), C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathcal{Y}). \tag{12.6}$$

12.6 Embedding d-orbifolds with corners into orbifolds

Section 4.4 studied embeddings of d-manifolds into manifolds. There were two main classes of results: firstly, given a d-manifold $\boldsymbol{X}$ satisfying an extra condition such as $\boldsymbol{X}$ compact, we proved the existence of embeddings $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{\mathbb{R}^n}$ in **dMan** for $n \gg 0$. Secondly, given an embedding $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in **dMan** with $\boldsymbol{X}$ a d-manifold and Y a manifold, we showed that $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ for V open in Y. Combining the two gave strong results on when d-manifolds are principal.

We generalized this to d-manifolds with corners in §7.7. The main new issue was that for the first class, one can construct embeddings $\boldsymbol{f} : \mathbf{X} \to \boldsymbol{\mathbb{R}^n}$ in $\mathbf{dMan^c}$, but for the second class, to show that $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$ one needs an sf-embedding $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dMan^c}$ with $\mathbf{X}$ a d-manifold with corners and Y a manifold with corners. So we need to bridge the gap between embeddings and sf-embeddings. This was done in the proof of Theorem 7.47, in which given an embedding $\boldsymbol{f} : \mathbf{X} \to \boldsymbol{\mathbb{R}^n}$ for n large enough, we constructed an sf-embedding $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ for Z a submanifold of $\mathbb{R}^n$, with corners, with $\dim Z = n$.

In §10.5 we considered generalizations of §4.4 to d-orbifolds. We found that the second class of results extends nicely to d-orbifolds, but we were unable to extend the first class in a satisfactory way, that is, we do not have useful criteria guaranteeing the existence of embeddings of d-orbifolds into orbifolds (though see Proposition 10.34). Since much of the interest in §4.4 came from combining the two classes, this meant the results of §10.5 were somewhat disappointing.

For d-orbifolds with corners, the situation is similar. For the first class of results, we still lack useful criteria guaranteeing the existence of embeddings of d-orbifolds with corners into orbifolds. For the second class, here is the analogue of Theorems 4.34, 7.48 and 10.32, proved in the same way.

Theorem 12.37. *Suppose $\mathcal{X}$ is a d-orbifold with corners, $\mathcal{Y}$ an orbifold with corners, and $\boldsymbol{f} : \mathcal{X} \to \boldsymbol{\mathcal{Y}}$ an sf-embedding, in the sense of Definition* 12.25. *Then there exist an open suborbifold $\mathcal{V} \subseteq \mathcal{Y}$ with $\boldsymbol{f}(\mathcal{X}) \subseteq \boldsymbol{\mathcal{V}}$, a vector bundle $\mathcal{E}$ on $\mathcal{V}$, and a section $s \in C^\infty(\mathcal{E})$ fitting into a* 2*-Cartesian diagram in* $\mathbf{dOrb^c}$:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{f}"'] \arrow[dr, phantom, "\Uparrow"] & \boldsymbol{\mathcal{V}} \arrow[d, "\mathbf{0}"] \\ \boldsymbol{\mathcal{V}} \arrow[r, "\boldsymbol{s}"] & \boldsymbol{\mathcal{E}}, \end{tikzcd}$$

where $\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{E}}, \boldsymbol{s}, \mathbf{0} = F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}\bigl(\mathcal{Y}, \mathcal{V}, \mathrm{Tot^c}(\mathcal{E}), \mathrm{Tot^c}(s), \mathrm{Tot^c}(0)\bigr)$, in the notation of Definition 8.19. *Thus $\mathcal{X}$ is equivalent to the 'standard model' $\mathbf{S}_{\mathcal{V},\mathcal{E},s}$ of Definition* 12.2, *and is a principal d-orbifold with corners.*

Here is one possibly useful criterion for the existence of sf-embeddings into orbifolds with corners. To prove it, follow the proof of Proposition 10.34 to construct an embedding of $\mathcal{X}$ into the total space $\mathrm{Tot^c}(\mathcal{E})$ of a vector bundle $\mathcal{E}$ on $\mathcal{Y}$. Then adapt the proof of Theorem 7.47 to promote this to an sf-embedding $\boldsymbol{g} : \mathcal{X} \to \boldsymbol{\mathcal{Z}}$ for $\mathcal{Z}$ an embedded suborbifold of $\mathrm{Tot^c}(\mathcal{E})$ with $\dim \mathcal{Z} = \dim \mathrm{Tot^c}(\mathcal{E})$. We can increase the rank of $\mathcal{E}$ if necessary to ensure an orbifold analogue of the condition $n \geqslant 2\bigl(\dim T_x^*\underline{X} + |i_{\mathbf{X}}^{-1}(x)|\bigr) + 1$ for $x \in \underline{X}$ in Theorem 7.47 holds.

Proposition 12.38. *Suppose $\mathcal{X}$ is a compact d-orbifold with corners, $\mathcal{Y}$ an effective orbifold with corners, and $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ a 1-morphism in* $\mathbf{dOrb^c}$ *with $f:\mathcal{X}\to\mathcal{Y}$ representable, where $\boldsymbol{\mathcal{Y}}=F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}(\mathcal{Y})$. Then there exists an sf-embedding $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Z}}$ in* $\mathbf{dOrb^c}$, *where $\boldsymbol{\mathcal{Z}}=F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}(\mathcal{Z})$ for an orbifold with corners $\mathcal{Z}$. Hence $\boldsymbol{\mathcal{X}}\simeq\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is a principal d-orbifold with corners by Theorem* 12.37.

12.7 Orientations on d-orbifolds with corners

Section 4.6 discussed orientations on d-manifolds. This was extended to d-manifolds with corners in §7.8, and to d-orbifolds in §10.6. We combine all these ideas to study orientations on d-orbifolds with corners. There are no significant new issues; everything important has already been said in §4.6, §7.8 and §10.6. Here is the analogue of Definitions 4.44, 7.51 and 10.36.

Definition 12.39. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold with corners. Then the virtual cotangent bundle $T^*\boldsymbol{\mathcal{X}}=(\mathcal{E}_{\mathcal{X}},\mathcal{F}_{\mathcal{X}},\phi_{\mathcal{X}})$ is a virtual vector bundle on $\mathcal{X}$ as in Definition 12.5, so Definition 10.35 constructs a line bundle $\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}$ on the Deligne–Mumford C^∞-stack $\mathcal{X}$. We call $\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}$ the *orientation line bundle* of $\boldsymbol{\mathcal{X}}$. An *orientation* ω on $\boldsymbol{\mathcal{X}}$ is an orientation on $\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}$. An *oriented d-orbifold with corners* is a pair $(\boldsymbol{\mathcal{X}},\omega)$ where $\boldsymbol{\mathcal{X}}$ is a d-orbifold with corners and ω an orientation on $\boldsymbol{\mathcal{X}}$. If $\omega=[\tau]$ is an orientation on $\boldsymbol{\mathcal{X}}$, the *opposite orientation* is $-\omega=[-\tau]$. Usually we will refer to $\boldsymbol{\mathcal{X}}$ as an oriented d-orbifold with corners, leaving ω implicit, and then $-\boldsymbol{\mathcal{X}}$ will mean $\boldsymbol{\mathcal{X}}$ with the opposite orientation.

All of the results of §4.6, §7.8 and §10.6 now extend to d-orbifolds in the obvious way. Here is the analogue of Theorems 4.50, 7.52 and 10.37, proved in the same way as Theorem 10.37.

Theorem 12.40. *Work in the situation of Theorem* 12.31, *so that $\boldsymbol{\mathcal{W}},\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}$ are d-orbifolds with corners with $\boldsymbol{\mathcal{W}}=\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}$ for $\boldsymbol{g},\boldsymbol{h}$ bd-transverse, where $\boldsymbol{e}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{X}}$, $\boldsymbol{f}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{Y}}$ are the projections. Then we have orientation line bundles $\mathcal{L}_{T^*\boldsymbol{\mathcal{W}}},\ldots,\mathcal{L}_{T^*\boldsymbol{\mathcal{Z}}}$ on $\mathcal{W},\ldots,\mathcal{Z}$, so $\mathcal{L}_{T^*\boldsymbol{\mathcal{W}}},e^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}),f^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Y}}}),(g\circ e)^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Z}}})$ are line bundles on $\mathcal{W}$. With a suitable choice of orientation convention, there is a canonical isomorphism*

$$\Phi:\mathcal{L}_{T^*\boldsymbol{\mathcal{W}}}\longrightarrow e^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}})\otimes_{\mathcal{O}_{\mathcal{W}}}f^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Y}}})\otimes_{\mathcal{O}_{\mathcal{W}}}(g\circ e)^*(\mathcal{L}_{T^*\boldsymbol{\mathcal{Z}}})^*.$$

Hence, if $\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}$ are oriented d-orbifolds with corners, then $\boldsymbol{\mathcal{W}}$ has a natural orientation.

Here is the analogue of Propositions 4.52 and 7.53, with the same proof.

Proposition 12.41. *Suppose $\boldsymbol{\mathcal{V}},\ldots,\boldsymbol{\mathcal{Z}}$ are oriented d-orbifolds with corners, $\boldsymbol{e},\ldots,\boldsymbol{h}$ are* 1*-morphisms, and all fibre products below are bd-transverse. Then the following hold, in oriented d-orbifolds with corners:*

(a) *For $\boldsymbol{g}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Z}}$ and $\boldsymbol{h}:\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{Z}}$ we have*

$$\boldsymbol{\mathcal{X}}\times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}}\boldsymbol{\mathcal{Y}}\simeq(-1)^{(\operatorname{vdim}\boldsymbol{\mathcal{X}}-\operatorname{vdim}\boldsymbol{\mathcal{Z}})(\operatorname{vdim}\boldsymbol{\mathcal{Y}}-\operatorname{vdim}\boldsymbol{\mathcal{Z}})}\boldsymbol{\mathcal{Y}}\times_{\boldsymbol{h},\boldsymbol{\mathcal{Z}},\boldsymbol{g}}\boldsymbol{\mathcal{X}}.$$

In particular, when $\mathcal{Z} = *$ *so that* $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} = \mathcal{X} \times \mathcal{Y}$ *we have*

$$\mathcal{X} \times \mathcal{Y} \simeq (-1)^{\mathrm{vdim}\,\mathcal{X}\,\mathrm{vdim}\,\mathcal{Y}} \mathcal{Y} \times \mathcal{X}.$$

(b) *For* $\boldsymbol{e} : \mathcal{V} \to \mathcal{Y}$, $\boldsymbol{f} : \mathcal{W} \to \mathcal{Y}$, $\boldsymbol{g} : \mathcal{W} \to \mathcal{Z}$, *and* $\boldsymbol{h} : \mathcal{X} \to \mathcal{Z}$ *we have*

$$\mathcal{V} \times_{\boldsymbol{e},\mathcal{Y},\boldsymbol{f}\circ\boldsymbol{\pi}_{\mathcal{W}}} \big(\mathcal{W} \times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}} \mathcal{X}\big) \simeq \big(\mathcal{V} \times_{\boldsymbol{e},\mathcal{Y},\boldsymbol{f}} \mathcal{W}\big) \times_{\boldsymbol{g}\circ\boldsymbol{\pi}_{\mathcal{W}},\mathcal{Z},\boldsymbol{h}} \mathcal{X}.$$

(c) *For* $\boldsymbol{e} : \mathcal{V} \to \mathcal{Y}$, $\boldsymbol{f} : \mathcal{V} \to \mathcal{Z}$, $\boldsymbol{g} : \mathcal{W} \to \mathcal{Y}$, *and* $\boldsymbol{h} : \mathcal{X} \to \mathcal{Z}$ *we have*

$$\begin{aligned}&\mathcal{V} \times_{(\boldsymbol{e},\boldsymbol{f}),\mathcal{Y}\times\mathcal{Z},\boldsymbol{g}\times\boldsymbol{h}} (\mathcal{W} \times \mathcal{X}) \simeq \\ &\quad (-1)^{\mathrm{vdim}\,\mathcal{Z}(\mathrm{vdim}\,\mathcal{Y}+\mathrm{vdim}\,\mathcal{W})} (\mathcal{V} \times_{\boldsymbol{e},\mathcal{Y},\boldsymbol{g}} \mathcal{W}) \times_{\boldsymbol{f}\circ\boldsymbol{\pi}_{\mathcal{V}},\mathcal{Z},\boldsymbol{h}} \mathcal{X}.\end{aligned}$$

Here are the analogues of Theorems 7.54 and 7.56, proved in the same way.

Theorem 12.42. *Let* $\mathcal{X}$ *be a d-orbifold with corners. Then* $\partial\mathcal{X}$ *is also a d-orbifold with corners, so we have orientation line bundles* $\mathcal{L}_{T^*\mathcal{X}}$ *on* $\mathcal{X}$ *and* $\mathcal{L}_{T^*(\partial\mathcal{X})}$ *on* $\partial\mathcal{X}$. *With a suitable choice of orientation convention, there is a canonical isomorphism*

$$\Psi : \mathcal{L}_{T^*(\partial\mathcal{X})} \longrightarrow i_{\mathcal{X}}^*(\mathcal{L}_{T^*\mathcal{X}}) \otimes \mathcal{N}_{\mathcal{X}}^* \tag{12.7}$$

of line bundles on $\partial\mathcal{X}$, *where* $\mathcal{N}_{\mathcal{X}}$ *is the conormal bundle of* $\boldsymbol{\partial\mathcal{X}}$ *in* $\boldsymbol{\mathcal{X}}$ *from Definition* 11.1, *and* $\mathcal{N}_{\mathcal{X}}^*$ *its dual line bundle.*

Now $\mathcal{N}_{\mathcal{X}}$ *comes with an orientation* $\omega_{\mathcal{X}}$ *in* $\mathcal{X} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i}_{\mathcal{X}}, \omega_{\mathcal{X}})$. *Hence, if* $\mathcal{X}$ *is an oriented d-orbifold with corners, then* $\partial\mathcal{X}$ *also has a natural orientation, by combining the orientations on* $\mathcal{L}_{T^*\mathcal{X}}$ *and* $\mathcal{N}_{\mathcal{X}}^*$ *to get an orientation on* $\mathcal{L}_{T^*(\partial\mathcal{X})}$ *using* (12.7).

Theorem 12.43. *Let* $\boldsymbol{g} : \mathcal{X} \to \mathcal{Z}$ *and* $\boldsymbol{h} : \mathcal{Y} \to \mathcal{Z}$ *be* 1*-morphisms of oriented d-orbifolds with corners. Then the following hold in oriented d-orbifolds with corners, where by Theorem* 12.32 *all fibre products in* (12.8)–(12.14) *are cd-transverse, and so exist, and the orientations on cd-transverse fibre products and boundaries are determined by Theorems* 12.40 *and* 12.42:

(a) *If* $\mathcal{Z}$ *is an orbifold without boundary then there is an equivalence*

$$\begin{aligned}\partial\big(\mathcal{X} \times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}} \mathcal{Y}\big) \simeq{}& \big(\partial\mathcal{X} \times_{\boldsymbol{g}\circ\boldsymbol{i}_{\mathcal{X}},\mathcal{Z},\boldsymbol{h}} \mathcal{Y}\big) \\ &\amalg (-1)^{\mathrm{vdim}\,\mathcal{X}+\dim\mathcal{Z}} \big(\mathcal{X} \times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\mathcal{Y}}} \partial\mathcal{Y}\big).\end{aligned} \tag{12.8}$$

(b) *If* $\boldsymbol{g}$ *is a w-submersion then there is an equivalence*

$$\begin{aligned}\partial\big(\mathcal{X} \times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}} \mathcal{Y}\big) \simeq{}& \big(\partial_+^{\boldsymbol{g}}\mathcal{X} \times_{\boldsymbol{g}_+,\mathcal{Z},\boldsymbol{h}} \mathcal{Y}\big) \\ &\amalg (-1)^{\mathrm{vdim}\,\mathcal{X}+\mathrm{vdim}\,\mathcal{Z}} \big(\mathcal{X} \times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\mathcal{Y}}} \partial\mathcal{Y}\big).\end{aligned} \tag{12.9}$$

(c) *If* $\boldsymbol{g}$ *is an sw-submersion then there is an equivalence*

$$\partial\big(\mathcal{X} \times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}} \mathcal{Y}\big) \simeq (-1)^{\mathrm{vdim}\,\mathcal{X}+\mathrm{vdim}\,\mathcal{Z}} \mathcal{X} \times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\mathcal{Y}}} \partial\mathcal{Y}. \tag{12.10}$$

(d) *If $\boldsymbol{h}$ is a w-submersion then there is an equivalence*

$$\begin{aligned}\partial(\mathcal{X}\times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}}\mathcal{Y})\simeq{}&(\partial\mathcal{X}\times_{\boldsymbol{g}\circ\boldsymbol{i}_{\mathcal{X}},\mathcal{Z},\boldsymbol{h}}\mathcal{Y})\\&\amalg(-1)^{\mathrm{vdim}\,\mathcal{X}+\mathrm{vdim}\,\mathcal{Z}}(\mathcal{X}\times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}_+}\partial_+^{\boldsymbol{h}}\mathcal{Y}).\end{aligned}\tag{12.11}$$

(e) *If $\boldsymbol{h}$ is an sw-submersion then there is an equivalence*

$$\partial(\mathcal{X}\times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}}\mathcal{Y})\simeq\partial\mathcal{X}\times_{\boldsymbol{g}\circ\boldsymbol{i}_{\mathcal{X}},\mathcal{Z},\boldsymbol{h}}\mathcal{Y}.\tag{12.12}$$

(f) *If both $\boldsymbol{g}$ and $\boldsymbol{h}$ are w-submersions then there is an equivalence*

$$\begin{aligned}\partial(\mathcal{X}\times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}}\mathcal{Y})\simeq{}&(\partial_+^{\boldsymbol{g}}\mathcal{X}\times_{\boldsymbol{g}_+,\mathcal{Z},\boldsymbol{h}}\mathcal{Y})\\\amalg(-1)^{\mathrm{vdim}\,\mathcal{X}+\mathrm{vdim}\,\mathcal{Z}}&(\mathcal{X}\times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}_+}\partial_+^{\boldsymbol{h}}\mathcal{Y})\amalg(\partial_-^{\boldsymbol{g}}\mathcal{X}\times_{\boldsymbol{g}_-,\partial\mathcal{Z},\boldsymbol{h}_-}\partial_-^{\boldsymbol{h}}\mathcal{Y}).\end{aligned}\tag{12.13}$$

(g) *If both $\boldsymbol{g}$ and $\boldsymbol{h}$ are sw-submersions then there is an equivalence*

$$\partial(\mathcal{X}\times_{\boldsymbol{g},\mathcal{Z},\boldsymbol{h}}\mathcal{Y})\simeq\partial\mathcal{X}\times_{\boldsymbol{g}_-,\partial\mathcal{Z},\boldsymbol{h}_-}\partial\mathcal{Y}.\tag{12.14}$$

12.8 Orbifold strata of d-orbifolds with corners

We studied orbifold strata of orbifolds in §8.4, of orbifolds with corners in §8.9, of d-stacks in §9.6, of d-orbifolds in §10.7, and of d-stacks with corners in §11.7. We now discuss orbifold strata of d-orbifolds with corners. We will see that when $\mathcal{X}$ is a d-orbifold with corners and Γ a finite group, then the orbifold strata $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\ldots,\hat{\mathcal{X}}^\Gamma_\circ$ are objects in $\mathbf{d\check{O}rb^c}$, and decompose naturally as $\mathcal{X}^\Gamma=\coprod_{\lambda\in\Lambda^\Gamma}\mathcal{X}^{\Gamma,\lambda}$ into d-orbifolds with corners $\mathcal{X}^{\Gamma,\lambda},\tilde{\mathcal{X}}^{\Gamma,\mu},\ldots,\hat{\mathcal{X}}^{\Gamma,\mu}_\circ$. Here is the analogue of Definitions 8.5 and 10.38.

Definition 12.44. Let Γ be a finite group, and use the notation $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$, $\Lambda^\Gamma=K_0\bigl(\mathrm{Rep}_{\mathrm{nt}}(\Gamma)\bigr)$, $\Lambda^\Gamma_+\subseteq\Lambda^\Gamma$ and $\dim:\Lambda^\Gamma\to\mathbb{Z}$ of Definition 8.5. Let $R_0,R_1,\ldots,R_k$ be representatives for the isomorphism classes of irreducible Γ-representations, with $R_0=\mathbb{R}$ the trivial irreducible representation, so that $R_1,\ldots,R_k$ are nontrivial. Then Λ^Γ is freely generated over $\mathbb{Z}$ by $[R_1],\ldots,[R_k]$, so that (8.1) gives isomorphisms $\Lambda^\Gamma\cong\mathbb{Z}^k$, $\Lambda^\Gamma_+\cong\mathbb{N}^k$.

Let $\mathcal{X}$ be a d-orbifold with corners. Then Definition 11.28 defines a d-stack with corners $\mathcal{X}^\Gamma$ and a 1-morphism $\boldsymbol{O}^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\to\mathcal{X}$. As in Definition 10.38, $O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})$ has a natural Γ-action, and splits as $O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})\cong\bigoplus_{i=0}^k(T^*\boldsymbol{\mathcal{X}})_i^\Gamma\otimes R_i$, where $(T^*\boldsymbol{\mathcal{X}})_i^\Gamma$ are virtual vector bundles of mixed rank on $\mathcal{X}^\Gamma$. For each $\lambda\in\Lambda^\Gamma$, define $\mathcal{X}^{\Gamma,\lambda}$ to be the open and closed d-substack in $\mathcal{X}^\Gamma$ with $\mathrm{rank}\bigl((T^*\boldsymbol{\mathcal{X}})_1^\Gamma\bigr)[R_1]+\cdots+\mathrm{rank}\bigl((T^*\boldsymbol{\mathcal{X}})_k^\Gamma\bigr)[R_k]=\lambda$ in Λ^Γ. Then $\mathcal{X}^\Gamma=\coprod_{\lambda\in\Lambda^\Gamma}\mathcal{X}^{\Gamma,\lambda}$, as for (10.14). We will see in Proposition 12.45 that $\mathcal{X}^{\Gamma,\lambda}$ is a d-orbifold with corners, with $\mathrm{vdim}\,\mathcal{X}^{\Gamma,\lambda}=\mathrm{vdim}\,\mathcal{X}-\dim\lambda$.

Write $\boldsymbol{O}^{\Gamma,\lambda}(\mathcal{X})=\boldsymbol{O}^\Gamma(\mathcal{X})|_{\mathcal{X}^{\Gamma,\lambda}}:\mathcal{X}^{\Gamma,\lambda}\to\mathcal{X}$. Then $\boldsymbol{O}^{\Gamma,\lambda}(\mathcal{X})$ is a proper w-immersion of d-orbifolds with corners, in the sense of §12.4. Define $\mathcal{X}^{\Gamma,\lambda}_\circ=$

$\mathcal{X}_\circ^\Gamma \cap \mathcal{X}^{\Gamma,\lambda}$, and $\boldsymbol{O}_\circ^{\Gamma,\lambda}(\mathcal{X}) = \boldsymbol{O}_\circ^\Gamma(\mathcal{X})|_{\mathcal{X}_\circ^{\Gamma,\lambda}} : \mathcal{X}_\circ^{\Gamma,\lambda} \to \mathcal{X}$. Then $\boldsymbol{\mathcal{X}}_\circ^{\Gamma,\lambda}$ is a d-orbifold with corners with $\operatorname{vdim} \mathcal{X}_\circ^{\Gamma,\lambda} = \operatorname{vdim} \mathcal{X} - \dim \lambda$, and $\mathcal{X}_\circ^\Gamma = \coprod_{\lambda \in \Lambda^\Gamma} \mathcal{X}_\circ^{\Gamma,\lambda}$, and $\boldsymbol{O}_\circ^{\Gamma,\lambda}(\mathcal{X})$ is a w-immersion, but need not be proper.

As for $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$ in Definition 10.38, for each $\mu \in \Lambda^\Gamma / \operatorname{Aut}(\Gamma)$ we may define $\tilde{\mathcal{X}}^{\Gamma,\mu} = \bigl[(\coprod_{\lambda \in \mu} \mathcal{X}^{\Gamma,\lambda}) / \operatorname{Aut}(\Gamma)\bigr]$, an open and closed d-substack of $\tilde{\mathcal{X}}^\Gamma = [\mathcal{X}^\Gamma / \operatorname{Aut}(\Gamma)]$, and $\tilde{\mathcal{X}}_\circ^{\Gamma,\mu} = \tilde{\mathcal{X}}_\circ^\Gamma \cap \tilde{\mathcal{X}}^{\Gamma,\mu}$. Then $\tilde{\mathcal{X}}^{\Gamma,\mu}, \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}$ are d-orbifolds with corners with $\operatorname{vdim} \tilde{\mathcal{X}}^{\Gamma,\mu} = \operatorname{vdim} \tilde{\mathcal{X}}_\circ^{\Gamma,\mu} = \operatorname{vdim} \mathcal{X} - \dim \mu$, with

$$\tilde{\mathcal{X}}^\Gamma = \textstyle\coprod_{\mu \in \Lambda^\Gamma / \operatorname{Aut}(\Gamma)} \tilde{\mathcal{X}}^{\Gamma,\mu} \quad \text{and} \quad \tilde{\mathcal{X}}_\circ^\Gamma = \textstyle\coprod_{\mu \in \Lambda^\Gamma / \operatorname{Aut}(\Gamma)} \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}.$$

Set $\tilde{\boldsymbol{O}}^{\Gamma,\mu}(\mathcal{X}) = \tilde{\boldsymbol{O}}^\Gamma(\mathcal{X})|_{\tilde{\mathcal{X}}^{\Gamma,\mu}} : \tilde{\mathcal{X}}^{\Gamma,\mu} \to \mathcal{X}$, $\tilde{\boldsymbol{O}}_\circ^{\Gamma,\mu}(\mathcal{X}) = \tilde{\boldsymbol{O}}_\circ^\Gamma(\mathcal{X})|_{\tilde{\mathcal{X}}_\circ^{\Gamma,\mu}} : \tilde{\mathcal{X}}_\circ^{\Gamma,\mu} \to \mathcal{X}$. Then $\tilde{\boldsymbol{O}}^{\Gamma,\mu}(\mathcal{X}), \tilde{\boldsymbol{O}}_\circ^{\Gamma,\mu}(\mathcal{X})$ are w-immersions, with $\tilde{\boldsymbol{O}}^{\Gamma,\mu}(\mathcal{X})$ proper.

The 1-morphism $\hat{\boldsymbol{\Pi}}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \to \hat{\mathcal{X}}^\Gamma$ induces a homeomorphism of topological spaces by Theorem C.49(e), so it maps open and closed d-substacks of $\tilde{\mathcal{X}}^\Gamma$ to open and closed d-substacks of $\hat{\mathcal{X}}^\Gamma$. Let $\hat{\mathcal{X}}^{\Gamma,\mu} = \hat{\boldsymbol{\Pi}}^\Gamma(\mathcal{X})(\tilde{\mathcal{X}}^{\Gamma,\mu})$ for each $\mu \in \Lambda^\Gamma / \operatorname{Aut}(\Lambda)$, and write $\hat{\mathcal{X}}_\circ^{\Gamma,\mu} = \hat{\mathcal{X}}_\circ^\Gamma \cap \hat{\mathcal{X}}^{\Gamma,\mu}$. Then $\hat{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ are d-orbifolds with corners of virtual dimension $\operatorname{vdim} \boldsymbol{\mathcal{X}} - \dim \mu$, with

$$\hat{\mathcal{X}}^\Gamma = \textstyle\coprod_{\mu \in \Lambda^\Gamma / \operatorname{Aut}(\Gamma)} \hat{\mathcal{X}}^{\Gamma,\mu} \quad \text{and} \quad \hat{\mathcal{X}}_\circ^\Gamma = \textstyle\coprod_{\mu \in \Lambda^\Gamma / \operatorname{Aut}(\Gamma)} \hat{\mathcal{X}}_\circ^{\Gamma,\mu}.$$

Also $\hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ is a d-manifold, that is, it lies in $\mathbf{d\hat{M}an^c}$.

We can generalize Example 10.39 to the corners case, to describe the orbifold strata $(\mathcal{S}_{\mathcal{V},\mathcal{E},s})^\Gamma$ of the 'standard model' d-orbifold with corners $\mathcal{S}_{\mathcal{V},\mathcal{E},s}$ of Definition 10.5 as a disjoint union of standard models $\mathcal{S}_{\mathcal{V}^{\Gamma,\lambda_1,\lambda_2},\mathcal{E}^{\Gamma,\lambda_1,\lambda_2},s^{\Gamma,\lambda_1,\lambda_2}}$, and similarly for $(\widetilde{\mathcal{S}_{\mathcal{V},\mathcal{E},s}})^\Gamma, \ldots, (\widehat{\mathcal{S}_{\mathcal{V},\mathcal{E},s}})_\circ^\Gamma$. Thus as for Corollary 10.40 we deduce:

Proposition 12.45. *In Definition* 12.44, $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}_\circ^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}, \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ *are d-orbifolds with corners of virtual dimensions* $\operatorname{vdim} \mathcal{X} - \dim \lambda$, $\operatorname{vdim} \mathcal{X} - \dim \mu$. *Hence* $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$ *are objects in* $\mathbf{d\check{O}rb^c}$.

Applying Corollaries 11.32 and 11.34 to a d-orbifold with corners $\mathcal{X}$, by considering local models we see that $\boldsymbol{J}^\Gamma(\mathcal{X}) : (\partial \mathcal{X})^\Gamma \to \partial(\mathcal{X}^\Gamma)$ maps $(\partial \mathcal{X})^{\Gamma,\lambda} \to \partial(\mathcal{X}^{\Gamma,\lambda})$ for each $\lambda \in \Lambda^\Gamma$, and similarly for $\tilde{\boldsymbol{J}}^\Gamma(\mathcal{X}), \hat{\boldsymbol{J}}^\Gamma(\mathcal{X})$. Thus as for Corollaries 8.34 and 8.36, we deduce:

Proposition 12.46. *Let* $\mathcal{X}$ *be a d-orbifold with corners, and* Γ *a finite group. Then Corollary* 11.32 *gives* 1*-morphisms* $\boldsymbol{J}^\Gamma(\mathcal{X}) : (\partial \mathcal{X})^\Gamma \to \partial(\mathcal{X}^\Gamma)$, $\tilde{\boldsymbol{J}}^\Gamma(\mathcal{X}) : (\widetilde{\partial \mathcal{X}})^\Gamma \to \partial(\tilde{\mathcal{X}}^\Gamma)$, $\hat{\boldsymbol{J}}^\Gamma(\mathcal{X}) : (\widehat{\partial \mathcal{X}})^\Gamma \to \partial(\hat{\mathcal{X}}^\Gamma)$ *in* $\mathbf{d\check{O}rb^c}$, *which are equivalences with open and closed subobjects in* $\partial(\mathcal{X}^\Gamma), \partial(\tilde{\mathcal{X}}^\Gamma), \partial(\hat{\mathcal{X}}^\Gamma)$.

These restrict to 1*-morphisms* $\boldsymbol{J}^{\Gamma,\lambda}(\mathcal{X}) : (\partial \mathcal{X})^{\Gamma,\lambda} \to \partial(\mathcal{X}^{\Gamma,\lambda})$ *in* $\mathbf{dOrb^c}$ *for* $\lambda \in \Lambda^\Gamma$ *and* $\tilde{\boldsymbol{J}}^{\Gamma,\mu}(\mathcal{X}) : (\widetilde{\partial \mathcal{X}})^{\Gamma,\mu} \to \partial(\tilde{\mathcal{X}}^{\Gamma,\mu})$, $\hat{\boldsymbol{J}}^{\Gamma,\mu}(\mathcal{X}) : (\widehat{\partial \mathcal{X}})^{\Gamma,\mu} \to \partial(\hat{\mathcal{X}}^{\Gamma,\mu})$ *for* $\mu \in \Lambda^\Gamma / \operatorname{Aut}(\Lambda)$, *which are equivalences with open and closed d-suborbifolds. Hence, if* $\mathcal{X}^{\Gamma,\lambda} = \emptyset$ *then* $(\partial \mathcal{X})^{\Gamma,\lambda} = \emptyset$, *and similarly for* $\tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}$.

Now suppose $\mathcal{X}$ *is straight, in the sense of Definition* 11.33, *for instance* $\mathcal{X}$ *could be a d-orbifold with boundary. Then* $\boldsymbol{J}^\Gamma(\mathcal{X}), \ldots, \hat{\boldsymbol{J}}^{\Gamma,\mu}(\mathcal{X})$ *are equivalences, so that* $(\partial \mathcal{X})^\Gamma \simeq \partial(\mathcal{X}^\Gamma)$, $(\partial \mathcal{X})^{\Gamma,\lambda} \simeq \partial(\mathcal{X}^{\Gamma,\lambda})$, *and so on.*

Lemma 10.41 and Propositions 10.43 and 10.44 on orbifold strata of d-orbifolds and orientations extend to d-orbifolds with corners without change.

12.9 Kuranishi neighbourhoods, good coordinate systems

In §10.8 we defined type A and type B Kuranishi neighbourhoods, coordinate changes, and good coordinate systems, on d-orbifolds. We now generalize these to d-orbifolds with corners. This will be important in Chapters 13 and 14.

The definitions in the corners case are obtained by replacing $\mathbf{Man}, \mathbf{Orb}, \mathbf{dMan}, \mathbf{dOrb}$ by $\mathbf{Man^c}, \mathbf{Orb^c}, \mathbf{dMan^c}, \mathbf{dOrb^c}$ throughout, and making a few other easy changes such as taking the e_{ij} to be sf-embeddings in Definitions 10.46(c) and 10.51(b). For brevity we will not write the definitions out again, but just indicate the differences. We begin with the 'type A' material of §10.8.1.

Definition 12.47. Let $\mathcal{X}$ be a d-orbifold with corners. Define a *type A Kuranishi neighbourhood* $(V, E, \Gamma, s, \boldsymbol{\psi})$ on $\mathcal{X}$ following Definition 10.45, but taking V to be a manifold with corners, and using Example 12.16 to define the principal d-orbifold with corners $[\mathbf{S}_{V,E,s}/\Gamma]$, rather than Example 10.11.

If $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ are type A Kuranishi neighbourhoods on $\mathcal{X}$ with $\emptyset \neq \boldsymbol{\psi}_i([\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\mathbf{S}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \mathcal{X}$, define a *type A coordinate change* $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$ *from* $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ *to* $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ following Definition 10.46, but taking $e_{ij} : V_{ij} \to V_j$ to be an sf-embedding of manifolds with corners, and using Example 12.17 to define the quotient 1-morphism $[\mathbf{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}]$ in (10.19) rather than Example 10.12, and using Corollary 12.22 rather than Corollary 10.18 in (e).

Define a *type A good coordinate system* on $\mathcal{X}$ following Definition 10.47, but using Definition 11.7 rather than Example 10.13 to define the 2-morphism $\boldsymbol{\eta}_{ijk} = [0, \gamma_{ijk}]$ in $\mathbf{dOrb^c}$ in (10.21). Let Y be a manifold with corners, and $\boldsymbol{h} : \mathcal{X} \to \mathcal{Y}$ a 1-morphism in $\mathbf{dOrb^c}$, where $\mathcal{Y} = F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y)$. Define a *type A good coordinate system for* $\boldsymbol{h} : \mathcal{X} \to \mathcal{Y}$ following Definition 10.47, using Example 12.17 rather than Example 10.12 in (f).

Here is the analogue of Theorem 10.48. It will be proved in Appendix D.

Theorem 12.48. *Suppose $\mathcal{X}$ is a d-orbifold with corners. Then there exists a type A good coordinate system $\bigl(I,<,(V_i,E_i,\Gamma_i,s_i,\boldsymbol{\psi}_i),(V_{ij},e_{ij},\hat{e}_{ij},\rho_{ij},\boldsymbol{\eta}_{ij}),\gamma_{ijk}\bigr)$ for $\mathcal{X}$, in the sense of Definition 12.47. If $\mathcal{X}$ is compact, we may take I to be finite. If $\{\mathcal{U}_j : j \in J\}$ is an open cover of $\mathcal{X}$, we may take $\mathcal{X}_i = \boldsymbol{\psi}_i([\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i]) \subseteq \mathcal{U}_{j_i}$ for each $i \in I$ and some $j_i \in J$. If $\mathcal{X}$ is a d-orbifold with boundary, we may take the V_i to be manifolds with boundary.*

Now let Y be a manifold with corners and $\boldsymbol{h} : \mathcal{X} \to \mathcal{Y} = F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y)$ a semisimple, flat 1-morphism in $\mathbf{dOrb^c}$. Then all the above extends to type A good coordinate systems for $\boldsymbol{h} : \mathcal{X} \to \mathcal{Y}$, and we may take the $g_i : V_i \to Y$ to be submersions in $\mathbf{Man^c}$.

Note that we make the extra assumption that $\boldsymbol{h}$ is semisimple and flat in the last part. This happens automatically if Y is without boundary. Since submersions in $\mathbf{Man^c}$ are automatically semisimple and flat, $\boldsymbol{h}$ being semisimple and flat is a necessary condition for the $g_i : V_i \to Y$ to be submersions.

Next we extend the 'type B' material of §10.8.2.

Definition 12.49. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold with corners. Define a *type B Kuranishi neighbourhood* $(\mathcal{V}, \mathcal{E}, s, \boldsymbol{\psi})$ on $\boldsymbol{\mathcal{X}}$ following Definition 10.50, but taking $\mathcal{V}$ to be an effective orbifold with corners, and using Definition 12.2 to define the principal d-orbifold with corners $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$, rather than Definition 10.5.

If $(\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i), (\mathcal{V}_j, \mathcal{E}_j, s_j, \boldsymbol{\psi}_j)$ are type B Kuranishi neighbourhoods on $\boldsymbol{\mathcal{X}}$ with $\emptyset \neq \boldsymbol{\psi}_i(\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}) \cap \boldsymbol{\psi}_j(\boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j}) \subseteq \boldsymbol{\mathcal{X}}$, define a *type B coordinate change* $(\mathcal{V}_{ij}, e_{ij}, \hat{e}_{ij}, \boldsymbol{\eta}_{ij})$ *from* $(\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i)$ *to* $(\mathcal{V}_j, \mathcal{E}_j, s_j, \boldsymbol{\psi}_j)$ following Definition 10.51, but taking $e_{ij} : \mathcal{V}_{ij} \to \mathcal{V}_j$ to be an sf-embedding of orbifolds with corners, in the sense of Definition 8.28, and using Definition 12.13 to define the 1-morphism $\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}$ in (10.22) rather than Definition 10.9, and using Corollary 12.22 rather than Corollary 10.18 in (e).

Define a *type B good coordinate system* on $\boldsymbol{\mathcal{X}}$ following Definition 10.52, but using Proposition 12.15 rather than Proposition 10.10 to define the 2-morphism $\boldsymbol{\eta}_{ijk}$ in $\mathbf{dOrb^c}$ in (d). Let $\mathcal{Y}$ be an effective orbifold with corners, and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ a 1-morphism in $\mathbf{dOrb^c}$, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}(\mathcal{Y})$. Define a *type B good coordinate system for* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ following Definition 10.52.

As in Definition 10.55, a type B good coordinate system $\bigl(I, <, (\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i), \ldots, \eta_{ijk}\bigr)$ on $\boldsymbol{\mathcal{X}}$ or for $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is called a *very good coordinate system* if $I \subset \mathbb{N} = \{0, 1, 2, \ldots\}$, and the order $<$ on I is the restriction of $<$ on $\mathbb{N}$, and $\dim \mathcal{V}^i = i$ for all $i \in I$.

Here is the analogue of Theorems 10.54 and 10.56. It follows from Theorem 12.48 as Theorems 10.54 and 10.56 follow from Theorem 10.48.

Theorem 12.50. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold with corners. Then there exists a type B good coordinate system $\bigl(I, <, (\mathcal{V}_i, \mathcal{E}_i, s_i, \boldsymbol{\psi}_i), (\mathcal{V}_{ij}, e_{ij}, \hat{e}_{ij}, \boldsymbol{\eta}_{ij}), \eta_{ijk}\bigr)$ for $\boldsymbol{\mathcal{X}}$. If $\boldsymbol{\mathcal{X}}$ is compact, we may take I to be finite. If $\{\boldsymbol{\mathcal{U}}_j : j \in J\}$ is an open cover of $\boldsymbol{\mathcal{X}}$, we may take $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i(\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}) \subseteq \boldsymbol{\mathcal{U}}_{j_i}$ for each $i \in I$ and some $j_i \in J$. If $\boldsymbol{\mathcal{X}}$ is a d-orbifold with boundary, we may take the $\mathcal{V}_i$ to be orbifolds with boundary.*

Now let $\mathcal{Y}$ be an effective orbifold with corners and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}(\mathcal{Y})$ a semisimple, flat 1-morphism in $\mathbf{dOrb^c}$. Then all the above extends to type B good coordinate systems for $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, and we may take the $g_i : \mathcal{V}_i \to \mathcal{Y}$ to be submersions in $\mathbf{Orb^c}$.

We may also take $\bigl(I, <, \ldots, \eta_{ijk}\bigr)$ to be a very good coordinate system in both cases (though not requiring $\boldsymbol{\mathcal{X}}_i \subseteq \boldsymbol{\mathcal{U}}_{j_i}$ as above), with I finite if $\boldsymbol{\mathcal{X}}$ is compact.

12.10 Semieffective and effective d-orbifolds with corners

In §10.9 we discussed effective and semieffective d-orbifolds. All this material extends to d-orbifolds with corners essentially unchanged, so we will not write it out again. We define *semieffective* and *effective* d-orbifolds with corners $\boldsymbol{\mathcal{X}}$ following Definition 10.57. The analogues of Propositions 10.58, 10.64 and 10.65

and Lemmas 10.60–10.63 then hold, with (d-)orbifolds replaced by (d-)orbifolds with corners throughout. Here is a new result in the corners case.

Proposition 12.51. *Let $\boldsymbol{\mathcal{X}}$ be a effective (or semieffective) d-orbifold with corners. Then $\partial^k\boldsymbol{\mathcal{X}}$ is also effective (or semieffective), for all $k \geqslant 0$.*

Proof. It is enough to prove $\partial\boldsymbol{\mathcal{X}}$ is (semi)effective. Let $[x'] \in \partial\mathcal{X}_{\rm top}$ with $i_{\boldsymbol{\mathcal{X}}}([x']) = [x] \in \mathcal{X}_{\rm top}$, so that $x = i_{\boldsymbol{\mathcal{X}}} \circ x' : \bar{\underline{*}} \to \partial\mathcal{X}$. Consider the commutative diagram in qcoh$(\bar{\underline{*}})$, where the (exact) rows are (10.24) for $[x], \boldsymbol{\mathcal{X}}$ and $[x'], \partial\boldsymbol{\mathcal{X}}$:

$$\begin{tikzcd}
0 \arrow[r] & K_{[x]} \arrow[r] \arrow[d, dotted, "\alpha"] & x^*(\mathcal{E}_{\mathcal{X}}) \arrow[rr, "x^*(\phi_{\mathcal{X}})"'] \arrow[d, "x^*(i''_{\boldsymbol{\mathcal{X}}})\circ I_{x,i_{\boldsymbol{\mathcal{X}}}}(\mathcal{E}_{\mathcal{X}})", "\cong"'] && x^*(\mathcal{F}_{\mathcal{X}}) \arrow[rr, "x^*(\psi_{\mathcal{X}})"'] \arrow[d, "x^*(i^2_{\boldsymbol{\mathcal{X}}})\circ I_{x,i_{\boldsymbol{\mathcal{X}}}}(\mathcal{F}_{\mathcal{X}})"] && T^*_x\mathcal{X} \arrow[r] \arrow[d, "\substack{x^*(\Omega_{i_{\boldsymbol{\mathcal{X}}}})\circ \\ I_{x,i_{\boldsymbol{\mathcal{X}}}}(T^*\mathcal{X})}"] & 0 \\
0 \arrow[r] & K_{[x']} \arrow[r] & (x')^*(\mathcal{E}_{\partial\mathcal{X}}) \arrow[rr, "(x')^*(\phi_{\partial\mathcal{X}})"] && (x')^*(\mathcal{F}_{\partial\mathcal{X}}) \arrow[rr, "(x')^*(\psi_{\partial\mathcal{X}})"] && T^*_{x'}(\partial\mathcal{X}) \arrow[r] & 0.
\end{tikzcd} \tag{12.15}$$

Exactness implies that there is a unique morphism $\alpha : K_{[x']} \to K_{[x]}$ making the diagram commute. The second column is an isomorphism by Definition 11.1(b), and as (11.1) is split exact, the third column is surjective with kernel $\mathbb{R}$. Therefore there are two possibilities:

(a) $K_{[x']} \cong K_{[x]}$ and $T^*_x\mathcal{X} \cong T^*_{x'}(\partial\mathcal{X}) \oplus \mathbb{R}$; or

(b) $K_{[x']} \cong K_{[x]} \oplus \mathbb{R}$ and $T^*_x\mathcal{X} \cong T^*_{x'}(\partial\mathcal{X})$.

As in Definition 10.57, the top line of (12.15) is in $\mathrm{Iso}_{\mathcal{X}}([x])$-representations, with $K_{[x]}$ a trivial representation as $\boldsymbol{\mathcal{X}}$ is semieffective, and $T^*_x\mathcal{X}$ effective if $\boldsymbol{\mathcal{X}}$ is effective. Also the bottom line of (12.15) is in $\mathrm{Iso}_{\partial\mathcal{X}}([x'])$-representations. The morphism $(i_{\boldsymbol{\mathcal{X}}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ is injective, as $i_{\boldsymbol{\mathcal{X}}}$ is representable, so we can use it to make (12.15) into a commutative diagram of representations of $\mathrm{Iso}_{\partial\mathcal{X}}([x'])$, where again $K_{[x]}$ is trivial, and $T^*_x\mathcal{X}$ is an effective $\mathrm{Iso}_{\partial\mathcal{X}}([x'])$-representation if $\boldsymbol{\mathcal{X}}$ is effective.

Parts (a) or (b) now hold in $\mathrm{Iso}_{\partial\mathcal{X}}([x'])$-representations, with $\mathbb{R}$ the trivial representation. So $K_{[x']}$ is a trivial representation, and $T^*_{x'}(\partial\mathcal{X})$ is effective if $\boldsymbol{\mathcal{X}}$ is effective. The proposition follows from Definition 10.57. $\square$

However, $\boldsymbol{\mathcal{X}}$ (semi)effective does not imply $C_k(\boldsymbol{\mathcal{X}})$ (semi)effective, as for orbifolds with corners in Definition 8.37.

13 Bordism for d-manifolds and d-orbifolds

As a sample application of the results of Chapters 2–12, we now study *bordism groups* for d-manifolds and d-orbifolds. In the d-manifold case, for a fixed manifold Y, the *d-manifold bordism group* $dB_k(Y)$ whose elements are bordism classes $[\boldsymbol{X}, \boldsymbol{f}]$ of pairs $(\boldsymbol{X}, \boldsymbol{f})$, where $\boldsymbol{X}$ is a compact, oriented d-manifold, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$ a 1-morphism, and pairs $(\boldsymbol{X}, \boldsymbol{f})$, $(\boldsymbol{X}', \boldsymbol{f}')$ are bordant if there exists a compact, oriented d-manifold with boundary $\mathbf{W}$ and a 1-morphism $\boldsymbol{e} : \mathbf{W} \to \boldsymbol{Y}$ with $\partial\mathbf{W} \simeq -\boldsymbol{X} \amalg \boldsymbol{X}'$ and $\boldsymbol{e} \circ \boldsymbol{i}_{\mathbf{W}} \cong \boldsymbol{f} \amalg \boldsymbol{f}'$.

Our main result is that the natural projection $\Pi_{\rm bo}^{\rm dbo} : B_k(Y) \to dB_k(Y)$ from classical bordism $B_*(Y)$ is an isomorphism. This holds as every d-manifold can be perturbed to a manifold. One consequence is that *compact oriented d-manifolds have virtual classes*, that is, there is a natural *virtual class map* $\Pi_{\rm dbo}^{\rm hom} : dB_k(Y) \to H_k(Y;\mathbb{Z})$. So d-manifolds and d-orbifolds are suitable for use in enumerative invariant problems involving virtual cycles on moduli spaces.

In the d-orbifold case things are more complicated, since not every d-orbifold can be perturbed to an orbifold. For an orbifold $\mathcal{Y}$ we define d-orbifold bordism groups $dB_*^{\rm orb}(\mathcal{Y}), dB_*^{\rm sef}(\mathcal{Y}), dB_*^{\rm eff}(\mathcal{Y})$ with elements $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$ for $\boldsymbol{\mathcal{X}}$ a compact, oriented d-orbifold which is arbitrary, or semieffective, or effective, respectively. Then $dB_k^{\rm sef}(\mathcal{Y}), dB_k^{\rm eff}(\mathcal{Y})$ are isomorphic to 'classical' orbifold bordism groups $B_k^{\rm orb}(\mathcal{Y}), B_k^{\rm eff}(\mathcal{Y})$, but $dB_k^{\rm orb}(\mathcal{Y})$ is not, and generally has infinite rank even for $k < 0$. We define Gromov–Witten type invariants in $dB_*^{\rm orb}(\overline{\mathcal{M}}_{g,m} \times X^m)$ for (X, ω) a compact symplectic manifold, and we anticipate these ideas will be useful in Gromov–Witten theory, in particular for integrality questions.

13.1 Classical bordism groups for manifolds

Bordism is an invariant of topological spaces, which shares many features with homology. The subject began with the work of Thom [97]. Bordism groups were introduced by Atiyah [6], and Conner [24, §I] gives a good introduction. Other useful references are Stong [96] and Conner and Floyd [25, 26]. We will define bordism only for manifolds, and in a not quite standard way.

Definition 13.1. Let Y be a manifold, and $k \in \mathbb{Z}$. Consider pairs (X, f), where X is a compact, oriented manifold without boundary with $\dim X = k$, and $f : X \to Y$ is a smooth map. We make the convention that the empty set $\emptyset$ is an oriented manifold of any dimension $k \in \mathbb{Z}$, including $k < 0$, and the trivial map $\emptyset : \emptyset \to Y$ is smooth, so $(\emptyset, \emptyset)$ is allowed as a pair (X, f), and is the only such pair when $k < 0$.

Define a binary relation $\sim$ between such pairs by $(X, f) \sim (X', f')$ if there exists a compact, oriented $(k+1)$-manifold with boundary W, a smooth map $e : W \to Y$, and a diffeomorphism of oriented manifolds $j : -X \amalg X' \to \partial W$, such that $f \amalg f' = e \circ i_W \circ j$, where $-X$ is X with the opposite orientation, and the orientation of ∂W is induced from that of W. Proposition 13.3 shows $\sim$ is an equivalence relation, which is called *bordism*.

Write $[X, f]$ for the $\sim$-equivalence class (*bordism class*) of a pair (X, f). For each $k \in \mathbb{Z}$, define the k^{th} *bordism group* $B_k(Y)$ of Y to be the set of all such bordism classes $[X, f]$ with $\dim X = k$. We give $B_k(Y)$ the structure of an abelian group, with zero element $0_Y = [\emptyset, \emptyset]$, and addition given by $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$, and additive inverses $-[X, f] = [-X, f]$. It is easy to show that these operations are well defined and satisfy the group axioms. If $k < 0$ then $B_k(Y) = 0$, as the only element is $0_Y = [\emptyset, \emptyset]$.

Remark 13.2. Let Y be a topological space. Then Atiyah [6, §2] and Conner [24, §I.4] define *bordism groups* $MSO_k(Y)$ as in Definition 13.1, with X, W manifolds, but with $f : X \to Y$ and $e : W \to Y$ *continuous* maps of topological spaces, rather than smooth maps of manifolds. When Y is a manifold, Conner [24, §I.9] defines *differential bordism groups* $B_k(Y)$ as in Definition 13.1, and then proves [24, Th. I.9.1] that the natural projection $B_k(Y) \to MSO_k(Y)$ is an isomorphism. Hence our groups $B_k(Y)$ are isomorphic to the usual definition of bordism groups $MSO_k(Y)$.

The next proof follows Conner [24, Th. I.2.1].

Proposition 13.3. *In Definition* 13.1, $\sim$ *is an equivalence relation.*

Proof. We must show $\sim$ in is reflexive, symmetric, and transitive. If (X, f) is a pair as above then considering $W = X \times [0, 1]$ and $e = f \circ \pi_X : W \to Y$ shows that $(X, f) \sim (X, f)$, and $\sim$ is reflexive. Suppose $(X, f) \sim (X', f')$. Then there exist $e : W \to Y$ and $j : -X \amalg X' \to \partial W$ as above with $f \amalg f' = e \circ i_W \circ j$. Replacing W by $-W$ gives $(X', f') \sim (X, f)$, so $\sim$ is symmetric.

Suppose $(X, f) \sim (X', f')$ and $(X', f') \sim (X'', f'')$. Then there exist smooth $e : W \to Y$, $e' : W' \to Y$, and oriented diffeomorphisms $j : -X \amalg X' \to \partial W$, $j' : -X' \amalg X'' \to \partial W'$ with $f \amalg f' = e \circ i_W \circ j$ and $f' \amalg f'' = e' \circ i_{W'} \circ j'$. By the *Differentiable Collaring Theorem* [24, Th. I.1.2], there are open sets $U \subset W$ and $U' \subset W'$ with oriented diffeomorphisms $(-\epsilon, 0] \times X' \cong U$ and $[0, \epsilon) \times X' \cong U'$ for small $\epsilon > 0$, such that the induced oriented diffeomorphisms $X' \cong \partial U \subset \partial W$ and $-X' \cong \partial U' \subset \partial W'$ are $j|_{X'}$ and $j'|_{-X'}$.

We may therefore glue W, W' along X' to get a new compact manifold with boundary $W'' = W \amalg_{j(X') = j'(-X')} W'$, in which the subsets $(-\epsilon, 0] \times X' \cong U \subset W$ and $[0, \epsilon) \times X' \cong U' \subset W'$ are joined to give an open subset of W'' diffeomorphic to $(-\epsilon, \epsilon) \times X'$. We have an oriented diffeomorphism $j'' : -X \amalg X'' \to \partial W''$ induced by j on $-X$ and j' on X''. Choose smooth perturbations $\tilde{e} : W \to Y$, $\tilde{e}' : W' \to Y$ of e, e' such that $\tilde{e} = e$ on $W \setminus U$ and $\tilde{e}' = e'$ on $W' \setminus U'$, and on the subsets of U, U' identified with $(-\frac{1}{2}\epsilon, 0] \times X'$ and $[0, -\frac{1}{2}\epsilon) \times X'$ we have $\tilde{e}, \tilde{e}' \cong f' \circ \pi_{X'}$.

Define $e'' : W'' \to Y$ by $e''|_W = \tilde{e}$ and $e''|_{W'} = \tilde{e}'$. Then e'' is smooth. Also $e'' = \tilde{e} = e$ near $j''(-X)$ in W'', so $e'' \circ i_{W''} \circ j''|_{-X} = f$, and similarly $e'' \circ i_{W''} \circ j''|_{X''} = f''$, so $f \amalg f'' = e'' \circ i_{W''} \circ j''$. Therefore e'', W'', j'' imply that $(X, f) \sim (X'', f'')$, and $\sim$ is transitive. $\square$

We define intersection products $\bullet$, fundamental classes $[Y]$, pushforwards g_*, and projections to homology, for bordism $B_*(Y)$.

Definition 13.4. Suppose Y is an oriented manifold of dimension n. Define the *intersection product* $\bullet : B_k(Y) \times B_l(Y) \to B_{k+l-n}(Y)$ as follows. Given classes $[X, f], [X', f']$, we perturb f, f' in their bordism classes to make $f : X \to Y$ and $f' : X' \to Y$ transverse smooth maps, and then

$$[X, f] \bullet [X', f'] = [X \times_{f,Y,f'} X', f \circ \pi_X]. \tag{13.1}$$

Here the fibre product $X \times_{f,Y,f'} X'$ exists in **Man** as f, f' are transverse. The orientations on X, X', Y combine to give an orientation on $X \times_{f,Y,f'} X'$. The associativity and commutativity properties of oriented fibre products, imply that $\bullet$ is biadditive, associative and supercommutative.

If Y is compact and oriented, we define the *fundamental class* $[Y] \in B_n(Y)$ by $[Y] = [Y, \mathrm{id}_Y]$. It is the identity for $\bullet$ on $B_*(Y)$.

Let $g : Y \to Z$ be a smooth map of manifolds. Define the *pushforward* $g_* : B_k(Y) \to B_k(Z)$ by $g_* : [X, f] \mapsto [X, g \circ f]$.

Define morphisms $\Pi_{\mathrm{bo}}^{\mathrm{hom}} : B_k(Y) \to H_k(Y; \mathbb{Z})$ by $\Pi_{\mathrm{bo}}^{\mathrm{hom}} : [X, f] \mapsto f_*([X])$, where $[X] \in H_k(X; \mathbb{Z})$ is the fundamental class of X in homology.

When Y is a point $*$, $B_*(*)$ is known as the *bordism ring*. It has been completely computed. In the first major work on bordism, Thom [97] proved:

Theorem 13.5 (Thom [97])**.** *$B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the free commutative $\mathbb{Q}$-algebra generated by $\zeta_{4k} = [\mathbb{CP}^{2k}, \pi] \in B_{4k}(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ for $k \geqslant 1$. Hence $B_n(*) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$ if and only if $n = 4k$ for $k = 0, 1, 2, \ldots$.*

Work of Milnor, Wall and others describes the full ring $B_*(*)$, as in [24, §I.2].

Remark 13.6. **(a)** As in [24, §I.5 & §I.13], bordism is a *generalized homology theory*, that is, it satisfies all the Eilenberg–Steenrod axioms for homology except the dimension axiom. (The dimension axiom would say that $B_k(*) = 0$ for $k \neq 0$, which is false by Theorem 13.5.) This gives some information on bordism groups of general spaces Y: for any generalized homology theory $GH_*(Y)$, there is a spectral sequence from $H_k\big(Y; GH_l(*)\big)$ converging to $GH_{k+l}(Y)$, so for instance we may deduce that $B_m(\mathcal{S}^n) \cong \bigoplus_{k+l=m} H_k\big(\mathcal{S}^n; B_l(*)\big)$.
(b) There is also a generalized cohomology theory dual to bordism, called *cobordism*, as in Atiyah [6] and Conner [24, §13]. For general (sufficiently nice) topological spaces Y, the cobordism groups $MSO^k(Y)$ for $k \in \mathbb{Z}$ have a complicated definition involving homotopy theory, direct limits of k-fold suspensions, and classifying spaces, very unlike the elementary definition of $MSO_k(Y)$.

If Y is a compact oriented n-manifold, there are Poincaré duality isomorphisms $MSO^k(Y) \cong MSO_{n-k}(Y)$ for $k \in \mathbb{Z}$, [6, Th. 3.6], [24, Th. 13.4]. Thus, cobordism is essentially the same as bordism for (compact, oriented) manifolds.

We can use this to give a definition of cobordism parallel to Definition 13.1. Suppose for simplicity that Y is a compact manifold, not necessarily oriented, with $\dim Y = n$. Define the k^{th} *cobordism group* $B^k(Y)$ to be the set of $\sim$-equivalence classes $[X, f]$ of pairs (X, f), where now X is a compact manifold with $\dim X = n - k$, and $f : X \to Y$ is a *cooriented* smooth map, that is, we are given an orientation on the line bundle $\Lambda^{n-k}T^*X \otimes f^*(\Lambda^n T^*Y)^*$ over X.

There is a *cup product* $\cup : B^k(Y) \times B^l(Y) \to B^{k+l}(Y)$ defined as for $\bullet$ in (13.1). The *identity element* is $1_Y = [Y, \mathrm{id}_Y] \in B^0(Y)$, where $\mathrm{id}_Y : Y \to Y$ has the natural coorientation. These make $B^*(Y)$ into an associative, supercommutative ring. There is a *cap product* $\cap : B^k(Y) \times B_l(Y) \to B_{l-k}(Y)$, defined as in (13.1), making $B_*(Y)$ into a module over $B^*(Y)$. If $g : Y \to Z$ is smooth we define *pullbacks* $g^* : B^k(Z) \to B^k(Y)$ by $g^*([X, f]) = [X \times_{f,Z,g} Y, \pi_Y]$, provided f, g are transverse, which can be achieved by perturbing f in its cobordism class.

If Y is compact and oriented then we have a fundamental class $[Y] \in B_n(Y)$, and so $\alpha \mapsto \alpha \cap [Y]$ gives an isomorphism $\cap [Y] : B^k(Y) \to B_{n-k}(Y)$. This is *Poincaré duality* for (co)bordism of compact oriented manifolds.

In a similar way, all the material on bordism and d-bordism of manifolds and orbifolds below has easy analogues for cobordism and d-cobordism of manifolds and orbifolds. For brevity we will discuss only bordism and d-bordism.

(c) For (nice) noncompact topological spaces, there are actually two versions of homology: *homology* $H_*(Y;\mathbb{Z})$ and *homology with arbitrary support* $H_*^\infty(Y;\mathbb{Z})$, where $H_*^\infty(Y;\mathbb{Z})$ is the relative homology $H_*(Y \amalg \{\infty\}, \{\infty\}; \mathbb{Z})$, with $Y \amalg \{\infty\}$ the one-point compactification of Y. There are also two versions of cohomology, *cohomology* $H^*(Y;\mathbb{Z})$ and *compactly-supported cohomology* $H^*_{\mathrm{cs}}(Y;\mathbb{Z})$, where $H^*_{\mathrm{cs}}(Y;\mathbb{Z}) \cong H^*(Y \amalg \{\infty\}, \{\infty\}; \mathbb{Z})$. If Y is compact then $H_*(Y;\mathbb{Z}) \cong H_*^\infty(Y;\mathbb{Z})$ and $H^*(Y;\mathbb{Z}) \cong H^*_{\mathrm{cs}}(Y;\mathbb{Z})$. If Y is an oriented n-manifold, Poincaré duality gives $H^k(Y;\mathbb{Z}) \cong H^\infty_{n-k}(Y;\mathbb{Z})$ and $H^k_{\mathrm{cs}}(Y;\mathbb{Z}) \cong H_{n-k}(Y;\mathbb{Z})$.

Motivated by this, if Y is a noncompact manifold we define can two kinds of bordism $B_*(Y), B_*^\infty(Y)$ and cobordism $B^*(Y), B^*_{\mathrm{cs}}(Y)$, following Definition 13.1 and (b) above. For $B_*(Y), B^*_{\mathrm{cs}}(Y)$ we take X, W compact, as in Definition 13.1, but for $B_*^\infty(Y), B^*(Y)$ we do not require W, X compact, but instead take $f : X \to Y$ and $e : W \to Y$ proper. The material below has easy generalizations to (d-)(co)bordism with arbitrary support, but for brevity we omit them.

(d) Bordism groups are written $MSO_k(Y)$, as for $[X, f]$ in $MSO_k(Y)$ we may take X to be an oriented Riemannian k-manifold, so that TX has structure group $\mathrm{SO}(k)$. In fact there are bordism theories $MSO_*(Y), MO_*(Y)$, $MU_*(Y), MSU_*(Y), MSp_*(Y)$ for each series of classical groups $\mathrm{SO}(k), \mathrm{O}(k)$, $\mathrm{U}(k), \mathrm{SU}(k), \mathrm{Sp}(k)$, in which TX is given a (stable) $\mathrm{SO}(k), \ldots, \mathrm{Sp}(k)$-structure. There are also cobordism theories $MSO^*(Y), MO^*(Y), MU^*(Y), MSU^*(Y)$, $MSp^*(Y)$. For more details see Conner and Floyd [25, 26] and Stong [96].

The simplest of these other bordism theories is *unoriented bordism* $MO_*(Y)$, defined as in Definition 13.1, but without taking X, W oriented. Unoriented bordism groups have been completely determined: as in [24, §I.8], [96, §VI], for the point $MO_*(*)$ is the free commutative $\mathbb{Z}_2$-algebra with generators in dimension $i = 2, 4, 5, \ldots$ for all i not of the form $2^k - 1$, and $MO_*(Y) \cong H_*(Y;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} MO_*(*)$ for any CW-complex Y.

Unitary bordism $MU_*(Y)$ [25, §III], [96, §VII] is defined using pairs (X, f) in which X has a *stable almost complex structure*, that is, a complex vector bundle structure on $TX \oplus \mathbb{R}^k$ over X for $k \gg 0$. For a point, $MU_*(*)$ is the free commutative ring with generators in dimension $2k$ for all $k > 0$.

13.2 D-manifold bordism groups

We define *d-bordism* by replacing manifolds X in $[X, f]$ by d-manifolds $\boldsymbol{X}$ throughout §13.1. This section is based on Spivak [94, §6.2] and [95, §3.1] on (unoriented) 'derived cobordism' for his derived manifolds. As in §7.1, we have isomorphic 2-categories $\mathbf{dMan}$, with objects $\boldsymbol{X}$, and $\mathbf{d\bar{M}an}$, with objects $\mathbf{X} = (\boldsymbol{X}, \emptyset, \emptyset, \emptyset)$, with isomorphism $\mathbf{dMan} \to \mathbf{d\bar{M}an}$ mapping $\boldsymbol{X} \mapsto (\boldsymbol{X}, \emptyset, \emptyset, \emptyset)$, where $\mathbf{d\bar{M}an} \subset \mathbf{dMan^b} \subset \mathbf{dMan^c}$. For simplicity we identify $\mathbf{dMan}$ and $\mathbf{d\bar{M}an}$, writing objects of both as $\boldsymbol{X}$.

Definition 13.7. Let Y be a manifold, and $k \in \mathbb{Z}$. Consider pairs $(\boldsymbol{X}, \boldsymbol{f})$, where $\boldsymbol{X} \in \mathbf{dMan}$ is a compact, oriented d-manifold without boundary with $\mathrm{vdim}\, \boldsymbol{X} = k$, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in $\mathbf{dMan}$, where $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$.

Define a binary relation $\sim$ between such pairs by $(\boldsymbol{X}, \boldsymbol{f}) \sim (\boldsymbol{X}', \boldsymbol{f}')$ if there exists a compact, oriented d-manifold with boundary $\mathbf{W}$ with $\mathrm{vdim}\, \mathbf{W} = k + 1$, a 1-morphism $\boldsymbol{e} : \mathbf{W} \to \boldsymbol{Y}$ in $\mathbf{dMan^b}$, an equivalence of oriented d-manifolds $\boldsymbol{j} : -\boldsymbol{X} \amalg \boldsymbol{X}' \to \partial \mathbf{W}$, and a 2-morphism $\eta : \boldsymbol{f} \amalg \boldsymbol{f}' \Rightarrow \boldsymbol{e} \circ \boldsymbol{i}_{\mathbf{W}} \circ \boldsymbol{j}$. Proposition 13.8 shows $\sim$ is an equivalence relation, which we call *d-bordism*.

Write $[\boldsymbol{X}, \boldsymbol{f}]$ for the $\sim$-equivalence class (*d-bordism class*) of a pair $(\boldsymbol{X}, \boldsymbol{f})$. For each $k \in \mathbb{Z}$, define the k^{th} *d-manifold bordism group*, or *d-bordism group*, $dB_k(Y)$ of Y to be the set of all such d-bordism classes $[\boldsymbol{X}, \boldsymbol{f}]$ with $\mathrm{vdim}\, \boldsymbol{X} = k$. As for $B_k(Y)$, we give $dB_k(Y)$ the structure of an abelian group, with zero element $0_Y = [\boldsymbol{\emptyset}, \boldsymbol{\emptyset}]$, addition $[\boldsymbol{X}, \boldsymbol{f}] + [\boldsymbol{X}', \boldsymbol{f}'] = [\boldsymbol{X} \amalg \boldsymbol{X}', \boldsymbol{f} \amalg \boldsymbol{f}']$, and additive inverses $-[\boldsymbol{X}, \boldsymbol{f}] = [-\boldsymbol{X}, \boldsymbol{f}]$.

Here is the analogue of Proposition 13.3, similar to Spivak [94, Prop. 6.2.12].

Proposition 13.8. *In Definition* 13.7, $\sim$ *is an equivalence relation.*

Proof. Let $\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{X}''$ be compact oriented d-manifolds of dimension $k \in \mathbb{Z}$, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{f}' : \boldsymbol{X}' \to \boldsymbol{Y}$ and $\boldsymbol{f}'' : \boldsymbol{X}'' \to \boldsymbol{Y}$ be 1-morphisms for $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$, and $\sim$ be as in Definition 13.7. The proofs that $\sim$ is reflexive and symmetric are as in Proposition 13.3: to show that $(\boldsymbol{X}, \boldsymbol{f}) \sim (\boldsymbol{X}, \boldsymbol{f})$ we consider $\boldsymbol{W} = [\mathbf{0}, \mathbf{1}] \times \boldsymbol{X}$ and $\boldsymbol{e} = \boldsymbol{f} \circ \boldsymbol{\pi}_{\boldsymbol{X}}$, and if $(\boldsymbol{X}, \boldsymbol{f}) \sim (\boldsymbol{X}', \boldsymbol{f}')$, so that there exist $\boldsymbol{W}, \boldsymbol{e}, \boldsymbol{j}, \eta$, then using $-\boldsymbol{W}, \boldsymbol{e}, \boldsymbol{j}, \eta$ we see that $(\boldsymbol{X}', \boldsymbol{f}') \sim (\boldsymbol{X}, \boldsymbol{f})$.

Suppose that $(\boldsymbol{X}, \boldsymbol{f}) \sim (\boldsymbol{X}', \boldsymbol{f}')$ and $(\boldsymbol{X}', \boldsymbol{f}') \sim (\boldsymbol{X}'', \boldsymbol{f}'')$. Then there exist compact, oriented d-manifolds with boundary $\mathbf{W}, \mathbf{W}'$ of dimension $k + 1$, 1-morphisms $\boldsymbol{e} : \mathbf{W} \to \boldsymbol{Y}$, $\boldsymbol{e}' : \mathbf{W}' \to \boldsymbol{Y}$, oriented equivalences $\boldsymbol{j} : -\boldsymbol{X} \amalg \boldsymbol{X}' \to \partial \mathbf{W}$, $\boldsymbol{j}' : -\boldsymbol{X}' \amalg \boldsymbol{X}'' \to \partial \mathbf{W}'$, and 2-morphisms $\eta : \boldsymbol{f} \amalg \boldsymbol{f}' \Rightarrow \boldsymbol{e} \circ \boldsymbol{i}_{\mathbf{W}} \circ \boldsymbol{j}$ and $\eta' : \boldsymbol{f}' \amalg \boldsymbol{f}'' \Rightarrow \boldsymbol{e}' \circ \boldsymbol{i}_{\mathbf{W}'} \circ \boldsymbol{j}'$. The new issue in this proof is that the Differentiable Collaring Theorem [24, Th. I.1.2] is *false* for d-manifolds with boundary. That is, in general $\mathbf{W}$ and $\mathbf{W}'$ are not equivalent to $(-\boldsymbol{\epsilon}, \mathbf{0}] \times \boldsymbol{X}'$ and $[\mathbf{0}, \boldsymbol{\epsilon}) \times \boldsymbol{X}'$ near $\boldsymbol{j}(\boldsymbol{X}')$ and $\boldsymbol{j}'(\boldsymbol{X}')$, so we cannot simply glue $\mathbf{W}, \mathbf{W}'$ along their common boundary components $\boldsymbol{X}'$ to make a new d-manifold with boundary $\mathbf{W}''$, as we did for manifolds in the proof of Proposition 13.3.

Generalizing the proof of Proposition 7.45, we may construct simple, flat 1-morphisms $\boldsymbol{g} : \mathbf{W} \to [\mathbf{0}, \mathbf{1}]$, $\boldsymbol{g}' : \mathbf{W}' \to [\mathbf{1}, \mathbf{2}]$ in $\mathbf{dMan^b}$, where $[\mathbf{0}, \mathbf{1}], [\mathbf{1}, \mathbf{2}] =$

$F_{\mathbf{Man^b}}^{\mathbf{dMan^c}}([0,1],[1,2])$, fitting into 2-Cartesian diagrams in $\mathbf{dMan^b}$:

$$
\begin{tikzcd}
\boldsymbol{X} \arrow[r, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{W}}\circ \boldsymbol{j}|_{\boldsymbol{X}}"'] \arrow[dr, phantom, "\Uparrow \mathrm{id}_{\boldsymbol{0}\circ\boldsymbol{\pi}}"] & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\
\mathbf{W} \arrow[r, "\boldsymbol{g}"] & \mathbf{[0,1]},
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{X}' \arrow[r, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{W}}\circ \boldsymbol{j}|_{\boldsymbol{X}'}"'] \arrow[dr, phantom, "\Uparrow \mathrm{id}_{\boldsymbol{1}\circ\boldsymbol{\pi}}"] & \boldsymbol{*} \arrow[d, "\boldsymbol{1}"] \\
\mathbf{W} \arrow[r, "\boldsymbol{g}"] & \mathbf{[0,1]},
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boldsymbol{X}' \arrow[r, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{W}'}\circ \boldsymbol{j}'|_{\boldsymbol{X}'}"'] \arrow[dr, phantom, "\Uparrow \mathrm{id}_{\boldsymbol{1}\circ\boldsymbol{\pi}}"] & \boldsymbol{*} \arrow[d, "\boldsymbol{1}"] \\
\mathbf{W}' \arrow[r, "\boldsymbol{g}'"] & \mathbf{[1,2]},
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{X}'' \arrow[r, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{i}_{\mathbf{W}'}\circ \boldsymbol{j}'|_{\boldsymbol{X}''}"'] \arrow[dr, phantom, "\Uparrow \mathrm{id}_{\boldsymbol{2}\circ\boldsymbol{\pi}}"] & \boldsymbol{*} \arrow[d, "\boldsymbol{2}"] \\
\mathbf{W}' \arrow[r, "\boldsymbol{g}'"] & \mathbf{[1,2]}.
\end{tikzcd}
\tag{13.2}
$$

Choose a smooth function $h:[0,\frac{4}{3})\rightarrow[0,\frac{4}{3})$ satisfying $h(x)=x$ for $x\in[0,\frac{1}{3}]$, $h(x)\in(\frac{1}{3},1)$ and $\frac{\mathrm{d}h}{\mathrm{d}x}(x)>0$ for $x\in(\frac{1}{3},\frac{2}{3})$, and $h(x)=1$ for $x\in[\frac{2}{3},\frac{4}{3})$. Choose another smooth function $h':(\frac{2}{3},2]\rightarrow(\frac{2}{3},2]$ satisfying $h'(x)=1$ for $x\in(\frac{2}{3},\frac{4}{3}]$, $h'(x)\in(1,\frac{5}{3})$ and $\frac{\mathrm{d}h'}{\mathrm{d}x}(x)>0$ for $x\in(\frac{4}{3},\frac{5}{3})$, and $h'(x)=x$ for $x\in[\frac{5}{3},2]$. Note that although h actually maps $h:[0,\frac{4}{3})\rightarrow[0,1]$, it is not a smooth map of manifolds with boundary into $[0,1]$, as neither of Definition 5.5(i),(ii) holds at $x=\frac{2}{3}$. Similarly, h' maps $h':(\frac{2}{3},2]\rightarrow[1,2]$, but is not a smooth map into $[1,2]$ near $x=\frac{4}{3}$.

Roughly speaking, the idea of the next part of the proof is to replace $\mathbf{W}$ and $\mathbf{W}'$ by $\tilde{\mathbf{W}}=\mathbf{W}\times_{\boldsymbol{g},\mathbf{[0,1]},\boldsymbol{h}}\mathbf{[0,\frac{4}{3})}$ and $\tilde{\mathbf{W}}'=\mathbf{W}'\times_{\boldsymbol{g}',\mathbf{[1,2]},\boldsymbol{h}'}\mathbf{(\frac{2}{3},2]}$, and then note that $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{W}}'$ both contain an open d-submanifold equivalent to $\mathbf{(\frac{2}{3},\frac{4}{3})}\times\boldsymbol{X}'$ as $h(x)=h'(x)=1$ for $x\in(\frac{2}{3},\frac{4}{3})$, so we can glue $\tilde{\mathbf{W}},\tilde{\mathbf{W}}'$ on these equivalent open d-submanifolds to obtain a d-manifold with corners $\mathbf{W}''$.

However, the fibre products $\mathbf{W}\times_{\boldsymbol{g},\mathbf{[0,1]},\boldsymbol{h}}\mathbf{[0,\frac{4}{3})}$ and $\mathbf{W}'\times_{\boldsymbol{g}',\mathbf{[1,2]},\boldsymbol{h}'}\mathbf{(\frac{2}{3},2]}$ do not make sense in $\mathbf{dMan^b}$, since $\boldsymbol{h}:\mathbf{[0,\frac{4}{3})}\rightarrow\mathbf{[0,1]}$ and $\boldsymbol{h}':\mathbf{(\frac{2}{3},2]}\rightarrow\mathbf{[1,2]}$ are not 1-morphisms in $\mathbf{dMan^b}$ for the same reason that h,h' are not smooth as maps into $[0,1]$ and $[1,2]$. So we will construct $\tilde{\mathbf{W}},\tilde{\mathbf{W}}'$ in a more ad hoc way.

In the d-manifold with boundary $\tilde{\mathbf{W}}=(\tilde{\boldsymbol{W}},\boldsymbol{\partial}\tilde{\boldsymbol{W}},\boldsymbol{i}_{\tilde{\mathbf{W}}},\omega_{\tilde{\mathbf{W}}})$, we define the d-space $\tilde{\boldsymbol{W}}$ to be a d-space fibre product $\boldsymbol{W}\times_{\boldsymbol{g},\mathbf{[0,1]},\boldsymbol{h}}\mathbf{[0,\frac{4}{3})}$ in $\mathbf{dSpa}$, which exists as in §2.5, with projections $\tilde{\boldsymbol{e}}:\tilde{\mathbf{W}}\rightarrow\boldsymbol{W}$ and $\tilde{\boldsymbol{f}}:\tilde{\mathbf{W}}\rightarrow\mathbf{[0,\frac{4}{3})}$. Note however that this specifies $\tilde{\boldsymbol{W}}$ only up to equivalence in $\mathbf{dSpa}$, and the conditions on $\tilde{\boldsymbol{W}},\boldsymbol{\partial}\tilde{\boldsymbol{W}},\boldsymbol{i}_{\tilde{\mathbf{W}}}$ for $\tilde{\mathbf{W}}$ to be a d-space with boundary depend on $\tilde{\boldsymbol{W}}$ up to 1-isomorphism, not just up to equivalence.

Since $h(x)=x$ for $x\in[0,\frac{1}{3})$, over $\mathbf{[0,\frac{1}{3})}$ the fibre product is the same as $\boldsymbol{W}\times_{\boldsymbol{g},\mathbf{[0,1]},\mathbf{id}_{[0,1]}}\mathbf{[0,1]}\simeq\boldsymbol{W}$. Using Theorem 2.28, we can show that we can take $\tilde{\boldsymbol{W}}$ to be actually *equal* to $\boldsymbol{W}$ over the smaller open subset $\mathbf{[0,\frac{1}{4})}$. That is, the open d-spaces $(\boldsymbol{g}\circ\tilde{\boldsymbol{e}})^{-1}(\mathbf{[0,\frac{1}{4})})\subseteq\tilde{\boldsymbol{W}}$ and $\boldsymbol{g}^{-1}(\mathbf{[0,\frac{1}{4})})\subseteq\boldsymbol{W}$ are the same, and $\tilde{\boldsymbol{e}}|_{(\boldsymbol{g}\circ\tilde{\boldsymbol{e}})^{-1}(\mathbf{[0,\frac{1}{4})})}=\mathrm{id}_{\boldsymbol{g}^{-1}(\mathbf{[0,\frac{1}{4})})}$.

As $\boldsymbol{j}:-\boldsymbol{X}\amalg\boldsymbol{X}'\rightarrow\partial\mathbf{W}$ is an equivalence, we have $\boldsymbol{\partial W}=\boldsymbol{j}(\boldsymbol{X})\amalg\boldsymbol{j}(\boldsymbol{X}')$, where $\boldsymbol{j}(\boldsymbol{X}),\boldsymbol{j}(\boldsymbol{X}')$ are open and closed d-subspaces of $\boldsymbol{\partial W}$. Define $\boldsymbol{\partial}\tilde{\boldsymbol{W}}=\boldsymbol{j}(\boldsymbol{X})$, and $\boldsymbol{i}_{\tilde{\mathbf{W}}}=\boldsymbol{i}_{\mathbf{W}}|_{\boldsymbol{j}(\boldsymbol{X})}$, and $\omega_{\tilde{\mathbf{W}}}=\omega_{\mathbf{W}}|_{\boldsymbol{j}(\boldsymbol{X})}$. Since $\boldsymbol{g}\circ\boldsymbol{i}_{\mathbf{W}}\circ\boldsymbol{j}|_{\boldsymbol{X}}=\mathbf{0}\circ\boldsymbol{\pi}$ by (13.2), we see that $\boldsymbol{i}_{\mathbf{W}}|_{\boldsymbol{j}(\boldsymbol{X})}$ maps $\boldsymbol{j}(\boldsymbol{X})$ to $\boldsymbol{g}^{-1}(\mathbf{0})\subseteq\boldsymbol{g}^{-1}(\mathbf{[0,\frac{1}{4})})\subseteq\boldsymbol{W}$. Since $\mathrm{id}_{\boldsymbol{g}^{-1}(\mathbf{[0,\frac{1}{4})})}=(\boldsymbol{g}\circ\tilde{\boldsymbol{e}})^{-1}(\mathbf{[0,\frac{1}{4})})\subseteq\tilde{\boldsymbol{W}}$, we may regard $\boldsymbol{i}_{\tilde{\mathbf{W}}}=\boldsymbol{i}_{\mathbf{W}}|_{\boldsymbol{j}(\boldsymbol{X})}$ as a 1-morphism $\boldsymbol{j}(\boldsymbol{X})\rightarrow\tilde{\boldsymbol{W}}$ in $\mathbf{dSpa}$. Similarly, $\omega_{\tilde{\mathbf{W}}}$ makes sense.

Because $\tilde{\boldsymbol{W}}, \boldsymbol{\partial}\tilde{\boldsymbol{W}}, \boldsymbol{i}_{\tilde{\mathbf{W}}}, \omega_{\tilde{\mathbf{W}}}$ are equal to $\boldsymbol{W}, \boldsymbol{\partial W}, \boldsymbol{i}_{\mathbf{W}}, \omega_{\mathbf{W}}$ over $[\mathbf{0}, \frac{\mathbf{1}}{\mathbf{4}})$, and all of $\boldsymbol{\partial}\tilde{\boldsymbol{W}}$ lies over $\mathbf{0} \in [\mathbf{0}, \frac{\mathbf{1}}{\mathbf{4}})$, Definition 6.1 holds for $\tilde{\mathbf{W}}$ as it does for $\mathbf{W}$, so $\tilde{\mathbf{W}}$ is a d-space with boundary. We claim it is a d-manifold with boundary, with $\operatorname{vdim} \tilde{\mathbf{W}} = k+1$. Near $\partial\tilde{\mathbf{W}}$ this is immediate, as $\tilde{\mathbf{W}}$ coincides with $\mathbf{W}$ there.

Away from $\partial\tilde{\mathbf{W}}$, note that $\boldsymbol{g}$ and $\boldsymbol{h}$ are d-transverse, since $\boldsymbol{g}$ simple and flat implies that $\boldsymbol{g}$ is a submersion near $\boldsymbol{g}^{-1}(\mathbf{1})$, and $h^{-1}\big([0,1)\big) = [0, \frac{2}{3})$ and $\frac{\mathrm{d}h}{\mathrm{d}x}(x) > 0$ for $x \in [0, \frac{2}{3})$ imply that $\boldsymbol{h}$ is a submersion on $\boldsymbol{h}^{-1}\big([\mathbf{0}, \mathbf{1})\big)$, so one of $\boldsymbol{g}$ or $\boldsymbol{h}$ is a submersion over every point of $[\mathbf{0}, \mathbf{1}]$. Thus, it more-or-less follows from Theorem 4.21 that $\tilde{\boldsymbol{W}}$ is a d-manifold with $\operatorname{vdim} \tilde{\boldsymbol{W}} = k+1$ away from $\partial\tilde{\mathbf{W}}$. This is not quite true as $\boldsymbol{W}$ and $[\mathbf{0}, \mathbf{1}]$ are not d-manifolds without boundary near $\boldsymbol{i}_{\mathbf{W}} \circ \boldsymbol{j}(\boldsymbol{X}')$ and $\mathbf{1}$ respectively, but considering local models shows that the boundaries do not affect $\tilde{\boldsymbol{W}}$ being a d-manifold.

Thus, $\tilde{\mathbf{W}}$ is a d-manifold with boundary. Since $h(x) = 1$ for $x \in (\frac{2}{3}, \frac{4}{3})$, and $\boldsymbol{W} \times_{\boldsymbol{g},[\mathbf{0},\mathbf{1}],\mathbf{1}} \boldsymbol{*} \simeq \boldsymbol{X}'$ by (13.2), and $\tilde{\boldsymbol{W}} = \boldsymbol{W} \times_{\boldsymbol{g},[\mathbf{0},\mathbf{1}],\boldsymbol{h}} [\mathbf{0}, \frac{\mathbf{4}}{\mathbf{3}})$, we see there is an equivalence $\tilde{\boldsymbol{W}} \supseteq \tilde{\boldsymbol{f}}^{-1}\big((\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}})\big) \simeq (\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}}) \times \boldsymbol{X}'$ in $\mathbf{dMan}$. Similarly, we define a d-manifold with boundary $\tilde{\mathbf{W}}'$ with $\operatorname{vdim} \tilde{\boldsymbol{W}}' = k+1$, with projections $\tilde{\boldsymbol{e}}' : \tilde{\mathbf{W}}' \to \boldsymbol{W}'$ and $\tilde{\boldsymbol{f}}' : \tilde{\mathbf{W}}' \to (\frac{\mathbf{2}}{\mathbf{3}}, \mathbf{1}]$, and an equivalence $\tilde{\boldsymbol{W}}' \supseteq (\tilde{\boldsymbol{f}}')^{-1}\big((\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}})\big) \simeq (\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}}) \times \boldsymbol{X}'$. We then apply Theorem 6.25 to glue $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{W}}'$ on their equivalent open d-subspaces $\tilde{\boldsymbol{f}}^{-1}\big((\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}})\big), (\tilde{\boldsymbol{f}}')^{-1}\big((\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}})\big)$ to obtain a d-manifold with boundary $\mathbf{W}''$, with $\operatorname{vdim} \mathbf{W}'' = k+1$.

We also have 1-morphisms $\boldsymbol{e} \circ \tilde{\boldsymbol{e}} : \tilde{\mathbf{W}} \to \boldsymbol{Y}$ and $\boldsymbol{e}' \circ \tilde{\boldsymbol{e}}' : \tilde{\mathbf{W}}' \to \boldsymbol{Y}$, which on the d-subspaces $\tilde{\boldsymbol{f}}^{-1}\big((\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}})\big), (\tilde{\boldsymbol{f}}')^{-1}\big((\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}})\big)$ are identified up to 2-isomorphism with $\boldsymbol{f}' \circ \boldsymbol{\pi}_{\boldsymbol{X}'} : (\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}}) \times \boldsymbol{X}' \to \boldsymbol{Y}$ by the equivalences $\tilde{\boldsymbol{f}}^{-1}\big((\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}})\big) \simeq (\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}}) \times \boldsymbol{X}' \simeq (\tilde{\boldsymbol{f}}')^{-1}\big((\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}})\big)$. Thus, the second part of Theorem 6.25 gives a 1-morphism $\boldsymbol{e}'' : \mathbf{W}'' \to \boldsymbol{Y}$ identified up to 2-isomorphism and equivalences on open d-subspaces with $\boldsymbol{e} \circ \tilde{\boldsymbol{e}}$ and $\boldsymbol{e}' \circ \tilde{\boldsymbol{e}}'$.

The orientations on $\mathbf{W}, \mathbf{W}'$ and induce orientations on $\tilde{\mathbf{W}}, \tilde{\mathbf{W}}'$ which are compatible on $\tilde{\boldsymbol{f}}^{-1}\big((\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}})\big), (\tilde{\boldsymbol{f}}')^{-1}\big((\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{3}})\big)$, and so descend to an orientation on $\mathbf{W}''$. The boundary $\partial\mathbf{W}''$ is equivalent to the disjoint union of the boundaries $\boldsymbol{j}(-\mathbf{X})$ of $\tilde{\mathbf{W}}$ and $\boldsymbol{j}'(\mathbf{X}'')$ of $\tilde{\mathbf{W}}'$, so there is an orientation-preserving equivalence $\boldsymbol{j}'' : -\boldsymbol{X} \amalg \boldsymbol{X}'' \to \partial\mathbf{W}''$. Using the 2-morphisms $\eta|_{\boldsymbol{X}}$ and $\eta'|_{\boldsymbol{X}''}$ we see there is a 2-morphism $\eta'' : \boldsymbol{f} \amalg \boldsymbol{f}'' \Rightarrow \boldsymbol{e}'' \circ \boldsymbol{i}_{\mathbf{W}''} \circ \boldsymbol{j}''$. Hence $(\boldsymbol{X}, \boldsymbol{f}) \sim (\boldsymbol{X}'', \boldsymbol{f}'')$ in the sense of Definition 13.7, so $\sim$ is an equivalence relation. □

Here is the analogue of Definition 13.4.

Definition 13.9. Suppose Y is an oriented manifold of dimension n. As in (13.1), define the *intersection product* $\bullet : dB_k(Y) \times dB_l(Y) \to dB_{k+l-n}(Y)$ by

$$[\boldsymbol{X}, \boldsymbol{f}] \bullet [\boldsymbol{X}', \boldsymbol{f}'] = [\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Y},\boldsymbol{f}'} \boldsymbol{X}', \boldsymbol{f} \circ \boldsymbol{\pi}_{\boldsymbol{X}}]. \tag{13.3}$$

Here the fibre product $\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Y},\boldsymbol{f}'} \boldsymbol{X}'$ exists in $\mathbf{dMan}$ by Theorem 4.22(a), and is oriented by Theorem 4.50. Using Proposition 4.52, one can show that $\bullet$ is biadditive, supercommutative and associative.

If Y is compact and oriented, we define the *fundamental class* $[Y] \in dB_n(Y)$ by $[Y] = [\boldsymbol{Y}, \mathbf{id}_{\boldsymbol{Y}}]$. It is the identity for $\bullet$ on $dB_*(Y)$.

Let $g : Y \to Z$ be a smooth map of manifolds. Define the *pushforward* $g_* : dB_k(Y) \to dB_k(Z)$ by $g_* : [\boldsymbol{X}, \boldsymbol{f}] \mapsto [\boldsymbol{X}, \boldsymbol{g} \circ \boldsymbol{f}]$, where $\boldsymbol{g} = F_{\mathbf{Man}}^{\mathbf{dMan}}(g)$.

Define a projection $\Pi_{\rm bo}^{\rm dbo} : B_k(Y) \to dB_k(Y)$ by $\Pi_{\rm bo}^{\rm dbo} : [X, f] \mapsto [\boldsymbol{X}, \boldsymbol{f}]$, where $\boldsymbol{X}, \boldsymbol{f} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X, f)$. As in Example 4.45, orientations on X correspond to orientations on $\boldsymbol{X}$. Then $\Pi_{\rm bo}^{\rm dbo}$ takes $\bullet, [Y]$ in $B_*(Y)$ to $\bullet, [Y]$ in $dB_*(Y)$, and commutes with pushforwards $g_* : B_*(Y) \to B_*(Z)$, $g_* : dB_*(Y) \to dB_*(Z)$.

Remark 13.10. **(a)** In §13.1, to define $[X, f] \bullet [X', f']$, it was necessary to first perturb f, f' to make $f : X \to Y$, $f' : X' \to Y$ transverse. In contrast, the definition of $[\boldsymbol{X}, \boldsymbol{f}] \bullet [\boldsymbol{X}', \boldsymbol{f}']$ in d-bordism works for all $(\boldsymbol{X}, \boldsymbol{f}), (\boldsymbol{X}', \boldsymbol{f}')$ without perturbation, since $\boldsymbol{f}, \boldsymbol{f}'$ are automatically d-transverse as $\boldsymbol{Y}$ is a manifold.

(b) Since there exist nontrivial d-manifolds $\boldsymbol{X}$ with $\operatorname{vdim} \boldsymbol{X} < 0$, it is not obvious that $dB_k(Y) = 0$ for $k < 0$. But we will prove this in Theorem 13.11.

(c) As in Remark 13.6(d), there are other classical bordism theories $MO_*(Y)$, $MU_*(Y), MSU_*(Y), MSp_*(Y)$, and we can try to define d-manifold versions of these. Generalizing unoriented bordism $MO_*(Y)$ is easy: we just omit orientations on $\boldsymbol{X}, \mathbf{W}$ in Definition 13.7, and then the proof of Theorem 13.11 shows that unoriented d-manifold bordism $dB_*^{\rm unor}(Y)$ is isomorphic to classical unoriented bordism $MO_*(Y)$. To define *unitary d-manifold bordism* generalizing $MU_*(Y)$ will require a suitable notion of stable almost complex structure on a d-manifold $\boldsymbol{X}$. We return to this in Remark 13.28.

The next theorem is the main result of this section. Spivak [94, Th. 6.2.24] and [95, Th. 2.6] gives similar results for his derived manifolds. The theorem implies that we may define projections

$$\Pi_{\rm dbo}^{\rm hom} : dB_k(Y) \longrightarrow H_k(Y; \mathbb{Z}) \text{ by } \Pi_{\rm dbo}^{\rm hom} = \Pi_{\rm bo}^{\rm hom} \circ (\Pi_{\rm bo}^{\rm dbo})^{-1}. \tag{13.4}$$

We think of these $\Pi_{\rm dbo}^{\rm hom}$ as *virtual class maps.* Virtual classes (or virtual cycles, or virtual chains) are used in several areas of geometry to construct enumerative invariants using moduli spaces. Our main point is that *compact oriented d-manifolds admit virtual classes.*

Theorem 13.11. *For any manifold Y, the morphisms $\Pi_{\rm bo}^{\rm dbo} : B_k(Y) \to dB_k(Y)$ above are isomorphisms for all $k \in \mathbb{Z}$, and preserve the structures $\bullet, [Y], g_*$.*

Proof. Suppose $[\boldsymbol{X}, \boldsymbol{f}] \in dB_k(Y)$. Then Theorem 4.29 gives an embedding $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{\mathbb{R}^n}$ for $n \gg 0$, as $\boldsymbol{X}$ is compact. The direct product $(\boldsymbol{f}, \boldsymbol{g}) : \boldsymbol{X} \to \boldsymbol{Y} \times \boldsymbol{\mathbb{R}^n}$ is also an embedding. Theorem 4.34 thus gives an open set $V \subseteq Y \times \mathbb{R}^n$, a vector bundle $E \to V$, a smooth section $s \in C^\infty(E)$, an equivalence $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{S}_{V,E,s}$, and a 2-morphism $\eta : \boldsymbol{S}_{\pi_Y,0} \circ \boldsymbol{i} \Rightarrow \boldsymbol{f}$, where $\pi_Y : V \to Y$ is the restriction to V of $\pi_Y : Y \times \mathbb{R}^n \to Y$, and $\boldsymbol{S}_{\pi_Y,0} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{Y,0,0} = \boldsymbol{Y}$ is as in Definition 3.30.

There is a unique orientation on $\boldsymbol{S}_{V,E,s}$ such that $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{S}_{V,E,s}$ is orientation-preserving. As in Definition 4.48, orientations on $\boldsymbol{S}_{V,E,s}$ correspond to orientations on the line bundle $\Lambda^{\operatorname{rank} E} \otimes \Lambda^{\dim V} T^*V$ over V near $s^{-1}(0)$. Making V smaller, we can suppose $\Lambda^{\operatorname{rank} E} \otimes \Lambda^{\dim V} T^*V$ is oriented on V.

Now $\boldsymbol{X}$ is compact, so $s^{-1}(0) \subseteq V$ is compact, but V may be noncompact. Choose an open neighbourhood U of $s^{-1}(0)$ in V whose closure $\bar{U}$ in V is compact. Let $\tilde{s}: V \to E$ be a generic smooth section of E such that $|\tilde{s}-s| \leqslant \frac{1}{2}|s|$ on $V \setminus U$, where $|\,.\,|$ is computed using some choice of metric on the fibres of E. Then $\tilde{s}^{-1}(0)$ is closed in V and contained in the compact subset $\bar{U}$, so $\tilde{s}^{-1}(0)$ is compact. Hence $\boldsymbol{S}_{V,E,\tilde{s}}$ is a compact d-manifold, which is oriented using the orientation on the fibres of $\Lambda^{\mathrm{rank}\,E} \otimes \Lambda^{\dim V} T^*V$, as in Definition 4.48.

Since $\tilde{s}$ is generic, it is transverse, so $\tilde{X} = \tilde{s}^{-1}(0)$ is a compact submanifold of V of dimension $k = \dim V - \mathrm{rank}\, E$, and $\tilde{f} = \pi_Y|_{\tilde{X}} : \tilde{X} \to Y$ is a smooth map. We have an equivalence $\tilde{\boldsymbol{\imath}} : \tilde{\boldsymbol{X}} \to \boldsymbol{S}_{V,E,\tilde{s}}$ in $\mathbf{dMan}$ and a 2-morphism $\tilde{\eta} : \boldsymbol{S}_{\pi_Y,0} \circ \tilde{\boldsymbol{\imath}} \Rightarrow \tilde{\boldsymbol{f}}$, where $\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{f}} = F_{\mathbf{Man}}^{\mathbf{dMan}}(\tilde{X}, \tilde{f})$. Also the orientation on $\boldsymbol{S}_{V,E,\tilde{s}}$ is identified with an orientation on $\tilde{\boldsymbol{X}}$ by $\tilde{\boldsymbol{\imath}}$, which in turn is identified with a unique orientation on $\tilde{X}$ by Example 4.45. Hence $[\tilde{X}, \tilde{f}] \in B_k(Y)$.

Define $W = V \times [0,1]$, as a manifold with boundary, and $F = \pi_V^*(E)$ as a vector bundle $F \to W$, and a smooth section $t : W \to F$ by $t = (1-z)\pi_V^*(s) + z\pi_V^*(\tilde{s})$, where z is the coordinate on $[0,1]$. Then $\partial W \cong (V \times \{0\}) \amalg (V \times \{1\})$, the disjoint union of two copies of V, and $F|_{\partial W} \cong E$ on each copy of V, and $t|_{\partial W} \cong s \in C^\infty(E)$ on $V \times \{0\}$, and $t|_{\partial W} \cong \tilde{s} \in C^\infty(E)$ on $V \times \{1\}$. Using the orientations on the fibres of $\Lambda^{\mathrm{rank}\,E} \otimes \Lambda^{\dim V} T^*V$ and on $[0,1]$, as in Definition 4.48 we obtain orientations on $\mathbf{S}_{W,F,t}$ and $\boldsymbol{S}_{V,E,\tilde{s}}$, and then $\partial \mathbf{S}_{W,F,t} \simeq -\boldsymbol{S}_{V,E,s} \amalg \boldsymbol{S}_{V,E,\tilde{s}}$ in oriented d-manifolds. Since $|\tilde{s}-s| \leqslant \frac{1}{2}|s|$ on $V \setminus U$ we see that $t^{-1}(0) \subseteq U \times [0,1]$, so $t^{-1}(0)$ is closed in $V \times [0,1]$ and contained in the compact subset $\bar{U} \times [0,1]$, and is compact. Thus $\mathbf{S}_{W,F,t}$ is compact.

We now have a compact, oriented d-manifold with boundary $\mathbf{S}_{W,F,t}$, and a 1-morphism $\mathbf{S}_{\pi_Y,0} : \mathbf{S}_{W,F,t} \to \boldsymbol{Y}$. We have equivalences of oriented d-manifolds $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$, $\tilde{\boldsymbol{X}} \simeq \boldsymbol{S}_{V,E,\tilde{s}}$ and $\partial \mathbf{S}_{W,F,t} \simeq -\boldsymbol{S}_{V,E,s} \amalg \boldsymbol{S}_{V,E,\tilde{s}}$, so putting these together gives an equivalence $\boldsymbol{j} : -\boldsymbol{X} \amalg \tilde{\boldsymbol{X}} \to \partial \mathbf{S}_{W,F,t}$. The 2-morphism $\eta : \boldsymbol{S}_{\pi_Y,0} \circ \boldsymbol{i} \Rightarrow \boldsymbol{f}$ and definition $\tilde{f} = \pi_Y|_{\tilde{X}}$ imply that there exists a 2-morphism $\tilde{\eta} : \boldsymbol{f} \amalg \tilde{\boldsymbol{f}} \Rightarrow \mathbf{S}_{\pi_Y,0} \circ \boldsymbol{i}_{\mathbf{S}_{W,F,t}} \circ \boldsymbol{j}$. Hence $(\boldsymbol{X}, \boldsymbol{f}) \simeq (\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{f}})$ by Definition 13.7, so $[\boldsymbol{X}, \boldsymbol{f}] = [\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{f}}] = \Pi_{\mathrm{bo}}^{\mathrm{dbo}}\bigl([\tilde{X}, \tilde{f}]\bigr)$. Thus $\Pi_{\mathrm{bo}}^{\mathrm{dbo}} : B_k(Y) \to dB_k(Y)$ is surjective.

Next, suppose $[X,f], [X',f'] \in B_k(Y)$ with $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}\bigl([X,f]\bigr) = \Pi_{\mathrm{bo}}^{\mathrm{dbo}}\bigl([X',f']\bigr)$. Then $(\boldsymbol{X}, \boldsymbol{f}) \sim (\boldsymbol{X}', \boldsymbol{f}')$, so there exist a compact, oriented d-manifold with boundary $\mathbf{W}$ with $\mathrm{vdim}\, \mathbf{W} = k+1$, a 1-morphism $\boldsymbol{e} : \mathbf{W} \to \boldsymbol{Y}$, an equivalence of oriented d-manifolds $\boldsymbol{j} : -\boldsymbol{X} \amalg \boldsymbol{X}' \to \partial \mathbf{W}$, and a 2-morphism $\eta : \boldsymbol{f} \amalg \boldsymbol{f}' \Rightarrow \boldsymbol{e} \circ \boldsymbol{i}_{\mathbf{W}} \circ \boldsymbol{j}$, where $\boldsymbol{X}, \boldsymbol{f}, \boldsymbol{X}', \boldsymbol{f}', \boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X, f, X', f', Y)$.

By Corollary 7.46 there exists an sf-embedding $\boldsymbol{g} : \mathbf{W} \to \mathbb{R}_{\mathbf{1}}^{\boldsymbol{n}}$ for $n \gg 0$, as $\mathbf{W}$ is compact with boundary. The direct product $(\boldsymbol{e}, \boldsymbol{g}) : \mathbf{W} \to \boldsymbol{Y} \times \mathbb{R}_{\mathbf{1}}^{\boldsymbol{n}}$ is also an sf-embedding, as $\partial \boldsymbol{Y} = \emptyset$. Theorem 7.48 thus gives an open set $V \subseteq Y \times \mathbb{R}_1^n$, a vector bundle $E \to V$, a smooth section $s \in C^\infty(E)$, an equivalence $\boldsymbol{i} : \mathbf{W} \to \mathbf{S}_{V,E,s}$, and a 2-morphism $\zeta : \mathbf{S}_{\pi_Y,0} \circ \boldsymbol{i} \Rightarrow \boldsymbol{e}$, where $\pi_Y : V \to Y$ is the restriction to V of $\pi_Y : Y \times \mathbb{R}_1^n \to Y$. As in the first part, making V smaller we can suppose $\Lambda^{\mathrm{rank}\,E} \otimes \Lambda^{\dim V} T^*V$ is oriented, so that $\mathbf{S}_{V,E,s}$ is oriented, and $\boldsymbol{i} : \mathbf{W} \to \mathbf{S}_{V,E,s}$ is orientation-preserving.

Now $\boldsymbol{X}, \boldsymbol{X}'$ are manifolds, and $\boldsymbol{j} : -\boldsymbol{X} \amalg \boldsymbol{X}' \to \partial \mathbf{W}$, $\boldsymbol{i}_- : \partial \mathbf{W} \to \partial \mathbf{S}_{V,E,s} \simeq \mathbf{S}_{\partial V, E|_{\partial V}, s|_{\partial V}}$ are equivalences. So $\mathbf{S}_{\partial V, E|_{\partial V}, s|_{\partial V}}$ is a manifold, which implies that $s|_{\partial V}$ is a transverse section of $E|_{\partial V}$, and hence s is transverse near ∂V. As above, choose open $s^{-1}(0) \subseteq U \subseteq V$ with $\bar{U}$ compact. Let $\tilde{s}$ be a transverse perturbation of s in $C^\infty(E)$ such that $\tilde s=s$ near ∂V where s is transverse, and $|\tilde s-s|\leqslant\frac12|s|$ on $V\setminus U$. Let $\tilde W=\tilde s^{-1}(0)$. Then $\tilde W$ is a $(k+1)$-submanifold of V as $\tilde s$ is transverse, and is compact as it is a closed subset of $\bar U$.

Define a smooth map $\tilde e:\tilde W\to Y$ by $\tilde e=\pi_Y|_{\tilde W}$. The orientation on the fibres of $\Lambda^{\operatorname{rank}E}\otimes\Lambda^{\dim V}T^*V$ induces an orientation on $\tilde W$. As $\tilde s=s$ near ∂V, we have $\partial\tilde W=s|_{\partial V}^{-1}(0)$, so $F_{\mathbf{Man}}^{\mathbf{dMan}}(\partial\tilde W)\cong\mathbf{S}_{\partial V,E|_{\partial V},s|_{\partial V}}\simeq-\boldsymbol{X}\amalg\boldsymbol{X}'$. Thus, $\boldsymbol{i}_-\circ\boldsymbol{j}$ induces an orientation-preserving diffeomorphism $\tilde\jmath:-X\amalg X'\to\partial\tilde W$, and $\eta:\boldsymbol{f}\amalg\boldsymbol{f}'\Rightarrow\boldsymbol{e}\circ\boldsymbol{i}_{\mathbf{W}}\circ\boldsymbol{j}$ implies that $f\amalg f'=\tilde e\circ i_W\circ\tilde\jmath$. Hence $\tilde W,\tilde e,\tilde\jmath$ imply that $(X,f)\sim(X',f')$ in the notation of Definition 13.1, so $[X,f]=[X',f']\in B_k(Y)$, and $\Pi_{\rm bo}^{\rm dbo}$ is injective. Therefore $\Pi_{\rm bo}^{\rm dbo}:B_k(Y)\to dB_k(Y)$ is an isomorphism. The last part is immediate from the definitions. □

13.3 Classical bordism for orbifolds

We generalize Definitions 13.1 and 13.4 to orbifolds $\mathcal{Y}$.

Definition 13.12. The 2-categories of orbifolds **Orb**, written $\mathcal{W},\mathcal{X},\mathcal{Y},\ldots$ and orbifolds with boundary $\mathbf{Orb^b}$, written $\mathfrak{W},\mathfrak{X},\mathfrak{Y},\ldots$, were defined in §8.2 and §8.5. As in Definition 8.15, the 2-functor $F_{\mathbf{Orb}}^{\mathbf{Orb^b}}:\mathbf{Orb}\to\mathbf{Orb^b}$ mapping $\mathcal{X}\mapsto\mathfrak{X}=(\mathcal{X},\emptyset,\emptyset)$ is a strict isomorphism from **Orb** to the 2-subcategory $\dot{\mathbf{O}}\mathbf{rb}$ of $\mathbf{Orb^b}$. For brevity we will identify the 2-categories **Orb** and $\dot{\mathbf{O}}\mathbf{rb}$ by $F_{\mathbf{Orb}}^{\mathbf{Orb^b}}$.

Let $\mathcal{Y}$ be an orbifold, and $k\in\mathbb{Z}$. Consider pairs $(\mathcal{X},f)$, where $\mathcal{X}$ is a compact, oriented orbifold (without boundary) with $\dim\mathcal{X}=k$, and $f:\mathcal{X}\to\mathcal{Y}$ is a 1-morphism in **Orb**. We allow $\emptyset$ as an oriented orbifold of any dimension $k\in\mathbb{Z}$, so $(\emptyset,\emptyset)$ is allowed as a pair $(\mathcal{X},f)$, and is the only such pair when $k<0$.

Define a binary relation $\sim$ between such pairs by $(\mathcal{X},f)\sim(\mathcal{X}',f')$ if there exists a compact, oriented $(k+1)$-orbifold with boundary $\mathfrak{W}$, a 1-morphism $e:\mathfrak{W}\to\mathcal{Y}$ in $\mathbf{Orb^b}$, an orientation-preserving equivalence $j:-\mathcal{X}\amalg\mathcal{X}'\to\partial\mathfrak{W}$, and a 2-morphism $\eta:f\amalg f'\Rightarrow e\circ i_{\mathfrak{W}}\circ j$ in $\mathbf{Orb^b}$, where $-\mathcal{X}$ is $\mathcal{X}$ with the opposite orientation, and the orientation of $\partial\mathfrak{W}$ is induced from that of $\mathfrak{W}$. Then $\sim$ is an equivalence relation, which is called *orbifold bordism*.

Write $[\mathcal{X},f]$ for the $\sim$-equivalence class (*bordism class*) of a pair $(\mathcal{X},f)$. For each $k\in\mathbb{Z}$, define the k^{th} *orbifold bordism group* $B_k^{\rm orb}(\mathcal{Y})$ of $\mathcal{Y}$ to be the set of all such bordism classes $[\mathcal{X},f]$ with $\dim\mathcal{X}=k$. It is an abelian group, with zero $0_{\mathcal{Y}}=[\emptyset,\emptyset]$, addition $[\mathcal{X},f]+[\mathcal{X}',f']=[\mathcal{X}\amalg\mathcal{X}',f\amalg f']$, and additive inverses $-[\mathcal{X},f]=[-\mathcal{X},f]$. If $k<0$ then $B_k^{\rm orb}(\mathcal{Y})=0$.

Define *effective orbifold bordism* $B_k^{\rm eff}(\mathcal{Y})$ in the same way, but requiring both orbifolds $\mathcal{X}$ and orbifolds with boundary $\mathfrak{W}$ to be effective (as in §8.4 and §8.9) in pairs $(\mathcal{X},f)$ and the definition of $\sim$.

Definition 13.13. Suppose $\mathcal{Y}$ is an oriented orbifold of dimension n which is a manifold, that is, the orbifold groups $\operatorname{Iso}_{\mathcal{Y}}([y])$ are trivial for all $[y]\in\mathcal{Y}_{\rm top}$. Define *intersection products* $\bullet:B_k^{\rm orb}(\mathcal{Y})\times B_l^{\rm orb}(\mathcal{Y})\to B_{k+l-n}^{\rm orb}(\mathcal{Y})$ and $\bullet:B_k^{\rm eff}(\mathcal{Y})\times B_l^{\rm eff}(\mathcal{Y})\to B_{k+l-n}^{\rm eff}(\mathcal{Y})$ as follows. Given classes $[\mathcal{X},f],[\mathcal{X}',f']$, we perturb f,f' in their bordism classes to make $f:\mathcal{X}\to\mathcal{Y}$ and $f':\mathcal{X}'\to\mathcal{Y}$

transverse 1-morphisms, and then as in (13.1) we set

$$[\mathcal{X},f]\bullet[\mathcal{X}',f']=[\mathcal{X}\times_{f,\mathcal{Y},f'}\mathcal{X}',f\circ\pi_{\mathcal{X}}]. \tag{13.5}$$

The associativity and commutativity properties of oriented fibre products imply that $\bullet$ is biadditive, associative and supercommutative on $B_*^{\rm orb}(\mathcal{Y})$ and $B_*^{\rm eff}(\mathcal{Y})$.

The reason we suppose $\mathcal{Y}$ is a manifold is that otherwise it may not be possible to perturb f,f' to be transverse. For example, consider the orbifold 1-morphisms $f,f':[\mathbb{R}/\{\pm1\}]\to[\mathbb{R}^3/\{\pm1\}]$ acting by $f:\pm x\mapsto\pm(x,0,0)$ and $f':\pm x\mapsto\pm(0,x,0)$. All perturbations of f,f' must map $0\mapsto0$, and so are non-transverse over $0\in[\mathbb{R}^3/\pm1]$. Also, for $\bullet$ on $B_*^{\rm eff}(\mathcal{Y})$, if $\mathcal{X},\mathcal{X}'$ are effective, f,f' are transverse, but $\mathcal{Y}$ is not a manifold, then $\mathcal{X}\times_{f,\mathcal{Y},f'}\mathcal{X}'$ need not be an effective orbifold, so (13.5) might not define a class in $B_*^{\rm eff}(\mathcal{Y})$.

If $\mathcal{Y}$ is a compact, oriented orbifold of dimension n, we define the *fundamental class* $[\mathcal{Y}]\in B_n^{\rm orb}(\mathcal{Y})$ by $[\mathcal{Y}]=[\mathcal{Y},{\rm id}_{\mathcal{Y}}]$. If $\mathcal{Y}$ is also effective we define $[\mathcal{Y}]=[\mathcal{Y},{\rm id}_{\mathcal{Y}}]\in B_n^{\rm eff}(\mathcal{Y})$. If $\mathcal{Y}$ is a manifold then $[\mathcal{Y}]$ is the identity for $\bullet$ on $B_*^{\rm orb}(\mathcal{Y})$ and $B_*^{\rm eff}(\mathcal{Y})$. Let $g:\mathcal{Y}\to\mathcal{Z}$ be a 1-morphism of orbifolds. Define *pushforwards* $g_*:B_k^{\rm orb}(\mathcal{Y})\to B_k^{\rm orb}(\mathcal{Z})$, $g_*:B_k^{\rm eff}(\mathcal{Y})\to B_k^{\rm eff}(\mathcal{Z})$ by $g_*:[\mathcal{X},f]\mapsto[\mathcal{X},g\circ f]$.

If $\mathcal{Y}$ is an orbifold, define group morphisms

$$\begin{aligned}\Pi_{\rm eff}^{\rm orb}:B_k^{\rm eff}(\mathcal{Y})\longrightarrow B_k^{\rm orb}(\mathcal{Y}),\;\;\Pi_{\rm orb}^{\rm hom}:B_k^{\rm orb}(\mathcal{Y})\longrightarrow H_k(\mathcal{Y}_{\rm top};\mathbb{Q})\\ \text{and}\qquad \Pi_{\rm eff}^{\rm hom}:B_k^{\rm eff}(\mathcal{Y})\longrightarrow H_k(\mathcal{Y}_{\rm top};\mathbb{Z})\end{aligned} \tag{13.6}$$

by $\Pi_{\rm eff}^{\rm orb}:[\mathcal{X},f]\mapsto[\mathcal{X},f]$ and $\Pi_{\rm orb}^{\rm hom},\Pi_{\rm eff}^{\rm hom}:[\mathcal{X},f]\mapsto(f_{\rm top})_*([\mathcal{X}])$, where $[\mathcal{X}]$ is the fundamental class of the compact, oriented k-orbifold $\mathcal{X}$, which lies in $H_k(\mathcal{X}_{\rm top};\mathbb{Q})$ for general $\mathcal{X}$, and in $H_k(\mathcal{X}_{\rm top};\mathbb{Z})$ for effective $\mathcal{X}$. The 1-morphism $f:\mathcal{X}\to\mathcal{Y}$ gives a continuous map $f_{\rm top}:\mathcal{X}_{\rm top}\to\mathcal{Y}_{\rm top}$, and so induces morphisms of homology groups $(f_{\rm top})_*:H_k(\mathcal{X}_{\rm top};\mathbb{Q})\to H_k(\mathcal{Y}_{\rm top};\mathbb{Q})$ and $(f_{\rm top})_*:H_k(\mathcal{X}_{\rm top};\mathbb{Z})\to H_k(\mathcal{Y}_{\rm top};\mathbb{Z})$.

If Y is a manifold and $\mathcal{Y}=F_{\mathbf{Man}}^{\mathbf{Orb}}(Y)$, define a morphism

$$\Pi_{\rm bo}^{\rm eff}:B_k(Y)\longrightarrow B_k^{\rm eff}(\mathcal{Y})\;\text{ by }\;\Pi_{\rm bo}^{\rm eff}:[X,f]\longmapsto\bigl[F_{\mathbf{Man}}^{\mathbf{Orb}}(X),F_{\mathbf{Man}}^{\mathbf{Orb}}(f)\bigr]. \tag{13.7}$$

The morphisms (13.6)–(13.7) commute with pushforwards g_*, and preserve intersection products $\bullet$ and fundamental classes $[\mathcal{Y}]$ when these are defined.

Remark 13.14. (a) In Remark 13.6(b), we defined cobordism groups $B^*(Y)$ for a compact manifold Y, and explained that cobordism is a generalized cohomology theory, with cup product $\cup:B^k(Y)\times B^l(Y)\to B^{k+l}(Y)$, identity $1_Y=[Y,{\rm id}_Y]\in B^0(Y)$, cap product $\cap:B^k(Y)\times B_l(Y)\to B_{l-k}(Y)$, and pullbacks $g^*:B^k(Z)\to B^k(Y)$. We now generalize this to orbifolds.

Consider the following condition on 1-morphisms $f:\mathcal{X}\to\mathcal{Y}$ in **Orb**:

(†) As in §C.6 we have a morphism $\Omega_f:f^*(T^*\mathcal{Y})\to T^*\mathcal{X}$ in ${\rm vect}(\mathcal{X})$. Let Γ be a finite group and $\lambda\in\Lambda_+^\Gamma$, so that we have an orbifold stratum $\mathcal{X}^{\Gamma,\lambda}$ and 1-morphism $O^{\Gamma,\lambda}(\mathcal{X}):\mathcal{X}^{\Gamma,\lambda}\to\mathcal{X}$. Thus we may form the pullback $O^{\Gamma,\lambda}(\mathcal{X})^*(\Omega_f):O^{\Gamma,\lambda}(\mathcal{X})^*(f^*(T^*\mathcal{Y}))\to O^{\Gamma,\lambda}(\mathcal{X})^*(T^*\mathcal{X})$. As in §C.9, Γ

acts on $O^{\Gamma,\lambda}(\mathcal{X})^*(f^*(T^*\mathcal{Y}))$ and $O^{\Gamma,\lambda}(\mathcal{X})^*(T^*\mathcal{X})$, decomposing them into trivial and nontrivial parts, and $O^{\Gamma,\lambda}(\mathcal{X})^*(\Omega_f)$ is Γ-equivariant, so it has a nontrivial part $O^{\Gamma,\lambda}(\mathcal{X})^*(\Omega_f)_{\rm nt} : f^*(T^*\mathcal{Y})_{\rm nt} \rightarrow (T^*\mathcal{X})_{\rm nt}$. We require that $O^{\Gamma,\lambda}(\mathcal{X})^*(\Omega_f)_{\rm nt}$ should be injective, for all such Γ,λ.

Roughly speaking, condition (†) says that if f maps an orbifold stratum $\mathcal{X}^{\Gamma,\lambda}$ to $\mathcal{Y}^{\Gamma,\lambda'}$ locally, then $f(\mathcal{X})$ intersects $\mathcal{Y}^{\Gamma,\lambda'}$ transversely in $\mathcal{Y}$.

The following properties are easy to verify:

(i) If $f,g : \mathcal{X} \rightarrow \mathcal{Y}$ are 1-morphisms and $\eta : f \Rightarrow g$ a 2-morphism in **Orb**, then f satisfies (†) if and only if g does.

(ii) If $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $g : \mathcal{Y} \rightarrow \mathcal{Z}$ satisfy (†) then $g \circ f : \mathcal{X} \rightarrow \mathcal{Z}$ satisfies (†).

(iii) If $\mathcal{X}$ or $\mathcal{Y}$ is a manifold then any $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies (†).

(iv) Suppose $g : \mathcal{X} \rightarrow \mathcal{Z}$ and $h : \mathcal{Y} \rightarrow \mathcal{Z}$ are transverse 1-morphisms in **Orb**, so that a fibre product $\mathcal{W} = \mathcal{X} \times_{g,\mathcal{Z},h} \mathcal{Y}$ exists with projections $e : \mathcal{W} \rightarrow \mathcal{X}$, $f : \mathcal{W} \rightarrow \mathcal{Y}$. If g satisfies (†) then f does, and if h satisfies (†) then e does.

(v) If $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies (†), then any small perturbation $\tilde{f} : \mathcal{X} \rightarrow \mathcal{Y}$ of f also satisfies (†). That is, (†) is an open condition on 1-morphisms $f : \mathcal{X} \rightarrow \mathcal{Y}$.

(vi) Suppose $g : \mathcal{X} \rightarrow \mathcal{Z}$ and $h : \mathcal{Y} \rightarrow \mathcal{Z}$ are 1-morphisms in **Orb** and g satisfies (†). If $\tilde{g} : \mathcal{X} \rightarrow \mathcal{Z}$ is a small generic perturbation of g, then $\tilde{g}$ and h are transverse, so that the fibre product $\mathcal{X} \times_{\tilde{g},\mathcal{Z},h} \mathcal{Y}$ exists in **Orb**.

Let $\mathcal{Y}$ be a compact orbifold of dimension n, not necessarily oriented. Consider pairs $(\mathcal{X}, f)$, where $\mathcal{X}$ is a compact orbifold with $\dim \mathcal{X} = n-k$, and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a 1-morphism satisfying (†) which is *cooriented*, that is, we are given an orientation on the line bundle $\Lambda^{n-k}T^*\mathcal{X} \otimes f^*(\Lambda^n T^*\mathcal{Y})^*$ over $\mathcal{X}$. Define an equivalence relation $\sim$ on such pairs by $(\mathcal{X}, f) \sim (\mathcal{X}', f')$ if there exists an orbifold with corners $\mathcal{W}$ and a cooriented 1-morphism $e : \mathcal{W} \rightarrow \mathcal{Y}$ satisfying (†) such that $e \circ i_{\mathcal{W}} : \partial\mathcal{W} \rightarrow \mathcal{Y}$ is equivalent to $-(f : \mathcal{X} \rightarrow \mathcal{Y}) \amalg (f' : \mathcal{X}' \rightarrow \mathcal{Y})$ in cooriented 1-morphisms. Define the *orbifold cobordism group* $B^k_{\rm orb}(\mathcal{Y})$ to be the set of $\sim$-equivalence classes $[\mathcal{X}, f]$ of such pairs $(\mathcal{X}, f)$.

Call a 1-morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ in **Orb** *coeffective* if whenever $[x] \in \mathcal{X}_{\rm top}$ with $f_{\rm top}([x]) = [y]$, so that we have morphisms $f_* : {\rm Iso}_{\mathcal{X}}([x]) \rightarrow {\rm Iso}_{\mathcal{Y}}([y])$ and $df|_x : T_x\mathcal{X} \rightarrow T_y\mathcal{Y}$, then ${\rm Ker}\bigl(f_* : {\rm Iso}_{\mathcal{X}}([x]) \rightarrow {\rm Iso}_{\mathcal{Y}}([y])\bigr)$ acts effectively on ${\rm Ker}\bigl(df|_x : T_x\mathcal{X} \rightarrow T_y\mathcal{Y}\bigr)$. Define *effective orbifold cobordism* $B^*_{\rm eff}(\mathcal{Y})$ as for $B^*_{\rm orb}(\mathcal{Y})$, but requiring $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $e : \mathcal{W} \rightarrow \mathcal{Y}$ to be coeffective.

We can now define *cup products* $\cup : B^k_{\rm orb}(\mathcal{Y}) \times B^l_{\rm orb}(\mathcal{Y}) \rightarrow B^{k+l}_{\rm orb}(\mathcal{Y})$ and $\cup : B^k_{\rm eff}(\mathcal{Y}) \times B^l_{\rm eff}(\mathcal{Y}) \rightarrow B^{k+l}_{\rm eff}(\mathcal{Y})$ as for $\bullet$ in (13.5), without requiring $\mathcal{Y}$ oriented. The important point in defining $[\mathcal{X}, f] \cup [\mathcal{X}', f']$ is that by (vi) above, as f, f' satisfy (†), we can perturb them in their cobordism classes so that they are transverse, and (13.5) makes sense. Also $\pi_{\mathcal{X}}$ and $f \circ \pi_{\mathcal{X}}$ satisfy (†) by (ii),(iv).

We define *identity elements* $1_{\mathcal{Y}} = [\mathcal{Y}, {\rm id}_{\mathcal{Y}}]$ in $B^0_{\rm orb}(\mathcal{Y}), B^0_{\rm eff}(\mathcal{Y})$. Then $\cup, 1_{\mathcal{Y}}$ make $B^*_{\rm orb}(\mathcal{Y}), B^*_{\rm eff}(\mathcal{Y})$ into associative, supercommutative rings. We define *cap products* $\cap : B^k_{\rm orb}(\mathcal{Y}) \times B^{\rm orb}_l(\mathcal{Y}) \rightarrow B^{\rm orb}_{l-k}(\mathcal{Y})$ and $\cap : B^k_{\rm eff}(\mathcal{Y}) \times B^{\rm eff}_l(\mathcal{Y}) \rightarrow B^{\rm eff}_{l-k}(\mathcal{Y})$

mixing bordism and cobordism as for $\cup$. These make $B_*^{\rm orb}(\mathcal{Y}), B_*^{\rm eff}(\mathcal{Y})$ into modules over $B^*_{\rm orb}(\mathcal{Y}), B^*_{\rm eff}(\mathcal{Y})$.

If $g : \mathcal{Y} \to \mathcal{Z}$ is a 1-morphism compact of orbifolds, we define *pullbacks* $g^* : B^k_{\rm orb}(\mathcal{Z}) \to B^k_{\rm orb}(\mathcal{Y})$ and $g^* : B^k_{\rm eff}(\mathcal{Z}) \to B^k_{\rm eff}(\mathcal{Y})$ by $g^*([\mathcal{X}, f]) = [\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}, \pi_{\mathcal{Y}}]$, provided f, g are transverse. This can be achieved by perturbing f in its cobordism class, by (vi) above.

Note that the condition (†) on f in classes $[\mathcal{X}, f] \in B^*_{\rm orb}(\mathcal{Y}), B^*_{\rm eff}(\mathcal{Y})$ is essential in defining $\cup, \cap$ and g^*, since otherwise we would not be able to perturb f in its cobordism class so that f, f' or f, g are transverse.

(b) If $\mathcal{Y}$ is a noncompact orbifold, then as in Remark 13.6(c) we can define compactly-supported and non-compactly-supported versions of all four (effective) orbifold (co)bordism theories.

(c) Let $\mathcal{Y}$ be a compact, oriented orbifold of dimension n, so that we have fundamental classes $[\mathcal{Y}] \in B_n^{\rm orb}(\mathcal{Y}), B_n^{\rm eff}(\mathcal{Y})$. Thus we have morphisms $\cap [\mathcal{Y}] : B^k_{\rm orb}(\mathcal{Y}) \to B^{\rm orb}_{n-k}(\mathcal{Y})$ and $\cap [\mathcal{Y}] : B^k_{\rm eff}(\mathcal{Y}) \to B^{\rm eff}_{n-k}(\mathcal{Y})$ mapping $\alpha \mapsto \alpha \cap [\mathcal{Y}]$. Essentially these map $[\mathcal{X}, f] \mapsto [\mathcal{X}, f]$, using the orientation on $\mathcal{Y}$ to convert the coorientation on f for $B^k_{\rm orb}(\mathcal{Y})$ to an orientation on $\mathcal{X}$ for $B^{\rm orb}_{n-k}(\mathcal{Y})$.

In Remark 13.6(b) we saw that for Y a compact oriented manifold, $\cap [Y] : B^k(Y) \to B_{n-k}(Y)$ is an isomorphism, giving Poincaré duality for (co)bordism of manifolds. In the orbifold case, $\cap [\mathcal{Y}] : B^k_{\rm orb}(\mathcal{Y}) \to B^{\rm orb}_{n-k}(\mathcal{Y})$ and $\cap [\mathcal{Y}] : B^k_{\rm eff}(\mathcal{Y}) \to B^{\rm eff}_{n-k}(\mathcal{Y})$ may not be isomorphisms, since f must satisfy (†) for $[\mathcal{X}, f]$ in $B^k_{\rm orb}(\mathcal{Y}), B^k_{\rm eff}(\mathcal{Y})$, but need not satisfy (†) for $[\mathcal{X}, f]$ in $B^{\rm orb}_{n-k}(\mathcal{Y}), B^{\rm eff}_{n-k}(\mathcal{Y})$. Thus, *Poincaré duality fails for (effective) orbifold (co)bordism for general compact oriented orbifolds* $\mathcal{Y}$. We could have corrected this by including condition (†) on f in the definition of $[\mathcal{X}, f] \in B_*^{\rm orb}(\mathcal{Y})$, but then pushforwards $g_* : B_*^{\rm orb}(\mathcal{Y}) \to B_*^{\rm orb}(\mathcal{Z})$ would be defined only for $g : \mathcal{Y} \to \mathcal{Z}$ satisfying (†).

If the orbifold $\mathcal{Y}$ is a manifold, then the condition (†) for $f : \mathcal{X} \to \mathcal{Y}$ is trivial, so $\cap [\mathcal{Y}] : B^k_{\rm orb}(\mathcal{Y}) \to B^{\rm orb}_{n-k}(\mathcal{Y})$ and $\cap [\mathcal{Y}] : B^k_{\rm eff}(\mathcal{Y}) \to B^{\rm eff}_{n-k}(\mathcal{Y})$ are isomorphisms in this case. So Poincaré duality does hold for (effective) orbifold (co)bordism for general compact oriented manifolds $\mathcal{Y}$.

This gives a perspective on why we could define $\bullet$ on $B_*^{\rm orb}(\mathcal{Y}), B_*^{\rm eff}(\mathcal{Y})$ in Definition 13.13 only when $\mathcal{Y}$ is a manifold. For manifolds Y, the intersection product $\bullet$ on $B_*(Y)$ is the image of the cup product $\cup$ on $B^*(Y)$ under the Poincaré duality isomorphism $B^k(Y) \cong B_{n-k}(Y)$. Thus we should only expect to define $\bullet$ when Poincaré duality holds.

(d) The projections $\Pi^{\rm hom}_{\rm orb}, \Pi^{\rm hom}_{\rm eff}$ in (13.6) map to the homology of the underlying topological space $\mathcal{Y}_{\rm top}$ of $\mathcal{Y}$. There is also an alternative notion of homology of an orbifold. Given an orbifold $\mathcal{Y}$, as in [2, §1.4], [84, §4] one can define a topological space $\mathcal{Y}_{\rm cla}$ called the *classifying space* of $\mathcal{Y}$, with a projection $\pi : \mathcal{Y}_{\rm cla} \to \mathcal{Y}_{\rm top}$. This has the following properties:

(i) $\mathcal{Y}_{\rm cla}$ is canonical *only up to homotopy.* Hence the homology $H_*(\mathcal{Y}_{\rm cla}; R)$ over any commutative ring R is canonical up to isomorphism. The projection $\pi_* : H_*(\mathcal{Y}_{\rm cla}; R) \to H_*(\mathcal{Y}_{\rm top}; R)$ is also canonical.

(ii) The fibre of $\pi : \mathcal{Y}_{\rm cla} \to \mathcal{Y}_{\rm top}$ over $[y] \in \mathcal{Y}_{\rm top}$ is a classifying space $B\,{\rm Iso}_{\mathcal{Y}}([y])$ for ${\rm Iso}_{\mathcal{Y}}([y])$. Since $H_*(BG;\mathbb{Q}) \cong H_*(*;\mathbb{Q})$ for any finite group G, this implies that $\pi_* : H_*(\mathcal{Y}_{\rm cla};\mathbb{Q}) \to H_*(\mathcal{Y}_{\rm top};\mathbb{Q})$ is an isomorphism. But $\pi_* : H_*(\mathcal{Y}_{\rm cla};\mathbb{Z}) \to H_*(\mathcal{Y}_{\rm top};\mathbb{Z})$ is not an isomorphism in general.

(iii) If $\mathcal{Y}$ is a manifold then $\pi : \mathcal{Y}_{\rm cla} \to \mathcal{Y}_{\rm top}$ is a homotopy equivalence, so $\pi_* : H_*(\mathcal{Y}_{\rm cla};R) \to H_*(\mathcal{Y}_{\rm top};R)$ is an isomorphism.

(iv) If $\mathcal{Y}$ is a compact, oriented, effective orbifold of dimension n then it has a fundamental class $[\mathcal{Y}]$ in $H_n(\mathcal{Y}_{\rm cla};\mathbb{Z})$, which projects to the fundamental class $[\mathcal{Y}] \in H_n(\mathcal{Y}_{\rm top};\mathbb{Z})$ used in Definition 13.13.

(v) If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism in **Orb** then there is a continuous map $f_{\rm cla} : \mathcal{X}_{\rm cla} \to \mathcal{Y}_{\rm cla}$, canonical only up to homotopy. So $(f_{\rm cla})_* : H_*(\mathcal{X}_{\rm cla};R) \to H_*(\mathcal{Y}_{\rm cla};R)$ is canonical.

Thus, as for $\Pi_{\rm eff}^{\rm hom}$ in (13.6), we can define $\Pi_{\rm eff}^{\rm cla} : B_k^{\rm eff}(\mathcal{Y}) \to H_k(\mathcal{Y}_{\rm cla};\mathbb{Z})$ by $\Pi_{\rm eff}^{\rm cla} : [\mathcal{X}, f] \mapsto (f_{\rm cla})_*([\mathcal{X}])$, and then $\Pi_{\rm eff}^{\rm hom} = \pi_* \circ \Pi_{\rm eff}^{\rm cla}$. There is no benefit in using $\mathcal{Y}_{\rm cla}$ rather than $\mathcal{Y}_{\rm top}$ for $\Pi_{\rm orb}^{\rm hom}$, since $H_*(\mathcal{Y}_{\rm cla};\mathbb{Q}) \cong H_*(\mathcal{Y}_{\rm top};\mathbb{Q})$.

Suppose $\mathcal{X}$ is a compact, effective orbifold, and $\mathcal{W}$ a compact orbifold with boundary, not necessarily effective, with $\partial\mathcal{W} \simeq \mathcal{X}$. Let $\mathcal{W}'$ be the maximal effective open suborbifold in $\mathcal{W}$. Then $\mathcal{W}'$ is a compact, effective orbifold with boundary, open and closed in $\mathcal{W}$, with $\partial\mathcal{W}' = \partial\mathcal{W} \simeq \mathcal{X}$. Using this we deduce:

Lemma 13.15. $\Pi_{\rm eff}^{\rm orb} : B_*^{\rm eff}(\mathcal{Y}) \to B_*^{\rm orb}(\mathcal{Y})$ *is injective for any orbifold* $\mathcal{Y}$.

Example 13.16. We will compute $B_0^{\rm orb}(*), B_0^{\rm eff}(*)$ when $\mathcal{Y}$ is the point $*$. One can show that compact, oriented orbifolds $\mathcal{X}$ of dimension 0 are equivalent to finite disjoint unions of orbifolds $\pm[\underline{*}/\Gamma]$ for Γ a finite group, where $\Gamma = \{1\}$ if $\mathcal{X}$ is effective. Similarly, compact, oriented orbifolds with boundary $\mathcal{W}$ of dimension 1 are equivalent to finite disjoint unions of components $[0,1] \times [\underline{*}/\Gamma]$ and $\mathcal{S}^1 \times [\underline{*}/\Gamma]$ for Γ a finite group, with $\Gamma = \{1\}$ for $\mathcal{W}$ effective. Therefore

$$B_0^{\rm orb}(*) = \bigoplus_{\text{iso. classes of finite groups } \Gamma} \mathbb{Z} \cdot \bigl[[\underline{*}/\Gamma], \pi\bigr], \quad B_0^{\rm eff}(*) = \mathbb{Z} \cdot [*, \pi], \tag{13.8}$$

writing $\pi : \mathcal{X} \to *$ for the unique 1-morphism for any orbifold $\mathcal{X}$.

Thus $B_0^{\rm orb}(*)$ is of infinite rank. This illustrates the principle that orbifold bordism groups $B^*_{\rm orb}(\mathcal{Y}), B^*_{\rm eff}(\mathcal{Y})$ tend to be larger than bordism groups $B_*(Y)$, because of the extra information stored in the orbifold strata $\mathcal{X}^{\Gamma,\lambda}$ for classes $[\mathcal{X}, f]$. We define functors $\Pi_{\rm orb}^{\Gamma,\lambda}, \tilde\Pi_{\rm orb}^{\Gamma,\mu}$ that extract some of this information.

Definition 13.17. Let Γ be a finite group, use the notation of Definitions 8.5 and 8.8, and choose orientations on $R_1, \ldots, R_k$ for representatives $(R_1, \rho_1), \ldots, (R_k, \rho_k)$ of the nontrivial, irreducible, even-dimensional Γ-representations as in §8.4.2. For each orbifold $\mathcal{Y}$, finite group Γ, and $\lambda \in \Lambda^\Gamma_{\rm ev,+}$, define a morphism

$$\Pi_{\rm orb}^{\Gamma,\lambda} : B_k^{\rm orb}(\mathcal{Y}) \to B_{k-\dim\lambda}^{\rm orb}(\mathcal{Y}) \text{ by } \Pi_{\rm orb}^{\Gamma,\lambda} : [\mathcal{X}, f] \mapsto [\mathcal{X}^{\Gamma,\lambda}, f \circ O^{\Gamma,\lambda}(\mathcal{X})],$$

where $\mathcal{X}^{\Gamma,\lambda}$ has the orientation given by Proposition 8.9. Note that $\mathcal{X}^{\Gamma,\lambda}$ is compact as $\mathcal{X}$ is and $O^{\Gamma,\lambda}(\mathcal{X}) : \mathcal{X}^{\Gamma,\lambda} \to \mathcal{X}$ is proper. If $(\mathcal{X}, f) \sim (\mathcal{X}', f')$ using $\mathcal{W}, e, j$ as in Definition 13.12, then using $\mathcal{W}^{\Gamma,\lambda}, e \circ O^{\Gamma,\lambda}(\mathcal{W}), j^{\Gamma,\lambda}$ we see that $\bigl(\mathcal{X}^{\Gamma,\lambda}, f \circ O^{\Gamma,\lambda}(\mathcal{X})\bigr) \sim \bigl(\mathcal{X}'^{\Gamma,\lambda}, f' \circ O^{\Gamma,\lambda}(\mathcal{X}')\bigr)$, so $\Pi^{\Gamma,\lambda}_{\mathrm{orb}}$ is well defined.

Now suppose $\lambda \in \Lambda^{\Gamma}_{\mathrm{ev},+}$ with $\Phi^{\Gamma}(\delta, \lambda) = 1$ for all $\delta \in \mathrm{Aut}(\Gamma)$ with $\lambda \cdot \delta = \lambda$. Set $\mu = \lambda \cdot \mathrm{Aut}(\Gamma) \in \Lambda^{\Gamma}_{\mathrm{ev},+}/\,\mathrm{Aut}(\Gamma)$, and define a morphism

$$\tilde{\Pi}^{\Gamma,\mu}_{\mathrm{orb}} : B^{\mathrm{orb}}_{k}(\mathcal{Y}) \to B^{\mathrm{orb}}_{k-\dim \mu}(\mathcal{Y}) \;\text{ by }\; \tilde{\Pi}^{\Gamma,\mu}_{\mathrm{orb}} : [\mathcal{X}, f] \mapsto [\tilde{\mathcal{X}}^{\Gamma,\mu}, f \circ \tilde{O}^{\Gamma,\mu}(\mathcal{X})],$$

where $\tilde{\mathcal{X}}^{\Gamma,\mu}$ has the orientation given by Proposition 8.10. The $\Pi^{\Gamma,\lambda}_{\mathrm{orb}}, \tilde{\Pi}^{\Gamma,\mu}_{\mathrm{orb}}$ commute with pushforwards $g_* : B^{\mathrm{orb}}_*(\mathcal{Y}) \to B^{\mathrm{orb}}_*(\mathcal{Z})$.

Remark 13.18. (a) We do not define functors $\Pi^{\Gamma,\lambda}_{\mathrm{eff}}, \tilde{\Pi}^{\Gamma,\mu}_{\mathrm{eff}} : B^{\mathrm{eff}}_*(\mathcal{Y}) \to B^{\mathrm{eff}}_*(\mathcal{Y})$, since $\mathcal{X}$ effective does not imply $\mathcal{X}^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}$ effective. But we can consider $\Pi^{\Gamma,\lambda}_{\mathrm{orb}} \circ \Pi^{\mathrm{orb}}_{\mathrm{eff}}$ and $\tilde{\Pi}^{\Gamma,\mu}_{\mathrm{orb}} \circ \Pi^{\mathrm{orb}}_{\mathrm{eff}} : B^{\mathrm{eff}}_*(\mathcal{Y}) \to B^{\mathrm{orb}}_*(\mathcal{Y})$.
(b) Suppose $\lambda \in \Lambda^{\Gamma}_{\mathrm{ev},+}$ with $\Phi^{\Gamma}(\delta, \lambda) = -1$ for some $\delta \in \mathrm{Aut}(\Gamma)$ with $\lambda \cdot \delta = \lambda$, and $\mu = \lambda \cdot \mathrm{Aut}(\Lambda)$. Then we have defined $\Pi^{\Gamma,\lambda}_{\mathrm{orb}}$, but not $\tilde{\Pi}^{\Gamma,\mu}_{\mathrm{orb}}$. In this case, for $[\mathcal{X}, f] \in B^{\mathrm{orb}}_*(\mathcal{Y})$, from §C.8 we see that $L^{\Gamma}(\delta, \mathcal{X})|_{\mathcal{X}^{\Gamma,\lambda}} : \mathcal{X}^{\Gamma,\lambda} \to \mathcal{X}^{\Gamma,\lambda}$ is an orientation-reversing 1-isomorphism with $O^{\Gamma,\lambda}(\mathcal{X}) \circ L^{\Gamma}(\delta, \mathcal{X})|_{\mathcal{X}^{\Gamma,\lambda}} = O^{\Gamma,\lambda}(\mathcal{X})$. It follows that $\Pi^{\Gamma,\lambda}_{\mathrm{orb}}([\mathcal{X}, f]) = -\Pi^{\Gamma,\lambda}_{\mathrm{orb}}([\mathcal{X}, f])$, so $\Pi^{\Gamma,\lambda}_{\mathrm{orb}}$ maps only to torsion elements of order 2 in $B^{\mathrm{orb}}_*(\mathcal{Y})$.
(c) In Definition 13.17, we can consider the functors

$$\begin{aligned} \Pi^{\mathrm{hom}}_{\mathrm{orb}} \circ \Pi^{\Gamma,\lambda}_{\mathrm{orb}} &: B^{\mathrm{orb}}_{k}(\mathcal{Y}) \longrightarrow H_{k-\dim \lambda}(\mathcal{Y}_{\mathrm{top}}; \mathbb{Q}), \\ \Pi^{\mathrm{hom}}_{\mathrm{orb}} \circ \tilde{\Pi}^{\Gamma,\mu}_{\mathrm{orb}} &: B^{\mathrm{orb}}_{k}(\mathcal{Y}) \longrightarrow H_{k-\dim \mu}(\mathcal{Y}_{\mathrm{top}}; \mathbb{Q}). \end{aligned} \tag{13.9}$$

Let $\lambda \in \Lambda^{\Gamma}_{\mathrm{ev},+}$. If $\Phi^{\Gamma}(\delta, \lambda) = -1$ for some $\delta \in \mathrm{Aut}(\Gamma)$ with $\lambda \cdot \delta = \lambda$ then (b) implies that $\Pi^{\mathrm{hom}}_{\mathrm{orb}} \circ \Pi^{\Gamma,\lambda}_{\mathrm{orb}} = 0$. Otherwise, setting $\mu = \lambda \cdot \mathrm{Aut}(\Lambda)$, as $\tilde{\mathcal{X}}^{\Gamma,\mu} \simeq [\mathcal{X}^{\Gamma,\lambda}/\Delta]$ for $\Delta = \{\delta \in \mathrm{Aut}(\Lambda) : \lambda \cdot \delta = \lambda\}$, we see that

$$\Pi^{\mathrm{hom}}_{\mathrm{orb}} \circ \tilde{\Pi}^{\Gamma,\mu}_{\mathrm{orb}} = \frac{1}{|\{\delta \in \mathrm{Aut}(\Lambda) : \lambda \cdot \delta = \lambda\}|} \cdot \Pi^{\mathrm{hom}}_{\mathrm{orb}} \circ \Pi^{\Gamma,\lambda}_{\mathrm{orb}}.$$

Thus the $\Pi^{\mathrm{hom}}_{\mathrm{orb}} \circ \tilde{\Pi}^{\Gamma,\mu}_{\mathrm{orb}}$ contain the same information as the $\Pi^{\mathrm{hom}}_{\mathrm{orb}} \circ \Pi^{\Gamma,\lambda}_{\mathrm{orb}}$. By showing that $\Pi^{\mathrm{hom}}_{\mathrm{orb}} \circ \Pi^{\Gamma,\lambda}_{\mathrm{orb}}$ and $\Pi^{\mathrm{hom}}_{\mathrm{orb}} \circ \Pi^{\Gamma,\lambda}_{\mathrm{orb}} \circ \Pi^{\mathrm{orb}}_{\mathrm{eff}}$ are nonzero for many Γ, λ, we can show that $B^{\mathrm{orb}}_*(\mathcal{Y}), B^{\mathrm{eff}}_*(\mathcal{Y})$ are large.

In §13.1 we discussed the determination of the bordism ring $B_*(*)$ of the point $*$ by Thom, Milnor, Wall and others. Druschel [28, 29] and Angel [3–5] study the effective orbifold bordism ring $B^{\mathrm{eff}}_*(*)$. We summarize their results:

Theorem 13.19. (a) (Druschel [28]). *The morphism $\Pi^{\mathrm{eff}}_{\mathrm{bo}} : B_*(*) \to B^{\mathrm{eff}}_*(*)$ from* (13.7) *induces a $\mathbb{Q}$-algebra morphism $B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q} \to B^{\mathrm{eff}}_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ is described in Theorem* 13.5. *As a $B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$-module we have*

$$B^{\mathrm{eff}}_*(*) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigl(B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}\bigr) \otimes_{\mathbb{Q}} \textstyle\bigoplus_{\Gamma \subset \mathrm{SO}(n)} H_*\bigl(B(N_{\mathrm{O}(n)}(\Gamma)/\Gamma); \hat{Q}\bigr)^*. \tag{13.10}$$

Here the sum is over conjugacy classes of finite subgroups $\Gamma \subset \mathrm{SO}(n)$ for $n \geqslant 0$ with $(\mathbb{R}^n)^\Gamma = \{0\}$, and $N_{\mathrm{O}(n)}(\Gamma)$ is the normalizer of Γ in $\mathrm{O}(n)$, and $B(N_{\mathrm{O}(n)}(\Gamma)/\Gamma)$ is the classifying space of the quotient subgroup $N_{\mathrm{O}(n)}(\Gamma)/\Gamma$, and $\hat{Q}$ is a local system on $B(N_{\mathrm{O}(n)}(\Gamma)/\Gamma)$ with fibre $\mathbb{Q}$ induced by orientations on the fibres of the universal $\mathbb{R}^n/\Gamma$-bundle over $B(N_{\mathrm{O}(n)}(\Gamma)/\Gamma)$.

Furthermore, $B_*^{\mathrm{eff}}(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a free commutative algebra over $B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$, generated by bases for $H_*\big(B(N_{\mathrm{O}(n)}(\Gamma)/\Gamma);\hat{Q}\big)$ for those $\Gamma \subset \mathrm{SO}(n)$ which do not split as $\Gamma_1 \times \Gamma_2$ for $\{1\} \neq \Gamma_1 \subset \mathrm{SO}(k)$ and $\{1\} \neq \Gamma_2 \subset \mathrm{SO}(n-k)$.

(b) (Druschel [28]). *$B_{2k+1}^{\mathrm{eff}}(*) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for all $k \geqslant 0$. In contrast to the manifold case, $B_{4k+2}^{\mathrm{eff}}(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ is not always zero, for example $B_{70}^{\mathrm{eff}}(*) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$.*

(c) (Druschel [29]). *$B_k^{\mathrm{eff}}(*) = \{0\}$ for $k = 1,2,3$.*

(d) (Angel [5]). *The kernel of $\Pi_{\mathrm{bo}}^{\mathrm{eff}} : B_*(*) \to B_*^{\mathrm{eff}}(*)$ is exactly the torsion (elements of finite order) in $B_*(*)$.*

Here is how part (a) is proved. Druschel [28, §2] defines Γ-*characteristic numbers.* These are additive maps $B_*(*) \to \mathbb{Q}$ which (loosely) map $[\mathcal{X},\pi]$ to the integral over an orbifold stratum $\tilde{\mathcal{X}}^{\Gamma,\mu}$ of a product of Pontryagin classes of $T\tilde{\mathcal{X}}^{\Gamma,\mu}$, and characteristic classes of the normal bundle $(\widetilde{T\mathcal{X}})_{\mathrm{nt}}^{\Gamma,\mu}$ of $\tilde{\mathcal{X}}^{\Gamma,\mu}$ in $\mathcal{X}$. These characteristic classes of $(\widetilde{T\mathcal{X}})_{\mathrm{nt}}^{\Gamma,\mu}$ are classified by $H_*\big(B(N_{\mathrm{O}(n)}(\Gamma)/\Gamma);\hat{Q}\big)$ in (13.10). A simple example of a Γ-characteristic number is the projection $\Pi_{\mathrm{orb}}^{\mathrm{hom}} \circ \tilde{\Pi}_{\mathrm{orb}}^{\Gamma,\mu} : B_{\dim \mu}^{\mathrm{orb}}(*) \to H_0(*;\mathbb{Q}) \cong \mathbb{Q}$ from (13.9).

Druschel then shows that a certain set O_n of n-dimensional Γ-characteristic numbers are linearly independent on $B_n^{\mathrm{eff}}(*) \otimes_{\mathbb{Z}} \mathbb{Q}$, and the vanishing of these on $[\mathcal{X},\pi] \in B_n^{\mathrm{eff}}(*)$ is a necessary and sufficient condition for $[\mathcal{X},\pi]$ to be torsion in $B_n^{\mathrm{eff}}(*)$. Hence O_n is a basis for $(B_n^{\mathrm{eff}}(*) \otimes_{\mathbb{Z}} \mathbb{Q})^*$, and (13.10) follows.

The last part of (b) is proved by producing an explicit class $[\mathcal{X},\pi] \in B_{70}(*)$ with $\Pi_{\mathrm{orb}}^{\mathrm{hom}} \circ \tilde{\Pi}_{\mathrm{orb}}^{S_8,\mu}([\mathcal{X},\pi]) \neq 0$ in $H_0(*;\mathbb{Q})$, where μ is the class of the S_8-representation $(\mathbb{R}^{70},\rho)$ in Example 8.12.

Remark 13.20. We can also consider orbifold versions of the other bordism theories $MO_*(Y), MU_*(Y), MSU_*(Y), MSp_*(Y)$ of Remark 13.6(d). For unoriented bordism $MO_*(Y)$, note that any compact orbifold $\mathcal{X}$ is the boundary of the compact orbifold with boundary $\mathcal{X}\times([-1,1]/\mathbb{Z}_2)$. Thus, unoriented orbifold bordism of arbitrary compact (effective) orbifolds is zero. So to get a nontrivial theory, we should impose extra conditions. Angel [3, 4] studies the bordism rings of unoriented but locally orientable effective orbifolds, and of unoriented effective orbifolds with orbifold groups of odd order.

There are also orbifold generalizations $MU_*^{\mathrm{orb}}(\mathcal{Y}), MSU_*^{\mathrm{orb}}(\mathcal{Y}), MSp_*^{\mathrm{orb}}(\mathcal{Y})$. In §8.4.2 we saw that the orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}$ of an oriented orbifold $\mathcal{X}$ are oriented only under conditions on Γ,λ,μ, so the operators $\Pi_{\mathrm{orb}}^{\Gamma,\lambda}, \tilde{\Pi}_{\mathrm{orb}}^{\Gamma,\mu}$ in Definition 13.17 are defined only under these conditions. For $MU_*^{\mathrm{orb}}(\mathcal{Y})$, in which the orbifold $\mathcal{X}$ in $[\mathcal{X},f]$ has a stable almost complex structure J in $\mathrm{End}(T\mathcal{X} \oplus \mathbb{R}^k)$, we can use J to orient all orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}$. Thus $\Pi_{\mathrm{orb}}^{\Gamma,\lambda}, \tilde{\Pi}_{\mathrm{orb}}^{\Gamma,\mu}$ are defined on $MU_*^{\mathrm{orb}}(\mathcal{Y})$ for all Γ,λ,μ.

13.4 Bordism for d-orbifolds

We combine the ideas of §13.2 and §13.3 to define bordism for d-orbifolds.

Definition 13.21. As in §12.1, we have isomorphic 2-categories $\mathbf{dOrb}$, with objects $\boldsymbol{\mathcal{X}}$, and $\mathbf{d\bar{O}rb}$, with objects $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \emptyset, \emptyset, \emptyset)$, with isomorphism $\mathbf{dOrb} \to \mathbf{d\bar{O}rb}$ mapping $\boldsymbol{\mathcal{X}} \mapsto (\boldsymbol{\mathcal{X}}, \emptyset, \emptyset, \emptyset)$, where $\mathbf{d\bar{O}rb} \subset \mathbf{dOrb^b} \subset \mathbf{dOrb^c}$. For simplicity we identify $\mathbf{dOrb}$ and $\mathbf{d\bar{O}rb}$, writing objects of both as $\boldsymbol{\mathcal{X}}$.

Let $\mathcal{Y}$ be an orbifold, and $k \in \mathbb{Z}$. Consider pairs $(\boldsymbol{\mathcal{X}}, \boldsymbol{f})$, where $\boldsymbol{\mathcal{X}} \in \mathbf{dOrb}$ is a compact, oriented d-orbifold without boundary with $\operatorname{vdim} \boldsymbol{\mathcal{X}} = k$, and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism in $\mathbf{dOrb}$, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$.

Define a binary relation $\sim$ between such pairs by $(\boldsymbol{\mathcal{X}}, \boldsymbol{f}) \sim (\boldsymbol{\mathcal{X}}', \boldsymbol{f}')$ if there exists a compact, oriented d-orbifold with boundary $\boldsymbol{\mathcal{W}}$ with $\operatorname{vdim} \boldsymbol{\mathcal{W}} = k+1$, a 1-morphism $\boldsymbol{e} : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb^b}$, an equivalence of oriented d-orbifolds $\boldsymbol{j} : -\boldsymbol{\mathcal{X}} \amalg \boldsymbol{\mathcal{X}}' \to \partial \boldsymbol{\mathcal{W}}$, and a 2-morphism $\eta : \boldsymbol{f} \amalg \boldsymbol{f}' \Rightarrow \boldsymbol{e} \circ \boldsymbol{i_{\mathcal{W}}} \circ \boldsymbol{j}$. As in Proposition 13.8 we can show $\sim$ is an equivalence relation, which we call *d-bordism*.

Write $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$ for the $\sim$-equivalence class (*d-bordism class*) of a pair $(\boldsymbol{\mathcal{X}}, \boldsymbol{f})$. For each $k \in \mathbb{Z}$, define the k^{th} *d-orbifold bordism group* $dB_k^{\mathrm{orb}}(\mathcal{Y})$ of $\mathcal{Y}$ to be the set of all such d-bordism classes $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$ with $\operatorname{vdim} \boldsymbol{\mathcal{X}} = k$. We give $dB_k^{\mathrm{orb}}(\mathcal{Y})$ the structure of an abelian group, with zero element $0_{\mathcal{Y}} = [\boldsymbol{\emptyset}, \boldsymbol{\emptyset}]$, addition $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}] + [\boldsymbol{\mathcal{X}}', \boldsymbol{f}'] = [\boldsymbol{\mathcal{X}} \amalg \boldsymbol{\mathcal{X}}', \boldsymbol{f} \amalg \boldsymbol{f}']$, and additive inverses $-[\boldsymbol{\mathcal{X}}, \boldsymbol{f}] = [-\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$.

Similarly, define the *semieffective d-orbifold bordism group* $dB_k^{\mathrm{sef}}(\mathcal{Y})$ and the *effective d-orbifold bordism group* $dB_k^{\mathrm{eff}}(\mathcal{Y})$ as above, but taking $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{W}}$ to be semieffective, or effective, respectively, in the sense of §10.9 and §12.10.

Definition 13.22. Let $\mathcal{Y}$ be an oriented orbifold of dimension n. As in (13.3), define the *intersection product* $\bullet : dB_k^{\mathrm{orb}}(\mathcal{Y}) \times dB_l^{\mathrm{orb}}(\mathcal{Y}) \to dB_{k+l-n}^{\mathrm{orb}}(\mathcal{Y})$ by

$$[\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \bullet [\boldsymbol{\mathcal{X}}', \boldsymbol{f}'] = [\boldsymbol{\mathcal{X}} \times_{\boldsymbol{f}, \boldsymbol{\mathcal{Y}}, \boldsymbol{f}'} \boldsymbol{\mathcal{X}}', \boldsymbol{f} \circ \boldsymbol{\pi_{\mathcal{X}}}]. \tag{13.11}$$

Here $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{f}, \boldsymbol{\mathcal{Y}}, \boldsymbol{f}'} \boldsymbol{\mathcal{X}}'$ exists in $\mathbf{dOrb}$ by Theorem 10.28(a), and is oriented by Theorem 10.37. Then $\bullet$ is biadditive, supercommutative and associative.

Now $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{X}}'$ (semi)effective do not imply $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{\mathcal{Y}}} \boldsymbol{\mathcal{X}}'$ is (semi)effective unless $\mathcal{Y}$ is a manifold. So, as for $\bullet$ on $B_*^{\mathrm{orb}}(\mathcal{Y}), B_*^{\mathrm{eff}}(\mathcal{Y})$ in Definition 13.13, suppose $\mathcal{Y}$ is a manifold, and define intersection products $\bullet : dB_k^{\mathrm{sef}}(\mathcal{Y}) \times dB_l^{\mathrm{sef}}(\mathcal{Y}) \to dB_{k+l-n}^{\mathrm{sef}}(\mathcal{Y})$ and $\bullet : dB_k^{\mathrm{eff}}(\mathcal{Y}) \times dB_l^{\mathrm{eff}}(\mathcal{Y}) \to dB_{k+l-n}^{\mathrm{eff}}(\mathcal{Y})$ by (13.11).

If $\mathcal{Y}$ is compact and oriented, define the *fundamental class* $[\mathcal{Y}]$ in $dB_n^{\mathrm{orb}}(\mathcal{Y})$ and $dB_n^{\mathrm{sef}}(\mathcal{Y})$ by $[\mathcal{Y}] = [\boldsymbol{\mathcal{Y}}, \mathbf{id}_{\boldsymbol{\mathcal{Y}}}]$. If $\mathcal{Y}$ is also effective, define $[\mathcal{Y}] = [\boldsymbol{\mathcal{Y}}, \mathrm{id}_{\boldsymbol{\mathcal{Y}}}] \in dB_n^{\mathrm{eff}}(\mathcal{Y})$. Then $[\mathcal{Y}]$ is the identity for $\bullet$, when this is defined.

Let $g : \mathcal{Y} \to \mathcal{Z}$ be a 1-morphism of orbifolds. Define *pushforwards* $g_* : dB_k^{\mathrm{orb}}(\mathcal{Y}) \to dB_k^{\mathrm{orb}}(\mathcal{Z})$, $g_* : dB_k^{\mathrm{sef}}(\mathcal{Y}) \to dB_k^{\mathrm{sef}}(\mathcal{Z})$ and $g_* : dB_k^{\mathrm{eff}}(\mathcal{Y}) \to dB_k^{\mathrm{eff}}(\mathcal{Z})$ by $g_* : [\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \mapsto [\boldsymbol{\mathcal{X}}, \boldsymbol{g} \circ \boldsymbol{f}]$, where $\boldsymbol{g} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(g)$.

If $\mathcal{Y}$ is an orbifold, define group morphisms

$$\begin{aligned} \Pi_{\mathrm{orb}}^{\mathrm{sef}} &: B_k^{\mathrm{orb}}(\mathcal{Y}) \longrightarrow dB_k^{\mathrm{sef}}(\mathcal{Y}), \quad \Pi_{\mathrm{eff}}^{\mathrm{deff}} : B_k^{\mathrm{eff}}(\mathcal{Y}) \longrightarrow dB_k^{\mathrm{eff}}(\mathcal{Y}), \\ \Pi_{\mathrm{deff}}^{\mathrm{sef}} &: dB_k^{\mathrm{eff}}(\mathcal{Y}) \longrightarrow dB_k^{\mathrm{sef}}(\mathcal{Y}), \quad \Pi_{\mathrm{deff}}^{\mathrm{dorb}} : dB_k^{\mathrm{eff}}(\mathcal{Y}) \longrightarrow dB_k^{\mathrm{orb}}(\mathcal{Y}), \\ &\text{and} \qquad \Pi_{\mathrm{sef}}^{\mathrm{dorb}} : dB_k^{\mathrm{sef}}(\mathcal{Y}) \longrightarrow dB_k^{\mathrm{orb}}(\mathcal{Y}) \end{aligned} \tag{13.12}$$

by $\Pi_{\rm orb}^{\rm sef}, \Pi_{\rm eff}^{\rm deff} : [\mathcal{X}, f] \mapsto [\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$, where $\boldsymbol{\mathcal{X}}, \boldsymbol{f} = F_{\bf Orb}^{\bf dOrb}(\mathcal{X}, f)$, and $\Pi_{\rm deff}^{\rm sef}, \Pi_{\rm deff}^{\rm dorb}$, $\Pi_{\rm sef}^{\rm dorb} : [\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \mapsto [\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$. The morphisms (13.12) commute with pushforwards g_* and preserve $\bullet, [\mathcal{Y}]$ when these are defined.

Here is the main result of this section, an orbifold analogue of Theorem 13.11. It will be proved in §13.5. The key idea is that semieffective (or effective) d-orbifolds can be perturbed to (effective) orbifolds, as in §10.9; to make this rigorous, we use good coordinate systems on $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{W}}$, as in §10.8 and §12.9.

Theorem 13.23. *For any orbifold $\mathcal{Y}$, the maps $\Pi_{\rm orb}^{\rm sef} : B_k^{\rm orb}(\mathcal{Y}) \to dB_k^{\rm sef}(\mathcal{Y})$ and $\Pi_{\rm eff}^{\rm deff} : B_k^{\rm eff}(\mathcal{Y}) \to dB_k^{\rm eff}(\mathcal{Y})$ in* (13.12) *are isomorphisms for all $k \in \mathbb{Z}$.*

Combined with §13.3, this gives us some understanding of (semi)effective d-orbifold bordism $dB_*^{\rm sef}(\mathcal{Y}), dB_*^{\rm eff}(\mathcal{Y})$, and complete descriptions of $dB_0^{\rm sef}(*)$ and $dB_*^{\rm eff}(*)$. As for (13.4), the theorem implies that we may define projections

$$\begin{gathered} \Pi_{\rm sef}^{\rm hom} : dB_k^{\rm sef}(\mathcal{Y}) \to H_k(\mathcal{Y}_{\rm top}; \mathbb{Q}), \ \ \Pi_{\rm deff}^{\rm hom} : dB_k^{\rm eff}(\mathcal{Y}) \to H_k(\mathcal{Y}_{\rm top}; \mathbb{Z}) \\ \text{by } \Pi_{\rm sef}^{\rm hom} = \Pi_{\rm orb}^{\rm hom} \circ (\Pi_{\rm orb}^{\rm sef})^{-1} \ \text{ and } \ \Pi_{\rm deff}^{\rm hom} = \Pi_{\rm eff}^{\rm hom} \circ (\Pi_{\rm eff}^{\rm deff})^{-1}. \end{gathered} \tag{13.13}$$

We think of these $\Pi_{\rm sef}^{\rm hom}, \Pi_{\rm deff}^{\rm hom}$ as *virtual class maps* on $dB_*^{\rm sef}(\mathcal{Y}), dB_*^{\rm eff}(\mathcal{Y})$. In fact, with more work, one can also define virtual class maps on $dB_*^{\rm orb}(\mathcal{Y})$:

$$\Pi_{\rm dorb}^{\rm hom} : dB_k^{\rm orb}(\mathcal{Y}) \longrightarrow H_k(\mathcal{Y}_{\rm top}; \mathbb{Q}), \tag{13.14}$$

satisfying $\Pi_{\rm dorb}^{\rm hom} \circ \Pi_{\rm sef}^{\rm dorb} = \Pi_{\rm sef}^{\rm hom}$, for instance following the method of Fukaya et al. [34, §6], [32, §A1] for virtual cycles of Kuranishi spaces using 'multisections'.

Virtual classes (or virtual cycles, or virtual chains) are used in many areas of geometry to construct enumerative invariants using moduli spaces. In algebraic geometry, Behrend and Fantechi [12] construct virtual classes for schemes with obstruction theories. In symplectic geometry, there are many versions — see for example Fukaya et al. [34, §6], [32, §A1], Hofer et al. [46], and McDuff [77].

The main message we want to draw from this is that *compact oriented d-orbifolds admit virtual classes*. Thus, we can use d-manifolds and d-orbifolds as the geometric structure on moduli spaces in enumerative invariant problems such as Gromov–Witten invariants, Lagrangian Floer cohomology, Donaldson–Thomas invariants, ..., as this structure is strong enough to contain all the 'counting' information.

In future work the author intends to define a virtual chain construction for d-manifolds and d-orbifolds, expressed in terms of new (co)homology theories whose (co)chains are built from d-manifolds or d-orbifolds, as for the 'Kuranishi (co)homology' described in [53, 54].

Here is the analogue of Definition 13.17 for d-orbifold bordism.

Definition 13.24. Let Γ be a finite group, use the notation of §8.4.1–§8.4.2, and choose orientations on $R_1, \ldots, R_k$ for $(R_1, \rho_1), \ldots, (R_k, \rho_k)$ the nontrivial, irreducible, even-dimensional Γ-representations. Let $\mathcal{Y}$ be an orbifold.

For each finite group Γ with $|\Gamma|$ odd and each $\lambda \in \Lambda^\Gamma$, define a morphism

$$\Pi_{\rm dorb}^{\Gamma,\lambda} : dB_k^{\rm orb}(\mathcal{Y}) \to dB_{k-\dim\lambda}^{\rm orb}(\mathcal{Y}) \ \text{ by } \ \Pi_{\rm dorb}^{\Gamma,\lambda} : [\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \mapsto [\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \boldsymbol{f} \circ \boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}})],$$

where $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ has the orientation given by Proposition 10.43. Suppose also that $\Phi^\Gamma(\delta,\lambda)=1$ for all $\delta\in\mathrm{Aut}(\Gamma)$ with $\lambda\cdot\delta=\lambda$. Set $\mu=\lambda\cdot\mathrm{Aut}(\Gamma)\in\Lambda^\Gamma/\mathrm{Aut}(\Gamma)$, and define a morphism

$$\tilde\Pi^{\Gamma,\mu}_{\rm dorb}:dB^{\rm orb}_k(\mathcal{Y})\to dB^{\rm orb}_{k-\dim\mu}(\mathcal{Y})\;\text{ by }\;\tilde\Pi^{\Gamma,\mu}_{\rm dorb}:[\boldsymbol{\mathcal{X}},\boldsymbol{f}]\mapsto[\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\boldsymbol{f}\circ\tilde{\boldsymbol{O}}^{\Gamma,\mu}(\boldsymbol{\mathcal{X}})],$$

where $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$ has the orientation given by Proposition 10.44.

For each finite group Γ (allowing $|\Gamma|$ even) and each $\lambda\in\Lambda^\Gamma_{\rm ev,+}$, define

$$\Pi^{\Gamma,\lambda}_{\rm sef}:dB^{\rm sef}_k(\mathcal{Y})\to dB^{\rm sef}_{k-\dim\lambda}(\mathcal{Y})\;\text{ by }\;\Pi^{\Gamma,\lambda}_{\rm sef}:[\boldsymbol{\mathcal{X}},\boldsymbol{f}]\mapsto[\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\boldsymbol{f}\circ\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}})],$$

where $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ has the orientation given by Proposition 10.64, noting that $\boldsymbol{\mathcal{X}}$ is semieffective. Suppose also that $\Phi^\Gamma(\delta,\lambda)=1$ for all $\delta\in\mathrm{Aut}(\Gamma)$ with $\lambda\cdot\delta=\lambda$. Set $\mu=\lambda\cdot\mathrm{Aut}(\Gamma)\in\Lambda^\Gamma_{\rm ev,+}/\mathrm{Aut}(\Gamma)$, and define a morphism

$$\tilde\Pi^{\Gamma,\mu}_{\rm sef}:dB^{\rm sef}_k(\mathcal{Y})\to dB^{\rm sef}_{k-\dim\mu}(\mathcal{Y})\;\text{ by }\;\tilde\Pi^{\Gamma,\mu}_{\rm sef}:[\boldsymbol{\mathcal{X}},\boldsymbol{f}]\mapsto[\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu},\boldsymbol{f}\circ\tilde{\boldsymbol{O}}^{\Gamma,\mu}(\boldsymbol{\mathcal{X}})],$$

where $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$ has the orientation given by Proposition 10.65.

These $\Pi^{\Gamma,\lambda}_{\rm dorb},\tilde\Pi^{\Gamma,\mu}_{\rm dorb},\Pi^{\Gamma,\lambda}_{\rm sef},\tilde\Pi^{\Gamma,\mu}_{\rm sef}$ commute with pushforwards g_*.

As in Remark 13.18(a), we do not define $\Pi^{\Gamma,\lambda}_{\rm deff},\tilde\Pi^{\Gamma,\mu}_{\rm deff}:dB^{\rm eff}_*(\mathcal{Y})\to dB^{\rm eff}_*(\mathcal{Y})$, since $\boldsymbol{\mathcal{X}}$ effective does not imply $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda},\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$ effective.

Example 13.25. Let Γ be a finite group with $|\Gamma|$ odd, and choose $\lambda\in\Lambda^\Gamma_+$ with $\Phi^\Gamma(\delta,\lambda)=1$ for all $\delta\in\mathrm{Aut}(\Gamma)$ with $\lambda\cdot\delta=\lambda$, for instance, $\lambda=2\lambda'$ for any $\lambda'\in\Lambda^\Gamma_+$ will do. Set $\mu=\lambda\cdot\mathrm{Aut}(\Gamma)\in\Lambda^\Gamma/\mathrm{Aut}(\Gamma)$. Let (R,ρ) be a nontrivial Γ-representation with $[(R,\rho)]=\lambda$, and choose an orientation on R.

Form the fibre product $*\times_{\boldsymbol{0},\boldsymbol{R},\boldsymbol{0}}*$ in **dMan**, $0:*\to R$ maps $0:*\to 0$ and $*,\boldsymbol{0},\boldsymbol{R}=F^{\mathbf{dMan}}_{\mathbf{Man}}(*,0,R)$. This is a single point with obstruction space R, and is a compact oriented d-manifold with virtual dimension $-\dim\mu$. The Γ-action ρ on R and the trivial action of Γ on 0 induce Γ-action on $*\times_{\boldsymbol{0},\boldsymbol{R},\boldsymbol{0}}*$ preserving orientations, so $\boldsymbol{\mathcal{X}}_{\Gamma,\mu}:=[*\times_{\boldsymbol{0},\boldsymbol{R},\boldsymbol{0}}*/\Gamma]$ is a compact oriented d-orbifold, with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}_{\Gamma,\mu}=-\dim\lambda$. Hence $[\boldsymbol{\mathcal{X}}_{\Gamma,\mu},\boldsymbol{\pi}]\in dB^{\rm orb}_{-\dim\mu}(*)$, where $\boldsymbol{\pi}:\boldsymbol{\mathcal{X}}_{\Gamma,\mu}\to *$. The orbifold stratum $(\widetilde{\boldsymbol{\mathcal{X}}_{\Gamma,\mu}})^{\Gamma,\mu}$ is $[*/\Gamma]$, so $\tilde\Pi^{\Gamma,\mu}_{\rm dorb}:dB^{\rm orb}_*(*)\to dB^{\rm orb}_*(*)$ in Definition 13.24 maps $\tilde\Pi^{\Gamma,\mu}_{\rm dorb}:[\boldsymbol{\mathcal{X}}_{\Gamma,\mu},\boldsymbol{\pi}]\mapsto\big[[*/\Gamma],\boldsymbol{\pi}\big]\in dB^{\rm orb}_0(*)$.

We now *assume* there is a virtual class map $\Pi^{\rm hom}_{\rm dorb}:dB^{\rm orb}_0(*)\to H_0(*;\mathbb{Q})\cong\mathbb{Q}$ as in (13.14) satisfying $\Pi^{\rm hom}_{\rm dorb}\circ\Pi^{\rm dorb}_{\rm sef}=\Pi^{\rm hom}_{\rm sef}:dB^{\rm sef}_0(*)\to H_0(*;\mathbb{Q})$, although we have not proved this. It then follows that

$$\Pi^{\rm hom}_{\rm dorb}\circ\tilde\Pi^{\Gamma,\mu}_{\rm dorb}\big([\boldsymbol{\mathcal{X}}_{\Gamma,\mu},\boldsymbol{\pi}]\big)=|\Gamma|^{-1}\in\mathbb{Q}\cong H_0(*;\mathbb{Q}).\tag{13.15}$$

Hence $[\boldsymbol{\mathcal{X}}_{\Gamma,\mu},\boldsymbol{\pi}]$ is nonzero in $dB^{\rm orb}_{-\dim\mu}(*)$, and also in $dB^{\rm orb}_{-\dim\mu}(*)\otimes_{\mathbb{Z}}\mathbb{Q}$.

Let Γ',λ',μ' be alternative choices for Γ,λ,μ with $\dim\mu'=\dim\mu$. Then

$$\Pi^{\rm hom}_{\rm dorb}\circ\tilde\Pi^{\Gamma',\mu'}_{\rm dorb}\big([\boldsymbol{\mathcal{X}}_{\Gamma,\mu},\boldsymbol{\pi}]\big)=0\;\text{ unless }|\Gamma'|<|\Gamma|\text{ or }(\Gamma,\mu)\cong(\Gamma',\mu'),\tag{13.16}$$

since if $|\Gamma'|>|\Gamma|$ or $|\Gamma'|=|\Gamma|$ and $(\Gamma,\mu)\not\cong(\Gamma',\mu')$ then $(\widetilde{\boldsymbol{\mathcal{X}}_{\Gamma,\mu}})^{\Gamma',\mu'}=\boldsymbol{\emptyset}$.

Equations (13.15)–(13.16) imply:

Corollary 13.26. *Taken over all isomorphism classes of finite groups* Γ *with* $|\Gamma|$ *odd, and all* $\mu = \lambda \cdot \mathrm{Aut}(\Gamma) \in \Lambda_+^\Gamma / \mathrm{Aut}(\Gamma)$ *with* $\Phi^\Gamma(\delta, \lambda) = 1$ *for all* $\delta \in \mathrm{Aut}(\Gamma)$ *with* $\lambda \cdot \delta = \lambda$, *the elements* $[\boldsymbol{\mathcal{X}}_{\Gamma,\mu}, \boldsymbol{\pi}] \in dB^{\mathrm{orb}}_{-\dim \mu}(*)$ *are linearly independent over* $\mathbb{Z}$ *in* $dB^{\mathrm{orb}}_*(*)$. *When* $\mu = 0$, *we have* $[\boldsymbol{\mathcal{X}}_{\Gamma,0}, \boldsymbol{\pi}] = [*/\Gamma, \boldsymbol{\pi}] \in dB^{\mathrm{orb}}_0(*)$.

This implies that $dB^{\mathrm{orb}}_{4k}(*)$ and $dB^{\mathrm{orb}}_{4k}(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ have infinite rank for all $k \leqslant 0$. In contrast, $dB^{\mathrm{sef}}_k(*) = dB^{\mathrm{eff}}_k(*) = 0$ for all $k < 0$ by Theorem 13.23.

Example 13.27. Let $V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 1\}$, as a manifold with boundary, let $E = \mathbb{R} \times V \to V$, and define $s \in C^\infty(E)$ by $s(x_1, x_2) = x_1 x_2$. Define actions $r, \hat{r}$ of $\mathbb{Z}_2 = \{1, \sigma\}$ on V and $E = \mathbb{R} \times V$ by

$$r(\sigma) : (x_1, x_2) \longmapsto (x_1, -x_2), \quad \hat{r}(\sigma) : (e, x_1, x_2) \longmapsto (-e, x_1, -x_2).$$

Then $\mathbf{S}_{V,E,s}$ is a compact oriented d-manifold with boundary of virtual dimension 1. The $\mathbb{Z}_2$-actions $r, \hat{r}$ induce a $\mathbb{Z}_2$-action on $\boldsymbol{S}_{V,E,s}$ preserving the orientation, so $\boldsymbol{\mathcal{W}} = [\mathbf{S}_{V,E,s}/\mathbb{Z}_2]$ is a compact oriented d-orbifold with boundary.

The boundary $\partial \mathbf{S}_{V,E,s}$ has underlying topological space

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1, \ x_1 x_2 = 0\} = \{(1,0), (-1,0), (0,1), (0,-1)\},$$

and as $\mathrm{d}s \neq 0$ at these four points, $\partial \mathbf{S}_{V,E,s}$ is equivalent in **dMan** to four points $*$, where $(\pm 1, 0)$ are positively oriented, and $(0, \pm 1)$ are negatively oriented. The $\mathbb{Z}_2$-action fixes $(\pm 1, 0)$ and swaps $(0,1)$ and $(0,-1)$. Hence in compact oriented d-orbifolds we have $\partial \boldsymbol{\mathcal{W}} \simeq [*/\mathbb{Z}_2] \amalg [*/\mathbb{Z}_2] \amalg -*$. Therefore, using this $\boldsymbol{\mathcal{W}}$ and $\boldsymbol{e} = \boldsymbol{\pi} : \boldsymbol{\mathcal{W}} \to *$ in the definition of $\sim$ in Definition 13.21, we see that

$$2\big[[*/\mathbb{Z}_2], \boldsymbol{\pi}\big] = [*, \boldsymbol{\pi}] \in dB^{\mathrm{orb}}_0(*).$$

From (13.8) and Theorem 13.23 we see that

$$dB^{\mathrm{sef}}_0(*) = \bigoplus_{\text{iso. classes of finite groups } \Gamma} \mathbb{Z} \cdot \big[[*/\Gamma], \boldsymbol{\pi}\big], \quad dB^{\mathrm{eff}}_0(*) = \mathbb{Z} \cdot [*, \boldsymbol{\pi}].$$

In particular, $[[*/\mathbb{Z}_2], \boldsymbol{\pi}]$ and $[*, \boldsymbol{\pi}]$ are linearly independent over $\mathbb{Z}$ in $dB^{\mathrm{sef}}_0(*)$. Therefore $2[[*/\mathbb{Z}_2], \boldsymbol{\pi}] - [*, \boldsymbol{\pi}] \in dB^{\mathrm{sef}}_0(*)$ is a nonzero element of the kernel of $\Pi^{\mathrm{dorb}}_{\mathrm{sef}} : dB^{\mathrm{sef}}_0(*) \to dB^{\mathrm{orb}}_0(*)$ in (13.12). Thus $\Pi^{\mathrm{dorb}}_{\mathrm{sef}}$ need not be injective, in contrast to Lemma 13.15. This example works as $\boldsymbol{\mathcal{W}}$ is not semieffective.

Remark 13.28. We can also consider d-orbifold versions of the other bordism theories $MO_*(Y), MU_*(Y), MSU_*(Y), MSp_*(Y)$, as in Remarks 13.6(d), 13.10(c) and 13.20. For unoriented bordism $MO_*(Y)$, we saw in Remark 13.20 that bordism using arbitrary compact, unoriented orbifolds is zero, but bordism using *locally orientable* compact, unoriented orbifolds is nontrivial.

Example 13.27 implies that the bordism ring of compact, locally orientable d-orbifolds is zero, as the identity in this ring is $[*, \boldsymbol{\pi}] = 2\big[[*/\mathbb{Z}_2], \boldsymbol{\pi}\big] = 0$.

Restricting to semieffective (or effective) locally oriented d-orbifolds will give the same bordism rings as for (effective) orbifolds, as in Theorem 13.23.

The author hopes in future work to define and study *unitary d-orbifold bordism groups* $dBU_*^{\mathrm{orb}}(\mathcal{Y})$ of orbifolds $\mathcal{Y}$. Elements of $dBU_k^{\mathrm{orb}}(\mathcal{Y})$ should be equivalence classes $[\boldsymbol{\mathcal{X}},\boldsymbol{J},\boldsymbol{f}]$, where $\boldsymbol{\mathcal{X}}$ is a compact oriented d-orbifold with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}=k$, and $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ is a 1-morphism, and $\boldsymbol{J}$ is some suitable notion of 'stable almost complex structure' on $T^*\boldsymbol{\mathcal{X}}$.

This may be a useful tool for studying *Gromov–Witten invariants* of symplectic and complex manifolds. Let (X,ω) be a compact symplectic manifold, J a compatible almost complex structure on X, $\beta\in H_2(X;\mathbb{Z})$ and $g,m\geqslant 0$. Then one can define compact moduli spaces $\overline{\mathcal{M}}_{g,m}(X,J,\beta)$ of stable J-holomorphic curves in X with class β, genus g and m marked points, with 'evaluation maps' $\mathrm{ev}:\overline{\mathcal{M}}_{g,m}(X,J,\beta)\to\overline{\mathcal{M}}_{g,m}\times X^m$. These are oriented Kuranishi spaces in the work of [34], or polyfolds in the framework of Hofer et al. [43–46].

As in Chapter 14, we can make $\overline{\mathcal{M}}_{g,m}(X,J,\beta)$ into a compact oriented d-orbifold, with a 1-morphism $\mathbf{ev}:\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)\to\overline{\boldsymbol{\mathcal{M}}}_{g,m}\times\boldsymbol{X}^m$ in $\mathbf{dOrb}$. Fukaya and Ono define 'stably almost complex' Kuranishi spaces [34, Def. 5.17], and prove $\overline{\mathcal{M}}_{g,m}(X,J,\beta)$ is stably almost complex [34, Prop. 16.5]. By generalizing this to d-orbifolds, it should be possible to define Gromov–Witten type invariants $\bigl[\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta),\boldsymbol{K},\mathbf{ev}\bigr]$ in $dBU_*^{\mathrm{orb}}(\overline{\mathcal{M}}_{g,m}\times X^m)$, which are independent of the choice of J, and lift the conventional Gromov–Witten invariants $\bigl[\overline{\mathcal{M}}_{g,m}(X,J,\beta),\mathrm{ev}\bigr]_{\mathrm{virt}}$ in $H_*(\overline{\mathcal{M}}_{g,m}\times X^m;\mathbb{Q})$.

Since $dBU_*^{\mathrm{orb}}(\overline{\mathcal{M}}_{g,m}\times X^m)$ will be much larger than $H_*(\overline{\mathcal{M}}_{g,m}\times X^m;\mathbb{Q})$, these new invariants will contain more information than conventional Gromov–Witten invariants, and may be helpful for studying *integrality properties* of Gromov–Witten invariants.

13.5 The proof of Theorem 13.23

Let $\mathcal{Y}$ be an orbifold. We first show that $\Pi_{\mathrm{orb}}^{\mathrm{sef}}:B_k^{\mathrm{orb}}(\mathcal{Y})\to dB_k^{\mathrm{sef}}(\mathcal{Y})$ is surjective. Suppose $[\boldsymbol{\mathcal{X}},\boldsymbol{f}]\in dB_k^{\mathrm{sef}}(\mathcal{Y})$, so that $\boldsymbol{\mathcal{X}}$ is a compact, oriented, semieffective d-orbifold with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}=k$, and $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ is a 1-morphism. By Theorem 10.56, we can choose a very good coordinate system $\bigl(I,<,(\mathcal{V}_i,\mathcal{E}_i,s_i,\boldsymbol{\psi}_i),(\mathcal{V}_{ij},e_{ij},\hat e_{ij},\boldsymbol{\eta}_{ij}),\eta_{ijk},g_i,\boldsymbol{\zeta}_i,\zeta_{ij}\bigr)$ for $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$, with $I\subset\mathbb{N}$ finite.

For each $i\in I$, there is a unique orientation on $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$ such that $\boldsymbol{\psi}_i:\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}\to\boldsymbol{\mathcal{X}}_i\subseteq\boldsymbol{\mathcal{X}}$ is orientation-preserving. As in Definition 4.48, this means that the line bundle $\Lambda^{\mathrm{rank}\,\mathcal{E}_i}\mathcal{E}_i\otimes\Lambda^{\dim\mathcal{V}_i}T^*\mathcal{V}_i$ on $\mathcal{V}_i$ is oriented near $s_i^{-1}(0)\subseteq\mathcal{V}_i$. Making the $\mathcal{V}_i,\mathcal{V}_{ij}$ smaller if necessary, we can suppose that $\Lambda^{\mathrm{rank}\,\mathcal{E}_i}\mathcal{E}_i\otimes\Lambda^{\dim\mathcal{V}_i}T^*\mathcal{V}_i$ is oriented on all of $\mathcal{V}_i$ for each $i\in I$.

Choose open suborbifolds $\mathcal{U}_i$ in $\mathcal{V}_i$ for each $i\in I$ such that the closure $\overline{\mathcal{U}}_{i,\mathrm{top}}$ of $\mathcal{U}_{i,\mathrm{top}}$ in $\mathcal{V}_{i,\mathrm{top}}$ is compact, and $\mathcal{X}_{\mathrm{top}}=\bigcup_{i\in I}\psi_{i,\mathrm{top}}\bigl(s_i^{-1}(0)\cap\mathcal{U}_{i,\mathrm{top}}\bigr)$. This is possible as $\mathcal{X}_{\mathrm{top}}$ is compact and each $\mathcal{V}_{i,\mathrm{top}}$ is locally compact. If $i<j$ in I with $\boldsymbol{\mathcal{X}}_i\cap\boldsymbol{\mathcal{X}}_j\neq\boldsymbol{\emptyset}$, set $\mathcal{U}_{ij}=\mathcal{V}_{ij}\cap\mathcal{U}_i\cap e_{ij}^{-1}(\mathcal{U}_j)$. We claim that the closure $\overline{\mathcal{U}}_{ij,\mathrm{top}}$ of $\mathcal{U}_{ij,\mathrm{top}}$ in $\mathcal{V}_{ij,\mathrm{top}}$ is compact. To see this, note that the quotient topological space $\mathcal{V}_{i,\mathrm{top}}\amalg_{\mathcal{V}_{ij,\mathrm{top}}}\mathcal{V}_{j,\mathrm{top}}$ is Hausdorff by Definition 10.51(f). The images of $\overline{\mathcal{U}}_{i,\mathrm{top}}$ and $\overline{\mathcal{U}}_{j,\mathrm{top}}$ in this topological space are compact. Their intersection is homeomorphic

to $\mathcal{V}_{ij} \cap \overline{\mathcal{U}}_i \cap e_{ij}^{-1}(\overline{\mathcal{U}}_j)$, and is compact as the intersection of compact subsets of a Hausdorff topological space is compact. Therefore $\overline{\mathcal{U}}_{ij,\rm top}$ is compact, as it is a closed subset of the compact space $\mathcal{V}_{ij} \cap \overline{\mathcal{U}}_i \cap e_{ij}^{-1}(\overline{\mathcal{U}}_j)$.

By induction on increasing $k \in I$, we will choose $N \gg 0$ and sections $\dot{s}_k^a \in C^\infty(\mathcal{E}_k)$ for $a = 1, \ldots, N$ with the following properties:

(a) Let $[v] \in \mathcal{V}_{k,\rm top}$ with $s_k|_v = 0$, and regard $\mathrm{d}s_k|_v$ as a linear map $T_v\mathcal{V}_k \to \mathcal{E}_k|_v$. Then $\mathcal{E}_k|_v = \mathrm{d}s_k|_v(T_v\mathcal{V}_k) + \langle \dot{s}_k^1|_v, \ldots, \dot{s}_k^N|_v \rangle$.

(b) For all $i < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_k \neq \emptyset$ we have $\hat{e}_{ik} \circ \dot{s}_i^a|_{\mathcal{V}_{ik}} = e_{ik}^*(\dot{s}_k^a)$ on $\overline{\mathcal{U}}_{ik}$ in $\mathcal{V}_{ik}$ for all $a = 1, \ldots, N$.

Choose $N \geqslant 0$ such that $N \geqslant \operatorname{rank} \mathcal{E}_k + \dim \mathcal{V}_k$ for all $k \in I$, which is possible as I is finite. We first show that if $\dot{s}_k^1, \ldots, \dot{s}_k^N$ are generic elements of $C^\infty(\mathcal{E}_k)$ for $k \in I$, then (a) holds. As $\boldsymbol{\psi}_k : \boldsymbol{\mathcal{S}}_{\mathcal{V}_k,\mathcal{E}_k,s_k} \to \boldsymbol{\mathcal{X}}_k \subseteq \boldsymbol{\mathcal{X}}$ is an equivalence and $\boldsymbol{\mathcal{X}}$ is semieffective, $\boldsymbol{\mathcal{S}}_{\mathcal{V}_k,\mathcal{E}_k,s_k}$ is semieffective. Let $[v] \in \mathcal{V}_k$ with $s_k|_v = 0$. Then $\mathrm{d}s_k|_v : T_v\mathcal{V}_k \to \mathcal{E}_k|_v$ is a morphism of $\mathrm{Iso}_{\mathcal{V}_k}([v])$-representations. Equation (10.24) for $\boldsymbol{\mathcal{S}}_{\mathcal{V}_k,\mathcal{E}_k,s_k}$ at $[v]$ is

$$0 \longrightarrow K_{[v]} \longrightarrow \mathcal{E}_k|_v^* \xrightarrow{\mathrm{d}s_k|_v^*} T_v^*\mathcal{V}_k \longrightarrow T_v^*\mathcal{S}_{\mathcal{V}_k,\mathcal{E}_k,s_k} \longrightarrow 0. \tag{13.17}$$

As $\boldsymbol{\mathcal{S}}_{\mathcal{V}_k,\mathcal{E}_k,s_k}$ is semieffective, the representation of $\mathrm{Iso}_{\mathcal{V}_k}([v])$ on $K_{[v]}$ is trivial.

That is, the $\mathrm{Iso}_{\mathcal{V}_k}([v])$-representation on the cokernel of $\mathrm{d}s_k|_v : T_v\mathcal{V}_k \to \mathcal{E}_k|_v$ is trivial. But $s|_v$ takes values in the trivial $\mathrm{Iso}_{\mathcal{V}_k}([v])$-subrepresentation of $\mathcal{E}_k|_v$ for any $s \in C^\infty(\mathcal{E}_k)$, and such $s|_v$ span the whole of this trivial representation. Therefore at any point $[v] \in \mathcal{V}_{k,\rm top}$ with $s_k|_v = 0$, as $N \geqslant \operatorname{rank} \mathcal{E}_k$, we can choose $\dot{s}_k^1, \ldots, \dot{s}_k^N \in C^\infty(\mathcal{E}_k)$ with $\mathcal{E}_k|_v = \mathrm{d}s_k|_v(T_v\mathcal{V}_k) + \langle \dot{s}_k^1|_v, \ldots, \dot{s}_k^N|_v \rangle$. A dimension-counting argument now shows that as $N \geqslant \operatorname{rank} \mathcal{E}_k + \dim \mathcal{V}_k$, generic choices of $\dot{s}_k^1, \ldots, \dot{s}_k^N$ in $C^\infty(\mathcal{E}_k)$ satisfy (a).

For the inductive step, let $k \in I$, and suppose we have chosen $\dot{s}_i^a \in C^\infty(\mathcal{E}_i)$ satisfying (a),(b) for all $i \in I$ with $i < k$. We will choose $\dot{s}_k^a$ satisfying (a),(b). Note that if $i < k$ with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_k \neq \emptyset$, then $\dot{s}_i^a$ has been chosen in a previous inductive step, so (b) determines $\dot{s}_k^a$ on $e_{ik}(\overline{\mathcal{U}}_{ik}) \subseteq e_{ik}(\mathcal{V}_{ik})$ for $a = 1, \ldots, N$.

There are two issues in choosing $\dot{s}_k^a \in C^\infty(\mathcal{E}_k)$ with these prescribed values on $e_{ik}(\overline{\mathcal{U}}_{ik})$ for all $i < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_k \neq \emptyset$. Firstly, for just one such i, we must show that the prescribed values of $\dot{s}_k^a$ on $e_{ik}(\overline{\mathcal{U}}_{ik})$ admit a smooth extension to $\mathcal{V}_k$. Secondly, given two (or more) such i, j, we must show that the prescribed values of $\dot{s}_k^a$ on $e_{ik}(\overline{\mathcal{U}}_{ik})$ and $e_{jk}(\overline{\mathcal{U}}_{jk})$ are consistent on $e_{ik}(\overline{\mathcal{U}}_{ik}) \cap e_{jk}(\overline{\mathcal{U}}_{jk})$.

For the first issue, consider the following analogy. Let $X = (0, \infty)$, $Y = \mathbb{R}^2$ and $i : X \to Y$ maps $i : x \mapsto (x, 0)$, so that $i(X)$ is an embedded submanifold in Y. Given a smooth function $f : X \to \mathbb{R}$, when can we choose a smooth function $g : Y \to \mathbb{R}$ with $g \circ i = f$? This is not always possible, for example, if $f(x) = x^{-1}$ then no continuous g exists near $(0, 0) \in Y$ with $g \circ i = f$. Here $i(X)$ is not closed in Y, and problems occur at points of $\overline{i(X)} \setminus i(X)$.

However, if we only require $g \circ i = f$ to hold on some compact subset $W \subset X$, then for any smooth $f : X \to \mathbb{R}$ there exists a smooth $g : Y \to \mathbb{R}$ with $g \circ i|_W = f|_W$. This is because $i(W)$ is closed in Y, as it is compact, so there are no points of $\overline{i(W)} \setminus i(W)$ to cause problems. In a similar way, since in

(b) above we require $\hat{e}_{ik}\circ\dot{s}_i^a|_{\mathcal{V}_{ik}}=e_{ik}^*(\dot{s}_k^a)$ to hold only on the compact subset $\overline{\mathcal{U}}_{ik}$ in $\mathcal{V}_{ik}$, where $e_{ik}:\mathcal{V}_{ik}\hookrightarrow\mathcal{V}_k$ is an embedding, one can show that for any smooth choice of $\dot{s}_i^a$, there exists a smooth $\dot{s}_k^a$ for which (b) holds.

For the second issue, suppose $i<j<k$ in I and $[v_i]\in\overline{\mathcal{U}}_{ik,\mathrm{top}}\subseteq\mathcal{V}_{ik,\mathrm{top}}$, $[v_j]\in\overline{\mathcal{U}}_{jk,\mathrm{top}}\subseteq\mathcal{V}_{jk,\mathrm{top}}$ with $e_{ik,\mathrm{top}}([v_i])=e_{jk,\mathrm{top}}([v_j])=[v_k]\in\mathcal{V}_{k,\mathrm{top}}$. Then Definition 10.52(e) implies that $[v_i]\in\mathcal{V}_{ij,\mathrm{top}}$ with $[v_j]=e_{ij,\mathrm{top}}([v_i])$. Also $[v_i]\in\overline{\mathcal{U}}_{i,\mathrm{top}}$, $[v_j]\in\overline{\mathcal{U}}_{j,\mathrm{top}}$ imply that $[v_i]\in\overline{\mathcal{U}}_{ij,\mathrm{top}}$. Hence the previous inductive step implies that $\hat{e}_{ij}\circ\dot{s}_i^a|_{\mathcal{V}_{ij}}=e_{ij}^*(\dot{s}_j^a)$ at $[v_i]$. Suppose that some choice of $\dot{s}_k^a$ satisfies $\hat{e}_{jk}\circ\dot{s}_j^a|_{\mathcal{V}_{jk}}=e_{jk}^*(\dot{s}_k^a)$ at $[v_j]$. Then

$$
\begin{aligned}
\hat{e}_{ik}\circ\dot{s}_i^a|_{[v_i]}&\cong e_{ij}^*(\hat{e}_{jk})\circ\hat{e}_{ij}\circ\dot{s}_i^a|_{[v_i]}=e_{ij}^*(\hat{e}_{jk})\circ e_{ij}^*(\dot{s}_j^a)|_{[v_i]}=e_{ij}^*(\hat{e}_{jk}\circ\dot{s}_j^a|_{\mathcal{V}_{jk}})|_{[v_i]}\\
&=e_{ij}^*\circ e_{jk}^*(\dot{s}_k^a)|_{[v_i]}\cong(e_{jk}\circ e_{ij})^*(\dot{s}_k^a)|_{[v_i]}\cong e_{ik}^*(\dot{s}_k^a)|_{[v_i]},
\end{aligned}
$$

where '$\cong$' denotes that we have omitted canonical isomorphisms $I_{*,*}(*),\eta_{ijk}^*(\mathcal{E}_k)$ for simplicity. Therefore (b) for j,k at $[v_j]$ implies (b) for i,k at $[v_i]$.

Therefore for any $[v]\in\mathcal{V}_{k,\mathrm{top}}$, if $j\in I$ is largest in the order $<$ such that $j<k$ and $[v]\in e_{jk,\mathrm{top}}(\overline{\mathcal{U}}_{jk,\mathrm{top}})$, then (b) for these j,k near $[v]$ implies (b) for all other i,k near $[v]$. That is, the prescribed values of $\dot{s}_k^a$ on $e_{ik}(\overline{\mathcal{U}}_{ik})$ and $e_{jk}(\overline{\mathcal{U}}_{jk})$ are automatically consistent on $e_{ik}(\overline{\mathcal{U}}_{ik})\cap e_{jk}(\overline{\mathcal{U}}_{jk})$, and near any point $[v]\in\mathcal{V}_{k,\mathrm{top}}$, it is enough to check (b) for just one $i<k$ in I.

We can now complete the inductive step. Given $k\in I$ and $\dot{s}_i^a\in C^\infty(\mathcal{E}_i)$ satisfying (a),(b) for all $i\in I$ with $i<k$, we first choose $\dot{s}_k^a\in C^\infty(\mathcal{E}_k)$ satisfying (b) for all $i<k$ in I. The argument above shows this is possible near any point $[v]\in\mathcal{V}_{k,\mathrm{top}}$, so joining local choices by a partition of unity using the ideas of Remark C.27 and Example C.33, it is possible globally.

Next, we claim that if $i<k$ and $[v_i]\in\overline{\mathcal{U}}_{ik,\mathrm{top}}$ with $[v_k]=e_{ik,\mathrm{top}}([v_k])$, then (a) for $\dot{s}_i^a$ at $[v_i]$, which holds by assumption, implies (a) for $\dot{s}_k^a$ at $[v_k]$. This is because it follows from the definition of type B coordinate change $(\mathcal{V}_{ik},e_{ik},\hat{e}_{ik},\boldsymbol{\eta}_{ik})$ in Definition 10.51 that the cokernels of $\mathrm{d}s_i|_{v_i}:T_{v_i}\mathcal{V}_i\to\mathcal{E}_i|_{v_i}$ and $\mathrm{d}s_k|_{v_k}:T_{v_k}\mathcal{V}_k\to\mathcal{E}_k|_{v_k}$ are isomorphic.

Now (a) is an open condition on $[v]\in\mathcal{V}_{k,\mathrm{top}}$. Therefore (a) holds for $\dot{s}_k^a\in C^\infty(\mathcal{E}_k)$ for all $[v]$ in an open neighbourhood of $\bigcup_{i<k}e_{ik,\mathrm{top}}(\overline{\mathcal{U}}_{ik,\mathrm{top}})$ in $\mathcal{V}_{k,\mathrm{top}}$. By making a generic perturbation of $\dot{s}_k^a$ away from the compact (hence closed) set $\bigcup_{i<k}e_{ik,\mathrm{top}}(\overline{\mathcal{U}}_{ik,\mathrm{top}})$, we can make (a) hold in all of $\mathcal{V}_k$, while (b) still holds. Hence by induction, we can choose $\dot{s}_k^a$ for all a,k satisfying (a),(b).

Now fix $(\epsilon^1,\ldots,\epsilon^N)\in\mathbb{R}^N$ to be chosen later, and define:

(i) For each $i\in I$, define $\mathcal{W}_i=\mathcal{U}_i\times[0,1]$, as an effective orbifold with boundary. Define a vector bundle $\mathcal{F}_i$ on $\mathcal{W}_i$ by $\mathcal{F}_i=\pi_{\mathcal{U}_i}^*(\mathcal{E}_i)$, where $\pi_{\mathcal{U}_i}:\mathcal{W}_i=\mathcal{U}_i\times[0,1]\to\mathcal{U}_i$ is the projection. Define $t_i\in C^\infty(\mathcal{F}_i)$ by $t_i=\pi_{\mathcal{U}_i}^*(s_i)+\sum_{a=1}^N x\,\epsilon^a\cdot\pi_{\mathcal{U}_i}^*(\dot{s}_i^a)$, where x is the coordinate on $[0,1]$. Then Definition 12.2 defines a d-orbifold with boundary $\boldsymbol{S}_{\mathcal{W}_i,\mathcal{F}_i,t_i}$.

From above, the line bundle $\Lambda^{\mathrm{rank}\,\mathcal{E}_i}\mathcal{E}_i\otimes\Lambda^{\dim\mathcal{V}_i}T^*\mathcal{V}_i$ is oriented. Together with the orientation on $[0,1]$, this induces an orientation on $\Lambda^{\mathrm{rank}\,\mathcal{F}_i}\mathcal{F}_i\otimes\Lambda^{\dim\mathcal{W}_i}T^*\mathcal{W}_i$. Hence $\boldsymbol{S}_{\mathcal{W}_i,\mathcal{F}_i,t_i}$ is an oriented d-orbifold. Define $\tilde{s}_i=$

$s_i+\sum_{a=1}^N \epsilon^a \dot s_i^a$ in $C^\infty(\mathcal{E}_i)$. Then in oriented d-orbifolds we have

$$\partial \boldsymbol{\mathcal{S}}_{\mathcal{W}_i,\mathcal{F}_i,t_i} \simeq -\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i} \amalg \boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,\tilde s_i}, \tag{13.18}$$

since $\partial\mathcal{W}_i \simeq \mathcal{V}_i\times\{0\}\amalg\mathcal{V}_i\times\{1\}$ with $t_i|_{\mathcal{V}_i\times\{0\}}\cong s_i$ and $t_i|_{\mathcal{V}_i\times\{1\}}\cong \tilde s_i$.

(ii) For all $i<j$ in I with $\boldsymbol{\mathcal{X}}_i\cap\boldsymbol{\mathcal{X}}_j\neq\boldsymbol{\emptyset}$ and $\mathcal{U}_{ij}\neq\emptyset$, define $\mathcal{W}_{ij}=\mathcal{U}_{ij}\times[0,1]$, an open suborbifold in $\mathcal{W}_i$, and define a simple, flat 1-morphism $f_{ij}:\mathcal{W}_{ij}\to\mathcal{W}_j$ by $f_{ij}=e_{ij}|_{\mathcal{U}_{ij}}\times\mathrm{id}_{[0,1]}$ and a morphism of vector bundles $\hat f_{ij}:\mathcal{F}_i|_{\mathcal{W}_{ij}}\to f_{ij}^*(\mathcal{F}_j)$ by $\hat f_{ij}=I_{\pi_{\mathcal{U}_{ij}},e_{ij}}(\mathcal{E}_j)^{-1}\circ\pi_{\mathcal{U}_{ij}}^*(\hat e_{ij})$. Then $\hat e_{ij}\circ s_i|_{\mathcal{V}_{ij}}=e_{ij}^*(s_j)$ and $\hat e_{ij}\circ\dot s_i^a|_{\mathcal{V}_{ij}}=e_{ij}^*(\dot s_j^a)$ imply that $\hat f_{ij}\circ t_i|_{\mathcal{W}_{ij}}=f_{ij}^*(t_j)$, so as in Definition 12.13 we have a 1-morphism in $\mathbf{dOrb^b}$:

$$\boldsymbol{\mathcal{S}}_{f_{ij},\hat f_{ij}}:\boldsymbol{\mathcal{S}}_{\mathcal{W}_{ij},\mathcal{F}_i|_{\mathcal{W}_{ij}},t_i|_{\mathcal{W}_{ij}}}\longrightarrow\boldsymbol{\mathcal{S}}_{\mathcal{W}_j,\mathcal{F}_j,t_j}, \tag{13.19}$$

where $\boldsymbol{\mathcal{S}}_{\mathcal{W}_{ij},\mathcal{F}_i|_{\mathcal{W}_{ij}},t_i|_{\mathcal{W}_{ij}}}$ is an open d-suborbifold in $\boldsymbol{\mathcal{S}}_{\mathcal{W}_i,\mathcal{F}_i,t_i}$.

(iii) Define a quotient topological space Z by $Z=\bigcup_{i\in I}t_i^{-1}(0)/\sim$, where $t_i^{-1}(0)=\bigl\{[w]\in\mathcal{W}_{i,\mathrm{top}}:t_i(w)=0\bigr\}\subseteq\mathcal{W}_{i,\mathrm{top}}$, and $\sim$ is the equivalence relation generated by $[w_i]\sim[w_j]$ if $[w_i]=[v_i,x]$ and $[w_j]=[v_j,x]$ for $[v_i]\in\mathcal{U}_{ij,\mathrm{top}}\subseteq\mathcal{U}_{i,\mathrm{top}}$, $[v_j]=e_{ij,\mathrm{top}}([v_i])\in\mathcal{U}_{j,\mathrm{top}}$, and $x\in[0,1]$. Write $\psi_i:(\mathcal{S}_{\mathcal{W}_i,\mathcal{F}_i,t_i})_{\mathrm{top}}=t_i^{-1}(0)\to Z$ for the inclusion, and $\hat Z_i=\psi_i(t_i^{-1}(0))$.

We now claim:

(A) If $\epsilon^1,\ldots,\epsilon^N$ are sufficiently small, then the topological space Z is compact.

(B) If $\epsilon^1,\ldots,\epsilon^N$ are sufficiently small, then (13.19) are equivalences from $\boldsymbol{\mathcal{S}}_{\mathcal{W}_{ij},\mathcal{F}_i|_{\mathcal{W}_{ij}},t_i|_{\mathcal{W}_{ij}}}$ to an open d-suborbifold in $\boldsymbol{\mathcal{S}}_{\mathcal{W}_j,\mathcal{F}_j,t_j}$ for all i,j.

(C) If $\epsilon^1,\ldots,\epsilon^N$ are sufficiently small and generic, then the sections $\tilde s_i$ in $C^\infty(\mathcal{E}_i)$ are transverse, and the d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,\tilde s_i}$ in (13.18) are orbifolds, for all $i\in I$.

(D) If $\epsilon^1,\ldots,\epsilon^N$ are sufficiently small then $\boldsymbol{\mathcal{S}}_{\mathcal{W}_i,\mathcal{F}_i,t_i}$ is semieffective for $i\in I$.

To prove (A), identify $\mathcal{W}_{i,\mathrm{top}}\cong\mathcal{U}_{i,\mathrm{top}}\times[0,1]$, and consider the inclusions $Z\subseteq\bigcup_{i\in I}\mathcal{U}_{i,\mathrm{top}}\times[0,1]/\sim\subseteq\bigcup_{i\in I}\overline{\mathcal{U}}_{i,\mathrm{top}}\times[0,1]/\sim$. Here Z is closed in $\bigcup_{i\in I}\mathcal{U}_{i,\mathrm{top}}\times[0,1]/\sim$, as $t_i^{-1}(0)$ is closed in $\mathcal{U}_{i,\mathrm{top}}\times[0,1]$, and $\bigcup_{i\in I}\overline{\mathcal{U}}_{i,\mathrm{top}}\times[0,1]/\sim$ is compact as $\overline{\mathcal{U}}_{i,\mathrm{top}}$ is compact and I is finite. Therefore Z is compact if it has no limit points in $\bigl(\bigcup_{i\in I}\overline{\mathcal{U}}_{i,\mathrm{top}}\times[0,1]/\sim\bigr)\setminus\bigl(\bigcup_{i\in I}\mathcal{U}_{i,\mathrm{top}}\times[0,1]/\sim\bigr)$.

Suppose $j\in I$ and $[v_j]\in\overline{\mathcal{U}}_{i,\mathrm{top}}\setminus\mathcal{U}_{i,\mathrm{top}}$. Since $\boldsymbol{\mathcal{X}}$ is compact and $\mathcal{X}_{\mathrm{top}}=\bigcup_{i\in I}\psi_{i,\mathrm{top}}\bigl(s_i^{-1}(0)\cap\mathcal{U}_{i,\mathrm{top}}\bigr)$, there are three possibilities: $(*)$ $s_j(v_j)\neq0$; $(**)$ $s_j(v_j)=0$ and $[v_j]\in\mathcal{V}_{jk,\mathrm{top}}$ for $j<k$ in I with $e_{jk,\mathrm{top}}([v_j])=[v_k]\in\mathcal{U}_{k,\mathrm{top}}$ with $s_k(v_k)=0$; and $(*{*}*)$ $s_j(v_j)=0$ and $[v_j]=e_{ij,\mathrm{top}}([v_i])$ for $i<j$ in I and $[v_i]\in\mathcal{U}_{i,\mathrm{top}}\cap\mathcal{V}_{ij,\mathrm{top}}$ with $s_i(v_i)=0$.

In case $(*)$, there exists an open neighbourhood $N_{[v_j]}$ of $[v_j]$ in $\overline{\mathcal{U}}_{j,\mathrm{top}}\setminus\mathcal{U}_{j,\mathrm{top}}$ and $\delta_{[v_j]}>0$ such that if $[v']\in N_{[v_j]}$, $x\in[0,1]$ and $(\epsilon^1,\ldots,\epsilon^N)\in\mathbb{R}^N$ with $\bigl|(\epsilon^1,\ldots,\epsilon^N)\bigr|<\delta_{[v_j]}$, then $(s_j+\sum_{a=1}^N x\,\epsilon^a\dot s_j^a)|_{v'}\neq0$. Hence t_j is nonzero on

$N_{[v_j]} \times [0,1]$. In case $(**)$, there exists an open neighbourhood $N_{[v_j]}$ of $[v_j]$ in $\overline{\mathcal{U}}_{j,\mathrm{top}} \setminus \mathcal{U}_{j,\mathrm{top}}$ with $N_{[v_j]} \subseteq \mathcal{V}_{jk,\mathrm{top}}$ and $e_{jk,\mathrm{top}}(N_{[v_j]}) \subseteq \mathcal{U}_{k,\mathrm{top}}$.

In case $(***)$, equation (10.23) is an isomorphism at $[v_i],[v_j]$ by Definition 10.51(d). This is an open condition in s_i, s_j. Hence there exists $\delta_{[v_j]} > 0$ such that for all $x \in [0,1]$ and $(\epsilon^1,\ldots,\epsilon^N) \in \mathbb{R}^N$ with $\bigl|(\epsilon^1,\ldots,\epsilon^N)\bigr| < \delta_{[v_j]}$, equation (10.23) is an isomorphism with $(s_i + \sum_{a=1}^N x\,\epsilon^a \dot{s}_i^a), (s_j + \sum_{a=1}^N x\,\epsilon^a \dot{s}_j^a)$ in place of s_i, s_j. It follows that $(s_j + \sum_{a=1}^N x\,\epsilon^a \dot{s}_j^a)^{-1}(0)$ coincides with $e_{ij,\mathrm{top}}\bigl((s_i + \sum_{a=1}^N x\,\epsilon^a \dot{s}_i^a)^{-1}(0)\bigr)$ close to $[v_j]$. Thus there exists an open neighbourhood $N_{[v_j]}$ of $[v_j]$ in $\overline{\mathcal{U}}_{j,\mathrm{top}} \setminus \mathcal{U}_{j,\mathrm{top}}$ and $\delta_{[v_j]} > 0$ such that if $[v'] \in N_{[v_j]}$, $x \in [0,1]$ and $(\epsilon^1,\ldots,\epsilon^N) \in \mathbb{R}^N$ with $\bigl|(\epsilon^1,\ldots,\epsilon^N)\bigr| < \delta_{[v_j]}$ and $(s_j + \sum_{a=1}^N x\,\epsilon^a \dot{s}_j^a)(v') = 0$, then $[v'] = e_{ij,\mathrm{top}}([v])$ for some $[v] \in \mathcal{U}_{i,\mathrm{top}} \cap \mathcal{V}_{ij,\mathrm{top}}$.

For each $[v_j]$ in $\overline{\mathcal{U}}_{j,\mathrm{top}} \setminus \mathcal{U}_{j,\mathrm{top}}$ we get an open neighbourhood $N_{[v_j]}$. As $\overline{\mathcal{U}}_{j,\mathrm{top}} \setminus \mathcal{U}_{j,\mathrm{top}}$ is compact, we can choose a finite number of such neighbourhoods $N_{[v_j]}$ to cover $\overline{\mathcal{U}}_{j,\mathrm{top}} \setminus \mathcal{U}_{j,\mathrm{top}}$. Do this for each j in I, which is finite. This gives a finite set of bounds $\delta_{[v_j]} > 0$ for $\bigl|(\epsilon^1,\ldots,\epsilon^N)\bigr|$ corresponding to the finite covers $N_{[v_j]}$ chosen in cases $(*), (***)$. Let $\delta > 0$ be the minimum of these finite set of $\delta_{[v_j]}$. Following the definitions, we see that if $\bigl|(\epsilon^1,\ldots,\epsilon^N)\bigr| < \delta$ then Z has no limit points in $\bigl(\bigcup_{i\in I} \overline{\mathcal{U}}_{i,\mathrm{top}} \times [0,1]/\sim\bigr) \setminus \bigl(\bigcup_{i\in I} \mathcal{U}_{i,\mathrm{top}} \times [0,1]/\sim\bigr)$, so Z is compact. This proves (A).

For (B), we use a similar argument. For each $[v_i] \in \overline{\mathcal{U}}_{ij,\mathrm{top}}$, we can choose an open neighbourhood $N_{[v_i]}$ of $[v_i]$ in $\overline{\mathcal{U}}_{ij,\mathrm{top}}$ and $\delta_{[v_i]} > 0$ such that *either* $s_i(v_i) \neq 0$, and if $[v] \in N_{[v_i]}$, $x \in [0,1]$ and $(\epsilon^1,\ldots,\epsilon^N) \in \mathbb{R}^N$ with $\bigl|(\epsilon^1,\ldots,\epsilon^N)\bigr| < \delta_{[v_i]}$, then $(s_i + \sum_{a=1}^N x\,\epsilon^a \dot{s}_i^a)|_v \neq 0$; *or* $s_i(v_i) = 0$, and if $[v] \in N_{[v_i]}$, $x \in [0,1]$ and $(\epsilon^1,\ldots,\epsilon^N) \in \mathbb{R}^N$ with $\bigl|(\epsilon^1,\ldots,\epsilon^N)\bigr| < \delta_{[v_i]}$, then (10.23) is an isomorphism with $[v], e_{ij,\mathrm{top}}([v]), s_i + \sum_{a=1}^N x\,\epsilon^a \dot{s}_i^a, s_j + \sum_{a=1}^N x\,\epsilon^a \dot{s}_j^a$ in place of $[v_i],[v_i],s_i,s_j$.

Thus as $\overline{\mathcal{U}}_{ij,\mathrm{top}}$ is compact, we can choose a finite open cover of such $N_{[v_i]}$, and define $\delta > 0$ to be the minimum of the corresponding $\delta_{[v_i]}$. Then for all $[v_i] \in \mathcal{U}_{ij,\mathrm{top}}$, $x \in [0,1]$ and $(\epsilon^1,\ldots,\epsilon^N) \in \mathbb{R}^N$ with $\bigl|(\epsilon^1,\ldots,\epsilon^N)\bigr| < \delta$, either $t_i(v,x) \neq 0$, or $t_i(v,x) = 0$ and (12.1) for (13.19) is an isomorphism at $([v],x)$. Therefore (13.19) is a local equivalence by Corollary 12.22, proving (B).

For (C), note that (a) above implies that for each $[v] \in \mathcal{V}_{i,\mathrm{top}}$ with $s_i|_v = 0$ and all generic $(\epsilon^1,\ldots,\epsilon^N)$, the section $\tilde{s}_i = s_i + \sum_{a=1}^N \epsilon^a \dot{s}_i^a$ is transverse near $[v]$, so that $\tilde{s}_i^{-1}(0)$ is an orbifold near $[v]$. Using a similar argument covering $\overline{\mathcal{U}}_{i,\mathrm{top}}$ by finitely many open $N_{[v_i]}$ in which $\tilde{s}_i$ is transverse provided $(\epsilon^1,\ldots,\epsilon^N)$ lies in a dense open subset of $\bigl\{(\epsilon^1,\ldots,\epsilon^N) \in \mathbb{R}^N : \bigl|(\epsilon^1,\ldots,\epsilon^N)\bigr| < \delta_{[v_i]}\bigr\}$ for some small $\delta_{[v_i]} > 0$, part (C) follows.

For (D), from above the representation of $\mathrm{Iso}_{\mathcal{V}_k}([v])$ on $K_{[v]}$ in (13.17) is trivial for each $[v] \in \mathcal{V}_{k,\mathrm{top}}$ with $s_k(v) = 0$. This is an open condition for small perturbations of $[v]$ and s_k. So by a similar argument covering the compact $\overline{\mathcal{U}}_{k,\mathrm{top}}$ by open $N_{[v_k]}$ with corresponding $\delta_{[v_k]}$, we can find $\delta > 0$ such that for each $[v] \in \overline{\mathcal{U}}_{k,\mathrm{top}}$, $x \in [0,1]$ and $(\epsilon^1,\ldots,\epsilon^N) \in \mathbb{R}^N$ with $\bigl|(\epsilon^1,\ldots,\epsilon^N)\bigr| < \delta$ with $\bigl(s_k + \sum_{a=1}^N x\,\epsilon^a \dot{s}_k^a\bigr)|_v = 0$, the representation of $\mathrm{Iso}_{\mathcal{V}_k}([v])$ on $K_{[v]}$ is trivial in

(13.17) with $\mathrm{d}(s_k + \sum_{a=1}^N x\,\epsilon^a \dot s_k^a)$ in place of $\mathrm{d}s_k$. This implies $\boldsymbol{\mathcal{S}}_{\mathcal{W}_k,\mathcal{F}_k,t_k}$ is semieffective, proving (D).

Now let $(\epsilon^1,\ldots,\epsilon^N)\in\mathbb{R}^N$ be chosen to satisfy (A)–(D) above. We will apply Theorem 12.23 to glue the d-orbifolds with boundary $\boldsymbol{\mathcal{S}}_{\mathcal{W}_i,\mathcal{F}_i,t_i}$ by the local equivalences (13.19) to make a d-orbifold with boundary $\boldsymbol{\mathcal{Z}}$ with underlying topological space Z as above, with $\operatorname{vdim}\boldsymbol{\mathcal{Z}}=\operatorname{vdim}\boldsymbol{\mathcal{X}}+1=k+1$. For Theorem 12.23(b), Z Hausdorff follows from Definition 10.51(f) and I finite, and Z second countable from $Z=\bigcup_{i\in I}\hat Z_i$ with $\hat Z_i\cong t_i^{-1}(0)$ second countable and I finite, and the rest of (a)–(e) are clear. Also Theorem 12.23(i) is obvious, (ii) follows from (B) above, and (iii) follows from Definition 10.52(d). Thus Theorem 12.23 gives a d-orbifold with boundary $\boldsymbol{\mathcal{Z}}$. Also, the second part with $g_i\circ\pi_{\mathcal{U}_i}:\mathcal{W}_i=\mathcal{U}_i\times[0,1]\to\mathcal{Y}$ in place of g_i gives a 1-morphism $\boldsymbol{h}:\boldsymbol{\mathcal{Z}}\to\boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb^b}$.

As the $\boldsymbol{\mathcal{S}}_{\mathcal{W}_i,\mathcal{F}_i,t_i}$ are oriented, and (13.19) are orientation-preserving, $\boldsymbol{\mathcal{Z}}$ is an oriented d-orbifold with boundary, and is compact by (A), and semieffective by (D). The boundary $\partial\boldsymbol{\mathcal{Z}}$ is the result of gluing the $\partial\boldsymbol{\mathcal{S}}_{\mathcal{W}_i,\mathcal{F}_i,t_i}$ by equivalences. From (13.18) we may write $\partial\boldsymbol{\mathcal{Z}}=\partial_0\boldsymbol{\mathcal{Z}}\amalg\partial_1\boldsymbol{\mathcal{Z}}$, where $\partial_0\boldsymbol{\mathcal{Z}}$ comes from gluing the $-\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,\tilde s_i}$ by equivalences at $x=0$ in $[0,1]$, and $\partial_1\boldsymbol{\mathcal{Z}}$ from gluing the $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$ by equivalences at $x=1$ in $[0,1]$. But $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,\tilde s_i}$ is an orbifold by (C) for each $i\in I$, so $\partial_0\boldsymbol{\mathcal{Z}}\simeq -F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\tilde X)$ for some oriented orbifold $\tilde X$ with $\dim\tilde{\mathcal{X}}=k$.

Since the $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$ are part of a very good coordinate system for $\boldsymbol{\mathcal{X}}$, and $\partial_1\boldsymbol{\mathcal{Z}}$ is made by gluing the $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$ by equivalences, $\boldsymbol{\mathcal{S}}_{e_{ij},\hat e_{ij}}$, uniqueness up to equivalence in Theorem 12.23 implies that $\partial_1\boldsymbol{\mathcal{Z}}\simeq\boldsymbol{\mathcal{X}}$. Thus we have an equivalence $\partial\boldsymbol{\mathcal{Z}}\simeq -F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\tilde X)\amalg\boldsymbol{\mathcal{X}}$ of oriented d-orbifolds. Also, as the g_i used to define $\boldsymbol{h}:\boldsymbol{\mathcal{Z}}\to\boldsymbol{\mathcal{Y}}$ are part of a very good coordinate system for $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$, the 1-morphism $\boldsymbol{h}:\boldsymbol{\mathcal{Z}}\to\boldsymbol{\mathcal{Y}}$ yields $\boldsymbol{h}\circ\boldsymbol{i}_{\boldsymbol{\mathcal{Z}}}:\partial\boldsymbol{\mathcal{Z}}\to\boldsymbol{\mathcal{Y}}$, which is identified up to 2-isomorphism by the equivalence $\partial\boldsymbol{\mathcal{Z}}\simeq -F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\tilde X)\amalg\boldsymbol{\mathcal{X}}$ with the disjoint union of 1-morphisms $F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\tilde f):F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\tilde X)\to\boldsymbol{\mathcal{Y}}$, for some 1-morphism $\tilde f:\tilde{\mathcal{X}}\to\mathcal{Y}$ in $\mathbf{Orb}$, and $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$.

Then $[\tilde X,\tilde f]$ lies in $B_k^{\mathrm{orb}}(\mathcal{Y})$, so $\Pi_{\mathrm{orb}}^{\mathrm{sef}}\bigl([\tilde X,\tilde f]\bigr)=\bigl[F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\tilde X),F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\tilde f)\bigr]$ in $dB_k^{\mathrm{sef}}(\mathcal{Y})$. The definition of $\sim$ for $dB_k^{\mathrm{sef}}(\mathcal{Y})$ in Definition 13.21 with $\boldsymbol{\mathcal{Z}},\boldsymbol{h}$ in place of $\boldsymbol{\mathcal{W}},\boldsymbol{e}$ implies that $\bigl(F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\tilde X),F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\tilde f)\bigr)\sim(\boldsymbol{\mathcal{X}},\boldsymbol{f})$, so $\Pi_{\mathrm{orb}}^{\mathrm{sef}}\bigl([\tilde X,\tilde f]\bigr)=[\boldsymbol{\mathcal{X}},\boldsymbol{f}]$. This proves that $\Pi_{\mathrm{orb}}^{\mathrm{sef}}:B_k^{\mathrm{orb}}(\mathcal{Y})\to dB_k^{\mathrm{sef}}(\mathcal{Y})$ is surjective, as we want.

Next we prove $\Pi_{\mathrm{orb}}^{\mathrm{sef}}:B_k^{\mathrm{orb}}(\mathcal{Y})\to dB_k^{\mathrm{sef}}(\mathcal{Y})$ is injective. Suppose $[\mathcal{X},f]\in B_k^{\mathrm{orb}}(\mathcal{Y})$ with $\Pi_{\mathrm{orb}}^{\mathrm{sef}}\bigl([\mathcal{X},f]\bigr)=0$. Then by Definitions 13.21 and 13.22 there exists a compact, oriented, semieffective d-orbifold with boundary $\boldsymbol{\mathcal{W}}$ with $\operatorname{vdim}\boldsymbol{\mathcal{W}}=k+1$, a 1-morphism $\boldsymbol{e}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb^b}$, an equivalence of oriented d-orbifolds $\boldsymbol{j}:\boldsymbol{\mathcal{X}}\to\partial\boldsymbol{\mathcal{W}}$, and a 2-morphism $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{e}\circ\boldsymbol{i}_{\boldsymbol{\mathcal{W}}}\circ\boldsymbol{j}$, where $\boldsymbol{\mathcal{X}},\boldsymbol{f}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}(X,f)$.

Theorem 12.27 yields a very good coordinate system $\bigl(I,<,(\mathcal{V}_i,\mathcal{E}_i,s_i,\boldsymbol{\psi}_i),(\mathcal{V}_{ij},e_{ij},\hat e_{ij},\boldsymbol{\eta}_{ij}),\eta_{ijk},g_i,\boldsymbol{\zeta}_i,\zeta_{ij}\bigr)$ for $\boldsymbol{e}:\boldsymbol{\mathcal{W}}\to\boldsymbol{\mathcal{Y}}$, with $I\subset\mathbb{N}$ finite. Here $\mathcal{V}_i$ are orbifolds with boundary, and as $\partial\boldsymbol{\mathcal{W}}\simeq\boldsymbol{\mathcal{X}}$ is an orbifold, s_i is transverse on $\partial\mathcal{V}_i$, and so near $\partial\mathcal{V}_i$, for each $i\in I$.

We now use a similar argument to the above to choose transverse perturbations $\tilde s_i$ of s_i for $i\in I$, with $\tilde s_i=s_i$ near $\partial\mathcal{V}_i$, such that the $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,\tilde s_i}$ may be glued using Theorem 12.23 to get a compact, oriented d-orbifold with boundary $\boldsymbol{\mathcal{Z}}$ with a 1-morphism $\boldsymbol{h}:\boldsymbol{\mathcal{Z}}\to\boldsymbol{\mathcal{Y}}$, such that $\boldsymbol{\mathcal{Z}}\simeq F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{W})$ and $\boldsymbol{h}\cong F_{\mathbf{Orb}}^{\mathbf{dOrb}}(e)$ for some oriented orbifold $\tilde{\mathcal{W}}$ and 1-morphism $\tilde{e}:\tilde{\mathcal{W}}\to\mathcal{Y}$ with $\partial\tilde{\mathcal{W}}\simeq\mathcal{X}$ and $\tilde{e}\circ i_{\tilde{\mathcal{W}}}\cong f$. Then $[\mathcal{X},f]=0$ in $B_k^{\rm orb}(\mathcal{Y})$ by Definition 13.12, so $\Pi_{\rm orb}^{\rm sef}:B_k^{\rm orb}(\mathcal{Y})\to dB_k^{\rm sef}(\mathcal{Y})$ is injective, and thus an isomorphism, proving the first part of Theorem 13.23.

For $\Pi_{\rm eff}^{\rm deff}:B_k^{\rm eff}(\mathcal{Y})\to dB_k^{\rm eff}(\mathcal{Y})$, we modify the above arguments as follows. For the first part of the proof, we take $[\boldsymbol{\mathcal{X}},\boldsymbol{f}]\in dB_k^{\rm eff}(\mathcal{Y})$, so that $\boldsymbol{\mathcal{X}}$ is effective, and show that $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,\tilde{s}_i},\boldsymbol{\mathfrak{S}}_{\mathcal{W}_i,\mathcal{F}_i,t_i}$ are effective in (C),(D), and $\boldsymbol{\mathfrak{Z}},\tilde{X}$ are both effective. Hence $\Pi_{\rm eff}^{\rm deff}:B_k^{\rm eff}(\mathcal{Y})\to dB_k^{\rm eff}(\mathcal{Y})$ is surjective. For the second part, we suppose $[\mathcal{X},f]\in B_k^{\rm eff}(\mathcal{Y})$ with $\Pi_{\rm eff}^{\rm deff}\bigl([\mathcal{X},f]\bigr)=0$, so that $\mathcal{X},\boldsymbol{\mathcal{W}}$ are both effective, and then we show we can take $\boldsymbol{\mathfrak{Z}},\tilde{\mathcal{W}}$ to be effective, so that $[\mathcal{X},f]=0$ in $B_k^{\rm eff}(\mathcal{Y})$, and $\Pi_{\rm eff}^{\rm deff}$ is injective, giving the second part of Theorem 13.23.

14 Relating d-manifolds and d-orbifolds to other classes of spaces in mathematics

We now explain the relationships between d-manifolds and d-orbifolds, and several other classes of geometric spaces in mathematics: zero sets of Fredholm sections of Banach vector bundles over Banach manifolds or Banach orbifolds, the polyfolds of Hofer, Wysocki and Zehnder [41–48], the Kuranishi spaces of Fukaya, Oh, Ohta and Ono [32, 34], $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-stacks with obstruction theories as in Behrend and Fantechi [12], quasi-smooth derived $\mathbb{C}$-schemes and derived Deligne–Mumford $\mathbb{C}$-stacks as in Toën and Vezzosi [100–102], and Spivak's derived manifolds [94, 95]. These include all the major structures used in differential geometry and algebraic geometry to define virtual classes and virtual chains in enumerative invariant problems over $\mathbb{R}$ or $\mathbb{C}$.

In each case (except for Kuranishi spaces, for which morphisms are not defined), we will define truncation functors from categories of these geometric spaces to the homotopy categories $\mathrm{Ho}(\mathbf{dMan}),\ldots,\mathrm{Ho}(\mathbf{dOrb^c})$ of d-manifolds and d-orbifolds, possibly with corners. Therefore, by quoting theorems from the literature on existence of one these geometric structures on moduli spaces, we deduce that many important classes of moduli spaces in differential and algebraic geometry may be given the structure of d-manifolds or d-orbifolds.

Since truncation functors forget information, one moral is that d-manifolds and d-orbifolds are actually simpler, more basic objects than the other geometric spaces we discuss. For example, compared to d-orbifolds, polyfolds contain a huge amount of extra information, which for the purposes of defining virtual classes and virtual chains is redundant, and actually makes the problem rather more difficult: there are subtle analytic issues about abstract perturbations of polyfolds which disappear at the level of d-orbifolds.

14.1 Fredholm sections on Banach manifolds and solution spaces of nonlinear elliptic equations

Definition 14.1. The theory of manifolds and differential geometry can be developed, with little extra effort, using open sets in (possibly infinite-dimensional) Banach spaces as local models. This yields the theory of *Banach manifolds.* A good reference is Lang [65]. We require Banach manifolds to be Hausdorff and second countable. Ordinary manifolds are examples of Banach manifolds.

We will use the ideas of *smooth maps* between Banach manifolds, *Banach vector bundles* E over Banach manifolds V, which are fibre bundles $E \to V$ whose fibres E_x are Banach spaces, and *smooth sections* $s : V \to E$. Let s be a smooth section of a Banach vector bundle E over a Banach manifold V. We call s *Fredholm* if for each $x \in V$ with $s(v) = 0$, the derivative $\mathrm{d}s|_x : T_xV \to E|_x$, which is a continuous linear map between Banach spaces, is Fredholm. Note that $\mathrm{d}s|_x$ has an *index* $\mathrm{ind}(\mathrm{d}s|_x) = \dim \mathrm{Ker}(\mathrm{d}s|_x) - \dim \mathrm{Coker}(\mathrm{d}s|_x)$ in $\mathbb{Z}$.

If V is a Banach manifold, $E, F \to V$ are Banach vector bundles, and $s \in C^\infty(E)$, $t_1, t_2 \in C^\infty(F)$ are smooth sections, then the notation $t_1 = t_2 + O(s)$

and $t_1 = t_2 + O(s^2)$ in Definition 3.29 makes sense in this infinite-dimensional setting. Similarly, if W is another Banach manifold and $f, g : V \to W$ are smooth maps then the notation $f = g + O(s)$ and $f = g + O(s^2)$ in Definition 3.29 also makes sense in this infinite-dimensional setting.

In Definitions 3.13 and 3.30 we defined 'standard model' d-manifolds $\boldsymbol{S}_{V,E,s}$ and 1-morphisms $\boldsymbol{S}_{f,\hat{f}}$. The next theorem generalizes these to Banach manifolds. Part (a) is related to Fukaya et al. [32, Ex. A1.7] for Kuranishi spaces. We will prove (a),(b) in §14.1.1. In (b), we interpret Υ as follows: ignoring the fact that V_2, E_2, s_2 may be infinite-dimensional, think of $(f|_{W_1}, \hat{f}|_{W_1}), (g, \hat{g}) : (W_1, F_1, t_1) \to (V_2, E_2, s_2)$ as defining 'standard model' 1-morphisms as in Definition 3.30, and $\Upsilon : (f|_{W_1}, \hat{f}|_{W_1}) \Rightarrow (g, \hat{g})$ as defining a 'standard model' 2-morphism as in Definition 3.35, where (14.2)–(14.3) correspond to (3.30).

Part (c) is easy: we must show $\Pi^{\mathbf{dMan}}_{\mathbf{BManFS}}(\mathrm{id}_V, \mathrm{id}_E) = [\boldsymbol{\mathrm{id}}_{\boldsymbol{X}}]$ and $\Pi^{\mathbf{dMan}}_{\mathbf{BManFS}}$ preserves composition, but these follow from the characterization of the 1-morphism $\boldsymbol{h}$ in part (b). We leave (d) as an exercise for the reader.

Theorem 14.2. **(a)** *Let V be a Banach manifold, $E \to V$ a Banach vector bundle, and $s : V \to E$ a Fredholm section. Set $X = s^{-1}(0) \subseteq V$, and suppose $\mathrm{d}s|_x : T_xV \to E|_x$ has Fredholm index $n \in \mathbb{Z}$ for all $x \in X$. Then we may construct a d-manifold $\boldsymbol{X}$, natural up to equivalence in* **dMan**, *with underlying topological space X and virtual dimension n.*

The d-manifold structure on $\boldsymbol{X}$ may be characterized as follows: suppose W is a finite-dimensional embedded submanifold of V, and $F \to W$ a finite rank vector subbundle of $E|_W$, such that $t := s|_W$ lies in $C^\infty(F) \subseteq C^\infty(E|_W)$, and for every $x \in \hat{X} := W \cap X$, the map $(\mathrm{d}s|_x)_*$ in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_xW \arrow[r, "\mathrm{inc}"'] \arrow[d, "\mathrm{d}t|_x"'] & T_xV \arrow[r] \arrow[d, "\mathrm{d}s|_x"'] & T_xV/T_xW \arrow[r] \arrow[d, "(\mathrm{d}s|_x)_*"'] & 0 \\ 0 \arrow[r] & F|_x \arrow[r, "\mathrm{inc}"] & E|_x \arrow[r] & E|_x/F|_x \arrow[r] & 0 \end{tikzcd} \tag{14.1}$$

is an isomorphism. Then $\hat{X} := W \cap X = t^{-1}(0)$ is open in X, and there is an equivalence $\boldsymbol{\psi} : \boldsymbol{S}_{W,F,t} \to \hat{\boldsymbol{X}} \subseteq \boldsymbol{X}$ in **dMan**, *which acts as the identity map $t^{-1}(0) \to \hat{X}$ on topological spaces, where $\boldsymbol{S}_{W,F,t}$ is given in Definition 3.13.*

(b) *Let V_a, E_a, s_a, X_a, n_a and $\boldsymbol{X}_a$ be as in* **(a)** *for $a = 1, 2$. Suppose $f : V_1 \to V_2$ is a smooth map of Banach manifolds, and $\hat{f} : E_1 \to f^*(E_2)$ is a morphism of Banach vector bundles on V_1 satisfying $\hat{f} \circ s_1 = f^*(s_2) + O(s_1^2)$ in $C^\infty(f^*(E_2))$, as in (3.22). Then we may construct a 1-morphism $\boldsymbol{h} : \boldsymbol{X}_1 \to \boldsymbol{X}_2$ in* **dMan**, *natural up to 2-isomorphism, with continuous map $h = f|_{X_1} : X_1 \to X_2$.*

This 1-morphism $\boldsymbol{h}$ may be characterized as follows: for $a = 1, 2$ suppose $W_a \subseteq V_a$ is a finite-dimensional embedded submanifold and $F_a \to W_a$ a finite rank vector subbundle of $E_a|_{W_a}$ with $t_a := s_a|_{W_a} \in C^\infty(F_a) \subseteq C^\infty(E_a|_{W_a})$ and $(\mathrm{d}s_a|_x)_*$ in (14.1) an isomorphism for all $x \in W_a \cap X_a$. Then **(a)** *defines equivalences $\boldsymbol{\psi}_a : \boldsymbol{S}_{W_a,F_a,t_a} \to \hat{\boldsymbol{X}}_a \subseteq \boldsymbol{X}_a$ for $a = 1, 2$.*

Suppose $g : W_1 \to W_2$ is smooth and $\hat{g} : F_1 \to g^*(F_2)$, $\Upsilon : F_1 \to f|_{W_1}^*(TV_2)$ are morphisms of vector bundles on W_1 satisfying, as in (3.30)*:*

$$g = f|_{W_1} + \Upsilon \circ t_1 + O(t_1^2), \tag{14.2}$$

$$\hat{g} = \hat{f}|_{F_1} + f|_{W_1}^*(\mathrm{d}s_2) \circ \Upsilon + O(t_1), \tag{14.3}$$

where (14.2) *holds in smooth maps $W_1 \to V_2$, and* (14.3) *in vector bundle morphisms $F_1 \to f|_{W_1}^*(E_2)$ or $F_1 \to g^*(E_2)$. Then $\hat{g} \circ t_1 = g^*(t_2) + O(t_1^2)$ in $C^\infty(g^*(F_2))$, so that Definition* 3.30 *gives a* 1*-morphism $\boldsymbol{S}_{g,\hat{g}} : \boldsymbol{S}_{W_1,F_1,t_1} \to \boldsymbol{S}_{W_2,F_2,t_2}$, and we have a* 2*-commutative diagram in* **dMan***:*

$$\begin{tikzcd} \boldsymbol{S}_{W_1,F_1,t_1} \arrow[rr, "\boldsymbol{S}_{g,\hat{g}}"] \arrow[d, "\boldsymbol{\psi}_1"'] \arrow[drr, phantom, "\Uparrow"] & & \boldsymbol{S}_{W_2,F_2,t_2} \arrow[d, "\boldsymbol{\psi}_2"] \\ \boldsymbol{X}_1 \arrow[rr, "\boldsymbol{h}"] & & \boldsymbol{X}_2. \end{tikzcd} \tag{14.4}$$

(c) *Define a category* **BManFS** *of 'Banach manifolds with Fredholm sections' to have objects (V, E, s) as in* **(a)***, and morphisms $(V_1, E_1, s_1) \to (V_2, E_2, s_2)$ pairs $(f, \hat{f})$ as in* **(b)***, with composition $(g, \hat{g}) \circ (f, \hat{f}) = \big(g \circ f, \hat{f} \circ f^*(\hat{g})\big)$. Define a functor $\Pi_{\mathbf{BManFS}}^{\mathbf{dMan}} : \mathbf{BManFS} \to \mathrm{Ho}(\mathbf{dMan})$ as follows, where* Ho(**dMan**) *is the homotopy category of the* 2*-category* **dMan***.*

For each (V, E, s) in **BManFS***, choose a d-manifold $\boldsymbol{X}$ in the equivalence class in* **dMan** *given by* **(a)***, and set $\Pi_{\mathbf{BManFS}}^{\mathbf{dMan}}(V, E, s) = \boldsymbol{X}$. For each morphism $(f, \hat{f}) : (V_1, E_1, s_1) \to (V_2, E_2, s_2)$, part* **(b)** *defines a* 1*-morphism $\boldsymbol{h} : \Pi_{\mathbf{BManFS}}^{\mathbf{dMan}}(V_1, E_1, s_1) = \boldsymbol{X}_1 \to \boldsymbol{X}_2 = \Pi_{\mathbf{BManFS}}^{\mathbf{dMan}}(V_2, E_2, s_2)$ in* **dMan** *unique up to* 2*-isomorphism, so the morphism $[\boldsymbol{h}] : \boldsymbol{X}_1 \to \boldsymbol{X}_2$ in* Ho(**dMan**) *is uniquely defined. Set $\Pi_{\mathbf{BManFS}}^{\mathbf{dMan}}(f, \hat{f}) = [\boldsymbol{h}]$. Then $\Pi_{\mathbf{BManFS}}^{\mathbf{dMan}}$ is a functor.*

(d) *Analogues of* **(a)**–**(c)** *also hold with Banach orbifolds, orbifolds, and d-orbifolds in place of Banach manifolds, manifolds, and d-manifolds, yielding a functor $\Pi_{\mathbf{BOrbFS}}^{\mathbf{dOrb}} : \mathrm{Ho}(\mathbf{BOrbFS}) \to \mathrm{Ho}(\mathbf{dOrb})$.*

Similarly, analogues of all the above hold involving Banach manifolds (or orbifolds) with corners, manifolds (or orbifolds) with corners, and d-manifolds (or d-orbifolds) with corners, yielding functors $\Pi_{\mathbf{BMan^cFS}}^{\mathbf{dMan^c}} : \mathbf{BMan^cFS} \to \mathrm{Ho}(\mathbf{dMan^c})$ and $\Pi_{\mathbf{BOrb^cFS}}^{\mathbf{dOrb^c}} : \mathrm{Ho}(\mathbf{BOrb^cFS}) \to \mathrm{Ho}(\mathbf{dOrb^c})$.

In (b), a useful case is when W is a finite-dimensional manifold and $F = t = 0$, so that $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{W} = F_{\mathbf{Man}}^{\mathbf{dMan}}(W)$. We write Ho(**BOrbFS**) in (d) as triples $(\mathcal{V}, \mathcal{E}, s)$ of a Banach orbifold $\mathcal{V}$, Banach vector bundle $\mathcal{E}$ over $\mathcal{V}$, and Fredholm section $s \in C^\infty(\mathcal{E})$, naturally form a 2-category, since Banach orbifolds are.

It is well known that smooth nonlinear elliptic equations on compact manifolds, when written in Hölder spaces $C^{k,\alpha}$ or Sobolev spaces L_k^p for sufficiently large k, yield Fredholm sections of Banach vector bundles over Banach manifolds. So we may deduce the following (somewhat informal) corollary. In it, by 'smooth nonlinear elliptic equation' we exclude problems which involve dividing out by a group of symmetries, and moduli spaces which are compactified by including singular solutions. Both require more sophisticated techniques.

By ‘with fixed topological invariants’, we mean that at each point x of the solution set $s^{-1}(0)$, the index of the linear elliptic operator $\mathrm{d}s|_x$ may be computed using the Atiyah–Singer Index Theorem in terms of certain characteristic classes on the compact manifold. We require that these characteristic classes are fixed, so that the virtual dimension is constant on the solution set.

Corollary 14.3. *Any solution set of a smooth nonlinear elliptic equation with fixed topological invariants on a compact manifold naturally has the structure of a d-manifold, uniquely up to equivalence in* **dMan**.

Here are two examples. The first is generalized in Corollary 14.7 below.

Example 14.4. Let (Σ, j) be a compact Riemann surface of genus g, X a $2n$-manifold, and J an almost complex structure on X. Let $\beta \in H_2(X;\mathbb{Z})$, and consider the family $\mathcal{M}_\Sigma(X,J,\beta)$ of *J-holomorphic maps* $u : \Sigma \to X$ with $u_*([\Sigma]) = \beta$ in $H_2(X;\mathbb{Z})$, where u J-holomorphic means that $J \circ \mathrm{d}u = \mathrm{d}u \circ j : T\Sigma \to u^*(TX)$. This is a smooth, nonlinear first-order elliptic equation.

We write this in terms of Banach manifolds as follows: for $k \geqslant 1$ and $\alpha \in (0,1)$ we write $V = C^{k,\alpha}(\Sigma, X)_\beta$ for the Banach manifold of Hölder $C^{k,\alpha}$ maps $u : \Sigma \to X$ with $u_*([\Sigma]) = \beta \in H_2(X;\mathbb{Z})$. We define $E \to V$ to be the Banach vector bundle with fibre $C^{k-1,\alpha}\big(u^*(TX) \otimes_{\mathbb{C}} \Lambda^{0,1}T^*\Sigma\big)$ at each $u \in V$. We define $s : V \to E$ to map $u \mapsto \bar\partial u = \frac{1}{2}(\mathrm{d}u + J \circ \mathrm{d}u \circ j)$. Then $s(u) = 0$ if and only if u is J-holomorphic. If $u \in V$ with $\bar\partial u = 0$, then the linearization $\mathrm{d}_u s : C^{k,\alpha}\big(u^*(TX)\big) \to C^{k-1,\alpha}\big(u^*(TX) \otimes_{\mathbb{C}} \Lambda^{0,1}T^*\Sigma\big)$ is a $\bar\partial$-operator, whose index, computed using the Atiyah–Singer Index Theorem, is $2\big(c_1(X)\cdot\beta + n(1-g)\big)$, where $c_1(X) \in H^2(X;\mathbb{Z})$ is the first Chern class of (X,J).

Thus Theorem 14.2(a) gives $\mathcal{M}_\Sigma(X,J,\beta) = s^{-1}(0)$ the structure of a d-manifold $\boldsymbol{\mathcal{M}}_\Sigma(X,J,\beta)$, with virtual dimension $2\big(c_1(X)\cdot\beta + n(1-g)\big)$. Note that elliptic regularity implies that if $u \in C^{k,\alpha}(\Sigma,X)_\beta$ with $s(u) = 0$ then $u \in C^\infty(\Sigma,X)_\beta$, so $s^{-1}(0)$ is independent of the choice of k,α, and in fact $\boldsymbol{\mathcal{M}}_\Sigma(X,J,\beta)$ is also independent of k,α up to equivalence in **dMan**.

Let $p \in \Sigma$. Then evaluation at p gives a smooth map of Banach manifolds $\mathrm{ev}_p : V \to X$ mapping $\mathrm{ev}_p : u \mapsto u(p)$. So Theorem 14.2(b) with $W = X$ and $F = t = 0$ gives a 1-morphism $\mathbf{ev}_p : \boldsymbol{\mathcal{M}}_\Sigma(X,J,\beta) \to \boldsymbol{X} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X)$.

Example 14.5. Let (M,g) and (N,h) be Riemannian manifolds, with M compact. Consider the family $\mathcal{H}_{M,N}$ of *harmonic maps* $f : M \to N$. This is a nonlinear second-order elliptic equation on f. Its linearizations are self-adjoint operators of Laplacian type, with index zero. Hence Corollary 14.3 makes $\mathcal{H}_{M,N}$ into a d-manifold $\boldsymbol{\mathcal{H}}_{M,N}$ with $\mathrm{vdim}\,\boldsymbol{\mathcal{H}}_{M,N} = 0$. If $M = \mathcal{S}^1$, then $\boldsymbol{\mathcal{H}}_{M,N}$ is the moduli space of *parametrized closed geodesics* in (N,h).

14.1.1 The proof of Theorem 14.2(a),(b)

In the proof below we have many equations such as (14.5)–(14.6) involving error terms $O(t_1)$ and $O(t_1^2)$, using the notation of Definition 3.29. To interpret these, one should first choose local trivializations (coordinates) of the Banach manifolds V, V_1, V_2 or Banach vector bundles E, E_1, E_2 involved, so that each term in the equation takes values in the same Banach space, and it makes sense to add and subtract them. Choosing different local trivializations gives different equations, but the differences are absorbed in the $O(t_1)$ or $O(t_1^2)$ error terms, so whether the equation holds or not is independent of the choice.

For example, (14.5) at $w \in W_{12}$ gives $e_{12}(w) = f(w) + \Upsilon|_w \circ t_1|_w + O(t_1|_w^2)$. Here $e_{12}(w), f(w)$ are points in V_2, and $\Upsilon|_w \circ t_1|_w \in T_{f(w)}V_2$. If we identify the Banach manifold V_2 and its tangent space $T_{f(w)}V_2$ locally with a Banach space B, then the equation makes sense in B. Similarly, (14.6) at $w \in W_{12}$ applied to $\alpha \in F_1|_w$ gives $\hat{e}_{12}|_w(\alpha) = \hat{f}|_w(\alpha) + \mathrm{d}s_2|_{f(w)} \circ \Upsilon|_w(\alpha) + O(t_1|_w)$. Here $\hat{e}_{12}|_w(\alpha)$ lies in $F_2|_{e_{12}(w)} \subseteq E_2|_{e_{12}(w)}$, and $\hat{f}|_w(\alpha), \mathrm{d}s_2|_{f(w)} \circ \Upsilon|_w(\alpha)$ in $E_2|_{f(w)}$. If we trivialize the Banach vector bundle E_2 locally then $E_2|_{e_{12}(w)}$ and $E_2|_{f(w)}$ are identified, and the equation makes sense.

We will prove Theorem 14.2(a),(b) in the following four steps:

Step 1. Let $V, E, s, X = s^{-1}(0)$ and $n \in \mathbb{Z}$ be as in Theorem 14.2(a). We say that a triple (W, F, t) *satisfies condition* $(*)$ if:

$(*)$ W is a finite-dimensional embedded submanifold of V, so that we may regard W as a subset of V, and $F \to W$ is a finite rank vector subbundle of $E|_W$, and $t := s|_W$ lies in $C^\infty(F) \subseteq C^\infty(E|_W)$, and $W \cap X \neq \emptyset$, and for all $x \in W \cap X$, the map $(\mathrm{d}s|_x)_*$ in (14.1) is an isomorphism.

We show that if (W, F, t) satisfies $(*)$ then $\dim W - \operatorname{rank} F = n$, and $W \cap X$ is open in X. Hence the 'standard model' d-manifold $\boldsymbol{S}_{W,F,t}$ from Definition 3.13 has virtual dimension n, and its topological space $t^{-1}(0) = W \cap X$ is open in X. We prove that for all $x \in X$, there exists (W, F, t) satisfying $(*)$ with $x \in W$.

Step 2. Let $V_1, E_1, s_1, X_1,\ V_2, E_2, s_2, X_2,\ f : V_1 \to V_2$ and $\hat{f} : E_1 \to f^*(E_2)$ with $\hat{f} \circ s_1 = f^*(s_2) + O(s_1^2)$ be as in Theorem 14.2(b). Let (W_1, F_1, s_1) satisfy $(*)$ in (V_1, E_1, s_1) and (W_2, F_2, s_2) satisfy $(*)$ in (V_2, E_2, s_2). We say that $(W_{12}, e_{12}, \hat{e}_{12}) : (W_1, F_1, s_1) \to (W_2, F_2, s_2)$ *satisfies condition* $(+)$ if:

$(+)$ $W_{12} \subseteq W_1$ is open with $W_{12} \cap X_1 = W_1 \cap f^{-1}(W_2) \cap X_1$, and $e_{12} : W_{12} \to W_2$ is smooth, and $\hat{e}_{12} : F_1|_{W_{12}} \to e_{12}^*(F_2)$ is a morphism of vector bundles on W_{12}, and there exists a morphism of Banach vector bundles $\Upsilon : F_1|_{W_{12}} \to f^*(TV_2)|_{W_{12}}$ on W_{12} satisfying

$$e_{12} = f|_{W_{12}} + \Upsilon \circ t_1|_{W_{12}} + O(t_1^2), \tag{14.5}$$

$$\hat{e}_{12} = \hat{f}|_{F_1|_{W_{12}}} + f^*(\mathrm{d}s_2)|_{W_{12}} \circ \Upsilon + O(t_1), \tag{14.6}$$

where (14.5) holds in smooth maps $W_{12} \to V_2$, and (14.6) in vector bundle morphisms $F_1|_{W_{12}} \to f^*(E_2)|_{W_{12}}$, and $f^*(\mathrm{d}s_2)|_{W_{12}}$ in (14.6) lies in $\mathrm{Hom}\big(f^*(TV_2)|_{W_{12}}, f^*(E_2)|_{W_{12}}\big)$.

We will prove:

(a) Suppose that, as well as the data above, we are given $V_3, E_3, s_3, X_3,\ g : V_2 \to V_3$ and $\hat{g} : E_2 \to g^*(E_3)$ with $\hat{g} \circ s_2 = g^*(s_3) + O(s_2^2)$ as in Theorem

14.2(b). Then $h := g \circ f : V_1 \to V_2$ and $\hat{h} := f^*(\hat{g}) \circ \hat{f} : E_1 \to h^*(E_3)$ satisfy $\hat{h} \circ s_1 = h^*(s_3) + O(s_1^2)$, as in Theorem 14.2(b).

If $(W_{12}, e_{12}, \hat{e}_{12}) : (W_1, F_1, s_1) \to (W_2, F_2, s_2)$ satisfies (+) for $(f, \hat{f}) : (V_1, E_1, s_1) \to (V_2, E_2, s_2)$, and $(W_{23}, e_{23}, \hat{e}_{23}) : (W_2, F_2, s_2) \to (W_3, F_3, s_3)$ satisfies (+) for $(g, \hat{g}) : (V_2, E_2, s_2) \to (V_3, E_3, s_3)$, then

$$\left(e_{12}^{-1}(W_{23}), e_{23} \circ e_{12}|_{e_{12}^{-1}(W_{23})}, e_{12}|^*_{e_{12}^{-1}(W_{23})}(\hat{e}_{23}) \circ \hat{e}_{12}|_{e_{12}^{-1}(W_{23})}\right) \tag{14.7}$$

satisfies (+) for $(h, \hat{h}) : (V_1, E_1, s_1) \to (V_3, E_3, s_3)$. That is, condition (+) is closed under composition in a suitable sense.

(b) If $(W_{12}, e_{12}, \hat{e}_{12}) : (W_1, F_1, s_1) \to (W_2, F_2, s_2)$ satisfies (+) then

$$\hat{e}_{12} \circ t_1|_{W_{12}} = e_{12}^*(t_2) + O(t_1^2). \tag{14.8}$$

Hence Definition 3.30 defines a 'standard model' 1-morphism in **dMan**:

$$\boldsymbol{S}_{e_{12},\hat{e}_{12}} : \boldsymbol{S}_{W_{12},F_1|_{W_{12}},t_1|_{W_{12}}} \longrightarrow \boldsymbol{S}_{W_2,F_2,t_2}. \tag{14.9}$$

(c) If $(W_{12}, e_{12}, \hat{e}_{12}), (W'_{12}, e'_{12}, \hat{e}'_{12}) : (W_1, F_1, s_1) \to (W_2, F_2, s_2)$ both satisfy (+) then there exists a morphism $\Lambda : F_1|_{W_{12} \cap W'_{12}} \to e_{12}|^*_{W_{12} \cap W'_{12}}(TW_2)$ of vector bundles on $W_{12} \cap W'_{12}$ satisfying

$$e'_{12}|_{W_{12} \cap W'_{12}} = e_{12}|_{W_{12} \cap W'_{12}} + \Lambda \circ t_1|_{W_{12} \cap W'_{12}} + O(t_1^2), \tag{14.10}$$
$$\hat{e}'_{12}|_{W_{12} \cap W'_{12}} = \hat{e}_{12}|_{W_{12} \cap W'_{12}} + (e_{12}|^*_{W_{12} \cap W'_{12}}(\mathrm{d}t_2)) \circ \Lambda + O(t_1). \tag{14.11}$$

As $W_{12} \cap t_1^{-1}(0) = W'_{12} \cap t_1^{-1}(0) = W_1 \cap f^{-1}(W_2) \cap X_1$, we can replace W_{12}, W'_{12} by $W_{12} \cap W'_{12}$ without changing the 1-morphisms (14.8) for $e_{12}, \hat{e}_{12}$ and $e'_{12}, \hat{e}'_{12}$. Thus Definition 3.35 gives a 'standard model' 2-morphism

$$\boldsymbol{S}_\Lambda : \boldsymbol{S}_{e_{12},\hat{e}_{12}} \Longrightarrow \boldsymbol{S}_{e'_{12},\hat{e}'_{12}}. \tag{14.12}$$

(d) If (W_1, F_1, s_1) satisfies $(*)$ in (V_1, E_1, s_1) and (W_2, F_2, s_2) satisfies $(*)$ in (V_2, E_2, s_2), then there exists $(W_{12}, e_{12}, \hat{e}_{12})$ satisfying (+).

Step 3. Using Step 1, we choose an indexing set I, and for each $i \in I$ a triple (W_i, F_i, t_i) satisfying $(*)$, such that $\{W_i \cap X : i \in I\}$ is an open cover of X. Using Step 2(d) with $(V_1, E_1, s_1) = (V_2, E_2, s_2) = (V, E, s)$, $f = \mathrm{id}_V$ and $\hat{f} = \mathrm{id}_E$ for all $i, j \in I$, we choose $(W_{ij}, e_{ij}, \hat{e}_{ij}) : (W_i, F_i, t_i) \to (W_j, F_j, t_j)$ satisfying (+). We then use Theorem 3.42 to construct a d-manifold $\boldsymbol{X}$ with topological space X and $\operatorname{vdim} \boldsymbol{X} = n$ by gluing the d-manifolds $\boldsymbol{S}_{W_i,F_i,t_i}$ for $i \in I$ on overlaps using $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}$ for $i, j \in I$, which are equivalences of open d-submanifolds. We show that $\boldsymbol{X}$ is independent of choices up to equivalence in **dMan**, and satisfies the condition in the second part of Theorem 14.2(a) for any such (W, F, t).

Step 4. Using Step 3, we construct a d-manifold $\boldsymbol{X}_1$ using triples (W_i, F_i, t_i) satisfying $(*)$ for $i \in I$ with $\{W_i \cap X_1 : i \in I\}$ an open cover of X_1, with equivalences $\boldsymbol{\psi}_i : \boldsymbol{S}_{W_i,F_i,t_i} \to \hat{\boldsymbol{X}}_{1,i} \subseteq \boldsymbol{X}_1$, and a d-manifold $\boldsymbol{X}_2$ using triples (W'_k, F'_k, t'_k) satisfying $(*)$ for $k \in K$ with $\{W_k \cap X_2 : k \in K\}$ an open cover of X_2, with equivalences $\boldsymbol{\psi}'_k : \boldsymbol{S}_{W'_k,F'_k,t'_k} \to \hat{\boldsymbol{X}}_{2,k} \subseteq \boldsymbol{X}_2$.

We choose these so that for each $i \in I$ there exists $k_i \in K$ with $f(W_i \cap X_1) \subseteq W'_{k_i} \cap X_2$, and Step 2(d) gives a triple $(W_i, g_{ik_i}, \hat{g}_{ik_i}) : (W_i, F_i, t_i) \to (W'_{k_i}, F'_{k_i}, t'_{k_i})$ satisfying $(+)$. Then $\hat{g}_{ik_i} \circ t_i = g^*_{ik_i}(t'_{k_i}) + O(t_i^2)$ by Step 2(b), so Definition 3.30 gives a 1-morphism

$$\boldsymbol{S}_{g_{ik_i},\hat{g}_{ik_i}} : \boldsymbol{S}_{W_i,F_i,t_i} \longrightarrow \boldsymbol{S}_{W'_{k_i},F'_{k_i},t'_{k_i}}. \tag{14.13}$$

Composing with $\boldsymbol{\psi}'_{k_i}$ gives a 1-morphism

$$\boldsymbol{\psi}'_{k_i} \circ \boldsymbol{S}_{g_{ik_i},\hat{g}_{ik_i}} : \boldsymbol{S}_{W_i,F_i,t_i} \longrightarrow \boldsymbol{X}_2. \tag{14.14}$$

Now $\boldsymbol{X}_1$ is constructed, as a d-space, by gluing the d-spaces $\boldsymbol{S}_{W_i,F_i,t_i}$ for i in I by equivalences on overlaps, using the first part of Theorem 2.33. We show that the 1-morphisms (14.14) satisfy the conditions on the $\boldsymbol{g}_i$ in the second part of Theorem 2.33, so we can glue them to give a 1-morphism $\boldsymbol{h} : \boldsymbol{X}_1 \to \boldsymbol{X}_2$, unique up to 2-isomorphism, such that $\boldsymbol{h} \circ \boldsymbol{\psi}_i \cong \boldsymbol{\psi}'_k \circ \boldsymbol{S}_{g_{ik_i},\hat{g}_{ik_i}}$ for all $i \in I$. We show that $\boldsymbol{h}$ is independent of choices up to 2-isomorphism, and satisfies the conditions in the second part of Theorem 14.2(b) for all such $W_1, F_1, t_1, W_2, F_2, t_2, g, \hat{g}$. This then completes the proof.

For Step 1, let (W, F, t) satisfy $(*)$, and let $x \in W \cap X \neq \emptyset$. Then as $(\mathrm{d}s|_x)_*$ in (14.1) is an isomorphism, we have an exact sequence

$$\mathrm{Ker}(\mathrm{d}s|_x) \longrightarrow T_xW \xrightarrow{\mathrm{d}t|_x} F|_x \longrightarrow \mathrm{Coker}(\mathrm{d}s|_x)$$

of finite-dimensional vector spaces. Therefore, as we have to prove,

$$\begin{aligned}\dim W - \operatorname{rank} F &= \dim T_xW - \dim F|_x \\ &= \dim \mathrm{Ker}(\mathrm{d}s|_x) - \dim \mathrm{Coker}(\mathrm{d}s|_x) = \mathrm{ind}(\mathrm{d}s|_x) = n.\end{aligned}$$

On an open neighbourhood U of x in V we can choose a decomposition $E|_U \cong F|_U \oplus G$ where G is a Banach vector subbundle of $E|_U$, and $G|_x \cong E_x/F_x$. Then $s|_U = p \oplus q$ for $p \in C^\infty(F|_U)$ and $p \in C^\infty(G)$, and $\mathrm{d}q|_x : T_xV \to G|_x \cong E_x/F_x$ is surjective with kernel T_xW, since in (14.1) the top row is exact and $(\mathrm{d}s|_x)_*$ is an isomorphism. We have $p|_{U\cap W} = t|_{U\cap W}$ and $q|_{U\cap W} = 0$. Hence $U \cap W \subseteq q^{-1}(0)$.

Using $\mathrm{d}q|_x : T_xV \to G|_x \cong E_x/F_x$ surjective with kernel T_xW and the Implicit Function Theorem for Banach spaces, we can show that $U \cap W$ coincides with $q^{-1}(0)$ near x. Hence $U \cap W \cap X = t|_{U\cap W}^{-1}(0)$ coincides with $p^{-1}(0) \cap q^{-1}(0) = s|_U^{-1}(0) = U \cap X$ near x. Thus $W \cap X$ is open in X.

Now let $x \in X$. We will construct (W, F, t) satisfying $(*)$ with $x \in W$. By assumption $\mathrm{d}s|_x : T_xV \to E|_x$ is Fredholm. Choose an open neighbourhood U of x in V and $e_1, \ldots, e_k \in C^\infty(E|_U)$ such that $e_1, \ldots, e_k$ are linearly independent on U and $e_1|_x, \ldots, e_k|_x$ generate $\mathrm{Coker}(\mathrm{d}s|_x)$. Then $\langle e_1, \ldots, e_k\rangle$ is a rank k vector subbundle of $E|_U$. The quotient $E|_U/\langle e_1, \ldots, e_k\rangle$ is a Banach vector bundle over U, and s induces $s_* \in C^\infty(E|_U/\langle e_1, \ldots, e_k\rangle)$, which is also Fredholm.

The linearization $\mathrm{d}s_*|_x : T_xV \to E|_x/\langle e_1,\ldots,e_k\rangle|_x$ at x is surjective by choice of $e_1,\ldots,e_k$, and has kernel of dimension $k+\mathrm{ind}(\mathrm{d}s|_x)=k+n$. This is an open condition, so making U smaller we can suppose that $\mathrm{d}s_*|_w$ is surjective with kernel of dimension $k+n$ for all $w\in W:=(s_*)^{-1}(0)\subseteq U$. The Implicit Function Theorem for Banach spaces now implies that W is a nonsingular embedded submanifold of $U\subseteq V$, with finite dimension $k+n$. Set $F=\langle e_1,\ldots,e_k\rangle|_W$, as a rank k vector subbundle of $E|_W$. Then $t:=s|_W\in C^\infty(F)\subseteq C^\infty(E|_W)$, as $s_*|_W=0$. Also $\mathrm{d}s_*|_w$ and $T_wW=\mathrm{Ker}(\mathrm{d}s_*)$ implies that $(\mathrm{d}s|_w)_*$ in (14.1) is an isomorphism for each $w\in W$. So (W,F,t) satisfies $(*)$. This completes Step 1.

For Step 2(a), if $e_{12},\hat e_{12}$ satisfy (14.5)–(14.6) with $\Upsilon : F_1|_{W_{12}}\to f^*(TV_2)|_{W_{12}}$, and $e_{23},\hat e_{23}$ satisfy (14.5)–(14.6) with $\Upsilon : F_2|_{W_{23}}\to g^*(TV_3)|_{W_{23}}$, then it is not difficult to show that (14.7) satisfies (+) using $\Upsilon'' : F_1|_{e_{12}^{-1}(W_{23})}\to (g\circ f)^*(TV_3)|_{e_{12}^{-1}(W_{23})}$ in (14.5)–(14.6), where

$$\Upsilon''=(f^*(\mathrm{d}g)\circ\Upsilon)|_{e_{12}^{-1}(W_{23})}+e_{12}|^*_{e_{12}^{-1}(W_{23})}(\Upsilon')\circ\hat e_{12}|_{e_{12}^{-1}(W_{23})}+O(t_1).$$

For Step 2(b), work near a point $w\in W_{12}$ close to X_1, so that $t_1(w)$ is small and $f(w)$ and $e_{12}(w)$ are close in V_2 by (14.5), and choose local trivializations of V_2 and E_2 near $f(w)\approx e_{12}(w)$, so that locally $V_2\cong T_{f(w)}V_2$, and $E_2|_{f(w)}\cong E_2|_{e_{12}(w)}$. Then using (14.5)–(14.6) and $\hat f\circ s_1=f^*(s_2)+O(s_1^2)$ we have

$$\begin{aligned}
e_{12}^*(t_2)|_w&=t_2|_{e_{12}(w)}=s_2|_{e_{12}(w)}=s_2|_{f(w)+(\Upsilon\circ t_1)|_w+O(t_1|_w^2)}\\
&=s_2|_{f(w)}+\mathrm{d}s_2|_{f(w)}\circ(\Upsilon\circ t_1)|_w+O(t_1|_w^2)\\
&=\hat f_1|_w\circ s_1|_w+O(s_1|_w^2)+(f^*(\mathrm{d}s_2)\circ\Upsilon\circ t_1)|_w+O(t_1|_w^2)\\
&=\hat f_1|_w\circ t_1|_w+(f^*(\mathrm{d}s_2)\circ\Upsilon\circ t_1)|_w+O(t_1|_w^2),\\
\hat e_{12}\circ t_1|_w&=\bigl(\hat f|_w+(f^*(\mathrm{d}s_2)\circ\Upsilon)|_w+O(t_1|_w)\bigr)\circ t_1|_w\\
&=\hat f_1|_w\circ t_1|_w+(f^*(\mathrm{d}s_2)\circ\Upsilon\circ t_1)|_w+O(t_1|_w^2).
\end{aligned}$$

Comparing the two equations proves (14.8).

For Step 2(c), let (14.5)–(14.6) hold for $e_{12},\hat e_{12}$ with Υ and for $e'_{12},\hat e'_{12}$ with Υ'. Then restricting (14.5)–(14.6) for $e_{12},\hat e_{12}$ and $e'_{12},\hat e'_{12}$ to $W_{12}\cap W'_{12}$ and subtracting them yields

$$e'_{12}|_{W_{12}\cap W'_{12}}-e_{12}|_{W_{12}\cap W'_{12}}=(\Upsilon'-\Upsilon)|_{W_{12}\cap W'_{12}}\circ t_1|_{W_{12}\cap W'_{12}}+O(t_1^2),\quad(14.15)$$
$$\hat e'_{12}|_{W_{12}\cap W'_{12}}-\hat e_{12}|_{W_{12}\cap W'_{12}}=f^*(\mathrm{d}s_2)\circ(\Upsilon'-\Upsilon)|_{W_{12}\cap W'_{12}}+O(t_1).\quad(14.16)$$

Since $f|_{W_{12}}=e_{12}+O(t_1)$ by (14.5) we have $f^*(\mathrm{d}s_2)|_{W_{12}}=e_{12}^*(\mathrm{d}s_2)+O(t_1)$. We also identify $f^*(TV_2)|_{W_{12}}\cong e_{12}^*(TV_2)$ using our local trivialization of V_2, and thus regard $(\Upsilon'-\Upsilon)|_{W_{12}\cap W'_{12}}$ as a morphism $F_1|_{W_{12}\cap W'_{12}}\to e_{12}^*(TV_2)|_{W_{12}\cap W'_{12}}$. Thus (14.16) becomes

$$\hat e'_{12}|_{W_{12}\cap W'_{12}}-\hat e_{12}|_{W_{12}\cap W'_{12}}=e_{12}^*(\mathrm{d}s_2)\circ(\Upsilon'-\Upsilon)|_{W_{12}\cap W'_{12}}+O(t_1).\quad(14.17)$$

Using $(\mathrm{d}s_2|_x)_*$ an isomorphism in (14.1), near $t_1^{-1}(0)$ in $W_{12}\cap W'_{12}$ we may choose a Banach vector bundle $G \to W_{12} \cap W'_{12}$ and decompositions

$$\begin{aligned} e_{12}|^*_{W_{12}\cap W'_{12}}(TV_2) &= e_{12}|^*_{W_{12}\cap W'_{12}}(TW_2) \oplus G, \\ e_{12}|^*_{W_{12}\cap W'_{12}}(E_2) &= e_{12}|^*_{W_{12}\cap W'_{12}}(F_2) \oplus G, \end{aligned}$$

such that $e_{12}|^*_{W_{12}\cap W'_{12}}(\mathrm{d}s_2) = \left(\begin{smallmatrix} e^*_{12}(\mathrm{d}t_2) & 0 \\ 0 & \mathrm{id}_G \end{smallmatrix}\right)$. Write $(\Upsilon' - \Upsilon)|_{W_{12}\cap W'_{12}} = \Lambda \oplus M$ for $\Lambda : F_1|_{W_{12}\cap W'_{12}} \to e_{12}|^*_{W_{12}\cap W'_{12}}(TW_2)$ and $M : F_1|_{W_{12}\cap W'_{12}} \to G$. Then (14.15), (14.17) become

$$e'_{12}|_{W_{12}\cap W'_{12}} - e_{12}|_{W_{12}\cap W'_{12}} = \Lambda\circ t_1|_{W_{12}\cap W'_{12}} + M\circ t_1|_{W_{12}\cap W'_{12}} + O(t_1^2), \quad (14.18)$$

$$\hat e'_{12}|_{W_{12}\cap W'_{12}} - \hat e_{12}|_{W_{12}\cap W'_{12}} = e^*_{12}(\mathrm{d}t_2)|_{W_{12}\cap W'_{12}} \circ \Lambda + M + O(t_1). \quad (14.19)$$

Now in (14.19), $\hat e'_{12}|_{\cdots}$, $\hat e_{12}|_{\cdots}, e^*_{12}(\mathrm{d}t_2)|_{\cdots}$ are morphisms $F_1 \to TW_2$ and M a morphism $F_1 \to G$. Thus taking components of (14.19) in G gives $M = O(t_1)$, so $M \circ t_1|_{\cdots} = O(t_1^2)$. Substituting these into equations (14.18)–(14.19) proves (14.10)–(14.11), and Step 2(c).

For Step 2(d), choose an open neighbourhood U of $X_2 \cap W_2$ in V_2, with $U \cap W_2$ closed in U, and a smooth map $\pi : U \to U \cap W_2$ with $\pi|_{U\cap W_2} = \mathrm{id}_{U\cap W_2}$. For instance, U could be a tubular neighbourhood of W_2 in V_2. Then $\mathrm{d}\pi : TU \to \pi^*(TW_2)$ is surjective on $U \cap W_2$, so making U smaller we can take $\mathrm{d}\pi$ surjective on U, and $\mathrm{Ker}\,\mathrm{d}\pi$ a vector subbundle of TU.

Making U smaller, we can choose a splitting $E_2|_U \cong G \oplus H$, where G has finite rank rank F_2 with $G|_{U\cap W_2} = F_2|_{U\cap W_2}$, and H is a Banach vector subbundle. Write $s_2|_U = p \oplus q$ for $p \in C^\infty(G)$ and $q \in C^\infty(H)$. Then $q|_{U\cap W_2} = 0$, since $s|_{U\cap W_2} = t_2|_{U\cap W_2} \in C^\infty(F_2|_{U\cap W_2}) = C^\infty(G|_{U\cap W_2})$. The vector bundles G and $\pi^*(F_2)$ are isomorphic to $F_2|_{U\cap W_2}$ on $U \cap W_2 \subseteq U$, so making U smaller we can choose an isomorphism $i : \pi^*(F_2) \to G$ with $i|_{U\cap W_2} = \mathrm{id}_{F_2|_{U\cap W_2}}$.

Choose connections ∇^G, ∇^H on the vector bundles G, H. At $x \in U \cap X_2 \cap W_2$, as $(\mathrm{d}s_2|_x)_*$ is an isomorphism in (14.1), we see that $\nabla^H q|_x : T_xU \to H|_x$ is surjective with kernel T_xW_2. But $T_xU = T_xW_2 \oplus \mathrm{Ker}\,\mathrm{d}\pi|_x$, so $(\nabla^H q|_{\mathrm{Ker}\,\mathrm{d}\pi})|_x : \mathrm{Ker}\,\mathrm{d}\pi|_x \to H|_x$ is an isomorphism. This is an open condition, so making U smaller, we can suppose that $\nabla^H q|_{\mathrm{Ker}\,\mathrm{d}\pi} : \mathrm{Ker}\,\mathrm{d}\pi \to H$ is an isomorphism of Banach vector bundles on U, and also that $q^{-1}(0) = U \cap W_2$.

Let $\gamma \in C^\infty(H, G)$, so that $\gamma \oplus \mathrm{id}_H : H \to G \oplus H = E_2|_U$ is a vector bundle embedding, and replace H by the vector subbundle $H' = (\gamma \oplus \mathrm{id}_H)(H)$. This fixes q, and replaces p by $p' = p - \gamma \circ q$. So

$$\nabla^G p'|_{\mathrm{Ker}\,\mathrm{d}\pi} = \nabla^G p|_{\mathrm{Ker}\,\mathrm{d}\pi} - (\nabla^{G,H}\gamma|_{\mathrm{Ker}\,\mathrm{d}\pi}) \circ q - \gamma \circ \nabla^H q|_{\mathrm{Ker}\,\mathrm{d}\pi}.$$

Thus, taking $\gamma = \nabla^G p|_{\mathrm{Ker}\,\mathrm{d}\pi} \circ (\nabla^H q|_{\mathrm{Ker}\,\mathrm{d}\pi})^{-1}$ we have $\nabla^G p|_{\mathrm{Ker}\,\mathrm{d}\pi} = O(q)$.

Define $W_{12} = W_1 \cap f^{-1}(U)$, and set $e_{12} = \pi \circ f|_{W_{12}} : W_{12} \to W_2$. Define $\hat e_{12} : F_1|_{W_{12}} \to e^*_{12}(F_2)$ by the commutative diagram

$$\begin{tikzcd} F_1|_{W_{12}} \arrow[r, "\mathrm{inc}"'] \arrow[d, "\hat e_{12}"] & E_1|_{W_{12}} \arrow[r, "\hat f|_{W_{12}}"'] & f|^*_{W_{12}}(E_2) = f|^*_{W_{12}}(G) \oplus f|^*_{W_{12}}(H) \arrow[d, "\pi_{f^*(G)}"'] \\ e^*_{12}(F_2) = f|^*_{W_{12}}(\pi^*(F_2)) & & f|^*_{W_{12}}(G). \arrow[ll, "f|^*_{W_{12}}(i|^{-1}_{W_{12}})"'] \end{tikzcd} \quad (14.20)$$

Define $\Upsilon : F_1|_{W_{12}} \to f^*(TV_2)$ by the commutative diagram

$$\begin{tikzcd}
F_1|_{W_{12}} \arrow[r, "\mathrm{inc}"'] \arrow[d, "\Upsilon"] & E_1|_{W_{12}} \arrow[r, "\hat f|_{W_{12}}"'] & f|_{W_{12}}^*(E_2) \arrow[r, equal] & f|_{W_{12}}^*(G) \oplus f|_{W_{12}}^*(H) \arrow[d, "\pi_{f^*(H)}"'] \\
f^*(TV_2) & & f^*(\operatorname{Ker} \mathrm{d}\pi) \arrow[ll, "\mathrm{inc}"'] & f|_{W_{12}}^*(H). \arrow[l, "-\nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi}^{-1}"']
\end{tikzcd} \tag{14.21}$$

We will show these $W_{12}, e_{12}, \hat e_{12}, \Upsilon$ satisfy (14.5)–(14.6).

Let $w \in W_{12}$ be close to X_1. Then $f(w)$ and $e_{12}(w)$ are close together in the same fibre of $\pi : U \to U \cap W_2$, so we can choose a smooth path $\gamma : [0,1] \to U$ with $\gamma(0) = f(w)$, $\gamma(1) = e_{12}(w)$ and $\pi \circ \gamma = e_{12}(w)$. We can also make γ approximate a straight line segment in a local trivialization of U. Then we have

$$\begin{aligned}
\Upsilon \circ t_1|_w &= -\nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi}|_{f(w)}^{-1} \circ \pi_H|_{f(w)} \circ \hat f(t_1)|_w \\
&= -\nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi}|_{f(w)}^{-1} \circ \pi_H|_{f(w)} \circ s_2|_{f(w)} + O(t_1|_w^2) \\
&= \nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi}|_{e_{12}(w)}^{-1} \circ q|_{e_{12}(w)} - \nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi}|_{f(w)}^{-1} \circ q|_{f(w)} + O(t_1|_w^2) \\
&= \int_0^1 \frac{d}{\mathrm{d}y}\Bigl(\nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi}|_{\gamma(y)}^{-1} \circ q|_{\gamma(y)}\Bigr)\,\mathrm{d}y + O(t_1|_w^2) \\
&= \int_0^1 \nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi}|_{\gamma(y)}^{-1} \circ \nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi}|_{\gamma(y)} \circ \dot\gamma(y)\,\mathrm{d}y + O(t_1|_w^2) \\
&= \int_0^1 \dot\gamma(y)\,\mathrm{d}y + O(t_1|_w^2) = \gamma(1) - \gamma(0) + O(t_1|_w^2) = e_{12}(w) - f(w) + O(t_1|_w^2).
\end{aligned} \tag{14.22}$$

Here in the first step we use (14.21), in the second $\hat f \circ s_1 = f^*(s_2) + O(s_1^2)$, in the third $s_2|_U = p \oplus q$, $\pi_H(p \oplus q) = q$, and $q|_{e_{12}(w)} = 0$ as $q|_{U \cap W_2} = 0$, and in the fourth $\gamma(0) = f(w)$ and $\gamma(1) = e_{12}(w)$. In the fifth we neglect terms in $\nabla^H\bigl(\nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi}|_{\gamma(y)}^{-1}\bigr)$, as these are $O(t_1|_w^2)$. Rearranging (14.22) proves (14.5).

Similarly, if $\alpha \in F_1|_w$ then

$$\begin{aligned}
\hat e_{12}|_w(\alpha) - \hat f|_w(\alpha) &= i|_{f(w)}^{-1} \circ \pi_G|_{f(w)} \circ \hat f|_w(\alpha) - \hat f|_w(\alpha) \\
&= \pi_G|_{f(w)} \circ \hat f|_w(\alpha) - \hat f|_w(\alpha) + O(t_1|_w) = -\pi_H|_{f(w)} \circ \hat f|_w(\alpha) + O(t_1|_w) \\
&= (\nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi})|_{f(w)} \circ (-\nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi})|_{f(w)}^{-1} \circ \pi_H|_{f(w)} \circ \hat f|_w(\alpha) + O(t_1|_w) \\
&= (\nabla^H q|_{\operatorname{Ker} \mathrm{d}\pi})|_{f(w)} \circ \Upsilon|_w(\alpha) + O(t_1|_w) = \nabla^H q|_{f(w)} \circ \Upsilon|_w(\alpha) + O(t_1|_w) \\
&= (\nabla^G p \oplus \nabla^H q)|_w \circ \Upsilon|_w(\alpha) + O(t_1|_w) = (f^*(\mathrm{d}s_2) \circ \Upsilon)|_w(\alpha) + O(t_1|_w),
\end{aligned}$$

using (14.20) in the first step, $i = \mathrm{id} + O(s_2)$ so that $f^*(i) = \mathrm{id} + O(t_1)$ in the second, $\pi_G + \pi_H = \mathrm{id}$ in the third, (14.21) in the fifth, $\Upsilon|_w(\alpha) \in \operatorname{Ker} \mathrm{d}\pi$ in the sixth, and $\nabla^G p|_{\operatorname{Ker} \mathrm{d}\pi} = O(q) = O(s_2)$ so that $f^*(\nabla^G p|_{\operatorname{Ker} \mathrm{d}\pi}) = O(t_1)$ in the seventh. This proves (14.6) at w, α, and completes Step 2.

For Step 3, we first choose an indexing set I, and for each $i \in I$ a triple (W_i, F_i, t_i) satisfying $(*)$, with $\dim W_i - \operatorname{rank} F_i = n$, such that $\{W_i \cap X : i \in I\}$ is an open cover of X. This is possible by Step 1, since $W_i \cap X$ is open in X, and for any $x \in X$ there exists (W_i, F_i, t_i) with $x \in W_i \cap X$. We define

$\psi_i : t_i^{-1}(0) \to X$ to be the inclusion $W_i \cap X \hookrightarrow X$. Next, we apply Step 2 with $(V_1, E_1, s_1) = (V_2, E_2, s_2) = (V, E, s)$, $f = \mathrm{id}_V$ and $\hat{f} = \mathrm{id}_E$. For all $i, j \in I$ with $W_i \cap W_j \cap X \neq \emptyset$, we choose $(W_{ij}, e_{ij}, \hat{e}_{ij}) : (W_i, F_i, t_i) \to (W_j, F_j, t_j)$ satisfying $(+)$. This is possible by Step 2(d). We also choose a total order $<$ on I.

We claim that all this data satisfies the hypotheses of the first part of Theorem 3.42. As the Banach manifold V is Hausdorff and second countable, so is the subspace X. For (iii), let $i, j, k \in I$. Then the following satisfy $(+)$ with $(V_1, E_1, s_1) = (V_2, E_2, s_2) = (V, E, s)$, $f = \mathrm{id}_V$ and $\hat{f} = \mathrm{id}_E$:

$$\begin{aligned}
&\bigl(W_{ij} \cap W_{ik} \cap e_{ij}^{-1}(W_{jk}), e_{jk} \circ e_{ij}|_{\cdots}, e_{ij}|_{\cdots}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{\cdots}\bigr) : \\
&\qquad\qquad \bigl(W_{ij} \cap W_{ik} \cap e_{ij}^{-1}(W_{jk}), F_i|_{\cdots}, t_i|_{\cdots}\bigr) \longrightarrow (W_k, F_k, t_k), \\
&\bigl(W_{ij} \cap W_{ik} \cap e_{ij}^{-1}(W_{jk}), e_{ik}|_{\cdots}, \hat{e}_{ik}|_{\cdots}\bigr) : \\
&\qquad\qquad \bigl(W_{ij} \cap W_{ik} \cap e_{ij}^{-1}(W_{jk}), F_i|_{\cdots}, t_i|_{\cdots}\bigr) \longrightarrow (W_k, F_k, t_k),
\end{aligned}$$

the first by Step 2(a), the second by choice of $(W_{ik}, e_{ik}, \hat{e}_{ik})$. Therefore Step 2(c) gives Λ_{ijk} such that (3.34)–(3.35) hold. When $i < j < k$ this proves Theorem 3.42(iii). Also, when $k = i$ we may take $W_{ii} = W_i$, $e_{ii} = \mathrm{id}_{W_i}$ and $\hat{e}_{ii} = \mathrm{id}_{F_i}$, and (14.12) gives a 2-morphism

$$\boldsymbol{S}_{\Lambda_{iji}} : \boldsymbol{S}_{e_{ji},\hat{e}_{ji}} \circ \boldsymbol{S}_{e_{ij},\hat{e}_{ij}}|_{X \cap W_i \cap W_j} \Longrightarrow \mathbf{id}_{\boldsymbol{S}_{W_i,F_i,t_i}|_{X \cap W_i \cap W_j}}.$$

Exchanging i, j shows that $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}$ and $\boldsymbol{S}_{e_{ji},\hat{e}_{ji}}$ are quasi-inverse on $X \cap W_i \cap W_j$, so both are equivalences. This implies Theorem 3.42(ii), by Theorem 3.39.

Theorem 3.42 now constructs a d-manifold $\boldsymbol{X}$ with topological space X and $\mathrm{vdim}\, \boldsymbol{X} = n$. The theorem shows that having chosen $I, <, (W_i, F_i, t_i), \ldots$, then $\boldsymbol{X}$ is unique up to equivalence. We claim that $\boldsymbol{X}$ is also independent of the choices of $I, <, (W_i, F_i, t_i), \ldots$ up to equivalence. To see this, note that if $\tilde{I}, \tilde{<}, (\tilde{W}_i, \tilde{F}_i, \tilde{t}_i), \ldots$ are alternative choices yielding $\tilde{\boldsymbol{X}}$, we can also do the construction starting with the indexing set $\check{I} = I \amalg \tilde{I}$ and data $(W_i, F_i, t_i) : i \in I$, $(\tilde{W}_i, \tilde{F}_i, \tilde{t}_i) : i \in \tilde{I}$, yielding $\check{\boldsymbol{X}}$. Then on $X \cap W_i$, both $\boldsymbol{X}$ and $\check{\boldsymbol{X}}$ are equivalent to $\boldsymbol{S}_{W_i,F_i,t_i}$, so $\boldsymbol{X}, \check{\boldsymbol{X}}$ are locally equivalent. These local equivalences can be glued using a partition of unity to get a global equivalence $\boldsymbol{X} \simeq \check{\boldsymbol{X}}$. Similarly locally $\tilde{\boldsymbol{X}} \simeq \boldsymbol{S}_{\tilde{W}_i,\tilde{F}_i,\tilde{t}_i} \simeq \check{\boldsymbol{X}}$, so $\tilde{\boldsymbol{X}} \simeq \check{\boldsymbol{X}}$, and thus $\boldsymbol{X} \simeq \tilde{\boldsymbol{X}}$.

Thus, $\boldsymbol{X}$ is independent of all choices up to equivalence in **dMan**. To see that the second part of Theorem 14.2(a) holds, note that given any (W, F, t) satisfying $(*)$, we can include (W, F, t) as one of the (W_i, F_i, t_i) for $i \in I$ above, and then Theorem 3.42 gives a 1-morphism $\boldsymbol{\psi} : \boldsymbol{S}_{W,F,t} \to \boldsymbol{X}$ which is an equivalence with the d-submanifold $\hat{\boldsymbol{X}} \subseteq \boldsymbol{X}$ with subspace $\hat{X} = W \cap X$. This completes the proof of Step 3, and Theorem 14.2(a).

For Step 4, let V_1, E_1, s_1, X_1, n_1, V_2, E_2, s_2, X_2, n_2, $f : V_1 \to V_2$ and $\hat{f} : E_1 \to f^*(E_2)$ satisfying $\hat{f} \circ s_1 = f^*(s_2) + O(s_1^2)$ be as in Theorem 14.2(b).

First we apply Step 3 to construct a d-manifold $\boldsymbol{X}_2$ from V_2, E_2, s_2, X_2, n_2, using an ordered indexing set $(K, <)$, and triples (W'_k, F'_k, t'_k) for each $k \in K$ satisfying $(*)$, such that $\{W'_k \cap X_2 : k \in K\}$ is an open cover of X_2, and triples $(W'_{kl}, e'_{kl}, \hat{e}'_{kl}) : (W'_k, F'_k, t'_k) \to (W'_l, F'_l, t'_l)$ satisfying $(+)$ for all $k, l \in$

K. The application of Theorem 3.42 in Step 3 also yields 1-morphisms $\boldsymbol{\psi}'_k : \boldsymbol{S}_{W'_k,F'_k,t'_k} \to \boldsymbol{X}_2$ for all $k \in K$ which are equivalences with open $\hat{\boldsymbol{X}}_{2,k} \subseteq \boldsymbol{X}_2$, and 2-morphisms $\eta'_{kl} : \boldsymbol{\psi}'_l \circ \boldsymbol{S}_{e'_{kl},\hat{e}'_{kl}} \Rightarrow \boldsymbol{\psi}'_k \circ \boldsymbol{i}_{W'_{kl},W'_k}$ for $k < l$ in K.

Now $f|_{X_1} : X_1 \to X_2$ is continuous, and $\{W'_k \cap X_2 : k \in K\}$ is an open cover of X_2, so $\{f^{-1}(W'_k) \cap X_1 : k \in K\}$ is an open cover of X_1. We choose an indexing set I, and for each $i \in I$ a triple (W_i, F_i, t_i) satisfying $(*)$ for V_1, E_1, s_1, with $\dim W_i - \operatorname{rank} F_i = n_1$, such that $\{W_i \cap X_1 : i \in I\}$ is an open cover of X_1 subordinate to $\{f^{-1}(W'_k) \cap X_1 : k \in K\}$. That is, for each $i \in I$ there exists $k_i \in K$ such that $W_i \cap X_1 \subseteq f^{-1}(W'_{k_i}) \cap X_1$, or equivalently, $f(W_i \cap X_1) \subseteq W'_{k_i}$. We also choose a total order $<$ on I such that $i \leqslant j$ in I implies $k_i \leqslant k_j$ in K.

For each $i \in I$, (W_i, F_i, t_i) satisfies $(*)$ in (V_1, E_1, s_1), and $(W'_{k_i}, F'_{k_i}, t'_{k_i})$ satisfies $(*)$ in (V_2, E_2, s_2), and $f(W_i \cap X_1) \subseteq W'_{k_i}$. Thus Step 2(d) gives a triple $(\tilde{W}_{ik_i}, g_{ik_i}, \hat{g}_{ik_i})$ satisfying $(+)$, where $\tilde{W}_{ik_i} \subseteq W_i$ is open with $\tilde{W}_{ik_i} \cap X_1 = W_i \cap f^{-1}(W_{k_i}) \cap X_1 = W_i \cap X_1$. Replacing W_i, F_i, t_i by $\tilde{W}_{ik_i}, F_i|_{\tilde{W}_{ik_i}}, t_i|_{\tilde{W}_{ik_i}}$, noting that this does not change $W_i \cap X_1$, we may assume that $\tilde{W}_{ik_i} = W_i$. Thus, $g_{ik_i} : W_i \to W'_{k_i}$ is smooth, and $\hat{g}_{ik_i} : F_i \to g^*_{ik_i}(F'_{k_i})$ is a morphism of vector bundles on W_i. Step 2(b) implies that $\hat{g}_{ik_i} \circ t_i = g^*_{ik_i}(t'_{k_i}) + O(t_i^2)$, so Definition 3.30 gives a 1-morphism $\boldsymbol{S}_{g_{ik_i},\hat{g}_{ik_i}}$ as in (14.13).

Next, as in Step 3, we choose $(W_{ij}, e_{ij}, \hat{e}_{ij}) : (W_i, F_i, t_i) \to (W_j, F_j, t_j)$ satisfying $(+)$ for all $i, j \in I$. Then $W_{ij} \subseteq W_i$ is open with $W_{ij} \cap X_1 = W_i \cap W_j \cap X_1$. Now $g_{ik_i}^{-1}(W'_{k_ik_j}) \subseteq W_i$ is open with $g_{ik_i}^{-1}(W'_{k_ik_j}) \cap X_1 = g_{ik_i}^{-1}(W'_{k_ik_j} \cap X_2) = f(W'_{k_i} \cap W_{k'_j} \cap X_2) \supseteq W_i \cap W_j \cap X_1$. Thus, making W_{ij} smaller if necessary, we can suppose that $W_{ij} \subseteq g_{ik_i}^{-1}(W'_{k_ik_j})$, and so $g_{ik_i}|_{W_{ij}}$ maps $W_{ij} \to W'_{k_ik_j}$. With these choices of $I, <, (W_i, F_i, t_i)$ and $(W_{ij}, e_{ij}, \hat{e}_{ij})$, we now complete Step 3 to construct a d-manifold $\boldsymbol{X}_1$ from V_1, E_1, s_1, X_1, n_1, with equivalences $\boldsymbol{\psi}_i : \boldsymbol{S}_{W_i,F_i,t_i} \to \hat{\boldsymbol{X}}_{1,i} \subseteq \boldsymbol{X}_1$ for all $i \in I$ and 2-morphisms $\eta_{ij} : \boldsymbol{\psi}_j \circ \boldsymbol{S}_{e_{ij},\hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i \circ \boldsymbol{i}_{W_{ij},W_i}$ for $i < j$ in I.

Now let $i < j \in I$. Then we have diagrams of triples satisfying $(*)$ and $(+)$:

$$(W_{ij}, F_i|_{W_{ij}}, t_i|_{W_{ij}}) \xrightarrow{(W_{ij},e_{ij},\hat{e}_{ij})} (W_j, F_j, t_j) \xrightarrow{(W_{ij},g_{jk_j},\hat{g}_{jk_j})} (W'_{k_j}, F'_{k_j}, t'_{k_j}),$$
$$(W_{ij}, F_i|_{W_{ij}}, t_i|_{W_{ij}}) \xrightarrow{(W_{ij},g_{ik_i}|_{W_{ij}},\hat{g}_{ik_i}|_{W_{ij}})} (W'_{k_ik_j}, F'_i|_{W'_{k_ik_j}}, t'_i|_{W'_{k_ik_j}}) \xrightarrow{(W'_{k_ik_j},e'_{k_ik_j},\hat{e}'_{k_ik_j})} (W'_{k_j}, F'_{k_j}, t'_{k_j}).$$

Step 2(a) shows the compositions of both lines are triples $(W_{ij}, F_i|_{W_{ij}}, t_i|_{W_{ij}}) \to (W'_{k_j}, F'_{k_j}, t'_{k_j})$ satisfying $(+)$. Thus Step 2(c) for these compositions gives Λ_{ij} satisfying analogues of (14.10)–(14.11), and yielding a 2-morphism $\boldsymbol{S}_{\Lambda_{ij}}$ as in

(14.12). Consider the 2-commutative diagram in **dMan**:

$$
\begin{tikzcd}
& \boldsymbol{S}_{W_i,F_i,t_i} \arrow[rr, "\boldsymbol{S}_{g_{ik_i},\hat{g}_{ik_i}}"'] & \arrow[d, phantom, "\Uparrow \mathrm{id}"] & \boldsymbol{S}_{W'_{k_i},F'_{k_i},t'_{k_i}} \arrow[dr, "\boldsymbol{\psi}'_{k_i}"] \arrow[dd, phantom, "\Uparrow \eta'_{k_ik_j}"] & \\
\boldsymbol{S}_{W_{ij},F_i|_{W_{ij}},t_i|_{W_{ij}}} \arrow[ur, "\boldsymbol{i}_{W_{ij},W_i}"] \arrow[rr, "\boldsymbol{S}_{g_{ik_i}|_{W_{ij}},\hat{g}_{ik_i}|_{W_{ij}}}"] \arrow[dr, "\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}"'] & & \boldsymbol{S}_{W'_{k_ik_j},F'_i|\ldots,t'_i|\ldots} \arrow[ur, "\boldsymbol{i}_{W'_{k_ik_j},W'_{k_i}}"] \arrow[dr, "\boldsymbol{S}_{e'_{k_ik_j},\hat{e}'_{k_ik_j}}"] & & \boldsymbol{X}_2. \\
& \boldsymbol{S}_{W_j,F_j,t_j} \arrow[rr, "\boldsymbol{S}_{g_{jk_j},\hat{g}_{jk_j}}"] & \arrow[u, phantom, "\Uparrow \boldsymbol{S}_{\Lambda_{ij}}"] & \boldsymbol{S}_{W'_{k_j},F'_{k_j},t'_{k_j}} \arrow[ur, "\boldsymbol{\psi}'_{k_j}"'] &
\end{tikzcd}
$$

Composing 2-morphisms across this diagram yields a 2-morphism

$$\zeta_{ij} : \big(\boldsymbol{\psi}'_{k_j} \circ \boldsymbol{S}_{g_{jk_j},\hat{g}_{jk_j}}\big) \circ \boldsymbol{S}_{e_{ij},\hat{e}_{ij}} \Longrightarrow \big(\boldsymbol{\psi}'_{k_i} \circ \boldsymbol{S}_{g_{ik_i},\hat{g}_{ik_i}}\big) \circ \boldsymbol{i}_{W_{ij},W_i}. \tag{14.23}$$

The d-manifold $\boldsymbol{X}_1$ was constructed in Theorem 3.42, which was proved by using Theorem 2.33 to glue the d-manifolds $\boldsymbol{S}_{W_i,F_i,t_i}$ for i in I by equivalences $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}$ on overlaps. We have constructed 1-morphisms $\boldsymbol{\psi}'_{k_i} \circ \boldsymbol{S}_{g_{ik_i},\hat{g}_{ik_i}} : \boldsymbol{S}_{W_i,F_i,t_i} \to \boldsymbol{X}_2$ in (3.14) for each $i \in I$, and 2-morphisms ζ_{ij} in (14.23) for all $i < j$ in I. Thus, the last part of Theorem 2.33 gives a 1-morphism $\boldsymbol{h} : \boldsymbol{X}_1 \to \boldsymbol{X}_2$, with 2-morphisms $\zeta_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \boldsymbol{\psi}'_k \circ \boldsymbol{S}_{g_{ik_i},\hat{g}_{ik_i}}$ for all $i \in I$. We use Theorem 2.33 rather than Theorem 3.42 as Theorem 3.42 only allows the target $\boldsymbol{X}_2$ to be a manifold, not a d-manifold or a d-space.

Theorem 2.33 says that having chosen $I, <, (W_i, F_i, t_i), \ldots$, then $\boldsymbol{h}$ is unique up to 2-isomorphism. The method of Step 3, involving taking alternative choices $\tilde{I}, <, (\tilde{W}_j, \tilde{F}_j, \tilde{t}_j), \ldots$ and doing the construction using $I \amalg \tilde{I}$ and both (W_i, F_i, t_i) for $i \in I$ and $(\tilde{W}_j, \tilde{F}_j, \tilde{t}_j)$ for $j \in \tilde{I}$ shows that $\boldsymbol{h}$ is also independent up to 2-isomorphism of the choices of $I, <, (W_i, F_i, t_i), \ldots$.

To prove the last part of Theorem 14.2(b), if $(W_1, F_1, t_1), (W_2, F_2, t_2), g, \hat{g}$ are as in Theorem 14.2(b), then we can choose the $I, (W_i, F_i, t_i)$ and $K, (W'_k, F'_k, t'_k)$ above such that $(W_{i_0}, F_{i_0}, t_{i_0}) = (W_1, F_1, t_1)$ for $i_0 \in I$, and $(W'_{k_{i_0}}, F'_{k_{i_0}}, t'_{k_{i_0}}) = (W_2, F_2, t_2)$, and $(g_{i_0k_{i_0}}, \hat{g}_{i_0k_{i_0}}) = (g, \hat{g})$. The construction yields $\boldsymbol{\psi}_1 := \boldsymbol{\psi}_{i_0} : \boldsymbol{S}_{W_1,F_1,t_1} \to \hat{\boldsymbol{X}}_{\boldsymbol{1}} \subseteq \boldsymbol{X}_1$ and $\boldsymbol{\psi}_2 := \boldsymbol{\psi}'_{k_{i_0}} : \boldsymbol{S}_{W_2,F_2,t_2} \to \hat{\boldsymbol{X}}_{\boldsymbol{2}} \subseteq \boldsymbol{X}_2$ as we want, and then $\zeta_{i_0} : \boldsymbol{h} \circ \boldsymbol{\psi}_1 \Rightarrow \boldsymbol{\psi}_2 \circ \boldsymbol{S}_{g,\hat{g}}$ above is a 2-morphism making (14.4) 2-commute. This completes Step 4, and Theorem 14.2(b).

14.2 Hofer–Wysocki–Zehnder's polyfolds

The *polyfold programme* of Hofer, Wysocki and Zehnder [41–48] is a functional-analytic framework for describing the structure of (compactified) moduli spaces in differential geometry, especially moduli spaces of J-holomorphic curves in symplectic geometry. The objects in this programme related to d-manifolds with corners are triples (V, E, s), where V is an *M-polyfold*, E a *strong M-polyfold bundle* over V, and $s : V \to E$ a *Fredholm section* of E. The objects related to d-orbifolds with corners are triples $(\mathcal{V}, \mathcal{E}, s)$, where $\mathcal{V}$ is a *polyfold*, $\mathcal{E}$ a *strong polyfold bundle* over $\mathcal{V}$, and $s : \mathcal{V} \to \mathcal{E}$ a *Fredholm section* of $\mathcal{E}$. Here polyfolds are orbifold versions of M-polyfolds, where 'M' stands for 'manifold'.

The set-up is really very complex, and to give proper definitions of these concepts and an outline of the basic ideas would take many pages. So for brevity we will try only to give a flavour of the theory and its motivations, and refer interested readers to the survey [42] as a good starting point. Consider the following 'classical' problem. Let (X, ω) be a compact symplectic manifold, and J an almost complex structure on X compatible with ω. We wish to study the moduli space $\mathcal{M}$ of all J-holomorphic maps $u : \mathbb{CP}^1 \to X$.

One way to do this is to consider the Banach manifold $V = L^2_k(\mathbb{CP}^1, X)$ of all maps $u : \mathbb{CP}^1 \to X$ that are Sobolev L^2_k in local coordinates on $\mathbb{CP}^1, X$ for some $k \geqslant 1$, and the Banach vector bundle $E \to V$ whose fibre over $u \in L^2_k(\mathbb{CP}^1, X)$ is the Banach space $L^2_{k-1}(u^*(TX) \otimes_{\mathbb{C}} \Lambda^{0,1}T^*\mathbb{CP}^1)$ of Sobolev L^2_{k-1} sections of the complex vector bundle $u^*(TX) \otimes_{\mathbb{C}} \Lambda^{0,1}T^*\mathbb{CP}^1$ over Σ, and the Fredholm section $s : V \to E$ mapping $s : u \mapsto \overline{\partial}_J u$. Thus, $\mathcal{M}$ is the zeroes of a Fredholm section s of a Banach vector bundle E over a Banach manifold V, as in §14.1.

Now consider two generalizations of this situation. Each causes a problem, and the solutions to these problems are the two main innovations in the polyfold theory. Both problems are strictly infinite-dimensional issues.

(A) The automorphism group $\mathrm{Aut}(\mathbb{CP}^1) \cong \mathrm{PSL}(2, \mathbb{C})$ of $\mathbb{CP}^1$ acts on $\mathcal{M}, V, E$, so we can study the quotient moduli space $\mathcal{M}/\mathrm{Aut}(\mathbb{CP}^1)$. This is the zeros of a section s_* of a bundle $E/\mathrm{Aut}(\mathbb{CP}^1)$ over $V/\mathrm{Aut}(\mathbb{CP}^1)$.

    However, the action of $\mathrm{Aut}(\mathbb{CP}^1)$ on the Banach manifold $V = L^2_k(\mathbb{CP}^1, X)$ is *not smooth*, but only continuous. This is because if $u \in V$ and $w \in \mathfrak{aut}(\mathbb{CP}^1)$ then the Lie derivative $\mathcal{L}_w u$ lies in $L^2_{k-1}(u^*(TX))$, rather than in $T_u V = L^2_k(u^*(TX))$. So $V/\mathrm{Aut}(\mathbb{CP}^1)$ is not a Banach manifold.

(B) Consider the family of Riemann surfaces $\Sigma_t = \{[x, y, z] \in \mathbb{CP}^2 : xy = tz^2\}$ for $t \in \mathbb{C}$. Then Σ_t is nonsingular $\mathbb{CP}^1$ for $t \neq 0$, and Σ_0 is two $\mathbb{CP}^1$'s meeting in a node at $[0, 0, 1]$. For each $t \in \mathbb{C}$ we have a moduli space $\mathcal{M}_t$ of J-holomorphic maps $u_t : \Sigma_t \to X$, the zeroes of a Fredholm section s_t of a Banach vector bundle E_t over the Banach manifold $V_t = L^2_k(\Sigma_t, X)$.

    We would like to combine these moduli spaces $\mathcal{M}_t$ into a big moduli space $\mathfrak{M} = \coprod_{t \in \mathbb{C}} \mathcal{M}_t$, with a global geometric structure. The obvious way to do this is to try and make $\mathfrak{V} = \coprod_{t \in \mathbb{C}} V_t$ into a Banach manifold, and $\mathfrak{E} = \coprod_{t \in \mathbb{C}} E_t$ into a Banach vector bundle over $\mathfrak{V}$ with section $\mathfrak{s} = \coprod_{t \in \mathbb{C}} s_t$.

    However, although the Banach manifolds V_t for $t \neq 0$ form a smooth family, because of the topology change of Σ_t at $t = 0$, the Banach manifold V_0 looks different to V_t for $t \neq 0$, and the V_t for $t \in \mathbb{C}$ are not a smooth family. So we cannot make $\mathfrak{V}$ into a Banach manifold.

To deal with problem (A), Hofer at al. introduce the notion of an *sc-structure* on V, which is a sequence $V = V_0 \supset V_1 \supset V_2 \supset \cdots$, where each V_i is a Banach manifold, with its own Banach topology (not that induced from V), such that the inclusion $V_i \hookrightarrow V_{i+1}$ is compact, and $V_\infty = \bigcap_{i=0}^\infty V_i$ is dense in each V_i. In our example $V = L^2_1(\mathbb{CP}^1, X)$, and $V_i = L^2_{i+1}(\mathbb{CP}^1, X)$ for all $i \geqslant 0$. They then define *sc-smooth* maps $f : V \to W$ between spaces V, W with sc-structures. The basic idea is that the k^{th} derivative $\nabla^k f$ maps $\otimes^k TV_{i+k} \to TW_i$ for $i \geqslant 0$.

To deal with problem (B), Hofer at al. introduce *splicing cores*, or (more generally) *sc^∞-retracts*. The basic idea is this: given a Banach space B, one can consider families of projections $\pi_t : B \to B$ for $t \in \mathbb{R}^n$, with $\pi_t^2 = \pi_t$, which depend smoothly on $t \in \mathbb{R}^n$ in a weak sense. Then the image $\operatorname{Im} \pi_t$ is a Banach subspace of B, but can vary discontinuously with $t \in \mathbb{R}^n$. (This is possible only in infinite dimensions.) In our example, one can find $\pi_t : B \to B$ for $t \in \mathbb{C}$ such that $L^2_k(\Sigma_t, X)$ is locally modelled on $\operatorname{Im} \pi_t$ for all $t \in \mathbb{C}$, and the discontinuous change in $L^2_k(\Sigma_t, X)$ between $t = 0$ and $t \neq 0$ can be understood as the discontinuous change in $\operatorname{Im} \pi_t$.

They then define *M-polyfolds*, which may be thought of as generalizations of Banach manifolds with corners – generally infinite-dimensional, though finite-dimensional manifolds with corners are examples. Local models for M-polyfolds are $\coprod_{t \in [0,\infty)^k \times \mathbb{R}^{n-k}} \operatorname{Im} \pi_t$ with $\pi_t : B \to B$ a smooth family of projections for $t \in [0,\infty)^k \times \mathbb{R}^{n-k}$, where B is a Banach space with sc-smooth structure. *Polyfolds* are orbifold versions of M-polyfolds, generalizations of Banach orbifolds with corners. There are also generalizations of Banach vector bundles over Banach manifolds, called *strong (M-)polyfold bundles*, and of Fredholm sections, called *Fredholm sections*.

Perhaps the most compelling reason to be interested in polyfolds is:

> **Claim (Hofer at al. [43, §1]).** *Basically every moduli space of stable J-holomorphic curves, with or without boundary, of interest in symplectic geometry, has the structure of the zeroes of a Fredholm section of a strong polyfold bundle over a polyfold.*
>
> *The same should also hopefully be true of other nonlinear, elliptic, compactified moduli space problems in differential geometry, for instance, instanton or monopole-type equations from gauge theory.*

This claim has so far been proved and published only for moduli of stable, closed J-holomorphic curves with marked points in symplectic manifolds [48], as in Gromov–Witten theory, though proofs have been announced for other classes of curves. Much effort has first been expended in developing polyfolds as an abstract theory, prior to constructing such structures on moduli spaces.

The next theorem defines truncation functors from (M-)polyfolds with Fredholm sections to d-orbifolds (or d-manifolds) with corners. It is closely modelled on Theorem 14.2. We will prove (a),(b) in §14.2.1. Part (c) is easy: we must show $\Pi^{\mathbf{dMan}}_{\mathbf{MPolFS}}(\mathrm{id}_V, \mathrm{id}_E) = [\mathbf{id}_{\boldsymbol{X}}]$ and $\Pi^{\mathbf{dMan}}_{\mathbf{MPolFS}}$ preserves composition, but these follow from the characterization of the 1-morphism $\boldsymbol{h}$ in (b). We leave (d) as an exercise for the reader. The author also expects that the truncation functors will commute with fibre products over manifolds or orbifolds, and with boundaries $(V, E, s) \mapsto (\partial V, E|_{\partial V}, s|_{\partial V})$, $\mathbf{X} \mapsto \partial \mathbf{X}$ in the corners case, but these constructions were not yet available for polyfolds at the time of writing.

Theorem 14.6. **(a)** *Let V be an M-polyfold without boundary* [43, §3.3], *E a fillable strong M-polyfold bundle over V* [43, §4.3], *and $s : V \to E$ an sc-smooth*

Fredholm section of E [43, §4.4], [44, §3]. *Set* $X = s^{-1}(0) \subseteq V$, *and suppose the linearization* $\mathrm{d}s|_x : T_xV \to E|_x$ [43, §4.4] *has Fredholm index* $n \in \mathbb{Z}$ *for all* $x \in X$. *Then we may construct a d-manifold* $\boldsymbol{X}$, *natural up to equivalence in* **dMan**, *with underlying topological space* X *and virtual dimension* n.

The d-manifold structure on $\boldsymbol{X}$ *may be characterized as follows: let* W *be a manifold,* $F \to W$ *a vector bundle, and* $t : W \to F$ *a smooth section. Regard* W *as an M-polyfold, and* $F \to W$ *as a strong M-polyfold bundle. Suppose* $i : W \to V$ *is an sc-smooth embedding* [43, §3.3], *so that the pullback* $i^*(E)$ *is a strong M-polyfold bundle over* W [43, Prop. 4.11] *and* $i^*(s)$ *is an sc-smooth section of* $i^*(E)$, *and* $\hat{\imath} : F \to i^*(E)$ *is an* sc$_\triangleleft$*-smooth embedding of strong M-polyfold bundles* [43, §4.1–§4.3], *such that* $\hat{\imath} \circ t = i^*(s)$, *and for every* $w \in W$ *with* $t(w) = 0$ *and* $x = i(w) \in V$, *the map* $(\mathrm{d}s|_x)_*$ *in the commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & T_wW \arrow[r, "\mathrm{d}i|_w"] \arrow[d, "\mathrm{d}t|_w"'] & T_xV \arrow[r] \arrow[d, "\mathrm{d}s|_x"'] & T_xV/\mathrm{d}i|_w(T_wW) \arrow[r] \arrow[d, "(\mathrm{d}s|_x)_*"'] & 0 \\ 0 \arrow[r] & F|_w \arrow[r, "\hat{\imath}|_w"] & E|_x \arrow[r] & E|_x/\hat{\imath}|_w(F|_w) \arrow[r] & 0 \end{tikzcd} \tag{14.24}$$

is an isomorphism. Then $\hat{X} := i(W) \cap X$ *is open in* X, *and there is an equivalence* $\boldsymbol{\psi} : \boldsymbol{S}_{W,F,t} \to \hat{\boldsymbol{X}} \subseteq \boldsymbol{X}$ *in* **dMan**, *which acts as* $i|_{t^{-1}(0)} : t^{-1}(0) \to \hat{X}$ *on topological spaces, where* $\boldsymbol{S}_{W,F,t}$ *is given in Definition* 3.13.

Suppose that the Fredholm section s *is oriented in the sense of* [44, §6.4]. *Then one can construct a natural orientation on* $\boldsymbol{X}$.

(b) *Let* V_a, E_a, s_a, X_a, n_a *and* $\boldsymbol{X}_a$ *be as in* **(a)** *for* $a = 1, 2$. *Suppose* $f : V_1 \to V_2$ *is an sc-smooth map of M-polyfolds, and* $\hat{f} : E_1 \to f^*(E_2)$ *is an* sc$_\triangleleft$*-smooth morphism of strong M-polyfold bundles on* V_1 *satisfying* $\hat{f} \circ s_1 = f^*(s_2) + O(s_1^2)$. *Then we may construct a* 1*-morphism* $\boldsymbol{h} : \boldsymbol{X}_1 \to \boldsymbol{X}_2$ *in* **dMan**, *natural up to* 2*-isomorphism, with continuous map* $h = f|_{X_1} : X_1 \to X_2$.

This 1*-morphism* $\boldsymbol{h}$ *may be characterized as follows: let* W_a, F_a, t_a, $i_a : W_a \to V_a$, $\hat{\imath}_a : F_a \to i_a^*(E_a)$ *be as in* **(a)** *for* $a = 1, 2$, *so that* **(a)** *defines equivalences* $\boldsymbol{\psi}_a : \boldsymbol{S}_{W_a,F_a,t_a} \to \hat{\boldsymbol{X}}_a \subseteq \boldsymbol{X}_a$. *Suppose* $g : W_1 \to W_2$ *is smooth,* $\hat{g} : F_1 \to g^*(F_2)$ *is a vector bundle morphism and* $\Upsilon : F_1 \to (f \circ i_1)^*(TV_2)$ *an* sc$_\triangleleft$*-smooth strong M-polyfold bundle morphism on* W_1 *satisfying*

$$i_2 \circ g = f \circ i_1 + \Upsilon \circ t_1 + O(t_1^2), \tag{14.25}$$

$$\hat{\imath}_2 \circ \hat{g} = \hat{f} \circ \hat{\imath}_1 + (f \circ i_1)^*(\mathrm{d}s_2) \circ \Upsilon + O(t_1), \tag{14.26}$$

where (14.25) *holds in sc-smooth maps* $W_1 \to V_2$, *and* (14.26) *in strong M-polyfold bundle morphisms* $F_1 \to (f \circ i_1)^*(E_2)$ *or* $F_1 \to (i_2 \circ g)^*(E_2)$. *Then* $\hat{g} \circ t_1 = g^*(t_2) + O(t_1^2)$ *in* $C^\infty(g^*(F_2))$, *so that Definition* 3.30 *gives a* 1*-morphism* $\boldsymbol{S}_{g,\hat{g}} : \boldsymbol{S}_{W_1,F_1,t_1} \to \boldsymbol{S}_{W_2,F_2,t_2}$, *and we have a* 2*-commutative diagram in* **dMan**:

$$\begin{tikzcd} \boldsymbol{S}_{W_1,F_1,t_1} \arrow[rr, "\boldsymbol{S}_{g,\hat{g}}"'] \arrow[d, "\boldsymbol{\psi}_1"] & {} \arrow[d, phantom, "\Uparrow"] & \boldsymbol{S}_{W_2,F_2,t_2} \arrow[d, "\boldsymbol{\psi}_2"'] \\ \boldsymbol{X}_1 \arrow[rr, "\boldsymbol{h}"] & {} & \boldsymbol{X}_2. \end{tikzcd} \tag{14.27}$$

(c) *Define a category* **MPolFS** *of 'M-polyfolds without boundary with Fredholm sections' to have objects* (V, E, s) *as in* **(a)**, *and morphisms* $(V_1, E_1, s_1) \to (V_2, E_2, s_2)$ *pairs* $(f, \hat{f})$ *as in* **(b)**, *with composition* $(g, \hat{g}) \circ (f, \hat{f}) = \big(g \circ f, \hat{f} \circ f^*(\hat{g})\big)$. *Define a functor* $\Pi^{\mathbf{dMan}}_{\mathbf{MPolFS}} : \mathbf{MPolFS} \to \mathrm{Ho}(\mathbf{dMan})$ *as follows, where* $\mathrm{Ho}(\mathbf{dMan})$ *is the homotopy category of the* 2*-category* **dMan**.

For each (V, E, s) *in* **MPolFS**, *choose a d-manifold* $\boldsymbol{X}$ *in the equivalence class in* **dMan** *given by* **(a)**, *and set* $\Pi^{\mathbf{dMan}}_{\mathbf{MPolFS}}(V, E, s) = \boldsymbol{X}$. *For each morphism* $(f, \hat{f}) : (V_1, E_1, s_1) \to (V_2, E_2, s_2)$, *part* **(b)** *defines a* 1*-morphism* $\boldsymbol{h} : \Pi^{\mathbf{dMan}}_{\mathbf{MPolFS}}(V_1, E_1, s_1) = \boldsymbol{X}_1 \to \boldsymbol{X}_2 = \Pi^{\mathbf{dMan}}_{\mathbf{MPolFS}}(V_2, E_2, s_2)$ *in* **dMan** *unique up to* 2*-isomorphism, so the morphism* $[\boldsymbol{h}] : \boldsymbol{X}_1 \to \boldsymbol{X}_2$ *in* $\mathrm{Ho}(\mathbf{dMan})$ *is uniquely defined. Set* $\Pi^{\mathbf{dMan}}_{\mathbf{MPolFS}}(f, \hat{f}) = [\boldsymbol{h}]$. *Then* $\Pi^{\mathbf{dMan}}_{\mathbf{MPolFS}}$ *is a functor.*

(d) *Analogues of* **(a)**–**(c)** *also hold with polyfolds without boundary, orbifolds, and d-orbifolds in place of M-polyfolds without boundary, manifolds, and d-manifolds, yielding a functor* $\Pi^{\mathbf{dOrb}}_{\mathbf{PolFS}} : \mathrm{Ho}(\mathbf{PolFS}) \to \mathrm{Ho}(\mathbf{dOrb})$.

Similarly, analogues of all the above hold involving M-polyfolds (or polyfolds) with corners, manifolds (or orbifolds) with corners, and d-manifolds (or d-orbifolds) with corners, yielding functors $\Pi^{\mathbf{dMan^c}}_{\mathbf{MPol^cFS}} : \mathbf{MPol^cFS} \to \mathrm{Ho}(\mathbf{dMan^c})$ *and* $\Pi^{\mathbf{dOrb^c}}_{\mathbf{Pol^cFS}} : \mathrm{Ho}(\mathbf{Pol^cFS}) \to \mathrm{Ho}(\mathbf{dOrb^c})$.

Now we give some applications of these ideas to symplectic geometry. In [48], Hofer, Wysocki and Zehnder prove that Gromov–Witten moduli spaces of J-holomorphic stable maps of Riemann surfaces with marked points are zeros of a Fredholm section of a strong polyfold bundle over a polyfold. Combining Theorem 14.6 with [48, Th.s 1.7–1.11] yields the following corollary.

The first two paragraphs recall standard ideas in Gromov–Witten theory, which can be found in [34] or [48]. For the last part, Hofer et al. [48] define a polyfold without boundary $\mathcal{V}$ and fillable strong polyfold bundle $\mathcal{E} \to \mathcal{V}$ independent of J, and for each complex structure J_t they define an oriented Fredholm section $s_t : \mathcal{V} \to \mathcal{E}$. As the J_t depend smoothly on $t \in [0, 1]$, so do the s_t. So the s_t for $t \in [0, 1]$ combine to give an oriented Fredholm section s of the strong polyfold bundle $\mathcal{E} \times [0, 1]$ over the polyfold with boundary $\mathcal{V} \times [0, 1]$. We then apply Theorem 14.6(d) to $\mathcal{V} \times [0, 1], \mathcal{E} \times [0, 1], s$.

Corollary 14.7. *Let* (X, ω) *be a compact symplectic manifold of dimension* $2n$, *and* J *an almost complex structure on* X *compatible with* ω. *For* $\beta \in H_2(X, \mathbb{Z})$ *and* $g, m \geqslant 0$, *write* $\overline{\mathcal{M}}_{g,m}(X, J, \beta)$ *for the moduli space of* J*-holomorphic stable maps* $(\Sigma, \vec{z}, u)$ *to* X *from a prestable Riemann surface* Σ *with genus* g *and* m *marked points* $\vec{z} = (z_1, \ldots, z_m)$, *with* $[u(\Sigma)] = \beta$ *in* $H_2(X, \mathbb{Z})$. *It is a compact topological space, with Gromov's* C^∞*-topology.*

Define evaluation maps $\mathrm{ev}_j : \overline{\mathcal{M}}_{g,m}(X, J, \beta) \to X$ *for* $j = 1, \ldots, m$ *by* $\mathrm{ev}_j : [\Sigma, \vec{z}, u] \mapsto u(z_j)$. *When* $2g+m \geqslant 3$, *write* $\overline{\mathcal{M}}_{g,m}$ *for the moduli space of Deligne–Mumford stable Riemann surfaces of genus* g *with* m *marked points, which is a compact orbifold of real dimension* $2(m+3g-3)$. *Define* $\pi_{g,m} : \overline{\mathcal{M}}_{g,m}(X, J, \beta) \to \overline{\mathcal{M}}_{g,m}$ *by* $\mathrm{ev}_j : [\Sigma, \vec{z}, w] \mapsto [\tilde{\Sigma}, \tilde{\vec{z}}\,]$, *where* $(\tilde{\Sigma}, \tilde{\vec{z}}\,)$ *is the stabilization of* $(\Sigma, \vec{z})$.

Then $\overline{\mathcal{M}}_{g,m}(X, J, \beta)$ *may be given the structure of a compact, oriented d-*

orbifold without boundary $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)$, *with*

$$\operatorname{vdim}\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)=2\bigl(c_1(X)\cdot\beta+(n-3)(1-g)+m\bigr). \tag{14.28}$$

This depends on the choice of 'gluing profile' $\varphi(r)=e^{1/r}-e$, *and sequence* $0<\delta_0<\delta_1<\cdots<2\pi$, *used to define the smooth structure on* $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)$ *near singular curves. With these choices made,* $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)$ *is unique up to equivalence in* **dOrb**.

Also the ev_j *extend to* 1*-morphisms* $\mathbf{ev}_j:\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)\to F_{\mathbf{Man}}^{\mathbf{dOrb}}(X)$ *in* **dOrb**, *and* $\pi_{g,m}$ *extends to a* 1*-morphism* $\boldsymbol{\pi}_{g,m}:\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)\to\bar{\boldsymbol{\mathcal{M}}}_{g,m}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\bar{\mathcal{M}}_{g,m})$ *in* **dOrb**, *both natural up to* 2*-isomorphism.*

Now let $J_t:t\in[0,1]$ *be a smooth family of almost complex structures on* X *compatible with* ω. *Then we may define a compact, oriented d-orbifold with boundary* $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J_t:t\in[0,1],\beta)$, *with virtual dimension*

$$\operatorname{vdim}\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J_t:t\in[0,1],\beta)=2\bigl(c_1(X)\cdot\beta+(n-3)(1-g)+m\bigr)+1,$$

underlying topological space $\coprod_{t\in[0,1]}\bar{\mathcal{M}}_{g,m}(X,J_t,\beta)$, *and boundary*

$$\partial\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J_t:t\in[0,1],\beta)\simeq-\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J_0,\beta)\amalg\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J_1,\beta) \tag{14.29}$$

as an oriented d-orbifold. The 1*-morphisms* $\mathbf{ev}_j$ *for* $j=1,\ldots,m$ *and* $\boldsymbol{\pi}_{g,m}$ *for* $2g+m\geqslant 3$ *are also defined for* $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J_t:t\in[0,1],\beta)$, *and restrict to* $\mathbf{ev}_j$, $\boldsymbol{\pi}_{g,m}$ *on* $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J_0,\beta)$, $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J_1,\beta)$ *under* (14.29).

We use Corollary 14.7 to define new Gromov–Witten type invariants in the d-orbifold bordism groups of §13.4:

Definition 14.8. Let (X,ω) be a compact symplectic manifold, and g,m be nonnegative integers, and $\beta\in H_2(X;\mathbb{Z})$. Define the *Gromov–Witten d-orbifold bordism invariant* $GW_{g,m}^{\mathrm{dorb}}(X,\omega,\beta)$ by

$$GW_{g,m}^{\mathrm{dorb}}(X,\omega,\beta)=$$
$$\begin{cases}\bigl[\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta),\mathbf{ev}_1\times\cdots\times\mathbf{ev}_m\bigr]\in dB_{2k}^{\mathrm{orb}}(X^m), & 2g+m<3,\\ \bigl[\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta),\mathbf{ev}_1\times\cdots\times\mathbf{ev}_m\times\boldsymbol{\pi}_{g,m}\bigr]\in dB_{2k}^{\mathrm{orb}}(X^m\times\bar{\mathcal{M}}_{g,m}), & 2g+m\geqslant 3.\end{cases}$$

Here J is an almost complex structure on X compatible with ω, $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)$, $\mathbf{ev}_j,\boldsymbol{\pi}_{g,m}$ are as in Corollary 14.7, and $k=c_1(X)\cdot\beta+(n-3)(1-g)+m$ as in (14.28). The second part of Corollary 14.7 and the definition of d-orbifold bordism in §13.4 imply that $GW_{g,m}^{\mathrm{dorb}}(X,\omega,\beta)$ is *independent of the choice of almost complex structure* J.

Remark 14.9. **(a)** As in (13.14), there are virtual class maps $\Pi_{\mathrm{dorb}}^{\mathrm{hom}}$ mapping $dB_{2k}^{\mathrm{orb}}(X^m)$ and $dB_{2k}^{\mathrm{orb}}(X^m\times\bar{\mathcal{M}}_{g,m})$ to $H_{2k}(X^m;\mathbb{Q})$ and $H_{2k}(X^m\times\bar{\mathcal{M}}_{g,m};\mathbb{Q})$. Applying these to the new invariants $GW_{g,m}^{\mathrm{dorb}}(X,\omega,\beta)$ will recover the conventional symplectic Gromov–Witten invariants of [34] or [48, Th. 1.12].

As in §13.4, d-orbifold bordism groups are far larger than homology groups, and can have infinite rank even in negative degrees. Therefore the new invariants $GW_{g,m}^{\rm dorb}(X,\omega,\beta)$ *contain more information than conventional Gromov–Witten invariants*, in particular, on the orbifold strata of the moduli spaces $\bar{\boldsymbol{\mathcal M}}_{g,m}(X,J,\beta)$, which can be recovered by applying the functors $\Pi_{\rm dorb}^{\Gamma,\lambda}$ of §13.4. Since the $dB_*^{\rm orb}(-)$ are defined over $\mathbb{Z}$ rather than $\mathbb{Q}$, these new invariants may be good tools for studying *integrality properties* of Gromov–Witten invariants.

As in Remark 13.28, it may be a useful idea to define a theory of *unitary d-orbifold bordism* $dBU_*^{\rm orb}(\mathcal{Y})$ for orbifolds $\mathcal{Y}$, and define similar Gromov–Witten type invariants in $dBU_{2k}^{\rm orb}(X^m)$ and $dBU_{2k}^{\rm orb}(X^m\times\bar{\mathcal M}_{g,m})$.

(b) Polyfolds are a tool designed by functional analysts, and they abstract the overall analytic structure of compactified J-holomorphic curve moduli problems. In the author's view, their principal value is as a framework for proving the existence of geometric structures on such moduli spaces.

After the existence of the structure on moduli spaces is proved, it may be preferable not to use polyfolds in subsequent geometric applications to Gromov–Witten Theory, Lagrangian Floer Theory, Symplectic Field Theory, ... Instead, one should truncate to some simpler and more friendly structure, such as d-orbifolds, to do the geometry, virtual chain computations, and so on.

Polyfolds comprise a huge amount of information even in the simplest cases, and so may be impractical to work with. For example, consider the case $g=0$, $m=3$ and $\beta=0$ in Corollary 14.7, the moduli space of constant rational curves in X with 3 marked points. The d-orbifold $\bar{\boldsymbol{\mathcal M}}_{0,3}(X,J,0)$ is just the manifold X, that is, $\bar{\boldsymbol{\mathcal M}}_{0,3}(X,J,0)\simeq F_{\bf Man}^{\bf dOrb}(X)$ in $\mathbf{dOrb}$, and the evaluation maps $\mathbf{ev}_1,\mathbf{ev}_2,\mathbf{ev}_3$ are all $\mathbf{id}_{\boldsymbol M}$. But the corresponding polyfold $\mathcal{V}$ still contains infinitely many nested infinite-dimensional Banach manifolds.

14.2.1 The proof of Theorem 14.6(a),(b)

The proof is modelled closely on that of Theorem 14.2(a),(b) in §14.1.1. So rather than giving the proof in full, we will just explain the modifications to the proof in §14.1.1 required to replace Banach manifolds and Banach vector bundles by M-polyfolds without boundary and strong M-polyfold bundles. Here are some introductory remarks on M-polyfolds, to establish notation:

(a) An *sc-Banach space* B [43, §2] is a sequence $B=B_0\supseteq B_1\supseteq\cdots$, where B_m is a Banach space with Banach norm $\|\,.\,\|_m$ for $m=0,1,\ldots$, and if $m<n$ then the inclusion $B_n\hookrightarrow B_m$ is a compact linear operator, and $B_\infty:=\bigcap_{m=0}^\infty B_m$ is dense in B_m for all $m\geqslant 0$. Subsets of B are called *open* if they are open in the Banach topology on B_0.

(b) Let B,C be sc-Banach spaces, and $T\subseteq B$, $U\subseteq C$ be open, and write $T_m=T\cap B_m$, $U_m=U\cap C_m$. A map $\varphi:T\to U$ is called sc^0 if $\varphi(T_m)\subseteq U_m$ and $\varphi_m:=\varphi|_{T_m}:T_m\to U_m$ is continuous in the Banach norms $\|\,.\,\|_m$ on B_m,C_m for all $m\geqslant 0$. Hofer et al. [43, §2.3] define notions of derivatives of φ, and when φ is *sc-smooth*. If φ is sc-smooth, then $\varphi_{m+k}:T_{m+k}\to U_m$

is a C^k map of Banach manifolds for all $m, k \geqslant 0$. But note that φ sc-smooth does *not* imply that $\varphi : T_m \to U_m$ is a smooth map of Banach manifolds for any $m \geqslant 0$.

(c) Let B, C be sc-Banach spaces, $T \subseteq B$ an open set, and $\pi_t : C \to C$ for $t \in T$ be a family of linear projections on C, with $\pi_t \circ \pi_t = \pi_t$, such that $\Pi : T \times C \to T \times C$ mapping $(t, c) \mapsto (t, \pi_t(c))$ is sc-smooth. The associated *splicing core* (without boundary) is $K = \{(t, c) \in T \times C : \pi_t(c) = c\} = \operatorname{Im} \Pi$. We write $K_m = K \cap (T_m \times C_m)$ for $m = 0, 1, \ldots, \infty$.

M-polyfolds (without boundary) in [43–46] are locally modelled on open subsets of splicing cores (without boundary). In [47] Hofer et al. introduce more general local models called *sc-retracts*, but this will make little difference to our proof.

Note in particular that the splicing core K is embedded in the Banach space $B_0 \times C_0$. Thus, M-polyfolds are locally subsets of Banach manifolds. Similarly, strong M-polyfold bundles over M-polyfolds are locally subsets of Banach vector bundles over Banach manifolds.

(d) We introduced notation $O(s), O(s^2)$ for manifolds and vector bundles in Definition 3.29, and noted in Definition 14.1 that it also works for Banach manifolds and Banach vector bundles. Since M-polyfolds and strong M-polyfold bundles are locally subsets of Banach manifolds and Banach vector bundles, as in (c), the notation $O(s), O(s^2)$ also makes sense for M-polyfolds, as used in (14.25)–(14.26) in Theorem 14.6(b), for instance.

(e) A *finite-dimensional* sc-Banach space B has $B = B_0 = B_1 = \cdots = B_\infty$. Thus, if B, C, T, U, φ are as in (b) with B finite-dimensional, then $\varphi(T) \subseteq U_\infty \subseteq U$, and as $\varphi : T = T_{m+k} \to U_m$ is C^k for all $k \geqslant 0$, we see that $\varphi : T \to U_m$ is a smooth map of Banach manifolds for all $m \geqslant 0$.

Hence, if W is a (finite-dimensional) manifold, regarded as an M-polyfold, V is an M-polyfold, $f : W \to V$ is sc-smooth, and V is locally modelled on a splicing core $K \subseteq B \times C$ as in (c), then locally $f(W) \subseteq K_\infty$ and $f : W \to B_m \times C_m$ is a smooth map of Banach manifolds for all $m \geqslant 0$.

One moral is that for sc-smooth maps $f : W \to V$ from manifolds W into M-polyfolds V, on which our proof is based, the subtleties of sc-smoothness become irrelevant, and we can treat V like a Banach manifold.

We now describe the modifications to the proof in §14.1.1 to replace Banach manifolds by M-polyfolds. Here is the analogue of Step 1 in §14.1.1:

Step 1′. Let $V, E, s, X = s^{-1}(0)$ and $n \in \mathbb{Z}$ be as in Theorem 14.6(a). We say that a triple $(W, F, t, i, \hat{\imath})$ *satisfies condition* $(*')$ if:

$(*')$ W is a manifold, $F \to W$ a vector bundle, $t : W \to F$ a smooth section, $i : W \to V$ is an sc-smooth embedding of M-polyfolds, and $\hat{\imath} : F \to i^*(E)$ an $\mathrm{sc}_\triangleleft$-smooth embedding of strong M-polyfold bundles, such that $\hat{\imath} \circ t = i^*(s)$, and $i(W) \cap X \neq \emptyset$, and for every $w \in W$ with $t(w) = 0$ and $x = i(w) \in V$, the map $(\mathrm{d}s|_x)_*$ in (14.24) is an isomorphism.

We show that if $(W, F, t, i, \hat{\imath})$ satisfies $(*')$ then $\dim W - \operatorname{rank} F = n$, and $i(W) \cap X$ is open in X. Hence the 'standard model' d-manifold $\boldsymbol{S}_{W,F,t}$ from Definition 3.13 has virtual dimension n, and its topological space $t^{-1}(0)$ is homeomorphic to the open set $\hat{X} = i(W) \cap X$ in X. We prove that for all $x \in X$, there exists $(W, F, t, i, \hat{\imath})$ satisfying $(*')$ with $x \in i(W)$.

To prove Step $1'$, showing $\dim W - \operatorname{rank} F = n$ is the same as in §14.5.1. For the rest, if $x \in X \subseteq V$ then V is modelled near x on an open subset O in a splicing core $K \subset B \times C$ as in (c) above, so we can regard V near x as a subset of the Banach manifold $\bar{V} = B_0 \times C_0$. Also E is modelled near x on a subset of a Banach vector bundle $\bar{E}$ over $\bar{V}$, and s is modelled near x on a Fredholm section $\bar{s}$ of $\bar{E} \to \bar{V}$ called a *filled version* of s [43, Def. 4.16]. In a similar way to (14.24) we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_xV \arrow[r, "\text{inc}"] \arrow[d, "\mathrm{d}s|_x"'] & T_x\bar{V} \arrow[r] \arrow[d, "\mathrm{d}\bar{s}|_x"'] & T_x\bar{V}/T_xV \arrow[r] \arrow[d, "(\mathrm{d}\bar{s}|_x)_*"', "\cong"] & 0 \\
0 \arrow[r] & E|_x \arrow[r, "\text{inc}"] & \bar{E}|_x \arrow[r] & \bar{E}|_x/E|_x \arrow[r] & 0,
\end{tikzcd}$$

with $(\mathrm{d}\bar{s}|_x)_*$ an isomorphism, and $s^{-1}(0) = \bar{s}^{-1}(0)$ near x.

The proof in §14.5.1 shows that $i(W) \cap \bar{s}^{-1}(0)$ is open in $\bar{s}^{-1}(0)$, so $i(W) \cap X$ is open in X as $s^{-1}(0) = \bar{s}^{-1}(0)$ near x. To construct $(W, F, t, i, \hat{\imath})$ satisfying $(*')$ with $x \in i(W)$, we follow the proof in §14.5.1 for $\bar{V}, \bar{E}, \bar{s}$, choosing an open $x \in \bar{U} \subseteq \bar{V}$ and $\bar{e}_1, \ldots, \bar{e}_k \in C^\infty(\bar{E}|_{\bar{U}})$, with the additional requirements that $\bar{e}_i$ is sc-smooth and $\bar{e}_i|_{\bar{U} \cap V} \in C^\infty(E|_{\bar{U} \cap V}) \subseteq C^\infty(\bar{E}_{\bar{U} \cap V})$ for $i = 1, \ldots, k$. It is easy to produce such sections. Then the construction of §14.5.1 yields a finite-dimensional submanifold $W \subseteq \bar{V}$, where in fact $W \subseteq V_\infty \subseteq V \subseteq \bar{V}$, and we take $i : W \to V$ to be the inclusion. Similarly, §14.5.1 yields a vector subbundle $F \subseteq \bar{E}|_W$ with $s|_W = \bar{s}|_W = t \in C^\infty(F) \subseteq C^\infty(\bar{E}|_W)$, where $F \subseteq E|_W$, and we take $\hat{\imath} : F \to E|_W = i^*(E)$ to be the inclusion. This completes Step $1'$.

The analogue of the first part of Step 2 is:

Step 2′. Let $V_1, E_1, s_1, X_1, V_2, E_2, s_2, X_2, f, \hat{f}$ be as in Theorem 14.6(b). Suppose $(W_a, F_a, s_a, i_a, \hat{\imath}_a)$ satisfies $(*')$ in (V_a, E_a, s_a) for $a = 1, 2$. We say that $(W_{12}, e_{12}, \hat{e}_{12}) : (W_1, F_1, s_1, i_1, \hat{\imath}_1) \to (W_2, F_2, s_2, i_2, \hat{\imath}_2)$ *satisfies condition* $(+')$ if:

$(+')$ $W_{12} \subseteq W_1$ is open with $i_1(W_{12}) \cap X_1 = i_1(W_1) \cap f^{-1}(i_2(W_2)) \cap X_1$, and $e_{12} : W_{12} \to W_2$ is smooth, and $\hat{e}_{12} : F_1|_{W_{12}} \to e_{12}^*(F_2)$ is a morphism of vector bundles on W_{12}, and there exists a morphism of strong M-polyfold bundles $\Upsilon : F_1|_{W_{12}} \to (f \circ i_1)^*(TV_2)|_{W_{12}}$ on W_{12} satisfying

$$i_2 \circ e_{12} = f \circ i_1|_{W_{12}} + \Upsilon \circ t_1|_{W_{12}} + O(t_1^2), \tag{14.30}$$

$$e_{12}^*(\hat{\imath}_2)|_{W_{12}} \circ \hat{e}_{12} = i_1^*(\hat{f}) \circ \hat{\imath}_1|_{W_{12}} + (f \circ i_1)^*(\mathrm{d}s_2)|_{W_{12}} \circ \Upsilon + O(t_1), \tag{14.31}$$

where (14.30) holds in smooth maps $W_{12} \to V_2$, and (14.31) in vector bundle morphisms $F_1|_{W_{12}} \to (f \circ i_1)^*(E_2)|_{W_{12}}$ or $F_1|_{W_{12}} \to (i_2 \circ e_{12})^*(E_2)|_{W_{12}}$.

Then parts (a)–(d) of Step 2 in §14.1.1 transfer essentially without change.

For the proof of Step $2'$(a)–(d), note as above that locally near a point $x \in X_a \subseteq V_a$ for $a = 1, 2$ we have inclusions $V_a \subseteq \bar V_a$, $E_a \subseteq \bar E_a|_{V_a} \subseteq \bar E_a$ and $s_a = \bar s_a|_{V_a}$, where $\bar V_a$ is a Banach manifold, $\bar E_a \to \bar V_a$ a Banach vector bundle, and $\bar s_a : \bar V_a \to \bar E_a$ a Fredholm section. Then $(*')$ above in (V_a, E_a, s_a) implies $(*)$ in §14.5.1 in $(\bar V_a, \bar E_a, \bar s_a)$ near x. Also $(f, \hat f) : (V_1, E_1, s_1) \to (V_2, E_2, s_2)$ as in Theorem 14.6(b) extends locally near x to $(\bar f, \hat{\bar f}) : (\bar V_1, \bar E_1, \bar s_1) \to (\bar V_2, \bar E_2, \bar s_2)$, and $(+')$ above for $(f, \hat f)$ implies $(+)$ in §14.5.1 for $(\bar f, \hat{\bar f})$.

Thus, in the situation of Step $2'$(a)–(d), we deduce that Step 2(a)–(d) in §14.1.1 hold for $(\bar f, \hat{\bar f}) : (\bar V_1, \bar E_1, \bar s_1) \to (\bar V_2, \bar E_2, \bar s_2)$. These immediately imply Step $2'$(a)–(d), except for one issue: in (d), the construction in §14.5.1 yields a morphism of Banach vector bundles $\Upsilon : F_1|_{W_{12}} \to (f \circ i_1)^*(T\bar V_2)|_{W_{12}}$ rather than a morphism of strong M-polyfold bundles $\Upsilon : F_1|_{W_{12}} \to (f \circ i_1)^*(TV_2)|_{W_{12}}$.

Recall that $\bar V_2$ is an sc-Banach manifold, so that $\bar V_2 = \bar V_{2,0} \supseteq \bar V_{2,1} \supseteq \cdots$, where $\bar V_{2,m}$ are Banach manifold for $m = 0, 1, \ldots$, and in applying Step 2(a)–(d) we are using only the Banach manifold $\bar V_{2,0}$. Now $\Pi_2 : \bar V_2 \to \bar V_2$ is sc-smooth. So $\Pi_2 : \bar V_{2,0} \to \bar V_{2,0}$ is a continuous map of Banach manifolds, but need not be differentiable. However, on restricting to $\bar V_{2,1}$ we do have a derivative $\mathrm{d}\Pi_2 : \Pi_2^*(T\bar V_{2,0})|_{\bar V_{2,1}} \to T\bar V_{2,0}|_{\bar V_{2,1}}$. Since $f \circ i_1 : W_1 \to V_2$ maps to $V_{2,\infty} \subseteq \bar V_{2,\infty} \subseteq \bar V_{2,1}$, the pullback $(f \circ i_1)^*(\mathrm{d}\Pi_2) : (f \circ i_1)^* \circ \Pi_2^*(T\bar V_2) \to (f \circ i_1)^*(T\bar V_2)$ exists as a morphism of Banach vector bundles on W_1. Also $(f \circ i_1)^* \circ \Pi_2^*(T\bar V_2) = (f \circ i_1)^*(T\bar V_2)$ as $\Pi_2 \circ f \circ i_1 = f \circ i_1$ since $f \circ i_1(W_1) \subseteq V_2$ and $\Pi_2|_{V_2} = \mathrm{id}_{V_2}$.

Thus, $(f \circ i_1)^*(\mathrm{d}\Pi_2) : (f \circ i_1)^*(T\bar V_2) \to (f \circ i_1)^*(T\bar V_2)$ is a projection as Π_2 is, and maps to $(f \circ i_1)^*(TV_2)$ as Π_2 maps to V_2. Hence $\Upsilon' := (f \circ i_1)^*(\mathrm{d}\Pi_2)|_{W_{12}} \circ \Upsilon$ is a morphism $\Upsilon' : F_1|_{W_{12}} \to (f \circ i_1)^*(TV_2)|_{W_{12}}$ as required, and composing (14.5)–(14.6) for Υ with the given projections $\Pi_2 : \bar V_2 \to V_2$, $\tilde\Pi_2 : \bar E_2 \to E_2$ implies (14.30)–(14.31) for Υ'. This proves Step $2'$(d), and completes Step $2'$.

Steps 3 and 4 and their proofs in §14.1.1 now extend to M-polyfolds with only cosmetic changes, replacing Steps 1,2, $(*), (+)$ by Steps $1', 2'$, $(*'), (+')$, and inserting inclusions $i : W \hookrightarrow V$ and $\hat\imath : F \hookrightarrow i^*(E)$. These proofs use only the results of Steps 1,2, plus the fact that Banach manifolds are Hausdorff and second countable, which is also true of M-polyfolds.

This proves Theorem 14.6(a),(b), except for the last part of Theorem 14.6(a) on orientations of $s, \boldsymbol{X}$, which has no analogue in Theorem 14.6(a). In [44, §6.4], Hofer et al. explain that for V, E, s, X as in Theorem 14.6(a), so that s is a Fredholm section and $X = s^{-1}(0)$, then one can define a topological line bundle $L^{\det}$ over X called the *determinant line bundle*, whose fibre at $x \in X$ is $L^{\det}|_x = \Lambda^{\mathrm{top}}(\operatorname{Ker} \mathrm{d}s|_x) \otimes \Lambda^{\mathrm{top}}(\operatorname{Coker} \mathrm{d}s|_x)^*$. In fact they construct L not just over $X = s^{-1}(0) \subseteq V_\infty \subseteq V$, but also over V_∞, even though $\mathrm{d}s|_x : T_xV \to E|_x$ is not canonically defined at $x \in V_\infty \setminus X$. But we need $L^{\det}$ only over X.

Let $W, F, t, i, \hat\imath$ and $\boldsymbol{\psi} : \boldsymbol{S}_{W,F,t} \to \boldsymbol{X}$ as in the second part of Theorem 14.6(a), and let $w \in W$ with $t(w) = 0$ and $x = i(w) \in X$. Then $(\mathrm{d}s|_x)_*$ an isomorphism in (14.24) implies that we have an exact sequence

$$0 \longrightarrow \operatorname{Ker}(\mathrm{d}s|_x) \longrightarrow T_wW \longrightarrow E|_w \longrightarrow \operatorname{Coker}(\mathrm{d}s|_x) \longrightarrow 0. \qquad (14.32)$$

We now have isomorphisms

$$\begin{aligned} L^{\det}|_x &= \Lambda^{\mathrm{top}}(\operatorname{Ker} \mathrm{d}s|_x) \otimes \Lambda^{\mathrm{top}}(\operatorname{Coker} \mathrm{d}s|_x)^* \\ &\cong \Lambda^{\mathrm{top}} T_w W \otimes \Lambda^{\mathrm{top}}(E|_w)^* \cong \mathcal{L}_{T^*\boldsymbol{S}_{W,F,t}}|_w \cong \mathcal{L}_{T^*\boldsymbol{X}}|_x, \end{aligned} \tag{14.33}$$

where $\mathcal{L}_{T^*\boldsymbol{S}_{W,F,t}}$ and $\mathcal{L}_{T^*\boldsymbol{X}}$ are the orientation line bundles of the d-manifolds $\boldsymbol{S}_{W,F,t}, \boldsymbol{X}$, as in §4.6, and in the second step of (14.33) we use (14.32), in the third Definition 4.48, and in the fourth we use the restriction to w of the isomorphism $\mathcal{L}_{\boldsymbol{\psi}} : \underline{\psi}^*(\mathcal{L}_{T^*\boldsymbol{X}}) \to \mathcal{L}_{T^*\boldsymbol{S}_{W,F,t}}$ from Definition 4.46, as $\boldsymbol{\psi}$ is étale.

Thus, on X we have two topological real line bundles: the determinant line bundle $L^{\det}$ from the M-polyfold set-up in [44, §6.4], and the orientation line bundle $\mathcal{L}_{T^*\boldsymbol{X}}$ on the d-manifold $\boldsymbol{X}$ from §4.6. At each point $x \in X$ we have constructed an isomorphism $L^{\det}|_x \cong \mathcal{L}_{T^*\boldsymbol{X}}|_x$ in (14.33). One can show this is independent of choices, and over all $x \in X$ gives an isomorphism of topological line bundles $L^{\det}|_X \cong \mathcal{L}_{T^*\boldsymbol{X}}$. An orientation for s is an orientation on $L^{\det}$, which gives an orientation on $L^{\det}|_X$, and thus on $\mathcal{L}_{T^*\boldsymbol{X}}$, and so on $\boldsymbol{X}$.

14.3 Fukaya–Oh–Ohta–Ono's Kuranishi spaces

Kuranishi spaces are used in Fukaya and Ono [34] and Fukaya, Oh, Ohta and Ono [32] as the geometric structure on moduli spaces of J-holomorphic curves, to develop theories of Gromov–Witten invariants and Lagrangian Floer cohomology in symplectic geometry. We begin with some words of warning:

Remark 14.10. In the opinion of the author (and some of his friends), there are some problems with the theories of Kuranishi spaces in [32,34], and the constructions of virtual cycles and chains, and the proofs of existence of Kuranishi structures on moduli spaces of J-holomorphic curves. Recently, McDuff and Wehrheim [78] and Fukaya et al. [33] have addressed some of these issues.

Probably most of these problems can be fixed by changing definitions in [32], and giving more detailed proofs, as in [33, 78]. But this is not our purpose. We argue that instead of repairing the theory of Kuranishi spaces, one should replace it with the theory of d-orbifolds with corners. Providing a good substitute for Kuranishi spaces was one of the main reasons the author wrote this book.

Our aim here is to explain the relations between Kuranishi spaces and d-orbifolds, without getting bogged down in correcting or rewriting [32]. We take the following approach. The definitions we give are essentially those in [32, §A1], but with minor changes of notation to improve compatibility with our d-orbifolds material. In this section only, we will assume results stated in [32], and use them to prove Theorem 14.17 and Corollary 14.18. So these two results are only as trustworthy as their source [32]. They will not be used elsewhere in the book.

Our definition of Kuranishi space follows that of 'Kuranishi space with a tangent bundle' in [32, §A1.1].

Definition 14.11. Let X be a topological space, and $p \in X$. A *Kuranishi neighbourhood* of p in X is a quintuple $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ where V_p is a manifold (possibly with boundary, or with corners), $E_p \to V_p$ a vector bundle, Γ_p a finite group acting smoothly and locally effectively on V_p, E_p preserving the vector bundle structure, $s_p : V_p \to E_p$ a smooth, Γ_p-equivariant section of E_p, and $\psi_p : s_p^{-1}(0)/\Gamma_p \to X$ is a homeomorphism from $s_p^{-1}(0)/\Gamma_p$ to an open neighbourhood of p in X. We write the Γ_p-actions on V_p, E_p as $r_p(\gamma) : V_p \to V_p$ and $\hat{r}_p(\gamma) : E_p \to r_p(\gamma)^*(E_p)$ for $\gamma \in \Gamma_p$.

Definition 14.12. Let X be a topological space, and $(V_p, E_p, \Gamma_p, s_p, \psi_p), (V_q, E_q, \Gamma_q, s_q, \psi_q)$ be Kuranishi neighbourhoods of $p, q \in X$ with $p \in \psi_q(s_q^{-1}(0)/\Gamma_q)$. A *coordinate change* from $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ to $(V_q, E_q, \Gamma_q, s_q, \psi_q)$ is a quadruple $(V_{pq}, e_{pq}, \hat{e}_{pq}, \rho_{pq})$ satisfying:

(a) $\emptyset \neq V_{pq} \subseteq V_p$ is a Γ_p-invariant open submanifold, with

$$p \in \psi_p\big(s_p|_{V_{pq}}^{-1}(0)/\Gamma_p\big) \subseteq \psi_q(s_q^{-1}(0)/\Gamma_q) \subseteq X.$$

(b) $\rho_{pq} : \Gamma_p \to \Gamma_q$ is an injective group morphism.

(c) $e_{pq} : V_{pq} \to V_q$ is an embedding of manifolds with $e_{pq} \circ r_p(\gamma) = r_q(\rho_{pq}(\gamma)) \circ e_{pq} : V_{pq} \to V_q$ for all $\gamma \in \Gamma_p$. If $v_p, v_p' \in V_{pq}$ and $\delta \in \Gamma_q$ with $r_q(\delta) \circ e_{pq}(v_p') = e_{pq}(v_p)$, there exists $\gamma \in \Gamma_p$ with $\rho_{pq}(\gamma) = \delta$ and $r_p(\gamma)(v_p') = v_p$.

(d) $\hat{e}_{pq} : E_p|_{V_{pq}} \to e_{pq}^*(E_q)$ is an embedding of vector bundles, such that $\hat{e}_{pq} \circ s_p|_{V_{pq}} = e_{pq}^*(s_q)$ and $r_p(\gamma)^*(\hat{e}_{pq}) \circ \hat{r}_p(\gamma) = e_{pq}^*(\hat{r}_q(\rho_{pq}(\gamma))) \circ \hat{e}_{pq} : E_p|_{V_{pq}} \to (e_{pq} \circ r_p(\gamma))^*(E_q)$ for all $\gamma \in \Gamma_p$.

(e) If $v_p \in V_{pq}$ with $s_p(v_p) = 0$ and $v_q = e_{pq}(v_p) \in V_q$ then the following linear map is an isomorphism:

$$\big(\mathrm{d}s_q(v_q)\big)_* : \big(T_{v_q} V_q\big) \big/ \big(\mathrm{d}e_{pq}(v_p)[T_{v_p} V_p]\big) \to \big(E_q|_{v_q}\big) \big/ \big(\hat{e}_{pq}(v_p)[E_p|_{v_p}]\big).$$

(f) $\psi_q \circ (e_{pq})_*|_{s_p|_{V_{pq}}^{-1}(0)/\Gamma_p} = \psi_p|_{s_p|_{V_{pq}}^{-1}(0)/\Gamma_p} : s_p|_{V_{pq}}^{-1}(0)/\Gamma_p \to X$, where $(e_{pq})_* : V_{pq}/\Gamma_p \to V_q/\Gamma_q$ is induced by $e_{pq} : V_{pq} \to V_q$ by equivariance in (c).

Definition 14.13. Let X be a Hausdorff, second countable topological space. A *Kuranishi structure* κ on X of dimension $n \in \mathbb{Z}$ assigns a Kuranishi neighbourhood $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ for each $p \in X$ with $\dim V_p - \operatorname{rank} E_p = n$, and a coordinate change $(V_{pq}, e_{pq}, \hat{e}_{pq}, \rho_{pq})$ for all $p, q \in X$ with $p \in \psi_q(s_q^{-1}(0)/\Gamma_q)$, such that if $p, q, r \in X$ with $p \in \psi_q(s_q^{-1}(0)/\Gamma_q)$ and $p, q \in \psi_r(s_r^{-1}(0)/\Gamma_r)$, so that we have coordinate changes $(V_{pq}, e_{pq}, \hat{e}_{pq}, \rho_{pq})$, $(V_{qr}, e_{qr}, \hat{e}_{qr}, \rho_{qr})$ and $(V_{pr}, e_{pr}, \hat{e}_{pr}, \rho_{pr})$, then there exists a unique $\gamma_{pqr} \in \Gamma_r$ satisfying $\rho_{pr}(\gamma) = \gamma_{pqr}\, \rho_{qr}(\rho_{pq}(\gamma))\, \gamma_{pqr}^{-1}$ for all $\gamma \in \Gamma_p$, and as for (10.20) we have

$$\begin{aligned} e_{pr}|_{V_{pr} \cap e_{pq}^{-1}(V_{qr})} &= r_r(\gamma_{pqr}) \circ e_{qr} \circ e_{pq}|_{V_{pr} \cap e_{pq}^{-1}(V_{qr})}, \\ \hat{e}_{pr}|_{V_{pr} \cap e_{pq}^{-1}(V_{qr})} &= \big(e_{pq}^*(e_{qr}^*(\hat{r}_r(\gamma_{pqr}))) \circ e_{pq}^*(\hat{e}_{qr}) \circ \hat{e}_{pq}\big)|_{V_{pr} \cap e_{pq}^{-1}(V_{qr})}. \end{aligned} \tag{14.34}$$

A *Kuranishi space* (X, κ) *of virtual dimension* n is a Hausdorff, second countable topological space X with a Kuranishi structure κ of dimension n. We call

(X, κ) a Kuranishi space *without boundary*, or *with boundary*, or *with corners*, if the manifolds V_p in the Kuranishi neighbourhoods $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ in κ are without boundary, or with boundary, or with corners, respectively.

Fukaya et al. do not define morphisms between Kuranishi spaces (though see [32, Rem. A1.44(2)]), so Kuranishi spaces are not (presently) a category. But they do define morphisms from a Kuranishi space to a manifold [32, Def. A1.13]:

Definition 14.14. Let (X, κ) be a Kuranishi space, and Y a manifold. A *strongly smooth map* $(f, \lambda) : (X, \kappa) \to Y$ is a continuous map $f : X \to Y$ of topological spaces, together with extra data λ which assigns a Γ_p-invariant smooth map $f_p : V_p \to Y$ for each Kuranishi neighbourhood $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ in κ for $p \in X$, such that $f \circ \psi_p = (f_p)_* : s_p^{-1}(0)/\Gamma_p \to Y$, and $f_q \circ e_{pq} = f_p|_{V_{pq}}$ for all coordinate changes $(V_{pq}, e_{pq}, \hat{e}_{pq}, \rho_{pq})$ in κ. We call (f, λ) *weakly submersive* if $f_p : V_p \to Y$ is a submersion for each $p \in X$.

The next remark outlines the main elements of the theory of Kuranishi spaces in Fukaya et al. [32], and compares them to d-orbifolds with corners.

Remark 14.15. **(i)** Let (X, κ) be a Kuranishi space. An *orientation* on (X, κ) [32, Def. A1.17] assigns an orientation of the line bundle $\Lambda^{\mathrm{top}} E_p \otimes \Lambda^{\mathrm{top}} T^* V_p$ on V_p for each $p \in X$, compatible under coordinate changes $(V_{pq}, e_{pq}, \hat{e}_{pq}, \rho_{pq})$.

This corresponds to orientations on $\boldsymbol{S}_{V,E,s}$ in Definition 4.48.

(ii) If (X, κ) is a Kuranishi space, then as in [32, Def. A1.30] one can define the *boundary* $(\partial X, \kappa_\partial)$, a Kuranishi space with $\mathrm{vdim}(\partial X, \kappa_\partial) = \mathrm{vdim}(X, \kappa) - 1$. The Kuranishi neighbourhoods in κ_∂ are $(\partial V_p, E_p|_{\partial V_p}, \Gamma_p, s_p|_{\partial V_p}, \psi_p|_{\partial V_p})$ for $p \in X$. This is analogous to boundaries of d-orbifolds with corners in §11.3.

(iii) Fukaya et al. [32, Lem. A1.11] define *good coordinate systems* on Kuranishi spaces, and claim (without proof) that they exist on every (compact) Kuranishi space. They are very close to our type A good coordinate systems on d-orbifolds and d-orbifolds with corners in §10.8.1 and §12.9.

(iv) Suppose $(X, \kappa), (X, \kappa')$ are Kuranishi spaces, Y is a manifold, and $(f, \lambda) : (X, \kappa) \to Y$, $(f', \lambda') : (X', \kappa') \to Y$ are weakly submersive, strongly smooth maps. Then Fukaya et al. [32, §A1.2] define a 'fibre product' Kuranishi space $(X, \kappa) \times_Y (X', \kappa')$. It has topological space $X \times_{f,Y,f'} X'$ and Kuranishi neighbourhoods $\bigl(V_p \times_{f_p,Y,f'_q} V'_q, \pi_{V_p}^*(E_p) \oplus \pi_{V'_q}^*(E'_q), \Gamma_p \times \Gamma'_q, \pi_{V_p}^*(s_p) \oplus \pi_{V'_q}^*(s'_q), \psi_{(p,q)}\bigr)$ for $(p, q) \in X \times_Y X'$. Note that this *not* a fibre product in the sense of being characterized by a universal property in a (higher) category of Kuranishi spaces.

Orientations on $(X, \kappa), (X, \kappa'), Y$ induce orientations on $(X, \kappa) \times_Y (X', \kappa')$.

When $(X, \kappa), (X', \kappa')$ (but not Y) have corners, Fukaya et al. observe that $\partial\bigl[(X, \kappa) \times_Y (X', \kappa')\bigr] \cong \bigl[(\partial(X, \kappa)) \times_Y (X', \kappa')\bigr] \amalg \bigl[(X, \kappa) \times_Y (\partial(X', \kappa'))\bigr]$, as for (6.132). In [32, §8.2] they give results on orientations of 'fibre products' of Kuranishi spaces, and their boundaries, analogous to Proposition 12.41 and Theorem 12.43 for d-orbifolds with corners.

(v) Let (X, κ) be a compact, oriented Kuranishi space of virtual dimension n, Y a manifold, and $(f, \lambda) : (X, \kappa) \to Y$ a strongly smooth map. Then after choosing some extra data, Fukaya et al. [32, Th. A1.23] define a *virtual chain* $[(X,\kappa)]_{\rm virt}$ for (X,κ) in the singular chains $C_n^{\rm si}(Y;\mathbb{Q})$. If $\partial(X,\kappa)=\emptyset$ then $\partial[(X,\kappa)]_{\rm virt}=0$, and $\bigl[[(X,\kappa)]_{\rm virt}\bigr]\in H_n^{\rm si}(Y;\mathbb{Q})$ is a *virtual class* for (X,κ). The proof uses good coordinate systems, and is similar to the proof of Theorem 13.23 in §13.5.

This should be compared with our explanation in §13.2 and §13.4 that compact, oriented d-manifolds and d-orbifolds admit virtual classes.

(vi) Let (X,ω) be a compact symplectic manifold, and J an almost complex structure on X compatible with ω. Then Fukaya and Ono [34, §12-§16] construct oriented Kuranishi structures (but with a different definition of Kuranishi space) on moduli spaces $\overline{\mathcal M}_{g,h}(X,J,\beta)$ of stable J-holomorphic curves in X with genus g and h marked points.

Now let Y be a compact embedded Lagrangian in X. In [32, §7] Fukaya et al. construct Kuranishi structures on moduli spaces $\overline{\mathcal M}_k(X,Y,J,\beta)$ of stable J-holomorphic discs in X with k boundary marked points, with boundary in Y. In [32, §8], given a 'relative spin structure' for (X,Y), they define orientations on these Kuranishi spaces. They define strongly smooth 'evaluation maps' ${\rm ev}_i:\overline{\mathcal M}_k(X,Y,J,\beta)\to Y$ for $i=1,\ldots,k$.

From the point of view of applications to symplectic geometry, the most important point is the combination of (v) and (vi), to show that moduli spaces of J-holomorphic curves have virtual chains or virtual classes.

Here are some more technical remarks about the definitions:

Remark 14.16. **(a)** We have changed notation from [32, §A]. In particular, in Definition 14.12 we have reversed the order of p,q compared to [32], so that Fukaya et al. would write $(\hat e_{qp},e_{qp},\rho_{qp})$ rather than $(V_{pq},e_{pq},\hat e_{pq},\rho_{pq})$.

(b) Definition 14.12(e) is not part of the definition of coordinate change in [32, Def. A1.3], but is given separately in the definition of 'Kuranishi space with a tangent bundle' in [32, Def. A1.14]. The author knows of no applications for Kuranishi spaces without tangent bundles, so we have combined the two.

(c) Fukaya et al. [32, 34] require their topological spaces X in Kuranishi spaces to be *compact and metrizable*, rather than Hausdorff and second countable, as we have assumed in Definition 14.13. A compact topological space is metrizable if and only if it is Hausdorff and second countable. So our condition is equivalent to theirs in the compact case, but also allows X noncompact.

(d) Note the close correspondence between Definitions 10.45, 10.46(a)–(f), and 10.47(d), and Definitions 14.11, 14.12(a)–(f), and 14.13, respectively. Also note the similarity of Definitions 14.11–14.13 with the hypotheses of Theorem 10.21. This is because the author based the material of §10.2 and §10.8 on Kuranishi spaces, partly to make the proof of Theorem 14.17 easy.

(e) The definition of Kuranishi space in Fukaya and Ono [34, §5] is not equivalent to that in [32, §A]. In particular, in [34] the sections s_p in Kuranishi neighbourhoods are only *continuous*, not smooth. This is *not compatible* with our use of C^∞-schemes and C^∞-stacks, which needs smooth sections.

Here is the main result of this section. It will be proved, assuming material from Fukaya et al. [32], in §14.3.1.

Theorem 14.17. **(a)** *Let (X,κ) be a Kuranishi space. Then we can construct a d-orbifold with corners $\mathcal{X}$ from (X,κ), with the same underlying topological space and virtual dimension, which is unique up to equivalence in* $\mathbf{dOrb^c}$.

Suppose $(f,\lambda):(X,\kappa)\to Y$ is a strongly smooth map, for Y a manifold. Then we can construct a 1*-morphism $\boldsymbol{f}:\mathcal{X}\to\mathcal{Y}=F_{\mathbf{Man}}^{\mathbf{dOrb^c}}(Y)$ in $\mathbf{dOrb^c}$, with the same continuous map $f:X\to Y$, uniquely up to* 2*-isomorphism in* $\mathbf{dOrb^c}$.

Orientations on (X,κ) correspond naturally to orientations on $\mathcal{X}$.

The construction commutes with boundaries, up to equivalence in $\mathbf{dOrb^c}$.

Suppose $(f,\lambda):(X,\kappa)\to Y$ and $(f',\lambda'):(X',\kappa')\to Y$ are weakly submersive, strongly smooth maps, and (X'',κ'') be the 'fibre product' $(X,\kappa)\times_Y(X',\kappa')$ of [32, §A1.2]*. Let $\boldsymbol{f}:\mathcal{X}\to\mathcal{Y}$, $\boldsymbol{f}':\mathcal{X}'\to\mathcal{Y}$ and $\mathcal{X}''$ be the corresponding objects and* 1*-morphisms in $\mathbf{dOrb^c}$. Then $\mathcal{X}''\simeq\mathcal{X}\times_{\boldsymbol{f},\mathcal{Y},\boldsymbol{f}'}\mathcal{X}'$. That is, the construction commutes with fibre products, up to equivalence in* $\mathbf{dOrb^c}$.

(b) *Let $\mathcal{X}$ be a d-orbifold with corners. Then we can construct a Kuranishi space (X,κ) from $\mathcal{X}$, with the same topological space $X=\mathcal{X}_{\rm top}$ and virtual dimension. The Kuranishi structure κ depends on many arbitrary choices.*

Let $\boldsymbol{f}:\mathcal{X}\to\mathcal{Y}=F_{\mathbf{Man}}^{\mathbf{dOrb^c}}(Y)$ be a 1*-morphism in $\mathbf{dOrb^c}$. Then we can construct a strongly smooth map $(f,\lambda):(X,\kappa)\to Y$, with the same underlying continuous map $f=f_{\rm top}:\mathcal{X}_{\rm top}\to Y$, where λ depends on many choices.*

(c) *The construction of* **(a)** *is left inverse to that of* **(b)**, *up to equivalence in* $\mathbf{dOrb^c}$. *That is, if $\mathcal{X}$ is a d-orbifold with corners, applying* **(b)** *gives (X,κ), and applying* **(a)** *gives $\mathcal{X}'$, then there exists an equivalence $\boldsymbol{j}:\mathcal{X}\to\mathcal{X}'$ in $\mathbf{dOrb^c}$, which is natural up to* 2*-isomorphism. Similarly, if $\boldsymbol{f}:\mathcal{X}\to\mathcal{Y}=F_{\mathbf{Man}}^{\mathbf{dOrb^c}}(Y)$ maps to $(f,\lambda):(X,\kappa)\to Y$ under* **(b)**, *which maps to $\boldsymbol{f}':\mathcal{X}'\to\mathcal{Y}$ under* **(a)**, *then there is a* 2*-isomorphism $\boldsymbol{f}'\circ\boldsymbol{j}\Rightarrow\boldsymbol{f}$.*

Assuming both Theorem 14.17, and the construction of Kuranishi structures on moduli spaces of J-holomorphic discs in Fukaya et al. [32, §7–§8], we deduce:

Corollary 14.18. *Suppose (X,ω) is a compact symplectic manifold, J an almost complex structure on X compatible with ω, and Y a compact, embedded Lagrangian submanifold in X. For $\beta\in H_2(X,Y;\mathbb{Z})$ and $k\geqslant 0$, write $\bar{\mathcal{M}}_k(X,Y,J,\beta)$ for the moduli space of J-holomorphic stable maps $(\Sigma,\vec z,u)$ to X from a prestable holomorphic disc Σ with k boundary marked points $\vec z=(z_1,\ldots,z_k)$, with $u(\partial\Sigma)\subseteq Y$ and $[u(\Sigma)]=\beta$ in $H_2(X,Y;\mathbb{Z})$. Define evaluation maps $\mathrm{ev}_i:\bar{\mathcal{M}}_k(X,Y,J,\beta)\to Y$ for $i=1,\ldots,k$ by $\mathrm{ev}_i:[\Sigma,\vec z,u]\mapsto u(z_i)$.*

Then $\bar{\mathcal{M}}_k(X,Y,J,\beta)$ may be given the structure of a compact d-orbifold with corners $\bar{\boldsymbol{\mathcal{M}}}_k(X,Y,J,\beta)$, with $\mathrm{vdim}\,\bar{\boldsymbol{\mathcal{M}}}_k(X,Y,J,\beta)=\mu_Y(\beta)+k+n-3$, where $\dim X=2n$ and $\mu_Y:H_2(X,Y;\mathbb{Z})\to\mathbb{Z}$ is the Maslov index. The maps ev_i extend to 1*-morphisms $\mathbf{ev}_i:\bar{\boldsymbol{\mathcal{M}}}_k(X,Y,J,\beta)\to\boldsymbol{Y}=F_{\mathbf{Man}}^{\mathbf{dOrb^c}}(Y)$. Given a relative spin structure for (X,Y), we can define an orientation on $\bar{\boldsymbol{\mathcal{M}}}_k(X,Y,J,\beta)$.*

We draw some conclusions:

Remark 14.19. **(i)** It should by now be clear that Kuranishi spaces, and polyfolds in §14.2, do exactly the same job. Both are intended primarily as geometric structures on moduli spaces of J-holomorphic curves, for applications in symplectic geometry. There are virtual chain constructions for Kuranishi spaces [32, §A1.1] and polyfolds [46] providing the bridge to homological algebra, based on the same idea of multi-valued transverse perturbations.

The philosophical difference between the two is that in a moduli problem, Kuranishi spaces (and even more so, d-orbifolds) remember essentially only the minimal amount of information needed to form virtual cycles. Polyfolds remember very much more information — arguably, an excessive amount.

Hofer, Wysocki and Zehnder in [41–48], and the author in this book, have similar relationships with the work of Fukaya et al. [32,34]: we are both inspired by it, and reacting against it. The polyfold programme originated, in part, as an attempt to replace unsatisfactory proofs in [32,34] of the existence of Kuranishi structures on moduli spaces of J-holomorphic curves. This book originated, in part, as an attempt to find a better theory of Kuranishi spaces.

Fukaya, Oh, Ohta and Ono are pioneers in their field, and had the courage and tenacity to attack some devastatingly difficult geometric and analytic problems. We should honour their contributions and their overall vision, which have been pivotal to the area, even if we criticize some of the details of their work.

(ii) In the author's view, Kuranishi spaces should be regarded as an *incomplete* theory. Definitions 14.11–14.14, and the material of Remark 14.15(i)–(v), are sufficient for the applications in [32]. But for a proper understanding of Kuranishi spaces as geometric spaces in their own right, one wants good notions of morphisms between Kuranishi spaces making them into a (higher) category, and theorems on existence of fibre products, gluing by equivalences, etc., involving this categorical structure, and all this is lacking.

We claim that our theory of d-manifolds and d-orbifolds with corners remedies these deficiencies. In brief, as a slogan, we assert that:

The 'right' way to define Kuranishi spaces is as d-orbifolds with corners.

Theorem 14.17 justifies the idea that the theories of Kuranishi spaces and d-orbifolds with corners are roughly equivalent to the extent that the theory of Kuranishi spaces has been developed, that is, on the level of objects, and morphisms to manifolds. In the work of Fukaya, Oh, Ohta and Ono [32], one can essentially replace Kuranishi spaces by d-orbifolds with corners throughout. This book can thus be regarded as a prequel to [32].

14.3.1 The proof of Theorem 14.17, assuming results from [32]

For (a), given a Kuranishi space (X,κ), by [32, Lem. A1.11] we can choose a good coordinate system on (X,κ). The data of this good coordinate system satisfies the hypotheses of the first part of Theorem 12.24. So Theorem 12.24 constructs a d-orbifold with corners $\mathfrak{X}$, uniquely up to equivalence in $\mathbf{dOrb^c}$.

Similarly, given a strongly smooth map $(f,\lambda):(X,\kappa)\rightarrow Y$, we choose a good coordinate system for it, and the data satisfies the hypotheses of both parts of Theorem 12.24. So Theorem 12.24 constructs a d-orbifold with corners $\mathfrak{X}$, uniquely up to equivalence in $\mathbf{dOrb^c}$, and a 1-morphism $\boldsymbol{f}:\mathfrak{X}\rightarrow\mathfrak{Y}=F_{\mathbf{Man}}^{\mathbf{dOrb^c}}(Y)$, unique up to 2-isomorphism and equivalences of $\mathfrak{X}$. The claims about orientations, boundaries, and fibre products follow by comparing the definitions for Kuranishi spaces and d-orbifolds with corners.

For (b), let $\boldsymbol{\mathcal{X}}$ be a d-orbifold with corners. Theorem 12.48 gives a type A good coordinate system $\big(I,<,(V_i,E_i,\Gamma_i,s_i,\boldsymbol{\psi}_i),(V_{ij},e_{ij},\hat e_{ij},\rho_{ij},\boldsymbol{\eta}_{ij}),\gamma_{ijk}\big)$ for $\boldsymbol{\mathcal{X}}$. We define a Kuranishi structure κ on the topological space $X=\mathcal{X}_{\rm top}$ as follows. We have $\mathcal{X}_{\rm top}=\bigcup_{i\in I}\hat{\mathcal{X}}_{i,\rm top}$, where $\psi_i:[\underline{S}_{V_i,E_i,s_i}/\Gamma_i]\rightarrow\mathcal{X}$ is an equivalence with the open C^∞-substack $\hat{\mathcal{X}}_i\subseteq\mathcal{X}$. Let $p\in\mathcal{X}_{\rm top}$. Then $\{i\in I:p\in\hat{\mathcal{X}}_{i,\rm top}\}$ is a nonempty subset of I, where $(I,<)$ is a well-ordered set. Hence there is a unique least element i_p in $\{i\in I:p\in\hat{\mathcal{X}}_{i,\rm top}\}$ in the order $<$. Define a Kuranishi neighbourhood $(V_p,E_p,\Gamma_p,s_p,\psi_p)$ of p in $\mathcal{X}_{\rm top}$, in the sense of Definition 14.11, by $(V_p,E_p,\Gamma_p,s_p,\psi_p)=(V_{i_p},E_{i_p},\Gamma_{i_p},s_{i_p},\psi_{i_p,\rm top})$.

Suppose $p,q\in\mathcal{X}_{\rm top}$ with $p\in\psi_q(s_q^{-1}(0)/\Gamma_q)$. As $\psi_q(s_q^{-1}(0)/\Gamma_q)=\hat{\mathcal{X}}_{i_q,\rm top}$ we have $p\in\hat{\mathcal{X}}_{i_q,\rm top}$, so i_q lies in $\{i\in I:p\in\hat{\mathcal{X}}_{i,\rm top}\}$, of which i_p is the least element. Hence $i_p\leqslant i_q$ in I. If $i_p=i_q$ we set $(V_{pq},e_{pq},\hat e_{pq},\rho_{pq})=(V_p,{\rm id}_{V_p},{\rm id}_{E_p},{\rm id}_{\Gamma_p})$. If $i_p<i_q$ the good coordinate system includes a type A coordinate change $(V_{i_pi_q},e_{i_pi_q},\hat e_{i_pi_q},\rho_{i_pi_q},\boldsymbol{\eta}_{i_pi_q})$ from $(V_{i_p},E_{i_p},\Gamma_{i_p},s_{i_p},\boldsymbol{\psi}_{i_p})$ to $(V_{i_q},E_{i_q},\Gamma_{i_q},s_{i_q},\boldsymbol{\psi}_{i_q})$. Define $(V_{pq},e_{pq},\hat e_{pq},\rho_{pq})=(V_{i_pi_q},e_{i_pi_q},\hat e_{i_pi_q},\rho_{i_pi_q})$. Then $(V_{pq},e_{pq},\hat e_{pq},\rho_{pq})$ is a coordinate change from $(V_p,E_p,\Gamma_p,s_p,\psi_p)$ to $(V_q,E_q,\Gamma_q,s_q,\psi_q)$, in the sense of Definition 14.12.

The associativity property (14.34) of coordinate changes follows from Definition 10.47(d), with $\gamma_{pqr}=\gamma_{i_pi_qi_r}$. Therefore this data $(V_p,E_p,\Gamma_p,s_p,\psi_p)$, $(V_{pq},e_{pq},\hat e_{pq},\rho_{pq})$ for all p,q comprises a Kuranishi structure κ on $X=\mathcal{X}_{\rm top}$, and (X,κ) is a Kuranishi space, of virtual dimension ${\rm vdim}\,\boldsymbol{\mathcal{X}}$.

Similarly, if $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}=F_{\bf Man}^{\bf dOrb^c}(Y)$ is a 1-morphism in $\bf dOrb^c$, then $\boldsymbol{f}$ is semisimple and flat as $\partial Y=\emptyset$, so Theorem 12.48 gives a type A good coordinate system $\big(I,<,(V_i,E_i,\Gamma_i,s_i,\boldsymbol{\psi}_i),(V_{ij},e_{ij},\hat e_{ij},\rho_{ij},\boldsymbol{\eta}_{ij}),\gamma_{ijk},g_i,\boldsymbol{\zeta}_i\big)$ for $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\rightarrow\boldsymbol{\mathcal{Y}}$. We construct the Kuranishi space (X,κ) as above. Define a continuous map $f=f_{\rm top}:\mathcal{X}_{\rm top}\rightarrow\mathcal{Y}_{\rm top}\cong Y$. For each $p\in X=\mathcal{X}_{\rm top}$, define a smooth map $f_p:V_p\rightarrow Y$ by $f_p=g_{i_p}$. Then Definition 10.47(f),(g) imply that this data f,f_p for $p\in X$ comprise a strongly smooth map $(f,\lambda):(X,\kappa)\rightarrow Y$.

Part (c) can be deduced from the constructions above. Roughly, it just says that given a d-orbifold with corners $\boldsymbol{\mathcal{X}}$, if we use Theorem 12.48 to choose a type A good coordinate system $\big(I,<,(V_i,E_i,\Gamma_i,s_i,\boldsymbol{\psi}_i),(V_{ij},e_{ij},\hat e_{ij},\rho_{ij},\boldsymbol{\eta}_{ij}),\gamma_{ijk}\big)$ for $\boldsymbol{\mathcal{X}}$, and then apply Theorem 12.24 to construct a d-orbifold with corners $\boldsymbol{\mathcal{X}}'$ from the data $\big(I,<,(V_i,E_i,\Gamma_i,s_i,\psi_{i,\rm top}),(V_{ij},e_{ij},\hat e_{ij},\rho_{ij}),\gamma_{ijk}\big)$, then $\boldsymbol{\mathcal{X}}'\simeq\boldsymbol{\mathcal{X}}$. This is true as $\boldsymbol{\mathcal{X}}$ is a possible choice for $\boldsymbol{\mathcal{X}}'$ in Theorem 12.24, and Theorem 12.24 gives uniqueness of $\boldsymbol{\mathcal{X}}'$ up to equivalence.

14.4 Derived algebraic geometry and derived schemes

Derived algebraic geometry is a generalization of conventional algebraic geometry. It studies new classes of geometric objects, *derived schemes* and *derived stacks*, which are enriched versions of classical schemes and stacks, with more structure. The foundational ideas were introduced by Deligne, Drinfel'd, and Kontsevich. Systematic treatments are provided by Toën and Vezzosi [101,102] and Lurie [70–72]. It is a difficult, technical area, requiring extensive background — even more so than polyfolds. Toën [100] gives a good introduction.

One major motivation for introducing derived algebraic geometry was to study moduli spaces, and deformation/obstruction theory. For example, let X be a smooth projective scheme, and $\mathcal{M}$ a moduli scheme of stable coherent sheaves on X. Then for each point $[E] \in \mathcal{M}$ representing $E \in \mathrm{coh}(X)$, we have $T^*_{[E]}\mathcal{M} \cong \mathrm{Ext}^1(E,E)^*$. As in §3.3 one can define an 'obstruction space' $O_{[E]}\mathcal{M}$ measuring how singular $\mathcal{M}$ is at $[E]$, and there is a surjective map $\mathrm{Ext}^2(E,E)^* \to O_{[E]}\mathcal{M}$. But $\mathcal{M}$ encodes essentially no information on $\mathrm{Ext}^i(E,E)$ for $i > 2$. The idea was to form a 'derived moduli scheme' $\mathcal{M}^{\mathrm{der}}$ whose 'cotangent space' $T^*_{[E]}\mathcal{M}^{\mathrm{der}}$ encodes $\mathrm{Ext}^i(E,E)^*$ for all $i > 0$.

Such derived moduli schemes are useful in enumerative invariant problems such as Gromov–Witten or Donaldson–Thomas theory. Here one has to define 'virtual classes' of moduli schemes. Classical moduli schemes and stacks contain insufficient information for this, as one needs to know $\mathrm{Ext}^i(E,E)$ for all i. But one can define virtual classes for 'quasi-smooth' derived moduli schemes.

A characteristic feature of derived algebraic geometry is that it involves ∞-*categories*. There are several theories of ∞-categories which are roughly equivalent: simplicial categories, quasicategories, model categories, Segal categories, dg-categories, and A_∞-categories. In examples, ∞-categories are often constructed from an ordinary category $\mathcal{C}$ by inverting a class of morphisms $\mathcal{W}$ in $\mathcal{C}$, called *quasi-isomorphisms* or *weak equivalences*. This is called *localization*. One can localize to get a category $\mathcal{C}[\mathcal{W}^{-1}]$, but this may lose too much information, and an ∞-category localization $\mathcal{C}[\mathcal{W}^{-1}]_\infty$ may be better behaved — for instance, may have the 'correct' (homotopy) fibre products.

Our theory of d-manifolds and d-orbifolds is a theory of 'derived differential geometry'. It differs from derived algebraic geometry in two major respects:

(A) We use C^∞-rings, C^∞-schemes and C^∞-stacks in place of rings, schemes and stacks, as for Spivak's derived manifolds [94, 95] in §14.6.

(B) Our d-spaces, d-manifolds, d-stacks, and d-orbifolds form 2-categories, not ∞-categories. Our 2-categories are defined very explicitly, and not by localization. Our kind of 'derived' geometry is really very much simpler than that of Toën and Vezzosi [101, 102], Lurie [70–72], or Spivak [94, 95].

We will discuss connections between derived algebraic geometry and d-manifolds or d-orbifolds further in §14.5.4–§14.5.5 and §14.6. In the rest of this section we will discuss (B), considering the questions:

(i) How can our '2-category style derived geometry' be expressed, in its simplest form, in the language of conventional derived algebraic geometry?

(ii) Why is it that we can develop a successful 'derived differential geometry' in a simple 2-category framework, whereas 'derived algebraic geometry' requires a much more complex ∞-category set-up?

(iii) Do we lose anything by working in 2-categories instead of ∞-categories? Supposing our theory is a 2-category truncation of an ∞-category version of 'derived differential geometry', what is forgotten by this truncation?

For Question (i), recall our definition of d-spaces $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ in §2.2, where $\underline{X} = (X, \mathcal{O}_X)$. Forgetting about C^∞-rings, regard $\mathcal{O}_X$ and $\mathcal{O}'_X$ as sheaves of $\mathbb{R}$-algebras on X. Define $\mathcal{A}_0 = \mathcal{O}'_X$, as a sheaf of $\mathbb{R}$-algebras on X, and $\mathcal{A}_{-1} = \mathcal{E}_X$, as a sheaf of $\mathcal{A}_0$-modules on X, and define a sheaf morphism $\mathrm{d} : \mathcal{A}_{-1} \to \mathcal{A}_0$ by $\mathrm{d} = \kappa_X \circ \jmath_X$. Then $\mathcal{A}_{-1} \xrightarrow{\mathrm{d}} \mathcal{A}_0$ is a *sheaf of commutative differential graded algebras* (*dg-algebras*) on X. Roughly speaking, this makes $\bigl(X, \mathcal{A}_{-1} \xrightarrow{\mathrm{d}} \mathcal{A}_0\bigr)$ into a *derived scheme*. Note that as (2.18) is exact, the data $\mathcal{O}_X, \imath_X$ in $\boldsymbol{X}$ is the cokernel of $\mathrm{d} : \mathcal{A}_{-1} \to \mathcal{A}_0$, that is, $\mathcal{O}_X \cong h^0(\mathcal{A}_{-1} \xrightarrow{\mathrm{d}} \mathcal{A}_0)$, and $\underline{X}$ is the classical scheme underlying the derived scheme $\bigl(X, \mathcal{A}_{-1} \xrightarrow{\mathrm{d}} \mathcal{A}_0\bigr)$.

The dg-algebras here are of a simple kind:

Definition 14.20. A (*nonpositively graded*) *commutative differential graded algebra* (*dg-algebra*) (A_*, d) over a field $\mathbb{K}$ is a $\mathbb{K}$-algebra $\bigoplus_{k \leqslant 0} A_k$ graded in degrees $k = 0, -1, -2, \ldots$, with $\mathbb{K}$-linear differentials $\mathrm{d} : A_k \to A_{k+1}$ satisfying $\mathrm{d}^2 = 0$ and $ab = (-1)^{kl} ba$, $\mathrm{d}(ab) = (\mathrm{d}a)b + (-1)^k a(\mathrm{d}b)$ for all $a \in A_k$ and $b \in A_l$.

We call (A_*, d) *square zero* if $A_k = 0$ for $k \neq 0, -1$ and $A_{-1} \cdot \mathrm{d}(A_{-1}) = 0$. This implies that $\mathrm{d}(A_{-1})$ is a square zero ideal in A_0.

Thus, the analogue of our '2-category derived geometry' in derived algebraic geometry would be to study *square zero derived schemes*, that is, derived schemes in which the dg-algebras in the structure sheaf are all square zero.

There is a natural truncation functor from commutative dg-algebras to square zero dg-algebras, which maps a dg-algebra (A_*, d) to the square zero dg-algebra

$$A_{-1}/\bigl(\mathrm{d}A_{-2} + A_{-1} \cdot \mathrm{d}(A_{-1})\bigr) \xrightarrow{\mathrm{d}_*} A_0/\bigl(\mathrm{d}(A_{-1})\bigr)^2.$$

Applying this after a suitable fibrant replacement, one could try to define a truncation functor from derived schemes to square zero derived schemes.

For example, let V be a manifold, $E \to V$ a vector bundle of rank k, and $s : V \to E$ a smooth section. In conventional derived algebraic geometry, one would model the 'derived manifold' $V \times_{s,E,0} V$ by the dg-algebra:

$$C^\infty(\Lambda^k E^*) \xrightarrow{s\,\cdot} C^\infty(\Lambda^{k-1} E^*) \xrightarrow{s\,\cdot} \cdots \xrightarrow{s\,\cdot} C^\infty(\Lambda^1 E^*) \xrightarrow{s\,\cdot} C^\infty(V).$$

In our 2-category version, we model $V \times_{s,E,0} V$ by the square zero dg-algebra:

$$C^\infty(E^*)/(C^\infty(E^*) \cdot I_s) \xrightarrow{s\,\cdot} C^\infty(V)/I_s^2,$$

where $I_s = s \cdot C^\infty(E^*)$ is the ideal in $C^\infty(V)$ generated by s.

Remark 14.21. We have been using the term 'derived scheme' above rather loosely. Early work in derived algebraic geometry involved the *dg-schemes* of Ciocan-Fontanine and Kapranov [23]. These have largely been superseded by the *derived stacks* of Toen and Vezzosi [100–102], which include *derived schemes*

[100, §4.2]. Both dg-schemes and derived schemes X may be covered by Zariski open subschemes of the form $\mathrm{Spec}(A_*, \mathrm{d})$ for a dg-algebra (A_*, d). One difference between the two theories is that for dg-schemes the underlying classical scheme and topological space is $\mathrm{Spec}\, A_0$, but for derived schemes the underlying classical scheme and topological space is $\mathrm{Spec}\, h^0(A_*, \mathrm{d})$.

For the case of a dg-scheme or a derived scheme defined as the zeroes of a section s of a vector bundle E over a smooth scheme V, similar to the 'standard model' d-manifold $\boldsymbol{S}_{V,E,s}$ of Definition 3.13, the dg-scheme would have classical scheme V, and the derived scheme would have classical scheme $s^{-1}(0) \subset V$. Thus, Toën–Vezzosi's derived schemes [100, §4.2] are a better analogue for d-manifolds than Ciocan-Fontanine–Kapranov's dg-schemes [23].

For Question (ii), note that our '2-category derived geometry' is not intended as a substitute for derived algebraic geometry in full generality, but only for *quasi-smooth derived schemes* (which correspond to d-manifolds) and *quasi-smooth derived Deligne–Mumford stacks* (which correspond to d-orbifolds). If a derived scheme $\mathcal{M}$ is quasi-smooth then its cotangent complex $\mathbb{L}_{\mathcal{M}}$ is concentrated in degrees $-1, 0$, as for square zero dg-algebras. But the structure sheaf $\mathcal{O}_{\mathcal{M}}$ may be nonzero in many negative degrees. As in Toën [100, §4.4.3], virtual classes are only expected to exist for proper, quasi-smooth derived schemes and Deligne–Mumford stacks.

Behrend [8, 9] studied a version of derived algebraic geometry involving a 2-category of differential graded schemes, rather than an ∞-category. He did not restrict to the quasi-smooth case, and his '2-category geometry' is different from ours. Behrend's 2-category does not have all the good properties one might hope for. In particular, his results [9, §2.3] on gluing by equivalences are quite weak, as they work only over an affine open cover of an affine base.

One nice feature of C^∞-algebraic geometry, that does not hold in conventional algebraic geometry, is the existence of *partitions of unity* on suitable C^∞-schemes $\underline{X}$. This implies that quasicoherent sheaves $\mathcal{E}$ on $\underline{X}$, are *fine*, or *soft* sheaves, which have good properties, as in Proposition B.37 for instance.

In the author's view, the main reason why '2-category derived geometry' works well for d-manifolds and d-orbifolds, is this existence of partitions of unity, and consequent softness of sheaves. In particular, the proofs of Proposition 2.27 and Theorems 2.28–2.33 on gluing d-spaces by equivalences in §2.4 use partitions of unity in an essential way. Note that gluing by equivalences was problematic in Behrend's proposal [8, 9] for 2-categorical derived algebraic geometry.

For Question (iii), for the applications the author has in mind (principally to do with moduli spaces, enumerative invariants, Floer theory, etc.), as far as the author can see nothing important is lost by working in a 2-category rather than an ∞-category, and there are some significant benefits. But there are some constructions in derived algebraic geometry which should work in a suitable ∞-category **DerMan** of derived manifolds, such as Spivak's derived manifolds in §14.6, but will fail in our 2-category **dMan**.

Here is an example. De Rham theory on a smooth manifold X has the following nice interpretation in derived algebraic geometry. The 'loop space'

$\mathcal{L}X$ of X in derived algebraic geometry is the fibre product $X \times_{\Delta_X, X\times X, \Delta_X} X$ in **DerMan**, where $\Delta_X : X \to X \times X$ is the diagonal map. Then the dg-algebra of functions on the derived manifold $\mathcal{L}X$ is the algebra $\Omega^*(X)$ of exterior forms on X. The exterior derivative $\mathrm{d} : \Omega^*(X) \to \Omega^{*+1}(X)$ is interpreted as the Lie derivative of functions on $\mathcal{L}X$ by rotation around the loop, and de Rham cohomology $H^*(X;\mathbb{R})$ is interpreted as the $\mathcal{S}^1$-invariant functions on $\mathcal{L}X$.

In **dMan**, the 'functions' on the fibre product $X \times_{\Delta_X, X\times X, \Delta_X} X$ are $\Omega^0(X) \oplus \Omega^1(X)$, so the effect of truncating to 2-categories is to forget about k-forms on X for $k \geqslant 2$. In general, exterior forms on d-manifolds do not work that well.

14.5 $\mathbb{C}$-schemes and $\mathbb{C}$-stacks with obstruction theories, and quasi-smooth derived $\mathbb{C}$-schemes and $\mathbb{C}$-stacks

In enumerative problems in algebraic geometry, such as Gromov–Witten [7], Donaldson–Thomas [59, 97], or Pandharipande–Thomas invariants [88], a standard method is to show that the moduli space of interest is a scheme or Deligne–Mumford stack equipped with a perfect obstruction theory in the sense of [12]. It then has a virtual class in Chow homology [12], which 'counts' the points in the moduli space, and is used to define the enumerative invariants.

The main result of this section, Theorem 14.27, defines truncation functors from $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-stacks with perfect obstruction theories to d-manifolds and d-orbifolds. Thus, many interesting moduli spaces in complex algebraic geometry have natural d-manifold or d-orbifold structures.

In the derived algebraic geometry of Toën and Vezzosi [100–102] there are notions of quasi-smooth derived schemes and stacks, whose classical truncations are schemes and stacks with perfect obstruction theories. So we obtain truncation functors from quasi-smooth derived $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-stacks to d-manifolds and d-orbifolds.

We begin in §14.5.1–§14.5.4 by summarizing some background material on cotangent complexes, perfect obstruction theories, and quasi-smooth derived schemes and stacks. The new results are stated in §14.5.5 and proved in §14.5.6.

14.5.1 Introduction to cotangent complexes

Suppose $f : X \to Y$ is a morphism of $\mathbb{C}$-schemes. Then one can define the *cotangent sheaf* (or *sheaf of relative differentials*) $\Omega_{X/Y}$ in the abelian category $\mathrm{coh}(X)$ of coherent sheaves on X, as in Hartshorne [38, §II.8]. When $Y = \operatorname{Spec}\mathbb{C}$ and $f : X \to \operatorname{Spec}\mathbb{C}$ is the unique projection, we write Ω_X rather than $\Omega_{X/\operatorname{Spec}\mathbb{C}}$. If X is a smooth $\mathbb{C}$-scheme then Ω_X is the cotangent bundle T^*X, a vector bundle (locally free sheaf) of rank $\dim X$ on X.

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms of $\mathbb{C}$-schemes then there is an exact sequence

$$f^*(\Omega_{Y/Z}) \xrightarrow{\Omega_f} \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0 \tag{14.35}$$

in $\mathrm{coh}(X)$. Note that the morphism $f^*(\Omega_{Y/Z}) \to \Omega_{X/Z}$ need not be injective, that is, (14.35) may not be a short exact sequence. Morally speaking, this says that $f \mapsto \Omega_{X/Y}$ is a right exact functor, but may not be left exact.

Cotangent complexes are derived versions of cotangent sheaves, for which (14.35) is replaced by a distinguished triangle (14.36), making it fully exact. The *cotangent complex* $\mathbb{L}_{X/Y}$ of a morphism $f : X \to Y$ is an object in the (unbounded) derived category $D(\mathrm{qcoh}(X))$ of quasicoherent sheaves on X, constructed by Illusie [50]; a helpful review is given in Illusie [51, §1]. When $Y = \operatorname{Spec}\mathbb{C}$ and $\phi : X \to \operatorname{Spec}\mathbb{C}$ is the projection, we write $\mathbb{L}_X$ rather than $\mathbb{L}_{X/\operatorname{Spec}\mathbb{C}}$. Here are some properties of cotangent complexes:

(a) $h^i(\mathbb{L}_{X/Y}) = 0$ for $i > 0$, and $h^0(\mathbb{L}_{X/Y}) \cong \Omega_{X/Y}$. If $f : X \to Y$ is smooth then $\mathbb{L}_{X/Y} \cong \Omega_{X/Y}$.

(b) Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms of $\mathbb{C}$-schemes. Then there is a distinguished triangle in $D(\mathrm{qcoh}(X))$, [50, §2.1], [51, §1.2]:

$$f^*(\mathbb{L}_{Y/Z}) \xrightarrow{\mathbb{L}_f} \mathbb{L}_{X/Z} \longrightarrow \mathbb{L}_{X/Y} \longrightarrow f^*(\mathbb{L}_{Y/Z})[1]. \tag{14.36}$$

Here $f^* : D(\mathrm{qcoh}(Y)) \to D(\mathrm{qcoh}(X))$ is the left derived pullback functor.

(c) Suppose we have a commutative diagram of morphisms of $\mathbb{C}$-schemes:

$$\begin{tikzcd} U \arrow[r, "e"'] \arrow[d, "f"] & V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r, "g"] & Y \arrow[r] & Z. \end{tikzcd}$$

Then we get a commutative diagram in $D(\mathrm{qcoh}(U))$, [50, §2.1]:

$$\begin{tikzcd} e^*(\mathbb{L}_{V/W}) \arrow[r] & \mathbb{L}_{U/W} \arrow[r] & \mathbb{L}_{U/V} \arrow[r] & e^*(\mathbb{L}_{V/W})[1] \\ f^*\bigl(g^*(\mathbb{L}_{Y/Z})\bigr) \arrow[r] \arrow[u] & f^*(\mathbb{L}_{X/Z}) \arrow[r] \arrow[u] & f^*(\mathbb{L}_{X/Z}) \arrow[r] \arrow[u] & f^*\bigl(g^*(\mathbb{L}_{Y/Z})[1]\bigr), \arrow[u] \end{tikzcd}$$

where the rows come from (14.36) for $U \to V \to W$ and $X \to Y \to Z$.

(d) Suppose we have a Cartesian diagram of $\mathbb{C}$-schemes:

$$\begin{tikzcd} W \arrow[r, "f"'] \arrow[d, "e"'] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z. \end{tikzcd}$$

If g or h is flat then we have *base change isomorphisms* [50, §2.2], [51, §1.3]:

$$\mathbb{L}_{W/Y} \cong e^*(\mathbb{L}_{X/Z}), \;\; \mathbb{L}_{W/X} \cong f^*(\mathbb{L}_{Y/Z}), \;\; \mathbb{L}_{W/Z} \cong e^*(\mathbb{L}_{X/Z}) \oplus f^*(\mathbb{L}_{Y/Z}).$$

(e) There are *truncation functors* $\tau_{<k}, \tau_{\geqslant k} : D(\mathrm{qcoh}(X)) \to D(\mathrm{qcoh}(X))$ for each $k \in \mathbb{Z}$. For any $E^\bullet$ in $D(\mathrm{qcoh}(X))$ and $i, k \in \mathbb{Z}$, these satisfy

$$h^i(\tau_{<k}(E^\bullet)) \cong \begin{cases} h^i(E^\bullet), & i < k, \\ 0, & i \geqslant k, \end{cases} \quad h^i(\tau_{\geqslant k}(E^\bullet)) \cong \begin{cases} 0, & i < k, \\ h^i(E^\bullet), & i \geqslant k, \end{cases}$$

and there is a distinguished triangle

$$\tau^{<k}E^\bullet \xrightarrow{\tau^{<k}} E^\bullet \xrightarrow{\tau^{\geqslant k}} \tau^{\geqslant k}E^\bullet \longrightarrow (\tau^{<k}E^\bullet)[1].$$

For some problems involving cotangent complexes $\mathbb{L}_X$, it is sufficient to consider the truncation $\tau_{\geqslant -1}(\mathbb{L}_X)$. Although cotangent complexes $\mathbb{L}_X, \mathbb{L}_{X/Y}$ are generally difficult to compute unless X or $f : X \to Y$ is smooth, there is a useful explicit expression for $\tau_{\geqslant -1}(\mathbb{L}_X)$. Suppose $j : X \hookrightarrow W$ is an embedding of X as a $\mathbb{C}$-subscheme in a smooth $\mathbb{C}$-scheme W. Then we have an exact sequence of sheaves on X

$$0 \longrightarrow I \longrightarrow j^{-1}(\mathcal{O}_W) \xrightarrow{j^\sharp} \mathcal{O}_X \longrightarrow 0, \tag{14.37}$$

where I is a sheaf of ideals in $j^{-1}(\mathcal{O}_W)$. There is an isomorphism

$$\tau_{\geqslant -1}(\mathbb{L}_X) \cong \bigl[\, I/I^2 \xrightarrow{\alpha} j^*(T^*W) \,\bigr] \tag{14.38}$$

in $D(\mathrm{qcoh}(X))$, where I/I^2 is in degree -1 and $j^*(T^*W)$ in degree 0, and the morphism α maps $\alpha : f + I^2 \mapsto j^*(\mathrm{d}f)$.

All the above also holds with Deligne–Mumford $\mathbb{C}$-stacks in place of $\mathbb{C}$-schemes.

14.5.2 Perfect obstruction theories

We now define (*perfect*) *obstruction theories*. These are tools used in algebraic geometry to construct virtual cycles on moduli spaces, and hence to define enumerative invariants such as Gromov–Witten and Donaldson–Thomas invariants. Obstruction theories were introduced by Behrend and Fantechi [12]. They defined obstruction theories as morphisms $\phi : E^\bullet \to \mathbb{L}_X$. However, we follow Huybrechts and Thomas [49] weaker definition of a morphism $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$.

If $\phi : E^\bullet \to \mathbb{L}_X$ is an obstruction theory in the sense of [12] then $\tau_{\geqslant -1} \circ \phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$ is an obstruction theory in the sense of [49]. Behrend and Fantechi's proof of Theorem 14.23 below uses only properties of the truncation $\tau_{\geqslant -1}(\mathbb{L}_X)$, rather than the full cotangent complex $\mathbb{L}_X$, and so also works for Huybrechts and Thomas' definition. Obstruction theories in Huybrechts and Thomas' sense are sometimes easier to construct. We follow Huybrechts and Thomas because as their definition is weaker, it makes our Theorem 14.27 below stronger.

Definition 14.22. Let X be a $\mathbb{C}$-scheme or Deligne–Mumford $\mathbb{C}$-stack.

(a) A complex $E^\bullet \in D(\mathrm{qcoh}(X))$ is *perfect of amplitude contained in* $[a, b]$, if locally on X (Zariski locally if X is a $\mathbb{C}$-scheme, and étale locally if X is a Deligne–Mumford $\mathbb{C}$-stack), $E^\bullet$ is quasi-isomorphic to a complex $F^\bullet$ of vector bundles (locally free sheaves) of finite rank in degrees $a, a+1, \ldots, b$.

The *virtual rank* of $E^\bullet$ is a locally constant function $\mathrm{rank}\, E^\bullet : X \to \mathbb{Z}$ defined (Zariski or étale) locally by $\mathrm{rank}\, E^\bullet = \sum_{k=a}^{b} (-1)^k \,\mathrm{rank}\, F^k$, for $F^\bullet$ as above. We say that $E^\bullet$ has *constant rank* $n \in \mathbb{Z}$ if $\mathrm{rank}\, E^\bullet = n$.

(b) An *obstruction theory* for X is a morphism $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$ in $D(\mathrm{qcoh}(X))$, where $\mathbb{L}_X$ is the cotangent complex of X, and $\tau_{\geqslant -1}(\mathbb{L}_X)$ its truncation, as in §14.5.1, and E satisfies:

  (i) $h^i(E^\bullet) = 0$ for all $i > 0$;
  (ii) $h^i(E^\bullet)$ is coherent for $i = 0, -1$;
  (iii) $h^0(\phi) : h^0(E^\bullet) \to h^0(\tau_{\geqslant -1}(\mathbb{L}_X)) \cong h^0(\mathbb{L}_X)$ is an isomorphism; and
  (iv) $h^{-1}(\phi) : h^{-1}(E^\bullet) \to h^{-1}(\tau_{\geqslant -1}(\mathbb{L}_X)) \cong h^{-1}(\mathbb{L}_X)$ is surjective.

(c) An obstruction theory $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$ is called *perfect* if $E^\bullet$ is perfect of amplitude contained in $[-1, 0]$.

If instead $f : X \to Y$ is a morphism of $\mathbb{C}$-schemes, we define *relative (perfect) obstruction theories* $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_{X/Y})$ in the same way.

Theorem 14.23 (Behrend and Fantechi [12, §5]). *Suppose X is a proper $\mathbb{C}$-scheme or Deligne–Mumford $\mathbb{C}$-stack, and $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$ is a perfect obstruction theory on X, where $E^\bullet$ has constant rank $n \in \mathbb{Z}$. Then one can construct a **virtual fundamental class**, or **virtual class**, $[X]_{\mathrm{virt}}$ in the Chow homology group $A_n(X)$. If X is smooth of dimension n and ϕ is $\mathrm{id}_{T^*X} : T^*X \to \tau_{\geqslant -1}(\mathbb{L}_X) \cong T^*X$, then $[X]_{\mathrm{virt}}$ is the usual fundamental class of X.*

Behrend and Fantechi's virtual classes have other important properties, which we will not state formally. In particular, they are invariant under continuous deformations of $X, E^\bullet, \phi$. Thus, for instance, Gromov–Witten invariants defined using Behrend and Fantechi's virtual classes are unchanged under deformations of the underlying projective complex manifold.

Theorem 14.24. *Many interesting and important moduli schemes or moduli stacks in algebraic geometry have natural obstruction theories. Some of these are perfect, and so have virtual fundamental classes. In particular, the following moduli problems have been proved to carry perfect obstruction theories:*

(a) *Deligne–Mumford moduli $\mathbb{C}$-stacks of stable morphisms from curves with marked points to smooth complex projective varieties, as in Behrend* [7].

(b) *Deligne–Mumford moduli $\mathbb{C}$-stacks of stable morphisms from curves to complex projective $K3$ surfaces, with 'reduced' obstruction theories, as in Maulik, Thomas and Pandharipande* [81].

(c) *Stable moduli $\mathbb{C}$-schemes of vector bundles and coherent sheaves E over complex algebraic surfaces X, as in Mochizuki* [83]. *Also stable moduli $\mathbb{C}$-schemes of 'L-Bradlow pairs' $\phi : L \to E$ on X, for L a line bundle on X,* [83].

(d) *Stable moduli $\mathbb{C}$-schemes of coherent sheaves E on a complex Calabi–Yau 3-fold or smooth Fano 3-fold X, as in Thomas* [98]. *Also stable moduli $\mathbb{C}$-schemes of morphisms $\phi : \mathcal{O}(-n) \to E$ on a complex Calabi–Yau 3-fold X for $n \gg 0$, as in Joyce and Song* [59].

(e) *Moduli $\mathbb{C}$-schemes of 'stable PT pairs' (C, D) in a smooth complex projective 3-fold X, where $C \subset X$ is a curve and $D \subset C$ is a divisor, as in Pandharipande and Thomas* [88].

(f) *Separated moduli $\mathbb{C}$-schemes of simple perfect complexes in the derived category $D^b \operatorname{coh}(X)$ for X a complex Calabi–Yau 3-fold, as in Huybrechts and Thomas* [49].

The combination of Theorems 14.23 and 14.24 means perfect obstruction theories are the standard tool in enumerative problems in algebraic geometry.

14.5.3 $\mathbb{C}$-schemes with perfect obstruction theories as a category

We make $\mathbb{C}$-schemes X with perfect obstruction theories $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$ into a category, and Deligne–Mumford $\mathbb{C}$-stacks with perfect obstruction theories into a 2-category. We impose the extra conditions that X is separated and second countable and $E^\bullet$ has constant virtual rank in order to have functors from these categories to d-manifolds and d-orbifolds in §14.5.5.

Definition 14.25. Define a category $\mathbf{Sch_{\mathbb{C}}Obs}$ to have objects $(X, E^\bullet, \phi)$, where X is a separated, second countable $\mathbb{C}$-scheme and $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$ is a perfect obstruction theory on X with constant rank. Given objects $(X, E^\bullet, \phi)$, $(Y, F^\bullet, \psi)$, define morphisms $(f, \hat f) : (X, E^\bullet, \phi) \to (Y, F^\bullet, \psi)$ to be a pair $(f, \hat f)$ of a morphism $f : X \to Y$ of $\mathbb{C}$-schemes, and a morphism $\hat f : f^*(F^\bullet) \to E^\bullet$ in $D(\operatorname{qcoh}(X))$ that makes the following diagram commute:

$$\begin{tikzcd} f^*(F^\bullet) \arrow[rr, "\hat f"] \arrow[d, "f^*(\psi)"'] && E^\bullet \arrow[d, "\phi"] \\ f^*(\tau_{\geqslant -1}(\mathbb{L}_Y)) \arrow[rr, "\tau_{\geqslant -1}(\mathbb{L}_f)"] && \tau_{\geqslant -1}(\mathbb{L}_X). \end{tikzcd} \tag{14.39}$$

If $(f, \hat f) : (X, E^\bullet, \phi) \to (Y, F^\bullet, \psi)$ and $(g, \hat g) : (Y, F^\bullet, \psi) \to (Z, G^\bullet, \chi)$ are morphisms, in a similar way to (2.23) the composition is

$$(g, \hat g) \circ (f, \hat f) = \bigl(g \circ f, \hat f \circ f^*(\hat g) \circ I_{f,g}(G^\bullet)\bigr),$$

where $I_{f,g}(G^\bullet) : (g \circ f)^*(G^\bullet) \to f^*(g^*(G^\bullet))$ is the canonical isomorphism. The identity morphism for $(X, E^\bullet, \phi)$ is $(\mathrm{id}_X, \delta_{E^\bullet})$, where $\delta_{E_X} : \mathrm{id}_X^*(E^\bullet) \to E^\bullet$ is the natural isomorphism. It is easy to check that $\mathbf{Sch_{\mathbb{C}}Obs}$ is a category.

Similarly, we define a 2-category $\mathbf{Sta_{\mathbb{C}}Obs}$ to have objects $(X, E^\bullet, \phi)$, where X is a separated, second countable Deligne–Mumford $\mathbb{C}$-stack and $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$ a perfect obstruction theory on X with constant rank. Define 1-morphisms, composition, and identities in $\mathbf{Sta_{\mathbb{C}}Obs}$ as for $\mathbf{Sch_{\mathbb{C}}Obs}$.

Let $(f, \hat f), (g, \hat g) : (X, E^\bullet, \phi) \to (Y, F^\bullet, \psi)$ be 1-morphisms in $\mathbf{Sta_{\mathbb{C}}Obs}$. Then $f, g : X \to Y$ are 1-morphisms in the 2-category $\mathbf{DMSta_{\mathbb{C}}}$ of Deligne–Mumford $\mathbb{C}$-stacks. A 2-morphism $\eta : (f, \hat f) \Rightarrow (g, \hat g)$ in $\mathbf{Sta_{\mathbb{C}}Obs}$ is a 2-morphism $\eta : f \Rightarrow g$ in $\mathbf{DMSta_{\mathbb{C}}}$ such that $\hat g \circ \eta^*(F^\bullet) = \hat f$, where $\eta^*(F^\bullet) : f^*(F^\bullet) \to g^*(F^\bullet)$ is the natural isomorphism in $D(\operatorname{qcoh}(X))$. Horizontal and vertical composition, and identity 2-morphisms, are as in $\mathbf{DMSta_{\mathbb{C}}}$. Then $\mathbf{Sta_{\mathbb{C}}Obs}$ is a 2-category.

Although we will not use it in the sequel, we briefly describe related results of Manolache [74], in which schemes and stacks with obstruction theories are treated as a category. See the references in [74] for similar work. Suppose X, Y are $\mathbb{C}$-schemes, and $\phi : E^\bullet \to \mathbb{L}_X$, $\psi : F^\bullet \to \mathbb{L}_Y$ are obstruction theories in the sense of [12] with virtual ranks m, n, and $f : X \to Y$ is a morphism, and $\hat{f} : f^*(F^\bullet) \to E^\bullet$ is a morphism in $D(\mathrm{qcoh}(X))$ such that $\phi \circ \hat{f} = \mathbb{L}_f \circ \psi$, that is, the analogue of (14.39) without truncations commutes.

Consider the commutative diagram in $D(\mathrm{qcoh}(X))$:

$$\begin{tikzcd}
f^*(F^\bullet) \arrow[r, "\hat{f}"'] \arrow[d, "f^*(\psi)"] & E^\bullet \arrow[r, dotted] \arrow[d, "\phi"'] & G^\bullet \arrow[r, dotted] \arrow[d, dotted, "\chi"'] & f^*(F^\bullet)[1] \arrow[d, "f^*(\psi)[1]"'] \\
f^*(\mathbb{L}_Y) \arrow[r, "\mathbb{L}_f"] & \mathbb{L}_X \arrow[r] & \mathbb{L}_{X/Y} \arrow[r] & f^*(\mathbb{L}_Y)[1],
\end{tikzcd}$$

where the rows are distinguished triangles, with $G^\bullet$ being the cone of $\hat{f}$. As in [74, §3.2], $\chi : G^\bullet \to \mathbb{L}_{X/Y}$ is a relative obstruction theory.

Suppose $G^\bullet$ is perfect. Then Behrend and Fantechi [12] define the virtual class $[X/Y]_{\mathrm{virt}}$ of χ in the relative Chow group $A_{m-n}(X/Y)$. Manolache [74] regards $[X/Y]_{\mathrm{virt}}$ as a 'virtual pull-back map' $f^!_{\mathrm{virt}} : A_*(Y) \to A_{*+m-n}(X)$. She proves various functoriality properties, including $f^!_{\mathrm{virt}}([Y]_{\mathrm{virt}}) = [X]_{\mathrm{virt}}$, where $[X]_{\mathrm{virt}}, [Y]_{\mathrm{virt}}$ are the virtual classes of X, Y defined using ϕ, ψ.

14.5.4 Quasi-smooth derived schemes and stacks

We now explain the relation of §14.5.1–§14.5.3 to *derived algebraic geometry*, which was discussed in §14.4. We use the framework of Toën and Vezzosi [100–102], though a similar story should hold in that of Lurie [70–72].

Toën and Vezzosi define notions of *derived* $\mathbb{C}$*-schemes* and *derived* $\mathbb{C}$*-stacks* X, including *derived Deligne–Mumford* $\mathbb{C}$*-stacks*. Derived $\mathbb{C}$-schemes and $\mathbb{C}$-stacks X have cotangent complexes $\mathbb{L}_X$ [101, §1.4], with properties as in §14.5.1. Each such X has a *classical truncation* $X_0 = t_0(X)$, which is a classical scheme or stack, forgetting the derived structure. There is an inclusion morphism $i : X_0 \to X$. Thus as in §14.5.1 we obtain a morphism $\mathbb{L}_i : i^*(\mathbb{L}_X) \to \mathbb{L}_{X_0}$, which is an obstruction theory on X_0 in the sense of [12]. So $\tau_{\geqslant -1} \circ \mathbb{L}_i : i^*(\mathbb{L}_X) \to \tau_{\geqslant -1}(\mathbb{L}_{X_0})$ is an obstruction theory on X_0 in the sense of §14.5.2.

As in Toën [100, §4.4.3] or Schürg et al. [92, §1], a derived $\mathbb{C}$-scheme or Deligne–Mumford $\mathbb{C}$-stack X is called *quasi-smooth* if $\mathbb{L}_X$ is perfect of amplitude contained in $[-1, 0]$. Then $\mathbb{L}_i : i^*(\mathbb{L}_X) \to \mathbb{L}_{X_0}$ is a perfect obstruction theory on X_0, so if X_0 is proper then Theorem 14.23 gives a virtual class in $A_*(X_0)$.

For many interesting moduli problems in which one can construct a classical moduli scheme or stack $\mathcal{M}_0$, one can also construct a *derived moduli scheme* or *derived moduli stack* $\mathcal{M}$, of which $\mathcal{M}_0 = t_0(\mathcal{M})$ is the classical truncation. Furthermore, the cotangent complex $\mathbb{L}_{\mathcal{M}}$ of the derived moduli scheme or stack $\mathcal{M}$ is usually simply related to the obstruction theory of the moduli problem, and may be perfect of amplitude contained in $[a, b]$ for a, b depending on the dimension of the moduli problem.

In 'classical' situations in which one constructs an obstruction theory on a moduli scheme or stack $\mathcal{M}_0$, such as those in Theorem 14.24, very often there also exists a natural derived moduli scheme or moduli stack $\mathcal{M}$, and the obstruction theory is $\mathbb{L}_i : i^*(\mathbb{L}_{\mathcal{M}}) \to \mathbb{L}_{\mathcal{M}_0}$. So we can think of these obstruction theories as a 'classical shadow' of the derived moduli spaces. Perfect obstruction theories generally come from quasi-smooth moduli schemes or stacks.

Using these ideas and material from Toën and Vezzosi [100–102], we deduce, in a similar way to Schürg [91, Prop. 3.70] and Schürg, Toën and Vezzosi [92, §1]:

Theorem 14.26. *Write* $\mathbf{QsDSch}_{\mathbb{C}}$ *for the Segal category of separated, second countable, quasi-smooth derived* $\mathbb{C}$*-schemes of constant dimension, in the sense of Toën and Vezzosi* [100–102], *and* $\mathrm{Ho}(\mathbf{QsDSch}_{\mathbb{C}})$ *for its homotopy category, and let* $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}, \mathbf{Sta}_{\mathbb{C}}\mathbf{Obs}$ *be as in* §14.5.3. *There is a natural functor* $\Pi_{\mathbf{QsDSch}}^{\mathbf{SchObs}} : \mathrm{Ho}(\mathbf{QsDSch}_{\mathbb{C}}) \to \mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$ *mapping* $X \mapsto \bigl(X_0, i^*(\mathbb{L}_X), \tau_{\geqslant -1} \circ \mathbb{L}_i\bigr)$ *on objects* X, *where* $X_0 = t_0(X)$ *is the classical truncation of* X *and* $i : X_0 \to X$ *the inclusion, and mapping* $f \mapsto \bigl(f_0, \tau_{\geqslant -1}(i^*(\mathbb{L}_f))\bigr)$ *on morphisms* $f : X \to Y$, *where* $f_0 = t_0(f) : X_0 \to Y_0$ *is the classical truncation of* f.

Similarly, writing $\mathbf{QsDSta}_{\mathbb{C}}$ *for the Segal category of separated, second countable, quasi-smooth derived Deligne–Mumford* $\mathbb{C}$*-stacks of constant dimension, there is a functor* $\Pi_{\mathbf{QsDSta}}^{\mathbf{StaObs}} : \mathrm{Ho}(\mathbf{QsDSta}_{\mathbb{C}}) \to \mathrm{Ho}(\mathbf{Sta}_{\mathbb{C}}\mathbf{Obs})$.

Again, the assumptions that X is separated, second countable and of constant virtual dimension are imposed so that we will get functors from these categories to d-manifolds and d-orbifolds in §14.5.5. Schürg [91] gives partial results on the construction of an inverse functor $\mathbf{Sta}_{\mathbb{C}}\mathbf{Obs} \to \mathbf{QsDSta}_{\mathbb{C}}$.

14.5.5 Truncation functors from schemes and stacks with perfect obstruction theories to d-manifolds and d-orbifolds

We can now state our main result, which roughly says that $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-stacks with perfect obstruction theories may be given the structure of d-manifolds and d-orbifolds. It is modelled on Theorems 14.2 and 14.6. We will prove parts (a),(b) in §14.5.6, using ideas from Behrend [10] and Schürg [91]. Part (c) is easy: as for Theorems 14.2(c) and 14.6(c), we must show $\Pi_{\mathbf{SchObs}}^{\mathbf{dMan}}$ preserves identities and composition, and this follows from the construction. We leave (d) as an exercise for the reader.

In (a), we suppose X is separated and second countable so that $X(\mathbb{C})$ is Hausdorff and second countable, which are required for $\boldsymbol{X}$ to be a d-manifold. Note too that if X is a complex manifold of (complex) dimension n, then X is also an oriented real manifold of (real) dimension $2n$. This is why we pass from virtual rank n to virtual dimension $2n$, and include an orientation on $\boldsymbol{X}$.

Theorem 14.27. **(a)** *Suppose* X *is a separated, second countable* $\mathbb{C}$*-scheme and* $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$ *is a perfect obstruction theory on* X, *with virtual rank* $n \in \mathbb{Z}$. *Then we may construct an oriented d-manifold* $\boldsymbol{X}$ *with* $\mathrm{vdim}\,\boldsymbol{X} = 2n$, *natural up to oriented equivalence in* $\mathbf{dMan}$, *whose underlying topological space is the set* $X(\mathbb{C})$ *of* $\mathbb{C}$*-points of* X, *with the complex analytic topology.*

(b) *Let* $(f,\hat f):(X_1,E_1^\bullet,\phi_1)\to(X_2,E_2^\bullet,\phi_2)$ *be a morphism in* $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$ *from* §14.5.3, *and* $\boldsymbol{X}_1,\boldsymbol{X}_2$ *be (choices of) the d-manifolds constructed from* $X_1,E_1^\bullet,\phi_1$ *and* $X_2,E_2^\bullet,\phi_2$ *in* **(a)**. *Then we may construct a* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}_1\to\boldsymbol{X}_2$ *in* **dMan**, *natural up to* 2*-isomorphism, whose underlying continuous map is the map* $f(\mathbb{C}):X_1(\mathbb{C})\to X_2(\mathbb{C})$ *induced by* f *on the sets of* $\mathbb{C}$*-points of* X_1,X_2.

(c) *Using* **(a)**,**(b)** *we define a functor* $\Pi^{\mathbf{dMan}}_{\mathbf{SchObs}}:\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}\to\mathrm{Ho}(\mathbf{dMan})$, *where* $\mathrm{Ho}(\mathbf{dMan})$ *is the homotopy category of the* 2*-category* **dMan**. *For each* $(X,E^\bullet,\phi)$ *in* $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$, *choose a d-manifold* $\boldsymbol{X}$ *in the equivalence class in* **dMan** *given by* **(a)**, *using the Axiom of Choice, and set* $\Pi^{\mathbf{dMan}}_{\mathbf{SchObs}}(X,E^\bullet,\phi)=\boldsymbol{X}$. *For each morphism* $(f,\hat f):(X_1,E_1^\bullet,\phi_1)\to(X_2,E_2^\bullet,\phi_2)$ *in* $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$, *part* **(b)** *defines a* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}_1\to\boldsymbol{X}_2$ *in* **dMan** *unique up to* 2*-isomorphism, so the morphism* $[\boldsymbol{f}]:\boldsymbol{X}_1\to\boldsymbol{X}_2$ *in* $\mathrm{Ho}(\mathbf{dMan})$ *is uniquely defined. Set* $\Pi^{\mathbf{dMan}}_{\mathbf{SchObs}}(f,\hat f)=[\boldsymbol{f}]$. *Then* $\Pi^{\mathbf{dMan}}_{\mathbf{SchObs}}$ *is a functor.*

(d) *Analogues of* **(a)**–**(c)** *also hold for separated, second countable Deligne–Mumford* $\mathbb{C}$*-stacks* X *with perfect obstruction theories* $\phi:E^\bullet\to\tau_{\geqslant -1}(\mathbb{L}_X)$ *of virtual rank* $n\in\mathbb{Z}$, *and oriented d-orbifolds* $\boldsymbol{\mathcal{X}}$ *with* $\operatorname{vdim}\boldsymbol{\mathcal{X}}=2n$, *yielding a functor* $\Pi^{\mathbf{dOrb}}_{\mathbf{StaObs}}:\mathrm{Ho}(\mathbf{Sta}_{\mathbb{C}}\mathbf{Obs})\to\mathrm{Ho}(\mathbf{dOrb})$.

Combining Theorems 14.24 and 14.27 yields existence of d-manifold and d-orbifold structures on many important moduli spaces in complex algebraic geometry. By 'with fixed topological invariants' below we mean, for example, that the genus, number of marked points and homology class of the curve should be fixed in Theorem 14.24(a), and the Chern character of E be fixed in Theorem 14.24(c). This implies that the obstruction theories have constant rank, as required by Theorem 14.27(a). The separated and second countable conditions are automatic in Theorem 14.24(a)–(f), noting that (f) assumes separated.

Corollary 14.28. *The moduli spaces described in Theorem* 14.24(a)–(f), *with fixed topological invariants, all have the structure of oriented d-orbifolds* (*for* (a),(b)) *or oriented d-manifolds* (*for* (c)–(f)), *naturally up to equivalence.*

Composing the truncation functors in Theorems 14.26 and 14.27 gives:

Corollary 14.29. *There are natural truncation functors*

$$\begin{aligned}\Pi^{\mathbf{dMan}}_{\mathbf{QsDSch}}&=\Pi^{\mathbf{dMan}}_{\mathbf{SchObs}}\circ\Pi^{\mathbf{SchObs}}_{\mathbf{QsDSch}}:\mathrm{Ho}(\mathbf{QsDSch}_{\mathbb{C}})\longrightarrow\mathrm{Ho}(\mathbf{dMan}),\\ \Pi^{\mathbf{dOrb}}_{\mathbf{QsDSta}}&=\Pi^{\mathbf{dOrb}}_{\mathbf{StaObs}}\circ\Pi^{\mathbf{StaObs}}_{\mathbf{QsDSta}}:\mathrm{Ho}(\mathbf{QsDSta}_{\mathbb{C}})\longrightarrow\mathrm{Ho}(\mathbf{dOrb})\end{aligned}\tag{14.40}$$

from separated, second countable, quasi-smooth derived $\mathbb{C}$*-schemes and Deligne–Mumford* $\mathbb{C}$*-stacks of constant dimension to d-manifolds and d-orbifolds.*

Remark 14.30. (a) In the proof of Theorem 14.27(a), it is striking how close the correspondence is between the data provided by the $\mathbb{C}$-scheme with perfect obstruction theory, and the data required to define a d-manifold, as in the definition of $\boldsymbol{S}_{V,E,s}$ in Definition 3.13, for instance. The mathematics of perfect obstruction theories, and their connection with square zero extensions as in [12, Th. 4.5], was part of the author's motivation in inventing d-manifolds.

(b) Combining Corollary 14.28 with the material on virtual classes for compact, oriented d-manifolds and d-orbifolds in §13.2 and §13.4 allows us to define virtual classes for the moduli schemes and stacks in Theorem 14.24(a)–(f). The author expects that these will yield the same values for the invariants as if we used Behrend and Fantechi's virtual cycles [12].

As discussed in Remarks 13.28 and 14.9(a), we can also use the d-manifold and d-orbifold structures on moduli spaces in Corollary 3.22 to define classes in d-manifold and d-orbifold bordism groups. It seems plausible that these may contain more information than the 'classical' enumerative invariants.

(c) Let X be a projective complex manifold. Embedding X in some $\mathbb{CP}^n$, it becomes a compact Kähler manifold (X, J, ω), and hence a compact symplectic manifold, with an integrable almost complex structure J. Consider the moduli spaces $\bar{\mathcal{M}}_{g,m}(X, J, \beta)$ of stable J-holomorphic curves in X with genus g, m marked points, and homology class $\beta \in H_2(X; \mathbb{Z})$, which are used to define Gromov–Witten invariants, as in [7, 34, 48].

In symplectic geometry, Hofer et al. [48] realize $\bar{\mathcal{M}}_{g,m}(X, J, \beta)$ as the zeroes of a Fredholm section over a polyfold, so Corollary 14.7 makes $\bar{\mathcal{M}}_{g,m}(X, J, \beta)$ into a compact, oriented d-orbifold $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm sym}$. In algebraic geometry, Behrend [7] makes $\bar{\mathcal{M}}_{g,m}(X, J, \beta)$ into a Deligne–Mumford $\mathbb{C}$-stack with a perfect obstruction theory, as in Theorem 14.24(a), so Corollary 14.28 again makes $\bar{\mathcal{M}}_{g,m}(X, J, \beta)$ into a compact, oriented d-orbifold $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm alg}$.

We can ask: what is the relation between these two d-orbifold structures $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm sym}, \bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm alg}$ on the same moduli space $\bar{\mathcal{M}}_{g,m}(X, J, \beta)$? Are the Gromov–Witten invariants of X defined using the symplectic and algebraic theories the same? For related work see Li and Tian [68] and Siebert [93], who both show that a symplectic and an algebraic definition of Gromov–Witten invariants coincide, though not the definitions we have discussed.

The author expects that the open d-suborbifolds of $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm sym}$ and $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm alg}$ parametrizing nonsingular curves will be equivalent in **dOrb**, but $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm sym}, \bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm alg}$ will in general *not* be equivalent near singular curves. This is because the smooth structure of the moduli spaces near singular curves depends on a choice of *gluing profile* $\varphi : (0, 1] \to [0, \infty)$, in the language of Hofer et al. [41, Def. 1.19], [47, §4.2], [48, §2.1]. As in [48, §2.1], the gluing profiles used to construct $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm sym}$ and $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm alg}$ are $\varphi(r) = e^{1/r} - e$ and $\varphi(r) = -\frac{1}{2\pi} \ln r$, respectively.

The author expects there to be a 1-morphism $\boldsymbol{i} : \bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm sym} \to \bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm alg}$, natural up to 2-isomorphism in **dOrb**, which is the identity on the topological space $\bar{\mathcal{M}}_{g,m}(X, J, \beta)$, but need not be étale near singular curves. Also, $\bar{\mathcal{M}}_{g,m}(X, J, \beta) \times [0, 1]$ should become an oriented d-orbifold with boundary $-\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm sym} \amalg \bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm alg}$. Thus d-orbifold bordism classes defined using $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm sym}$ and $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)_{\rm alg}$ as in Remark 14.9(a) are the same, and the Gromov–Witten invariants coincide.

(d) Corollary 14.29 defines functors (14.40) from $\mathrm{Ho}(\mathbf{QsDSch}_{\mathbb{C}}), \mathrm{Ho}(\mathbf{QsDSta}_{\mathbb{C}})$ to $\mathrm{Ho}(\mathbf{dMan}), \mathrm{Ho}(\mathbf{dOrb})$. Using homotopy categories forgets information in the 2-morphisms of $\mathbf{dMan}, \mathbf{dOrb}$. It seems likely that (14.40) are truncations of (higher) functors $\mathbf{QsDSch}_{\mathbb{C}} \to \mathbf{dMan}$ and $\mathbf{QsDSta}_{\mathbb{C}} \to \mathbf{dOrb}$. But it would take more work to prove this.

14.5.6 The proof of Theorem 14.27(a),(b)

The proof is modelled on that of Theorem 14.2(a),(b) in §14.5.1. Note that Step $2''$(a),(b),(c) below are analogous to Step 2(a),(c),(d) in §14.5.1, and Step 2(b) in §14.5.1 is incorporated into $(+'')$ below. We will prove Theorem 14.27(a),(b) in the following four steps. The proofs of the last part of Step $1''$ and of Step $2''(c)$ are modelled on Behrend [10, Prop. 3.13] and Schürg [91, Prop. 5.20].

As we are working with $\mathbb{C}$-schemes, we can use the Zariski topology. To prove the analogous result for Deligne–Mumford $\mathbb{C}$-stacks, as for Theorem 14.27(d), one should use the étale topology. Note that the notation $t_1 = t_2 + O(s)$ and $t_1 = t_2 + O(s^2)$ in Definition 3.29 also makes sense for smooth $\mathbb{C}$-schemes and vector bundles, as used in (14.43), (14.46) and (14.47) below.

Step $1''$. Let X, $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$ and $n \in \mathbb{Z}$ be as in Theorem 14.27(a). We say that a sextuple (U, W, j, F, t, χ) *satisfies condition* $(*'')$ if:

$(*'')$ $U \subseteq X$ is a nonempty affine (Zariski) open $\mathbb{C}$-subscheme of X, and W is a smooth affine $\mathbb{C}$-scheme with $\dim_{\mathbb{C}} W = k$, and $j : U \hookrightarrow W$ is a closed embedding of U in W, and F is a trivializable vector bundle (locally free sheaf) on W with $\operatorname{rank}_{\mathbb{C}} F = l$, and $t \in H^0(F)$ with $t^{-1}(0) = j(U)$ as $\mathbb{C}$-subschemes of W. Let I be the sheaf of ideals in $j^{-1}(\mathcal{O}_W)$ from the embedding $j : U \hookrightarrow W$, as in (14.37), so that (14.38) gives an isomorphism from the complex $\bigl[I/I^2 \to j^*(T^*W)\bigr]$ in degrees $[-1,0]$ in $D(\mathrm{qcoh}(U))$ to $\tau_{\geqslant -1}(\mathbb{L}_U) = \tau_{\geqslant -1}(\mathbb{L}_X)|_U$. Then χ should be an isomorphism in $D(\mathrm{qcoh}(U))$ from the complex $\bigl[j^*(F^*) \xrightarrow{j^*(\mathrm{d}t)} j^*(T^*W)\bigr]$ in degrees $[-1,0]$ to $E^\bullet|_U$ such that the following diagram in $D(\mathrm{qcoh}(U))$ commutes:

$$\begin{tikzcd}
\bigl[j^*(F^*) \arrow[r, "j^*(\mathrm{d}t)"] \arrow[d, "j^*(t)\cdot"] & j^*(T^*W)\bigr] \arrow[rr, "\chi", "\cong"'] \arrow[d, "\mathrm{id}"] && E^\bullet|_U \arrow[d, "\phi|_U"] \\
\bigl[I/I^2 \arrow[r, "f+I^2\mapsto j^*(\mathrm{d}f)"] & j^*(T^*W)\bigr] \arrow[rr, "(14.38)", "\cong"'] && \tau_{\geqslant -1}(\mathbb{L}_X)|_U.
\end{tikzcd} \tag{14.41}$$

We show that if (U, W, j, F, t, χ) satisfies $(*'')$ then $k - l = n$, and that for all $x \in X(\mathbb{C})$, there exists (U, W, j, F, t, χ) satisfying $(*'')$ with $x \in U(\mathbb{C})$.

Step $2''$. Let $(f, \hat{f}) : (X_1, E_1^\bullet, \phi_1) \to (X_2, E_2^\bullet, \phi_2)$ be a morphism in $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$, as in §14.5.3, and $(U_a, W_a, j_a, F_a, t_a, \chi_a)$ satisfy $(*'')$ in $(X_a, E_a^\bullet, \phi_a)$ for $a = 1, 2$. We say that $(U_{12}, W_{12}, e_{12}, \hat{e}_{12}) : (U_1, W_1, j_1, F_1, t_1, \chi_1) \to (U_2, W_2, j_2, F_2, t_2, \chi_2)$ *satisfies condition* $(+'')$ if:

$(+'')$ $U_{12} \subseteq U_1 \subseteq X_1$ and $W_{12} \subseteq W_1$ are open $\mathbb{C}$-subschemes with $U_{12} = U_1 \cap f^{-1}(U_2) = j_1^{-1}(W_{12})$, and $e_{12} : W_{12} \to W_2$ is a morphism of $\mathbb{C}$-schemes and $\hat{e}_{12} : F_1|_{W_{12}} \to e_{12}^*(F_2)$ a morphism of vector bundles on

W_{12}, satisfying

$$e_{12} \circ j_1|_{U_{12}} = j_2 \circ f|_{U_{12}} : U_{12} \longrightarrow W_2, \tag{14.42}$$

$$\hat{e}_{12} \circ t_1|_{W_{12}} = e_{12}^*(t_2) + O(t_1^2), \tag{14.43}$$

such that the following diagram in $D(\mathrm{qcoh}(U_{12}))$ commutes:

$$\begin{tikzcd}
f|_{U_{12}}^*\bigl(\bigl[j_2^*(F_2^*) \arrow[r, "j_2^*(\mathrm{d}t_2)"] \arrow[d, "j_1|_{U_{12}}^*(\hat{e}_{12}^*)\circ I_{j_1|_{U_{12}},e_{12}}(F_2^*)\circ I_{f|_{U_{12}},j_2}(F_2^*)^{-1}"] & j_2^*(T^*W_2)\bigr]\bigr) \arrow[r, "f|_{U_{12}}^*(\chi_2)", "\cong"'] \arrow[d, "j_1|_{U_{12}}^*(\mathrm{d}e_{12}^*)\circ I_{j_1|_{U_{12}},e_{12}}(T^*W_2)\circ I_{f|_{U_{12}},j_2}(T^*W_2)^{-1}"] & f|_{U_{12}}^*(E_2^\bullet) \arrow[d, "\hat{f}|_{U_{12}}"] \\
\bigl[j_1^*(F_1^*) \arrow[r, "j_1^*(\mathrm{d}t_1)"] & j_1^*(T^*W_1)\bigr]\bigr|_{U_{12}} \arrow[r, "\chi_1|_{U_{12}}", "\cong"'] & E_1^\bullet|_{U_{12}}.
\end{tikzcd} \tag{14.44}$$

Here the two left hand columns are well defined and form a commutative square, and thus a morphism in $D(\mathrm{qcoh}(U_{12}))$, by (14.42)–(14.43).

We will prove:

(a) Suppose that, as well as the data above, we are given a morphism $(g, \hat{g}) : (X_2, E_2^\bullet, \phi_2) \to (X_3, E_3^\bullet, \phi_3)$ in $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$, and $(U_3, W_3, j_3, F_3, t_3, \chi_3)$ satisfying $(*'')$ in $(X_3, E_3^\bullet, \phi_3)$, and $(U_{23}, W_{23}, e_{23}, \hat{e}_{23}) : (U_2, W_2, j_2, F_2, t_2, \chi_2) \to (U_3, W_3, j_3, F_3, t_3, \chi_3)$ satisfying $(+'')$ for $(g, \hat{g})$. Then

$$\begin{aligned}\bigl(U_{12} \cap f^{-1}(U_{23}), e_{12}^{-1}(W_{23}), e_{23} \circ e_{12}|_{e_{12}^{-1}(W_{23})},\\ e_{12}|_{e_{12}^{-1}(W_{23})}^*(\hat{e}_{23}) \circ \hat{e}_{12}|_{e_{12}^{-1}(W_{23})}\bigr)\end{aligned} \tag{14.45}$$

satisfies $(+'')$ for $(g, \hat{g}) \circ (f, \hat{f}) : (X_1, E_1^\bullet, \phi_1) \to (X_3, E_3^\bullet, \phi_3)$. That is, condition $(+'')$ is closed under composition in a suitable sense.

(b) If $(U_{12}, W_{12}, e_{12}, \hat{e}_{12}), (U'_{12}, W'_{12}, e'_{12}, \hat{e}'_{12}) : (U_1, W_1, j_1, F_1, t_1, \chi_1) \to (U_2, W_2, j_2, F_2, t_2, \chi_2)$ both satisfy $(+'')$, then there exists a (Zariski) open neighbourhood W''_{12} of $j_1(U_{12} \cap U'_{12})$ in $W_{12} \cap W'_{12}$ and a vector bundle morphism $\Lambda : F_1|_{W''_{12}} \to e_{12}|_{W''_{12}}^*(TW_2)$ on W''_{12} satisfying

$$e'_{12}|_{W''_{12}} = e_{12}|_{W''_{12}} + \Lambda \circ t_1|_{W''_{12}} + O(t_1^2), \tag{14.46}$$

$$\hat{e}'_{12}|_{W''_{12}} = \hat{e}_{12}|_{W''_{12}} + (e_{12}|_{W''_{12}}^*(\mathrm{d}t_2)) \circ \Lambda + O(t_1). \tag{14.47}$$

(c) If $(U_1, W_1, j_1, F_1, t_1, \chi_1)$ satisfies $(*'')$ in $(X_1, E_1^\bullet, \phi_1)$ and $(U_2, W_2, j_2, F_2, t_2, \chi_2)$ satisfies $(*'')$ in $(X_2, E_2^\bullet, \phi_2)$, then there exists $(U_{12}, W_{12}, e_{12}, \hat{e}_{12}) : (U_1, W_1, j_1, F_1, t_1, \chi_1) \to (U_2, W_2, j_2, F_2, t_2, \chi_2)$ satisfying $(+'')$.

Step 3$''$. Using Step 1$''$, we choose an indexing set I, and $(U_i, W_i, j_i, F_i, t_i, \chi_i)$ satisfying $(*'')$, with $\dim_{\mathbb{C}} W_i = k_i$ and $\mathrm{rank}_{\mathbb{C}} F_i = l_i$ for each $i \in I$, such that $\{U_i : i \in I\}$ is an open cover of X. Using Step 2$''$(c) with $(X_1, E_1^\bullet, \phi_1) = (X_2, E_2^\bullet, \phi_2) = (X, E^\bullet, \phi)$, $f = \mathrm{id}_X$ and $\hat{f} = \mathrm{id}_{E^\bullet}$, we choose $(U_{ij}, W_{ij}, e_{ij}, \hat{e}_{ij}) : (U_i, W_i, j_i, F_i, t_i, \chi_i) \to (U_j, W_j, j_j, F_j, t_j, \chi_j)$ satisfying $(+'')$ for all $i, j \in I$.

So far we have been working in the algebraic geometry world of $\mathbb{C}$-schemes and locally free sheaves. We now transfer to the differential geometry world of smooth manifolds and smooth vector bundles. For each $i \in I$, W_i is a smooth affine $\mathbb{C}$-scheme with $\dim_{\mathbb{C}} W_i = k_i$, so its set of $\mathbb{C}$-points $W_i(\mathbb{C})$, with the complex analytic topology, has the structure of a complex manifold of complex dimension k_i, and thus of a real manifold of real dimension $2k_i$.

Similarly, F_i is an algebraic vector bundle (locally free sheaf) on W_i with $\mathrm{rank}_{\mathbb{C}} F_i = l_i$, so the set of $\mathbb{C}$-points $F_i(\mathbb{C})$ of its total space, with the complex analytic topology, is a holomorphic vector bundle over the complex manifold $W_i(\mathbb{C})$ with complex rank l_i, and thus a real vector bundle over the real manifold $W_i(\mathbb{C})$ with real rank $2l_i$. Also t_i is a section of F_i over W_i, so $t_i(\mathbb{C})$ is a holomorphic section of $F_i(\mathbb{C})$ as a holomorphic vector bundle, and thus a smooth section of $F_i(\mathbb{C})$ as a real vector bundle.

From now on, regard $W_i(\mathbb{C})$ as a real manifold, $F_i(\mathbb{C}) \to W_i(\mathbb{C})$ as a real vector bundle, and $t_i(\mathbb{C}) : W_i(\mathbb{C}) \to F_i(\mathbb{C})$ as a smooth section. Then Definition 3.13 defines a 'standard model' d-manifold $\boldsymbol{S}_{W_i(\mathbb{C}),F_i(\mathbb{C}),t_i(\mathbb{C})}$, which has virtual dimension $2k_i - 2l_i = 2n$ by Step $1''$. Its underlying topological space is $t_i(\mathbb{C})^{-1}(0)$. As $j_i : U_i \to t^{-1}(0)$ is a $\mathbb{C}$-scheme isomorphism by $(*'')$, it follows that $j_i(\mathbb{C}) : U_i(\mathbb{C}) \to t_i(\mathbb{C})^{-1}(0)$ is a homeomorphism, where $U_i(\mathbb{C}) \subseteq X(\mathbb{C})$ is open, and $X(\mathbb{C})$ has the complex analytic topology. Define a homeomorphism $\psi_i = j_i(\mathbb{C})^{-1} : t_i(\mathbb{C})^{-1}(0) \to U_i(\mathbb{C}) \subseteq X(\mathbb{C})$.

Definition 4.48 shows that an orientation on the line bundle $\Lambda^{\mathrm{top}} F_i(\mathbb{C}) \otimes \Lambda^{\mathrm{top}} T^* W_i(\mathbb{C})$ induces an orientation on the d-manifold $\boldsymbol{S}_{W_i(\mathbb{C}),F_i(\mathbb{C}),t_i(\mathbb{C})}$. But the complex structures on $W_i(\mathbb{C}), F_i(\mathbb{C})$ induce orientations on $\Lambda^{\mathrm{top}} T^* W_i(\mathbb{C})$ and $\Lambda^{\mathrm{top}} F_i(\mathbb{C})$, using the convention that if V is a finite-dimensional complex vector space and $v_1, \ldots, v_k$ a basis for V over $\mathbb{C}$, then $v_1, Jv_1, v_2, Jv_2, \ldots, v_k, Jv_k$ is an oriented basis for V over $\mathbb{R}$, where J is the complex structure on V. Hence $\boldsymbol{S}_{W_i(\mathbb{C}),F_i(\mathbb{C}),t_i(\mathbb{C})}$ has a natural orientation.

If $i, j \in I$ then $W_{ij}(\mathbb{C}) \subseteq W_i(\mathbb{C})$ is open, and $e_{ij}(\mathbb{C}) : W_{ij}(\mathbb{C}) \to W_j(\mathbb{C})$ is a holomorphic map of complex manifolds, and hence a smooth map of real manifolds, and $\hat{e}_{ij}(\mathbb{C}) : F_i(\mathbb{C})|_{W_{ij}(\mathbb{C})} \to e_{ij}(\mathbb{C})^*(F_j(\mathbb{C}))$ a morphism of holomorphic and hence of smooth vector bundles on $W_{12}(\mathbb{C})$, and (14.43) implies that $\hat{e}_{ij}(\mathbb{C}) \circ t_i(\mathbb{C})|_{W_{ij}(\mathbb{C})} = e_{ij}(\mathbb{C})^*\bigl(t_j(\mathbb{C})\bigr) + O\bigl(t_i(\mathbb{C})^2\bigr)$, in the sense of Definition 3.29. Therefore Definition 3.30 defines a 'standard model' 1-morphism

$$\boldsymbol{S}_{e_{ij}(\mathbb{C}),\hat{e}_{ij}(\mathbb{C})} : \boldsymbol{S}_{W_{ij}(\mathbb{C}),F_i(\mathbb{C})|_{W_{ij}(\mathbb{C})},t_i(\mathbb{C})|_{W_{ij}(\mathbb{C})}} \longrightarrow \boldsymbol{S}_{W_j(\mathbb{C}),F_j(\mathbb{C}),t_j(\mathbb{C})}. \tag{14.48}$$

We show that (14.48) is an equivalence with its image, and identifies the orientations on $\boldsymbol{S}_{W_i(\mathbb{C}),F_i(\mathbb{C}),t_i(\mathbb{C})}$, $\boldsymbol{S}_{W_j(\mathbb{C}),F_j(\mathbb{C}),t_j(\mathbb{C})}$.

We then use Theorem 3.42 to construct a d-manifold $\boldsymbol{X}$ with topological space $X(\mathbb{C})$ and $\mathrm{vdim}\, \boldsymbol{X} = 2n$ by gluing the d-manifolds $\boldsymbol{S}_{W_i(\mathbb{C}),F_i(\mathbb{C}),t_i(\mathbb{C})}$ for $i \in I$ on overlaps using the equivalences (14.48) for $i, j \in I$. As the $\boldsymbol{S}_{W_i(\mathbb{C}),F_i(\mathbb{C}),t_i(\mathbb{C})}$ are oriented and the equivalences (14.48) preserve orientations, $\boldsymbol{X}$ is an oriented d-manifold. We show that $\boldsymbol{X}$ is independent of choices up to oriented equivalence in **dMan**. This proves Theorem 14.27(a).

Step $4''$. By Step $3''$, we construct a d-manifold $\boldsymbol{X}_1$ using $(U_i, W_i, j_i, F_i, t_i, \chi_i)$

satisfying $(*'')$ for $i \in I$ with $\{U_i : i \in I\}$ an open cover of X_1, with equivalences $\boldsymbol{\psi}_i : \boldsymbol{S}_{W_i(\mathbb{C}),F_i(\mathbb{C}),t_i(\mathbb{C})} \to \hat{\boldsymbol{X}}_{1,i} \subseteq \boldsymbol{X}_1$, and a d-manifold $\boldsymbol{X}_2$ using $(U'_k, W'_k, j'_k, F'_k, t'_k, \chi'_k)$ satisfying $(*'')$ for $k \in K$ with $\{U'_k : k \in K\}$ an open cover of X_2, with equivalences $\boldsymbol{\psi}'_k : \boldsymbol{S}_{W'_k(\mathbb{C}),F'_k(\mathbb{C}),t'_k(\mathbb{C})} \to \hat{\boldsymbol{X}}_{2,k} \subseteq \boldsymbol{X}_2$.

We choose these so that for each $i \in I$ there exists $k_i \in K$ with $f(U_i) \subseteq U'_{k_i} \subseteq X_2$, and Step $2''$(c) gives $(U_i, W_i, g_{ik_i}, \hat{g}_{ik_i}) : (U_i, W_i, j_i, F_i, t_i, \chi_i) \to (U'_{k_i}, W'_{k_i}, j'_{k_i}, F'_{k_i}, t'_{k_i}, \chi'_{k_i})$ satisfying $(+'')$. Then $\hat{g}_{ik_i} \circ t_i = g^*_{ik_i}(t'_{k_i}) + O(t_i^2)$ by (14.43), so Definition 3.30 gives a 1-morphism

$$\boldsymbol{S}_{g_{ik_i}(\mathbb{C}),\hat{g}_{ik_i}(\mathbb{C})} : \boldsymbol{S}_{W_i(\mathbb{C}),F_i(\mathbb{C}),t_i(\mathbb{C})} \longrightarrow \boldsymbol{S}_{W'_{k_i}(\mathbb{C}),F'_{k_i}(\mathbb{C}),t'_{k_i}(\mathbb{C})}. \tag{14.49}$$

Composing with $\boldsymbol{\psi}'_{k_i}$ gives a 1-morphism

$$\boldsymbol{\psi}'_{k_i} \circ \boldsymbol{S}_{g_{ik_i}(\mathbb{C}),\hat{g}_{ik_i}(\mathbb{C})} : \boldsymbol{S}_{W_i(\mathbb{C}),F_i(\mathbb{C}),t_i(\mathbb{C})} \longrightarrow \boldsymbol{X}_2. \tag{14.50}$$

Now $\boldsymbol{X}_1$ is constructed, as a d-space, by gluing the d-spaces $\boldsymbol{S}_{W_i(\mathbb{C}),F_i(\mathbb{C}),t_i(\mathbb{C})}$ for i in I by equivalences (14.48) on overlaps, using the first part of Theorem 2.33. We show that the 1-morphisms (14.50) satisfy the conditions on the $\boldsymbol{g}_i$ in the second part of Theorem 2.33, so we can glue them to give a 1-morphism $\boldsymbol{f} : \boldsymbol{X}_1 \to \boldsymbol{X}_2$, unique up to 2-isomorphism, such that $\boldsymbol{f} \circ \boldsymbol{\psi}_i \cong \boldsymbol{\psi}'_k \circ \boldsymbol{S}_{g_{ik_i}(\mathbb{C}),\hat{g}_{ik_i}(\mathbb{C})}$ for all $i \in I$. We show that $\boldsymbol{f}$ is independent of choices up to 2-isomorphism, and satisfies Theorem 14.27(b). This then completes the proof.

For Step $1''$, suppose (U, W, j, F, t, χ) satisfies $(*'')$. Then $U \neq \emptyset$ and $E^\bullet|_U$ is quasi-isomorphic to the complex $\big[j^*(F^*) \to j^*(T^*W)\big]$ in degrees $[-1, 0]$, where $j^*(F^*), j^*(T^*W)$ are vector bundles on U. Hence

$$n = \operatorname{rank} E^\bullet|_U = \operatorname{rank}_{\mathbb{C}} j^*(TW) - \operatorname{rank}_{\mathbb{C}} j^*(F^*) = \dim_{\mathbb{C}} W - \operatorname{rank}_{\mathbb{C}} F = k - l,$$

by definition of $\operatorname{rank} E^\bullet$ in Definition 14.22(a). So $k - l = n$, as we want.

Let $x \in X(\mathbb{C})$. We will construct (U, W, j, F, t, χ) satisfying $(*'')$ with $x \in U(\mathbb{C}) \subseteq X(\mathbb{C})$, loosely following Behrend's proof of [10, Prop. 3.13]. Choose an affine, (Zariski) open neighbourhood U of x in X, and a closed embedding $j : U \hookrightarrow W$, where W is a smooth $\mathbb{C}$-scheme of dimension $\dim_{\mathbb{C}} T^*_x X$. By Definition 14.22(a),(c), $E^\bullet$ is (Zariski) locally quasi-isomorphic to a complex of vector bundles in degrees $-1, 0$. So making U, W smaller, we have an isomorphism

$$\big[G^{-1} \xrightarrow{\beta} G^0\big] \xrightarrow{\quad\chi\quad} E^\bullet|_U \tag{14.51}$$

in $D(\operatorname{qcoh}(U))$, where G^{-1}, G^0 are vector bundles on U.

Definition 14.22(b)(iii) gives $\operatorname{Coker} \beta|_x \cong T^*_x X$. Thus, (Zariski) locally near $x \in U$ we can find splittings $G^{-1} = \tilde{G}^{-1} \oplus H$, $G^0 = \tilde{G}^0 \oplus H$, such that $\operatorname{rank} \tilde{G}^0 = \dim T^*_x X$, and $\beta = \left(\begin{smallmatrix} \tilde{\beta} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right)$ with $\tilde{\beta}|_x = 0$. So making U, W smaller and replacing G^{-1}, G^0, β by $\tilde{G}^{-1}, \tilde{G}^0, \tilde{\beta}$, we can suppose $\operatorname{rank} G^0 = \dim T^*_x X = \dim_{\mathbb{C}} W$, and $\beta|_x = 0$, and also that G^{-1}, G^0 are trivializable.

Consider the commutative diagram of morphisms in $D(\mathrm{qcoh}(U))$:

$$\begin{tikzcd}[\big[G^{-1} \xrightarrow{\beta} G^0\big] \arrow[rr, "\chi", "\cong"'] \arrow[d, "\theta"] & & E^\bullet|_U \arrow[d, "\phi|_U"'] \\ \big[I/I^2 \xrightarrow{\alpha} j^*(T^*W)\big] & & \tau_{\geqslant -1}(\mathbb{L}_X)|_U, \arrow[ll, "(14.38)"', "\cong"] \end{tikzcd} \tag{14.52}$$

where I is the sheaf of ideals in $j^{-1}(\mathcal{O}_W)$ defining $j(U)$ as a subscheme in W, and θ is the composition of the other morphisms. Since G^{-1}, G^0 are vector bundles and U is affine, we may represent θ by a morphism of complexes

$$\begin{tikzcd}\big[G^{-1} \arrow[r, "\beta"'] \arrow[d, "\theta^{-1}"] & G^0\big] \arrow[d, "\theta^0"'] \\ \big[I/I^2 \arrow[r, "\alpha"] & j^*(T^*W)\big]. \end{tikzcd} \tag{14.53}$$

Now $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective by Definition 14.22(b), so $h^0(\theta)$ is an isomorphism and $h^{-1}(\theta)$ is surjective as the other morphisms in (14.52) are isomorphisms. As $\beta|_x = 0$ and $\operatorname{rank} G^0 = \dim W = \operatorname{rank} j^*(T^*W)$, $h^0(\theta)$ an isomorphism implies that $\theta^0|_x$ is an isomorphism, so θ^0 is an isomorphism of vector bundles near x. Making U, W smaller we can suppose θ^0 is an isomorphism on U, and identify $G^0 \cong j^*(T^*W)$ and $\theta^0 = \mathrm{id}_{j^*(T^*W)}$.

Since G^{-1} is trivializable we may identify it with $j^*(F^*)$, for F a (trivial) vector bundle over W. Thus (14.53) becomes

$$\begin{tikzcd}\big[j^*(F^*) \arrow[r, "\beta"'] \arrow[d, "\theta^{-1}"] & j^*(T^*V)\big] \arrow[d, "\mathrm{id}_{j^*(T^*W)}"'] \\ \big[I/I^2 \arrow[r, "\alpha"] & j^*(T^*W)\big]. \end{tikzcd} \tag{14.54}$$

Making U, W smaller, we can lift $\theta^{-1} : j^*(F^*) \to I/I^2$ to a morphism $\hat{\theta}^{-1} : j^*(F^*) \to I$. But $I \subset j^{-1}(\mathcal{O}_W)$ is an ideal of germs at $j(U)$ of functions on W, so we may regard $\hat{\theta}^{-1}$ as a germ at $j(U)$ of sections of F on W. Thus, making W smaller, we can choose $t \in H^0(F)$ whose germ at $j(U)$ is $\hat{\theta}^{-1}$. As $h^{-1}(\theta)$ is surjective, $\hat{\theta}^{-1}$ generates the ideal I, so $t^{-1}(0)$ is the closed subscheme $j(U)$ in W near $j(U)$. Making W smaller gives $t^{-1}(0) = j(U)$. Then (14.54) becomes

$$\begin{tikzcd}\big[j^*(F^*) \arrow[r, "j^*(\mathrm{d}t)"'] \arrow[d, "j^{-1}(t)\cdot"] & j^*(T^*W)\big] \arrow[d, "\mathrm{id}_{j^*(T^*W)}"'] \\ \big[I/I^2 \arrow[r, "\alpha"] & j^*(T^*W)\big], \end{tikzcd} \tag{14.55}$$

where $\beta = j^*(\mathrm{d}t)$ follows from (14.55) commutative and $\alpha : f + I^2 \mapsto i^*(\mathrm{d}f)$. From (14.51)–(14.55) we see that (14.41) commutes in $D(\mathrm{qcoh}(U))$. Hence (U, W, j, F, t, χ) satisfies $(*'')$, proving Step $1''$.

Step $2''$(a) follows by the commutative diagram in $D(\mathrm{qcoh}(U_{12}\cap f^{-1}(U_{23})))$

$$\begin{tikzcd}[column sep=small]
(g\circ f)|^*_{\cdots}\big(\big[j_3^*(F_3^*) \arrow[r, "j_3^*(\mathrm{d}t_3)"] \arrow[d, "I_{f|\ldots,g}(j_3^*(F_3^*))"'] & j_3^*(T^*W_3)\big]\big) \arrow[rr, "(g\circ f)|^*_{\cdots}(\chi_3)", "\cong"'] \arrow[d, "I_{f|\ldots,g}(j_3^*(T^*W_3))"] && (g\circ f)|^*_{U_{12}\cap f^{-1}(U_{23})}\big(E_3^\bullet\big) \arrow[d, "I_{f|\ldots,g}(E_3^\bullet)"'] \\
f|^*_{\cdots}\circ g|^*_{\cdots}\big(\big[j_3^*(F_3^*) \arrow[r, "j_3^*(\mathrm{d}t_3)"] \arrow[d, "\substack{j_2|^*_{\cdots}(\hat e_{23}^*)\circ \\ I_{j_2|\ldots,e_{23}}(F_3^*)\circ \\ I_{g|\ldots,j_3}(F_3^*)^{-1}}"'] & j_3^*(T^*W_3)\big]\big) \arrow[rr, "f|^*_{\cdots}\circ g|^*_{\cdots}(\chi_3)", "\cong"'] \arrow[d, "\substack{j_2|^*_{\cdots}(\mathrm{d}e_{23}^*)\circ \\ I_{j_2|\ldots,e_{23}}(T^*W_3)\circ \\ I_{g|\ldots,j_3}(T^*W_3)^{-1}}"] && f|^*_{U_{12}\cap f^{-1}(U_{23})}\big(g|^*_{\cdots}\big(E_3^\bullet\big)\big) \arrow[d, "f|^*_{\cdots}(\hat g)"'] \\
f|^*_{\cdots}\big(\big[j_2^*(F_2^*) \arrow[r, "j_2^*(\mathrm{d}t_2)"] \arrow[d, "\substack{j_1|^*_{\cdots}(\hat e_{12}^*)\circ \\ I_{j_1|\ldots,e_{12}}(F_2^*)\circ \\ I_{f|\ldots,j_2}(F_2^*)^{-1}}"'] & j_2^*(T^*W_2)\big]\big) \arrow[rr, "f|^*_{\cdots}(\chi_2)", "\cong"'] \arrow[d, "\substack{j_1|^*_{\cdots}(\mathrm{d}e_{12}^*)\circ \\ I_{j_1|\ldots,e_{12}}(T^*W_2)\circ \\ I_{f|\ldots,j_2}(T^*W_2)^{-1}}"] && f|^*_{U_{12}\cap f^{-1}(U_{23})}\big(E_2^\bullet\big) \arrow[d, "\hat f|_{\cdots}"'] \\
\big[j_1^*(F_1^*) \arrow[r, "j_1^*(\mathrm{d}t_1)"] & j_1^*(T^*W_1)\big]\big|_{\cdots} \arrow[rr, "\chi_1|_{\cdots}", "\cong"'] && E_1^\bullet|_{U_{12}\cap f^{-1}(U_{23})},
\end{tikzcd}$$

where the second–to–fourth rows commute by (14.44) for $(U_{12},W_{12},e_{12},\hat e_{12})$ and $(U_{23},W_{23},e_{23},\hat e_{23})$, and the vertical composition of the whole diagram is equation (14.44) for the data (14.45).

For Step $2''$(b), suppose $(U_{12},W_{12},e_{12},\hat e_{12})$ and $(U_{12}',W_{12}',e_{12}',\hat e_{12}')$ both satisfy $(+'')$. Then (14.44) commutes for $(U_{12},W_{12},e_{12},\hat e_{12})$ on U_{12}, and for $(U_{12}',W_{12}',e_{12}',\hat e_{12}')$ on U_{12}'. Restrict both of these to $U_{12}\cap U_{12}'$. As the right hand side of (14.44) for both is the same, and χ_1 is an isomorphism, it follows that the left hand column for both is the same. That is, the left and right hand sides below are equal, as morphisms in $D(\mathrm{qcoh}(U_{12}\cap U_{12}'))$:

$$\begin{tikzcd}[column sep=tiny]
f|^*_{U_{12}\cap U_{12}'}\big(\big[j_2^*(F_2^*) \arrow[r, "j_2^*(\mathrm{d}t_2)"] \arrow[d, "\substack{j_1|^*_{\cdots}(\hat e_{12}^*)\circ \\ I_{j_1|\ldots,e_{12}}(F_2^*)\circ \\ I_{f|\ldots,j_2}(F_2^*)^{-1}}"'] & j_2^*(T^*W_2)\big]\big) \arrow[d, "\substack{j_1|^*_{\cdots}(\mathrm{d}e_{12}^*)\circ \\ I_{j_1|\ldots,e_{12}}(T^*W_2)\circ \\ I_{f|\ldots,j_2}(T^*W_2)^{-1}}"] & f|^*_{U_{12}\cap U_{12}'}\big(\big[j_2^*(F_2^*) \arrow[r, "j_2^*(\mathrm{d}t_2)"] \arrow[d, "\substack{j_1|^*_{\cdots}(\hat e_{12}'^*)\circ \\ I_{j_1|\ldots,e_{12}'}(F_2^*)\circ \\ I_{f|\ldots,j_2}(F_2^*)^{-1}}"'] & j_2^*(T^*W_2)\big]\big) \arrow[d, "\substack{j_1|^*_{\cdots}(\mathrm{d}e_{12}'^*)\circ \\ I_{j_1|\ldots,e_{12}'}(T^*W_2)\circ \\ I_{f|\ldots,j_2}(T^*W_2)^{-1}}"] \\
\big[j_1^*(F_1^*) \arrow[r, "j_1^*(\mathrm{d}t_1)"] & j_1^*(T^*W_1)\big]\big|_{U_{12}\cap U_{12}'}, & \big[j_1^*(F_1^*) \arrow[r, "j_1^*(\mathrm{d}t_1)"] & j_1^*(T^*W_1)\big]\big|_{U_{12}\cap U_{12}'}.
\end{tikzcd} \tag{14.56}$$

The derived category $D(\mathrm{qcoh}(U_{12}\cap U_{12}'))$ is constructed as in Weibel [104, §10]. Write $C(\mathrm{qcoh}(U_{12}\cap U_{12}'))$ for the dg-category of (unbounded) cochain complexes in $\mathrm{qcoh}(U_{12}\cap U_{12}')$. Define $K(\mathrm{qcoh}(U_{12}\cap U_{12}'))$ to be the category whose objects are objects of $C(\mathrm{qcoh}(U_{12}\cap U_{12}'))$, and whose morphisms are chain homotopy equivalence classes of morphisms in $C(\mathrm{qcoh}(U_{12}\cap U_{12}'))$. Finally one defines $D(\mathrm{qcoh}(U_{12}\cap U_{12}'))$ by localizing quasi-isomorphisms in $K(\mathrm{qcoh}(U_{12}\cap U_{12}'))$.

The columns in (14.56) are written as morphisms in $C(\mathrm{qcoh}(U_{12}\cap U_{12}'))$. Now $U_{12}\cap U_{12}'$ is affine, and $f|^*_{U_{12}\cap U_{12}'}\circ j_2^*(F_2^*), f|^*_{U_{12}\cap U_{12}'}\circ j_2^*(T^*W_2)$ are vector bundles on $U_{12}\cap U_{12}'$, so they are projective objects in $\mathrm{qcoh}(U_{12}\cap U_{12}')$. Hence Weibel [104, Cor. 10.4.7] implies that morphism groups in $D(\mathrm{qcoh}(U_{12}\cap U_{12}'))$

and $K(\mathrm{qcoh}(U_{12}\cap U_{12}'))$ for the objects in (14.56) coincide. Therefore the left and right hand sides of (14.56) agree as morphisms in $K(\mathrm{qcoh}(U_{12}\cap U_{12}'))$. Thus, as morphisms in $C(\mathrm{qcoh}(U_{12}\cap U_{12}'))$, they differ by a chain homotopy. That is, there exists a morphism $\lambda: f|^*_{U_{12}\cap U_{12}'}\circ j_2^*(T^*W_2)\to j_1^*(F_1^*)|_{U_{12}\cap U_{12}'}$ satisfying

$$\begin{aligned}&j_1|^*_{U_{12}\cap U_{12}'}(\mathrm{d}e'^*_{12})\circ I_{j_1|_{U_{12}\cap U_{12}'},e_{12}'}(T^*W_2)\circ I_{f|_{U_{12}\cap U_{12}'},j_2}(T^*W_2)^{-1}\\&\quad=j_1|^*_{\dots}(\mathrm{d}e^*_{12})\circ I_{j_1|\dots,e_{12}}(T^*W_2)\circ I_{f|\dots,j_2}(T^*W_2)^{-1}+j_1^*(\mathrm{d}t_1)|_{\dots}\circ\lambda,\end{aligned}\tag{14.57}$$

$$\begin{aligned}&j_1|^*_{U_{12}\cap U_{12}'}(\hat e'^*_{12})\circ I_{j_1|_{U_{12}\cap U_{12}'},e_{12}'}(F_2^*)\circ I_{f|_{U_{12}\cap U_{12}'},j_2}(F_2^*)^{-1}\\&\quad=j_1|^*_{\dots}(\hat e^*_{12})\circ I_{j_1|\dots,e_{12}}(F_2^*)\circ I_{f|\dots,j_2}(F_2^*)^{-1}+\lambda\circ f|^*_{\dots}\circ j_2^*(\mathrm{d}t_2).\end{aligned}\tag{14.58}$$

Now $j_1|_{U_{12}\cap U_{12}'}:U_{12}\cap U_{12}'\to W_{12}\cap W_{12}'$ is an embedding with $U_{12}\cap U_{12}'$ affine, and λ is a morphism of pullbacks by $j_1|_{U_{12}\cap U_{12}'}$ of trivializable vector bundles on $W_{12}\cap W_{12}'$, so λ is (Zariski) locally the pullback of a vector bundle morphism on $W_{12}\cap W_{12}'$. Hence we can choose a (Zariski) open neighbourhood W_{12}'' of $j_1(U_{12}\cap U_{12}')$ in $W_{12}\cap W_{12}'$ and a vector bundle morphism $\Lambda: F_1|_{W_{12}''}\to e_{12}|^*_{W_{12}''}(TW_2)$ such that the following commutes, using (14.42):

$$\begin{tikzcd}
j_1|^*_{U_{12}\cap U_{12}'}\circ e_{12}^*(T^*W_2) \arrow[r, "I_{j_1|\dots,e_{12}}(T^*W_2)"'] \arrow[d, "j_1|^*_{\dots}(\Lambda^*)"] & (e_{12}\circ j_1)|^*_{U_{12}\cap U_{12}'}(T^*W_2) \arrow[d, "I_{f|\dots,j_2}(T^*W_2)"'] \\
j_1|^*_{U_{12}\cap U_{12}'}(F_1) & f|^*_{U_{12}\cap U_{12}'}\circ j_2^*(T^*W_2). \arrow[l, "\lambda"']
\end{tikzcd}$$

Then (14.46)–(14.47) follow from (14.57)–(14.58). This proves Step 2″(b).

For Step 2″(c), let $(f,\hat f):(X_1,E_1^\bullet,\phi_1)\to(X_2,E_2^\bullet,\phi_2)$ be a morphism in $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$, and $(U_a,W_a,j_a,F_a,t_a,\chi_a)$ satisfy $(*'')$ in $(X_a,E_a^\bullet,\phi_a)$ for $a=1,2$. We will construct $(U_{12},W_{12},e_{12},\hat e_{12})$ satisfying $(+'')$. Set $U_{12}=U_1\cap f^{-1}(U_2)$. Then $j_1(U_{12})$ is an affine, locally closed $\mathbb{C}$-subscheme of X_1, with isomorphism $j_1|_{U_{12}}:U_{12}\to j_1(U_{12})$, and $j_2\circ f\circ j_1|^{-1}_{U_{12}}:j_1(U_{12})\to W_2$ is a morphism into a smooth $\mathbb{C}$-scheme W_{12}, so it extends Zariski locally to a morphism $W_1\to W_2$. That is, there exists an open $W_{12}\subseteq W_1$ with $j_1(U_{12})\subseteq W_{12}$ and $j_1(U_{12})$ closed in W_{12}, and a $\mathbb{C}$-scheme morphism $e_{12}:W_{12}\to W_2$ with $e_{12}|_{j_1(U_{12})}=j_2\circ f\circ j_1|^{-1}_{U_{12}}$, or equivalently $e_{12}\circ j_1|_{U_{12}}=j_2\circ f|_{U_{12}}$, so that (14.42) holds.

The next part is based on Schürg [91, Lem. 5.18 & Prop. 5.20]. Consider the commutative diagram in $D(\mathrm{qcoh}(U_{12}))$ (or $K(\mathrm{qcoh}(U_{12}))$, or $C(\mathrm{qcoh}(U_{12}))$):

$$\begin{tikzcd}[column sep=small]
f|_{U_{12}}^*\big(\big[j_2^*(F_2^*) \arrow[r, "j_2^*(\mathrm{d}t_2)"] \arrow[d, dotted, "\theta^{-1}"'] \arrow[ddr, dotted, "f|_{U_{12}}^*\circ j_2^{-1}(t_2\cdot\,)"] & j_2^*(T^*W_2)\big]\big) \arrow[rr, "f|_{U_{12}}^*(\chi_2)", "\cong"'] \arrow[d, dotted, "\theta^0"'] \arrow[ddr, dotted, "\mathrm{id}_{f|_{U_{12}}^*\circ j_2^*(T^*W_2)}"] & & f|_{U_{12}}^*\big(E_2^\bullet\big) \arrow[d, "\hat f|_{U_{12}}"'] \arrow[ddr, "f|_{U_{12}}^*(\phi_2)"] & \\
\big[j_1^*(F_1^*) \arrow[r, "j_1^*(\mathrm{d}t_1)"] \arrow[ddr, dotted, "j_1^{-1}(t_1\cdot\,)|_{U_{12}}"'] & j_1^*(T^*W_1)\big]\big|_{U_{12}} \arrow[rr, "\chi_1|_{U_{12}}", "\cong"'] \arrow[ddr, dotted, "\mathrm{id}_{j_1^*(T^*W_1)|_{U_{12}}}"] & & E_1^\bullet|_{U_{12}} \arrow[ddr, "\phi_1|_{U_{12}}"] & \\
& f|_{U_{12}}^*\big(\big[I_2/I_2^2 \arrow[r, "\alpha_2"] \arrow[d, dotted, "{\begin{array}{l} j_1|_{\dots}^{-1}(e_{12}^\sharp)\circ \\ I_{j_1|\dots,e_{12}}(\mathcal{O}_{W_2})_*\circ \\ I_{f|\dots,j_2}(\mathcal{O}_{W_2})_*^{-1}\end{array}}"] & j_2^*(T^*W_2)\big]\big) \arrow[rr, "f|_{U_{12}}^*(14.38)", "\cong"'] \arrow[d, dotted, "{\begin{array}{l} j_1|_{\dots}^*(\mathrm{d}e_{12}^*)\circ \\ I_{j_1|\dots,e_{12}}(T^*W_2)\circ \\ I_{f|\dots,j_2}(T^*W_2)^{-1}\end{array}}"] & & f|_{U_{12}}^*\big(\tau_{\geqslant -1}(\mathbb{L}_{X_2})\big) \arrow[d, "\tau_{\geqslant -1}(\mathbb{L}_f)|_{U_{12}}"'] \\
& \big[I_1/I_1^2 \arrow[r, "\alpha_1"] & j_1^*(T^*W_1)\big]\big|_{U_{12}} \arrow[rr, "(14.38)", "\cong"'] & & \tau_{\geqslant -1}(\mathbb{L}_{X_1})|_{U_{12}}.
\end{tikzcd} \tag{14.59}$$

Here $[* \to *]$ are complexes in degrees $[-1, 0]$, objects in the derived category. Other arrows '$\rightarrow$' are morphisms in $D(\mathrm{qcoh}(U_{12}))$. Parallel pairs of arrows '$\dashrightarrow$' are morphisms of complexes, so morphisms in $C(\mathrm{qcoh}(U_{12}))$, which induce morphisms in $K(\mathrm{qcoh}(U_{12}))$ and $D(\mathrm{qcoh}(U_{12}))$. All morphisms in (14.59) are known except θ^{-1}, θ^0, which remain to be determined. Our goal is to choose θ^{-1}, θ^0 so that the entire diagram commutes in $D(\mathrm{qcoh}(U_{12}))$, and also the subdiagram of morphisms '$\dashrightarrow$' commutes in $C(\mathrm{qcoh}(U_{12}))$.

There is a unique morphism $\theta = \chi_1|_{U_{12}}^{-1} \circ \hat f|_{U_{12}} \circ f|_{U_{12}}^*(\chi_2)$ in $D(\mathrm{qcoh}(U_{12}))$ making the upper rectangle in (14.59) commute in $D(\mathrm{qcoh}(U_{12}))$. The argument in Step 2″(b) using Weibel [104, Cor. 10.4.7] shows that θ lifts to a unique morphism in $K(\mathrm{qcoh}(U_{12}))$, and thus lifts to a morphism in $C(\mathrm{qcoh}(U_{12}))$, uniquely up to chain homotopy. Hence we can choose morphisms θ^{-1}, θ^0 as shown in (14.59) such that the upper left hand side is a morphism in $C(\mathrm{qcoh}(U_{12}))$, inducing the morphism θ in $D(\mathrm{qcoh}(U_{12}))$ making the upper rectangle in (14.59) commute in $D(\mathrm{qcoh}(U_{12}))$. The two diamonds in (14.59) commute in $D(\mathrm{qcoh}(U_{12}))$ by (14.41), and the lower rectangle in (14.59) commutes in $D(\mathrm{qcoh}(U_{12}))$ by functoriality of (14.38). As the lower horizontal morphisms are isomorphisms, it now follows that the whole of (14.59) commutes in $D(\mathrm{qcoh}(U_{12}))$.

The argument using [104, Cor. 10.4.7] above shows that morphisms from the top left hand corner in (14.59) in $D(\mathrm{qcoh}(U_{12}))$ and $K(\mathrm{qcoh}(U_{12}))$ agree. Hence the subdiagram of morphisms '$\dashrightarrow$' in (14.59) commutes in $K(\mathrm{qcoh}(U_{12}))$. Thus, it commutes in $C(\mathrm{qcoh}(U_{12}))$ up to chain homotopy. That is, there exists $\eta : f|_{U_{12}}^* \circ j_2^*(T^*W_2) \to I_1/I_1^2|_{U_{12}}$ such that

$$\begin{aligned}
&j_1|_{U_{12}}^{-1}(e_{12}^\sharp) \circ I_{j_1|_{U_{12}},e_{12}}(\mathcal{O}_{W_2})_* \circ I_{f|_{U_{12}},j_2}(\mathcal{O}_{W_2})_*^{-1} \circ f|_{U_{12}}^* \circ j_2^{-1}(t_2\cdot\,) \\
&\quad = j_1^{-1}(t_1\cdot\,)|_{U_{12}} \circ \theta^{-1} + \eta \circ f|_{U_{12}}^* \circ j_2^*(\mathrm{d}t_2), \\
&j_1|_{U_{12}}^*(\mathrm{d}e_{12}^*) \circ I_{j_1|_{U_{12}},e_{12}}(T^*W_2) \circ I_{f|_{U_{12}},j_2}(T^*W_2)^{-1} \circ \mathrm{id}_{f|_{U_{12}}^*\circ j_2^*(T^*W_2)} \\
&\quad = \mathrm{id}_{j_1^*(T^*W_1)|_{U_{12}}} \circ \theta^0 + \alpha_1|_{U_{12}} \circ \eta.
\end{aligned}$$

Following Schürg [91, Lem. 5.18], as $j_1^{-1}(t_1\cdot)|_{U_{12}} : j_1^*(F_1^*)|_{U_{12}} \to I_1/I_1^2|_{U_{12}}$ is surjective and $f|_{U_{12}}^* \circ j_2^*(T^*W_2)$ is projective in $\mathrm{qcoh}(U_{12})$, there exists $\zeta : f|_{U_{12}}^* \circ j_2^*(T^*W_2) \to j_1^*(F_1^*)|_{U_{12}}$ with $\eta = j_1^{-1}(t_1\cdot)|_{U_{12}} \circ \zeta$. Set $\tilde\theta^{-1} = \theta^{-1} + \zeta \circ f|_{U_{12}}^* \circ j_2^*(\mathrm{d}t_2)$ and $\tilde\theta^0 = \theta^0 + j_1^*(\mathrm{d}t_1)|_{U_{12}} \circ \zeta$. Replacing θ^{-1}, θ^0 by $\tilde\theta^{-1}, \tilde\theta^0$, we find that (14.59) commutes in $D(\mathrm{qcoh}(U_{12}))$, and the subdiagram of morphisms '$\dashrightarrow$' commutes in $C(\mathrm{qcoh}(U_{12}))$. Note too that

$$\theta^0 = j_1|_{U_{12}}^*(\mathrm{d}e_{12}^*) \circ I_{j_1|_{U_{12}}, e_{12}}(T^*W_2) \circ I_{f|_{U_{12}}, j_2}(T^*W_2)^{-1}. \tag{14.60}$$

Now consider the morphism

$$\theta^{-1} \circ I_{f|_{U_{12}}, j_2}(F_2^*) \circ I_{j_1|_{U_{12}}, e_{12}}(F_2^*)^{-1} : j_1|_{U_{12}}^*(e_{12}^*(F_2^*)) \longrightarrow j_1|_{U_{12}}^*(F_1^*). \tag{14.61}$$

We may regard this as a morphism $e_{12}^*(F_2^*)|_{j_1(U_{12})} \to F_1^*|_{j_1(U_{12})}$ of trivializable vector bundles on the closed affine subscheme $j_1(U_{12})$ in W_{12}, or dually as a morphism $F_1|_{j_1(U_{12})} \to e_{12}^*(F_2)|_{j_1(U_{12})}$. This extends Zariski locally to a morphism on W_{12}. Thus, making W_{12} smaller if necessary, we may choose a morphism $\hat e_{12} : F_1|_{W_{12}} \to e_{12}^*(F_2)$ of vector bundles on W_{12} such that $j_1|_{U_{12}}^*(\hat e_{12}^*)$ is (14.61), or equivalently

$$\theta^{-1} = j_1|_{U_{12}}^*(\hat e_{12}^*) \circ I_{j_1|_{U_{12}}, e_{12}}(F_2^*) \circ I_{f|_{U_{12}}, j_2}(F_2^*)^{-1}. \tag{14.62}$$

We already chose e_{12} to satisfy (14.42). Equation (14.43) holds as the left hand diamond of morphisms '$\dashrightarrow$' in (14.59) commutes in $\mathrm{qcoh}(U_{12})$, noting that elements of I_1^2 are $O(t_1^2)$. Equation (14.44) commutes in $D(\mathrm{qcoh}(U_{12}))$ by the top rectangle of (14.59) and equations (14.60) and (14.62). Therefore $(U_{12}, W_{12}, e_{12}, \hat e_{12})$ satisfies $(+'')$, proving Step $2''$(c).

Steps $3''$ and $4''$ are either self-explanatory, or follow the proofs of Steps 3 and 4 of the proof of Theorem 14.2(a),(b) in §14.1.1 closely, so we leave these as an exercise. This completes the proof of Theorem 14.27(a),(b).

14.6 The derived manifolds of Spivak and Borisov–Noel

The fifth volume of Jacob Lurie's mammoth work on derived algebraic geometry concludes with a few lines on 'derived differential geometry' [72, §4.5], explaining how Lurie's abstract framework can be used to define an ∞-category of objects one might call 'derived C^∞-stacks', including manifolds, orbifolds, C^∞-schemes, C^∞-stacks, and derived versions of all four.

These ideas were taken forward by Lurie's student David Spivak [94,95], who defined and studied an ∞-category of 'derived manifolds'. Spivak's construction is very complicated, using the full weight of Lurie's machinery. Borisov and Noel [17] showed that an equivalent ∞-category of derived manifolds can be defined in a much simpler way. Borisov [16] defined a truncation functor from Spivak's derived manifolds to our d-manifolds, and considered what information this functor forgets. Sections 14.6.1–14.6.3 will review [94, 95], [17], and [16].

14.6.1 Spivak's derived manifolds

In his thesis [94] and a subsequent journal paper [95], David Spivak gave two different constructions of an ∞-category (simplicial category) of *derived manifolds*, which we will write as **DerMan**. Primarily we will follow [95], but we will also discuss some material appearing in [94] but not in [95].

Spivak's construction of **DerMan** in [95] is complicated and uses a lot of sophisticated mathematical technology, but has the same basic outline as our construction of **dMan** in Chapters 2 and 3, since the author followed Spivak in this. Spivak begins by defining an ∞-category $\mathbf{LC^\infty RS}$ of local C^∞-ringed spaces, essentially an ∞-categorical version of our d-spaces in Chapter 2.

A *local C^∞-ringed space* $\boldsymbol{X} = (X, \mathcal{O}_X)$ is a compactly generated Hausdorff topological space X equipped with a homotopy sheaf $\mathcal{O}_X$ of homotopy simplicial C^∞-rings satisfying a locality condition on stalks. Here the ∞-category of *homotopy simplicial C^∞-rings* (*lax simplicial C^∞-rings*) is obtained by localizing the ∞-category of simplicial C^∞-rings at a suitable class of quasi-isomorphisms. The *homotopy sheaf* $\mathcal{O}_X$ on X assigns a homotopy simplicial C^∞-ring $\mathcal{O}_X(U)$ for each open $U \subseteq X$, with restriction morphisms $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ for $V \subseteq U \subseteq X$ which satisfy the sheaf axioms not strictly, but only up to (specified) homotopies. Thus, Spivak works up to homotopy twice.

In the analogue of Theorem 2.36 for d-spaces, Spivak shows [95, Prop. 8.14] that all (homotopy) fibre products in $\mathbf{LC^\infty RS}$. As for $F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \to \mathbf{dSpa}$ in §2.2, Spivak [95, Prop. 6.11] constructs a full and faithful functor $\boldsymbol{i} : \mathbf{Man} \to \mathbf{LC^\infty RS}$. He defines [95, Def. 6.15] an *affine derived manifold* $\boldsymbol{U}$ to be a homotopy fibre product $\boldsymbol{i}(\mathbb{R}^n) \times_{\boldsymbol{i}(f),\boldsymbol{i}(\mathbb{R}^m),\boldsymbol{i}(0)} \boldsymbol{i}(*)$ in $\mathbf{LC^\infty RS}$. He then defines **DerMan** to be the full ∞-subcategory of objects $\boldsymbol{X}$ in $\mathbf{LC^\infty RS}$ which can be covered by open subspaces $\boldsymbol{U} \subseteq \boldsymbol{X}$ which are affine derived manifolds.

Spivak does not require these open $\boldsymbol{U}$ to have a fixed virtual dimension, as we do, so his derived manifolds can have connected components of different dimensions. We will write $\mathbf{DerMan^{pd}}$ for the full ∞-subcategory of derived manifolds $\boldsymbol{X}$ of *pure dimension* in **DerMan**, that is, those $\boldsymbol{X}$ with open covers by affine derived manifolds $\boldsymbol{U}$ with $\operatorname{vdim} \boldsymbol{U} = n$ for some fixed $n \in \mathbb{Z}$.

Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism in **DerMan**, with $\boldsymbol{Y}$ a manifold. Spivak [95, Def. 2.1] defines $\boldsymbol{f}$ to be an *embedding* if $\boldsymbol{f}$ is locally modelled on a projection $\boldsymbol{Y} \times_{\boldsymbol{g},\mathbb{R}^k,\boldsymbol{0}} * \to \boldsymbol{Y}$ for some morphisms $\boldsymbol{g} : \boldsymbol{Y} \to \mathbb{R}^k$. (This should be compared with Proposition 4.27 for embeddings of d-manifolds.)

The next theorem summarizes some of Spivak's results [95]:

Theorem 14.31. *There is an ∞-category* **DerMan** *of derived manifolds, and a full and faithful functor* $\boldsymbol{i} : \mathbf{Man} \to \mathbf{DerMan}$, *with the following properties:*

(a) *Let $\boldsymbol{X}, \boldsymbol{Y}$ be derived manifolds, $\boldsymbol{U} \subseteq \boldsymbol{X}$, $\boldsymbol{V} \subseteq \boldsymbol{Y}$ be open, and $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{Y}$ an equivalence in* **DerMan**. *Suppose the topological space $Z = X \amalg_f Y$ obtained by gluing X, Y along U, V using f is Hausdorff. Then there exists a derived manifold $\boldsymbol{Z}$ obtained by gluing $\boldsymbol{X}, \boldsymbol{Y}$ along $\boldsymbol{U}, \boldsymbol{V}$ by the equivalence $\boldsymbol{f}$, which is a homotopy pushout $\boldsymbol{X} \amalg_{\mathbf{id}_{\boldsymbol{U}},\boldsymbol{U},\boldsymbol{f}} \boldsymbol{Y}$ in* **DerMan**.

(b) $\boldsymbol{i} : \mathbf{Man} \to \mathbf{DerMan}$ *preserves transverse fibre products in* **Man**.

(c) *Let* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{g}:\boldsymbol{Y}\to\boldsymbol{Z}$ *be morphisms in* **DerMan**, *where* $\boldsymbol{Z}$ *is a manifold. Then a homotopy fibre product* $\boldsymbol{X}\times_{\boldsymbol{f},\boldsymbol{Z},\boldsymbol{g}}\boldsymbol{Y}$ *exists in* **DerMan**.

(d) *Suppose* $\boldsymbol{X}$ *is a compact derived manifold. Then there exists an embedding* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{\mathbb{R}^n}$ *for some* $n\gg 0$.

(e) *Suppose* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ *is an embedding in* **DerMan**, *with* $\boldsymbol{Y}=\boldsymbol{i}(Y)$ *a manifold. Then there exist an open neighbourhood* V *of* $f(X)$ *in* Y, *a vector bundle* $E\to V$, *and a smooth section* $s:V\to E$ *of* E *fitting into a homotopy Cartesian diagram in* **DerMan**, *where* $0:V\to E$ *is the zero section and* $\boldsymbol{V},\boldsymbol{E},\boldsymbol{s},\boldsymbol{0}=\boldsymbol{i}(V,E,s,0)$:

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{f}"'] \arrow[dr, phantom, "\Uparrow"] & \boldsymbol{V} \arrow[d, "\boldsymbol{0}"] \\
\boldsymbol{V} \arrow[r, "\boldsymbol{s}"] & \boldsymbol{E}.
\end{tikzcd}$$

(f) *Let* Y *be a manifold,* $B_*^{\mathrm{un}}(Y)$ *the unoriented bordism ring of* Y, *and* $B_*^{\mathrm{un}}(Y)^{\mathrm{der}}$ *the unoriented bordism ring of* Y *defined using derived manifolds, in a similar way to* §13.1–§13.2. *Then* $\boldsymbol{i}:$ **Man** $\to$ **DerMan** *induces a morphism* $\boldsymbol{i}_*:B_*^{\mathrm{un}}(Y)\to B_*^{\mathrm{un}}(Y)^{\mathrm{der}}$, *which is an isomorphism.*

Here part (a) should be compared with Theorems 2.29 and 3.41, (b) with Theorem 2.42, (c) with Theorem 4.22(a), (d) with Theorem 4.29, (e) with Theorem 4.34, and (f) with Theorem 13.11. Thus, many of our important results on d-manifolds are modelled on Spivak's results for his derived manifolds.

In [94, §6.2], Spivak defines an ∞-category $\mathbf{DerMan^b}$ of *derived manifolds with boundary.* Like our d-manifolds with boundary $\mathbf{X}=(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i_{\mathbf{X}}},\omega_{\mathbf{X}})$ in Chapter 7, Spivak's derived manifolds with boundary are triples $(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i_X})$, where $\boldsymbol{X}=(X,\mathcal{O}_X)$ is a local C^∞-ringed space, $\boldsymbol{\partial X}$ a derived manifold, and $\boldsymbol{i_X}:\boldsymbol{\partial X}\to\boldsymbol{X}$ a morphism in $\mathbf{LC^\infty RS}$. He requires that there should exist a derived manifold without boundary $\tilde{\boldsymbol{X}}=(\tilde X,\mathcal{O}_{\tilde X})$ and a morphism $\boldsymbol{b}:\tilde{\boldsymbol{X}}\to\boldsymbol{\mathbb{R}}$ similar to our boundary defining functions, such that $X=b^{-1}([0,\infty))\subseteq\tilde X$ and $\mathcal{O}_X=\mathcal{O}_{\tilde X}|_X$, and there is a homotopy Cartesian diagram in **DerMan**

$$\begin{tikzcd}
\boldsymbol{\partial X} \arrow[r, "\boldsymbol{\pi}"] \arrow[d, "\boldsymbol{j}\circ\boldsymbol{i_X}"'] \arrow[dr, phantom, "\Uparrow"] & \boldsymbol{*} \arrow[d, "\boldsymbol{0}"] \\
\tilde{\boldsymbol{X}} \arrow[r, "\boldsymbol{b}"] & \boldsymbol{\mathbb{R}},
\end{tikzcd}$$

where $\boldsymbol{j}:\boldsymbol{X}\hookrightarrow\tilde{\boldsymbol{X}}$ is the inclusion. Note that $\tilde{\boldsymbol{X}},\boldsymbol{b}$ must exist globally, not just locally on $\boldsymbol{\partial X},\boldsymbol{X}$. Morphisms $(\boldsymbol{f},\boldsymbol{\partial f}):(\boldsymbol{X},\boldsymbol{\partial X},\boldsymbol{i_X})\to(\boldsymbol{Y},\boldsymbol{\partial Y},\boldsymbol{i_Y})$ are morphisms $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ and $\boldsymbol{\partial f}:\boldsymbol{\partial X}\to\boldsymbol{\partial Y}$ in $\mathbf{LC^\infty RS}$ with $\boldsymbol{f}\circ\boldsymbol{i_X}\cong\boldsymbol{i_Y}\circ\boldsymbol{\partial f}$.

Spivak's approach to boundaries is different to ours (and somewhat cruder). He does not include orientations $\omega_{\mathbf{X}}$ as we do, as he does not define orientations, and studies unoriented bordism. Defining $\mathcal{O}_X=\mathcal{O}_{\tilde X}|_X$ for $\tilde{\boldsymbol{X}}=(\tilde X,\mathcal{O}_{\tilde X})$ a derived manifold without boundary means that the morphisms in $\mathbf{DerMan^b}$ are not quite what you expect even for classical manifolds with boundary. For example, morphisms $(\boldsymbol{f},\boldsymbol{\partial f}):\boldsymbol{[0,1]}\to\boldsymbol{\mathbb{R}}$ in $\mathbf{DerMan^b}$ correspond not to smooth maps $f:[0,1]\to\mathbb{R}$ in our sense, but to germs at $[0,1]$ of smooth maps $f:\mathbb{R}\to\mathbb{R}$. Roughly, $(\boldsymbol{f},\boldsymbol{\partial f})$ is a smooth map $(-\epsilon,1+\epsilon)\to\mathbb{R}$ for small $\epsilon>0$.

14.6.2 Borisov and Noel's simplified version

Borisov and Noel [17] showed that Spivak's construction of **DerMan** can be significantly simplified, that is, one can define an equivalent ∞-category $\hat{\mathbf{D}}\mathbf{erMan}$ in a much less painful way. Their simplification works because of the existence of partitions of unity in C^∞-geometry, which implies that the structure sheaves $\mathcal{O}_X$ are soft sheaves. In §14.4 we suggested that our 2-category style 'derived geometry' works for the same reason. Borisov and Noel prove:

(a) Spivak's ∞-category $\mathbf{LC^\infty RS}$ of local C^∞-ringed spaces $\boldsymbol{X} = (X, \mathcal{O}_X)$ have $\mathcal{O}_X$ a *homotopy* sheaf of *homotopy* simplicial C^∞-rings on X.

    Borisov and Noel define an ∞-category $\hat{\mathbf{L}}\mathbf{C^\infty RS}$ of spaces $\boldsymbol{X} = (X, \mathcal{O}_X)$ with $\mathcal{O}_X$ a (strict) sheaf of (strict) simplicial C^∞-rings on X, and show that the natural inclusion functor $I : \hat{\mathbf{L}}\mathbf{C^\infty RS} \hookrightarrow \mathbf{LC^\infty RS}$ is an equivalence of ∞-categories.

(b) Write $\mathbf{sC^\infty Rings}$ for the ∞-category of simplicial C^∞-rings. There is a spectrum functor $\mathrm{Spec} : \mathbf{sC^\infty Rings^{op}} \to \hat{\mathbf{L}}\mathbf{C^\infty RS}$. Borisov and Noel define a notion of *finite type* for objects in $\hat{\mathbf{L}}\mathbf{C^\infty RS}, \mathbf{sC^\infty Rings}$, which essentially means that the underlying C^∞-scheme $\underline{X}$ is a fair affine C^∞-scheme. Write $\hat{\mathbf{L}}\mathbf{C^\infty RS_{ft}}, \mathbf{sC^\infty Rings_{ft}}$ for the full ∞-subcategories of finite type objects in $\hat{\mathbf{L}}\mathbf{C^\infty RS}, \mathbf{sC^\infty Rings}$. Then $\mathrm{Spec} : \mathbf{sC^\infty Rings_{ft}^{op}} \to \hat{\mathbf{L}}\mathbf{C^\infty RS_{ft}}$ is an equivalence of ∞-categories.

Thus, Borisov and Noel eliminate both of Spivak's uses of homotopy, and also his use of sheaves. Their simplification has the advantage that one can write down examples of simplicial C^∞-rings explicitly, and do computations with them, which would be difficult in Spivak's set-up.

Combining (a) and (b) yields Borisov and Noel's main result [17, Th. 1]:

Theorem 14.32. *Write* $\hat{\mathbf{D}}\mathbf{erMan_{ft}}$ *for the full* ∞*-subcategory of the opposite* ∞*-category* $\mathbf{sC^\infty Rings^{op}}$ *of simplicial* C^∞*-rings whose objects are finite type simplicial* C^∞*-rings which are locally equivalent to homotopy fibre products* $C^\infty(\mathbb{R}^m) \times_{C^\infty(\mathbb{R}^n)} \mathbb{R}$*. Then* $I \circ \mathrm{Spec} : \hat{\mathbf{D}}\mathbf{erMan_{ft}} \to \mathbf{DerMan_{ft}}$ *is an equivalence of* ∞*-categories, where* $\mathbf{DerMan_{ft}}$ *is the full* ∞*-subcategory of finite type objects in Spivak's* ∞*-category* **DerMan***.*

A derived manifold $\boldsymbol{X}$ is of finite type if and only if it admits an embedding $\boldsymbol{f} : \boldsymbol{X} \hookrightarrow \boldsymbol{\mathbb{R}^n}$ for some n. For d-manifolds, we gave a necessary and sufficient condition for the existence of embeddings $\boldsymbol{f} : \boldsymbol{X} \hookrightarrow \boldsymbol{\mathbb{R}^n}$ in Theorem 4.33, and an example of a d-manifold $\boldsymbol{X}$ with no embedding $\boldsymbol{f} : \boldsymbol{X} \hookrightarrow \boldsymbol{\mathbb{R}^n}$ in Example 4.31.

14.6.3 A truncation functor from derived manifolds to d-manifolds

Let $\boldsymbol{\mathcal{C}}$ be a simplicial model category. Then one can define a natural 2-category truncation $\pi_1(\boldsymbol{\mathcal{C}})$ of $\boldsymbol{\mathcal{C}}$, such that objects X, Y of $\pi_1(\boldsymbol{\mathcal{C}})$ are fibrant-cofibrant objects X, Y of $\boldsymbol{\mathcal{C}}$, and the category of 1- and 2-morphisms $\mathrm{Hom}_{\pi_1(\boldsymbol{\mathcal{C}})}(X, Y)$ is the fundamental groupoid $\pi_1\big(\mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(X, Y)\big)$ of the simplicial set $\mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(X, Y)$. This $\pi_1(\boldsymbol{\mathcal{C}})$ is a strict 2-category, with all 2-morphisms invertible.

Using this notation and the material of §14.6.2, Borisov [16] proves:

Theorem 14.33. *There is a strict* 2*-functor* $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}} : \pi_1\bigl(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}\bigr) \rightarrow \mathbf{dMan}_{\mathbf{ft}}$ *from the* 2*-category truncation* $\pi_1\bigl(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}\bigr)$ *of the full* ∞*-subcategory* $\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}$ *of finite type derived manifolds* $\boldsymbol{X}$ *of pure dimension in Spivak's* ∞*-category* $\mathbf{DerMan}$, *to the full* 2*-subcategory* $\mathbf{dMan}_{\mathbf{pr}}$ *of principal d-manifolds in* $\mathbf{dMan}$, *with the following properties:*

(a) $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}$ *induces a bijection between equivalence classes of objects in the* 2*-categories* $\pi_1(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}})$ *and* $\mathbf{dMan}_{\mathbf{pr}}$.

(b) *For all objects* $\boldsymbol{X}, \boldsymbol{Y}$ *in* $\pi_1\bigl(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}\bigr)$, $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}$ *induces a surjective map between* 2*-isomorphism classes of* 1*-morphisms* $\boldsymbol{f} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ *in* $\pi_1\bigl(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}\bigr)$ *and* $\boldsymbol{f}' : \Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}(\boldsymbol{X}) \rightarrow \Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}(\boldsymbol{Y})$ *in* $\mathbf{dMan}_{\mathbf{pr}}$, *but this map may not be injective.*

(c) $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}$ *recognizes equivalences, that is,* $\boldsymbol{f} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ *in* $\pi_1(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}})$ *is an equivalence if and only if* $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}(\boldsymbol{f})$ *is an equivalence.*

(d) *If* $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ *are* 1*-morphisms in* $\pi_1(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}})$, *the induced map on* 2*-morphisms* $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}} : \mathrm{Hom}(\boldsymbol{f}, \boldsymbol{g}) \rightarrow \mathrm{Hom}\bigl(\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}(\boldsymbol{f}), \Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}(\boldsymbol{g})\bigr)$ *need not be either injective or surjective.*

(e) *The* 2*-functors* $F_{\mathbf{Man}}^{\mathbf{dMan}}$ *and* $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}} \circ F_{\mathbf{Man}}^{\mathbf{DerMan}} : \mathbf{Man} \rightarrow \mathbf{dMan}$ *are naturally isomorphic.*

Parts **(a)**–**(c)** *imply the induced functor* $\mathrm{Ho}(\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}) : \mathrm{Ho}(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}) \rightarrow \mathrm{Ho}(\mathbf{dMan}_{\mathbf{pr}})$ *on homotopy categories is essentially surjective, full but not faithful, and recognizes isomorphisms.*

In fact Borisov defines the target 2-category $\mathbf{dMan}_{\mathbf{pr}}$ to consist of *finite type* d-manifolds $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$, by which he means that $\underline{X}$ is a fair affine C^∞-scheme. But one can use Corollary 4.35 to show that $\boldsymbol{X}$ is of finite type if and only if it is principal.

To prove $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}$ is not injective on 2-isomorphism classes of 1-morphisms in part (b), Borisov [16, §3.3] considers the following example:

Example 14.34. Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ map $g : (x, y) \mapsto (x^2, y^2, xy)$, and define derived manifolds $\boldsymbol{X}, \boldsymbol{Y}$ in $\pi_1\bigl(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}\bigr)$ by the 2-Cartesian diagrams

$$\begin{tikzcd} \boldsymbol{X} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Uparrow"] & & * \arrow[d, "\mathbf{0}"'] \\ \boldsymbol{\mathbb{R}^2} \arrow[rr, "\boldsymbol{g}"] & & \boldsymbol{\mathbb{R}^3}, \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{Y} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Uparrow"] & & * \arrow[d, "\mathbf{0}"'] \\ * \arrow[rr, "\mathbf{0}"] & & \boldsymbol{\mathbb{R}}, \end{tikzcd}$$

where $*, \boldsymbol{\mathbb{R}}, \boldsymbol{\mathbb{R}^2}, \boldsymbol{\mathbb{R}^3}, \boldsymbol{g}, \mathbf{0} = F_{\mathbf{Man}}^{\mathbf{DerMan}}(*, \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, g, 0)$. Define $h_1, h_2 : \mathbb{R}^3 \rightarrow \mathbb{R}$ by $h_1(u, v, w) = 0$ and $h_2(u, v, w) = uv - w^2$. Then $h_1 \circ g = h_2 \circ g = 0$. Properties of fibre products now give 1-morphisms $\boldsymbol{f}_1, \boldsymbol{f}_2 : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ in $\pi_1\bigl(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}\bigr)$, unique up to 2-isomorphism, such that the following diagram 2-commutes for

$\boldsymbol{f}_1, \boldsymbol{h}_1$ and for $\boldsymbol{f}_2, \boldsymbol{h}_2$:

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r] \arrow[drr, shift left, "\boldsymbol{f}_2"] \arrow[drr, shift right, "\boldsymbol{f}_1"'] \arrow[dd] & {*} \arrow[drr] \arrow[dd, "\boldsymbol{0}"'] & & \\
& & \boldsymbol{Y} \arrow[r] \arrow[dd] & {*} \arrow[dd, "\boldsymbol{0}"] \\
\mathbb{R}^{\boldsymbol{2}} \arrow[r, "\boldsymbol{g}"] \arrow[drr] & \mathbb{R}^{\boldsymbol{3}} \arrow[drr, shift left, "\boldsymbol{h}_2"] \arrow[drr, shift right, "\boldsymbol{h}_1"'] & & \\
& & {*} \arrow[r, "\boldsymbol{0}"'] & \mathbb{R}.
\end{tikzcd}$$

By calculations of homotopy groups of simplicial C^∞-rings, Borisov shows that $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are not 2-isomorphic in $\pi_1\big(\mathbf{DerMan}^{\mathbf{pd}}_{\mathbf{ft}}\big)$, but $\Pi^{\mathbf{dMan}}_{\mathbf{DerMan}}(\boldsymbol{f}_1)$ and $\Pi^{\mathbf{dMan}}_{\mathbf{DerMan}}(\boldsymbol{f}_2)$ are 2-isomorphic in $\mathbf{dMan}_{\mathbf{ft}}$.

Remark 14.35. We have two different higher categories of 'derived manifolds': Spivak's ∞-category **DerMan**, and our 2-category of d-manifolds **dMan**.

The functor $\Pi^{\mathbf{dMan}}_{\mathbf{DerMan}}$ from $\mathbf{DerMan}^{\mathbf{pd}}_{\mathbf{ft}}$ to **dMan** may be regarded as truncating in *two different ways*, forgetting two different kinds of information. Firstly, in passing from $\mathbf{DerMan}^{\mathbf{pd}}_{\mathbf{ft}}$ to $\pi_1\big(\mathbf{DerMan}^{\mathbf{pd}}_{\mathbf{ft}}\big)$ we truncate from ∞-categories to 2-categories, and so forget information in n-morphisms in $\mathbf{DerMan}^{\mathbf{pd}}_{\mathbf{ft}}$ for $n > 2$.

Secondly, as our definition of d-spaces in §2.1–§2.2 involves square zero ideals, working up to ideals, or squares of ideals, is built into our theory. For the 'standard model' d-manifolds $\boldsymbol{S}_{V,E,s}$, 1-morphisms $\boldsymbol{S}_{f,\hat f} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ and 2-morphisms $\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat f} \Rightarrow \boldsymbol{S}_{g,\hat g}$ of §3.2 and §3.4, this manifests itself as follows:

- Two d-manifolds $\boldsymbol{S}_{V,E,s_1}, \boldsymbol{S}_{V,E,s_2}$ are equal if $s_1 - s_2 = O(s_1^2) = O(s_2^2)$.
- Two 1-morphisms $\boldsymbol{S}_{f_1,\hat f_1}, \boldsymbol{S}_{f_2,\hat f_2} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ are equal if $f_1 = f_2 + O(s^2)$ and $\hat f_1 = \hat f_2 + O(s)$.
- Two 2-morphisms $\boldsymbol{S}_{\Lambda_1}, \boldsymbol{S}_{\Lambda_2} : \boldsymbol{S}_{f,\hat f} \Rightarrow \boldsymbol{S}_{g,\hat g}$ are equal if $\Lambda_1 = \Lambda_2 + O(s)$.

Thus, for the 'standard model' d-manifold $\boldsymbol{S}_{V,E,s}$, $\Pi^{\mathbf{dMan}}_{\mathbf{DerMan}}$ forgets all information on V which is $O(s^2)$, that is, which is zero modulo the square I_s^2 of the ideal I_s in $C^\infty(V)$ defined by $s \in C^\infty(E)$. This second kind of truncation is the reason why $\Pi^{\mathbf{dMan}}_{\mathbf{DerMan}}$ is not an equivalence of 2-categories in Theorem 14.33.

Consider the question: do there also exist other higher categories of 'derived manifolds' $\widetilde{\mathbf{DerMan}}$ different from both **DerMan** and **dMan**? For such a category $\widetilde{\mathbf{DerMan}}$ to be reasonable, we should require that it satisfies a list of properties similar to Spivak's 'axioms for derived manifolds' [95, §2], for instance: $\widetilde{\mathbf{DerMan}}$ should contain **Man** as a full subcategory, derived manifolds should have (Hausdorff, second countable) underlying topological spaces and have sheaf-like properties over open covers, all d-transverse fibre products should exist, there should be a good notion of bordism in $\widetilde{\mathbf{DerMan}}$ yielding isomorphic groups to bordism in **Man** — basically, most of the main theorems for **dMan** above and for **DerMan** in Spivak [95] should also hold in $\widetilde{\mathbf{DerMan}}$.

It seems likely that there are many such 'reasonable' higher categories of derived manifolds $\widetilde{\mathbf{DerMan}}$. For example, in terms of the two kinds of truncation above, for any $m, n = 2, 3, \ldots, \infty$ we could try to define an m-category $\mathbf{DerMan}_{m,n}$ by truncating $\mathbf{DerMan}$ to an m-category, and forgetting all information which is $O(s^n)$ on V in 'standard models' $\boldsymbol{S}_{V,E,s}$.

Out of all such 'reasonable' higher categories $\widetilde{\mathbf{DerMan}}$, the author expects that Spivak's ∞-category $\mathbf{DerMan}$ should be (heuristically at least) in some sense the 'largest' or 'most complex', and our 2-category $\mathbf{dMan}$ the 'smallest' or 'simplest'. Since Spivak's $\mathbf{DerMan}$ is essentially built by a universal construction, there should exist a truncation functor $\mathbf{DerMan} \to \widetilde{\mathbf{DerMan}}$ for any $\widetilde{\mathbf{DerMan}}$ satisfying appropriate axioms. And the author expects that any sufficiently well-behaved $\widetilde{\mathbf{DerMan}}$ should have a truncation functor $\widetilde{\mathbf{DerMan}} \to \mathbf{dMan}$, or at least a truncation functor $\widetilde{\mathbf{DerMan}} \to \mathrm{Ho}(\mathbf{dMan})$.

A Categories and 2-categories

We now explain the background in category theory we need. Some good references are Behrend et al. [11, App. B], and MacLane [73] for §A.1–§A.2.

A.1 Basics of category theory

For completeness, here are the basic definitions in category theory, as in [73, §I].

Definition A.1. A *category* (or 1-*category*) $\mathcal{C}$ consists of a proper class of *objects* $\mathrm{Obj}(\mathcal{C})$, and for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a set $\mathrm{Hom}(X, Y)$ of *morphisms* f from X to Y, written $f : X \to Y$, and for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a *composition map* $\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, written $(f, g) \mapsto g \circ f$. Composition must be *associative*, that is, if $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$ are morphisms in $\mathcal{C}$ then $(h \circ g) \circ f = h \circ (g \circ f)$. For each $X \in \mathrm{Obj}(\mathcal{C})$ there must exist an *identity morphism* $\mathrm{id}_X : X \to X$ such that $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for all $f : X \to Y$ in $\mathcal{C}$.

A morphism $f : X \to Y$ is an *isomorphism* if there exists $f^{-1} : Y \to X$ with $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. A category $\mathcal{C}$ is called a *groupoid* if every morphism is an isomorphism. In a (small) groupoid $\mathcal{C}$, for each $X \in \mathrm{Obj}(\mathcal{C})$ the set $\mathrm{Hom}(X, X)$ of morphisms $f : X \to X$ form a group.

If $\mathcal{C}$ is a category, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ is $\mathcal{C}$ with the directions of all morphisms reversed. That is, we define $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathcal{C})$, and for all $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$ we define $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, and for $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$ we define $f \circ_{\mathcal{C}^{\mathrm{op}}} g = g \circ_{\mathcal{C}} f$, and $\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} X = \mathrm{id}_{\mathcal{C}} X$.

Given categories $\mathcal{C}, \mathcal{D}$, the *product category* $\mathcal{C} \times \mathcal{D}$ has objects (W, X) in $\mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$ and morphisms $f \times g : (W, X) \to (Y, Z)$ when $f : W \to Y$ is a morphism in $\mathcal{C}$ and $g : X \to Z$ is a morphism in $\mathcal{D}$, in the obvious way.

We call $\mathcal{D}$ a *subcategory* of $\mathcal{C}$ if $\mathrm{Obj}(\mathcal{D}) \subseteq \mathrm{Obj}(\mathcal{C})$, and $\mathrm{Hom}_{\mathcal{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Obj}(\mathcal{D})$. We call $\mathcal{D}$ a *full* subcategory if also $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all X, Y.

Definition A.2. Let $\mathcal{C}, \mathcal{D}$ be categories. A (*covariant*) *functor* $F : \mathcal{C} \to \mathcal{D}$ gives, for all objects X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, and for all morphisms $f : X \to Y$ in $\mathcal{C}$ a morphism $F(f) : F(X) \to F(Y)$, such that $F(g \circ f) = F(g) \circ F(f)$ for all $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Obj}(\mathcal{C})$. A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Functors compose in the obvious way. Each category $\mathcal{C}$ has an obvious *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ with $\mathrm{id}_{\mathcal{C}}(X) = X$ and $\mathrm{id}_{\mathcal{C}}(f) = f$ for all X, f. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *full* if the maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, $f \mapsto F(f)$ are surjective for all $X, Y \in \mathrm{Obj}(\mathcal{C})$, and *faithful* if the maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ are injective for all $X, Y \in \mathrm{Obj}(\mathcal{C})$.

Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ gives, for all objects X in $\mathcal{C}$, a morphism $\eta(X) : F(X) \to G(X)$ such that if $f : X \to Y$ is a morphism in $\mathcal{C}$ then $\eta(Y) \circ F(f) = G(f) \circ \eta(X)$ as a morphism $F(X) \to G(Y)$ in $\mathcal{D}$. We call η a *natural isomorphism* if $\eta(X)$ is an isomorphism for all $X \in \mathrm{Obj}(\mathcal{C})$.

An *equivalence* between categories $\mathcal{C}, \mathcal{D}$ consists of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $\eta : G \circ F \Rightarrow \mathrm{id}_{\mathcal{C}}$, $\zeta : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$.

It is a fundamental principle of category theory that equivalent categories $\mathcal{C}, \mathcal{D}$ should be thought of as being 'the same', and naturally isomorphic functors $F, G : \mathcal{C} \to \mathcal{D}$ should be thought of as being 'the same'. Note that equivalence of categories $\mathcal{C}, \mathcal{D}$ is much weaker than strict isomorphism: isomorphism classes of objects in $\mathcal{C}$ are naturally in bijection with isomorphism classes of objects in $\mathcal{D}$, but there is no relation between the sizes of the isomorphism classes, so that $\mathcal{C}$ could have many more objects than $\mathcal{D}$, for instance.

A.2 Limits, colimits and fibre products in categories

We shall be interested in various kinds of *limits* and *colimits* in our categories of spaces. These are objects in the category with a universal property with respect to some class of diagrams. For an introduction to limits and colimits in category theory, see MacLane [73, §III]. Here are the basic definitions:

Definition A.3. Let $\mathcal{C}$ be a category. A *diagram* Δ in $\mathcal{C}$ is a class of objects S_i in $\mathcal{C}$ for $i \in I$, and a class of morphisms $\rho_j : S_{b(j)} \to S_{e(j)}$ in $\mathcal{C}$ for $j \in J$, where $b, e : J \to I$. The diagram is called *small* if I, J are sets (rather than something too large to be a set), and *finite* if I, J are finite sets.

A *limit* of the diagram Δ is an object L in $\mathcal{C}$ and morphisms $\pi_i : L \to S_i$ for $i \in I$ such that $\rho_j \circ \pi_{b(j)} = \pi_{e(j)}$ for all $j \in J$, which has the universal property that given $L' \in \mathcal{C}$ and $\pi'_i : L' \to S_i$ for $i \in I$ with $\rho_j \circ \pi'_{b(j)} = \pi'_{e(j)}$ for all $j \in J$, there is a unique morphism $\lambda : L' \to L$ with $\pi'_i = \pi_i \circ \lambda$ for all $i \in I$.

A *fibre product* is a limit of a diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$. The limit object W is often written $X \times_{g,Z,h} Y$ or $X \times_Z Y$, and the diagram

$$\begin{tikzcd} W \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d, "h"'] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

is called a *Cartesian square* in the category $\mathcal{C}$. A *terminal object* is a limit of the empty diagram.

A *colimit* of the diagram Δ is an object L in $\mathcal{C}$ and morphisms $\lambda_i : S_i \to L$ for $i \in I$ such that $\lambda_{b(j)} = \lambda_{e(j)} \circ \rho_j$ for all $j \in J$, which has the universal property that given $L' \in \mathcal{C}$ and $\lambda'_i : S_i \to L'$ for $i \in I$ with $\lambda'_{b(j)} = \lambda'_{e(j)} \circ \rho_j$ for all $j \in J$, there is a unique morphism $\pi : L \to L'$ with $\lambda'_i = \pi \circ \lambda_i$ for all $i \in I$.

A *pushout* is a colimit of a diagram $X \xleftarrow{e} W \xrightarrow{f} Y$. An *initial object* is a colimit of the empty diagram.

If a limit or colimit exists, it is unique up to unique isomorphism in $\mathcal{C}$. We say that *all limits*, or *all small limits*, or *all finite limits*, or *all fibre products exist in $\mathcal{C}$*, if limits exist for all diagrams, or all small diagrams, or all finite diagrams, or all diagrams $X \xrightarrow{g} Z \xleftarrow{h} Y$ respectively; and similarly for colimits.

By category theory general nonsense, one can prove:

Proposition A.4. *Suppose a category $\mathcal{C}$ has a terminal object, and all fibre products exist in $\mathcal{C}$. Then all finite limits exist in $\mathcal{C}$.*

Our categories of spaces (manifolds, C^∞-schemes, ...) will always have a terminal object, the point $*$. So we will generally concentrate on the existence or not of fibre products.

A.3 2-categories

Next we discuss 2-*categories*. A good reference for our purposes is Behrend et al. [11, App. B], and Kelly and Street [61] is also helpful.

Definition A.5. A 2-*category* $\boldsymbol{\mathcal{C}}$ (also called a *strict* 2-*category*) consists of a proper class of *objects* $\mathrm{Obj}(\boldsymbol{\mathcal{C}})$, for all $X,Y\in\mathrm{Obj}(\boldsymbol{\mathcal{C}})$ a category $\mathrm{Hom}(X,Y)$, for all $X\in\mathrm{Obj}(\boldsymbol{\mathcal{C}})$ an object id_X in $\mathrm{Hom}(X,X)$ called the *identity* 1-*morphism*, and for all $X,Y,Z\in\mathrm{Obj}(\boldsymbol{\mathcal{C}})$ a functor $\mu_{X,Y,Z}:\mathrm{Hom}(X,Y)\times\mathrm{Hom}(Y,Z)\rightarrow\mathrm{Hom}(X,Z)$. These must satisfy the *identity property*, that $\mu_{X,X,Y}(\mathrm{id}_X,-)=\mu_{X,Y,Y}(-,\mathrm{id}_Y)=\mathrm{id}_{\mathrm{Hom}(X,Y)}$ as functors $\mathrm{Hom}(X,Y)\rightarrow\mathrm{Hom}(X,Y)$, and the *associativity property*, that $\mu_{W,Y,Z}\circ(\mu_{W,X,Y}\times\mathrm{id}_{\mathrm{Hom}(Y,Z)})=\mu_{W,X,Z}\circ(\mathrm{id}_{\mathrm{Hom}(W,X)}\times\mu_{X,Y,Z})$ as functors $\mathrm{Hom}(W,X)\times\mathrm{Hom}(X,Y)\times\mathrm{Hom}(Y,Z)\rightarrow\mathrm{Hom}(W,X)$, for all W,X,Y,Z.

Objects f of $\mathrm{Hom}(X,Y)$ are called 1-*morphisms*, written $f:X\rightarrow Y$. For 1-morphisms $f,g:X\rightarrow Y$, morphisms $\eta\in\mathrm{Hom}_{\mathrm{Hom}(X,Y)}(f,g)$ are called 2-*morphisms*, written $\eta:f\Rightarrow g$. Thus, a 2-category has objects X, and two kinds of morphisms, 1-morphisms $f:X\rightarrow Y$ between objects, and 2-morphisms $\eta:f\Rightarrow g$ between 1-morphisms. In many examples, all 2-morphisms are 2-isomorphisms (i.e. have an inverse), so that the categories $\mathrm{Hom}(X,Y)$ are groupoids. Such 2-categories are called (2,1)-*categories*.

This is quite a complicated structure. There are three kinds of composition in a 2-category, satisfying various associativity relations. If $f:X\rightarrow Y$ and $g:Y\rightarrow Z$ are 1-morphisms then $\mu_{X,Y,Z}(f,g)$ is the *composition of* 1-*morphisms*, written $g\circ f:X\rightarrow Z$. If $f,g,h:X\rightarrow Y$ are 1-morphisms and $\eta:f\Rightarrow g$, $\zeta:g\Rightarrow h$ are 2-morphisms then composition of η,ζ in the category $\mathrm{Hom}(X,Y)$ gives the *vertical composition of* 2-*morphisms* of η,ζ, written $\zeta\odot\eta:f\Rightarrow h$, as a diagram

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=50, "h"'{name=H}] \arrow[from=F, to=G, phantom, "\Downarrow\eta"] \arrow[from=G, to=H, phantom, "\Downarrow\zeta"] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, bend left, "f"{name=F}] \arrow[r, bend right, "h"'{name=H}] \arrow[from=F, to=H, phantom, "\Downarrow\zeta\odot\eta"] & Y. \end{tikzcd} \tag{A.1}$$

And if $f,\tilde f:X\rightarrow Y$ and $g,\tilde g:Y\rightarrow Z$ are 1-morphisms and $\eta:f\Rightarrow\tilde f$, $\zeta:g\Rightarrow\tilde g$ are 2-morphisms then $\mu_{X,Y,Z}(\eta,\zeta)$ is the *horizontal composition of*

2-*morphisms*, written $\zeta * \eta : g \circ f \Rightarrow \tilde{g} \circ \tilde{f}$, as a diagram

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "\tilde{f}"'] \arrow[r, phantom, "\Downarrow\eta"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "\tilde{g}"'] \arrow[r, phantom, "\Downarrow\zeta"] & Z \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, bend left, "g\circ f"] \arrow[r, bend right, "\tilde{g}\circ\tilde{f}"'] \arrow[r, phantom, "\Downarrow\zeta*\eta"] & Z. \end{tikzcd} \tag{A.2}$$

There are also two kinds of identity: *identity* 1-*morphisms* $\mathrm{id}_X : X \to X$ and *identity* 2-*morphisms* $\mathrm{id}_f : f \Rightarrow f$.

A basic example is the 2-*category of categories* $\mathfrak{Cat}$, with objects categories $\mathcal{C}$, 1-morphisms functors $F : \mathcal{C} \to \mathcal{D}$, and 2-morphisms natural transformations $\eta : F \Rightarrow G$ for functors $F, G : \mathcal{C} \to \mathcal{D}$. Orbifolds naturally form a 2-category, as do Deligne–Mumford stacks and Artin stacks in algebraic geometry.

In a 2-category $\boldsymbol{\mathcal{C}}$, there are three notions of when objects X, Y in $\boldsymbol{\mathcal{C}}$ are 'the same': *equality* $X = Y$, and *isomorphism*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, and *equivalence*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$. Usually equivalence is the most useful notion. By [11, Prop. B.8], we can also choose η, ζ to satisfy the extra identities $\mathrm{id}_f * \eta = \zeta * \mathrm{id}_f$ and $\mathrm{id}_g * \zeta = \eta * \mathrm{id}_g$:

Proposition A.6. *Let $\boldsymbol{\mathcal{C}}$ be a* 2*-category, and $f : X \to Y$ be an equivalence in $\boldsymbol{\mathcal{C}}$. Then there exist a* 1*-morphism $g : Y \to X$ and* 2*-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$ with $\mathrm{id}_f * \eta = \zeta * \mathrm{id}_f$ as* 2*-isomorphisms $f \circ g \circ f \Rightarrow f$, and $\mathrm{id}_g * \zeta = \eta * \mathrm{id}_g$ as* 2*-isomorphisms $g \circ f \circ g \Rightarrow g$.*

Let $\boldsymbol{\mathcal{C}}$ be a 2-category. The *homotopy category* $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$ of $\boldsymbol{\mathcal{C}}$ is the category whose objects are objects of $\boldsymbol{\mathcal{C}}$, and whose morphisms $[f] : X \to Y$ are 2-isomorphism classes $[f]$ of 1-morphisms $f : X \to Y$ in $\boldsymbol{\mathcal{C}}$. Then equivalences in $\boldsymbol{\mathcal{C}}$ become isomorphisms in $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$, 2-commutative diagrams in $\boldsymbol{\mathcal{C}}$ become commutative diagrams in $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$, and so on.

As in Borceux [15, §7.7], there is also a second kind of 2-category, called a *weak* 2*-category* (or *bicategory*), which we will not define in detail. In a weak 2-category, compositions of 1-morphisms need only be associative up to (specified) 2-isomorphisms. That is, part of the data of a weak 2-category $\boldsymbol{\mathcal{C}}$ is a 2-isomorphism $\alpha(f, g, h) : (h \circ g) \circ f \Rightarrow h \circ (g \circ f)$ for all 1-morphisms $f : W \to X$, $g : X \to Y$, $h : Y \to Z$ in $\boldsymbol{\mathcal{C}}$. A strict 2-category $\boldsymbol{\mathcal{C}}$ can be made into a weak 2-category by putting $\alpha(f, g, h) = \mathrm{id}_{h \circ g \circ f}$ for all f, g, h.

Some categorical constructions naturally yield weak 2-categories rather than strict 2-categories, e.g. the weak 2-categories of orbifolds defined by Pronk [89] and Lerman [67, §3.3] mentioned in §9.1. Every weak 2-category is equivalent as a weak 2-category to a strict 2-category (that is, weak 2-categories can be strictified), so we lose little by working only with strict 2-categories.

A.4 Fibre products in 2-categories

Commutative diagrams in 2-categories should in general only commute *up to (specified)* 2*-isomorphisms*, rather than strictly. Then we say the diagram 2-

commutes. A simple example of a commutative diagram in a 2-category $\boldsymbol{\mathcal{C}}$ is

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\eta"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z, \end{tikzcd}$$

which means that X, Y, Z are objects of $\boldsymbol{\mathcal{C}}$, $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ are 1-morphisms in $\boldsymbol{\mathcal{C}}$, and $\eta : g \circ f \Rightarrow h$ is a 2-isomorphism.

The generalizations of *limit* and *colimit* to 2-categories turn out to be rather complicated. As in [15, §7] there are many different kinds — 2-limits, bilimits, pseudolimits, lax limits, and weighted limits (or indexed limits), depending on whether one considers diagrams to commute on the nose, up to 2-isomorphism, or up to 2-morphisms, and what kind of universal property one requires. Our definition of *fibre product*, following Behrend et al. [11, Def. B.13], is actually an example of a pseudolimit.

Definition A.7. Let $\boldsymbol{\mathcal{C}}$ be a 2-category and $g : X \to Z$, $h : Y \to Z$ be 1-morphisms in $\boldsymbol{\mathcal{C}}$. A *fibre product* $X \times_Z Y$ in $\boldsymbol{\mathcal{C}}$ consists of an object W, 1-morphisms $e : W \to X$ and $f : W \to Y$ and a 2-isomorphism $\eta : g \circ e \Rightarrow h \circ f$ in $\boldsymbol{\mathcal{C}}$, so that we have a 2-commutative diagram

$$\begin{tikzcd} W \arrow[r, "f"'] \arrow[d, "e"] \arrow[dr, phantom, "\eta\Uparrow"] & Y \arrow[d, "h"'] \\ X \arrow[r, "g"] & Z \end{tikzcd} \tag{A.3}$$

with the following universal property: suppose $e' : W' \to X$ and $f' : W' \to Y$ are 1-morphisms and $\eta' : g \circ e' \Rightarrow h \circ f'$ is a 2-isomorphism in $\boldsymbol{\mathcal{C}}$. Then there should exist a 1-morphism $b : W' \to W$ and 2-isomorphisms $\zeta : e \circ b \Rightarrow e'$, $\theta : f \circ b \Rightarrow f'$ such that the following diagram of 2-isomorphisms commutes:

$$\begin{tikzcd} g \circ e \circ b \arrow[r, Rightarrow, "\eta * \mathrm{id}_b"'] \arrow[d, Rightarrow, "\mathrm{id}_g * \zeta"'] & h \circ f \circ b \arrow[d, Rightarrow, "\mathrm{id}_h * \theta"] \\ g \circ e' \arrow[r, Rightarrow, "\eta'"] & h \circ f'. \end{tikzcd} \tag{A.4}$$

Furthermore, if $\tilde{b}, \tilde{\zeta}, \tilde{\theta}$ are alternative choices of b, ζ, θ then there should exist a unique 2-isomorphism $\epsilon : \tilde{b} \Rightarrow b$ with

$$\tilde{\zeta} = \zeta \odot (\mathrm{id}_e * \epsilon) \quad \text{and} \quad \tilde{\theta} = \theta \odot (\mathrm{id}_f * \epsilon). \tag{A.5}$$

We call such a fibre product diagram (A.3) a 2-*Cartesian square*.

If a fibre product $X \times_Z Y$ in $\boldsymbol{\mathcal{C}}$ exists then it is unique up to equivalence in $\boldsymbol{\mathcal{C}}$. If $\boldsymbol{\mathcal{C}}$ is a category, that is, all 2-morphisms are identities $\mathrm{id}_f : f \Rightarrow f$, this definition of fibre products in $\boldsymbol{\mathcal{C}}$ coincides with that in §A.2.

Orbifolds, and stacks in algebraic geometry, form 2-categories, and Definition A.7 is the right way to define fibre products of orbifolds or stacks, as in [11]. One can also define *pushouts* in a 2-category $\boldsymbol{\mathcal{C}}$ in a dual way to Definition A.7, reversing directions of morphisms.

B Algebraic Geometry over C^∞-rings

If X is a manifold then the $\mathbb{R}$-algebra $C^\infty(X)$ of smooth functions $c : X \to \mathbb{R}$ is a C^∞-*ring*. That is, for each smooth function $f : \mathbb{R}^n \to \mathbb{R}$ there is an n-fold operation $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ acting by $\Phi_f : c_1, \ldots, c_n \mapsto f(c_1, \ldots, c_n)$, and these operations Φ_f satisfy many natural identities. Thus, $C^\infty(X)$ actually has a far richer algebraic structure than the obvious $\mathbb{R}$-algebra structure.

In [56] the author set out the foundations of a version of algebraic geometry in which rings or algebras are replaced by C^∞-rings. We now summarize material from [56, §1–§6] on C^∞-*schemes*, a category of geometric objects which generalize manifolds, and whose morphisms generalize smooth maps, and *quasicoherent sheaves* on C^∞-schemes. Most of the material on C^∞-schemes was already known in synthetic differential geometry, see in particular Dubuc [30] and Moerdijk and Reyes [86]. Appendix C will discuss *Deligne–Mumford C^∞-stacks*, a 2-category of geometric objects which generalize orbifolds.

B.1 C^∞-rings

Definition B.1. A C^∞-*ring* is a set $\mathfrak{C}$ together with operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all $n \geqslant 0$ and smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, where by convention when $n = 0$ we define $\mathfrak{C}^0$ to be the single point $\{\emptyset\}$. These operations must satisfy the following relations: suppose $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ are smooth functions. Define a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g\big(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)\big),$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g\big(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)\big).$$

We also require that for all $1 \leqslant j \leqslant n$, defining $\pi_j : \mathbb{R}^n \to \mathbb{R}$ by $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$, we have $\Phi_{\pi_j}(c_1, \ldots, c_n) = c_j$ for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$.

Usually we refer to $\mathfrak{C}$ as the C^∞-ring, leaving the operations Φ_f implicit.

A *morphism* between C^∞-rings $\big(\mathfrak{C}, (\Phi_f)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\big)$, $\big(\mathfrak{D}, (\Psi_f)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\big)$ is a map $\phi : \mathfrak{C} \to \mathfrak{D}$ such that $\Psi_f\big(\phi(c_1), \ldots, \phi(c_n)\big) = \phi \circ \Phi_f(c_1, \ldots, c_n)$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$ and $c_1, \ldots, c_n \in \mathfrak{C}$. We will write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Here is the motivating example:

Example B.2. Let X be a manifold, which may have boundary or corners. Write $C^\infty(X)$ for the set of smooth functions $c : X \to \mathbb{R}$. For $n \geqslant 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ smooth, define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by

$$\big(\Phi_f(c_1, \ldots, c_n)\big)(x) = f\big(c_1(x), \ldots, c_n(x)\big), \tag{B.1}$$

for all $c_1, \ldots, c_n \in C^\infty(X)$ and $x \in X$. It is easy to see that $C^\infty(X)$ and the operations Φ_f form a C^∞-ring.

Example B.3. Take $X = \{0\}$ in Example B.2. Then $C^\infty(\{0\}) = \mathbb{R}$, with operations $\Phi_f : \mathbb{R}^n \to \mathbb{R}$ given by $\Phi_f(x_1, \dots, x_n) = f(x_1, \dots, x_n)$. This makes $\mathbb{R}$ into the simplest nonzero example of a C^∞-ring.

Note that C^∞-rings are far more general than those coming from manifolds. For example, if X is any topological space we could define a C^∞-ring $C^0(X)$ to be the set of *continuous* $c : X \to \mathbb{R}$ with operations Φ_f defined as in (B.1). For X a manifold with $\dim X > 0$, the C^∞-rings $C^\infty(X)$ and $C^0(X)$ are different. We will introduce *colimits* in §A.2. We have:

Proposition B.4. *In the category* $\mathbf{C^\infty Rings}$ *of* C^∞*-rings, all small colimits exist, and so in particular pushouts and all finite colimits exist.*

We will write $\mathfrak{D} \amalg_{\phi,\mathfrak{C},\psi} \mathfrak{E}$ or $\mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$ for the pushout of morphisms $\phi : \mathfrak{C} \to \mathfrak{D}$, $\psi : \mathfrak{C} \to \mathfrak{E}$ in $\mathbf{C^\infty Rings}$. When $\mathfrak{C} = \mathbb{R}$, the initial object in $\mathbf{C^\infty Rings}$, pushouts $\mathfrak{D} \amalg_{\mathbb{R}} \mathfrak{E}$ are called *coproducts* and are written $\mathfrak{D} \hat{\otimes} \mathfrak{E}$, as they are a kind of completed tensor product. (For $\mathbb{R}$-algebras A, B the coproduct is $A \otimes B$.)

Definition B.5. Let $\mathfrak{C}$ be a C^∞-ring. Then we may give $\mathfrak{C}$ the structure of a *commutative* $\mathbb{R}$*-algebra*. Define addition '+' on $\mathfrak{C}$ by $c + c' = \Phi_f(c, c')$ for $c, c' \in \mathfrak{C}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is $f(x, y) = x + y$. Define multiplication '$\cdot$' on $\mathfrak{C}$ by $c \cdot c' = \Phi_g(c, c')$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is $g(x, y) = xy$. Define scalar multiplication by $\lambda \in \mathbb{R}$ by $\lambda c = \Phi_{\lambda'}(c)$, where $\lambda' : \mathbb{R} \to \mathbb{R}$ is $\lambda'(x) = \lambda x$. Define elements 0 and 1 in $\mathfrak{C}$ by $0 = \Phi_{0'}(\emptyset)$ and $1 = \Phi_{1'}(\emptyset)$, where $0' : \mathbb{R}^0 \to \mathbb{R}$ and $1' : \mathbb{R}^0 \to \mathbb{R}$ are the maps $0' : \emptyset \mapsto 0$ and $1' : \emptyset \mapsto 1$. One can then show using the relations on the Φ_f that all the axioms of a commutative $\mathbb{R}$-algebra are satisfied. In Example B.2, this yields the obvious $\mathbb{R}$-algebra structure on the smooth functions $c : X \to \mathbb{R}$.

An *ideal* I in $\mathfrak{C}$ is an ideal $I \subset \mathfrak{C}$ in $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra. Then we make the quotient $\mathfrak{C}/I$ into a C^∞-ring as follows. If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, define $\Phi_f^I : (\mathfrak{C}/I)^n \to \mathfrak{C}/I$ by

$$\bigl(\Phi_f^I(c_1 + I, \dots, c_n + I)\bigr)(x) = f\bigl(c_1(x), \dots, c_n(x)\bigr) + I.$$

To show this is well-defined, we must show it is independent of the choice of representatives $c_1, \dots, c_n$ in $\mathfrak{C}$ for $c_1 + I, \dots, c_n + I$ in $\mathfrak{C}/I$. By Hadamard's Lemma there exist smooth functions $g_i : \mathbb{R}^{2n} \to \mathbb{R}$ for $i = 1, \dots, n$ with

$$f(y_1, \dots, y_n) - f(x_1, \dots, x_n) = \textstyle\sum_{i=1}^n (y_i - x_i) g_i(x_1, \dots, x_n, y_1, \dots, y_n)$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$. If $c_1', \dots, c_n'$ are alternative choices for $c_1, \dots, c_n$, so that $c_i' + I = c_i + I$ for $i = 1, \dots, n$ and $c_i' - c_i \in I$, we have

$$\begin{aligned} &f\bigl(c_1'(x), \dots, c_n'(x)\bigr) - f\bigl(c_1(x), \dots, c_n(x)\bigr) \\ &\qquad = \textstyle\sum_{i=1}^n (c_i' - c_i) g_i\bigl(c_1'(x), \dots, c_n'(x), c_1(x), \dots, c_n(x)\bigr). \end{aligned}$$

The second line lies in I as $c_i' - c_i \in I$ and I is an ideal, so Φ_f^I is well-defined, and clearly $\bigl(\mathfrak{C}/I, (\Phi_f^I)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\bigr)$ is a C^∞-ring.

We will use the notation $(f_a : a \in A)$ to denote the ideal in a C^∞-ring $\mathfrak{C}$ generated by a collection of elements $f_a \in \mathfrak{C}$, $a \in A$. That is,

$$(f_a : a \in A) = \left\{\textstyle\sum_{i=1}^n f_{a_i} \cdot c_i : n \geqslant 0,\, a_1, \ldots, a_n \in A,\, c_1, \ldots, c_n \in \mathfrak{C}\right\}.$$

B.2 Special classes of C^∞-ring

We define *finitely generated*, *finitely presented*, *local*, and *fair* C^∞-rings.

Definition B.6. A C^∞-ring $\mathfrak{C}$ is called *finitely generated* if there exist $c_1, \ldots, c_n$ in $\mathfrak{C}$ which generate $\mathfrak{C}$ over all C^∞-operations. That is, for each $c \in \mathfrak{C}$ there exists smooth $f : \mathbb{R}^n \to \mathbb{R}$ with $c = \Phi_f(c_1, \ldots, c_n)$. Given such $\mathfrak{C}, c_1, \ldots, c_n$, define $\phi : C^\infty(\mathbb{R}^n) \to \mathfrak{C}$ by $\phi(f) = \Phi_f(c_1, \ldots, c_n)$ for smooth $f : \mathbb{R}^n \to \mathbb{R}$, where $C^\infty(\mathbb{R}^n)$ is as in Example B.2 with $X = \mathbb{R}^n$. Then ϕ is a surjective morphism of C^∞-rings, so $I = \operatorname{Ker}\phi$ is an ideal in $C^\infty(\mathbb{R}^n)$, and $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ as a C^∞-ring. Thus, $\mathfrak{C}$ is finitely generated if and only if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some $n \geqslant 0$ and ideal I in $C^\infty(\mathbb{R}^n)$.

An ideal I in $C^\infty(\mathbb{R}^n)$ is called *finitely generated* if $I = (f_1, \ldots, f_k)$ for some $f_1, \ldots, f_k \in C^\infty(\mathbb{R}^n)$. A C^∞-ring $\mathfrak{C}$ is called *finitely presented* if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some $n \geqslant 0$, where I is a finitely generated ideal in $C^\infty(\mathbb{R}^n)$.

A difference with conventional algebraic geometry is that $C^\infty(\mathbb{R}^n)$ is not noetherian, so ideals in $C^\infty(\mathbb{R}^n)$ may not be finitely generated, and $\mathfrak{C}$ finitely generated does not imply $\mathfrak{C}$ finitely presented.

Definition B.7. A C^∞-ring $\mathfrak{C}$ is called a C^∞-*local ring* if regarded as an $\mathbb{R}$-algebra, as in Definition B.5, $\mathfrak{C}$ is a local $\mathbb{R}$-algebra with residue field $\mathbb{R}$. That is, $\mathfrak{C}$ has a unique maximal ideal $\mathfrak{m}_{\mathfrak{C}}$ with $\mathfrak{C}/\mathfrak{m}_{\mathfrak{C}} \cong \mathbb{R}$.

If $\mathfrak{C}, \mathfrak{D}$ are C^∞-local rings with maximal ideals $\mathfrak{m}_{\mathfrak{C}}, \mathfrak{m}_{\mathfrak{D}}$, and $\phi : \mathfrak{C} \to \mathfrak{D}$ is a morphism of C^∞ rings, then using the fact that $\mathfrak{C}/\mathfrak{m}_{\mathfrak{C}} \cong \mathbb{R} \cong \mathfrak{D}/\mathfrak{m}_{\mathfrak{D}}$ we see that $\phi^{-1}(\mathfrak{m}_{\mathfrak{D}}) = \mathfrak{m}_{\mathfrak{C}}$, that is, ϕ is a *local* morphism of C^∞-local rings. Thus, there is no difference between morphisms and local morphisms.

Example B.8. For $n \geqslant 0$ and $p \in \mathbb{R}^n$, define $C^\infty_p(\mathbb{R}^n)$ to be the set of germs of smooth functions $c : \mathbb{R}^n \to \mathbb{R}$ at $p \in \mathbb{R}^n$. That is, $C^\infty_p(\mathbb{R}^n)$ is the quotient of the set of pairs (U, c) with $p \in U \subset \mathbb{R}^n$ open and $c : U \to \mathbb{R}$ smooth by the equivalence relation $(U, c) \sim (U', c')$ if there exists $p \in V \subseteq U \cap U'$ open with $c|_V = c'|_V$. Define operations $\Phi_f : (C^\infty_p(\mathbb{R}^n))^m \to C^\infty_p(\mathbb{R}^n)$ for $f : \mathbb{R}^m \to \mathbb{R}$ smooth by (B.1). Then $C^\infty_p(\mathbb{R}^n)$ is a C^∞-local ring, with maximal ideal $\mathfrak{m} = \{[(U, c)] : c(p) = 0\}$.

Definition B.9. An ideal I in $C^\infty(\mathbb{R}^n)$ is called *fair* if for each $f \in C^\infty(\mathbb{R}^n)$, f lies in I if and only if $\pi_p(f)$ lies in $\pi_p(I) \subseteq C^\infty_p(\mathbb{R}^n)$ for all $p \in \mathbb{R}^n$, where $C^\infty_p(\mathbb{R}^n)$ is as in Example B.8 and $\pi_p : C^\infty(\mathbb{R}^n) \to C^\infty_p(\mathbb{R}^n)$ is the natural projection $\pi_p : c \mapsto [(\mathbb{R}^n, c)]$. A C^∞-ring $\mathfrak{C}$ is called *fair* if it is isomorphic to $C^\infty(\mathbb{R}^n)/I$, where I is a fair ideal.

As in [56, §2.4], if $C^\infty(\mathbb{R}^m)/I \cong C^\infty(\mathbb{R}^n)/J$ then I is finitely generated, or fair, if and only if J is. Thus, to decide whether a C^∞-ring $\mathfrak{C}$ is finitely presented, or fair, it is enough to test one presentation $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$. Also, $\mathfrak{C}$ finitely presented implies $\mathfrak{C}$ fair implies $\mathfrak{C}$ finitely generated. Write $\mathbf{C^\infty Rings^{fp}}$, $\mathbf{C^\infty Rings^{fa}}$ and $\mathbf{C^\infty Rings^{fg}}$ for the full subcategories of finitely presented, fair, and finitely generated C^∞-rings in $\mathbf{C^\infty Rings}$, respectively. Then

$$\mathbf{C^\infty Rings^{fp}} \subset \mathbf{C^\infty Rings^{fa}} \subset \mathbf{C^\infty Rings^{fg}} \subset \mathbf{C^\infty Rings}.$$

From [56, Prop.s 2.23 & 2.25] we have:

Proposition B.10. *The subcategories* $\mathbf{C^\infty Rings^{fg}}, \mathbf{C^\infty Rings^{fp}}$ *are closed under pushouts and all finite colimits in* $\mathbf{C^\infty Rings}$, *but* $\mathbf{C^\infty Rings^{fa}}$ *is not. Nonetheless, pushouts and finite colimits exist in* $\mathbf{C^\infty Rings^{fa}}$, *though they may not coincide with pushouts and finite colimits in* $\mathbf{C^\infty Rings}$.

B.3 Sheaves on topological spaces

Sheaves are a fundamental concept in algebraic geometry. They are necessary even to define schemes, since a scheme is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$. In this book, sheaves of abelian groups, sheaves of C^∞-rings, and sheaves of modules over a sheaf of C^∞-rings, all play a fundamental rôle.

We now summarize some basics of sheaf theory, following Hartshorne [38, §II.1]. A more detailed reference is Godement [35]. We concentrate on sheaves of abelian groups; to define sheaves of C^∞-rings, etc., one replaces abelian groups with C^∞-rings, etc., throughout.

Definition B.11. Let X be a topological space. A *presheaf of abelian groups* $\mathcal{E}$ on X consists of the data of an abelian group $\mathcal{E}(U)$ for every open set $U \subseteq X$, and a morphism of abelian groups $\rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ called the *restriction map* for every inclusion $V \subseteq U \subseteq X$ of open sets, satisfying the conditions that

(i) $\mathcal{E}(\emptyset) = 0$;

(ii) $\rho_{UU} = \mathrm{id}_{\mathcal{E}(U)} : \mathcal{E}(U) \to \mathcal{E}(U)$ for all open $U \subseteq X$; and

(iii) $\rho_{UW} = \rho_{VW} \circ \rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(W)$ for all open $W \subseteq V \subseteq U \subseteq X$.

A presheaf of abelian groups $\mathcal{E}$ on X is called a *sheaf* if it also satisfies

(iv) If $U \subseteq X$ is open, $\{V_i : i \in I\}$ is an open cover of U, and $s \in \mathcal{E}(U)$ has $\rho_{UV_i}(s) = 0$ in $\mathcal{E}(V_i)$ for all $i \in I$, then $s = 0$ in $\mathcal{E}(U)$; and

(v) If $U \subseteq X$ is open, $\{V_i : i \in I\}$ is an open cover of U, and we are given elements $s_i \in \mathcal{E}(V_i)$ for all $i \in I$ such that $\rho_{V_i(V_i \cap V_j)}(s_i) = \rho_{V_j(V_i \cap V_j)}(s_j)$ in $\mathcal{E}(V_i \cap V_j)$ for all $i, j \in I$, then there exists $s \in \mathcal{E}(U)$ with $\rho_{UV_i}(s) = s_i$ for all $i \in I$. This s is unique by (iv).

Suppose $\mathcal{E}, \mathcal{F}$ are presheaves or sheaves of abelian groups on X. A *morphism* $\phi : \mathcal{E} \to \mathcal{F}$ consists of a morphism of abelian groups $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ for all open $U \subseteq X$, such that the following diagram commutes for all open $V \subseteq U \subseteq X$

$$\begin{tikzcd}
\mathcal{E}(U) \arrow[rr, "\phi(U)"] \arrow[d, "\rho_{UV}"'] && \mathcal{F}(U) \arrow[d, "\rho'_{UV}"'] \\
\mathcal{E}(V) \arrow[rr, "\phi(V)"] && \mathcal{F}(V).
\end{tikzcd}$$

where ρ_{UV} is the restriction map for $\mathcal{E}$, and ρ'_{UV} the restriction map for $\mathcal{F}$.

Definition B.12. Let $\mathcal{E}$ be a presheaf of abelian groups on X. For each $x \in X$, the *stalk* $\mathcal{E}_x$ is the direct limit of the groups $\mathcal{E}(U)$ for all $x \in U \subseteq X$, via the restriction maps ρ_{UV}. It is an abelian group. A morphism $\phi : \mathcal{E} \to \mathcal{F}$ induces morphisms $\phi_x : \mathcal{E}_x \to \mathcal{F}_x$ for all $x \in X$. If $\mathcal{E}, \mathcal{F}$ are sheaves then ϕ is an isomorphism if and only if ϕ_x is an isomorphism for all $x \in X$.

Sheaves of abelian groups on X form an *abelian category* $\mathrm{Sh}(X)$. Thus we have (category-theoretic) notions of when a morphism $\phi : \mathcal{E} \to \mathcal{F}$ in $\mathrm{Sh}(X)$ is *injective* or *surjective*, and when a sequence $\mathcal{E} \to \mathcal{F} \to \mathcal{G}$ in $\mathrm{Sh}(X)$ is *exact*. It turns out that $\phi : \mathcal{E} \to \mathcal{F}$ is injective if and only if $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ is injective for all open $U \subseteq X$. However $\phi : \mathcal{E} \to \mathcal{F}$ surjective does not imply that $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ is surjective for all open $U \subseteq X$. Instead, ϕ is surjective if and only if $\phi_x : \mathcal{E}_x \to \mathcal{F}_x$ is surjective for all $x \in X$.

Definition B.13. Let $\mathcal{E}$ be a presheaf of abelian groups on X. A *sheafification* of $\mathcal{E}$ is a sheaf of abelian groups $\hat{\mathcal{E}}$ on X and a morphism $\pi : \mathcal{E} \to \hat{\mathcal{E}}$, such that whenever $\mathcal{F}$ is a sheaf of abelian groups on X and $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism, there is a unique morphism $\hat{\phi} : \hat{\mathcal{E}} \to \mathcal{F}$ with $\phi = \hat{\phi} \circ \pi$. As in [38, Prop. II.1.2], a sheafification always exists, and is unique up to canonical isomorphism; one can be constructed explicitly using the stalks $\mathcal{E}_x$ of $\mathcal{E}$.

Next we discuss *pushforwards* and *pullbacks* of sheaves by continuous maps.

Definition B.14. Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{E}$ a sheaf of abelian groups on X. Define the *pushforward* (*direct image*) sheaf $f_*(\mathcal{E})$ on Y by $\bigl(f_*(\mathcal{E})\bigr)(U) = \mathcal{E}\bigl(f^{-1}(U)\bigr)$ for all open $U \subseteq V$, with restriction maps $\rho'_{UV} = \rho_{f^{-1}(U)f^{-1}(V)} : \bigl(f_*(\mathcal{E})\bigr)(U) \to \bigl(f_*(\mathcal{E})\bigr)(V)$ for all open $V \subseteq U \subseteq Y$. Then $f_*(\mathcal{E})$ is a sheaf of abelian groups on Y.

If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism in $\mathrm{Sh}(X)$ we define $f_*(\phi) : f_*(\mathcal{E}) \to f_*(\mathcal{F})$ by $\bigl(f_*(\phi)\bigr)(u) = \phi\bigl(f^{-1}(U)\bigr)$ for all open $U \subseteq Y$. Then $f_*(\phi)$ is a morphism in $\mathrm{Sh}(Y)$, and f_* is a functor $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. It is a left exact functor between abelian categories, but in general is not exact. For continuous maps $f : X \to Y$, $g : Y \to Z$ we have $(g \circ f)_* = g_* \circ f_*$.

Definition B.15. Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{E}$ a sheaf of abelian groups on Y. Define a presheaf $\mathcal{P}f^{-1}(\mathcal{E})$ on X by $\bigl(\mathcal{P}f^{-1}(\mathcal{E})\bigr)(U) = \lim_{A \supseteq f(U)} \mathcal{E}(A)$ for open $A \subseteq X$, where the direct limit is taken over all open $A \subseteq Y$ containing $f(U)$, using the restriction maps ρ_{AB} in $\mathcal{E}$. For open $V \subseteq U \subseteq X$, define $\rho'_{UV} : \bigl(\mathcal{P}f^{-1}(\mathcal{E})\bigr)(U) \to \bigl(\mathcal{P}f^{-1}(\mathcal{E})\bigr)(V)$ as the direct limit of the morphisms ρ_{AB} in $\mathcal{E}$ for $B \subseteq A \subseteq Y$ with $f(U) \subseteq A$

and $f(V) \subseteq B$. Then we define the *pullback* (*inverse image*) $f^{-1}(\mathcal{E})$ to be the sheafification of the presheaf $\mathcal{P}f^{-1}(\mathcal{E})$.

Pullbacks $f^{-1}(\mathcal{E})$ are only unique up to canonical isomorphism, rather than unique. By convention we choose once and for all a pullback $f^{-1}(\mathcal{E})$ for all $X, Y, f, \mathcal{E}$, using the Axiom of Choice if necessary. If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism in $\mathrm{Sh}(Y)$, one can define a pullback morphism $f^{-1}(\phi) : f^{-1}(\mathcal{E}) \to f^{-1}(\mathcal{F})$. Then $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is an exact functor between abelian categories.

We compare pushforwards and pullbacks:

Remark B.16. (a) There are two kinds of pullback, with slightly different notation. The first kind, written $f^{-1}(\mathcal{E})$ as in Definition B.15, is used for sheaves of abelian groups or C^∞-rings. The second kind, written $f^*(\mathcal{E})$ or $\underline{f}^*(\mathcal{E})$ and discussed in §B.7, is used for sheaves of $\mathcal{O}_Y$-modules $\mathcal{E}$.

(b) The definition of pushforward sheaves $f_*(\mathcal{E})$ is wholly elementary. In contrast, the definition of pullbacks $f^{-1}(\mathcal{E})$ is complex, involving a direct limit followed by a sheafification, and includes arbitrary choices.

Pushforwards f_* are strictly functorial in the continuous map $f : X \to Y$, that is, for continuous $f : X \to Y$, $g : Y \to Z$ we have $(g \circ f)_* = g_* \circ f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Z)$. However, pullbacks f^{-1} are only weakly functorial in f: if $\mathcal{E} \in \mathrm{Sh}(Z)$ then we need not have $(g \circ f)^{-1}(\mathcal{E}) = f^{-1}(g^{-1}(\mathcal{E}))$. This is because pullbacks are only natural up to canonical isomorphism, and we make an arbitrary choice for each pullback. So although $f^{-1}(g^{-1}(\mathcal{E}))$ is a possible pullback for $\mathcal{E}$ by $g \circ f$, it may not be the one we chose.

Thus, there is a canonical isomorphism $(g \circ f)^{-1}(\mathcal{E}) \cong f^{-1}(g^{-1}(\mathcal{E}))$, which we will write as $I_{f,g}(\mathcal{E}) : (g \circ f)^{-1}(\mathcal{E}) \to f^{-1}(g^{-1}(\mathcal{E}))$. The $I_{f,g}(\mathcal{E})$ for all $\mathcal{E} \in \mathrm{Sh}(Z)$ comprise a natural isomorphism of functors $I_{f,g} : (g \circ f)^{-1} \Rightarrow f^{-1} \circ g^{-1}$. Similarly, for $\mathcal{E} \in \mathrm{Sh}(X)$ we may not have $\mathrm{id}_X^{-1}(\mathcal{E}) = \mathcal{E}$, but instead there are canonical isomorphisms $\delta_X(\mathcal{E}) : \mathrm{id}_X^{-1}(\mathcal{E}) \to \mathcal{E}$, which make up a natural isomorphism $\delta_X : \mathrm{id}_X^{-1} \Rightarrow \mathrm{id}_{\mathrm{Sh}(X)}$.

Many authors ignore the natural isomorphisms $I_{f,g}, \delta_X$ entirely. We are careful to keep track of them, in part because by including them in our constructions we can make d-manifolds and d-orbifolds into strict 2-categories, rather than some weaker structure.

(c) Let $f : X \to Y$ be a continuous map of topological spaces. Then we have functors $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, and $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$. As in [38, Ex. II.1.18], f_* is right adjoint to f^{-1}. That is, there is a natural bijection

$$\mathrm{Hom}_X\big(f^{-1}(\mathcal{E}), \mathcal{F}\big) = \mathrm{Hom}_Y\big(\mathcal{E}, f_*(\mathcal{F})\big) \tag{B.2}$$

for all $\mathcal{E} \in \mathrm{Sh}(Y)$ and $\mathcal{F} \in \mathrm{Sh}(X)$, with functorial properties.

Because of the adjoint property (B.2), statements can often be formulated equivalently using either pushforwards or pullbacks. Our policy is always to express things in terms of pullbacks, despite the disadvantages noted in **(b)**. One reason is that when working with quasicoherent sheaves, coherent sheaves and vector bundles are preserved by pullbacks, but not by pushforwards.

B.4 C^∞-schemes

Next we summarize material in [56, §4] on C^∞-schemes.

Definition B.17. A *C^∞-ringed space* $\underline{X}=(X,\mathcal{O}_X)$ is a topological space X with a sheaf $\mathcal{O}_X$ of C^∞-rings on X, as in Definition B.11.

A *morphism* $\underline{f}=(f,f^\sharp):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ of C^∞ ringed spaces is a continuous map $f:X\to Y$ and a morphism $f^\sharp:f^{-1}(\mathcal{O}_Y)\to\mathcal{O}_X$ of sheaves of C^∞-rings on X, for $f^{-1}(\mathcal{O}_Y)$ as in Definition B.15. There is another way to write the data $f^\sharp$: since as in Remark B.16(c) pushforward f_* is right adjoint to pullback f^{-1}, as in (B.2) there is a natural bijection

$$\mathrm{Hom}_X\bigl(f^{-1}(\mathcal{O}_Y),\mathcal{O}_X\bigr)\cong\mathrm{Hom}_Y\bigl(\mathcal{O}_Y,f_*(\mathcal{O}_X)\bigr). \tag{B.3}$$

Write $f_\sharp:\mathcal{O}_Y\to f_*(\mathcal{O}_X)$ for the morphism of sheaves of C^∞-rings on Y corresponding to $f^\sharp$ under (B.3), so that

$$f^\sharp:f^{-1}(\mathcal{O}_Y)\longrightarrow\mathcal{O}_X \quad\rightsquigarrow\quad f_\sharp:\mathcal{O}_Y\longrightarrow f_*(\mathcal{O}_X). \tag{B.4}$$

Depending on the application, either $f^\sharp$ or $f_\sharp$ may be more useful. We choose to regard $f^\sharp$ as primary and write morphisms as $\underline{f}=(f,f^\sharp)$ rather than $(f,f_\sharp)$, because we find it convenient to work uniformly using pullbacks, rather than mixing pullbacks and pushforwards.

If $\underline{f}:\underline{X}\to\underline{Y}$ and $\underline{g}:\underline{Y}\to\underline{Z}$ are C^∞-scheme morphisms, the composition is

$$\underline{g}\circ\underline{f}=\bigl(g\circ f,(g\circ f)^\sharp\bigr)=\bigl(g\circ f,f^\sharp\circ f^{-1}(g^\sharp)\circ I_{f,g}(\mathcal{O}_Z)\bigr), \tag{B.5}$$

where $I_{f,g}(\mathcal{O}_Z):(g\circ f)^{-1}(\mathcal{O}_Z)\to f^{-1}(g^{-1}(\mathcal{O}_Z))$ is the canonical isomorphism. In terms of $f_\sharp:\mathcal{O}_Y\to f_*(\mathcal{O}_X)$, composition is

$$(g\circ f)_\sharp=g_*(f_\sharp)\circ g_\sharp:\mathcal{O}_Z\longrightarrow(g\circ f)_*(\mathcal{O}_X)=g_*\circ f_*(\mathcal{O}_X). \tag{B.6}$$

A *local C^∞-ringed space* $\underline{X}=(X,\mathcal{O}_X)$ is a C^∞-ringed space for which the stalks $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at x are C^∞-local rings for all $x\in X$. Since morphisms of C^∞-local rings are automatically local morphisms, morphisms of local C^∞-ringed spaces $(X,\mathcal{O}_X),(Y,\mathcal{O}_Y)$ are just morphisms of C^∞-ringed spaces, without any additional locality condition.

Write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces, and $\mathbf{LC^\infty RS}$ for the full subcategory of local C^∞-ringed spaces.

For brevity, we will use the notation that underlined upper case letters $\underline{X},\underline{Y},\underline{Z},\ldots$ represent C^∞-ringed spaces $(X,\mathcal{O}_X),(Y,\mathcal{O}_Y),(Z,\mathcal{O}_Z),\ldots$, and underlined lower case letters $\underline{f},\underline{g},\ldots$ represent morphisms of C^∞-ringed spaces $(f,f^\sharp),(g,g^\sharp),\ldots$. When we write '$x\in\underline{X}$' we mean that $\underline{X}=(X,\mathcal{O}_X)$ and $x\in X$. When we write '*$\underline{U}$ is open in $\underline{X}$*' we mean that $\underline{U}=(U,\mathcal{O}_U)$ and $\underline{X}=(X,\mathcal{O}_X)$ with $U\subseteq X$ an open set and $\mathcal{O}_U=\mathcal{O}_X|_U$.

Definition B.18. Write $\mathbf{C^\infty Rings}^{\mathrm{op}}$ for the opposite category of $\mathbf{C^\infty Rings}$. The *global sections functor* $\Gamma:\mathbf{LC^\infty RS}\to\mathbf{C^\infty Rings}^{\mathrm{op}}$ acts on objects $(X,\mathcal{O}_X)$

in $\mathbf{LC^\infty RS}$ by $\Gamma:(X,\mathcal{O}_X)\mapsto\mathcal{O}_X(X)$ and on morphisms $(f,f^\sharp):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ by $\Gamma:(f,f^\sharp)\mapsto f_\sharp(Y)$, for $f_\sharp:\mathcal{O}_X\to f_*(\mathcal{O}_Y)$ as in (B.4). As in [30, Th. 8] there is a *spectrum functor* Spec : $\mathbf{C^\infty Rings}^{\rm op}\to\mathbf{LC^\infty RS}$, defined explicitly in [56, Def. 4.12], which is a right adjoint to Γ, that is, for all $\mathfrak{C}\in\mathbf{C^\infty Rings}$ and $\underline{X}\in\mathbf{LC^\infty RS}$ there are functorial isomorphisms

$$\mathrm{Hom}_{\mathbf{C^\infty Rings}}(\mathfrak{C},\Gamma(\underline{X}))\cong\mathrm{Hom}_{\mathbf{LC^\infty RS}}(\underline{X},\mathrm{Spec}\,\mathfrak{C}).\tag{B.7}$$

For any C^∞-ring $\mathfrak{C}$ there is a natural morphism of C^∞-rings $\Phi_{\mathfrak{C}}:\mathfrak{C}\to\Gamma(\mathrm{Spec}\,\mathfrak{C})$ corresponding to $\underline{\mathrm{id}}_{\underline{X}}$ in (B.7) with $\underline{X}=\mathrm{Spec}\,\mathfrak{C}$. By [30, Th. 13], the restriction of Spec to $(\mathbf{C^\infty Rings^{fa}})^{\rm op}$ is full and faithful.

A local C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme* if it is isomorphic in $\mathbf{LC^\infty RS}$ to $\mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. We call $\underline{X}$ a *finitely presented*, or *fair*, affine C^∞-scheme if $X\cong\mathrm{Spec}\,\mathfrak{C}$ for $\mathfrak{C}$ that kind of C^∞-ring.

Let $\underline{X}=(X,\mathcal{O}_X)$ be a local C^∞-ringed space. We call $\underline{X}$ a *C^∞-scheme* if X can be covered by open sets $U\subseteq X$ such that $(U,\mathcal{O}_X|_U)$ is an affine C^∞-scheme. We call a C^∞-scheme $\underline{X}$ *locally fair*, or *locally finitely presented*, if X can be covered by open $U\subseteq X$ with $(U,\mathcal{O}_X|_U)$ a fair, or finitely presented, affine C^∞-scheme, respectively.

Write $\mathbf{C^\infty Sch^{lf}},\mathbf{C^\infty Sch^{lfp}},\mathbf{C^\infty Sch}$ for the full subcategories of locally fair, and locally finitely presented, and all, C^∞-schemes in $\mathbf{LC^\infty RS}$, respectively.

We call a C^∞-scheme $\underline{X}$ *separated, second countable, compact, locally compact,* or *paracompact*, if the underlying topological space X is Hausdorff, second countable, compact, locally compact, or paracompact, respectively. Write $\mathbf{C^\infty Sch^{lf}_{ssc}}$ for the full subcategory of separated, second countable, locally fair C^∞-schemes in $\mathbf{LC^\infty RS}$.

By [56, Prop.s 4.10, 4.11, 4.32, Cor. 4.18 & Th. 4.33], and deducing the $\mathbf{C^\infty Sch^{lf}_{ssc}}$ case in (b) from the $\mathbf{C^\infty Sch^{lf}}$ case, we have:

Theorem B.19. **(a)** *All finite limits exist in the category* $\mathbf{C^\infty RS}$.

(b) *The full subcategories* $\mathbf{C^\infty Sch},\mathbf{C^\infty Sch^{lfp}},\mathbf{C^\infty Sch^{lf}},\mathbf{C^\infty Sch^{lf}_{ssc}},\mathbf{LC^\infty RS}$ *in* $\mathbf{C^\infty RS}$ *are closed under all finite limits in* $\mathbf{C^\infty RS}$*. Hence, fibre products and all finite limits exist in each of these subcategories.*

(c) *If* $\mathfrak{C}$ *is a finitely generated C^∞-ring then* $\mathrm{Spec}\,\mathfrak{C}$ *is a fair affine C^∞-scheme.*

(d) *Let* $(X,\mathcal{O}_X)$ *be a locally finitely presented, locally fair, or general, C^∞-scheme, and $U\subseteq X$ be open. Then $(U,\mathcal{O}_X|_U)$ is also a locally finitely presented, or locally fair, or general, C^∞-scheme, respectively.*

In [56, Def. 4.34 & Prop. 4.35] we discuss *partitions of unity* on C^∞-schemes.

Definition B.20. Let $\underline{X}=(X,\mathcal{O}_X)$ be a C^∞-scheme. Consider a formal sum $\sum_{a\in A}c_a$, where A is an indexing set and $c_a\in\mathcal{O}_X(X)$ for $a\in A$. We say $\sum_{a\in A}c_a$ is a *locally finite sum on* $\underline{X}$ if X can be covered by open $U\subseteq X$ such that for all but finitely many $a\in A$ we have $\rho_{XU}(c_a)=0$ in $\mathcal{O}_X(U)$.

By the sheaf axioms for $\mathcal{O}_X$, if $\sum_{a\in A}c_a$ is a locally finite sum there exists a unique $c\in\mathcal{O}_X(X)$ such that for all open $U\subseteq X$ such that $\rho_{XU}(c_a)=0$

in $\mathcal{O}_X(U)$ for all but finitely many $a \in A$, we have $\rho_{XU}(c) = \sum_{a\in A} \rho_{XU}(c_a)$ in $\mathcal{O}_X(U)$, where the sum makes sense as there are only finitely many nonzero terms. We call c the *limit* of $\sum_{a\in A} c_a$, written $\sum_{a\in A} c_a = c$.

Let $c \in \mathcal{O}_X(X)$. Suppose $V_i \subseteq X$ is open and $\rho_{XV_i}(c) = 0 \in \mathcal{O}_X(V_i)$ for $i \in I$, and let $V = \bigcup_{i\in I} V_i$. Then $V \subseteq X$ is open, and $\rho_{XV}(c) = 0 \in \mathcal{O}_X(V)$ as $\mathcal{O}_X$ is a sheaf. Thus taking the union of all open $V \subseteq X$ with $\rho_{XV}(c) = 0$ gives a unique maximal open set $V_c \subseteq X$ such that $\rho_{XV_c}(c) = 0 \in \mathcal{O}_X(V_c)$. Define the *support* $\operatorname{supp} c$ of c to be $X \setminus V_c$, so that $\operatorname{supp} c$ is closed in X. If $U \subseteq X$ is open, we say that *c is supported in U* if $\operatorname{supp} c \subseteq U$.

Let $\{U_a : a \in A\}$ be an open cover of X. A *partition of unity on $\underline{X}$ subordinate to* $\{U_a : a \in A\}$ is $\{\eta_a : a \in A\}$ with $\eta_a \in \mathcal{O}_X(X)$ supported in U_a for $a \in A$, such that $\sum_{a\in A} \eta_a$ is a locally finite sum on $\underline{X}$ with $\sum_{a\in A} \eta_a = 1$.

Proposition B.21. *Suppose $\underline{X}$ is a separated, paracompact, locally fair C^∞-scheme, and $\{\underline{U}_a : a \in A\}$ an open cover of $\underline{X}$. Then there exists a partition of unity $\{\eta_a : a \in A\}$ on $\underline{X}$ subordinate to $\{\underline{U}_a : a \in A\}$.*

Here are some differences between ordinary schemes and C^∞-schemes:

Remark B.22. (i) If A is a ring or algebra, then points of the corresponding scheme $\operatorname{Spec} A$ are prime ideals in A. However, if $\mathfrak{C}$ is a C^∞-ring then (by definition) points of $\operatorname{Spec}\mathfrak{C}$ are maximal ideals in $\mathfrak{C}$ with residue field $\mathbb{R}$, or equivalently, $\mathbb{R}$-algebra morphisms $x : \mathfrak{C} \to \mathbb{R}$. This has the effect that if X is a manifold then points of $\operatorname{Spec} C^\infty(X)$ are just points of X.

(ii) In conventional algebraic geometry, affine schemes are a restrictive class. Central examples such as $\mathbb{CP}^n$ are not affine, and affine schemes are not closed under open subsets, so that $\mathbb{C}^2$ is affine but $\mathbb{C}^2 \setminus \{0\}$ is not. In contrast, affine C^∞-schemes are already general enough for many purposes. For example:

- All manifolds are affine C^∞-schemes.
- Open C^∞-subschemes of fair affine C^∞-schemes are fair and affine.
- Separated, second countable, locally fair C^∞-schemes are affine.

Affine C^∞-schemes are always separated (Hausdorff), so we need general C^∞-schemes to include non-Hausdorff behaviour.

(iii) In conventional algebraic geometry the Zariski topology is too coarse for many purposes, so one has to introduce the étale topology. In C^∞-algebraic geometry there is no need for this, as affine C^∞-schemes are Hausdorff.

(iv) Even very basic C^∞-rings such as $C^\infty(\mathbb{R}^n)$ for $n > 0$ are not noetherian as $\mathbb{R}$-algebras. So C^∞-schemes should be compared to non-noetherian schemes in conventional algebraic geometry.

B.5 Manifolds as C^∞-rings and C^∞-schemes

Here is [56, Prop. 3.1]:

Proposition B.23. **(a)** *If X is a manifold without boundary then the C^∞-ring $C^\infty(X)$ of Example* B.2 *is finitely presented.*

(b) *If X is a manifold with boundary, or with corners, and $\partial X \neq \emptyset$, then the C^∞-ring $C^\infty(X)$ of Example* B.2 *is fair, but is not finitely presented.*

Definition B.24. Define functors

$$\begin{aligned} F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}} &: \mathbf{Man} \longrightarrow (\mathbf{C^\infty Rings^{fp}})^{\mathrm{op}} \subset (\mathbf{C^\infty Rings})^{\mathrm{op}}, \\ F_{\mathbf{Man^b}}^{\mathbf{C^\infty Rings}} &: \mathbf{Man^b} \longrightarrow (\mathbf{C^\infty Rings^{fa}})^{\mathrm{op}} \subset (\mathbf{C^\infty Rings})^{\mathrm{op}}, \\ F_{\mathbf{Man^c}}^{\mathbf{C^\infty Rings}} &: \mathbf{Man^c} \longrightarrow (\mathbf{C^\infty Rings^{fa}})^{\mathrm{op}} \subset (\mathbf{C^\infty Rings})^{\mathrm{op}}, \end{aligned}$$

as follows. On objects $F_{\mathbf{Man^*}}^{\mathbf{C^\infty Rings}}$ map $X \mapsto C^\infty(X)$, for $C^\infty(X)$ as in Example B.2. On morphisms, if $f : X \to Y$ is smooth then $f^* : C^\infty(Y) \to C^\infty(X)$ mapping $c \mapsto c \circ f$ is a morphism of C^∞-rings, so that $f^* : C^\infty(X) \to C^\infty(Y)$ is a morphism in $(\mathbf{C^\infty Rings})^{\mathrm{op}}$, and $F_{\mathbf{Man^*}}^{\mathbf{C^\infty Rings}}$ map $f \mapsto f^*$. Define functors

$$\begin{aligned} F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}} &: \mathbf{Man} \longrightarrow \mathbf{C^\infty Sch^{lfp}} \subset \mathbf{C^\infty Sch}, \\ F_{\mathbf{Man^b}}^{\mathbf{C^\infty Sch}} &: \mathbf{Man^b} \longrightarrow \mathbf{C^\infty Sch^{lf}} \subset \mathbf{C^\infty Sch}, \\ F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}} &: \mathbf{Man^c} \longrightarrow \mathbf{C^\infty Sch^{lf}} \subset \mathbf{C^\infty Sch}, \end{aligned}$$

by $F_{\mathbf{Man^*}}^{\mathbf{C^\infty Sch}} = \mathrm{Spec} \circ F_{\mathbf{Man^*}}^{\mathbf{C^\infty Rings}}$.

If $X, Y, \ldots$ are manifolds, or $f, g, \ldots$ are (weakly) smooth maps, we will often use $\underline{X}, \underline{Y}, \ldots, \underline{f}, \underline{g}, \ldots$ to denote $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(X, Y, \ldots, f, g, \ldots)$. So for instance we will write $\underline{\mathbb{R}}^n$ and $\underline{[0,\infty)}$ for $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(\mathbb{R}^n)$ and $F_{\mathbf{Man^b}}^{\mathbf{C^\infty Sch}}\bigl([0,\infty)\bigr)$.

We can describe the C^∞-ringed space $\underline{X} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(X)$ for a manifold X.

Example B.25. Let X be a manifold, which may have boundary or corners. Define a C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$ to have topological space X and $\mathcal{O}_X(U) = C^\infty(U)$ for each open $U \subseteq X$, where $C^\infty(U)$ is the C^∞-ring of smooth maps $c : U \to \mathbb{R}$, and if $V \subseteq U \subseteq X$ are open define $\rho_{UV} : C^\infty(U) \to C^\infty(V)$ by $\rho_{UV} : c \mapsto c|_V$. Then $\underline{X} = (X, \mathcal{O}_X)$ is a local C^∞-ringed space. It is canonically isomorphic to $\mathrm{Spec}\, C^\infty(X) = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(X)$, and so is an affine C^∞-scheme.

As in §5.2, for manifolds with boundary or corners X, Y we have two classes of morphisms $f : X \to Y$, called *weakly smooth* and *smooth* maps. If f is only weakly smooth then $f^* : C^\infty(Y) \to C^\infty(X)$ in Definition B.24 is still a morphism of C^∞-rings, so $\mathrm{Spec}\, f^* : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes. By [56, Prop. 3.3 & Th. 4.16] we have:

Proposition B.26. *Let X, Y be manifolds with corners, and $\underline{X}, \underline{Y}$ the associated C^∞-schemes. Then the map $f \mapsto \underline{f} = \mathrm{Spec}(f^*)$ from weakly smooth maps $f : X \to Y$ to morphisms of C^∞-schemes $\underline{f} : \underline{X} \to \underline{Y}$ is a* 1-1 *correspondence.*

As for manifolds with boundary or corners the smooth maps are a proper subset of the weakly smooth maps, we deduce [56, Cor. 4.21]:

Corollary B.27. *The functor* $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}:\mathbf{Man}\rightarrow\mathbf{C^\infty Sch}$ *is full and faithful. However, the functors* $F_{\mathbf{Man^b}}^{\mathbf{C^\infty Sch}}:\mathbf{Man^b}\rightarrow\mathbf{C^\infty Sch}$ *and* $F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}:\mathbf{Man^c}\rightarrow\mathbf{C^\infty Sch}$ *are faithful, but not full.*

From [56, Cor. 4.21] we have:

Theorem B.28. *The functors* $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}},F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}$ *take transverse fibre products in* $\mathbf{Man},\mathbf{Man^c}$, *in the sense of §5.6, to fibre products in* $\mathbf{C^\infty Sch}$.

B.6 Modules over C^∞-rings, and cotangent modules

In [56, §5] we discuss *modules* over C^∞-rings.

Definition B.29. Let $\mathfrak{C}$ be a C^∞-ring. A *module* (M,μ) *over* $\mathfrak{C}$, or $\mathfrak{C}$-*module*, is a module over $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra as in Definition B.5. That is, M is a vector space over $\mathbb{R}$ equipped with a bilinear map $\mu:\mathfrak{C}\times M\rightarrow M$, satisfying $\mu(c_1\cdot c_2,m)=\mu\bigl(c_1,\mu(c_2,m)\bigr)$ and $\mu(1,m)=m$ for all $c_1,c_2\in\mathfrak{C}$ and $m\in M$. A *morphism* $\alpha:(M,\mu)\rightarrow(M',\mu')$ of $\mathfrak{C}$-modules $(M,\mu),(M',\mu')$ is a linear map $\alpha:M\rightarrow M'$ such that $\alpha\circ\mu=\mu'\circ(\mathrm{id}_{\mathfrak{C}}\times\alpha):\mathfrak{C}\times M\rightarrow M'$. Then $\mathfrak{C}$-modules form an *abelian category*, which we write as $\mathfrak{C}$-mod. Often we write M for the $\mathfrak{C}$-module, leaving μ implicit.

Now $\mathfrak{C}\otimes_{\mathbb{R}}V$ is a $\mathfrak{C}$-module for any real vector space V. A $\mathfrak{C}$-module (M,μ) is called *finitely presented* if there is an exact sequence $(\mathfrak{C}\otimes_{\mathbb{R}}\mathbb{R}^m,\mu_{\mathbb{R}^m})\rightarrow(\mathfrak{C}\otimes_{\mathbb{R}}\mathbb{R}^n,\mu_{\mathbb{R}^n})\rightarrow(M,\mu)\rightarrow 0$ in $\mathfrak{C}$-mod for some $m,n\geqslant 0$. We write $\mathfrak{C}\text{-mod}^{\mathrm{fp}}$ for the full subcategory of finitely presented $\mathfrak{C}$-modules in $\mathfrak{C}$-mod. Then $\mathfrak{C}\text{-mod}^{\mathrm{fp}}$ is closed under cokernels and extensions in $\mathfrak{C}$-mod. But it may not be closed under kernels, so $\mathfrak{C}\text{-mod}^{\mathrm{fp}}$ may not be an abelian category.

Let $\phi:\mathfrak{C}\rightarrow\mathfrak{D}$ be a morphism of C^∞-rings. If (M,μ) is a $\mathfrak{C}$-module then $\phi_*(M,\mu)=\bigl(M\otimes_{\mathfrak{C}}\mathfrak{D},\mu_{\mathfrak{D}}\bigr)$ is a $\mathfrak{D}$-module, where $\mu_{\mathfrak{D}}=\mu_{\mathfrak{C}}\times\mathrm{id}_{\mathfrak{D}}:\mathfrak{D}\times M\otimes_{\mathfrak{C}}\mathfrak{D}\cong\mathfrak{C}\otimes_{\mathfrak{C}}\mathfrak{D}\times M\otimes_{\mathfrak{C}}\mathfrak{D}\rightarrow M\otimes_{\mathfrak{C}}\mathfrak{D}$, and this induces a functor $\phi_*:\mathfrak{C}\text{-mod}\rightarrow\mathfrak{D}\text{-mod}$, which maps $\mathfrak{C}\text{-mod}^{\mathrm{fp}}\rightarrow\mathfrak{D}\text{-mod}^{\mathrm{fp}}$.

Vector bundles E over manifolds X give examples of modules over $C^\infty(X)$.

Example B.30. Let X be a manifold, which may have boundary or corners. Let $E\rightarrow X$ be a vector bundle, and $C^\infty(E)$ the vector space of smooth sections e of E. Define $\mu_E:C^\infty(X)\times C^\infty(E)\rightarrow C^\infty(E)$ by $\mu_E(c,e)=c\cdot e$. Then $\bigl(C^\infty(E),\mu_E\bigr)$ is a $C^\infty(X)$-module. One can show it is finitely presented.

Let $E,F\rightarrow X$ be vector bundles over X and $\lambda:E\rightarrow F$ a morphism of vector bundles. Then $\lambda_*:C^\infty(E)\rightarrow C^\infty(F)$ defined by $\lambda_*:e\mapsto\lambda\circ e$ is a morphism of $C^\infty(X)$-modules.

Now let X,Y be manifolds and $f:X\rightarrow Y$ a (weakly) smooth map. Then $f^*:C^\infty(Y)\rightarrow C^\infty(X)$ is a morphism of C^∞-rings. If $E\rightarrow Y$ is a vector bundle over Y, then $f^*(E)$ is a vector bundle over X. Under the functor $(f^*)_*:C^\infty(Y)\text{-mod}\rightarrow C^\infty(X)\text{-mod}$ of Definition B.29, we see that $(f^*)_*\bigl(C^\infty(E)\bigr)=C^\infty(E)\otimes_{C^\infty(Y)}C^\infty(X)$ is isomorphic as a $C^\infty(X)$-module to $C^\infty\bigl(f^*(E)\bigr)$.

In [56, §5.3] we define the *cotangent module* $(\Omega_{\mathfrak{C}},\mu_{\mathfrak{C}})$ of a C^∞-ring $\mathfrak{C}$.

Definition B.31. Suppose $\mathfrak{C}$ is a C^∞-ring, and (M,μ) a $\mathfrak{C}$-module. A C^∞-*derivation* is an $\mathbb{R}$-linear map $\mathrm{d}:\mathfrak{C}\rightarrow M$ such that whenever $f:\mathbb{R}^n\rightarrow\mathbb{R}$ is a smooth map and $c_1,\ldots,c_n\in\mathfrak{C}$, we have

$$\mathrm{d}\Phi_f(c_1,\ldots,c_n)=\textstyle\sum_{i=1}^n\mu\bigl(\Phi_{\frac{\partial f}{\partial x_i}}(c_1,\ldots,c_n),\mathrm{d}c_i\bigr). \tag{B.8}$$

Note that d is *not* a morphism of $\mathfrak{C}$-modules. We call such a pair $(M,\mu),\mathrm{d}$ a *cotangent module* for $\mathfrak{C}$ if it has the universal property that for any $\mathfrak{C}$-module (M',μ') and C^∞-derivation $\mathrm{d}':\mathfrak{C}\rightarrow M'$, there exists a unique morphism of $\mathfrak{C}$-modules $\phi:(M,\mu)\rightarrow(M',\mu')$ with $\mathrm{d}'=\phi\circ\mathrm{d}$.

There is a natural construction for a cotangent module: we take (M,μ) to be the quotient of the free $\mathfrak{C}$-module with basis of symbols $\mathrm{d}c$ for $c\in\mathfrak{C}$ by the $\mathfrak{C}$-submodule spanned by all expressions of the form $\mathrm{d}\Phi_f(c_1,\ldots,c_n)-\sum_{i=1}^n\mu\bigl(\Phi_{\frac{\partial f}{\partial x_i}}(c_1,\ldots,c_n),\mathrm{d}c_i\bigr)$ for $f:\mathbb{R}^n\rightarrow\mathbb{R}$ smooth and $c_1,\ldots,c_n\in\mathfrak{C}$. Thus cotangent modules exist, and are unique up to unique isomorphism. When we speak of 'the' cotangent module, we mean that constructed above. We will write $(\Omega_{\mathfrak{C}},\mu_{\mathfrak{C}})$, $\mathrm{d}_{\mathfrak{C}}:\mathfrak{C}\rightarrow\Omega_{\mathfrak{C}}$ for this cotangent module for $\mathfrak{C}$.

Let $\mathfrak{C},\mathfrak{D}$ be C^∞-rings with cotangent modules $(\Omega_{\mathfrak{C}},\mu_{\mathfrak{C}}),\mathrm{d}_{\mathfrak{C}}$, $(\Omega_{\mathfrak{D}},\mu_{\mathfrak{D}}),\mathrm{d}_{\mathfrak{D}}$, and $\phi:\mathfrak{C}\rightarrow\mathfrak{D}$ be a morphism of C^∞-rings. Then the action $\mu_{\mathfrak{C}}\circ(\phi\times\mathrm{id}_{\Omega_{\mathfrak{D}}})$ makes $\Omega_{\mathfrak{D}}$ into a $\mathfrak{C}$-module, and $\mathrm{d}_{\mathfrak{D}}\circ\phi:\mathfrak{C}\rightarrow\Omega_{\mathfrak{D}}$ is a C^∞-derivation. Thus by the universal property of $\Omega_{\mathfrak{C}}$, there exists a unique morphism of $\mathfrak{C}$-modules $\Omega_\phi:\Omega_{\mathfrak{C}}\rightarrow\Omega_{\mathfrak{D}}$ with $\mathrm{d}_{\mathfrak{D}}\circ\phi=\Omega_\phi\circ\mathrm{d}_{\mathfrak{C}}$. This then induces a morphism of $\mathfrak{D}$-modules $(\Omega_\phi)_*:\Omega_{\mathfrak{C}}\otimes_{\mathfrak{C}}\mathfrak{D}\rightarrow\Omega_{\mathfrak{D}}$ with $(\Omega_\phi)_*\circ(\mathrm{d}_{\mathfrak{C}}\otimes\mathrm{id}_{\mathfrak{D}})=\mathrm{d}_{\mathfrak{D}}$ as a composition $\mathfrak{D}=\mathfrak{C}\otimes_{\mathfrak{C}}\mathfrak{D}\rightarrow\Omega_{\mathfrak{C}}\otimes_{\mathfrak{C}}\mathfrak{D}\rightarrow\Omega_{\mathfrak{D}}$. If $\phi:\mathfrak{C}\rightarrow\mathfrak{D}$, $\psi:\mathfrak{D}\rightarrow\mathfrak{E}$ are morphisms of C^∞-rings then $\Omega_{\psi\circ\phi}=\Omega_\psi\circ\Omega_\phi:\Omega_{\mathfrak{C}}\rightarrow\Omega_{\mathfrak{E}}$.

Example B.32. Let X be a manifold. Then the cotangent bundle T^*X is a vector bundle over X, so as in Example B.30 it yields a $C^\infty(X)$-module $C^\infty(T^*X)$. The exterior derivative $\mathrm{d}:C^\infty(X)\rightarrow C^\infty(T^*X)$, $\mathrm{d}:c\mapsto\mathrm{d}c$ is then a C^∞-derivation, since equation (B.8) follows from

$$\mathrm{d}\bigl(f(c_1,\ldots,c_n)\bigr)=\textstyle\sum_{i=1}^n\frac{\partial f}{\partial x_i}(c_1,\ldots,c_n)\,\mathrm{d}c_n$$

for $f:\mathbb{R}^n\rightarrow\mathbb{R}$ smooth and $c_1,\ldots,c_n\in C^\infty(X)$, which holds by the chain rule. It is easy to show that $\bigl(C^\infty(T^*X),\mu_{T^*X}\bigr),\mathrm{d}$ have the universal property in Definition B.31, and so form a cotangent module for $C^\infty(X)$.

Now let X,Y be manifolds, and $f:X\rightarrow Y$ a (weakly) smooth map. Then $f^*(TY),TX$ are vector bundles over X, and the derivative of f is a vector bundle morphism $\mathrm{d}f:TX\rightarrow f^*(TY)$. The dual of this morphism is $(\mathrm{d}f)^*:f^*(T^*Y)\rightarrow T^*X$. This induces a morphism of $C^\infty(X)$-modules $((\mathrm{d}f)^*)_*:C^\infty\bigl(f^*(T^*Y)\bigr)\rightarrow C^\infty(T^*X)$. This $((\mathrm{d}f)^*)_*$ is identified with $(\Omega_{f^*})_*$ under the natural isomorphism $C^\infty\bigl(f^*(T^*Y)\bigr)\cong C^\infty(T^*Y)\otimes_{C^\infty(Y)}C^\infty(X)$, where we identify $C^\infty(Y),C^\infty(X),f^*$ with $\mathfrak{C},\mathfrak{D},\phi$ in Definition B.31.

Definition B.31 abstracts the notion of cotangent bundle of a manifold in a way that makes sense for any C^∞-ring. From [56, Th.s 5.13 & 5.16] we have:

Theorem B.33. **(a)** *Suppose $\mathfrak{C}$ is a finitely presented C^∞-ring. Then $\Omega_{\mathfrak{C}}$ is a finitely presented $\mathfrak{C}$-module.*

(b) *Suppose we are given a pushout diagram of finitely generated C^∞-rings:*

$$\begin{tikzcd}
\mathfrak{C} \arrow[r, "\beta"] \arrow[d, "\alpha"] & \mathfrak{E} \arrow[d, "\delta"] \\
\mathfrak{D} \arrow[r, "\gamma"] & \mathfrak{F},
\end{tikzcd}$$

so that $\mathfrak{F} = \mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$. Then the following sequence of $\mathfrak{F}$-modules is exact:

$$\Omega_{\mathfrak{C}} \otimes_{\mu_{\mathfrak{C}},\mathfrak{C},\gamma\circ\alpha} \mathfrak{F} \xrightarrow{\substack{(\Omega_\alpha)_* \oplus \\ -(\Omega_\beta)_*}} \begin{array}{c} \Omega_{\mathfrak{D}} \otimes_{\mu_{\mathfrak{D}},\mathfrak{D},\gamma} \mathfrak{F} \oplus \\ \Omega_{\mathfrak{E}} \otimes_{\mu_{\mathfrak{E}},\mathfrak{E},\delta} \mathfrak{F} \end{array} \xrightarrow{(\Omega_\gamma)_* \oplus (\Omega_\delta)_*} \Omega_{\mathfrak{F}} \longrightarrow 0.$$

Here $(\Omega_\alpha)_* : \Omega_{\mathfrak{C}} \otimes_{\mu_{\mathfrak{C}},\mathfrak{C},\gamma\circ\alpha} \mathfrak{F} \to \Omega_{\mathfrak{D}} \otimes_{\mu_{\mathfrak{D}},\mathfrak{D},\gamma} \mathfrak{F}$ is induced by $\Omega_\alpha : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{D}}$, and so on.

B.7 Quasicoherent sheaves on C^∞-schemes

In [56, §6] we discuss *sheaves of modules* on C^∞-schemes.

Definition B.34. Let $(X, \mathcal{O}_X)$ be a C^∞-ringed space. A *sheaf of $\mathcal{O}_X$-modules*, or simply an *$\mathcal{O}_X$-module*, $\mathcal{E}$ on X assigns a module $\mathcal{E}(U) = (M_U, \mu_U)$ over the C^∞-ring $\mathcal{O}_X(U)$ for each open set $U \subseteq X$, and a linear map $\mathcal{E}_{UV} : M_U \to M_V$ for each inclusion of open sets $V \subseteq U \subseteq X$, such that the following commutes

$$\begin{tikzcd}
\mathcal{O}_X(U) \times M_U \arrow[r, "\mu_U"] \arrow[d, "\rho_{UV} \times \mathcal{E}_{UV}"] & M_U \arrow[d, "\mathcal{E}_{UV}"] \\
\mathcal{O}_X(V) \times M_V \arrow[r, "\mu_V"] & M_V,
\end{tikzcd} \tag{B.9}$$

and all this data $\mathcal{E}(U), \mathcal{E}_{UV}$ satisfies the sheaf axioms in Definition B.11.

A *morphism of sheaves of $\mathcal{O}_X$-modules* $\phi : \mathcal{E} \to \mathcal{F}$ assigns a morphism of $\mathcal{O}_X(U)$-modules $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ for each open set $U \subseteq X$, such that $\phi(V) \circ \mathcal{E}_{UV} = \mathcal{F}_{UV} \circ \phi(U)$ for each inclusion of open sets $V \subseteq U \subseteq X$. Then $\mathcal{O}_X$-modules form an abelian category, which we write as $\mathcal{O}_X$-mod.

Let $\underline{f} = (f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of C^∞-ringed spaces, and $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Define the *pullback* $\underline{f}^*(\mathcal{E})$ by $\underline{f}^*(\mathcal{E}) = f^{-1}(\mathcal{E}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, where $f^{-1}(\mathcal{E})$ is as in Definition B.15, a sheaf of modules over the sheaf of C^∞-rings $f^{-1}(\mathcal{O}_Y)$ on X, and the tensor product uses the morphism $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of $\mathcal{O}_Y$-modules we have an induced morphism of $\mathcal{O}_X$-modules $\underline{f}^*(\phi) = f^{-1}(\phi) \otimes \mathrm{id}_{\mathcal{O}_X} : \underline{f}^*(\mathcal{E}) \to \underline{f}^*(\mathcal{F})$. Then $\underline{f}^* : \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod}$ is a right exact functor between abelian categories.

Pullbacks $\underline{f}^*(\mathcal{E})$ are a kind of fibre product, and may be characterized by a universal property. So they should be regarded as being *unique up to canonical isomorphism*, rather than unique. We choose $\underline{f}^*(\mathcal{E})$ for all $\underline{f}, \mathcal{E}$, and so speak of 'the' pullback $\underline{f}^*(\mathcal{E})$. However, it may not be possible to make these choices functorial in $\underline{f}$. That is, if $\underline{f} : \underline{X} \to \underline{Y}$, $\underline{g} : \underline{Y} \to \underline{Z}$ are morphisms and $\mathcal{E} \in$

$\mathcal{O}_Z$-mod then $(\underline{g}\circ\underline{f})^*(\mathcal{E})$, $\underline{f}^*(\underline{g}^*(\mathcal{E}))$ are canonically isomorphic in $\mathcal{O}_X$-mod, but may not be equal. As in Remark B.16(b), we will write $I_{\underline{f},\underline{g}}(\mathcal{E}) : (\underline{g}\circ\underline{f})^*(\mathcal{E}) \to \underline{f}^*(\underline{g}^*(\mathcal{E}))$ for these canonical isomorphisms. Then $I_{\underline{f},\underline{g}} : (\underline{g}\circ\underline{f})^* \Rightarrow \underline{f}^*\circ\underline{g}^*$ is a natural isomorphism of functors.

When $\underline{f}$ is the identity $\underline{\mathrm{id}}_{\underline{X}} : \underline{X} \to \underline{X}$ and $\mathcal{E}\in\mathcal{O}_X$-mod we do not require $\underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E}) = \mathcal{E}$, but as $\mathcal{E}$ is a possible pullback for $\underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E})$ there is a canonical isomorphism $\delta_{\underline{X}}(\mathcal{E}) : \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E}) \to \mathcal{E}$, and then $\delta_{\underline{X}} : \underline{\mathrm{id}}_{\underline{X}}^* \Rightarrow \mathrm{id}_{\mathcal{O}_X\text{-mod}}$ is a natural isomorphism of functors.

Definition B.35. Definition B.18 discussed the spectrum $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{LC^\infty RS}$. As in [56, §6.2] this has a counterpart for modules: if $\mathfrak{C}$ is a C^∞-ring and $(X,\mathcal{O}_X) = \mathrm{Spec}\,\mathfrak{C}$ we can define a functor $\mathrm{MSpec} : \mathfrak{C}\text{-mod} \to \mathcal{O}_X\text{-mod}$.

Suppose now that $\mathfrak{C}$ is a fair C^∞-ring. As Spec is full and faithful on fair C^∞-rings, the morphism of C^∞-rings $\Phi_{\mathfrak{C}} : \mathfrak{C} \to \Gamma(\mathrm{Spec}\,\mathfrak{C})$ in Definition B.18 is an isomorphism. Define the *global sections functor* $\Gamma : \mathcal{O}_X\text{-mod} \to \mathfrak{C}\text{-mod}$ on objects by $\Gamma : \mathcal{E} \mapsto \mathcal{E}(X)$, where the $\mathcal{O}_X(X)$-module $\mathcal{E}(X)$ is regarded as a $\mathfrak{C}$-module using $\Phi_{\mathfrak{C}}^{-1}$, and on morphisms $\alpha : \mathcal{E}\to\mathcal{F}$ in $\mathcal{O}_X$-mod by $\Gamma : \alpha \mapsto \alpha(X)$. Then Γ is a *right adjoint* to MSpec, that is, as in (B.7) for all $M\in\mathfrak{C}$-mod and $\mathcal{E}\in\mathcal{O}_X$-mod there are functorial isomorphisms

$$\mathrm{Hom}_{\mathfrak{C}\text{-mod}}(M,\Gamma(\mathcal{E})) \cong \mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\mathrm{MSpec}\,M,\mathcal{E}). \tag{B.10}$$

Taking $\mathcal{E} = \mathrm{MSpec}\,M$, we obtain a natural morphism of $\mathfrak{C}$-modules $\Phi_M : M \to \Gamma(\mathrm{MSpec}\,M)$ corresponding to $\mathrm{id}_{\mathrm{MSpec}\,M}$ in (B.10). A $\mathfrak{C}$-module M is called *complete* if Φ_M is an isomorphism. Write $\mathfrak{C}\text{-mod}^{\mathrm{co}}$ for the full abelian subcategory of complete $\mathfrak{C}$-modules in $\mathfrak{C}$-mod. Then $\mathrm{MSpec}\,|_{\mathfrak{C}\text{-mod}^{\mathrm{co}}} : \mathfrak{C}\text{-mod}^{\mathrm{co}} \to \mathcal{O}_X\text{-mod}$ is an equivalence of categories.

Let $\underline{X} = (X,\mathcal{O}_X)$ be a C^∞-scheme, and $\mathcal{E}$ a sheaf of $\mathcal{O}_X$-modules. We call $\mathcal{E}$ *quasicoherent* if X can be covered by open subsets U with $(U,\mathcal{O}_X|_U)\cong\mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$, and under this identification $\mathcal{E}|_U \cong \mathrm{MSpec}\,M$ for some $\mathfrak{C}$-module M. We call $\mathcal{E}$ *coherent* if we can take these $\mathfrak{C}$-modules M to be finitely presented. We call $\mathcal{E}$ a *vector bundle of rank* $n\geqslant 0$ if X may be covered by open U such that $\mathcal{E}|_U \cong \mathcal{O}_X|_U\otimes_{\mathbb{R}}\mathbb{R}^n$. Vector bundles are coherent sheaves. Write $\mathrm{qcoh}(\underline{X}), \mathrm{coh}(\underline{X})$, and $\mathrm{vect}(\underline{X})$ for the full subcategories of quasicoherent sheaves, coherent sheaves, and vector bundles in $\mathcal{O}_X$-mod, respectively.

The next theorem comes from [56, Cor. 6.11 & Prop. 6.12]. In part (a), the reason $\mathrm{coh}(\underline{X})$ is not closed under kernels is that the C^∞-rings we are interested in are generally *not noetherian* as commutative $\mathbb{R}$-algebras, and this causes problems with coherence; in conventional algebraic geometry, one usually only considers coherent sheaves over noetherian schemes.

Theorem B.36. (a) *Let* $\underline{X}$ *be a* C^∞*-scheme. Then* $\mathrm{qcoh}(\underline{X})$ *is closed under kernels, cokernels and extensions in* $\mathcal{O}_X$-mod, *so it is an abelian category. Also* $\mathrm{coh}(\underline{X})$ *is closed under cokernels and extensions in* $\mathcal{O}_X$-mod, *but may not be closed under kernels in* $\mathcal{O}_X$-mod, *so* $\mathrm{coh}(\underline{X})$ *may not be an abelian category.*

(b) *Suppose $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes. Then pullback $\underline{f}^*$: $\mathcal{O}_Y$-mod $\to \mathcal{O}_X$-mod maps* $\mathrm{qcoh}(\underline{Y}) \to \mathrm{qcoh}(\underline{X})$ *and* $\mathrm{coh}(\underline{Y}) \to \mathrm{coh}(\underline{X})$ *and* $\mathrm{vect}(\underline{Y}) \to \mathrm{vect}(\underline{X})$. *Also $\underline{f}^* : \mathrm{qcoh}(\underline{Y}) \to \mathrm{qcoh}(\underline{X})$ is a right exact functor.*

(c) *Let $\underline{X}$ be a locally fair C^∞-scheme. Then every $\mathcal{O}_X$-module $\mathcal{E}$ on $\underline{X}$ is quasicoherent, that is,* $\mathrm{qcoh}(\underline{X}) = \mathcal{O}_X$-mod.

Let $\underline{X}$ be a separated, paracompact, locally fair C^∞-scheme. Then partitions of unity exist on $\underline{X}$ subordinate to any open cover by Proposition B.21. As in [56, §6.3], this shows that quasicoherent sheaves $\mathcal{E}$ on $\underline{X}$ are *fine*, in the sense of Godement [35, §II.3.7], which implies that their cohomology groups $H^i(\mathcal{E})$ are zero for all $i > 0$. In [56, Prop. 6.13] we deduce an exactness property for sections of quasicoherent sheaves on $\underline{X}$:

Proposition B.37. *Suppose $\underline{X} = (X, \mathcal{O}_X)$ is a separated, paracompact, locally fair C^∞-scheme, and $\cdots \to \mathcal{E}^i \xrightarrow{\phi^i} \mathcal{E}^{i+1} \xrightarrow{\phi^{i+1}} \mathcal{E}^{i+2} \to \cdots$ an exact sequence in* $\mathrm{qcoh}(\underline{X})$. *Then $\cdots \to \mathcal{E}^i(U) \xrightarrow{\phi^i(U)} \mathcal{E}^{i+1}(U) \xrightarrow{\phi^{i+1}(U)} \mathcal{E}^{i+2}(U) \to \cdots$ is an exact sequence of $\mathcal{O}_X(U)$-modules for each open $U \subseteq X$.*

We define cotangent sheaves, the sheaf version of cotangent modules in §B.6.

Definition B.38. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-ringed space. Define $\mathcal{P}T^*\underline{X}$ to associate to each open $U \subseteq X$ the cotangent module $(\Omega_{\mathcal{O}_X(U)}, \mu_{\mathcal{O}_X(U)})$ of Definition B.31, regarded as a module over the C^∞-ring $\mathcal{O}_X(U)$, and to each inclusion of open sets $V \subseteq U \subseteq X$ the morphism of $\mathcal{O}_X(U)$-modules $\Omega_{\rho_{UV}}$: $\Omega_{\mathcal{O}_X(U)} \to \Omega_{\mathcal{O}_X(V)}$ associated to the morphism of C^∞-rings $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. Then $\mathcal{P}T^*\underline{X}$ is a presheaf of $\mathcal{O}_X$-modules on $\underline{X}$. Define the *cotangent sheaf $T^*\underline{X}$ of $\underline{X}$* to be the sheaf of $\mathcal{O}_X$-modules associated to $\mathcal{P}T^*\underline{X}$.

If $U \subseteq X$ is open then we have an equality of sheaves of $\mathcal{O}_X|_U$-modules

$$T^*(U, \mathcal{O}_X|_U) = T^*\underline{X}|_U.$$

Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of C^∞-schemes. Then by Definition B.34, $\underline{f}^*\bigl(T^*\underline{Y}\bigr) = f^{-1}(T^*\underline{Y}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, where $T^*\underline{Y}$ is the sheafification of the presheaf $V \mapsto \Omega_{\mathcal{O}_Y(V)}$, and $f^{-1}(T^*\underline{Y})$ the sheafification of the presheaf $U \mapsto \lim_{V \supseteq f(U)} (T^*\underline{Y})(V)$, and $f^{-1}(\mathcal{O}_Y)$ the sheafification of the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathcal{O}_Y(V)$. The three sheafifications combine into one, so that $\underline{f}^*\bigl(T^*\underline{Y}\bigr)$ is the sheafification of the presheaf $\mathcal{P}(\underline{f}^*(T^*\underline{Y}))$ acting by

$$U \longmapsto \mathcal{P}(\underline{f}^*(T^*\underline{Y}))(U) = \lim_{V \supseteq f(U)} \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U).$$

Define a morphism of presheaves $\mathcal{P}\Omega_{\underline{f}} : \mathcal{P}(\underline{f}^*(T^*\underline{Y})) \to \mathcal{P}T^*\underline{X}$ on X by

$$(\mathcal{P}\Omega_{\underline{f}})(U) = \lim_{V \supseteq f(U)} (\Omega_{\rho_{f^{-1}(V)\,U} \circ f_\sharp(V)})_*,$$

where $(\Omega_{\rho_{f^{-1}(V)\,U} \circ f_\sharp(V)})_* : \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) \to \Omega_{\mathcal{O}_X(U)} = (\mathcal{P}T^*\underline{X})(U)$ is constructed as in Definition B.31 from the C^∞-ring morphisms $f_\sharp(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ from $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ corresponding to $f^\sharp$ in $\underline{f}$ as in (B.4), and $\rho_{f^{-1}(V)\,U} : \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_X(U)$ in $\mathcal{O}_X$. Define $\Omega_{\underline{f}} : \underline{f}^*\bigl(T^*\underline{Y}\bigr) \to T^*\underline{X}$ to be the induced morphism of the associated sheaves.

Here [56, Th.s 6.16 & 6.17] are some properties of cotangent sheaves.

Theorem B.39. **(a)** *Suppose X is an n-manifold, which may have boundary or corners, and $\underline{X} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(X)$ in the notation of* §B.5. *Then $T^*\underline{X}$ is a rank n vector bundle on $\underline{X}$, with $(T^*\underline{X})(U) \cong C^\infty(T^*X|_U)$ for all open $U \subseteq X$.*

(b) *Let $\underline{f} : \underline{X} \to \underline{Y}$ and $\underline{g} : \underline{Y} \to \underline{Z}$ be morphisms of C^∞-schemes. Then*

$$\Omega_{\underline{g}\circ\underline{f}} = \Omega_{\underline{f}} \circ \underline{f}^*(\Omega_{\underline{g}}) \circ I_{\underline{f},\underline{g}}(T^*\underline{Z})$$

*as morphisms $(\underline{g}\circ\underline{f})^*(T^*\underline{Z}) \to T^*\underline{X}$ in $\mathcal{O}_X$-*mod. *Here $\Omega_{\underline{g}} : \underline{g}^*(T^*\underline{Z}) \to T^*\underline{Y}$ in $\mathcal{O}_Y$-*mod, *so applying $\underline{f}^*$ gives $\underline{f}^*(\Omega_{\underline{g}}) : \underline{f}^*(\underline{g}^*(T^*\underline{Z})) \to \underline{f}^*(T^*\underline{Y})$ in $\mathcal{O}_X$-*mod, *and $I_{\underline{f},\underline{g}}(T^*\underline{Z}) : (\underline{g}\circ\underline{f})^*(T^*\underline{Z}) \to \underline{f}^*(\underline{g}^*(T^*\underline{Z}))$ is as in Definition* B.34.

(c) *Suppose $\underline{W}, \underline{X}, \underline{Y}, \underline{Z}$ are locally fair C^∞-schemes with a Cartesian square*

$$\begin{tikzcd} \underline{W} \arrow[r, "\underline{f}"] \arrow[d, "\underline{e}"] & \underline{Y} \arrow[d, "\underline{h}"] \\ \underline{X} \arrow[r, "\underline{g}"] & \underline{Z} \end{tikzcd}$$

in $\mathbf{C^\infty Sch^{lf}}$, *so that $\underline{W} = \underline{X} \times_{\underline{Z}} \underline{Y}$. Then the following is exact in* qcoh$(\underline{W})$:

$$(\underline{g}\circ\underline{e})^*(T^*\underline{Z}) \xrightarrow{\substack{\underline{e}^*(\Omega_{\underline{g}})\circ I_{\underline{e},\underline{g}}(T^*\underline{Z})\oplus \\ -\underline{f}^*(\Omega_{\underline{h}})\circ I_{\underline{f},\underline{h}}(T^*\underline{Z})}} \underline{e}^*(T^*\underline{X})\oplus\underline{f}^*(T^*\underline{Y}) \xrightarrow{\Omega_{\underline{e}}\oplus\Omega_{\underline{f}}} T^*\underline{W} \longrightarrow 0. \quad \text{(B.11)}$$

Definition B.40. Let $\underline{X}$ be a C^∞-scheme. A quasicoherent sheaf $\mathcal{L}$ on $\underline{X}$ is called a *line bundle* if it is a vector bundle of rank 1. The structure sheaf $\mathcal{O}_X$ is an example of a line bundle, which we call the *trivial line bundle*. A *trivialization* of a line bundle $\mathcal{L}$ is an isomorphism $\tau : \mathcal{O}_X \to \mathcal{L}$. A line bundle is called *trivializable* if it admits a trivialization. If $\tau, \tau' : \mathcal{O}_X \to \mathcal{L}$ are trivializations then $\tau' = c\cdot\tau$ for some unique invertible function $c \in \mathcal{O}_X(X)$, where $c \in \mathcal{O}_X(X)$ is invertible if and only if $c(x) \neq 0 \in \mathbb{R}$ for all $x \in X$.

Call a function $c \in \mathcal{O}_X(X)$ *positive* if $c(x) > 0$ for all $x \in X$. Positive functions are invertible. An *orientation* ω on a line bundle $\mathcal{L}$ on $\underline{X}$ is an equivalence class $[\tau]$ of isomorphisms $\tau : \mathcal{O}_X \to \mathcal{L}$, where τ, τ' are equivalent if $\tau' = c\cdot\tau$ for some positive function $c \in \mathcal{O}_X(X)$.

Let $\underline{X}, \underline{Y}$ be C^∞-schemes, $\underline{f} : \underline{X} \to \underline{Y}$ a morphism, and $\mathcal{L} \in$ qcoh$(\underline{Y})$ a line bundle on $\underline{Y}$. Then $\underline{f}^*(\mathcal{L})$ is a line bundle on $\underline{X}$. Suppose ω is an orientation on $\mathcal{L}$. Then we can define the *pullback orientation* $\underline{f}^*(\omega)$ on $\underline{f}^*(\mathcal{L})$ by $\underline{f}^*(\omega) = [\underline{f}^*(\tau)\circ\iota_{\underline{f}}]$, where $\tau : \mathcal{O}_Y \to \mathcal{L}$ is an isomorphism representing ω, so that $\underline{f}^*(\tau) : \underline{f}^*(\mathcal{O}_Y) \to \underline{f}^*(\mathcal{L})$ is an isomorphism in qcoh$(\underline{X})$, and $\iota_{\underline{f}} : \mathcal{O}_X \to \underline{f}^*(\mathcal{O}_Y)$ is the canonical isomorphism. This $\underline{f}^*(\omega)$ is independent of the choice of τ.

C Deligne–Mumford C^∞-stacks

We now explain the theory of *C^∞-stacks* (the analogues of Artin stacks in algebraic geometry), focussing mainly on *Deligne–Mumford C^∞-stacks* (the analogues of Deligne–Mumford stacks in algebraic geometry, and of orbifolds in differential geometry). Deligne–Mumford C^∞-stacks are the foundation of Chapters 8–12. C^∞-stacks were developed by the author in [56, §7–§11].

Some important points needed to understand Chapters 8–12 are these:

- Deligne–Mumford C^∞-stacks form a 2-category $\mathbf{DMC^\infty Sta}$. That is, we have objects $\mathcal{X}, \mathcal{Y}$, 1-morphisms $f, g : \mathcal{X} \to \mathcal{Y}$, and 2-morphisms $\eta : f \Rightarrow g$. All 2-morphisms are invertible, that is, they are 2-isomorphisms.
- There is a full and faithful functor $F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}} : \mathbf{C^\infty Sch} \to \mathbf{DMC^\infty Sta}$ which embeds C^∞-schemes as a 2-subcategory of Deligne–Mumford C^∞-stacks. Thus, we regard C^∞-schemes as examples of C^∞-stacks.

  If $\underline{X}, \underline{Y}, \ldots$ are C^∞-schemes and $\underline{f}, \underline{g}, \ldots$ are morphisms of C^∞-schemes, as a shorthand we will write $\bar{\underline{X}}, \bar{\underline{Y}}, \ldots, \bar{\underline{f}}, \bar{\underline{g}}, \ldots$ for the corresponding C^∞-stacks and 1-morphisms, so that $\bar{\underline{X}}, \bar{\underline{Y}}, \bar{\underline{f}}, \bar{\underline{g}} = F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}}(\underline{X}, \underline{Y}, \underline{f}, \underline{g})$.
- If $\underline{U}$ is a C^∞-scheme and G a finite group acting on $\underline{U}$ by isomorphisms, we may form a *quotient C^∞-stack* $[\underline{U}/G]$, as in Definition C.17 below. Deligne–Mumford C^∞-stacks $\mathcal{X}$ are locally modelled on such $[\underline{U}/G]$. That is, we may cover $\mathcal{X}$ by Zariski open C^∞-substacks $\mathcal{U}$ equivalent in $\mathbf{C^\infty Sta}$ to $[\underline{U}/G]$, for $\underline{U}$ an affine C^∞-scheme and G a finite group.
- All fibre products exist in $\mathbf{DMC^\infty Sta}$.
- Each C^∞-stack $\mathcal{X}$ has an *associated topological space* $\mathcal{X}_{\mathrm{top}}$, where points of $\mathcal{X}_{\mathrm{top}}$ are 2-isomorphism classes $[x]$ of 1-morphisms $x : \bar{\underline{*}} \to \mathcal{X}$, with $\bar{\underline{*}}$ the point in $\mathbf{DMC^\infty Sta}$. Each point $[x] \in \mathcal{X}_{\mathrm{top}}$ has an *orbifold group* $\mathrm{Iso}_{\mathcal{X}}([x])$, a finite group.
- 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ induce continuous maps $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$, with $f_{\mathrm{top}} = g_{\mathrm{top}}$ if f, g are 2-isomorphic. They also induce morphisms of orbifold groups $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ for each $[x] \in \mathcal{X}_{\mathrm{top}}$.

The author knows of no other work on C^∞-stacks in the sense of [56, 57], which are stacks on the site $(\mathbf{C^\infty Sch}, \mathcal{J})$, where $\mathcal{J}$ is the Grothendieck topology of open covers in $\mathbf{C^\infty Sch}$. The closest are the 'differentiable stacks' of Behrend and Xu [13] and the 'smooth stacks' of Metzler [82], which are both stacks on the site $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$. Some useful references on various kinds of stack are Behrend et al. [11], Gomez [36], Laumon and Moret-Bailly [64], and Metzler [82].

C.1 C^∞-stacks

Stacks are a rather technical subject which take a lot of work and many pages to set up properly, so for brevity we will give less detail than in Appendix B.

Definition C.1. Define a *Grothendieck topology* $\mathcal{J}$ on the category $\mathbf{C^\infty Sch}$ of C^∞-schemes to have coverings $\{\underline{i}_a : \underline{U}_a \to \underline{U}\}_{a\in A}$ where $V_a = i_a(U_a)$ is open in U with $\underline{i}_a : \underline{U}_a \to (V_a, \mathcal{O}_U|_{V_a})$ an isomorphism for all $a \in A$, and $U = \bigcup_{a\in A} V_a$. Up to isomorphisms of the $\underline{U}_a$, the coverings $\{\underline{i}_a : \underline{U}_a \to \underline{U}\}_{a\in A}$ of $\underline{U}$ correspond exactly to open covers $\{V_a : a \in A\}$ of U. Then $(\mathbf{C^\infty Sch}, \mathcal{J})$ is a *site*.

The *stacks on* $(\mathbf{C^\infty Sch}, \mathcal{J})$ form a 2-category $\mathbf{Sta}_{(\mathbf{C^\infty Sch},\mathcal{J})}$, with all 2-morphisms invertible. As the site $(\mathbf{C^\infty Sch}, \mathcal{J})$ is *subcanonical*, there is a natural, fully faithful functor $\mathbf{C^\infty Sch} \to \mathbf{Sta}_{(\mathbf{C^\infty Sch},\mathcal{J})}$, defined explicitly below, which we write as $\underline{X} \mapsto \bar{\underline{X}}$ on objects and $\underline{f} \mapsto \bar{\underline{f}}$ on morphisms. A C^∞-*stack* is a stack $\mathcal{X}$ on $(\mathbf{C^\infty Sch}, \mathcal{J})$ such that the diagonal 1-morphism $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, and there exists a surjective 1-morphism $\Pi : \bar{\underline{U}} \to \mathcal{X}$ called an *atlas* for some C^∞-scheme $\underline{U}$. Write $\mathbf{C^\infty Sta}$ for the full 2-subcategory of C^∞-stacks in $\mathbf{Sta}_{(\mathbf{C^\infty Sch},\mathcal{J})}$. The functor $\mathbf{C^\infty Sch} \to \mathbf{Sta}_{(\mathbf{C^\infty Sch},\mathcal{J})}$ above maps into $\mathbf{C^\infty Sta}$, so we also write it as $F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}} : \mathbf{C^\infty Sch} \to \mathbf{C^\infty Sta}$.

Formally, a C^∞-stack is a category $\mathcal{X}$ with a functor $p_{\mathcal{X}} : \mathcal{X} \to \mathbf{C^\infty Sch}$, where $\mathcal{X}, p_{\mathcal{X}}$ must satisfy many complicated conditions, including sheaf-like conditions for all open covers in $\mathbf{C^\infty Sch}$. A 1-*morphism* $f : \mathcal{X} \to \mathcal{Y}$ of C^∞-stacks is a functor $f : \mathcal{X} \to \mathcal{Y}$ with $p_{\mathcal{Y}} \circ f = p_{\mathcal{X}} : \mathcal{X} \to \mathbf{C^\infty Sch}$. Given 1-morphisms $f, g : \mathcal{X} \to \mathcal{Y}$, a 2-*morphism* $\eta : f \Rightarrow g$ is an isomorphism of functors $\eta : f \Rightarrow g$ with $\mathrm{id}_{p_{\mathcal{Y}}} * \eta = \mathrm{id}_{p_{\mathcal{X}}} : p_{\mathcal{Y}} \circ f \Rightarrow p_{\mathcal{Y}} \circ g$.

If $\underline{X}$ is a C^∞-scheme, the corresponding C^∞-stack $\bar{\underline{X}} = F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}}(\underline{X})$ is the category with objects $(\underline{U}, \underline{u})$ for $\underline{u} : \underline{U} \to \underline{X}$ a morphism in $\mathbf{C^\infty Sch}$, and morphisms $\underline{h} : (\underline{U}, \underline{u}) \to (\underline{V}, \underline{v})$ for $\underline{h} : \underline{U} \to \underline{V}$ a morphism in $\mathbf{C^\infty Sch}$ with $\underline{v} \circ \underline{h} = \underline{u}$. The functor $p_{\bar{\underline{X}}} : \bar{\underline{X}} \to \mathbf{C^\infty Sch}$ maps $p_{\bar{\underline{X}}} : (\underline{U}, \underline{u}) \mapsto \underline{U}$ and $p_{\bar{\underline{X}}} : \underline{h} \mapsto \underline{h}$. When we say that a C^∞-stack $\mathcal{X}$ *is a* C^∞*-scheme*, we mean that $\mathcal{X} \simeq \bar{\underline{X}}$ in $\mathbf{C^\infty Sta}$ for some C^∞-scheme $\underline{X}$.

If $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes, the corresponding 1-morphism $\bar{\underline{f}} = F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}}(\underline{f}) : \bar{\underline{X}} \to \bar{\underline{Y}}$ maps $\bar{\underline{f}} : (\underline{U}, \underline{u}) \mapsto (\underline{U}, \underline{f} \circ \underline{u})$ on objects $(\underline{U}, \underline{u})$ and $\bar{\underline{f}} : \underline{h} \mapsto \underline{h}$ on morphisms $\underline{h}$ in $\bar{\underline{X}}$. This defines a functor $\bar{\underline{f}} : \bar{\underline{X}} \to \bar{\underline{Y}}$ with $p_{\bar{\underline{Y}}} \circ \bar{\underline{f}} = p_{\bar{\underline{X}}} : \mathcal{X} \to \mathbf{C^\infty Sch}$, so $\bar{\underline{f}}$ is a 1-morphism $\bar{\underline{f}} : \bar{\underline{X}} \to \bar{\underline{Y}}$ in $\mathbf{C^\infty Sta}$.

Definition C.2. A *groupoid object* $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$ in $\mathbf{C^\infty Sch}$, or simply *groupoid* in $\mathbf{C^\infty Sch}$, consists of objects $\underline{U}, \underline{V}$ in $\mathbf{C^\infty Sch}$ and morphisms $\underline{s}, \underline{t} : \underline{V} \to \underline{U}$, $\underline{u} : \underline{U} \to \underline{V}$, $\underline{i} : \underline{V} \to \underline{V}$ and $\underline{m} : \underline{V} \times_{\underline{s},\underline{U},\underline{t}} \underline{V} \to \underline{V}$ satisfying the identities

$$\underline{s} \circ \underline{u} = \underline{t} \circ \underline{u} = \mathrm{id}_{\underline{U}}, \;\; \underline{s} \circ \underline{i} = \underline{t}, \;\; \underline{t} \circ \underline{i} = \underline{s}, \;\; \underline{s} \circ \underline{m} = \underline{s} \circ \underline{\pi}_2, \;\; \underline{t} \circ \underline{m} = \underline{t} \circ \underline{\pi}_1,$$
$$\underline{m} \circ (\underline{i} \times \underline{\mathrm{id}}_{\underline{V}}) = \underline{u} \circ \underline{s}, \;\; \underline{m} \circ (\underline{\mathrm{id}}_{\underline{V}} \times \underline{i}) = \underline{u} \circ \underline{t},$$
$$\underline{m} \circ (\underline{m} \times \underline{\mathrm{id}}_{\underline{V}}) = \underline{m} \circ (\underline{\mathrm{id}}_{\underline{V}} \times \underline{m}) : \underline{V} \times_{\underline{U}} \underline{V} \times_{\underline{U}} \underline{V} \longrightarrow \underline{V},$$
$$\underline{m} \circ (\underline{\mathrm{id}}_{\underline{V}} \times \underline{u}) = \underline{m} \circ (\underline{u} \times \underline{\mathrm{id}}_{\underline{V}}) : \underline{V} = \underline{V} \times_{\underline{U}} \underline{U} \longrightarrow \underline{V}.$$

We write groupoids in $\mathbf{C^\infty Sch}$ as $\underline{V} \rightrightarrows \underline{U}$ for short, to emphasize the morphisms $\underline{s}, \underline{t} : \underline{V} \to \underline{U}$. To any such groupoid we can associate a *groupoid stack* $[\underline{V} \rightrightarrows \underline{U}]$, which is a C^∞-stack. Conversely, if $\mathcal{X}$ is a C^∞-stack and $\Pi : \bar{\underline{U}} \to \mathcal{X}$ is an atlas one can construct a groupoid $\underline{V} \rightrightarrows \underline{U}$ in $\mathbf{C^\infty Sch}$, and $\mathcal{X}$ is equivalent (in the 2-category sense, as in §A.3) to $[\underline{V} \rightrightarrows \underline{U}]$. Thus, every C^∞-stack is equivalent to a groupoid stack.

From [56, Th. 8.5] we have:

Theorem C.3. *All fibre products exist in the* 2*-category* $\mathbf{C^\infty Sta}$.

Here fibre products in a 2-category are defined in §A.4. We define some classes of morphisms of C^∞-schemes, following [56, §8.2].

Definition C.4. Let $\underline{f} = (f, f^\sharp) : \underline{X} = (X, \mathcal{O}_X) \to \underline{Y} = (Y, \mathcal{O}_Y)$ be a morphism in $\mathbf{C^\infty Sch}$. Then:

- We call $\underline{f}$ an *open embedding* if $V = f(X)$ is an open subset in Y and $(f, f^\sharp) : (X, \mathcal{O}_X) \to (V, \mathcal{O}_Y|_V)$ is an isomorphism.
- We call $\underline{f}$ a *closed embedding* if $f : X \to Y$ is a homeomorphism with a closed subset of $\underline{Y}$, and $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ is a surjective morphism of sheaves of C^∞-rings.
- We call $\underline{f}$ an *embedding* if we may write $\underline{f} = \underline{g} \circ \underline{h}$ where $\underline{h}$ is an open embedding and $\underline{g}$ is a closed embedding.
- We call $\underline{f}$ *étale* if each $x \in X$ has an open neighbourhood U in X such that $V = f(U)$ is open in Y and $(f|_U, f^\sharp|_U) : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is an isomorphism. That is, $\underline{f}$ is a local isomorphism.
- We call $\underline{f}$ *proper* if $f : X \to Y$ is a proper map of topological spaces, that is, if $S \subseteq Y$ is compact then $f^{-1}(S) \subseteq X$ is compact.
- We call $\underline{f}$ *separated* if $f : X \to Y$ is a separated map of topological spaces, that is, $\Delta_X = \{(x, x) : x \in X\}$ is a closed subset of the topological fibre product $X \times_{f,Y,f} X = \{(x, x') \in X \times X : f(x) = f(x')\}$.
- We call $\underline{f}$ *universally closed* if whenever $\underline{g} : \underline{W} \to \underline{Y}$ is a morphism then $\underline{\pi}_{\underline{W}} : \underline{X} \times_{\underline{f},\underline{Y},\underline{g}} \underline{W} \to \underline{W}$ is a closed morphism, that is, π_W is a closed map of topological spaces.
- We call $\underline{f}$ a *submersion* if for all $x \in X$ with $f(x) = y$, there exists an open neighbourhood U of y in Y and a morphism $\underline{g} = (g, g^\sharp) : (U, \mathcal{O}_Y|_U) \to (X, \mathcal{O}_X)$ with $g(y) = x$ and $\underline{f} \circ \underline{g} = \mathrm{id}_{(U, \mathcal{O}_Y|_U)}$.

Each one is invariant under base change and local in the target in $(\mathbf{C^\infty Sch}, \mathcal{J})$. Thus, they are also defined for representable 1-morphisms of C^∞-stacks.

Definition C.5. Let $\mathcal{X}$ be a C^∞-stack. We say that $\mathcal{X}$ is *separated* if the diagonal 1-morphism $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is universally closed. If $\mathcal{X} = \bar{\underline{X}}$ for some C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ then $\mathcal{X}$ is separated if and only if $\Delta_X : X \to X \times X$ is closed, that is, if and only if X is Hausdorff, so $\underline{X}$ is separated.

Definition C.6. Let $\mathcal{X}$ be a C^∞-stack. A C^∞*-substack* $\mathcal{Y}$ in $\mathcal{X}$ is a strictly full subcategory $\mathcal{Y}$ in $\mathcal{X}$ such that $p_{\mathcal{Y}} := p_{\mathcal{X}}|_{\mathcal{Y}} : \mathcal{Y} \to \mathbf{C^\infty Sch}$ is also a C^∞-stack. It has a natural inclusion 1-morphism $i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{X}$. We call $\mathcal{Y}$ an *open* C^∞*-substack* of $\mathcal{X}$ if $i_{\mathcal{Y}}$ is a representable open embedding.

An *open cover* $\{\mathcal{U}_a : a \in A\}$ of $\mathcal{X}$ is a family of open C^∞-substacks $\mathcal{U}_a$ in $\mathcal{X}$ with $\coprod_{a \in A} i_{\mathcal{U}_a} : \coprod_{a \in A} \mathcal{U}_a \to \mathcal{X}$ surjective. We write $\mathcal{U} \subseteq \mathcal{X}$ when $\mathcal{U}$ is an open C^∞-substack of $\mathcal{X}$, and $\bigcup_{a \in A} \mathcal{U} = \mathcal{X}$ to mean that $\coprod_{a \in A} i_{\mathcal{U}_a}$ is surjective.

A C^∞-stack $\mathcal{X}$ has an *underlying topological space* $\mathcal{X}_{\rm top}$, [56, §8.4].

Definition C.7. Let $\mathcal{X}$ be a C^∞-stack. Write $\underline{*}$ for the point $\operatorname{Spec}\mathbb{R}$ in $\mathbf{C^\infty Sch}$, and $\bar{\underline{*}}$ for the associated point in $\mathbf{C^\infty Sta}$. Define $\mathcal{X}_{\rm top}$ to be the set of 2-isomorphism classes $[x]$ of 1-morphisms $x:\bar{\underline{*}}\rightarrow\mathcal{X}$. If $\mathcal{U}\subseteq\mathcal{X}$ is an open C^∞-substack then any 1-morphism $x:\bar{\underline{*}}\rightarrow\mathcal{U}$ is also a 1-morphism $x:\bar{\underline{*}}\rightarrow\mathcal{X}$, and $\mathcal{U}_{\rm top}$ is a subset of $\mathcal{X}_{\rm top}$. Define $\mathcal{T}_{\mathcal{X}_{\rm top}}=\bigl\{\mathcal{U}_{\rm top}:\mathcal{U}\subseteq\mathcal{X}$ is an open C^∞-substack in $\mathcal{X}\bigr\}$. Then $\mathcal{T}_{\mathcal{X}_{\rm top}}$ is a set of subsets of $\mathcal{X}_{\rm top}$ which is a topology on $\mathcal{X}_{\rm top}$, so $(\mathcal{X}_{\rm top},\mathcal{T}_{\mathcal{X}_{\rm top}})$ is a topological space, which we call the *underlying topological space* of $\mathcal{X}$, and usually write as $\mathcal{X}_{\rm top}$.

If $f:\mathcal{X}\rightarrow\mathcal{Y}$ is a 1-morphism of C^∞-stacks then there is a natural continuous map $f_{\rm top}:\mathcal{X}_{\rm top}\rightarrow\mathcal{Y}_{\rm top}$ defined by $f_{\rm top}([x])=[f\circ x]$. If $f,g:\mathcal{X}\rightarrow\mathcal{Y}$ are 1-morphisms and $\eta:f\Rightarrow g$ is a 2-isomorphism then $f_{\rm top}=g_{\rm top}$. Mapping $\mathcal{X}\mapsto\mathcal{X}_{\rm top}$, $f\mapsto f_{\rm top}$ and 2-morphisms to identities defines a 2-functor $F_{\mathbf{C^\infty Sta}}^{\mathbf{Top}}:\mathbf{C^\infty Sta}\rightarrow\mathbf{Top}$, where the category of topological spaces $\mathbf{Top}$ is regarded as a 2-category with only identity 2-morphisms.

If $\underline{X}=(X,\mathcal{O}_X)$ is a C^∞-scheme, so that $\bar{\underline{X}}$ is a C^∞-stack, then $\bar{\underline{X}}_{\rm top}$ is naturally homeomorphic to X, and we will identify $\bar{\underline{X}}_{\rm top}$ with X. If $\underline{f}=(f,f^\sharp):\underline{X}=(X,\mathcal{O}_X)\rightarrow\underline{Y}=(Y,\mathcal{O}_Y)$ is a morphism of C^∞-schemes, so that $\bar{\underline{f}}:\bar{\underline{X}}\rightarrow\bar{\underline{Y}}$ is a 1-morphism of C^∞-stacks, then $\bar{\underline{f}}_{\rm top}:\bar{\underline{X}}_{\rm top}\rightarrow\bar{\underline{Y}}_{\rm top}$ is $f:X\rightarrow Y$.

Definition C.8. Let $\mathcal{X}$ be a C^∞-stack, and $[x]\in\mathcal{X}_{\rm top}$. Pick a representative x for $[x]$, so that $x:\bar{\underline{*}}\rightarrow\mathcal{X}$ is a 1-morphism. Let G be the group of 2-morphisms $\eta:x\Rightarrow x$. There is a natural C^∞-scheme $\underline{G}=(G,\mathcal{O}_G)$ with $\bar{\underline{G}}\cong\bar{\underline{*}}\times_{x,\mathcal{X},x}\bar{\underline{*}}$, which makes $\underline{G}$ into a C^∞-*group* (a group object in $\mathbf{C^\infty Sch}$, just as a Lie group is a group object in $\mathbf{Man}$). With $[x]$ fixed, this C^∞-group $\underline{G}$ is independent of choices up to noncanonical isomorphism; roughly, $\underline{G}$ is canonical up to conjugation in $\underline{G}$. We define the *orbifold group* (or *isotropy group*, or *stabilizer group*) $\operatorname{Iso}_{\mathcal{X}}([x])$ of $[x]$ to be this C^∞-group $\underline{G}$, regarded as a C^∞-group up to noncanonical isomorphism.

If $f:\mathcal{X}\rightarrow\mathcal{Y}$ is a 1-morphism of C^∞-stacks and $[x]\in\mathcal{X}_{\rm top}$ with $f_{\rm top}([x])=[y]\in\mathcal{Y}_{\rm top}$, for $y=f\circ x$, then we define $f_*:\operatorname{Iso}_{\mathcal{X}}([x])\rightarrow\operatorname{Iso}_{\mathcal{Y}}([y])$ by $f_*(\eta)=\mathrm{id}_f*\eta$. Then f_* is a group morphism, and extends to a C^∞-group morphism. It is independent of the choice of $x\in[x]$ up to conjugation in $\operatorname{Iso}_{\mathcal{Y}}([y])$.

C.2 Gluing C^∞-stacks by equivalences

Here are some results on gluing C^∞-stacks by equivalences, taken from [56, §8.5]. They are used in §9.4 to glue d-stacks by equivalences.

Proposition C.9. *Suppose $\mathcal{X},\mathcal{Y}$ are C^∞-stacks, $\mathcal{U}\subseteq\mathcal{X}$, $\mathcal{V}\subseteq\mathcal{Y}$ are open C^∞-substacks, and $f:\mathcal{U}\rightarrow\mathcal{V}$ is an equivalence in* $\mathbf{C^\infty Sta}$. *Then there exist a C^∞-stack $\mathcal{Z}$, open C^∞-substacks $\hat{\mathcal{X}},\hat{\mathcal{Y}}$ in $\mathcal{Z}$ with $\mathcal{Z}=\hat{\mathcal{X}}\cup\hat{\mathcal{Y}}$, equivalences $g:\mathcal{X}\rightarrow\hat{\mathcal{X}}$ and $h:\mathcal{Y}\rightarrow\hat{\mathcal{Y}}$ such that $g|_{\mathcal{U}}$ and $h|_{\mathcal{V}}$ are both equivalences with $\hat{\mathcal{X}}\cap\hat{\mathcal{Y}}$, and a 2-morphism $\eta:g|_{\mathcal{U}}\Rightarrow h\circ f:\mathcal{U}\rightarrow\hat{\mathcal{X}}\cap\hat{\mathcal{Y}}$ in* $\mathbf{C^\infty Sta}$. *Furthermore, $\mathcal{Z}$ is independent of choices up to equivalence.*

Proposition C.10. *Suppose $\mathcal{X},\mathcal{Y}$ are C^∞-stacks, $\mathcal{U},\mathcal{V}\subseteq\mathcal{X}$ are open C^∞-substacks with $\mathcal{X}=\mathcal{U}\cup\mathcal{V}$, $f:\mathcal{U}\to\mathcal{Y}$ and $g:\mathcal{V}\to\mathcal{Y}$ are 1-morphisms, and $\eta:f|_{\mathcal{U}\cap\mathcal{V}}\Rightarrow g|_{\mathcal{U}\cap\mathcal{V}}$ is a 2-morphism in* $\mathbf{C^\infty Sta}$. *Then there exists a 1-morphism $h:\mathcal{X}\to\mathcal{Y}$ and 2-morphisms $\zeta:h|_{\mathcal{U}}\Rightarrow f$, $\theta:h|_{\mathcal{V}}\Rightarrow g$ such that $\theta|_{\mathcal{U}\cap\mathcal{V}}=\eta\odot\zeta|_{\mathcal{U}\cap\mathcal{V}}:h|_{\mathcal{U}\cap\mathcal{V}}\Rightarrow g|_{\mathcal{U}\cap\mathcal{V}}$. This h is unique up to 2-isomorphism.*

*In general, h is **not** independent up to 2-isomorphism of the choice of η.*

Here is an example in which h is not independent of η up to 2-isomorphism in the last part of Proposition C.10.

Example C.11. Let $\mathcal{X}$ be the C^∞-stack associated to the circle $X=\{(x,y)\in\mathbb{R}^2:x^2+y^2=1\}$, and $\mathcal{U},\mathcal{V}\subseteq\mathcal{X}$ the substacks associated to the open sets $U=\{(x,y)\in X:x>-\frac{1}{2}\}$ and $V=\{(x,y)\in X:x<\frac{1}{2}\}$. Let $\mathcal{Y}$ be the quotient C^∞-stack $[\underline{*}/\mathbb{Z}_2]$, as in §C.4. Then 1-morphisms $f:\mathcal{X}\to\mathcal{Y}$ correspond to principal $\mathbb{Z}_2$-bundles $P_f\to X$, and for 1-morphisms $f,g:\mathcal{X}\to\mathcal{Y}$ with principal $\mathbb{Z}_2$-bundles $P_f,P_g\to X$, a 2-morphism $\eta:f\Rightarrow g$ corresponds to an isomorphism of principal $\mathbb{Z}_2$-bundles $P_f\cong P_g$. The same holds for 1-morphisms $\mathcal{U},\mathcal{V},\mathcal{U}\cup\mathcal{V}\to\mathcal{Y}$ and their 2-morphisms.

Let $f:\mathcal{U}\to\mathcal{Y}$ and $g:\mathcal{V}\to\mathcal{Y}$ be the 1-morphisms corresponding to the trivial $\mathbb{Z}_2$-bundles $P_f=\mathbb{Z}_2\times U\to U$, $P_g=\mathbb{Z}_2\times V\to V$. Then 2-morphisms $\eta:f|_{\mathcal{U}\cap\mathcal{V}}\Rightarrow g|_{\mathcal{U}\cap\mathcal{V}}$ correspond to automorphisms of the trivial $\mathbb{Z}_2$-bundle $\mathbb{Z}_2\times(U\cap V)\to U\cap V$, that is, to continuous maps $U\cap V\to\mathbb{Z}_2$. Note that $U\cap V$ has two connected components $\{(x,y)\in X:-\frac{1}{2}<x<\frac{1}{2},\ y>0\}$ and $\{(x,y)\in X:-\frac{1}{2}<x<\frac{1}{2},\ y<0\}$.

Define 2-morphisms $\eta_1,\eta_2:f|_{\mathcal{U}\cap\mathcal{V}}\Rightarrow g|_{\mathcal{U}\cap\mathcal{V}}$ such that η_1 corresponds to the map $1:(U\cap V)\to\mathbb{Z}_2=\{\pm1\}$, and η_1 corresponds to the map $\mathrm{sign}(y):(U\cap V)\to\mathbb{Z}_2=\{\pm1\}$. Then Proposition C.10 gives 1-morphisms $h_1,h_2:\mathcal{X}\to\mathcal{Y}$ from η_1,η_2. The associated principal $\mathbb{Z}_2$-bundles P_{h_1},P_{h_2} over X come from gluing P_f,P_g over U,V using the transition functions $1,\mathrm{sign}(y)$. Therefore P_{h_1} is the trivial $\mathbb{Z}_2$-bundle over $X=\mathcal{S}^1$, and P_{h_2} the nontrivial $\mathbb{Z}_2$-bundle. Hence P_{h_1},P_{h_2} are not isomorphic as principal $\mathbb{Z}_2$-bundles, and h_1,h_2 are not 2-isomorphic. Hence in this example, h is not independent up to 2-isomorphism of the choice of η.

C.3 Strongly representable 1-morphisms of C^∞-stacks

Strongly representable 1-morphisms, discussed in [56, §8.6], will be important in the definitions of orbifolds with corners, d-stacks with corners, and d-orbifolds with corners in Chapters 8, 11 and 12.

Definition C.12. Let $\mathcal{Y},\mathcal{Z}$ be C^∞-stacks, and $g:\mathcal{Y}\to\mathcal{Z}$ a 1-morphism. Then $\mathcal{Y},\mathcal{Z}$ are categories with functors $p_{\mathcal{Y}}:\mathcal{Y}\to\mathbf{C^\infty Sch}$, $p_{\mathcal{Z}}:\mathcal{Z}\to\mathbf{C^\infty Sch}$, and $g:\mathcal{Y}\to\mathcal{Z}$ is a functor with $p_{\mathcal{Z}}\circ g=p_{\mathcal{Y}}$.

We call g *strongly representable* if whenever $A\in\mathcal{Y}$ with $p_{\mathcal{Y}}(A)=\underline{U}\in\mathbf{C^\infty Sch}$, so that $B=g(A)\in\mathcal{Z}$ with $p_{\mathcal{Z}}(B)=\underline{U}$, and $b:B\to B'$ is an isomorphism in $\mathcal{Z}$ with $p_{\mathcal{Z}}(B')=\underline{U}$ and $p_{\mathcal{Z}}(b)=\underline{\mathrm{id}}_U$, then there exist a unique object A' and isomorphism $a:A\to A'$ in $\mathcal{Y}$ with $g(A')=B'$ and $g(a)=b$.

Note that this definition is purely category-theoretic, with nothing to do with C^∞-geometry, and also makes sense for other kinds of stacks. It is related to the notion of *isofibration* in category theory. The next four propositions are [56, Prop.s 8.25–8.28]. The first is the important property of strongly representable 1-morphisms, which will sometimes allow us to work with 1-morphisms up to equality, rather than just up to 2-isomorphism.

Proposition C.13. *Suppose $\mathcal{X},\mathcal{Y},\mathcal{Z}$ are C^∞-stacks, $f:\mathcal{X}\to\mathcal{Y}$, $g:\mathcal{Y}\to\mathcal{Z}$, $h:\mathcal{X}\to\mathcal{Z}$ are 1-morphisms with g strongly representable, and $\eta:g\circ f\Rightarrow h$ is a 2-morphism in $\mathbf{C^\infty Sta}$. Then as in the diagram below there exist a 1-morphism $f':\mathcal{X}\to\mathcal{Y}$ with $g\circ f'=h$, and a 2-morphism $\zeta:f\Rightarrow f'$ with $\mathrm{id}_g*\zeta=\eta$, and f',ζ are unique under these conditions.*

$$\begin{tikzcd}
 & \mathcal{Y} \arrow[dr, "g"] \arrow[d, phantom, "\eta\Downarrow"] & \\
\mathcal{X} \arrow[ur, bend left=40, "f'"] \arrow[ur, "f"'] \arrow[ur, phantom, bend left=20, "\zeta\Uparrow"] \arrow[rr, "h"] & {} & \mathcal{Z}.
\end{tikzcd}$$

Parts (a),(b) of the next proposition justify the term 'strongly representable'. For (a) we include the assumption that $\mathcal{Y},\mathcal{Z}$ are Deligne–Mumford.

Proposition C.14. (a) *Let $g:\mathcal{Y}\to\mathcal{Z}$ be a strongly representable 1-morphism of Deligne–Mumford C^∞-stacks. Then g is representable.*

(b) *Suppose $g:\mathcal{Y}\to\mathcal{Z}$ is a representable 1-morphism of C^∞-stacks. Then there exist a C^∞-stack $\mathcal{Y}'$, an equivalence $i:\mathcal{Y}\to\mathcal{Y}'$, and a strongly representable 1-morphism $g':\mathcal{Y}'\to\mathcal{Z}$ with $g=g'\circ i$. Also $\mathcal{Y}'$ is unique up to canonical 1-isomorphism in $\mathbf{C^\infty Sta}$.*

Here is an explicit construction of fibre products $\mathcal{X}\times_{g,\mathcal{Z},h}\mathcal{Y}$ in $\mathbf{C^\infty Sta}$ when g is strongly representable, yielding a strictly commutative 2-Cartesian square.

Proposition C.15. *Let $g:\mathcal{X}\to\mathcal{Z}$ and $h:\mathcal{Y}\to\mathcal{Z}$ be 1-morphisms of C^∞-stacks with g strongly representable. Define a category $\mathcal{W}$ to have objects pairs (A,B) for $A\in\mathcal{X}$, $B\in\mathcal{Y}$ with $g(A)=h(B)$ in $\mathcal{Z}$, so that $p_{\mathcal{X}}(A)=p_{\mathcal{Y}}(B)$ in $\mathbf{C^\infty Sch}$, and morphisms pairs $(a,b):(A,B)\to(A',B')$ with $a:A\to A'$, $b:B\to B'$ morphisms in $\mathcal{X},\mathcal{Y}$ with $p_{\mathcal{X}}(a)=p_{\mathcal{Y}}(b)$ in $\mathbf{C^\infty Sch}$.*

Define functors $p_{\mathcal{W}}:\mathcal{W}\to\mathbf{C^\infty Sch}$, $e:\mathcal{W}\to\mathcal{X}$, $f:\mathcal{W}\to\mathcal{Y}$ by $p_{\mathcal{W}}:(A,B)\mapsto p_{\mathcal{X}}(A)=p_{\mathcal{Y}}(B)$, $e:(A,B)\mapsto A$, $f:(A,B)\mapsto B$ on objects and $p_{\mathcal{W}}:(a,b)\mapsto p_{\mathcal{X}}(a)=p_{\mathcal{Y}}(b)$, $e:(a,b)\mapsto a$, $f:(a,b)\mapsto b$ on morphisms. Then $\mathcal{W}$ is a C^∞-stack and $e:\mathcal{W}\to\mathcal{X}$, $f:\mathcal{W}\to\mathcal{Y}$ are 1-morphisms, with f strongly representable, and $g\circ e=h\circ f$. Furthermore, the following diagram in $\mathbf{C^\infty Sta}$ is 2-Cartesian:

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\mathrm{id}_{g\circ e}\Uparrow"] & \mathcal{Y} \arrow[d, "h"] \\
\mathcal{X} \arrow[r, "g"] & \mathcal{Z}.
\end{tikzcd}$$

If also h is strongly representable, then e is strongly representable.

Propositions C.13–C.15 show that when working with strongly representable 1-morphisms, we can often take 2-morphisms to be identities. Morphisms of C^∞-schemes naturally map to strongly representable 1-morphisms of C^∞-stacks.

Proposition C.16. *Suppose $\underline{g} : \underline{Y} \to \underline{Z}$ is a morphism in* $\mathbf{C^\infty Sch}$. *Then the corresponding C^∞-stack 1-morphism $\bar{\underline{g}} : \bar{\underline{Y}} \to \bar{\underline{Z}}$ is strongly representable.*

C.4 Quotient C^∞-stacks

We now define quotient C^∞-stacks $[\underline{X}/G]$, and their 1- and 2-morphisms, following [56, §9.1]. The definitions are discussed in Remark C.20.

Definition C.17. Let $\underline{X}$ be a C^∞-scheme, G a finite group, and $\underline{r} : G \to \mathrm{Aut}(\underline{X})$ an action of G on $\underline{X}$ by isomorphisms. We will define the *quotient C^∞-stack* $\mathcal{X} = [\underline{X}/G]$. Define a category $\mathcal{X}$ to have objects septuples $(A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})$, where A is a finite group, $\mu : A \to G$ is a group morphism, $\underline{T}, \underline{U}$ are C^∞-schemes, $\underline{t} : A \to \mathrm{Aut}(\underline{T})$ is a free action of A on $\underline{T}$ by isomorphisms, $\underline{u} : \underline{T} \to \underline{X}$ is a morphism with $\underline{u} \circ \underline{t}(\alpha) = \underline{r}(\mu(\alpha)) \circ \underline{u} : \underline{T} \to \underline{X}$ for all $\alpha \in A$, and $\underline{v} : \underline{T} \to \underline{U}$ is a morphism which makes $\underline{T}$ into a principal A-bundle over $\underline{U}$, that is, $\underline{v}$ is proper, étale and surjective, and its fibres are A-orbits in $\underline{T}$.

Given such $(A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})$, define commuting, free actions $\hat{\mu} : A \to \mathrm{Aut}(G)$, $\nu : G \to \mathrm{Aut}(G)$ of A, G on G as a set by $\hat{\mu}(\alpha) : \gamma \mapsto \gamma\mu(\alpha^{-1})$ and $\nu(\gamma) : \delta \mapsto \gamma\delta$ for $\alpha \in A$ and $\gamma, \delta \in G$. Regard $\underline{T} \times G$ as a C^∞-scheme. Then $\underline{t} \times \hat{\mu} : A \to \mathrm{Aut}(\underline{T} \times G)$ and $\underline{\mathrm{id}}_{\underline{T}} \times \nu : G \to \mathrm{Aut}(\underline{T} \times G)$ are commuting, free actions of A, G on $\underline{T} \times G$. So we can define the quotient C^∞-scheme $\underline{T} \times_{\underline{t}, A, \hat{\mu}} G$ or $\underline{T} \times_A G := (\underline{T} \times G)/A$, and $\underline{\mathrm{id}}_{\underline{T}} \times \nu$ descends to a free G-action $\tilde{\underline{t}} : G \to \mathrm{Aut}(\underline{T} \times_A G)$. The morphism $\underline{T} \times G \to \underline{X}$ acting as $\underline{r}(\gamma) \circ \underline{u}$ on $\underline{T} \times \{\gamma\}$ is A-invariant and G-equivariant, so it descends to a G-equivariant morphism $\tilde{\underline{u}} : \underline{T} \times_A G \to \underline{X}$. Also $\underline{v} \circ \underline{\pi}_{\underline{T}} : \underline{T} \times G \to \underline{U}$ is A-invariant, and descends to $\tilde{\underline{v}} : \underline{T} \times_A G \to \underline{U}$. Then $\tilde{\underline{t}}, \tilde{\underline{v}}$ make $\underline{T} \times_A G$ into a principal G-bundle over $\underline{U}$, and $(G, \mathrm{id}_G, \underline{T} \times_A G, \underline{U}, \tilde{\underline{t}}, \tilde{\underline{u}}, \tilde{\underline{v}})$ is an object in $\mathcal{X}$. Write $\tilde{\underline{p}} : \underline{T} \to \underline{T} \times_A G$ for the composition of $\underline{\mathrm{id}}_{\underline{T}} \times 1 : \underline{T} \to \underline{T} \times \{1\} \subseteq \underline{T} \times G$ with the projection $\underline{T} \times G \to \underline{T} \times_A G$. Then $\tilde{\underline{t}}(\gamma) \circ \tilde{\underline{p}} = \tilde{\underline{p}} \circ \underline{t}(\gamma)$ for $\gamma \in G$, and $\underline{u} = \tilde{\underline{u}} \circ \tilde{\underline{p}}$, and $\underline{v} = \tilde{\underline{v}} \circ \tilde{\underline{p}}$.

Let $(A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})$ and $(A', \mu', \underline{T}', \underline{U}', \underline{t}', \underline{u}', \underline{v}')$ be objects in $\mathcal{X}$, and define $\underline{T} \times_A G, \tilde{\underline{t}}, \tilde{\underline{u}}, \tilde{\underline{v}}$ and $\underline{T}' \times_{A'} G, \tilde{\underline{t}}', \tilde{\underline{u}}', \tilde{\underline{v}}'$ as above. A morphism $(\underline{a}, \tilde{\underline{a}}) : (A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \to (A', \mu', \underline{T}', \underline{U}', \underline{t}', \underline{u}', \underline{v}')$ is a pair of morphisms $\underline{a} : \underline{U} \to \underline{U}'$ and $\tilde{\underline{a}} : \underline{T} \times_A G \to \underline{T}' \times_{A'} G$ such that $\tilde{\underline{a}} \circ \tilde{\underline{t}}(\gamma) = \tilde{\underline{t}}'(\gamma) \circ \tilde{\underline{a}}$ for $\gamma \in G$, and $\tilde{\underline{u}} = \tilde{\underline{u}}' \circ \tilde{\underline{a}}$, and $\underline{a} \circ \tilde{\underline{v}} = \tilde{\underline{v}}' \circ \tilde{\underline{a}}$. Composition is $(\underline{b}, \tilde{\underline{b}}) \circ (\underline{a}, \tilde{\underline{a}}) = (\underline{b} \circ \underline{a}, \tilde{\underline{b}} \circ \tilde{\underline{a}})$, and identities are $\mathrm{id}_{(A, \ldots, \underline{v})} = (\underline{\mathrm{id}}_{\underline{U}}, \underline{\mathrm{id}}_{\underline{T} \times_A G})$.

This defines the category $\mathcal{X}$. The functor $p_{\mathcal{X}} : \mathcal{X} \to \mathbf{C^\infty Sch}$ acts by $p_{\mathcal{X}} : (A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \mapsto \underline{U}$ on objects, and $p_{\mathcal{X}} : (\underline{a}, \tilde{\underline{a}}) \mapsto \underline{a}$ on morphisms. Then $\mathcal{X}$ is a C^∞-stack, which we also write as $[\underline{X}/G]$.

From Definition C.1, the C^∞-stack $\bar{\underline{X}}$ has objects $(\underline{U}, \underline{f})$ for $\underline{f} : \underline{U} \to \underline{X}$ a morphism in $\mathbf{C^\infty Sch}$, and morphisms $\underline{g} : (\underline{U}, \underline{f}) \to (\underline{U}', \underline{f}')$ for $\underline{g} : \underline{U} \to \underline{U}'$ with $\underline{f}' \circ \underline{g} = \underline{f}$. Define a functor $\pi_{[\underline{X}/G]} : \bar{\underline{X}} \to [\underline{X}/G]$ by $\pi_{[\underline{X}/G]} : (\underline{U}, \underline{f}) \mapsto (\{1\}, \mu, \underline{U}, \underline{U}, \underline{\mathrm{id}}_{\underline{U}}, \underline{f}, \underline{\mathrm{id}}_{\underline{U}})$ on objects, where $\mu : \{1\} \to G$ maps $\mu : 1 \mapsto 1$, and

$\pi_{[\underline{X}/G]} : \underline{g} \mapsto (\underline{g}, \underline{g} \times \mathrm{id}_G)$. Then $\pi_{[\underline{X}/G]} : \bar{\underline{X}} \to [\underline{X}/G]$ is a representable 1-morphism, and makes $\bar{\underline{X}}$ into a principal G-bundle over $[\underline{X}/G]$.

Definition C.18. Let $\underline{X}, \underline{Y}$ be C^∞-schemes acted on by finite groups G, H with actions $\underline{r} : G \to \mathrm{Aut}(\underline{X})$, $\underline{s} : H \to \mathrm{Aut}(\underline{Y})$, so that we have quotient C^∞-stacks $\mathcal{X} = [\underline{X}/G]$ and $\mathcal{Y} = [\underline{Y}/H]$ as in Definition C.17. Suppose we have morphisms $\underline{f} : \underline{X} \to \underline{Y}$ of C^∞-schemes and $\rho : G \to H$ of groups, with $\underline{f} \circ \underline{r}(\gamma) = \underline{s}(\rho(\gamma)) \circ \underline{f}$ for all $\gamma \in G$. We will define a *quotient* 1*-morphism* $[\underline{f}, \rho] : \mathcal{X} \to \mathcal{Y}$.

Define a functor $[\underline{f}, \rho] : \mathcal{X} \to \mathcal{Y}$ by $[\underline{f}, \rho] : (A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \mapsto (A, \rho \circ \mu, \underline{T}, \underline{U}, \underline{t}, \underline{f} \circ \underline{u}, \underline{v})$ on objects. For a morphism $(\underline{a}, \tilde{\underline{a}}) : (A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \to (A', \mu', \underline{T}', \underline{U}', \underline{t}', \underline{u}', \underline{v}')$ in $\mathcal{X}$, let $\underline{T} \times_A G, \tilde{\underline{t}}, \tilde{\underline{u}}, \tilde{\underline{v}}, \underline{T} \times_{A'} G, \tilde{\underline{t}}', \tilde{\underline{u}}', \tilde{\underline{v}}'$ be as in Definition C.17, and similarly $\underline{T} \times_A H, \dot{\underline{t}}, \dot{\underline{u}}, \dot{\underline{v}}$ and $\underline{T} \times_{A'} H, \dot{\underline{t}}', \dot{\underline{u}}', \dot{\underline{v}}'$ for the objects $(A, \rho \circ \mu, \underline{T}, \underline{U}, \underline{t}, \underline{f} \circ \underline{u}, \underline{v}), (A', \rho \circ \mu', \underline{T}', \underline{U}', \underline{t}', \underline{f} \circ \underline{u}', \underline{v}')$ in $\mathcal{Y}$. Then $\underline{T} \times_A H \cong (\underline{T} \times_A G) \times_G H$ and $\underline{T} \times_{A'} H \cong (\underline{T} \times_{A'} G) \times_G H$, so the morphism $\tilde{\underline{a}} : \underline{T} \times_A G \to \underline{T}' \times_{A'} G$ and $\mathrm{id}_H : H \to H$ induce a morphism $\dot{\underline{a}} : \underline{T} \times_A H \to \underline{T}' \times_{A'} H$. Define $[\underline{f}, \rho] : (\underline{a}, \tilde{\underline{a}}) \mapsto (\underline{a}, \dot{\underline{a}})$ on morphisms. Then $[\underline{f}, \rho] : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks, which we write as $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$.

We have $[\underline{f}, \rho] \circ \pi_{[\underline{X}/G]} = \pi_{[\underline{Y}/H]} \circ \bar{\underline{f}}$, and if $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$, $[\underline{g}, \sigma] : [\underline{Y}/H] \to [\underline{Z}/I]$ are 1-morphisms then $[\underline{g}, \sigma] \circ [\underline{f}, \rho] = [\underline{g} \circ \underline{f}, \sigma \circ \rho]$.

Definition C.19. Let $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$ and $[\underline{g}, \sigma] : [\underline{X}/G] \to [\underline{Y}/H]$ be quotient 1-morphisms, so that $\underline{f}, \underline{g} : \underline{X} \to \underline{Y}$ and $\rho, \sigma : G \to H$ are morphisms. Suppose $\delta \in H$ satisfies $\sigma(\gamma) = \delta\, \rho(\gamma)\, \delta^{-1}$ for all $\gamma \in G$, and $\underline{g} = \underline{s}(\delta) \circ \underline{f}$. We will define a 2-morphism $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$, which we call a *quotient* 2*-morphism*. Let $(A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})$ be an object in $[\underline{X}/G]$. Define an isomorphism in $[\underline{Y}/H]$:

$$
\begin{aligned}
&[\delta]\big((A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})\big) = (\underline{\mathrm{id}}_{\underline{U}}, \underline{i}_\delta) : [\underline{f}, \rho]\big((A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})\big) = \\
&(A, \rho \circ \mu, \underline{T}, \underline{U}, \underline{t}, \underline{f} \circ \underline{u}, \underline{v}) \to [\underline{g}, \sigma]\big((A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})\big) = (A, \sigma \circ \mu, \underline{T}, \underline{U}, \underline{t}, \underline{g} \circ \underline{u}, \underline{v}),
\end{aligned}
$$

where the isomorphism $\underline{i}_\delta : \underline{T} \times_{\underline{t}, A, \widehat{\rho \circ \mu}} H \to \underline{T} \times_{\underline{t}, A, \widehat{\sigma \circ \mu}} H$ is induced by the isomorphism $\underline{T} \times H \to \underline{T} \times H$ acting as $\underline{\mathrm{id}}_{\underline{T}}$ on $\underline{T}$ and as $\zeta \mapsto \zeta\, \delta^{-1}$ on H, for $\zeta \in H$. Then $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$ is a 2-morphism in $\mathbf{C^\infty Sta}$.

Quotient 2-morphisms have the obvious, strongly functorial properties under vertical and horizontal composition of 2-morphisms. For instance, if $[\underline{f}, \rho], [\underline{g}, \sigma], [\underline{h}, \tau] : [\underline{X}/G] \to [\underline{Y}/H]$ are quotient 1-morphisms and $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$, $[\epsilon] : [\underline{g}, \sigma] \Rightarrow [\underline{h}, \tau]$ are quotient 2-morphisms then $[\epsilon] \odot [\delta] = [\epsilon\delta] : [\underline{f}, \rho] \Rightarrow [\underline{h}, \tau]$.

Remark C.20. (a) Quotient C^∞-stacks $[\underline{X}/G]$ in Definition C.19 are special examples of groupoid stacks $[\underline{V} \rightrightarrows \underline{U}]$ in Definition C.2, with $\underline{U} = \underline{X}$ and $\underline{V} = \underline{X} \times G$. However, Definition C.19 specifies the C^∞-stack $[\underline{X}/G]$ uniquely, whereas the definition of $[\underline{V} \rightrightarrows \underline{U}]$ involves stackification, and so is unique only up to equivalence in $\mathbf{C^\infty Sta}$.

(b) There are several different ways to define $[\underline{X}/G]$, which yield equivalent C^∞-stacks. Definition C.17 is more complicated than it need be. In particular, the category $\mathcal{X} = [\underline{X}/G]$ is equivalent to the full subcategory $\mathcal{X}'$ of objects $(G, \mathrm{id}_G, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})$, in which $A = G$ and $\mu = \mathrm{id}_G : G \to G$. So objects in

$\mathcal{X}'$ can just be written $(\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})$. For morphisms $(\underline{a},\tilde{\underline{a}}):(\underline{T},\underline{U},\underline{t},\underline{u},\underline{v})\to(\underline{T}',\underline{U}',\underline{t}',\underline{u}',\underline{v}')$ in $\mathcal{X}'$, the morphism $\tilde{\underline{a}}:\underline{T}\times_G G\to\underline{T}'\times_G G$ is effectively a morphism $\tilde{\underline{a}}:\underline{T}\to\underline{T}'$ with $\underline{a}\circ\underline{t}(\gamma)=\underline{t}'(\gamma)\circ\tilde{\underline{a}}$ for $\gamma\in G$, and $\underline{u}=\underline{u}'\circ\tilde{\underline{a}}$, and $\underline{a}\circ\underline{v}=\underline{v}'\circ\tilde{\underline{a}}$. This gives a simpler definition of an equivalent C^∞-stack $\mathcal{X}'$.

Our more complicated definition has the advantage that quotient 1- and 2-morphisms in Definitions C.18 and C.19 are *strictly functorial.* In particular, for quotient 1-morphisms $[\underline{f},\rho]:[\underline{X}/G]\to[\underline{Y}/H]$, $[\underline{g},\sigma]:[\underline{Y}/H]\to[\underline{Z}/I]$ we have an equality of 1-morphisms $[\underline{g},\sigma]\circ[\underline{f},\rho]=[\underline{g}\circ\underline{f},\sigma\circ\rho]:[\underline{X}/G]\to[\underline{Z}/I]$, not just a 2-isomorphism. Also, we have an equality $[\underline{f},\rho]\circ\pi_{[\underline{X}/G]}=\pi_{[\underline{Y}/H]}\circ\bar{\underline{f}}$.

(c) Studying quotient C^∞-stacks $[\underline{X}/G]$ and their 1- and 2-morphisms is a good way to develop geometric intuition about Deligne–Mumford C^∞-stacks (including orbifolds) and their 1- and 2-morphisms.

(d) If $[\underline{X}/G],[\underline{Y}/H]$ are quotient C^∞-stacks, then general 1-morphisms $f:[\underline{X}/G]\to[\underline{Y}/H]$ in $\mathbf{C^\infty Sta}$ need not be quotient 1-morphisms $[\underline{f},\rho]$, or even 2-isomorphic to $[\underline{f},\rho]$. But Theorem C.25(b) says that $f\cong[\underline{f},\rho]$ locally in $[\underline{X}/G]$.

(e) If $[\underline{f},\rho],[\underline{g},\sigma]:[\underline{X}/G]\to[\underline{Y}/H]$ are quotient 1-morphisms, and $[\underline{X}/G]$ is connected, then Theorem C.25(d) says that all 2-morphisms $\eta:[\underline{f},\rho]\Rightarrow[\underline{g},\sigma]$ are quotient 2-morphisms $[\delta]:[\underline{f},\rho]\Rightarrow[\underline{g},\sigma]$.

(f) Quotient 1-morphisms $[\underline{f},\rho]:[\underline{X}/G]\to[\underline{Y}/H]$ with $\rho:G\to H$ an isomorphism are strongly representable, in the sense of §C.3.

C.5 Deligne–Mumford C^∞-stacks

Deligne–Mumford stacks in algebraic geometry are locally modelled on quotient stacks $[X/G]$ for X an affine scheme and G a finite group. This motivates:

Definition C.21. A *Deligne–Mumford C^∞-stack* is a C^∞-stack $\mathcal{X}$ which admits an open cover $\{\mathcal{Y}_a:a\in A\}$ with each $\mathcal{Y}_a$ equivalent to a quotient stack $[\underline{U}_a/G_a]$ in §C.4 for $\underline{U}_a$ an affine C^∞-scheme and G_a a finite group. We call $\mathcal{X}$ a *locally fair*, or *locally finitely presented*, Deligne–Mumford C^∞-stack if it has such an open cover with each $\underline{U}_a$ a fair, or finitely presented, affine C^∞-scheme, respectively. We call $\mathcal{X}$ *second countable, compact, locally compact,* or *paracompact*, if the underlying topological space $\mathcal{X}_{\rm top}$ of Definition C.7 is second countable, compact, locally compact, or paracompact, respectively.

A Deligne–Mumford C^∞-stack $\mathcal{X}$ is separated, in the sense of Definition C.5, if and only if the topological space $\mathcal{X}_{\rm top}$ of Definition C.7 is Hausdorff.

Write $\mathbf{DMC^\infty Sta}$ for the full 2-subcategory of Deligne–Mumford C^∞-stacks in $\mathbf{C^\infty Sta}$. Write $\mathbf{DMC^\infty Sta^{lfp}}$, $\mathbf{DMC^\infty Sta^{lf}}$, and $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$ for the full 2-subcategories of locally finitely presented Deligne–Mumford C^∞-stacks, and of locally fair Deligne–Mumford C^∞-stacks, and of separated, second countable, locally fair Deligne–Mumford C^∞-stacks, respectively.

The next theorem comes from [56, Th.s 9.10, 9.17 & Prop. 9.6], except for the parts about $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$, which follow from the case of $\mathbf{DMC^\infty Sta^{lf}}$.

Theorem C.22. **(a)** $\mathbf{DMC^\infty Sta}, \mathbf{DMC^\infty Sta^{lfp}}, \mathbf{DMC^\infty Sta^{lf}}, \mathbf{DMC^\infty Sta^{lf}_{ssc}}$ *are closed under fibre products in* $\mathbf{C^\infty Sta}$.

(b) $\mathbf{DMC^\infty Sta}, \mathbf{DMC^\infty Sta^{lfp}}, \mathbf{DMC^\infty Sta^{lf}}$ *and* $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$ *are closed under taking open C^∞-substacks in* $\mathbf{C^\infty Sta}$.

(c) *A C^∞-stack $\mathcal{X}$ is separated and Deligne–Mumford if and only if it is equivalent to a groupoid stack $[\underline{V} \rightrightarrows \underline{U}]$ where $\underline{U}, \underline{V}$ are separated C^∞-schemes, $\underline{s} : \underline{V} \to \underline{U}$ is étale, and $\underline{s} \times \underline{t} : \underline{V} \to \underline{U} \times \underline{U}$ is universally closed.*

(d) *A C^∞-stack $\mathcal{X}$ is separated, Deligne–Mumford and locally fair (or locally finitely presented) if and only if it is equivalent to a groupoid stack $[\underline{V} \rightrightarrows \underline{U}]$ with $\underline{U}, \underline{V}$ separated, locally fair (or locally finitely presented) C^∞-schemes, $\underline{s} : \underline{V} \to \underline{U}$ étale, and $\underline{s} \times \underline{t} : \underline{V} \to \underline{U} \times \underline{U}$ proper.*

If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack then the orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$ in Definition C.8 is a finite group for all $[x]$ in $\mathcal{X}_{\rm top}$. Here is [56, Th. 9.20]:

Theorem C.23. *Suppose $\mathcal{X}$ is a Deligne–Mumford C^∞-stack with $\mathrm{Iso}_{\mathcal{X}}([x]) \cong \{1\}$ for all $[x] \in \mathcal{X}_{\rm top}$. Then $\mathcal{X}$ is equivalent to $\bar{\underline{X}}$ for some C^∞-scheme $\underline{X}$.*

Recall that a 1-morphism of C^∞-stacks $f : \mathcal{X} \to \mathcal{Y}$ is *representable* if whenever $\underline{U}$ is a C^∞-scheme and $g : \bar{\underline{U}} \to \mathcal{Y}$ is a 1-morphism then the fibre product $\mathcal{W} = \mathcal{X} \times_{f,\mathcal{Y},g} \bar{\underline{U}}$ in $\mathbf{C^\infty Sta}$ is equivalent to a C^∞-scheme $\bar{\underline{V}}$. Using Theorem C.23 we may deduce, as in [56, Cor. 9.21]:

Corollary C.24. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks. Then f is representable if and only if $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$ in Definition C.8 is injective for all $[x] \in \mathcal{X}_{\rm top}$ with $f_{\rm top}([x]) = [y] \in \mathcal{Y}_{\rm top}$.*

In [56, Th. 9.18 & Prop. 9.19] we show that Deligne–Mumford C^∞-stacks and their 1- and 2-morphisms are locally modelled on quotient C^∞-stacks, quotient 1-morphisms and quotient 2-morphisms from §C.4.

Theorem C.25. **(a)** *Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack and $[x] \in \mathcal{X}_{\rm top}$, and write $G = \mathrm{Iso}_{\mathcal{X}}([x])$. Then there exists a quotient C^∞-stack $[\underline{U}/G]$ for $\underline{U}$ an affine C^∞-scheme, and a 1-morphism $i : [\underline{U}/G] \to \mathcal{X}$ which is an equivalence with an open C^∞-substack $\mathcal{U}$ in $\mathcal{X}$, such that $i_{\rm top} : [u] \mapsto [x] \in \mathcal{U}_{\rm top} \subseteq \mathcal{X}_{\rm top}$ for some fixed point u of G in $\underline{U}$.*

(b) *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks, and $[x] \in \mathcal{X}_{\rm top}$ with $f_{\rm top} : [x] \mapsto [y] \in \mathcal{Y}_{\rm top}$, and write $G = \mathrm{Iso}_{\mathcal{X}}([x])$ and $H = \mathrm{Iso}_{\mathcal{Y}}([y])$. Part* **(a)** *gives 1-morphisms $i : [\underline{U}/G] \to \mathcal{X}$, $j : [\underline{V}/H] \to \mathcal{Y}$ which are equivalences with open $\mathcal{U} \subseteq \mathcal{X}$, $\mathcal{V} \subseteq \mathcal{Y}$, such that $i_{\rm top} : [u] \mapsto [x] \in \mathcal{U}_{\rm top} \subseteq \mathcal{X}_{\rm top}$, $j_{\rm top} : [v] \mapsto [y] \in \mathcal{V}_{\rm top} \subseteq \mathcal{Y}_{\rm top}$ for u, v fixed points of G, H in $\underline{U}, \underline{V}$.*

Then there exists a G-invariant open neighbourhood $\underline{U}'$ of u in $\underline{U}$ and a quotient 1-morphism $[\underline{f}, \rho] : [\underline{U}'/G] \to [\underline{V}/H]$ such that $\underline{f}(u) = v$, and $\rho : G \to H$ is $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$, fitting into a 2-commutative diagram:

$$\begin{tikzcd} {[\underline{U}'/G]} \arrow[rr, "{[\underline{f},\rho]}"] \arrow[d, "i|_{[\underline{U}'/G]}"] \arrow[drr, phantom, "\zeta\Uparrow"] && {[\underline{V}/H]} \arrow[d, "j"] \\ \mathcal{X} \arrow[rr, "f"] && \mathcal{Y}. \end{tikzcd}$$

(c) *Let $f,g:\mathcal{X}\to\mathcal{Y}$ be 1-morphisms of Deligne–Mumford C^∞-stacks and $\eta:f\Rightarrow g$ a 2-morphism, let $[x]\in\mathcal{X}_{\rm top}$ with $f_{\rm top}:[x]\mapsto[y]\in\mathcal{Y}_{\rm top}$, and write $G=\mathrm{Iso}_{\mathcal{X}}([x])$ and $H=\mathrm{Iso}_{\mathcal{Y}}([y])$. Part* **(a)** *gives $i:[\underline{U}/G]\to\mathcal{X}$, $j:[\underline{V}/H]\to\mathcal{Y}$ which are equivalences with open $\mathcal{U}\subseteq\mathcal{X}$, $\mathcal{V}\subseteq\mathcal{Y}$ and map $i_{\rm top}:[u]\mapsto[x]$, $j_{\rm top}:[v]\mapsto[y]$ for u,v fixed points of G,H.*

By making $\underline{U}'$ smaller, we can take the same $\underline{U}'$ in **(b)** *for both f,g. Thus part* **(b)** *gives a G-invariant open $\underline{U}'\subseteq\underline{U}$, quotient morphisms $[\underline{f},\rho]:[\underline{U}'/G]\to[\underline{V}/H]$ and $[\underline{g},\sigma]:[\underline{U}'/G]\to[\underline{V}/H]$ with $\underline{f}(u)=\underline{g}(u)=v$ and $\rho=f_*:\mathrm{Iso}_{\mathcal{X}}([x])\to\mathrm{Iso}_{\mathcal{Y}}([y])$, $\sigma=g_*:\mathrm{Iso}_{\mathcal{X}}([x])\to\mathrm{Iso}_{\mathcal{Y}}([y])$, and 2-morphisms $\zeta:f\circ i|_{[\underline{U}'/G]}\Rightarrow j\circ[\underline{f},\rho]$, $\theta:g\circ i|_{[\underline{U}'/G]}\Rightarrow j\circ[\underline{g},\sigma]$.*

Then there exists a G-invariant open neighbourhood $\underline{U}''$ of u in $\underline{U}'$ and $\delta\in H$ such that $\sigma(\gamma)=\delta\,\rho(\gamma)\circ\delta^{-1}$ for all $\gamma\in G$ and $\underline{g}|_{\underline{U}''}=\underline{s}(\delta)\circ\underline{f}|_{\underline{U}''}$, so that $[\delta]:[\underline{f}|_{\underline{U}''},\rho]\Rightarrow[\underline{g}|_{\underline{U}''},\sigma]$ is a quotient 2-morphism, and the following diagram of 2-morphisms in $\mathbf{C^\infty Sta}$ *commutes:*

$$\begin{tikzcd}[column sep=huge]
f\circ i|_{[\underline{U}''/G]} \arrow[r, Rightarrow, "\eta*\mathrm{id}_{i|_{[\underline{U}''/G]}}"'] \arrow[d, Rightarrow, "\zeta|_{[\underline{U}''/G]}"] & g\circ i|_{[\underline{U}''/G]} \arrow[d, Rightarrow, "\theta|_{[\underline{U}''/G]}"'] \\
j\circ[\underline{f}|_{\underline{U}''},\rho] \arrow[r, Rightarrow, "\mathrm{id}_j*[\delta]"] & j\circ[\underline{g}|_{\underline{U}''},\sigma].
\end{tikzcd}$$

(d) *Let $[\underline{f},\rho],[\underline{g},\sigma]:[\underline{X}/G]\to[\underline{Y}/H]$ be quotient 1-morphisms of quotient C^∞-stacks, and suppose $[\underline{X}/G]$ is connected, that is, X/G is connected as a topological space. Then every 2-morphism $\eta:[\underline{f},\rho]\Rightarrow[\underline{g},\sigma]$ in* $\mathbf{C^\infty Sta}$ *is a quotient 2-morphism $[\delta]:[\underline{f},\rho]\Rightarrow[\underline{g},\sigma]$ from Definition C.19, for some unique $\delta\in H$.*

In [56, Def. 8.17 & Th. 9.11] we define the *coarse moduli C^∞-scheme* of a locally fair Deligne–Mumford C^∞-stack.

Theorem C.26. *Let $\mathcal{X}$ be a locally fair Deligne–Mumford C^∞-stack. Then the topological space $\mathcal{X}_{\rm top}$ of Definition C.7 may in a unique way be given the structure of a locally fair C^∞-scheme $\underline{\mathcal{X}}_{\rm top}=(\mathcal{X}_{\rm top},\mathcal{O}_{\mathcal{X}_{\rm top}})$ called the **coarse moduli C^∞-scheme**, with a 1-morphism $\pi:\mathcal{X}\to\bar{\underline{\mathcal{X}}}_{\rm top}$ called the **structural 1-morphism**, with the universal property that if $f:\mathcal{X}\to\bar{\underline{Y}}$ is a 1-morphism in* $\mathbf{C^\infty Sta}$ *for any C^∞-scheme $\underline{Y}$ then $f\cong\bar{\underline{g}}\circ\pi$ for some unique C^∞-scheme morphism $\underline{g}:\underline{\mathcal{X}}_{\rm top}\to\underline{Y}$.*

If $\mathcal{X}$ is locally finitely presented, or separated, or paracompact, or second countable, then $\underline{\mathcal{X}}_{\rm top}$ is also locally finitely presented, or separated, or paracompact, or second countable, respectively.

Remark C.27. In §B.4 we discussed *partitions of unity* on C^∞-schemes. We can use Theorem C.26 to extend these ideas to Deligne–Mumford C^∞-stacks.

Let $\mathcal{X}$ be a separated, paracompact, locally fair Deligne–Mumford C^∞-stack, and $\{\mathcal{V}_a:a\in A\}$ an open cover of $\mathcal{X}$. Then Theorem C.26 gives a coarse moduli C^∞-scheme $\underline{\mathcal{X}}_{\rm top}$, which is separated, paracompact, and locally fair, with structural 1-morphism $\pi:\mathcal{X}\to\bar{\underline{\mathcal{X}}}_{\rm top}$, and $\{\underline{\mathcal{V}}_{a,\rm top}:a\in A\}$ is an open cover of $\underline{\mathcal{X}}_{\rm top}$. Thus Proposition B.21 gives a partition of unity $\{\eta_a:a\in A\}$ on $\underline{\mathcal{X}}_{\rm top}$ subordinate to $\{\underline{\mathcal{V}}_{a,\rm top}:a\in A\}$.

Then $\{\pi^*(\eta_a) : a \in A\}$ is (in a suitable sense) a partition of unity on $\mathcal{X}$ subordinate to $\{\mathcal{V}_a : a \in A\}$, where we may interpret $\pi^*(\eta_a)$ as a global section of the structure sheaf $\mathcal{O}_{\mathcal{X}}$ of Example C.42 supported on $\mathcal{V}_a$. We will return to these ideas in Example C.33.

In [56, §9.5] we discuss effective Deligne–Mumford C^∞-stacks.

Definition C.28. A Deligne–Mumford C^∞-stack $\mathcal{X}$ is called *effective* if whenever $[x] \in \mathcal{X}_{\mathrm{top}}$ and $\mathcal{X}$ near $[x]$ is locally modelled near $[x]$ on a quotient C^∞-stack $[\underline{U}/G]$ near $[u]$, where $G = \mathrm{Iso}_{\mathcal{X}}([x])$ and $u \in \underline{U}$ is fixed by G, as in Theorem C.25(a), then G acts effectively on $\underline{U}$ near u. That is, for each $1 \neq \gamma \in G$, we have $\underline{r}(\gamma) \not\equiv \underline{\mathrm{id}}_{\underline{U}}$ near u in $\underline{U}$, where $\underline{r} : G \to \mathrm{Aut}(\underline{U})$ is the G-action.

Here the C^∞-scheme $\underline{U}$ in Theorem C.25(a) is determined by $\mathcal{X}, [x]$ up to G-equivariant isomorphism locally near u. Hence to test whether $\mathcal{X}$ is effective, it is enough to consider one choice of $[\underline{U}/G]$ for each $[x] \in \mathcal{X}_{\mathrm{top}}$.

A quotient C^∞-stack $[\underline{X}/G]$ is effective if and only if the action $\underline{r} : G \to \mathrm{Aut}(\underline{X})$ of G on $\underline{X}$ is *locally effective*, that is, if for each $1 \neq \gamma \in G$ we have $\underline{r}(\gamma)|_{\underline{U}} \not\equiv \underline{\mathrm{id}}_{\underline{U}}$ for every nonempty open C^∞-subscheme $\underline{U} \subseteq \underline{X}$. If a Deligne–Mumford C^∞-stack $\mathcal{X}$ is a C^∞-scheme, it is automatically effective. Quotients $[\underline{*}/G]$ for $G \neq \{1\}$ are not effective.

Effective orbifolds are important in Chapters 8–13. Here [56, Prop. 9.24] is a uniqueness property of 2-morphisms of effective Deligne–Mumford C^∞-stacks. It will be useful in §9.4, when we show that gluing effective d-stacks by equivalences is simpler than gluing general d-stacks by equivalences. Embeddings and submersions of C^∞-stacks are defined in §C.1.

Proposition C.29. *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of Deligne–Mumford C^∞-stacks. Suppose any one of the following conditions hold:*

- **(i)** *$\mathcal{X}$ is effective and f is an embedding of C^∞-stacks (this implies $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is an isomorphism for each $[x] \in \mathcal{X}_{\mathrm{top}}$);*
- **(ii)** *$\mathcal{Y}$ is effective and f is a submersion; or*
- **(iii)** *$\mathcal{Y}$ is a C^∞-scheme.*

Then there exists at most one 2-morphism $\eta : f \Rightarrow g$.

A similar proof shows that if $f, g : \mathcal{X} \to \mathcal{Y}$ are arbitrary 1-morphisms of Deligne–Mumford C^∞-stacks with $\mathcal{X}$ connected, then there are at most finitely many 2-morphisms $\eta : f \Rightarrow g$.

C.6 Quasicoherent sheaves on C^∞-stacks

In [56, §10] the author studied sheaves on Deligne–Mumford C^∞-stacks. We begin by discussing sheaves of $\mathcal{O}_{\mathcal{X}}$-modules, and quasicoherent sheaves.

Definition C.30. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Define a category $\mathcal{C}_{\mathcal{X}}$ to have objects pairs $(\underline{U}, u)$ where $\underline{U}$ is a C^∞-scheme and $u : \bar{\underline{U}} \to \mathcal{X}$ is an étale 1-morphism, and morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ where $\underline{f}:\underline{U}\to\underline{V}$ is an étale morphism of C^∞-schemes, and $\eta:u\Rightarrow v\circ\bar{\underline{f}}$ is a 2-isomorphism. If $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ and $(\underline{g},\zeta):(\underline{V},v)\to(\underline{W},w)$ are morphisms in $\mathcal{C}_{\mathcal{X}}$ then we define the composition $(\underline{g},\zeta)\circ(\underline{f},\eta)$ to be $(\underline{g}\circ\underline{f},\theta):(\underline{U},u)\to(\underline{W},w)$, where θ is the composition of 2-morphisms across the diagram:

$$\begin{tikzcd}
\bar{\underline{U}} \arrow[dr, "\bar{\underline{f}}"] \arrow[drr, bend left, "u"] \arrow[dd, "\overline{\underline{g}\circ\underline{f}}"'] \arrow[dr, phantom, bend left=50, "\Downarrow\eta"] & & \\
\arrow[r, phantom, "\Leftarrow_{\mathrm{id}}"] & \bar{\underline{V}} \arrow[r, "v"] \arrow[dl, "\bar{\underline{g}}"] & \mathcal{X}. \\
\bar{\underline{W}} \arrow[urr, bend right, "w"'] \arrow[ur, phantom, bend right=50, "\Downarrow\zeta"] & &
\end{tikzcd}$$

Define a *sheaf of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{E}$*, or just an *$\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$*, to assign a sheaf of $\mathcal{O}_U$-modules $\mathcal{E}(\underline{U},u)$ on $\underline{U}=(U,\mathcal{O}_U)$ for all objects $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$, and an isomorphism of $\mathcal{O}_U$-modules $\mathcal{E}_{(\underline{f},\eta)}:\underline{f}^*(\mathcal{E}(\underline{V},v))\to\mathcal{E}(\underline{U},u)$ for all morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ in $\mathcal{C}_{\mathcal{X}}$, such that for all $(\underline{f},\eta),(\underline{g},\zeta),(\underline{g}\circ\underline{f},\theta)$ as above the following diagram of isomorphisms of sheaves of $\mathcal{O}_U$-modules commutes:

$$\begin{tikzcd}
(\underline{g}\circ\underline{f})^*\bigl(\mathcal{E}(\underline{W},w)\bigr) \arrow[rrr, "\mathcal{E}_{(\underline{g}\circ\underline{f},\theta)}"'] \arrow[dr, "I_{\underline{f},\underline{g}}(\mathcal{E}(\underline{W},w))"'] & & & \mathcal{E}(\underline{U},u), \\
& \underline{f}^*\bigl(\underline{g}^*(\mathcal{E}(\underline{W},w))\bigr) \arrow[r, "\underline{f}^*(\mathcal{E}_{(\underline{g},\zeta)})"] & \underline{f}^*\bigl(\mathcal{E}(\underline{V},v)\bigr) \arrow[ur, "\mathcal{E}_{(\underline{f},\eta)}"'] &
\end{tikzcd} \tag{C.1}$$

for $I_{\underline{f},\underline{g}}(\mathcal{E}(\underline{W},w))$ as in Definition B.34.

A *morphism of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules* $\phi:\mathcal{E}\to\mathcal{F}$ assigns a morphism of $\mathcal{O}_U$-modules $\phi(\underline{U},u):\mathcal{E}(\underline{U},u)\to\mathcal{F}(\underline{U},u)$ for each object $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$, such that for all morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ in $\mathcal{C}_{\mathcal{X}}$ the following commutes:

$$\begin{tikzcd}
\underline{f}^*\bigl(\mathcal{E}(\underline{V},v)\bigr) \arrow[r, "\mathcal{E}_{(\underline{f},\eta)}"'] \arrow[d, "\underline{f}^*(\phi(\underline{V},v))"'] & \mathcal{E}(\underline{U},u) \arrow[d, "\phi(\underline{U},u)"] \\
\underline{f}^*\bigl(\mathcal{F}(\underline{V},v)\bigr) \arrow[r, "\mathcal{F}_{(\underline{f},\eta)}"] & \mathcal{F}(\underline{U},u).
\end{tikzcd} \tag{C.2}$$

We call $\mathcal{E}$ *quasicoherent*, or *coherent*, or a *vector bundle of rank n*, if $\mathcal{E}(\underline{U},u)$ is quasicoherent, or coherent, or a vector bundle of rank n, respectively, for all $(\underline{U},u)\in\mathcal{C}_{\mathcal{X}}$. Write $\mathcal{O}_{\mathcal{X}}$-mod for the category of $\mathcal{O}_{\mathcal{X}}$-modules, and $\mathrm{qcoh}(\mathcal{X})$, $\mathrm{coh}(\mathcal{X})$, $\mathrm{vect}(\mathcal{X})$ for the full subcategories of quasicoherent sheaves, coherent sheaves, and vector bundles, respectively.

Here are [56, Prop. 10.3 & Ex. 10.4].

Proposition C.31. *Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Then* $\mathcal{O}_{\mathcal{X}}$-mod *is an abelian category, and* $\mathrm{qcoh}(\mathcal{X})$ *is closed under kernels, cokernels and extensions in* $\mathcal{O}_{\mathcal{X}}$-mod, *so it is also an abelian category. Also* $\mathrm{coh}(\mathcal{X})$ *is closed under cokernels and extensions in* $\mathcal{O}_{\mathcal{X}}$-mod, *but it may not be closed under kernels in* $\mathcal{O}_{\mathcal{X}}$-mod, *so may not be abelian. If $\mathcal{X}$ is locally fair then* $\mathrm{qcoh}(\mathcal{X})=\mathcal{O}_{\mathcal{X}}$-mod.

Example C.32. Let $\underline{X}$ be a C^∞-scheme. Then $\mathcal{X} = \bar{\underline{X}}$ is a Deligne–Mumford C^∞-stack. We will define an *inclusion functor* $\mathcal{I}_{\underline{X}} : \mathcal{O}_X\text{-mod} \to \mathcal{O}_{\mathcal{X}}\text{-mod}$. Let $\mathcal{E}$ be an object in $\mathcal{O}_X$-mod. If $(\underline{U}, u)$ is an object in $\mathcal{C}_{\mathcal{X}}$ then $u : \bar{\underline{U}} \to \mathcal{X} = \bar{\underline{X}}$ is 2-isomorphic to $\bar{\underline{u}} : \bar{\underline{U}} \to \bar{\underline{X}}$ for some unique morphism $\underline{u} : \underline{U} \to \underline{X}$. Define $\mathcal{E}'(\underline{U}, u) = \underline{u}^*(\mathcal{E})$. If $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ is a morphism in $\mathcal{C}_{\mathcal{X}}$ and $\underline{u}, \underline{v}$ are associated to u, v as above, so that $\underline{u} = \underline{v} \circ \underline{f}$, then define

$$\mathcal{E}'_{(\underline{f},\eta)} = I_{\underline{f},\underline{v}}(\mathcal{E})^{-1} : \underline{f}^*(\mathcal{E}'(\underline{V}, v)) = \underline{f}^*\bigl(\underline{v}^*(\mathcal{E})\bigr) \longrightarrow (\underline{v} \circ \underline{f})^*(\mathcal{E}) = \mathcal{E}'(\underline{U}, u).$$

Then (C.1) commutes for all $(\underline{f}, \eta), (\underline{g}, \zeta)$, so $\mathcal{E}'$ is an $\mathcal{O}_{\mathcal{X}}$-module.

If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of $\mathcal{O}_X$-modules then we define a morphism $\phi' : \mathcal{E}' \to \mathcal{F}'$ in $\mathcal{O}_{\mathcal{X}}$-mod by $\phi'(\underline{U}, u) = \underline{u}^*(\phi)$ for $\underline{u}$ associated to u as above. Then defining $\mathcal{I}_{\underline{X}} : \mathcal{E} \mapsto \mathcal{E}'$, $\mathcal{I}_{\underline{X}} : \phi \mapsto \phi'$ gives a functor $\mathcal{O}_X\text{-mod} \to \mathcal{O}_{\mathcal{X}}\text{-mod}$, which induces equivalences between the categories $\mathcal{O}_X\text{-mod}, \mathrm{qcoh}(\underline{X}), \mathrm{coh}(\underline{X})$ defined in §B.7 and $\mathcal{O}_{\mathcal{X}}\text{-mod}, \mathrm{qcoh}(\mathcal{X}), \mathrm{coh}(\mathcal{X})$ above.

We explain how to use *partitions of unity* to join morphisms of quasicoherent sheaves $\mathcal{E}, \mathcal{F}$ defined on an open cover of $\mathcal{X}$, using the ideas of Remark C.27. This is used in proofs about gluing d-orbifolds by equivalences in §9.4 and §11.4.

Example C.33. Let $\mathcal{X}$ be a separated, paracompact, locally fair Deligne–Mumford C^∞-stack, and $\{\mathcal{V}_a : a \in A\}$ an open cover of $\mathcal{X}$. Then as in Remark C.27 we have a coarse moduli C^∞-scheme $\underline{\mathcal{X}}_{\mathrm{top}}$, with structural 1-morphism $\pi : \mathcal{X} \to \bar{\underline{\mathcal{X}}}_{\mathrm{top}}$, and an open cover $\bigl\{\underline{\mathcal{V}}_{a,\mathrm{top}} : a \in A\bigr\}$ of $\underline{\mathcal{X}}_{\mathrm{top}}$, and we may choose a partition of unity $\{\eta_a : a \in A\}$ on $\underline{\mathcal{X}}_{\mathrm{top}}$ subordinate to $\bigl\{\underline{\mathcal{V}}_{a,\mathrm{top}} : a \in A\bigr\}$.

Suppose $\mathcal{E}, \mathcal{F} \in \mathrm{qcoh}(\mathcal{X})$, and $\alpha_a : \mathcal{E}|_{\mathcal{V}_a} \to \mathcal{F}|_{\mathcal{V}_a}$ is a morphism in $\mathrm{qcoh}(\mathcal{V}_a)$ for each $a \in A$. We will construct a morphism $\beta : \mathcal{E} \to \mathcal{F}$ in $\mathrm{qcoh}(\mathcal{X})$ which is morally given by the locally finite sum

$$\beta = \textstyle\sum_{a\in A} \pi^*(\eta_a) \cdot \alpha_a, \tag{C.3}$$

where the morphism $\pi^*(\eta_a) \cdot \alpha_a : \mathcal{E} \to \mathcal{F}$ in $\mathrm{qcoh}(\mathcal{X})$ is supported on $\mathcal{V}_a$.

Use the notation of Definition C.30. For each $(\underline{U}, u) \in \mathcal{C}_{\mathcal{X}}$, there is a unique C^∞-scheme morphism $\underline{v} : \underline{U} \to \underline{\mathcal{X}}_{\mathrm{top}}$ with $\bar{\underline{v}} \cong \pi \circ u : \bar{\underline{U}} \to \bar{\underline{\mathcal{X}}}_{\mathrm{top}}$. Then $\bigl\{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}}) : a \in A\bigr\}$ is an open cover of $\underline{U}$. We have a morphism $v_\sharp : \mathcal{O}_{\mathcal{X}_{\mathrm{top}}} \to v_*(\mathcal{O}_U)$ of sheaves of C^∞-rings related to $v^\sharp$ in $\underline{v} = (v, v^\sharp)$ as in (B.4), so $v_\sharp(\mathcal{X}_{\mathrm{top}}) : \mathcal{O}_{\mathcal{X}_{\mathrm{top}}}(\mathcal{X}_{\mathrm{top}}) \to \mathcal{O}_U(U)$ is a morphism of C^∞-rings. As $\eta_a \in \mathcal{O}_{\mathcal{X}_{\mathrm{top}}}(\mathcal{X}_{\mathrm{top}})$ for $a \in A$, we see that $v_\sharp(\mathcal{X}_{\mathrm{top}})(\eta_a) \in \mathcal{O}_U(U)$, and $\bigl\{v_\sharp(\mathcal{X}_{\mathrm{top}})(\eta_a) : a \in A\bigr\}$ is a partition of unity on $\underline{U}$ subordinate to the open cover $\bigl\{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}}) : a \in A\bigr\}$.

Now $\bigl(\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}}), u|_{\overline{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})}}\bigr)$ lies in $\mathcal{C}_{\mathcal{V}_a}$ for each $a \in A$, with

$$\begin{aligned}
\mathcal{E}\bigl(\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}}), u|_{\overline{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})}}\bigr) &= \mathcal{E}(\underline{U}, u)|_{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})}, \\
\mathcal{F}\bigl(\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}}), u|_{\overline{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})}}\bigr) &= \mathcal{F}(\underline{U}, u)|_{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})}.
\end{aligned}$$

Hence $\alpha_a : \mathcal{E}|_{\mathcal{V}_a} \to \mathcal{F}|_{\mathcal{V}_a}$ induces a morphism in $\mathrm{qcoh}\bigl(\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})\bigr)$:

$$\alpha_a\bigl(\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}}), u|_{\overline{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})}}\bigr) : \mathcal{E}(\underline{U}, u)|_{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})} \longrightarrow \mathcal{F}(\underline{U}, u)|_{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})}.$$

Define a morphism $\beta(\underline{U},u):\mathcal{E}(\underline{U},u)\rightarrow\mathcal{F}(\underline{U},u)$ in $\mathrm{qcoh}(\underline{U})$ by

$$\beta(\underline{U},u)=\textstyle\sum_{a\in A}v_\sharp(\mathcal{X}_{\mathrm{top}})(\eta_a)\cdot\alpha_a\big(\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}}),u\vert_{\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})}\big). \tag{C.4}$$

Here $v_\sharp(\mathcal{X}_{\mathrm{top}})(\eta_a)$ is defined on $\underline{U}$ but supported on $\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})$, and $\alpha_a(\cdots)$ is defined on $\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})$, so the product makes sense as a morphism $\mathcal{E}(\underline{U},u)\rightarrow\mathcal{F}(\underline{U},u)$ on $\underline{U}$ supported on $\underline{v}^{-1}(\underline{\mathcal{V}}_{a,\mathrm{top}})$. The sum $\sum_{a\in A}\cdots$ in (C.4) is locally finite, and so has a unique limit in the sheaf of morphisms $\mathcal{E}\rightarrow\mathcal{F}$. Thus $\beta(\underline{U},u)$ in (C.4) is well defined. These $\beta(\underline{U},u)$ for all $(\underline{U},u)\in\mathcal{C}_{\mathcal{X}}$ form a morphism $\beta:\mathcal{E}\rightarrow\mathcal{F}$ in $\mathrm{qcoh}(\mathcal{X})$, and (C.3) holds as (C.4) is (C.3) at $(\underline{U},u)\in\mathcal{C}_{\mathcal{X}}$.

In [56, §10.2] we explain how to describe sheaves of $\mathcal{O}_{\mathcal{X}}$-modules on a Deligne–Mumford C^∞-stack $\mathcal{X}$ in terms of sheaves on $\underline{U}$ for an étale atlas $\Pi:\bar{\underline{U}}\rightarrow\mathcal{X}$ for $\mathcal{X}$. Here are [56, Def. 10.5 & Th. 10.6].

Definition C.34. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Then $\mathcal{X}$ admits an étale atlas $\Pi:\bar{\underline{U}}\rightarrow\mathcal{X}$, and as in Definition C.2 from Π we can construct a groupoid $(\underline{U},\underline{V},\underline{s},\underline{t},\underline{u},\underline{i},\underline{m})$ in $\mathbf{C^\infty Sch}$, with $\underline{s},\underline{t}:\underline{V}\rightarrow\underline{U}$ étale, such that $\mathcal{X}$ is equivalent to the groupoid stack $[\underline{V}\rightrightarrows\underline{U}]$. Define a $(\underline{V}\rightrightarrows\underline{U})$-*module* to be a pair (E,Φ) where E is an $\mathcal{O}_U$-module and $\Phi:\underline{s}^*(E)\rightarrow\underline{t}^*(E)$ is an isomorphism of $\mathcal{O}_V$-modules, such that

$$\begin{aligned}I_{\underline{m},\underline{t}}(E)^{-1}\circ\underline{m}^*(\Phi)\circ I_{\underline{m},\underline{s}}(E)=\big(&I_{\underline{\pi}_1,\underline{t}}(E)^{-1}\circ\underline{\pi}_1^*(\Phi)\circ I_{\underline{\pi}_1,\underline{s}}(E)\big)\circ\\ &\big(I_{\underline{\pi}_2,\underline{t}}(E)^{-1}\circ\underline{\pi}_2^*(\Phi)\circ I_{\underline{\pi}_2,\underline{s}}(E)\big)\end{aligned} \tag{C.5}$$

in morphisms of $\mathcal{O}_W$-modules $(\underline{s}\circ\underline{m})^*(E)\rightarrow(\underline{t}\circ\underline{m})^*(E)$, where $\underline{W}=\underline{V}\times_{\underline{s},\underline{U},\underline{t}}\underline{V}$ and $\underline{\pi}_1,\underline{\pi}_2:\underline{W}\rightarrow\underline{V}$ are the projections. Define a *morphism of* $(\underline{V}\rightrightarrows\underline{U})$-*modules* $\phi:(E,\Phi)\rightarrow(F,\Psi)$ to be a morphism of $\mathcal{O}_U$-modules $\phi:E\rightarrow F$ such that $\Psi\circ\underline{s}^*(\phi)=\underline{t}^*(\phi)\circ\Phi:\underline{s}^*(E)\rightarrow\underline{t}^*(F)$. Then $(\underline{V}\rightrightarrows\underline{U})$-modules form an abelian category $(\underline{V}\rightrightarrows\underline{U})$-mod. Write $\mathrm{qcoh}(\underline{V}\rightrightarrows\underline{U})$ and $\mathrm{coh}(\underline{V}\rightrightarrows\underline{U})$ for the full subcategories of (E,Φ) in $(\underline{V}\rightrightarrows\underline{U})$-mod with E quasicoherent, or coherent, respectively. Then $\mathrm{qcoh}(\underline{V}\rightrightarrows\underline{U})$ is abelian. Define a functor $F_\Pi:\mathcal{O}_{\mathcal{X}}\text{-mod}\rightarrow(\underline{V}\rightrightarrows\underline{U})\text{-mod}$ by $F_\Pi:\mathcal{E}\mapsto\big(\mathcal{E}(\underline{U},\Pi),\mathcal{E}_{(\underline{t},\eta)}^{-1}\circ\mathcal{E}_{(\underline{s},\mathrm{id}_{\Pi\circ\underline{s}})}\big)$ and $F_\Pi:\phi\mapsto\phi(\underline{U},\Pi)$. As in [56, §10.2], $F_\Pi(\mathcal{E})$ does satisfy (C.5) and so lies in $(\underline{V}\rightrightarrows\underline{U})$-mod, and it also maps $\mathrm{qcoh},\mathrm{coh}(\mathcal{X})$ to $\mathrm{qcoh},\mathrm{coh}(\underline{V}\rightrightarrows\underline{U})$.

Theorem C.35. *The functor F_Π above induces equivalences between* $\mathcal{O}_{\mathcal{X}}$-mod, $\mathrm{qcoh}(\mathcal{X}),\mathrm{coh}(\mathcal{X})$ *and* $(\underline{V}\rightrightarrows\underline{U})$-mod, $\mathrm{qcoh}(\underline{V}\rightrightarrows\underline{U}),\mathrm{coh}(\underline{V}\rightrightarrows\underline{U})$, *respectively.*

In §B.7, for a morphism of C^∞-schemes $\underline{f}:\underline{X}\rightarrow\underline{Y}$ we defined a right exact pullback functor $\underline{f}^*:\mathcal{O}_Y\text{-mod}\rightarrow\mathcal{O}_X\text{-mod}$. Pullbacks may not be strictly functorial in $\underline{f}$, that is, we do not have $\underline{f}^*(\underline{g}^*(\mathcal{E}))=(\underline{g}\circ\underline{f})^*(\mathcal{E})$ for all $\underline{f}:\underline{X}\rightarrow\underline{Y}$, $\underline{g}:\underline{Y}\rightarrow\underline{Z}$ and $\mathcal{E}\in\mathcal{O}_Z$-mod, but instead we have canonical isomorphisms $I_{\underline{f},\underline{g}}(\mathcal{E}):(\underline{g}\circ\underline{f})^*(\mathcal{E})\rightarrow\underline{f}^*(\underline{g}^*(\mathcal{E}))$. We now generalize this to Deligne–Mumford C^∞-stacks. We must interpret pullback for 2-morphisms as well as 1-morphisms.

Definition C.36. Let $f:\mathcal{X}\rightarrow\mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks, and $\mathcal{F}$ be an $\mathcal{O}_{\mathcal{Y}}$-module. A *pullback* of $\mathcal{F}$ to $\mathcal{X}$ is an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$, together with the following data: if $\underline{U},\underline{V}$ are C^∞-schemes and $u:\bar{\underline{U}}\to\mathcal{X}$ and $v:\bar{\underline{V}}\to\mathcal{Y}$ are étale 1-morphisms, then there is a C^∞-scheme $\underline{W}$ and morphisms $\underline{\pi}_{\underline{U}}:\underline{W}\to\underline{U}$, $\underline{\pi}_{\underline{V}}:\underline{W}\to\underline{V}$ giving a 2-Cartesian diagram:

$$\begin{tikzcd} \bar{\underline{W}} \arrow[r, "\bar{\underline{\pi}}_{\underline{V}}"'] \arrow[d, "\bar{\underline{\pi}}_{\underline{U}}"'] \arrow[dr, phantom, "\zeta\Nearrow"] & \bar{\underline{V}} \arrow[d, "v"] \\ \bar{\underline{U}} \arrow[r, "f\circ u"] & \mathcal{Y}. \end{tikzcd} \tag{C.6}$$

Then an isomorphism $i(\mathcal{F},f,u,v,\zeta):\underline{\pi}_{\underline{U}}^*\big(\mathcal{E}(\underline{U},u)\big)\to\underline{\pi}_{\underline{V}}^*\big(\mathcal{F}(\underline{V},v)\big)$ of $\mathcal{O}_W$-modules should be given, which is functorial in $(\underline{U},u)$ in $\mathcal{C}_\mathcal{X}$ and $(\underline{V},v)$ in $\mathcal{C}_\mathcal{Y}$ and the 2-isomorphism ζ in (C.6). We usually write pullbacks $\mathcal{E}$ as $f^*(\mathcal{F})$. By [56, Prop. 10.9], pullbacks $f^*(\mathcal{F})$ exist, and are unique up to unique isomorphism. Using the Axiom of Choice, we choose a pullback $f^*(\mathcal{F})$ for all such $f:\mathcal{X}\to\mathcal{Y}$ and $\mathcal{F}$.

Let $f:\mathcal{X}\to\mathcal{Y}$ be such a 1-morphism, and $\phi:\mathcal{E}\to\mathcal{F}$ be a morphism in $\mathcal{O}_\mathcal{Y}$-mod. Then $f^*(\mathcal{E}),f^*(\mathcal{F})\in\mathcal{O}_\mathcal{X}$-mod. Define the *pullback morphism* $f^*(\phi):f^*(\mathcal{E})\to f^*(\mathcal{F})$ to be the unique morphism in $\mathcal{O}_\mathcal{X}$-mod such that whenever $u:\bar{\underline{U}}\to\mathcal{X}$, $v:\bar{\underline{V}}\to\mathcal{Y}$, $\underline{W},\underline{\pi}_{\underline{U}},\underline{\pi}_{\underline{V}}$ are as above, the following diagram of morphisms of $\mathcal{O}_W$-modules commutes:

$$\begin{tikzcd} \underline{\pi}_{\underline{U}}^*\big(f^*(\mathcal{E})(\underline{U},u)\big) \arrow[r, "{i(\mathcal{E},f,u,v,\zeta)}"'] \arrow[d, "{\underline{\pi}_{\underline{U}}^*(f^*(\phi)(\underline{U},u))}"'] & \underline{\pi}_{\underline{V}}^*\big(\mathcal{E}(\underline{V},v)\big) \arrow[d, "{\underline{\pi}_{\underline{V}}^*(\phi(\underline{V},v))}"] \\ \underline{\pi}_{\underline{U}}^*\big(f^*(\mathcal{F})(\underline{U},u)\big) \arrow[r, "{i(\mathcal{F},f,u,v,\zeta)}"] & \underline{\pi}_{\underline{V}}^*\big(\mathcal{F}(\underline{V},v)\big). \end{tikzcd}$$

This defines a functor $f^*:\mathcal{O}_\mathcal{Y}\text{-mod}\to\mathcal{O}_\mathcal{X}\text{-mod}$, which also maps $\mathrm{qcoh}(\mathcal{Y})\to\mathrm{qcoh}(\mathcal{X})$ and $\mathrm{coh}(\mathcal{Y})\to\mathrm{coh}(\mathcal{X})$. It is right exact by [56, Prop. 10.12].

Let $f:\mathcal{X}\to\mathcal{Y}$ and $g:\mathcal{Y}\to\mathcal{Z}$ be 1-morphisms of Deligne–Mumford C^∞-stacks, and $\mathcal{E}\in\mathcal{O}_\mathcal{Z}$-mod. Then $(g\circ f)^*(\mathcal{E})$ and $f^*(g^*(\mathcal{E}))$ both lie in $\mathcal{O}_\mathcal{X}$-mod. One can show that $f^*(g^*(\mathcal{E}))$ is a possible pullback of $\mathcal{E}$ by $g\circ f$. Thus as in Definition B.34, we have a canonical isomorphism $I_{f,g}(\mathcal{E}):(g\circ f)^*(\mathcal{E})\to f^*(g^*(\mathcal{E}))$. This defines a natural isomorphism of functors $I_{f,g}:(g\circ f)^*\Rightarrow f^*\circ g^*$.

Let $f,g:\mathcal{X}\to\mathcal{Y}$ be 1-morphisms of Deligne–Mumford C^∞-stacks, $\eta:f\Rightarrow g$ a 2-morphism, and $\mathcal{E}\in\mathcal{O}_\mathcal{Y}$-mod. Then we have $\mathcal{O}_\mathcal{X}$-modules $f^*(\mathcal{E}),g^*(\mathcal{E})$. Define $\eta^*(\mathcal{E}):f^*(\mathcal{E})\to g^*(\mathcal{E})$ to be the unique isomorphism such that whenever $\underline{U},\underline{V},\underline{W},u,v,\underline{\pi}_{\underline{U}},\underline{\pi}_{\underline{V}}$ are as above, so that we have 2-Cartesian diagrams

$$\begin{tikzcd} \bar{\underline{W}} \arrow[r, "\bar{\underline{\pi}}_{\underline{V}}"'] \arrow[d, "\bar{\underline{\pi}}_{\underline{U}}"'] \arrow[dr, phantom, "\zeta\odot(\eta*\mathrm{id}_{u\circ\bar{\underline{\pi}}_{\underline{U}}})\Nearrow"] & \bar{\underline{V}} \arrow[d, "v"] \\ \bar{\underline{U}} \arrow[r, "f\circ u"] & \mathcal{Y}, \end{tikzcd} \qquad \begin{tikzcd} \bar{\underline{W}} \arrow[r, "\bar{\underline{\pi}}_{\underline{V}}"'] \arrow[d, "\bar{\underline{\pi}}_{\underline{U}}"'] \arrow[dr, phantom, "\zeta\Nearrow"] & \bar{\underline{V}} \arrow[d, "v"] \\ \bar{\underline{U}} \arrow[r, "g\circ u"] & \mathcal{Y}, \end{tikzcd}$$

as in (C.6), where in $\zeta\odot(\eta*\mathrm{id}_{u\circ\bar{\underline{\pi}}_{\underline{U}}})$ '$*$' is horizontal and '$\odot$' vertical composition of 2-morphisms, then we have commuting isomorphisms of $\mathcal{O}_W$-modules:

$$\begin{tikzcd}
\underline{\pi}_{\underline{U}}^*\big(f^*(\mathcal{E})(\underline{U},u)\big) \arrow[d, "\underline{\pi}_{\underline{U}}^*((\eta^*(\mathcal{E}))(\underline{U},u))"'] \arrow[dr, "i(\mathcal{E},f,u,v,\zeta\odot(\eta*\mathrm{id}_{u\circ\underline{\pi}_{\underline{U}}}))"] & \\
\underline{\pi}_{\underline{U}}^*\big(g^*(\mathcal{E})(\underline{U},u)\big) \arrow[r, "i(\mathcal{E},g,u,v,\zeta)"'] & \underline{\pi}_{\underline{V}}^*\big(\mathcal{E}(\underline{V},v)\big).
\end{tikzcd}$$

This defines a natural isomorphism $\eta^* : f^* \Rightarrow g^*$.

If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack with identity 1-morphism $\mathrm{id}_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ then for each $\mathcal{E} \in \mathcal{O}_{\mathcal{X}}$-mod, $\mathcal{E}$ is a possible pullback $\mathrm{id}_{\mathcal{X}}^*(\mathcal{E})$, so we have a canonical isomorphism $\delta_{\mathcal{X}}(\mathcal{E}) : \mathrm{id}_{\mathcal{X}}^*(\mathcal{E}) \to \mathcal{E}$. These define a natural isomorphism $\delta_{\mathcal{X}} : \mathrm{id}_{\mathcal{X}}^* \Rightarrow \mathrm{id}_{\mathcal{O}_{\mathcal{X}}\text{-mod}}$.

Here is [56, Th. 10.11]. For *pseudofunctors* see [15, §7.5] or [11, §B.4].

Theorem C.37. *Mapping $\mathcal{X}$ to $\mathcal{O}_{\mathcal{X}}$-mod for objects $\mathcal{X}$ in* $\mathbf{DMC^\infty Sta}$, *and mapping 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ to $f^* : \mathcal{O}_{\mathcal{Y}}\text{-mod} \to \mathcal{O}_{\mathcal{X}}\text{-mod}$, and mapping 2-morphisms $\eta : f \Rightarrow g$ to $\eta^* : f^* \Rightarrow g^*$ for 1-morphisms $f, g : \mathcal{X} \to \mathcal{Y}$, and the natural isomorphisms $I_{f,g} : (g \circ f)^* \Rightarrow f^* \circ g^*$ for all 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ in* $\mathbf{DMC^\infty Sta}$, *and $\delta_{\mathcal{X}}$ for all $\mathcal{X} \in \mathbf{DMC^\infty Sta}$, together make up a **weak 2-functor** or **pseudofunctor** $(\mathbf{DMC^\infty Sta})^{\mathrm{op}} \to \mathbf{AbCat}$, where* $\mathbf{AbCat}$ *is the 2-category of abelian categories. That is, they satisfy:*

(a) *If $f : \mathcal{W} \to \mathcal{X}$, $g : \mathcal{X} \to \mathcal{Y}$, $h : \mathcal{Y} \to \mathcal{Z}$ are 1-morphisms in* $\mathbf{DMC^\infty Sta}$ *and $\mathcal{E} \in \mathcal{O}_{\mathcal{Z}}$-mod then the following diagram commutes in $\mathcal{O}_{\mathcal{X}}$-mod :*

$$\begin{tikzcd}
(h\circ g\circ f)^*(\mathcal{E}) \arrow[r, "I_{f,h\circ g}(\mathcal{E})"'] \arrow[d, "I_{g\circ f,h}(\mathcal{E})"'] & f^*\big((h\circ g)^*(\mathcal{E})\big) \arrow[d, "f^*(I_{g,h}(\mathcal{E}))"] \\
(g\circ f)^*\big(h^*(\mathcal{E})\big) \arrow[r, "I_{f,g}(h^*(\mathcal{E}))"] & f^*\big(g^*(h^*(\mathcal{E}))\big).
\end{tikzcd}$$

(b) *If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism in* $\mathbf{DMC^\infty Sta}$ *and $\mathcal{E} \in \mathcal{O}_{\mathcal{Y}}$-mod then the following pairs of morphisms in $\mathcal{O}_{\mathcal{X}}$-mod are inverse:*

$$\begin{tikzcd}
\substack{f^*(\mathcal{E}) = \\ (f\circ\mathrm{id}_{\mathcal{X}})^*(\mathcal{E})} \arrow[r, shift left, "I_{\mathrm{id}_{\mathcal{X}},f}(\mathcal{E})"] & \mathrm{id}_{\mathcal{X}}^*(f^*(\mathcal{E})), \arrow[l, shift left, "\delta_{\mathcal{X}}(f^*(\mathcal{E}))"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\substack{f^*(\mathcal{E}) = \\ (\mathrm{id}_{\mathcal{Y}}\circ f)^*(\mathcal{E})} \arrow[r, shift left, "I_{f,\mathrm{id}_{\mathcal{Y}}}(\mathcal{E})"] & f^*(\mathrm{id}_{\mathcal{Y}}^*(\mathcal{E})). \arrow[l, shift left, "f^*(\delta_{\mathcal{Y}}(\mathcal{E}))"]
\end{tikzcd}$$

Also $(\mathrm{id}_f)^*(\mathrm{id}_{\mathcal{E}}) = \mathrm{id}_{f^*(\mathcal{E})} : f^*(\mathcal{E}) \to f^*(\mathcal{E})$.

(c) *If $f, g, h : \mathcal{X} \to \mathcal{Y}$ are 1-morphisms and $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms in* $\mathbf{DMC^\infty Sta}$, *so that $\zeta \odot \eta : f \Rightarrow h$ is the vertical composition, and $\mathcal{E} \in \mathcal{O}_{\mathcal{Y}}$-mod, then*

$$\zeta^*(\mathcal{F}) \circ \eta^*(\mathcal{E}) = (\zeta \odot \eta)^*(\mathcal{E}) : f^*(\mathcal{E}) \to h^*(\mathcal{E}) \quad \textit{in } \mathcal{O}_{\mathcal{X}}\text{-mod}.$$

(d) *If $f, \tilde{f} : \mathcal{X} \to \mathcal{Y}$, $g, \tilde{g} : \mathcal{Y} \to \mathcal{Z}$ are 1-morphisms and $\eta : f \Rightarrow f'$, $\zeta : g \Rightarrow g'$ 2-morphisms in* $\mathbf{DMC^\infty Sta}$, *so that $\zeta * \eta : g \circ f \Rightarrow \tilde{g} \circ \tilde{f}$ is the horizontal* *composition, and* $\mathcal{E} \in \mathcal{O}_{\mathcal{Z}}$-mod, *then the following commutes in* $\mathcal{O}_{\mathcal{X}}$-mod :

$$\begin{tikzcd}
(g \circ f)^*(\mathcal{E}) \arrow[rr, "(\zeta * \eta)^*(\mathcal{E})"'] \arrow[d, "I_{f,g}(\mathcal{E})"'] & & (\tilde g \circ \tilde f)^*(\mathcal{E}) \arrow[d, "I_{\tilde f,\tilde g}(\mathcal{E})"] \\
f^*(g^*(\mathcal{E})) \arrow[r, "f^*(\zeta^*(\mathcal{E}))"] & f^*(\tilde g^*(\mathcal{E})) \arrow[r, "\eta^*(\tilde g^*(\mathcal{E}))"] & \tilde f^*(\tilde g^*(\mathcal{E})).
\end{tikzcd}$$

Definition C.38. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Define an $\mathcal{O}_{\mathcal{X}}$-module $T^*\mathcal{X}$ called the *cotangent sheaf* of $\mathcal{X}$ by $(T^*\mathcal{X})(\underline{U}, u) = T^*\underline{U}$ for all objects $(\underline{U}, u)$ in $\mathcal{C}_{\mathcal{X}}$ and $(T^*\mathcal{X})_{(\underline{f},\eta)} = \Omega_{\underline{f}} : \underline{f}^*(T^*\underline{V}) \to T^*\underline{U}$ for all morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_{\mathcal{X}}$, where $T^*\underline{U}$ and $\Omega_{\underline{f}}$ are as in §B.7.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks. Then $f^*(T^*\mathcal{Y}), T^*\mathcal{X}$ are $\mathcal{O}_{\mathcal{X}}$-modules. Define $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ to be the unique morphism characterized as follows. Let $u : \bar{\underline{U}} \to \mathcal{X}$, $v : \bar{\underline{V}} \to \mathcal{Y}$, $\underline{W}, \underline{\pi}_{\underline{U}}, \underline{\pi}_{\underline{V}}$ be as in Definition C.36, with (C.6) 2-Cartesian. Then the following diagram of morphisms of $\mathcal{O}_W$-modules commutes:

$$\begin{tikzcd}
\underline{\pi}_{\underline{U}}^*\bigl(f^*(T^*\mathcal{Y})(\underline{U}, u)\bigr) \arrow[r, "i(T^*\mathcal{Y}, f, u, v, \zeta)"'] \arrow[d, "\underline{\pi}_{\underline{U}}^*(\Omega_f(\underline{U}, u))"'] & \underline{\pi}_{\underline{V}}^*\bigl((T^*\mathcal{Y})(\underline{V}, v)\bigr) \arrow[r, equal] & \underline{\pi}_{\underline{V}}^*(T^*\underline{V}) \arrow[d, "\Omega_{\underline{\pi}_{\underline{V}}}"'] \\
\underline{\pi}_{\underline{U}}^*\bigl((T^*\mathcal{X})(\underline{U}, u)\bigr) \arrow[r, "(T^*\mathcal{X})_{(\underline{\pi}_{\underline{U}}, \mathrm{id}_{u \circ \underline{\pi}_{\underline{U}}})}"] & (T^*\mathcal{X})(\underline{W}, u \circ \underline{\pi}_{\underline{U}}) \arrow[r, equal] & T^*\underline{W}.
\end{tikzcd}$$

If $\Pi : \bar{\underline{U}} \to \mathcal{X}$, $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$ and $F_\Pi : \mathcal{O}_{\mathcal{X}}\text{-mod} \to (\underline{V} \rightrightarrows \underline{U})\text{-mod}$ are as in Definition C.34 then by definition $F_\Pi(T^*\mathcal{X}) = (T^*\underline{U}, \Omega_{\underline{t}}{}^{-1} \circ \Omega_{\underline{s}})$, and so we write $T^*(\underline{V} \rightrightarrows \underline{U}) = (T^*\underline{U}, \Omega_{\underline{t}}{}^{-1} \circ \Omega_{\underline{s}})$ in $(\underline{V} \rightrightarrows \underline{U})$-mod.

Here [56, Th. 10.15] is the analogue of Theorem B.39.

Theorem C.39. **(a)** *Let* $f : \mathcal{X} \to \mathcal{Y}$ *and* $g : \mathcal{Y} \to \mathcal{Z}$ *be* 1*-morphisms of Deligne–Mumford* C^∞*-stacks. Then in morphisms* $(g \circ f)^*(T^*\mathcal{Z}) \to T^*\mathcal{X}$ *in* $\mathcal{O}_{\mathcal{X}}$-mod, *we have*

$$\Omega_{g \circ f} = \Omega_f \circ f^*(\Omega_g) \circ I_{f,g}(T^*\mathcal{Z}).$$

(b) *Let* $f, g : \mathcal{X} \to \mathcal{Y}$ *be* 1*-morphisms of Deligne–Mumford* C^∞*-stacks and* $\eta : f \Rightarrow g$ *a* 2*-morphism. Then* $\Omega_f = \Omega_g \circ \eta^*(T^*\mathcal{Y}) : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$.

(c) *Suppose* $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ *are locally fair Deligne–Mumford* C^∞*-stacks with a* 2*-Cartesian square*

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "f"'] \arrow[d, "e"] \arrow[dr, phantom, "\eta\Uparrow"] & \mathcal{Y} \arrow[d, "h"'] \\
\mathcal{X} \arrow[r, "g"] & \mathcal{Z}
\end{tikzcd}$$

in $\mathbf{DMC^\infty Sta^{lf}}$, *so that* $\mathcal{W} \simeq \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. *Then the following is exact in* $\mathrm{qcoh}(\mathcal{W})$:

$$(g \circ e)^*(T^*\mathcal{Z}) \xrightarrow{\substack{e^*(\Omega_g) \circ I_{e,g}(T^*\mathcal{Z}) \oplus \\ -f^*(\Omega_h) \circ I_{f,h}(T^*\mathcal{Z}) \circ \eta^*(T^*\mathcal{Z})}} \begin{matrix} e^*(T^*\mathcal{X}) \oplus \\ f^*(T^*\mathcal{Y}) \end{matrix} \xrightarrow{\Omega_e \oplus \Omega_f} T^*\mathcal{W} \longrightarrow 0.$$

As in Definition B.40, we can define *line bundles* on C^∞-stacks, *orientations* on line bundles, and *pullback orientations*, in the obvious way.

C.7 Sheaves of abelian groups and C^∞-rings on C^∞-stacks

In [56, §10.5] we generalize §C.6 to *sheaves of abelian groups* and *sheaves of C^∞-rings* on Deligne–Mumford C^∞-stacks, as these will be needed in Chapters 9–12. Here is the analogue of Definition C.30. We use the same notation of the category $\mathcal{C}_{\mathcal{X}}$ with objects $(\underline{U}, u)$ and morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$.

Definition C.40. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Following Definition C.30, we define a *sheaf of abelian groups* $\mathcal{E}$ on $\mathcal{X}$ to assign a sheaf of abelian groups $\mathcal{E}(\underline{U}, u)$ on U for all objects $(\underline{U}, u)$ in $\mathcal{C}_{\mathcal{X}}$ with $\underline{U} = (U, \mathcal{O}_U)$, and an isomorphism of sheaves of abelian groups $\mathcal{E}_{(\underline{f},\eta)} : f^{-1}(\mathcal{E}(\underline{V}, v)) \to \mathcal{E}(\underline{U}, u)$ for all morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_{\mathcal{X}}$ with $\underline{f} = (f, f^\sharp)$, such that for all $(\underline{f}, \eta), (\underline{g}, \zeta), (\underline{g} \circ \underline{f}, \theta)$ the analogue of (C.1) commutes:

$$
\begin{tikzcd}
(g \circ f)^{-1}\bigl(\mathcal{E}(\underline{W}, w)\bigr) \arrow[rr, "\mathcal{E}_{(\underline{g}\circ\underline{f},\theta)}"'] \arrow[dr, "I_{f,g}(\mathcal{E}(\underline{W},w))"'] & & \mathcal{E}(\underline{U}, u). \\
& f^{-1}\bigl(g^{-1}(\mathcal{E}(\underline{W}, w)\bigr) \arrow[r, "f^{-1}(\mathcal{E}_{(\underline{g},\zeta)})"] & f^{-1}\bigl(\mathcal{E}(\underline{V}, v)\bigr) \arrow[u, "\mathcal{E}_{(\underline{f},\eta)}"']
\end{tikzcd}
$$

Here $I_{f,g}(\mathcal{E}(\underline{W}, w))$ is the natural isomorphism, as for the isomorphisms $I_{\underline{f},\underline{g}}(\mathcal{E})$ in Definition B.34. Note that we use pullbacks f^{-1} for sheaves of abelian groups, as in Definition B.15, rather than pullbacks $\underline{f}^*$ or f^* for sheaves of modules as in Definitions B.34 and C.36.

A *morphism of sheaves of abelian groups* $\phi : \mathcal{E} \to \mathcal{F}$ assigns a morphism of sheaves of abelian groups $\phi(\underline{U}, u) : \mathcal{E}(\underline{U}, u) \to \mathcal{F}(\underline{U}, u)$ on U for each $(\underline{U}, u)$ in $\mathcal{C}_{\mathcal{X}}$ with $\underline{U} = (U, \mathcal{O}_U)$, such that for all $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_{\mathcal{X}}$ the analogue of (C.2) commutes:

$$
\begin{tikzcd}
f^{-1}\bigl(\mathcal{E}(\underline{V}, v)\bigr) \arrow[r, "\mathcal{E}_{(\underline{f},\eta)}"'] \arrow[d, "f^{-1}(\phi(\underline{V},v))"'] & \mathcal{E}(\underline{U}, u) \arrow[d, "\phi(\underline{U},u)"] \\
f^{-1}\bigl(\mathcal{F}(\underline{V}, v)\bigr) \arrow[r, "\mathcal{F}_{(\underline{f},\eta)}"] & \mathcal{F}(\underline{U}, u).
\end{tikzcd}
$$

Sheaves of C^∞-rings on $\mathcal{X}$, and their morphisms, are defined in the same way, replacing sheaves of abelian groups by sheaves of C^∞-rings throughout.

Remark C.41. On a C^∞-scheme $\underline{X}$, a quasicoherent sheaf $\mathcal{E}$ has an underlying sheaf of abelian groups, which we also write as $\mathcal{E}$, by regarding $\mathcal{E}(U)$ as an abelian group for open $U \subseteq X$ and forgetting about its $\mathcal{O}_X(U)$-module structure. In the same way, a quasicoherent sheaf $\mathcal{E}$ on a Deligne–Mumford C^∞-stack $\mathcal{X}$ has an underlying sheaf of abelian groups, which we also write as $\mathcal{E}$. There is a minor difference in the morphisms $\mathcal{E}_{(\underline{f},\eta)}$: for $\mathcal{E}$ to be a quasicoherent sheaf we need $\mathcal{E}_{(\underline{f},\eta)} : \underline{f}^*(\mathcal{E}(\underline{V}, v)) \to \mathcal{E}(\underline{U}, u)$ Definition C.30, but for $\mathcal{E}$ to be a sheaf of abelian groups we need $\mathcal{E}_{(\underline{f},\eta)} : f^{-1}(\mathcal{E}(\underline{V}, v)) \to \mathcal{E}(\underline{U}, u)$ in Definition C.40. The two are related by the morphism

$$
\begin{aligned}
(\mathrm{id} \otimes f^\sharp) : f^{-1}(\mathcal{E}(\underline{V}, v)) &= f^{-1}(\mathcal{E}(\underline{V}, v)) \otimes_{f^{-1}(\mathcal{O}_V)} f^{-1}(\mathcal{O}_V) \\
&\longrightarrow f^{-1}(\mathcal{E}(\underline{V}, v)) \otimes_{f^{-1}(\mathcal{O}_V)} \mathcal{O}_U = \underline{f}^*(\mathcal{E}(\underline{V}, v)),
\end{aligned}
$$

where the tensor products use the $\mathcal{O}_V$-module structure on $\mathcal{E}(\underline{V},v)\in \mathrm{qcoh}(\underline{V})$.

Example C.42. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. The *structure sheaf* $\mathcal{O}_\mathcal{X}$ is a sheaf of C^∞-rings on $\mathcal{X}$ defined by $\mathcal{O}_\mathcal{X}(\underline{U},u)=\mathcal{O}_U$ for $(\underline{U},u)$ in $\mathcal{C}_\mathcal{X}$ with $\underline{U}=(U,\mathcal{O}_U)$, and $(\mathcal{O}_\mathcal{X})_{(\underline{f},\eta)}=f^\sharp:f^{-1}(\mathcal{O}_V)\to\mathcal{O}_U$ for all $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ in $\mathcal{C}_\mathcal{X}$ with $\underline{f}=(f,f^\sharp)$. We may also regard $\mathcal{O}_\mathcal{X}$ as a quasicoherent sheaf on $\mathcal{X}$, using the ideas of Remark C.41.

Here is the analogue of Definition C.36:

Definition C.43. Let $f:\mathcal{X}\to\mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks, and $\mathcal{F}$ be a sheaf of abelian groups or C^∞-rings on $\mathcal{Y}$. We define a *pullback* $f^{-1}(\mathcal{F})$ of $\mathcal{F}$ to $\mathcal{X}$ to be a sheaf $\mathcal{E}$ of abelian groups or C^∞-rings on $\mathcal{X}$, together with the following data: if $\underline{U},\underline{V}$ are C^∞-schemes and $u:\bar{U}\to\mathcal{X}$ and $v:\bar{V}\to\mathcal{Y}$ are étale 1-morphisms, then there is a C^∞-scheme $\underline{W}$ and morphisms $\underline{\pi}_U:\underline{W}\to\underline{U}$, $\underline{\pi}_V:\underline{W}\to\underline{V}$ giving a 2-Cartesian diagram (C.6) in $\mathbf{C^\infty Sta}$. Then an isomorphism $i(\mathcal{F},f,u,v,\zeta):\pi_U^{-1}\big(\mathcal{E}(\underline{U},u)\big)\to\pi_V^{-1}\big(\mathcal{F}(\underline{V},v)\big)$ of sheaves of abelian groups or C^∞-rings on W should be given, which is functorial in $(\underline{U},u)\in\mathcal{C}_\mathcal{X}$, $(\underline{V},v)\in\mathcal{C}_\mathcal{Y}$ and ζ. As for sheaves of $\mathcal{O}_\mathcal{X}$-modules, pullbacks $f^{-1}(\mathcal{F})$ always exist, and are unique up to unique isomorphism. From now on we suppose we have chosen a pullback $f^{-1}(\mathcal{F})$ for all such $f:\mathcal{X}\to\mathcal{Y}$ and $\mathcal{F}$.

Given 1-morphisms $f:\mathcal{X}\to\mathcal{Y}$, $g:\mathcal{Y}\to\mathcal{Z}$ and a sheaf $\mathcal{E}$ of abelian groups or C^∞-rings on $\mathcal{Z}$ we have a canonical isomorphism $I_{f,g}(\mathcal{E}):(g\circ f)^{-1}(\mathcal{E})\to f^{-1}\circ g^{-1}(\mathcal{E})$. For 1-morphisms $f,g:\mathcal{X}\to\mathcal{Y}$, a 2-morphism $\eta:f\Rightarrow g$ and a sheaf $\mathcal{E}$ of abelian groups or C^∞-rings on $\mathcal{Y}$ we have a canonical isomorphism $\eta^{-1}(\mathcal{E}):f^{-1}(\mathcal{E})\to g^{-1}(\mathcal{E})$. For a sheaf $\mathcal{E}$ of abelian groups or C^∞-rings on $\mathcal{X}$ we have a canonical isomorphism $\delta_\mathcal{X}(\mathcal{E}):\mathrm{id}_\mathcal{X}^{-1}(\mathcal{E})\to\mathcal{E}$. These all satisfy some natural identities.

If $\underline{f}:\underline{X}\to\underline{Y}$ is a morphism of C^∞-schemes then $\underline{f}=(f,f^\sharp)$, $\underline{X}=(X,\mathcal{O}_X)$ and $\underline{Y}=(Y,\mathcal{O}_Y)$ with $f^\sharp:f^{-1}(\mathcal{O}_Y)\to\mathcal{O}_X$ a morphism of sheaves of C^∞-rings on X. Here is an analogue of this for C^∞-stacks.

Example C.44. Let $f:\mathcal{X}\to\mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks. Then $f^{-1}(\mathcal{O}_\mathcal{Y})$ and $\mathcal{O}_\mathcal{X}$ are sheaves of C^∞-rings on $\mathcal{X}$, by Example C.42. There is a unique morphism $f^\sharp:f^{-1}(\mathcal{O}_\mathcal{Y})\to\mathcal{O}_\mathcal{X}$ of sheaves of C^∞-rings on $\mathcal{X}$, characterized by the following property: for all $(\underline{U},u),(\underline{V},v),\underline{W},\zeta$ as in Definition C.43, the following diagram of sheaves of C^∞-rings on W commutes:

$$\begin{tikzcd}
\pi_U^{-1}\big(f^{-1}(\mathcal{O}_\mathcal{Y})(\underline{U},u)\big) \arrow[rr, "\pi_U^{-1}(f^\sharp(\underline{U},u))"'] \arrow[d, "\cong"', "i(\mathcal{O}_\mathcal{Y},f,u,v,\zeta)"] & & \pi_U^{-1}\big((\mathcal{O}_\mathcal{X})(\underline{U},u)\big) \arrow[r, equal] & \pi_U^{-1}(\mathcal{O}_U) \arrow[d, "\pi_U^\sharp"', "\cong"] \\
\pi_V^{-1}\big(\mathcal{O}_\mathcal{Y}(\underline{V},v)\big) \arrow[r, equal] & \pi_V^{-1}(\mathcal{O}_V) \arrow[rr, "\pi_V^\sharp"] & & \mathcal{O}_W,
\end{tikzcd}$$

where $\underline{\pi}_U=(\pi_U,\pi_U^\sharp)$ and $\underline{\pi}_V=(\pi_V,\pi_V^\sharp)$.

C.8 Orbifold strata of C^∞-stacks

In [56, §11] we study orbifold strata of Deligne–Mumford C^∞-stacks. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, with topological space $\mathcal{X}_{\rm top}$. Then each point $[x]$ in $\mathcal{X}_{\rm top}$ has an orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$, a finite group defined up to isomorphism. For each finite group Γ we write $\tilde{\mathcal{X}}^\Gamma_{\circ,\rm top}=\bigl\{[x]\in\mathcal{X}_{\rm top}:\mathrm{Iso}_{\mathcal{X}}([x])\cong\Gamma\bigr\}$. This is a locally closed subset of $\mathcal{X}_{\rm top}$, coming from a locally closed C^∞-substack $\tilde{\mathcal{X}}^\Gamma_\circ$ of $\mathcal{X}$ with inclusion $\tilde O^\Gamma_\circ(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma_\circ\rightarrow\mathcal{X}$, with

$$\mathcal{X}_{\rm top}=\textstyle\coprod_{\substack{\text{isomorphism classes}\\ \text{of finite groups }\Gamma}}\tilde{\mathcal{X}}^\Gamma_{\circ,\rm top}. \tag{C.7}$$

One can show that for each Γ, the closure $\overline{\tilde{\mathcal{X}}}{}^\Gamma_{\circ,\rm top}$ of $\tilde{\mathcal{X}}^\Gamma_{\circ,\rm top}$ in $\mathcal{X}_{\rm top}$ satisfies

$$\overline{\tilde{\mathcal{X}}}{}^\Gamma_{\circ,\rm top}\subseteq\textstyle\coprod_{\substack{\text{isomorphism classes of finite groups }\Delta:\\ \Gamma\text{ is isomorphic to a subgroup of }\Delta}}\tilde{\mathcal{X}}^\Delta_{\circ,\rm top}.$$

Thus (C.7) is a stratification of $\mathcal{X}_{\rm top}$, and the $\tilde{\mathcal{X}}^\Gamma_\circ$ are called *orbifold strata* of $\mathcal{X}$.

In fact we will define six variations of this idea, Deligne–Mumford C^∞-stacks written $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma$, and open C^∞-substacks $\mathcal{X}^\Gamma_\circ\subseteq\mathcal{X}^\Gamma$, $\tilde{\mathcal{X}}^\Gamma_\circ\subseteq\tilde{\mathcal{X}}^\Gamma$, $\hat{\mathcal{X}}^\Gamma_\circ\subseteq\hat{\mathcal{X}}^\Gamma$. The geometric points and orbifold groups of $\mathcal{X}^\Gamma,\ldots,\hat{\mathcal{X}}^\Gamma_\circ$ are given by:

(i) Points of $\mathcal{X}^\Gamma$ are isomorphism classes $[x,\rho]$, where $[x]\in\mathcal{X}_{\rm top}$ and $\rho:\Gamma\rightarrow\mathrm{Iso}_{\mathcal{X}}([x])$ is an injective morphism, and $\mathrm{Iso}_{\mathcal{X}^\Gamma}([x,\rho])$ is the centralizer of $\rho(\Gamma)$ in $\mathrm{Iso}_{\mathcal{X}}([x])$. Points of $\mathcal{X}^\Gamma_\circ\subseteq\mathcal{X}^\Gamma$ are $[x,\rho]$ with ρ an isomorphism, and $\mathrm{Iso}_{\mathcal{X}^\Gamma_\circ}([x,\rho])\cong C(\Gamma)$, the centre of Γ.

(ii) Points of $\tilde{\mathcal{X}}^\Gamma$ are pairs $[x,\Delta]$, where $[x]\in\mathcal{X}_{\rm top}$ and $\Delta\subseteq\mathrm{Iso}_{\mathcal{X}}([x])$ is isomorphic to Γ, and $\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])$ is the normalizer of Δ in $\mathrm{Iso}_{\mathcal{X}}([x])$. Points of $\tilde{\mathcal{X}}^\Gamma_\circ\subseteq\tilde{\mathcal{X}}^\Gamma$ are $[x,\Delta]$ with $\Delta=\mathrm{Iso}_{\mathcal{X}}([x])$, and $\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma_\circ}([x,\Delta])\cong\Gamma$.

(iii) Points $[x,\Delta]$ of $\hat{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma_\circ$ are the same as for $\tilde{\mathcal{X}}^\Gamma,\tilde{\mathcal{X}}^\Gamma_\circ$, but with orbifold groups $\mathrm{Iso}_{\hat{\mathcal{X}}^\Gamma}([x,\Delta])\cong\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])/\Delta$ and $\mathrm{Iso}_{\hat{\mathcal{X}}^\Gamma_\circ}([x,\Delta])\cong\{1\}$.

There are 1-morphisms $O^\Gamma(\mathcal{X}),\ldots,\hat\Pi^\Gamma_\circ(\mathcal{X})$ forming a strictly commutative diagram, where the columns are inclusions of open C^∞-substacks:

$$\begin{tikzcd}
\mathcal{X}^\Gamma_\circ \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde\Pi^\Gamma_\circ(\mathcal{X})"] \arrow[dr, "O^\Gamma_\circ(\mathcal{X})"'] \arrow[dd, "\subset"'] & & \tilde{\mathcal{X}}^\Gamma_\circ \arrow[r, "\hat\Pi^\Gamma_\circ(\mathcal{X})"] \arrow[dl, "\tilde O^\Gamma_\circ(\mathcal{X})"] \arrow[dd, "\subset"] & \hat{\mathcal{X}}^\Gamma_\circ\simeq\underline{\hat{X}}{}^\Gamma_\circ \arrow[dd, "\subset"] \\
& \mathcal{X} & & \\
\mathcal{X}^\Gamma \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde\Pi^\Gamma(\mathcal{X})"'] \arrow[ur, "O^\Gamma(\mathcal{X})"] & & \tilde{\mathcal{X}}^\Gamma \arrow[r, "\hat\Pi^\Gamma(\mathcal{X})"'] \arrow[ul, "\tilde O^\Gamma(\mathcal{X})"'] & \hat{\mathcal{X}}^\Gamma.
\end{tikzcd} \tag{C.8}$$

Also $\mathrm{Aut}(\Gamma)$ acts on $\mathcal{X}^\Gamma,\mathcal{X}^\Gamma_\circ$, with $\tilde{\mathcal{X}}^\Gamma\simeq[\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$, $\tilde{\mathcal{X}}^\Gamma_\circ\simeq[\mathcal{X}^\Gamma_\circ/\mathrm{Aut}(\Gamma)]$.

Note that there are in general no natural 1-morphisms from $\hat{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma_\circ$ to any of $\mathcal{X},\mathcal{X}^\Gamma,\mathcal{X}^\Gamma_\circ,\tilde{\mathcal{X}}^\Gamma,\tilde{\mathcal{X}}^\Gamma_\circ$. Although $\tilde{\mathcal{X}}^\Gamma_\circ$ or $\underline{\hat{X}}{}^\Gamma_\circ$ correspond most closely to the usual idea of orbifold stratum, we will find that $\mathcal{X}^\Gamma$ and $\tilde{\mathcal{X}}^\Gamma$ are most useful in applications to d-orbifold bordism in Chapter 13, in which it is vital that $O^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\rightarrow\mathcal{X}$ and $\tilde O^\Gamma(\tilde{\mathcal{X}}):\tilde{\mathcal{X}}^\Gamma\rightarrow\tilde{\mathcal{X}}$ are proper.

We now define $\mathcal{X}^\Gamma,\ldots,\hat{\mathcal{X}}^\Gamma_\circ$ and study their properties, following [56, §11.1].

Definition C.45. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, and Γ a finite group. We will explicitly define another Deligne–Mumford C^∞-stack $\mathcal{X}^\Gamma$. Since $\mathcal{X}$ is a stack on the site $(\mathbf{C^\infty Sch}, \mathcal{J})$, $\mathcal{X}$ is a category with a functor $p_\mathcal{X} : \mathcal{X} \to \mathbf{C^\infty Sch}$ satisfying many conditions. To define $\mathcal{X}^\Gamma$ we must define another category $\mathcal{X}^\Gamma$ and a functor $p_{\mathcal{X}^\Gamma} : \mathcal{X}^\Gamma \to \mathbf{C^\infty Sch}$.

Define objects of the category $\mathcal{X}^\Gamma$ to be pairs (A, ρ) satisfying:

(a) A is an object in $\mathcal{X}$, with $p_\mathcal{X}(A) = \underline{U}$ for some object $\underline{U} \in \mathbf{C^\infty Sch}$;

(b) $\rho : \Gamma \to \mathrm{Aut}(A)$ is a group morphism, where $\mathrm{Aut}(A)$ is the group of isomorphisms $a : A \to A$ in $\mathcal{X}$, and $p_\mathcal{X} \circ \rho(\gamma) = \underline{\mathrm{id}}_U$ for all $\gamma \in \Gamma$; and

(c) Let u be a point in $\underline{U}$, and $\underline{u} : \underline{*} \to \underline{U}$ the corresponding morphism in $\mathbf{C^\infty Sch}$. Since $p_\mathcal{X} : \mathcal{X} \to \mathbf{C^\infty Sch}$ is a category fibred in groupoids [56, Def. 7.4], there exists a morphism $a_u : A_u \to A$ in $\mathcal{X}$ with $p_\mathcal{X}(A_u) = \underline{*}$ and $p_\mathcal{X}(a_u) = \underline{u}$, where A_u is unique up to isomorphism in $\mathcal{X}$.

    Having fixed A_u, a_u, [56, Def. 7.4] also implies that for each $\gamma \in \Gamma$ there is a unique isomorphism $\rho_u(\gamma) : A_u \to A_u$ such that $a_u \circ \rho_u(\gamma) = \rho(\gamma) \circ a_u : A_u \to A$, and $p_\mathcal{X}(\rho_u(\gamma)) = \underline{\mathrm{id}}_*$. Then $\rho_u : \Gamma \to \mathrm{Aut}(A_u)$ is a group morphism. We require that $\rho_u : \Gamma \to \mathrm{Aut}(A_u)$ should be injective for all $u \in \underline{U}$. This condition is independent of the choice of A_u, a_u.

Define morphisms $c : (A, \rho) \to (B, \sigma)$ of the category $\mathcal{X}^\Gamma$ to be morphisms $c : A \to B$ in $\mathcal{X}$ satisfying $\sigma(\gamma) \circ c = c \circ \rho(\gamma) : A \to B$ in $\mathcal{X}$ for all $\gamma \in \Gamma$. Given morphisms $c : (A, \rho) \to (B, \sigma)$, $d : (B, \sigma) \to (C, \tau)$ in $\mathcal{X}^\Gamma$, define composition $d \circ c : (A, \rho) \to (C, \tau)$ in $\mathcal{X}^\Gamma$ to be the composition $d \circ c : A \to C$ in $\mathcal{X}$. For each object (A, ρ) in $\mathcal{X}^\Gamma$, define the identity morphism $\mathrm{id}_{(A,\rho)} : (A, \rho) \to (A, \rho)$ in $\mathcal{X}^\Gamma$ to be $\mathrm{id}_A : A \to A$ in $\mathcal{X}$. Define a functor $p_{\mathcal{X}^\Gamma} : \mathcal{X}^\Gamma \to \mathbf{C^\infty Sch}$ by $p_{\mathcal{X}^\Gamma} : (A, \rho) \mapsto \underline{U} = p_\mathcal{X}(A)$ on objects and $p_{\mathcal{X}^\Gamma} : c \mapsto p_\mathcal{X}(c)$ on morphisms.

Define $\mathcal{X}^\Gamma_\circ$ to be the full subcategory of objects (A, ρ) in $\mathcal{X}^\Gamma$ such that $\rho_u : \Gamma \to \mathrm{Aut}(A_u)$ in (c) above is an isomorphism for all $u \in \underline{U}$. Define a functor $p_{\mathcal{X}^\Gamma_\circ} = p_\mathcal{X}|_{\mathcal{X}^\Gamma_\circ} : \mathcal{X}^\Gamma_\circ \to \mathbf{C^\infty Sch}$. By Theorem C.49(a) below, $\mathcal{X}^\Gamma$ is a Deligne–Mumford C^∞-stack, and $\mathcal{X}^\Gamma_\circ$ is an open C^∞-substack in $\mathcal{X}^\Gamma$.

Definition C.46. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, and Γ a finite group. Define a category $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$ to have objects pairs (A, Δ) satisfying:

(a) A is an object in $\mathcal{X}$, with $p_\mathcal{X}(A) = \underline{U}$ for some object $\underline{U} \in \mathbf{C^\infty Sch}$;

(b) $\Delta \subseteq \mathrm{Aut}(A)$ is a subgroup isomorphic to Γ, where $\mathrm{Aut}(A)$ is the group of isomorphisms $a : A \to A$ in $\mathcal{X}$, and $p_\mathcal{X}(\delta) = \underline{\mathrm{id}}_U$ for all $\delta \in \Delta$; and

(c) Let u be a point in $\underline{U}$, and $\underline{u} : \underline{*} \to \underline{U}$ the corresponding morphism in $\mathbf{C^\infty Sch}$. Since $p_\mathcal{X} : \mathcal{X} \to \mathbf{C^\infty Sch}$ is a category fibred in groupoids, there exists a morphism $a_u : A_u \to A$ in $\mathcal{X}$ with $p_\mathcal{X}(A_u) = \underline{*}$ and $p_\mathcal{X}(a_u) = \underline{u}$, where A_u is unique up to isomorphism in $\mathcal{X}$. For each $\delta \in \Delta$ there is a unique isomorphism $\delta_u : A_u \to A_u$ such that $a_u \circ \delta_u = \delta \circ a_u : A_u \to A$, and $p_\mathcal{X}(\delta_u) = \underline{\mathrm{id}}_*$. Then $\{\delta_u : \delta \in \Delta\}$ is a subgroup of $\mathrm{Aut}(A_u)$, and $\delta \mapsto \delta_u$ is a group morphism. We require that the map $\delta \mapsto \delta_u$ should be injective for all $u \in \underline{U}$.

Define morphisms $(A,\Delta)\to(A',\Delta')$ of $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$ to be pairs (c,ι), where $c: A\to A'$ is a morphism in $\mathcal{X}$ and $\iota:\Delta\to\Delta'$ is a group isomorphism, satisfying $\iota(\delta)\circ c=c\circ\delta:A\to A'$ for all $\delta\in\Delta$. Given morphisms $(c,\iota):(A,\Delta)\to(A',\Delta')$, $(c',\iota'):(A',\Delta')\to(A'',\Delta'')$ in $\mathcal{P}\mathcal{X}^\Gamma$, define composition $(c',\iota')\circ(c,\iota)=(c'\circ c,\iota'\circ\iota)$. Define identities $\mathrm{id}_{(A,\Delta)}=(\mathrm{id}_A,\mathrm{id}_\Delta):(A,\Delta)\to(A,\Delta)$.

Define a functor $p_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma}:\mathcal{P}\tilde{\mathcal{X}}^\Gamma\to\mathbf{C^\infty Sch}$ by $p_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma}:(A,\Delta)\mapsto \underline{U}=p_{\mathcal{X}}(A)$ on objects and $p_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma}:(c,\iota)\mapsto p_{\mathcal{X}}(c)$ on morphisms. Define $\mathcal{P}\tilde{\mathcal{X}}^\Gamma_\circ$ to be the full subcategory of objects (A,Δ) in $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$ with $\{\delta_u:\delta\in\Delta\}=\mathrm{Aut}(A_u)$ in (c) above for all $u\in\underline{U}$. Define a functor $p_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma_\circ}=p_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma}|_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma_\circ}:\mathcal{P}\tilde{\mathcal{X}}^\Gamma_\circ\to\mathbf{C^\infty Sch}$.

Although $\mathcal{P}\tilde{\mathcal{X}}^\Gamma,\mathcal{P}\tilde{\mathcal{X}}^\Gamma_\circ$ are in general not C^∞-stacks, they are prestacks on the site $(\mathbf{C^\infty Sch},\mathcal{J})$ in the sense of [56, Def. 7.5] (that is, morphisms in $\mathcal{P}\tilde{\mathcal{X}}^\Gamma,\mathcal{P}\tilde{\mathcal{X}}^\Gamma_\circ$ satisfy a sheaf-like condition over $(\mathbf{C^\infty Sch},\mathcal{J})$, but objects may not). Thus, $\mathcal{P}\tilde{\mathcal{X}}^\Gamma,\mathcal{P}\tilde{\mathcal{X}}^\Gamma_\circ$ have stackifications $\tilde{\mathcal{X}}^\Gamma,\tilde{\mathcal{X}}^\Gamma_\circ$, defined up to equivalence, which are stacks on the site $(\mathbf{C^\infty Sch},\mathcal{J})$. By Theorem C.49(a) below, $\tilde{\mathcal{X}}^\Gamma$ is a Deligne–Mumford C^∞-stack, and $\tilde{\mathcal{X}}^\Gamma_\circ$ is an open C^∞-substack in $\tilde{\mathcal{X}}^\Gamma$.

This specifies $\tilde{\mathcal{X}}^\Gamma,\tilde{\mathcal{X}}^\Gamma_\circ$ only up to equivalence. In Definition C.47 we will explain how to choose $\tilde{\mathcal{X}}^\Gamma,\tilde{\mathcal{X}}^\Gamma_\circ$ within their equivalence classes in order to make (C.8) strictly commute, and to make the 1-morphisms $\tilde{O}^\Gamma(\mathcal{X}),\tilde{\Pi}^\Gamma(\mathcal{X}),\tilde{O}^\Gamma_\circ(\mathcal{X})$, $\tilde{\Pi}^\Gamma_\circ(\mathcal{X})$ below strongly representable.

Let $(A,\Delta),(A',\Delta')$ be objects in $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$. Define a right action of Δ on morphisms $(c,\iota):(A,\Delta)\to(A',\Delta')$ in $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$ by $(c,\iota)\cdot\delta=(c\circ\delta,\iota^\delta)$, where $\iota^\delta:\Delta\to\Delta'$ maps $\iota^\delta:\epsilon\mapsto\iota(\delta\circ\epsilon\circ\delta^{-1})$. If $(c',\iota'):(A',\Delta')\to(A'',\Delta'')$ is another morphism and $\delta'\in\Delta'$, it is easy to show that

$$\bigl((c',\iota')\cdot\delta'\bigr)\circ\bigl((c,\iota)\cdot\delta\bigr)=\bigl((c',\iota')\circ(c,\iota)\bigr)\cdot(\iota^{-1}(\delta')\circ\delta). \tag{C.9}$$

Define a category $\mathcal{P}\hat{\mathcal{X}}^\Gamma$ to have objects (A,Δ) as in $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$, and to have morphisms $(c,\iota)\Delta:(A,\Delta)\to(A',\Delta')$ for morphisms $(c,\iota):(A,\Delta)\to(A',\Delta')$ in $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$, where $(c,\iota)\Delta=\{(c,\iota)\cdot\delta:\delta\in\Delta\}$ is the Δ-orbit of (c,ι). Define composition of morphisms in $\mathcal{P}\hat{\mathcal{X}}^\Gamma$ by $\bigl((c',\iota')\Delta'\bigr)\circ\bigl((c,\iota)\Delta\bigr)=\bigl((c',\iota')\circ(c,\iota)\bigr)\Delta$, where $(c',\iota')\circ(c,\iota)$ is composition of morphisms in $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$. Equation (C.9) shows this is well-defined. Define identity morphisms $\mathrm{id}_{(A,\Delta)}=(\mathrm{id}_A,\mathrm{id}_\Delta)\Delta:(A,\Delta)\to(A,\Delta)$ in $\mathcal{P}\hat{\mathcal{X}}^\Gamma$. Define a functor $p_{\mathcal{P}\hat{\mathcal{X}}^\Gamma}:\mathcal{P}\hat{\mathcal{X}}^\Gamma\to\mathbf{C^\infty Sch}$ to map $(A,\Delta)\mapsto p_{\mathcal{X}}(A)$ on objects and $(c,\iota)\Delta\mapsto p_{\mathcal{X}}(c)$ on morphisms.

Define $\mathcal{P}\hat{\mathcal{X}}^\Gamma_\circ$ to be the full subcategory of $\mathcal{P}\hat{\mathcal{X}}^\Gamma$ whose objects are objects of $\mathcal{P}\tilde{\mathcal{X}}^\Gamma_\circ$, and define $p_{\mathcal{P}\hat{\mathcal{X}}^\Gamma_\circ}=p_{\mathcal{P}\hat{\mathcal{X}}^\Gamma}|_{\mathcal{P}\hat{\mathcal{X}}^\Gamma_\circ}:\mathcal{P}\hat{\mathcal{X}}^\Gamma_\circ\to\mathbf{C^\infty Sch}$. Then as for $\mathcal{P}\hat{\mathcal{X}}^\Gamma,\mathcal{P}\hat{\mathcal{X}}^\Gamma_\circ$ are prestacks on $(\mathbf{C^\infty Sch},\mathcal{J})$, and by Theorem C.49(a) their stackifications $\hat{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma_\circ$ are Deligne–Mumford C^∞-stacks. Furthermore, by Theorem C.49(g) below $\hat{\mathcal{X}}^\Gamma_\circ$ has trivial orbifold groups, so by Theorem C.23 there is a C^∞-scheme $\underline{\hat{X}}^\Gamma_\circ$, unique up to isomorphism, such that $\hat{\mathcal{X}}^\Gamma_\circ\simeq\bar{\underline{\hat{X}}}^\Gamma_\circ$.

Next, we define all the 1-morphisms in (C.8).

Definition C.47. In Definitions C.45 and C.46, for $\Lambda\in\mathrm{Aut}(\Gamma)$ define functors

$$L^\Gamma(\Lambda,\mathcal{X}):\mathcal{X}^\Gamma\longrightarrow\mathcal{X}^\Gamma,\quad O^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\longrightarrow\mathcal{X},\quad \mathcal{P}\tilde{O}^\Gamma(\mathcal{X}):\mathcal{P}\tilde{\mathcal{X}}^\Gamma\longrightarrow\mathcal{X},$$
$$\mathcal{P}\tilde{\Pi}^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\longrightarrow\mathcal{P}\tilde{\mathcal{X}}^\Gamma\quad\text{and}\quad \mathcal{P}\hat{\Pi}^\Gamma(\mathcal{X}):\mathcal{P}\tilde{\mathcal{X}}^\Gamma\longrightarrow\mathcal{P}\hat{\mathcal{X}}^\Gamma$$

on objects by

$$L^\Gamma(\Lambda,\mathcal{X}):(A,\rho)\mapsto(A,\rho\circ\Lambda^{-1}),\ O^\Gamma(\mathcal{X}):(A,\rho)\mapsto A,\ \mathcal{P}\tilde O^\Gamma(\mathcal{X}):(A,\Delta)\mapsto A,$$
$$\mathcal{P}\tilde\Pi^\Gamma(\mathcal{X}):(A,\rho)\longmapsto\bigl(A,\rho(\Gamma)\bigr)\quad\text{and}\quad\mathcal{P}\hat\Pi^\Gamma(\mathcal{X}):(A,\Delta)\longmapsto(A,\Delta),$$

and on morphisms by

$$\begin{gathered}L^\Gamma(\Lambda,\mathcal{X}):c\longmapsto c,\quad O^\Gamma(\mathcal{X}):c\longmapsto c,\quad \mathcal{P}\tilde O^\Gamma(\mathcal{X}):(c,\iota)\longmapsto c,\\ \mathcal{P}\tilde\Pi^\Gamma(\mathcal{X}):c\mapsto(c,\sigma\circ\rho^{-1})\quad\text{on } c:(A,\rho)\to(B,\sigma),\text{ and}\\ \mathcal{P}\hat\Pi^\Gamma(\mathcal{X}):(c,\iota)\mapsto(c,\iota)\Delta\quad\text{ on }(c,\iota):(A,\Delta)\to(A',\Delta').\end{gathered}$$

It is trivial to check that these are all functors, and commute with the projections $p_{\mathcal{X}},p_{\mathcal{X}^\Gamma},p_{\tilde{\mathcal{X}}^\Gamma},p_{\hat{\mathcal{X}}^\Gamma}$ to $\mathbf{C^\infty Sch}$. Hence $L^\Gamma(\Lambda,\mathcal{X}),O^\Gamma(\mathcal{X})$ are 1-morphisms of C^∞-stacks. Note that $L^\Gamma(\Lambda,\mathcal{X})\circ L^\Gamma(\Lambda',\mathcal{X})=L^\Gamma(\Lambda\circ\Lambda',\mathcal{X})$ and $L^\Gamma(\Lambda^{-1},\mathcal{X})=L^\Gamma(\Lambda,\mathcal{X})^{-1}$ for $\Lambda,\Lambda'\in\mathrm{Aut}(\Gamma)$, so $L^\Gamma(-,\mathcal{X})$ is an action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ by 1-isomorphisms.

Now $\mathcal{P}\tilde O^\Gamma(\mathcal{X}),\mathcal{P}\tilde\Pi^\Gamma(\mathcal{X}),\mathcal{P}\hat\Pi^\Gamma(\mathcal{X})$ are 1-morphisms of prestacks, so stackifying gives 1-morphisms of C^∞-stacks $\tilde O^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\to\mathcal{X}$, $\tilde\Pi^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\to\tilde{\mathcal{X}}^\Gamma$, $\hat\Pi^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\to\hat{\mathcal{X}}^\Gamma$. Define 1-morphisms of C^∞-stacks

$$\begin{gathered}L^\Gamma_\circ(\Lambda,\mathcal{X}):\mathcal{X}^\Gamma_\circ\longrightarrow\mathcal{X}^\Gamma_\circ,\quad O^\Gamma_\circ(\mathcal{X}):\mathcal{X}^\Gamma_\circ\longrightarrow\mathcal{X},\quad \tilde O^\Gamma_\circ(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma_\circ\longrightarrow\mathcal{X},\\ \tilde\Pi^\Gamma_\circ(\mathcal{X}):\mathcal{X}^\Gamma_\circ\longrightarrow\tilde{\mathcal{X}}^\Gamma_\circ\quad\text{and}\quad\hat\Pi^\Gamma_\circ(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma_\circ\longrightarrow\hat{\mathcal{X}}^\Gamma_\circ,\end{gathered}$$

to be the restrictions of $L^\Gamma(\Lambda,\mathcal{X}),\ldots,\hat\Pi^\Gamma(\mathcal{X})$ to the open C^∞-substacks $\mathcal{X}^\Gamma_\circ,\tilde{\mathcal{X}}^\Gamma_\circ$. Then $L^\Gamma_\circ(-,\mathcal{X})$ is an action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma_\circ$ by 1-isomorphisms.

It is easy to see that the analogue of (C.8) with prestacks $\mathcal{P}\tilde{\mathcal{X}}^\Gamma,\ldots,\mathcal{P}\hat{\mathcal{X}}^\Gamma_\circ$ and prestack 1-morphisms $\mathcal{P}\tilde O^\Gamma(\mathcal{X}),\ldots,\mathcal{P}\hat\Pi^\Gamma_\circ(\mathcal{X})$ is strictly commutative, i.e. 2-commutative with identity 2-morphisms. Thus on stackifying, (C.8) commutes weakly up to 2-isomorphisms, with some choice of 2-morphisms.

Theorem C.49(f) shows that $\tilde O^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\to\mathcal{X}$ is representable. Thus by Proposition C.14(b), we can replace $\tilde{\mathcal{X}}^\Gamma$ by an equivalent C^∞-stack to make $\tilde O^\Gamma(\mathcal{X})$ strongly representable. Since $\tilde{\mathcal{X}}^\Gamma$ was only defined up to equivalence in Definition C.46 anyway, we may take this replacement to be $\tilde{\mathcal{X}}^\Gamma$, and then $\tilde O^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\to\mathcal{X}$ is strongly representable, and this determines $\tilde{\mathcal{X}}^\Gamma$ uniquely up to 1-isomorphism.

Similarly, the 1-morphism $\tilde\Pi^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\to\tilde{\mathcal{X}}^\Gamma$ is defined by stackification, and so is unique up to 2-isomorphism, and we have a 2-isomorphism $\tilde O^\Gamma(\mathcal{X})\circ\tilde\Pi^\Gamma(\mathcal{X})\Rightarrow O^\Gamma(\mathcal{X})$. Proposition C.13 now shows that we can choose $\tilde\Pi^\Gamma(\mathcal{X})$ uniquely within its 2-isomorphism class so that $\tilde O^\Gamma(\mathcal{X})\circ\tilde\Pi^\Gamma(\mathcal{X})=O^\Gamma(\mathcal{X})$. Thus the lower triangle in (C.8) strictly commutes. The rest of (C.8) then strictly commutes, since $O^\Gamma_\circ(\mathcal{X}),\ldots,\hat\Pi^\Gamma_\circ(\mathcal{X})$ are the restrictions of $O^\Gamma(\mathcal{X}),\ldots,\hat\Pi^\Gamma(\mathcal{X})$ to open C^∞-substacks.

Definition C.48. Let the 1-morphisms $O^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\to\mathcal{X}$, $O^\Gamma_\circ(\mathcal{X}):\mathcal{X}^\Gamma_\circ\to\mathcal{X}$ be as in Definition C.47. We will define actions of Γ on $O^\Gamma(\mathcal{X}),O^\Gamma_\circ(\mathcal{X})$ by 2-morphisms. For each $\gamma\in\Gamma$ and $(A,\rho)\in\mathcal{X}^\Gamma$, define an isomorphism

$E^\Gamma(\gamma,\mathcal{X})(A,\rho):O^\Gamma(\mathcal{X})(A,\rho)\to O^\Gamma(\mathcal{X})(A,\rho)$ in $\mathcal{X}$ by $E^\Gamma(\gamma,\mathcal{X})=\rho(\gamma):A\to A$. If $c:(A,\rho)\to(B,\sigma)$ is a morphism in $\mathcal{X}^\Gamma$ then

$$O^\Gamma(\mathcal{X})(c)\circ E^\Gamma(\gamma,\mathcal{X})(A,\rho)=c\circ\rho(\gamma)=\sigma(\gamma)\circ\rho=E^\Gamma(\gamma,\mathcal{X})(B,\sigma)\circ O^\Gamma(\mathcal{X})(c).$$

Hence $E^\Gamma(\gamma,\mathcal{X}):O^\Gamma(\mathcal{X})\Rightarrow O^\Gamma(\mathcal{X})$ is a natural isomorphism of functors. Since $p_{\mathcal{X}}(E^\Gamma(\gamma,\mathcal{X})(A,\rho))=p_{\mathcal{X}}(\rho(\gamma))=\underline{\mathrm{id}}_{p_{\mathcal{X}}(A)}$ for all (A,ρ), we have $p_{\mathcal{X}}*E^\Gamma(\gamma,\mathcal{X})=p_{\mathcal{X}^\Gamma}$, so $E^\Gamma(\gamma,\mathcal{X}):O^\Gamma(\mathcal{X})\Rightarrow O^\Gamma(\mathcal{X})$ is a 2-morphism of C^∞-stacks. Clearly $E^\Gamma(1,\mathcal{X})=\mathrm{id}_{O^\Gamma(\mathcal{X})}$ and $E^\Gamma(\gamma,\mathcal{X})\odot E^\Gamma(\delta,\mathcal{X})=E^\Gamma(\gamma\delta,\mathcal{X})$ for all $\gamma,\delta\in\Gamma$, so $E^\Gamma(-,\mathcal{X}):\Gamma\to\mathrm{Aut}\big(O^\Gamma(\mathcal{X})\big)$ is a group morphism. We define 2-morphisms $E^\Gamma_\circ(\gamma,\mathcal{X}):O^\Gamma_\circ(\mathcal{X})\Rightarrow O^\Gamma_\circ(\mathcal{X})$ for $\gamma\in\Gamma$ in the same way.

Here are some basic properties of these definitions, [56, Th. 11.5].

Theorem C.49. **(a)** *$\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma$ are Deligne–Mumford C^∞-stacks, and $\mathcal{X}^\Gamma_\circ\subseteq\mathcal{X}^\Gamma$, $\tilde{\mathcal{X}}^\Gamma_\circ\subseteq\tilde{\mathcal{X}}^\Gamma$, $\hat{\mathcal{X}}^\Gamma_\circ\subseteq\hat{\mathcal{X}}^\Gamma$ are open C^∞-substacks. Also $\tilde{\mathcal{X}}^\Gamma\simeq[\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$ and $\tilde{\mathcal{X}}^\Gamma_\circ\simeq[\mathcal{X}^\Gamma_\circ/\mathrm{Aut}(\Gamma)]$, where the $\mathrm{Aut}(\Gamma)$-actions are $L^\Gamma(-,\mathcal{X})$ and $L^\Gamma_\circ(-,\mathcal{X})$..*

(b) *If $\mathcal{X}$ is separated, locally fair, locally finitely presented, or second countable, then $\mathcal{X}^\Gamma,\mathcal{X}^\Gamma_\circ,\tilde{\mathcal{X}}^\Gamma,\tilde{\mathcal{X}}^\Gamma_\circ,\hat{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma_\circ$ are separated, locally fair, locally finitely presented, or second countable respectively.*

If $\mathcal{X}$ is compact then $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma$ are compact.

(c) *Points of $\mathcal{X}^\Gamma_{\mathrm{top}}$ are equivalence classes $[x,\rho]$ of pairs (x,ρ), where $x:\underline{*}\to\mathcal{X}$ is a 1-morphism and $\rho:\Gamma\to\mathrm{Aut}(x)$ is an injective group morphism into the group $\mathrm{Aut}(x)$ of 2-isomorphisms $\eta:x\Rightarrow x$, and pairs $(x,\rho),(x',\rho')$ are equivalent if there exists $\zeta:x\Rightarrow x'$ with $\zeta\odot\rho(\gamma)=\rho'(\gamma)\odot\zeta:x\Rightarrow x'$ for all $\gamma\in\Gamma$. They have orbifold groups*

$$\mathrm{Iso}_{\mathcal{X}^\Gamma}([x,\rho])=\big\{\eta\in\mathrm{Aut}(x):\rho(\gamma)=\eta\rho(\gamma)\eta^{-1}\ \ \forall\gamma\in\Gamma\big\}.$$

Points of $\mathcal{X}^\Gamma_{\circ,\mathrm{top}}$ are $[x,\rho]$ with $\rho:\Gamma\to\mathrm{Aut}(x)$ an isomorphism, and have canonical isomorphisms $\mathrm{Iso}_{\mathcal{X}^\Gamma_\circ}([x,\rho])\cong C(\Gamma)$, where $C(\Gamma)$ is the centre of Γ.

(d) *Points of $\tilde{\mathcal{X}}^\Gamma_{\mathrm{top}}$ are equivalence classes $[x,\Delta]$ of pairs (x,Δ), where $x:\underline{*}\to\mathcal{X}$ is a 1-morphism and $\Delta\subseteq\mathrm{Aut}(x)$ is a subgroup isomorphic to Γ, and pairs $(x,\Delta),(x',\Delta')$ are equivalent if there exists a 2-isomorphism $\zeta:x\Rightarrow x'$ with $\Delta'=\zeta\odot\Delta\odot\zeta^{-1}$. They have orbifold groups*

$$\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])\cong\big\{\eta\in\mathrm{Aut}(x):\Delta=\eta\Delta\eta^{-1}\big\}.$$

Points of $\tilde{\mathcal{X}}^\Gamma_{\circ,\mathrm{top}}$ are $[x,\Delta]$ with $\Delta=\mathrm{Aut}(x)$, and have non-canonical isomorphisms $\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma_\circ}([x,\Delta])\cong\Gamma$.

(e) *As topological spaces $\hat{\mathcal{X}}^\Gamma_{\mathrm{top}}=\tilde{\mathcal{X}}^\Gamma_{\mathrm{top}}$ and $\hat{\mathcal{X}}^\Gamma_{\circ,\mathrm{top}}=\tilde{\mathcal{X}}^\Gamma_{\circ,\mathrm{top}}$, and $\hat{\Pi}^\Gamma(\mathcal{X})_{\mathrm{top}}$, $\hat{\Pi}^\Gamma_\circ(\mathcal{X})_{\mathrm{top}}$ are the identity maps. For $[x,\Delta]\in\hat{\mathcal{X}}^\Gamma_{\mathrm{top}}$ we have*

$$\mathrm{Iso}_{\hat{\mathcal{X}}^\Gamma}([x,\Delta])\cong\big\{\eta\in\mathrm{Aut}(x):\Delta=\eta\Delta\eta^{-1}\big\}/\Delta.$$

Also $\mathrm{Iso}_{\hat{\mathcal{X}}^\Gamma_\circ}([x,\Delta])=\{1\}$ for all $[x,\Delta]\in\hat{\mathcal{X}}^\Gamma_{\circ,\mathrm{top}}$, so $\hat{\mathcal{X}}^\Gamma_\circ$ is a C^∞-scheme.

(f) $L^\Gamma(\Lambda,\mathcal{X})$, $L^\Gamma_\circ(\Lambda,\mathcal{X})$, $O^\Gamma(\mathcal{X})$, $O^\Gamma_\circ(\mathcal{X})$, $\tilde O^\Gamma(\mathcal{X})$, $\tilde O^\Gamma_\circ(\mathcal{X})$, $\tilde\Pi^\Gamma(\mathcal{X})$, $\tilde\Pi^\Gamma_\circ(\mathcal{X})$ *are all strongly representable, but* $\hat\Pi^\Gamma(\mathcal{X}),\hat\Pi^\Gamma_\circ(\mathcal{X})$ *in general are not representable.*

(g) $L^\Gamma(\Lambda,\mathcal{X})$, $L^\Gamma_\circ(\Lambda,\mathcal{X})$, $O^\Gamma(\mathcal{X})$, $\tilde O^\Gamma(\mathcal{X})$, $\tilde\Pi^\Gamma(\mathcal{X})$, $\tilde\Pi^\Gamma_\circ(\mathcal{X})$, $\hat\Pi^\Gamma(\mathcal{X})$, $\hat\Pi^\Gamma_\circ(\mathcal{X})$ *are all proper, but* $O^\Gamma_\circ(\mathcal{X}),\tilde O^\Gamma_\circ(\mathcal{X})$ *in general are not.*

(h) $O^\Gamma_\circ(\mathcal{X})_{\rm top}:\mathcal{X}^\Gamma_{\circ,\rm top}\to\mathcal{X}_{\rm top}$ *takes* $|\operatorname{Aut}(\Gamma)|\cdot|C(\Gamma)|/|\Gamma|$ *points* $[x,\rho]$ *of* $\mathcal{X}^\Gamma_{\circ,\rm top}$ *to each point* $[x]\in\mathcal{X}_{\rm top}$ *with* $\mathrm{Iso}_{\mathcal{X}}([x])\cong\Gamma$*. Also* $\tilde O^\Gamma_\circ(\mathcal{X})_{\rm top}:\tilde{\mathcal{X}}^\Gamma_{\circ,\rm top}\to\mathcal{X}_{\rm top}$ *is a bijection with the subset of* $[x]\in\mathcal{X}_{\rm top}$ *with* $\mathrm{Iso}_{\mathcal{X}}([x])\cong\Gamma$*.*

Example C.50. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. The *inertia stack* $I_{\mathcal{X}}$ of $\mathcal{X}$ is the fibre product $\mathcal{X}\times_{\Delta_{\mathcal{X}},\mathcal{X}\times\mathcal{X},\Delta_{\mathcal{X}}}\mathcal{X}$, where $\Delta_{\mathcal{X}}:\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ is the diagonal 1-morphism. There is a natural equivalence

$$\mathcal{I}_{\mathcal{X}}=\mathcal{X}\times_{\Delta_{\mathcal{X}},\mathcal{X}\times\mathcal{X},\Delta_{\mathcal{X}}}\mathcal{X}\simeq\textstyle\coprod_{k\geqslant 1}\mathcal{X}^{\mathbb{Z}_k}.$$

To see this, note that points of $\mathcal{I}_{\mathcal{X}}$ are equivalence classes $[x,\eta]$, where $[x]\in\mathcal{X}_{\rm top}$ and $\eta\in\mathrm{Iso}_{\mathcal{X}}([x])$. Since $\mathcal{X}$ is Deligne–Mumford, $\mathrm{Iso}_{\mathcal{X}}([x])$ is a finite group, so each $\eta\in\mathrm{Iso}_{\mathcal{X}}([x])$ has some finite order $k\geqslant 1$, and generates an injective morphism $\rho:\mathbb{Z}_k\to\mathrm{Iso}_{\mathcal{X}}([x])$ mapping $\rho:a\mapsto\eta^a$. We may identify $\mathcal{X}^{\mathbb{Z}_k}$ with the open and closed C^∞-substack of $[x,\eta]$ in $\mathcal{I}_{\mathcal{X}}$ for which η has order k.

As in [56, §11.2], the construction of $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma$ extends functorially to 1- and 2-morphisms.

Definition C.51. Let $\mathcal{X},\mathcal{Y}$ be Deligne–Mumford C^∞-stacks, Γ a finite group, and $f:\mathcal{X}\to\mathcal{Y}$ a representable 1-morphism, so that $f:\mathcal{X}\to\mathcal{Y}$ is a functor with $p_{\mathcal{Y}}\circ f=p_{\mathcal{X}}$. We will define a representable 1-morphism $f^\Gamma:\mathcal{X}^\Gamma\to\mathcal{Y}^\Gamma$.

On objects (A,ρ) in $\mathcal{X}^\Gamma$, define $f^\Gamma(A,\rho)=(f(A),f\circ\rho)$, and on morphisms $c:(A,\rho)\to(B,\sigma)$ in $\mathcal{X}^\Gamma$, define $f^\Gamma(c):f^\Gamma(A,\rho)\to f^\Gamma(B,\sigma)$ by $f^\Gamma(c)=f(c):f(A)\to f(B)$. Then $f^\Gamma:\mathcal{X}^\Gamma\to\mathcal{Y}^\Gamma$ is a 1-morphism of C^∞-stacks. It is the unique such 1-morphism with $O^\Gamma(\mathcal{Y})\circ f^\Gamma=f\circ O^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\to\mathcal{Y}$. Also, f^Γ is injective on morphisms, as f is, so f^Γ is representable.

Now let $f,g:\mathcal{X}\to\mathcal{Y}$ be representable, and $\eta:f\Rightarrow g$ be a 2-morphism. Then $f,g:\mathcal{X}\to\mathcal{Y}$ are functors, and $\eta:f\Rightarrow g$ is a natural isomorphism. Define $\eta^\Gamma:f^\Gamma\Rightarrow g^\Gamma$ by taking the isomorphism $\eta^\Gamma(A,\rho):f^\Gamma(A,\rho)\to g^\Gamma(A,\rho)$ in $\mathcal{Y}^\Gamma$ for each object (A,ρ) in $\mathcal{X}^\Gamma$ to be the isomorphism $\eta^\Gamma(A,\rho)=\eta(A):f(A)\to g(A)$ in $\mathcal{Y}$. Then $\eta^\Gamma:f^\Gamma\Rightarrow g^\Gamma$ is a 2-morphism in $\mathbf{DMC^\infty Sta}$. It is the unique such 2-morphism with $\mathrm{id}_{O^\Gamma(\mathcal{Y})}*\eta^\Gamma=\eta*\mathrm{id}_{O^\Gamma(\mathcal{X})}$.

Similarly, if $f:\mathcal{X}\to\mathcal{Y}$ is representable we define functors $\mathcal{P}\tilde f^\Gamma:\mathcal{P}\tilde{\mathcal{X}}^\Gamma\to\mathcal{P}\tilde{\mathcal{Y}}^\Gamma$ mapping $(A,\Delta)\mapsto(f(A),f(\Delta))$ on objects and $(c,\iota)\mapsto(f(c),f\circ\iota\circ f|_\Delta^{-1})$ on morphisms, and $\mathcal{P}\hat f^\Gamma:\mathcal{P}\hat{\mathcal{X}}^\Gamma\to\mathcal{P}\hat{\mathcal{Y}}^\Gamma$ mapping $(A,\Delta)\mapsto(f(A),f(\Delta))$ and $(c,\iota)\Delta\mapsto(f(c),f\circ\iota\circ f|_\Delta^{-1})f(\Delta)$. Then $\mathcal{P}\tilde f^\Gamma,\mathcal{P}\hat f^\Gamma$ are 1-morphisms of prestacks, so stackifying gives 1-morphisms $\tilde f^\Gamma:\tilde{\mathcal{X}}^\Gamma\to\tilde{\mathcal{Y}}^\Gamma$ and $\hat f^\Gamma:\hat{\mathcal{X}}^\Gamma\to\hat{\mathcal{Y}}^\Gamma$.

Stackifications of 1-morphisms of prestacks involve arbitrary choices, and are unique only up to 2-isomorphism. One consequence of this is that strict equalities of 1-morphisms of prestacks translate, on stackification, to 2-isomorphisms of their stackifications, rather than strict equalities. In prestack 1-morphisms we have $\mathcal{P}\tilde{O}^\Gamma(\mathcal{Y})\circ\mathcal{P}\tilde{f}^\Gamma = f\circ\mathcal{P}\tilde{O}^\Gamma(\mathcal{X}):\mathcal{P}\tilde{\mathcal{X}}^\Gamma\to\mathcal{Y}$. Thus, stackification gives a 2-morphism $\zeta:\tilde{O}^\Gamma(\mathcal{Y})\circ\tilde{f}^\Gamma\Rightarrow f\circ\tilde{O}^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\to\mathcal{Y}$, which need not be the identity. Since $\tilde{O}^\Gamma(\mathcal{Y})$ is strongly representable by Theorem C.49(f), Proposition C.13 shows that we may choose $\tilde{f}^\Gamma$ uniquely within its 2-isomorphism class so that $\tilde{O}^\Gamma(\mathcal{Y})\circ\tilde{f}^\Gamma = f\circ\tilde{O}^\Gamma(\mathcal{X})$, and we do this.

We cannot fix $\hat{f}^\Gamma$ uniquely in a similar way, it is natural up to 2-isomorphism.

If $f,g:\mathcal{X}\to\mathcal{Y}$ are representable, and $\eta:f\Rightarrow g$ is a 2-morphism, we define $\mathcal{P}\tilde{\eta}^\Gamma:\mathcal{P}\tilde{f}^\Gamma\Rightarrow\mathcal{P}\tilde{g}^\Gamma$ and $\mathcal{P}\hat{\eta}^\Gamma:\mathcal{P}\hat{f}^\Gamma\Rightarrow\mathcal{P}\hat{g}^\Gamma$ by $\mathcal{P}\tilde{\eta}^\Gamma:(A,\Delta)\mapsto(\eta(A),\iota^\eta)$, where $\iota^\eta:f(\Delta)\to g(\Delta)$ maps $\iota^\eta:f(\delta)\mapsto g(\delta)=\eta(A)\circ f(\delta)\circ\eta(A)^{-1}$ for $\delta\in\Delta$, and $\mathcal{P}\hat{\eta}^\Gamma:(A,\Delta)\mapsto(\eta(A),\iota^\eta)f(\Delta)$. Then $\mathcal{P}\tilde{\eta}^\Gamma,\mathcal{P}\hat{\eta}^\Gamma$ are 2-morphisms of prestacks, so stackifying gives 2-morphisms $\tilde{\eta}^\Gamma:\tilde{f}^\Gamma\Rightarrow\tilde{g}^\Gamma$ and $\hat{\eta}^\Gamma:\hat{f}^\Gamma\Rightarrow\hat{g}^\Gamma$.

The 1-morphisms in (C.8) are compatible with $f^\Gamma,\tilde{f}^\Gamma,\hat{f}^\Gamma$ by

$$O^\Gamma(\mathcal{Y})\circ f^\Gamma = f\circ O^\Gamma(\mathcal{X}),\;\; \tilde{O}^\Gamma(\mathcal{Y})\circ\tilde{f}^\Gamma = f\circ\tilde{O}^\Gamma(\mathcal{X}),\;\; \tilde{\Pi}^\Gamma(\mathcal{Y})\circ f^\Gamma = \tilde{f}^\Gamma\circ\tilde{\Pi}^\Gamma(\mathcal{X}).$$

We have $\mathcal{P}\hat{\Pi}^\Gamma(\mathcal{Y})\circ\mathcal{P}\tilde{f}^\Gamma = \mathcal{P}\hat{f}^\Gamma\circ\mathcal{P}\hat{\Pi}^\Gamma(\mathcal{X})$, so stackifying gives a 2-morphism $\zeta:\hat{\Pi}^\Gamma(\mathcal{Y})\circ\tilde{f}^\Gamma\Rightarrow\hat{f}^\Gamma\circ\hat{\Pi}^\Gamma(\mathcal{X})$.

We can express all this in terms of (strict or weak) 2-functors. Write $\mathbf{DMC^\infty Sta^{re}}$ for the 2-subcategory of $\mathbf{DMC^\infty Sta}$ with only representable 1-morphisms. Define $F^\Gamma,\tilde{F}^\Gamma:\mathbf{DMC^\infty Sta^{re}}\to\mathbf{DMC^\infty Sta^{re}}$ by $F^\Gamma:\mathcal{X}\mapsto F^\Gamma(\mathcal{X})=\mathcal{X}^\Gamma$ on objects, $F^\Gamma:f\mapsto F^\Gamma(f)=f^\Gamma$ on representable 1-morphisms, and $F^\Gamma:\eta\mapsto F^\Gamma(\eta)=\eta^\Gamma$ on 2-morphisms, and similarly for $\tilde{F}^\Gamma$. Then $F^\Gamma,\tilde{F}^\Gamma$ are strict 2-functors, so that for example $F^\Gamma(g\circ f)=F^\Gamma(g)\circ F^\Gamma(f)$ for representable $f:\mathcal{X}\to\mathcal{Y}$, $g:\mathcal{Y}\to\mathcal{Z}$.

For the orbifold strata $\hat{\mathcal{X}}^\Gamma$, the situation is more complicated. For example, if $f:\mathcal{X}\to\mathcal{Y}$, $g:\mathcal{Y}\to\mathcal{Z}$ are representable then the prestack 1-morphisms $\mathcal{P}\hat{f}^\Gamma:\mathcal{P}\hat{\mathcal{X}}^\Gamma\to\mathcal{P}\hat{\mathcal{Y}}^\Gamma$, $\mathcal{P}\hat{g}^\Gamma:\mathcal{P}\hat{\mathcal{X}}^\Gamma\to\mathcal{P}\hat{\mathcal{Y}}^\Gamma$, $\mathcal{P}(\widehat{g\circ f})^\Gamma:\mathcal{P}\hat{\mathcal{X}}^\Gamma\to\mathcal{P}\hat{\mathcal{Y}}^\Gamma$ satisfy $\mathcal{P}(\widehat{g\circ f})^\Gamma=\mathcal{P}\hat{g}^\Gamma\circ\mathcal{P}\hat{f}^\Gamma$. However, stackifying involves arbitrary choices, so we need not have $(\widehat{g\circ f})^\Gamma=\hat{g}^\Gamma\circ\hat{f}^\Gamma$, but instead there is a natural 2-isomorphism $\hat{F}^\Gamma(f,g):(\widehat{g\circ f})^\Gamma\Rightarrow\hat{g}^\Gamma\circ\hat{f}^\Gamma$.

The correct structure here is a *weak 2-functor* or *pseudofunctor* [15, §7.5], [11, §B.4], as in Theorem C.37, a class of 2-functors preserving composition of 1-morphisms up to (specified) 2-isomorphisms. Defining $\hat{F}^\Gamma:\mathbf{DMC^\infty Sta^{re}}\to\mathbf{DMC^\infty Sta^{re}}$ by $\hat{F}^\Gamma:\mathcal{X}\mapsto\hat{F}^\Gamma(\mathcal{X})=\hat{\mathcal{X}}^\Gamma$ on objects, $\hat{F}^\Gamma:f\mapsto\hat{F}^\Gamma(f)=\tilde{f}^\Gamma$ on representable 1-morphisms, $\tilde{F}^\Gamma:\eta\mapsto\hat{F}^\Gamma(\eta)=\hat{\eta}^\Gamma$ on 2-morphisms, and $\hat{F}^\Gamma(f,g):\hat{F}^\Gamma(g\circ f)\Rightarrow\hat{F}^\Gamma(g)\circ\hat{F}^\Gamma(f)$ on composable 1-morphisms, one can show that $\hat{F}^\Gamma:\mathbf{DMC^\infty Sta^{re}}\to\mathbf{DMC^\infty Sta^{re}}$ is a weak 2-functor.

Remark C.52. For $f:\mathcal{X}\to\mathcal{Y}$ and Γ as above, the restriction $f^\Gamma|_{\mathcal{X}^\Gamma_\circ}$ need not map $\mathcal{X}^\Gamma_\circ\to\mathcal{Y}^\Gamma_\circ$, but only $\mathcal{X}^\Gamma_\circ\to\mathcal{Y}^\Gamma$, unless f induces isomorphisms on orbifold groups. Thus we do not define a 1-morphism $f^\Gamma_\circ:\mathcal{X}^\Gamma_\circ\to\mathcal{Y}^\Gamma_\circ$, or a 2-functor $F^\Gamma_\circ:\mathbf{DMC^\infty Sta^{re}}\to\mathbf{DMC^\infty Sta^{re}}$. The same applies for the actions of f on orbifold strata $\tilde{\mathcal{X}}^\Gamma_\circ,\hat{\mathcal{X}}^\Gamma_\circ$.

The next theorem [56, Th. 11.9] describes $\mathcal{X}^\Gamma,\ldots,\hat{\underline{X}}^\Gamma_\circ$ explicitly when $\mathcal{X}$ is a quotient stack $[\underline{X}/G]$, as in §C.4.

Theorem C.53. *Let $\underline{X}$ be a separated C^∞-scheme and G a finite group acting on $\underline{X}$ by isomorphisms, and write $\mathcal{X}=[\underline{X}/G]$ for the quotient C^∞-stack, which is a Deligne–Mumford C^∞-stack. Let Γ be a finite group. Then there are equivalences of C^∞-stacks*

$$\mathcal{X}^\Gamma \simeq \bigl[\bigl(\textstyle\coprod_{\textit{injective group morphisms } \rho\,:\,\Gamma\to G}\underline{X}^{\rho(\Gamma)}\bigr)/G\bigr], \tag{C.10}$$

$$\mathcal{X}^\Gamma_\circ \simeq \bigl[\bigl(\textstyle\coprod_{\textit{injective group morphisms } \rho\,:\,\Gamma\to G}\underline{X}^{\rho(\Gamma)}_\circ\bigr)/G\bigr], \tag{C.11}$$

$$\tilde{\mathcal{X}}^\Gamma \simeq \bigl[\bigl(\textstyle\coprod_{\textit{subgroups } \Delta\subseteq G:\ \Delta\cong\Gamma}\underline{X}^{\Delta}\bigr)/G\bigr], \tag{C.12}$$

$$\tilde{\mathcal{X}}^\Gamma_\circ \simeq \bigl[\bigl(\textstyle\coprod_{\textit{subgroups } \Delta\subseteq G:\ \Delta\cong\Gamma}\underline{X}^{\Delta}_\circ\bigr)/G\bigr], \tag{C.13}$$

where for each subgroup $\Delta\subseteq G$, we write $\underline{X}^\Delta$ for the closed C^∞-subscheme in $\underline{X}$ fixed by Δ in G, and $\underline{X}^\Delta_\circ$ for the open C^∞-subscheme in $\underline{X}^\Delta$ of points in $\underline{X}$ whose stabilizer group in G is exactly Δ.

Here the action of G on $\coprod_\rho\underline{X}^{\rho(\Gamma)}$ in (C.10) *is defined as follows. Let $g\in G$ and $\rho:\Gamma\to G$ be an injective morphism. Define another injective morphism $\rho^g:\Gamma\to G$ by $\rho^g:\gamma\mapsto g\rho(\gamma)g^{-1}$. Then $g(\underline{X}^{\rho(\Gamma)})=\underline{X}^{\rho^g(\Gamma)}$, as C^∞-subschemes of $\underline{X}$, and the action of g on $\coprod_\rho\underline{X}^{\rho(\Gamma)}$ maps $\underline{X}^{\rho(\Gamma)}\to\underline{X}^{\rho^g(\Gamma)}$ by the restriction of $g:\underline{X}\to\underline{X}$ to $\underline{X}^{\rho(\Gamma)}$. The G-actions for* (C.11)*–*(C.13) *are similar.*

We can also rewrite equations (C.10)*–*(C.13) *as*

$$\mathcal{X}^\Gamma \simeq \coprod_{\substack{\textit{conjugacy classes } [\rho] \textit{ of injective}\\ \textit{group morphisms } \rho\,:\,\Gamma\to G}}\bigl[\underline{X}^{\rho(\Gamma)}/\{g\in G: g\rho(\gamma)=\rho(\gamma)g\ \ \forall\gamma\in\Gamma\}\bigr], \tag{C.14}$$

$$\mathcal{X}^\Gamma_\circ \simeq \coprod_{\substack{\textit{conjugacy classes } [\rho] \textit{ of injective}\\ \textit{group morphisms } \rho\,:\,\Gamma\to G}}\bigl[\underline{X}^{\rho(\Gamma)}_\circ/\{g\in G: g\rho(\gamma)=\rho(\gamma)g\ \ \forall\gamma\in\Gamma\}\bigr], \tag{C.15}$$

$$\tilde{\mathcal{X}}^\Gamma \simeq \coprod_{\textit{conjugacy classes } [\Delta] \textit{ of subgroups } \Delta\subseteq G \textit{ with } \Delta\cong\Gamma}\bigl[\underline{X}^{\Delta}/\{g\in G:\Delta=g\Delta g^{-1}\}\bigr], \tag{C.16}$$

$$\tilde{\mathcal{X}}^\Gamma_\circ \simeq \coprod_{\textit{conjugacy classes } [\Delta] \textit{ of subgroups } \Delta\subseteq G \textit{ with } \Delta\cong\Gamma}\bigl[\underline{X}^{\Delta}_\circ/\{g\in G:\Delta=g\Delta g^{-1}\}\bigr]. \tag{C.17}$$

Here morphisms $\rho,\rho':\Gamma\to G$ are conjugate if $\rho'=\rho^g$ for some $g\in G$, and subgroups $\Delta,\Delta'\subseteq G$ are conjugate if $\Delta=g\Delta'g^{-1}$ for some $g\in G$. In (C.14)*–*(C.17) *we sum over one representative ρ or Δ for each conjugacy class.*

In the notation of (C.16)*–*(C.17)*, there are equivalences of C^∞-stacks*

$$\hat{\mathcal{X}}^\Gamma \simeq \coprod_{\textit{conjugacy classes } [\Delta] \textit{ of subgroups } \Delta\subseteq G \textit{ with } \Delta\cong\Gamma}\bigl[\underline{X}^{\Delta}/\bigl(\{g\in G:\Delta=g\Delta g^{-1}\}/\Delta\bigr)\bigr], \tag{C.18}$$

$$\hat{\mathcal{X}}^\Gamma_\circ \simeq \coprod_{\textit{conjugacy classes } [\Delta] \textit{ of subgroups } \Delta\subseteq G \textit{ with } \Delta\cong\Gamma}\bigl[\underline{X}^{\Delta}_\circ/\bigl(\{g\in G:\Delta=g\Delta g^{-1}\}/\Delta\bigr)\bigr]. \tag{C.19}$$

Under the equivalences (C.10)*–*(C.19)*, the* 1*-morphisms in* (C.8) *are identified up to* 2*-isomorphism with* 1*-morphisms between quotient C^∞-stacks induced by natural C^∞-scheme morphisms between $\coprod_\rho\underline{X}^{\rho(\Gamma)},\underline{X},\ldots.$ For example, the* *disjoint union over ρ of the inclusion $\underline{X}^{\rho(\Gamma)} \hookrightarrow \underline{X}$ is a G-equivariant morphism $\coprod_\rho \underline{X}^{\rho(\Gamma)} \to \underline{X}$, inducing a 1-morphism $[\coprod_\rho \underline{X}^{\rho(\Gamma)}/G] \to [\underline{X}/G]$. This is identified with $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$ by* (C.10). *Similarly, $\tilde{\Pi}^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \tilde{\mathcal{X}}^\Gamma$ is identified by* (C.10), (C.12) *with the* 1*-morphism $[\coprod_\rho \underline{X}^{\rho(\Gamma)}/G] \to [\coprod_\Delta \underline{X}^\Delta/G]$ induced by the C^∞-scheme morphism $\coprod_\rho \underline{X}^{\rho(\Gamma)} \to \coprod_\Delta \underline{X}^\Delta$ mapping morphisms ρ to subgroups $\Delta = \rho(\Gamma)$, and acting by $\underline{\mathrm{id}}_{\underline{X}^\Delta} : \underline{X}^{\rho(\Gamma)} \to \underline{X}^\Delta$ for $\Delta = \rho(\Gamma)$.*

C.9 Sheaves on orbifold strata

Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, Γ a finite group, and $\mathcal{E} \in \mathrm{qcoh}(\mathcal{X})$, so that $\mathcal{E}^\Gamma := O^\Gamma(\mathcal{X})^*(\mathcal{E}) \in \mathrm{qcoh}(\mathcal{X}^\Gamma)$. In [56, §11.4] we show that there is a natural representation of Γ on $\mathcal{E}^\Gamma$, and also the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ lifts to $\mathcal{E}^\Gamma$, so that $\mathrm{Aut}(\Gamma) \ltimes \Gamma$ acts equivariantly on $\mathcal{E}^\Gamma$.

Definition C.54. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, and Γ a finite group, so that §C.8 defines the orbifold stratum $\mathcal{X}^\Gamma$, a 1-morphism $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$, an action of $\mathrm{Aut}(\Gamma)$ on $O^\Gamma(\mathcal{X})$ by 2-isomorphisms $E^\Gamma(\gamma, \mathcal{X}) : O^\Gamma(\mathcal{X}) \Rightarrow O^\Gamma(\mathcal{X})$, and an action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ by 1-isomorphisms $L^\Gamma(\Lambda, \mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}^\Gamma$.

Suppose $\mathcal{E}$ is a quasicoherent sheaf on $\mathcal{X}$, and write $\mathcal{E}^\Gamma$ for the pullback sheaf $O^\Gamma(\mathcal{X})^*(\mathcal{E})$ in $\mathrm{qcoh}(\mathcal{X}^\Gamma)$. Using the notation of Definition C.36, for each $\gamma \in \Gamma$ and $\Lambda \in \mathrm{Aut}(\Gamma)$ define morphisms $R^\Gamma(\gamma, \mathcal{E}) : \mathcal{E}^\Gamma \to \mathcal{E}^\Gamma$ and $S^\Gamma(\Lambda, \mathcal{E}) : L^\Gamma(\Lambda, \mathcal{X})^*(\mathcal{E}^\Gamma) \to \mathcal{E}^\Gamma$ in $\mathrm{qcoh}(\mathcal{X}^\Gamma)$ by

$$\begin{aligned} &R^\Gamma(\gamma, \mathcal{E}) = E^\Gamma(\gamma, \mathcal{X})^*(\mathcal{E}) : O^\Gamma(\mathcal{X})^*(\mathcal{E}) \longrightarrow O^\Gamma(\mathcal{X})^*(\mathcal{E}) \qquad \text{and} \\ &S^\Gamma(\Lambda, \mathcal{E}) = I_{L^\Gamma(\Lambda,\mathcal{X}), O^\Gamma(\mathcal{X})}(\mathcal{E})^{-1} : L^\Gamma(\Lambda, \mathcal{X})^* \circ O^\Gamma(\mathcal{X})^*(\mathcal{E}) \longrightarrow O^\Gamma(\mathcal{X})^*(\mathcal{E}), \end{aligned}$$

where the definition of $S^\Gamma(\Lambda, \mathcal{E})$ uses $O^\Gamma(\mathcal{X}) \circ L^\Gamma(\Lambda, \mathcal{X}) = O^\Gamma(\mathcal{X})$.

In [56, §11.4] we prove that $R^\Gamma(-, \mathcal{E})$ is an action of Γ on $\mathcal{E}^\Gamma$ by isomorphisms, and the $S^\Gamma(\Lambda, \mathcal{E})$ define a lift of the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ to $\mathcal{E}^\Gamma$, that is, $\mathcal{E}^\Gamma$ is an $\mathrm{Aut}(\Gamma)$-equivariant sheaf on $\mathcal{X}^\Gamma$, and these Γ- and $\mathrm{Aut}(\Gamma)$-actions are compatible for all $\gamma \in \Gamma$ and $\Lambda \in \mathrm{Aut}(\Gamma)$ by

$$R^\Gamma(\gamma, \mathcal{E}) \circ S^\Gamma(\Lambda, \mathcal{E}) = S^\Gamma(\Lambda, \mathcal{E}) \circ L^\Gamma(\Lambda, \mathcal{X})^*(R^\Gamma(\Lambda(\gamma), \mathcal{E})). \tag{C.20}$$

Let $\alpha : \mathcal{E}_1 \to \mathcal{E}_2$ be a morphism in $\mathrm{qcoh}(\mathcal{X})$. Then $\alpha^\Gamma := O^\Gamma(\mathcal{X})^*(\alpha) : \mathcal{E}_1^\Gamma \to \mathcal{E}_2^\Gamma$ is a morphism in $\mathrm{qcoh}(\mathcal{X}^\Gamma)$. We have

$$\begin{aligned} &\alpha^\Gamma \circ R^\Gamma(\gamma, \mathcal{E}_1) = R^\Gamma(\gamma, \mathcal{E}_2) \circ \alpha^\Gamma \quad \text{for } \gamma \in \Gamma, \\ &\alpha^\Gamma \circ S^\Gamma(\Lambda, \mathcal{E}_1) = S^\Gamma(\Lambda, \mathcal{E}_2) \circ L^\Gamma(\Lambda, \mathcal{X})^*(\alpha^\Gamma) \quad \text{for } \Lambda \in \mathrm{Aut}(\Gamma). \end{aligned}$$

Thus $R(\gamma, -)$ and $S(\Lambda, -)$ are natural isomorphisms of functors.

Now let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism of C^∞-stacks, so that as in §C.8 we have $f^\Gamma : \mathcal{X}^\Gamma \to \mathcal{Y}^\Gamma$. Let $\mathcal{F} \in \mathrm{qcoh}(\mathcal{Y})$. Then we may form $f^*(\mathcal{F}) \in \mathrm{qcoh}(\mathcal{X})$ and hence $f^*(\mathcal{F})^\Gamma = O^\Gamma(\mathcal{X})^*(f^*(\mathcal{F})) \in \mathrm{qcoh}(\mathcal{X}^\Gamma)$, or we may form $\mathcal{F}^\Gamma = O^\Gamma(\mathcal{Y})^*(\mathcal{F}) \in \mathrm{qcoh}(\mathcal{Y}^\Gamma)$ and hence $(f^\Gamma)^*(\mathcal{F}^\Gamma) \in \mathrm{qcoh}(\mathcal{X}^\Gamma)$. Since $O^\Gamma(\mathcal{Y}) \circ f^\Gamma = f \circ O^\Gamma(\mathcal{X})$, these are related by the canonical isomorphism

$$T^\Gamma(f, \mathcal{F}) := I_{f^\Gamma, O^\Gamma(\mathcal{Y})}(\mathcal{F}) \circ I_{O^\Gamma(\mathcal{X}), f}(\mathcal{F})^{-1} : f^*(\mathcal{F})^\Gamma \longrightarrow (f^\Gamma)^*(\mathcal{F}^\Gamma). \tag{C.21}$$

These $T^\Gamma(f,\mathcal{F})$ identify the $(\mathrm{Aut}(\Gamma)\ltimes\Gamma)$-actions on $f^*(\mathcal{F})^\Gamma$ and $(f^\Gamma)^*(\mathcal{F}^\Gamma)$.

Now let $\mathcal{X},\Gamma,\mathcal{X}^\Gamma,\mathcal{E}$ and $\mathcal{E}^\Gamma$ be as above, and write $R_0,\ldots,R_k$ for the irreducible representations of Γ over $\mathbb{R}$ (that is, we choose one representative R_i in each isomorphism class of irreducible representations), with $R_0=\mathbb{R}$ the trivial representation. Then since $R^\Gamma(-,\mathcal{E})$ is an action of Γ on $\mathcal{E}^\Gamma$ by isomorphisms, by elementary representation theory we have a canonical decomposition

$$\mathcal{E}^\Gamma\cong\textstyle\bigoplus_{i=0}^k\mathcal{E}_i^\Gamma\otimes R_i\quad\text{for }\mathcal{E}_0^\Gamma,\ldots,\mathcal{E}_k^\Gamma\in\mathrm{qcoh}(\mathcal{X}^\Gamma).\tag{C.22}$$

We will be interested in splitting $\mathcal{E}^\Gamma$ into *trivial* and *nontrivial* representations of Γ, denoted by subscripts 'tr' and 'nt'. So we write

$$\mathcal{E}^\Gamma=\mathcal{E}^\Gamma_{\mathrm{tr}}\oplus\mathcal{E}^\Gamma_{\mathrm{nt}},\tag{C.23}$$

where $\mathcal{E}^\Gamma_{\mathrm{tr}},\mathcal{E}^\Gamma_{\mathrm{nt}}$ are the subsheaves of $\mathcal{E}^\Gamma$ corresponding to the factors $\mathcal{E}_0^\Gamma\otimes R_0$ and $\bigoplus_{i=1}^k\mathcal{E}_i^\Gamma\otimes R_i$ respectively.

If Γ acts on R_i by $\rho_i:\Gamma\rightarrow\mathrm{Aut}(R_i)$, and $\Lambda\in\mathrm{Aut}(\Gamma)$, then $\rho_i\circ\Lambda^{-1}:\Gamma\rightarrow\mathrm{Aut}(R_i)$ is also an irreducible representation of Γ, and so is isomorphic to $R_{\Lambda(i)}$ for some unique $\Lambda(i)=0,\ldots,k$. This defines an action of $\mathrm{Aut}(\Gamma)$ on $\{0,\ldots,k\}$ by permutations. One can show using (C.20) that $S^\Gamma(\Lambda,\mathcal{E})$ acts on the splitting (C.22) by mapping $L^\Gamma(\Lambda,\mathcal{X})^*(\mathcal{E}_i^\Gamma\otimes R_i)\rightarrow\mathcal{E}^\Gamma_{\Lambda^{-1}(i)}\otimes R_{\Lambda^{-1}(i)}$. Since $\Lambda(0)=0$, it follows that $S^\Gamma(\Lambda,\mathcal{E})$ maps $L^\Gamma(\Lambda,\mathcal{X})^*(\mathcal{E}^\Gamma_{\mathrm{tr}})\rightarrow\mathcal{E}^\Gamma_{\mathrm{tr}}$ and $L^\Gamma(\Lambda,\mathcal{X})^*(\mathcal{E}^\Gamma_{\mathrm{nt}})\rightarrow\mathcal{E}^\Gamma_{\mathrm{nt}}$, that is, $S^\Gamma(\Lambda,\mathcal{E})$ preserves the splitting (C.23). Also $T^\Gamma(f,\mathcal{F})$ maps $f^*(\mathcal{F})^\Gamma_{\mathrm{tr}}\rightarrow(f^\Gamma)^*(\mathcal{F}^\Gamma_{\mathrm{tr}})$ and $f^*(\mathcal{F})^\Gamma_{\mathrm{nt}}\rightarrow(f^\Gamma)^*(\mathcal{F}^\Gamma_{\mathrm{nt}})$ in (C.23).

The next two definitions explain to what extent this generalizes to $\tilde{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma$.

Definition C.55. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, and Γ a finite group, so that §C.8 defines the orbifold strata $\mathcal{X}^\Gamma,\tilde{\mathcal{X}}^\Gamma$ with $\tilde{\mathcal{X}}^\Gamma\simeq[\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$, and 1-morphisms $O^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\rightarrow\mathcal{X}$, $\tilde{O}^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\rightarrow\mathcal{X}$ and $\tilde{\Pi}^\Gamma(\mathcal{X}):\mathcal{X}^\Gamma\rightarrow\tilde{\mathcal{X}}^\Gamma$ with $\tilde{O}^\Gamma(\mathcal{X})\circ\tilde{\Pi}^\Gamma(\mathcal{X})=O^\Gamma(\mathcal{X})$.

How much of the structure on $\mathcal{E}^\Gamma$ in Definition C.54 descends to $\tilde{\mathcal{E}}^\Gamma$? It turns out that $\tilde{\mathcal{E}}^\Gamma$ does not have natural representations of Γ or $\mathrm{Aut}(\Gamma)$, since we do not have actions of Γ on $\tilde{O}^\Gamma(\mathcal{X})$ by 2-isomorphisms or of $\mathrm{Aut}(\Gamma)$ on $\tilde{\mathcal{X}}^\Gamma$ by 1-isomorphisms. In effect, taking the quotient by $\mathrm{Aut}(\Gamma)$ in $\tilde{\mathcal{X}}^\Gamma\simeq[\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$ destroys both these actions.

However, at least part of the natural decompositions (C.22)–(C.23) descends to $\tilde{\mathcal{E}}^\Gamma$. As in Definition C.54, write $R_0,\ldots,R_k$ for the irreducible representations of Γ, so that $\mathrm{Aut}(\Gamma)$ acts on the indexing set $\{0,\ldots,k\}$. Form the quotient set $\{0,\ldots,k\}/\mathrm{Aut}(\Gamma)$, so that points of $\{0,\ldots,k\}/\mathrm{Aut}(\Gamma)$ are orbits O of $\mathrm{Aut}(\Gamma)$ in $\{0,\ldots,k\}$. Then we may rewrite (C.22) as

$$\mathcal{E}^\Gamma\cong\textstyle\bigoplus_{O\in\{0,\ldots,k\}/\mathrm{Aut}(\Gamma)}\bigl[\bigoplus_{i\in O}\mathcal{E}_i^\Gamma\otimes R_i\bigr].$$

Since $S^\Gamma(\Lambda,\mathcal{E})$ maps $L^\Gamma(\Lambda,\mathcal{X})^*(\mathcal{E}_i^\Gamma\otimes R_i)\rightarrow\mathcal{E}^\Gamma_{\Lambda^{-1}(i)}\otimes R_{\Lambda^{-1}(i)}$, we see that

$$S^\Gamma(\Lambda,\mathcal{E}):L^\Gamma(\Lambda,\mathcal{X})^*\bigl(\textstyle\bigoplus_{i\in O}\mathcal{E}_i^\Gamma\otimes R_i\bigr)\longrightarrow\textstyle\bigoplus_{i\in O}\mathcal{E}_i^\Gamma\otimes R_i$$

for each $O \in \{0,\ldots,k\}/\operatorname{Aut}(\Gamma)$. Now the $S^\Gamma(\Lambda,\mathcal{E})$ lift the action of $\operatorname{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ to $\mathcal{E}^\Gamma$, and $\tilde{\mathcal{E}}^\Gamma$ is essentially the quotient of $\mathcal{E}^\Gamma$ by this lifted action of $\operatorname{Aut}(\Gamma)$ under the equivalence $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma/\operatorname{Aut}(\Gamma)]$. Therefore any decomposition of $\mathcal{E}^\Gamma$ which is invariant under $S^\Gamma(\Lambda,\mathcal{E})$ for all $\Lambda \in \operatorname{Aut}(\Gamma)$ corresponds to a decomposition of $\tilde{\mathcal{E}}^\Gamma$. Hence there is a canonical splitting

$$\begin{aligned} \tilde{\mathcal{E}}^\Gamma = \textstyle\bigoplus_{O\in\{0,\ldots,k\}/\operatorname{Aut}(\Gamma)} \tilde{\mathcal{E}}^\Gamma_O, \quad \text{where} \\ I_{\tilde{\Pi}^\Gamma(\mathcal{X}),\tilde{O}^\Gamma(\mathcal{X})}(\mathcal{E})^{-1}\bigl[\tilde{\Pi}^\Gamma(\mathcal{X})^*(\tilde{\mathcal{E}}^\Gamma_O)\bigr] \cong \textstyle\bigoplus_{i\in O} \mathcal{E}^\Gamma_i \otimes R_i \quad \text{under (C.22).} \end{aligned} \tag{C.24}$$

As for (C.23) we define the *trivial* and *nontrivial* parts of $\tilde{\mathcal{E}}^\Gamma$ by $\tilde{\mathcal{E}}^\Gamma_{\rm tr} = \tilde{\mathcal{E}}^\Gamma_{\{0\}}$ and $\tilde{\mathcal{E}}^\Gamma_{\rm nt} = \bigoplus_{O\in\{1,\ldots,k\}/\operatorname{Aut}(\Gamma)} \tilde{\mathcal{E}}^\Gamma_O$. Then

$$\begin{aligned} \tilde{\mathcal{E}}^\Gamma = \tilde{\mathcal{E}}^\Gamma_{\rm tr} \oplus \tilde{\mathcal{E}}^\Gamma_{\rm nt}, \ \text{ where } I_{\tilde{\Pi}^\Gamma(\mathcal{X}),\tilde{O}^\Gamma(\mathcal{X})}(\mathcal{E})^{-1}\bigl[\tilde{\Pi}^\Gamma(\mathcal{X})^*(\tilde{\mathcal{E}}^\Gamma_{\rm tr})\bigr] = \mathcal{E}^\Gamma_{\rm tr} \\ \text{and } I_{\tilde{\Pi}^\Gamma(\mathcal{X}),\tilde{O}^\Gamma(\mathcal{X})}(\mathcal{E})^{-1}\bigl[\tilde{\Pi}^\Gamma(\mathcal{X})^*(\tilde{\mathcal{E}}^\Gamma_{\rm nt})\bigr] = \mathcal{E}^\Gamma_{\rm nt}. \end{aligned} \tag{C.25}$$

Each point $[x,\Delta]$ of $\tilde{\mathcal{X}}^\Gamma_{\rm top}$ has orbifold group $\operatorname{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])$ with a distinguished subgroup Δ with a noncanonical isomorphism $\Delta \cong \Gamma$. The fibre of $\tilde{\mathcal{E}}^\Gamma$ at $[x,\Delta]$ is a representation of $\operatorname{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])$, and hence a representation of Δ. Equation (C.25) corresponds to splitting the fibre of $\tilde{\mathcal{E}}^\Gamma$ at $[x,\Delta]$ into trivial and nontrivial representations of Δ. Equation (C.24) corresponds to decomposing the fibre of $\tilde{\mathcal{E}}^\Gamma$ at $[x,\Delta]$ into families of irreducible representations of $\Delta \cong \Gamma$ that are independent of the choice of isomorphism $\Delta \cong \Gamma$.

Now let $f:\mathcal{X}\to\mathcal{Y}$ be a representable 1-morphism of C^∞-stacks, so that as in §C.8 we have a representable 1-morphism $\tilde{f}^\Gamma:\tilde{\mathcal{X}}^\Gamma\to\tilde{\mathcal{Y}}^\Gamma$ with $f\circ\tilde{O}^\Gamma(\mathcal{X}) = \tilde{O}^\Gamma(\mathcal{Y})\circ\tilde{f}^\Gamma$. Let $\mathcal{F}\in\operatorname{qcoh}(\mathcal{Y})$, so that $\tilde{\mathcal{F}}^\Gamma\in\operatorname{qcoh}(\tilde{\mathcal{Y}}^\Gamma)$, $f^*(\mathcal{F})\in\operatorname{qcoh}(\mathcal{X})$, and $\widetilde{f^*(\mathcal{F})}^\Gamma\in\operatorname{qcoh}(\tilde{\mathcal{X}}^\Gamma)$. As for (C.21), we have a canonical isomorphism

$$\tilde{T}^\Gamma(f,\mathcal{F}) := I_{\tilde{f}^\Gamma,\tilde{O}^\Gamma(\mathcal{Y})}(\mathcal{F})\circ I_{\tilde{O}^\Gamma(\mathcal{X}),f}(\mathcal{F})^{-1} : \widetilde{f^*(\mathcal{F})}^\Gamma \longrightarrow (\tilde{f}^\Gamma)^*(\tilde{\mathcal{F}}^\Gamma).$$

It maps $\widetilde{f^*(\mathcal{F})}^\Gamma_{\rm tr}\to(\tilde{f}^\Gamma)^*(\tilde{\mathcal{F}}^\Gamma_{\rm tr})$ and $\widetilde{f^*(\mathcal{F})}^\Gamma_{\rm nt}\to(\tilde{f}^\Gamma)^*(\tilde{\mathcal{F}}^\Gamma_{\rm nt})$ in (C.25).

Definition C.56. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, and Γ a finite group, so that §C.8 defines the orbifold strata $\tilde{\mathcal{X}}^\Gamma,\hat{\mathcal{X}}^\Gamma$ and 1-morphisms $\tilde{O}^\Gamma(\mathcal{X}):\tilde{\mathcal{X}}^\Gamma\to\mathcal{X}$ and $\hat{\Pi}^\Gamma:\tilde{\mathcal{X}}^\Gamma\to\hat{\mathcal{X}}^\Gamma$, where $\hat{\Pi}^\Gamma$ is non-representable, with fibre $[\bar{*}/\Gamma]$.

Suppose $\mathcal{E}$ is a quasicoherent sheaf on $\mathcal{X}$. Since we have no 1-morphism $\hat{\mathcal{X}}^\Gamma\to\mathcal{X}$, we cannot pull $\mathcal{E}$ back to $\hat{\mathcal{X}}^\Gamma$ to define $\hat{\mathcal{E}}^\Gamma$ in $\operatorname{qcoh}(\hat{\mathcal{X}}^\Gamma)$. But we do have $\tilde{\mathcal{E}}^\Gamma=\tilde{O}^\Gamma(\mathcal{X})^*(\mathcal{E})$ in $\operatorname{qcoh}(\tilde{\mathcal{X}}^\Gamma)$, with splitting $\tilde{\mathcal{E}}^\Gamma=\tilde{\mathcal{E}}^\Gamma_{\rm tr}\oplus\tilde{\mathcal{E}}^\Gamma_{\rm nt}$ as in (C.25), so we can form the pushforward $\hat{\Pi}^\Gamma_*(\tilde{\mathcal{E}}^\Gamma)$ in $\operatorname{qcoh}(\hat{\mathcal{X}}^\Gamma)$. Now pushforwards take global sections of a sheaf on the fibres of the 1-morphism. The fibres of $\hat{\Pi}^\Gamma$ are $[\bar{*}/\Gamma]$. Quasicoherent sheaves on $[\bar{*}/\Gamma]$ correspond to Γ-representations, and the global sections correspond to the trivial (Γ-invariant) part.

As the Γ-invariant part of $\tilde{\mathcal{E}}^\Gamma$ is $\tilde{\mathcal{E}}^\Gamma_{\rm tr}$, we see that $\hat{\Pi}^\Gamma_*(\tilde{\mathcal{E}}^\Gamma_{\rm nt})=0$, that is, $\mathcal{E}^\Gamma_{\rm nt}$ *and* $\tilde{\mathcal{E}}^\Gamma_{\rm nt}$ *do not descend to* $\hat{\mathcal{X}}^\Gamma$. Define $\hat{\mathcal{E}}^\Gamma_{\rm tr}=\hat{\Pi}^\Gamma_*(\tilde{\mathcal{E}}^\Gamma_{\rm tr})$ in $\operatorname{qcoh}(\hat{\mathcal{X}}^\Gamma)$. This is the natural analogue of $\mathcal{E}^\Gamma_{\rm tr},\tilde{\mathcal{E}}^\Gamma_{\rm tr}$ on $\hat{\mathcal{X}}^\Gamma$, and has a canonical isomorphism

$$(\hat{\Pi}^\Gamma)^*(\hat{\mathcal{E}}^\Gamma_{\rm tr})\cong\tilde{\mathcal{E}}^\Gamma_{\rm tr}. \tag{C.26}$$

Now let $f:\mathcal{X}\to\mathcal{Y}$ be a representable 1-morphism of C^∞-stacks, so that as in §C.8 we have a representable 1-morphism $\tilde f^\Gamma:\tilde{\mathcal{X}}^\Gamma\to\tilde{\mathcal{Y}}^\Gamma$. Then there is a canonical isomorphism

$$\hat T^\Gamma_{\rm tr}(f,\mathcal{F}):\widehat{f^*(\mathcal{F})}^\Gamma_{\rm tr}\longrightarrow(\hat f^\Gamma)^*(\hat{\mathcal{F}}^\Gamma_{\rm tr}),$$

the composition of the natural isomorphism $\hat\Pi^\Gamma_*\circ(\tilde f^\Gamma)^*(\tilde{\mathcal{F}}^\Gamma_{\rm tr})\to(\hat f^\Gamma)^*\circ\hat\Pi^\Gamma_*(\tilde{\mathcal{F}}^\Gamma_{\rm tr})$ with $\hat\Pi^\Gamma_*(\tilde T^\Gamma(f,\mathcal{F})|_{\widetilde{f^*(\mathcal{F})}^\Gamma_{\rm tr}})$.

In the next theorem [56, Th. 11.13] we take $\mathcal{X}=[\underline{X}/G]$, and use the explicit description of $\mathcal{X}^\Gamma$ in Theorem C.53 to give an alternative formula for the action $R^\Gamma(-,\mathcal{E})$ of Γ on $\mathcal{E}^\Gamma$ in Definition C.54. This then allows us to understand the splittings (C.22)–(C.26) in terms of sheaves on $\underline{X}$.

Theorem C.57. *Let $\underline{X}$ be a separated C^∞-scheme, G a finite group, $\underline{r}:G\to\mathrm{Aut}(\underline{X})$ an action of G on $\underline{X}$, and $\mathcal{X}=[\underline{X}/G]$ the quotient Deligne–Mumford C^∞-stack. Then* (C.10) *gives an equivalence $\mathcal{X}^\Gamma\simeq[\coprod_{injective\ \rho:\Gamma\to G}\underline{X}^{\rho(\Gamma)}/G]$.*

Write $\mathrm{qcoh}^G(\underline{X})$ for the abelian category of G-equivariant quasicoherent sheaves on $\underline{X}$, with objects pairs $(\mathcal{E},\Phi)$ for $\mathcal{E}\in\mathrm{qcoh}(\underline{X})$ and $\Phi(g):\underline{r}(g)^*(\mathcal{E})\to\mathcal{E}$ is an isomorphism in $\mathrm{qcoh}(\underline{X})$ for all $g\in G$ satisfying $\Phi(1)=\delta_{\underline{X}}(\mathcal{E})$ and

$$\Phi(gh)=\Phi(h)\circ\underline{r}(h)^*(\Phi(g))\circ I_{\underline{r}(h),\underline{r}(g)}(\mathcal{E})\quad\textit{for all } g,h\in G,$$

and morphisms $\alpha:(\mathcal{E},\Phi)\to(\mathcal{F},\Psi)$ in $\mathrm{qcoh}^G(\underline{X})$ are morphisms $\alpha:\mathcal{E}\to\mathcal{F}$ in $\mathrm{qcoh}(\underline{X})$ with $\alpha\circ\Phi(g)=\Psi(g)\circ\underline{r}(g)^*(\alpha)$ for all $g\in G$.

Then $\mathrm{qcoh}^G(\underline{X})$ is isomorphic to $\mathrm{qcoh}(G\times\underline{X}\rightrightarrows\underline{X})$ in Definition C.34, *so Theorem* C.35 *gives an equivalence of categories $F_\Pi:\mathrm{qcoh}(\mathcal{X})\to\mathrm{qcoh}^G(\underline{X})$. Using* (C.10) *we also get an equivalence $F_\Pi^\Gamma:\mathrm{qcoh}(\mathcal{X}^\Gamma)\to\mathrm{qcoh}^G(\coprod_\rho\underline{X}^{\rho(\Gamma)})$. These categories and functors fit into a 2-commutative diagram:*

$$\begin{tikzcd}
\mathrm{qcoh}(\mathcal{X}) \arrow[rr, "F_\Pi"'] \arrow[d, "O^\Gamma(\mathcal{X})^*"] & \arrow[d, phantom, "\Uparrow N^\Gamma(\mathcal{X})"] & \mathrm{qcoh}^G(\underline{X}) \arrow[d, "\underline{i}_{\underline{X}}^*"'] \\
\mathrm{qcoh}(\mathcal{X}^\Gamma) \arrow[rr, "F_\Pi^\Gamma"] & {} & \mathrm{qcoh}^G(\coprod_\rho\underline{X}^{\rho(\Gamma)}),
\end{tikzcd}\tag{C.27}$$

where $\underline{i}_{\underline{X}}:\coprod_\rho\underline{X}^{\rho(\Gamma)}\to\underline{X}$ is the union over ρ of the inclusion morphisms $\underline{X}^{\rho(\Gamma)}\to\underline{X}$, which is G-equivariant and so induces a pullback functor $\underline{i}_{\underline{X}}^*$ as shown, and $N^\Gamma(\mathcal{X})$ is a natural isomorphism of functors.

Let $(E,\Phi)\in\mathrm{qcoh}^G(\underline{X})$, so that $\underline{i}_{\underline{X}}^*(E,\Phi)\in\mathrm{qcoh}^G(\coprod_\rho\underline{X}^{\rho(\Gamma)})$. Define $\bar R^\Gamma\bigl(\gamma,(E,\Phi)\bigr):\underline{i}_{\underline{X}}^*(E,\Phi)\to\underline{i}_{\underline{X}}^*(E,\Phi)$ in $\mathrm{qcoh}^G(\coprod_\rho\underline{X}^{\rho(\Gamma)})$ for $\gamma\in\Gamma$ such that

$$\begin{aligned}&\bar R^\Gamma\bigl(\gamma,(E,\Phi)\bigr)|_{\underline{X}^{\rho(\Gamma)}}:\underline{i}_{\underline{X}}|^*_{\underline{X}^{\rho(\Gamma)}}(E)\longrightarrow\underline{i}_{\underline{X}}|^*_{\underline{X}^{\rho(\Gamma)}}(E)\ \textit{ is given by}\\ &\bar R^\Gamma\bigl(\gamma,(E,\Phi)\bigr)|_{\underline{X}^{\rho(\Gamma)}}=\underline{i}_{\underline{X}}|^*_{\underline{X}^{\rho(\Gamma)}}(\Phi(\rho(\gamma^{-1})))\circ I_{\underline{i}_{\underline{X}}|_{\underline{X}^{\rho(\Gamma)}},\underline{r}(\rho(\gamma^{-1}))}(\mathcal{E})\end{aligned}$$

for each ρ, noting that $\underline{r}(\rho(\gamma^{-1}))\circ\underline{i}_{\underline{X}}|_{\underline{X}^{\rho(\Gamma)}}=\underline{i}_{\underline{X}}|_{\underline{X}^{\rho(\Gamma)}}$. Then $\bar R^\Gamma\bigl(-,(E,\Phi)\bigr)$ is an action of Γ on $\underline{i}_{\underline{X}}|^*_{\underline{X}^{\rho(\Gamma)}}(E)$ by isomorphisms. Furthermore, for each $\mathcal{E}$ in

$\mathrm{qcoh}(\mathcal{X})$ *and* γ *in* Γ, *the following diagram in* $\mathrm{qcoh}^G(\coprod_\rho \underline{X}^{\rho(\Gamma)})$ *commutes:*

$$\begin{tikzcd} F_\Pi^\Gamma(\mathcal{E}^\Gamma) \arrow[rr, "F_\Pi^\Gamma(R^\Gamma(\gamma,\mathcal{E}))"'] \arrow[d, "N^\Gamma(\mathcal{X})(\mathcal{E})"] && F_\Pi^\Gamma(\mathcal{E}^\Gamma) \arrow[d, "N^\Gamma(\mathcal{X})(\mathcal{E})"'] \\ \underline{i}_{\underline{X}}^* \circ F_\Pi(\mathcal{E}) \arrow[rr, "\bar{R}^\Gamma(\gamma,F_\Pi(\mathcal{E}))"] && \underline{i}_{\underline{X}}^* \circ F_\Pi(\mathcal{E}). \end{tikzcd}$$

That is, the equivalences of categories F_Π, F_Π^Γ *in* (C.27) *identify the* Γ*-actions* $R^\Gamma(-,-)$ *on* $O^\Gamma(\mathcal{X})^*$ *and* $\bar{R}^\Gamma(-,-)$ *on* $\underline{i}_{\underline{X}}^*$ *by natural isomorphisms.*

In [56, Th. 11.14] we apply these ideas to write the cotangent sheaves of $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$ in terms of the pullbacks of $T^*\mathcal{X}$. The theorem illustrates the principle that when passing to orbifold strata, it is often natural to restrict to the trivial parts $\mathcal{E}^\Gamma_{\mathrm{tr}}, \tilde{\mathcal{E}}^\Gamma_{\mathrm{tr}}, \hat{\mathcal{E}}^\Gamma_{\mathrm{tr}}$ of the pullbacks of $\mathcal{E}$. The nontrivial parts $(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}, (\widetilde{T^*\mathcal{X}})^\Gamma_{\mathrm{nt}}$ should be interpreted as the *conormal sheaves* of $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma$ in $\mathcal{X}$.

Theorem C.58. *Let* $\mathcal{X}$ *be a locally fair Deligne–Mumford* C^∞*-stack and* Γ *a finite group, so that* §C.8 *defines* $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$. *As in Definition* C.38 *we have cotangent sheaves* $T^*\mathcal{X}, T^*(\mathcal{X}^\Gamma)$ *and a morphism* $\Omega_{O^\Gamma(\mathcal{X})} : O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) \to T^*(\mathcal{X}^\Gamma)$ *in* $\mathrm{qcoh}(\mathcal{X}^\Gamma)$. *But* $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) = (T^*\mathcal{X})^\Gamma$, *so by* (C.23) *we have a splitting* $(T^*\mathcal{X})^\Gamma = (T^*\mathcal{X})^\Gamma_{\mathrm{tr}} \oplus (T^*\mathcal{X})^\Gamma_{\mathrm{nt}}$. *Then* $\Omega_{O^\Gamma(\mathcal{X})}|_{(T^*\mathcal{X})^\Gamma_{\mathrm{tr}}} : (T^*\mathcal{X})^\Gamma_{\mathrm{tr}} \to T^*(\mathcal{X}^\Gamma)$ *is an isomorphism, and* $\Omega_{O^\Gamma(\mathcal{X})}|_{(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}} = 0$.

Similarly, using the 1*-morphism* $\tilde{O}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \to \mathcal{X}$ *and the splitting* (C.25) *for* $(\widetilde{T^*\mathcal{X}})^\Gamma$ *we find that* $\Omega_{\tilde{O}^\Gamma(\mathcal{X})}|_{(\widetilde{T^*\mathcal{X}})^\Gamma_{\mathrm{tr}}} : (\widetilde{T^*\mathcal{X}})^\Gamma_{\mathrm{tr}} \to T^*(\tilde{\mathcal{X}}^\Gamma)$ *is an isomorphism, and* $\Omega_{\tilde{O}^\Gamma(\mathcal{X})}|_{(\widetilde{T^*\mathcal{X}})^\Gamma_{\mathrm{nt}}} = 0$.

Also, there is a natural isomorphism $(\widehat{T^*\mathcal{X}})^\Gamma_{\mathrm{tr}} \cong T^*(\hat{\mathcal{X}}^\Gamma)$ *in* $\mathrm{qcoh}(\hat{\mathcal{X}}^\Gamma)$.

D Existence of good coordinate systems

We now prove Theorem 10.48 in §10.8 and Theorem 12.48 in §12.9 on the existence of *type A good coordinate systems* on d-orbifolds and d-orbifolds with corners. It is enough to prove Theorem 12.48, as Theorem 10.48 follows by omitting boundaries and corners throughout.

D.1 Outline of the proof of Theorem 12.48

We will prove Theorem 12.48 in five steps, outlined here and carried out in detail in §D.2–§D.5. Let $\mathfrak{X}$ be a d-orbifold with corners. Steps 1–4 prove the first part of Theorem 12.48, constructing a type A good coordinate system for $\mathfrak{X}$. Step 5 extends this to a type A good coordinate system for $\boldsymbol{h}:\mathfrak{X}\rightarrow\mathfrak{Y}=F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y)$.

Step 1. We choose the following data satisfying conditions:

(i) A countable indexing set J, and a total order $\prec$ on J such that $(J,\prec)$ is well-ordered. If $\mathfrak{X}$ is compact, we may choose J finite.

(ii) For each $a\in J$, open d-suborbifolds $\mathfrak{X}^a\subseteq\hat{\mathfrak{X}}^a\subseteq\mathfrak{X}$ satisfying:

  (a) The closure of $\mathfrak{X}^a$ in $\hat{\mathfrak{X}}^a$ is compact.

  (b) $\{\mathfrak{X}^a:a\in J\}$ is an open cover of $\mathfrak{X}$ (and hence so is $\{\hat{\mathfrak{X}}^a:a\in J\}$).

  (c) For each $a\in J$, there are only finitely many $b\in J$ with $\hat{\mathfrak{X}}^a\cap\hat{\mathfrak{X}}^b\neq\boldsymbol{\emptyset}$.

If $\{\mathfrak{U}_k:k\in K\}$ is an open cover of $\mathfrak{X}$, we may take $\mathfrak{X}^a\subseteq\mathfrak{U}_{k_a}$ for each $a\in J$ and some $k_a\in K$.

(iii) For each $a\in J$, a principal d-manifold with corners $\hat{\mathbf{Z}}^a$, a finite indexing set C^a, a decomposition $\partial\hat{\mathbf{Z}}^a=\coprod_{c\in C^a}\partial^c\hat{\mathbf{Z}}^a$, a finite group Γ^a, an action $\boldsymbol{t}^a:\Gamma^a\rightarrow\mathrm{Aut}(\hat{\mathbf{Z}}^a)$ of Γ^a on $\hat{\mathbf{Z}}^a$ by 1-isomorphisms, an equivalence $\boldsymbol{i}^a:[\hat{\mathbf{Z}}^a/\Gamma^a]\rightarrow\hat{\mathfrak{X}}^a\subseteq\mathfrak{X}$ in $\mathbf{dOrb^c}$ for $[\hat{\mathbf{Z}}^a/\Gamma^a]$ as in §11.2, an action $p^a:\Gamma^a\rightarrow\mathrm{Aut}(C^a)$ of Γ^a on C^a, and 1-morphism $\boldsymbol{b}^{ac}:\hat{\boldsymbol{Z}}^a\rightarrow\boldsymbol{[0,\infty)}$ in $\mathbf{dSpa}$ for each $c\in C^a$. These satisfy:

  (a) $\boldsymbol{\emptyset}\neq\partial^c\hat{\mathbf{Z}}^a\subseteq\partial\hat{\mathbf{Z}}^a$ is open and closed for each $c\in C^a$.

  (b) For each $\gamma\in\Gamma^a$, the 1-isomorphism $\boldsymbol{t}^a(\gamma):\hat{\mathbf{Z}}^a\rightarrow\hat{\mathbf{Z}}^a$ is simple, and so induces a 1-isomorphism $\boldsymbol{t}^a(\gamma)_-:\partial\hat{\mathbf{Z}}^a\rightarrow\partial\hat{\mathbf{Z}}^a$ as in §6.3. We require that $\boldsymbol{t}^a(\gamma)_-(\partial^c\hat{\mathbf{Z}}^a)=\partial^{p^a(\gamma)(c)}\hat{\mathbf{Z}}^a$ for all $c\in C^a$.

  (c) $(\hat{\boldsymbol{Z}}^a,\boldsymbol{b}^{ac})$ is a boundary defining function for $\hat{\mathbf{Z}}^a$ at $z'\in\partial\hat{\mathbf{Z}}^a$ in the sense of §6.1 if and only if $z'\in\partial^c\hat{\mathbf{Z}}^a$, for each $c\in C^a$.

  (d) $\boldsymbol{b}^{ac}=\boldsymbol{b}^{ap^a(\gamma)(c)}\circ\boldsymbol{t}^a(\gamma)$ for all $c\in C^a$ and $\gamma\in\Gamma^a$.

Note that (b) determines p^a uniquely, as $\partial^c\hat{\mathbf{Z}}^a\neq\boldsymbol{\emptyset}$. Write $\mathbf{Z}^a\subseteq\hat{\mathbf{Z}}^a$ for the Γ^a-invariant open d-submanifold with $\boldsymbol{i}^a([\mathbf{Z}^a/\Gamma^a])=\mathfrak{X}^a\subseteq\hat{\mathfrak{X}}^a$.

(iv) Suppose $a,b\in J$ with $a\prec b$ and $\hat{\mathfrak{X}}^a\cap\hat{\mathfrak{X}}^b\neq\boldsymbol{\emptyset}$. Then we are given a subgroup $\Gamma^{ab}\subseteq\Gamma^a$ and an open d-submanifold $\hat{\mathbf{Z}}^{ab}\subseteq\hat{\mathbf{Z}}^a$ satisfying:

  (a) $\hat{\mathbf{Z}}^{ab}$ is invariant under Γ^{ab}.

(b) if $\gamma \in \Gamma^a \setminus \Gamma^{ab}$ then $\hat{\mathbf{Z}}^{ab} \cap \boldsymbol{t}^a(\gamma)(\hat{\mathbf{Z}}^{ab}) = \boldsymbol{\emptyset}$.

(c) parts (a),(b) imply that $\coprod_{\gamma\Gamma^{ab} \in \Gamma^a/\Gamma^{ab}} \boldsymbol{t}^a(\gamma)(\hat{\mathbf{Z}}^{ab})$ is a Γ^a-invariant open d-submanifold of $\hat{\mathbf{Z}}^a$, so that $\bigl[\coprod_{\gamma\Gamma^{ab} \in \Gamma^a/\Gamma^{ab}} \boldsymbol{t}^a(\gamma)(\hat{\mathbf{Z}}^{ab})/\Gamma^a\bigr]$ is an open d-suborbifold of $[\hat{\mathbf{Z}}^a/\Gamma^a]$. We require that $\boldsymbol{i}^a$ identifies this open d-suborbifold with the open d-suborbifold $\hat{\mathfrak{X}}^a \cap \hat{\mathfrak{X}}^b$ in $\hat{\mathfrak{X}}^a$.

Also we are given a subgroup $\Gamma^{ba} \subseteq \Gamma^b$ and an open $\hat{\mathbf{Z}}^{ba} \subseteq \hat{\mathbf{Z}}^b$ satisfying the analogues of (a)–(c).

We should be given an isomorphism $\rho^{ab} : \Gamma^{ab} \to \Gamma^{ba}$ and an equivalence $\boldsymbol{j}^{ab} : \hat{\mathbf{Z}}^{ab} \to \hat{\mathbf{Z}}^{ba}$ satisfying $\boldsymbol{j}^{ab} \circ \boldsymbol{t}^a(\gamma)|_{\hat{\mathbf{Z}}^{ab}} = \boldsymbol{t}^b(\rho^{ab}(\gamma)) \circ \boldsymbol{j}^{ab}$ for all $\gamma \in \Gamma^{ab}$. Using quotient d-orbifolds with corners and 1-morphisms as in §11.2, we should be given a 2-morphism $\boldsymbol{\zeta}^{ab}$ in $\mathbf{dOrb^c}$ to make the following diagram 2-commute:

$$\begin{tikzcd}
{[\hat{\mathbf{Z}}^{ab}/\Gamma^{ab}]} \arrow[rr, "{[\boldsymbol{j}^{ab},\rho^{ab}]}"'] \arrow[d, "{[\mathbf{inc},\mathrm{inc}]}"] & \arrow[d, phantom, "\Downarrow\boldsymbol{\zeta}^{ab}"] & {[\hat{\mathbf{Z}}^{ba}/\Gamma^{ba}]} \arrow[d, "{[\mathbf{inc},\mathrm{inc}]}"'] \\
{[\hat{\mathbf{Z}}^a/\Gamma^a]} \arrow[r, "\boldsymbol{i}^a"] & \mathfrak{X} & {[\hat{\mathbf{Z}}^b/\Gamma^b]}, \arrow[l, "\boldsymbol{i}^b"']
\end{tikzcd} \tag{D.1}$$

where for example $[\mathbf{inc}, \mathrm{inc}] : [\hat{\mathbf{Z}}^{ab}/\Gamma^{ab}] \to [\hat{\mathbf{Z}}^a/\Gamma^a]$ is the quotient 1-morphism induced by the inclusions $\mathbf{inc} : \hat{\mathbf{Z}}^{ab} \to \hat{\mathbf{Z}}^a$ and $\mathrm{inc} : \Gamma^{ab} \to \Gamma^a$.

Write $C^{ab} = \{c \in C^a : \partial\hat{\mathbf{Z}}^{ab} \cap \partial^c\hat{\mathbf{Z}}^a \neq \boldsymbol{\emptyset}\}$ and $C^{ba} = \{c' \in C^b : \partial\hat{\mathbf{Z}}^{ba} \cap \partial^{c'}\hat{\mathbf{Z}}^b \neq \boldsymbol{\emptyset}\}$. Then we should be given a bijection $q^{ab} : C^{ab} \to C^{ba}$ such that $\boldsymbol{j}^{ab}_- : \partial\hat{\mathbf{Z}}^{ab} \to \partial\hat{\mathbf{Z}}^{ba}$ satisfies $\boldsymbol{j}^{ab}_-\bigl(\partial\hat{\mathbf{Z}}^{ab} \cap \partial^c\hat{\mathbf{Z}}^a\bigr) = \partial\hat{\mathbf{Z}}^{ba} \cap \partial^{q^{ab}(c)}\hat{\mathbf{Z}}^b$ for all $c \in C^{ab}$. This determines q^{ab} uniquely. It is automatic that C^{ab}, C^{ba} are invariant under Γ^{ab}, Γ^{ba}, and q^{ab} is equivariant under $\rho^{ab} : \Gamma^{ab} \to \Gamma^{ba}$.

(v) Suppose $a, b, c \in C$ with $a \prec b \prec c$ and $\alpha \in \Gamma^a$, $\beta \in \Gamma^b$ with

$$\hat{\mathbf{Z}}^{abc}_{\alpha\beta} := \hat{\mathbf{Z}}^{ab} \cap [\boldsymbol{t}^a(\alpha)]^{-1}(\hat{\mathbf{Z}}^{ac}) \cap [\boldsymbol{t}^b(\beta) \circ \boldsymbol{j}^{ab}]^{-1}(\hat{\mathbf{Z}}^{bc}) \neq \boldsymbol{\emptyset}. \tag{D.2}$$

Then we should be given $\gamma^{abc}_{\alpha\beta} \in \Gamma^c$ and a 2-morphism $\lambda^{abc}_{\alpha\beta}$ in $\mathbf{dMan^c}$:

$$\lambda^{abc}_{\alpha\beta} : \boldsymbol{j}^{bc} \circ \boldsymbol{t}^b(\beta) \circ \boldsymbol{j}^{ab}|_{\hat{\mathbf{Z}}^{abc}_{\alpha\beta}} \Longrightarrow \boldsymbol{t}^c((\gamma^{abc}_{\alpha\beta})^{-1}) \circ \boldsymbol{j}^{ac} \circ \boldsymbol{t}^a(\alpha)|_{\hat{\mathbf{Z}}^{abc}_{\alpha\beta}}, \tag{D.3}$$

such that the compositions of 2-morphisms across the following two diagrams are equal:

$$\begin{tikzcd}
{[\hat{\mathbf{Z}}^{abc}_{\alpha\beta}/\{1\}]} \arrow[r, "{[\mathbf{inc},\mathrm{inc}]}"'] \arrow[dr, "{[\boldsymbol{j}^{ac}\circ\boldsymbol{t}^a(\beta)|_{\hat{\mathbf{Z}}^{abc}_{\alpha\beta}},\iota]}"'] \arrow[dd, "{[\boldsymbol{t}^b(\beta)\circ\boldsymbol{j}^{ab}\circ|_{\hat{\mathbf{Z}}^{abc}_{\alpha\beta}},\iota]}"] & {[\hat{\mathbf{Z}}^a/\Gamma^a]} \arrow[dr, bend left, "\boldsymbol{i}^a"] \arrow[d, phantom, "{[\lambda^{abc}_{\alpha\beta},\gamma^{abc}_{\alpha\beta}]\Uparrow}"] & \arrow[d, phantom, "\boldsymbol{\zeta}^{ac}*\mathbf{id}_{[\mathbf{inc},\mathrm{inc}]}\Uparrow"] \\
& {[\hat{\mathbf{Z}}^c/\Gamma^c]} \arrow[r, "\boldsymbol{i}^c"] & \mathfrak{X}, \arrow[d, phantom, "\boldsymbol{\zeta}^{bc}\Downarrow"] \\
{[\hat{\mathbf{Z}}^{bc}/\Gamma^{bc}]} \arrow[ur, "{[\boldsymbol{j}^{bc},\rho^{bc}]}"'] \arrow[r, "{[\mathbf{inc},\mathrm{inc}]}"] & {[\hat{\mathbf{Z}}^b/\Gamma^b]} \arrow[ur, bend right, "\boldsymbol{i}^b"'] & {}
\end{tikzcd} \tag{D.4}$$

$$
\begin{tikzcd}[column sep=huge]
{[\hat{\mathbf{Z}}^{abc}_{\alpha\beta}/\{1\}]} \arrow[r, "{[\mathbf{inc},\mathrm{inc}]}"'] \arrow[dr, "{[\mathbf{inc},\mathrm{inc}]}"'] \arrow[dd, "{[\boldsymbol{t}^b(\beta)\circ\boldsymbol{j}^{ab}|_{\hat{\mathbf{Z}}^{abc}_{\alpha\beta}},\iota]}"] & {[\hat{\mathbf{Z}}^a/\Gamma^a]} \arrow[dr, bend left, "\boldsymbol{i}^a"] \arrow[d, phantom, "\mathbf{id}\ \Downarrow"'] & \\
\arrow[r, phantom, "\overset{[\mathrm{id},\beta]}{\Leftarrow}"] & {[\hat{\mathbf{Z}}^{ab}/\Gamma^{ab}]} \arrow[u, "{[\mathbf{inc},\mathrm{inc}]}"'] \arrow[d, "{[\boldsymbol{j}^{ab},\rho^{ab}]}"'] \arrow[r, phantom, "\boldsymbol{\zeta}^{ab}\Uparrow"] & \boldsymbol{\mathcal{X}}. \\
{[\hat{\mathbf{Z}}^{bc}/\Gamma^{bc}]} \arrow[r, "{[\mathbf{inc},\mathrm{inc}]}"] & {[\hat{\mathbf{Z}}^b/\Gamma^b]} \arrow[ur, bend right, "\boldsymbol{i}^b"'] &
\end{tikzcd}
\tag{D.5}
$$

Here $\iota : \{1\} \to G$ maps $1 \mapsto 1$ for any group G. Since $\boldsymbol{i}^c$ is a local equivalence, this determines $[\lambda^{abc}_{\alpha\beta}, \gamma^{abc}_{\alpha\beta}]$ and hence $\gamma^{abc}_{\alpha\beta}, \lambda^{abc}_{\alpha\beta}$ uniquely.

Step 2. We choose a continuous partition of unity $\{\theta^a : a \in J\}$ on $\mathcal{X}_{\mathrm{top}}$ subordinate to $\{\mathcal{X}^a_{\mathrm{top}} : a \in J\}$, so that $\theta^a : \mathcal{X}_{\mathrm{top}} \to [0,1]$ is continuous and supported in $\mathcal{X}^a_{\mathrm{top}} \subseteq \mathcal{X}_{\mathrm{top}}$, and $\sum_{a\in J} \theta^a = 1$. For each finite set $\emptyset \neq A \subseteq J$, define $\boldsymbol{\mathcal{X}}_A \subseteq \boldsymbol{\mathcal{X}}$ to be the open d-suborbifold with underlying topological space

$$
\begin{aligned}
\mathcal{X}_{A,\mathrm{top}} = \bigl\{[x] \in \mathcal{X}_{\mathrm{top}} : {}&\textstyle\sum_{a\in A} \theta^a([x]) > 1 - \frac{2}{4^{|A|}}, \text{ and} \\
&\text{for all } B \subset A,\ \emptyset \neq B \neq A,\ \textstyle\sum_{b\in B} \theta^b([x]) < 1 - \frac{1}{4^{|B|}}\bigr\}.
\end{aligned}
\tag{D.6}
$$

Define I to be the set of finite subsets $A \subseteq J$ with $\boldsymbol{\mathcal{X}}_A \neq \boldsymbol{\emptyset}$. We show that:

(A) $\{\boldsymbol{\mathcal{X}}_A : A \in I\}$ is an open cover of $\boldsymbol{\mathcal{X}}$;

(B) $\boldsymbol{\mathcal{X}}_A \subseteq \bigcap_{a\in A} \boldsymbol{\mathcal{X}}^a \subseteq \bigcap_{a\in A} \hat{\boldsymbol{\mathcal{X}}}^a$ for each $A \in I$; and

(C) if $A, B \in I$ then $\boldsymbol{\mathcal{X}}_A \cap \boldsymbol{\mathcal{X}}_B \neq \boldsymbol{\emptyset}$ only if $A \subseteq B$ or $B \subseteq A$.

Since J is countable, I is countable. If $\boldsymbol{\mathcal{X}}$ is compact we can choose J finite, and then I is finite. We also construct a total order $<$ on I such that $(I, <)$ is well-ordered, and if $A, B \in I$ with $A < B$ and $\boldsymbol{\mathcal{X}}_A \cap \boldsymbol{\mathcal{X}}_B \neq \boldsymbol{\emptyset}$ then $B \subsetneq A$.

Next, we choose the following data satisfying conditions:

(i) For each $A \in I$, a principal d-manifold with corners $\mathbf{Z}_A$, a finite indexing set C_A, a decomposition $\partial\mathbf{Z}_A = \coprod_{c\in C_A} \partial_c\mathbf{Z}_A$, a finite group Γ_A, an action $\boldsymbol{t}_A : \Gamma_A \to \mathrm{Aut}(\mathbf{Z}_A)$ of Γ_A on $\mathbf{Z}_A$ by 1-isomorphisms, an equivalence $\boldsymbol{i}_A : [\mathbf{Z}_A/\Gamma_A] \to \boldsymbol{\mathcal{X}}_a \subseteq \boldsymbol{\mathcal{X}}$ in $\mathbf{dOrb^c}$, an action $p_A : \Gamma_A \to \mathrm{Aut}(C_A)$ of Γ_A on C_A, and 1-morphism $\boldsymbol{b}_{Ac} : \boldsymbol{Z}_A \to \mathbf{[0,\infty)}$ in $\mathbf{dSpa}$ for each $c \in C_A$. These satisfy:

   (a) $\partial^c\mathbf{Z}_A \subseteq \partial\mathbf{Z}_A$ is open and closed for each $c \in C_A$.

   (b) $\boldsymbol{t}_A(\gamma)_-(\partial_c\mathbf{Z}_A) = \partial^{p_A(\gamma)(c)}\mathbf{Z}_A$ for all $c \in C_A$ and $\gamma \in \Gamma_A$.

   (c) $(\boldsymbol{Z}_A, \boldsymbol{b}_{Ac})$ is a boundary defining function for $\mathbf{Z}_A$ at $z' \in \partial\mathbf{Z}_A$ if and only if $z' \in \partial_c\mathbf{Z}_A$, for each $c \in C_A$.

   (d) $\boldsymbol{b}_{Ac} = \boldsymbol{b}_{Ap_A(\gamma)(c)} \circ \boldsymbol{t}_A(\gamma)$ for all $c \in C_A$ and $\gamma \in \Gamma_A$.

   Note that in contrast to Step 1(iii)(a) we allow $\partial_c\mathbf{Z}_A = \boldsymbol{\emptyset}$ for $c \in C_A$. So (b) may not determine p_A.

(ii) For all $A, B \in I$ with $A < B$ and $\mathfrak{X}_A \cap \mathfrak{X}_B \neq \emptyset$, a Γ_A-invariant open d-submanifold $\mathbf{Z}_{AB} \subseteq \mathbf{Z}_A$, an injective group morphism $\rho_{AB} : \Gamma_A \to \Gamma_B$, and a 1-morphism $\boldsymbol{j}_{AB} : \mathbf{Z}_{AB} \to \mathbf{Z}_B$ satisfying $\boldsymbol{j}_{AB} \circ \boldsymbol{t}_A(\gamma) = \boldsymbol{t}_B(\rho_{AB}(\gamma)) \circ \boldsymbol{j}_{AB}$ for all $\gamma \in \Gamma_{AB}$. As in §11.2 this induces a quotient 1-morphism $[\boldsymbol{j}_{AB}, \rho_{AB}] : [\mathbf{Z}_{AB}/\Gamma_A] \to [\mathbf{Z}_B/\Gamma_B]$, where $[\mathbf{Z}_{AB}/\Gamma_A] \subseteq [\mathbf{Z}_A/\Gamma_A]$ is an open d-suborbifold. We should be given a 2-morphism $\boldsymbol{\zeta}_{AB} : \boldsymbol{i}_B \circ [\boldsymbol{j}_{AB}, \rho_{AB}] \Rightarrow \boldsymbol{i}_A|_{[\mathbf{Z}_{AB}/\Gamma_A]}$ in $\mathbf{dOrb^c}$.

We are given an injective map $q_{AB} : C_A \to C_B$ with $q_{AB} \circ p_A(\gamma) = p_B(\rho_{AB}(\gamma)) \circ q_{AB} : C_A \to C_B$ for all $\gamma \in \Gamma_A$, such that $(\boldsymbol{j}_{AB})_-\bigl(\partial\mathbf{Z}_{AB} \cap \partial_c \mathbf{Z}_A\bigr) = (\boldsymbol{j}_{AB})_-\bigl(\partial\mathbf{Z}_{AB}\bigr) \cap \partial_{q_{AB}(c)}\mathbf{Z}_B$ for all $c \in C_A$.

(iii) For all $A, B, C \in I$ with $A < B < C$ and $\mathfrak{X}_A \cap \mathfrak{X}_B \cap \mathfrak{X}_C \neq \emptyset$, we are given $\gamma_{ABC} \in \Gamma_C$ satisfying $\rho_{AC}(\gamma) = \gamma_{ABC}\, \rho_{BC}(\rho_{AB}(\gamma))\, \gamma_{ABC}^{-1}$ for $\gamma \in \Gamma_A$, and

$$\boldsymbol{j}_{AC}|_{\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}} = \boldsymbol{t}_C(\gamma_{ABC}) \circ \boldsymbol{j}_{BC} \circ \boldsymbol{j}_{AB}|_{\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}}. \tag{D.7}$$

Note that these 1-morphisms should be equal, not just 2-isomorphic. Thus as in §11.2 we have a quotient 2-morphism

$$\begin{aligned} \bigl[\mathrm{id}, \gamma_{ABC}\bigr] : \bigl[\boldsymbol{j}_{BC}, \rho_{BC}\bigr] \circ \bigl[\boldsymbol{j}_{AB}|_{\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}}, \rho_{AB}\bigr] \\ \Longrightarrow \bigl[\boldsymbol{j}_{AC}|_{\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}}, \rho_{AC}\bigr]. \end{aligned} \tag{D.8}$$

The following diagram of 2-morphisms in $\mathbf{dOrb^c}$ should commute:

$$\begin{tikzcd} \begin{array}{l} \boldsymbol{i}_C \circ [\boldsymbol{j}_{BC}, \rho_{BC}] \circ \\ {[\boldsymbol{j}_{AB}|_{\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}}, \rho_{AB}]} \end{array} \arrow[r, Rightarrow, "\mathbf{id}_{\boldsymbol{i}_C} * [\mathrm{id}, \gamma_{ABC}]"'] \arrow[d, Rightarrow, "\boldsymbol{\zeta}_{BC} * \mathbf{id}_{[\boldsymbol{j}_{AB}|\cdots, \rho_{AB}]}"] & \boldsymbol{i}_C \circ [\boldsymbol{j}_{AC}|_{\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}}, \rho_{AC}] \arrow[d, Rightarrow, "\boldsymbol{\zeta}_{AC}|\cdots"'] \\ \boldsymbol{i}_B \circ [\boldsymbol{j}_{AB}|_{\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}}, \rho_{AB}] \arrow[r, Rightarrow, "\boldsymbol{\zeta}_{AB}|\cdots"] & \boldsymbol{i}_A|_{[(\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC})/\Gamma_A]}. \end{tikzcd} \tag{D.9}$$

In a condition similar to (D.7), we also require that

$$q_{AC} = p_C(\gamma_{ABC}) \circ q_{BC} \circ q_{AB} : C_A \longrightarrow C_C. \tag{D.10}$$

(iv) If $A, B, C, D \in I$ with $A < B < C < D$ and $\mathfrak{X}_A \cap \mathfrak{X}_B \cap \mathfrak{X}_C \cap \mathfrak{X}_D \neq \emptyset$, then $\gamma_{ACD}\, \rho_{CD}(\gamma_{ABC}) = \gamma_{ABD}\, \gamma_{BCD}$.

Note that all this data $\mathfrak{X}_A, \mathbf{Z}_A, \Gamma_A, \ldots$ is similar to the data $\mathfrak{X}^a, \mathbf{Z}^a, \Gamma^a, \ldots$ of Step 1, but has some better properties. In particular, in Step 1(iv) $\hat{\boldsymbol{Z}}^{ab}$ is invariant under a subgroup $\Gamma^{ab} \subseteq \Gamma^a$ and $\boldsymbol{j}^{ab}$ is Γ^{ab}-equivariant, but in Step 2(ii) $\boldsymbol{Z}_{AB}$ is invariant under the full group Γ_A, and $\boldsymbol{j}_{AB}$ is Γ_A-equivariant. Also, in Step 1 we have a 2-morphism (D.3) relating $\boldsymbol{j}^{ab}, \boldsymbol{j}^{ac}, \boldsymbol{j}^{bc}$, but in Step 2 we have an equality (D.7) relating $\boldsymbol{j}_{AB}, \boldsymbol{j}_{AC}, \boldsymbol{j}_{BC}$.

Here is how the new data relates to the data of Step 1. We may write each $A \in I$ as $\{a_1, \ldots, a_n\}$ with $a_1, \ldots, a_n \in J$ and $a_1 \prec a_2 \prec \cdots \prec a_n$. Then in (i), Γ_A is a subgroup of Γ^{a_1}, and $\mathbf{Z}_A$ an open d-submanifold of $\mathbf{Z}^{a_1}$ invariant under $\boldsymbol{t}^{a_1}(\Gamma_A)$, and $\boldsymbol{t}_A = \boldsymbol{t}^{a_1}|_{\Gamma_A}$, and $\boldsymbol{i}_A = \boldsymbol{i}^{a_1} \circ [\mathbf{inc}, \mathrm{inc}]$, and C_A is a Γ_A-invariant subset of C^{a_1} with $p_A = (p^{a_1}|_{\Gamma_A})|_{C_A}$, and $\boldsymbol{b}_{Ac} = b^{a_1 c}|_{\boldsymbol{Z}_A}$.

In (ii), if $A, B \in I$ with $A < B$ and $\mathfrak{X}_A \cap \mathfrak{X}_B \neq \emptyset$ then $B \subsetneq A$ as above. So we write $B = \{a_{b_1}, \ldots, a_{b_k}\}$ for $\{b_1, \ldots, b_k\} \subsetneq \{1, \ldots, n\}$ with $b_1 < b_2 < \cdots < b_k$. Then $\rho_{AB} : \Gamma_A \hookrightarrow \Gamma_B$ is of the form $\gamma \mapsto \beta\rho^{a_1 a_{b_1}}(\alpha\gamma\alpha^{-1})\beta^{-1}$ for some $\alpha \in \Gamma^{a_1}$ and $\beta \in \Gamma^{a_{b_1}}$ with $\alpha\Gamma_A\alpha^{-1} \subseteq \Gamma^{a_1 a_{b_1}}$, and $q_{AB} = p^{a_{b_1}}(\beta) \circ q^{a_1 a_{b_1}} \circ p^{a_1}(\alpha)|_{C_A}$, and $\boldsymbol{j}_{AB} : \mathbf{Z}_{AB} \to \mathbf{Z}_B$ is defined by a 2-morphism $\epsilon_{AB} : \boldsymbol{t}^{a_{b_1}}(\beta) \circ \boldsymbol{j}^{a_1 a_{b_1}} \circ \boldsymbol{t}^{a_1}(\alpha)|_{\mathbf{Z}_{AB}} \Rightarrow \boldsymbol{j}_{AB}$. The 2-morphisms ϵ_{AB} are chosen to ensure equality in (D.7).

Note that in Step 1(ii), the closure of $\mathfrak{X}^a$ in $\hat{\mathfrak{X}}^a$ is compact, so the closure of $\mathbf{Z}^a$ in $\hat{\mathbf{Z}}^a$ is compact. Since $\mathbf{Z}_A \subseteq \mathbf{Z}^{a_1}$, the closure of $\mathbf{Z}_A$ in $\hat{\mathbf{Z}}^{a_1}$ is compact for each A in I. Also the closure of $\mathbf{Z}_{AB}$ in $\hat{\mathbf{Z}}^{a_1 a_{b_1}}$ is compact.

Step 3. We choose the following data satisfying conditions:

(i) For each $A \in I$, a finite-dimensional real vector space T_A, an effective representation $r_A : \Gamma_A \to \mathrm{Aut}(T_A)$ of Γ_A on T_A, and linear maps $\tau_{Ac} : T_A \to \mathbb{R}$ for $c \in C_A$, satisfying:

   (a) τ_{Ac} for $c \in C_A$ are linearly independent in T_A^*.
   (b) $\tau_{Ac} = \tau_{Ap_A(\gamma)(c)} \circ r_A(\gamma)$ for all $c \in C_A$ and $\gamma \in \Gamma_A$.

(ii) For each $A \in I$, define $U_A = \{t \in T_A : \tau_{Ac}(t) \geqslant -1 \text{ for all } c \in C_A\}$. Then part (i)(a) implies that T_A is a manifold with corners, isomorphic to $\mathbb{R}^n_k$ for $n = \dim T_A$ and $k = |C_A|$. Also part (i)(b) implies that U_A is a Γ_A-invariant subset of T_A, so $\tilde{r}_A := r_A|_{U_A} : \Gamma_A \to \mathrm{Aut}(U_A)$ is an action of Γ_A on U_A by diffeomorphisms.

   We have a natural decomposition $\partial U_A = \coprod_{c \in C_A} \partial_c U_A$, where $\partial_c U_A \subseteq \partial U_A$ is open and closed with $\tau_{Ac} \circ i_{U_A}|_{\partial_c U_A} = 0$, and $\upsilon_{Ac} := (\tau_{Ac} + 1)|_{U_A} : U_A \to [0, \infty)$ is a boundary defining function for U_A at each $u' \in \partial_c U_A$. Write $\mathbf{U}_A, \tilde{\boldsymbol{r}}_A, \boldsymbol{\upsilon}_{Ac} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(U_A, \tilde{r}_A, \upsilon_{Ac})$.

   We should be given an embedding $\boldsymbol{f}_A : \mathbf{Z}_A \to \mathbf{U}_A$ in $\mathbf{dMan^c}$, satisfying:

   (a) $\tilde{\boldsymbol{r}}_A(\gamma) \circ \boldsymbol{f}_A = \boldsymbol{f}_A \circ \boldsymbol{t}_A(\gamma)$ for all $\gamma \in \Gamma_A$.
   (b) $(\boldsymbol{Z}_A, \boldsymbol{\upsilon}_{Ac} \circ \boldsymbol{f}_A)$ is a boundary defining function for $\mathbf{Z}_A$ at $z' \in \partial \mathbf{Z}_A$ if and only if $z' \in \partial_c \mathbf{Z}_A$, for each $c \in C_A$.

   Part (a) implies that $\boldsymbol{f}_A$ induces a quotient 1-morphism

$$[\boldsymbol{f}_A, \mathrm{id}_{\Gamma_A}] : [\mathbf{Z}_A/\Gamma_A] \longrightarrow [\mathbf{U}_A/\Gamma_A].$$

   Part (b) and $\partial \mathbf{Z}_A = \coprod_{c \in C_A} \partial_c \mathbf{Z}_A$ imply that $\boldsymbol{f}_A$ is simple and flat. Thus $\boldsymbol{f}_A$ is an *sf-embedding*, in the sense of §7.5.

(iii) For all $A, B \in I$ with $A < B$ and $\mathfrak{X}_A \cap \mathfrak{X}_B \neq \emptyset$, an injective linear map $l_{AB} : T_A \to T_B$ satisfying:

   (a) $l_{AB} \circ r_A(\gamma) = r_B(\rho_{AB}(\gamma)) \circ l_{AB}$ for all $\gamma \in \Gamma_A$.
   (b) $\tau_{Ac} = \tau_{Bq_{AB}(c)} \circ l_{AB}$ for all $c \in C_A$.
   (c) $\tau_{Bc'} \circ l_{AB} = 0$ for all $c \in C_B \setminus q_{AB}(C_A)$.

(d) Parts (b),(c) imply that $l_{AB}(U_A) \subseteq U_B$, and $\tilde{l}_{AB} := l_{AB}|_{U_A} : U_A \to U_B$ is a smooth map in $\mathbf{Man^c}$, which is an sf-embedding, and is ρ_{AB}-equivariant by (a). Write $\tilde{\boldsymbol{l}}_{AB} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(\tilde{l}_{AB}) : \mathbf{U}_A \to \mathbf{U}_B$. We require that $\boldsymbol{f}_B \circ \boldsymbol{j}_{AB} = \tilde{\boldsymbol{l}}_{AB} \circ \boldsymbol{f}_A|_{\mathbf{Z}_{AB}} : \mathbf{Z}_{AB} \to \mathbf{U}_B$. Note that these 1-morphisms are equal, not just 2-isomorphic. Thus

$$[\boldsymbol{f}_B, \mathrm{id}_{\Gamma_B}] \circ [\boldsymbol{j}_{AB}, \rho_{AB}] = [\tilde{\boldsymbol{l}}_{AB}, \rho_{AB}] \circ [\boldsymbol{f}_A|_{\mathbf{Z}_{AB}}, \mathrm{id}_{\Gamma_A}].$$

(iv) Let $A, B, C \in I$ with $A < B < C$ and $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B \cap \boldsymbol{\mathfrak{X}}_C \neq \boldsymbol{\emptyset}$. Then Step 2(iii) gives $\gamma_{ABC} \in \Gamma_C$. We require that $l_{AC} = r_C(\gamma_{ABC}) \circ l_{BC} \circ l_{AB} : T_A \to T_C$. Hence we have a commutative diagram of 2-morphisms:

$$\begin{tikzcd}[column sep=small]
\begin{array}{c}[\tilde{\boldsymbol{l}}_{BC}, \rho_{BC}] \circ [\tilde{\boldsymbol{l}}_{AB}, \rho_{AB}] \\ \circ [\boldsymbol{f}_A|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}, \mathrm{id}_{\Gamma_A}]\end{array} \arrow[r, equal] \arrow[d, Rightarrow, "{[\mathrm{id},\gamma_{ABC}]*\mathbf{id}_{[\boldsymbol{f}_A|\ldots,\mathrm{id}_{\Gamma_A}]}}"] & \begin{array}{c}[\tilde{\boldsymbol{l}}_{BC}, \rho_{BC}] \circ [\boldsymbol{f}_B, \mathrm{id}_{\Gamma_B}] \\ \circ [\boldsymbol{j}_{AB}|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}, \rho_{AB}]\end{array} \arrow[r, equal] & \begin{array}{c}[\boldsymbol{f}_C, \mathrm{id}_{\Gamma_C}] \circ [\boldsymbol{j}_{BC}, \rho_{BC}] \\ \circ [\boldsymbol{j}_{AB}|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}, \rho_{AB}]\end{array} \arrow[d, Rightarrow, "{\mathbf{id}_{[\boldsymbol{f}_C,\mathrm{id}_{\Gamma_C}]}*[\mathrm{id},\gamma_{ABC}]}"'] \\
{[\tilde{\boldsymbol{l}}_{AC}, \rho_{AC}] \circ [\boldsymbol{f}_A|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}, \mathrm{id}_{\Gamma_A}]} \arrow[rr, equal] & & {[\boldsymbol{f}_C, \mathrm{id}_{\Gamma_C}] \circ [\boldsymbol{j}_{AC}|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}, \rho_{AC}].}
\end{tikzcd}$$

Note that in part (ii) we define U_A by the inequalities $\tau_{Ac}(t) \geqslant -1$ in T_A, rather than $\tau_{Ac}(t) \geqslant 0$. This is to ensure the maps $\tilde{l}_{AB} : U_A \to U_B$ in part (iii)(d) have the correct behaviour. In particular, if $c' \in C_B \setminus q_{AB}(C_A)$ then $\tau_{Bc'} \geqslant -1$ is one of the defining inequalities of U_B, and l_{AB} maps U_A to the hyperplane $\tau_{Bc'} = 0$ in U_B by (iii)(c). Defining U_B using $\tau_{Bc'} \geqslant -1$ means that $l_{AB}(U_A)$ does not intersect the boundary component $\tau_{Bc'} = -1$ in U_B. If we had defined U_A by $\tau_{Ac}(t) \geqslant 0$, then l_{AB} would map U_A to the boundary of U_B, and $\tilde{l}_{AB}$ would not be flat.

Using Step 2(i)(c) and the material of §4.4 and §7.7, it is not difficult to construct sf-embeddings $\boldsymbol{f}_A : \mathbf{Z}_A \to \mathbf{U}_A \cong \mathbb{R}^{\boldsymbol{n}}_{\boldsymbol{k}}$ for $k = |C_A|$ and $n \gg 0$ satisfying parts (i) and (ii), with $\boldsymbol{v}_{Ac} \circ \boldsymbol{f}_A = \boldsymbol{b}_{Ac} : \mathbf{Z}_A \to \boldsymbol{[0,\infty)}$. We can also easily make $\boldsymbol{f}_A$ equivariant with respect to an effective representation of Γ_A on $\mathbb{R}^n = T_A$. The problem is to choose such $T_A, U_A, \boldsymbol{f}_A$ with the required compatibilities over double and triple overlaps $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B$ and $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B \cap \boldsymbol{\mathfrak{X}}_C$.

Step 4. We choose the following data satisfying conditions:

(i) For each $A \in I$, we are given:

(a) a Γ_A-invariant open neighbourhood V_A of $\boldsymbol{f}_A(\mathbf{Z}_A)$ in U_A;

(b) a vector bundle E_A on V_A;

(c) a lift $\hat{r}_A : \Gamma_A \to \mathrm{Aut}(E_A)$ of the Γ_A-action $\tilde{r}_A|_{V_A}$ on V_A to E_A, so that $\hat{r}_A(\gamma) : E_A \to \tilde{r}_A(\gamma)|_{V_A}^*(E_A)$ is a isomorphism of vector bundles on V_A for each $\gamma \in \Gamma_A$, and $\hat{r}_A(\gamma\delta) = \tilde{r}_A(\delta)|_{V_A}^*(\hat{r}_A(\gamma)) \circ \hat{r}_A(\delta)$ for all $\gamma, \delta \in \Gamma_A$. We may equivalently think of $\hat{r}_A$ as an action of Γ_A on the total space of E_A by diffeomorphisms.

(d) a smooth section $s_A : V_A \to E_A$ which is Γ_A-equivariant, that is, $\tilde{r}_A(\gamma)|_{V_A}^*(s_A) = \hat{r}_A(\gamma)(s_A)$ for all $\gamma \in \Gamma_A$; and

(e) a 2-morphism θ_A fitting into a 2-Cartesian diagram in $\mathbf{dMan^c}$:

$$\begin{tikzcd} \mathbf{Z}_A \arrow[r, "\boldsymbol{f}_A"] \arrow[d, "\boldsymbol{f}_A"'] \arrow[dr, phantom, "\theta_A \Uparrow"] & \mathbf{V}_A \arrow[d, "\mathbf{0}"] \\ \mathbf{V}_A \arrow[r, "\boldsymbol{s}_A"] & \mathbf{E}_A. \end{tikzcd} \tag{D.11}$$

This θ_A should be equivariant under the Γ_A-actions $\boldsymbol{t}_A, \tilde{\boldsymbol{r}}_A|_{\mathbf{V}_A}, \hat{\boldsymbol{r}}_A$ on $\mathbf{Z}_A, \mathbf{V}_A, \mathbf{E}_A$, that is, $\mathrm{id}_{\hat{\boldsymbol{r}}_A(\gamma)} * \theta_A = \theta_A * \mathrm{id}_{\boldsymbol{t}_A(\gamma)}$ for all $\gamma \in \Gamma_A$.

As (D.11) is 2-Cartesian we have equivalences $\mathbf{Z}_A \simeq \mathbf{V}_A \times_{\boldsymbol{s}_A, \mathbf{E}_A, \mathbf{0}} \mathbf{V}_A \simeq \mathbf{S}_{V_A, E_A, s_A}$, so we may choose an equivalence $\boldsymbol{k}_A : \mathbf{S}_{V_A,E_A,s_A} \to \mathbf{Z}_A$. Since θ_A is Γ_A-equivariant we may choose $\boldsymbol{k}_A$ to be Γ_A-equivariant. Thus we have a quotient 1-morphism in $\mathbf{dOrb^c}$

$$[\boldsymbol{k}_A, \mathrm{id}_{\Gamma_A}] : [\mathbf{S}_{V_A,E_A,s_A}/\Gamma_A] \longrightarrow [\mathbf{Z}_A/\Gamma_A], \tag{D.12}$$

which is an equivalence. Define an equivalence in $\mathbf{dOrb^c}$:

$$\boldsymbol{\psi}_A = \boldsymbol{i}_A \circ [\boldsymbol{k}_A, \mathrm{id}_{\Gamma_A}] : [\mathbf{S}_{V_A,E_A,s_A}/\Gamma_A] \longrightarrow \mathfrak{X}_A \subseteq \mathfrak{X}. \tag{D.13}$$

Then $(V_A, E_A, \Gamma_A, s_A, \boldsymbol{\psi}_A)$ is a type A Kuranishi neighbourhood on $\mathfrak{X}$.

(ii) For all $A, B \in I$ with $A < B$ and $\mathfrak{X}_A \cap \mathfrak{X}_B \neq \boldsymbol{\emptyset}$, set $V_{AB} = V_A \cap \tilde{l}_{AB}^{-1}(V_B)$, where the intersection is in U_A, and define $e_{AB} = \tilde{l}_{AB}|_{V_{AB}}$. Then V_{AB} is a Γ_A-invariant open submanifold of V_A, and $e_{AB} : V_{AB} \to V_B$ is an sf-embedding (as $\tilde{l}_{AB}$ is) with $e_{AB} \circ r_A(\gamma)|_{V_{AB}} = r_B(\rho_{AB}(\gamma)) \circ e_{AB}$ for all $\gamma \in \Gamma_A$ (by Step 3(iii)).

We should be given an embedding of vector bundles $\hat{e}_{AB} : E_A|_{V_{AB}} \to e_{AB}^*(E_B)$ on V_{AB} and a 2-morphism

$$\boldsymbol{\eta}_{AB} : \boldsymbol{\psi}_B \circ [\mathbf{S}_{e_{AB},\hat{e}_{AB}}, \rho_{AB}] \Rightarrow \boldsymbol{\psi}_A|_{[\mathbf{S}_{V_{AB},E_A|_{V_{AB}},s_A|_{V_{AB}}}/\Gamma_A]}$$

such that $(V_{AB}, e_{AB}, \hat{e}_{AB}, \rho_{AB}, \boldsymbol{\eta}_{AB})$ is a type A coordinate change from $(V_A, E_A, \Gamma_A, s_A, \boldsymbol{\psi}_A)$ to $(V_B, E_B, \Gamma_B, s_B, \boldsymbol{\psi}_B)$, as in Definition 12.47.

(iii) For all $A, B, C \in I$ with $A < B < C$ and $\mathfrak{X}_A \cap \mathfrak{X}_B \cap \mathfrak{X}_C \neq \boldsymbol{\emptyset}$, Step 2(iii) gives $\gamma_{ABC} \in \Gamma_C$ satisfying $\rho_{AC}(\gamma) = \gamma_{ABC}\, \rho_{BC}(\rho_{AB}(\gamma))\, \gamma_{ABC}^{-1}$ for $\gamma \in \Gamma_A$. Step 3(iv) implies that $e_{AC}|_{V_{AC} \cap e_{AB}^{-1}(V_{BC})} = r_C(\gamma_{ABC}) \circ e_{BC} \circ e_{AB}|_{V_{AC} \cap e_{AB}^{-1}(V_{BC})}$, which is the first equation of (10.20). We require that

$$\begin{aligned} &\hat{e}_{AC}|_{V_{AC} \cap e_{AB}^{-1}(V_{BC})} = \\ &\bigl(e_{AB}^*(e_{BC}^*(\hat{r}_C(\gamma_{ABC}))) \circ e_{AB}^*(\hat{e}_{BC}) \circ \hat{e}_{AB}\bigr)|_{V_{AC} \cap e_{AB}^{-1}(V_{BC})}, \end{aligned} \tag{D.14}$$

which is the second equation of (10.20).

Note that much of part (i) follows almost immediately from Theorem 7.48.

We have now constructed data $\bigl(I, <, (V_A, E_A, \Gamma_A, s_A, \boldsymbol{\psi}_A), (V_{AB}, e_{AB}, \hat{e}_{AB}, \rho_{AB}, \boldsymbol{\eta}_{AB}), \gamma_{ABC}\bigr)$ satisfying the corners analogues of Definition 10.47(a)–(d).

We show that it also satisfies part (e), and so is a type A good coordinate system for $\boldsymbol{\mathcal{X}}$. This proves the first part of Theorem 12.48.

Step 5. For the second part of Theorem 12.48, let Y be a manifold with corners, $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y)$, and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a semisimple, flat 1-morphism. Then we modify the proof in Steps 1–4 above as follows. Since $\boldsymbol{h}$ is semisimple, we have a decomposition $\partial\boldsymbol{\mathcal{X}} = \partial_+^{\boldsymbol{h}}\boldsymbol{\mathcal{X}} \amalg \partial_-^{\boldsymbol{h}}\boldsymbol{\mathcal{X}}$ as in §11.3. For $\hat{\mathbf{Z}}^a, \boldsymbol{i}^a$ and $\mathbf{Z}_A, \boldsymbol{i}_A$ as in Steps 1(iii) and 2(i), we also have semisimple, flat 1-morphisms $\boldsymbol{h}^a : \hat{\mathbf{Z}}^a \to \mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(Y)$ and $\boldsymbol{h}_A : \mathbf{Z}_A \to \mathbf{Y}$ in $\mathbf{dMan^c}$ which are the unique lifts to $\mathbf{dMan^c}$ of the compositions

$$F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}}(\hat{\mathbf{Z}}^a) = [\hat{\mathbf{Z}}^a/\{1\}] \xrightarrow[{[\mathbf{id}_{\hat{\mathbf{Z}}^a},\iota]}]{} [\hat{\mathbf{Z}}^a/\Gamma^a] \xrightarrow[\boldsymbol{i}^a]{} \boldsymbol{\mathcal{X}} \xrightarrow[\boldsymbol{h}]{} \boldsymbol{\mathcal{Y}} = F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y),$$

$$F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}}(\mathbf{Z}_A) = [\mathbf{Z}_A/\{1\}] \xrightarrow{[\mathbf{id}_{\mathbf{Z}_A},\iota]} [\mathbf{Z}_A/\Gamma_A] \xrightarrow{\boldsymbol{i}_A} \boldsymbol{\mathcal{X}} \xrightarrow{\boldsymbol{h}} \boldsymbol{\mathcal{Y}} = F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y).$$

So as in §6.3 we have $\partial\hat{\mathbf{Z}}^a = \partial_+^{\boldsymbol{h}^a}\hat{\mathbf{Z}}^a \amalg \partial_-^{\boldsymbol{h}^a}\hat{\mathbf{Z}}^a$ and $\partial\mathbf{Z}_A = \partial_+^{\boldsymbol{h}_A}\mathbf{Z}_A \amalg \partial_-^{\boldsymbol{h}_A}\mathbf{Z}_A$.

We modify Steps 1–3 above by replacing $\partial\hat{\mathbf{Z}}^a$ and $\partial\mathbf{Z}_A$ with $\partial_+^{\boldsymbol{h}^a}\hat{\mathbf{Z}}^a$ and $\partial_+^{\boldsymbol{h}_A}\mathbf{Z}_A$ throughout. That is, in Step 1(iii) we should be given a decomposition $\partial_+^{\boldsymbol{h}^a}\hat{\mathbf{Z}}^a = \coprod_{c\in C^a}\partial^c\hat{\mathbf{Z}}^a$ with $\partial^c\hat{\mathbf{Z}}^a$ open and closed in $\partial_+^{\boldsymbol{h}^a}\hat{\mathbf{Z}}^a$, and in Step 2(i) we should similarly be given a decomposition $\partial_+^{\boldsymbol{h}_A}\mathbf{Z}_A = \coprod_{c\in C_A}\partial_c\mathbf{Z}_A$. In Step 3(ii) we still take $\boldsymbol{f}_A : \mathbf{Z}_A \to \mathbf{U}_A$ to be an embedding, and in 3(ii)(b) we still require $(\boldsymbol{Z}_A, \boldsymbol{v}_{Ac}\circ\boldsymbol{f}_A)$ to be a boundary defining function for $\mathbf{Z}_A$ at $z' \in \partial\mathbf{Z}_A$ if and only if $z' \in \partial_c\mathbf{Z}_A$, for each $c \in C_A$.

Since we now have $\partial_+^{\boldsymbol{h}_A}\mathbf{Z}_A = \coprod_{c\in C_A}\partial_c\mathbf{Z}_A$ rather than $\partial\mathbf{Z}_A = \coprod_{c\in C_A}\partial_c\mathbf{Z}_A$, the final deduction in Step 3(ii) that $\boldsymbol{f}_A$ is an sf-embedding no longer holds. In fact $\boldsymbol{f}_A$ is semisimple and flat, but the morphism $\underline{s}_{\boldsymbol{f}_A} : \underline{S}_{\boldsymbol{f}_A} \to \underline{\partial Z}_A$ from §6.1 has image $\underline{\partial}_+^{\boldsymbol{h}_A}\underline{Z}_A \subseteq \underline{\partial Z}_A$, so $\underline{s}_{\boldsymbol{f}_A}$ need not be surjective, and $\boldsymbol{f}_A$ need not be simple, in the sense of §6.3.

Consider the direct product 1-morphism $(\boldsymbol{f}_A, \boldsymbol{h}_A) : \mathbf{Z}_A \to \mathbf{U}_A \times \mathbf{Y}$. This is an embedding as $\boldsymbol{f}_A$ is, and it is flat as $\boldsymbol{f}_A$ and $\boldsymbol{h}_A$ are. We claim that $(\boldsymbol{f}_A, \boldsymbol{h}_A)$ is also simple. To see this, note that $\boldsymbol{f}_A, \boldsymbol{h}_A$ are semisimple, so $\underline{s}_{\boldsymbol{f}_A} : \underline{S}_{\boldsymbol{f}_A} \to \underline{\partial Z}_A$ and $\underline{s}_{\boldsymbol{h}_A} : \underline{S}_{\boldsymbol{h}_A} \to \underline{\partial Z}_A$ are injective, and we have $\underline{s}_{\boldsymbol{f}_A}(\underline{S}_{\boldsymbol{f}_A}) = \underline{\partial}_+^{\boldsymbol{h}_A}\underline{Z}_A$ from above, and $\underline{s}_{\boldsymbol{h}_A}(\underline{S}_{\boldsymbol{h}_A}) = \underline{\partial}_-^{\boldsymbol{h}_A}\underline{Z}_A$ by definition of $\underline{\partial}_-^{\boldsymbol{h}_A}\underline{Z}_A$.

By properties of products we have $\partial(\mathbf{U}_A\times\mathbf{Y}) \cong ((\partial\mathbf{U}_A)\times\mathbf{Y})\amalg(\mathbf{U}_A\times(\partial\mathbf{Y}))$. Using this we can show that for the direct product $(\boldsymbol{f}_A, \boldsymbol{h}_A)$ we have a canonical isomorphism $\underline{S}_{(\boldsymbol{f}_A,\boldsymbol{h}_A)} \cong \underline{S}_{\boldsymbol{f}_A} \amalg \underline{S}_{\boldsymbol{h}_A}$, which identifies the projection $\underline{s}_{(\boldsymbol{f}_A,\boldsymbol{h}_A)} : \underline{S}_{(\boldsymbol{f}_A,\boldsymbol{h}_A)} \to \underline{\partial Z}_A$ with the disjoint unions of the projections $\underline{s}_{\boldsymbol{f}_A} : \underline{S}_{\boldsymbol{f}_A} \to \underline{\partial Z}_A$ and $\underline{s}_{\boldsymbol{h}_A} : \underline{S}_{\boldsymbol{h}_A} \to \underline{\partial Z}_A$. Since $\underline{s}_{\boldsymbol{f}_A}$ is injective with image $\underline{\partial}_+^{\boldsymbol{h}_A}\underline{Z}_A$, and $\underline{s}_{\boldsymbol{h}_A}$ is injective with image $\underline{\partial}_-^{\boldsymbol{h}_A}\underline{Z}_A$, and $\underline{\partial Z}_A = \underline{\partial}_+^{\boldsymbol{h}_A}\underline{Z}_A \amalg \underline{\partial}_-^{\boldsymbol{h}_A}\underline{Z}_A$, it follows that $\underline{s}_{(\boldsymbol{f}_A,\boldsymbol{h}_A)}$ is a bijection, and $(\boldsymbol{f}_A, \boldsymbol{h}_A)$ is simple. Hence $(\boldsymbol{f}_A, \boldsymbol{h}_A) : \mathbf{Z}_A \to \mathbf{U}_A \times \mathbf{Y}$ is an sf-embedding.

We now modify Step 4 by replacing $\boldsymbol{f}_A : \mathbf{Z}_A \to \mathbf{U}_A$ by the sf-embedding $(\boldsymbol{f}_A, \boldsymbol{h}_A) : \mathbf{Z}_A \to \mathbf{U}_A\times\mathbf{Y}$ throughout. We also replace U_A, $\tilde r_A : \Gamma_A \to \mathrm{Aut}(U_A)$, and $\tilde l_{AB} : U_A \to U_B$ by $U_A \times Y$, $\tilde r_A \times \mathrm{id}_Y : \Gamma_A \to \mathrm{Aut}(U_A \times Y)$, and $\tilde l_{AB} \times \mathrm{id}_Y :$

$U_A\times Y\rightarrow U_B\times Y$, respectively. Thus we take V_A to be open in $U_A\times Y$ rather than in U_A, and set $V_{AB}=V_A\cap(\tilde l_{AB}\times\mathrm{id}_Y)^{-1}(V_B)$ and $e_{AB}=(\tilde l_{AB}\times\mathrm{id}_Y)|_{V_{AB}}$.

The maps $g_A:V_A\rightarrow Y$ in Definition 10.47(f) are defined by $g_A=\pi_Y|_{V_A}$, where $\pi_Y:U_A\times Y\rightarrow Y$ is the projection. Then g_A is a submersion, as π_Y is, and the conditions $g_A\circ(\tilde r_A\times\mathrm{id}_Y)|_{V_A}(\gamma)=g_A$ in Definition 10.47(f) and $g_B\circ e_{AB}=g_A|_{V_{AB}}$ in Definition 10.47(g) are immediate. The 2-morphism $\boldsymbol\zeta_A:\boldsymbol h\circ\boldsymbol\psi_i\Rightarrow[\mathbf S_{g_A,0},\pi]$ in Definition 10.47(f) exists as we have 2-isomorphisms $\boldsymbol h\circ\boldsymbol\psi_i\cong[\boldsymbol h_A\circ\boldsymbol k_A,\pi]$ in $\mathbf{dOrb^c}$ by definition of $\boldsymbol h_A$, and $\mathbf S_{g_A,0}\cong\boldsymbol h_A\circ\boldsymbol k_A$ in $\mathbf{dMan^c}$ by construction, so $\boldsymbol h\circ\boldsymbol\psi_i\cong[\boldsymbol h_A\circ\boldsymbol k_A,\pi]\cong[\mathbf S_{g_A,0},\pi]$ in $\mathbf{dOrb^c}$. Hence $\big(I,<,(V_A,E_A,\Gamma_A,s_A,\boldsymbol\psi_A),g_A,(V_{AB},e_{AB},\hat e_{AB},\rho_{AB},\boldsymbol\eta_{AB}),\gamma_{ABC}\big)$ is a type A good coordinate system for $\boldsymbol h:\boldsymbol{\mathcal X}\rightarrow\boldsymbol{\mathcal Y}$, as we want.

In the rest of the section we explain Steps 1–4 in more detail. Step 5 needs no further explanation.

D.2 Step 1: Choose an open cover of $\boldsymbol{\mathcal X}$ by $\boldsymbol{\mathcal X}^a\simeq[\mathbf Z^a/\Gamma^a]$

Let $\boldsymbol{\mathcal X}$ be a d-orbifold with corners. Several times in what follows here and in Step 2 we will choose an open cover $\{\boldsymbol{\mathcal X}^a:a\in J\}$ for $\boldsymbol{\mathcal X}$, and then replace $\{\boldsymbol{\mathcal X}^a:a\in J\}$ by a refinement $\{\boldsymbol{\mathcal X}^{a_\star}_\star:a_\star\in J_\star\}$ with better properties. Here $\{\boldsymbol{\mathcal X}^{a_\star}_\star:a_\star\in J_\star\}$ is a *refinement* of $\{\boldsymbol{\mathcal X}^a:a\in J\}$ if $\{\boldsymbol{\mathcal X}^{a_\star}_\star:a_\star\in J_\star\}$ is also an open cover of $\boldsymbol{\mathcal X}$, and for each $a_\star\in J_\star$ we have $\boldsymbol{\mathcal X}^{a_\star}_\star\subseteq\boldsymbol{\mathcal X}^{a_{a_\star}}$ for some $a_{a_\star}\in J$.

Generally we will be working not just with one open cover, $\{\boldsymbol{\mathcal X}^a:a\in J\}$, but with two compatible open covers $\{\boldsymbol{\mathcal X}^a:a\in J\}$, $\{\hat{\boldsymbol{\mathcal X}}^a:a\in J\}$ and extra data such as equivalences $\boldsymbol i^a:[\hat{\mathbf Z}^a/\Gamma^a]\rightarrow\hat{\boldsymbol{\mathcal X}}^a$, and a total order $\prec$ on J. In this case, we choose simultaneous, compatible refinements $\{\boldsymbol{\mathcal X}^{a_\star}_\star:a_\star\in J_\star\}$, $\{\hat{\boldsymbol{\mathcal X}}^{a_\star}_\star:a_\star\in J_\star\}$ of $\{\boldsymbol{\mathcal X}^a:a\in J\}$, $\{\hat{\boldsymbol{\mathcal X}}^a:a\in J\}$, and we induce extra data on the refinements in the obvious way, so that $\boldsymbol i^{a_\star}_\star:[\hat{\mathbf Z}^{a_\star}_\star/\Gamma^{a_\star}_\star]\rightarrow\hat{\boldsymbol{\mathcal X}}^{a_\star}_\star$ is $\boldsymbol i^{a_{a_\star}}|_{\cdots}:[\hat{\mathbf Z}^{a_{a_\star}}_\star/\Gamma^{a_{a_\star}}]\rightarrow\hat{\boldsymbol{\mathcal X}}^{a_\star}_\star\subseteq\hat{\boldsymbol{\mathcal X}}^{a_{a_\star}}$, where $\hat{\mathbf Z}^{a_{a_\star}}_\star$ is the $\Gamma^{a_{a_\star}}$-invariant open d-submanifold of $\hat{\mathbf Z}^{a_{a_\star}}$ with $\boldsymbol i^{a_{a_\star}}\big([\hat{\mathbf Z}^{a_\star}_\star/\Gamma^{a_\star}_\star]\big)=\hat{\boldsymbol{\mathcal X}}^{a_\star}_\star$, and we choose the new total order $\prec_\star$ on $J_\star$ such that $a_{a_\star}\prec a_{b_\star}$ implies $a_\star\prec_\star b_\star$.

We first show that for each point $[x]\in\mathcal X_{\rm top}$ we can choose an open neighbourhood $\hat{\boldsymbol{\mathcal X}}^{[x]}$ of $[x]$ in $\boldsymbol{\mathcal X}$ and data $\hat{\mathbf Z}^{[x]},C^{[x]}$, $\partial\hat{\mathbf Z}^{[x]}=\coprod_{c\in C^{[x]}}\partial^c\hat{\mathbf Z}^{[x]}$, $\Gamma^{[x]},\boldsymbol t^{[x]}$, $\boldsymbol i^{[x]},p^{[x]},\boldsymbol b^{[x]c}$ satisfying the analogue of Step 1(iii) in §D.1. Since $\boldsymbol{\mathcal X}$ is a d-orbifold with corners, by Theorems 9.16(a) and 11.8 every point $[x]$ in $\mathcal X_{\rm top}$ has an open neighbourhood $\hat{\boldsymbol{\mathcal X}}^{[x]}\subseteq\boldsymbol{\mathcal X}$ with an equivalence $\boldsymbol i^{[x]}:[\hat{\mathbf Z}^{[x]}/\Gamma^{[x]}]\rightarrow\hat{\boldsymbol{\mathcal X}}^{[x]}$, where $\hat{\mathbf Z}^{[x]}$ is a principal d-manifold with corners and $\Gamma^{[x]}=\mathrm{Iso}_{\mathcal X}([x])$ acts on $\hat{\mathbf Z}^{[x]}$ with action $\boldsymbol t^{[x]}:\Gamma^{[x]}\rightarrow\mathrm{Aut}(\hat{\mathbf Z}^{[x]})$.

We may take $\hat{\mathbf Z}^{[x]}=\mathbf S_{V^{[x]},E^{[x]},s^{[x]}}$ from Definition 7.2, where Γ acts on $V^{[x]},E^{[x]}$ with $s^{[x]}$, and $[x]\in\mathcal X_{\rm top}$ is identified with a $\Gamma^{[x]}$-invariant point $v^{[x]}\in V^{[x]}$ with $s^{[x]}(v^{[x]})=0$. Making $\hat{\boldsymbol{\mathcal X}}^{[x]},\hat{\mathbf Z}^{[x]},V^{[x]}$ smaller if necessary, we can take $\Gamma^{[x]}$ to act linearly on $\mathbb R^n$ preserving the subset $\mathbb R^n_k\subseteq\mathbb R^n$, and $V^{[x]}\in\mathbf{Man^c}$ to be a $\Gamma^{[x]}$-invariant open neighbourhood of $v^{[x]}=0$ in $\mathbb R^n_k$.

Writing $(x_1,\ldots,x_n)$ for the coordinates in $\mathbb R^n_k\supseteq V^{[x]}$, so that $x_1,\ldots,x_k\in[0,\infty)$, we see that $\partial V^{[x]}=\coprod_{c=1}^k\partial^cV^{[x]}$, where $\partial^cV^{[x]}\subseteq\partial V^{[x]}\subseteq\partial(\mathbb R^n_k)$ is the open and closed subset of $\partial V^{[x]}$ coming from the boundary component $x_c = 0$. This induces a decomposition $\hat{\mathbf{Z}}^{[x]} = \coprod_{c \in C^{[x]}} \partial^c \hat{\mathbf{Z}}^{[x]}$, for $\boldsymbol{\emptyset} \neq \partial^c \hat{\mathbf{Z}}^{[x]} \subseteq \partial^c \hat{\mathbf{Z}}^{[x]}$ open and closed in $\partial^c \hat{\mathbf{Z}}^{[x]}$ as in (iii)(a), where $C^{[x]} = \{1, \ldots, k\}$. The linear action of $\Gamma^{[x]}$ on $\mathbb{R}^n$ preserving $\mathbb{R}^n_k$ must permute the coordinates $x_1, \ldots, x_k$, so as $C^{[x]} = \{1, \ldots, k\}$, this induces the action $p^{[x]} : \Gamma^{[x]} \to \mathrm{Aut}(C^{[x]})$ required.

For each $c \in C^{[x]} = \{1, \ldots, k\}$, the coordinate $x_c : V^{[x]} \to [0, \infty)$ is a boundary defining function for $V^{[x]}$ at each point $v' \in \partial^c V^{[x]}$. As $\hat{\mathbf{Z}}^{[x]} = \mathbf{S}_{V^{[x]}, E^{[x]}, s^{[x]}}$, the function $x_c : V^{[x]} \to [0, \infty)$ induces a 1-morphism $\boldsymbol{b}^{[x]c} : \hat{\boldsymbol{Z}}^{[x]} \to \boldsymbol{[0, \infty)}$ in **dSpa** such that $(\hat{\boldsymbol{Z}}^{[x]}, \boldsymbol{b}^{[x]c})$ is a boundary defining function for $\hat{\mathbf{Z}}^{[x]}$ at every point z' in $\partial^c \hat{\mathbf{Z}}^{[x]}$, as in (iii)(c). Also (iii)(b),(d) are immediate.

As $\mathcal{X}_{\mathrm{top}}$ is locally compact, we can choose an open neighbourhood $\boldsymbol{\mathfrak{X}}^{[x]}$ of $[x]$ in $\hat{\boldsymbol{\mathfrak{X}}}^{[x]}$ such that the closure of $\boldsymbol{\mathfrak{X}}^{[x]}$ in $\hat{\boldsymbol{\mathfrak{X}}}^{[x]}$ is compact. Choose such $\hat{\boldsymbol{\mathfrak{X}}}^{[x]}, \boldsymbol{\mathfrak{X}}^{[x]}, \hat{\mathbf{Z}}^{[x]}, C^{[x]}, \partial \hat{\mathbf{Z}}^{[x]} = \coprod_{c \in C^{[x]}} \partial^c \hat{\mathbf{Z}}^{[x]}, \Gamma^{[x]}, \boldsymbol{t}^{[x]}, \boldsymbol{i}^{[x]}, p^{[x]}, \boldsymbol{b}^{[x]c}$ for each $[x] \in \mathcal{X}_{\mathrm{top}}$. Then $\{\boldsymbol{\mathfrak{X}}^{[x]} : [x] \in \mathcal{X}_{\mathrm{top}}\}$ and $\{\hat{\boldsymbol{\mathfrak{X}}}^{[x]} : [x] \in \mathcal{X}_{\mathrm{top}}\}$ are open covers of $\boldsymbol{\mathfrak{X}}$.

Recall that an open cover $\{U_i : i \in I\}$ of a topological space Y is called *star-finite* if for all $i \in I$, there are only finitely many $j \in I$ with $U_i \cap U_j = \emptyset$. A topological space Y is called *strongly paracompact* if every open cover of Y has a star-finite refinement. If Y is Hausdorff, paracompact, and locally compact, then it is strongly paracompact. In our case, as $\boldsymbol{\mathfrak{X}}$ is a d-stack with corners, $\mathcal{X}$ is separated, second countable, and locally fair, so the underlying topological space $\mathcal{X}_{\mathrm{top}}$ is Hausdorff, paracompact, and locally compact. Thus $\mathcal{X}_{\mathrm{top}}$ is strongly paracompact.

Therefore we can choose simultaneous refinements $\{\boldsymbol{\mathfrak{X}}^a : a \in J\}$, $\{\hat{\boldsymbol{\mathfrak{X}}}^a : a \in J\}$ of $\{\boldsymbol{\mathfrak{X}}^{[x]} : [x] \in \mathcal{X}_{\mathrm{top}}\}$ and $\{\hat{\boldsymbol{\mathfrak{X}}}^{[x]} : [x] \in \mathcal{X}_{\mathrm{top}}\}$ satisfying Step 1(ii)(a)–(c) in §D.1, where (c) follows from strong paracompactness. As $\mathcal{X}_{\mathrm{top}}$ is second countable, Step 1(ii)(c) implies that J is countable. If $\boldsymbol{\mathfrak{X}}$ is compact, we may also choose J finite. If $\{\boldsymbol{\mathfrak{U}}_k : k \in K\}$ is an open cover of $\boldsymbol{\mathfrak{X}}$, we may take $\boldsymbol{\mathfrak{X}}^a \subseteq \boldsymbol{\mathfrak{U}}_{k_a}$ for each $a \in J$ and some $k_a \in K$.

By construction $\hat{\boldsymbol{\mathfrak{X}}}^a \simeq [\hat{\mathbf{Z}}^a / \Gamma^a]$ for each $a \in J$, with $\hat{\mathbf{Z}}^a$ a principal d-manifold with corners. Choose equivalences $\boldsymbol{i}^a : [\hat{\mathbf{Z}}^a / \Gamma^a] \to \hat{\boldsymbol{\mathfrak{X}}}^a$ and $\boldsymbol{k}^a : \hat{\boldsymbol{\mathfrak{X}}}^a \to [\hat{\mathbf{Z}}^a / \Gamma^a]$ and a 2-morphism $\boldsymbol{\eta}^a : \boldsymbol{k}^a \circ \boldsymbol{i}^a \Rightarrow \mathbf{id}_{[\hat{\mathbf{Z}}^a / \Gamma^a]}$. Write $\boldsymbol{t}^a : \Gamma^a \to \mathrm{Aut}(\hat{\mathbf{Z}}^a)$ for the action of Γ^a on $\hat{\mathbf{Z}}^a$ by 1-isomorphisms, and $\mathbf{Z}^a \subseteq \hat{\mathbf{Z}}^a$ for the Γ^a-invariant open d-submanifold with $\boldsymbol{i}^a([\mathbf{Z}^a / \Gamma^a]) = \boldsymbol{\mathfrak{X}}^a \subseteq \hat{\boldsymbol{\mathfrak{X}}}^a$. Choose data C^a, $\partial \hat{\mathbf{Z}}^a = \coprod_{c \in C^a} \partial^c \hat{\mathbf{Z}}^a$, $p^a, \boldsymbol{b}^{ac}$ satisfying Step 1(iii) as above. Choose an arbitrary total order $\prec$ on J such that $(J, \prec)$ is well-ordered. We have now completed parts (i)–(iii) of Step 1. But we will have to modify these choices to satisfy (iv).

Suppose now that $a, b \in J$ with $a \prec b$ and $\hat{\boldsymbol{\mathfrak{X}}}^a \cap \hat{\boldsymbol{\mathfrak{X}}}^b \neq \boldsymbol{\emptyset}$. Write $\hat{\mathbf{Z}}^a_{ab} \subseteq \hat{\mathbf{Z}}^a$ for the Γ^a-invariant open d-submanifold with $\boldsymbol{i}^a([\hat{\mathbf{Z}}^a_{ab} / \Gamma^a]) = \hat{\boldsymbol{\mathfrak{X}}}^a \cap \hat{\boldsymbol{\mathfrak{X}}}^b \subseteq \hat{\boldsymbol{\mathfrak{X}}}^a$, and similarly for $\hat{\mathbf{Z}}^b_{ab} \subseteq \hat{\mathbf{Z}}^b$. Consider the 2-Cartesian square:

$$\begin{tikzcd}
F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\mathbf{W}^{ab}) \arrow[rr, "F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\boldsymbol{f}^{ab})"] \arrow[d, "F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\boldsymbol{e}^{ab})"'] \arrow[drr, phantom, "\Downarrow \boldsymbol{\omega}^{ab}"] & & F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\hat{\mathbf{Z}}^b_{ab}) \arrow[d, "\boldsymbol{i}^b \circ \boldsymbol{\pi}_{[\hat{\mathbf{Z}}^b_{ab}/\Gamma^b]}"] \\
F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\hat{\mathbf{Z}}^a_{ab}) \arrow[rr, "\boldsymbol{i}^a \circ \boldsymbol{\pi}_{[\hat{\mathbf{Z}}^a_{ab}/\Gamma^a]}"] & & \hat{\boldsymbol{\mathfrak{X}}}^a \cap \hat{\boldsymbol{\mathfrak{X}}}^b.
\end{tikzcd} \tag{D.15}$$

Here $\boldsymbol{\pi}_{[\hat{\mathbf{Z}}^a_{ab}/\Gamma^a]}:F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\hat{\mathbf{Z}}^a_{ab})\rightarrow[\hat{\mathbf{Z}}^a_{ab}/\Gamma^a]$ is the natural projection, and similarly for $\boldsymbol{\pi}_{[\hat{\mathbf{Z}}^b_{ab}/\Gamma^b]}$. As $\boldsymbol{i}^a\circ\boldsymbol{\pi}_{[\hat{\mathbf{Z}}^a_{ab}/\Gamma^a]}$ is étale it is a submersion, so a fibre product $\mathcal{W}=F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\hat{\mathbf{Z}}^a_{ab})\times_{\hat{\mathfrak{X}}^a\cap\hat{\mathfrak{X}}^b}F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\hat{\mathbf{Z}}^b_{ab})$ exists in $\mathbf{dOrb^c}$ by Theorem 12.32(b), with projections $\boldsymbol{e}:\mathcal{W}\rightarrow F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\hat{\mathbf{Z}}^a_{ab})$, $\boldsymbol{f}:\mathcal{W}\rightarrow F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\hat{\mathbf{Z}}^b_{ab})$. As $F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\hat{\mathbf{Z}}^a_{ab}),F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\hat{\mathbf{Z}}^b_{ab})$ are d-manifolds with corners, $\mathcal{W}$ is a d-manifold in $\mathbf{dOrb^c}$, so changing it up to equivalence we can take $\mathcal{W}=F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\mathbf{W}^{ab})$ for some d-manifold with corners $\mathbf{W}^{ab}$ in $\mathbf{dMan^c}$, and changing $\boldsymbol{e},\boldsymbol{f}$ up to 2-isomorphism we can take $\boldsymbol{e}=F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\boldsymbol{e}^{ab})$, $\boldsymbol{f}=F^{\mathbf{dOrb^c}}_{\mathbf{dMan^c}}(\boldsymbol{f}^{ab})$ for 1-morphisms $\boldsymbol{e}^{ab}:\mathbf{W}^{ab}\rightarrow\hat{\mathbf{Z}}^a_{ab}$, $\boldsymbol{f}^{ab}:\mathbf{W}^{ab}\rightarrow\hat{\mathbf{Z}}^b_{ab}$ in $\mathbf{dMan^c}$.

Now Γ^a acts by 2-morphisms $\boldsymbol{\pi}_{[\hat{\mathbf{Z}}^a_{ab}/\Gamma^a]}\Rightarrow\boldsymbol{\pi}_{[\hat{\mathbf{Z}}^a_{ab}/\Gamma^a]}$, and similarly for Γ^b. Using these and the 2-Cartesian property of (D.15) we can construct a natural action of $\Gamma^a\times\Gamma^b$ on $\mathbf{W}^{ab}$ by 1-isomorphisms, such that $\boldsymbol{e}^{ab}:\mathbf{W}^{ab}\rightarrow\hat{\mathbf{Z}}^a_{ab}$ is Γ^a-equivariant and Γ^b-invariant, and $\boldsymbol{f}^{ab}:\mathbf{W}^{ab}\rightarrow\hat{\mathbf{Z}}^b_{ab}$ is Γ^a-invariant and Γ^b-equivariant. These make $\boldsymbol{e}^{ab}:\mathbf{W}^{ab}\rightarrow\hat{\mathbf{Z}}^a_{ab}$ into a principal Γ^b-bundle, and $\boldsymbol{f}^{ab}:\mathbf{W}^{ab}\rightarrow\hat{\mathbf{Z}}^b_{ab}$ into a principal Γ^a-bundle.

Since $\boldsymbol{e}^{ab},\boldsymbol{f}^{ab}$ are principal, they are trivializable over sufficiently small open d-submanifolds in $\hat{\mathbf{Z}}^a_{ab},\hat{\mathbf{Z}}^b_{ab}$, and more generally, trivializable over the preimages in $\hat{\mathbf{Z}}^a_{ab},\hat{\mathbf{Z}}^b_{ab}$ of sufficiently small open d-suborbifolds in $\hat{\mathfrak{X}}^a$. Regarding $a\in J$ as fixed and b as varying, using Step 1(ii)(c) we see that $\hat{\mathfrak{X}}^a$ can be covered by small open $\mathcal{U}$ such that $e^{ab}:\mathbf{W}^{ab}\rightarrow\hat{\mathbf{Z}}^a_{ab}$ and $\boldsymbol{f}^{ab}:\mathbf{W}^{ab}\rightarrow\hat{\mathbf{Z}}^b_{ab}$ are trivializable over the preimage of $\mathcal{U}\cap\hat{\mathfrak{X}}^a\cap\hat{\mathfrak{X}}^b$ for all $a\prec b\in J$ with $\mathcal{U}\cap\hat{\mathfrak{X}}^a\cap\hat{\mathfrak{X}}^b\neq\emptyset$.

Replace $\{\mathfrak{X}^a:a\in J\}$, $\{\hat{\mathfrak{X}}^a:a\in J\}$ by refinements $\{\mathfrak{X}^{a_\star}_\star:a_\star\in J_\star\}$, $\{\hat{\mathfrak{X}}^{a_\star}_\star:a_\star\in J_\star\}$ which still satisfy Step 1(i)–(iii) in §D.1, but such that $\hat{\mathfrak{X}}^{a_\star}_\star\subseteq\mathcal{U}\subseteq\hat{\mathfrak{X}}^{a_{a_\star}}$ for each $a_\star\in J_\star$, for some small open $\mathcal{U}\subseteq\hat{\mathfrak{X}}^{a_{a_\star}}$ as above. Then the new versions of $\mathfrak{X}^a,\hat{\mathfrak{X}}^a,\mathcal{Z}^a,\Gamma^a,\ldots$ have the extra property that $e^{ab}:\mathbf{W}^{ab}\rightarrow\hat{\mathbf{Z}}^a_{ab}$ and $\boldsymbol{f}^{ab}:\mathbf{W}^{ab}\rightarrow\hat{\mathbf{Z}}^b_{ab}$ are trivializable principal Γ^b- and Γ^a-bundles for all $a\prec b\in J$ with $\hat{\mathfrak{X}}^a\cap\hat{\mathfrak{X}}^b\neq\emptyset$. Choosing such trivializations gives $\Gamma^a\times\Gamma^b$-equivariant equivalences $\hat{\mathbf{Z}}^a_{ab}\times\Gamma^b\simeq\mathbf{W}^{ab}\simeq\Gamma^a\times\hat{\mathbf{Z}}^b_{ab}$, which we combine into a $\Gamma^a\times\Gamma^b$-equivariant equivalence $\boldsymbol{p}^{ab}:\hat{\mathbf{Z}}^a_{ab}\times\Gamma^b\rightarrow\Gamma^a\times\hat{\mathbf{Z}}^b_{ab}$.

Write the $\Gamma^a\times\Gamma^b$-actions on $\hat{\mathbf{Z}}^a_{ab}\times\Gamma^b$ and $\Gamma^a\times\hat{\mathbf{Z}}^b_{ab}$ as $\sigma^{ab}:\Gamma^a\times\Gamma^b\rightarrow\mathrm{Aut}(\hat{\mathbf{Z}}^a_{ab}\times\Gamma^b)$ and $\tau^{ab}:\Gamma^a\times\Gamma^b\rightarrow\mathrm{Aut}(\Gamma^a\times\hat{\mathbf{Z}}^b_{ab})$. On points these act by

$$\begin{aligned}\sigma^{ab}(\gamma,\delta)&:(z,\epsilon)\mapsto\bigl(\boldsymbol{t}^a(\gamma)z,\delta\epsilon\alpha^{ab}(z,\gamma)\bigr),\quad z\in\hat{\mathbf{Z}}^a_{ab},\ \gamma\in\Gamma^a,\ \delta,\epsilon\in\Gamma^b,\\ \tau^{ab}(\gamma,\delta)&:(\epsilon,\tilde z)\mapsto\bigl(\gamma\epsilon\beta^{ab}(\tilde z,\delta),\boldsymbol{t}^b(\gamma)\tilde z\bigr),\quad \tilde z\in\hat{\mathbf{Z}}^b_{ab},\ \gamma,\epsilon\in\Gamma^a,\ \delta\in\Gamma^b,\end{aligned}\tag{D.16}$$

for some $\alpha^{ab}(z,\gamma)\in\Gamma^b$ and $\beta^{ab}(\tilde z,\delta)\in\Gamma^a$. To see this, note that as $\mathbf{W}^{ab}\simeq\hat{\mathbf{Z}}^a_{ab}\times\Gamma^b$ trivializes $\mathbf{W}^{ab}$ as a principal Γ^b-bundle, $\sigma^{ab}(1,\delta)$ acts by $\sigma^{ab}(1,\delta):(z,\epsilon)\mapsto(z,\delta\epsilon)$. Also the projection $\hat{\mathbf{Z}}^a_{ab}\times\Gamma^b\rightarrow\hat{\mathbf{Z}}^a_{ab}$ is Γ^a-equivariant, so $\sigma^{ab}(\gamma,\delta)$ must act on the first term z in (z,γ) by $z\mapsto\boldsymbol{t}^a(\gamma)z$. Define $\alpha^{ab}(z,\gamma)$ by $\sigma^{ab}(\gamma,1):(z,1)\mapsto\bigl(\boldsymbol{t}^a(\gamma)z,\alpha^{ab}(z,\gamma)\bigr)$. As $\sigma^{ab}(\gamma,1)$ commutes with $\sigma^{ab}(1,\delta)$, $\sigma^{ab}(1,\epsilon)$ the first line of (D.16) follows, and the second is similar.

As σ^{ab},τ^{ab} are continuous, the maps $z\mapsto\alpha^{ab}(z,\gamma)$ and $\tilde z\mapsto\beta^{ab}(\tilde z,\delta)$ are locally constant on $\hat{\mathbf{Z}}^a_{ab}$ and $\hat{\mathbf{Z}}^b_{ab}$, but need not be globally constant. The conditions that σ^{ab}, τ^{ab} are group actions are

$$\begin{aligned} \alpha^{ab}(z,\gamma\delta) &= \alpha^{ab}(z,\delta)\alpha^{ab}\bigl(\boldsymbol{t}^a(\delta)(z),\gamma\bigr), && \text{all } z\in\hat{\mathbf{Z}}{}^a_{ab} \text{ and } \gamma,\delta\in\Gamma^a,\\ \beta^{ab}(\tilde z,\gamma\delta) &= \beta^{ab}(\tilde z,\delta)\beta^{ab}\bigl(\boldsymbol{t}^b(\delta)(\tilde z),\gamma\bigr), && \text{all } \tilde z\in\hat{\mathbf{Z}}{}^b_{ab} \text{ and } \gamma,\delta\in\Gamma^b. \end{aligned} \tag{D.17}$$

Write $\mathrm{Map}_1(\Gamma^a,\Gamma^b)$ for the set of maps $\phi:\Gamma^a\to\Gamma^b$ with $\phi(1)=1$ (we do not require ϕ to be a group morphism), and $\mathrm{Map}_1(\Gamma^b,\Gamma^a)$ for the set of $\phi:\Gamma^b\to\Gamma^a$ with $\phi(1)=1$. Define a map

$$\begin{gathered} \Phi^{ab}:\hat{\mathbf{Z}}{}^a_{ab}\times\Gamma^b\longrightarrow\Gamma^a\times\Gamma^b\times\mathrm{Map}_1(\Gamma^a,\Gamma^b)\times\mathrm{Map}_1(\Gamma^b,\Gamma^a) \text{ by}\\ \Phi^{ab}:(z,\delta)\longmapsto\bigl(\gamma,\delta,\epsilon\mapsto\alpha^{ab}(z,\epsilon),\zeta\mapsto\beta^{ab}(\tilde z,\zeta)\bigr) \text{ if } \boldsymbol{p}^{ab}(z,\delta)=(\gamma,\tilde z). \end{gathered} \tag{D.18}$$

Define an action υ^{ab} of $\Gamma^a\times\Gamma^b$ on $\Gamma^a\times\Gamma^b\times\mathrm{Map}_1(\Gamma^a,\Gamma^b)\times\mathrm{Map}_1(\Gamma^b,\Gamma^a)$ by

$$\begin{aligned} &\upsilon^{ab}(\gamma,\delta):(\epsilon,\zeta,\phi,\psi)\longmapsto\\ &\qquad\bigl(\gamma\,\epsilon\,\psi(\delta),\delta\,\zeta\,\phi(\gamma),\theta\mapsto\phi(\gamma)^{-1}\phi(\theta\gamma),\theta\mapsto\psi(\delta)^{-1}\psi(\theta\delta)\bigr). \end{aligned} \tag{D.19}$$

Then combining (D.16)–(D.19) and the $\Gamma^a\times\Gamma^b$-equivariance of $\boldsymbol{p}^{ab}$ shows that $\upsilon^{ab}(\gamma,\delta)\circ\Phi^{ab}=\Phi^{ab}\circ\sigma^{ab}(\gamma,\delta)$, that is, Φ^{ab} is $\Gamma^a\times\Gamma^b$-equivariant.

Given any point $[x]$ in $\hat{\mathfrak{X}}^a\cap\hat{\mathfrak{X}}^b$, the preimages (z,δ) of $[x]$ in $\hat{\mathbf{Z}}{}^a_{ab}\times\Gamma^b$ are a $\Gamma^a\times\Gamma^b$-orbit in $\hat{\mathbf{Z}}{}^a_{ab}\times\Gamma^b$, and so their images $\Phi^{ab}(z,\delta)$ are a $\Gamma^a\times\Gamma^b$-orbit in $\Gamma^a\times\Gamma^b\times\mathrm{Map}_1(\Gamma^a,\Gamma^b)\times\mathrm{Map}_1(\Gamma^b,\Gamma^a)$. Since Φ^{ab} is locally constant, for any $[x']$ close to $[x]$ in $\hat{\mathfrak{X}}^a\cap\hat{\mathfrak{X}}^b$ the preimages (z',γ') of $[x']$ in $\hat{\mathbf{Z}}{}^a_{ab}$ have $\Phi^{ab}(z',\gamma')$ in the same $\Gamma^a\times\Gamma^b$-orbit. Thus, we can cover $\hat{\mathfrak{X}}^a\cap\hat{\mathfrak{X}}^b$ by open $\boldsymbol{\mathcal{U}}$ such that if $(z,\gamma)\in\hat{\mathbf{Z}}{}^a_{ab}\times\Gamma^b$ has image in $\boldsymbol{\mathcal{U}}$, then $\Phi^{ab}(z,\gamma)$ lies in a fixed $\Gamma^a\times\Gamma^b$-orbit. So by the argument above, replacing $\{\mathfrak{X}^a:a\in J\}$, $\{\hat{\mathfrak{X}}^a:a\in J\}$ by refinements $\{\mathfrak{X}^{a_\star}_\star:a_\star\in J_\star\}$, $\{\hat{\mathfrak{X}}^{a_\star}_\star:a_\star\in J_\star\}$, we can suppose that Φ^{ab} has image a single $\Gamma^a\times\Gamma^b$-orbit for all $a,b\in J$ with $a\prec b$ and $\hat{\mathfrak{X}}^a\cap\hat{\mathfrak{X}}^b\neq\emptyset$.

For all such a,b, choose $(\gamma^{ab},\delta^{ab},\phi^{ab},\psi^{ab})\in\Gamma^a\times\Gamma^b\times\mathrm{Map}_1(\Gamma^a,\Gamma^b)\times\mathrm{Map}_1(\Gamma^b,\Gamma^a)$ in the image of Φ^{ab}. Consider the subgroup of $\Gamma^a\times\Gamma^b$ fixing $(\gamma^{ab},\delta^{ab},\phi^{ab},\psi^{ab})$ under the action υ^{ab}. From the first two terms of (D.19) we see that $\upsilon^{ab}(\Gamma^a\times\{1\})$ and $\upsilon^{ab}(\{1\}\times\Gamma^b)$ act freely on $\Gamma^a\times\Gamma^b\times\mathrm{Map}_1(\Gamma^a,\Gamma^b)\times\mathrm{Map}_1(\Gamma^b,\Gamma^a)$, so the stabilizer subgroup contains no elements $(\gamma,1)$ or $(1,\delta)$ for $\gamma\neq 1\neq\delta$, and the projections from the stabilizer subgroup to Γ^a and Γ^b are injective. Therefore there exist unique subgroups $\Gamma^{ab}\subseteq\Gamma^a$ and $\Gamma^{ba}\subseteq\Gamma^b$ and an isomorphism $\rho^{ab}:\Gamma^{ab}\to\Gamma^{ba}$ such that the stabilizer subgroup of $(\gamma^{ab},\delta^{ab},\phi^{ab},\psi^{ab})$ in $\Gamma^a\times\Gamma^b$ is $\bigl\{(\gamma,\rho^{ab}(\gamma)):\gamma\in\Gamma^{ab}\bigr\}=(\mathrm{id}\times\rho^{ab})(\Gamma^{ab})$.

Define $\hat{\mathbf{Z}}{}^{ab}$ to be the open and closed d-submanifold in $\hat{\mathbf{Z}}{}^a_{ab}$ such that $\Phi^{ab}=(\gamma^{ab},\delta^{ab},\phi^{ab},\psi^{ab})$ on $\hat{\mathbf{Z}}{}^{ab}\times\{\delta^{ab}\}\subseteq\hat{\mathbf{Z}}{}^a_{ab}\times\Gamma^b$, and $\hat{\mathbf{Z}}{}^{ba}$ to be the open and closed d-submanifold in $\hat{\mathbf{Z}}{}^b_{ab}$ such that $\boldsymbol{p}^{ab}\bigl(\hat{\mathbf{Z}}{}^{ab}\times\{\delta^{ab}\}\bigr)=\{\gamma^{ab}\}\times\hat{\mathbf{Z}}{}^{ba}$, and define $\boldsymbol{j}^{ab}:\hat{\mathbf{Z}}{}^{ab}\to\hat{\mathbf{Z}}{}^{ba}$ to be the unique equivalence which lifts to $\boldsymbol{p}^{ab}|_{\hat{\mathbf{Z}}^{ab}\times\{\delta^{ab}\}}:\hat{\mathbf{Z}}{}^{ab}\times\{\delta^{ab}\}\to\{\gamma^{ab}\}\times\hat{\mathbf{Z}}{}^{ba}$. Then the open and closed d-submanifold $\hat{\mathbf{Z}}{}^{ab}\times\{\delta^{ab}\}$ in $\hat{\mathbf{Z}}{}^a_{ab}\times\Gamma^b$ is invariant under the subgroup $(\mathrm{id}\times\rho^{ab})(\Gamma^{ab})$ in $\Gamma^a\times\Gamma^b$, with

$$\hat{\mathbf{Z}}{}^a_{ab}\times\Gamma^b=\textstyle\coprod_{(\gamma,\delta)(\mathrm{id}\times\rho^{ab})(\Gamma^{ab})\in(\Gamma^a\times\Gamma^b)/(\mathrm{id}\times\rho^{ab})(\Gamma^{ab})}\sigma^{ab}(\gamma,\delta)(\hat{\mathbf{Z}}{}^{ab}\times\{\delta^{ab}\}),$$

where the sum is over a choice of representative (γ,δ) for each coset $(\gamma,\delta)(\mathrm{id}\times\rho^{ab})(\Gamma^{ab})$ in $(\Gamma^a\times\Gamma^b)/(\mathrm{id}\times\rho^{ab})(\Gamma^{ab})$. Projecting to $\hat{\mathbf{Z}}^a_{ab}$, on which Γ^b acts trivially, this implies that $\hat{\mathbf{Z}}^{ab}$ is invariant under $\boldsymbol{t}^a(\Gamma^{ab})$, and

$$\hat{\mathbf{Z}}^a_{ab}=\textstyle\coprod_{\gamma\Gamma^{ab}\in\Gamma^a/\Gamma^{ab}}\boldsymbol{t}^a(\gamma)(\hat{\mathbf{Z}}^{ab}).$$

It follows that

$$[\hat{\mathbf{Z}}^{ab}/\Gamma^{ab}]\cong[\hat{\mathbf{Z}}^a_{ab}/\Gamma^a]\simeq\hat{\mathcal{X}}^a\cap\hat{\mathcal{X}}^b.$$

This proves Step 1(iv)(a)–(c) in §D.1 for $\mathbf{Z}^{ab},\Gamma^{ab}$, and the proof for $\mathbf{Z}^{ba},\Gamma^{ba}$ is the same. To construct the 2-morphism $\boldsymbol{\zeta}^{ab}$ in (D.1), note that in (D.15) we have a 2-morphism ω^{ab}, which is equivariant under the action of $\Gamma^a\times\Gamma^b$ on $\mathbf{W}^{ab}$. Under the equivalence $\mathbf{W}^{ab}\simeq\hat{\mathbf{Z}}^a_{ab}\times\Gamma^b$ we may restrict ω^{ab} to the open d-submanifold of $\mathbf{W}^{ab}$ corresponding to $\hat{\mathbf{Z}}^{ab}\times\{\delta^{ab}\}\cong\hat{\mathbf{Z}}^{ab}$, and this is Γ^{ab}-equivariant, and so descends to the quotient $[\hat{\mathbf{Z}}^{ab}/\Gamma^{ab}]$ to give $\boldsymbol{\zeta}^{ab}$ in (D.1).

For the last part of Step 1(iv), involving the bijection $q^{ab}:C^{ab}\to C^{ba}$, note that if $z\in\hat{\mathbf{Z}}^{ab}$ then the set of $c\in C^a$ with $\partial^c\hat{\mathbf{Z}}^a$ nonempty near z is in bijection with $\boldsymbol{i}^{-1}_{\hat{\mathbf{Z}}^a}(z)$. Similarly, the set of $c'\in C^b$ with $\partial^{c'}\hat{\mathbf{Z}}^b$ nonempty near $\boldsymbol{j}^{ab}(z)$ is in bijection with $\boldsymbol{i}^{-1}_{\hat{\mathbf{Z}}^b}(\boldsymbol{j}^{ab}(z))$. But $\boldsymbol{j}^{ab}_-$ induces a bijection $\boldsymbol{i}^{-1}_{\hat{\mathbf{Z}}^a}(z)\to\boldsymbol{i}^{-1}_{\hat{\mathbf{Z}}^b}(\boldsymbol{j}^{ab}(z))$. If we were to replace $\hat{\mathbf{Z}}^a,\hat{\mathbf{Z}}^b$ by sufficiently small neighbourhoods of the Γ^a- and Γ^b-orbits of $z,\boldsymbol{j}^{ab}(z)$, then we would have $C^{ab}\cong\boldsymbol{i}^{-1}_{\hat{\mathbf{Z}}^a}(z)$ and $C^{ba}\cong\boldsymbol{i}^{-1}_{\hat{\mathbf{Z}}^b}(\boldsymbol{j}^{ab}(z))$, so the bijection $\boldsymbol{j}^{ab}_-|_{\cdots}:\boldsymbol{i}^{-1}_{\hat{\mathbf{Z}}^a}(z)\to\boldsymbol{i}^{-1}_{\hat{\mathbf{Z}}^b}(\boldsymbol{j}^{ab}(z))$ induces the bijection $q^{ab}:C^{ab}\to C^{ba}$ that we want. From this we see that replacing $\{\mathcal{X}^a:a\in J\}$, $\{\hat{\mathcal{X}}^a:a\in J\}$ by refinements $\{\mathcal{X}^{a_\star}_\star:a_\star\in J_\star\}$, $\{\hat{\mathcal{X}}^{a_\star}_\star:a_\star\in J_\star\}$, we can make the last part of Step 1(iv) hold. This completes part (iv) of Step 1.

For part (v), suppose $\hat{\mathbf{Z}}^{abc}_{\alpha\beta}\neq\boldsymbol{\emptyset}$ in (D.2), and consider the diagrams (D.4)–(D.5), where $[\lambda^{abc}_{\alpha\beta},\gamma^{abc}_{\alpha\beta}]$ in (D.4) has yet to be defined. As $\boldsymbol{i}^c$ is an equivalence with an open d-suborbifold of $\mathcal{X}$, there exists a unique 2-morphism in $\mathbf{dOrb^c}$

$$\boldsymbol{\eta}:[\boldsymbol{j}^{bc},\rho^{bc}]\circ[\boldsymbol{t}^b(\beta)\circ\boldsymbol{j}^{ab}\circ|_{\hat{\mathbf{Z}}^{abc}_{\alpha\beta}},\iota]\Longrightarrow[\boldsymbol{j}^{ac}\circ\boldsymbol{t}^a(\beta)|_{\hat{\mathbf{Z}}^{abc}_{\alpha\beta}},\iota],$$

such that (D.4) with $\boldsymbol{\eta}$ in place of $[\lambda^{abc}_{\alpha\beta},\gamma^{abc}_{\alpha\beta}]$, and (D.5), have the same composition.

Applying Theorems 9.16(c) and 11.8 with $\mathcal{X}=[\boldsymbol{X}/G]$, $\mathcal{Y}=[\boldsymbol{Y}/G]$, $\tilde{\boldsymbol{f}}=[\boldsymbol{f},\rho]$, $\tilde{\boldsymbol{g}}=[\boldsymbol{g},\sigma]$ and $\boldsymbol{i},\boldsymbol{j},\boldsymbol{\zeta},\boldsymbol{\theta}$ identities shows that if $[\boldsymbol{f},\rho],[\boldsymbol{g},\sigma]:[\boldsymbol{X}/G]\to[\boldsymbol{Y}/H]$ are quotient 1-morphisms and $\boldsymbol{\eta}:[\boldsymbol{f},\rho]\Rightarrow[\boldsymbol{g},\sigma]$ is a 2-morphism in $\mathbf{dSta}$ then $\boldsymbol{\eta}$ is of the form $[\lambda,\gamma]$ from Definition 11.7 *locally in* $[\boldsymbol{X}/G]$, that is, we can cover $\boldsymbol{X}$ by open G-invariant $\mathbf{U}\subseteq\boldsymbol{X}$ such that $\boldsymbol{\eta}|_{[\mathbf{U}/G]}=[\lambda,\gamma]:[\boldsymbol{f}|_{\mathbf{U}},\rho]\Rightarrow[\boldsymbol{g}|_{\mathbf{U}},\sigma]$, and $\boldsymbol{\eta}|_{[\mathbf{U}/G]}$ determines λ,γ uniquely provided $\mathbf{U}\neq\emptyset$. Thus we can cover $\hat{\mathbf{Z}}^{abc}_{\alpha\beta}$ by open $\mathbf{U}$ such that $\boldsymbol{\eta}|_{[\mathbf{U}/\{1\}]}=[\lambda^{abc}_{\alpha\beta},\gamma^{abc}_{\alpha\beta}]$ for some unique $\lambda^{abc}_{\alpha\beta},\gamma^{abc}_{\alpha\beta}$.

Replacing $\{\mathcal{X}^a:a\in J\}$, $\{\hat{\mathcal{X}}^a:a\in J\}$ by refinements $\{\mathcal{X}^{a_\star}_\star:a_\star\in J_\star\}$, $\{\hat{\mathcal{X}}^{a_\star}_\star:a_\star\in J_\star\}$, we can suppose that just one $\mathbf{U}$ is sufficient to cover all of $\hat{\mathbf{Z}}^{abc}_{\alpha\beta}$, for all such a,b,c,α,β. So there exist unique $\gamma^{abc}_{\alpha\beta}\in\Gamma^c$ and $\gamma^{abc}_{\alpha\beta}$ in (D.3) such that (D.4) and (D.5) have the same composition. This completes Step 1.

D.3 Step 2: Modify to a better open cover

The Deligne–Mumford C^∞-stack $\mathcal{X}$ is separated, paracompact, and locally fair. So by Theorem C.26, $\mathcal{X}$ has a coarse moduli C^∞-scheme $\underline{\mathcal{X}}_{\rm top}$, with underlying topological space $\mathcal{X}_{\rm top}$, and $\underline{\mathcal{X}}_{\rm top}$ is also separated, paracompact, and locally fair. By Proposition B.21, any open cover of $\underline{\mathcal{X}}_{\rm top}$ admits a subordinate smooth partition of unity, as in Remark C.27. Thus we can choose a continuous partition of unity $\{\theta^a : a \in J\}$ on $\mathcal{X}_{\rm top}$ subordinate to $\{\mathcal{X}^a_{\rm top} : a \in J\}$, and in fact we can take the θ^a to be smooth on $\underline{\mathcal{X}}_{\rm top}$, though we do not need this. For each finite $A \subseteq J$, define $\boldsymbol{\mathcal{X}}_A \subseteq \boldsymbol{\mathcal{X}}$ by (D.6), and let I be the set of finite $A \subseteq J$ with $\boldsymbol{\mathcal{X}}_A \neq \boldsymbol{\emptyset}$. We will prove $\{\boldsymbol{\mathcal{X}}_A : A \in I\}$ satisfy Step 2(A)–(C) in §D.1.

For (A), let $[x] \in \mathcal{X}_{\rm top}$, and set $A' = \{a \in J : [x] \in \mathcal{X}^a_{\rm top}\}$. Then A' is finite and nonempty by Step 1(ii). Since $\{\theta^a : a \in J\}$ is subordinate to $\{\mathcal{X}^a_{\rm top} : a \in J\}$, we see that $\sum_{a\in A'} \theta^a([x]) = 1$. Choose $\emptyset \neq A \subseteq J$ finite with $|A|$ least such that $\sum_{a\in A} \theta^a([x]) > 1 - \frac{2}{4^{|A|}}$. Since A' satisfies this condition, such A exist, and we can choose A with $|A|$ least. If $[x] \notin \mathcal{X}_{A,\rm top}$ then by (D.6) there exists $\emptyset \neq B \subsetneq A$ with $\sum_{b\in B} \theta^b([x]) \geqslant 1 - \frac{1}{4^{|B|}}$, so $\sum_{b\in B} \theta^b([x]) > 1 - \frac{2}{4^{|B|}}$, and $|B| < |A|$, contradicting the choice of A with $|A|$ least. Hence $[x] \in \mathcal{X}_{A,\rm top}$, and $A \in I$ as $\boldsymbol{\mathcal{X}}_A \neq \boldsymbol{\emptyset}$. Therefore $\boldsymbol{\mathcal{X}} = \bigcup_{A\in I} \boldsymbol{\mathcal{X}}_A$, and $\{\boldsymbol{\mathcal{X}}_A : A \in I\}$ is an open cover of $\boldsymbol{\mathcal{X}}$, proving part (A).

For (B), suppose $A \in I$ and $[x] \in \mathcal{X}_{A,\rm top}$. Then $\theta^a([x]) > 0$ for each $a \in A$, as otherwise putting $B = A \setminus \{a\}$ in (D.6) gives a contradiction. Hence $[x] \in \mathcal{X}^a_{\rm top}$ as θ^a is supported on $\mathcal{X}^a_{\rm top}$. Thus $[x] \in \bigcap_{a\in A} \mathcal{X}^a_{\rm top}$, so $\boldsymbol{\mathcal{X}}_A \subseteq \bigcap_{a\in A} \boldsymbol{\mathcal{X}}^a$.

For (C), suppose $A \neq B \in I$ with $\boldsymbol{\mathcal{X}}_A \cap \boldsymbol{\mathcal{X}}_B \neq \boldsymbol{\emptyset}$, and write $C = A \cap B$. Let $[x] \in \mathcal{X}_{A,\rm top} \cap \mathcal{X}_{B,\rm top}$. Then (D.6) implies that

$$\begin{aligned} 1 - \tfrac{2}{4^{|A|}} + 1 - \tfrac{2}{4^{|B|}} &< \textstyle\sum_{a\in A} \theta^a([x]) + \sum_{b\in B} \theta^b([x]) \\ &= \textstyle\sum_{a\in A\cup B} \theta^a([x]) + \sum_{c\in C} \theta^c([x]) \leqslant 1 + \sum_{c\in C} \theta^c([x]) < 1 - \tfrac{1}{4^{|C|}}, \end{aligned}$$

where in the last step we note that $A \neq C$ or $B \neq C$ as $A \neq B$. Hence $\frac{1}{4^{|C|}} < \frac{2}{4^{|A|}} + \frac{2}{4^{|B|}}$. If $|C| < |A|, |B|$ this is a contradiction. Hence either $|C| = |A|$, giving $A \subseteq B$, or $|C| = |B|$, giving $B \subseteq A$.

Next we construct the total order $<$ on I. For each $A \in I$, define the *depth* $\mathrm{depth}(A) \geqslant 0$ to be the maximum number n of distinct elements $c_1, \ldots, c_n$ in $J \setminus A$ such that $\bigcap_{a\in A} \boldsymbol{\mathcal{X}}^a \cap \bigcap_{i=1}^n \boldsymbol{\mathcal{X}}^{c_i} \neq \boldsymbol{\emptyset}$. Step 1(ii)(c) implies that this number n is bounded, so $\mathrm{depth}(A)$ is well-defined. Observe that if $A, B \in I$ with $A \subsetneq B$ then $\mathrm{depth}(A) > \mathrm{depth}(B)$, since if $\mathrm{depth}(B) = n$ there exist $c_1, \ldots, c_n \in J \setminus B$ with $\bigcap_{b\in B} \boldsymbol{\mathcal{X}}^b \cap \bigcap_{i=1}^n \boldsymbol{\mathcal{X}}^{c_i} \neq \boldsymbol{\emptyset}$, so writing $B \setminus A = \{c_{n+1}, \ldots, c_{n+k}\}$ we have $\bigcap_{a\in A} \boldsymbol{\mathcal{X}}^a \cap \bigcap_{i=1}^{n+k} \boldsymbol{\mathcal{X}}^{c_i} \neq \boldsymbol{\emptyset}$, and thus $\mathrm{depth}(A) \geqslant n + k > \mathrm{depth}(B)$.

Define a total order $<$ on I by $A < B$ if either $\mathrm{depth}(A) < \mathrm{depth}(B)$, or $\mathrm{depth}(A) = \mathrm{depth}(B)$ and A precedes B in the lexicographic order on subsets of J induced by the total order $\prec$. Since $\mathbb{N}$ and $(J, \prec)$ are well-ordered, $(I, <)$ is well-ordered. Suppose $A, B \in I$ with $A < B$ and $\boldsymbol{\mathcal{X}}_A \cap \boldsymbol{\mathcal{X}}_B \neq \boldsymbol{\emptyset}$. Then $A \neq B$, so $A \subsetneq B$ or $B \subsetneq A$ by Step 2(C). But $A \subsetneq B$ implies $\mathrm{depth}(A) > \mathrm{depth}(B)$, contradicting $A < B$. Hence $B \subsetneq A$.

Now let $A \in I$. We can write A uniquely as $\{a_1, a_2, \ldots, a_n\}$ for $a_1, \ldots, a_n \in J$

with $a_1 \prec a_2 \prec \cdots \prec a_n$. Suppose $[x] \in \bigcap_{i=1}^n \hat{\mathcal{X}}^{a_i}$. Then $[x] \in \hat{\mathcal{X}}^{a_1} \simeq [\hat{\mathbf{Z}}^{a_1}/\Gamma^{a_1}]$, so we can choose a lift $z \in \hat{\mathbf{Z}}^{a_1}$ for $[x]$. For each $i = 2, \ldots, n$, as $[x] \in \hat{\mathcal{X}}^{a_1} \cap \hat{\mathcal{X}}^{a_i}$, $[x]$ has a lift to $\hat{\mathbf{Z}}^{a_1 a_i}$, which lies in the Γ^{a_1}-orbit of z. Therefore we can choose $\alpha_A^{a_1 a_i} \in \Gamma^{a_1}$ with $\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_i})(z) \in \hat{\mathbf{Z}}^{a_1 a_i}$. Hence $\boldsymbol{j}^{a_1 a_i} \circ \boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_i})(z) \in \hat{\mathbf{Z}}^{a_i a_1} \subseteq \hat{\mathbf{Z}}^{a_i}$ is a lift of $[x]$ to $\hat{\mathbf{Z}}^{a_i}$.

Let $1 < i < j \leqslant n$. Then $[x] \in \hat{\mathcal{X}}^{a_i} \cap \hat{\mathcal{X}}^{a_j}$, so $[x]$ has a lift to $\hat{\mathbf{Z}}^{a_i a_j} \subseteq \hat{\mathbf{Z}}^{a_i}$, which lies in the Γ^{a_i}-orbit of $\boldsymbol{j}^{a_1 a_i} \circ \boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_i})(z)$. Hence we can choose $\alpha_A^{a_i a_j} \in \Gamma^{a_i}$ with $\boldsymbol{t}^{a_i}(\alpha_A^{a_i a_j}) \circ \boldsymbol{j}^{a_1 a_i} \circ \boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_i})(z) \in \hat{\mathbf{Z}}^{a_i a_j}$. This shows that we can choose $\alpha_A^{a_i a_j} \in \Gamma^{a_i}$ for $1 \leqslant i < j \leqslant n$ such that the lift z of $[x]$ to $\hat{\mathbf{Z}}^{a_1}$ lies in

$$\begin{aligned} \hat{\mathbf{Z}}_A = & \textstyle\bigcap_{1<i\leqslant n} [\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_i})]^{-1}(\hat{\mathbf{Z}}^{a_1 a_i}) \\ & \cap \textstyle\bigcap_{1<i<j\leqslant n} [\boldsymbol{t}^{a_i}(\alpha_A^{a_i a_j}) \circ \boldsymbol{j}^{a_1 a_i} \circ \boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_i})]^{-1}(\hat{\mathbf{Z}}^{a_i a_j}). \end{aligned} \tag{D.20}$$

Define a subgroup $\Gamma_A \subseteq \Gamma^{a_1}$ by

$$\begin{aligned} \Gamma_A = & \Gamma^{a_1} \cap \textstyle\bigcap_{1<i\leqslant n} (\alpha_A^{a_1 a_i})^{-1} \Gamma^{a_1 a_i} \alpha_A^{a_1 a_i} \\ & \cap \textstyle\bigcap_{1<i<j\leqslant n} (\alpha_A^{a_1 a_i})^{-1} (\rho^{a_1 a_i})^{-1} [\Gamma^{a_1 a_i} \cap (\alpha_A^{a_i a_j})^{-1} \Gamma^{a_i a_j} \alpha_A^{a_i a_j}]\, \alpha_A^{a_1 a_i}. \end{aligned}$$

Since $\Gamma^{a_i a_j}$ preserves $\hat{\mathbf{Z}}^{a_i a_j}$, we see that $\hat{\mathbf{Z}}_A$ is invariant under $\boldsymbol{t}^{a_1}(\Gamma_A)$. So we may define an action $\boldsymbol{t}_A : \Gamma_A \to \mathrm{Aut}(\hat{\mathbf{Z}}_A)$ by $\boldsymbol{t}_A(\gamma) = \boldsymbol{t}^{a_1}(\gamma)|_{\hat{\mathbf{Z}}_A}$ for $\gamma \in \Gamma_A$, and we have a quotient d-orbifold with corners $[\hat{\mathbf{Z}}_A/\Gamma_A]$.

As $\boldsymbol{t}^{a_i}(\gamma)[\hat{\mathbf{Z}}^{a_i a_j}] \cap \hat{\mathbf{Z}}^{a_i a_j} = \emptyset$ for all $\gamma \in \Gamma^{a_i} \setminus \Gamma^{a_i a_j}$, we see that $\boldsymbol{t}^{a_1}(\gamma)[\hat{\mathbf{Z}}_A] \cap \hat{\mathbf{Z}}_A = \emptyset$ for all $\gamma \in \Gamma^{a_1} \setminus \Gamma_A$. Hence $[\mathbf{Z}_A/\Gamma_A] \simeq \big[\boldsymbol{t}^{a_1}(\Gamma^{a_1})[\mathbf{Z}_A]/\Gamma^{a_1}\big]$. But $\boldsymbol{t}^{a_1}(\Gamma^{a_1})[\mathbf{Z}_A]$ is Γ^{a_1}-invariant and open in $\hat{\mathbf{Z}}^{a_1}$, so $\big[\boldsymbol{t}^{a_1}(\Gamma^{a_1})[\mathbf{Z}_A]/\Gamma^{a_1}\big]$ is an open d-suborbifold of $[\hat{\mathbf{Z}}^{a_1}/\Gamma^{a_1}]$, which is equivalent under $\boldsymbol{i}^{a_1}$ to an open d-suborbifold of $\hat{\mathcal{X}}^{a_1}$. Define $\hat{\boldsymbol{\imath}}_A : [\hat{\mathbf{Z}}_A/\Gamma_A] \to \mathcal{X}$ to be the composition

$$[\hat{\mathbf{Z}}_A/\Gamma_A] \xrightarrow{[\mathbf{inc},\mathrm{inc}]} [\hat{\mathbf{Z}}^{a_1}/\Gamma^{a_1}] \xrightarrow{\boldsymbol{i}^{a_1}} \hat{\mathcal{X}}^{a_1} \subseteq \mathcal{X}.$$

Then $\hat{\boldsymbol{\imath}}_A$ is an equivalence with an open d-suborbifold $\hat{\boldsymbol{\imath}}_A([\hat{\mathbf{Z}}_A/\Gamma_A])$ in $\bigcap_{i=1}^n \hat{\mathcal{X}}^{a_i}$.

We have shown that for each $[x] \in \bigcap_{i=1}^n \hat{\mathcal{X}}^{a_i}$ we can choose $\alpha_A^{a_i a_j} \in \Gamma^{a_i}$ for $1 \leqslant i < j \leqslant n$ such that an open neighbourhood of $[x]$ in $\bigcap_{i=1}^n \hat{\mathcal{X}}^{a_i}$ is equivalent to $[\hat{\mathbf{Z}}_A/\Gamma_A]$ for $\hat{\mathbf{Z}}_A, \Gamma_A$ constructed using the $\alpha_A^{a_i a_j}$. Using the argument of Step 1 in §D.2, we may replace $\{\mathcal{X}^a : a \in J\}$, $\{\hat{\mathcal{X}}^a : a \in J\}$ by refinements $\{\mathcal{X}_\star^{a_\star} : a_\star \in J_\star\}$, $\{\hat{\mathcal{X}}_\star^{a_\star} : a_\star \in J_\star\}$ to ensure that one choice of data $\alpha_A^{a_i a_j} \in \Gamma^{a_i}$ for $1 \leqslant i < j \leqslant n$ works for all $[x] \in \bigcap_{i=1}^n \hat{\mathcal{X}}^{a_i}$. Then $\hat{\boldsymbol{\imath}}_A([\hat{\mathbf{Z}}_A/\Gamma_A]) = \bigcap_{i=1}^n \hat{\mathcal{X}}^{a_i}$, so $\hat{\boldsymbol{\imath}}_A : [\hat{\mathbf{Z}}_A/\Gamma_A] \to \bigcap_{i=1}^n \hat{\mathcal{X}}^{a_i}$ is an equivalence.

Define $\mathbf{Z}_A \subseteq \hat{\mathbf{Z}}_A$ to be the unique Γ_A-invariant open d-submanifold with $\hat{\boldsymbol{\imath}}_A([\mathbf{Z}_A/\Gamma_A]) = \mathcal{X}_A \subseteq \bigcap_{i=1}^n \hat{\mathcal{X}}^{a_i}$, and define $\boldsymbol{i}_A = \hat{\boldsymbol{\imath}}_A|_{[\mathbf{Z}_A/\Gamma_A]}$. Then $\boldsymbol{i}_A : [\mathbf{Z}_A/\Gamma_A] \to \mathcal{X}_A$ is an equivalence, as we want. Define

$$C_A = \{c \in C^{a_1} : \partial\hat{\mathbf{Z}}_A \cap \partial^c \hat{\mathbf{Z}}^{a_1} \neq \emptyset\}, \tag{D.21}$$

and define $\partial_c\mathbf{Z}_A=\partial\mathbf{Z}_A\cap\partial^c\hat{\mathbf{Z}}^{a_1}$. As we used $\hat{\mathbf{Z}}_A$ rather than $\mathbf{Z}_A$ in (D.21) we may have $\partial_c\mathbf{Z}_A=\boldsymbol{\emptyset}$ for $c\in C_A$. Since $\hat{\mathbf{Z}}_A$ is invariant under $\Gamma_A\subseteq\Gamma^{a_1}$, it follows from Step 1(iii)(a),(b) that the subset C_A in C^{a_1} is invariant under $p^{a_1}|_{\Gamma_A}:\Gamma_A\rightarrow\mathrm{Aut}(C^{a_1})$. Define $p_A:\Gamma_A\rightarrow\mathrm{Aut}(C_A)$ by $p_A=\bigl(p^{a_1}|_{\Gamma_A}\bigr)|_{C_A}$. Set $\boldsymbol{b}_{Ac}=\boldsymbol{b}^{a_1c}|_{\boldsymbol{Z}_A}:\boldsymbol{Z}_A\rightarrow[\mathbf{0},\boldsymbol{\infty})$ for each $c\in C_A$. Part (i) of Step 2 now follows easily from Step 1(iii).

For part (ii), suppose $A,B\in I$ with $A<B$ and $\boldsymbol{\mathfrak{X}}_A\cap\boldsymbol{\mathfrak{X}}_B\neq\boldsymbol{\emptyset}$. The end of Step 1 shows that $B\subsetneq A$. Write $A=\{a_1,\ldots,a_n\}$ with $a_1\prec\cdots\prec a_n$ as above. Then $B=\{a_{b_1},\ldots,a_{b_k}\}$ for $\{b_1,\ldots,b_k\}\subsetneq\{1,\ldots,n\}$ with $b_1<b_2<\cdots<b_k$. Write $\mathbf{Z}_{AB}\subseteq\mathbf{Z}_A$ for the Γ_A-invariant open d-submanifold in $\mathbf{Z}_A$ with $\boldsymbol{i}_A([\mathbf{Z}_{AB}/\Gamma_A])=\boldsymbol{\mathfrak{X}}_A\cap\boldsymbol{\mathfrak{X}}_B$. We divide into two cases (a) $b_1=1$ and (b) $b_1>1$.

In case (a), let $[x]\in\bigcap_{i=1}^n\hat{\boldsymbol{\mathfrak{X}}}^{a_i}\subseteq\bigcap_{j=1}^k\hat{\boldsymbol{\mathfrak{X}}}^{a_{b_j}}$, and pick lifts $z_A\in\hat{\mathbf{Z}}_A\subseteq\hat{\mathbf{Z}}^{a_1}$, $z_B\in\hat{\mathbf{Z}}_B\subseteq\hat{\mathbf{Z}}^{a_1}$ for $[x]$. Then z_A,z_B lie in the same Γ^{a_1}-orbit in $\hat{\mathbf{Z}}^{a_1}$, so $z_B=\boldsymbol{t}^{a_1}(\beta_{AB})(z_A)$ for some $\beta_{AB}\in\Gamma^{a_1}$. We claim that

$$\hat{\mathbf{Z}}_A\subseteq(\boldsymbol{t}^{a_1}(\beta_{AB}))^{-1}[\hat{\mathbf{Z}}_B]\quad\text{and}\quad\mathbf{Z}_{AB}\subseteq(\boldsymbol{t}^{a_1}(\beta_{AB}))^{-1}[\mathbf{Z}_B].\tag{D.22}$$

The second equation follows from the first, as $\mathbf{Z}_{AB}$ is the subset of $\hat{\mathbf{Z}}_A$ mapping to $\boldsymbol{\mathfrak{X}}_A\cap\boldsymbol{\mathfrak{X}}_B$, and $\mathbf{Z}_B$ the subset of $\hat{\mathbf{Z}}_B$ mapping to $\boldsymbol{\mathfrak{X}}_B$. To prove the first equation of (D.22), use (D.20) to write each side as a multiple intersection, and show that each term in the right hand intersection also occurs in the left. For example, let $1<i\leqslant k$. Then both $\boldsymbol{t}^{a_1}(\alpha_A^{a_1a_{b_i}})(z_A)$ and

$$\begin{aligned}\boldsymbol{t}^{a_1}(\alpha_B^{a_1a_{b_i}})(z_B)&=\boldsymbol{t}^{a_1}(\alpha_B^{a_1a_{b_i}}\beta_{AB})(z_A)\\&=\boldsymbol{t}^{a_1}(\alpha_B^{a_1a_{b_i}}\beta_{AB}(\alpha_A^{a_1a_{b_i}})^{-1})\circ\boldsymbol{t}^{a_1}(\alpha_A^{a_1a_{b_i}})(z_A)\end{aligned}$$

lie in $\hat{\mathbf{Z}}^{a_1a_{b_i}}$. Therefore $\alpha_B^{a_1a_{b_i}}\beta_{AB}(\alpha_A^{a_1a_{b_i}})^{-1}\in\Gamma^{a_1a_{b_i}}$, so it fixes $\hat{\mathbf{Z}}^{a_1a_{b_i}}$, giving

$$(\boldsymbol{t}^{a_1}(\alpha_A^{a_1a_{b_i}}))^{-1}[\hat{\mathbf{Z}}^{a_1a_{b_i}}]=(\boldsymbol{t}^{a_1}(\beta_{AB}))^{-1}\bigl[(\boldsymbol{t}^{a_1}(\alpha_B^{a_1a_{b_i}}))^{-1}[\hat{\mathbf{Z}}^{a_1a_{b_i}}]\bigr].$$

Here the left hand set is in the intersection for $\hat{\mathbf{Z}}_A$, and the right hand in the intersection for $(\boldsymbol{t}^{a_1}(\beta_{AB}))^{-1}[\hat{\mathbf{Z}}_B]$. A parallel proof for groups shows that

$$\beta_{AB}\Gamma_A\beta_{AB}^{-1}\subseteq\Gamma_B.\tag{D.23}$$

Define $\boldsymbol{j}_{AB}:\mathbf{Z}_{AB}\rightarrow\mathbf{Z}_B$ by $\boldsymbol{j}_{AB}=\boldsymbol{t}^{a_1}(\beta_{AB})|_{\mathbf{Z}_{AB}}$ and $\rho_{AB}:\Gamma_A\rightarrow\Gamma_B$ by $\rho_{AB}:\gamma\mapsto\beta_{AB}\gamma\beta_{AB}^{-1}$. These are well-defined by (D.22)–(D.23), and induce a quotient 1-morphism $[\boldsymbol{j}_{AB},\rho_{AB}]:[\mathbf{Z}_{AB}/\Gamma_A]\rightarrow[\mathbf{Z}_B/\Gamma_B]$. We define $\boldsymbol{\zeta}_{AB}:\boldsymbol{i}_B\circ[\boldsymbol{j}_{AB},\rho_{AB}]\Rightarrow\boldsymbol{i}_A|_{[\mathbf{Z}_{AB}/\Gamma_A]}$ to be the composition of 2-morphisms in:

$$\begin{tikzcd}
{[\mathbf{Z}_{AB}/\Gamma_A]} \arrow[dr, "{[\mathbf{inc},\mathrm{inc}]}"'] \arrow[drr, bend left, "\boldsymbol{i}_A|_{[\mathbf{Z}_{AB}/\Gamma_A]}"] \arrow[dd, "{[\boldsymbol{j}_{AB},\rho_{AB}]}"'] & {} \arrow[d, phantom, "\mathbf{id}\Downarrow"] & \\
{} \arrow[r, phantom, "\overset{[\mathrm{id},\beta_{AB}]}{\Leftarrow}"] & {[\hat{\mathbf{Z}}^{a_1}/\Gamma^{a_1}]} \arrow[r, "\boldsymbol{i}^{a_1}"] & \boldsymbol{\mathfrak{X}}. \\
{[\mathbf{Z}_B/\Gamma_B]} \arrow[ur, "{[\mathbf{inc},\mathrm{inc}]}"] \arrow[urr, bend right, "\boldsymbol{i}_B"'] & {} \arrow[u, phantom, "\mathbf{id}\Uparrow"] &
\end{tikzcd}$$

Step 1(iii)(b), equation (D.21), and the first equation of (D.22) imply that the subsets $C_A, C_B \subseteq C^{a_1}$ satisfy $p^{a_1}(\beta_{AB})(C_A) \subseteq C_B$. Define an injective map $q_{AB} : C_A \to C_B$ by $q_{AB} = p^{a_1}(\beta_{AB})|_{C_A}$. Then $q_{AB} \circ p_A(\gamma) = p_B(\rho_{AB}(\gamma)) \circ q_{AB}$ for $\gamma \in \Gamma_A \subseteq \Gamma^{a_1}$ and $(\boldsymbol{j}_{AB})_-\big(\partial \mathbf{Z}_{AB} \cap \partial_c \mathbf{Z}_A\big) = (\boldsymbol{j}_{AB})_-\big(\partial \mathbf{Z}_{AB}\big) \cap \partial_{q_{AB}(c)} \hat{\mathbf{Z}}_B$ for $c \in C_A$ are immediate from the definitions.

In case (b), let $[x] \in \bigcap_{i=1}^n \hat{\boldsymbol{\mathfrak{X}}}^{a_i} \subseteq \bigcap_{j=1}^k \hat{\boldsymbol{\mathfrak{X}}}^{a_{b_j}}$, and pick lifts $z_A \in \hat{\mathbf{Z}}_A \subseteq \hat{\mathbf{Z}}^{a_1}$, $z_B \in \hat{\mathbf{Z}}_B \subseteq \hat{\mathbf{Z}}^{a_{b_1}}$ for $[x]$. Then $\boldsymbol{j}^{a_1 a_{b_1}} \circ \boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{b_1}})(z_A), z_B$ lie in the same $\Gamma^{a_{b_1}}$-orbit in $\hat{\boldsymbol{\mathfrak{X}}}^{a_{b_1}}$, so $z_B = \boldsymbol{t}^{a_{b_1}}(\beta_{AB}) \circ \boldsymbol{j}^{a_1 a_{b_1}} \circ \boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{b_1}})(z_A)$ for some $\beta_{AB} \in \Gamma^{a_{b_1}}$. Similar proofs to (D.22)–(D.23), but also using Step 1(v) to relate $\boldsymbol{j}^{a_1 a_{b_i}}$ to $\boldsymbol{j}^{a_{b_1} a_{b_i}} \circ \boldsymbol{t}^{a_{b_1}}(\alpha_A^{a_{b_1} a_{b_i}}) \circ \boldsymbol{j}^{a_1 a_{b_1}}$, imply that

$$
\begin{aligned}
\hat{\mathbf{Z}}_A &\subseteq \big(\boldsymbol{t}^{a_{b_1}}(\beta_{AB}) \circ \boldsymbol{j}^{a_1 a_{b_1}} \circ \boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{b_1}})\big)^{-1}[\hat{\mathbf{Z}}_B],\\
\mathbf{Z}_{AB} &\subseteq \big(\boldsymbol{t}^{a_{b_1}}(\beta_{AB}) \circ \boldsymbol{j}^{a_1 a_{b_1}} \circ \boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{b_1}})\big)^{-1}[\mathbf{Z}_B],\\
\text{and}\quad & \beta_{AB}\rho^{a_1 a_{b_1}}\big(\alpha_A^{a_1 a_{b_1}}\,\Gamma_A\,(\alpha_A^{a_1 a_{b_1}})^{-1}\big)\beta_{AB}^{-1} \subseteq \Gamma_B.
\end{aligned}
\tag{D.24}
$$

Define $\tilde{\boldsymbol{\jmath}}_{AB} : \mathbf{Z}_{AB} \to \mathbf{Z}_B$ and $\rho_{AB} : \Gamma_A \to \Gamma_B$ by

$$
\begin{aligned}
\tilde{\boldsymbol{\jmath}}_{AB} &= \boldsymbol{t}^{a_{b_1}}(\beta_{AB}) \circ \boldsymbol{j}^{a_1 a_{b_1}} \circ \boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{b_1}})|_{\mathbf{Z}_{AB}},\\
\rho_{AB} &: \gamma \mapsto \beta_{AB}\rho^{a_1 a_{b_1}}\big(\alpha_A^{a_1 a_{b_1}}\,\gamma\,(\alpha_A^{a_1 a_{b_1}})^{-1}\big)\beta_{AB}^{-1}.
\end{aligned}
$$

These are well-defined by (D.24). Let $\epsilon_{AB} : \underline{\tilde{\jmath}}_{AB}^*(\mathcal{F}_{Z_B}) \to \mathcal{E}_{Z_{AB}}$ be a morphism in $\mathrm{qcoh}(\underline{Z}_{AB})$ which we will choose shortly, during the proof of (iii). By Proposition 2.17 there is a unique 1-morphism $\boldsymbol{j}_{AB} : \mathbf{Z}_{AB} \to \mathbf{Z}_B$ in $\mathbf{dMan^c}$ such that $\epsilon_{AB} : \tilde{\boldsymbol{\jmath}}_{AB} \Rightarrow \boldsymbol{j}_{AB}$ is a 2-morphism.

Now $\tilde{\boldsymbol{\jmath}}_{AB}$ is equivariant under $\rho_{AB} : \Gamma_A \to \Gamma_B$, and we choose ϵ_{AB} to be equivariant, so that $\boldsymbol{j}_{AB}$ is also equivariant. So we have quotient 1-morphisms $[\tilde{\boldsymbol{\jmath}}_{AB}, \rho_{AB}], [\boldsymbol{j}_{AB}, \rho_{AB}] : [\mathbf{Z}_{AB}/\Gamma_A] \to [\mathbf{Z}_B/\Gamma_B]$, and a 2-morphism $[\epsilon_{AB}, 1] : [\tilde{\boldsymbol{\jmath}}_{AB}, \rho_{AB}] \Rightarrow [\boldsymbol{j}_{AB}, \rho_{AB}]$. We define $\boldsymbol{\zeta}_{AB} : \boldsymbol{i}_B \circ [\boldsymbol{j}_{AB}, \rho_{AB}] \Rightarrow \boldsymbol{i}_A|_{[\mathbf{Z}_{AB}/\Gamma_A]}$ to be the composition of 2-morphisms in:

$$
\begin{tikzcd}[column sep=huge, row sep=large]
{[\mathbf{Z}_{AB}/\Gamma_A]} \arrow[r, "{[\mathbf{inc},\mathrm{inc}]}"'] \arrow[dr, "{[\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{b_1}})|_{\ldots},\ \mathrm{Ad}(\alpha_A^{a_1 a_{b_1}})]}"' near start] \arrow[dd, bend right=50, "{[\boldsymbol{j}_{AB},\ \rho_{AB}]}"'] \arrow[dd, bend left=50, "{[\tilde{\boldsymbol{\jmath}}_{AB},\ \rho_{AB}]}"] \arrow[dd, phantom, "{[\epsilon_{AB},1]\ \Leftarrow}"] & {[\hat{\mathbf{Z}}^{a_1}/\Gamma^{a_1}]} \arrow[dr, bend left, "\boldsymbol{i}^{a_1}"] \arrow[d, phantom, "{[\mathrm{id},\alpha_A^{a_1 a_{b_1}}]\ \Downarrow}" near start] & \\
 & {[\hat{\mathbf{Z}}^{a_1 a_{b_1}}/\Gamma^{a_1 a_{b_1}}]} \arrow[u, "{[\mathbf{inc},\mathrm{inc}]}"'] \arrow[d, "{[\boldsymbol{j}^{a_1 a_{b_1}},\rho^{a_1 a_{b_1}}]}"] \arrow[r, phantom, "\boldsymbol{\zeta}^{a_1 a_{b_1}}\ \Uparrow"] \arrow[dl, phantom, "{[\mathrm{id},\beta_{AB}]\ \Leftarrow}"] & \boldsymbol{\mathfrak{X}}. \\
{[\mathbf{Z}_B/\Gamma_B]} \arrow[r, "{[\mathbf{inc},\mathrm{inc}]}"] & {[\hat{\mathbf{Z}}^{a_{b_1}}/\Gamma^{a_{b_1}}]} \arrow[ur, bend right, "\boldsymbol{i}^{a_{b_1}}"'] &
\end{tikzcd}
\tag{D.25}
$$

Step 1(iii)(b),(iv), equation (D.21), and the first equation of (D.24) imply that $C_A \subseteq C^{a_1}$, $C_B \subseteq C^{a_{b_1}}$ satisfy $p^{a_{b_1}}(\beta_{AB}) \circ q^{a_1 a_{b_1}} \circ p^{a_1}(\alpha_A^{a_1 a_{b_1}})[C_A] \subseteq C_B$, where $p^{a_1}(\alpha_A^{a_1 a_{b_1}})[C_A] \subseteq C^{a_1 a_{b_1}} \subseteq C^{a_1}$. Define an injective map $q_{AB} : C_A \to C_B$ by $q_{AB} = p^{a_{b_1}}(\beta_{AB}) \circ q^{a_1 a_{b_1}} \circ p^{a_1}(\alpha_A^{a_1 a_{b_1}})|_{C_A}$. Then $q_{AB} \circ p_A(\gamma) = p_B(\rho_{AB}(\gamma)) \circ q_{AB}$ and $(\boldsymbol{j}_{AB})_-\big(\partial \mathbf{Z}_{AB} \cap \partial_c \mathbf{Z}_A\big) = (\boldsymbol{j}_{AB})_-\big(\partial \mathbf{Z}_{AB}\big) \cap \partial_{q_{AB}(c)} \hat{\mathbf{Z}}_B$ follow from the definitions. This completes part (ii) of Step 2.

For part (iii), suppose $A, B, C \in I$ with $A < B < C$ and $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B \cap \boldsymbol{\mathfrak{X}}_C \neq \emptyset$. Then $C \subsetneq B \subsetneq A$. Write $A = \{a_1, \ldots, a_n\}$ with $a_1 \prec \cdots \prec a_n$, $B =$

$\{a_{b_1},\dots,a_{b_k}\}$ and $C=\{a_{c_1},\dots,a_{c_l}\}$ for $\{c_1,\dots,c_l\}\subsetneq\{b_1,\dots,b_k\}\subsetneq\{1,\dots,n\}$ with $b_1<\cdots<b_k$ and $c_1<\cdots<c_l$. As for cases (a),(b) in part (ii), divide into cases (a) $c_1=b_1=1$, (b) $c_1=b_1>1$, (c) $c_1>b_1=1$ and (d) $c_1>b_1>1$. In case (a), define $\gamma_{ABC}=\beta_{AC}\beta_{AB}^{-1}\beta_{BC}^{-1}$. In case (b), define $\gamma_{ABC}=\beta_{AC}\beta_{AB}^{-1}\beta_{BC}^{-1}$, and also require that $\epsilon_{AB},\epsilon_{AC}$ in Step (ii) satisfy

$$\epsilon_{AC}|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}=\mathbf{id}_{\boldsymbol{t}^{a_{b_1}}(\beta_{AC}\beta_{AB}^{-1})}*\epsilon_{AB}|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}\quad\text{when } c_1=b_1>1.\tag{D.26}$$

In case (c), define $\gamma_{ABC}=\beta_{AC}\rho^{a_1a_{c_1}}\bigl(\alpha_A^{a_1a_{c_1}}\beta_{AB}^{-1}(\alpha_B^{a_1a_{c_1}})^{-1}\bigr)\beta_{BC}^{-1}$, and also require that $\epsilon_{AC},\epsilon_{BC}$ in Step (ii) satisfy

$$\begin{aligned}\epsilon_{AC}|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}&=\mathbf{id}_{\boldsymbol{t}^{a_{c_1}}(\beta_{AC}\beta_{BC}^{-1})}*\epsilon_{BC}\\&\quad*\mathbf{id}_{\boldsymbol{t}^{a_1}((\alpha_B^{a_1a_{c_1}})^{-1}\alpha_A^{a_1a_{c_1}})|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}}\quad\text{when } c_1>b_1=1.\end{aligned}\tag{D.27}$$

For these three cases, it is easy to check from the definitions that $\rho_{AC}(\gamma)=\gamma_{ABC}\,\rho_{BC}(\rho_{AB}(\gamma))\,\gamma_{ABC}^{-1}$ for $\gamma\in\Gamma_A$, and (D.7) and (D.10) hold, and (D.9) commutes, as we have to prove.

In case (d), we have to do more work. Apply Step 1(v) with $a=a_1$, $b=b_1$, $c=c_1$, $\alpha=\alpha_A^{a_1a_{c_1}}(\alpha_A^{a_1a_{b_1}})^{-1}$ and $\beta=\alpha_B^{a_{b_1}a_{c_1}}\beta_{AB}$. Then $\boldsymbol{t}^{a_1}(\alpha_A^{a_1a_{b_1}})(\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC})\neq\emptyset$ is contained in $\hat{\mathbf{Z}}_{\alpha\beta}^{a_1a_{b_1}a_{c_1}}$ in (D.2) for these α,β, so Step 1(v) gives $\gamma_{\alpha\beta}^{a_1a_{b_1}a_{c_1}}\in\Gamma^{a_{c_1}}$ and a 2-morphism

$$\begin{aligned}\lambda_{\alpha\beta}^{a_1a_{b_1}a_{c_1}}:{}&\boldsymbol{j}^{a_{b_1}a_{c_1}}\circ\boldsymbol{t}^{a_{b_1}}(\alpha_B^{a_{b_1}a_{c_1}})\circ\boldsymbol{t}^{a_{b_1}}(\beta_{AB})\circ\boldsymbol{j}^{a_1a_{b_1}}|_{\hat{\mathbf{Z}}_{\alpha\beta}^{a_1a_{b_1}a_{c_1}}}\\&\Longrightarrow\boldsymbol{t}^{a_{c_1}}((\gamma_{\alpha\beta}^{a_1a_{b_1}a_{c_1}})^{-1})\circ\boldsymbol{j}^{a_1a_{c_1}}\circ\boldsymbol{t}^{a_1}(\alpha_A^{a_1a_{c_1}})\circ\boldsymbol{t}^{a_1}((\alpha_A^{a_1a_{b_1}})^{-1})|_{\hat{\mathbf{Z}}_{\alpha\beta}^{a_1a_{b_1}a_{c_1}}}.\end{aligned}$$

Consider the 2-commutative diagram:

$$\begin{tikzcd}[column sep=large, row sep=large]
{[(\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC})/\Gamma_A]} \arrow[dr, "{[\boldsymbol{t}^{a_1}(\alpha_A^{a_1a_{b_1}})|_{\cdots},\ \mathrm{Ad}(\alpha_A^{a_1a_{b_1}})]}"] \arrow[drr, bend left, "{[\boldsymbol{j}_{AB}|_{\cdots},\rho_{AB}]}"] \arrow[dd, "{[\boldsymbol{j}_{AC}|_{\cdots},\rho_{AC}]}"'] & {[\epsilon_{AB},1]\Uparrow} & \\
{[\mathbf{id}_{\boldsymbol{t}^{a_1}(\beta_{AC}^{-1})}*\epsilon_{AC},\beta_{AC}]\Leftarrow} & {[\boldsymbol{t}^{a_1}(\alpha_A^{a_1a_{b_1}})(\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC})/\alpha_A^{a_1a_{b_1}}\Gamma_A(\alpha_A^{a_1a_{b_1}})^{-1}]} \arrow[r, "{[\boldsymbol{t}^{a_{b_1}}(\beta_{AB})\circ\boldsymbol{j}^{a_1a_{b_1}}|_{\cdots},\ \mathrm{Ad}(\beta_{AB})\circ\rho^{a_1a_{b_1}}]}"] \arrow[dl, "{[\boldsymbol{j}^{a_1a_{c_1}}\circ\boldsymbol{t}^{a_1}(\alpha_A^{a_1a_{c_1}}(\alpha_A^{a_1a_{b_1}})^{-1}),\ \rho^{a_1a_{c_1}}\circ\mathrm{Ad}(\alpha_A^{a_1a_{c_1}}(\alpha_A^{a_1a_{b_1}})^{-1})]}"'] \arrow[d, phantom, "{[\lambda_{\alpha\beta}^{a_1a_{b_1}a_{c_1}},\gamma_{\alpha\beta}^{a_1a_{b_1}a_{c_1}}]\Leftarrow}"] & {[\mathbf{Z}_{BC}/\Gamma_B].} \arrow[dll, "{[\boldsymbol{j}^{a_{b_1}a_{c_1}}\circ\boldsymbol{t}^{a_{b_1}}(\alpha_B^{a_{b_1}a_{c_1}})|_{\cdots},\ \rho^{a_{b_1}a_{c_1}}\circ\mathrm{Ad}(\alpha_B^{a_{b_1}a_{c_1}})]}"] \arrow[dll, bend left, "{[\boldsymbol{j}_{BC},\rho_{BC}]}"] \\
{[\hat{\mathbf{Z}}^{a_{c_1}}/\Gamma^{a_{c_1}}]} & {[\mathbf{id}_{\boldsymbol{t}^{a_{c_1}}(\beta_{BC}^{-1})}*\epsilon_{BC},\beta_{BC}]\Downarrow} &
\end{tikzcd}\tag{D.28}$$

Here uniqueness of $\gamma_{\alpha\beta}^{a_1a_{b_1}a_{c_1}},\lambda_{\alpha\beta}^{a_1a_{b_1}a_{c_1}}$ in Step 1(v) implies that they are equivariant under $\alpha_A^{a_1a_{b_1}}\Gamma_A(\alpha_A^{a_1a_{b_1}})^{-1}$, and so descend to a quotient 1-morphism $[\lambda_{\alpha\beta}^{a_1a_{b_1}a_{c_1}},\gamma_{\alpha\beta}^{a_1a_{b_1}a_{c_1}}]$ as shown.

The composition of 2-morphisms across (D.28) gives a quotient 2-morphism

$$[\lambda_{ABC},\gamma_{ABC}]:[\boldsymbol{j}_{BC},\rho_{BC}]\circ[\boldsymbol{j}_{AB}|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}},\rho_{AB}]\Longrightarrow[\boldsymbol{j}_{AC}|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}},\rho_{AC}],$$

where $\gamma_{ABC}=\beta_{AC}\gamma_{\alpha\beta}^{a_1 a_{b_1} a_{c_1}}\beta_{BC}^{-1}$ and $\lambda_{ABC}=\mathrm{id}_{\dots}$ if and only if

$$\begin{aligned}\epsilon_{AC}|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}&=\mathbf{id}_{\boldsymbol{t}^{a_{c_1}}(\gamma_{ABC}\beta_{BC})\circ\boldsymbol{j}^{a_{b_1}a_{c_1}}\circ\boldsymbol{t}^{a_{b_1}}(\alpha_B^{a_{b_1}a_{c_1}})}*\epsilon_{AB}|_{\dots}\\&\quad+\mathbf{id}_{\boldsymbol{t}^{a_{c_1}}(\gamma_{ABC})}*\epsilon_{BC}*\mathbf{id}_{\boldsymbol{t}^{a_{b_1}}(\beta_{AB})\circ\boldsymbol{j}^{a_1 a_{b_1}}\circ\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{b_1}})|_{\dots}}\\&\quad-\mathbf{id}_{\boldsymbol{t}^{a_{c_1}}(\gamma_{ABC}\beta_{BC})}*\lambda_{\alpha\beta}^{a_1 a_{b_1} a_{c_1}}*\mathbf{id}_{\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{b_1}})|_{\dots}}\quad\text{when } c_1>b_1>1.\end{aligned}\tag{D.29}$$

Supposing (D.29) holds, we have a quotient 1-morphism (D.8), which implies that $\rho_{AC}(\gamma)=\gamma_{ABC}\,\rho_{BC}(\rho_{AB}(\gamma))\,\gamma_{ABC}^{-1}$ for $\gamma\in\Gamma_A$ and (D.7) holds. Composing (D.28) with the projection to $\boldsymbol{\mathfrak{X}}$ and using Step 1(v) and the definitions of $\boldsymbol{\zeta}_{AB},\boldsymbol{\zeta}_{AC},\boldsymbol{\zeta}_{BC}$ in (D.25), we find that (D.9) commutes. One can also prove (D.10) for this γ_{ABC} using the definitions and equations (D.3), (D.21).

This proves part (iii), provided we can choose the ρ_{AB}-equivariant morphisms ϵ_{AB} in part (ii) to satisfy (D.26), (D.27) and (D.29). We do this by choosing ϵ_{AB} by induction on $|A|-|B|$, that is, at the k^{th} inductive step we choose ϵ_{AB} for all $B\subsetneq A$ with $\boldsymbol{\mathfrak{X}}_A\cap\boldsymbol{\mathfrak{X}}_B\neq\boldsymbol{\emptyset}$ and $|A|-|B|=k$. For the first step when $k=1$, we define $\epsilon_{AB}=0$ for all A,B with $|A|-|B|=1$.

For the inductive step, let $k>1$, suppose we have chosen ρ_{AB}-equivariant ϵ_{AB} satisfying (D.26), (D.27) and (D.29) for all $A,B\in I$ with $|A|-|B|<k$, and let $C\subsetneq A$ with $\boldsymbol{\mathfrak{X}}_A\cap\boldsymbol{\mathfrak{X}}_C\neq\boldsymbol{\emptyset}$ and $|A|-|C|=k$. Suppose $B\in I$ with $C\subsetneq B\subsetneq A$. Then $|A|-|B|<k$ and $|B|-|C|<k$, so $\epsilon_{AB},\epsilon_{BC}$ have already been chosen if $\boldsymbol{\mathfrak{X}}_A\cap\boldsymbol{\mathfrak{X}}_B\neq\boldsymbol{\emptyset}$, $\boldsymbol{\mathfrak{X}}_B\cap\boldsymbol{\mathfrak{X}}_C\neq\boldsymbol{\emptyset}$. Thus in (D.26), (D.27), (D.29) the right hand sides are already determined. So our problem is to choose ϵ_{AC} on $\mathbf{Z}_{AC}$ which is ρ_{AC}-equivariant, and takes prescribed values on the open d-submanifolds $\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}\subseteq\mathbf{Z}_{AC}$ for all $B\in I$ with $C\subsetneq B\subsetneq A$.

First we show that these prescribed values on $\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}$ are consistent on overlaps. Suppose $B\neq B'\in I$ with $C\subsetneq B\subsetneq A$ and $C\subsetneq B'\subsetneq A$. Then either (A) $\boldsymbol{\mathfrak{X}}_B\cap\boldsymbol{\mathfrak{X}}_{B'}=\boldsymbol{\emptyset}$, (B) $B\subsetneq B'$, or (C) $B'\subsetneq B$. In case (A) $(\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC})\cap(\mathbf{Z}_{AB'}\cap\mathbf{Z}_{AC})=\boldsymbol{\emptyset}$, so the prescribed values for $\epsilon_{AC}|_{\mathbf{Z}_{AB}\cap\mathbf{Z}_{AC}}$ and $\epsilon_{AC}|_{\mathbf{Z}_{AB'}\cap\mathbf{Z}_{AC}}$ are trivially consistent. In case (B) we have $C\subsetneq B\subsetneq B'$, so the previous inductive steps give conditions on $\epsilon_{B'C},\epsilon_{BC},\epsilon_{B'B}$, and $B\subsetneq B'\subsetneq A$, so the previous inductive steps give conditions on $\epsilon_{AB},\epsilon_{B'B},\epsilon_{AB}$. Combining these implies that the prescribed values for ϵ_{AC} from B,B' on $\mathbf{Z}_{AB}\cap\mathbf{Z}_{AB'}\cap\mathbf{Z}_{AC}$ are equal. Case (C) is similar.

Thus, the problem is to choose ρ_{AC}-equivariant ϵ_{AC} taking a prescribed value on the open d-submanifold $\mathbf{Z}_{AC}\cap\bigcup_{B:C\subsetneq B\subsetneq A}\mathbf{Z}_{AB}$ in $\mathbf{Z}_{AC}$. This prescribed value is automatically ρ_{AC}-equivariant, because of the equivariance of all the ingredients. Using partitions of unity, one can show that a (ρ_{AC}-equivariant) extension ϵ_{AC} of this prescribed value exists if and only if an extension exists to some open neighbourhood of the closure of $\mathbf{Z}_{AC}\cap\bigcup_{B:C\subsetneq B\subsetneq A}\mathbf{Z}_{AB}$ in $\mathbf{Z}_{AC}$.

Now ϵ_{AC} is defined using $\tilde{\boldsymbol{j}}_{AC}=\boldsymbol{t}^{a_{c_1}}(\beta_{AC})\circ\boldsymbol{j}^{a_1 a_{c_1}}\circ\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{c_1}})|_{\mathbf{Z}_{AC}}$, which is defined not just on $\mathbf{Z}_{AC}$ but on $\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{c_1}})^{-1}[\hat{\mathbf{Z}}^{a_1 a_{c_1}}]\subseteq\hat{\mathbf{Z}}^{a_1}$. Make the convention that we always choose ϵ_{AC} so that it extends smoothly to an open neighbourhood of the closure of $\mathbf{Z}_{AC}$ in $\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{c_1}})^{-1}[\hat{\mathbf{Z}}^{a_1 a_{c_1}}] \subseteq \hat{\mathbf{Z}}^{a_1}$. We claim that with this convention, the convention for $\epsilon_{AB}, \epsilon_{BC}$ imply that the prescribed values (D.26), (D.27), (D.29) for ϵ_{AC} automatically extend to an open neighbourhood of the closure of $\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}$ in $\mathbf{Z}_{AC}$, so that choosing ϵ_{AC} is possible in the inductive step, and the induction works.

We prove this claim using the fact in Step 1(ii)(a) that the closure of $\mathfrak{X}^a$ in $\hat{\mathfrak{X}}^a$ is compact for $a \in J$. Since $\mathfrak{X}$ is separated, this also implies that the closures of $\mathfrak{X}^a$ in $\hat{\mathfrak{X}}^a$ and in $\mathfrak{X}$ are equal. One can use this and the properness of the projections $\hat{\mathbf{Z}}^a \to \hat{\mathfrak{X}}^a$ to prove many other statements on equality of closures of subsets in different spaces. For example, in (D.26), by induction ϵ_{AB} extends to a neighbourhood of the closure of $\mathbf{Z}_{AB}$ in $\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{b_1}})^{-1}[\hat{\mathbf{Z}}^{a_1 a_{b_1}}]$. But the (compact) closures of $\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}$ in $\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{b_1}})^{-1}[\hat{\mathbf{Z}}^{a_1 a_{b_1}}]$ and in $\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{c_1}})^{-1}[\hat{\mathbf{Z}}^{a_1 a_{c_1}}]$ are the same. So the r.h.s. of (D.26) extends to a neighbourhood of the closure of $\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}$ in $\boldsymbol{t}^{a_1}(\alpha_A^{a_1 a_{c_1}})^{-1}[\hat{\mathbf{Z}}^{a_1 a_{c_1}}]$, as we need. This completes the inductive step, and the proof of part (iii) of Step 2.

For part (iv), suppose $A, B, C, D \in I$ with $A < B < C < D$ and $\mathfrak{X}_A \cap \mathfrak{X}_B \cap \mathfrak{X}_C \cap \mathfrak{X}_D \neq \emptyset$. Consider the diagram of 2-morphisms in $\mathbf{dOrb^c}$:

$$
\begin{tikzcd}
\substack{\boldsymbol{i}_D \circ [\boldsymbol{j}_{CD}, \rho_{CD}] \circ [\boldsymbol{j}_{BC}, \rho_{BC}] \\ \circ [\boldsymbol{j}_{AB}|_{\dots}, \rho_{AB}]} \arrow[rrr, Rightarrow, "\mathbf{id}*[\mathrm{id},\gamma_{ABC}]"'] \arrow[dr, Rightarrow, "\boldsymbol{\zeta}_{CD}*\mathbf{id}"'] \arrow[ddd, Rightarrow, "\mathbf{id}*[\mathrm{id},\gamma_{BCD}]*\mathbf{id}"] & & & \substack{\boldsymbol{i}_D \circ [\boldsymbol{j}_{CD}, \rho_{CD}] \\ \circ [\boldsymbol{j}_{AC}|_{\dots}, \rho_{AC}]} \arrow[dl, Rightarrow, "\boldsymbol{\zeta}_{CD}*\mathbf{id}"] \arrow[ddd, Rightarrow, "\mathbf{id}*[\mathrm{id},\gamma_{ACD}]"'] \\
& \substack{\boldsymbol{i}_C \circ [\boldsymbol{j}_{BC}, \rho_{BC}] \\ \circ [\boldsymbol{j}_{AB}|_{\dots}, \rho_{AB}]} \arrow[r, Rightarrow, "\mathbf{id}*[\mathrm{id},\gamma_{ABC}]"] \arrow[d, Rightarrow, "\boldsymbol{\zeta}_{BC}*\mathbf{id}"] & \boldsymbol{i}_C \circ [\boldsymbol{j}_{AC}|_{\dots}, \rho_{AC}] \arrow[d, Rightarrow, "\boldsymbol{\zeta}_{AC}|_{\cdots}"'] & \\
& \boldsymbol{i}_B \circ [\boldsymbol{j}_{AB}|_{\dots}, \rho_{AB}] \arrow[r, Rightarrow, "\boldsymbol{\zeta}_{AB}|_{\cdots}"'] & \boldsymbol{i}_A|_{\dots} & \\
\substack{\boldsymbol{i}_D \circ [\boldsymbol{j}_{BD}, \rho_{BD}] \\ \circ [\boldsymbol{j}_{AB}|_{\dots}, \rho_{AB}]} \arrow[ur, Rightarrow, "\boldsymbol{\zeta}_{BD}*\mathbf{id}"] \arrow[rrr, Rightarrow, "\mathbf{id}*[\mathrm{id},\gamma_{ABD}|_{\dots}]"] & & & \boldsymbol{i}_D \circ [\boldsymbol{j}_{AD}|_{\dots}, \rho_{AD}]. \arrow[ul, Rightarrow, "\boldsymbol{\zeta}_{AD}|_{\cdots}"']
\end{tikzcd}
$$

Here the domain of every 1-morphism is $[\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC} \cap \mathbf{Z}_{AD}/\Gamma_A]$. The top quadrilateral commutes by compatibility of horizontal and vertical composition. The other four small quadrilaterals commute by (D.9). Hence the whole diagram commutes. Since $\boldsymbol{i}_D$ is an equivalence with an open d-suborbifold, we can omit '$\boldsymbol{i}_D \circ$' from the outer rectangle, so that the following commutes

$$
\begin{tikzcd}
{[\boldsymbol{j}_{CD}, \rho_{CD}] \circ [\boldsymbol{j}_{BC}, \rho_{BC}] \circ [\boldsymbol{j}_{AB}|_{\dots}, \rho_{AB}]} \arrow[r, Rightarrow, "\mathbf{id}_{[\boldsymbol{j}_{CD}, \rho_{CD}]}*[\mathrm{id},\gamma_{ABC}]"'] \arrow[d, Rightarrow, "{[\mathrm{id},\gamma_{BCD}]*\mathbf{id}_{[\boldsymbol{j}_{AB}|_{\dots}, \rho_{AB}]}}"] & {[\boldsymbol{j}_{CD}, \rho_{CD}] \circ [\boldsymbol{j}_{AC}|_{\dots}, \rho_{AC}]} \arrow[d, Rightarrow, "{[\mathrm{id},\gamma_{ACD}]}"'] \\
{[\boldsymbol{j}_{BD}, \rho_{BD}] \circ [\boldsymbol{j}_{AB}|_{\dots}, \rho_{AB}]} \arrow[r, Rightarrow, "{[\mathrm{id},\gamma_{ABD}]}"] & {[\boldsymbol{j}_{AD}|_{\dots}, \rho_{AD}].}
\end{tikzcd}
$$

Therefore $\gamma_{ACD}\, \rho_{CD}(\gamma_{ABC}) = \gamma_{ABD}\, \gamma_{BCD}$. This completes Step 2.

Remark D.1. In part (ii) we defined $\rho_{AB}, \boldsymbol{j}_{AB}, \boldsymbol{\zeta}_{AB}, q_{AB}$ on a region $\mathbf{Z}_{AB}$ in $\hat{\mathbf{Z}}^{a_1}$ lying over $\mathfrak{X}_A \cap \mathfrak{X}_B$, so we assumed $\mathfrak{X}_A \cap \mathfrak{X}_B \neq \emptyset$. Similarly, in (iii) we defined γ_{ABC} and proved (D.7)–(D.10) on a region $\mathbf{Z}_{AB} \cap \mathbf{Z}_{AC}$ in $\hat{\mathbf{Z}}^{a_1}$ lying over $\mathfrak{X}_A \cap \mathfrak{X}_B \cap \mathfrak{X}_C$, so we assumed $\mathfrak{X}_A \cap \mathfrak{X}_B \cap \mathfrak{X}_C \neq \emptyset$, and in (iv) we proved $\gamma_{ACD}\, \rho_{CD}(\gamma_{ABC}) = \gamma_{ABD}\, \gamma_{BCD}$ assuming $\mathfrak{X}_A \cap \mathfrak{X}_B \cap \mathfrak{X}_C \cap \mathfrak{X}_D \neq \emptyset$.

In fact in part (ii) the only place the assumption $\mathfrak{X}_A \cap \mathfrak{X}_B \neq \emptyset$ is used is to deduce that $B \subsetneq A$. We do pick points $[x] \in \bigcap_{i=1}^n \hat{\mathfrak{X}}^{a_i}$, $z_A \in \hat{\mathbf{Z}}_A$, $z_B \in \hat{\mathbf{Z}}_B$, but $\bigcap_{i=1}^n \hat{\mathfrak{X}}^{a_i}$, $\hat{\mathbf{Z}}_A, \hat{\mathbf{Z}}_B$ are automatically nonempty as $A, B \in I$. Thus, in (ii) we can take $\rho_{AB} : \Gamma_A \to \Gamma_B$ and $q_{AB} : C_A \to C_B$ to be defined whenever $A, B \in I$ with $B \subsetneq A$, without assuming $\mathfrak{X}_A \cap \mathfrak{X}_B \neq \emptyset$.

Similarly, in (iii) we can take $\gamma_{ABC} \in \Gamma_C$ with $q_{AC} = p_C(\gamma_{ABC}) \circ q_{BC} \circ q_{AB}$ and $\rho_{AC}(\gamma) = \gamma_{ABC}\, \rho_{BC}(\rho_{AB}(\gamma))\, \gamma_{ABC}^{-1}$ for $\gamma \in \Gamma_A$ to be defined whenever $A, B, C \in I$ with $C \subsetneq B \subsetneq A$, without assuming $\mathfrak{X}_A \cap \mathfrak{X}_B \cap \mathfrak{X}_C \neq \emptyset$, and in (iv) we can suppose $\gamma_{ACD}\, \rho_{CD}(\gamma_{ABC}) = \gamma_{ABD}\, \gamma_{BCD}$ whenever $A, B, C, D \in I$ with $D \subsetneq C \subsetneq B \subsetneq A$, without assuming $\mathfrak{X}_A \cap \mathfrak{X}_B \cap \mathfrak{X}_C \cap \mathfrak{X}_D \neq \emptyset$. We will need all this in Step 3.

D.4 Step 3: Choose sf-embeddings $\boldsymbol{f}_A : \mathbf{Z}_A \to \mathbf{U}_A$, $A \in I$

As in Remark D.1 we will suppose ρ_{AB}, q_{AB} in Step 2(ii) are defined whenever $A, B \in I$ with $B \subsetneq A$, and similarly for γ_{ABC} in Step 2(iii),(iv). Let $A \in I$, and write $A = \{a_1, \ldots, a_n\}$ with $a_1 \prec \cdots \prec a_n$. Then $\mathbf{Z}_A \subseteq \hat{\mathbf{Z}}^{a_1}$, and $\hat{\mathbf{Z}}^{a_1}$ is a principal d-manifold with corners. So by Corollary 7.49, $\dim T_z^* \underline{Z}^{a_1}$ is bounded above for $z \in \underline{Z}^{a_1}$. Choose an integer $n_A > 0$ such that $n_A \geqslant 2 \dim T_z^* \underline{Z}^{a_1} + 1$ for $z \in \underline{Z}^{a_1}$. Then there exist embeddings $\hat{\mathbf{Z}}^{a_1} \hookrightarrow \mathbb{R}^{n_A}$ and $\mathbf{Z}_A \hookrightarrow \mathbb{R}^{n_A}$ by Theorem 7.44.

For each $B \in I$, write R_B for the real vector space with basis of symbols $|\gamma\rangle$ for $\gamma \in \Gamma_B$, and S_B for the real vector space with basis of symbols $|c\rangle$ for $c \in C_B$. Define a finite-dimensional real vector space T_B by

$$T_B = \left(\textstyle\bigoplus_{A \in I:\, B \subseteq A} \mathbb{R}^{n_A}\right) \otimes R_B \oplus S_B. \tag{D.30}$$

Step 1(ii)(c) implies that this direct sum is finite. Write elements of T_B as $\sum_{B \subseteq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\gamma\rangle + \sum_{c \in C_B} u_B^c |c\rangle$, for $t_A^\gamma \in \mathbb{R}^{n_A}$ and $u_B^c \in \mathbb{R}$. Define an effective representation r_B of Γ_B on T_B by

$$\begin{aligned} r_B(\gamma) : {}& \textstyle\sum_{B \subseteq A \in I,\, \delta \in \Gamma_B} t_A^\delta \otimes |\delta\rangle + \sum_{c \in C_B} u_B^c |c\rangle \\ &\longmapsto \textstyle\sum_{B \subseteq A \in I,\, \delta \in \Gamma_B} t_A^\delta \otimes |\gamma\delta\rangle + \sum_{c \in C_B} u_B^c |p_B(\gamma)(c)\rangle. \end{aligned} \tag{D.31}$$

For $c \in C_B$, define a linear map $\tau_{Bc} : T_B \to \mathbb{R}$ by

$$\tau_{Bc} : \textstyle\sum_{B \subseteq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\gamma\rangle + \sum_{c \in C_B} u_B^c |c\rangle \longmapsto u_B^c. \tag{D.32}$$

Then $U_B = \bigl\{\sum_{B \subseteq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\gamma\rangle + \sum_{c \in C_B} u_B^c |c\rangle \in T_B : u_B^c \geqslant -1\ \forall c \in C_B\bigr\}$.

Suppose $B, C \in I$ with $B < C$ and $\mathfrak{X}_B \cap \mathfrak{X}_C \neq \emptyset$. Then $C \subsetneq B$, as in Step 2. Define a linear map $l_{BC} : T_B \to T_C$ by

$$\begin{aligned} l_{BC} : \sum_{B \subseteq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\gamma\rangle + \sum_{c \in C_B} u_B^c |c\rangle \longmapsto \sum_{\gamma \in \Gamma_B} t_B^\gamma \otimes |\rho_{BC}(\gamma)\rangle \\ + \sum_{B \subsetneq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\rho_{BC}(\gamma)\, \gamma_{ABC}^{-1}\rangle + \sum_{c \in C_B} u_B^c |q_{BC}(c)\rangle. \end{aligned} \tag{D.33}$$

In (D.33) we use γ_{ABC} assuming only that $C \subsetneq B \subsetneq A$, rather than $\mathfrak{X}_A \cap \mathfrak{X}_B \cap \mathfrak{X}_C \neq \emptyset$, as in Remark D.1. Step 3(i)(a),(b) and Step 3(iii)(a)–(c) now follow easily from (D.30)–(D.33).

Suppose $B, C, D \in I$ with $B < C < D$ and $\mathfrak{X}_B \cap \mathfrak{X}_C \cap \mathfrak{X}_D \neq \emptyset$. Then $D \subsetneq C \subsetneq B$. Let $\sum_{B \subseteq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\gamma\rangle + \sum_{c \in C_B} u_B^c |c\rangle$ lie in T_B. Then

$$\begin{aligned}
&r_D(\gamma_{BCD}) \circ l_{CD} \circ l_{BC}\bigl(\textstyle\sum_{B \subseteq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\gamma\rangle + \sum_{c \in C_B} u_B^c |c\rangle\bigr)\\
&\quad = r_D(\gamma_{BCD}) \circ l_{CD}\bigl(\textstyle\sum_{\gamma \in \Gamma_B} t_B^\gamma \otimes |\rho_{BC}(\gamma)\rangle\\
&\qquad + \textstyle\sum_{B \subsetneq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\rho_{BC}(\gamma)\,\gamma_{ABC}^{-1}\rangle + \sum_{c \in C_B} u_B^c |q_{BC}(c)\rangle\bigr)\\
&\quad = r_D(\gamma_{BCD})\bigl(\textstyle\sum_{\gamma \in \Gamma_B} t_B^\gamma \otimes |\rho_{CD}(\rho_{BC}(\gamma))\,\gamma_{BCD}^{-1}\rangle\\
&\qquad + \sum_{B \subsetneq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\rho_{CD}(\rho_{BC}(\gamma)\,\gamma_{ABC}^{-1})\,\gamma_{ACD}^{-1}\rangle + \sum_{c \in C_B} u_B^c |q_{CD} \circ q_{BC}(c)\rangle\bigr)\\
&\quad = \textstyle\sum_{\gamma \in \Gamma_B} t_B^\gamma \otimes |\gamma_{BCD}\,\rho_{CD}(\rho_{BC}(\gamma))\,\gamma_{BCD}^{-1}\rangle\\
&\qquad + \textstyle\sum_{B \subsetneq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\gamma_{BCD}\,\rho_{CD}(\rho_{BC}(\gamma)\,\gamma_{ABC}^{-1})\,\gamma_{ACD}^{-1}\rangle)\\
&\qquad + \textstyle\sum_{c \in C_B} u_B^c \bigl|p_D(\gamma_{BCD})(q_{CD} \circ q_{BC}(c))\bigr\rangle\\
&\quad = \textstyle\sum_{\gamma \in \Gamma_B} t_B^\gamma \otimes |\rho_{BD}(\gamma)\rangle + \sum_{B \subsetneq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\rho_{BD}(\gamma)\,\gamma_{ABD}^{-1}\rangle\\
&\qquad + \textstyle\sum_{c \in C_B} u_B^c |q_{BD}(c)\rangle\\
&\quad = l_{BD}\bigl(\textstyle\sum_{B \subseteq A \in I,\, \gamma \in \Gamma_B} t_A^\gamma \otimes |\gamma\rangle + \sum_{c \in C_B} u_B^c |c\rangle\bigr).
\end{aligned}$$

Here we use (D.33) in the first, second and fifth steps, (D.31) in the third, and $\rho_{BD}(\gamma) = \gamma_{BCD}\,\rho_{CD}(\rho_{BC}(\gamma))\,\gamma_{BCD}^{-1}$, $\gamma_{ACD}\,\rho_{CD}(\gamma_{ABC}) = \gamma_{ABD}\,\gamma_{BCD}$ and (D.10) in the fourth, which hold by Step 2(iii),(iv) extended as in Remark D.1. This proves that $l_{BD} = r_D(\gamma_{BCD}) \circ l_{CD} \circ l_{BC} : T_A \to T_C$, as in Step 3(iv).

We have now constructed the data $T_A, r_A, \tau_{Ac}, l_{AB}$ in Step 3(i),(iii) and shown it satisfies Steps 3(i)(a),(b), 3(iii)(a)–(c) and 3(iv). The beginning of Steps 3(ii) and 3(iii)(d) now construct manifolds with corners U_A, actions $\tilde r_A : \Gamma_A \to \mathrm{Aut}(U_A)$, boundary defining functions $\upsilon_{Ac} : U_A \to [0, \infty)$, and sf-embeddings $\tilde l_{AB} : U_A \to U_B$, and write $\mathbf{U}_A, \tilde{\boldsymbol{r}}_A, \boldsymbol{\upsilon}_{Ac}, \tilde{\boldsymbol{l}}_{AB} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(U_A, \tilde r_A, \upsilon_{Ac}, \tilde l_{AB})$. Next we construct the sf-embeddings $\boldsymbol{f}_B : \mathbf{Z}_B \to \mathbf{U}_B$ for each $B \in I$.

Let $\boldsymbol{g}_{AB} : \mathbf{Z}_B \to \mathbb{R}^{n_A}$ for each $A \in I$ with $B \subseteq A$ and $\boldsymbol{d}_{Bc} : \mathbf{Z}_B \to \boldsymbol{(0, \infty)}$ for each $c \in C_B$ be 1-morphisms in $\mathbf{dMan^c}$ satisfying conditions we will explain later. Define $\boldsymbol{f}_B : \mathbf{Z}_B \to \mathbf{U}_B$ by

$$\begin{aligned}
\boldsymbol{f}_B = \bigl(\textstyle\prod_{A \in I:\, B \subseteq A} \prod_{\gamma \in \Gamma_B} (\boldsymbol{g}_{AB} \circ \boldsymbol{t}_B(\gamma^{-1})) \otimes |\gamma\rangle\bigr)&\\
\times \bigl(\textstyle\prod_{c \in C_B} (\boldsymbol{d}_{Bc} \cdot \boldsymbol{b}_{Bc} - \mathbf{1}) \cdot |c\rangle\bigr)&,
\end{aligned} \tag{D.34}$$

where $\boldsymbol{b}_{Bc} : \mathbf{Z}_A \to \boldsymbol{[0, \infty)}$ is as in Step 2(i). That is, the component of $\boldsymbol{f}_B$ mapping into the factor $\mathbb{R}^{n_A} \otimes |\gamma\rangle$ in $\mathbf{U}_B$ is $\boldsymbol{g}_{AB} \circ \boldsymbol{t}_B(\gamma^{-1})$, and the component mapping into $[-1, \infty) \cdot |c\rangle$ is $\boldsymbol{d}_{Bc} \cdot \boldsymbol{b}_{Bc} - \mathbf{1}$.

For these $\boldsymbol{f}_B$ to satisfy Step 3(ii)(a), they must be equivariant under the Γ_B-actions on $\mathbf{Z}_B, \mathbf{U}_B$. The first line of (D.34) is already Γ_B-equivariant, so 3(ii)(b) imposes no conditions on the $\boldsymbol{g}_{AB}$. Also the $\boldsymbol{b}_{Bc}$ are Γ_B-equivariant by

Step 2(i)(d), so the $\boldsymbol{d}_{Bc}\cdot\boldsymbol{b}_{Bc}-\boldsymbol{1}$ terms are Γ_B-equivariant provided the $\boldsymbol{d}_{Bc}$ terms are. Hence Step 3(ii)(a) follows from

$$\boldsymbol{d}_{Bc}=\boldsymbol{b}_{Bp_B(\gamma)(c)}\circ\boldsymbol{t}_A(\gamma)\quad\text{for all } c\in C_B \text{ and } \gamma\in\Gamma_B. \tag{D.35}$$

For Step 3(ii)(b), equations (D.32), (D.34) and $\upsilon_{Bc}=\tau_{Bc}+1$ imply that $\boldsymbol{\upsilon}_{Bc}\circ\boldsymbol{f}_B=\boldsymbol{d}_{Bc}\cdot\boldsymbol{b}_{Bc}$ for $c\in C_B$. So Step 2(i)(c), Proposition 6.6(d) and $\boldsymbol{d}_{Bc}:\mathbf{Z}_B\to\boldsymbol{(0,\infty)}$ show that Step 3(ii)(b) holds.

Let $B,C\in I$ with $B<C$ and $\boldsymbol{\mathfrak{X}}_B\cap\boldsymbol{\mathfrak{X}}_C\neq\boldsymbol{\emptyset}$. Then by (D.33) and (D.34), the condition $\boldsymbol{f}_C\circ\boldsymbol{j}_{BC}=\tilde{\boldsymbol{l}}_{BC}\circ\boldsymbol{f}_B|_{\mathbf{Z}_{BC}}$ in Step 3(iii)(d) is equivalent to:

$$\boldsymbol{g}_{BC}\circ\boldsymbol{j}_{BC}=\boldsymbol{g}_{BB}|_{\mathbf{Z}_{BC}}, \tag{D.36}$$

$$\boldsymbol{g}_{AC}\circ\boldsymbol{t}_C(\gamma_{ABC})\circ\boldsymbol{j}_{BC}=\boldsymbol{g}_{AB}|_{\mathbf{Z}_{BC}}\qquad\text{for all } A\in I \text{ with } B\subsetneq A, \tag{D.37}$$

$$\boldsymbol{g}_{BC}|_{(\boldsymbol{t}_C(\gamma)\circ\boldsymbol{j}_{BC})(\mathbf{Z}_{BC})}=\boldsymbol{0}\qquad\text{for all } \gamma\in\Gamma_C\setminus\rho_{BC}(\Gamma_B), \tag{D.38}$$

$$\begin{aligned}\boldsymbol{g}_{AC}|_{(\boldsymbol{t}_C(\gamma)\circ\boldsymbol{j}_{BC})(\mathbf{Z}_{BC})}=\boldsymbol{0}\quad&\text{for all } A\in I \text{ with } B\subsetneq A\\ &\text{and } \gamma\in\Gamma_C\setminus[\gamma_{ABC}\cdot\rho_{BC}(\Gamma_B)],\end{aligned} \tag{D.39}$$

$$\begin{aligned}\boldsymbol{g}_{AC}|_{(\boldsymbol{t}_C(\gamma)\circ\boldsymbol{j}_{BC})(\mathbf{Z}_{BC})}=\boldsymbol{0}\quad&\text{for all } A\in I \text{ with } C\subseteq A,\ B\not\subseteq A\\ &\text{and } \gamma\in\Gamma_C,\end{aligned} \tag{D.40}$$

$$\boldsymbol{d}_{Bc}|_{\mathbf{Z}_{BC}}\cdot\boldsymbol{b}_{Bc}|_{\mathbf{Z}_{BC}}=\bigl(\boldsymbol{d}_{Cq_{BC}(c)}\cdot\boldsymbol{b}_{Cq_{BC}(c)}\bigr)\circ\boldsymbol{j}_{BC}\quad\text{for all } c\in C_B, \tag{D.41}$$

$$\boldsymbol{d}_{Cc}|_{\boldsymbol{j}_{BC}(\mathbf{Z}_{BC})}=\boldsymbol{b}_{Cc}|^{-1}_{\boldsymbol{j}_{BC}(\mathbf{Z}_{BC})}\quad\text{for all } c\in C_C\setminus q_{BC}(C_B). \tag{D.42}$$

Here (D.42) is equivalent to $(\boldsymbol{d}_{Cc}\cdot\boldsymbol{b}_{Cc}-\boldsymbol{1})\circ\boldsymbol{j}_{BC}=\boldsymbol{0}$.

Fix $A\in I$, and write $A=\{a_1,\ldots,a_n\}$ with $a_1\prec\cdots\prec a_n$ as usual. Define an open d-submanifold $\mathbf{W}_A\subseteq\mathbf{Z}_A\subseteq\hat{\mathbf{Z}}^{a_1}$ by

$$\mathbf{W}_A=\textstyle\bigcup_{E\in I:A\subsetneq E,\ \boldsymbol{\mathfrak{X}}_A\cap\boldsymbol{\mathfrak{X}}_E\neq\boldsymbol{\emptyset}}\boldsymbol{r}_A(\Gamma_A)[\boldsymbol{j}_{EA}(\mathbf{Z}_{EA})].$$

Choose a 1-morphism $\boldsymbol{h}_A:\hat{\mathbf{Z}}^{a_1}\to\mathbb{R}^{n_A}$ such that $\boldsymbol{h}_A|_{\mathbf{W}_A}=0$, and $\boldsymbol{h}_A|_{\hat{\mathbf{Z}}^{a_1}\setminus\overline{\mathbf{W}}_A}$ is an embedding $\hat{\mathbf{Z}}^{a_1}\setminus\overline{\mathbf{W}}_A\hookrightarrow\mathbb{R}^{n_A}\setminus\{0\}$, where $\overline{\mathbf{W}}_A$ is the closure of $\mathbf{W}_A$ in $\hat{\mathbf{Z}}^{a_1}$. This is possible by a modification of Theorem 7.44, and the choice of n_A.

We will choose the morphisms $\boldsymbol{g}_{AC}:\mathbf{Z}_C\to\mathbb{R}^{n_A}$ for $C\in I$ with $C\subsetneq A$ and $\boldsymbol{\mathfrak{X}}_A\cap\boldsymbol{\mathfrak{X}}_C\neq\boldsymbol{\emptyset}$ by induction on increasing $|C|$, where we write $C=\{a_{c_1},\ldots,a_{c_l}\}$ with $c_1<\cdots<c_l$. Each such choice $\boldsymbol{g}_{AC}$ must satisfy the following conditions:

(a) When $C=A$ we have $\boldsymbol{g}_{AA}|_{\mathbf{Z}_{AB}}=\boldsymbol{g}_{AB}\circ\boldsymbol{j}_{AB}$ whenever $B\in I$ with $B\subsetneq A$ and $\boldsymbol{\mathfrak{X}}_A\cap\boldsymbol{\mathfrak{X}}_B\neq\boldsymbol{\emptyset}$. This ensures (D.36) holds.

(b) When $C\neq A$ we have $\boldsymbol{g}_{AC}|_{\mathbf{Z}_{CD}}=\boldsymbol{g}_{AD}\circ\boldsymbol{t}_D(\gamma_{ACD})\circ\boldsymbol{j}_{CD}$ whenever $D\in I$ with $D\subsetneq C$ and $\boldsymbol{\mathfrak{X}}_C\cap\boldsymbol{\mathfrak{X}}_D\neq\boldsymbol{\emptyset}$. This ensures (D.37) holds.

(c) When $C\neq A$ we have $\boldsymbol{g}_{AC}|_{(\boldsymbol{t}_C(\gamma)\circ\boldsymbol{j}_{AC})(\mathbf{Z}_{AC})}=\boldsymbol{0}$ for all $\gamma\in\Gamma_C\setminus\rho_{AC}(\Gamma_A)$. This ensures (D.38) holds.

(d) When $C\neq A$ we have $\boldsymbol{g}_{AC}|_{(\boldsymbol{t}_C(\gamma)\circ\boldsymbol{j}_{BC})(\mathbf{Z}_{BC})}=\boldsymbol{0}$ whenever $B\in I$ with $C\subsetneq B\subsetneq A$, $\boldsymbol{\mathfrak{X}}_B\cap\boldsymbol{\mathfrak{X}}_C\neq\boldsymbol{\emptyset}$ and $\gamma\in\Gamma_C\setminus[\gamma_{ABC}\cdot\rho_{BC}(\Gamma_B)]$. This ensures (D.39) holds.

(e) In both cases $C = A$ and $C \neq A$, we have $\boldsymbol{g}_{AC}|_{(\boldsymbol{t}_C(\gamma)\circ\boldsymbol{j}_{BC})(\mathbf{Z}_{BC})} = \mathbf{0}$ whenever $B \in I$ with $C \subsetneq B$, $B \not\subseteq A$, $\mathfrak{X}_B \cap \mathfrak{X}_C \neq \emptyset$ and $\gamma \in \Gamma_C$. This ensures (D.40) holds.

(f) $\boldsymbol{g}_{AC}$ extends to a 1-morphism $\mathbf{U} \to \mathbb{R}^{n_A}$ for some open neighbourhood $\mathbf{U}$ of the closure of $\mathbf{Z}_C$ in $\hat{\mathbf{Z}}^{a_{c_1}}$.

(g) When $C \neq A$, $\boldsymbol{g}_{AC} \circ \boldsymbol{j}_{AC}$ is 2-isomorphic to $\boldsymbol{h}_A|_{\mathbf{Z}_{AC}}$, and this also holds in a neighbourhood of the closure of $\mathbf{Z}_{AC}$ in $\hat{\mathbf{Z}}^{a_{c_1}}$, noting that $\boldsymbol{j}_{AC}$ extends to such a neighbourhood. Since $\boldsymbol{j}_{AC} : \mathbf{Z}_{AC} \to \boldsymbol{j}_{AC}(\mathbf{Z}_{AC}) \subseteq \mathbf{Z}_C$ is an equivalence, this determines $\boldsymbol{g}_{AC}|_{\boldsymbol{j}_{AC}(\mathbf{Z}_{AC})}$ uniquely up to 2-isomorphism.

(h) When $C = A$, $\boldsymbol{g}_{AA}$ is 2-isomorphic to $\boldsymbol{h}_A|_{\mathbf{Z}_A}$.

In a similar way to the choice of the ϵ_{AC} in the proof of Step 2(iii), one can show using previous inductive steps that these conditions on $\boldsymbol{g}_{AC}$ are consistent on overlaps, so at each inductive step we can choose $\boldsymbol{g}_{AC}$ satisfying (a)–(h). Hence by induction we can choose $\boldsymbol{g}_{AC}$ for all A, C such that (D.36)–(D.40) hold.

By a similar argument we can choose the $\boldsymbol{d}_{Bc} : \mathbf{Z}_B \to \mathbf{(0,\infty)}$ for all B, c by induction on increasing $|B|$ such that (D.35), (D.41) and (D.42) hold. Let $B \in I$, and suppose we have chosen $\boldsymbol{d}_{Cc'}$ for all $C \in I$ with $|C|^2 < |B|$ and $c' \in C_C$. Then in the inductive step we must choose $\boldsymbol{d}_{Bc}$ for $c \in C_B$. Equation (D.42) implies that we must have

$$\boldsymbol{d}_{Bc}|_{\mathbf{W}_{Bc}} = \boldsymbol{b}_{Bc}|_{\mathbf{W}_{Bc}}^{-1}, \;\text{ where }\; \mathbf{W}_{Bc} = \bigcup_{A \in I: A<B,\; \mathfrak{X}_A \cap \mathfrak{X}_B \neq \emptyset,\; c \in C_B \setminus q_{AB}(C_A)} \boldsymbol{j}_{AB}(\mathbf{Z}_{AB}).$$

Here $\mathbf{W}_{Bc}$ is open in $\mathbf{Z}_B$. If $c \in C_B \setminus q_{AB}(C_A)$ then $\partial(\boldsymbol{j}_{AB}(\mathbf{Z}_{AB})) \cap \partial_c \mathbf{Z}_B = \emptyset$, so $\partial \mathbf{W}_{Bc} \cap \partial_c \mathbf{Z}_B = \emptyset$. But $\boldsymbol{b}_{Bc} : \mathbf{Z}_B \to \mathbf{[0,\infty)}$ is zero exactly on $\boldsymbol{i}_{\mathbf{Z}_B}(\partial_c \mathbf{Z}_B)$. Hence $\boldsymbol{b}_{Bc} > 0$ on $\mathbf{W}_{Bc}$, and $\boldsymbol{b}_{Bc}|_{\mathbf{W}_{Bc}}^{-1}$ is well-defined.

Let $C \in I$ with $B < C$ and $\mathfrak{X}_B \cap \mathfrak{X}_C \neq \emptyset$, and let $c \in C_B$. Then (D.41) essentially determines $\boldsymbol{d}_{Bc}|_{\mathbf{Z}_{BC}}$, noting that $\boldsymbol{d}_{Cq_{BC}(c)}$ is already chosen in a previous inductive step. Write $\mathbf{Z}_{BC}^c$ for the open d-submanifold in $\mathbf{Z}_{BC}$ where $\boldsymbol{b}_{Bc} > 0$. Then (D.41) forces

$$\boldsymbol{d}_{Bc}|_{\mathbf{Z}_{BC}^c} = \boldsymbol{b}_{Bc}|_{\mathbf{Z}_{BC}^c}^{-1} \cdot \big(\boldsymbol{d}_{Cq_{BC}(c)} \cdot \boldsymbol{b}_{Cq_{BC}(c)}\big) \circ \boldsymbol{j}_{BC}|_{\mathbf{Z}_{BC}^c}.$$

It is not immediately obvious whether (D.41) can be solved for $\boldsymbol{d}_{Bc}|_{\mathbf{Z}_{BC}}$ near points $z \in \mathbf{Z}_{BC}$ with $\boldsymbol{b}_{Bc}(z) = 0$, since $\boldsymbol{b}_{Bc}|_{\mathbf{Z}_{BC}}^{-1}$ does not make sense at such z.

To prove that it can be solved, note that $z = \boldsymbol{i}_{\mathbf{Z}_B}(z')$ for a unique $z' \in \partial_c \mathbf{Z}_B$, and $\boldsymbol{j}_{BC}(z) = \boldsymbol{i}_{\mathbf{Z}_C}(z'')$ for a unique $z'' \in \partial_{q_{BC}(c)} \mathbf{Z}_C$, and (z', z'') lies in $\underline{S}_{\boldsymbol{j}_{BC}}$. As $(\boldsymbol{Z}_C, \boldsymbol{b}_{Cq_{BC}(c)})$ is a boundary defining function for $\mathbf{Z}_C$ at z'' and $\boldsymbol{d}_{Cq_{BC}(c)} : \boldsymbol{Z}_C \to \mathbf{(0,\infty)}$, Proposition 6.6(d) implies that $(\boldsymbol{Z}_C, \boldsymbol{d}_{Cq_{BC}(c)} \cdot \boldsymbol{b}_{Cq_{BC}(c)})$ is a boundary defining function for $\mathbf{Z}_C$ at z''. As $(z', z'') \in \underline{S}_{\boldsymbol{j}_{BC}}$, Definition 6.2(i) implies that $(\boldsymbol{W}, \big(\boldsymbol{d}_{Cq_{BC}(c)} \cdot \boldsymbol{b}_{Cq_{BC}(c)}\big) \circ \boldsymbol{j}_{BC}|_{\boldsymbol{W}})$ is a boundary defining function for $\mathbf{Z}_B$ at z' for some open neighbourhood $\boldsymbol{W}$ of z in $\boldsymbol{Z}_{BC} \subseteq \boldsymbol{Z}$.

Now $(\boldsymbol{Z}_B, \boldsymbol{b}_{Bc})$ is also a boundary defining function for $\mathbf{Z}_B$ at z'. Therefore Proposition 6.6(c) shows that there exists an open neighbourhood $\boldsymbol{W}'$ of z in $\boldsymbol{W}$ and a 1-morphism $\boldsymbol{e} : \boldsymbol{W}' \to \mathbf{(0,\infty)}$ such that $\boldsymbol{e} \cdot \boldsymbol{b}_{Bc}|_{\boldsymbol{W}'} = \big(\boldsymbol{d}_{Cq_{BC}(c)} \cdot \boldsymbol{b}_{Cq_{BC}(c)}\big) \circ$

$\boldsymbol{j}_{BC}|_{\boldsymbol{W}'}$. This $\boldsymbol{e}$ is a possible choice for $\boldsymbol{d}_{Bc}$ near z in $\mathbf{Z}_{BC}$ satisfying (D.41). Thus, (D.41) is solvable for $\boldsymbol{d}_{Bc}$ near each $z \in \mathbf{Z}_{BC}$, and the solution is unique on the open subset $\mathbf{Z}^c_{BC} \subseteq \mathbf{Z}_{BC}$. By combining local choices for $\boldsymbol{d}_{Bc}$ with a partition of unity, we can find a global choice for $\boldsymbol{d}_{Bc}$ on $\mathbf{Z}_{BC}$ satisfying (D.41).

We have shown that in the inductive step when we choose $\boldsymbol{d}_{Bc} : \mathbf{Z}_B \to \boldsymbol{(0,\infty)}$, to satisfy (D.42) we must have $\boldsymbol{d}_{Bc}|_{\mathbf{W}_{Bc}} = \boldsymbol{b}_{Bc}|^{-1}_{\mathbf{W}_{Bc}}$ for $\mathbf{W}_{Bc} \subseteq \mathbf{Z}_B$ open, and to satisfy (D.41) we must prescribe $\boldsymbol{d}_{Bc}|_{\mathbf{Z}_{BC}}$ for each $C \in I$ with $B < C$ and $\mathfrak{X}_B \cap \mathfrak{X}_C \neq \boldsymbol{\emptyset}$, uniquely on $\mathbf{Z}^c_{BC} \subseteq \mathbf{Z}_{BC}$. As in the induction for $\boldsymbol{g}_{AC}$, previous steps imply these conditions are consistent on overlaps, so we can choose $\boldsymbol{d}_{Bc}$ globally on $\mathbf{Z}_B$ satisfying (D.41) and (D.42). Do this for all $c \in C_B$. If the resulting family of $\boldsymbol{d}_{Bc}$ do not satisfy the Γ_B-equivariance condition (D.35), then average them over the Γ_B-action, and they will. Hence by induction we can choose $\boldsymbol{d}_{Bc}$ for all B, c to satisfy (D.35), (D.41) and (D.42).

We have now proved that we can choose $\boldsymbol{f}_B$ to satisfy Steps 3(ii)(a),(b) and 3(iii)(d) in §D.1. Furthermore part (h) and the definition of $\boldsymbol{h}_C$ implies that $\boldsymbol{g}_{CC}$ is an embedding on $\mathbf{Z}_C \setminus \overline{\mathbf{W}}_C$, and part (g) implies that $\boldsymbol{g}_{AC}$ is an embedding on $\boldsymbol{j}_{AC}(\mathbf{Z}_{AC} \setminus \overline{\mathbf{W}}_A)$, and furthermore $\boldsymbol{j}_{AC}$ extends to $\tilde{\boldsymbol{\jmath}}_{AC} : \mathbf{U}_{AC} \to \mathbf{Z}_C$ for some open neighbourhood $\mathbf{U}_{AC}$ of the closure of $\mathbf{Z}_{AC}$ in $\mathbf{Z}_A$, and $\boldsymbol{g}_{AC}$ is an embedding on $\tilde{\boldsymbol{\jmath}}_{AC}(\mathbf{U}_{AC} \setminus \overline{\mathbf{W}}_A)$.

Now $\mathbf{Z}_C \setminus \overline{\mathbf{W}}_C$ together with $\boldsymbol{t}_C(\gamma)\bigl[\tilde{\boldsymbol{\jmath}}_{AC}(\mathbf{U}_{AC} \setminus \overline{\mathbf{W}}_A)\bigr]$ for all $A \in I$ with $C \subsetneq A$ and $\mathfrak{X}_A \cap \mathfrak{X}_C \neq \boldsymbol{\emptyset}$ and $\gamma \in \Gamma_C$ form an open cover of $\mathbf{Z}_C$. Hence $\mathbf{Z}_C$ is covered by open d-submanifolds on which one of the terms $(\boldsymbol{g}_{AC} \circ \boldsymbol{t}_C(\gamma^{-1})) \otimes |\gamma\rangle$ in the definition (D.34) of $\boldsymbol{f}_C$ is an embedding. Therefore $\boldsymbol{f}_C$ is an immersion. But using the condition that $\boldsymbol{h}_A|_{\hat{\mathbf{Z}}^{a_1} \setminus \overline{\mathbf{W}}_A}$ maps to $\mathbb{R}^{n_A} \setminus \{0\}$ above we can show that $\boldsymbol{f}_C$ is injective, so $\boldsymbol{f}_C$ is an embedding. This completes Step 3.

D.5 Step 4: Construct the good coordinate system

For part (i), let $A \in I$, so that Step 3(ii) gives a manifold with corners U_A and an sf-embedding $\boldsymbol{f}_A : \mathbf{Z}_A \to \mathbf{U}_A$, which is equivariant under the actions $\boldsymbol{t}_A, \tilde{\boldsymbol{r}}_A$ of Γ_A on $\mathbf{Z}_A, \mathbf{U}_A$. Furthermore, writing $A = \{a_1, \ldots, a_n\}$ with $a_1 \prec \cdots \prec a_n$, so that $\mathbf{Z}_A \subseteq \hat{\mathbf{Z}}^{a_1}$ is open with compact closure $\overline{\mathbf{Z}}_A$ in $\hat{\mathbf{Z}}^{a_1}$, the embedding $\boldsymbol{f}_A$ extends to an open neighbourhood of $\overline{\mathbf{Z}}_A$ in $\hat{\mathbf{Z}}^{a_1}$.

Apply Theorem 7.48 to the sf-embedding $\boldsymbol{f}_A : \mathbf{Z}_A \to \mathbf{U}_A$. This implies that there exist an open subset V_A in U_A with $\boldsymbol{f}_A(\mathbf{Z}_A) \subseteq \mathbf{V}_A$, a vector bundle $E_A \to V_A$, a smooth section $s_A : V_A \to E_A$ of E, and a 2-morphism θ_A fitting into a 2-Cartesian diagram (D.11) in $\mathbf{dMan^c}$. The proof of Theorem 7.48 is easily extended to work equivariantly under the Γ_A-actions on $\mathbf{Z}_A, U_A$, as $\boldsymbol{f}_A$ is Γ_A-equivariant. Thus we may choose V_A, E_A, s_A, θ_A to that V_A is Γ_A-invariant, and there is a lift $\hat{r}_A : \Gamma_A \to \mathrm{Aut}(E_A)$ of the Γ_A-action on V_A to E_A, so that s_A, θ_A are Γ_A-equivariant. Since $\boldsymbol{f}_A$ extends to an open neighbourhood of $\overline{\mathbf{Z}}_A$ in $\hat{\mathbf{Z}}^{a_1}$, we can also choose $E_A, s_A, \theta_A, \hat{r}_A$ to extend to a Γ_A-invariant open neighbourhood of $\overline{\mathbf{Z}}_A$ in $\hat{\mathbf{Z}}^{a_1}$. This proves Step 4(i)(a)–(e).

Now both $\mathbf{Z}_A$ and $\mathbf{S}_{V_A,E_A,s_A}$ are fibre products $\mathbf{V}_A \times_{\boldsymbol{s}_A,\mathbf{E}_A,\mathbf{0}} \mathbf{V}_A$, where the projections to the first factor are $\boldsymbol{f}_A : \mathbf{Z}_A \to \mathbf{V}_A$ and $\mathbf{S}_{\mathrm{id}_{V_A},0} : \mathbf{S}_{V_A,E_A,s_A} \to \mathbf{S}_{V_A,0,0} = \mathbf{V}_A$. Hence we can choose an equivalence $\boldsymbol{k}_A : \mathbf{S}_{V_A,E_A,s_A} \to \mathbf{Z}_A$

and a 2-morphism $\xi_A : \boldsymbol{f}_B \circ \boldsymbol{k}_A \Rightarrow \mathbf{S}_{\mathrm{id}_{V_A},0}$. We can also choose $\boldsymbol{k}_A, \xi_A$ to be Γ_A-equivariant. Then $\boldsymbol{k}_A$ descends to $[\boldsymbol{k}_A, \mathrm{id}_{\Gamma_A}]$ as in (D.12), and we define $\boldsymbol{\psi}_A : [\mathbf{S}_{V_A,E_A,s_A}/\Gamma_A] \to \boldsymbol{\mathfrak{X}}_A$ by (D.13).

Then $(V_A, E_A, \Gamma_A, s_A, \boldsymbol{\psi}_A)$ is a type A Kuranishi neighbourhood on $\boldsymbol{\mathfrak{X}}$, by Definition 12.47. This completes part (i) of Step 4. During the next steps we may need to make V_A smaller, that is, we replace V_A by a Γ_A-invariant open neighbourhood V'_A of $\boldsymbol{f}_A(\mathbf{Z}_A)$ in V_A, and replace E_A, s_A, θ_A by $E'_A = E_A|_{V'_A}$, $s'_A = s_A|_{V'_A}$, $\theta'_A = \theta_A|_{V'_A}$. Although this take places during an induction argument with infinitely many steps, because of the finiteness condition Step 1(ii)(c) for each $A \in I$ we will need to make V_A smaller only finitely many times, so the final answer for V_A is well-defined.

For part (ii), let $A, B \in I$ with $A < B$ and $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B \neq \boldsymbol{\emptyset}$, and set $V_{AB} = V_A \cap l_{AB}^{-1}(V_B)$. Then V_{AB} is Γ_A-invariant, and $\boldsymbol{\psi}_A|_{\cdots} : [\mathbf{S}_{V_{AB},E_A|_{V_{AB}},s_A|_{V_{AB}}}/\Gamma_A] \to \boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B$ and $\boldsymbol{k}_A|_{\cdots} : \mathbf{S}_{V_{AB},E_A|_{V_{AB}},s_A|_{V_{AB}}} \to \mathbf{Z}_{AB}$ are equivalences. Consider the 2-commutative diagram in $\mathbf{dMan^c}$:

$$\begin{tikzcd}
\mathbf{S}_{V_{AB},E_A|_{V_{AB}},s_A|_{V_{AB}}} \arrow[r, "\boldsymbol{k}_A|_{\cdots}"'] \arrow[d, "\boldsymbol{h}_{AB}"] \arrow[rr, bend left, "\mathbf{S}_{\mathrm{id}_{V_{AB}},0}"] \arrow[dr, phantom, "\chi_{AB}\Uparrow"] & \mathbf{Z}_{AB} \arrow[r, "\boldsymbol{f}_A|_{\mathbf{Z}_{AB}}"'] \arrow[d, "\boldsymbol{j}_{AB}"] \arrow[u, phantom, "\xi_A|_{\cdots}\Uparrow"] \arrow[dr, phantom, "\mathrm{id}\Uparrow"] & \mathbf{V}_A \arrow[d, "\tilde{\boldsymbol{l}}_{AB}"] \\
\mathbf{S}_{V_B,E_B,s_B} \arrow[r, "\boldsymbol{k}_B"] \arrow[rr, bend right, "\mathbf{S}_{\mathrm{id}_{V_B},0}"'] & \mathbf{Z}_B \arrow[r, "\boldsymbol{f}_B"] \arrow[d, phantom, "\xi_B\Downarrow"] & \mathbf{V}_B.
\end{tikzcd} \tag{D.43}$$

The right hand square 2-commutes by Step 3(iii), and the upper and lower semicircles by the definition of $\boldsymbol{k}_A, \xi_A$ above. Since $\boldsymbol{k}_B$ is an equivalence we can choose a 1-morphism $\boldsymbol{h}_{AB}$ and 2-morphism χ_{AB} to make the left hand square 2-commute, where $\boldsymbol{h}_{AB}$ is unique up to 2-isomorphism, and as $\boldsymbol{k}_A|_{\ldots}, \boldsymbol{k}_B, \boldsymbol{j}_{AB}$ are equivariant under $\Gamma_A, \Gamma_B, \rho_{AB}$ we can also choose $\boldsymbol{h}_{AB}, \chi_{AB}$ to be equivariant.

Composing 2-morphisms across (D.43) gives a 2-morphism $\eta : \tilde{\boldsymbol{l}}_{AB} \circ \mathbf{S}_{\mathrm{id}_{V_{AB}},0} \Rightarrow \mathbf{S}_{\mathrm{id}_{V_B},0} \circ \boldsymbol{h}_{AB}$, so as in §2.2, η is a morphism $(\underline{S}_{\mathrm{id}_{V_B},0} \circ \underline{h}_{AB})^*(\mathcal{F}_{V_B}) \to \mathcal{E}_{S_{V_{AB},E_{AB}|\ldots,s_{AB}|\ldots}}$ in $\mathrm{qcoh}(\underline{S}_{V_{AB},E_{AB}|\ldots,s_{AB}|\ldots})$. There are natural isomorphisms

$$(\underline{S}_{\mathrm{id}_{V_B},0} \circ \underline{h}_{AB})^*(\mathcal{F}_{V_B}) \cong (\underline{S}_{\mathrm{id}_{V_B},0} \circ \underline{h}_{AB})^*(T^*\underline{V}_B) \cong \underline{h}_{AB}^*(\mathcal{F}_{S_{V_B,E_B,s_B}}).$$

Let $\eta' : \underline{h}_{AB}^*(\mathcal{F}_{S_{V_B,E_B,s_B}}) \to \mathcal{E}_{S_{V_{AB},E_{AB}|\ldots,s_{AB}|\ldots}}$ correspond to $-\eta$ under this isomorphism. Since η is a 2-morphism in $\mathbf{dMan^c}$, η' satisfies (6.9) and (6.10), so by Proposition 6.9 there is a unique 1-morphism $\boldsymbol{h}'_{AB} : \mathbf{S}_{V_{AB},E_A|_{V_{AB}},s_A|_{V_{AB}}} \to \mathbf{S}_{V_B,E_B,s_B}$ in $\mathbf{dMan^c}$ such that $\eta' : \boldsymbol{h}_{AB} \Rightarrow \boldsymbol{h}'_{AB}$ is a 2-morphism. Then (D.43) with $\boldsymbol{h}'_{AB}$ and $\chi'_{AB} = \chi_{AB} \odot (\mathrm{id}_{\boldsymbol{k}_B} * (-\eta'))$ in place of $\boldsymbol{h}_{AB}, \chi_{AB}$ 2-commutes, and has composition $(\mathrm{id}_{\mathbf{S}_{\mathrm{id}_{V_B},0}} * \eta') \odot \eta = 0 = \mathrm{id}$. This shows that we can choose $\boldsymbol{h}_{AB}, \chi_{AB}$ so that the composition of 2-morphisms (D.43) is the identity, so that $\mathbf{S}_{\mathrm{id}_{V_B},0} \circ \boldsymbol{h}_{AB} = \tilde{\boldsymbol{l}}_{AB} \circ \mathbf{S}_{\mathrm{id}_{V_{AB}},0}$, and this in fact determines $\boldsymbol{h}_{AB}$ uniquely. We choose $\boldsymbol{h}_{AB}, \chi_{AB}$ in this way.

Next we apply Theorem 7.19 to $\boldsymbol{h}_{AB} : \mathbf{S}_{V_{AB},E_A|_{V_{AB}},s_A|_{V_{AB}}} \to \mathbf{S}_{V_B,E_B,s_B}$. This shows that we may choose an open neighbourhood $\tilde{V}_{AB}$ of $s_A|_{V_{AB}}^{-1}(0)$ in

V_{AB}, a smooth map $e_{AB} : \tilde V_{AB} \to V_B$, and a morphism of vector bundles $\hat e_{AB} : E_A|_{\tilde V_{AB}} \to e^*_{AB}(E_B)$ with $\hat e_{AB} \circ s_A|_{\tilde V_{AB}} = e^*_{AB}(s_B)$, such that $\boldsymbol{h}_{AB} = \mathbf{S}_{e_{AB},\hat e_{AB}} \circ \boldsymbol{i}^{-1}_{\tilde V_{AB},V_{AB}}$. We want $\tilde V_{AB}, e_{AB}, \hat e_{AB}$ to satisfy some extra conditions:

(a) We should have $e_{AB} = l_{AB}|_{\tilde V_{AB}}$.

(b) $\tilde V_{AB}$ should be Γ_A-invariant, and $\hat e_{AB}$ should be ρ_{AB}-equivariant, that is, $\tilde r_A(\gamma)|^*_{\tilde V_{AB}}(\hat e_{AB}) \circ \hat r_A(\gamma)|_{\tilde V_{AB}} = e^*_{AB}(\hat r_B(\rho_{AB}(\gamma))) \circ \hat e_{AB}$ for all $\gamma \in \Gamma_A$.

(c) $\hat e_{AB}$ should be an embedding of vector bundles, that is, it has a left inverse.

(d) If $v, v' \in \tilde V_{AB}$ and $\delta \in \Gamma_B$ with $\tilde r_B(\delta) \circ e_{AB}(v') = e_{AB}(v)$, then there exists $\gamma \in \Gamma_A$ with $\rho_{AB}(\gamma) = \delta$ and $\tilde r_A(\gamma)(v') = v$.

(e) We should have $\tilde V_{AB} = V_{AB}$, so that $\boldsymbol{h}_{AB} = \mathbf{S}_{e_{AB},\hat e_{AB}}$.

Here is how to modify the construction to achieve this. For (a), the proof of Theorem 7.19 following that of Theorem 3.34 first chooses $\tilde V = \tilde V_{AB}$ and $f = e_{AB}$ such that (3.23) commutes. Since $\mathbf{S}_{\mathrm{id}_{V_B},0} \circ \boldsymbol{h}_{AB} = \tilde{\boldsymbol{l}}_{AB} \circ \mathbf{S}_{\mathrm{id}_{V_{AB}},0}$, equation (3.23) commutes with $f = l_{AB}|_{\tilde V_{AB}}$, so $l_{AB}|_{\tilde V_{AB}}$ is a possible choice for e_{AB}. For (b), we can make $\tilde V_{AB}$ Γ_A-invariant by replacing it by $\tilde V'_{AB} = \bigcap_{\gamma \in \Gamma_A} \tilde r_A(\gamma)[\tilde V_{AB}]$, and restricting $\hat e_{AB}$ to $\tilde V'_{AB}$. Having done this, we can make $\hat e_{AB}$ equivariant under ρ_{AB} by replacing it by $\hat e'_{AB} = \frac{1}{|\Gamma_A|}\sum_{\gamma\in\Gamma_A} e^*_{AB}(\hat r_B(\rho_{AB}(\gamma)))^{-1} \circ \tilde r_A(\gamma)^*(\hat e_{AB}) \circ \hat r_A(\gamma)$. Then $\hat e'_{AB}$ is ρ_{AB}-equivariant, satisfies $\hat e'_{AB} \circ s_A|_{\tilde V_{AB}} = e^*_{AB}(s_B)$, and is a possible choice in the proof of Theorem 7.19.

For (c), since $\boldsymbol{j}_{AB}$ is an equivalence with an open d-submanifold, so are $\boldsymbol{h}_{AB}$ and $\mathbf{S}_{e_{AB},\hat e_{AB}}$. Therefore Theorem 7.21 shows that for all $v \in \tilde V_{AB}$ with $s_{AB}(v) = 0$ and $w = e_{AB}(v)$, the following sequence is exact:

$$0 \longrightarrow T_v V_A \xrightarrow{\mathrm{d}s_A(v) \oplus \mathrm{d}e_{AB}(v)} E_A|_v \oplus T_w V_B \xrightarrow{\hat e_{AB}(v) \oplus -\mathrm{d}s_B(w)} E_B|_{v_B} \longrightarrow 0. \quad \text{(D.44)}$$

As $e_{AB} = l_{AB}|_{\tilde V_{AB}}$ and l_{AB} is injective, $\mathrm{d}e_{AB}(v)$ is injective, so exactness implies that $\hat e_{AB}(v) : E_A|_v \to E_B|_w$ is injective. This is an open condition, so $\hat e_{AB}$ is injective (has a left inverse) on an open neighbourhood $\tilde V'_{AB}$ of $s_A|^{-1}_{\tilde V_{AB}}(0)$. Replacing $\tilde V_{AB}$ by a Γ_A-invariant choice of $\tilde V'_{AB}$ proves (c).

For (d), note that using Step 1(iv)(b) and the definition of $\mathbf{Z}_{AB}$ in Step 2 we can show that $\boldsymbol{j}_{AB}(\mathbf{Z}_{AB}) \cap \tilde{\boldsymbol{r}}_B(\delta)[\boldsymbol{j}_{AB}(\mathbf{Z}_{AB})] = \boldsymbol{\emptyset}$ for all $\delta \in \Gamma_B \setminus \rho_{AB}(\Gamma_A)$, where the intersection is in $\mathbf{Z}_B$. Since $\boldsymbol{f}_B$ identifies $\mathbf{Z}_B$ with $\{w \in V_B : s_B(w) = 0\}$ and $\boldsymbol{j}_{AB}(\mathbf{Z}_{AB})$ with $\{e_{AB}(v) : v \in \tilde V_{AB},\ s_A(v) = 0\}$, we see that if $v, v' \in \tilde V_{AB}$ with $s_A(v) = s_A(v') = 0$ and $\delta \in \Gamma_B$ with $\tilde r_B(\delta) \circ e_{AB}(v') = e_{AB}(v)$, then there exists $\gamma \in \Gamma_A$ with $\rho_{AB}(\gamma) = \delta$ and $\tilde r_A(\gamma)(v') = v$. This is an open condition, so it also holds in an open neighbourhood of $s_A|^{-1}_{\tilde V_{AB}}(0)$ in $\tilde V_{AB}$. Thus by making $\tilde V_{AB}$ smaller, we can make (d) hold.

For (e), the idea is to replace V_B by a Γ_B-invariant open neighbourhood V'_B of $s_B^{-1}(0)$ in V_B with the property that $V_A \cap l^{-1}_{AB}(V'_B) \subseteq \tilde V_{AB}$. Then we can replace E_B by $E'_B = E_B|_{V'_B}$, s_B by $s'_B = s_B|_{V'_B}$, $\tilde V_{AB}$ by $V'_{AB} = V_A \cap l^{-1}_{AB}(V'_B)$, e_{AB} by $e'_{AB} = e_{AB}|_{V'_{AB}}$, and $\hat e_{AB}$ by $\hat e'_{AB} = \hat e_{AB}|_{V'_{AB}}$, and these new choices satisfy (a)–(e), as we want.

To show that such a V'_B exists, let $w \in s_B^{-1}(0)$, and consider the three cases:

(A) $w \in e_{AB}(s_A|_{\tilde{V}_{AB}}^{-1}(0))$,

(B) $w \in s_B^{-1}(0) \setminus \overline{e_{AB}(s_A|_{\tilde{V}_{AB}}^{-1}(0))}$, and

(C) $w \in \overline{e_{AB}(s_A|_{\tilde{V}_{AB}}^{-1}(0))} \setminus e_{AB}(s_A|_{\tilde{V}_{AB}}^{-1}(0))$,

where $\overline{e_{AB}(s_A|_{\tilde{V}_{AB}}^{-1}(0))}$ is the closure of $e_{AB}(s_A|_{\tilde{V}_{AB}}^{-1}(0))$ in $s_B^{-1}(0) \subseteq V_B$.

In case (A), as l_{AB} is an embedding, there is an open neighbourhood U of w in V_B such that $V_A \cap l_{AB}^{-1}(U) \subseteq \tilde{V}_{AB}$. In case (B), as l_{AB} is continuous, there is an open neighbourhood U of w in V_B such that $V_A \cap l_{AB}^{-1}(U) = \emptyset \subseteq \tilde{V}_{AB}$. In case (C), the fact that $\tilde{V}_{AB}$ is an open neighbourhood of $s_A|_{V_{AB}}^{-1}(0)$ in V_{AB} does not guarantee that w has an open neighbourhood U in V_B with $V_A \cap l_{AB}^{-1}(U) \subseteq \tilde{V}_{AB}$. However, as above $\boldsymbol{f}_A$ extends to an open neighbourhood of $\bar{\mathbf{Z}}_A$ in $\hat{\mathbf{Z}}^{a_1}$, and similarly $\boldsymbol{f}_B, \boldsymbol{j}_{AB}$ extend to open neighbourhoods of $\bar{\mathbf{Z}}_A, \bar{\mathbf{Z}}_{AB}$. So we can suppose $V_A, \ldots, \tilde{V}_{AB}, e_{AB}$ have extensions $\dot{V}_A, \ldots, \dot{\tilde{V}}_{AB}, \dot{e}_{AB}$ with $\overline{e_{AB}(s_A|_{\tilde{V}_{AB}}^{-1}(0))} \subseteq \dot{e}_{AB}(\dot{s}_A|_{\dot{\tilde{V}}_{AB}}^{-1}(0))$. Then (C) follows from (A). We can now take V_B' to be the union of such a neighbourhood U for each $w \in s_B^{-1}(0)$.

Our construction of $V_{AB}, e_{AB}, \hat{e}_{AB}$ involves making V_B smaller, so we should check that this can be done consistently with all other choices. We do this by an inductive procedure, working by transfinite induction on $A \in I$ in the order $<$, which is valid as $(I, <)$ is a well-ordered set. At the inductive step A we choose $V_{AB}, e_{AB}, \hat{e}_{AB}$ for all $B \in I$ with $A < B$ and $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B \neq \boldsymbol{\emptyset}$, as above, and making V_B smaller if necessary. Making V_B smaller also modifies $V_{A'B}, e_{A'B}, \hat{e}_{A'B}$ for $A' < A$ with $\boldsymbol{\mathfrak{X}}_{A'} \cap \boldsymbol{\mathfrak{X}}_B \neq \boldsymbol{\emptyset}$, but this is not a problem. At step A we only modify V_B for $B > A$, so at step A, $V_{A'}$ is in its final form for all $A' \leqslant A$. Step 1(ii)(c) implies that for any $B \in I$ there are only finitely many $A \in I$ with $A < B$ and $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B \neq \boldsymbol{\emptyset}$. Hence for any $B \in I$, we only make V_B smaller finitely many times during the induction. Thus the process works, and by induction we can choose $V_A, E_A, s_A, \hat{r}_A, V_{AB}, e_{AB}, \hat{e}_{AB}$ satisfying Step 4(i), the first part of Step 4(ii), and (a)–(e) above.

Suppose $\hat{e}_{AB}$ and $\hat{e}'_{AB}$ are two possible choices for $\hat{e}_{AB}$. Then $\mathbf{S}_{e_{AB}, \hat{e}'_{AB}} = \boldsymbol{h}_{AB} = \mathbf{S}_{e_{AB}, \hat{e}_{AB}}$, so Lemma 7.16 gives $\hat{e}'_{AB} = \hat{e}_{AB} + O(s_A)$. Conversely, if $\hat{e}_{AB}$ is as above and $\hat{e}'_{AB}$ satisfies $\hat{e}'_{AB} = \hat{e}_{AB} + O(s_A)$ and is ρ_{AB}-equivariant and injective, then $\hat{e}'_{AB}$ is an alternative choice for $\hat{e}_{AB}$. We will use the freedom to change $\hat{e}_{AB}$ by an $O(s_A)$ term in the proof of part (iii).

For all such A, B, define a 2-morphism $\boldsymbol{\eta}_{AB} : \boldsymbol{\psi}_B \circ [\mathbf{S}_{e_{AB}, \hat{e}_{AB}}, \rho_{AB}] \Rightarrow \boldsymbol{\psi}_A|_{[\mathbf{S}_{V_{AB}, E_A|_{V_{AB}}, s_A|_{V_{AB}}}/\Gamma_A]}$ to be the composition across the diagram

$$\begin{tikzcd}
{[\mathbf{S}_{V_{AB}, E_A|_{V_{AB}}, s_A|_{V_{AB}}}/\Gamma_A]} \arrow[r, "{[\boldsymbol{k}_A|_{\ldots}, \mathrm{id}_{\Gamma_A}]}"'] \arrow[d, "{[\mathbf{S}_{e_{AB}, \hat{e}_{AB}}, \rho_{AB}]}"] \arrow[dr, phantom, "{[\chi_{AB}, 1] \Uparrow}"] & {[\mathbf{Z}_{AB}/\Gamma_A]} \arrow[rd, "\boldsymbol{i}_A|_{\ldots}"] \arrow[d, "{[\boldsymbol{j}_{AB}, \rho_{AB}]}"] \arrow[rd, phantom, "\boldsymbol{\zeta}_{AB} \Uparrow"'] & \\
{[\mathbf{S}_{V_B, E_B, s_B}/\Gamma_B]} \arrow[r, "{[\boldsymbol{k}_B, \mathrm{id}_{\Gamma_B}]}"] & {[\mathbf{Z}_B/\Gamma_B]} \arrow[r, "\boldsymbol{i}_B"'] & \boldsymbol{\mathfrak{X}},
\end{tikzcd}$$

where χ_{AB} comes from (D.43), and $\boldsymbol{\zeta}_{AB}$ from Step 2(ii).

The material so far now shows $(V_{AB}, e_{AB}, \hat{e}_{AB}, \rho_{AB}, \boldsymbol{\eta}_{AB})$ satisfies the corners analogues of Definition 10.46(a)–(g), where the second part of Definition

10.46(c) is (d) above, and Definition 10.46(e) follows from (D.44) exact and $\mathrm{d}e_{AB}(v), \hat{e}_{AB}(v)$ injective, and Definition 10.46(g) from $V_{AB} = V_A \cap l_{AB}^{-1}(V_B)$, as then $V_A \amalg_{V_{AB}} V_B$ is homeomorphic to $l_{AB}(V_A) \cup V_B$, where the union is in U_B, and so is Hausdorff. Therefore $(V_{AB}, e_{AB}, \hat{e}_{AB}, \rho_{AB}, \boldsymbol{\eta}_{AB})$ is a type A coordinate change from $(V_A, E_A, \Gamma_A, s_A, \boldsymbol{\psi}_A)$ to $(V_B, E_B, \Gamma_B, s_B, \boldsymbol{\psi}_B)$. This completes Step 4(ii).

For part (iii), let $A, B, C \in I$ with $A < B < C$ and $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B \cap \boldsymbol{\mathfrak{X}}_C \neq \emptyset$. Then using (D.7) and the 2-morphisms $\chi_{AB}, \chi_{AC}, \chi_{BC}$ from (D.44) for AB, AC, BC we can write down a 2-morphism

$$\begin{aligned} \omega_{ABC} : \mathbf{S}_{\tilde{r}_C(\gamma_{ABC}), \hat{r}_C(\gamma_{ABC})} \circ \boldsymbol{h}_{BC} \circ \boldsymbol{h}_{AB}|_{\mathbf{S}_{V_{AC} \cap e_{AB}^{-1}(V_{BC}), E_A|\dots, s_A|\dots}} \\ \Longrightarrow \boldsymbol{h}_{AC}|_{\mathbf{S}_{V_{AC} \cap e_{AB}^{-1}(V_{BC}), E_A|\dots, s_A|\dots}}. \end{aligned} \tag{D.45}$$

Using the identities $\mathbf{S}_{\mathrm{id}_{V_B},0} \circ \boldsymbol{h}_{AB} = \tilde{\boldsymbol{l}}_{AB} \circ \mathbf{S}_{\mathrm{id}_{V_{AB}},0}$ from above and $l_{AC} = r_C(\gamma_{ABC}) \circ l_{BC} \circ l_{AB}$ from Step 3(iv) we find that

$$\begin{aligned} &\mathbf{S}_{\mathrm{id}_{V_C},0} \circ \mathbf{S}_{\tilde{r}_C(\gamma_{ABC}), \hat{r}_C(\gamma_{ABC})} \circ \boldsymbol{h}_{BC} \circ \boldsymbol{h}_{AB}|_{\mathbf{S}_{V_{AC} \cap e_{AB}^{-1}(V_{BC}), E_A|\dots, s_A|\dots}} \\ &\quad = \tilde{\boldsymbol{r}}_C(\gamma_{ABC}) \circ \mathbf{S}_{\mathrm{id}_{V_C},0} \circ \boldsymbol{h}_{BC} \circ \boldsymbol{h}_{AB}|_{\mathbf{S}_{V_{AC} \cap e_{AB}^{-1}(V_{BC}), E_A|\dots, s_A|\dots}} \\ &\quad = \tilde{\boldsymbol{r}}_C(\gamma_{ABC}) \circ \tilde{\boldsymbol{l}}_{BC} \circ \mathbf{S}_{\mathrm{id}_{V_B},0} \circ \boldsymbol{h}_{AB}|_{\mathbf{S}_{V_{AC} \cap e_{AB}^{-1}(V_{BC}), E_A|\dots, s_A|\dots}} \\ &\quad = \tilde{\boldsymbol{r}}_C(\gamma_{ABC}) \circ \tilde{\boldsymbol{l}}_{BC} \circ \tilde{\boldsymbol{l}}_{AB} \circ \mathbf{S}_{\mathrm{id}_{V_{AC} \cap e_{AB}^{-1}(V_{BC})},0} \\ &\quad = \tilde{\boldsymbol{l}}_{AC} \circ \mathbf{S}_{\mathrm{id}_{V_{AC} \cap e_{AB}^{-1}(V_{BC})},0} = \mathbf{S}_{\mathrm{id}_{V_C},0} \circ \boldsymbol{h}_{AC}|_{\mathbf{S}_{V_{AC} \cap e_{AB}^{-1}(V_{BC}), E_A|\dots, s_A|\dots}}. \end{aligned} \tag{D.46}$$

In (D.43) we chose $\boldsymbol{h}_{AC}$ to be the unique 1-morphism in its 2-isomorphism class with $\mathbf{S}_{\mathrm{id}_{V_C},0} \circ \boldsymbol{h}_{AC} = \tilde{\boldsymbol{l}}_{AC} \circ \mathbf{S}_{\mathrm{id}_{V_{AC}},0}$. Thus (D.45)–(D.46) imply that

$$\begin{aligned} &\boldsymbol{h}_{AC}|_{\mathbf{S}_{V_{AC} \cap e_{AB}^{-1}(V_{BC}), E_A|\dots, s_A|\dots}} = \\ &\qquad \mathbf{S}_{\tilde{r}_C(\gamma_{ABC}), \hat{r}_C(\gamma_{ABC})} \circ \boldsymbol{h}_{BC} \circ \boldsymbol{h}_{AB}|_{\mathbf{S}_{V_{AC} \cap e_{AB}^{-1}(V_{BC}), E_A|\dots, s_A|\dots}}, \end{aligned}$$

an analogue of (D.7). Substituting $\boldsymbol{h}_{AB} = \mathbf{S}_{e_{AB}, \hat{e}_{AB}}$, etc., gives

$$\mathbf{S}_{e_{AC}|_{V_{AC} \cap e_{AB}^{-1}(V_{BC})}, \hat{e}_{AC}|\dots} = \mathbf{S}_{\substack{\tilde{r}_C(\gamma_{ABC}) \circ e_{BC} \circ e_{AB}|_{V_{AC} \cap e_{AB}^{-1}(V_{BC})}, \\ (e_{AB}^*(e_{BC}^*(\hat{r}_C(\gamma_{ABC}))) \circ e_{AB}^*(\hat{e}_{BC}) \circ \hat{e}_{AB})|\dots}}.$$

Thus Lemma 7.16 implies that

$$\begin{aligned} &\hat{e}_{AC}|_{V_{AC} \cap e_{AB}^{-1}(V_{BC})} = \\ &\qquad \bigl(e_{AB}^*(e_{BC}^*(\hat{r}_C(\gamma_{ABC}))) \circ e_{AB}^*(\hat{e}_{BC}) \circ \hat{e}_{AB}\bigr)|_{V_{AC} \cap e_{AB}^{-1}(V_{BC})} + O(s_A), \end{aligned}$$

which apart from the error term $O(s_A)$ is equation (D.14).

In the proof of Step 2(iii), our definition of $\boldsymbol{j}_{AB}$ involved a 2-morphism $\epsilon_{AB} : \tilde{\boldsymbol{j}}_{AB} \Rightarrow \boldsymbol{j}_{AB}$, and we chose the ϵ_{AB} by induction on $|A| - |B|$ to satisfy some conditions which ensured that (D.7) holds exactly, not just up to 2-isomorphism.

In a very similar way, we can arrange to modify the $\hat{e}_{AB}$ by induction on $|A|-|B|$ so that (D.14) holds exactly, not just up to errors $O(s_A)$.

In the inductive step when we modify $\hat{e}_{AC}$, if $B \in I$ with $A < B < C$ and $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B \cap \boldsymbol{\mathfrak{X}}_C \neq \boldsymbol{\emptyset}$ then $|A| - |B|, |B| - |C| < |A| - |C|$, so we have already chosen the final values of $\hat{e}_{AB}, \hat{e}_{BC}$ in previous inductive steps. We choose a new value of $\hat{e}_{AC}$ such that (D.14) holds exactly for each such B. This prescribes $\hat{e}_{AC}$ on $V_{AC} \cap e_{AB}^{-1}(V_{BC})$. From above, the new and old values of $\hat{e}_{AC}$ differ by $O(s_A)$, so the new value of $\hat{e}_{AC}$ is also an allowed choice. In a similar way to Step 2(iii), the prescribed values agree on overlaps between $V_{AC} \cap e_{AB}^{-1}(V_{BC})$ and $V_{AC} \cap e_{AB'}^{-1}(V_{B'C})$ for different B, B'. Using ideas above on extensions to closures, and making the V_A smaller if necessary, we can show the induction works. This completes part (iii).

We have now constructed data $\bigl(I, <, (V_A, E_A, \Gamma_A, s_A, \boldsymbol{\psi}_A), (V_{AB}, e_{AB}, \hat{e}_{AB}, \rho_{AB}, \boldsymbol{\eta}_{AB}), \gamma_{ABC}\bigr)$ satisfying the corners analogues of Definition 10.47(a)–(d). To make it satisfy (e), note that it follows from Step 2 that if $A < B < C$ in I with $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_C \neq \boldsymbol{\emptyset}$ and $\boldsymbol{\mathfrak{X}}_B \cap \boldsymbol{\mathfrak{X}}_C \neq \boldsymbol{\emptyset}$ and $z \in \mathbf{Z}_{AC}$, $z' \in \mathbf{Z}_{BC}$, $\delta \in \Gamma_C$ with $\boldsymbol{j}_{BC}(v') = \boldsymbol{t}_C(\delta) \circ \boldsymbol{j}_{AC}(v)$ in $\mathbf{Z}_C$, then $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B \cap \boldsymbol{\mathfrak{X}}_C \neq \boldsymbol{\emptyset}$, and $v \in \mathbf{Z}_{AB}$, and there exists $\gamma \in \Gamma_B$ with $\rho_{BC}(\gamma) = \delta\, \gamma_{ABC}$ and $v' = \boldsymbol{t}_B(\gamma) \circ \boldsymbol{j}_{AB}(v)$.

Since $\boldsymbol{f}_B$ identifies $\mathbf{Z}_B$ with $\{w \in V_B : s_B(w) = 0\}$ and $\boldsymbol{j}_{AB}(\mathbf{Z}_{AB})$ with $\{e_{AB}(v) : v \in \tilde{V}_{AB},\ s_A(v) = 0\}$, and so on, this implies that if $A < B < C$ in I with $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_C \neq \boldsymbol{\emptyset}$ and $\boldsymbol{\mathfrak{X}}_B \cap \boldsymbol{\mathfrak{X}}_C \neq \boldsymbol{\emptyset}$ and $v \in V_{AC}$, $v' \in V_{BC}$ with $s_A(v) = s_B(v') = 0$ and $\delta \in \Gamma_C$ with $e_{BC}(v') = \tilde{r}_C(\delta) \circ e_{AC}(v)$ in V_C, then $\boldsymbol{\mathfrak{X}}_A \cap \boldsymbol{\mathfrak{X}}_B \cap \boldsymbol{\mathfrak{X}}_C \neq \boldsymbol{\emptyset}$, and $v \in V_{AB}$, and there exists $\gamma \in \Gamma_B$ with $\rho_{BC}(\gamma) = \delta\, \gamma_{ABC}$ and $v' = \tilde{r}_B(\gamma) \circ e_{AB}(v)$. That is, Definition 10.47(e) holds provided v, v' satisfy the extra conditions $s_A(v) = s_B(v') = 0$. As this is an open condition, replacing V_A, V_B by open neighbourhoods V'_A, V'_B of $s_A^{-1}(0), s_B^{-1}(0)$ in V_A, V_B, we can make Definition 10.47(e) hold. Thus $\bigl(I, <, \ldots, \gamma_{ABC}\bigr)$ is a type A good coordinate system, proving the first part of Theorem 12.48.

References

[1] M. Adachi, *Embeddings and immersions*, Translations of mathematical monographs 124, A.M.S., Providence, RI, 1993.

[2] A. Adem, J. Leida and Y. Ruan, *Orbifolds and Stringy Topology*, Cambridge Tracts in Math. 171, Cambridge University Press, Cambridge, 2007.

[3] A. Angel, *Orbifold cobordism*, preprint, 2009.

[4] A. Angel, *A spectral sequence for orbifold cobordism*, pages 141–154 in M. Golasiński et al., editors, *Algebraic topology — old and new*, Banach Center publications 85, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 2009.

[5] A. Angel, *When is a differentiable manifold the boundary of an orbifold?*, pages 330–343 in H. Ocampo et al., editors, *Geometric and topological methods for quantum field theory*, Cambridge University Press, Cambridge, 2010.

[6] M.F. Atiyah, *Bordism and cobordism*, Proc. Camb. Phil. Soc. 57 (1961), 200–208.

[7] K. Behrend, *Gromov–Witten invariants in algebraic geometry*, Invent. Math. 127 (1997), 601–617. alg-geom/9601011.

[8] K. Behrend, *Differential graded schemes I: Perfect resolving algebras*, math.AG/0212225, 2002.

[9] K. Behrend, *Differential graded schemes II: The* 2*-category of differential graded schemes*, math.AG/0212226, 2002.

[10] K. Behrend, *Donaldson–Thomas type invariants via microlocal geometry*, Annals of Mathematics 170 (2009), 1307–1338. math.AG/0507523.

[11] K. Behrend, D. Edidin, B. Fantechi, W. Fulton, L. Göttsche and A. Kresch, *Introduction to stacks*, book in preparation, 2010.

[12] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. 128 (1997), 45–88.

[13] K. Behrend and P. Xu, *Differentiable stacks and gerbes*, J. Symplectic Geom. 9 (2011), 285–341. math.DG/0605694.

[14] J.E. Bergner, *A survey of* $(\infty,1)$*-categories*, math.AT/0610239.

[15] F. Borceux, *Handbook of categorical algebra 1. Basic category theory*, Encyclopedia of Mathematics and its Applications 50, Cambridge University Press, 1994.

[16] D. Borisov, *Derived manifolds and d-manifolds*, arXiv:1212.1153, 2012.

[17] D. Borisov and J. Noel, *Simplicial approach to derived differential manifolds*, arXiv:1112.0033, 2011.

[18] G.E. Bredon, *Topology and Geometry*, Graduate Texts in Math. 139, Springer-Verlag, New York, 1993.

[19] A.S. Cattaneo and F. Schätz, *Introduction to supergeometry*, Rev. Math. Phys. 23 (2011), 669–690. arXiv:1011.3401.

[20] J. Cerf, *Topologies de certains espaces de plongements*, Bull. Soc. Math. France 89 (1961), 227–380.

[21] W. Chen and Y. Ruan, *Orbifold Gromov–Witten theory*, pages 25–86 in A. Adem, J. Morava and Y. Ruan, editors, *Orbifolds in mathematics and physics*, Cont. Math. 310, A.M.S., Providence, RI, 2002. math.AG/0103156.

[22] W. Chen and Y. Ruan, *A new cohomology theory of orbifold*, Commun. Math. Phys. 248 (2004), 1–31. math.AG/0004129.

[23] I. Ciocan-Fontanine and M.M. Kapranov, *Derived Quot schemes*, Ann. Sci. École Norm. Sup. 34 (2001), 3, 403–440. math.AG/9905174.

[24] P.E. Conner, *Differentiable Periodic Maps*, second edition, Springer Lecture Notes in Mathematics 738, Springer-Verlag, Berlin, 1979.

[25] P.E. Conner and E.E. Floyd, *The relation of cobordism to K-theories*, Springer Lecture Notes in Mathematics 28, Springer-Verlag, Berlin, 1966.

[26] P.E. Conner and E.E. Floyd, *Torsion in SU-bordism*, Memoirs of the A.M.S. 60, A.M.S., Providence, RI, 1966.

[27] A. Douady, *Variétés à bord anguleux et voisinages tubulaires*, Séminaire Henri Cartan 14 (1961-2), exp. 1, 1–11.

[28] K.S. Druschel, *Oriented orbifold cobordism*, Pacific J. Math. 164 (1994), 299–319.

[29] K.S. Druschel, *The cobordism of oriented three dimensional orbifolds*, Pacific J. Math. 193 (2000), 45–55.

[30] E.J. Dubuc, C^∞*-schemes*, Amer. J. Math. 103 (1981), 683–690.

[31] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to Symplectic Field Theory*, Geom. Funct. Anal. 2000, Special Volume, Part II, 560–673. math.SG/0010059.

[32] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory — anomaly and obstruction*, Parts I & II. AMS/IP Studies in Advanced Mathematics, 46.1 & 46.2, A.M.S./International Press, 2009.

[33] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Technical details on Kuranishi structure and virtual fundamental chain*, arXiv:1209.4410, 2012.

[34] K. Fukaya and K. Ono, *Arnold Conjecture and Gromov–Witten invariant*, Topology 38 (1999), 933–1048.

[35] R. Godement, *Topologie algébrique et theórie des faisceaux*, Hermann, Paris, 1958.

[36] T.L. Gómez, *Algebraic stacks*, Proc. Indian Acad. Sci. Math. Sci. 111 (2001), 1–31. math.AG/9911199.

[37] A. Grothendieck, *Elements de Géométrie Algébrique I*, Publ. Math. IHES 4, 1960.

[38] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. 52, Springer, New York, 1977.

[39] A. Henriques and D.S. Metzler, *Presentations of noneffective orbifolds*, Trans. A.M.S. 356 (2004), 2481–2499. math.AT/0302182.

[40] K. Hess, *Model categories in algebraic topology*, Appl. Categ. Structures 10 (2002), 195–220.

[41] H. Hofer, *A general Fredholm theory and applications*, in D. Jerison et al., editors, *Current Developments in Mathematics*, International Press, 2006. math.SG/0509366.

[42] H. Hofer, *Polyfolds and a general Fredholm theory*, arXiv:0809.3753, 2008.

[43] H. Hofer, K. Wysocki and E. Zehnder, *A general Fredholm theory I: A splicing-based differential geometry*, J. Eur. Math. Soc. 9 (2007), 841–876. math.FA/0612604.

[44] H. Hofer, K. Wysocki and E. Zehnder, *A general Fredholm theory II: implicit function theorems*, Geom. Funct. Anal. 19 (2009), 206–293. arXiv:0705.1310.

[45] H. Hofer, K. Wysocki and E. Zehnder, *A general Fredholm theory III: Fredholm functors and polyfolds*, Geom. Topol. 13 (2009), 2279–2387. arXiv:0810.0736.

[46] H. Hofer, K. Wysocki and E. Zehnder, *Integration theory for zero sets of polyfold Fredholm sections*, Math. Ann. 346 (2010), 139–198. arXiv:0711.0781.

[47] H. Hofer, K. Wysocki and E. Zehnder, *Sc-smoothness, retractions and new models for smooth spaces*, arXiv:1002.3381, 2010.

[48] H. Hofer, K. Wysocki and E. Zehnder, *Applications of polyfold theory I: the polyfolds of Gromov–Witten theory*, arXiv:1107.2097, 2011.

[49] D. Huybrechts and R.P. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes*, Math. Ann. 346 (2010), 545–569. arXiv:0805.3527.

[50] L. Illusie, *Complexe cotangent et déformations. I*, Springer Lecture Notes in Math. 239, Springer-Verlag, Berlin, 1971.

[51] L. Illusie, *Cotangent complex and deformations of torsors and group schemes*, pages 159–189 in Springer Lecture Notes in Math. 274, Springer-Verlag, Berlin, 1972.

[52] K. Jänich, *On the classification of $O(n)$-manifolds*, Math. Ann. 176 (1968), 53–76.

[53] D. Joyce, *Kuranishi homology and Kuranishi cohomology*, arXiv:0707.3572, version 5, 2008. 290 pages.

[54] D. Joyce, *Kuranishi homology and Kuranishi cohomology: a User's Guide*, arXiv:0710.5634, version 2, 2008. 29 pages.

[55] D. Joyce, *On manifolds with corners*, pages 225–258 in S. Janeczko, J. Li and D.H. Phong, editors, *Advances in Geometric Analysis*, Advanced Lectures in Mathematics 21, International Press, Boston, 2012. arXiv:0910.3518.

[56] D. Joyce, *Algebraic Geometry over C^∞-rings*, arXiv:1001.0023, 2010.

[57] D. Joyce, *An introduction to C^∞-schemes and C^∞-algebraic geometry*, pages 299–325 in H.-D. Cao and S.-T. Yau, editors, *In memory of C.C. Hsiung: Lectures given at the JDG symposium, Lehigh University, June 2010*, Surveys in Differential Geometry 17, 2012. arXiv:1104.4951.

[58] D. Joyce, *An introduction to d-manifolds and derived differential geometry*, arXiv:1206.4207, 2012.

[59] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, Memoirs of the A.M.S., 2011. arXiv:0810.5645.

[60] T. Kawasaki, *The signature theorem for V-manifolds*, Topology 17 (1978), 75–83.

[61] G.M. Kelly and R.H. Street, *Review of the elements of 2-categories*, pages 75–103 in Lecture Notes in Math. 420, Springer-Verlag, New York, 1974.

[62] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, volume I*, John Wiley & sons, New York, 1963.

[63] M. Kontsevich, *Enumeration of rational curves via torus actions*, pages 335–368 in R. Dijkgraaf, C. Faber and G. van der Geer, editors, *The moduli space of curves*, Progr. Math. 129, Birkhäuser, 1995. hep-th/9405035.

[64] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergeb. der Math. und ihrer Grenzgebiete 39, Springer-Verlag, Berlin, 2000.

[65] S. Lang, *Differentiable and Riemannian manifolds*, Springer, 1995.

[66] G. Laures, *On cobordism of manifolds with corners*, Trans. A.M.S. 352 (2000), 5667–5688.

[67] E. Lerman, *Orbifolds as stacks?*, Enseign. Math. 56 (2010), 315–363. arXiv:0806.4160.

[68] J. Li and G. Tian, *Comparison of algebraic and symplectic Gromov–Witten invariants*, Asian J. Math. 3 (1999), 689-728. alg-geom/9712035.

[69] E. Lupercio and B. Uribe, *Gerbes over orbifolds and twisted K-theory*, Comm. Math. Phys. 245 (2004), 449–489. math.AT/01050393.

[70] J. Lurie, *Higher Topos Theory*, Annals of Math. Studies, 170, Princeton University Press, Princeton, NJ, 2009. math.CT/0608040.

[71] J. Lurie, *Derived Algebraic Geometry I: Stable ∞-categories*, math.CT/0608228, 2006.

[72] J. Lurie, *Derived Algebraic Geometry V: Structured spaces*, arXiv:0905.0459, 2009.

[73] S. MacLane, *Categories for the Working Mathematician*, second edition, Graduate Texts in Math. 5, Springer, New York, 1998.

[74] C. Manolache, *Virtual pullbacks*, arXiv:0805.2065, 2008.

[75] W.S. Massey, *Homology and Cohomology Theory*, Graduate Texts in Math. 70, Marcel Dekker, New York, 1978.

[76] W.S. Massey, *Singular Homology Theory*, Graduate Texts in Math. 70, Springer-Verlag, New York, 1980.

[77] D. McDuff, *The virtual moduli cycle*, pages 73–102 in Y. Eliashberg, D. Fuchs, T. Ratiu and A. Weinstein, editors, *Northern California Symplectic Geometry Seminar*, A.M.S. Translations 196, A.M.S., Providence, RI, 1999.

[78] D. McDuff and K. Wehrheim, *Smooth Kuranishi structures with trivial isotopy*, arXiv:1208.1340, 2012.

[79] R.B. Melrose, *The Atiyah–Patodi–Singer Index Theorem*, A.K. Peters, Wellesley, MA, 1993.

[80] R.B. Melrose, *Differential Analysis on Manifolds with Corners*, unfinished book available at `http://math.mit.edu/~rbm`, 1996.

[81] D. Maulik, R. Pandharipande and R.P. Thomas, *Curves on $K3$ surfaces and modular forms*, J. Topol. 3 (2010), 937–996. arXiv:1001.2719.

[82] D.S. Metzler, *Topological and smooth stacks*, math.DG/0306176, 2003.

[83] T. Mochizuki, *Donaldson type invariants for algebraic surfaces*, Lecture Notes in Math. 1972, Spinger, 2009.

[84] I. Moerdijk, *Orbifolds as groupoids: an introduction*, pages 205–222 in A. Adem, J. Morava and Y. Ruan, editors, *Orbifolds in Mathematics and Physics*, Cont. Math. 310, A.M.S., Providence, RI, 2002. math.DG/0203100.

[85] I. Moerdijk and D.A. Pronk, *Orbifolds, sheaves and groupoids*, K-theory 12 (1997), 3–21.

[86] I. Moerdijk and G.E. Reyes, *Models for smooth infinitesimal analysis*, Springer-Verlag, New York, 1991.

[87] B. Monthubert, *Groupoids and pseudodifferential calculus on manifolds with corners*, J. Funct. Anal. 199 (2003), 243–286.

[88] R. Pandharipande and R.P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. math. 178 (2009) 407–447. arXiv:0707.2348.

[89] D. Pronk, *Etendues and stacks as bicategories of fractions*, Compositio Math. 102 (1996), 243–303.

[90] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 359–363.

[91] T. Schürg, *Deriving Deligne–Mumford stacks with perfect obstruction theories*, PhD thesis, Johannes Gutenberg Universität, Mainz, 2011.

[92] T. Schürg, B. Toën and G. Vezzosi, *Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes*, arXiv:1102.1150, 2011.

[93] B. Siebert, *Algebraic and symplectic Gromov–Witten invariants coincide*, Ann. Inst. Fourier (Grenoble) 49 (1999), 1743-1795. math.AG/9804108.

[94] D.I. Spivak, *Quasi-smooth derived manifolds*, PhD thesis, University of California, Berkeley, 2007.

[95] D.I. Spivak, *Derived smooth manifolds*, Duke Math. J. 153 (2010), 55–128. arXiv:0810.5174.

[96] R.E. Stong, *Notes on cobordism theory*, Princeton University Press, Princeton, NJ, 1968.

[97] R. Thom, *Quelques propriétés globales des variétés differentiables*, Comment. Math. Helv. 28 (1954), 18–88.

[98] R.P. Thomas, *A holomorphic Casson invariant for Calabi–Yau* 3*-folds, and bundles on* $K3$ *fibrations*, J. Diff. Geom. 54 (2000), 367–438. math.AG/9806111.

[99] W. Thurston, *The Geometry and Topology of Three-manifolds*, Princeton lecture notes, Princeton, 1980. Available at *http://library.msri.org/books/gt3m.*

[100] B. Toën, *Higher and derived stacks: a global overview*, pages 435–487 in Proc. Symp. Pure Math. vol 80, part 1, A.M.S., 2009. math.AG/0604504.

[101] B. Toën and G. Vezzosi, *Homotopical Algebraic Geometry II: Geometric Stacks and Applications*, Memoirs of the A.M.S. vol. 193, no. 902, 2008. math.AG/0404373.

[102] B. Toën and G. Vezzosi, *From HAG to DAG: derived moduli stacks*, pages 173–216 in *Axiomatic, enriched and motivic homotopy theory*, NATO Sci. Ser. II Math. Phys. Chem., 131, Kluwer, Dordrecht, 2004. math.AG/0210407.

[103] A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, pages 1–104 in B. Fantechi, Barbara, L. Göttsche, L. Illusie, S.L. Kleiman, N. Nitsure and A. Vistoli, *Fundamental algebraic geometry: Grothendieck's FGA explained*, Math. Surveys and Monographs, 123, A.M.S., Providence, RI, 2005. math.AG/0404512.

[104] C.A. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics 38, C.U.P., Cambridge, 1994.

[105] G.W. Whitehead, *Generalized homology theories*, Trans. A.M.S. 102 (1962), 292–311.

[106] H. Whitney, *Differentiable manifolds*, Ann. Math. 37 (1936), 645–680.

Glossary of Notation

Generally we give two page references, the first in Chapter 1, and the second in the remainder of the book.

$B_k(Y)$ classical bordism group of manifold Y, 142, 571

$B_k^{\mathrm{orb}}(\mathcal{Y})$ orbifold bordism group of orbifold $\mathcal{Y}$, 144, 580

$B_k^{\mathrm{eff}}(\mathcal{Y})$ effective orbifold bordism group of orbifold $\mathcal{Y}$, 144, 580

$C, \hat{C} : \mathbf{Man^c} \to \check{\mathbf{Man}}^{\mathbf{c}}$ 'corner functors' for manifolds with corners, 44, 287, 288

$C, \hat{C} : \mathbf{dSpa^c} \to \mathbf{dSpa^c}$ 'corner functors' for d-spaces with corners, 53, 343, 345

$C, \hat{C} : \mathbf{Orb^c} \to \check{\mathbf{Orb}}^{\mathbf{c}}$ 'corner functors' for orbifolds with corners, 118, 452

$C, \hat{C} : \mathbf{dSta^c} \to \mathbf{dSta^c}$ 'corner functors' for d-stacks with corners, 127, 541

$\mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \ldots$ C^∞-rings, 9, 659

$\mathfrak{C} \amalg_{\mathfrak{D}} \mathfrak{E}$ pushout of C^∞-rings $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$, 660

$\mathfrak{C} \hat{\otimes} \mathfrak{D}$ coproduct of C^∞-rings $\mathfrak{C}, \mathfrak{D}$, 660

$\mathfrak{C}$-mod abelian category of modules over a C^∞-ring $\mathfrak{C}$, 13, 669

$\mathfrak{C}$-mod$^{\mathrm{co}}$ abelian subcategory of complete modules in $\mathfrak{C}$-mod for $\mathfrak{C}$ fair, 14, 672

$\mathfrak{C}$-mod$^{\mathrm{fp}}$ subcategory of finitely presented modules in $\mathfrak{C}$-mod, 669

coh$(\underline{X})$ category of coherent sheaves on C^∞-scheme $\underline{X}$, 672

coh$(\mathcal{X})$ category of coherent sheaves on Deligne–Mumford C^∞-stack $\mathcal{X}$, 687

coh$(\underline{V} \rightrightarrows \underline{U})$ category of coherent modules on a groupoid $\underline{V} \rightrightarrows \underline{U}$, 689

$\mathbf{C^\infty Rings}$ category of C^∞-rings, 9, 659

$\mathbf{C^\infty Rings^{fa}}$ category of fair C^∞-rings, 662

$\mathbf{C^\infty Rings^{fg}}$ category of finitely generated C^∞-rings, 662

$\mathbf{C^\infty Rings^{fp}}$ category of finitely presented C^∞-rings, 662

$\mathbf{C^\infty RS}$ category of C^∞-ringed spaces, 11, 665

$\mathbf{C^\infty Sch}$ category of C^∞-schemes, 11, 666

$\mathbf{C^\infty Sch^{lf}}$ category of locally fair C^∞-schemes, 11, 666

$\mathbf{C^\infty Sch^{lfp}}$ category of locally finitely presented C^∞-schemes, 666

$\mathbf{C^\infty Sch^{lf}_{ssc}}$ category of separated, second countable, locally fair C^∞-schemes, 19, 666

$\mathbf{C^\infty Sta}$ 2-category of C^∞-stacks, 69, 676

$dB_k(Y)$ d-manifold bordism group of manifold Y, 142, 575

$dB_k^{\mathrm{orb}}(\mathcal{Y})$ d-orbifold bordism group of orbifold $\mathcal{Y}$, 145, 587

$dB_k^{\mathrm{sef}}(\mathcal{Y})$ semieffective d-orbifold bordism group of orbifold $\mathcal{Y}$, 145, 587

$dB_k^{\mathrm{eff}}(\mathcal{Y})$ effective d-orbifold bordism group of orbifold $\mathcal{Y}$, 145, 587

$[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$ quotient 2-morphism of quotient 1-morphisms, 73, 682

DerMan Spivak's ∞-category of derived manifolds, 148, 648

$\delta_X(\mathcal{E}) : \mathrm{id}_X^{-1}(\mathcal{E}) \to \mathcal{E}$ canonical isomorphism of pullback sheaves, 664

$\delta_{\underline{X}}(\mathcal{E}) : \underline{\mathrm{id}}_X^*(\mathcal{E}) \to \mathcal{E}$ canonical isomorphism of pullbacks in $\mathcal{O}_X$-mod, 15, 672

$\partial_\pm^f X$ sets of decomposition $\partial X = \partial_+^f X \amalg \partial_-^f X$ of boundary ∂X induced by $f : X \to Y$ in $\mathbf{Man^c}$, 43, 285

$\partial_\pm^{\boldsymbol{f}} \mathbf{X}$ sets of decomposition $\partial \mathbf{X} = \partial_+^{\boldsymbol{f}} \mathbf{X} \amalg \partial_-^{\boldsymbol{f}} \mathbf{X}$ of boundary $\partial \mathbf{X}$ induced by 1-morphism $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dSpa^c}$, 50, 321

$\partial_\pm^f \mathcal{X}$ sets of decomposition $\partial \mathcal{X} = \partial_+^f \mathcal{X} \amalg \partial_-^f \mathcal{X}$ of boundary $\partial \mathcal{X}$ induced by 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{Orb^c}$, 116, 447

$\partial_\pm^{\boldsymbol{f}} \boldsymbol{\mathcal{X}}$ sets of decomposition $\partial \boldsymbol{\mathcal{X}} = \partial_+^{\boldsymbol{f}} \boldsymbol{\mathcal{X}} \amalg \partial_-^{\boldsymbol{f}} \boldsymbol{\mathcal{X}}$ of boundary $\partial \boldsymbol{\mathcal{X}}$ induced by 1-morphism $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dSta^c}$, 125, 536

dMan 2-category of d-manifolds, 24, 214

$\mathbf{d\bar{M}an}$ 2-subcategory of d-manifolds with corners equivalent to d-manifolds, 58, 388

$\mathbf{d\hat{M}an}$ 2-subcategory of d-orbifolds equivalent to d-manifolds, 97, 489

$\mathbf{dMan^b}$ 2-category of d-manifolds with boundary, 58, 388

$\mathbf{dMan^c}$ 2-category of d-manifolds with corners, 58, 388

$\mathbf{d\check{M}an^c}$ 2-category of disjoint unions of d-manifolds with corners of different dimensions, 59, 389

$\mathbf{d\hat{M}an^c}$ 2-subcategory of d-orbifolds with corners equivalent to d-manifolds with corners, 131, 552

$\mathbf{DMC^\infty Sta}$ 2-category of Deligne–Mumford C^∞-stacks, 73, 683

$\mathbf{DMC^\infty Sta^{lf}}$ 2-category of locally fair Deligne–Mumford C^∞-stacks, 73, 683

$\mathbf{DMC^\infty Sta^{lfp}}$ 2-category of locally finitely presented Deligne–Mumford C^∞-stacks, 683

$\mathbf{DMC^\infty Sta^{lf}_{ssc}}$ 2-category of separated, second countable, locally fair Deligne–Mumford C^∞-stacks, 73, 683

$\mathbf{DMC^\infty Sta^{re}}$ 2-category of Deligne–Mumford C^∞-stacks with representable 1-morphisms, 701

dOrb 2-category of d-orbifolds, 97, 489

$\mathbf{d\bar{O}rb}$ 2-subcategory of d-orbifolds with corners equivalent to d-orbifolds, 131, 551

$\mathbf{dOrb^b}$ 2-category of d-orbifolds with boundary, 131, 551

$\mathbf{dOrb^c}$ 2-category of d-orbifolds with corners, 131, 551

$\mathbf{d\check{O}rb^c}$ 2-category of disjoint unions of d-orbifolds with corners of different dimensions, 132, 553

$\mathbf{dSpa}$ 2-category of d-spaces, 19, 161
$\mathbf{d\bar{S}pa}$ 2-subcategory of d-spaces with corners equivalent to d-spaces, 49, 302
$\mathbf{d\hat{S}pa}$ 2-subcategory of d-stacks equivalent to d-spaces, 90, 470
$\mathbf{dSpa^b}$ 2-category of d-spaces with boundary, 49, 302
$\mathbf{dSpa^c}$ 2-category of d-spaces with corners, 48, 302
$\mathbf{d\hat{S}pa^c}$ 2-subcategory of d-stacks with corners equivalent to d-spaces with corners, 123, 532
$\mathbf{d\widetilde{Spa}^c}$ alternative 2-category of d-spaces with corners, 304
$\mathbf{dSta}$ 2-category of d-stacks, 89, 470
$\mathbf{d\bar{S}ta}$ 2-subcategory of d-stacks with corners equivalent to d-stacks, 122, 531
$\mathbf{dSta^b}$ 2-category of d-stacks with boundary, 122, 531
$\mathbf{dSta^c}$ 2-category of d-stacks with corners, 122, 531
∂X boundary of a manifold with corners X, 40, 282
$\partial\mathbf{X}$ boundary of a d-space with corners $\mathbf{X}$, 49, 314
$\partial\mathcal{X}$ boundary of an orbifold with corners $\mathcal{X}$, 115, 446
$\partial\boldsymbol{\mathfrak{X}}$ boundary of a d-stack with corners $\boldsymbol{\mathfrak{X}}$, 123, 535
$(\mathcal{E}^\bullet,\phi)$ virtual quasicoherent sheaf, or virtual vector bundle, 28, 205
$F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}}:\mathbf{C^\infty Sch}\to\mathbf{C^\infty Sta}$ inclusion from C^∞-schemes to C^∞-stacks, 69, 676
$f_*(\mathcal{E})$ pushforward (direct image) sheaf, 663
$f^*(\mathcal{E})$ pullback (inverse image) sheaf, 664
$\underline{f}^*(\mathcal{E})$ pullback of sheaf of $\mathcal{O}_Y$-modules under $\underline{f}:\underline{X}\to\underline{Y}$, 15, 671
$f^*(\mathcal{E})$ pullback of sheaf of $\mathcal{O}_{\mathcal{Y}}$-modules under $f:\mathcal{X}\to\mathcal{Y}$, 76, 690
$[\underline{f},\rho]:[\underline{X}/G]\to[\underline{Y}/H]$ quotient 1-morphism of quotient C^∞-stacks, 73, 682
$f^\sharp:f^{-1}(\mathcal{O}_Y)\to\mathcal{O}_X$ morphism of sheaves of C^∞-rings in $\underline{f}:\underline{X}\to\underline{Y}$, 11, 665
$f_\sharp:\mathcal{O}_Y\to f_*(\mathcal{O}_X)$ morphism of sheaves of C^∞-rings in $\underline{f}:\underline{X}\to\underline{Y}$, 11, 665
$f^\sharp:f^{-1}(\mathcal{O}_{\mathcal{Y}})\to\mathcal{O}_{\mathcal{X}}$ morphism of sheaves of C^∞-rings on $\mathcal{X}$ from a 1-morphism $f:\mathcal{X}\to\mathcal{Y}$ of Deligne–Mumford C^∞-stacks $\mathcal{X},\mathcal{Y}$, 77, 694
$\Gamma:\mathbf{LC^\infty RS}\to\mathbf{C^\infty Rings}^{\rm op}$ global sections functor on C^∞-ringed spaces, 665
$\Gamma:\mathcal{O}_X\text{-mod}\to\mathfrak{C}\text{-mod}$ global sections functor on $\mathcal{O}_X$-modules, $\underline{X}=\operatorname{Spec}\mathfrak{C}$, 672
$\mathrm{Ho}(\mathbf{Orb})$ homotopy category of the 2-category of orbifolds $\mathbf{Orb}$, 83, 427
$I_{f,g}(\mathcal{E}):(g\circ f)^{-1}(\mathcal{E})\to f^{-1}(g^{-1}(\mathcal{E}))$ isomorphism of pullback sheaves, 664
$I_{\underline{f},\underline{g}}(\mathcal{E}):(\underline{g}\circ\underline{f})^*(\mathcal{E})\to\underline{f}^{-1}(\underline{g}^{-1}(\mathcal{E}))$ isomorphism of pullbacks in $\mathcal{O}_X$-mod, 15, 672
$\boldsymbol{i}_{\tilde V,V}:\boldsymbol{S}_{\tilde V,\tilde E,\tilde s}\to\boldsymbol{S}_{V,E,s}$ inclusion of open set in ‘standard model’ d-manifold, 26, 223

$\boldsymbol{i}_{\tilde V,V}:\mathbf{S}_{\tilde V,\tilde E,\tilde s}\to\mathbf{S}_{V,E,s}$ inclusion of open set in 'standard model' d-manifold with corners, 60, 395

$\boldsymbol{i}_{\tilde{\mathcal V},\mathcal V}:\boldsymbol{\mathcal S}_{\tilde{\mathcal V},\tilde{\mathcal E},\tilde s}\to\boldsymbol{\mathcal S}_{\mathcal V,\mathcal E,s}$ inclusion of open set in 'standard model' d-orbifold, 98, 493

$\boldsymbol{i}_{\tilde{\mathcal V},\mathcal V}:\mathbf{\mathfrak S}_{\tilde{\mathcal V},\tilde{\mathcal E},\tilde s}\to\mathbf{\mathfrak S}_{\mathcal V,\mathcal E,s}$ inclusion of open set in 'standard model' d-orbifold with corners, 133, 554

$\mathcal{I}_{\underline{X}}:\mathcal{O}_X\text{-mod}\to\mathcal{O}_{\mathcal X}\text{-mod}$ inclusion functor from sheaves on a C^∞-scheme $\underline{X}$ to sheaves on the associated Deligne–Mumford C^∞-stack $\mathcal{X}=\bar{\underline{X}}$, 75, 688

$i_X:\partial X\to X$ inclusion of boundary ∂X into a manifold with corners X, 40, 282

$\boldsymbol{i}_{\mathbf X}:\partial\mathbf{X}\to\mathbf{X}$ inclusion of boundary $\partial\mathbf{X}$ into a d-space with corners $\mathbf{X}$, 48, 320

$\boldsymbol{i}_{\mathfrak X}:\partial\mathfrak{X}\to\mathfrak{X}$ inclusion of boundary $\partial\mathfrak{X}$ into an orbifold with corners $\mathfrak{X}$, 112, 446

$\boldsymbol{i}_{\boldsymbol{\mathfrak X}}:\partial\boldsymbol{\mathfrak X}\to\boldsymbol{\mathfrak X}$ inclusion of boundary $\partial\boldsymbol{\mathfrak X}$ into a d-stack with corners $\boldsymbol{\mathfrak X}$, 122, 535

$\boldsymbol{j}_{\boldsymbol X,\Gamma}:\boldsymbol{X}^\Gamma\hookrightarrow\boldsymbol{X}$ inclusion of Γ-fixed d-subspace $\boldsymbol{X}^\Gamma$ in a d-space $\boldsymbol{X}$, 23, 200

$j_{X,\Gamma}:X^\Gamma\hookrightarrow X$ inclusion of Γ-fixed subset X^Γ in a manifold with corners X, 47, 289

$\boldsymbol{j}_{\mathbf X,\Gamma}:\mathbf{X}^\Gamma\hookrightarrow\mathbf{X}$ inclusion of Γ-fixed d-subspace $\mathbf{X}^\Gamma$ in a d-space with corners $\mathbf{X}$, 57, 379

$\lambda_{\boldsymbol f}:\underline{u}_{\boldsymbol f}^*(\mathcal{N}_{\mathbf Y})\to\underline{s}_{\boldsymbol f}^*(\mathcal{N}_{\mathbf X})$ morphism of conormal line bundles from 1-morphism of d-spaces with corners $\boldsymbol{f}:\mathbf{X}\to\mathbf{Y}$, 307

$\lambda_{\boldsymbol f}:u_{\boldsymbol f}^*(\mathcal{N}_{\mathfrak Y})\to s_{\boldsymbol f}^*(\mathcal{N}_{\mathfrak X})$ morphism of conormal line bundles from 1-morphism of d-stacks with corners $\boldsymbol{f}:\boldsymbol{\mathfrak X}\to\boldsymbol{\mathfrak Y}$, 529

Λ^Γ lattice generated by nontrivial representations of a finite group Γ, 85, 432

Λ^Γ_+ 'positive cone' of classes of Γ-representations in lattice Λ^Γ, 85, 432

$\mathbf{LC^\infty RS}$ category of local C^∞-ringed spaces, 665

$\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ orientation line bundle of a virtual vector bundle $(\mathcal{E}^\bullet,\phi)$, 38, 269, 506

$\mathcal{L}_{T^*\boldsymbol X}$ orientation line bundle of a d-manifold $\boldsymbol{X}$, 38, 272

$\mathcal{L}_{T^*\mathbf X}$ orientation line bundle of a d-manifold with corners $\mathbf{X}$, 67, 420

$\mathcal{L}_{T^*\boldsymbol{\mathcal X}}$ orientation line bundle of a d-orbifold $\boldsymbol{\mathcal X}$, 106, 506

$\mathcal{L}_{T^*\mathfrak X}$ orientation line bundle of a d-orbifold with corners $\mathfrak{X}$, 139, 564

$\mathbb{L}_X,\mathbb{L}_{X/Y}$ cotangent complexes of a scheme X and a morphism $f:X\to Y$, 631

$\mathbf{Man}$ category of manifolds, 10, 283

$\hat{\mathbf{Man}}$ 2-subcategory of d-spaces equivalent to manifolds, 19, 161

$\bar{\mathbf{Man}}$ 2-subcategory of d-spaces with corners equivalent to manifolds without boundary, 329

$\mathbf{Man^b}$ category of manifolds with boundary, 41, 283

$\bar{\mathbf{Man}}^{\mathbf{b}}$ 2-subcategory of d-spaces with corners equivalent to manifolds with boundary, 329

$\mathbf{Man^c}$ category of manifolds with corners, 41, 283

$\check{\mathbf{Man}}^{\mathbf{c}}$ category of disjoint unions of manifolds with corners of different dimensions, 44, 287

$\bar{\mathbf{Man}}^{\mathbf{c}}$ 2-subcategory of d-spaces with corners equivalent to manifolds with corners, 52, 329

$\mathrm{MSpec} : \mathfrak{C}\text{-mod} \to \mathcal{O}_X\text{-mod}$ spectrum functor on $\mathfrak{C}$-modules, $\underline{X} = \operatorname{Spec}\mathfrak{C}$, 14, 672

$\mu_{\boldsymbol{f}} : \underline{u}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial Y}) \to \underline{s}_{\boldsymbol{f}}^*(\mathcal{F}_{\partial X})$ morphism of boundary cotangent sheaves from 1-morphism of d-spaces with corners $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$, 307

$\mu_{\boldsymbol{f}} : u_{\boldsymbol{f}}^*(\mathcal{F}_{\partial \mathcal{Y}}) \to s_{\boldsymbol{f}}^*(\mathcal{F}_{\partial \mathcal{X}})$ morphism of boundary cotangent sheaves from 1-morphism of d-stacks with corners $\boldsymbol{f} : \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathfrak{Y}}$, 529

$\mathcal{N}_{\mathbf{X}}$ conormal line bundle of $\partial\mathbf{X}$ in $\mathbf{X}$ for a d-space with corners $\mathbf{X}$, 48, 299

$\mathcal{N}_{\boldsymbol{\mathfrak{X}}}$ conormal line bundle of $\partial\boldsymbol{\mathfrak{X}}$ in $\boldsymbol{\mathfrak{X}}$ for a d-stack with corners $\boldsymbol{\mathfrak{X}}$, 122, 525

$O(s)$ an error term in the ideal generated by a section $s \in C^\infty(E)$, 25, 222

$O(s^2)$ an error term in the ideal generated by $s \otimes s$ for $s \in C^\infty(E)$, 25, 222

$O^\Gamma(\mathcal{X}), \tilde{O}^\Gamma(\mathcal{X}), O_\circ^\Gamma(\mathcal{X}), \tilde{O}_\circ^\Gamma(\mathcal{X})$ 1-morphisms of orbifold strata $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}_\circ^\Gamma$ of a Deligne–Mumford C^∞-stack $\mathcal{X}$, 78, 695

$\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}), \tilde{\boldsymbol{O}}^\Gamma(\boldsymbol{\mathcal{X}}), \boldsymbol{O}_\circ^\Gamma(\boldsymbol{\mathcal{X}}), \tilde{\boldsymbol{O}}_\circ^\Gamma(\boldsymbol{\mathcal{X}})$ 1-morphisms of orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma, \ldots, \hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma$ of a d-stack $\boldsymbol{\mathcal{X}}$, 94, 480

$O^\Gamma(\mathfrak{X}), \tilde{O}^\Gamma(\mathfrak{X}), O_\circ^\Gamma(\mathfrak{X}), \tilde{O}_\circ^\Gamma(\mathfrak{X})$ 1-morphisms of orbifold strata $\mathfrak{X}^\Gamma, \ldots, \hat{\mathfrak{X}}_\circ^\Gamma$ of an orbifold with corners $\mathfrak{X}$, 120, 458

$\boldsymbol{O}^\Gamma(\boldsymbol{\mathfrak{X}}), \tilde{\boldsymbol{O}}^\Gamma(\boldsymbol{\mathfrak{X}}), \boldsymbol{O}_\circ^\Gamma(\boldsymbol{\mathfrak{X}}), \tilde{\boldsymbol{O}}_\circ^\Gamma(\boldsymbol{\mathfrak{X}})$ 1-morphisms of orbifold strata $\boldsymbol{\mathfrak{X}}^\Gamma, \ldots, \hat{\boldsymbol{\mathfrak{X}}}_\circ^\Gamma$ of a d-stack with corners $\boldsymbol{\mathfrak{X}}$, 129, 547

$\omega_{\mathbf{X}}$ orientation on line bundle $\mathcal{N}_{\mathbf{X}}$ for a d-space with corners $\mathbf{X}$, 48, 300

$\omega_{\boldsymbol{\mathfrak{X}}}$ orientation on line bundle $\mathcal{N}_{\boldsymbol{\mathfrak{X}}}$ for a d-stack with corners $\boldsymbol{\mathfrak{X}}$, 122, 525

$\mathbf{Orb}$ 2-category of orbifolds, 82, 427

$\hat{\mathbf{Orb}}$ 2-subcategory of d-stacks equivalent to orbifolds, 90, 470

$\check{\mathbf{Orb}}$ 2-category of disjoint unions of orbifolds of different dimensions, 431

$\dot{\mathbf{Orb}}$ 2-subcategory of orbifolds with corners equivalent to orbifolds, 113, 440

$\mathbf{Orb^b}$ 2-category of orbifolds with boundary, 113, 440

$\mathbf{Orb^c}$ 2-category of orbifolds with corners, 113, 440

$\bar{\mathbf{Orb}}^{\mathbf{c}}$ 2-subcategory of d-stacks with corners equivalent to orbifolds with corners, 123, 532

$\check{\mathbf{O}}\mathbf{rb^c}$ 2-category of disjoint unions of orbifolds with corners of different dimensions, 117, 451

$\mathcal{O}_X$-mod abelian category of $\mathcal{O}_X$-modules on C^∞-scheme $\underline{X}$, 14, 671

$\mathcal{O}_{\mathcal{X}}$-mod abelian category of $\mathcal{O}_{\mathcal{X}}$-modules on Deligne–Mumford C^∞-stack $\mathcal{X}$, 75, 687

$\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ operations on C^∞-ring $\mathfrak{C}$, for smooth $f : \mathbb{R}^n \to \mathbb{R}$, 9, 659

$\tilde\Pi^\Gamma(\mathcal{X}), \hat\Pi^\Gamma(\mathcal{X}), \tilde\Pi^\Gamma_\circ(\mathcal{X}), \hat\Pi^\Gamma_\circ(\mathcal{X})$ 1-morphisms of orbifold strata $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ of a Deligne–Mumford C^∞-stack $\mathcal{X}$, 78, 695

$\tilde{\mathbf{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}}), \hat{\mathbf{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}}), \tilde{\mathbf{\Pi}}^\Gamma_\circ(\boldsymbol{\mathcal{X}}), \hat{\mathbf{\Pi}}^\Gamma_\circ(\boldsymbol{\mathcal{X}})$ 1-morphisms of orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma, \ldots, \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ of a d-stack $\boldsymbol{\mathcal{X}}$, 94, 480

$\tilde\Pi^\Gamma(\mathfrak{X}), \hat\Pi^\Gamma(\mathfrak{X}), \tilde\Pi^\Gamma_\circ(\mathfrak{X}), \hat\Pi^\Gamma_\circ(\mathfrak{X})$ 1-morphisms of orbifold strata $\mathfrak{X}^\Gamma, \ldots, \hat{\mathfrak{X}}^\Gamma_\circ$ of an orbifold with corners $\mathfrak{X}$, 120, 458

$\tilde{\mathbf{\Pi}}^\Gamma(\boldsymbol{\mathfrak{X}}), \hat{\mathbf{\Pi}}^\Gamma(\boldsymbol{\mathfrak{X}}), \tilde{\mathbf{\Pi}}^\Gamma_\circ(\boldsymbol{\mathfrak{X}}), \hat{\mathbf{\Pi}}^\Gamma_\circ(\boldsymbol{\mathfrak{X}})$ 1-morphisms of orbifold strata $\boldsymbol{\mathfrak{X}}^\Gamma, \ldots, \hat{\boldsymbol{\mathfrak{X}}}^\Gamma_\circ$ of a d-stack with corners $\boldsymbol{\mathfrak{X}}$, 129, 547

$\mathrm{qcoh}(\underline{X})$ abelian category of quasicoherent sheaves on C^∞-scheme $\underline{X}$, 14, 672

$\mathrm{qcoh}(\mathcal{X})$ abelian category of quasicoherent sheaves on Deligne–Mumford C^∞-stack $\mathcal{X}$, 75, 687

$\mathrm{qcoh}^G(\underline{X})$ abelian category of G-equivariant quasicoherent sheaves on a C^∞-scheme $\underline{X}$ acted on by a finite group G, 706

$\mathrm{qcoh}(\underline{V} \rightrightarrows \underline{U})$ category of quasicoherent modules on a groupoid $\underline{V} \rightrightarrows \underline{U}$, 689

$S_f \subseteq \partial X \times_Y \partial Y$ set associated to smooth map $f : X \to Y$ in $\mathbf{Man^c}$, 41, 284

$\underline{S}_{\boldsymbol{f}} \subseteq \underline{\partial X} \times_{\underline{Y}} \underline{\partial Y}$ C^∞-scheme associated to 1-morphism $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dSpa^c}$, 49, 301

$\mathcal{S}_f \subseteq \partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$ C^∞-stack associated to 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{Orb^c}$, 115, 444

$\mathcal{S}_{\boldsymbol{f}} \subseteq \partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$ C^∞-stack associated to 1-morphism $\boldsymbol{f} : \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathfrak{Y}}$ in $\mathbf{dSta^c}$, 124, 528

$\boldsymbol{S}_{f,\hat f} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ 'standard model' 1-morphism in $\mathbf{dMan}$, 26, 223

$\mathbf{S}_{f,\hat f} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ 'standard model' 1-morphism in $\mathbf{dMan^c}$, 60, 395

$\boldsymbol{\mathcal{S}}_{f,\hat f} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ 'standard model' 1-morphism in $\mathbf{dOrb}$, 98, 492

$\boldsymbol{\mathfrak{S}}_{f,\hat f} : \boldsymbol{\mathfrak{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathfrak{S}}_{\mathcal{W},\mathcal{F},t}$ 'standard model' 1-morphism in $\mathbf{dOrb^c}$, 133, 554

$[\boldsymbol{S}_{f,\hat f}, \rho] : [\boldsymbol{S}_{V,E,s}/\Gamma] \to [\boldsymbol{S}_{W,F,t}/\Delta]$ 'standard model' 1-morphism in $\mathbf{dOrb}$, 100, 494

$[\mathbf{S}_{f,\hat f}, \rho] : [\mathbf{S}_{V,E,s}/\Gamma] \to [\mathbf{S}_{W,F,t}/\Delta]$ 'standard model' 1-morphism in $\mathbf{dOrb^c}$, 133, 555

$\mathrm{Sh}(X)$ category of sheaves of abelian groups on topological space X, 663

$S^k(X)$ depth k stratum of a manifold with corners X, 40, 282

$\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat f} \Rightarrow \boldsymbol{S}_{g,\hat g}$ 'standard model' 2-morphism in $\mathbf{dMan}$, 27, 229

$[\boldsymbol{S}_\Lambda,\delta]:[\boldsymbol{S}_{f,\hat f},\rho]\Rightarrow[\boldsymbol{S}_{g,\hat g},\sigma]$ 'standard model' 2-morphism in **dOrb**, 100, 494
Spec : $\mathbf{C^\infty Rings}^{\rm op}\rightarrow\mathbf{LC^\infty RS}$ spectrum functor on C^∞-rings, 11, 666
$\mathbf{Sta}_{(\mathcal{C},\mathcal{J})}$ 2-category of stacks on a site $(\mathcal{C},\mathcal{J})$, 69, 676
$\boldsymbol{S}_{V,E,s}$ 'standard model' d-manifold, 24, 211
$\mathbf{S}_{V,E,s}$ 'standard model' d-manifold with corners, 60, 385
$\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ 'standard model' d-orbifold, 98, 490
$\boldsymbol{\mathscr{S}}_{\mathscr{V},\mathscr{E},s}$ 'standard model' d-orbifold with corners, 133, 550
$[\boldsymbol{S}_{V,E,s}/\Gamma]$ alternative 'standard model' d-orbifold, 99, 494
$[\mathbf{S}_{V,E,s}/\Gamma]$ alternative 'standard model' d-orbifold with corners, 133, 554
$T^*\boldsymbol{X}$ virtual cotangent sheaf of a d-space $\boldsymbol{X}$, 29, 206
$T^*\mathbf{X}$ virtual cotangent sheaf of a d-space with corners $\mathbf{X}$, 58, 388
$T^*\boldsymbol{\mathcal{X}}$ virtual cotangent sheaf of a d-stack $\boldsymbol{\mathcal{X}}$, 96, 488
$T^*\boldsymbol{\mathfrak{X}}$ virtual cotangent sheaf of a d-stack with corners $\boldsymbol{\mathfrak{X}}$, 132, 551
$T_f\subseteq X\times_Y\partial Y$ set associated to smooth map $f:X\rightarrow Y$ in $\mathbf{Man^c}$, 41, 284
$\underline{T}_{\boldsymbol{f}}\subseteq\underline{X}\times_{\underline{Y}}\partial\underline{Y}$ C^∞-scheme associated to 1-morphism $\boldsymbol{f}:\mathbf{X}\rightarrow\mathbf{Y}$ in $\mathbf{dSpa^c}$, 49, 301
$\mathcal{T}_f\subseteq\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ C^∞-stack associated to 1-morphism $f:\mathscr{X}\rightarrow\mathscr{Y}$ in $\mathbf{Orb^c}$, 116, 444
$\mathcal{T}_{\boldsymbol{f}}\subseteq\mathcal{X}\times_{\mathcal{Y}}\partial\mathcal{Y}$ C^∞-stack associated to 1-morphism $\boldsymbol{f}:\boldsymbol{\mathfrak{X}}\rightarrow\boldsymbol{\mathfrak{Y}}$ in $\mathbf{dSta^c}$, 125, 528
vect$(\underline{X})$ category of vector bundles on C^∞-scheme $\underline{X}$, 14, 672
vect$(\mathcal{X})$ category of vector bundles on Deligne–Mumford C^∞-stack $\mathcal{X}$, 75, 687
vqcoh$(\underline{X})$ 2-category of virtual quasicoherent sheaves on a C^∞-scheme $\underline{X}$, 28, 205
vqcoh$(\mathcal{X})$ 2-category of virtual quasicoherent sheaves on a Deligne–Mumford C^∞-stack $\mathcal{X}$, 96, 488
$(\underline{V}\rightrightarrows\underline{U})$-mod category of modules on a groupoid $\underline{V}\rightrightarrows\underline{U}$ in $\mathbf{C^\infty Sch}$, 689
vvect$(\underline{X})$ 2-category of virtual vector bundles on a C^∞-scheme $\underline{X}$, 29, 206
vvect$(\mathcal{X})$ 2-category of virtual vector bundles on a Deligne–Mumford C^∞-stack $\mathcal{X}$, 96, 488
$\underline{W},\underline{X},\underline{Y},\underline{Z},\ldots$ C^∞-schemes, 11, 666
$\boldsymbol{W},\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\ldots$ d-spaces, including d-manifolds, 17, 158
$\mathcal{W},\mathcal{X},\mathcal{Y},\mathcal{Z},\ldots$ Deligne–Mumford C^∞-stacks, including orbifolds, 69, 683
$\boldsymbol{\mathcal{W}},\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}},\ldots$ d-stacks, including d-orbifolds, 87, 467
$\mathscr{W},\mathscr{X},\mathscr{Y},\mathscr{Z},\ldots$ orbifolds with corners, 113, 439
$\boldsymbol{\mathfrak{W}},\boldsymbol{\mathfrak{X}},\boldsymbol{\mathfrak{Y}},\boldsymbol{\mathfrak{Z}},\ldots$ d-stacks with corners, including d-orbifolds with corners, 122, 525

$\bar{\underline{X}}$ C^∞-stack associated to a C^∞-scheme $\underline{X}$, 69, 675

$[\underline{X}/G]$ quotient C^∞-stack, 72, 681

$\boldsymbol{X}^\Gamma$ fixed d-subspace of group Γ acting on a d-space $\boldsymbol{X}$, 23, 200

X^Γ fixed subset of a group Γ acting on a manifold with corners X, 47, 288

$\mathbf{X}^\Gamma$ fixed d-subspace of group Γ acting on a d-space with corners $\mathbf{X}$, 57, 379

$\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma, \mathcal{X}^\Gamma_\circ, \tilde{\mathcal{X}}^\Gamma_\circ, \hat{\mathcal{X}}^\Gamma_\circ$ orbifold strata of a Deligne–Mumford C^∞-stack $\mathcal{X}$, 78, 695

$\mathcal{X}^{\Gamma,\lambda}, \tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \mathcal{X}^{\Gamma,\lambda}_\circ, \tilde{\mathcal{X}}^{\Gamma,\mu}_\circ, \hat{\mathcal{X}}^{\Gamma,\mu}_\circ$ orbifold strata of an orbifold $\mathcal{X}$, 85, 432

$\boldsymbol{\mathcal{X}}^\Gamma, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \hat{\boldsymbol{\mathcal{X}}}^\Gamma, \boldsymbol{\mathcal{X}}^\Gamma_\circ, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ, \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ$ orbifold strata of a d-stack $\boldsymbol{\mathcal{X}}$, 94, 480

$\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ, \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ orbifold strata of a d-orbifold $\boldsymbol{\mathcal{X}}$, 107, 509

$\mathfrak{X}^{\Gamma,\lambda}, \tilde{\mathfrak{X}}^{\Gamma,\mu}, \hat{\mathfrak{X}}^{\Gamma,\mu}, \mathfrak{X}^{\Gamma,\lambda}_\circ, \tilde{\mathfrak{X}}^{\Gamma,\mu}_\circ, \hat{\mathfrak{X}}^{\Gamma,\mu}_\circ$ orbifold strata of an orbifold with corners $\mathfrak{X}$, 120, 457

$\boldsymbol{\mathfrak{X}}^\Gamma, \tilde{\boldsymbol{\mathfrak{X}}}^\Gamma, \hat{\boldsymbol{\mathfrak{X}}}^\Gamma, \boldsymbol{\mathfrak{X}}^\Gamma_\circ, \tilde{\boldsymbol{\mathfrak{X}}}^\Gamma_\circ, \hat{\boldsymbol{\mathfrak{X}}}^\Gamma_\circ$ orbifold strata of a d-stack with corners $\boldsymbol{\mathfrak{X}}$, 129, 546

$\boldsymbol{\mathfrak{X}}^{\Gamma,\lambda}, \tilde{\boldsymbol{\mathfrak{X}}}^{\Gamma,\mu}, \hat{\boldsymbol{\mathfrak{X}}}^{\Gamma,\mu}, \boldsymbol{\mathfrak{X}}^{\Gamma,\lambda}_\circ, \tilde{\boldsymbol{\mathfrak{X}}}^{\Gamma,\mu}_\circ, \hat{\boldsymbol{\mathfrak{X}}}^{\Gamma,\mu}_\circ$ orbifold strata of a d-orbifold with corners $\boldsymbol{\mathfrak{X}}$, 139, 566

$\mathcal{X}_{\rm top}$ underlying topological space of a C^∞-stack $\mathcal{X}$, 70, 678

$\underline{\mathcal{X}}_{\rm top}$ coarse moduli C^∞-scheme of a Deligne–Mumford C^∞-stack $\mathcal{X}$, 685

Index

∞-category, 6–8, 20, 22, 36, 147, 186, 210, 258, 626–629, 647–653
2-category, 17–19, 28, 48–49, 69, 73, 82, 87–89, 112–113, 122, 158–163, 205, 298–302, 426, 439–440, 467–470, 525–531, 627, 656–658, 675
 1-morphism, 18, 69, 88, 113, 159, 468, 656, 675, 676
  composition, 18, 89, 159, 468, 656
 2-Cartesian square, 22, 351–366, 372–378, 658, 680
  locally 2-Cartesian, 314, 319
 2-commutative diagram, 71, 113, 658
 2-morphism, 18, 69, 89, 113, 160, 469, 656, 675, 676
  horizontal composition, 19, 89, 161, 302, 470, 531, 657, 690
  vertical composition, 19, 89, 160, 302, 470, 531, 656, 679, 682, 690, 699
 colimit, 658
 equivalence in, 20, 30, 50, 148, 164, 206, 657
 fibre products in, 22, 54–56, 74, 82, 94, 119, 127–129, 137, 186, 426, 455, 479, 543–546, 561, 657–658
 homotopy category, 82–84, 147–148, 186, 278–280, 426, 427, 439, 478, 498, 518, 598, 600, 614, 636–638, 657
 limit, 658
 pseudofunctor, 691–692, 701
 pushout, 21, 178–179, 658
 strict, 15, 656
 strict 2-functor, 29, 52, 59, 89, 90, 96, 113, 122, 131, 132, 206
 weak, 15, 82, 83, 426, 427, 657
 weak 2-functor, 95, 458, 485, 691, 701
abelian category, 13–15, 75, 85, 205, 663, 672, 687, 689, 706
 split short exact sequence, 30, 32, 61, 100, 134, 166–170, 197, 237, 398, 496, 555
adjoint functor, 664–666, 672
algebraic space, 74
Atiyah–Singer Index Theorem, 434, 601
atlas, 69, 676
Axiom of Choice, 76, 637, 664, 690

b-transversality, 54–56, 127–129, 345–366, 541–543
Banach manifold, 147, 598–610, 616–617
 Banach vector bundle, 598–600, 616–617
  Fredholm section, 598–600
 Implicit Function Theorem, 604–605
 smooth map, 598–600
Banach orbifold, 600
bd-transversality, 64, 137, 409–414, 561–562
bordism, 141–146, 571–597
 classical bordism, 141–142, 571–574
  fundamental class, 573
  intersection product, 142, 573
  pushforward, 573
  unitary, 574
  unoriented, 574, 578, 649
 cobordism, *see* cobordism
 d-manifold bordism, 142–143, 575–580, 638
  fundamental class, 577
  intersection product, 143, 577
  pushforward, 578
  unitary, 578
  unoriented, 578
 d-orbifold bordism, 111, 112, 145–146, 521, 587–591, 638

    and orbifold strata, 146, 588–589
    effective, 145–146, 587–588
    fundamental class, 587
    intersection product, 145, 587
    pushforward, 587
    semieffective, 145–146, 587–588
    unitary, 591, 616
    unoriented, 590
  for Spivak's derived manifolds, 143, 575, 578, 649
  orbifold bordism, 143–145, 580–586
    and orbifold strata, 144–145, 584–586
    effective, 144, 580
    fundamental class, 581
    intersection product, 144, 580
    Poincaré duality fails, 583
    pushforward, 581
    unitary, 586
    unoriented, 586
    with arbitrary support, 583
  Poincaré duality, 573, 574, 583
  projection to homology, 142–144, 146, 573, 578, 581, 588
  with arbitrary support, 574
bordism ring, 573
boundary
  of a d-space with corners, 49, 314–321
  of a d-stack with corners, 123, 535–537
  of a manifold with corners, 40, 282
  of an orbifold with corners, 115, 446

C^∞-algebraic geometry, 9–17, 69–81
C^∞-group, 678
C^∞-ring, 9–10, 659–662
  as commutative $\mathbb{R}$-algebra, 660, 669
  C^∞-derivation, 670
  C^∞-local ring, 306, 661, 665
  colimit, 660
  coproduct, 660
  cotangent module $\Omega_{\mathfrak{C}}$, 13–14, 151–155, 669–671
  definition, 659
  fair, 661–662, 668
  finitely generated, 10, 661–662, 666, 671
  finitely presented, 661–662, 668, 671
  ideal, *see* ideal in C^∞-ring
  module, *see* module over C^∞-ring
  not noetherian, 661, 667, 672
  of a manifold X, 667
  simplicial, 648, 650, 652
  square zero extension, 150–155, 212
C^∞-ringed space, 10, 665–666
  cotangent sheaf, 673
  local, 665, 666
  morphism, 665
  sheaves of $\mathcal{O}_X$-modules on, 671–674
    pullback, 671
C^∞-scheme, 9–13, 665–669
  affine, 11, 666
  C^∞-group, 678
  closed embedding, 677
  coherent sheaves on, 15, 672
  compact, 666
  cotangent sheaf, 16–17, 673–674
  cotangent space $T_x^*\underline{X}$, 214–216
  definition, 666
  embedding, 677
  étale morphism, 69, 677
  fair affine, 666
  fibre products, 11–12, 666
  finitely presented affine, 210, 666
  is a manifold, 216
  line bundle on, 674
    orientation, 674
  locally compact, 666
  locally fair, 11, 666, 667, 673, 674
  locally finitely presented, 213–216, 666
  morphism, 665
  obstruction space $O_x\underline{X}$, 214–216
  open embedding, 69, 677
  paracompact, 666, 667, 673
  proper morphism, 70, 677
  quasicoherent sheaves on, 14–17, 671–674

fine, 673
pullback, 15, 155
second countable, 666
separated, 666, 667, 673, 706
separated morphism, 677
sheaves of $\mathcal{O}_X$-modules on, 671–674
spectrum functor, 11, 14, 59, 155, 650, 666, 672
square zero extension, 155–158, 164, 317
submersion, 677
universally closed morphism, 70, 677
vector bundles on, 14, 672
C^∞-stack, 69–74, 675–683
1-morphism, 69, 676
2-morphism, 69, 676
associated to a groupoid, 676, 682, 684, 689
C^∞-substack, 70, 677
open, 70, 677, 684, 695–697, 699
closed embedding, 677
definition, 676
Deligne–Mumford, *see* Deligne–Mumford C^∞-stack
embedding, 677, 686
étale 1-morphism, 70, 677
fibre products, 71, 74, 677, 680, 684
gluing by equivalences, 678–679
is a C^∞-scheme, 676, 686, 699
isotropy group $\mathrm{Iso}_{\mathcal{X}}([x])$, *see* orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$
open cover, 70, 677, 688
open embedding, 70, 677
orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$, 70, 74, 78, 97, 99, 111, 114, 490, 494, 520, 678, 684, 686, 695, 699, 705
proper 1-morphism, 70, 677, 695, 700
quotients $[\underline{X}/G]$, 72–73, 79, 449, 675, 679, 681–685
definition, 72, 681
orbifold strata, 706–707
quotient 1-morphism, 73, 682–685
quotient 2-morphism, 73, 682–685
strictly functorial, 683
representable 1-morphism, 677, 680, 682, 684, 700–701
separated, 70, 677, 684, 699
separated 1-morphism, 677
stabilizer group $\mathrm{Iso}_{\mathcal{X}}([x])$, *see* orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$
strongly representable 1-morphism, 71–72, 112, 114–116, 122, 123, 439, 442–444, 447, 450, 456, 525, 528, 533, 535, 536, 538, 547, 679–681, 683, 697, 700, 701
submersion, 677, 686
underlying topological space $\mathcal{X}_{\mathrm{top}}$, 70, 78, 675, 678, 683
universally closed 1-morphism, 70, 677, 684
C^∞-substack, 677
open, 677, 684, 695–697, 699
c-transversality, 54–56, 127–129, 345–351, 541–543
Calabi–Yau 3-fold, 149, 633
Cantor set, 24
Cartesian square, 291, 367, 655, 674
category, 654–656
2-category, *see* 2-category
abelian, *see* abelian category
Cartesian square, 55, 655
colimit, 655–656, 660
equivalence of, 475, 655, 689, 706
fibre product, 655, 666, 671
functor, 69, 654
faithful, 654
full, 654
natural isomorphism, 69, 654
natural transformation, 654
groupoid, 654, 656
isofibration, 680
limit, 655–656, 666
morphism, 654
opposite, 654

pushout, 177–178, 215, 278, 655, 660, 662, 671
subcategory, 654
full, 654
terminal object, 655, 656
universal property, 655
cd-transversality, 64, 137, 409–414, 561–562
cobordism
classical cobordism, 573–574, 581
cap product, 574
cup product, 574
pullback, 574
compactly-supported, 574
orbifold cobordism, 581–583
cap product, 582
compactly-supported, 583
cup product, 582
effective, 582
Poincaré duality fails, 583
pullback, 583
Poincaré duality, 573, 574, 583
cohomology, 574
compactly-supported, 574
colimit, 660
contact homology, 6
coproduct, 660
cotangent complex, 7, 29, 88, 150, 206, 214, 629–633

d-manifold, 23–40, 205–280
and Banach manifolds with Fredholm sections, 147, 598–610
and dg-manifolds, 149
and M-polyfolds, 147, 612–614
and quasi-smooth derived schemes, 147, 636–639
and schemes with obstruction theories, 147, 636–647
and solutions of elliptic equations, 148, 600–601
and Spivak's derived manifolds, 148, 650–653
as d-manifold with corners, 58, 388
bordism, 142–143, 513, 575–580, 638
fundamental class, 577
intersection product, 143, 577
pushforward, 578
unitary, 578
unoriented, 578
d-submanifold, 33, 240
d-transverse 1-morphisms, 34–36, 247–259, 274, 277, 279, 409, 507
definition, 24, 213
embedding, 33–34, 36, 240–247, 258–259, 279, 401
into manifolds, 36–37, 138, 259–266, 563
equivalence, 30–33, 229–235, 241
étale 1-morphism, 30, 229–232, 241, 273, 279
example which is not principal, 37, 266
fibre products, 34–36, 247–259, 274, 277, 279
d-transverse, 35
orientations on, 39–40
fixed point loci, 246–247
gluing by equivalences, 31–32, 233–235
homotopy category, 278–280
immersion, 33–34, 36, 37, 240–247, 259, 262, 279, 401
is a manifold, 24, 29, 35, 214, 221, 246, 257, 279
local properties, 214–221
open d-submanifold, 33, 214, 240
orientation line bundle, 38, 272, 274
orientations, 38–40, 147, 149, 272–278, 636
principal, 23–24, 36–37, 58, 210–213, 265–266, 385, 563, 651
standard model, 24–28, 30–32, 37, 149, 211–212, 221–229, 234, 259, 385, 392, 599–600, 602–610, 629, 652

1-morphism, 26, 223–229, 232–234, 242, 599–600, 652
2-morphism, 27, 228–229, 234, 652
orientations on, 39, 273–274
submersion, 33–35, 240–247, 257, 279, 401
virtual class, 143, 578, 638
virtual cotangent bundle, 29, 213, 214, 221, 266
virtual dimension, 24, 35, 213, 250, 256, 601
w-embedding, 33–34, 240–247, 279, 401
w-immersion, 33–34, 240–247, 279, 401
w-submersion, 33–35, 240–247, 257, 279, 401
why **dMan** is a 2-category, 36, 258
d-manifold with boundary, 58, 66, 142, 388, 571, 575–577, 590, 649
embedding
into manifolds, 415–416
d-manifold with corners, 58–68, 385–424
and Banach manifolds with Fredholm sections, 147, 600
and M-polyfolds, 147, 612–614
bd-transverse 1-morphisms, 64, 409–414, 420
boundary, 59, 388
conormal bundle $\mathcal{N}_{\mathbf{X}}$, 59, 60, 68, 387, 389, 390, 393, 421
orientation on, 68, 303, 421
cd-transverse 1-morphisms, 64, 68, 409–414
corner functors, 59, 389, 402
d-submanifold, 63, 401
definition, 58–59, 385–388
embedding, 62–65, 401–409, 414
into manifolds, 65–67, 414–419
equivalence, 61–62, 398–401, 414
étale 1-morphism, 61, 398–399, 403
fibre products, 59, 64–65, 409–414
bd-transverse, 64, 420
boundary of, 423
cd-transverse, 423, 424
not bd-transverse, 369
orientations on, 68
fixed point loci, 409
flat 1-morphism, 65, 398, 413, 415, 424
gluing by equivalences, 61–62, 399–401
group action on, 409
immersion, 62–65, 401–409, 414
include d-manifolds, 58, 388
is a manifold, 59, 64, 388, 393, 413, 423
k-corners $C_k(\mathbf{X})$, 388
not orientable, 422–423
local properties, 389–394
of mixed dimension, 59, 389
open d-submanifold, 63, 388, 401, 413
orientation line bundle, 67, 420
orientations, 67–68, 376, 419–424
principal, 58, 65–67, 385, 388, 419
s-embedding, 62–65, 401–409, 413
s-immersion, 62–65, 401–409, 413
s-submersion, 62–64, 401–409
semisimple 1-morphism, 65, 402, 413, 424
sf-embedding, 62–67, 401–409, 413, 416–419
sf-immersion, 62–65, 401–409, 413, 416
sfw-embedding, 62–64, 401–409
sfw-immersion, 62–64, 401–409
simple 1-morphism, 398, 415, 424
standard model, 60–61, 65–67, 385–387, 390–393, 399, 400, 419
1-morphism, 60–61, 394–400, 553
boundary of, 60, 387
corners of, 387
submersion, 62–64, 401–409
sw-embedding, 62–64, 401–409
sw-immersion, 62–64, 401–409
sw-submersion, 62–64, 401–409, 423–424
virtual cotangent bundle, 58, 388
virtual dimension, 58, 64, 388

w-embedding, 62–64, 401–409
w-immersion, 62–64, 401–409
w-submersion, 62–64, 401–409, 413, 423
d-orbifold, 96–112, 488–524
and Banach orbifolds with Fredholm sections, 147, 600
and Deligne–Mumford stacks with obstruction theories, 147, 636–647
and Kuranishi spaces, 111, 147, 622–626
and polyfolds, 147, 612–614
and quasi-smooth derived Deligne–Mumford stacks, 147, 636–639
as d-orbifold with corners, 131, 552
bordism, 111, 112, 145–146, 521, 587–591, 638
and orbifold strata, 146, 588–589
effective, 145–146, 587–588
fundamental class, 587
intersection product, 145, 587
pushforward, 587
semieffective, 145–146, 587–588
unitary, 591, 616
unoriented, 590
d-suborbifold, 104, 500
d-transverse 1-morphisms, 104–105, 501–504
definition, 96–97, 489
effective, 111–112, 146, 511, 520–524, 571, 588, 589
orbifold strata of, 112, 523–524
embedding, 103–105, 500–501, 504
into orbifolds, 105–106, 504–505
equivalence, 99–101, 491, 495–496
étale 1-morphism, 100–101, 104, 495–496, 500, 507
fibre products, 104–105, 501–504
gluing by equivalences, 100–103, 496–499
good coordinate system, 32, 108–112, 146, 235, 498, 513–522, 588, 591–597, 708–737
type A, 514–516
type B, 517–520
immersion, 103–105, 500–501, 504
is a d-manifold, 97, 107, 489, 490, 509
is an orbifold, 97, 104, 105, 489, 492, 501, 503, 504
Kuranishi neighbourhood, 108–111, 513–520
coordinate change, 108–109, 233, 514–518
local properties, 97–100, 490–492
locally orientable, 511
open d-suborbifold, 104, 490, 500
orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$, 99, 111, 490, 494, 520
orbifold strata, 106–108, 146, 247, 508–513, 588–589
orientations on, 107–108, 112, 511–513, 524
orientation line bundle, 106, 506
orientations, 106–108, 148, 149, 506–508, 511–513
perturbing to orbifolds, 111–112, 146, 520–523, 588, 591–597
principal, 97, 105–106, 111, 489–490, 494, 504–505
quotients $[\boldsymbol{X}/G]$, 105, 493–495, 504
semieffective, 111–112, 146, 511, 520–524, 571, 588, 590
orbifold strata of, 112, 523–524
standard model $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$, 97–99, 101–102, 105, 110, 490–492, 505, 517–520
1-morphism, 98, 101, 492–493, 496–498, 500–501
orbifold strata, 509–510
standard model $[\boldsymbol{S}_{V,E,s}/\Gamma]$, 99–100, 102–103, 108–110, 493–495, 498–499, 511, 514–516
1-morphism, 100, 494, 498–499
2-morphism, 100, 494
submersion, 103–105, 500–501, 504
very good coordinate system, 519–520, 522, 591
virtual class, 146, 588, 589, 615, 638

virtual cotangent bundle, 97, 106, 489, 492, 506
virtual dimension, 97, 104, 105, 489, 502, 504
w-embedding, 103–104, 500–501
w-immersion, 103–104, 500–501
w-submersion, 103–104, 500–501, 504
d-orbifold with boundary, 131, 140, 145, 551, 567–569, 587, 590, 615, 638
d-orbifold with corners, 130–141, 550–570
and Banach orbifolds with Fredholm sections, 147, 600
and Kuranishi spaces, 147, 622–626
and polyfolds, 147, 612–614
bd-transverse 1-morphisms, 137, 561–562, 564
boundary, 71, 132, 552, 570
conormal bundle $\mathcal{N}_{\mathfrak{X}}$, 133, 551, 553, 565
orientation on, 565
cd-transverse 1-morphisms, 137, 561–562
corner functors, 132, 553
d-suborbifold, 137, 559
definition, 131–132, 551
effective, 141, 569–570
embedding, 136–137, 558–561
into manifolds, 728–732
into orbifolds, 138–139, 563–564
equivalence, 133–136, 555–558, 567
étale 1-morphism, 133–134, 137, 555, 559
fibre products, 137–138, 561–562, 564
bd-transverse, 137, 561
boundary of, 562, 565
cd-transverse, 137, 562, 565
corners, 562
flat 1-morphism, 133, 138, 141, 554, 555, 562, 568
gluing by equivalences, 134–136, 556–558
good coordinate system, 140–141, 558, 568–569, 626, 708–737
type A, 568–569
type B, 569
immersion, 136–137, 558–561
include d-orbifolds, 131, 552
is a d-manifold, 131, 132, 552
is an orbifold, 131, 137, 138, 552, 553, 561, 562, 565
k-corners $C_k(\mathfrak{X})$, 132, 141, 552, 570
Kuranishi neighbourhood, 140–141, 568–569
coordinate change, 140, 568, 569
local properties, 132–133, 553–555
of mixed dimension, 132, 553, 567
open cover, 716–728
open d-suborbifold, 137, 552, 559
orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$, 555
orbifold strata, 139–140, 566–568
boundaries of, 139, 567
orientations on, 139, 568
orientation line bundle, 139
orientations, 139, 564–566
principal, 131, 133, 138–139, 550–551, 554, 563–564
quotients $[\mathbf{X}/G]$, 555
representable 1-morphism, 555, 559, 564
s-embedding, 136–138, 558–562
s-immersion, 136–138, 558–562
s-submersion, 136–137, 558–561
semieffective, 141, 569–570
semisimple 1-morphism, 138, 141, 554, 562, 568
sf-embedding, 136–139, 558–564, 728–732
sf-immersion, 136–138, 558–562
sfw-embedding, 136–137, 558–561
sfw-immersion, 136–137, 558–561
simple 1-morphism, 133, 554, 555
standard model $\mathbf{S}_{\mathcal{V},\mathcal{E},s}$, 132–133, 135, 138–139, 141, 550–551, 557, 563–564, 567, 569
1-morphism, 133, 134, 553–556, 560–561
boundary, 133, 551

corners, 551
standard model $[\mathbf{S}_{V,E,s}/\Gamma]$, 133, 136, 140–141, 554–555, 558, 568
1-morphism, 133, 554–555
straight, 140, 567
submersion, 136–138, 558–562
sw-embedding, 136–137, 558–561
sw-immersion, 136–137, 558–561
sw-submersion, 136–137, 558–561, 565–566
very good coordinate system, 569, 596
virtual cotangent bundle, 132, 139, 551, 553, 564
virtual dimension, 131, 137, 139, 551, 552, 561, 566, 567
w-embedding, 136–137, 558–561
w-immersion, 136–137, 558–561, 566–567
w-submersion, 136–137, 558–562, 565–566
d-space, 17–23, 150–204
1-morphism, 18, 159
2-morphism, 18, 160–161
as d-space with corners, 49, 302
definition, 17, 158–161
equivalence, 20, 164–171
étale 1-morphism, 170, 337
fibre products, 22, 186–198, 279
fibre products of manifolds, 198–199
fixed point loci, 23, 200–204, 246–247, 485–486, 546
gluing by equivalences, 20–22, 171–186
is a C^∞-scheme, 19, 161, 170, 209
is a manifold, 19, 161
open cover, 20, 170
open d-subspace, 20, 170, 214
products, 196–197
virtual cotangent sheaf, 18, 29, 158, 206–207, 209
d-space with boundary, 49, 300
d-space with corners, 48–57, 298–384
1-morphism
definition, 300–301
2-morphism
definition, 301–302
alternative definitions, 304–305, 321, 371
b-transverse 1-morphisms, 54–56, 345–366, 376, 385, 409
boundary, 49, 314–321
conormal bundle $\mathcal{N}_{\mathbf{X}}$, 48, 49, 51, 52, 54, 299–300, 302, 303, 307, 311, 319, 320, 329, 345, 351, 379
strictly functorial, 51, 303, 321
boundary defining function, 300, 305–306, 308, 323, 331, 370, 381–382
c-transverse 1-morphisms, 54–56, 345–351, 367, 376, 409
corner functors, 52–54, 56, 59, 339–345, 371–376, 383
definition, 48–49, 298–302
equivalence, 52, 332–339
étale 1-morphism, 336–337, 383
fibre products, 54–56, 345–378
b-transverse, 55, 303, 304, 361
boundary, 56, 371–378
c-transverse, 371–378
corners, 56, 371–376
may not exist, 54, 369–370
not b-transverse, 368–371
fixed point loci, 57, 378–384, 409, 546
flat 1-morphism, 49–52, 55, 321–328, 332–337, 347, 376–378
gluing by equivalences, 52, 337–339
group action on, 378, 409
include d-spaces, 49, 302
include manifolds with corners, 51–52, 329–332
is a manifold, 52, 329
k-corners $C_k(\mathbf{X})$, 52–54, 339–345
open cover, 52, 300
open d-subspace, 52, 300, 337
products, 366
semisimple 1-morphism, 49–52, 55, 321–328, 332, 343, 347, 376–

378
simple 1-morphism, 49–52, 303, 321–328, 332–337, 376–378
weak 2-morphism, 304, 321
d-stack, 87–96, 463–487
1-morphism, 88, 468
2-morphism, 89, 469
definition, 87–90, 467–470
equivalence, 92–95, 471–472, 476–478, 485
étale 1-morphism, 92, 472
fibre products, 94, 479–480, 502
gluing by equivalences, 22, 92–94, 476–478
conditions on overlaps, 93, 476–478, 498
is a C^∞-stack, 89, 470, 479
is an orbifold, 90, 470
open cover, 92, 472
open d-substack, 92, 472, 482
orbifold strata, 23, 94–96, 200, 480–487
lifting 1- and 2-morphisms to, 95, 483–485
quotients $[\boldsymbol{X}/G]$, 23, 90–92, 95, 105, 200, 472–476, 485–486, 493–495, 504
quotient 1-morphism, 90, 473–476
quotient 2-morphism, 91, 474–476
representable 1-morphism, 95, 483
virtual cotangent sheaf, 88, 96, 468, 488
of orbifold strata, 96, 486
with boundary, *see* d-stack with boundary
with corners, *see* d-stack with corners
d-stack with boundary, 122, 130, 526, 549, 551
d-stack with corners, 122–130, 525–549
1-morphism, 527–529
2-morphism, 529–531
b-transverse 1-morphisms, 127–129, 131, 541–543, 550
boundary, 71, 123, 535–537
conormal bundle $\mathcal{N}_{\mathfrak{X}}$, 122, 124, 126, 127, 525–527, 529, 531–533, 542
strictly functorial, 123, 533, 536–537
boundary defining function, 533
c-transverse 1-morphisms, 127–129, 541–543
corner functors, 126–127, 129, 132, 540–541, 545, 549, 553
definition, 122–123, 525–531
equivalence, 125–126, 537–540, 548, 549
étale 1-morphism, 124, 537
étale locally modelled on d-spaces with corners, 122, 526, 527, 532, 533, 535, 536, 538, 540, 547, 554
fibre products, 127–129, 541–546
b-transverse, 128, 543
boundary, 129, 545–546
corners, 129, 545–546
flat 1-morphism, 125, 127, 128, 131, 536–537, 541, 543, 550
gluing by equivalences, 125–126, 537–540
conditions on overlaps, 539
include d-stacks, 122, 531
is a d-space, 123, 532, 547
is an orbifold, 123, 532
k-corners $C_k(\mathfrak{X})$, 126–127, 540–541
open cover, 126, 527
open d-substack, 126, 527, 537
orbifold strata, 129–130, 546–549
products, 545
quotients $[\mathbf{X}/G]$, 57, 123–124, 133, 533–535, 548, 555
representable 1-morphism, 537, 547
semisimple 1-morphism, 125, 127, 128, 536–537, 541, 543
simple 1-morphism, 125, 127, 131, 536–537, 541, 550
straight, 130, 140, 549, 567
virtual cotangent sheaf, 551

d-transversality, 34–36, 104–105, 247–259, 501–504
Deligne–Mumford C^∞-stack, 69–81, 675–707
  coarse moduli C^∞-scheme $\underline{\mathcal{X}}_{\mathrm{top}}$, 477, 538, 685, 688, 721
  coherent sheaves on, 687
  compact, 683
  cotangent sheaf, 75, 77–78, 81, 692
  definition, 73, 683
  effective, 523, 539, 686
  fibre products, 74
  inertia stack, 79
  line bundle on, 692
    orientation, 692
  locally compact, 683
  locally fair, 73, 463, 683–685, 687, 688, 692, 699, 707
  locally finitely presented, 489, 491, 683–685, 699
  orbifold strata, 78–81, 430, 546, 695–707
    cotangent sheaves, 707
    functoriality, 79
    lifting 1- and 2-morphisms to, 700–701
    of quotient C^∞-stacks, 706–707
    sheaves on, 431, 703–707
  paracompact, 683, 685, 688
  partition of unity on, 477, 538, 593, 685–686, 688–689, 721
  quasicoherent sheaves on, 74–78, 428, 463–467, 686–692, 703–707
    pullbacks, 76–77
    restriction to orbifold strata, 80
  representable 1-morphism, 95, 104, 137, 458, 483, 500, 559
  second countable, 427, 683, 685, 699
  separated, 427, 683, 685, 688
  sheaves of $\mathcal{O}_{\mathcal{X}}$-modules on, 686–692
    pullback, 689–692
  sheaves of abelian groups on, 75, 693–694
    pullback, 694
  sheaves of C^∞-rings on, 75, 693–694
    pullback, 694
  square zero extension, 88, 463–467, 481, 491
    morphism, 464
  structure sheaf $\mathcal{O}_{\mathcal{X}}$, 75, 694
  vector bundles on, 75, 428, 687
    of mixed rank, 432
Deligne–Mumford stack with obstruction theory, 149, 630, 638
  and d-orbifolds, 147, 636–647
derived algebraic geometry, 6–8, 19–20, 626–630, 635–639
derived category, 7, 149, 631, 634, 644, 646
derived Deligne–Mumford stack
  quasi-smooth, 629
    and d-orbifolds, 147, 636–639
derived manifold, *see* Spivak's derived manifolds
derived scheme, 19–20, 626–630
  cotangent complex, 629
  quasi-smooth, 8, 627, 629
    and d-manifolds, 147, 636–639
  square zero, 628
derived stack, 626–630
dg-algebra, 20, 628, 629
  square zero, 20, 628, 629
dg-manifold, 149
dg-scheme, 6, 8, 149, 628–629
Differentiable Collaring Theorem, 572, 575
Donaldson–Thomas invariants, 143, 588, 627, 630, 632

elliptic equations, 148, 600–601
étale topology, 13, 74, 92, 106, 472, 500, 667

Fano 3-fold, 149, 633
fibre product, 666, 671
  definition, 655
  in 2-category, 657–658

of C^∞-schemes, 11
of C^∞-stacks, 74
of d-manifolds, 35, 247–259
of d-manifolds with corners, 64, 409–414
of d-orbifolds, 104, 502
of d-orbifolds with corners, 137, 561
of d-spaces, 22, 186–198
of d-spaces with corners, 55, 345–378
of d-stacks, 94, 479
of d-stacks with corners, 128, 543
of orbifolds with corners, 119, 455
fine sheaf, 8, 629, 673
fractal, 24
Fukaya categories, 6
functor, 654
adjoint, 664–666, 672
exact, 663, 664
faithful, 10, 11, 19, 52, 69, 82, 90, 113, 131, 161, 163, 302, 329, 330, 427, 441, 471, 532, 552, 654, 666, 669
full, 10, 11, 19, 52, 69, 82, 90, 113, 131, 161, 163, 302, 329, 330, 427, 441, 471, 532, 552, 654, 666, 669
left exact, 663
natural isomorphism, 654
natural transformation, 654
right exact, 689
truncation, 6, 146–148, 598–653

generalized cohomology theory, 573, 581
generalized homology theory, 142, 144, 573
global sections functor Γ, 665, 672
good coordinate system, 32, 108–112, 140–141, 146, 235, 498, 513–520, 558, 568–569, 588, 591–597, 622, 623, 708–737
type A, 514–516, 568–569
type B, 517–520, 569
very good, 519–520, 522, 569, 591, 596
Gromov–Witten invariants, 6, 143, 588, 591, 616, 620, 627, 630, 632, 633, 638
in d-orbifold bordism, 571, 591, 615–616
integrality properties, 591, 616
Grothendieck topology, 69, 82, 426, 675, 676
groupoid, 654

Hadamard's Lemma, 10, 153, 218, 244, 249, 660
harmonic maps, 148, 601
homology, 142–144, 146, 573, 574, 578, 581, 588
with arbitrary support, 574
homotopy category, 82–84, 147–148, 186, 278–280, 426, 427, 439, 478, 498, 518, 598, 600, 614, 636–638, 651, 657

ideal in C^∞-ring, 660–661
fair, 661
finitely generated, 661
inertia stack, 700

Kuranishi (co)homology, 146, 588
Kuranishi space, 6, 110–111, 148, 297, 591, 599, 620–626
and d-orbifolds, 111, 147, 514, 622–626
and polyfolds, 624–625
boundary, 622, 624
coordinate change, 233, 621
'fibre products' over manifolds, 39, 256, 276, 277, 423, 622
good coordinate system, 32, 108, 110, 513, 516, 519, 522, 622, 623, 625
Kuranishi neighbourhood, 111, 211, 620
orientation, 39, 274, 277, 421, 423, 622, 624
stably almost complex, 591
strongly smooth map, 622, 624, 625
weakly submersive, 622, 624

virtual chain, 522, 623
virtual class, 146, 522, 588, 623
virtual dimension, 621, 624
with a tangent bundle, 620, 623

Lagrangian Floer cohomology, 6, 143, 588, 616, 620, 629
Lagrangian submanifold, 7, 148, 624
locally 2-Cartesian, 314, 319
locally effective group action, 462, 557, 686

manifold
as C^∞-scheme, 11, 667–669
as d-space, 19, 161
C^∞-ring of, 659
cotangent bundle, 282, 670, 674
definition, 281
embedding, 36, 241, 259
immersion, 36, 236, 241, 259
orientation, 38, 273
submersion, 236, 241
tangent space, 282
transverse fibre products, 12, 22, 34, 198, 247, 255, 257, 276–277, 669
vector bundles on, 669
with boundary, *see* manifold with boundary
with corners, *see* manifold with corners
manifold with boundary, 40–48, 140, 142, 231, 281–297, 568, 571
definition, 281
manifold with corners, 40–48, 281–297
as C^∞-scheme, 667–669
as d-space with corners, 51–52, 329–332
boundary, 40, 282
boundary defining function, 41, 282, 331
C^∞-ring of, 659
corner functors, 44–46, 287–289, 291, 371
cotangent bundle, 282, 674
definition, 281
diffeomorphism, 42
embedding, 42–43, 292–294, 401, 414
fixed point loci, 47–48
flat map, 42–43, 52, 284–286, 290, 332, 399
immersion, 42–43, 292–294, 401
k-corners $C_k(X)$, 43, 286–289
not orientable, 422–423
local boundary component, 40, 282
orientations, 46–47, 294–297
s-embedding, 42–43, 292–294, 401, 414
s-immersion, 42–43, 292–294, 401
s-submersion, 42–43, 284–286
semisimple map, 42–43, 46, 52, 284–286, 290–291, 332
sf-embedding, 42–43, 140, 292–294, 401, 414, 568
sf-immersion, 42–43, 292–294, 401
simple map, 42–43, 52, 284–286, 290, 332, 399
smooth map, 41, 282–284
action on corners, 283–284
definition, 283
strongly transverse maps, 45–46, 291–292, 345, 367, 371
submanifold, 42, 292
submersion, 42–43, 141, 284–286, 289–290, 424, 568–569
tangent space, 282
transverse fibre products, 45–46, 289–292, 345, 366–368, 669
boundaries of, 45, 290–291
orientations on, 46, 296–297
vector bundles on, 669
weakly smooth map, 41, 48, 49, 282, 294, 331, 398, 668–670
module over C^∞-ring, 13–14, 151–155, 669–671
C^∞-derivation, 670
complete, 14, 672
cotangent module $\Omega_{\mathfrak{C}}$, 669–671
finitely presented, 669, 671, 672
moduli space, 32, 143, 148–149, 235, 588, 598, 627, 629

of algebraic curves, 149
of coherent sheaves on a 3-fold, 149
of coherent sheaves on a surface, 149
of harmonic maps, 148, 601
of J-holomorphic curves, 103, 110, 136, 148, 401, 498, 558, 591, 601, 610, 612, 614–616, 620, 623–624, 638
of perfect complexes on a 3-fold, 149
of PT pairs on a 3-fold, 149, 634
of solutions of nonlinear elliptic equations, 148, 600–601

orbifold, 81–87, 425–439, 683
a category or a 2-category?, 82, 426, 439
as Deligne–Mumford C^∞-stack, 82, 427–428
as groupoid in **Man**, 81–83, 426–427, 443
as orbifold with corners, 113, 440, 441
as stack on **Man**, 82, 426–427
classifying space $\mathcal{X}_{\mathrm{cla}}$, 583–584
cotangent bundle, 84, 428
different definitions, 81–83, 425–427
effective, 83–84, 86, 98, 101–102, 112, 434, 438–439, 491, 497–498, 505, 519–523
embedding, 83, 428, 439
étale 1-morphism, 83, 428
immersion, 83, 428, 433
is a manifold, 144, 427, 433, 439, 580
locally orientable, 86, 434, 590
orbifold bordism, 143–145, 580–586
and orbifold strata, 144–145, 584–586
effective, 144, 580
fundamental class, 581
intersection product, 144, 580
Poincaré duality fails, 583
pushforward, 581
unitary, 586
unoriented, 586
with arbitrary support, 583
orbifold cobordism, 581–583
cap product, 582
compactly-supported, 583
cup product, 582
effective, 582
Poincaré duality fails, 583
pullback, 583
orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$, 428–431, 439
orbifold strata, 84–87, 144, 430–439, 584–586
orientations on, 86–87, 434–438
orientations, 84, 86, 434–438
our definition, 82, 427
representable 1-morphism, 83, 428, 433, 505
smooth map, 427
submersion, 83, 428, 439
suborbifolds, 84
tangent bundle, 428
transverse fibre products, 82–84, 94, 426, 427, 480, 658
vector bundles on, 84, 428–430
pullback, 428
smooth section, 428
total space functor Tot, 84, 105, 429–430, 490, 505
with boundary, *see* orbifold with boundary
with corners, *see* orbifold with corners
orbifold strata
of d-orbifolds, 106–108, 146, 247, 508–513, 588–589
of d-orbifolds with corners, 139–140, 566–568
of d-stacks, 23, 94–96, 200, 480–487
of d-stacks with corners, 129–130, 546–549
of Deligne–Mumford C^∞-stacks, 78–81

of orbifolds, 84–87, 144–145, 430–439, 584–586
of orbifolds with corners, 119–121, 455–462
orbifold with boundary, 112–121, 144, 439–462, 580, 584, 586, 596
definition, 440
orbifold with corners, 112–121, 439–462
boundary, 71, 115, 446
conormal line bundle $\mathcal{N}_{\mathfrak{X}}$, 116, 440, 445
strictly functorial, 114, 442
corner functors, 117–118, 451–454
definition, 112–113, 439
effective, 115, 133, 462, 551, 556–558, 564
embedding, 118, 454
flat 1-morphism, 115–116, 118, 131, 446–448, 454, 550, 554, 555
immersion, 118, 454
is a manifold, 441, 457
k-corners $C_k(\mathfrak{X})$, 117–118, 448–454
open cover, 113, 441
open suborbifold, 113, 441, 447
orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$, 442, 454, 457, 461
orbifold strata, 119–121, 455–462
quotients $[X/G]$, 113, 118
representable 1-morphism, 555
s-embedding, 118, 454
s-immersion, 118, 454
s-submersion, 118, 454
semisimple 1-morphism, 115–116, 118, 446–448, 454, 554
sf-embedding, 118, 454, 556
sf-immersion, 118, 454
simple 1-morphism, 115–116, 118, 131, 441, 446–448, 454, 550, 554, 555
straight, 114, 121, 443, 455, 461–462
strongly transverse 1-morphisms, 119, 129, 454–455, 545
submersion, 118, 454
transverse fibre products, 119, 128, 454–455, 545
vector bundles on, 114, 445–446, 553, 556
smooth section, 445
total space functor $\mathrm{Tot}^{\mathrm{c}}$, 114, 131, 139, 445–446, 550, 563
orientation convention, 39, 46, 68, 272, 274, 276, 295–296, 303, 420, 421, 507, 564, 565
orientation line bundle, 38, 39, 67, 68, 106, 139, 266–272, 420, 506, 564

Pandharipande–Thomas invariants, 149, 630, 634
partition of unity, 8, 12–13, 21, 34, 170–172, 174, 178, 181, 226, 228, 248, 264, 294, 336, 337, 363, 365, 397, 415, 417–419, 476–477, 538, 593, 608, 629, 650, 666–667, 673, 685–686, 688–689, 710, 721, 726, 732
polyfold, 6, 148, 591, 598, 610–620, 624–625, 627, 638
and d-manifolds, 147, 612–614
and d-orbifolds, 147, 612, 614
and Kuranishi spaces, 624–625
gluing profile, 615, 638
M-polyfold, 610–614, 616–620
sc-Banach space, 616–617
sc^{∞}-retract, 612
sc-smooth map, 612, 616
sc-structure, 611
splicing core, 612, 617
strong M-polyfold bundle, 610–614, 616–620
Fredholm section, 610–614, 616–620
strong polyfold bundle, 610–614
Fredholm section, 610–614
Pontryagin class, 221
presheaf, 662, 673
sheafification, 663, 664, 673
prestack
stackification, 697–698, 700–701

principal d-manifold, 23–24, 36–37, 210–213, 265–266, 563, 651
principal d-manifold with corners, 58, 65–67, 385, 388, 419
principal d-orbifold, 97, 105–106, 111, 489–490, 494, 504–505
principal d-orbifold with corners, 131, 133, 138–139, 550–551, 554, 563–564
pseudofunctor, 691–692, 701
pushout, 21, 177–179, 215, 648, 655, 658, 660, 662, 671

quasi-smooth, 8, 147, 635–636
quotient C^∞-stack, 72–73, 449, 675, 679, 681–685
  1-morphism, 682–685
  2-morphism, 682–685
  definition, 72, 681
  orbifold strata
    sheaves on, 706–707
  strictly functorial, 683
quotient d-orbifold, 493–495
quotient d-stack, 23, 200, 493–495

scheme with obstruction theory, 143, 149–150, 588, 630
  and d-manifolds, 147, 636–647
  as a category, 147, 634–637
Schur's Lemma, 435
sheaf, 662–664
  definition, 662–663
  direct image, 663
  fine, 8, 629, 673
  inverse image, 664
  of abelian groups, 662, 693–694
  of C^∞-rings, 665–666, 693–694
  on topological space, 662–664
  presheaf, 662
  pullback, 664
  pushforward, 663
  soft, 8, 629, 650
  stalk, 663, 665
site, 82, 426, 675, 676, 696, 697
  subcanonical, 676
soft sheaf, 8, 629, 650
spectral sequence, 142, 573
Spivak's derived manifolds, 6, 7, 22, 36, 173, 186, 210, 233, 258, 627, 629, 647–653
  affine, 648
  and d-manifolds, 148, 650–653
  derived cobordism, 143, 575, 578, 649
  embedding into manifolds, 37, 259, 648, 649
  homotopy fibre products, 35, 198, 256, 649
  of finite type, 650, 651
  of pure dimension, 648, 651
  with boundary, 649
split short exact sequence, 30, 32, 61, 100, 134, 166–170, 197, 208, 230, 237, 299, 398, 496, 555
square zero extension, 88, 150–158, 164, 463–467, 637
square zero ideal, 17, 20, 25, 88, 150, 317, 652
stack, 6, 69, 73, 74, 82, 147, 148, 426, 626–630, 635–639
String Theory, 434
String Topology, 7
Symplectic Field Theory, 6, 616
symplectic geometry, 6–7, 110, 143, 148, 588, 591, 610, 620, 623, 625
synthetic differential geometry, 9, 659

topological space, 19–21, 24, 53, 70, 78, 79, 110, 115, 117, 126, 141, 574
  closed map, 70
  fibre product, 41, 45, 284
  Hausdorff, 11, 17, 31, 61, 101, 102, 134, 135, 158, 173, 666
  locally compact, 11, 17, 158, 666
  normal, 173
  paracompact, 11, 17, 158, 173, 666
  proper map, 70
  second countable, 11, 17, 31, 61, 101, 102, 134, 135, 158, 181, 182, 666

truncation functor, 6, 146–148, 598–653

virtual chain, 6, 110, 143, 146, 149, 522, 578, 588, 598, 616, 625
virtual class, 110, 143, 149, 150, 598, 627, 629, 630
for d-manifolds, 143, 571, 578, 638
for d-orbifolds, 111, 146, 588, 589, 615, 638
for Kuranishi spaces, 146, 522, 588
for quasi-smooth derived schemes, 627
for schemes with obstruction theory, 143, 588, 633
virtual cotangent bundle, 29, 38, 58, 67, 97, 106, 132, 139, 213, 214, 266, 388, 420, 489, 506, 508, 564
virtual quasicoherent sheaf, 28–29, 96–97, 205–210
is a quasicoherent sheaf, 209
is a vector bundle, 209
on C^∞-scheme, 28, 205
on Deligne–Mumford C^∞-stack, 96, 488
virtual vector bundle, 28–29, 58, 67, 96–97, 131, 136, 139, 205–210, 236–240, 387, 420, 551, 558, 564
injective 1-morphism, 32, 103, 237–240, 500
is a vector bundle, 29, 221, 492
of mixed rank, 107, 389, 393, 508, 566
on a C^∞-scheme, 28, 206
on a Deligne–Mumford C^∞-stack, 96, 103, 488, 500
orientation generator, 267
orientation line bundle of, 38, 106, 266–272, 389, 506
surjective 1-morphism, 32, 103, 237–240, 500
weakly injective 1-morphism, 32, 103, 237–240, 248, 500
weakly surjective 1-morphism, 32, 103, 237–240, 500

well-ordered set $(I,<)$, 515, 516, 518

Zariski topology, 13, 74, 92, 104, 106, 137, 429, 472, 500, 667