D-orbifolds, Kuranishi spaces, and polyfolds.

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based on

arXiv:0910.3518,

arXiv:1001.0023,

and work in progress.

See also

arXiv:0707.3572v5 and

arXiv:0710.5634v2.

These slides available at

`www.maths.ox.ac.uk/∼joyce/talks.html`

1. Introduction

Several important areas in Symplectic Geometry — open and closed Gromov–Witten invariants, Lagrangian Floer cohomology, Symplectic Field Theory, etc. — are concerned with *moduli spaces of stable J-holomorphic curves*. Here (M,ω) is a symplectic manifold, J is a choice of almost complex structure on M compatible with ω. We have to consider *moduli spaces* $\bar{\mathcal{M}}(M,J,\beta)$ of J-holomorphic curves $u:\Sigma\rightarrow M$ in M, where Σ is a Riemann surface, possibly with boundary or marked points, which may have nodal singularities, and β is some topological data we fix, e.g. the genus and number of marked points of Σ and the homology class $[u(\Sigma)]\in H_2(M;\mathbb{Z})$.

Then $\bar{\mathcal{M}}(M, J, \beta)$ is a topological space. By including 'stable' curves with nodal singularities, we can usually make $\bar{\mathcal{M}}(M, J, \beta)$ compact. The idea is to 'count' the moduli space $\bar{\mathcal{M}}(M, J, \beta)$ to get a 'number' of J-holomorphic curves with topological data β. This 'number' could be in $\mathbb{Z}$, or in $\mathbb{Q}$, or a chain or homology class in some homology theory. To do this 'counting' we need an extra geometric structure on $\bar{\mathcal{M}}(M, J, \beta)$, which makes it behave like a compact oriented manifold (possibly with boundary or corners). The goal is to make the 'number' of J-holomorphic curves independent of choice of J (possibly up to some kind of homotopy in a homology theory), and so is an *invariant* of (M, ω).

Any such J-holomorphic curve programme must solve four basic problems:

(a) Define the kind of geometric structure $\mathcal{G}$ you want to put on moduli spaces $\bar{\mathcal{M}}(M, J, \beta)$.

(b) Prove that moduli spaces $\bar{\mathcal{M}}(M, J, \beta)$ really do have geometric structure $\mathcal{G}$. Prove that relationships between curve moduli spaces (e.g. boundary formulae) lift to relations between the structures $\mathcal{G}$.

(c) Prove that given a compact topological space T with structure $\mathcal{G}$ you can define a 'number of points' in T, in $\mathbb{Z}$ or $\mathbb{Q}$ or some homology theory. (This is a *virtual cycle* or *virtual chain* construction.)

(d) Derive some interesting consequences in symplectic geometry – define G–W invariants, prove the Arnold Conjecture, etc.

Of these, to do problem (b) in full generality is definitely the worst: to model moduli spaces $\bar{\mathcal{M}}(M, J, \beta)$ near singular curves Σ involves gruesome analytic problems which take decades to sort out properly. In the beginning (the 1990s), it was not done properly. There was a race to get down to (d) and claim the geometric theorems first, so the treatment of (a)–(c) by some groups was rushed and unsatisfactory.
I am not going to talk about problem (b), in fact, I don't want to go anywhere near it. My interest today is in problem (a) (and problem (c)).

Oversimplifying rather, there are broadly three approaches in the literature:

(i) (The Fukaya school). Geometric structure $\mathcal{G}$ is called a *Kuranishi space*. (Actually, an 'oriented Kuranishi space with virtual tangent bundle'.) Fukaya–Ono 1999, Fukaya–Oh–Ohta–Ono 2003–2009.

(ii) (The Hofer school). Geometric structure $\mathcal{G}$ is called a *polyfold*. (Actually, a 'Fredholm section of a polyfold bundle over a polyfold'.) Hofer, Wysocki and Zehnder 2005–2020 (?). Seems to be rigorous.

(iii) (The rest of the world). Make strong assumptions on geometry, e.g. (M,ω) exact, J generic. Then ensure that moduli spaces $\bar{\mathcal{M}}(M,J,\beta)$ are manifolds (or at least 'pseudomanifolds').

Kuranishi spaces and polyfolds are philosophically opposed: Kuranishi spaces remember only minimal information about the moduli problem, but polyfolds remember essentially everything. E.g. for J-holomorphic curves, the polyfold moduli space remembers the Banach manifolds of (not necessarily J-holomorphic) C^k-maps $u : \Sigma \rightarrow M$ for all $k = 0, 1, \ldots, \infty$, and the inclusion relations between them.

So there should be a *truncation functor* from polyfolds to Kuranishi spaces.

This means that we can take the Hofer et al. proof of existence of polyfold structures on moduli spaces (when it is finished), and deduce existence of Kuranishi structures on moduli spaces, to replace the Fukaya et al. proof (which has holes in, I'm told).

In the Fukaya picture, there are also problems with the *definition* of Kuranishi spaces, which changes a bit with each version. No definition (including mine in 2007) is really satisfactory. For example, until the 2009 edition of [FOOO] the definition of 'virtual tangent bundle' was too weak for the construction of virtual cycles to be correct.

I believe I can now fix this problem, and give a rigorous, 'correct' definition of Kuranishi spaces – or at least, what the definition of Kuranishi spaces ought to have been. I call these new objects 'd-orbifolds'. They are based on ideas coming from a completely different direction: David Spivak's 'derived manifolds', arXiv:0810.5174, which come out of Jacob Lurie's Derived Algebraic Geometry programme.

Spivak's 'derived manifolds' are complex objects, which form an ∞-category (a simplicial category), i.e. they have nontrivial n-morphisms for all $n \geqslant 1$. My 'd-manifolds' are (I believe) a truncated version of derived manifolds, with less information, which form a 2-category.

'D-orbifolds' are an orbifold version of d-manifolds. Kuranishi spaces should really be 'd-orbifolds with corners'.

I think d-manifolds are better than derived manifolds for applications in Symplectic Geometry: they are simpler, they have enough structure to form virtual cycles, and I believe that the Fukaya/Hofer proofs will be enough to deduce a d-orbifold structure on J-holomorphic curve moduli spaces, but not sufficient to deduce a derived orbifold structure.

** Advertizing feature: please buy my (co)homology theory **

In arXiv:0707.3572v5, arXiv:0710.5634v2 I define (co)homology theories in which the (co)chains are Kuranishi spaces (with extra data). These are parked for the moment while I sort out definition of Kuranishi spaces; I will rewrite them using d-orbifolds instead of Kuranishi spaces, and then they will (I claim) be rigorous.

They are a novel solution to problem (c). Forming virtual chains or virtual cycles in these (co)homology theories is trivial, as the moduli space is its own virtual chain. There is no need to perturb moduli spaces. This gives a huge simplification to [FOOO]-style Lagrangian Floer cohomology.

2. D-manifolds

To get the idea across I am going to explain only the simplest class of these objects: *d-manifolds without boundary*. I also study *d-orbifolds* (locally modelled on $\boldsymbol{X}/G$ for $\boldsymbol{X}$ a d-manifold and G a finite group), and d-manifolds and d-orbifolds *with boundary*, or *with corners*.

They are founded on the ideas of C^∞-*ring* and C^∞-*scheme* from Synthetic Differential Geometry, an approach to smooth manifolds using the tools of Algebraic Geometry, going back to Lawvere, Dubuc, Moerdijk and Reyes,... in the 1960s–1980s.

I have also nicked ideas from David Spivak's derived manifolds, and Jacob Lurie.

2.1. C^∞-rings

Let X be a manifold, and write $C^\infty(X)$ for the smooth functions $c : X \to \mathbb{R}$. Then $C^\infty(X)$ is an $\mathbb{R}$-*algebra*: we can add smooth functions $(c,d) \mapsto c+d$, and multiply them $(c,d) \mapsto cd$, and multiply by $\lambda \in \mathbb{R}$.

But there are many more operations on $C^\infty(X)$ than this, e.g. if $c : X \to \mathbb{R}$ is smooth then $\exp(c) : X \to \mathbb{R}$ is smooth, giving $\exp : C^\infty(X) \to C^\infty(X)$, which is algebraically independent of addition and multiplication.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by $\Phi_f(c_1,\ldots,c_n)(x) = f\bigl(c_1(x),\ldots,c_n(x)\bigr)$ for all $x \in X$. Then addition comes from $f : \mathbb{R}^2 \to \mathbb{R}$, $f : (x,y) \mapsto x+y$, multiplication from $(x,y) \mapsto xy$, etc.

Definition 1. A C^∞*-ring* is a set $\mathfrak{C}$ together with n-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying the following conditions:
Let $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathfrak{C}$ we have

$$\begin{aligned}&\Phi_h(c_1, \ldots, c_n) = \\ &\Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).\end{aligned}$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have $\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j$.
A *morphism* of C^∞-rings is $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Then $C^\infty(X)$ is a C^∞-ring for any manifold X, and from $C^\infty(X)$ we can recover X up to canonical isomorphism.

If $f : X \to Y$ is smooth then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings; conversely, if $\phi : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings then $\phi = f^*$ for some unique smooth $f : X \to Y$. This gives a *full and faithful functor* $F : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ by $F : X \mapsto C^\infty(X)$, $F : f \mapsto f^*$.

Thus, we can think of manifolds as examples of C^∞-rings, and C^∞-rings as generalizations of manifolds. But there are many more C^∞-rings than manifolds. For example, $C^0(X)$ is a C^∞-ring for any topological space X.

2.2. C^∞-schemes

We can now develop the whole machinery of scheme theory in Algebraic Geometry, replacing rings or algebras by C^∞-rings throughout — see my arXiv:1001.0023.

A C^∞-*ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf of C^∞-rings $\mathcal{O}_X$. Write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces.

The *global sections functor* $\Gamma : \mathbf{C^\infty RS} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ maps $\Gamma : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$. It has a right adjoint, the *spectrum functor* $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{C^\infty RS}$. That is, for each C^∞-ring $\mathfrak{C}$ we construct a C^∞-ringed space $\mathrm{Spec}\,\mathfrak{C}$. On the subcategory of *fair* C^∞-rings, Spec is full and faithful.

A C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme* if $\underline{X} \cong \operatorname{Spec} \mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. We call $\underline{X}$ a *C^∞-scheme* if X can be covered by open subsets U with $(U, \mathcal{O}_X|_U)$ an affine C^∞-scheme. Write $\mathbf{C^\infty Sch}$ for the full subcategory of C^∞-schemes in $\mathbf{C^\infty RS}$.

If X is a manifold, define a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ by $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. Then $\underline{X} \cong \operatorname{Spec} C^\infty(X)$. This defines a full and faithful embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$. So we can regard manifolds as examples of C^∞-schemes.

All *fibre products* exist in $\mathbf{C^\infty Sch}$. But in manifolds $\mathbf{Man}$, fibre products $X \times_{f,Z,g} Y$ need exist only if $f : X \to Z$ and $g : Y \to Z$ are transverse. When f, g are not transverse, the fibre product $X \times_{f,Z,g} Y$ exists in $\mathbf{C^\infty Sch}$ as a C^∞-scheme, but may not be a manifold.

We also define *vector bundles*, *coherent sheaves* and *quasicoherent sheaves* on a C^∞-scheme $\underline{X}$, and write $\mathrm{coh}(\underline{X}), \mathrm{qcoh}(\underline{X})$ for the categories of coherent and quasicoherent sheaves. Then $\mathrm{qcoh}(\underline{X})$ is an abelian category. A C^∞-scheme $\underline{X}$ has a well-behaved *cotangent sheaf* $T^*\underline{X}$.

If $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes, we define *pullback* $\underline{f}^* : \mathrm{coh}(\underline{Y}) \to \mathrm{coh}(\underline{X})$ and $\underline{f}^* : \mathrm{qcoh}(\underline{Y}) \to \mathrm{qcoh}(\underline{X})$. It is a right exact functor.

Differences with Algebraic Geometry

The topology on C^∞-schemes is finer than the Zariski topology on schemes – affine schemes are always Hausdorff. No need to introduce the étale topology.

Can find smooth functions supported on (almost) any open set. (Almost) any open cover has a subordinate partition of unity.

Our C^∞-rings $\mathfrak{C}$ are generally *not noetherian* as $\mathbb{R}$-algebras. So ideals I in $\mathfrak{C}$ may not be finitely generated, even in $C^\infty(\mathbb{R}^n)$.

This also causes problems with coherent sheaves: if $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of coherent sheaves on $\underline{X}$ then $\operatorname{Coker}\phi$ (in $\operatorname{qcoh}(\underline{X})$) is coherent, but $\operatorname{Ker}\phi$ need not be coherent, so $\operatorname{coh}(\underline{X})$ is not an abelian category.

2.3. The 2-category of d-spaces

We define d-manifolds as a 2-subcategory of a larger 2-category of *d-spaces*. These are 'derived' versions of C^∞-schemes.

Definition. A *d-space* is a is a quintuple $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ where $\underline{X} = (X, \mathcal{O}_X)$ is a separated, second countable, locally fair C^∞-scheme, $\mathcal{O}'_X$ is a second sheaf of C^∞-rings on X, and $\mathcal{E}_X$ is a quasicoherent sheaf on $\underline{X}$, and $\imath_X : \mathcal{O}'_X \to \mathcal{O}_X$ is a surjective morphism of sheaves of C^∞-rings whose kernel $\mathcal{I}_X$ is a sheaf of *square zero ideals* in $\mathcal{O}'_X$, and $\jmath_X : \mathcal{E}_X \to \mathcal{I}_X$ is a surjective morphism in qcoh($\underline{X}$), so we have an exact sequence of sheaves on X:

$$\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{O}'_X \xrightarrow{\imath_X} \mathcal{O}_X \longrightarrow 0.$$

A 1-*morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is a triple $\boldsymbol{f}=(\underline{f},f',f'')$, where $\underline{f}=(f,f^\sharp):\underline{X}\to\underline{Y}$ is a morphism of C^∞-schemes and $f':f^{-1}(\mathcal{O}'_Y)\to\mathcal{O}'_X$, $f'':\underline{f}^*(\mathcal{E}_Y)\to\mathcal{E}_X$ are sheaf morphisms such that the following commutes:

$$\begin{tikzcd}
f^{-1}(\mathcal{E}_Y) \arrow[r, "f^{-1}(\jmath_Y)"'] \arrow[d, "f''"] & f^{-1}(\mathcal{O}'_Y) \arrow[r, "f^{-1}(\imath_Y)"'] \arrow[d, "f'"] & f^{-1}(\mathcal{O}_X) \arrow[r] \arrow[d, "f^\sharp"] & 0 \\
\mathcal{E}_X \arrow[r, "\jmath_X"] & \mathcal{O}'_X \arrow[r, "\imath_X"] & \mathcal{O}_X \arrow[r] & 0.
\end{tikzcd}$$

Let $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$ be 1-morphisms with $\boldsymbol{f}=(\underline{f},f',f'')$, $\boldsymbol{f}=(\underline{g},g',g'')$. Suppose $\underline{f}=\underline{g}$. A 2-*morphism* $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is a morphism

$$\eta: f^{-1}(\Omega_{\mathcal{O}'_Y})\otimes_{f^{-1}(\mathcal{O}'_Y)}\mathcal{O}_X\longrightarrow\mathcal{E}_X$$

in qcoh($\underline{X}$), where $\Omega_{\mathcal{O}'_Y}$ is the sheaf of cotangent modules of $\mathcal{O}'_Y$, such that $g'=f'+\jmath_X\circ\eta\circ\Pi_{XY}$ and $g''=f''+\eta\circ\underline{f}^*(\phi_Y)$, for natural morphisms Π_{XY},ϕ_Y.

Theorem 1. *This defines a strict 2-category of d-spaces* $\mathbf{dSpa}$.

Theorem 2. *Let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms of d-spaces. Then an explicit 2-category* **fibre product** *exists in* $\mathbf{dSpa}$*. That is, we can construct an object $\boldsymbol{W}$, 1-morphisms $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ and a 2-morphism $\eta : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$, satisfying a universal property.*

We can map $\mathbf{C^\infty Sch}$ into $\mathbf{dSpa}$ by taking a C^∞-scheme $\underline{X}$ to the d-space $\boldsymbol{X} = (\underline{X}, \mathcal{O}_X, 0, \mathrm{id}_{\mathcal{O}_X}, 0)$, with exact sequence

$$0 \xrightarrow{0} \mathcal{O}_X \xrightarrow{\mathrm{id}_{\mathcal{O}_X}} \mathcal{O}_X \longrightarrow 0.$$

This embeds $\mathbf{C^\infty Sch}$, and hence manifolds $\mathbf{Man}$, as discrete 2-subcategories of $\mathbf{dSpa}$. For *transverse* fibre products of manifolds, the fibre products in $\mathbf{Man}$ and $\mathbf{dSpa}$ agree.

2.4. The 2-subcategory of d-manifolds

Definition. A d-space $\boldsymbol{X}$ is a *d-manifold of dimension* $n \in \mathbb{Z}$ if $\boldsymbol{X}$ may be covered by open d-subspaces $\boldsymbol{Y}$ equivalent in $\mathbf{dSpa}$ to a fibre product $\boldsymbol{U} \times_{\boldsymbol{W}} \boldsymbol{V}$, where $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are manifolds without boundary and $\dim \boldsymbol{U} + \dim \boldsymbol{V} - \dim \boldsymbol{W} = n$. We allow $n < 0$.

Think of a d-manifold $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ as a 'classical' C^∞-scheme $\underline{X}$, with extra 'derived' data $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$.

Write $\mathbf{dMan}$ for the full 2-subcategory of d-manifolds in $\mathbf{dSpa}$. It is not closed under fibre products in $\mathbf{dSpa}$, but we can say:

Theorem 3. *All fibre products of the form* $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ *with* $\boldsymbol{X}, \boldsymbol{Y}$ *d-manifolds and* $\boldsymbol{Z}$ *a manifold exist in the 2-category* $\mathbf{dMan}$.

Here is how to relate d-manifolds and Kuranishi spaces. Let V be a manifold, and $E \to V$ a vector bundle (the *obstruction bundle*), and $s : V \to E$ a smooth section (the *Kuranishi map*). We define a d-manifold $\boldsymbol{X}$ from the data (V, E, s), called a *principal d-manifold*, of dimension $\dim V - \operatorname{rank} E$. Any d-manifold $\boldsymbol{W}$ is covered by principal d-manifolds $\boldsymbol{X}$.

The orbifold version of this is basically the *Kuranishi neighbourhoods* in the Fukaya–Ono definition of Kuranishi spaces: a Kuranishi neighbourhood (V, E, s, ψ) on a topological space X is an orbifold V, a vector bundle $E \to V$, a smooth section $s : V \to E$, and a map $\psi : s^{-1}(0) \hookrightarrow X$ which is a homeomorphism with an open set in X.

The problems in the definition of Kuranishi spaces are mainly to do with saying when two Kuranishi neighbourhoods are compatible, how Kuranishi neighbourhoods are glued on double overlaps, triple overlaps, and so on. In our theory this is answered by saying that Kuranishi neighbourhoods are glued on overlaps by giving an equivalence in $\mathbf{dSpa}$ between open subsets of the corresponding principal d-manifolds.

There should also be a version of this story in Algebraic Geometry, using rings or algebras rather than C^∞-rings. Let us call these *algebraic d-manifolds*. They should be related to objects people already study. I expect that algebraic d-manifolds should be some kind of 2-category truncation of the ∞-category of *quasi-smooth derived schemes* in the sense of Lurie or Toen. Also, an algebraic d-manifold should be roughly the same thing as a scheme with a perfect obstruction theory.

So maybe there is no real need to introduce algebraic d-manifolds. Still, the claim (unproved) that they form a 2-category with good fibre products seems attractive.