Constructing compact manifolds with exceptional holonomy

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Riemannian geometry

Let M^n be a manifold of dimension n. Let $x \in M$. Then $T_x M$ is the *tangent space* to M at x.

Let g be a Riemannian metric on M.

Let ∇ be the *Levi-Civita connection* of g.

Let $R(g)$ be the *Riemann curvature* of g.

Holonomy groups

Fix $x \in M$. The *holonomy group* $\mathrm{Hol}(g)$ of g is the set of isometries of T_xM given by *parallel transport* using ∇ about closed loops γ in M based at x. It is a subgroup of $O(n)$. Up to conjugation, it is independent of the base-point x.

Berger's classification

Let M be simply-connected and g be irreducible and nonsymmetric. Then Hol(g) is one of $SO(m)$, $U(m)$, $SU(m)$, $Sp(m)$, $Sp(m)Sp(1)$ for $m \geq 2$, or G_2 or $Spin(7)$. We call G_2 and $Spin(7)$ the *exceptional holonomy groups*. Dim(M) is 7 when Hol(g) is G_2 and 8 when Hol(g) is $Spin(7)$.

Understanding Berger's list

The four *inner product algebras* are

$\mathbb{R}$ — *real numbers*.

$\mathbb{C}$ — *complex numbers*.

$\mathbb{H}$ — *quaternions*.

$\mathbb{O}$ — *octonions*,

or *Cayley numbers*.

Here $\mathbb{C}$ is not ordered,

$\mathbb{H}$ is not commutative,

and $\mathbb{O}$ is not associative.

Also we have $\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{H} \cong \mathbb{R}^4$

and $\mathbb{O} \cong \mathbb{R}^8$, with $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$.

Group	**Acts on**
$SO(m)$	$\mathbb{R}^m$
$O(m)$	$\mathbb{R}^m$
$SU(m)$	$\mathbb{C}^m$
$U(m)$	$\mathbb{C}^m$
$Sp(m)$	$\mathbb{H}^m$
$Sp(m)Sp(1)$	$\mathbb{H}^m$
G_2	$\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$
$Spin(7)$	$\mathbb{O} \cong \mathbb{R}^8$

Thus there are two holonomy groups for each of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

The goal of the talk

To discuss constructions of examples of compact manifolds of holonomy G_2 and $Spin(7)$.

Why is this difficult?

In many problems in geometry the simplest examples are symmetric. But G_2- and $Spin(7)$-manifolds have no continuous symmetries.

Why is this interesting?

- Such manifolds are Ricci-flat.
- They are important to physicists working in String Theory.
- They have beautiful geometrical properties.

Geometry of G_2

The action of G_2 on $\mathbb{R}^7$ preserves the metric g_0 and a 3-form φ_0 on $\mathbb{R}^7$. Let g be a metric and φ a 3-form on M^7. We call (φ, g) a G_2-*structure* if $(\varphi, g) \cong (\varphi_0, g_0)$ at each $x \in M$. We call $\nabla\varphi$ the *torsion* of (φ, g).

If $\nabla\varphi = 0$ then (φ, g) is *torsion-free*. Also $\nabla\varphi = 0$ iff $\mathrm{d}\varphi = \mathrm{d}^*\varphi = 0$. If (φ, g) is torsion-free then $\mathrm{Hol}(g) \subseteq G_2$. Conversely, if g is a metric on M^7 then $\mathrm{Hol}(g) \subseteq G_2$ iff there is a G_2-structure (φ, g) with $\nabla\varphi = 0$. If M is compact and $\mathrm{Hol}(g) \subseteq G_2$ then $\mathrm{Hol}(g) = G_2$ iff $\pi_1(M)$ is finite.

The construction, 1

First we choose a compact 7-manifold M. We write down an explicit G_2-structure (φ, g) on M with *small torsion*. Then we use analysis to deform to a nearby G_2-structure $(\tilde{\varphi}, \tilde{g})$ with *zero torsion.* If $\pi_1(M)$ is finite then $\mathrm{Hol}(\tilde{g}) = G_2$ as we want.

The construction, 2

It is not easy to find G_2-structures with small torsion! Here is one way to do it, in 4 steps.

Step 1. Choose a finite group Γ of isometries of the 7-torus T^7, and a flat, Γ-invariant G_2-structure (φ_0, g_0) on T^7. Then T^7/Γ is compact, with a torsion-free G_2-structure (φ_0, g_0).

Step 2. However, T^7/Γ is an *orbifold*. We repair its singularities to get a compact 7-manifold M. We can resolve *complex* orbifolds using algebraic geometry.

If the singularities of T^7/Γ locally resemble $\mathcal{S}^1 \times \mathbb{C}^3/G$ for $G \subset SU(3)$, then we model M on a *crepant resolution* X of $\mathbb{C}^3/G$.

Step 3. M is made by gluing patches $\mathcal{S}^1 \times X$ into T^7/Γ. Now X carries ALE metrics of holonomy $SU(3)$. As $SU(3) \subset G_2$, these give torsion-free G_2-structures on $\mathcal{S}^1 \times X$.

We join them to (φ_0, g_0) on T^7/Γ to get a family $\{(\varphi_t, g_t) : t \in (0, \epsilon)\}$ of G_2-structures on M.

Step 4. This (φ_t, g_t) has $\nabla\varphi_t = O(t^4)$. So $\nabla\varphi_t$ is small for small t. But $R(g_t) = O(t^{-2})$ and the injectivity radius $\delta(g_t) = O(t)$, since g_t becomes singular as $t \to 0$.

For small t we deform (φ_t, g_t) to $(\tilde\varphi_t, \tilde g_t)$ with $\nabla\tilde\varphi_t = 0$, using analysis. Then $\mathrm{Hol}(\tilde g_t) = G_2$ if $\pi_1(M)$ is finite.

Steps in the analysis proof:

- Arrange that $\mathrm{d}\varphi_t = 0$ and $\mathrm{d}^*\varphi_t = \mathrm{d}^*\psi_t$, where $\psi_t = O(t^4)$.
- Set $\tilde{\varphi}_t = \varphi_t + \mathrm{d}\eta_t$, where $\mathrm{d}^*\eta_t = 0$.
- Then $(\tilde{\varphi}_t, \tilde{g}_t)$ is torsion-free iff

$$(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)(\eta_t) = \mathrm{d}^*\psi_t + \mathrm{d}F(\mathrm{d}\eta_t),$$

where F is nonlinear with $F(\chi) = O\bigl(|\chi|^2\bigr)$.

- Integrating by parts gives $\|d\eta_t\|_{L^2} \leq 2\|\psi_t\|_{L^2}$ when $\|d\eta_t\|_{C^0}$ is small.
- Solve by contraction method in $L^{14}_2(\Lambda^2 T^*M)$, using elliptic regularity of $d^*d + dd^*$, balls of radius t and Sobolev embedding.

The construction, 3

Using different groups Γ acting on T^7 or T^8, and resolving T^k/Γ in more than one way, we get many compact manifolds with holonomy G_2 and $Spin(7)$. We can generalize the construction by replacing T^7 or T^8 with another space made from a Calabi-Yau manifold.

Geometry of $Spin(7)$

The action of $Spin(7)$ on $\mathbb{R}^8$ preserves the metric g_0 and a 4-form Ω_0 on $\mathbb{R}^8$. Let g be a metric and Ω a 4-form on M^8. We call (Ω, g) a $Spin(7)$-*structure* if $(\Omega, g) \cong (\Omega_0, g_0)$ at each $x \in M$. We call $\nabla\Omega$ the *torsion* of (Ω, g).

If $\nabla\Omega = 0$ then (Ω, g) is *torsion-free*. Also $\nabla\Omega = 0$ iff $\mathrm{d}\Omega = 0$. If $\nabla\Omega = 0$ then $\mathrm{Hol}(g) \subseteq Spin(7)$. If g is a metric on M^8 then $\mathrm{Hol}(g) \subseteq Spin(7)$ iff there is a $Spin(7)$-structure (Ω, g) with $\nabla\Omega = 0$. If M is compact and $\mathrm{Hol}(g) \subseteq Spin(7)$ then g has holonomy $Spin(7)$ iff $\pi_1(M) = \{1\}$, $\hat{A}(M) = 1$.

Compact examples

The first examples of compact 8-manifolds with holonomy $Spin(7)$ were constructed by me in 1995. Here is how.

Let T^8 be a torus with flat $Spin(7)$-structure (Ω_0, g_0), and let Γ be a finite group acting on T^8 preserving (Ω_0, g_0). Then T^8/Γ is an *orbifold*.

We choose Γ so that the singularities of T^8/Γ are locally modelled on $\mathbb{C}^4/G$, for $G \subset SU(4)$.
Then we use complex algebraic geometry to resolve T^8/Γ, giving a compact 8-manifold M.
Finally we use analysis to construct metrics on M with holonomy $Spin(7)$.

A new construction

We shall describe a new way of making compact 8-manifolds with holonomy $Spin(7)$, where we start not with a torus T^8 but with a *Calabi-Yau 4-orbifold* Y with isolated singular points $p_1, \ldots, p_k$.

Instead of a group Γ we use an antiholomorphic, isometric involution σ on Y fixing only the p_j. Then $Z = Y/\langle\sigma\rangle$ is a real 8-orbifold with singular points $p_1, \ldots, p_k$. We resolve the p_j to get a compact 8-manifold M, and construct holonomy Spin(7) metrics on M.

Calabi–Yau orbifolds

A *Calabi–Yau orbifold* is a compact complex orbifold with a Kähler metric of holonomy $SU(m)$. One can find many examples using algebraic geometry and Yau's proof of the Calabi conjecture.

The construction

Let Y be a Calabi-Yau 4-orbifold with only isolated singular points $p_1, \ldots, p_k$, each modelled on $\mathbb{C}^4/\mathbb{Z}_4$, where the generator of $\mathbb{Z}_4$ acts by

$$(z_1, \ldots, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4).$$

We call this a singular point of type $\frac{1}{4}(1,1,1,1)$.

Pick an antiholomorphic, isometric involution σ on Y, fixing only $p_1, \ldots, p_k$, and let $Z = Y/\langle\sigma\rangle$. As $SU(4) \subset Spin(7)$ and Y has holonomy $SU(4)$, there is a torsion-free $Spin(7)$-structure (Ω,g) on Y. We can choose (Ω, g) to be σ-invariant, so (Ω,g) pushes down to Z. Thus Z is a $Spin(7)$-orbifold.

All the singularities p_j of Z are modelled on $\mathbb{R}^8/G$, where $G=\langle\alpha,\sigma\rangle$ is a non-abelian group of order 8, and α,σ act on $\mathbb{R}^8=\mathbb{C}^4$ by

$$\alpha:(z_1,\ldots,z_4)\mapsto (iz_1,iz_2,iz_3,iz_4),$$

$$\sigma:(z_1,\ldots,z_4)\mapsto (\bar z_2,-\bar z_1,\bar z_4,-\bar z_3).$$

There are two different ways to resolve $\mathbb{R}^8/G$ within holonomy Spin(7).

The first way is to take a crepant resolution W_1 of $\mathbb{C}^4/\langle\alpha\rangle$, and lift σ to a free antiholomorphic involution of W_1.
Then $X_1 = W_1/\langle\sigma\rangle$ is a resolution of $\mathbb{R}^8/G$. There is an ALE metric with holonomy $SU(4)$ on W_1 which pushes down to a metric on $W_1/\langle\sigma\rangle$ with holonomy $\mathbb{Z}_2 \ltimes SU(4)$.

But there is a second complex structure on $\mathbb{R}^8$, so that σ is holomorphic and α anti-holomorphic. Resolve $\mathbb{C}^4/\langle\sigma\rangle$ to get W_2, lift α to W_2, and $X_2 = W_2/\langle\alpha\rangle$ is a resolution of $\mathbb{R}^8/G$, with ALE metrics of holonomy $\mathbb{Z}_2 \ltimes SU(4)$. Note that we have used two different inclusions of $\mathbb{Z}_2 \ltimes SU(4)$ in $Spin(7)$.

We resolve each point p_j in Z using either X_1 or X_2, to get a compact 8-manifold M. Now Z, X_1 and X_2 carry torsion-free $Spin(7)$-structures.

We glue these together to get a $Spin(7)$-structure (Ω_t, g_t) on M for $t \in (0, \epsilon)$, with torsion $O(t^{24/5})$.

For small t we can deform (Ω_t, g_t) to a torsion-free $Spin(7)$-structure $(\tilde{\Omega}, \tilde{g})$ on M. If we resolve using X_1 for all p_j then $\pi_1(M) = \mathbb{Z}_2$ and $\mathrm{Hol}(\tilde{g}) = \mathbb{Z}_2 \ltimes SU(4)$. If we use X_2 for any p_j then $\pi_1(M) = \{1\}$ and $\mathrm{Hol}(\tilde{g}) = Spin(7)$. This is what we want.

An example

Let Y be the degree 12 hypersurface in the weighted projective space $\mathbb{C}P^5_{1,1,1,1,4,4}$ given by

$$\Big\{[z_0,\ldots,z_5]\in\mathbb{C}P^5_{1,\ldots,4}: z_0^{12}+z_1^{12}+z_2^{12}+z_3^{12} +z_4^3+z_5^3=0\Big\}.$$

Then $c_1(Y)=0$, so Y is a Calabi-Yau 4-orbifold. It has 3 singularities p_1,p_2,p_3, of type $\frac{1}{4}(1,1,1,1)$.

Define $\sigma : Y \to Y$ by
$\sigma : [z_0, \ldots, z_5] \mapsto$
$[\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4]$.
Then σ is an anti-holomorphic involution, fixing only p_1, p_2, p_3. We apply our construction to Y and σ, to get a compact 8-manifold M with holonomy Spin(7) and Betti numbers $b^2 = 0$, $b^3 = 0$ and $b^4 = 2446$.

Conclusions

Using hypersurfaces in other weighted projective spaces, and dividing by finite groups, we can find many new examples of compact 8-manifolds with holonomy Spin(7). Here are some of their Betti numbers.

Betti numbers (b^2, b^3, b^4)

$(4, 33, 200)$	$(3, 33, 202)$
$(2, 33, 204)$	$(1, 33, 206)$
$(0, 33, 208)$	$(1, 0, 908)$
$(0, 0, 910)$	$(1, 0, 1292)$
$(0, 0, 1294)$	$(1, 0, 2444)$
$(0, 0, 2446)$	$(0, 6, 3730)$
$(0, 0, 4750)$	$(0, 0, 11\,662)$

Note that b^4 tends to be rather large — bigger than in the first construction, where $b^4 \approx 100$-200.