COMPACT RIEMANNIAN 7-MANIFOLDS WITH HOLONOMY G_2. II

DOMINIC D. JOYCE

published as J. Differential Geometry 43 (1996), 329–375.

1. Introduction

This is the second of two papers about metrics of holonomy G_2 on compact 7-manifolds. In our first paper [15] we established the existence of a family of metrics of holonomy G_2 on a single, compact, simply-connected 7-manifold M, using three general results, Theorems A, B and C. Our purpose in this paper is to explore the theory of compact Riemannian 7-manifolds with holonomy G_2 in greater detail. By relying on Theorems A-C we will be able to avoid the emphasis on analysis that characterized [15], so that this sequel will have a more topological flavour.

The paper has four chapters. The first chapter consists of introductory material. Section 1.1 gives some elementary geometric and topological material on compact 7-manifolds with torsion-free G_2- structures. Then §1.2 describes the holonomy groups $SU(2)$ and $SU(3)$, and §1.3 explains the concept of asymptotically locally Euclidean Riemannian manifolds (shortened to ALE spaces) with special holonomy.

Recall that in [15], a compact 7-manifold M was defined by desingularizing a quotient T^7/Γ of the 7-torus by a finite group of isometries $\Gamma \cong \mathbb{Z}_2^3$. The subject of Chapters 2 and 3 is a generalization of this idea. Chapter 2 defines a general construction for compact 7-manifolds with torsion-free G_2- structures, which works by desingularizing quotients T^7/Γ for finite groups Γ. The ALE spaces with holonomy $SU(2)$ and $SU(3)$ discussed in §1.3 are an essential ingredient in performing this desingularization.

The central result of Chapter 2 is Theorem 2.2.3, which states that given a suitable finite group Γ and certain other data, one may construct a compact 7-manifold M from T^7/Γ that admits torsion-free G_2- structures. This result is proved using Theorems A-C of [15]. Chapter 3 is devoted entirely to examples of this construction. We give many examples of compact 7-manifolds with holonomy G_2, and determine their basic topological invariants — the Betti numbers and fundamental group. Finally, in Chapter 4 we discuss some areas of interest, and give a number of open problems.

This paper is not written to be read independently of [15]. The language and results of [15] will be used freely, in particular the introductory material in [15, §1.1]. For reference we reproduce here the model 3- and 4- forms φ, $*\varphi$ defining the flat G_2- structure on $\mathbb{R}^7$, as given in [15, §1.1]:

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$$\begin{aligned}\varphi =&y_1\wedge y_2\wedge y_7+y_1\wedge y_3\wedge y_6+y_1\wedge y_4\wedge y_5+y_2\wedge y_3\wedge y_5\\&-y_2\wedge y_4\wedge y_6+y_3\wedge y_4\wedge y_7+y_5\wedge y_6\wedge y_7,\end{aligned} \tag{1}$$

$$\begin{aligned}*\varphi =&y_1\wedge y_2\wedge y_3\wedge y_4+y_1\wedge y_2\wedge y_5\wedge y_6-y_1\wedge y_3\wedge y_5\wedge y_7+y_1\wedge y_4\wedge y_6\wedge y_7\\&+y_2\wedge y_3\wedge y_6\wedge y_7+y_2\wedge y_4\wedge y_5\wedge y_7+y_3\wedge y_4\wedge y_5\wedge y_6.\end{aligned} \tag{2}$$

Here $y_1,\ldots,y_7$ is an oriented orthonormal basis of $(\mathbb{R}^7)^*$.

1.1. Topological properties of compact 7-manifolds with holonomy G_2. In this section we give a number of elementary general facts about compact 7-manifolds with torsion-free G_2- structures. Some of these will be used later, and some are just for interest. Most of the results can be found in [5], or are otherwise already known.

Proposition 1.1.1. *Let M be a compact 7-manifold, let φ be a torsion-free G_2-structure on M, and let g be the metric associated to φ. Then the holonomy group of g is G_2 if and only if the fundamental group $\pi_1(M)$ is finite. The holonomy group of g is $F\ltimes SU(3)$ for some finite group F if and only if M admits a finite cover $N\times\mathcal{S}^1$ for some compact, simply-connected 6-manifold N. The holonomy group of g is $F\ltimes SU(2)$ for some finite group F if and only if M admits a finite cover $N\times T^3$ for some compact, simply-connected 4-manifold N.*

Proof. The Cheeger-Gromoll splitting Theorem [3, Cor. 6.67, p. 168] deals with the global topology of Riemannian manifolds of nonnegative Ricci curvature, and one of its conclusions is that if (M,g) is a compact, Ricci-flat Riemannian manifold, then M has a finite cover isometric to a product $N\times T^k$, where N is a compact, simply-connected, Ricci-flat Riemannian manifold, and T^k is a flat Riemannian torus. Suppose that g has holonomy contained in G_2. Then g is Ricci-flat from above, so that M has a finite cover isometric to $N\times T^k$, where N is simply-connected. Let g' be the lift of g to $N\times T^k$. As $N\times T^k$ is a Riemannian product, the holonomy group H' of g' is the product of the holonomy group of N and the trivial group. But $N\times T^k$ is a finite cover of M, and therefore the holonomy group H of g is a semidirect product $H=F\ltimes H'$ for some finite group F.

Now N is a compact, simply-connected Riemannian manifold of dimension $7-k$, with holonomy group H'. Since N is simply-connected, H' is connected. Suppose for a contradiction that the holonomy representation of H' on $\mathbb{R}^{7-k}$ has a summand $\mathbb{R}^l$ on which H' acts trivially. Then g' is locally a Riemannian product of $\mathbb{R}^l$, and some metric in dimension $7-k-l$, so that the universal cover of N contains a factor $\mathbb{R}^l$. But this contradicts the fact that N is compact and simply-connected. Therefore the holonomy representation of H' on $\mathbb{R}^{7-k}$ has no trivial summand.

By examining Berger's classification of Riemannian holonomy groups, it can be shown that the only connected subgroups of G_2 that can be Riemannian holonomy groups in the standard representation of G_2 are $\{1\}$, $SU(2)$, $SU(3)$ and G_2. Therefore there are four possible cases, the case $H'=\{1\}$ and $k=7$, the case $H'=SU(2)$ and $k=3$, the case $H'=SU(3)$ and $k=1$, and the case $H'=G_2$ and $k=0$. Suppose that $H=G_2$. Then $k=0$ as $H'=G_2$, so M has a finite, simply-connected cover N, and $\pi_1(M)$ is finite, as we have to prove. Conversely, if $\pi_1(M)$ is finite, then $k=0$ as $N\times T^k$ covers M, so $H'=G_2$, and thus $H=G_2$, as we have to prove.

Suppose that $H = F \ltimes SU(3)$ for some finite group F. Then $H' = SU(3)$, so $k = 1$, and M has a finite cover $N \times \mathcal{S}^1$. Conversely, if M admits a finite cover $N^6 \times \mathcal{S}^1$ with N^6 simply-connected, then $k = 1$ and $H' = SU(3)$, so that $H = F \ltimes SU(3)$ for some finite F, as we have to prove. Similarly, $H = F \ltimes SU(2)$ for some finite group F if and only if M admits a finite cover $N^4 \times T^3$ for N^4 simply-connected. This completes the Proposition. $\square$

The Weitzenbock formula for k- forms [24, p. 55, Proposition 4.10] states that if ξ is a k- form on a Riemannian manifold M with metric g, then

$$(d^* d + dd^*)\xi = \nabla^* \nabla \xi - 2\tilde{R}\xi. \tag{3}$$

Here ∇ is the Levi-Civita connection of g, and $\tilde{R} : \Lambda^k T^* M \to \Lambda^k T^* M$ is a linear map defined using the Riemann curvature R of g, as follows. Let $p \in M$, and let $G(p) \subset \mathrm{End}(T_p M)$ be the holonomy group of g at p, and $\mathfrak{g}$ its Lie algebra. Then $R(p) \in \odot^2 \mathfrak{g}$. Now $G(p)$ has a natural representation on $\Lambda^k T_p^* M$, which gives a natural map $\rho : \mathfrak{g} \to \mathrm{End}(\Lambda^k T_p^* M)$. So ρ acts on $R(p)$ to give an element of $\odot^2 \mathrm{End}(\Lambda^k T_p^* M)$, and then composition in $\mathrm{End}(\Lambda^k T_p^* M)$ yields an element of $\mathrm{End}(\Lambda^k T_p^* M)$. Define $\tilde{R}(p)$ to be this element, and let $\tilde{R}$ have value $\tilde{R}(p)$ at each $p \in M$. Then $\tilde{R}$ is a smooth endomorphism of $\Lambda^k T^* M$, as we require.

Now let M be compact. Then by Hodge theory, we have

$$H^k(M, \mathbb{R}) \cong \left\{\xi \in C^\infty(\Lambda^k T^* M) : d\xi = d^*\xi = 0\right\} = \left\{\xi \in C^\infty(\Lambda^k T^* M) : (d^* d + dd^*)\xi = 0\right\}. \tag{4}$$

Suppose that $\Lambda^k T^* M$ splits into the direct sum of subbundles Λ^k_l under the action of the holonomy group G. Then both $\nabla^* \nabla$ and $\tilde{R}$ preserve these subbundles, so by (3) the operator $d^* d + dd^*$ preserves these subbundles too. Thus $\mathrm{Ker}(d^* d + dd^*)$ splits into the direct sum over l of $\mathrm{Ker}(d^* d + dd^*)|_{\Lambda^k_l}$, and (4) gives $H^k(M, \mathbb{R}) = \bigoplus_l H^k_l$, where

$$H^k_l = \left\{\xi \in C^\infty(\Lambda^k_l) : d\xi = d^*\xi = 0\right\}. \tag{5}$$

Define $b^k_l = \dim H^k_l$. Then $b^k = \Sigma_l b^k_l$. There is also another important conclusion. Since the operator $\nabla^* \nabla - 2\tilde{R}$ acting on Λ^k_l depends only on the representation of G, if two subbundles Λ^i_j and Λ^k_l come from isomorphic representations of G, then $H^i_j \cong H^k_l$, and $b^i_j = b^k_l$.

The discussion above holds for any Riemannian holonomy group; for instance, in the case of holonomy $U(n)$ this is the familiar splitting of the cohomology into the Dolbeault cohomology groups $H^{p,q}$ of a Kähler manifold. We are interested in the case that M is 7-dimensional and G is a subgroup of G_2. In this case, using the G- invariant splittings of [15, Proposition 1.1.1], we write $b^0 = b^0_1$, $b^1 = b^1_7$, $b^2 = b^2_7 + b^2_{14}$ and $b^3 = b^3_1 + b^3_7 + b^3_{27}$. Isomorphisms of representations give the equalities $b^k_l = b^{7-k}_l$, $b^0_1 = b^3_1$ and $b^1_7 = b^2_7 = b^3_7$. For a compact, connected 7-manifold M with holonomy G_2, we have $b^0 = 1$ and $b^1 = 0$ by Proposition 1.1.1. Thus M has only two nontrivial Betti-type invariants, b^2_{14} and b^3_{27}.

Lemma 1.1.2. *Let (M, g) be a compact Riemannian 7-manifold with holonomy G_2. Then the pontryagin class $p_1(M) \in H^4(M, \mathbb{Z})$ is nonzero. Let φ be the G_2-structure of g and $[\varphi]$ be its cohomology class in $H^3(M, \mathbb{R})$. Then $[\varphi]$ satisfies $p_1(M) \cup [\varphi] < 0$, and $\sigma \cup \sigma \cup \varphi < 0$ whenever σ is a nonzero class in $H^2(M, \mathbb{R})$.*

Proof. The pontryagin class $p_1(M)$ is represented by a 4-form $R \wedge R$ made from the Riemann curvature $R^i{}_{jkl}$ of g. Because the Levi-Civita connection of g preserves the G_2- structure φ, the 2-form part of R lies in the subbundle of $\Lambda^2 T^*M$ associated to the Lie algebra of G_2. This subbundle is Λ^2_{14}. Now if $\xi \in \Lambda^2_{14}$, then $\xi \wedge \xi \wedge \varphi = -|\xi|^2 \,\mathrm{vol}$. It follows that $R \wedge R \wedge \varphi = -|R|^2 \,\mathrm{vol}$ (up to some positive constant). Therefore $p_1(M) \cup [\varphi] = \int_M R \wedge R \wedge \varphi = -\|R\|_2^2$ (up to some positive constant). But R must be nonzero because the holonomy group of g is nontrivial. Thus $p_1(M) \cup [\varphi] < 0$, as we have to prove, and $p_1(M)$ is nonzero.

Since M has holonomy G_2, $\pi_1(M)$ is finite by Proposition 1.1.1, and thus $b^1(M) = 0$. But $b^1 = b^1_7 = b^2_7$, so $H^2_7 = \{0\}$, and $H^2(M,\mathbb{R}) = H^2_{14}$. It follows that if $\sigma \in H^2(M,\mathbb{R})$ is nonzero, then σ is represented by some nonzero $\xi \in C^\infty(\Lambda^2_{14})$. Thus $\sigma \cup \sigma \cup [\varphi] = \int_M \xi \wedge \xi \wedge \varphi = -\|\xi\|_2^2 < 0$, as ξ is nonzero. So $\sigma \cup \sigma \cup [\varphi] < 0$, as we have to prove. $\square$

Bryant and Harvey [5] also show that if M is a compact Riemannian 7-manifold with holonomy G_2, then M cannot be diffeomorphic to a product of lower-dimensional manifolds.

Lemma 1.1.3. *Let M be a compact Riemannian 7-manifold admitting torsion-free G_2- structures, and let L be the subset of $H^3(M,\mathbb{R}) \times H^4(M,\mathbb{R})$ defined by*

$$L = \Big\{ \big([\varphi], [\Theta(\varphi)]\big) : \varphi \text{ is a torsion-free } G_2\text{- structure on } M \Big\}. \tag{6}$$

Then L is locally a Lagrangian submanifold of $H^3(M,\mathbb{R}) \times H^4(M,\mathbb{R})$ with its natural symplectic structure.

Proof. By [15, Theorem C], the set of diffeomorphism classes of torsion-free G_2- structures is a manifold locally isomorphic to $H^3(M,\mathbb{R})$ (considering only diffeomorphisms isotopic to the identity). Therefore L is locally a manifold of dimension $b^3(M)$, with nonsingular projection to $H^3(M,\mathbb{R})$. Regarding $H^4(M,\mathbb{R})$ as $H^3(M,\mathbb{R})^*$, we will show that L may be written locally in the form $(x, df(x))$, where f is a smooth real function on $H^3(M,\mathbb{R})$ defined by $f\big([\varphi]\big) = \frac{3}{7}[\varphi] \cup [\Theta(\varphi)]$. It then follows immediately that L is a Lagrangian submanifold of $H^3(M,\mathbb{R}) \times H^4(M,\mathbb{R})$.

Let $\big\{\varphi_t : t \in (-\epsilon, \epsilon)\big\}$ be a smooth family of torsion-free G_2- structures on M. Then

$$\frac{\partial f([\varphi_t])}{\partial t} = \frac{3}{7}\int_M \Big\{ \frac{\partial \varphi_t}{\partial t} \wedge \Theta(\varphi_t) + \varphi_t \wedge \frac{\partial \Theta(\varphi_t)}{\partial t} \Big\} = \int_M \frac{\partial \varphi_t}{\partial t} \wedge \Theta(\varphi_t) = \frac{\partial [\varphi_t]}{\partial t} \cup [\Theta(\varphi_t)], \tag{7}$$

using the fact that $\pi_1\big(\Theta(\varphi + \chi)\big) = *\varphi + \frac{4}{3} * \pi_1(\chi) + O(|\chi|^2)$. Since (7) holds for all smooth families φ_t with $\varphi_0 = \varphi$, we deduce that $df([\varphi]) = [\Theta(\varphi)]$, interpreting $H^4(M,\mathbb{R})$ as $H^3(M,\mathbb{R})^*$. Therefore L is the locally the graph of an exact 1-form on $H^3(M,\mathbb{R})$, so that L is a Lagrangian submanifold. $\square$

The author does not know whether L has any special properties other than being a Lagrangian submanifold, but this does appear to be a question worth researching. In the case $M = K3 \times T^3$, it is possible to define L entirely explicitly, using the very strong results known about the moduli space of metrics of holonomy $SU(2)$ on the $K3$ case by Todorov and others, which can be found in the survey paper [2].

1.2. The holonomy groups $SU(2)$ and $SU(3)$. Let Z be a compact, complex n-manifold, let g be a Kähler metric on Z, and let ω be the Kähler form of g.

Then g has holonomy contained in $U(n)$. The Ricci curvature of g is equivalent to the curvature of the Levi-Civita connection on the complex line bundle of complex volume forms. Therefore g is Ricci-flat if and only if this connection is flat, i.e. if and only if the line bundle of complex volume forms admits a local constant section Ω. But when the Levi-Civita connection preserves some geometric structure, the holonomy group of the metric preserves the same structure. In this case, we see that g is Ricci-flat if and only if the holonomy group of g lies locally in $SU(n)$. Globally, the holonomy group of g is contained in $F \ltimes SU(n)$ for some finite group F (as Z is compact), and if Ω exists globally then the holonomy group of g is contained in $SU(n)$.

There is a celebrated conjecture by Calabi ([7], [3, Chapter 11]) about the Ricci forms realized by Kähler metrics on a compact complex manifold. It has been proved by Yau [28]. One consequence of Yau's proof of the Calabi conjecture is the following theorem:

Theorem 1.2.1 [28]. *Suppose Z is a compact, simply-connected, Kähler, complex n-fold with vanishing real first Chern class $c_1(Z)$. Then Z admits a Ricci-flat Kähler metric g with holonomy group contained in $SU(n)$.*

As the hypotheses of the Theorem are simple and algebraic, one can find many examples of compact complex manifolds with metrics of holonomy $SU(n)$. We shall be interested only in the cases $n = 2, 3$. For the case $n = 2$, the Kähler form ω is a real 2-form and the complex volume form Ω is a complex 2-form. Let us write $\omega = \omega_1$ and $\Omega = \omega_2 + i\omega_3$ for real 2-forms $\omega_1, \omega_2, \omega_3$. Then $\omega_1, \omega_2, \omega_3$ may be written

$$\omega_1 = y_1 \wedge y_2 + y_3 \wedge y_4, \qquad \omega_2 = y_1 \wedge y_3 - y_2 \wedge y_4, \qquad \omega_3 = y_1 \wedge y_4 + y_2 \wedge y_3, \tag{8}$$

with respect to an oriented orthonormal basis $(y_1, \ldots, y_4)$ for $(\mathbb{R}^4)^*$. If $\omega_1, \omega_2, \omega_3$ are 2-forms on a 4-manifold M that may be written in the form (8) at each point, then they define an $SU(2)$- structure on M, and this $SU(2)$- structure is torsion-free if and only if $d\omega_j = 0$ for $j = 1, 2, 3$.

The forms φ and $*\varphi$ of (1) and (2) may be written in terms of $\omega_1, \omega_2, \omega_3$ as follows:

$$\varphi = \omega_1 \wedge \delta_1 + \omega_2 \wedge \delta_2 + \omega_3 \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3, \tag{9}$$

$$*\varphi = \omega_1 \wedge \delta_2 \wedge \delta_3 + \omega_2 \wedge \delta_3 \wedge \delta_1 + \omega_3 \wedge \delta_1 \wedge \delta_2 + \tfrac{1}{2}\omega_1 \wedge \omega_1. \tag{10}$$

Here $\delta_1 = y_7$, $\delta_2 = y_6$ and $\delta_3 = y_5$. As $\omega_i \wedge \omega_i = \omega_j \wedge \omega_j$, these equations are preserved by the cyclic permutation of the indices 1,2,3. Since $SU(2)$ preserves $\omega_1, \omega_2, \omega_3$, it preserves φ and $*\varphi$ by (9) and (10), and this defines an inclusion $SU(2) \subset G_2$.

The Kähler form ω and complex 3-form Ω making up an $SU(3)$- structure may be written

$$\omega = y_1 \wedge y_2 + y_3 \wedge y_4 + y_5 \wedge y_6, \qquad \Omega = (y_1 + iy_2) \wedge (y_3 + iy_4) \wedge (y_5 + iy_6), \tag{11}$$

where $(y_1, \ldots, y_6)$ is an oriented orthonormal basis of $(\mathbb{R}^6)^*$. The subgroup of $GL(6, \mathbb{R})$ preserving ω and Ω is $SU(3)$. In terms of ω and Ω, the 3- and 4-forms φ, $*\varphi$ of (1) and (2) are

$$\varphi = \omega \wedge y_7 + \mathrm{Im}(\Omega), \qquad *\varphi = \tfrac{1}{2}\omega \wedge \omega - \mathrm{Re}(\Omega) \wedge y_7. \tag{12}$$

The subgroup of $GL(7,\mathbb{R})$ preserving φ and $*\varphi$ is G_2, and it can be seen from (12) that the subgroup of $GL(7,\mathbb{R})$ preserving φ, $*\varphi$ and y_7 is the subgroup of $GL(6,\mathbb{R})$ preserving ω and Ω, which is $SU(3)$. This gives an inclusion $SU(3) \subset G_2$. In fact $SU(3)$ is the maximal connected subgroup of G_2. Let Z be a compact, Kähler, complex 3-manifold with holonomy $SU(3)$. Then using (12) we may define a torsion-free G_2-structure φ on $Z \times \mathbb{R}$, putting $y_7 = dx$, where x is the coordinate in $\mathbb{R}$.

1.3. Asymptotically locally Euclidean metrics with special holonomy. In §2.1 we will define the idea of a *generalized Kummer construction*, which is a desingularization of the quotient T^n/Γ of a torus T^n by a finite group Γ. Under favourable conditions the singular set of T^n/Γ is a disjoint union of singular tori T^l modelled on the singular set of $T^l \times (\mathbb{R}^{n-l}/G)$, for some finite subgroup G of $SO(n-l)$. To desingularize T^n/Γ one then replaces $\mathbb{R}^{n-l}/G$ by a complete Riemannian manifold X that has only one end, which is asymptotic to $\mathbb{R}^{n-l}/G$ in some sense. To achieve some particular holonomy group H on the desingularized manifold, it is natural to require the holonomy group of X to be the subgroup of H acting trivially on $\mathbb{R}^l \subset \mathbb{R}^n$.

Such spaces X are called *asymptotically locally Euclidean* manifolds, which will be shortened to *ALE spaces*. An ALE space X is a complete Riemannian manifold with one end modelled on the end of $\mathbb{R}^n/G$, such that the metric g of X is asymptotic to the Euclidean metric h on $\mathbb{R}^n/G$ in a sense to be given below. Here G is a nontrivial finite subgroup of $SO(n)$ that acts freely on $\mathbb{R}^n \setminus \{0\}$. We may observe at once that n must be even, since if n is odd then every nontrivial element of $SO(n)$ has fixed points on $\mathbb{R}^n \setminus \{0\}$, so that there can be no candidates for the finite group G. Also, since we are only interested in Ricci-flat g we may exclude the case $n = 2$, since a Ricci-flat 2-manifold is flat, and thus X would be isometric to $\mathbb{R}^2$. So we shall consider X of even dimension 4 and above.

We require the metric g on X to be asymptotic to the Euclidean metric h on $\mathbb{R}^n/G$ in the following sense. There should exist a continuous, surjective map $\phi : X \to \mathbb{R}^n/G$ that is smooth in the appropriate sense, such that $\phi^{-1}(0)$ is a connected, simply-connected, finite union of compact submanifolds of X, and ϕ induces a diffeomorphism from $X \setminus \phi^{-1}(0)$ to $(\mathbb{R}^n \setminus \{0\})/G$. Under this diffeomorphism, $\phi_*(g)$ should satisfy

$$\phi_*(g) - h = O(r^{-4}), \qquad \partial\phi_*(g) = O(r^{-5}), \qquad \partial^2\phi_*(g) = O(r^{-6}) \tag{13}$$

for large r, where r is the distance from the origin in $\mathbb{R}^n/G$, and ∂ is the flat connection on $\mathbb{R}^n/G$. The powers of r in this definition are chosen to match the power t^4 in condition (i) of [15, Theorem B].

In this paper we shall only be interested in the cases $n = 4$ when g has holonomy $SU(2)$, and $n = 6$ when g has holonomy $SU(3)$. In these cases we shall impose the following extra condition. The $SU(2)$- and $SU(3)$- structures are defined by closed 2- and 3-forms, and we require that if ω is one of the closed k-forms defining the $SU(n/2)$- structure and $\hat{\omega}$ is the corresponding constant k-form on $\mathbb{R}^n/G$, then $\phi_*(\omega) - \hat{\omega} = d\rho$ on $(\mathbb{R}^n \setminus \{0\})/G$, where ρ is a $k-1$-form on $(\mathbb{R}^n \setminus \{0\})/G$, and

$$\rho = O(r^{-3}), \qquad \partial\rho = O(r^{-4}), \qquad \partial^2\rho = O(r^{-5}), \qquad \partial^3\rho = O(r^{-6}) \tag{14}$$

for large r. This condition will be used later to prove condition (i) of [15, Theorem B].

The Eguchi-Hanson space of [15, §1.3] is an example of an ALE space with $SU(2)$ holonomy in the above sense, and the properties above were essential to the example of [15]. Now ALE spaces with holonomy $SU(2)$ have been well studied, and it is known that there are families of ALE spaces with holonomy $SU(2)$ for all finite subgroups $G \subset SU(2)$. A complete construction and classification of these has been given by Kronheimer [16], [17]. For cyclic groups, the metrics are given explicitly in [11].

For ALE spaces with holonomy $SU(3)$ there also exists a good theory, which is less explicit than the $SU(2)$ case. The theory splits into two parts. The first part is to show that if Γ is a finite subgroup of $SU(3)$ acting freely on $\mathbb{C}^3 \setminus \{0\}$, then the singularity of $\mathbb{C}^3/\Gamma$ admits a complex resolution X with $c_1(X) = 0$. This part has been proved for several different types of group Γ by Markushevich et al. [19, Appendix], by Roan [23, pp. 527-8] and by Ito [14]. In particular, if Γ is abelian then such resolutions always exist, and their topology can be found using toric geometry.

The second part of the theory is to show that these candidate noncompact complex manifolds possess ALE metrics with holonomy $SU(3)$. To prove this we draw on the results of Tian and Yau [26]. Let X be a resolution of the singularities of $\mathbb{C}^3/\Gamma$ with $c_1(X) = 0$. Then we may write $X = M \setminus D$, where M is a compact complex orbifold, and D is a divisor associated to the line bundle $K_M^{-\beta}$ for some β. (In fact, D is an orbifold of $\mathbb{CP}^2$.) Tian and Yau show that $X = M \setminus D$ has a Ricci-flat Kähler metric satisfying certain asymptotic conditions near D. This is the ALE metric with holonomy $SU(3)$ that we want.

The only problem that remains is to show that this metric has the asymptotic behaviour (13) and (14). Tian and Yau prove only that the Riemann curvature satisfies $|R| = O(r^{-3})$. But by a refinement of their arguments for this particular case, it does not appear difficult to show that (13) and (14) both hold.

There is one ALE space with holonomy $SU(3)$ for which the metric is known explicitly. Let (z_1, z_2, z_3) be complex coordinates on $\mathbb{C}^3$. Define an action of $\mathbb{Z}_3$ on $\mathbb{C}^3$, generated by the map $z_j \mapsto e^{2\pi i/3} z_j$. Let X be the blow-up of $\mathbb{C}^3/\mathbb{Z}_3$ at the origin. Then Calabi [8, eq. (4.14), p. 285] gives an explicit Kähler potential for a metric of holonomy $SU(3)$ on X. It is easy to verify from the Kähler potential that (13) and (14) hold in this case.

2. A construction for compact 7-manifolds with holonomy G_2

This chapter describes a generalization of the method used in [15] to make a compact 7-manifold M admitting metrics with holonomy G_2. Since the idea of [15] was based on the Kummer construction for the $K3$ surface, we have chosen to call our method the *generalized Kummer construction.* The idea of a generalized Kummer construction is defined in §2.1, as a means of constructing smooth, compact n-manifolds by desingularizing a quotient T^n/Γ of the n-torus by a finite group Γ of automorphisms of T^n.

In §2.2 we prove that given a finite group Γ acting on T^7 and satisfying some (rather restrictive) conditions, there exists a generalized Kummer construction that desingularizes T^7/Γ to give a compact 7-manifold M, and that M has a family of torsion-free G_2- structures. The proof of this result uses Theorems A-C of [15], and is a simple generalization of the material of [15, Chapter 2]. The chapter finishes in §2.3 with an explanation of how to calculate the fundamental group and Betti numbers of the 7-manifolds M produced in this way.

2.1. The generalized Kummer construction. The $K3$ surface is a compact, simply-connected complex surface with $c_1 = 0$. By Yau's solution of the Calabi conjecture [28], the $K3$ surface admits a family of metrics of holonomy $SU(2)$. An approximate description of some of these metrics was given by Page [22], which employs an idea known as the Kummer construction. Roan [23] made use of the same idea to find compact complex 3-manifolds with holonomy $SU(3)$. We shall now define a 'generalized Kummer construction' which extracts the essential features of the $K3$ case.

Definition 2.1.1. Let T^n be an n-torus with a flat Riemannian metric. Let Γ be a finite group of isometries of T^n. Then T^n/Γ is a manifold with singularities. Let S be the singular set of T^n/Γ. Let M be a compact, smooth n-manifold, and let $\Phi : M \to T^n/\Gamma$ be a continuous, surjective map that is smooth except at S. We say that the quadruple (T^n, Γ, M, Φ) is a *generalized Kummer construction* if it has the following three properties:

(i) Φ is injective on $\Phi^{-1}\bigl(M \setminus S\bigr)$,

(ii) $\Phi^{-1}(S)$ is a finite union of compact submanifolds of M, and

(iii) for each $s \in S$, $\Phi^{-1}(s)$ is a connected, simply-connected, finite union of compact submanifolds of M.

Now let us consider the question of how to choose a group Γ and define a compact manifold M. There are good and bad types of singular point in the singular set S of T^n/Γ. The 'bad' types of singular point are where two or more (singular) submanifolds of S intersect, and they are bad because it is not very clear how to resolve them. For simplicity we shall usually choose groups Γ that have no bad singular points in this sense. Here is a condition that ensures this.

Condition 2.1.2. Let T^n be an n-torus with a flat Riemannian metric, and let Γ be a finite group of orientation-preserving isometries of T^n. Suppose that whenever γ_1, γ_2 are nonidentity elements of Γ that have fixed points in T^n, then either $\gamma_1\gamma_2$ has no fixed points on T^n, or $\gamma_1 = \gamma_2^k$ for some integer k.

Lemma 2.1.3. *Suppose that T^n is an n-torus with a flat Riemannian metric, and that Γ is a finite group of isometries of T^n satisfying Condition 2.1.2. Let S be the singular set of T^n/Γ. Then S is a disjoint union of connected components, and each connected component has a neighbourhood isometric to a neighbourhood of the singular set in $\bigl\{T^{n-2l} \times (\mathbb{R}^{2l}/\mathbb{Z}_p)\bigr\}/F$. Here l is a positive integer with $2l \le n$, $\mathbb{R}^{2l}/\mathbb{Z}_p$ is the quotient of $\mathbb{R}^{2l}$ by some nontrivial, prime, cyclic subgroup $\mathbb{Z}_p$ of $SO(2l)$ acting freely on $\mathbb{R}^{2l} \setminus \{0\}$, and F is a finite group of isometries of $T^{n-2l} \times (\mathbb{R}^{2l}/\mathbb{Z}_p)$ that acts freely on T^{n-2l}.*

Proof. Let S' be the subset of T^n of points that are fixed points of some nonidentity element of Γ. Then the singular set S is $S = S'/\Gamma$. It is easy to see that the fixed point set $S'(\gamma)$ of any nonidentity element $\gamma \in \Gamma$ is a disjoint union of tori T^m in T^n. Let γ_1, γ_2 be nonidentity elements with fixed points on T^n. If $S'(\gamma_1) \cap S'(\gamma_2) \neq \emptyset$

then $\gamma_1\gamma_2$ has fixed points, so by Condition 2.1.2, γ_1 is a power of γ_2. Exchanging γ_1 and γ_2 shows that γ_2 is also a power of γ_1, and therefore $S'(\gamma_1) = S'(\gamma_2)$.

We deduce that S' is a *disjoint* union of tori T^m. It follows that S is a disjoint union of components, each of which is isometric to the quotient of a torus T^m by a finite group F; moreover, F acts freely on T^m, as a fixed point of F on T^m would lead to a contradiction of Condition 2.1.2. Consider a single component T^m/F of S. Its preimage in S' is a finite union of isomorphic tori T^m, of which we select one. The nonidentity elements of Γ fixing the points of this torus T^m are powers of each other by Condition 2.1.2, so the subgroup of Γ fixing the points of T^m is a nontrivial, cyclic group $\mathbb{Z}_p$ of prime order.

Because the torus T^m is isolated in S', $\mathbb{Z}_p$ must act freely on the normal bundle $\mathbb{R}^{n-m}$ of T^m in T^n, and $\mathbb{Z}_p$ preserves the orientation of $\mathbb{R}^{n-m}$ since Γ preserves the orientation of T^n. Thus $\mathbb{Z}_p$ is a subgroup of $SO(n-m)$ acting freely on $\mathbb{R}^{n-m}\setminus\{0\}$, as we have to prove. If $n-m$ is odd, then all elements of $SO(n-m)$ have fixed points, so there are no nontrivial groups $\mathbb{Z}_p \subset SO(n-m)$ acting freely on $\mathbb{R}^{n-m}\setminus\{0\}$. Thus $n-m$ is even, and we may write $n-m=2l$ for some positive integer l with $2l \le n$, and $m = n-2l$. The remainder of the Lemma is now clear. $\square$

2.2. Torsion-free G_2- structures on compact 7-manifolds. The general plan for constructing compact manifolds with interesting metrics via the generalized Kummer construction runs as follows. One chooses a group Γ acting on a flat Riemannian torus T^n, such that the singularities of T^n/Γ are not too severe, for instance if Γ satisfies Condition 2.1.2. In particularly good cases the singular set of T^n/Γ is composed of disjoint components modelled on $T^{n-2l}\times(\mathbb{R}^{2l}/\mathbb{Z}_p)$. To construct M in this case one chooses an ALE space X for the group $\mathbb{Z}_p \subset SO(2l)$, for each component of S.

A compact n-manifold M can then be constructed in a natural way by using $T^{n-2l}\times X$ to desingularize the singular component modelled on $T^{n-2l}\times(\mathbb{R}^{2l}/\mathbb{Z}_p)$. In this paper we consider only the case $n=7$ and $l=2$ or 3, and we aim to write down G_2- structures on M that have small torsion. This is done in the next Theorem, which covers only the case of very well-behaved singularities. The conclusions of the Theorem are the same as the hypotheses of [15, Theorem B], which is the reason for the Theorem.

Theorem 2.2.1. *Let $\hat\varphi$ be a flat G_2- structure on T^7, and let Γ be a finite group of diffeomorphisms of T^7 preserving $\hat\varphi$. Let $S_1,\ldots,S_k$ be the connected components of the singular set S of T^7/Γ. Suppose that each S_j either has a neighbourhood isometric to a neighbourhood of the singular set of $T^3\times\mathbb{C}^2/G_j$, where T^3 is a flat Riemannian torus and G_j a finite subgroup of $SU(2)$, or has a neighbourhood isometric to a neighbourhood of the singular set of $\mathcal{S}^1\times\mathbb{C}^3/G_j$, where G_j is a finite subgroup of $SU(3)$ acting freely except at 0. For each j, let X_j be an ALE space with holonomy $SU(2)$ or $SU(3)$ asymptotic to $\mathbb{C}^2/G_j$ or $\mathbb{C}^3/G_j$ as appropriate, in the sense of §1.3.*

Then there exists a compact 7-manifold M constructed from T^7/Γ and $X_1,\ldots,X_k$, a positive constant θ, and a family $\{\varphi_t : t\in(0,\theta]\}$ of smooth, closed sections of Λ^3_+M. Let g_t be the metric on M associated to φ_t. There exists a family $\{\psi_t : t\in(0,\theta]\}$ of smooth 3-forms on M with $d^*\psi_t = d^*\varphi_t$, where d^* is defined using g_t. There exist positive constants $D_1,\ldots,D_5$ independent of t, such that the following five conditions hold for each $t\in(0,\theta]$, where all norms are calculated using g_t.

(i) $\|\psi_t\|_2 \le D_1 t^4$ *and* $\|\psi_t\|_{C^2} \le D_1 t^4$,

(ii) *the injectivity radius* $\delta(g_t)$ *satisfies* $\delta(g_t) \ge D_2 t$,

(iii) *the Riemann curvature* $R(g_t)$ *of* g_t *satisfies* $\|R(g_t)\|_{C^0} \le D_3 t^{-2}$,

(iv) *the volume* $\mathrm{vol}(M)$ *satisfies* $\mathrm{vol}(M) \ge D_4$, *and*

(v) *the diameter* $\mathrm{diam}(M)$ *satisfies* $\mathrm{diam}(M) \le D_5$.

Proof. The proof follows [15, §2.2]. Let ζ be a positive constant sufficiently small that if T is the open set of all points a distance less than ζ from S, then each connected component of T contains exactly one component of S, the closures of these components of T do not intersect, and the component containing S_j is isometric to $T^3 \times (B^4_\zeta/G_j)$ or $\mathcal{S}^1 \times (B^6_\zeta/G_j)$ as appropriate, where B^4_ζ and B^6_ζ are the open balls of radius ζ about 0 in $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. Let T_j be the component of T containing S_j.

The compact 7-manifold M will be defined by modifying T^7/Γ in each neighbourhood T_j, following the procedure below, and on M a family of 3-forms $\{\varphi_t : t \in (0,\theta]\}$ and a family of 4-forms $\{\upsilon_t : t \in (0,\theta]\}$ will be defined. On $(T^7/\Gamma) \setminus T$ we define φ_t and υ_t by $\varphi_t = \hat\varphi$ and $\upsilon_t = *\hat\varphi$. Thus it suffices to work on a single component T_j, and explain how to define the corresponding part of M, and the forms φ_t, υ_t upon it. There are two cases, the case $T_j \cong T^3 \times (B^4_\zeta/G_j)$ and the case $T_j \cong \mathcal{S}^1 \times (B^6_\zeta/G_j)$, and we will deal with them in this order.

For the first case, suppose that T_j is isometric to $T^3 \times (B^4_\zeta/G_j)$, where G_j is a finite subgroup of $SU(2)$. By the hypotheses of the Theorem we are given an ALE space X_j with holonomy $SU(2)$, asymptotic to $\mathbb{C}^2/G_j$. By the definition of ALE space in §1.3, X_j comes with a map $\phi_j : X_j \to \mathbb{C}^2/G_j$ satisfying certain conditions. For $t > 0$, define $U_{j,t}$ to be the open subset $\phi_j^{-1}(B^4_{\zeta/t}/G_j)$ of X_j, and define a map $\phi_{j,t} : U_{j,t} \to B^4_\zeta/G_j$ by $\phi_{j,t}(u) = t\phi_j(u)$. Then $\phi_{j,t}$ is surjective, as ϕ_j is surjective. So $\phi_{j,t}$ induces a map $\Phi_j : T^3 \times U_{j,t} \to T^3 \times B^4_\zeta/G_j \cong T_j$, which is a resolution of the singularities S_j of T_j. Define the part of the compact 7-manifold M coming from T_j in the obvious way, using this map Φ_j to resolve the singularities S_j, and define the map $\Phi : M \to T^7/\Gamma$ on this part of M by $\Phi = \Phi_j$. The 7-manifold M is independent of t, as a smooth 7-manifold.

Now we must define the forms φ_t, υ_t on $T^3 \times U_{j,t}$. Following equations (9), (10) of §1.2, we may write $\hat\varphi$ and $*\hat\varphi$ on $T_j \cong T^3 \times B^4_\zeta$ by

$$\hat\varphi = \hat\omega_1 \wedge \delta_1 + \hat\omega_2 \wedge \delta_2 + \hat\omega_3 \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3, \tag{15}$$

$$*\hat\varphi = \hat\omega_1 \wedge \delta_2 \wedge \delta_3 + \hat\omega_2 \wedge \delta_3 \wedge \delta_1 + \hat\omega_3 \wedge \delta_1 \wedge \delta_2 + \tfrac{1}{2}\hat\omega_1 \wedge \hat\omega_1, \tag{16}$$

where δ_1, δ_2 and δ_3 are constant orthonormal 1-forms on $S_j \cong T^3$, and $\hat\omega_1, \hat\omega_2, \hat\omega_3$ are constant 2-forms on $B^4_\zeta/\{\pm 1\}$ that can be written in the form (8) of §1.2. Now the $SU(2)$- structure on X_j defines 2-forms $\omega_1, \omega_2, \omega_3$ on X_j, which are closed because the $SU(2)$- structure is torsion-free. The 2-forms $t^2\omega_i$ define a torsion-free $SU(2)$-structure on $U_{j,t}$, and the definition of ALE space in §1.3 implies that there is a 1-form ρ_i on $\big(B^4_\zeta \setminus \{0\}\big)/G_j$ that satisfies $(\phi_{j,t})_*(t^2\omega_i) - \hat\omega_i = d\rho_i$. The powers of r in (14) have been chosen such that scaling by the appropriate powers of t in (14) gives $\partial^l \rho_i = O(t^4)$ outside $B^4_{\zeta/4}/G_j$, for $l = 0,1,2,3$.

Let r be the lift to $U_{j,t}$ by $\phi_{j,t}$ of the radius function on B^4_ζ/G_j. Let $\tau : [0,\zeta] \to [0,1]$ be a fixed, smooth, nondecreasing function with $\tau(x) = 0$ for $x \le \zeta/4$ and

$\tau(x) = 1$ for $x \geq \zeta/2$. Then $\tau(r)$ is a smooth real function on $U_{j,t}$. Define closed 2-forms $\tilde\omega_1, \tilde\omega_2, \tilde\omega_3$ on $U_{j,t}$ by $\tilde\omega_i = t^2\omega_i - d\big(\tau(r)\phi^*_{j,t}(\rho_i)\big)$. Then $\tilde\omega_i$ is a smooth, closed interpolation between the 2-form $t^2\omega_i$ on the interior of $U_{j,t}$, and the 2-form $\hat\omega_i$ near the boundary of $U_{j,t}$. Define 3-form φ_t and a 4-form υ_t on $T^3 \times U_{j,t}$ by

$$\varphi_t = \tilde\omega_1 \wedge \delta_1 + \tilde\omega_2 \wedge \delta_2 + \tilde\omega_3 \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3, \tag{17}$$

$$\upsilon_t = \tilde\omega_1 \wedge \delta_2 \wedge \delta_3 + \tilde\omega_2 \wedge \delta_3 \wedge \delta_1 + \tilde\omega_3 \wedge \delta_1 \wedge \delta_2 + \tfrac{1}{2}\tilde\omega_1 \wedge \tilde\omega_1, \tag{18}$$

following (15) and (16). Since by definition the 2-forms $\tilde\omega_i$ and $\hat\omega_i$ agree near the boundary of $U_{j,t}$, φ_t and υ_t extend smoothly over the boundary of $U_{j,t}$, and as $\tilde\omega_i$, δ_i are closed, φ_t and υ_t are closed.

What is happening here is this. Define A_j to be the open subset of $T^3 \times U_{j,t}$ for which $r \in (\zeta/4, \zeta/2)$. We think of A_j as an annulus, a transition zone between two regions of M. On one side of A_j, φ_t, υ_t are equal to $\hat\varphi$ and $*\hat\varphi$ respectively, and so they are the 3- and 4-forms of a flat, torsion-free G_2- structure. On the other side of A_j, φ_t and υ_t are also the 3- and 4- forms of a torsion-free G_2- structure, which is the product of a flat 3-torus and a torsion-free $SU(2)$- structure on X_j. The metric of X_j has been scaled by a homothety multiplying distances by t. On A_j itself, φ_t and υ_t interpolate smoothly between the two.

Now the derivative of $\tau(r)$ is nonzero on A_j, and the terms in $\tilde\omega_i$ introduced by this derivative mean that the 2-forms $\tilde\omega_i$ need not be in the form (8). Because of this, φ_t and υ_t need not be the the 3- and 4-forms of the same G_2- structure on A_j. However, the estimates above on ρ_i and $\partial\rho_i$ show that $\varphi_t - \hat\varphi = O(t^4)$ on A_j. Therefore for small t, φ_t is a section of $\Lambda^3_+ M$ on A_j. The estimates $\partial^l\rho_i = O(t^4)$ on A_j for $l = 0, 1, 2, 3$ given above show that $\Theta(\varphi_t) - \upsilon_t$ and its first two derivatives are $O(t^4)$ on A_j for small t. This completes our treatment of the first case, for which $T_j \cong T^3 \times B^4_\zeta/G_j$.

The second case is handled in a very similar way. The main differences are that the analogues of (15), (16) and (17), (18) should be modelled on (12) instead of on (9) and (10). One can define the 7-manifold M and 3- and 4-forms φ_t, υ_t just as in the first case; the details will be left for the reader. Thus we conclude that we may define a compact 7-manifold M, a family of smooth, closed 3-forms φ_t on M, and a family of smooth, closed 4-forms υ_t on M. On the complement of some annular regions $A_1, \dots, A_k$, φ_t and υ_t are the 3- and 4-forms of a torsion-free G_2- structure. For small t, φ_t is also a G_2- structure on the annuli A_j, and $\Theta(\varphi_t) - \upsilon_t$ and each of its derivatives are $O(t^4)$.

For small t, let g_t be the metric associated to the G_2- structure φ_t, and define a 3-form ψ_t on M by $\psi_t = \varphi_t - *\upsilon_t$, where $*$ is the Hodge star of g_t. Then $d^*\psi_t = d^*\varphi_t$, as $d\upsilon_t = 0$. Also, ψ_t is zero outside $A_1, \dots, A_k$ as $\upsilon_t = \Theta(\varphi_t)$ there, and on each A_j, $\psi_t = O(t^4)$. Therefore there exists some positive θ such that for each $t \in (0, \theta]$, φ_t is a smooth, closed G_2- structure on M, ψ_t is a smooth 3-form on M, and ψ_t satisfies $\|\psi_t\|_2 \leq D_1 t^4$ and $\|\psi_t\|_{C^2} \leq D_1 t^4$ for some positive constant D_1. This gives part (i) of the Theorem, as we have to prove.

The remaining parts (ii)-(v) of the Theorem are elementary, if we allow ourselves to make θ smaller if necessary. Since the metric g_t is made by scaling the metric on the ALE spaces X_j by a homothety multiplying distances by t, it follows that for small t, the injectivity radius of g_t on the regions of M coming from X_j must be proportional to t. Thus it is easy to see that part (ii) holds for some positive constant D_2. To estimate the Riemann curvature $R(g_t)$ of g_t, we have to consider two sources of curvature: firstly, the curvature of the metrics of the ALE spaces X_j, and secondly, the extra curvature on the regions A_j introduced by the derivatives of $\tau(r)$.

Equation (13) ensures that the Riemann curvature is bounded on X_j even though X_j is a noncompact manifold. It can be shown that scaling distances on X_j by t has the effect of scaling $|R|$ by t^{-2}. Thus the first source of curvature has C^0-norm proportional to t^{-2}, as we want. It can also be shown that the second source of curvature only contributes terms that are $O(t^4)$. Therefore part (iii) of the Theorem holds for some positive constant D_3. When t is small, the volume and diameter of M with the metric g_t are close to the volume and diameter of T^7/Γ. Thus positive constants D_4, D_5 must exist such that parts (iv) and (v) hold as well. This completes the Theorem. $\square$

Recall how Lemma 2.1.3 described the singular set of T^n/Γ for Γ satisfying Condition 2.1.2. For simplicity, Theorem 2.2.1 covered only the case where the groups F arising in Lemma 2.1.3 are trivial. Here is the extension of the result to nontrivial groups F.

Theorem 2.2.2. *Let $\hat{\varphi}$ be a flat G_2- structure on T^7, and let Γ be a finite group of diffeomorphisms of T^7 preserving $\hat{\varphi}$. Let $S_1, \ldots, S_k$ be the connected components of the singular set S of T^7/Γ. Suppose that for each $j = 1, \ldots, k$, either*

(i) *S_j has a neighbourhood isometric to a neighbourhood of the singular set of $\{T^3 \times \mathbb{C}^2/G_j\}/F_j$, where T^3 is a flat Riemannian torus, G_j a finite subgroup of $SU(2)$, and F_j is a group of isometries of $T^3 \times \mathbb{C}^2/G_j$ acting freely on T^3. There is an ALE space X_j with holonomy $SU(2)$ asymptotic to $\mathbb{C}^2/G_j$ in the sense of §1.3, and an action of F_j on X_j such that $\{T^3 \times X_j\}/F_j$ is asymptotic to $\{T^3 \times \mathbb{C}^2/G_j\}/F_j$ in the obvious way, or*

(ii) *S_j has a neighbourhood isometric to a neighbourhood of the singular set of $\{\mathcal{S}^1 \times \mathbb{C}^3/G_j\}/F_j$, where G_j is a finite subgroup of $SU(3)$ acting freely except at 0, and F_j is a group of isometries of $\mathcal{S}^1 \times \mathbb{C}^3/G_j$ acting freely on $\mathcal{S}^1$. There is an ALE space X_j with holonomy $SU(3)$ asymptotic to $\mathbb{C}^3/G_j$ in the sense of §1.3, and an action of F_j on X_j such that $\{\mathcal{S}^1 \times X_j\}/F_j$ is asymptotic to $\{\mathcal{S}^1 \times \mathbb{C}^3/G_j\}/F_j$ in the obvious way.*

Then the conclusions of Theorem 2.2.1 hold.

Proof. The proof of this Theorem is obtained from the proof of Theorem 2.2.1 by making everything F_j- invariant or F_j- equivariant, which just increases the amount of confusing notation. The forms ρ_j must have a suitable form of F_j- invariance or F_j- equivariance. Forms ρ_j with this property can be obtained from forms ρ'_j without it by taking the average of the images of the ρ'_j under the elements of F_j. The details will be left to the reader. $\square$

Now we can give the main result of this section.

Theorem 2.2.3. *Let M be the compact 7-manifold constructed in Theorem 2.2.1 or in Theorem 2.2.2. Then M admits a smooth family of torsion-free G_2-structures of dimension $b^3(M)$.*

Proof. The Theorem follows from [15, Theorems A-C]. The conclusions of Theorem 2.2.1 are the hypotheses of Theorem B, and the conclusions of Theorem B are the hypotheses of Theorem A. Therefore Theorem A of [15] shows that for sufficiently small t, the G_2- structure φ_t on M may be deformed to a torsion-free G_2-

structure $\tilde{\varphi}_t$ on M. Thus there exist torsion-free G_2- structures on the compact 7-manifolds M of Theorem 2.2.1 and 2.2.2. By [15, Theorem C], the family of torsion-free G_2- structures on M is a smooth manifold of dimension $b^3(M)$. $\square$

2.3. How to calculate topological invariants of the manifolds M. Section 2.2 showed that a class of 7-manifolds generated by the generalized Kummer construction admit torsion-free G_2- structures. The next chapter will be dedicated to giving explicit examples of this. For such a 7-manifold M we will want to know the most basic topological invariants— the fundamental group $\pi_1(M)$, and the Betti numbers $b^1(M)$, $b^2(M)$ and $b^3(M)$. The fundamental group is useful because it determines the holonomy group of the metrics, by Proposition 1.1.1, and also $b^3(M)$ determines the dimension of the family of torsion-free G_2- structures. And all of the invariants are useful to distinguish different manifolds, so that we can show that there are many distinct compact 7-manifolds with metrics of holonomy G_2, for instance.

Determining the topological invariants of a 7-manifold presented as a generalized Kummer construction is an elementary though sometimes slightly tricky calculation. We will in general omit these calculations, or make only brief remarks, and state the results. This section will explain how the calculations are performed, so that the reader may verify them for herself, and will be able to make similar calculations for finite groups Γ not given as examples in this paper.

We begin with the fundamental group. Suppose that (T^7, Γ, M, Φ) is a generalized Kummer construction in the sense of §2.1. From part (iii) of Definition 2.1.1 we deduce that $\pi_1(M) = \pi_1(T^7/\Gamma)$. It is easy to show that the mapping $T^7 \to T^7/\Gamma$ induces a surjective group homomorphism

$$\rho : \Gamma \ltimes \mathbb{Z}^7 \to \pi_1(T^7/\Gamma). \tag{19}$$

The kernel $\operatorname{Ker}\rho$ of this homomorphism is a normal subgroup of $\Gamma \ltimes \mathbb{Z}^7$, so that $\pi_1(M) \cong \left(\Gamma \ltimes \mathbb{Z}^7\right) / \operatorname{Ker}\rho$. Thus the problem is to determine $\operatorname{Ker}\rho$.

There is a normal subgroup $\rho(\mathbb{Z}^7)$ of $\pi_1(T^7/\Gamma)$, and dividing out by it gives a group homomorphism

$$\rho' : \Gamma \to \pi_1(T^7/\Gamma)/\rho(\mathbb{Z}^7). \tag{20}$$

Define Γ' to be the kernel of ρ'. Now if $\gamma \in \Gamma$ has fixed points, then a path $p : [0,1] \to T^7$ with $p(0) = \gamma(p(1))$ may be deformed through paths with the same property to get a loop with $p(0) = p(1)$, so that p defines an element of $\mathbb{Z}^7$. The definition of ρ' therefore shows that $\rho'(\gamma) = 1$, so $\gamma \in \operatorname{Ker}\rho'$. Therefore Γ' contains the subgroup of Γ generated by the elements γ with fixed points on T^7.

It can be shown that Γ' is in fact equal to this subgroup, and moreover that M admits a finite cover M' and Φ a finite cover Φ' such that $(T^7, \Gamma', M', \Phi')$ is a generalized Kummer construction, and M is the quotient of M' by Γ/Γ', which acts freely. Therefore $\pi_1(M) = (\Gamma/\Gamma') \ltimes \pi_1(M')$, and it is sufficient to calculate $\pi_1(M')$, which reduces the calculation to the case $\Gamma' = \Gamma$. So suppose that $\Gamma' = \Gamma$. Then $\operatorname{Im}\rho' = \{1\}$, and since ρ' is surjective, this shows that $\pi_1(T^7/\Gamma) = \rho(\mathbb{Z}^7)$. Thus $\pi_1(T^7/\Gamma)$ is abelian.

It follows that the image of any commutator in $\Gamma \ltimes \mathbb{Z}^7$ is zero. Regard $\mathbb{Z}^7$ as $\pi_1(T^7)$ with its natural action of Γ. Let $\mathbb{Z}^l$ be the subspace of $\mathbb{Z}^7$ that is invariant under Γ, and let $\mathbb{Z}^{7-l}$ be the subspace of $\mathbb{Z}^7$ orthogonal to $\mathbb{Z}^l$ using the flat metric on T^7. Then Γ acts on $\mathbb{Z}^{7-l}$, and $\mathbb{Z}^l \oplus \mathbb{Z}^{7-l}$ is a sublattice of finite index in $\mathbb{Z}^7$. Since the action of Γ on $\mathbb{Z}^{7-l}$ fixes only zero, the commutators of elements of Γ with elements of $\mathbb{Z}^{7-l}$ generate a sublattice $(\mathbb{Z}^{7-l})'$ of finite index in $\mathbb{Z}^{7-l}$, and $(\mathbb{Z}^{7-l})'$ is contained in $\operatorname{Ker}\rho$. On the other hand, it is easy to see that the intersection of $\mathbb{Z}^l$ with $\operatorname{Ker}\rho$ is zero.

We deduce that $\operatorname{Ker}\rho|_{\mathbb{Z}^7}$ is a sublattice of $\mathbb{Z}^7$ of dimension $7-l$ over $\mathbb{Z}$. Therefore $\pi_1(T^7/\Gamma) \cong \mathbb{Z}^l \times A$, where A is some finite, abelian group. This group A does not always vanish. It can be determined from a careful study of the singular set of T^7/Γ, but the author knows of no easy way to do this calculation. This concludes our study of the fundamental group.

Next we shall explain how to calculate the Betti numbers $b^i(M)$. In the case of Theorem 2.2.2 this may be done using the formula

$$b^i(M) = b^i(T^7/\Gamma) + \sum_{j=1}^{k} b^i_c(Y_j) \tag{21}$$

for $i = 1, 2, 3$. Here $Y_j = \{T^3 \times X_j\}/F_j$ in case (i) and $Y_j = \{\mathcal{S}^1 \times X_j\}/F_j$ in case (ii) of Theorem 2.2.2, and $b^i_c(Y_j) = \dim H^i_c(Y_j, \mathbb{R})$, where H^i_c is *de Rham cohomology with compact supports*. For the case of Theorem 2.2.1, one omits the groups F_j in the definition of Y_j. The cohomology group $H^i(T^7/\Gamma, \mathbb{R})$ is isomorphic to the vector space of constant i-forms on T^7 that are invariant under Γ. Regarding the vector space of constant i-forms on T^7 as a representation of Γ, it is easy to calculate $b^i(T^7/\Gamma)$.

Since X_j is simply-connected by assumption, $b^1_c(Y_j) = 0$ for each j, and so $b^1(M) = b^1(T^7/\Gamma)$ by (21). To calculate $b^i_c(Y_j)$ for $i = 2, 3$ in the case $F_j = \{1\}$, one may use the equations

$$b^2_c(T^3 \times X_j) = b^2_c(X_j), \quad b^3_c(T^3 \times X_j) = 3b^2_c(X_j), \quad b^2_c(\mathcal{S}^1 \times X_j) = b^2_c(X_j), \quad b^3_c(\mathcal{S}^1 \times X_j) = b^2_c(X_j). \tag{22}$$

Here we have used the fact that $b^i_c(X_j)$ is zero for odd i, because the homology of X_j is usually generated by compact, complex submanifolds, which are even-dimensional. The case $F_j \neq \{1\}$ is more complicated, since one must calculate the F_j- invariant part of $H^i_c(T^3 \times X_j, \mathbb{R})$ or $H^i_c(\mathcal{S}^1 \times X_j, \mathbb{R})$ as appropriate. Finally, since the noncompact end of X_j is trivial in real cohomology, we can show that $H^i_c(X_j, \mathbb{R}) \cong H^i(X_j, \mathbb{R})$ unless $i = 0$ or $i = \dim X_j - 1$. Using the material above, and a good understanding of the topology of the ALE spaces X_j, one may calculate the Betti numbers $b^i(M)$.

3. Examples of compact 7-manifolds with holonomy G_2

This chapter is devoted to examples of the construction of Chapter 2. We shall construct compact 7-manifolds admitting families of metrics of holonomy G_2, and determine their Betti numbers and fundamental group. In this way we will prove that there are a number of topologically distinct compact 7-manifolds with metrics of holonomy G_2. Section 3.1 begins with the finite group $\Gamma \cong \mathbb{Z}_2^3$ used in [15], and considers small modifications of its action on T^7. The resulting quotients T^7/Γ are still desingularized using the Eguchi-Hanson space as in [15], but there are several different cases giving different manifolds. The result is also used to prove the existence of metrics of holonomy $SU(2)$ and $SU(3)$ on compact 4- and 6-manifolds.

In §3.2 we consider a different class of finite groups Γ based on the splitting $\mathbb{R}^7 \cong \mathbb{C}^3 \oplus \mathbb{R}$, and produce more examples of compact 7-manifolds with holonomy G_2, which can have quite small Betti numbers. The examples of this section are still desingularized using only the Eguchi-Hanson space. Section 3.3 finishes the chapter with some examples of quotients T^7/Γ that require other ALE spaces for their desingularization.

3.1. Variations on a theme. Chapter 2 of [15] defined a compact 7-manifold M by a generalized Kummer construction using a group $\Gamma \cong \mathbb{Z}_2^3$ of automorphisms of T^7 preserving the standard G_2- structure $\hat{\varphi}$ on T^7. In this section we will consider simple modifications of this group Γ. Let $(x_1, \ldots, x_7)$ be coordinates on $T^7 = \mathbb{R}^7/\mathbb{Z}^7$, where $x_i \in \mathbb{R}/\mathbb{Z}$. Define a section $\hat{\varphi}$ of $\Lambda^3_+ T^7$ by equation (1), where y_i is replaced by dx_i. Let α, β and γ be the involutions of T^7 defined by

$$\alpha\big((x_1, \ldots, x_7)\big) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7), \tag{23}$$

$$\beta\big((x_1, \ldots, x_7)\big) = (b_1 - x_1, b_2 - x_2, x_3, x_4, -x_5, -x_6, x_7), \tag{24}$$

$$\gamma\big((x_1, \ldots, x_7)\big) = (c_1 - x_1, x_2, c_3 - x_3, x_4, c_5 - x_5, x_6, -x_7), \tag{25}$$

where b_1, b_2, c_1, c_3 and c_5 are constants equal to 0 or $\frac{1}{2}$. By inspection, α, β and γ preserve $\hat{\varphi}$, because of the careful choice of exactly which signs to change.

As in [15], we have $\alpha^2 = \beta^2 = \gamma^2 = 1$, and α, β and γ commute. Define Γ to be the finite group $\langle \alpha, \beta, \gamma \rangle$ of isometries of T^7. Then $\Gamma \cong \mathbb{Z}_2^3$. Calculation shows that the Betti numbers $b^j(T^7/\Gamma)$ are independent of b_i and c_i, and are given by $b^1(T^7/\Gamma) = b^2(T^7/\Gamma) = 0$ and $b^3(T^7/\Gamma) = 7$. Throughout this section, let X denote the Eguchi-Hanson space of [15, §1.3], which is an ALE space with holonomy $SU(2)$ asymptotic to $\mathbb{C}^2/\{\pm 1\}$.

Example 1. Here is a simple application of our results, to prove known facts about the $K3$ surface. Let us ignore β, γ and consider the quotient $T^7/\langle \alpha \rangle$. Clearly $T^7/\langle \alpha \rangle$ is a product $T^4/\mathbb{Z}_2 \times T^3$, and since $T^4/\mathbb{Z}_2$ has 16 singular points, $T^7/\langle \alpha \rangle$ has 16 singular 3-tori. Desingularizing each of these using the Eguchi-Hanson space gives a compact 7-manifold $M \cong K3 \times T^3$. It can be shown that $b^3(M) = 67$, so that $K3 \times T^3$ admits a 67-parameter family of torsion-free G_2- structures, by Theorems 2.2.1 and 2.2.3. The fundamental group is $\pi_1(M) = \mathbb{Z}^3$, so Proposition 1.1.1 shows that the holonomy group of the underlying metrics is $SU(2)$.

Now the torsion-free G_2- structures on M do not all yield distinct metrics. The reason for this is that to define a compatible G_2- structure on $\mathbb{R}^4 \oplus \mathbb{R}^3$, where $\mathbb{R}^4$ and $\mathbb{R}^3$ are oriented Euclidean vector spaces, we require an identification of $\mathbb{R}^3$ with $\Lambda^2_+ \mathbb{R}^4$. The family of such identifications is $SO(3)$, and has dimension 3. Therefore every metric on $K3 \times T^3$ with holonomy $SU(2) \times \{1\}$ gives rise to a 3-dimensional family of torsion-free G_2- structures on M. So by subtraction, the family of metrics on $K3 \times T^3$ with holonomy $SU(2) \times \{1\}$ is of dimension 64.

As the family of flat metrics on T^3 is of dimension 6, it follows that the family of metrics of holonomy $SU(2)$ on $K3$ is of dimension 58. We have shown that the Kummer construction for the $K3$ surface [22] does give metrics of holonomy $SU(2)$ on $K3$. Proofs of this have been given by Topiwala [27], and LeBrun and Singer [18]. Their proofs use the deformation theory of singular complex manifolds, and are rather different to our analytic approach.

Example 2. We can also construct metrics with holonomy $SU(3)$ on a 6-manifold N. Set $(b_1, b_2) = (\frac{1}{2}, 0)$ and let β be defined by (24). Then the nonidentity elements of $\langle \alpha, \beta \rangle$ with fixed points on T^7 are α and β. The singular set of $T^7/\langle \alpha, \beta \rangle$ consists of 16 copies of T^3, each with a neighbourhood isometric to $T^3 \times \left(B^4_\zeta/\{\pm 1\}\right)$. Desingularizing these using the Eguchi-Hanson space yields a compact 7-manifold M, which carries torsion-free G_2- structures by Theorem 2.2.3. Since both α and β fix the coordinate x_7 on T^7, $M = N \times \mathcal{S}^1$, where N is a compact 6-manifold and $\mathcal{S}^1$ has coordinate x_7. It can be shown that N is simply-connected. Therefore $\pi_1(M) = \mathbb{Z}$, so that the holonomy group of the metrics underlying the torsion-free G_2- structures is $SU(3) \times \{1\}$, by Proposition 1.1.1.

Thus N is a compact, simply-connected 6-manifold with a family of metrics of holnomy $SU(3)$. Let us calculate the Betti numbers of N. We start with the Betti numbers $b^1(T^7/\langle \alpha, \beta \rangle) = 1$, $b^2(T^7/\langle \alpha, \beta \rangle) = 3$ and $b^3(T^7/\langle \alpha, \beta \rangle) = 11$. Each of the 16 components of the desingularization adds 1 to b^2 and 3 to b^3, so that $b^1(M) = 1$, $b^2(M) = 19$ and $b^3(M) = 59$. From these we deduce that $b^1(N) = 0$, $b^2(N) = 19$ and $b^3(N) = 40$. Note that the Euler characteristic $\chi(N)$ is zero. The family of torsion-free G_2- structures on M has dimension 59.

In a similar way to Example 1, each metric gives rise to a 1-parameter family of torsion-free G_2- structures, so that the family of metrics on M of holonomy $SU(3) \times \{1\}$ has dimension $59 - 1 = 58$. As the family of metrics on $\mathcal{S}^1$ has dimension 1, the family of metrics on N of holonomy $SU(3)$ has dimension 57. This can also be proved using Yau's solution of the Calabi conjecture [28].

Example 3. Here is the case considered in [15]. Set $(b_1, b_2, c_1, c_3, c_5) = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$. Then the singular set S of T^7/Γ has 12 T^3 components modelled on the singular set of $T^3 \times \left(B^4_\zeta/\{\pm 1\}\right)$. By Theorem 2.2.1 we may desingularize T^7/Γ to get a compact 7-manifold M, using the ALE space X for each component S_j. The topological invariants are $\pi_1(M) = \{1\}$, $b^1(M) = 0$, $b^2(M) = 12$ and $b^3(M) = 43$. By Theorem 2.2.3, M admits a 43-dimensional family of torsion-free G_2- structures. By Proposition 1.1.1, the holonomy group of the associated metrics is G_2, because $\pi_1(M)$ is finite.

Example 4. Here is a more complex example. Set $(b_1, b_2, c_1, c_3, c_5) = (0, \frac{1}{2}, \frac{1}{2}, 0, 0)$. An elementary calculation following [15, Lemma 2.1.1] shows that the only non-identity elements of Γ with fixed points on T^7 are α, β and γ. Therefore $\Gamma' = \Gamma$, Γ satisfies Condition 2.1.2, and Lemma 2.1.3 gives the general form of the singular set S of T^7/Γ. The subset S' of T^7 of points that are fixed points of some nonidentity element of Γ, is therefore the disjoint union of 48 copies of T^3, 16 from each of α, β and γ.

It can be shown, following the proof of [15, Lemma 2.1.1], that the group $\langle \beta, \gamma \rangle$ acts freely on the set of 16 3-tori fixed by α, and the group $\langle \alpha, \gamma \rangle$ acts freely on the set of 16 3-tori fixed by β. Therefore the fixed points of α and β each contribute 4 copies of T^3 to S, each of which has a neighbourhood isometric to $T^3 \times \left(B^4_\zeta/\{\pm 1\}\right)$. This is just as in Example 3. However, the element $\alpha\beta$ acts trivially on the set of 16 3-tori fixed by γ.

Because of this, it can be seen that the fixed points of γ contribute 8 copies of $T^3/\mathbb{Z}_2$ to S, where $\mathbb{Z}_2 = \langle \alpha\beta \rangle$, and each copy has a neighbourhood isometric to $\left\{T^3 \times B^4_\zeta/\{\pm 1\}\right\}/\mathbb{Z}_2$, where $\alpha\beta \in \mathbb{Z}_2$ acts on $T^3 \times B^4_\zeta/\{\pm 1\}$ by

$$(y_1, y_2, y_3, z_1, z_2) \mapsto (\tfrac{1}{2} + y_1, -y_2, -y_3, z_1, -z_2). \tag{26}$$

Here y_1, y_2, y_3 are coordinates on T^3 with $y_i \in \mathbb{R}/\mathbb{Z}$, and z_1, z_2 are complex coordinates on $B^4_\zeta/\{\pm 1\}$, so that (z_1, z_2) and $(-z_1, -z_2)$ are equivalent. Notice that $\mathbb{Z}_2$ acts freely on T^3, as it should. To apply Theorems 2.2.2 and 2.2.3, we must assign a suitable ALE space X_j to each component S_j of S. For the 8 copies of T^3 from the fixed points of α and β, we put $X_j = X$, the Eguchi-Hanson space, as in Example 3.

For the 8 copies of $T^3/\mathbb{Z}_2$ from the fixed points of γ we shall also put $X_j = X$, but we also require a suitable action of $\mathbb{Z}_2$ on X_j, asymptotic to the action $(z_1, z_2) \mapsto (z_1, -z_2)$ of $\mathbb{Z}_2$ on $\mathbb{C}^2/\{\pm 1\}$, taken from (26). It turns out that there are two *topologically distinct* ways of doing this. The Eguchi-Hanson space X is equivalent, as a complex manifold, to the cotangent bundle $T^*\mathbb{CP}^1$. Let $[w_1, w_2]$ be projective coordinates on $\mathbb{CP}^1$, and consider the two involutions $[w_1, w_2] \mapsto [w_1, -w_2]$ and $[w_1, w_2] \mapsto [\overline{w}_1, \overline{w}_2]$ of $\mathbb{CP}^1$. These induce involutions of $T^*\mathbb{CP}^1$ that are clearly topologically distinct, because one preserves the homology class of the zero section (which generates $H^2(X, \mathbb{R})$), and the other changes its sign.

Corresponding to these involutions there are two distinct, isometric actions of $\mathbb{Z}_2$ on X, asymptotic to the action $(z_1, z_2) \mapsto (z_1, -z_2)$ as we require. Therefore for each of the 8 $T^3/\mathbb{Z}_2$ components of S there is a choice of two alternatives for X_j. This gives 256 possible choices, which do not all lead to distinct 7-manifolds M. Let l be an integer with $0 \leq l \leq 8$, and consider the 7-manifold M made with l of the X_j with $\mathbb{Z}_2$- action modelled on the first involution, and $8 - l$ of the X_j with $\mathbb{Z}_2$- action modelled on the second involution. Calculation shows that this 7-manifold is simply-connected. By Theorems 2.2.2 and 2.2.3, M admits torsion-free G_2- structures, and by Proposition 1.1.1, the underlying metrics have holonomy G_2, as $\pi_1(M)$ is finite.

Let us find the Betti numbers of M. We begin with $b^2(T^7/\Gamma) = 0$ and $b^3(T^7/\Gamma) = 7$. Each of the 8 singular T^3 from α and β adds 1 to b^2 and 3 to b^3 as in Example 3. It can be shown that each of the l X_j with $\mathbb{Z}_2$- action modelled on the first involution contributes 1 to b^2 and 1 to b^3, and that each of the $8 - l$ X_j with $\mathbb{Z}_2$- action modelled on the second involution contributes 0 to b^2 and 2 to b^3. Therefore the Betti numbers of M are

$$b^2(M) = 8 + l, \qquad b^3(M) = 47 - l, \qquad l = 0, 1, \ldots, 8. \tag{27}$$

Thus this example gives at least 9 topologically distinct, simply-connected, compact 7-manifolds with metrics of holonomy G_2.

It is natural to wonder whether for other choices of b_i, c_i, we can generate any more simply-connected manifolds with holonomy G_2. In fact, a careful analysis shows that Examples 3 and 4 are essentially the only interesting cases. It can be shown that if the only nonidentity elements of Γ with fixed points are α, β and γ, then Γ is similar to either Example 3 or to Example 4, but perhaps singling out α or β rather than γ. If fewer nonidentity elements have fixed points then the holonomy group is not G_2, and if more nonidentity elements of Γ have fixed points then the singular set seems always to contain 'bad' singular points. For different choices of b_i, c_i and involutions on X, there are many ways of generating 7-manifolds M with the same Betti numbers. The author does not know whether these 7-manifolds are always diffeomorphic.

Example 5. We shall modify Example 4 by adding a finite group of translations. Let Γ be the group of Example 4, and define isometries $\sigma_1, \sigma_2, \sigma_3$ of T^7 by

$$\sigma_1\big((x_1,\ldots,x_7)\big) = (x_1, x_2, \tfrac12 + x_3, \tfrac12 + x_4, \tfrac12 + x_5, x_6, x_7), \tag{28}$$

$$\sigma_2\big((x_1,\ldots,x_7)\big) = (x_1, x_2, x_3, \tfrac12 + x_4, x_5, \tfrac12 + x_6, x_7), \tag{29}$$

$$\sigma_3\big((x_1,\ldots,x_7)\big) = (x_1, \tfrac12 + x_2, \tfrac12 + x_3, x_4, x_5, x_6, \tfrac12 + x_7). \tag{30}$$

Then σ_1, σ_2 and σ_3 commute with Γ. We claim that $\langle\sigma_1,\sigma_2,\sigma_3\rangle$ acts freely on T^7/Γ. To prove this claim, it is enough to show that the only nonidentity elements of $\Gamma \times \langle\sigma_1,\sigma_2,\sigma_3\rangle$ that have fixed points in T^7 are α, β and γ. For instance, $\alpha\sigma_1$ has no fixed points on T^7 because it acts on x_5 as $x_5 \mapsto \frac12 + x_5$. Reasoning in this way, the claim is easily proved.

Let A be a subgroup of $\langle\sigma_1,\sigma_2,\sigma_3\rangle$, and define $\tilde\Gamma = \Gamma \times A$. Then $T^7/\tilde\Gamma$ is a singular 7-manifold with fundamental group A. It can be shown that the singular set of $T^7/\tilde\Gamma$ breaks up into disjoint components of the form T^3 and $T^3/\mathbb{Z}_2$, as in Example 4. Therefore we may desingularize $T^7/\tilde\Gamma$ as in Example 4, to get a compact 7-manifold $\tilde M$ admitting metrics with holonomy G_2, with $\pi_1(\tilde M) = A$. Each of the $T^3/\mathbb{Z}_2$ components in the singular set may be resolved in two distinct ways, giving a number of different topological types for $\tilde M$.

We present in the following table the form of the singular set for six choices of the group A, and the fundamental group and Betti numbers b^2, b^3 of the resulting 7-manifold $\tilde M$. As in Example 4, these Betti numbers depend on an integer l, which is the number of $T^3/\mathbb{Z}_2$ components of S that are resolved using the first $\mathbb{Z}_2$-action on the Eguchi-Hanson space. The range of l is also given in the table. The calculations to determine the singular set of $T^7/\tilde\Gamma$ are elementary but long.

Group A	Singular set	$\pi_1(\tilde M)$	$b^2(\tilde M)$	$b^3(\tilde M)$	Range l
$\{1\}$	$8T^3 + 8T^3/\mathbb{Z}_2$	$\{1\}$	$8+l$	$47-l$	$0,\ldots,8$
$\langle\sigma_2\rangle$	$4T^3 + 8T^3/\mathbb{Z}_2$	$\mathbb{Z}_2$	$4+l$	$35-l$	$0,\ldots,8$
$\langle\sigma_3\rangle$	$2T^3 + 8T^3/\mathbb{Z}_2$	$\mathbb{Z}_2$	$2+l$	$29-l$	$0,\ldots,8$
$\langle\sigma_2,\sigma_3\rangle$	$T^3 + 6T^3/\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$1+l$	$22-l$	$0,\ldots,6$
$\langle\sigma_1\sigma_2,\sigma_3\rangle$	$6T^3/\mathbb{Z}_2$	$\mathbb{Z}_2^2$	l	$19-l$	$0,\ldots,6$
$\langle\sigma_1,\sigma_2,\sigma_3\rangle$	$4T^3/\mathbb{Z}_2$	$\mathbb{Z}_2^3$	l	$15-l$	$0,\ldots,4$

Table 1.

When $A = \{1\}$ we retrieve the case of Example 4. The other cases yield at least 37 distinct non-simply-connected 7-manifolds admitting metrics with holonomy G_2.

Example 6. Here is another example in which we supplement the group Γ by some translations on T^7. Set $(b_1, b_2, c_1, c_3, c_5) = (\frac12, 0, \frac12, 0, \frac12)$, and let α, β and γ be defined by (23), (24) and (25). Define δ on T^7 by

$$\delta\big((x_1,\ldots,x_7)\big) = (\tfrac12 + x_1, x_2, \tfrac12 + x_3, \tfrac12 + x_4, \tfrac12 + x_5, x_6, x_7), \tag{31}$$

and define $\tilde\Gamma = \langle\alpha,\beta,\gamma,\delta\rangle$. Then $\tilde\Gamma \cong \mathbb{Z}_2^4$, and $T^7/\tilde\Gamma$ is a singular 7-manifold. Calculation shows that the only nonidentity elements of $\tilde\Gamma$ with fixed points on T^7

are α, β, γ and $\alpha\beta\delta$. Thus $\tilde{\Gamma}$ satisfies Condition 2.1.2, and Lemma 2.1.3 gives the form of the singular set S of $T^7/\tilde{\Gamma}$.

It can be shown that $T^7/\tilde{\Gamma}$ is simply-connected, that the fixed points of α and β each contribute 4 copies of $T^3/\mathbb{Z}_2$ to S, and that the fixed points of γ and $\alpha\beta\delta$ each contribute 2 copies of T^3 to S. As in Examples 4 and 5, we may desingularize $T^7/\tilde{\Gamma}$ in a number of different ways to get a simply-connected, compact 7-manifold $\tilde{M}$. Working out the Betti numbers of these 7-manifolds $\tilde{M}$ as in Examples 4 and 5 gives $b^2(\tilde{M}) = 4 + l$ and $b^3(\tilde{M}) = 35 - l$ for $l = 0, 1, \ldots, 8$. Thus we have at least 9 simply-connected, compact 7-manifolds with metrics of holonomy G_2. They are distinct from the manifolds of Example 4, as the Betti numbers are different.

3.2. Another class of finite groups. We shall now give further examples of compact 7-manifolds with metrics of holonomy G_2, using Theorem 2.2.3. Let z_1, z_2, z_3 be the complex coordinates on $\mathbb{C}^3$ with its Euclidean, hermitian metric. Let ω be the Kähler form associated to this metric, and define $\Omega = dz_1 \wedge dz_2 \wedge dz_3$. Let x be the coordinate on $\mathbb{R}$. From §1.2, ω and Ω may be written in the form (11), and we may define a flat G_2- structure φ and its dual $*\varphi$ on $\mathbb{C}^3 \times \mathbb{R}$ by the formula (12), where y_7 is replaced by dx. Let $\Lambda \subset \mathbb{C}^3$ be a lattice in $\mathbb{C}^3$ that is isomorphic to $\mathbb{Z}^6$, and consider the 7-torus $T^7 = \mathbb{C}^3 \times \mathbb{R}/\Lambda \times \mathbb{Z}$, where $\mathbb{Z} \subset \mathbb{R}$ in the obvious way. Then T^7 is equipped with a flat G_2- structure φ and its dual $*\varphi$.

Let u, v be unit complex numbers, and suppose that $u^a = v^a = 1$, where a is a positive integer, and the least positive integer for which this holds. Define isometries α, β of $\mathbb{C}^3 \times \mathbb{R}$ by

$$\alpha\big((z_1, z_2, z_3, x)\big) = (uz_1, vz_2, \overline{uv}z_3, x + \tfrac{1}{a}), \tag{32}$$

$$\beta\big((z_1, z_2, z_3, x)\big) = (-\overline{z}_1, -\overline{z}_2, -\overline{z}_3, -x). \tag{33}$$

Suppose the lattice Λ is preserved by α and β. Then α and β push down to isometries α, β of T^7, and on T^7 they satisfy $\alpha^a = \beta^2 = 1$ and $\alpha\beta = \beta\alpha^{-1}$. Therefore α, β generate a finite group $\Gamma = \langle \alpha, \beta \rangle$ of isometries of T^7, which is isomorphic to the dihedral group with $2a$ elements. By inspection, α and β preserve the flat G_2- structure φ on T^7.

Now let us consider the singular set S of T^7/Γ. We may write

$$\Gamma = \{1, \alpha, \alpha^2, \ldots, \alpha^{a-1}, \beta, \beta\alpha, \ldots, \beta\alpha^{a-1}\}. \tag{34}$$

It is clear that $\alpha, \ldots, \alpha^{a-1}$ have no fixed points, because they translate the x-coordinate. On the other hand, it can easily be shown that all of $\beta, \beta\alpha, \ldots, \beta\alpha^{a-1}$ have fixed points. Thus Γ satisfies Condition 2.1.2. The fixed point sets of $\beta\alpha^j$ and $\beta\alpha^l$ are disjoint if $\alpha^j \neq \alpha^l$, since α^{j-l} has no fixed points. Let $S'(\beta\alpha^j)$ be the fixed point set of $\beta\alpha^j$ in T^7, and let S' be the union of all the $S'(\beta\alpha^j)$. Then $S = S'/\Gamma$. We shall divide into two cases, the case a odd and the case a even.

Suppose first that a is odd. Then for each $j = 0, 1, \ldots, a-1$, there exists an integer l such that $j + 2l \equiv 0 \mod a$. We have $\alpha^{-l}(\beta\alpha^j)\alpha^l = (\alpha^{-l}\beta)\alpha^{j+l} = \beta\alpha^{j+2l} = \beta$. Therefore every $\beta\alpha^j$ is conjugate to β in Γ. It follows that the singularities of T^7/Γ are in one-to-one correspondence with the fixed points of β on T^7. As the fixed points of β on T^7 consist of some finite number k of copies of T^3, it follows that the singular set S of T^7/Γ consists of k copies of T^3, each with a neighbourhood of the form $T^3 \times \big(B^4_\zeta/\{\pm 1\}\big)$. These can be desingularized using the Eguchi-Hanson space X, as in the examples of §3.1.

Now suppose that a is even, so that $a = 2b$. In this case, there are two conjugacy classes of elements $\beta\alpha^j$, as for j even $\beta\alpha^j$ is conjugate to β, and for j odd $\beta\alpha^j$ is conjugate to $\beta\alpha$. This means that S is a disjoint union of a contribution from the fixed points of β and a contribution from the fixed points of $\beta\alpha$. Now α^b commutes with β. It follows that S is isomorphic to the disjoint union of $S'(\beta)/\langle\alpha^b\rangle$ and $S'(\beta\alpha)/\langle\alpha^b\rangle$. The set $S'(\beta)$ splits into two disjoint, isomorphic parts, the part with $x = 0$ and the part with $x = \frac{1}{2}$, and α^b interchanges the two. Similarly, $S'(\beta\alpha)$ splits into two disjoint, isomorphic parts which are exchanged by α^b.

Therefore S is the disjoint union of half of the fixed point set of β, and half of the fixed point set of $\beta\alpha$. As in the previous case, S is a disjoint union of k copies of T^3, each with a neighbourhood of the form $T^3 \times \big(B^4_\zeta/\{\pm 1\}\big)$. We have shown that in both cases, the singular set of T^7/Γ consists of k copies of T^3, with neighbourhoods of the standard form. So by Theorems 2.2.1 and 2.2.3, there is a compact 7-manifold M admitting torsion-free G_2- structures. This manifold M has $\pi_1(M) \cong \pi_1(T^7/\Gamma)$, $b^2(M) = b^2(T^7/\Gamma) + k$, and $b^3(M) = b^3(T^7/\Gamma) + 3k$.

Here are some examples of this construction.

Example 7. Set $u = v = e^{2\pi i/3}$, so that $a = 3$, and let $\Lambda = \mathbb{Z}^3 \oplus e^{2\pi i/3}\mathbb{Z}^3 \subset \mathbb{C}^3$. Then Λ is preserved by α and β. The fixed point set of β is 2 copies of T^3, given by $\operatorname{Im} z_1 = \operatorname{Im} z_2 = \operatorname{Im} z_3 = 0$, $x \in \{0, \frac{1}{2}\}$. Therefore $k = 2$, as a is odd. Calculation shows that $\pi_1(T^7/\Gamma) = \{1\}$, $b^2(T^7/\Gamma) = 3$ and $b^3(T^7/\Gamma) = 7$. Thus M is simply-connected, and $b^2(M) = 5$, $b^3(M) = 13$. The metrics on M have holonomy G_2 by Proposition 1.1.1, as $\pi_1(M)$ is finite.

Example 8. Set $u = v = e^{\pi i/3}$, so that $a = 6$, and let Λ be as in Example 7. Then Λ is preserved by α and β. Calculation shows that $\pi_1(T^7/\Gamma) = \{1\}$, $b^2(T^7/\Gamma) = 1$, and $b^3(T^7/\Gamma) = 5$. The fixed point set of β is 2 copies of T^3, given by $\operatorname{Im} z_1 = \operatorname{Im} z_2 = \operatorname{Im} z_3 = 0$, $x \in \{0, \frac{1}{2}\}$, so that β contributes 1 copy of T^3 to S. Similarly, $\beta\alpha$ contributes 1 copy of T^3 to S. Therefore $k = 2$, and M is simply-connected with $b^2(M) = 3$ and $b^3(M) = 11$. Again, the metrics on M have holonomy G_2.

Example 9. Set $u = v = i$, so that $a = 4$, and let $\Lambda = \mathbb{Z}^3 \oplus i\mathbb{Z}^3 \subset \mathbb{C}^3$. Then Λ is preserved by α and β. The fixed point set of β is 16 copies of T^3, as $\operatorname{Im} z_1, \operatorname{Im} z_2, \operatorname{Im} z_3$ and x must lie in $\{0, \frac{1}{2}\}$. Because of the action of α^2, β contributes 8 copies of T^3 to S. The action of $\beta\alpha$ on T^7 is

$$\beta\alpha : (z_1, z_2, z_3, x) \to (-i\bar{z}_1, -i\bar{z}_2, -\bar{z}_3, \tfrac{3}{4} - x), \tag{35}$$

and so the fixed point set of $\beta\alpha$ is given by the equations $\operatorname{Re} z_1 + \operatorname{Im} z_1 = 0$, $\operatorname{Re} z_2 + \operatorname{Im} z_2 = 0$, $\operatorname{Re} z_3 \in \{0, \frac{1}{2}\}$ and $x \in \{\frac{3}{8}, \frac{7}{8}\}$. This breaks up into only 4 copies of T^3, which are identified in pairs by α^2. Thus $\beta\alpha$ contributes 2 copies of T^3 to S, and $k = 10$.

Calculation shows that $\pi_1(T^7/\Gamma) = \{1\}$, $b^2(T^7/\Gamma) = 1$ and $b^3(T^7/\Gamma) = 6$. Thus M is simply-connected and has $b^2(M) = 11$ and $b^3(M) = 36$. The metrics on M have holonomy G_2.

Example 10. The involution $(z_1, z_2, z_3, x) \mapsto (z_1, z_2, z_3 + \frac{1+i}{2}, x)$ acts freely on the singular manifold T^7/Γ of Example 9. Dividing by this involution gives $k = 5$, so that we construct a 7-manifold M with $\pi_1(M)=\mathbb{Z}_2$, $b^2(M)=6$ and $b^3(M)=21$, that carries metrics of holonomy G_2.

Example 11. Set $u=e^{\pi i/3}$ and $v=e^{2\pi i/3}$, so that $a=6$, and define Λ by

$$\Lambda=(\mathbb{Z}+e^{2\pi i/3}\mathbb{Z})\oplus(\mathbb{Z}+e^{2\pi i/3}\mathbb{Z})\oplus(\mathbb{Z}+i\mathbb{Z}). \tag{36}$$

Then $\Lambda\cong\mathbb{Z}^3$ and α,β preserve Λ. Calculation shows that $\pi_1(T^7/\Gamma)=\{1\}$, $b^2(T^7/\Gamma)=0$ and $b^3(T^7/\Gamma)=5$. The fixed points of β are given by $\operatorname{Im}z_1=\operatorname{Im}z_2=0$, $\operatorname{Im}z_3\in\{0,\frac12\}$ and $x\in\{0,\frac12\}$, which divides into 4 copies of T^3. Therefore the fixed points of β contribute 2 copies of T^3 to S. Similarly, the fixed points of $\beta\alpha$ contribute 2 copies of T^3 to S. So $k=4$, and M is simply-connected with $b^2(M)=4$ and $b^3(M)=17$. The metrics on M have holonomy G_2.

Example 12. As in Example 10, we may add the involution $(z_1,z_2,z_3,x)\mapsto(z_1,z_2,z_3+\frac{1+i}{2},x)$ to the situation of Example 11. We find that $k=2$, so that we produce a 7-manifold M with $\pi_1(M)=\mathbb{Z}_2$, $b^2(M)=2$ and $b^3(M)=11$, which has metrics of holonomy G_2.

Example 13. Let $p=e^{2\pi i/7}$. Set $u=p$ and $v=p^2$, so that $a=7$. Define $\Lambda\subset\mathbb{C}^3$ by

$$\Lambda=\big\langle(p^j,p^{2j},p^{4j})\in\mathbb{C}^3 : j=1,2,3,4,5,6\big\rangle. \tag{37}$$

Then $\Lambda\cong\mathbb{Z}^6$, and Λ is preserved by α and β. Calculation shows that $\pi_1(T^7/\Gamma)=\{1\}$, $b^2(T^7/\Gamma)=0$ and $b^3(T^7/\Gamma)=4$. The fixed point set of β is 2 copies of T^3, so that $k=2$. Thus M is simply-connected and $b^2(M)=2$, $b^3(M)=10$. The metrics on M have holonomy G_2.

Here is an example of a modified version of the construction.

Example 14. Following Example 7, set $\Lambda=\mathbb{Z}^3\oplus e^{2\pi i/3}\mathbb{Z}^3\subset\mathbb{C}^3$, let $T^7=(\mathbb{C}^3\times\mathbb{R})/(\Lambda\times\mathbb{Z})$. Define isometries α,β,γ of T^7 by

$$\alpha\big((z_1,z_2,z_3,x)\big)=(e^{2\pi i/3}z_1,e^{2\pi i/3}z_2,e^{2\pi i/3}z_3,x+\tfrac13), \tag{38}$$

$$\beta\big((z_1,z_2,z_3,x)\big)=(\overline{z}_1,\overline{z}_2,\overline{z}_3,-x), \tag{39}$$

$$\gamma\big((z_1,z_2,z_3,x)\big)=(e^{2\pi i/3}z_1,e^{4\pi i/3}z_2,z_3+\tfrac{i}{\sqrt3},x). \tag{40}$$

It can be shown that α,γ commute, that $\alpha^3=\gamma^3=1$, that $\alpha\beta=\beta\alpha^{-1}$, and that $\gamma\beta=\beta\gamma^{-1}$. Thus $\Gamma=\langle\alpha,\beta,\gamma\rangle$ is a finite group of order 18, similar to a dihedral group. Since α changes the x-coordinate and γ changes the z_3-coordinate, the only elements of Γ with fixed points are the 9 elements $\beta\alpha^j\gamma^l$, which are all conjugate.

The fixed points of β are 2 copies of T^3, as in Example 7. It follows that T^7/Γ has just 2 copies of T^3 in its singular set, each with a neighbourhood of the usual form. Calculation shows that $\pi_1(T^7/\Gamma)=\{1\}$, $b^2(T^7/\Gamma)=0$ and $b^3(T^7/\Gamma)=4$. Therefore M is simply-connected, and has $b^2(M)=2$ and $b^3(M)=10$, as in Example 13. The metrics on M have holonomy G_2.

3.3. More examples. The singular spaces T^7/Γ described in the previous two sections were all desingularized in exactly the same way — up to a finite cover, each component of the singular set was modelled on $T^3\times\big(\mathbb{R}^4/\{\pm1\}\big)$, and is desingularized using the Eguchi-Hanson space. Now §1.3 suggests that there are a number of ALE spaces that might be used to desingularize quotients T^7/Γ to get a Riemannian 7-manifold M with holonomy G_2. Thus it is clearly of interest to show that one can find examples of the construction of Chapter 2 involving other, more complicated ALE spaces, and it is the purpose of this section to give such examples. For some reason, they were quite difficult to find.

Example 15. Let T^7 be $\mathbb{R}^7/\mathbb{Z}^7$ with the metric, G_2- structure $\hat{\varphi}$ and coordinates $(x_1,\ldots,x_7)$ used in §3.1. Let α, β and σ_1 be the isometries of T^7 defined by

$$\alpha\big((x_1,\ldots,x_7)\big)=(x_3,x_4,x_5,x_6,x_1,x_2,x_7), \tag{41}$$

$$\beta\big((x_1,\ldots,x_7)\big)=(-x_1,\tfrac{1}{2}+x_2,-x_3,\tfrac{1}{2}+x_4,-x_5,\tfrac{1}{2}+x_6,-x_7), \tag{42}$$

$$\sigma_1\big((x_1,\ldots,x_7)\big)=(x_1,\tfrac{1}{2}+x_2,-x_3,\tfrac{1}{2}-x_4,-x_5,-x_6,x_7). \tag{43}$$

Define $\sigma_2=\alpha^2\sigma_1\alpha$ and $\sigma_3=\alpha\sigma_1\alpha^2$. It can be shown that $\{1,\sigma_1,\sigma_2,\sigma_3\}$ is a group isomorphic to $\mathbb{Z}_2^2$. Let Γ be the group $\langle\alpha,\beta,\sigma_1\rangle$. From the definitions we see that β,σ_1,σ_2 and σ_3 are of order 2 and commute, and that $\sigma_1\sigma_2\sigma_3=1$, so that $\langle\beta,\sigma_1,\sigma_2,\sigma_3\rangle\cong\mathbb{Z}_2^3$. Also $\alpha^3=1$, α and β commute, and conjugation with α induces a cyclic permutation of $\sigma_1,\sigma_2,\sigma_3$. Therefore Γ is a finite group of order 24, with a normal subgroup $\mathbb{Z}_2^3$ of order 8. All elements of Γ preserve $\hat{\varphi}$.

Let us consider the singular points of T^7/Γ. First we shall determine the elements of Γ with fixed points. It can be shown that the only nonidentity elements of $\langle\beta,\sigma_1,\sigma_2,\sigma_3\rangle$ with fixed points are $\beta\sigma_1,\beta\sigma_2,\beta\sigma_3$, and these are all conjugate under α,α^2. Now α,α^2 have fixed points, but $\alpha\beta,\alpha^2\beta$ have none because they take $x_2+x_4+x_6$ to $x_2+x_4+x_6+\frac{1}{2}$. The 16 elements of $\Gamma\setminus\mathbb{Z}_2^3$ are divided into sets of 4 elements conjugate to $\alpha,\alpha^2,\alpha\beta$ and $\alpha^2\beta$ under some element of $\langle\sigma_1,\sigma_2,\sigma_3\rangle\cong\mathbb{Z}_2^2$. Thus the only elements of Γ with fixed points are $\beta\sigma_1,\beta\sigma_2,\beta\sigma_3$, the 4 elements conjugate to α, and the 4 elements conjugate to α^2.

Although Γ does not satisfy Condition 2.1.2, nevertheless it can be shown that the singular set S of T^7/Γ is of the form given in Lemma 2.1.3. In fact, S is the disjoint union of a contribution from the fixed points of $\beta\sigma_1$ and a contribution from the fixed points of α. The fixed points of $\beta\sigma_1$ are a disjoint union of 16 copies of T^3. Since $\langle\sigma_2,\sigma_3\rangle$ acts freely on these 16 copies, the fixed points of $\beta\sigma_1$ contribute 4 copies of T^3 to S, each with a neighbourhood isometric to $T^3\times\big(B^4_\zeta/\{\pm1\}\big)$.

The fixed points of α are one copy of T^3. However, because β commutes with α we must take into account the action of β on the fixed point set of α. The result is that the fixed points of α contribute one copy of $T^3/\mathbb{Z}_2$ to S, which has a neighbourhood isometric to $\big\{T^3\times B^4_\zeta/\mathbb{Z}_3\big\}/\mathbb{Z}_2$, where $\beta\in\mathbb{Z}_2$ acts on $T^3\times B^4_\zeta/\mathbb{Z}_3$ by

$$(y_1,y_2,y_3,z_1,z_2)\mapsto(\tfrac{1}{2}+y_1,-y_2,-y_3,z_1,-z_2). \tag{44}$$

Here y_1,y_2,y_3 are coordinates on T^3 with $y_i\in\mathbb{R}/\mathbb{Z}$, and z_1,z_2 are complex coordinates on $B^4_\zeta/\mathbb{Z}_3$, so that (z_1,z_2) and $(e^{2\pi i/3}z_1,e^{4\pi i/3}z_2)$ are equivalent. Notice that $\mathbb{Z}_2$ acts freely on T^3, as it should.

We shall apply Theorems 2.2.2 and 2.2.3 to get a nonsingular 7-manifold M with metrics of holonomy G_2. To do this we require suitable ALE spaces $X_1,\ldots,X_5$ for the 5 components $S_1, \ldots, S_5$ of S. For the 4 T^3 components, $X_1, \ldots, X_4$ should be the Eguchi-Hanson space, as in §§3.1 and 3.2. For the $T^3/\mathbb{Z}_2$ component, X_5 should be an ALE space with holonomy $SU(2)$ asymptotic to the Euclidean metric on $\mathbb{C}^2/\mathbb{Z}_3$. As in §1.3, there exists a family of ALE spaces with holonomy $SU(2)$ for each cyclic subgroup of $SU(2)$, which are given explicitly in [11] (see also [16], [17]). Therefore we may choose X_5 from the family for $\mathbb{Z}_3 \subset SU(2)$.

Now to apply Theorem 2.2.2, we also require an isometric action of $\mathbb{Z}_2$ on X_5 that is asymptotic to the action of $\mathbb{Z}_2$ on $\mathbb{C}^2/\mathbb{Z}_3$ given by $(z_1, z_2) \mapsto (z_1, -z_2)$, from (44). As in Example 4 of §3.1, there are in fact two topologically distinct actions of $\mathbb{Z}_2$ on X_5 with the right asymptotic behaviour, and for each of these actions there are metrics of holonomy $SU(2)$ in the family invariant under the action. The first of these $\mathbb{Z}_2$- actions acts trivially on $H^2(X_j, \mathbb{R}) \cong \mathbb{R}^2$, and the second $\mathbb{Z}_2$- action has one eigenvalue 1 and one eigenvalue -1 on $H^2(X_j, \mathbb{R})$. Thus we produce two compact 7-manifolds M. It can be shown that $\pi_1(T^7/\Gamma) = \{1\}$, so both 7-manifolds are simply-connected. By Theorem 2.2.3 and Proposition 1.1.1, both 7-manifolds carry metrics with holonomy G_2.

Calculation shows that $b^2(T^7/\Gamma) = 0$ and $b^3(T^7/\Gamma) = 3$. Each of the 4 T^3 components of S adds 1 to b^2 and 3 to b^3 as usual. Choosing the first $\mathbb{Z}_2$- action on X_5 adds 2 to b^2 and 2 to b^3, and choosing the second $\mathbb{Z}_2$- action adds 1 to b^2 and 3 to b^3. Therefore the Betti numbers are $b^2(M) = 6$ and $b^3(M) = 17$ in the first case, and $b^2(M) = 5$ and $b^3(M) = 18$ in the second.

Example 16. Let $\mathbb{R}^7$ have coordinates $(x_1, \ldots, x_7)$, let $\hat\varphi$ be the flat G_2- structure on $\mathbb{R}^7$, and define a lattice $\Lambda \cong \mathbb{Z}^7$ in $\mathbb{R}^7$ by

$$\begin{aligned}\Lambda = \big\langle &(1,0,0,0,0,0,0), (0,1,0,0,0,0,0), (0,0,1,0,0,0,0), (0,0,0,1,0,0,0),\\ &(0,0,0,0,2,0,0), (0,0,0,0,1,1,0), (0,0,0,0,1,0,1)\big\rangle.\end{aligned} \tag{45}$$

Then Λ is invariant under permutations of x_5, x_6, x_7. Let $T^7 = \mathbb{R}^7/\Lambda$, and define isometries $\sigma_1, \sigma_2, \sigma_3$ and α of T^7 by

$$\sigma_1\big((x_1, \ldots, x_7)\big) = (x_4, x_3, -x_2, -x_1, \tfrac12 + x_5, -\tfrac12 - x_6, -x_7), \tag{46}$$

$$\sigma_2\big((x_1, \ldots, x_7)\big) = (x_3, -x_4, -x_1, x_2, -x_5, \tfrac12 + x_6, -\tfrac12 - x_7), \tag{47}$$

$$\sigma_3\big((x_1, \ldots, x_7)\big) = (x_2, -x_1, x_4, -x_3, -\tfrac12 - x_5, -x_6, \tfrac12 + x_7), \tag{48}$$

$$\alpha\big((x_1, \ldots, x_7)\big) = (-x_1, -x_2, -x_3, -x_4, x_5 + 1, x_6 + 1, x_7 + 1). \tag{49}$$

Note that because $(0,0,0,0,1,1,1) \notin \Lambda$, α acts nontrivially on coordinates x_5, x_6, x_7, and thus α has no fixed points. Now it is easy to show that $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_1\sigma_2\sigma_3 = \alpha$, because of the definition of Λ. Therefore $\langle \sigma_1, \sigma_2, \sigma_3\rangle$ is a finite, non-abelian group of order 8. Moreover, no nonidentity element of $\langle \sigma_1, \sigma_2, \sigma_3\rangle$ has fixed points on T^7.

Define an isometry β of T^7 by

$$\beta\big((x_1, \ldots, x_7)\big) = (x_1, x_4, x_2, x_3, x_6, x_7, x_5). \tag{50}$$

Then $\beta^3 = 1$, and conjugation with β induces a cyclic permutation of $\sigma_1, \sigma_2, \sigma_3$. Let $\Gamma = \langle \beta, \sigma_1, \sigma_2, \sigma_3 \rangle$. Then Γ is a finite group of order 24. Since $\beta, \sigma_1, \sigma_2$ and σ_3 preserve $\hat{\varphi}$, Γ preserves $\hat{\varphi}$. It is easy to see that Γ consists of the group $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$, together with 4 elements conjugate to β, 4 elements conjugate to β^2, 4 elements of order 6 conjugate to $\beta\alpha$, and 4 elements of order 6 conjugate to $\beta^2\alpha$.

Now $\beta\alpha$ and $\beta^2\alpha$ have no fixed points, since both take $x_5 + x_6 + x_7 \mapsto x_5 + x_6 + x_7 + 3$, whereas all elements of Λ add only even integers to $x_5 + x_6 + x_7$. Therefore the only nonidentity elements of Γ with fixed points are the 4 elements conjugate to β and the 4 elements conjugate to β^2. Although Γ does not satisfy Condition 2.1.2, the form of the singular set is still given by Lemma 2.1.3. The fixed points of β are one copy of T^3, on which α acts freely, so the singular set S of T^7/Γ is one copy of $T^3/\mathbb{Z}_2$, and it has a neighbourhood isometric to $\{T^3 \times B^4_\zeta/\mathbb{Z}_3\}/\mathbb{Z}_2$, where the actions of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are as in (44) of Example 15.

As in Example 15, there are two different ways of desingularizing the singularity to get a 7-manifold M. Calculation shows that $\pi_1(T^7/\Gamma) = \{1\}$, $b^2(T^7/\Gamma) = 1$ and $b^3(T^7/\Gamma) = 4$. Thus the first desingularization yields a simply-connected 7-manifold M with $b^2(M) = 3$ and $b^3(M) = 6$, and the second desingularization yields a simply-connected M with $b^2(M) = 2$ and $b^3(M) = 7$. Both these 7-manifolds carry metrics with holonomy G_2, by Theorems 2.2.2 and 2.2.3 and Proposition 1.1.1.

Example 17. Let $\mathbb{R}^7$ have coordinates $(x_1, \ldots, x_7)$, let $\hat{\varphi}$ be the flat G_2- structure on $\mathbb{R}^7$, and define a lattice $\Lambda \cong \mathbb{Z}^7$ in $\mathbb{R}^7$ by

$$\begin{aligned} \Lambda = \big\langle &(2,0,0,0,0,0,0), (0,2,0,0,0,0,0), (0,0,2,0,0,0,0), (1,1,1,1,0,0,0), \\ &(0,0,0,0,2,0,0), (0,0,0,0,1,1,0), (0,0,0,0,1,0,1) \big\rangle. \end{aligned} \tag{51}$$

Let $T^7 = \mathbb{R}^7/\Lambda$, and define isometries $\sigma_1, \sigma_2, \sigma_3$ and α of T^7 by equations (46)-(49). Then σ_i and α preserve Λ, and satisfy $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_1\sigma_2\sigma_3 = \alpha$, so that $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is a finite, nonabelian group of order 8 as in Example 16, and again, no nonidentity element of $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ has fixed points on T^7.

Define an isometry β of T^7 by

$$\beta\big((x_1, \ldots, x_7)\big) = \Big(\tfrac{-x_1+x_2+x_3+x_4}{2}, \tfrac{-x_1-x_2+x_3-x_4}{2}, \tfrac{-x_1-x_2-x_3+x_4}{2}, \tfrac{-x_1+x_2-x_3-x_4}{2}, x_6, x_7, x_5\Big). \tag{52}$$

To understand the action on $x_1, \ldots, x_4$, it is helpful to think of $x_1 + x_2 i + x_3 j + x_4 k$ as an element of $\mathbb{H}$, and then β is left multiplication by $-\frac{1}{2}(1 + i + j + k)$. It can be shown that $\beta^3 = 1$, and that conjugation by β induces a cyclic permutation of $\sigma_1, \sigma_2, \sigma_3$. Therefore $\Gamma = \langle \beta, \sigma_1, \sigma_2, \sigma_3 \rangle$ is a finite group of order 24, which is isomorphic to the group Γ of Example 16. Just as in Example 16, it can be shown that Γ preserves $\hat{\varphi}$, and that the only nonidentity elements of Γ with fixed points are the conjugates of β and β^2. Thus the singular set S of T^7/Γ is the image of the fixed set of β in T^7, and must be divided by the action of α.

A delicate calculation shows that the fixed points of β are given by $(x_1, \ldots, x_4) = k(0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}) + l(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}, 0)$, $x_5 = x_6 = x_7$, for $k, l \in \{0, 1, 2\}$. Thus the fixed points of β are 9 copies of $\mathcal{S}^1$ in T^7. But α acts on these 9 copies of $\mathcal{S}^1$, identifying 8 of them in pairs, and fixing the copy $k = l = 0$. Therefore the singular set S of T^7/Γ consists of 4 copies of $\mathcal{S}^1$ with neighbourhoods isometric to $\mathcal{S}^1 \times \big(B^6_\zeta/\mathbb{Z}_3\big)$, and 1 copy of $\mathcal{S}^1$ with a neighbourhood isometric to $\{\mathcal{S}^1 \times B^6_\zeta/\mathbb{Z}_3\}/\mathbb{Z}_2$, where the action of $\mathbb{Z}_3$ on B^6_ζ is generated by

$$(z_1, z_2, z_3) \mapsto (e^{2\pi i/3}z_1, e^{2\pi i/3}z_2, e^{2\pi i/3}z_3) \tag{53}$$

for complex coordinates z_1, z_2, z_3 on B^6_ζ, and the action of $\mathbb{Z}_2$ on $\mathcal{S}^1 \times B^6_\zeta/\mathbb{Z}_3$ is generated by $(y, z_1, z_2, z_3) \mapsto (y + \frac{1}{2}, -z_1, -z_2, z_3)$, where y is the coordinate on $\mathcal{S}^1 = \mathbb{R}/\mathbb{Z}$.

Now in §1.3 we described an explicit ALE space X with holonomy $SU(3)$ asymptotic to $\mathbb{C}^3/\mathbb{Z}_3$, due to Calabi. We may use this to desingularize all 5 components of S, as there is an appropriate $\mathbb{Z}_2$- action on X. A careful computation shows that $\pi_1(T^7/\Gamma) = \{1\}$, $b^2(T^7/\Gamma) = 3$ and $b^3(T^7/\Gamma) = 2$. Since $H^2(X, \mathbb{R}) \cong \mathbb{R}$, it is easy to see that desingularizing each component of S adds 1 to b^2 and 1 to b^3. Therefore Theorems 2.2.2 and 2.2.3 yield a compact, simply-connected 7-manifold M with $b^2(M) = 8$ and $b^3(M) = 7$. This 7-manifold admits metrics with holonomy G_2. Note that this is the first example with $b^2(M) > b^3(M)$.

Example 18. Let T^7 be $\mathbb{R}^7/\mathbb{Z}^7$ with the metric, G_2- structure $\hat{\varphi}$ and coordinates $(x_1, \dots, x_7)$ used in §3.1. Let α, β be the isometries of T^7 defined by

$$\alpha\bigl((x_1, \dots, x_7)\bigr) = (x_2, x_3, x_7, -x_6, -x_4, x_1, x_5), \tag{54}$$

$$\beta\bigl((x_1, \dots, x_7)\bigr) = (\tfrac{1}{2} - x_1, \tfrac{1}{2} - x_2, -x_3, -x_4, \tfrac{1}{2} + x_5, \tfrac{1}{2} + x_6, x_7). \tag{55}$$

It can be shown that α, β preserve $\hat{\varphi}$, and they generate a finite group Γ of order 56, with normal subgroup $\mathbb{Z}_2^3$ of order 8 consisting of the conjugates of β and 1. The 48 elements of $\Gamma \setminus \mathbb{Z}_2^3$ are all conjugate to α^j for some $j = 1, \dots, 6$, and these are the only nonidentity elements of Γ with fixed points.

The singular set S of T^7/Γ consists of one copy of $\mathcal{S}^1$, and it has a neighbourhood isometric to $\mathcal{S}^1 \times (B^6_\zeta/\mathbb{Z}_7)$, where the action of $\mathbb{Z}_7$ on B^6_ζ is generated by

$$(z_1, z_2, z_3) \mapsto (e^{2\pi i/7}z_1, e^{4\pi i/7}z_2, e^{8\pi i/7}z_3), \tag{56}$$

for complex coordinates (z_1, z_2, z_3) on B^6_ζ. To apply Theorems 2.2.1 and 2.2.3 to T^7/Γ, we require a suitable ALE space X with holonomy $SU(3)$. Now in [19, Example 3, pp. 269-271], Markushevich constructs an explicit toric variety X that desingularizes $\mathbb{C}^3/\mathbb{Z}_7$ with the $\mathbb{Z}_7$- action (56), which has $b^1(X) = b^3(X) = 0$ and $b^2(X) = 3$. By the discussion in §1.3, this X must carry ALE metrics with holonomy $SU(3)$. Calculation shows that $\pi_1(T^7/\Gamma) = \{1\}$, $b^2(T^7/\Gamma) = 0$ and $b^3(T^7/\Gamma) = 1$. Therefore, Theorems 2.2.1 and 2.2.3 yield a compact, simply-connected 7-manifold M with $b^2(M) = 3$ and $b^3(M) = 4$, that admits metrics of holonomy G_2. This example has the smallest Betti numbers we have found so far.

4. Directions for further research

The purpose of this final chapter is to bring together a number of ideas and questions which in the opinion of the author may be interesting and worth pursuing. In §4.1 we give a graph of the Betti numbers of the compact 7-manifolds with holonomy G_2 presented as examples in this paper. From this we draw some conclusions, and ask a number of questions about the topology of compact 7-manifolds with holonomy G_2. Section 4.2 discusses submanifolds of 7-manifolds with holonomy G_2 with special properties, drawing on work by Harvey and Lawson, and McLean.

For a 7-manifold with torsion-free G_2- structure there are two special sorts of submanifold of dimension 3 and 4, christened *associative* and *coassociative* respectively by Harvey and Lawson, which are analogous to complex submanifolds of complex manifolds. Examples are given of compact associative and coassociative submanifolds in the 7-manifolds M of §3.1. Finally, section 4.3 sketches a construction for compact 7-manifolds with holonomy G_2 starting from a 6-manifold with holonomy $SU(3)$ with an antiholomorphic involution, and describes two classes of complete, noncompact 7-manifolds with holonomy G_2.

4.1. Betti numbers of compact 7-manifolds with holonomy G_2. We will now give a table of values of Betti numbers $b^2(M)$, $b^3(M)$ of compact Riemannian 7-manifolds M with holonomy G_2 from Examples 1-18. In the table, the symbols '$\bullet$', '$*$' and '$+$' represent the Betti numbers of compact 7-manifolds M admitting metrics of holonomy G_2, '$\bullet$' denoting a simply-connected M, '$*$' denoting an M with finite, nontrivial fundamental group, and '$+$' denoting Betti numbers occurring in both simply-connected and non-simply-connected examples.

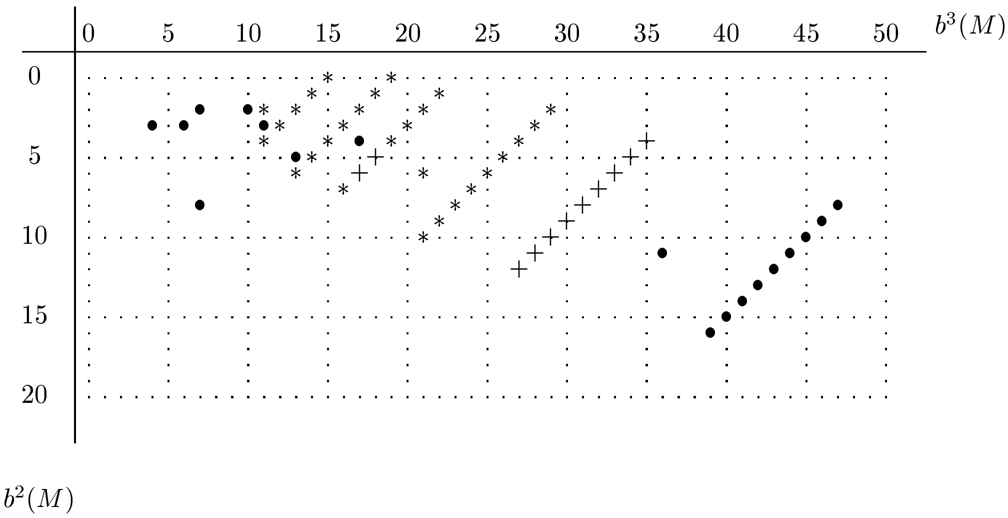

Table 2.

For some holonomy groups H, there are topological restrictions upon the Betti numbers $b^i(M)$ or the refined Betti numbers $b^i_j(M)$ introduced in §1.1, of a compact Riemannian manifold M with holonomy H. These restrictions can be divided into equations, inequalities, and divisibility properties. By an *equation* we mean a linear equation in the Betti numbers. For instance, Salamon [25] shows that the Betti numbers of a compact, Riemannian $4n$-manifold with holonomy contained in $\mathrm{Sp}(n)$ satisfy a certain equation. By an *inequality* we mean a linear inequality in the Betti numbers.

By a *divisibility property*, we mean that some linear combination of the Betti numbers, with integer coefficients having no common factor, should be divisible by some integer $p > 1$. A well-known example of a divisibility property occurs for Kähler manifolds (with holonomy $U(n)$), since the identity $h^{p,q} = h^{q,p}$ on the

Hodge numbers forces certain of the b^i_j to be even. Also, the odd Betti numbers of a manifold with holonomy $\mathrm{Sp}(n)$ are divisible by 4. From §1.1, a compact Riemannian 7-manifold with holonomy G_2 has only two independent refined Betti numbers, which are b^2_{14} and b^3_{27}, and these are determined by b^2 and b^3. From Table 2 we may observe that there are *no nontrivial equations* and *no nontrivial divisibility properties* for compact 7-manifolds with holonomy G_2, even for simply-connected manifolds.

In the last few years, some very strange ideas about compact 6-manifolds with holonomy $SU(3)$ have emerged from the unexpected direction of String Theory, a branch of theoretical Physics. Central to these ideas is the phenomenon of 'mirror symmetry'; see for example [9], [1] and references therein. The author has had many fascinating discussions with the physicists S. Shatashvili, C. Vafa and M. Roček, and it appears that much of the mirror symmetry story holds in a modified form for compact 7-manifolds with holonomy G_2, and also for compact 8-manifolds with holonomy $\mathrm{Spin}(7)$.

The idea, as I understand it, is that to a 7-manifold with holonomy G_2 one associates a Conformal Field Theory, with an algebra of operators satisfying some relations determined by the G_2 geometry. It can turn out that for two topologically distinct 7-manifolds with holonomy G_2 the conformal field theories are identical, or related in some way, and this provides a mysterious relation between the different 7-manifolds. In particular, some of the examples in this paper appear to have this property. This field may well soon produce some exciting results and conjectures about 7-manifolds with holonomy G_2. One would like, for instance, to know the appropriate 'mirror conjecture'.

It seems likely that certain patterns in Table 2 have not arisen merely by chance, but actually have heuristic explanations in terms of string theory. For instance, a striking feature of Table 2 is the arrangement of points into lines with $b^2 + b^3$ constant. Also, the great majority of the points have $b^2 + b^3 \equiv 3 \mod 4$. These could simply be spurious regularities introduced by our particular choice of finite groups Γ. However, some recent work of C. Vafa and S. Shatashvili, communicated privately to the author, indicates that it is natural from the point of view of string theory to collect together 7-manifolds with holonomy G_2 and constant $b^2 + b^3$.

In addition, the line $b^3 = 3b^2 + 7$ looks a bit like an 'axis of symmetry' of the graph. The significance of this is not clear, but Coilin Nunan (under the guidance of Simon Salamon) has proved an inequality on the Betti numbers of orbifolds of tori T^n/Γ [21, §3.5], which in the case $n = 7$ yields $5b^1 + b^3 \leq 3b^2 + 7$. For comparison, a graph of Betti numbers of Calabi-Yau 3-folds is given in [9], and shows beautiful regularities, including an approximate 'axis of symmetry'. We await further developments in these areas with great interest.

Questions 4.1.1. Are there finitely many or infinitely many compact 7-manifolds admitting metrics with holonomy G_2? What is the set of Betti numbers $b^2(M), b^3(M)$ realized by such 7-manifolds? Can one prove nontrivial inequalities on the Betti numbers by topological means? (For instance, the data in Table 2 is consistent with the inequalities $b^2(M) + b^3(M) \leq 55$ and $b^2(M) \leq 2b^3(M)$.) Classify all compact 7-manifolds with holonomy G_2 arising as generalized Kummer constructions in the sense of Chapter 2. Do all compact 7-manifolds with holonomy G_2 arise this way? Do there exist patterns or symmetries in the graph of Betti numbers $b^2(M), b^3(M)$ of such 7-manifolds?

For the first of these questions, the author's conjecture is that there exist only finitely many compact 7-manifolds admitting metrics with holonomy G_2. It can certainly be shown that the number of 7-manifolds obtained by the construction of Chapter 2 is finite. In contrast, there are infinitely many compact, simply-connected 7-manifolds that admit G_2- structures — it can readily be shown that if M_1, M_2 admit G_2- structures, then so does the connected sum $M_1 \# M_2$.

As a first step in classifying finite groups Γ acting on T^7 preserving a G_2- structure, one may classify elements α of G_2 acting on $\mathbb{R}^7$ and preserving some lattice $\Lambda \cong \mathbb{Z}^7$ in $\mathbb{R}^7$. Such an α is conjugate in $GL(7, \mathbb{R})$ to an element of $SL(7, \mathbb{Z})$, and therefore the characteristic polynomial $\det(\alpha - tI)$ must have integer coefficients. Nunan [21, §2.3] has classified all such $\alpha \in G_2$ explicitly, with the aid of a computer. He finds that the order of α must be 1, 2, 3, 4, 6, 7, 8 or 12.

4.2. Special submanifolds of a 7-manifold with holonomy G_2. If X is a complex manifold, then the complex submanifolds Y of X are a special class of submanifolds of X with interesting geometry attached to them. By analogy, we may ask whether there exist any special classes of submanifolds N of a 7-manifold M equipped with a torsion-free G_2- structure. In this section we will discuss two classes of submanifolds, which have been studied by Harvey and Lawson [13] and McLean [20]. First we will explain the idea of a calibration, which was defined and studied by Harvey and Lawson in their seminal paper on calibrated geometries [13].

A *calibration* is a closed k-form φ on a Riemannian n-manifold M, such that the restriction of φ to each tangent k-plane is less than or equal to the volume of the k-plane. A *calibrated submanifold* N is a submanifold N of dimension k such that $\varphi|_N$ is equal to the volume form of the induced metric on N. Therefore for any compact submanifold N of dimension k, $\int_N \varphi \leq \mathrm{vol}(N)$, with equality if and only if N is a calibrated submanifold. However, since φ is closed, $\int_N \varphi$ depends only on the homology class of N. Thus a (compact) calibrated submanifold minimizes volume in its homology class.

Now let M be a 7-manifold with a torsion-free G_2- structure φ. We have two closed forms, φ and $*\varphi$, to use as calibrations. Harvey and Lawson define an *associative submanifold* to be a 3-dimensional submanifold N of M that is a calibrated submanifold of M w.r.t. the calibration φ, and define a *coassociative submanifold* to be a 4-dimensional submanifold N of M that is a calibrated submanifold of M w.r.t. the calibration $*\varphi$. An alternative definition of a coassociative submanifold is a 4-dimensional submanifold N of M such that $\varphi|_N = 0$. Associative and coassociative submanifolds are studied in [13, §§IV.2.A and IV.2.B].

McLean [20] considered the problem of deforming a given associative or coassociative submanifold N within a 7-manifold M with a fixed, torsion-free G_2- structure. For associative submanifolds the deformation problem is elliptic, with index zero. Probably this means that in the generic case, compact associative submanifolds admit no deformations. Using the sign and orientation conventions of this paper (McLean's are different), in [20, §4] he proves that the deformations of a coassociative submanifold N are locally given by closed, self-dual 2-forms on N, where N is oriented such that $*\varphi|_N$ is a positive 4-form. Further, he proves that the moduli space of coassociative manifolds is locally a smooth manifold with dimension $b^2_+(N)$.

Given a 7-manifold M with a torsion-free G_2- structure, one natural way to find associative and coassociative submanifolds of M is to look at the fixed point sets of isometries of M preserving the G_2- structure. If some component of the fixed point set has dimension 3, then it will in general be an associative submanifold. Let us also allow orientation-reversing isometries of M, that take φ to $-\varphi$. Such an isometry can have components of dimension 4 in its fixed point set, and these will in general be coassociative submanifolds. Here are some examples of this.

Example A. In Example 3 of §3.1, consider the isometry σ of T^7 defined by

$$\sigma\big((x_1,\ldots,x_7)\big)=(\tfrac{1}{2}-x_1,\tfrac{1}{2}-x_2,-x_3,-x_4,x_5,x_6,x_7). \tag{57}$$

Then σ preserves $\hat\varphi$ and commutes with Γ, and the fixed points of σ on T^7 are 16 copies of T^3, that are disjoint from the fixed points of α,β and γ. Since Γ acts freely on the set of 16 fixed 3-tori of σ, it follows that the fixed points of σ on T^7/Γ are 2 disjoint copies of T^3.

Now the desingularization of T^7/Γ to give a compact 7-manifold M with a torsion-free G_2- structure φ may be done in a σ-invariant way. Therefore there exists a family of σ-invariant torsion-free G_2- structures on M. But the fixed points of σ in M are 2 disjoint copies of T^3, and is is easy to see that these must be associative submanifolds of (M,φ).

Example B. In Example 3 of §3.1, consider the isometry σ of T^7 defined by

$$\sigma\big((x_1,\ldots,x_7)\big)=(-x_1,-x_2,x_3,x_4,\tfrac{1}{2}-x_5,\tfrac{1}{2}-x_6,x_7). \tag{58}$$

Everything works as in Example A, except that this time the fixed points of α and σ in T^7 intersect. Therefore the fixed points of σ in T^7/Γ are 4 disjoint copies of $T^3/\mathbb{Z}_2$, where $\alpha\in\mathbb{Z}_2$ acts by $(y_1,y_2,y_3)\mapsto(-y_1,-y_2,y_3)$, for coordinates $y_1,y_2,y_3\in\mathbb{R}/\mathbb{Z}$ on T^3.

Now $T^3/\mathbb{Z}_2$ is homeomorphic to $\mathcal{S}^2\times\mathcal{S}^1$, and performing a σ-invariant desingularization does not change this topology. Therefore the 7-manifold M of Example 3 admits σ-invariant torsion-free G_2- structures φ, and the fixed points of σ are 4 (smooth) copies of $\mathcal{S}^2\times\mathcal{S}^1$, which are associative submanifolds of (M,φ).

Example C. In Example 4 of §3.1, consider the isometry σ of T^7 defined by

$$\sigma\big((x_1,\ldots,x_7)\big)=(x_1,x_2,-x_3,-x_4,-x_5,-x_6,x_7). \tag{59}$$

Then σ preserves $\hat\varphi$ and commutes with Γ. This time, the fixed points of σ intersect the fixed points of α, β and γ. Thus the fixed points of σ on T^7/Γ are 16 copies of $T^3/\mathbb{Z}_2^3$. But since $\alpha\beta$ acts on T^3 as a translation, we may regard each component of the fixed set of σ on T^7/Γ as a copy of $T^3/\mathbb{Z}_2^2$, where the generators α,γ of $\mathbb{Z}_2^2$ act by

$$\alpha:(y_1,y_2,y_3)\mapsto(-y_1,-y_2,y_3)\qquad\text{and}\qquad\gamma:(y_1,y_2,y_3)\mapsto(\tfrac{1}{2}-y_1,y_2,-y_3). \tag{60}$$

Here $y_1,y_2,y_3\in\mathbb{R}/\mathbb{Z}$ are coordinates on T^3.

It can be shown that $T^3/\mathbb{Z}_2^2$ is homeomorphic to $\mathcal{S}^3$. Performing a σ-invariant desingularization does not change the topology, so that the 7-manifold M of Example 4 admits a family of σ-invariant torsion-free G_2- structures, and the fixed points of σ are 16 (smooth) copies of $\mathcal{S}^3$, which are associative submanifolds of (M,φ).

Example D. In Example 3 of §3.1, consider the orientation-reversing isometry σ of T^7 defined by

$$\sigma\big((x_1,\ldots,x_7)\big)=(\tfrac12-x_1,\tfrac12-x_2,x_3,x_4,x_5,x_6,\tfrac12-x_7). \tag{61}$$

Then σ commutes with Γ, and takes $\hat\varphi$ to $-\hat\varphi$. The fixed points of σ in T^7 are 8 copies of T^4. These copies of T^4 are disjoint from the fixed points of α, β and γ, and Γ acts freely on the set of 8 4-tori fixed by σ. Therefore the fixed points of σ in T^7/Γ are 1 copy of T^4, which avoids all the singular points.

Now the desingularization of T^7/Γ to give a 7-manifold M with torsion-free G_2-structure φ may be done in a σ-equivariant way, so that $\sigma^*(\varphi)=-\varphi$. Therefore there exists a family of σ-equivariant torsion-free G_2- structures on the manifold M of Example 3. The fixed point set of σ is one copy of T^4 in M, and it is easy to see that this is a coassociative submanifold N of M. Since $b^2_+(T^4)=3$, there is a 3-parameter family of coassociative 4-tori in M close to N, by McLean's result described above.

Example E. In Example 3 of §3.1, consider the orientation-reversing isometry σ of T^7 defined by

$$\sigma\big((x_1,\ldots,x_7)\big)=(x_1,x_2,x_3,x_4,\tfrac12-x_5,\tfrac12-x_6,\tfrac12-x_7). \tag{62}$$

Then σ commutes with Γ, and takes $\hat\varphi$ to $-\hat\varphi$. This time, though, the fixed points of σ intersect with those of α, and so the fixed points of σ in T^7/Γ are 2 copies of $T^4/\mathbb{Z}_2$, where $\alpha\in\mathbb{Z}_2$ acts by $(y_1,\ldots,y_4)\mapsto(-y_1,\ldots,-y_4)$, for $y_1,\ldots,y_4\in\mathbb{R}/\mathbb{Z}$ coordinates on T^4. Performing a σ-equivariant desingularization, the fixed set of σ in M is 2 copies of the $K3$ surface of [15, §1.3]. Thus M admits a family of σ-equivariant G_2- structures, and the fixed points of σ give 2 coassociative $K3$'s in M. Since $b^2_+(K3)=3$, each of these admits a 3-parameter family of deformations.

Example F. In Example 3 of §3.1, consider the orientation-reversing isometry σ of T^7 defined by

$$\sigma\big((x_1,\ldots,x_7)\big)=(x_1,x_2,x_3,x_4,\tfrac12-x_5,-x_6,-x_7). \tag{63}$$

This time, the fixed points of σ intersect the fixed points of α, but in addition $\beta\gamma$ acts freely on each of the 8 fixed T^4 of σ. Therefore the fixed points of σ in T^7/Γ are 4 copies of $T^4/\mathbb{Z}_2^2$, where the generators $\alpha,\beta\gamma$ of $\mathbb{Z}_2^2$ act by

$$\alpha:(y_1,\ldots,y_4)\mapsto(-y_1,\ldots,-y_4)\qquad\text{and}\qquad \beta\gamma:(y_1,\ldots,y_4)\mapsto(\tfrac12+y_1,\tfrac12-y_2,\tfrac12-y_3,y_4). \tag{64}$$

When T^7/Γ is desingularized in a σ-equivariant way, these yield 4 coassociative copies of $K3/\mathbb{Z}_2$ in M, where $\mathbb{Z}_2$ acts freely on $K3$. Calculation shows that $b^2_+(K3/\mathbb{Z}_2)=1$, so each of the 4 coassociative submanifolds admits a 1-parameter family of deformations.

Now Examples D and E yield coassociative submanifolds with a 3-parameter family of deformations. Thus the ambient space is locally fibred by coassociative submanifolds. It seems possible that the whole manifold is globally fibred by coassociative submanifolds, with some singular fibres. (This idea was suggested by MacLean [20, p. 25].) If this is the case, the 7-manifold would be a sort of 7-dimensional analogue of an elliptic surface, i.e. a complex surface fibred by elliptic curves, with some singular fibres.

Questions 4.2.1. Suppose that M has a coassociative submanifold N diffeomorphic to T^4 or to the $K3$ surface. Is M foliated by coassociative 4-tori or $K3$ surfaces, allowing for some singular leaves? If so, what can be said about the topology of M? Study the properties of the moduli spaces of compact associative and coassociative submanifolds of a compact 7-manifold M with torsion-free G_2- structure φ. Are the moduli spaces compact? How do the moduli spaces change under small and under large deformations of φ?

4.3. Connections between the structure groups $SU(3)$ and G_2. We begin by describing a possible construction for compact 7-manifolds with holonomy G_2. Suppose that N is a compact Riemannian 6-manifold with holonomy $SU(3)$, with Kähler form ω and holomorphic volume form Ω, and suppose that $\sigma : N \to N$ is an antiholomorphic involution of N, such that σ is an isometry, $\sigma^*(\omega) = -\omega$, and $\sigma^*(\Omega) = -\overline{\Omega}$. Examples of such N can be constructed in a similar way to Example 2 of §3.1. Using (12) we may define a torsion-free G_2- structure φ on the 7-manifold $M = N \times \mathcal{S}^1$. Let x be the coordinate on $\mathcal{S}^1$, and define an involution σ' of M by

$$\sigma'\big((n,x)\big) = \big(\sigma(n), -x\big). \tag{65}$$

Then σ' preserves the G_2- structure φ on M.

Let S be the fixed point set of σ in N. It is easy to show that S is a (possibly empty) compact submanifold of N of dimension 3. So the fixed points of σ' in M are $S \times \{0, \frac{1}{2}\}$. Therefore the quotient $M/\langle\sigma'\rangle$ is a compact, singular 7-manifold with a torsion-free G_2- structure, whose singular set is 2 copies of S. Each singular point is modelled on the singular points of $\mathbb{R}^3 \times \big(\mathbb{R}^4/\{\pm 1\}\big)$. As in Chapters 2 and 3, it is natural to try and desingularize $M/\langle\sigma'\rangle$ using the Eguchi-Hanson space at each point, to get a compact 7-manifold $\tilde{M}$, and then to define metrics with holonomy G_2 on $\tilde{M}$.

Calculations by the author indicate that the following condition is important in this problem.

Condition 4.3.1. Suppose there exists a smooth 1-form α on S that is nonzero at every point of S, and that α is closed and coclosed w.r.t. the metric on S induced by the metric on N.

This condition seems to be the necessary and sufficient condition for a family of metrics of holonomy G_2 to exist on $\tilde{M}$, desingularizing the singular structure on $M/\langle\sigma'\rangle$. The proof that it is a sufficient condition has two parts. The first part is to write down a family of G_2- structures φ_t on $\tilde{M}$ using the 1-form α of Condition 4.3.1, roughly following the method of Chapter 2. The second part is to show that the hypotheses of [15, Theorem B] apply to $(\tilde{M}, \varphi_t)$ for sufficiently small t, and to apply [15, Theorems A and B]. The author has a sketch of such a proof, but it will not be given here.

Questions 4.3.2. Develop a rigorous construction for compact 7-manifolds $\tilde{M}$ with holonomy G_2 by desingularizing $(N \times \mathcal{S}^1)/\langle\sigma'\rangle$, where N is a compact 6-manifold with holonomy $SU(3)$, and σ' is an involution given by (65). Can one produce many new 7-manifolds $\tilde{M}$ this way? The proof of the Calabi conjecture yields a large supply of candidates for N. Do these yield 7-manifolds $\tilde{M}$ that do not also arise by a generalized Kummer construction?

We finish the paper by making some remarks on two types of complete, non-compact Riemannian 7-manifolds with holonomy G_2. There appear to the author to be two interesting types of noncompact ends for 7-manifolds with holonomy G_2, which we shall refer to as 'cone' ends and 'cylinder' ends respectively. Let N be a compact Riemannian 6-manifold with metric g_N. For a 'cone' end, the metric g on the noncompact 7-manifold M should be asymptotic to the metric $t^2 g_N + dt^2$ on $N \times (0, \infty)$, as $t \to +\infty$. For a 'cylinder' end, g should be asymptotic to the metric $g_N + dt^2$ on $N \times (0, \infty)$ as $t \to +\infty$.

In both cases the G_2- structure on M induces an $SU(3)$- structure on N. For the 'cone' case, the $SU(3)$- structure on N is required to be Einstein with scalar curvature $+1$, and also *nearly Kähler* in the sense of Gray [12], which means that if J is the almost complex structure and ∇ the Levi-Civita connection of g_N, then $(\nabla_X J)X = 0$ for all vector fields X on N. Gray shows that in dimension 6, if the $SU(3)$- structure is nearly Kähler and not Kähler, then it is automatically Einstein. Three examples of complete metrics with holonomy G_2 and 'cone' ends are given by Bryant and Salamon [6].

For the 'cylinder' end case, g_N is required to have holonomy contained in $SU(3)$. By considering compact 7-manifolds M with several boundary components $N_1, \ldots, N_k$ we have the attractive possibility of developing a *cobordism theory for compact 6-manifolds with holonomy $SU(3)$*. Cobordisms with just one end may be constructed by modifying the construction given above. Instead of considering $(N \times \mathcal{S}^1)/\langle\sigma'\rangle$, we may attempt to desingularize the singular, noncompact 7-manifold $(N \times \mathbb{R})/\langle\sigma'\rangle$, which has one infinite end of the form $N \times (0, \infty)$. It seems likely that Condition 4.3.1 is the necessary and sufficient condition for this construction to work.

Acknowledgements

I would like to thank Simon Salamon for many helpful conversations and suggestions during the composition of this paper. I would also like to thank Robert Bryant, Coilin Nunan, Cumrun Vafa, Samson Shatashvili and David Morrison for interesting conversations and ideas, and to thank Christ Church, Oxford and the Institute for Advanced Study, Princeton for hospitality. This work was partially supported by NSF grant no. DMS 9304580.

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Christ Church, Oxford, OX1 1DP, England