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The hypercomplex quotient and the quaternionic quotient

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1 Introduction

When a symplectic manifold M is acted on by a compact Lie group of isometries F, then a new symplectic manifold of dimension $\dim M - 2\dim F$ can be defined, called the Marsden-Weinstein reduction of M by F [MW]. Kähler manifolds are important examples of symplectic manifolds, and in this case the Marsden-Weinstein reduction yields a new Kähler manifold, which as a complex manifold is the quotient of the set of stable points of M by the complexified action of F. This is called the Kähler quotient.

Recently, these constructions have been extended to two other classes of manifolds. In the classification of Riemannian manifolds by holonomy [S 2], Kähler manifolds are manifolds with holonomy $U(n)$, and related to these are hyperkähler manifolds with holonomy $Sp(n)$, and quaternionic Kähler manifolds with holonomy $Sp(n)Sp(1)$. A quotient process for hyperkähler manifolds has been described by Hitchin et al. [HKLR] that reduces dimension by $4\dim H$, and this was generalised by Galicki and Lawson [GL] to a quotient for quaternionic Kähler manifolds.

Now in parallel with the classification of Riemannian manifolds by holonomy there is a theory [B] that classifies manifolds with torsion-free connections by holonomy. Kähler, hyperkähler, and quaternionic Kähler manifolds have analogues in this theory: the analogue of a Kähler manifold is a complex manifold [with holonomy $GL(n, \mathbb{C})$], the analogue of a hyperkähler manifold is a hypercomplex manifold [with holonomy $GL(n, \mathbb{H})$], and the analogue of a quaternionic Kähler manifold is a quaternionic manifold [with holonomy $GL(n, \mathbb{H})GL(1, \mathbb{H})$].

The purpose of this paper is to present quotient constructions for hypercomplex and quaternionic manifolds that are analogous to those already known for hyperkähler and quaternionic Kähler manifolds. There is an essential difference between the new constructions and the known ones, which will now be explained.

The Marsden-Weinstein reduction and the other reductions above are two-stage processes. First, a *moment map* is defined, which is a map from the manifold

M into a vector space or vector bundle satisfying a certain differential equation. Under reasonable conditions it is shown that the moment map exists and is unique, up to at most the addition of a constant vector. Second, it is shown that the quotient of the zero set of the moment map by the group F inherits the structure of the original manifold.

In the processes to be described, a moment map will be defined, but it will not be possible to prove either existence or uniqueness. However, given a moment map for a particular group action, the second stage of defining structure on the quotient of the zero set presents no problems.

Thus in some cases it is not possible to define the reduction of a hypercomplex or quaternionic manifold by a respectable group because no moment map exists, whereas in others there may be moduli spaces of distinct reductions of a manifold by a fixed group that far exceed the freedom to add a constant vector in the hyperkähler quotient.

One special class of quaternionic manifolds are Kähler surfaces with zero scalar curvature; they are quaternionic because they are conformally anti-self-dual. (See for instance [L 1].) In Sect. 7 it will be shown that the zero-scalar-curvature Kähler condition fits in well with the quotient picture for quaternionic manifolds, higher-dimensional analogues will be defined [manifolds with holonomy $SL(n,\mathbb{H})U(1)$] and a quotient process described.

The principal examples of the reductions we consider are quotients of flat spaces by finite-dimensional Lie groups. Since in dimension four, quaternionic manifolds are just self-dual conformal manifolds, the quaternionic quotient gives a way of producing conformally self-dual 4-manifolds. In particular, LeBrun's metrics on $n\mathbb{CP}^2$ [L 2] are quaternionic quotients of $\mathbb{HP}^{n+1}$ by $U(1)^n$. The simplest case of this correspondence, the construction of Poon's metrics on $\mathbb{CP}^2 \# \mathbb{CP}^2$ as quaternionic quotients, will be given as an example in Sect. 6 of this paper. A further paper has been written to describe the general case.

The quotient formalism is convenient for writing down monads for instantons on quotients of $\mathbb{HP}^n$, and using existing ADHM constructions for $\mathcal{S}^4$, $\tilde{\mathbb{C}}^2$ and the ALE spaces as models, we have been able to find monads for instantons on weighted projective spaces and LeBrun's metrics on $n\mathbb{CP}^2$.

2 Review of necessary theory

We shall begin by briefly recalling the basic properties of hypercomplex, hyperkähler, quaternionic, and quaternionic Kähler manifolds, and explain how to each quaternionic or quaternionic Kähler manifold M one can associate a bundle $\mathscr{U}(M)$ with fibre $\mathbb{H}/\{\pm 1\}$, the total space of which carries respectively a hypercomplex structure and a pseudo-hyperkähler structure.

We will then describe the connection between the hyperkähler and quaternionic Kähler quotients found by Swann [Sw], that the quaternionic Kähler quotient is simply the hyperkähler quotient in the associated bundle $\mathscr{U}(M)$. Having defined the hypercomplex quotient, this model will then enable us to define a quotient construction for quaternionic manifolds.

A hypercomplex manifold M is defined to be a $4n$-dimensional smooth manifold M with 3 integrable complex structures I_1, I_2, I_3 that satisfy the quaternionic relations $I_1I_2 = I_3, I_3I_1 = I_2, I_2I_3 = I_1$. Each hypercomplex manifold has a unique torsion-free connection ∇^M called the *Obata connection* [S 3, Sect. 6] satisfying $\nabla^M I_i = 0$.

Conversely, a manifold M that has three almost complex structures I_1, I_2, I_3 satisfying $I_1 I_2 = I_3$ and a torsion-free connection ∇^M with $\nabla^M I_i = 0$ is hypercomplex, for $\nabla^M I_i = 0$ implies that I_i is integrable. A (pseudo-)hyperkähler manifold is then a hypercomplex manifold together with a (pseudo-)Riemannian metric g that is Kähler with respect to each complex structure.

In dimensions greater than four, a quaternionic manifold is a manifold M with a subbundle $\mathscr{G}$ of $\mathrm{End}(TM)$ allowing at each point a basis I_1, I_2, I_3 of almost complex structures satisfying $I_1 I_2 = I_3$, that admits a torsion-free connection ∇^M preserving $\mathscr{G}$. A quaternionic Kähler manifold is then a quaternionic manifold M with a metric g preserved by the complex structures in $\mathscr{G}$, such that ∇^M is the Levi-Civita connection of g. In four dimensions we must make the special definitions that a quaternionic manifold is a self-dual conformal manifold and a quaternionic Kähler manifold is a Riemannian manifold that is self-dual and Einstein.

These types of manifold may alternatively be defined and described in the language of G-structures. If G is a Lie group, a G-structure Q on a manifold M is a principal bundle Q over M for the group G, that is a subbundle of the frame bundle of M. For each of the four types of manifold above, we let G be the group of automorphisms of the tangent plane at a point that preserve the structures defined on it.

Thus for the hypercomplex manifolds G must preserve three complex structures, so $G = GL(n, \mathbb{H})$, for the hyperkähler manifolds G preserves three complex structures and a metric, so $G = Sp(n)$, and in the quaternionic and quaternionic Kähler cases we allow elements of G also to act on the family of complex structures in $\mathscr{G}$, so G is $GL(n, \mathbb{H}) GL(1, \mathbb{H})$ and $Sp(n)Sp(1)$, respectively.

So the structures on the four families of manifolds are encapsulated by a G-structure Q on M for the four families of groups. The additional integrability conditions can then be summed up by saying that there should exist a torsion-free connection ∇^M on M that preserves Q. A good reference for the above material is [S 2, Chaps. 8, 9].

Given a principal bundle Q over M for the group G, to each representation V of G one may associate a bundle $\mathscr{V}$ over M defined by $\mathscr{V} = Q \times_G V$. The bundle $\mathscr{V}$ has fibre V. Now the groups $GL(n, \mathbb{H}) GL(1, \mathbb{H})$ and $Sp(n)Sp(1)$ have double covers $GL(n, \mathbb{H}) \times Sp(1)$ and $Sp(n) \times Sp(1)$. Thus for each quaternionic or quaternionic Kähler manifold M with G-structure Q, locally there is a double cover $\tilde{Q}$ which is a principal bundle with group $GL(n, \mathbb{H}) \times Sp(1)$ or $Sp(n) \times Sp(1)$. ($\tilde{Q}$ need not exist globally.)

We shall now take V to be the natural representation of $GL(1, \mathbb{H})$ or $Sp(1)$ on the right on the quaternions $\mathbb{H}$, with $GL(n, \mathbb{H})$ or $Sp(n)$ acting trivially. Actually, in the quaternionic case it is necessary to let the scalars $\mathbb{R}^* \subset GL(n, \mathbb{H}) GL(1, \mathbb{H})$ act to some prescribed power on V; this corresponds to regarding $GL(n, \mathbb{H}) GL(1, \mathbb{H})$ not as $GL(n, \mathbb{H}) Sp(1)$ or as $SL(n, \mathbb{H}) GL(1, \mathbb{H})$ but as a mixture of the two.

Forming the bundle associated to the local principal bundle $\tilde{Q}$ gives a local fibre bundle $\tilde{\mathscr{V}}$ over M with fibre $\mathbb{H}$, called the *natural quaternionic line bundle*. To make a bundle that exists globally it is necessary to divide $\tilde{\mathscr{V}}$ by ± 1, as a double cover of Q may not exist, and this gives a global bundle $\mathscr{U}(M)$ over M with fibre $\mathbb{H}/\{\pm 1\}$, called the *associated bundle*.

By projectivising the fibre $\mathbb{H}/\{\pm 1\}$ with respect to any of the left actions of $\mathbb{C}$, we get a fibre bundle Z over M with fibre $\mathbb{CP}^1$ called the *twistor space*. It is well known that the twistor space of a conformal 4-manifold has an almost complex structure that is complex if and only if the manifold is self-dual.

A large part of the theory of quaternionic and quaternionic Kähler manifolds is an extension and a generalisation of this fact. In [S3, Corollary 7.4], Salamon proves that quaternionic manifolds have complex twistor spaces by first proving that the total space of the associated bundle, which he calls Y, is hypercomplex. This then implies that the twistor space Z is complex. Quaternionic manifolds can therefore be regarded as the most general sort of manifolds that admit a complex twistor space in analogy with the four-dimensional case (see, e.g. [S2, p. 135]).

As a quaternionic Kähler manifold is quaternionic, its associated bundle will certainly be hypercomplex. However, Swann goes on to prove [Sw, Theorem 3.5] that the Riemannian metric on M induces a pseudo-Riemannian metric on $\mathscr{U}(M)$ which together with the hypercomplex structure makes it pseudo-hyperkähler.

Now if a compact Lie group F acts freely and smoothly on a quaternionic Kähler manifold M preserving the quaternionic Kähler structure, Galicki and Lawson have shown [GL] that there exists a quaternionic Kähler reduction of M by F. But there is an induced action of F on the associated bundle $\mathscr{U}(M)$ preserving the complex structures and the metric, and so one can do a pseudo-hyperkähler quotient of $\mathscr{U}(M)$ by F. Swann shows [Sw, Theorem 4.6] that these two processes are the same. By constraining the moment map to vanish on the zero section of the fibration, the moment map is uniquely defined and the resulting pseudo-hyperkähler quotient is the associated bundle of the quaternionic Kähler quotient of M by F.

3 The hypercomplex quotient

Let M be a hypercomplex manifold with complex structures I_1, I_2, and I_3 and F be a compact Lie group acting smoothly and freely on M preserving I_i. Let the Lie algebra of F be $\mathfrak{F}$. Then F acts on $\mathfrak{F}$ by the adjoint action.

We define a hypercomplex moment map to be a triple $\mu=(\mu_1,\mu_2,\mu_3)$ of F-equivariant maps $\mu_i: M\to\mathfrak{F}^*$ satisfying the following two conditions:

(i) μ satisfies the "Cauchy-Riemann equations"

$$I_1 d\mu_1 = I_2 d\mu_2 = I_3 d\mu_3 , \tag{1}$$

where I_i acts on the cotangent bundle of $\mathfrak{F}^*$-valued 1-forms, $T^*M\otimes\mathfrak{F}^*$.

(ii) Let $X:\mathfrak{F}\to\Gamma(TM)$ be the map assigning to each $f\in\mathfrak{F}$ the vector field it induces on M. Then for every non-zero f in $\mathfrak{F}$, μ must satisfy the "transversality condition"

$$(I_1 d\mu_1(f))(X(f)) \text{ does not vanish on } M . \tag{2}$$

We make two remarks about these conditions. Firstly, an equivalent formulation of condition (i) is

$$\begin{aligned} I_1(d\mu_2+id\mu_3) &= -i(d\mu_2+id\mu_3), \\ I_2(d\mu_3+id\mu_1) &= -i(d\mu_3+id\mu_1), \\ I_3(d\mu_1+id\mu_2) &= -i(d\mu_1+id\mu_2), \end{aligned} \tag{3}$$

and these three are the Cauchy-Riemann conditions for $\mu_2+i\mu_3$ to be a holomorphic function with respect to the complex structure I_1, $\mu_3+i\mu_1$ to be holomorphic w.r.t. I_2 and $\mu_1+i\mu_2$ to be holomorphic w.r.t. I_3. This is why they were called Cauchy-Riemann conditions.

Secondly, (ii) can actually be replaced with the weaker but more complicated condition that for every non-zero $f \in \mathfrak{F}$ and $m \in M$ there should exist some f' in $\mathfrak{F}$ such that

$$(I_1 d\mu_1(f'))_m (X(f))_m \neq 0 .$$

We shall prove the following

Proposition 3.1. *Let M, F, and μ be as above, and let $\zeta_1, \zeta_2, \zeta_3$ be elements of the centre of $\mathfrak{F}^*$. Define $P = \{m \in M\colon \mu_1(m) = \zeta_1,\ \mu_2(m) = \zeta_2,\ \mu_3(m) = \zeta_3\}$ and $N = P/F$. Then N has a natural hypercomplex structure.*

Two proofs will be given, the first one being the informal proof that led us to the result, and the second being more technically satisfactory as it does not rely on complexifying group actions. The second method of proof extends to the quaternionic quotient, where it has the advantage of being direct, and not an application of the hypercomplex quotient in the associated bundle.

First proof. To show that the quotient has three integrable complex structures, observe that restricting to the solutions of $\mu_j = \zeta_j$ $(j = 1, 2, 3)$ and dividing by F is locally equivalent to restricting to the solutions of $\mu_2 + i\mu_3 = \zeta_2 + i\zeta_3$ and dividing by the complexification of F by I_1. This is because condition (ii) ensures that locally each orbit of the complexification of F meets the solutions of $\mu_1 = \zeta_1$ in only one orbit of F.

But by (i), $\mu_2 + i\mu_3 = \zeta_2 + i\zeta_3$ is a holomorphic condition w.r.t. I_1, and so the quotient N is equivalent to the quotient of a complex manifold by a complex group, and is complex with complex structure I_1. Thus N has three complex structures upon it.

To show that these satisfy the relation $I_1 I_2 = I_3$, for each p in P let V_p be defined by

$$\begin{aligned} V_p &= \{v \in T_pM\colon d\mu_1(v) = d\mu_2(v) = d\mu_3(v) = (I_1 d\mu_1)(v) = 0\} \\ &= \{v \in T_pP\colon (I_1 d\mu_1)(v) = 0\} . \end{aligned} \tag{4}$$

This defines a vector bundle V over P that is a subbundle of $TP \subset TM|_P$. Now by (ii) the condition $(I_1 d\mu_1)(v) = 0$ is transverse to the infinitesimal action of F, and thus there is a natural isomorphism between V_p and $T_{\pi(p)}N$.

So there is an isomorphism between $\pi^*(TN)$ and the subbundle V of TP. But condition (i) implies that V, considered as a subbundle of $TM|_P$, is closed under I_1, I_2, I_3. As I_1, I_2, I_3 are F-invariant, this defines actions of I_1, I_2, I_3 on TN which are clearly the same as the ones above defining integrable complex structures on N. Because I_1, I_2, I_3 satisfy $I_1 I_2 = I_3$ on V, this relation also holds on N. □

Second proof. Using ∇^M, I_1, I_2, and I_3, a connection ∇^N and three almost complex structures I_1, I_2, I_3 will be defined on N. It will then be shown that ∇^N is torsion-free and satisfies $\nabla^N I_i = 0$. From Sect. 2, this will imply that N is hypercomplex.

As F acts freely the map $X_m\colon \mathfrak{F} \to T_mM$ is an injection for each m in M. By abuse of notation $\mathfrak{F}$ will be identified with its image $X_m(\mathfrak{F})$ in each T_mM. Now by (ii) and the definition of μ,

$$TM|_P = TP \oplus I_1\mathfrak{F} \oplus I_2\mathfrak{F} \oplus I_3\mathfrak{F} , \tag{5}$$

where both sides are vector bundles over P.

Because of the transversality condition on the moment map, $\mathfrak{F}$ is transverse to the annihilator of $I_1 d\mu_1$ in TP. So there is a direct sum decomposition $TP = V \oplus \mathfrak{F}$, where V is the bundle defined above.

Thus P satisfies $TM|_P = TP \oplus \mathrm{Im}\,\mathbb{H} \cdot \mathfrak{F}$ and $TP = V \oplus \mathfrak{F}$, where V is some $\mathbb{H}$-invariant vector subbundle of $TM|_P$. It will be shown that under these conditions, $N = P/F$ is a hypercomplex manifold.

Lemma 3.2. *Let M be hypercomplex and acted on smoothly and freely by a compact Lie group F preserving the structure. Suppose P is an F-invariant submanifold of M satisfying $TM|_P = TP \oplus \mathrm{Im}\,\mathbb{H} \cdot \mathfrak{F}$ and $TP = V \oplus \mathfrak{F}$, where V is an $\mathbb{H}$-invariant vector subbundle of $TM|_P$. Then there is a natural hypercomplex structure on $N = P/F$.*

Proof. A torsion-free connection ∇^N on N and three almost complex structures I_1, I_2, I_3 will be defined on N and it will be shown that $\nabla^N I_i = 0$. Let π be the projection from P to N. Observe that V_p is identified with $T_{\pi(p)}N$ by π. Now the complex structures I_1, I_2, I_3 on M act on V_p, and therefore also on $T_{\pi(p)}N$. The actions are F-invariant, and so descend to give three almost complex structures I_1, I_2, I_3 on N.

To define the connection on N, let v_1, v_2 be vector fields on N. They lift uniquely to give F-invariant sections $\tilde{v}_1, \tilde{v}_2$ of V over P. We shall think of $\tilde{v}_1$ as a section of TP and $\tilde{v}_2$ as a section of $TM|_P$.

Now if w_1, w_2 are vector fields on M, the vector field $\nabla^M_{w_1} w_2$ may be formed. This action of ∇^M can be restricted to P: if w_1 is a section of TP and w_2 is a section of $TM|_P$, then $\nabla^{M|P}_{w_1} w_2$ is a section of $TM|_P$.

Thus $\nabla^{M|P}_{\tilde{v}_1} \tilde{v}_2$ is defined as a section of $TM|_P$. As ∇^M is unique, it is F-invariant, and so is this section. To get a vector field on N, project to V and then push down. So ∇^N is defined by the equation

$$\widetilde{(\nabla^N_{v_1} v_2)} = \varrho(\nabla^{M|P}_{\tilde{v}_1} \tilde{v}_2), \tag{6}$$

where ϱ is projection to the first factor in the vector bundle decomposition $TM|_P = V \oplus \mathbb{H} \cdot \mathfrak{F}$.

This definition gives a connection ∇^N on N. Note that in the hyperkähler case where there are metrics, this definition is the same as usual one involving orthogonal projection. It will now be shown that the connection is torsion-free.

It is sufficient to show that whenever α is a 1-form on N, then the anti-symmetric part of $\nabla^N \alpha$ is the same as $d\alpha$. Using the direct-sum decomposition, lift α to a unique F-invariant section $\tilde{\alpha}$ of $T^*M|_P$ [different from $\pi^*(\alpha)$, which is a section of T^*P]. Choose a section β of T^*M in a neighbourhood of P such that $\beta|_P = \tilde{\alpha}$. (As this is a local question, we need not worry about global existence.)

Let A denote anti-symmetrisation. As ∇^M is torsion-free, $A(\nabla^M \beta) = d\beta$. Restrict this identity to P and take horizontal parts. On the right there is $(d\beta)|_P = d(\beta|_P) = d\pi^*(\alpha)$, which is horizontal. On the left, restriction is in two stages, first restricting to $T^*M|_P$, giving $\nabla^{M|P}\tilde{\alpha}$, and then projecting $T^*M|_P$ to T^*P. By definition of $\nabla^N \alpha$, the horizontal part of this double restriction is $\pi^*(\nabla^N \alpha)$.

Now anti-symmetrisation commutes with restriction to P and taking horizontal parts, and so $A(\pi^*(\nabla^N \alpha)) = d\pi^*(\alpha)$. Pushing down to N gives $A(\nabla^N \alpha) = d\alpha$, and ∇^N is torsion-free.

Finally, it will be shown that if $\nabla^M I_i = 0$, then $\nabla^N I_i = 0$. This is equivalent to the statement that whenever v_1, v_2 are vector fields on N, then $\nabla^N_{v_1}(I_i v_2) = I_i \nabla^N_{v_1} v_2$.

Lifting to $TM|_P$, this equation is

$$\varrho(\nabla_{\tilde{v}_1}^{M|P}(I_i\tilde{v}_2)) = I_i\varrho(\nabla_{\tilde{v}_1}^{M|P}\tilde{v}_2). \tag{7}$$

But since ϱ commutes with I_i, this is equivalent to showing that

$$\varrho(\nabla_{\tilde{v}_1}^{M|P}(I_i\tilde{v}_2)) = \varrho(I_i\nabla_{\tilde{v}_1}^{M|P}\tilde{v}_2), \tag{8}$$

which is an immediate consequence of the fact that $\nabla^M I_i = 0$. Therefore, $\nabla^N I_i = 0$. □

This completes the second proof of Proposition 3.1.

It is necessary to assume that F is compact to ensure that the quotient N is Hausdorff. One can remove this assumption by instead assuming that N or M/G is a manifold. An example of a group action for which no moment map exists is the dilation action of $U(1)$ on the Hopf surface.

The hypercomplex quotient construction can be used to show that for all $n \geqq 2$ there exist compact, nonsingular, simply-connected $4n$-dimensional hypercomplex manifolds that are not (even locally) hyperkähler, and are not (even locally) products of other hypercomplex manifolds. The examples we have found are nontrivial fibre bundles over compact nonsingular quaternionic manifolds with fibre a Hopf surface, and will be described in a later paper. This contrasts with the four-dimensional case, where it is known [Bo] that the only compact hypercomplex four-manifolds are the hyperkähler K3 surfaces and conformally flat examples, that is, tori and Hopf surfaces. So all compact hypercomplex four-manifolds are locally hyperkähler.

4 The quaternionic moment map

One would at first except the torsion-free connection on a quaternionic manifold M to define a connection on $\mathcal{G}$, and that as in the quaternionic Kähler case of [GL], a moment map on a quaternionic manifold should be defined as a section of $\mathcal{G} \otimes \mathfrak{F}^*$, where F is the quotient group.

For technical reasons this is not quite true. As explained in [S 3, Sect. 5] to define invariant differential operators on vector bundles over quaternionic manifolds it is usually necessary to tensor through by some power of the real line bundle of volume forms on the manifold, because otherwise the operators defined will not be independent of the choice of connection.

Thus to define "moment maps" which can be differentiated in a meaningful fashion we work not with the bundle $\mathcal{G}$, but with the bundle $\tilde{\mathcal{G}} = \mathcal{G} \otimes e$, which is $\mathcal{G}$ tensored with a non-zero power e of the real line bundle of volume forms. (In fact, in the notation of [S 3], the bundle $\mathcal{G}$ is S^2H, and the condition imposed on the moment maps is that their image under the operator $D: S^2H \to E \otimes S^3H$ should be zero. By Corollary 5.4 of [S 3], this operator can only be defined from S^2H' to $E' \otimes S^3H'$, where $E' = d^m E$, $H' = d^{-m}H$ with m non-zero, and d is a real line bundle whose $4n^{\text{th}}$ power is the bundle of volume forms.)

On $\tilde{\mathcal{G}}$ there is a connection ∇^M induced from a torsion-free connection on M. The condition we write down in terms of ∇^M will be independent of the choice of connection on M.

Suppose F is a connected Lie group acting smoothly and freely on M, and let $\mathfrak{F}$ be its Lie algebra. Define a quaternionic moment map to be an F-equivariant section μ of $\mathscr{G}\otimes\mathfrak{F}^*$ which satisfies the following two conditions:

(i) For v some section of $T^*M\otimes e\otimes\mathfrak{F}^*$,

$$\nabla^M\mu = I_1\otimes(I_1 v)+I_2\otimes(I_2 v)+I_3\otimes(I_3 v). \tag{9}$$

(ii) Define $X:\mathfrak{F}\to\Gamma(TM)$ to be the map assigning to each $f\in\mathfrak{F}$ the vector field induced by f. Then for each non-zero $f\in\mathfrak{F}$ the section v must satisfy

$$v(f)X(f) \text{ does not vanish on } M\,, \tag{10}$$

where $v(f)X(f)$ is a section of e.

These conditions are direct translations of conditions (i) and (ii) of Sect. 3 into the quaternionic context. Condition (i) is independent of the local basis I_1, I_2, I_3 of $\mathscr{G}$ because it says that $\nabla^M\mu$ should be the contraction of Ω and v, where Ω is $I_1\otimes I_1 + I_2\otimes I_2 + I_3\otimes I_3$, which is independent of the choice of I_1, I_2, I_3.

Another way of writing (i) is this: certainly $\nabla^M\mu = I_1\otimes v_1 + I_2\otimes v_2 + I_3\otimes v_3$, where v_1, v_2, v_3 are sections of $T^*M\otimes e\otimes\mathfrak{F}^*$. Then v_1, v_2, v_3 must satisfy the condition $I_1v_1 = I_2v_2 = I_3v_3$. As v_1, v_2, v_3 are the analogues of $d\mu_1, d\mu_2, d\mu_3$ in the hypercomplex case, this is a translation of $I_1 d\mu_1 = I_2 d\mu_2 = I_3 d\mu_3$. As in the hypercomplex case, (ii) may be replaced by a weaker condition.

Because the class of hypercomplex manifolds is included in the class of quaternionic manifolds, this new definition also defines quaternionic moment maps on hypercomplex manifolds. However, in the original definition μ is a section of a trivial vector bundle, and in the new it is a section of a trivial vector bundle tensored with e. Therefore, the two definitions are only consistent if the connection on e is flat, that is, if the holonomy of the hypercomplex manifold reduces to $SL(n, \mathbb{H})$. It can be shown that in the non-trivial case, the moment map equation in sections of e has no non-zero solutions.

From [GL], the condition on the quaternionic Kähler moment map is that for each $f\in\mathfrak{F}$, $\nabla^M\mu(f) = i_{X(f)}\Omega$, where $i_{X(f)}$ is contraction with $X(f)$ using the metric. This can be put in a neater form by observing that X defines a section v' of $T^*M\otimes\mathfrak{F}^*$ assigning to each f the covector field associated by the metric to the vector field $X(f)$. Then the condition on μ is that $\nabla^M\mu$ should be the contraction of v' and Ω, as in the quaternionic case above.

Thus the difference between the quaternionic and the quaternionic Kähler case is that in the quaternionic case $\nabla^M\mu$ may be the contraction of Ω with any section v of $T^*M\otimes e\otimes\mathfrak{F}^*$, whereas in the quaternionic Kähler case it must be the contraction with a particular section v' given by the group action and the metric. This holds for the hypercomplex and hyperkähler quotients, where for the former $I_1 d\mu_1$ can be any suitable section of a bundle and for the latter it must be a particular section given by the group action and the metric.

The definition of quaternionic moment map given above may be related to our previous definition of hypercomplex moment map on the associated bundle. A moment map μ on the associated bundle consists of three $\mathfrak{F}^*$-valued functions on the associated bundle which are quadratic on each fibre. By restricting (1) to the fibre, one finds that on each fibre the solutions μ form $\mathfrak{F}^*$ tensored with a three-dimensional vector space, and this vector space is simply S^2H', the fibre of $\mathscr{G}$.

5 The quaternionic quotient

In the previous section the quaternionic moment map has been defined. It will now be shown that if such a moment map exists, then the quotient of the zero set by the group has a natural quaternionic structure.

The first proof we give is by applying the hypercomplex quotient to the associated bundle of a quaternionic manifold, following the example of Swann in the quaternionic Kähler case (Sect. 2 above), and was what led us to the result.

However, to understand the process on the level of quaternionic manifolds, a second proof will be given that does not use associated bundles. Again, the first proof is informal and the second more technical.

Proposition 5.1. *Let M be a quaternionic manifold acted on freely and smoothly by a connected Lie group F preserving the structure. Suppose that there exists a moment map μ for the action of the group that is not everywhere zero. Let P be the zero set of μ in M and let $N=P/F$. Then N has a natural quaternionic structure.*

First proof. From [S 2, p. 135], we know that the torsion-free connection ∇^M is not unique, but can be made unique by choosing a volume form for it to preserve. Choose an F-invariant volume form on M. Then there is a torsion-free connection ∇^M preserving the quaternionic structure and the volume form, and as it is unique it is F-invariant.

The choice of connection gives a hypercomplex structure on $\mathscr{U}(M)$, and as ∇^M is F-invariant, the induced action of F on $\mathscr{U}(M)$ must preserve the hypercomplex structure.

Observe that μ is simply a hypercomplex moment map for the induced action of F on the associated bundle. By the results of Sect. 3, one can perform a hypercomplex quotient. The new hypercomplex manifold is easily seen to fibre over N with fibre $\mathbb{H}/\{\pm 1\}$. It is therefore the associated bundle for a quaternionic structure on N, which can be reconstructed from the hypercomplex structure of the bundle. N is therefore a quaternionic manifold. □

Second proof. As above, choose ∇^M to be F-invariant. Let I_1, I_2, I_3 be a local F-invariant basis for the bundle of almost complex structures that satisfies $I_1 I_2 = I_3$, and s be a local, smooth, non-vanishing F-invariant section of e. Then $\mu=\mu_1 s\otimes I_1+\mu_2 s\otimes I_2+\mu_3 s\otimes I_3$ where μ_1, μ_2, μ_3 are scalar functions. At the points where $\mu=0$, i.e. on P, we have

$$\nabla^M \mu=(d\mu_1)\otimes s\otimes I_1+(d\mu_2)\otimes s\otimes I_2+(d\mu_3)\otimes s\otimes I_3\,, \tag{11}$$

which is not generally true away from P because $\nabla^M(s\otimes I_i)$ need not vanish. Then condition (i) becomes $I_1 d\mu_1=I_2 d\mu_2=I_3 d\mu_3$, as in the hypercomplex case. As P is defined by the vanishing of the scalar functions μ_1, μ_2, μ_3, the vector bundle V may be defined as in the hypercomplex case. Therefore, $TM|_P=TP\oplus \mathrm{Im}\,\mathbb{H}\cdot\mathfrak{F}$ and $TP=V\oplus\mathfrak{F}$, where V is a subbundle of TP that is invariant under I_1, I_2, I_3 as a subbundle of $TM|_P$.

The proof of Proposition 5.1 will therefore be completed by the

Lemma 5.2. *Suppose M, F, and ∇^M are as above, and that P is an F-invariant submanifold of M satisfying $TM|_P=TP\oplus\mathrm{Im}\,\mathbb{H}\cdot\mathfrak{F}$ and $TP=V\oplus\mathfrak{F}$, where V is a subbundle of TP that is invariant under I_1, I_2, I_3 as a subbundle of $TM|_P$. Then $N=P/F$ has a natural quaternionic structure.*

Proof. A connection ∇^N and three almost complex structures I_1, I_2, and I_3 can be defined on N exactly as in the proof of Lemma 3.2, with the proviso that I_i are only local, and the proof there that ∇^N is torsion-free also transfers unchanged to this situation. It will now be shown that ∇^N preserves the family of almost complex structures on N. This then implies that N is quaternionic, except in four dimensions, as the definition of quaternionic manifold is stronger in this case.

On M the I_i satisfy $\nabla^M I_i = \alpha_{ij} \otimes I_j$, using the summation convention, where (α_{ij}) is an anti-symmetric 3×3 matrix of 1-forms on M.

Choosing the local almost complex structures I_1, I_2, I_3 to be F-invariant makes the 1-forms α_{ij} F-invariant. The horizontal parts of the α_{ij} project down to give an anti-symmetric matrix of 1-forms (α_{ij}) on N. It will be shown that $\nabla^N I_i = \alpha_{ij} \otimes I_j$.

It is sufficient to show that $\nabla^N_{v_1}(I_i v_2) = I_i \nabla^N_{v_1} v_2 + \alpha_{ij}(v_1) I_j v_2$, where v_1, v_2 are any vector fields on N and $\alpha(v)$ is the contraction of the 1-form α with the vector field v. We lift this equation to $TM|_P$. The fields v_1, v_2 lift uniquely to give F-invariant sections $\tilde{v}_1, \tilde{v}_2$ of V over P. We shall again think of $\tilde{v}_1$ as a section of TP and $\tilde{v}_2$ as a section of $TM|_P$.

By definition of ∇^N, the first two terms lift to $\varrho(\nabla^{M|P}_{\tilde{v}_1}(I_i \tilde{v}_2))$ and $I_i \varrho(\nabla^{M|P}_{\tilde{v}_1} \tilde{v}_2)$. The scalar field $\alpha_{ij}(v_1)$ lifts to $\alpha_{ij}(\tilde{v}_1)$, because although the vertical part of α_{ij} is lost on projection to N, $\tilde{v}_1$ is horizontal and so this does not matter. Also $I_j v_2$ lifts to $I_j \tilde{v}_2$.

Thus we must demonstrate that

$$\varrho(\nabla^{M|P}_{\tilde{v}_1}(I_i \tilde{v}_2)) = I_i \varrho(\nabla^{M|P}_{\tilde{v}_1} \tilde{v}_2) + \alpha_{ij}(\tilde{v}_1) I_j \tilde{v}_2 . \tag{12}$$

But this is just the application of ϱ to the equation

$$\nabla^{M|P}_{\tilde{v}_1}(I_i \tilde{v}_2) = I_i \nabla^{M|P}_{\tilde{v}_1} \tilde{v}_2 + \alpha_{ij}(\tilde{v}_1) I_j \tilde{v}_2 , \tag{13}$$

which follows from $\nabla^M I_i = \alpha_{ij} \otimes I_j$. □

As with the hypercomplex quotient, we have been able to show that the quaternionic quotient facilitates the construction of compact, nonsingular, simply-connected quaternionic manifolds in all dimensions, that are not (even locally) quaternionic Kähler or hypercomplex, and whose associated bundles are not (even locally) products of other associated bundles. We believe these are the first examples of this kind, and will describe them in a later paper.

6 An example of the quaternionic quotient: Poon's metrics on $\mathbb{CP}^2 \# \mathbb{CP}^2$

In [P], Poon describes a family of self-dual metrics on $\mathbb{CP}^2 \# \mathbb{CP}^2$ parametrised by an open interval of the real line. This is done by showing that the intersection of two quadrics in $\mathbb{CP}^5$ can be given a real structure and desingularised so that it is the twistor space of a nonsingular manifold. Poon's description of the twistor space is (from p. 114 of [P]) a small resolution of the intersection of the two quadrics

$$2(z_0^2 + z_1^2) + \lambda z_2^2 + \tfrac{3}{2} z_3^2 + (z_4^2 + z_5^2) = 0 , \tag{Q_0}$$

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0 \tag{Q_∞}$$

in $\mathbb{CP}^5$, with the real structure

$$(z_0, z_1, z_2, z_3, z_4, z_5) \mapsto (\bar{z}_0, \bar{z}_1, \bar{z}_2, -\bar{z}_3, -\bar{z}_4, -\bar{z}_5) . \tag{R}$$

Consider the quaternionic quotient of $\mathbb{HP}^3$ by the group $U(1)\times U(1)$. For convenience in what follows the complex structure I_1 will be singled out, and everything will be written in complex coordinates with respect to I_1. We choose complex coordinates $(x_1, l_1, x_2, l_2, x_3, l_3, x_4, l_4)$ on $\mathbb{H}^4$, the associated bundle of $\mathbb{HP}^3$, with the other complex structures given by the antilinear action of I_2:

$$I_2((x_1,l_1,x_2,l_2,x_3,l_3,x_4,l_4))=(\bar{l}_1,-\bar{x}_1,\bar{l}_2,-\bar{x}_2,\bar{l}_3,-\bar{x}_3,\bar{l}_4,-\bar{x}_4). \tag{14}$$

The action of $U(1)\times U(1)$ is

$$(x_1,l_1,x_2,l_2,x_3,l_3,x_4,l_4)\overset{(u,v)}{\longmapsto}(ux_1,u^{-1}l_1,ux_2,u^{-1}l_2,vx_3,v^{-1}l_3,vx_4,v^{-1}l_4),$$
$$(u,v)\in U(1)\times U(1), \tag{15}$$

which preserves I_1, I_2 and $I_3=I_1I_2$, and the quaternionic moment maps we choose are

$$\mu_1=\begin{pmatrix}|x_1|^2-|l_1|^2+|x_2|^2-|l_2|^2+\alpha(|x_3|^2-|l_3|^2-|x_4|^2+|l_4|^2)\\ |x_3|^2-|l_3|^2+|x_4|^2-|l_4|^2+\alpha(|x_1|^2-|l_1|^2-|x_2|^2+|l_2|^2)\end{pmatrix}, \tag{16}$$

$$\mu_2+i\mu_3=2i\begin{pmatrix}x_1l_1+x_2l_2+\alpha(x_3l_3-x_4l_4)\\ x_3l_3+x_4l_4+\alpha(x_1l_1-x_2l_2)\end{pmatrix}, \tag{17}$$

in coordinates on the associated bundle, where α is a real parameter which lies in the interval $(0,1)$.

We will prove that the twistor spaces described by Poon and the twistor spaces of the quaternionic quotient above are isomorphic.

Proposition 6.1. *The twistor space of the quaternionic quotient above with parameter α is biholomorphic to Poon's description of the twistor space with parameter $\lambda=\dfrac{2+\alpha^4}{1+\alpha^4}$ in a way that identifies the real structures.*

Proof. A holomorphic map will be defined from the twistor space of the quaternionic quotient, which is the projectivisation of the associated bundle with respect to I_1, to $\mathbb{CP}^5$ with homogeneous coordinates $(z_0,\dots,z_5)$. The image will be seen to satisfy (Q_0) and (Q_∞) and the real structure on the quotient twistor space will induce the real structure (R) on $\mathbb{CP}^5$. This map is the required biholomorphism.

We make the following string of definitions:

let $w_0=x_1l_2$, $w_1=-x_2l_1$, $w_2=\alpha x_3l_4$, $w_3=-\alpha x_4l_3$, $z_0=\dfrac{w_0+w_1}{2}$, $z_1=\dfrac{w_0-w_1}{2i}$, $z_4=\dfrac{w_2-w_3}{2}$, and $z_5=\dfrac{w_2+w_3}{2i}$.

This gives $w_0w_1=-(x_1l_2)(x_2l_1)=z_0^2+z_1^2$, $-w_2w_3=\alpha^2(x_3l_4)(x_4l_3)=z_4^2+z_5^2$, and the action of $I_2:(z_0,z_1,z_4,z_5)\mapsto(\bar{z}_0,\bar{z}_1,-\bar{z}_4,-\bar{z}_5)$.

Define $z_2=\dfrac{i}{2}(1+\alpha^4)^{1/2}\cdot(x_1l_1-x_2l_2)$, $z_3=\dfrac{1}{\sqrt{2}}(x_1l_1+x_2l_2)$.

Then $I_2:(z_2,z_3)\mapsto(\bar{z}_2,-\bar{z}_3)$. Let $\lambda=\dfrac{2+\alpha^4}{1+\alpha^4}$. Note that all the new variables above are not acted upon by the quotient variables of the quaternionic quotient, and thus descend to functions on the associated bundle of the quotient that are holomorphic w.r.t. I_1.

Then

$$(z_0^2+z_1^2)+(\lambda-1)z_2^2+\frac{1}{2}z_3^2=-(x_1l_1)(x_2l_2)+\frac{1}{1+\alpha^4}\left(\frac{i}{2}(1+\alpha^4)^{1/2}\cdot(x_1l_1-x_2l_2)\right)^2$$
$$+\frac{1}{2}\left(\frac{1}{\sqrt{2}}(x_1l_1+x_2l_2)\right)^2$$
$$=0 \tag{18}$$

and

$$(2-\lambda)z_2^2+\frac{1}{2}z_3^2+(z_4^2+z_5^2)=\frac{\alpha^4}{1+\alpha^4}\left(\frac{i}{2}(1+\alpha^4)^{1/2}\cdot(x_1l_1-x_2l_2)\right)^2$$
$$+\frac{1}{2}\left(\frac{1}{\sqrt{2}}(x_1l_1+x_2l_2)\right)^2+\alpha^2(x_3l_3)(x_4l_4)$$
$$=-\frac{\alpha^4}{4}(x_1l_1-x_2l_2)^2+\frac{1}{4}(x_1l_1+x_2l_2)^2+\alpha^2(x_3l_3)(x_4l_4). \tag{19}$$

Now, from the quaternionic moment map equations,

$$x_3l_3=-\frac{(1+\alpha^2)x_1l_1+(1-\alpha^2)x_2l_2}{2\alpha}, \tag{20}$$

$$x_4l_4=\frac{(1-\alpha^2)x_1l_1+(1+\alpha^2)x_2l_2}{2\alpha}, \tag{21}$$

so $\alpha^2(x_3l_3)(x_4l_4)=-\frac{1}{4}((x_1l_1+x_2l_2)^2-\alpha^4(x_1l_1-x_2l_2)^2)$. Therefore,

$$(2-\lambda)z_2^2+\tfrac{1}{2}z_3^2+(z_4^2+z_5^2)=-\tfrac{1}{4}\alpha^4(x_1l_1-x_2l_2)^2+\tfrac{1}{4}(x_1l_1+x_2l_2)^2$$
$$-\tfrac{1}{4}((x_1l_1+x_2l_2)^2-\alpha^4(x_1l_1-x_2l_2)^2)$$
$$=0. \tag{22}$$

Collecting all the above information together, we find that there is a map from the twistor space of the quaternionic quotient to $\mathbb{CP}^5$ with homogeneous coordinates $(z_0,\ldots,z_5)$, such that the action of I_2 induces the involution

$$(z_0,z_1,z_2,z_3,z_4,z_5)\mapsto(\bar{z}_0,\bar{z}_1,\bar{z}_2,-\bar{z}_3,-\bar{z}_4,-\bar{z}_5), \tag{23}$$

that is, Eq. (R), and such that the image of the twistor space in $\mathbb{CP}^5$ satisfies (18) and (22); equivalently, the image satisfies 2(18)+(22) and (18)+(22), which are

$$2(z_0^2+z_1^2)+\lambda z_2^2+\tfrac{3}{2}z_3^2+(z_4^2+z_5^2)=0$$

and

$$z_0^2+z_1^2+z_2^2+z_3^2+z_4^2+z_5^2=0,$$

that is, (Q_0) and (Q_∞).

Thus it has been shown that the quotient model of the twistor space of $\mathbb{CP}^2\#\mathbb{CP}^2$ can be mapped directly to Poon's model of the twistor space with the value $\lambda=\dfrac{2+\alpha^4}{1+\alpha^4}$. Also, the interval of the parameter α, which is $(0,1)$, is mapped bijectively to the interval of Poon's parameter λ, which is $(\frac{3}{2}, 2)$. This completes the proof. □

We were able to write down a quaternionic quotient for metrics on $\mathbb{CP}^2 \# \mathbb{CP}^2$ because there is a general method for finding quotients for self-dual metrics on generalised connected sums of weighted projective spaces at orbifold points. This involves building up the quotient and its moment maps from smaller quotients and moment maps in a formal way.

A quotient may then be seen to be composed of basic building blocks, which are the self-dual metrics on weighted projective spaces described by Galicki and Lawson [GL]. Altering the expressions for the quaternionic moment maps corresponds to deforming the metrics. Then for sufficiently small generic changes to the moment maps from the connected sum form, the metrics are non-singular.

The result is a quaternionic quotient of $\mathbb{HP}^{n+1}$ by $U(1)^n$. An approach of this sort is necessary because, although it is easy to write down such a quotient, it is in general difficult to tell if the quotient is non-singular.

As mentioned in the Introduction, I have recently been able to show that LeBrun's self-dual metrics on $n\mathbb{CP}^2$ [L 2] are examples of this method. Other distinct families of self-dual metrics can also be constructed on $n\mathbb{CP}^2$, which have $h^0(K_Z^{-1/2})=2$ and symmetry groups of identity component $U(1)\times U(1)$.

7 Quaternionic complex manifolds

Let M be a quaternionic 4-manifold. The complex structures at a point x in M compatible with the quaternionic structure are parametrised by the points of the fibre of the twistor space Z of M over the point x. Thus an almost complex structure I on M compatible with the quaternionic structure is a section of the fibration $Z \to M$; I is integrable whenever the section is a complex hypersurface in Z.

By convention a quaternionic 4-manifold is conformally self-dual. However, a complex surface has a natural orientation, and we wish to consider metrics that are anti-self-dual with respect to this. So the complex orientation is opposite to the usual quaternionic orientation. This should not cause any confusion.

We begin by quoting a result of Pontecorvo [Pt, Theorem 2.1], that if M is Hermitian with metric g and complex structure I and anti-self-dual, then g is conformal to a Kähler metric if and only if the line bundle defined by the divisor $[X]$ is isomorphic to $K_Z^{-1/2}$.

Here K_Z is the canonical line bundle over Z and $[X]$ is the sum of the hypersurface Σ in Z defined by the complex structure I and the hypersurface $\bar{\Sigma}$ defined by $-I$. So the Kähler metrics in the conformal class of M are exactly given by real holomorphic sections of $K_Z^{-1/2}$.

Using a calculation in [S 1, Theorem 4.3], we find that a real holomorphic section of $K_Z^{-1/2}$ is a complex function ψ on the associated bundle that is quadratic, holomorphic with respect to I_1, and satisfies the reality condition $\psi(h)=\overline{\psi(I_2 h)}$ for h in the associated bundle.

As ψ is quadratic it has two zeros in each fibre of the twistor space Z, which are interchanged by the real structure σ on Z because ψ is real. So ψ defines a complex structure I and its conjugate $-I$ on M. Also, the "norm" of ψ on each fibre lies in a non-zero power of the volume forms, which gives a volume form on M. So M has a complex structure, a conformal structure and a volume form, which together make M Hermitian. Pontecorvo's result is that M is in fact Kähler with zero scalar curvature.

We shall make an observation that will enable us to put this information in a form that does not single out the complex structure I_1 on the associated bundle.

The quaternionic moment maps defined in Sect. 4 were interpreted on the associated bundle as triples of L_G^*-valued quadratic functions μ_1, μ_2, μ_3 with $\mu_2 + i\mu_3$ holomorphic w.r.t. I_1. The remaining conditions are equivalent to the reality condition $(\mu_2 + i\mu_3)(h) = (\mu_2 - i\mu_3)(I_2 h)$.

So a real holomorphic section ψ of $K_Z^{-1/2}$ is equivalent to a triple μ of quadratic scalar functions μ_1, μ_2, μ_3 on the associated bundle satisfying (1). On M this is a section μ of $\tilde{\mathscr{G}}$ satisfying (9).

Define a *twistor function* μ on a quaternionic manifold M to be a section μ of $\tilde{\mathscr{G}}$ satisfying (9). We call them twistor functions because they are in the kernel of a differential operator called the *(quadratic) twistor operator*. (See [S 3, Sect. 5] for the theory of invariant differential operators on quaternionic manifolds and [S 1, Lemma 6.4] for the definition of the quadratic twistor operator D in the quaternionic Kähler case.) Pontecorvo's result now states that the Kähler metrics in the conformal class of a quaternionic 4-manifold M are exactly given by the non-vanishing twistor functions μ on M.

A generalisation to higher dimensions is now apparent. The manifolds are quaternionic with a preferred complex structure, so we shall adopt the same *quaternionic complex* for them. We define a quaternionic complex manifold to be a quaternionic manifold M together with a twistor function μ that vanishes nowhere on M. It is sometimes convenient to allow μ to vanish at points on M, and these will be called *singular* points of the quaternionic complex manifold, so in general a quaternionic complex manifold will be an open set of a singular quaternionic complex manifold.

The quaternionic quotient generalises very simply to the quaternionic complex case: if one does a quaternionic quotient of a quaternionic complex manifold by a group preserving the twistor function μ then it is easy to see that μ descends to a twistor function on the quotient, which will again be non-vanishing. (However, when dealing with singular quaternionic complex manifolds it is important to ensure that μ does not lie in the span of the moment maps chosen.)

We remark that it is easily shown that the only solutions of the twistor equation on a connected open set in $\mathbb{H}^n$, and hence $\mathbb{HP}^n$, are polynomials of degree at most two.

The main result that we will prove about quaternionic complex manifolds is that they can alternatively be described as manifolds with a $SL(n,\mathbb{H})U(1)$-structure preserved by a torsion-free connection, and that as in the hypercomplex case this connection is unique. [The structure group is $SL(n,\mathbb{H})U(1)$ because, as in four dimensions, the twistor function gives a complex structure and a volume form, and the group preserving a quaternionic structure, a complex structure and a volume form is $SL(n,\mathbb{H})U(1)$.]

We note that in the classification by Berger [B] of holonomy groups of manifolds with torsion-free connections, $SL(n,\mathbb{H})U(1)$ is given as a possible holonomy group in Theorem 4, p. 320; in Berger's notation, $SL(n,\mathbb{H})U(1)$ is $\mathbf{T}^1 \times \mathbf{SU}^*(2n)$.

Theorem 7.1. *Let M be a quaternionic complex manifold. Then M has a natural $SL(n,\mathbb{H})U(1)$-structure Q, and there is a unique torsion-free connection ∇^M preserving Q.*

Proof. As M is quaternionic, it has a $GL(n,\mathbb{H})GL(1,\mathbb{H})$-structure Q'. A point q' in the fibre of Q' over $m\in M$ is an isomorphism of vector spaces $q':\mathbb{H}^n\to T_mM$ inducing isomorphisms on the families of complex structures on $\mathbb{H}^n$ and T_mM.

But the twistor function on M gives a non-zero volume form θ on T_mM and selects one of the complex structures, denoted I. Define the subset Q of Q' as those $q'\in Q'$ taking I_1 to I and the standard volume form on $\mathbb{H}^n$ to θ. Clearly Q fibres over M with fibre $SL(n,\mathbb{H})U(1)$, so Q is an $SL(n,\mathbb{H})U(1)$-structure on M.

To show that there exists a unique connection ∇^M on M preserving Q, it is sufficient to find a unique ∇^M preserving Q', I, and θ. Recall that from [S 2, p. 135] the torsion-free connection on a quaternionic manifold may be uniquely defined by giving a volume form for it to preserve. Let ∇^M be the torsion-free connection on M preserving Q' and the volume form θ. We will show that $\nabla^M I=0$.

Set $I_1=I$ and choose I_2, I_3 locally in $\mathscr{G}$ such that $I_1I_2=I_3$. Then $\mu=s\otimes I_1$, where s is a non-vanishing section of e. As $\nabla^M\theta=0$ and θ is some non-zero power of s, we have $\nabla^M s=0$. Also as ∇^M preserves Q' we have $\nabla^M I_i=\alpha_{ij}I_j$, where (α_{ij}) is an antisymmetric matrix of 1-forms. Thus $\nabla^M\mu=s\otimes\alpha_{1j}\otimes I_j$.

However, $\nabla^M\mu$ satisfies the twistor equation, and writing $\nabla^M\mu=s\otimes v_j\otimes I_j$ gives $I_1v_1=I_2v_2=I_3v_3$. But $v_1=\alpha_{11}=0$, as (α_{ij}) is anti-symmetric. So $v_2=v_3=0$ and $\nabla^M\mu=0$.

Therefore, $\nabla^M I=0$ and there is a unique torsion-free connection ∇^M preserving the quaternionic structure Q' of M, the complex structure I and the volume form θ. □

As a corollary we reprove Pontecorvo's result quoted above.

Corollary 7.2 [Pt, Theorem 2.1]. *A four-dimensional quaternionic complex manifold is exactly a Kähler surface of zero scalar curvature.*

Proof. Let M be a four-dimensional quaternionic complex manifold. Then M has the structure of a Hermitian manifold, with Riemannian metric g and compatible complex structure I. Since ∇^M is torsion-free and preserves g it must be the Levi-Civita connection, and as I satisfies $\nabla^M I=0$, M is by definition Kähler. But M is conformally anti-self-dual, so it must have zero scalar curvature.

Conversely, if M is a zero-scalar-curvature Kähler surface, it is quaternionic, and the volume form and complex structure together make up a section μ of $\mathscr{G}$ satisfying $\nabla^M\mu=0$, and a fortiori the twistor equation. Thus M is quaternionic complex. □

We also have an alternative definition for quaternionic complex manifolds:

Corollary 7.3. *In $4n$ dimensions with $n>1$, a quaternionic complex manifold is a manifold with a $SL(n,\mathbb{H})U(1)$-structure preserved by a torsion-free connection. In four dimensions a quaternionic complex manifold is a Kähler surface with zero scalar curvature.*

Using Poon's metrics of the last section as an example, we shall show that there are Kähler surfaces with zero scalar curvature that have no infinitesimal isometries.

To find Kähler metrics in the conformal class of the Poon metrics, we look for twistor functions. These descend from twistor functions on $\mathbb{HP}^3$ which are invariant under the quotient group. By inspection, there is a 6 real-dimensional vector space of twistor functions. The conformal isometry group of the metrics is $U(1) \times U(1)$, and only 4-dimensional subspaces of the twistor functions are invariant by more than a discrete subgroup of $U(1) \times U(1)$.

Thus the Kähler metrics conformal to Poon's metrics with a $U(1)$ isometry are of codimension 2 in the space of all Kähler metrics conformal to Poon's metrics, and we have constructed zero-scalar-curvature Kähler metrics which have no $U(1)$ isometries, and indeed no infinitesimal isometries.

This contrasts with the quaternionic Kähler case, for which we have Salamon's result [S 1, Lemma 6.5] that the space of twistor functions (my notation) is isomorphic to the space of infinitesimal isometries of the manifold when the scalar curvature is non-zero. So every zero-scalar-curvature Kähler surface that is conformal to a quaternionic Kähler manifold with non-zero scalar curvature has a preferred infinitesimal isometry.

A curious aspect of this work is that although we have a quotient for a type of Kähler manifold, it is not a Kähler quotient. This is because the higher dimensional manifolds do not have metrics. I have also found a pseudo-Kähler quotient for the zero-scalar-curvature Kähler surfaces given as examples of the quaternionic complex quotient in the next section. But the two quotients seem almost unrelated and I do not know if there is a systematic way of producing zero-scalar-curvature Kähler surfaces as Kähler quotients.

8 LeBrun's metrics on line bundles over $\mathbb{CP}^1$

In this section, as an example of the quaternionic complex quotient, we shall consider the two-dimensional weighted projective spaces with at most one singular point. There is just one family of these, those of the form $\mathbb{CP}^2_{1,1,k}$. It will be shown that they admit a quaternionic structure which is $U(2)$-symmetric. The family are in fact adaptations of the weighted projective spaces considered by Galicki and Lawson [GL].

Then using the results of Sect. 7, the Kähler metrics of zero scalar curvature that are conformal to these manifolds can be simply described, and it will be seen that there is one such metric that is $U(2)$-symmetric. It has a single pole at the orbifold point, close to which it is asymptotically flat.

Thus each member of the family carries an asymptotically flat, non-singular Kähler metric of zero scalar curvature with $U(2)$-symmetry. But all such metrics have been classified by LeBrun in his paper [L 1]. He finds that for each $k>0$, the total space of the line bundle L^{-k} over $\mathbb{CP}^1$ admits a Kähler metric with zero scalar curvature that is asymptotically flat, $U(2)$-symmetric and unique up to homothety, and that these comprise all the cases. Here L is the line bundle over $\mathbb{CP}^1$ that has Chern class $+1$.

It is clear on topological grounds how the two descriptions correspond, for $\mathbb{CP}^2_{1,1,k}$ is the compactification of the total space of the line bundle L^{-k} over $\mathbb{CP}^1$.

A *weighted projective space* is a singular complex manifold that generalises the ordinary notion of projective space. We shall consider weighted projective spaces of two complex dimensions, denoted $\mathbb{CP}^2_{p,q,r}$, where p, q, r are coprime positive integers.

This is defined as the quotient of $\mathbb{C}^3\backslash\{(0,0,0)\}$ by an action of $\mathbb{C}^*$, given by

$$(f,g,h)\overset{u}{\mapsto}(u^p f, u^q g, u^r h), \qquad u\in\mathbb{C}^*. \tag{24}$$

Thus $\mathbb{CP}^2$ is $\mathbb{CP}^2_{1,1,1}$. The only weighted projective spaces with exactly one singular point are $\mathbb{CP}^2_{1,1,k}$ for $k>1$.

Now a quaternionic structure on $\mathbb{CP}^2_{1,1,k}$ will be given as a quaternionic quotient of $\mathbb{HP}^2$ by the group $U(1)$. For convenience we single out the complex structure I_1 and work in coordinates that are complex with respect to it. The other complex structures are then given by the action of I_2.

The associated bundle of $\mathbb{HP}^2$ is $\mathbb{H}^3$, represented by complex coordinates (x,y,z,l,m,n), with the action of the second complex structure being

$$I_2((x,y,z,l,m,n))=(\bar{l},\bar{m},\bar{n},-\bar{x},-\bar{y},-\bar{z}), \tag{25}$$

and the action of the group is

$$(x,y,z,l,m,n)\overset{u}{\mapsto}(u^k x, u^k y, u^{2-k}z, u^{-k}l, u^{-k}m, u^{k-2}n), \qquad u\in U(1). \tag{26}$$

The moment maps we choose are

$$\mu_1=|x|^2+|y|^2+|z|^2-|l|^2-|m|^2-|n|^2 \tag{27}$$

and

$$\mu_2+i\mu_3=2i(xl+ym+zn). \tag{28}$$

The important point about these moment maps is the relative choice of signs between the terms in x,y,l,m and z,n. For $k>2$ the signs are consistent with the quotient being a quaternionic Kähler quotient of the non-compact dual of $\mathbb{HP}^2$, which has as associated bundle $\mathbb{H}^3$ with an indefinite metric, positive on the first two pairs of variables x,l and y,m but negative on the third pair z,n.

The difference that the signs make is this: with this group action but with the other choice of signs there would be singularities when $x=y=z=0$ and when $l=m=n=0$, which would lie above the singular line in $\mathbb{CP}^2_{k-1,k-1,k}$.

However, $x=y=z=0$ and $l=m=n=0$ are now excluded by the equation $\mu_1=0$, and in their place we see that the group acts freely, up to ± 1, on the sets $x=y=n=0$ and $l=m=z=0$. This is because the line $h=0$ in $\mathbb{CP}^2_{1,1,k}$ is not actually singular.

So doing a quaternionic quotient by $U(1)$ with the above moment maps gives a four-dimensional quaternionic manifold, with one singular point, given in associated bundle coordinates by $z=n=0$. To show that this manifold is $\mathbb{CP}^2_{1,1,k}$, a map $\pi(\phi)$ from one to the other will be given.

Consider the vector product of the three-dimensional complex vectors (x,y,z), $(\bar{l},\bar{m},\bar{n})$. This induces a map from $\mathbb{H}^3$ to $\mathbb{C}^3$. The map is fixed by complex multiplication by I_1 and I_2. So up to multiplication by positive real constants, the map from $\mathbb{H}^3$ to $\mathbb{C}^3$ is constant on quaternionic lines in $\mathbb{H}^3$, and thus induces a map $\phi: U\subset\mathbb{HP}^2\to\mathscr{S}^5$, where U is the set of points in $\mathbb{HP}^2$ for which $(x,y,z)\wedge(\bar{l},\bar{m},\bar{n})\neq 0$.

It is clear that the zero set of the moment maps in $\mathbb{HP}^2$ lies inside U, for if (x,y,z,l,m,n) in $\mathbb{HP}^2$ is not in U then (x,y,z) is proportional to $(\bar{l},\bar{m},\bar{n})$, and the moment maps then force $x=\ldots=n=0$, which is a contradiction.

Now consider the effect of the $U(1)$ action on the image under ϕ of a point of $\mathbb{HP}^2$. Clearly it is

$$(f,g,h)\overset{u}{\mapsto}(u^2 f, u^2 g, u^{2k}h), \qquad u\in U(1). \tag{29}$$

Pushing ϕ down to the quotients of both spaces gives a map $\pi(\phi)$. Thus the image under $\pi(\phi)$ of a point in the quotient of $\mathbb{HP}^2$ by $U(1)$ lies in the quotient of $\mathcal{S}^5 \subset \mathbb{C}^3$ by the action (29) of $U(1)$ on $\mathbb{C}^3$, i.e. in the weighted projective space $\mathbb{CP}^2_{1,1,k}$.

From Sect. 7, the Kähler metrics in the conformal class of these manifolds are given by twistor functions. There is an obvious twistor function on the spaces we consider given on the associated bundle by $\mu_1 = |z|^2 - |n|^2$, $\mu_2 + i\mu_3 = 2izn$. As this is a moment map for a $U(1)$-action it clearly satisfies the twistor equation, and it vanishes only at the orbifold point $z = n = 0$, so it represents a Kähler metric which is asymptotically flat at the orbifold point and has no other poles. It is up to a constant the unique twistor function preserved by the symmetry group $U(2)$ of the weighted projective space, so the Kähler metric it represents has symmetry group $U(2)$.

Thus the quaternionic metric on $\mathbb{CP}^2_{1,1,k}$ is conformal to a $U(2)$-symmetric, nonsingular, complete Kähler metric of zero scalar curvature that is asymptotically flat near the orbifold point. But by [L 1], this must be one of the metrics defined on line bundles over $\mathbb{CP}^1$ by LeBrun. From the topology, it is the metric on the line bundle L^{-k}.

The first two cases are $k = 1$, where the quaternionic manifold is $\mathbb{CP}^2$ and the zero-scalar-curvature Kähler metric is the Burns metric, and $k = 2$, which is the familiar Eguchi-Hanson space with its hyperkähler metric (see [L 1]).

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