Generalized Donaldson–Thomas invariants

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1. Introduction

Let X be a Calabi–Yau 3-fold, and $\mathcal{A} = \text{coh}(X)$ the abelian category of *coherent sheaves* on X. Write $K(\mathcal{A})$ for the *numerical Grothendieck group* of $\mathcal{A}$. If E is a coherent sheaf on X, write $[E]$ for its class in $K(\mathcal{A})$. The *Chern character* $\text{ch}(E)$ lies in $H^{\text{even}}(X;\mathbb{Q})$. It descends to a group morphism $\text{ch} : K(\mathcal{A}) \to H^{\text{even}}(X;\mathbb{Q})$. So $K(\mathcal{A})$ is a finite rank lattice $\mathbb{Z}^n$, a subgroup of $H^{\text{even}}(X;\mathbb{Q})$. The *Euler form* is $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$, antisymmetric and biadditive. Using Serre duality gives

$$\begin{aligned} &\dim \text{Hom}(E,F) - \dim \text{Ext}^1(E,F) \\ &- \dim \text{Hom}(F,E) + \dim \text{Ext}^1(F,E) = \chi([E],[F]). \end{aligned} \tag{1}$$

Choose an ample line bundle $\mathcal{L}$ on X. This induces a notion of *Gieseker stability* on $\mathcal{A} = \mathrm{coh}(X)$. Write τ for the stability condition coming from $\mathcal{L}$. It depends on $\mathcal{L}$, so a different ample line bundle $\tilde{\mathcal{L}}$ induces a different stability condition $\tilde{\tau}$.

Given $\alpha \in K(\mathcal{A})$, we can form the moduli spaces $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau), \mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$ of τ-(semi)stable sheaves E in $\mathcal{A}$ with $[E] = \alpha$ in $K(\mathcal{A})$. We can regard these as *schemes*, with points of $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$ being S-equivalence classes of τ-semistable sheaves, rather than isomorphism classes. Alternatively, we can regard them as *Artin stacks*, as open constructible subsets in the moduli stack $\mathfrak{M}$ of all coherent sheaves.

Donaldson–Thomas invariants $\mathrm{DT}^\alpha(\tau)$ are integer-valued invariants 'counting' τ-(semi) stable sheaves in class $\alpha \in K(\mathcal{A})$. They are defined only in the case when $\mathcal{M}_{\mathrm{st}}^\alpha(\tau) = \mathcal{M}_{\mathrm{ss}}^\alpha(\tau)$, that is, when there are no strictly semistable sheaves in class α.

The interesting property of Donaldson–Thomas invariants is that they are unchanged by continuous deformations of the underlying Calabi–Yau 3-fold X, that is, they are independent of the complex structure J of X up to deformation. This is a strong statement, as deforming X can change $\mathcal{A}$ and $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ radically.

Until now, it was not known how $\mathrm{DT}^\alpha(\tau)$ depends on τ, that is, on the choice of ample line bundle L.

Kai Behrend showed that $\mathrm{DT}^\alpha(\tau)$ can be written as a *weighted Euler characteristic*

$$\mathrm{DT}^\alpha(\tau) = \int_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)} \nu \, \mathrm{d}\chi, \qquad (2)$$

where ν is the 'microlocal function', a $\mathbb{Z}$-valued constructible function on $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ depending only on the scheme structure of $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$. We think of ν as a *multiplicity function*. If $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ is a k-fold point $\operatorname{Spec}\mathbb{C}[z]/(z^k)$ then $\nu \equiv k$. If $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ is smooth of dimension d then $\nu \equiv (-1)^d$.

In a series of previous papers, I defined a different set of invariants $J^\alpha(\tau) \in \mathbb{Q}$ 'counting' τ-semistable sheaves in class α. They are defined for all $\alpha \in K(\mathcal{A})$, including classes with strictly semistables. If $\mathcal{M}^\alpha_{\mathrm{st}}(\tau) = \mathcal{M}^\alpha_{\mathrm{ss}}(\tau)$ then $J^\alpha(\tau)$ is the (unweighted) Euler characteristic $\chi\big(\mathcal{M}^\alpha_{\mathrm{st}}(\tau)\big) \in \mathbb{Z}$.

The important property of the $J^\alpha(\tau)$ is that their transformation law under change of stability condition is known: we can write $J^\alpha(\tilde\tau)$ as a sum of products of $J^\beta(\tau)$, with combinatorial coefficients.

However, the $J^\alpha(\tau)$ are not invariant under deformations of the underlying Calabi-Yau 3-fold. This is because they do not count points in $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ with multiplicity, so a k-fold point $\operatorname{Spec}\mathbb{C}[z]/(z^k)$ in $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ would contribute 1 to $J^\alpha(\tau)$, for instance.

The goal of the project

We will define a family of *generalized D–T invariants* $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$ defined for all $\alpha \in K(\mathcal{A})$, combining the good properties of both the D–T invariants $\mathrm{DT}^\alpha(\tau)$, and my invariants $J^\alpha(\tau)$. That is:

- $\bar{DT}^\alpha(\tau)$ is unchanged by deformations of the underlying Calabi–Yau 3-fold.
- If $\mathcal{M}^\alpha_{\mathrm{st}}(\tau) = \mathcal{M}^\alpha_{\mathrm{ss}}(\tau)$ then $\bar{DT}^\alpha(\tau) = \mathrm{DT}^\alpha(\tau)$.
- The $\bar{DT}^\alpha(\tau)$ transform according to a known transformation law under change of stability condition. (As for the $J^\alpha(\tau)$, but with sign changes).

The general method is fairly obvious: we define $\bar{DT}^\alpha(\tau)$ by inserting Behrend's microlocal function ν as a weight in the definition of my $J^\alpha(\tau)$, so that the $\bar{DT}^\alpha(\tau)$ count sheaves with the correct multiplicity. But the details are complex.

2. Constructible functions on stacks

Fix $\mathbb{K}$ algebraically closed of characteristic zero. *Artin* $\mathbb{K}$*-stacks* $\mathfrak{F}$ are a very general kind of space in algebraic geometry. They include $\mathbb{K}$*-schemes.* Write $\mathfrak{F}(\mathbb{K})$ for the set of *geometric points* of $\mathfrak{F}$. Each $x \in \mathfrak{F}(\mathbb{K})$ has a *stabilizer group* $\mathrm{Iso}_{\mathbb{K}}(x)$, an algebraic $\mathbb{K}$-group, with $\mathrm{Iso}_{\mathbb{K}}(x) = \{1\}$ if $\mathfrak{F}$ is a scheme. Examples are *quotient stacks* $[X/G]$, for X a $\mathbb{K}$-scheme acted on by an algebraic $\mathbb{K}$-group G.

Call $S \subseteq \mathfrak{F}(\mathbb{K})$ *constructible* if $S = \cup_{i=1}^{n} \mathfrak{G}_i(\mathbb{K})$ for finite type $\mathbb{K}$-substacks $\mathfrak{G}_i \subseteq \mathfrak{F}$.

Call $f : \mathfrak{F}(\mathbb{K}) \to \mathbb{Q}$ *constructible* if $f(\mathfrak{F}(\mathbb{K}))$ is finite and $f^{-1}(c)$ is constructible for all $0 \neq c \in \mathbb{Q}$. Write $\mathrm{CF}(\mathfrak{F})$ for the $\mathbb{Q}$-algebra of constructible functions on $\mathfrak{F}$.

For $\phi : \mathfrak{F} \to \mathfrak{G}$ a *finite type* 1-morphism and $f \in \mathrm{CF}(\mathfrak{G})$ define the *pullback* $\phi^*(g) = g \circ \phi_*$, where $\phi_* : \mathfrak{F}(\mathbb{K}) \to \mathfrak{G}(\mathbb{K})$. Then $\phi^* : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$ is a $\mathbb{Q}$-algebra morphism, and $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

Defining *pushforwards* is more difficult. Let $\phi : \mathfrak{F} \to \mathfrak{G}$ be a *representable* morphism (injective on stabilizer groups). Define $\mathrm{CF}^{\mathrm{stk}}(\phi) : \mathrm{CF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{G})$ by

$$\big(\mathrm{CF}^{\mathrm{stk}}(\phi) f\big)(y) = \int_{\phi_*^{-1}(y)} m_\phi \cdot f \,\mathrm{d}\chi,$$

integrating using Euler characteristic as measure, where

$$m_\phi(x) = \chi\Big(\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))\Big).$$

Constructible functions on stacks satisfy:

- To each $\mathbb{K}$-stack $\mathfrak{F}$ with affine stabilizers, associate a $\mathbb{Q}$-algebra $\mathrm{CF}(\mathfrak{F})$.
- Constructible $S \subseteq \mathfrak{F}(\mathbb{K})$ have *characteristic functions* $\delta_S \in \mathrm{CF}(\mathfrak{F})$.
- To each finite type 1-morphism $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$ associate a *pullback* algebra morphism $\phi^* : \mathrm{CF}(\mathfrak{G}) \rightarrow \mathrm{CF}(\mathfrak{F})$, with $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
- To each representable 1-morphism $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$ associate a linear *pushforward* $\mathrm{CF}^{\mathrm{stk}}(\phi) : \mathrm{CF}(\mathfrak{G}) \rightarrow \mathrm{CF}(\mathfrak{F})$, with $\mathrm{CF}^{\mathrm{stk}}(\psi \circ \phi) = \mathrm{CF}^{\mathrm{stk}}(\psi) \circ \mathrm{CF}^{\mathrm{stk}}(\phi)$.
- In a *Cartesian square* of Artin $\mathbb{K}$-stacks $\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G} \\ \downarrow \theta & & \psi \downarrow \\ \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H}, \end{array}$ the following commutes: $\begin{array}{ccc} \mathrm{CF}(\mathfrak{E}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\eta)} & \mathrm{CF}(\mathfrak{G}) \\ \uparrow \theta^* & & \psi^* \uparrow \\ \mathrm{CF}(\mathfrak{F}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\phi)} & \mathrm{CF}(\mathfrak{H}). \end{array}$

3. Stack functions

Stack functions are a universal theory with this package of properties. Fix an algebraically closed field $\mathbb{K}$. Let $\mathfrak{F}$ be an Artin $\mathbb{K}$-stack with affine stabilizers. Consider pairs $(\mathfrak{R}, \rho)$, where $\mathfrak{R}$ is finite type with affine stabilizers and $\rho : \mathfrak{R} \to \mathfrak{F}$ a representable 1-morphism. Call $(\mathfrak{R}, \rho)$, $(\mathfrak{R}', \rho')$ *equivalent* if there is a 1-isomorphism $\iota : \mathfrak{R} \to \mathfrak{R}'$ with $\rho' \circ \iota$ and ρ 2-isomorphic 1-morphisms $\mathfrak{R} \to \mathfrak{F}$.

Write $[(\mathfrak{R}, \rho)]$ for the equivalence class of $(\mathfrak{R}, \rho)$. Define $\mathrm{SF}(\mathfrak{F})$ to be the $\mathbb{Q}$-vector space generated by such $[(\mathfrak{R}, \rho)]$ with for each closed $\mathbb{K}$-substack $\mathfrak{S}$ of $\mathfrak{R}$ a relation $[(\mathfrak{R}, \rho)] = [(\mathfrak{S}, \rho|_{\mathfrak{S}})] + [(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R}\setminus\mathfrak{S}})]$.

Define *multiplication* '$\cdot$' on $\mathrm{SF}(\mathfrak{F})$ by

$$[(\mathfrak{R},\rho)]\cdot[(\mathfrak{S},\sigma)]=[(\mathfrak{R}\times_{\rho,\mathfrak{F},\sigma}\mathfrak{S},\rho\circ\pi_{\mathfrak{R}})].$$

For $\mathfrak{G}\subseteq\mathfrak{F}$ a finite type $\mathbb{K}$-substack with inclusion $\iota:\mathfrak{G}\to\mathfrak{F}$ define the *characteristic function* $\bar{\delta}_{\mathfrak{G}(\mathbb{K})}=[(\mathfrak{G},\iota)]$.

For $\phi:\mathfrak{F}\to\mathfrak{G}$ of finite type define the *pullback* $\phi^*:\mathrm{SF}(\mathfrak{G})\to\mathrm{SF}(\mathfrak{F})$ by

$$\phi^*:[(\mathfrak{R},\rho)]\mapsto[(\mathfrak{R}\times_{\rho,\mathfrak{G},\phi}\mathfrak{F},\pi_{\mathfrak{F}})].$$

For $\phi:\mathfrak{F}\to\mathfrak{G}$ representable define the *pushforward* $\phi_*:\mathrm{SF}(\mathfrak{F})\to\mathrm{SF}(\mathfrak{G})$ by

$$\phi_*:[(\mathfrak{R},\rho)]\mapsto[(\mathfrak{R},\phi\circ\rho)].$$

These satisfy the same properties as the constructible functions operations. When $\mathrm{char}\,\mathbb{K}=0$, define $\pi^{\mathrm{stk}}_{\mathfrak{F}}:\mathrm{SF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{F})$ by

$$\pi^{\mathrm{stk}}_{\mathfrak{F}}:[(\mathfrak{R},\rho)]\mapsto\mathrm{CF}^{\mathrm{stk}}(\rho)1,$$

Then $\pi^{\mathrm{stk}}_{\mathfrak{F}}$ takes '$\cdot$',$\bar{\delta}_S$,ϕ^*,ϕ_* on $\mathrm{SF}(\cdots)$ to '$\cdot$',δ_S,ϕ^*,ϕ_* on $\mathrm{CF}(\cdots)$.

4. 'Virtual rank' and projections $\Pi_n^{\rm vi}$

If G is an algebraic $\mathbb{K}$-group, it has a maximal torus T^G, unique up to conjugation. The *rank* of G is the dimension of T^G. So $\mathrm{GL}(n,\mathbb{K})$ has rank n, for instance.

For $n \geqslant 0$ we can define projections $\Pi_n^{\rm vi}$: $\mathrm{SF}(\mathfrak{F}) \rightarrow \mathrm{SF}(\mathfrak{F})$ which project to stack functions with 'virtual rank n'. The rough idea is this: if $\Pi_n^{\rm vi}$ takes $[\mathfrak{R},\rho]$ to $[\mathfrak{R}_n,\rho]$, where $\mathfrak{R}_n$ is the substack of points in $\mathfrak{R}$ whose stabilizer groups have rank n. In fact this is true only if the stabilizer groups of $\mathfrak{R}$ are abelian.

These satisfy $(\Pi_n^{\rm vi})^2 = \Pi_n^{\rm vi}$, so that $\Pi_n^{\rm vi}$ is a projection, and $\Pi_m^{\rm vi} \circ \Pi_n^{\rm vi} = 0$ for $m \neq n$. They commute with pushforwards.

The $\Pi_n^{\rm vi}$ are important for Ringel–Hall algebras because they have a deep compatibility with the Ringel–Hall multiplication.

If $\mathfrak{R}$ is a quotient stack $[X/G]$ and G has maximal torus T^G and Weyl group $W(G,T^G)$, then $\Pi_n^{\rm vi}([\mathfrak{R},\rho])$ has a complicated expression

$$\Pi_n^{\rm vi}\bigl([(\mathfrak{R},\rho)]\bigr)= \int_{t\in T^G}\frac{|\{w\in W(G,T^G):w\cdot t=t\}|}{|W(G,T^G)|} \Bigl[\Bigl([X^{\{t\}}/C_G(\{t\})],\rho\circ\iota^{\{t\}}\Bigr)\Bigr]{\rm d}\mu_n,$$

where $X^{\{t\}}$ is the subscheme of X fixed by t, and $C_G(\{t\})$ the subgroup of G commuting with t, and μ_n is a certain measure on constructible subsets of T^G. See the handout for more details.

5. Ringel–Hall algebras of constructible functions

Let X be a Calabi–Yau 3-fold, and $\mathcal{A} = \mathrm{coh}(X)$. Write $\mathfrak{M}$ for the moduli stack of objects in $\mathcal{A}$, and $\mathfrak{Exact}$ for the moduli stack of exact sequences $0 \to E_1 \to E_2 \to E_3 \to 0$ in $\mathcal{A}$. There are 1-morphisms $\pi_j : \mathfrak{Exact} \to \mathfrak{M}$ projecting $0 \to E_1 \to E_2 \to E_3 \to 0$ to E_j for $j = 1, 2, 3$.

Define a binary operation $*$ on $\mathrm{CF}(\mathfrak{M})$ by

$$f * g = \mathrm{CF}^{\mathrm{stk}}(\pi_2)\big((\pi_1 \times \pi_3)^*(f \otimes g)\big).$$

Essentially, this says that

$$(f * g)(E_2) = \int_{0 \to E_1 \to E_2 \to E_3 \to 0} f(E_1) g(E_3) \mathrm{d}\chi,$$

integrating over exact sequences w.r.t. the Euler characteristic.

Then $*$ is *associative*. We prove this using diagrams of Cartesian squares of stacks involving $\mathfrak{M}, \mathfrak{Exact}$, and the Cartesian square property of $CF^{\rm stk}$ in §2.

Write $CF^{\rm ind}(\mathfrak{M})$ for the subspace of f in $\mathrm{CF}(\mathfrak{M})$ *supported on indecomposables*, that is, $f(E) = 0$ unless E is indecomposable ($E \neq 0$, $E \not\cong E_1 \oplus E_2$ for $E_1, E_2 \neq 0$; equivalently, $\mathrm{Aut}(E)$ has rank 1.) Then $CF^{\rm ind}(\mathfrak{M})$ is a *Lie subalgebra* of $\mathrm{CF}(\mathfrak{M})$ under the Lie bracket $[f,g] = f*g - g*f$. To see this, note that $f*g$ is supported on E with one or two indecomposable factors, and for E_1, E_2 indecomposable

$$(f*g)(E_1\oplus E_2) = f(E_1)g(E_2)+f(E_2)g(E_1),$$

symmetric in f, g, so $[f,g](E_1 \oplus E_2) = 0$.

6. Ringel–Hall algebras of stack functions

Similarly, define a binary operation $*$ on $\mathrm{SF}(\mathfrak{M})$ by

$$f * g = (\pi_2)_*\big((\pi_1 \times \pi_3)^*(f \otimes g)\big).$$

This is associative, by the same proof as for $\mathrm{CF}(\mathfrak{M})$.

The analogue of constructible functions supported on indecomposables is *stack functions with virtual rank 1*, that is $f \in \mathrm{SF}(\mathfrak{M})$ with $\Pi_1^{\mathrm{vi}}(f) = f$. (Actually, need to restrict to stack functions 'with algebra stabilizers' $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$, but ignore this.) Define $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}) = \{f \in \mathrm{SF}(\mathfrak{M}) : \Pi_1^{\mathrm{vi}}(f) = f\}$. Then $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$ is a *Lie subalgebra* of $\mathrm{SF}(\mathfrak{M})$ with $[f,g] = f*g - g*f$. The proof is complicated: it involves a deep compatibility between $*$ and Π_n^{vi}.

$\pi_{\mathfrak{M}}^{\mathrm{stk}} : \mathrm{SF}(\mathfrak{M}) \to \mathrm{CF}(\mathfrak{M})$ is an algebra morphism, and takes $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$ to $CF^{\mathrm{ind}}(\mathfrak{M})$.

Let τ be a stability condition on $\mathcal{A} = \mathrm{coh}(X)$, e.g. Gieseker stability. For $\alpha \in K(\mathcal{A})$, write $\mathfrak{M}^\alpha_{\mathrm{ss}}(\tau)$ for the substack of $\mathfrak{M}$ of τ-semistable sheaves in class α. Write $\delta^\alpha_{\mathrm{ss}}(\tau)$ for the characteristic function of $\mathfrak{M}^\alpha_{\mathrm{ss}}(\tau)$ in $CF(\mathfrak{M})$. Write $\bar\delta^\alpha_{\mathrm{ss}}(\tau)$ for the stack function $[(\mathfrak{M}^\alpha_{\mathrm{ss}}(\tau), \mathrm{inc})]$ in $\mathrm{SF}(\mathfrak{M})$.

For $\alpha \in C(\mathcal{A})$, define

$$\epsilon^\alpha(\tau) = \sum_{\substack{\alpha_1,\ldots,\alpha_n \in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha,\\ \tau(\alpha_i)=\tau(\alpha),\,\forall i}} \frac{(-1)^{n-1}}{n} \cdot \delta^{\alpha_1}_{\mathrm{ss}}(\tau) * \cdots * \delta^{\alpha_n}_{\mathrm{ss}}(\tau). \tag{3}$$

This is *invertible* combinatorially: we have

$$\delta^\alpha_{\mathrm{ss}}(\tau) = \sum_{\substack{\alpha_1,\ldots,\alpha_n \in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha,\\ \tau(\alpha_i)=\tau(\alpha),\,\forall i}} \frac{1}{n!} \cdot \epsilon^{\alpha_1}(\tau) * \cdots * \epsilon^{\alpha_n}(\tau). \tag{4}$$

Similarly, define stack functions $\bar\epsilon^\alpha(\tau)$ in $\mathrm{SF}(\mathfrak{M})$ by the analogue of (3) using $\bar\delta^{\alpha_i}_{\mathrm{ss}}(\tau)$; they satisfy the analogue of (4).

For $[E]\in\mathfrak{M}(\mathbb{K})$ in class $\alpha\in K(\mathcal{A})$ we have

- $\epsilon^\alpha(\tau)([E])=1$ if E is τ-*stable*,
- $\epsilon^\alpha(\tau)([E])=0$ is E is τ-*unstable* or *decomposable*,
- $\epsilon^\alpha(\tau)([E])\in\mathbb{Q}$ if E is *strictly* τ-*semistable* and *indecomposable*.

Hence $\epsilon^\alpha(\tau)$ is *supported on indecomposables*, so $\epsilon^\alpha(\tau)$ lies in the Lie algebra $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{M})$. Also $\bar\epsilon^\alpha(\tau)$ lies in the Lie algebra $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$. (Proof very nontrivial.) The $\delta^\alpha_{\mathrm{ss}}(\tau),\epsilon^\alpha(\tau)$ and $\bar\delta^\alpha_{\mathrm{ss}}(\tau),\bar\epsilon^\alpha(\tau)$ satisfy many interesting universal algebra identities in $\mathrm{CF}(\mathfrak{M}),\mathrm{SF}(\mathfrak{M})$. Those involving only the $\epsilon^\alpha(\tau),\bar\epsilon^\alpha(\tau)$ can be written as Lie algebra identities. For example, there are *change of stability condition formulae* writing $\delta^\alpha_{\mathrm{ss}}(\tilde\tau),\ldots,\bar\epsilon^\alpha(\tilde\tau)$ in terms of $\delta^\beta_{\mathrm{ss}}(\tau),\ldots,\bar\epsilon^\beta(\tau)$ for two stability conditions $\tilde\tau,\tau$.

7. Algebra morphisms from Ringel–Hall algebras

The heart of the Kontsevich–Soibelman paper, as Tom told us, is this: let X be a Calabi-Yau 3-fold over $\mathbb{K}$. Let Υ be a multiplicative motivic invariant of quasiprojective $\mathbb{K}$-varieties taking values in a commutative $\mathbb{Q}$-algebra Λ. Suppose $\Upsilon(\mathbb{K}) = \wp^2$ for some $\wp \in \Lambda$ with $\wp$ and $\wp^{2k} - 1$ invertible in Λ for $k = 1, 2, \ldots$. For example, let $\Upsilon(Y)$ be the virtual Poincaré polynomial $P_Y(t)$, $\Lambda = \mathbb{Q}(t)$ be rational functions in t, and $\wp = t$.

Define a Λ-algebra $A(X)$ to have Λ-basis a^α for $\alpha \in K(\mathcal{A})$, and multiplication

$$a^\alpha \star a^\beta = \wp^{\chi(\alpha,\beta)} a^{\alpha+\beta}.$$

Conjecture (K–S). *There is a natural algebra morphism* $\Phi : \mathrm{SF}(\mathfrak{M}) \to A(X)$*.*

That is, Φ is a morphism from a very large, universal algebra $\mathrm{SF}(\mathfrak{M})$ to a much smaller, explicit algebra $A(X)$. We could define *motivic Donaldson–Thomas invariants* $DT^\alpha_\Upsilon(\tau)\in\Lambda$ by $\Phi(\bar\delta^\alpha_{\mathrm{ss}}(\tau))=DT^\alpha_\Upsilon(\tau)a^\alpha$. Then as Φ is an algebra morphism, identities on the $\bar\delta^\alpha_{\mathrm{ss}}(\tau)$ in $\mathrm{SF}(\mathfrak{M})$, such as change of stability condition, translate to multiplicative identities on the $DT^\alpha_\Upsilon(\tau)$.

However, this approach does not work when Υ is the Euler characteristic, since $\Upsilon(\mathbb{K})=\wp^2=1$, so $\wp^{2k}-1$ is not invertible.

In general, morphisms from Ringel–Hall (Lie) algebras to smaller, explicit (Lie) algebras are a powerful way of defining invariants with multiplicative properties – see my configurations papers.

8. A Lie algebra morphism $\Psi : \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}) \to C(X)$

Define a $\mathbb{Q}$-Lie algebra $C(X)$ to have basis, as a $\mathbb{Q}$-vector space, symbols c^α for $\alpha \in K(\mathcal{A})$, and Lie bracket

$$[c^\alpha, c^\beta] = \chi(\alpha,\beta)\, c^{\alpha+\beta}, \qquad (5)$$

where χ is the Euler form. As χ is antisymmetric this satisfies the Jacobi identity. In my configurations paper II (2005) I defined a *Lie algebra morphism* $\Psi : \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}) \to C(X)$ by $\Psi(f) = \sum_{\alpha\in K(\mathcal{A})} \overline{\chi}(f|_{\mathcal{M}^\alpha_{\mathcal{A}}})c^\alpha$, where $\mathfrak{M}^\alpha$ is the substack of sheaves in class α in $\mathfrak{M}$, and $\overline{\chi}$ is a kind of stack-theoretic Euler characteristic.

Here $\overline{\chi}$ is not easy to define. The natural Euler characteristic of a quotient stack $[X/G]$ should be $\overline{\chi}([X/G]) = \chi(X)/\chi(G)$, but $\chi(G) = 0$ for any algebraic group of positive rank, so we have to divide by zero.

The point about $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$ is that we can write $f \in \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$ using only $[X/G]$ with $\mathrm{rank}(G) = 1$, and then set $\overline{\chi}([X/G]) = \chi(X)/\chi(G/\mathbb{C}^\times)$, where $\mathbb{C}^\times$ is the maximal torus of G, and $\chi(G/\mathbb{C}^\times) \neq 0$.
Using the Calabi-Yau 3-fold property, equation (1), we can show that $\Psi : \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}) \to C(X)$ is a *Lie algebra morphism*.
We then define invariants $J^\alpha(\tau) \in \mathbb{Q}$ by $\Psi(\overline{\epsilon}^\alpha(\tau)) = J^\alpha(\tau)c^\alpha$ for all $\alpha \in K(\mathcal{A})$.
Since the $\overline{\epsilon}^\alpha(\tau)$ satisfy a universal transformation law in the Lie algebra $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$ under change of stability condition, and Ψ is a Lie algebra morphism, the images $J^\alpha(\tau)c^\alpha$ satisfy the same transformation law in the Lie algebra $C(X)$.
Note that the $J^\alpha(\tau)$ do not count sheaves with multiplicity, as Ψ does not include Behrend functions. So they will not be unchanged under deformations of X.

9. Another Lie algebra morphism $\tilde\Psi : \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}) \to \tilde C(X)$

We can now explain our new work. We want to modify the Lie algebra morphism Ψ by inserting Behrend's microlocal function ν as a weight in its definition of Ψ, to get a new Lie algebra morphism $\tilde\Psi$. As ν is a 'multiplicity function', the new *generalized D–T invariants* $\bar{DT}{}^\alpha(\tau)$ we define using $\tilde\Psi$ will count sheaves with multiplicity, and so they will be unchanged under deformations of X.

Surprisingly, we also have to change the signs in the Lie algebra $C(X)$.

Define a Lie algebra $\tilde C(X)$ to have basis, as a $\mathbb{Q}$-vector space, symbols $\tilde c^\alpha$ for $\alpha \in K(\mathcal{A})$, and Lie bracket

$$[\tilde c^\alpha, \tilde c^\beta] = (-1)^{\chi(\alpha,\beta)}\chi(\alpha,\beta)\,\tilde c^{\alpha+\beta}, \qquad (6)$$

which is (5) with an extra factor $(-1)^{\chi(\alpha,\beta)}$.

Define a linear map $\tilde{\Psi} : \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}) \to \tilde{C}(X)$ by $\tilde{\Psi}(f) = \sum_{\alpha \in K(\mathcal{A})} \overline{\chi}(f|_{\mathcal{M}^\alpha_{\mathcal{A}}}, \nu)\tilde{c}^\alpha$, where $\overline{\chi}(\cdots, \nu)$ is $\overline{\chi}$ weighted by ν.

Theorem. $\tilde{\Psi} : \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}) \to \tilde{C}(X)$ *is a Lie algebra morphism*.

This follows from my previous proof that Ψ is a Lie algebra morphism, together with two multiplicative identities for the Behrend function ν, that is

$$\nu(E_1 \oplus E_2) = (-1)^{\chi([E_1],[E_2])}\nu(E_1)\nu(E_2), \quad (7)$$

$$\int_{\epsilon \in P(\mathrm{Ext}^1(E_2,E_1))} \nu(F)\mathrm{d}\chi - \int_{\epsilon \in P(\mathrm{Ext}^1(E_1,E_2))} \nu(F)\mathrm{d}\chi = \quad (8)$$
$$\Big(\dim \mathrm{Ext}^1(E_2, E_1) - \dim \mathrm{Ext}^1(E_1, E_2)\Big)\nu(E_1 \oplus E_2),$$

where in the first integral in (8), F is defined in terms of ϵ such that the exact sequence $0 \to E_1 \to F \to E_2 \to 0$ corresponds to $\epsilon \in P(\mathrm{Ext}^1(E_2, E_1))$, and similarly for the second integral.

10. Proving the Behrend function identities (7),(8)

Let $\mathfrak{F}$ be a $\mathbb{C}$-scheme or Artin $\mathbb{C}$-stack, locally of finite type. The Behrend function $\nu_{\mathfrak{F}}$ is a $\mathbb{Z}$-valued constructible function on $\mathfrak{F}$ which measures the 'multiplicity' of $\mathfrak{F}$ at each point. In general it is difficult to compute. But there is a special case in which we can give an explicit formula for $\nu_{\mathfrak{F}}$: suppose $\mathfrak{F}$ is a $\mathbb{C}$-scheme, U is a complex manifold, $f : U \to \mathbb{C}$ is holomorphic, and $\mathfrak{F}$ is locally isomorphic (in the analytic topology) to $\mathrm{Crit}(f)$ as a complex analytic space. Then

$$\nu_{\mathfrak{F}}(x) = (-1)^{\dim U}\bigl(1 - \chi(MF_f(x))\bigr),$$

with $MF_f(x)$ the *Milnor fibre* of f at x.

Our proof of (7),(8) involves first showing that we can write an atlas for the moduli stack $\mathfrak{M}$ of coherent sheaves on a Calabi–Yau 3-fold X over $\mathbb{C}$ in the form Crit(f) locally in the analytic topology, for f a holomorphic function on a complex manifold U. Note that f,U are *not* algebraic, they are constructed by transcendental, gauge-theoretic methods. Our proof works only over $\mathbb{C}$, not for more general fields $\mathbb{K}$.

The proof has three steps:
(a) Show that the moduli stack $\mathfrak{M}$ of coherent sheaves on X is locally isomorphic (in the Zariski topology) to the moduli stack $\mathfrak{Vect}$ of vector bundles on X. (This works for Calabi–Yau m-folds X over $\mathbb{K}$ for any $m, \mathbb{K}$.)
(b) Show that an atlas for $\mathfrak{Vect}$ near $[E]$ can be locally written in the form Crit(f) for $f : U \to \mathbb{C}$, where f, U are invariant under at least the maximal compact subgroup of Aut(E).
(c) Prove (7),(8) using an atlas near $E = E_1 \oplus E_2$ and localizing under the action of the $U(1)$ group $\{\mathrm{id}_{E_1} + \lambda\,\mathrm{id}_{E_2} : \lambda \in U(1)\}$.

For (a), one uses Fourier–Mukai transforms (Seidel–Thomas twists) to show the local equivalence of moduli of sheaves and vector bundles. Seidel–Thomas twists work for all Calabi-Yau m-folds, X. Given an integer n, the Seidel–Thomas twist with $\mathcal{O}_X(-n)$, T_n, is the Fourier-Mukai transform from $D(X)$ to $D(X)$ with kernel:

$$\text{cone}(\mathcal{O}_X(n) \boxtimes \mathcal{O}_X(-n) \to \mathcal{O}_\Delta).$$

Given $E \in \mathfrak{M}$. For large enough n, $F = T_n(E)$ is a sheaf and we have:

$$0 \to F \to \mathcal{O}_X(-n) \otimes H^0(E(n)) \to E \to 0.$$

To show local isomorphisms of moduli at E and F, one needs to check that T_n induces an equivalence of functors between Def_E and Def_F over the category of noetherian henselian local $\mathbb{K}$-algebras with residue field $\mathbb{K}$ (the category of analytic germs at a point).

Then we can define T_n inductively, with integers $n_1, \ldots, n_m$. Let $F_i = T_{n_i} \circ T_{n_{i-1}} \ldots \circ T_{n_1}(E)$. Then we get an exact sequence:

$$\begin{aligned} 0 \to \quad & F_m \to \mathcal{O}_X(-n_m) \otimes H^0(F_{m-1}(m)) \to \\ \ldots \to \quad & \mathcal{O}_X(-n_1) \otimes H^0(E(n_1)) \to E \to 0 \end{aligned}$$

By the Hilbert Syzygy Theorem, F_m is a vector bundle.

For (b), we use an idea of Richard Thomas. Let $E \to X$ be a fixed complex (not holomorphic) vector bundle. The holomorphic structures on E are $\bar\partial$-operators $\bar\partial_E : C^\infty(E) \to C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1}T^*X)$. The set of such $\bar\partial$-operators is an infinite-dimensional affine space $\mathcal{A}$. A $\bar\partial$-operator $\bar\partial_E$ is a holomorphic structure iff the $(0,2)$-curvature $\bar\partial_E^2$ is zero. Gauge transformations $\mathcal{G} = C^\infty(\mathrm{Aut}(E))$ act on $\mathcal{A}$. Thus, the moduli space (stack) of holomorphic structures on E up to isomorphisms is

$$\mathcal{M}_E = \{\bar\partial_E \in \mathcal{A} : \bar\partial_E^2 = 0\}/\mathcal{G}.$$

Richard observed that $\{\bar\partial_E \in \mathcal{A} : \bar\partial_E^2 = 0\}$ is $\mathrm{Crit}(CS)$, in some infinite-dimensional manifold sense, where $CS : \mathcal{A} \to \mathbb{C}$ is the *holomorphic Chern–Simons functional*.

To prove (b), we show that an atlas for $\mathfrak{Vect}$ near $(E, \bar\partial_E)$ can be written locally as Crit$(CS|_U)$, where U is a finite-dimensional complex submanifold of $\mathcal{A}$, which is roughly speaking transverse to the orbit of $\mathcal{G}$ through $\bar\partial_E$. We use results of Miyajima and others which locally identify the moduli spaces of holomorphic structures on E, and of analytic vector bundles on X, and of algebraic vector bundles on X.

To prove (c): let $E = E_1 \oplus E_2$ be a coherent sheaf on X. Then (a),(b) show that we can write an atlas for $\mathfrak{M}$ near E as Crit(f) near 0, where f is a holomorphic function defined near 0 on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$, and f is invariant under the action of $T = \{\mathrm{id}_{E_1} + \lambda\,\mathrm{id}_{E_2} : \lambda \in U(1)\}$ on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ by conjugation.
The fixed points of T on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ are $\mathrm{Ext}^1(E_1, E_1) \oplus \mathrm{Ext}^1(E_2, E_2)$, and that the restriction of f to these fixed points is $f_1 + f_2$, where f_j is defined near 0 in $\mathrm{Ext}^1(E_j, E_j)$, and Crit(f_j) is an atlas for $\mathfrak{M}$ near E_j.

The Milnor fibre $MF_f(0)$ is invariant under T, so by localization we have

$$\chi(MF_f(0)) = \chi(MF_f(0)^T) = \chi(MF_{f_1+f_2}(0)).$$

The Thom–Sebastiani theorem gives

$$1 - \chi(MF_{f_1+f_2}(0)) = (1 - \chi(MF_{f_1}(0)) \\ (1 - \chi(MF_{f_2}(0)).$$

Equation (7) then follows easily from

$$\nu_{\mathfrak{M}}(E) = (-1)^{\dim \mathrm{Ext}^1(E,E) - \dim \mathrm{Hom}(E,E)} \\ (1 - \chi(MF_f(0)),$$

and the analogues for E_1, E_2. Equation (8) uses a more involved argument to do with Milnor fibres of f at non-fixed points of the $U(1)$-action.

11. Generalized D–T invariants

We then define invariants $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$ by $\tilde\Psi(\bar\epsilon^\alpha(\tau)) = \bar{DT}^\alpha(\tau)\tilde c^\alpha$ for all $\alpha \in K(\mathcal{A})$. Since $\tilde\Psi$ is a Lie algebra morphism, and the $\bar\epsilon^\alpha(\tau)$ satisfy a universal transformation law under change of stability condition, it follows that the $\bar{DT}^\alpha(\tau)$ satisfy a known transformation law under change of stability condition. When $\mathcal{M}^\alpha_{\mathrm{st}}(\tau) = \mathcal{M}^\alpha_{\mathrm{ss}}(\tau)$ we have $\bar\epsilon^\alpha(\tau) = \bar\delta^\alpha_{\mathrm{ss}}(\tau)$, giving

$$\bar{DT}^\alpha(\tau) = \int_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)} \nu \,\mathrm{d}\chi = \mathrm{DT}^\alpha(\tau) \qquad (9)$$

by (2). Thus, the $\bar{DT}^\alpha(\tau)$ are generalizations of Donaldson–Thomas invariants.

It remains to show that the $\bar{DT}^\alpha(\tau)$ are unchanged under deformations of the underlying Calabi–Yau 3-fold X.

In algebraic geometry, to get invariants invariant under deformation, one uses intersection numbers, because of the law of "conservation of numbers". It is more or less standard now to use virtual fundamental class and perfect obstruction theory to get counting invariants from moduli spaces which are invariant under deformation (Li–Tian, Behrend–Fantechi).

One needs dim Aut to be constant to have perfectness. For our moduli space, $\mathfrak{M}^{\alpha}_{ss}$, the deformation/obstruction at a point $[E]$ is given by $\mathrm{Ext}^i(E, E)$. The infinitesimal automorphism is given by $\mathrm{Hom}(E, E)$. When E is stable, $\dim \mathrm{Hom}(E, E) = 1$. When E is strictly semistable, $\dim \mathrm{Hom}(E, E) \geq 1$.

One can also try to rigidify $\mathfrak{M}_{ss}^{\alpha}$ to a moduli of sheaves with extra structures. A natural choice is the moduli of stable pairs $\mathfrak{M}_{stp}^{\alpha,n}$ which parametrizes equivalence classes of non-zero stable pairs. A pair of class (α, n) is a morphism: $s : \mathcal{O}_X(-n) \to E$ where E is a coherent sheaf of class α.

Let $\tau(E) = \frac{P_E(m)}{\text{rank}(E)}$ be the Gieseker stability on coherent sheaves, then $\tau'(s) = \frac{P_E(m)+q}{\text{rank}(E)}$ with rational number q is the new stability condition on pairs. Note that the stability condition depends on a parameter q. We can pick q for $0 < q \ll 1$ so that if a pair is stable then E is Gieseker semistable. And if a pairs is semistable, then it is stable. Consequently, we have

$$\pi : \mathfrak{M}_{stp}^{\alpha,n} \to \mathfrak{M}_{ss}^{\alpha},$$

where π is the forgetful map. Here π is surjective and smooth. And $\mathfrak{M}_{stp}^{\alpha,n}$ is a fine moduli scheme. In fact, $\mathfrak{M}_{stp}^{\alpha,n}$ is an atlas for $\mathfrak{M}_{ss}^{\alpha}$.

The deformation/obstruction theory of

$$\mathbb{I} : \mathcal{O}_X(-n) \to E$$

is governed by $\mathrm{Ext}^i(\mathbb{I}, E)$, $i = -1, 0, \ldots$. It is not perfect. It is the idea of PT to replace $\mathrm{Ext}^i(\mathbb{I}, E)$ by $\mathrm{Ext}^i(\mathbb{I}, \mathbb{I})$, which is perfect and symmetric. $\mathrm{Ext}^i(\mathbb{I}, \mathbb{I})$ is the deformation theory of $\mathbb{I}$ as complexes. PT can identify arbitrary infinitesimal deformation of $\mathbb{I}$ as pairs with deformations as complexes. We can't do that. Instead, we show that deformation spaces are isomorphic and the obstruction space of pairs maps injectively into the obstruction space of complexes. These facts come from a long exact sequence obtained by applying $\mathrm{Hom}(\mathbb{I}, *)$ to

$$\mathbb{I} \to \mathcal{O}_X(-n) \to E$$

$$\longrightarrow \mathrm{Ext}^{-1}(\mathbb{I}, E)$$

$$\longrightarrow \mathrm{Hom}(\mathbb{I}, \mathbb{I}) \longrightarrow \mathrm{Hom}(\mathbb{I}, \mathcal{O}_X(-n)) \longrightarrow \mathrm{Hom}(\mathbb{I}, E)$$

$$\longrightarrow \mathrm{Ext}^1(\mathbb{I}, \mathbb{I}) \longrightarrow \mathrm{Ext}^1(\mathbb{I}, \mathcal{O}_X(-n)) \longrightarrow \mathrm{Ext}^1(\mathbb{I}, E)$$

$$\longrightarrow \mathrm{Ext}^2(\mathbb{I}, \mathbb{I}).$$

Here, $\mathrm{Ext}^{-1}(\mathbb{I}, E) = 0$, because of stability. Also

$$\mathrm{Hom}(\mathbb{I}, \mathbb{I}) \cong \mathrm{Hom}(\mathbb{I}, \mathcal{O}_X(-n)) \cong H^0(\mathcal{O}_X)$$

and

$$\mathrm{Ext}^1(\mathbb{I}, \mathcal{O}_X(-n)) = H^1(\mathcal{O}_X) = 0$$

by assumption. Therefore, $\mathrm{Hom}(\mathbb{I}, E) \cong \mathrm{Ext}^1(\mathbb{I}, \mathbb{I})$ and $\mathrm{Ext}^1(\mathbb{I}, E)$ maps injectively into $\mathrm{Ext}^2(\mathbb{I}, \mathbb{I})$.

Consequently, $\text{Ext}^i(\mathbb{I}, \mathbb{I})$ gives us a symmetric perfect obstruction theory in the sense of [BF] on $\mathfrak{M}_{stp}^{\alpha,n}$. The resulting virtual class has dimension zero. We can define invariants of stable pairs using the virtual fundamental class.

$$PI^{\alpha,n} = \int_{[\mathfrak{M}_{stp}^{\alpha,n}]^{\text{vir}}} 1 = \chi(\mathfrak{M}_{stp}^{\alpha,n}, \nu_{\mathfrak{M}_{stp}^{\alpha,n}}).$$

We can do everything in families, so we get deformation invariance.

Write $PI^\alpha(N,\tau)\in\mathbb{Z}$ for the invariant counting pairs (E,s) with E a semistable sheaf in class α and $s\in H^0(E\otimes\mathcal{L}^N)$, for $N\gg 0$, where $\mathcal{L}$ is the ample line bundle used to define τ. Yinan has proved that $PI^\alpha(N,\tau)$ is unchanged by deformations of X.

We claim that $PI^\alpha(N,\tau)$ can be written in terms of the $\bar{DT}^\beta(\tau)$ by

$$PI^\alpha(N,\tau)=\sum_{\substack{\alpha_1,\ldots,\alpha_n\in K(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha,\\ \tau(\alpha_i)=\tau(\alpha)\ \forall i}}\frac{(-1)^n}{n!}\prod_{i=1}^{n}(-1)^{\chi([\mathcal{L}^{-N}]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i)}\cdot\chi([\mathcal{L}^{-N}]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i)\bar{DT}^{\alpha_i}(\tau). \tag{10}$$

To prove (10) for fixed $N \gg 0$ we introduce an auxiliary abelian category $\mathcal{B}$, whose objects are triples (V, E, ϕ) for V a finite-dimensional $\mathbb{C}$-vector space, E a coherent sheaf, and $\phi : V \to H^0(E \otimes \mathcal{L}^N)$ a linear map. Then $K(\mathcal{B}) = \mathbb{Z} \oplus K(\mathcal{A})$, with $[(V, E, \phi)] = (\dim V, [E])$. If $\alpha \in K(\mathcal{A})$ then objects in class $(1, \alpha)$ in $\mathcal{B}$ are (V, E, ϕ) with $\dim V = 1$ and $[E] = \alpha$, so we can identify $V = \mathbb{C}$ and ϕ with $\phi(1) = s \in H^0(E \otimes \mathcal{L}^N)$. Thus, objects in class $(1, \alpha)$ are pairs (E, s) of the kind $PI^\alpha(N, \tau)$ counts.

From τ on $\mathcal{A}$ we produce two stability conditions τ', τ'' on $\mathcal{B}$. Elements $(0, E, 0)$ in class $(0, \alpha)$ are τ'- and also τ''-(semi)stable iff E is τ-(semi)stable. Also $(\mathbb{C}, 0, 0)$ is both τ'- and also τ''-stable. All $(\mathbb{C}, E, s)$ for $E \neq 0$ are τ'-unstable, and are τ''-stable iff (E, s) is a τ-stable pair in Yinan's sense.

We can now make invariants $\bar{DT}^{(k,\alpha)}(\tau')$, $\bar{DT}^{(k,\alpha)}(\tau'')$ in $\mathcal{B}$ as for $\bar{DT}^{\alpha}(\tau)$ in $\mathcal{A}$. We find that $\bar{DT}^{(1,0)}(\tau')=1$, and for $\alpha\neq 0$ in $K(\mathcal{A})$ we have $\bar{DT}^{(0,\alpha)}(\tau')=\bar{DT}^{\alpha}(\tau)$ and $\bar{DT}^{(1,\alpha)}(\tau')=0$.
Similarly $\bar{DT}^{(1,0)}(\tau'')=1$, and for $\alpha\neq 0$ in $K(\mathcal{A})$ we have $\bar{DT}^{(0,\alpha)}(\tau'')=\bar{DT}^{\alpha}(\tau)$ and $\bar{DT}^{(1,\alpha)}(\tau'')=PI^{\alpha}(N,\tau)$. Using the change of stability condition formula in $\mathcal{B}$ to write $\bar{DT}^{(1,\alpha)}(\tau'')$ in terms of $\bar{DT}^{(0,\beta)}(\tau')$, $\bar{DT}^{(1,\beta)}(\tau')$ yields (10).

Finally, we note that (10) implies that

$$PI^{\alpha}(N,\tau) = (-1)^{\chi([\mathcal{L}^{-N}],\alpha)}\chi([\mathcal{L}^{-N}],\alpha)\bar{DT}^{\alpha}(\tau)+\cdots,$$

where the lower order terms '$\cdots$' involve only $\bar{DT}^{\beta}(\tau)$ with $\dim\beta = \dim\alpha$ and $\operatorname{rank}\beta < \operatorname{rank}\alpha$.

Also $\chi([\mathcal{L}^{-N}],\alpha) = \dim H^0(E\otimes\mathcal{L}^N) > 0$ for $N \gg 0$. Thus, fixing $\dim\alpha$ and arguing by induction on $\operatorname{rank}\alpha$, since $PI^{\alpha}(N,\tau)$ is deformation-invariant, we see that $\bar{DT}^{\alpha}(\tau)$ is deformation-invariant.

Integrality properties of the invariants

Suppose E is stable and rigid in class α. Then $kE = E\oplus\cdots\oplus E$ is strictly semistable in class $k\alpha$, for $k \geqslant 2$. Calculations show that E contributes 1 to $\bar{DT}^{\alpha}(\tau)$, and kE contributes $1/k^2$ to $\bar{DT}^{k\alpha}(\tau)$. So we do not expect the $\bar{DT}^{\alpha}(\tau)$ to be integers, in general.

Define new invariants $KS^{\alpha}(\tau) \in \mathbb{Q}$ by

$$\bar{DT}^{\alpha}(\tau) = \sum_{k\geqslant 1:k \text{ divides } \alpha} \frac{1}{k^2} KS^{\alpha/k}(\tau).$$

Then the kE for $k \geqslant 1$ above contribute 1 to $KS^{\alpha}(\tau)$ and 0 to $KS^{k\alpha}(\tau)$ for $k > 1$.

Conjecture. *Suppose τ is generic, in the sense that $\tau(\alpha) = \tau(\beta)$ implies $\chi(\alpha,\beta) =$ 0. Then $KS^{\alpha}(\tau) \in \mathbb{Z}$ for all $\alpha \in K(\mathcal{A})$.*

These $KS^{\alpha}(\tau)$ may coincide with invariants conjectured by Kontsevich–Soibelman, and in String Theory should perhaps be interpreted as 'numbers of BPS states'.