Kuranishi spaces and Symplectic Geometry

Volume I.
Basic theory of (m-)Kuranishi spaces

Dominic Joyce

The Mathematical Institute,
Radcliffe Observatory Quarter,
Woodstock Road,
Oxford,
OX2 6GG,
U.K.
joyce@maths.ox.ac.uk

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Kuranishi spaces and Symplectic Geometry. Volume II.
Differential Geometry of (m-)Kuranishi spaces

Introduction to the series

On the foundations of Symplectic Geometry

Several important areas of Symplectic Geometry involve 'counting' moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves in a symplectic manifold (S,ω) satisfying some conditions, where J is an almost complex structure on S compatible with ω, and using the 'numbers of curves' to build some interesting theory, which is then shown to be independent of the choice of J. Areas of this type include Gromov–Witten theory [12, 39, 52, 68, 73, 79, 102, 104], Quantum Cohomology [68, 79], Lagrangian Floer cohomology [2, 21, 24, 29, 92, 109], Fukaya categories [18, 98, 100], Symplectic Field Theory [9, 15, 16], Contact Homology [14, 94], and Symplectic Cohomology [99].

Setting up the foundations of these areas, rigorously and in full generality, is a very long and difficult task, comparable to the work of Grothendieck and his school on the foundations of Algebraic Geometry, or the work of Lurie and Toën–Vezzosi on the foundations of Derived Algebraic Geometry. Any such foundational programme for Symplectic Geometry can be divided into five steps:

(i) We must define a suitable class of geometric structures $\mathcal{G}$ to put on the moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves we wish to 'count'. This must satisfy both (ii) and (iii) below.

(ii) Given a compact space X with geometric structure $\mathcal{G}$ and an 'orientation', we must define a 'virtual class' $[[X]_{\mathrm{virt}}]$ in some homology group, or a 'virtual chain' $[X]_{\mathrm{virt}}$ in the chains of the homology theory, which 'counts' X.

Actually, usually one studies a compact, oriented $\mathcal{G}$-space X with a 'smooth map' $f : X \to Y$ to a manifold Y, and defines $[[X]_{\mathrm{virt}}]$ or $[X]_{\mathrm{virt}}$ in a suitable (co)homology theory of Y, such as singular homology or de Rham cohomology. These virtual classes/(co)chains must satisfy a package of properties, including a deformation-invariance property.

(iii) We must prove that all the moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves that will be used in our theory have geometric structure $\mathcal{G}$, preferably in a natural way. Note that in order to make the moduli spaces $\bar{\mathcal{M}}$ compact (necessary for existence of virtual classes/chains), we have to include *singular* J-holomorphic curves in $\bar{\mathcal{M}}$. This makes construction of the $\mathcal{G}$-structure on $\bar{\mathcal{M}}$ significantly more difficult.

(iv) We combine (i)–(iii) to study the situation in Symplectic Geometry we are interested in, e.g. to define Lagrangian Floer cohomology $HF^*(L_1, L_2)$ for compact Lagrangians L_1, L_2 in a compact symplectic manifold (S, ω).

To do this we choose an almost complex structure J on (S, ω) and define a collection of moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves relevant to the problem. By (iii) these have structure $\mathcal{G}$, so by (ii) they have virtual classes/(co)chains $[\bar{\mathcal{M}}]_{\rm virt}$ in some (co)homology theory.

There will be geometric relationships between these moduli spaces – for instance, boundaries of moduli spaces may be written as sums of fibre products of other moduli spaces. By the package of properties in (ii), these geometric relationships should translate to algebraic relationships between the virtual classes/(co)chains, e.g. the boundaries of virtual cochains may be written as sums of cup products of other virtual cochains.

We use the virtual classes/(co)chains, and the algebraic identities they satisfy, and homological algebra, to build the theory we want – Quantum Cohomology, Lagrangian Floer Theory, and so on. We show the result is independent of the choice of almost complex structure J using the deformation-invariance properties of virtual classes/(co)chains.

(v) We apply our new machine to do something interesting in Symplectic Geometry, e.g. prove the Arnold Conjecture.

Many authors have worked on programmes of this type, since the introduction of J-holomorphic curve techniques into Symplectic Geometry by Gromov [42] in 1985. Oversimplifying somewhat, we can divide these approaches into three main groups, according to their answer to (i) above:

(A) **(Kuranishi-type spaces.)** In the work of Fukaya, Oh, Ohta and Ono [19–39], moduli spaces are given the structure of *Kuranishi spaces* (we will call their definition *FOOO Kuranishi spaces*).

Several other groups also work with Kuranishi-type spaces, including McDuff and Wehrheim [77, 78, 80–83], Pardon [94, 95], and the author in [60, 62] and this series.

(B) **(Polyfolds.)** In the work of Hofer, Wysocki and Zehnder [46–53], moduli spaces are given the structure of *polyfolds*.

(C) **(The rest of the world.)** One makes restrictive assumptions on the symplectic geometry – for instance, consider only noncompact, exact symplectic manifolds, and exact Lagrangians in them – takes J to be generic, and arranges that all the moduli spaces $\bar{\mathcal{M}}$ we are interested in are smooth manifolds (or possibly 'pseudomanifolds', manifolds with singularities in codimension 2). Then we form virtual classes/chains as for fundamental classes of manifolds. A good example of this approach is Seidel's construction [100] of Fukaya categories of Liouville domains.

We have not given complete references here, much important work is omitted.

Although Kuranishi-type spaces in (A), and polyfolds in (B), do exactly the same job, there is an important philosophical difference between them. Kuranishi spaces basically remember the minimal information needed to form virtual cycles/chains, and no more. Kuranishi spaces contain about the same amount of data as smooth manifolds, and include manifolds as examples.

In contrast, polyfolds remember the entire functional-analytic moduli problem, forgetting nothing. Any polyfold curve moduli space, even a moduli space of constant curves, is a hugely infinite-dimensional object, a vast amount of data.

Approach (C) makes one's life a lot simpler, but this comes at a cost. Firstly, one can only work in rather restricted situations, such as exact symplectic manifolds. And secondly, one must go through various contortions to ensure all the moduli spaces $\bar{\mathcal{M}}$ are manifolds, such as using domain-dependent almost complex structures, which are unnecessary in approaches (A),(B).

The aim and scope of the series, and its novel features

The aim of this series of books is to set up the foundations of these areas of Symplectic Geometry built using J-holomorphic curves following approach (A) above, using the author's own definition of Kuranishi space. We will do this starting from the beginning, rigorously, in detail, and as the author believes the subject ought to be done. The author hopes that in future, the series will provide a complete framework which symplectic geometers can refer to for theorems and proofs, and use large parts as a 'black box'.

The author currently plans four or more volumes, as follows:

Volume I. **Basic theory of (m-)Kuranishi spaces.** Definitions of the category $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ of μ-Kuranishi spaces, and the 2-categories $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ of m-Kuranishi spaces and $\dot{\mathbf{K}}\mathbf{ur}$ of Kuranishi spaces, over a category of 'manifolds' $\dot{\mathbf{M}}\mathbf{an}$ such as classical manifolds $\mathbf{Man}$ or manifolds with corners $\mathbf{Man^c}$. Boundaries, corners, and corner (2-)functors for (m- and μ-)Kuranishi spaces with corners. Relation to similar structures in the literature, including Fukaya–Oh–Ohta–Ono's Kuranishi spaces, and Hofer–Wysocki–Zehnder's polyfolds. 'Kuranishi moduli problems', our approach to putting Kuranishi structures on moduli spaces, canonical up to equivalence.

Volume II. **Differential Geometry of (m-)Kuranishi spaces.** Tangent and obstruction spaces for (m- and μ-)Kuranishi spaces. Canonical bundles and orientations. (W-)transversality, (w-)submersions, and existence of w-transverse fibre products in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ and $\dot{\mathbf{K}}\mathbf{ur}$. M-(co)homology of manifolds and orbifolds [63], virtual (co)chains and virtual (co)cycles for compact, oriented (m-)Kuranishi spaces in M-(co)homology. Orbifold strata of Kuranishi spaces. Bordism and cobordism for (m-)Kuranishi spaces.

Volume III. **Kuranishi structures on moduli spaces of J-holomorphic curves.** For very many moduli spaces of J-holomorphic curves $\bar{\mathcal{M}}$ of interest in Symplectic Geometry, including singular curves,

curves with Lagrangian boundary conditions, marked points, etc., we show that $\bar{\mathcal{M}}$ can be made into a Kuranishi space $\bar{\boldsymbol{\mathcal{M}}}$, uniquely up to equivalence in $\dot{\mathbf{Kur}}$. We do this by a new method using 2-categories, similar to Grothendieck's representable functor approach to moduli spaces in Algebraic Geometry. We do the same for many other classes of moduli problems for nonlinear elliptic p.d.e.s, including gauge theory moduli spaces. Natural relations between moduli spaces, such as maps $F_i : \bar{\mathcal{M}}_{k+1} \to \bar{\mathcal{M}}_k$ forgetting a marked point, correspond to relations between the Kuranishi spaces, such as a 1-morphism $\boldsymbol{F}_i : \bar{\boldsymbol{\mathcal{M}}}_{k+1} \to \bar{\boldsymbol{\mathcal{M}}}_k$ in $\dot{\mathbf{Kur}}$. We discuss orientations on Kuranishi moduli spaces.

Volumes IV– **Big theories in Symplectic Geometry.** To include Gromov–Witten invariants, Quantum Cohomology, Lagrangian Floer cohomology, and Fukaya categories.

For steps (i)–(v) above, (i)–(iii) will be tackled in volumes I–III respectively, and (iv)–(v) in volume IV onwards.

Readers familiar with the field will probably have noticed that our series sounds a lot like the work of Fukaya, Oh, Ohta and Ono [19–39], in particular, their 2009 two-volume book [24] on Lagrangian Floer cohomology. And it is very similar. On the large scale, and in a lot of the details, we have taken many ideas from Fukaya–Oh–Ohta–Ono, which the author acknowledges with thanks. Actually this is true of most foundational projects in this field: Fukaya, Oh, Ohta and Ono were the pioneers, and enormously creative, and subsequent authors have followed in their footsteps to a great extent.

However, there are features of our presentation that are genuinely new, and here we will highlight three:

(a) The use of *Derived Differential Geometry* in our Kuranishi space theory.

(b) The use of *M-(co)homology* to form virtual cycles and chains.

(c) The use of *'Kuranishi moduli problems'*, similar to Grothendieck's representable functor approach to moduli spaces in Algebraic Geometry, to prove moduli spaces of J-holomorphic curves have Kuranishi structures.

We discuss these in turn.

(a) Derived Differential Geometry

Derived Algebraic Geometry, developed by Lurie [74] and Toën–Vezzosi [106, 107], is the study of 'derived schemes' and 'derived stacks', enhanced versions of classical schemes and stacks with a richer geometric structure. They were introduced to study moduli spaces in Algebraic Geometry. Roughly, a classical moduli space $\mathcal{M}$ of objects E knows about the infinitesimal deformations of E, but not the obstructions to deformations. The corresponding derived moduli space $\boldsymbol{\mathcal{M}}$ remembers the deformations, obstructions, and higher obstructions.

Derived Algebraic Geometry has a less well-known cousin, Derived Differential Geometry, the study of 'derived' versions of smooth manifolds. Probably the first reference to Derived Differential Geometry is a short final paragraph in Lurie [74, §4.5]. Lurie's ideas were developed further in 2008 by his student David Spivak [103], who defined an ∞-category $\mathbf{DerMan_{Spi}}$ of 'derived manifolds'.

When I read Spivak's thesis [103], armed with a good knowledge of Fukaya–Oh–Ohta–Ono's Kuranishi space theory [24], I had a revelation:

Kuranishi spaces are really derived smooth orbifolds.

This should not be surprising, as derived schemes and Kuranishi spaces are both geometric structures designed to remember the obstructions in moduli problems.

This has important consequences for Symplectic Geometry: to understand Kuranishi spaces properly, we should use the insights and methods of Derived Algebraic Geometry. Fukaya–Oh–Ohta–Ono could not do this, as their Kuranishi spaces predate Derived Algebraic Geometry by several years. Since they lacked essential tools, their FOOO Kuranishi spaces are not really satisfactory as geometric spaces, though they are adequate for their applications. For example, they give no definition of morphism of FOOO Kuranishi spaces.

A very basic fact about Derived Algebraic Geometry is that it always happens in higher categories, usually ∞-categories. We have written our theory in terms of 2-categories, which are much simpler than ∞-categories. There are special features of our situation which mean that 2-categories are enough for our purposes. Firstly, the existence of partitions of unity in Differential Geometry means that structure sheaves are soft, and have no higher cohomology. Secondly, we are only interested in 'quasi-smooth' derived spaces, which have deformations and obstructions, but no higher obstructions. As we are studying Kuranishi spaces with deformations and obstructions – two levels of tangent directions – these spaces need to live in a higher category $\boldsymbol{\mathcal{C}}$ with at least two levels of morphism, 1- and 2-morphisms, so $\boldsymbol{\mathcal{C}}$ needs to be at least a 2-category.

Our Kuranishi spaces form a weak 2-category $\mathbf{\dot{K}ur}$. One can take the homotopy category $\mathrm{Ho}(\mathbf{\dot{K}ur})$ to get an ordinary category, but this loses important information. For example:

- 1-morphisms $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ in $\mathbf{\dot{K}ur}$ are a 2-sheaf (stack) on $\boldsymbol{X}$, but morphisms $[\boldsymbol{f}]:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ in $\mathrm{Ho}(\mathbf{\dot{K}ur})$ are not a sheaf on $\boldsymbol{X}$, they are not 'local'. This is probably one reason why Fukaya et al. do not define morphisms for FOOO Kuranishi spaces, as higher category techniques would be needed.
- As in Chapter 11 of volume II, there is a good notion of (w-)transverse 1-morphisms $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ in $\mathbf{\dot{K}ur}$, and (w-)transverse fibre products $\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exist in $\mathbf{\dot{K}ur}$, characterized by a universal property involving the 2-morphisms in $\mathbf{\dot{K}ur}$. In $\mathrm{Ho}(\mathbf{\dot{K}ur})$ this universal property makes no sense, and (w-)transverse fibre products may not exist.

Derived Differential Geometry will be discussed in §4.8 of volume I.

(b) M-(co)homology and virtual cycles

In Fukaya–Oh–Ohta–Ono's Lagrangian Floer theory [24], a lot of extra complexity and hard work is due to the fact that their homology theory for forming virtual chains (singular homology) does not play nicely with FOOO Kuranishi spaces. For example, they deal with moduli spaces $\bar{\boldsymbol{\mathcal{M}}}_k(\alpha)$ of stable J-holomorphic discs Σ in (S,ω) with boundary in a Lagrangian L, with homology class $[\Sigma]=\alpha$ in $H_2(S,L;\mathbb{Z})$, and k boundary marked points. These satisfy boundary equations

$$\partial\bar{\boldsymbol{\mathcal{M}}}_k(\alpha)\simeq\textstyle\coprod_{\alpha=\beta+\gamma,\ k=i+j}\bar{\boldsymbol{\mathcal{M}}}_{i+1}(\beta)\times_{\mathbf{ev}_{i+1},L,\mathbf{ev}_{j+1}}\bar{\boldsymbol{\mathcal{M}}}_{j+1}(\gamma).$$

One would like to choose virtual chains $[\bar{\boldsymbol{\mathcal{M}}}_k(\alpha)]_{\rm virt}$ in homology satisfying

$$\partial[\bar{\boldsymbol{\mathcal{M}}}_k(\alpha)]_{\rm virt}=\textstyle\sum_{\alpha=\beta+\gamma,\ k=i+j}[\bar{\boldsymbol{\mathcal{M}}}_{i+1}(\beta)]_{\rm virt}\bullet_L[\bar{\boldsymbol{\mathcal{M}}}_{j+1}(\gamma)]_{\rm virt},$$

where $\bullet_L$ is a chain-level intersection product/cup product on the (co)homology of L. But singular homology has no chain-level intersection product.

In their later work [27, §12], [33], Fukaya et al. define virtual cochains in de Rham cohomology, which does have a cochain-level cup product. But there are disadvantages to this too, for example, one is forced to work in (co)homology over $\mathbb{R}$, rather than $\mathbb{Z}$ or $\mathbb{Q}$.

As in Chapter 12 of volume II, the author [63] defined new (co)homology theories $MH_*(X;R),MH^*(X;R)$ of manifolds and orbifolds X, called 'M-homology' and 'M-cohomology'. They satisfy the Eilenberg–Steenrod axioms, and so are canonically isomorphic to usual (co)homology $H_*(X;R),H^*(X;R)$, e.g. singular homology $H_*^{\rm si}(X;R)$. They are specially designed for forming virtual (co)chains for (m-)Kuranishi spaces, and have very good (co)chain-level properties.

In Chapter 13 of volume II we will explain how to form virtual (co)cycles and (co)chains for (m-)Kuranishi spaces in M-(co)homology. There is no need to perturb the (m-)Kuranishi space to do this. Our construction has a number of technical advantages over competing theories: we can make infinitely many compatible choices of virtual (co)chains, which can be made strictly compatible with relations between (m-)Kuranishi spaces, such as boundary formulae.

These technical advantages mean that applying our machinery to define some theory like Lagrangian Floer cohomology, Fukaya categories, or Symplectic Field Theory, will be significantly easier. Identities which only hold up to homotopy in the Fukaya–Oh–Ohta–Ono model, often hold on the nose in our version.

(c) Kuranishi moduli problems

The usual approaches to moduli spaces in Differential Geometry, and in Algebraic Geometry, are very different. In Differential Geometry, one defines a moduli space (e.g. of J-holomorphic curves, or instantons on a 4-manifold), initially as a set $\mathcal{M}$ of isomorphism classes of the objects of interest, and then adds extra structure: first a topology, and then an atlas of charts on $\mathcal{M}$ making the moduli space into a manifold or Kuranishi-type space. The individual charts are defined by writing the p.d.e. as a nonlinear Fredholm operator between Sobolev or Hölder spaces, and using the Implicit Function Theorem for Banach spaces.

In Algebraic Geometry, following Grothendieck, one begins by defining a functor F called the *moduli functor*, which encodes the behaviour of families of objects in the moduli problem. This might be of the form $F:(\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}}\rightarrow\mathbf{Sets}$

(to define a moduli $\mathbb{C}$-scheme) or $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ (to define a moduli $\mathbb{C}$-stack), where $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}, \mathbf{Sets}, \mathbf{Groupoids}$ are the categories of affine $\mathbb{C}$-schemes, and sets, and groupoids, and $(\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}}$ is the opposite category of $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$. Here if S is an affine $\mathbb{C}$-scheme then $F(S)$ is the set or groupoid of families of objects in the moduli problem over the base $\mathbb{C}$-scheme S.

We say that the moduli functor F is *representable* if there exists a $\mathbb{C}$-scheme $\mathcal{M}$ such that F is naturally isomorphic to $\mathrm{Hom}(-, \mathcal{M}) : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Sets}$, or an Artin $\mathbb{C}$-stack $\mathcal{M}$ such that F is naturally equivalent to $\mathbf{Hom}(-, \mathcal{M}) : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$. Then $\mathcal{M}$ is unique up to canonical isomorphism or canonical equivalence, and is called the *moduli scheme* or *moduli stack*.

As in Gomez [41, §2.1–§2.2], there are two equivalent ways to encode stacks, or moduli problems, as functors: either as a functor $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ as above, or as a *category fibred in groupoids* $G : \mathcal{C} \to \mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$, that is, a category $\mathcal{C}$ with a functor G to $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$ satisfying some lifting properties of morphisms in $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$ to morphisms in $\mathcal{C}$.

We introduce a new approach to constructing Kuranishi structures on Differential-Geometric moduli problems, including moduli of J-holomorphic curves, which is a 2-categorical analogue of the 'category fibred in groupoids' version of moduli functors in Algebraic Geometry. Our analogue of $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$ is the 2-category $\mathbf{G\dot{K}N}$ of *global Kuranishi neighbourhoods* (V, E, Γ, s), which are basically Kuranishi spaces $\boldsymbol{X}$ covered by a single chart (V, E, Γ, s, ψ).

We define a *Kuranishi moduli problem* (*KMP*) to be a 2-functor $F : \boldsymbol{\mathcal{C}} \to \mathbf{G\dot{K}N}$ satisfying some lifting properties, where $\boldsymbol{\mathcal{C}}$ is a 2-category. For example, if $\boldsymbol{\mathcal{M}} \in \mathbf{\dot{K}ur}$ is a Kuranishi space we can define a 2-category $\boldsymbol{\mathcal{C}}_{\boldsymbol{\mathcal{M}}}$ with objects $\big((V, E, \Gamma, s), \boldsymbol{f}\big)$ for $(V, E, \Gamma, s) \in \mathbf{G\dot{K}N}$ and $\boldsymbol{f} : (s^{-1}(0)/\Gamma, (V, E, \Gamma, s, \mathrm{id}_{s^{-1}(0)/\Gamma}))$ $\to \boldsymbol{\mathcal{M}}$ a 1-morphism, and a 2-functor $F_{\boldsymbol{\mathcal{M}}} : \boldsymbol{\mathcal{C}}_{\boldsymbol{\mathcal{M}}} \to \mathbf{G\dot{K}N}$ acting by $F_{\boldsymbol{\mathcal{M}}} : ((V, E, \Gamma, s), \boldsymbol{f}) \mapsto (V, E, \Gamma, s)$ on objects. A KMP $F : \boldsymbol{\mathcal{C}} \to \mathbf{G\dot{K}N}$ is called *representable* if it is equivalent in a certain sense to $F_{\boldsymbol{\mathcal{M}}} : \boldsymbol{\mathcal{C}}_{\boldsymbol{\mathcal{M}}} \to \mathbf{G\dot{K}N}$ for some $\boldsymbol{\mathcal{M}}$ in $\mathbf{\dot{K}ur}$, which is unique up to equivalence. Then Kuranishi moduli problems form a 2-category $\mathbf{\dot{K}MP}$, and the full 2-subcategory $\mathbf{\dot{K}MP^{re}}$ of representable KMP's is equivalent to $\mathbf{\dot{K}ur}$.

To construct a Kuranishi structure on some moduli space $\mathcal{M}$, e.g. a moduli space of J-holomorphic curves in some (S, ω), we carry out three steps:

1. Define a 2-category $\boldsymbol{\mathcal{C}}$ and 2-functor $F : \boldsymbol{\mathcal{C}} \to \mathbf{G\dot{K}N}$, where objects A in $\boldsymbol{\mathcal{C}}$ with $F(A) = (V, E, \Gamma, s)$ correspond to families of objects in the moduli problem over the base Kuranishi neighbourhood (V, E, Γ, s).
2. Prove that $F : \boldsymbol{\mathcal{C}} \to \mathbf{G\dot{K}N}$ is a Kuranishi moduli problem.
3. Prove that $F : \boldsymbol{\mathcal{C}} \to \mathbf{G\dot{K}N}$ is representable.

Here step (1) is usually fairly brief — far shorter than constructions of curve moduli spaces in [24, 39, 52], for instance. Step (2) is also short and uses standard arguments. The major effort is in (3). Step (3) has two parts: firstly we must show that a topological space $\mathcal{M}$ naturally associated to the KMP is Hausdorff and second countable (often we can quote this from the literature), and secondly we must prove that every point of $\mathcal{M}$ admits a Kuranishi neighbourhood with a certain universal property.

We compare our approach to moduli problems with other current approaches, such as those of Fukaya–Oh–Ohta–Ono or Hofer–Wysocki–Zehnder:

- Rival approaches are basically very long ad hoc constructions, the effort is in the definition itself. In our approach we have a short-ish definition, followed by a theorem (representability of the KMP) with a long proof.
- Rival approaches may involve making many arbitrary choices to construct the moduli space. In our approach the definition of the KMP is natural, with no arbitrary choices. If the KMP is representable, the corresponding Kuranishi space $\boldsymbol{\mathcal{M}}$ is unique up to canonical equivalence in $\dot{\mathbf{K}}\mathbf{ur}$.
- In our approach, morphisms between moduli spaces, e.g. forgetting a marked point, are usually easy and require almost no work to construct.

Kuranishi moduli problems are introduced in Chapter 8 of volume I, and volume III is dedicated to constructing Kuranishi structures on moduli spaces using the KMP method.

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Chapter 1

Introduction to volume I

Kuranishi spaces were introduced in the work of Fukaya, Oh, Ohta and Ono [19–39], as the geometric structure on moduli spaces of J-holomorphic curves, which was to be used to define virtual cycles and virtual chains for such moduli spaces, for applications in Symplectic Geometry such as Gromov–Witten invariants, Lagrangian Floer cohomology, and Symplectic Field Theory.

Something which has consistently been a problem with Kuranishi spaces, since their introduction by Fukaya and Ono [39, §5] in 1999, has been to find a satisfactory definition, preferably as a category (or higher category) of geometric spaces, with a well-behaved notion of morphism, and good functorial properties. The definition used by Fukaya et al. has changed several times as their work has evolved [19–39], and others including McDuff and Wehrheim [77, 78, 80–83] have proposed their own variations.

This first volume will develop a theory of Kuranishi spaces. We use a new, more complex definition of Kuranishi space, first introduced by the author [60] in 2014, which form a 2-category **Kur**. They are *not* the same as the Kuranishi spaces of Fukaya–Oh–Ohta–Ono [19–39] (which we will call *FOOO Kuranishi spaces*), but we prove in §7.5 that any FOOO Kuranishi space $\boldsymbol{X}$ can be made into a Kuranishi space $\boldsymbol{X}'$ in our sense, uniquely up to equivalence in **Kur**. Therefore their work may be easily translated into our new language.

In fact, we give three variations on the notion of Kuranishi space:

(i) a simple 'manifold' version, 'μ-Kuranishi spaces', with trivial isotropy groups, which form an ordinary category $\boldsymbol{\mu}$**Kur** in Chapter 5;

(ii) a more complicated 'manifold' version, 'm-Kuranishi spaces', with trivial isotropy groups, which form a weak 2-category **mKur** in Chapter 4; and

(iii) the full 'orbifold' version, 'Kuranishi spaces', with finite isotropy groups, which form a weak 2-category **Kur** in Chapter 6.

These are related by an equivalence of categories $\boldsymbol{\mu}$**Kur** $\simeq$ Ho(**mKur**), where Ho(**mKur**) is the homotopy category of **mKur**, and by a full and faithful embedding **mKur** $\hookrightarrow$ **Kur**. Symplectic geometry will need Kuranishi spaces, since we allow J-holomorphic curves with finite symmetry groups, which cause finite isotropy groups at the corresponding point in the moduli space.

Our definitions start with a category of 'manifolds' $\dot{\mathbf{Man}}$ satisfying some assumptions given in Chapter 3, and yield corresponding (2-)categories of '(m- and μ-)Kuranishi spaces' $\mathbf{m\dot{K}ur}, \boldsymbol{\mu}\mathbf{\dot{K}ur}, \mathbf{\dot{K}ur}$. Here $\dot{\mathbf{Man}}$ can be the category of classical manifolds $\mathbf{Man}$, but there are many other possibilities, including the categories $\mathbf{Man^c}, \mathbf{Man^c_{st}}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$ of manifolds with corners, and generalizations, discussed in Chapter 2. This gives many different (2-)categories $\mathbf{mKur^c}, \mathbf{mKur^c_{st}}, \ldots, \boldsymbol{\mu}\mathbf{Kur^c}, \boldsymbol{\mu}\mathbf{Kur^c_{st}}, \ldots, \mathbf{Kur^c}, \mathbf{Kur^c_{st}}, \ldots$ of variations on the theme of (m- and μ-)Kuranishi spaces, useful in different problems.

Like manifolds, an (m- or μ-)Kuranishi space $\boldsymbol{X} = (X, \mathcal{K})$ is a Hausdorff, second countable topological space X with an 'atlas of charts' $\mathcal{K}$. For m- and μ-Kuranishi spaces the 'charts' are (V_i, E_i, s_i, ψ_i) for V_i a manifold, $E_i \to V_i$ a vector bundle, $s_i : V_i \to E_i$ a smooth section, and $\psi_i : s_i^{-1}(0) \to X$ a homeomorphism with an open set $\operatorname{Im}\psi_i \subseteq X$. For Kuranishi spaces the charts are $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ for V_i, E_i, s_i as above, Γ_i a finite group acting on V_i, E_i with s_i equivariant, and $\psi_i : s_i^{-1}(0)/\Gamma_i \to \operatorname{Im}\psi_i$ a homeomorphism.

As in Chapter 7, this is also true for other definitions of Kuranishi-type spaces due to Fukaya–Oh–Ohta and Ono [30, §4] and McDuff and Wehrheim [77, 78, 80–83]. The main technical innovation in our definition is our treatment of *coordinate changes* between the (m- or μ-)Kuranishi neighbourhoods on X — the 'transition functions' between the charts in the atlas.

For μ-Kuranishi spaces, coordinate changes and more general morphisms $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ are germs $[V_{ij}, \phi_{ij}, \hat{\phi}_{ij}]$ of equivalence classes of triples $(V_{ij}, \phi_{ij}, \hat{\phi}_{ij})$, where $(V_{ij}, \phi_{ij}, \hat{\phi}_{ij})$ is a generalized Fukaya–Oh–Ohta–Ono-style coordinate change, and the equivalence relation is not obvious. They have the property that *coordinate changes* $(V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ *form a sheaf on* $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$. Also, *coordinate changes are exactly the invertible morphisms between μ-Kuranishi neighbourhoods.*

For (m-)Kuranishi spaces, we have FOOO-style coordinate changes and more general 1-morphisms $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ between Kuranishi neighbourhoods, but we also introduce 2-*morphisms* $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ *between* 1-*morphisms* Φ_{ij}, Φ'_{ij}, involving germs of equivalence classes, and making (m-)Kuranishi neighbourhoods on X into a 2-category. This 2-category has the property that *coordinate changes* $(V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ *form a 2-sheaf (stack) on* $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$. Also, *coordinate changes are 1-morphisms of Kuranishi neighbourhoods which are invertible up to 2-isomorphism.*

These sheaf/stack properties of (m- and μ-)Kuranishi neighbourhoods are crucial in our theory. For example, they are essential in defining compositions $\boldsymbol{g} \circ \boldsymbol{f}$ of (1-)morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ between (m- or μ-)Kuranishi spaces $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, so that we can make (m- and μ-)Kuranishi spaces into well behaved (2-)categories $\mathbf{mKur}, \boldsymbol{\mu}\mathbf{Kur}, \mathbf{Kur}$. The lack of such a sheaf property in the Fukaya–Oh–Ohta–Ono picture is why they have no good notion of morphism between FOOO Kuranishi spaces $\boldsymbol{X}, \boldsymbol{Y}$.

An (m- or μ-)Kuranishi space $\boldsymbol{X}$ has a *virtual dimension* $\operatorname{vdim}\boldsymbol{X} \in \mathbb{Z}$, which may be negative, where $\operatorname{vdim} X = \dim V_i - \operatorname{rank} E_i$ for any (m- or μ-)Kuranishi neighbourhood (V_i, E_i, s_i, ψ_i) or $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ on $\boldsymbol{X}$.

We begin in Chapter 2 with background material on categories of manifolds with corners, of which there are several versions $\mathbf{Man^c}, \mathbf{Man^c_{st}}, \mathbf{Man^{gc}}, \ldots$. Chapter 3 states assumptions on categories $\dot{\mathbf{M}}\mathbf{an}, \dot{\mathbf{M}}\mathbf{an^c}$ of 'manifolds' and 'manifolds with corners', and explains how these assumptions allow us to do differential geometry in $\dot{\mathbf{M}}\mathbf{an}, \dot{\mathbf{M}}\mathbf{an^c}$, defining vector bundles, $E \to X$, tangent and cotangent bundles (sheaves) $\mathcal{T}X, \mathcal{T}^*X$, and so on. Detailed definitions and proofs from Chapter 3 are postponed to Appendix B.

Given a category $\dot{\mathbf{M}}\mathbf{an}$ or $\dot{\mathbf{M}}\mathbf{an^c}$ satisfying the assumptions of Chapter 3, Chapters 4–6 define (2-)categories $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}, \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}, \dot{\mathbf{K}}\mathbf{ur}$ or $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur^c}, \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}, \dot{\mathbf{K}}\mathbf{ur^c}$ of m-Kuranishi spaces, μ-Kuranishi spaces, and Kuranishi spaces, respectively. Taking $\dot{\mathbf{M}}\mathbf{an}, \dot{\mathbf{M}}\mathbf{an^c}$ to be different examples yields a large number of interesting (2-)categories $\mathbf{mKur}, \mathbf{mKur^c}, \mathbf{mKur^c_{st}}, \mathbf{mKur^{gc}}, \ldots$. We also study topics such as interesting classes of (1-)morphisms in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}, \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}, \dot{\mathbf{K}}\mathbf{ur}$, and boundaries and corners in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur^c}, \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}, \dot{\mathbf{K}}\mathbf{ur^c}$, and isotropy groups in $\dot{\mathbf{K}}\mathbf{ur}$.

Chapter 7 explains the relation of our Kuranishi spaces with other Kuranishi-type spaces defined by Fukaya, Oh, Ohta and Ono [19–39] and McDuff and Wehrheim [77, 78, 80–83]. Chapter 8 introduces *Kuranishi moduli problems*, which will be our principal tool in volume III for proving that moduli spaces of J-holomorphic curves have Kuranishi structures, and proves some theorems about them. We illustrate their use by defining a truncation functor from the polyfold theory of Hofer, Wysocki and Zehnder [46–53] to our Kuranishi spaces.

Appendix A gives background on categories and 2-categories, and Appendix B gives more detail and proofs on the differential geometry in $\dot{\mathbf{M}}\mathbf{an}, \dot{\mathbf{M}}\mathbf{an^c}$ that was outlined in Chapter 3.

Chapter 2

Manifolds with corners

We begin with background material about manifolds, manifolds with boundary, and manifolds with corners. We define the category of ordinary manifolds **Man** in §2.2 as a subcategory of the category of manifolds with corners $\mathbf{Man^c}$, and generally we treat manifolds as special cases of manifolds with corners. Some references on manifolds are Lee [71] and Lang [70], and on manifolds with boundary and corners are Melrose [85, 86] and the author [59, 64].

2.1 The definition of manifolds with corners

Definition 2.1. Use the notation $\mathbb{R}^m_k = [0,\infty)^k \times \mathbb{R}^{m-k}$ for $0 \leqslant k \leqslant m$, and write points of $\mathbb{R}^m_k$ as $u = (x_1,\ldots,x_m)$ for $x_1,\ldots,x_k \in [0,\infty)$, $x_{k+1},\ldots,x_m \in \mathbb{R}$. Let $U \subseteq \mathbb{R}^m_k$ and $V \subseteq \mathbb{R}^n_l$ be open, and $f = (f_1,\ldots,f_n) : U \to V$ be a continuous map, so that $f_j = f_j(x_1,\ldots,x_m)$ maps $U \to [0,\infty)$ for $j = 1,\ldots,l$ and $U \to \mathbb{R}$ for $j = l+1,\ldots,n$. Then we say:

(a) f is *weakly smooth* if all derivatives $\frac{\partial^{a_1+\cdots+a_m}}{\partial x_1^{a_1}\cdots\partial x_m^{a_m}} f_j(x_1,\ldots,x_m) : U \to \mathbb{R}$ exist and are continuous for all $j = 1,\ldots,n$ and $a_1,\ldots,a_m \geqslant 0$, including one-sided derivatives where $x_i = 0$ for $i = 1,\ldots,k$.

(b) f is *smooth* if it is weakly smooth and every $u = (x_1,\ldots,x_m) \in U$ has an open neighbourhood $\tilde U$ in U such that for each $j = 1,\ldots,l$, either:

  (i) we may uniquely write $f_j(\tilde x_1,\ldots,\tilde x_m) = F_j(\tilde x_1,\ldots,\tilde x_m)\cdot \tilde x_1^{a_{1,j}} \cdots \tilde x_k^{a_{k,j}}$ for all $(\tilde x_1,\ldots,\tilde x_m) \in \tilde U$, where $F_j : \tilde U \to (0,\infty)$ is weakly smooth and $a_{1,j},\ldots,a_{k,j} \in \mathbb{N} = \{0,1,2,\ldots\}$, with $a_{i,j} = 0$ if $x_i \neq 0$; or

  (ii) $f_j|_{\tilde U} = 0$.

(c) f is *interior* if it is smooth, and case (b)(ii) does not occur.

(d) f is *b-normal* if it is interior, and in case (b)(i), for each $i = 1,\ldots,k$ we have $a_{i,j} > 0$ for at most one $j = 1,\ldots,l$.

(e) f is *strongly smooth* if it is smooth, and in case (b)(i), for each $j = 1,\ldots,l$ we have $a_{i,j} = 1$ for at most one $i = 1,\ldots,k$, and $a_{i,j} = 0$ otherwise.

(f) f is *simple* if it is interior, and in case (b)(i), for each $i = 1, \ldots, k$ with $x_i = 0$ we have $a_{i,j} = 1$ for exactly one $j = 1, \ldots, l$ and $a_{i,j} = 0$ otherwise, and for all $j = 1, \ldots, l$ we have $a_{i,j} = 1$ for at most one $i = 1, \ldots, k$.

(g) f is a *diffeomorphism* if it is a smooth bijection with smooth inverse.

All the classes (a)–(g) include identities and are closed under composition.

Definition 2.2. Let X be a second countable Hausdorff topological space. An *m-dimensional chart on X* is a pair (U, ϕ), where $U \subseteq \mathbb{R}^m_k$ is open for some $0 \leqslant k \leqslant m$, and $\phi : U \to X$ is a homeomorphism with an open set $\phi(U) \subseteq X$.

Let $(U, \phi), (V, \psi)$ be m-dimensional charts on X. We call (U, ϕ) and (V, ψ) *compatible* if $\psi^{-1} \circ \phi : \phi^{-1}\bigl(\phi(U) \cap \psi(V)\bigr) \to \psi^{-1}\bigl(\phi(U) \cap \psi(V)\bigr)$ is a diffeomorphism between open subsets of $\mathbb{R}^m_k, \mathbb{R}^m_l$, in the sense of Definition 2.1(g).

An *m-dimensional atlas* for X is a system $\{(U_a, \phi_a) : a \in A\}$ of pairwise compatible m-dimensional charts on X with $X = \bigcup_{a \in A} \phi_a(U_a)$. We call such an atlas *maximal* if it is not a proper subset of any other atlas. Any atlas $\{(U_a, \phi_a) : a \in A\}$ is contained in a unique maximal atlas, the set of all charts (U, ϕ) of this type on X which are compatible with (U_a, ϕ_a) for all $a \in A$.

An *m-dimensional manifold with corners* is a second countable Hausdorff topological space X equipped with a maximal m-dimensional atlas. Usually we refer to X as the manifold, leaving the atlas implicit, and by a *chart* (U, ϕ) *on* X, we mean an element of the maximal atlas.

Now let X, Y be manifolds with corners of dimensions m, n, and $f : X \to Y$ a continuous map. We call f *weakly smooth*, or *smooth*, or *interior*, or *b-normal*, or *strongly smooth*, or *simple*, if whenever $(U, \phi), (V, \psi)$ are charts on X, Y with $U \subseteq \mathbb{R}^m_k$, $V \subseteq \mathbb{R}^n_l$ open, then

$$\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}(\psi(V)) \longrightarrow V \tag{2.1}$$

is weakly smooth, or smooth, ..., or simple, respectively, as maps between open subsets of $\mathbb{R}^m_k, \mathbb{R}^n_l$ in the sense of Definition 2.1.

We write $\mathbf{Man^c}$ for the category with objects manifolds with corners X, Y, and morphisms smooth maps $f : X \to Y$ in the sense above. We will also write $\mathbf{Man^c_{in}}$, $\mathbf{Man^c_{bn}}$, $\mathbf{Man^c_{st}}$, $\mathbf{Man^c_{st,in}}$, $\mathbf{Man^c_{st,bn}}$, $\mathbf{Man^c_{si}}$ for the subcategories of $\mathbf{Man^c}$ with morphisms interior maps, and b-normal maps, and strongly smooth maps, and strongly smooth interior maps, and strongly smooth b-normal maps, and simple maps, respectively.

We write $\mathbf{Man^c_{we}}$ for the category with objects manifolds with corners and morphisms weakly smooth maps.

Remark 2.3. There are several non-equivalent definitions of categories of manifolds with corners. Just as objects, without considering morphisms, most authors define manifolds with corners as in Definition 2.2. However, Melrose [84–86] imposes an extra condition: in §2.2 we will define the boundary ∂X of a manifold with corners X, with an immersion $i_X : \partial X \to X$. Melrose requires that $i_X|_C : C \to X$ should be injective for each connected component C of ∂X (such X are sometimes called *manifolds with faces*).

There is no general agreement in the literature on how to define smooth maps, or morphisms, of manifolds with corners:

(i) Our smooth maps are due to Melrose [86, §1.12], [84, §1], who calls them *b-maps*. Interior and b-normal maps are also due to Melrose.

(ii) The author [59] defined and studied strongly smooth maps above (which were just called 'smooth maps' in [59]).

(iii) Monthubert's *morphisms of manifolds with corners* [91, Def. 2.8] coincide with our strongly smooth b-normal maps.

(iv) Most other authors, such as Cerf [11, §I.1.2], define smooth maps of manifolds with corners to be weakly smooth maps, in our notation.

2.2 Boundaries and corners of manifolds with corners

The material of this section broadly follows the author [59, 64].

Definition 2.4. Let $U \subseteq \mathbb{R}^m_k$ be open. For each $u = (x_1, \ldots, x_m)$ in U, define the *depth* $\operatorname{depth}_U u$ of u in U to be the number of $x_1, \ldots, x_k$ which are zero. That is, $\operatorname{depth}_U u$ is the number of boundary faces of U containing u.

Let X be an m-manifold with corners. For $x \in X$, choose a chart (U, ϕ) on the manifold X with $\phi(u) = x$ for $u \in U$, and define the *depth* $\operatorname{depth}_X x$ of x in X by $\operatorname{depth}_X x = \operatorname{depth}_U u$. This is independent of the choice of (U, ϕ). For each $l = 0, \ldots, m$, define the *depth l stratum* of X to be

$$S^l(X) = \bigl\{x \in X : \operatorname{depth}_X x = l\bigr\}.$$

Then $X = \coprod_{l=0}^m S^l(X)$ and $\overline{S^l(X)} = \bigcup_{k=l}^m S^k(X)$. The *interior* of X is $X^\circ = S^0(X)$. Each $S^l(X)$ has the structure of an $(m-l)$-manifold without boundary.

The following lemma is easy to prove from Definition 2.1(b).

Lemma 2.5. *Let $f : X \to Y$ be a smooth map of manifolds with corners. Then f* ***is compatible with the depth stratifications*** *$X = \coprod_{k\geqslant 0} S^k(X)$, $Y = \coprod_{l\geqslant 0} S^l(Y)$ in Definition 2.4, in the sense that if $\emptyset \neq W \subseteq S^k(X)$ is a connected subset for some $k \geqslant 0$, then $f(W) \subseteq S^l(Y)$ for some unique $l \geqslant 0$.*

The analogue of Lemma 2.5 is false for weakly smooth maps, so the functorial properties of corners below are false for $\mathbf{Man^c_{we}}$.

Definition 2.6. Let X be an m-manifold with corners, $x \in X$, and $k = 0, 1, \ldots, m$. A *local k-corner component γ of X at x* is a local choice of connected component of $S^k(X)$ near x. That is, for each small open neighbourhood V of x in X, γ gives a choice of connected component W of $V \cap S^k(X)$ with $x \in \overline{W}$, and any two such choices V, W and V', W' must be compatible in that $x \in \overline{(W \cap W')}$. When $k = 1$, we call γ a *local boundary component*.

As sets, define the *boundary* ∂X and *k-corners*Imanifold with corners!k-corners Ck(X)@k-corners $C_k(X)$!definition $C_k(X)$ for $k = 0, 1, \ldots, m$ by

$$\partial X = \{(x, \beta) : x \in X,\ \beta \text{ is a local boundary component of } X \text{ at } x\},$$
$$C_k(X) = \{(x, \gamma) : x \in X,\ \gamma \text{ is a local } k\text{-corner component of } X \text{ at } x\}.$$

Define $i_X : \partial X \to X$ and $\Pi_k : C_k(X) \to X$ by $i_X : (x, \beta) \mapsto x$, $\Pi_k : (x, \gamma) \mapsto x$.

If (U, ϕ) is a chart on X with $U \subseteq \mathbb{R}^m_k$ open, then for each $i = 1, \ldots, k$ we can define a chart (U_i, ϕ_i) on ∂X by

$$U_i = \{(x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}_{k-1} : (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{m-1}) \in U \subseteq \mathbb{R}^m_k\},$$
$$\phi_i : (x_1, \ldots, x_{m-1}) \longmapsto \big(\phi(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{m-1}), \phi_*(\{x_i = 0\})\big).$$

The set of all such charts on ∂X forms an atlas, making ∂X into a manifold with corners of dimension $m - 1$, and $i_X : \partial X \to X$ into a smooth (but not interior) map. Similarly, we make $C_k(X)$ into an $(m - k)$-manifold with corners, and $\Pi_k : C_k(X) \to X$ into a smooth map. We have $\partial X = C_1(X)$.

We call X a *manifold without boundary* (or just a *manifold*) if $\partial X = \emptyset$, and a *manifold with boundary* if $\partial^2 X = \emptyset$. We write $\mathbf{Man}$ and $\mathbf{Man^b}$ for the full subcategories of $\mathbf{Man^c}$ with objects manifolds without boundary, and manifolds with boundary, so that $\mathbf{Man} \subset \mathbf{Man^b} \subset \mathbf{Man^c}$. This definition of $\mathbf{Man}$ is equivalent to the usual definition of the category of manifolds. We also write $\mathbf{Man^b_{in}}, \mathbf{Man^b_{si}}$ for the subcategories of $\mathbf{Man^b}$ with morphisms interior maps, and simple maps.

For X a manifold with corners and $k \geqslant 0$, there are natural identifications

$$\begin{aligned}\partial^k X \cong \{&(x, \beta_1, \ldots, \beta_k) : x \in X,\ \beta_1, \ldots, \beta_k \text{ are distinct} \\ &\text{local boundary components for } X \text{ at } x\},\end{aligned} \tag{2.2}$$

$$\begin{aligned}C_k(X) \cong \{&(x, \{\beta_1, \ldots, \beta_k\}) : x \in X,\ \beta_1, \ldots, \beta_k \text{ are distinct} \\ &\text{local boundary components for } X \text{ at } x\}.\end{aligned} \tag{2.3}$$

There is a natural, free, smooth action of the symmetric group S_k on $\partial^k X$, by permutation of $\beta_1, \ldots, \beta_k$ in (2.2), and (2.2)–(2.3) give a natural diffeomorphism

$$C_k(X) \cong \partial^k X / S_k. \tag{2.4}$$

Corners commute with boundaries: there are natural isomorphisms

$$\begin{aligned}\partial C_k(X) \cong C_k(\partial X) \cong \{&(x, \{\beta_1, \ldots, \beta_k\}, \beta_{k+1}) : x \in X,\ \beta_1, \ldots, \beta_{k+1} \\ &\text{are distinct local boundary components for } X \text{ at } x\}.\end{aligned} \tag{2.5}$$

For products of manifolds with corners we have natural diffeomorphisms

$$\partial(X \times Y) \cong (\partial X \times Y) \amalg (X \times \partial Y), \tag{2.6}$$
$$C_k(X \times Y) \cong \textstyle\coprod_{i,j \geqslant 0,\ i+j=k} C_i(X) \times C_j(Y). \tag{2.7}$$

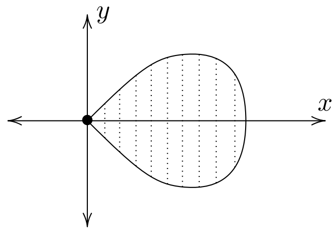


Figure 2.1: The teardrop, a 2-manifold with corners

Example 2.7. The *teardrop* $T = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ y^2 \leqslant x^2 - x^4\}$, shown in Figure 2.1, is a manifold with corners of dimension 2. The boundary ∂T is diffeomorphic to $[0, 1]$, and so is connected, but $i_T : \partial T \to T$ is not injective. Thus T is not a manifold with faces, in the sense of Remark 2.3.

It is *not* true that general smooth $f : X \to Y$ induce maps $\partial f : \partial X \to \partial Y$ or $C_k(f) : C_k(Y) \to C_k(Y)$, though this is true for simple maps f. For example, if $f : X \to Y$ is the inclusion $[0, \infty) \hookrightarrow \mathbb{R}$ then no map $\partial f : \partial X \to \partial Y$ exists, as $\partial X \neq \emptyset$ and $\partial Y = \emptyset$. However, by working in an enlarged category $\check{\mathbf{Man}}^{\mathbf{c}}$ of manifolds with corners of mixed dimension and considering $C(X) = \coprod_{k \geqslant 0} C_k(X)$, we can define a functor.

Definition 2.8. Write $\check{\mathbf{Man}}^{\mathbf{c}}$ for the category whose objects are disjoint unions $\coprod_{m=0}^{\infty} X_m$, where X_m is a manifold with corners of dimension m, allowing $X_m = \emptyset$, and whose morphisms are continuous maps $f : \coprod_{m=0}^{\infty} X_m \to \coprod_{n=0}^{\infty} Y_n$, such that $f|_{X_m \cap f^{-1}(Y_n)} : X_m \cap f^{-1}(Y_n) \to Y_n$ is a smooth map of manifolds with corners for all $m, n \geqslant 0$. Objects of $\check{\mathbf{Man}}^{\mathbf{c}}$ will be called *manifolds with corners of mixed dimension*. We will also write $\check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}}, \check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{st}}$ for the subcategories of $\check{\mathbf{Man}}^{\mathbf{c}}$ with morphisms interior maps, and strongly smooth maps.

Definition 2.9. Define the *corners* $C(X)$ of a manifold with corners X by

$$\begin{aligned} C(X) &= \textstyle\coprod_{k=0}^{\dim X} C_k(X) \\ &= \{(x, \gamma) : x \in X,\ \gamma \text{ is a local } k\text{-corner component of } X \text{ at } x,\ k \geqslant 0\}, \end{aligned}$$

considered as an object of $\check{\mathbf{Man}}^{\mathbf{c}}$ in Definition 2.8, a manifold with corners of mixed dimension. Define $\Pi : C(X) \to X$ by $\Pi : (x, \gamma) \mapsto x$. This is smooth (i.e. a morphism in $\check{\mathbf{Man}}^{\mathbf{c}}$) as the maps $\Pi_k : C_k(X) \to X$ are smooth for $k \geqslant 0$.

Let $f : X \to Y$ be a smooth map of manifolds with corners, and suppose γ is a local k-corner component of X at $x \in X$. For each sufficiently small open neighbourhood V of x in X, γ gives a choice of connected component W of $V \cap S^k(X)$ with $x \in \overline{W}$, so by Lemma 2.5 $f(W) \subseteq S^l(Y)$ for some $l \geqslant 0$. As f is continuous, $f(W)$ is connected, and $f(x) \in \overline{f(W)}$. Thus there is a unique local l-corner component $f_*(\gamma)$ of Y at $f(x)$, such that if $\tilde{V}$ is a sufficiently small open neighbourhood of $f(x)$ in Y, then the connected component $\tilde{W}$ of $\tilde{V} \cap S^l(Y)$ given by $f_*(\gamma)$ has $f(W) \cap \tilde{W} \neq \emptyset$. This $f_*(\gamma)$ is independent of the choice of sufficiently small $V, \tilde{V}$, so is well-defined.

Define a map $C(f) : C(X) \to C(Y)$ by $C(f) : (x, \gamma) \mapsto (f(x), f_*(\gamma))$. Then $C(f)$ is an interior morphism in $\check{\mathbf{M}}\mathbf{an^c}$. If $g : Y \to Z$ is another smooth map of manifolds with corners then $C(g \circ f) = C(g) \circ C(f) : C(X) \to C(Z)$, so $C : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c_{in}} \subset \check{\mathbf{M}}\mathbf{an^c}$ is a functor, which we call a *corner functor*.

From [64, Prop. 2.11] we have:

Proposition 2.10. *Let $f : X \to Y$ be a morphism in* $\mathbf{Man^c}$. *Then*

(a) *f is interior if and only if $C(f)$ maps $C_0(X) \to C_0(Y)$.*

(b) *f is b-normal if and only if $C(f)$ maps $C_k(X) \to \coprod_{l=0}^{k} C_l(Y)$ for all k.*

(c) *If f is simple then $C(f)$ maps $C_k(X) \to C_k(Y)$ for all $k \geqslant 0$, and $C_k(f) := C(f)|_{C_k(X)} : C_k(X) \to C_k(Y)$ is also a simple map.*

Thus we have a ***boundary functor*** $\partial : \mathbf{Man^c_{si}} \to \mathbf{Man^c_{si}}$ *mapping $X \mapsto \partial X$ on objects and $f \mapsto \partial f := C(f)|_{C_1(X)} : \partial X \to \partial Y$ on (simple) morphisms $f : X \to Y$, and for all $k \geqslant 0$ a* ***k-corner functor*** $C_k : \mathbf{Man^c_{si}} \to \mathbf{Man^c_{si}}$ *mapping $X \mapsto C_k(X)$ on objects and $f \mapsto C_k(f) := C(f)|_{C_k(X)} : C_k(X) \to C_k(Y)$ on (simple) morphisms.*

As in [59, Def. 4.5] there is also a second corner functor on $\mathbf{Man^c}$, which we write as $C' : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$.

Definition 2.11. Define $C'(X) = C(X)$ in $\check{\mathbf{M}}\mathbf{an^c}$ for each X in $\mathbf{Man^c}$.

Let $f : X \to Y$ be a smooth map of manifolds with corners. Define a map $C'(f) : C'(X) \to C'(Y)$ by $C'(f) : (x, \gamma) \mapsto (y, \delta)$, where $y = f(x)$ in Y, and δ is the unique maximal local corner component of Y at y with the property that if V is an open neighbourhood of y in Y and $a : V \to [0, \infty)$ is smooth with $a(y) = a \circ f(x) = 0$ and $a \circ f|_\gamma = 0$ then $a|_\delta = 0$.

Here δ is *maximal* means that if $\tilde\delta$ is any other local corner component with this property then $\dim \delta \geqslant \dim \tilde\delta$ (so that $\operatorname{codim} \delta \leqslant \operatorname{codim} \tilde\delta$) and $\tilde\delta$ is contained in the closure of δ. By considering local models in coordinates we can show that $C'(f) : C'(X) \to C'(Y)$ is a morphism in $\check{\mathbf{M}}\mathbf{an^c}$, and that this defines a functor $C' : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$, which we also call a *corner functor*.

The next proposition is easy:

Proposition 2.12. *Let $f : X \to Y$ be a morphism in* $\mathbf{Man^c}$. *Then $C'(f)$ maps $C_0(X) \to C_0(Y)$, and $C'(f) = C(f)$ if and only if f is interior.*

By Proposition 2.10(c), *this implies that if f is simple (hence interior) then $C'(f) = C(f)$ maps $C_k(X) \to C_k(Y)$ for all $k \geqslant 0$, and $C_k(f) := C'(f)|_{C_k(X)} : C_k(X) \to C_k(Y)$ is also a simple map.*

Equations (2.5) and (2.7) imply that if X, Y are manifolds with corners, we have natural isomorphisms

$$\partial C(X) \cong C(\partial X), \tag{2.8}$$

$$C(X \times Y) \cong C(X) \times C(Y). \tag{2.9}$$

The corner functors C, C' *preserve products and direct products*. That is, if $f : W \to Y$, $g : X \to Y$, $h : X \to Z$ are smooth then the following commute

$$\begin{tikzcd} C(W \times X) \arrow[r, "C(f\times h)"'] \arrow[d, "\cong"] & C(Y \times Z) \arrow[d, "\cong"'] \\ C(W)\times C(X) \arrow[r, "C(f)\times C(h)"] & C(Y)\times C(Z), \end{tikzcd} \qquad \begin{tikzcd} & C(Y \times Z) \arrow[d, "\cong"] \\ C(X) \arrow[ur, "C((g,h))"] \arrow[r, "(C(g),C(h))"'] & C(Y)\times C(Z), \end{tikzcd}$$

$$\begin{tikzcd} C'(W \times X) \arrow[r, "C'(f\times h)"'] \arrow[d, "\cong"] & C'(Y \times Z) \arrow[d, "\cong"'] \\ C'(W)\times C'(X) \arrow[r, "C'(f)\times C'(h)"] & C'(Y)\times C'(Z), \end{tikzcd} \qquad \begin{tikzcd} & C'(Y \times Z) \arrow[d, "\cong"] \\ C'(X) \arrow[ur, "C'((g,h))"] \arrow[r, "(C'(g),C'(h))"'] & C'(Y)\times C'(Z), \end{tikzcd}$$

where the columns are the isomorphisms (2.9).

Example 2.13. **(a)** Let $X = [0,\infty)$, $Y = [0,\infty)^2$, and define $f : X \to Y$ by $f(x) = (x, x)$. We have

$$\begin{aligned} &C_0(X) \cong [0,\infty), \qquad C_1(X) \cong \{0\}, \qquad C_0(Y) \cong [0,\infty)^2, \\ &C_1(Y) \cong \bigl(\{0\} \times [0,\infty)\bigr) \amalg \bigl([0,\infty) \times \{0\}\bigr), \qquad C_2(Y) \cong \{(0,0)\}. \end{aligned}$$

Then $C(f)$ maps $C_0(X) \to C_0(Y)$, $x \mapsto (x, x)$, and $C_1(X) \to C_2(Y)$, $0 \mapsto (0, 0)$. Also $C'(f) = C(f)$, as f is interior.

(b) Let $X = *$, $Y = [0,\infty)$ and define $f : X \to Y$ by $f(*) = 0$. Then $C_0(X) \cong *$, $C_0(Y) \cong [0,\infty)$, $C_1(Y) \cong \{0\}$, and $C(f)$ maps $C_0(X) \to C_1(Y)$, $* \mapsto 0$, but $C'(f)$ maps $C_0(X) \to C_0(Y)$, $* \mapsto 0$, so $C'(f) \neq C(f)$.

Note that $C(f), C'(f)$ need not map $C_k(X) \to C_k(Y)$.

2.3 Tangent bundles and b-tangent bundles

Manifolds with corners X have two notions of tangent bundle with functorial properties, the (*ordinary*) *tangent bundle* TX, the obvious generalization of tangent bundles of manifolds without boundary, and the *b-tangent bundle* ${}^bTX$ introduced by Melrose [84, §2], [85, §2.2], [86, §I.10]. Taking duals gives two notions of cotangent bundle T^*X, ${}^bT^*X$. First we discuss vector bundles:

Definition 2.14. Let X be a manifold with corners. A *vector bundle* $E \to X$ *of rank* k is a manifold with corners E and a smooth map $\pi : E \to X$, such that each fibre $E_x := \pi^{-1}(x)$ for $x \in X$ is given the structure of a real vector space of dimension k, and X may be covered by open $U \subseteq X$ with diffeomorphisms $\pi^{-1}(U) \cong U \times \mathbb{R}^k$ identifying $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ with the projection $U \times \mathbb{R}^k \to \mathbb{R}^k$, and the vector space structure on E_x with that on $\{x\} \times \mathbb{R}^k \cong \mathbb{R}^k$, for each $x \in U$. A *section* of E is a smooth map $s : X \to E$ with $\pi \circ s = \mathrm{id}_X$.

We write $\Gamma^\infty(E)$ for the vector space of smooth sections of E, and $C^\infty(X)$ for the $\mathbb{R}$-algebra of smooth functions $X \to \mathbb{R}$. Then $\Gamma^\infty(E)$ is a $C^\infty(X)$-module.

Morphisms of vector bundles, dual vector bundles, tensor products of vector bundles, exterior products, and so on, all work as usual.

Definition 2.15. Let X be an m-manifold with corners. The *tangent bundle* $\pi : TX \to X$ and *b-tangent bundle* $\pi : {}^bTX \to X$ are natural rank m vector bundles on X, with a vector bundle morphism $I_X : {}^bTX \to TX$. The fibres of $TX, {}^bTX$ at $x \in X$ are written $T_xX, {}^bT_xX$. We may describe $TX, {}^bTX, I_X$ in local coordinates as follows.

If (U, ϕ) is a chart on X, with $U \subseteq \mathbb{R}^m_k$ open, and $(x_1, \ldots, x_m)$ are the coordinates on U, then over $\phi(U)$, TX is the trivial vector bundle with basis of sections $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}$, and ${}^bTX$ is the trivial vector bundle with basis of sections $x_1 \frac{\partial}{\partial x_1}, \ldots, x_k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_m}$.

We have corresponding charts $(TU, T\phi)$ on TX and $({}^bTU, {}^bT\phi)$ on ${}^bTX$, where $TU = {}^bTU = U \times \mathbb{R}^m \subseteq \mathbb{R}^{2m}_k$, such that $(x_1, \ldots, x_m, q_1, \ldots, q_m)$ in TU represents the vector $q_1 \frac{\partial}{\partial x_1} + \cdots + q_m \frac{\partial}{\partial x_m}$ over $\phi(x_1, \ldots, x_m) \in X$, and $(x_1, \ldots, x_m, r_1, \ldots, r_m)$ in ${}^bTU$ represents $r_1 x_1 \frac{\partial}{\partial x_1} + \cdots + r_k x_k \frac{\partial}{\partial x_k} + r_{k+1} \frac{\partial}{\partial x_{k+1}} + \cdots + r_m \frac{\partial}{\partial x_m}$ over $\phi(x_1, \ldots, x_m)$ in X, and I_X maps $(x_1, \ldots, x_m, r_1, \ldots, r_m)$ in ${}^bTU$ to $(x_1, \ldots, x_m, x_1 r_1, \ldots, x_k r_k, r_{k+1}, \ldots, r_m)$ in TU.

Under change of coordinates $(x_1, \ldots, x_m) \rightsquigarrow (\tilde{x}_1, \ldots, \tilde{x}_m)$ from (U, ϕ) to $(\tilde{U}, \tilde{\phi})$, the corresponding change $(x_1, \ldots, x_m, q_1, \ldots, q_m) \rightsquigarrow (\tilde{x}_1, \ldots, \tilde{q}_m)$ from $(TU, T\phi)$ to $(T\tilde{U}, T\tilde{\phi})$ is determined by $\frac{\partial}{\partial x_i} = \sum_{j=1}^m \frac{\partial \tilde{x}_j}{\partial x_i}(x_1, \ldots, x_m) \cdot \frac{\partial}{\partial \tilde{x}_j}$, so that $\tilde{q}_j = \sum_{i=1}^m \frac{\partial \tilde{x}_j}{\partial x_i}(x_1, \ldots, x_m) q_i$, and similarly for $({}^bTU, {}^bT\phi), ({}^bT\tilde{U}, {}^bT\tilde{\phi})$.

Elements of $\Gamma^\infty(TX)$ are called *vector fields*, and of $\Gamma^\infty({}^bTX)$ are called *b-vector fields*. The map $(I_X)_* : \Gamma^\infty({}^bTX) \to \Gamma^\infty(TX)$ is injective, and identifies $\Gamma^\infty({}^bTX)$ with the vector subspace of $v \in \Gamma^\infty(TX)$ such that $v|_{S^k(X)}$ is tangent to $S^k(X)$ for all $k = 1, \ldots, \dim X$.

Taking duals gives two notions of cotangent bundle $T^*X, {}^bT^*X$. The fibres of $T^*X, {}^bT^*X$ at $x \in X$ are written $T^*_xX, {}^bT^*_xX$.

Now suppose $f : X \to Y$ is a smooth map of manifolds with corners. Then there is a natural smooth map $Tf : TX \to TY$ so that the following commutes:

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "\pi"] & TY \arrow[d, "\pi"'] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Let (U, ϕ) and (V, ψ) be coordinate charts on X, Y with $U \subseteq \mathbb{R}^m_k$, $V \subseteq \mathbb{R}^n_l$, with coordinates $(x_1, \ldots, x_m) \in U$ and $(y_1, \ldots, y_n) \in V$, and let $(TU, T\phi)$, $(TV, T\psi)$ be the corresponding charts on TX, TY, with coordinates $(x_1, \ldots, x_m, q_1, \ldots, q_m) \in TU$ and $(y_1, \ldots, y_n, r_1, \ldots, r_n) \in TV$. Equation (2.1) defines a map $\psi^{-1} \circ f \circ \phi$ between open subsets of U, V. Write $\psi^{-1} \circ f \circ \phi = (f_1, \ldots, f_n)$, for $f_j = f_j(x_1, \ldots, x_m)$. Then the corresponding $T\psi^{-1} \circ Tf \circ T\phi$ maps

$$\begin{aligned} &T\psi^{-1} \circ Tf \circ T\phi : (x_1, \ldots, x_m, q_1, \ldots, q_m) \longmapsto \bigl(f_1(x_1, \ldots, x_m), \ldots, \\ &f_n(x_1, \ldots, x_m), \textstyle\sum_{i=1}^m \frac{\partial f_1}{\partial x_i}(x_1, \ldots, x_m) q_i, \ldots, \sum_{i=1}^m \frac{\partial f_n}{\partial x_i}(x_1, \ldots, x_m) q_i\bigr). \end{aligned}$$

We can also regard Tf as a vector bundle morphism $\mathrm{d}f : TX \to f^*(TY)$ on X, which has dual morphism $\mathrm{d}f : f^*(T^*Y) \to T^*X$. If $x \in X$ with $f(x) = y$ in Y we have linear maps $T_xf : T_xX \to T_yY$ and $T^*_xf : T^*_yY \to T^*_xX$ on the fibres.

If $g: Y \to Z$ is smooth then $T(g \circ f) = Tg \circ Tf : TX \to TZ$, and $T(\mathrm{id}_X) = \mathrm{id}_{TX} : TX \to TX$. Thus, the assignment $X \mapsto TX$, $f \mapsto Tf$ is a functor, the *tangent functor* $T : \mathbf{Man^c} \to \mathbf{Man^c}$. It restricts to $T : \mathbf{Man^c_{in}} \to \mathbf{Man^c_{in}}$.

As in [84, §2], the analogue of the morphisms $Tf : TX \to TY$ for b-tangent bundles works only for *interior* maps $f : X \to Y$. So let $f : X \to Y$ be an interior map of manifolds with corners. If f is interior, there is a unique interior map ${}^bTf : {}^bTX \to {}^bTY$ so that the following commutes:

$$\begin{tikzcd} {}^bTX \arrow[rrr, "{}^bTf"'] \arrow[dr, "I_X"] \arrow[ddr, "\pi"'] & & & {}^bTY \arrow[dr, "I_Y"] \arrow[ddr, "\pi"'] & \\ & TX \arrow[rrr, "Tf"] \arrow[d, "\pi"] & & & TY \arrow[d, "\pi"] \\ & X \arrow[rrr, "f"] & & & Y. \end{tikzcd} \tag{2.10}$$

The assignment $X \mapsto {}^bTX$, $f \mapsto {}^bTf$ is a functor, the *b-tangent functor* ${}^bT : \mathbf{Man^c_{in}} \to \mathbf{Man^c_{in}}$. The maps $I_X : {}^bTX \to TX$ give a natural transformation $I : {}^bT \to T$ of functors $\mathbf{Man^c_{in}} \to \mathbf{Man^c_{in}}$.

We can also regard ${}^bTf$ as a vector bundle morphism ${}^b\mathrm{d}f : {}^bTX \to f^*({}^bTY)$ on X, with dual morphism ${}^b\mathrm{d}f : f^*({}^bT^*Y) \to {}^bT^*X$. If $x \in X$ with $f(x) = y$ in Y we have linear maps ${}^bT_xf : {}^bT_xX \to {}^bT_yY$ and ${}^bT^*_xf : {}^bT^*_yY \to {}^bT^*_xX$.

Note that if $f : X \to Y$ is a smooth map in $\mathbf{Man^c}$ then $C(f) : C(X) \to C(Y)$ is interior, so ${}^bTC(f) : {}^bTC(X) \to {}^bTC(Y)$ is well defined, and we can use this as a substitute for ${}^bTf : {}^bTX \to {}^bTY$ when f is not interior.

Let X be a manifold with corners, and $k \geqslant 0$. Then we have an exact sequence of vector bundles on $C_k(X)$:

$$0 \longrightarrow T(C_k(X)) \xrightarrow{\mathrm{d}\Pi_k} \Pi_k^*(TX) \longrightarrow N_{C_k(X)} \longrightarrow 0, \tag{2.11}$$

where $N_{C_k(X)}$ is the *normal bundle of* $C_k(X)$ *in* X, a natural rank k vector bundle on $C_k(X)$. When $k = 1$ this becomes

$$0 \longrightarrow T(\partial X) \xrightarrow{\mathrm{d}i_X} i_X^*(TX) \longrightarrow N_{\partial X} \longrightarrow 0. \tag{2.12}$$

Here the normal line bundle $N_{\partial X}$ has a natural orientation on its fibres, by outward-pointing vectors. Using (2.12) and the orientation on $N_{\partial X}$, we can show that an orientation on X induces an orientation on ∂X, as in §2.6.

For b-tangent bundles, as in [64, Prop. 2.22] there is an analogue of (2.11):

$$0 \longrightarrow {}^bN_{C_k(X)} \longrightarrow \Pi_k^*({}^bTX) \xrightarrow{I_X^\diamond} {}^bT(C_k(X)) \longrightarrow 0, \tag{2.13}$$

where ${}^bN_{C_k(X)}$ is the *b-normal bundle of* $C_k(X)$ *in* X, a rank k vector bundle with a natural flat connection. Note that (2.13) goes in the opposite direction to (2.11). There is no natural map ${}^b\mathrm{d}\Pi_k : {}^bT(C_k(X)) \to \Pi_k^*({}^bTX)$ for $k > 0$, as Π_k is not interior. We can define $I_X^\diamond$ in (2.13) by noting that $(I_X)_* : \Gamma^\infty({}^bTX) \to \Gamma^\infty(TX)$ identifies $\Gamma^\infty({}^bTX)$ with the vector subspace of v in $\Gamma^\infty(TX)$ with $v|_{S^l(X)}$ tangent to $S^l(X)$ for all l, as in Definition 2.15, and under this identification, $I_X^\diamond$ is just restriction/pullback of vector fields from X to $C_k(X)$. When $k=1$, ${}^bN_{C_1(X)}$ is naturally trivial, giving an exact sequence

$$0 \longrightarrow \mathcal{O}_{\partial X} \longrightarrow i_X^*({}^bTX) \xrightarrow{I_X^\diamond} {}^bT(\partial X) \longrightarrow 0, \qquad (2.14)$$

where $\mathcal{O}_{\partial X} = \partial X \times \mathbb{R} \to \partial X$ is the trivial line bundle on ∂X.

Here is some similar notation to $N_{C_k(X)}, {}^bN_{C_k(X)}$, but working over X rather than $C(X)$, taken from [64, Def. 2.25].

Definition 2.16. Let X be a manifold with corners. For $x \in S^k(X) \subseteq X$, we have a natural exact sequence of real vector spaces

$$0 \longrightarrow T_x(S^k(X)) \xrightarrow{\iota_x X} T_xX \xrightarrow{\pi_x X} \tilde{N}_xX \longrightarrow 0, \qquad (2.15)$$

where $\dim \tilde{N}_xX = k$. We call $\tilde{N}_xX$ the *stratum normal space*. There is a unique point $x' \in C_k(X)$ with $\Pi_k(x') = x$, and then $\tilde{N}_xX \cong N_{C_k(X)}|_{x'}$, and $T_x(S^k(X)) \cong {}^bT(C_k(X))|_{x'}$, and (2.15) is canonically isomorphic to the restriction of (2.11) to x'.

Let $f : X \to Y$ be a morphism in $\mathbf{Man^c}$, and let $x \in S^k(X) \subseteq X$ with $f(x) = y \in S^l(Y) \subseteq Y$. Then f maps $S^k(X) \to S^l(Y)$ near x by Lemma 2.5. There is a unique linear map $\tilde{N}_xf : \tilde{N}_xX \to \tilde{N}_yY$, the *stratum normal map*, fitting into the following commutative diagram, where the rows are (2.15):

$$\begin{tikzcd}
0 \arrow[r] & T_x(S^k(X)) \arrow[r, "\iota_x X"'] \arrow[d, "T_x(f|_{S^k(X)})"] & T_xX \arrow[r, "\pi_x X"'] \arrow[d, "T_xf"] & \tilde{N}_xX \arrow[r] \arrow[d, dotted, "\tilde{N}_xf"] & 0 \\
0 \arrow[r] & T_y(S^l(Y)) \arrow[r, "\iota_y Y"] & T_yY \arrow[r, "\pi_y Y"] & \tilde{N}_yY \arrow[r] & 0.
\end{tikzcd} \qquad (2.16)$$

These morphisms $\tilde{N}_xf$ are functorial in f and x. That is, if $g : Y \to Z$ is another morphism in $\mathbf{Man^c}$ then $\tilde{N}_x(g \circ f) = \tilde{N}_yg \circ \tilde{N}_xf$.

There is also a 'b-tangent' version. Let X be a manifold with corners. For each $x \in S^k(X) \subseteq X$, we have a natural exact sequence of real vector spaces

$$0 \longrightarrow {}^b\tilde{N}_xX \xrightarrow{{}^b\iota_x X} {}^bT_xX \xrightarrow{\Pi_x X} T_x(S^k(X)) \longrightarrow 0, \qquad (2.17)$$

where $\dim {}^b\tilde{N}_xX = k$. We call ${}^b\tilde{N}_xX$ the *stratum b-normal space*. There is a unique point $x' \in C_k(X)$ with $\Pi_k(x') = x$, and then ${}^b\tilde{N}_xX \cong {}^bN_{C_k(X)}|_{x'}$, and $T_x(S^k(X)) \cong {}^bT(C_k(X))|_{x'}$, and (2.17) is canonically isomorphic to the restriction of (2.13) to x'.

Note that the $\tilde{N}_xX, {}^b\tilde{N}_xX$ for $x \in X$ are not the fibres of vector bundles on X, as $\dim \tilde{N}_xX, \dim {}^b\tilde{N}_xX$ are only upper semicontinuous in x.

If $(x_1, \ldots, x_m) \in \mathbb{R}^m_k$ are local coordinates on X near x then we have

$${}^b\tilde{N}_xX = \bigl\langle x_1 \tfrac{\partial}{\partial x_1}, \ldots, x_k \tfrac{\partial}{\partial x_k} \bigr\rangle_{\mathbb{R}}, \qquad T_x(S^k(X)) = \bigl\langle \tfrac{\partial}{\partial x_{k+1}}, \ldots, \tfrac{\partial}{\partial x_m} \bigr\rangle_{\mathbb{R}},$$
$$\text{and} \qquad {}^bT_xX = \bigl\langle x_1 \tfrac{\partial}{\partial x_1}, \ldots, x_k \tfrac{\partial}{\partial x_k}, \tfrac{\partial}{\partial x_{k+1}}, \ldots, \tfrac{\partial}{\partial x_m} \bigr\rangle_{\mathbb{R}}.$$

Using these identifications, define a subset $\tilde M_xX \subseteq {}^b\tilde N_xX$ by

$$\tilde M_xX = \bigl\{b_1 \cdot x_1 \tfrac{\partial}{\partial x_1} + \cdots + b_k \cdot x_k \tfrac{\partial}{\partial x_k} : b_1, \dots, b_k \in \mathbb{N}\bigr\},$$

so that $\tilde M_xX \cong \mathbb{N}^k$. This is independent of the choice of coordinates. We consider $\tilde M_xX$ to be a commutative monoid under addition in ${}^b\tilde N_xX$, as in Definition 2.17 below.

Now let $f : X \to Y$ be an interior map in $\mathbf{Man^c}$, and let $x \in S^k(X) \subseteq X$ with $f(x) = y \in S^l(Y) \subseteq Y$. Then f maps $S^k(X) \to S^l(Y)$ near x by Lemma 2.5. There is a unique linear map ${}^b\tilde N_xf : {}^b\tilde N_xX \to {}^b\tilde N_yY$, the *stratum b-normal map*, fitting into the following commutative diagram, where the rows are (2.17):

$$\begin{tikzcd} 0 \arrow[r] & {}^b\tilde N_xX \arrow[r] \arrow[d, dotted, "{}^b\tilde N_xf"] & {}^bT_xX \arrow[r] \arrow[d, "{}^bT_xf"] & T_x(S^k(X)) \arrow[r] \arrow[d, "T_x(f|_{S^k(X)})"] & 0 \\ 0 \arrow[r] & {}^b\tilde N_yY \arrow[r] & {}^bT_yY \arrow[r] & T_y(S^l(Y)) \arrow[r] & 0. \end{tikzcd} \tag{2.18}$$

We have ${}^b\tilde N_xf(\tilde M_xX) \subseteq \tilde M_yY$, so we define a monoid morphism $\tilde M_xf : \tilde M_xX \to \tilde M_yY$ by $\tilde M_xf = {}^b\tilde N_xf|_{\tilde M_xX}$. These morphisms ${}^b\tilde N_xf, \tilde M_xf$ are functorial in f and x. That is, if $g : Y \to Z$ is another interior morphism in $\mathbf{Man^c}$ then ${}^b\tilde N_x(g \circ f) = {}^b\tilde N_yg \circ {}^b\tilde N_xf$ and $\tilde M_x(g \circ f) = \tilde M_yg \circ \tilde M_xf$.

We have canonical isomorphisms ${}^b\tilde N_xX \cong \tilde M_xX \otimes_{\mathbb{N}} \mathbb{R}$ for all x, X, which identify ${}^b\tilde N_xf : {}^b\tilde N_xX \to {}^b\tilde N_yY$ with $\tilde M_xf \otimes \mathrm{id}_{\mathbb{R}} : \tilde M_xX \otimes_{\mathbb{N}} \mathbb{R} \to \tilde M_yY \otimes_{\mathbb{N}} \mathbb{R}$.

An interior map $f : X \to Y$ is b-normal if ${}^b\tilde N_xf$ is surjective for all $x \in X$.

In §10.1.5 and §10.3 we will refer to $\tilde N_xX, {}^b\tilde N_xX, \tilde M_xX$ as *quasi-tangent spaces*, as they behave quite like tangent spaces.

2.4 Generalizations of manifolds with corners

We briefly discuss the categories $\mathbf{Man^{gc}}$ of *manifolds with g-corners* from [64] and $\mathbf{Man^{ac}}$ of *manifolds with a-corners* from [66].

2.4.1 Manifolds with generalized corners

In [64] the author introduced an extension of manifolds with corners called *manifolds with generalized corners*, or *manifolds with g-corners*. They are locally modelled on certain spaces X_P for P a weakly toric monoid.

Definition 2.17. A (*commutative*) *monoid* $(P, +, 0)$ is a set P with a commutative, associative operation $+ : P \times P \to P$ and an identity element $0 \in P$. Monoids are like abelian groups, but without inverses. They form a category **Mon**. Some examples of monoids are the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$, the integers $\mathbb{Z}$, any abelian group G, and $[0, \infty) = \bigl([0, \infty), \cdot, 1\bigr)$.

A monoid P is called *weakly toric* if for some $m, k \geqslant 0$ and $c_i^j \in \mathbb{Z}$ for $i = 1, \dots, m$, $j = 1, \dots, k$ we have

$$P \cong \bigl\{(l_1, \dots, l_m) \in \mathbb{Z}^m : c_1^j l_1 + \cdots + c_m^j l_m \geqslant 0,\ \ j = 1, \dots, k\bigr\}.$$

The *rank* of a weakly toric monoid P is $\operatorname{rank} P = \dim_{\mathbb{R}}(P \otimes_{\mathbb{N}} \mathbb{R})$. A weakly toric monoid P is called *toric* if $0 \in P$ is the only invertible element.

Let P be a weakly toric monoid. Define X_P to be the set of monoid morphisms $x : P \to [0,\infty)$, where $\bigl([0,\infty),\cdot,1\bigr)$ is the monoid $[0,\infty)$ with operation multiplication and identity 1. Define the *interior* $X_P^\circ \subset X_P$ of X_P to be the subset of x with $x(P) \subseteq (0,\infty) \subset [0,\infty)$.

For each $p \in P$, define a function $\lambda_p : X_P \to [0,\infty)$ by $\lambda_p(x) = x(p)$. Then $\lambda_{p+q} = \lambda_p \cdot \lambda_q$ for $p,q \in P$, and $\lambda_0 = 1$. Define a topology on X_P to be the weakest such that $\lambda_p : X_P \to [0,\infty)$ is continuous for all $p \in P$. If $U \subseteq X_P$ is open, define the *interior* U° of U to be $U^\circ = U \cap X_P^\circ$.

Choose generators $p_1,\ldots,p_m$ for P, and a generating set of relations for $p_1,\ldots,p_m$ of the form

$$a_1^j p_1 + \cdots + a_m^j p_m = b_1^j p_1 + \cdots + b_m^j p_m \quad \text{in } P \text{ for } j = 1,\ldots,k,$$

where $a_i^j, b_i^j \in \mathbb{N}$ for $i = 1,\ldots,m$ and $j = 1,\ldots,k$. Here as P is integral (i.e. a submonoid of an abelian group) we may suppose that $a_i^j = 0$ or $b_i^j = 0$ for all i,j. Then $\lambda_{p_1} \times \cdots \times \lambda_{p_m} : X_P \to [0,\infty)^m$ is a homeomorphism from X_P to

$$X_P' = \bigl\{(x_1,\ldots,x_m) \in [0,\infty)^m : x_1^{a_1^j} \cdots x_m^{a_m^j} = x_1^{b_1^j} \cdots x_m^{b_m^j},\ j = 1,\ldots,k\bigr\},$$

regarding X_P' as a closed subset of $[0,\infty)^m$ with the induced topology.

Let $U \subseteq X_P$ be open, and $U' = (\lambda_{p_1} \times \cdots \times \lambda_{p_m})(U)$ be the corresponding open subset of X_P'. We say that a continuous function $f : U \to \mathbb{R}$ or $f : U \to [0,\infty)$ is *smooth* if there exists an open neighbourhood W of U' in $[0,\infty)^m$ and a smooth function $g : W \to \mathbb{R}$ or $g : W \to [0,\infty)$ in the sense of manifolds with (ordinary) corners in §2.1–§2.3, such that $f = g \circ (\lambda_{p_1} \times \cdots \times \lambda_{p_m})$. This definition turns out to be independent of the choice of generators $p_1,\ldots,p_m$.

Now let Q be another weakly toric monoid, $V \subseteq X_Q$ be open, and $f : U \to V$ be continuous. We say that f is *smooth* if $\lambda_q \circ f : U \to [0,\infty)$ is smooth in the sense above for all $q \in Q$. We call a smooth map $f : U \to V$ *interior* if $f(U^\circ) \subseteq V^\circ$, and a *diffeomorphism* if f has a smooth inverse $f^{-1} : V \to U$.

With these definitions, for any weakly toric monoid P, the interior X_P° is naturally a manifold of dimension $\operatorname{rank} P$, diffeomorphic to $\mathbb{R}^{\operatorname{rank} P}$.

Example 2.18. Let P be the weakly toric monoid $\mathbb{N}^k \times \mathbb{Z}^{m-k}$ for $0 \leqslant k \leqslant m$. Then points of X_P are monoid morphisms $x : \mathbb{N}^k \times \mathbb{Z}^{m-k} \to \bigl([0,\infty),\cdot,1\bigr)$, which may be written uniquely in the form

$$x(p_1,\ldots,p_m) = y_1^{p_1} \cdots y_k^{p_k} e^{p_{k+1}y_{k+1} + \cdots + p_m y_m}$$

for $(y_1,\ldots,y_m) \in \mathbb{R}_k^m$. This gives a bijection $X_P \cong \mathbb{R}_k^m = [0,\infty)^k \times \mathbb{R}^{m-k}$. As in [64, §3.2], this bijection identifies the topologies on $\mathbb{R}_k^m, X_P$, and identifies the notions of smooth map between open subsets of $\mathbb{R}_k^m, \mathbb{R}_l^n$ and between open subsets of X_P, X_Q in Definitions 2.1 and 2.17. Thus, the X_P for general weakly toric monoids P are a class of smooth spaces generalizing the spaces $\mathbb{R}_k^m$ used as local models for manifolds with corners in §2.1.

In [64, §3.3] we use this to define the category $\mathbf{Man^{gc}}$ of *manifolds with g-corners*, by generalizing Definition 2.2. A *manifold with g-corners of dimension m* is a Hausdorff, second countable topological space X equipped with a maximal atlas $\{(P_a, U_a, \phi_a) : a \in A\}$ of charts (P_a, U_a, ϕ_a), such that P_a is a weakly toric monoid with $\operatorname{rank} P_a = m$, and $U_a \subseteq X_{P_a}$ is open, and $\phi_a : U_a \to X$ is a homeomorphism with an open set $\phi_a(U_a) \subseteq X$. Any two such charts $(P_a, U_a, \phi_a), (P_b, U_b, \phi_b)$ are required to be pairwise compatible, in that the transition map $\phi_b^{-1} \circ \phi_a : \phi_a^{-1}(\phi_b(U_b)) \to \phi_b^{-1}(\phi_a(U_a))$ must be a diffeomorphism between open subsets of X_{P_a}, X_{P_b} in the sense of Definition 2.17. For set-theoretic reasons we require the P_a to be submonoids of some $\mathbb{Z}^k$.

Morphisms $f : X \to Y$ in $\mathbf{Man^{gc}}$, called *smooth maps*, are continuous maps $f : X \to Y$ such that for all charts $(P_a, U_a, \phi_a), (Q_b, V_b, \psi_b)$ on X, Y, the transfer map $\psi_b^{-1} \circ f \circ \phi_a$ is a smooth map between open subsets of X_{P_a}, X_{Q_b} in the sense of Definition 2.17. We call f *interior* if the $\psi_b^{-1} \circ f \circ \phi_a : (f \circ \phi_a)^{-1}(\psi_b(V_b)) \to V_b$ are interior maps for all a, b, in the sense of Definition 2.17, and we write $\mathbf{Man^{gc}_{in}}$ for the subcategory of $\mathbf{Man^{gc}}$ with morphisms interior maps.

Generalizing Definition 2.16, in [64, Def. 3.51], if $X \in \mathbf{Man^{gc}}$, for each $x \in S^k(X) \subseteq X$ we define a real vector space ${}^b\tilde{N}_x X$ with $\dim {}^b\tilde{N}_x X = k$ in a natural exact sequence (2.17), and a subset $\tilde{M}_x X \subseteq {}^b\tilde{N}_x X$ which is a commutative monoid under addition in ${}^b\tilde{N}_x X$. But now $\tilde{M}_x X$ is a toric monoid of rank k, such that if $\tilde{M}_x X = P$ then X near x is locally modelled on $X_P \times \mathbb{R}^{\dim X - \operatorname{rank} P}$ near $(\delta_0, 0)$, and $X \in \mathbf{Man^c} \subset \mathbf{Man^{gc}}$ if and only if $\tilde{M}_x X \cong \mathbb{N}^k$ for all $x \in X$.

If $f : X \to Y$ is an interior map in $\mathbf{Man^{gc}}$ and $x \in S^k(X) \subseteq X$ with $f(x) = y \in S^l(Y) \subseteq Y$, there is a unique linear map ${}^b\tilde{N}_x f : {}^b\tilde{N}_x X \to {}^b\tilde{N}_y Y$ making (2.18) commute. Then ${}^b\tilde{N}_x f(\tilde{M}_x X) \subseteq \tilde{M}_y Y$, so we define a monoid morphism $\tilde{M}_x f : \tilde{M}_x X \to \tilde{M}_y Y$ by $\tilde{M}_x f = {}^b\tilde{N}_x f|_{\tilde{M}_x X}$, as in Definition 2.16.

We call an interior map $f : X \to Y$ *simple* if $\tilde{M}_x f$ is an isomorphism for all $x \in X$. Write $\mathbf{Man^{gc}_{si}}$ for the subcategory of $\mathbf{Man^{gc}}$ with simple morphisms. We call an interior map $f : X \to Y$ *b-normal* if ${}^b\tilde{N}_x f$ is surjective for all $x \in X$. We write $\mathbf{Man^{gc}_{bn}}$ for the subcategory of $\mathbf{Man^{gc}}$ with morphisms b-normal maps.

Using Example 2.18 to view $\mathbb{R}^m_k$ as a space X_P, we obtain a full embedding $\mathbf{Man^c} \subset \mathbf{Man^{gc}}$, which restricts to a full embedding $\mathbf{Man^c_{in}} \subset \mathbf{Man^{gc}_{in}}$. By an abuse of notation we will regard $\mathbf{Man^c}$ as a full subcategory of $\mathbf{Man^{gc}}$, closed under isomorphisms in $\mathbf{Man^{gc}}$, so that Proposition 3.21(b) below holds. We could modify the definitions of $\mathbf{Man^c}, \mathbf{Man^{gc}}$ to make this true.

Example 2.19. The simplest manifold with g-corners which is not a manifold with corners is $X = \{(x_1, x_2, x_3, x_4) \in [0, \infty)^4 : x_1 x_2 = x_3 x_4\}$. We have $X \cong X_P$, where P is the monoid $P = \{(a, b, c) \in \mathbb{N}^3 : c \leqslant a + b\}$.

Then X is 3-dimensional, and has four 2-dimensional boundary faces

$$\begin{aligned} X_{13} &= \{(x_1, 0, x_3, 0) : x_1, x_3 \in [0, \infty)\}, \quad X_{14} = \{(x_1, 0, 0, x_4) : x_1, x_4 \in [0, \infty)\}, \\ X_{23} &= \{(0, x_2, x_3, 0) : x_2, x_3 \in [0, \infty)\}, \quad X_{24} = \{(0, x_2, 0, x_4) : x_2, x_4 \in [0, \infty)\}, \end{aligned}$$

and four 1-dimensional edges

$$X_1 = \bigl\{(x_1,0,0,0) : x_1 \in [0,\infty)\bigr\}, \qquad X_2 = \bigl\{(0,x_2,0,0) : x_2 \in [0,\infty)\bigr\},$$
$$X_3 = \bigl\{(0,0,x_3,0) : x_3 \in [0,\infty)\bigr\}, \qquad X_4 = \bigl\{(0,0,0,x_4) : x_4 \in [0,\infty)\bigr\},$$

all meeting at the vertex $(0,0,0,0) \in X$. In a 3-manifold with (ordinary) corners such as $[0,\infty)^3$, three 2-dimensional boundary faces and three 1-dimensional edges meet at each vertex, so X has an exotic corner structure at $(0,0,0,0)$.

As in [64, §3.4–§3.6], the theory of §2.2–§2.3 extends to manifolds with g-corners, but with some important differences:

- As in §2.2, boundaries ∂X, k-corners $C_k(X)$, and the first corner functor $C : \mathbf{Man^{gc}} \to \check{\mathbf{M}}\mathbf{an^{gc}_{in}} \subset \check{\mathbf{M}}\mathbf{an^{gc}}$ in Definition 2.9 work for manifolds with g-corners, where $\check{\mathbf{M}}\mathbf{an^{gc}_{in}}, \check{\mathbf{M}}\mathbf{an^{gc}}$ are the extensions of $\mathbf{Man^{gc}_{in}}, \mathbf{Man^{gc}}$ with objects disjoint unions $\coprod_{m=0}^{\infty} X_m$, where X_m is a manifold with g-corners of dimension m. However, equations (2.2)–(2.5) and (2.8) are false for manifolds with g-corners X: for $k > 2$ there is no natural S_k-action on $\partial^k X$, and no natural diffeomorphism $C_k(X) \cong \partial^k X/S_k$.
- The second corner functor C' in Definition 2.11 does not extend to $\mathbf{Man^{gc}}$, as the maximal local corner component δ there may not be unique.
- B-(co)tangent bundles ${}^bTX, {}^bT^*X$ and the functor ${}^bT : \mathbf{Man^{gc}_{in}} \to \mathbf{Man^{gc}_{in}}$ work nicely for manifolds with g-corners X. But ordinary (co)tangent bundles TX, T^*X are not well defined. One can define tangent spaces T_xX for $x \in X$, but $\dim T_xX$ is only upper semicontinuous in x, and the T_xX do not form a vector bundle on X.

As discussed in §2.5.3, transverse fibre products exist in $\mathbf{Man^{gc}}$ and $\mathbf{Man^{gc}_{in}}$ under weak conditions, and this is an important reason for working with $\mathbf{Man^{gc}}$. We can think of $\mathbf{Man^{gc}}$ as a closure of $\mathbf{Man^{c}}$ under transverse fibre products.

2.4.2 Manifolds with analytic corners

In [66] the author introduced yet another variation on manifolds with corners, called *manifolds with analytic corners* or *manifolds with a-corners*, which form a category $\mathbf{Man^{ac}}$. They have applications to some classes of analytic problems.

The motivating idea is that a manifold with corners X has two tangent bundles $TX, {}^bTX$, as in §2.3. Now the definition of smooth functions on X in §2.1 favours TX, as $f : X \to \mathbb{R}$ is smooth if $\nabla^k f$ exists as a continuous section of $\bigotimes^k T^*X$ for all $k = 0, 1, \ldots$. For manifolds with a-corners X we define 'a-smooth functions' and 'a-smooth maps' using ${}^bTX$, so that roughly speaking $f : X \to \mathbb{R}$ is a-smooth if ${}^b\nabla^k f$ exists as a section of $\bigotimes^k {}^bT^*X$ for all $k = 0, 1, \ldots$. This gives a different smooth structure even for $X = [0,\infty)$. For example, $x^\alpha : [0,\infty) \to \mathbb{R}$ is a-smooth for all real $\alpha > 0$.

Here are the a-smooth versions of Definition 2.1(b)–(g):

Definition 2.20. As in §2.1 write $\mathbb{R}^m_k = [0,\infty)^k \times \mathbb{R}^{m-k}$ for $0 \leqslant k \leqslant m$, let $U \subseteq \mathbb{R}^m_k$ be open, and $f : U \to \mathbb{R}$ be continuous. We say that f is *a-smooth* if for all $a_1, \ldots, a_m \in \mathbb{N}$ and for any compact subset $S \subseteq U$, there exist positive constants C, α such that

$$\left| \frac{\partial^{a_1+\cdots+a_m}}{\partial x_1^{a_1} \cdots \partial x_m^{a_m}} f(x_1, \ldots, x_m) \right| \leqslant C \prod_{i=1,\ldots,k:\, a_i>0} x_i^{\alpha - a_i}$$

for all $(x_1, \ldots, x_m) \in S$ with $x_i > 0$ if $i = 1, \ldots, k$ with $a_i > 0$, where continuous partial derivatives must exist at the required points.

Now let $U \subseteq \mathbb{R}^m_k$ and $V \subseteq \mathbb{R}^n_l$ be open, and $f = (f_1, \ldots, f_n) : U \to V$ be a continuous map, so that $f_j = f_j(x_1, \ldots, x_m)$ maps $U \to [0,\infty)$ for $j = 1, \ldots, l$ and $U \to \mathbb{R}$ for $j = l+1, \ldots, n$. Then we say that

(a) f is *a-smooth* if $f_j : U \to \mathbb{R}$ is a-smooth as above for $j = l+1, \ldots, n$, and every $u = (x_1, \ldots, x_m) \in U$ has an open neighbourhood $\tilde U$ in U such that for each $j = 1, \ldots, l$, either:

  (i) we may uniquely write $f_j(\tilde x_1, \ldots, \tilde x_m) = F_j(\tilde x_1, \ldots, \tilde x_m) \cdot \tilde x_1^{a_{1,j}} \cdots \tilde x_k^{a_{k,j}}$ for all $(\tilde x_1, \ldots, \tilde x_m) \in \tilde U$, where $F_j : \tilde U \to (0,\infty) \subset \mathbb{R}$ is a-smooth as above, and $a_{1,j}, \ldots, a_{k,j} \in [0,\infty)$, with $a_{i,j} = 0$ if $x_i \neq 0$; or

  (ii) $f_j|_{\tilde U} = 0$.

(b) f is *interior* if it is a-smooth, and case (a)(ii) does not occur.

(c) f is *b-normal* if it is interior, and in case (a)(i), for each $i = 1, \ldots, k$ we have $a_{i,j} > 0$ for at most one $j = 1, \ldots, l$.

(d) f is *strongly a-smooth* if it is a-smooth, and in case (a)(i), for each $j = 1, \ldots, l$ we have $a_{i,j} > 0$ for at most one $i = 1, \ldots, k$.

(e) f is *simple* if it is interior, and in case (a)(i), for each $i = 1, \ldots, k$ with $x_i = 0$ we have $a_{i,j} > 0$ for exactly one $j = 1, \ldots, l$, and for all $j = 1, \ldots, l$ we have $a_{i,j} > 0$ for at most one $i = 1, \ldots, k$.

(f) f is an *a-diffeomorphism* if it is an a-smooth bijection with a-smooth inverse.

As in [66, §3.2], we define the category $\mathbf{Man^{ac}}$ of *manifolds with a-corners* as for $\mathbf{Man^{ac}}$ in Definition 2.2, but replacing Definition 2.1(b)–(g) by Definition 2.17(a)–(f). We define subcategories $\mathbf{Man^{ac}_{in}}, \mathbf{Man^{ac}_{bn}}, \mathbf{Man^{ac}_{st}}, \mathbf{Man^{ac}_{st,in}}$, $\mathbf{Man^{ac}_{st,bn}}$ and $\mathbf{Man^{ac}_{si}}$ of $\mathbf{Man^{ac}}$ with interior, b-normal, strongly a-smooth, strongly a-smooth interior, strongly a-smooth b-normal, and simple morphisms, respectively. As in [66, §3], there is an (obvious) functor $F_{\mathbf{Man^c}}^{\mathbf{Man^{ac}}} : \mathbf{Man^c} \to \mathbf{Man^{ac}}$, and a (non-obvious and nontrivial) functor $F_{\mathbf{Man^{ac}_{st}}}^{\mathbf{Man^c_{st}}} : \mathbf{Man^{ac}_{st}} \to \mathbf{Man^c_{st}}$.

We also define a category $\mathbf{Man^{c,ac}}$ of *manifolds with corners and a-corners*, including $\mathbf{Man^c}$, $\mathbf{Man^{ac}}$ as full subcategories, and subcategories $\mathbf{Man^{c,ac}_{in}}$,

$\mathbf{Man^{c,ac}_{bn}}, \mathbf{Man^{c,ac}_{st}}, \mathbf{Man^{c,ac}_{st,in}}, \mathbf{Man^{c,ac}_{st,bn}}, \mathbf{Man^{c,ac}_{si}}$ of $\mathbf{Man^{c,ac}}$ with interior, b-normal, strongly a-smooth, strongly a-smooth interior, strongly a-smooth b-normal, and simple morphisms, respectively. There are functors $F_{\mathbf{Man^{c,ac}}}^{\mathbf{Man^{ac}}}$: $\mathbf{Man^{c,ac}} \to \mathbf{Man^{ac}}$ and $F_{\mathbf{Man^{c,ac}_{st}}}^{\mathbf{Man^{c}_{st}}} : \mathbf{Man^{c,ac}_{st}} \to \mathbf{Man^{c}_{st}}$.

As in [66, §4], the theory of §2.2–§2.3 extends to manifolds with a-corners $\mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$, including both corner functors C, C' in Definitions 2.9 and 2.11, with the difference that we do not define ordinary tangent bundles TX for manifolds with a-corners X, but only b-tangent bundles ${}^bTX$.

If X lies in $\mathbf{Man^{ac}}$ or $\mathbf{Man^{c,ac}}$, so that we have the k-corners $C_k(X)$ with a projection $\Pi_k : C_k(X) \to X$, then as in (2.13) there is a rank k bundle ${}^bN_{C_k(X)}$ on $C_k(X)$ in an exact sequence (2.13). When $k = 1$, for $\mathbf{Man^c}$ and $\mathbf{Man^{gc}}$ this ${}^bN_{C_1(X)}$ was naturally trivial, ${}^bN_{C_1(X)} = \mathcal{O}_{\partial X}$, giving an exact sequence (2.14) on ∂X. However, for X in $\mathbf{Man^{ac}}$ or $\mathbf{Man^{c,ac}}$ this ${}^bN_{C_1(X)} = {}^bN_{\partial X}$ may not be naturally trivial, so that instead of (2.14) we have an exact sequence on ∂X:

$$0 \longrightarrow {}^bN_{\partial X} \longrightarrow i_X^*({}^bTX) \xrightarrow{I_X^\diamond} {}^bT(\partial X) \longrightarrow 0. \qquad (2.19)$$

Here ${}^bN_{\partial X} \to \partial X$ is a line bundle which has a natural orientation on its fibres, by outward-pointing vectors. Also ${}^bN_{\partial X}$ has a natural flat connection.

2.5 Transversality, submersions, and fibre products

Fibre products in categories are defined in §A.1. Transversality and submersions are about giving useful criteria for existence of fibre products of manifolds. If we work in some category of manifolds $\dot{\mathbf{M}}\mathbf{an}$ such as $\mathbf{Man}, \mathbf{Man^c_{st}}, \mathbf{Man^{gc}_{in}}, \mathbf{Man^{gc}}$, $\mathbf{Man^c_{in}}, \mathbf{Man^c}$, then we would like the properties:

(i) If $g : X \to Z$ and $h : Y \to Z$ are 'transverse' then a fibre product $W = X \times_{g,Z,h} Y$ exists in $\dot{\mathbf{M}}\mathbf{an}$, with $\dim W = \dim X + \dim Y - \dim Z$.

(ii) If $g : X \to Z$ is a 'submersion' then g, h are transverse for any $h : Y \to Z$.

We would also like the definitions of 'transverse' and 'submersion' to be easy to check, and not to be too restrictive. Chapter 11 in volume II will extend the results of this section to (m-)Kuranishi spaces.

2.5.1 Transversality and submersions in Man

The next definition and theorem are well known, see for instance Lee [71, §4, §6] and Lang [70, §II.2].

Definition 2.21. Let $g : X \to Z$ and $h : Y \to Z$ be smooth maps of manifolds. We call g, h *transverse* if $T_xg \oplus T_yh : T_xX \oplus T_yY \to T_zZ$ is surjective for all $x \in X$ and $y \in Y$ with $g(x) = h(y) = z$ in Z. We call g a *submersion* if $T_xg : T_xX \to T_zZ$ is surjective for all $x \in X$ with $g(x) = z$ in Z.

Theorem 2.22. **(a)** *Suppose $g : X \to Z$ and $h : Y \to Z$ are transverse smooth maps of manifolds. Then a fibre product $W = X \times_{g,Z,h} Y$ exists in* **Man**, *with* $\dim W = \dim X + \dim Y - \dim Z$, *in a Cartesian square in* **Man**:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z. \end{tikzcd} \tag{2.20}$$

We may write

$$W = \bigl\{(x, y) \in X \times Y : g(x) = h(y) \text{ in } Z\bigr\} \tag{2.21}$$

as an embedded submanifold of $X \times Y$, where $e : W \to X$ and $f : W \to Y$ act by $e : (x, y) \mapsto x$ and $f : (x, y) \mapsto y$. If $w \in W$ with $e(w) = x \in X$, $f(w) = y \in Y$ and $g(x) = h(y) = z \in Z$ then the following sequence is exact:

$$0 \longrightarrow T_wW \xrightarrow{T_we \oplus T_wf} T_xX \oplus T_yY \xrightarrow{T_xg \oplus -T_yh} T_zZ \longrightarrow 0. \tag{2.22}$$

(b) *Suppose $g : X \to Z$ is a submersion in* **Man**. *Then g, h are transverse for any morphism $h : Y \to Z$ in* **Man**.

(c) *Let $g : X \to Z$ be a morphism in* **Man**. *Then g is a submersion if and only if the following condition holds: for each $x \in X$ with $g(x) = z$, there should exist open neighbourhoods X', Z' of x, z in X, Z with $g(X') = Z'$, a manifold Y' with $\dim X = \dim Y' + \dim Z$, and a diffeomorphism $X' \cong Y' \times Z'$, such that $g|_{X'} : X' \to Z'$ is identified with $\pi_{Z'} : Y' \times Z' \to Z'$.*

Part (c) gives an alternative definition of submersions in **Man**: submersions are local projections. Here are some examples of non-transverse fibre products in **Man**. They illustrate the facts that: (i) non-transverse fibre products need not exist; (ii),(iii) a fibre product $W = X \times_Z Y$ may exist, but have $\dim W \neq \dim X + \dim Y - \dim Z$; and (iv) a fibre product $W = X \times_Z Y$ may exist, but may not be homeomorphic to (2.21) as a topological space.

Example 2.23. **(i)** Define manifolds $X = \mathbb{R}^2$, $Y = \{*\}$, $Z = \mathbb{R}$, and smooth maps $g : X \to Z$, $h : Y \to Z$ by $g(x, y) = xy$ and $h(*) = 0$. Then g, h are not transverse at $(0, 0) \in X$ and $* \in Y$. In this case no fibre product $X \times_{g,Z,h} Y$ exists in **Man**. Roughly this is because the fibre product ought to be $\bigl\{(x, y) \in \mathbb{R}^2 : xy = 0\bigr\}$, which is not a manifold near $(0, 0)$.

(ii) Set $X = Y = \{*\}$, $Z = \mathbb{R}$, and define $g : X \to Z$, $h : Y \to Z$ by $g(*) = h(*) = 0$. Then g, h are not transverse at $* \in X$ and $* \in Y$. A fibre product $W = X \times_{g,Z,h} Y$ exists in **Man**, where $W = \{*\}$ with projections $e : W \to X$, $f : W \to Y$ given by $e(*) = f(*) = *$. Note that $\dim W > \dim X + \dim Y - \dim Z$, so W has larger than the expected dimension.

(iii) Set $X = \mathbb{R}^2$, $Y = \{*\}$, $Z = \mathbb{R}$, and define $g : X \to Z$, $h : Y \to Z$ by $g(x, y) = x^2 + y^2$ and $h(*) = 0$. Then g, h are not transverse at $(0, 0) \in X$ and $* \in Y$. A fibre product $W = X \times_{g,Z,h} Y$ exists in **Man**, where $W = \{*\}$ with $e : W \to X$, $f : W \to Y$ given by $e(*) = (0, 0)$ and $f(*) = *$. Note that $\dim W < \dim X + \dim Y - \dim Z$, so W has smaller than expected dimension.

(iv) Set $X = \mathbb{R}^2$, $Y = \{*\}$, $Z = \mathbb{R}$, and define smooth $g : X \to Z$, $h : Y \to Z$ by

$$g(x,y) = \begin{cases} e^{-1/x^2}(y - \sin(1/x)), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad h(*) = 0.$$

Then g,h are not transverse at $(0,y) \in X$ and $* \in Y$ for $y \in \mathbb{R}$. A fibre product $W = X \times_{g,Z,h} Y$ exists in **Man**. It is the disjoint union $W = (-\infty,0) \amalg (0,\infty) \amalg \mathbb{R}$, where $e : W \to X$, $f : W \to Y$ act by $e(x) = (x, \sin(1/x))$ for $x \in (-\infty,0) \amalg (0,\infty)$ and $e(y) = (0,y)$ for $y \in \mathbb{R}$, and $f \equiv *$.

We can also form the fibre product in topological spaces **Top**, which is

$$X_{\rm top} \times_{Z_{\rm top}} Y_{\rm top} \cong \bigl\{(x,y) \in \mathbb{R}^2 : x \neq 0 \text{ and } y = \sin(1/x), \text{ or } x = 0\bigr\}.$$

Note that the fibre products in **Man** and **Top** coincide at the level of sets, but not at the level of topological spaces, since $X \times_Z Y$ has three connected components but $X_{\rm top} \times_{Z_{\rm top}} Y_{\rm top}$ has only one.

2.5.2 Transversality and submersions in $\mathbf{Man^c_{st}}$ and $\mathbf{Man^c}$

The author [59] studied transverse fibre products and submersions in the category $\mathbf{Man^c_{st}}$ of manifolds with corners and strongly smooth maps. The next definition is equivalent to [59, Def.s 3.2, 6.1 & 6.10]:

Definition 2.24. Let $g : X \to Z$ and $h : Y \to Z$ be morphisms in $\mathbf{Man^c_{st}}$. We call g,h *s-transverse* if for all $x \in S^j(X) \subseteq X$ and $y \in S^k(Y) \subseteq Y$ with $g(x) = h(y) = z \in S^l(Z) \subseteq Z$, the following morphisms are surjective:

$$\begin{aligned} T_x g|_{T_x S^j(X)} \oplus T_y h|_{T_y S^k(Y)} : T_x S^j(X) \oplus T_y S^k(Y) &\longrightarrow T_z S^l(Z), \\ \tilde N_x g \oplus \tilde N_y h : \tilde N_x X \oplus \tilde N_y Y &\longrightarrow \tilde N_z Z. \end{aligned} \tag{2.23}$$

This is an open condition on $x \in X$ and $y \in Y$. That is, if (2.23) holds for some x,y,z, then there are open neighbourhoods $x \in X' \subseteq X$ and $y \in Y' \subseteq Y$ such that (2.23) also holds for all $x' \in X'$ and $y' \in Y'$ with $g(x') = h(y') = z'$ in Z, even though j,k,l may not be constant.

We call g,h *t-transverse* if they are s-transverse, and if $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$, then for all $\boldsymbol{x} \in C_j(X)$ and $\boldsymbol{y} \in C_k(Y)$ with $\Pi_j(\boldsymbol{x}) = x$, $\Pi_k(\boldsymbol{y}) = y$ and $C(g)\boldsymbol{x} = C(h)\boldsymbol{y} = \boldsymbol{z}$ in $C_l(Z)$, we have $j + k \geqslant l$, and there is exactly one triple $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ with $j + k = l$. This is an open condition on $x \in X$ and $y \in Y$.

We call g an *s-submersion* if for all $x \in S^j(X) \subseteq X$ with $g(x) = z \in S^l(Z) \subseteq Z$, the following morphisms are surjective:

$$T_x g|_{T_x S^j(X)} : T_x S^j(X) \longrightarrow T_z S^l(Z), \quad \tilde N_x g : \tilde N_x X \longrightarrow \tilde N_z Z. \tag{2.24}$$

These imply that s-submersions are interior and b-normal. Again, (2.24) is an open condition on $x \in X$.

Theorem 2.25. (a) *Suppose $g : X \to Z$ and $h : Y \to Z$ are s-transverse smooth maps in* $\mathbf{Man^c_{st}}$. *Then a fibre product $W = X \times_{g,Z,h} Y$ exists in* $\mathbf{Man^c_{st}}$, *with* $\dim W = \dim X + \dim Y - \dim Z$, *in a Cartesian square* (2.20) *in* $\mathbf{Man^c_{st}}$, *which is also a Cartesian square in* $\mathbf{Man^c}$. *We may define W by* (2.21) *as an embedded submanifold of $X \times Y$, where $e : W \to X$ and $f : W \to Y$ act by $e : (x, y) \mapsto x$ and $f : (x, y) \mapsto y$.*

If $w \in S^i(W)$ with $e(w) = x \in S^j(X)$, $f(w) = y \in S^k(Y)$ and $g(x) = h(y) = z \in S^l(Z)$ then the following sequences are exact:

$$0 \longrightarrow T_wW \xrightarrow{T_we \oplus T_wf} T_xX \oplus T_yY \xrightarrow{T_xg \oplus -T_yh} T_zZ \longrightarrow 0, \quad (2.25)$$

$$0 \to T_wS^i(W) \xrightarrow{T_we \oplus T_wf|_{\cdots}} T_xS^j(X) \oplus T_yS^k(Y) \xrightarrow{T_xg \oplus -T_yh|_{\cdots}} T_zS^l(Z) \to 0, \quad (2.26)$$

$$0 \longrightarrow \tilde N_wW \xrightarrow{\tilde N_we \oplus \tilde N_wf} \tilde N_xX \oplus \tilde N_yY \xrightarrow{\tilde N_xg \oplus -\tilde N_yh} \tilde N_zZ \longrightarrow 0. \quad (2.27)$$

(b) *In* **(a)**, *g, h are t-transverse if and only if the following are s-transverse (and indeed t-transverse) Cartesian squares in* $\check{\mathbf{Man}}{}^{\mathbf{c}}_{\mathbf{st}}$ *from Definition* 2.8:

$$\begin{tikzcd} C(W) \arrow[r, "C(f)"] \arrow[d, "C(e)"] & C(Y) \arrow[d, "C(h)"] \\ C(X) \arrow[r, "C(g)"] & C(Z), \end{tikzcd} \quad (2.28)$$

$$\begin{tikzcd} C(W) \arrow[r, "C'(f)"] \arrow[d, "C'(e)"] & C(Y) \arrow[d, "C'(h)"] \\ C(X) \arrow[r, "C'(g)"] & C(Z). \end{tikzcd} \quad (2.29)$$

Here in (2.28) *if $\boldsymbol{w} \in C_i(W)$ with $C(e)(\boldsymbol{w}) = \boldsymbol{x}$ in $C_j(X)$, $C(f)(\boldsymbol{w}) = \boldsymbol{y}$ in $C_k(Y)$ and $C(g)(\boldsymbol{x}) = C(h)(\boldsymbol{y}) = \boldsymbol{z}$ in $C_l(Z)$ then $i = j + k - l$. Hence we have*

$$C_i(W) \cong \coprod_{\substack{j,k,l \geqslant 0: \\ i=j+k-l}} \begin{array}{l} \bigl(C_j(X) \cap C(g)^{-1}(C_l(Z))\bigr) \times_{C(g),C_l(Z),C(h)} \\ \qquad \bigl(C_k(Y) \cap C(h)^{-1}(C_l(Z))\bigr) \end{array} \quad (2.30)$$

for $i \geqslant 0$. When $i = 1$, this computes the boundary ∂W. The analogue holds for the second corner functor C' in Definition 2.11, *using* (2.29). *Also* (2.28) *and* (2.29) *are Cartesian in* $\check{\mathbf{Man}}{}^{\mathbf{c}}$. *If g is an s-submersion then $C(g), C(f), C'(g)$ and $C'(f)$ are s-submersions in* $\check{\mathbf{Man}}{}^{\mathbf{c}}_{\mathbf{st}}$.

(c) *Let $g : X \to Z$ be a morphism in* $\mathbf{Man^c_{st}}$. *Then g is an s-submersion if and only if the following condition holds: for each $x \in X$ with $g(x) = z$, there should exist open neighbourhoods X', Z' of x, z in X, Z with $g(X') = Z'$, a manifold with corners Y' with $\dim X = \dim Y' + \dim Z$, and a diffeomorphism $X' \cong Y' \times Z'$, such that $g|_{X'} : X' \to Z'$ is identified with $\pi_{Z'} : Y' \times Z' \to Z'$.*

(d) *Suppose $g : X \to Z$ is an s-submersion, and $h : Y \to Z$ is any morphism in* $\mathbf{Man^c}$, *which need not be strongly smooth. Then a fibre product $W = X \times_{g,Z,h} Y$ exists in* $\mathbf{Man^c}$, *in a Cartesian square* (2.20) *in* $\mathbf{Man^c}$, *with* $\dim W = \dim X +$

$\dim Y - \dim Z$, *and is given by* (2.21). *Also f is an s-submersion, and* (2.28)–(2.29) *are Cartesian in* $\check{\mathbf{Man}}^{\mathbf{c}}$, *and* (2.30) *holds. If h is strongly smooth then e is strongly smooth, and g,h are s- and t-transverse, and* (2.20) *is Cartesian in* $\mathbf{Man^c_{st}}$, *and* (2.28)–(2.29) *are Cartesian in* $\check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{st}}$.

Proof. For (a), [59, Th. 6.4] shows that a fibre product $W = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^c_{st}}$, with $\dim W = \dim X + \dim Y - \dim Z$, given by (2.21) as an embedded submanifold of $X \times Y$. This embedded submanifold property implies that (2.20) is also Cartesian in $\mathbf{Man^c}$. Exactness of (2.25)–(2.27) may be deduced from Theorem 2.22(a) and the proof of [59, Th. 6.4]. Part (b) in $\check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{st}}$ is proved in [59, Th. 6.11], and in $\check{\mathbf{Man}}^{\mathbf{c}}$ follows from the embedded submanifold property. Part (c) is proved in [59, Prop. 5.1]. Part (d) follows easily from (a)–(c). $\square$

Example 2.26. Set $X = Y = [0,\infty)$ and $Z = [0,\infty)^2$, and define strongly smooth $g : X \to Z$, $h : Y \to Z$ by $g(x) = (x, 2x)$ and $h(y) = (2y, y)$. Then g,h are s-transverse. However

$$C(g)\bigl(0,X\bigr) = C(h)\bigl(0,Y\bigr) = \bigl((0,0),Z\bigr),$$

where $(0,X) \in C_0(X)$, $(0,Y) \in C_0(Y)$, $((0,0),Z) \in C_0(Z)$, and

$$C(g)\bigl(0,\{x=0\}\bigr) = C(h)\bigl(0,\{y=0\}\bigr) = \bigl((0,0),\{x=y=0\}\bigr),$$

with $\bigl(0,\{x=0\}\bigr)$ in $C_1(X)$, $\bigl(0,\{y=0\}\bigr)$ in $C_1(Y)$ and $\bigl((0,0),\{x=y=0\}\bigr)$ in $C_2(Z)$, so there are two triples $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ with $j+k=l$ over $(x,y,z) = (0,0,(0,0))$, and g,h are not t-transverse in Definition 2.24.

The fibre product $W = X_{g,Z,h}Y$ in $\mathbf{Man^c_{st}}$ is a single point $*$. In (2.30) when $i = 0$ the left hand side is one point, and the right hand side is two points, so (2.30) does not hold. For $i \neq 0$, both sides of (2.30) are empty.

2.5.3 Transversality and submersions in $\mathbf{Man^{gc}_{in}}$ and $\mathbf{Man^{gc}}$

In [64, §4.3] the author studied transverse fibre products of manifolds with g-corners $\mathbf{Man^{gc}_{in}}, \mathbf{Man^{gc}}$ in §2.4.1. The next definition is equivalent to [64, Def.s 4.3 & 4.24], except for c-fibrations in (e), which are new. The corresponding names and definitions of b-transverse, b-normal and b-fibrations in $\mathbf{Man^c}$ are due to Melrose [84, §I], [85, §2], [87, §2.4].

Definition 2.27. Let $g : X \to Z$ and $h : Y \to Z$ be interior morphisms in $\mathbf{Man^{gc}}$. Then:

(a) We call g,h *b-transverse* if ${}^bT_xg \oplus {}^bT_yh : {}^bT_xX \oplus {}^bT_yY \to {}^bT_zZ$ is surjective for all $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$.

(b) We call g,h *c-transverse* if they are b-transverse, and whenever there are points $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$ in $C_j(X), C_k(Y), C_l(Z)$ with $C(g)\boldsymbol{x} = C(h)\boldsymbol{y} = \boldsymbol{z}$, we have either $j+k > l$ or $j = k = l = 0$, for $C : \mathbf{Man^{gc}} \to \check{\mathbf{Man}}^{\mathbf{gc}}$ as in §2.4.1.

(c) We call g a *b-submersion* if ${}^bT_xg : {}^bT_xX \to {}^bT_zZ$ is surjective for all $x \in X$ with $g(x) = z$ in Z.

(d) We call g a *b-fibration* if it is a b-normal b-submersion. Here g is *b-normal* if whenever there are $\boldsymbol{x}, \boldsymbol{z}$ in $C_j(X), C_l(Z)$ with $C(g)\boldsymbol{x} = \boldsymbol{z}$, we have $j \geqslant l$.

(e) We call g a *c-fibration* if it is a b-fibration, and if $x \in X$ and $\boldsymbol{z} \in C_l(Z)$ with $g(x) = \Pi_l(\boldsymbol{z}) = z \in Z$, then there is exactly one $\boldsymbol{x} \in C_l(X)$ with $\Pi_l(\boldsymbol{x}) = x$ and $C(g)\boldsymbol{x} = \boldsymbol{z}$.

Theorem 2.28. **(a)** *Let $g : X \to Z$ and $h : Y \to Z$ be b-transverse morphisms in $\mathbf{Man^{gc}_{in}}$. Then a fibre product $W = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^{gc}_{in}}$, in a Cartesian square* (2.20) *in $\mathbf{Man^{gc}_{in}}$, with $\dim W = \dim X + \dim Y - \dim Z$.*

Explicitly, we may write

$$W^\circ = \bigl\{(x, y) \in X^\circ \times Y^\circ : g(x) = h(y) \text{ in } Z^\circ\bigr\}, \tag{2.31}$$

and take W to be the closure $\overline{W^\circ}$ of W° in $X \times Y$. Then W is a submanifold of $X \times Y$, and $e : W \to X$, $f : W \to Y$ act by $e : (x, y) \mapsto x$, $f : (x, y) \mapsto y$.

If $w \in W$ with $e(w) = x \in X$, $f(w) = y \in Y$ and $g(x) = h(y) = z \in Z$ then the following sequence is exact:

$$0 \longrightarrow {}^bT_wW \xrightarrow{{}^bT_we \oplus {}^bT_wf} {}^bT_xX \oplus {}^bT_yY \xrightarrow{{}^bT_xg \oplus -{}^bT_yh} {}^bT_zZ \longrightarrow 0. \tag{2.32}$$

(b) *In* **(a)**, *if g, h are c-transverse then W is also a fibre product in $\mathbf{Man^{gc}}$, and is given by* (2.21). *Furthermore,* (2.28) *is Cartesian in $\check{\mathbf{M}}\mathbf{an^{gc}}$, and* (2.30) *holds. If g is a b-fibration (or c-fibration) then $C(g)$ and $C(f)$ are b-fibrations (or c-fibrations) in $\check{\mathbf{M}}\mathbf{an^{gc}}$.*

(c) *Let $g : X \to Z$ be a b-submersion. Then g, h are b-transverse for any $h : Y \to Z$ in $\mathbf{Man^{gc}_{in}}$, and in the Cartesian square* (2.20), *f is a b-submersion.*

(d) *Let $g : X \to Z$ be a b-fibration. Then g, h are c-transverse for any $h : Y \to Z$ in $\mathbf{Man^{gc}_{in}}$, and in the Cartesian square* (2.20), *f is a b-fibration.*

(e) *Let $g : X \to Z$ be a c-fibration, and $h : Y \to Z$ be any morphism in $\mathbf{Man^{gc}}$, which need not be interior. Then a fibre product $W = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^{gc}}$, in a Cartesian square* (2.20) *in $\mathbf{Man^{gc}}$, with $\dim W = \dim X + \dim Y - \dim Z$, and is given by* (2.21). *Also f is a c-fibration, and* (2.28) *is Cartesian in $\check{\mathbf{M}}\mathbf{an^{gc}}$, and* (2.30) *holds.*

Proof. Part (a) is proved in [64, Th. 4.27], apart from exactness of (2.32), which may be deduced from the proof. Part (b) is [64, Th. 4.28]. The first parts of (c),(d) are in [64, Def. 4.24 & Prop. 4.25]. That f is a b-submersion in (c) follows from exactness of (2.32) and g a b-submersion. Then in (d), f is a b-submersion, and we can show f is b-normal using g b-normal and (2.28) Cartesian at the level of sets, so f is a b-fibration.

For part (e), as g is a b-fibration, $C(g) : C(X) \to C(Z)$ is a b-fibration, and $C(h) : C(Y) \to C(Z)$ is interior even if h is not, so $C(g), C(h)$ are b-transverse, and a fibre product $C(X) \times_{C(g),C(Z),C(h)} C(Y)$ exists in $\check{\mathbf{M}}\mathbf{an^{gc}_{in}}$ by the analogue of (a) in $\check{\mathbf{M}}\mathbf{an^{gc}_{in}}$. Write W for the component of $C(X) \times_{C(Z)} C(Y)$ of dimension $\dim X + \dim Y - \dim Z$. Then using the ideas of [64, §4] and the c-fibration condition, we can show W satisfies (e). □

This is a strong result, and means that $\mathbf{Man^{gc}}$ is useful for problems in 'manifolds with corners' in which we want transverse fibre products to exist.

In contrast to Theorems 2.22(c) and 2.25(c), b-submersions and b-fibrations in $\mathbf{Man^{gc}}$ need not be local projections. For example, $g:[0,\infty)^2\rightarrow[0,\infty)$, $g(x,y)=xy$, is a b-fibration, but is not a local projection near $(0,0)$.

Example 2.29. Set $X=Y=[0,\infty)^2$ and $Z=[0,\infty)$, and define $g:X\rightarrow Z$, $h:Y\rightarrow Z$ by $g(x_1,x_2)=x_1x_2$ and $h(x_3,x_4)=x_3x_4$. Then g,h are interior and c-transverse, so a fibre product $W=X\times_{g,Z,h}Y$ exists in $\mathbf{Man^{gc}_{in}}$ by Theorem 2.28(a),(b), and is also a fibre product in $\mathbf{Man^{gc}}$. We may write

$$W=\bigl\{(x_1,x_2,x_3,x_4)\in[0,\infty)^4:x_1x_2=x_3x_4\bigr\},$$

which as in Example 2.19 is a manifold with g-corners, but not a manifold with corners. Thus, $\mathbf{Man^{c}}$ is not closed under c-transverse fibre products in $\mathbf{Man^{gc}}$.

Example 2.30. Define $X=[0,\infty)^2$, $Z=[0,\infty)$ and a smooth map $g:X\rightarrow Z$ by $g(x,y)=xy$. Then g is a b-fibration, but not a c-fibration, since over $x=(0,0)\in X$ with $g(x)=z=0$ in Z and $\boldsymbol{z}=(0,\{z=0\})$ in $C_1(Z)$ with $\Pi_1(\boldsymbol{z})=z$, we have two points $\boldsymbol{x}=\bigl((0,0),\{x_1=0\}\bigr)$ and $\boldsymbol{x}'=\bigl((0,0),\{x_1=0\}\bigr)$ in $C_1(X)$ with $\Pi_1(\boldsymbol{x})=\Pi_1(\boldsymbol{x}')=x$ and $C(g)\boldsymbol{x}=C(g)\boldsymbol{x}'=\boldsymbol{z}$.

Set $Y=*$ and define $h:Y\rightarrow Z$ by $h:*\mapsto 0$, so that h is not interior. No fibre product $W=X\times_{g,Z,h}Y$ exists in $\mathbf{Man^{gc}}$.

2.5.4 Transversality and submersions in $\mathbf{Man^{c}_{in}}$ and $\mathbf{Man^{c}}$

We can also consider fibre products in $\mathbf{Man^{c}_{in}}$ and $\mathbf{Man^{c}}$. The appropriate definition of transversality is rather complicated (in particular, b- or c-transversality are not sufficient conditions). It is helpful to regard such fibre products as special cases of fibre products in $\mathbf{Man^{gc}_{in}},\mathbf{Man^{gc}}$, as in §2.5.3.

Definition 2.31. Let $g:X\rightarrow Z$ and $h:Y\rightarrow Z$ be morphisms in $\mathbf{Man^{c}_{in}}$. We can consider g,h as morphisms in $\mathbf{Man^{gc}_{in}}$, so Definition 2.27 makes sense. We call g,h *strictly b-transverse* (*sb-transverse*) or *strictly c-transverse* (*sc-transverse*) if they are b-transverse or c-transverse, respectively, and for all $x\in X$ and $y\in Y$ with $g(x)=h(y)=z\in Z$, the toric monoid

$$\tilde M_xX\times_{\tilde M_zZ}\tilde M_yY=\bigl\{(\lambda,\mu)\in\tilde M_xX\times\tilde M_yY:\tilde M_xg(\lambda)=\tilde M_yh(\mu)\bigr\}\qquad(2.33)$$

is isomorphic to $\mathbb{N}^n$, for $n\in\mathbb{N}$ depending on x,y,z.

Here given morphisms $g:X\rightarrow Z$, $h:Y\rightarrow Z$ in $\mathbf{Man^{c}_{in}}$ or $\mathbf{Man^{c}}$, we first require them to be b- or c-transverse, so that a fibre product $W=X\times_{g,Z,h}Y$ exists in $\mathbf{Man^{gc}_{in}}$ or $\mathbf{Man^{gc}}$ by Theorem 2.28(a),(b). We have $\tilde M_{(x,y)}W\cong\tilde M_xX\times_{\tilde M_zZ}\tilde M_yY$, so W lies in $\mathbf{Man^{c}}\subset\mathbf{Man^{gc}}$ if and only if $\tilde M_xX\times_{\tilde M_zZ}\tilde M_yY\cong\mathbb{N}^k$ for all x,y,z. Since $\mathbf{Man^{c}_{in}}\subset\mathbf{Man^{gc}_{in}}$, $\mathbf{Man^{c}}\subset\mathbf{Man^{gc}}$ are full subcategories, W is then a fibre product in $\mathbf{Man^{c}_{in}}$ or $\mathbf{Man^{c}}$. This proves:

Theorem 2.32. *Let $g : X \to Z$ and $h : Y \to Z$ be sb-transverse morphisms in $\mathbf{Man^c_{in}}$. Then a fibre product $W = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^c_{in}}$, with $\dim W = \dim X + \dim Y - \dim Z$. Explicitly, we may define W° by* (2.31)*, and take W to be the closure $\overline{W^\circ}$ of W° in $X \times Y$. Also* (2.32) *is exact for all $w \in W$.*

If g, h are sc-transverse then W is also a fibre product in $\mathbf{Man^c}$, and is given by (2.21)*. Also* (2.28) *is Cartesian in $\check{\mathbf{Man}}^{\mathbf{c}}$, and* (2.30) *holds.*

Kottke and Melrose [69, §11] study fibre products in $\mathbf{Man^c}$, and the sc-transverse case in Theorem 2.32 is essentially equivalent to [69, Th. 11.5].

The case when $\partial Z = \emptyset$ is simpler. The next theorem follows from [59, 64]:

Theorem 2.33. *Suppose $g : X \to Z$ and $h : Y \to Z$ are b-transverse morphisms in $\mathbf{Man^c}$ with $\partial Z = \emptyset$. Then a fibre product $W = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^c}$, with $\dim W = \dim X + \dim Y - \dim Z$, and is given by* (2.21) *as an embedded submanifold of $X \times Y$. It is also a fibre product in $\mathbf{Man^c_{st}}$ and $\mathbf{Man^c_{in}}$. Furthermore, $g \circ i_X, h$ and $g, h \circ i_Y$ are also b-transverse, and there is a natural diffeomorphism*

$$\partial(X \times_{g,Z,h} Y) \cong \bigl(\partial X \times_{g \circ i_X,Z,h} Y\bigr) \amalg \bigl(X \times_{g,Z,h \circ i_Y} \partial Y\bigr). \tag{2.34}$$

We would also like classes of 'submersions' $g : X \to Z$ in $\mathbf{Man^c}$, such that g, h are sb- or sc-transverse for all (interior) $h : Y \to Z$ in $\mathbf{Man^c}$. In both cases, the appropriate notion is s-submersions from Definition 2.24.

Example 2.34. Let X, Y, Z, g, h be as in Example 2.29. Then g, h are c-transverse, but they are not sc-transverse, as in (2.33) we have

$$\tilde M_{(0,0)} X \times_{\tilde M_0 Z} \tilde M_{(0,0)} Y \cong \bigl\{(n_1, n_2, n_3, n_4) \in \mathbb{N}^4 : n_1 + n_2 = n_3 + n_4\bigr\},$$

which is not isomorphic to $\mathbb{N}^k$ for any $k \geqslant 0$. A fibre product $W = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^{gc}_{in}}$ and $\mathbf{Man^{gc}}$, but not in $\mathbf{Man^c_{in}}$ or $\mathbf{Man^c}$.

Example 2.35. Let $X = [0, \infty) \times \mathbb{R}$, $Y = [0, \infty)$ and $Z = [0, \infty)^2$. Define $g : X \to Z$ by $g(x_1, x_2) = (x_1, x_1 e^{x_2})$ and $h : Y \to Z$ by $h(y) = (y, y)$. Then g is a b-submersion and h is interior, so g, h are b-transverse by Theorem 2.28(c), and in fact g, h are sb-transverse. But g, h are not c-transverse, since we have $\bigl((0, x_2), \{x_1 = 0\}\bigr)$ in $C_1(X)$ and $\bigl(0, \{y = 0\}\bigr)$ in $C_1(Y)$ with $C(g)\bigl((0, x_2), \{x_1 = 0\}\bigr) = C(h)\bigl(0, \{y = 0\}\bigr) = \bigl((0, 0), \{z_1 = z_2 = 0\}\bigr)$ in $C_2(Z)$.

Theorem 2.32 gives a fibre product $W = X \times_{g,Z,h} Y$ in $\mathbf{Man^c_{in}}$, where

$$W = \bigl\{\bigl((w, 0), w\bigr) : w \in [0, \infty)\bigr\} \cong [0, \infty).$$

It is also a fibre product in $\mathbf{Man^{gc}_{in}}$. Note that W is not given by the usual formula (2.21) which also contains points $\bigl((0, x_2), 0\bigr)$ for $0 \neq x_2 \in \mathbb{R}$, that is, *$W$ is not a fibre product at the level of topological spaces.* In this case no fibre product $X \times_Z Y$ exists in $\mathbf{Man^c}$ or $\mathbf{Man^{gc}}$.

Example 2.36. Let $X = Y = [0,\infty)$ and $Z = [0,\infty)^2$, and define $g : X \to Z$, $h : Y \to Z$ by $g(x) = (x,x)$, $h(y) = (y,y^2)$. Then g,h are sb-transverse. However, they are not c-transverse, since we have $\bigl(0,\{x=0\}\bigr)$ in $C_1(X)$ and $\bigl(0,\{y=0\}\bigr)$ in $C_1(Y)$ with $C(g)\bigl(0,\{x=0\}\bigr) = C(h)\bigl(0,\{y=0\}\bigr) = \bigl((0,0),\{z_1=z_2=0\}\bigr)$ in $C_2(Z)$.

The fibre product $W = X \times_{g,Z,h} Y$ in $\mathbf{Man^c_{in}}$ given by Theorem 2.32 is $W = \{(1,1)\}$, a single point. Although g,h are not c- or sc-transverse, in this case a fibre product $W' = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^c}$ with $W' = \{(0,0),(1,1)\}$. So *fibre products* $X \times_{g,Z,h} Y$ *in* $\mathbf{Man^c_{in}}$ *and* $\mathbf{Man^c}$ *exist, but do not coincide.*

Remark 2.37. Suppose we have some category of 'manifolds' $\dot{\mathbf{Man}}$ such as $\mathbf{Man}, \mathbf{Man^c}, \mathbf{Man^c_{in}}, \ldots$, and morphisms $g : X \to Z$, $h : Y \to Z$ in $\dot{\mathbf{Man}}$ for which a fibre product $W = X \times_{g,Z,h} Y$ exists in $\dot{\mathbf{Man}}$. When should we expect W to be given, either as a set or as a topological space, by the usual formula

$$W = \bigl\{(x,y) \in X \times Y : g(x) = h(y) \text{ in } Z\bigr\}? \tag{2.35}$$

From §2.5.1–§2.5.4 we observe that:

(i) Theorems 2.22(a), 2.25(a), 2.28(b) and 2.32 show that (2.35) holds in topological spaces for transverse fibre products in $\mathbf{Man}$, and s-transverse fibre products in $\mathbf{Man^c_{st}}$, and c-transverse fibre products in $\mathbf{Man^{gc}}$, and sc-transverse fibre products in $\mathbf{Man^c}$.

(ii) Theorems 2.28(a) and 2.32 show that b- and sb-transverse fibre products in $\mathbf{Man^{gc}_{in}}$ and $\mathbf{Man^c_{in}}$ are given by a different formula to (2.35), and in Examples 2.35 and 2.36 equation (2.35) is false at the level of sets.

(iii) Example 2.23(iv) gives a non-transverse fibre product in $\mathbf{Man}$ such that (2.35) holds at the level of sets, but not at the level of topological spaces.

For some categories $\dot{\mathbf{Man}}$, there is a 1-1 correspondence between morphisms $f : \{*\} \to X$ in $\dot{\mathbf{Man}}$, and points $x \in X$ of the underlying topological space, by $f \leftrightarrow f(*) = x$. This holds when $\dot{\mathbf{Man}} = \mathbf{Man}, \mathbf{Man^c_{st}}, \mathbf{Man^{gc}}, \mathbf{Man^c}$. For such $\dot{\mathbf{Man}}$, the universal property of fibre products in Definition A.3 applied to $W' = \{*\}$ shows that (2.35) holds automatically *at the level of sets*, though not necessarily for topological spaces, as Example 2.23(iv) shows. In $\mathbf{Man^{gc}_{in}}$ and $\mathbf{Man^c_{in}}$, morphisms $f : \{*\} \to X$ correspond not to $x \in X$, but to $x \in X^\circ$. Then (2.35) can be false even for sets, as Examples 2.35 and 2.36 show.

2.6 Orientations

Orientations on manifolds are discussed by Lee [71, §15], and on manifolds with boundary and corners by the author [59, §7], [57] and Fukaya et al. [24, §8.2].

Definition 2.38. An *orientation* o_X on a manifold X is an equivalence class $[\omega]$ of top-degree forms $\omega \in \Gamma^\infty(\Lambda^{\dim X} T^*X)$ with $\omega|_x \neq 0$ for all $x \in X$, where two such ω, ω' are equivalent if $\omega' = K \cdot \omega$ for $K : X \to (0,\infty)$ smooth. The *opposite* *orientation* is $-o_X=[-\omega]$. Then we call (X,o_X) an *oriented manifold*. Usually we suppress the orientation o_X, and just refer to X as an oriented manifold, and then we write $-X$ for X with the opposite orientation. A nonvanishing top-degree form ω on X is called *positive* if $[\omega]=o_X$, and *negative* if $[\omega]=-o_X$.

If $x\in X$ and $(v_1,\ldots,v_m)$ is a basis for T_xX, then we call $(v_1,\ldots,v_m)$ *oriented* if $\omega|_x\cdot v_1\wedge\cdots\wedge v_m>0$, and *anti-oriented* otherwise.

We will refer to the real line bundle $\Lambda^{\dim X}T^*X\rightarrow X$ as the *canonical bundle* K_X of X, following common practice in algebraic geometry. Then an orientation on X is an orientation on the fibres of K_X.

Let $f:X\rightarrow Y$ be a smooth map of manifolds. A *coorientation* c_f for f is a an equivalence class $[\gamma]$ of $\gamma\in\Gamma^\infty\bigl(\Lambda^{\dim X}T^*X\otimes f^*(\Lambda^{\dim Y}T^*Y)^*\bigr)$ with $\gamma|_x\neq 0$ for all $x\in X$, where γ,γ' are equivalent if $\gamma'=K\cdot\gamma$ for $K:X\rightarrow(0,\infty)$ smooth. The *opposite coorientation* is $-c_f=[-\gamma]$. If Y is oriented then coorientations on f are equivalent to orientations on X. Orientations on X are equivalent to coorientations on $\pi:X\rightarrow *$, for $*$ the point.

All the above also works for manifolds with boundary $\mathbf{Man^b}$ and corners $\mathbf{Man^c}$, their subcategories $\mathbf{Man^c_{in}},\ldots,$ and $\mathbf{Man^{gc}},\mathbf{Man^{ac}}$ in §2.4. For $\mathbf{Man^c}$ we can define orientations using either $\Lambda^{\dim X}T^*X$ or $\Lambda^{\dim X}({}^bT^*X)$, and they yield equivalent notions of orientation, since an orientation o_X on X is determined by its restriction to $X^\circ|_X$, and $T^*X|_{X^\circ}={}^bT^*X|_{X^\circ}$.

Operations on manifolds with corners $X,Y,Z,\ldots$ such as products $X\times Y$, transverse fibre products $X\times_{g,Z,h}Y$, and boundaries ∂X, can be lifted to oriented manifolds with corners. To do this requires a choice of *orientation convention*. Ours are equivalent to those of Fukaya et al. [24, §8.2], see also [59, §7].

Convention 2.39. **(a)** Let X,Y be oriented manifolds. Then there is a natural orientation on $X\times Y$, such that if $x\in X$, $y\in Y$ and $(u_1,\ldots,u_m),(v_1,\ldots,v_n)$ are oriented bases for T_xX,T_yY then $(u_1,\ldots,u_m,v_1,\ldots,v_n)$ is an oriented basis for $T_{(x,y)}(X\times Y)=T_xX\oplus T_yY$. This also works for manifolds with boundary, corners, g-corners, $\ldots$, using T_xX,T_xY or ${}^bT_xX,{}^bT_xY$.

(b) Let X,Y,Z be oriented manifolds, $g:X\rightarrow Z$, $h:Y\rightarrow Z$ be transverse smooth maps, and $W=X\times_{g,Z,h}Y$ be the fibre product as in §2.5.1, with projections $e:W\rightarrow X$, $f:W\rightarrow Y$. Then there is a natural orientation on W, such that if $w\in W$ with $e(w)=x\in X$, $f(w)=y\in Y$ and $g(x)=h(y)=z\in Z$, so that we have an exact sequence of tangent spaces

$$0 \longrightarrow T_wW \xrightarrow{T_we\oplus T_wf} T_xX\oplus T_yY = T_{(x,y)}(X\times Y) \xrightarrow{T_xg\oplus -T_yh} T_zZ \longrightarrow 0,$$

then if $(u_1,\ldots,u_m)$ is an oriented basis for T_wW, and

$$\bigl((T_we\oplus T_wf)(u_1),\ldots,(T_we\oplus T_wf)(u_m),v_1,\ldots,v_n\bigr)$$

is an oriented basis for $T_{(x,y)}(X\times Y)$ using the orientation from **(a)**, then

$$\bigl((-1)^{\dim Y\dim Z}(T_xg\oplus -T_yh)(v_1),(T_xg\oplus -T_yh)(v_2),\ldots,(T_xg\oplus -T_yh)(v_n)\bigr)$$

is an oriented basis for T_zZ. This also works for manifolds with corners, etc.

(c) Let X be an oriented manifold with boundary, or corners (etc.). Then there is a natural orientation on the boundary ∂X, such that if $(x_1,\ldots,x_m)$ in $[0,\infty)\times\mathbb{R}^{m-1}$ are local coordinates on X near $x\in S^1(X)$ and $(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_m})$ are an oriented basis of T_xX, or equivalently $(x_1\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_m})$ are an oriented basis of ${}^bT_xX$, then $(\frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_m})$ are an anti-oriented basis of $T_{(x,\{x_1=0\})}(\partial X)$, or equivalently ${}^bT_{(x,\{x_1=0\})}(\partial X)$. We can also explain this using (2.12) or (2.14).

If X is an oriented manifold with corners then part **(c)** gives orientations on $\partial X,\partial^2X,\ldots,\partial^{\dim X}X$. Note however that the free S_k-action on ∂^kX does not preserve orientations for $k\geqslant 2$, so we cannot define an orientation on $C_k(X)\cong\partial^kX/S_k$ in (2.4), and $C_k(X)$ can be non-orientable for $k\geqslant 2$.

There are often canonical diffeomorphisms between expressions involving fibre products and boundaries of manifolds with corners. When we promote these to oriented manifolds with corners using Convention 2.39, there will be some sign relating the orientations on each side.

For example, in Theorem 2.33, if X,Y,Z are oriented then in oriented manifolds with corners, as in [59, Prop. 7.4], equation (2.34) becomes

$$\partial(X\times_{g,Z,h}Y)\cong\bigl(\partial X\times_{g\circ i_X,Z,h}Y\bigr)\amalg(-1)^{\dim X+\dim Z}\bigl(X\times_{g,Z,h\circ i_Y}\partial Y\bigr). \tag{2.36}$$

Here [59, Prop. 7.5] are some more identities on orientations:

Proposition 2.40. **(a)** *If $g:X\to Z$, $h:Y\to Z$ are transverse smooth maps of oriented manifolds with corners then in oriented manifolds we have*

$$X\times_{g,Z,h}Y\cong(-1)^{(\dim X-\dim Z)(\dim Y-\dim Z)}Y\times_{h,Z,g}X. \tag{2.37}$$

(b) *If $e:V\to Y$, $f:W\to Y$, $g:W\to Z$, $h:X\to Z$ are smooth maps of oriented manifolds with corners then in oriented manifolds we have*

$$V\times_{e,Y,f\circ\pi_W}\bigl(W\times_{g,Z,h}X\bigr)\cong\bigl(V\times_{e,Y,f}W\bigr)\times_{g\circ\pi_W,Z,h}X, \tag{2.38}$$

provided all four fibre products are transverse.

(c) *If $e:V\to Y$, $f:V\to Z$, $g:W\to Y$, $h:X\to Z$ are smooth maps of oriented manifolds with corners then in oriented manifolds we have*

$$\begin{aligned}&V\times_{(e,f),Y\times Z,g\times h}(W\times X)\cong\\&\quad(-1)^{\dim Z(\dim Y+\dim W)}(V\times_{e,Y,g}W)\times_{f\circ\pi_V,Z,h}X,\end{aligned} \tag{2.39}$$

provided all three fibre products are transverse.

Chapter 3

Assumptions about 'manifolds'

In Chapters 4–6, starting from a category $\dot{\mathbf{Man}}$ of 'manifolds' satisfying some assumptions, we will construct 2-categories $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}, \dot{\mathbf{K}}\mathbf{ur}$ of '(m-)Kuranishi spaces', and a category $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ of 'μ-Kuranishi spaces' associated to $\dot{\mathbf{Man}}$.

When $\dot{\mathbf{Man}}$ is the usual category of smooth manifolds $\mathbf{Man}$, this will yield our usual (2-)categories of (m- or μ-)Kuranishi spaces $\mathbf{mKur}, \boldsymbol{\mu}\mathbf{Kur}, \mathbf{Kur}$. But there are many other possibilities for $\dot{\mathbf{Man}}$.

Sections 3.1–3.3 set out our basic assumptions and additional structures on the category $\dot{\mathbf{Man}}$, give examples of categories $\dot{\mathbf{Man}}$ satisfying these conditions, explain some consequences of them, and define notation to be used later.

If $\dot{\mathbf{Man}}$ satisfies the assumptions of §3.1, much of conventional differential geometry for classical manifolds $\mathbf{Man}$ can be extended to $\dot{\mathbf{Man}}$ — smooth functions and partitions of unity, vector bundles, tangent and cotangent bundles, connections, and so on. To streamline our presentation, we will do this extension in detail in Appendix B, and summarize the results in §3.3.

Section 3.4 extends §3.1–§3.3 to categories $\dot{\mathbf{Man}}^{\mathbf{c}}$ of 'manifolds with corners'. In fact §3.1–§3.3 already apply without change to $\dot{\mathbf{Man}} = \dot{\mathbf{Man}}^{\mathbf{c}}$, as the basic assumptions on $\dot{\mathbf{Man}}$ in §3.1 are weak enough to include the categories of manifolds with corners $\dot{\mathbf{Man}}^{\mathbf{c}}$ we are interested in. So the material of §3.1–§3.3 and Chapters 4–6 does not need to be repeated, and our focus in §3.4 is on issues special to the corners case, such as interior maps, simple maps, boundaries ∂X, corners $C_k(X)$, and the corner functor $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}_{\mathbf{in}}$.

3.1 Core assumptions on 'manifolds'

This section gives seven assumptions, Assumptions 3.1–3.7, which we will make on all our categories of 'manifolds'. They are the minimal assumptions we will need to define nicely behaved (2-)categories $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}, \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}, \dot{\mathbf{K}}\mathbf{ur}$ of (m- and μ-)Kuranishi spaces in Chapters 4–6.

Some assumptions require us to give data, and others require this data to have certain properties. The essential data we have to provide is:

- A category $\dot{\mathbf{Man}}$ in Assumption 3.1.

- A faithful functor $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}} : \dot{\mathbf{Man}} \to \mathbf{Top}$ to the category of topological spaces **Top** in Assumption 3.2.
- An inclusion $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$ of the category of classical manifolds **Man** as a full subcategory in Assumption 3.4.

Some examples to have in mind when reading this section, which satisfy all the assumptions, are the category **Man** of classical manifolds, and the categories of manifolds with corners $\mathbf{Man^c_{we}}$, $\mathbf{Man^c}$, $\mathbf{Man^c_{in}}$, $\mathbf{Man^c_{st}}$, $\mathbf{Man^c_{st,in}}$, $\mathbf{Man^{gc}}$, $\mathbf{Man^{gc}_{in}}$, $\mathbf{Man^{ac}}$, $\mathbf{Man^{ac}_{in}}, \ldots$ from Chapter 2.

3.1.1 General properties

Assumption 3.1. (Category-theoretic properties.) **(a)** We are given a category $\dot{\mathbf{Man}}$. For simplicity, from Chapter 4 onwards, objects X in $\dot{\mathbf{Man}}$ will be called *manifolds* (although they may in examples not be manifolds, but some kind of singular space), and morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ will be called *smooth maps* (although they may in examples be non-smooth).

Isomorphisms in $\dot{\mathbf{Man}}$ are called *diffeomorphisms.*

(b) There is an object $\emptyset \in \dot{\mathbf{Man}}$ called the *empty set*, which is an initial object in $\dot{\mathbf{Man}}$ (i.e. every $X \in \dot{\mathbf{Man}}$ has a unique morphism $\emptyset \to X$).

(c) There is an object $* \in \dot{\mathbf{Man}}$ called the *point*, which is a terminal object in $\dot{\mathbf{Man}}$ (i.e. every $X \in \dot{\mathbf{Man}}$ has a unique morphism $\pi : X \to *$).

(d) Each object X in $\dot{\mathbf{Man}}$ has a *dimension* $\dim X \in \mathbb{N} = \{0, 1, \ldots\}$, except that $\dim \emptyset$ is undefined, or allowed to take any value. We have $\dim * = 0$.

(e) *Products* $X \times Y$ of objects $X, Y \in \dot{\mathbf{Man}}$ exist in $\dot{\mathbf{Man}}$, in the sense of category theory (fibre products over $*$), with projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. They have $\dim(X \times Y) = \dim X + \dim Y$. Hence *products* $f \times g : W \times X \to Y \times Z$ of morphisms $f : W \to Y$, $g : X \to Z$, and *direct products* $(f, g) : X \to Y \times Z$ of $f : X \to Y$, $g : X \to Z$, exist in $\dot{\mathbf{Man}}$.

(f) If $X, Y \in \dot{\mathbf{Man}}$ with $\dim X = \dim Y$ there is a *disjoint union* $X \amalg Y$ in $\dot{\mathbf{Man}}$ with inclusion morphisms $\iota_X : X \hookrightarrow X \amalg Y$, $\iota_Y : Y \hookrightarrow X \amalg Y$. It is a coproduct in the sense of category theory, with $\dim(X \amalg Y) = \dim X = \dim Y$.

Assumption 3.2. (Underlying topological spaces.) **(a)** There is a faithful functor $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}} : \dot{\mathbf{Man}} \to \mathbf{Top}$ from $\dot{\mathbf{Man}}$ to the category of topological spaces **Top**, mapping objects $X \in \dot{\mathbf{Man}}$ to the *underlying topological space* $X_{\mathrm{top}} := F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}(X)$, and morphisms $f : X \to Y$ to $f_{\mathrm{top}} := F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}(f) : X_{\mathrm{top}} \to Y_{\mathrm{top}}$.

So we can think of objects X of $\dot{\mathbf{Man}}$ as 'topological spaces X_{top} with extra structure'. Since $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}$ is faithful (injective on morphisms), so that $f_{\mathrm{top}} : X_{\mathrm{top}} \to Y_{\mathrm{top}}$ determines $f : X \to Y$, we can think of morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ as 'continuous maps f_{top} satisfying conditions'.

(b) Underlying topological spaces X_{top} are Hausdorff, locally compact, and second countable, and $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}(\emptyset) = \emptyset$, and $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}(*)$ is a point.

(c) $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}$ takes products and disjoint unions in $\dot{\mathbf{Man}}$ functorially to products and disjoint unions in **Top**.

(d) If $X \in \dot{\mathbf{Man}}$ and $U' \subseteq X_{\rm top}$ is open with inclusion $i' : U' \hookrightarrow X_{\rm top}$, there is a natural object U in $\dot{\mathbf{Man}}$ called an *open submanifold* with $U_{\rm top} = U'$ and $\dim U = \dim X$, and an *inclusion morphism* $i : U \hookrightarrow X$ with $i_{\rm top} = i'$. If $U' = \emptyset$ then $U = \emptyset$. Inclusion morphisms are functorial under inclusions of open sets $U' \hookrightarrow V' \hookrightarrow X_{\rm top}$. Given a morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$, we often write $f|_U : U \to Y$ instead of $f \circ i : U \to Y$.

If $f : W \to X$ is a morphism in $\dot{\mathbf{Man}}$ with $f_{\rm top}(W_{\rm top}) \subseteq U_{\rm top} \subseteq X_{\rm top}$ then f factorizes uniquely as $f = i \circ f'$ for a morphism $f' : W \to U$ in $\dot{\mathbf{Man}}$. If f is an open submanifold then so is f'.

Inclusions $\iota_X : X \hookrightarrow X \amalg Y$, $\iota_Y : Y \hookrightarrow X \amalg Y$ are open submanifolds.

(e) Suppose $X \in \dot{\mathbf{Man}}$, and Y' is a topological space, and $\psi : X_{\rm top} \to Y'$ is a homeomorphism. Then there exists an object $Y \in \dot{\mathbf{Man}}$ and a diffeomorphism $\phi : X \to Y$ such that $Y_{\rm top} = Y'$ and $\phi_{\rm top} = \psi$.

In later chapters we will generally drop the distinction between X and $X_{\rm top}$, and write $x \in X$ rather than $x \in X_{\rm top}$, identify open submanifolds $i : U \hookrightarrow X$ with open sets $U \subseteq X$, and so on, just as one does for ordinary manifolds in differential geometry.

We suppose morphisms and objects in $\dot{\mathbf{Man}}$ can be glued over open covers.

Assumption 3.3. (Sheaf-theoretic properties.) **(a)** Let X, Y be objects in $\dot{\mathbf{Man}}$, and $f' : X_{\rm top} \to Y_{\rm top}$ be a continuous map, and $\{U'_a : a \in A\}$ be an open cover of $X_{\rm top}$. Write $i_a : U_a \hookrightarrow X$ for the open submanifold with $U_{a,\rm top} = U'_a$, and suppose there is a morphism $f_a : U_a \to Y$ in $\dot{\mathbf{Man}}$ with $f_{a,\rm top} = f' \circ i_{a,\rm top} : U_{a,\rm top} \to Y_{\rm top}$ for each $a \in A$. Then there is a morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$ with $f_{\rm top} = f'$ and $f \circ i_a = f_a$ for all $a \in A$. Note that f_a, f must be unique by faithfulness in Assumption 3.2(a).

This implies that morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ form a sheaf on X.

(b) Let X' be a Hausdorff, second countable topological space, $\{U'_a : a \in A\}$ an open cover of X', and $\{U_a : a \in A\}$ a family of objects in $\dot{\mathbf{Man}}$ with $U_{a,\rm top} = U'_a$ and $\dim U_a = m$ for all $a \in A$, with $m \in \mathbb{N}$. For $a, b \in A$ write $i_{ab} : U_{ab} \hookrightarrow U_a$ for the open submanifold associated to $U'_a \cap U'_b \subset U'_a = U_{a,\rm top}$.

Suppose that there is a (necessarily unique) diffeomorphism $j_{ab} : U_{ab} \to U_{ba}$ in $\dot{\mathbf{Man}}$ with $j_{ab,\rm top} = \mathrm{id}_{U'_a \cap U'_b}$ for all $a, b \in A$. Then there exists an object X in $\dot{\mathbf{Man}}$ with $X_{\rm top} = X'$ and $\dim X = m$, unique up to diffeomorphism, covered by open submanifolds $i_a : U_a \hookrightarrow X$ for $a \in A$, for U_a as above.

3.1.2 Relation with classical manifolds

Assumption 3.4. (Inclusion of ordinary manifolds.) The usual category **Man** of smooth manifolds and smooth maps between them is included as a full subcategory $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$.

Dimensions of objects in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$ are as usual in $\mathbf{Man}$. Products and disjoint unions in $\dot{\mathbf{Man}}$ of $X, Y \in \mathbf{Man}$ agree with those in $\mathbf{Man}$. The empty set $\emptyset$ and point $*$ in Assumption 3.1(b),(c) lie in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$.

The underlying topological space functor $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}$ is as usual on $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$. Open submanifolds in $\mathbf{Man}, \dot{\mathbf{Man}}$ agree. We will often use that $\mathbb{R}^n$ is an object of $\dot{\mathbf{Man}}$ for $n = 0, 1, \ldots,$ since $\mathbb{R}^n \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}$. We generally write $\mathbb{R}^n$ rather than $\mathbb{R}^n_{\mathrm{top}}$, and X rather than X_{top} when $X \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}$.

From Chapter 4 onwards, by an abuse of notation we will usually refer to objects X of $\dot{\mathbf{Man}}$ as 'manifolds', and morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ as 'smooth maps'. When we need to refer to objects $X \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}$ we will call them 'classical manifolds', and morphisms $f : X \to Y$ in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$ 'classical smooth maps'.

Assumption 3.5. (Hadamard's Lemma.) Suppose X is an object in $\dot{\mathbf{Man}}$, and $i : U \hookrightarrow X \times \mathbb{R}^n$ is an open submanifold with $(x, 0, \ldots, 0) \in U_{\mathrm{top}}$ for all $x \in X_{\mathrm{top}}$, and $f : U \to \mathbb{R}$ is a morphism in $\dot{\mathbf{Man}}$. Then there exist morphisms $g_1, \ldots, g_n : U \to \mathbb{R}$ in $\dot{\mathbf{Man}}$ with

$$f_{\mathrm{top}}(x, t_1, \ldots, t_n) = f_{\mathrm{top}}(x, 0, \ldots, 0) + \textstyle\sum_{i=1}^n t_i \cdot g_{i,\mathrm{top}}(x, t_1, \ldots, t_n) \qquad (3.1)$$

for all $(x, t_1, \ldots, t_n) \in U_{\mathrm{top}}$, so that $x \in X_{\mathrm{top}}$ and $t_1, \ldots, t_n \in \mathbb{R}$.

Note that this has strong implications for the differentiability of functions in $\dot{\mathbf{Man}}$. For example, taking partial derivatives of (3.1) in $t_1, \ldots, t_n$ at $t_1 = \cdots = t_n = 0$ and noting that $g_{1,\mathrm{top}}, \ldots, g_{n,\mathrm{top}}$ are continuous implies that

$$\tfrac{\partial f_{\mathrm{top}}}{\partial t_i}(x, 0, \ldots, 0) = g_{i,\mathrm{top}}(x, 0, \ldots, 0) \qquad (3.2)$$

for all $x \in X_{\mathrm{top}}$, where the partial derivative exists. A more complicated argument shows that there exist unique morphisms $h_1, \ldots, h_n : U \to \mathbb{R}$ in $\dot{\mathbf{Man}}$ with $h_{i,\mathrm{top}}(x, t_1, \ldots, t_n) = \frac{\partial f_{\mathrm{top}}}{\partial t_i}(x, t_1, \ldots, t_n)$ for all $(x, t_1, \ldots, t_n) \in U_{\mathrm{top}}$.

The next assumption means that for $X \in \dot{\mathbf{Man}}$, the topology on X_{top} is generated by open subsets $f_{\mathrm{top}}^{-1}((0, \infty)) \subseteq X_{\mathrm{top}}$ for smooth functions $f : X \to \mathbb{R}$.

Assumption 3.6. (Topology is generated by smooth functions to $\mathbb{R}$.) Let X be an object of $\dot{\mathbf{Man}}$. As $\mathbb{R} \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}$, we can consider morphisms $f : X \to \mathbb{R}$ in $\dot{\mathbf{Man}}$. Suppose $U' \subseteq X_{\mathrm{top}}$ is open and $x \in U'$. Then there should exist $f : X \to \mathbb{R}$ in $\dot{\mathbf{Man}}$ with $f_{\mathrm{top}}(x) > 0$ and $f_{\mathrm{top}}|_{X_{\mathrm{top}} \setminus U'} \leqslant 0$.

3.1.3 Extension properties of smooth maps

Assumptions 3.1–3.6 hold for many categories of manifold-like spaces, including some which are not suitable for defining Kuranishi spaces. Though its significance is probably not clear on a first reading, our next assumption makes many features of ordinary manifolds work in $\dot{\mathbf{Man}}$, and is vital for much that we do in this book. For example, we show in §B.4 that Assumption 3.7(a) allows us to define a 'tangent sheaf $\mathcal{T}X$' for objects $X \in \dot{\mathbf{Man}}$, a substitute for the tangent bundle $TX \to X$ for $X \in \mathbf{Man}$.

Assumption 3.7. (Extension properties of smooth maps.) **(a)** Let X, Y be objects in $\dot{\mathbf{Man}}$, and $k \geqslant 2$, $n > 0$. Suppose

$$U_i \hookrightarrow X \times (\mathbb{R}^n)^{k-1}$$

is an open submanifold for $i = 1, \ldots, k$ with $X_{\mathrm{top}} \times \{(0, \ldots, 0)\} \subset U_{i,\mathrm{top}}$, and $f_i : U_i \to Y$ is a morphism in $\dot{\mathbf{Man}}$ for $i = 1, \ldots, k$ such that

$$\begin{aligned} f_{i,\mathrm{top}}(x,& \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{i-1}, \boldsymbol{z}_{i+1}, \ldots, \boldsymbol{z}_{j-1}, 0, \boldsymbol{z}_{j+1}, \ldots, \boldsymbol{z}_k) \\ &= f_{j,\mathrm{top}}(x, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{i-1}, 0, \boldsymbol{z}_{i+1}, \ldots, \boldsymbol{z}_{j-1}, \boldsymbol{z}_{j+1}, \ldots, \boldsymbol{z}_k) \end{aligned}$$

for all $1 \leqslant i < j \leqslant k$, $x \in X_{\mathrm{top}}$ and $\boldsymbol{z}_a \in \mathbb{R}^n$ for $a = 1, \ldots, k$, $a \neq i, j$, such that $(x, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{i-1}, \boldsymbol{z}_{i+1}, \ldots, \boldsymbol{z}_{j-1}, 0, \boldsymbol{z}_{j+1}, \ldots, \boldsymbol{z}_k) \in U_{i,\mathrm{top}}$ and $(x, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{i-1}, 0, \boldsymbol{z}_{i+1}, \ldots, \boldsymbol{z}_{j-1}, \boldsymbol{z}_{j+1}, \ldots, \boldsymbol{z}_k) \in U_{j,\mathrm{top}}$. Then there should exist an open submanifold $V \hookrightarrow X \times (\mathbb{R}^n)^k$ with $X_{\mathrm{top}} \times \{(0, \ldots, 0)\} \subset V_{\mathrm{top}}$, and a morphism $g : V \to Y$ in $\dot{\mathbf{Man}}$ such that

$$f_{i,\mathrm{top}}(x, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{i-1}, \boldsymbol{z}_{i+1}, \ldots, \boldsymbol{z}_k) = g_{\mathrm{top}}(x, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{i-1}, 0, \boldsymbol{z}_{i+1}, \ldots, \boldsymbol{z}_k)$$

for all $i = 1, \ldots, k$, $x \in X_{\mathrm{top}}$ and $\boldsymbol{z}_a \in \mathbb{R}^n$ for $a = 1, \ldots, k$, $a \neq i$, with $(x, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{i-1}, \boldsymbol{z}_{i+1}, \ldots, \boldsymbol{z}_k) \in U_{i,\mathrm{top}}$, $(x, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{i-1}, 0, \boldsymbol{z}_{i+1}, \ldots, \boldsymbol{z}_k) \in V_{\mathrm{top}}$.

(b) In part **(a)**, suppose in addition that $s : X \to \mathbb{R}^n$ and $h : X \to Y$ are morphisms in $\dot{\mathbf{Man}}$ with

$$f_{i,\mathrm{top}}(x, t_1 \cdot s_{\mathrm{top}}(x), \ldots, t_{i-1} \cdot s_{\mathrm{top}}(x), t_{i+1} \cdot s_{\mathrm{top}}(x), \ldots, t_k \cdot s_{\mathrm{top}}(x)) = h_{\mathrm{top}}(x)$$

for all $i = 1, \ldots, k$, $x \in X_{\mathrm{top}}$ and $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k \in \mathbb{R}$ with $t_1 + \cdots + t_{i-1} + t_{i+1} + \cdots + t_k = 1$ and $(x, t_1 \cdot s_{\mathrm{top}}(x), \ldots, t_{i-1} \cdot s_{\mathrm{top}}(x), t_{i+1} \cdot s_{\mathrm{top}}(x), \ldots, t_k \cdot s_{\mathrm{top}}(x)) \in U_{i,\mathrm{top}}$. Then we can choose g to satisfy

$$g_{\mathrm{top}}(x, t_1 \cdot s_{\mathrm{top}}(x), \ldots, t_k \cdot s_{\mathrm{top}}(x)) = h_{\mathrm{top}}(x)$$

for all $x \in X_{\mathrm{top}}$ and $t_1, \ldots, t_k \in \mathbb{R}$ with $t_1 + \cdots + t_k = 1$ and $(x, t_1 \cdot s_{\mathrm{top}}(x), \ldots, t_k \cdot s_{\mathrm{top}}(x)) \in V_{\mathrm{top}}$.

(c) In both **(a)** and **(b)**, suppose the whole situation is invariant/equivariant under a finite group Γ, which acts on X, Y by diffeomorphisms in $\dot{\mathbf{Man}}$, and acts linearly on $\mathbb{R}^n$, and may also act on $\{1, \ldots, k\}$ by permutations, and hence permute the $U_i, f_i, \boldsymbol{z}_i, t_i$ for $i = 1, \ldots, k$, in addition to the Γ-actions on $X, Y, \mathbb{R}^n$. Then we can choose V to be Γ-invariant, and $g : V \to Y$ to be Γ-equivariant.

3.2 Examples of categories satisfying the assumptions

Here are some examples of categories $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7.

Example 3.8. **(i)** The usual category of manifolds **Man** from Chapter 2 satisfies all assumptions in §3.1.

(ii) In Chapter 2 we discussed many categories of manifolds with corners. Of these, the following satisfy all assumptions in §3.1:

$$\begin{gathered}\mathbf{Man^c_{we}}, \mathbf{Man^c}, \mathbf{Man^c_{in}}, \mathbf{Man^c_{bn}}, \mathbf{Man^c_{st}}, \mathbf{Man^c_{st,in}}, \\ \mathbf{Man^{gc}}, \mathbf{Man^{gc}_{in}}, \mathbf{Man^{gc}_{bn}}, \mathbf{Man^{ac}}, \mathbf{Man^{ac}_{in}}, \mathbf{Man^{ac}_{bn}}, \mathbf{Man^{ac}_{st}}, \\ \mathbf{Man^{ac}_{st,in}}, \mathbf{Man^{c,ac}}, \mathbf{Man^{c,ac}_{in}}, \mathbf{Man^{c,ac}_{bn}}, \mathbf{Man^{c,ac}_{st}}, \mathbf{Man^{c,ac}_{st,in}}.\end{gathered} \tag{3.3}$$

Example 3.9. In §6.6 we will define the 2-category of orbifolds **Orb**. Define a 2-subcategory $\mathbf{Orb^{eff}_{sur}} \subset \mathbf{Orb}$ with objects $\mathfrak{X}$ effective orbifolds, and with 1-morphisms $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ whose morphisms of isotropy groups $G_x\mathfrak{f}: G_x\mathfrak{X} \to G_y\mathfrak{Y}$ are surjective for all $x \in \mathfrak{X}$ with $\mathfrak{f}(x) = y \in \mathfrak{Y}$, and with arbitrary 2-morphisms. Consider the homotopy category $\mathrm{Ho}(\mathbf{Orb^{eff}_{sur}})$. The combination of the effective and surjective conditions means that $\mathbf{Orb^{eff}_{sur}}$ is a *discrete* 2-category (i.e. there is at most one 2-morphism $\eta: \mathfrak{f} \Rightarrow \mathfrak{g}$ between any two 1-morphisms $\mathfrak{f}, \mathfrak{g}: \mathfrak{X} \to \mathfrak{Y}$ in $\mathbf{Orb^{eff}_{sur}}$). So $\mathbf{Orb^{eff}_{sur}}$ is equivalent to $\mathrm{Ho}(\mathbf{Orb^{eff}_{sur}})$ as a 2-category, and passing to the homotopy category does not lose any important information.

Any orbifold $\mathfrak{X}$ has a natural locally closed stratification $\mathfrak{X} = \coprod_{k=0}^{\dim \mathfrak{X}} \mathfrak{X}_k$, where $\mathfrak{X}_k$ is the disjoint union of the orbifold strata of $\mathfrak{X}$ with codimension k, and $\mathfrak{X}_k$ has the structure of a manifold of dimension $\dim \mathfrak{X} - k$. Because of the surjectivity on isotropy groups condition, 1-morphisms $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ are compatible with these stratifications in the sense of Lemma 2.5, and locally in $\mathfrak{X}_k$ induce smooth maps $\mathfrak{f}|_{\mathfrak{X}_k}: \mathfrak{X}_k \to \mathfrak{Y}_l$ between manifolds.

One can now show that the category $\dot{\mathbf{Man}} = \mathrm{Ho}(\mathbf{Orb^{eff}_{sur}})$ satisfies Assumptions 3.1–3.7. There are a few subtle points in the proof. For Assumption 3.3 we use stack-theoretic properties of **Orb** and the fact that $\mathbf{Orb^{eff}_{sur}}$ is a discrete 2-category, so that we get sheaves and not just presheaves when we pass to the homotopy category.

We can also consider 'corners' versions of $\mathrm{Ho}(\mathbf{Orb^{eff}_{sur}})$ modelled on one of the categories in (3.3). These all work without any problems.

Remark 3.10. Here are some categories of manifolds which fail parts of Assumptions 3.1–3.7, and so are excluded from our theory:

(a) The category $\mathbf{Man}_{\mathrm{ra}}$ of real analytic manifolds and real analytic maps fails Assumption 3.4, as there is no inclusion $\mathbf{Man} \subseteq \mathbf{Man}_{\mathrm{ra}}$.

Partitions of unity will be important in our theory, but they do not exist in $\mathbf{Man}_{\mathrm{ra}}$. So we will not define real analytic Kuranishi spaces.

(b) The category $\mathbf{Man}_{C^k}$ of C^k-manifolds for $k \geqslant 0$ fails Assumption 3.5, since in general maps $g_1, \ldots, g_n: U \to \mathbb{R}$ satisfying (3.1) would have to be only C^{k-1}, and so would not be morphisms in $\mathbf{Man}_{C^k}$.

(c) The category $\mathbf{Man^b}$ of manifolds with boundary is not closed under products such as $[0,1] \times [0,1]$, so Assumption 3.1(e) fails. To include this example we should embed $\mathbf{Man^b} \subset \mathbf{Man^c}$ and take $\dot{\mathbf{Man}} = \mathbf{Man^c}$.

(d) As in Remark 2.3, Melrose [84–86] works in the full subcategory $\mathbf{Man^c_{fa}} \subset \mathbf{Man^c}$ of 'manifolds with faces' X, for which $i_X : \partial X \to X$ is injective on each connected component of ∂X. Since this is not a local condition on X, Assumption 3.3(b) fails for $\mathbf{Man^c_{fa}}$. Again, we should take $\dot{\mathbf{Man}} = \mathbf{Man^c}$.

(e) The categories $\mathbf{Man^c_{si}}, \mathbf{Man^{gc}_{si}}, \mathbf{Man^{ac}_{si}}, \mathbf{Man^{c,ac}_{si}}$ in Chapter 2 of various kinds of manifolds with corners, and *simple* maps, fail Assumption 3.6, since if X lies in one of these categories with $\partial X \neq \emptyset$ then no map $f : X \to \mathbb{R}$ is simple, so almost all of §3.3 does not work within $\mathbf{Man^c_{si}}, \ldots$.

However, these categories will play an important rôle in our treatment of (m- and μ-)Kuranishi spaces with corners in §3.4, §4.6, §5.4 and §6.3.

3.3 Differential geometry in $\dot{\mathbf{Man}}$

Suppose $\dot{\mathbf{Man}}$ is a category satisfying Assumptions 3.1–3.7 in §3.1. Much of conventional differential geometry for classical manifolds **Man** can be extended to $\dot{\mathbf{Man}}$ — smooth functions and partitions of unity, vector bundles, tangent and cotangent bundles, connections, and so on. To avoid a lengthy diversion in our narrative, we will explain the extension to $\dot{\mathbf{Man}}$ in detail in Appendix B, and summarize it here. Readers primarily interested in the conventional cases $\dot{\mathbf{Man}} = \mathbf{Man}$ or $\dot{\mathbf{Man}} = \mathbf{Man^c}$ should not need to look at Appendix B.

Here are two important differences with conventional differential geometry:

- If $X \in \dot{\mathbf{Man}}$ is a 'manifold', we will define a *tangent sheaf* $\mathcal{T}X$ and *cotangent sheaf* $\mathcal{T}^*X$, which are our substitutes for the (co)tangent bundles TX, T^*X of a classical manifold. These $\mathcal{T}X, \mathcal{T}^*X$ may not be vector bundles for general $\dot{\mathbf{Man}}$, but are sheaves of modules over the *structure sheaf* $\mathcal{O}_X$ of smooth functions $X \to \mathbb{R}$. Also $\mathcal{T}X, \mathcal{T}^*X$ may not be dual to each other, though there is a natural pairing $\mu_X : \mathcal{T}X \times \mathcal{T}^*X \to \mathcal{O}_X$.
- If $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$, we will define a *relative tangent sheaf* $\mathcal{T}_fY$ of $\mathcal{O}_X$-modules on X, with $\mathcal{T}_fY = \mathcal{T}X$ when $X = Y$ and $f = \mathrm{id}_X$. When $\dot{\mathbf{Man}} = \mathbf{Man}$, $\mathcal{T}_fY$ is the sheaf of sections of the pullback vector bundle $f^*(TY) \to X$, but in general we may have $\mathcal{T}_fY \ncong f^*(\mathcal{T}Y)$.

In §3.3.5 we describe some '$O(s)$' and '$O(s^2)$' notation, explained in detail in §B.5, which will be important in Chapters 4–6.

3.3.1 Smooth functions and the structure sheaf

We summarize the material of §B.1:

(a) For each $X \in \dot{\mathbf{Man}}$, write $C^\infty(X)$ for the set of morphisms $a : X \to \mathbb{R}$ in $\dot{\mathbf{Man}}$. We show that $C^\infty(X)$ has the structure of a commutative $\mathbb{R}$-algebra, and also of a C^∞-*ring*, in the sense of C^∞-algebraic geometry as in the author [56, 65] or Dubuc [13].

(b) We define a sheaf $\mathcal{O}_X$ of commutative $\mathbb{R}$-algebras or C^∞-rings on the topological space $X_{\rm top}$, called the *structure sheaf*, with $\mathcal{O}_X(U_{\rm top}) = C^\infty(U)$ for all open submanifolds $U \hookrightarrow X$. Sheaves are explained in §A.5.

(c) We show that $(X_{\rm top}, \mathcal{O}_X)$ is an *affine C^∞-scheme* in the sense of [13, 56, 65]. If $f : X \to Y$ is a morphism in $\dot{\mathbf{M}}\mathbf{an}$, we define a morphism $(f_{\rm top}, f^\sharp) : (X_{\rm top}, \mathcal{O}_X) \to (Y_{\rm top}, \mathcal{O}_Y)$ of affine C^∞-schemes. This defines a functor $F_{\dot{\mathbf{M}}\mathbf{an}}^{\mathbf{C^\infty Sch}} : \dot{\mathbf{M}}\mathbf{an} \to \mathbf{C^\infty Sch^{aff}}$ to the category of affine C^∞-schemes, which is faithful, but need not be full.

(d) We show that partitions of unity exist in $\mathcal{O}_X$ subordinate to any open cover $\{U_a : a \in A\}$ of X. Thus, $\mathcal{O}_X$ is a *fine sheaf.*

When $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$ all this is standard material.

3.3.2 Vector bundles and sections

In §B.2 we discuss *vector bundles* $E \to X$ in $\dot{\mathbf{M}}\mathbf{an}$, and (smooth) *sections* $s : X \to E$, and we write $\Gamma^\infty(E)$ for the $C^\infty(X)$-module of sections s of E. The usual definitions and operations on vector bundles and sections in differential geometry also work for vector bundles in $\dot{\mathbf{M}}\mathbf{an}$, in exactly the same way with no surprises, so for instance if $E, F \to X$ are vector bundles we can define vector bundles $E^* \to X$, $E \oplus F \to X$, $E \otimes F \to X$, $\Lambda^k E \to X$, and so on, and if $f : X \to Y$ is a morphism in $\dot{\mathbf{M}}\mathbf{an}$ and $G \to Y$ is a vector bundle we can define a pullback vector bundle $f^*(G) \to X$.

If $E \to X$ is a vector bundle, we write $\mathcal{E}$ for the sheaf of sections of E, as a sheaf of modules over $\mathcal{O}_X$. Morphisms of vector bundles $\theta : E \to F$ are in natural 1-1 correspondence with morphisms of $\mathcal{O}_X$-modules $\tilde{\theta} : \mathcal{E} \to \mathcal{F}$.

3.3.3 The cotangent sheaf $\mathcal{T}^*X$, and connections ∇

In §B.3, for each $X \in \dot{\mathbf{M}}\mathbf{an}$ we define the *cotangent sheaf* $\mathcal{T}^*X$, a sheaf of $\mathcal{O}_X$-modules on $X_{\rm top}$. We also define the *de Rham differential* $\mathrm{d} : \mathcal{O}_X \to \mathcal{T}^*X$, a morphism of sheaves of $\mathbb{R}$-vector spaces which is a universal C^∞-derivation. We do this by noting that $(X_{\rm top}, \mathcal{O}_X)$ is an affine C^∞-scheme in the sense of [13, 56, 65], as in §3.3.1 and §B.1, and then using cotangent sheaves of C^∞-schemes from the author [65, §5].

Example 3.11. (a) If $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$ then $\mathcal{T}^*X$ is the sheaf of sections of the usual cotangent bundle $T^*X \to X$ in differential geometry. The same holds if $X \in \mathbf{Man} \subseteq \dot{\mathbf{M}}\mathbf{an}$ for general $\dot{\mathbf{M}}\mathbf{an}$.

(b) If $\dot{\mathbf{M}}\mathbf{an}$ is one of the following categories from Chapter 2:

$$\mathbf{Man^c}, \mathbf{Man^c_{in}}, \mathbf{Man^c_{st}}, \mathbf{Man^c_{st,in}}, \mathbf{Man^c_{we}},$$

then as in §2.3 there are two notions of cotangent bundle $T^*X, {}^bT^*X$ of X in $\dot{\mathbf{M}}\mathbf{an}$. It turns out that $\mathcal{T}^*X$ is isomorphic to the sheaf of sections of T^*X.

(c) If $\dot{\mathbf{Man}}$ is one of the following categories from §2.4:

$$\mathbf{Man^{gc}}, \mathbf{Man^{gc}_{in}}, \mathbf{Man^{ac}}, \mathbf{Man^{ac}_{in}}, \mathbf{Man^{ac}_{st}},$$
$$\mathbf{Man^{ac}_{st,in}}, \mathbf{Man^{c,ac}}, \mathbf{Man^{c,ac}_{in}}, \mathbf{Man^{c,ac}_{st}}, \mathbf{Man^{c,ac}_{st,in}},$$

then the cotangent bundle T^*X of $X \in \dot{\mathbf{Man}}$ may not be defined, though the b-cotangent bundle ${}^bT^*X$ is. It turns out that $\mathcal{T}^*X$ need not be isomorphic to the sheaf of sections of any vector bundle in these cases.

Let $E \to X$ be a vector bundle in $\dot{\mathbf{Man}}$, and $\mathcal{E}$ the $\mathcal{O}_X$-module of sections of E as in §3.3.2. We define a *connection* ∇ on E to be a morphism $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X$ of sheaves of $\mathbb{R}$-vector spaces on X_{top}, satisfying the Leibniz rule $\nabla(a \cdot e) = a \cdot (\nabla e) + e \otimes (\mathrm{d}a)$ for all local sections a of $\mathcal{O}_X$ and e of $\mathcal{E}$. We show that connections ∇ on E always exist, and if ∇, ∇' are two connections then $\nabla' = \nabla + \Gamma$ for $\Gamma : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X$ an $\mathcal{O}_X$-module morphism.

3.3.4 Tangent sheaves $\mathcal{T}X$, and relative tangent sheaves $\mathcal{T}_fY$

We summarize the material of §B.4:

(a) For each $X \in \dot{\mathbf{Man}}$ we define the *tangent sheaf* $\mathcal{T}X$, as a sheaf of $\mathcal{O}_X$-modules on X_{top}.

(b) If $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$ we define the *relative tangent sheaf* $\mathcal{T}_fY$, as an $\mathcal{O}_X$-module on X_{top}. There is a natural $\mathcal{O}_X$-module morphism

$$f^\flat \otimes \mathrm{id}_{\mathcal{O}_X} : f^*(\mathcal{T}Y) = f_{\mathrm{top}}^{-1}(\mathcal{T}Y) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \longrightarrow \mathcal{T}_fY. \tag{3.4}$$

If $g : Y \to Z$ is a morphism in $\dot{\mathbf{Man}}$ we have an $\mathcal{O}_X$-module morphism

$$f^\flat \otimes \mathrm{id}_{\mathcal{O}_X} : f^*(\mathcal{T}_gZ) = f_{\mathrm{top}}^{-1}(\mathcal{T}_gZ) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \longrightarrow \mathcal{T}_{g\circ f}Z. \tag{3.5}$$

Neither of (3.4) or (3.5) need be isomorphisms.

(c) If $f : X \to Y$, $g : Y \to Z$ are morphisms in $\dot{\mathbf{Man}}$ then we define an $\mathcal{O}_X$-module morphism $\mathcal{T}g : \mathcal{T}_fY \to \mathcal{T}_{g\circ f}Z$.

(d) If $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$ and $E, F \to X$ are vector bundles then we define *morphisms* $\theta : E \to \mathcal{T}_fY$, $\phi : \mathcal{T}_fY \to F$. These are just $\mathcal{O}_X$-module morphisms $\theta : \mathcal{E} \to \mathcal{T}_fY$, $\phi : \mathcal{T}_fY \to \mathcal{F}$, for $\mathcal{E}, \mathcal{F}$ the $\mathcal{O}_X$-modules of sections of E, F.

We can compose such morphisms by composing $\mathcal{O}_X$-module morphisms, so that $\phi \circ \theta : \mathcal{E} \to \mathcal{F}$ is a vector bundle morphism $E \to F$.

(e) We define a natural pairing $\mu_X : \mathcal{T}X \times \mathcal{T}^*X \to \mathcal{O}_X$ between tangent and cotangent sheaves.

(f) Let $E \to X$ be a vector bundle in $\dot{\mathbf{Man}}$, ∇ a connection on E, and $s \in \Gamma^\infty(E)$, so that $\nabla s \in \Gamma(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X)$ as in §3.3.3. Using the pairing μ_X in (e) we can regard ∇s as a morphism $\nabla s : \mathcal{T}X \to E$.

(g) Let $f : X \to Y$, $g : Y \to Z$ be morphisms in $\dot{\mathbf{Man}}$, $F \to Y$ be a vector bundle, and $\theta : F \to \mathcal{T}_g Z$ be a morphism on Y, as in (d). We define a morphism $f^*(\theta) : f^*(F) \to \mathcal{T}_{g\circ f} Z$ by composing (3.5) with the pullback of θ under f_{top}. This is something of an abuse of notation: we will treat $\mathcal{T}_{g\circ f} Z$ as if it were the pullback $f^*(\mathcal{T}_g Z)$, although (3.5) may not be an isomorphism. Incorporating (3.5) in the definition of $f^*(\theta)$ allows us to omit $f^\flat \otimes \mathrm{id}_{\mathcal{O}_X}$ in (3.5) from our notation.

(h) Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$, $F \to Y$ be a vector bundle, ∇ a connection on F, and $t \in \Gamma^\infty(F)$, so that $\nabla t \in \Gamma(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{T}^* Y)$. We define a morphism $f^*(\nabla t) : \mathcal{T}_f Y \to f^*(F)$. This is not done by pulling back the morphism $\nabla t : \mathcal{T} Y \to F$ in (f) along f, since the morphism (3.4) goes the wrong way, but by a different method.

Example 3.12. Let $\dot{\mathbf{Man}} = \mathbf{Man}$. Then $\mathcal{T} X$ in (a) is the sheaf of sections of the usual tangent bundle $TX \to X$ in differential geometry, and $\mathcal{T}_f Y$ in (b) is the sheaf of sections of $f^*(TY) \to X$, and (3.4)–(3.5) are isomorphisms. In (c), $\mathcal{T} g$ is the pullback $f^*(Tg) : f^*(TY) \to (g \circ f)^*(TZ)$ of the derivative map $Tg : TY \to g^*(TZ)$. In (d), morphisms are vector bundle morphisms $\theta : E \to f^*(TY)$, $\phi : f^*(TY) \to F$. In (e), μ_X is the usual dual pairing $TX \times T^*X \to \mathcal{O}_X$. In (g),(h), $f^*(\theta), f^*(\nabla t)$ are the usual pullbacks in differential geometry.

The moral is that when $\dot{\mathbf{Man}} = \mathbf{Man}$, we should remember that $\mathcal{T}_f Y$ means $f^*(TY)$, all the sheaves $\mathcal{O}_X, \mathcal{T}^* X, \mathcal{T} X, \mathcal{T}_f Y$ are vector bundles, and all of (a)–(h) are standard differential geometry of classical manifolds.

Example 3.13. Let $\dot{\mathbf{Man}}$ be one of the following categories from Chapter 2:

$$\mathbf{Man^c_{in}}, \mathbf{Man^c_{st,in}}, \mathbf{Man^{gc}_{in}}, \mathbf{Man^{ac}_{in}}, \mathbf{Man^{ac}_{st,in}}, \mathbf{Man^{c,ac}_{in}}, \mathbf{Man^{c,ac}_{st,in}}.$$

Then $\mathcal{T} X$ in (a) is the sheaf of sections of the b-tangent bundle ${}^bTX \to X$ from §2.3, and $\mathcal{T}_f Y$ in (b) is the sheaf of sections of $f^*({}^bTY) \to X$, and (3.4)–(3.5) are isomorphisms. Note that in these cases $\mathcal{T} X$ and $\mathcal{T}^* X$ may not be dual, since as in Example 3.11(b),(c) either $\mathcal{T}^* X$ is the sheaf of sections of $T^*X \to X$ (not ${}^bT^*X \to X$), or $\mathcal{T}^* X$ may not be a vector bundle.

Example 3.14. Let $\dot{\mathbf{Man}}$ be one of the following categories from Chapter 2:

$$\mathbf{Man^c}, \mathbf{Man^c_{st}}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{ac}_{st}}, \mathbf{Man^{c,ac}}, \mathbf{Man^{c,ac}_{st}}.$$

Then $\mathcal{T} X$ in (a) is the sheaf of sections of the b-tangent bundle ${}^bTX \to X$, but $\mathcal{T}_f Y$ in (b) is the sheaf of sections of the vector bundle of mixed rank $C(f)^*({}^bTC(Y))|_{C_0(X)} \to X$, using the corner functor $C(f) : C(X) \to C(Y)$ and the identification $X \cong C_0(X)$ from §2.2. Also (3.4)–(3.5) may not be isomorphisms, and $\mathcal{T} X$ and $\mathcal{T}^* X$ may not be dual.

3.3.5 The $O(s)$ and $O(s^2)$ notation

Section B.5 defines some '$O(s)$' and '$O(s^2)$' notation, which will be important in §4.1, §5.1 and §6.1. Here is an informal version of Definition B.36:

Definition 3.15. Let X be an object in $\dot{\mathbf{Man}}$, and $\pi : E \to X$ be a vector bundle, and $s \in \Gamma^\infty(E)$ be a section. Then:

(i) If $F \to X$ is a vector bundle and $t_1, t_2 \in \Gamma^\infty(F)$, we write $t_2 = t_1 + O(s)$ if there exists a morphism $\alpha : E \to F$ such that $t_2 = t_1 + \alpha \circ s$ in $\Gamma^\infty(F)$.

Similarly, we write $t_2 = t_1 + O(s^2)$ if there exists $\beta : E \otimes E \to F$ such that $t_2 = t_1 + \beta \circ (s \otimes s)$ in $\Gamma^\infty(F)$. This implies that $t_2 = t_1 + O(s)$.

We can also apply this $O(s), O(s^2)$ notation to morphisms of vector bundles $\theta_1, \theta_2 : F \to G$, by regarding θ_1, θ_2 as sections of $F^* \otimes G$.

(ii) If $F \to X$ is a vector bundle, $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$, and $\Lambda_1, \Lambda_2 : F \to \mathcal{T}_f Y$ are morphisms as in §3.3.4(d), we define a notion of when $\Lambda_2 = \Lambda_1 + O(s)$. Basically this says that locally near $0_{E,\mathrm{top}}(s_{\mathrm{top}}^{-1}(0)) \subseteq E_{\mathrm{top}}$, there should exist $\mathrm{M} : \pi^*(F) \to \mathcal{T}_{f\circ\pi} Y$ on E with $0_E^*(\mathrm{M}) = \Lambda_1$ and $s^*(\mathrm{M}) = \Lambda_2$, where $0_E : X \to E$ is the zero section.

(iii) If $f, g : X \to Y$ are morphisms, we define a notion of when $g = f + O(s)$. Basically this says that locally near $0_{E,\mathrm{top}}(s_{\mathrm{top}}^{-1}(0)) \subseteq E_{\mathrm{top}}$, there should exist a morphism $v : E \to Y$ with $v \circ 0_E = f$ and $v \circ s = g$.

(iv) Let $f, g : X \to Y$ with $g = f + O(s)$ be as in (iii), and $F \to X$, $G \to Y$ be vector bundles, and $\theta_1 : F \to f^*(G)$, $\theta_2 : F \to g^*(G)$ be morphisms. We wish to compare θ_1, θ_2, though they map to *different* vector bundles.

We define a notion of when $\theta_2 = \theta_1 + O(s)$. Basically this says that locally near $0_{E,\mathrm{top}}(s_{\mathrm{top}}^{-1}(0)) \subseteq E_{\mathrm{top}}$, there should exist a morphism $v : E \to Y$ with $v \circ 0_E = f$ and $v \circ f = g$ as in (iii), and a morphism $\phi : \pi^*(F) \to v^*(G)$ on E with $0_E^*(\phi) = \theta_1$ and $s^*(\phi) = \theta_2$.

(v) Let $f, g : X \to Y$ with $g = f + O(s)$ be as in (iii), and $F \to X$ be a vector bundle, and $\Lambda_1 : F \to \mathcal{T}_f Y$, $\Lambda_2 : F \to \mathcal{T}_g Y$ be morphisms, as in §3.3.4(d). We wish to compare Λ_1, Λ_2, though they map to *different* sheaves.

We define a notion of when $\Lambda_2 = \Lambda_1 + O(s)$. Basically this says that locally near $0_{E,\mathrm{top}}(s_{\mathrm{top}}^{-1}(0)) \subseteq E_{\mathrm{top}}$, there should exist a morphism $v : E \to Y$ with $v \circ 0_E = f$ and $v \circ s = g$ as in (iii), and a morphism $\mathrm{M} : \pi^*(F) \to \mathcal{T}_v Y$ on E with $0_E^*(\mathrm{M}) = \Lambda_1$ and $s^*(\mathrm{M}) = \Lambda_2$.

(vi) Suppose $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$, and $F \to X$, $G \to Y$ are vector bundles, and $t \in \Gamma^\infty(G)$ with $f^*(t) = O(s)$ in the sense of (i), and $\Lambda : F \to \mathcal{T}_f Y$ is a morphism, as in §3.3.4(d), and $\theta : F \to f^*(G)$ is a vector bundle morphism. We write $\theta = f^*(\mathrm{d}t) \circ \Lambda + O(s)$ if whenever ∇ is a connection on G we have $\theta = f^*(\nabla t) \circ \Lambda + O(s)$ in the sense of (i), where $f^*(\nabla t) : \mathcal{T}_f Y \to f^*(G)$ is as in §3.3.4(h), so that $f^*(\nabla t) \circ \Lambda : F \to f^*(G)$ is a vector bundle morphism as in §3.3.4(d).

Here a connection ∇ on G exists as in §3.3.4, and the condition $\theta = f^*(\nabla t) \circ \Lambda + O(s)$ is independent of the choice of connection ∇. The notation '$\mathrm{d}t$' in $\theta = f^*(\mathrm{d}t) \circ \Lambda + O(s)$ is intended to suggest that the condition is natural, and independent of the choice of connection.

(vii) Let $f, g : X \to Y$ with $g = f + O(s)$ be as in (iii), and $\Lambda : E \to \mathcal{T}_f Y$ be a morphism in the sense of §3.3.4(d). We define a notion of when $g = f + \Lambda \circ s + O(s^2)$. Basically this says that locally near $0_{E,\mathrm{top}}(s_{\mathrm{top}}^{-1}(0)) \subseteq E_{\mathrm{top}}$, there should exist a morphism $v : E \to Y$ with $v \circ 0_E = f$ and $v \circ s = g$ as in (iii), and the normal derivative of v at the zero section $0_E(X) \subseteq E$ should be Λ. Making sense of this formally needs the details of the definition of $\mathcal{T}_f Y$ in §B.4, which we have not explained.

Here are equivalent but simpler definitions when $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$. We combine Definition 3.15(i),(ii) into Definition 3.16(i), and Definition 3.15(iv),(v) into Definition 3.16(iii), since the sheaf $\mathcal{T}_f Y = f^*(TY)$ is a vector bundle when $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$, and does not need separate treatment.

Definition 3.16. Let X be a classical manifold, $E \to X$ a vector bundle, and $s \in \Gamma^\infty(E)$ a smooth section.

(i) If $F \to X$ is another vector bundle and $t_1, t_2 \in \Gamma^\infty(F)$ are smooth sections, we write $t_2 = t_1 + O(s)$ if there exists $\alpha \in \Gamma^\infty(E^* \otimes F)$ such that $t_2 = t_1 + \alpha \cdot s$ in $\Gamma^\infty(F)$, where the contraction $\alpha \cdot s$ is formed using the natural pairing of vector bundles $(E^* \otimes F) \times E \to F$ over X.

Similarly, we write $t_2 = t_1 + O(s^2)$ if there exists $\alpha \in \Gamma^\infty(E^* \otimes E^* \otimes F)$ such that $t_2 = t_1 + \alpha \cdot (s \otimes s)$ in $\Gamma^\infty(F)$.

(ii) Suppose $f, g : X \to Y$ are smooth maps of classical manifolds. We write $g = f + O(s)$ if whenever $a : Y \to \mathbb{R}$ is a smooth map, there exists $\beta \in \Gamma^\infty(E^*)$ such that $a \circ g = a \circ f + \beta \cdot s$.

(iii) Let $f, g : X \to Y$ with $g = f + O(s)$ be as in (ii), and $F \to X$, $G \to Y$ be vector bundles, and $\theta_1 : F \to f^*(G)$, $\theta_2 : F \to g^*(G)$ be morphisms. We wish to compare θ_1, θ_2, though they map to *different* vector bundles.

We write $\theta_2 = \theta_1 + O(s)$ if for all $\alpha \in \Gamma^\infty(F)$ and $\beta \in \Gamma^\infty(G^*)$ we have $g^*(\beta) \cdot (\theta_2 \circ \alpha) = f^*(\beta) \cdot (\theta_1 \circ \alpha) + O(s)$ in $C^\infty(X)$, in the sense of (i).

(iv) Suppose $f : X \to Y$ is a smooth map of classical manifolds, $F \to X$, $G \to Y$ are vector bundles, $t \in \Gamma^\infty(G)$ with $f^*(t) = O(s)$ in the sense of (i), and $\Lambda : F \to f^*(TY)$, $\theta : F \to f^*(G)$ are vector bundle morphisms. We write $\theta = f^*(\mathrm{d}t) \circ \Lambda + O(s)$ if $\theta = f^*(\nabla t) \circ \Lambda + O(s)$ in the sense of (i) when ∇ is a connection on G, so that $\nabla t \in \Gamma^\infty(T^*Y \otimes G)$ and $f^*(\nabla t) : f^*(TY) \to f^*(G)$ is a vector bundle morphism. This condition is independent of the choice of connection ∇ on G.

(v) Let $f, g : X \to Y$ with $g = f + O(s)$ be as in (ii), and $\Lambda : E \to f^*(TY)$ be a vector bundle morphism. We write $g = f + \Lambda \circ s + O(s^2)$ if whenever $a : Y \to \mathbb{R}$ is a smooth map, there exists β in $\Gamma^\infty(E^* \otimes E^*)$ such that $a \circ g = a \circ f + \Lambda \cdot (s \otimes f^*(\mathrm{d}h)) + \beta \cdot (s \otimes s)$. Here $s \otimes f^*(\mathrm{d}h)$ lies in $\Gamma^\infty\big(E \otimes f^*(T^*Y)\big)$, and so pairs with Λ.

When $\dot{\mathbf{Man}} = \mathbf{Man}$ we can interpret the $O(s)$ and $O(s^2)$ conditions in Definitions 3.15–3.16 in terms of C^∞-algebraic geometry, as in [56, 65]. A manifold X corresponds to a C^∞-scheme $\underline{X}$. Given a vector bundle $E \to X$ and $s \in \Gamma^\infty(E)$, we have closed C^∞-subschemes $\underline{S}_1 \subseteq \underline{S}_2 \subseteq \underline{X}$, where $\underline{S}_1$ is defined by $s = 0$, and $\underline{S}_2$ by $s \otimes s = 0$. Roughly, an equation on X holds up to $O(s)$ if when translated into C^∞-scheme language, the restriction of the equation to $\underline{S}_1 \subseteq \underline{X}$ holds, and it holds up to $O(s^2)$ if its restriction to $\underline{S}_2 \subseteq \underline{X}$ holds. For example, $t_2 = t_1 + O(s) \Leftrightarrow t_2|_{\underline{S}_1} = t_1|_{\underline{S}_1}$ and $t_2 = t_1 + O(s^2) \Leftrightarrow t_2|_{\underline{S}_2} = t_1|_{\underline{S}_2}$ in Definition 3.15(i), and $g = f + O(s) \Leftrightarrow \underline{g}|_{\underline{S}_1} = \underline{f}|_{\underline{S}_1}$ in Definition 3.15(iii).

The next theorem gives the properties of this $O(s)$ and $O(s^2)$ notation we will need for our (m- and μ-)Kuranishi space theories. It will be proved in §B.9.

Theorem 3.17. *Work in the situation of Definition* 3.15. *Then:*

(a) *All the '$O(s)$' and '$O(s^2)$' conditions above are local on $s_{\rm top}^{-1}(0) \subseteq X_{\rm top}$. That is, each condition holds on all of $X_{\rm top}$ if and only if it holds on a family of open subsets of $X_{\rm top}$ covering $s_{\rm top}^{-1}(0)$.*

(b) *In Definition* 3.15(i),(ii),(iv)–(vi) *the conditions are $C^\infty(X)$-linear in $t, t_1, t_2, \theta, \theta_1, \theta_2, \Lambda, \Lambda_1, \Lambda_2$. For example, in* (i) *if $t_2 = t_1 + O(s)$, $t_2' = t_1' + O(s)$ and $a, b \in C^\infty(X)$ then $(at_2 + bt_2') = (at_1 + bt_1') + O(s)$.*

(c) *In Definition* 3.15(i)–(iii) *the conditions are equivalence relations. For example, in* (iii) *if $f, g, h : X \to Y$ are morphisms in* $\dot{\mathbf{Man}}$, *then $f = f + O(s)$, and $g = f + O(s)$ implies that $f = g + O(s)$, and $g = f + O(s)$, $h = g + O(s)$ imply that $h = f + O(s)$.*

(d) *In Definition* 3.15(iv),(v) *the conditions are equivalence relations relative to the equivalence relation of* (iii). *For example, if $f, g, h : X \to Y$ are morphisms in* $\dot{\mathbf{Man}}$ *with $g = f + O(s)$, $h = g + O(s)$, and $F \to X$, $G \to Y$ are vector bundles, and $\theta_1 : F \to f^*(G)$, $\theta_2 : F \to g^*(G)$, $\theta_3 : F \to h^*(G)$ with $\theta_2 = \theta_1 + O(s)$ (using $g = f + O(s)$) and $\theta_3 = \theta_2 + O(s)$ (using $h = g + O(s)$) as in* (iv), *then $h = f + O(s)$ by* **(c)**, *and $\theta_3 = \theta_1 + O(s)$ (using $h = f + O(s)$) as in* (iv).

(e) *Let $X_a \hookrightarrow X$ for $a \in A$ be open submanifolds with $s_{\rm top}^{-1}(0) \subseteq \bigcup_{a \in A} X_{a,\rm top}$. Write $X_{ab} \hookrightarrow X$ for the open submanifold with $X_{ab,\rm top} = X_{a,\rm top} \cap X_{b,\rm top}$ for $a, b \in A$. Suppose we are given morphisms $f_a : X_a \to Y$ in* $\dot{\mathbf{Man}}$ *for all $a \in A$ with $f_a|_{X_{ab}} = f_b|_{X_{ab}} + O(s)$ on X_{ab} for all $a, b \in A$. Then there exist an open submanifold $j : X' \hookrightarrow X$ with $s_{\rm top}^{-1}(0) \subseteq X'_{\rm top}$ and a morphism $g : X' \to Y$ such that $g|_{X' \cap X_a} = f_a|_{X' \cap X_a} + O(s)$ for all $a \in A$.*

Suppose also that a finite group Γ acts on X, Y by diffeomorphisms in $\dot{\mathbf{Man}}$, *and that the $X_a \hookrightarrow X$ are Γ-invariant, and the $f_a : X_a \to Y$ are Γ-equivariant, for all $a \in A$. Then we can choose X' to be Γ-invariant, and g to be Γ-equivariant.*

(f) *Let $X, E, s, f, g, F, G, \theta_1$ be as in Definition* 3.15(iv). *Then there exists $\theta_2 : F \to g^*(G)$ with $\theta_2 = \theta_1 + O(s)$, as in* (iv). *If $\tilde\theta_2$ is an alternative choice for θ_2 then $\tilde\theta_2 = \theta_2 + O(s)$, as in* (i).

(g) *Let $X, E, s, f, g, F, G, \Lambda_1$ be as in Definition* 3.15(v)*. Then there exists $\Lambda_2 : F \to \mathcal{T}_g Y$ with $\Lambda_2 = \Lambda_1 + O(s)$ as in* (v)*. If $\tilde\Lambda_2$ is an alternative choice for Λ_2 then $\tilde\Lambda_2 = \Lambda_2 + O(s)$, as in* (ii)*.*

(h) *Let $X, E, s, f, Y, F, G, t, \Lambda$ be as in* (vi)*. Then there exists a vector bundle morphism $\theta : F \to f^*(G)$ on X such that $\theta = f^*(\mathrm{d}t) \circ \Lambda + O(s)$, in the sense of* (vi)*. If $\tilde\theta$ is an alternative choice for θ then $\tilde\theta = \theta + O(s)$ as in* (i)*, regarding $\theta, \tilde\theta$ as sections of $F^* \otimes f^*(G)$.*

(i) *Suppose $f, g : X \to Y$ are morphisms with $g = f + O(s)$ as in* (iii)*. Then there exists $\Lambda : E \to \mathcal{T}_f Y$ with $g = f + \Lambda \circ s + O(s^2)$ as in* (vii)*.*

(j) *Let $X, E, s, f, g, Y, \Lambda$ with $g = f + \Lambda \circ s + O(s^2)$ be as in* (vii)*, and $\tilde\Lambda : E \to \mathcal{T}_f Y$ be a morphism with $\tilde\Lambda = \Lambda + O(s)$ as in* (ii)*. Then $g = f + \tilde\Lambda \circ s + O(s^2)$.*

(k) *Let $X, E, s, f, g, Y, \Lambda$ with $g = f + \Lambda \circ s + O(s^2)$ be as in* (vii)*. Part* **(g)** *gives $\tilde\Lambda : F \to \mathcal{T}_g Y$ with $\tilde\Lambda = \Lambda + O(s)$ as in* (v)*, where $\tilde\Lambda$ is unique up to $O(s)$. Then $f = g + (-\tilde\Lambda) \circ s + O(s^2)$ as in* (vii)*.*

(l) *Let $f, g, h : X \to Y$ be morphisms in* $\dot{\mathbf{Man}}$ *with $g = f + O(s)$, $h = g + O(s)$, so that $h = f + O(s)$ by* **(c)***, and $\Lambda_1 : E \to \mathcal{T}_f Y$, $\Lambda_2 : E \to \mathcal{T}_g Y$ be morphisms with $g = f + \Lambda_1 \circ s + O(s^2)$ and $h = g + \Lambda_2 \circ s + O(s^2)$ be as in* (vii)*. Part* **(g)** *gives $\tilde\Lambda_2 : E \to \mathcal{T}_f Y$ with $\tilde\Lambda_2 = \Lambda_2 + O(s)$ as in* (v)*, unique up to $O(s)$. Then $h = f + (\Lambda_1 + \tilde\Lambda_2) \circ s + O(s^2)$ as in* (vii)*.*

(m) *Let $f, g : X \to Y$ be morphisms in* $\dot{\mathbf{Man}}$ *with $g = f + O(s)$, and $\Lambda_1, \ldots, \Lambda_k : E \to \mathcal{T}_f Y$ be morphisms with $g = f + \Lambda_a \circ s + O(s^2)$ for $a = 1, \ldots, k$ as in* (vii)*, and $\alpha_1, \ldots, \alpha_k \in C^\infty(X)$ with $\alpha_1 + \cdots + \alpha_k = 1$. Then $g = f + (\alpha_1 \cdot \Lambda_1 + \cdots + \alpha_k \cdot \Lambda_k) \circ s + O(s^2)$ as in* (vii)*.*

(n) *Let $f : X \to Y$ be a morphism in* $\dot{\mathbf{Man}}$*, and $F, G \to Y$ be vector bundles, $t \in \Gamma^\infty(F)$ with $f^*(t) = O(s)$, and $u_1, u_2 \in \Gamma^\infty(G)$.*

If $u_2 = u_1 + O(t)$ as in (i) *then $f^*(u_2) = f^*(u_1) + O(s)$, and if $u_2 = u_1 + O(t^2)$ as in* (i) *then $f^*(u_2) = f^*(u_1) + O(s^2)$.*

(o) *Let $f : X \to Y$, $g : Y \to Z$ be morphisms in* $\dot{\mathbf{Man}}$*, and $F, G \to Y$ be vector bundles, and $t \in \Gamma^\infty(F)$ with $f^*(t) = O(s)$, and $\Lambda_1, \Lambda_2 : G \to \mathcal{T}_g Z$ with $\Lambda_2 = \Lambda_1 + O(t)$ be as in* (ii)*. Then $f^*(\Lambda_2) = f^*(\Lambda_1) + O(s)$ as in* (ii)*, where $f^*(\Lambda_1), f^*(\Lambda_2) : f^*(G) \to \mathcal{T}_{g \circ f} Z$ are as in* §3.3.4(g)*.*

(p) *Suppose $f : X \to Y$ and $g, h : Y \to Z$ are morphisms in* $\dot{\mathbf{Man}}$*, and $F \to Y$ is a vector bundle, and $t \in \Gamma^\infty(F)$ with $f^*(t) = O(s)$.*

If $h = g + O(t)$ as in (iii) *then $h \circ f = g \circ f + O(s)$.*

If $h = g + \Lambda \circ t + O(t^2)$ as in (vii) *for $\Lambda : F \to \mathcal{T}_g Z$, and $\theta : E \to f^*(F)$ is a morphism with $\theta \circ s = f^*(t) + O(s^2)$ as in* (i)*, then*

$$h \circ f = g \circ f + [f^*(\Lambda) \circ \theta] \circ s + O(s^2),$$

where $f^*(\Lambda) \circ \theta : E \to \mathcal{T}_{g \circ f} Z$ is as in §3.3.4(d),(g)*.*

(q) *Let $f : X \to Y$, $g, h : Y \to Z$ be morphisms in* $\dot{\mathbf{Man}}$, *and $F, G \to Y$, $H \to Z$ be vector bundles, and $t \in \Gamma^\infty(F)$ with $f^*(t) = O(s)$ and $h = g + O(t)$, and $\theta_1 : G \to g^*(H)$, $\theta_2 : G \to h^*(H)$ with $\theta_2 = \theta_1 + O(t)$ be as in* (iv). *Then $f^*(\theta_2) = f^*(\theta_1) + O(s)$ as in* (iv).

(r) *Let $f : X \to Y$, $g, h : Y \to Z$ be morphisms in* $\dot{\mathbf{Man}}$, *and $F, G \to Y$ be vector bundles, and $t \in \Gamma^\infty(F)$ with $f^*(t) = O(s)$, $h = g + O(t)$, and $\Lambda_1 : G \to \mathcal{T}_g Z$, $\Lambda_2 : G \to \mathcal{T}_h Z$ with $\Lambda_2 = \Lambda_1 + O(t)$ be as in* (v). *Then $f^*(\Lambda_2) = f^*(\Lambda_1) + O(s)$ as in* (v), *where $f^*(\Lambda_1) : f^*(G) \to \mathcal{T}_{g\circ f} Z$ and $f^*(\Lambda_2) : f^*(G) \to \mathcal{T}_{h\circ f} Z$ are as in* §3.3.4(g).

(s) *Let $f : X \to Y$, $g : Y \to Z$ be morphisms in* $\dot{\mathbf{Man}}$, *and $F, G \to Y$, $H \to Z$ be vector bundles, and $t \in \Gamma^\infty(F)$, $u \in \Gamma^\infty(H)$ with $f^*(t) = O(s)$, $g^*(u) = O(t)$ as in* (i), *and $\Lambda : G \to \mathcal{T}_g Z$, $\theta : G \to g^*(H)$ with $\theta = g^*(\mathrm{d}u) \circ \Lambda + O(t)$ be as in* (vi). *Then $f^*(\theta) = (g \circ f)^*(\mathrm{d}u) \circ f^*(\Lambda) + O(s)$ as in* (vi), *where $f^*(\Lambda) : f^*(G) \to \mathcal{T}_{g\circ f} Z$ is as in* §3.3.4(g).

(t) *Let $f : X \to Y$, $g : Y \to Z$ be morphisms in* $\dot{\mathbf{Man}}$, *and $F \to Y$ be a vector bundle, and $\Lambda_1, \Lambda_2 : F \to \mathcal{T}_f Y$ with $\Lambda_2 = \Lambda_1 + O(s)$ be as in* (ii). *Then $\mathcal{T}g \circ \Lambda_2 = \mathcal{T}g \circ \Lambda_1 + O(s)$ as in* (ii), *where $\mathcal{T}g \circ \Lambda_1, \mathcal{T}g \circ \Lambda_2 : F \to \mathcal{T}_{g\circ f} Z$ are as in* §3.3.4(c),(d).

(u) *Let $f, g : X \to Y$, $h : Y \to Z$ be morphisms in* $\dot{\mathbf{Man}}$. *If $g = f + O(s)$ as in* (iii) *then $h \circ g = h \circ f + O(s)$. If $g = f + \Lambda \circ s + O(s^2)$ as in* (vii) *for $\Lambda : E \to \mathcal{T}_f Y$, then $h \circ g = h \circ f + [\mathcal{T}h \circ \Lambda] \circ s + O(s^2)$, where $\mathcal{T}h \circ \Lambda : E \to \mathcal{T}_{h\circ f} Z$ is as in* §3.3.4(c),(d).

(v) *Let $f, g : X \to Y$, $h : Y \to Z$ be morphisms in* $\dot{\mathbf{Man}}$ *with $g = f + O(s)$ as in* (iii), *so that $h \circ g = h \circ f + O(s)$ by* **(u)**. *Suppose $F \to X$ is a vector bundle, and $\Lambda_1 : F \to \mathcal{T}_f Y$, $\Lambda_2 : F \to \mathcal{T}_g Y$ are morphisms with $\Lambda_2 = \Lambda_1 + O(s)$ as in* (v). *Then $\mathcal{T}h \circ \Lambda_2 = \mathcal{T}h \circ \Lambda_1 + O(s)$ as in* (v), *where $\mathcal{T}h \circ \Lambda_1 : E \to \mathcal{T}_{h\circ f} Z$ and $\mathcal{T}h \circ \Lambda_2 : E \to \mathcal{T}_{h\circ g} Z$ are as in* §3.3.4(c),(d).

3.3.6 Discrete properties of morphisms in $\dot{\mathbf{Man}}$

Section B.6 defines a condition for classes of morphisms in $\dot{\mathbf{Man}}$ to lift nicely to classes of (1-)morphisms in $\mathbf{m\dot{K}ur}$, $\boldsymbol{\mu}\mathbf{\dot{K}ur}$, $\mathbf{\dot{K}ur}$ in Chapters 4–6.

Definition 3.18. Let $\boldsymbol{P}$ be a property of morphisms in $\dot{\mathbf{Man}}$, so that for any morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$, either f is $\boldsymbol{P}$, or f is not $\boldsymbol{P}$. For example, if $\dot{\mathbf{Man}}$ is $\mathbf{Man^c}$ from §2.1, then $\boldsymbol{P}$ could be interior, or b-normal.

We call $\boldsymbol{P}$ a *discrete* property of morphisms in $\dot{\mathbf{Man}}$ if:

(i) All diffeomorphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ are $\boldsymbol{P}$.

(ii) All open submanifolds $i : U \hookrightarrow X$ in $\dot{\mathbf{Man}}$ are $\boldsymbol{P}$.

(iii) If $f : X \to Y$ and $g : Y \to Z$ in $\dot{\mathbf{Man}}$ are $\boldsymbol{P}$ then $g \circ f : X \to Z$ is $\boldsymbol{P}$.

(iv) For a morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$ to be $\boldsymbol{P}$ is a *local property on* X, in the sense that if we can cover X by open submanifolds $i : U \hookrightarrow X$ such that $f \circ i : U \to Y$ is $\boldsymbol{P}$, then f is $\boldsymbol{P}$.

Some notation: if $f : X \to Y$ in $\dot{\mathbf{Man}}$ and $S \subseteq X_{\rm top}$ then we say that f *is* $\boldsymbol{P}$ *near* S if there exists an open submanifold $i : U \hookrightarrow X$ such that $S \subseteq U_{\rm top} \subseteq X_{\rm top}$ and $f \circ i : U \to Y$ is $\boldsymbol{P}$. This is a well behaved notion as $\boldsymbol{P}$ is a local property, e.g. f is $\boldsymbol{P}$ if and only if f is $\boldsymbol{P}$ near each $x \in X_{\rm top}$.

(v) All morphisms in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$ are $\boldsymbol{P}$.

(vi) Suppose $f : X \times \mathbb{R} \to Y$ is a morphism in $\dot{\mathbf{Man}}$. If f is $\boldsymbol{P}$ near $X_{\rm top} \times \{0\}$ in $X_{\rm top} \times \mathbb{R}$, then f is $\boldsymbol{P}$.

(vii) Suppose $E \to X$ is a vector bundle in $\dot{\mathbf{Man}}$, and $s \in \Gamma^\infty(E)$, so that $s_{\rm top}^{-1}(0) \subseteq X_{\rm top}$, and $f, g : X \to Y$ are morphisms in $\dot{\mathbf{Man}}$ with $g = f + O(s)$ in the sense of Definition 3.15(iii). Then f is $\boldsymbol{P}$ near $s_{\rm top}^{-1}(0)$ if and only if g is $\boldsymbol{P}$ near $s_{\rm top}^{-1}(0)$.

(viii) Suppose we are given a diagram in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd} U' \arrow[r, hook, "i'"] \arrow[dr, "f'"' near start] & U \arrow[r, hook, "i"] \arrow[dr, "f"' near start] & X \\ V' \arrow[r, hook, "j'"'] \arrow[ur, "g'" near start] & V \arrow[r, hook, "j"'] \arrow[ur, "g" near start] & Y, \end{tikzcd} \tag{3.6}$$

where i, i', j, j' are open submanifolds in $\dot{\mathbf{Man}}$, and $f \circ i' = j \circ f' : U' \to Y$, $g \circ j' = i \circ g' : V' \to X$, and we are given points $x \in U'_{\rm top} \subseteq U_{\rm top} \subseteq X_{\rm top}$ and $y \in V'_{\rm top} \subseteq V_{\rm top} \subseteq Y_{\rm top}$ such that $f_{\rm top}(x) = y$ and $g_{\rm top}(y) = x$. Suppose too that there are vector bundles $E \to U'$ and $F \to V'$ and sections $s \in \Gamma^\infty(E)$, $t \in \Gamma^\infty(F)$ with $s(x) = t(y) = 0$, such that $g \circ f' = i \circ i' + O(s)$ on U' and $f \circ g' = j \circ j' + O(t)$ on V' in the sense of Definition 3.15(iii). Then f, f' are $\boldsymbol{P}$ near x, and g, g' are $\boldsymbol{P}$ near y.

Parts (i),(iii) imply that we have a subcategory $\dot{\mathbf{Man}}_{\boldsymbol{P}} \subseteq \dot{\mathbf{Man}}$ containing all objects X, Y in $\dot{\mathbf{Man}}$, and all morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ which are $\boldsymbol{P}$.

Example 3.19. **(a)** When $\dot{\mathbf{Man}}$ is $\mathbf{Man^c}$ from §2.1, the following properties of morphisms in $\mathbf{Man^c}$ are discrete: interior, b-normal, strongly smooth, simple.

(b) When $\dot{\mathbf{Man}}$ is $\mathbf{Man^{gc}}$ from §2.4.1, the following properties of morphisms in $\mathbf{Man^{gc}}$ are discrete: interior, b-normal, simple.

(c) When $\dot{\mathbf{Man}}$ is $\mathbf{Man^{ac}}$ or $\mathbf{Man^{c,ac}}$ from §2.4.2, the following properties of morphisms in $\dot{\mathbf{Man}}$ are discrete: interior, b-normal, strongly a-smooth, simple.

3.3.7 Comparing different categories $\dot{\mathbf{Man}}$

In §B.7 we discuss how to compare different categories $\dot{\mathbf{Man}}, \ddot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7. Here is Condition B.40:

Condition 3.20. Suppose $\dot{\mathbf{Man}}, \ddot{\mathbf{Man}}$ satisfy Assumptions 3.1–3.7, and $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \to \ddot{\mathbf{Man}}$ is a functor in a commutative diagram

$$\begin{tikzcd} & \dot{\mathbf{Man}} \arrow[dd, "F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}"] \arrow[dr, "F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}"] & \\ \mathbf{Man} \arrow[ur, "\subset"] \arrow[dr, "\subset"'] & & \mathbf{Top}, \\ & \ddot{\mathbf{Man}} \arrow[ur, "F_{\ddot{\mathbf{Man}}}^{\mathbf{Top}}"'] & \end{tikzcd} \tag{3.7}$$

where the functors $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}, F_{\ddot{\mathbf{Man}}}^{\mathbf{Top}}$ are as in Assumption 3.2, and the inclusions $\mathbf{Man} \hookrightarrow \dot{\mathbf{Man}}, \ddot{\mathbf{Man}}$ as in Assumption 3.4. We require $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ to take products, disjoint unions, and open submanifolds in $\dot{\mathbf{Man}}$ to products, disjoint unions, and open submanifolds in $\ddot{\mathbf{Man}}$, and to preserve dimensions.

Note that $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ must be faithful (injective on morphisms), as $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}$ is.

In §B.7 we explain that given a functor $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ satisfying Condition 3.20, *all* the geometry of §B.1–§B.5 in $\dot{\mathbf{Man}}$ from §3.3.1–§3.3.5 maps functorially to its analogue in $\ddot{\mathbf{Man}}$. We chose the definitions in Appendix B to ensure this. For example, if $\dot{X} \in \dot{\mathbf{Man}}$ and $\ddot{X} = F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}(\dot{X})$ there are natural sheaf morphisms

$$\mathcal{O}_{\dot{X}} \longrightarrow \mathcal{O}_{\ddot{X}}, \quad \mathcal{T}\dot{X} \longrightarrow \mathcal{T}\ddot{X}, \quad \mathcal{T}^*\dot{X} \longrightarrow \mathcal{T}^*\ddot{X}$$

on the common topological space $\dot{X}_{\mathrm{top}} = \ddot{X}_{\mathrm{top}}$.

Proposition B.43 discusses inclusions of subcategories $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$:

Proposition 3.21. *Suppose $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \hookrightarrow \ddot{\mathbf{Man}}$ is an inclusion of subcategories satisfying Condition 3.20, and either:*

(a) *All objects of $\ddot{\mathbf{Man}}$ are objects of $\dot{\mathbf{Man}}$, and all morphisms $f : X \to \mathbb{R}$ in $\ddot{\mathbf{Man}}$ are morphisms in $\dot{\mathbf{Man}}$, and for a morphism $f : X \to Y$ in $\ddot{\mathbf{Man}}$ to lie in $\dot{\mathbf{Man}}$ is a* ***discrete*** *condition, as in Definition 3.18; or*

(b) *$\dot{\mathbf{Man}}$ is a full subcategory of $\ddot{\mathbf{Man}}$ closed under isomorphisms in $\ddot{\mathbf{Man}}$.*

Then all the material of §3.3.1–§3.3.5 for $\dot{\mathbf{Man}}$ is ***exactly the same*** *if computed in $\dot{\mathbf{Man}}$ or $\ddot{\mathbf{Man}}$, and the functorial maps from geometry in $\dot{\mathbf{Man}}$ to geometry in $\ddot{\mathbf{Man}}$ discussed above are the identity maps. For example, if $f : X \to Y$ lies in $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$ then the relative tangent sheaves $(\mathcal{T}_f Y)_{\dot{\mathbf{Man}}}, (\mathcal{T}_f Y)_{\ddot{\mathbf{Man}}}$ on X_{top} from §3.3.4 computed in $\dot{\mathbf{Man}}$ and $\ddot{\mathbf{Man}}$ are not just canonically isomorphic, but actually the same sheaf.*

For example, Figure 3.1 gives a diagram of functors from Chapter 2 which satisfy Condition 3.20. Arrows '$\to$' are inclusions of subcategories satisfying Proposition 3.21(a) or (b). Arrows marked '$\star$' involve the non-obvious functor $F_{\mathbf{Man^{c,ac}_{st}}}^{\mathbf{Man^c_{st}}} : \mathbf{Man^{c,ac}_{st}} \to \mathbf{Man^c_{st}}$ from §2.4.2; some cycles in Figure 3.1 including arrows '$\star$' do not commute.

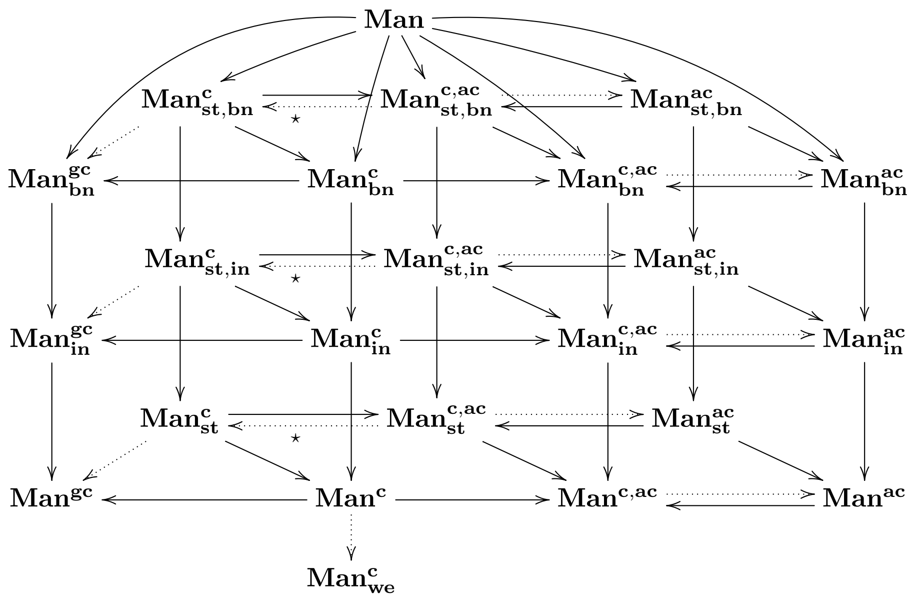


Figure 3.1: Functors satisfying Condition 3.20.
Arrows '$\rightarrow$' satisfy Proposition 3.21(a) or (b).

Chapters 4–6 will associate (2-)categories $\mathbf{m\dot{K}ur}, \boldsymbol{\mu}\mathbf{\dot{K}ur}, \mathbf{\dot{K}ur}$ of (m- or μ-) Kuranishi spaces to each such category $\mathbf{\dot{M}an}$. When Condition 3.20 holds, by mapping geometry in $\mathbf{\dot{M}an}$ to $\mathbf{\ddot{M}an}$ as above, we will define natural (2)-functors

$$F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}} : \mathbf{m\dot{K}ur} \longrightarrow \mathbf{m\ddot{K}ur},\ F_{\boldsymbol{\mu}\mathbf{\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\ddot{K}ur}} : \boldsymbol{\mu}\mathbf{\dot{K}ur} \longrightarrow \boldsymbol{\mu}\mathbf{\ddot{K}ur},\ F_{\mathbf{\dot{K}ur}}^{\mathbf{\ddot{K}ur}} : \mathbf{\dot{K}ur} \longrightarrow \mathbf{\ddot{K}ur}$$

between the (2-)categories $\mathbf{m\dot{K}ur}, \boldsymbol{\mu}\mathbf{\dot{K}ur}, \mathbf{\dot{K}ur}$ and $\mathbf{m\ddot{K}ur}, \boldsymbol{\mu}\mathbf{\ddot{K}ur}, \mathbf{\ddot{K}ur}$ associated to $\mathbf{\dot{M}an}$ and $\mathbf{\ddot{M}an}$. When Proposition 3.21(a) or (b) holds, these are inclusions of (2)-subcategories.

3.4 Extension to 'manifolds with corners'

The assumptions of §3.1 include many categories of manifolds with corners, as in Example 3.8(ii), giving corresponding (2-)categories of (m- or μ-)Kuranishi spaces in Chapters 4–6. So to study '(m- or μ-)Kuranishi spaces with corners' we do not need to start again. Instead, we give extra assumptions about special features of manifolds with corners: boundaries ∂X, k-corners $C_k(X)$, and the corner functor C. We change notation from $\mathbf{\dot{M}an}$ in §3.1–§3.3 to $\mathbf{\dot{M}an^c}$.

3.4.1 Core assumptions on 'manifolds with corners'

Assumption 3.22. (a) We are given a category $\mathbf{\dot{M}an^c}$. For simplicity, objects X in $\mathbf{\dot{M}an^c}$ will be called *manifolds with corners*, and morphisms $f : X \to Y$ in

$\dot{\mathbf{Man}}^{\mathbf{c}}$ will be called *smooth maps*.

(b) The category $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumptions 3.1–3.7 with $\dot{\mathbf{Man}}^{\mathbf{c}}$ in place of $\dot{\mathbf{Man}}$. The functor in Assumption 3.2 will be written $F_{\dot{\mathbf{Man}}^{\mathbf{c}}}^{\mathbf{Top}} : \dot{\mathbf{Man}}^{\mathbf{c}} \to \mathbf{Top}$.

(c) We are given a class of morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$ called *simple maps*. To be simple is a discrete property in the sense of §3.3.6. We write $\dot{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{si}} \subseteq \dot{\mathbf{Man}}^{\mathbf{c}}$ for the subcategory of $\dot{\mathbf{Man}}^{\mathbf{c}}$ with all objects, and simple morphisms.

(d) For each object X in $\dot{\mathbf{Man}}^{\mathbf{c}}$ and each $k = 0, \ldots, \dim X$, we are given an object $C_k(X)$ in $\dot{\mathbf{Man}}^{\mathbf{c}}$ called the *k-corners of* X with $\dim C_k(X) = \dim X - k$, and a morphism $\Pi_k : C_k(X) \to X$ in $\dot{\mathbf{Man}}^{\mathbf{c}}$, such that $\Pi_{k,\mathrm{top}} : C_k(X)_{\mathrm{top}} \to X_{\mathrm{top}}$ is proper, with finite fibres $\Pi_{k,\mathrm{top}}^{-1}(x)$, $x \in X_{\mathrm{top}}$.

We write $C_k(X) = \emptyset$ for $k > \dim X$.

When $k = 0$, $\Pi_0 : C_0(X) \to X$ is a diffeomorphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, so we can identify $C_0(X)$ with X. When $k = 1$ we write $\partial X = C_1(X)$ and call ∂X the *boundary of* X. We also write $i_X : \partial X \to X$ for $\Pi_1 : C_1(X) \to X$.

(e) If $X \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}^{\mathbf{c}}$ then $C_k(X) = \emptyset$ for $k > 0$, so that $\partial X = \emptyset$.

(f) For all X in $\dot{\mathbf{Man}}^{\mathbf{c}}$ and $k, l \geqslant 0$ with $k + l \leqslant \dim X$ there is a natural morphism $I_{k,l} : C_k(C_l(X)) \to C_{k+l}(X)$ such that the following commutes:

$$\begin{tikzcd} C_k(C_l(X)) \arrow[r, "\Pi_k"'] \arrow[d, "I_{k,l}"] & C_l(X) \arrow[d, "\Pi_l"'] \\ C_{k+l}(X) \arrow[r, "\Pi_{k+l}"] & X. \end{tikzcd}$$

Also $I_{k,l}$ is étale, that is, a local diffeomorphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and surjective.

(g) As for $\check{\mathbf{Man}}^{\mathbf{c}}$ in Definition 2.8, construct a category $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}$ from $\dot{\mathbf{Man}}^{\mathbf{c}}$, such that $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}$ has objects $\vec{X} = \coprod_{m=0}^{\infty} X_m$, for X_m an object of $\dot{\mathbf{Man}}^{\mathbf{c}}$ with $\dim X_m = m$, allowing $X_m = \emptyset$, and $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}$ has morphisms

$$\vec{f} = \textstyle\coprod_{m,n=0}^{\infty} f_{mn} : \vec{X} = \coprod_{m=0}^{\infty} X_m \longrightarrow \vec{Y} = \coprod_{n=0}^{\infty} Y_n,$$

where for each $m = 0, 1, \ldots$ we have a disjoint union $X_m = \coprod_{n=0}^{\infty} X_{mn}$ in $\dot{\mathbf{Man}}^{\mathbf{c}}$, with X_{mn} open and closed in X_m, allowing $X_{mn} = \emptyset$, and $f_{mn} : X_{mn} \to Y_n$ is a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$. Composition and identities are defined in the obvious way. We write $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}_{\mathbf{si}}$ for the subcategory of $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}$ in which the f_{mn} are simple.

There is an obvious full and faithful *inclusion functor* $\mathrm{Inc} : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}^{\mathbf{c}}$, which maps X to $\coprod_{m=0}^{\infty} X_m$ with $X_m = X$ if $m = \dim X$ and $X_m = \emptyset$ otherwise.

Then we are given a functor $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}^{\mathbf{c}}$ called the *corner functor*, which on objects acts as $C(X) = \coprod_{k=0}^{\dim X} C_k(X)$, for $C_k(X)$ the k-corners of X as in **(d)**. The morphisms $\Pi_k : C_k(X) \to X$ in $\dot{\mathbf{Man}}^{\mathbf{c}}$ for $k = 0, \ldots, \dim X$ from **(d)** give a morphism $\Pi = \coprod_{k \geqslant 0} \Pi_k : C(X) \to \mathrm{Inc}(X)$ in $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}$, and over all X these comprise a natural transformation $\Pi : C \Rightarrow \mathrm{Inc}$ of functors $\dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}^{\mathbf{c}}$. That is, we have $\Pi \circ C(f) = f \circ \Pi : C(X) \to Y$ for all morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}^{\mathbf{c}}$.

We may extend C to a functor $\check{C} : \check{\dot{\mathbf{Man}}}{}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$ in the obvious way. Then the morphisms $I_{k,l}$ in **(f)** induce a natural transformation $I : \check{C} \circ C \Rightarrow C$ of functors $\dot{\mathbf{Man}}{}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$.

(h) For all $X, Y \in \dot{\mathbf{Man}}{}^{\mathbf{c}}$ and $k \geqslant 0$ there are natural diffeomorphisms

$$C_k(X \times Y) \cong \textstyle\coprod_{i,j \geqslant 0,\ i+j=k} C_i(X) \times C_j(Y).$$

By part **(g)** these combine to give a diffeomorphism (isomorphism) in $\check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$

$$C(X \times Y) \cong C(X) \times C(Y). \tag{3.8}$$

The corner functor C in **(g)** *preserves products and direct products.* That is, if $f : W \to Y$, $g : X \to Y$, $h : X \to Z$ are smooth then the following commute

$$\begin{tikzcd} C(W \times X) \arrow[r, "C(f\times h)"'] \arrow[d, "\cong"] & C(Y \times Z) \arrow[d, "\cong"'] \\ C(W) \times C(X) \arrow[r, "C(f)\times C(h)"] & C(Y) \times C(Z), \end{tikzcd} \qquad \begin{tikzcd} & C(Y \times Z) \arrow[d, "\cong"] \\ C(X) \arrow[ur, "C((g,h))"] \arrow[r, "(C(g),C(h))"'] & C(Y) \times C(Z), \end{tikzcd}$$

where the columns are the isomorphisms (3.8).

(i) Suppose $f : X \to Y$ is a simple map in $\dot{\mathbf{Man}}{}^{\mathbf{c}}$. Then $C(f) : C(X) \to C(Y)$ in **(g)** lies in $\check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}_{\mathbf{si}}$ and maps $C_k(X) \to C_k(Y)$ for all $k = 0, \dots, \dim X$. Hence we have functors $C_k : \dot{\mathbf{Man}}{}^{\mathbf{c}}_{\mathbf{si}} \to \dot{\mathbf{Man}}{}^{\mathbf{c}}_{\mathbf{si}}$ for $k = 0, 1, \dots$, called the *k-corner functors*, which on objects map X to $C_k(X)$, and on morphisms map $f : X \to Y$ to the component $C_k(f)$ of $C(f) : C(X) \to C(Y)$ mapping $C_k(X) \to C_k(Y)$. We also write $\partial = C_1 : \dot{\mathbf{Man}}{}^{\mathbf{c}}_{\mathbf{si}} \to \dot{\mathbf{Man}}{}^{\mathbf{c}}_{\mathbf{si}}$, and call it the *boundary functor*.

(j) Let $i : U \hookrightarrow X$ be an open submanifold in $\dot{\mathbf{Man}}{}^{\mathbf{c}}$. Then i is simple by Definition 3.18(ii), as simple is a discrete property by **(c)**, so we have morphisms $C_k(i) : C_k(U) \to C_k(X)$ in $\dot{\mathbf{Man}}{}^{\mathbf{c}}$ for $k = 0, \dots, \dim X$ by **(i)**. We require these $C_k(i)$ to be open submanifolds in $\dot{\mathbf{Man}}{}^{\mathbf{c}}$, with topological spaces $C_k(U)_{\mathrm{top}} = \Pi_{k,\mathrm{top}}^{-1}(U_{\mathrm{top}}) \subseteq C_k(X)_{\mathrm{top}}$.

(k) Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}{}^{\mathbf{c}}$ with $\partial X = \partial Y = \emptyset$. Then f is simple.

Remark 3.23. For the corner functor $C : \dot{\mathbf{Man}}{}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$ in Assumption 3.22(g), we shall be interested in cases in which there is a discrete property $\boldsymbol{P}$ of morphisms in $\dot{\mathbf{Man}}{}^{\mathbf{c}}$ such that C maps to the subcategory $\check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}_{\boldsymbol{P}}$ of $\check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$ whose morphisms are $\boldsymbol{P}$. For example, for $\mathbf{Man^c}$ in §2.2 we have $C : \mathbf{Man^c} \to \check{\mathbf{Man}}{}^{\mathbf{c}}_{\mathbf{in}} \subset \check{\mathbf{Man}}{}^{\mathbf{c}}$, with $\boldsymbol{P}$ interior morphisms in $\mathbf{Man^c}$.

3.4.2 Examples of categories satisfying the assumptions

Here are some examples satisfying Assumption 3.22:

Example 3.24. **(a)** The standard example is to take $\dot{\mathbf{Man}}{}^{\mathbf{c}}$ to be $\mathbf{Man^c}$ from §2.1, and to define simple maps as in §2.1, and k-corners $C_k(X)$, projections

$\Pi_k : C_k(X) \to X$, and the corner functor $C : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$ from Definition 2.9 as in §2.2. Note that C maps to $\check{\mathbf{M}}\mathbf{an^c_{in}} \subset \check{\mathbf{M}}\mathbf{an^c}$, as in Remark 3.23.

(b) We can also take $\dot{\mathbf{M}}\mathbf{an^c}$ to be $\mathbf{Man^c}$ and simple maps, $C_k(X), \Pi_k$ as in **(a)**, but use the second corner functor $C' : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$ from Definition 2.11.

(c) We can take $\dot{\mathbf{M}}\mathbf{an^c}$ to be $\mathbf{Man^c_{st}}$ from §2.1, with simple maps, $C_k(X), \Pi_k$ as in §2.1–§2.2, and either corner functor $C : \mathbf{Man^c_{st}} \to \check{\mathbf{M}}\mathbf{an^c_{st,in}} \subset \check{\mathbf{M}}\mathbf{an^c_{st}}$ or $C' : \mathbf{Man^c_{st}} \to \check{\mathbf{M}}\mathbf{an^c_{st}}$.

(d) We can take $\dot{\mathbf{M}}\mathbf{an^c} = \mathbf{Man^{ac}}$ with simple maps, $C_k(X), \Pi_k$ as in §2.4.2, and either $C : \mathbf{Man^{ac}} \to \check{\mathbf{M}}\mathbf{an^{ac}_{in}} \subset \check{\mathbf{M}}\mathbf{an^{ac}}$ or $C' : \mathbf{Man^{ac}} \to \check{\mathbf{M}}\mathbf{an^{ac}}$.

(e) We can take $\dot{\mathbf{M}}\mathbf{an^c} = \mathbf{Man^{ac}_{st}}$ with simple maps, $C_k(X), \Pi_k$ as in §2.4.2, and either $C : \mathbf{Man^{ac}_{st}} \to \check{\mathbf{M}}\mathbf{an^{ac}_{st,in}} \subset \check{\mathbf{M}}\mathbf{an^{ac}_{st}}$ or $C' : \mathbf{Man^{ac}_{st}} \to \check{\mathbf{M}}\mathbf{an^{ac}_{st}}$.

(f) We can take $\dot{\mathbf{M}}\mathbf{an^c} = \mathbf{Man^{c,ac}}$ with simple maps, $C_k(X), \Pi_k$ as in §2.4.2, and either $C : \mathbf{Man^{c,ac}} \to \check{\mathbf{M}}\mathbf{an^{c,ac}_{in}} \subset \check{\mathbf{M}}\mathbf{an^{c,ac}}$ or $C' : \mathbf{Man^{c,ac}} \to \check{\mathbf{M}}\mathbf{an^{c,ac}}$.

(g) We can take $\dot{\mathbf{M}}\mathbf{an^c} = \mathbf{Man^{c,ac}_{st}}$ with simple maps, $C_k(X), \Pi_k$ as in §2.4.2, and either $C : \mathbf{Man^{c,ac}_{st}} \to \check{\mathbf{M}}\mathbf{an^{c,ac}_{st,in}} \subset \check{\mathbf{M}}\mathbf{an^{c,ac}_{st}}$ or $C' : \mathbf{Man^{c,ac}_{st}} \to \check{\mathbf{M}}\mathbf{an^{c,ac}_{st}}$.

(h) We can take $\dot{\mathbf{M}}\mathbf{an^c} = \mathbf{Man^{gc}}$ with simple maps, $C_k(X), \Pi_k$ and $C : \mathbf{Man^{gc}} \to \check{\mathbf{M}}\mathbf{an^{gc}_{in}} \subset \check{\mathbf{M}}\mathbf{an^{gc}}$ as in §2.4.1. The second corner functor C' does not work on $\mathbf{Man^{gc}}$.

(i) A trivial example: if $\dot{\mathbf{M}}\mathbf{an}$ satisfies Assumptions 3.1–3.7, such as $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$, we can set $\dot{\mathbf{M}}\mathbf{an^c} = \dot{\mathbf{M}}\mathbf{an}$, define all morphisms in $\dot{\mathbf{M}}\mathbf{an^c}$ to be simple, and for each X in $\dot{\mathbf{M}}\mathbf{an^c}$ we put $C_0(X) = X$, $\partial X = \emptyset$ and $C_k(X) = \emptyset$ for $k > 0$. Then Assumption 3.22 holds. This allows us for example to take $\dot{\mathbf{M}}\mathbf{an^c} = \mathbf{Man^c}$, but to have $\partial X = \emptyset$ and $C_k(X) = \emptyset$ for $k > 0$, for all X in $\mathbf{Man^c}$.

Note that Example 3.24 does not include the category $\mathbf{Man^c_{we}}$ of manifolds with corners and weakly smooth maps from §2.1. This is because Lemma 2.5 is false for $\mathbf{Man^c_{we}}$, so the corner functor C in §2.2 cannot be defined for $\mathbf{Man^c_{we}}$, and Assumption 3.22 fails.

3.4.3 Pulling back morphisms $\theta : E \to \mathcal{T}_f Y$ by $\Pi : C(X) \to X$

Suppose throughout this section that $\dot{\mathbf{M}}\mathbf{an^c}$ satisfies Assumption 3.22 in §3.4.1. In §B.8.1, given a morphism $\theta : E \to \mathcal{T}_f Y$ on X we define a 'pullback' morphism $\Pi^\diamond(\theta) : \Pi^*(E) \to \mathcal{T}_{C(f)} C(Y)$ on $C(X)$. This does not follow from the material of §3.3.1–§3.3.5, it is a new feature for manifolds with corners $\dot{\mathbf{M}}\mathbf{an^c}$.

Definition 3.25. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{M}}\mathbf{an^c}$, and $E \to X$ be a vector bundle on X, and $\theta : E \to \mathcal{T}_f Y$ be a morphism on X in the sense of §3.3.4 and §B.4.8. Then we have a morphism $C(f) : C(X) \to C(Y)$ in $\check{\mathbf{M}}\mathbf{an^c}$, and pulling back by $\Pi : C(X) \to X$ gives a vector bundle $\Pi^*(E) \to C(X)$. Definition B.45 in §B.8.1 defines a morphism $\Pi^\diamond(\theta) : \Pi^*(E) \to \mathcal{T}_{C(f)} C(Y)$ on $C(X)$, in the sense of §3.3.4 and §B.4.8.

We think of $\Pi^\diamond(\theta)$ as a kind of pullback of θ by $\Pi : C(X) \to X$.

We write the restriction $\Pi^\diamond(\theta)|_{C_k(X)}$ for $k=0,1,\ldots$ as $\Pi_k^\diamond(\theta)$. Thus if $f:X\to Y$ is simple, so that $C(f)$ maps $C_k(X)\to C_k(Y)$ by Assumption 3.22(i), we have morphisms $\Pi_k^\diamond(\theta):\Pi_k^*(E)\to\mathcal{T}_{C_k(f)}C_k(Y)$ for $k=0,1,\ldots$.

Example 3.26. Take $\dot{\mathbf{Man}}^{\mathbf{c}}=\mathbf{Man}^{\mathbf{c}}$, and let $f:X\to Y$ be an interior map in $\mathbf{Man}^{\mathbf{c}}$, and $E\to X$ be a vector bundle. Then $\mathcal{T}_fY$ is the sheaf of sections of $f^*({}^bTY)\to X$, as in Example 3.13, so morphisms $\theta:E\to\mathcal{T}_fY$ correspond to vector bundle morphisms $\tilde\theta:E\to f^*({}^bTY)$ on X. Then $\Pi^\diamond(\theta)$ corresponds to the composition of vector bundle morphisms on $C(X)$

$$\Pi^*(E)\xrightarrow{\Pi^*(\tilde\theta)}\Pi^*\circ f^*({}^bTY)=C(f)^*\circ\Pi^*({}^bTY)\xrightarrow{C(f)^*(I_Y^\diamond)}C(f)^*({}^bTC(Y)),$$

where $I_Y^\diamond:\Pi^*({}^bTY)\to{}^bTC(Y)$ is as in (2.13).

Here is Theorem B.47, giving properties of the morphisms $\Pi^\diamond(\theta)$:

Theorem 3.27. **(a)** *Let $f:X\to Y$ be a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $E\to X$ be a vector bundle, and $\theta:E\to\mathcal{T}_fY$ be a morphism, in the sense of* §3.3.4(d). *Then the following diagram of sheaves on $C(X)_{\rm top}$ commutes:*

$$\begin{tikzcd}
\Pi^*(E) \arrow[rr, "\Pi^\diamond(\theta)"'] \arrow[d, "\Pi^*(\theta)"] & & \mathcal{T}_{C(f)}C(Y) \arrow[d, "\mathcal{T}\Pi"'] \\
\mathcal{T}_{f\circ\Pi}Y \arrow[rr, equal] & & \mathcal{T}_{\Pi\circ C(f)}Y,
\end{tikzcd}$$

where $\mathcal{T}\Pi$ and $\Pi^*(\theta)$ are as in §3.3.4(c),(g).

(b) *Let $f:X\to Y$ be a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, $D,E\to X$ be vector bundles, $\lambda:D\to E$ a vector bundle morphism, and $\theta:E\to\mathcal{T}_fY$ a morphism. Then*

$$\Pi^\diamond(\theta\circ\lambda)=\Pi^\diamond(\theta)\circ\Pi^*(\lambda):\Pi^*(D)\longrightarrow\mathcal{T}_{C(f)}C(Y).$$

(c) *Let $f:X\to Y$, $g:Y\to Z$ be morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $E\to X$ be a vector bundle, and $\theta:E\to\mathcal{T}_fY$ be a morphism. Then the following diagram of sheaves on $C(X)_{\rm top}$ commutes:*

$$\begin{tikzcd}
\Pi^*(E) \arrow[rr, "\Pi^\diamond(\theta)"'] \arrow[d, "\Pi^\diamond(\mathcal{T}g\circ\theta)"] & & \mathcal{T}_{C(f)}C(Y) \arrow[d, "\mathcal{T}C(g)"'] \\
\mathcal{T}_{C(g\circ f)}C(Z) \arrow[rr, equal] & & \mathcal{T}_{C(g)\circ C(f)}C(Z).
\end{tikzcd}$$

(d) *Let $f:X\to Y$, $g:Y\to Z$ be morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $F\to Y$ be a vector bundle, and $\phi:F\to\mathcal{T}_gZ$ be a morphism. Then*

$$\begin{aligned}C(f)^*(\Pi^\diamond(\phi))=\Pi^\diamond(f^*(\phi)):C(f)^*\circ\Pi^*(F)&=\Pi^*\circ f^*(F)\\&\longrightarrow\mathcal{T}_{C(g)\circ C(f)}C(Z)=\mathcal{T}_{C(g\circ f)}C(Z).\end{aligned}$$

Here is Theorem B.48, which shows that the $O(s), O(s^2)$ notation of Definition 3.15(i)–(vii) on X pulls back under $\Pi : C(X) \to X$ to the corresponding $O(\Pi(s)), O(\Pi(s)^2)$ notation, using $\Pi^\diamond$ to pull back morphisms $\Lambda : E \to \mathcal{T}_f Y$.

Theorem 3.28. *Let X be an object in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $E \to X$ be a vector bundle, and $s \in \Gamma^\infty(E)$ be a section. Then:*

(i) *Suppose $F \to X$ is a vector bundle and $t_1, t_2 \in \Gamma^\infty(F)$ with $t_2 = t_1 + O(s)$ (or $t_2 = t_1 + O(s^2)$) on X as in Definition* 3.15(i). *Then $\Pi^*(t_2) = \Pi^*(t_1) + O(\Pi^*(s))$ (or $\Pi^*(t_2) = \Pi^*(t_1) + O(\Pi^*(s)^2)$) on $C(X)$.*

(ii) *Suppose $F \to X$ is a vector bundle, $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $\Lambda_1, \Lambda_2 : F \to \mathcal{T}_f Y$ are morphisms with $\Lambda_2 = \Lambda_1 + O(s)$ on X as in Definition* 3.15(ii). *Then Definition* 3.25 *gives morphisms $\Pi^\diamond(\Lambda_1), \Pi^\diamond(\Lambda_2) : \Pi^*(F) \to \mathcal{T}_{C(f)} C(Y)$ on $C(X)$, which satisfy $\Pi^\diamond(\Lambda_2) = \Pi^\diamond(\Lambda_1) + O(\Pi^*(s))$ on $C(X)$.*

(iii) *Suppose $f, g : X \to Y$ are morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$ with $g = f + O(s)$ on X as in Definition* 3.15(iii). *Then $C(g) = C(f) + O(\Pi^*(s))$ on $C(X)$.*

(iv) *Suppose $f, g : X \to Y$ with $g = f + O(s)$ are in* **(iii)**, *and $F \to X$, $G \to Y$ are vector bundles, and $\theta_1 : F \to f^*(G)$, $\theta_2 : F \to g^*(G)$ are morphisms with $\theta_2 = \theta_1 + O(s)$ on X as in Definition* 3.15(iv). *Then $\Pi^*(\theta_2) = \Pi^*(\theta_1) + O(\Pi^*(s))$ on $C(X)$.*

(v) *Suppose $f, g : X \to Y$ with $g = f + O(s)$ are in* **(iii)**, *and $F \to X$ is a vector bundle, and $\Lambda_1 : F \to \mathcal{T}_f Y$, $\Lambda_2 : F \to \mathcal{T}_g Y$ are morphisms with $\Lambda_2 = \Lambda_1 + O(s)$ on X as in Definition* 3.15(v). *Then $C(g) = C(f) + O(\Pi^*(s))$ on $C(X)$ by* **(iii)**, *and Definition* 3.25 *gives morphisms $\Pi^\diamond(\Lambda_1) : \Pi^*(F) \to \mathcal{T}_{C(f)} C(Y)$, $\Pi^\diamond(\Lambda_2) : \Pi^*(F) \to \mathcal{T}_{C(g)} C(Y)$, which satisfy $\Pi^\diamond(\Lambda_2) = \Pi^\diamond(\Lambda_1) + O(\Pi^*(s))$ on $C(X)$.*

(vi) *Suppose $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $F \to X$, $G \to Y$ are vector bundles, and $t \in \Gamma^\infty(G)$ with $f^*(t) = O(s)$, and $\Lambda : F \to \mathcal{T}_f Y$ is a morphism, and $\theta : F \to f^*(G)$ is a vector bundle morphism with $\theta = f^*(\mathrm{d}t) \circ \Lambda + O(s)$ on X as in Definition* 3.15(vi). *Then $\Pi^*(\theta) = C(f)^*(\mathrm{d}\Pi^*(t)) \circ \Pi^\diamond(\Lambda) + O(\Pi^*(s))$ on $C(X)$.*

(vii) *Suppose $f, g : X \to Y$ with $g = f + O(s)$ are in* **(iii)**, *and $\Lambda : E \to \mathcal{T}_f Y$ is a morphism with $g = f + \Lambda \circ s + O(s^2)$ on X as in Definition* 3.15(vii). *Then $C(g) = C(f) + \Pi^\diamond(\Lambda) \circ \Pi^*(s) + O(\Pi^*(s)^2)$ on $C(X)$.*

3.4.4 Comparing different categories $\dot{\mathbf{Man}}^{\mathbf{c}}$

Condition 3.20 in §3.3.7 and §B.7 compared two categories $\dot{\mathbf{Man}}, \ddot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7. Here is Condition B.49 in §B.8.2, the corners analogue:

Condition 3.29. Let $\dot{\mathbf{M}}\mathbf{an^c}, \ddot{\mathbf{M}}\mathbf{an^c}$ satisfy Assumption 3.22, and $F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}} : \dot{\mathbf{M}}\mathbf{an^c} \to \ddot{\mathbf{M}}\mathbf{an^c}$ be a functor in the commutative diagram, as in (3.7)

$$\begin{tikzcd}
 & \dot{\mathbf{M}}\mathbf{an^c} \arrow[d, "F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}"] \arrow[dr, "F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\mathbf{Top}}"] & \\
\mathbf{Man} \arrow[ur, "\subset"] \arrow[dr, "\subset"'] & \ddot{\mathbf{M}}\mathbf{an^c} \arrow[r, "F_{\ddot{\mathbf{M}}\mathbf{an^c}}^{\mathbf{Top}}"'] & \mathbf{Top}.
\end{tikzcd}$$

We also require:

(i) $F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}$ should take products, disjoint unions, open submanifolds, and simple maps in $\dot{\mathbf{M}}\mathbf{an^c}$ to products, disjoint unions, open submanifolds, and simple maps in $\ddot{\mathbf{M}}\mathbf{an^c}$, and preserve dimensions.

(ii) There are canonical isomorphisms $F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}(C_k(X)) \cong C_k(F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}(X))$ for all X in $\dot{\mathbf{M}}\mathbf{an^c}$ and $k \geqslant 0$, so $k=1$ gives $F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}(\partial X) \cong \partial(F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}(X))$.

These isomorphisms commute with the projections $\Pi : C_k(X) \to X$ and $I_{k,l} : C_k(C_l(X)) \to C_{k+l}(X)$ in $\dot{\mathbf{M}}\mathbf{an^c}$ and $\ddot{\mathbf{M}}\mathbf{an^c}$, and induce a natural isomorphism $F_{\check{\dot{\mathbf{M}}}\mathbf{an^c}}^{\check{\ddot{\mathbf{M}}}\mathbf{an^c}} \circ C \Rightarrow C \circ F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}$ of functors $\dot{\mathbf{M}}\mathbf{an^c} \to \check{\ddot{\mathbf{M}}}\mathbf{an^c}$.

As for Figure 3.1, Figure 3.2 gives a diagram of functors from Chapter 2 which satisfy Condition 3.29, with the first corner functor C from Definition 2.9. With the second corner functor C' from Definition 2.9 we get the same diagram omitting $\mathbf{Man^{gc}}$. Arrows '$\to$' satisfy Proposition 3.21(a) or (b). The arrow marked '$\star$' is the non-obvious functor $F_{\mathbf{Man^{ac}_{st}}}^{\mathbf{Man^{c}_{st}}} : \mathbf{Man^{ac}_{st}} \to \mathbf{Man^{c}_{st}}$ from §2.4.2.

$$\begin{tikzcd}
 & \mathbf{Man^{c}_{st}} \arrow[rr, shift left] \arrow[dl, dotted] \arrow[dr] & & \mathbf{Man^{c,ac}_{st}} \arrow[ll, dotted, shift left, "\star"] \arrow[rr, dotted, shift left] \arrow[dr] & & \mathbf{Man^{ac}_{st}} \arrow[ll, shift left] \arrow[dr] & \\
\mathbf{Man^{gc}} & & \mathbf{Man^{c}} \arrow[ll] \arrow[rr] & & \mathbf{Man^{c,ac}} \arrow[rr, dotted, shift right] & & \mathbf{Man^{ac}} \arrow[ll, shift right]
\end{tikzcd}$$

Figure 3.2: Functors satisfying Condition 3.29, with the first corner functor C. Arrows '$\to$' satisfy Proposition 3.21(a) or (b).

Condition 3.29 implies that $F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}} : \dot{\mathbf{M}}\mathbf{an^c} \to \ddot{\mathbf{M}}\mathbf{an^c}$ satisfies Condition 3.20. Thus §3.3.7 applies, so that all the material of §3.3.1–§3.3.5 in $\dot{\mathbf{M}}\mathbf{an^c}$ maps functorially to its analogue in $\ddot{\mathbf{M}}\mathbf{an^c}$. Remark B.50 explains that the morphisms $\Pi^{\diamond}(\theta)$ in §3.4.3 are also compatible with these functorial maps.

Chapter 4

M-Kuranishi spaces

Throughout this chapter we suppose we are given a category $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7 in §3.1. Examples of such categories are given in §3.2. The primary example is the category $\mathbf{Man}$ of ordinary manifolds, and the assumptions are almost all well-known differential-geometric facts in this case. To each such category $\dot{\mathbf{Man}}$ we will associate a 2-category $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ of 'm-Kuranishi spaces'. The possibilities for $\dot{\mathbf{Man}}$ include many categories of manifolds with corners, such as $\mathbf{Man^c}$ in §2.1. In §4.6, to discuss the corners case, we switch notation from $\dot{\mathbf{Man}}$ to a category $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfying Assumption 3.22, with a corresponding 2-category $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ of 'm-Kuranishi spaces with corners'.

We will use the notation of Appendix B for differential geometry in $\dot{\mathbf{Man}}$ throughout, which is summarized in §3.3. In particular, readers should familiarize themselves with 'relative tangent sheaves' $\mathcal{T}_f Y$ in §3.3.4 and §B.4, and the '$O(s)$' and '$O(s^2)$' notation in §3.3.5 and §B.5, before proceeding.

By an abuse of notation we will often refer to objects X of $\dot{\mathbf{Man}}$ as 'manifolds' (though they may in examples have singularities, corners, etc.), and morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ as 'smooth maps' (though they may in examples be non-smooth). As in Assumption 3.4 we have an inclusion $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$. We will call objects $X \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}$ 'classical manifolds', and call morphisms $f : X \to Y$ in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$ 'classical smooth maps'.

In Chapter 3 we distinguished between objects X, Y and morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$, and the corresponding topological spaces $X_{\mathrm{top}}, Y_{\mathrm{top}}$ and continuous maps $f_{\mathrm{top}} : X_{\mathrm{top}} \to Y_{\mathrm{top}}$. We will now drop this distinction, and just write X, Y, f in place of $X_{\mathrm{top}}, Y_{\mathrm{top}}, f_{\mathrm{top}}$, as usual in differential geometry. We will also treat open submanifolds $i : U \hookrightarrow X$ in Assumption 3.2(d) just as open subsets $U \subseteq X$.

On a first reading it may be helpful to take $\dot{\mathbf{Man}} = \mathbf{Man}$. For an introduction to 2-categories, see Appendix A.

4.1 The strict 2-category of m-Kuranishi neighbourhoods

We work throughout in a category $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7.

Definition 4.1. Let X be a topological space. An *m-Kuranishi neighbourhood* on X is a quadruple (V,E,s,ψ) such that:

(a) V is a manifold (object in $\dot{\mathbf{Man}}$). We allow $V=\emptyset$.

(b) $\pi:E\rightarrow V$ is a vector bundle over V, called the *obstruction bundle*.

(c) $s:V\rightarrow E$ is a section of E, called the *Kuranishi section*.

(d) ψ is a homeomorphism from $s^{-1}(0)$ to an open subset $\operatorname{Im}\psi$ in X, where $\operatorname{Im}\psi=\{\psi(x):x\in s^{-1}(0)\}$ is the image of ψ, and is called the *footprint* of (V,E,s,ψ).

If $S\subseteq X$ is open, by an *m-Kuranishi neighbourhood over* S, we mean an m-Kuranishi neighbourhood (V,E,s,ψ) on X with $S\subseteq\operatorname{Im}\psi\subseteq X$.

We call (V,E,s,ψ) a *global m-Kuranishi neighbourhood* if $\operatorname{Im}\psi=X$.

Definition 4.2. Let X,Y be topological spaces, $f:X\rightarrow Y$ a continuous map, (V_i,E_i,s_i,ψ_i), (V_j,E_j,s_j,ψ_j) be m-Kuranishi neighbourhoods on X,Y respectively, and $S\subseteq\operatorname{Im}\psi_i\cap f^{-1}(\operatorname{Im}\psi_j)\subseteq X$ be an open set. A 1-*morphism* $\Phi_{ij}:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ *of m-Kuranishi neighbourhoods over* (S,f) is a triple $\Phi_{ij}=(V_{ij},\phi_{ij},\hat\phi_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i. We do not require that $V_{ij}\cap s_i^{-1}(0)=\psi_i^{-1}(S)$, only that $\psi_i^{-1}(S)\subseteq V_{ij}\cap s_i^{-1}(0)\subseteq V_{ij}$.

(b) $\phi_{ij}:V_{ij}\rightarrow V_j$ is a smooth map.

(c) $\hat\phi_{ij}:E_i|_{V_{ij}}\rightarrow\phi_{ij}^*(E_j)$ is a morphism of vector bundles on V_{ij}.

(d) $\hat\phi_{ij}(s_i|_{V_{ij}})=\phi_{ij}^*(s_j)+O(s_i^2)$, in the sense of Definition 3.15(i).

(e) $f\circ\psi_i=\psi_j\circ\phi_{ij}$ on $s_i^{-1}(0)\cap V_{ij}$.

When $X=Y$ and $f=\mathrm{id}_X$ we just call Φ_{ij} a 1-*morphism over* S. In this case, the *identity* 1-*morphism* $\mathrm{id}_{(V_i,E_i,s_i,\psi_i)}:(V_i,E_i,s_i,\psi_i)\rightarrow(V_i,E_i,s_i,\psi_i)$ over S is $\mathrm{id}_{(V_i,E_i,s_i,\psi_i)}=(V_i,\mathrm{id}_{V_i},\mathrm{id}_{E_i})$.

Definition 4.3. Let $f:X\rightarrow Y$ be a continuous map of topological spaces, (V_i,E_i,s_i,ψ_i), (V_j,E_j,s_j,ψ_j) be m-Kuranishi neighbourhoods on X,Y, and $\Phi_{ij},\Phi'_{ij}:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ be 1-morphisms of m-Kuranishi neighbourhoods over (S,f) for $S\subseteq\operatorname{Im}\psi_i\cap f^{-1}(\operatorname{Im}\psi_j)\subseteq X$ open, where $\Phi_{ij}=(V_{ij},\phi_{ij},\hat\phi_{ij})$ and $\Phi'_{ij}=(V'_{ij},\phi'_{ij},\hat\phi'_{ij})$. Consider pairs $(\acute V_{ij},\hat\lambda_{ij})$ satisfying:

(a) $\acute V_{ij}$ is an open neighbourhood of $\psi_i^{-1}(S)$ in $V_{ij}\cap V'_{ij}$.

(b) $\hat\lambda_{ij}:E_i|_{\acute V_{ij}}\rightarrow\mathcal{T}_{\phi_{ij}}V_j|_{\acute V_{ij}}$ is a morphism in the notation of §3.3.4, with

$$\phi'_{ij}=\phi_{ij}+\hat\lambda_{ij}\circ s_i+O(s_i^2)\ \text{ and }\ \hat\phi'_{ij}=\hat\phi_{ij}+\phi_{ij}^*(\mathrm{d}s_j)\circ\hat\lambda_{ij}+O(s_i)\ \text{ on }\acute V_{ij},\quad(4.1)$$

in the sense of Definition 3.15(iv),(vi),(vii).

Define a binary relation $\sim$ on such pairs by $(\acute{V}_{ij}, \hat{\lambda}_{ij}) \sim (\acute{V}'_{ij}, \hat{\lambda}'_{ij})$ if there exists an open neighbourhood $\ddot{V}_{ij}$ of $\psi_i^{-1}(S)$ in $\acute{V}_{ij} \cap \acute{V}'_{ij}$ with

$$\hat{\lambda}_{ij}|_{\ddot{V}_{ij}} = \hat{\lambda}'_{ij}|_{\ddot{V}_{ij}} + O(s_i) \quad \text{on } \ddot{V}_{ij}, \tag{4.2}$$

in the sense of Definition 3.15(ii). We see from Theorem 3.17(c) that $\sim$ is an equivalence relation. We also write $\sim_S$ in place of $\sim$ if we want to emphasize the open set $S \subseteq X$.

Write $[\acute{V}_{ij}, \hat{\lambda}_{ij}]$ for the $\sim$-equivalence class of $(\acute{V}_{ij}, \hat{\lambda}_{ij})$. We say that $[\acute{V}_{ij}, \hat{\lambda}_{ij}] : \Phi_{ij} \Rightarrow \Phi'_{ij}$ is a *2-morphism of 1-morphisms of m-Kuranishi neighbourhoods on X over* (S, f), or just a *2-morphism over* (S, f). We often write $\Lambda_{ij} = [\acute{V}_{ij}, \hat{\lambda}_{ij}]$.

When $X = Y$ and $f = \mathrm{id}_X$ we just call Λ_{ij} a *2-morphism over* S.

The *identity 2-morphism* of Φ_{ij} over (S, f) is $\mathrm{id}_{\Phi_{ij}} = [V_{ij}, 0] : \Phi_{ij} \Rightarrow \Phi_{ij}$.

Definition 4.4. Let X, Y, Z be topological spaces, $f : X \to Y$, $g : Y \to Z$ be continuous maps, $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j), (V_k, E_k, s_k, \psi_k)$ be m-Kuranishi neighbourhoods on X, Y, Z respectively, and $T \subseteq \mathrm{Im}\,\psi_j \cap g^{-1}(\mathrm{Im}\,\psi_k) \subseteq Y$ and $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(T) \subseteq X$ be open. Suppose $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat{\phi}_{ij}) : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a 1-morphism of m-Kuranishi neighbourhoods over (S, f), and $\Phi_{jk} = (V_{jk}, \phi_{jk}, \hat{\phi}_{jk}) : (V_j, E_j, s_j, \psi_j) \to (V_k, E_k, s_k, \psi_k)$ is a 1-morphism of m-Kuranishi neighbourhoods over (T, g).

Define the *composition of 1-morphisms* to be $\Phi_{jk} \circ \Phi_{ij} = (V_{ik}, \phi_{ik}, \hat{\phi}_{ik})$, where $V_{ik} = \phi_{ij}^{-1}(V_{jk}) \subseteq V_{ij} \subseteq V_i$, and $\phi_{ik} : V_{ik} \to V_k$ is $\phi_{ik} = \phi_{jk} \circ \phi_{ij}|_{V_{ik}}$, and $\hat{\phi}_{ik} : E_i|_{V_{ik}} \to \phi_{ik}^*(E_k)$ is $\hat{\phi}_{ik} = \phi_{ij}|_{V_{ik}}^*(\hat{\phi}_{jk}) \circ \hat{\phi}_{ij}|_{V_{ik}}$.

It is easy to check that $\Phi_{jk} \circ \Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_k, E_k, s_k, \psi_k)$ is a 1-morphism of m-Kuranishi neighbourhoods over $(S, g \circ f)$, using Theorem 3.17(n) to prove that Definition 4.2(d) holds.

An important special case is when $X = Y = Z$, $f = g = \mathrm{id}_X$, and $S = T$, so that Φ_{ij}, Φ_{jk} and $\Phi_{jk} \circ \Phi_{ij}$ are all 1-morphisms over $S \subseteq X$.

Clearly, composition of 1-morphisms is *strictly associative*, that is,

$$(\Phi_{kl} \circ \Phi_{jk}) \circ \Phi_{ij} = \Phi_{kl} \circ (\Phi_{jk} \circ \Phi_{ij}) : (V_i, E_i, s_i, \psi_i) \longrightarrow (V_l, E_k, s_l, \psi_l).$$

So we generally leave the brackets out of such compositions. Also,

$$\Phi_{ij} \circ \mathrm{id}_{(V_i, E_i, s_i, \psi_i)} = \mathrm{id}_{(V_j, E_j, s_j, \psi_j)} \circ \Phi_{ij} = \Phi_{ij}$$

for a 1-morphism $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ over (S, f).

Definition 4.5. Let X, Y be topological spaces, $f : X \to Y$ be continuous, (V_i, E_i, s_i, ψ_i), (V_j, E_j, s_j, ψ_j) be m-Kuranishi neighbourhoods on X, Y, $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\psi_j) \subseteq X$ be open, and $\Phi_{ij}, \Phi'_{ij}, \Phi''_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ be 1-morphisms over (S, f) with $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat{\phi}_{ij})$, $\Phi'_{ij} = (V'_{ij}, \phi'_{ij}, \hat{\phi}'_{ij})$, $\Phi''_{ij} = (V''_{ij}, \phi''_{ij}, \hat{\phi}''_{ij})$. Suppose $\Lambda_{ij} = [\acute{V}_{ij}, \hat{\lambda}_{ij}] : \Phi_{ij} \Rightarrow \Phi'_{ij}$ and $\Lambda'_{ij} = [\acute{V}'_{ij}, \hat{\lambda}'_{ij}] : \Phi'_{ij} \Rightarrow \Phi''_{ij}$ are 2-morphisms over (S, f). We will define the *vertical composition of 2-morphisms*, written

$$\Lambda'_{ij} \odot \Lambda_{ij} = [\acute{V}'_{ij}, \hat{\lambda}'_{ij}] \odot [\acute{V}_{ij}, \hat{\lambda}_{ij}] : \Phi_{ij} \Longrightarrow \Phi''_{ij} \qquad \text{over } (S, f).$$

Choose representatives $(\acute{V}_{ij}, \hat{\lambda}_{ij}), (\acute{V}'_{ij}, \hat{\lambda}'_{ij})$ in the $\sim$-equivalence classes Λ_{ij}, Λ'_{ij}. Define $\acute{V}''_{ij} = \acute{V}_{ij} \cap \acute{V}'_{ij} \subseteq V_i$. Since $\phi'_{ij}|_{\acute{V}''_{ij}} = \phi_{ij}|_{\acute{V}''_{ij}} + O(s_i)$ by (4.1), Theorem 3.17(g) shows that there exists $\check{\lambda}'_{ij} : E_i|_{\acute{V}''_{ij}} \to \mathcal{T}_{\phi_{ij}} V_j|_{\acute{V}''_{ij}}$, unique up to $O(s_i)$, with $\check{\lambda}'_{ij} = \hat{\lambda}'_{ij}|_{\acute{V}''_{ij}} + O(s_i)$ in the sense of Definition 3.15(v).

Define $\hat{\lambda}''_{ij} : E_i|_{\acute{V}''_{ij}} \to \mathcal{T}_{\phi_{ij}} V_j|_{\acute{V}''_{ij}}$ by $\hat{\lambda}''_{ij} = \hat{\lambda}_{ij}|_{\acute{V}''_{ij}} + \check{\lambda}'_{ij}$. Then Theorem 3.17(b),(c),(d),(g),(j),(l) imply $(\acute{V}''_{ij}, \hat{\lambda}''_{ij})$ satisfies Definition 4.3(b) for Φ_{ij}, Φ''_{ij}. Hence $\Lambda''_{ij} = [\acute{V}''_{ij}, \hat{\lambda}''_{ij}] : \Phi_{ij} \Rightarrow \Phi''_{ij}$ is a 2-morphism over (S, f). Since $\check{\lambda}'_{ij}$ is unique up to $O(s_i)$ in Theorem 3.17(f), the equivalence class $\Lambda''_{ij} = [\acute{V}''_{ij}, \hat{\lambda}''_{ij}]$ is independent of choices. We define $\Lambda'_{ij} \odot \Lambda_{ij} = \Lambda''_{ij}$, and call this the *vertical composition of* 2*-morphisms over* (S, f). When $X = Y$ and $f = \mathrm{id}_X$ we call it *vertical composition of* 2*-morphisms over* S.

Let $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ be a 2-morphism over (S, f), and choose a representative $(\acute{V}_{ij}, \hat{\lambda}_{ij})$ for $\Lambda_{ij} = [\acute{V}_{ij}, \hat{\lambda}_{ij}]$. Now $\phi'_{ij}|_{\acute{V}_{ij}} = \phi_{ij}|_{\acute{V}_{ij}} + O(s_i)$ by (4.1), so Theorem 3.17(f) gives $\hat{\lambda}'_{ij} : E_i|_{\acute{V}_{ij}} \to \mathcal{T}_{\phi'_{ij}} V_j|_{\acute{V}_{ij}}$, unique up to $O(s_i)$, with $\hat{\lambda}'_{ij} = -\hat{\lambda}_{ij} + O(s_i)$, in the sense of Definition 3.15(v). We can then show that $\Lambda'_{ij} = [\acute{V}_{ij}, \hat{\lambda}'_{ij}] : \Phi'_{ij} \Rightarrow \Phi_{ij}$ is a 2-morphism over (S, f), and is a two-sided inverse Λ_{ij}^{-1} for Λ_{ij} under vertical composition. Thus, *all* 2*-morphisms over* (S, f) *are invertible under vertical composition, that is, they are* 2*-isomorphisms.*

Definition 4.6. Let X, Y, Z be topological spaces, $f : X \to Y$, $g : Y \to Z$ be continuous maps, $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j), (V_k, E_k, s_k, \psi_k)$ be m-Kuranishi neighbourhoods on X, Y, Z, and $T \subseteq \mathrm{Im}\,\psi_j \cap g^{-1}(\mathrm{Im}\,\psi_k) \subseteq Y$ and $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(T) \subseteq X$ be open. Suppose $\Phi_{ij}, \Phi'_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ are 1-morphisms of m-Kuranishi neighbourhoods over (S, f), and $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ is a 2-morphism over (S, f), and $\Phi_{jk}, \Phi'_{jk} : (V_j, E_j, s_j, \psi_j) \to (V_k, E_k, s_k, \psi_k)$ are 1-morphisms of m-Kuranishi neighbourhoods over (T, g), and $\Lambda_{jk} : \Phi_{jk} \Rightarrow \Phi'_{jk}$ is a 2-morphism over (T, g).

We will define the *horizontal composition of* 2*-morphisms*, written

$$\Lambda_{jk} * \Lambda_{ij} : \Phi_{jk} \circ \Phi_{ij} \Longrightarrow \Phi'_{jk} \circ \Phi'_{ij} \qquad \text{over } (S, g \circ f).$$

Use our usual notation for $\Phi_{ij}, \ldots, \Lambda_{jk}$, and write $(V_{ik}, \phi_{ik}, \hat{\phi}_{ik}) = \Phi_{jk} \circ \Phi_{ij}$, $(V'_{ik}, \phi'_{ik}, \hat{\phi}'_{ik}) = \Phi'_{jk} \circ \Phi'_{ij}$, as in Definition 4.4. Choose representatives $(\acute{V}_{ij}, \hat{\lambda}_{ij})$, $(\acute{V}_{jk}, \hat{\lambda}_{jk})$ for $\Lambda_{ij}, \Lambda_{jk}$.

Set $\acute{V}_{ik} = \acute{V}_{ij} \cap \phi_{ij}^{-1}(\acute{V}_{jk}) \subseteq V_i$. Define a morphism on $\acute{V}_{ik}$

$$\hat{\lambda}_{ik} : E_i|_{\acute{V}_{ik}} \longrightarrow \mathcal{T}_{\phi_{ik}} V_k|_{\acute{V}_{ik}} \quad \text{by} \quad \hat{\lambda}_{ik} = \mathcal{T}\phi_{jk} \circ \hat{\lambda}_{ij} + \phi_{ij}|^*_{\acute{V}_{ik}}(\hat{\lambda}_{jk}) \circ \hat{\phi}_{ij}|_{\acute{V}_{ik}}.$$

We can now check using Theorem 3.17(b),(c),(d),(g),(j),(l),(n),(p),(q),(t),(u) that $(\acute{V}_{ik}, \hat{\lambda}_{ik})$ satisfies Definition 4.3(b) for $\Phi_{jk} \circ \Phi_{ij}, \Phi'_{jk} \circ \Phi'_{ij}$, so $\Lambda_{ik} = [\acute{V}_{ik}, \hat{\lambda}_{ik}]$ is a 2-morphism over $(S, g \circ f)$, which is independent of choices. We define *horizontal composition of* 2*-morphisms* to be $\Lambda_{jk} * \Lambda_{ij} = \Lambda_{ik}$.

When $X = Y = Z$, $f = g = \mathrm{id}_X$ and $S = T$ we call this *horizontal composition of* 2*-morphisms over* S.

We have now defined all the structures of a strict 2-category, as in §A.2: objects (m-Kuranishi neighbourhoods on X over open $S \subseteq X$), 1- and 2-morphisms, their three kinds of composition, and two kinds of identities. The next theorem has a long but straightforward proof, using Theorem 3.17 at some points, and we leave it as an exercise.

Theorem 4.7. *The structures in Definitions* 4.1–4.6 *satisfy the axioms of a strict* 2*-category in* §A.2.

We define three 2-categories of m-Kuranishi neighbourhoods:

Definition 4.8. Write $\mathbf{m\dot{K}N}$ for the *strict* 2*-category of m-Kuranishi neighbourhoods* defined using $\mathbf{\dot{M}an}$, where:

- Objects of $\mathbf{m\dot{K}N}$ are triples $(X, S, (V, E, s, \psi))$, where X is a topological space, $S \subseteq X$ is open, and (V, E, s, ψ) is an m-Kuranishi neighbourhood over S, as in Definition 4.1.
- 1-morphisms $(f, \Phi_{ij}) : (X, S, (V_i, E_i, s_i, \psi_i)) \to (Y, T, (V_j, E_j, s_j, \psi_j))$ of $\mathbf{m\dot{K}N}$ are a pair of a continuous map $f : X \to Y$ with $S \subseteq f^{-1}(T) \subseteq X$ and a 1-morphism $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ over (S, f), as in Definition 4.2.
- For 1-morphisms $(f, \Phi_{ij}), (f, \Phi'_{ij}) : (X, S, (V_i, E_i, s_i, \psi_i)) \to (Y, T, (V_j, E_j, s_j, \psi_j))$ with the same continuous map $f : X \to Y$, a 2-morphism $\Lambda_{ij} : (f, \Phi_{ij}) \Rightarrow (f, \Phi'_{ij})$ of $\mathbf{m\dot{K}N}$ is a 2-morphism $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ over (S, f), as in Definition 4.3.
- Identities, and the three kinds of composition of 1- and 2-morphisms, are defined in the obvious way using Definitions 4.2–4.6.

Define $\mathbf{Gm\dot{K}N}$ to be the full 2-subcategory of $\mathbf{m\dot{K}N}$ with objects $(s^{-1}(0), s^{-1}(0), (V, E, s, \mathrm{id}_{s^{-1}(0)}))$ for which $X = S = s^{-1}(0)$ and $\psi = \mathrm{id}_{s^{-1}(0)}$. We call $\mathbf{Gm\dot{K}N}$ the *strict* 2*-category of global m-Kuranishi neighbourhoods*. For brevity we usually write objects of $\mathbf{Gm\dot{K}N}$ as (V, E, s) rather than $(s^{-1}(0), s^{-1}(0), (V, E, s, \mathrm{id}_{s^{-1}(0)}))$. For a 1-morphism in $\mathbf{Gm\dot{K}N}$

$$\begin{aligned}(f, \Phi_{ij}) : (s_i^{-1}(0), s_i^{-1}(0), (V_i, E_i, s_i, \mathrm{id}_{s_i^{-1}(0)})) \longrightarrow \\ (s_j^{-1}(0), s_j^{-1}(0), (V_j, E_j, s_j, \mathrm{id}_{s_j^{-1}(0)}))\end{aligned}$$

with $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat{\phi}_{ij})$ we must have $f = \phi_{ij}|_{s_i^{-1}(0)} : s_i^{-1}(0) \to s_j^{-1}(0)$ by Definition 4.2(e), so f is determined by Φ_{ij}, and we write 1-morphisms of $\mathbf{Gm\dot{K}N}$ as $\Phi_{ij} : (V_i, E_i, s_i) \to (V_j, E_j, s_j)$ rather than as (f, Φ_{ij}). Similarly, we write 2-morphisms of $\mathbf{Gm\dot{K}N}$ as $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$.

Let X be a topological space and $S \subseteq X$ be open. Write $\mathbf{m\dot{K}N}_S(X)$ for the 2-subcategory of $\mathbf{m\dot{K}N}$ with objects $(X, S, (V, E, s, \psi))$ for X, S as given, 1-morphisms $(\mathrm{id}_X, \Phi_{ij}) : (X, S, (V_i, E_i, s_i, \psi_i)) \to (X, S, (V_j, E_j, s_j, \psi_j))$

for $f = \mathrm{id}_X$, and all 2-morphisms $\Lambda_{ij} : (\mathrm{id}_X, \Phi_{ij}) \Rightarrow (\mathrm{id}_X, \Phi'_{ij})$. We call $\mathbf{m\dot{K}N}_S(X)$ the *strict* 2*-category of m-Kuranishi neighbourhoods over* $S \subseteq X$.

We generally write objects of $\mathbf{m\dot{K}N}_S(X)$ as (V, E, s, ψ), omitting X, S, and 1-morphisms of $\mathbf{m\dot{K}N}_S(X)$ as Φ_{ij}, omitting id_X. That is, objects, 1- and 2-morphisms of $\mathbf{m\dot{K}N}_S(X)$ are just m-Kuranishi neighbourhoods over S and 1- and 2-morphisms over S as in Definitions 4.2–4.4.

The accent '$\dot{}$' in $\mathbf{m\dot{K}N}, \mathbf{Gm\dot{K}N}, \mathbf{m\dot{K}N}_S(X)$ is because they are constructed using $\mathbf{\dot{M}an}$. For particular $\mathbf{\dot{M}an}$ we modify the notation in the obvious way, e.g. if $\mathbf{\dot{M}an} = \mathbf{Man}$ we write $\mathbf{mKN}, \mathbf{GmKN}, \mathbf{mKN}_S(X)$, and if $\mathbf{\dot{M}an} = \mathbf{Man^c}$ we write $\mathbf{mKN^c}, \mathbf{GmKN^c}, \mathbf{mKN}^{\mathbf{c}}_S(X)$.

If $f : X \to Y$ is continuous, $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)$ are m-Kuranishi neighbourhoods on X, Y, and $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\psi_j) \subseteq X$ is open, write $\mathbf{Hom}_{S,f}((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$ for the groupoid with objects 1-morphisms $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ over (S, f), and morphisms 2-morphisms $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ over (S, f).

If $X = Y$ and $f = \mathrm{id}_X$, we write $\mathbf{Hom}_S((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$ in place of $\mathbf{Hom}_{S,f}((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$.

Theorem 4.7 and the last part of Definition 4.5 imply:

Corollary 4.9. *In Definition* 4.8, $\mathbf{m\dot{K}N}, \mathbf{Gm\dot{K}N}$ *and* $\mathbf{m\dot{K}N}_S(X)$ *are strict* 2*-categories, and in fact* $(2,1)$*-categories, as all* 2*-morphisms are invertible.*

Definition 4.10. Let X be a topological space, and $S \subseteq X$ be open, and $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ be a 1-morphism of m-Kuranishi neighbourhoods on X over S. Then Φ_{ij} is a 1-morphism in the 2-category $\mathbf{m\dot{K}N}_S(X)$ of Definition 4.8. We call Φ_{ij} a *coordinate change over* S if it is an equivalence in $\mathbf{m\dot{K}N}_S(X)$. That is, Φ_{ij} is a coordinate change if there exist a 1-morphism $\Phi_{ji} : (V_j, E_j, s_j, \psi_j) \to (V_i, E_i, s_i, \psi_i)$ and 2-(iso)morphisms $\eta : \Phi_{ji} \circ \Phi_{ij} \Rightarrow \mathrm{id}_{(V_i, E_i, s_i, \psi_i)}$ and $\zeta : \Phi_{ij} \circ \Phi_{ji} \Rightarrow \mathrm{id}_{(V_j, E_j, s_j, \psi_j)}$ over S. Write

$$\mathbf{Equ}_S\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big) \subseteq \mathbf{Hom}_S\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)$$

for the subgroupoid with objects coordinate changes over S.

Theorems 10.57 and 10.58 in §10.5.1 give criteria for when 1-morphisms of m-Kuranishi neighbourhoods are coordinate changes.

Definition 4.11. Let $T \subseteq S \subseteq X$ be open. Define the *restriction* 2*-functor* $|_T : \mathbf{m\dot{K}N}_S(X) \to \mathbf{m\dot{K}N}_T(X)$ to map objects (V_i, E_i, s_i, ψ_i) to exactly the same objects, and 1-morphisms Φ_{ij} to exactly the same 1-morphisms but regarded as 1-morphisms over T, and 2-morphisms $\Lambda_{ij} = [\acute{V}_{ij}, \hat{\lambda}_{ij}]$ over S to $\Lambda_{ij}|_T = [\acute{V}_{ij}, \hat{\lambda}_{ij}]|_T$, where $[\acute{V}_{ij}, \hat{\lambda}_{ij}]|_T$ is the $\sim_T$-equivalence class of any representative $(\acute{V}_{ij}, \hat{\lambda}_{ij})$ for the $\sim_S$-equivalence class $[\acute{V}_{ij}, \hat{\lambda}_{ij}]$.

Then $|_T : \mathbf{m\dot{K}N}_S(X) \to \mathbf{m\dot{K}N}_T(X)$ commutes with all the structure, so it is a strict 2-functor of strict 2-categories as in §A.3. If $U \subseteq T \subseteq S \subseteq X$ are open then $|_U \circ |_T = |_U : \mathbf{m\dot{K}N}_S(X) \to \mathbf{m\dot{K}N}_U(X)$.

Now let $f : X \to Y$ be continuous, $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)$ be m-Kuranishi neighbourhoods on X, Y, and $T \subseteq S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j) \subseteq X$ be open. Then as for $|_T$ on 1- and 2-morphisms above, we define a functor

$$\begin{aligned} |_T : \mathbf{Hom}_{S,f}\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big) \longrightarrow \\ \mathbf{Hom}_{T,f}\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big). \end{aligned} \tag{4.3}$$

Convention 4.12. So far we have discussed 1- and 2-morphisms of m-Kuranishi neighbourhoods, and coordinate changes, *over a specified open set* $S \subseteq X$, or over (S, f). We now make the convention that *when we do not specify a domain* S *for a* 1*-morphism,* 2*-morphism, or coordinate change, the domain should be as large as possible.* For example, if we say that $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a 1-morphism (or a 1-morphism over $f : X \to Y$) without specifying S, we mean that $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ (or $S = \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$).

Similarly, if we write a formula involving several 2-morphisms (possibly defined on different domains), without specifying the domain S, we make the convention *that the domain where the formula holds should be as large as possible.* That is, the domain S is taken to be the intersection of the domains of each 2-morphism in the formula, and we implicitly restrict each morphism in the formula to S as in Definition 4.11, so that it makes sense.

4.2 The stack property of m-Kuranishi neighbourhoods

In §A.6 we define *stacks on topological spaces*, a 2-category version of sheaves on topological spaces discussed in §A.5. The next theorem follows from the orbifold version Theorem 6.16, proved in §6.7, by taking $\Gamma_i = \Gamma_j = \{1\}$. It is very important in our theory. We call it the *stack property.* We will use it in §4.3 to construct compositions of 1- and 2-morphisms of m-Kuranishi spaces.

Theorem 4.13. *Let $f : X \to Y$ be a continuous map of topological spaces, and $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)$ be m-Kuranishi neighbourhoods on X, Y. For each open $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j) \subseteq X$, define a groupoid*

$$\begin{aligned} \boldsymbol{\mathcal{H}om}_f\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)(S) \\ = \mathbf{Hom}_{S,f}\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big), \end{aligned}$$

as in Definition 4.8*, for all open $T \subseteq S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ define a functor*

$$\begin{aligned} \rho_{ST} : \boldsymbol{\mathcal{H}om}_f\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)(S) \longrightarrow \\ \boldsymbol{\mathcal{H}om}_f\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)(T) \end{aligned}$$

between groupoids by $\rho_{ST} = |_T$, as in (4.3)*, and for all open $U \subseteq T \subseteq S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ take the obvious isomorphism $\eta_{STU} = \mathrm{id}_{\rho_{SU}} : \rho_{TU} \circ \rho_{ST} \Rightarrow \rho_{SU}$. Then $\boldsymbol{\mathcal{H}om}_f((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$ is a* ***stack*** *on the open subset $\operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ in X, as in* §A.6.

When $X = Y$ and $f = \mathrm{id}_X$ we write $\boldsymbol{\mathcal{H}om}((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$ rather than $\boldsymbol{\mathcal{H}om}_f((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$. Then coordinate changes $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ also form a stack $\boldsymbol{\mathcal{E}qu}((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$ on $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$, a substack of $\boldsymbol{\mathcal{H}om}((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$.

Here it is clear that $\boldsymbol{\mathcal{H}om}_f(\cdots)$ is a prestack on $\mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\psi_j)$, but not at all obvious that it is a stack; the point is that 1- and 2-morphisms of m-Kuranishi neighbourhoods have important gluing properties over open covers.

4.3 The weak 2-category of m-Kuranishi spaces

We can now at last give one of the main definitions of the book:

Definition 4.14. Let X be a Hausdorff, second countable topological space, and $n \in \mathbb{Z}$. An *m-Kuranishi structure $\mathcal{K}$ on X of virtual dimension n* is data $\mathcal{K} = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \Lambda_{ijk,\ i,j,k\in I}\bigr)$, where:

(a) I is an indexing set (not necessarily finite).

(b) (V_i, E_i, s_i, ψ_i) is an m-Kuranishi neighbourhood on X for each $i \in I$, with $\dim V_i - \mathrm{rank}\, E_i = n$.

(c) $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat\phi_{ij}) : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a coordinate change for all $i, j \in I$ (over $S = \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$, as in Convention 4.12).

(d) $\Lambda_{ijk} = [\acute V_{ijk}, \hat\lambda_{ijk}] : \Phi_{jk} \circ \Phi_{ij} \Rightarrow \Phi_{ik}$ is a 2-morphism for all $i, j, k \in I$ (over $S = \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \cap \mathrm{Im}\,\psi_k$, as in Convention 4.12).

(e) $\bigcup_{i\in I} \mathrm{Im}\,\psi_i = X$.

(f) $\Phi_{ii} = \mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$ for all $i \in I$.

(g) $\Lambda_{iij} = \Lambda_{ijj} = \mathrm{id}_{\Phi_{ij}}$ for all $i, j \in I$.

(h) The following diagram of 2-morphisms over $S = \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \cap \mathrm{Im}\,\psi_k \cap \mathrm{Im}\,\psi_l$ commutes for all $i, j, k, l \in I$:

$$\begin{tikzcd}[column sep=huge] \Phi_{kl} \circ \Phi_{jk} \circ \Phi_{ij} \arrow[r, Rightarrow, "\Lambda_{jkl} * \mathrm{id}_{\Phi_{ij}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{kl}} * \Lambda_{ijk}"] & \Phi_{jl} \circ \Phi_{ij} \arrow[d, Rightarrow, "\Lambda_{ijl}"'] \\ \Phi_{kl} \circ \Phi_{ik} \arrow[r, Rightarrow, "\Lambda_{ikl}"] & \Phi_{il}. \end{tikzcd} \tag{4.4}$$

We call $\boldsymbol{X} = (X, \mathcal{K})$ an *m-Kuranishi space*, of *virtual dimension* $\mathrm{vdim}\,\boldsymbol{X} = n$. When we write $x \in \boldsymbol{X}$, we mean that $x \in X$.

Remark 4.15. Our basic assumption on the topological space X of an m-Kuranishi space $\boldsymbol{X} = (X, \mathcal{K})$ is that X should be *Hausdorff and second countable*, following the usual topological assumptions on manifolds, and the definitions of d-manifolds in [57, 58, 61]. Here is how this relates to other conditions.

Since X can be covered by open sets $\mathrm{Im}\,\psi_i \cong s_i^{-1}(0)/\Gamma$, it is automatically *locally compact*, *locally second countable*, and *regular*. Hausdorff, second countable, and locally compact imply *paracompact*. Hausdorff, second countable, and regular imply *metrizable*. Compact and locally second countable, imply second countable. Metrizable implies Hausdorff.

Thus, if $\boldsymbol{X} = (X, \mathcal{K})$ is an m-Kuranishi space in our sense, then X is also Hausdorff, second countable, locally compact, regular, paracompact, and metrizable. Paracompactness is very useful.

The usual topological assumption in previous papers on Kuranishi spaces [24, 30, 39, 77, 78, 80–83, 110–112] is that X is *compact and metrizable*. Since X is automatically locally second countable as it can be covered by m-Kuranishi neighbourhoods, this implies that X is Hausdorff and second countable.

Example 4.16. Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section, so that (V, E, s) is an object in $\mathbf{Gm\dot{K}N}$ from Definition 4.8. Set $X = s^{-1}(0) \subseteq V$, as a topological space with the subspace topology. Then X is Hausdorff and second countable, as V is.

Define an m-Kuranishi structure $\mathcal{K} = \big(\{0\}, (V_0, E_0, s_0, \psi_0), \Phi_{00}, \Lambda_{000}\big)$ on X with indexing set $I = \{0\}$, one m-Kuranishi neighbourhood (V_0, E_0, s_0, ψ_0) with $V_0 = V$, $E_0 = E$, $s_0 = s$ and $\psi_0 = \mathrm{id}_X$, one coordinate change $\Phi_{00} = \mathrm{id}_{(V_0,E_0,s_0,\psi_0)}$, and one 2-morphism $\Lambda_{000} = \mathrm{id}_{\Phi_{00}}$. Then $\boldsymbol{X} = (X, \mathcal{K})$ is an m-Kuranishi space, with $\mathrm{vdim}\, \boldsymbol{X} = \dim V - \mathrm{rank}\, E$. We write $\boldsymbol{S}_{V,E,s} = \boldsymbol{X}$.

We will need notation to distinguish m-Kuranishi neighbourhoods, coordinate changes, and 2-morphisms on different m-Kuranishi spaces. We will often use the following notation for m-Kuranishi spaces $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$:

$$
\begin{aligned}
&\boldsymbol{W} = (W, \mathcal{H}), && \mathcal{H} = \big(H, (T_h, C_h, q_h, \varphi_h)_{h\in H}, && (4.5)\\
&\quad \Sigma_{hh'} = (T_{hh'}, \sigma_{hh'}, \hat\sigma_{hh'})_{h,h'\in H},\ \mathrm{I}_{hh'h''} = [\acute T_{hh'h''}, \hat\iota_{hh'h''}]_{h,h',h''\in H}\big),\\
&\boldsymbol{X} = (X, \mathcal{I}), && \mathcal{I} = \big(I, (U_i, D_i, r_i, \chi_i)_{i\in I}, && (4.6)\\
&\quad \mathrm{T}_{ii'} = (U_{ii'}, \tau_{ii'}, \hat\tau_{ii'})_{i,i'\in I},\ \mathrm{K}_{ii'i''} = [\acute U_{ii'i''}, \hat\kappa_{ii'i''}]_{i,i',i''\in I}\big),\\
&\boldsymbol{Y} = (Y, \mathcal{J}), && \mathcal{J} = \big(J, (V_j, E_j, s_j, \psi_j)_{j\in J}, && (4.7)\\
&\quad \Upsilon_{jj'} = (V_{jj'}, \upsilon_{jj'}, \hat\upsilon_{jj'})_{j,j'\in J},\ \Lambda_{jj'j''} = [\acute V_{jj'j''}, \hat\lambda_{jj'j''}]_{j,j',j''\in J}\big),\\
&\boldsymbol{Z} = (Z, \mathcal{K}), && \mathcal{K} = \big(K, (W_k, F_k, t_k, \omega_k)_{k\in K}, && (4.8)\\
&\quad \Phi_{kk'} = (W_{kk'}, \phi_{kk'}, \hat\phi_{kk'})_{k,k'\in K},\ \mathrm{M}_{kk'k''} = [\acute W_{kk'k''}, \hat\mu_{kk'k''}]_{k,k',k''\in K}\big).
\end{aligned}
$$

The rest of the section until Theorem 4.28 will make m-Kuranishi spaces into a weak 2-category, as in §A.2. We first define 1- and 2-morphisms of m-Kuranishi spaces. Note a possible confusion: we will be defining 1-*morphisms of m-Kuranishi spaces* $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ and 2-*morphisms of m-Kuranishi spaces* $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$, but these will be built out of 1-*morphisms of m-Kuranishi neighbourhoods* $\boldsymbol{f}_{ij}, \boldsymbol{g}_{ij} : (U_i, D_i, r_i, \chi_i) \to (V_j, E_j, s_j, \psi_j)$ and 2-*morphisms of m-Kuranishi neighbourhoods* $\boldsymbol{\eta}_{ij} : \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij}$ in the sense of §4.1, so '1-morphism' and '2-morphism' can mean two different things.

Definition 4.17. Let $\boldsymbol{X} = (X, \mathcal{I})$ and $\boldsymbol{Y} = (Y, \mathcal{J})$ be m-Kuranishi spaces, with notation (4.6)–(4.7). A 1-*morphism of m-Kuranishi spaces* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is data

$$
\boldsymbol{f} = \big(f, \boldsymbol{f}_{ij,\ i\in I,\ j\in J},\ \boldsymbol{F}_{ii',\ i,i'\in I}^{j,\ j\in J},\ \boldsymbol{F}_{i,\ i\in I}^{jj',\ j,j'\in J}\big), \tag{4.9}
$$

satisfying the conditions:

(a) $f : X \to Y$ is a continuous map.

(b) $\boldsymbol{f}_{ij} = (U_{ij}, f_{ij}, \hat{f}_{ij}) : (U_i, D_i, r_i, \chi_i) \to (V_j, E_j, s_j, \psi_j)$ is a 1-morphism of m-Kuranishi neighbourhoods over f for all $i \in I$, $j \in J$ (defined over $S = \mathrm{Im}\,\chi_i \cap f^{-1}(\mathrm{Im}\,\psi_j)$, as usual).

(c) $\boldsymbol{F}_{ii'}^j = [\acute{U}_{ii'}^j, \hat{F}_{ii'}^j] : \boldsymbol{f}_{i'j} \circ \mathrm{T}_{ii'} \Rightarrow \boldsymbol{f}_{ij}$ is a 2-morphism over f for all $i, i' \in I$ and $j \in J$ (defined over $S = \mathrm{Im}\,\chi_i \cap \mathrm{Im}\,\chi_{i'} \cap f^{-1}(\mathrm{Im}\,\psi_j)$).

(d) $\boldsymbol{F}_i^{jj'} = [\acute{U}_i^{jj'}, \hat{F}_i^{jj'}] : \Upsilon_{jj'} \circ \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{f}_{ij'}$ is a 2-morphism over f for all $i \in I$ and $j, j' \in J$ (defined over $S = \mathrm{Im}\,\chi_i \cap f^{-1}(\mathrm{Im}\,\psi_j \cap \mathrm{Im}\,\psi_{j'})$).

(e) $\boldsymbol{F}_{ii}^j = \boldsymbol{F}_i^{jj} = \mathrm{id}_{\boldsymbol{f}_{ij}}$ for all $i \in I$, $j \in J$.

(f) The following commutes for all $i, i', i'' \in I$ and $j \in J$:

$$\begin{tikzcd}
\boldsymbol{f}_{i''j} \circ \mathrm{T}_{i'i''} \circ \mathrm{T}_{ii'} \arrow[r, Rightarrow, "\boldsymbol{F}_{i'i''}^j * \mathrm{id}_{\mathrm{T}_{ii'}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{f}_{i''j}} * \mathrm{K}_{ii'i''}"] & \boldsymbol{f}_{i'j} \circ \mathrm{T}_{ii'} \arrow[d, Rightarrow, "\boldsymbol{F}_{ii'}^j"'] \\
\boldsymbol{f}_{i''j} \circ \mathrm{T}_{ii''} \arrow[r, Rightarrow, "\boldsymbol{F}_{ii''}^j"] & \boldsymbol{f}_{ij}.
\end{tikzcd} \tag{4.10}$$

(g) The following commutes for all $i, i' \in I$ and $j, j' \in J$:

$$\begin{tikzcd}
\Upsilon_{jj'} \circ \boldsymbol{f}_{i'j} \circ \mathrm{T}_{ii'} \arrow[r, Rightarrow, "\boldsymbol{F}_{i'}^{jj'} * \mathrm{id}_{\mathrm{T}_{ii'}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\Upsilon_{jj'}} * \boldsymbol{F}_{ii'}^j"] & \boldsymbol{f}_{i'j'} \circ \mathrm{T}_{ii'} \arrow[d, Rightarrow, "\boldsymbol{F}_{ii'}^{j'}"'] \\
\Upsilon_{jj'} \circ \boldsymbol{f}_{ij} \arrow[r, Rightarrow, "\boldsymbol{F}_i^{jj'}"] & \boldsymbol{f}_{ij'}.
\end{tikzcd} \tag{4.11}$$

(h) The following commutes for all $i \in I$ and $j, j', j'' \in J$:

$$\begin{tikzcd}
\Upsilon_{j'j''} \circ \Upsilon_{jj'} \circ \boldsymbol{f}_{ij} \arrow[r, Rightarrow, "\Lambda_{jj'j''} * \mathrm{id}_{\boldsymbol{f}_{ij}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\Upsilon_{j'j''}} * \boldsymbol{F}_i^{jj'}"] & \Upsilon_{jj''} \circ \boldsymbol{f}_{ij} \arrow[d, Rightarrow, "\boldsymbol{F}_i^{jj''}"'] \\
\Upsilon_{j'j''} \circ \boldsymbol{f}_{ij'} \arrow[r, Rightarrow, "\boldsymbol{F}_i^{j'j''}"] & \boldsymbol{f}_{ij''}.
\end{tikzcd} \tag{4.12}$$

If $x \in \boldsymbol{X}$ (i.e. $x \in X$), we will write $\boldsymbol{f}(x) = f(x) \in \boldsymbol{Y}$.

When $\boldsymbol{Y} = \boldsymbol{X}$, define the *identity* 1-*morphism* $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ by

$$\mathbf{id}_{\boldsymbol{X}} = \bigl(\mathrm{id}_X, \mathrm{T}_{ij,\ i,j \in I},\ \mathrm{K}_{ii'j,\ i,i' \in I}^{\ j \in I},\ \mathrm{K}_{ijj',\ i \in I}^{\ j,j' \in I}\bigr). \tag{4.13}$$

Then Definition 4.14(h) implies that (f)–(h) above hold.

Definition 4.18. Let $\boldsymbol{X} = (X, \mathcal{I})$ and $\boldsymbol{Y} = (Y, \mathcal{J})$ be m-Kuranishi spaces, with notation as in (4.6)–(4.7), and $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms, with notation (4.9). Suppose the continuous maps $f, g : X \to Y$ in $\boldsymbol{f}, \boldsymbol{g}$ satisfy $f = g$. A 2-*morphism of m-Kuranishi spaces* $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is data $\boldsymbol{\eta} = \bigl(\boldsymbol{\eta}_{ij,\ i \in I,\ j \in J}\bigr)$, where $\boldsymbol{\eta}_{ij} = [\acute{U}_{ij}, \hat{\eta}_{ij}] : \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij}$ is a 2-morphism of m-Kuranishi neighbourhoods over $f = g$ (defined over $S = \mathrm{Im}\,\chi_i \cap f^{-1}(\mathrm{Im}\,\psi_j)$, as usual), satisfying the conditions:

(a) $\boldsymbol{G}_{ii'}^{j} \odot (\boldsymbol{\eta}_{i'j} * \mathrm{id}_{\mathrm{T}_{ii'}}) = \boldsymbol{\eta}_{ij} \odot \boldsymbol{F}_{ii'}^{j} : \boldsymbol{f}_{i'j} \circ \mathrm{T}_{ii'} \Rightarrow \boldsymbol{g}_{ij}$ for all $i, i' \in I$, $j \in J$.

(b) $\boldsymbol{G}_{i}^{jj'} \odot (\mathrm{id}_{\Upsilon_{jj'}} * \boldsymbol{\eta}_{ij}) = \boldsymbol{\eta}_{ij'} \odot \boldsymbol{F}_{i}^{jj'} : \Upsilon_{jj'} \circ \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij'}$ for all $i \in I$, $j, j' \in J$.

Note that by definition, 2-morphisms $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ only exist if $f = g$.

If $\boldsymbol{f} = \boldsymbol{g}$, the *identity* 2*-morphism* is $\mathbf{id}_{\boldsymbol{f}} = \bigl(\mathrm{id}_{\boldsymbol{f}_{ij},\, i\in I,\, j\in J}\bigr) : \boldsymbol{f} \Rightarrow \boldsymbol{f}$.

Next we will define composition of 1-morphisms. We must use the stack property in Theorem 4.13 to construct compositions of 1-morphisms $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$, and $\boldsymbol{g} \circ \boldsymbol{f}$ is only unique up to 2-isomorphism.

In the next proposition, part (a) constructs candidates $\boldsymbol{h}$ for $\boldsymbol{g} \circ \boldsymbol{f}$, part (b) shows such $\boldsymbol{h}$ are unique up to canonical 2-isomorphism, and part (c) that $\boldsymbol{g}$ and $\boldsymbol{f}$ are allowed candidates for $\boldsymbol{g} \circ \mathbf{id}_{\boldsymbol{Y}}$, $\mathbf{id}_{\boldsymbol{Y}} \circ \boldsymbol{f}$ respectively.

Proposition 4.19. **(a)** *Let $\boldsymbol{X} = (X, \mathcal{I})$, $\boldsymbol{Y} = (Y, \mathcal{J})$, and $\boldsymbol{Z} = (Z, \mathcal{K})$ be m-Kuranishi spaces with notation* (4.6)–(4.8), *and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be* 1*-morphisms, with $\boldsymbol{f} = \bigl(f, \boldsymbol{f}_{ij}, \boldsymbol{F}_{ii'}^{j}, \boldsymbol{F}_{i}^{jj'}\bigr)$, $\boldsymbol{g} = \bigl(g, \boldsymbol{g}_{jk}, \boldsymbol{G}_{jj'}^{k}, \boldsymbol{G}_{j}^{kk'}\bigr)$. Then there exists a* 1*-morphism $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Z}$ with $\boldsymbol{h} = \bigl(h, \boldsymbol{h}_{ik}, \boldsymbol{H}_{ii'}^{k}, \boldsymbol{H}_{i}^{kk'}\bigr)$, such that $h = g \circ f : X \to Z$, and for all $i \in I$, $j \in J$, $k \in K$ we have* 2*-morphisms of m-Kuranishi neighbourhoods over h*

$$\Theta_{ijk} : \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \Longrightarrow \boldsymbol{h}_{ik}, \tag{4.14}$$

where as usual (4.14) *holds over $S = \mathrm{Im}\,\chi_i \cap f^{-1}(\mathrm{Im}\,\psi_j) \cap h^{-1}(\mathrm{Im}\,\omega_k)$, and for all $i, i' \in I$, $j, j' \in J$, $k, k' \in K$ the following commute:*

$$\begin{tikzcd}
\boldsymbol{g}_{jk} \circ \boldsymbol{f}_{i'j} \circ \mathrm{T}_{ii'} \arrow[r, Rightarrow, "\Theta_{i'jk} * \mathrm{id}_{\mathrm{T}_{ii'}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{jk}} * \boldsymbol{F}_{ii'}^{j}"] & \boldsymbol{h}_{i'k} \circ \mathrm{T}_{ii'} \arrow[d, Rightarrow, "\boldsymbol{H}_{ii'}^{k}"'] \\
\boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \arrow[r, Rightarrow, "\Theta_{ijk}"] & \boldsymbol{h}_{ik},
\end{tikzcd} \tag{4.15}$$

$$\begin{tikzcd}
\boldsymbol{g}_{j'k} \circ \Upsilon_{jj'} \circ \boldsymbol{f}_{ij} \arrow[r, Rightarrow, "\boldsymbol{G}_{jj'}^{k} * \mathrm{id}_{\boldsymbol{f}_{ij}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{j'k}} * \boldsymbol{F}_{i}^{jj'}"] & \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \arrow[d, Rightarrow, "\Theta_{ijk}"'] \\
\boldsymbol{g}_{j'k} \circ \boldsymbol{f}_{ij'} \arrow[r, Rightarrow, "\Theta_{ij'k}"] & \boldsymbol{h}_{ik},
\end{tikzcd} \tag{4.16}$$

$$\begin{tikzcd}
\Phi_{kk'} \circ \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \arrow[r, Rightarrow, "\boldsymbol{G}_{j}^{kk'} * \mathrm{id}_{\boldsymbol{f}_{ij}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{kk'}} * \Theta_{ijk}"] & \boldsymbol{g}_{jk'} \circ \boldsymbol{f}_{ij} \arrow[d, Rightarrow, "\Theta_{ijk'}"'] \\
\Phi_{kk'} \circ \boldsymbol{h}_{ik} \arrow[r, Rightarrow, "\boldsymbol{H}_{i}^{kk'}"] & \boldsymbol{h}_{ik'}.
\end{tikzcd} \tag{4.17}$$

(b) *If $\tilde{\boldsymbol{h}} = \bigl(h, \tilde{\boldsymbol{h}}_{ik}, \tilde{\boldsymbol{H}}_{ii'}^{k}, \tilde{\boldsymbol{H}}_{i}^{kk'}\bigr), \tilde{\Theta}_{ijk}$ are alternative choices for $\boldsymbol{h}, \Theta_{ijk}$ in* **(a)**, *then there is a unique* 2*-morphism of m-Kuranishi spaces $\boldsymbol{\eta} = (\boldsymbol{\eta}_{ik}) : \boldsymbol{h} \Rightarrow \tilde{\boldsymbol{h}}$ satisfying $\boldsymbol{\eta}_{ik} \odot \Theta_{ijk} = \tilde{\Theta}_{ijk} : \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \Rightarrow \tilde{\boldsymbol{h}}_{ik}$ for all $i \in I$, $j \in J$, $k \in K$.*

(c) *If $\boldsymbol{X} = \boldsymbol{Y}$ and $\boldsymbol{f} = \mathbf{id}_{\boldsymbol{Y}}$ in* **(a)**, *so that $I = J$, then a possible choice for $\boldsymbol{h}, \Theta_{ijk}$ in* **(a)** *is $\boldsymbol{h} = \boldsymbol{g}$ and $\Theta_{ijk} = \boldsymbol{G}_{ij}^{k}$.*

Similarly, if $\boldsymbol{Z} = \boldsymbol{Y}$ and $\boldsymbol{g} = \mathbf{id}_{\boldsymbol{Y}}$ in **(a)**, *so that $K = J$, then a possible choice for $\boldsymbol{h}, \Theta_{ijk}$ in* **(a)** *is $\boldsymbol{h} = \boldsymbol{f}$ and $\Theta_{ijk} = \boldsymbol{F}_{i}^{jk}$.*

Proof. For (a), define $h = g \circ f : X \to Z$. Let $i \in I$ and $k \in K$, and set $S = \operatorname{Im}\chi_i \cap h^{-1}(\operatorname{Im}\omega_k)$, so that S is open in X. We want to choose a 1-morphism $\boldsymbol{h}_{ik} : (U_i, D_i, r_i, \chi_i) \to (W_k, F_k, t_k, \omega_k)$ of m-Kuranishi neighbourhoods over (S, h). Since $\{\operatorname{Im}\psi_j : j \in J\}$ is an open cover of Y and f is continuous, $\bigl\{S \cap f^{-1}(\operatorname{Im}\psi_j) : j \in J\bigr\}$ is an open cover of S. For all $j, j' \in J$ we have a 2-morphism over $S \cap f^{-1}(\operatorname{Im}\psi_j \cap \operatorname{Im}\psi_{j'}), h$

$$\begin{aligned}&(\mathrm{id}_{\boldsymbol{g}_{j'k}} * \boldsymbol{F}_i^{jj'}) \odot (\boldsymbol{G}_{jj'}^k * \mathrm{id}_{\boldsymbol{f}_{ij}})^{-1} : \\ &\boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij}|_{S \cap f^{-1}(\operatorname{Im}\psi_j \cap \operatorname{Im}\psi_{j'})} \Longrightarrow \boldsymbol{g}_{j'k} \circ \boldsymbol{f}_{ij'}|_{S \cap f^{-1}(\operatorname{Im}\psi_j \cap \operatorname{Im}\psi_{j'})}.\end{aligned} \tag{4.18}$$

For $j, j', j'' \in J$, consider the diagram of 2-morphisms of 1-morphisms $(U_i, D_i, r_i, \chi_i) \to (W_k, F_k, t_k, \omega_k)$ over $S \cap f^{-1}(\operatorname{Im}\psi_j \cap \operatorname{Im}\psi_{j'} \cap \operatorname{Im}\psi_{j''}), h$:

$$\begin{tikzcd}
\boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} & \boldsymbol{g}_{j'k} \circ \Upsilon_{jj'} \circ \boldsymbol{f}_{ij} \arrow[l, Rightarrow, "\boldsymbol{G}_{jj'}^k * \mathrm{id}_{\boldsymbol{f}_{ij}}"'] \arrow[dr, Rightarrow, bend left, "\mathrm{id}_{\boldsymbol{g}_{j'k}} * \boldsymbol{F}_i^{jj'}"] & \\
\boldsymbol{g}_{j''k} \circ \Upsilon_{jj''} \circ \boldsymbol{f}_{ij} \arrow[u, Rightarrow, "\boldsymbol{G}_{jj''}^k * \mathrm{id}_{\boldsymbol{f}_{ij}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{j''k}} * \boldsymbol{F}_i^{jj''}"] & \boldsymbol{g}_{j''k} \circ \Upsilon_{j'j''} \circ \Upsilon_{jj'} \circ \boldsymbol{f}_{ij} \arrow[l, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{j''k}} * \Lambda_{jj'j''} * \mathrm{id}_{\boldsymbol{f}_{ij}}"'] \arrow[u, Rightarrow, "\boldsymbol{G}_{j'j''}^k * \mathrm{id}_{\Upsilon_{jj'}} * \mathrm{id}_{\boldsymbol{f}_{ij}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{j''k}} * \mathrm{id}_{\Upsilon_{j'j''}} * \boldsymbol{F}_i^{jj'}"] & \boldsymbol{g}_{j'k} \circ \boldsymbol{f}_{ij'}. \\
\boldsymbol{g}_{j''k} \circ \boldsymbol{f}_{ij''} & \boldsymbol{g}_{j''k} \circ \Upsilon_{j'j''} \circ \boldsymbol{f}_{ij'} \arrow[l, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{j''k}} * \boldsymbol{F}_i^{j'j''}"'] \arrow[ur, Rightarrow, bend right, "\boldsymbol{G}_{j'j''}^k * \mathrm{id}_{\boldsymbol{f}_{ij'}}"'] &
\end{tikzcd} \tag{4.19}$$

Here the top left rectangle of (4.19) commutes by Definition 4.17(f) for $\boldsymbol{g}$ composed with $\mathrm{id}_{\boldsymbol{f}_{ij}}$, the bottom left rectangle by Definition 4.17(h) for $\boldsymbol{f}$ composed with $\mathrm{id}_{\boldsymbol{g}_{j''k}}$, and the right hand quadrilateral commutes by properties of strict 2-categories. Thus (4.19) commutes. This implies that

$$\begin{aligned}&\bigl((\mathrm{id}_{\boldsymbol{g}_{j''k}} * \boldsymbol{F}_i^{j'j''}) \odot (\boldsymbol{G}_{j'j''}^k * \mathrm{id}_{\boldsymbol{f}_{ij'}})^{-1}\bigr) \odot \bigl((\mathrm{id}_{\boldsymbol{g}_{j'k}} * \boldsymbol{F}_i^{jj'}) \odot (\boldsymbol{G}_{jj'}^k * \mathrm{id}_{\boldsymbol{f}_{ij}})^{-1}\bigr) \\ &\quad = (\mathrm{id}_{\boldsymbol{g}_{j''k}} * \boldsymbol{F}_i^{jj''} \odot (\boldsymbol{G}_{jj''}^k * \mathrm{id}_{\boldsymbol{f}_{ij}})^{-1}.\end{aligned} \tag{4.20}$$

Now Theorem 4.13 says that 1- and 2-morphisms from (U_i, D_i, r_i, χ_i) to $(W_k, F_k, t_k, \omega_k)$ over h form a stack on S, so applying Definition A.17(v) to the open cover $\bigl\{S \cap f^{-1}(\operatorname{Im}\psi_j) : j \in J\bigr\}$ of S with $\boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij}$ in place of A_j, (4.18) in place of $\alpha_{jj'}$, and (4.20), shows that there exist a 1-morphism $\boldsymbol{h}_{ik} : (U_i, D_i, r_i, \chi_i) \to (W_k, F_k, t_k, \omega_k)$ over (S, h), and 2-morphisms

$$\Theta_{ijk} : \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij}|_{S \cap f^{-1}(\operatorname{Im}\psi_j)} \Longrightarrow \boldsymbol{h}_{ik}|_{S \cap f^{-1}(\operatorname{Im}\psi_j)}$$

for all $j \in J$, satisfying for all $j, j' \in J$

$$\Theta_{ijk}|_{S \cap f^{-1}(\operatorname{Im}\psi_j \cap \operatorname{Im}\psi_{j'})} = \Theta_{ij'k} \odot (\mathrm{id}_{\boldsymbol{g}_{j'k}} * \boldsymbol{F}_i^{jj'}) \odot (\boldsymbol{G}_{jj'}^k * \mathrm{id}_{\boldsymbol{f}_{ij}})^{-1}. \tag{4.21}$$

Observe that (4.21) is equivalent to equation (4.16) in the proposition.

So far we have chosen the data $h, \boldsymbol{h}_{ik}$ for all i, k in $\boldsymbol{h} = \bigl(h, \boldsymbol{h}_{ik}, \boldsymbol{H}_{ii'}^k, \boldsymbol{H}_i^{kk'}\bigr)$, where $\boldsymbol{h}_{ik}$ involved an arbitrary choice. To define $\boldsymbol{H}_{ii'}^k$ for $i, i' \in I$ and $k \in K$, note that for each $j \in J$, equation (4.15) of the proposition implies that

$$\begin{aligned} &\boldsymbol{H}_{ii'}^k|_{\mathrm{Im}\,\chi_i \cap \mathrm{Im}\,\chi_{i'} \cap f^{-1}(\mathrm{Im}\,\psi_j) \cap h^{-1}(\mathrm{Im}\,\omega_k)} \\ &\qquad = \Theta_{ijk} \odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \boldsymbol{F}_{ii'}^j) \odot \bigl(\Theta_{i'jk} * \mathrm{id}_{\mathrm{T}_{ii'}}\bigr)^{-1}. \end{aligned} \tag{4.22}$$

Using (4.21) for i, i' and a similar commutative diagram to (4.19), we can show that the prescribed values (4.22) for $j, j' \in J$ agree when restricted to $\mathrm{Im}\,\chi_i \cap \mathrm{Im}\,\chi_{i'} \cap f^{-1}(\mathrm{Im}\,\psi_j \cap \mathrm{Im}\,\psi_{j'}) \cap h^{-1}(\mathrm{Im}\,\omega_k)$. Therefore the stack property Theorem 4.13 and Definition A.17(iii),(iv) show that there is a unique 2-morphism $\boldsymbol{H}_{ii'}^k : \boldsymbol{h}_{i'k} \circ \mathrm{T}_{ii'} \Rightarrow \boldsymbol{h}_{ik}$ over h satisfying (4.22) for all $j \in J$, or equivalently, satisfying (4.15) for all $j \in J$. Similarly, there is a unique 2-morphism $\boldsymbol{H}_i^{kk'} : \Phi_{kk'} \circ \boldsymbol{h}_{ik} \Rightarrow \boldsymbol{h}_{ik'}$ over h satisfying (4.17) for all $j \in J$.

We now claim that $\boldsymbol{h} = \bigl(h, \boldsymbol{h}_{ik}, \boldsymbol{H}_{ii'}^k, \boldsymbol{H}_i^{kk'}\bigr)$ is a 1-morphism $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Z}$. It remains to show Definition 4.17(f)–(h) hold for $\boldsymbol{h}$. To prove this, we first fix $j \in J$ and prove the restrictions of (f)–(h) to the intersections of their domains with $f^{-1}(\mathrm{Im}\,\psi_j)$. For instance, for part (f), for $i, i', i'' \in I$ and $k \in K$ we have

$$\begin{aligned} &\bigl(\boldsymbol{H}_{ii''}^k \odot (\mathrm{id}_{\boldsymbol{h}_{i''k}} * \mathrm{K}_{ii'i''})\bigr)|_{\mathrm{Im}\,\chi_i \cap \cdots \cap h^{-1}(\mathrm{Im}\,\omega_k)} \\ &= \bigl[\Theta_{ijk} \odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \boldsymbol{F}_{ii''}^j) \odot \bigl(\Theta_{i''jk} * \mathrm{id}_{\mathrm{T}_{ii''}}\bigr)^{-1}\bigr] \odot (\mathrm{id}_{\boldsymbol{h}_{i''k}} * \mathrm{K}_{ii'i''}) \\ &= \Theta_{ijk} \odot \bigl(\mathrm{id}_{\boldsymbol{g}_{jk}} * (\boldsymbol{F}_{ii''}^j \odot (\mathrm{id}_{\boldsymbol{f}_{ij''}} * \mathrm{K}_{ii'i''}))\bigr) \odot \bigl((\Theta_{i''jk}^{-1} * \mathrm{id}_{\mathrm{T}_{i'i''}}) * \mathrm{id}_{\mathrm{T}_{ii'}}\bigr) \\ &= \Theta_{ijk} \odot \bigl(\mathrm{id}_{\boldsymbol{g}_{jk}} * (\boldsymbol{F}_{ii'}^j \odot (\boldsymbol{F}_{i'i''}^j * \mathrm{id}_{\mathrm{T}_{ii'}}))\bigr) \odot \bigl((\Theta_{i''jk}^{-1} * \mathrm{id}_{\mathrm{T}_{i'i''}}) * \mathrm{id}_{\mathrm{T}_{ii'}}\bigr) \\ &= \bigl[\Theta_{ijk} \odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \boldsymbol{F}_{ii'}^j) \odot \bigl(\Theta_{i'jk} * \mathrm{id}_{\mathrm{T}_{ii'}}\bigr)^{-1}\bigr] \\ &\quad \odot \bigl(\bigl[\bigl(\Theta_{i'jk} \odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \boldsymbol{F}_{i'i''}^j)\bigr) \odot \bigl(\Theta_{i''jk} * \mathrm{id}_{\mathrm{T}_{i'i''}}\bigr)^{-1}\bigr] * \mathrm{id}_{\mathrm{T}_{ii'}}\bigr) \\ &= \bigl(\boldsymbol{H}_{ii'}^k \odot (\boldsymbol{H}_{i'i''}^k * \mathrm{id}_{\mathrm{T}_{ii'}})\bigr)|_{\mathrm{Im}\,\chi_i \cap \mathrm{Im}\,\chi_{i'} \cap \mathrm{Im}\,\chi_{i''} \cap f^{-1}(\mathrm{Im}\,\psi_j) \cap h^{-1}(\mathrm{Im}\,\omega_k)}, \end{aligned}$$

using (4.22) in the first and fifth steps, Definition 4.17(f) for $\boldsymbol{f}$ in the third, and properties of strict 2-categories. Then we use the stack property Theorem 4.13 and Definition A.17(iii) to deduce that as Definition 4.17(f)–(h) for $\boldsymbol{h}$ hold on the sets of an open cover, they hold globally. Therefore $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Z}$ is a 1-morphism of m-Kuranishi spaces satisfying (4.15)–(4.17), proving (a).

For (b), if $\tilde{\boldsymbol{h}}, \tilde{\Theta}_{ijk}$ are alternatives, then $\boldsymbol{h}_{ik}, \tilde{\boldsymbol{h}}_{ik}$ are alternative solutions to the application of Theorem 4.13 and Definition A.17(v) above, for all $i \in I$ and $k \in K$. Thus, the last part of Definition A.17(v) implies that there is a unique 2-morphism $\boldsymbol{\eta}_{ik} : \boldsymbol{h}_{ik} \Rightarrow \tilde{\boldsymbol{h}}_{ik}$ over h such that for all $j \in J$ we have

$$\boldsymbol{\eta}_{ik}|_{\mathrm{Im}\,\chi_i \cap f^{-1}(\mathrm{Im}\,\psi_j) \cap h^{-1}(\mathrm{Im}\,\omega_k)} = \tilde{\Theta}_{ijk} \odot \Theta_{ijk}^{-1}. \tag{4.23}$$

This implies that $\boldsymbol{\eta}_{ik} \odot \Theta_{ijk} = \tilde{\Theta}_{ijk}$, as in (b). For each $j \in J$ we have

$$\begin{aligned} &\bigl(\tilde{\boldsymbol{H}}_{ii'}^k \odot (\boldsymbol{\eta}_{i'k} * \mathrm{id}_{\mathrm{T}_{ii'}})\bigr)|_{\mathrm{Im}\,\chi_i \cap \mathrm{Im}\,\chi_{i'} \cap f^{-1}(\mathrm{Im}\,\psi_j) \cap h^{-1}(\mathrm{Im}\,\omega_k)} \\ &= \bigl[\tilde{\Theta}_{ijk} \odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \boldsymbol{F}_{ii'}^j) \odot \bigl(\tilde{\Theta}_{i'jk} * \mathrm{id}_{\mathrm{T}_{ii'}}\bigr)^{-1}\bigr] \odot \bigl[\bigl(\tilde{\Theta}_{i'jk} \odot \Theta_{i'jk}^{-1}\bigr) * \mathrm{id}_{\mathrm{T}_{ii'}}\bigr] \\ &= \bigl[\tilde{\Theta}_{ijk} \odot \Theta_{ijk}^{-1}\bigr] \odot \bigl[\Theta_{ijk} \odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \boldsymbol{F}_{ii'}^j) \odot \bigl(\Theta_{i'jk} * \mathrm{id}_{\mathrm{T}_{ii'}}\bigr)^{-1}\bigr] \\ &= \bigl(\boldsymbol{\eta}_{ik} \odot \boldsymbol{H}_{ii'}^k\bigr)|_{\mathrm{Im}\,\chi_i \cap \mathrm{Im}\,\chi_{i'} \cap f^{-1}(\mathrm{Im}\,\psi_j) \cap h^{-1}(\mathrm{Im}\,\omega_k)}, \end{aligned}$$

using (4.22) and (4.23) in the first and third steps. So by Definition A.17(iii) we deduce that $\tilde{\boldsymbol{H}}^k_{ii'} \odot (\boldsymbol{\eta}_{i'k} * \mathrm{id}_{\mathrm{T}_{ii'}}) = \boldsymbol{\eta}_{ik} \odot \boldsymbol{H}^k_{ii'}$, which is Definition 4.18(a) for $\boldsymbol{\eta} = (\boldsymbol{\eta}_{ik}) : \boldsymbol{h} \Rightarrow \tilde{\boldsymbol{h}}$. Similarly Definition 4.18(b) holds, so $\boldsymbol{\eta} : \boldsymbol{h} \Rightarrow \tilde{\boldsymbol{h}}$ is a 2-morphism of m-Kuranishi spaces. This proves (b). Part (c) is immediate, using Definition 4.17(f)–(h) for $\boldsymbol{f}, \boldsymbol{g}$ to prove (4.15)–(4.17) hold for the given choices of $\boldsymbol{h}$ and Θ_{ijk}. This completes the proof of Proposition 4.19. $\square$

Proposition 4.19(a) gives possible values $\boldsymbol{h}$ for the composition $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$. Since there is no distinguished choice, we choose $\boldsymbol{g} \circ \boldsymbol{f}$ arbitrarily.

Definition 4.20. For all pairs of 1-morphisms of m-Kuranishi spaces $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$, use the Axiom of Global Choice (see Remark 4.21) to choose possible values of $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Z}$ and Θ_{ijk} in Proposition 4.19(a), and write $\boldsymbol{g} \circ \boldsymbol{f} = \boldsymbol{h}$, and for $i \in I$, $j \in J$, $k \in K$ write

$$\Theta^{\boldsymbol{g},\boldsymbol{f}}_{ijk} = \Theta_{ijk} : \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \Longrightarrow (\boldsymbol{g} \circ \boldsymbol{f})_{ik}. \tag{4.24}$$

We call $\boldsymbol{g} \circ \boldsymbol{f}$ the *composition of* 1*-morphisms of m-Kuranishi spaces.*

For general $\boldsymbol{f}, \boldsymbol{g}$ we make these choices arbitrarily. However, if $\boldsymbol{X} = \boldsymbol{Y}$ and $\boldsymbol{f} = \mathbf{id}_{\boldsymbol{Y}}$ then we choose $\boldsymbol{g} \circ \mathbf{id}_{\boldsymbol{Y}} = \boldsymbol{g}$ and $\Theta^{\boldsymbol{g},\mathbf{id}_{\boldsymbol{Y}}}_{jj'k} = \boldsymbol{G}^k_{jj'}$, and if $\boldsymbol{Z} = \boldsymbol{Y}$ and $\boldsymbol{g} = \mathbf{id}_{\boldsymbol{Y}}$ then we choose $\mathbf{id}_{\boldsymbol{Y}} \circ \boldsymbol{f} = \boldsymbol{f}$ and $\Theta^{\mathbf{id}_{\boldsymbol{Y}},\boldsymbol{f}}_{ijj'} = \boldsymbol{F}^{jj'}_i$. This is allowed by Proposition 4.19(c).

The definition of a weak 2-category in Appendix A includes 2-isomorphisms $\boldsymbol{\beta}_{\boldsymbol{f}} : \boldsymbol{f} \circ \mathbf{id}_{\boldsymbol{X}} \Rightarrow \boldsymbol{f}$ and $\boldsymbol{\gamma}_{\boldsymbol{f}} : \mathbf{id}_{\boldsymbol{Y}} \circ \boldsymbol{f} \Rightarrow \boldsymbol{f}$ in (A.10), since one does not require $\boldsymbol{f} \circ \mathbf{id}_{\boldsymbol{X}} = \boldsymbol{f}$ and $\mathbf{id}_{\boldsymbol{Y}} \circ \boldsymbol{f} = \boldsymbol{f}$ in a general weak 2-category. We define

$$\boldsymbol{\beta}_{\boldsymbol{f}} = \mathbf{id}_{\boldsymbol{f}} : \boldsymbol{f} \circ \mathbf{id}_{\boldsymbol{X}} \Longrightarrow \boldsymbol{f}, \quad \boldsymbol{\gamma}_{\boldsymbol{f}} = \mathbf{id}_{\boldsymbol{f}} : \mathbf{id}_{\boldsymbol{Y}} \circ \boldsymbol{f} \Longrightarrow \boldsymbol{f}. \tag{4.25}$$

Remark 4.21. As in Shulman [101, §7] or Herrlick and Strecker [45, §1.2], the *Axiom of Global Choice*, or *Axiom of Choice for classes*, used in Definition 4.20, is a strong form of the Axiom of Choice.

As in Jech [54], in Set Theory one distinguishes between sets, and 'classes', which are like sets but may be larger. We are not allowed to consider things like 'the set of all sets', or 'the set of all manifolds', as this would lead to paradoxes such as 'the set of all sets which are not members of themselves'. Instead sets, manifolds, ... form classes, upon which more restrictive operations are allowed.

The Axiom of Choice says that if $\{S_i : i \in I\}$ is a family of nonempty sets, with I a set, then we can simultaneously choose an element $s_i \in S_i$ for all $i \in I$. The Axiom of Global Choice says the same thing, but allowing I (and possibly also the S_i) to be classes rather than sets. As in [101, §7], the Axiom of Global Choice follows from the axioms of von Neumann–Bernays–Gödel Set Theory.

The Axiom of Global Choice is used, implicitly or explicitly, in the proofs of important results in category theory in their most general form, for example, Adjoint Functor Theorems, or that every category has a skeleton, or that every weak 2-category can be strictified.

We need to use the Axiom of Global Choice above because we make an arbitrary choice of $\boldsymbol{g} \circ \boldsymbol{f}$ for all $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ in $\mathbf{m\dot{K}ur}$, and as we have defined things, the collection of all such $(\boldsymbol{f}, \boldsymbol{g})$ may be a proper class, not a set. We could avoid this by arranging our foundations differently. For example, if we required $\dot{\mathbf{M}}\mathbf{an}$ and **Top** to be small categories, then the collection of all $(\boldsymbol{f}, \boldsymbol{g})$ would be a set, and the usual Axiom of Choice would suffice.

If we did not make arbitrary choices of compositions $\boldsymbol{g} \circ \boldsymbol{f}$ at all, then $\mathbf{m\dot{K}ur}$ would not be a weak 2-category in Theorem 4.28 below, since for 1-morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ in $\mathbf{m\dot{K}ur}$ we would not be given a unique composition $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$, but only a nonempty family of possible choices for $\boldsymbol{g} \circ \boldsymbol{f}$, which are all 2-isomorphic. Such structures appear in the theory of *quasi-categories*, as in Boardman and Vogt [5] or Joyal [55], which are a form of ∞-category, and $\mathbf{m\dot{K}ur}$ would be an example of a 3-coskeletal quasi-category.

Since composition of 1-morphisms $\boldsymbol{g} \circ \boldsymbol{f}$ is natural only up to canonical 2-isomorphism, as in Proposition 4.19(b), composition is associative only up to canonical 2-isomorphism. Note that the 2-isomorphisms $\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}}$ in (4.26) are part of the definition of a weak 2-category in §A.2, as in (A.7).

Proposition 4.22. *Let $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms of m-Kuranishi spaces, and define composition of 1-morphisms as in Definition 4.20. Then using notation (4.5)–(4.8), there is a unique 2-morphism*

$$\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}} : (\boldsymbol{g} \circ \boldsymbol{f}) \circ \boldsymbol{e} \Longrightarrow \boldsymbol{g} \circ (\boldsymbol{f} \circ \boldsymbol{e}) \tag{4.26}$$

with the property that for all $h \in H$, $i \in I$, $j \in J$ and $k \in K$ we have

$$(\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}})_{hk} \odot \Theta_{hik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}} \odot (\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}} * \mathrm{id}_{\boldsymbol{e}_{hi}}) = \Theta_{hjk}^{\boldsymbol{g},\boldsymbol{f}\circ\boldsymbol{e}} \odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \Theta_{hij}^{\boldsymbol{f},\boldsymbol{e}}). \tag{4.27}$$

Proof. The proof uses similar ideas to that of Proposition 4.19, so we will be brief. Note that for $h \in H$, $i \in I$, $j \in J$, $k \in K$, equation (4.27) implies that

$$\begin{aligned}&(\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}})_{hk}|_{\mathrm{Im}\,\varphi_h \cap e^{-1}(\mathrm{Im}\,\chi_i)\cap(f\circ e)^{-1}(\mathrm{Im}\,\psi_j)\cap(g\circ f\circ e)^{-1}(\mathrm{Im}\,\omega_k)}\\ &\quad= \Theta_{hjk}^{\boldsymbol{g},\boldsymbol{f}\circ\boldsymbol{e}} \odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \Theta_{hij}^{\boldsymbol{f},\boldsymbol{e}}) \odot (\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}} * \mathrm{id}_{\boldsymbol{e}_{hi}})^{-1} \odot (\Theta_{hik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}})^{-1}.\end{aligned} \tag{4.28}$$

We show that for $i' \in I$, $j' \in J$, the right hand sides of (4.28) for h, i, j, k and for h, i', j', k agree on the overlap of their domains, using the properties (4.15)–(4.17) of the $\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}}$. Then we use the stack property Theorem 4.13 and Definition A.17(iii),(iv) to deduce that there is a unique 2-morphism $(\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}})_{hk}$ satisfying (4.28) for all $i \in I$, $j \in J$.

We prove the restrictions of Definition 4.18(a),(b) for $\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}} = ((\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}})_{hk})$ to the intersection of their domains with $e^{-1}(\mathrm{Im}\,\chi_i) \cap (f \circ e)^{-1}(\mathrm{Im}\,\psi_j)$, for all $i \in I$ and $j \in J$, using (4.28) and properties of the $\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}}$. Since these intersections form an open cover of the domains, Theorem 4.13 and Definition A.17(iii) imply that Definition 4.18(a),(b) for $\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}}$ hold on the correct domains, so $\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}}$ is a 2-morphism, as in (4.26). Uniqueness follows from uniqueness of $(\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}})_{hk}$ above. This completes the proof. □

We define vertical and horizontal composition of 2-morphisms:

Definition 4.23. Let $\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ be 1-morphisms of m-Kuranishi spaces, using notation (4.6)–(4.7), and $\boldsymbol{\eta}=(\boldsymbol{\eta}_{ij}):\boldsymbol{f}\Rightarrow\boldsymbol{g}$, $\boldsymbol{\zeta}=(\boldsymbol{\zeta}_{ij}):\boldsymbol{g}\Rightarrow\boldsymbol{h}$ be 2-morphisms. Define the *vertical composition of* 2*-morphisms* $\boldsymbol{\zeta}\odot\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{h}$ by

$$\boldsymbol{\zeta}\odot\boldsymbol{\eta}=\big(\boldsymbol{\zeta}_{ij}\odot\boldsymbol{\eta}_{ij},\ i\in I,\ j\in J\big). \tag{4.29}$$

To see that $\boldsymbol{\zeta}\odot\boldsymbol{\eta}$ satisfies Definition 4.18(a),(b), for (a) note that for all $i,i'\in I$ and $j\in J$, by Definition 4.18(a) for $\boldsymbol{\eta},\boldsymbol{\zeta}$ we have

$$\begin{aligned}\boldsymbol{H}^j_{ii'}\odot((\boldsymbol{\zeta}_{i'j}\odot\boldsymbol{\eta}_{i'j})*\mathrm{id}_{\mathrm{T}_{ii'}})&=\boldsymbol{H}^j_{ii'}\odot(\boldsymbol{\zeta}_{i'j}*\mathrm{id}_{\mathrm{T}_{ii'}})\odot(\boldsymbol{\eta}_{i'j}*\mathrm{id}_{\mathrm{T}_{ii'}})\\&=\boldsymbol{\zeta}_{ij}\odot\boldsymbol{G}^j_{ii'}\odot(\boldsymbol{\eta}_{i'j}*\mathrm{id}_{\mathrm{T}_{ii'}})=(\boldsymbol{\zeta}_{ij}\odot\boldsymbol{\eta}_{ij})\odot\boldsymbol{F}^j_{ii'},\end{aligned}$$

and Definition 4.18(b) for $\boldsymbol{\zeta}\odot\boldsymbol{\eta}$ is proved similarly.

Clearly, vertical composition of 2-morphisms of m-Kuranishi spaces is associative, $(\boldsymbol{\theta}\odot\boldsymbol{\zeta})\odot\boldsymbol{\eta}=\boldsymbol{\theta}\odot(\boldsymbol{\zeta}\odot\boldsymbol{\eta})$, since vertical composition of 2-morphisms of m-Kuranishi neighbourhoods is associative.

If $\boldsymbol{g}=\boldsymbol{h}$ and $\boldsymbol{\zeta}=\mathbf{id}_{\boldsymbol{g}}$ then $\mathbf{id}_{\boldsymbol{g}}\odot\boldsymbol{\eta}=(\mathrm{id}_{\boldsymbol{g}_{ij}}\odot\boldsymbol{\eta}_{ij})=(\boldsymbol{\eta}_{ij})=\boldsymbol{\eta}$, and similarly $\boldsymbol{\zeta}\odot\mathbf{id}_{\boldsymbol{g}}=\boldsymbol{\zeta}$, so identity 2-morphisms behave as expected under $\odot$.

If $\boldsymbol{\eta}=(\boldsymbol{\eta}_{ij}):\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is a 2-morphism of m-Kuranishi spaces, then as 2-morphisms $\boldsymbol{\eta}_{ij}$ of m-Kuranishi neighbourhoods are invertible, we may define $\boldsymbol{\eta}^{-1}=(\boldsymbol{\eta}_{ij}^{-1}):\boldsymbol{g}\Rightarrow\boldsymbol{f}$. It is easy to check that $\boldsymbol{\eta}^{-1}$ is a 2-morphism, and $\boldsymbol{\eta}^{-1}\odot\boldsymbol{\eta}=\mathbf{id}_{\boldsymbol{f}}$, $\boldsymbol{\eta}\odot\boldsymbol{\eta}^{-1}=\mathbf{id}_{\boldsymbol{g}}$. Thus, all 2-morphisms of m-Kuranishi spaces are 2-isomorphisms.

Definition 4.24. Let $\boldsymbol{e},\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ and $\boldsymbol{g},\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ be 1-morphisms of m-Kuranishi spaces, using notation (4.6)–(4.8), and $\boldsymbol{\eta}=(\boldsymbol{\eta}_{ij}):\boldsymbol{e}\Rightarrow\boldsymbol{f}$, $\boldsymbol{\zeta}=(\boldsymbol{\zeta}_{jk}):\boldsymbol{g}\Rightarrow\boldsymbol{h}$ be 2-morphisms. We claim there is a unique 2-morphism $\boldsymbol{\theta}=(\boldsymbol{\theta}_{ik}):\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$, such that for all $i\in I$, $j\in J$, $k\in K$, we have

$$\boldsymbol{\theta}_{ik}|_{\mathrm{Im}\,\chi_i\cap e^{-1}(\mathrm{Im}\,\psi_j)\cap(g\circ e)^{-1}(\mathrm{Im}\,\omega_k)}=\Theta^{\boldsymbol{h},\boldsymbol{f}}_{ijk}\odot(\boldsymbol{\zeta}_{jk}*\boldsymbol{\eta}_{ij})\odot(\Theta^{\boldsymbol{g},\boldsymbol{e}}_{ijk})^{-1}. \tag{4.30}$$

To prove this, suppose $j,j'\in J$, and consider the diagram of 2-morphisms over $\mathrm{Im}\,\chi_i\cap e^{-1}(\mathrm{Im}\,\psi_j\cap\mathrm{Im}\,\psi_{j'})\cap(g\circ e)^{-1}(\mathrm{Im}\,\omega_k)$:

$$\begin{tikzcd}
& \boldsymbol{g}_{jk}\circ\boldsymbol{e}_{ij} \arrow[r, Rightarrow, "\boldsymbol{\zeta}_{jk}*\boldsymbol{\eta}_{ij}"'] & \boldsymbol{h}_{jk}\circ\boldsymbol{f}_{ij} \arrow[dr, Rightarrow, bend left, "\Theta^{\boldsymbol{h},\boldsymbol{f}}_{ijk}"] & \\
(\boldsymbol{g}\circ\boldsymbol{e})_{ik} \arrow[ur, Rightarrow, bend left, "(\Theta^{\boldsymbol{g},\boldsymbol{e}}_{ijk})^{-1}"] \arrow[dr, Rightarrow, bend right, "(\Theta^{\boldsymbol{g},\boldsymbol{e}}_{ij'k})^{-1}"'] & \boldsymbol{g}_{j'k}\circ\Upsilon_{jj'}\circ\boldsymbol{e}_{ij} \arrow[u, Rightarrow, "\boldsymbol{G}^k_{jj'}*\mathrm{id}_{\boldsymbol{e}_{ij}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{j'k}}*\boldsymbol{E}^{jj'}_i"'] \arrow[r, Rightarrow, "\boldsymbol{\zeta}_{j'k}*\mathrm{id}_{\Upsilon_{jj'}}*\boldsymbol{\eta}_{ij}"] & \boldsymbol{h}_{j'k}\circ\Upsilon_{jj'}\circ\boldsymbol{f}_{ij} \arrow[u, Rightarrow, "\boldsymbol{H}^k_{jj'}*\mathrm{id}_{\boldsymbol{f}_{ij}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{j'k}}*\boldsymbol{E}^{jj'}_i"] & (\boldsymbol{h}\circ\boldsymbol{f})_{ik}. \\
& \boldsymbol{g}_{j'k}\circ\boldsymbol{e}_{ij'} \arrow[r, Rightarrow, "\boldsymbol{\zeta}_{j'k}*\boldsymbol{\eta}_{ij'}"] & \boldsymbol{h}_{j'k}\circ\boldsymbol{f}_{ij'} \arrow[ur, Rightarrow, bend right, "\Theta^{\boldsymbol{h},\boldsymbol{f}}_{ij'k}"'] &
\end{tikzcd} \tag{4.31}$$

Here the left and right quadrilaterals commute by (4.16), and the central rectangles commute by Definition 4.18(a),(b) for $\boldsymbol{\zeta},\boldsymbol{\eta}$. Hence (4.31) commutes.

The two routes round the outside of (4.31) imply that the prescribed values (4.30) for $\boldsymbol{\theta}_{ik}$ agree on overlaps between open sets for j,j'. As the $\mathrm{Im}\,\chi_i\cap e^{-1}(\mathrm{Im}\,\psi_j)\cap(g\circ e)^{-1}(\mathrm{Im}\,\omega_k)$ for $j\in J$ form an open cover of the correct domain

$\operatorname{Im}\chi_i \cap (g\circ e)^{-1}(\operatorname{Im}\omega_k)$, by Theorem 4.13 and Definition A.17(iii),(iv), there is a unique 2-morphism $\boldsymbol{\theta}_{ik}:(\boldsymbol{g}\circ\boldsymbol{e})_{ik}\Rightarrow(\boldsymbol{h}\circ\boldsymbol{f})_{ik}$ satisfying (4.30) for all $j\in J$.

To show $\boldsymbol{\theta}=(\boldsymbol{\theta}_{ik}):\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ is a 2-morphism, we must verify Definition 4.18(a),(b) for $\boldsymbol{\theta}$. We do this by first showing that (a),(b) hold on the intersections of their domains with $e^{-1}(\operatorname{Im}\psi_j)$ for $j\in J$ using (4.15), (4.17), (4.30), and Definition 4.18 for $\boldsymbol{\eta},\boldsymbol{\zeta}$, and then use Theorem 4.13 and Definition A.17(iii) to deduce that Definition 4.18(a),(b) for $\boldsymbol{\theta}$ hold on their whole domains. So $\boldsymbol{\theta}$ is a 2-morphism of m-Kuranishi spaces.

Define the *horizontal composition of 2-morphisms* $\boldsymbol{\zeta}*\boldsymbol{\eta}:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ to be $\boldsymbol{\zeta}*\boldsymbol{\eta}=\boldsymbol{\theta}$. By (4.30), for all $i\in I$, $j\in J$, $k\in K$ we have

$$(\boldsymbol{\zeta}*\boldsymbol{\eta})_{ik}\odot\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{e}}=\Theta_{ijk}^{\boldsymbol{h},\boldsymbol{f}}\odot(\boldsymbol{\zeta}_{jk}*\boldsymbol{\eta}_{ij}), \tag{4.32}$$

and this characterizes $\boldsymbol{\zeta}*\boldsymbol{\eta}$ uniquely.

We have now defined all the structures of a *weak 2-category of m-Kuranishi spaces* $\mathbf{m\dot{K}ur}$, as in Appendix A: objects $\boldsymbol{X},\boldsymbol{Y}$, 1-morphisms $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$, 2-morphisms $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$, identity 1- and 2-morphisms, composition of 1-morphisms, vertical and horizontal composition of 2-morphisms, 2-isomorphisms $\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}}$ in (4.26) for associativity of 1-morphisms, and $\boldsymbol{\beta}_{\boldsymbol{f}},\boldsymbol{\gamma}_{\boldsymbol{f}}$ in (4.25) for identity 1-morphisms. To show that $\mathbf{m\dot{K}ur}$ is a weak 2-category, it remains only to prove the 2-morphism identities (A.6), (A.8), (A.9), (A.11) and (A.12). Of these, (A.11)–(A.12) are easy as $\boldsymbol{\beta}_{\boldsymbol{f}}=\boldsymbol{\gamma}_{\boldsymbol{f}}=\mathbf{id}_{\boldsymbol{f}}$, and we leave them as an exercise. The next three propositions prove (A.6), (A.8) and (A.9) hold.

Proposition 4.25. *Let $\boldsymbol{f},\dot{\boldsymbol{f}},\ddot{\boldsymbol{f}}:\boldsymbol{X}\to\boldsymbol{Y}$, $\boldsymbol{g},\dot{\boldsymbol{g}},\ddot{\boldsymbol{g}}:\boldsymbol{Y}\to\boldsymbol{Z}$ be 1-morphisms of m-Kuranishi spaces, and $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\dot{\boldsymbol{f}}$, $\dot{\boldsymbol{\eta}}:\dot{\boldsymbol{f}}\Rightarrow\ddot{\boldsymbol{f}}$, $\boldsymbol{\zeta}:\boldsymbol{g}\Rightarrow\dot{\boldsymbol{g}}$, $\dot{\boldsymbol{\zeta}}:\dot{\boldsymbol{g}}\Rightarrow\ddot{\boldsymbol{g}}$ be 2-morphisms. Then*

$$(\dot{\boldsymbol{\zeta}}\odot\boldsymbol{\zeta})*(\dot{\boldsymbol{\eta}}\odot\boldsymbol{\eta})=(\dot{\boldsymbol{\zeta}}*\dot{\boldsymbol{\eta}})\odot(\boldsymbol{\zeta}*\boldsymbol{\eta}):\boldsymbol{g}\circ\boldsymbol{f}\Longrightarrow\ddot{\boldsymbol{g}}\circ\ddot{\boldsymbol{f}}. \tag{4.33}$$

Proof. Use notation (4.6)–(4.8) for $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$. For $i\in I$, $j\in J$, $k\in K$ we have

$$\begin{aligned}
&\bigl[(\dot{\boldsymbol{\zeta}}\odot\boldsymbol{\zeta})*(\dot{\boldsymbol{\eta}}\odot\boldsymbol{\eta})\bigr]_{ik}\big|_{\operatorname{Im}\chi_i\cap f^{-1}(\operatorname{Im}\psi_j)\cap(g\circ f)^{-1}(\operatorname{Im}\omega_k)}\\
&=\Theta_{ijk}^{\ddot{\boldsymbol{g}},\ddot{\boldsymbol{f}}}\odot\bigl((\dot{\boldsymbol{\zeta}}_{jk}\odot\boldsymbol{\zeta}_{jk})*(\dot{\boldsymbol{\eta}}_{ij}\odot\boldsymbol{\eta}_{ij})\bigr)\odot(\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}})^{-1}\\
&=\Theta_{ijk}^{\ddot{\boldsymbol{g}},\ddot{\boldsymbol{f}}}\odot\bigl((\dot{\boldsymbol{\zeta}}_{jk}*\dot{\boldsymbol{\eta}}_{ij})\odot(\boldsymbol{\zeta}_{jk}*\boldsymbol{\eta}_{ij})\bigr)\odot(\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}})^{-1}\\
&=\bigl[\Theta_{ijk}^{\ddot{\boldsymbol{g}},\ddot{\boldsymbol{f}}}\odot(\dot{\boldsymbol{\zeta}}_{jk}*\dot{\boldsymbol{\eta}}_{ij})\odot(\Theta_{ijk}^{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}}})^{-1}\bigr]\odot\bigl[\Theta_{ijk}^{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}}}\odot(\boldsymbol{\zeta}_{jk}*\boldsymbol{\eta}_{ij})\odot(\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}})^{-1}\bigr]\\
&=\bigl[(\dot{\boldsymbol{\zeta}}*\dot{\boldsymbol{\eta}})\odot(\boldsymbol{\zeta}*\boldsymbol{\eta})\bigr]_{ik}\big|_{\operatorname{Im}\chi_i\cap f^{-1}(\operatorname{Im}\psi_j)\cap(g\circ f)^{-1}(\operatorname{Im}\omega_k)},
\end{aligned}$$

using (4.29) and (4.32) in the first and fourth steps, and compatibility of vertical and horizontal composition for 2-morphisms of m-Kuranishi neighbourhoods in the second. Since the $\operatorname{Im}\chi_i\cap f^{-1}(\operatorname{Im}\psi_j)\cap(g\circ f)^{-1}(\operatorname{Im}\omega_k)$ for all $j\in J$ form an open cover of the domain $\operatorname{Im}\chi_i\cap(g\circ f)^{-1}(\operatorname{Im}\omega_k)$, Theorem 4.13 and Definition A.17(iii) imply that $\bigl[(\dot{\boldsymbol{\zeta}}\odot\boldsymbol{\zeta})*(\dot{\boldsymbol{\eta}}\odot\boldsymbol{\eta})\bigr]_{ik}=\bigl[(\dot{\boldsymbol{\zeta}}*\dot{\boldsymbol{\eta}})\odot(\boldsymbol{\zeta}*\boldsymbol{\eta})\bigr]_{ik}$. As this holds for all $i\in I$ and $k\in K$, equation (4.33) follows. $\square$

Proposition 4.26. *Suppose $\boldsymbol{e}, \dot{\boldsymbol{e}} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f}, \dot{\boldsymbol{f}} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g}, \dot{\boldsymbol{g}} : \boldsymbol{Y} \to \boldsymbol{Z}$ are 1-morphisms of m-Kuranishi spaces, and $\boldsymbol{\epsilon} : \boldsymbol{e} \Rightarrow \dot{\boldsymbol{e}}$, $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \dot{\boldsymbol{f}}$, $\boldsymbol{\zeta} : \boldsymbol{g} \Rightarrow \dot{\boldsymbol{g}}$ are 2-morphisms. Then the following diagram of 2-morphisms commutes:*

$$\begin{tikzcd}
(\boldsymbol{g} \circ \boldsymbol{f}) \circ \boldsymbol{e} \arrow[rrr, Rightarrow, "\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}}"'] \arrow[d, Rightarrow, "(\boldsymbol{\zeta}*\boldsymbol{\eta})*\boldsymbol{\epsilon}"] & & & \boldsymbol{g} \circ (\boldsymbol{f} \circ \boldsymbol{e}) \arrow[d, Rightarrow, "\boldsymbol{\zeta}*(\boldsymbol{\eta}*\boldsymbol{\epsilon})"'] \\
(\dot{\boldsymbol{g}} \circ \dot{\boldsymbol{f}}) \circ \dot{\boldsymbol{e}} \arrow[rrr, Rightarrow, "\boldsymbol{\alpha}_{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}},\dot{\boldsymbol{e}}}"] & & & \dot{\boldsymbol{g}} \circ (\dot{\boldsymbol{f}} \circ \dot{\boldsymbol{e}}).
\end{tikzcd} \tag{4.34}$$

Proof. Use notation (4.5)–(4.8) for $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$. For $h \in H$, $i \in I$, $j \in J$, $k \in K$ we have

$$\begin{aligned}
&\bigl[(\boldsymbol{\zeta} * (\boldsymbol{\eta} * \boldsymbol{\epsilon})) \odot \boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}}\bigr]_{hk}\big|_{\mathrm{Im}\,\varphi_h \cap e^{-1}(\mathrm{Im}\,\chi_i)\cap(f\circ e)^{-1}(\mathrm{Im}\,\psi_j)\cap(g\circ f\circ e)^{-1}(\mathrm{Im}\,\omega_k)} \\
&= \bigl[\Theta_{hjk}^{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}}\circ\dot{\boldsymbol{e}}} \odot \bigl[\boldsymbol{\zeta}_{jk} * \bigl(\Theta_{hij}^{\dot{\boldsymbol{f}},\dot{\boldsymbol{e}}} \odot (\boldsymbol{\eta}_{ij} * \boldsymbol{\epsilon}_{hi}) \odot (\Theta_{hij}^{\boldsymbol{f},\boldsymbol{e}})^{-1}\bigr)\bigr] \odot (\Theta_{hjk}^{\boldsymbol{g},\boldsymbol{f}\circ\boldsymbol{e}})^{-1}\bigr] \\
&\quad \odot \bigl[\Theta_{hjk}^{\boldsymbol{g},\boldsymbol{f}\circ\boldsymbol{e}} \odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \Theta_{hij}^{\boldsymbol{f},\boldsymbol{e}}) \odot (\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}} * \mathrm{id}_{\boldsymbol{e}_{hi}})^{-1} \odot (\Theta_{hik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}})^{-1}\bigr] \\
&= \Theta_{hjk}^{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}}\circ\dot{\boldsymbol{e}}} \odot (\mathrm{id}_{\dot{\boldsymbol{g}}_{jk}} * \Theta_{hij}^{\dot{\boldsymbol{f}},\dot{\boldsymbol{e}}}) \odot (\boldsymbol{\zeta}_{jk} * \boldsymbol{\eta}_{ij} * \boldsymbol{\epsilon}_{hi}) \odot (\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}} * \mathrm{id}_{\boldsymbol{e}_{hi}})^{-1} \odot (\Theta_{hik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}})^{-1} \\
&= \bigl[\Theta_{hjk}^{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}}\circ\dot{\boldsymbol{e}}} \odot (\mathrm{id}_{\dot{\boldsymbol{g}}_{jk}} * \Theta_{hij}^{\dot{\boldsymbol{f}},\dot{\boldsymbol{e}}}) \odot (\Theta_{ijk}^{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}}} * \mathrm{id}_{\dot{\boldsymbol{e}}_{hi}})^{-1} \odot (\Theta_{hik}^{\dot{\boldsymbol{g}}\circ\dot{\boldsymbol{f}},\dot{\boldsymbol{e}}})^{-1}\bigr] \\
&\quad \odot \bigl[\Theta_{hik}^{\dot{\boldsymbol{g}}\circ\dot{\boldsymbol{f}},\dot{\boldsymbol{e}}} \odot \bigl[\bigl(\Theta_{ijk}^{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}}} \odot (\boldsymbol{\zeta}_{jk} * \boldsymbol{\eta}_{ij}) \odot (\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}})^{-1}\bigr) * \boldsymbol{\epsilon}_{hi}\bigr] \odot (\Theta_{hik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}})^{-1}\bigr] \\
&= \bigl[\boldsymbol{\alpha}_{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}},\dot{\boldsymbol{e}}} \odot ((\boldsymbol{\zeta} * \boldsymbol{\eta}) * \boldsymbol{\epsilon})\bigr]_{hk}\big|_{\mathrm{Im}\,\varphi_h \cap e^{-1}(\mathrm{Im}\,\chi_i)\cap(f\circ e)^{-1}(\mathrm{Im}\,\psi_j)\cap(g\circ f\circ e)^{-1}(\mathrm{Im}\,\omega_k)},
\end{aligned}$$

using (4.27) and (4.32) in the first and fourth steps, and properties of strict 2-categories in the second and third. This proves the restriction of the 'hk' component of (4.34) to $\mathrm{Im}\,\varphi_h \cap e^{-1}(\mathrm{Im}\,\chi_i) \cap (f\circ e)^{-1}(\mathrm{Im}\,\psi_j) \cap (g\circ f\circ e)^{-1}(\mathrm{Im}\,\omega_k)$ commutes. Since these subsets for all i, j form an open cover of the domain, Theorem 4.13 and Definition A.17(iii) imply that the 'hk' component of (4.34) commutes for all $h \in H$, $k \in K$, so (4.34) commutes. $\square$

Proposition 4.27. *Let $\boldsymbol{d} : \boldsymbol{V} \to \boldsymbol{W}$, $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms of m-Kuranishi spaces. Then in 2-morphisms we have*

$$\begin{aligned}
\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}\circ\boldsymbol{d}} \odot \boldsymbol{\alpha}_{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e},\boldsymbol{d}} = (\mathbf{id}_{\boldsymbol{g}} * \boldsymbol{\alpha}_{\boldsymbol{f},\boldsymbol{e},\boldsymbol{d}}) \odot \boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f}\circ\boldsymbol{e},\boldsymbol{d}} \odot (\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}} * \mathbf{id}_{\boldsymbol{d}}) : \\
((\boldsymbol{g} \circ \boldsymbol{f}) \circ \boldsymbol{e}) \circ \boldsymbol{d} \Longrightarrow \boldsymbol{g} \circ (\boldsymbol{f} \circ (\boldsymbol{e} \circ \boldsymbol{d})).
\end{aligned} \tag{4.35}$$

Proof. Use notation (4.5)–(4.8) for $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, and take G to be the indexing set for $\boldsymbol{V}$. Then for $g \in G$, $h \in H$, $i \in I$, $j \in J$, $k \in K$, on $\mathrm{Im}\,\upsilon_g \cap d^{-1}(\mathrm{Im}\,\varphi_h)$

$\cap\,(e\circ d)^{-1}(\operatorname{Im}\chi_i)\cap(f\circ e\circ d)^{-1}(\operatorname{Im}\psi_j)\cap(g\circ f\circ e\circ d)^{-1}(\operatorname{Im}\omega_k)$ we have

$$
\begin{aligned}
&\bigl[(\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}\circ\boldsymbol{d}})\odot(\boldsymbol{\alpha}_{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e},\boldsymbol{d}})\bigr]_{gk}\big|_{\ldots}\\
&=\bigl\{\Theta_{gjk}^{\boldsymbol{g},\boldsymbol{f}\circ(\boldsymbol{e}\circ\boldsymbol{d})}\odot(\mathrm{id}_{\boldsymbol{g}_{jk}}*\Theta_{gij}^{\boldsymbol{f},\boldsymbol{e}\circ\boldsymbol{d}})\odot(\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}}*\mathrm{id}_{(\boldsymbol{e}\circ\boldsymbol{d})_{gi}})^{-1}\odot(\Theta_{gik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}\circ\boldsymbol{d}})^{-1}\bigr\}\\
&\quad\odot\bigl\{\Theta_{gik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}\circ\boldsymbol{d}}\odot(\mathrm{id}_{(\boldsymbol{g}\circ\boldsymbol{f})_{ik}}*\Theta_{ghi}^{\boldsymbol{e},\boldsymbol{d}})\odot(\Theta_{hik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}}*\mathrm{id}_{\boldsymbol{d}_{gh}})^{-1}\odot(\Theta_{ghk}^{(\boldsymbol{g}\circ\boldsymbol{f})\circ\boldsymbol{e},\boldsymbol{d}})^{-1}\bigr\}\\
&=\Theta_{gjk}^{\boldsymbol{g},\boldsymbol{f}\circ(\boldsymbol{e}\circ\boldsymbol{d})}\odot(\mathrm{id}_{\boldsymbol{g}_{jk}}*\Theta_{gij}^{\boldsymbol{f},\boldsymbol{e}\circ\boldsymbol{d}})\odot((\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}})^{-1}*\Theta_{ghi}^{\boldsymbol{e},\boldsymbol{d}})\\
&\quad\odot(\Theta_{hik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}}*\mathrm{id}_{\boldsymbol{d}_{gh}})^{-1}\odot(\Theta_{ghk}^{(\boldsymbol{g}\circ\boldsymbol{f})\circ\boldsymbol{e},\boldsymbol{d}})^{-1}\\
&=\bigl\{\Theta_{gjk}^{\boldsymbol{g},\boldsymbol{f}\circ(\boldsymbol{e}\circ\boldsymbol{d})}\odot\bigl(\mathrm{id}_{\boldsymbol{g}_{jk}}*\bigl[\Theta_{gij}^{\boldsymbol{f},\boldsymbol{e}\circ\boldsymbol{d}}\odot(\mathrm{id}_{\boldsymbol{f}_{ij}}*\Theta_{ghi}^{\boldsymbol{e},\boldsymbol{d}})\odot(\Theta_{hij}^{\boldsymbol{f},\boldsymbol{e}}*\mathrm{id}_{\boldsymbol{d}_{gh}})^{-1}\odot(\Theta_{ghj}^{\boldsymbol{f}\circ\boldsymbol{e},\boldsymbol{d}})^{-1}\bigr]\bigr)\\
&\quad\odot(\Theta_{gjk}^{\boldsymbol{g},(\boldsymbol{f}\circ\boldsymbol{e})\circ\boldsymbol{d}})^{-1}\bigr\}\odot\bigl\{\Theta_{gjk}^{\boldsymbol{g},(\boldsymbol{f}\circ\boldsymbol{e})\circ\boldsymbol{d}}\odot(\mathrm{id}_{\boldsymbol{g}_{jk}}*\Theta_{ghj}^{\boldsymbol{f}\circ\boldsymbol{e},\boldsymbol{d}})\odot(\Theta_{hjk}^{\boldsymbol{g},\boldsymbol{f}\circ\boldsymbol{e}}*\mathrm{id}_{\boldsymbol{d}_{gh}})^{-1}\\
&\quad\odot(\Theta_{ghk}^{\boldsymbol{g}\circ(\boldsymbol{f}\circ\boldsymbol{e}),\boldsymbol{d}})^{-1}\bigr\}\odot\bigl\{\Theta_{ghk}^{\boldsymbol{g}\circ(\boldsymbol{f}\circ\boldsymbol{e}),\boldsymbol{d}})\odot\bigl(\bigl[\Theta_{hjk}^{\boldsymbol{g},\boldsymbol{f}\circ\boldsymbol{e}}\odot(\mathrm{id}_{\boldsymbol{g}_{jk}}*\Theta_{hij}^{\boldsymbol{f},\boldsymbol{e}})\\
&\quad\odot(\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}}*\mathrm{id}_{\boldsymbol{e}_{hi}})^{-1}\odot(\Theta_{hik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}})^{-1}\bigr]*\mathrm{id}_{\boldsymbol{d}_{gh}}\bigr)\odot(\Theta_{ghk}^{(\boldsymbol{g}\circ\boldsymbol{f})\circ\boldsymbol{e},\boldsymbol{d}})^{-1}\bigr\}\\
&=\bigl[(\mathbf{id}_{\boldsymbol{g}}*\boldsymbol{\alpha}_{\boldsymbol{f},\boldsymbol{e},\boldsymbol{d}})\odot\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f}\circ\boldsymbol{e},\boldsymbol{d}}\odot(\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}}*\mathbf{id}_{\boldsymbol{d}})\bigr]_{gk}\big|_{\ldots},
\end{aligned}
$$

using (4.27) and (4.32) in the first and fourth steps, and properties of strict 2-categories in the second and third. This proves the restriction of the 'gk' component of (4.35) to the subset $\operatorname{Im}v_g\cap d^{-1}(\operatorname{Im}\varphi_h)\cap(e\circ d)^{-1}(\operatorname{Im}\chi_i)\cap(f\circ e\circ d)^{-1}(\operatorname{Im}\psi_j)\cap(g\circ f\circ e\circ d)^{-1}(\operatorname{Im}\omega_k)$. Since these subsets for all h,i,j form an open cover of the domain, Theorem 4.13 and Definition A.17(iii) imply that the 'gk' component of (4.35) commutes for all $g\in G$ and $k\in K$, so (4.35) commutes. $\square$

We summarize the work of this section in the following:

Theorem 4.28. *The definitions and propositions above define a* ***weak*** **2*-category of m-Kuranishi spaces*** $\mathbf{m\dot{K}ur}$.

Definition 4.29. In Theorem 4.28 we write $\mathbf{m\dot{K}ur}$ for the 2-category of m-Kuranishi spaces constructed from our chosen category $\mathbf{\dot{M}an}$ satisfying Assumptions 3.1–3.7 in §3.1. By Example 3.8, the following categories from Chapter 2 are possible choices for $\mathbf{\dot{M}an}$:

$$\mathbf{Man},\mathbf{Man^c},\mathbf{Man^c_{we}},\mathbf{Man^{gc}},\mathbf{Man^{ac}},\mathbf{Man^{c,ac}}. \tag{4.36}$$

We write the corresponding 2-categories of m-Kuranishi spaces as follows:

$$\mathbf{mKur},\mathbf{mKur^c},\mathbf{mKur^c_{we}},\mathbf{mKur^{gc}},\mathbf{mKur^{ac}},\mathbf{mKur^{c,ac}}. \tag{4.37}$$

Objects of $\mathbf{mKur^c},\mathbf{mKur^{gc}},\mathbf{mKur^{ac}},\mathbf{mKur^{c,ac}}$ will be called *m-Kuranishi spaces with corners*, and *with g-corners*, and *with a-corners*, and *with corners and a-corners*, respectively.

Actually, Example 3.8 gives lots more categories satisfying Assumptions 3.1–3.7, such as $\mathbf{Man^c_{in}}\subset\mathbf{Man^c}$, but we will not define notation for corresponding 2-categories of m-Kuranishi spaces $\mathbf{mKur^c_{in}},\ldots$ here. Instead, in §4.5 we will define the 2-categories $\mathbf{mKur^c_{in}},\ldots$ as 2-subcategories of the 2-categories in (4.37). The reason for this is explained in Remark 4.38.

Example 4.30. We will define a weak 2-functor $F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}} : \dot{\mathbf{Man}} \to \mathbf{m\dot{K}ur}$. Weak 2-functors are explained in §A.3. Since $\mathbf{m\dot{K}ur}$ is a weak 2-category, no other kind of functor to $\mathbf{m\dot{K}ur}$ makes sense.

If $X \in \dot{\mathbf{Man}}$, define an m-Kuranishi space $F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(X) = \boldsymbol{X} = (X, \mathcal{K})$ with topological space X and m-Kuranishi structure

$$\mathcal{K} = \big(\{0\}, (V_0, E_0, s_0, \psi_0), \Phi_{00}, \Lambda_{000}\big),$$

with indexing set $I = \{0\}$, one m-Kuranishi neighbourhood (V_0, E_0, s_0, ψ_0) with $V_0 = X$, $E_0 \to V_0$ the zero vector bundle, $s_0 = 0$, and $\psi_0 = \mathrm{id}_X$, one coordinate change $\Phi_{00} = \mathrm{id}_{(V_0,E_0,s_0,\psi_0)}$, and one 2-morphism $\Lambda_{000} = \mathrm{id}_{\Phi_{00}}$.

On 1-morphisms, if $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$ and $\boldsymbol{X} = F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(X)$, $\boldsymbol{Y} = F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(Y)$, define a 1-morphism $F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(f) = \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ by $\boldsymbol{f} = (f, \boldsymbol{f}_{00}, \boldsymbol{F}_{00}^{0}, \boldsymbol{F}_{0}^{00})$, where $\boldsymbol{f}_{00} = (U_{00}, f_{00}, \hat{f}_{00})$ with $U_{00} = X$, $f_{00} = f$, and $\hat{f}_{00}$ is the zero map on zero vector bundles, and $\boldsymbol{F}_{00}^{0} = \boldsymbol{F}_{0}^{00} = \mathrm{id}_{\boldsymbol{f}_{00}}$.

On 2-morphisms, regarding $\dot{\mathbf{Man}}$ as a 2-category, the only 2-morphisms are identity morphisms $\mathrm{id}_f : f \Rightarrow f$ for (1-)morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$. We define $F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(\mathrm{id}_f) = \mathbf{id}_{F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(f)}$.

If $\boldsymbol{X} = F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(X)$ and $\boldsymbol{Y} = F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(Y)$ for $X, Y \in \dot{\mathbf{Man}}$, it is easy to check that the only 1-morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$ are those of the form $F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(f)$ for morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$, and the only 2-morphisms $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\mathbf{m\dot{K}ur}$ for any 1-morphisms $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ are identity 2-morphisms $\mathbf{id}_{\boldsymbol{f}} : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ when $\boldsymbol{f} = \boldsymbol{g}$.

Suppose $f : X \to Y$, $g : Y \to Z$ are (1-)morphisms in $\dot{\mathbf{Man}}$, and write $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{f}, \boldsymbol{g}$ for the images of X, Y, Z, f, g under $F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}$. Then Definition 4.20 defines the composition $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$, by making an arbitrary choice. But the uniqueness property of 1-morphisms above implies that the only possibility is $\boldsymbol{g} \circ \boldsymbol{f} = F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(g \circ f)$. Define

$$(F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}})_{g,f} := \mathbf{id}_{F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(g\circ f)} : F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(g) \circ F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(f) \Longrightarrow F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(g \circ f).$$

For any object X in $\dot{\mathbf{Man}}$ with $\boldsymbol{X} = F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(X)$, define

$$(F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}})_X := \mathbf{id}_{\mathbf{id}_{\boldsymbol{X}}} : F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(\mathrm{id}_X) \Longrightarrow \mathbf{id}_{F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(X)}.$$

We have defined all the data of a weak 2-functor $F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}} : \dot{\mathbf{Man}} \to \mathbf{m\dot{K}ur}$ in Definition A.8. It is easy to check that $F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}$ is a weak 2-functor, which is full and faithful, and so embeds $\dot{\mathbf{Man}}$ as a full 2-subcategory of $\mathbf{m\dot{K}ur}$.

We say that an m-Kuranishi space $\boldsymbol{X}$ *is a manifold* if $\boldsymbol{X} \simeq F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(X')$ in $\mathbf{m\dot{K}ur}$, for some $X' \in \dot{\mathbf{Man}}$. Theorem 10.45 in §10.4.2 gives a necessary and sufficient criterion for when $\boldsymbol{X}$ is a manifold.

Assumption 3.4 gives a full subcategory $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$. Define a full and faithful weak 2-functor $F_{\mathbf{Man}}^{\mathbf{m\dot{K}ur}} = F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}|_{\mathbf{Man}} : \mathbf{Man} \to \mathbf{m\dot{K}ur}$, which embeds

Man as a full 2-subcategory of $\mathbf{m\dot{K}ur}$. We say that an m-Kuranishi space $\boldsymbol{X}$ *is a classical manifold* if $\boldsymbol{X} \simeq F_{\mathbf{Man}}^{\mathbf{m\dot{K}ur}}(X')$ in $\mathbf{m\dot{K}ur}$, for some $X' \in \mathbf{Man}$.

In a similar way to Example 4.30, we can define a weak 2-functor $\mathbf{Gm\dot{K}N} \to \mathbf{m\dot{K}ur}$ which is an equivalence from the 2-category $\mathbf{Gm\dot{K}N}$ of global m-Kuranishi neighbourhoods in Definition 4.8 to the full 2-subcategory of objects $(X, \mathcal{K})$ in $\mathbf{m\dot{K}ur}$ for which $\mathcal{K}$ contains only one m-Kuranishi neighbourhood. It acts by $(V, E, s) \mapsto \boldsymbol{S}_{V,E,s}$ on objects, for $\boldsymbol{S}_{V,E,s}$ as in Example 4.16.

The next example defines *products* $\boldsymbol{X} \times \boldsymbol{Y}$ of m-Kuranishi spaces $\boldsymbol{X}, \boldsymbol{Y}$. We discuss products further in §11.2.3, as examples of fibre products $\boldsymbol{X} \times_{\boldsymbol{*}} \boldsymbol{Y}$.

Example 4.31. Let $\boldsymbol{X} = (X, \mathcal{I})$, $\boldsymbol{Y} = (Y, \mathcal{J})$ be m-Kuranishi spaces in $\mathbf{m\dot{K}ur}$, with notation (4.6)–(4.7). Define the *product* to be $\boldsymbol{X} \times \boldsymbol{Y} = (X \times Y, \mathcal{K})$, where

$$\mathcal{K} = \big(I \times J, (W_{(i,j)}, F_{(i,j)}, t_{(i,j)}, \omega_{(i,j)})_{(i,j)\in I\times J},\ \Phi_{(i,j)(i',j'),\ (i,j),(i',j')\in I\times J},\\ \mathrm{M}_{(i,j)(i',j')(i'',j''),\ (i,j),(i',j'),(i'',j'')\in I\times J}\big).$$

Here for all $(i,j) \in I \times J$ we set $W_{(i,j)} = U_i \times V_j$, $F_{(i,j)} = \pi_{U_i}^*(D_i) \oplus \pi_{V_j}^*(E_j)$, and $t_{(i,j)} = \pi_{U_i}^*(r_i) \oplus \pi_{V_j}^*(s_j)$ so that $t_{(i,j)}^{-1}(0) = r_i^{-1}(0) \times s_j^{-1}(0)$, and $\omega_{(i,j)} = \chi_i \times \psi_j : r_i^{-1}(0) \times s_j^{-1}(0) \to X \times Y$. Also

$$\Phi_{(i,j)(i',j')} = \mathrm{T}_{ii'} \times \Upsilon_{jj'} = \big(U_{ii'} \times V_{jj'}, \tau_{ii'} \times \upsilon_{jj'}, \pi_{U_{ii'}}^*(\hat{\tau}_{ii'}) \oplus \pi_{V_{jj'}}^*(\hat{\upsilon}_{jj'})\big),$$

and $\mathrm{M}_{(i,j)(i',j')(i'',j'')} = \mathrm{K}_{ii'i''} \times \Lambda_{jj'j''}$ is defined as a product 2-morphism in the obvious way. It is easy to check that $\boldsymbol{X} \times \boldsymbol{Y}$ is an m-Kuranishi space, with $\mathrm{vdim}(\boldsymbol{X} \times \boldsymbol{Y}) = \mathrm{vdim}\,\boldsymbol{X} + \mathrm{vdim}\,\boldsymbol{Y}$.

We can also define explicit projection 1-morphisms $\boldsymbol{\pi_X} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{X}$, $\boldsymbol{\pi_Y} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{Y}$, where

$$\boldsymbol{\pi_X} = \big(\pi_X, \boldsymbol{\pi}_{(i,j)i',\ (i,j)\in I\times J,\ i'\in I},\ \boldsymbol{\Pi}_{(i,j)(i',j'),\ (i,j),(i',j')\in I\times J}^{i'',\ i''\in I},\ \boldsymbol{\Pi}_{(i,j),\ (i,j)\in I\times J}^{i',i'',\ i',i''\in I}\big),$$

with $\boldsymbol{\pi}_{(i,j)i'} = (U_{ii'} \times V_j, \tau_{ii'} \circ \pi_{U_{ii'}}, \pi_{U_{ii'}}^*(\hat{\tau}_{ii'}) \circ \pi_{\pi_{U_i}^*(D_i)})$, and $\boldsymbol{\Pi}_{(i,j)(i',j')}^{i''}, \boldsymbol{\Pi}_{(i,j)}^{i',i''}$ are the basically the compositions of the 2-morphism $\mathrm{K}_{ii'i''}$ in $\mathcal{I}$ with the projection $U_i \times V_j \to U_i$. We define $\boldsymbol{\pi_Y}$ in the same way.

We will show in §11.2.3 that $\boldsymbol{X} \times \boldsymbol{Y}, \boldsymbol{\pi_X}, \boldsymbol{\pi_Y}$ have the universal property of products in a 2-category. That is, $\boldsymbol{X} \times \boldsymbol{Y}$ is a fibre product $\boldsymbol{X} \times_{\boldsymbol{*}} \boldsymbol{Y}$ over the point (terminal object) $\boldsymbol{*}$ in $\mathbf{m\dot{K}ur}$, as in §A.4, in a 2-Cartesian square

$$\begin{tikzcd} \boldsymbol{X} \times \boldsymbol{Y} \arrow[rr, "\boldsymbol{\pi_Y}"] \arrow[d, "\boldsymbol{\pi_X}"] & \arrow[d, phantom, "\mathrm{id}\Uparrow"] & \boldsymbol{Y} \arrow[d] \\ \boldsymbol{X} \arrow[rr] & & \boldsymbol{*}. \end{tikzcd}$$

Products are commutative and associative up to canonical equivalence, and in fact (with the above definition) up to canonical 1-isomorphism. That is, if $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ are m-Kuranishi spaces, we have canonical 1-isomorphisms in $\mathbf{m\dot{K}ur}$

$$\boldsymbol{Y} \times \boldsymbol{X} \cong \boldsymbol{X} \times \boldsymbol{Y} \qquad \text{and} \qquad (\boldsymbol{X} \times \boldsymbol{Y}) \times \boldsymbol{Z} \cong \boldsymbol{X} \times (\boldsymbol{Y} \times \boldsymbol{Z}). \tag{4.38}$$

We can also define products and direct products of 1-morphisms. That is, if $\boldsymbol{f}: \boldsymbol{W} \to \boldsymbol{Y}$, $\boldsymbol{g}: \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{h}: \boldsymbol{X} \to \boldsymbol{Z}$ are 1-morphisms in $\mathbf{m\dot{K}ur}$ then we have a *product* 1-*morphism* $\boldsymbol{f} \times \boldsymbol{h}: \boldsymbol{W} \times \boldsymbol{X} \to \boldsymbol{Y} \times \boldsymbol{Z}$ and a *direct product* 1-*morphism* $(\boldsymbol{g}, \boldsymbol{h}): \boldsymbol{X} \to \boldsymbol{Y} \times \boldsymbol{Z}$ in $\mathbf{m\dot{K}ur}$, both easy to write down explicitly. The existence of $\boldsymbol{f} \times \boldsymbol{h}, (\boldsymbol{g}, \boldsymbol{h})$ is also guaranteed by the universal property of products, uniquely up to canonical 2-isomorphism.

4.4 Comparing m-Kuranishi spaces from different $\dot{\mathbf{M}}\mathbf{an}$

Using the ideas of §3.3.7 and §B.7, we explain how to lift a functor $F_{\dot{\mathbf{M}}\mathbf{an}}^{\ddot{\mathbf{M}}\mathbf{an}}: \dot{\mathbf{M}}\mathbf{an} \to \ddot{\mathbf{M}}\mathbf{an}$ satisfying Condition 3.20 to a corresponding weak 2-functor $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}: \mathbf{m\dot{K}ur} \to \mathbf{m\ddot{K}ur}$ between the 2-categories of m-Kuranishi spaces $\mathbf{m\dot{K}ur}, \mathbf{m\ddot{K}ur}$ associated to $\dot{\mathbf{M}}\mathbf{an}, \ddot{\mathbf{M}}\mathbf{an}$.

Definition 4.32. Suppose $\dot{\mathbf{M}}\mathbf{an}, \ddot{\mathbf{M}}\mathbf{an}$ satisfy Assumptions 3.1–3.7, and $F_{\dot{\mathbf{M}}\mathbf{an}}^{\ddot{\mathbf{M}}\mathbf{an}}: \dot{\mathbf{M}}\mathbf{an} \to \ddot{\mathbf{M}}\mathbf{an}$ is a functor satisfying Condition 3.20. Then in §3.3.7 and §B.7 we explain how all the material of §3.3 on differential geometry in $\dot{\mathbf{M}}\mathbf{an}$ maps functorially to its analogue in $\ddot{\mathbf{M}}\mathbf{an}$ under $F_{\dot{\mathbf{M}}\mathbf{an}}^{\ddot{\mathbf{M}}\mathbf{an}}$.

Write $\mathbf{m\dot{K}ur}, \mathbf{m\ddot{K}ur}$ for the 2-categories of m-Kuranishi spaces constructed from $\dot{\mathbf{M}}\mathbf{an}, \ddot{\mathbf{M}}\mathbf{an}$ in §4.3. We will define a weak 2-functor $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}: \mathbf{m\dot{K}ur} \to \mathbf{m\ddot{K}ur}$. The basic idea is obvious: we apply $F_{\dot{\mathbf{M}}\mathbf{an}}^{\ddot{\mathbf{M}}\mathbf{an}}$ to turn the m-Kuranishi neighbourhoods and their 1- and 2-morphisms over $\dot{\mathbf{M}}\mathbf{an}$ used in $\mathbf{m\dot{K}ur}$, into their analogues over $\ddot{\mathbf{M}}\mathbf{an}$ used in $\mathbf{m\ddot{K}ur}$.

As in §B.7, we will use accents ‘ $\dot{}$ ’ and ‘ $\ddot{}$ ’ to denote objects associated to $\dot{\mathbf{M}}\mathbf{an}$ and $\ddot{\mathbf{M}}\mathbf{an}$, respectively. When something is independent of $\dot{\mathbf{M}}\mathbf{an}$ or $\ddot{\mathbf{M}}\mathbf{an}$ (such as the underlying topological space X in $\dot{\boldsymbol{X}}$) we omit the accent.

Let $\dot{\boldsymbol{X}} = (X, \dot{\mathcal{K}})$ be an object in $\mathbf{m\dot{K}ur}$, where

$$\dot{\mathcal{K}} = \big(I, (\dot{V}_i, \dot{E}_i, \dot{s}_i, \psi_i)_{i \in I},\ \ \dot{\Phi}_{ij,\ i,j \in I},\ \ \dot{\Lambda}_{ijk,\ i,j,k \in I}\big),$$

with $\dot{\Phi}_{ij} = (\dot{V}_{ij}, \dot{\phi}_{ij}, \hat{\dot{\phi}}_{ij}): (\dot{V}_i, \dot{E}_i, \dot{s}_i, \psi_i) \to (\dot{V}_j, \dot{E}_j, \dot{s}_j, \psi_j)$ and $\Lambda_{ijk} = [\acute{\dot{V}}_{ijk}, \hat{\dot{\lambda}}_{ijk}]$ for all $i, j, k \in I$. Define $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\dot{\boldsymbol{X}}) = \ddot{\boldsymbol{X}} = (X, \ddot{\mathcal{K}})$ in $\mathbf{m\ddot{K}ur}$, where

$$\ddot{\mathcal{K}} = \big(I, (\ddot{V}_i, \ddot{E}_i, \ddot{s}_i, \psi_i)_{i \in I},\ \ \ddot{\Phi}_{ij,\ i,j \in I},\ \ \ddot{\Lambda}_{ijk,\ i,j,k \in I}\big),$$

with $\ddot{\Phi}_{ij} = (\ddot{V}_{ij}, \ddot{\phi}_{ij}, \hat{\ddot{\phi}}_{ij}): (\ddot{V}_i, \ddot{E}_i, \ddot{s}_i, \psi_i) \to (\ddot{V}_j, \ddot{E}_j, \ddot{s}_j, \psi_j)$ and $\Lambda_{ijk} = [\acute{\ddot{V}}_{ijk}, \hat{\ddot{\lambda}}_{ijk}]$ for all $i, j, k \in I$. Here $\ddot{V}_i, \ddot{E}_i, \ddot{s}_i, \ddot{V}_{ij}, \ddot{\phi}_{ij}, \hat{\ddot{\phi}}_{ij}, \acute{\ddot{V}}_{ijk}, \hat{\ddot{\lambda}}_{ijk}$ are the images of $\dot{V}_i, \dot{E}_i$, $\dot{s}_i, \dot{V}_{ij}, \dot{\phi}_{ij}, \hat{\dot{\phi}}_{ij}, \acute{\dot{V}}_{ijk}, \hat{\dot{\lambda}}_{ijk}$ under $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}$, respectively, as in §B.7.

Similarly, if $\dot{\boldsymbol{f}}: \dot{\boldsymbol{X}} \to \dot{\boldsymbol{Y}}$ is a 1-morphism in $\mathbf{m\dot{K}ur}$ we define a 1-morphism $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\dot{\boldsymbol{f}}) = \ddot{\boldsymbol{f}}: \ddot{\boldsymbol{X}} \to \ddot{\boldsymbol{Y}}$ in $\mathbf{m\ddot{K}ur}$, and if $\dot{\boldsymbol{\eta}}: \dot{\boldsymbol{f}} \Rightarrow \dot{\boldsymbol{g}}$ is a 2-morphism in $\mathbf{m\dot{K}ur}$ we define a 2-morphism $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\dot{\boldsymbol{\eta}}) = \ddot{\boldsymbol{\eta}}: \ddot{\boldsymbol{f}} \Rightarrow \ddot{\boldsymbol{g}}$ in $\mathbf{m\ddot{K}ur}$, by applying $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}$ to all the $\dot{\mathbf{M}}\mathbf{an}$ structures in $\dot{\boldsymbol{f}}, \dot{\boldsymbol{\eta}}$, in the obvious way.

Let $\dot{\boldsymbol{f}}: \dot{\boldsymbol{X}} \to \dot{\boldsymbol{Y}}$ and $\dot{\boldsymbol{g}}: \dot{\boldsymbol{Y}} \to \dot{\boldsymbol{Z}}$ be 1-morphisms in $\mathbf{m\dot{K}ur}$, and write $\ddot{\boldsymbol{f}}: \ddot{\boldsymbol{X}} \to \ddot{\boldsymbol{Y}}$, $\ddot{\boldsymbol{g}}: \ddot{\boldsymbol{Y}} \to \ddot{\boldsymbol{Z}}$ for their images in $\mathbf{m\ddot{K}ur}$ under $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}$. Then Definition 4.20 defined $\dot{\boldsymbol{g}} \circ \dot{\boldsymbol{f}}: \dot{\boldsymbol{X}} \to \dot{\boldsymbol{Z}}$ in $\mathbf{m\dot{K}ur}$ and $\ddot{\boldsymbol{g}} \circ \ddot{\boldsymbol{f}}: \ddot{\boldsymbol{X}} \to \ddot{\boldsymbol{Z}}$ in $\mathbf{m\ddot{K}ur}$, by making arbitrary choices. Since these choices may not be consistent, we need not have $\ddot{\boldsymbol{g}} \circ \ddot{\boldsymbol{f}} = F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\dot{\boldsymbol{g}} \circ \dot{\boldsymbol{f}})$. However, because $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\dot{\boldsymbol{g}} \circ \dot{\boldsymbol{f}})$ is one of the possible choices for $\ddot{\boldsymbol{g}} \circ \ddot{\boldsymbol{f}}$, Proposition 4.19(b) gives a canonical 2-morphism

$$(F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}})_{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}}}: F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\dot{\boldsymbol{g}}) \circ F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\dot{\boldsymbol{f}}) = \ddot{\boldsymbol{g}} \circ \ddot{\boldsymbol{f}} \Longrightarrow F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\dot{\boldsymbol{g}} \circ \dot{\boldsymbol{f}})$$

in $\mathbf{m\ddot{K}ur}$, using the data $\Theta_{ijk}^{\dot{\boldsymbol{g}},\dot{\boldsymbol{f}}}$ and their images under $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}$.

For $\dot{\boldsymbol{X}}$ in $\mathbf{m\dot{K}ur}$ with $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\dot{\boldsymbol{X}}) = \ddot{\boldsymbol{X}}$ in $\mathbf{m\ddot{K}ur}$, we see using (4.13) that $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\mathbf{id}_{\dot{\boldsymbol{X}}}) = \mathbf{id}_{\ddot{\boldsymbol{X}}}$. Define

$$(F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}})_{\dot{\boldsymbol{X}}} = \mathbf{id}_{\mathbf{id}_{\ddot{\boldsymbol{X}}}}: F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\mathbf{id}_{\dot{\boldsymbol{X}}}) \Longrightarrow \mathbf{id}_{F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}(\dot{\boldsymbol{X}})}.$$

This defines all the data of a weak 2-functor $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}: \mathbf{m\dot{K}ur} \to \mathbf{m\ddot{K}ur}$, as in §A.3. It is easy to check that the weak 2-functor axioms hold.

Now suppose that $F_{\mathbf{\dot{M}an}}^{\mathbf{\ddot{M}an}}: \mathbf{\dot{M}an} \hookrightarrow \mathbf{\ddot{M}an}$ is an inclusion of subcategories $\mathbf{\dot{M}an} \subseteq \mathbf{\ddot{M}an}$ satisfying either Proposition 3.21(a) or (b). Then Proposition 3.21 says that the maps $F_{\mathbf{\dot{M}an}}^{\mathbf{\ddot{M}an}}$ in §3.3.7 from geometry in $\mathbf{\dot{M}an}$ to geometry in $\mathbf{\ddot{M}an}$ used above are identity maps. Hence $\mathbf{m\dot{K}ur}$ is actually a 2-subcategory of $\mathbf{m\ddot{K}ur}$, and the 2-functor $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}$ is the inclusion $\mathbf{m\dot{K}ur} \subseteq \mathbf{m\ddot{K}ur}$.

For the case of Proposition 3.21(b), when $\mathbf{\dot{M}an}$ is a full subcategory of $\mathbf{\ddot{M}an}$, then $\mathbf{m\dot{K}ur}$ is a full 2-subcategory of $\mathbf{m\ddot{K}ur}$. That is, if $\boldsymbol{X}, \boldsymbol{Y}$ are objects of $\mathbf{m\dot{K}ur}$ then all 1-morphisms $\boldsymbol{f}, \boldsymbol{g}: \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{m\ddot{K}ur}$ are 1-morphisms in $\mathbf{m\dot{K}ur}$, and all 2-morphisms $\boldsymbol{\eta}: \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\mathbf{m\ddot{K}ur}$ are 2-morphisms in $\mathbf{m\dot{K}ur}$.

$$\begin{tikzcd}
 & & \mathbf{mKur} \arrow[dll, bend right] \arrow[dl] \arrow[d] \arrow[dr, bend left] & \\
\mathbf{mKur^{gc}} & \mathbf{mKur^{c}} \arrow[l] \arrow[r] \arrow[d, dotted] & \mathbf{mKur^{c,ac}} \arrow[r, dotted, shift left] & \mathbf{mKur^{ac}} \arrow[l, shift left] \\
 & \mathbf{mKur^{c}_{we}} & &
\end{tikzcd}$$

Figure 4.1: 2-functors between 2-categories of m-Kuranishi spaces from Definition 4.32. Arrows '$\to$' are inclusions of 2-subcategories.

Applying Definition 4.32 to the parts of the diagram Figure 3.1 of functors $F_{\mathbf{\dot{M}an}}^{\mathbf{\ddot{M}an}}$ involving the categories (4.36) yields a diagram Figure 4.1 of 2-functors $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}$. Arrows '$\to$' are inclusions of 2-subcategories.

4.5 Discrete properties of 1-morphisms in $\mathbf{m\dot{K}ur}$

In §3.3.6 and §B.6 we defined when a property $\boldsymbol{P}$ of morphisms in $\dot{\mathbf{Man}}$ is *discrete*. For example, when $\dot{\mathbf{Man}} = \mathbf{Man^c}$ from §2.1, for a morphism $f : X \to Y$ in $\mathbf{Man^c}$ to be interior, or simple, are both discrete conditions.

We will now show that a discrete property $\boldsymbol{P}$ of morphisms in $\dot{\mathbf{Man}}$ lifts to a corresponding property $\boldsymbol{P}$ of 1-morphisms in $\mathbf{m\dot{K}ur}$, in a well behaved way. We first define $\boldsymbol{P}$ for 1-morphisms of m-Kuranishi neighbourhoods, as in §4.1.

Definition 4.33. Let $\boldsymbol{P}$ be a discrete property of morphisms in $\dot{\mathbf{Man}}$. Suppose $f : X \to Y$ is a continuous map and $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat\phi_{ij}) : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a 1-morphism of m-Kuranishi neighbourhoods over (S, f), for $S \subseteq X$ open. We say that Φ_{ij} *is* $\boldsymbol{P}$ if $\phi_{ij} : V_{ij} \to V_j$ is $\boldsymbol{P}$ near $\psi_i^{-1}(S)$ in V_{ij}. That is, there should exist an open submanifold $\iota : U \hookrightarrow V_{ij}$ with $\psi_i^{-1}(S) \subseteq U \subseteq V_{ij}$ such that $\phi_{ij} \circ \iota : U \to V_j$ has property $\boldsymbol{P}$ in $\dot{\mathbf{Man}}$.

Proposition 4.34. *Let $\boldsymbol{P}$ be a discrete property of morphisms in $\dot{\mathbf{Man}}$. Then:*

(a) *Let $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ be a 1-morphism of m-Kuranishi neighbourhoods over (S, f) for $f : X \to Y$ continuous and $S \subseteq X$ open. If Φ_{ij} is $\boldsymbol{P}$ and $T \subseteq S$ is open then $\Phi_{ij}|_T$ is $\boldsymbol{P}$. If $\{T_a : a \in A\}$ is an open cover of S and $\Phi_{ij}|_{T_a}$ is $\boldsymbol{P}$ for all $a \in A$ then Φ_{ij} is $\boldsymbol{P}$.*

(b) *Let $\Phi_{ij}, \Phi'_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ be 1-morphisms over (S, f) and $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ a 2-morphism. Then Φ_{ij} is $\boldsymbol{P}$ if and only if Φ'_{ij} is $\boldsymbol{P}$.*

(c) *Let $f : X \to Y$, $g : Y \to Z$ be continuous, $T \subseteq Y$, $S \subseteq f^{-1}(T) \subseteq X$ be open, $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ be a 1-morphism over (S, f), and $\Phi_{jk} : (V_j, E_j, s_j, \psi_j) \to (V_k, E_k, s_k, \psi_k)$ be a 1-morphism over (T, g), so that $\Phi_{jk} \circ \Phi_{ij}$ is a 1-morphism over $(S, g \circ f)$. If Φ_{ij}, Φ_{jk} are $\boldsymbol{P}$ then $\Phi_{jk} \circ \Phi_{ij}$ is $\boldsymbol{P}$.*

(d) *Let $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ be a coordinate change of m-Kuranishi neighbourhoods over $S \subseteq X$. Then Φ_{ij} is $\boldsymbol{P}$.*

Proof. Part (a) follows from Definition 3.18(iv), and part (b) from Definitions 3.18(vii) and 4.3(b), and part (c) from Definitions 3.18(iii) and 4.4.

For (d), as Φ_{ij} is a coordinate change there exist a 1-morphism $\Phi_{ji} : (V_j, E_j, s_j, \psi_j) \to (V_i, E_i, s_i, \psi_i)$ and 2-morphisms $\Lambda_{ii} : \Phi_{ji} \circ \Phi_{ij} \Rightarrow \mathrm{id}_{(V_i, E_i, s_i, \psi_i)}$, $\Lambda_{jj} : \Phi_{ij} \circ \Phi_{ji} \Rightarrow \mathrm{id}_{(V_j, E_j, s_j, \psi_j)}$. Write $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat\phi_{ij})$, $\Phi_{ji} = (V_{ji}, \phi_{ji}, \hat\phi_{ji})$, and as in (3.8) consider the diagram in $\dot{\mathbf{Man}}$:

$$
\begin{tikzcd}
\phi_{ij}^{-1}(V_{ji}) \arrow[r, hook] \arrow[dr, "\phi_{ij}|_{\phi_{ij}^{-1}(V_{ji})}"' near start] & V_{ij} \arrow[r, hook] \arrow[dr, "\phi_{ij}" near start] & V_i \\
\phi_{ji}^{-1}(V_{ij}) \arrow[r, hook] \arrow[ur, "\phi_{ji}|_{\phi_{ji}^{-1}(V_{ij})}" near start] & V_{ji} \arrow[r, hook] \arrow[ur, "\phi_{ji}"' near start] & V_j.
\end{tikzcd}
$$

For each $x \in S$ let $v_i = \psi_i^{-1}(x) \in \phi_{ij}^{-1}(V_{ji}) \subseteq V_{ij} \subseteq V_i$ and $v_j = \psi_j^{-1}(x) \in \phi_{ji}^{-1}(V_{ij}) \subseteq V_{ji} \subseteq V_j$, so that $\phi_{ij}(v_i) = v_j$ and $\phi_{ji}(v_j) = v_i$ by Definition 4.2(e) for Φ_{ij}, Φ_{ji}. Definition 4.3(b) for $\Lambda_{ii}, \Lambda_{jj}$ implies that $\phi_{ji} \circ \phi_{ij} = \mathrm{id}_{V_i} + O(s_i)$ on $\phi_{ij}^{-1}(V_{ji})$ and $\phi_{ij} \circ \phi_{ji} = \mathrm{id}_{V_j} + O(s_j)$ on $\phi_{ji}^{-1}(V_{ij})$. Therefore Definition 3.18(viii) implies that ϕ_{ij} is $\boldsymbol{P}$ near v_i. As this holds for all $x \in S$, Definition 3.18(iv) shows that ϕ_{ij} is $\boldsymbol{P}$ near $\psi_i^{-1}(S)$, so Φ_{ij} is $\boldsymbol{P}$. $\square$

Definition 4.35. Let $\boldsymbol{P}$ be a discrete property of morphisms in $\dot{\mathbf{Man}}$. Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in $\mathbf{m\dot{K}ur}$, and use notation (4.6), (4.7), (4.9) for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{f}$. We say that $\boldsymbol{f}$ *is* $\boldsymbol{P}$ if $\boldsymbol{f}_{ij}$ is $\boldsymbol{P}$ in the sense of Definition 4.33 for all $i \in I$ and $j \in J$.

Proposition 4.36. *Let $\boldsymbol{P}$ be a discrete property of morphisms in $\dot{\mathbf{Man}}$. Then:*

(a) *Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms in $\mathbf{m\dot{K}ur}$ and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism. Then $\boldsymbol{f}$ is $\boldsymbol{P}$ if and only if $\boldsymbol{g}$ is $\boldsymbol{P}$.*

(b) *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms in $\mathbf{m\dot{K}ur}$. If $\boldsymbol{f}$ and $\boldsymbol{g}$ are $\boldsymbol{P}$ then $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ is $\boldsymbol{P}$.*

(c) *Identity 1-morphisms $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$ are $\boldsymbol{P}$. Equivalences $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$ are $\boldsymbol{P}$.*

Parts **(b)**,**(c)** *imply that we have a 2-subcategory $\mathbf{m\dot{K}ur}_{\boldsymbol{P}} \subseteq \mathbf{m\dot{K}ur}$ containing all objects in $\mathbf{m\dot{K}ur}$, and all 1-morphisms $\boldsymbol{f}$ in $\mathbf{m\dot{K}ur}$ which are $\boldsymbol{P}$, and all 2-morphisms $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\mathbf{m\dot{K}ur}$ between 1-morphisms $\boldsymbol{f}, \boldsymbol{g}$ which are $\boldsymbol{P}$.*

Proof. For (a), use notation (4.6), (4.7), (4.9) for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{f}, \boldsymbol{g}$. Then we have 2-morphisms of m-Kuranishi neighbourhoods $\boldsymbol{\eta}_{ij} : \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij}$ for all i, j, so Proposition 4.34(b) implies that $\boldsymbol{f}_{ij}$ is $\boldsymbol{P}$ if and only if $\boldsymbol{g}_{ij}$ is $\boldsymbol{P}$, and (a) follows.

For (b), use the notation of Definition 4.20, and suppose $\boldsymbol{f}, \boldsymbol{g}$ are $\boldsymbol{P}$. Then for all $i \in I$, $j \in J$, $k \in K$ we have 2-morphisms $\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}} : \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \Rightarrow (\boldsymbol{g} \circ \boldsymbol{f})_{ik}$ over $(T_j, g \circ f)$ for $T_j = \mathrm{Im}\, \chi_i \cap f^{-1}(\mathrm{Im}\, \psi_j) \cap (g \circ f)^{-1}(\mathrm{Im}\, \omega_k)$. As $\boldsymbol{f}, \boldsymbol{g}$ are $\boldsymbol{P}$, $\boldsymbol{f}_{ij}, \boldsymbol{g}_{jk}$ are $\boldsymbol{P}$, so $\boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij}$ is $\boldsymbol{P}$ by Proposition 4.34(c), and thus $(\boldsymbol{g} \circ \boldsymbol{f})_{ik}$ is $\boldsymbol{P}$ over $(T_j, g \circ f)$ by Proposition 4.34(b). Since this holds for all $j \in J$, Proposition 4.34(a) implies that $(\boldsymbol{g} \circ \boldsymbol{f})_{ik}$ is $\boldsymbol{P}$ over $(S, g \circ f)$ for $S = \bigcup_{j \in J} T_j = \mathrm{Im}\, \chi_i \cap (g \circ f)^{-1}(\mathrm{Im}\, \omega_k)$, which is the domain we want. As this holds for all $i \in I$ and $k \in K$, $\boldsymbol{g} \circ \boldsymbol{f}$ is $\boldsymbol{P}$.

For (c), that $\mathbf{id}_{\boldsymbol{X}}$ is $\boldsymbol{P}$ follows from (4.13) and Proposition 4.34(d), as the T_{ij} are coordinate changes. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be an equivalence in $\mathbf{m\dot{K}ur}$, and use notation (4.6), (4.7), (4.9). Then there exist a 1-morphism $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{X}$ and 2-morphisms $\boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \mathbf{id}_{\boldsymbol{X}}$, $\boldsymbol{\zeta} : \boldsymbol{f} \circ \boldsymbol{g} \Rightarrow \mathbf{id}_{\boldsymbol{Y}}$. Using the proof of Proposition 4.34(d) with $\boldsymbol{f}_{ij}, \boldsymbol{g}_{ji}, \boldsymbol{\eta}_{ii}, \boldsymbol{\zeta}_{jj}$ in place of $\Phi_{ij}, \Phi_{ji}, \Lambda_{ii}, \Lambda_{jj}$ shows that $\boldsymbol{f}_{ij}$ is $\boldsymbol{P}$, for all $i \in I$ and $j \in J$, so $\boldsymbol{f}$ is $\boldsymbol{P}$. $\square$

Definition 4.37. **(a)** Taking $\dot{\mathbf{Man}} = \mathbf{Man^c}$ from §2.1 gives the 2-category of m-Kuranishi spaces $\mathbf{mKur^c}$ from Definition 4.29. We write

$$\mathbf{mKur^c_{in}}, \mathbf{mKur^c_{bn}}, \mathbf{mKur^c_{st}}, \mathbf{mKur^c_{st,in}}, \mathbf{mKur^c_{st,bn}}, \mathbf{mKur^c_{si}}$$

for the 2-subcategories of $\mathbf{mKur^c}$ with 1-morphisms which are *interior*, and *b-normal*, and *strongly smooth*, and *strongly smooth-interior*, and *strongly smooth-b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^c}$ are discrete by Example 3.19(a), so as in Definition 4.35 and Proposition 4.36 we have corresponding notions of interior, ..., simple 1-morphisms in $\mathbf{mKur^c}$.

(b) Taking $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^{gc}}$ from §2.4.1 gives the 2-category of m-Kuranishi spaces with g-corners $\mathbf{mKur^{gc}}$ from Definition 4.29. We write

$$\mathbf{mKur^{gc}_{in}}, \mathbf{mKur^{gc}_{bn}}, \mathbf{mKur^{gc}_{si}}$$

for the 2-subcategories of $\mathbf{mKur^{gc}}$ with 1-morphisms which are *interior*, and *b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^{gc}}$ are discrete by Example 3.19(b), so we have corresponding notions for 1-morphisms in $\mathbf{mKur^{gc}}$.

(c) Taking $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^{ac}}$ from §2.4.2 gives the 2-category of m-Kuranishi spaces with a-corners $\mathbf{mKur^{ac}}$ from Definition 4.29. We write

$$\mathbf{mKur^{ac}_{in}}, \mathbf{mKur^{ac}_{bn}}, \mathbf{mKur^{ac}_{st}}, \mathbf{mKur^{ac}_{st,in}}, \mathbf{mKur^{ac}_{st,bn}}, \mathbf{mKur^{ac}_{si}}$$

for the 2-subcategories of $\mathbf{mKur^{ac}}$ with 1-morphisms which are *interior*, and *b-normal*, and *strongly a-smooth*, and *strongly a-smooth-interior*, and *strongly a-smooth-b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^{ac}}$ are discrete by Example 3.19(c), so we have corresponding notions for 1-morphisms in $\mathbf{mKur^{ac}}$.

(d) Taking $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^{c,ac}}$ from §2.4.2 gives the 2-category of m-Kuranishi spaces with corners and a-corners $\mathbf{mKur^{c,ac}}$ from Definition 4.29. We write

$$\mathbf{mKur^{c,ac}_{in}}, \mathbf{mKur^{c,ac}_{bn}}, \mathbf{mKur^{c,ac}_{st}}, \mathbf{mKur^{c,ac}_{st,in}}, \mathbf{mKur^{c,ac}_{st,bn}}, \mathbf{mKur^{c,ac}_{si}}$$

for the 2-subcategories of $\mathbf{mKur^{c,ac}}$ with 1-morphisms which are *interior*, and *b-normal*, and *strongly a-smooth*, and *strongly a-smooth-interior*, and *strongly a-smooth-b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^{c,ac}}$ are discrete by Example 3.19(c), so we have corresponding notions for 1-morphisms in $\mathbf{mKur^{c,ac}}$.

Figure 4.1 gives inclusions between the 2-categories in (4.37). Combining this with the inclusions between the 2-subcategories in Definition 4.37 we get a diagram Figure 4.2 of inclusions of 2-subcategories of m-Kuranishi spaces.

Remark 4.38. (i) Most of the 2-categories $\mathbf{mKur^c_{in}}, \mathbf{mKur^c_{bn}}, \ldots$ in Definition 4.37 come from categories $\mathbf{Man^c_{in}}, \mathbf{Man^c_{bn}}, \ldots$ satisfying Assumptions 3.1–3.7, so we could have applied §4.3 to construct 2-categories of m-Kuranishi spaces $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur^c}$ directly from $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^c_{in}}, \mathbf{Man^c_{bn}}, \ldots$. But what we actually did was slightly different. We explain this for $\mathbf{Man^c_{in}}$ and $\mathbf{mKur^c_{in}}$, though it applies to all the 2-categories above except those with simple 1-morphisms.

If $\boldsymbol{X} = (X, \mathcal{I})$ lies in $\mathbf{mKur^c}$, with notation (4.6), each $\mathrm{T}_{ii'}$ in $\mathcal{I}$ includes a morphism $\tau_{ii'} : U_{ii'} \to U_{i'}$ in $\mathbf{Man^c}$. Then $\boldsymbol{X}$ lies in $\mathbf{mKur^c_{in}}$ as defined above

$$
\begin{tikzcd}[column sep=small]
 & & & \mathbf{mKur} \arrow[dlll, bend right=10] \arrow[dl] \arrow[dr] \arrow[drrr, bend left=10] & & & \\
\mathbf{mKur}^{\mathbf{gc}}_{\mathbf{si}} \arrow[dd] & & \mathbf{mKur}^{\mathbf{c}}_{\mathbf{si}} \arrow[ll] \arrow[rr] \arrow[dd] \arrow[dl] & & \mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{si}} \arrow[dd] \arrow[dl] & & \mathbf{mKur}^{\mathbf{ac}}_{\mathbf{si}} \arrow[ll] \arrow[dd] \arrow[dl] \\
 & \mathbf{mKur}^{\mathbf{c}}_{\mathbf{st,bn}} \arrow[rr] \arrow[dl] \arrow[dr] \arrow[dd] & & \mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{st,bn}} \arrow[dr] \arrow[dd] & & \mathbf{mKur}^{\mathbf{ac}}_{\mathbf{st,bn}} \arrow[ll] \arrow[dr] \arrow[dd] & \\
\mathbf{mKur}^{\mathbf{gc}}_{\mathbf{bn}} \arrow[dd] & & \mathbf{mKur}^{\mathbf{c}}_{\mathbf{bn}} \arrow[ll] \arrow[rr] \arrow[dd] & & \mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{bn}} \arrow[dd] & & \mathbf{mKur}^{\mathbf{ac}}_{\mathbf{bn}} \arrow[ll] \arrow[dd] \\
 & \mathbf{mKur}^{\mathbf{c}}_{\mathbf{st,in}} \arrow[rr] \arrow[dl] \arrow[dr] \arrow[dd] & & \mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{st,in}} \arrow[dr] \arrow[dd] & & \mathbf{mKur}^{\mathbf{ac}}_{\mathbf{st,in}} \arrow[ll] \arrow[dr] \arrow[dd] & \\
\mathbf{mKur}^{\mathbf{gc}}_{\mathbf{in}} \arrow[dd] & & \mathbf{mKur}^{\mathbf{c}}_{\mathbf{in}} \arrow[ll] \arrow[rr] \arrow[dd] & & \mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{in}} \arrow[dd] & & \mathbf{mKur}^{\mathbf{ac}}_{\mathbf{in}} \arrow[ll] \arrow[dd] \\
 & \mathbf{mKur}^{\mathbf{c}}_{\mathbf{st}} \arrow[rr] \arrow[dl] \arrow[dr] & & \mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{st}} \arrow[dr] & & \mathbf{mKur}^{\mathbf{ac}}_{\mathbf{st}} \arrow[ll] \arrow[dr] & \\
\mathbf{mKur}^{\mathbf{gc}} & & \mathbf{mKur}^{\mathbf{c}} \arrow[ll] \arrow[rr] & & \mathbf{mKur}^{\mathbf{c,ac}} & & \mathbf{mKur}^{\mathbf{ac}} \arrow[ll]
\end{tikzcd}
$$

Figure 4.2: Inclusions of 2-categories of m-Kuranishi spaces.

if $\tau_{ii'}$ is interior near $\chi_i^{-1}(\operatorname{Im}\chi_{i'})$ for all $i,i'\in I$, as in Definition 4.33. But $\boldsymbol{X}$ lies in the 2-category $\mathbf{m\dot{K}ur}$ associated to $\dot{\mathbf{Man}}=\mathbf{Man^c_{in}}$ in §4.3 if the $\tau_{ii'}$ are interior on all of $U_{ii'}$. Similarly, if $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ in (4.9) is a 1-morphism in $\mathbf{mKur^c}$ then $\boldsymbol{f}$ lies in $\mathbf{mKur^c_{in}}$ above if the $f_{ij}:U_{ij}\to V_j$ in $\boldsymbol{f}_{ij}$ are interior near $(f\circ\chi_i)^{-1}(\operatorname{Im}\psi_j)$, but $\boldsymbol{f}$ lies in $\mathbf{m\dot{K}ur}$ if the f_{ij} are interior on all of U_{ij}.

We have $\mathbf{m\dot{K}ur}\subseteq\mathbf{mKur^c_{in}}\subseteq\mathbf{mKur^c}$, where the inclusion $\mathbf{m\dot{K}ur}\subseteq\mathbf{mKur^c_{in}}$ is an equivalence of 2-categories, but $\mathbf{m\dot{K}ur}$ is not closed in $\mathbf{mKur^c}$ under either equivalences of objects or under 2-isomorphism of 1-morphisms, but $\mathbf{mKur^c_{in}}$ is closed in $\mathbf{mKur^c}$ under both of these. This closure is a useful property, which is why we prefer this definition of $\mathbf{mKur^c_{in}},\ldots.$

(ii) In §2.4.2 we mentioned a functor $F_{\mathbf{Man^{ac}_{st}}}^{\mathbf{Man^c_{st}}}:\mathbf{Man^{ac}_{st}}\to\mathbf{Man^c_{st}}$ from [66, §3]. Taking this to be $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}:\dot{\mathbf{Man}}\to\ddot{\mathbf{Man}}$ and applying §4.4 gives a 2-functor $F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}}:\mathbf{m\dot{K}ur}\to\mathbf{m\ddot{K}ur}$. This does not map $\mathbf{mKur^{ac}_{st}}\to\mathbf{mKur^c_{st}}$, with the notation above, since $\mathbf{m\dot{K}ur}\subset\mathbf{mKur^{ac}_{st}}$, $\mathbf{m\ddot{K}ur}\subset\mathbf{mKur^c_{st}}$ are proper but equivalent 2-subcategories, as in **(i)**. However, we can get a 2-functor $F_{\mathbf{mKur^{ac}_{st}}}^{\mathbf{mKur^c_{st}}}:\mathbf{mKur^{ac}_{st}}\to\mathbf{mKur^c_{st}}$ by composing with a quasi-inverse for $\mathbf{m\dot{K}ur}\hookrightarrow\mathbf{mKur^{ac}_{st}}$. The same applies to $F_{\mathbf{Man^{c,ac}_{st}}}^{\mathbf{Man^c_{st}}}:\mathbf{Man^{c,ac}_{st}}\to\mathbf{Man^c_{st}}$ in §2.4.2.

4.6 M-Kuranishi spaces with corners. Boundaries, k-corners, and the corner 2-functor

We now change notation from $\dot{\mathbf{Man}}$ in §3.1–§3.3 to $\dot{\mathbf{Man}}^{\mathbf{c}}$, and from $\mathbf{m\dot{K}ur}$ in §4.3–§4.5 to $\mathbf{m\dot{K}ur^c}$. Suppose throughout this section that $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumption 3.22 in §3.4.1. Then $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumptions 3.1–3.7, so §4.3 constructs a 2-category $\mathbf{m\dot{K}ur^c}$ of m-Kuranishi spaces associated to $\dot{\mathbf{Man}}^{\mathbf{c}}$. For instance, $\mathbf{m\dot{K}ur^c}$ could be $\mathbf{mKur^c}, \mathbf{mKur^{gc}}, \mathbf{mKur^{ac}}$ or $\mathbf{mKur^{c,ac}}$ from Definition 4.29. We will refer to objects of $\mathbf{m\dot{K}ur^c}$ as *m-Kuranishi spaces with corners*. We also write $\mathbf{m\dot{K}ur^c_{si}}$ for the 2-subcategory of $\mathbf{m\dot{K}ur^c}$ with simple 1-morphisms in the sense of §4.5, noting that simple is a discrete property of morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$ by Assumption 3.22(c).

Generalizing §2.2 for ordinary manifolds with corners $\mathbf{Man^c}$, we will define the *boundary* $\partial\boldsymbol{X}$ and *k-corners* $C_k(\boldsymbol{X})$ for each $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur^c}$, and the *corner 2-functor* $C : \mathbf{m\dot{K}ur^c} \to \mathbf{m\check{\dot{K}}ur^c}$. The definitions below are rather long, mechanical, heavy on notation, and boring. Despite this, the underlying ideas are straightforward, with little subtlety — everything just works, mostly in the obvious way. The principle is to apply $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}^{\mathbf{c}}$ in Assumption 3.22(g) to everything in sight, and use the ideas of §3.4.3 on how differential geometry lifts along $\Pi_k : C_k(X) \to X$.

4.6.1 Definition of the k-corners $C_k(\boldsymbol{X})$

Definition 4.39. Let $\boldsymbol{X} = (X,\mathcal{K})$ in $\mathbf{m\dot{K}ur^c}$ be an m-Kuranishi space with corners with $\operatorname{vdim}\boldsymbol{X} = n$, and as in Definition 4.14 write $\mathcal{K} = \bigl(I, (V_i,E_i,s_i,\psi_i)_{i\in I}, \Phi_{ij,\, i,j\in I}, \Lambda_{hij,\, h,i,j\in I}\bigr)$ with $\Phi_{ij} = (V_{ij},\phi_{ij},\hat\phi_{ij})$ and $\Lambda_{hij} = [\acute V_{hij},\hat\lambda_{hij}]$. Let $k\in\mathbb{N}$. We will define an m-Kuranishi space with corners $C_k(\boldsymbol{X})$ in $\mathbf{m\dot{K}ur^c}$ called the *k-corners of* $\boldsymbol{X}$, with $\operatorname{vdim} C_k(\boldsymbol{X}) = n-k$, and a 1-morphism $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ in $\mathbf{m\dot{K}ur^c}$.

Explicitly we write $C_k(\boldsymbol{X}) = (C_k(X),\mathcal{K}_k)$ with

$$
\begin{aligned}
\mathcal{K}_k &= \bigl(\{k\}\times I, (V_{(k,i)},E_{(k,i)},s_{(k,i)},\psi_{(k,i)})_{i\in I}, \Phi_{(k,i),(k,j),\, i,j\in I}, \Lambda_{(k,h)(k,i)(k,j),\, h,i,j\in I}\bigr)\\
\text{with}\quad & \Phi_{(k,i)(k,j)} = \bigl(V_{(k,i)(k,j)},\phi_{(k,i)(k,j)},\hat\phi_{(k,i)(k,j)}\bigr)\\
\text{and}\quad & \Lambda_{(k,h)(k,i)(k,j)} = [\acute V_{(k,h)(k,i)(k,j)},\hat\lambda_{(k,h)(k,i)(k,j)}],
\end{aligned}
$$

where $\mathcal{K}_k$ has indexing set $\{k\}\times I$ with elements (k,i) for $i\in I$, for reasons that will become clear in §4.6.2, and as in (4.9) we write

$$
\begin{aligned}
\boldsymbol{\Pi}_k &= \bigl(\Pi_k, \boldsymbol{\Pi}_{(k,i)j,\, i,j\in I}, \boldsymbol{\Pi}^{j,\, j\in I}_{(k,i)(k,i'),\, i,i'\in I}, \boldsymbol{\Pi}^{jj',\, j,j'\in I}_{(k,i),\, i\in I}\bigr), \quad \text{where}\\
\boldsymbol{\Pi}_{(k,i)j} &= (V_{(k,i)j},\Pi_{(k,i)j},\hat\Pi_{(k,i)j}) : (V_{(k,i)},E_{(k,i)},s_{(k,i)},\psi_{(k,i)})\\
&\qquad\longrightarrow (V_j,E_j,s_j,\psi_j),\\
\boldsymbol{\Pi}^j_{(k,i)(k,i')} &= [\acute V^j_{(k,i)(k,i')},\hat\Pi^j_{(k,i)(k,i')}] : \boldsymbol{\Pi}_{(k,i')j}\circ\Phi_{(k,i)(k,i')} \Longrightarrow \boldsymbol{\Pi}_{(k,i)j},\\
\boldsymbol{\Pi}^{jj'}_{(k,i)} &= [\acute V^{jj'}_{(k,i)},\hat\Pi^{jj'}_{(k,i)}] : \Phi_{jj'}\circ\boldsymbol{\Pi}_{(k,i)j} \Longrightarrow \boldsymbol{\Pi}_{(k,i)j'}.
\end{aligned}
$$

The hardest part is to define the topological space $C_k(X)$ and the continuous maps $\Pi_k : C_k(X) \to X$, $\psi_{(k,i)} : s_{(k,i)}^{-1}(0) \to C_k(X)$, and we do these last.

For each $i \in I$, define $V_{(k,i)} = C_k(V_i)$ to be the k-corners of V_i from Assumption 3.22(d). Define $E_{(k,i)} \to V_{(k,i)}$ to be the pullback vector bundle $\Pi_k^*(E_i)$, where $\Pi_k : V_{(k,i)} = C_k(V_i) \to V_i$ is as in Assumption 3.22(d), and let $s_{(k,i)} = \Pi_k^*(s_i)$ in $\Gamma^\infty(E_{(k,i)})$ be the pullback section. Using Assumption 3.22 we can show these are equivalent to $E_{(k,i)} = C_k(E_i)$, $s_{(k,i)} = C_k(s_i)$, where $s_i : V_i \to E_i$ is simple. Note that

$$\dim V_{(k,i)} - \operatorname{rank} E_{(k,i)} = \dim C_k(V_i) - \operatorname{rank} E_i = \dim V_i - k - \operatorname{rank} E_i = n - k,$$

by Assumption 3.22(d), as required in Definition 4.14(b) for $C_k(\boldsymbol{X})$.

Although we have not yet defined $C_k(X)$ and $\psi_{(k,i)} : s_{(k,i)}^{-1}(0) \to C_k(X)$, the definition we later give will have the property that for $i, j \in I$ we have

$$\psi_{(k,i)}^{-1}(\operatorname{Im}\psi_{(k,j)}) = (\Pi_k \circ \psi_{(k,i)})^{-1}(\operatorname{Im}\psi_j) = \Pi_k^{-1}(\psi_i^{-1}(\operatorname{Im}\psi_j)), \tag{4.39}$$

where $\psi_i^{-1}(\operatorname{Im}\psi_j) \subseteq s_i^{-1}(0) \subseteq V_i$ and $\Pi_k : V_{(k,i)} = C_k(V_i) \to V_i$, and the definition of $s_{(k,i)}$ implies that $s_{(k,i)}^{-1}(0) = \Pi_k^{-1}(s_i^{-1}(0))$.

Let $i, j \in I$. Since simple maps are a discrete property in $\dot{\mathbf{Man}}^{\mathbf{c}}$ by Assumption 3.22(c), Definition 4.33 and Proposition 4.34(d) imply that $\phi_{ij} : V_{ij} \to V_j$ is simple near $\psi_i^{-1}(\operatorname{Im}\psi_j) \subseteq V_{ij}$. Let $V'_{ij} \subseteq V_{ij}$ be the maximal open set on which ϕ_{ij} is simple, so that $\psi_i^{-1}(\operatorname{Im}\psi_j) \subseteq V'_{ij}$. Write $\phi'_{ij}, \hat{\phi}'_{ij}$ for the restrictions of $\phi_{ij}, \hat{\phi}_{ij}$ to V'_{ij}. Define

$$V_{(k,i)(k,j)} = C_k(V'_{ij}). \tag{4.40}$$

Then $V_{(k,i)(k,j)}$ is open in $V_{(k,i)}$ by Assumption 3.22(j), as $V'_{ij} \subseteq V_i$ is open, and $\psi_{(k,i)}^{-1}(\operatorname{Im}\psi_{(k,i)} \cap \operatorname{Im}\psi_{(k,j)}) \subseteq V_{(k,i)(k,j)}$ as required in Definition 4.2(a) for $\Phi_{(k,i)(k,j)}$ follows from (4.39) and $\psi_i^{-1}(\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j) \subseteq V'_{ij}$. As $\phi'_{ij} : V'_{ij} \to V_j$ is simple, Assumption 3.22(d) gives a morphism $C_k(\phi'_{ij}) : C_k(V'_{ij}) \to C_k(V_j)$ in $\dot{\mathbf{Man}}^{\mathbf{c}}$. Define

$$\phi_{(k,i)(k,j)} = C_k(\phi'_{ij}) : V_{(k,i)(k,j)} \longrightarrow V_{(k,j)}. \tag{4.41}$$

Assumption 3.22(g) implies that $\phi'_{ij} \circ \Pi_k = \Pi_k \circ C_k(\phi'_{ij}) : C_k(V'_{ij}) \to V_j$. Thus we may define

$$\begin{aligned} \hat{\phi}_{(k,i)(k,j)} = \Pi_k^*(\hat{\phi}'_{ij}) : E_{(k,i)}|_{V_{(k,i)(k,j)}} = \Pi_k^*(E_i|_{V'_{ij}}) &\longrightarrow \Pi_k^* \circ \phi'^*_{ij}(E_j) \\ &= (\phi'_{ij} \circ \Pi_k)^*(E_j) = (\Pi_k \circ C_k(\phi'_{ij}))^*(E_j) \\ &= C_k(\phi'_{ij})^* \circ \Pi_k^*(E_j) = \phi_{(k,i)(k,j)}^*(E_{(k,j)}). \end{aligned} \tag{4.42}$$

We have $\hat{\phi}_{ij}(s_i|_{V_{ij}}) = \phi_{ij}^*(s_j) + O(s_i^2)$ by Definition 4.2(d) for Φ_{ij}, so pulling back by $\Pi_k : V_{(k,i)(k,j)} = C_k(V'_{ij}) \to V'_{ij} \subseteq V_{ij}$ using Theorem 3.28(i) yields

$$\hat{\phi}_{(k,i)(k,j)}(s_{(k,i)}|_{V_{(k,i)(k,j)}}) = \phi_{(k,i)(k,j)}^*(s_{(k,j)}) + O(s_{(k,i)}^2),$$

giving Definition 4.2(d) for Φ_{ij}.

For $\mathbf{\Pi}_{(k,i)j}$, define

$$\begin{aligned}V_{(k,i)j}&=C_k(V_{ij}),\qquad \text{and}\\ \Pi_{(k,i)j}&=\phi_{ij}\circ\Pi_k:V_{(k,i)j}=C_k(V_{ij})\longrightarrow V_j,\\ \hat\Pi_{(k,i)j}&=\Pi_k^*(\hat\phi_{ij}):E_{(k,i)}|_{V_{(k,i)j}}=\Pi_k^*(E_i|_{V_{ij}})\longrightarrow\\ &\qquad\Pi_k^*\circ\phi_{ij}^*(E_j)=(\phi_{ij}\circ\Pi_k)^*(E_j)=\Pi_{(k,i)j}^*(E_j).\end{aligned}\tag{4.43}$$

We verify Definition 4.2(a),(d) for $\mathbf{\Pi}_{(k,i)j}$ as for Φ_{ij}.

We have now completely defined the 1-morphisms $\Phi_{(k,i)(k,j)},\mathbf{\Pi}_{(k,i)j}$, although we have not yet defined the data $C_k(X)$ or $\Pi_k:C_k(X)\to X$ or $\psi_{(k,i)}$ in $(V_{(k,i)},E_{(k,i)},s_{(k,i)},\psi_{(k,i)})$, and have not yet verified condition Definition 4.2(e) for $\Phi_{(k,i)(k,j)},\mathbf{\Pi}_{(k,i)j}$ which involves $C_k(X),\Pi_k,\psi_{(k,i)},\psi_{(k,j)}$. The definition of the 2-morphisms $\Lambda_{(k,h)(k,i)(k,j)},\mathbf{\Pi}^j_{(k,i)(k,i')},\mathbf{\Pi}^{jj'}_{(k,i)}$ in Definition 4.3 does not involve $C_k(X),\Pi_k,\psi_{(k,i)}$, so we can do these next.

For $h,i,j\in I$, choose a representative $(\acute V_{hij},\hat\lambda_{hij})$ for the $\sim$-equivalence class Λ_{hij}. Then $\acute V_{hij}\subseteq V_{hi}\cap\phi_{hi}^{-1}(V_{ij})\cap V_{hj}\subseteq V_h$ is open, and $\hat\lambda_{hij}:E_h|_{\acute V_{hij}}\to\mathcal{T}_{\phi_{ij}\circ\phi_{hi}}V_j|_{\acute V_{hij}}$ is a morphism. Set $\acute V'_{hij}=\acute V_{hij}\cap V'_{hi}\cap\phi_{hi}^{-1}(V'_{ij})\cap V'_{hj}$. Define

$$\acute V_{(k,h)(k,i)(k,j)}=C_k(\acute V'_{hij})\subseteq C_k(V_h)=V_{(k,h)}.\tag{4.44}$$

Define a morphism

$$\begin{aligned}&\hat\lambda_{(k,h)(k,i)(k,j)}=\Pi_k^\diamond(\hat\lambda_{hij}):E_{(k,h)}|_{\acute V_{(k,h)(k,i)(k,j)}}=\Pi_k^*(E_h|_{\acute V'_{hij}})\\ &\quad\longrightarrow\mathcal{T}_{\phi_{(k,i)(k,j)}\circ\phi_{(k,h)(k,i)}}V_{(k,j)}|_{\acute V_{(k,h)(k,i)(k,j)}}=\mathcal{T}_{C_k(\phi_{ij}\circ\phi_{hi}|_{\acute V'_{hij}})}C_k(V_j),\end{aligned}\tag{4.45}$$

where $\Pi_k^\diamond(\hat\lambda_{hij})$ is as in §3.4.3 and §B.8.1.

Now Definition 4.3(a) for Λ_{hij} gives

$$\psi_h^{-1}(\operatorname{Im}\psi_h\cap\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j)\subseteq\acute V'_{hij}.$$

Applying Π_k^{-1} to this and using (4.39) (which we assume for now) yields

$$\psi_{(k,h)}^{-1}(\operatorname{Im}\psi_{(k,h)}\cap\operatorname{Im}\psi_{(k,i)}\cap\operatorname{Im}\psi_{(k,j)})\subseteq\acute V_{(k,h)(k,i)(k,j)},\tag{4.46}$$

which is Definition 4.3(a) for $(\acute V_{(k,h)(k,i)(k,j)},\hat\lambda_{(k,h)(k,i)(k,j)})$ for the domain $S=\operatorname{Im}\psi_{(k,h)}\cap\operatorname{Im}\psi_{(k,i)}\cap\operatorname{Im}\psi_{(k,j)}$ for $\Lambda_{(k,h)(k,i)(k,j)}$ in Definition 4.14(d) for $C_k(\boldsymbol{X})$. Definition 4.3(b) for Λ_{hij} gives

$$\begin{aligned}\phi_{hj}&=\phi_{ij}\circ\phi_{hi}+\hat\lambda_{hij}\circ s_h+O(s_h^2),\\ \hat\phi_{hj}&=\phi_{hi}^*(\hat\phi_{ij})\circ\hat\phi_{hi}+(\phi_{ij}\circ\phi_{hi})^*(\mathrm{d}s_j)\circ\hat\lambda_{hij}+O(s_h).\end{aligned}$$

Pulling both equations back by $\Pi_k:\acute V_{(k,h)(k,i)(k,j)}=C_k(\acute V'_{hij})\to\acute V'_{hij}$ and using Theorem 3.28(vi),(vii) yields

$$\begin{aligned}\phi_{(k,h)(k,j)}&=\phi_{(k,i)(k,j)}\circ\phi_{(k,h)(k,i)}+\hat\lambda_{(k,h)(k,i)(k,j)}\circ s_{(k,h)}+O(s_{(k,h)}^2),\\ \hat\phi_{(k,h)(k,j)}&=\phi'^*_{hi}(\hat\phi_{(k,i)(k,j)})\circ\hat\phi_{(k,h)(k,i)}\\ &\quad+(\phi_{(k,i)(k,j)}\circ\phi_{(k,h)(k,i)})^*(\mathrm{d}s_{(k,j)})\circ\hat\lambda_{(k,h)(k,i)(k,j)}+O(s_{(k,h)}),\end{aligned}\tag{4.47}$$

which is Definition 4.3(b) for $(\acute{V}_{(k,h)(k,i)(k,j)}, \hat{\lambda}_{(k,h)(k,i)(k,j)})$.

Write $\Lambda_{(k,h)(k,i)(k,j)} = [\acute{V}_{(k,h)(k,i)(k,j)}, \hat{\lambda}_{(k,h)(k,i)(k,j)}]$ for the $\sim$-equivalence class of $(\acute{V}_{(k,h)(k,i)(k,j)}, \hat{\lambda}_{(k,h)(k,i)(k,j)})$, as in Definition 4.3. Theorem 3.28(ii) implies that equivalence $\sim$ on pairs $(\acute{V}_{hij}, \hat{\lambda}_{hij})$ lifts to $\sim$ on pairs $(\acute{V}_{(k,h)(k,i)(k,j)}, \hat{\lambda}_{(k,h)(k,i)(k,j)})$, so $\Lambda_{(k,h)(k,i)(k,j)}$ depends only on $\Lambda_{hij} = [\acute{V}_{hij}, \hat{\lambda}_{hij}]$, and (once we define $C_k(X), \psi_{(k,i)}$ and verify the $\Phi_{(k,i)(k,j)}$ are 1-morphisms), we have a well defined 2-morphism of m-Kuranishi neighbourhoods

$$\Lambda_{(k,h)(k,i)(k,j)} : \Phi_{(k,i)(k,j)} \circ \Phi_{(k,h)(k,i)} \Longrightarrow \Phi_{(k,h)(k,j)}.$$

Next, for $i, i', j \in I$ and $i, j, j' \in I$, choose representatives $(\acute{V}_{ii'j}, \hat{\lambda}_{ii'j})$ and $(\acute{V}_{ijj'}, \hat{\lambda}_{ijj'})$ for $\Lambda_{ii'j} = [\acute{V}_{ii'j}, \hat{\lambda}_{ii'j}]$ and $\Lambda_{ijj'} = [\acute{V}_{ijj'}, \hat{\lambda}_{ijj'}]$, define $\acute{V}^j_{(k,i)(k,i')} = C_k(\acute{V}_{ii'j})$ and $\acute{V}^{jj'}_{(k,i)} = C_k(\acute{V}_{ijj'})$, and define morphisms $\hat{\Pi}^j_{(k,i)(k,i')}, \hat{\Pi}^{jj'}_{(k,i)}$ by the commutative diagrams

$$\begin{tikzcd}
E_{(k,i)}|_{\acute{V}^j_{(k,i)(k,i')}} \arrow[r, equal] \arrow[d, "\hat{\Pi}^j_{(k,i)(k,i')}"'] & \Pi_k^*(E_i|_{\acute{V}_{ii'j}}) \arrow[d, "\Pi_k^*(\hat{\lambda}_{ii'j})"] \\
\mathcal{T}_{\Pi_{(k,i')j}\circ\phi_{(k,i)(k,i')}} V_j|_{\acute{V}^j_{(k,i)(k,i')}} \arrow[r, equal] & \mathcal{T}_{\phi_{i'j}\circ\phi_{ii'}\circ\Pi_k} V_j,
\end{tikzcd}$$

$$\begin{tikzcd}
E_{(k,i)}|_{\acute{V}^{jj'}_{(k,i)}} \arrow[r, equal] \arrow[d, "\hat{\Pi}^{jj'}_{(k,i)}"'] & \Pi_k^*(E_i|_{\acute{V}_{ijj'}}) \arrow[d, "\Pi_k^*(\hat{\lambda}_{ijj'})"] \\
\mathcal{T}_{\phi_{jj'}\circ\Pi_{(k,i)j}} V_j|_{\acute{V}^{jj'}_{(k,i)}} \arrow[r, equal] & \mathcal{T}_{\phi_{jj'}\circ\phi_{ij}\circ\Pi_k} V_j,
\end{tikzcd}$$

where $\Pi_k^*(\hat{\lambda}_{ii'j}), \Pi_k^*(\hat{\lambda}_{ijj'})$ are as in §3.3.4(g).

Definition 4.3(a),(b) for $(\acute{V}^j_{(k,i)(k,i')}, \hat{\Pi}^j_{(k,i)(k,i')})$ and $(\acute{V}^{jj'}_{(k,i)}, \hat{\Pi}^{jj'}_{(k,i)})$ follow from Definition 4.3(a),(b) for $(\acute{V}_{ii'j}, \hat{\lambda}_{ii'j})$ and $(\acute{V}_{ijj'}, \hat{\lambda}_{ijj'})$, as for (4.46)–(4.47). Write $\boldsymbol{\Pi}^j_{(k,i)(k,i')} = [\acute{V}^j_{(k,i)(k,i')}, \hat{\Pi}^j_{(k,i)(k,i')}]$ and $\boldsymbol{\Pi}^{jj'}_{(k,i)} = [\acute{V}^{jj'}_{(k,i)}, \hat{\Pi}^{jj'}_{(k,i)}]$ for the $\sim$-equivalence classes of $(\acute{V}^j_{(k,i)(k,i')}, \hat{\Pi}^j_{(k,i)(k,i')})$ and $(\acute{V}^{jj'}_{(k,i)}, \hat{\Pi}^{jj'}_{(k,i)})$, in the sense of Definition 4.3. These depend only on $\Lambda_{ii'j}$ and $\Lambda_{ijj'}$, and (once we define $C_k(X), \Pi_k$, $\psi_{(k,i)}$ and verify the $\boldsymbol{\Pi}_{(k,i)j}, \Phi_{(k,i)(k,j)}$ are 1-morphisms), $\boldsymbol{\Pi}^j_{(k,i)(k,i')} : \boldsymbol{\Pi}_{(k,i')j} \circ \Phi_{(k,i)(k,i')} \Rightarrow \boldsymbol{\Pi}_{(k,i)j}$ and $\boldsymbol{\Pi}^{jj'}_{(k,i)} : \Phi_{jj'} \circ \boldsymbol{\Pi}_{(k,i)j} \Rightarrow \boldsymbol{\Pi}_{(k,i)j'}$ are 2-morphisms of m-Kuranishi neighbourhoods.

It remains to define the topological space $C_k(X)$ and the continuous maps $\psi_{(k,i)} : s^{-1}_{(k,i)}(0) \to C_k(X)$, $\Pi_k : C_k(X) \to X$. Define a binary relation $\approx$ on $\coprod_{i\in I} s^{-1}_{(k,i)}(0)$ by $v_i \approx v_j$ if $i, j \in I$ and $v_i \in V_{(k,i)(k,j)} \cap s^{-1}_{(k,i)}(0)$ with $\phi_{(k,i)(k,j)}(v_i) = v_j$ in $s^{-1}_{(k,j)}(0)$. We claim that $\approx$ is an equivalence relation on $\coprod_{i\in I} s^{-1}_{(k,i)}(0)$.

To prove this, suppose $h, i, j \in I$ and $v_h \in s^{-1}_{(k,h)}(0)$, $v_i \in s^{-1}_{(k,i)}(0)$, $v_j \in$

$s^{-1}_{(k,j)}(0)$ with $v_h \approx v_i$ and $v_i \approx v_j$. Then

$$v_h \in s^{-1}_{(k,h)}(0) \cap V_{(k,h)(k,i)} = \Pi_k^{-1}(s_h^{-1}(0) \cap V_{hi}) = \Pi_k^{-1}(\psi_h^{-1}(\operatorname{Im}\psi_h \cap \operatorname{Im}\psi_i)),$$
$$v_i \in s^{-1}_{(k,i)}(0) \cap V_{(k,i)(k,j)} = \Pi_k^{-1}(s_i^{-1}(0) \cap V_{ij}) = \Pi_k^{-1}(\psi_i^{-1}(\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j)),$$

with $\phi_{(k,h)(k,i)}(v_h) = v_i$, $\phi_{(k,i)(k,j)}(v_i) = v_j$. Hence

$$\begin{aligned}\psi_h \circ \Pi_k(v_h) &= \psi_i \circ \phi_{hi} \circ \Pi_k(v_h) = \psi_i \circ \Pi_k \circ \phi_{(k,h)(k,i)}(v_h)\\ &= \psi_i \circ \Pi_k(v_i) \in \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j,\end{aligned}$$

using Definition 4.2(e) for Φ_{hi}. Thus

$$v_h \in \Pi_k^{-1}(\psi_h^{-1}(\operatorname{Im}\psi_h \cap \operatorname{Im}\psi_j)) = \Pi_k^{-1}(s_h^{-1}(0) \cap V'_{hj}) = s^{-1}_{(k,h)}(0) \cap V_{(k,h)(k,j)},$$

and $\phi_{(k,h)(k,j)}(v_h)$ is defined. The first equation of (4.47) and $s_{(k,h)}(v_h) = 0$ imply that $\phi_{(k,h)(k,j)}(v_h) = \phi_{(k,i)(k,j)} \circ \phi_{(k,h)(k,i)}(v_h) = \phi_{(k,i)(k,j)}(v_i) = v_j$. Hence $v_h \approx v_j$, and $v_h \approx v_i$, $v_i \approx v_j$ imply that $v_h \approx v_j$.

Taking $j = h$ and noting that $\phi_{(k,h)(k,h)} = \mathrm{id}_{V_{(k,h)}}$, we see that

$$\begin{aligned}\phi_{(k,h)(k,i)}|_{\cdots} &: s^{-1}_{(k,h)}(0) \cap V_{(k,h)(k,i)} \longrightarrow s^{-1}_{(k,i)}(0) \cap V_{(k,i)(k,h)},\\ \phi_{(k,i)(k,h)}|_{\cdots} &: s^{-1}_{(k,i)}(0) \cap V_{(k,i)(k,h)} \longrightarrow s^{-1}_{(k,h)}(0) \cap V_{(k,h)(k,i)},\end{aligned} \tag{4.48}$$

are inverse maps. Hence $v_h \approx v_i$ implies that $v_i \approx v_h$. And $v_h \approx v_h$ for any $v_h \in s^{-1}_{(k,h)}(0)$ as $\phi_{(k,h)(k,h)} = \mathrm{id}_{V_{(k,h)}}$. Therefore $\approx$ is an equivalence relation.

Now define $C_k(X)$ to be the topological space, with the quotient topology,

$$C_k(X) = \left[\textstyle\coprod_{i \in I} s^{-1}_{(k,i)}(0)\right] / \approx . \tag{4.49}$$

For each $i \in I$ define $\psi_{(k,i)} : s^{-1}_{(k,i)}(0) \to C_k(X)$ by $\psi_{(k,i)} : v_i \mapsto [v_i]$, where $[v_i]$ is the $\approx$-equivalence class of v_i. Define $\Pi_k : C_k(X) \to X$ by $\Pi_k([v_i]) = \psi_i \circ \Pi_k(v_i)$ for $i \in I$ and $v_i \in s^{-1}_{(k,i)}(0)$, so that $\Pi_k(v_i) \in s_i^{-1}(0)$ and $\psi_i \circ \Pi_k(v_i) \in X$. To show this is well defined, suppose $[v_i] = [v_j]$, so that $i, j \in I$ and $v_i \in s^{-1}_{(k,i)}(0)$, $v_j \in s^{-1}_{(k,j)}(0)$ with $v_i \approx v_j$. Then $v_i \in V_{(k,i)(k,j)}$ with $\phi_{(k,i)(k,j)}(v_i) = v_j$, so that

$$\psi_j \circ \Pi_k(v_j) = \psi_j \circ \Pi_k \circ \phi_{(k,i)(k,j)}(v_i) = \psi_j \circ \phi_{ij} \circ \Pi_k(v_i) = \psi_i \circ \Pi_k(v_i),$$

using Definition 4.2(e) for Φ_{ij} in the last step. Hence Π_k is well defined. Observe that (4.39) follows easily from the definitions of $C_k(X), \Pi_k, \psi_{(k,i)}$ above.

We have now defined all the data in $C_k(\boldsymbol{X}) = (C_k(X), \mathcal{K}_k)$. It remains to verify the conditions of Definition 4.14. As $C_k(X)$ is made by gluing the topological spaces $s^{-1}_{(k,i)}(0)$ for $i \in I$ by an equivalence relation $v_h \approx v_i$ for $v_h \in s^{-1}_{(k,h)}(0)$, $v_i \in s^{-1}_{(k,i)}(0)$ which identifies open sets $s^{-1}_{(k,h)}(0) \cap V_{(k,h)(k,i)}$ in $s^{-1}_{(k,h)}(0)$ and $s^{-1}_{(k,i)}(0) \cap V_{(k,i)(k,h)}$ in $s^{-1}_{(k,i)}(0)$ by a homeomorphism (since $\phi_{(k,h)(k,i)}|_{\cdots}, \phi_{(k,i)(k,h)}|_{\cdots}$ in (4.48) are continuous, inverse maps), it follows that

$\psi_{(k,i)} : s_{(k,i)}^{-1}(0) \to C_k(X)$ is a homeomorphism with an open set $\operatorname{Im}\psi_{(k,i)}$ in $C_k(X)$ for $i \in I$, giving Definition 4.1(d) for $(V_{(k,i)}, E_{(k,i)}, s_{(k,i)}, \psi_{(k,i)})$, so $(V_{(k,i)}, E_{(k,i)}, s_{(k,i)}, \psi_{(k,i)})$ is an m-Kuranishi neighbourhood on $C_k(X)$ for $i \in I$.

Because $\psi_{(k,i)} : s_{(k,i)}^{-1}(0) \to \operatorname{Im}\psi_{(k,i)}$ is a homeomorphism, we see that

$$\Pi_k|_{\operatorname{Im}\psi_{(k,i)}} = \psi_i \circ \Pi_k \circ (\psi_{(k,i)})^{-1} : \operatorname{Im}\psi_{(k,i)} \longrightarrow X,$$

which is clearly continuous. As the $\operatorname{Im}\psi_{(k,i)}$, $i \in I$ cover $C_k(X)$, this proves that $\Pi_k : C_k(X) \to X$ is continuous. Also $\operatorname{Im}\psi_{(k,i)} = \Pi_k^{-1}(\operatorname{Im}\psi_i)$, and $\Pi_k|_{\operatorname{Im}\psi_{(k,i)}} : \operatorname{Im}\psi_{(k,i)} \to \operatorname{Im}\psi_i$ is isomorphic to $\Pi_k|_{\cdots} : \Pi_k^{-1}(s_i^{-1}(0)) \to s_i^{-1}(0)$. Since $\Pi_k : C_k(V_i) \to V_i$ is proper with finite fibres by Assumption 3.22(d), we see that $\Pi_k|_{\cdots} : \Pi_k^{-1}(\operatorname{Im}\psi_i) \to \operatorname{Im}\psi_i$ is proper with finite fibres. As the $\operatorname{Im}\psi_i : i \in I$ cover X, it follows that $\Pi_k : C_k(X) \to X$ is proper with finite fibres.

Suppose $x_1' \neq x_2' \in C_k(X)$, and set $x_1 = \Pi_k(x_1')$, $x_2 = \Pi_k(x_2')$ in X. If $x_1 \neq x_2$ then as X is Hausdorff there exist open $x_1 \in U_1 \subseteq X$, $x_2 \in U_2 \subseteq X$ with $U_1 \cap U_2 = \emptyset$, and then $U_1' := \Pi_k^{-1}(U_1)$, $U_2' := \Pi_k^{-1}(U_2)$ are open in X with $x_1' \in U_1'$, $x_2' \in U_2'$ and $U_1' \cap U_2' = \emptyset$. If $x_1 = x_2$ then $x_1, x_2 \in \operatorname{Im}\psi_i \subseteq X$ for some $i \in I$, so $x_1', x_2' \in \operatorname{Im}\psi_{(k,i)} \subseteq C_k(X)$. But $\operatorname{Im}\psi_{(k,i)}$ is open in $C_k(X)$ and is homeomorphic to $s_{(k,i)}^{-1}(0) \subseteq V_{(k,i)}$, which is Hausdorff by Assumption 3.2(b) for $V_{(k,i)}$. Hence there exist open $x_1' \in U_1' \subseteq \operatorname{Im}\psi_{(k,i)} \subseteq C_k(X)$ and $x_2' \in U_2' \subseteq \operatorname{Im}\psi_{(k,i)} \subseteq C_k(X)$ with $U_1' \cap U_2' = \emptyset$. Therefore $C_k(X)$ is Hausdorff.

As X is second countable and the $\operatorname{Im}\psi_i$, $i \in I$ cover X, there exists a countable subset $J \subseteq I$ with $X = \bigcup_{i \in J} \operatorname{Im}\psi_i$. Therefore $C_k(X) = \bigcup_{i \in J} \operatorname{Im}\psi_{(k,i)}$. But each $\operatorname{Im}\psi_{(k,i)}$ is homeomorphic to $s_{(k,i)}^{-1}(0) \subseteq V_{(k,i)}$, which is second countable by Assumption 3.2(b) for $V_{(k,i)}$. So $C_k(X)$ is a countable union of second countable open subspaces, and is second countable.

For all of Definition 4.14(a)–(h) for $C_k(\boldsymbol{X})$, either we have proved them above, or they follow from Definition 4.14(a)–(h) for $\boldsymbol{X}$ by pulling back by Π_k and using Theorems 3.27–3.28. (In (c), that $\Phi_{(k,i)(k,j)}$ is a coordinate change follows from $\Phi_{(k,i)(k,j)}$ a 1-morphism and (d),(f).) Hence $C_k(\boldsymbol{X}) = (C_k(X), \mathcal{K}_k)$ is an m-Kuranishi space with corners in $\mathbf{m\dot{K}ur^c}$, with $\operatorname{vdim} C_k(\boldsymbol{X}) = n - k$.

Similarly, for Definition 4.17(a)–(h) for $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$, either we have proved them above, or they follow from Definition 4.14 for $\boldsymbol{X}$ using Theorems 3.27–3.28, where we deduce Definition 4.17(f)–(h) for $\boldsymbol{\Pi}_k$ from Definition 4.14(h) for $\boldsymbol{X}$. Thus $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ is a 1-morphism in $\mathbf{m\dot{K}ur^c}$.

When $k = 1$ we also write $\partial\boldsymbol{X} = C_1(\boldsymbol{X})$ and call it the *boundary* of $\boldsymbol{X}$, and we write $\boldsymbol{i_X} : \partial\boldsymbol{X} \to \boldsymbol{X}$ in place of $\boldsymbol{\Pi}_1 : C_1(\boldsymbol{X}) \to \boldsymbol{X}$.

We summarize Definition 4.39 in:

Theorem 4.40. *For each $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur^c}$ and $k = 0, 1, \ldots$ we have defined the k-**corners** $C_k(\boldsymbol{X})$, an object in $\mathbf{m\dot{K}ur^c}$ with $\operatorname{vdim} C_k(\boldsymbol{X}) = \operatorname{vdim}\boldsymbol{X} - k$, and a 1-morphism $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ in $\mathbf{m\dot{K}ur^c}$, whose underlying continuous map $\Pi_k : C_k(X) \to X$ is proper with finite fibres. We also write $\partial\boldsymbol{X} = C_1(\boldsymbol{X})$, called the **boundary** of $\boldsymbol{X}$, and we write $\boldsymbol{i_X} = \boldsymbol{\Pi}_1 : \partial\boldsymbol{X} \to \boldsymbol{X}$.*

Remark 4.41. **(a)** The definitions of $C_k(\boldsymbol{X})$ and $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ in Definition 4.39 involve the notions of simple maps in $\dot{\mathbf{M}}\mathbf{an}^{\mathbf{c}}$, and the functor $C_k : \mathbf{Man}^{\mathbf{c}}_{\mathbf{si}} \to \mathbf{Man}^{\mathbf{c}}_{\mathbf{si}}$, and the projections $\Pi_k : C_k(V) \to V$ for $V \in \dot{\mathbf{M}}\mathbf{an}^{\mathbf{c}}$. Apart from these, they do not involve the corner functor $C : \dot{\mathbf{M}}\mathbf{an}^{\mathbf{c}} \to \check{\mathbf{M}}\mathbf{an}^{\mathbf{c}}$.

As in Example 3.24, when $\dot{\mathbf{M}}\mathbf{an}^{\mathbf{c}}$ is $\mathbf{Man}^{\mathbf{c}}, \mathbf{Man}^{\mathbf{c}}_{\mathbf{st}}, \mathbf{Man}^{\mathbf{ac}}, \mathbf{Man}^{\mathbf{ac}}_{\mathbf{st}}, \mathbf{Man}^{\mathbf{c,ac}}$ or $\mathbf{Man}^{\mathbf{c,ac}}_{\mathbf{st}}$ there are two possibilities C, C' for $C : \dot{\mathbf{M}}\mathbf{an}^{\mathbf{c}} \to \check{\mathbf{M}}\mathbf{an}^{\mathbf{c}}$. In each case, simple maps, the functor C_k, and projections Π_k, are the same for C, C'. Therefore $C_k(\boldsymbol{X})$ and $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ are the same for C and C'.

(b) Definition 4.39 is similar to Fukaya, Oh, Ohta and Ono [24, Def. A1.30] for FOOO Kuranishi spaces — see §7.1 for more details.

4.6.2 The corner 2-functor $C : \mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}} \to \mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$

Definition 4.42. Define the 2-category $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$ by following the definition of $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ in §4.3, but with the following modifications. In Definition 4.14, for objects $\boldsymbol{X} = (X, \mathcal{K})$ in $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$, rather than taking $\operatorname{vdim} \boldsymbol{X}$ to be an integer n, it is a locally constant function $\operatorname{vdim} : X \to \mathbb{Z}$. In part (b), we omit $\dim V_i - \operatorname{rank} E_i = n$, but instead we require that $\operatorname{vdim}|_{\operatorname{Im}\psi_i} = \dim V_i - \operatorname{rank} E_i$, for all $i \in I$. This determines $\operatorname{vdim} : X \to \mathbb{Z}$, so it is not extra data. Objects of $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$ will be called *m-Kuranishi spaces with corners of mixed dimension.*

Then $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ embeds as a full 2-subcategory $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}} \subset \mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$ in the obvious way. Any $\boldsymbol{X}$ in $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$ may be uniquely written as $\boldsymbol{X} = \coprod_{n\in\mathbb{Z}} \boldsymbol{X}_n$, where $\boldsymbol{X}_n \subseteq \boldsymbol{X}$ is open and closed with topological space $X_n = \operatorname{vdim}^{-1}(n)$, and $\boldsymbol{X}_n \in \mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}} \subset \mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$ with $\operatorname{vdim} \boldsymbol{X}_n = n \in \mathbb{Z}$.

If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$ with $\boldsymbol{X} = \coprod_{m\in\mathbb{Z}} \boldsymbol{X}_m$, $\boldsymbol{Y} = \coprod_{n\in\mathbb{Z}} \boldsymbol{Y}_n$ for $\boldsymbol{X}_m, \boldsymbol{Y}_n$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ with $\operatorname{vdim} \boldsymbol{X}_m = m$, $\operatorname{vdim} \boldsymbol{Y}_n = n$, then $\boldsymbol{f}|_{\boldsymbol{X}_{mn}} : \boldsymbol{X}_{mn} \to \boldsymbol{Y}_n$ is a 1-morphism in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ for all $m, n \in \mathbb{Z}$, where $\boldsymbol{X}_{mn} := \boldsymbol{X}_m \cap \boldsymbol{f}^{-1}(\boldsymbol{Y}_n)$ is open and closed in $\boldsymbol{X}_m \subseteq \boldsymbol{X}$, with $\boldsymbol{X}_m = \coprod_{n\in\mathbb{Z}} \boldsymbol{X}_{mn}$.

An alternative way to construct $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$ from $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ is to say that objects of $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$ are $\coprod_{n\in\mathbb{Z}} \boldsymbol{X}_n$ for $\boldsymbol{X}_n$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ with $\operatorname{vdim} \boldsymbol{X}_n = n$ as above, and a 1-morphism $\boldsymbol{f} : \coprod_{m\in\mathbb{Z}} \boldsymbol{X}_m \to \coprod_{n\in\mathbb{Z}} \boldsymbol{Y}_n$ in $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$ assigns a decomposition $\boldsymbol{X}_m = \coprod_{n\in\mathbb{Z}} \boldsymbol{X}_{mn}$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ for $m \in \mathbb{Z}$ with $\boldsymbol{X}_{mn} \subseteq \boldsymbol{X}_m$ open and closed, and 1-morphisms $\boldsymbol{f}_{mn} : \boldsymbol{X}_{mn} \to \boldsymbol{Y}_n$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ for all $m, n \in \mathbb{Z}$, and so on.

We write $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}_{\mathbf{si}}$ for the 2-subcategory of $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$ with the same objects, and with simple 1-morphisms, and all 2-morphisms between 1-morphisms in $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}_{\mathbf{si}}$. For the examples of $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}_{\mathbf{si}} \subseteq \mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$ in §4.3 and §4.5 we use the obvious notation for the corresponding 2-categories $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}_{\mathbf{si}} \subseteq \mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$, so for instance we enlarge $\mathbf{mKur}^{\mathbf{c}}$ associated to $\dot{\mathbf{M}}\mathbf{an}^{\mathbf{c}} = \mathbf{Man}^{\mathbf{c}}$ to $\mathbf{m}\check{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$.

Definition 4.43. We will define a weak 2-functor $C : \mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}} \to \mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$, the *corner 2-functor*. On objects $\boldsymbol{X}$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$, define $C(\boldsymbol{X}) = \coprod_{k=0}^{\infty} C_k(\boldsymbol{X})$ in $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$. Extending the notation of Definition 4.39, we regard $C(\boldsymbol{X}) = (C(X), \mathcal{K}_{\mathbb{N}})$ as a single object in $\mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}$, where $\mathcal{K}_{\mathbb{N}}$ has indexing set $\mathbb{N} \times I$, and the part of $C(\boldsymbol{X})$ with indexing set $\{k\} \times I \subset \mathbb{N} \times I$ for $k \in \mathbb{N}$ is $C_k(\boldsymbol{X}) \subset$

$C(\boldsymbol{X})$. Define a 1-morphism $\boldsymbol{\Pi}: C(\boldsymbol{X}) \to \boldsymbol{X}$ in $\mathbf{m\check{\dot{K}}ur^c}$ by $\boldsymbol{\Pi} = \coprod_{k=0}^{\infty} \boldsymbol{\Pi}_k$, for $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ as in Definition 4.39.

Let $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in $\mathbf{m\dot{K}ur^c}$, and use notation (4.6), (4.7) and (4.9) for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{f}$. Thus as above we write

$$
\begin{aligned}
&C(\boldsymbol{X}) = (C(X), \mathcal{I}_{\mathbb{N}}), \;\; \mathcal{I}_{\mathbb{N}} = \bigl(\mathbb{N} \times I, (U_{(k,i)}, D_{(k,i)}, r_{(k,i)}, \chi_{(k,i)})_{(k,i)\in\mathbb{N}\times I},\\
&\;\; \mathrm{T}_{(k,i)(k',i')} = (U_{(k,i)(k',i')}, \tau_{(k,i)(k',i')}, \hat\tau_{(k,i)(k',i')})_{(k,i),(k',i')\in\mathbb{N}\times I},\\
&\;\; \mathrm{K}_{(k,i)(k',i')(k'',i'')} = [\acute U_{(k,i)(k',i')(k'',i'')}, \hat\kappa_{(k,i)(k',i')(k'',i'')}]_{(k,i),(k',i'),(k'',i'')\in\mathbb{N}\times I}\bigr),\\
&C(\boldsymbol{Y}) = (C(Y), \mathcal{J}_{\mathbb{N}}), \;\; \mathcal{J}_{\mathbb{N}} = \bigl(\mathbb{N} \times J, (V_{(l,j)}, E_{(l,j)}, s_{(l,j)}, \psi_{(l,j)})_{(l,j)\in\mathbb{N}\times J},\\
&\;\; \Upsilon_{(l,j)(l',j')} = (V_{(l,j)(l',j')}, \upsilon_{(l,j)(l',j')}, \hat\upsilon_{(l,j)(l',j')})_{(l,j),(l',j')\in\mathbb{N}\times J},\\
&\;\; \Lambda_{(l,j)(l',j')(l'',j'')} = [\acute V_{(l,j)(l',j')(l'',j'')}, \hat\lambda_{(l,j)(l',j')(l'',j'')}]_{(l,j),(l',j'),(l'',j'')\in\mathbb{N}\times J}\bigr).
\end{aligned}
$$

We will define a 1-morphism $C(\boldsymbol{f}): C(\boldsymbol{X}) \to C(\boldsymbol{Y})$ in $\mathbf{m\check{\dot{K}}ur^c}$, where

$$
\begin{aligned}
C(\boldsymbol{f}) = \bigl(C(f), &\boldsymbol{f}_{(k,i)(l,j),\; (k,i)\in\mathbb{N}\times I,\; (l,j)\in\mathbb{N}\times J},\\
&\boldsymbol{F}^{(l,j),\; (l,j)\in\mathbb{N}\times J}_{(k,i)(k',i'),\; (k,i),(k',i')\in\mathbb{N}\times I}, \boldsymbol{F}^{(l,j)(l',j'),\; (l,j),(l',j')\in\mathbb{N}\times J}_{(k,i),\; (k,i)\in\mathbb{N}\times I}\bigr).
\end{aligned}
\tag{4.50}
$$

First we define the map $C(f): C(X) \to C(Y)$. Suppose $x' \in C_k(X) \subseteq C(X)$ with $\Pi_k(x') = x \in X$, and let $y = f(x) \in Y$. Choose $i \in I$ and $j \in J$ with $x \in \operatorname{Im}\chi_i$ and $y \in \operatorname{Im}\psi_j$, so that $x' \in \operatorname{Im}\chi_{(k,i)}$. Write $u_i = \chi_i^{-1}(x) \in r_i^{-1}(0) \subseteq U_i$, $u_i' = \chi_{(k,i)}^{-1}(x') \in r_{(k,i)}^{-1}(0) \subseteq U_{(k,i)} = C_k(U_i)$, so that $\Pi_k(u_i') = u_i$, and write $v_j = \psi_j^{-1}(y) \in s_j^{-1}(0) \subseteq V_j$. Then $f_{ij}(v_i) = v_j$ by Definition 4.2(e) for $\boldsymbol{f}_{ij}$.

In $\boldsymbol{f}$ we have $\boldsymbol{f}_{ij} = (U_{ij}, f_{ij}, \hat f_{ij}) : (U_i, D_i, r_i, \chi_i) \to (V_j, E_j, s_j, \psi_j)$, and $u_i \in U_{ij} \subseteq U_i$, so that $u_i' \in C_k(U_{ij}) \subseteq C_k(U_i)$. Then $f_{ij} : U_{ij} \to V_j$ is a morphism in $\mathbf{\dot{M}an^c}$, so $C(f_{ij}) : C(U_{ij}) \to C(V_j)$ is a morphism in $\mathbf{\check{\dot{M}}an^c}$ by Assumption 3.22(g). Write $v_j' = C(f_{ij})(u_i') \in C_l(V_j) \subseteq C(V_j)$. Then

$$
\Pi_l(v_j') = \Pi_l \circ C(f_{ij})(u_i') = f_{ij} \circ \Pi_k(u_i') = f_{ij}(u_i) = v_j \in s_j^{-1}(0),
$$

so $v_j' \in \Pi_l^{-1}(s_j^{-1}(0)) = s_{(l,j)}^{-1}(0)$. Define $C(f)(x') = \psi_{(l,j)}(v_j') \in C_l(Y) \subseteq C(Y)$.

To show this well defined, let $\tilde\imath \in I$, $\tilde\jmath \in J$ be alternative choices with $x \in \operatorname{Im}\chi_{\tilde\imath}$, $y \in \operatorname{Im}\psi_{\tilde\jmath}$, and write $u_{\tilde\imath}, u_{\tilde\imath}', v_{\tilde\jmath}, v_{\tilde\jmath}'$ for the alternative u_i, u_i', v_j, v_j'. We have coordinate changes $\mathrm{T}_{\tilde\imath i} = (U_{\tilde\imath i}, \tau_{\tilde\imath i}, \hat\tau_{\tilde\imath i})$, $\Upsilon_{j\tilde\jmath} = (V_{j\tilde\jmath}, \upsilon_{j\tilde\jmath}, \hat\upsilon_{j\tilde\jmath})$ in $\boldsymbol{X}, \boldsymbol{Y}$. Then

$$
\begin{aligned}
\psi_{(l,j)}(v_j') &= \psi_{(l,j)} \circ C(f_{ij})(u_i') = \psi_{(l,\tilde\jmath)} \circ C(\upsilon_{j\tilde\jmath}) \circ C(f_{ij}) \circ C(\tau_{\tilde\imath i})(u_{\tilde\imath}')\\
&= \psi_{(l,\tilde\jmath)} \circ C(\upsilon_{j\tilde\jmath} \circ f_{ij} \circ \tau_{\tilde\imath i})(u_{\tilde\imath}') = \psi_{(l,\tilde\jmath)} \circ C(f_{\tilde\imath\tilde\jmath})(u_{\tilde\imath}') = \psi_{(l,\tilde\jmath)}(v_{\tilde\jmath}').
\end{aligned}
$$

Here in the first and fifth steps we use the definitions of $v_j', v_{\tilde\jmath}'$, in the second the definition of $C_k(X), C_l(Y)$ in (4.49) with $\tau_{\tilde\imath i}, \upsilon_{j\tilde\jmath}$ simple near $u_{\tilde\imath}, v_{\tilde\jmath}$ so that k, l do not change, in the third that $C: \mathbf{\dot{M}an^c} \to \mathbf{\check{\dot{M}}an^c}$ is a functor, and in the fourth Definition 4.17(g) for $\boldsymbol{f}$. Hence $C(f)(x')$ is well defined.

We now have a commutative diagram

$$
\begin{tikzcd}
u_i' \in r_{(k,i)}^{-1}(0) \cap C_k(U_{ij}) \cap C(f_{ij})^{-1}(C_l(V_j)) \arrow[r, "C(f_{ij})|_{\cdots}"'] \arrow[d, "\chi_{(k,i)}|_{\cdots}"] & s_{(l,j)}^{-1}(0) \arrow[d, "\psi_{(l,j)}"'] \\
x' \in C(X) \arrow[r, "C(f)"] & C(Y).
\end{tikzcd}
\tag{4.51}
$$

As the top row is continuous, and the columns are homeomorphisms with open subsets of $C(X),C(Y)$, we see that $C(f)$ is continuous in an open neighbourhood of x' in $C(X)$. As this holds for all x', $C(f)$ is continuous.

If $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is simple then $f_{ij}:U_{ij}\to V_j$ is simple near $r_i^{-1}(0)$ for all i,j, so $C(f_{ij})$ maps $C_k(U_{ij})\to C_k(V_j)$ near $r_{(k,i)}^{-1}(0)$ for all $k=0,1,\ldots,$ and hence $C(f)$ maps $C_k(X)\to C_k(Y)$ for all $k=0,1,\ldots.$

For $(k,i)\in\mathbb{N}\times I$ and $(l,j)\in\mathbb{N}\times J$, with $\boldsymbol{f}_{ij}=(U_{ij},f_{ij},\hat f_{ij})$, define

$$
\begin{aligned}
&U_{(k,i)(l,j)}=C_k(U_{ij})\cap C(f_{ij})^{-1}(C_l(V_j))\subseteq U_{(k,i)}=C_k(U_i),\\
&f_{(k,i)(l,j)}=C(f_{ij})|_{U_{(k,i)(l,j)}}:U_{(k,i)(l,j)}\longrightarrow V_{(l,j)}=C_l(V_j),\\
&\hat f_{(k,i)(l,j)}=\Pi_k^*(\hat f_{ij})|_{U_{(k,i)(l,j)}}:D_{(k,i)}|_{U_{(k,i)(l,j)}}=\Pi_k|^*_{U_{(k,i)(l,j)}}(D_i)\\
&\quad\longrightarrow\Pi_k|^*_{U_{(k,i)(l,j)}}\circ f_{ij}^*(E_j)=C_k(f_{ij})|^*_{U_{(k,i)(l,j)}}\circ\Pi_k^*(E_j)=f^*_{(k,i)(l,j)}(E_{(l,j)}).
\end{aligned}
$$

Then we have a 1-morphism of m-Kuranishi neighbourhoods

$$
\begin{aligned}
\boldsymbol{f}_{(k,i)(l,j)}=(U_{(k,i)(l,j)},f_{(k,i)(l,j)},\hat f_{(k,i)(l,j)}):(U_{(k,i)},D_{(k,i)},r_{(k,i)},\chi_{(k,i)})&\\
\longrightarrow(V_{(l,j)},E_{(l,j)},s_{(l,j)},\psi_{(l,j)})&
\end{aligned}
\tag{4.52}
$$

over $C(f):C(X)\to C(Y)$ and $S=\operatorname{Im}\chi_{(k,i)}\cap C(f)^{-1}(\operatorname{Im}\psi_{(l,j)})$. Here Definition 4.2(a)–(c) for $\boldsymbol{f}_{(k,i)(l,j)}$ are immediate, (d) follows by applying Π_k^* to (d) for $\boldsymbol{f}_{ij}$ and using Theorem 3.28(i), and (e) holds by (4.51).

Let $i,i'\in I$ and $j,j'\in J$, and choose representatives $(\acute U^j_{ii'},\hat F^j_{ii'})$, $(\acute U^{jj'}_i,\hat F^{jj'}_i)$ for $\boldsymbol{F}^j_{ii'}=[\acute U^j_{ii'},\hat F^j_{ii'}]$, $\boldsymbol{F}^{jj'}_i=[\acute U^{jj'}_i,\hat F^{jj'}_i]$ in $\boldsymbol{f}$. For $k,l\in\mathbb{N}$, define

$$
\begin{aligned}
\acute U^{(l,j)}_{(k,i)(k,i')}&=C_k(\acute U^j_{ii'})\cap C(f_{i'j}\circ\tau_{ii'})^{-1}(C_l(V_j)),\\
\acute U^{(l,j)(l,j')}_{(k,i)}&=C_k(\acute U^{jj'}_i)\cap C(\upsilon_{jj'}\circ f_{ij})^{-1}(C_l(V_{j'})).
\end{aligned}
\tag{4.53}
$$

As for (4.45), define morphisms

$$
\begin{aligned}
&\hat F^{(l,j)}_{(k,i)(k,i')}=\Pi_k^\diamond(\hat F^j_{ii'})|_{\acute U^{(l,j)}_{(k,i)(k,i')}}:D_{(k,i)}|_{\acute U^{(l,j)}_{(k,i)(k,i')}}=\Pi_k^*(D_i)|_{\acute U^{(l,j)}_{(k,i)(k,i')}}\\
&\quad\longrightarrow\mathcal{T}_{f_{(k,i')(l,j)}\circ\tau_{(k,i)(k,i')}}V_{(l,j)}|_{\acute U^{(l,j)}_{(k,i)(k,i')}}=\mathcal{T}_{C(f_{i'j}\circ\tau_{ii'})|_{\acute U^{(l,j)}_{(k,i)(k,i')}}}C_l(V_j),
\end{aligned}
\tag{4.54}
$$

$$
\begin{aligned}
&\hat F^{(l,j)(l,j')}_{(k,i)}=\Pi_k^\diamond(\hat F^{jj'}_i)|_{\acute U^{(l,j)(l,j')}_{(k,i)}}:D_{(k,i)}|_{\acute U^{(l,j)(l,j')}_{(k,i)}}=\Pi_k^*(D_i)|_{\acute U^{(l,j)(l,j')}_{(k,i)}}\\
&\quad\longrightarrow\mathcal{T}_{\upsilon_{(l,j)(l,j')}\circ f_{(k,i)(l,j)}}V_{(l,j')}|_{\acute U^{(l,j)(l,j')}_{(k,i)}}=\mathcal{T}_{C(\upsilon_{jj'}\circ f_{ij})|_{\acute U^{(l,j)(l,j')}_{(k,i)}}}C_l(V_{j'}),
\end{aligned}
\tag{4.55}
$$

where $\Pi_k^\diamond(\hat F^j_{ii'}),\Pi_k^\diamond(\hat F^{jj'}_i)$ are as in §3.4.3.

Now define 2-morphisms of m-Kuranishi neighbourhoods

$$
\begin{aligned}
\boldsymbol{F}^{(l,j)}_{(k,i)(k,i')}&=\bigl[\acute U^{(l,j)}_{(k,i)(k,i')},\hat F^{(l,j)}_{(k,i)(k,i')}\bigr]:\boldsymbol{f}_{(k,i')(l,j)}\circ\mathrm{T}_{(k,i)(k,i')}\Longrightarrow\boldsymbol{f}_{(k,i)(l,j)},\\
\boldsymbol{F}^{(l,j)(l,j')}_{(k,i)}&=\bigl[\acute U^{(l,j)(l,j')}_{(k,i)},\hat F^{(l,j)(l,j')}_{(k,i)}\bigr]:\Upsilon_{(l,j)(l,j')}\circ\boldsymbol{f}_{(k,i)(l,j)}\Longrightarrow\boldsymbol{f}_{(k,i)(l,j')}.
\end{aligned}
$$

Definition 4.3(a),(b) for $\boldsymbol{F}_{(k,i)(k,i')}^{(l,j)}, \boldsymbol{F}_{(k,i)}^{(l,j)(l,j')}$ follow from Definition 4.3(a),(b) for $\boldsymbol{F}_{ii'}^{j}, \boldsymbol{F}_{i}^{jj'}$, as for (4.46)–(4.47). The equivalences $\sim$ on pairs $(\acute{U}_{ii'}^{j}, \hat{F}_{ii'}^{j})$, $(\acute{U}_{i}^{jj'}, \hat{F}_{i}^{jj'})$ lift to $\sim$ on pairs $(\acute{U}_{(k,i)(k,i')}^{(l,j)}, \hat{F}_{(k,i)(k,i')}^{(l,j)})$, $(\acute{U}_{(k,i)}^{(l,j)(l,j')}, \hat{F}_{(k,i)}^{(l,j)(l,j')})$ by Theorem 3.28(ii), so $\boldsymbol{F}_{(k,i)(k,i')}^{(l,j)}, \boldsymbol{F}_{(k,i)}^{(l,j)(l,j')}$ depend only on $\boldsymbol{F}_{ii'}^{j}, \boldsymbol{F}_{i}^{jj'}$.

If $k \neq k'$ and $l \neq l'$ we define

$$\begin{aligned}\boldsymbol{F}_{(k,i)(k',i')}^{(l,j)} &= [\emptyset, 0] : \boldsymbol{f}_{(k',i')(l,j)} \circ \mathrm{T}_{(k,i)(k',i')} \Longrightarrow \boldsymbol{f}_{(k,i)(l,j)},\\ \boldsymbol{F}_{(k,i)}^{(l,j)(l',j')} &= [\emptyset, 0] : \Upsilon_{(l,j)(l',j')} \circ \boldsymbol{f}_{(k,i)(l,j)} \Longrightarrow \boldsymbol{f}_{(k,i)(l',j')}.\end{aligned}$$

This makes sense as $\mathrm{T}_{(k,i)(k',i')}, \Upsilon_{(l,j)(l',j')}$ are trivial, since

$$\operatorname{Im}\chi_{(k,i)} \cap \operatorname{Im}\chi_{(k',i')} = \operatorname{Im}\psi_{(l,j)} \cap \operatorname{Im}\psi_{(l',j')} = \emptyset$$

as $C_k(X) \cap C_{k'}(X) = \emptyset$, $C_l(Y) \cap C_{l'}(Y) = \emptyset$.

We have now defined all the data in $C(\boldsymbol{f})$ in (4.50), and verified Definition 4.17(a)–(d) for $C(\boldsymbol{f})$. We deduce (e)–(h) from Definition 4.17(e)–(h) for $\boldsymbol{f}$ by pulling back by $\Pi_k : C_k(V_i) \to V_i$ using Theorems 3.27–3.28. This proves $C(\boldsymbol{f}) : C(\boldsymbol{X}) \to C(\boldsymbol{Y})$ is a 1-morphism in $\mathbf{m\dot{\check{K}}ur^c}$.

If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is simple (that is, a 1-morphism in $\mathbf{m\dot{K}ur^c_{si}}$) then $C(f)$ maps $C_k(X) \to C_k(Y)$ for $k = 0, 1, \ldots$. Also as $f_{ij} : U_{ij} \to V_j$ is simple near $r_i^{-1}(0)$, $C(f_{ij}) : C(U_{ij}) \to C(V_j)$ is simple near $r_{(k,i)}^{-1}(0)$ by Assumption 3.22(i), so $f_{(k,i)(l,j)}$ and $\boldsymbol{f}_{(k,i)(l,j)}$ in (4.52) are simple. Therefore $C(\boldsymbol{f}) : C(\boldsymbol{X}) \to C(\boldsymbol{Y})$ is simple and decomposes as $C(\boldsymbol{f}) = \coprod_{k=0}^{\infty} C_k(\boldsymbol{f})$ for $C_k(\boldsymbol{f}) : C_k(\boldsymbol{X}) \to C_k(\boldsymbol{Y})$ in $\mathbf{m\dot{K}ur^c_{si}}$.

Now let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism in $\mathbf{m\dot{K}ur^c}$. Use the notation above for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{f}, C(\boldsymbol{X}), C(\boldsymbol{Y}), C(\boldsymbol{f})$, and the obvious extensions to $\boldsymbol{g}, C(\boldsymbol{g})$, and write $\boldsymbol{\eta} = \big(\boldsymbol{\eta}_{ij,\, i\in I,\, j\in J}\big)$. For $i \in I$ and $j \in J$, choose a representative $(\acute{U}_{ij}, \hat{\eta}_{ij})$ for $\boldsymbol{\eta}_{ij} = [\acute{U}_{ij}, \hat{\eta}_{ij}] : \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij}$. Let $k, l \in \mathbb{N}$. As in (4.53)–(4.55), define

$$\begin{aligned}\acute{U}_{(k,i)(l,j)} &= C_k(\acute{U}_{ij}) \cap C(f_{ij})^{-1}(C_l(V_j)) \qquad\qquad \text{and}\\ \hat{\eta}_{(k,i)(l,j)} &= \Pi_k^{\diamond}(\hat{\eta}_{ij})|_{\acute{U}_{(k,i)(l,j)}} : D_{(k,i)}|_{\acute{U}_{(k,i)(l,j)}} = \Pi_k|^*_{\acute{U}_{(k,i)(l,j)}}(D_i)\\ &\qquad \longrightarrow \mathcal{T}_{f_{(k,i')(l,j)}} V_{(l,j)}|_{\acute{U}_{(k,i)(l,j)}} = \mathcal{T}_{C(f_{ij})|_{\acute{U}_{(k,i)(l,j)}}} C_l(V_j),\end{aligned}$$

where $\Pi_k^{\diamond}(\hat{\eta}_{ij})$ is as in §3.4.3. The same proof as for $\boldsymbol{F}_{(k,i)(k,i')}^{(l,j)}, \boldsymbol{F}_{(k,i)}^{(l,j)(l,j')}$ shows

$$\boldsymbol{\eta}_{(k,i)(l,j)} = [\acute{U}_{(k,i)(l,j)}, \hat{\eta}_{(k,i)(l,j)}] : \boldsymbol{f}_{(k,i)(l,j)} \Longrightarrow \boldsymbol{g}_{(k,i)(l,j)}$$

is a 2-morphism of m-Kuranishi neighbourhoods, and is independent of the choice of $(\acute{U}_{ij}, \hat{\eta}_{ij})$. Define

$$C(\boldsymbol{\eta}) = \big(\boldsymbol{\eta}_{(k,i)(l,j),\ (k,i)\in\mathbb{N}\times I,\ (l,j)\in\mathbb{N}\times J}\big) : C(\boldsymbol{f}) \Longrightarrow C(\boldsymbol{g}).$$

We can deduce Definition 4.18(a),(b) for $C(\boldsymbol{\eta})$ from Definition 4.18(a),(b) for $\boldsymbol{\eta}$, by pulling back by $\Pi_k : C_k(V_i) \to V_i$ using Theorems 3.27–3.28. Hence $C(\boldsymbol{\eta}) : C(\boldsymbol{f}) \Rightarrow C(\boldsymbol{g})$ is a 2-morphism in $\mathbf{m\check{\dot{K}}ur^c}$.

Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms in $\mathbf{m\dot{K}ur^c}$, with notation (4.6)–(4.9). Definition 4.20 defines the composition $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ in $\mathbf{m\dot{K}ur^c}$, by making an arbitrary choice, with 1-morphisms $\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}} : \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \Rightarrow (\boldsymbol{g} \circ \boldsymbol{f})_{ik}$ in (4.24) making (4.15)–(4.17) commute. The constructions above now give $C(\boldsymbol{f}) : C(\boldsymbol{X}) \to C(\boldsymbol{Y})$ and $C(\boldsymbol{g}) : C(\boldsymbol{Y}) \to C(\boldsymbol{Z})$ and $C(\boldsymbol{g} \circ \boldsymbol{f}) : C(\boldsymbol{X}) \to C(\boldsymbol{Z})$ in $\mathbf{m\check{\dot{K}}ur^c}$. Definition 4.20 also defines the composition $C(\boldsymbol{g}) \circ C(\boldsymbol{f}) : C(\boldsymbol{X}) \to C(\boldsymbol{Z})$ in $\mathbf{m\check{\dot{K}}ur^c}$, by making an arbitrary choice.

Since the choices in $\boldsymbol{g} \circ \boldsymbol{f}$ and $C(\boldsymbol{g}) \circ C(\boldsymbol{f})$ may not be consistent, we need not have $C(\boldsymbol{g}) \circ C(\boldsymbol{f}) = C(\boldsymbol{g} \circ \boldsymbol{f})$. However, by applying the corner functor to the 2-morphisms $\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}}$ as for $\Lambda_{hij}, \boldsymbol{F}_{ii'}^{j}, \ldots$ above, we can show that $C(\boldsymbol{g} \circ \boldsymbol{f})$ is one of the possible choices for $C(\boldsymbol{g}) \circ C(\boldsymbol{f})$. Hence Proposition 4.19(b) gives a canonical 2-morphism $C_{\boldsymbol{g},\boldsymbol{f}} : C(\boldsymbol{g}) \circ C(\boldsymbol{f}) \Rightarrow C(\boldsymbol{g} \circ \boldsymbol{f})$ in $\mathbf{m\check{\dot{K}}ur^c}$.

For any $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur^c}$ we can show from the definitions that $C(\mathbf{id}_X) = \mathbf{id}_{C(\boldsymbol{X})}$. Define a 2-morphism $C_{\boldsymbol{X}} = \mathbf{id}_{\mathbf{id}_{C(\boldsymbol{X})}} : C(\mathbf{id}_{\boldsymbol{X}}) \Rightarrow \mathbf{id}_{C(\boldsymbol{X})}$ in $\mathbf{m\check{\dot{K}}ur^c}$. This defines all the data of a weak 2-functor $C : \mathbf{m\dot{K}ur^c} \to \mathbf{m\check{\dot{K}}ur^c}$, as in §A.3. It is easy to check that the weak 2-functor axioms hold.

As above, if $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ lies in $\mathbf{m\dot{K}ur^c_{si}}$ then $C(\boldsymbol{f}) = \coprod_{k=0}^{\infty} C_k(\boldsymbol{f})$ for $C_k(\boldsymbol{f}) : C_k(\boldsymbol{X}) \to C_k(\boldsymbol{Y})$ 1-morphisms in $\mathbf{m\dot{K}ur^c_{si}}$. Hence $C|_{\mathbf{m\dot{K}ur^c_{si}}}$ decomposes as $C|_{\mathbf{m\dot{K}ur^c_{si}}} = \coprod_{k=0}^{\infty} C_k$ where $C_k : \mathbf{m\dot{K}ur^c_{si}} \to \mathbf{m\dot{K}ur^c_{si}}$ is a weak 2-functor. Let the *boundary* 2*-functor* be $\partial = C_1 : \mathbf{m\dot{K}ur^c_{si}} \to \mathbf{m\dot{K}ur^c_{si}}$.

If for some discrete property $\boldsymbol{P}$ of morphisms in $\mathbf{\dot{M}an^c}$ the corner functor $C : \mathbf{\dot{M}an^c} \to \mathbf{\check{\dot{M}}an^c}$ in Assumption 3.22(g) maps to the subcategory $\mathbf{\check{\dot{M}}an^c_{\boldsymbol{P}}}$ of $\mathbf{\check{\dot{M}}an^c}$ whose morphisms are $\boldsymbol{P}$, then in the definition of $C(\boldsymbol{f})$ above the 1-morphisms $\boldsymbol{f}_{(k,i)(l,j)}$ are $\boldsymbol{P}$, so that $C : \mathbf{m\dot{K}ur^c} \to \mathbf{m\check{\dot{K}}ur^c}$ maps to the 2-subcategory $\mathbf{m\check{\dot{K}}ur^c_{\boldsymbol{P}}}$ of $\mathbf{m\check{\dot{K}}ur^c}$ whose 1-morphisms are $\boldsymbol{P}$.

We summarize Definition 4.43 in:

Theorem 4.44. *We have defined a weak* 2*-functor* $C : \mathbf{m\dot{K}ur^c} \to \mathbf{m\check{\dot{K}}ur^c}$ *called the* ***corner*** **2*-functor****. It acts on objects* $\boldsymbol{X}$ *in* $\mathbf{m\dot{K}ur^c}$ *by* $C(\boldsymbol{X}) = \coprod_{k=0}^{\infty} C_k(\boldsymbol{X})$*. If* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is simple then* $C(\boldsymbol{f}) : C(\boldsymbol{X}) \to C(\boldsymbol{Y})$ *is simple and maps* $C_k(\boldsymbol{X}) \to C_k(\boldsymbol{Y})$ *for* $k = 0, 1, \ldots$*. Thus* $C|_{\mathbf{m\dot{K}ur^c_{si}}}$ *decomposes as* $C|_{\mathbf{m\dot{K}ur^c_{si}}} = \coprod_{k=0}^{\infty} C_k$*, where* $C_k : \mathbf{m\dot{K}ur^c_{si}} \to \mathbf{m\dot{K}ur^c_{si}}$ *is a weak* 2*-functor acting on objects by* $\boldsymbol{X} \mapsto C_k(\boldsymbol{X})$*, for* $C_k(\boldsymbol{X})$ *as in* §4.6.1*. We also write* $\partial = C_1 : \mathbf{m\dot{K}ur^c_{si}} \to \mathbf{m\dot{K}ur^c_{si}}$*, and call it the* ***boundary*** **2*-functor****.*

If for some discrete property $\boldsymbol{P}$ *of morphisms in* $\mathbf{\dot{M}an^c}$ *the corner functor* $C : \mathbf{\dot{M}an^c} \to \mathbf{\check{\dot{M}}an^c}$ *maps to the subcategory* $\mathbf{\check{\dot{M}}an^c_{\boldsymbol{P}}}$ *of* $\mathbf{\check{\dot{M}}an^c}$ *whose morphisms are* $\boldsymbol{P}$*, then* $C : \mathbf{m\dot{K}ur^c} \to \mathbf{m\check{\dot{K}}ur^c}$ *maps to the* 2*-subcategory* $\mathbf{m\check{\dot{K}}ur^c_{\boldsymbol{P}}}$ *of* $\mathbf{m\check{\dot{K}}ur^c}$ *whose* 1*-morphisms are* $\boldsymbol{P}$*.*

4.6.3 Examples, and easy consequences

Example 4.45. Example 3.24(a)–(h) give examples of data $\dot{\mathbf{Man}}^{\mathbf{c}}$, simple maps, corner functors $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$, etc. satisfying Assumption 3.22, where the corner functors are written either C as in Definition 2.9 or C' as in Definition 2.11. Definitions 4.29 and 4.37 give our notation for the corresponding 2-categories of m-Kuranishi spaces $\mathbf{mKur^c}, \mathbf{mKur^c_{st}}, \ldots$ from §4.3 and §4.5. Applying the constructions of §4.6.1–§4.6.2 to this data $\dot{\mathbf{Man}}^{\mathbf{c}}, \ldots$ gives $C_k(\boldsymbol{X}), \partial\boldsymbol{X}$ and 1-morphisms $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$, $\boldsymbol{i_X} : \partial\boldsymbol{X} \to \boldsymbol{X}$ for $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur^c}$, and corner 2-functors $C : \mathbf{m\dot{K}ur^c} \to \mathbf{m\check{\dot{K}}ur^c}$.

We write the corner 2-functors coming from Example 3.24(a)–(h) as:

$$
\begin{aligned}
&C : \mathbf{mKur^c} \longrightarrow \mathbf{m\check{K}ur^c_{in}} \subset \mathbf{m\check{K}ur^c}, && C' : \mathbf{mKur^c} \longrightarrow \mathbf{m\check{K}ur^c},\\
&C : \mathbf{mKur^c_{st}} \longrightarrow \mathbf{m\check{K}ur^c_{st,in}} \subset \mathbf{m\check{K}ur^c_{st}}, && C' : \mathbf{mKur^c_{st}} \longrightarrow \mathbf{m\check{K}ur^c_{st}},\\
&C : \mathbf{mKur^{ac}} \longrightarrow \mathbf{m\check{K}ur^{ac}_{in}} \subset \mathbf{m\check{K}ur^{ac}}, && C' : \mathbf{mKur^{ac}} \longrightarrow \mathbf{m\check{K}ur^{ac}},\\
&C : \mathbf{mKur^{ac}_{st}} \longrightarrow \mathbf{m\check{K}ur^{ac}_{st,in}} \subset \mathbf{m\check{K}ur^{ac}_{st}}, && C' : \mathbf{mKur^{ac}_{st}} \longrightarrow \mathbf{m\check{K}ur^{ac}_{st}},\\
&C : \mathbf{mKur^{c,ac}} \longrightarrow \mathbf{m\check{K}ur^{c,ac}_{in}} \subset \mathbf{m\check{K}ur^{c,ac}}, && C' : \mathbf{mKur^{c,ac}} \longrightarrow \mathbf{m\check{K}ur^{c,ac}},\\
&C : \mathbf{mKur^{c,ac}_{st}} \longrightarrow \mathbf{m\check{K}ur^{c,ac}_{st,in}} \subset \mathbf{m\check{K}ur^{c,ac}_{st}}, && C' : \mathbf{mKur^{c,ac}_{st}} \longrightarrow \mathbf{m\check{K}ur^{c,ac}_{st}},\\
&C : \mathbf{mKur^{gc}} \longrightarrow \mathbf{m\check{K}ur^{gc}_{in}} \subset \mathbf{m\check{K}ur^{gc}}.
\end{aligned}
\tag{4.56}
$$

As in Example 3.24(h) and §2.4.1, there is no second corner functor C' on $\mathbf{Man^{gc}}$, and so no 2-functor C' on $\mathbf{mKur^{gc}}$. The functors C map to interior morphisms in $\check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}, \ldots$, where interior is a discrete property as in §3.3.6, so the last part of Theorem 4.44 implies that the corresponding 2-functors C map to interior 1-morphisms in $\mathbf{m\check{K}ur^c}$.

Remark 4.41(a) explains that the notions of boundary $\partial\boldsymbol{X}$, k-corners $C_k(\boldsymbol{X})$, and 1-morphisms $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ in $\mathbf{mKur^c}, \mathbf{mKur^c_{st}}, \mathbf{mKur^{ac}}, \mathbf{mKur^{ac}_{st}}$, $\mathbf{mKur^{c,ac}}$ and $\mathbf{mKur^{c,ac}_{st}}$ are independent of whether we choose C or C' in Assumption 3.22. So in each of the first six lines of (4.56), the 2-functors C and C' agree on objects, but differ on 1- and 2-morphisms.

As in Proposition 2.10(a),(b), all of the functors $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$ in Example 3.24(a)–(h) (though not the functors C') have the property that a morphism $f : X \to Y$ is interior if and only if $C(f) : C(X) \to C(Y)$ maps $C_0(X) \to C_0(Y)$, and f is b-normal if and only if $C(f)$ maps $C_k(X) \to \coprod_{l=0}^{k} C_l(Y)$ for all $k = 0, \ldots, \dim X$, where interior and b-normal are discrete properties. Applying this to the definition of $C(\boldsymbol{f})$ in Definition 4.43, we easily deduce:

Proposition 4.46. *For all of the* 2*-functors C in* (4.56) *(though not the* 2*-functors C'), a* 1*-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is interior (or b-normal) if and only if $C(\boldsymbol{f})$ maps $C_0(\boldsymbol{X}) \to C_0(\boldsymbol{Y})$ (or $C(\boldsymbol{f})$ maps $C_k(\boldsymbol{X}) \to \coprod_{l=0}^{k} C_l(\boldsymbol{Y})$ for all $k = 0, 1, \ldots,$ respectively).*

The boundary $\partial\boldsymbol{X}$ and k-corners $C_k(\boldsymbol{X})$ of $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur^c}$ depend, up to equivalence in $\mathbf{m\check{\dot{K}}ur^c}$, only on $\boldsymbol{X}$ up to equivalence in $\mathbf{m\dot{K}ur^c}$. In applications m-Kuranishi spaces with corners $\boldsymbol{X}$ are usually only natural up to equivalence in $\mathbf{m\dot{K}ur^c}$, so this is important for boundaries and corners to be well behaved.

Proposition 4.47. *Let $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ be an equivalence in $\mathbf{m\dot{K}ur^c}$. Then $\boldsymbol{f}$ is simple by Proposition 4.36(c), and $C_k(\boldsymbol{f}):C_k(\boldsymbol{X})\rightarrow C_k(\boldsymbol{Y})$ for $k=0,1,\ldots$ and $\partial\boldsymbol{f}:\partial\boldsymbol{X}\rightarrow\partial\boldsymbol{Y}$ are also equivalences in $\mathbf{m\dot{K}ur^c}$.*

Proof. As $\boldsymbol{f}$ is an equivalence there exist a 1-morphism $\boldsymbol{g}:\boldsymbol{Y}\rightarrow\boldsymbol{X}$ and 2-morphisms $\boldsymbol{\eta}:\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$, $\boldsymbol{\zeta}:\boldsymbol{f}\circ\boldsymbol{g}\Rightarrow\mathbf{id}_{\boldsymbol{Y}}$ in $\mathbf{m\dot{K}ur^c}$, where $\boldsymbol{g}$ is also an equivalence, and so simple. For $k\geqslant 0$ we can apply the 2-functor $C_k:\mathbf{m\dot{K}ur^c_{si}}\rightarrow\mathbf{m\dot{K}ur^c_{si}}$ to $\boldsymbol{f},\boldsymbol{g},\boldsymbol{\eta},\boldsymbol{\zeta}$. The compositions of 2-morphisms

$$C_k(\boldsymbol{g})\circ C_k(\boldsymbol{f})\xRightarrow{(C_k)_{\boldsymbol{g},\boldsymbol{f}}}C_k(\boldsymbol{g}\circ\boldsymbol{f})\xRightarrow{C_k(\boldsymbol{\eta})}C_k(\mathbf{id}_X)=\!=\!=\!=\mathbf{id}_{C_k(\boldsymbol{X})},$$

$$C_k(\boldsymbol{f})\circ C_k(\boldsymbol{g})\xRightarrow{(C_k)_{\boldsymbol{f},\boldsymbol{g}}}C_k(\boldsymbol{f}\circ\boldsymbol{g})\xRightarrow{C_k(\boldsymbol{\zeta})}C_k(\mathbf{id}_Y)=\!=\!=\!=\mathbf{id}_{C_k(\boldsymbol{Y})},$$

show $C_k(\boldsymbol{f})$ is an equivalence, so putting $k=1$ shows $\partial\boldsymbol{f}$ is an equivalence. □

Definition 4.48. As in Definition 4.29 we write $\mathbf{mKur^c}$ for the 2-category of m-Kuranishi spaces with corners associated to $\mathbf{\dot{M}an^c}=\mathbf{Man^c}$. An object $\boldsymbol{X}$ in $\mathbf{mKur^c}$ is called an *m-Kuranishi space with boundary* if $\partial(\partial\boldsymbol{X})=\emptyset$. Write $\mathbf{mKur^b}$ for the full 2-subcategory of m-Kuranishi spaces with boundary in $\mathbf{mKur^c}$, and write $\mathbf{mKur^b_{si}}\subseteq\mathbf{mKur^b_{in}}\subseteq\mathbf{mKur^b}$ for the 2-subcategories of $\mathbf{mKur^b}$ with simple and interior 1-morphisms.

If $V\in\mathbf{Man^c}$ then $\partial(\partial V)=\emptyset$ if and only if $C_k(V)=\emptyset$ for all $k>1$. (For any $\mathbf{\dot{M}an^c}$ satisfying Assumption 3.22, surjectivity of $I_{k,l}$ in (f) implies that the same holds in $\mathbf{\dot{M}an^c}$). Using this we can show that $\boldsymbol{X}\in\mathbf{mKur^c}$ is an m-Kuranishi space with boundary if and only if $C_k(\boldsymbol{X})=\emptyset$ for all $k>1$.

4.7 M-Kuranishi neighbourhoods on m-Kuranishi spaces

At the beginning of differential geometry, one defines manifolds X and smooth maps $f:X\rightarrow Y$ in terms of an atlas $\bigl\{(V_i,\psi_i):i\in I\bigr\}$ of charts on X, and transition functions $\psi_{ij}=\psi_j^{-1}\circ\psi_i|_{\psi_i^{-1}(\operatorname{Im}\psi_j)}$ between charts $(V_i,\psi_i),(V_j,\psi_j)$. However, one quickly comes to regard actually choosing an atlas on X or working explicitly with atlases as unnatural and inelegant, so we generally suppress them, working with 'local coordinates' on X if we really need to reduce things to $\mathbb{R}^n$.

We now wish to advocate a similar philosophy for working with m-Kuranishi spaces $\boldsymbol{X}=(X,\mathcal{K})$, in which, like atlases, actually choosing or working explicitly with m-Kuranishi structures $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},\,\Phi_{ij,\;i,j\in I},\,\Lambda_{ijk,\;i,j,k\in I}\bigr)$ is regarded as inelegant and to be avoided where possible, and $\boldsymbol{X}$ is understood to exist as a geometric space independently of any choices of $I,(V_i,E_i,s_i,\psi_i),\ldots.$ Our analogue of 'local coordinates' will be 'm-Kuranishi neighbourhoods on m-Kuranishi spaces'.

4.7.1 Defining m-Kuranishi neighbourhoods on m-Kuranishi spaces

Definition 4.49. Suppose $\boldsymbol{X} = (X, \mathcal{K})$ is an m-Kuranishi space, where $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\, i,j\in I}, \Lambda_{ijk,\, i,j,k\in I}\big)$. An *m-Kuranishi neighbourhood on the m-Kuranishi space* $\boldsymbol{X}$ is data (V_a, E_a, s_a, ψ_a), $\Phi_{ai,\, i\in I}$ and $\Lambda_{aij,\, i,j\in I}$, where (V_a, E_a, s_a, ψ_a) is an m-Kuranishi neighbourhood on the topological space X in the sense of Definition 4.1, and $\Phi_{ai} : (V_a, E_a, s_a, \psi_a) \to (V_i, E_i, s_i, \psi_i)$ is a coordinate change for each $i \in I$ (over $S = \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_i$, as usual) as in Definition 4.10, and $\Lambda_{aij} : \Phi_{ij} \circ \Phi_{ai} \Rightarrow \Phi_{aj}$ is a 2-morphism (over $S = \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, as usual) as in Definition 4.3 for all $i, j \in I$, such that $\Lambda_{aii} = \mathrm{id}_{\Phi_{ai}}$ for all $i \in I$, and as in Definition 4.14(h), for all $i, j, k \in I$ we have

$$\Lambda_{ajk} \odot (\mathrm{id}_{\Phi_{jk}} * \Lambda_{aij}) = \Lambda_{aik} \odot (\Lambda_{ijk} * \mathrm{id}_{\Phi_{ai}}) : \Phi_{jk} \circ \Phi_{ij} \circ \Phi_{ai} \Longrightarrow \Phi_{ak}, \quad (4.57)$$

where (4.57) holds over $S = \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$ by our usual convention.

Here the subscript 'a' in (V_a, E_a, s_a, ψ_a) is just a label used to distinguish m-Kuranishi neighbourhoods, generally not in I. If we omit a we will write '$*$' in place of 'a' in Φ_{ai}, Λ_{aij}, giving $\Phi_{*i} : (V, E, s, \psi) \to (V_i, E_i, s_i, \psi_i)$ and $\Lambda_{*ij} : \Phi_{ij} \circ \Phi_{*i} \Rightarrow \Phi_{*j}$.

We will usually just say (V_a, E_a, s_a, ψ_a) or (V, E, s, ψ) *is an m-Kuranishi neighbourhood on* $\boldsymbol{X}$, leaving the data Φ_{ai}, Λ_{aij} or Φ_{*i}, Λ_{*ij} implicit. We call such a (V, E, s, ψ) a *global* m-Kuranishi neighbourhood on $\boldsymbol{X}$ if $\operatorname{Im}\psi = X$.

Example 4.50. Let $\boldsymbol{X} = (X, \mathcal{K})$ be as in Definition 4.49, and let $a \in I$. Then (V_a, E_a, s_a, ψ_a) is an m-Kuranishi neighbourhood on $\boldsymbol{X}$, with data $\Phi_{ai,\, i\in I}$, $\Lambda_{aij,\, i,j\in I}$ as in $\mathcal{K}$, where (4.57) follows from Definition 4.14(h) for $\boldsymbol{X}$. Thus, all the m-Kuranishi neighbourhoods in $\mathcal{K}$ are m-Kuranishi neighbourhoods on $\boldsymbol{X}$.

Definition 4.51. Using the same notation, suppose $(V_a, E_a, s_a, \psi_a), \Phi_{ai,\, i\in I}$, $\Lambda_{aij,\, i,j\in I}$ and (V_b, E_b, s_b, ψ_b), $\Phi_{bi,\, i\in I}$, $\Lambda_{bij,\, i,j\in I}$ are m-Kuranishi neighbourhoods on $\boldsymbol{X}$, and $S \subseteq \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b$ is open. A *coordinate change from* (V_a, E_a, s_a, ψ_a) *to* (V_b, E_b, s_b, ψ_b) *over* S *on the m-Kuranishi space* $\boldsymbol{X}$ is data Φ_{ab}, $\Lambda_{abi,\, i\in I}$, where $\Phi_{ab} : (V_a, E_a, s_a, \psi_a) \to (V_b, E_b, s_b, \psi_b)$ is a coordinate change over S as in Definition 4.10, and $\Lambda_{abi} : \Phi_{bi} \circ \Phi_{ab} \Rightarrow \Phi_{ai}$ is a 2-morphism over $S \cap \operatorname{Im}\psi_i$ as in Definition 4.3 for each $i \in I$, such that for $i, j \in I$ we have

$$\Lambda_{aij} \odot (\mathrm{id}_{\Phi_{ij}} * \Lambda_{abi}) = \Lambda_{abj} \odot (\Lambda_{bij} * \mathrm{id}_{\Phi_{ab}}) : \Phi_{ij} \circ \Phi_{bi} \circ \Phi_{ab} \Longrightarrow \Phi_{aj}, \quad (4.58)$$

where (4.58) holds over $S \cap \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$.

We will usually just say that $\Phi_{ab} : (V_a, E_a, s_a, \psi_a) \to (V_b, E_b, s_b, \psi_b)$ *is a coordinate change over* S *on* $\boldsymbol{X}$, leaving the data $\Lambda_{abi,\, i\in I}$ implicit. If we do not specify S, we mean that S is as large as possible, that is, $S = \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b$.

Suppose $\Phi_{ab} : (V_a, E_a, s_a, \psi_a) \to (V_b, E_b, s_b, \psi_b)$, $\Lambda_{abi,\, i\in I}$ and $\Phi_{bc} : (V_b, E_b, s_b, \psi_b) \to (V_c, E_c, s_c, \psi_c)$, $\Lambda_{bci,\, i\in I}$ are such coordinate changes over $S \subseteq \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b \cap \operatorname{Im}\psi_c$. Define $\Phi_{ac} = \Phi_{bc} \circ \Phi_{ab} : (V_a, E_a, s_a, \psi_a) \to (V_c, E_c, s_c, \psi_c)$ and $\Lambda_{aci} = \Lambda_{abi} \odot (\Lambda_{bci} * \mathrm{id}_{\Phi_{ab}}) : \Phi_{ci} \circ \Phi_{ac} \Rightarrow \Phi_{ai}$ for all $i \in I$. It is easy to show that $\Phi_{ac} = \Phi_{bc} \circ \Phi_{ab}$, $\Lambda_{aci,\, i\in I}$ is a coordinate change from (V_a, E_a, s_a, ψ_a) to (V_c, E_c, s_c, ψ_c) over S on $\boldsymbol{X}$. We call this *composition of coordinate changes.*

Example 4.52. Let $\boldsymbol{X} = (X, \mathcal{K})$ be as in Definition 4.49, and let $a, b \in I$. Then (V_a, E_a, s_a, ψ_a) and (V_b, E_b, s_b, ψ_b) are m-Kuranishi neighbourhoods on $\boldsymbol{X}$ as in Example 4.50. The coordinate change $\Phi_{ab} : (V_a, E_a, s_a, \psi_a) \to (V_b, E_b, s_b, \psi_b)$ in $\mathcal{K}$ is a coordinate change over $\operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b$ on $\boldsymbol{X}$, with data $\Lambda_{abi,\, i\in I}$ as in $\mathcal{K}$.

Example 4.53. Let $\boldsymbol{X}, \boldsymbol{Y}$ be m-Kuranishi spaces in $\mathbf{m\dot{K}ur}$, and (U, D, r, χ) and (V, E, s, ψ) be m-Kuranishi neighbourhoods on $\boldsymbol{X}, \boldsymbol{Y}$. Example 4.31 defined the product m-Kuranishi space $\boldsymbol{X} \times \boldsymbol{Y}$. It is easy to construct a product m-Kuranishi neighbourhood $(U \times V, \pi_U^*(D) \oplus \pi_V^*(E), \pi_U^*(r) \oplus \pi_V^*(s), \chi \times \psi)$ on $\boldsymbol{X} \times \boldsymbol{Y}$.

Definition 4.54. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism of m-Kuranishi spaces, and use notation (4.6)–(4.7) for $\boldsymbol{X}, \boldsymbol{Y}$, and (4.9) for $\boldsymbol{f}$. Suppose (U_a, D_a, r_a, χ_a), $\mathrm{T}_{ai,\, i\in I}$, $\mathrm{K}_{aii',\, i,i'\in I}$ is an m-Kuranishi neighbourhood on $\boldsymbol{X}$, and (V_b, E_b, s_b, ψ_b), $\Upsilon_{bj,\, j\in J}$, $\Lambda_{bjj',\, j,j'\in J}$ an m-Kuranishi neighbourhood on $\boldsymbol{Y}$, as in Definition 4.49. Let $S \subseteq \operatorname{Im}\chi_a \cap f^{-1}(\operatorname{Im}\psi_b)$ be open. A 1-*morphism from* (U_a, D_a, r_a, χ_a) *to* (V_b, E_b, s_b, ψ_b) *over* $(S, \boldsymbol{f})$ *on the m-Kuranishi spaces* $\boldsymbol{X}, \boldsymbol{Y}$ is data $\boldsymbol{f}_{ab}$, $\boldsymbol{F}_{ai,\, i\in I}^{bj,\, j\in J}$, where $\boldsymbol{f}_{ab} : (U_a, D_a, r_a, \chi_a) \to (V_b, E_b, s_b, \psi_b)$ is a 1-morphism of m-Kuranishi neighbourhoods over (S, f) in the sense of Definition 4.2, and $\boldsymbol{F}_{ai}^{bj} : \Upsilon_{bj} \circ \boldsymbol{f}_{ab} \Rightarrow \boldsymbol{f}_{ij} \circ \mathrm{T}_{ai}$ is a 2-morphism over $S \cap \operatorname{Im}\chi_i \cap f^{-1}(\operatorname{Im}\psi_j), f$ as in Definition 4.3 for all $i \in I$, $j \in J$, such that for all $i, i' \in I$, $j, j' \in J$ we have

$$\begin{aligned} (\boldsymbol{F}_{ai}^{bj})^{-1} \odot (\boldsymbol{F}_{ii'}^{j} * \mathrm{id}_{\mathrm{T}_{ai}}) &= (\boldsymbol{F}_{ai'}^{bj})^{-1} \odot (\mathrm{id}_{\boldsymbol{f}_{i'j}} * \mathrm{K}_{aii'}) : \\ &\qquad (\boldsymbol{f}_{i'j} \circ \mathrm{T}_{ii'}) \circ \mathrm{T}_{ai} \Longrightarrow \Upsilon_{bj} \circ \boldsymbol{f}_{ab}, \\ \boldsymbol{F}_{ai}^{bj'} \odot (\Lambda_{bjj'} * \mathrm{id}_{\boldsymbol{f}_{ab}}) &= (\boldsymbol{F}_{i}^{jj'} * \mathrm{id}_{\mathrm{T}_{ai}}) \odot (\mathrm{id}_{\Upsilon_{jj'}} * \boldsymbol{F}_{ai}^{bj}) : \\ &\qquad (\Upsilon_{jj'} \circ \Upsilon_{bj}) \circ \boldsymbol{f}_{ab} \Longrightarrow \boldsymbol{f}_{ij'} \circ \mathrm{T}_{ai}. \end{aligned} \tag{4.59}$$

We will usually just say that $\boldsymbol{f}_{ab} : (U_a, D_a, r_a, \chi_a) \to (V_b, E_b, s_b, \psi_b)$ *is a* 1-*morphism of m-Kuranishi neighbourhoods over* $(S, \boldsymbol{f})$ *on* $\boldsymbol{X}, \boldsymbol{Y}$, leaving the data $\boldsymbol{F}_{ai,\, i\in I}^{bj,\, j\in J}$ implicit.

Suppose $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ is another 1-morphism of m-Kuranishi spaces, using notation (4.8) for $\boldsymbol{Z}$, and $(W_c, F_c, t_c, \omega_c)$ is an m-Kuranishi neighbourhood on $\boldsymbol{Z}$, and $T \subseteq \operatorname{Im}\psi_b \cap g^{-1}(\operatorname{Im}\omega_c)$, $S \subseteq \operatorname{Im}\chi_a \cap f^{-1}(T)$ are open, $\boldsymbol{f}_{ab} : (U_a, D_a, r_a, \chi_a) \to (V_b, E_b, s_b, \psi_b)$ is a 1-morphism of m-Kuranishi neighbourhoods over $(S, \boldsymbol{f})$ on $\boldsymbol{X}, \boldsymbol{Y}$, and $\boldsymbol{g}_{bc} : (V_b, E_b, s_b, \psi_b) \to (W_c, F_c, t_c, \omega_c)$ is a 1-morphism of m-Kuranishi neighbourhoods over $(T, \boldsymbol{g})$ on $\boldsymbol{Y}, \boldsymbol{Z}$.

Define $\boldsymbol{h} = \boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$, so that Definition 4.20 gives 2-morphisms

$$\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}} : \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \Longrightarrow \boldsymbol{h}_{ik}$$

for all $i \in I$, $j \in J$ and $k \in K$. Set $\boldsymbol{h}_{ac} = \boldsymbol{g}_{bc} \circ \boldsymbol{f}_{ab} : (U_a, D_a, r_a, \chi_a) \to (W_c, F_c, t_c, \omega_c)$. Using the stack property Theorem 4.13, one can show that for all $i \in I$, $k \in K$ there is a unique 2-morphism $\boldsymbol{H}_{ai}^{ck} : \Phi_{ck} \circ \boldsymbol{h}_{ac} \Rightarrow \boldsymbol{h}_{ik} \circ \mathrm{T}_{ai}$ over $S \cap \operatorname{Im}\chi_i \cap h^{-1}(\operatorname{Im}\omega_k), h$, such that for all $j \in J$ we have

$$\begin{aligned} &\boldsymbol{H}_{ai}^{ck}|_{S\cap\operatorname{Im}\chi_i\cap f^{-1}(\operatorname{Im}\psi_j)\cap h^{-1}(\operatorname{Im}\omega_k)} \\ &\qquad = (\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}} * \mathrm{id}_{\mathrm{T}_{ai}}) \odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \boldsymbol{F}_{ai}^{bj}) \odot (\boldsymbol{G}_{bj}^{ck} * \mathrm{id}_{\boldsymbol{f}_{ab}}). \end{aligned} \tag{4.60}$$

It is then easy to prove that $\boldsymbol{h}_{ac}=\boldsymbol{g}_{bc}\circ\boldsymbol{f}_{ab}$, $\boldsymbol{H}_{ai,\; i\in I}^{ck,\; k\in K}$ is a 1-morphism from (U_a,D_a,r_a,χ_a) to (W_c,F_c,t_c,ω_c) over $(S,\boldsymbol{h})$ on $\boldsymbol{X},\boldsymbol{Z}$. We call this *composition of* 1*-morphisms*.

Example 4.55. Let $\boldsymbol{X}=(X,\mathcal{I}),\boldsymbol{Y}=(Y,\mathcal{J}),\boldsymbol{f}$ be as in Definition 4.54, and let $a\in I$ and $b\in J$. Then (U_a,D_a,r_a,χ_a) in $\mathcal{I}$ and (V_b,E_b,s_b,ψ_b) in $\mathcal{J}$ are m-Kuranishi neighbourhoods on $\boldsymbol{X},\boldsymbol{Y}$ by Example 4.50. The 1-morphism $\boldsymbol{f}_{ab}:(U_a,D_a,r_a,\chi_a)\to(V_b,E_b,s_b,\psi_b)$ in $\boldsymbol{f}$ is a 1-morphism over $(\operatorname{Im}\chi_a\cap f^{-1}(\psi_b),\boldsymbol{f})$, with extra data $\boldsymbol{F}_{ai,\; i\in I}^{bj,\; j\in J}$, where for $\boldsymbol{F}_{ai}^{j},\boldsymbol{F}_{a}^{bj}$ as in $\boldsymbol{f}$ we have

$$\boldsymbol{F}_{ai}^{bj}=(\boldsymbol{F}_{ai}^{j})^{-1}\odot\boldsymbol{F}_{a}^{bj}:\Upsilon_{bj}\circ\boldsymbol{f}_{ab}\Longrightarrow\boldsymbol{f}_{ij}\circ\mathrm{T}_{ai}.$$

The next theorem can be proved using the stack property Theorem 4.13 by very similar methods to Propositions 4.19, 4.22, 4.25, 4.26 and 4.27, so we leave the proof as an exercise for the reader.

Theorem 4.56. **(a)** *Let $\boldsymbol{X}=(X,\mathcal{K})$ be an m-Kuranishi space, where $\mathcal{K}=\big(I,(V_i,E_i,s_i,\psi_i)_{i\in I},\,\Phi_{ij},\,\Lambda_{ijk}\big)$, and (V_a,E_a,s_a,ψ_a), (V_b,E_b,s_b,ψ_b) be m-Kuranishi neighbourhoods on $\boldsymbol{X}$, in the sense of Definition* 4.49, *and $S\subseteq\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b$ be open. Then there exists a coordinate change $\Phi_{ab}:(V_a,E_a,s_a,\psi_a)\to(V_b,E_b,s_b,\psi_b),\Lambda_{abi,\; i\in I}$ over S on $\boldsymbol{X}$, in the sense of Definition* 4.51. *If $\Phi_{ab},\tilde\Phi_{ab}$ are two such coordinate changes, there is a unique* 2*-morphism $\Xi_{ab}:\Phi_{ab}\Rightarrow\tilde\Phi_{ab}$ over S as in Definition* 4.3, *such that for all $i\in I$ we have*

$$\Lambda_{abi}=\tilde\Lambda_{abi}\odot(\mathrm{id}_{\Phi_{bi}}*\Xi_{ab}):\Phi_{bi}\circ\Phi_{ab}\Longrightarrow\Phi_{ai},\tag{4.61}$$

which holds over $S\cap\operatorname{Im}\psi_i$ by our usual convention.

(b) *Let $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ be a* 1*-morphism of m-Kuranishi spaces, and use notation* (4.6), (4.7), (4.9)*. Let (U_a,D_a,r_a,χ_a), (V_b,E_b,s_b,ψ_b) be m-Kuranishi neighbourhoods on $\boldsymbol{X},\boldsymbol{Y}$ respectively in the sense of Definition* 4.49, *and let $S\subseteq\operatorname{Im}\chi_a\cap f^{-1}(\operatorname{Im}\psi_b)$ be open. Then there exists a* 1*-morphism $\boldsymbol{f}_{ab}:(U_a,D_a,r_a,\chi_a)\to(V_b,E_b,s_b,\psi_b)$ of m-Kuranishi neighbourhoods over $(S,\boldsymbol{f})$ on $\boldsymbol{X},\boldsymbol{Y}$, in the sense of Definition* 4.54.

(c) *Let $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$ be* 1*-morphisms of m-Kuranishi spaces and $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a* 2*-morphism, and use notation* (4.6), (4.7), (4.9) *and $\boldsymbol{\eta}=\big(\boldsymbol{\eta}_{ij,\; i\in I,\; j\in J}\big)$. Suppose (U_a,D_a,r_a,χ_a), (V_b,E_b,s_b,ψ_b) are m-Kuranishi neighbourhoods on $\boldsymbol{X},\boldsymbol{Y}$, and $S\subseteq\operatorname{Im}\chi_a\cap f^{-1}(\operatorname{Im}\psi_b)$ is open, and $\boldsymbol{f}_{ab},\boldsymbol{g}_{ab}:(U_a,D_a,r_a,\chi_a)\to(V_b,E_b,s_b,\psi_b)$ are* 1*-morphisms over $(S,\boldsymbol{f}),(S,\boldsymbol{g})$ respectively. Then there is a unique* 2*-morphism $\boldsymbol{\eta}_{ab}:\boldsymbol{f}_{ab}\Rightarrow\boldsymbol{g}_{ab}$ over (S,f) as in Definition* 4.3, *such that the following commutes over $S\cap\operatorname{Im}\chi_i\cap f^{-1}(\operatorname{Im}\psi_j)$ for all $i\in I$ and $j\in J$:*

$$\begin{tikzcd}
\Upsilon_{bj}\circ\boldsymbol{f}_{ab} \arrow[rr, Rightarrow, "\boldsymbol{F}_{ai}^{bj}"] \arrow[d, Rightarrow, "\mathrm{id}_{\Upsilon_{bj}}*\boldsymbol{\eta}_{ab}"'] && \boldsymbol{f}_{ij}\circ\mathrm{T}_{ai} \arrow[d, Rightarrow, "\boldsymbol{\eta}_{ij}*\mathrm{id}_{\mathrm{T}_{ai}}"] \\
\Upsilon_{bj}\circ\boldsymbol{g}_{ab} \arrow[rr, Rightarrow, "\boldsymbol{G}_{ai}^{bj}"] && \boldsymbol{g}_{ij}\circ\mathrm{T}_{ai}.
\end{tikzcd}\tag{4.62}$$

(d) *The unique* 2*-morphisms in* **(c)** *are compatible with vertical and horizontal composition and identities. For example, if* $\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Y}$ *are* 1*-morphisms in* $\mathbf{m\dot{K}ur}$, *and* $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$, $\boldsymbol{\zeta}:\boldsymbol{g}\Rightarrow\boldsymbol{h}$ *are* 2*-morphisms with* $\boldsymbol{\theta}=\boldsymbol{\zeta}\odot\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{h}$, *and* (U_a,D_a,r_a,χ_a), (V_b,E_b,s_b,ψ_b) *are m-Kuranishi neighbourhoods on* $\boldsymbol{X},\boldsymbol{Y}$, *and* $\boldsymbol{f}_{ab},\boldsymbol{g}_{ab},\boldsymbol{h}_{ab}:(U_a,D_a,r_a,\chi_a)\to(V_b,E_b,s_b,\psi_b)$ *are* 1*-morphisms over* $(S,\boldsymbol{f})$, $(S,\boldsymbol{g})$, $(S,\boldsymbol{h})$, *and* $\boldsymbol{\eta}_{ab}:\boldsymbol{f}_{ab}\Rightarrow\boldsymbol{g}_{ab}$, $\boldsymbol{\zeta}_{ab}:\boldsymbol{g}_{ab}\Rightarrow\boldsymbol{h}_{ab}$, $\boldsymbol{\theta}_{ab}:\boldsymbol{f}_{ab}\Rightarrow\boldsymbol{h}_{ab}$ *come from* $\boldsymbol{\eta},\boldsymbol{\zeta},\boldsymbol{\theta}$ *as in* **(c)**, *then* $\boldsymbol{\theta}_{ab}=\boldsymbol{\zeta}_{ab}\odot\boldsymbol{\eta}_{ab}$.

Remark 4.57. Note that we make the (potentially confusing) distinction between *m-Kuranishi neighbourhoods* (V_i,E_i,s_i,ψ_i) *on a topological space* X, as in Definition 4.1, and *m-Kuranishi neighbourhoods* (V_a,E_a,s_a,ψ_a) *on an m-Kuranishi space* $\boldsymbol{X}=(X,\mathcal{K})$, which are as in Definition 4.49, and come equipped with the extra implicit data $\Phi_{ai,\ i\in I}$, $\Lambda_{aij,\ i,j\in I}$ giving the compatibility with the m-Kuranishi structure $\mathcal{K}$ on X.

We also distinguish between *coordinate changes* $\Phi_{ij}:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ *between m-Kuranishi neighbourhoods on a topological space* X, which are as in Definition 4.10 and for which there may be many choices or none, and *coordinate changes* $\Phi_{ab}:(V_a,E_a,s_a,\psi_a)\to(V_b,E_b,s_b,\psi_b)$ *between m-Kuranishi neighbourhoods on an m-Kuranishi space* $\boldsymbol{X}$, which are as in Definition 4.51, and come equipped with implicit extra data $\Lambda_{abi,\ i\in I}$, and which by Theorem 4.56(a) always exist, and are unique up to unique 2-isomorphism.

Similarly, we distinguish between 1*-morphisms* $\boldsymbol{f}_{ij}:(U_i,D_i,r_i,\chi_i)\to(V_j,E_j,s_j,\psi_j)$ *of m-Kuranishi neighbourhoods over a continuous map of topological spaces* $f:X\to Y$, which are as in Definition 4.2 and for which there may be many choices or none, and 1*-morphisms* $\boldsymbol{f}_{ab}:(U_a,D_a,r_a,\chi_a)\to(V_b,E_b,s_b,\psi_b)$ *of m-Kuranishi neighbourhoods over a* 1*-morphism of m-Kuranishi spaces* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$, which are as Definition 4.54, and come equipped with implicit extra data $\boldsymbol{F}_{ai,\ i\in I}^{bj,\ j\in J}$, and which by Theorem 4.56(b),(c) always exist, and are unique up to unique 2-isomorphism.

4.7.2 Constructing equivalent m-Kuranishi structures

We can use m-Kuranishi neighbourhoods on $\boldsymbol{X}=(X,\mathcal{K})$ to construct alternative m-Kuranishi structures $\mathcal{K}'$ on X.

Theorem 4.58. *Let* $\boldsymbol{X}=(X,\mathcal{K})$ *be an m-Kuranishi space, and* $\{(V_a,E_a,s_a,\psi_a):a\in A\}$ *a family of m-Kuranishi neighbourhoods on* $\boldsymbol{X}$ *with* $X=\bigcup_{a\in A}\mathrm{Im}\,\psi_a$. *For all* $a,b\in A$, *let* $\Phi_{ab}:(V_a,E_a,s_a,\psi_a)\to(V_b,E_b,s_b,\psi_b)$ *be a coordinate change over* $S=\mathrm{Im}\,\psi_a\cap\mathrm{Im}\,\psi_b$ *on* $\boldsymbol{X}$ *given by Theorem* 4.56(a), *which is unique up to* 2*-isomorphism; when* $a=b$ *we choose* $\Phi_{aa}=\mathrm{id}_{(V_a,E_a,s_a,\psi_a)}$ *and* $\Lambda_{aai}=\mathrm{id}_{\Phi_{ai}}$ *for* $i\in I$, *which is allowed by Theorem* 4.56(a).

For all $a,b,c\in A$, *both* $\Phi_{bc}\circ\Phi_{ab}|_S$ *and* $\Phi_{ac}|_S$ *are coordinate changes* $(V_a,E_a,s_a,\psi_a)\to(V_c,E_c,s_c,\psi_c)$ *over* $S=\mathrm{Im}\,\psi_a\cap\mathrm{Im}\,\psi_b\cap\mathrm{Im}\,\psi_c$ *on* $\boldsymbol{X}$, *so Theorem* 4.56(a) *gives a unique* 2*-morphism* $\Lambda_{abc}:\Phi_{bc}\circ\Phi_{ab}|_S\Rightarrow\Phi_{ac}|_S$. *Then* $\mathcal{K}'=\bigl(A,(V_a,E_a,s_a,\psi_a)_{a\in A},\Phi_{ab,\ a,b\in A},\Lambda_{abc,\ a,b,c\in A}\bigr)$ *is an m-Kuranishi structure on* X, *and* $\boldsymbol{X}'=(X,\mathcal{K}')$ *is canonically equivalent to* $\boldsymbol{X}$ *in* $\mathbf{m\dot{K}ur}$, *in the sense of Definition* A.7.

Proof. Write $\mathcal{K} = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\ i,j\in I}, \Lambda_{ijk,\ i,j,k\in I}\bigr)$, and let $\mathcal{K}'$ be as in the theorem. We claim that $\mathcal{K}'$ is an m-Kuranishi structure on X. Definition 4.14(a)–(f) are immediate. For (g), if $a, b \in A$ then we have a 2-morphism $\Lambda_{aab} : \Phi_{ab} \circ \Phi_{aa} \Rightarrow \Phi_{ab}$, with the defining property, from (4.61), that

$$\Lambda_{aai} \odot (\Lambda_{abi} * \mathrm{id}_{\Phi_{aa}}) = \Lambda_{abi} \odot (\mathrm{id}_{\Phi_{bi}} * \Lambda_{aab}) : \Phi_{bi} \circ \Phi_{ab} \Longrightarrow \Phi_{ai}. \tag{4.63}$$

Here the left hand side is the 2-morphism $\tilde{\Lambda}_{abi}$ from Definition 4.51 for the composition $\tilde{\Phi}_{ab} = \Phi_{ab} \circ \Phi_{aa}$. Since by definition $\Phi_{aa} = \mathrm{id}_{(V_a,E_a,s_a,\psi_a)}$ and $\Lambda_{aai} = \mathrm{id}_{\Phi_{ai}}$, equation (4.63) is satisfied by $\Lambda_{aab} = \mathrm{id}_{\Phi_{ab}}$ for all $i \in I$, so by uniqueness in Theorem 4.56(a) we have $\Lambda_{aab} = \mathrm{id}_{\Phi_{ab}}$. Similarly $\Lambda_{abb} = \mathrm{id}_{\Phi_{ab}}$, proving Definition 4.14(g) for $\mathcal{K}'$.

For (h), let $a, b, c, d \in A$ and $i \in I$, and consider the diagram of 2-morphisms

$$\begin{tikzcd}
\Phi_{di}\circ\Phi_{cd}\circ\Phi_{bc}\circ\Phi_{ab} \arrow[rrr, Rightarrow, "\mathrm{id}_{\Phi_{di}}*\Lambda_{bcd}*\mathrm{id}_{\Phi_{ab}}"'] \arrow[dr, Rightarrow, "\Lambda_{cdi}*\mathrm{id}_{\Phi_{bc}}*\mathrm{id}_{\Phi_{ab}}"] \arrow[ddd, Rightarrow, "\mathrm{id}_{\Phi_{di}}*\mathrm{id}_{\Phi_{cd}}*\Lambda_{abc}"] & & & \Phi_{di}\circ\Phi_{bd}\circ\Phi_{ab} \arrow[dl, Rightarrow, "\Lambda_{bdi}*\mathrm{id}_{\Phi_{ab}}"'] \arrow[ddd, Rightarrow, "\mathrm{id}_{\Phi_{di}}*\Lambda_{abd}"'] \\
& \Phi_{ci}\circ\Phi_{bc}\circ\Phi_{ab} \arrow[r, Rightarrow, "\Lambda_{bci}*\mathrm{id}_{\Phi_{ab}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{ci}}*\Lambda_{abc}"] & \Phi_{bi}\circ\Phi_{ab} \arrow[d, Rightarrow, "\Lambda_{abi}"] & \\
& \Phi_{ci}\circ\Phi_{ac} \arrow[r, Rightarrow, "\Lambda_{aci}"'] & \Phi_{ai} & \\
\Phi_{di}\circ\Phi_{cd}\circ\Phi_{ac} \arrow[ur, Rightarrow, "\Lambda_{cdi}*\mathrm{id}_{\Phi_{ac}}"'] \arrow[rrr, Rightarrow, "\mathrm{id}_{\Phi_{di}}*\Lambda_{acd}"] & & & \Phi_{di}\circ\Phi_{ad}. \arrow[ul, Rightarrow, "\Lambda_{adi}"]
\end{tikzcd}$$

Here each small quadrilateral commutes by definition of Λ_{abc}. Thus the outer quadrilateral commutes. But the outer quadrilateral is '$\Phi_{di}\circ$' on 1-morphisms and '$\mathrm{id}_{\Phi_{di}}*$' on 2-morphisms applied to (4.4) with a, b, c, d in place of i, j, k, l. As Φ_{di} is a coordinate change, this implies (4.4) commutes, restricted to the intersection of its domain with $\mathrm{Im}\,\psi_i$. As this holds for all $i \in I$, we deduce Definition 4.14(h) for $\mathcal{K}'$. So $\boldsymbol{X}'$ is an m-Kuranishi space.

To show $\boldsymbol{X}', \boldsymbol{X}$ are equivalent in $\mathbf{m\dot{K}ur}$, we must construct 1-morphisms $\boldsymbol{f} : \boldsymbol{X}' \to \boldsymbol{X}$, $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{X}'$ and 2-morphisms $\boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \mathbf{id}_{\boldsymbol{X}'}$, $\boldsymbol{\zeta} : \boldsymbol{f} \circ \boldsymbol{g} \Rightarrow \mathbf{id}_{\boldsymbol{X}}$. As in (4.9), define

$$\boldsymbol{f} = \bigl(\mathrm{id}_X, \Phi_{ai,\ a\in A,\ i\in I},\ (\Lambda_{aa'i})^{i\in I}_{a,a'\in A},\ (\Lambda_{aii'})^{i,i'\in I}_{a\in A}\bigr),$$

where the $\Lambda_{aii'}, \Lambda_{aa'i}$ are from Definitions 4.49–4.51. We can check using (4.57)–(4.61) that Definition 4.17(a)–(h) hold, so $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{X}'$ is a 1-morphism.

For $\boldsymbol{g}$, as $\Phi_{ai} : (V_a, E_a, s_a, \psi_a) \to (V_i, E_i, s_i, \psi_i)$ is a coordinate change, there exist a 1-morphism $\Psi_{ia} : (V_i, E_i, s_i, \psi_i) \to (V_a, E_a, s_a, \psi_a)$, and 2-morphisms $\xi_{ia} : \Psi_{ia} \circ \Phi_{ai} \Rightarrow \mathrm{id}_{(V_a,E_a,s_a,\psi_a)}$ and $\chi_{ia} : \Phi_{ai} \circ \Psi_{ia} \Rightarrow \mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$. By Proposition A.5, we can choose these to satisfy $\xi_{ia} * \mathrm{id}_{\Psi_{ia}} = \mathrm{id}_{\Psi_{ia}} * \chi_{ia}$ and $\chi_{ia} * \mathrm{id}_{\Phi_{ai}} = \mathrm{id}_{\Phi_{ai}} * \xi_{ia}$. Define

$$\boldsymbol{g} = \bigl(\mathrm{id}_X, \Psi_{ia,\ i\in I,\ a\in A},\ (\mathrm{M}_{ii'a})^{a\in A}_{i,i'\in I},\ (\mathrm{M}_{iaa'})^{a,a'\in A}_{i\in I}\bigr),$$

where $\mathrm{M}_{ii'a}, \mathrm{M}_{iaa'}$ are defined by the commutative diagrams

$$\begin{tikzcd}[column sep=large]
\Psi_{i'a}\circ\Phi_{ii'}\circ\Phi_{ai}\circ\Psi_{ia} \arrow[r, Rightarrow, "\mathrm{id}_{\Psi_{i'a}}*\Lambda_{aii'}*\mathrm{id}_{\Psi_{ia}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\Psi_{i'a}}*\mathrm{id}_{\Phi_{ii'}}*\chi_{ia}"] & \Psi_{i'a}\circ\Phi_{ai'}\circ\Psi_{ia} \arrow[r, Rightarrow, "\xi_{i'a}*\mathrm{id}_{\Psi_{ia}}"'] & \mathrm{id}_{(V_a,E_a,s_a,\psi_a)}\circ\Psi_{ia} \arrow[d, equal] \\
\Psi_{i'a}\circ\Phi_{ii'}\circ\mathrm{id}_{(V_i,E_i,s_i,\psi_i)} \arrow[r, equal] & \Psi_{i'a}\circ\Phi_{ii'} \arrow[r, Rightarrow, "\mathrm{M}_{ii'a}"] & \Psi_{ia},
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
\Psi_{ia'}\circ\Phi_{a'i}\circ\Phi_{aa'}\circ\Psi_{ia} \arrow[r, Rightarrow, "\mathrm{id}_{\Psi_{ia'}}*\Lambda_{aa'i}*\mathrm{id}_{\Psi_{ia}}"'] \arrow[d, Rightarrow, "\xi_{ia'}*\mathrm{id}_{\Phi_{aa'}}*\mathrm{id}_{\Psi_{ia}}"] & \Psi_{ia'}\circ\Phi_{ai}\circ\Psi_{ia} \arrow[r, Rightarrow, "\mathrm{id}_{\Psi_{ia'}}*\chi_{ia}"'] & \Psi_{ia'}\circ\mathrm{id}_{(V_i,E_i,s_i,\psi_i)} \arrow[d, equal] \\
\mathrm{id}_{(V_{a'},E_{a'},s_{a'},\psi_{a'})}\circ\Phi_{aa'}\circ\Psi_{ia} \arrow[r, equal] & \Phi_{aa'}\circ\Psi_{ia} \arrow[r, Rightarrow, "\mathrm{M}_{iaa'}"] & \Psi_{ia'}.
\end{tikzcd}$$

Using the various identities we can show that $\boldsymbol{g}:\boldsymbol{X}'\rightarrow\boldsymbol{X}$ is a 1-morphism.

Definition 4.20 defines the compositions $\boldsymbol{g}\circ\boldsymbol{f}$, $\boldsymbol{f}\circ\boldsymbol{g}$, and some 2-morphisms of m-Kuranishi neighbourhoods $\Theta^{\boldsymbol{g},\boldsymbol{f}}_{aia'}$ and $\Theta^{\boldsymbol{f},\boldsymbol{g}}_{iai'}$. For all $a,a'\in A$, there is a unique 2-morphism $\boldsymbol{\eta}_{aa'}:(\boldsymbol{g}\circ\boldsymbol{f})_{aa'}\Rightarrow(\mathbf{id}_{\boldsymbol{X}'})_{aa'}=\Phi_{aa'}$ of m-Kuranishi neighbourhoods over $\mathrm{Im}\,\psi_a\cap\mathrm{Im}\,\psi_{a'}$ such that for all $i\in I$, the following commutes:

$$\begin{tikzcd}[column sep=large]
\Psi_{ia'}\circ\Phi_{a'i}\circ\Phi_{aa'} \arrow[rr, Rightarrow, "\xi_{ia'}*\mathrm{id}_{\Phi_{aa'}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\Psi_{ia'}}*\Lambda_{aa'i}"] & & \mathrm{id}_{(V_{a'},E_{a'},s_{a'},\psi_{a'})}\circ\Phi_{aa'} \arrow[d, equal] \\
\Psi_{ia'}\circ\Phi_{ai} \arrow[r, Rightarrow, "\Theta^{\boldsymbol{g},\boldsymbol{f}}_{aia'}"] & (\boldsymbol{g}\circ\boldsymbol{f})_{aa'} \arrow[r, Rightarrow, "\boldsymbol{\eta}_{aa'}\vert_{\mathrm{Im}\,\psi_a\cap\mathrm{Im}\,\psi_{a'}\cap\mathrm{Im}\,\psi_i}"] & \Phi_{aa'}.
\end{tikzcd}\tag{4.64}$$

To prove this we show that the prescribed values for $i,i'\in I$ agree on the intersection $\mathrm{Im}\,\psi_a\cap\mathrm{Im}\,\psi_{a'}\cap\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_{i'}$, and use the stack property Theorem 4.13 to prove there is a unique $\boldsymbol{\eta}_{aa'}$ such that (4.64) commutes for all $i\in I$. Then we show that $\boldsymbol{\eta}=(\boldsymbol{\eta}_{aa',\,a,a'\in A})$ is a 2-morphism $\boldsymbol{\eta}:\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{X}'}$ in $\mathbf{m\dot{K}ur}$.

Similarly, we construct a 2-morphism $\boldsymbol{\zeta}=(\boldsymbol{\zeta}_{ii',\,i,i'\in I}):\boldsymbol{f}\circ\boldsymbol{g}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$, where $\boldsymbol{\zeta}_{ii'}$ fits into a commuting diagram for all $a\in A$

$$\begin{tikzcd}[column sep=large]
\Phi_{ii'}\circ\Phi_{ai}\circ\Psi_{ia} \arrow[rr, Rightarrow, "\mathrm{id}_{\Phi_{ii'}}*\chi_{ia}"'] \arrow[d, Rightarrow, "\Lambda_{aii'}*\mathrm{id}_{\Psi_{ia}}"] & & \Phi_{ii'}\circ\mathrm{id}_{(V_i,E_i,s_i,\psi_i)} \arrow[d, equal] \\
\Phi_{ai'}\circ\Psi_{ia} \arrow[r, Rightarrow, "\Theta^{\boldsymbol{f},\boldsymbol{g}}_{iai'}"] & (\boldsymbol{f}\circ\boldsymbol{g})_{ii'} \arrow[r, Rightarrow, "\boldsymbol{\zeta}_{ii'}\vert_{\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_{i'}\cap\mathrm{Im}\,\psi_a}"] & \Phi_{ii'}.
\end{tikzcd}$$

Thus $\boldsymbol{X}'$ and $\boldsymbol{X}$ are equivalent in $\mathbf{m\dot{K}ur}$. The equivalence $\boldsymbol{f}:\boldsymbol{X}'\rightarrow\boldsymbol{X}$ is actually independent of choices, so its quasi-inverse $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{X}'$ is canonical up to 2-isomorphism. $\square$

As the m-Kuranishi neighbourhoods (V_i,E_i,s_i,ψ_i) in the m-Kuranishi structure on $\boldsymbol{X}$ are m-Kuranishi neighbourhoods on $\boldsymbol{X}$, we deduce:

Corollary 4.59. *Let $\boldsymbol{X}=(X,\mathcal{K})$ be an m-Kuranishi space with $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},\Phi_{ij,\,i,j\in I},\Lambda_{ijk,\,i,j,k\in I}\bigr)$. Suppose $J\subseteq I$ with $\bigcup_{j\in J}\mathrm{Im}\,\psi_j=X$. Then $\mathcal{K}'=\bigl(J,(V_i,E_i,s_i,\psi_i)_{i\in J},\Phi_{ij,\,i,j\in J},\Lambda_{ijk,\,i,j,k\in J}\bigr)$ is an m-Kuranishi structure on X, and $\boldsymbol{X}'=(X,\mathcal{K}')$ is canonically equivalent to $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$.*

Thus, adding or subtracting extra m-Kuranishi neighbourhoods to or from the m-Kuranishi structure of $\boldsymbol{X}$ leaves $\boldsymbol{X}$ unchanged up to equivalence.

4.7.3 M-Kuranishi neighbourhoods on boundaries and corners

Now suppose $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumption 3.22, so that as in §4.6 we have a 2-category $\mathbf{m\dot{K}ur^c}$ of m-Kuranishi spaces with corners $\boldsymbol{X}$, which have boundaries $\partial\boldsymbol{X}$ and k-corners $C_k(\boldsymbol{X})$. We will show that m-Kuranishi neighbourhoods (V_a,E_a,s_a,ψ_a) on $\boldsymbol{X}$ lift to m-Kuranishi neighbourhoods on $\partial\boldsymbol{X}$ and $C_k(\boldsymbol{X})$.

Definition 4.60. Let $\boldsymbol{X}=(X,\mathcal{K})$ be an m-Kuranishi space with corners in $\mathbf{m\dot{K}ur^c}$ with $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \Lambda_{hij,\ h,i,j\in I}\bigr)$. Then for each $k\in\mathbb{N}$, Definition 4.39 defines an object $C_k(\boldsymbol{X})=(C_k(X),\mathcal{K}_k)$ and a 1-morphism $\boldsymbol{\Pi}_k:C_k(\boldsymbol{X})\to\boldsymbol{X}$ in $\mathbf{m\dot{K}ur^c}$, where

$$\begin{aligned}\mathcal{K}_k&=\bigl(\{k\}\times I,(V_{(k,i)},E_{(k,i)},s_{(k,i)},\psi_{(k,i)})_{i\in I},\Phi_{\substack{(k,i),(k,j),\\ i,j\in I}},\Lambda_{\substack{(k,h)(k,i)(k,j),\\ h,i,j\in I}}\bigr),\\ \boldsymbol{\Pi}_k&=\bigl(\Pi_k,\boldsymbol{\Pi}_{(k,i)j,\ i,j\in I},\ \boldsymbol{\Pi}^{j,\ j\in I}_{(k,i)(k,i'),\ i,i'\in I},\ \boldsymbol{\Pi}^{jj',\ j,j'\in I}_{(k,i),\ i\in I}\bigr).\end{aligned}$$

Let (V_a,E_a,s_a,ψ_a), $\Phi_{ai,\ i\in I}$, $\Lambda_{aij,\ i,j\in I}$ be an m-Kuranishi neighbourhood on $\boldsymbol{X}$, as in Definition 4.49. We will define a corresponding m-Kuranishi neighbourhood $(V_{(k,a)},E_{(k,a)},s_{(k,a)},\psi_{(k,a)}),\Phi_{(k,a),(k,i),\ i\in I},\Lambda_{(k,a)(k,i)(k,j),\ i,j\in I}$ on $C_k(\boldsymbol{X})$, with $V_{(k,a)}=C_k(V_a)$, $E_{(k,a)}=C_k(E_a)$, and $s_{(k,a)}=C_k(s_a)$. When $k=1$ this is an m-Kuranishi neighbourhood on $\partial\boldsymbol{X}=C_1(\boldsymbol{X})$. Almost all the hard work has been done already in Definition 4.39.

We take $(V_{(k,a)},E_{(k,a)},s_{(k,a)},\psi_{(k,a)})$ to be the m-Kuranishi neighbourhood on $C_k(X)$ constructed from (V_a,E_a,s_a,ψ_a) in the same way that $(V_{(k,i)},E_{(k,i)},s_{(k,i)},\psi_{(k,i)})$ is constructed from (V_i,E_i,s_i,ψ_i) in Definition 4.39, except that $\psi_{(k,a)}$ is defined as we explain shortly. Also $\Phi_{(k,a),(k,i)}$, $\Lambda_{(k,a)(k,i)(k,j)}$ are constructed from Φ_{ai}, Λ_{aij} in exactly the same way that $\Phi_{(k,i),(k,j)}$, $\Lambda_{(k,h)(k,i)(k,j)}$ are constructed from Φ_{ij}, Λ_{hij} in Definition 4.39, though we postpone the proof of Definition 4.2(e) for $\Phi_{(k,a),(k,i)}$.

To define $\psi_{(k,a)}:s_{(k,a)}^{-1}(0)\to C_k(X)$, let $v'\in s_{(k,a)}^{-1}(0)\subseteq V_{(k,a)}=C_k(V_a)$ with $\Pi_k(v')=v\in s_a^{-1}(0)\subseteq V_a$, where $\Pi_k:C_k(V_a)\to V_a$. Then $x=\psi_a(v)\in X$, so there exists $i\in I$ with $x\in\operatorname{Im}\psi_i$, and thus $v\in V_{ai}\cap s_a^{-1}(0)$, which implies that $v'\in V_{(k,a)(k,i)}\cap s_{(k,a)}^{-1}(0)$, so $\phi_{(k,a)(k,i)}(v')\in s_{(k,i)}^{-1}(0)\subseteq V_{(k,i)}$, and $\psi_{(k,i)}\circ\phi_{(k,a)(k,i)}(v')\in C_k(X)$. Define $\psi_{(k,a)}(v')=\psi_{(k,i)}\circ\phi_{(k,a)(k,i)}(v')$. If also $x\in\operatorname{Im}\psi_j$ for $j\in I$ then the 1- and 2-morphisms

$$\begin{aligned}\Phi_{(k,i)(k,j)}&:(V_{(k,i)},E_{(k,i)},s_{(k,i)},\psi_{(k,i)})\longrightarrow(V_{(k,j)},E_{(k,j)},s_{(k,j)},\psi_{(k,j)})\\ \Lambda_{(k,a)(k,i)(k,j)}&:\Phi_{(k,i)(k,j)}\circ\Phi_{(k,a)(k,i)}\Longrightarrow\Phi_{(k,a)(k,j)}\end{aligned}$$

imply that

$$\psi_{(k,i)}\circ\phi_{(k,a)(k,i)}(v')=\psi_{(k,j)}\circ\phi_{(k,i)(k,j)}\circ\phi_{(k,a)(k,i)}(v')=\psi_{(k,k)}\circ\phi_{(k,a)(k,k)}(v').$$

Thus $\psi_{(k,a)}(v')$ is independent of the choice of $i\in I$ with $x\in\operatorname{Im}\psi_i$, and is well defined. We show $\psi_{(k,a)}$ is a homeomorphism with its open image as in Definition 4.39. Therefore $(V_{(k,a)},E_{(k,a)},s_{(k,a)},\psi_{(k,a)})$ is an m-Kuranishi neighbourhood on $C_k(X)$. Definition 4.2(e) for $\Phi_{(k,a),(k,i)}$ follows from $\psi_{(k,a)}(v')=\psi_{(k,i)}\circ\phi_{(k,a)(k,i)}(v')$ above. Hence $\Phi_{(k,a),(k,i)}$, $\Lambda_{(k,a)(k,i)(k,j)}$ are 1- and 2-morphisms of m-Kuranishi neighbourhoods, as required. The condition (4.57) for the $\Lambda_{(k,a)(k,i)(k,j)}$ follows from (4.57) for the Λ_{aij} in the same way that Definition 4.14(h) for the $\Lambda_{(k,h)(k,i)(k,j)}$ is proved in Definition 4.39. This shows that $(V_{(k,a)},E_{(k,a)},s_{(k,a)},\psi_{(k,a)})$ with data $\Phi_{(k,a),(k,i),\ i\in I},\Lambda_{(k,a)(k,i)(k,j),\ i,j\in I}$ is an m-Kuranishi neighbourhood on $C_k(\boldsymbol{X})$, as in §4.6.

Very much like $\boldsymbol{\Pi}_{(k,i)i}$ in Definition 4.39, we can show that that

$$\boldsymbol{\Pi}_{(k,a)a}=(V_{(k,a)},\Pi_k,\mathrm{id}_{E_{(k,a)}}):(V_{(k,a)},E_{(k,a)},s_{(k,a)},\psi_{(k,a)})\longrightarrow(V_a,E_a,s_a,\psi_a)$$

is a 1-morphism of m-Kuranishi neighbourhoods over $\boldsymbol{\Pi}_k:C_k(\boldsymbol{X})\to\boldsymbol{X}$, in the sense of Definition 4.54.

Definition 4.61. Let $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ be a 1-morphism in $\mathbf{m\dot{K}ur^c}$, with notation (4.6), (4.7), (4.9), suppose we are given m-Kuranishi neighbourhoods $(U_a,D_a,r_a,\chi_a),\mathrm{T}_{ai,\ i\in I},\mathrm{K}_{aii',\ i,i'\in I}$ on $\boldsymbol{X}$ and $(V_b,E_b,s_b,\psi_b),\Upsilon_{bj,\ j\in J},\Lambda_{bjj',\ j,j'\in J}$ on $\boldsymbol{Y}$, and let $\boldsymbol{f}_{ab}$, $\boldsymbol{F}_{ai,\ i\in I}^{bj,\ j\in J}$ be a 1-morphism $\boldsymbol{f}_{ab}:(U_a,D_a,r_a,\chi_a)\to(V_b,E_b,s_b,\psi_b)$ of m-Kuranishi neighbourhoods over $(\mathrm{Im}\,\chi_a\cap f^{-1}(\mathrm{Im}\,\psi_b),\boldsymbol{f})$ on $\boldsymbol{X},\boldsymbol{Y}$, as in Definition 4.54 and Theorem 4.56(b), with $\boldsymbol{f}_{ab}=(U_{ab},f_{ab},\hat f_{ab})$.

Let $k,l\in\mathbb{N}$, so that Definition 4.60 gives m-Kuranishi neighbourhoods $(U_{(k,a)},D_{(k,a)},r_{(k,a)},\chi_{(k,a)})$, $\mathrm{T}_{(k,a),(k,i),\ i\in I},\mathrm{K}_{(k,a)(k,i)(k,i'),\ i,i'\in I}$ on $C_k(\boldsymbol{X})$ and $(V_{(l,b)},E_{(l,b)},s_{(l,b)},\psi_{(l,b)})$, $\Upsilon_{(l,a),(l,j),\ j\in J},\Lambda_{(l,a)(l,j)(l,j'),\ j,j'\in J}$ on $C_l(\boldsymbol{Y})$. Then exactly as for (4.52) in Definition 4.43, from $\boldsymbol{f}_{ab}$ we define a 1-morphism of m-Kuranishi neighbourhoods

$$\begin{aligned}\boldsymbol{f}_{(k,a)(l,b)}=(U_{(k,a)(l,b)},f_{(k,a)(l,b)},\hat f_{(k,a)(l,b)}):(U_{(k,a)},D_{(k,a)},r_{(k,a)},\chi_{(k,a)})&\\ \longrightarrow(V_{(l,b)},E_{(l,b)},s_{(l,b)},\psi_{(l,b)})&\end{aligned}$$

over $C(f):C(X)\to C(Y)$ and $S=\mathrm{Im}\,\chi_{(k,a)}\cap C(f)^{-1}(\mathrm{Im}\,\psi_{(l,b)})$, where

$$\begin{aligned}U_{(k,a)(l,b)}&=C_k(U_{ab})\cap C(f_{ab})^{-1}(C_l(V_b))\subseteq U_{(k,a)}=C_k(U_a),\\ f_{(k,a)(l,b)}&=C(f_{ab})|_{U_{(k,a)(l,b)}}:U_{(k,a)(l,b)}\longrightarrow V_{(l,b)}=C_l(V_b),\\ \hat f_{(k,a)(l,b)}&=\Pi_k^*(\hat f_{ab})|_{U_{(k,a)(l,b)}}:D_{(k,a)}|_{U_{(k,a)(l,b)}}\longrightarrow f_{(k,a)(l,b)}^*(E_{(l,b)}).\end{aligned}$$

We also define 2-morphisms $\boldsymbol{F}_{(k,a)(k,i)}^{(l,b)(l,j)}:\Upsilon_{(l,b)(l,j)}\circ\boldsymbol{f}_{(k,a)(l,b)}\Rightarrow\boldsymbol{f}_{(k,i)(l,j)}\circ\mathrm{T}_{(k,a)(k,i)}$ from the $\boldsymbol{F}_{ai}^{bj}$ as for $\boldsymbol{F}_{(k,i)(k,i')}^{(l,j)}$ in Definition 4.43. Then (4.59) for the $\boldsymbol{F}_{(k,a)(k,i)}^{(l,b)(l,j)}$ follows from (4.59) for the $\boldsymbol{F}_{ai}^{bj}$ by applying the corner functor. Hence $\boldsymbol{f}_{(k,a)(l,b)}$, $\boldsymbol{F}_{(k,a)(k,i),\ i\in I}^{(l,b)(l,j),\ j\in J}$ is a 1-morphism of m-Kuranishi neighbourhoods $\boldsymbol{f}_{(k,a)(l,b)}:(U_{(k,a)},D_{(k,a)},r_{(k,a)},\chi_{(k,a)})\to(V_{(l,b)},E_{(l,b)},s_{(l,b)},\psi_{(l,b)})$ over $(\mathrm{Im}\,\chi_{(k,a)}\cap C(f)^{-1}(\mathrm{Im}\,\psi_{(l,b)}),C(\boldsymbol{f}))$ on $C(\boldsymbol{X}),C(\boldsymbol{Y})$, as in Definition 4.54.

A special case of this construction is when $\boldsymbol{X}=\boldsymbol{Y}$, $\boldsymbol{f}=\mathbf{id}_{\boldsymbol{X}}$, and $k=l$, and $\boldsymbol{f}_{ab}:(U_a,D_a,r_a,\chi_a)\to(V_b,E_b,s_b,\psi_b)$ is a coordinate change of m-Kuranishi neighbourhoods on $\boldsymbol{X}$. Then $\boldsymbol{f}_{(k,a)(k,b)}:(U_{(k,a)},D_{(k,a)},r_{(k,a)},\chi_{(k,a)})\to(V_{(k,b)},E_{(k,b)},s_{(k,b)},\psi_{(k,b)})$ is a coordinate change on $C_k(\boldsymbol{X})$.

4.7.4 A philosophical digression

We can now state our:

Philosophy for working with m-Kuranishi spaces. *A good way to think about the 'real' geometric structure on m-Kuranishi spaces is as follows:*

(i) *Every m-Kuranishi space* $\boldsymbol{X}$ *has an underlying topological space* X, *and a large collection of 'm-Kuranishi neighbourhoods'* (V_a, E_a, s_a, ψ_a) *on* $\boldsymbol{X}$, *which are m-Kuranishi neighbourhoods on* X *in the sense of* §4.1, *but with an additional compatibility with the m-Kuranishi structure on* $\boldsymbol{X}$.

We think of (V_a, E_a, s_a, ψ_a) *as a choice of 'local coordinates' on* $\boldsymbol{X}$.

(ii) *For any two m-Kuranishi neighbourhoods* $(V_a, E_a, s_a, \psi_a), (V_b, E_b, s_b, \psi_b)$ *on* $\boldsymbol{X}$, *there is a coordinate change* $\Phi_{ab} : (V_a, E_a, s_a, \psi_a) \to (V_b, E_b, s_b, \psi_b)$, *natural up to canonical* 2*-isomorphism.*

(iii) *A* 1*-morphism of m-Kuranishi spaces* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *has an underlying continuous map* $f : X \to Y$. *If* $(U_a, D_a, r_a, \chi_a), (V_b, E_b, s_b, \psi_b)$ *are m-Kuranishi neighbourhoods on* $\boldsymbol{X}, \boldsymbol{Y}$, *there is a* 1*-morphism* $\boldsymbol{f}_{ab} : (U_a, D_a, r_a, \chi_a) \to (V_b, E_b, s_b, \psi_b)$ *over* f, *natural up to canonical* 2*-isomorphism.*

(iv) *The coordinate changes and* 1*-morphisms in* (ii),(iii) *behave in the obvious functorial ways under compositions and identities, up to canonical* 2*-isomorphisms.*

(v) *The family of m-Kuranishi neighbourhoods on* $\boldsymbol{X}$ *is closed under several natural constructions. For example:*

   (a) *If* (V, E, s, ψ) *is an m-Kuranishi neighbourhood on* $\boldsymbol{X}$ *and* $V' \subseteq V$ *is open then* $\bigl(V', E|_{V'}, s|_{V'}, \psi|_{V' \cap s^{-1}(0)}\bigr)$ *is an m-Kuranishi neighbourhood on* $\boldsymbol{X}$.

   (b) *If* (V, E, s, ψ) *is an m-Kuranishi neighbourhood on* $\boldsymbol{X}$ *and* $\pi : F \to V$ *is a vector bundle then* $\bigl(F, \pi^*(E) \oplus \pi^*(F), \pi^*(s) \oplus \mathrm{id}_F, \psi \circ \pi|_{\dots}\bigr)$ *is an m-Kuranishi neighbourhood on* $\boldsymbol{X}$.

(vi) *The collection of all m-Kuranishi neighbourhoods* (V_a, E_a, s_a, ψ_a) *on* $\boldsymbol{X}$ *will usually be much larger than a particular atlas* $\bigl\{(V_i, E_i, s_i, \psi_i) : i \in I\bigr\}$.

There are so many m-Kuranishi neighbourhoods on $\boldsymbol{X}$ *that we can often choose them to satisfy extra conditions. For example, in* §10.4 *we discuss m-Kuranishi neighbourhoods on* $\boldsymbol{X}$ *which are 'minimal at* x *in* $\boldsymbol{X}$*'.*

We will be guided by this philosophy from Chapter 7 onwards, where we will usually frame our definitions and results in terms of m-Kuranishi neighbourhoods on $\boldsymbol{X} = (X, \mathcal{K})$, rather than in terms of the particular m-Kuranishi neighbourhoods (V_i, E_i, s_i, ψ_i) in the m-Kuranishi structure $\mathcal{K}$, which we try not to use.

4.8 M-Kuranishi spaces and derived manifolds

We now take $\dot{\mathbf{Man}} = \mathbf{Man}$, and work with the corresponding 2-category of m-Kuranishi spaces $\mathbf{mKur}$.

Derived Differential Geometry is the study of 'derived smooth manifolds', where 'derived' is in the sense of the Derived Algebraic Geometry of Lurie [74] and Toën–Vezzosi [106, 107]. There are several different models of Derived Differential Geometry in the literature, all closely related:

- Probably the first reference to Derived Differential Geometry is a short final paragraph in Lurie [74, §4.5], outlining how to define an ∞-category of 'derived C^∞-schemes', and an ∞-subcategory of 'derived manifolds'.
- Lurie's ideas were developed further by his student David Spivak [103], who defined an ∞-category $\mathbf{DerMan_{Spi}}$ of 'derived manifolds'. Spivak's construction was rather complicated.
- Borisov and Noel [8] gave a simpler ∞-category $\mathbf{DerMan_{BN}}$ of 'derived manifolds', with an ∞-category equivalence $\mathbf{DerMan_{BN}} \simeq \mathbf{DerMan_{Spi}}$.
- The author [57, 58, 61] defined a strict 2-category $\mathbf{dMan}$ of 'd-manifolds', and studied their differential geometry in detail.
- Borisov [7] relates the derived manifolds of [8, 103] with the d-manifolds of [57, 58, 61]. Borisov constructs a 2-functor
$$\Pi : \pi_1(\mathbf{DerMan_{BN}}) \longrightarrow \mathbf{dMan} \tag{4.65}$$
from the 2-category truncation $\pi_1(\mathbf{DerMan_{BN}})$ of $\mathbf{DerMan_{BN}}$. This 2-functor Π is not an equivalence of 2-categories, but it is fairly close to being an equivalence. Reducing to homotopy categories, the functor
$$\mathrm{Ho}(\Pi) : \mathrm{Ho}(\mathbf{DerMan_{BN}}) \longrightarrow \mathrm{Ho}(\mathbf{dMan}) \tag{4.66}$$
is full but not faithful, and induces a 1-1 correspondence between isomorphism classes of objects.
- Wallbridge [108] defines a rather general ∞-category of 'derived manifolds', which we prefer to think of as 'derived C^∞-schemes', and then extends them to an Artin stack version, 'derived smooth stacks'.
- Macpherson [76] states a universal property of an '∞-category of derived manifolds', and argues that $\mathbf{DerMan_{Spi}}$ and $\mathbf{DerMan_{BN}}$ satisfy his universal property. This universal property explains the existence of Borisov's 2-functor (4.65), and of (4.67) below.

The next theorem will be proved in [57]:

Theorem 4.62. *There is an equivalence of* 2*-categories* $\mathbf{dMan} \simeq \mathbf{mKur}$, *where* $\mathbf{dMan}$ *is the strict* 2*-category of d-manifolds from* [57, 58, 61], *and* $\mathbf{mKur}$ *is as in* §4.3 *for* $\dot{\mathbf{Man}} = \mathbf{Man}$.

Combining with Borisov's 2-functor (4.65) gives a 2-functor

$$\pi_1(\mathbf{DerMan_{Spi}}) \simeq \pi_1(\mathbf{DerMan_{BN}}) \longrightarrow \mathbf{mKur}, \tag{4.67}$$

which is close to being an equivalence.

Remark 4.63. **(a)** The author carefully designed the definitions of §4.1–§4.3 using facts about d-manifolds from [57, 58, 61], in order to make Theorem 4.62 hold.

(b) The definitions of m-Kuranishi spaces above, and of (μ-)Kuranishi spaces in Chapters 5 and 6, are also very much inspired by Fukaya–Oh–Ohta–Ono's Kuranishi spaces [19–39] in Symplectic Geometry (which we call *FOOO Kuranishi spaces*), and by related structures such as McDuff–Wehrheim's Kuranishi atlases [77, 78, 80–83], all of which are geometric structures put on moduli spaces of J-holomorphic curves. From this we can draw an important conclusion:

> **Fukaya–Oh–Ohta–Ono's Kuranishi spaces [19–39], and similar geometric structures in Symplectic Geometry, are actually a prototype kind of derived orbifold.**

This is not surprising, as FOOO Kuranishi spaces and derived schemes were invented to do more-or-less the same job, namely to be a geometric structure on moduli spaces which encodes the obstructions in deformation theory of objects.

(c) We now have two different approaches to derived manifolds:

(i) Spivak [103], Borisov–Noel [7, 8] and the author [57, 58, 61] all define a derived manifold $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ as a topological space X with a (homotopy) sheaf of derived C^∞-rings $\boldsymbol{\mathcal{O}}_X$. The differences between [103], [7, 8], and [57, 58, 61] are in the notions of sheaf and derived C^∞-ring used.

(ii) M-Kuranishi spaces $(X, \mathcal{K})$ above are a topological space X with an atlas $\mathcal{K}$ of m-Kuranishi neighbourhoods $(V_i, E_i, s_i, \psi_i)_{i\in I}$, plus coordinate changes and 2-morphisms between them.

For comparison, here are two equivalent ways to define classical manifolds:

(i) A manifold $(X, \mathcal{O}_X)$ is a Hausdorff, second countable topological space X with a sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras or C^∞-rings, such that $(X, \mathcal{O}_X)$ is locally modelled on $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$, for $\mathcal{O}_{\mathbb{R}^n}$ the sheaf of smooth functions $\mathbb{R}^n \to \mathbb{R}$.

(ii) A manifold $(X, \mathcal{A})$ is a Hausdorff, second countable topological space X with an atlas $\mathcal{A}$ of charts $(V_i, \psi_i)_{i\in I}$, where $V_i \subseteq \mathbb{R}^n$ is open and $\psi_i : V_i \to X$ is a homeomorphism with an open set $\operatorname{Im}\psi_i \subseteq X$, and charts (V_i, ψ_i), (V_j, ψ_j) for $i, j \in I$ are compatible (i.e. coordinate changes are smooth).

These two approaches (i) and (ii) to derived differential geometry are broadly equivalent, but each has advantages for different purposes. In approach (i), derived manifolds are embedded in a much larger ∞- or 2-category of *derived*

C^∞*-schemes* (the 2-category of *d-spaces* $\mathbf{dSpa}$ in [57, 58, 61]), which may be useful.

An advantage of approach (ii) is that we can replace the base category $\mathbf{Man}$ with a variation, such as manifolds with corners $\mathbf{Man^c}$, and so define a 2-category $\mathbf{mKur^c}$, or whatever. We have done this already, by defining $\mathbf{m\dot{K}ur}$ starting from a category $\mathbf{\dot{M}an}$ of 'manifolds' satisfying some basic assumptions, leading to many different (2-)categories of 'derived manifolds', as in (4.37). This would be much more difficult to do in approach (i).

Chapter 5

μ-Kuranishi spaces

Throughout this chapter we suppose we are given a category $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7 in §3.1. To each such $\dot{\mathbf{Man}}$ we will associate a category $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ of 'μ-Kuranishi spaces', a simplified version of the 2-category of m-Kuranishi spaces $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ from Chapter 4.

We will prove that $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ is equivalent to the homotopy category $\mathrm{Ho}(\mathbf{m}\dot{\mathbf{K}}\mathbf{ur})$. Given this, the reader may wonder if there is any point in studying $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$, as we could just consider $\mathrm{Ho}(\mathbf{m}\dot{\mathbf{K}}\mathbf{ur})$ instead. Some reasons are that the definition of $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ is a lot simpler than those of $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ or $\mathrm{Ho}(\mathbf{m}\dot{\mathbf{K}}\mathbf{ur})$, involving categories rather than 2-categories, and sheaves rather than stacks. Also, $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ has better geometrical properties than one would expect of $\mathrm{Ho}(\mathbf{m}\dot{\mathbf{K}}\mathbf{ur})$: morphisms $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ form a sheaf on $\boldsymbol{X}$, when one would only expect morphisms $[\boldsymbol{f}]:\boldsymbol{X}\to\boldsymbol{Y}$ in $\mathrm{Ho}(\mathbf{m}\dot{\mathbf{K}}\mathbf{ur})$ to form a presheaf on $\boldsymbol{X}$.

Nonetheless, the 2-category structure in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ contains important information, which is lost in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$, so that $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ is better than $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ for some purposes. In particular, the fibre products $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ discussed in §11.2 are characterized by a universal property involving 2-morphisms, which makes no sense in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$. As in §11.4, the corresponding fibre products in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ may not exist, or may exist but be the wrong answer for applications.

We begin in §5.1 by discussing linearity properties of 2-morphisms of m-Kuranishi neighbourhoods from §4.1. We can glue such 2-morphisms using a partition of unity. Because of this, we show in §5.2 that the homotopy category of the 2-category of m-Kuranishi neighbourhoods in §4.1 forms a sheaf rather than just a presheaf, which is what we need to make the definition of μ-Kuranishi spaces work in §5.3, and in particular to define composition of morphisms of μ-Kuranishi spaces.

For the orbifold analogue, Kuranishi neighbourhoods in §6.1, the results of §5.1 would be false, and therefore we will not define an orbifold version of μ-Kuranishi spaces. The good properties of $\mathrm{Ho}(\mathbf{m}\dot{\mathbf{K}}\mathbf{ur})$ mentioned above do not hold for $\mathrm{Ho}(\dot{\mathbf{K}}\mathbf{ur})$ in Chapter 5, in particular, morphisms $[\boldsymbol{f}]:\boldsymbol{X}\to\boldsymbol{Y}$ in $\mathrm{Ho}(\dot{\mathbf{K}}\mathbf{ur})$ form a presheaf on $\boldsymbol{X}$, but generally not a sheaf.

5.1 Linearity properties of 2-morphisms of m-Kuranishi neighbourhoods

We explain some linearity properties of 2-morphisms of m-Kuranishi neighbourhoods. The set $\mathrm{Hom}_S(\Phi_{ij},\Phi'_{ij})$ of 2-morphisms $\Lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}$ over (S,f) is a real affine space, and a real vector space when $\Phi_{ij}=\Phi'_{ij}$. We can also multiply 2-morphisms $\Lambda_{ij}:\Phi_{ij}\Rightarrow\Phi_{ij}$ by smooth functions on V_{ij}, and combine 2-morphisms $\Lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}$ using a partition of unity.

Definition 5.1. Let $f:X\rightarrow Y$ be continuous, $(V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)$ be m-Kuranishi neighbourhoods on X,Y, and $S\subseteq\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\subseteq X$ be open, and $\Phi_{ij},\Phi'_{ij}:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ be 1-morphisms over (S,f), with $\Phi_{ij}=(V_{ij},\phi_{ij},\hat\phi_{ij})$ and $\Phi'_{ij}=(V'_{ij},\phi'_{ij},\hat\phi'_{ij})$. Write

$$\mathrm{Hom}_S(\Phi_{ij},\Phi'_{ij})=\bigl\{\Lambda_{ij}\colon\ \Lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}\text{ is a 2-morphism over }(S,f)\bigr\}. \tag{5.1}$$

We will show that $\mathrm{Hom}_S(\Phi_{ij},\Phi'_{ij})$ naturally has the structure of a real affine space, and $\mathrm{Hom}_S(\Phi_{ij},\Phi_{ij})$ the structure of a real vector space. Write

$$\mathrm{Hom}(E_i|_{V_{ij}},\mathcal{T}_{\phi_{ij}}V_j|_{\acute V_{ij}})_{\psi_i^{-1}(S)} \tag{5.2}$$

for the real vector space of germs at $\psi_i^{-1}(S)\subseteq V_{ij}$ of morphisms $E_i|_{V_{ij}}\rightarrow\mathcal{T}_{\phi_{ij}}V_j|_{V_{ij}}$ in the sense of §3.3.4. That is, an element of (5.3) is an equivalence class $[\acute V_{ij},\hat\lambda_{ij}]$ of pairs $(\acute V_{ij},\hat\lambda_{ij})$, where $\acute V_{ij}$ is an open neighbourhood of $\psi_i^{-1}(S)$ in V_{ij} and $\hat\lambda_{ij}:E_i|_{\acute V_{ij}}\rightarrow\mathcal{T}_{\phi_{ij}}V_j|_{\acute V_{ij}}$ is a morphism, and pairs $(\acute V_{ij},\hat\lambda_{ij}),(\acute V'_{ij},\hat\lambda'_{ij})$ are equivalent if there exists an open neighbourhood $\acute V''_{ij}$ of $\psi_i^{-1}(S)$ in $\acute V_{ij}\cap\acute V'_{ij}$ with $\hat\lambda_{ij}|_{\acute V''_{ij}}=\hat\lambda'_{ij}|_{\acute V''_{ij}}$. Then by Definition 4.3 we have:

$$\begin{aligned}&\mathrm{Hom}_S(\Phi_{ij},\Phi'_{ij})\cong\\&\quad\frac{\begin{array}{l}\bigl\{[\acute V_{ij},\hat\lambda_{ij}]\in\mathrm{Hom}(E_i|_{V_{ij}},\mathcal{T}_{\phi_{ij}}V_j|_{\acute V_{ij}})_{\psi_i^{-1}(S)}:\\ \phi'_{ij}=\phi_{ij}+\hat\lambda_{ij}\circ s_i+O(s_i^2),\ \ \hat\phi'_{ij}=\hat\phi_{ij}+\phi_{ij}^*(\mathrm{d}s_j)\circ\hat\lambda_{ij}+O(s_i)\bigr\}\end{array}}{\sim\,:\ [\acute V_{ij},\hat\lambda_{ij}]\sim[\acute V'_{ij},\hat\lambda'_{ij}]\text{ if }\hat\lambda'_{ij}-\hat\lambda_{ij}=O(s_i)}.\end{aligned} \tag{5.3}$$

We claim that the equations on $\hat\lambda_{ij}$ in the numerator of (5.3) are linear in $[\acute V_{ij},\hat\lambda_{ij}]$ if $\Phi'_{ij}=\Phi_{ij}$, and affine linear for general Φ'_{ij}. To prove this, noting that $\hat\lambda_{ij}=0$ is a solution when $\Phi'_{ij}=\Phi_{ij}$, it is enough to show that if $[\acute V_{ij},\hat\lambda_{ij}]$ and $[\acute V'_{ij},\hat\lambda'_{ij}]$ satisfy the equations and $\alpha\in\mathbb{R}$ then $\alpha\cdot[\acute V_{ij},\hat\lambda_{ij}]+(1-\alpha)[\acute V'_{ij},\hat\lambda'_{ij}]$ also satisfy the equations. For the first equation, as we have

$$\phi'_{ij}=\phi_{ij}+\hat\lambda_{ij}\circ s_i+O(s_i^2)\quad\text{and}\quad\phi'_{ij}=\phi_{ij}+\hat\lambda'_{ij}\circ s_i+O(s_i^2), \tag{5.4}$$

so Theorem 3.17(m) with $k=2$ gives

$$\phi'_{ij}=\phi_{ij}+[\alpha\cdot\hat\lambda_{ij}+(1-\alpha)\cdot\hat\lambda'_{ij}]\circ s_i+O(s_i^2),$$

as we want. For the second equation $\hat\phi'_{ij}=\hat\phi_{ij}+\phi_{ij}^*(\mathrm{d}s_j)\circ\hat\lambda_{ij}+O(s_i)$, affine linearity is immediate from Definition 3.15(vi) and Theorem 3.17(b).

The equivalence relation $\sim$ on the denominator of (5.3) is the quotient by a vector subspace of $\mathrm{Hom}(E_i|_{V_{ij}},\mathcal{T}_{\phi_{ij}}V_j|_{\acute V_{ij}})_{\psi_i^{-1}(S)}$ acting by translation. Hence $\mathrm{Hom}_S(\Phi_{ij},\Phi'_{ij})$ is the quotient of a real affine space (or a real vector space if $\Phi'_{ij}=\Phi_{ij}$) by a vector subspace acting by translations, and is a real affine space (or a real vector space if $\Phi'_{ij}=\Phi_{ij}$).

This proves the first part of the next result, the second is straightforward:

Proposition 5.2. *Let $f:X\to Y$ be continuous, $(V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)$ be m-Kuranishi neighbourhoods on X,Y, and $S\subseteq\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\subseteq X$ be open, and $\Phi_{ij},\Phi'_{ij}:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ be 1-morphisms over (S,f). Then the set $\mathrm{Hom}_S(\Phi_{ij},\Phi'_{ij})$ of 2-morphisms $\Lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}$ over (S,f) naturally has the structure of a real affine space, and $\mathrm{Hom}_S(\Phi_{ij},\Phi_{ij})$ the structure of a real vector space.*

These vector space and affine space structures are compatible with vertical and horizontal composition, identities, and inverses, in the obvious ways. Thus, the strict 2-categories $\mathbf{m\dot{K}N},\mathbf{Gm\dot{K}N},\mathbf{m\dot{K}N}_S(X)$ of §4.1 have a real linear structure at the level of 2-morphisms.

In any 2-category $\boldsymbol{\mathcal{C}}$, if $\Phi:A\to B$ is a 1-morphism in $\boldsymbol{\mathcal{C}}$ then the set $\mathrm{Hom}(\Phi,\Phi)$ of 2-morphisms $\Lambda:\Phi\to\Phi$ is a monoid under vertical composition $\odot$. For the 2-categories $\mathbf{m\dot{K}N},\mathbf{Gm\dot{K}N},\mathbf{m\dot{K}N}_S(X)$ of §4.1, this monoid is a real vector space, and in particular an abelian group.

The next lemma holds as (5.2) is clearly a module over both $C^\infty(V_i)$ and $C^\infty(V_i)_{\psi_i^{-1}(S)}$, and the conditions in (5.3) for $\Phi'_{ij}=\Phi_{ij}$ are $C^\infty(V_i)$-linear, by Theorem 3.17(b),(m), so the actions of $C^\infty(V_i),C^\infty(V_i)_{\psi_i^{-1}(S)}$ on (5.2) descend to (5.3).

Lemma 5.3. *Let $f:X\to Y$ be continuous, $(V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)$ be m-Kuranishi neighbourhoods on X,Y, and $S\subseteq\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\subseteq X$ be open, and $\Phi_{ij}:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ be a 1-morphism over (S,f). Then the vector space $\mathrm{Hom}_S(\Phi_{ij},\Phi_{ij})$ is naturally a module over $C^\infty(V_i)$, and also over $C^\infty(V_i)_{\psi_i^{-1}(S)}$, the $\mathbb{R}$-algebra of germs at $\psi_i^{-1}(S)$ of smooth functions $V_i\to\mathbb{R}$.*

That is, if $\Lambda:\Phi_{ij}\Rightarrow\Phi_{ij}$ is a 2-morphism over (S,f) then we can define another 2-morphism $\alpha\cdot\Lambda:\Phi_{ij}\Rightarrow\Phi_{ij}$ for any $\alpha\in C^\infty(V_i)$, or more generally any $\alpha\in C^\infty(\acute V_i)$ for $\acute V_i$ an open neighbourhood of $\psi_i^{-1}(S)$ in V_i. Next we explain how to glue 2-morphisms $\Lambda^a:\Phi_{ij}\Rightarrow\Phi'_{ij}$ using a partition of unity.

Definition 5.4. Let $f:X\to Y$ be continuous, $(V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)$ be m-Kuranishi neighbourhoods on X,Y, and $S\subseteq\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\subseteq X$ be open, and $\Phi_{ij},\Phi'_{ij}:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ be 1-morphisms over (S,f), with $\Phi_{ij}=(V_{ij},\phi_{ij},\hat\phi_{ij})$ and $\Phi'_{ij}=(V'_{ij},\phi'_{ij},\hat\phi'_{ij})$.

Suppose $\{T^a:a\in A\}$ is an open cover of S, and $\Lambda^a:\Phi_{ij}\Rightarrow\Phi'_{ij}$ is a 2-morphism over (T^a,f). Choose representatives $(\acute V^a,\hat\lambda^a)$ for $\Lambda^a=[\acute V^a,\hat\lambda^a]$

for $a \in A$, so that $\acute{V}^a$ is an open neighbourhood of $\psi_i^{-1}(T^a)$ in $V_{ij} \cap V'_{ij}$. Set $\acute{V}_{ij} = \bigcup_{a\in A} \acute{V}^a$, so that $\acute{V}_{ij}$ is an open neighbourhood of $\psi_i^{-1}(S)$ in $V_{ij} \cap V'_{ij}$. Then $\{\acute{V}^a : a \in A\}$ is an open cover of $\acute{V}_{ij}$. Choose a partition of unity $\{\eta^a : a \in A\}$ on $\acute{V}_{ij}$ subordinate to $\{\acute{V}^a : a \in A\}$, as in §3.3.1(d). Define a morphism on $\acute{V}_{ij}$:

$$\hat{\lambda}_{ij} : E_i|_{\acute{V}_{ij}} \longrightarrow \mathcal{T}_{\phi_{ij}} V_j|_{\acute{V}_{ij}} \quad \text{by} \quad \hat{\lambda}_{ij} = \textstyle\sum_{a\in A} \eta^a \cdot \hat{\lambda}^a. \tag{5.5}$$

Here $\hat{\lambda}^a$ is only defined on $\acute{V}^a \subseteq \acute{V}_{ij}$, but as $\operatorname{supp} \eta^a \subseteq \acute{V}^a$, we can extend $\eta^a \cdot \hat{\lambda}^a$ by zero on $\acute{V}_{ij} \setminus \acute{V}^a$, and so make $\eta^a \cdot \hat{\lambda}^a$ defined on all of $\acute{V}_{ij}$. As $\{\eta^a : a \in A\}$ is locally finite, the sum $\sum_{a\in A} \cdots$ in (5.5) is locally finite, and so is well defined as we are working with sheaves. Thus $\hat{\lambda}_{ij}$ is well defined.

We now claim that $(\acute{V}_{ij}, \hat{\lambda}_{ij})$ satisfies Definition 4.3, so that $\Lambda_{ij} := [\acute{V}_{ij}, \hat{\lambda}_{ij}] : \Phi_{ij} \Rightarrow \Phi'_{ij}$ is a 2-morphism over (S, f). To see this, note that as the conditions on $\hat{\lambda}_{ij}$ in (5.3) are affine linear, combining a family of solutions using a partition of unity as in (5.5) gives another solution. Informally we write

$$\Lambda_{ij} = \textstyle\sum_{a\in A} \eta^a \cdot \Lambda^a, \qquad \text{in 2-morphisms } \Phi_{ij} \Longrightarrow \Phi'_{ij}. \tag{5.6}$$

That is, we can combine 2-morphisms $\Lambda^a : \Phi_{ij} \Rightarrow \Phi'_{ij}$ over (T^a, f) for $a \in A$ using a partition of unity, to get a 2-morphism over (S, f) for $S = \bigcup_{a\in A} T^a$.

5.2 The category of μ-Kuranishi neighbourhoods

Recall from §A.2 that the *homotopy category* $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$ of a 2-category $\boldsymbol{\mathcal{C}}$ is the category whose objects are objects of $\boldsymbol{\mathcal{C}}$, and whose morphisms $[f] : X \to Y$ are 2-isomorphism classes $[f]$ of 1-morphisms $f : X \to Y$ in $\boldsymbol{\mathcal{C}}$. In §5.2–§5.3 we define a simplified version of m-Kuranishi spaces, called *μ-Kuranishi spaces*, in which we reduce from 2-categories to categories by taking homotopy categories.

Here is the analogue of Definitions 4.1–4.6 and 4.8.

Definition 5.5. Define the *category of μ-Kuranishi neighbourhoods* to be the homotopy category of the 2-category of m-Kuranishi neighbourhoods from §4.1. In more detail:

(a) Let X be a topological space, and $S \subseteq X$ be open. A *μ-Kuranishi neighbourhood (V, E, s, ψ) on X* (or *over S*) is just an m-Kuranishi neighbourhood on X (or over S), in the sense of Definition 4.1.

(b) Let $f : X \to Y$ be a continuous map, $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)$ be μ-Kuranishi neighbourhoods (hence m-Kuranishi neighbourhoods) on X, Y, and $S \subseteq \operatorname{Im} \psi_i \cap f^{-1}(\operatorname{Im} \psi_j)$ be open. A *morphism* $[\Phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ *of μ-Kuranishi neighbourhoods over* (S, f) is an equivalence class $[\Phi_{ij}]$ of 1-morphisms $\Phi_{ij}, \Phi'_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ of m-Kuranishi neighbourhoods over (S, f), where 1-morphisms Φ_{ij}, Φ'_{ij} are equivalent (written $\Phi_{ij} \approx_S \Phi'_{ij}$) if there exists a 2-morphism $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ of m-Kuranishi neighbourhoods over (S, f).

When $X=Y$ and $f=\mathrm{id}_X$ we call $[\Phi_{ij}]$ a *morphism over* S. In this case, the *identity morphism* $\mathrm{id}_{(V_i,E_i,s_i,\psi_i)}:(V_i,E_i,s_i,\psi_i)\rightarrow(V_i,E_i,s_i,\psi_i)$ over S is $[\mathrm{id}_{(V_i,E_i,s_i,\psi_i)}]$, for $\mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$ as in §4.1.

If $\Phi_{ij}=(V_{ij},\phi_{ij},\hat\phi_{ij})$, we write $[\Phi_{ij}]=[V_{ij},\phi_{ij},\hat\phi_{ij}]$.

(c) Let $f:X\rightarrow Y$, $g:Y\rightarrow Z$ be continuous, (V_i,E_i,s_i,ψ_i), (V_j,E_j,s_j,ψ_j), (V_k,E_k,s_k,ψ_k) be μ-Kuranishi neighbourhoods on X,Y,Z respectively, and $T\subseteq\mathrm{Im}\,\psi_j\cap g^{-1}(\mathrm{Im}\,\psi_k)\subseteq Y$ and $S\subseteq\mathrm{Im}\,\psi_i\cap f^{-1}(T)\subseteq X$ be open. Suppose $[\Phi_{ij}]:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ is a morphism of μ-Kuranishi neighbourhoods over (S,f), and $[\Phi_{jk}]:(V_j,E_j,s_j,\psi_j)\rightarrow(V_k,E_k,s_k,\psi_k)$ a morphism of μ-Kuranishi neighbourhoods over (T,g).

Define the *composition of morphisms* to be

$$[\Phi_{jk}]\circ[\Phi_{ij}]=[\Phi_{jk}\circ\Phi_{ij}]:(V_i,E_i,s_i,\psi_i)\longrightarrow(V_k,E_k,s_k,\psi_k),$$

as a morphism of μ-Kuranishi neighbourhoods over $(S,g\circ f)$. Here we choose representatives Φ_{ij},Φ_{jk} for the equivalence classes $[\Phi_{ij}],[\Phi_{jk}]$, and use the composition of 1-morphisms $\Phi_{jk}\circ\Phi_{ij}$ from §4.1. Properties of 2-categories imply that $[\Phi_{jk}\circ\Phi_{ij}]$ is independent of the choice of Φ_{ij},Φ_{jk}.

Definition 4.8 defined a strict 2-category $\mathbf{m\dot{K}N}$ and 2-subcategories $\mathbf{Gm\dot{K}N}$ and $\mathbf{m\dot{K}N}_S(X)$ for $S\subseteq X$ open. In the same way, we define the *category of μ-Kuranishi neighbourhoods* $\boldsymbol{\mu}\mathbf{\dot{K}N}$, where:

- Objects of $\boldsymbol{\mu}\mathbf{\dot{K}N}$ are triples $(X,S,(V,E,s,\psi))$, with X a topological space, $S\subseteq X$ open, and (V,E,s,ψ) a μ-Kuranishi neighbourhood over S.
- Morphisms $(f,[\Phi_{ij}]):(X,S,(V_i,E_i,s_i,\psi_i))\rightarrow(Y,T,(V_j,E_j,s_j,\psi_j))$ of $\boldsymbol{\mu}\mathbf{\dot{K}N}$ are a pair of a continuous map $f:X\rightarrow Y$ with $S\subseteq f^{-1}(T)\subseteq X$ and a morphism $[\Phi_{ij}]:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ of μ-Kuranishi neighbourhoods over (S,f).
- Identities and composition are defined in the obvious way, using (b),(c).

Define the *category of global μ-Kuranishi neighbourhoods* $\mathbf{G}\boldsymbol{\mu}\mathbf{\dot{K}N}$ to be the full subcategory of $\boldsymbol{\mu}\mathbf{\dot{K}N}$ with objects $(s^{-1}(0),s^{-1}(0),(V,E,s,\mathrm{id}_{s^{-1}(0)}))$ for which $X=S=s^{-1}(0)$ and $\psi=\mathrm{id}_{s^{-1}(0)}$. We usually write objects of $\mathbf{G}\boldsymbol{\mu}\mathbf{\dot{K}N}$ as (V,E,s) rather than $(s^{-1}(0),s^{-1}(0),(V,E,s,\mathrm{id}_{s^{-1}(0)}))$, and we write morphisms of $\mathbf{G}\boldsymbol{\mu}\mathbf{\dot{K}N}$ as $[\Phi_{ij}]:(V_i,E_i,s_i)\rightarrow(V_j,E_j,s_j)$ rather than as $(f,[\Phi_{ij}])$, since $f=\phi_{ij}|_{s_i^{-1}(0)}$ is determined by $[\Phi_{ij}]$ as in Definition 4.8.

Let X be a topological space and $S\subseteq X$ be open. Write $\boldsymbol{\mu}\mathbf{\dot{K}N}_S(X)$ for the subcategory of $\boldsymbol{\mu}\mathbf{\dot{K}N}$ with objects $(X,S,(V,E,s,\psi))$ for X,S as given and morphisms $(\mathrm{id}_X,[\Phi_{ij}]):(X,S,(V_i,E_i,s_i,\psi_i))\rightarrow(X,S,(V_j,E_j,s_j,\psi_j))$ for $f=\mathrm{id}_X$. We call $\boldsymbol{\mu}\mathbf{\dot{K}N}_S(X)$ the *category of μ-Kuranishi neighbourhoods over* $S\subseteq X$. We generally write objects of $\boldsymbol{\mu}\mathbf{\dot{K}N}_S(X)$ as (V,E,s,ψ), omitting X,S, and morphisms of $\mathbf{m\dot{K}N}_S(X)$ as $[\Phi_{ij}]$, omitting id_X.

Then we have equalities $\dot{\boldsymbol{\mu}\mathbf{KN}} = \mathrm{Ho}(\mathbf{m\dot{K}N})$, $\mathbf{G}\boldsymbol{\mu}\mathbf{\dot{K}N} = \mathrm{Ho}(\mathbf{Gm\dot{K}N})$, $\boldsymbol{\mu}\mathbf{\dot{K}N}_S(X) = \mathrm{Ho}(\mathbf{m\dot{K}N}_S(X))$ with the homotopy categories of the strict 2-categories $\mathbf{m\dot{K}N}, \mathbf{Gm\dot{K}N}, \mathbf{m\dot{K}N}_S(X)$ of §4.1.

The accent '$\dot{}$' in $\boldsymbol{\mu}\mathbf{\dot{K}N}, \mathbf{G}\boldsymbol{\mu}\mathbf{\dot{K}N}, \boldsymbol{\mu}\mathbf{\dot{K}N}_S(X)$ is because they are constructed using $\mathbf{\dot{M}an}$. For particular $\mathbf{\dot{M}an}$ we modify the notation in the obvious way, e.g. if $\mathbf{\dot{M}an} = \mathbf{Man}$ we write $\boldsymbol{\mu}\mathbf{KN}, \mathbf{G}\boldsymbol{\mu}\mathbf{KN}, \boldsymbol{\mu}\mathbf{KN}_S(X)$, and if $\mathbf{\dot{M}an} = \mathbf{Man^c}$ we write $\boldsymbol{\mu}\mathbf{KN^c}, \mathbf{G}\boldsymbol{\mu}\mathbf{KN^c}, \boldsymbol{\mu}\mathbf{KN}^{\mathbf{c}}_S(X)$.

If $f : X \to Y$ is continuous, $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)$ are μ-Kuranishi neighbourhoods on X, Y, and $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\psi_j) \subseteq X$ is open, write $\mathrm{Hom}_{S,f}((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$ for the set of morphisms $[\Phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ over (S, f).

If $X = Y$ and $f = \mathrm{id}_X$, we write $\mathrm{Hom}_S((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$ in place of $\mathrm{Hom}_{S,f}((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$.

Remark 5.6. **(a)** In §4.1, for m-Kuranishi neighbourhoods (V_i, E_i, s_i, ψ_i) over S, or 1-morphisms Φ_{ij} over (S, f), the open set $S \subseteq X$ appears only as a condition on (V_i, E_i, s_i, ψ_i) or Φ_{ij}, as we need $S \subseteq \mathrm{Im}\,\psi_i$ or $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\psi_j)$. Thus m-Kuranishi neighbourhoods and their 1-morphisms make sense without knowing S. However, 2-morphisms $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ over (S, f) are equivalence classes under $\sim_S$ depending on S, so do not make sense without specifying S.

Similarly, μ-Kuranishi neighbourhoods (V_i, E_i, s_i, ψ_i) make sense without knowing S, but their morphisms $[\Phi_{ij}]$ are equivalence classes under $\approx_S$ depending on S, so do not make sense without specifying S.

(b) If we define μ-Kuranishi neighbourhoods and their morphisms directly, rather than via m-Kuranishi neighbourhoods and their 1- and 2-morphisms, the definitions and proofs can be simplified a bit. For example, the equivalence relation $\sim_S$ in Definition 4.3 is not needed for the μ-Kuranishi case.

Here are the analogues of Definitions 4.10, 4.11 and Convention 4.12:

Definition 5.7. Let X be a topological space, and $S \subseteq X$ be open, and $[\Phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ be a morphism of μ-Kuranishi neighbourhoods on X over S. Then $[\Phi_{ij}]$ is a morphism in the category $\boldsymbol{\mu}\mathbf{\dot{K}N}_S(X)$ of Definition 5.5. We call $[\Phi_{ij}]$ a *coordinate change over S* if it is an isomorphism in $\boldsymbol{\mu}\mathbf{\dot{K}N}_S(X)$. This holds if and only if any representative Φ_{ij} is an equivalence in $\mathbf{m\dot{K}N}_S(X)$, that is, if and only if Φ_{ij} is a coordinate change of m-Kuranishi neighbourhoods over S, as in Definition 4.10. Write

$$\mathrm{Iso}_S\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big) \subseteq \mathrm{Hom}_S\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)$$

for the subset of coordinate changes $[\Phi_{ij}]$ over S.

Definition 5.8. Let $T \subseteq S \subseteq X$ be open. Define the *restriction functor* $|_T : \boldsymbol{\mu}\mathbf{\dot{K}N}_S(X) \to \boldsymbol{\mu}\mathbf{\dot{K}N}_T(X)$ to map objects (V_i, E_i, s_i, ψ_i) to exactly the same objects, and morphisms $[\Phi_{ij}]$ to $[\Phi_{ij}]|_T$, where $[\Phi_{ij}]|_T$ is the $\approx_T$-equivalence class of any representative Φ_{ij} of the $\approx_S$-equivalence class $[\Phi_{ij}]$. Then $|_T$:

$\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{N}_S(X) \to \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{N}_T(X)$ commutes with all the structure, so it is a functor. If $U \subseteq T \subseteq S \subseteq X$ are open then $|_U \circ |_T = |_U : \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{N}_S(X) \to \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{N}_U(X)$.

Now let $f : X \to Y$ be continuous, $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)$ be μ-Kuranishi neighbourhoods on X, Y, and $T \subseteq S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j) \subseteq X$ be open. Then as for $|_T$ on morphisms above, we define a map

$$\begin{aligned} |_T : \operatorname{Hom}_{S,f}\bigl((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\bigr) \longrightarrow \\ \operatorname{Hom}_{T,f}\bigl((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\bigr). \end{aligned} \tag{5.7}$$

Convention 5.9. *When we do not specify a domain S for a morphism, or coordinate change, of μ-Kuranishi neighbourhoods, the domain should be as large as possible.* For example, if we say that $[\Phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a morphism (or a morphism over $f : X \to Y$) without specifying S, we mean that $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ (or $S = \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$).

Similarly, if we write a formula involving several morphisms or coordinate changes (possibly defined on different domains), without specifying the domain S, we make the convention that *the domain where the formula holds should be as large as possible.* That is, the domain S is taken to be the intersection of the domains of each morphism in the formula, and we implicitly restrict each morphism in the formula to S as in Definition 5.8, to make it make sense.

For example, if we say that $[\Phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$, $[\Phi_{jk}] : (V_j, E_j, s_j, \psi_j) \to (V_k, E_k, s_k, \psi_k)$ and $[\Phi_{ik}] : (V_i, E_i, s_i, \psi_i) \to (V_k, E_k, s_k, \psi_k)$ are morphisms of μ-Kuranishi neighbourhoods on X, and

$$[\Phi_{ik}] = [\Phi_{jk}] \circ [\Phi_{ij}], \tag{5.8}$$

we mean that $[\Phi_{ij}]$ is defined over $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, and $[\Phi_{jk}]$ over $\operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$, and $[\Phi_{ik}]$ over $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_k$, and (5.8) holds over $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$, that is, (5.8) is equivalent to

$$[\Phi_{ik}]|_{\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k} = [\Phi_{jk}]|_{\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k} \circ [\Phi_{ij}]|_{\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k}.$$

Note in particular the potentially confusing point that (5.8) *does not determine* $[\Phi_{ik}]$ *on* $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_k$, *but only on* $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$.

The next theorem is proved by combining Theorem 4.13 and the ideas of §5.1.

Theorem 5.10. *Let $f : X \to Y$ be a continuous map of topological spaces, and $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)$ be μ-Kuranishi neighbourhoods on X, Y. For each open $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j) \subseteq X$, as in Definition 5.5 define a set*

$$\begin{aligned} &\mathcal{H}om_f\bigl((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\bigr)(S) \\ &\qquad = \operatorname{Hom}_{S,f}\bigl((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\bigr), \end{aligned}$$

and for open $T \subseteq S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ as in Definition 5.8 define a map

$$\begin{aligned} \rho_{ST} : \mathcal{H}om_f\bigl((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\bigr)(S) \longrightarrow \\ \mathcal{H}om_f\bigl((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\bigr)(T) \end{aligned}$$

by $\rho_{ST} = |_T$ in (5.7)*. Then $\mathcal{H}om_f((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j))$ is a **sheaf of sets** on the open subset $\operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ in X, as in Definition* A.12.

When $X = Y$ and $f = \mathrm{id}_X$ we write $\mathcal{H}om((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j))$ instead of $\mathcal{H}om_f((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j))$. Then coordinate changes $[\Phi_{ij}] : (V_i,E_i,s_i,\psi_i) \to (V_j,E_j,s_j,\psi_j)$ also form a sheaf $\mathcal{I}so((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j))$ on $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, a subsheaf of $\mathcal{H}om((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j))$.

Proof. For the first part, we must show $\mathcal{H}om_f((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j))$ satisfies the sheaf axioms Definition A.12(i)–(v). Parts (i)–(iii), the presheaf axioms, are immediate. For (iv)–(v), let $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j) \subseteq X$ be open, and $\{T^a : a \in A\}$ be an open cover of S.

For (iv), suppose $[\Phi_{ij}],[\Phi'_{ij}] : (V_i,E_i,s_i,\psi_i) \to (V_j,E_j,s_j,\psi_j)$ are morphisms of μ-Kuranishi neighbourhoods over (S,f), and $[\Phi_{ij}]|_{T^a} = [\Phi'_{ij}]|_{T^a}$ for all $a \in A$. Choose representatives Φ_{ij},Φ'_{ij} for $[\Phi_{ij}],[\Phi'_{ij}]$, so that Φ_{ij},Φ'_{ij} are 1-morphisms of m-Kuranishi neighbourhoods over (S,f). Since $[\Phi_{ij}]|_{T^a} = [\Phi'_{ij}]|_{T^a}$, there exists a 2-morphism $\Lambda^a : \Phi_{ij} \Rightarrow \Phi'_{ij}$ of m-Kuranishi neighbourhoods over (T^a,f) for all $a \in A$. Then Definition 5.4 constructs a 2-morphism $\Lambda_{ij} = \sum_{a\in A} \eta^a \cdot \Lambda^a : \Phi_{ij} \Rightarrow \Phi'_{ij}$ of m-Kuranishi neighbourhoods over (S,f), using a partition of unity $\{\eta^a : a \in A\}$. So Λ_{ij} implies that $[\Phi_{ij}] = [\Phi'_{ij}]$ in morphisms of μ-Kuranishi neighbourhoods over (S,f). Hence Definition A.12(iv) holds.

For (v), suppose $[\Phi^a_{ij}] : (V_i,E_i,s_i,\psi_i) \to (V_j,E_j,s_j,\psi_j)$ are morphisms of μ-Kuranishi neighbourhoods over (T^a,f) for $a \in A$, and $[\Phi^a_{ij}]|_{T^a\cap T^b} = [\Phi^b_{ij}]|_{T^a\cap T^b}$ for all $a,b \in A$. Choose representatives $\Phi^a_{ij} = (V^a_{ij},\phi^a_{ij},\hat\phi^a_{ij})$ for $[\Phi^a_{ij}]$ for $a \in A$, so that Φ^a_{ij} is a 1-morphism of m-Kuranishi neighbourhoods over (T^a,f). Since $[\Phi^a_{ij}]|_{T^a\cap T^b} = [\Phi^b_{ij}]|_{T^a\cap T^b}$, there exists a 2-morphism $\Lambda^{ab} : \Phi^a_{ij} \Rightarrow \Phi^b_{ij}$ of m-Kuranishi neighbourhoods over $(T^a \cap T^b,f)$ for all $a,b \in A$. Choose representatives $(\acute V^{ab},\hat\lambda^{ab})$ for $\Lambda^{ab} = [\acute V^{ab},\hat\lambda^{ab}]$ for $a,b \in A$, so that $\acute V^{ab}$ is an open neighbourhood of $\psi_i^{-1}(T^a \cap T^b)$ in $V^a_{ij} \cap V^b_{ij} \subseteq V_i$.

Define $V_{ij} = \bigcup_{a\in A} V^a_{ij}$, so that V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i. Then $\{V^a_{ij} : a \in A\}$ is an open cover of V_{ij}. Choose a partition of unity $\{\eta^a : a \in A\}$ on V_{ij} subordinate to $\{V^a_{ij} : a \in A\}$, as in §B.1.4. Now for all $a,b,c \in A$, we have a 2-morphism $(\Lambda^{bc})^{-1} \odot \Lambda^{ac} : \Phi^a_{ij} \Rightarrow \Phi^b_{ij}$ of m-Kuranishi neighbourhoods over $(T^a \cap T^b \cap T^c,f)$. And $\{T^a \cap T^b \cap T^c : c \in A\}$ is an open cover of $T^a \cap T^b$. So by Definition 5.4, as in (5.6) we can form a 2-morphism

$$\tilde\Lambda^{ab} = \textstyle\sum_{c\in A} \eta^c \cdot ((\Lambda^{bc})^{-1} \odot \Lambda^{ac}) : \Phi^a_{ij} \Longrightarrow \Phi^b_{ij}$$

over $(T^a \cap T^b,f)$. We claim that these $\tilde\Lambda^{ab}$ satisfy

$$\tilde\Lambda^{bc}|_{T^a\cap T^b\cap T^c} \odot \tilde\Lambda^{ab}|_{T^a\cap T^b\cap T^c} = \tilde\Lambda^{ac}|_{T^a\cap T^b\cap T^c} \quad \text{for all } a,b,c \in A. \tag{5.9}$$

To see this, note that $\tilde\Lambda^{ab} = [\tilde V^{ab}_{ij},\tilde\lambda^{ab}_{ij}]$ with $\tilde\lambda^{ab}_{ij} = \sum_{c\in A} \eta^c \cdot (-\hat\lambda^{bc} + \hat\lambda^{ac})$, and thus on $\tilde V^{ab}_{ij} \cap \tilde V^{bc}_{ij}$ we have

$$\begin{aligned}\tilde\lambda^{bc}_{ij} + \tilde\lambda^{ab}_{ij} &= \left(\textstyle\sum_{d\in A} \eta^d \cdot (-\hat\lambda^{cd} + \hat\lambda^{bd})\right) + \left(\textstyle\sum_{d\in A} \eta^d \cdot (-\hat\lambda^{bd} + \hat\lambda^{ad})\right)\\ &= \textstyle\sum_{d\in A} \eta^d \cdot (-\hat\lambda^{cd} + \hat\lambda^{ad}) = \tilde\lambda^{ac}_{ij}.\end{aligned}$$

Theorem 4.13 says $\boldsymbol{\mathcal{H}om}_f((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j))$ is a stack. Applying Definition A.17(v) to the 1-morphisms $\Phi_{ij}^a:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ over (T^a,f) and 2-morphisms $\tilde\Lambda^{ab}:\Phi_{ij}^a\Rightarrow\Phi_{ij}^b$ over $(T^a\cap T^b,f)$ satisfying (5.9) shows that there exist a 1-morphism $\Phi_{ij}:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ over (S,f) and 2-morphisms $\tilde\Lambda^a:\Phi_{ij}^a\Rightarrow\Phi_{ij}$ over (T^a,f) for $a\in A$ satisfying $\tilde\Lambda^a|_{T^a\cap T^b}=\tilde\Lambda^b|_{T^a\cap T^b}\odot\tilde\Lambda^{ab}$ for all $a,b\in A$. Then $[\Phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ is a morphism of μ-Kuranishi neighbourhoods over (S,f), and $\tilde\Lambda^a:\Phi_{ij}^a\Rightarrow\Phi_{ij}$ implies that $[\Phi_{ij}]|_{T^a}=[\Phi_{ij}^a]$ for all $a\in A$. Hence Definition A.12(v) holds, and $\mathcal{H}om_f((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j))$ is a sheaf. $\square$

We call Theorem 5.10 the *sheaf property.* We will use it in §5.3 to construct compositions of morphisms of μ-Kuranishi spaces.

5.3 The category of μ-Kuranishi spaces

5.3.1 The definition of the category $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$

We give the analogue of §4.3 for μ-Kuranishi spaces. This is much simpler, as we do not have to deal with 2-morphisms.

Definition 5.11. Let X be a Hausdorff, second countable topological space, and $n\in\mathbb{Z}$. A *μ-Kuranishi structure $\mathcal{K}$ on X of virtual dimension n* is data $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},[\Phi_{ij}]_{i,j\in I}\bigr)$, where:

(a) I is an indexing set.

(b) (V_i,E_i,s_i,ψ_i) is a μ-Kuranishi neighbourhood on X for each $i\in I$, with $\dim V_i-\operatorname{rank}E_i=n$.

(c) $[\Phi_{ij}]=[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ is a coordinate change for all $i,j\in I$ (as in Convention 5.9, defined on $S=\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j$).

(d) $\bigcup_{i\in I}\operatorname{Im}\psi_i=X$.

(e) $[\Phi_{ii}]=[\mathrm{id}_{(V_i,E_i,s_i,\psi_i)}]$ for all $i\in I$.

(f) $[\Phi_{jk}]\circ[\Phi_{ij}]=[\Phi_{ik}]$ for all $i,j,k\in I$ (as in Convention 5.9, this holds on $S=\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\cap\operatorname{Im}\psi_k$).

We call $\boldsymbol{X}=(X,\mathcal{K})$ a *μ-Kuranishi space*, of *virtual dimension* $\operatorname{vdim}\boldsymbol{X}=n$. When we write $x\in\boldsymbol{X}$, we mean that $x\in X$.

Example 5.12. Let V be a manifold (object in $\dot{\mathbf{M}}\mathbf{an}$), $E\to V$ a vector bundle, and $s:V\to E$ a smooth section, so that (V,E,s) is an object in $\mathbf{G}\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{N}$ from Definition 5.5. Set $X=s^{-1}(0)$, as a closed subset of V with the induced topology. Then X is Hausdorff and second countable, as V is. Define a μ-Kuranishi structure $\mathcal{K}=\bigl(\{0\},(V_0,E_0,s_0,\psi_0),\Phi_{00}\bigr)$ on X with indexing set $I=\{0\}$, one μ-Kuranishi neighbourhood (V_0,E_0,s_0,ψ_0) with $V_0=V$, $E_0=E$, $s_0=s$ and $\psi_0=\mathrm{id}_X$, and one coordinate change $\Phi_{00}=\mathrm{id}_{(V_0,E_0,s_0,\psi_0)}$. Then $\boldsymbol{X}=(X,\mathcal{K})$ is a μ-Kuranishi space, with $\operatorname{vdim}\boldsymbol{X}=\dim V-\operatorname{rank}E$. We write $\boldsymbol{S}_{V,E,s}=\boldsymbol{X}$.

When we are discussing several μ-Kuranishi spaces at once, we need notation to distinguish μ-Kuranishi neighbourhoods and coordinate changes on the different spaces. As for (4.5)–(4.8), one choice we will often use for μ-Kuranishi spaces $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ is

$$\begin{aligned} \boldsymbol{W} = (W, \mathcal{H}),\ \mathcal{H} = \bigl(H, (T_h, C_h, q_h, \varphi_h)_{h\in H},& \\ [\Sigma_{hh'}] = [T_{hh'}, \sigma_{hh'}, \hat\sigma_{hh'}]_{h,h'\in H}\bigr),& \end{aligned} \tag{5.10}$$

$$\boldsymbol{X} = (X, \mathcal{I}),\ \mathcal{I} = \bigl(I, (U_i, D_i, r_i, \chi_i)_{i\in I}, [\mathrm{T}_{ii'}] = [U_{ii'}, \tau_{ii'}, \hat\tau_{ii'}]_{i,i'\in I}\bigr), \tag{5.11}$$

$$\boldsymbol{Y} = (Y, \mathcal{J}),\ \mathcal{J} = \bigl(J, (V_j, E_j, s_j, \psi_j)_{j\in J}, [\Upsilon_{jj'}] = [V_{jj'}, \upsilon_{jj'}, \hat\upsilon_{jj'}]_{j,j'\in J}\bigr), \tag{5.12}$$

$$\begin{aligned} \boldsymbol{Z} = (Z, \mathcal{K}),\ \mathcal{K} = \bigl(K, (W_k, F_k, t_k, \omega_k)_{k\in K},& \\ [\Phi_{kk'}] = [W_{kk'}, \phi_{kk'}, \hat\phi_{kk'}]_{k,k'\in K}\bigr).& \end{aligned} \tag{5.13}$$

Definition 5.13. Let $\boldsymbol{X} = (X, \mathcal{I})$ and $\boldsymbol{Y} = (Y, \mathcal{J})$ be μ-Kuranishi spaces, with notation (5.11)–(5.12). A *morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is $\boldsymbol{f} = \bigl(f, [\boldsymbol{f}_{ij}]_{i\in I,\ j\in J}\bigr)$, where $f : X \to Y$ is a continuous map, and $[\boldsymbol{f}_{ij}] = [U_{ij}, f_{ij}, \hat f_{ij}] : (U_i, D_i, r_i, \chi_i) \to (V_j, E_j, s_j, \psi_j)$ is a morphism of μ-Kuranishi neighbourhoods over f for all $i \in I$, $j \in J$ (defined over $S = \operatorname{Im}\chi_i \cap f^{-1}(\operatorname{Im}\psi_j)$, by Convention 5.9), satisfying:

(a) If $i, i' \in I$ and $j \in J$ then in morphisms over f we have

$$[\boldsymbol{f}_{i'j}] \circ [\mathrm{T}_{ii'}] = [\boldsymbol{f}_{ij}], \tag{5.14}$$

where (5.14) holds over $S = \operatorname{Im}\chi_i \cap \operatorname{Im}\chi_{i'} \cap f^{-1}(\operatorname{Im}\psi_j)$ by Convention 5.9, and each term in (5.14) is implicitly restricted to S. In particular, (5.14) *does not determine* $\boldsymbol{f}_{ij}$, but only its restriction $[\boldsymbol{f}_{ij}]|_S$.

(b) If $i \in I$ and $j, j' \in J$ then interpreted as for (5.14), we have

$$[\Upsilon_{jj'}] \circ [\boldsymbol{f}_{ij}] = [\boldsymbol{f}_{ij'}]. \tag{5.15}$$

If $x \in \boldsymbol{X}$ (i.e. $x \in X$), we will write $\boldsymbol{f}(x) = f(x) \in \boldsymbol{Y}$.

When $\boldsymbol{Y} = \boldsymbol{X}$, so that $J = I$, define $\mathbf{id}_{\boldsymbol{X}} = \bigl(\mathrm{id}_X, [\mathrm{T}_{ij}]_{i,j\in I}\bigr)$. Then Definition 5.11(f) implies that (a),(b) hold, so $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ is a morphism of μ-Kuranishi spaces, which we call the *identity morphism*.

In the next theorem, we use the sheaf property of morphisms of μ-Kuranishi neighbourhoods in Theorem 5.10 to construct compositions $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ of morphisms of μ-Kuranishi spaces $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$, and hence show that μ-Kuranishi spaces form a category $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$.

In §4.3 we made arbitrary choices to define composition of 1-morphisms of m-Kuranishi spaces. For μ-Kuranishi spaces, composition is canonical.

Theorem 5.14. **(a)** *Let* $\boldsymbol{X} = (X, \mathcal{I}), \boldsymbol{Y} = (Y, \mathcal{J}), \boldsymbol{Z} = (Z, \mathcal{K})$ *be μ-Kuranishi spaces with notation* (5.11)–(5.13), *and* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ *be morphisms, where* $\boldsymbol{f} = \bigl(f, [\boldsymbol{f}_{ij}]_{i\in I,\ j\in J}\bigr)$, $\boldsymbol{g} = \bigl(g, [\boldsymbol{g}_{jk}]_{j\in J,\ k\in K}\bigr)$. *Then there exists a unique* *morphism* $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Z}$, *where* $\boldsymbol{h} = \bigl(h, [\boldsymbol{h}_{ik}]_{i\in I,\, k\in K}\bigr)$ *such that* $h = g \circ f : X \to Z$, *and for all* $i \in I$, $j \in J$, $k \in K$ *we have*

$$[\boldsymbol{h}_{ik}] = [\boldsymbol{g}_{jk}] \circ [\boldsymbol{f}_{ij}], \tag{5.16}$$

where by Convention 5.9, (5.16) *holds over* $\operatorname{Im}\chi_i \cap f^{-1}(\operatorname{Im}\psi_j) \cap h^{-1}(\operatorname{Im}\omega_k)$, *and so may not determine* $[\boldsymbol{h}_{ik}]$ *over* $\operatorname{Im}\chi_i \cap h^{-1}(\operatorname{Im}\omega_k)$.

We write $\boldsymbol{g} \circ \boldsymbol{f} = \boldsymbol{h}$, *so that* $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ *is a morphism of* μ*-Kuranishi spaces, and call* $\boldsymbol{g} \circ \boldsymbol{f}$ *the* ***composition*** *of* $\boldsymbol{f}, \boldsymbol{g}$.

(b) *Composition of morphisms is associative, that is, if* $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{Z}$ *is another morphism of* μ*-Kuranishi spaces then* $(\boldsymbol{g} \circ \boldsymbol{f}) \circ \boldsymbol{e} = \boldsymbol{g} \circ (\boldsymbol{f} \circ \boldsymbol{e})$.

(c) *Composition is compatible with identities, that is,* $\boldsymbol{f} \circ \mathbf{id}_{\boldsymbol{X}} = \mathbf{id}_{\boldsymbol{Y}} \circ \boldsymbol{f} = \boldsymbol{f}$ *for all morphisms of* μ*-Kuranishi spaces* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$.

Thus μ*-Kuranishi spaces form a category, which we write as* $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$.

Proof. For (a), define $h = g \circ f : X \to Z$. Let $i \in I$ and $k \in K$, and set $S = \operatorname{Im}\chi_i \cap h^{-1}(\operatorname{Im}\omega_k)$, so that S is open in X. We want to define a morphism $[\boldsymbol{h}_{ik}] : (U_i, D_i, r_i, \chi_i) \to (W_k, F_k, t_k, \omega_k)$ of μ-Kuranishi neighbourhoods over (S, h). Equation (5.16) means that for each $j \in J$ we must have

$$[\boldsymbol{h}_{ik}]|_{S\cap f^{-1}(\operatorname{Im}\psi_j)} = [\boldsymbol{g}_{jk}] \circ [\boldsymbol{f}_{ij}]|_{S\cap f^{-1}(\operatorname{Im}\psi_j)}. \tag{5.17}$$

As $\{\operatorname{Im}\psi_j : j \in J\}$ is an open cover of Y and f is continuous, $\bigl\{S \cap f^{-1}(\operatorname{Im}\psi_j) : j \in J\bigr\}$ is an open cover of S. For all $j, j' \in J$ we have

$$\begin{aligned}[\boldsymbol{g}_{jk}] \circ [\boldsymbol{f}_{ij}]|_{S\cap f^{-1}(\operatorname{Im}\psi_j)\cap f^{-1}(\operatorname{Im}\psi_{j'})} &= [\boldsymbol{g}_{j'k}] \circ [\Upsilon_{jj'}] \circ [\boldsymbol{f}_{ij}]|_{\cdots} \\ &= [\boldsymbol{g}_{j'k}] \circ [\boldsymbol{f}_{ij'}]|_{S\cap f^{-1}(\operatorname{Im}\psi_j)\cap f^{-1}(\operatorname{Im}\psi_{j'})},\end{aligned} \tag{5.18}$$

using (5.14) for $\boldsymbol{g}$ in the first step, and (5.15) for $\boldsymbol{f}$ in the second.

Now the right hand side of (5.17) prescribes values for a morphism over h on the sets of an open cover $\{S \cap f^{-1}(\operatorname{Im}\psi_j) : j \in J\}$ of S. Equation (5.18) shows that these values agree on overlaps $(S \cap f^{-1}(\operatorname{Im}\psi_j)) \cap (S \cap f^{-1}(\operatorname{Im}\psi_{j'}))$. Therefore the sheaf property Theorem 5.10 shows that there is a unique morphism $[\boldsymbol{h}_{ik}]$ over (S, h) satisfying (5.17) for all $j \in J$.

We have now defined $\boldsymbol{h} = \bigl(h, [\boldsymbol{h}_{ik}]_{i\in I,\, k\in K}\bigr)$. To show $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Z}$ is a morphism, we must verify Definition 5.13(a),(b). For (a), suppose $i, i' \in I$, $j \in J$ and $k \in K$. Then we have

$$\begin{aligned}[\boldsymbol{h}_{i'k}] \circ [\mathrm{T}_{ii'}]|_{\operatorname{Im}\chi_i\cap\operatorname{Im}\chi_{i'}\cap f^{-1}(\operatorname{Im}\psi_j)\cap h^{-1}(\operatorname{Im}\omega_k)} &= [\boldsymbol{g}_{jk}] \circ [\boldsymbol{f}_{i'j}] \circ [\mathrm{T}_{ii'}]|_{\cdots} \\ = [\boldsymbol{g}_{jk}] \circ [\boldsymbol{f}_{ij}]|_{\cdots} &= [\boldsymbol{h}_{ij}]|_{\operatorname{Im}\chi_i\cap\operatorname{Im}\chi_{i'}\cap f^{-1}(\operatorname{Im}\psi_j)\cap h^{-1}(\operatorname{Im}\omega_k)},\end{aligned}$$

using (5.17) with i' in place of i in the first step, (5.14) for $\boldsymbol{f}$ in the second, and (5.17) in the third. This proves the restriction of (5.14) for $\boldsymbol{h}, i, i', k$ to $\operatorname{Im}\chi_i \cap \operatorname{Im}\chi_{i'} \cap f^{-1}(\operatorname{Im}\psi_j) \cap h^{-1}(\operatorname{Im}\omega_k)$, for each $j \in J$.

Since the $\operatorname{Im}\chi_i \cap \operatorname{Im}\chi_{i'} \cap f^{-1}(\operatorname{Im}\psi_j) \cap h^{-1}(\operatorname{Im}\omega_k)$ for $j \in J$ form an open cover of $\operatorname{Im}\chi_i \cap \operatorname{Im}\chi_{i'} \cap h^{-1}(\operatorname{Im}\omega_k)$, Theorem 5.10 implies that (5.14) holds for

$\boldsymbol{h}, i, i', k$ on the correct domain $\operatorname{Im}\chi_i \cap \operatorname{Im}\chi_{i'} \cap h^{-1}(\operatorname{Im}\omega_k)$, yielding Definition 5.13(a) for $\boldsymbol{h}$. Definition 5.13(b) follows by a similar argument, involving (5.15) for $\boldsymbol{g}$. Hence $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Z}$ is a morphism, proving part (a).

For (b), in notation (5.10)–(5.13), if $h \in H$, $i \in I$, $j \in J$, $k \in K$ we find that

$$\begin{aligned}
&[((\boldsymbol{g} \circ \boldsymbol{f}) \circ \boldsymbol{e})_{h,k}]|_{\operatorname{Im}\varphi_h \cap e^{-1}(\operatorname{Im}\chi_i) \cap (f\circ e)^{-1}(\operatorname{Im}\psi_j) \cap (g\circ f\circ e)^{-1}(\operatorname{Im}\omega_k)} \\
&\qquad = [\boldsymbol{g}_{jk}] \circ [\boldsymbol{f}_{ij}] \circ [\boldsymbol{e}_{hi}] \\
&\qquad = [(\boldsymbol{g} \circ (\boldsymbol{f} \circ \boldsymbol{e}))_{h,k}]|_{\operatorname{Im}\varphi_h \cap e^{-1}(\operatorname{Im}\chi_i) \cap (f\circ e)^{-1}(\operatorname{Im}\psi_j) \cap (g\circ f\circ e)^{-1}(\operatorname{Im}\omega_k)},
\end{aligned}$$

where the middle step makes sense without brackets by associativity of composition of morphisms of μ-Kuranishi neighbourhoods. Since $\operatorname{Im}\varphi_h \cap e^{-1}(\operatorname{Im}\chi_i) \cap (f \circ e)^{-1}(\operatorname{Im}\psi_j) \cap (g \circ f \circ e)^{-1}(\operatorname{Im}\omega_k)$ for all $i \in I$, $j \in J$ form an open cover of $\operatorname{Im}\varphi_h \cap (g \circ f \circ e)^{-1}(\operatorname{Im}\omega_k)$, Theorem 5.10 implies that $\big[((\boldsymbol{g} \circ \boldsymbol{f}) \circ \boldsymbol{e})_{h,k}\big] = \big[(\boldsymbol{g} \circ (\boldsymbol{f} \circ \boldsymbol{e}))_{h,k}\big]$ over the correct domain $\operatorname{Im}\varphi_h \cap (g \circ f \circ e)^{-1}(\operatorname{Im}\omega_k)$, so that $(\boldsymbol{g} \circ \boldsymbol{f}) \circ \boldsymbol{e} = \boldsymbol{g} \circ (\boldsymbol{f} \circ \boldsymbol{e})$, proving (b).

For (c), let $i \in I$ and $j \in J$. Then we have

$$[(\boldsymbol{f} \circ \mathbf{id}_{\boldsymbol{X}})_{i,j}] = [\boldsymbol{f}_{ij}] \circ [\mathrm{T}_{ii}] = [\boldsymbol{f}_{ij}] \circ [\mathrm{id}_{(U_i,D_i,r_i,\chi_i)}] = [\boldsymbol{f}_{ij}],$$

using (5.16) and the definition of $\mathbf{id}_{\boldsymbol{X}}$ in the first step, and Definition 5.11(e) in the second. Thus $\boldsymbol{f} \circ \mathbf{id}_{\boldsymbol{X}} = \boldsymbol{f}$. We show that $\mathbf{id}_{\boldsymbol{Y}} \circ \boldsymbol{f} = \boldsymbol{f}$ in the same way. This completes the proof. □

5.3.2 Examples of categories $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$

Here are the analogues of Definition 4.29 and Example 4.30:

Definition 5.15. In Theorem 5.14 we write $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ for the category of μ-Kuranishi spaces constructed from our chosen category $\dot{\mathbf{M}}\mathbf{an}$ satisfying Assumptions 3.1–3.7 in §3.1. By Example 3.8, the following categories from Chapter 2 are possible choices for $\dot{\mathbf{M}}\mathbf{an}$:

$$\mathbf{Man}, \mathbf{Man^c_{we}}, \mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}. \tag{5.19}$$

We write the corresponding categories of μ-Kuranishi spaces as follows:

$$\boldsymbol{\mu}\mathbf{Kur}, \boldsymbol{\mu}\mathbf{Kur^c_{we}}, \boldsymbol{\mu}\mathbf{Kur^c}, \boldsymbol{\mu}\mathbf{Kur^{gc}}, \boldsymbol{\mu}\mathbf{Kur^{ac}}, \boldsymbol{\mu}\mathbf{Kur^{c,ac}}. \tag{5.20}$$

Example 5.16. We will define a functor $F_{\dot{\mathbf{M}}\mathbf{an}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}} : \dot{\mathbf{M}}\mathbf{an} \to \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$. On objects, if $X \in \dot{\mathbf{M}}\mathbf{an}$ define a μ-Kuranishi space $F_{\dot{\mathbf{M}}\mathbf{an}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}(X) = \boldsymbol{X} = (X, \mathcal{K})$ with topological space X and μ-Kuranishi structure $\mathcal{K} = (\{0\}, (V_0, E_0, s_0, \psi_0), [\Phi_{00}])$, with indexing set $I = \{0\}$, one μ-Kuranishi neighbourhood (V_0, E_0, s_0, ψ_0) with $V_0 = X$, $E_0 \to V_0$ the zero vector bundle, $s_0 = 0$, and $\psi_0 = \mathrm{id}_X$, and one coordinate change $[\Phi_{00}] = [\mathrm{id}_{(V_0,E_0,s_0,\psi_0)}]$.

On morphisms, if $f : X \to Y$ is a morphism in $\dot{\mathbf{M}}\mathbf{an}$ and $\boldsymbol{X} = F_{\dot{\mathbf{M}}\mathbf{an}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}(X)$, $\boldsymbol{Y} = F_{\dot{\mathbf{M}}\mathbf{an}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}(Y)$, define a morphism $F_{\dot{\mathbf{M}}\mathbf{an}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}(f) = \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ by $\boldsymbol{f} = (f, [\boldsymbol{f}_{00}])$, where $[\boldsymbol{f}_{00}] = [V_{00}, f_{00}, \hat{f}_{00}]$ with $V_{00} = X$, $f_{00} = f$, and $\hat{f}_{00}$ is the zero map on zero vector bundles.

It is now easy to check that $F_{\dot{\mathbf{Man}}}^{\boldsymbol{\mu}\dot{\mathbf{Kur}}}$ is a functor, which is full and faithful, and thus embeds $\dot{\mathbf{Man}}$ as a full subcategory of $\boldsymbol{\mu}\dot{\mathbf{Kur}}$. So we can identify $\dot{\mathbf{Man}}$ with its image in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$. We say that a μ-Kuranishi space $\boldsymbol{X}$ *is a manifold* if $\boldsymbol{X} \cong F_{\dot{\mathbf{Man}}}^{\boldsymbol{\mu}\dot{\mathbf{Kur}}}(X')$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$, for some $X' \in \dot{\mathbf{Man}}$.

Assumption 3.4 gives a full subcategory $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$. Define a full and faithful functor $F_{\mathbf{Man}}^{\boldsymbol{\mu}\dot{\mathbf{Kur}}} = F_{\dot{\mathbf{Man}}}^{\boldsymbol{\mu}\dot{\mathbf{Kur}}}|_{\mathbf{Man}} : \mathbf{Man} \to \boldsymbol{\mu}\dot{\mathbf{Kur}}$, which embeds $\mathbf{Man}$ as a full subcategory of $\boldsymbol{\mu}\dot{\mathbf{Kur}}$. We say that a μ-Kuranishi space $\boldsymbol{X}$ *is a classical manifold* if $\boldsymbol{X} \cong F_{\mathbf{Man}}^{\boldsymbol{\mu}\dot{\mathbf{Kur}}}(X')$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$, for some $X' \in \mathbf{Man}$.

In a similar way to Example 5.16, we can define a functor $\mathbf{G}\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{N} \to \boldsymbol{\mu}\dot{\mathbf{Kur}}$ which is an equivalence from the category $\mathbf{G}\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{N}$ of global μ-Kuranishi neighbourhoods in Definition 5.5 to the full subcategory of objects $(X, \mathcal{K})$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ for which $\mathcal{K}$ contains only one μ-Kuranishi neighbourhood. It acts by $(V, E, s) \mapsto \boldsymbol{S}_{V,E,s}$ on objects, where $\boldsymbol{S}_{V,E,s}$ is as in Example 5.12.

Example 5.17. As in Example 4.31, if $\boldsymbol{X}, \boldsymbol{Y}$ are μ-Kuranishi spaces in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ with notation (5.11)–(5.12), we can define an explicit *product* $\boldsymbol{X} \times \boldsymbol{Y}$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ with $\operatorname{vdim}(\boldsymbol{X} \times \boldsymbol{Y}) = \operatorname{vdim} \boldsymbol{X} + \operatorname{vdim} \boldsymbol{Y}$, such that $\boldsymbol{X} \times \boldsymbol{Y} = (X \times Y, \mathcal{K})$ with

$$\mathcal{K} = \big(I \times J, (W_{(i,j)}, F_{(i,j)}, t_{(i,j)}, \omega_{(i,j)})_{(i,j) \in I \times J},\ [\Phi_{(i,j)(i',j')}]_{(i,j),(i',j') \in I \times J}\big)$$

for $(W_{(i,j)}, F_{(i,j)}, t_{(i,j)}, \omega_{(i,j)}), \Phi_{(i,j)(i',j')}$ as in Example 4.31. There are natural projection morphisms $\boldsymbol{\pi_X} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{X}$, $\boldsymbol{\pi_Y} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{Y}$. These have the universal property of products in an ordinary category, that is, $\boldsymbol{X} \times \boldsymbol{Y}$ is a fibre product $\boldsymbol{X} \times_{\boldsymbol{*}} \boldsymbol{Y}$ over the point (terminal object) $\boldsymbol{*}$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$.

Products are commutative and associative up to canonical isomorphism. We can also define products and direct products of morphisms. That is, if $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Z}$ are morphisms in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ then we have a *product morphism* $\boldsymbol{f} \times \boldsymbol{h} : \boldsymbol{W} \times \boldsymbol{X} \to \boldsymbol{Y} \times \boldsymbol{Z}$ and a *direct product morphism* $(\boldsymbol{g}, \boldsymbol{h}) : \boldsymbol{X} \to \boldsymbol{Y} \times \boldsymbol{Z}$ in $\mathbf{m}\dot{\mathbf{Kur}}$, both easy to write down explicitly.

5.3.3 Comparing μ-Kuranishi spaces from different $\dot{\mathbf{Man}}$

As in §4.4, following Definition 4.32, we easily prove:

Proposition 5.18. *Suppose* $\dot{\mathbf{Man}}, \ddot{\mathbf{Man}}$ *are categories satisfying Assumptions* 3.1–3.7, *and* $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \to \ddot{\mathbf{Man}}$ *is a functor satisfying Condition* 3.20. *Then we can define a natural functor* $F_{\boldsymbol{\mu}\dot{\mathbf{Kur}}}^{\boldsymbol{\mu}\ddot{\mathbf{Kur}}} : \boldsymbol{\mu}\dot{\mathbf{Kur}} \to \boldsymbol{\mu}\ddot{\mathbf{Kur}}$.

If $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \hookrightarrow \ddot{\mathbf{Man}}$ *is an inclusion of subcategories* $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$ *satisfying either Proposition* 3.21(a) *or* (b), *then* $F_{\boldsymbol{\mu}\dot{\mathbf{Kur}}}^{\boldsymbol{\mu}\ddot{\mathbf{Kur}}} : \boldsymbol{\mu}\dot{\mathbf{Kur}} \hookrightarrow \boldsymbol{\mu}\ddot{\mathbf{Kur}}$ *is also an inclusion of subcategories* $\boldsymbol{\mu}\dot{\mathbf{Kur}} \subseteq \boldsymbol{\mu}\ddot{\mathbf{Kur}}$.

As for Figure 4.1, applying Proposition 5.18 to the parts of the diagram Figure 3.1 of functors $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ involving the categories (5.19) yields a diagram Figure 5.1 of functors $F_{\boldsymbol{\mu}\dot{\mathbf{Kur}}}^{\boldsymbol{\mu}\ddot{\mathbf{Kur}}}$. Arrows '$\rightarrow$' are inclusions of subcategories.

$$\begin{tikzcd}
 & & \boldsymbol{\mu}\mathbf{Kur} \arrow[dll, bend right=10] \arrow[dl] \arrow[dr] \arrow[drr, bend left=10] & & \\
\boldsymbol{\mu}\mathbf{Kur^{gc}} & \boldsymbol{\mu}\mathbf{Kur^{c}} \arrow[l] \arrow[rr] \arrow[d, dotted] & & \boldsymbol{\mu}\mathbf{Kur^{c,ac}} \arrow[r, dotted, shift left] & \boldsymbol{\mu}\mathbf{Kur^{ac}} \arrow[l, shift left] \\
 & \boldsymbol{\mu}\mathbf{Kur^{c}_{we}} & & &
\end{tikzcd}$$

Figure 5.1: Functors between categories of μ-Kuranishi spaces from Proposition 5.18. Arrows '$\rightarrow$' are inclusions of subcategories.

5.3.4 Discrete properties of morphisms in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$

In §3.3.6 and §B.6 we defined when a property $\boldsymbol{P}$ of morphisms in $\dot{\mathbf{Man}}$ is *discrete*. Section 4.5 explained how to extend discrete properties of morphisms in $\dot{\mathbf{Man}}$ to corresponding properties of 1-morphisms in $\mathbf{m}\dot{\mathbf{Kur}}$. We now do the same for $\boldsymbol{\mu}\dot{\mathbf{Kur}}$. Here are the analogues of Definition 4.35, and Proposition 4.36(b),(c), proved in the same way, and Definition 4.37.

Definition 5.19. Let $\boldsymbol{P}$ be a discrete property of morphisms in $\dot{\mathbf{Man}}$. Suppose $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ is a morphism in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$. Use notation (5.11)–(5.12) for $\boldsymbol{X},\boldsymbol{Y}$, and write $\boldsymbol{f}=\bigl(f,[\boldsymbol{f}_{ij}]_{i\in I,\,j\in J}\bigr)$ as in Definition 5.13. We say that $\boldsymbol{f}$ *is* $\boldsymbol{P}$ if $\boldsymbol{f}_{ij}$ is $\boldsymbol{P}$ in the sense of Definition 4.33 for all $i\in I$ and $j\in J$. This is independent of the choice of representative $\boldsymbol{f}_{ij}$ for $[\boldsymbol{f}_{ij}]$ in $\boldsymbol{f}$ by Proposition 4.34(b).

Proposition 5.20. *Let $\boldsymbol{P}$ be a discrete property of morphisms in $\dot{\mathbf{Man}}$. Then:*

- **(a)** *Let $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ and $\boldsymbol{g}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ be morphisms in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$. If $\boldsymbol{f}$ and $\boldsymbol{g}$ are $\boldsymbol{P}$ then $\boldsymbol{g}\circ\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$ is $\boldsymbol{P}$.*
- **(b)** *Identity morphisms $\mathbf{id}_{\boldsymbol{X}}:\boldsymbol{X}\rightarrow\boldsymbol{X}$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ are $\boldsymbol{P}$. Isomorphisms $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ are $\boldsymbol{P}$.*

Parts **(a)**,**(b)** *imply that we have a subcategory $\boldsymbol{\mu}\dot{\mathbf{Kur}}_{\boldsymbol{P}}\subseteq\boldsymbol{\mu}\dot{\mathbf{Kur}}$ containing all objects in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$, and all morphisms $\boldsymbol{f}$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ which are $\boldsymbol{P}$.*

Definition 5.21. **(a)** Taking $\dot{\mathbf{Man}}=\mathbf{Man^c}$ from §2.1 gives the category of μ-Kuranishi spaces with corners $\boldsymbol{\mu}\mathbf{Kur^c}$ from Definition 5.15. We write

$$\boldsymbol{\mu}\mathbf{Kur^c_{in}},\boldsymbol{\mu}\mathbf{Kur^c_{bn}},\boldsymbol{\mu}\mathbf{Kur^c_{st}},\boldsymbol{\mu}\mathbf{Kur^c_{st,in}},\boldsymbol{\mu}\mathbf{Kur^c_{st,bn}},\boldsymbol{\mu}\mathbf{Kur^c_{si}}$$

for the subcategories of $\boldsymbol{\mu}\mathbf{Kur^c}$ with morphisms which are *interior*, and *b-normal*, and *strongly smooth*, and *strongly smooth-interior*, and *strongly smooth-b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^c}$ are discrete by Example 3.19(a), so as in Definition 5.19 and Proposition 5.20 we have corresponding notions of interior, ..., simple morphisms in $\boldsymbol{\mu}\mathbf{Kur^c}$.

(b) Taking $\dot{\mathbf{Man}} = \mathbf{Man^{gc}}$ from §2.4.1 gives the category of μ-Kuranishi spaces with g-corners $\boldsymbol{\mu}\mathbf{Kur^{gc}}$ from Definition 5.15. We write

$$\boldsymbol{\mu}\mathbf{Kur^{gc}_{in}}, \boldsymbol{\mu}\mathbf{Kur^{gc}_{bn}}, \boldsymbol{\mu}\mathbf{Kur^{gc}_{si}}$$

for the subcategories of $\boldsymbol{\mu}\mathbf{Kur^{gc}}$ with morphisms which are *interior*, and *b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^{gc}}$ are discrete by Example 3.19(b), so we have corresponding notions in $\boldsymbol{\mu}\mathbf{Kur^{gc}}$.

(c) Taking $\dot{\mathbf{Man}} = \mathbf{Man^{ac}}$ from §2.4.2 gives the category of μ-Kuranishi spaces with a-corners $\boldsymbol{\mu}\mathbf{Kur^{ac}}$ from Definition 5.15. We write

$$\boldsymbol{\mu}\mathbf{Kur^{ac}_{in}}, \boldsymbol{\mu}\mathbf{Kur^{ac}_{bn}}, \boldsymbol{\mu}\mathbf{Kur^{ac}_{st}}, \boldsymbol{\mu}\mathbf{Kur^{ac}_{st,in}}, \boldsymbol{\mu}\mathbf{Kur^{ac}_{st,bn}}, \boldsymbol{\mu}\mathbf{Kur^{ac}_{si}}$$

for the subcategories of $\boldsymbol{\mu}\mathbf{Kur^{ac}}$ with morphisms which are *interior*, and *b-normal*, and *strongly a-smooth*, and *strongly a-smooth-interior*, and *strongly a-smooth-b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^{ac}}$ are discrete by Example 3.19(c), so we have corresponding notions for morphisms in $\boldsymbol{\mu}\mathbf{Kur^{ac}}$.

(d) Taking $\dot{\mathbf{Man}} = \mathbf{Man^{c,ac}}$ from §2.4.2 gives the category of μ-Kuranishi spaces with corners and a-corners $\boldsymbol{\mu}\mathbf{Kur^{c,ac}}$ from Definition 5.15. We write

$$\boldsymbol{\mu}\mathbf{Kur^{c,ac}_{in}}, \boldsymbol{\mu}\mathbf{Kur^{c,ac}_{bn}}, \boldsymbol{\mu}\mathbf{Kur^{c,ac}_{st}}, \boldsymbol{\mu}\mathbf{Kur^{c,ac}_{st,in}}, \boldsymbol{\mu}\mathbf{Kur^{c,ac}_{st,bn}}, \boldsymbol{\mu}\mathbf{Kur^{c,ac}_{si}}$$

for the subcategories of $\boldsymbol{\mu}\mathbf{Kur^{c,ac}}$ with morphisms which are *interior*, and *b-normal*, and *strongly a-smooth*, and *strongly a-smooth-interior*, and *strongly a-smooth-b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^{c,ac}}$ are discrete by Example 3.19(c), so we have corresponding notions for morphisms in $\boldsymbol{\mu}\mathbf{Kur^{c,ac}}$.

Figure 5.1 gives inclusions between the categories in (5.20). Combining this with the inclusions between the subcategories in Definition 5.21 we get a diagram Figure 5.2 of inclusions of subcategories of μ-Kuranishi spaces, as for Figure 4.2.

5.3.5 μ-Kuranishi spaces and m-Kuranishi spaces

Next we relate μ-Kuranishi spaces to m-Kuranishi spaces in §4.3.

Definition 5.22. We will define a functor $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}} : \mathrm{Ho}(\mathbf{m\dot{K}ur}) \to \boldsymbol{\mu}\mathbf{\dot{K}ur}$, where $\mathrm{Ho}(\mathbf{m\dot{K}ur})$ is the homotopy category of the weak 2-category $\mathbf{m\dot{K}ur}$ as in §A.2, that is, the category with objects $\boldsymbol{X}, \boldsymbol{Y}$ objects of $\mathbf{m\dot{K}ur}$, and morphisms $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ are 2-isomorphism classes $[\boldsymbol{f}]$ of 1-morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$.

Let $\boldsymbol{X} = (X, \mathcal{K})$ be an object of $\mathbf{m\dot{K}ur}$, with $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i \in I}, \Phi_{ij,\, i,j \in I},\ \Lambda_{ijk,\, i,j,k \in I}\big)$. Then (V_i, E_i, s_i, ψ_i) is a μ-Kuranishi neighbourhood on X for each $i \in I$, and taking the $\approx_S$-equivalence class $[\Phi_{ij}]$ of Φ_{ij} over $S = \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$ as in Definition 5.5(b) gives a coordinate change $[\Phi_{ij}]$:

$$
\begin{tikzcd}[column sep=small]
 & & & \boldsymbol{\mu}\mathbf{Kur} \arrow[dlll, bend right] \arrow[dl] \arrow[dr] \arrow[drrr, bend left] & & & \\
\boldsymbol{\mu}\mathbf{Kur}^{\mathbf{gc}}_{\mathbf{si}} \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c}}_{\mathbf{si}} \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{si}} \arrow[dl] \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{ac}}_{\mathbf{si}} \arrow[ll] \arrow[dl] \arrow[dd] \\
 & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c}}_{\mathbf{st,bn}} \arrow[rr] \arrow[dl] \arrow[dr] \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{st,bn}} \arrow[dr] \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{ac}}_{\mathbf{st,bn}} \arrow[ll] \arrow[dr] \arrow[dd] & \\
\boldsymbol{\mu}\mathbf{Kur}^{\mathbf{gc}}_{\mathbf{bn}} \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c}}_{\mathbf{bn}} \arrow[ll] \arrow[rr] \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{bn}} \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{ac}}_{\mathbf{bn}} \arrow[ll] \arrow[dd] \\
 & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c}}_{\mathbf{st,in}} \arrow[rr] \arrow[dl] \arrow[dr] \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{st,in}} \arrow[dr] \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{ac}}_{\mathbf{st,in}} \arrow[ll] \arrow[dr] \arrow[dd] & \\
\boldsymbol{\mu}\mathbf{Kur}^{\mathbf{gc}}_{\mathbf{in}} \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c}}_{\mathbf{in}} \arrow[ll] \arrow[rr] \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{in}} \arrow[dd] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{ac}}_{\mathbf{in}} \arrow[ll] \arrow[dd] \\
 & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c}}_{\mathbf{st}} \arrow[rr] \arrow[dl] \arrow[dr] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{st}} \arrow[dr] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{ac}}_{\mathbf{st}} \arrow[ll] \arrow[dr] & \\
\boldsymbol{\mu}\mathbf{Kur}^{\mathbf{gc}} & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c}} \arrow[ll] \arrow[rr] & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{c,ac}} & & \boldsymbol{\mu}\mathbf{Kur}^{\mathbf{ac}} \arrow[ll]
\end{tikzcd}
$$

Figure 5.2: Inclusions of categories of μ-Kuranishi spaces.

$(V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ for $i, j \in I$. Write $\mathcal{K}' = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, [\Phi_{ij}]_{i,j\in I}\bigr)$ and $\boldsymbol{X}' = (X, \mathcal{K}')$. Then Definition 5.11(d)–(f) follow from Definition 4.14(e),(f),(d), so $\boldsymbol{X}'$ is a μ-Kuranishi space. Define $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{X}) = \boldsymbol{X}'$.

Next let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in $\mathbf{m\dot{K}ur}$, using notation (4.6), (4.7), (4.9) for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{f}$, and set $\boldsymbol{X}' = F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{X})$ and $\boldsymbol{Y}' = F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{Y})$. Taking the $\approx_S$-equivalence class $[\boldsymbol{f}_{ij}]$ of $\boldsymbol{f}_{ij}$ over $S = \operatorname{Im}\chi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ as in Definition 5.5(b) we find that

$$\boldsymbol{f}' = \bigl(f, [\boldsymbol{f}_{ij}]_{i\in I,\, j\in J}\bigr) : \boldsymbol{X}' \to \boldsymbol{Y}' \tag{5.21}$$

is a morphism in $\boldsymbol{\mu}\mathbf{\dot{K}ur}$, as Definition 5.13(a),(b) for $\boldsymbol{f}'$ follow from Definition 4.17(c),(d) for $\boldsymbol{f}$. Define $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}([\boldsymbol{f}]) = \boldsymbol{f}'$.

To show this is well-defined, let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism in $\mathbf{m\dot{K}ur}$, where $\boldsymbol{g} = \bigl(g, \boldsymbol{g}_{ij,\, i\in I,\, j\in J},\, \boldsymbol{G}^{j,\, j\in J}_{ii',\, i,i'\in I},\, \boldsymbol{G}^{jj',\, j,j'\in J}_{i,\, i\in I}\bigr)$ and $\boldsymbol{\eta} = \bigl(\boldsymbol{\eta}_{ij,\, i\in I,\, j\in J}\bigr)$. Then $f = g : X \to Y$, and $\boldsymbol{\eta}_{ij} : \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij}$ is a 2-morphism of m-Kuranishi neighbourhoods over (S, f) for $S = \operatorname{Im}\chi_i \cap f^{-1}(\operatorname{Im}\psi_j)$, so $[\boldsymbol{f}_{ij}] = [\boldsymbol{g}_{ij}]$ in morphisms of μ-Kuranishi neighbourhoods over (S, f). Therefore $\boldsymbol{f}'$ in (5.21) is independent of the choice of representative $\boldsymbol{f}$ for the morphism $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ in $\operatorname{Ho}(\mathbf{m\dot{K}ur})$, so $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}([\boldsymbol{f}])$ is well defined.

Comparing Proposition 4.19 and Definition 4.20 with Theorem 5.14(a) we see that $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}$ preserves composition of morphisms, and comparing Definitions 4.17 and 5.13 we see that $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}$ preserves identities. Hence $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}} : \operatorname{Ho}(\mathbf{m\dot{K}ur}) \to \boldsymbol{\mu}\mathbf{\dot{K}ur}$ is a functor.

The next theorem will be proved in §5.6.

Theorem 5.23. *The functor* $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}:\mathrm{Ho}(\mathbf{m\dot{K}ur})\rightarrow\boldsymbol{\mu}\mathbf{\dot{K}ur}$ *in Definition* 5.22 *is an equivalence of categories.*

Section 4.8 related m-Kuranishi spaces to the derived manifolds of Spivak [103], Borisov–Noel [7, 8] and the author [57, 58, 61]. Theorems 4.62 and 5.23 imply:

Corollary 5.24. *There is an equivalence of categories* $\mathrm{Ho}(\mathbf{dMan})\simeq\boldsymbol{\mu}\mathbf{Kur}$, *where* $\mathbf{dMan}$ *is the strict* 2*-category of d-manifolds from* [57, 58, 61], *and* $\boldsymbol{\mu}\mathbf{Kur}$ *is as above for* $\mathbf{\dot{M}an}=\mathbf{Man}$.

Combining this with Borisov's functor (4.66) gives a functor

$$\mathrm{Ho}(\mathbf{DerMan_{Spi}})\simeq\mathrm{Ho}(\mathbf{DerMan_{BN}})\longrightarrow\boldsymbol{\mu}\mathbf{Kur},$$

which is close to being an equivalence (it is full but not faithful, and induces a 1-1 correspondence between isomorphism classes of objects).

5.4 μ-Kuranishi spaces with corners. Boundaries, k-corners, and the corner functor

We now change notation from $\mathbf{\dot{M}an}$ in §3.1–§3.3 to $\mathbf{\dot{M}an^c}$, and from $\boldsymbol{\mu}\mathbf{\dot{K}ur}$ in §5.3 to $\boldsymbol{\mu}\mathbf{\dot{K}ur^c}$. Suppose throughout this section that $\mathbf{\dot{M}an^c}$ satisfies Assumption 3.22 in §3.4.1. Then $\mathbf{\dot{M}an^c}$ satisfies Assumptions 3.1–3.7, so §5.3 constructs a category $\boldsymbol{\mu}\mathbf{\dot{K}ur^c}$ of μ-Kuranishi spaces associated to $\mathbf{\dot{M}an^c}$. For instance, $\boldsymbol{\mu}\mathbf{\dot{K}ur^c}$ could be $\boldsymbol{\mu}\mathbf{Kur^c},\boldsymbol{\mu}\mathbf{Kur^{gc}},\boldsymbol{\mu}\mathbf{Kur^{ac}}$ or $\boldsymbol{\mu}\mathbf{Kur^{c,ac}}$ from Definition 5.15. We will refer to objects of $\boldsymbol{\mu}\mathbf{\dot{K}ur^c}$ as *μ-Kuranishi spaces with corners.* We also write $\boldsymbol{\mu}\mathbf{\dot{K}ur^c_{si}}$ for the subcategory of $\boldsymbol{\mu}\mathbf{\dot{K}ur^c}$ with simple morphisms in the sense of §5.3.4, noting that simple is a discrete property of morphisms in $\mathbf{\dot{M}an^c}$ by Assumption 3.22(c).

In §4.6, for each $\boldsymbol{X}\in\mathbf{m\dot{K}ur^c}$ we defined the k-corners $C_k(\boldsymbol{X})$ in $\mathbf{m\dot{K}ur^c}$, with $\partial\boldsymbol{X}=C_1(\boldsymbol{X})$. We constructed a 2-category $\mathbf{m\check{K}ur^c}$ from $\mathbf{m\dot{K}ur^c}$ with objects $\coprod_{n\in\mathbb{Z}}\boldsymbol{X}_n$ for $\boldsymbol{X}_n\in\mathbf{m\dot{K}ur^c}$ with $\mathrm{vdim}\,\boldsymbol{X}_n=n$, and defined the corner 2-functor $C:\mathbf{m\dot{K}ur^c}\rightarrow\mathbf{m\check{K}ur^c}$.

We will now extend all this to μ-Kuranishi spaces with corners. This is a simplification of §4.6. Here is the analogue of Definition 4.39:

Definition 5.25. Let $\boldsymbol{X}=(X,\mathcal{K})$ in $\boldsymbol{\mu}\mathbf{\dot{K}ur^c}$ be a μ-Kuranishi space with corners, and write $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},[\Phi_{ij}]_{i,j\in I}\bigr)$ as in Definition 5.11. Choose representatives $\Phi_{ij}=(V_{ij},\phi_{ij},\hat\phi_{ij})$ for $[\Phi_{ij}]$ for all $i,j\in I$, so that $\Phi_{ij}:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ is a 1-morphism of m-Kuranishi neighbourhoods. Since $[\Phi_{ij}]\circ[\Phi_{hi}]=[\Phi_{hj}]$ for $h,i,j\in I$ by Definition 5.11(f), we can choose a 2-morphism $\Lambda_{hij}:\Phi_{ij}\circ\Phi_{hi}\Rightarrow\Phi_{hj}$. We are now in the situation of the beginning of Definition 4.39, except that the Λ_{hij} need not satisfy Definition 4.14(g),(h). This will not matter to us.

Let $k \in \mathbb{N}$. We will define a μ-Kuranishi space with corners $C_k(\boldsymbol{X})$ in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ called the *k-corners of* $\boldsymbol{X}$, and a morphism $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$. Explicitly we write $C_k(\boldsymbol{X}) = (C_k(X), \mathcal{K}_k)$ with

$$\mathcal{K}_k = \bigl(\{k\} \times I, (V_{(k,i)}, E_{(k,i)}, s_{(k,i)}, \psi_{(k,i)})_{i\in I}, [\Phi_{(k,i),(k,j)}]_{i,j\in I}\bigr)$$
$$\text{with} \qquad \Phi_{(k,i)(k,j)} = (V_{(k,i)(k,j)}, \phi_{(k,i)(k,j)}, \hat{\phi}_{(k,i)(k,j)}),$$

where $\mathcal{K}_k$ has indexing set $\{k\} \times I$, and as in Definition 5.13 we write

$$\boldsymbol{\Pi}_k = \bigl(\Pi_k, [\boldsymbol{\Pi}_{(k,i)j}]_{i,j\in I}\bigr), \quad \text{where}$$
$$\boldsymbol{\Pi}_{(k,i)j} = (V_{(k,i)j}, \Pi_{(k,i)j}, \hat{\Pi}_{(k,i)j}) : (V_{(k,i)}, E_{(k,i)}, s_{(k,i)}, \psi_{(k,i)}) \to (V_j, E_j, s_j, \psi_j).$$

We follow Definition 4.39 closely. For all $i, j \in I$, define $\Phi_{(k,i)(k,j)} = (V_{(k,i)(k,j)}, \phi_{(k,i)(k,j)}, \hat{\phi}_{(k,i)(k,j)})$ by (4.40)–(4.42), and $\boldsymbol{\Pi}_{(k,i)j}$ by (4.43). Define the topological space $C_k(X)$ by $C_k(X) = \bigl[\coprod_{i\in I} s_{(k,i)}^{-1}(0)\bigr]/\approx$ and the continuous maps $\psi_{(k,i)} : s_{(k,i)}^{-1}(0) \to C_k(X)$, $\Pi_k : C_k(X) \to X$ as in Definition 4.39. Here the proof that $\approx$ is an equivalence relation involves the existence of the 2-morphism $\Lambda_{hij} : \Phi_{ij} \circ \Phi_{hi} \Rightarrow \Phi_{hj}$ as above, but not Definition 4.14(g),(h).

The proofs in Definition 4.39 show that $C_k(X)$ is Hausdorff and second countable, and $\Pi_k : C_k(X) \to X$ is continuous and proper with finite fibres, and $(V_{(k,i)}, E_{(k,i)}, s_{(k,i)}, \psi_{(k,i)})$ is an m-Kuranishi neighbourhood (hence a μ-Kuranishi neighbourhood) on $C_k(X)$ for $i \in I$, and

$$\Phi_{(k,i)(k,j)} : (V_{(k,i)}, E_{(k,i)}, s_{(k,i)}, \psi_{(k,i)}) \longrightarrow (V_{(k,j)}, E_{(k,j)}, s_{(k,j)}, \psi_{(k,j)}),$$
$$\boldsymbol{\Pi}_{(k,i)j} : (V_{(k,i)}, E_{(k,i)}, s_{(k,i)}, \psi_{(k,i)}) \longrightarrow (V_j, E_j, s_j, \psi_j),$$

are 1-morphisms of m-Kuranishi neighbourhoods (over Π_k). Thus

$$[\Phi_{(k,i)(k,j)}] : (V_{(k,i)}, E_{(k,i)}, s_{(k,i)}, \psi_{(k,i)}) \longrightarrow (V_{(k,j)}, E_{(k,j)}, s_{(k,j)}, \psi_{(k,j)}),$$
$$[\boldsymbol{\Pi}_{(k,i)j}] : (V_{(k,i)}, E_{(k,i)}, s_{(k,i)}, \psi_{(k,i)}) \longrightarrow (V_j, E_j, s_j, \psi_j),$$

are morphisms of μ-Kuranishi neighbourhoods (over Π_k).

To see $[\Phi_{(k,i)(k,j)}]$, $[\boldsymbol{\Pi}_{(k,i)j}]$ are independent of the choice of representative Φ_{ij} for $[\Phi_{ij}]$, and so are well defined, note that if Φ'_{ij} is an alternative choice giving $\Phi'_{(k,i)(k,j)}, \boldsymbol{\Pi}'_{(k,i)j}$ then there is a 2-morphism $\boldsymbol{\eta}_{ij} = [\dot{V}_{ij}, \hat{\eta}_{ij}] : \Phi_{ij} \Rightarrow \Phi'_{ij}$. As for $\Lambda_{hij}, \Lambda_{(k,h)(k,i)(k,j)}$ and $\boldsymbol{\Pi}^j_{(k,i)(k,i')}$ in Definition 4.39 we define 2-morphisms

$$[C_k(\dot{V}_{ij}), \Pi_k^\diamond(\hat{\eta}_{ij})] : \Phi_{(k,i)(k,j)} \Longrightarrow \Phi'_{(k,i)(k,j)},$$
$$[C_k(\dot{V}_{ij}), \Pi_k^*(\hat{\eta}_{ij})] : \boldsymbol{\Pi}_{(k,i)j} \Longrightarrow \boldsymbol{\Pi}'_{(k,i)j},$$

so that $[\Phi_{(k,i)(k,j)}] = [\Phi'_{(k,i)(k,j)}]$ and $[\boldsymbol{\Pi}_{(k,i)j}] = [\boldsymbol{\Pi}'_{(k,i)j}]$.

We have now defined all the data in $C_k(\boldsymbol{X})$ and $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$. We can check that $C_k(\boldsymbol{X})$ and $\boldsymbol{\Pi}_k$ satisfy the conditions of Definitions 5.11 and 5.13, with $\operatorname{vdim} C_k(\boldsymbol{X}) = \operatorname{vdim} \boldsymbol{X} - k$, in the same way as in Definition 4.39, where for example to show that $[\Phi_{(k,i)(k,j)}] \circ [\Phi_{(k,h)(k,i)}] = [\Phi_{(k,h)(k,j)}]$ in Definition 5.11(f) for $C_k(\boldsymbol{X})$ we construct a 2-morphism $\Lambda_{(k,h)(k,i)(k,j)} : \Phi_{(k,i)(k,j)} \circ \Phi_{(k,h)(k,i)} \Rightarrow \Phi_{(k,h)(k,j)}$ from Λ_{hij} as in Definition 4.39.

This proves the analogue of Theorem 4.40:

Theorem 5.26. *For each* $\boldsymbol{X}$ *in* $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ *and* $k=0,1,\ldots$ *we have defined the* k-***corners*** $C_k(\boldsymbol{X})$, *an object in* $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ *with* $\operatorname{vdim} C_k(\boldsymbol{X})=\operatorname{vdim}\boldsymbol{X}-k$, *and a morphism* $\boldsymbol{\Pi}_k: C_k(\boldsymbol{X})\rightarrow\boldsymbol{X}$ *in* $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$, *whose underlying continuous map* $\Pi_k: C_k(X)\rightarrow X$ *is proper with finite fibres. We also write* $\partial\boldsymbol{X}=C_1(\boldsymbol{X})$, *called the* ***boundary*** *of* $\boldsymbol{X}$, *and we write* $\boldsymbol{i_X}=\boldsymbol{\Pi}_1:\partial\boldsymbol{X}\rightarrow\boldsymbol{X}$.

Modifying Definition 4.42 we construct categories $\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c_{si}}\subseteq\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c}$ from $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}\subseteq\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ in the obvious way, with objects $\coprod_{n\in\mathbb{Z}}\boldsymbol{X}_n$ for $\boldsymbol{X}_n$ in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ with $\operatorname{vdim}\boldsymbol{X}_n=n$, where $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}},\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ embed as full subcategories of $\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c_{si}},\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c}$. For the examples of $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}\subseteq\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ in Definitions 5.15, 5.21 we use the obvious notation for the corresponding categories $\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c_{si}}\subseteq\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c}$, so for instance we enlarge $\boldsymbol{\mu}\mathbf{Kur^c}$ associated to $\dot{\mathbf{M}}\mathbf{an^c}=\mathbf{Man^c}$ to $\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^c}$.

Then following Definition 4.43, but modifying it as in Definition 5.25, we define the corner functor $C:\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}\rightarrow\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c}$. This is straightforward and involves no new ideas, so we leave it as an exercise for the reader. This proves the analogue of Theorem 4.44:

Theorem 5.27. *We can define a functor* $C:\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}\rightarrow\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c}$ *called the* ***corner functor***. *It acts on objects* $\boldsymbol{X}$ *in* $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ *by* $C(\boldsymbol{X})=\coprod_{k=0}^{\infty}C_k(\boldsymbol{X})$. *If* $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ *is simple then* $C(\boldsymbol{f}):C(\boldsymbol{X})\rightarrow C(\boldsymbol{Y})$ *is simple and maps* $C_k(\boldsymbol{X})\rightarrow C_k(\boldsymbol{Y})$ *for* $k=0,1,\ldots.$ *Thus* $C|_{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}}$ *decomposes as* $C|_{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}}=\coprod_{k=0}^{\infty}C_k$, *where* $C_k:\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}\rightarrow\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}$ *is a functor acting on objects by* $\boldsymbol{X}\mapsto C_k(\boldsymbol{X})$, *for* $C_k(\boldsymbol{X})$ *as in Definition* 5.25. *We also write* $\partial=C_1:\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}\rightarrow\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}$, *and call it the* ***boundary functor***.

If for some discrete property $\boldsymbol{P}$ *of morphisms in* $\dot{\mathbf{M}}\mathbf{an^c}$ *the corner functor* $C:\dot{\mathbf{M}}\mathbf{an^c}\rightarrow\check{\dot{\mathbf{M}}}\mathbf{an^c}$ *maps to the subcategory* $\check{\dot{\mathbf{M}}}\mathbf{an}^{\mathbf{c}}_{\boldsymbol{P}}$ *of* $\check{\dot{\mathbf{M}}}\mathbf{an^c}$ *whose morphisms are* $\boldsymbol{P}$, *then* $C:\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}\rightarrow\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c}$ *maps to the subcategory* $\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}_{\boldsymbol{P}}$ *of* $\boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c}$ *whose morphisms are* $\boldsymbol{P}$.

As for Example 4.45, applying Theorem 5.27 to the data $\dot{\mathbf{M}}\mathbf{an^c},\ldots$ in Example 3.24(a)–(h) gives corner functors:

$$
\begin{aligned}
&C:\boldsymbol{\mu}\mathbf{Kur^c}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^c_{in}}\subset\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^c}, && C':\boldsymbol{\mu}\mathbf{Kur^c}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^c},\\
&C:\boldsymbol{\mu}\mathbf{Kur^c_{st}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^c_{st,in}}\subset\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^c_{st}}, && C':\boldsymbol{\mu}\mathbf{Kur^c_{st}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^c_{st}},\\
&C:\boldsymbol{\mu}\mathbf{Kur^{ac}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{ac}_{in}}\subset\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{ac}}, && C':\boldsymbol{\mu}\mathbf{Kur^{ac}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{ac}},\\
&C:\boldsymbol{\mu}\mathbf{Kur^{ac}_{st}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{ac}_{st,in}}\subset\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{ac}_{st}}, && C':\boldsymbol{\mu}\mathbf{Kur^{ac}_{st}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{ac}_{st}},\\
&C:\boldsymbol{\mu}\mathbf{Kur^{c,ac}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{c,ac}_{in}}\subset\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{c,ac}}, && C':\boldsymbol{\mu}\mathbf{Kur^{c,ac}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{c,ac}},\\
&C:\boldsymbol{\mu}\mathbf{Kur^{c,ac}_{st}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{c,ac}_{st,in}}\subset\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{c,ac}_{st}}, && C':\boldsymbol{\mu}\mathbf{Kur^{c,ac}_{st}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{c,ac}_{st}},\\
&C:\boldsymbol{\mu}\mathbf{Kur^{gc}}\longrightarrow\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{gc}_{in}}\subset\boldsymbol{\mu}\check{\mathbf{K}}\mathbf{ur^{gc}}.
\end{aligned}
\tag{5.22}
$$

As for Propositions 4.46 and 4.47, we prove:

Proposition 5.28. *For all of the functors C in* (5.22) *(though not the functors C'), a morphism $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ is interior (or b-normal) if and only if $C(\boldsymbol{f})$ maps $C_0(\boldsymbol{X})\rightarrow C_0(\boldsymbol{Y})$ (or $C(\boldsymbol{f})$ maps $C_k(\boldsymbol{X})\rightarrow\coprod_{l=0}^k C_l(\boldsymbol{Y})$ for all $k=0,1,\ldots$).*

Proposition 5.29. *Let $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ be an isomorphism in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$. Then $\boldsymbol{f}$ is simple by Proposition* 5.20(b)*, and $C_k(\boldsymbol{f}):C_k(\boldsymbol{X})\rightarrow C_k(\boldsymbol{Y})$ for $k=0,1,\ldots$ and $\partial\boldsymbol{f}:\partial\boldsymbol{X}\rightarrow\partial\boldsymbol{Y}$ are also isomorphisms in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$.*

Here is the analogue of Definition 4.48:

Definition 5.30. As in Definition 5.15 we write $\boldsymbol{\mu}\mathbf{Kur^c}$ for the category of μ-Kuranishi spaces with corners associated to $\dot{\mathbf{M}}\mathbf{an^c}=\mathbf{Man^c}$. An object $\boldsymbol{X}$ in $\boldsymbol{\mu}\mathbf{Kur^c}$ is called a *μ-Kuranishi space with boundary* if $\partial(\partial\boldsymbol{X})=\emptyset$. Write $\boldsymbol{\mu}\mathbf{Kur^b}$ for the full subcategory of μ-Kuranishi spaces with boundary in $\boldsymbol{\mu}\mathbf{Kur^c}$, and write $\boldsymbol{\mu}\mathbf{Kur^b_{si}}\subseteq\boldsymbol{\mu}\mathbf{Kur^b_{in}}\subseteq\boldsymbol{\mu}\mathbf{Kur^b}$ for the subcategories of $\boldsymbol{\mu}\mathbf{Kur^b}$ with simple and interior morphisms. We can show that $\boldsymbol{X}\in\boldsymbol{\mu}\mathbf{Kur^c}$ is a μ-Kuranishi space with boundary if and only if $C_k(\boldsymbol{X})=\emptyset$ for all $k>1$.

5.5 μ-Kuranishi neighbourhoods on μ-Kuranishi spaces

We now give the 'μ-Kuranishi' analogue of the ideas of §4.7.

Definition 5.31. Suppose $\boldsymbol{X}=(X,\mathcal{K})$ is a μ-Kuranishi space, where $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},[\Phi_{ij}]_{i,j\in I}\bigr)$. A *$\mu$-Kuranishi neighbourhood on $\boldsymbol{X}$* is data (V_a,E_a,s_a,ψ_a) and $[\Phi_{ai}]_{i\in I}$, where (V_a,E_a,s_a,ψ_a) is a μ-Kuranishi neighbourhood on the topological space X as in Definition 5.5(a), and $[\Phi_{ai}]:(V_a,E_a,s_a,\psi_a)\rightarrow(V_i,E_i,s_i,\psi_i)$ is a coordinate change for each $i\in I$ as in Definition 5.7 (over $S=\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_i$, as usual), such that for all $i,j\in I$ we have

$$[\Phi_{ij}]\circ[\Phi_{ai}]=[\Phi_{aj}],\tag{5.23}$$

where (5.23) holds over $S=\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j$ by Convention 5.9.

Here the subscript 'a' in (V_a,E_a,s_a,ψ_a) is just a label used to distinguish μ-Kuranishi neighbourhoods, generally not in I. If we omit a we will write '$*$' in place of 'a' in $[\Phi_{ai}]$, giving $[\Phi_{*i}]:(V,E,s,\psi)\rightarrow(V_i,E_i,s_i,\psi_i)$.

We will usually just say (V_a,E_a,s_a,ψ_a) or (V,E,s,ψ) *is a μ-Kuranishi neighbourhood on $\boldsymbol{X}$*, leaving the data $[\Phi_{ai}]_{i\in I}$ or $[\Phi_{*i}]_{i\in I}$ implicit. We call such a (V,E,s,ψ) a *global* μ-Kuranishi neighbourhood on $\boldsymbol{X}$ if $\operatorname{Im}\psi=X$.

The next theorem can be proved using the sheaf property Theorem 5.10 by very similar methods to Theorem 5.14, noting that (5.24)–(5.25) imply that

$$\begin{aligned}[\Phi_{ab}]|_{\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b\cap\operatorname{Im}\psi_i}&=[\Phi_{bi}]^{-1}\circ[\Phi_{ai}],\\ [\boldsymbol{f}_{ab}]|_{\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_i\cap f^{-1}(\operatorname{Im}\psi_b\cap\operatorname{Im}\psi_j)}&=[\Phi_{bj}]^{-1}\circ[\boldsymbol{f}_{ij}]\circ[\mathrm{T}_{bi}],\end{aligned}$$

so we leave the proof as an exercise for the reader.

Theorem 5.32. **(a)** *Let $\boldsymbol{X} = (X, \mathcal{K})$ be a μ-Kuranishi space, where $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i \in I}, [\Phi_{ij}]_{i,j \in I}\big)$, and $(V_a, E_a, s_a, \psi_a), (V_b, E_b, s_b, \psi_b)$ be μ-Kuranishi neighbourhoods on $\boldsymbol{X}$, in the sense of Definition* 5.31*. Then there is a unique coordinate change $[\Phi_{ab}] : (V_a, E_a, s_a, \psi_a) \to (V_b, E_b, s_b, \psi_b)$ in the sense of Definition* 5.7 *such that for all $i \in I$ we have*

$$[\Phi_{bi}] \circ [\Phi_{ab}] = [\Phi_{ai}], \tag{5.24}$$

which holds on $\operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b \cap \operatorname{Im}\psi_i$ by Convention 5.9*. We will call $[\Phi_{ab}]$ the* ***coordinate change between the μ-Kuranishi neighbourhoods*** *$(V_a, E_a, s_a, \psi_a), (V_b, E_b, s_b, \psi_b)$* ***on the μ-Kuranishi space $\boldsymbol{X}$****.*

(b) *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism of μ-Kuranishi spaces, with notation* (5.11)–(5.12)*, and let $(U_a, D_a, r_a, \chi_a), (V_b, E_b, s_b, \psi_b)$ be μ-Kuranishi neighbourhoods on $\boldsymbol{X}, \boldsymbol{Y}$ respectively, in the sense of Definition* 5.31*. Then there is a unique morphism $[\boldsymbol{f}_{ab}] : (U_a, D_a, r_a, \chi_a) \to (V_b, E_b, s_b, \psi_b)$ of μ-Kuranishi neighbourhoods over f as in Definition* 5.5(b)*, such that for all $i \in I$ and $j \in J$ we have*

$$[\Phi_{bj}] \circ [\boldsymbol{f}_{ab}] = [\boldsymbol{f}_{ij}] \circ [\mathrm{T}_{bi}]. \tag{5.25}$$

We will call $[\boldsymbol{f}_{ab}]$ the ***morphism of μ-Kuranishi neighbourhoods*** *$(V_a, E_a, s_a, \psi_a), (V_b, E_b, s_b, \psi_b)$* ***over $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$****.*

Remark 5.33. Note that we make the (potentially confusing) distinction between *μ-Kuranishi neighbourhoods (V_i, E_i, s_i, ψ_i) on a topological space X*, as in Definition 5.5(a), and *μ-Kuranishi neighbourhoods (V_a, E_a, s_a, ψ_a) on a μ-Kuranishi space $\boldsymbol{X} = (X, \mathcal{K})$*, which are as in Definition 5.31, and come equipped with the extra implicit data $[\Phi_{ai}]_{i \in I}$ giving the compatibility with the μ-Kuranishi structure $\mathcal{K}$ on X. Similarly, we distinguish between coordinate changes of μ-Kuranishi neighbourhoods over X or $\boldsymbol{X}$, and between morphisms of μ-Kuranishi neighbourhoods over $f : X \to Y$ or $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$.

Theorem 5.34. *Let $\boldsymbol{X} = (X, \mathcal{K})$ be a μ-Kuranishi space, and $\big\{(V_a, E_a, s_a, \psi_a) : a \in A\big\}$ a family of μ-Kuranishi neighbourhoods on $\boldsymbol{X}$ with $X = \bigcup_{a \in A} \operatorname{Im}\psi_a$. For all $a, b \in A$, let $[\Phi_{ab}] : (V_a, E_a, s_a, \psi_a) \to (V_b, E_b, s_b, \psi_b)$ be the coordinate change from Theorem* 5.32(a)*. Then $\mathcal{K}' = \big(A, (V_a, E_a, s_a, \psi_a)_{a \in A}, [\Phi_{ab}]_{a,b \in A}\big)$ is a μ-Kuranishi structure on X, and $\boldsymbol{X}' = (X, \mathcal{K}')$ is canonically isomorphic to $\boldsymbol{X}$ in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$.*

Proof. Write $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i \in I}, [\Phi_{ij}]_{i,j \in I}\big)$, and let $\mathcal{K}'$ be as in the theorem. Definition 5.11(a)–(d) for $\mathcal{K}'$ are immediate. For part (e), note that $[\Phi_{aa}], [\mathrm{id}_{(V_a, E_a, s_a, \psi_a)}] : (V_a, E_a, s_a, \psi_a) \to (V_a, E_a, s_a, \psi_a)$ both satisfy the conditions of Theorem 5.32(a) with $a = b$, so by uniqueness we have $[\Phi_{aa}] = [\mathrm{id}_{(V_a, E_a, s_a, \psi_a)}]$. Similarly, for $a, b, c \in A$ we can show that $[\Phi_{bc}] \circ [\Phi_{ab}]$ and $[\Phi_{ac}]$ are coordinate changes $(V_a, E_a, s_a, \psi_a) \to (V_c, E_c, s_c, \psi_c)$ over $\operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b \cap \operatorname{Im}\psi_c$ satisfying the conditions of Theorem 5.32(a), so uniqueness gives $[\Phi_{bc}] \circ [\Phi_{ab}] = [\Phi_{ac}]$, proving (f). Hence $\mathcal{K}'$ is a μ-Kuranishi structure.

To show $\boldsymbol{X}, \boldsymbol{X}'$ are canonically isomorphic, note that each (V_a, E_a, s_a, ψ_a) comes equipped with implicit extra data $[\Phi_{ai}]_{i \in I}$. Define morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{X}'$

and $\boldsymbol{g}:\boldsymbol{X}'\to\boldsymbol{X}$ by $\boldsymbol{f}=\bigl(\mathrm{id}_X,[\Phi_{ai}]_{a\in A,\,i\in I}\bigr)$ and $\boldsymbol{g}=\bigl(\mathrm{id}_X,[\Phi_{ai}]^{-1}_{i\in I,\,a\in A}\bigr)$. It is easy to check that $\boldsymbol{f},\boldsymbol{g}$ are morphisms in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ with $\boldsymbol{g}\circ\boldsymbol{f}=\mathbf{id}_{\boldsymbol{X}}$ and $\boldsymbol{f}\circ\boldsymbol{g}=\mathbf{id}_{\boldsymbol{X}'}$. So $\boldsymbol{f},\boldsymbol{g}$ are canonical isomorphisms. $\square$

As the μ-Kuranishi neighbourhoods (V_i,E_i,s_i,ψ_i) in the μ-Kuranishi structure on $\boldsymbol{X}$ are μ-Kuranishi neighbourhoods on $\boldsymbol{X}$, we deduce:

Corollary 5.35. *Let $\boldsymbol{X}=(X,\mathcal{K})$ be a μ-Kuranishi space with $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},[\Phi_{ij}]_{i,j\in I}\bigr)$. Suppose $J\subseteq I$ with $\bigcup_{j\in J}\mathrm{Im}\,\psi_j=X$. Then $\mathcal{K}'=\bigl(J,(V_i,E_i,s_i,\psi_i)_{i\in J},[\Phi_{ij}]_{i,j\in J}\bigr)$ is a μ-Kuranishi structure on X, and $\boldsymbol{X}'=(X,\mathcal{K}')$ is canonically isomorphic to $\boldsymbol{X}$ in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$.*

Thus, adding or subtracting extra μ-Kuranishi neighbourhoods to the μ-Kuranishi structure of $\boldsymbol{X}$ leaves $\boldsymbol{X}$ unchanged up to canonical isomorphism.

As in §4.7.3, if $\dot{\mathbf{M}}\mathbf{an^c}$ satisfies Assumption 3.22 then we can lift μ-Kuranishi neighbourhoods (V_a,E_a,s_a,ψ_a) on $\boldsymbol{X}$ in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ to μ-Kuranishi neighbourhoods $(V_{(k,a)},E_{(k,a)},s_{(k,a)},\psi_{(k,a)})$ on the k-corners $C_k(\boldsymbol{X})$ from §5.4, and we can lift morphisms $[\boldsymbol{f}_{ab}]:(U_a,D_a,r_a,\chi_a)\to(V_b,E_b,s_b,\psi_b)$ of μ-Kuranishi neighbourhoods over $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ to morphisms $[\boldsymbol{f}_{(k,a)(l,b)}]:(U_{(k,a)},D_{(k,a)},r_{(k,a)},\chi_{(k,a)})\to(V_{(l,b)},E_{(l,b)},s_{(l,b)},\psi_{(l,b)})$ over $C(\boldsymbol{f}):C(\boldsymbol{X})\to C(\boldsymbol{Y})$. We leave the details to the reader. As in §4.7.4, we could now state our philosophy for working with μ-Kuranishi spaces, but we will not.

5.6 Proof of Theorem 5.23

Use the notation of Definition 5.22. To show $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}:\mathrm{Ho}(\mathbf{m}\dot{\mathbf{K}}\mathbf{ur})\to\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ is an equivalence of categories, we have to prove three things: that $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}$ is faithful (injective on morphisms), and full (surjective on morphisms), and surjective on isomorphism classes of objects.

The proofs of these will involve gluing together 2-morphisms of m-Kuranishi neighbourhoods using families of partitions of unity, so we begin by showing that partitions of unity with the properties we need exist.

5.6.1 A lemma on partitions of unity on $\boldsymbol{X}$ in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$

Let $\boldsymbol{X}=(X,\mathcal{I})$ be a μ-Kuranishi space, with $\mathcal{I}=\bigl(I,(U_i,D_i,r_i,\chi_i)_{i\in I},\mathrm{T}_{ij}=[U_{ij},\tau_{ij},\hat\tau_{ij}]_{i,j\in I}\bigr)$, as in (5.11). Then $\bigl\{\mathrm{Im}\,\chi_i:i\in I\bigr\}$ is an open cover of X, with $\chi_i:r_i^{-1}(0)\to\mathrm{Im}\,\chi_i$ a homeomorphism for each $i\in I$.

Roughly speaking, we want to define a smooth partition of unity $\{\eta_i:i\in I\}$ on X subordinate to $\bigl\{\mathrm{Im}\,\chi_i:i\in I\bigr\}$, so that $\eta_i:X\to\mathbb{R}$ is smooth with $\eta_i(X)\subseteq[0,1]$ and $\sum_{i\in I}\eta_i=1$. However, X is not a manifold, so naïvely '$\eta_i:X\to\mathbb{R}$ is smooth' does not make sense.

In fact we will not work with 'smooth functions' η_i on X directly, apart from in the proof of Lemma 5.36. Instead, for each $i\in I$ we want a partition of unity $\{\eta_{ij}:j\in I\}$ on U_i in the sense of §3.3.1(d), such that $\eta_{ij}|_{r_i^{-1}(0)}=\eta_j\circ\chi_i$ for each

$j \in I$. The fact that $\eta_{ik} : U_i \to \mathbb{R}$ and $\eta_{jk} : U_j \to \mathbb{R}$ both come from the same $\eta_k : X \to \mathbb{R}$ is expressed in the condition $\eta_{ik} = \eta_{jk} \circ \tau_{ij} + O(r_i)$ on $U_{ij} \subseteq U_i$ for all $i, j \in I$. So our result Lemma 5.36 is stated using only smooth functions on manifolds (objects in $\dot{\mathbf{Man}}$).

But to prove Lemma 5.36, it is convenient to first choose a 'smooth partition of unity' $\{\eta_i : i \in I\}$ on X subordinate to $\{\mathrm{Im}\,\chi_i : i \in I\}$, so that $\{\eta_j \circ \chi_i : j \in I\}$ is a partition of unity on $r_i^{-1}(0) \subseteq U_i$, and then extend this from $r_i^{-1}(0)$ to U_i. To do this we have to interpret X and $r_i^{-1}(0)$ as some kind of 'smooth space'. We do this using C^∞-*schemes* and C^∞-*algebraic geometry*, as in [56, 65], which are the foundation of the author's theory of d-manifolds and d-orbifolds in [57, 58, 61].

Lemma 5.36. *Let $\boldsymbol{X} = (X, \mathcal{I})$ be a μ-Kuranishi space, with notation* (5.11) *for $\mathcal{I}$, and let $\mathrm{T}_{ij} = (U_{ij}, \tau_{ij}, \hat{\tau}_{ij})$ represent $[\mathrm{T}_{ij}]$ for $i, j \in I$, with $(U_{ii}, \tau_{ii}, \hat{\tau}_{ii}) = (U_i, \mathrm{id}_{U_i}, \mathrm{id}_{D_i})$. Then for all $i \in I$ we can choose a partition of unity $\{\eta_{ij} : j \in I\}$ on U_i subordinate to the open cover $\{U_{ij} : j \in I\}$ of U_i, as in* §3.3.1(d) *and* §B.1.4, *such that for all $i, j, k \in I$ we have*

$$\eta_{ik}|_{U_{ij}} = \eta_{jk} \circ \tau_{ij} + O(r_i) \quad \text{on } U_{ij} \subseteq U_i, \tag{5.26}$$

in the sense of Definition 3.15(i).

Proof. We use notation and results on C^∞-schemes and C^∞-algebraic geometry from [65], in which C^∞-schemes are written $\underline{X} = (X, \mathcal{O}_X)$ for X a topological space and $\mathcal{O}_X$ a sheaf of C^∞-rings on X, satisfying certain conditions.

For each $i \in I$, as in §3.3.1(c) and §B.1.3 the manifold U_i in $\dot{\mathbf{Man}}$ naturally becomes an affine C^∞-scheme $\underline{U}_i$, and $r_i^{-1}(0) \subseteq U_i$ becomes the closed C^∞-subscheme $\underline{r}_i^{-1}(0)$ in $\underline{U}_i$ defined by $r_i = 0$. If $i, j \in I$ and $(U_{ij}, \tau_{ij}, \hat{\tau}_{ij})$ represents T_{ij}, then $\hat{\tau}_{ij}(r_i|_{U_{ij}}) = \tau_{ij}^*(r_j) + O(r_i^2)$ on U_{ij} by Definition 4.2(d). This implies that $\underline{\tau}_{ij} : \underline{U}_{ij} \to \underline{U}_j$ restricts to an isomorphism of C^∞-schemes

$$\underline{\tau}_{ij}|_{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0)} : \underline{U}_{ij} \cap \underline{r}_i^{-1}(0) \to \underline{U}_{ji} \cap \underline{r}_j^{-1}(0). \tag{5.27}$$

We now have a topological space X, an open cover $\{\mathrm{Im}\,\chi_i : i \in I\}$ on X, C^∞-schemes $\underline{r}_i^{-1}(0)$ with underlying topological spaces $r_i^{-1}(0)$ and homeomorphisms $\chi_i : r_i^{-1}(0) \to \mathrm{Im}\,\chi_i \subseteq X$ for all $i \in I$, and isomorphisms of C^∞-schemes (5.27) lifting the homeomorphisms $\chi_j^{-1} \circ \chi_i : U_{ij} \cap r_i^{-1}(0) \to U_{ji} \cap r_j^{-1}(0)$ over double overlaps $\mathrm{Im}\,\chi_i \cap \mathrm{Im}\,\chi_j \subseteq X$. From $\mathrm{T}_{jk} \circ \mathrm{T}_{ij} = \mathrm{T}_{ik}$ in Definition 5.11(f), we deduce that the isomorphisms (5.27) have the obvious composition property $\underline{\tau}_{jk}|_{\cdots} \circ \underline{\tau}_{ij}|_{\cdots} = \underline{\tau}_{ik}|_{\cdots}$ over triple overlaps $\mathrm{Im}\,\chi_i \cap \mathrm{Im}\,\chi_j \cap \mathrm{Im}\,\chi_k \subseteq X$.

Standard results on schemes (actually, just the fact that sheaves of C^∞-rings on X form a stack on X) imply that X may be made into a C^∞-scheme $\underline{X}$, uniquely up to unique isomorphism, and the homeomorphisms $\chi_i : r_i^{-1}(0) \to \mathrm{Im}\,\chi_i \subseteq X$ upgraded to C^∞-scheme morphisms $\underline{\chi}_i : \underline{r}_i^{-1}(0) \to \underline{X}$ which are isomorphisms with open C^∞-subschemes $\mathrm{Im}\,\underline{\chi}_i \subseteq \underline{X}$ for $i \in I$, such that

$$\underline{\chi}_j \circ \underline{\tau}_{ij}|_{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0)} = \underline{\chi}_i|_{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0)} \qquad \text{for all } i, j \in I. \tag{5.28}$$

Since X is Hausdorff, second countable, and regular, as in Remark 4.15, [65, Cor. 4.42] implies that $\underline{X}$ is an affine C^∞-scheme, and [65, Th. 4.40] says that $\mathcal{O}_X$ is *fine*, that is, there exists a locally finite partition of unity in $\mathcal{O}_X$ subordinate to any open cover of $\underline{X}$. Thus we can choose a partition of unity $\{\underline{\eta}_i : i \in I\}$ on $\underline{X}$ subordinate to $\{\mathrm{Im}\,\underline{\chi}_i : i \in I\}$.

Then for each $i \in I$, $\{\underline{\eta}_j \circ \underline{\chi}_i : j \in I\}$ is a partition of unity on the C^∞-scheme $\underline{r}_i^{-1}(0)$ subordinate to the open cover $\bigl\{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0) : j \in J\bigr\}$. From the proof of the existence of partitions of unity on C^∞-schemes in [65, §4.7], we see that a partition of unity on $\underline{r}_i^{-1}(0) \subseteq \underline{U}_i$ subordinate to $\bigl\{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0) : j \in J\bigr\}$ can be extended to a partition of unity on $\underline{U}_i$ subordinate to $\bigl\{\underline{U}_{ij} : j \in J\bigr\}$, which is equivalent to a partition of unity on U_i in the sense of §B.1.4.

Thus, for all $i \in I$ we can choose a partition of unity $\{\eta_{ij} : j \in I\}$ on U_i subordinate to $\{U_{ij} : j \in I\}$, such that $\eta_{ij}|_{\underline{r}_i^{-1}(0)} = \underline{\eta}_j \circ \underline{\chi}_i$ for all $j \in I$, in the sense of C^∞-schemes. If $i, j, k \in I$ then

$$\eta_{ik}|_{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0)} = \underline{\eta}_k \circ \underline{\chi}_i|_{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0)} = \underline{\eta}_k \circ \underline{\chi}_j \circ \underline{\tau}_{ij}|_{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0)} = \eta_{jk} \circ \tau_{ij}|_{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0)},$$

using (5.28). But $f|_{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0)} = g|_{\underline{U}_{ij} \cap \underline{r}_i^{-1}(0)}$ for smooth $f, g : U_{ij} \to \mathbb{R}$ is equivalent to $f = g + O(r_i)$ on U_{ij}, so equation (5.26) follows. $\square$

5.6.2 $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}$ is faithful

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms in $\mathbf{m\dot{K}ur}$, so that $[\boldsymbol{f}], [\boldsymbol{g}] : \boldsymbol{X} \to \boldsymbol{Y}$ are morphisms in $\mathrm{Ho}(\mathbf{m\dot{K}ur})$. Write $\boldsymbol{X}', \boldsymbol{Y}', \boldsymbol{f}', \boldsymbol{g}'$ for the images of $\boldsymbol{X}, \boldsymbol{Y}, [\boldsymbol{f}], [\boldsymbol{g}]$ under $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}$. Suppose $\boldsymbol{f}' = \boldsymbol{g}'$. We must show that $[\boldsymbol{f}] = [\boldsymbol{g}]$, that is, that there exists a 2-morphism $\boldsymbol{\mu} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\mathbf{m\dot{K}ur}$.

Use notation (4.6), (4.7), (4.9) for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{f}$, and write $\boldsymbol{g} = \bigl(g, \boldsymbol{g}_{ij,\; i \in I,\; j \in J}, \boldsymbol{G}_{ii',\; i,i' \in I}^{j,\; j \in J}, \boldsymbol{G}_{i,\; i \in I}^{jj',\; j,j' \in J}\bigr)$. Then $\boldsymbol{f}' = \boldsymbol{g}'$ means that $f = g$, and $[\boldsymbol{f}_{ij}] = [\boldsymbol{g}_{ij}]$ for all $i \in I$, $j \in J$ as morphisms $(U_i, D_i, r_i, \chi_i) \to (V_j, E_j, s_j, \psi_j)$ of μ-Kuranishi neighbourhoods over (S, f) in the sense of §5.2, where $S = \mathrm{Im}\,\chi_i \cap f^{-1}(\mathrm{Im}\,\psi_j)$. Hence there exists a 2-morphism $\boldsymbol{\lambda}_{ij} : \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij}$ of m-Kuranishi neighbourhoods over (S, f) in the sense of §4.1.

We would like $\boldsymbol{\lambda} = \bigl(\boldsymbol{\lambda}_{ij,\; i \in I,\; j \in J}\bigr) : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ to be a 2-morphism of m-Kuranishi spaces, but there is a problem: as the $\boldsymbol{\lambda}_{ij}$ are chosen arbitrarily, they have no compatibility with the $\boldsymbol{F}_{ii'}^j, \boldsymbol{F}_i^{jj'}, \boldsymbol{G}_{ii'}^j, \boldsymbol{G}_i^{jj'}$, so Definition 4.18(a),(b) may not hold for $\boldsymbol{\lambda}$. We will define a modified version $\boldsymbol{\mu} = \bigl(\boldsymbol{\mu}_{ij,\; i \in I,\; j \in J}\bigr)$ of $\boldsymbol{\lambda}$ which does have the required compatibility.

For $i, \tilde{\imath} \in I$ and $j, \tilde{\jmath} \in I$, define $\boldsymbol{\lambda}_{i\tilde{\imath}}^{\tilde{\jmath}j}$ to be the horizontal composition of 2-morphisms over $S = \mathrm{Im}\,\chi_i \cap \mathrm{Im}\,\chi_{\tilde{\imath}} \cap f^{-1}(\mathrm{Im}\,\psi_j \cap \mathrm{Im}\,\psi_{\tilde{\jmath}})$ and $f : X \to Y$

$$\boldsymbol{f}_{ij} \xRightarrow[=(\boldsymbol{F}_{i\tilde{\imath}}^{j} \odot (\boldsymbol{F}_{\tilde{\imath}}^{\tilde{\jmath}j} * \mathrm{id})^{-1}]{(\boldsymbol{F}_i^{\tilde{\jmath}j} \odot (\mathrm{id} * \boldsymbol{F}_{i\tilde{\imath}}^{\tilde{\jmath}}))^{-1}} \begin{matrix} \Upsilon_{\tilde{\jmath}j} \circ \\ \boldsymbol{f}_{\tilde{\imath}\tilde{\jmath}} \circ \mathrm{T}_{i\tilde{\imath}} \end{matrix} \xRightarrow{\mathrm{id} * \boldsymbol{\lambda}_{\tilde{\imath}\tilde{\jmath}} * \mathrm{id}} \begin{matrix} \Upsilon_{\tilde{\jmath}j} \circ \\ \boldsymbol{g}_{\tilde{\imath}\tilde{\jmath}} \circ \mathrm{T}_{i\tilde{\imath}} \end{matrix} \xRightarrow[=\boldsymbol{G}_{i\tilde{\imath}}^{j} \odot (\boldsymbol{G}_{\tilde{\imath}}^{\tilde{\jmath}j} * \mathrm{id}))]{\boldsymbol{G}_i^{\tilde{\jmath}j} \odot (\mathrm{id} * \boldsymbol{G}_{i\tilde{\imath}}^{\tilde{\jmath}})} \boldsymbol{g}_{ij}, \qquad (5.29)$$

where the alternative expressions for the first and third 2-morphisms come from Definition 4.17(g).

Apply Lemma 5.36 to $\boldsymbol{X}' = F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{X})$, using $(U_{ii'}, \tau_{ii'}, \hat{\tau}_{ii'})$ to represent $\mathrm{T}'_{ii'}$. This gives a partition of unity $\{\eta_{i\tilde{\imath}} : \tilde{\imath} \in I\}$ on U_i subordinate to $\{U_{i\tilde{\imath}} : \tilde{\imath} \in I\}$ for each $i \in I$, such that for all $i, i', \tilde{\imath} \in I$ we have

$$\eta_{i\tilde{\imath}}|_{U_{ii'}} = \eta_{i'\tilde{\imath}} \circ \tau_{ii'} + O(r_i) \quad \text{on } U_{ii'} \subseteq U_i.$$

Similarly, applying Lemma 5.36 to $\boldsymbol{Y}' = F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{Y})$ gives a partition of unity $\{\zeta_{j\tilde{\jmath}} : \tilde{\jmath} \in J\}$ on V_j subordinate to $\{V_{j\tilde{\jmath}} : \tilde{\jmath} \in J\}$ for each $j \in J$, such that for all $j, j', \tilde{\jmath} \in J$ we have

$$\zeta_{j\tilde{\jmath}}|_{V_{jj'}} = \zeta_{j'\tilde{\jmath}} \circ \upsilon_{jj'} + O(s_j) \quad \text{on } V_{jj'} \subseteq V_j.$$

Now, using the notation of (5.6) in Definition 5.4, for $i \in I$ and $j \in J$ define a 2-morphism $\boldsymbol{\mu}_{ij} : \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij}$ over (S, f) with $\boldsymbol{f}_{ij} = (V_{ij}, f_{ij}, \hat{f}_{ij})$ by

$$\boldsymbol{\mu}_{ij} = \textstyle\sum_{\tilde{\imath}\in I} \sum_{\tilde{\jmath}\in J} \eta_{i\tilde{\imath}} \cdot f_{ij}^*(\zeta_{j\tilde{\jmath}}) \cdot \boldsymbol{\lambda}_{i\tilde{\imath}}^{\tilde{\jmath}j}. \tag{5.30}$$

We will show that $\boldsymbol{\mu} = \bigl(\boldsymbol{\mu}_{ij,\ i\in I,\ j\in J}\bigr) : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism in $\mathbf{m\dot{K}ur}$. For $i, i', \tilde{\imath} \in I$ and $j, \tilde{\jmath} \in J$ consider the diagram

$$\begin{tikzcd}
\boldsymbol{f}_{i'j} \circ \mathrm{T}_{ii'} \arrow[rrr, Rightarrow, "\boldsymbol{F}_{ii'}^j"'] \arrow[ddddd, Rightarrow, "\boldsymbol{\lambda}_{i'\tilde{\imath}}^{\tilde{\jmath}j}*\mathrm{id}"'] & & & \boldsymbol{f}_{ij} \arrow[ddddd, Rightarrow, "\boldsymbol{\lambda}_{i\tilde{\imath}}^{\tilde{\jmath}j}"] \\
& \boldsymbol{f}_{\tilde{\imath}j} \circ \mathrm{T}_{i'\tilde{\imath}} \circ \mathrm{T}_{ii'} \arrow[ul, Rightarrow, "\boldsymbol{F}_{i'\tilde{\imath}}^j*\mathrm{id}"] \arrow[r, Rightarrow, "\mathrm{id}*\mathrm{K}_{ii'\tilde{\imath}}"] & \boldsymbol{f}_{\tilde{\imath}j} \circ \mathrm{T}_{i\tilde{\imath}} \arrow[ur, Rightarrow, "\boldsymbol{F}_{i\tilde{\imath}}^j"] & \\
& \Upsilon_{\tilde{\jmath}j} \circ \boldsymbol{f}_{\tilde{\imath}\tilde{\jmath}} \circ \mathrm{T}_{i'\tilde{\imath}} \circ \mathrm{T}_{ii'} \arrow[u, Rightarrow, "\boldsymbol{F}_{\tilde{\imath}}^{\tilde{\jmath}j}*\mathrm{id}"'] \arrow[r, Rightarrow, "\mathrm{id}*\mathrm{K}_{ii'\tilde{\imath}}"] \arrow[d, Rightarrow, "\mathrm{id}*\boldsymbol{\lambda}_{\tilde{\imath}\tilde{\jmath}}*\mathrm{id}"'] & \Upsilon_{\tilde{\jmath}j} \circ \boldsymbol{f}_{\tilde{\imath}\tilde{\jmath}} \circ \mathrm{T}_{i\tilde{\imath}} \arrow[u, Rightarrow, "\boldsymbol{F}_{\tilde{\imath}}^{\tilde{\jmath}j}*\mathrm{id}"] \arrow[d, Rightarrow, "\mathrm{id}*\boldsymbol{\lambda}_{\tilde{\imath}\tilde{\jmath}}*\mathrm{id}"] & \\
& \Upsilon_{\tilde{\jmath}j} \circ \boldsymbol{g}_{\tilde{\imath}\tilde{\jmath}} \circ \mathrm{T}_{i'\tilde{\imath}} \circ \mathrm{T}_{ii'} \arrow[r, Rightarrow, "\mathrm{id}*\mathrm{K}_{ii'\tilde{\imath}}"] \arrow[d, Rightarrow, "\boldsymbol{G}_{\tilde{\imath}}^{\tilde{\jmath}j}*\mathrm{id}"'] & \Upsilon_{\tilde{\jmath}j} \circ \boldsymbol{g}_{\tilde{\imath}\tilde{\jmath}} \circ \mathrm{T}_{i\tilde{\imath}} \arrow[d, Rightarrow, "\boldsymbol{G}_{\tilde{\imath}}^{\tilde{\jmath}j}*\mathrm{id}"] & \\
& \boldsymbol{g}_{\tilde{\imath}j} \circ \mathrm{T}_{i'\tilde{\imath}} \circ \mathrm{T}_{ii'} \arrow[dl, Rightarrow, "\boldsymbol{G}_{i'\tilde{\imath}}^j*\mathrm{id}"] \arrow[r, Rightarrow, "\mathrm{id}*\mathrm{K}_{ii'\tilde{\imath}}"] & \boldsymbol{g}_{\tilde{\imath}j} \circ \mathrm{T}_{i\tilde{\imath}} \arrow[dr, Rightarrow, "\boldsymbol{G}_{i\tilde{\imath}}^j"'] & \\
\boldsymbol{g}_{i'j} \circ \mathrm{T}_{ii'} \arrow[rrr, Rightarrow, "\boldsymbol{G}_{ii'}^j"] & & & \boldsymbol{g}_{ij}.
\end{tikzcd} \tag{5.31}$$

Here the hexagons commute by the definition (5.29) of $\boldsymbol{\lambda}_{i\tilde{\imath}}^{\tilde{\jmath}j}$, the top and bottom quadrilaterals by Definition 4.17(f) for $\boldsymbol{f}, \boldsymbol{g}$, and the central rectangles by compatibility of horizontal and vertical composition. Thus (5.31) commutes.

We now have

$$\begin{aligned}
\boldsymbol{G}_{ii'}^j \odot (\boldsymbol{\mu}_{i'j} * \mathrm{id}) &= \boldsymbol{G}_{ii'}^j \odot \bigl(\textstyle\sum_{\tilde{\imath}\in I} \sum_{\tilde{\jmath}\in J} \eta_{i'\tilde{\imath}} \cdot f_{i'j}^*(\zeta_{j\tilde{\jmath}}) \cdot \boldsymbol{\lambda}_{i'\tilde{\imath}}^{\tilde{\jmath}j}\bigr) * \mathrm{id} \\
&= \textstyle\sum_{\tilde{\imath}\in I} \sum_{\tilde{\jmath}\in J} \tau_{ii'}^*(\eta_{i'\tilde{\imath}}) \cdot (f_{i'j} \circ \tau_{ii'})^*(\zeta_{j\tilde{\jmath}}) \cdot \boldsymbol{G}_{ii'}^j \odot (\boldsymbol{\lambda}_{i'\tilde{\imath}}^{\tilde{\jmath}j} * \mathrm{id}) \\
&= \textstyle\sum_{\tilde{\imath}\in I} \sum_{\tilde{\jmath}\in J} \eta_{i\tilde{\imath}} \cdot (f_{i'j} \circ \tau_{ii'})^*(\zeta_{j\tilde{\jmath}}) \cdot (\boldsymbol{\lambda}_{i\tilde{\imath}}^{\tilde{\jmath}j} \odot \boldsymbol{F}_{ii'}^j) \\
&= \bigl(\textstyle\sum_{\tilde{\imath}\in I} \sum_{\tilde{\jmath}\in J} \eta_{i\tilde{\imath}} \cdot f_{ij}^*(\zeta_{j\tilde{\jmath}}) \cdot \boldsymbol{\lambda}_{i\tilde{\imath}}^{\tilde{\jmath}j}\bigr) \odot \boldsymbol{F}_{ii'}^j = \boldsymbol{\mu}_{ij} \odot \boldsymbol{F}_{ii'}^j,
\end{aligned}$$

where the first and fifth steps use (5.30), and the third uses (5.26), (5.31), and the fact that $\boldsymbol{\mu}_{ij}$ in (5.30) only depends on $\eta_{i\tilde{\imath}}$ up to $O(r_i)$. This proves Definition 4.18(a) for $\boldsymbol{\mu}$, and part (b) is similar. Hence $\boldsymbol{\mu} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism, so $[\boldsymbol{f}] = [\boldsymbol{g}]$ as morphisms in $\mathrm{Ho}(\mathbf{m\dot{K}ur})$, and $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}$ is faithful, as we want.

5.6.3 $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}$ is full

Let $\boldsymbol{X},\boldsymbol{Y}$ be objects in $\mathbf{m\dot{K}ur}$, and write $\boldsymbol{X}'=F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{X})$, $\boldsymbol{Y}'=F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{Y})$. Suppose $\boldsymbol{f}':\boldsymbol{X}'\to\boldsymbol{Y}'$ is a morphism in $\boldsymbol{\mu}\mathbf{\dot{K}ur}$. We must show that there exists a 1-morphism $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$ with $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}([\boldsymbol{f}])=\boldsymbol{f}'$.

Use notation (4.6)–(4.7) for $\boldsymbol{X},\boldsymbol{Y}$, as in §5.3 write $\boldsymbol{f}'=\bigl(f,[\boldsymbol{f}_{ij}]_{i\in I,\,j\in J}\bigr)$, and let $\boldsymbol{f}_{ij}:(U_i,D_i,r_i,\chi_i)\to(V_j,E_j,s_j,\psi_j)$ be a 1-morphism of m-Kuranishi neighbourhoods representing $[\boldsymbol{f}_{ij}]$ for all $i\in I$ and $j\in J$. Then Definition 5.13(a),(b) for $\boldsymbol{f}'$ imply that $[\boldsymbol{f}_{\tilde\imath j}]\circ[\mathrm{T}_{i\tilde\imath}]=[\boldsymbol{f}_{ij}]$ and $[\Upsilon_{j\tilde\jmath}]\circ[\boldsymbol{f}_{ij}]=[\boldsymbol{f}_{i\tilde\jmath}]$ for all $i,\tilde\imath\in I$ and $j,\tilde\jmath\in J$, so that $[\Upsilon_{\tilde\jmath j}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i\tilde\imath}]=[\Upsilon_{\tilde\jmath j}]\circ[\boldsymbol{f}_{\tilde\imath\tilde\jmath}]\circ[\mathrm{T}_{i\tilde\imath}]=[\boldsymbol{f}_{ij}]$. Hence we may choose 2-morphisms of m-Kuranishi neighbourhoods over f

$$\boldsymbol{\lambda}_{i\tilde\imath}^{\tilde\jmath j}:\Upsilon_{\tilde\jmath j}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i\tilde\imath}\Longrightarrow\boldsymbol{f}_{ij}$$

for all $i,\tilde\imath\in I$ and $j,\tilde\jmath\in J$. For $i,i',\tilde\imath\in I$ and $j,j',\tilde\jmath\in J$, define 2-morphisms $\boldsymbol{F}_{ii'(\tilde\imath)}^{j(\tilde\jmath)}:\boldsymbol{f}_{i'j}\circ\mathrm{T}_{ii'}\Rightarrow\boldsymbol{f}_{ij}$ over (S,f) for $S=\mathrm{Im}\,\chi_i\cap\mathrm{Im}\,\chi_{i'}\cap\mathrm{Im}\,\chi_{\tilde\imath}\cap f^{-1}(\mathrm{Im}\,\psi_j\cap\mathrm{Im}\,\psi_{\tilde\jmath})$ and $\boldsymbol{F}_{i(\tilde\imath)}^{jj'(\tilde\jmath)}:\Upsilon_{jj'}\circ\boldsymbol{f}_{ij}\Rightarrow\boldsymbol{f}_{ij'}$ over (S,f) for $S=\mathrm{Im}\,\chi_i\cap\mathrm{Im}\,\chi_{\tilde\imath}\cap f^{-1}(\mathrm{Im}\,\psi_j\cap\mathrm{Im}\,\psi_{j'}\cap\mathrm{Im}\,\psi_{\tilde\jmath})$ by the commutative diagrams

$$\begin{tikzcd}
\boldsymbol{f}_{i'j}\circ\mathrm{T}_{ii'} \arrow[rr, Rightarrow, "\boldsymbol{F}_{ii'(\tilde\imath)}^{j(\tilde\jmath)}"'] \arrow[d, Rightarrow, "(\boldsymbol{\lambda}_{i'\tilde\imath}^{\tilde\jmath j})^{-1}*\mathrm{id}_{\mathrm{T}_{ii'}}"] & & \boldsymbol{f}_{ij} \\
\Upsilon_{\tilde\jmath j}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i'\tilde\imath}\circ\mathrm{T}_{ii'} \arrow[rr, Rightarrow, "\mathrm{id}_{\Upsilon_{\tilde\jmath j}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}}*\mathrm{K}_{ii'\tilde\imath}"] & & \Upsilon_{\tilde\jmath j}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i\tilde\imath}, \arrow[u, Rightarrow, "\boldsymbol{\lambda}_{i\tilde\imath}^{\tilde\jmath j}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\Upsilon_{jj'}\circ\boldsymbol{f}_{ij} \arrow[rr, Rightarrow, "\boldsymbol{F}_{i(\tilde\imath)}^{jj'(\tilde\jmath)}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\Upsilon_{jj'}}*(\boldsymbol{\lambda}_{i\tilde\imath}^{\tilde\jmath j'})^{-1}"] & & \boldsymbol{f}_{ij'} \\
\Upsilon_{jj'}\circ\Upsilon_{\tilde\jmath j}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i\tilde\imath} \arrow[rr, Rightarrow, "\Lambda_{\tilde\jmath jj'}*\mathrm{id}_{\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i\tilde\imath}}"] & & \Upsilon_{\tilde\jmath j'}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i\tilde\imath}. \arrow[u, Rightarrow, "\boldsymbol{\lambda}_{i\tilde\imath}^{\tilde\jmath j'}"']
\end{tikzcd} \tag{5.32}$$

Apply Lemma 5.36 to $\boldsymbol{X}'$, using $\mathrm{T}_{ij}=(U_{ii'},\tau_{ii'},\hat\tau_{ii'})$ from $\boldsymbol{X}$ to represent $[\mathrm{T}_{ij}]$. This gives a partition of unity $\{\eta_{i\tilde\imath}:\tilde\imath\in I\}$ on U_i subordinate to $\{U_{i\tilde\imath}:\tilde\imath\in I\}$ for each $i\in I$ satisfying (5.26). Similarly, applying Lemma 5.36 to $\boldsymbol{Y}'$ gives a partition of unity $\{\zeta_{j\tilde\jmath}:\tilde\jmath\in J\}$ on V_j subordinate to $\{V_{j\tilde\jmath}:\tilde\jmath\in J\}$ for $j\in J$.

As in (5.30), using the notation of (5.6) in Definition 5.4, for $i,i'\in I$ and $j,j'\in J$ define 2-morphisms $\boldsymbol{F}_{ii'}^{j}:\boldsymbol{f}_{i'j}\circ\mathrm{T}_{ii'}\Rightarrow\boldsymbol{f}_{ij}$ over (S,f) for $S=\mathrm{Im}\,\chi_i\cap\mathrm{Im}\,\chi_{i'}\cap f^{-1}(\mathrm{Im}\,\psi_j)$, and $\boldsymbol{F}_{i}^{jj'}:\Upsilon_{jj'}\circ\boldsymbol{f}_{ij}\Rightarrow\boldsymbol{f}_{ij'}$ over (S,f) for $S=\mathrm{Im}\,\chi_i\cap f^{-1}(\mathrm{Im}\,\psi_j\cap\mathrm{Im}\,\psi_{j'})$ by

$$\begin{aligned}\boldsymbol{F}_{ii'}^{j}&=\textstyle\sum_{\tilde\imath\in I}\sum_{\tilde\jmath\in J}\eta_{i\tilde\imath}\cdot f_{ij}^*(\zeta_{j\tilde\jmath})\cdot\boldsymbol{F}_{ii'(\tilde\imath)}^{j(\tilde\jmath)},\\ \boldsymbol{F}_{i}^{jj'}&=\textstyle\sum_{\tilde\imath\in I}\sum_{\tilde\jmath\in J}\eta_{i\tilde\imath}\cdot f_{ij'}^*(\zeta_{j'\tilde\jmath})\cdot\boldsymbol{F}_{i(\tilde\imath)}^{jj'(\tilde\jmath)}.\end{aligned} \tag{5.33}$$

We now claim that $\boldsymbol{f}=\bigl(f,\boldsymbol{f}_{ij,\,i\in I,\,j\in J},\,\boldsymbol{F}_{ii',\,i,i'\in I}^{j,\,j\in J},\,\boldsymbol{F}_{i,\,i\in I}^{jj',\,j,j'\in J}\bigr)$ is a 1-morphism $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$. We must verify Definition 4.17(a)–(h).

Parts (a)–(d) are immediate. For (e), if $i=i'$ then $\boldsymbol{F}_{ii(\tilde\imath)}^{j(\tilde\jmath)}$ in (5.32) is $\mathrm{id}_{\boldsymbol{f}_{ij}}$, giving $\boldsymbol{F}_{ii}^{j}=\mathrm{id}_{\boldsymbol{f}_{ij}}$ in (5.33). Similarly $\boldsymbol{F}_{i}^{jj}=\mathrm{id}_{\boldsymbol{f}_{ij}}$, proving Definition 4.17(e).

To prove part (f), let $\tilde\imath\in I$, $\tilde\jmath\in J$ and consider the diagram

$$
\begin{tikzcd}
\boldsymbol{f}_{i''j}\circ \mathrm{T}_{i'i''}\circ \mathrm{T}_{ii'} \arrow[rrr, Rightarrow, "\boldsymbol{F}_{i'i''(\tilde\imath)}^{j(\tilde\jmath)}*\mathrm{id}"'] \arrow[ddd, Rightarrow, "\mathrm{id}*\mathrm{K}_{ii'i''}"] & & & \boldsymbol{f}_{i'j}\circ \mathrm{T}_{ii'} \arrow[ddd, Rightarrow, "\boldsymbol{F}_{ii'(\tilde\imath)}^{j(\tilde\jmath)}"'] \\
& \Upsilon_{\tilde\jmath j}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i''\tilde\imath}\circ\mathrm{T}_{i'i''}\circ\mathrm{T}_{ii'} \arrow[ul, Rightarrow, "\boldsymbol{\lambda}_{i''\tilde\imath}^{\tilde\jmath j}*\mathrm{id}"'] \arrow[r, Rightarrow, "\mathrm{id}*\mathrm{K}_{i'i''\tilde\imath}*\mathrm{id}"'] \arrow[d, Rightarrow, "\mathrm{id}*\mathrm{K}_{ii'i''}"] & \Upsilon_{\tilde\jmath j}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i'\tilde\imath}\circ\mathrm{T}_{ii'} \arrow[ur, Rightarrow, "\boldsymbol{\lambda}_{i'\tilde\imath}^{\tilde\jmath j}*\mathrm{id}"] \arrow[d, Rightarrow, "\mathrm{id}*\mathrm{K}_{ii'\tilde\imath}"'] & \\
& \Upsilon_{\tilde\jmath j}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i''\tilde\imath}\circ\mathrm{T}_{ii''} \arrow[r, Rightarrow, "\mathrm{id}*\mathrm{K}_{ii''\tilde\imath}"] \arrow[dl, Rightarrow, "\boldsymbol{\lambda}_{i''\tilde\imath}^{\tilde\jmath j}*\mathrm{id}"] & \Upsilon_{\tilde\jmath j}\circ\boldsymbol{f}_{\tilde\imath\tilde\jmath}\circ\mathrm{T}_{i\tilde\imath} \arrow[dr, Rightarrow, "\boldsymbol{\lambda}_{i\tilde\imath}^{\tilde\jmath j}"'] & \\
\boldsymbol{f}_{i''j}\circ \mathrm{T}_{ii''} \arrow[rrr, Rightarrow, "\boldsymbol{F}_{ii''(\tilde\imath)}^{j(\tilde\jmath)}"] & & & \boldsymbol{f}_{ij}.
\end{tikzcd}
\tag{5.34}
$$

Here the top, bottom and right quadrilaterals commute by (5.32), the central rectangle by Definition 4.14(h) for $\boldsymbol{X}$, and the left quadrilateral by compatibility of horizontal and vertical composition. Thus (5.34) commutes.

We now have

$$
\begin{aligned}
&\boldsymbol{F}_{ii''}^{j}\odot(\mathrm{id}_{\boldsymbol{f}_{i''j}}*\mathrm{K}_{ii'i''})=\Bigl(\sum_{\tilde\imath\in I}\sum_{\tilde\jmath\in J}\eta_{i\tilde\imath}\cdot f_{ij}^{*}(\zeta_{j\tilde\jmath})\cdot\boldsymbol{F}_{ii''(\tilde\imath)}^{j(\tilde\jmath)}\Bigr)\odot(\mathrm{id}_{\boldsymbol{f}_{i''j}}*\mathrm{K}_{ii'i''})\\
&=\sum_{\tilde\imath\in I}\sum_{\tilde\jmath\in J}\eta_{i\tilde\imath}\cdot f_{ij}^{*}(\zeta_{j\tilde\jmath})\cdot\bigl(\boldsymbol{F}_{ii''(\tilde\imath)}^{j(\tilde\jmath)}\odot(\mathrm{id}_{\boldsymbol{f}_{i''j}}*\mathrm{K}_{ii'i''})\bigr)\\
&=\sum_{\tilde\imath\in I}\sum_{\tilde\jmath\in J}\eta_{i\tilde\imath}\cdot f_{ij}^{*}(\zeta_{j\tilde\jmath})\cdot\bigl(\boldsymbol{F}_{ii'(\tilde\imath)}^{j(\tilde\jmath)}\odot(\boldsymbol{F}_{i'i''(\tilde\imath)}^{j(\tilde\jmath)}*\mathrm{id}_{\mathrm{T}_{ii'}})\bigr)\\
&=\Bigl(\sum_{\tilde\imath\in I}\sum_{\tilde\jmath\in J}\eta_{i\tilde\imath}\cdot f_{ij}^{*}(\zeta_{j\tilde\jmath})\cdot\boldsymbol{F}_{ii'(\tilde\imath)}^{j(\tilde\jmath)}\Bigr)\odot\Bigl(\sum_{\tilde\imath\in I}\sum_{\tilde\jmath\in J}\eta_{i\tilde\imath}\cdot f_{ij}^{*}(\zeta_{j\tilde\jmath})\cdot(\boldsymbol{F}_{i'i''(\tilde\imath)}^{j(\tilde\jmath)}*\mathrm{id}_{\mathrm{T}_{ii'}})\Bigr)\\
&=\boldsymbol{F}_{ii'}^{j}\odot\Bigl(\sum_{\tilde\imath\in I}\sum_{\tilde\jmath\in J}\tau_{ii'}^{*}(\eta_{i'\tilde\imath})\cdot(f_{i'j}\circ\tau_{ii'})^{*}(\zeta_{j\tilde\jmath})\cdot(\boldsymbol{F}_{i'i''(\tilde\imath)}^{j(\tilde\jmath)}*\mathrm{id}_{\mathrm{T}_{ii'}})\Bigr)\\
&=\boldsymbol{F}_{ii'}^{j}\odot\Bigl(\Bigl(\sum_{\tilde\imath\in I}\sum_{\tilde\jmath\in J}\eta_{i'\tilde\imath}\cdot f_{i'j}^{*}(\zeta_{j\tilde\jmath})\cdot\boldsymbol{F}_{i'i''(\tilde\imath)}^{j(\tilde\jmath)}\Bigr)*\mathrm{id}_{\mathrm{T}_{ii'}}\Bigr)=\boldsymbol{F}_{ii'}^{j}\odot(\boldsymbol{F}_{i'i''}^{j}*\mathrm{id}_{\mathrm{T}_{ii'}}).
\end{aligned}
\tag{5.35}
$$

Here we use (5.33) in the first and seventh steps, and (5.34) in the third. In the fourth step, it may be surprising that one sum $\sum_{\tilde\imath}\sum_{\tilde\jmath}$ turns into two sums composed with $\odot$. This is because $\odot$ in Definition 4.5 is basically an operation of addition, not multiplication, so sums (5.6) are distributive over $\odot$. In the fifth step we use (5.26) for the $\eta_{i\tilde\imath}$, and (5.33), and $f_{i'j}\circ\tau_{ii'}=f_{ij}+O(r_i)$, and the fact that $\boldsymbol{F}_{ii'}^{j}$ in (5.33) only depends on $\eta_{i\tilde\imath},f_{ij}^{*}(\zeta_{j\tilde\jmath})$ up to $O(r_i)$.

Equation (5.35) proves Definition 4.17(f) for $\boldsymbol{f}$. Parts (g),(h) are similar. Hence $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is a 1-morphism in $\mathbf{m\dot{K}ur}$. By construction $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}([\boldsymbol{f}])=\boldsymbol{f}'$, so $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}$ is full, as we have to prove.

5.6.4 $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}$ is surjective on isomorphism classes

Let $\boldsymbol{X}' = (X, \mathcal{K}')$ be a μ-Kuranishi space, with $\mathcal{K}' = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, [\Phi_{ij}]_{i,j\in I}\bigr)$. To show $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}$ is surjective on isomorphism classes, we must construct an object $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$ with $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{X}) \cong \boldsymbol{X}'$ in $\boldsymbol{\mu}\mathbf{\dot{K}ur}$. Actually we will arrange that $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{X}) = \boldsymbol{X}'$.

Then (V_i, E_i, s_i, ψ_i) is an m-Kuranishi neighbourhood on X for $i \in I$. Choose a representative Φ_{ij} for $[\Phi_{ij}]$ for $i, j \in I$, where as $[\Phi_{ii}] = [\mathrm{id}_{(V_i,E_i,s_i,\psi_i)}]$ we take $\Phi_{ii} = \mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$. As $[\Phi_{jk}] \circ [\Phi_{ij}] = [\Phi_{ik}]$ for $i, j, k \in I$ by Definition 5.11(f) for $\boldsymbol{X}'$, there exists a 2-morphism of m-Kuranishi neighbourhoods

$$\mathrm{K}_{ijk} : \Phi_{jk} \circ \Phi_{ij} \Longrightarrow \Phi_{ik}$$

over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$, where as Φ_{ii}, Φ_{jj} are identities we choose $\mathrm{K}_{iij} = \mathrm{K}_{ijj} = \mathrm{id}_{\Phi_{ij}}$ for $i, j \in I$. Therefore $\mathrm{K}_{iji} : \Phi_{ji} \circ \Phi_{ij} \Rightarrow \mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$, $\mathrm{K}_{jij} : \Phi_{ij} \circ \Phi_{ji} \Rightarrow \mathrm{id}_{(V_j,E_j,s_j,\psi_j)}$ imply that Φ_{ij} is an equivalence in $\mathbf{\dot{K}N}_S(X)$ for $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, and so a coordinate change over S, for all $i, j \in I$.

Let $\tilde\imath, i, j, k \in I$. Then Lemma A.6 in the 2-category $\mathbf{\dot{K}N}_S(X)$ and $\Phi_{\tilde\imath i}$ an equivalence implies that there is a unique 2-morphism

$$\mathrm{K}_{ijk}^{(\tilde\imath)} : \Phi_{jk} \circ \Phi_{ij} \Longrightarrow \Phi_{ik}$$

over $S = \operatorname{Im}\psi_{\tilde\imath} \cap \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$ making the following diagram commute:

$$\begin{tikzcd}
\Phi_{jk}\circ\Phi_{ij}\circ\Phi_{\tilde\imath i} \arrow[rr, Rightarrow, "\mathrm{K}_{ijk}^{(\tilde\imath)}*\mathrm{id}_{\Phi_{\tilde\imath i}}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{jk}}*\mathrm{K}_{\tilde\imath ij}"] & & \Phi_{ik}\circ\Phi_{\tilde\imath i} \\
\Phi_{jk}\circ\Phi_{\tilde\imath j} \arrow[rr, Rightarrow, "\mathrm{K}_{\tilde\imath jk}"] & & \Phi_{\tilde\imath k}. \arrow[u, Rightarrow, "\mathrm{K}_{\tilde\imath ik}^{-1}"']
\end{tikzcd} \tag{5.36}$$

Apply Lemma 5.36 to $\boldsymbol{X}'$, using $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat\phi_{ij})$ to represent $[\Phi_{ij}]$. This gives a partition of unity $\{\eta_{i\tilde\imath} : \tilde\imath \in I\}$ on V_i subordinate to $\{V_{i\tilde\imath} : \tilde\imath \in I\}$ for each $i \in I$, satisfying (5.26). As in (5.30) and (5.33), using the notation of (5.6) in Definition 5.4, for all $i, j, k \in I$ define a 2-morphism $\Lambda_{ijk} : \Phi_{jk} \circ \Phi_{ij} \Rightarrow \Phi_{ik}$ over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$ by

$$\Lambda_{ijk} = \textstyle\sum_{\tilde\imath\in I} \eta_{i\tilde\imath} \cdot \mathrm{K}_{ijk}^{(\tilde\imath)}. \tag{5.37}$$

Define $\mathcal{K} = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \Lambda_{ijk,\ i,j,k\in I}\bigr)$. We will show that $\boldsymbol{X} = (X, \mathcal{K})$ is an m-Kuranishi space with $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{X}) = \boldsymbol{X}'$. Definition 4.14(a)–(f) for $\mathcal{K}$ are immediate. For (g), as $\mathrm{K}_{iij} = \mathrm{K}_{ijj} = \mathrm{id}_{\Phi_{ij}}$, equation (5.36) implies that $\mathrm{K}_{iij}^{(\tilde\imath)} = \mathrm{K}_{ijj}^{(\tilde\imath)} = \mathrm{id}_{\Phi_{ij}}$, so (5.37) gives $\Lambda_{iij} = \Lambda_{ijj} = \mathrm{id}_{\Phi_{ij}}$, as we want.

To prove Definition 4.14(h) for $\mathcal{K}$, let $\tilde{\imath}, i, j, k, l \in I$, and consider the diagram

$$
\begin{tikzcd}
\Phi_{kl}\circ\Phi_{jk}\circ\Phi_{ij}\circ\Phi_{\tilde{\imath}i} \arrow[rrr, Rightarrow, "\mathrm{K}^{(\tilde{\imath})}_{jkl}*\mathrm{id}"'] \arrow[dr, Rightarrow, "\mathrm{id}*\mathrm{K}_{\tilde{\imath}ij}"] \arrow[ddd, Rightarrow, "\mathrm{id}*\mathrm{K}^{(\tilde{\imath})}_{ijk}*\mathrm{id}"] & & & \Phi_{jl}\circ\Phi_{ij}\circ\Phi_{\tilde{\imath}i} \arrow[dl, Rightarrow, "\mathrm{id}*\mathrm{K}_{\tilde{\imath}ij}"'] \arrow[ddd, Rightarrow, "\mathrm{K}^{(\tilde{\imath})}_{ijl}*\mathrm{id}"'] \\
 & \Phi_{kl}\circ\Phi_{jk}\circ\Phi_{\tilde{\imath}j} \arrow[r, Rightarrow, "\mathrm{K}^{(\tilde{\imath})}_{jkl}*\mathrm{id}"'] \arrow[d, Rightarrow, "\mathrm{id}*\mathrm{K}_{\tilde{\imath}jk}"] & \Phi_{jl}\circ\Phi_{\tilde{\imath}j} \arrow[d, Rightarrow, "\mathrm{K}_{\tilde{\imath}jl}"'] & \\
 & \Phi_{kl}\circ\Phi_{\tilde{\imath}k} \arrow[r, Rightarrow, "\mathrm{K}_{\tilde{\imath}kl}"] & \Phi_{\tilde{\imath}l} & \\
\Phi_{kl}\circ\Phi_{ik}\circ\Phi_{\tilde{\imath}i} \arrow[ur, Rightarrow, "\mathrm{id}*\mathrm{K}_{\tilde{\imath}ik}"'] \arrow[rrr, Rightarrow, "\mathrm{K}^{(\tilde{\imath})}_{ikl}*\mathrm{id}"] & & & \Phi_{il}\circ\Phi_{\tilde{\imath}i} \arrow[ul, Rightarrow, "\mathrm{K}_{\tilde{\imath}il}"]
\end{tikzcd}
\tag{5.38}
$$

over $S = \operatorname{Im}\psi_{\tilde{\imath}} \cap \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k \cap \operatorname{Im}\psi_l$. Here the top quadrilateral commutes by compatibility of horizontal and vertical composition, and the other four quadrilaterals commute by (5.36). Hence (5.38) commutes.

Applying Lemma A.6 to the outer rectangle of (5.38) and using $\Phi_{\tilde{\imath}i}$ an equivalence shows that over $S = \operatorname{Im}\psi_{\tilde{\imath}} \cap \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k \cap \operatorname{Im}\psi_l$ we have

$$
\mathrm{K}^{(\tilde{\imath})}_{ikl} \odot \big(\mathrm{id}_{\Phi_{kl}} * \mathrm{K}^{(\tilde{\imath})}_{ijk}\big) = \mathrm{K}^{(\tilde{\imath})}_{ijl} \odot \big(\mathrm{K}^{(\tilde{\imath})}_{jkl} * \mathrm{id}_{\Phi_{ij}}\big) : \Phi_{kl}\circ\Phi_{jk}\circ\Phi_{ij} \Longrightarrow \Phi_{il}. \tag{5.39}
$$

Now

$$
\begin{aligned}
\Lambda_{ikl} \odot (\mathrm{id}_{\Phi_{kl}} * \Lambda_{ijk}) &= \Big(\sum_{\tilde{\imath}\in I} \eta_{i\tilde{\imath}} \cdot \mathrm{K}^{(\tilde{\imath})}_{ikl}\Big) \odot \Big(\mathrm{id}_{\Phi_{kl}} * \Big(\sum_{\tilde{\imath}\in I} \eta_{i\tilde{\imath}} \cdot \mathrm{K}^{(\tilde{\imath})}_{ijk}\Big)\Big) \\
&= \sum_{\tilde{\imath}\in I} \eta_{i\tilde{\imath}} \cdot \big(\mathrm{K}^{(\tilde{\imath})}_{ikl} \odot \big(\mathrm{id}_{\Phi_{kl}} * \mathrm{K}^{(\tilde{\imath})}_{ijk}\big)\big) = \sum_{\tilde{\imath}\in I} \eta_{i\tilde{\imath}} \cdot \big(\mathrm{K}^{(\tilde{\imath})}_{ijl} \odot \big(\mathrm{K}^{(\tilde{\imath})}_{jkl} * \mathrm{id}_{\Phi_{ij}}\big)\big) \\
&= \Big(\sum_{\tilde{\imath}\in I} \eta_{i\tilde{\imath}} \cdot \mathrm{K}^{(\tilde{\imath})}_{ijl}\Big) \odot \Big(\sum_{\tilde{\imath}\in I} \eta_{i\tilde{\imath}} \cdot \big(\mathrm{K}^{(\tilde{\imath})}_{jkl} * \mathrm{id}_{\Phi_{ij}}\big)\Big) \\
&= \Lambda_{ijl} \odot \Big(\sum_{\tilde{\imath}\in I} \phi_{ij}^*(\eta_{j\tilde{\imath}}) \cdot \big(\mathrm{K}^{(\tilde{\imath})}_{jkl} * \mathrm{id}_{\Phi_{ij}}\big)\Big) \\
&= \Lambda_{ijl} \odot \Big(\Big(\sum_{\tilde{\imath}\in I} \eta_{j\tilde{\imath}} \cdot \mathrm{K}^{(\tilde{\imath})}_{jkl}\Big) * \mathrm{id}_{\Phi_{ij}}\Big) = \Lambda_{ijl} \odot (\Lambda_{jkl} * \mathrm{id}_{\Phi_{ij}}).
\end{aligned}
\tag{5.40}
$$

Here we use (5.37) in the first and seventh steps, and (5.39) in the third. In the second and fourth steps we use the fact that sums (5.6) are distributive over $\odot$, as in the proof of (5.35). In the fifth step we use (5.37), and (5.26) for the $\eta_{i\tilde{\imath}}$, and the fact that Λ_{ijk} in (5.37) only depends on $\eta_{i\tilde{\imath}}$ up to $O(s_i)$.

Equation (5.40) proves Definition 4.14(h) for $\mathcal{K}$. Hence $\boldsymbol{X} = (X, \mathcal{K})$ is an m-Kuranishi space. By construction $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}(\boldsymbol{X}) = \boldsymbol{X}'$. Therefore $F_{\mathbf{m\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\dot{K}ur}}$ is surjective on isomorphism classes. This completes the proof of Theorem 5.23.

Chapter 6

Kuranishi spaces, and orbifolds

Throughout this chapter we suppose we are given a category $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7 in §3.1 (though defining the 2-category of orbifolds $\dot{\mathbf{Orb}}$ in §6.6 only needs Assumptions 3.1–3.3). As in Chapter 4, we will usually refer to objects $X \in \dot{\mathbf{Man}}$ as 'manifolds', and morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ as 'smooth maps'. We will call objects X in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$ 'classical manifolds', and call morphisms $f : X \to Y$ in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$ 'classical smooth maps'.

Classical orbifolds $\mathfrak{X}$ are generalizations of classical manifolds which are locally modelled on $\mathbb{R}^n/\Gamma$ for Γ a finite group acting linearly on $\mathbb{R}^n$. Kuranishi spaces are an orbifold version of m-Kuranishi spaces in Chapter 4, and as in §4.8 should be regarded as 'derived orbifolds'. From the category $\dot{\mathbf{Man}}$ we will construct a weak 2-category of 'Kuranishi spaces' $\dot{\mathbf{Kur}}$, with a full and faithful embedding $\mathbf{m}\dot{\mathbf{Kur}} \hookrightarrow \dot{\mathbf{Kur}}$ of $\mathbf{m}\dot{\mathbf{Kur}}$ from §4.3.

Sections 6.1–6.4 follow §4.1–§4.7 closely, but including extra finite groups Γ_i throughout. Section 6.5 discusses isotropy groups, and §6.6 relates orbifolds and Kuranishi spaces. The proof of Theorem 6.16 is deferred until §6.7.

6.1 The weak 2-category of Kuranishi neighbourhoods

The next seven definitions are the orbifold analogues of Definitions 4.1–4.6:

Definition 6.1. Let X be a topological space. A *Kuranishi neighbourhood* on X is a quintuple (V, E, Γ, s, ψ) such that:

(a) V is a manifold (object in $\dot{\mathbf{Man}}$). We allow $V = \emptyset$.

(b) $\pi : E \to V$ is a vector bundle over V, called the *obstruction bundle*.

(c) Γ is a finite group with a smooth action on V (that is, an action by isomorphisms in $\dot{\mathbf{Man}}$), and a compatible action on E preserving the vector bundle structure. We do not assume the Γ-actions are effective.

(d) $s : V \to E$ is a Γ-equivariant smooth section of E, called the *Kuranishi section*.

(e) ψ is a homeomorphism from $s^{-1}(0)/\Gamma$ to an open subset $\operatorname{Im}\psi = \{\psi(\Gamma v) : v \in s^{-1}(0)\}$ in X, called the *footprint* of (V, E, Γ, s, ψ).

We will write $\bar\psi : s^{-1}(0) \to \operatorname{Im}\psi \subseteq X$ for the composition of ψ with the projection $s^{-1}(0) \to s^{-1}(0)/\Gamma$.

Definition 6.2. Let X, Y be topological spaces, $f : X \to Y$ a continuous map, $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, $(V_j, E_j, \Gamma_j, s_j, \psi_j)$ be Kuranishi neighbourhoods on X, Y respectively, and $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j) \subseteq X$ be an open set. A 1-*morphism* $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ *of Kuranishi neighbourhoods over* (S, f) is a quadruple $(P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij})$ satisfying:

(a) P_{ij} is a manifold (object in $\dot{\mathbf{Man}}$), with commuting smooth actions of Γ_i, Γ_j (that is, with a smooth action of $\Gamma_i \times \Gamma_j$), with the Γ_j-action free.

(b) $\pi_{ij} : P_{ij} \to V_i$ is a smooth map (morphism in $\dot{\mathbf{Man}}$) which is Γ_i-equivariant, Γ_j-invariant, and étale (a local diffeomorphism). The image $V_{ij} := \pi_{ij}(P_{ij})$ is a Γ_i-invariant open neighbourhood of $\bar\psi_i^{-1}(S)$ in V_i (that is, $V_{ij} \subseteq V_i$ is an open submanifold in $\dot{\mathbf{Man}}$), and the fibres $\pi_{ij}^{-1}(v)$ of π_{ij} for $v \in V_{ij}$ are Γ_j-orbits, so that $\pi_{ij} : P_{ij} \to V_{ij}$ is a principal Γ_j-bundle.

We do not require $\bar\psi_i^{-1}(S) = V_{ij} \cap s_i^{-1}(0)$, only that $\bar\psi_i^{-1}(S) \subseteq V_{ij} \cap s_i^{-1}(0)$.

(c) $\phi_{ij} : P_{ij} \to V_j$ is a Γ_i-invariant and Γ_j-equivariant smooth map, that is, $\phi_{ij}(\gamma_i \cdot p) = \phi_{ij}(p)$, $\phi_{ij}(\gamma_j \cdot p) = \gamma_j \cdot \phi_{ij}(p)$ for all $\gamma_i \in \Gamma_i$, $\gamma_j \in \Gamma_j$, $p \in P_{ij}$.

(d) $\hat\phi_{ij} : \pi_{ij}^*(E_i) \to \phi_{ij}^*(E_j)$ is a Γ_i- and Γ_j-equivariant morphism of vector bundles on P_{ij}, where the Γ_i, Γ_j-actions are induced by the given Γ_i-action and the trivial Γ_j-action on E_i, and vice versa for E_j.

(e) $\hat\phi_{ij}(\pi_{ij}^*(s_i)) = \phi_{ij}^*(s_j) + O(\pi_{ij}^*(s_i)^2)$, as in Definition 3.15(i).

(f) $f \circ \bar\psi_i \circ \pi_{ij} = \bar\psi_j \circ \phi_{ij}$ on $\pi_{ij}^{-1}(s_i^{-1}(0)) \subseteq P_{ij}$.

If $X = Y$ and $f = \mathrm{id}_X$ then we call Φ_{ij} a 1-*morphism of Kuranishi neighbourhoods over* S, or just a 1-*morphism over* S.

Definition 6.3. Let $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ be a Kuranishi neighbourhood on X, and $S \subseteq \operatorname{Im}\psi_i$ be open. We will define the *identity* 1-*morphism*

$$\mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)} = (P_{ii}, \pi_{ii}, \phi_{ii}, \hat\phi_{ii}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_i, E_i, \Gamma_i, s_i, \psi_i). \quad (6.1)$$

Since P_{ii} must have two different actions of Γ_i, for clarity we write $\Gamma_i^1 = \Gamma_i^2 = \Gamma_i$, where Γ_i^1 and Γ_i^2 mean the copies of Γ_i acting on the domain and target of the 1-morphism in (6.1), respectively.

Define $P_{ii} = V_i \times \Gamma_i$, and let Γ_i^1 act on P_{ii} by $\gamma^1 : (v, \gamma) \mapsto (\gamma^1 \cdot v, \gamma(\gamma^1)^{-1})$ and Γ_i^2 act on P_{ii} by $\gamma^2 : (v, \gamma) \mapsto (v, \gamma^2\gamma)$. Define $\pi_{ii}, \phi_{ii} : P_{ii} \to V_i$ by $\pi_{ii} : (v, \gamma) \mapsto v$ and $\phi_{ii} : (v, \gamma) \mapsto \gamma \cdot v$. Then π_{ii} is Γ_i^1-equivariant and Γ_i^2-invariant, and is a Γ_i^2-principal bundle, and ϕ_{ii} is Γ_i^1-invariant and Γ_i^2-equivariant.

At $(v, \gamma) \in P_{ii}$, the morphism $\hat\phi_{ii} : \pi_{ii}^*(E_i) \to \phi_{ii}^*(E_i)$ must map $E_i|_v \to E_i|_{\gamma \cdot v}$. We have such a map, the lift of the γ-action on V_i to E_i. So we define $\hat\phi_{ii}$ on

$V_i \times \{\gamma\} \subseteq P_{ii}$ to be the lift to E_i of the γ-action on V_i, for each $\gamma \in \Gamma$. It is now easy to check that $(P_{ii}, \pi_{ii}, \phi_{ii}, \hat{\phi}_{ii})$ satisfies Definition 6.2(a)–(f), so (6.1) is a 1-morphism over S.

Definition 6.4. Suppose X, Y are topological spaces, $f : X \to Y$ is a continuous map, $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, $(V_j, E_j, \Gamma_j, s_j, \psi_j)$ are Kuranishi neighbourhoods on X, Y respectively, $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j) \subseteq X$ is open, and $\Phi_{ij}, \Phi'_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ are two 1-morphisms over (S, f), with $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat{\phi}_{ij})$ and $\Phi'_{ij} = (P'_{ij}, \pi'_{ij}, \phi'_{ij}, \hat{\phi}'_{ij})$.

Consider triples $(\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij})$ satisfying:

(a) $\acute{P}_{ij}$ is a Γ_i- and Γ_j-invariant open neighbourhood of $\pi_{ij}^{-1}(\bar{\psi}_i^{-1}(S))$ in P_{ij}.

(b) $\lambda_{ij} : \acute{P}_{ij} \to P'_{ij}$ is a Γ_i- and Γ_j-equivariant smooth map with $\pi'_{ij} \circ \lambda_{ij} = \pi_{ij}|_{\acute{P}_{ij}}$. This implies that λ_{ij} is an isomorphism of principal Γ_j-bundles over $\acute{V}_{ij} := \pi_{ij}(\acute{P}_{ij})$, so λ_{ij} is a diffeomorphism with a Γ_i- and Γ_j-invariant open set $\lambda_{ij}(\acute{P}_{ij})$ in P'_{ij}.

(c) $\hat{\lambda}_{ij} : \pi_{ij}^*(E_i)|_{\acute{P}_{ij}} \to \mathcal{T}_{\phi_{ij}} V_j|_{\acute{P}_{ij}}$ is a morphism in the notation of §3.3.4, which is Γ_i- and Γ_j-equivariant, and satisfies

$$\begin{aligned} &\phi'_{ij} \circ \lambda_{ij} = \phi_{ij}|_{\acute{P}_{ij}} + \hat{\lambda}_{ij} \circ \pi_{ij}^*(s_i) + O\big(\pi_{ij}^*(s_i)^2\big) \ \text{ and} \\ &\lambda_{ij}^*(\hat{\phi}'_{ij}) = \hat{\phi}_{ij}|_{\acute{P}_{ij}} + \phi_{ij}^*(\mathrm{d}s_j) \circ \hat{\lambda}_{ij} + O\big(\pi_{ij}^*(s_i)\big) \ \text{ on } \acute{P}_{ij}, \end{aligned} \tag{6.2}$$

in the sense of Definition 3.15(iv),(vi),(vii).

Define a binary relation $\sim$ on such triples by $(\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}) \sim (\acute{P}'_{ij}, \lambda'_{ij}, \hat{\lambda}'_{ij})$ if there exists an open neighbourhood $\ddot{P}_{ij}$ of $\pi_{ij}^{-1}(\bar{\psi}_i^{-1}(S))$ in $\acute{P}_{ij} \cap \acute{P}'_{ij}$ with

$$\lambda_{ij}|_{\ddot{P}_{ij}} = \lambda'_{ij}|_{\ddot{P}_{ij}} \quad \text{and} \quad \hat{\lambda}_{ij}|_{\ddot{P}_{ij}} = \hat{\lambda}'_{ij}|_{\ddot{P}_{ij}} + O\big(\pi_{ij}^*(s_i)\big) \quad \text{on } \ddot{P}_{ij}, \tag{6.3}$$

in the sense of Definition 3.15(ii). We see from Theorem 3.17(c) that $\sim$ is an equivalence relation. We also write $\sim_S$ in place of $\sim$ if we want to emphasize the open set $S \subseteq X$.

Write $[\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}]$ for the $\sim$-equivalence class of $(\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij})$. We say that $[\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}] : \Phi_{ij} \Rightarrow \Phi'_{ij}$ is a *2-morphism of 1-morphisms of Kuranishi neighbourhoods on X over (S, f)*, or just a *2-morphism over (S, f)*. We often write $\Lambda_{ij} = [\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}]$.

If $X = Y$ and $f = \mathrm{id}_X$ then we call Λ_{ij} a *2-morphism of Kuranishi neighbourhoods over S*, or just a *2-morphism over S*.

For a 1-morphism $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat{\phi}_{ij})$, define the *identity 2-morphism*

$$\mathrm{id}_{\Phi_{ij}} = [P_{ij}, \mathrm{id}_{P_{ij}}, 0] : \Phi_{ij} \Longrightarrow \Phi_{ij}. \tag{6.4}$$

Definition 6.5. Let X, Y, Z be topological spaces, $f : X \to Y$, $g : Y \to Z$ be continuous maps, $(V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j), (V_k, E_k, \Gamma_k, s_k, \psi_k)$ be Kuranishi neighbourhoods on X, Y, Z respectively, and $T \subseteq \operatorname{Im}\psi_j \cap g^{-1}(\operatorname{Im}\psi_k) \subseteq Y$ and $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(T) \subseteq X$ be open. Suppose $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat{\phi}_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ is a 1-morphism of Kuranishi neighbourhoods over (S, f), and $\Phi_{jk} = (P_{jk}, \pi_{jk}, \phi_{jk}, \hat{\phi}_{jk}) : (V_j, E_j, \Gamma_j, s_j, \psi_j) \to (V_k, E_k, \Gamma_k, s_k, \psi_k)$ is a 1-morphism of Kuranishi neighbourhoods over (T, g).

Consider the diagram in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd}
 & & & P_{ij}\times_{V_j}P_{jk} \arrow[loop above, "\Gamma_i\times\Gamma_j\times\Gamma_k"] \arrow[dll, "\pi_{P_{ij}}"'] \arrow[drr, "\pi_{P_{jk}}"] & & & \\
 & P_{ij} \arrow[loop left, "\Gamma_i\times\Gamma_j"] \arrow[dl, "\pi_{ij}"] \arrow[drr, "\phi_{ij}"'] & & & & P_{jk} \arrow[loop right, "\Gamma_j\times\Gamma_k"] \arrow[dll, "\pi_{jk}"] \arrow[dr, "\phi_{jk}"'] & \\
V_i \arrow[loop left, "\Gamma_i"] & & & V_j \arrow[loop above, "\Gamma_j"] & & & V_k. \arrow[loop right, "\Gamma_k"]
\end{tikzcd} \tag{6.5}$$

Here as π_{jk} is étale one can show that the fibre product $P_{ij} \times_{V_j} P_{jk}$ exists in $\dot{\mathbf{Man}}$ using Assumptions 3.2(e) and 3.3(b). We have shown the actions of various combinations of $\Gamma_i, \Gamma_j, \Gamma_k$ on each space. In fact $\Gamma_i \times \Gamma_j \times \Gamma_k$ acts on the whole diagram, with all maps equivariant, but we have omitted the trivial actions (for instance, Γ_j, Γ_k act trivially on V_i).

As Γ_j acts freely on P_{ij}, it also acts freely on $P_{ij} \times_{V_j} P_{jk}$. Using Assumption 3.3 and the facts that $P_{ij} \times_{V_j} P_{jk}$ is Hausdorff and Γ_j is finite, we can show that the quotient $P_{ik} := (P_{ij} \times_{V_j} P_{jk})/\Gamma_j$ exists in $\dot{\mathbf{Man}}$, with projection $\Pi : P_{ij} \times_{V_j} P_{jk} \to P_{ik}$. The commuting actions of Γ_i, Γ_k on $P_{ij} \times_{V_j} P_{jk}$ descend to commuting actions of Γ_i, Γ_k on P_{ik}, such that Π is Γ_i- and Γ_k-equivariant. As $\pi_{ij} \circ \pi_{P_{ij}} : P_{ij} \times_{V_j} P_{jk} \to V_i$ and $\phi_{jk} \circ \pi_{P_{jk}} : P_{ij} \times_{V_j} P_{jk} \to V_k$ are Γ_j-invariant, they factor through Π, so there are unique smooth maps $\pi_{ik} : P_{ik} \to V_i$ and $\phi_{ik} : P_{ik} \to V_k$ such that $\pi_{ij} \circ \pi_{P_{ij}} = \pi_{ik} \circ \Pi$ and $\phi_{jk} \circ \pi_{P_{jk}} = \phi_{ik} \circ \Pi$.

Consider the diagram of vector bundles on $P_{ij} \times_{V_j} P_{jk}$:

$$\begin{tikzcd}
\Pi^*\circ\pi_{ik}^*(E_i) \arrow[rrr, dotted, "\Pi^*(\hat{\phi}_{ik})"'] \arrow[d, equal] & & & \Pi^*\circ\phi_{ik}^*(E_k) \arrow[d, equal] \\
\pi_{P_{ij}}^*\circ\pi_{ij}^*(E_i) \arrow[r, "\pi_{P_{ij}}^*(\hat{\phi}_{ij})"] & \pi_{P_{ij}}^*\circ\phi_{ij}^*(E_j) \arrow[r, equal] & \pi_{P_{jk}}^*\circ\pi_{jk}^*(E_j) \arrow[r, "\pi_{P_{jk}}^*(\hat{\phi}_{jk})"] & \pi_{P_{jk}}^*\circ\phi_{jk}^*(E_k).
\end{tikzcd}$$

There is a unique morphism on the top line making the diagram commute. As $\hat{\phi}_{ij}, \hat{\phi}_{jk}$ are Γ_j-equivariant, this is Γ_j-equivariant, so it is the pullback under Π^* of a unique morphism $\hat{\phi}_{ik} : \pi_{ik}^*(E_i) \to \phi_{ik}^*(E_k)$, as shown. It is now easy to check that $(P_{ik}, \pi_{ik}, \phi_{ik}, \hat{\phi}_{ik})$ satisfies Definition 6.2(a)–(f), and is a 1-morphism $\Phi_{ik} = (P_{ik}, \pi_{ik}, \phi_{ik}, \hat{\phi}_{ik}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_k, E_k, \Gamma_k, s_k, \psi_k)$ over $(S, g \circ f)$. We write $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$, and call it the *composition of* 1*-morphisms*.

If we have three such 1-morphisms $\Phi_{ij}, \Phi_{jk}, \Phi_{kl}$, define

$$\lambda_{ijkl} : \bigl[P_{ij}\times_{V_j}\bigl((P_{jk}\times_{V_k}P_{kl})/\Gamma_k\bigr)\bigr]/\Gamma_j \to \bigl[\bigl((P_{ij}\times_{V_j}P_{jk})/\Gamma_j\bigr)\times_{V_k}P_{kl}\bigr]/\Gamma_k \tag{6.6}$$

to be the natural identification. Then we have a 2-isomorphism

$$\begin{aligned}\boldsymbol{\alpha}_{\Phi_{kl},\Phi_{jk},\Phi_{ij}} := \big[[P_{ij}\times_{V_j}((P_{jk}\times_{V_k}P_{kl})/\Gamma_k)]/\Gamma_j,\lambda_{ijkl},0\big]:\\ (\Phi_{kl}\circ\Phi_{jk})\circ\Phi_{ij}\Longrightarrow\Phi_{kl}\circ(\Phi_{jk}\circ\Phi_{ij}).\end{aligned}\tag{6.7}$$

That is, composition of 1-morphisms is associative up to canonical 2-isomorphism, as for weak 2-categories in §A.2.

For $\Phi_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ as above, define

$$\begin{aligned}\mu_{ij}:((V_i\times\Gamma_i)\times_{V_i}P_{ij})/\Gamma_i\longrightarrow P_{ij},\\ \nu_{ij}:(P_{ij}\times_{V_j}(V_j\times\Gamma_j))/\Gamma_j\longrightarrow P_{ij},\end{aligned}$$

to be the natural identifications. Then we have 2-isomorphisms

$$\begin{aligned}\boldsymbol{\beta}_{\Phi_{ij}} &:= \big[((V_i\times\Gamma_i)\times_{V_i}P_{ij})/\Gamma_i,\mu_{ij},0\big]:\Phi_{ij}\circ\mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)}\Longrightarrow\Phi_{ij},\\ \boldsymbol{\gamma}_{\Phi_{ij}} &:= \big[(P_{ij}\times_{V_j}(V_j\times\Gamma_j))/\Gamma_j,\nu_{ij},0\big]:\mathrm{id}_{(V_j,E_j,\Gamma_j,s_j,\psi_j)}\circ\Phi_{ij}\Longrightarrow\Phi_{ij},\end{aligned}\tag{6.8}$$

so identity 1-morphisms behave as they should up to canonical 2-isomorphism, as for weak 2-categories in §A.2.

Definition 6.6. Let X,Y be topological spaces, $f:X\rightarrow Y$ be continuous, $(V_i,E_i,\Gamma_i,s_i,\psi_i)$, $(V_j,E_j,\Gamma_j,s_j,\psi_j)$ be Kuranishi neighbourhoods on X,Y, $S\subseteq\operatorname{Im}\psi_i\cap f^{-1}(\operatorname{Im}\psi_j)\subseteq X$ be open, and $\Phi_{ij},\Phi'_{ij},\Phi''_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ be 1-morphisms over (S,f) with $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$, $\Phi'_{ij}=(P'_{ij},\pi'_{ij},\phi'_{ij},\hat\phi'_{ij})$, $\Phi''_{ij}=(P''_{ij},\pi''_{ij},\phi''_{ij},\hat\phi''_{ij})$. Suppose $\Lambda_{ij}=[\acute P_{ij},\lambda_{ij},\hat\lambda_{ij}]:\Phi_{ij}\Rightarrow\Phi'_{ij}$ and $\Lambda'_{ij}=[\acute P'_{ij},\lambda'_{ij},\hat\lambda'_{ij}]:\Phi'_{ij}\Rightarrow\Phi''_{ij}$ are 2-morphisms over (S,f). We will define the *vertical composition of 2-morphisms over* (S,f), written

$$\Lambda'_{ij}\odot\Lambda_{ij}=[\acute P'_{ij},\lambda'_{ij},\hat\lambda'_{ij}]\odot[\acute P_{ij},\lambda_{ij},\hat\lambda_{ij}]:\Phi_{ij}\Longrightarrow\Phi''_{ij}.$$

When $X=Y$ and $f=\mathrm{id}_X$ we call it *vertical composition over S*.

Choose representatives $(\acute P_{ij},\lambda_{ij},\hat\lambda_{ij}),(\acute P'_{ij},\lambda'_{ij},\hat\lambda'_{ij})$ in the $\sim$-equivalence classes $[\acute P_{ij},\lambda_{ij},\hat\lambda_{ij}],[\acute P'_{ij},\lambda'_{ij},\hat\lambda'_{ij}]$. Define $\acute P''_{ij}=\lambda_{ij}^{-1}(\acute P_{ij})\subseteq\acute P_{ij}\subseteq P_{ij}$, and $\lambda''_{ij}=\lambda'_{ij}\circ\lambda_{ij}|_{\acute P''_{ij}}$. Consider the sheaf morphism on $\acute P''_{ij}$:

$$\pi_{ij}^*(E_i)|_{\acute P''_{ij}}=\lambda_{ij}^*\circ\pi'^*_{ij}(E_i)|_{\acute P''_{ij}}\xrightarrow{\lambda_{ij}^*(\hat\lambda'_{ij})}\mathcal{T}_{\phi'_{ij}\circ\lambda_{ij}}V_j|_{\acute P''_{ij}},$$

using the notation of §3.3.4. Since $\phi'_{ij}\circ\lambda_{ij}|_{\acute P''_{ij}}=\phi_{ij}|_{\acute P''_{ij}}+O(\pi_{ij}^*(s_i))$ by (6.2), Theorem 3.17(g) shows that there exists a morphism $\check\lambda'_{ij}:\pi_{ij}^*(E_i)|_{\acute P''_{ij}}\rightarrow\mathcal{T}_{\phi_{ij}}V_j|_{\acute P''_{ij}}$, unique up to $O(\pi_{ij}^*(s_i))$, with

$$\check\lambda'_{ij}=\lambda_{ij}^*(\hat\lambda'_{ij})+O(\pi_{ij}^*(s_i)),\tag{6.9}$$

as in Definition 3.15(v). By averaging over the $\Gamma_i\times\Gamma_j$-action we can suppose $\check\lambda'_{ij}$ is Γ_i- and Γ_j-equivariant, as $\hat\lambda'_{ij}$ is.

Define $\hat\lambda''_{ij} : \pi_{ij}^*(E_i)|_{\acute P''_{ij}} \to \mathcal{T}_{\phi_{ij}} V_j|_{\acute P''_{ij}}$ by $\hat\lambda''_{ij} = \hat\lambda_{ij}|_{\acute P''_{ij}} + \check\lambda'_{ij}$. We can prove using Theorem 3.17 that $(\acute P''_{ij}, \lambda''_{ij}, \hat\lambda''_{ij})$ satisfies Definition 6.4(a)–(c) for Φ_{ij}, Φ''_{ij}, using (6.2) for $\hat\lambda_{ij}, \hat\lambda'_{ij}$ and (6.9) to prove (6.2) for $\hat\lambda''_{ij}$. Hence $\Lambda''_{ij} = [\acute P''_{ij}, \lambda''_{ij}, \hat\lambda''_{ij}] : \Phi_{ij} \Rightarrow \Phi''_{ij}$ is a 2-morphism over (S,f). It is independent of choices. We define $[\acute P'_{ij}, \lambda'_{ij}, \hat\lambda'_{ij}] \odot [\acute P_{ij}, \lambda_{ij}, \hat\lambda_{ij}] = [\acute P''_{ij}, \lambda''_{ij}, \hat\lambda''_{ij}]$, or $\Lambda'_{ij} \odot \Lambda_{ij} = \Lambda''_{ij}$.

Let $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ be a 2-morphism over (S,f), and choose a representative $(\acute P_{ij}, \lambda_{ij}, \hat\lambda_{ij})$ for $\Lambda_{ij} = [\acute P_{ij}, \lambda_{ij}, \hat\lambda_{ij}]$. Define $\acute P'_{ij} = \lambda_{ij}(\acute P_{ij})$, so that $\acute P'_{ij} \subseteq P'_{ij}$ is open and $\lambda_{ij} : \acute P_{ij} \to \acute P'_{ij}$ is a diffeomorphism. Set $\lambda'_{ij} = \lambda_{ij}^{-1} : \acute P'_{ij} \to \acute P_{ij} \subseteq P_{ij}$. Then $\acute P'_{ij}$ is Γ_i- and Γ_j-invariant, and λ'_{ij} is Γ_i- and Γ_j-equivariant.

Now $\phi'_{ij} = \phi_{ij} \circ \lambda'_{ij} + O(\pi'^*_{ij}(s_i))$, so Theorem 3.17(g) gives $\hat\lambda'_{ij} : \pi'^*_{ij}(E_i)|_{\acute P'_{ij}} \to \mathcal{T}_{\phi'_{ij}} V_j|_{\acute P'_{ij}}$, unique up to $O(\pi'^*_{ij}(s_i))$, with $\hat\lambda'_{ij} = -\lambda'^*_{ij}(\hat\lambda_{ij}) + O(\pi'^*_{ij}(s_i))$, as in Definition 3.15(v). Since $\hat\lambda_{ij}$ is Γ_i, Γ_j-equivariant, by averaging $\hat\lambda'_{ij}$ over the $\Gamma_i \times \Gamma_j$-action we can suppose $\hat\lambda'_{ij}$ is Γ_i, Γ_j-equivariant. We can then show that $(\acute P'_{ij}, \lambda'_{ij}, \hat\lambda'_{ij})$ satisfies Definition 6.4(a)–(c), so that $\Lambda'_{ij} = [\acute P'_{ij}, \lambda'_{ij}, \hat\lambda'_{ij}] : \Phi'_{ij} \Rightarrow \Phi_{ij}$ is a 2-morphism over (S,f). This Λ'_{ij} is a two-sided inverse Λ_{ij}^{-1} for Λ_{ij} under vertical composition. Thus, *all* 2*-morphisms over* (S,f) *are invertible under vertical composition, that is, they are* 2*-isomorphisms.*

Definition 6.7. Let X, Y, Z be topological spaces, $f : X \to Y$, $g : Y \to Z$ be continuous maps, $(V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j), (V_k, E_k, \Gamma_k, s_k, \psi_k)$ be Kuranishi neighbourhoods on X, Y, Z, and $T \subseteq \operatorname{Im}\psi_j \cap g^{-1}(\operatorname{Im}\psi_k) \subseteq Y$ and $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(T) \subseteq X$ be open. Suppose $\Phi_{ij}, \Phi'_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ are 1-morphisms of Kuranishi neighbourhoods over (S,f), and $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ is a 2-morphism over (S,f), and $\Phi_{jk}, \Phi'_{jk} : (V_j, E_j, \Gamma_j, s_j, \psi_j) \to (V_k, E_k, \Gamma_k, s_k, \psi_k)$ are 1-morphisms of Kuranishi neighbourhoods over (T,g), and $\Lambda_{jk} : \Phi_{jk} \Rightarrow \Phi'_{jk}$ is a 2-morphism over (T,g).

We will define the *horizontal composition of* 2*-morphisms*, written

$$\Lambda_{jk} * \Lambda_{ij} : \Phi_{jk} \circ \Phi_{ij} \Longrightarrow \Phi'_{jk} \circ \Phi'_{ij} \qquad \text{over } (S, g \circ f). \tag{6.10}$$

Use our usual notation for $\Phi_{ij}, \ldots, \Lambda_{jk}$, and write $(P_{ik}, \pi_{ik}, \phi_{ik}, \hat\phi_{ik}) = \Phi_{jk} \circ \Phi_{ij}$, $(P'_{ik}, \pi'_{ik}, \phi'_{ik}, \hat\phi'_{ik}) = \Phi'_{jk} \circ \Phi'_{ij}$, as in Definition 6.5. Choose representatives $(\acute P_{ij}, \lambda_{ij}, \hat\lambda_{ij})$, $(\acute P_{jk}, \lambda_{jk}, \hat\lambda_{jk})$ for $\Lambda_{ij} = [\acute P_{ij}, \lambda_{ij}, \hat\lambda_{ij}]$ and $\Lambda_{jk} = [\acute P_{jk}, \lambda_{jk}, \hat\lambda_{jk}]$.

Then $P_{ik} = (P_{ij} \times_{V_j} P_{jk})/\Gamma_j$, and $\acute P_{ij} \subseteq P_{ij}$, $\acute P_{jk} \subseteq P_{jk}$ are open and Γ_j-invariant, so $\acute P_{ij} \times_{V_j} \acute P_{jk}$ is open and Γ_j-invariant in $P_{ij} \times_{V_j} P_{jk}$. Define $\acute P_{ik} = (\acute P_{ij} \times_{V_j} \acute P_{jk})/\Gamma_j$, as an open subset of P_{ik}. It is Γ_i- and Γ_k-invariant, as $\acute P_{ij}$, $\acute P_{jk}$ are Γ_i- and Γ_k-invariant, respectively.

The maps $\lambda_{ij} : \acute P_{ij} \to P'_{ij}$, $\lambda_{jk} : \acute P_{jk} \to P'_{jk}$ satisfy $\phi'_{ij} \circ \lambda_{ij} = \phi_{ij}|_{\acute P_{ij}} : \acute P_{ij} \to V_j$ and $\pi'_{jk} \circ \lambda_{jk} = \pi_{jk}|_{\acute P_{jk}} : \acute P_{jk} \to V_j$. Hence by properties of fibre products they induce a unique smooth map $\tilde\lambda_{ik} : \acute P_{ij} \times_{\phi_{ij}, V_j, \pi_{jk}} \acute P_{jk} \to P'_{ij} \times_{\phi'_{ij}, V_j, \pi'_{jk}} P'_{jk}$ with $\pi_{P'_{ij}} \circ \tilde\lambda_{ik} = \lambda_{ij} \circ \pi_{\acute P_{ij}}$ and $\pi_{P'_{jk}} \circ \tilde\lambda_{ik} = \lambda_{jk} \circ \pi_{\acute P_{jk}}$. As everything is Γ_j-equivariant,

$\tilde{\lambda}_{ik}$ descends to the quotients by Γ_j. Thus we obtain a unique smooth map

$$\lambda_{ik} : \acute{P}_{ik} = (\acute{P}_{ij} \times_{V_j} \acute{P}_{jk})/\Gamma_j \longrightarrow (P'_{ij} \times_{V_j} P'_{jk})/\Gamma_j = P'_{ik}$$

with $\lambda_{ik} \circ \Pi = \Pi' \circ \tilde{\lambda}_{ik}$, for $\Pi : \acute{P}_{ij} \times_{V_j} \acute{P}_{jk} \to (\acute{P}_{ij} \times_{V_j} \acute{P}_{jk})/\Gamma_j$, $\Pi' : P'_{ij} \times_{V_j} P'_{jk} \to (P'_{ij} \times_{V_j} P'_{jk})/\Gamma_j$ the projections.

Define a morphism of sheaves on $\acute{P}_{ij} \times_{V_j} \acute{P}_{jk}$

$$\begin{aligned} \check{\lambda}_{ik} &: \Pi^* \circ \pi_{ik}^*(E_i) = (\pi_{ij} \circ \pi_{\acute{P}_{ij}})^*(E_i) \longrightarrow \Pi^*(\mathcal{T}_{\phi_{ik}} V_k) \qquad \text{by} \\ \check{\lambda}_{ik} &= (\Pi_*^\flat)^{-1} \circ \mathcal{T}\phi_{jk} \circ (\mathcal{T}\pi_{jk})^{-1} \circ \pi_{\acute{P}_{ij}}^*(\hat{\lambda}_{ij}) \\ &\qquad + (\Pi_*^\flat)^{-1} \circ \pi_{\acute{P}_{jk}}^*(\hat{\lambda}_{jk}) \circ \pi_{\acute{P}_{ij}}^*(\hat{\phi}_{ij}), \end{aligned}$$

where the morphisms are given in the diagram

$$\begin{tikzcd}
(\pi_{ij} \circ \pi_{\acute{P}_{ij}})^*(E_i) \arrow[r, "\pi_{\acute{P}_{ij}}^*(\hat{\phi}_{ij})"'] \arrow[dd, "\pi_{\acute{P}_{ij}}^*(\hat{\lambda}_{ij})"'] & (\phi_{ij} \circ \pi_{\acute{P}_{ij}})^*(E_j) \arrow[rr, equal] & & (\pi_{jk} \circ \pi_{\acute{P}_{jk}})^*(E_j) \arrow[dd, "\pi_{\acute{P}_{jk}}^*(\hat{\lambda}_{jk})"] \\
& \Pi^*(\mathcal{T}_{\phi_{ik}} V_k) \arrow[r, shift left, "\Pi_*^\flat"] & \mathcal{T}_{\phi_{ik} \circ \Pi} V_k \arrow[l, shift left, "(\Pi_*^\flat)^{-1}"] \arrow[dr, equal] & \\
\mathcal{T}_{\phi_{ij} \circ \pi_{\acute{P}_{ij}}} V_j \arrow[r, equal] & \mathcal{T}_{\pi_{jk} \circ \pi_{\acute{P}_{jk}}} V_j \arrow[r, shift left, "(\mathcal{T}\pi_{jk})^{-1}"] & \mathcal{T}_{\pi_{\acute{P}_{jk}}} \acute{P}_{jk} \arrow[l, shift left, "\mathcal{T}\pi_{jk}"] \arrow[r, "\mathcal{T}\phi_{jk}"] & \mathcal{T}_{\phi_{jk} \circ \pi_{\acute{P}_{jk}}} V_k.
\end{tikzcd}$$

Here $\mathcal{T}\pi_{jk} : \mathcal{T}_{\pi_{\acute{P}_{jk}}} \acute{P}_{jk} \to \mathcal{T}_{\pi_{jk} \circ \pi_{\acute{P}_{jk}}} V_j$ and $\Pi_*^\flat : \Pi^*(\mathcal{T}_{\phi_{ik}} V_k) \to \mathcal{T}_{\phi_{ik} \circ \Pi} V_k$ are invertible as π_{jk}, Π are étale. As all the ingredients are $\Gamma_i, \Gamma_j, \Gamma_k$-invariant or equivariant, $\check{\lambda}_{ik}$ is Γ_j-invariant, and so descends to $\acute{P}_{ik} = (\acute{P}_{ij} \times_{V_j} \acute{P}_{jk})/\Gamma_j$. That is, there is a unique morphism $\hat{\lambda}_{ik} : \pi_{ik}^*(E_i)|_{\acute{P}_{ik}} \to \mathcal{T}_{\phi_{ik}} V_k|_{\acute{P}_{ik}}$ of sheaves on $\acute{P}_{ik}$ with $\Pi^*(\hat{\lambda}_{ik}) = \check{\lambda}_{ik}$. As $\check{\lambda}_{ik}$ is Γ_i- and Γ_k-equivariant, so is $\hat{\lambda}_{ik}$.

One can now check that $(\acute{P}_{ik}, \lambda_{ik}, \hat{\lambda}_{ik})$ satisfies Definition 6.4(a)–(c), where (6.2) for $\hat{\lambda}_{ik}$ follows from adding the pullbacks to $\acute{P}_{ij} \times_{V_j} \acute{P}_{jk}$ of (6.2) for $\hat{\lambda}_{ij}, \hat{\lambda}_{jk}$, so $\Lambda_{ik} = [\acute{P}_{ik}, \lambda_{ik}, \hat{\lambda}_{ik}]$ is a 2-morphism as in (6.10), which is independent of choices of $(\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij})$, $(\acute{P}_{jk}, \lambda_{jk}, \hat{\lambda}_{jk})$. We define $\Lambda_{jk} * \Lambda_{ij} = \Lambda_{ik}$ in (6.10).

We have now defined all the structures of a weak 2-category: objects (Kuranishi neighbourhoods), 1- and 2-morphisms, their three kinds of composition, two kinds of identities, and the coherence 2-isomorphisms (6.7), (6.8). The next theorem, the analogue of Theorem 4.7, has a long but straightforward proof using Theorem 3.17 at some points, and we leave it as an exercise.

Theorem 6.8. *The structures in Definitions* 6.1–6.7 *satisfy the axioms of a weak* 2*-category in* §A.2.

Here are the analogues of Definition 4.8 and Corollary 4.9:

Definition 6.9. Write $\dot{\mathbf{K}}\mathbf{N}$ for the *weak* 2*-category of Kuranishi neighbourhoods* defined using $\dot{\mathbf{M}}\mathbf{an}$, where:

- Objects of $\dot{\mathbf{KN}}$ are triples $(X,S,(V,E,\Gamma,s,\psi))$, where X is a topological space, $S\subseteq X$ is open, and (V,E,Γ,s,ψ) is a Kuranishi neighbourhood over S, as in Definition 6.1.
- 1-morphisms $(f,\Phi_{ij}):(X,S,(V_i,E_i,\Gamma_i,s_i,\psi_i))\rightarrow(Y,T,(V_j,E_j,\Gamma_j,s_j,\psi_j))$ of $\dot{\mathbf{KN}}$ are a pair of a continuous map $f:X\rightarrow Y$ with $S\subseteq f^{-1}(T)\subseteq X$ and a 1-morphism $\Phi_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ over (S,f), as in Definition 6.2.
- For 1-morphisms $(f,\Phi_{ij}),(f,\Phi'_{ij}):(X,S,(V_i,E_i,\Gamma_i,s_i,\psi_i))\rightarrow(Y,T,(V_j,E_j,\Gamma_j,s_j,\psi_j))$ with the same continuous map $f:X\rightarrow Y$, a 2-morphism of $\dot{\mathbf{KN}}$ is a 2-morphism $\Lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}$ over (S,f), as in Definition 6.4.
- Identities, the three kinds of composition of 1- and 2-morphisms, and the coherence 2-isomorphisms $\alpha_{g,f,e},\beta_f,\gamma_f$ are defined in the obvious way using Definitions 6.3 and 6.5–6.7.

Write $\mathbf{G}\dot{\mathbf{KN}}$ for the full 2-subcategory of $\dot{\mathbf{KN}}$ with objects $(s^{-1}(0)/\Gamma,s^{-1}(0)/\Gamma,(V,E,\Gamma,s,\mathrm{id}_{s^{-1}(0)/\Gamma}))$ for which $X=S=s^{-1}(0)/\Gamma$ and $\psi=\mathrm{id}_{s^{-1}(0)/\Gamma}$. We call $\mathbf{G}\dot{\mathbf{KN}}$ the *weak* 2*-category of global Kuranishi neighbourhoods.* We usually write objects of $\mathbf{G}\dot{\mathbf{KN}}$ as (V,E,Γ,s) rather than $(s^{-1}(0)/\Gamma,s^{-1}(0)/\Gamma,(V,E,\Gamma,s,\mathrm{id}_{s^{-1}(0)/\Gamma}))$. Similarly, we write 1-morphisms of $\mathbf{G}\dot{\mathbf{KN}}$ as $\Phi_{ij}:(V_i,E_i,\Gamma_i,s_i)\rightarrow(V_j,E_j,\Gamma_j,s_j)$ rather than as (f,Φ_{ij}), since f is determined by Φ_{ij} as in Definition 4.8, and we write 2-morphisms of $\mathbf{G}\dot{\mathbf{KN}}$ as $\Lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}$.

Let X be a topological space and $S\subseteq X$ be open. Write $\dot{\mathbf{KN}}_S(X)$ for the 2-subcategory of $\dot{\mathbf{KN}}$ with objects $(X,S,(V,E,\Gamma,s,\psi))$ for X,S as given, 1-morphisms $(\mathrm{id}_X,\Phi_{ij}):(X,S,(V_i,E_i,\Gamma_i,s_i,\psi_i))\rightarrow(X,S,(V_j,E_j,\Gamma_j,s_j,\psi_j))$ for $f=\mathrm{id}_X$, and all 2-morphisms $\Lambda_{ij}:(\mathrm{id}_X,\Phi_{ij})\Rightarrow(\mathrm{id}_X,\Phi'_{ij})$. We call $\dot{\mathbf{KN}}_S(X)$ the *weak* 2*-category of Kuranishi neighbourhoods over* $S\subseteq X$.

We generally write objects of $\dot{\mathbf{KN}}_S(X)$ as (V,E,Γ,s,ψ), omitting X,S, and 1-morphisms of $\dot{\mathbf{KN}}_S(X)$ as Φ_{ij}, omitting id_X. That is, objects, 1- and 2-morphisms of $\dot{\mathbf{KN}}_S(X)$ are just Kuranishi neighbourhoods over S and 1- and 2-morphisms over S as in Definitions 6.1, 6.2 and 6.4.

The accent '$\dot{\ }$' in $\dot{\mathbf{KN}},\mathbf{G}\dot{\mathbf{KN}},\dot{\mathbf{KN}}_S(X)$ is because they are constructed using $\dot{\mathbf{Man}}$. For particular $\dot{\mathbf{Man}}$ we modify the notation in the obvious way, e.g. if $\dot{\mathbf{Man}}=\mathbf{Man}$ we write $\mathbf{KN},\mathbf{GKN},\mathbf{KN}_S(X)$, and if $\dot{\mathbf{Man}}=\mathbf{Man^c}$ we write $\mathbf{KN^c},\mathbf{GKN^c},\mathbf{KN}^{\mathbf{c}}_S(X)$.

If $f:X\rightarrow Y$ is continuous, $(V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)$ are Kuranishi neighbourhoods on X,Y, and $S\subseteq\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\subseteq X$ is open, write $\mathbf{Hom}_{S,f}((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j))$ for the groupoid with objects 1-morphisms $\Phi_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ over (S,f), and morphisms 2-morphisms $\Lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}$ over (S,f).

If $X=Y$ and $f=\mathrm{id}_X$, we write $\mathbf{Hom}_S((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j))$ in place of $\mathbf{Hom}_{S,f}((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j))$.

Corollary 6.10. *In Definition* 6.9, $\dot{\mathbf{KN}},\mathbf{G}\dot{\mathbf{KN}}$ *and* $\dot{\mathbf{KN}}_S(X)$ *are weak* 2*-categories, and in fact* $(2,1)$*-categories, as all* 2*-morphisms are invertible.*

Here are the analogues of Definitions 4.10–4.11 and Convention 4.12:

Definition 6.11. Let X be a topological space, and $S \subseteq X$ be open, and $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be a 1-morphism of Kuranishi neighbourhoods on X over S. Then Φ_{ij} is a 1-morphism in the 2-category $\dot{\mathbf{KN}}_S(X)$ of Definition 6.9. We call Φ_{ij} a *coordinate change over* S if it is an equivalence in $\dot{\mathbf{KN}}_S(X)$. Write

$$\begin{aligned}\mathbf{Equ}_S&\bigl((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\bigr)\\ &\subseteq \mathbf{Hom}_S\bigl((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\bigr)\end{aligned}$$

for the subgroupoid with objects coordinate changes over S.

Here is Theorem 10.65(a)–(c) from §10.5.3 in volume II, which gives criteria for when a 1-morphism of Kuranishi neighbourhoods on X is a coordinate change when $\dot{\mathbf{Man}}$ is $\mathbf{Man}, \mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}$ or $\mathbf{Man^{c,ac}}$.

Theorem 6.12. *Working in a category* $\dot{\mathbf{Man}}$ *which we specify in* **(a)**–**(c)** *below, let* $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ *be a 1-morphism of Kuranishi neighbourhoods on a topological space* X *over an open subset* $S \subseteq X$. *Let* $p \in \pi_{ij}^{-1}(\bar\psi_i^{-1}(S)) \subseteq P_{ij}$, *set* $v_i = \pi_{ij}(p) \in V_i$ *and* $v_j = \phi_{ij}(p) \in V_j$, *and consider the morphism of finite groups*

$$\begin{aligned}&\rho_p : \bigl\{(\gamma_i, \gamma_j) \in \Gamma_i \times \Gamma_j : (\gamma_i, \gamma_j) \cdot p = p\bigr\} \longrightarrow \bigl\{\gamma_j \in \Gamma_j : \gamma_j \cdot v_j = v_j\bigr\},\\ &\rho_p : (\gamma_i, \gamma_j) \longmapsto \gamma_j.\end{aligned} \tag{6.11}$$

Then:

(a) *If* $\dot{\mathbf{Man}} = \mathbf{Man}$ *then* Φ_{ij} *is a coordinate change over* S *if and only if for all* $p \in \pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *equation* (6.11) *is an isomorphism, and the following is exact:*

$$0 \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}_{v_i}s_i \oplus (T_p\phi_{ij} \circ (T_p\pi_{ij})^{-1})} E_i|_{v_i} \oplus T_{v_j}V_j \xrightarrow{-\hat\phi_{ij}|_p \oplus \mathrm{d}_{v_j}s_j} E_j|_{v_j} \longrightarrow 0. \tag{6.12}$$

(b) *If* $\dot{\mathbf{Man}} = \mathbf{Man^c}$ *then* Φ_{ij} *is a coordinate change over* S *if and only if* ϕ_{ij} *is simple near* $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *as in* §2.1, *and for all* $p \in \pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *equation* (6.11) *is an isomorphism and* (6.12) *is exact.*

(c) *If* $\dot{\mathbf{Man}}$ *is one of* $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}$ *or* $\mathbf{Man^{c,ac}}$ *then* Φ_{ij} *is a coordinate change over* S *if and only if* ϕ_{ij} *is simple near* $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *and using b-tangent spaces from* §2.3, *for all* $p \in \pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *equation* (6.11) *is an isomorphism and the following is exact:*

$$0 \longrightarrow {}^bT_{v_i}V_i \xrightarrow{{}^b\mathrm{d}_{v_i}s_i \oplus ({}^bT_p\phi_{ij} \circ ({}^bT_p\pi_{ij})^{-1})} E_i|_{v_i} \oplus {}^bT_{v_j}V_j \xrightarrow{-\hat\phi_{ij}|_p \oplus {}^b\mathrm{d}_{v_j}s_j} E_j|_{v_j} \longrightarrow 0.$$

Definition 6.13. Let $T \subseteq S \subseteq X$ be open. Define the *restriction 2-functor* $|_T : \dot{\mathbf{K}}\mathbf{N}_S(X) \to \dot{\mathbf{K}}\mathbf{N}_T(X)$ to map objects $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ to exactly the same objects, and 1-morphisms Φ_{ij} to exactly the same 1-morphisms but regarded as 1-morphisms over T, and 2-morphisms Λ_{ij} over S to $\Lambda_{ij}|_T$, where $\Lambda_{ij}|_T$ is the $\sim_T$-equivalence class of any representative $(\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij})$ for the $\sim_S$-equivalence class Λ_{ij}. We take the 2-morphisms $F_{g,f}, F_X$ in Definition A.8 to be identities. Then $|_T : \dot{\mathbf{K}}\mathbf{N}_S(X) \to \dot{\mathbf{K}}\mathbf{N}_T(X)$ is a weak 2-functor of weak 2-categories as in §A.3. If $U \subseteq T \subseteq S \subseteq X$ are open then $|_U \circ |_T = |_U : \dot{\mathbf{K}}\mathbf{N}_S(X) \to \dot{\mathbf{K}}\mathbf{N}_U(X)$.

Now let $f : X \to Y$ be continuous, $(V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be Kuranishi neighbourhoods on X, Y, and $T \subseteq S \subseteq \operatorname{Im} \psi_i \cap f^{-1}(\operatorname{Im} \psi_j) \subseteq X$ be open. Then as for $|_T$ on 1- and 2-morphisms above, we define a functor

$$|_T : \mathbf{Hom}_{S,f}\big((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\big) \longrightarrow \mathbf{Hom}_{T,f}\big((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\big).$$

Convention 6.14. *When we do not specify a domain S for a morphism, or coordinate change, of Kuranishi neighbourhoods, the domain should be as large as possible.* For example, if we say that $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ is a 1-morphism (or a 1-morphism over $f : X \to Y$) without specifying S, we mean that $S = \operatorname{Im} \psi_i \cap \operatorname{Im} \psi_j$ (or $S = \operatorname{Im} \psi_i \cap f^{-1}(\operatorname{Im} \psi_j)$).

Similarly, if we write a formula involving several 2-morphisms (possibly defined on different domains), without specifying the domain S, we make the convention *that the domain where the formula holds should be as large as possible.* That is, the domain S is taken to be the intersection of the domains of each 2-morphism in the formula, and we implicitly restrict each morphism in the formula to S as in Definition 6.13, so that it makes sense.

Remark 6.15. (i) Our coordinate changes in Definition 6.11 are closely related to coordinate changes between Kuranishi neighbourhoods in the theory of Fukaya, Oh, Ohta and Ono [19–39], as described in §7.1. We explain the connection in §7.1. One of the most important innovations in our theory is to introduce the notion of 2-morphism between coordinate changes.

(ii) Our 1-morphisms of Kuranishi neighbourhoods involve $V_{ij} \xleftarrow{\pi_{ij}} P_{ij} \xrightarrow{\phi_{ij}} V_j$ with π_{ij} a Γ_i-equivariant principal Γ_j-bundle, and ϕ_{ij} Γ_i-invariant and Γ_j-equivariant. As in §6.6, this is a known way of writing 1-morphisms of orbifolds $[V_{ij}/\Gamma_i] \to [V_j/\Gamma_j]$, called *Hilsum–Skandalis morphisms.* So the data $P_{ij}, \pi_{ij}, \phi_{ij}$ in $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat{\phi}_{ij})$ is very natural from the orbifold point of view.

(iii) In the definition of 2-morphisms $\Lambda_{ij} = [\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}]$ in Definition 6.4, by restricting to arbitrarily small open neighbourhoods $\acute{P}_{ij}$ of $\pi_{ij}^{-1}(\bar{\psi}_i^{-1}(S))$ in P_{ij} and then taking equivalence classes, we are in effect taking *germs* about $\bar{\psi}_i^{-1}(S)$ in V_i, or germs about $\pi_{ij}^{-1}(\bar{\psi}_i^{-1}(S))$ in P_{ij}. Fukaya–Ono's first definition of Kuranishi space [39, §5] involved germs of Kuranishi neighbourhoods at points. We take germs at larger subsets $\bar{\psi}_i^{-1}(S)$ in 2-morphisms.

Here is the analogue of Theorem 4.13, proved in §6.7, which is very important in our theory. We will call Theorem 6.16 the *stack property.* We will use it in §6.2 to construct compositions of 1- and 2-morphisms of Kuranishi spaces.

Theorem 6.16. *Let $f : X \to Y$ be a continuous map of topological spaces, and $(V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be Kuranishi neighbourhoods on X, Y. For each open $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j) \subseteq X$, define a groupoid*

$$\begin{aligned}\boldsymbol{\mathcal{H}om}_f\bigl((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\bigr)(S)\\ = \mathbf{Hom}_{S,f}\bigl((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\bigr),\end{aligned}$$

as in Definition 6.9, *for all open $T \subseteq S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ define a functor*

$$\begin{aligned}\rho_{ST} : \boldsymbol{\mathcal{H}om}_f\bigl((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\bigr)(S) \longrightarrow\\ \boldsymbol{\mathcal{H}om}_f\bigl((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\bigr)(T)\end{aligned}$$

between groupoids by $\rho_{ST} = |_T$, as in Definition 6.13, *and for all open $U \subseteq T \subseteq S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ take the obvious isomorphism $\eta_{STU} = \mathrm{id}_{\rho_{SU}} : \rho_{TU} \circ \rho_{ST} \Rightarrow \rho_{SU}$. Then $\boldsymbol{\mathcal{H}om}_f((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j))$ is a* ***stack*** *on the open subset $\operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ in X, as in* §A.6.

When $X = Y$, $f = \mathrm{id}_X$ we write $\boldsymbol{\mathcal{H}om}((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j))$ rather than $\boldsymbol{\mathcal{H}om}_f((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j))$. Coordinate changes $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ also form a stack $\boldsymbol{\mathcal{E}qu}((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j))$ on $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, a substack of $\boldsymbol{\mathcal{H}om}((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j))$.

6.2 The weak 2-category of Kuranishi spaces

6.2.1 The definition of the 2-category $\dot{\mathbf{K}}\mathbf{ur}$

We now define the weak 2-category of Kuranishi spaces $\dot{\mathbf{K}}\mathbf{ur}$. We follow the definition of $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ in §4.3 closely, with the difference that m-Kuranishi neighbourhoods in §4.1 are a strict 2-category, but Kuranishi neighbourhoods in §6.1 are a weak 2-category. So we cannot omit brackets in compositions of 1-morphisms such as $\Phi_{kl} \circ \Phi_{jk} \circ \Phi_{ij}$ in (4.4), we must write $(\Phi_{kl} \circ \Phi_{jk}) \circ \Phi_{ij}$ or $\Phi_{kl} \circ (\Phi_{jk} \circ \Phi_{ij})$ as in (6.13), and we have to insert extra coherence 2-morphisms $\boldsymbol{\alpha}_{\Phi_{kl},\Phi_{jk},\Phi_{ij}}, \boldsymbol{\beta}_{\Phi_{ij}}, \boldsymbol{\gamma}_{\Phi_{ij}}$ from (6.7)–(6.8) throughout.

For example, compare (4.4), (4.10), (4.11), and (4.12) above with (6.13), (6.19), (6.20), and (6.21) below, noting the extra $\boldsymbol{\alpha}_{*,*,*}$, and compare Definitions 4.14(g) and 6.17(g), noting the extra $\boldsymbol{\beta}_*, \boldsymbol{\gamma}_*$.

Since every weak 2-category is equivalent as a weak 2-category to a strict 2-category, we can guarantee that any proof which works in strict 2-categories can be extended to a proof in weak 2-categories by including extra 2-morphisms $\boldsymbol{\alpha}_{*,*,*}, \boldsymbol{\beta}_*, \boldsymbol{\gamma}_*$, although diagrams such as (4.19) and (4.31) become rather more complicated. So we omit proofs in this section, referring to those in §4.3.

Here is the analogue of Definition 4.14.

Definition 6.17. Let X be a Hausdorff, second countable topological space, and $n \in \mathbb{Z}$. A *Kuranishi structure* $\mathcal{K}$ *on* X *of virtual dimension* n is data $\mathcal{K} = \bigl(I, (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i \in I},\ \Phi_{ij,\ i,j \in I},\ \Lambda_{ijk,\ i,j,k \in I}\bigr)$, where:

(a) I is an indexing set (not necessarily finite).

(b) $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ is a Kuranishi neighbourhood on X for each $i \in I$, with $\dim V_i - \operatorname{rank} E_i = n$.

(c) $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat{\phi}_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ is a coordinate change for all $i, j \in I$ (as usual, defined over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$).

(d) $\Lambda_{ijk} = [\acute{P}_{ijk}, \lambda_{ijk}, \hat{\lambda}_{ijk}] : \Phi_{jk} \circ \Phi_{ij} \Rightarrow \Phi_{ik}$ is a 2-morphism for all $i, j, k \in I$ (as usual, defined over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$).

(e) $\bigcup_{i \in I} \operatorname{Im}\psi_i = X$.

(f) $\Phi_{ii} = \mathrm{id}_{(V_i, E_i, \Gamma_i, s_i, \psi_i)}$ for all $i \in I$.

(g) $\Lambda_{iij} = \boldsymbol{\beta}_{\Phi_{ij}}$ and $\Lambda_{ijj} = \boldsymbol{\gamma}_{\Phi_{ij}}$ for all $i, j \in I$, for $\boldsymbol{\beta}_{\Phi_{ij}}, \boldsymbol{\gamma}_{\Phi_{ij}}$ as in (6.8).

(h) The following diagram of 2-morphisms over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k \cap \operatorname{Im}\psi_l$ commutes for all $i, j, k, l \in I$, for $\boldsymbol{\alpha}_{\Phi_{kl}, \Phi_{jk}, \Phi_{ij}}$ as in (6.7):

$$\begin{tikzcd}
(\Phi_{kl} \circ \Phi_{jk}) \circ \Phi_{ij} \arrow[rr, Rightarrow, "\Lambda_{jkl} * \mathrm{id}_{\Phi_{ij}}"'] \arrow[d, Rightarrow, "\boldsymbol{\alpha}_{\Phi_{kl}, \Phi_{jk}, \Phi_{ij}}"] & & \Phi_{jl} \circ \Phi_{ij} \arrow[d, Rightarrow, "\Lambda_{ijl}"'] \\
\Phi_{kl} \circ (\Phi_{jk} \circ \Phi_{ij}) \arrow[r, Rightarrow, "\mathrm{id}_{\Phi_{kl}} * \Lambda_{ijk}"] & \Phi_{kl} \circ \Phi_{ik} \arrow[r, Rightarrow, "\Lambda_{ikl}"] & \Phi_{il}.
\end{tikzcd} \tag{6.13}$$

We call $\boldsymbol{X} = (X, \mathcal{K})$ a *Kuranishi space*, of *virtual dimension* $\operatorname{vdim} \boldsymbol{X} = n$. When we write $x \in \boldsymbol{X}$, we mean that $x \in X$.

Here is the analogue of Example 4.16.

Example 6.18. Let V be a manifold, $E \to V$ a vector bundle, Γ a finite group with a smooth action on V and a compatible action on E preserving the vector bundle structure, and $s : V \to E$ a Γ-equivariant smooth section, so that (V, E, Γ, s) is an object in $\mathbf{G\dot{K}N}$ from Definition 6.9. Set $X = s^{-1}(0)/\Gamma$, with the quotient topology induced from the closed subset $s^{-1}(0) \subseteq V$. Then X is Hausdorff and second countable, as V is and Γ is finite.

Define a Kuranishi structure $\mathcal{K} = \bigl(\{0\}, (V_0, E_0, \Gamma_0, s_0, \psi_0), \Phi_{00}, \Lambda_{000}\bigr)$ on X with indexing set $I = \{0\}$, one Kuranishi neighbourhood $(V_0, E_0, \Gamma_0, s_0, \psi_0)$ with $V_0 = V$, $E_0 = E$, $\Gamma_0 = \Gamma$, $s_0 = s$ and $\psi_0 = \mathrm{id}_X$, one coordinate change $\Phi_{00} = \mathrm{id}_{(V_0, E_0, \Gamma_0, s_0, \psi_0)}$, and one 2-morphism $\Lambda_{000} = \mathrm{id}_{\Phi_{00}}$. Then $\boldsymbol{X} = (X, \mathcal{K})$ is a Kuranishi space, with $\operatorname{vdim} \boldsymbol{X} = \dim V - \operatorname{rank} E$. We write $\boldsymbol{S}_{V,E,\Gamma,s} = \boldsymbol{X}$.

We will need notation to distinguish Kuranishi neighbourhoods, coordinate changes, and 2-morphisms on different Kuranishi spaces. As for (4.5)–(4.8), we will often use the following notation for Kuranishi spaces $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$:

$$\begin{aligned}
\boldsymbol{W} = (W, \mathcal{H}), \quad \mathcal{H} = \bigl(H, (T_h, C_h, \mathrm{A}_i, q_h, \varphi_h)_{h \in H},\ \Sigma_{hh'} = (O_{hh'}, \pi_{hh'}, \sigma_{hh'}, \\
\hat{\sigma}_{hh'})_{h,h' \in H},\ \mathrm{I}_{hh'h''} = [\acute{O}_{hh'h''}, \iota_{hh'h''}, \hat{\iota}_{hh'h''}]_{h,h',h'' \in H}\bigr),
\end{aligned} \tag{6.14}$$

$$\begin{aligned}\boldsymbol{X}=(X,\mathcal{I}),\quad &\mathcal{I}=\bigl(I,(U_i,D_i,\mathrm{B}_i,r_i,\chi_i)_{i\in I},\ \mathrm{T}_{ii'}=(P_{ii'},\pi_{ii'},\tau_{ii'},\\ &\hat\tau_{ii'})_{i,i'\in I},\ \mathrm{K}_{ii'i''}=[\acute P_{ii'i''},\kappa_{ii'i''},\hat\kappa_{ii'i''}]_{i,i',i''\in I}\bigr),\end{aligned}\tag{6.15}$$

$$\begin{aligned}\boldsymbol{Y}=(Y,\mathcal{J}),\quad &\mathcal{J}=\bigl(J,(V_j,E_j,\Gamma_j,s_j,\psi_j)_{j\in J},\ \Upsilon_{jj'}=(Q_{jj'},\pi_{jj'},\upsilon_{jj'},\\ &\hat\upsilon_{jj'})_{j,j'\in J},\ \Lambda_{jj'j''}=[\acute Q_{jj'j''},\lambda_{jj'j''},\hat\lambda_{jj'j''}]_{j,j',j''\in J}\bigr),\end{aligned}\tag{6.16}$$

$$\begin{aligned}\boldsymbol{Z}=(Z,\mathcal{K}),\quad &\mathcal{K}=\bigl(K,(W_k,F_k,\Delta_k,t_k,\omega_k)_{k\in K},\ \Phi_{kk'}=(R_{kk'},\pi_{kk'},\phi_{kk'},\\ &\hat\phi_{kk'})_{k,k'\in K},\ \mathrm{M}_{kk'k''}=[\acute R_{kk'k''},\mu_{kk'k''},\hat\mu_{kk'k''}]_{k,k',k''\in K}\bigr).\end{aligned}\tag{6.17}$$

Here are the analogues of Definitions 4.17 and 4.18.

Definition 6.19. Let $\boldsymbol{X}=(X,\mathcal{I})$ and $\boldsymbol{Y}=(Y,\mathcal{J})$ be Kuranishi spaces, with notation (6.15)–(6.16). A 1-*morphism of Kuranishi spaces* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is data

$$\boldsymbol{f}=\bigl(f,\boldsymbol{f}_{ij,\ i\in I,\ j\in J},\ \boldsymbol{F}_{ii',\ i,i'\in I}^{j,\ j\in J},\ \boldsymbol{F}_{i,\ i\in I}^{jj',\ j,j'\in J}\bigr),\tag{6.18}$$

satisfying the conditions:

(a) $f:X\to Y$ is a continuous map.

(b) $\boldsymbol{f}_{ij}=(P_{ij},\pi_{ij},f_{ij},\hat f_{ij}):(U_i,D_i,\mathrm{B}_i,r_i,\chi_i)\to(V_j,E_j,\Gamma_j,s_j,\psi_j)$ is a 1-morphism of Kuranishi neighbourhoods over f for all $i\in I$, $j\in J$ (defined over $S=\operatorname{Im}\chi_i\cap f^{-1}(\operatorname{Im}\psi_j)$, as usual).

(c) $\boldsymbol{F}_{ii'}^{j}=[\acute P_{ii'}^{j},F_{ii'}^{j},\hat F_{ii'}^{j}]:\boldsymbol{f}_{i'j}\circ\mathrm{T}_{ii'}\Rightarrow\boldsymbol{f}_{ij}$ is a 2-morphism over f for all $i,i'\in I$ and $j\in J$ (defined over $S=\operatorname{Im}\chi_i\cap\operatorname{Im}\chi_{i'}\cap f^{-1}(\operatorname{Im}\psi_j)$).

(d) $\boldsymbol{F}_{i}^{jj'}=[\acute P_{i}^{jj'},F_{i}^{jj'},\hat F_{i}^{jj'}]:\Upsilon_{jj'}\circ\boldsymbol{f}_{ij}\Rightarrow\boldsymbol{f}_{ij'}$ is a 2-morphism over f for all $i\in I$ and $j,j'\in J$ (defined over $S=\operatorname{Im}\chi_i\cap f^{-1}(\operatorname{Im}\psi_j\cap\operatorname{Im}\psi_{j'})$).

(e) $\boldsymbol{F}_{ii}^{j}=\boldsymbol{\beta}_{\boldsymbol{f}_{ij}}$ and $\boldsymbol{F}_{i}^{jj}=\boldsymbol{\gamma}_{\boldsymbol{f}_{ij}}$ for all $i\in I$, $j\in J$, for $\boldsymbol{\beta}_{\boldsymbol{f}_{ij}},\boldsymbol{\gamma}_{\boldsymbol{f}_{ij}}$ as in (6.8).

(f) The following commutes for all $i,i',i''\in I$ and $j\in J$:

$$\begin{tikzcd}[column sep=large]
(\boldsymbol{f}_{i''j}\circ\mathrm{T}_{i'i''})\circ\mathrm{T}_{ii'} \arrow[rr, Rightarrow, "\boldsymbol{F}_{i'i''}^{j}*\mathrm{id}_{\mathrm{T}_{ii'}}"'] \arrow[d, Rightarrow, "\boldsymbol{\alpha}_{\boldsymbol{f}_{i''j},\mathrm{T}_{i'i''},\mathrm{T}_{ii'}}"] & & \boldsymbol{f}_{i'j}\circ\mathrm{T}_{ii'} \arrow[d, Rightarrow, "\boldsymbol{F}_{ii'}^{j}"'] \\
\boldsymbol{f}_{i''j}\circ(\mathrm{T}_{i'i''}\circ\mathrm{T}_{ii'}) \arrow[r, Rightarrow, "\mathrm{id}_{\boldsymbol{f}_{i''j}}*\mathrm{K}_{ii'i''}"] & \boldsymbol{f}_{i''j}\circ\mathrm{T}_{ii''} \arrow[r, Rightarrow, "\boldsymbol{F}_{ii''}^{j}"] & \boldsymbol{f}_{ij}.
\end{tikzcd}\tag{6.19}$$

(g) The following commutes for all $i,i'\in I$ and $j,j'\in J$:

$$\begin{tikzcd}[column sep=large]
(\Upsilon_{jj'}\circ\boldsymbol{f}_{i'j})\circ\mathrm{T}_{ii'} \arrow[rr, Rightarrow, "\boldsymbol{F}_{i'}^{jj'}*\mathrm{id}_{\mathrm{T}_{ii'}}"'] \arrow[d, Rightarrow, "\boldsymbol{\alpha}_{\Upsilon_{jj'},\boldsymbol{f}_{i'j},\mathrm{T}_{ii'}}"] & & \boldsymbol{f}_{i'j'}\circ\mathrm{T}_{ii'} \arrow[d, Rightarrow, "\boldsymbol{F}_{ii'}^{j'}"'] \\
\Upsilon_{jj'}\circ(\boldsymbol{f}_{i'j}\circ\mathrm{T}_{ii'}) \arrow[r, Rightarrow, "\mathrm{id}_{\Upsilon_{jj'}}*\boldsymbol{F}_{ii'}^{j}"] & \Upsilon_{jj'}\circ\boldsymbol{f}_{ij} \arrow[r, Rightarrow, "\boldsymbol{F}_{i}^{jj'}"] & \boldsymbol{f}_{ij'}.
\end{tikzcd}\tag{6.20}$$

(h) The following commutes for all $i\in I$ and $j,j',j''\in J$:

$$\begin{tikzcd}[column sep=large]
(\Upsilon_{j'j''}\circ\Upsilon_{jj'})\circ\boldsymbol{f}_{ij} \arrow[rr, Rightarrow, "\Lambda_{jj'j''}*\mathrm{id}_{\boldsymbol{f}_{ij}}"'] \arrow[d, Rightarrow, "\boldsymbol{\alpha}_{\Upsilon_{j'j''},\Upsilon_{jj'},\boldsymbol{f}_{ij}}"] & & \Upsilon_{jj''}\circ\boldsymbol{f}_{ij} \arrow[d, Rightarrow, "\boldsymbol{F}_{i}^{jj''}"'] \\
\Upsilon_{j'j''}\circ(\Upsilon_{jj'}\circ\boldsymbol{f}_{ij}) \arrow[r, Rightarrow, "\mathrm{id}_{\Upsilon_{j'j''}}*\boldsymbol{F}_{i}^{jj'}"] & \Upsilon_{j'j''}\circ\boldsymbol{f}_{ij'} \arrow[r, Rightarrow, "\boldsymbol{F}_{i}^{j'j''}"] & \boldsymbol{f}_{ij''}.
\end{tikzcd}\tag{6.21}$$

If $x \in \boldsymbol{X}$ (i.e. $x \in X$), we will write $\boldsymbol{f}(x) = f(x) \in \boldsymbol{Y}$.

When $\boldsymbol{Y} = \boldsymbol{X}$, define the *identity* 1-*morphism* $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ by

$$\mathbf{id}_{\boldsymbol{X}} = \bigl(\mathrm{id}_X, \mathrm{T}_{ij,\ i,j\in I},\ \mathrm{K}_{ii'j,\ i,i'\in I}^{\ j\in I},\ \mathrm{K}_{ijj',\ i\in I}^{\ j,j'\in I}\bigr). \tag{6.22}$$

Then Definition 6.17(h) implies that (f)–(h) above hold.

Definition 6.20. Let $\boldsymbol{X} = (X, \mathcal{I})$ and $\boldsymbol{Y} = (Y, \mathcal{J})$ be Kuranishi spaces, with notation as in (6.15)–(6.16), and $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms. Suppose the continuous maps $f, g : X \to Y$ in $\boldsymbol{f}, \boldsymbol{g}$ satisfy $f = g$. A 2-*morphism of Kuranishi spaces* $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is data $\boldsymbol{\eta} = \bigl(\boldsymbol{\eta}_{ij,\ i\in I,\ j\in J}\bigr)$, where $\boldsymbol{\eta}_{ij} = [\acute{P}_{ij}, \eta_{ij}, \hat{\eta}_{ij}] : \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij}$ is a 2-morphism of Kuranishi neighbourhoods over $f = g$ (defined over $S = \operatorname{Im}\chi_i \cap f^{-1}(\operatorname{Im}\psi_j)$, as usual), satisfying the conditions:

(a) $\boldsymbol{G}_{ii'}^{j} \odot (\boldsymbol{\eta}_{i'j} * \mathrm{id}_{\mathrm{T}_{ii'}}) = \boldsymbol{\eta}_{ij} \odot \boldsymbol{F}_{ii'}^{j} : \boldsymbol{f}_{i'j} \circ \mathrm{T}_{ii'} \Rightarrow \boldsymbol{g}_{ij}$ for all $i, i' \in I$, $j \in J$.

(b) $\boldsymbol{G}_{i}^{jj'} \odot (\mathrm{id}_{\Upsilon_{jj'}} * \boldsymbol{\eta}_{ij}) = \boldsymbol{\eta}_{ij'} \odot \boldsymbol{F}_{i}^{jj'} : \Upsilon_{jj'} \circ \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij'}$ for all $i \in I$, $j, j' \in J$.

Note that by definition, 2-morphisms $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ only exist if $f = g$.

If $\boldsymbol{f} = \boldsymbol{g}$, the *identity* 2-*morphism* is $\mathbf{id}_{\boldsymbol{f}} = \bigl(\mathrm{id}_{\boldsymbol{f}_{ij},\ i\in I,\ j\in J}\bigr) : \boldsymbol{f} \Rightarrow \boldsymbol{f}$.

As for m-Kuranishi spaces in §4.3, given 1-morphisms of Kuranishi spaces $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$, we must use the stack property in Theorem 6.16 to define the composition $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$, where $\boldsymbol{g} \circ \boldsymbol{f}$ is only unique up to 2-isomorphism, so we must make an arbitrary choice.

Here is the analogue of Proposition 4.19. It is proved in the same way, but inserting extra 2-morphisms $\boldsymbol{\alpha}_{*,*,*}, \boldsymbol{\beta}_*, \boldsymbol{\gamma}_*$ as we are now working in a weak 2-category.

Proposition 6.21. (a) *Let $\boldsymbol{X} = (X, \mathcal{I}), \boldsymbol{Y} = (Y, \mathcal{J}), \boldsymbol{Z} = (Z, \mathcal{K})$ be Kuranishi spaces with notation* (6.15)–(6.17), *and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be* 1*-morphisms, with $\boldsymbol{f} = \bigl(f, \boldsymbol{f}_{ij}, \boldsymbol{F}_{ii'}^{j}, \boldsymbol{F}_{i}^{jj'}\bigr)$, $\boldsymbol{g} = \bigl(g, \boldsymbol{g}_{jk}, \boldsymbol{G}_{jj'}^{k}, \boldsymbol{G}_{j}^{kk'}\bigr)$. Then there exists a* 1*-morphism $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Z}$ with $\boldsymbol{h} = \bigl(h, \boldsymbol{h}_{ik}, \boldsymbol{H}_{ii'}^{k}, \boldsymbol{H}_{i}^{kk'}\bigr)$, such that $h = g \circ f : X \to Z$, and for all $i \in I$, $j \in J$, $k \in K$ we have* 2*-morphisms over h*

$$\Theta_{ijk} : \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \Longrightarrow \boldsymbol{h}_{ik}, \tag{6.23}$$

where as usual (6.23) *holds over $S = \operatorname{Im}\chi_i \cap f^{-1}(\operatorname{Im}\psi_j) \cap h^{-1}(\operatorname{Im}\omega_k)$, and for* *all $i,i'\in I$, $j,j'\in J$, $k,k'\in K$ the following commute:*

$$
\begin{tikzcd}
(\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{i'j})\circ \mathrm{T}_{ii'} \arrow[rr, Rightarrow, "\Theta_{i'jk}*\mathrm{id}_{\mathrm{T}_{ii'}}"'] \arrow[d, Rightarrow, "\boldsymbol{\alpha}_{\boldsymbol{g}_{jk},\boldsymbol{f}_{i'j},\mathrm{T}_{ii'}}"] & & \boldsymbol{h}_{i'k}\circ \mathrm{T}_{ii'} \arrow[d, Rightarrow, "\boldsymbol{H}^k_{ii'}"'] \\
\boldsymbol{g}_{jk}\circ(\boldsymbol{f}_{i'j}\circ \mathrm{T}_{ii'}) \arrow[r, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{jk}}*\boldsymbol{F}^j_{ii'}"] & \boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij} \arrow[r, Rightarrow, "\Theta_{ijk}"] & \boldsymbol{h}_{ik},
\end{tikzcd}
\tag{6.24}
$$

$$
\begin{tikzcd}
(\boldsymbol{g}_{j'k}\circ\Upsilon_{jj'})\circ \boldsymbol{f}_{ij} \arrow[rr, Rightarrow, "\boldsymbol{G}^k_{jj'}*\mathrm{id}_{\boldsymbol{f}_{ij}}"'] \arrow[d, Rightarrow, "\boldsymbol{\alpha}_{\boldsymbol{g}_{j'k},\Upsilon_{jj'},\boldsymbol{f}_{ij}}"] & & \boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij} \arrow[d, Rightarrow, "\Theta_{ijk}"'] \\
\boldsymbol{g}_{j'k}\circ(\Upsilon_{jj'}\circ\boldsymbol{f}_{ij}) \arrow[r, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{j'k}}*\boldsymbol{F}^{jj'}_i"] & \boldsymbol{g}_{j'k}\circ\boldsymbol{f}_{ij'} \arrow[r, Rightarrow, "\Theta_{ij'k}"] & \boldsymbol{h}_{ik},
\end{tikzcd}
\tag{6.25}
$$

$$
\begin{tikzcd}
(\Phi_{kk'}\circ\boldsymbol{g}_{jk})\circ \boldsymbol{f}_{ij} \arrow[rr, Rightarrow, "\boldsymbol{G}^{kk'}_j*\mathrm{id}_{\boldsymbol{f}_{ij}}"'] \arrow[d, Rightarrow, "\boldsymbol{\alpha}_{\Phi_{kk'},\boldsymbol{g}_{jk},\boldsymbol{f}_{ij}}"] & & \boldsymbol{g}_{jk'}\circ\boldsymbol{f}_{ij} \arrow[d, Rightarrow, "\Theta_{ijk'}"'] \\
\Phi_{kk'}\circ(\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}) \arrow[r, Rightarrow, "\mathrm{id}_{\Phi_{kk'}}*\Theta_{ijk}"] & \Phi_{kk'}\circ\boldsymbol{h}_{ik} \arrow[r, Rightarrow, "\boldsymbol{H}^{kk'}_i"] & \boldsymbol{h}_{ik'}.
\end{tikzcd}
\tag{6.26}
$$

(b) *If $\tilde{\boldsymbol{h}}=\bigl(h,\tilde{\boldsymbol{h}}_{ik},\tilde{\boldsymbol{H}}^k_{ii'},\tilde{\boldsymbol{H}}^{kk'}_i\bigr),\tilde\Theta_{ijk}$ are alternative choices for $\boldsymbol{h},\Theta_{ijk}$ in* **(a)**, *then there is a unique 2-morphism of Kuranishi spaces $\boldsymbol{\eta}=(\boldsymbol{\eta}_{ik}):\boldsymbol{h}\Rightarrow\tilde{\boldsymbol{h}}$ satisfying $\boldsymbol{\eta}_{ik}\odot\Theta_{ijk}=\tilde\Theta_{ijk}:\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}\Rightarrow\tilde{\boldsymbol{h}}_{ik}$ for all $i\in I$, $j\in J$, $k\in K$.*

(c) *If $\boldsymbol{X}=\boldsymbol{Y}$ and $\boldsymbol{f}=\mathbf{id}_{\boldsymbol{Y}}$ in* **(a)**, *so that $I=J$, then a possible choice for $\boldsymbol{h},\Theta_{ijk}$ in* **(a)** *is $\boldsymbol{h}=\boldsymbol{g}$ and $\Theta_{ijk}=\boldsymbol{G}^k_{ij}$.*

Similarly, if $\boldsymbol{Z}=\boldsymbol{Y}$ and $\boldsymbol{g}=\mathbf{id}_{\boldsymbol{Y}}$ in **(a)**, *so that $K=J$, then a possible choice for $\boldsymbol{h},\Theta_{ijk}$ in* **(a)** *is $\boldsymbol{h}=\boldsymbol{f}$ and $\Theta_{ijk}=\boldsymbol{F}^{jk}_i$.*

Here is the analogue of Definition 4.20.

Definition 6.22. For all pairs of 1-morphisms of Kuranishi spaces $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ and $\boldsymbol{g}:\boldsymbol{Y}\to\boldsymbol{Z}$, use the Axiom of Global Choice (see Remark 4.21) to choose possible values of $\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Z}$ and Θ_{ijk} in Proposition 6.21(a), and write $\boldsymbol{g}\circ\boldsymbol{f}=\boldsymbol{h}$, and for $i\in I$, $j\in J$, $k\in K$

$$\Theta^{\boldsymbol{g},\boldsymbol{f}}_{ijk}=\Theta_{ijk}:\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}\Longrightarrow(\boldsymbol{g}\circ\boldsymbol{f})_{ik}.$$

We call $\boldsymbol{g}\circ\boldsymbol{f}$ the *composition of* 1*-morphisms of Kuranishi spaces.*

For general $\boldsymbol{f},\boldsymbol{g}$ we make these choices arbitrarily. However, if $\boldsymbol{X}=\boldsymbol{Y}$ and $\boldsymbol{f}=\mathbf{id}_{\boldsymbol{Y}}$ then we choose $\boldsymbol{g}\circ\mathbf{id}_{\boldsymbol{Y}}=\boldsymbol{g}$ and $\Theta^{\boldsymbol{g},\mathbf{id}_{\boldsymbol{Y}}}_{jj'k}=\boldsymbol{G}^k_{jj'}$, and if $\boldsymbol{Z}=\boldsymbol{Y}$ and $\boldsymbol{g}=\mathbf{id}_{\boldsymbol{Y}}$ then we choose $\mathbf{id}_{\boldsymbol{Y}}\circ\boldsymbol{f}=\boldsymbol{f}$ and $\Theta^{\mathbf{id}_{\boldsymbol{Y}},\boldsymbol{f}}_{ijj'}=\boldsymbol{F}^{jj'}_i$. This is allowed by Proposition 6.21(c).

The definition of a weak 2-category in Appendix A includes 2-isomorphisms $\boldsymbol{\beta}_{\boldsymbol{f}}:\boldsymbol{f}\circ\mathbf{id}_{\boldsymbol{X}}\Rightarrow\boldsymbol{f}$ and $\boldsymbol{\gamma}_{\boldsymbol{f}}:\mathbf{id}_{\boldsymbol{Y}}\circ\boldsymbol{f}\Rightarrow\boldsymbol{f}$ in (A.10), since one does not require $\boldsymbol{f}\circ\mathbf{id}_{\boldsymbol{X}}=\boldsymbol{f}$ and $\mathbf{id}_{\boldsymbol{Y}}\circ\boldsymbol{f}=\boldsymbol{f}$ in a general weak 2-category. We define

$$\boldsymbol{\beta}_{\boldsymbol{f}}=\mathbf{id}_{\boldsymbol{f}}:\boldsymbol{f}\circ\mathbf{id}_{\boldsymbol{X}}\Longrightarrow\boldsymbol{f},\quad \boldsymbol{\gamma}_{\boldsymbol{f}}=\mathbf{id}_{\boldsymbol{f}}:\mathbf{id}_{\boldsymbol{Y}}\circ\boldsymbol{f}\Longrightarrow\boldsymbol{f}. \tag{6.27}$$

Here is the analogue of Proposition 4.22. It is proved in the same way, but inserting extra 2-morphisms $\boldsymbol{\alpha}_{\boldsymbol{g}_{jk},\boldsymbol{f}_{ij},\boldsymbol{e}_{hi}}$ of Kuranishi neighbourhoods.

Proposition 6.23. *Let* $\boldsymbol{e}:\boldsymbol{W}\rightarrow\boldsymbol{X}$, $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$, $\boldsymbol{g}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ *be* 1*-morphisms of Kuranishi spaces, and define composition of* 1*-morphisms as in Definition* 6.22*. Then using notation* (6.14)–(6.17)*, there is a unique* 2*-morphism*

$$\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}}:(\boldsymbol{g}\circ\boldsymbol{f})\circ\boldsymbol{e}\Longrightarrow\boldsymbol{g}\circ(\boldsymbol{f}\circ\boldsymbol{e}) \tag{6.28}$$

with the property that for all $h\in H$, $i\in I$, $j\in J$ *and* $k\in K$ *we have*

$$(\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}})_{hk}\odot\Theta_{hik}^{\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{e}}\odot(\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}}*\mathrm{id}_{\boldsymbol{e}_{hi}})=\Theta_{hjk}^{\boldsymbol{g},\boldsymbol{f}\circ\boldsymbol{e}}\odot(\mathrm{id}_{\boldsymbol{g}_{jk}}*\Theta_{hij}^{\boldsymbol{f},\boldsymbol{e}})\odot\boldsymbol{\alpha}_{\boldsymbol{g}_{jk},\boldsymbol{f}_{ij},\boldsymbol{e}_{hi}}. \tag{6.29}$$

Here are the analogues of Definitions 4.23 and 4.24.

Definition 6.24. Let $\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ be 1-morphisms of Kuranishi spaces, using notation (6.15)–(6.16), and $\boldsymbol{\eta}=(\boldsymbol{\eta}_{ij}):\boldsymbol{f}\Rightarrow\boldsymbol{g}$, $\boldsymbol{\zeta}=(\boldsymbol{\zeta}_{ij}):\boldsymbol{g}\Rightarrow\boldsymbol{h}$ be 2-morphisms. Define the *vertical composition of* 2*-morphisms* $\boldsymbol{\zeta}\odot\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{h}$ by

$$\boldsymbol{\zeta}\odot\boldsymbol{\eta}=\bigl(\boldsymbol{\zeta}_{ij}\odot\boldsymbol{\eta}_{ij},\ i\in I,\ j\in J\bigr). \tag{6.30}$$

To see that $\boldsymbol{\zeta}\odot\boldsymbol{\eta}$ satisfies Definition 6.20(a),(b), for (a) note that for all $i,i'\in I$ and $j\in J$, by Definition 6.20(a) for $\boldsymbol{\eta},\boldsymbol{\zeta}$ we have

$$\begin{aligned}\boldsymbol{H}_{ii'}^{j}\odot((\boldsymbol{\zeta}_{i'j}\odot\boldsymbol{\eta}_{i'j})*\mathrm{id}_{\mathrm{T}_{ii'}})&=\boldsymbol{H}_{ii'}^{j}\odot(\boldsymbol{\zeta}_{i'j}*\mathrm{id}_{\mathrm{T}_{ii'}})\odot(\boldsymbol{\eta}_{i'j}*\mathrm{id}_{\mathrm{T}_{ii'}})\\&=\boldsymbol{\zeta}_{ij}\odot\boldsymbol{G}_{ii'}^{j}\odot(\boldsymbol{\eta}_{i'j}*\mathrm{id}_{\mathrm{T}_{ii'}})=(\boldsymbol{\zeta}_{ij}\odot\boldsymbol{\eta}_{ij})\odot\boldsymbol{F}_{ii'}^{j},\end{aligned}$$

and Definition 6.20(b) for $\boldsymbol{\zeta}\odot\boldsymbol{\eta}$ is proved in a similar way.

Clearly, vertical composition of 2-morphisms of Kuranishi spaces is associative, $(\boldsymbol{\theta}\odot\boldsymbol{\zeta})\odot\boldsymbol{\eta}=\boldsymbol{\theta}\odot(\boldsymbol{\zeta}\odot\boldsymbol{\eta})$, since vertical composition of 2-morphisms of Kuranishi neighbourhoods is associative.

If $\boldsymbol{g}=\boldsymbol{h}$ and $\boldsymbol{\zeta}=\mathbf{id}_{\boldsymbol{g}}$ then $\mathbf{id}_{\boldsymbol{g}}\odot\boldsymbol{\eta}=(\mathrm{id}_{\boldsymbol{g}_{ij}}\odot\boldsymbol{\eta}_{ij})=(\boldsymbol{\eta}_{ij})=\boldsymbol{\eta}$, and similarly $\boldsymbol{\zeta}\odot\mathbf{id}_{\boldsymbol{g}}=\boldsymbol{\zeta}$, so identity 2-morphisms behave as expected under $\odot$.

If $\boldsymbol{\eta}=(\boldsymbol{\eta}_{ij,\ i\in I,\ j\in J}):\boldsymbol{f}\Rightarrow\boldsymbol{g}$ is a 2-morphism of Kuranishi spaces, then as 2-morphisms $\boldsymbol{\eta}_{ij}$ of Kuranishi neighbourhoods are invertible, we may define $\boldsymbol{\eta}^{-1}=(\boldsymbol{\eta}_{ij,\ j\in J,\ i\in I}^{-1}):\boldsymbol{g}\Rightarrow\boldsymbol{f}$. It is easy to check that $\boldsymbol{\eta}^{-1}$ is a 2-morphism, and $\boldsymbol{\eta}^{-1}\odot\boldsymbol{\eta}=\mathbf{id}_{\boldsymbol{f}}$, $\boldsymbol{\eta}\odot\boldsymbol{\eta}^{-1}=\mathbf{id}_{\boldsymbol{g}}$. Thus, all 2-morphisms of Kuranishi spaces are 2-isomorphisms.

Definition 6.25. Let $\boldsymbol{e},\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ and $\boldsymbol{g},\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ be 1-morphisms of Kuranishi spaces, using notation (6.15)–(6.17), and $\boldsymbol{\eta}=(\boldsymbol{\eta}_{ij}):\boldsymbol{e}\Rightarrow\boldsymbol{f}$, $\boldsymbol{\zeta}=(\boldsymbol{\zeta}_{jk}):\boldsymbol{g}\Rightarrow\boldsymbol{h}$ be 2-morphisms. We claim there is a unique 2-morphism $\boldsymbol{\theta}=(\boldsymbol{\theta}_{ik}):\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$, such that for all $i\in I$, $j\in J$, $k\in K$, we have

$$\boldsymbol{\theta}_{ik}|_{\mathrm{Im}\,\chi_i\cap e^{-1}(\mathrm{Im}\,\psi_j)\cap(g\circ e)^{-1}(\mathrm{Im}\,\omega_k)}=\Theta_{ijk}^{\boldsymbol{h},\boldsymbol{f}}\odot(\boldsymbol{\zeta}_{jk}*\boldsymbol{\eta}_{ij})\odot(\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{e}})^{-1}. \tag{6.31}$$

To prove this, suppose $j, j' \in J$, and consider the diagram of 2-morphisms over $\operatorname{Im}\chi_i \cap e^{-1}(\operatorname{Im}\psi_j \cap \operatorname{Im}\psi_{j'}) \cap (g \circ e)^{-1}(\operatorname{Im}\omega_k)$:

$$
\begin{tikzcd}
& \boldsymbol{g}_{jk} \circ \boldsymbol{e}_{ij} \arrow[r, Rightarrow, "\boldsymbol{\zeta}_{jk} * \boldsymbol{\eta}_{ij}"'] & \boldsymbol{h}_{jk} \circ \boldsymbol{f}_{ij} \arrow[ddr, Rightarrow, "\Theta^{\boldsymbol{h},\boldsymbol{f}}_{ijk}"] & \\
& (\boldsymbol{g}_{j'k} \circ \boldsymbol{\Upsilon}_{jj'}) \circ \boldsymbol{e}_{ij} \arrow[u, Rightarrow, "\boldsymbol{G}^k_{jj'} * \mathrm{id}_{\boldsymbol{e}_{ij}}"'] \arrow[r, Rightarrow, "(\boldsymbol{\zeta}_{j'k} * \mathrm{id}_{\boldsymbol{\Upsilon}_{jj'}}) * \boldsymbol{\eta}_{ij}"] \arrow[dd, Rightarrow, "\boldsymbol{\alpha}_{\boldsymbol{g}_{j'k}, \boldsymbol{\Upsilon}_{jj'}, \boldsymbol{e}_{ij}}"] & (\boldsymbol{h}_{j'k} \circ \boldsymbol{\Upsilon}_{jj'}) \circ \boldsymbol{f}_{ij} \arrow[u, Rightarrow, "\boldsymbol{H}^k_{jj'} * \mathrm{id}_{\boldsymbol{f}_{ij}}"] \arrow[dd, Rightarrow, "\boldsymbol{\alpha}_{\boldsymbol{h}_{j'k}, \boldsymbol{\Upsilon}_{jj'}, \boldsymbol{f}_{ij}}"'] & \\
(\boldsymbol{g} \circ \boldsymbol{e})_{ik} \arrow[uur, Rightarrow, "(\Theta^{\boldsymbol{g},\boldsymbol{e}}_{ijk})^{-1}"] \arrow[ddr, Rightarrow, "(\Theta^{\boldsymbol{g},\boldsymbol{e}}_{ij'k})^{-1}"'] & & & (\boldsymbol{h} \circ \boldsymbol{f})_{ik}. \\
& \boldsymbol{g}_{j'k} \circ (\boldsymbol{\Upsilon}_{jj'} \circ \boldsymbol{e}_{ij}) \arrow[r, Rightarrow, "\boldsymbol{\zeta}_{j'k} * (\mathrm{id}_{\boldsymbol{\Upsilon}_{jj'}} * \boldsymbol{\eta}_{ij})"] \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{g}_{j'k}} * \boldsymbol{E}^{jj'}_i"] & \boldsymbol{h}_{j'k} \circ (\boldsymbol{\Upsilon}_{jj'} \circ \boldsymbol{f}_{ij}) \arrow[d, Rightarrow, "\mathrm{id}_{\boldsymbol{h}_{j'k}} * \boldsymbol{F}^{jj'}_i"'] & \\
& \boldsymbol{g}_{j'k} \circ \boldsymbol{e}_{ij'} \arrow[r, Rightarrow, "\boldsymbol{\zeta}_{j'k} * \boldsymbol{\eta}_{ij'}"] & \boldsymbol{h}_{j'k} \circ \boldsymbol{f}_{ij'} \arrow[uur, Rightarrow, "\Theta^{\boldsymbol{h},\boldsymbol{f}}_{ij'k}"'] &
\end{tikzcd}
\tag{6.32}
$$

Here the left and right polygons commute by (6.25), the top and bottom rectangles commute by Definition 6.20(a),(b) for $\boldsymbol{\zeta}, \boldsymbol{\eta}$, and the central rectangle commutes by properties of weak 2-categories. Hence (6.32) commutes.

The two routes round the outside of (6.32) imply that the prescribed values (6.31) for $\boldsymbol{\theta}_{ik}$ agree on overlaps between open sets for j, j'. As the $\operatorname{Im}\chi_i \cap e^{-1}(\operatorname{Im}\psi_j) \cap (g \circ e)^{-1}(\operatorname{Im}\omega_k)$ for $j \in J$ form an open cover of the correct domain $\operatorname{Im}\chi_i \cap (g \circ e)^{-1}(\operatorname{Im}\omega_k)$, by Theorem 6.16 and Definition A.17(iii),(iv), there is a unique 2-morphism $\boldsymbol{\theta}_{ik} : (\boldsymbol{g} \circ \boldsymbol{e})_{ik} \Rightarrow (\boldsymbol{h} \circ \boldsymbol{f})_{ik}$ satisfying (6.31) for all $j \in J$.

To show $\boldsymbol{\theta} = (\boldsymbol{\theta}_{ik}) : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$ is a 2-morphism, we must verify Definition 6.20(a),(b) for $\boldsymbol{\theta}$. We do this by first showing that (a),(b) hold on the intersections of their domains with $e^{-1}(\operatorname{Im}\psi_j)$ for $j \in J$ using (6.24), (6.26), (6.31), and Definition 6.20 for $\boldsymbol{\eta}, \boldsymbol{\zeta}$, and then use Theorem 6.16 and Definition A.17(iii) to deduce that Definition 6.20(a),(b) for $\boldsymbol{\theta}$ hold on their whole domains. So $\boldsymbol{\theta}$ is a 2-morphism of Kuranishi spaces.

Define the *horizontal composition of* 2-*morphisms* $\boldsymbol{\zeta} * \boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$ to be $\boldsymbol{\zeta} * \boldsymbol{\eta} = \boldsymbol{\theta}$. By (6.31), for all $i \in I$, $j \in J$, $k \in K$ we have

$$
(\boldsymbol{\zeta} * \boldsymbol{\eta})_{ik} \odot \Theta^{\boldsymbol{g},\boldsymbol{e}}_{ijk} = \Theta^{\boldsymbol{h},\boldsymbol{f}}_{ijk} \odot (\boldsymbol{\zeta}_{jk} * \boldsymbol{\eta}_{ij}), \tag{6.33}
$$

and this characterizes $\boldsymbol{\zeta} * \boldsymbol{\eta}$ uniquely.

We have now defined all the structures of a *weak* 2-*category of Kuranishi spaces* $\dot{\mathbf{K}}\mathbf{ur}$, as in §A.2: objects $\boldsymbol{X}, \boldsymbol{Y}$, 1-morphisms $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$, 2-morphisms $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$, identity 1- and 2-morphisms, composition of 1-morphisms, vertical and horizontal composition of 2-morphisms, 2-isomorphisms $\boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{f},\boldsymbol{e}}$ in (6.28) for associativity of 1-morphisms, and $\boldsymbol{\beta}_{\boldsymbol{f}}, \boldsymbol{\gamma}_{\boldsymbol{f}}$ in (6.27) for identity 1-morphisms. Following the proofs of Propositions 4.25–4.27 in §4.3, but including extra 2-morphisms $\boldsymbol{\alpha}_{*,*,*}, \boldsymbol{\beta}_*, \boldsymbol{\gamma}_*$, as in Theorem 4.28 we prove:

Theorem 6.26. *The definitions and propositions above define a* ***weak* 2*-category of Kuranishi spaces*** $\dot{\mathbf{K}}\mathbf{ur}$.

Remark 6.27. (a) We proved in §6.1 that Kuranishi neighbourhoods over $S \subseteq X$ form a weak 2-category $\dot{\mathbf{K}}\mathbf{N}_S(X)$, and now we have shown that Kuranishi spaces also form a weak 2-category $\dot{\mathbf{K}}\mathbf{ur}$. But morally, $\dot{\mathbf{K}}\mathbf{N}_S(X)$ is closer to being a strict 2-category. In $\dot{\mathbf{K}}\mathbf{N}_S(X)$ there is a natural notion of composition of 1-morphisms $\Phi_{jk}\circ\Phi_{ij}$, but it just fails to be strictly associative, as the canonical isomorphism of fibre products λ_{ijkl} in (6.6) is not the identity. The analogue $\mathbf{m}\dot{\mathbf{K}}\mathbf{N}_S(X)$ for m-Kuranishi spaces in §4.1 is a strict 2-category.

In $\dot{\mathbf{K}}\mathbf{ur}$, there is no natural notion of composition of 1-morphisms $\boldsymbol{g}\circ\boldsymbol{f}$, so as in Definition 6.22 we have to choose $\boldsymbol{g}\circ\boldsymbol{f}$ using the Axiom of Global Choice, and composition of 1-morphisms in $\dot{\mathbf{K}}\mathbf{ur}$ is far from being strictly associative.

(b) We can define a weak 2-functor $\mathbf{G}\dot{\mathbf{K}}\mathbf{N}\to\dot{\mathbf{K}}\mathbf{ur}$ which is an equivalence from the 2-category $\mathbf{G}\dot{\mathbf{K}}\mathbf{N}$ of global Kuranishi neighbourhoods in Definition 6.9 to the full 2-subcategory of objects $(X,\mathcal{K})$ in $\dot{\mathbf{K}}\mathbf{ur}$ for which $\mathcal{K}$ contains only one Kuranishi neighbourhood. It acts by $(V,E,\Gamma,s)\mapsto \boldsymbol{S}_{V,E,\Gamma,s}$ on objects, for $\boldsymbol{S}_{V,E,\Gamma,s}$ as in Example 6.18.

Here is the analogue of Examples 4.31 and 5.17:

Example 6.28. Let $\boldsymbol{X}=(X,\mathcal{I})$, $\boldsymbol{Y}=(Y,\mathcal{J})$ be Kuranishi spaces in $\dot{\mathbf{K}}\mathbf{ur}$, with notation (6.15)–(6.16). Define the *product* to be $\boldsymbol{X}\times\boldsymbol{Y}=(X\times Y,\mathcal{K})$, where

$$\mathcal{K}=\bigl(I\times J,(W_{(i,j)},F_{(i,j)},\Delta_{(i,j)},t_{(i,j)},\omega_{(i,j)})_{(i,j)\in I\times J},\ \Phi_{(i,j)(i',j'),\ (i,j),(i',j')\in I\times J},\\ \mathrm{M}_{(i,j)(i',j')(i'',j''),\ (i,j),(i',j'),(i'',j'')\in I\times J}\bigr).$$

Here for all $(i,j)\in I\times J$ we set $W_{(i,j)}=U_i\times V_j$, $F_{(i,j)}=\pi_{U_i}^*(D_i)\oplus\pi_{V_j}^*(E_j)$, $\Delta_{(i,j)}=\mathrm{B}_i\times\Gamma_j$, and $t_{(i,j)}=\pi_{U_i}^*(r_i)\oplus\pi_{V_j}^*(s_j)$ so that $t_{(i,j)}^{-1}(0)=r_i^{-1}(0)\times s_j^{-1}(0)$, and $\omega_{(i,j)}=\chi_i\times\psi_j:(r_i^{-1}(0)\times s_j^{-1}(0))/(\mathrm{B}_i\times\Gamma_j)\to X\times Y$. Also

$$\Phi_{(i,j)(i',j')}=\mathrm{T}_{ii'}\times\Upsilon_{jj'}=\bigl(P_{ii'}\times Q_{jj'},\pi_{ii'}\times\pi_{jj'},\tau_{ii'}\times\upsilon_{jj'},\pi_{P_{ii'}}^*(\hat\tau_{ii'})\oplus\pi_{Q_{jj'}}^*(\hat\upsilon_{jj'})\bigr),$$

and $\mathrm{M}_{(i,j)(i',j')(i'',j'')}=\mathrm{K}_{ii'i''}\times\Lambda_{jj'j''}$ is defined as a product 2-morphism in the obvious way. Then $\boldsymbol{X}\times\boldsymbol{Y}$ is a Kuranishi space, with $\mathrm{vdim}(\boldsymbol{X}\times\boldsymbol{Y})=\mathrm{vdim}\,\boldsymbol{X}+\mathrm{vdim}\,\boldsymbol{Y}$. As in Example 4.31 we define explicit projection 1-morphisms $\boldsymbol{\pi_X}:\boldsymbol{X}\times\boldsymbol{Y}\to\boldsymbol{X}$ and $\boldsymbol{\pi_Y}:\boldsymbol{X}\times\boldsymbol{Y}\to\boldsymbol{Y}$.

Then $\boldsymbol{X}\times\boldsymbol{Y},\boldsymbol{\pi_X},\boldsymbol{\pi_Y}$ have the universal property of products in a 2-category, as in §11.5 in volume II. Products are commutative and associative up to canonical equivalence. If $\boldsymbol{f}:\boldsymbol{W}\to\boldsymbol{Y}$, $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Y}$, $\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Z}$ are 1-morphisms in $\dot{\mathbf{K}}\mathbf{ur}$ then we have a *product* 1-*morphism* $\boldsymbol{f}\times\boldsymbol{h}:\boldsymbol{W}\times\boldsymbol{X}\to\boldsymbol{Y}\times\boldsymbol{Z}$ and a *direct product* 1-*morphism* $(\boldsymbol{g},\boldsymbol{h}):\boldsymbol{X}\to\boldsymbol{Y}\times\boldsymbol{Z}$ in $\dot{\mathbf{K}}\mathbf{ur}$, both easy to write down explicitly.

6.2.2 Examples of 2-categories $\dot{\mathbf{K}}\mathbf{ur}$, and 2-functors of them

Here is the analogue of Definition 4.29:

Definition 6.29. In Theorem 6.26 we write $\dot{\mathbf{K}}\mathbf{ur}$ for the 2-category of Kuranishi spaces constructed from a category $\dot{\mathbf{M}}\mathbf{an}$ satisfying Assumptions 3.1–3.7. By Example 3.8, the following categories from Chapter 2 are possible choices for $\dot{\mathbf{M}}\mathbf{an}$:

$$\mathbf{Man},\mathbf{Man^c_{we}},\mathbf{Man^c},\mathbf{Man^{gc}},\mathbf{Man^{ac}},\mathbf{Man^{c,ac}}. \tag{6.34}$$

We write the corresponding 2-categories of Kuranishi spaces as follows:

$$\mathbf{Kur}, \mathbf{Kur^c_{we}}, \mathbf{Kur^c}, \mathbf{Kur^{gc}}, \mathbf{Kur^{ac}}, \mathbf{Kur^{c,ac}}. \tag{6.35}$$

Objects of $\mathbf{Kur^c}, \mathbf{Kur^{gc}}, \mathbf{Kur^{ac}}, \mathbf{Kur^{c,ac}}$ will be called *Kuranishi spaces with corners*, and *with g-corners*, and *with a-corners*, and *with corners and a-corners*, respectively.

In §4.4 we showed that any functor $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \to \ddot{\mathbf{Man}}$ satisfying Condition 3.20 induces a weak 2-functor $F_{\mathbf{m}\dot{\mathbf{Kur}}}^{\mathbf{m}\ddot{\mathbf{Kur}}} : \mathbf{m}\dot{\mathbf{Kur}} \to \mathbf{m}\ddot{\mathbf{Kur}}$, and under the hypotheses of Proposition 3.21 this is an inclusion of 2-subcategories. The same arguments work for Kuranishi spaces, proving:

Proposition 6.30. *Suppose* $\dot{\mathbf{Man}}, \ddot{\mathbf{Man}}$ *are categories satisfying Assumptions* 3.1–3.7, *and* $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \to \ddot{\mathbf{Man}}$ *is a functor satisfying Condition* 3.20. *Then we can define a natural weak* 2*-functor* $F_{\dot{\mathbf{Kur}}}^{\ddot{\mathbf{Kur}}} : \dot{\mathbf{Kur}} \to \ddot{\mathbf{Kur}}$.

If $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \hookrightarrow \ddot{\mathbf{Man}}$ *is an inclusion of subcategories* $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$ *satisfying either Proposition* 3.21(a) *or* (b), *then* $F_{\dot{\mathbf{Kur}}}^{\ddot{\mathbf{Kur}}} : \dot{\mathbf{Kur}} \hookrightarrow \ddot{\mathbf{Kur}}$ *is also an inclusion of* 2*-subcategories* $\dot{\mathbf{Kur}} \subseteq \ddot{\mathbf{Kur}}$.

$$\begin{tikzcd}
 & & \mathbf{Kur} \arrow[dll, bend right] \arrow[dl] \arrow[dr] \arrow[drr, bend left] & & \\
\mathbf{Kur^{gc}} & \mathbf{Kur^c} \arrow[l] \arrow[rr] \arrow[d, dotted] & & \mathbf{Kur^{c,ac}} \arrow[r, dotted, shift left] & \mathbf{Kur^{ac}} \arrow[l, shift left] \\
 & \mathbf{Kur^c_{we}} & & &
\end{tikzcd}$$

Figure 6.1: 2-functors between 2-categories of Kuranishi spaces from Definition 6.29. Arrows '$\to$' are inclusions of 2-subcategories.

Applying Definition 4.32 to the parts of the diagram Figure 3.1 of functors $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ involving the categories (6.34) yields a diagram Figure 6.1 of 2-functors $F_{\dot{\mathbf{Kur}}}^{\ddot{\mathbf{Kur}}}$. Arrows '$\to$' are inclusions of 2-subcategories.

6.2.3 Discrete properties of 1-morphisms in $\dot{\mathbf{Kur}}$

In §3.3.6 and §B.6 we defined when a property $\boldsymbol{P}$ of morphisms in $\dot{\mathbf{Man}}$ is *discrete*. Section 4.5 explained how to extend discrete properties of morphisms in $\dot{\mathbf{Man}}$ to corresponding properties of 1-morphisms in $\mathbf{m}\dot{\mathbf{Kur}}$. We now do the same for $\dot{\mathbf{Kur}}$. Here are the analogues of Definitions 4.33, 4.35, and 4.37 and Propositions 4.34 and 4.36, proved in a very similar way.

Definition 6.31. Let $\boldsymbol{P}$ be a discrete property of morphisms in $\dot{\mathbf{Man}}$. Suppose $f : X \to Y$ is a continuous map and $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat{\phi}_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ is a 1-morphism of Kuranishi neighbourhoods over (S, f), for $S \subseteq X$ open. We say that Φ_{ij} *is* $\boldsymbol{P}$ if $\phi_{ij} : P_{ij} \to V_j$ is $\boldsymbol{P}$ near $(\bar{\psi}_i \circ \pi_{ij})^{-1}(S)$

in P_{ij}. That is, there should exist an open submanifold $\iota : U \hookrightarrow P_{ij}$ with $(\bar{\psi}_i \circ \pi_{ij})^{-1}(S) \subseteq U \subseteq P_{ij}$ such that $\phi_{ij} \circ \iota : U \to V_j$ has property $\boldsymbol{P}$ in $\dot{\mathbf{Man}}$.

Proposition 6.32. *Let $\boldsymbol{P}$ be a discrete property of morphisms in $\dot{\mathbf{Man}}$. Then:*

(a) *Let $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be a 1-morphism of Kuranishi neighbourhoods over (S, f) for $f : X \to Y$ continuous and $S \subseteq X$ open. If Φ_{ij} is $\boldsymbol{P}$ and $T \subseteq S$ is open then $\Phi_{ij}|_T$ is $\boldsymbol{P}$. If $\{T_a : a \in A\}$ is an open cover of S and $\Phi_{ij}|_{T_a}$ is $\boldsymbol{P}$ for all $a \in A$ then Φ_{ij} is $\boldsymbol{P}$.*

(b) *Let $\Phi_{ij}, \Phi'_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be 1-morphisms over (S, f) and $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ a 2-morphism. Then Φ_{ij} is $\boldsymbol{P}$ if and only if Φ'_{ij} is $\boldsymbol{P}$.*

(c) *Let $f : X \to Y$, $g : Y \to Z$ be continuous, $T \subseteq Y$, $S \subseteq f^{-1}(T) \subseteq X$ be open, $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be a 1-morphism over (S, f), and $\Phi_{jk} : (V_j, E_j, \Gamma_j, s_j, \psi_j) \to (V_k, E_k, \Gamma_k, s_k, \psi_k)$ be a 1-morphism over (T, g), so that $\Phi_{jk} \circ \Phi_{ij}$ is a 1-morphism over $(S, g \circ f)$. If Φ_{ij}, Φ_{jk} are $\boldsymbol{P}$ then $\Phi_{jk} \circ \Phi_{ij}$ is $\boldsymbol{P}$.*

(d) *Let $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be a coordinate change of Kuranishi neighbourhoods over $S \subseteq X$. Then Φ_{ij} is $\boldsymbol{P}$.*

Definition 6.33. Let $\boldsymbol{P}$ be a discrete property of morphisms in $\dot{\mathbf{Man}}$. Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in $\dot{\mathbf{Kur}}$, and use notation (6.15), (6.16), (6.18) for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{f}$. We say that $\boldsymbol{f}$ *is* $\boldsymbol{P}$ if $\boldsymbol{f}_{ij}$ is $\boldsymbol{P}$ in the sense of Definition 6.31 for all $i \in I$ and $j \in J$.

Proposition 6.34. *Let $\boldsymbol{P}$ be a discrete property of morphisms in $\dot{\mathbf{Man}}$. Then:*

(a) *Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms in $\dot{\mathbf{Kur}}$ and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism. Then $\boldsymbol{f}$ is $\boldsymbol{P}$ if and only if $\boldsymbol{g}$ is $\boldsymbol{P}$.*

(b) *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms in $\dot{\mathbf{Kur}}$. If $\boldsymbol{f}$ and $\boldsymbol{g}$ are $\boldsymbol{P}$ then $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ is $\boldsymbol{P}$.*

(c) *Identity 1-morphisms $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ in $\dot{\mathbf{Kur}}$ are $\boldsymbol{P}$. Equivalences $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\dot{\mathbf{Kur}}$ are $\boldsymbol{P}$.*

Parts **(b)**,**(c)** *imply that we have a 2-subcategory $\dot{\mathbf{Kur}}_{\boldsymbol{P}} \subseteq \dot{\mathbf{Kur}}$ containing all objects in $\dot{\mathbf{Kur}}$, and all 1-morphisms $\boldsymbol{f}$ in $\dot{\mathbf{Kur}}$ which are $\boldsymbol{P}$, and all 2-morphisms $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\dot{\mathbf{Kur}}$ between 1-morphisms $\boldsymbol{f}, \boldsymbol{g}$ which are $\boldsymbol{P}$.*

Definition 6.35. **(a)** Taking $\dot{\mathbf{Man}} = \mathbf{Man^c}$ from §2.1 gives the 2-category of Kuranishi spaces $\mathbf{Kur^c}$ from Definition 6.29. We write

$$\mathbf{Kur^c_{in}}, \mathbf{Kur^c_{bn}}, \mathbf{Kur^c_{st}}, \mathbf{Kur^c_{st,in}}, \mathbf{Kur^c_{st,bn}}, \mathbf{Kur^c_{si}}$$

for the 2-subcategories of $\mathbf{Kur^c}$ with 1-morphisms which are *interior*, and *b-normal*, and *strongly smooth*, and *strongly smooth-interior*, and *strongly smooth-b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^c}$ are discrete by Example 3.19(a), so as in Definition 6.33 and Proposition 6.34 we have corresponding notions of interior, ..., simple 1-morphisms in $\mathbf{Kur^c}$.

(b) Taking $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^{gc}}$ from §2.4.1 gives the 2-category of Kuranishi spaces with g-corners $\mathbf{Kur^{gc}}$ from Definition 6.29. We write

$$\mathbf{Kur^{gc}_{in}}, \mathbf{Kur^{gc}_{bn}}, \mathbf{Kur^{gc}_{si}}$$

for the 2-subcategories of $\mathbf{Kur^{gc}}$ with 1-morphisms which are *interior*, and *b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^{gc}}$ are discrete by Example 3.19(b), so we have corresponding notions for 1-morphisms in $\mathbf{Kur^{gc}}$.

(c) Taking $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^{ac}}$ from §2.4.2 gives the 2-category of Kuranishi spaces with a-corners $\mathbf{Kur^{ac}}$ from Definition 6.29. We write

$$\mathbf{Kur^{ac}_{in}}, \mathbf{Kur^{ac}_{bn}}, \mathbf{Kur^{ac}_{st}}, \mathbf{Kur^{ac}_{st,in}}, \mathbf{Kur^{ac}_{st,bn}}, \mathbf{Kur^{ac}_{si}}$$

for the 2-subcategories of $\mathbf{Kur^{ac}}$ with 1-morphisms which are *interior*, and *b-normal*, and *strongly a-smooth*, and *strongly a-smooth-interior*, and *strongly a-smooth-b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^{ac}}$ are discrete by Example 3.19(c), so we have corresponding notions for 1-morphisms in $\mathbf{Kur^{ac}}$.

(d) Taking $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^{c,ac}}$ from §2.4.2 gives the 2-category of Kuranishi spaces with corners and a-corners $\mathbf{Kur^{c,ac}}$ from Definition 6.29. We write

$$\mathbf{Kur^{c,ac}_{in}}, \mathbf{Kur^{c,ac}_{bn}}, \mathbf{Kur^{c,ac}_{st}}, \mathbf{Kur^{c,ac}_{st,in}}, \mathbf{Kur^{c,ac}_{st,bn}}, \mathbf{Kur^{c,ac}_{si}}$$

for the 2-subcategories of $\mathbf{Kur^{c,ac}}$ with 1-morphisms which are *interior*, and *b-normal*, and *strongly a-smooth*, and *strongly a-smooth-interior*, and *strongly a-smooth-b-normal*, and *simple*, respectively. These properties of morphisms in $\mathbf{Man^{c,ac}}$ are discrete by Example 3.19(c), so we have corresponding notions for 1-morphisms in $\mathbf{Kur^{c,ac}}$.

Figure 6.1 gives inclusions between the 2-categories in (6.35). Combining this with the inclusions between the 2-subcategories in Definition 6.35 we get a diagram Figure 6.2 of inclusions of 2-subcategories of Kuranishi spaces.

6.2.4 Kuranishi spaces and m-Kuranishi spaces

We relate m-Kuranishi spaces in Chapter 4 to Kuranishi spaces above.

Example 6.36. Let $\mathbf{m\dot{K}ur}$ and $\mathbf{\dot{K}ur}$ be the weak 2-categories constructed in §4.3 and above from the same category of 'manifolds' $\dot{\mathbf{M}}\mathbf{an}$. We will define a full and faithful weak 2-functor $F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}} : \mathbf{m\dot{K}ur} \hookrightarrow \mathbf{\dot{K}ur}$, as in §A.3.

First we explain how to map m-Kuranishi neighbourhoods and their 1- and 2-morphisms to Kuranishi neighbourhoods and their 1- and 2-morphisms. An m-Kuranishi neighbourhood (V_i, E_i, s_i, ψ_i) on X maps to the Kuranishi neighbourhood $(V_i, E_i, \{1\}, s_i, \psi_i)$ on X, that is, to $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ with group $\Gamma_i = \{1\}$.

$$
\begin{tikzcd}[column sep=small]
 & & & \mathbf{Kur} \arrow[dl] \arrow[dr] \arrow[dlll, bend right] \arrow[drrr, bend left] & & & \\
\mathbf{Kur}^{\mathbf{gc}}_{\mathbf{si}} \arrow[dd] & & \mathbf{Kur}^{\mathbf{c}}_{\mathbf{si}} \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{si}} \arrow[dl] \arrow[dd] & & \mathbf{Kur}^{\mathbf{ac}}_{\mathbf{si}} \arrow[ll] \arrow[dl] \arrow[dd] \\
 & \mathbf{Kur}^{\mathbf{c}}_{\mathbf{st,bn}} \arrow[rr] \arrow[dl] \arrow[dr] \arrow[dd] & & \mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{st,bn}} \arrow[dr] \arrow[dd] & & \mathbf{Kur}^{\mathbf{ac}}_{\mathbf{st,bn}} \arrow[ll] \arrow[dr] \arrow[dd] & \\
\mathbf{Kur}^{\mathbf{gc}}_{\mathbf{bn}} \arrow[dd] & & \mathbf{Kur}^{\mathbf{c}}_{\mathbf{bn}} \arrow[ll] \arrow[rr] \arrow[dd] & & \mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{bn}} \arrow[dd] & & \mathbf{Kur}^{\mathbf{ac}}_{\mathbf{bn}} \arrow[ll] \arrow[dd] \\
 & \mathbf{Kur}^{\mathbf{c}}_{\mathbf{st,in}} \arrow[rr] \arrow[dl] \arrow[dr] \arrow[dd] & & \mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{st,in}} \arrow[dr] \arrow[dd] & & \mathbf{Kur}^{\mathbf{ac}}_{\mathbf{st,in}} \arrow[ll] \arrow[dr] \arrow[dd] & \\
\mathbf{Kur}^{\mathbf{gc}}_{\mathbf{in}} \arrow[dd] & & \mathbf{Kur}^{\mathbf{c}}_{\mathbf{in}} \arrow[ll] \arrow[rr] \arrow[dd] & & \mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{in}} \arrow[dd] & & \mathbf{Kur}^{\mathbf{ac}}_{\mathbf{in}} \arrow[ll] \arrow[dd] \\
 & \mathbf{Kur}^{\mathbf{c}}_{\mathbf{st}} \arrow[rr] \arrow[dl] \arrow[dr] & & \mathbf{Kur}^{\mathbf{c,ac}}_{\mathbf{st}} \arrow[dr] & & \mathbf{Kur}^{\mathbf{ac}}_{\mathbf{st}} \arrow[ll] \arrow[dr] & \\
\mathbf{Kur}^{\mathbf{gc}} & & \mathbf{Kur}^{\mathbf{c}} \arrow[ll] \arrow[rr] & & \mathbf{Kur}^{\mathbf{c,ac}} & & \mathbf{Kur}^{\mathbf{ac}} \arrow[ll]
\end{tikzcd}
$$

Figure 6.2: Inclusions of 2-categories of Kuranishi spaces.

A 1-morphism $\Phi_{ij}=(V_{ij},\phi_{ij},\hat\phi_{ij}):(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ of m-Kuranishi neighbourhoods over (S,f) maps to the 1-morphism $\tilde\Phi_{ij}=(V_{ij},\mathrm{id}_{V_{ij}},\phi_{ij},\hat\phi_{ij}):(V_i,E_i,\{1\},s_i,\psi_i)\to(V_j,E_j,\{1\},s_j,\psi_j)$ of Kuranishi neighbourhoods over (S,f). That is, in $\tilde\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$, $\pi_{ij}:P_{ij}\to V_{ij}\subseteq V_i$ must be a principal Γ_j-bundle for $\Gamma_j=\{1\}$, so we take $P_{ij}=V_{ij}$ and $\pi_{ij}=\mathrm{id}_{V_{ij}}$.

Given 1-morphisms $\Phi_{ij},\Phi'_{ij}:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ of m-Kuranishi neighbourhoods over (S,f) and corresponding 1-morphisms $\tilde\Phi_{ij},\tilde\Phi'_{ij}:(V_i,E_i,\{1\},s_i,\psi_i)\to(V_j,E_j,\{1\},s_j,\psi_j)$ of Kuranishi neighbourhoods over (S,f), a 2-morphism $[\acute V_{ij},\hat\lambda_{ij}]:\Phi_{ij}\Rightarrow\Phi'_{ij}$ of m-Kuranishi neighbourhoods maps to the 2-morphism $[\acute V_{ij},\mathrm{id}_{\acute V_{ij}},\hat\lambda_{ij}]:\tilde\Phi_{ij}\Rightarrow\tilde\Phi'_{ij}$ of Kuranishi neighbourhoods.

To define $F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}$, we apply this process to all m-Kuranishi neighbourhoods, 1- and 2-morphisms in the structures on $\mathbf{m\dot{K}ur}$. On objects, let $\boldsymbol X=(X,\mathcal K)$ be an m-Kuranishi space, with $\mathcal K=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \Lambda_{ijk,\ i,j,k\in I}\bigr)$, where $\Phi_{ij}=(V_{ij},\phi_{ij},\hat\phi_{ij})$ and $\Lambda_{ijk}=[\acute V_{ijk},\hat\lambda_{ijk}]$. Define $\tilde{\mathcal K}=\bigl(I,(V_i,E_i,\{1\},s_i,\psi_i)_{i\in I},\ \tilde\Phi_{ij,\ i,j\in I},\ \tilde\Lambda_{ijk,\ i,j,k\in I}\bigr)$, where $\tilde\Phi_{ij}=(V_{ij},\mathrm{id}_{V_{ij}},\phi_{ij},\hat\phi_{ij})$ and $\tilde\Lambda'_{ijk}=[\acute V_{ijk},\mathrm{id}_{\acute V_{ijk}},\hat\lambda_{ijk}]$. Then $\tilde{\boldsymbol X}=(X,\tilde{\mathcal K})$ is a Kuranishi space, and we set $F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\boldsymbol X)=\tilde{\boldsymbol X}$. Similarly we define 1- and 2-morphisms $F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\boldsymbol f),F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\boldsymbol\eta)$ in $\mathbf{\dot{K}ur}$ for all 1- and 2-morphisms $\boldsymbol f,\boldsymbol\eta$ in $\mathbf{m\dot{K}ur}$.

Let $\boldsymbol f:\boldsymbol X\to\boldsymbol Y$ and $\boldsymbol g:\boldsymbol Y\to\boldsymbol Z$ be 1-morphisms in $\mathbf{m\dot{K}ur}$, and write $\tilde{\boldsymbol X},\tilde{\boldsymbol Y},\tilde{\boldsymbol Z},\tilde{\boldsymbol f},\tilde{\boldsymbol g}$ for the images of $\boldsymbol X,\boldsymbol Y,\boldsymbol Z,\boldsymbol f,\boldsymbol g$ under $F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}$. Then Definition 4.20 defines $\boldsymbol g\circ\boldsymbol f:\boldsymbol X\to\boldsymbol Z$ in $\mathbf{m\dot{K}ur}$, and Definition 6.22 defines $\tilde{\boldsymbol g}\circ\tilde{\boldsymbol f}:\tilde{\boldsymbol X}\to\tilde{\boldsymbol Z}$ in $\mathbf{\dot{K}ur}$, both by making an arbitrary choice. As these choices may not be compatible, we need not have $F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\boldsymbol g\circ\boldsymbol f)=\tilde{\boldsymbol g}\circ\tilde{\boldsymbol f}$. But $F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\boldsymbol g\circ\boldsymbol f)$ is a possible choice for $\tilde{\boldsymbol{g}} \circ \tilde{\boldsymbol{f}}$, so as in Proposition 6.23 there is a canonical 2-isomorphism $(F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}})_{\boldsymbol{g},\boldsymbol{f}} : F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\boldsymbol{g}) \circ F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\boldsymbol{f}) \Rightarrow F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\boldsymbol{g} \circ \boldsymbol{f})$. We also write $(F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}})_{\boldsymbol{X}} : F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\mathbf{id}_{\boldsymbol{X}}) \Rightarrow \mathbf{id}_{F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\boldsymbol{X})}$ for the obvious 2-morphism.

This defines all the data of a weak 2-functor $F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}} : \mathbf{m\dot{K}ur} \hookrightarrow \mathbf{\dot{K}ur}$, as in §A.3. It is easy to check that $F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}$ satisfies the conditions for a weak 2-functor, and that it is full and faithful, and so embeds $\mathbf{m\dot{K}ur}$ as a full 2-subcategory of $\mathbf{\dot{K}ur}$. It is an equivalence between $\mathbf{m\dot{K}ur}$ and the full 2-subcategory of objects $\boldsymbol{X} = (X, \mathcal{K})$ in $\mathbf{\dot{K}ur}$ with $\Gamma_i = \{1\}$ for all Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ in $\mathcal{K}$.

6.3 Kuranishi spaces with corners. Boundaries, k-corners, and the corner 2-functor

We now change notation from $\mathbf{\dot{M}an}$ in §3.1–§3.3 to $\mathbf{\dot{M}an^c}$ in §3.4, and from $\mathbf{\dot{K}ur}$ in §6.2 to $\mathbf{\dot{K}ur^c}$. Suppose throughout this section that $\mathbf{\dot{M}an^c}$ satisfies Assumption 3.22 in §3.4.1. Then $\mathbf{\dot{M}an^c}$ satisfies Assumptions 3.1–3.7, so §6.2 constructs a 2-category $\mathbf{\dot{K}ur^c}$ of Kuranishi spaces associated to $\mathbf{\dot{M}an^c}$. For instance, $\mathbf{\dot{K}ur^c}$ could be $\mathbf{Kur^c}, \mathbf{Kur^{gc}}, \mathbf{Kur^{ac}}$ or $\mathbf{Kur^{c,ac}}$ from Definition 6.29. We will refer to objects of $\mathbf{\dot{K}ur^c}$ as *Kuranishi spaces with corners*. We also write $\mathbf{\dot{K}ur^c_{si}}$ for the 2-subcategory of $\mathbf{\dot{K}ur^c}$ with simple 1-morphisms in the sense of §6.2.3, noting that simple is a discrete property of morphisms in $\mathbf{\dot{M}an^c}$ by Assumption 3.22(c).

In §4.6, for each $\boldsymbol{X} \in \mathbf{m\dot{K}ur^c}$ we defined the k-corners $C_k(\boldsymbol{X})$ in $\mathbf{m\dot{K}ur^c}$ for $k = 0, 1, \ldots$, with $\partial\boldsymbol{X} = C_1(\boldsymbol{X})$. We constructed a 2-category $\mathbf{m\check{\dot{K}}ur^c}$ from $\mathbf{m\dot{K}ur^c}$ with objects $\coprod_{n\in\mathbb{Z}} \boldsymbol{X}_n$ for $\boldsymbol{X}_n \in \mathbf{m\dot{K}ur^c}$ with $\operatorname{vdim} \boldsymbol{X}_n = n$, and defined the corner 2-functor $C : \mathbf{m\dot{K}ur^c} \to \mathbf{m\check{\dot{K}}ur^c}$.

We will now extend all this to Kuranishi spaces with corners. We have to work with the more complicated notions of Kuranishi neighbourhoods and their 1- and 2-morphisms from §6.1, rather than m-Kuranishi neighbourhoods from §4.1, but apart from this the definitions and proofs are essentially the same. Here is the analogue of Definition 4.39:

Definition 6.37. Let $\boldsymbol{X} = (X, \mathcal{K})$ in $\mathbf{\dot{K}ur^c}$ be a Kuranishi space with corners with $\operatorname{vdim} \boldsymbol{X} = n$, and as in Definition 6.17 write $\mathcal{K} = \bigl(I, (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\; i,j\in I}, \Lambda_{hij,\; h,i,j\in I}\bigr)$ with $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij})$ and $\Lambda_{hij} = [\acute{P}_{hij}, \lambda_{hij}, \hat\lambda_{hij}]$. Let $k \in \mathbb{N}$. We will define a Kuranishi space with corners $C_k(\boldsymbol{X})$ in $\mathbf{\dot{K}ur^c}$ called the *k-corners of $\boldsymbol{X}$*, with $\operatorname{vdim} C_k(\boldsymbol{X}) = n - k$, and a 1-morphism $\boldsymbol{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ in $\mathbf{\dot{K}ur^c}$.

Explicitly we write $C_k(\boldsymbol{X}) = (C_k(X), \mathcal{K}_k)$ with

$$\mathcal{K}_k = \bigl(\{k\}\times I, (V_{(k,i)}, E_{(k,i)}, \Gamma_{(k,i)}, s_{(k,i)}, \psi_{(k,i)})_{i\in I}, \Phi_{(k,i),(k,j),}{}_{i,j\in I}, \Lambda_{(k,h)(k,i)(k,j),}{}_{h,i,j\in I}\bigr)$$

$$\text{with} \qquad \Phi_{(k,i)(k,j)} = (P_{(k,i)(k,j)}, \pi_{(k,i)(k,j)}, \phi_{(k,i)(k,j)}, \hat\phi_{(k,i)(k,j)})$$

$$\text{and} \quad \Lambda_{(k,h)(k,i)(k,j)} = [\acute{P}_{(k,h)(k,i)(k,j)}, \lambda_{(k,h)(k,i)(k,j)}, \hat\lambda_{(k,h)(k,i)(k,j)}],$$

where $\mathcal{K}_k$ has indexing set $\{k\}\times I$, and as in (6.18) we write

$$\begin{aligned}
\boldsymbol{\Pi}_k &= \bigl(\Pi_k, \boldsymbol{\Pi}_{(k,i)j,\ i,j\in I},\ \boldsymbol{\Pi}^{j,\ j\in I}_{(k,i)(k,i'),\ i,i'\in I},\ \boldsymbol{\Pi}^{jj',\ j,j'\in I}_{(k,i),\ i\in I}\bigr), \quad \text{where}\\
\boldsymbol{\Pi}_{(k,i)j} &= (P_{(k,i)j}, \pi_{(k,i)j}, \Pi_{(k,i)j}, \hat\Pi_{(k,i)j}) : (V_{(k,i)}, E_{(k,i)}, \Gamma_{(k,i)}, s_{(k,i)}, \psi_{(k,i)})\\
&\qquad\qquad \longrightarrow (V_j, E_j, \Gamma_j, s_j, \psi_j),\\
\boldsymbol{\Pi}^j_{(k,i)(k,i')} &= [\acute{P}^j_{(k,i)(k,i')}, \Pi^j_{(k,i)(k,i')}, \hat\Pi^j_{(k,i)(k,i')}] : \boldsymbol{\Pi}_{(k,i')j}\circ\Phi_{(k,i)(k,i')} \Longrightarrow \boldsymbol{\Pi}_{(k,i)j},\\
\boldsymbol{\Pi}^{jj'}_{(k,i)} &= [\acute{P}^{jj'}_{(k,i)}, \Pi^{jj'}_{(k,i)}, \hat\Pi^{jj'}_{(k,i)}] : \Phi_{jj'}\circ\boldsymbol{\Pi}_{(k,i)j} \Longrightarrow \boldsymbol{\Pi}_{(k,i)j'}.
\end{aligned}$$

As in Definition 4.39, for each $i\in I$, define $V_{(k,i)} = C_k(V_i)$ to be the k-corners of V_i from Assumption 3.22(d). Define $E_{(k,i)} \to V_{(k,i)}$ to be the pullback vector bundle $\Pi_k^*(E_i)$, where $\Pi_k : V_{(k,i)} = C_k(V_i) \to V_i$ is as in Assumption 3.22(d), and let $s_{(k,i)} = \Pi_k^*(s_i)$ in $\Gamma^\infty(E_{(k,i)})$ be the pullback section. These are equivalent to $E_{(k,i)} = C_k(E_i)$, $s_{(k,i)} = C_k(s_i)$, where $s_i : V_i \to E_i$ is simple. Note that

$$\dim V_{(k,i)} - \operatorname{rank} E_{(k,i)} = \dim C_k(V_i) - \operatorname{rank} E_i = \dim V_i - k - \operatorname{rank} E_i = n-k,$$

by Assumption 3.22(d), as required in Definition 6.17(b) for $C_k(\boldsymbol{X})$.

Define a finite group $\Gamma_{(k,i)} = \Gamma_i$. As in Definition 6.1(c), Γ_i acts on V_i by diffeomorphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and we write these as $\rho(\gamma) : V_i \to V_i$ for $\gamma\in\Gamma_i$. Then $\rho(\gamma)$ is simple by Definition 3.18(i) as simple maps are discrete, so Assumption 3.22(i) gives morphisms $C_k\circ\rho(\gamma) : V_{(k,i)} = C_k(V_i) \to V_{(k,i)} = C_k(V_i)$ for $\gamma\in\Gamma_{(k,i)} = \Gamma_i$, and these form a smooth action of $\Gamma_{(k,i)}$ on $V_{(k,i)}$. Similarly the Γ_i-action on E_i lifts to a $\Gamma_{(k,i)}$-action on $E_{(k,i)} = C_k(E_i)$ preserving the vector bundle structure, and $s_{(k,i)} = C_k(s_i) : V_{(k,i)} \to E_{(k,i)}$ is $\Gamma_{(k,i)}$-equivariant as $s_i : V_i \to E_i$ is Γ_i-equivariant. This defines the data $V_{(k,i)}, E_{(k,i)}, \Gamma_{(k,i)}, s_{(k,i)}$ in $(V_{(k,i)}, E_{(k,i)}, \Gamma_{(k,i)}, s_{(k,i)}, \psi_{(k,i)})$, and verifies Definition 6.1(a)–(d).

Let $i,j\in I$. Since simple maps are a discrete property in $\dot{\mathbf{Man}}^{\mathbf{c}}$ by Assumption 3.22(c), Definition 6.31 and Proposition 6.32(d) imply that $\phi_{ij} : P_{ij} \to V_j$ is simple near $(\bar\psi_i\circ\pi_{ij})^{-1}(\operatorname{Im}\psi_j) \subseteq P_{ij}$. Note too that $\pi_{ij} : P_{ij} \to V_i$ is always simple, by Definition 3.18(i),(iv) and discreteness of simple maps, as π_{ij} is étale by Definition 6.2(b). Let $P'_{ij} \subseteq P_{ij}$ be the maximal open set on which ϕ_{ij} is simple, so that $(\bar\psi_i\circ\pi_{ij})^{-1}(\operatorname{Im}\psi_j) \subseteq P'_{ij}$. Write $\pi'_{ij}, \phi'_{ij}, \hat\phi'_{ij}$ for the restrictions of $\pi_{ij}, \phi_{ij}, \hat{\phi}_{ij}$ to P'_{ij}, so π'_{ij}, ϕ'_{ij} are simple. Generalizing (4.40)–(4.43), define

$$\begin{aligned}
P_{(k,i)(k,j)} &= C_k(P'_{ij}),\\
\pi_{(k,i)(k,j)} &= C_k(\pi'_{ij}) : P_{(k,i)(k,j)} = C_k(P'_{ij}) \longrightarrow V_{(k,i)} = C_k(V_i),\\
\phi_{(k,i)(k,j)} &= C_k(\phi'_{ij}) : P_{(k,i)(k,j)} = C_k(P'_{ij}) \longrightarrow V_{(k,j)} = C_k(V_j),\\
\hat{\phi}_{(k,i)(k,j)} &= \Pi_k^*(\hat{\phi}'_{ij}) : \pi^*_{(k,i)(k,j)}(E_{(k,i)}) = C_k(\pi'_{ij})^* \circ \Pi_k^*(E_i) = \Pi_k^* \circ \pi'^*_{ij}(E_i)\\
&\longrightarrow \Pi_k^* \circ \phi'^*_{ij}(E_j) = C_k(\phi'_{ij})^* \circ \Pi_k^*(E_j) = \phi^*_{(k,i)(k,j)}(E_{(k,j)}),\\
P_{(k,i)j} &= C_k(P_{ij}),\\
\pi_{(k,i)j} &= C_k(\pi_{ij}) : P_{(k,i)j} = C_k(P_{ij}) \longrightarrow V_{(k,i)} = C_k(V_i),\\
\Pi_{(k,i)j} &= \phi_{ij} \circ \Pi_k : V_{(k,i)j} = C_k(V_{ij}) \longrightarrow V_j,\\
\hat{\Pi}_{(k,i)j} &= \Pi_k^*(\hat{\phi}_{ij}) : \pi^*_{(k,i)j}(E_{(k,i)}) = C_k(\pi_{ij})^* \circ \Pi_k^*(E_i) = \Pi_k^* \circ \pi^*_{ij}(E_i)\\
&\longrightarrow \Pi_k^* \circ \phi^*_{ij}(E_j) = (\phi_{ij} \circ \Pi_k)^*(E_j) = \Pi^*_{(k,i)j}(E_j).
\end{aligned}$$

This defines $\Phi_{(k,i)(k,j)}$ and $\mathbf{\Pi}_{(k,i)j}$. We can verify Definition 6.2(a)–(e) for $\Phi_{(k,i)(k,j)}, \mathbf{\Pi}_{(k,i)j}$ (except for $\bar{\psi}_i^{-1}(S) \subseteq V_{ij}$ in Definition 6.2(b), as $\psi_{(k,i)}$ is not yet defined) by applying C_k to Definition 6.2(a)–(e) for Φ_{ij} and using Theorem 3.28 as in Definition 4.39.

For $h, i, j \in I$, choose a representative $(\acute{P}_{hij}, \lambda_{hij}, \hat{\lambda}_{hij})$ for the $\sim$-equivalence class $\Lambda_{hij} = [\acute{P}_{hij}, \lambda_{hij}, \hat{\lambda}_{hij}]$ in Definition 6.4. Here $\Lambda_{hij} : \Phi_{ij} \circ \Phi_{hi} \Rightarrow \Phi_{hj}$ is a 2-morphism, where $\Phi_{ij} \circ \Phi_{hi}$ is defined in Definition 6.5. From the definitions, $\acute{P}_{hij} \subseteq (P_{hi} \times_{\phi_{hi}, V_i, \pi_{ij}} P_{ij})/\Gamma_i$ is open, and λ_{hij} maps $\acute{P}_{hij} \to P_{hj}$. Set

$$\acute{P}'_{hij} = \acute{P}_{hij} \cap \left[(P'_{hi} \times_{\phi'_{hi}, V_i, \pi'_{ij}} P'_{ij})/\Gamma_i\right] \cap \lambda^{-1}_{hij}(P'_{hj}).$$

Let $\lambda'_{hij}, \hat{\lambda}'_{hij}$ be the restrictions of $\lambda_{hij}, \hat{\lambda}_{hij}$ to $\acute{P}'_{hij}$. Generalizing (4.44), define

$$\begin{aligned}
&\acute{P}_{(k,h)(k,i)(k,j)} = C_k(\acute{P}'_{hij}) \subseteq (P_{(k,h)(k,i)} \times_{\phi_{(k,h)(k,i)}, V_{(k,i)}, \pi_{(k,i)(k,j)}} P_{(k,i)(k,j)})/\Gamma_{(k,i)}\\
&\quad = (C_k(P'_{hi}) \times_{C_k(\phi'_{hi}), C_k(V_i), C_k(\pi'_{ij})} C_k(P'_{ij}))/\Gamma_i = C_k\bigl((P'_{hi} \times_{\phi'_{hi}, V_i, \pi'_{ij}} P'_{ij})/\Gamma_i\bigr),
\end{aligned}$$

where as ϕ'_{ij}, π'_{ij} are simple with π'_{ij} étale, the corner functor C_k commutes with the fibre products and group quotients. Generalizing (4.45), define

$$\begin{aligned}
&\lambda_{(k,h)(k,i)(k,j)} = C_k(\lambda'_{hij}) : \acute{P}_{(k,h)(k,i)(k,j)} = C_k(\acute{P}'_{hij}) \longrightarrow P_{(k,h)(k,j)} = C_k(P'_{hj}),\\
&\hat{\lambda}_{(k,h)(k,i)(k,j)} = \Pi_k^\diamond(\hat{\lambda}'_{hij}) : \pi^*_{V_{(k,h)}}(E_{(k,h)}) = \pi^*_{V_{(k,h)}} \circ \Pi_k^*(E_h) = \Pi_k^* \circ \pi^*_{V_h}(E_h)\\
&\quad \longrightarrow \mathcal{T}_{\phi_{(k,i)(k,j)} \circ \pi_{P_{(k,i)(k,j)}}/\Gamma_{(k,i)}} V_{(k,j)} = \mathcal{T}_{C_k(\phi'_{ij} \circ \pi_{P'_{ij}}/\Gamma_i)} C_k(V_j).
\end{aligned}$$

We check $(\acute{P}_{(k,h)(k,i)(k,j)}, \lambda_{(k,h)(k,i)(k,j)}, \hat{\lambda}_{(k,h)(k,i)(k,j)})$ satisfies Definition 6.4(a)–(c) (except for $\pi_{ij}^{-1}(\bar{\psi}_i^{-1}(S)) \subseteq \acute{P}_{ij}$ in (a), as $\psi_{(k,i)}$ is not yet defined) by applying C_k to Definition 6.4(a)–(c) for $(\acute{P}_{hij}, \lambda_{hij}, \hat{\lambda}_{hij})$ and using Theorem 3.28 as in Definition 4.39.

Write $\Lambda_{(k,h)(k,i)(k,j)} = [\acute{P}_{(k,h)(k,i)(k,j)}, \lambda_{(k,h)(k,i)(k,j)}, \hat{\lambda}_{(k,h)(k,i)(k,j)}]$ for the $\sim$-equivalence class of $(\acute{P}_{(k,h)(k,i)(k,j)}, \lambda_{(k,h)(k,i)(k,j)}, \hat{\lambda}_{(k,h)(k,i)(k,j)})$, as in Definition 6.4. Theorem 3.28(ii) implies that equivalence $\sim$ on triples $(\acute{P}_{hij}, \lambda_{hij}, \hat{\lambda}_{hij})$ lifts to $\sim$ on triples $(\acute{P}_{(k,h)(k,i)(k,j)}, \lambda_{(k,h)(k,i)(k,j)}, \hat{\lambda}_{(k,h)(k,i)(k,j)})$, so $\Lambda_{(k,h)(k,i)(k,j)}$ depends only on $\Lambda_{hij} = [\acute{P}_{hij}, \lambda_{hij}, \hat{\lambda}_{hij}]$, and (once we define $C_k(X), \psi_{(k,i)}$ and verify the $\Phi_{(k,i)(k,j)}$ are 1-morphisms), we have a well defined 2-morphism of Kuranishi neighbourhoods

$$\Lambda_{(k,h)(k,i)(k,j)} : \Phi_{(k,i)(k,j)} \circ \Phi_{(k,h)(k,i)} \Longrightarrow \Phi_{(k,h)(k,j)}.$$

We define the 2-morphisms $\mathbf{\Pi}^j_{(k,i)(k,i')}, \mathbf{\Pi}^{jj'}_{(k,i)}$ in $\mathbf{\Pi}_k$ by generalizing the m-Kuranishi case in Definition 4.39 as for $\Lambda_{(k,h)(k,i)(k,j)}$ above.

It remains to define the topological space $C_k(X)$ and the continuous maps $\psi_{(k,i)} : s^{-1}_{(k,i)}(0)/\Gamma_{(k,i)} \to C_k(X)$, $\Pi_k : C_k(X) \to X$. Define a binary relation $\approx$ on $\coprod_{i\in I} s^{-1}_{(k,i)}(0)/\Gamma_{(k,i)}$ by $v_i\Gamma_{(k,i)} \approx v_j\Gamma_{(k,j)}$ if $v_i \in s^{-1}_{(k,i)}(0)$, $v_j \in s^{-1}_{(k,j)}(0)$ for $i, j \in I$ and there exists $p_{ij} \in P_{(k,i)(k,j)}$ with $\pi_{(k,i)(k,j)}(p_{ij}) = v_i$ and $\phi_{(k,i)(k,j)}(p_{ij}) = v_j$. We can prove that $\approx$ is an equivalence relation on $\coprod_{i\in I} s^{-1}_{(k,i)}(0)/\Gamma_{(k,i)}$ by generalizing the proof in Definition 4.39, using the 2-morphism $\Lambda_{(k,h)(k,i)(k,j)}$ above to show that $v_h\Gamma_{(k,h)} \approx v_i\Gamma_{(k,i)}$ and $v_i\Gamma_{(k,i)} \approx v_j\Gamma_{(k,j)}$ imply that $v_h\Gamma_{(k,h)} \approx v_j\Gamma_{(k,j)}$.

Generalizing (4.49), define $C_k(X)$ to be the topological space

$$C_k(X) = \left[\textstyle\coprod_{i\in I} s^{-1}_{(k,i)}(0)/\Gamma_{(k,i)}\right]/\approx,$$

with the quotient topology. For each $i \in I$ define $\psi_{(k,i)} : s^{-1}_{(k,i)}(0)/\Gamma_{(k,i)} \to C_k(X)$ by $\psi_{(k,i)} : v_i\Gamma_{(k,i)} \mapsto [v_i\Gamma_{(k,i)}]$, where $[v_i\Gamma_{(k,i)}]$ is the $\approx$-equivalence class of $v_i\Gamma_{(k,i)}$. Define $\Pi_k : C_k(X) \to X$ by $\Pi_k([v_i\Gamma_{(k,i)}]) = \bar{\psi}_i \circ \Pi_k(v_i)$ for $i \in I$ and $v_i \in s^{-1}_{(k,i)}(0)$, so that $\Pi_k(v_i) \in s_i^{-1}(0)$ and $\bar{\psi}_i \circ \Pi_k(v_i) \in X$.

We can show as in Definition 4.39 that $C_k(X)$ is Hausdorff and second countable, and $\Pi_k : C_k(X) \to X$ is well defined, continuous and proper with finite fibres, and $(V_{(k,i)}, E_{(k,i)}, \Gamma_{(k,i)}, s_{(k,i)}, \psi_{(k,i)})$ is a Kuranishi neighbourhood on $C_k(X)$ for $i \in I$.

For all of Definition 6.17(a)–(h) for $C_k(\boldsymbol{X})$, either we have proved them above, or they follow from Definition 6.17(a)–(h) for $\boldsymbol{X}$ by pulling back by Π_k and using Theorems 3.27–3.28, as in Definition 4.39. Hence $C_k(\boldsymbol{X})$ is a Kuranishi space with corners in $\dot{\mathbf{K}}\mathbf{ur^c}$, with $\operatorname{vdim} C_k(\boldsymbol{X}) = n - k$. Similarly, for Definition 6.19(a)–(h) for $\mathbf{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$, either we have proved them above, or they follow from Definition 6.17 for $\boldsymbol{X}$ using Theorems 3.27–3.28, as in Definition 4.39, where we deduce Definition 6.19(f)–(h) for $\mathbf{\Pi}_k$ from Definition 6.17(h) for $\boldsymbol{X}$. Thus $\mathbf{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ is a 1-morphism in $\dot{\mathbf{K}}\mathbf{ur^c}$.

When $k = 1$ we also write $\partial\boldsymbol{X} = C_1(\boldsymbol{X})$ and call it the *boundary* of $\boldsymbol{X}$, and we write $\boldsymbol{i_X} : \partial\boldsymbol{X} \to \boldsymbol{X}$ in place of $\mathbf{\Pi}_1 : C_1(\boldsymbol{X}) \to \boldsymbol{X}$.

This proves the analogue of Theorem 4.40:

Theorem 6.38. *For each $\boldsymbol{X}$ in $\dot{\mathbf{K}}\mathbf{ur^c}$ and $k = 0, 1, \ldots$ we have defined the k-**corners** $C_k(\boldsymbol{X})$, an object in $\dot{\mathbf{K}}\mathbf{ur^c}$ with $\operatorname{vdim} C_k(\boldsymbol{X}) = \operatorname{vdim} \boldsymbol{X} - k$, and a 1-morphism $\mathbf{\Pi}_k : C_k(\boldsymbol{X}) \to \boldsymbol{X}$ in $\dot{\mathbf{K}}\mathbf{ur^c}$, whose underlying continuous map $\Pi_k : C_k(X) \to X$ is proper with finite fibres. We also write $\partial \boldsymbol{X} = C_1(\boldsymbol{X})$, called the **boundary** of $\boldsymbol{X}$, and we write $\boldsymbol{i_X} = \mathbf{\Pi}_1 : \partial \boldsymbol{X} \to \boldsymbol{X}$.*

Definition 6.37 is similar to Fukaya, Oh, Ohta and Ono [24, Def. A1.30] for FOOO Kuranishi spaces — see §7.1 for more details.

Modifying Definition 4.42 we construct weak 2-categories $\check{\dot{\mathbf{K}}}\mathbf{ur^c_{si}} \subseteq \check{\dot{\mathbf{K}}}\mathbf{ur^c}$ from $\dot{\mathbf{K}}\mathbf{ur^c_{si}} \subseteq \dot{\mathbf{K}}\mathbf{ur^c}$ in the obvious way, with objects $\coprod_{n \in \mathbb{Z}} \boldsymbol{X}_n$ for $\boldsymbol{X}_n \in \dot{\mathbf{K}}\mathbf{ur^c}$ with $\operatorname{vdim} \boldsymbol{X}_n = n$, where $\dot{\mathbf{K}}\mathbf{ur^c_{si}}, \dot{\mathbf{K}}\mathbf{ur^c}$ embed as full 2-subcategories of $\check{\dot{\mathbf{K}}}\mathbf{ur^c_{si}}$ and $\check{\dot{\mathbf{K}}}\mathbf{ur^c}$. For the examples of $\dot{\mathbf{K}}\mathbf{ur^c_{si}} \subseteq \dot{\mathbf{K}}\mathbf{ur^c}$ in Definitions 6.29 and 6.35 we use the obvious notation for the corresponding 2-categories $\check{\dot{\mathbf{K}}}\mathbf{ur^c_{si}} \subseteq \check{\dot{\mathbf{K}}}\mathbf{ur^c}$, so for instance we enlarge $\mathbf{Kur^c}$ associated to $\dot{\mathbf{M}}\mathbf{an^c} = \mathbf{Man^c}$ to $\check{\mathbf{K}}\mathbf{ur^c}$.

Then following Definition 4.43, but modifying it as in Definition 6.37, we define the corner 2-functor $C : \dot{\mathbf{K}}\mathbf{ur^c} \to \check{\dot{\mathbf{K}}}\mathbf{ur^c}$. This is straightforward and involves no new ideas, so we leave it as an exercise for the reader. This proves the analogue of Theorem 4.44:

Theorem 6.39. *We can define a weak 2-functor $C : \dot{\mathbf{K}}\mathbf{ur^c} \to \check{\dot{\mathbf{K}}}\mathbf{ur^c}$ called the **corner 2-functor**. It acts on objects $\boldsymbol{X}$ in $\dot{\mathbf{K}}\mathbf{ur^c}$ by $C(\boldsymbol{X}) = \coprod_{k=0}^{\infty} C_k(\boldsymbol{X})$. If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is simple then $C(\boldsymbol{f}) : C(\boldsymbol{X}) \to C(\boldsymbol{Y})$ is simple and maps $C_k(\boldsymbol{X}) \to C_k(\boldsymbol{Y})$ for $k = 0, 1, \ldots$. Thus $C|_{\dot{\mathbf{K}}\mathbf{ur^c_{si}}}$ decomposes as $C|_{\dot{\mathbf{K}}\mathbf{ur^c_{si}}} = \coprod_{k=0}^{\infty} C_k$, where $C_k : \dot{\mathbf{K}}\mathbf{ur^c_{si}} \to \dot{\mathbf{K}}\mathbf{ur^c_{si}}$ is a weak 2-functor acting on objects by $\boldsymbol{X} \mapsto C_k(\boldsymbol{X})$, for $C_k(\boldsymbol{X})$ as in Definition 6.37. We also write $\partial = C_1 : \dot{\mathbf{K}}\mathbf{ur^c_{si}} \to \dot{\mathbf{K}}\mathbf{ur^c_{si}}$, and call it the **boundary 2-functor**.*

If for some discrete property $\boldsymbol{P}$ of morphisms in $\dot{\mathbf{M}}\mathbf{an^c}$ the corner functor $C : \dot{\mathbf{M}}\mathbf{an^c} \to \check{\dot{\mathbf{M}}}\mathbf{an^c}$ maps to the subcategory $\check{\dot{\mathbf{M}}}\mathbf{an^c_{\boldsymbol{P}}}$ of $\check{\dot{\mathbf{M}}}\mathbf{an^c}$ whose morphisms are $\boldsymbol{P}$, then $C : \dot{\mathbf{K}}\mathbf{ur^c} \to \check{\dot{\mathbf{K}}}\mathbf{ur^c}$ maps to the 2-subcategory $\check{\dot{\mathbf{K}}}\mathbf{ur^c_{\boldsymbol{P}}}$ of $\check{\dot{\mathbf{K}}}\mathbf{ur^c}$ whose 1-morphisms are $\boldsymbol{P}$.

As for Example 4.45, applying Theorem 6.39 to the data $\dot{\mathbf{M}}\mathbf{an^c}, \ldots$ in Example 3.24(a)–(h) gives corner functors:

$$
\begin{aligned}
&C : \mathbf{Kur^c} \longrightarrow \check{\mathbf{K}}\mathbf{ur^c_{in}} \subset \check{\mathbf{K}}\mathbf{ur^c}, && C' : \mathbf{Kur^c} \longrightarrow \check{\mathbf{K}}\mathbf{ur^c}, \\
&C : \mathbf{Kur^c_{st}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^c_{st,in}} \subset \check{\mathbf{K}}\mathbf{ur^c_{st}}, && C' : \mathbf{Kur^c_{st}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^c_{st}}, \\
&C : \mathbf{Kur^{ac}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^{ac}_{in}} \subset \check{\mathbf{K}}\mathbf{ur^{ac}}, && C' : \mathbf{Kur^{ac}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^{ac}}, \\
&C : \mathbf{Kur^{ac}_{st}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^{ac}_{st,in}} \subset \check{\mathbf{K}}\mathbf{ur^{ac}_{st}}, && C' : \mathbf{Kur^{ac}_{st}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^{ac}_{st}}, \\
&C : \mathbf{Kur^{c,ac}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^{c,ac}_{in}} \subset \check{\mathbf{K}}\mathbf{ur^{c,ac}}, && C' : \mathbf{Kur^{c,ac}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^{c,ac}}, \\
&C : \mathbf{Kur^{c,ac}_{st}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^{c,ac}_{st,in}} \subset \check{\mathbf{K}}\mathbf{ur^{c,ac}_{st}}, && C' : \mathbf{Kur^{c,ac}_{st}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^{c,ac}_{st}}, \\
&C : \mathbf{Kur^{gc}} \longrightarrow \check{\mathbf{K}}\mathbf{ur^{gc}_{in}} \subset \check{\mathbf{K}}\mathbf{ur^{gc}}.
\end{aligned}
\tag{6.36}
$$

As for Propositions 4.46 and 4.47, we prove:

Proposition 6.40. *For all of the 2-functors C in* (6.36) *(though not the 2-functors C'), a 1-morphism $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is interior (or b-normal) if and only if $C(\boldsymbol{f})$ maps $C_0(\boldsymbol{X})\to C_0(\boldsymbol{Y})$ (or $C(\boldsymbol{f})$ maps $C_k(\boldsymbol{X})\to\coprod_{l=0}^k C_l(\boldsymbol{Y})$ for all $k=0,1,\ldots,$ respectively).*

Proposition 6.41. *Let $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ be an equivalence in $\dot{\mathbf{Kur}}^{\mathbf{c}}$. Then $\boldsymbol{f}$ is simple by Proposition* 6.34(c)*, and $C_k(\boldsymbol{f}):C_k(\boldsymbol{X})\to C_k(\boldsymbol{Y})$ for $k=0,1,\ldots$ and $\partial\boldsymbol{f}:\partial\boldsymbol{X}\to\partial\boldsymbol{Y}$ are also equivalences in $\dot{\mathbf{Kur}}^{\mathbf{c}}$.*

6.4 Kuranishi neighbourhoods on Kuranishi spaces

In §4.7 we discussed 'm-Kuranishi neighbourhoods on m-Kuranishi spaces', and in §5.5 we explained the μ-Kuranishi analogue. Now we define 'Kuranishi neighbourhoods on Kuranishi spaces'. We follow §4.7 closely, with the difference that m-Kuranishi neighbourhoods in §4.1 are a strict 2-category, but Kuranishi neighbourhoods in §6.1 are a weak 2-category. So we cannot omit brackets in compositions of 1-morphisms such as $(\Phi_{jk}\circ\Phi_{ij})\circ\Phi_{ai}$ in (6.37), and we have to insert extra coherence 2-morphisms $\boldsymbol{\alpha}_{*,*,*},\boldsymbol{\beta}_*,\boldsymbol{\gamma}_*$ from (6.7)–(6.8) throughout.

Definition 6.42. Suppose $\boldsymbol{X}=(X,\mathcal{K})$ is a Kuranishi space, where $\mathcal{K}=\bigl(I,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \Lambda_{ijk,\ i,j,k\in I}\bigr)$. A *Kuranishi neighbourhood on the Kuranishi space $\boldsymbol{X}$* is data $(V_a,E_a,\Gamma_a,s_a,\psi_a)$, $\Phi_{ai,\ i\in I}$ and $\Lambda_{aij,\ i,j\in I}$ where $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ is a Kuranishi neighbourhood on the topological space X in the sense of Definition 6.1, and $\Phi_{ai}:(V_a,E_a,\Gamma_a,s_a,\psi_a)\to(V_i,E_i,\Gamma_i,s_i,\psi_i)$ is a coordinate change for each $i\in I$ (over $S=\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_i$, as usual) as in Definition 6.11, and $\Lambda_{aij}:\Phi_{ij}\circ\Phi_{ai}\Rightarrow\Phi_{aj}$ is a 2-morphism (over $S=\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j$, as usual) as in Definition 6.4 for all $i,j\in I$, such that $\Lambda_{aii}=\mathrm{id}_{\Phi_{ai}}$ for all $i\in I$, and as in Definition 6.17(h), for all $i,j,k\in I$ we have

$$\begin{aligned}\Lambda_{ajk}\odot(\mathrm{id}_{\Phi_{jk}}*\Lambda_{aij})\odot\boldsymbol{\alpha}_{\Phi_{jk},\Phi_{ij},\Phi_{ai}}=\Lambda_{aik}\odot(\Lambda_{ijk}*\mathrm{id}_{\Phi_{ai}}):\\ (\Phi_{jk}\circ\Phi_{ij})\circ\Phi_{ai}\Longrightarrow\Phi_{ak},\end{aligned}\tag{6.37}$$

where (6.37) holds over $S=\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\cap\operatorname{Im}\psi_k$ by Convention 6.14.

Here the subscript 'a' in $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ is just a label used to distinguish Kuranishi neighbourhoods, generally not in I. If we omit a we will write '$*$' in place of 'a' in Φ_{ai},Λ_{aij}, giving $\Phi_{*i}:(V,E,\Gamma,s,\psi)\to(V_i,E_i,\Gamma_i,s_i,\psi_i)$ and $\Lambda_{*ij}:\Phi_{ij}\circ\Phi_{*i}\Rightarrow\Phi_{*j}$.

We will usually just say $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ or (V,E,Γ,s,ψ) *is a Kuranishi neighbourhood on $\boldsymbol{X}$*, leaving the data Φ_{ai},Λ_{aij} or Φ_{*i},Λ_{*ij} implicit. We call such a (V,E,Γ,s,ψ) a *global* Kuranishi neighbourhood on $\boldsymbol{X}$ if $\operatorname{Im}\psi=X$.

Definition 6.43. Using the same notation, suppose $(V_a,E_a,\Gamma_a,s_a,\psi_a),\Phi_{ai,\ i\in I}$, $\Lambda_{aij,\ i,j\in I}$ and $(V_b,E_b,\Gamma_b,s_b,\psi_b)$, $\Phi_{bi,\ i\in I}$, $\Lambda_{bij,\ i,j\in I}$ are Kuranishi neighbourhoods on $\boldsymbol{X}$, and $S\subseteq\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b$ is open. A *coordinate change from $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ to $(V_b,E_b,\Gamma_b,s_b,\psi_b)$ over S on the Kuranishi space $\boldsymbol{X}$* is data Φ_{ab}, $\Lambda_{abi,\ i\in I}$, where $\Phi_{ab}:(V_a,E_a,\Gamma_a,s_a,\psi_a)\to(V_b,E_b,\Gamma_b,s_b,\psi_b)$ is a coordinate change over S as in Definition 6.11, and $\Lambda_{abi}:\Phi_{bi}\circ\Phi_{ab}\Rightarrow\Phi_{ai}$ is a

2-morphism over $S\cap\operatorname{Im}\psi_i$ as in Definition 6.4 for each $i\in I$, such that for all $i,j\in I$ we have

$$\begin{aligned}\Lambda_{aij}\odot(\mathrm{id}_{\Phi_{ij}}*\Lambda_{abi})\odot\boldsymbol{\alpha}_{\Phi_{ij},\Phi_{bi},\Phi_{ab}}=\Lambda_{abj}\odot(\Lambda_{bij}*\mathrm{id}_{\Phi_{ab}}):\\ (\Phi_{ij}\circ\Phi_{bi})\circ\Phi_{ab}\Longrightarrow\Phi_{aj},\end{aligned}\tag{6.38}$$

where (6.38) holds over $S\cap\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j$.

We will usually just say that $\Phi_{ab}:(V_a,E_a,\Gamma_a,s_a,\psi_a)\to(V_b,E_b,\Gamma_b,s_b,\psi_b)$ *is a coordinate change over S on $\boldsymbol{X}$*, leaving the data $\Lambda_{abi,\,i\in I}$ implicit. If we do not specify S, we mean that S is as large as possible, that is, $S=\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b$.

Suppose $\Phi_{ab}:(V_a,E_a,\Gamma_a,s_a,\psi_a)\to(V_b,E_b,\Gamma_b,s_b,\psi_b)$, $\Lambda_{abi,\,i\in I}$ and $\Phi_{bc}:(V_b,E_b,\Gamma_b,s_b,\psi_b)\to(V_c,E_c,\Gamma_c,s_c,\psi_c)$, $\Lambda_{bci,\,i\in I}$ are such coordinate changes over $S\subseteq\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b\cap\operatorname{Im}\psi_c$. Define $\Phi_{ac}=\Phi_{bc}\circ\Phi_{ab}:(V_a,E_a,\Gamma_a,s_a,\psi_a)\to(V_c,E_c,\Gamma_c,s_c,\psi_c)$ and $\Lambda_{aci}=\Lambda_{abi}\odot(\Lambda_{bci}*\mathrm{id}_{\Phi_{ab}})\odot\boldsymbol{\alpha}^{-1}_{\Phi_{ci},\Phi_{bc},\Phi_{ab}}:\Phi_{ci}\circ\Phi_{ac}\Rightarrow\Phi_{ai}$ for all $i\in I$. It is easy to show that $\Phi_{ac}=\Phi_{bc}\circ\Phi_{ab}$, $\Lambda_{aci,\,i\in I}$ is a coordinate change from $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ to $(V_c,E_c,\Gamma_c,s_c,\psi_c)$ over S on $\boldsymbol{X}$. We call this *composition of coordinate changes*.

Definition 6.44. Let $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ be a 1-morphism of Kuranishi spaces, and use notation (6.15)–(6.16) for $\boldsymbol{X},\boldsymbol{Y}$, and (6.18) for $\boldsymbol{f}$. Suppose $(U_a,D_a,\mathrm{B}_a,r_a,\chi_a)$, $\mathrm{T}_{ai,\,i\in I}$, $\mathrm{K}_{aii',\,i,i'\in I}$ is a Kuranishi neighbourhood on $\boldsymbol{X}$, and $(V_b,E_b,\Gamma_b,s_b,\psi_b)$, $\Upsilon_{bj,\,j\in J}$, $\Lambda_{bjj',\,j,j'\in J}$ a Kuranishi neighbourhood on $\boldsymbol{Y}$, as in Definition 6.42. Let $S\subseteq\operatorname{Im}\chi_a\cap f^{-1}(\operatorname{Im}\psi_b)$ be open. A 1-*morphism from $(U_a,D_a,\mathrm{B}_a,r_a,\chi_a)$ to $(V_b,E_b,\Gamma_b,s_b,\psi_b)$ over $(S,\boldsymbol{f})$ on the Kuranishi spaces $\boldsymbol{X},\boldsymbol{Y}$* is data $\boldsymbol{f}_{ab}$, $\boldsymbol{F}_{ai,\,i\in I}^{bj,\,j\in J}$, where $\boldsymbol{f}_{ab}:(U_a,D_a,\mathrm{B}_a,r_a,\chi_a)\to(V_b,E_b,\Gamma_b,s_b,\psi_b)$ is a 1-morphism of Kuranishi neighbourhoods over (S,f) in the sense of Definition 6.2, and $\boldsymbol{F}_{ai}^{bj}:\Upsilon_{bj}\circ\boldsymbol{f}_{ab}\Rightarrow\boldsymbol{f}_{ij}\circ\mathrm{T}_{ai}$ is a 2-morphism over $S\cap\operatorname{Im}\chi_i\cap f^{-1}(\operatorname{Im}\psi_j),f$ as in Definition 6.4 for all $i\in I$, $j\in J$, such that for all $i,i'\in I$, $j,j'\in J$ we have

$$\begin{aligned}(\boldsymbol{F}_{ai}^{bj})^{-1}\odot(\boldsymbol{F}_{ii'}^{j}*\mathrm{id}_{\mathrm{T}_{ai}})=(\boldsymbol{F}_{ai'}^{bj})^{-1}\odot(\mathrm{id}_{\boldsymbol{f}_{i'j}}*\mathrm{K}_{aii'})\odot\boldsymbol{\alpha}_{\boldsymbol{f}_{i'j},\mathrm{T}_{ii'},\mathrm{T}_{ai}}:\\ (\boldsymbol{f}_{i'j}\circ\mathrm{T}_{ii'})\circ\mathrm{T}_{ai}\Longrightarrow\Upsilon_{bj}\circ\boldsymbol{f}_{ab},\\ \boldsymbol{F}_{ai}^{bj'}\odot(\Lambda_{bjj'}*\mathrm{id}_{\boldsymbol{f}_{ab}})=(\boldsymbol{F}_{i}^{jj'}*\mathrm{id}_{\mathrm{T}_{ai}})\odot(\mathrm{id}_{\Upsilon_{jj'}}*\boldsymbol{F}_{ai}^{bj})\odot\boldsymbol{\alpha}_{\Upsilon_{jj'},\Upsilon_{bj},\boldsymbol{f}_{ab}}:\\ (\Upsilon_{jj'}\circ\Upsilon_{bj})\circ\boldsymbol{f}_{ab}\Longrightarrow\boldsymbol{f}_{ij'}\circ\mathrm{T}_{ai}.\end{aligned}$$

We will usually just say that $\boldsymbol{f}_{ab}:(U_a,D_a,\mathrm{B}_a,r_a,\chi_a)\to(V_b,E_b,\Gamma_b,s_b,\psi_b)$ *is a* 1-*morphism of Kuranishi neighbourhoods over $(S,\boldsymbol{f})$ on $\boldsymbol{X},\boldsymbol{Y}$*, leaving the data $\boldsymbol{F}_{ai,\,i\in I}^{bj,\,j\in J}$ implicit.

Suppose $\boldsymbol{g}:\boldsymbol{Y}\to\boldsymbol{Z}$ is another 1-morphism of Kuranishi spaces, using notation (6.17) for $\boldsymbol{Z}$, and $(W_c,F_c,\Delta_c,t_c,\omega_c)$ is a Kuranishi neighbourhood on $\boldsymbol{Z}$, and $T\subseteq\operatorname{Im}\psi_b\cap g^{-1}(\operatorname{Im}\omega_c)$, $S\subseteq\operatorname{Im}\chi_a\cap f^{-1}(T)$ are open, $\boldsymbol{f}_{ab}:(U_a,D_a,\mathrm{B}_a,r_a,\chi_a)\to(V_b,E_b,\Gamma_b,s_b,\psi_b)$ is a 1-morphism of Kuranishi neighbourhoods over $(S,\boldsymbol{f})$ on $\boldsymbol{X},\boldsymbol{Y}$, and $\boldsymbol{g}_{bc}:(V_b,E_b,\Gamma_b,s_b,\psi_b)\to(W_c,F_c,\Delta_c,t_c,\omega_c)$ is a 1-morphism of Kuranishi neighbourhoods over $(T,\boldsymbol{g})$ on $\boldsymbol{Y},\boldsymbol{Z}$.

Define $\boldsymbol{h}=\boldsymbol{g}\circ\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Z}$, so that Definition 6.22 gives 2-morphisms

$$\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}}:\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}\Longrightarrow\boldsymbol{h}_{ik}$$

for all $i \in I$, $j \in J$ and $k \in K$. Set $\boldsymbol{h}_{ac} = \boldsymbol{g}_{bc} \circ \boldsymbol{f}_{ab} : (U_a, D_a, \mathrm{B}_a, r_a, \chi_a) \to (W_c, F_c, \Delta_c, t_c, \omega_c)$. Using the stack property Theorem 6.16, one can show that for all $i \in I$, $k \in K$ there is a unique 2-morphism $\boldsymbol{H}_{ai}^{ck} : \Phi_{ck} \circ \boldsymbol{h}_{ac} \Rightarrow \boldsymbol{h}_{ik} \circ \mathrm{T}_{ai}$ over $S \cap \mathrm{Im}\,\chi_i \cap h^{-1}(\mathrm{Im}\,\omega_k), h$, such that for all $j \in J$ we have

$$\begin{aligned}\boldsymbol{H}_{ai}^{ck}|_{S\cap\mathrm{Im}\,\chi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\cap h^{-1}(\mathrm{Im}\,\omega_k)} &= (\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}} * \mathrm{id}_{\mathrm{T}_{ai}}) \odot \boldsymbol{\alpha}_{\boldsymbol{g}_{jk},\boldsymbol{f}_{ij},\mathrm{T}_{ai}}^{-1} \\ &\odot (\mathrm{id}_{\boldsymbol{g}_{jk}} * \boldsymbol{F}_{ai}^{bj}) \odot \boldsymbol{\alpha}_{\boldsymbol{g}_{jk},\Upsilon_{bj},\boldsymbol{f}_{ab}} \odot (\boldsymbol{G}_{bj}^{ck} * \mathrm{id}_{\boldsymbol{f}_{ab}}) \odot \boldsymbol{\alpha}_{\Phi_{ck},\boldsymbol{g}_{bc},\boldsymbol{f}_{ab}}^{-1}.\end{aligned}$$

It is then easy to prove that $\boldsymbol{h}_{ac} = \boldsymbol{g}_{bc} \circ \boldsymbol{f}_{ab}$, $\boldsymbol{H}_{ai,\, i\in I}^{ck,\, k\in K}$ is a 1-morphism from $(U_a, D_a, \mathrm{B}_a, r_a, \chi_a)$ to $(W_c, F_c, \Delta_c, t_c, \omega_c)$ over $(S, \boldsymbol{h})$ on $\boldsymbol{X}, \boldsymbol{Z}$. We call this *composition of* 1*-morphisms.*

As for Theorem 4.56, the next theorem can be proved using the stack property Theorem 6.16, and we leave the proof as an exercise for the reader.

Theorem 6.45. (a) *Let $\boldsymbol{X} = (X, \mathcal{K})$ be a Kuranishi space, where $\mathcal{K} = \big(I, (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i\in I},\ \Phi_{ij},\ \Lambda_{ijk}\big)$, and $(V_a, E_a, \Gamma_a, s_a, \psi_a)$, $(V_b, E_b, \Gamma_b, s_b, \psi_b)$ be Kuranishi neighbourhoods on $\boldsymbol{X}$, in the sense of Definition* 6.42, *and $S \subseteq \mathrm{Im}\,\psi_a \cap \mathrm{Im}\,\psi_b$ be open. Then there exists a coordinate change $\Phi_{ab} : (V_a, E_a, \Gamma_a, s_a, \psi_a) \to (V_b, E_b, \Gamma_b, s_b, \psi_b), \Lambda_{abi,\, i\in I}$ over S on $\boldsymbol{X}$, in the sense of Definition* 6.43. *If $\Phi_{ab}, \tilde\Phi_{ab}$ are two such coordinate changes, there is a unique* 2*-morphism $\Xi_{ab} : \Phi_{ab} \Rightarrow \tilde\Phi_{ab}$ over S as in Definition* 6.4, *such that for all $i \in I$ we have*

$$\Lambda_{abi} = \tilde\Lambda_{abi} \odot (\mathrm{id}_{\Phi_{bi}} * \Xi_{ab}) : \Phi_{bi} \circ \Phi_{ab} \Longrightarrow \Phi_{ai}, \tag{6.39}$$

which holds over $S \cap \mathrm{Im}\,\psi_i$ by our usual convention.

(b) *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a* 1*-morphism of Kuranishi spaces, and use notation* (6.15), (6.16), (6.18). *Let $(U_a, D_a, \mathrm{B}_a, r_a, \chi_a)$, $(V_b, E_b, \Gamma_b, s_b, \psi_b)$ be Kuranishi neighbourhoods on $\boldsymbol{X}, \boldsymbol{Y}$ respectively in the sense of Definition* 6.42, *and let $S \subseteq \mathrm{Im}\,\chi_a \cap f^{-1}(\mathrm{Im}\,\psi_b)$ be open. Then there exists a* 1*-morphism $\boldsymbol{f}_{ab} : (U_a, D_a, \mathrm{B}_a, r_a, \chi_a) \to (V_b, E_b, \Gamma_b, s_b, \psi_b)$ of Kuranishi neighbourhoods over $(S, \boldsymbol{f})$ on $\boldsymbol{X}, \boldsymbol{Y}$, in the sense of Definition* 6.44.

(c) *Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be* 1*-morphisms of Kuranishi spaces and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a* 2*-morphism, and use notation* (6.15), (6.16), (6.18) *and $\boldsymbol{\eta} = \big(\boldsymbol{\eta}_{ij,\, i\in I,\, j\in J}\big)$. Suppose $(U_a, D_a, \mathrm{B}_a, r_a, \chi_a)$, $(V_b, E_b, \Gamma_b, s_b, \psi_b)$ are Kuranishi neighbourhoods on $\boldsymbol{X}, \boldsymbol{Y}$, and $S \subseteq \mathrm{Im}\,\chi_a \cap f^{-1}(\mathrm{Im}\,\psi_b)$ is open, and $\boldsymbol{f}_{ab}, \boldsymbol{g}_{ab} : (U_a, D_a, \mathrm{B}_a, r_a, \chi_a) \to (V_b, E_b, \Gamma_b, s_b, \psi_b)$ are* 1*-morphisms over $(S, \boldsymbol{f}), (S, \boldsymbol{g})$. Then there is a unique* 2*-morphism $\boldsymbol{\eta}_{ab} : \boldsymbol{f}_{ab} \Rightarrow \boldsymbol{g}_{ab}$ over (S, f) as in Definition* 6.4, *such that the following commutes over $S \cap \mathrm{Im}\,\chi_i \cap f^{-1}(\mathrm{Im}\,\psi_j)$ for all $i \in I$ and $j \in J$:*

$$\begin{tikzcd} \Upsilon_{bj} \circ \boldsymbol{f}_{ab} \arrow[r, Rightarrow, "\boldsymbol{F}_{ai}^{bj}"] \arrow[d, Rightarrow, "\mathrm{id}_{\Upsilon_{bj}} * \boldsymbol{\eta}_{ab}"] & \boldsymbol{f}_{ij} \circ \mathrm{T}_{ai} \arrow[d, Rightarrow, "\boldsymbol{\eta}_{ij} * \mathrm{id}_{\mathrm{T}_{ai}}"] \\ \Upsilon_{bj} \circ \boldsymbol{g}_{ab} \arrow[r, Rightarrow, "\boldsymbol{G}_{ai}^{bj}"] & \boldsymbol{g}_{ij} \circ \mathrm{T}_{ai}. \end{tikzcd}$$

(d) *The unique* 2*-morphisms in* **(c)** *are compatible with vertical and horizontal composition and identities. For example, if $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Y}$ are* 1*-morphisms in*

$\dot{\mathbf{Kur}}$, *and* $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$, $\boldsymbol{\zeta}:\boldsymbol{g}\Rightarrow\boldsymbol{h}$ *are* 2*-morphisms with* $\boldsymbol{\theta}=\boldsymbol{\zeta}\odot\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{h}$, *and* $(U_a,D_a,\mathrm{B}_a,r_a,\chi_a)$, $(V_b,E_b,\Gamma_b,s_b,\psi_b)$ *are Kuranishi neighbourhoods on* $\boldsymbol{X},\boldsymbol{Y}$, *and* $\boldsymbol{f}_{ab},\boldsymbol{g}_{ab},\boldsymbol{h}_{ab}:(U_a,D_a,\mathrm{B}_a,r_a,\chi_a)\rightarrow(V_b,E_b,\Gamma_b,s_b,\psi_b)$ *are* 1*-morphisms over* $(S,\boldsymbol{f})$, $(S,\boldsymbol{g})$, $(S,\boldsymbol{h})$, *and* $\boldsymbol{\eta}_{ab}:\boldsymbol{f}_{ab}\Rightarrow\boldsymbol{g}_{ab}$, $\boldsymbol{\zeta}_{ab}:\boldsymbol{g}_{ab}\Rightarrow\boldsymbol{h}_{ab}$, $\boldsymbol{\theta}_{ab}:\boldsymbol{f}_{ab}\Rightarrow\boldsymbol{h}_{ab}$ *come from* $\boldsymbol{\eta},\boldsymbol{\zeta},\boldsymbol{\theta}$ *as in* **(c)**, *then* $\boldsymbol{\theta}_{ab}=\boldsymbol{\zeta}_{ab}\odot\boldsymbol{\eta}_{ab}$.

Remark 6.46. Note that we make the (potentially confusing) distinction between *Kuranishi neighbourhoods* $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ *on a topological space* X, as in Definition 6.1, and *Kuranishi neighbourhoods* $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ *on a Kuranishi space* $\boldsymbol{X}=(X,\mathcal{K})$, which are as in Definition 6.42, and come equipped with the extra implicit data $\Phi_{ai,\;i\in I}$, $\Lambda_{aij,\;i,j\in I}$ giving the compatibility with the Kuranishi structure $\mathcal{K}$ on X. Similarly, we distinguish between coordinate changes of Kuranishi neighbourhoods over X or $\boldsymbol{X}$, and between 1-morphisms of Kuranishi neighbourhoods over $f:X\rightarrow Y$ or $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$.

Here are the analogues of Theorem 4.58 and Corollary 4.59. They are proved in the same way, but extending from strict to weak 2-categories.

Theorem 6.47. *Let* $\boldsymbol{X}=(X,\mathcal{K})$ *be a Kuranishi space, and* $\bigl\{(V_a,E_a,\Gamma_a,s_a,\psi_a):a\in A\bigr\}$ *a family of Kuranishi neighbourhoods on* $\boldsymbol{X}$ *with* $X=\bigcup_{a\in A}\operatorname{Im}\psi_a$. *For all* $a,b\in A$, *let* $\Phi_{ab}:(V_a,E_a,\Gamma_a,s_a,\psi_a)\rightarrow(V_b,E_b,\Gamma_b,s_b,\psi_b)$ *be a coordinate change over* $S=\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b$ *on* $\boldsymbol{X}$ *given by Theorem* 6.45(a), *which is unique up to* 2*-isomorphism; when* $a=b$ *we choose* $\Phi_{ab}=\mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)}$ *and* $\Lambda_{aai}=\boldsymbol{\beta}_{\Phi_{ai}}$ *for* $i\in I$, *which is allowed by Theorem* 6.45(a).

For all $a,b,c\in A$, *both* $\Phi_{bc}\circ\Phi_{ab}|_S$ *and* $\Phi_{ac}|_S$ *are coordinate changes* $(V_a,E_a,\Gamma_a,s_a,\psi_a)\rightarrow(V_c,E_c,\Gamma_c,s_c,\psi_c)$ *over* $S=\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b\cap\operatorname{Im}\psi_c$ *on* $\boldsymbol{X}$, *so Theorem* 6.45(a) *gives a unique* 2*-morphism* $\Lambda_{abc}:\Phi_{bc}\circ\Phi_{ab}|_S\Rightarrow\Phi_{ac}|_S$. *Then* $\mathcal{K}'=\bigl(A,(V_a,E_a,\Gamma_a,s_a,\psi_a)_{a\in A},\Phi_{ab,\;a,b\in A},\Lambda_{abc,\;a,b,c\in A}\bigr)$ *is a Kuranishi structure on* X, *and* $\boldsymbol{X}'=(X,\mathcal{K}')$ *is canonically equivalent to* $\boldsymbol{X}$ *in* $\dot{\mathbf{Kur}}$.

Corollary 6.48. *Let* $\boldsymbol{X}=(X,\mathcal{K})$ *be a Kuranishi space with* $\mathcal{K}=\bigl(I,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\Phi_{ij,\;i,j\in I},\Lambda_{ijk,\;i,j,k\in I}\bigr)$. *Suppose* $J\subseteq I$ *with* $\bigcup_{j\in J}\operatorname{Im}\psi_j=X$. *Then* $\mathcal{K}'=\bigl(J,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in J},\Phi_{ij,\;i,j\in J},\Lambda_{ijk,\;i,j,k\in J}\bigr)$ *is a Kuranishi structure on* X, *and* $\boldsymbol{X}'=(X,\mathcal{K}')$ *is canonically equivalent to* $\boldsymbol{X}$ *in* $\dot{\mathbf{Kur}}$.

As in §4.7.3, if $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumption 3.22 then we can lift Kuranishi neighbourhoods $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ on $\boldsymbol{X}$ in $\dot{\mathbf{Kur}}^{\mathbf{c}}$ to Kuranishi neighbourhoods $(V_{(k,a)},E_{(k,a)},\Gamma_{(k,a)},s_{(k,a)},\psi_{(k,a)})$ on $C_k(\boldsymbol{X})$ from §6.3, with $\Gamma_{(k,a)}=\Gamma_a$, and we can lift 1-morphisms $\boldsymbol{f}_{ab}:(U_a,D_a,\mathrm{B}_a,r_a,\chi_a)\rightarrow(V_b,E_b,\Gamma_b,s_b,\psi_b)$ of Kuranishi neighbourhoods over $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ in $\dot{\mathbf{Kur}}^{\mathbf{c}}$ to 1-morphisms $\boldsymbol{f}_{(k,a)(l,b)}:(U_{(k,a)},D_{(k,a)},\mathrm{B}_{(k,a)},r_{(k,a)},\chi_{(k,a)})\rightarrow(V_{(l,b)},E_{(l,b)},\Gamma_{(l,b)},s_{(l,b)},\psi_{(l,b)})$ over $C(\boldsymbol{f}):C(\boldsymbol{X})\rightarrow C(\boldsymbol{Y})$. We leave the details to the reader. As in §4.7.4, we could now state our philosophy for working with Kuranishi spaces, but we will not.

6.5 Isotropy groups

Next we discuss *isotropy groups* of Kuranishi spaces (also called *orbifold groups*, or *stabilizer groups*). They are also studied for orbifolds, as in §6.6.

Definition 6.49. Let $\boldsymbol{X}=(X,\mathcal{K})$ be a Kuranishi space, with $\mathcal{K}=\bigl(I,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \Lambda_{ijk,\ i,j,k\in I}\bigr)$, and let $x\in\boldsymbol{X}$. Choose an arbitrary $i\in I$ with $x\in\operatorname{Im}\psi_i$, and choose $v_i\in s_i^{-1}(0)\subseteq V_i$ with $\bar\psi_i(v_i)=x$. Define a finite group $G_x\boldsymbol{X}$ called the *isotropy group of* $\boldsymbol{X}$ *at* x, as a subgroup of Γ_i, by

$$G_x\boldsymbol{X}=\bigl\{\gamma\in\Gamma_i:\gamma\cdot v_i=v_i\bigr\}=\mathrm{Stab}_{\Gamma_i}(v_i). \tag{6.40}$$

We explain to what extent $G_x\boldsymbol{X}$ depends on the arbitrary choice of i,v_i. Let j,v_j be alternative choices, giving another group $G'_x\boldsymbol{X}=\mathrm{Stab}_{\Gamma_j}(v_j)$. Then we have a coordinate change $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$ in $\mathcal{K}$. Consider the set

$$S_x=\bigl\{p\in P_{ij}:\pi_{ij}(p)=v_i,\ \ \phi_{ij}(p)=v_j\bigr\}. \tag{6.41}$$

In Lemma 6.50 below we show that $G_x\boldsymbol{X}$ and $G'_x\boldsymbol{X}$ have natural, commuting, free, transitive actions on S_x. Pick $p\in S_x$. Define an isomorphism of finite groups $I_x^G:G_x\boldsymbol{X}\to G'_x\boldsymbol{X}$ by $I_x^G(\gamma)=\gamma'$ if $\gamma\cdot p=(\gamma')^{-1}\cdot p$ in S_x, using the free, transitive actions of $G_x\boldsymbol{X},G'_x\boldsymbol{X}$ on S_x.

Suppose we instead picked $\tilde p\in S_x$, yielding $\tilde I_x^G:G_x\boldsymbol{X}\to G'_x\boldsymbol{X}$. Since $G'_x\boldsymbol{X}$ acts freely transitively on S_x, there is a unique $\delta\in G'_x\boldsymbol{X}$ with $\delta\cdot p=\tilde p$. Then we see that $\tilde I_x^G(\gamma)=\delta I_x^G(\gamma)\delta^{-1}$ for all $\gamma\in G_x\boldsymbol{X}$.

If k,v_k is a third choice for i,v_i, yielding a finite group $G''_x\boldsymbol{X}=\mathrm{Stab}_{\Gamma_k}(v_k)$, then as above by picking points $p\in S_x$ we can define isomorphisms

$$I_x^G:G_x\boldsymbol{X}\longrightarrow G'_x\boldsymbol{X},\quad \dot I_x^G:G'_x\boldsymbol{X}\longrightarrow G''_x\boldsymbol{X},\quad \ddot I_x^G:G_x\boldsymbol{X}\longrightarrow G''_x\boldsymbol{X}.$$

We can show that $\dot I_x^G\circ I_x^G$ and $\ddot I_x^G$ differ by the action of some canonical $\delta\in G''_x\boldsymbol{X}$, as for $I_x^G,\tilde I_x^G$ above. That is, $\dot I_x^G\circ I_x^G$ is a possible choice for $\ddot I_x^G$.

To summarize: $G_x\boldsymbol{X}$ is independent of the choice of i,v_i up to isomorphism, but not up to canonical isomorphism. There are isomorphisms $I_x^G:G_x\boldsymbol{X}\to G'_x\boldsymbol{X}$ between any two choices for $G_x\boldsymbol{X}$, which are canonical up to conjugation by an element of $G'_x\boldsymbol{X}$, and behave as expected under composition.

Lemma 6.50. *In Definition* 6.49, *the subset* $S_x\subseteq P_{ij}$ *in* (6.41) *is invariant under the commuting actions of* $G_x\boldsymbol{X}\subseteq\Gamma_i$ *and* $G'_x\boldsymbol{X}\subseteq\Gamma_j$ *on* P_{ij} *induced by the* Γ_i,Γ_j*-actions on* P_{ij}, *and* $G_x\boldsymbol{X},G'_x\boldsymbol{X}$ *each act freely transitively on* S_x.

Proof. If $\gamma\in G_x\boldsymbol{X}$ and $p\in S_x$ then $\pi_{ij}(\gamma\cdot p)=\gamma\cdot\pi_{ij}(p)=\gamma\cdot v_i=v_i$ (as π_{ij} is Γ_i-equivariant and $\gamma\in\mathrm{Stab}_{\Gamma_i}(v_i)$), and $\phi_{ij}(\gamma\cdot p)=\phi_{ij}(p)=v_j$ (as ϕ_{ij} is Γ_i-invariant). Hence $\gamma\cdot p\in S_x$, so S_x is $G_x\boldsymbol{X}$-invariant. If $\gamma'\in G'_x\boldsymbol{X}$ and $p\in S_x$ then $\pi_{ij}(\gamma'\cdot p)=\pi_{ij}(p)=v_i$ (as π_{ij} is Γ_j-invariant), and $\phi_{ij}(\gamma'\cdot p)=\gamma'\cdot\phi_{ij}(p)=\gamma'\cdot v_j=v_j$ (as ϕ_{ij} is Γ_j-equivariant and $\gamma'\in\mathrm{Stab}_{\Gamma_j}(v_j)$). Hence $\gamma'\cdot p\in S_x$, so S_x is $G'_x\boldsymbol{X}$-invariant. This proves the first part.

Next we prove that S_x is nonempty. As $\pi_{ij} : P_{ij} \to V_{ij} \subseteq V_i$ is a principal Γ_j-bundle and $v_i \in \bar\psi_i^{-1}(S) \subseteq V_{ij}$, there exists $p \in P_{ij}$ with $\pi_{ij}(p) = v_i$. Then $\bar\psi_j \circ \phi_{ij}(p) = \bar\psi_i \circ \pi_{ij}(p) = \bar\psi_i(v_i) = x$, so $\phi_{ij}(p) \in \bar\psi_j^{-1}(x)$. Since $\psi_j : V_j/\Gamma_j \to \operatorname{Im}\psi_j \subseteq X$ is a homeomorphism, $\bar\psi_j^{-1}(x)$ is a Γ_j-orbit in V_j, which contains $\phi_{ij}(p)$ and v_j. Hence $v_j = \gamma_j \cdot \phi_{ij}(p)$ for some $\gamma_j \in \Gamma_j$. But then $\pi_{ij}(\gamma_j \cdot p) = \pi_{ij}(p) = v_i$ (as π_{ij} is Γ_j-invariant) and $\phi_{ij}(\gamma_j \cdot p) = \gamma_j \cdot \phi_{ij}(p) = v_j$ (as ϕ_{ij} is Γ_j-equivariant). Thus $\gamma_j \cdot p \in S_x$, and $S_x \neq \emptyset$.

Suppose $p, p' \in S_x$. Then $p, p' \in \pi_{ij}^{-1}(v_i)$, where Γ_j acts freely and transitively on $\pi_{ij}^{-1}(v_i)$ as $\pi_{ij} : P_{ij} \to V_{ij} \subseteq V_i$ is a principal Γ_j-bundle. Thus there exists a unique $\gamma' \in \Gamma_j$ with $\gamma' \cdot p = p'$. But then

$$\gamma' \cdot v_j = \gamma' \cdot \phi_{ij}(p) = \phi_{ij}(\gamma' \cdot p) = \phi_{ij}(p') = v_j,$$

as $\phi_{ij}(p) = \phi_{ij}(p') = v_j$ and ϕ_{ij} is Γ_j-equivariant. Hence $\gamma' \in \operatorname{Stab}_{\Gamma_j}(v_j) = G'_x\boldsymbol{X}$. Therefore $G'_x\boldsymbol{X}$ acts freely and transitively on S_x.

Finally we show $G_x\boldsymbol{X}$ acts freely transitively on S_x. As Φ_{ij} is a coordinate change over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, there exist a 1-morphism $\Phi_{ji} = (P_{ji}, \pi_{ji}, \phi_{ji}, \hat\phi_{ji}) : (V_j, E_j, \Gamma_j, s_j, \psi_j) \to (V_i, E_i, \Gamma_i, s_i, \psi_i)$ and 2-morphisms $\Lambda_{ii} : \mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)} \Rightarrow \Phi_{ji} \circ \Phi_{ij}$, $\mathrm{M}_{jj} : \mathrm{id}_{(V_j,E_j,\Gamma_j,s_j,\psi_j)} \Rightarrow \Phi_{ij} \circ \Phi_{ji}$ over S. Choose representatives $(\acute P_{ii}, \lambda_{ii}, \hat\lambda_{ii})$ and $(\acute P_{jj}, \mu_{jj}, \hat\mu_{jj})$ for $\Lambda_{ii}, \mathrm{M}_{jj}$. Consider:

$$\begin{aligned}
\lambda_{ii}|_{\{v_i\}\times\Gamma_i} : \{v_i\}\times\Gamma_i &\xrightarrow{\cong} \bigl\{(p,q)\in P_{ij}\times P_{ji} : \pi_{ij}(p)=v_i,\ \phi_{ij}(p)=\pi_{ji}(q)\bigr\}/\Gamma_j \\
&\cong \bigl\{(p,q) \in P_{ij} \times P_{ji} : \pi_{ij}(p) = v_i,\ \phi_{ji}(p) = \pi_{ji}(q) = v_j\bigr\}/G'_x\boldsymbol{X} \\
&= \bigl\{(p,q) \in S_x \times P_{ji} : \pi_{ji}(q) = v_j\bigr\}/G'_x\boldsymbol{X}.
\end{aligned} \tag{6.42}$$

Here both $\mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)}$ and $\Phi_{ji} \circ \Phi_{ij}$ include a principal Γ_i-bundle over an open neighbourhood of $\bar\psi_i^{-1}(S)$ in V_i, and λ_{ii} is an isomorphism between them; the top line of (6.42) is this isomorphism restricted to the fibres over v_i. In the second line we use that $\phi_{ij}(p) = \pi_{ji}(q)$ lies in the Γ_j-orbit of v_j in V_j as $\pi_{ij}(p) = v_i$, and $\pi_{ji} : P_{ji} \to V_j$ is Γ_j-equivariant, and $G'_x\boldsymbol{X} = \operatorname{Stab}_{\Gamma_j}(v_j)$. In the third line we use (6.41). Similarly we show that

$$\begin{aligned}
\mu_{jj}|_{\{v_j\}\times\Gamma_j} : \{v_j\}\times\Gamma_j &\xrightarrow{\cong} \bigl\{(q,p)\in P_{ji}\times P_{ij} : \phi_{ij}(p)=v_j,\ \phi_{ji}(q)=\pi_{ij}(p)\bigr\}/\Gamma_i \\
&\cong \bigl\{(q,p) \in P_{ji} \times P_{ij} : \phi_{ij}(p) = v_j,\ \phi_{ji}(q) = \pi_{ij}(p) = v_i\bigr\}/G_x\boldsymbol{X} \\
&= \bigl\{(q,p) \in P_{ji} \times S_x : \phi_{ji}(q) = v_i\bigr\}/G_x\boldsymbol{X}.
\end{aligned} \tag{6.43}$$

Now the top line of (6.42) is equivariant under two commuting Γ_i-actions. On the left hand side these act by left and right Γ_i-multiplication on $\{v_i\} \times \Gamma_i$, so are free and transitive. On the right they act by Γ_i-multiplication on $P_{ij} \ni p$ and $P_{ji} \ni q$. Restricting the free Γ_i-action on P_{ij} to a free $G_x\boldsymbol{X}$-action, this free $G_x\boldsymbol{X}$-action descends to the second and third lines of (6.42), so we see that $G_x\boldsymbol{X}$ acts freely on S_x.

Similarly, the top line of (6.43) has two transitive actions of Γ_j. The action on $P_{ji} \ni q$ descends to a transitive Γ_j-action on the second and third lines. Therefore $\Gamma_j\backslash(\phi_{ji}^{-1}(v_i) \times S_x)/G_x\boldsymbol{X} \cong (\phi_{ji}^{-1}(v_i)/\Gamma_j) \times (S_x/G_x\boldsymbol{X})$ is a point, so $S_x/G_x\boldsymbol{X}$ is a point, and $G_x\boldsymbol{X}$ acts transitively on S_x. $\square$

We discuss functoriality of the $G_x\boldsymbol{X}$ under 1- and 2-morphisms.

Definition 6.51. Let $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ be a 1-morphism of Kuranishi spaces, with notation (6.15), (6.16), (6.18), and let $x\in\boldsymbol{X}$ with $\boldsymbol{f}(x)=y$ in $\boldsymbol{Y}$. Then Definition 6.49 gives isotropy groups $G_x\boldsymbol{X}$, defined using $i\in I$ and $u_i\in U_i$ with $\bar\chi_i(u_i)=x$, and $G_y\boldsymbol{Y}$, defined using $j\in J$ and $v_j\in V_j$ with $\bar\psi_j(v_j)=y$. In $\boldsymbol{f}$ we have a 1-morphism $\boldsymbol{f}_{ij}=(P_{ij},\pi_{ij},f_{ij},\hat f_{ij})$ over f. As in (6.41), define

$$S_{x,\boldsymbol{f}}=\bigl\{p\in P_{ij}:\pi_{ij}(p)=u_i,\;\; f_{ij}(p)=v_j\bigr\}. \tag{6.44}$$

Following the first part of the proof of Lemma 6.50, we find that $S_{x,\boldsymbol{f}}$ is invariant under the commuting actions of $G_x\boldsymbol{X}=\mathrm{Stab}_{\mathrm{B}_i}(u_i)\subseteq\mathrm{B}_i$ and $G_y\boldsymbol{Y}=\mathrm{Stab}_{\Gamma_j}(v_j)\subseteq\Gamma_j$ on P_{ij} induced by the B_i,Γ_j-actions on P_{ij}. But this time, $G_y\boldsymbol{Y}$ acts freely transitively on $S_{x,\boldsymbol{f}}$, but $G_x\boldsymbol{X}$ need not act freely or transitively.

Pick $p\in S_{x,\boldsymbol{f}}$. As for I_x^G in Definition 6.49, define a group morphism $G_x\boldsymbol{f}:G_x\boldsymbol{X}\rightarrow G_y\boldsymbol{Y}$ by $G_x\boldsymbol{f}(\gamma)=\gamma'$ if $\gamma\cdot p=(\gamma')^{-1}\cdot p$ in $S_{x,\boldsymbol{f}}$, using the actions of $G_x\boldsymbol{X},G_y\boldsymbol{Y}$ on $S_{x,\boldsymbol{f}}$ with $G_y\boldsymbol{Y}$ free and transitive.

If $\tilde p\in S_{x,\boldsymbol{f}}$ is an alternative choice for p, yielding $\tilde G_x\boldsymbol{f}:G_x\boldsymbol{X}\rightarrow G_y\boldsymbol{Y}$, there is a unique $\delta\in G_y\boldsymbol{Y}$ with $\delta\cdot p=\tilde p$, and then $\tilde G_x\boldsymbol{f}(\gamma)=\delta(G_x\boldsymbol{f}(\gamma))\delta^{-1}$ for all $\gamma\in G_x\boldsymbol{X}$. That is, the morphism $G_x\boldsymbol{f}:G_x\boldsymbol{X}\rightarrow G_y\boldsymbol{Y}$ is canonical up to conjugation by an element of $G_y\boldsymbol{Y}$.

Continuing with the same notation, suppose $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ is another 1-morphism and $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ a 2-morphism in $\dot{\mathbf{K}}\mathbf{ur}$. Then above we define $G_x\boldsymbol{g}$ by choosing an arbitrary point $q\in S_{x,\boldsymbol{g}}$, where

$$S_{x,\boldsymbol{g}}=\bigl\{q\in Q_{ij}:\pi_{ij}(q)=u_i,\;\; g_{ij}(q)=v_j\bigr\},$$

with $\boldsymbol{g}_{ij}=(Q_{ij},\pi_{ij},g_{ij},\hat g_{ij})$ in $\boldsymbol{g}$. In $\boldsymbol{\eta}$ we have $\boldsymbol{\eta}_{ij}=[\acute P_{ij},\eta_{ij},\hat\eta_{ij}]$ represented by $(\acute P_{ij},\eta_{ij},\hat\eta_{ij})$, where $\acute P_{ij}\subseteq P_{ij}$ and $\eta_{ij}:\acute P_{ij}\rightarrow Q_{ij}$. From the definitions we find that $S_{x,\boldsymbol{f}}\subseteq\acute P_{ij}$, and $\eta_{ij}|_{S_{x,\boldsymbol{f}}}:S_{x,\boldsymbol{f}}\rightarrow S_{x,\boldsymbol{g}}$ is a bijection. Since $G_y\boldsymbol{Y}$ acts freely and transitively on $S_{x,\boldsymbol{g}}$, there is a unique element $G_x\boldsymbol{\eta}\in G_y\boldsymbol{Y}$ with $G_x\boldsymbol{\eta}\cdot\eta_{ij}(p)=q$. One can now check that

$$G_x\boldsymbol{g}(\gamma)=(G_x\boldsymbol{\eta})(G_x\boldsymbol{f}(\gamma))(G_x\boldsymbol{\eta})^{-1}\quad\text{for all }\gamma\in G_x\boldsymbol{X}.$$

That is, $G_x\boldsymbol{g}$ is conjugate to $G_x\boldsymbol{f}$ under $G_x\boldsymbol{\eta}\in G_y\boldsymbol{Y}$, the same indeterminacy as in the definition of $G_x\boldsymbol{f}$.

Suppose instead that $\boldsymbol{g}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ is another 1-morphism of Kuranishi spaces and $\boldsymbol{g}(y)=z\in\boldsymbol{Z}$. Then in a similar way we can show there is a canonical element $G_{x,\boldsymbol{g},\boldsymbol{f}}\in G_z\boldsymbol{Z}$ such that for all $\gamma\in G_x\boldsymbol{X}$ we have

$$G_x(\boldsymbol{g}\circ\boldsymbol{f})(\gamma)=(G_{x,\boldsymbol{g},\boldsymbol{f}})((G_y\boldsymbol{g}\circ G_x\boldsymbol{f})(\gamma))(G_{x,\boldsymbol{g},\boldsymbol{f}})^{-1}.$$

That is, $G_x(\boldsymbol{g}\circ\boldsymbol{f})$ is conjugate to $G_y\boldsymbol{g}\circ G_x\boldsymbol{f}$ under $G_{x,\boldsymbol{g},\boldsymbol{f}}\in G_z\boldsymbol{Z}$.

Since 2-morphisms $\boldsymbol{\eta}:\boldsymbol{f}\Rightarrow\boldsymbol{g}$ relate $G_x\boldsymbol{f}$ and $G_x\boldsymbol{g}$ by isomorphisms, if $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ is an equivalence in $\dot{\mathbf{K}}\mathbf{ur}$ then $G_x\boldsymbol{f}$ is an isomorphism for all $x\in\boldsymbol{X}$.

Remark 6.52. The definitions of $G_x\boldsymbol{X}, G_x\boldsymbol{f}$ above depend on arbitrary choices. We could use the Axiom of (Global) Choice as in Remark 4.21 to choose particular values for $G_x\boldsymbol{X}, G_x\boldsymbol{f}$ for all $\boldsymbol{X}, x, \boldsymbol{f}$. But this is not really necessary, we can just bear the non-uniqueness in mind when working with them. All the definitions we make using $G_x\boldsymbol{X}, G_x\boldsymbol{f}$ will be independent of the arbitrary choices in Definitions 6.49 and 6.51.

Definition 6.53. **(a)** We call a 1-morphism $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ in $\dot{\mathbf{K}}\mathbf{ur}$ *representable* if $G_x\boldsymbol{f}:G_x\boldsymbol{X}\rightarrow G_{\boldsymbol{f}(x)}\boldsymbol{Y}$ is injective for all $x\in\boldsymbol{X}$.

(b) Write $\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}\subset\dot{\mathbf{K}}\mathbf{ur}$ for the full 2-subcategory of $\boldsymbol{X}$ in $\dot{\mathbf{K}}\mathbf{ur}$ with *trivial isotropy groups*, that is, with $G_x\boldsymbol{X}=\{1\}$ for all $x\in\boldsymbol{X}$.

In Example 6.36 we defined a weak 2-functor $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\dot{\mathbf{K}}\mathbf{ur}}:\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}\rightarrow\dot{\mathbf{K}}\mathbf{ur}$. If $\boldsymbol{X}\in\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ and $\boldsymbol{X}'=F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\dot{\mathbf{K}}\mathbf{ur}}(\boldsymbol{X})$ then $\boldsymbol{X}'$ has Kuranishi neighbourhoods $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ with $\Gamma_i=\{1\}$, so clearly $G_x\boldsymbol{X}'=\{1\}$ for all $x\in\boldsymbol{X}'$ as $G_x\boldsymbol{X}'\subseteq\Gamma_i$ for some $i\in I$, and thus $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\dot{\mathbf{K}}\mathbf{ur}}$ maps $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}\rightarrow\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}$, so we may write it as $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}}:\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}\rightarrow\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}$.

Theorem 6.54. *The weak* 2*-functor* $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}}:\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}\rightarrow\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}$ *from Example* 6.36 *is an equivalence of* 2*-categories.*

Proof. By construction, $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\dot{\mathbf{K}}\mathbf{ur}}$ is an equivalence from $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ to the full 2-subcategory $\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{tr}\boldsymbol{\Gamma}}\subset\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}\subset\dot{\mathbf{K}}\mathbf{ur}$ of Kuranishi spaces $\boldsymbol{X}=(X,\mathcal{K})$ such that all Kuranishi neighbourhoods $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ in $\mathcal{K}$ have $\Gamma_i=\{1\}$. Thus, to show that $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\dot{\mathbf{K}}\mathbf{ur}}:\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}\rightarrow\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}$ is an equivalence, it is enough to prove that the inclusion $\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{tr}\boldsymbol{\Gamma}}\subset\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}$ is an equivalence. That is, if $\boldsymbol{X}$ is an object of $\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}$, we must find $\boldsymbol{X}'$ in $\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{tr}\boldsymbol{\Gamma}}$ with $\boldsymbol{X}'\simeq\boldsymbol{X}$ in $\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}$.

Write $\boldsymbol{X}=(X,\mathcal{K})$ with $\mathcal{K}=\bigl(I,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\,\Phi_{ij,\,i,j\in I},\,\Lambda_{ijk,\,i,j,k\in I}\bigr)$. Let $x\in X$. Then there exists $i\in I$ with $x\in\operatorname{Im}\bar\psi_i$. Pick $v\in s_i^{-1}(0)\subseteq V_i$ with $\bar\psi_i(v)=x$. Then $\mathrm{Stab}_{\Gamma_i}(v)\cong G_x\boldsymbol{X}=\{1\}$, so Γ_i acts freely on V_i near v. Using Γ_i finite and V_i Hausdorff, we can choose an open neighbourhood W_x of v in V_i such that $W_x\cap(\gamma\cdot W_x)=\emptyset$ for all $1\neq\gamma\in\Gamma_i$. Set $F_x=E_i|_{W_x}$, and $\Delta_x=\{1\}$, and $t_x=s_i|_{W_x}$. Define $\omega_x:t_x^{-1}(0)\rightarrow X$ to be the composition

$$t_x^{-1}(0)\xrightarrow{\quad v'\mapsto v'\Gamma\quad}s_i^{-1}(0)/\Gamma_i\xrightarrow{\quad\psi_i\quad}X.\tag{6.45}$$

Since $W_x\cap(\gamma\cdot W_x)=\emptyset$ for all $1\neq\gamma\in\Gamma$, the first map in (6.45) is a homeomorphism with an open subset, and the second map ψ_i is too by Definition 6.1(e). Hence ω_x is a homeomorphism with an open subset $\operatorname{Im}\omega_x\subseteq X$. Thus $(W_x,F_x,\Delta_x,t_x,\omega_x)$ is a Kuranishi neighbourhood on X, with $x\in\operatorname{Im}\omega_x$.

Now define $Q_{xi}=W_x\times\Gamma_i$, considered as an object in $\dot{\mathbf{M}}\mathbf{an}$ which is the disjoint union of $|\Gamma_i|$ copies of W_x. Let Γ_i act on Q_{xi} by the trivial action on W_x and left action on Γ_i, and let $\Delta_x=\{1\}$ act trivially on Q_{xi}. Define morphisms $\pi_{xi}:W_x\times\Gamma_i\rightarrow W_x$ and $\upsilon_{xi}:W_x\times\Gamma_i\rightarrow V_i$ such that $\pi_{xi}:(v,\gamma)\mapsto v\in W_x$ and

$\pi_{xi} : (v, \gamma) \mapsto \gamma \cdot v \in V_i$ on points. That is, π_{xi} is the projection $W_x \times \Gamma_i \to W_x$, and on $W_x \times \{\gamma\}$, υ_{xi} is the composition of the inclusion $W_x \hookrightarrow V_i$ and the group action $\gamma \cdot : V_i \to V_i$, for each $\gamma \in \Gamma_i$. Define a vector bundle morphism $\hat{\upsilon}_{xi} : \pi_{xi}^*(F_x) \to \upsilon_{xi}^*(E_i)$ such that for each $\gamma \in \Gamma_i$, $\hat{\upsilon}_{xi}|_{W_x \times \{\gamma\}}$ is the action of γ on E_i, restricted to a map $\gamma \cdot : E_i|_{W_x} \to E_i|_{\gamma \cdot W_x}$.

It is now easy to check that $\Upsilon_{xi} := (Q_{xi}, \pi_{xi}, \upsilon_{xi}, \hat{\upsilon}_{xi})$ is a 1-morphism of Kuranishi neighbourhoods $\Upsilon_{xi} : (W_x, F_x, \Delta_x, t_x, \omega_x) \to (V_i, E_i, \Gamma_i, s_i, \psi_i)$ over $\operatorname{Im} \omega_x \subseteq X$. Furthermore, $\Upsilon_{ix} := (Q_{xi}, \upsilon_{xi}, \pi_{xi}, \hat{\upsilon}_{xi}^{-1})$ is a 1-morphism $\Upsilon_{ix} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (W_x, F_x, \Delta_x, t_x, \omega_x)$ over $\operatorname{Im} \omega_x$. There are obvious 2-morphisms $\eta_{xx} : \Upsilon_{ix} \circ \Upsilon_{xi} \Rightarrow \mathrm{id}_{(W_x, F_x, \Delta_x, t_x, \omega_x)}$ and $\zeta_{ii} : \Upsilon_{xi} \circ \Upsilon_{ix} \Rightarrow \mathrm{id}_{(V_i, E_i, \Gamma_i, s_i, \psi_i)}$ over $\operatorname{Im} \omega_x$. Hence $\Upsilon_{xi}, \Upsilon_{ix}$ are coordinate changes over $\operatorname{Im} \omega_x$.

Next we use the ideas of §6.4. For each $j \in I$ define a coordinate change $\Phi_{xj} := \Phi_{ij} \circ \Upsilon_{xi} : (W_x, F_x, \Delta_x, t_x, \omega_x) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ over $\operatorname{Im} \omega_x \cap \operatorname{Im} \psi_j \subseteq X$, and for all $j, k \in I$ define a 2-morphism $\Lambda_{xjk} : \Phi_{jk} \circ \Phi_{xj} \Rightarrow \Phi_{xk}$ by the commutative diagram

$$\begin{tikzcd}
\Phi_{jk} \circ \Phi_{xj} \arrow[rr, Rightarrow, "\Lambda_{xjk}"'] \arrow[d, equal] & & \Phi_{xk} \arrow[d, equal] \\
\Phi_{jk} \circ (\Phi_{ij} \circ \Upsilon_{xi}) \arrow[r, Rightarrow, "\boldsymbol{\alpha}^{-1}_{\Phi_{jk}, \Phi_{ij}, \Upsilon_{xi}}"] & (\Phi_{jk} \circ \Phi_{ij}) \circ \Upsilon_{xi} \arrow[r, Rightarrow, "\Lambda_{ijk} * \mathrm{id}_{\Upsilon_{xi}}"] & \Phi_{ik} \circ \Upsilon_{xi}.
\end{tikzcd}$$

Using Definition 6.17(h) for the Λ_{ijk} and properties of 2-categories we find that these Φ_{xj}, Λ_{xjk} satisfy (6.37), so that $(W_x, F_x, \Delta_x, t_x, \omega_x), \Phi_{xj}, \Lambda_{xjk}$ is a Kuranishi neighbourhood on the Kuranishi space $\boldsymbol{X}$, in the sense of Definition 6.42.

Thus we have a family $(W_x, F_x, \Delta_x, t_x, \omega_x)$ for $x \in X$ of Kuranishi neighbourhoods on $\boldsymbol{X}$ which cover X. Hence Theorem 6.47 constructs a Kuranishi space $\boldsymbol{X}' = (X, \mathcal{K}')$ equivalent to $\boldsymbol{X}$ in $\dot{\mathbf{K}}\mathbf{ur}$, such that $\mathcal{K}'$ has Kuranishi neighbourhoods $(W_x, F_x, \Delta_x, t_x, \omega_x)$ for $x \in X$. Since $\Delta_x = \{1\}$ for all x, this $\boldsymbol{X}'$ lies in $\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{tr\Gamma}} \subset \dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}$, which proves Theorem 6.54. □

6.6 Orbifolds and Kuranishi spaces

We have said that Kuranishi spaces are an orbifold version of m-Kuranishi spaces, and should be regarded as 'derived orbifolds', just as m-Kuranishi spaces are a kind of 'derived manifold', as in §4.8. We now explore the relationship between orbifolds and Kuranishi spaces in more detail. As we explain in §6.6.1, there are many different definitions of orbifolds in the literature, most of which are known to be equivalent at the level of categories or 2-categories.

To relate orbifolds and Kuranishi spaces, we find it convenient to give our own, new definition of a 2-category of orbifolds $\mathbf{Orb}_{\mathrm{Kur}}$ in §6.6.2, which is basically the 2-subcategory $\mathbf{Orb}_{\mathrm{Kur}} \subset \mathbf{Kur}$ of Kuranishi spaces $\boldsymbol{X}$ all of whose Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ have $E_i = s_i = 0$, and then to show $\mathbf{Orb}_{\mathrm{Kur}}$ is equivalent to the 2-categories of orbifolds defined by other authors.

6.6.1 Definitions of orbifolds in the literature

Orbifolds are generalizations of manifolds locally modelled on $\mathbb{R}^n/G$, for G a finite group acting linearly on $\mathbb{R}^n$. They were introduced by Satake [97], who called them 'V-manifolds'. Later they were studied by Thurston [105, Ch. 13] who gave them the name 'orbifold'.

As for Kuranishi spaces, defining orbifolds $\mathfrak{X}, \mathfrak{Y}$ and smooth maps $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ was initially problematic, and early definitions of ordinary categories of orbifolds [97, 105] had some bad differential-geometric behaviour (e.g. for some definitions, one cannot define pullbacks $\mathfrak{f}^*(\mathfrak{E})$ of orbifold vector bundles $\mathfrak{E} \to \mathfrak{Y}$). It is now generally agreed that it is best to define orbifolds to be a 2-category. See Lerman [72] for a good overview of ways to define orbifolds.

There are three main definitions of ordinary categories of orbifolds:

(a) Satake [97] and Thurston [105] defined an orbifold $\mathfrak{X}$ to be a Hausdorff topological space X with an atlas $\{(V_i, \Gamma_i, \psi_i) : i \in I\}$ of orbifold charts (V_i, Γ_i, ψ_i), where V_i is a manifold, Γ_i a finite group acting smoothly (and locally effectively) on V_i, and $\psi_i : V_i/\Gamma_i \to X$ a homeomorphism with an open set in X, and pairs of charts $(V_i, \Gamma_i, \phi_i), (V_j, \Gamma_j, \phi_j)$ satisfy compatibility conditions on their overlaps in X. Smooth maps $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ between orbifolds are continuous maps $f : X \to Y$ of the underlying spaces, which lift locally to smooth maps on the charts, giving a category $\mathbf{Orb}_{\mathrm{ST}}$.

(b) Chen and Ruan [12, §4] defined orbifolds $\mathfrak{X}$ in a similar way to [97, 105], but using germs of orbifold charts (V_p, Γ_p, ψ_p) for $p \in X$. Their morphisms $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ are called *good maps*, giving a category $\mathbf{Orb}_{\mathrm{CR}}$.

(c) Moerdijk and Pronk [89, 90] defined a category of orbifolds $\mathbf{Orb}_{\mathrm{MP}}$ as *proper étale Lie groupoids* in **Man**. Their definition of smooth map $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$, called *strong maps* [90, §5] is complicated: it is an equivalence class of diagrams $\mathfrak{X} \xleftarrow{\phi} \mathfrak{X}' \xrightarrow{\psi} \mathfrak{Y}$, where $\mathfrak{X}'$ is a third orbifold, and ϕ, ψ are morphisms of groupoids with ϕ an equivalence (loosely, a diffeomorphism).

A book on orbifolds in the sense of [12, 89, 90] is Adem, Leida and Ruan [1].

There are four main definitions of 2-categories of orbifolds:

(i) Pronk [96] defines a strict 2-category **LieGpd** of Lie groupoids in **Man** as in (c), with the obvious 1-morphisms of groupoids, and localizes by a class of weak equivalences $\mathcal{W}$ to get a weak 2-category $\mathbf{Orb}_{\mathrm{Pr}} = \mathbf{LieGpd}[\mathcal{W}^{-1}]$.

(ii) Lerman [72, §3.3] defines a weak 2-category $\mathbf{Orb}_{\mathrm{Le}}$ of Lie groupoids in **Man** as in (c), with a non-obvious notion of 1-morphism called 'Hilsum–Skandalis morphisms' involving 'bibundles', and does not need to localize.

Henriques and Metzler [44] also use Hilsum–Skandalis morphisms. We used Hilsum–Skandalis morphisms in our 1-morphisms of Kuranishi neighbourhoods in §6.1, as in Remark 6.15(ii).

(iii) Behrend and Xu [4, §2], Lerman [72, §4] and Metzler [88, §3.5] define a strict 2-category of orbifolds $\mathbf{Orb}_{\mathrm{ManSta}}$ as a class of Deligne–Mumford stacks on the site $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$ of manifolds with Grothendieck topology $\mathcal{J}_{\mathbf{Man}}$ coming from open covers.

(iv) The author [65] defines a strict 2-category of orbifolds $\mathbf{Orb}_{C^\infty\mathrm{Sta}}$ as a class of Deligne–Mumford stacks on the site $(\mathbf{C^\infty Sch}, \mathcal{J}_{\mathbf{C^\infty Sch}})$ of C^∞-schemes.

As in Behrend and Xu [4, §2.6], Lerman [72], Pronk [96], and the author [65, Th. 7.26], approaches (i)–(iv) give equivalent weak 2-categories $\mathbf{Orb}_{\mathrm{Pr}}, \mathbf{Orb}_{\mathrm{Le}}, \mathbf{Orb}_{\mathrm{ManSta}}, \mathbf{Orb}_{C^\infty\mathrm{Sta}}$. As they are equivalent, the differences between them are not of mathematical importance, but more a matter of convenience or taste. Properties of localization also imply that $\mathbf{Orb}_{\mathrm{MP}} \simeq \mathrm{Ho}(\mathbf{Orb}_{\mathrm{Pr}})$. Thus, all of (c) and (i)–(iv) are equivalent at the level of homotopy categories.

In §6.6.2 we give a fifth definition of a weak 2-category of orbifolds, similar to (ii) above, which is a special case of our definition of Kuranishi spaces.

6.6.2 The weak 2-category of orbifolds $\dot{\mathbf{O}}\mathbf{rb}$

In a similar way to (i)–(iv) in §6.6.1, we now give a fifth definition of a weak 2-category of orbifolds, essentially as a full 2-subcategory $\mathbf{Orb}_{\mathrm{Kur}} \subset \mathbf{Kur}$, and we will show that $\mathbf{Orb}_{\mathrm{Kur}}$ is equivalent to $\mathbf{Orb}_{\mathrm{Pr}}, \mathbf{Orb}_{\mathrm{Le}}, \mathbf{Orb}_{\mathrm{ManSta}}, \mathbf{Orb}_{C^\infty\mathrm{Sta}}$ in §6.6.1(i)–(iv). This provides a convenient way to relate orbifolds and Kuranishi spaces. Fukaya et al. [30, §9] and McDuff [78] also define (effective) orbifolds as special examples of their notions of Kuranishi space/Kuranishi atlas.

The basic idea is that orbifolds $\mathfrak{X}$ in $\mathbf{Orb}_{\mathrm{Kur}}$ are just Kuranishi spaces $\boldsymbol{X} = (X, \mathcal{K})$ with $\mathcal{K} = \bigl(I, (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i\in I},\ \Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij})_{i,j\in I}, \Lambda_{ijk} = [\acute P_{ijk}, \lambda_{ijk}, \hat\lambda_{ijk}]_{i,j,k\in I}\bigr)$, for which the obstruction bundles $E_i \to V_i$ are zero for all $i \in I$, so that the sections s_i are also zero. This allows us to simplify the notation a lot. Equations in §6.1 involving error terms $O\bigl(\pi_{ij}^*(s_i)\bigr)$ or $O\bigl(\pi_{ij}^*(s_i)^2\bigr)$ become exact, as $s_i = 0$.

As E_i, s_i are zero we can take 'orbifold charts' to be (V_i, Γ_i, ψ_i). As $\hat\phi_{ij} = 0$ we can take coordinate changes to be $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij})$, and we can also take $V_{ij} = \pi_{ij}(P_{ij})$ to be equal to $\bar\psi_i^{-1}(S)$, rather than just an open neighbourhood of $\bar\psi_i^{-1}(S)$ in V_i, since $\bar\psi_i^{-1}(S)$ is open in V_i when $s_i = 0$. For 2-morphisms $\Lambda_{ij} = [\acute P_{ij}, \lambda_{ij}, \hat\lambda_{ij}] : \Phi_{ij} \Rightarrow \Phi'_{ij}$ in §6.1, we have $\hat\lambda_{ij} = 0$, and we are forced to take $\acute P_{ij} = P_{ij}$, and the equivalence relation $\sim$ in Definition 6.4 becomes trivial, so we can take 2-morphisms to be just λ_{ij}.

Section 6.6.1 discussed only orbifolds modelled on classical manifolds, as almost all the literature on orbifolds concerns only these. However, we will construct a weak 2-category of 'orbifolds' $\dot{\mathbf{O}}\mathbf{rb}$ corresponding to any category of 'manifolds' $\dot{\mathbf{M}}\mathbf{an}$ satisfying Assumptions 3.1–3.3. When $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$ this gives a 2-category $\mathbf{Orb}_{\mathrm{Kur}}$ equivalent to the 2-categories of orbifolds discussed in §6.6.1. When $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^c}$ we get a 2-category $\mathbf{Orb^c}$ of orbifolds with corners, and so on. From here until Proposition 6.62, fix a category $\dot{\mathbf{M}}\mathbf{an}$ satisfying Assumptions 3.1–3.3. As usual we will call objects $X \in \dot{\mathbf{M}}\mathbf{an}$ 'manifolds', and morphisms $f : X \to Y$ in $\dot{\mathbf{M}}\mathbf{an}$ 'smooth maps'.

Definition 6.55. Let X be a topological space. An *orbifold chart* on X is a triple (V, Γ, ψ), where V is a manifold (object in $\dot{\mathbf{Man}}$), Γ is a finite group with a smooth action on V (that is, an action by isomorphisms in $\dot{\mathbf{Man}}$), and ψ is a homeomorphism from the topological space V/Γ to an open subset $\operatorname{Im}\psi$ in X. We write $\bar{\psi} : V \to X$ for the composition of ψ with the projection $V \to V/\Gamma$.

We call an orbifold chart (V, Γ, ψ) *effective* if the action of Γ on V is locally effective, that is, no nonempty open set $U \subseteq V$ is fixed by $1 \neq \gamma \in \Gamma$.

Definition 6.56. Let X, Y be topological spaces, $f : X \to Y$ a continuous map, $(V_i, \Gamma_i, \psi_i), (V_j, \Gamma_j, \psi_j)$ be orbifold charts on X, Y respectively, and $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j) \subseteq X$ be an open set. A 1-*morphism* $\Phi_{ij} : (V_i, \Gamma_i, \psi_i) \to (V_j, \Gamma_j, \psi_j)$ *of orbifold charts over* (S, f) is a triple $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij})$ satisfying:

(a) P_{ij} is a manifold (object in $\dot{\mathbf{Man}}$), with commuting smooth actions of Γ_i, Γ_j (that is, with a smooth action of $\Gamma_i \times \Gamma_j$), with the Γ_j-action free.

(b) $\pi_{ij} : P_{ij} \to V_i$ is a smooth map (morphism in $\dot{\mathbf{Man}}$) which is Γ_i-equivariant, Γ_j-invariant, and étale (a local diffeomorphism), with $\pi_{ij}(P_{ij}) = \bar{\psi}_i^{-1}(S)$. The fibres $\pi_{ij}^{-1}(v)$ of π_{ij} for $v \in \bar{\psi}_i^{-1}(S)$ are Γ_j-orbits, so that $\pi_{ij} : P_{ij} \to \bar{\psi}_i^{-1}(S)$ is a principal Γ_j-bundle, with $\bar{\psi}_i^{-1}(S)$ an open submanifold of V_i.

(c) $\phi_{ij} : P_{ij} \to V_j$ is a Γ_i-invariant and Γ_j-equivariant smooth map, that is, $\phi_{ij}(\gamma_i \cdot p) = \phi_{ij}(p)$, $\phi_{ij}(\gamma_j \cdot p) = \gamma_j \cdot \phi_{ij}(p)$ for all $\gamma_i \in \Gamma_i$, $\gamma_j \in \Gamma_j$, $p \in P_{ij}$.

(d) $f \circ \bar{\psi}_i \circ \pi_{ij} = \bar{\psi}_j \circ \phi_{ij} : P_{ij} \to Y$.

If $X = Y$ and $f = \mathrm{id}_X$ then we call Φ_{ij} a *coordinate change over* S if also:

(e) The Γ_i-action on P_{ij} is free, $\phi_{ij} : P_{ij} \to V_j$ is étale, and the fibres $\phi_{ij}^{-1}(v')$ of ϕ_{ij} for $v' \in \bar{\psi}_j^{-1}(S)$ are Γ_i-orbits, so that $\phi_{ij} : P_{ij} \to \bar{\psi}_j^{-1}(S)$ is a principal Γ_i-bundle, with $\bar{\psi}_j^{-1}(S)$ an open submanifold of V_j.

Then Φ_{ij} is a 'Hilsum–Skandalis morphism', as in §6.6.1. If $(P_{ij}, \pi_{ij}, \phi_{ij}) : (V_i, \Gamma_i, \psi_i) \to (V_j, \Gamma_j, \psi_j)$ is a coordinate change over S, then $(P_{ij}, \phi_{ij}, \pi_{ij}) : (V_j, \Gamma_j, \psi_j) \to (V_i, \Gamma_i, \psi_i)$ is also a coordinate change over S.

If $S \subseteq \operatorname{Im}\psi_i \subseteq X$ is open, we define the *identity coordinate change over* S

$$\mathrm{id}_{(V_i,\Gamma_i,\psi_i)} = (\bar{\psi}_i^{-1}(S) \times \Gamma_i, \pi_{ii}, \phi_{ii}) : (V_i, \Gamma_i, \psi_i) \longrightarrow (V_i, \Gamma_i, \psi_i),$$

where $\bar{\psi}_i^{-1}(S) \subseteq V_i$ is an open submanifold, and $\pi_{ii}, \phi_{ii} : \bar{\psi}_i^{-1}(S) \times \Gamma_i \to V_i$ map $\pi_{ii} : (v, \gamma) \mapsto v$ and $\phi_{ii} : (v, \gamma) \mapsto \gamma \cdot v$.

Definition 6.57. Let $\Phi_{ij}, \Phi'_{ij} : (V_i, \Gamma_i, \psi_i) \to (V_j, \Gamma_j, \psi_j)$ be 1-morphisms of orbifold charts over (S, f), where $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij})$ and $\Phi'_{ij} = (P'_{ij}, \pi'_{ij}, \phi'_{ij})$. A 2-*morphism* $\lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ is a Γ_i- and Γ_j-equivariant diffeomorphism $\lambda_{ij} : P_{ij} \to P'_{ij}$ with $\pi'_{ij} \circ \lambda_{ij} = \pi_{ij}$ and $\phi'_{ij} \circ \lambda_{ij} = \phi_{ij}$. That is, 2-morphisms are just isomorphisms preserving all the structure, in the most obvious way.

The *identity* 2-*morphism* $\mathrm{id}_{\Phi_{ij}} : \Phi_{ij} \Rightarrow \Phi_{ij}$ is $\mathrm{id}_{\Phi_{ij}} = \mathrm{id}_{P_{ij}} : P_{ij} \to P_{ij}$.

Definition 6.58. Let X, Y, Z be topological spaces, $f : X \to Y$, $g : Y \to Z$ be continuous maps, $(V_i, \Gamma_i, \psi_i), (V_j, \Gamma_j, \psi_j), (V_k, \Gamma_k, \psi_k)$ be orbifold charts on X, Y, Z respectively, and $T \subseteq \operatorname{Im} \psi_j \cap g^{-1}(\operatorname{Im} \psi_k) \subseteq Y$ and $S \subseteq \operatorname{Im} \psi_i \cap f^{-1}(T) \subseteq X$ be open. Suppose $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}) : (V_i, \Gamma_i, \psi_i) \to (V_j, \Gamma_j, \psi_j)$ is a 1-morphism of orbifold charts over (S, f), and $\Phi_{jk} = (P_{jk}, \pi_{jk}, \phi_{jk}) : (V_j, \Gamma_j, \psi_j) \to (V_k, \Gamma_k, \psi_k)$ is a 1-morphism of orbifold charts over (T, g).

Consider the diagram in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd}
& & & P_{ij}\times_{V_j}P_{jk} \arrow[dll, "\pi_{P_{ij}}"'] \arrow[drr, "\pi_{P_{jk}}"] \arrow[loop above, "\Gamma_i\times\Gamma_j\times\Gamma_k"] & & & \\
& P_{ij} \arrow[dl, "\pi_{ij}"'] \arrow[drr, "\phi_{ij}"'] \arrow[loop left, "\Gamma_i\times\Gamma_j"] & & & & P_{jk} \arrow[dll, "\pi_{jk}"] \arrow[dr, "\phi_{jk}"] \arrow[loop right, "\Gamma_j\times\Gamma_k"] & \\
V_i \arrow[loop left, "\Gamma_i"] & & & V_j \arrow[loop above, "\Gamma_j"] & & & V_k. \arrow[loop right, "\Gamma_k"]
\end{tikzcd}$$

Here as π_{jk} is étale one can show that the fibre product $P_{ij} \times_{V_j} P_{jk}$ exists in $\dot{\mathbf{Man}}$ using Assumptions 3.2(e) and 3.3(b). We have shown the actions of various combinations of $\Gamma_i, \Gamma_j, \Gamma_k$ on each space. In fact $\Gamma_i \times \Gamma_j \times \Gamma_k$ acts on the whole diagram, with all maps equivariant, but we have omitted the trivial actions (for instance, Γ_j, Γ_k act trivially on V_i).

As Γ_j acts freely on P_{ij}, it also acts freely on $P_{ij} \times_{V_j} P_{jk}$. Using Assumption 3.3 and the facts that $P_{ij} \times_{V_j} P_{jk}$ is Hausdorff and Γ_j is finite, we can show that the quotient $P_{ik} := (P_{ij} \times_{V_j} P_{jk})/\Gamma_j$ exists in $\dot{\mathbf{Man}}$, with projection $\Pi : P_{ij} \times_{V_j} P_{jk} \to P_{ik}$. The commuting actions of Γ_i, Γ_k on $P_{ij} \times_{V_j} P_{jk}$ descend to commuting actions of Γ_i, Γ_k on P_{ik}, such that Π is Γ_i- and Γ_k-equivariant. As $\pi_{ij} \circ \pi_{P_{ij}} : P_{ij} \times_{V_j} P_{jk} \to V_i$ and $\phi_{jk} \circ \pi_{P_{jk}} : P_{ij} \times_{V_j} P_{jk} \to V_k$ are Γ_j-invariant, they factor through Π, so there are unique smooth maps $\pi_{ik} : P_{ik} \to V_i$ and $\phi_{ik} : P_{ik} \to V_k$ such that $\pi_{ij} \circ \pi_{P_{ij}} = \pi_{ik} \circ \Pi$ and $\phi_{jk} \circ \pi_{P_{jk}} = \phi_{ik} \circ \Pi$.

It is now easy to check that $\Phi_{ik} = (P_{ik}, \pi_{ik}, \phi_{ik})$ satisfies Definition 6.56(a)–(d), and is a 1-morphism $\Phi_{ik} : (V_i, \Gamma_i, \psi_i) \to (V_k, \Gamma_k, \psi_k)$ over $(S, g \circ f)$. We write $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$, and call it the *composition of* 1*-morphisms*.

If we have three such 1-morphisms $\Phi_{ij}, \Phi_{jk}, \Phi_{kl}$, define

$$\begin{aligned}
\alpha_{\Phi_{kl},\Phi_{jk},\Phi_{ij}} : \bigl[P_{ij} \times_{V_j} \bigl((P_{jk} \times_{V_k} P_{kl})/\Gamma_k\bigr)\bigr]/\Gamma_j \\
\longrightarrow \bigl[\bigl((P_{ij} \times_{V_j} P_{jk})/\Gamma_j\bigr) \times_{V_k} P_{kl}\bigr]/\Gamma_k
\end{aligned}$$

to be the natural identification. Then $\alpha_{\Phi_{kl},\Phi_{jk},\Phi_{ij}}$ is a 2-isomorphism

$$\alpha_{\Phi_{kl},\Phi_{jk},\Phi_{ij}} : (\Phi_{kl} \circ \Phi_{jk}) \circ \Phi_{ij} \Longrightarrow \Phi_{kl} \circ (\Phi_{jk} \circ \Phi_{ij}).$$

That is, composition of 1-morphisms is associative up to canonical 2-isomorphism, as for weak 2-categories in §A.2.

For $\Phi_{ij} : (V_i, \Gamma_i, \psi_i) \to (V_j, \Gamma_j, \psi_j)$ a morphism over (S, f) as above with $S \subseteq \operatorname{Im} \psi_i \cap f^{-1}(\operatorname{Im} \psi_j)$, and for $T \subseteq \operatorname{Im} \psi_j \subseteq Y$ open with $S \subseteq f^{-1}(T)$, define

$$\begin{aligned}
\beta_{\Phi_{ij}} &: ((\bar\psi_i^{-1}(S) \times \Gamma_i) \times_{V_i} P_{ij})/\Gamma_i \longrightarrow P_{ij}, \\
\gamma_{\Phi_{ij}} &: (P_{ij} \times_{V_j} (\bar\psi_j^{-1}(T) \times \Gamma_j))/\Gamma_j \longrightarrow P_{ij},
\end{aligned}$$

to be the natural identifications. Then we have 2-isomorphisms

$$\beta_{\Phi_{ij}}:\Phi_{ij}\circ\mathrm{id}_{(V_i,\Gamma_i,\psi_i)}\Longrightarrow\Phi_{ij},$$
$$\gamma_{\Phi_{ij}}:\mathrm{id}_{(V_j,\Gamma_j,\psi_j)}\circ\Phi_{ij}\Longrightarrow\Phi_{ij},$$

where $\mathrm{id}_{(V_i,\Gamma_i,\psi_i)},\mathrm{id}_{(V_j,\Gamma_j,\psi_j)}$ are the identities over S,T, so identity 1-morphisms behave as they should up to canonical 2-isomorphism, as in §A.2.

Definition 6.59. Let $f:X\rightarrow Y$ be continuous, $(V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)$ be orbifold charts on X,Y, and $S\subseteq\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\subseteq X$ be open. Suppose $\Phi_{ij},\Phi'_{ij},\Phi''_{ij}:(V_i,\Gamma_i,\psi_i)\rightarrow(V_j,\Gamma_j,\psi_j)$ are 1-morphisms of orbifold charts over (S,f) with $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij})$, etc., and $\lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}$, $\lambda'_{ij}:\Phi'_{ij}\Rightarrow\Phi''_{ij}$ are 2-morphisms. The *vertical composition* $\lambda'_{ij}\odot\lambda_{ij}:\Phi_{ij}\Rightarrow\Phi''_{ij}$ is just the composition $\lambda'_{ij}\odot\lambda_{ij}=\lambda'_{ij}\circ\lambda_{ij}:P_{ij}\rightarrow P''_{ij}$ of morphisms in $\dot{\mathbf{Man}}$.

Now let $f:X\rightarrow Y$ and $g:Y\rightarrow Z$ be continuous, (V_i,Γ_i,ψ_i), (V_j,Γ_j,ψ_j), (V_k,Γ_k,ψ_k) be orbifold charts on X,Y,Z, and $T\subseteq\mathrm{Im}\,\psi_j\cap g^{-1}(\mathrm{Im}\,\psi_k)\subseteq Y$ and $S\subseteq\mathrm{Im}\,\psi_i\cap f^{-1}(T)\subseteq X$ be open. Suppose $\Phi_{ij},\Phi'_{ij}:(V_i,\Gamma_i,\psi_i)\rightarrow(V_j,\Gamma_j,\psi_j)$ are 1-morphisms of orbifold charts over (S,f), and $\Phi_{jk},\Phi'_{jk}:(V_j,\Gamma_j,\psi_j)\rightarrow(V_k,\Gamma_k,\psi_k)$ are 1-morphisms of orbifold charts over (T,g), with $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij})$, etc., and $\lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}$, $\lambda_{jk}:\Phi_{jk}\Rightarrow\Phi'_{jk}$ are 2-morphisms.

Write $\lambda_{jk}\times_{V_j}\lambda_{ij}:P_{ij}\times_{V_j}P_{jk}\rightarrow P'_{ij}\times_{V_j}P'_{jk}$ for the induced diffeomorphism of fibre products. It is Γ_j-equivariant, and so induces a unique diffeomorphism $\lambda_{jk}*\lambda_{ij}:P_{ik}=(P_{ij}\times_{V_j}P_{jk})/\Gamma_j\rightarrow(P'_{ij}\times_{V_j}P'_{jk})/\Gamma_j=P'_{ik}$. Then $\lambda_{jk}*\lambda_{ij}:\Phi_{jk}\circ\Phi_{ij}\Rightarrow\Phi'_{jk}\circ\Phi'_{ij}$ is a 2-morphism, *horizontal composition.*

As in Theorem 6.8, we have defined a weak 2-category, with objects orbifold charts. We can now follow §6.1–§6.2 from Definition 6.13 until Theorem 6.26, taking the $E_i,s_i,\hat\phi_{ijk},\hat\lambda_{ijk}$ to be zero throughout. This gives:

Theorem 6.60. *To any category* $\dot{\mathbf{Man}}$ *satisfying Assumptions* 3.1–3.3, *we can associate a corresponding weak* 2*-category* $\dot{\mathbf{Orb}}$ *of **Kuranishi orbifolds**, or just **orbifolds**. Objects of* $\dot{\mathbf{Orb}}$ *are* $\mathfrak{X}=(X,\mathcal{O})$ *for* X *a Hausdorff, second countable topological space and* $\mathcal{O}=\bigl(I,(V_i,\Gamma_i,\psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \lambda_{ijk,\ i,j,k\in I}\bigr)$ *an **orbifold structure on** X **of dimension*** $n\in\mathbb{N}$, *defined as in* §6.2 *but using orbifold charts, coordinate changes and* 2*-morphisms as above.*

Here is the analogue of Definition 4.29:

Definition 6.61. In Theorem 6.60 we write $\dot{\mathbf{Orb}}$ for the 2-category of orbifolds constructed from a category $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.3. By Example 3.8, the following categories from Chapter 2 are possible choices for $\dot{\mathbf{Man}}$:

$$\mathbf{Man},\mathbf{Man^c_{we}},\mathbf{Man^c},\mathbf{Man^{gc}},\mathbf{Man^{ac}},\mathbf{Man^{c,ac}}.$$

We write the corresponding 2-categories of orbifolds as follows:

$$\mathbf{Orb}_{\mathrm{Kur}},\mathbf{Orb^c_{we}},\mathbf{Orb^c},\mathbf{Orb^{gc}},\mathbf{Orb^{ac}},\mathbf{Orb^{c,ac}}.\tag{6.46}$$

Here we use '$\mathbf{Orb}_{\mathrm{Kur}}$' to distinguish it from the other (2-)categories of orbifolds discussed in §6.6.1.

In a similar way to Example 4.30, it is easy to prove:

Proposition 6.62. *There is a full, faithful weak* 2*-functor* $F_{\dot{\mathbf{Man}}}^{\dot{\mathbf{Orb}}} : \dot{\mathbf{Man}} \hookrightarrow \dot{\mathbf{Orb}}$ *embedding* $\dot{\mathbf{Man}}$ *as a full* (2-)*subcategory of* $\dot{\mathbf{Orb}}$*, which on objects maps* $F_{\dot{\mathbf{Man}}}^{\dot{\mathbf{Orb}}} : X \mapsto (X, \mathcal{O})$*, where* $\mathcal{O} = \bigl(\{0\}, (V_0, \Gamma_0, \psi_0), \Phi_{00}, \Lambda_{000}\bigr)$*, with indexing set* $I = \{0\}$*, one orbifold chart* (V_0, Γ_0, ψ_0) *with* $V_0 = X$*,* $\Gamma_0 = \{1\}$*, and* $\psi_0 = \mathrm{id}_X$*, one coordinate change* $\Phi_{00} = \mathrm{id}_{(V_0,\Gamma_0,\psi_0)}$*, and one* 2*-morphism* $\Lambda_{000} = \mathrm{id}_{\Phi_{00}}$*.*

We say that an orbifold $\mathfrak{X}$ ***is a manifold*** *if* $\mathfrak{X} \simeq F_{\dot{\mathbf{Man}}}^{\dot{\mathbf{Orb}}}(X)$ *in* $\dot{\mathbf{Orb}}$ *for some* $X \in \dot{\mathbf{Man}}$*.*

In §6.5, for a Kuranishi space $\boldsymbol{X}$, we defined the isotropy group $G_x\boldsymbol{X}$ for all $x \in \boldsymbol{X}$. In the same way, for an orbifold $\mathfrak{X}$ we have isotropy groups $G_x\mathfrak{X}$ for all $x \in \mathfrak{X}$. We use these to give a criterion for when an orbifold is a manifold.

Proposition 6.63. *An orbifold* $\mathfrak{X}$ *in* $\dot{\mathbf{Orb}}$ *is a manifold, in the sense of Proposition* 6.62*, if and only if* $G_x\mathfrak{X} = \{1\}$ *for all* $x \in \mathfrak{X}$*.*

Proof. The 'only if' part is obvious. For the 'if' part, suppose $\mathfrak{X} \in \dot{\mathbf{Orb}}$ with $G_x\mathfrak{X} = \{1\}$ for all $x \in \mathfrak{X}$. The proof of Theorem 6.54 in §6.5 implies that $\mathfrak{X} \simeq \mathfrak{X}'$ in $\dot{\mathbf{Orb}}$ for $\mathfrak{X}' = (X, \mathcal{O}')$ with $\mathcal{O}' = \bigl(I, (V_i, \Gamma_i, \psi_i)_{i\in I},\, \Phi_{ij,\, i,j\in I},\, \lambda_{ijk,\, i,j,k\in I}\bigr)$ an orbifold structure on X with $\Gamma_i = \{1\}$ for all $i \in I$.

Now X is a Hausdorff, second countable topological space, $\{\mathrm{Im}\,\psi_i : i \in I\}$ is an open cover of X, and $\{V_i : i \in I\}$ is a family of objects in $\dot{\mathbf{Man}}$ with $\psi_i : V_{i,\mathrm{top}} = V_{i,\mathrm{top}}/\{1\} \to \mathrm{Im}\,\psi_i$ a homeomorphism for $i \in I$. Using Assumption 3.2(e), we replace the V_i by diffeomorphic objects in $\dot{\mathbf{Man}}$ such that $V_{i,\mathrm{top}} = \mathrm{Im}\,\psi_i$, and $\psi_i : V_{i,\mathrm{top}} \to \mathrm{Im}\,\psi_i$ is the identity map for $i \in I$.

For $i,j \in I$, writing $V_{ij} \hookrightarrow V_i$ and $V_{ji} \hookrightarrow V_i$ for the open submanifolds with $V_{ij,\mathrm{top}} = V_{ji,\mathrm{top}} = \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$, using the coordinate change Φ_{ij} with $\Gamma_i = \Gamma_j = \{1\}$ we can show there is a unique diffeomorphism $\phi_{ij} : V_{ij} \to V_{ji}$ in $\dot{\mathbf{Man}}$ with $\phi_{ij,\mathrm{top}} = \mathrm{id}_{\mathrm{Im}\,\psi \cap \mathrm{Im}\,\psi_j}$. Therefore Assumption 3.3(b) makes X into an object in $\dot{\mathbf{Man}}$, such that $V_i \hookrightarrow X$ are open submanifolds for all $i \in I$. It is then easy to see that $\mathfrak{X}' \simeq F_{\dot{\mathbf{Man}}}^{\dot{\mathbf{Orb}}}(X)$ in $\dot{\mathbf{Orb}}$, and the proposition follows. □

Now let $\dot{\mathbf{Man}}$ satisfy all of Assumptions 3.1–3.7, not just Assumptions 3.1–3.3, so that we have both a 2-category of orbifolds $\dot{\mathbf{Orb}}$ above, and a 2-category of Kuranishi spaces $\dot{\mathbf{Kur}}$ from §6.2. In a similar way to Example 6.36 and Proposition 6.64, it is easy to prove:

Proposition 6.64. *There is a full, faithful weak* 2*-functor* $F_{\dot{\mathbf{Orb}}}^{\dot{\mathbf{Kur}}} : \dot{\mathbf{Orb}} \hookrightarrow \dot{\mathbf{Kur}}$ *embedding* $\dot{\mathbf{Orb}}$ *as a full* 2*-subcategory of* $\dot{\mathbf{Kur}}$*, which on objects maps* $F_{\dot{\mathbf{Orb}}}^{\dot{\mathbf{Kur}}} : (X, \mathcal{O}) \mapsto (X, \mathcal{K})$*, where for* $\mathcal{O}$ *as above,* $\mathcal{K} = \bigl(I, (V_i, 0, \Gamma_i, 0, \psi_i)_{i\in I},$ $(P_{ij}, \pi_{ij}, \phi_{ij}, 0)_{ij,\, i,j\in I},\, [P_{ijk}, \lambda_{ijk}, 0]_{i,j,k\in I}\bigr)$ *is the Kuranishi structure obtained by taking all the obstruction bundle data* $E_i, s_i, \hat\phi_{ijk}, \hat\lambda_{ijk}$ *to be zero.*

We say that a Kuranishi space $\boldsymbol{X}$ ***is an orbifold*** *if* $\boldsymbol{X} \simeq F_{\dot{\mathbf{Orb}}}^{\dot{\mathbf{Kur}}}(\mathfrak{X})$ *in* $\dot{\mathbf{Kur}}$ *for some* $\mathfrak{X} \in \dot{\mathbf{Orb}}$*.*

Theorem 10.52 in §10.4.4 gives a necessary and sufficient criterion for when a Kuranishi space $\boldsymbol{X}$ in $\dot{\mathbf{Kur}}$ is an orbifold.

6.6.3 Relation to previous definitions of orbifolds

We relate $\mathbf{Orb}_{\mathrm{Kur}}$ to previous definitions of (2-)categories of orbifolds.

Theorem 6.65. *The* 2*-category of Kuranishi orbifolds* $\mathbf{Orb}_{\mathrm{Kur}}$ *defined in Theorem* 6.60 *using* $\dot{\mathbf{Man}} = \mathbf{Man}$ *is equivalent as a weak* 2*-category to the* 2*-categories of orbifolds* $\mathbf{Orb}_{\mathrm{Pr}}$, $\mathbf{Orb}_{\mathrm{Le}}$, $\mathbf{Orb}_{\mathrm{ManSta}}$, $\mathbf{Orb}_{C^\infty\mathrm{Sta}}$ *in* [4, 65, 72, 88, 96] *described in* §6.6.1*. Also there is an equivalence of categories* $\mathrm{Ho}(\mathbf{Orb}_{\mathrm{Kur}}) \simeq \mathbf{Orb}_{\mathrm{MP}}$*, for* $\mathbf{Orb}_{\mathrm{MP}}$ *the category of orbifolds from Moerdijk and Pronk* [89, 90]*.*

Proof. Use the notation of §6.6.1. We will define a full and faithful weak 2-functor $F_{\mathbf{Orb}_{\mathrm{Kur}}}^{\mathbf{Orb}_{\mathrm{Le}}} : \mathbf{Orb}_{\mathrm{Kur}} \to \mathbf{Orb}_{\mathrm{Le}}$, which is an equivalence of 2-categories. Given an orbifold $\mathfrak{X} = (X, \mathcal{O})$ in our sense with $\mathcal{O} = \bigl(I, (V_i, \Gamma_i, \psi_i)_{i\in I},\ \Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij})_{i,j\in I},\ \lambda_{ijk,\ i,j,k\in I}\bigr)$, we define a natural proper étale Lie groupoid $[V \rightrightarrows U] = (U, V, s, t, u, i, m)$ in $\mathbf{Man}$ (that is, a groupoid-orbifold in the sense of [89, 90, 96] and [72, §3.3], as in §6.6.1(c),(i),(ii)) with $U = \coprod_{i\in I} V_i$, and $V = \coprod_{i,j\in I} P_{ij}$, and $s, t : V \to U$ given by $s = \coprod_{i,j\in I} \pi_{ij}$ and $t = \coprod_{i,j\in I} \phi_{ij}$, where the data $\lambda_{ijk,\ i,j,k\in I}$ gives the multiplication map $m : V \times_U V \to V$. We define $F_{\mathbf{Orb}_{\mathrm{Kur}}}^{\mathbf{Orb}_{\mathrm{Le}}}(\mathfrak{X}) = [V \rightrightarrows U]$.

By working through the definitions, it turns out that Lerman's definitions of 1- and 2-morphisms in $\mathbf{Orb}_{\mathrm{Le}}$ in terms of 'bibundles', when applied to groupoids $[V \rightrightarrows U]$ of the form $F_{\mathbf{Orb}_{\mathrm{Kur}}}^{\mathbf{Orb}_{\mathrm{Le}}}(\mathfrak{X})$, reduce exactly to 1- and 2-morphisms in $\mathbf{Orb}_{\mathrm{Kur}}$ as above. Thus, the definition of $F_{\mathbf{Orb}_{\mathrm{Kur}}}^{\mathbf{Orb}_{\mathrm{Le}}}$ on 1- and 2-morphisms, and that $F_{\mathbf{Orb}_{\mathrm{Kur}}}^{\mathbf{Orb}_{\mathrm{Le}}}$ is full and faithful, are immediate. The rest of the weak 2-functor data and conditions are straightforward. To show $F_{\mathbf{Orb}_{\mathrm{Kur}}}^{\mathbf{Orb}_{\mathrm{Le}}}$ is an equivalence, we need to show that every groupoid-orbifold $[V \rightrightarrows U]$ is equivalent in $\mathbf{Orb}_{\mathrm{Le}}$ to $F_{\mathbf{Orb}_{\mathrm{Kur}}}^{\mathbf{Orb}_{\mathrm{Le}}}(\mathfrak{X})$ for some $\mathfrak{X}$ in $\mathbf{Orb}_{\mathrm{Kur}}$. This can be done as in Moerdijk and Pronk [90, Proof of Th. 4.1].

The discussion in §6.6.1 now shows that our $\mathbf{Orb}_{\mathrm{Kur}}$ is equivalent as a weak 2-category to $\mathbf{Orb}_{\mathrm{Pr}}, \mathbf{Orb}_{\mathrm{Le}}, \mathbf{Orb}_{\mathrm{ManSta}}, \mathbf{Orb}_{C^\infty\mathrm{Sta}}$, and also that $\mathrm{Ho}(\mathbf{Orb}_{\mathrm{Kur}}) \simeq \mathbf{Orb}_{\mathrm{MP}}$ as categories. □

Combining Proposition 6.64 and Theorem 6.65 shows that the 2-categories of orbifolds $\mathbf{Orb}_{\mathrm{Pr}}, \mathbf{Orb}_{\mathrm{Le}}, \mathbf{Orb}_{\mathrm{ManSta}}, \mathbf{Orb}_{C^\infty\mathrm{Sta}}$ in [4, 65, 72, 88, 96] are equivalent to a full 2-subcategory of the 2-category of Kuranishi spaces $\mathbf{Kur}$. So (classical) orbifolds can be regarded as examples of Kuranishi spaces.

6.6.4 More about orbifolds, and orbifolds with corners

The material of §6.2.2, §6.2.3 and §6.3 for Kuranishi spaces (with corners) specializes easily to orbifolds (with corners). As in §6.6.2, this is a simplification, obtained by setting $E_i = s_i = 0$ in all Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$. Here are some brief comments on this:

(a) As in Proposition 6.30, if $\dot{\mathbf{Man}}, \ddot{\mathbf{Man}}$ satisfy Assumptions 3.1–3.3 and $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \to \ddot{\mathbf{Man}}$ satisfies Condition 3.20, we can define a natural weak 2-functor $F_{\dot{\mathbf{Orb}}}^{\ddot{\mathbf{Orb}}} : \dot{\mathbf{Orb}} \to \ddot{\mathbf{Orb}}$. As in Figure 6.1, we get a diagram Figure 6.3 of 2-functors between 2-categories of orbifolds.

(b) As in §6.2.3, if $\boldsymbol{P}$ is a discrete property of morphisms in $\dot{\mathbf{Man}}$, we can define when 1-morphisms in $\dot{\mathbf{Orb}}$ are $\boldsymbol{P}$, and the analogue of Proposition 6.34 holds. In the orbifold case, the definition of discrete properties $\boldsymbol{P}$ of morphisms in $\dot{\mathbf{Man}}$ is unnecessarily strong: we need only Definition 3.18(i)–(iv), not (v)–(viii), for a property $\boldsymbol{P}$ to lift nicely from $\dot{\mathbf{Man}}$ to $\dot{\mathbf{Orb}}$. For example, submersions in $\dot{\mathbf{Man}} = \mathbf{Man}$ satisfy (i)–(iv) but not (v)–(viii), and lift to a good notion of submersion in $\mathbf{Orb}_{\mathrm{Kur}}$.

Thus we can define many interesting 2-subcategories of the 2-categories of orbifolds in (6.46), as in Figure 6.2 for Kuranishi spaces.

(c) Suppose $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumption 3.22 in §3.4.1. (Actually, in Assumption 3.22(b) it is enough for $\dot{\mathbf{Man}}^{\mathbf{c}}$ to satisfy Assumptions 3.1–3.3, not Assumptions 3.1–3.7.) Then as in §6.6.2 we have a 2-category $\dot{\mathbf{Orb}}^{\mathbf{c}}$ of orbifolds associated to $\dot{\mathbf{Man}}^{\mathbf{c}}$. For instance, $\dot{\mathbf{Orb}}^{\mathbf{c}}$ could be $\mathbf{Orb}^{\mathbf{c}}, \mathbf{Orb}^{\mathbf{gc}}, \mathbf{Orb}^{\mathbf{ac}}$ or $\mathbf{Orb}^{\mathbf{c,ac}}$ from Definition 6.61. We will refer to objects of $\dot{\mathbf{Orb}}^{\mathbf{c}}$ as *orbifolds with corners.* We also write $\dot{\mathbf{Orb}}^{\mathbf{c}}_{\mathbf{si}}$ for the 2-subcategory of $\dot{\mathbf{Orb}}^{\mathbf{c}}$ with simple 1-morphisms, in the sense of (b).

As in §6.3, for any $\mathfrak{X}$ in $\dot{\mathbf{Orb}}^{\mathbf{c}}$ and $k = 0, \ldots, \dim \mathfrak{X}$ we can define the *k-corners* $C_k(\mathfrak{X})$, an object in $\dot{\mathbf{Orb}}^{\mathbf{c}}$ with $\dim C_k(\mathfrak{X}) = \dim \mathfrak{X} - k$, and a 1-morphism $\Pi_k : C_k(\mathfrak{X}) \to \mathfrak{X}$ in $\dot{\mathbf{Orb}}^{\mathbf{c}}$. We also write $\partial \mathfrak{X} = C_1(\mathfrak{X})$, the *boundary* of $\mathfrak{X}$, and we write $i_{\mathfrak{X}} = \Pi_1 : \partial \mathfrak{X} \to \mathfrak{X}$.

We define a 2-category $\check{\mathbf{Orb}}^{\mathbf{c}}$ from $\dot{\mathbf{Orb}}^{\mathbf{c}}$ with objects $\coprod_{n=0}^{\infty} \mathfrak{X}_n$ for $\mathfrak{X}_n$ in $\dot{\mathbf{Orb}}^{\mathbf{c}}$ with $\dim \mathfrak{X}_n = n$, and the *corner 2-functor* $C : \dot{\mathbf{Orb}}^{\mathbf{c}} \to \check{\mathbf{Orb}}^{\mathbf{c}}$. The restriction $C|_{\dot{\mathbf{Orb}}^{\mathbf{c}}_{\mathbf{si}}}$ decomposes as $C|_{\dot{\mathbf{Orb}}^{\mathbf{c}}_{\mathbf{si}}} = \coprod_{k=0}^{\infty} C_k$, where $C_k : \dot{\mathbf{Orb}}^{\mathbf{c}}_{\mathbf{si}} \to \dot{\mathbf{Orb}}^{\mathbf{c}}_{\mathbf{si}}$ is a weak 2-functor acting on objects by $\mathfrak{X} \mapsto C_k(\mathfrak{X})$. Examples of such corner 2-functors are given by the analogue of (6.36).

$$
\begin{tikzcd}
& \mathbf{Orb}_{\mathrm{Kur}} \arrow[dl, bend right] \arrow[d] \arrow[dr] \arrow[drr, bend left] & & \\
\mathbf{Orb}^{\mathbf{gc}} & \mathbf{Orb}^{\mathbf{c}} \arrow[l] \arrow[r] \arrow[d, dotted] & \mathbf{Orb}^{\mathbf{c,ac}} \arrow[r, dotted, shift left] & \mathbf{Orb}^{\mathbf{ac}} \arrow[l, shift left] \\
& \mathbf{Orb}^{\mathbf{c}}_{\mathbf{we}} & &
\end{tikzcd}
$$

Figure 6.3: 2-functors between 2-categories of orbifolds from Definition 6.61. Arrows '$\rightarrow$' are inclusions of 2-subcategories.

6.7 Proof of Theorems 4.13 and 6.16

Let $f : X \to Y$ be a continuous map of topological spaces, and $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, $(V_j, E_j, \Gamma_j, s_j, \psi_j)$ be Kuranishi neighbourhoods on X, Y. We must show that $\mathcal{H}om_f\big((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\big)$ from Theorem 6.16 is a stack on $\operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$, that is, that it satisfies Definition A.17(i)–(v). Parts (i),(ii) are immediate from the definition of restriction $|_T$ in Definition 6.13. When $\Gamma_i = \Gamma_j = \{1\}$ this will imply Theorem 4.13.

6.7.1 Definition A.17(iii) for $\mathcal{H}om_f((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j))$

For (iii), let $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ be open, $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij})$ and $\Phi'_{ij} = (P'_{ij}, \pi'_{ij}, \phi'_{ij}, \hat\phi'_{ij})$ be 1-morphisms $(V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ over (S, f), and $\Lambda_{ij}, \Lambda'_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ be 2-morphisms over (S, f). Suppose $\{T^a : a \in A\}$ is an open cover of S, such that $\Lambda_{ij}|_{T^a} = \Lambda'_{ij}|_{T^a}$ for all $a \in A$. Choose representatives $(\acute P_{ij}, \lambda_{ij}, \hat\lambda_{ij}), (\acute P'_{ij}, \lambda'_{ij}, \hat\lambda'_{ij})$ for $\Lambda_{ij}, \Lambda'_{ij}$. Then $\Lambda_{ij}|_{T^a} = \Lambda'_{ij}|_{T^a}$ means as in (6.3) that there exists an open neighbourhood $\ddot P^a_{ij}$ of $\pi_{ij}^{-1}(\bar\psi_i^{-1}(T^a))$ in $\acute P_{ij} \cap \acute P'_{ij}$ with

$$\lambda_{ij}|_{\ddot P^a_{ij}} = \lambda'_{ij}|_{\ddot P^a_{ij}} \quad \text{and} \quad \hat\lambda_{ij}|_{\ddot P^a_{ij}} = \hat\lambda'_{ij}|_{\ddot P^a_{ij}} + O\big(\pi^*_{ij}(s_i)\big) \quad \text{on } \ddot P^a_{ij}. \tag{6.47}$$

Set $\ddot P_{ij} = \bigcup_{a \in A} \ddot P^a_{ij}$, an open neighbourhood of $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$ in $\acute P_{ij} \cap \acute P'_{ij}$. Then (6.47) for all $a \in A$ implies (6.3) on $\ddot P_{ij}$ by Theorem 3.17(a), so $\Lambda_{ij} = \Lambda'_{ij}$. This proves Definition A.17(iii) for $\mathcal{H}om_f\big((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\big)$.

6.7.2 Definition A.17(iv) for $\mathcal{H}om_f((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j))$

For (iv), suppose S, Φ_{ij}, Φ'_{ij} are as in §6.7.1, $\{T^a : a \in A\}$ is an open cover of S, and $\Lambda^a_{ij} : \Phi_{ij}|_{T^a} \Rightarrow \Phi'_{ij}|_{T^a}$ are 2-morphisms over (T^a, f) for $a \in A$ with $\Lambda^a_{ij}|_{T^a \cap T^b} = \Lambda^b_{ij}|_{T^a \cap T^b}$ for all $a, b \in A$. Choose representatives $(\acute P^a_{ij}, \lambda^a_{ij}, \hat\lambda^a_{ij})$ for Λ^a_{ij} for $a \in A$, and making $\acute P^a_{ij}$ smaller if necessary, suppose that $\acute P^a_{ij} \cap \pi_{ij}^{-1}(s_i^{-1}(0)) = \pi_{ij}^{-1}(\bar\psi_i^{-1}(T^a))$. Then $\Lambda^a_{ij}|_{T^a \cap T^b} = \Lambda^b_{ij}|_{T^a \cap T^b}$ means there exists an open neighbourhood $\ddot P^{ab}_{ij}$ of $\pi_{ij}^{-1}(\bar\psi_i^{-1}(T^a \cap T^b))$ in $\acute P^a_{ij} \cap \acute P^b_{ij}$ with

$$\lambda^a_{ij}|_{\ddot P^{ab}_{ij}} = \lambda^b_{ij}|_{\ddot P^{ab}_{ij}} \quad \text{and} \quad \hat\lambda^a_{ij}|_{\ddot P^{ab}_{ij}} = \hat\lambda^b_{ij}|_{\ddot P^{ab}_{ij}} + O\big(\pi^*_{ij}(s_i)\big) \quad \text{on } \ddot P^{ab}_{ij}. \tag{6.48}$$

Here the second equation of (6.48) holds on $\acute P^a_{ij} \cap \acute P^b_{ij}$, as the $O(\pi^*_{ij}(s_i))$ condition is trivial away from $\pi_{ij}^{-1}(\bar\psi_i^{-1}(T^a \cap T^b))$.

Choose a partition of unity $\{\eta^a : a \in A\}$ on $\bigcup_{a \in A} \acute P^a_{ij} \subseteq P_{ij}$ subordinate to the open cover $\{\acute P^a_{ij} : a \in A\}$, as in §3.3.1(d). By averaging the η^a over the $\Gamma_i \times \Gamma_j$-action on P_{ij}, we suppose each η^a is Γ_i- and Γ_j-invariant. The *open* *support* of η^a is $\operatorname{supp}^\circ \eta^a = \{p \in \bigcup_{a' \in A} \acute{P}_{ij}^{a'} : \eta^a(p) > 0\}$, an open submanifold in $\bigcup_{a' \in A} \acute{P}_{ij}^{a'}$, and the *support* $\operatorname{supp} \eta^a = \overline{\operatorname{supp}^\circ \eta^a}$ of η^a is the closure of $\operatorname{supp}^\circ \eta^a$ in $\bigcup_{a' \in A} \acute{P}_{ij}^{a'}$. Consider the subset $\acute{P}_{ij} \subseteq P_{ij}$ given by

$$\begin{aligned} \acute{P}_{ij} = \big\{p \in \textstyle\bigcup_{a \in A} \acute{P}_{ij}^a : \text{if } a, b \in A \text{ with } p \in \operatorname{supp} \eta^a \cap \operatorname{supp} \eta^b \\ \text{then } \lambda_{ij}^a(p) = \lambda_{ij}^b(p)\big\}. \end{aligned} \tag{6.49}$$

We claim that $\acute{P}_{ij}$ is open in P_{ij}, and so an object in $\dot{\mathbf{Man}}$. To see this, note that $\acute{P}_{ij}$ is the complement in the open set $\bigcup_{a \in A} \acute{P}_{ij}^a \subseteq P_{ij}$ of the sets $S^{a,b}$ for all $a, b \in A$, where $S^{a,b} = \{p \in \operatorname{supp} \eta^a \cap \operatorname{supp} \eta^b : \lambda_{ij}^a(p) \neq \lambda_{ij}^b(p)\}$. Now $\lambda_{ij}^a, \lambda_{ij}^b : \acute{P}_{ij}^a \cap \acute{P}_{ij}^b \to P'_{ij}$ are smooth with $\pi'_{ij} \circ \lambda_{ij}^a = \pi'_{ij} \circ \lambda_{ij}^b$, where $\pi'_{ij} : P'_{ij} \to V_i$ is a principal Γ_j-bundle over $V'_{ij} \subseteq V_i$. Thus the condition $\lambda_{ij}^a \neq \lambda_{ij}^b$ is open and closed in $\acute{P}_{ij}^a \cap \acute{P}_{ij}^b$, so $S^{a,b}$ is open and closed in $\operatorname{supp} \eta^a \cap \operatorname{supp} \eta^b$, and closed in $\bigcup_{a \in A} \acute{P}_{ij}^a$. As $\{\eta^a : a \in A\}$ is locally finite, we see that $\acute{P}_{ij}$ is open.

Next we claim that $\acute{P}_{ij}$ contains $\pi_{ij}^{-1}(\bar{\psi}_i^{-1}(S))$. Let $p \in \pi_{ij}^{-1}(\bar{\psi}_i^{-1}(S))$. Then $p \in \pi_{ij}^{-1}(\bar{\psi}_i^{-1}(T^{a'})) \subseteq \acute{P}_{ij}^{a'}$ for some $a' \in A$ as $\bigcup_{a' \in A} T^{a'} = S$, so $p \in \acute{P}_{ij}^{a'} \subseteq \bigcup_{a \in A} \acute{P}_{ij}^a$. If $p \in \operatorname{supp} \eta^a \cap \operatorname{supp} \eta^b$ for $a, b \in A$ then $p \in \pi_{ij}^{-1}(\bar{\psi}_i^{-1}(T^a \cap T^b)) \subseteq \ddot{P}_{ij}^{ab}$, and the first equation of (6.48) gives $\lambda_{ij}^a(p) = \lambda_{ij}^b(p)$. Hence $p \in \acute{P}_{ij}$, proving the claim.

Define $\lambda_{ij} : \acute{P}_{ij} \to P'_{ij}$ by

$$\lambda_{ij}(p) = \lambda_{ij}^a(p) \quad \text{if } a \in A \text{ with } p \in \operatorname{supp} \eta^a. \tag{6.50}$$

This is well-defined by (6.49) as $\acute{P}_{ij} \subseteq \bigcup_{a \in A} \operatorname{supp} \eta^a$. As $\acute{P}_{ij}$ is covered by the open sets $\acute{P}_{ij} \cap \operatorname{supp}^\circ \eta^a$ for $a \in A$, and $\lambda_{ij} = \lambda_{ij}^a$ on $\acute{P}_{ij} \cap \operatorname{supp}^\circ \eta^a$ with λ_{ij}^a smooth and étale, λ_{ij} is smooth and étale by Assumption 3.3(a).

Define a morphism $\hat{\lambda}_{ij} : \pi_{ij}^*(E_i)|_{\acute{P}_{ij}} \to \mathcal{T}_{\phi_{ij}} V_j|_{\acute{P}_{ij}}$ by

$$\hat{\lambda}_{ij} = \textstyle\sum_{a \in A} \eta^a|_{\acute{P}_{ij}} \cdot \hat{\lambda}_{ij}^a, \tag{6.51}$$

where $\hat{\lambda}_{ij}^a$ is only defined on $\acute{P}_{ij} \cap \acute{P}_{ij}^a$, but $\eta^a \cdot \hat{\lambda}_{ij}^a$ is well-defined and smooth on $\acute{P}_{ij}$, being zero outside $\acute{P}_{ij}^a$.

For each $a \in A$, define $\ddot{P}_{ij}^a = \{p \in \acute{P}_{ij} \cap \acute{P}_{ij}^a : \lambda_{ij}(p) = \lambda_{ij}^a(p)\}$. As above this is open and closed in $\acute{P}_{ij} \cap \acute{P}_{ij}^a$ and so open in $\acute{P}_{ij} \cap \acute{P}_{ij}^a$, and contains $\pi_{ij}^{-1}(\bar{\psi}_i^{-1}(T^a))$, and by definition

$$\lambda_{ij}|_{\ddot{P}_{ij}^a} = \lambda_{ij}^a|_{\ddot{P}_{ij}^a}. \tag{6.52}$$

Using (6.51) in the first step, the second equation of (6.48) (which holds on $\acute{P}_{ij}^a \cap \acute{P}_{ij}^b$) in the second, and $\sum_{b \in A} \eta_b = 1$ in the fourth, we have

$$\begin{aligned} \hat{\lambda}_{ij}|_{\ddot{P}_{ij}^a} &= \textstyle\sum_{b \in A} \eta^b|_{\ddot{P}_{ij}^a} \cdot \hat{\lambda}_{ij}^b = \sum_{b \in A} \eta^b|_{\ddot{P}_{ij}^a} \cdot \big(\hat{\lambda}_{ij}^a + O(\pi_{ij}^*(s_i))\big) \\ &= \big(\textstyle\sum_{b \in A} \eta^b\big) \cdot \hat{\lambda}_{ij}^a|_{\ddot{P}_{ij}^a} + O(\pi_{ij}^*(s_i)) = \hat{\lambda}_{ij}^a|_{\ddot{P}_{ij}^a} + O(\pi_{ij}^*(s_i)). \end{aligned} \tag{6.53}$$

We now claim that $(\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij})$ satisfies Definition 6.4(a)–(c) over S. The Γ_i, Γ_j-equivariance of $\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}$ follows as the ingredients from which they are defined are Γ_i, Γ_j-equivariant. Equation (6.2) for $\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}$ on $\acute{P}_{ij} \cap \acute{P}^a_{ij}$ follows from (6.2) for $\acute{P}^a_{ij}, \lambda^a_{ij}, \hat{\lambda}^a_{ij}$, equation (6.53), and $\lambda_{ij} = \lambda^a_{ij}$ on $\ddot{P}^a_{ij}$, and the rest of (a)–(c) are already proved. Therefore $\Lambda_{ij} := [\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}]$ is a 2-morphism $\Phi_{ij} \Rightarrow \Phi'_{ij}$ over S. Equations (6.52)–(6.53) imply that $(\acute{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}) \sim_{T^a} (\acute{P}^a_{ij}, \lambda^a_{ij}, \hat{\lambda}^a_{ij})$ in the sense of Definition 6.4, so $\Lambda_{ij}|_{T^a} = \Lambda^a_{ij}$, for all $a \in A$. This proves Definition A.17(iv) for $\boldsymbol{\mathcal{H}om}_f\bigl((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\bigr)$.

6.7.3 Definition A.17(v) for $\boldsymbol{\mathcal{H}om}_f((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j))$

Let $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\psi_j)$ be open, and $\{T^a : a \in A\}$ be an open cover of S, and $\Phi^a_{ij} = (P^a_{ij}, \pi^a_{ij}, \phi^a_{ij}, \hat{\phi}^a_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be a 1-morphism of Kuranishi neighbourhoods over (T^a, f) for $a \in A$, and $\Lambda^{ab}_{ij} : \Phi^a_{ij}|_{T^a \cap T^b} \Rightarrow \Phi^b_{ij}|_{T^a \cap T^b}$ a 2-morphism over $(T^a \cap T^b, f)$ for all $a, b \in A$ such that $\Lambda^{bc}_{ij} \odot \Lambda^{ab}_{ij} = \Lambda^{ac}_{ij}$ over $(T^a \cap T^b \cap T^c, f)$ for all $a, b, c \in A$. Choose representatives $(\acute{P}^{ab}_{ij}, \lambda^{ab}_{ij}, \hat{\lambda}^{ab}_{ij})$ for Λ^{ab}_{ij} for all $a, b \in A$, so that (6.2) gives

$$\begin{aligned} \phi^b_{ij} \circ \lambda^{ab}_{ij} &= \phi^a_{ij}|_{\acute{P}^{ab}_{ij}} + \hat{\lambda}^{ab}_{ij} \circ (\pi^a_{ij})^*(s_i) + O\bigl((\pi^a_{ij})^*(s_i)^2\bigr) \text{ and} \\ (\lambda^{ab}_{ij})^*(\hat{\phi}^b_{ij}) &= \hat{\phi}^a_{ij}|_{\acute{P}^{ab}_{ij}} + (\phi^a_{ij})^*(\mathrm{d}s_j) \circ \hat{\lambda}^{ab}_{ij} + O\bigl((\pi^a_{ij})^*(s_i)\bigr) \text{ on } \acute{P}^{ab}_{ij}. \end{aligned} \tag{6.54}$$

Write $V^a_{ij} = \pi^a_{ij}(P^a_{ij})$, so that V^a_{ij} is an open neighbourhood of $\bar{\psi}_i^{-1}(T^a)$ in V_i for $a \in A$, and $\pi^a_{ij} : P^a_{ij} \to V^a_{ij}$ is a principal Γ_j-bundle, and similarly write $\acute{V}^{ab}_{ij} = \pi^a_{ij}(\acute{P}^{ab}_{ij})$ for $a, b \in A$. For simplicity, making P^a_{ij}, V^a_{ij} smaller if necessary, suppose that $V^a_{ij} \cap s_i^{-1}(0) = \bar{\psi}_i^{-1}(T^a)$.

From §6.1, $\Lambda^{bc}_{ij} \odot \Lambda^{ab}_{ij} = \Lambda^{ac}_{ij}$ means we can choose an open neighbourhood $\ddot{P}^{abc}_{ij}$ of $(\pi^a_{ij})^{-1}(\bar{\psi}_i^{-1}(T^a \cap T^b \cap T^c))$ in $(\lambda^{ab}_{ij})^{-1}(\acute{P}^{bc}_{ij}) \cap \acute{P}^{ac}_{ij} \subseteq P^a_{ij}$, such that

$$\begin{aligned} \lambda^{bc}_{ij} \circ \lambda^{ab}_{ij}|_{\ddot{P}^{abc}_{ij}} &= \lambda^{ac}_{ij}|_{\ddot{P}^{abc}_{ij}} \quad \text{and} \\ \hat{\lambda}^{ab}_{ij}|_{\ddot{P}^{abc}_{ij}} + \lambda^{ab}_{ij}|^*_{\ddot{P}^{abc}_{ij}}(\hat{\lambda}^{bc}_{ij}) &= \hat{\lambda}^{ac}_{ij}|_{\ddot{P}^{abc}_{ij}} + O\bigl((\pi^a_{ij})^*(s_i)\bigr) \quad \text{on } \ddot{P}^{abc}_{ij}. \end{aligned} \tag{6.55}$$

Choose a partition of unity $\{\eta^a : a \in A\}$ on $\bigcup_{a \in A} V^a_{ij} \subseteq V_i$ subordinate to the open cover $\{V^a_{ij} : a \in A\}$, as in §3.3.1(d). As in (6.49), define

$$\begin{aligned} V_{ij} = \bigl\{v \in \textstyle\bigcup_{a \in A} V^a_{ij} :{} & \text{if } a, b \in A \text{ with } v \in \operatorname{supp}\eta^a \cap \operatorname{supp}\eta^b \text{ then } v \in \acute{V}^{ab}_{ij}, \\ & \text{and if } a, b, c \in A \text{ with } v \in \operatorname{supp}\eta^a \cap \operatorname{supp}\eta^b \cap \operatorname{supp}\eta^c \\ & \text{then } \lambda^{bc}_{ij} \circ \lambda^{ab}_{ij} = \lambda^{ac}_{ij} \text{ on } (\pi^a_{ij})^{-1}(v)\bigr\}. \end{aligned} \tag{6.56}$$

As for the argument between (6.49) and (6.50), V_{ij} is an open neighbourhood of $\bar{\psi}_i^{-1}(S)$ in V_i, and is Γ_i-invariant as all the ingredients in (6.56) are.

Define $\dot P_{ij}$, initially as a topological space with the quotient topology, by

$$\dot P_{ij}=\bigl(\coprod_{a\in A}(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)\bigr)/\sim,\tag{6.57}$$

where $(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)\subseteq P_{ij}^a$ is open, and $\sim$ is the binary relation on $\coprod_{a\in A}(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)$ given by $p^a\sim p^b$ if $p^a\in(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)$ and $p^b\in(\pi_{ij}^b)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^b)$ for $a,b\in A$ with $p^b=\lambda_{ij}^{ab}(p^a)$. This is an equivalence relation by (6.56). Write $[p^a]$ for the $\sim$-equivalence class of p^a.

Define a map $\dot\pi_{ij}:\dot P_{ij}\to V_{ij}\subseteq V_i$ by $\dot\pi_{ij}:[p^a]\mapsto\pi_{ij}^a(p^a)$ for $p^a\in(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)$. This is well-defined as if $[p^a]=[p^b]$ then $p^a\sim p^b$, so $p^b=\lambda_{ij}^{ab}(p^a)$, and $\pi_{ij}^a(p^a)=\pi_{ij}^b(p^b)$ as $\pi_{ij}^b\circ\lambda_{ij}^{ab}=\pi_{ij}^a$ by Definition 6.4(b). The $\Gamma_i\times\Gamma_j$-actions on $(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)\subseteq P_{ij}^a$ induce a $\Gamma_i\times\Gamma_j$-action on $\dot P_{ij}$, and $\dot\pi_{ij}$ is Γ_i-equivariant and Γ_j-invariant.

Then $\dot\pi_{ij}:\dot P_{ij}\to V_{ij}$ is continuous and is a topological principal Γ_j-bundle, as it is built by gluing the topological principal Γ_j-bundles $\pi_{ij}^a:(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)\to V_{ij}\cap\operatorname{supp}^\circ\eta^a$ by the isomorphisms λ_{ij}^{ab} on overlaps $V_{ij}\cap\operatorname{supp}^\circ\eta^a\cap\operatorname{supp}^\circ\eta^b$, where the isomorphisms λ_{ij}^{ab} compose correctly by (6.56).

It follows that the natural morphisms $(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)\to\dot P_{ij}$ mapping $p^a\mapsto[p^a]$ for $a\in A$ are homeomorphisms with open subsets $\dot P_{ij}^a$ of $\dot P_{ij}$, and that $\dot P_{ij}$ is Hausdorff, and second countable, as $V_{ij}\in\dot{\mathbf{Man}}$ is by Assumption 3.2(b). Also the $(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)$ for $a\in A$ are objects in $\dot{\mathbf{Man}}$, and the gluing maps λ_{ij}^{ab} are diffeomorphisms between open submanifolds of $(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)$ and $(\pi_{ij}^b)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^b)$. Therefore Assumptions 3.2(e) and 3.3(b) make $\dot P_{ij}$ into an object in $\dot{\mathbf{Man}}$, with underlying topological space (6.57), such that the inclusion maps $(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)\to\dot P_{ij}$ are diffeomorphisms with open submanifolds $\dot P_{ij}^a$ of $\dot P_{ij}$ for $a\in A$, with $\{\dot P_{ij}^a:a\in A\}$ an open cover of $\dot P_{ij}$.

Furthermore, Assumption 3.3(a) now makes $\dot\pi_{ij}:\dot P_{ij}\to V_{ij}$ into a morphism in $\dot{\mathbf{Man}}$, locally modelled on $\pi_{ij}^a|_{\cdots}:(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)\to V_{ij}$, with $\dot P_{ij}^a=\dot\pi_{ij}^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)$. The topological $\Gamma_i\times\Gamma_j$-action on $\dot P_{ij}$ also lifts to a $\Gamma_i\times\Gamma_j$-action by morphisms in $\dot{\mathbf{Man}}$, where the $\dot P_{ij}^a$ are $\Gamma_i\times\Gamma_j$-invariant. As π_{ij}^a is étale, $\dot\pi_{ij}$ is étale, and as $\dot\pi_{ij}:\dot P_{ij}\to V_{ij}$ is a Γ_i-invariant topological principal Γ_j-bundle, it is a Γ_i-invariant principal Γ_j-bundle in $\dot{\mathbf{Man}}$.

Define $\lambda_{ij}^a:\dot P_{ij}^a\to P_{ij}^a$ in $\dot{\mathbf{Man}}$ to be the composition of the isomorphism $\dot P_{ij}^a\cong(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)$ with the inclusion $(\pi_{ij}^a)^{-1}(V_{ij}\cap\operatorname{supp}^\circ\eta^a)\hookrightarrow P_{ij}^a$. Then the definition of $\sim$ for $\dot P_{ij}$ in (6.57) implies that

$$\lambda_{ij}^{ab}\circ\lambda_{ij}^a|_{\dot P_{ij}^a\cap\dot P_{ij}^b}=\lambda_{ij}^b|_{\dot P_{ij}^a\cap\dot P_{ij}^b}:\dot P_{ij}^a\cap\dot P_{ij}^b\longrightarrow P_{ij}^b\qquad\text{for }a,b,\in A,\tag{6.58}$$

where $\lambda_{ij}^a(\dot P_{ij}^a\cap\dot P_{ij}^b)\subseteq\acute P_{ij}^{ab}$ by (6.56), so that $\lambda_{ij}^{ab}\circ\lambda_{ij}^a|_{\dot P_{ij}^a\cap\dot P_{ij}^b}$ is well defined.

We have smooth maps $\phi_{ij}^a\circ\lambda_{ij}^a:\dot P_{ij}^a\to V_j$ and morphisms $(\lambda_{ij}^a)^*(\hat\phi_{ij}^a):\dot\pi_{ij}^*(E_i)|_{\dot P_{ij}^a}\to\mathcal{T}_{\phi_{ij}^a\circ\lambda_{ij}^a}V_j$ for $a\in A$, such that for $a,b\in A$, applying $\circ\lambda_{ij}^a$ and

$(\lambda_{ij}^a)^*$ to the equations of (6.54) gives

$$(\phi_{ij}^b \circ \lambda_{ij}^b) = (\phi_{ij}^a \circ \lambda_{ij}^a) + (\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab}) \circ \dot{\pi}_{ij}^*(s_i) + O\bigl(\dot{\pi}_{ij}^*(s_i)^2\bigr), \tag{6.59}$$

$$(\lambda_{ij}^b)^*(\hat{\phi}_{ij}^b) = (\lambda_{ij}^a)^*(\hat{\phi}_{ij}^a) + (\phi_{ij}^a \circ \lambda_{ij}^a)^*(\mathrm{d}s_j) \circ (\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab}) + O\bigl(\dot{\pi}_{ij}^*(s_i)\bigr), \tag{6.60}$$

which hold on $\dot{P}_{ij}^a \cap \dot{P}_{ij}^b$ as $\lambda_{ij}^a(\dot{P}_{ij}^a \cap \dot{P}_{ij}^b) \subseteq \acute{P}_{ij}^{ab}$. For all $a, b, c \in A$, applying $(\lambda_{ij}^a)^*$ to the second equation of (6.55) and using (6.58) gives

$$(\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab}) + (\lambda_{ij}^b)^*(\hat{\lambda}_{ij}^{bc}) = (\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ac}) + O\bigl(\dot{\pi}_{ij}^*(s_i)\bigr) \tag{6.61}$$

on $(\lambda_{ij}^a)^{-1}(\ddot{P}_{ij}^{abc})$. In fact (6.61) holds on $\dot{P}_{ij}^a \cap \dot{P}_{ij}^b \cap \dot{P}_{ij}^c$, as the $O\bigl(\dot{\pi}_{ij}^*(s_i)\bigr)$ condition is trivial away from $\dot{\pi}_{ij}^{-1}(\bar{\psi}_i^{-1}(T^a \cap T^b \cap T^c))$.

Now (6.59) implies that $(\phi_{ij}^b \circ \lambda_{ij}^b) = (\phi_{ij}^a \circ \lambda_{ij}^a) + O(\dot{\pi}_{ij}^*(s_i))$ on $\dot{P}_{ij}^a \cap \dot{P}_{ij}^b$, where the $\dot{P}_{ij}^a$ are $\Gamma_i \times \Gamma_j$-invariant, and the $\phi_{ij}^a \circ \lambda_{ij}^a$ are $\Gamma_i \times \Gamma_j$-equivariant. Therefore by Theorem 3.17(c),(e) there exist a $\Gamma_i \times \Gamma_j$-invariant open neighbourhood $P_{ij} \hookrightarrow \dot{P}_{ij}$ of $\dot{\pi}_{ij}^*(s_i)^{-1}(0)$ in $\dot{P}_{ij}$, and a $\Gamma_i \times \Gamma_j$-equivariant morphism $\phi_{ij} : P_{ij} \to V_j$ in $\dot{\mathbf{Man}}$, such that for all $a \in A$ we have

$$\phi_{ij}^a \circ \lambda_{ij}^a|_{P_{ij} \cap \dot{P}_{ij}^a} = \phi_{ij}|_{P_{ij} \cap \dot{P}_{ij}^a} + O\bigl(\dot{\pi}_{ij}^*(s_i)\bigr) \quad \text{on } P_{ij} \cap \dot{P}_{ij}^a. \tag{6.62}$$

Define $\pi_{ij} = \dot{\pi}_{ij}|_{P_{ij}} : P_{ij} \to V_i$.

Applying Theorem 3.17(i) to (6.62) shows we may choose a morphism $\hat{\mu}_{ij}^a : \pi_{ij}^*(E_i)|_{P_{ij} \cap \dot{P}_{ij}^a} \to \mathcal{T}_{\phi_{ij}} V_j|_{P_{ij} \cap \dot{P}_{ij}^a}$ with

$$\phi_{ij}^a \circ \lambda_{ij}^a|_{P_{ij} \cap \dot{P}_{ij}^a} = \phi_{ij}|_{P_{ij} \cap \dot{P}_{ij}^a} + \hat{\mu}_{ij}^a \circ \pi_{ij}^*(s_i) + O\bigl(\pi_{ij}^*(s_i)^2\bigr) \text{ on } P_{ij} \cap \dot{P}_{ij}^a. \tag{6.63}$$

Since $\phi_{ij}^a \circ \lambda_{ij}^a|_{P_{ij} \cap \dot{P}_{ij}^a}$ and $\phi_{ij}|_{P_{ij} \cap \dot{P}_{ij}^a}$ are $\Gamma_i \times \Gamma_j$-equivariant, (6.63) also holds with $\hat{\mu}_{ij}^a$ replaced by $(\gamma_i, \gamma_j)^*(\hat{\mu}_{ij}^a)$ for $(\gamma_i, \gamma_j) \in \Gamma_i \times \Gamma_j$. Averaging $(\gamma_i, \gamma_j)^*(\hat{\mu}_{ij}^a)$ over $(\gamma_i, \gamma_j) \in \Gamma_i \times \Gamma_j$ and using Theorem 3.17(m), we see that we may take $\hat{\mu}_{ij}^a$ to be $\Gamma_i \times \Gamma_j$-equivariant.

Using the notation of Definition 3.15(v), and applying Theorem 3.17(g), we see that we can choose a morphism $\hat{\lambda}_{ij}^a : \pi_{ij}^*(E_i)|_{P_{ij} \cap \dot{P}_{ij}^a} \to \mathcal{T}_{\phi_{ij}} V_j|_{P_{ij} \cap \dot{P}_{ij}^a}$ with

$$\hat{\lambda}_{ij}^a = \hat{\mu}_{ij}^a + \sum_{b \in A} \pi_{ij}^*(\eta^b)|_{P_{ij} \cap \dot{P}_{ij}^a} \cdot \bigl(\hat{\mu}_{ij}^b - \hat{\mu}_{ij}^a - (\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab})\bigr) + O\bigl(\pi_{ij}^*(s_i)\bigr). \tag{6.64}$$

Here $(\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab})$ in (6.64) is a morphism $\pi_{ij}^*(E_i)|_{\cdots} \to \mathcal{T}_{\phi_{ij}^a \circ \lambda_{ij}^a} V_j|_{\cdots}$, but by (6.63) and Theorem 3.17(g) there exists $(\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab})' : \pi_{ij}^*(E_i)|_{\cdots} \to \mathcal{T}_{\phi_{ij}} V_j|_{\cdots}$, unique up to $O(\pi_{ij}^*(s_i))$, with $(\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab})' = (\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab}) + O(\pi_{ij}^*(s_i))$ as in Definition 3.15(v), and we replace $(\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab})$ in (6.64) by $(\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab})'$ to define $\hat{\lambda}_{ij}^a$. By averaging $\hat{\lambda}_{ij}^a$ over the $\Gamma_i \times \Gamma_j$-action, we can suppose it is $\Gamma_i \times \Gamma_j$-equivariant.

Combining (6.59) with (6.63) for a, b and using Theorem 3.17(l) to go from ϕ_{ij} to $\phi_{ij}^b \circ \lambda_{ij}^b$ to $\phi_{ij}^a \circ \lambda_{ij}^a$ to ϕ_{ij} we see that

$$\phi_{ij} = \phi_{ij} + \bigl(\hat{\mu}_{ij}^b - \hat{\mu}_{ij}^a - (\lambda_{ij}^a)^*(\hat{\lambda}_{ij}^{ab})\bigr) \circ \pi_{ij}^*(s_i) + O\bigl(\pi_{ij}^*(s_i)^2\bigr) \text{ on } P_{ij} \cap \dot{P}_{ij}^a \cap \dot{P}_{ij}^b.$$

Hence Theorem 3.17(a),(m) and local finiteness of $\{\pi_{ij}^*(\eta^b) : b \in A\}$ give

$$\phi_{ij} = \phi_{ij} + \Bigl(\sum_{b\in A} \pi_{ij}^*(\eta^b) \cdot \bigl(\hat\mu_{ij}^b - \hat\mu_{ij}^a - (\lambda_{ij}^a)^*(\hat\lambda_{ij}^{ab})\bigr)\Bigr) \circ \pi_{ij}^*(s_i) + O\bigl(\pi_{ij}^*(s_i)^2\bigr)$$

on P_{ij}. Combining this with (6.63), (6.64) and Theorem 3.17(m) shows that

$$\phi_{ij}^a \circ \lambda_{ij}^a|_{P_{ij}\cap \dot P_{ij}^a} = \phi_{ij}|_{P_{ij}\cap \dot P_{ij}^a} + \hat\lambda_{ij}^a \circ \pi_{ij}^*(s_i) + O\bigl(\pi_{ij}^*(s_i)^2\bigr) \text{ on } P_{ij} \cap \dot P_{ij}^a. \tag{6.65}$$

For all $a, b \in A$, on $P_{ij} \cap \dot P_{ij}^a \cap \dot P_{ij}^b$ we have

$$\begin{aligned}
\hat\lambda_{ij}^b - \hat\lambda_{ij}^a &= \hat\mu_{ij}^b - \hat\mu_{ij}^a + \sum_{c\in A} \pi_{ij}^*(\eta^c) \cdot \bigl(\hat\mu_{ij}^c - \hat\mu_{ij}^b - (\lambda_{ij}^b)^*(\hat\lambda_{ij}^{bc}) \\
&\qquad\qquad - \hat\mu_{ij}^c + \hat\mu_{ij}^a + (\lambda_{ij}^a)^*(\hat\lambda_{ij}^{ac})\bigr) + O\bigl(\pi_{ij}^*(s_i)\bigr) \\
&= \hat\mu_{ij}^b - \hat\mu_{ij}^a + \sum_{c\in A} \pi_{ij}^*(\eta^c) \cdot \bigl(-\hat\mu_{ij}^b + \hat\mu_{ij}^a + (\lambda_{ij}^a)^*(\hat\lambda_{ij}^{ab})\bigr) + O\bigl(\pi_{ij}^*(s_i)\bigr) \\
&= (\lambda_{ij}^a)^*(\hat\lambda_{ij}^{ab}) + O(\pi_{ij}^*(s_i)),
\end{aligned} \tag{6.66}$$

using (6.64) in the first step, (6.61) in the second, and $\sum_c \eta^c = 1$ in the third.

By Theorem 3.17(f),(h) we choose $\hat\phi_{ij}^{\prime a} : \pi_{ij}^*(E_i)|_{P_{ij}\cap \dot P_{ij}^a} \to \phi_{ij}^*(E_j)|_{P_{ij}\cap \dot P_{ij}^a}$ with

$$\hat\phi_{ij}^{\prime a} = (\lambda_{ij}^a)^*(\hat\phi_{ij}^a) - \phi_{ij}^*(\mathrm{d}s_j) \circ \hat\lambda_{ij}^a + O(\pi_{ij}^*(s_i)), \tag{6.67}$$

uniquely up to $O(\pi_{ij}^*(s_i))$. By averaging over the $\Gamma_i \times \Gamma_j$-action we can suppose $\hat\phi_{ij}^{\prime a}$ is Γ_i- and Γ_j-equivariant. Define a Γ_i- and Γ_j-equivariant morphism $\hat\phi_{ij} : \pi_{ij}^*(E_i) \to \phi_{ij}^*(E_j)$ on P_{ij} by

$$\hat\phi_{ij} = \textstyle\sum_{a\in A} \pi_{ij}^*(\eta^a) \cdot \hat\phi_{ij}^{\prime a}. \tag{6.68}$$

Then for each $a \in A$, on $P_{ij} \cap \dot P_{ij}^a$ we have

$$\begin{aligned}
(\lambda_{ij}^a)^*(\hat\phi_{ij}^a) &= \textstyle\sum_{b\in A} \pi_{ij}^*(\eta^b) \cdot \bigl[(\lambda_{ij}^a)^*(\hat\phi_{ij}^a) \\
&\quad + (\phi_{ij}^a \circ \lambda_{ij}^a)^*(\mathrm{d}s_j) \circ [(\lambda_{ij}^a)^*(\hat\lambda_{ij}^{ab}) - \hat\lambda_{ij}^b + \hat\lambda_{ij}^a]\bigr] + O(\pi_{ij}^*(s_i)) \\
&= \textstyle\sum_{b\in A} \pi_{ij}^*(\eta^b) \cdot \bigl[(\lambda_{ij}^b)^*(\hat\phi_{ij}^b) - \phi_{ij}^*(\mathrm{d}s_j) \circ \hat\lambda_{ij}^b + \phi_{ij}^*(\mathrm{d}s_j) \circ \hat\lambda_{ij}^a\bigr] + O(\pi_{ij}^*(s_i)) \\
&= \textstyle\sum_{b\in A} \pi_{ij}^*(\eta^b) \cdot \bigl[\hat\phi_{ij}^{\prime b} + \phi_{ij}^*(\mathrm{d}s_j) \circ \hat\lambda_{ij}^a\bigr] + O(\pi_{ij}^*(s_i)) \\
&= \hat\phi_{ij}|_{\dot P_{ij}^a} + \phi_{ij}^*(\mathrm{d}s_j) \circ \hat\lambda_{ij}^a + O(\pi_{ij}^*(s_i)),
\end{aligned} \tag{6.69}$$

using (6.66) and $\{\eta^b : b \in A\}$ a partition of unity in the first step, (6.60) and $\phi_{ij}^a \circ \lambda_{ij}^a = \phi_{ij}|_{\dot P_{ij}^a} + O(\pi_{ij}^*(s_i))$ from (6.65) in the second, (6.67) in the third, and (6.68) and $\{\eta^b : b \in A\}$ a partition of unity in the fourth.

We have already proved $\Phi_{ij} := (P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij})$ satisfies Definition 6.2(a)–(d). Parts (e),(f) hold on $P_{ij} \cap \dot P_{ij}^a \subseteq P_{ij}$ by (6.65), (6.69) and Definition 6.2(e),(f) for Φ_{ij}^a, for each $a \in A$, so they hold on $\bigcup_{a\in A}(P_{ij} \cap \dot P_{ij}^a) = P_{ij}$. Thus $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ is a 1-morphism over (S, f).

Equations (6.65) and (6.69) imply that $\Lambda_{ij}^a := [P_{ij} \cap \dot{P}_{ij}^a, \lambda_{ij}^a|_{P_{ij} \cap \dot{P}_{ij}^a}, \hat{\lambda}_{ij}^a]$ is a 2-morphism $\Phi_{ij}|_{T^a} \Rightarrow \Phi_{ij}^a$ over (T^a, f) for all $a \in A$. Equations (6.58) and (6.66) imply that $\Lambda_{ij}^b|_{T^a \cap T^b} = \Lambda_{ij}^{ab} \odot \Lambda_{ij}^a|_{T^a \cap T^b}$ for all $a, b \in A$. This proves Definition A.17(v), showing that $\boldsymbol{\mathcal{H}om}_f\big((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)\big)$ is a stack on $\mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\psi_j)$, and completes the first part of Theorem 6.16.

6.7.4 $\boldsymbol{\mathcal{E}qu}(\cdots)$ is a substack of $\boldsymbol{\mathcal{H}om}(\cdots)$

Now we take $X = Y$ and $f = \mathrm{id}_X$. In this subsection, we will by an abuse of notation treat the weak 2-category $\dot{\mathbf{K}}\mathbf{N}_S(X)$ defined in §6.1 as if it were a strict 2-category. That is, we will pretend the 2-morphisms $\boldsymbol{\alpha}_{\Phi_{kl},\Phi_{jk},\Phi_{ij}}, \boldsymbol{\beta}_{\Phi_{ij}}, \boldsymbol{\gamma}_{\Phi_{ij}}$ in (6.7) and (6.8) are identities or omit them, and we will omit brackets in compositions of 1-morphisms such as $\Phi_{kl} \circ \Phi_{jk} \circ \Phi_{ij}$. This is permissible as every weak 2-category can be strictified. We do it because otherwise diagrams such as Figure 6.4 would become too big.

Definition A.17(i)–(iv) for $\boldsymbol{\mathcal{E}qu}((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j))$ are immediate from (i)–(iv) for $\boldsymbol{\mathcal{H}om}((V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j))$. For (v), we must show that in the last part of the proof in §6.7.3, if the Φ_{ij}^a are coordinate changes over T^a (i.e. equivalences in $\dot{\mathbf{K}}\mathbf{N}_{T^a}(X)$), then the Φ_{ij} we construct with 2-morphisms $\Lambda_{ij}^a : \Phi_{ij}|_{T^a} \Rightarrow \Phi_{ij}^a$ for $a \in A$ is a coordinate change over S.

Let $S, \{T^a : a \in A\}, \Phi_{ij}^a, \Lambda_{ij}^{ab}, \Phi_{ij}, \Lambda_{ij}^a$ be as in §6.7.3, but with $X = Y$, $f = \mathrm{id}_X$ and all the Φ_{ij}^a coordinate changes. Since Φ_{ij}^a is an equivalence in $\dot{\mathbf{K}}\mathbf{N}_{T^a}(X)$, we may choose a coordinate change $\Phi_{ji}^a : (V_j, E_j, \Gamma_j, s_j, \psi_j) \to (V_i, E_i, \Gamma_i, s_i, \psi_i)$ over T^a and 2-morphisms $\mathrm{I}_i^a : \Phi_{ji}^a \circ \Phi_{ij}^a \Rightarrow \mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)}$ and $\mathrm{K}_j^a : \Phi_{ij}^a \circ \Phi_{ji}^a \Rightarrow \mathrm{id}_{(V_j,E_j,\Gamma_j,s_j,\psi_j)}$ for all $a \in A$. By Proposition A.5 we can suppose these satisfy

$$\mathrm{id}_{\Phi_{ij}^a} * \mathrm{I}_i^a = \mathrm{K}_j^a * \mathrm{id}_{\Phi_{ij}^a} \quad \text{and} \quad \mathrm{id}_{\Phi_{ji}^a} * \mathrm{K}_j^a = \mathrm{I}_i^a * \mathrm{id}_{\Phi_{ji}^a}. \tag{6.70}$$

Define 2-morphisms $\mathrm{M}_{ji}^{ab} : \Phi_{ji}^a|_{T^a \cap T^b} \Rightarrow \Phi_{ji}^b|_{T^a \cap T^b}$ over $T^a \cap T^b$ for all $a, b \in A$ to be the vertical composition

$$\Phi_{ji}^a|_{T^a \cap T^b} \xRightarrow{\mathrm{id}_{\Phi_{ji}^a} * (\mathrm{K}_j^b)^{-1}} \Phi_{ji}^a \circ \Phi_{ij}^b \circ \Phi_{ji}^b \xRightarrow{\mathrm{id}_{\Phi_{ji}^a} * (\Lambda_{ij}^{ab})^{-1} * \mathrm{id}_{\Phi_{ji}^b}} \Phi_{ji}^a \circ \Phi_{ij}^a \circ \Phi_{ji}^b \xRightarrow{\mathrm{I}_i^a * \mathrm{id}_{\Phi_{ji}^b}} \Phi_{ji}^b|_{T^a \cap T^b}. \tag{6.71}$$

For $a, b, c \in A$, consider the diagram Figure 6.4 of 2-morphisms over $T^a \cap T^b \cap T^c$. The three outer quadrilaterals commute by the definition (6.71) of M_{ji}^{ab}. Eight inner quadrilaterals commute by compatibility of horizontal and vertical composition, a 2-gon commutes by (6.70), and a triangle commutes as $\Lambda_{ij}^{bc} \odot \Lambda_{ij}^{ab} = \Lambda_{ij}^{ac}$. Hence Figure 6.4 commutes, which shows that $\mathrm{M}_{ji}^{bc} \odot \mathrm{M}_{ji}^{ab} = \mathrm{M}_{ji}^{ac}$ over $T^a \cap T^b \cap T^c$ for all $a, b, c \in A$.

Thus by Definition A.17(v) for $\boldsymbol{\mathcal{H}om}((V_j, E_j, \Gamma_j, s_j, \psi_j), (V_i, E_i, \Gamma_i, s_i, \psi_i))$, proved in §6.7.3, there exists a 1-morphism $\Phi_{ji} : (V_j, E_j, \Gamma_j, s_j, \psi_j) \to (V_i, E_i, \Gamma_i, s_i, \psi_i)$ over S and 2-morphisms $\mathrm{M}_{ji}^a : \Phi_{ji}|_{T^a} \Rightarrow \Phi_{ji}^a$ over T^a for $a \in A$, such that $\mathrm{M}_{ji}^b|_{T^a \cap T^b} = \mathrm{M}_{ji}^{ab} \odot \mathrm{M}_{ji}^a|_{T^a \cap T^b}$ over $T^a \cap T^b$ for all $a, b \in A$.

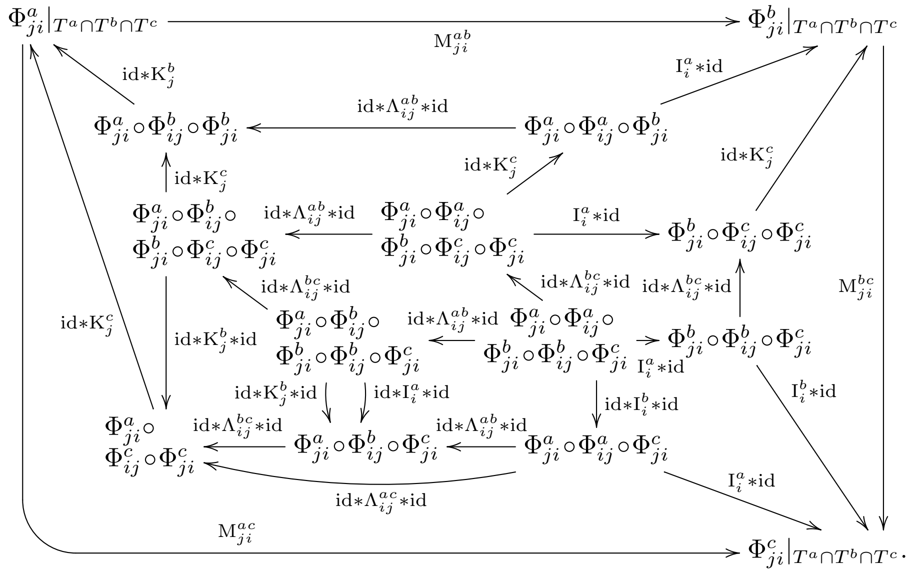
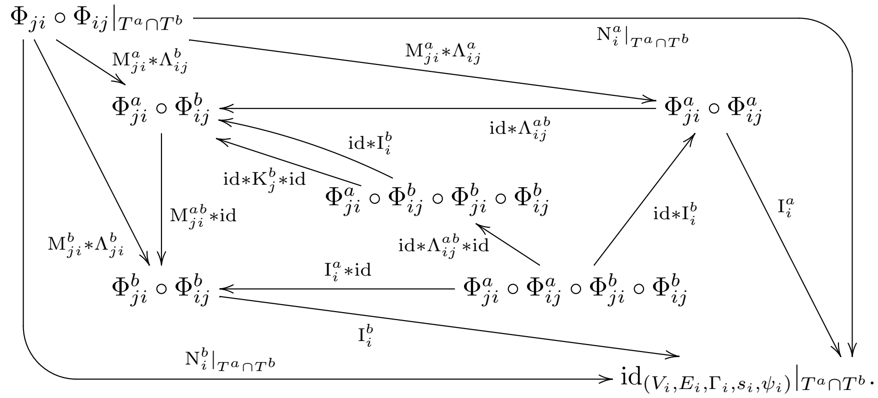


Figure 6.4: Proof that $\mathrm{M}_{ji}^{bc} \odot \mathrm{M}_{ji}^{ab} = \mathrm{M}_{ji}^{ac}$

For each $a \in A$, define a 2-morphism $\mathrm{N}_i^a : (\Phi_{ji} \circ \Phi_{ij})|_{T^a} \Rightarrow \mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)}|_{T^a}$ by the vertical composition

$$(\Phi_{ji} \circ \Phi_{ij})|_{T^a} \xRightarrow{\mathrm{M}_{ji}^a * \Lambda_{ij}^a} \Phi_{ji}^a \circ \Phi_{ij}^a \xRightarrow{\mathrm{I}_i^a} \mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)}|_{T^a}. \quad (6.72)$$

Then the following diagram commutes by (6.70), $\Lambda_{ij}^b = \Lambda_{ij}^{ab} \odot \Lambda_{ij}^a$, $\mathrm{M}_{ji}^b = \mathrm{M}_{ji}^{ab} \odot \mathrm{M}_{ji}^a$, the definitions of $\mathrm{M}_{ji}^{ab}, \mathrm{N}_i^a$ in (6.71) and (6.72), and compatibility of horizontal and vertical composition:



Hence $\mathrm{N}_i^a|_{T^a \cap T^b} = \mathrm{N}_i^b|_{T^a \cap T^b}$ for all $a, b \in A$. Therefore by Definition A.17(iv) for $\boldsymbol{\mathcal{H}om}((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_i,E_i,\Gamma_i,s_i,\psi_i))$, proved in §6.7.2, there is a unique 2-morphism $\mathrm{N}_i : \Phi_{ji} \circ \Phi_{ij} \Rightarrow \mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)}$ over S with $\mathrm{N}_i|_{T^a} = \mathrm{N}_i^a$ for all $a \in A$.

Similarly we construct $\mathrm{O}_j : \Phi_{ij} \circ \Phi_{ji} \Rightarrow \mathrm{id}_{(V_j,E_j,\Gamma_j,s_j,\psi_j)}$. These $\Phi_{ji}, \mathrm{N}_i, \mathrm{O}_j$ show Φ_{ij} is an equivalence in $\dot{\mathbf{K}}\mathbf{N}_S(X)$, and so a coordinate change. This gives Definition A.17(v) for $\boldsymbol{\mathcal{E}qu}((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j))$, which is thus a substack of $\boldsymbol{\mathcal{H}om}((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j))$, completing the proof of Theorem 6.16.

Chapter 7

Relation to other Kuranishi-type spaces (To be rewritten.)

We now compare our Kuranishi spaces in Chapter 6 with Kuranishi-type spaces developed by other authors. In §7.1–§7.4 we discuss various definitions of Kuranishi space, and of good coordinate system, in the work of Fukaya, Oh, Ohta and Ono [19–39], McDuff and Wehrheim [77, 78, 80–83], and Dingyu Yang [110–112]. We use Yang's work to connect our Kuranishi spaces with the polyfold theory of Hofer, Wysocki and Zehnder [46–53].

To improve compatibility with Chapter 6, we have made some small changes in notation compared to our sources, without changing the content. We hope the authors concerned will not mind this. Examples 7.2, 7.5, ... explain the relationship between the material we explain, and the definitions of §6.1. Section 7.5 will prove that all the structures we discuss can be converted to Kuranishi spaces in the sense of §6.2. The proof of Theorem 7.26 is deferred until §7.6.

7.1 Fukaya–Oh–Ohta–Ono's Kuranishi spaces

'Kuranishi spaces' are used in the work of Fukaya, Oh, Ohta and Ono [19–39] as the geometric structure on moduli spaces of J-holomorphic curves. Initially introduced by Fukaya and Ono [39, §5] in 1999, the definition has changed several times as their work has evolved.

This section explains their most recent definition of Kuranishi space, taken from [30, §4]. As in the rest of our book 'Kuranishi neighbourhood', 'coordinate change' and 'Kuranishi space' have a different meaning, we will use the terms 'FOOO Kuranishi neighbourhood', 'FOOO coordinate change' and 'FOOO Kuranishi space' below to refer to concepts from [30].

For the next definitions, let X be a compact, metrizable topological space.

Definition 7.1. A *FOOO Kuranishi neighbourhood* on X is a quintuple (V, E, Γ, s, ψ) such that:

(a) V is a classical manifold, or manifold with corners ($V \in$ **Man** or **Man^c**).

(b) E is a finite-dimensional real vector space.

(c) Γ is a finite group with a smooth, effective action on V, and a linear representation on E.

(d) $s : V \to E$ is a Γ-equivariant smooth map.

(e) ψ is a homeomorphism from $s^{-1}(0)/\Gamma$ to an open subset $\mathrm{Im}\,\psi$ in X, where $\mathrm{Im}\,\psi = \{\psi(x\Gamma) : x \in s^{-1}(0)\}$ is the image of ψ, and is called the *footprint* of (V, E, Γ, s, ψ).

We will write $\bar{\psi} : s^{-1}(0) \to \mathrm{Im}\,\psi \subseteq X$ for the composition of ψ with the projection $s^{-1}(0) \to s^{-1}(0)/\Gamma$.

Now let $p \in X$. A *FOOO Kuranishi neighbourhood of* p *in* X is a FOOO Kuranishi neighbourhood $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ with a distinguished point $o_p \in V_p$ such that o_p is fixed by Γ_p, and $s_p(o_p) = 0$, and $\psi_p([o_p]) = p$. Then o_p is unique.

Example 7.2. For our Kuranishi neighbourhoods $(V', E', \Gamma', s', \psi')$ in Definition 6.1, $\pi' : E' \to V'$ is a Γ'-equivariant vector bundle, and $s' : V' \to E'$ a Γ'-equivariant smooth section. Also Γ' is not required to act effectively on V'.

To make a FOOO Kuranishi neighbourhood (V, E, Γ, s, ψ) into one of our Kuranishi neighbourhoods $(V', E', \Gamma', s', \psi')$, take $V' = V$, $\Gamma' = \Gamma$, $\psi' = \psi$, let $\pi' : E' \to V'$ be the trivial vector bundle $\pi_V : V \times E \to V$ with fibre E, and $s' = (\mathrm{id}, s) : V \to V \times E$. Thus, FOOO Kuranishi neighbourhoods correspond to special examples of our Kuranishi neighbourhoods $(V', E', \Gamma', s', \psi')$, in which $\pi' : E' \to V'$ is a trivial vector bundle, and Γ' acts effectively on V'.

By an abuse of notation, we will sometimes identify FOOO Kuranishi neighbourhoods with the corresponding Kuranishi neighbourhoods in §6.1. That is, we will use E to denote both a vector space, and the corresponding trivial vector bundle over V, and s to denote both a map, and a section of a trivial bundle. Fukaya et al. [30, Def. 4.3(4)] also make the same abuse of notation.

Definition 7.3. Let $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, $(V_j, E_j, \Gamma_j, s_j, \psi_j)$ be FOOO Kuranishi neighbourhoods on X. Suppose $S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \subseteq X$ is an open subset of the intersection of the footprints $\mathrm{Im}\,\psi_i, \mathrm{Im}\,\psi_j \subseteq X$. We say a quadruple $\Phi_{ij} = (V_{ij}, h_{ij}, \varphi_{ij}, \hat{\varphi}_{ij})$ is a *FOOO coordinate change from* $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ *to* $(V_j, E_j, \Gamma_j, s_j, \psi_j)$ *over* S if:

(a) V_{ij} is a Γ_i-invariant open neighbourhood of $\bar{\psi}_i^{-1}(S)$ in V_i.

(b) $h_{ij} : \Gamma_i \to \Gamma_j$ is an injective group homomorphism.

(c) $\varphi_{ij} : V_{ij} \hookrightarrow V_j$ is an h_{ij}-equivariant smooth embedding, such that the induced map $(\varphi_{ij})_* : V_{ij}/\Gamma_i \to V_j/\Gamma_j$ is injective.

(d) $\hat{\varphi}_{ij} : V_{ij} \times E_i \hookrightarrow V_j \times E_j$ is an h_{ij}-equivariant embedding of vector bundles over $\varphi_{ij} : V_{ij} \hookrightarrow V_j$, viewing $V_{ij} \times E_i \to V_{ij}$, $V_j \times E_j \to V_j$ as trivial vector bundles.

(e) $\hat{\varphi}_{ij}(s_i|_{V_{ij}}) = \varphi_{ij}^*(s_j)$, in sections of $\varphi_{ij}^*(V_j \times E_j) \to V_{ij}$.

(f) $\psi_i = \psi_j \circ (\varphi_{ij})_*$ on $(s_i^{-1}(0) \cap V_{ij})/\Gamma_i$.

(g) h_{ij} restricts to an isomorphism $\mathrm{Stab}_{\Gamma_i}(v) \to \mathrm{Stab}_{\Gamma_j}(\varphi_{ij}(v))$ for all v in V_{ij}, where $\mathrm{Stab}_{\Gamma_i}(v)$ is the *stabilizer subgroup* $\{\gamma \in \Gamma_i : \gamma(v) = v\}$.

(h) For each $v \in s_i^{-1}(0) \cap V_{ij} \subseteq V_{ij} \subseteq V_i$ we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_v V_i \arrow[r, "\mathrm{d}\varphi_{ij}|_v"'] \arrow[d, "\mathrm{d}s_i|_v"] & T_{\varphi_{ij}(v)} V_j \arrow[r] \arrow[d, "\mathrm{d}s_j|_{\varphi_{ij}(v)}"] & N_{ij}|_v \arrow[r] \arrow[d, dotted, "\mathrm{d}_{\mathrm{fibre}} s_j|_v"] & 0 \\ 0 \arrow[r] & E_i|_v \arrow[r, "\hat\varphi_{ij}|_v"] & E_j|_{\varphi_{ij}(v)} \arrow[r] & F_{ij}|_v \arrow[r] & 0 \end{tikzcd} \tag{7.1}$$

with exact rows, where $N_{ij} \to V_{ij}$ is the normal bundle of V_{ij} in V_j, and $F_{ij} = \varphi_{ij}^*(E_j)/\hat\varphi_{ij}(E_i|_{V_{ij}})$ the quotient bundle. We require that the induced morphism $\mathrm{d}_{\mathrm{fibre}} s_j|_v$ in (7.1) should be an isomorphism.

Note that $\mathrm{d}_{\mathrm{fibre}} s_j|_v$ an isomorphism in (7.1) is equivalent to the following complex being exact:

$$0 \longrightarrow T_v V_i \xrightarrow{\mathrm{d}s_i|_v \oplus \mathrm{d}\varphi_{ij}|_v} E_i|_v \oplus T_{\varphi_{ij}(v)} V_j \xrightarrow{\hat\varphi_{ij}|_v \oplus - \mathrm{d}s_j|_{\varphi_{ij}(v)}} E_j|_{\varphi_{ij}(v)} \longrightarrow 0. \tag{7.2}$$

This should be compared to Theorem 6.12.

Now let $(V_p, E_p, \Gamma_p, s_p, \psi_p)$, $(V_q, E_q, \Gamma_q, s_q, \psi_q)$ be FOOO Kuranishi neighbourhoods of $p \in X$ and $q \in \operatorname{Im}\psi_p \subseteq X$, respectively. We say a quadruple $\Phi_{qp} = (V_{qp}, h_{qp}, \varphi_{qp}, \hat\varphi_{qp})$ is a *FOOO coordinate change* if it is a FOOO coordinate change from $(V_q, E_q, \Gamma_q, s_q, \psi_q)$ to $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ over S_{qp}, where S_{qp} is any open neighbourhood of q in $\operatorname{Im}\psi_q \cap \operatorname{Im}\psi_p$.

Remark 7.4. (a) We have changed notation slightly compared to [30], to improve compatibility with the rest of the book. Fukaya et al. [30, §4] write Kuranishi neighbourhoods as (V, E, Γ, ψ, s) rather than (V, E, Γ, s, ψ). Also, they write coordinate changes as $\Phi_{pq} = (\hat\varphi_{pq}, \varphi_{pq}, h_{pq})$, leaving V_{pq} implicit, rather than as $\Phi_{qp} = (V_{qp}, h_{qp}, \varphi_{qp}, \hat\varphi_{qp})$ as we do. Note that we have changed the order of p, q in the subscripts compared to [30].

Fukaya et al. do not require $\hat\varphi_{ij} : V_{ij} \times E_i \hookrightarrow \varphi_{ij}^*(V_j \times E_j)$ to come from an injective linear map of vector spaces $E_i \hookrightarrow E_j$. As in §7.3, McDuff and Wehrheim do require this.

Fukaya et al. only impose Definition 7.3(h) for Kuranishi spaces 'with a tangent bundle' in the sense of [24, 30, 39]. As the author knows of no reason for considering Kuranishi spaces 'without tangent bundles', and the notation appears to be merely historical, we will include 'with a tangent bundle' in our definitions of FOOO coordinate changes and FOOO Kuranishi spaces.

(b) Manifolds with corners were discussed in Chapter 2. When we allow the V_i in Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ to be manifolds with corners, it is important that the definition of *embedding* of manifolds with corners $\varphi_{ij} : V_{ij} \hookrightarrow V_j$ used in Definition 7.3(c) includes the condition that φ_{ij} be *simple*, in the sense of §2.1. For comparison, in our theory of Kuranishi spaces with corners in §6.3, it is important that coordinate changes Φ_{ij} are simple in the sense of Definition 6.31, as follows from Proposition 6.32(d).

We relate FOOO coordinate changes to coordinate changes in §6.1:

Example 7.5. Let $\Phi_{ij} = (V_{ij}, h_{ij}, \varphi_{ij}, \hat\varphi_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be a FOOO coordinate change over S, as in Definition 7.3. As in Example 7.2, regard the FOOO Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i), (V_j, E_j, \Gamma_j, s_j, \psi_j)$ as examples of Kuranishi neighbourhoods in the sense of §6.1.

Set $P_{ij} = V_{ij} \times \Gamma_j$. Let Γ_i act on P_{ij} by $\gamma_i : (v, \gamma) \mapsto (\gamma_i \cdot v, \gamma h_{ij}(\gamma_i)^{-1})$. Let Γ_j act on P_{ij} by $\gamma_j : (v, \gamma) \mapsto (v, \gamma_j\gamma)$. Define $\pi_{ij} : P_{ij} \to V_i$ and $\phi_{ij} : P_{ij} \to V_j$ by $\pi_{ij} : (v, \gamma) \mapsto v$ and $\phi_{ij} : (v, \gamma) \mapsto \gamma \cdot \varphi_{ij}(v)$. Then π_{ij} is Γ_i-equivariant and Γ_j-invariant. Since φ_{ij} is h_{ij}-equivariant, ϕ_{ij} is Γ_i-invariant, and Γ_j-equivariant.

We will define a vector bundle morphism $\hat\phi_{ij} : \pi_{ij}^*(E_i) \to \phi_{ij}^*(E_j)$. At $(v, \gamma) \in P_{ij}$, this $\hat\phi_{ij}$ must map $E_i|_v \to E_j|_{\gamma\cdot\varphi_{ij}(v)}$. We define $\hat\phi_{ij}|_{(v,\gamma)}$ to be the composition of $\hat\varphi_{ij}|_v : E_i|_v \to E_j|_{\varphi_{ij}(v)}$ with $\gamma\cdot : E_j|_{\varphi_{ij}(v)} \to E_j|_{\gamma\cdot\varphi_{ij}(v)}$ from the Γ_j-action on E_j. That is, $\hat\phi_{ij}|_{V_{ij}\times\{\gamma\}} = \gamma \cdot \hat\varphi_{ij}$ for each $\gamma \in \Gamma_j$.

It is now easy to see that $\tilde\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ is a 1-morphism over S, in the sense of §6.1. Using (7.2), Theorem 6.12(a),(b) show that $\tilde\Phi_{ij}$ is a coordinate change over S, as in §6.1, noting that φ_{ij} is simple in the corners case as in Remark 7.4(b).

Definition 7.6. A *FOOO Kuranishi structure* $\mathcal{K}$ on X of *virtual dimension* $n \in \mathbb{Z}$ in the sense of [30, §4], including the 'with a tangent bundle' condition, assigns a FOOO Kuranishi neighbourhood $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ for each $p \in X$ and a FOOO coordinate change $\Phi_{qp} = (V_{qp}, h_{qp}, \varphi_{qp}, \hat\varphi_{qp}) : (V_q, E_q, \Gamma_q, s_q, \psi_q) \to (V_p, E_p, \Gamma_p, s_p, \psi_p)$ for each $q \in \operatorname{Im}\psi_p$ such that the following holds:

(a) $\dim V_p - \operatorname{rank} E_p = n$ for all $p \in X$.

(b) If $q \in \operatorname{Im}\psi_p$, $r \in \psi_q((V_{qp} \cap s_q^{-1}(0))/\Gamma_q)$, then for each connected component $(\varphi_{rq}^{-1}(V_{qp}) \cap V_{rp})^\alpha$ of $\varphi_{rq}^{-1}(V_{qp}) \cap V_{rp}$ there exists $\gamma_{rqp}^\alpha \in \Gamma_p$ with

$$\begin{aligned} h_{qp} \circ h_{rq} = \gamma_{rqp}^\alpha \cdot h_{rp} \cdot (\gamma_{rqp}^\alpha)^{-1}, \qquad \varphi_{qp} \circ \varphi_{rq} = \gamma_{rqp}^\alpha \cdot \varphi_{rp}, \\ \text{and} \qquad \varphi_{rq}^*(\hat\varphi_{qp}) \circ \hat\varphi_{rq} = \gamma_{rqp}^\alpha \cdot \hat\varphi_{rp}, \end{aligned} \tag{7.3}$$

where the second and third equations hold on $(\varphi_{rq}^{-1}(V_{qp}) \cap V_{rp})^\alpha$.

If the V_p for $p \in X$ are classical manifolds, we call $\boldsymbol{X} = (X, \mathcal{K})$ a *FOOO Kuranishi space*, of *virtual dimension* $n \in \mathbb{Z}$, written $\operatorname{vdim}\boldsymbol{X} = n$. If the V_p are manifolds with corners, we call $\boldsymbol{X}$ a *FOOO Kuranishi space with corners.*

We prove in Theorem 7.29 below that a FOOO Kuranishi space $\boldsymbol{X}$ (with corners) can be made into a Kuranishi space $\boldsymbol{X}'$ (with corners) in the sense of §6.2. We will show that the elements $\gamma_{rqp}^\alpha \in \Gamma_p$ in Definition 7.6(b) correspond in the setting of §6.1 to a 2-morphism $\Lambda_{rqp} : \tilde\Phi_{qp} \circ \tilde\Phi_{rq} \Rightarrow \tilde\Phi_{rp}$.

Example 7.7. **(i)** In the Fukaya–Oh–Ohta–Ono theory [19–39], one often relates two FOOO coordinate changes in the following way. Let $\Phi_{ij} = (V_{ij}, h_{ij}, \varphi_{ij},$

$\hat\varphi_{ij}), \Phi'_{ij}=(V'_{ij},h'_{ij},\varphi'_{ij},\hat\varphi'_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ be FOOO coordinate changes over S. Suppose there exists $\gamma\in\Gamma_j$ such that

$$h_{ij}=\gamma\cdot h'_{ij}\cdot\gamma^{-1},\quad \phi_{ij}=\gamma\cdot\phi'_{ij},\quad\text{and}\quad \hat\phi_{ij}=\gamma\cdot\hat\phi'_{ij}, \tag{7.4}$$

where the second and third equations hold on $\acute V_{ij}:=V_{ij}\cap V'_{ij}$.

Let $\tilde\Phi_{ij},\tilde\Phi'_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ be the 1-morphisms in the sense of §6.1 corresponding to Φ_{ij},Φ'_{ij} in Example 7.5. Set $\acute P_{ij}=\acute V_{ij}\times\Gamma_j\subseteq P_{ij}$. Define $\lambda_{ij}:\acute P_{ij}=\acute V_{ij}\times\Gamma_j\rightarrow V'_{ij}\times\Gamma_j=P'_{ij}$ by $\lambda_{ij}:(v,\gamma')\mapsto(v,\gamma'\gamma)$, and $\hat\lambda_{ij}=0$. Then $(\acute P_{ij},\lambda_{ij},\hat\lambda_{ij})$ satisfies Definition 6.4(a)–(c), so we have defined a 2-morphism $\Lambda_{ij}=[\acute P_{ij},\lambda_{ij},\hat\lambda_{ij}]:\tilde\Phi_{ij}\Rightarrow\tilde\Phi'_{ij}$, in the sense of §6.1.

(ii) This enables us to interpret Definition 7.6(b) in terms of a 2-morphism. In the situation of Definition 7.6(b), the composition of the FOOO coordinate changes Φ_{rq},Φ_{qp} is $\Phi_{qp}\circ\Phi_{qp}=\bigl(\varphi_{rq}^{-1}(V_{qp}),h_{qp}\circ h_{rq},\varphi_{qp}\circ\varphi_{rq}|_{\varphi_{rq}^{-1}(V_{qp})},\varphi_{rq}^*(\hat\varphi_{qp})\circ\hat\varphi_{rq}|_{\varphi_{rq}^{-1}(V_{qp})}\bigr)$. Thus, (7.3) relates $\Phi_{qp}\circ\Phi_{rq}$ to Φ_{rp} in the same way that (7.4) relates Φ_{ij} to Φ'_{ij}, except for allowing γ_{rqp} to vary on different connected components. Hence, if $\tilde\Phi_{rq},\tilde\Phi_{qp},\tilde\Phi_{rp}$ are the coordinate changes in the sense of §6.1 associated to $\Phi_{rq},\Phi_{qp},\Phi_{rp}$ in Example 7.5, then the method of **(i)** defines a 2-morphism $\Lambda_{pqr}:\tilde\Phi_{qp}\circ\tilde\Phi_{rq}\Rightarrow\tilde\Phi_{rp}$, in the sense of §6.1.

(iii) In the situation of Definition 7.6(b), suppose $v\in(\varphi_{rq}^{-1}(V_{qp})\cap V_{rp})^\alpha$ is generic. Then $\mathrm{Stab}_{\Gamma_r}(v)=\{1\}$, as Γ_r acts (locally) effectively on V_r by Definition 7.1(c). Hence $\mathrm{Stab}_{\Gamma_p}(\varphi_{rp}(v))=\{1\}$ by Definition 7.3(g). Therefore the point $\gamma_{rqp}^\alpha\cdot\varphi_{rp}(v)=\varphi_{qp}\circ\varphi_{rq}(v)$ in V_p determines γ^α_{rqp} in Γ_p. So the second equation of (7.3) determines $\gamma^\alpha_{rqp}\in\Gamma_p$ uniquely, provided it exists. Thus the 2-morphism $\Lambda_{pqr}:\tilde\Phi_{qp}\circ\tilde\Phi_{rq}\Rightarrow\tilde\Phi_{rp}$ in **(ii)** is also determined uniquely.

Definition 7.8. Let $\boldsymbol X$ be a FOOO Kuranishi space (possibly with corners). Then for each $p\in X$, $q\in\operatorname{Im}\psi_p$ and $v\in s_q^{-1}(0)\cap V_{qp}$, we have an exact sequence (7.2). Taking top exterior powers in (7.2) yields an isomorphism

$$\bigl(\det T_vV_q\bigr)\otimes\det\bigl(E_p|_{\varphi_{qp}(v)}\bigr)\cong\bigl(\det E_q|_v\bigr)\otimes\bigl(T_{\varphi_{qp}(v)}V_p\bigr),$$

where $\det W$ means $\Lambda^{\dim W}W$, or equivalently, a canonical isomorphism

$$\bigl(\det T^*V_p\otimes\det E_p\bigr)|_{\varphi_{qp}(v)}\cong\bigl(\det T^*V_q\otimes\det E_q\bigr)|_v. \tag{7.5}$$

Defining the isomorphism (7.5) requires a suitable sign convention. Sign conventions are discussed in Fukaya et al. [24, §8.2] and McDuff and Wehrheim [82, §8.1]. An *orientation* on $\boldsymbol X$ is a choice of orientations on the line bundles

$$\det T^*V_p\otimes\det E_p\big|_{s_p^{-1}(0)}\longrightarrow s_p^{-1}(0)$$

for all $p\in X$, compatible with the isomorphisms (7.5). In §10.7 in volume II we will develop the analogue of these ideas for our (m- and μ-)Kuranishi spaces.

Definition 7.9. Let $\boldsymbol{X}$ be a FOOO Kuranishi space (possibly with corners), and Y a classical manifold. A *smooth map* $\boldsymbol{f} : \boldsymbol{X} \to Y$ is $\boldsymbol{f} = (f_p : p \in X)$ where $f_p : V_p \to Y$ is a Γ_p-invariant smooth map for all $p \in X$ (that is, f_p factors via $V_p \to V_p/\Gamma_p \to Y$), and $f_p \circ \varphi_{qp} = f_q|_{V_{qp}} : V_{qp} \to Y$ for all $q \in \operatorname{Im} \psi_p$. This induces a unique continuous map $f : X \to Y$ with $f_p|_{s_p^{-1}(0)} = f \circ \bar{\psi}_p$ for all $p \in X$. We call $\boldsymbol{f}$ *weakly submersive* if each f_p is a submersion.

Suppose $\boldsymbol{X}, \boldsymbol{X}'$ are FOOO Kuranishi spaces, Y is a classical manifold, and $\boldsymbol{f} : \boldsymbol{X} \to Y$, $\boldsymbol{f}' : \boldsymbol{X}' \to Y$ are weakly submersive. Then as in [24, §A1.2] one can define a 'fibre product' Kuranishi space $\boldsymbol{W} = \boldsymbol{X} \times_Y \boldsymbol{X}'$, with topological space $W = \{(p, p') \in X \times X' : f(p) = f'(p')\}$, and FOOO Kuranishi neighbourhoods $(V_{p,p'}, E_{p,p'}, \Gamma_{p,p'}, s_{p,p'}, \psi_{p,p'})$ for $(p, p') \in W$, where $V_{p,p'} = V_p \times_{f_p, Y, f'_{p'}} V'_{p'}$, $E_{p,p'} = \pi^*_{V_p}(E_p) \oplus \pi^*_{V'_{p'}}(E'_{p'})$, $\Gamma_{p,p'} = \Gamma_p \times \Gamma'_{p'}$, $s_{p,p'} = \pi^*_{V_p}(s_p) \oplus \pi^*_{V'_{p'}}(s'_{p'})$, and $\psi_{p,p'} = \psi_p \circ (\pi_{V_p})_* \times \psi'_{p'} \circ (\pi_{V'_{p'}})_*$. The weakly submersive condition ensures $V_{p,p'} = V_p \times_Y V'_{p'}$ is well-defined.

Remark 7.10. (i) Note that Fukaya et al. [19–39] *do not define morphisms between Kuranishi spaces*, but only morphisms $\boldsymbol{f} : \boldsymbol{X} \to Y$ from Kuranishi spaces $\boldsymbol{X}$ to classical manifolds Y. Thus, Kuranishi spaces in [19–39] *do not form a category.*

Observe however that Fukaya [19, §3, §5] (see also [35, §4.2]) works with a forgetful morphism $\mathfrak{forget} : \boldsymbol{\mathcal{M}}_{l,1}(\beta) \to \boldsymbol{\mathcal{M}}_{l,0}(\beta)$, which is clearly intended to be some kind of morphism of Kuranishi spaces, without defining the concept.

(ii) The 'fibre product' $\boldsymbol{X} \times_Y \boldsymbol{X}'$ in Definition 7.9 *is not a fibre product in the sense of category theory*, characterized by a universal property, since Fukaya et al. in [19–39] do not have a category (or higher category) of FOOO Kuranishi spaces in which to state such a universal property. Their 'fibre product' is really just an ad hoc construction. Chapter 11 in volume II will study w-transverse 2-category fibre products in our 2-categories of (m-)Kuranishi spaces $\mathbf{m\dot{K}ur}, \mathbf{\dot{K}ur}$.

7.2 Fukaya–Oh–Ohta–Ono's good coordinate systems

Good coordinate systems on Kuranishi spaces $\boldsymbol{X}$ in the work of Fukaya, Oh, Ohta and Ono [19, 24, 26, 27, 30, 33, 35–37, 39] are an open cover of $\boldsymbol{X}$ by FOOO Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ for i in a finite set I, with coordinate changes Φ_{ij} for $i, j \in I$, satisfying extra conditions. They are a tool for constructing virtual cycles for Kuranishi spaces using the method of 'perturbation by multisections', and the extra conditions are included to make this virtual cycle construction work.

As with Kuranishi spaces, since its introduction in [39, Def. 6.1] the definition of good coordinate system has changed several times during the evolution of [19, 24, 26, 27, 30, 33, 35–37, 39], see in chronological order [39, Def. 6.1], [24, Lem. A1.11], [26, §15], and [30, §5]. Of these, [30, 39] work with Kuranishi neighbourhoods $(\mathfrak{V}_i, \mathfrak{E}_i, s_i, \psi_i)$ where $\mathfrak{V}_i$ is an orbifold (which we do not want to do), and [24, 26] with Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ with V_i a manifold.

The definition we give below is a hybrid of those in [24, 26, 30, 36]. Essentially our 'FOOO weak good coordinate systems' follow the definitions in [24, 26], and our 'FOOO good coordinate systems' include extra conditions adapted from [30, 36]. We show in Theorem 7.31 below that given a FOOO weak good coordinate system on X, we can make X into a Kuranishi space $\boldsymbol{X}$ in the sense of §6.2.

Definition 7.11. Let X be a compact, metrizable topological space. A *FOOO weak good coordinate system* $\mathcal{G} = \big((I,\prec),(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\Phi_{ij,\; i\prec j \text{ in } I}\big)$ *on* X *of virtual dimension* $n\in\mathbb{Z}$ consists of a finite indexing set I, a partial order $\prec$ on I, FOOO Kuranishi neighbourhoods $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ for $i\in I$ with V_i a classical manifold, $\dim V_i - \operatorname{rank} E_i = n$, and $X=\bigcup_{i\in I}\operatorname{Im}\psi_i$, and FOOO coordinate changes $\Phi_{ij}=(V_{ij},h_{ij},\varphi_{ij},\hat\varphi_{ij})$ from $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ to $(V_j,E_j,\Gamma_j,s_j,\psi_j)$ over $S=\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j$ for all $i,j\in I$ with $i\prec j$ and $\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\neq\emptyset$, satisfying the two conditions:

(a) If $i\neq j\in I$ with $\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\neq\emptyset$ then either $i\prec j$ or $j\prec i$.

(b) If $i\prec j\prec k$ in I with $\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\cap\operatorname{Im}\psi_k\neq\emptyset$ then there exists $\gamma_{ijk}\in\Gamma_k$ such that as in (7.3) we have

$$\begin{gathered} h_{jk}\circ h_{ij}=\gamma_{ijk}\cdot h_{ik}\cdot\gamma_{ijk}^{-1},\qquad \varphi_{jk}\circ\varphi_{ij}=\gamma_{ijk}\cdot\varphi_{ik},\\ \text{and}\qquad \varphi_{ij}^*(\hat\varphi_{jk})\circ\hat\varphi_{ij}=\gamma_{ijk}\cdot\hat\varphi_{ik}, \end{gathered}\tag{7.6}$$

where the second and third equations hold on $V_{ij}\cap V_{ik}\cap\varphi_{ij}^{-1}(V_{jk})$. The γ_{ijk} are uniquely determined by (7.6) as in Example 7.7(iii).

If instead the V_i for $i\in I$ are manifolds with corners, we call $\mathcal{G}$ a *FOOO weak good coordinate system with corners.*

We call $\mathcal{G}$ a *FOOO good coordinate system on* X (*with corners*) if it also satisfies the extra conditions:

(c) If $i\prec j$ in I, $\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\neq\emptyset$ then $\psi_i\big((V_{ij}\cap s_i^{-1}(0))/\Gamma_i\big)=\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j$.

(d) If $i\prec j$ in I and $\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\neq\emptyset$ then $\operatorname{inc}\times\varphi_{ij}:V_{ij}\to V_i\times V_j$ is proper, where $\operatorname{inc}:V_{ij}\hookrightarrow V_i$ is the inclusion.

(e) If $i\prec j$, $i\prec k$ in I for $j\neq k$ and $\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\neq\emptyset\neq\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_k$, $V_{ij}\cap V_{ik}\neq\emptyset$, then $\operatorname{Im}\psi_j\cap\operatorname{Im}\psi_k\neq\emptyset$, and **either** $j\prec k$ and $V_{ij}\cap V_{ik}=\varphi_{ij}^{-1}(V_{jk})$, **or** $k\prec j$ and $V_{ij}\cap V_{ik}=\varphi_{ik}^{-1}(V_{kj})$.

(f) If $i\prec k$, $j\prec k$ in I for $i\neq j$ and $\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_k\neq\emptyset\neq\operatorname{Im}\psi_j\cap\operatorname{Im}\psi_k$ and $v_i\in V_{ik}$, $v_j\in V_{jk}$, $\delta\in\Gamma_k$ with $\varphi_{jk}(v_j)=\delta\cdot\varphi_{ik}(v_i)$ in V_k, then $\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\neq\emptyset$ and **either** $i\prec j$, $v_i\in V_{ij}$, and there exists $\gamma\in\Gamma_j$ with $h_{jk}(\gamma)=\delta\,\gamma_{ijk}$ and $v_j=\gamma\cdot\varphi_{ij}(v_i)$; **or** $j\prec i$, $v_j\in V_{ji}$, and there exists $\gamma\in\Gamma_i$ with $h_{ik}(\gamma)=\delta^{-1}\gamma_{jik}$ and $v_i=\gamma\cdot\varphi_{ji}(v_j)$, for $\gamma_{ijk},\gamma_{jik}$ as in (b).

As in [36], parts (c)–(f) are equivalent to:

(g) Define a symmetric, reflexive binary relation $\sim$ on $\coprod_{i\in I}V_i/\Gamma_i$ by $\Gamma_i v\sim\Gamma_j\varphi_{ij}(v_i)$ if $i\prec j$, $\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\neq\emptyset$ and $v\in V_{ij}$. Then $\sim$ is an equivalence relation, and $\big(\coprod_{i\in I}V_i/\Gamma_i\big)/\sim$ with the quotient topology is Hausdorff.

Now let $X, \mathcal{G}$ be as above (either weak or not), and Y be a classical manifold. As in Definition 7.9, a *smooth map* $(f_i,\, i \in I)$ *from* $(X, \mathcal{G})$ *to* Y is a Γ_i-invariant smooth map $f_i : V_i \to Y$ for $i \in I$, with $f_j \circ \varphi_{ij} = f_i|_{V_{ij}} : V_{ij} \to Y$ for all $i \prec j$ in I. This induces a unique continuous map $f : X \to Y$ with $f_i|_{s_i^{-1}(0)} = f \circ \bar{\psi}_i$ for $i \in I$.

Using elementary topology, Fukaya, Oh, Ohta and Ono [36] prove:

Theorem 7.12. *Suppose* $\mathcal{G} = \bigl((I, \prec), (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i \in I}, \Phi_{ij,\ i \prec j \text{ in } I}\bigr)$ *is a FOOO weak good coordinate system on* X*. Then we can construct a FOOO good coordinate system* $\mathcal{G}' = \bigl((I', \prec), (V_i', E_i', \Gamma_i', s_i', \psi_i')_{i \in I}, \Phi'_{ij,\ i \prec j \text{ in } I}\bigr)$ *on* X*, where* $I' \subseteq I$*,* $V_i' \subseteq V_i$*,* $V_{ij}' \subseteq V_{ij}$ *are open,* $\Gamma_i' = \Gamma_i$*,* $h_{ij}' = h_{ij}$*, and* $E_i', s_i', \psi_i', \varphi_{ij}', \hat{\varphi}_{ij}'$ *are obtained from* $E_i, \ldots, \hat{\varphi}_{ij}$ *by restricting from* V_i, V_{ij} *to* V_i', V_{ij}'.

In fact Fukaya et al. [36] work at the level of orbifolds V_i/Γ_i, V_{ij}/Γ_i rather than manifolds with finite group actions, but their result easily implies Theorem 7.12. The next definition is based on Fukaya et al. [30, Def. 7.2], but using $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ for V_i a manifold, rather than $(\mathfrak{V}_i, \mathfrak{E}_i, \mathfrak{s}_i, \psi_i)$ for $\mathfrak{V}_i$ an orbifold.

Definition 7.13. Let $\boldsymbol{X} = (X, \mathcal{K})$ be a FOOO Kuranishi space. A FOOO (weak) good coordinate system $\mathcal{G} = \bigl((I, \prec), (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i \in I}, \Phi_{ij,\ i \prec j \text{ in } I}\bigr)$ on the topological space X is called *compatible with the FOOO Kuranishi structure* $\mathcal{K}$ *on* $\boldsymbol{X}$ if for each $i \in I$ and each $p \in \operatorname{Im}\psi_i \subseteq X$ there exists a FOOO coordinate change Φ_{pi} from $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ to $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ on an open neighbourhood S_{pi} of p in $\operatorname{Im}\psi_p \cap \operatorname{Im}\psi_i$ (where $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ comes from $\mathcal{K}$ and $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ from the good coordinate system) such that

(a) If $q \in \operatorname{Im}\psi_p \cap \operatorname{Im}\psi_i$ then there exists $\gamma_{qpi} \in \Gamma_i$ such that

$$\begin{gathered} h_{pi} \circ h_{qp} = \gamma_{qpi} \cdot h_{qi} \cdot \gamma_{qpi}^{-1}, \qquad \varphi_{pi} \circ \varphi_{qp} = \gamma_{qpi} \cdot \varphi_{qi}, \\ \text{and} \qquad \varphi_{qp}^*(\hat{\varphi}_{pi}) \circ \hat{\varphi}_{qp} = \gamma_{qpi} \cdot \hat{\varphi}_{qi}, \end{gathered}$$

where the second and third equations hold on $\varphi_{qp}^{-1}(V_{pi}) \cap V_{qp} \cap V_{qi}$.

(b) If $i \prec j$ in I with $p \in \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ then there exists $\gamma_{pij} \in \Gamma_j$ such that

$$\begin{gathered} h_{ij} \circ h_{pi} = \gamma_{pij} \cdot h_{pj} \cdot \gamma_{pij}^{-1}, \qquad \varphi_{ij} \circ \varphi_{pi} = \gamma_{pij} \cdot \varphi_{pj}, \\ \text{and} \qquad \varphi_{pi}^*(\hat{\varphi}_{ij}) \circ \hat{\varphi}_{pi} = \gamma_{pij} \cdot \hat{\varphi}_{pj}, \end{gathered} \tag{7.7}$$

where the second and third equations hold on $\varphi_{pi}^{-1}(V_{ij}) \cap V_{pi} \cap V_{pj}$.

Remark 7.14. For the programme of [19–39], one would like to show:

(i) Any (oriented) FOOO Kuranishi space $\boldsymbol{X}$ (perhaps also with a smooth map $\boldsymbol{f} : \boldsymbol{X} \to Y$ to a manifold Y) admits a compatible (oriented) FOOO good coordinate system $\bigl((I, \prec), (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i \in I}, \Phi_{ij,\ i \prec j \text{ in } I}\bigr)$ (perhaps also with a smooth map $(f_i,\, i \in I)$ to Y).

(ii) Given a compact, metrizable topological space X with an oriented FOOO good coordinate system $\bigl((I,\prec),(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\Phi_{ij,\ i\prec j\text{ in }I}\bigr)$ (perhaps with a smooth map $(f_i,\ i\in I)$ to a classical manifold Y), we can construct a *virtual cycle* for X (perhaps in the singular homology $H_*(Y;\mathbb{Q})$ or de Rham cohomology $H^*_{\rm dR}(Y;\mathbb{R})$ of Y).

Producing such virtual cycles is, from the point of view of symplectic geometry, the sole reason for defining and studying Kuranishi spaces.

Statements (i), for various definitions of 'Kuranishi space', 'good coordinate system', and 'compatible', can be found in [39, Lem. 6.3] (with short proof), [24, Lem. A1.11] (with no proof), and [30, §7] (with long proof). Constructions (ii), again for various definitions, can be found in [39, §6], [24, §A1.1], [27, §12] (using de Rham cohomology), and [30, §6] (with long proof).

7.3 McDuff–Wehrheim's Kuranishi atlases

Next we discuss an approach to Kuranishi spaces developed by McDuff and Wehrheim [77, 78, 80–83]. Their main definition is that of a (*weak*) *Kuranishi atlas* on a topological space X. Here are [81, Def.s 2.2.2 & 2.2.8].

Definition 7.15. An *MW Kuranishi neighbourhood* (V,E,Γ,s,ψ) on a topological space X is the same as a FOOO Kuranishi neighbourhood in Definition 7.1, with V a classical manifold, except that Γ need not act effectively on V.

As in Example 7.2, by an abuse of notation we will regard MW Kuranishi neighbourhoods as examples of our Kuranishi neighbourhoods in §6.1.

Definition 7.16. Suppose $(V_B,E_B,\Gamma_B,s_B,\psi_B)$, $(V_C,E_C,\Gamma_C,s_C,\psi_C)$ are MW Kuranishi neighbourhoods on a topological space X, and $S\subseteq \operatorname{Im}\psi_B\cap\operatorname{Im}\psi_C\subseteq X$ is open. We say a quadruple $\Phi_{BC}=(\tilde V_{BC},\rho_{BC},\varpi_{BC},\hat\varphi_{BC})$ is an *MW coordinate change from* $(V_B,E_B,\Gamma_B,s_B,\psi_B)$ *to* $(V_C,E_C,\Gamma_C,s_C,\psi_C)$ *over* S if:

(a) $\tilde V_{BC}$ is a Γ_C-invariant embedded submanifold of V_C containing $\bar\psi_C^{-1}(S)$.

(b) $\rho_{BC}:\Gamma_C\to\Gamma_B$ is a surjective group morphism, with kernel $\Delta_{BC}\subseteq\Gamma_C$.

There should exist an isomorphism $\Gamma_C\cong\Gamma_B\times\Delta_{BC}$ identifying ρ_{BC} with the projection $\Gamma_B\times\Delta_{BC}\to\Gamma_B$.

(c) $\varpi_{BC}:\tilde V_{BC}\to V_B$ is a ρ_{BC}-equivariant étale map, with image $V_{BC}=\varpi_{BC}(\tilde V_{BC})$ a Γ_B-invariant open neighbourhood of $\bar\psi_B^{-1}(S)$ in V_B, such that $\varpi_{BC}:\tilde V_{BC}\to V_{BC}$ is a principal Δ_{BC}-bundle.

(d) $\hat\varphi_{BC}:E_B\to E_C$ is an injective Γ_C-equivariant linear map, where the Γ_C-action on E_B is induced from the Γ_B-action by ρ_{BC}, so in particular Δ_{BC} acts trivially on E_B.

(e) $\hat\varphi_{BC}\circ s_B\circ\varpi_{BC}=s_C|_{\tilde V_{BC}}:\tilde V_{BC}\to E_C$.

(f) $\psi_B\circ(\varpi_{BC})_*=\psi_C$ on $(s_C^{-1}(0)\cap\tilde V_{BC})/\Gamma_C$.

(g) For each $v \in \tilde{V}_{BC}$ we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_v\tilde{V}_{BC} \arrow[r, "\subset"] \arrow[d, "\mathrm{d}(\varpi_{BC}^*(s_B))|_v"] & T_vV_C \arrow[r] \arrow[d, "\mathrm{d}s_C|_v"] & N_{BC}|_v \arrow[r] \arrow[d, dotted, "\mathrm{d}_{\mathrm{fibre}}s_C|_v"] & 0 \\ 0 \arrow[r] & E_B \arrow[r, "\hat{\varphi}_{BC}"] & E_C \arrow[r] & E_C/\hat{\varphi}_{BC}(E_B) \arrow[r] & 0 \end{tikzcd} \tag{7.8}$$

with exact rows, where N_{BC} is the normal bundle of $\tilde{V}_{BC}$ in V_C. We require the induced morphism $\mathrm{d}_{\mathrm{fibre}}s_C|_v$ in (7.8) to be an isomorphism.

We relate MW coordinate changes to coordinate changes in §6.1:

Example 7.17. Let $\Phi_{BC} = (\tilde{V}_{BC}, \rho_{BC}, \varpi_{BC}, \hat{\varphi}_{BC}) : (V_B, E_B, \Gamma_B, s_B, \psi_B) \to (V_C, E_C, \Gamma_C, s_C, \psi_C)$ be an MW coordinate change over S, as in Definition 7.16. Regard $(V_B, E_B, \Gamma_B, s_B, \psi_B), (V_C, E_C, \Gamma_C, s_C, \psi_C)$ as Kuranishi neighbourhoods in the sense of §6.1, as in Example 7.2.

Set $P_{BC} = \tilde{V}_{BC} \times \Gamma_B$. Let Γ_B act on P_{BC} by $\gamma_B : (v, \gamma) \mapsto (v, \gamma_B\gamma)$. Let Γ_C act on P_{BC} by $\gamma_C : (v, \gamma) \mapsto (\gamma_C \cdot v, \gamma\rho_{BC}(\gamma_C)^{-1})$. Define $\pi_{BC} : P_{BC} \to V_B$ and $\phi_{BC} : P_{BC} \to V_C$ by $\pi_{BC} : (v, \gamma) \mapsto \gamma \cdot \varpi_{BC}(v)$ and $\phi_{BC} : (v, \gamma) \mapsto v$. Then π_{BC} is Γ_B-equivariant and Γ_C-invariant, and ϕ_{BC} is Γ_B-invariant and Γ_C-equivariant.

Define $\hat{\phi}_{BC} : \pi_{BC}^*(V_B \times E_B) \to \phi_{BC}^*(V_C \times E_C)$, as a morphism of trivial vector bundles with fibres E_B, E_C on $P_{BC} = \tilde{V}_{BC} \times \Gamma_B$, by $\hat{\phi}_{BC}|_{\tilde{V}_{BC}\times\{\gamma\}} = \hat{\varphi}_{BC} \circ (\gamma^{-1} \cdot -)$ for each $\gamma \in \Gamma_B$. It is easy to see that $\tilde{\Phi}_{BC} = (P_{BC}, \pi_{BC}, \phi_{BC}, \hat{\phi}_{BC}) : (V_B, E_B, \Gamma_B, s_B, \psi_B) \to (V_C, E_C, \Gamma_C, s_C, \psi_C)$ is a 1-morphism over S, in the sense of §6.1. Combining Definition 7.16(g) and Theorem 6.12(a) shows that $\tilde{\Phi}_{BC}$ is a coordinate change over S, in the sense of §6.1.

Definition 7.18. Let X be a compact, metrizable topological space. An *MW weak Kuranishi atlas* $\mathcal{K} = \bigl(A, I, (V_B, E_B, \Gamma_B, s_B, \psi_B)_{B\in I}, \Phi_{BC,\ B,C\in I,\ B\subsetneq C}\bigr)$ *on X of virtual dimension $n \in \mathbb{Z}$*, as in [81, Def. 2.3.1], consists of a finite indexing set A, a set I of nonempty subsets of A, MW Kuranishi neighbourhoods $(V_B, E_B, \Gamma_B, s_B, \psi_B)$ on X for all $B \in I$ with $\dim V_B - \operatorname{rank} E_B = n$ and $X = \bigcup_{B\in I} \operatorname{Im}\psi_B$, and MW coordinate changes $\Phi_{BC} = (\tilde{V}_{BC}, \rho_{BC}, \varpi_{BC}, \hat{\varphi}_{BC})$ from $(V_B, E_B, \Gamma_B, s_B, \psi_B)$ to $(V_C, E_C, \Gamma_C, s_C, \psi_C)$ on $S = \operatorname{Im}\psi_B \cap \operatorname{Im}\psi_C$ for all $B, C \in I$ with $B \subsetneq C$, satisfying the four conditions:

(a) We have $\{a\} \in I$ for all $a \in A$, and $I = \bigl\{\emptyset \neq B \subseteq A : \bigcap_{a\in B} \operatorname{Im}\psi_{\{a\}} \neq \emptyset\bigr\}$. Also $\operatorname{Im}\psi_B = \bigcap_{a\in B} \operatorname{Im}\psi_{\{a\}}$ for all $B \in I$.

(b) We have $\Gamma_B = \prod_{a\in B} \Gamma_{\{a\}}$ for all $B \in I$. If $B, C \in I$ with $B \subsetneq C$ then $\rho_{BC} : \Gamma_C \to \Gamma_B$ is the obvious projection $\prod_{a\in C} \Gamma_{\{a\}} \to \prod_{a\in B} \Gamma_{\{a\}}$, with kernel $\Delta_{BC} \cong \prod_{a\in C\setminus B} \Gamma_{\{a\}}$.

(c) We have $E_B = \prod_{a\in B} E_{\{a\}}$ for all $B \in I$, with the obvious representation of $\Gamma_B = \prod_{a\in B} \Gamma_{\{a\}}$. If $B \subsetneq C$ in I then $\hat{\varphi}_{BC} : E_B = \prod_{a\in B} E_{\{a\}} \to E_C = \prod_{a\in C} E_{\{a\}}$ is $\mathrm{id}_{E_{\{a\}}}$ for $a \in B$, and maps to zero in $E_{\{a\}}$ for $a \in C \setminus B$.

(d) If $B,C,D\in I$ with $B\subsetneq C\subsetneq D$ then $\varpi_{BC}\circ\varpi_{CD}=\varpi_{BD}$ on $\tilde V_{BCD}:=\tilde V_{BD}\cap\varpi_{CD}^{-1}(\tilde V_{BC})$. One can show using (b),(c) and Definition 7.16 that $\tilde V_{BD}$ and $\varpi_{CD}^{-1}(\tilde V_{BC})$ are both open subsets in $s_D^{-1}(\hat\varphi_{BD}(E_B))$, which is a submanifold of V_D, so $\tilde V_{BCD}$ is a submanifold of V_D.

We call $\mathcal{K}=\bigl(A,I,(V_B,E_B,\Gamma_B,s_B,\psi_B)_{B\in I},\Phi_{BC,\,B\subsetneq C}\bigr)$ an *MW Kuranishi atlas on* X, as in [81, Def. 2.3.1], if it also satisfies:

(e) If $B,C,D\in I$ with $B\subsetneq C\subsetneq D$ then $\varpi_{CD}^{-1}(\tilde V_{BC})\subseteq\tilde V_{BD}$.

McDuff and Wehrheim also define *orientations* on MW weak Kuranishi atlases, in a very similar way to Definition 7.8.

Two MW weak Kuranishi atlases $\mathcal{K},\mathcal{K}'$ on X are called *directly commensurate* if they are both contained in a third MW weak Kuranishi atlas $\mathcal{K}''$. They are called *commensurate* if there exist MW weak Kuranishi atlases $\mathcal{K}=\mathcal{K}_0,\mathcal{K}_1,\ldots,\mathcal{K}_m=\mathcal{K}'$ with $\mathcal{K}_{i-1},\mathcal{K}_i$ directly commensurate for $i=1,\ldots,m$. This is an equivalence relation on MW weak Kuranishi atlases on X.

We show in Theorem 7.33 below that given an MW weak Kuranishi atlas on X, we can make X into a Kuranishi space $\boldsymbol{X}$ in the sense of Chapter 6.

McDuff and Wehrheim argue that their concept of MW weak Kuranishi atlas is a more natural, or more basic, idea than a FOOO Kuranishi space, since in analytic moduli problems such as J-holomorphic curve moduli spaces, one has to construct an MW weak Kuranishi atlas (or something close to it) first, and then define the FOOO Kuranishi structure using this.

When one constructs an MW weak Kuranishi atlas $\mathcal{K}$ on a moduli space of J-holomorphic curves $\overline{\mathcal{M}}$, the construction involves many arbitrary choices, but McDuff and Wehrheim expect different choices $\mathcal{K},\mathcal{K}'$ to be commensurate. They prove this [82, Rem. 6.2.2] for their definition of MW weak Kuranishi atlases on moduli spaces of nonsingular genus zero Gromov–Witten curves in [82, §4.3].

We relate Definition 7.18(d) to 2-morphisms in §6.1:

Example 7.19. In the situation of Definition 7.18(d), let $\tilde\Phi_{BC},\tilde\Phi_{BD},\tilde\Phi_{CD}$ be the coordinate changes in the sense of §6.1 associated to the MW coordinate changes $\Phi_{BC},\Phi_{BD},\Phi_{CD}$ in Example 7.17. The composition coordinate change $\tilde\Phi_{CD}\circ\tilde\Phi_{BC}=(P_{BCD},\pi_{BCD},\phi_{BCD},\hat\phi_{BCD})$ from Definition 6.5 has

$$\begin{aligned}P_{BCD}&=\bigl[(\tilde V_{BC}\times\Gamma_B)\times_{V_C}(\tilde V_{CD}\times\Gamma_C)\bigr]/\Gamma_C\\&\cong(\tilde V_{BC}\times_{V_C}\tilde V_{CD})\times\Gamma_B\cong\varpi_{CD}^{-1}(\tilde V_{BC})\times\Gamma_B.\end{aligned}\tag{7.9}$$

Define $\acute P_{BCD}$ to be the open subset of P_{BCD} identified with $\tilde V_{BCD}\times\Gamma_B$ by (7.9), and $\lambda_{BCD}:\acute P_{BCD}\to P_{BD}=\tilde V_{BD}\times\Gamma_B$ to be the map identified by (7.9) with the inclusion $\tilde V_{BCD}\times\Gamma_B\hookrightarrow\tilde V_{BD}\times\Gamma_B$, and $\hat\lambda_{BCD}=0$. Then as in Example 7.7(i), we can show that $(\acute P_{BCD},\lambda_{BCD},\hat\lambda_{BCD})$ satisfies Definition 6.4(a)–(c), so we have defined a 2-morphism $\Lambda_{BCD}=[\acute P_{BCD},\lambda_{BCD},\hat\lambda_{BCD}]:\tilde\Phi_{CD}\circ\tilde\Phi_{BC}\Rightarrow\tilde\Phi_{BD}$ on $S_{BCD}=\operatorname{Im}\psi_B\cap\operatorname{Im}\psi_C\cap\operatorname{Im}\psi_D$, in the sense of §6.1.

McDuff and Wehrheim prove [82, Th. B], [81, Th. A]:

Theorem 7.20. *Let $\mathcal{K} = \bigl(A, I, (V_B, E_B, \Gamma_B, s_B, \psi_B)_{B\in I}, \Phi_{BC,\ B,C\in I,\ B\subsetneq C}\bigr)$ be an oriented MW weak Kuranishi atlas of dimension n on a compact, metrizable topological space X. Then $\mathcal{K}$ determines:*

- **(a)** *A* ***virtual moduli cycle*** $[X]_{\rm vmc}$ *in the cobordism group $\Omega_n^{\rm SO,\mathbb{Q}}$ of compact, oriented, n-dimensional '$\mathbb{Q}$-weighted manifolds' in the sense of* [81, §A].
- **(b)** *A* ***virtual fundamental class*** $[X]_{\rm vfc}$ *in $\check H_n(X;\mathbb{Q})$, where $\check H_*(-;\mathbb{Q})$ is Čech homology over $\mathbb{Q}$.*

Any two commensurate MW weak Kuranishi atlases $\mathcal{K},\mathcal{K}'$ on X yield the same virtual moduli cycle and virtual fundamental class.

If $\mathcal{K}$ has trivial isotropy (that is, $\Gamma_B = \{1\}$ for all $B \in I$) then we may instead take $[X]_{\rm vmc} \in \Omega_n^{\rm SO}$, where $\Omega_*^{\rm SO}$ is the usual oriented cobordism group, and $[X]_{\rm vfc} \in H_n^{\rm St}(X;\mathbb{Z})$, where $H_*^{\rm St}(-;\mathbb{Z})$ is Steenrod homology over $\mathbb{Z}$.

In part (a), the author expects that $\Omega_n^{\rm SO,\mathbb{Q}} \cong \Omega_n^{\rm SO} \otimes_{\mathbb{Z}} \mathbb{Q}$, so that $\Omega_*^{\rm SO,\mathbb{Q}} \cong \mathbb{Q}[x_4, x_8, \ldots]$ by results of Thom.

Theorem 7.20 is McDuff and Wehrheim's solution to the issues discussed in Remark 7.14. As an intermediate step in the proof of Theorem 7.20, they pass to a Kuranishi atlas with better properties (a 'reduction' of a 'tame, metrizable' Kuranishi atlas), which is similar to a FOOO good coordinate system.

7.4 Dingyu Yang's Kuranishi structures, and polyfolds

As part of a project to define a truncation functor from polyfolds to Kuranishi spaces, Dingyu Yang [110–112] writes down his own theory of Kuranishi spaces:

Definition 7.21. Let X be a compact, metrizable topological space. A *DY Kuranishi structure* $\mathcal{K}$ on X is a FOOO Kuranishi structure in the sense of Definition 7.6, satisfying the additional conditions [111, Def. 1.11]:

- (a) the *maximality condition*, which is essentially Definition 7.11(e),(f), but replacing $i \prec j$ by $q \in \operatorname{Im}\psi_p$.
- (b) the *topological matching condition*, which is related to Definition 7.11(d), but replacing $i \prec j$ by $q \in \operatorname{Im}\psi_p$.

There are a few other small differences — for instance, Yang does not require the vector bundles E_p in $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ to be trivial.

We show in Theorem 7.35 below that given a DY Kuranishi structure $\mathcal{K}$ on X, we can make X into a Kuranishi space $\boldsymbol{X}$ in the sense of §6.2.

Yang also defines his own notion of DY good coordinate system [111, Def. 2.4], which is almost the same as a FOOO good coordinate system in §7.2.

One reason for these modifications is that it simplifies the passage from Kuranishi spaces to good coordinate systems, as in Remark 7.14(i): Yang shows

[111, Th. 2.10] that given any DY Kuranishi space $\boldsymbol{X}$, one can construct a DY good coordinate system $\bigl((I,\prec),(V'_p,E'_p,\Gamma'_p,s'_p,\psi'_p)_{p\in I},\Phi'_{qp,\; q\prec p \text{ in } I}\bigr)$ in which $I\subseteq X$ is a finite subset, $V'_p\subseteq V_p$ is a Γ_p-invariant open subset, $\Gamma'_p=\Gamma_p$, and E'_p,s'_p,ψ'_p are the restrictions of E_p,s_p,ψ_p to V'_p for each $p\in I$, and the coordinate changes Φ'_{qp} for $q\prec p$ are obtained either by restricting Φ_{qp} to an open $V'_{qp}\subseteq V_{qp}$ if $q\in\operatorname{Im}\psi_p$, or in a more complicated way otherwise.

The next definition comes from Yang [110, §1.6], [111, §5], [112, §2.4].

Definition 7.22. Let $\mathcal{K},\mathcal{K}'$ be DY Kuranishi structures on a compact topological space X. An *embedding* $\epsilon:\mathcal{K}\hookrightarrow\mathcal{K}'$ is a choice of FOOO coordinate change $\epsilon_p:(V_p,E_p,\Gamma_p,s_p,\psi_p)\to(V'_p,E'_p,\Gamma'_p,s'_p,\psi'_p)$ with domain V_p for all $p\in X$, commuting with the FOOO coordinate changes Φ_{qp},Φ'_{qp} in $\mathcal{K},\mathcal{K}'$ up to elements of Γ'_p. An embedding is a *chart refinement* if the ϵ_p come from inclusions of Γ_p-invariant open sets $V_p\hookrightarrow V'_p$.

DY Kuranishi structures $\mathcal{K},\mathcal{K}'$ on X are called *R-equivalent* (or *equivalent*) if there is a diagram of DY Kuranishi structures on X

$$\mathcal{K}\xleftarrow{\;\sim\;}\mathcal{K}_1\Longrightarrow\mathcal{K}_2\Longleftarrow\mathcal{K}_3\xrightarrow{\;\sim\;}\mathcal{K}',$$

where arrows $\Longrightarrow$ are embeddings, and $\xrightarrow{\sim}$ are chart refinements. Using facts about existence of good coordinate systems, Yang proves [110, Th. 1.6.17], [111, §11.2] that R-equivalence is an equivalence relation on DY Kuranishi structures.

Yang emphasizes the idea, which he calls *choice independence*, that when one constructs a (DY) Kuranishi structure $\mathcal{K}$ on a moduli space $\overline{\mathcal{M}}$, it should be independent of choices up to R-equivalence.

One major goal of Yang's work is to relate the Kuranishi space theory of Fukaya, Oh, Ohta and Ono [19–39] to the polyfold theory of Hofer, Wysocki and Zehnder [46–53]. Here is a very brief introduction to this:

- An *sc-Banach space* $\mathcal{V}$ is a sequence $\mathcal{V}=(\mathcal{V}_0\supset\mathcal{V}_1\supset\mathcal{V}_2\supset\cdots)$, where the $\mathcal{V}_i$ are Banach spaces, the inclusions $\mathcal{V}_{i+1}\hookrightarrow\mathcal{V}_i$ are compact, bounded linear maps, and $\mathcal{V}_\infty=\bigcap_{i\geqslant 0}\mathcal{V}_i$ is dense in every $\mathcal{V}_i$.

  The *tangent space* $T\mathcal{V}$ is $T\mathcal{V}=(\mathcal{V}_1\oplus\mathcal{V}_0\supset\mathcal{V}_2\oplus\mathcal{V}_1\supset\cdots)$, an sc-Banach space. An *open set* $\mathcal{Q}$ in $\mathcal{V}$ is an open set $\mathcal{Q}\subset\mathcal{V}_0$, and we write $\mathcal{Q}_i=\mathcal{Q}\cap\mathcal{V}_i$ for $i\geqslant 0$. Its *tangent space* is $T\mathcal{Q}=\mathcal{Q}_1\oplus\mathcal{V}_0$, as an open set in $T\mathcal{V}$.

  An example to bear in mind is if M is a compact manifold, $E\to M$ a smooth vector bundle, $\alpha\in(0,1)$, and $\mathcal{V}_k=C^{k,\alpha}(E)$ for $k=0,1,\ldots$.

- Let $\mathcal{V}=(\mathcal{V}_0\supset\mathcal{V}_1\supset\cdots)$, $\mathcal{W}=(\mathcal{W}_0\supset\mathcal{W}_1\supset\cdots)$ be sc-Banach spaces and $\mathcal{Q}\subseteq\mathcal{V}$, $\mathcal{R}\subseteq\mathcal{W}$ be open. A map $f:\mathcal{Q}\to\mathcal{R}$ is called sc^0 if $f(\mathcal{Q}_i)\subseteq\mathcal{R}_i$ and $f|_{\mathcal{Q}_i}:\mathcal{Q}_i\to\mathcal{R}_i$ is a continuous map of Banach manifolds for all $i\geqslant 0$.

  An sc^0 map $f:\mathcal{Q}\to\mathcal{R}$ is called sc^1 if for each $q\in\mathcal{Q}_1$ there exists a bounded linear map $Df_q:\mathcal{V}_0\to\mathcal{W}_0$, such that $f|_{\mathcal{Q}_1}:\mathcal{Q}_1\to\mathcal{R}_0$ is a C^1 map of Banach manifolds with $\nabla f|_q=Df_q|_{\mathcal{V}_1}:\mathcal{V}_1\to\mathcal{W}_0$ for all $q\in\mathcal{Q}_1$, and $Tf:T\mathcal{Q}\to T\mathcal{R}$ mapping $Tf:(q,v)\mapsto(f(q),Df_q(v))$ is an sc^0 map.

By induction on k, we call $f : \mathcal{Q} \to \mathcal{R}$ an *sc*^{k} map for $k = 2, 3, \ldots$ if f is sc^1 and $Tf : T\mathcal{Q} \to T\mathcal{R}$ is an sc^{k-1} map. We call $f : \mathcal{Q} \to \mathcal{R}$ *sc-smooth*, or *sc*$^\infty$, if it is sc^k for all $k = 0, 1, \ldots$. This implies that $f|_{\mathcal{Q}_{i+k}} : \mathcal{Q}_{i+k} \to \mathcal{R}_i$ is a C^k-map of Banach manifolds for all $i, k \geqslant 0$.

- Let $\mathcal{V} = (\mathcal{V}_0 \supset \mathcal{V}_1 \supset \cdots)$ be an sc-Banach space and $\mathcal{Q} \subseteq \mathcal{V}$ be open. An *sc*$^\infty$*-retraction* is an sc-smooth map $r : \mathcal{Q} \to \mathcal{Q}$ with $r \circ r = r$. Set $\mathcal{O} = \operatorname{Im} r \subset \mathcal{V}$. We call $(\mathcal{O}, \mathcal{V})$ a *local sc-model.*

  If $\mathcal{V}$ is finite-dimensional then $\mathcal{O}$ is just a smooth manifold. But in infinite dimensions, new phenomena occur, and the tangent spaces $T_x\mathcal{O}$ can vary discontinuously with $x \in \mathcal{O}$. This is important for 'gluing'.
- An *M-polyfold chart* $(\mathcal{O}, \mathcal{V}, \psi)$ on a topological space Z is a local sc-model $(\mathcal{O}, \mathcal{V})$ and a homeomorphism $\psi : \mathcal{O} \to \operatorname{Im} \psi$ with an open set $\operatorname{Im} \psi \subset Z$.
- M-polyfold charts $(\mathcal{O}, \mathcal{V}, \psi), (\tilde{\mathcal{O}}, \tilde{\mathcal{V}}, \tilde{\psi})$ on Z are *compatible* if $\tilde{\psi}^{-1} \circ \psi \circ r : \mathcal{Q} \to \tilde{\mathcal{V}}$ and $\psi^{-1} \circ \tilde{\psi} \circ \tilde{r} : \tilde{\mathcal{Q}} \to \mathcal{V}$ are sc-smooth, where $\mathcal{Q} \subset \mathcal{V}$, $\tilde{\mathcal{Q}} \subset \tilde{\mathcal{V}}$ are open and $r : \mathcal{Q} \to \mathcal{Q}$, $\tilde{r} : \tilde{\mathcal{Q}} \to \tilde{\mathcal{Q}}$ are sc-smooth with $r \circ r = r$, $\tilde{r} \circ \tilde{r} = \tilde{r}$ and $\operatorname{Im} r = \psi^{-1}(\operatorname{Im} \tilde{\psi}) \subseteq \mathcal{O}$, $\operatorname{Im} \tilde{r} = \tilde{\psi}^{-1}(\operatorname{Im} \psi) \subseteq \tilde{\mathcal{O}}$.
- An *M-polyfold* is roughly a metrizable topological space Z with a maximal atlas of pairwise compatible M-polyfold charts.
- *Polyfolds* are the orbifold version of M-polyfolds, proper étale groupoids in M-polyfolds.
- A *polyfold Fredholm structure* $\mathcal{P}$ on a metrizable topological space X writes X as the zeroes of an sc-Fredholm section $\mathfrak{s} : \mathfrak{V} \to \mathfrak{E}$ of a strong polyfold vector bundle $\mathfrak{E} \to \mathfrak{V}$ over a polyfold $\mathfrak{V}$.

This is all rather complicated. The motivation for local sc-models $(\mathcal{O}, \mathcal{V})$ is that they can be used to describe functional-analytic problems involving 'gluing', 'bubbling', and 'neck-stretching', including moduli spaces of J-holomorphic curves with singularities of various kinds.

The polyfold programme [46–53] aims to show that moduli spaces of J-holomorphic curves in symplectic geometry may be given a polyfold Fredholm structure, and that compact spaces with oriented polyfold Fredholm structures have virtual chains and virtual classes. One can then use these virtual chains/classes to define big theories in symplectic geometry, such as Gromov–Witten invariants or Symplectic Field Theory. Constructing a polyfold Fredholm structure on a moduli space of J-holomorphic curves involves far fewer arbitrary choices than defining a Kuranishi structure. Fabert, Fish, Golovko and Wehrheim [17] survey the polyfold programme.

Yang proves [110, Th. 3.1.7] (see also [112, §2.6]):

Theorem 7.23. *Suppose we are given a 'polyfold Fredholm structure' $\mathcal{P}$ on a compact metrizable topological space X, that is, we write X as the zeroes of an sc-Fredholm section $\mathfrak{s} : \mathfrak{V} \to \mathfrak{E}$ of a strong polyfold vector bundle $\mathfrak{E} \to \mathfrak{V}$ over a polyfold $\mathfrak{V}$, where $\mathfrak{s}$ has constant Fredholm index $n \in \mathbb{Z}$. Then we can construct a DY Kuranishi structure $\mathcal{K}$ on X, of virtual dimension n, which is independent of choices up to R-equivalence.*

In the survey [112], Yang announces further results for which the proofs were not available at the time of writing. These include:

(a) Yang defines 'R-equivalence' of polyfold Fredholm structures on X [112, Def. 2.14], and claims [112, §2.8] that Theorem 7.23 extends to a 1-1 correspondence between R-equivalence classes of polyfold Fredholm structures on X, and R-equivalence classes of DY Kuranishi structures $\mathcal{K}$ on X.

(b) In [112, §2.4], Yang claims that R-equivalence extends as an equivalence relation to FOOO Kuranishi structures, and every R-equivalence class of FOOO Kuranishi structures contains a DY Kuranishi structure. Hence the 1-1 correspondence in (a) also extends to a 1-1 correspondence with R-equivalence classes of FOOO Kuranishi structures.

(c) Yang claims that virtual chains or virtual classes for polyfolds and for FOOO/DY Kuranishi spaces agree under (a),(b).

(d) Yang says [112, p. 26, p. 46] that in future work he will make spaces with DY Kuranishi structures into a category $\mathbf{Kur}_{\mathrm{DY}}$.

These results would enable a clean translation between the polyfold and Kuranishi approaches to symplectic geometry. It seems likely that in (d) there will be an equivalence of categories $\mathbf{Kur}_{\mathrm{DY}} \simeq \mathrm{Ho}(\mathbf{Kur})$, for $\mathbf{Kur}$ as in §6.2.

7.5 Relating our Kuranishi spaces to previous definitions

We now show that all of the Kuranishi-type structures discussed in §7.1–§7.3 can be made into a Kuranishi space $\boldsymbol{X}$ in our sense, uniquely up to equivalence in $\mathbf{Kur}$ or $\mathbf{Kur^c}$. We do this by defining a notion of 'fair coordinate system' $\mathcal{F}$ on a topological space X in §7.5.1 which is so general that it includes all of the structures of §7.1–§7.3 as special cases, and proving that given $X, \mathcal{F}$, we can construct a Kuranishi structure $\mathcal{K}$ on X uniquely up to equivalence.

In §7.5.1 we work over any category of 'manifolds' $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7, and then in §7.5.2–§7.5.5 we specialize to $\dot{\mathbf{Man}} = \mathbf{Man}$ or $\mathbf{Man^c}$, following our references [19–39, 77, 78, 80–83, 110–112].

Theorems 7.29, 7.31, 7.33, 7.35, and 7.36 below are important, as they show that the geometric structures on moduli spaces considered by Fukaya, Oh, Ohta and Ono [19–39], McDuff and Wehrheim [77, 78, 80–83], Yang [110–112], and Hofer, Wysocki and Zehnder [46–53], can all be transformed to Kuranishi spaces in our sense. Thus, large parts of the symplectic geometry literature can now be interpreted in our framework.

7.5.1 Fair coordinate systems and Kuranishi spaces

Our next definition is a kind of 'least common denominator' for the Kuranishi-type structures discussed in §7.1–§7.3. The name 'fair coordinate system' is intended to suggest something like the 'good coordinate systems' in §7.2, but not as strong. We work over a category $\dot{\mathbf{Man}}$ satisfying the assumptions of §3.1.

Definition 7.24. Let X be a Hausdorff, second countable topological space. A *fair coordinate system $\mathcal{F}$ on X, of virtual dimension $n \in \mathbb{Z}$*, is data $\mathcal{F} = \bigl(A, (V_a, E_a, \Gamma_a, s_a, \psi_a)_{a\in A},\ S_{ab}, \Phi_{ab,\ a,b\in A},\ S_{abc}, \Lambda_{abc,\ a,b,c\in A}\bigr)$, where:

(a) A is an indexing set (not necessarily finite).

(b) $(V_a, E_a, \Gamma_a, s_a, \psi_a)$ is a Kuranishi neighbourhood on X for each $a \in A$, with $\dim V_a - \operatorname{rank} E_a = n$, as in §6.1.

(c) $S_{ab} \subseteq \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b$ is an open set for all $a, b \in A$. (We can have $S_{ab} = \emptyset$.)

(d) $\Phi_{ab} = (P_{ab}, \pi_{ab}, \phi_{ab}, \hat\phi_{ab}) : (V_a, E_a, \Gamma_a, s_a, \psi_a) \to (V_b, E_b, \Gamma_b, s_b, \psi_b)$ is a coordinate change over S_{ab}, for all $a, b \in A$, as in §6.1.

(e) $S_{abc} \subseteq S_{ab} \cap S_{ac} \cap S_{bc} \subseteq \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b \cap \operatorname{Im}\psi_c$ is an open set for all $a, b, c \in A$. (We can have $S_{abc} = \emptyset$.)

(f) $\Lambda_{abc} = [\acute P_{abc}, \lambda_{abc}, \hat\lambda_{abc}] : \Phi_{bc}\circ\Phi_{ab} \Rightarrow \Phi_{ac}$ is a 2-morphism for all $a, b, c \in A$, defined over S_{abc}.

(g) $\bigcup_{a\in A} \operatorname{Im}\psi_a = X$.

(h) $S_{aa} = \operatorname{Im}\psi_a$ and $\Phi_{aa} = \mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)}$ for all $a \in A$.

(i) $S_{aab} = S_{abb} = S_{ab}$ and $\Lambda_{aab} = \boldsymbol\beta_{\Phi_{ab}}$, $\Lambda_{abb} = \boldsymbol\gamma_{\Phi_{ab}}$ for all $a, b \in A$.

(j) The following diagram of 2-morphisms over $S_{abc} \cap S_{abd} \cap S_{acd} \cap S_{bcd}$ commutes for all $a, b, c, d \in A$:

$$\begin{tikzcd}[column sep=large]
(\Phi_{cd}\circ\Phi_{bc})\circ\Phi_{ab} \arrow[rr, Rightarrow, "\Lambda_{bcd}*\mathrm{id}_{\Phi_{ab}}"'] \arrow[d, Rightarrow, "\boldsymbol\alpha_{\Phi_{cd},\Phi_{bc},\Phi_{ab}}"] & & \Phi_{bd}\circ\Phi_{ab} \arrow[d, Rightarrow, "\Lambda_{abd}"'] \\
\Phi_{cd}\circ(\Phi_{bc}\circ\Phi_{ab}) \arrow[r, Rightarrow, "\mathrm{id}_{\Phi_{cd}}*\Lambda_{abc}"] & \Phi_{cd}\circ\Phi_{ac} \arrow[r, Rightarrow, "\Lambda_{acd}"] & \Phi_{ad}.
\end{tikzcd}$$

Also, *either* condition (k) *or* condition (k)$'$ below hold, or both, where:

(k) Suppose $B \subseteq A$ is finite and nonempty, and $x \in \bigcap_{b\in B} \operatorname{Im}\psi_b \subseteq X$. Then there exists $a \in A$ such that $x \in S_{ab}$ for all $b \in B$, and if $b, c \in B$ with $x \in S_{bc}$ then $x \in S_{abc}$.

(k)$'$ Suppose $B \subseteq A$ is finite and nonempty, and $x \in \bigcap_{b\in B} \operatorname{Im}\psi_b \subseteq X$. Then there exists $d \in A$ such that $x \in S_{bd}$ for all $b \in B$, and if $b, c \in B$ with $x \in S_{bc}$ then $x \in S_{bcd}$.

Here (k),(k)$'$ are somewhat arbitrary. What we are trying to achieve by these conditions on the S_{ab}, S_{abc} is roughly that:

(A) If $x \in \operatorname{Im}\psi_b \cap \operatorname{Im}\psi_c$, one can map $(V_b, E_b, \Gamma_b, s_b, \psi_b) \to (V_c, E_c, \Gamma_c, s_c, \psi_c)$ near x by a finite chain of coordinate changes Φ_{ij} and their (quasi)inverses Φ_{ji}^{-1} — for (k) by $\Phi_{ac}\circ\Phi_{ab}^{-1}$, and for (k)$'$ by $\Phi_{cd}^{-1}\circ\Phi_{bd}$.

(B) Any two such chains of $\Phi_{ij}, \Phi_{ji}^{-1}$ near x are canonically 2-isomorphic near x using combinations of the 2-isomorphisms Λ_{ijk} and their inverses.

We chose (k),(k)$'$ as they hold in our examples, and there is a nice method to prove Theorem 7.26 using (k) or (k)$'$.

Example 7.25. Let $\boldsymbol{X}=(X,\mathcal{K})$ be a Kuranishi space in the sense of §6.2, with $\mathcal{K}=\bigl(I,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \Lambda_{ijk,\ i,j,k\in I}\bigr)$. Set $S_{ij}=\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j$ for all $i,j\in I$, and $S_{ijk}=\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\cap\operatorname{Im}\psi_k$ for all $i,j,k\in I$. Then $\mathcal{F}=\bigl(I,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\,S_{ij},\Phi_{ij,\ i,j\in I},\,S_{ijk},\Lambda_{ijk,\ i,j,k\in I}\bigr)$ is a fair coordinate system on X. Here Definition 7.24(a)–(j) are immediate from Definition 6.17(a)–(h), and both of Definition 7.24(k),(k)$'$ hold, where we can take $a\in B$ arbitrary in (k) and $d\in B$ arbitrary in (k)$'$.

The next theorem will be proved in §7.6. When we say $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ 'may be given the structure of a Kuranishi neighbourhood on the Kuranishi space $\boldsymbol{X}$', we mean that as in §6.4, we can choose implicit extra data $\Phi_{ai,\ i\in I}$, $\Lambda_{aij,\ i,j\in I}$ relating $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ to the Kuranishi structure $\mathcal{K}$ on X, and similarly, by 'Φ_{ab} may be given the structure of a coordinate change over S_{ab} on the Kuranishi space $\boldsymbol{X}$', we mean that we can choose implicit extra data $\Lambda_{abi,\ i\in I}$ relating Φ_{ab} to $\mathcal{K}$.

Theorem 7.26. *Suppose $\mathcal{F}=\bigl(A,(V_a,E_a,\Gamma_a,s_a,\psi_a)_{a\in A},\ S_{ab},\Phi_{ab,\ a,b\in A},\ S_{abc},$ $\Lambda_{abc,\ a,b,c\in A}\bigr)$ is a fair coordinate system of virtual dimension $n\in\mathbb{Z}$ on a Hausdorff, second countable topological space X, in the sense of Definition 7.24. Then we may make X into a Kuranishi space $\boldsymbol{X}=(X,\mathcal{K})$ in the sense of §6.2 with $\operatorname{vdim}\boldsymbol{X}=n$, such that $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ may be given the structure of a Kuranishi neighbourhood on the Kuranishi space $\boldsymbol{X}$ in the sense of §6.4 for all $a\in A$, and $\Phi_{ab}:(V_a,E_a,\Gamma_a,s_a,\psi_a)\to(V_b,E_b,\Gamma_b,s_b,\psi_b)$ may be given the structure of a coordinate change over S_{ab} on the Kuranishi space $\boldsymbol{X}$ in the sense of §6.4 for all $a,b\in A$, and $\Lambda_{abc}:\Phi_{bc}\circ\Phi_{ab}\Rightarrow\Phi_{ac}$ is the unique 2-morphism over S_{abc} given by Theorem 6.45(a) for all $a,b,c\in A$. This $\boldsymbol{X}$ is unique up to canonical equivalence in the 2-category $\dot{\mathbf{Kur}}$, as in Definition A.7.*

The next proposition follows easily from Corollary 6.48 and Theorem 7.26.

Proposition 7.27. *Let $\mathcal{F}=\bigl(A,(V_a,E_a,\Gamma_a,s_a,\psi_a)_{a\in A},\ S_{ab},\Phi_{ab,\ a,b\in A},\ S_{abc},$ $\Lambda_{abc,\ a,b,c\in A}\bigr)$ be a fair coordinate system on X. Suppose $\tilde A\subseteq A$ with $\bigcup_{a\in\tilde A}\operatorname{Im}\psi_a$ $=X$, and in Definition 7.24(k),(k)$'$, if $B\subseteq\tilde A\subseteq A$ then we can choose $a\in\tilde A$ in (k) and $d\in\tilde A$ in (k)$'$. Then $\tilde{\mathcal{F}}=\bigl(\tilde A,(V_a,E_a,\Gamma_a,s_a,\psi_a)_{a\in\tilde A},\ S_{ab},\Phi_{ab,\ a,b\in\tilde A},$ $S_{abc},\Lambda_{abc,\ a,b,c\in\tilde A}\bigr)$ is also a fair coordinate system on X. Let $\boldsymbol{X}=(X,\mathcal{K})$ and $\tilde{\boldsymbol{X}}=(X,\tilde{\mathcal{K}})$ be the Kuranishi spaces constructed from $\mathcal{F},\tilde{\mathcal{F}}$ in Theorem 7.26. Then $\boldsymbol{X},\tilde{\boldsymbol{X}}$ are canonically equivalent in $\dot{\mathbf{Kur}}$, as in Definition A.7.*

7.5.2 Fukaya–Oh–Ohta–Ono's Kuranishi spaces

Section 7.1 defined Fukaya–Oh–Ohta–Ono's 'FOOO Kuranishi spaces' (working over $\dot{\mathbf{Man}}=\mathbf{Man}$) and 'FOOO Kuranishi spaces with corners' (over $\dot{\mathbf{Man}}=\mathbf{Man^c}$). We now relate these to our notion of Kuranishi spaces.

Example 7.28. Let $\boldsymbol{X} = (X, \mathcal{K})$ be a FOOO Kuranishi space with vdim $\boldsymbol{X} = n$, in the sense of Definition 7.6. Then $\mathcal{K}$ gives a FOOO Kuranishi neighbourhood $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ for each $p \in X$, and for all $p, q \in X$ with $q \in \operatorname{Im}\psi_p$ it gives a FOOO coordinate change $\Phi_{qp} = (V_{qp}, h_{qp}, \varphi_{qp}, \hat{\varphi}_{qp}) : (V_q, E_q, \Gamma_q, s_q, \psi_q) \to (V_p, E_p, \Gamma_p, s_p, \psi_p)$ defined on an open neighbourhood S_{qp} of q in $\operatorname{Im}\psi_q \cap \operatorname{Im}\psi_p$, and for all $p, q, r \in X$ with $q \in \operatorname{Im}\psi_p$ and $r \in S_{qp}$, Definition 7.6(b) gives unique group elements $\gamma^\alpha_{rqp} \in \Gamma_p$ which relate $\Phi_{qp} \circ \Phi_{rq}$ to Φ_{rp} on $S_{rqp} := S_{qp} \cap S_{rp} \cap S_{rq}$.

We will define a fair coordinate system $\mathcal{F}$ on X, over $\dot{\mathbf{Man}} = \mathbf{Man}$. Take the indexing set A to be $A = X$, and for each $p \in A$, let the Kuranishi neighbourhood $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ be as in $\mathcal{K}$, regarded as a Kuranishi neighbourhood in the sense of §6.1 as in Example 7.2. If $p \neq q \in A$ with $q \in \operatorname{Im}\psi_p$, define $S_{qp} \subseteq \operatorname{Im}\psi_q \cap \operatorname{Im}\psi_p$ to be the domain of the FOOO coordinate change Φ_{qp} in $\mathcal{K}$. Define $\tilde{\Phi}_{qp} : (V_q, E_q, \Gamma_q, s_q, \psi_q) \to (V_p, E_p, \Gamma_p, s_p, \psi_p)$ to be the coordinate change over S_{qp} in the sense of §6.1 associated to the FOOO coordinate change Φ_{qp} in Example 7.5. Define $S_{pp} = \operatorname{Im}\psi_p$ and $\tilde{\Phi}_{pp} = \mathrm{id}_{(V_p, E_p, \Gamma_p, s_p, \psi_p)}$ for all $p \in A$. If $p \neq q \in A$ and $q \notin \operatorname{Im}\psi_p$, define $S_{qp} = \emptyset$ and $\tilde{\Phi}_{qp} = (\emptyset, \emptyset, \emptyset, \emptyset)$.

If $p \neq q \neq r \in A$ with $q \in \operatorname{Im}\psi_p$ and $r \in S_{qp}$, set $S_{rqp} = S_{qp} \cap S_{rp} \cap S_{rq}$, and define $\Lambda_{rqp} : \tilde{\Phi}_{qp} \circ \tilde{\Phi}_{rq} \Rightarrow \tilde{\Phi}_{rp}$ to be the 2-morphism over S_{rqp} defined in Example 7.7(ii) using the group elements $\gamma^\alpha_{rqp} \in \Gamma_p$ in Definition 7.6(b). If $p \neq q \neq r \in A$ with $q \notin \operatorname{Im}\psi_p$ or $r \notin S_{qp}$, define $S_{rqp} = \emptyset$ and $\Lambda_{rqp} = [\emptyset, \emptyset, \emptyset]$. Define $S_{qpp} = S_{qqp} = S_{qp}$ and $\Lambda_{qqp} = \boldsymbol{\beta}_{\tilde{\Phi}_{qp}}$, $\Lambda_{qpp} = \boldsymbol{\gamma}_{\tilde{\Phi}_{qp}}$ for all $p, q \in A$. This defines all the data in $\mathcal{F} = \bigl(A, (V_p, E_p, \Gamma_p, s_p, \psi_p)_{p \in A}, S_{qp}, \tilde{\Phi}_{qp,\, q,p \in A}, S_{rqp}, \Lambda_{rqp,\, r,q,p \in A}\bigr)$. We will show $\mathcal{F}$ satisfies Definition 7.24(a)–(k).

Parts (a)–(i) are immediate. For (j), if $p \neq q \neq r \neq s \in X$ with $q \in \operatorname{Im}\psi_p$ and $r \in S_{qp}$ and $s \in S_{rq} \cap S_{rp}$ then Definition 7.6(b) gives elements $\gamma^\alpha_{rqp}, \gamma^{\alpha'}_{sqp}, \gamma^{\alpha''}_{srp} \in \Gamma_p$ and $\gamma^{\alpha'''}_{srq} \in \Gamma_q$ satisfying (7.3). Using (7.3) four times we see that

$$\gamma^\alpha_{rqp}\gamma^{\alpha''}_{srp} \cdot \varphi_{sp} = \varphi_{qp} \circ \varphi_{rq} \circ \varphi_{sr} = h_{qp}(\gamma^{\alpha'''}_{srq})\gamma^{\alpha''}_{sqp} \cdot \varphi_{sp}, \tag{7.10}$$

where (7.10) holds on the domain

$$\begin{aligned}&\varphi_{sr}^{-1}\bigl((\varphi_{rq}^{-1}(V_{qp}) \cap V_{rq} \cap V_{rp})^\alpha\bigr) \cap (\varphi_{sq}^{-1}(V_{qp}) \cap V_{sq} \cap V_{sp})^{\alpha'} \cap \\ &(\varphi_{sr}^{-1}(V_{rp}) \cap V_{sr} \cap V_{sp})^{\alpha''} \cap (\varphi_{sr}^{-1}(V_{rq}) \cap V_{sr} \cap V_{sq})^{\alpha'''}.\end{aligned} \tag{7.11}$$

If (7.11) is nonempty, the argument of Example 7.7(iii) implies that $\gamma^\alpha_{rqp}\gamma^{\alpha''}_{srp} = h_{qp}(\gamma^{\alpha'''}_{srq})\gamma^{\alpha''}_{sqp}$. This is the condition required to verify $\Lambda_{srp} \odot (\Lambda_{rqp} * \mathrm{id}_{\tilde{\Phi}_{sr}}) = \Lambda_{sqp} \odot (\mathrm{id}_{\tilde{\Phi}_{qp}} * \Lambda_{srq}) \odot \boldsymbol{\alpha}_{\tilde{\Phi}_{qp}, \tilde{\Phi}_{rq}, \tilde{\Phi}_{sr}}$ on the component of $S_{srq} \cap S_{srp} \cap S_{sqp} \cap S_{rqp}$ corresponding to the connected components $\alpha, \alpha', \alpha'', \alpha'''$.

This proves Definition 7.24(j) in this case. If $p = q$ then (j) becomes

$$\begin{aligned}\Lambda_{srq} \odot (\boldsymbol{\gamma}_{\tilde{\Phi}_{rq}} * \mathrm{id}_{\tilde{\Phi}_{sr}}) = \boldsymbol{\gamma}_{\tilde{\Phi}_{sq}} &\odot (\mathrm{id}_{\mathrm{id}_{(V_q,E_q,\Gamma_q,s_q,\psi_q)}} * \Lambda_{srq}) \\ &\odot \boldsymbol{\alpha}_{\mathrm{id}_{(V_q,E_q,\Gamma_q,s_q,\psi_q)}, \tilde{\Phi}_{rq}, \tilde{\Phi}_{sr}},\end{aligned} \tag{7.12}$$

which holds trivially, and the cases $q = r$, $r = s$ are similar. In the remaining cases one of $S_{srq}, S_{srp}, S_{sqp}, S_{rqp}$ is empty, so (j) is vacuous. Thus (j) holds.

For (k), suppose $B \subseteq A$ is finite and nonempty, and $x \in \bigcap_{p \in B} \operatorname{Im} \psi_p \subseteq X$. Then $x \in S_{xp}$ for all $p \in B$, since S_{xp} is an open neighbourhood of x in $\operatorname{Im} \psi_x \cap \operatorname{Im} \psi_p$, and $x \in S_{xqp}$ for all $q, p \in B$ with $x \in S_{qp}$, since $S_{xqp} = S_{qp} \cap S_{xp} \cap S_{xq}$ in this case and $x \in S_{xp}$, $x \in S_{xq}$. Thus (k) holds with $a = x$, and $\mathcal{F}$ is a fair coordinate system on X, over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$.

If instead $\boldsymbol{X}$ is a FOOO Kuranishi space with corners, the same construction gives a fair coordinate system $\mathcal{F}$ on X over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^c}$.

Combining Example 7.28 and Theorem 7.26 yields:

Theorem 7.29. *Suppose $\boldsymbol{X} = (X, \mathcal{K})$ is a FOOO Kuranishi space, as in Definition* 7.6. *Then we can construct a Kuranishi space $\boldsymbol{X}' = (X, \mathcal{K}')$ over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$ in the sense of* §6.2 *with* $\operatorname{vdim} \boldsymbol{X}' = \operatorname{vdim} \boldsymbol{X}$, *with the same topological space X, and $\boldsymbol{X}'$ is unique up to canonical equivalence in* $\mathbf{Kur}$.

If instead $\boldsymbol{X}$ is a FOOO Kuranishi space with corners, the same holds over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^c}$, so that $\boldsymbol{X}'$ is unique up to canonical equivalence in $\mathbf{Kur^c}$.

One can also show that geometric data and constructions for FOOO Kuranishi spaces $\boldsymbol{X}$ such as orientations in Definition 7.8, smooth maps $\boldsymbol{f} : \boldsymbol{X} \to Y$ to a manifold Y and 'fibre products' $\boldsymbol{X} \times_Y \boldsymbol{X}'$ in Definition 7.9, and boundaries $\partial \boldsymbol{X}$ of FOOO Kuranishi spaces with corners $\boldsymbol{X}$ in [24, Def. A1.30], can be mapped to the corresponding notions in our theory.

7.5.3 Fukaya–Oh–Ohta–Ono's (weak) good coordinate systems

Section 7.2 discussed Fukaya–Oh–Ohta–Ono's 'FOOO (weak) good coordinate systems (with corners)'. We relate these to our Kuranishi spaces.

Example 7.30. Let $\mathcal{G} = \big((I, \prec), (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i \in I}, \Phi_{ij,\, i \prec j}\big)$ be a FOOO weak good coordinate system of virtual dimension $n \in \mathbb{Z}$ on a compact, metrizable topological space X, in the sense of Definition 7.11.

We will define a fair coordinate system $\mathcal{F}$ on X over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$. Take the indexing set A to be I, and the Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ for $i \in I$ to be as given. If $i \neq j \in I$ with $i \prec j$, define $S_{ij} = \operatorname{Im} \psi_i \cap \operatorname{Im} \psi_j$, and $\tilde{\Phi}_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ to be the coordinate change over S_{ij} in the sense of §6.1 associated to the FOOO coordinate change Φ_{ij} in Example 7.5. Define $S_{ii} = \operatorname{Im} \psi_i$ and $\tilde{\Phi}_{ii} = \mathrm{id}_{(V_i, E_i, \Gamma_i, s_i, \psi_i)}$ for all $i \in I$. If $i \neq j \in I$ and $i \not\prec j$, define $S_{ij} = \emptyset$ and $\tilde{\Phi}_{ij} = (\emptyset, \emptyset, \emptyset, \emptyset)$.

If $i \neq j \neq k \in I$ with $i \prec j \prec k$, set $S_{ijk} = \operatorname{Im} \psi_i \cap \operatorname{Im} \psi_j \cap \operatorname{Im} \psi_k$, and define $\Lambda_{ijk} : \tilde{\Phi}_{jk} \circ \tilde{\Phi}_{ij} \Rightarrow \tilde{\Phi}_{ik}$ to be the 2-morphism over S_{ijk} defined in Example 7.7(ii) using the unique group element $\gamma_{ijk} \in \Gamma_k$ in Definition 7.11(b). If $i \neq j \neq k \in I$ with $i \not\prec j$ or $j \not\prec k$, define $S_{ijk} = \emptyset$ and $\Lambda_{ijk} = [\emptyset, \emptyset, \emptyset]$. Set $S_{iij} = S_{ijj} = \operatorname{Im} \psi_i \cap \operatorname{Im} \psi_j$ and $\Lambda_{iij} = \boldsymbol{\beta}_{\tilde{\Phi}_{ij}}$, $\Lambda_{ijj} = \boldsymbol{\gamma}_{\tilde{\Phi}_{ij}}$ for all $i, j \in I$. This defines all the data in $\mathcal{F} = \big(I, (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i \in I},\ S_{ij}, \tilde{\Phi}_{ij,\, i,j \in I},\ S_{ijk}, \Lambda_{ijk,\, i,j,k \in I}\big)$. We shall show $\mathcal{F}$ satisfies Definition 7.24(a)–(k).

Parts (a)–(i) are immediate. For (j), if $i \neq j \neq k \neq l$ in I with $i \prec j \prec k \prec l$ and $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k \cap \operatorname{Im}\psi_l \neq \emptyset$ then the argument of (7.10)–(7.11) shows that $\gamma_{jkl}\gamma_{ijl} = h_{kl}(\gamma_{ijk})\gamma_{ikl}$, and so $\Lambda_{ijl} \odot (\Lambda_{jkl} * \mathrm{id}_{\tilde\Phi_{ij}}) = \Lambda_{ikl} \odot (\mathrm{id}_{\tilde\Phi_{kl}} * \Lambda_{ijk}) \odot \boldsymbol{\alpha}_{\tilde\Phi_{kl},\tilde\Phi_{jk},\tilde\Phi_{ij}}$ as we want. The cases $i = j$, $j = k$, $k = l$ hold as for (7.12), and in the remaining cases one of $S_{ijk}, S_{ijl}, S_{ikl}, S_{jkl}$ is empty, so (j) is vacuous. Thus (j) holds.

For (k) or (k)$'$, suppose $\emptyset \neq B \subseteq I$ is finite and $x \in \bigcap_{b\in B} \operatorname{Im}\psi_b$. Then for all $b \neq c \in B$ we have $x \in \operatorname{Im}\psi_b \cap \operatorname{Im}\psi_c \neq \emptyset$, so $b \prec c$ or $c \prec b$ by Definition 7.11(a). Thus the partial order $\prec$ restricted to B is a total order, and we may uniquely write $B = \{b_1, b_2, \ldots, b_m\}$ with $b_1 \prec b_2 \prec \cdots \prec b_m$. It is now easy to check that (k) holds with $a = b_1$, and also (k)$'$ holds with $d = b_m$. Therefore $\mathcal{F}$ is a fair coordinate system on X over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$.

If instead $\mathcal{G}$ is a FOOO weak good coordinate system with corners, the same construction gives a fair coordinate system $\mathcal{F}$ on X over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^c}$.

Combining Example 7.30 and Theorem 7.26 yields:

Theorem 7.31. *Suppose X is a compact, metrizable topological space with a FOOO weak good coordinate system $\mathcal{G} = \bigl((I,\prec),(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\Phi_{ij,\; i\prec j}\bigr)$, of virtual dimension $n \in \mathbb{Z}$, in the sense of Definition* 7.11*. Then we can make X into a Kuranishi space $\boldsymbol{X} = (X,\mathcal{K})$ over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$ in the sense of* §6.2 *with* $\operatorname{vdim}\boldsymbol{X} = n$, *and $\boldsymbol{X}$ is unique up to canonical equivalence in* $\mathbf{Kur}$.

If instead $\mathcal{G}$ is a FOOO weak good coordinate system with corners, the same holds over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^c}$, *so that $\boldsymbol{X}$ is an object in* $\mathbf{Kur^c}$.

7.5.4 McDuff–Wehrheim's (weak) Kuranishi atlases

Section 7.3 discussed McDuff–Wehrheim's 'MW (weak) Kuranishi atlases', working over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$. We relate these to our Kuranishi spaces.

Example 7.32. Let $\bigl(A, I, (V_B,E_B,\Gamma_B,s_B,\psi_B)_{B\in I}, \Phi_{BC,\; B,C\in I,\; B\subsetneq C}\bigr)$ be an MW weak Kuranishi atlas of virtual dimension $n \in \mathbb{Z}$ on a compact, metrizable topological space X, in the sense of Definition 7.18.

We will define a fair coordinate system $\mathcal{F}$ on X over $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$. Take the indexing set to be I, and the Kuranishi neighbourhoods $(V_B,E_B,\Gamma_B,s_B,\psi_B)$ for $B \in I$ to be as given. If $B, C \in I$ with $B \subsetneq C$, define $S_{BC} = \operatorname{Im}\psi_B \cap \operatorname{Im}\psi_C$, and $\tilde\Phi_{BC} : (V_B,E_B,\Gamma_B,s_B,\psi_B) \to (V_C,E_C,\Gamma_C,s_C,\psi_C)$ to be the coordinate change over S_{BC} in the sense of §6.1 associated to the MW coordinate change Φ_{BC} in Example 7.17. Define $S_{BB} = \operatorname{Im}\psi_B$ and $\tilde\Phi_{BB} = \mathrm{id}_{(V_B,E_B,\Gamma_B,s_B,\psi_B)}$ for all $B \in I$. If $B \not\subseteq C$ in I, define $S_{BC} = \emptyset$ and $\tilde\Phi_{BC} = (\emptyset,\emptyset,\emptyset,\emptyset)$.

If $B \subsetneq C \subsetneq D$ in I then Definition 7.18(b)–(d) say essentially that $\Phi_{CD} \circ \Phi_{BC} = \Phi_{BD}$ on the intersection of their domains. Example 7.19 defines a canonical 2-isomorphism $\Lambda_{BCD} : \tilde\Phi_{CD} \circ \tilde\Phi_{BC} \Rightarrow \tilde\Phi_{BD}$ on $S_{BCD} := \operatorname{Im}\psi_B \cap \operatorname{Im}\psi_C \cap \operatorname{Im}\psi_D$.

If $B \neq C \neq D \in I$ with $B \not\subset C$ or $C \not\subset D$, define $S_{BCD} = \emptyset$ and $\Lambda_{BCD} = [\emptyset,\emptyset,\emptyset]$. Set $S_{BBC} = S_{BCC} = S_{BC}$ and $\Lambda_{BBC} = \boldsymbol{\beta}_{\tilde\Phi_{BC}}$, $\Lambda_{BCC} = \boldsymbol{\gamma}_{\tilde\Phi_{BC}}$ for all $B, C \in I$. This defines all the data in $\mathcal{F} = \bigl(I, (V_B, E_B, \Gamma_B, s_B, \psi_B)_{B\in I}, S_{BC}, \tilde\Phi_{BC,\ B,C\in I}, S_{BCD}, \Lambda_{BCD,\ B,C,D\in I}\bigr)$.

We will show $\mathcal{F}$ satisfies Definition 7.24(a)–(j),(k)$'$. Parts (a)–(i) are immediate. For (j), if $B \subsetneq C \subsetneq D \subsetneq E$ in I then Definition 7.18(b)–(d) basically imply that

$$\Phi_{DE} \circ (\Phi_{CD} \circ \Phi_{BC}) = \Phi_{BE} = (\Phi_{DE} \circ \Phi_{CD}) \circ \Phi_{BC}$$

holds on the intersection of their domains, and from this we easily see that $\Lambda_{BDE} \odot (\Lambda_{CDE} * \mathrm{id}_{\tilde\Phi_{BC}}) = \Lambda_{BDE} \odot (\mathrm{id}_{\tilde\Phi_{DE}} * \Lambda_{BCD}) \odot \boldsymbol{\alpha}_{\tilde\Phi_{DE},\tilde\Phi_{CD},\tilde\Phi_{BC}}$, as we want. The remaining cases follow as in Examples 7.28 and 7.30. Thus (j) holds.

For (k)$'$, suppose $\emptyset \neq J \subseteq I$ is finite and $x \in \bigcap_{B\in J} \operatorname{Im}\psi_B \subseteq X$. Then Definition 7.18(a) says that $D = \bigcup_{B\in J} B$ lies in I, and $x \in \bigcap_{B\in J} \operatorname{Im}\psi_B \subseteq \operatorname{Im}\psi_D$. For any $B \in J$ we have $B \subseteq D$, so $S_{BD} = \operatorname{Im}\psi_B \cap \operatorname{Im}\psi_D \ni x$. If $B, C \in J$ with $x \in S_{BC}$ then $B \subseteq C$, as otherwise $S_{BC} = \emptyset$, so $B \subseteq C \subseteq D$ and $S_{BCD} = \operatorname{Im}\psi_B \cap \operatorname{Im}\psi_C \cap \operatorname{Im}\psi_D \ni x$. Therefore (k)$'$ holds with $d = D$, and $\mathcal{F}$ is a fair coordinate system on X over $\dot{\mathbf{Man}} = \mathbf{Man}$.

Theorem 7.33. *Suppose X is a compact, metrizable topological space with an MW weak Kuranishi atlas $\mathcal{K}$, of virtual dimension $n \in \mathbb{Z}$, in the sense of Definition 7.18. Then we can make X into a Kuranishi space $\boldsymbol{X}' = (X, \mathcal{K}')$ over $\dot{\mathbf{Man}} = \mathbf{Man}$ in the sense of §6.2 with $\operatorname{vdim}\boldsymbol{X}' = n$, and $\boldsymbol{X}'$ is unique up to canonical equivalence in the 2-category $\mathbf{Kur}$. Commensurate MW weak Kuranishi atlases $\mathcal{K}, \tilde{\mathcal{K}}$ on $\boldsymbol{X}$ yield equivalent Kuranishi spaces $\boldsymbol{X}', \tilde{\boldsymbol{X}}'$.*

Proof. The first part is immediate from Example 7.32 and Theorem 7.26. For the second part, note that as in Definition 7.18, if $\mathcal{K}, \tilde{\mathcal{K}}$ are commensurate then they are linked by a diagram of MW weak Kuranishi atlases

$$\begin{tikzcd}[column sep=small]
\mathcal{K} = \mathcal{K}_0 \arrow[dr] & & \mathcal{K}_1 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \mathcal{K}_{m-1} \arrow[dl] \arrow[dr] & & \mathcal{K}_m = \tilde{\mathcal{K}} \arrow[dl] \\
& \hat{\mathcal{K}}_1 & & \hat{\mathcal{K}}_1 \quad \cdots & & \hat{\mathcal{K}}_{m-1} & & \hat{\mathcal{K}}_m, &
\end{tikzcd} \tag{7.13}$$

where each arrow is an inclusion of MW weak Kuranishi atlases.

By Proposition 7.27, the construction of the first part applied to MW weak Kuranishi atlases $\mathcal{K}, \hat{\mathcal{K}}$ with $\mathcal{K} \subseteq \hat{\mathcal{K}}$ yields equivalent Kuranishi spaces, so (7.13) induces a corresponding diagram of equivalences in $\mathbf{Kur}$, and thus $\boldsymbol{X}', \tilde{\boldsymbol{X}}'$ are equivalent in $\mathbf{Kur}$. $\square$

7.5.5 Dingyu Yang's Kuranishi structures, and polyfolds

Section 7.4 discussed Dingyu Yang's 'DY Kuranishi structures', working over $\dot{\mathbf{Man}} = \mathbf{Man}$. We relate these to our Kuranishi spaces.

Example 7.34. Using the notation of §7.4, let X be a compact, metrizable topological space, and $\mathcal{K}$ a DY Kuranishi structure on X with $\operatorname{vdim}(X, \mathcal{K}) = n$, in the sense of Definition 7.21. Then exactly the same construction as in Example 7.28 yields a fair coordinate system $\mathcal{F}$ on X.

Theorem 7.35. *Suppose X is a compact, metrizable topological space with a DY Kuranishi structure $\mathcal{K}$, of virtual dimension $n\in\mathbb{Z}$, in the sense of Definition 7.21. Then we can construct a Kuranishi space $\boldsymbol{X}'=(X,\mathcal{K}')$ over $\dot{\mathbf{Man}}=\mathbf{Man}$ in the sense of §6.2 with $\operatorname{vdim}\boldsymbol{X}'=n$, with the same topological space X, and $\boldsymbol{X}'$ is unique up to canonical equivalence in the 2-category $\mathbf{Kur}$. R-equivalent DY Kuranishi structures $\mathcal{K},\tilde{\mathcal{K}}$ on $\boldsymbol{X}$ yield equivalent Kuranishi spaces $\boldsymbol{X}',\tilde{\boldsymbol{X}}'$.*

Proof. The first part is immediate from Example 7.34 and Theorem 7.26. For the second part, note that as in Definition 7.22, if $\mathcal{K},\tilde{\mathcal{K}}$ are R-equivalent then there is a diagram of embeddings of DY Kuranishi structures on X:

$$\mathcal{K} \xleftarrow{\ \sim\ } \mathcal{K}_1 \Longrightarrow \mathcal{K}_2 \Longleftarrow \mathcal{K}_3 \xrightarrow{\ \sim\ } \tilde{\mathcal{K}}. \tag{7.14}$$

If $\epsilon:\mathcal{K}_1\to\mathcal{K}_2$ is an embedding of DY Kuranishi structures, then following Example 7.28 we can define three fair coordinate systems $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_{12}$ on X, where $\mathcal{F}_1,\mathcal{F}_2$ come from $\mathcal{K}_1,\mathcal{K}_2$, and $\mathcal{F}_{12}$ contains the Kuranishi neighbourhoods from $\mathcal{K}_1$ and $\mathcal{K}_2$, and the coordinate changes from $\mathcal{K}_1,\mathcal{K}_2$ and ϵ, so that $\mathcal{F}_{12}$ contains $\mathcal{F}_1$ and $\mathcal{F}_2$. Theorem 7.26 then gives Kuranishi structures $\mathcal{K}_1',\mathcal{K}_2',\mathcal{K}_{12}'$ on X. Since $\mathcal{F}_1\subset\mathcal{F}_{12}$, $\mathcal{F}_2\subset\mathcal{F}_{12}$, by Proposition 7.27 we have equivalences $(X,\mathcal{K}_1')\to(X,\mathcal{K}_{12}')$, $(X,\mathcal{K}_2')\to(X,\mathcal{K}_{12}')$ in $\mathbf{Kur}$, and hence an equivalence $(X,\mathcal{K}_1')\to(X,\mathcal{K}_2')$ in $\mathbf{Kur}$. Therefore (7.14) induces a corresponding diagram of equivalences in $\mathbf{Kur}$, and thus $\boldsymbol{X}',\tilde{\boldsymbol{X}}'$ are equivalent in $\mathbf{Kur}$. □

Combining Theorem 7.35 with Yang's Theorem 7.23, [110, Th. 3.1.7], we relate Hofer–Wysocki–Zehnder's polyfold theory [46–53] to our Kuranishi spaces:

Theorem 7.36. *Suppose we are given a 'polyfold Fredholm structure' $\mathcal{P}$ on a compact metrizable topological space X, that is, we write X as the zeroes of a Fredholm section $\mathfrak{s}:\mathfrak{V}\to\mathfrak{E}$ of a strong polyfold vector bundle $\mathfrak{E}\to\mathfrak{V}$ over a polyfold $\mathfrak{V}$, where $\mathfrak{s}$ has constant Fredholm index $n\in\mathbb{Z}$. Then we can make X into a Kuranishi space $\boldsymbol{X}=(X,\mathcal{K})$ in the sense of §6.2 with $\operatorname{vdim}\boldsymbol{X}=n$, and $\boldsymbol{X}$ is unique up to canonical equivalence in the 2-category $\mathbf{Kur}$.*

7.6 Proof of Theorem 7.26

In this section, as in §6.7.4 we will by an abuse of notation treat the weak 2-category $\dot{\mathbf{KN}}_S(X)$ defined in §6.1 as if it were a strict 2-category, omitting 2-morphisms $\boldsymbol{\alpha}_{\Phi_{kl},\Phi_{jk},\Phi_{ij}},\boldsymbol{\beta}_{\Phi_{ij}},\boldsymbol{\gamma}_{\Phi_{ij}}$ in (6.7) and (6.8), and omitting brackets in compositions of 1-morphisms $\Phi_{kl}\circ\Phi_{jk}\circ\Phi_{ij}$. We do this because otherwise diagrams such as (7.17), (7.23), (7.25), ... would become too big.

Let $\mathcal{F}=\bigl(A,(V_a,E_a,\Gamma_a,s_a,\psi_a)_{a\in A},\ S_{ab},\Phi_{ab,\ a,b\in A},\ S_{abc},\Lambda_{abc,\ a,b,c\in A}\bigr)$ be a fair coordinate system of virtual dimension $n\in\mathbb{Z}$ on a Hausdorff, second countable topological space X, as in §7.5. Then $\mathcal{F}$ satisfies either Definition 7.24(k) or (k)$'$. We will suppose $\mathcal{F}$ satisfies Definition 7.24(k), and give the proof in this case. The proof for (k)$'$ is very similar, but the order of composition of 1-morphisms is reversed, and the order of horizontal composition of 2-morphisms is reversed (though vertical composition stays the same), and the order of subscripts

$a, b, c, \ldots$ is reversed, so Φ_{ab}, Λ_{abc} are replaced by Φ_{ba}, Λ_{cba}, and so on. We leave the details for case (k)$'$ to the interested reader.

Throughout the proof, we will use the following notation for multiple intersections of the open sets S_{ab} in X. For $a_1, \ldots, a_k \in A$, $k \geqslant 3$, write

$$\acute{S}_{a_1 a_2 \cdots a_k} = \bigcap\nolimits_{1 \leqslant i < j \leqslant k} S_{a_i a_j}.$$

More generally, if we enclose a group of consecutive indices $a_l a_{l+1} \cdots a_m$ in brackets, as in $\acute{S}_{a_1 \cdots a_{l-1}(a_l \cdots a_m) a_{m+1} \cdots a_k}$, we omit from the intersection any $S_{a_i a_j}$ with both a_i, a_j belonging to the bracketed group. So, for example

$$\acute{S}_{a(bc)} = S_{ab} \cap S_{ac}, \qquad \acute{S}_{(ab)(cd)} = S_{ac} \cap S_{ad} \cap S_{bc} \cap S_{bd},$$
$$\acute{S}_{a(bc)(de)} = S_{ab} \cap S_{ac} \cap S_{ad} \cap S_{ae} \cap S_{bd} \cap S_{be} \cap S_{cd} \cap S_{ce}.$$

In Definition 7.24, the 2-morphisms Λ_{abc} are defined on open sets $S_{abc} \subseteq S_{ab} \cap S_{ac} \cap S_{bc} \subseteq \operatorname{Im} \psi_a \cap \operatorname{Im} \psi_b \cap \operatorname{Im} \psi_c$. We begin by showing that we can extend the Λ_{abc} canonically to $\acute{S}_{abc} = S_{ab} \cap S_{ac} \cap S_{bc}$.

Lemma 7.37. *There exist unique* 2*-morphisms* $\tilde{\Lambda}_{abc} : \Phi_{bc} \circ \Phi_{ab} \Rightarrow \Phi_{ac}$ *defined over* $\acute{S}_{abc}$ *for all* $a, b, c \in A$, *such that* $\tilde{\Lambda}_{abc}|_{S_{abc}} = \Lambda_{abc}$, *and as in Definition* 7.24(j) *we have* $\tilde{\Lambda}_{acd} \odot (\mathrm{id}_{\Phi_{cd}} * \tilde{\Lambda}_{abc}) = \tilde{\Lambda}_{abd} \odot (\tilde{\Lambda}_{bcd} * \mathrm{id}_{\Phi_{ab}}) : \Phi_{cd} \circ \Phi_{bc} \circ \Phi_{ab} \Rightarrow \Phi_{ad}$ *over* $\acute{S}_{abcd}$, *for all* $a, b, c, d \in A$.

Proof. Fix $a, b, c \in A$. We will construct a 2-morphism $\tilde{\Lambda}_{abc} : \Phi_{bc} \circ \Phi_{ab} \Rightarrow \Phi_{ac}$ over $\acute{S}_{abc}$. For each $d \in A$, define

$$\tilde{S}^d_{abc} = S_{dab} \cap S_{dac} \cap S_{dbc} \subseteq \acute{S}_{abc}. \tag{7.15}$$

Then $\tilde{S}^d_{abc}$ is open in $\acute{S}_{abc}$. Definition 7.24(k) with $B = \{a, b, c\}$ implies that for each $x \in \acute{S}_{abc}$, there exists $d \in A$ with $x \in \tilde{S}^d_{abc}$. Thus, $\{\tilde{S}^d_{abc} : d \in A\}$ is an open cover of $\acute{S}_{abc}$.

Since Φ_{da} is an equivalence in the weak 2-category $\dot{\mathbf{K}}\mathbf{N}_{\tilde{S}^d_{abc}}(X)$ in Definition 6.9, as it is a coordinate change, Lemma A.6 implies that for each $d \in A$ there is a unique 2-morphism

$$\begin{aligned} \tilde{\Lambda}^d_{abc} : \Phi_{bc} \circ \Phi_{ab} \Longrightarrow \Phi_{ac} \quad \text{over } \tilde{S}^d_{abc}, \text{ such that} \\ \tilde{\Lambda}^d_{abc} * \mathrm{id}_{\Phi_{da}} = \Lambda^{-1}_{dac} \odot \Lambda_{dbc} \odot (\mathrm{id}_{\Phi_{bc}} * \Lambda_{dab}). \end{aligned} \tag{7.16}$$

For $d, e \in A$, we will show that $\tilde{\Lambda}^d_{abc}|_{\tilde{S}^d_{abc} \cap \tilde{S}^e_{abc}} = \tilde{\Lambda}^e_{abc}|_{\tilde{S}^d_{abc} \cap \tilde{S}^e_{abc}}$. Let $x \in \tilde{S}^d_{abc} \cap \tilde{S}^e_{abc}$. Then Definition 7.24(k) with $B = \{a, b, c, d, e\}$ gives $f \in A$ with $x \in S_{fab} \cap S_{fac} \cap S_{fbc} \cap S_{fda} \cap S_{fdb} \cap S_{fdc} \cap S_{fea} \cap S_{feb} \cap S_{fec} \cap \tilde{S}^d_{abc} \cap \tilde{S}^e_{abc}$.

Consider the diagram of 2-morphisms on this intersection:

$$
\begin{tikzcd}[column sep=small]
 & & \Phi_{bc}\circ\Phi_{ab}\circ\Phi_{fa} \arrow[d, "\mathrm{id}_{\Phi_{bc}}*\Lambda_{fab}"] \arrow[ddddd, bend right=90, looseness=1.6, "\tilde\Lambda^d_{abc}*\mathrm{id}_{\Phi_{fa}}"', end anchor=west] \arrow[ddddd, bend left=90, looseness=1.6, "\tilde\Lambda^e_{abc}*\mathrm{id}_{\Phi_{fa}}", end anchor=east] & & \\
\Phi_{bc}\circ\Phi_{ab}\circ\Phi_{da}\circ\Phi_{fd} \arrow[urr, "\mathrm{id}_{\Phi_{bc}\circ\Phi_{ab}}*\Lambda_{fda}"] \arrow[dr, "\mathrm{id}_{\Phi_{bc}}*\Lambda_{dab}*\mathrm{id}_{\Phi_{fd}}"] \arrow[dddd, "\Lambda_{abc}*\mathrm{id}_{\Phi_{da}\circ\Phi_{fd}}"] & & \Phi_{bc}\circ\Phi_{fb} \arrow[ddd, "\Lambda_{fbc}"] & & \Phi_{bc}\circ\Phi_{ab}\circ\Phi_{ea}\circ\Phi_{fe} \arrow[ull, "\mathrm{id}_{\Phi_{bc}\circ\Phi_{ab}}*\Lambda_{fea}"'] \arrow[dl, "\mathrm{id}_{\Phi_{bc}}*\Lambda_{eab}*\mathrm{id}_{\Phi_{fe}}"'] \arrow[dddd, "\Lambda_{abc}*\mathrm{id}_{\Phi_{ea}\circ\Phi_{fe}}"'] \\
 & \Phi_{bc}\circ\Phi_{db}\circ\Phi_{fd} \arrow[ur, "\mathrm{id}_{\Phi_{bc}}*\Lambda_{fdb}"'] \arrow[d, "\Lambda_{dbc}*\mathrm{id}_{\Phi_{fd}}"] & & \Phi_{bc}\circ\Phi_{eb}\circ\Phi_{fe} \arrow[ul, "\mathrm{id}_{\Phi_{bc}}*\Lambda_{feb}"] \arrow[d, "\Lambda_{ebc}*\mathrm{id}_{\Phi_{fe}}"'] & \\
 & \Phi_{dc}\circ\Phi_{fd} \arrow[dr, "\Lambda_{fdc}"] & & \Phi_{ec}\circ\Phi_{fe} \arrow[dl, "\Lambda_{fec}"'] & \\
\Phi_{ac}\circ\Phi_{da}\circ\Phi_{fd} \arrow[ur, "\Lambda_{dac}*\mathrm{id}_{\Phi_{fd}}"'] \arrow[drr, "\mathrm{id}_{\Phi_{ac}}*\Lambda_{fda}"] & & \Phi_{fc} & & \Phi_{ac}\circ\Phi_{ea}\circ\Phi_{fe} \arrow[ul, "\Lambda_{eac}*\mathrm{id}_{\Phi_{fe}}"] \arrow[dll, "\mathrm{id}_{\Phi_{ac}}*\Lambda_{fea}"'] \\
 & & \Phi_{ac}\circ\Phi_{fa}. \arrow[u, "\Lambda_{fac}"'] & &
\end{tikzcd}
\tag{7.17}
$$

Here the outer two quadrilaterals commute by (7.16), and the inner eight quadrilaterals commute by Definition 7.24(j). So (7.17) commutes.

Thus, for each $x \in \tilde{S}^d_{abc} \cap \tilde{S}^e_{abc}$, on an open neighbourhood of x we have $\tilde\Lambda^d_{abc} * \mathrm{id}_{\Phi_{fa}} = \tilde\Lambda^e_{abc} * \mathrm{id}_{\Phi_{fa}}$, so that on an open neighbourhood of x we have $\tilde\Lambda^d_{abc} = \tilde\Lambda^e_{abc}$ by Lemma A.6. Definition A.17(iii) and Theorem 6.16 now imply that $\tilde\Lambda^d_{abc} = \tilde\Lambda^e_{abc}$ on $\tilde{S}^d_{abc} \cap \tilde{S}^e_{abc}$. Since the $\tilde{S}^d_{abc}$ for $d \in A$ cover $\acute{S}_{abc}$, Definition A.17(iii),(iv) and Theorem 6.16 show that there exists a unique 2-morphism $\tilde\Lambda_{abc} : \Phi_{bc} \circ \Phi_{ab} \Rightarrow \Phi_{ac}$ over $\acute{S}_{abc}$ such that

$$
\tilde\Lambda_{abc}|_{\tilde{S}^d_{abc}} = \tilde\Lambda^d_{abc} \quad \text{for all } d \in A. \tag{7.18}
$$

When $d = a$, we see from (7.15)–(7.16) and Definition 7.24(h),(i) that $\tilde{S}^a_{abc} = S_{abc}$ and $\tilde\Lambda^a_{abc} = \Lambda_{abc}$. Hence $\tilde\Lambda_{abc}|_{S_{abc}} = \Lambda_{abc}$, as we have to prove.

Suppose $a, b, c, d \in A$, and $x \in \acute{S}_{abcd} = S_{ab} \cap S_{ac} \cap S_{ad} \cap S_{bc} \cap S_{bd} \cap S_{cd}$. Definition 7.24(k) with $B = \{a, b, c, d\}$ gives $e \in A$ with $x \in \tilde{S}^e_{abc} \cap \tilde{S}^e_{abd} \cap \tilde{S}^e_{acd} \cap \tilde{S}^e_{bcd}$. So, in an open neighbourhood of x we have

$$
\begin{aligned}
&\bigl[\tilde\Lambda_{acd} \odot (\mathrm{id}_{\Phi_{cd}} * \tilde\Lambda_{abc})\bigr] * \mathrm{id}_{\Phi_{ea}} = (\tilde\Lambda^e_{acd} * \mathrm{id}_{\Phi_{ea}}) \odot (\mathrm{id}_{\Phi_{cd}} * \tilde\Lambda^e_{abc} * \mathrm{id}_{\Phi_{ea}}) \\
&= \bigl(\Lambda^{-1}_{ead} \odot \Lambda_{ecd} \odot (\mathrm{id}_{\Phi_{cd}} * \Lambda_{eac})\bigr) \\
&\qquad \odot \bigl((\mathrm{id}_{\Phi_{cd}} * \Lambda^{-1}_{eac}) \odot (\mathrm{id}_{\Phi_{cd}} * \Lambda_{ebc}) \odot (\mathrm{id}_{\Phi_{cd}} * \mathrm{id}_{\Phi_{bc}} * \Lambda_{eab})\bigr) \\
&= \Lambda^{-1}_{ead} \odot \Lambda_{ebd} \odot \bigl(\Lambda^{-1}_{ebd} \odot \Lambda_{ecd} \odot (\mathrm{id}_{\Phi_{cd}} * \Lambda_{ebc})\bigr) \odot (\mathrm{id}_{\Phi_{cd}\circ\Phi_{bc}} * \Lambda_{eab}) \\
&= \bigl(\Lambda^{-1}_{ead} \odot \Lambda_{ebd} \odot (\mathrm{id}_{\Phi_{bd}} * \Lambda_{eab})\bigr) \\
&\quad \odot \bigl((\mathrm{id}_{\Phi_{bd}} * \Lambda^{-1}_{eab}) \odot (\tilde\Lambda^e_{bcd} * \mathrm{id}_{\Phi_{eb}}) \odot (\mathrm{id}_{\Phi_{cd}\circ\Phi_{bc}} * \Lambda_{eab})\bigr) \\
&= (\tilde\Lambda^e_{abd} * \mathrm{id}_{\Phi_{ea}}) \odot (\tilde\Lambda^e_{bcd} * \mathrm{id}_{\Phi_{ab}} * \mathrm{id}_{\Phi_{ea}}) = \bigl[\tilde\Lambda_{abd} \odot (\tilde\Lambda_{bcd} * \mathrm{id}_{\Phi_{ab}})\bigr] * \mathrm{id}_{\Phi_{ea}},
\end{aligned}
$$

using (7.18) in the first, fourth and sixth steps, and (7.16) in the second, third and fifth. Lemma A.6 now implies that $\tilde\Lambda_{acd} \odot (\mathrm{id}_{\Phi_{cd}} * \tilde\Lambda_{abc}) = \tilde\Lambda_{abd} \odot (\tilde\Lambda_{bcd} * \mathrm{id}_{\Phi_{ab}})$ holds near x. Applying Definition A.17(iii) and Theorem 6.16 again shows it holds on the correct domain $\acute{S}_{abcd}$. This completes the lemma. $\square$

Next, for all $a, b \in A$ we have a coordinate change $\Phi_{ab} : (V_a, E_a, \Gamma_a, s_a, \psi_a) \to (V_b, E_b, \Gamma_b, s_b, \psi_b)$ over $S_{ab} \subseteq \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b$. This is an equivalence in the 2-category $\dot{\mathbf{K}}\mathbf{N}_{S_{ab}}(X)$ by Definition 6.11. Thus we may choose a quasi-inverse $\check{\Phi}_{ba} : (V_b, E_b, \Gamma_b, s_b, \psi_b) \to (V_a, E_a, \Gamma_a, s_a, \psi_a)$, which is also a coordinate change over S_{ab}, and 2-morphisms

$$\eta_{ab} : \Phi_{ab} \circ \check{\Phi}_{ba} \Rightarrow \mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)}, \quad \zeta_{ab} : \check{\Phi}_{ba} \circ \Phi_{ab} \Rightarrow \mathrm{id}_{(V_b,E_b,\Gamma_b,s_b,\psi_b)}. \tag{7.19}$$

When $a = b$, so that $\Phi_{aa} = \mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)}$, we choose

$$\check{\Phi}_{aa} = \mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)} \quad \text{and} \quad \eta_{aa} = \zeta_{aa} = \mathrm{id}_{\mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)}}. \tag{7.20}$$

Now fix $a, b \in A$. For all $c \in A$, we have $\acute{S}_{c(ab)} = S_{ca} \cap S_{cb} \subseteq \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b$. From Definition 7.24(k) with $B = \{a, b\}$, we see that for each $x \in \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b$ there exists $c \in A$ with $x \in \acute{S}_{c(ab)}$, so $\{\acute{S}_{c(ab)} : c \in A\}$ is an open cover of $\operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b$. For each $c \in A$, define a 1-morphism $\Psi^c_{ab} : (V_a, E_a, \Gamma_a, s_a, \psi_a) \to (V_b, E_b, \Gamma_b, s_b, \psi_b)$ over $\acute{S}_{c(ab)}$ by $\Psi^c_{ab} = \Phi_{cb} \circ \check{\Phi}_{ac}$.

Lemma 7.38. *For all $a, b, c, d \in A$, there is a unique 2-morphism*

$$\mathrm{M}^{cd}_{ab} : \Psi^c_{ab} \Longrightarrow \Psi^d_{ab} \quad \textit{over } \acute{S}_{(cd)(ab)} = \acute{S}_{c(ab)} \cap \acute{S}_{d(ab)}, \tag{7.21}$$

such that for all $e \in A$, the following commutes on $\acute{S}_{e(cd)(ab)}$:

$$\begin{tikzcd}
\Phi_{cb} \circ \Phi_{ec} \arrow[r, Rightarrow, "\tilde{\Lambda}_{ecb}"'] \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{cb}} * \zeta_{ca}^{-1} * \mathrm{id}_{\Phi_{ac}}"] & \Phi_{eb} \arrow[rr, Rightarrow, "\tilde{\Lambda}_{edb}^{-1}"'] & & \Phi_{db} \circ \Phi_{ed} \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{db}} * \zeta_{da}^{-1} * \mathrm{id}_{\Phi_{ad}}"'] \\
\Phi_{cb} \circ \check{\Phi}_{ac} \circ \Phi_{ca} \circ \Phi_{ec} \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{cb} \circ \check{\Phi}_{ac}} * \tilde{\Lambda}_{eca}"] & & & \Phi_{db} \circ \check{\Phi}_{ad} \circ \Phi_{da} \circ \Phi_{ed} \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{db} \circ \check{\Phi}_{ad}} * \tilde{\Lambda}_{eda}"'] \\
\Phi_{cb} \circ \check{\Phi}_{ac} \circ \Phi_{ea} \arrow[r, equal] & \Psi^c_{ab} \circ \Phi_{ea} \arrow[r, Rightarrow, "\mathrm{M}^{cd}_{ab} * \mathrm{id}_{\Phi_{ea}}"] & \Psi^d_{ab} \circ \Phi_{ea} \arrow[r, equal] & \Phi_{db} \circ \check{\Phi}_{ad} \circ \Phi_{ea}.
\end{tikzcd} \tag{7.22}$$

Proof. Equation (7.22) determines $\mathrm{M}^{cd}_{ab} * \mathrm{id}_{\Phi_{ea}}$ over $\acute{S}_{e(cd)(ab)}$, and so by Lemma A.6, determines M^{cd}_{ab} over $\acute{S}_{e(cd)(ab)}$, as Φ_{ea} is an equivalence. Write $(\mathrm{M}^{cd}_{ab})^e$ for the value for M^{cd}_{ab} on $\acute{S}_{e(cd)(ab)}$ determined by (7.22). Observe that Definition 7.24(k) with $B = \{a, b, c, d\}$ implies that the $\acute{S}_{e(cd)(ab)}$ for $e \in A$ form an open cover of $\acute{S}_{(cd)(ab)}$.

Let $e, f \in A$, and $x \in \acute{S}_{(ef)(cd)(ab)} = \acute{S}_{e(cd)(ab)} \cap \acute{S}_{f(cd)(ab)}$. Applying Definition 7.24(k) with $B = \{a, b, c, d, e, f\}$ and this x gives $g \in A$ such that all the 1-

and 2-morphisms in the following diagram are defined on $x \in \acute{S}_{g(ef)(cd)(ab)}$:

$$
\begin{tikzcd}[column sep=small]
& & \Psi^c_{ab}\circ\Phi_{ga} \arrow[dll, "\tilde\Lambda^{-1}_{gea}"'] \arrow[drr, "\tilde\Lambda^{-1}_{gfa}"] \arrow[dd, "\tilde\Lambda^{-1}_{gca}"] \arrow[ddddddddddddd, bend right=60, "(\mathrm{M}^{cd}_{ab})^e*\mathrm{id}_{\Phi_{ga}}"'] \arrow[ddddddddddddd, bend left=60, "(\mathrm{M}^{cd}_{ab})^f*\mathrm{id}_{\Phi_{ga}}"] & & \\
\Phi_{cb}\circ\check{\Phi}_{ac}\circ\Phi_{ea}\circ\Phi_{ge} \arrow[dd, "\tilde\Lambda^{-1}_{eca}"] & & & & \Phi_{cb}\circ\check{\Phi}_{ac}\circ\Phi_{fa}\circ\Phi_{gf} \arrow[dd, "\tilde\Lambda^{-1}_{fca}"'] \\
& & \Phi_{cb}\circ\check{\Phi}_{ac}\circ\Phi_{ca}\circ\Phi_{gc} \arrow[dll, "\tilde\Lambda^{-1}_{gec}"'] \arrow[drr, "\tilde\Lambda^{-1}_{gfc}"] \arrow[dd, "\zeta_{ca}"] & & \\
\Phi_{cb}\circ\check{\Phi}_{ac}\circ\Phi_{ca}\circ\Phi_{ec}\circ\Phi_{ge} \arrow[dd, "\zeta_{ca}"] & & & & \Phi_{cb}\circ\check{\Phi}_{ac}\circ\Phi_{ca}\circ\Phi_{fc}\circ\Phi_{gf} \arrow[dd, "\zeta_{ca}"'] \\
& & \Phi_{cb}\circ\Phi_{gc} \arrow[dll, "\tilde\Lambda^{-1}_{gec}"'] \arrow[drr, "\tilde\Lambda^{-1}_{gfc}"] \arrow[dd, "\tilde\Lambda_{gcb}"] & & \\
\Phi_{cb}\circ\Phi_{ec}\circ\Phi_{ge} \arrow[dd, "\tilde\Lambda_{ecb}"] & & & & \Phi_{cb}\circ\Phi_{fc}\circ\Phi_{gf} \arrow[dd, "\tilde\Lambda_{fcb}"'] \\
& & \Phi_{gb} \arrow[dll, "\tilde\Lambda^{-1}_{geb}"'] \arrow[drr, "\tilde\Lambda^{-1}_{gfb}"] \arrow[dd, "\tilde\Lambda^{-1}_{gdb}"] & & \\
\Phi_{eb}\circ\Phi_{ge} \arrow[dd, "\tilde\Lambda^{-1}_{edb}"] & & & & \Phi_{fb}\circ\Phi_{gf} \arrow[dd, "\tilde\Lambda^{-1}_{fdb}"'] \\
& & \Phi_{db}\circ\Phi_{gd} \arrow[dll, "\tilde\Lambda^{-1}_{ged}"'] \arrow[drr, "\tilde\Lambda^{-1}_{gfd}"] \arrow[dd, "\zeta^{-1}_{da}"] & & \\
\Phi_{db}\circ\Phi_{ed}\circ\Phi_{ge} \arrow[dd, "\zeta^{-1}_{da}"] & & & & \Phi_{db}\circ\Phi_{fd}\circ\Phi_{gf} \arrow[dd, "\zeta^{-1}_{da}"'] \\
& & \Phi_{db}\circ\check{\Phi}_{ad}\circ\Phi_{da}\circ\Phi_{gd} \arrow[dll, "\tilde\Lambda^{-1}_{ged}"'] \arrow[drr, "\tilde\Lambda^{-1}_{gfd}"] \arrow[ddd, "\tilde\Lambda_{gda}"] & & \\
\Phi_{db}\circ\check{\Phi}_{ad}\circ\Phi_{da}\circ\Phi_{ed}\circ\Phi_{ge} \arrow[d, "\tilde\Lambda_{eda}"] & & & & \Phi_{db}\circ\check{\Phi}_{ad}\circ\Phi_{da}\circ\Phi_{fd}\circ\Phi_{gf} \arrow[d, "\tilde\Lambda_{fda}"'] \\
\Phi_{db}\circ\check{\Phi}_{ad}\circ\Phi_{ea}\circ\Phi_{ge} \arrow[drr, "\tilde\Lambda_{gea}"] & & & & \Phi_{db}\circ\check{\Phi}_{ad}\circ\Phi_{fa}\circ\Phi_{gf} \arrow[dll, "\tilde\Lambda_{gfa}"'] \\
& & \Psi^d_{ab}\circ\Phi_{ga}. & &
\end{tikzcd}
\tag{7.23}
$$

Here for clarity we have omitted all 'id$_{\cdots}*$' and '$*$id$_{\cdots}$' terms. The two outer nine-gons commute by (7.22), eight small quadrilaterals commute by Lemma 7.37, and four small quadrilaterals commute by compatibility of horizontal and vertical composition. Thus (7.23) commutes, and $(\mathrm{M}^{cd}_{ab})^e * \mathrm{id}_{\Phi_{ga}} = (\mathrm{M}^{cd}_{ab})^f * \mathrm{id}_{\Phi_{ga}}$ near x, so $(\mathrm{M}^{cd}_{ab})^e = (\mathrm{M}^{cd}_{ab})^f$ near x by Lemma A.6.

As this holds for all $x \in \acute{S}_{e(cd)(ab)} \cap \acute{S}_{f(cd)(ab)}$, Definition A.17(iii) and Theorem 6.16 show that $(\mathrm{M}^{cd}_{ab})^e = (\mathrm{M}^{cd}_{ab})^f$ on $\acute{S}_{e(cd)(ab)} \cap \acute{S}_{f(cd)(ab)}$. Since the $\acute{S}_{e(cd)(ab)}$ for $e \in A$ cover $\acute{S}_{(cd)(ab)}$, Definition A.17(iii),(iv) and Theorem 6.16 imply that there is a unique 2-morphism M^{cd}_{ab} as in (7.21) with $\mathrm{M}^{cd}_{ab}|_{\acute{S}_{e(cd)(ab)}} = (\mathrm{M}^{cd}_{ab})^e$. But by definition of $(\mathrm{M}^{cd}_{ab})^e$ this holds if and only if (7.22) commutes. This completes the lemma. $\square$

Lemma 7.39. *For all $a, b, c, d, e \in A$, we have*

$$
\begin{aligned}
&\mathrm{M}^{de}_{ab} \odot \mathrm{M}^{cd}_{ab} = \mathrm{M}^{ce}_{ab} : \Psi^c_{ab} \Longrightarrow \Psi^e_{ab}\\
&\textit{over } \acute{S}_{(cde)(ab)} = \acute{S}_{(cd)(ab)} \cap \acute{S}_{(ce)(ab)} \cap \acute{S}_{(de)(ab)}.
\end{aligned}
\tag{7.24}
$$

Proof. Let $x \in \acute{S}_{(cde)(ab)}$. Definition 7.24(k) with $B = \{a, b, c, d, e\}$ and this x gives $f \in A$ such that all the 1- and 2-morphisms in the following diagram are defined on $x \in \acute{S}_{f(cde)(ab)}$:

$$
\begin{tikzcd}
& \Phi_{fb} \arrow[dl, Rightarrow, "\tilde{\Lambda}^{-1}_{fcb}"'] \arrow[d, Rightarrow, "\tilde{\Lambda}^{-1}_{fdb}"] \arrow[dr, Rightarrow, "\tilde{\Lambda}^{-1}_{feb}"] & \\
\Phi_{cb}\circ\Phi_{fc} \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{cb}}*\zeta_{ca}^{-1}*\mathrm{id}_{\Phi_{ac}}"] & \Phi_{db}\circ\Phi_{fd} \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{db}}*\zeta_{da}^{-1}*\mathrm{id}_{\Phi_{ad}}"] & \Phi_{eb}\circ\Phi_{fe} \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{eb}}*\zeta_{ea}^{-1}*\mathrm{id}_{\Phi_{ae}}"'] \\
\Phi_{cb}\circ\check{\Phi}_{ac}\circ\Phi_{ca}\circ\Phi_{fc} \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{cb}\circ\check{\Phi}_{ac}}*\tilde{\Lambda}_{fca}"] & \Phi_{db}\circ\check{\Phi}_{ad}\circ\Phi_{da}\circ\Phi_{fd} \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{db}\circ\check{\Phi}_{ad}}*\tilde{\Lambda}_{fda}"] & \Phi_{eb}\circ\check{\Phi}_{ae}\circ\Phi_{ea}\circ\Phi_{fe} \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{eb}\circ\check{\Phi}_{ae}}*\tilde{\Lambda}_{fea}"'] \\
\Phi_{cb}\circ\check{\Phi}_{ac}\circ\Phi_{fa} \arrow[r, Rightarrow, "\mathrm{M}^{cd}_{ab}*\mathrm{id}_{\Phi_{fa}}"] \arrow[rr, Rightarrow, bend right, "\mathrm{M}^{ce}_{ab}*\mathrm{id}_{\Phi_{fa}}"'] & \Phi_{db}\circ\check{\Phi}_{ad}\circ\Phi_{fa} \arrow[r, Rightarrow, "\mathrm{M}^{de}_{ab}*\mathrm{id}_{\Phi_{fa}}"] & \Phi_{eb}\circ\check{\Phi}_{ae}\circ\Phi_{fa}.
\end{tikzcd}
\tag{7.25}
$$

Here the two inner and the outer septagons commute by (7.22). Thus (7.25) commutes, and compatibility of horizontal and vertical composition gives

$$(\mathrm{M}^{de}_{ab}\odot\mathrm{M}^{cd}_{ab})*\mathrm{id}_{\Phi_{fa}} = (\mathrm{M}^{de}_{ab}*\mathrm{id}_{\Phi_{fa}})\odot(\mathrm{M}^{de}_{ab}*\mathrm{id}_{\Phi_{fa}}) = \mathrm{M}^{ce}_{ab}*\mathrm{id}_{\Phi_{fa}}$$

near x, so (7.24) holds near x by Lemma A.6. As this is true for all $x \in \acute{S}_{(cde)(ab)}$, the lemma follows from Definition A.17(iii) and Theorem 6.16. $\square$

By Lemmas 7.38 and 7.39, as $\{\acute{S}_{c(ab)} : c \in A\}$ is an open cover of $\mathrm{Im}\,\psi_a \cap \mathrm{Im}\,\psi_b$, we may now apply Definition A.17(v) and Theorem 6.16 to show that for all $a, b \in A$, there exists a coordinate change $\Psi_{ab} : (V_a, E_a, \Gamma_a, s_a, \psi_a) \to (V_b, E_b, \Gamma_b, s_b, \psi_b)$ over $\mathrm{Im}\,\psi_a \cap \mathrm{Im}\,\psi_b$, and 2-morphisms $\epsilon^c_{ab} : \Psi^c_{ab} \Rightarrow \Psi_{ab}$ over $\acute{S}_{c(ab)}$ for all $c \in A$, such that for all $c, d \in A$ we have

$$\epsilon^d_{ab}\odot\mathrm{M}^{cd}_{ab} = \epsilon^c_{ab} : \Psi^c_{ab} \Longrightarrow \Psi_{ab} \qquad \text{over } \acute{S}_{(cd)(ab)} = \acute{S}_{c(ab)} \cap \acute{S}_{d(ab)}. \tag{7.26}$$

Furthermore Ψ_{ab} is unique up to 2-isomorphism.

In the case when $a = b$, we have $\Psi^a_{aa} = \Phi_{aa} = \check{\Phi}_{aa} = \mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)}$ and $\acute{S}_{a(aa)} = \mathrm{Im}\,\psi_a$, so $\epsilon^a_{aa} : \mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)} \Rightarrow \Psi_{aa}$ is a 2-morphism over $\mathrm{Im}\,\psi_a$. As we can choose Ψ_{aa} freely in its 2-isomorphism class, we choose

$$\Psi_{aa} = \mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)} \text{ and } \epsilon^a_{aa} = \mathrm{id}_{\mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)}}, \text{ for all } a \in A. \tag{7.27}$$

Lemma 7.40. *For all $a, b, c \in A$, there is a unique 2-morphism*

$$\mathrm{K}_{abc} : \Psi_{bc}\circ\Psi_{ab} \Longrightarrow \Psi_{ac} \quad \textit{over } \mathrm{Im}\,\psi_a \cap \mathrm{Im}\,\psi_b \cap \mathrm{Im}\,\psi_c,$$

such that for all $d \in A$, the following commutes over $\acute{S}_{d(abc)}$:

$$
\begin{tikzcd}
\Phi_{dc}\circ\check{\Phi}_{bd}\circ\Phi_{db}\circ\check{\Phi}_{ad} \arrow[r, equal] \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{dc}}*\zeta_{db}*\mathrm{id}_{\check{\Phi}_{ad}}"] & \Psi^d_{bc}\circ\Psi^d_{ab} \arrow[r, Rightarrow, "\epsilon^d_{bc}*\epsilon^d_{ab}"'] & \Psi_{bc}\circ\Psi_{ab} \arrow[d, Rightarrow, "\mathrm{K}_{abc}"'] \\
\Phi_{dc}\circ\check{\Phi}_{ad} \arrow[r, equal] & \Psi^d_{ac} \arrow[r, Rightarrow, "\epsilon^d_{ac}"] & \Psi_{ac}.
\end{tikzcd}
\tag{7.28}
$$

Proof. Fix $a, b, c \in A$. If $x \in \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b \cap \operatorname{Im}\psi_c$, then Definition 7.24(k) with $B = \{a, b, c\}$ and this x gives $d \in A$ with $x \in \acute{S}_{d(abc)}$. Hence $\bigl\{\acute{S}_{d(abc)} : d \in A\bigr\}$ is an open cover of $\operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b \cap \operatorname{Im}\psi_c$.

For each $d \in A$, write K^d_{abc} for the 2-morphism over $\acute{S}_{d(abc)}$ determined by (7.28) with K^d_{abc} in place of K_{abc}. We have to show that there is a unique 2-morphism K_{abc} over $\operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b \cap \operatorname{Im}\psi_c$ with $\mathrm{K}_{abc}|_{\acute{S}_{d(abc)}} = \mathrm{K}^d_{abc}$.

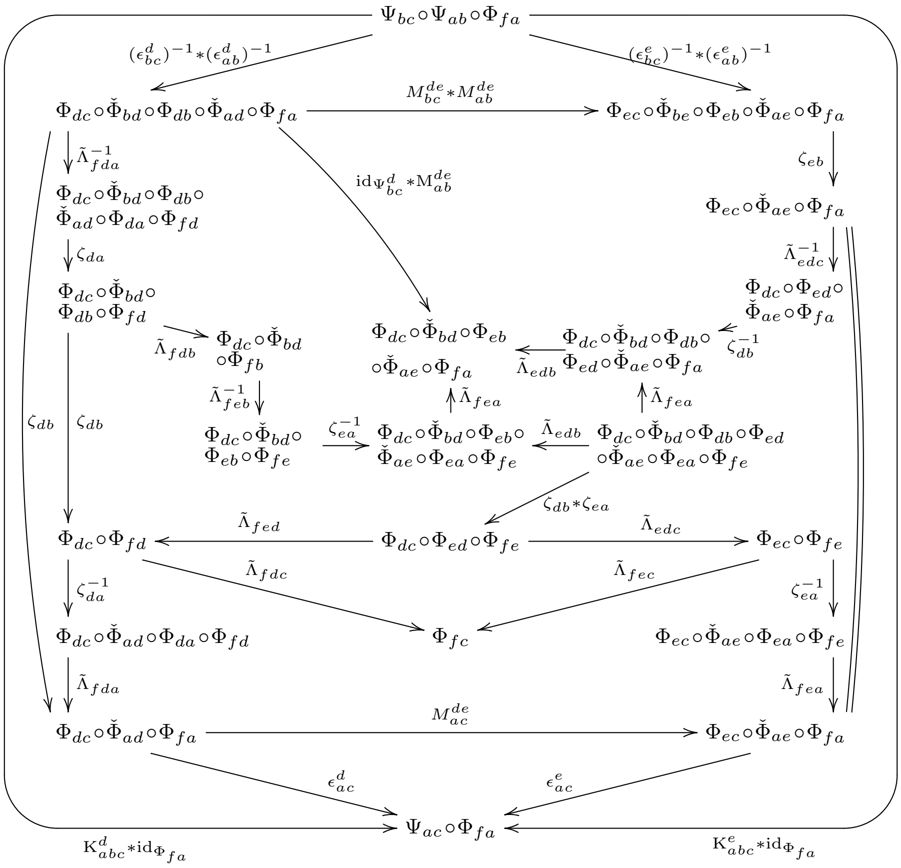


Figure 7.1: Proof that $\mathrm{K}^d_{abc} * \mathrm{id}_{\Phi_{fa}} = \mathrm{K}^e_{abc} * \mathrm{id}_{\Phi_{fa}}$

Let $d, e \in A$, and $x \in \acute{S}_{(de)(abc)} = \acute{S}_{d(abc)} \cap \acute{S}_{e(abc)}$. Definition 7.24(k) with $B = \{a, b, c, d, e\}$ and this x gives $f \in A$ with $x \in \acute{S}_{f(de)(abc)}$. Consider the diagram of 1- and 2-morphisms Figure 7.1. We have omitted most terms $*\mathrm{id}_{\cdots}$ and $\mathrm{id}_{\cdots}*$ in the 2-morphisms for clarity. The two outer crescent shapes are the definitions of $\mathrm{K}^d_{abc}, \mathrm{K}^e_{abc}$ in (7.28), composed with Φ_{fa}. The top and bottom triangles commute by (7.26). In the interior of the figure, the three polygons with sides involving $M^{de}_{ab}, M^{de}_{ac}, M^{de}_{bc}$ commute by (7.22). The remaining four polygons commute by Lemma 7.37 and compatibility of horizontal and vertical composition.

Thus Figure 7.1 commutes, which proves that $\mathrm{K}_{abc}^d * \mathrm{id}_{\Phi_{fa}} = \mathrm{K}_{abc}^e * \mathrm{id}_{\Phi_{fa}}$ on $\acute{S}_{f(de)(abc)}$. Lemma A.6 now shows that $\mathrm{K}_{abc}^d = \mathrm{K}_{abc}^e$ on $\acute{S}_{f(de)(abc)}$.

As the $\acute{S}_{f(de)(abc)}$ for $f \in A$ cover $\acute{S}_{d(abc)} \cap \acute{S}_{e(abc)}$, Definition A.17(iii) and Theorem 6.16 imply that $\mathrm{K}_{abc}^d = \mathrm{K}_{abc}^e$ on $\acute{S}_{d(abc)} \cap \acute{S}_{e(abc)}$. Since $\{\acute{S}_{d(abc)} : d \in A\}$ is an open cover of $\mathrm{Im}\,\psi_a \cap \mathrm{Im}\,\psi_b \cap \mathrm{Im}\,\psi_c$, Definition A.17(iii),(iv) and Theorem 6.16 show that there exists a unique 2-morphism K_{abc} over $\mathrm{Im}\,\psi_a \cap \mathrm{Im}\,\psi_b \cap \mathrm{Im}\,\psi_c$ such that $\mathrm{K}_{abc}|_{\acute{S}_{d(abc)}} = \mathrm{K}_{abc}^d$. Thus (7.28) commutes for all $d \in A$, by definition of K_{abc}^d. This completes the proof. $\square$

Putting a, a, b, a in place of a, b, c, d in (7.28) and using $\epsilon_{aa}^a, \zeta_{aa}$ identities by (7.20), (7.27), and similarly putting a, b, b, b in place of a, b, c, d and using $\epsilon_{bb}^b, \zeta_{bb}$ identities, yields

$$\mathrm{K}_{aab} = \mathrm{K}_{abb} = \mathrm{id}_{\Psi_{ab}}. \tag{7.29}$$

Lemma 7.41. *For all $a, b, c, d \in A$ we have $\mathrm{K}_{acd} \odot (\mathrm{id}_{\Psi_{cd}} * \mathrm{K}_{abc}) = \mathrm{K}_{abd} \odot (\mathrm{K}_{bcd} * \mathrm{id}_{\Psi_{ab}}) : \Psi_{cd} \circ \Psi_{bc} \circ \Psi_{ab} \Longrightarrow \Psi_{ad}$ over $\mathrm{Im}\,\psi_a \cap \mathrm{Im}\,\psi_b \cap \mathrm{Im}\,\psi_c \cap \mathrm{Im}\,\psi_d$.*

Proof. Let $x \in \mathrm{Im}\,\psi_a \cap \mathrm{Im}\,\psi_b \cap \mathrm{Im}\,\psi_c \cap \mathrm{Im}\,\psi_d$. Definition 7.24(k) with $B = \{a, b, c, d\}$ and this x gives $e \in A$ with $x \in \acute{S}_{e(abcd)}$. Consider the diagram

$$\begin{tikzcd}
\Psi_{cd} \circ \Psi_{bc} \circ \Psi_{ab} \arrow[rrr, Rightarrow, "\mathrm{id}_{\Psi_{cd}} * \mathrm{K}_{abc}"'] \arrow[dr, Rightarrow, "(\epsilon_{cd}^e * \epsilon_{bc}^e * \epsilon_{ab}^e)^{-1}"] \arrow[ddd, Rightarrow, "\mathrm{K}_{bcd} * \mathrm{id}_{\Psi_{ab}}"] & & & \Psi_{cd} \circ \Psi_{ac} \arrow[dl, Rightarrow, shift left, "(\epsilon_{cd}^e * \epsilon_{ac}^e)^{-1}"] \arrow[ddd, Rightarrow, "\mathrm{K}_{acd}"'] \\
& \begin{array}{l}\Phi_{ed} \circ \check{\Phi}_{ce} \circ \Phi_{ec} \circ \\ \check{\Phi}_{be} \circ \Phi_{eb} \circ \check{\Phi}_{ae}\end{array} \arrow[r, Rightarrow, "\begin{array}{l}\mathrm{id}_{\Phi_{ed} \circ \check{\Phi}_{ce} \circ \Phi_{ec}} * \\ \zeta_{eb} * \mathrm{id}_{\check{\Phi}_{ae}}\end{array}"] \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{ed}} * \zeta_{ec} * \mathrm{id}_{\check{\Phi}_{be} \circ \Phi_{eb} \circ \check{\Phi}_{ae}}"] & \begin{array}{l}\Phi_{ed} \circ \check{\Phi}_{ce} \circ \\ \Phi_{ec} \circ \check{\Phi}_{ae}\end{array} \arrow[ur, Rightarrow, shift left, "\epsilon_{cd}^e * \epsilon_{ac}^e"] \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{ed}} * \zeta_{ec} * \mathrm{id}_{\check{\Phi}_{ae}}"'] & \\
& \begin{array}{l}\Phi_{ed} \circ \check{\Phi}_{be} \circ \\ \Phi_{eb} \circ \check{\Phi}_{ae}\end{array} \arrow[r, Rightarrow, "\mathrm{id}_{\Phi_{ed}} * \zeta_{eb} * \mathrm{id}_{\check{\Phi}_{ae}}"'] \arrow[dl, Rightarrow, shift right, "\epsilon_{bd}^e * \epsilon_{ab}^e"'] & \Phi_{ed} \circ \check{\Phi}_{ae} \arrow[dr, Rightarrow, "\epsilon_{ad}^e"] & \\
\Psi_{bd} \circ \Psi_{ab} \arrow[ur, Rightarrow, shift right, "(\epsilon_{bd}^e * \epsilon_{ab}^e)^{-1}"'] \arrow[rrr, Rightarrow, "\mathrm{K}_{abd}"] & & & \Psi_{ad}.
\end{tikzcd} \tag{7.30}$$

Here the four outer quadrilaterals commute by (7.28), and the inner rectangle commutes by compatibility of horizontal and vertical multiplication. Thus (7.30) commutes, and the outer rectangle shows that $\mathrm{K}_{acd} \odot (\mathrm{id}_{\Psi_{cd}} * \mathrm{K}_{abc}) = \mathrm{K}_{abd} \odot (\mathrm{K}_{bcd} * \mathrm{id}_{\Psi_{ab}})$ holds over $\acute{S}_{e(abcd)}$. Since the $\acute{S}_{e(abcd)}$ for all $e \in A$ cover $\mathrm{Im}\,\psi_a \cap \mathrm{Im}\,\psi_b \cap \mathrm{Im}\,\psi_c \cap \mathrm{Im}\,\psi_d$, the lemma follows from Definition A.17(iii) and Theorem 6.16. $\square$

The definition of the Ψ_{ab} after Lemma 7.39, Lemmas 7.40–7.41, and equations (7.27) and (7.29), now imply that $\mathcal{K} = \big(A, (V_a, E_a, \Gamma_a, s_a, \psi_a)_{a \in A},\ \Psi_{ab,\ a,b \in A},\ \mathrm{K}_{abc,\ a,b,c \in A}\big)$ is a Kuranishi structure on X in the sense of §6.2, so $\boldsymbol{X} = (X, \mathcal{K})$ is a Kuranishi space with $\mathrm{vdim}\,\boldsymbol{X} = n$, as we have to prove.

To give $(V_a, E_a, \Gamma_a, s_a, \psi_a)$ the structure of a Kuranishi neighbourhood on the Kuranishi space $\boldsymbol{X}$ in the sense of §6.4 for $a \in A$, note that as $(V_a, E_a, \Gamma_a, s_a, \psi_a)$ is already part of the Kuranishi structure $\mathcal{K}$, we can take $\Psi_{ai,\, i\in A}$ and $\mathrm{K}_{aij,\, i,j\in A}$ to be the implicit extra data $\Phi_{ai,\, i\in I}$, $\Lambda_{aij,\, i,j\in I}$ in Definition 6.42.

To give $\Phi_{ab} : (V_a, E_a, \Gamma_a, s_a, \psi_a) \to (V_b, E_b, \Gamma_b, s_b, \psi_b)$ the structure of a coordinate change over S_{ab} on the Kuranishi space $\boldsymbol{X}$ as in §6.4 for $a, b \in A$, we need to specify implicit extra data $\mathrm{I}_{abi,\, i\in A}$ in place of $\Lambda_{abi,\, i\in A}$ in Definition 6.43, where $\mathrm{I}_{abi} : \Psi_{bi} \circ \Phi_{ab} \Rightarrow \Psi_{ai}$ is a 2-morphism over $S_{ab} \cap \operatorname{Im} \psi_i$ for all $i \in A$ satisfying (6.38) over $S_{ab} \cap \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ for all $i, j \in A$, which becomes

$$\mathrm{K}_{aij} \odot (\mathrm{id}_{\Psi_{ij}} * \mathrm{I}_{abi}) = \mathrm{I}_{abj} \odot (\mathrm{K}_{bij} * \mathrm{id}_{\Phi_{ab}}) : \Psi_{ij} \circ \Psi_{bi} \circ \Phi_{ab} \Longrightarrow \Psi_{aj}. \tag{7.31}$$

Since $\check\Phi_{aa} = \mathrm{id}_{(V_a,E_a,\Gamma_a,s_a,\psi_a)}$ by (7.20) we have $\Psi^a_{ab} = \Phi_{ab}$, so the definition of Ψ_{ab} gives a 2-morphism $\epsilon^a_{ab} : \Phi_{ab} \Rightarrow \Psi_{ab}$ over $S_{ab} \subseteq \operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b$. Define $\mathrm{I}_{abi} = \mathrm{K}_{abi} \odot (\mathrm{id}_{\Psi_{bi}} * (\epsilon^a_{ab})^{-1})$. Then (7.31) follows from vertically composing $\mathrm{id}_{\Psi_{ij}\circ\Psi_{bi}} * (\epsilon^a_{ab})^{-1}$ with Lemma 7.41 with i, j in place of c, d. This makes Φ_{ab} into a coordinate change over S_{ab} on $\boldsymbol{X}$, as we want.

Now let $a, b, c \in A$. To show that $\Lambda_{abc} : \Phi_{bc} \circ \Phi_{ab} \Rightarrow \Phi_{ac}$ is the unique 2-morphism over S_{abc} given by Theorem 6.45(a), we must prove that as in (6.39), for all $i \in A$, over $S_{abc} \cap \operatorname{Im}\psi_i$ we have

$$\mathrm{I}_{abi} \odot (\mathrm{I}_{bci} * \mathrm{id}_{\Phi_{ab}}) = \mathrm{I}_{aci} \odot (\mathrm{id}_{\Psi_{ci}} * \Lambda_{abc}) : \Psi_{ci} \circ \Phi_{bc} \circ \Phi_{ab} \Longrightarrow \Psi_{ai}. \tag{7.32}$$

To prove (7.32), consider the diagram of 2-morphisms over $S_{abc} \cap \operatorname{Im}\psi_i$:

$$\begin{tikzcd}
\Psi_{ci}\circ\Phi_{bc}\circ\Phi_{ab} \arrow[rrrr, Rightarrow, "\mathrm{id}_{\Psi_{ci}}*\Lambda_{abc}=\mathrm{id}_{\Psi_{ci}}*\tilde\Lambda_{abc}"'] \arrow[drr, Rightarrow, "\mathrm{id}_{\Psi_{ci}}*M^{ba}_{bc}*\mathrm{id}_{\Phi_{ab}}"'] \arrow[dddr, Rightarrow, "\mathrm{id}_{\Psi_{ci}}*\epsilon^b_{bc}*\epsilon^a_{ab}"] \arrow[ddddd, Rightarrow, "\mathrm{I}_{bci}*\mathrm{id}_{\Phi_{ab}}"] & & & & \Psi_{ci}\circ\Phi_{ac} \arrow[ddl, Rightarrow, "\mathrm{id}_{\Psi_{ci}}*\epsilon^a_{ac}"'] \arrow[ddddd, Rightarrow, "\mathrm{I}_{aci}"'] \\
 & & \Psi_{ci}\circ\Phi_{ac}\circ\check\Phi_{ba}\circ\Phi_{ab} \arrow[urr, Rightarrow, "\mathrm{id}_{\Psi_{ci}\circ\Phi_{ac}}*\zeta_{ab}"'] \arrow[ddl, Rightarrow, "\mathrm{id}_{\Psi_{ci}}*\epsilon^a_{bc}*\epsilon^a_{ab}"] & & \\
 & & & \Psi_{ci}\circ\Psi_{ac} \arrow[dddr, Rightarrow, "\mathrm{K}_{aci}"'] & \\
 & \Psi_{ci}\circ\Psi_{bc}\circ\Psi_{ab} \arrow[urr, Rightarrow, "\mathrm{id}_{\Psi_{ci}}*\mathrm{K}_{abc}"'] \arrow[d, Rightarrow, "\mathrm{K}_{bci}*\mathrm{id}_{\Psi_{ab}}"] & & & \\
 & \Psi_{bi}\circ\Psi_{ab} \arrow[drrr, Rightarrow, "\mathrm{K}_{abi}"] & & & \\
\Psi_{bi}\circ\Phi_{ab} \arrow[ur, Rightarrow, "\mathrm{id}_{\Psi_{bi}}*\epsilon^a_{ab}"] \arrow[rrrr, Rightarrow, "\mathrm{I}_{abi}"] & & & & \Psi_{ai}.
\end{tikzcd} \tag{7.33}$$

Here the bottom and rightmost triangles, and the leftmost quadrilateral, commute by definition of I_{abi}. The lower central quadrilateral commutes by Lemma 7.41, the upper central quadrilateral by (7.28) with $d = a$, the upper left triangle by (7.26), and the topmost triangle by (7.22) with b, c, b, a, a in place of a, b, c, d, e, noting that of the seven morphisms in (7.22), four are identities in this case, so we omit them. Also we use $\tilde\Lambda_{abc}|_{S_{abc}} = \Lambda_{abc}$ from Lemma 7.37. Thus (7.33) commutes, and the outer rectangle yields (7.32). Hence $\Lambda_{abc} : \Phi_{bc} \circ \Phi_{ab} \Rightarrow \Phi_{ac}$

is the unique 2-morphism over S_{abc} given by Theorem 6.45(a). This completes the proof of the first part of Theorem 7.26.

It remains to show that $\boldsymbol{X} = (X, \mathcal{K})$ is unique up to equivalence in $\dot{\mathbf{K}}\mathbf{ur}$. To prove this, we have to consider where in the proof above we made arbitrary choices, and show that if we made different choices yielding $\boldsymbol{X}' = (X, \mathcal{K}')$, then $\boldsymbol{X}$ and $\boldsymbol{X}'$ are equivalent in $\dot{\mathbf{K}}\mathbf{ur}$. There are two places in the construction of $\boldsymbol{X}$ where we made arbitrary choices: firstly the choice after Lemma 7.37 of a quasi-inverse $\check{\Phi}_{ba}$ for Φ_{ab} and 2-morphisms η_{ab}, ζ_{ab} in (7.19) (though in fact the η_{ab} were never used in the definition of $\boldsymbol{X}$), and secondly the choice after Lemma 7.39 of Ψ_{ab} and 2-morphisms ϵ_{ab}^c satisfying (7.26).

For the first, if $\check{\Phi}'_{ba}, \eta'_{ab}, \zeta'_{ab}$ are alternative choices for $\check{\Phi}_{ba}, \eta_{ab}, \zeta_{ab}$, for all $a, b \in A$, then there exist unique 2-morphisms $\alpha_{ab} : \check{\Phi}_{ba} \Rightarrow \check{\Phi}'_{ba}$ such that

$$\zeta_{ab} = \zeta'_{ab} \odot (\alpha_{ab} * \mathrm{id}_{\Phi_{ab}}) \quad \text{for all } a, b \in A, \tag{7.34}$$

and $\alpha_{aa} = \mathrm{id}_{\mathrm{id}_{(V_a, E_a, \Gamma_a, s_a, \psi_a)}}$. Then one can check that for the second choice we can keep Ψ_{ab} unchanged and replace ϵ_{ab}^c by

$$\epsilon'^c_{ab} = \epsilon_{ab}^c \odot (\mathrm{id}_{\Phi_{cb}} * (\alpha_{ac})^{-1}) \quad \text{for all } a, b, c \in A. \tag{7.35}$$

Using (7.34)–(7.35) to compare (7.28) for $\check{\Phi}_{ba}, \eta_{ab}, \zeta_{ab}, \epsilon_{ab}^c$ and $\check{\Phi}'_{ba}, \eta'_{ab}, \zeta'_{ab}, \epsilon'^c_{ab}$, we find that the two occurrences of α_{da} and of α_{db} cancel, so K_{abc} is unchanged. Thus, the family of possible outcomes for $\Psi_{ab}, \mathrm{K}_{abc}$ and $\boldsymbol{X}$ are independent of the first choice of $\check{\Phi}_{ba}, \eta_{ab}, \zeta_{ab}$ for $a, b \in A$.

Next, regard the $\check{\Phi}_{ba}, \eta_{ab}, \zeta_{ab}$ as fixed, and let $\Psi'_{ab}, \epsilon'^c_{ab}$ be alternative possibilities for $\Psi_{ab}, \epsilon_{ab}^c$ in the second choice, and K'_{abc} the corresponding 2-morphisms in Lemma 7.40. Then by Theorem 6.16 and the last part of Definition A.17(v), there are unique 2-morphisms $\beta_{ab} : \Psi_{ab} \Rightarrow \Psi'_{ab}$ for all $a, b \in A$, such that

$$\epsilon'^c_{ab} = \beta_{ab} \odot \epsilon_{ab}^c \quad \text{for all } a, b, c \in A. \tag{7.36}$$

Substituting (7.36) into (7.28) for $\Psi'_{ab}, \epsilon'^c_{ab}, \mathrm{K}'_{abc}$ and comparing with (7.28) for $\Psi_{ab}, \epsilon_{ab}^c, \mathrm{K}_{abc}$, we see that

$$\mathrm{K}'_{abc} = \beta_{ac} \odot \mathrm{K}_{abc} \odot (\beta_{bc}^{-1} * \beta_{ab}^{-1}).$$

Define 1-morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{X}'$, $\boldsymbol{g} : \boldsymbol{X}' \to \boldsymbol{X}$, in the notation of (6.18), by

$$\begin{aligned}
\boldsymbol{f} &= \bigl(\mathrm{id}_X, \Psi_{ab,\, a \in A,\, b \in A},\ (\mathrm{K}_{aa'b})^{b,\, b \in A}_{aa',\, a,a' \in A},\ (\mathrm{K}_{abb'} \odot (\beta_{bb'}^{-1} * \mathrm{id}_{\Psi_{ab}}))^{bb',\, b,b' \in A}_{a,\, a \in A}\bigr),\\
\boldsymbol{g} &= \bigl(\mathrm{id}_X, \Psi'_{ab,\, a \in A,\, b \in A},\ (\mathrm{K}'_{aa'b})^{b,\, b \in A}_{aa',\, a,a' \in A},\ (\mathrm{K}'_{abb'} \odot (\beta_{bb'} * \mathrm{id}_{\Psi'_{ab}}))^{bb',\, b,b' \in A}_{a,\, a \in A}\bigr).
\end{aligned}$$

One can check these satisfy Definition 6.19(a)–(h), and so are 1-morphisms of Kuranishi spaces. Definition 6.22 now gives a 1-morphism of Kuranishi spaces $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{X}$, and 2-morphisms of Kuranishi neighbourhoods for all $a, b, c \in A$

$$\Theta_{abc}^{\boldsymbol{g},\boldsymbol{f}} : \Psi'_{bc} \circ \Psi_{ab} \Longrightarrow (\boldsymbol{g} \circ \boldsymbol{f})_{ac}.$$

We claim that there is a unique 2-morphism $\boldsymbol{\varrho} = (\boldsymbol{\varrho}_{ac,\ a,c\in A}) : \boldsymbol{g}\circ\boldsymbol{f} \Rightarrow \mathbf{id}_{\boldsymbol{X}}$ of Kuranishi spaces such that for all $a,b,c\in A$ the following diagram of 2-morphisms of Kuranishi neighbourhoods over $\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b\cap\operatorname{Im}\psi_c$ commutes:

$$\begin{tikzcd}[column sep=huge]
\Psi'_{bc}\circ\Psi_{ab} \arrow[r, Rightarrow, "(\beta'_{bc})^{-1}*\mathrm{id}_{\Psi_{ab}}"'] \arrow[d, Rightarrow, "\Theta^{\boldsymbol{g},\boldsymbol{f}}_{abc}"] & \Psi_{bc}\circ\Psi_{ab} \arrow[d, Rightarrow, "\mathrm{K}_{abc}"'] \\
(\boldsymbol{g}\circ\boldsymbol{f})_{ac} \arrow[r, Rightarrow, "\boldsymbol{\varrho}_{ac}|_{\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b\cap\operatorname{Im}\psi_c}"] & \Psi_{ac}=(\mathbf{id}_{\boldsymbol{X}})_{ac}.
\end{tikzcd} \tag{7.37}$$

To prove this, note that (7.37) determines $\boldsymbol{\varrho}_{ac}$ on the open subset $\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b\cap\operatorname{Im}\psi_c\subseteq\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_c$. Using (6.24)–(6.26) for the $\Theta^{\boldsymbol{g},\boldsymbol{f}}_{abc}$ and Lemma 7.41 for the K_{abc}, we prove that these prescribed values for $\boldsymbol{\varrho}_{ac}$ agree on overlaps between $\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b\cap\operatorname{Im}\psi_c$ and $\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_{b'}\cap\operatorname{Im}\psi_c$, for all $b,b'\in A$. Thus, as the $\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b\cap\operatorname{Im}\psi_c$ for all $b\in A$ form an open cover of the correct domain $\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_c$ for $a,c\in A$, Theorem 6.16 and Definition A.17(iii),(iv) imply that there is a unique 2-morphism $\boldsymbol{\varrho}_{ac} : (\boldsymbol{g}\circ\boldsymbol{f})_{ac}\Rightarrow(\mathbf{id}_{\boldsymbol{X}})_{ac}$ such that (7.37) commutes for all $b\in A$.

We can then check that $\boldsymbol{\varrho} = (\boldsymbol{\varrho}_{ac,\ a,c\in A})$ satisfies Definition 6.20(a),(b), by proving that they hold on the restriction of their domains with $\operatorname{Im}\psi_b$ for each $b\in A$ using (7.37), (6.24)–(6.26) for the $\Theta^{\boldsymbol{g},\boldsymbol{f}}_{abc}$ and Lemma 7.41 for the K_{abc}, and then using Theorem 6.16 and Definition A.17(iii) to deduce that Definition 6.20(a),(b) hold on the correct domains. Therefore $\boldsymbol{\varrho} : \boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$ is a 2-morphism of Kuranishi spaces. Similarly, exchanging $\boldsymbol{X},\boldsymbol{X}'$ we construct a 2-morphism $\boldsymbol{\sigma} : \boldsymbol{f}\circ\boldsymbol{g}\Rightarrow\mathbf{id}_{\boldsymbol{X}'}$. Hence $\boldsymbol{f} : \boldsymbol{X}\to\boldsymbol{X}'$ is an equivalence, and $\boldsymbol{X},\boldsymbol{X}'$ are equivalent in the 2-category $\dot{\mathbf{K}}\mathbf{ur}$. This completes the proof of Theorem 7.26.

Chapter 8

(M-)Kuranishi spaces as stacks

Appendix A

Categories and 2-categories

We recall background material on categories, 2-categories, and sheaves and stacks on topological spaces. Some references are MacLane [75] for §A.1, and Borceux [6, §7], Kelly and Street [67], and Behrend et al. [3, App. B] for §A.2–§A.4, and Bredon [10], Godement [40], and Hartshorne [43, §II.1] for §A.5.

A.1 Basics of category theory

Here are the basic definitions in category theory, as in MacLane [75, §I].

Definition A.1. A *category* $\mathcal{C}$ consists of a class of *objects* $\mathrm{Obj}(\mathcal{C})$, and for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a set $\mathrm{Hom}(X, Y)$ of *morphisms* f from X to Y, written $f : X \to Y$, and for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a *composition map* $\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, written $(f, g) \mapsto g \circ f$. Composition must be *associative*, that is, if $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$ are morphisms in $\mathcal{C}$ then $(h \circ g) \circ f = h \circ (g \circ f)$. For each $X \in \mathrm{Obj}(\mathcal{C})$ there must exist an *identity morphism* $\mathrm{id}_X : X \to X$ such that $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for all $f : X \to Y$ in $\mathcal{C}$.

A morphism $f : X \to Y$ is an *isomorphism* if there exists $f^{-1} : Y \to X$ with $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. A category $\mathcal{C}$ is called a *groupoid* if every morphism is an isomorphism. In a groupoid $\mathcal{C}$, for each $X \in \mathrm{Obj}(\mathcal{C})$ the set $\mathrm{Hom}(X, X)$ of morphisms $f : X \to X$ form a group.

A category $\mathcal{C}$ is *small* if $\mathrm{Obj}(\mathcal{C})$ is a set, rather than a proper class. It is *essentially small* if the isomorphism classes $\mathrm{Obj}(\mathcal{C})/\cong$ of objects in $\mathcal{C}$ form a set, rather than a proper class.

If $\mathcal{C}$ is a category, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ is $\mathcal{C}$ with the directions of all morphisms reversed. That is, we define $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathcal{C})$, and for all $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$ we define $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, and for $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$ we define $f \circ_{\mathcal{C}^{\mathrm{op}}} g = g \circ_{\mathcal{C}} f$, and $\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} X = \mathrm{id}_{\mathcal{C}} X$.

Given categories $\mathcal{C}, \mathcal{D}$, the *product category* $\mathcal{C} \times \mathcal{D}$ has objects (W, X) in $\mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$ and morphisms $f \times g : (W, X) \to (Y, Z)$ when $f : W \to Y$ is a morphism in $\mathcal{C}$ and $g : X \to Z$ is a morphism in $\mathcal{D}$, in the obvious way.

We call $\mathcal{D}$ a *subcategory* of $\mathcal{C}$ if $\mathrm{Obj}(\mathcal{D}) \subseteq \mathrm{Obj}(\mathcal{C})$, and $\mathrm{Hom}_{\mathcal{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Obj}(\mathcal{D})$, and compositions and identities in $\mathcal{C}, \mathcal{D}$

agree. We call $\mathcal{D}$ a *full* subcategory if also $\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all X,Y in $\mathrm{Obj}(\mathcal{D})$.

Definition A.2. Let $\mathcal{C},\mathcal{D}$ be categories. A (*covariant*) *functor* $F : \mathcal{C} \to \mathcal{D}$ gives, for all objects X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, and for all morphisms $f : X \to Y$ in $\mathcal{C}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$, such that $F(g \circ f) = F(g) \circ F(f)$ for all $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Obj}(\mathcal{C})$. A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Functors compose in the obvious way. Each category $\mathcal{C}$ has an obvious *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ with $\mathrm{id}_{\mathcal{C}}(X) = X$ and $\mathrm{id}_{\mathcal{C}}(f) = f$ for all X, f. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *full* if the maps $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$, $f \mapsto F(f)$ are surjective for all $X,Y \in \mathrm{Obj}(\mathcal{C})$, and *faithful* if the maps $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ are injective for all $X,Y \in \mathrm{Obj}(\mathcal{C})$.

Let $\mathcal{C},\mathcal{D}$ be categories and $F,G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ gives, for all objects X in $\mathcal{C}$, a morphism $\eta(X) : F(X) \to G(X)$ in $\mathcal{D}$ such that if $f : X \to Y$ is a morphism in $\mathcal{C}$ then $\eta(Y) \circ F(f) = G(f) \circ \eta(X)$ as morphisms $F(X) \to G(Y)$ in $\mathcal{D}$. We call η a *natural isomorphism* if $\eta(X)$ is an isomorphism for all $X \in \mathrm{Obj}(\mathcal{C})$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence* if there exist a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : G \circ F \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $\zeta : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$. Then we call $\mathcal{C},\mathcal{D}$ *equivalent categories*.

It is a fundamental principle of category theory that equivalent categories $\mathcal{C},\mathcal{D}$ should be thought of as being 'the same', and naturally isomorphic functors $F,G : \mathcal{C} \to \mathcal{D}$ should be thought of as being 'the same'. Note that equivalence of categories $\mathcal{C},\mathcal{D}$ is much weaker than strict isomorphism: isomorphism classes of objects in $\mathcal{C}$ are naturally in bijection with isomorphism classes of objects in $\mathcal{D}$, but there need be no relation between the sizes of the isomorphism classes, so that $\mathcal{C}$ could have many more objects than $\mathcal{D}$, for instance.

Definition A.3. Let $\mathcal{C}$ be a category, and $g : X \to Z$, $h : Y \to Z$ be morphisms in $\mathcal{C}$. A *fibre product* of g,h in $\mathcal{C}$ is an object W and morphisms $e : W \to X$ and $f : W \to Y$ in $\mathcal{C}$, such that $g \circ e = h \circ f$, with the universal property that if $e' : W' \to X$ and $f' : W' \to Y$ are morphisms in $\mathcal{C}$ with $g \circ e' = h \circ f'$ then there is a unique morphism $b : W' \to W$ with $e' = e \circ b$ and $f' = f \circ b$. Then we write $W = X \times_{g,Z,h} Y$ or $W = X \times_Z Y$, and $e = \pi_X$, $f = \pi_Y$. The diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "h"'] \\ X \arrow[r, "g"] & Z \end{tikzcd} \tag{A.1}$$

is called a *Cartesian square*. Fibre products need not exist, but if they do exist they are unique up to canonical isomorphism in $\mathcal{C}$.

A.2 Strict and weak 2-categories

Definition A.4. A *strict* 2-*category* $\boldsymbol{\mathcal{C}}$ consists of a class of *objects* $\mathrm{Obj}(\boldsymbol{\mathcal{C}})$, for all $X,Y \in \mathrm{Obj}(\boldsymbol{\mathcal{C}})$ an essentially small category $\mathbf{Hom}(X,Y)$, for all X,Y,Z

in $\mathrm{Obj}(\boldsymbol{\mathcal{C}})$ a functor $\mu_{X,Y,Z} : \mathbf{Hom}(X,Y) \times \mathbf{Hom}(Y,Z) \to \mathbf{Hom}(X,Z)$ called *composition*, and for all X in $\mathrm{Obj}(\boldsymbol{\mathcal{C}})$ an object id_X in $\mathbf{Hom}(X,X)$ called the *identity* 1*-morphism.* These must satisfy the *associativity property*, that

$$\mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}_{\mathbf{Hom}(Y,Z)}) = \mu_{W,X,Z} \circ (\mathrm{id}_{\mathbf{Hom}(W,X)} \times \mu_{X,Y,Z}) \qquad \text{(A.2)}$$

as functors $\mathbf{Hom}(W,X) \times \mathbf{Hom}(X,Y) \times \mathbf{Hom}(Y,Z) \to \mathbf{Hom}(W,X)$, and the *identity property*, that

$$\mu_{X,X,Y}(\mathrm{id}_X, -) = \mu_{X,Y,Y}(-, \mathrm{id}_Y) = \mathrm{id}_{\mathbf{Hom}(X,Y)} \qquad \text{(A.3)}$$

as functors $\mathbf{Hom}(X,Y) \to \mathbf{Hom}(X,Y)$.

Objects f of $\mathbf{Hom}(X,Y)$ are called 1*-morphisms*, written $f : X \to Y$. For 1-morphisms $f, g : X \to Y$, morphisms η in $\mathrm{Hom}_{\mathbf{Hom}(X,Y)}(f,g)$ are called 2*-morphisms*, written $\eta : f \Rightarrow g$. Thus, a 2-category has objects X, and two kinds of morphisms: 1-morphisms $f : X \to Y$ between objects, and 2-morphisms $\eta : f \Rightarrow g$ between 1-morphisms.

A *weak* 2*-category*, or *bicategory*, is like a strict 2-category, except that the equations of functors (A.2), (A.3) are required to hold only up to specified natural isomorphisms. That is, a weak 2-category $\boldsymbol{\mathcal{C}}$ consists of data $\mathrm{Obj}(\boldsymbol{\mathcal{C}}), \mathbf{Hom}(X,Y), \mu_{X,Y,Z}, \mathrm{id}_X$ as above, but in place of (A.2), a natural isomorphism of functors

$$\alpha : \mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}_{\mathbf{Hom}(Y,Z)}) \Longrightarrow \mu_{W,X,Z} \circ (\mathrm{id}_{\mathbf{Hom}(W,X)} \times \mu_{X,Y,Z}), \quad \text{(A.4)}$$

and in place of (A.3), natural isomorphisms

$$\beta : \mu_{X,X,Y}(\mathrm{id}_X, -) \Longrightarrow \mathrm{id}_{\mathbf{Hom}(X,Y)}, \quad \gamma : \mu_{X,Y,Y}(-, \mathrm{id}_Y) \Longrightarrow \mathrm{id}_{\mathbf{Hom}(X,Y)}. \quad \text{(A.5)}$$

These α, β, γ must satisfy identities which we give below in (A.9) and (A.12).

A strict 2-category $\boldsymbol{\mathcal{C}}$ can be regarded as an example of a weak 2-category, in which the natural isomorphisms α, β, γ in (A.4)–(A.5) are the identities.

We now unpack Definition A.4, making it more explicit.

There are three kinds of composition in a 2-category, satisfying various associativity relations. If $f : X \to Y$ and $g : Y \to Z$ are 1-morphisms then $\mu_{X,Y,Z}(f,g)$ is the *composition of* 1*-morphisms*, written $g \circ f : X \to Z$. If $f, g, h : X \to Y$ are 1-morphisms and $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms then composition of η, ζ in $\mathbf{Hom}(X,Y)$ gives the *vertical composition of* 2*-morphisms*, written $\zeta \odot \eta : f \Rightarrow h$, as a diagram

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, "g"'] \arrow[r, bend right=50, "h"'] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow \zeta\odot\eta"] & Y. \end{tikzcd}$$

Vertical composition is associative.

If $f, \dot f : X \to Y$ and $g, \dot g : Y \to Z$ are 1-morphisms and $\eta : f \Rightarrow \dot f$, $\zeta : g \Rightarrow \dot g$ are 2-morphisms then $\mu_{X,Y,Z}(\eta, \zeta)$ is the *horizontal composition of* 2*-morphisms*, written $\zeta * \eta : g \circ f \Rightarrow \dot g \circ \dot f$, as a diagram

$$
\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "\dot f"'] \arrow[r, phantom, "\Downarrow\eta"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "\dot g"'] \arrow[r, phantom, "\Downarrow\zeta"] & Z
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
X \arrow[r, bend left, "g\circ f"] \arrow[r, bend right, "\dot g\circ \dot f"'] \arrow[r, phantom, "\Downarrow\zeta*\eta"] & Z.
\end{tikzcd}
$$

As $\mu_{X,Y,Z}$ is a functor, these satisfy *compatibility of vertical and horizontal composition*: given a diagram of 1- and 2-morphisms

$$
\begin{tikzcd}[column sep=large, row sep=large]
X \arrow[r, bend left=60, "f"] \arrow[r, "\dot f" description] \arrow[r, bend right=60, "\ddot f"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\dot\eta"] & Y \arrow[r, bend left=60, "g"] \arrow[r, "\dot g" description] \arrow[r, bend right=60, "\ddot g"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\zeta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\dot\zeta"] & Z,
\end{tikzcd}
$$

we have

$$
(\dot\zeta \odot \zeta) * (\dot\eta \odot \eta) = (\dot\zeta * \dot\eta) \odot (\zeta * \eta) : g \circ f \Longrightarrow \ddot g \circ \ddot f. \tag{A.6}
$$

There are also two kinds of identity: *identity* 1-*morphisms* $\mathrm{id}_X : X \to X$ and *identity* 2-*morphisms* $\mathrm{id}_f : f \Rightarrow f$.

In a strict 2-category $\boldsymbol{\mathcal{C}}$, composition of 1-morphisms is strictly associative, $(g \circ f) \circ e = g \circ (f \circ e)$, and horizontal composition of 2-morphisms is strictly associative, $(\zeta * \eta) * \epsilon = \zeta * (\eta * \epsilon)$. In a weak 2-category $\boldsymbol{\mathcal{C}}$, composition of 1-morphisms is associative up to specified 2-isomorphisms. That is, if $e : W \to X$, $f : X \to Y$, $g : Y \to Z$ are 1-morphisms in $\boldsymbol{\mathcal{C}}$ then the natural isomorphism α in (A.4) gives a 2-isomorphism

$$
\alpha_{g,f,e} : (g \circ f) \circ e \Longrightarrow g \circ (f \circ e). \tag{A.7}
$$

As α is a natural isomorphism, given 1-morphisms $e, \dot e : W \to X$, $f, \dot f : X \to Y$, $g, \dot g : Y \to Z$ and 2-morphisms $\epsilon : e \Rightarrow \dot e$, $\eta : f \Rightarrow \dot f$, $\zeta : g \Rightarrow \dot g$ in $\boldsymbol{\mathcal{C}}$, the following diagram of 2-morphisms must commute:

$$
\begin{tikzcd}[column sep=huge]
(g \circ f) \circ e \arrow[r, Rightarrow, "\alpha_{g,f,e}"'] \arrow[d, Rightarrow, "(\zeta*\eta)*\epsilon"] & g \circ (f \circ e) \arrow[d, Rightarrow, "\zeta*(\eta*\epsilon)"'] \\
(\dot g \circ \dot f) \circ \dot e \arrow[r, Rightarrow, "\alpha_{\dot g,\dot f,\dot e}"] & \dot g \circ (\dot f \circ \dot e).
\end{tikzcd} \tag{A.8}
$$

The $\alpha_{g,f,e}$ must satisfy the *associativity coherence axiom*: if $d : V \to W$ is another 1-morphism, then the following diagram of 2-morphisms must commute:

$$
\begin{tikzcd}
((g \circ f) \circ e) \circ d \arrow[r, Rightarrow, "\alpha_{g,f,e}*\mathrm{id}_d"'] \arrow[d, Rightarrow, "\alpha_{g\circ f,e,d}"] & (g \circ (f \circ e)) \circ d \arrow[r, Rightarrow, "\alpha_{g,f\circ e,d}"'] & g \circ ((f \circ e) \circ d) \arrow[d, Rightarrow, "\mathrm{id}_g*\alpha_{f,e,d}"'] \\
(g \circ f) \circ (e \circ d) \arrow[rr, Rightarrow, "\alpha_{g,f,d\circ e}"] & & g \circ (f \circ (e \circ d)).
\end{tikzcd} \tag{A.9}
$$

In a strict 2-category $\boldsymbol{\mathcal{C}}$, given a 1-morphism $f : X \to Y$, the identity 1-morphisms $\mathrm{id}_X, \mathrm{id}_Y$ satisfy $f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$. In a weak 2-category $\boldsymbol{\mathcal{C}}$, the natural isomorphisms β, γ in (A.5) give 2-isomorphisms

$$
\beta_f : f \circ \mathrm{id}_X \Longrightarrow f, \qquad \gamma_f : \mathrm{id}_Y \circ f \Longrightarrow f. \tag{A.10}
$$

As β, γ are natural isomorphisms, if $\eta : f \Rightarrow \dot{f}$ is a 2-morphism we must have

$$\begin{aligned} \eta \odot \beta_f &= \beta_{\dot{f}} \odot (\eta * \mathrm{id}_{\mathrm{id}_X}) : f \circ \mathrm{id}_X \Rightarrow \dot{f}, \\ \eta \odot \gamma_f &= \gamma_{\dot{f}} \odot (\mathrm{id}_{\mathrm{id}_Y} * \eta) : \mathrm{id}_Y \circ f \Rightarrow \dot{f}. \end{aligned} \tag{A.11}$$

The β_f, γ_f must satisfy the *identity coherence axiom*: if $g : Y \to Z$ is another 1-morphism, then the following diagram of 2-morphisms must commute:

$$\begin{tikzcd} (g \circ \mathrm{id}_Y) \circ f \arrow[d, Rightarrow, "\alpha_{g,\mathrm{id}_Y,f}"] \arrow[dr, Rightarrow, "\beta_g * \mathrm{id}_f"] & \\ g \circ (\mathrm{id}_Y \circ f) \arrow[r, Rightarrow, "\mathrm{id}_g * \gamma_f"'] & g \circ f. \end{tikzcd} \tag{A.12}$$

A 2-category $\boldsymbol{\mathcal{C}}$ is called a $(2,1)$-*category* if all 2-morphisms in $\boldsymbol{\mathcal{C}}$ are invertible under vertical composition.

A basic example of a strict 2-category is the 2-*category of categories* $\mathfrak{Cat}$, with objects small categories $\mathcal{C}$, 1-morphisms functors $F : \mathcal{C} \to \mathcal{D}$, and 2-morphisms natural transformations $\eta : F \Rightarrow G$ for functors $F, G : \mathcal{C} \to \mathcal{D}$. Orbifolds naturally form a 2-category (strict or weak, depending on the definition), and so do stacks in algebraic geometry.

In a 2-category $\boldsymbol{\mathcal{C}}$, there are three notions of when objects X, Y in $\boldsymbol{\mathcal{C}}$ are 'the same': *equality* $X = Y$, and 1-*isomorphism*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, and *equivalence*, that is, we have 1-morphisms $f : X \to Y$, $g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$. Usually equivalence is the correct notion. By [3, Prop. B.8], we can also choose η, ζ to satisfy some extra identities:

Proposition A.5. *Let $\boldsymbol{\mathcal{C}}$ be a weak 2-category, and $f : X \to Y$ be an equivalence in $\boldsymbol{\mathcal{C}}$. Then there exist a 1-morphism $g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$ with $\zeta * \mathrm{id}_f = (\mathrm{id}_f * \eta) \odot \alpha_{f,g,f}$ as 2-isomorphisms $(f \circ g) \circ f \Rightarrow f$, and $\eta * \mathrm{id}_g = (\mathrm{id}_g * \zeta) \odot \alpha_{g,f,g}$ as 2-isomorphisms $(g \circ f) \circ g \Rightarrow g$.*

The next elementary lemma about 2-categories is easy to prove.

Lemma A.6. *Suppose $f : X \to Y$ and $g, h : Y \to Z$ are 1-morphisms in a (strict or weak) 2-category $\boldsymbol{\mathcal{C}}$, with f an equivalence. Then the map $\eta \mapsto \eta * \mathrm{id}_f = \zeta$ induces a 1-1 correspondence between 2-morphisms $\eta : g \Rightarrow h$ and 2-morphisms $\zeta : g \circ f \Rightarrow h \circ f$ in $\boldsymbol{\mathcal{C}}$.*

Definition A.7. Let $\boldsymbol{\mathcal{C}}$ be a 2-category. When we say that objects X, Y in $\boldsymbol{\mathcal{C}}$ are *canonically equivalent*, we mean that there is a nonempty distinguished class $\mathcal{E}$ of equivalences $f : X \to Y$ in $\boldsymbol{\mathcal{C}}$, and given any f, g in $\mathcal{E}$ there is a 2-isomorphism $\eta : f \Rightarrow g$. Often there is a distinguished choice of such η.

When we say that an object X in $\boldsymbol{\mathcal{C}}$ is *unique up to canonical equivalence*, we mean that there is a nonempty class $\mathcal{O}$ of distinguished choices $X, X', X'', \ldots$ for X, and given any X, X' in $\mathcal{O}$ there is a nonempty distinguished class $\mathcal{E}_{X,X'}$ of equivalences $f : X \to X'$, and given any f, g in $\mathcal{E}_{X,X'}$ there is a 2-isomorphism $\eta : f \Rightarrow g$, such that $\mathrm{id}_X : X \to X$ lies in $\mathcal{E}_{X,X}$, and if $f : X \to X'$ lies in $\mathcal{E}_{X,X'}$ and $f' : X' \to X''$ in $\mathcal{E}_{X',X''}$ then $f' \circ f : X \to X''$ lies in $\mathcal{E}_{X,X''}$.

Commutative diagrams in 2-categories should in general only commute *up to (specified) 2-isomorphisms*, rather than strictly. A simple example of a commutative diagram in a 2-category $\mathcal{C}$ is

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\eta"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z, \end{tikzcd}$$

which means that X, Y, Z are objects of $\mathcal{C}$, $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ are 1-morphisms in $\mathcal{C}$, and $\eta : g \circ f \Rightarrow h$ is a 2-isomorphism.

Let $\mathcal{C}$ be a 2-category. The *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$ is the category whose objects are objects of $\mathcal{C}$, and whose morphisms $[f] : X \to Y$ are 2-isomorphism classes $[f]$ of 1-morphisms $f : X \to Y$ in $\mathcal{C}$. The condition in Definition A.4 that $\mathbf{Hom}(X,Y)$ is essentially small ensures that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y)$ is a set, rather than a proper class. Then equivalences in $\mathcal{C}$ become isomorphisms in $\mathrm{Ho}(\mathcal{C})$, 2-commutative diagrams in $\mathcal{C}$ become commutative diagrams in $\mathrm{Ho}(\mathcal{C})$, and so on.

A.3 2-functors, 2-natural transformations, modifications

Next we discuss 2-functors between 2-categories, following Borceux [6, §7.2, §7.5] and Behrend et al. [3, §B.4].

Definition A.8. Let $\mathcal{C}, \mathcal{D}$ be strict 2-categories. A *strict* 2*-functor* $F : \mathcal{C} \to \mathcal{D}$ assigns an object $F(X)$ in $\mathcal{D}$ for each object X in $\mathcal{C}$, a 1-morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ for each 1-morphism $f : X \to Y$ in $\mathcal{C}$, and a 2-morphism $F(\eta) : F(f) \Rightarrow F(g)$ in $\mathcal{D}$ for each 2-morphism $\eta : f \Rightarrow g$ in $\mathcal{C}$, such that F preserves all the structures on $\mathcal{C}, \mathcal{D}$, that is,

$$F(g \circ f) = F(g) \circ F(f), \quad F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \quad F(\zeta * \eta) = F(\zeta) * F(\eta), \quad \text{(A.13)}$$

$$F(\zeta \odot \eta) = F(\zeta) \odot F(\eta), \quad F(\mathrm{id}_f) = \mathrm{id}_{F(f)}. \quad \text{(A.14)}$$

Now let $\mathcal{C}, \mathcal{D}$ be weak 2-categories. Then strict 2-functors $F : \mathcal{C} \to \mathcal{D}$ are not well-behaved. To fix this, we need to relax (A.13) to hold only up to specified 2-isomorphisms. A *weak* 2*-functor* (or *pseudofunctor*) $F : \mathcal{C} \to \mathcal{D}$ assigns an object $F(X)$ in $\mathcal{D}$ for each object X in $\mathcal{C}$, a 1-morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ for each 1-morphism $f : X \to Y$ in $\mathcal{C}$, a 2-morphism $F(\eta) : F(f) \Rightarrow F(g)$ in $\mathcal{D}$ for each 2-morphism $\eta : f \Rightarrow g$ in $\mathcal{C}$, a 2-isomorphism $F_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ in $\mathcal{D}$ for all 1-morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, and a 2-isomorphism $F_X : F(\mathrm{id}_X) \Rightarrow \mathrm{id}_{F(X)}$ in $\mathcal{D}$ for all objects X in $\mathcal{C}$, such that (A.14) holds, and for all $e : W \to X$, $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$ the following diagram of 2-isomorphisms commutes in $\mathcal{D}$:

$$\begin{tikzcd} (F(g) \circ F(f)) \circ F(e) \arrow[r, Rightarrow, "F_{g,f} * \mathrm{id}_{F(e)}"'] \arrow[d, Rightarrow, "\alpha_{F(g),F(f),F(e)}"] & F(g \circ f) \circ F(e) \arrow[r, Rightarrow, "F_{g \circ f, e}"'] & F((g \circ f) \circ e) \arrow[d, Rightarrow, "F(\alpha_{g,f,e})"'] \\ F(g) \circ (F(f) \circ F(e)) \arrow[r, Rightarrow, "\mathrm{id}_{F(g)} * F_{f,e}"] & F(g) \circ F(f \circ e) \arrow[r, Rightarrow, "F_{g, f \circ e}"] & F(g \circ (f \circ e)), \end{tikzcd}$$

and for all 1-morphisms $f : X \to Y$ in $\boldsymbol{\mathcal{C}}$, the following commute in $\boldsymbol{\mathcal{D}}$:

$$
\begin{tikzcd}
F(f) \circ F(\mathrm{id}_X) \arrow[r, Rightarrow, "F_{f,\mathrm{id}_X}"'] \arrow[d, Rightarrow, "\mathrm{id}_{F(f)} * F_X"] & F(f \circ \mathrm{id}_X) \arrow[d, Rightarrow, "F(\beta_f)"'] \\
F(f) \circ \mathrm{id}_{F(X)} \arrow[r, Rightarrow, "\beta_{F(f)}"] & F(f),
\end{tikzcd}
\qquad
\begin{tikzcd}
F(\mathrm{id}_Y) \circ F(f) \arrow[r, Rightarrow, "F_{\mathrm{id}_Y, f}"'] \arrow[d, Rightarrow, "F_Y * \mathrm{id}_{F(f)}"] & F(\mathrm{id}_Y \circ f) \arrow[d, Rightarrow, "F(\gamma_f)"'] \\
\mathrm{id}_{F(Y)} \circ F(f) \arrow[r, Rightarrow, "\gamma_{F(f)}"] & F(f),
\end{tikzcd}
$$

and if $f, \dot{f} : X \to Y$ and $g, \dot{g} : Y \to Z$ are 1-morphisms and $\eta : f \Rightarrow \dot{f}$, $\zeta : g \Rightarrow \dot{g}$ are 2-morphisms in $\boldsymbol{\mathcal{C}}$ then the following commutes in $\boldsymbol{\mathcal{D}}$:

$$
\begin{tikzcd}
F(g) \circ F(f) \arrow[r, Rightarrow, "F_{g,f}"'] \arrow[d, Rightarrow, "F(\zeta) * F(\eta)"] & F(g \circ f) \arrow[d, Rightarrow, "F(\zeta * \eta)"'] \\
F(\dot{g}) \circ F(\dot{f}) \arrow[r, Rightarrow, "F_{\dot{g},\dot{f}}"] & F(\dot{g} \circ \dot{f}).
\end{tikzcd}
$$

There are obvious notions of *composition* $G \circ F$ of strict and weak 2-functors $F : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$, $G : \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{E}}$, *identity 2-functors* $\mathrm{id}_{\boldsymbol{\mathcal{C}}}$, and so on.

If $\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}}$ are strict 2-categories, then a strict 2-functor $F : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ can be made into a weak 2-functor by taking all $F_{g,f}, F_X$ to be identity 2-morphisms.

Here is the 2-category analogue of natural transformations of functors:

Definition A.9. Let $\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}}$ be weak 2-categories and $F, G : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ be weak 2-functors. A *weak 2-natural transformation* (or *pseudo-natural transformation*) $\Theta : F \Rightarrow G$ assigns a 1-morphism $\Theta(X) : F(X) \to G(X)$ in $\boldsymbol{\mathcal{D}}$ for all objects X in $\boldsymbol{\mathcal{C}}$ and a 2-isomorphism $\Theta(f) : \Theta(Y) \circ F(f) \Rightarrow G(f) \circ \Theta(X)$ in $\boldsymbol{\mathcal{D}}$ for all 1-morphisms $f : X \to Y$ in $\boldsymbol{\mathcal{C}}$, such that if $\eta : f \Rightarrow g$ is a 2-morphism in $\boldsymbol{\mathcal{C}}$ then

$$
\begin{aligned}
(G(\eta) * \mathrm{id}_{\Theta(X)}) \odot \Theta(f) = \Theta(g) \odot (\mathrm{id}_{\Theta(Y)} * F(\eta)) :& \\
\Theta(Y) \circ F(f) \longrightarrow G(g) \circ \Theta(X),&
\end{aligned}
$$

and if $f : X \to Y$, $g : Y \to Z$ are 1-morphisms in $\boldsymbol{\mathcal{C}}$ then the following diagram of 2-isomorphisms commutes in $\boldsymbol{\mathcal{D}}$:

$$
\begin{tikzcd}
(\Theta(Z) \circ F(g)) \circ F(f) \arrow[r, Rightarrow, "\alpha_{\Theta(Z),F(g),F(f)}"'] \arrow[d, Rightarrow, "\Theta(g) * \mathrm{id}_{F(f)}"] & \Theta(Z) \circ (F(g) \circ F(f)) \arrow[r, Rightarrow, "\mathrm{id}_{\Theta(Z)} * F_{g,f}"'] & \Theta(Z) \circ (F(g \circ f)) \arrow[d, Rightarrow, "\Theta(g \circ f)"'] \\
(G(g) \circ \Theta(Y)) \circ F(f) \arrow[d, Rightarrow, "\alpha_{G(g),\Theta(Y),F(f)}"] & & G(g \circ f) \circ \Theta(X) \\
G(g) \circ (\Theta(Y) \circ F(f)) \arrow[r, Rightarrow, "\mathrm{id}_{G(g)} * \Theta(f)"] & G(g) \circ (G(f) \circ \Theta(X)) \arrow[r, Rightarrow, "\alpha^{-1}_{G(g),G(f),\Theta(X)}"] & (G(g) \circ G(f)) \circ \Theta(X)), \arrow[u, Rightarrow, "G_{g,f} * \mathrm{id}_{\Theta(X)}"']
\end{tikzcd}
$$

and if $X \in \boldsymbol{\mathcal{C}}$ then the following diagram of 2-isomorphisms commutes in $\boldsymbol{\mathcal{D}}$:

$$
\begin{tikzcd}
\Theta(X) \circ F(\mathrm{id}_X) \arrow[r, Rightarrow, "\Theta(\mathrm{id}_X)"'] \arrow[d, Rightarrow, "\mathrm{id}_{\Theta(X)} * F_X"] & G(\mathrm{id}_X) \circ \Theta(X) \arrow[r, Rightarrow, "G_X * \mathrm{id}_{\Theta(X)}"'] & \mathrm{id}_{G(X)} \circ \Theta(X) \arrow[d, Rightarrow, "\gamma_{\Theta(X)}"'] \\
\Theta(X) \circ \mathrm{id}_{F(X)} \arrow[rr, Rightarrow, "\beta_{\Theta(X)}"] & & \Theta(X).
\end{tikzcd}
$$

Just as the 'category of (small) categories' is actually a (strict) 2-category, so the 'category of (weak) 2-categories' is actually a 3-category (which we will not define). The 3-morphisms in this 3-category, morphisms between weak 2-natural transformations, are called *modifications*.

Definition A.10. Let $\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}}$ be weak 2-categories, $F, G : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ be weak 2-functors, and $\Theta, \Phi : F \Rightarrow G$ be weak 2-natural transformations. A *modification* $\aleph : F \Rrightarrow G$ assigns a 2-isomorphism $\aleph(X) : \Theta(X) \Rightarrow \Phi(X)$ in $\boldsymbol{\mathcal{D}}$ for all objects X in $\boldsymbol{\mathcal{C}}$, such that for all 1-morphisms $f : X \to Y$ in $\boldsymbol{\mathcal{C}}$ we have

$$(\mathrm{id}_{G(f)} * \aleph(X)) \odot \Theta(f) = \Phi(f) \odot (\aleph(Y) * \mathrm{id}_{F(f)}) :$$
$$\Theta(Y) \circ F(f) \Longrightarrow \Phi(X) \circ G(Y).$$

There are obvious notions of composition of modifications, identity modifications, and so on.

A weak 2-natural transformation $\Theta : F \Rightarrow G$ is called an *equivalence of* 2*-functors* if there exist a weak 2-natural transformation $\Phi : G \Rightarrow F$ and modifications $\aleph : \Phi \circ \Theta \Rrightarrow \mathrm{id}_F$ and $\beth : \Theta \circ \Phi \Rrightarrow \mathrm{id}_G$. Equivalence of 2-functors is a good notion of when weak 2-functors $F, G : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ are 'the same'.

A weak 2-functor $F : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ is called an *equivalence of weak* 2*-categories* if there exists a weak 2-functor $G : \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{C}}$ and equivalences of 2-functors $\Theta : G \circ F \Rightarrow \mathrm{id}_{\boldsymbol{\mathcal{C}}}$, $\Phi : F \circ G \Rightarrow \mathrm{id}_{\boldsymbol{\mathcal{D}}}$. Equivalence of weak 2-categories is a good notion of when weak 2-categories $\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}}$ are 'the same'.

Here are some well-known facts about 2-categories:

(i) Every weak 2-category $\boldsymbol{\mathcal{C}}$ is equivalent as a weak 2-category to a strict 2-category $\boldsymbol{\mathcal{C}}'$, that is, weak 2-categories can always be strictified.

(ii) If $\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}}$ are strict 2-categories, and $F : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ is a weak 2-functor, it may not be true that F is equivalent to a strict 2-functor $F' : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ (though this does hold if $\boldsymbol{\mathcal{D}} = \mathfrak{Cat}$, the strict 2-category of categories). That is, weak 2-functors cannot necessarily be strictified.

Even if one is working with strict 2-categories, weak 2-functors are often the correct notion of functor between them.

(iii) A weak 2-functor $F : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ is an equivalence of weak 2-categories, as in Definition A.10, if and only if for all objects X, Y in $\boldsymbol{\mathcal{C}}$, the functor $F_{X,Y} : \mathbf{Hom}_{\boldsymbol{\mathcal{C}}}(X, Y) \to \mathbf{Hom}_{\boldsymbol{\mathcal{D}}}(F(X), F(Y))$ is an equivalence of categories, and the map induced by F from equivalence classes of objects in $\boldsymbol{\mathcal{C}}$ to equivalence classes of objects in $\boldsymbol{\mathcal{D}}$ is surjective (and hence a bijection).

A.4 Fibre products in 2-categories

Fibre products in ordinary categories were defined in Definition A.3. We now define fibre products in 2-categories, following Behrend et al. [3, Def. B.13].

Definition A.11. Let $\boldsymbol{\mathcal{C}}$ be a strict 2-category and $g : X \to Z$, $h : Y \to Z$ be 1-morphisms in $\boldsymbol{\mathcal{C}}$. A *fibre product* in $\boldsymbol{\mathcal{C}}$ consists of an object W, 1-morphisms $e : W \to X$ and $f : W \to Y$ and a 2-isomorphism $\eta : g \circ e \Rightarrow h \circ f$ in $\boldsymbol{\mathcal{C}}$, so that we have a 2-commutative diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\eta\Uparrow"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"'] & Z \end{tikzcd} \tag{A.15}$$

with the following universal property: suppose $e' : W' \to X$ and $f' : W' \to Y$ are 1-morphisms and $\eta' : g \circ e' \Rightarrow h \circ f'$ is a 2-isomorphism in $\boldsymbol{\mathcal{C}}$. Then there should exist a 1-morphism $b : W' \to W$ and 2-isomorphisms $\zeta : e \circ b \Rightarrow e'$, $\theta : f \circ b \Rightarrow f'$ such that the following diagram of 2-isomorphisms commutes:

$$\begin{tikzcd} g \circ e \circ b \arrow[r, Rightarrow, "\eta * \mathrm{id}_b"'] \arrow[d, Rightarrow, "\mathrm{id}_g * \zeta"] & h \circ f \circ b \arrow[d, Rightarrow, "\mathrm{id}_h * \theta"'] \\ g \circ e' \arrow[r, Rightarrow, "\eta'"] & h \circ f'. \end{tikzcd} \tag{A.16}$$

Furthermore, if $\tilde{b}, \tilde{\zeta}, \tilde{\theta}$ are alternative choices of b, ζ, θ then there should exist a unique 2-isomorphism $\epsilon : b \Rightarrow \tilde{b}$ with

$$\zeta = \tilde{\zeta} \odot (\mathrm{id}_e * \epsilon) \quad \text{and} \quad \theta = \tilde{\theta} \odot (\mathrm{id}_f * \epsilon).$$

We call such a fibre product diagram (A.15) a 2-*Cartesian square*. We often write $W = X \times_Z Y$ or $W = X \times_{g,Z,h} Y$, and call W the fibre product.

If a fibre product $X \times_Z Y$ in $\boldsymbol{\mathcal{C}}$ exists then it is unique up to canonical equivalence in $\boldsymbol{\mathcal{C}}$. If $\boldsymbol{\mathcal{C}}$ is an ordinary category, that is, all 2-morphisms are identities $\mathrm{id}_f : f \Rightarrow f$, this definition of fibre products in $\boldsymbol{\mathcal{C}}$ is equivalent to that in Definition A.3.

If instead $\boldsymbol{\mathcal{C}}$ is a weak 2-category, we must replace (A.16) by

$$\begin{tikzcd} (g \circ e) \circ b \arrow[r, Rightarrow, "\eta * \mathrm{id}_b"'] \arrow[d, Rightarrow, "\alpha_{g,e,b}"] & (h \circ f) \circ b \arrow[r, Rightarrow, "\alpha_{h,f,b}"'] & h \circ (f \circ b) \arrow[d, Rightarrow, "\mathrm{id}_h * \theta"'] \\ g \circ (e \circ b) \arrow[r, Rightarrow, "\mathrm{id}_g * \zeta"] & g \circ e' \arrow[r, Rightarrow, "\eta'"] & h \circ f'. \end{tikzcd} \tag{A.17}$$

Orbifolds, and stacks in algebraic geometry, form 2-categories, and Definition A.11 is the right way to define fibre products of orbifolds or stacks.

A.5 Sheaves on topological spaces

Next we discuss sheaves. These are a fundamental tool in Algebraic Geometry, as in Hartshorne [43, §II.1], for instance. Although Differential Geometers may not be familiar with sheaves, nonetheless they are everywhere in Differential Geometry, and one uses properties of sheaves all the time without noticing.

For something to be a sheaf on a space X just means that it is defined locally on X. For example, if X is a manifold then smooth functions $f : X \to \mathbb{R}$ form a sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras on X, since the condition that a function $f : X \to \mathbb{R}$ is smooth is a local condition near each $x \in X$. Some good references on sheaves are Bredon [10], Godement [40], and Hartshorne [43, §II.1].

Definition A.12. Let X be a topological space. A *presheaf of sets* $\mathcal{E}$ on X consists of the data of a set $\mathcal{E}(S)$ for every open set $S \subseteq X$, and a map $\rho_{ST} : \mathcal{E}(S) \to \mathcal{E}(T)$ called the *restriction map* for every inclusion $T \subseteq S \subseteq X$ of open sets, satisfying the conditions that:

(i) $\mathcal{E}(\emptyset) = *$ is a point.

(ii) $\rho_{SS} = \mathrm{id}_{\mathcal{E}(S)} : \mathcal{E}(S) \to \mathcal{E}(S)$ for all open $S \subseteq X$; and

(iii) $\rho_{SU} = \rho_{TU} \circ \rho_{ST} : \mathcal{E}(S) \to \mathcal{E}(U)$ for all open $U \subseteq T \subseteq S \subseteq X$.

A presheaf of sets $\mathcal{E}$ on X is called a *sheaf* if it also satisfies

(iv) If $S \subseteq X$ is open, $\{T_a : a \in A\}$ is an open cover of S, and $s, t \in \mathcal{E}(S)$ have $\rho_{ST_a}(s) = \rho_{ST_a}(t)$ in $\mathcal{E}(T_a)$ for all $a \in A$, then $s = t$ in $\mathcal{E}(S)$; and

(v) If $S \subseteq X$ is open, $\{T_a : a \in A\}$ is an open cover of S, and we are given elements $s_a \in \mathcal{E}(T_a)$ for all $a \in A$ such that $\rho_{T_a(T_a \cap T_b)}(s_a) = \rho_{T_b(T_a \cap T_b)}(s_b)$ in $\mathcal{E}(T_a \cap T_b)$ for all $a, b \in A$, then there exists $s \in \mathcal{E}(S)$ with $\rho_{ST_a}(s) = s_a$ for all $a \in A$. This s is unique by (iv).

Suppose $\mathcal{E}, \mathcal{F}$ are presheaves or sheaves of sets on X. A *morphism* $\phi : \mathcal{E} \to \mathcal{F}$ consists of a map $\phi(S) : \mathcal{E}(S) \to \mathcal{F}(S)$ for all open $S \subseteq X$, such that the following diagram commutes for all open $T \subseteq S \subseteq X$

$$\begin{tikzcd} \mathcal{E}(S) \arrow[r, "\phi(S)"'] \arrow[d, "\rho_{ST}"] & \mathcal{F}(S) \arrow[d, "\rho'_{ST}"'] \\ \mathcal{E}(T) \arrow[r, "\phi(T)"] & \mathcal{F}(T), \end{tikzcd}$$

where ρ_{ST} is the restriction map for $\mathcal{E}$, and ρ'_{ST} the restriction map for $\mathcal{F}$.

We have defined sheaves of sets, but one can also define sheaves of abelian groups, rings, modules, ..., by replacing sets by abelian groups, ..., throughout.

If $\mathcal{E}$ is a sheaf of sets, abelian groups,... on X then we write $\Gamma(\mathcal{E})$ for $\mathcal{E}(X)$, the *global sections of* $\mathcal{E}$, as a set, abelian group, ...

Definition A.13. Let $\mathcal{E}$ be a presheaf of sets on X. For each $x \in X$, the *stalk* $\mathcal{E}_x$ is the direct limit of the sets $\mathcal{E}(U)$ for all $x \in U \subseteq X$, via the restriction maps ρ_{UV}. A morphism $\phi : \mathcal{E} \to \mathcal{F}$ induces morphisms $\phi_x : \mathcal{E}_x \to \mathcal{F}_x$ for all $x \in X$. If $\mathcal{E}, \mathcal{F}$ are sheaves then ϕ is an isomorphism if and only if ϕ_x is an isomorphism for all $x \in X$.

Definition A.14. Let $\mathcal{E}$ be a presheaf of sets on X. A *sheafification* of $\mathcal{E}$ is a sheaf of sets $\hat{\mathcal{E}}$ on X and a morphism of presheaves $\pi : \mathcal{E} \to \hat{\mathcal{E}}$, such that whenever $\mathcal{F}$ is a sheaf of sets on X and $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism, there is a unique morphism $\hat{\phi} : \hat{\mathcal{E}} \to \mathcal{F}$ with $\phi = \hat{\phi} \circ \pi$. As in [43, Prop. II.1.2], a sheafification always exists, and is unique up to canonical isomorphism; one can be constructed explicitly using the stalks $\mathcal{E}_x$ of $\mathcal{E}$.

Next we discuss *pushforwards* and *pullbacks* of sheaves by continuous maps.

Definition A.15. Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{E}$ a sheaf of sets on X. Define the *pushforward* (*direct image*) sheaf $f_*(\mathcal{E})$ on Y by $\bigl(f_*(\mathcal{E})\bigr)(U) = \mathcal{E}\bigl(f^{-1}(U)\bigr)$ for all open $U \subseteq V$, with restriction maps $\rho'_{UV} = \rho_{f^{-1}(U)f^{-1}(V)} : \bigl(f_*(\mathcal{E})\bigr)(U) \to \bigl(f_*(\mathcal{E})\bigr)(V)$ for all open $V \subseteq U \subseteq Y$. Then $f_*(\mathcal{E})$ is a sheaf of sets on Y.

If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of sheaves we define a morphism $f_*(\phi) : f_*(\mathcal{E}) \to f_*(\mathcal{F})$ of sheaves on Y by $\bigl(f_*(\phi)\bigr)(u) = \phi\bigl(f^{-1}(U)\bigr)$ for all open $U \subseteq Y$. For continuous maps $f : X \to Y$, $g : Y \to Z$ we have $(g \circ f)_* = g_* \circ f_*$.

Definition A.16. Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{E}$ a sheaf of sets on Y. Define a presheaf $\mathcal{P}f^{-1}(\mathcal{E})$ on X by $\bigl(\mathcal{P}f^{-1}(\mathcal{E})\bigr)(U) = \lim_{A \supseteq f(U)} \mathcal{E}(A)$ for open $A \subseteq X$, where the direct limit is taken over all open $A \subseteq Y$ containing $f(U)$, using the restriction maps ρ_{AB} in $\mathcal{E}$. For open $V \subseteq U \subseteq X$, define $\rho'_{UV} : \bigl(\mathcal{P}f^{-1}(\mathcal{E})\bigr)(U) \to \bigl(\mathcal{P}f^{-1}(\mathcal{E})\bigr)(V)$ as the direct limit of the morphisms ρ_{AB} in $\mathcal{E}$ for $B \subseteq A \subseteq Y$ with $f(U) \subseteq A$ and $f(V) \subseteq B$. Then we define the *pullback* (*inverse image*) $f^{-1}(\mathcal{E})$ to be the sheafification of the presheaf $\mathcal{P}f^{-1}(\mathcal{E})$. It is unique up to canonical isomorphism.

If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of sheaves on Y, one can define a pullback morphism $f^{-1}(\phi) : f^{-1}(\mathcal{E}) \to f^{-1}(\mathcal{F})$ of sheaves on X. As in [43, Ex. II.1.18], pushforward f_* is right adjoint to f^{-1}. That is, there are natural bijections

$$\mathrm{Hom}_X\bigl(f^{-1}(\mathcal{E}), \mathcal{F}\bigr) \cong \mathrm{Hom}_Y\bigl(\mathcal{E}, f_*(\mathcal{F})\bigr) \tag{A.18}$$

for all sheaves $\mathcal{E}$ on Y and $\mathcal{F}$ on X, with functorial properties.

A.6 Stacks on topological spaces

In §A.5 we explained sheaves on topological spaces. We will also need a 2-category analogue of sheaves, called *stacks on a topological space.*

Definition A.17. Let X be a topological space. A *prestack* (or *prestack in groupoids*, or 2-*presheaf*) $\boldsymbol{\mathcal{E}}$ on X, consists of the data of a groupoid $\boldsymbol{\mathcal{E}}(S)$ for every open set $S \subseteq X$, and a functor $\rho_{ST} : \boldsymbol{\mathcal{E}}(S) \to \boldsymbol{\mathcal{E}}(T)$ called the *restriction map* for every inclusion $T \subseteq S \subseteq X$ of open sets, and a natural isomorphism of functors $\eta_{STU} : \rho_{TU} \circ \rho_{ST} \Rightarrow \rho_{SU}$ for all inclusions $U \subseteq T \subseteq S \subseteq X$ of open sets, satisfying the conditions that:

(i) $\rho_{SS} = \mathrm{id}_{\boldsymbol{\mathcal{E}}(S)} : \boldsymbol{\mathcal{E}}(S) \to \boldsymbol{\mathcal{E}}(S)$ for all open $S \subseteq X$, and $\eta_{SST} = \eta_{STT} = \mathrm{id}_{\rho_{ST}}$ for all open $T \subseteq S \subseteq X$; and

(ii) $\eta_{SUV} \odot (\mathrm{id}_{\rho_{UV}} * \eta_{STU}) = \eta_{STV} \odot (\eta_{TUV} * \mathrm{id}_{\rho_{ST}}) : \rho_{UV} \circ \rho_{TU} \circ \rho_{ST} \Longrightarrow \rho_{SV}$ for all open $V \subseteq U \subseteq T \subseteq S \subseteq X$.

A prestack $\boldsymbol{\mathcal{E}}$ on X is called a *stack* (or *stack in groupoids*, or 2-*sheaf*) on X if whenever $S \subseteq X$ is open and $\{T_a : a \in A\}$ is an open cover of S, and we write $T_{ab} = T_a \cap T_b$ and $T_{abc} = T_a \cap T_b \cap T_c$ for $a, b, c \in A$, then:

(iii) If $\epsilon, \zeta : E \to F$ are morphisms in $\boldsymbol{\mathcal{E}}(S)$ and $\rho_{ST_a}(\epsilon) = \rho_{ST_a}(\zeta) : \rho_{ST_a}(E) \to \rho_{ST_a}(F)$ in $\boldsymbol{\mathcal{E}}(T_a)$ for all $a \in A$, then $\epsilon = \zeta$.

(iv) If E, F are objects of $\boldsymbol{\mathcal{E}}(S)$ and $\epsilon_a : \rho_{ST_a}(E) \to \rho_{ST_a}(F)$ are morphisms in $\boldsymbol{\mathcal{E}}(T_a)$ for all $a \in A$ with

$$\begin{aligned}\eta_{ST_aT_{ab}}(F) \circ \rho_{T_aT_{ab}}(\epsilon_a) \circ \eta_{ST_aT_{ab}}(E)^{-1}\\ = \eta_{ST_bT_{ab}}(F) \circ \rho_{T_bT_{ab}}(\epsilon_b) \circ \eta_{ST_bT_{ab}}(E)^{-1}\end{aligned}$$

in $\boldsymbol{\mathcal{E}}(T_{ab})$ for all $a, b \in A$, then there exists $\epsilon : E \to F$ in $\boldsymbol{\mathcal{E}}(S)$ (necessarily unique by (iii)) with $\rho_{ST_a}(\epsilon) = \epsilon_a$ for all $a \in A$.

(v) If $E_a \in \boldsymbol{\mathcal{E}}(T_a)$ for $a \in A$ and $\epsilon_{ab} : \rho_{T_aT_{ab}}(E_a) \to \rho_{T_bT_{ab}}(E_b)$ are morphisms in $\boldsymbol{\mathcal{E}}(T_{ab})$ for all $a, b \in A$ satisfying

$$\begin{aligned}&\eta_{T_cT_{bc}T_{abc}}(E_c) \circ \rho_{T_{bc}T_{abc}}(\epsilon_{bc}) \circ \eta_{T_bT_{bc}T_{abc}}(E_b)^{-1}\\ &\circ \eta_{T_bT_{ab}T_{abc}}(E_b) \circ \rho_{T_{ab}T_{abc}}(\epsilon_{ab}) \circ \eta_{T_aT_{ab}T_{abc}}(E_a)^{-1}\\ &\quad = \eta_{T_cT_{ac}T_{abc}}(E_c) \circ \rho_{T_{ac}T_{abc}}(\epsilon_{ac}) \circ \eta_{T_aT_{ac}T_{abc}}(E_a)^{-1}\end{aligned}$$

for all $a, b, c \in A$, then there exist an object E in $\boldsymbol{\mathcal{E}}(S)$ and morphisms $\zeta_a : E_a \to \rho_{ST_a}(E)$ for $a \in A$ such that for all $a, b \in A$ we have

$$\eta_{ST_aT_{ab}}(E) \circ \rho_{T_aT_{ab}}(\zeta_a) = \eta_{ST_bT_{ab}}(E) \circ \rho_{T_bT_{ab}}(\zeta_b) \circ \epsilon_{ab}.$$

If $\tilde{E}, \tilde{\zeta}_a$ are alternative choices then (iii),(iv) imply there is a unique isomorphism $\theta : E \to \tilde{E}$ in $\boldsymbol{\mathcal{E}}(S)$ with $\rho_{ST_a}(\theta) = \tilde{\zeta}_a \circ \zeta_a^{-1}$ for all $a \in A$.

Remark A.18. **(a)** Actually the term 'stack' is used in Algebraic Geometry with a more general meaning, namely 'stack on a site', as in Olsson [93] for instance. Here a 'site' $\mathcal{S}$ is a generalization of a topological space. When $\mathcal{S}$ is the site of open subsets of a topological space X with the usual open covers, we recover Definition A.17. When $\mathcal{S}$ is the site $\mathrm{Sch}_{\mathbb{K}}$ of schemes over a field $\mathbb{K}$ with the étale or smooth topology, we obtain Deligne–Mumford or Artin $\mathbb{K}$-stacks in Algebraic Geometry. There are several equivalent ways to define stacks; we have chosen the definition which most obviously generalizes sheaves in §A.5.

(b) In the examples of stacks on topological spaces that will be important to us, we will have $\rho_{TU} \circ \rho_{ST} = \rho_{SU}$ and $\eta_{STU} = \mathrm{id}_{\rho_{SU}}$ for all open $U \subseteq T \subseteq S \subseteq X$. So (ii) is automatic, and all the $\eta_{\cdots}(\cdots)$ terms in (iv),(v) can be omitted.

Appendix B

Differential geometry in $\dot{\mathbf{Man}}$ and $\dot{\mathbf{Man}}^{\mathbf{c}}$

Suppose for the whole of §B.1–§B.6 that $\dot{\mathbf{Man}}$ satisfies Assumptions 3.1–3.7 in §3.1. Using the assumptions, we will define some notation and prove some results on differential geometry in $\dot{\mathbf{Man}}$. This is standard material for classical manifolds $\mathbf{Man}$, the main point is that it also works for any category $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7. In §B.7 we explain how to compare differential geometry in two categories $\dot{\mathbf{Man}}, \ddot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7 related by a functor $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \to \ddot{\mathbf{Man}}$. Sections B.1–B.7 are summarized in §3.3.

Section B.8 explains how to extend §B.1–§B.7 to a category of manifolds with corners $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfying Assumption 3.22 in §3.4. It is summarized in §3.4.3. Section B.9 proves Theorem 3.17.

B.1 Functions on manifolds, and the structure sheaf

B.1.1 The $\mathbb{R}$-algebra $C^\infty(X)$

Definition B.1. For each $X \in \dot{\mathbf{Man}}$, write $C^\infty(X)$ for the set of morphisms $a : X \to \mathbb{R}$ in $\dot{\mathbf{Man}}$. Faithfulness of $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}$ in Assumption 3.2(a) implies that we may identify $C^\infty(X)$ with a subset of the set $C^0(X_{\mathrm{top}})$ of continuous maps $a_{\mathrm{top}} : X_{\mathrm{top}} \to \mathbb{R}$. We will show that $C^\infty(X)$ has a natural commutative $\mathbb{R}$-algebra structure, a subalgebra of the obvious $\mathbb{R}$-algebra structure on $C^0(X_{\mathrm{top}})$.

Given $a, b \in C^\infty(X)$ and $\lambda \in \mathbb{R}$ we define $a + b, a \cdot b, \lambda \cdot a \in C^\infty(X)$ and the elements $0, 1 \in C^\infty(X)$ by the following commutative diagrams in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd} X \arrow[rr, "a+b"'] \arrow[dr, "(a,b)"'] & & \mathbb{R}, \\ & \mathbb{R}^2 \arrow[ur, "(x,y)\mapsto x+y"'] & \end{tikzcd} \quad \begin{tikzcd} X \arrow[rr, "a\cdot b"'] \arrow[dr, "(a,b)"'] & & \mathbb{R}, \\ & \mathbb{R}^2 \arrow[ur, "(x,y)\mapsto xy"'] & \end{tikzcd} \quad \begin{tikzcd} X \arrow[rr, "\lambda\cdot a"'] \arrow[dr, "a"'] & & \mathbb{R}, \\ & \mathbb{R} \arrow[ur, "x\mapsto \lambda x"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[rr, "0"'] \arrow[dr, "\pi"'] & & \mathbb{R}, \\ & * \arrow[ur, "0"'] & \end{tikzcd} \quad \begin{tikzcd} X \arrow[rr, "1"'] \arrow[dr, "\pi"'] & & \mathbb{R}. \\ & * \arrow[ur, "1"'] & \end{tikzcd}$$

Here $(x,y) \mapsto x+y$ and $(x,y) \mapsto xy$ mapping $\mathbb{R}^2 \to \mathbb{R}$ are morphisms in $\mathbf{Man} \subseteq \dot{\mathbf{M}}\mathbf{an}$, and similarly for $x \mapsto \lambda x$ and $0,1 : * \to \mathbb{R}$. The map $\pi : X \to *$ is as in Assumption 3.1(c).

One can now show that these operations make $C^\infty(X)$ into a commutative $\mathbb{R}$-algebra by straightforward diagram-chasing. For example, to show that multiplication is associative, consider the commutative diagram:

$$\begin{tikzcd}
 & \mathbb{R}^3 \arrow[r, "{(x,y,z)\mapsto(xy,z)}"] \arrow[dd, equal] & \mathbb{R}^2 \arrow[dr, "{(x,y)\mapsto xy}"] & \\
X \arrow[ur, "{(a,b,c)}"] \arrow[dr, "{(a,b,c)}"'] \arrow[urr, "{(ab,c)}"'] \arrow[drr, "{(a,bc)}"] \arrow[rrr, shift left, "(ab)c"] \arrow[rrr, shift right, "a(bc)"'] & & & \mathbb{R}. \\
 & \mathbb{R}^3 \arrow[r, "{(x,y,z)\mapsto(x,yz)}"'] & \mathbb{R}^2 \arrow[ur, "{(x,y)\mapsto xy}"'] &
\end{tikzcd}$$

If $f : X \to Y$ is a morphism in $\dot{\mathbf{M}}\mathbf{an}$, define $f^* : C^\infty(Y) \to C^\infty(X)$ by $f^* : a \mapsto a \circ f$. Then f is an $\mathbb{R}$-algebra morphism. If $g : Y \to Z$ is another morphism in $\dot{\mathbf{M}}\mathbf{an}$ then $(g \circ f)^* = f^* \circ g^* : C^\infty(Z) \to C^\infty(X)$.

B.1.2 Making $C^\infty(X)$ into a C^∞-ring

The subject of C^∞-*algebraic geometry* treats differential-geometric problems using the machinery of algebraic geometry, including sheaves, schemes and stacks. Some references are the author [56, 65] and Dubuc [13]. A key idea is C^∞-*rings*, which are a generalization of $\mathbb{R}$-algebras with a richer algebraic structure, such that if X is a smooth manifold then $C^\infty(X)$ is naturally a C^∞-ring.

Definition B.2. A C^∞-*ring* is a set $\mathfrak{C}$ together with operations

$$\Phi_f : \mathfrak{C}^n = \overbrace{\mathfrak{C} \times \cdots \times \mathfrak{C}}^{n \text{ copies}} \longrightarrow \mathfrak{C}$$

for all $n \geqslant 0$ and smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, where by convention when $n = 0$ we define $\mathfrak{C}^0$ to be the single point $\{\emptyset\}$. These operations must satisfy the following relations: suppose $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ are smooth functions. Define a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g\bigl(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)\bigr),$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g\bigl(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)\bigr).$$

We also require that for all $1 \leqslant j \leqslant n$, defining $\pi_j : \mathbb{R}^n \to \mathbb{R}$ by $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$, we have $\Phi_{\pi_j}(c_1, \ldots, c_n) = c_j$ for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$.

Usually we refer to $\mathfrak{C}$ as the C^∞-ring, leaving the operations Φ_f implicit.

A *morphism* between C^∞-rings $\bigl(\mathfrak{C}, (\Phi_f)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\bigr)$, $\bigl(\mathfrak{D}, (\Psi_f)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\bigr)$ is a map $\phi : \mathfrak{C} \to \mathfrak{D}$ such that $\Psi_f\bigl(\phi(c_1), \ldots, \phi(c_n)\bigr) = \phi \circ \Phi_f(c_1, \ldots, c_n)$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$ and $c_1, \ldots, c_n \in \mathfrak{C}$. We will write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings. As in [65, §2.2], every C^∞-ring $\mathfrak{C}$ has the structure of a commutative $\mathbb{R}$-algebra, in which addition and multiplication are the C^∞-ring operations Φ_f, Φ_g for $f, g : \mathbb{R}^2 \to \mathbb{R}$ mapping $f(x, y) = x + y$ and $g(x, y) = xy$.

A *module* M over a C^∞-ring $\mathfrak{C}$ is a module over $\mathfrak{C}$ as an $\mathbb{R}$-algebra.

As in [13, 56, 65], in C^∞-algebraic geometry one studies C^∞-*schemes* and C^∞-*stacks*, which are versions of schemes and stacks in Algebraic Geometry in which rings are replaced by C^∞-rings. C^∞-algebraic geometry has been used as the basis for Derived Differential Geometry, the study of 'derived smooth manifolds' and 'derived smooth orbifolds', by defining derived manifolds (or orbifolds) to be special examples of 'derived C^∞-schemes' or 'derived Deligne–Mumford C^∞-stacks'. See Spivak [103], Borisov and Noel [7, 8] and the author [57, 58, 61] for different notions of derived manifolds and derived orbifolds.

Our Kuranishi spaces are an alternative approach to Derived Differential Geometry, and the 2-categories $\mathbf{mKur}, \mathbf{Kur}$ of (m-)Kuranishi spaces defined in Chapters 4 and 6 using $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$ are equivalent to the 2-categories $\mathbf{dMan}$, $\mathbf{dOrb}$ of 'd-manifolds' and 'd-orbifolds' defined in [57, 58, 61] using C^∞-algebraic geometry.

Definition B.3. Let $X \in \dot{\mathbf{M}}\mathbf{an}$, and $C^\infty(X)$ be as in §B.1.1. Then we can give $C^\infty(X)$ the structure of a C^∞-ring, such that if $f : \mathbb{R}^n \to \mathbb{R}$ is smooth (and hence a morphism in $\dot{\mathbf{M}}\mathbf{an}$) and $a_1, \ldots, a_n \in C^\infty(X)$ then $\Phi_f(a_1, \ldots, a_n) \in C^\infty(X)$ is defined by the commutative diagram in $\dot{\mathbf{M}}\mathbf{an}$:

$$\begin{tikzcd} X \arrow[rr, "\Phi_f(a_1,\ldots,a_n)"'] \arrow[dr, "(a_1,\ldots,a_n)"'] & & \mathbb{R}. \\ & \mathbb{R}^n \arrow[ur, "f"'] & \end{tikzcd}$$

The method of proof in §B.1.1 that $C^\infty(X)$ is an $\mathbb{R}$-algebra now also shows that $C^\infty(X)$ is a C^∞-ring. The associated $\mathbb{R}$-algebra structure is that in §B.1.1.

B.1.3 The structure sheaf $\mathcal{O}_X$

Definition B.4. Let $X \in \dot{\mathbf{M}}\mathbf{an}$. Then for each open $U' \subseteq X_{\mathrm{top}}$, Assumption 3.2(d) gives a unique open submanifold $i : U \hookrightarrow X$ with $i_{\mathrm{top}}(U_{\mathrm{top}}) = U'$. Set $\mathcal{O}_X(U') = C^\infty(U)$, where $C^\infty(U)$ is regarded either as an $\mathbb{R}$-algebra as in §B.1.1, or as a C^∞-ring as in §B.1.2.

For open $V' \subseteq U' \subseteq X_{\mathrm{top}}$ we have open submanifolds $i : U \hookrightarrow X$, $j : V \hookrightarrow X$ with $\mathcal{O}_X(U') = C^\infty(U)$ and $\mathcal{O}_X(V') = C^\infty(V)$. Since $V_{\mathrm{top}} \subseteq U_{\mathrm{top}}$ Assumption 3.2(d) gives a unique $k : V \to U$ in $\dot{\mathbf{M}}\mathbf{an}$ with $i \circ k = j : V \to X$. Define $\rho_{U'V'} : \mathcal{O}_X(U') \to \mathcal{O}_X(V')$ by $\rho_{U'V'} : a \mapsto a \circ k$, for $a : U \to \mathbb{R}$ in $\dot{\mathbf{M}}\mathbf{an}$.

It is now easy to check that $\rho_{U'V'}$ is a morphism of $\mathbb{R}$-algebras, and of C^∞-rings, and so the data $\mathcal{O}_X(U'), \rho_{U'V'}$ defines a sheaf of $\mathbb{R}$-algebras or C^∞-rings $\mathcal{O}_X$ on X_{top}, as in Definition A.12(i)–(v), where the sheaf axiom (iv) follows from faithfulness in Assumption 3.2(a), and (v) from Assumption 3.3(a). We call $\mathcal{O}_X$ the *structure sheaf* of X.

If $f : X \to Y$ is a morphism in $\dot{\mathbf{M}}\mathbf{an}$, then $(f_{\mathrm{top}})_*(\mathcal{O}_X)$ and $\mathcal{O}_Y$ are sheaves of $\mathbb{R}$-algebras or C^∞-rings on Y. Define a morphism $f_\sharp : \mathcal{O}_Y \to (f_{\mathrm{top}})_*(\mathcal{O}_X)$ of sheaves of $\mathbb{R}$-algebras or C^∞-rings on $Y_{\rm top}$ as follows. Let $j : V \hookrightarrow Y$ be an open submanifold, and $i : U \hookrightarrow X$ the open submanifold with $U_{\rm top} = f_{\rm top}^{-1}(V_{\rm top}) \subseteq X_{\rm top}$, and $f' : U \to V$ the unique morphism with $j \circ f' = f \circ i : U \to Y$ from Assumption 3.2(d). Set

$$\begin{aligned} f_\sharp(V_{\rm top}) = f'^* : \mathcal{O}_Y(V_{\rm top}) = C^\infty(V) \longrightarrow C^\infty(U) = \mathcal{O}_X(U_{\rm top}) \\ = \mathcal{O}_X(f_{\rm top}^{-1}(V_{\rm top})) = (f_{\rm top})_*(\mathcal{O}_X)(V_{\rm top}). \end{aligned} \tag{B.1}$$

These $f_\sharp(V_{\rm top})$ for all open $j : V \hookrightarrow Y$ form a sheaf morphism $f_\sharp : \mathcal{O}_Y \to (f_{\rm top})_*(\mathcal{O}_X)$. Let $f^\sharp : f_{\rm top}^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ be the adjoint morphism of sheaves of $\mathbb{R}$-algebras or C^∞-rings on X under (A.18). Then $(f_{\rm top}, f^\sharp) : (X_{\rm top}, \mathcal{O}_X) \to (Y_{\rm top}, \mathcal{O}_Y)$ is a morphism of locally ringed spaces, or locally C^∞-ringed spaces.

Now results in [65, §4.8] give sufficient criteria for when a locally C^∞-ringed space $(X, \mathcal{O}_X)$ is an affine C^∞-scheme, and Assumptions 3.2(b) and 3.6 imply that these criteria hold. We then easily deduce:

Proposition B.5. (a) *Let X be an object of $\dot{\mathbf{Man}}$, so that $X_{\rm top}$ is a topological space and $\mathcal{O}_X$ a sheaf of C^∞-rings on $X_{\rm top}$. Then $(X_{\rm top}, \mathcal{O}_X)$ is an affine C^∞-scheme in the sense of* [13, 56, 65].

(b) *Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$. Then $(f_{\rm top}, f^\sharp) : (X_{\rm top}, \mathcal{O}_X) \to (Y_{\rm top}, \mathcal{O}_Y)$ is a morphism of affine C^∞-schemes in the sense of* [13, 56, 65].

(c) *Combining* **(a)**,**(b)** *we may define a functor $F_{\dot{\mathbf{Man}}}^{\mathbf{C^\infty Sch}} : \dot{\mathbf{Man}} \to \mathbf{C^\infty Sch^{aff}}$ to the category of affine C^∞-schemes, mapping $X \mapsto (X_{\rm top}, \mathcal{O}_X)$ on objects and $f \mapsto (f_{\rm top}, f^\sharp)$ on morphisms. This functor is faithful, but need not be full.*

This will help us to relate the (m-)Kuranishi spaces of Chapters 4 and 6 to the d-manifolds and d-orbifolds of [57, 58, 61].

B.1.4 Partitions of unity

Definition B.6. Let $X \in \dot{\mathbf{Man}}$. Then as in §B.1.1 we have an $\mathbb{R}$-algebra $C^\infty(X)$, which as in §B.1.3 is the global sections $C^\infty(X) = \mathcal{O}_X(X_{\rm top})$ of a sheaf of $\mathbb{R}$-algebras $\mathcal{O}_X$ on $X_{\rm top}$. Hence by sheaf theory each $\eta \in C^\infty(X)$ has a *support* $\operatorname{supp}\eta \subseteq X_{\rm top}$, a closed subset of $X_{\rm top}$, such that $X_{\rm top} \setminus \operatorname{supp} f$ is the largest open set $U' \subseteq X_{\rm top}$ with $\eta|_{U'} = 0$ in $\mathcal{O}_X(U')$.

Consider formal sums $\sum_{a\in A} \eta_a$ with $\eta_a \in C^\infty(X)$ for all a in a possibly infinite indexing set A. Such a sum is called *locally finite* if we can cover $X_{\rm top}$ by open $U' \subseteq X_{\rm top}$ such that $U' \cap \operatorname{supp}\eta_a = \emptyset$ for all but finitely many $a \in A$. By sheaf theory, for a locally finite sum $\sum_{a\in A} \eta_a$ there is a unique $\eta \in C^\infty(X)$ with $\sum_{a\in A} \eta_a|_{U'} = \eta|_{U'}$ whenever $U' \subseteq X_{\rm top}$ is open with $\eta_a|_{U'} = 0$ for all but finitely many $a \in A$, so that $\sum_{a\in A} \eta_a|_{U'}$ makes sense. We write $\sum_{a\in A} \eta_a = \eta$.

Let $\{U'_a : a \in A\}$ be an open cover of $X_{\rm top}$. A *partition of unity $\{\eta_a : a \in A\}$ on X subordinate to $\{U'_a : a \in A\}$* is $\eta_a \in C^\infty(X)$ with $\operatorname{supp}\eta_a \subseteq U'_a$ for all $a \in A$, with $\eta_{a,\rm top}(x) \geqslant 0$ in $\mathbb{R}$ for all $x \in X_{\rm top}$, such that $\sum_{a\in A} \eta_a$ is locally finite with $\sum_{a\in A} \eta_a = 1$ in $C^\infty(X)$.

The next proposition can be proved following the standard method for constructing partitions of unity on smooth manifolds, as in Lang [70, §II.3] or Lee [71, Th. 2.23], or alternatively follows from Proposition B.5 and results on partitions of unity on C^∞-schemes in [65, §4.7]. The important points are:

- By Assumption 3.2(b), $X_{\rm top}$ is Hausdorff, locally compact, and second countable, which is used in [70, Th. II.1] and [71, Th. 1.15].
- Let $U' \subseteq X_{\rm top}$ be open and $x \in U'$. Assumption 3.6 gives $a : X \to \mathbb{R}$ in $\dot{\mathbf{Man}}$ with $a_{\rm top}(x) > 0$ and $a_{\rm top}|_{X_{\rm top}\setminus U'} \leqslant 0$. Define $b : \mathbb{R} \to \mathbb{R}$ by $b(x) = e^{-1/x}$ for $x > 0$ and $b(x) = 0$ for $x \leqslant 0$. Then b is a morphism in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$ by Assumption 3.4, so $b \circ a : X \to \mathbb{R}$ is a morphism in $\dot{\mathbf{Man}}$. We have $(b \circ a)_{\rm top}(x) > 0$, and $(b \circ a)_{\rm top}(x') \geqslant 0$ for all $x' \in X_{\rm top}$, and $\mathrm{supp}(b \circ a) \subseteq U'$. Thus we can construct 'bump functions' on X.

This and Proposition B.5 are the main places we use Assumption 3.6.

Proposition B.7. *Let X be an object of $\dot{\mathbf{Man}}$, and $\{U'_a : a \in A\}$ be an open cover of $X_{\rm top}$. Then there exists a partition of unity $\{\eta_a : a \in A\}$ on X subordinate to $\{U'_a : a \in A\}$.*

Therefore $\mathcal{O}_X$ is a *fine sheaf*, and hence a *soft sheaf*, as in Godement [40, §II.3.7] or Bredon [10, §II.9], and all $\mathcal{O}_X$-modules $\mathcal{E}$ are also fine and soft.

B.2 Vector bundles

B.2.1 Vector bundles and sections

Definition B.8. Let X be an object in $\dot{\mathbf{Man}}$. A *vector bundle $E \to X$ of rank m* is a morphism $\pi : E \to X$ in $\dot{\mathbf{Man}}$, such that for each $x \in X_{\rm top}$ the topological fibre $E_{x,\rm top} := \pi_{\rm top}^{-1}(x) \subseteq E_{\rm top}$ is given the structure of a real vector space of dimension m, and X may be covered by open submanifolds $i : U \hookrightarrow X$, such that if $j : E_U \hookrightarrow E$ is the open submanifold corresponding to $\pi_{\rm top}^{-1}(U_{\rm top}) \subseteq E_{\rm top}$, and $k : E_U \to U$ is unique with $i \circ k = \pi \circ j : E_U \to X$ by Assumption 3.2(d), then there is an isomorphism $l : U \times \mathbb{R}^m \to E_U$ in $\dot{\mathbf{Man}}$ making the following diagram commute:

$$\begin{tikzcd} U \times \mathbb{R}^m \arrow[r, "l"'] \arrow[d, "\pi_U"] & E_U \arrow[r, hook, "j"'] \arrow[d, "k"] & E \arrow[d, "\pi"'] \\ U \arrow[r, equal] & U \arrow[r, hook, "i"] & X, \end{tikzcd}$$

and $l_{\rm top}$ identifies the vector space structure on $\{x\} \times \mathbb{R}^m \cong \mathbb{R}^m$ with that on $E_{x,\rm top}$, for each $x \in U_{\rm top}$.

The vector space structure on $E_{x,\rm top}$ may be encoded in morphisms $\mu_+, \mu_\cdot, z$ in $\dot{\mathbf{Man}}$ as follows. Addition '+' in $E_{x,\rm top}$ corresponds to a morphism $\mu_+ : E \times_{\pi,X,\pi} E \to E$, where the fibre product exists in $\dot{\mathbf{Man}}$, with $\mu_{+,\rm top}(v,w) = v+w$ for all $x \in X_{\rm top}$ and $v, w \in E_{x,\rm top}$. Multiplication by real numbers '$\cdot$' corresponds to a morphism $\mu_\cdot:\mathbb{R}\times E\to E$, with $\mu_{\cdot,\mathrm{top}}(\lambda,v)=\lambda\cdot v$ for all $\lambda\in\mathbb{R}$, $x\in X_{\mathrm{top}}$ and $v\in E_{x,\mathrm{top}}$. The zero element $0\in E_{x,\mathrm{top}}$ comes from $0_E:X\to E$ with $0_{E,\mathrm{top}}(x)=0\in E_{x,\mathrm{top}}$ for all $x\in X_{\mathrm{top}}$.

A *section* of E is a morphism $s:X\to E$ in $\dot{\mathbf{Man}}$ with $\pi\circ s=\mathrm{id}_X$. Write $\Gamma^\infty(E)$ for the set of sections of E. For $C^\infty(X)$ as in §B.1.1, if $a\in C^\infty(X)$ and $s,t\in\Gamma^\infty(E)$, we define $a\cdot s, s+t\in\Gamma^\infty(E)$ by the commutative diagrams

$$\begin{tikzcd}
 & \mathbb{R}\times E \arrow[d, "\mu_\cdot"] \\
X \arrow[ur, "{(a,s)}"] \arrow[r, "a\cdot s"'] & E,
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & E \arrow[dr, "\pi"] & \\
X \arrow[urrr, bend left=10, "s"] \arrow[rr, dotted, "u"] \arrow[drrr, "t"'] \arrow[dr, dotted, "s+t"'] & & E\times_{\pi,X,\pi}E \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[dl, "\mu_+"] & & X, \\
 & E & & E \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where the morphism u exists by the universal property of $E\times_X E$.

At each point $x\in X_{\mathrm{top}}$ we have

$$(a\cdot s)_{\mathrm{top}}(x)=a_{\mathrm{top}}(x)\cdot s_{\mathrm{top}}(x),\quad (s+t)_{\mathrm{top}}(x)=s_{\mathrm{top}}(x)+t_{\mathrm{top}}(x),$$

where on the right hand sides we use operations $\cdot,+$ in the $\mathbb{R}$-vector space $E_{x,\mathrm{top}}$. Thus for $a,b\in C^\infty(X)$ and $s,t,u\in\Gamma^\infty(E)$ we have

$$[a\cdot(b\cdot s)]_{\mathrm{top}}=[(a\cdot b)\cdot s]_{\mathrm{top}},\ \ [s+t]_{\mathrm{top}}=[t+s]_{\mathrm{top}},\ \ [s+(t+u)]_{\mathrm{top}}=[(s+t)+u]_{\mathrm{top}}$$

in maps $X_{\mathrm{top}}\to E_{\mathrm{top}}$, by identities in $E_{x,\mathrm{top}}$ for each $x\in X_{\mathrm{top}}$. Faithfulness in Assumption 3.2(a) implies the corresponding identities in $\dot{\mathbf{Man}}$. Therefore $\Gamma^\infty(E)$ is a $C^\infty(X)$-module, and hence an $\mathbb{R}$-vector space. We will write $0_E:X\to E$ for the zero section, the element $0\in\Gamma^\infty(E)$.

If $E,F\to X$ are vector bundles, a *morphism of vector bundles* $\theta:E\to F$ is a morphism $\theta:E\to F$ in $\dot{\mathbf{Man}}$ in a commutative diagram

$$\begin{tikzcd}
E \arrow[r, "\theta"'] \arrow[d, "\pi"] & F \arrow[d, "\pi"'] \\
X \arrow[r, equal] & X,
\end{tikzcd}$$

such that $\theta_{\mathrm{top}}|_{E_{x,\mathrm{top}}}:E_{x,\mathrm{top}}\to F_{x,\mathrm{top}}$ is a linear map for all $x\in X_{\mathrm{top}}$. We write $\mathrm{Hom}(E,F)$ for the set of vector bundle morphisms $\theta:E\to F$. As for $\Gamma^\infty(E)$, $\mathrm{Hom}(E,F)$ is naturally a $C^\infty(X)$-module, and hence an $\mathbb{R}$-vector space. If $\theta:E\to F$ is a vector bundle morphism and $s\in\Gamma^\infty(E)$ then $\theta\circ s\in\Gamma^\infty(F)$.

The usual operations on vector bundles and sections in differential geometry also work for vector bundles in $\dot{\mathbf{Man}}$, so for instance if $E,F\to X$ are vector bundles we can define vector bundles $E^*\to X$, $E\oplus F\to X$, $E\otimes F\to X$, $\Lambda^kE\to X$, and so on, and if $f:X\to Y$ is a morphism in $\dot{\mathbf{Man}}$ and $G\to Y$ is a vector bundle we can define a pullback vector bundle $f^*(G)\to X$. To construct $E^*,E\oplus F,\ldots$ as objects of $\dot{\mathbf{Man}}$, we build them using Assumptions 3.2(e) and 3.3(b) over an open cover $\{U_a:a\in A\}$ of X with $E,F\to X$ trivial over each U_a, by gluing together $U_a\times(\mathbb{R}^m)^*,U_a\times(\mathbb{R}^m\oplus\mathbb{R}^n),\ldots$ for all $a\in A$.

B.2.2 The sheaf of sections of a vector bundle

Definition B.9. Let X be an object in $\dot{\mathbf{Man}}$, and $E \to X$ be a vector bundle of rank r. Then for each open $U' \subseteq X_{\rm top}$, Assumption 3.2(d) gives an open submanifold $i : U \hookrightarrow X$ with $U_{\rm top} = U'$. Let $E|_U = i^*(E)$ as a vector bundle over U, and write $\mathcal{E}(U') = \Gamma^\infty(E|_U)$, considered as a module over $\mathcal{O}_X(U') = C^\infty(U)$.

For open $V' \subseteq U' \subseteq X_{\rm top}$ we have open submanifolds $i : U \hookrightarrow X$, $j : V \hookrightarrow X$ with $\mathcal{O}_X(U') = C^\infty(U)$ and $\mathcal{O}_X(V') = C^\infty(V)$. Since $V_{\rm top} \subseteq U_{\rm top}$ Assumption 3.2(d) gives a unique $k : V \to U$ in $\dot{\mathbf{Man}}$ with $i \circ k = j : V \to X$. Define $\rho_{U'V'} : \mathcal{E}(U') \to \mathcal{E}(V')$ by $\rho_{U'V'} : s \mapsto k^*(s) = s|_V$. Then as for $\mathcal{O}_X$ in §B.1.3, this defines a sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules on $X_{\rm top}$, which is locally free of rank r.

For brevity, sheaves of $\mathcal{O}_X$-modules will just be called *$\mathcal{O}_X$-modules.*

As for vector bundles in algebraic geometry, working with vector bundles $E, F \to X$ is equivalent to working with the corresponding $\mathcal{O}_X$-modules $\mathcal{E}, \mathcal{F}$, and one can easily translate between the two languages. In particular:

- There is a 1-1 correspondence, up to canonical isomorphism, between vector bundles $E \to X$ of rank r and locally free $\mathcal{O}_X$-modules $\mathcal{E}$ of rank r.
- If $E, F \to X$ are vector bundles, and $\mathcal{E}, \mathcal{F}$ the corresponding $\mathcal{O}_X$-modules, there is a natural identification $\mathrm{Hom}(E, F) \cong \mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\mathcal{E}, \mathcal{F})$ between vector bundle morphisms $\theta : E \to F$ and $\mathcal{O}_X$-module morphisms $\tilde\theta : \mathcal{E} \to \mathcal{F}$. These identifications preserve composition of morphisms.
- If $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$ and $E \to Y$ is a vector bundle, with $\mathcal{E}$ the corresponding $\mathcal{O}_Y$-module, then the vector bundle $f^*(E) \to X$ corresponds to the $\mathcal{O}_X$-module $f_{\rm top}^{-1}(\mathcal{E}) \otimes_{f_{\rm top}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, using the morphism $f^\sharp : f_{\rm top}^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves of $\mathbb{R}$-algebras on $X_{\rm top}$ from §B.1.3.

As in [65, §5], a module over a C^∞-ring is simply a module over the associated $\mathbb{R}$-algebra. So for sheaves of $\mathcal{O}_X$-modules, it makes no difference whether we consider $\mathcal{O}_X$ in §B.1.3 to be a sheaf of $\mathbb{R}$-algebras or a sheaf of C^∞-rings.

B.3 The cotangent sheaf, and connections

B.3.1 The cotangent sheaf $\mathcal{T}^*X$

In §B.1.2–§B.1.3 we showed that if X is an object of $\dot{\mathbf{Man}}$ then $(X_{\rm top}, \mathcal{O}_X)$ is an affine C^∞-scheme in the sense of [13, 56, 65]. As in [65, §5.6], C^∞-schemes have a good notion of cotangent sheaf, which we will use as a substitute for the cotangent bundle T^*X of a classical manifold X. The next two definitions are taken from [65, §5.2 & §5.6].

Definition B.10. Suppose $\mathfrak{C}$ is a C^∞-ring, as in Definition B.2, and M a $\mathfrak{C}$-module. A *C^∞-derivation* is an $\mathbb{R}$-linear map $\mathrm{d} : \mathfrak{C} \to M$ such that whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and $c_1, \ldots, c_n \in \mathfrak{C}$, we have

$$\mathrm{d}\Phi_f(c_1, \ldots, c_n) = \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i. \tag{B.2}$$

Note that d is *not* a morphism of $\mathfrak{C}$-modules. We call such a pair M, d a *cotangent module* for $\mathfrak{C}$ if it has the universal property that for any C^∞-derivation $\mathrm{d}' : \mathfrak{C} \to M'$, there exists a unique morphism of $\mathfrak{C}$-modules $\lambda : M \to M'$ with $\mathrm{d}' = \lambda \circ \mathrm{d}$.

There is a natural construction for a cotangent module: we take M to be the quotient of the free $\mathfrak{C}$-module with basis of symbols $\mathrm{d}c$ for $c \in \mathfrak{C}$ by the $\mathfrak{C}$-submodule spanned by all expressions of the form $\mathrm{d}\Phi_f(c_1, \ldots, c_n) - \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i$ for $f : \mathbb{R}^n \to \mathbb{R}$ smooth and $c_1, \ldots, c_n \in \mathfrak{C}$. Thus cotangent modules exist, and are unique up to unique isomorphism. When we speak of 'the' cotangent module, we mean that constructed above. We write $\mathrm{d}_{\mathfrak{C}} : \mathfrak{C} \to \Omega_{\mathfrak{C}}$ for the cotangent module of $\mathfrak{C}$.

Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings with cotangent modules $\Omega_{\mathfrak{C}}, \mathrm{d}_{\mathfrak{C}}$, $\Omega_{\mathfrak{D}}, \mathrm{d}_{\mathfrak{D}}$, and $\phi : \mathfrak{C} \to \mathfrak{D}$ be a morphism of C^∞-rings. Then we may regard $\Omega_{\mathfrak{D}}$ as a $\mathfrak{C}$-module, and $\mathrm{d}_{\mathfrak{D}} \circ \phi : \mathfrak{C} \to \Omega_{\mathfrak{D}}$ as a C^∞-derivation. Thus by the universal property of $\Omega_{\mathfrak{C}}$, there exists a unique morphism of $\mathfrak{C}$-modules $\Omega_\phi : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{D}}$ with $\mathrm{d}_{\mathfrak{D}} \circ \phi = \Omega_\phi \circ \mathrm{d}_{\mathfrak{C}}$. If $\phi : \mathfrak{C} \to \mathfrak{D}$, $\psi : \mathfrak{D} \to \mathfrak{E}$ are morphisms of C^∞-rings then $\Omega_{\psi\circ\phi} = \Omega_\psi \circ \Omega_\phi : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{E}}$.

Definition B.11. Let X be an object in $\dot{\mathbf{Man}}$, so that $(X_{\mathrm{top}}, \mathcal{O}_X)$ is an affine C^∞-scheme as in §B.1.3. Define $\mathcal{PT}^*X$ to associate to each open $U \subseteq X_{\mathrm{top}}$ the cotangent module $\Omega_{\mathcal{O}_X(U)}$ of Definition B.10, regarded as a module over the C^∞-ring $\mathcal{O}_X(U)$, and to each inclusion of open sets $V \subseteq U \subseteq X_{\mathrm{top}}$ the morphism of $\mathcal{O}_X(U)$-modules $\Omega_{\rho_{UV}} : \Omega_{\mathcal{O}_X(U)} \to \Omega_{\mathcal{O}_X(V)}$ associated to the morphism of C^∞-rings $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. Then the following commutes:

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \Omega_{\mathcal{O}_X(U)} \arrow[r, "\mu_{\mathcal{O}_X(U)}"'] \arrow[d, "\rho_{UV}\times\Omega_{\rho_{UV}}"] & \Omega_{\mathcal{O}_X(U)} \arrow[d, "\Omega_{\rho_{UV}}"'] \\
\mathcal{O}_X(V) \times \Omega_{\mathcal{O}_X(V)} \arrow[r, "\mu_{\mathcal{O}_X(V)}"] & \Omega_{\mathcal{O}_X(V)},
\end{tikzcd}$$

where $\mu_{\mathcal{O}_X(U)}, \mu_{\mathcal{O}_X(V)}$ are the module actions of $\mathcal{O}_X(U), \mathcal{O}_X(V)$ on $\Omega_{\mathcal{O}_X(U)}$, $\Omega_{\mathcal{O}_X(V)}$. Using this and functoriality of cotangent modules $\Omega_{\psi\circ\phi} = \Omega_\psi \circ \Omega_\phi$ in Definition B.10, we see that $\mathcal{PT}^*X$ is a presheaf of $\mathcal{O}_X$-modules on X_{top}. Define the *cotangent sheaf* $\mathcal{T}^*X$ *of* X to be the sheafification of $\mathcal{PT}^*X$.

Define a morphism $\mathcal{P}\mathrm{d} : \mathcal{O}_X \to \mathcal{PT}^*X$ of presheaves of $\mathbb{R}$-vector spaces by

$$\mathcal{P}\mathrm{d}(U) = \mathrm{d}_{\Omega_{\mathcal{O}_X(U)}} : \mathcal{O}_X(U) \longrightarrow \mathcal{PT}^*X(U) = \Omega_{\mathcal{O}_X(U)},$$

and define the *de Rham differential* $\mathrm{d} : \mathcal{O}_X \to \mathcal{T}^*X$ to be the corresponding morphism of sheaves of $\mathbb{R}$-vector spaces on X_{top}. It satisfies (B.2) on each open $U \subseteq X_{\mathrm{top}}$. Note that although $\mathcal{O}_X, \mathcal{T}^*X$ are $\mathcal{O}_X$-modules, d is not a morphism of $\mathcal{O}_X$-modules, as (B.2) is not compatible with $\mathcal{O}_X$-linearity.

Example B.12. **(a)** If $\dot{\mathbf{Man}} = \mathbf{Man}$ and $X \in \mathbf{Man}$ then $\mathcal{T}^*X$ is canonically isomorphic as an $\mathcal{O}_X$-module to the sheaf of sections of the usual cotangent bundle $T^*X \to X$, as in §B.2.2. For general $\dot{\mathbf{Man}}$, if $X \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}$ then as the definition of $\mathcal{T}^*X$ happens entirely inside $\mathbf{Man} \subseteq \dot{\mathbf{M}}\mathbf{an}$, again $\mathcal{T}^*X$ is isomorphic to the sheaf of sections of T^*X.

(b) If $\dot{\mathbf{M}}\mathbf{an}$ is one of the following categories from Chapter 2:

$$\mathbf{Man^c}, \mathbf{Man^c_{in}}, \mathbf{Man^c_{st}}, \mathbf{Man^c_{st,in}}, \mathbf{Man^c_{we}}, \tag{B.3}$$

then as in §2.3 there are two notions of cotangent bundle $T^*X, {}^bT^*X$ of X in $\dot{\mathbf{M}}\mathbf{an}$. It turns out that $\mathcal{T}^*X$ is isomorphic to the sheaf of sections of T^*X.

(c) If $\dot{\mathbf{M}}\mathbf{an}$ is one of the following categories from §2.4:

$$\mathbf{Man^{gc}}, \mathbf{Man^{gc}_{in}}, \mathbf{Man^{ac}}, \mathbf{Man^{ac}_{in}}, \mathbf{Man^{ac}_{st}},$$
$$\mathbf{Man^{ac}_{st,in}}, \mathbf{Man^{c,ac}}, \mathbf{Man^{c,ac}_{in}}, \mathbf{Man^{c,ac}_{st}}, \mathbf{Man^{c,ac}_{st,in}},$$

then the cotangent bundle T^*X of $X \in \dot{\mathbf{M}}\mathbf{an}$ may not be defined, though the b-cotangent bundle ${}^bT^*X$ is. It turns out that $\mathcal{T}^*X$ need not be isomorphic to the sheaf of sections of any vector bundle on X in these cases.

B.3.2 Connections on vector bundles

We can use cotangent sheaves in §B.3.1 to define a notion of connection.

Definition B.13. Let X be an object in $\dot{\mathbf{M}}\mathbf{an}$, and $E \to X$ a vector bundle, and $\mathcal{E}$ the $\mathcal{O}_X$-module of sections of E as in §B.2.2. A *connection* ∇ on E is a morphism of sheaves of $\mathbb{R}$-vector spaces on $X_{\rm top}$:

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X,$$

such that if $U \subseteq X_{\rm top}$ is open and $a \in \mathcal{O}_X(U)$, $e \in \mathcal{E}(U)$ then

$$\nabla(a \cdot e) = a \cdot (\nabla e) + e \otimes (\mathrm{d}(U)a) \qquad \text{in } (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X)(U), \tag{B.4}$$

where $\mathrm{d} : \mathcal{O}_X \to \mathcal{T}^*X$ is the de Rham differential from §B.3.1.

Note that although $\mathcal{E}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X$ are $\mathcal{O}_X$-modules, ∇ is not a morphism of $\mathcal{O}_X$-modules, as (B.4) is not $\mathcal{O}_X(U)$-linear.

Proposition B.14. *Let $X \in \dot{\mathbf{M}}\mathbf{an}$ and $E \to X$ be a vector bundle. Then:*

- **(a)** *There exists a connection ∇ on E.*
- **(b)** *If ∇, ∇' are connections on E then $\nabla' = \nabla + \Gamma$, for $\Gamma : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X$ an $\mathcal{O}_X$-module morphism on $X_{\rm top}$.*
- **(c)** *If ∇ is a connection on E and $\Gamma : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X$ is an $\mathcal{O}_X$-module morphism then $\nabla' = \nabla + \Gamma$ is a connection on E.*

Proof. For (a), first suppose E is trivial, say $E = X \times \mathbb{R}^k \to X$. Then we can define a connection ∇ on E by

$$\nabla(U) : (e_1, \ldots, e_k) \longmapsto (\mathrm{d}(U)e_1, \ldots, \mathrm{d}(U)e_k)$$

whenever $U \subseteq X_{\rm top}$ is open and $e_1, \ldots, e_k \in \mathcal{O}_X(U)$, using the obvious identifications $\mathcal{E}(U) \cong \mathcal{O}_X(U)^k$ and $(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X)(U) \cong \mathcal{T}^*X(U)^k$.

In the general case, choose an open cover $\{U_a : a \in A\}$ of X by open submanifolds $U_a \hookrightarrow X$ such that $E|_{U_a} \to U_a$ is trivial for each $a \in A$. Then there exists a connection ∇_a on $E|_{U_a}$. As in §B.1.4 we can choose a partition of unity $\{\eta_a : a \in A\}$ on X subordinate to $\{U_a : a \in A\}$. It is now easy to check that $\nabla = \sum_{a \in A} \eta_a \cdot \nabla_a$ is a well defined connection on E.

For (b), define $\Gamma = \nabla' - \nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X$, as a sheaf of morphisms of $\mathbb{R}$-vector spaces. If $U \subseteq X_{\rm top}$ is open and $a \in \mathcal{O}_X(U)$, $e \in \mathcal{E}(U)$ then subtracting (B.4) for ∇, ∇' implies that $\Gamma(a \cdot e) = a \cdot (\Gamma e)$ in $(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{T}^*X)(U)$, as the $e \otimes (\mathrm{d}(U)a)$ terms cancel. Hence Γ is $\mathcal{O}_X$-linear, and a morphism of $\mathcal{O}_X$-modules. Part (c) follows by the same argument in reverse. $\square$

Example B.15. If $\dot{\mathbf{Man}} = \mathbf{Man}$ then connections ∇ on a vector bundle $E \to X$ in the sense of Definition B.13 are in canonical 1-1 correspondence with the usual notion of connections on E in differential geometry, with (B.4) the usual Leibniz rule for connections. The same holds if $\dot{\mathbf{Man}}$ lies in (B.3).

B.4 Tangent sheaves

Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$, and $E \to X$ a vector bundle. To define 2-morphisms of m-Kuranishi neighbourhoods in Chapter 4, we will (roughly) need a notion of 'vector bundle morphism $\Lambda : E \to f^*(TY)$', where TY is the 'tangent bundle' of Y. For general categories $\dot{\mathbf{Man}}$, there are two problems with this. Firstly, objects X in $\dot{\mathbf{Man}}$ may not have tangent vector bundles $TX \to X$. And secondly, there are examples such as $\dot{\mathbf{Man}} = \mathbf{Man^c}$ in which tangent bundles do exist, but $f^*(TY)$ is the wrong thing for our purpose.

Our solution is to define 'TX', and '$f^*(TY)$', and '$\mathrm{Hom}(E, f^*(TY))$' as sheaves on X, rather than as vector bundles:

(i) For each $X \in \dot{\mathbf{Man}}$ we will define a sheaf $\mathcal{T}X$ of $\mathcal{O}_X$-modules on $X_{\rm top}$ called the *tangent sheaf* of X. Sections of $\mathcal{T}X$ parametrize infinitesimal deformations of $\mathrm{id}_X : X \to X$ as a morphism in $\dot{\mathbf{Man}}$. If $\dot{\mathbf{Man}} = \mathbf{Man}$ then $\mathcal{T}X$ is the sheaf of smooth sections of the usual tangent bundle TX.

(ii) For each morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$ we will define a sheaf $\mathcal{T}_fY$ of $\mathcal{O}_X$-modules on $X_{\rm top}$ called the *tangent sheaf* of f. Sections of $\mathcal{T}_fY$ parametrize infinitesimal deformations of $f : X \to Y$. If $\dot{\mathbf{Man}} = \mathbf{Man}$ then $\mathcal{T}_fY$ is the sheaf of smooth sections of $f^*(TY)$.

(iii) For each morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$ and vector bundle $E \to X$ we define morphisms $E \to \mathcal{T}_fY$ as morphisms of sheaves of $\mathcal{O}_X$-modules.

In §B.3.1 we defined the cotangent sheaf $\mathcal{T}^*X$. In general $\mathcal{T}X$ *and* $\mathcal{T}^*X$ *are not dual to each other*, though there is a natural pairing $\mathcal{T}X \times \mathcal{T}^*X \to \mathcal{O}_X$. We define $\mathcal{T}^*X$ using morphisms $X \to \mathbb{R}$ in $\dot{\mathbf{Man}}$, and $\mathcal{T}X$ using morphisms $X \times \mathbb{R} \to X$ in $\dot{\mathbf{Man}}$, so $\mathcal{T}X$ and $\mathcal{T}^*X$ depend on different data in $\dot{\mathbf{Man}}$.

B.4.1 Defining the f-vector fields just as a set $\Gamma(\mathcal{T}_f Y)$

Definition B.16. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$. Consider commutative diagrams in $\dot{\mathbf{Man}}$ of the form

$$\begin{tikzcd} & & X \arrow[dll, "{(\mathrm{id}_X,0)}"'] \arrow[d, "l"] \arrow[drr, "f"] & & \\ X \times \mathbb{R} & & U \arrow[ll, hook', "i"'] \arrow[rr, "u"] & & Y, \end{tikzcd} \tag{B.5}$$

where $i : U \hookrightarrow X \times \mathbb{R}$ is an open submanifold with $X_{\mathrm{top}} \times \{0\} \subseteq U_{\mathrm{top}} \subseteq X_{\mathrm{top}} \times \mathbb{R}$, and unique $l : X \to U$ with $i \circ l = (\mathrm{id}_X, 0)$ exists by Assumption 3.2(d), and $u : U \to Y$ is a morphism in $\dot{\mathbf{Man}}$ with $u \circ l = f$. We also require that U_{top} can be written as a union of subsets $X'_{\mathrm{top}} \times (-\epsilon, \epsilon)$ in $X_{\mathrm{top}} \times \mathbb{R}$ for $X'_{\mathrm{top}} \subseteq X_{\mathrm{top}}$ open and $\epsilon > 0$ (this condition will only be used in the proof of Proposition B.43). For brevity we write such a diagram as the pair (U, u).

Define a binary relation $\approx$ on such pairs (U, u) by $(U, u) \approx (U', u')$ if for all $\tilde{x} \in X_{\mathrm{top}}$ there exists an open submanifold $j : V \hookrightarrow X \times \mathbb{R}^2$ and a morphism $v : V \to Y$ satisfying

$$\begin{aligned} (\tilde{x}, 0, 0) \in V_{\mathrm{top}}, \qquad & v_{\mathrm{top}}(x, s, -s) = f_{\mathrm{top}}(x) \quad \forall (x, s, -s) \in V_{\mathrm{top}}, \\ v_{\mathrm{top}}(x, s, 0) = u_{\mathrm{top}}(x, s) \quad & \forall (x, s) \in U_{\mathrm{top}} \text{ with } (x, s, 0) \in V_{\mathrm{top}}, \\ v_{\mathrm{top}}(x, 0, s') = u'_{\mathrm{top}}(x, s') \quad & \forall (x, s') \in U'_{\mathrm{top}} \text{ with } (x, 0, s') \in V_{\mathrm{top}}. \end{aligned} \tag{B.6}$$

We will show $\approx$ is an equivalence relation. Suppose (U, u) is a pair, and let $j : V \hookrightarrow X \times \mathbb{R}^2$ be the open submanifold and $v : V \to Y$ the morphism with

$$V_{\mathrm{top}} = \bigl\{(x, s, s') \in X_{\mathrm{top}} \times \mathbb{R}^2 : (x, s+s') \in U_{\mathrm{top}}\bigr\}, \quad v_{\mathrm{top}} : (x, s, s') \mapsto u_{\mathrm{top}}(x, s+s').$$

Then (V, v) implies that $(U, u) \approx (U, u)$, so $\approx$ is reflexive. By exchanging the two factors of $\mathbb{R}$ in $X \times \mathbb{R}^2$ we see that $(U, u) \approx (U', u')$ for pairs $(U, u), (U', u')$ implies that $(U', u') \approx (U, u)$, so $\approx$ is symmetric. Suppose $(U, u) \approx (U', u')$ and $(U', u') \approx (U'', u'')$. Then for each $\tilde{x} \in X_{\mathrm{top}}$ there exist (V, v) as above for $(U, u) \approx (U', u')$, and (V', v') for $(U', u') \approx (U'', u'')$. Apply Assumption 3.7(a) with $k = 3$ and $n = 1$ to obtain an open submanifold $k : W \hookrightarrow X \times \mathbb{R}^3$ and a morphism $w : W \to Y$ such that $(\tilde{x}, 0, 0, 0) \in W_{\mathrm{top}}$, and $w_{\mathrm{top}}(x, s, s', 0) = v_{\mathrm{top}}(x, s, s')$ if $(x, s, s') \in V_{\mathrm{top}}$ with $(x, s, s', 0)$ in W_{top}, and $w_{\mathrm{top}}(x, 0, s', s'') = v'_{\mathrm{top}}(x, s', s'')$ if $(x, s', s'') \in V'_{\mathrm{top}}$ with $(x, 0, s', s'')$ in W_{top}, and $w_{\mathrm{top}}(x, s, s', s'') = f_{\mathrm{top}}(x)$ if $(x, s, s', s'') \in W_{\mathrm{top}}$ with $s + s' + s'' = 0$.

Here we change variables in $\mathbb{R}^3$ from (s, s', s'') to $(y_1, y_2, y_3) = (s+s'+s'', s, s'')$ to apply Assumption 3.7(a), so that $w_{\mathrm{top}}(x, s, s', s'') = f_{\mathrm{top}}(x)$ when $s+s'+s'' = 0$ prescribes w_{top} when $y_1 = 0$, and $w_{\mathrm{top}}(x, 0, s', s'') = v'_{\mathrm{top}}(x, s', s'')$ prescribes w_{top} when $y_2 = 0$, and $w_{\mathrm{top}}(x, s, s', 0) = v_{\mathrm{top}}(x, s, s')$ prescribes w_{top} when $y_3 = 0$. Making W smaller, we suppose that $(x, s, s', 0) \in W_{\mathrm{top}}$ implies that $(x, s, s') \in V_{\mathrm{top}}$, and $(x, 0, s', s'') \in W_{\mathrm{top}}$ implies that $(x, s', s'') \in V'_{\mathrm{top}}$.

Let $j'' : V'' \hookrightarrow X \times \mathbb{R}^2$ be the open submanifold with

$$V''_{\mathrm{top}} = \bigl\{(x, s, s'') \in X_{\mathrm{top}} \times \mathbb{R}^2 : (x, s, 0, s'') \in W_{\mathrm{top}}\bigr\}.$$

Then Assumption 3.2(d) applied to $(\mathrm{id}_X \times \mathrm{id}_{\mathbb{R}} \times 0 \times \mathrm{id}_{\mathbb{R}}) \circ j'' : V'' \to X \times \mathbb{R}^3$ gives a morphism $h : V'' \to W$ in $\dot{\mathbf{M}}\mathbf{an}$ with $h_{\mathrm{top}}(x, s, s'') = (x, s, 0, s'')$. Define $v'' = w \circ h : V'' \to Y$. Then such (V'', v'') for all $\tilde{x} \in X_{\mathrm{top}}$ establish that $(U, u) \approx (U'', u'')$, since $(\tilde{x}, 0, 0) \in V''_{\mathrm{top}}$, and $v''_{\mathrm{top}}(x, s, 0) = w_{\mathrm{top}}(x, s, 0, 0) = v_{\mathrm{top}}(x, s, 0) = u_{\mathrm{top}}(x, s)$, and $v''_{\mathrm{top}}(x, 0, s'') = w_{\mathrm{top}}(x, 0, 0, s'') = v'_{\mathrm{top}}(x, 0, s'') = u''_{\mathrm{top}}(x, s'')$. Thus $\approx$ is transitive, and is an equivalence relation.

Write $[U, u]$ for the $\approx$-equivalence class of pairs (U, u) as above. Write $\Gamma(\mathcal{T}_f Y)$ for the set of all such $\approx$-equivalence classes $[U, u]$. (In §B.4.5 we will define a sheaf of $\mathcal{O}_X$-modules $\mathcal{T}_f Y$ on X_{top} whose global sections are this set $\Gamma(\mathcal{T}_f Y)$, but for now $\Gamma(\mathcal{T}_f Y)$ is just our notation for the set of all $[U, u]$.)

When $Y = X$ and $f = \mathrm{id}_X$, we write $\Gamma(\mathcal{T} X) = \Gamma(\mathcal{T}_{\mathrm{id}_X} X)$.

Example B.17. Here is how to understand Definition B.16 in the case that $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$. Then we can use tangent spaces and derivatives of maps. Consider a diagram (B.5) in $\mathbf{Man}$. Write points in $U \subseteq X \times \mathbb{R}$ as (x, s) with $x \in X$ and $s \in \mathbb{R}$. Then for each $x \in X$ with $f(x) = y \in Y$ we have $u(x, 0) = y \in Y$ and $\frac{\partial u}{\partial s}(x, 0) \in T_y Y = f^*(TY)|_x$. The map $\hat{u} : x \mapsto \frac{\partial u}{\partial s}(x, 0)$ is a smooth section $\hat{u}$ of the vector bundle $f^*(TY) \to X$.

Now let (U, u), (U', u') be two such diagrams, and $\hat{u}, \hat{u}' \in \Gamma^\infty(f^*(TY))$ the corresponding sections. Suppose $(U, u) \approx (U', u')$, and let $\tilde{x} \in X$ with $\tilde{y} = f(\tilde{x})$, so that there exist $j : V \hookrightarrow X \times \mathbb{R}^2$ and $v : V \to Y$ satisfying (B.6). Considering points $(\tilde{x}, s, s') \in V$ with $v(\tilde{x}, s, s') \in Y$, we have $\frac{\partial v}{\partial s}(\tilde{x}, 0, 0), \frac{\partial v}{\partial s'}(\tilde{x}, 0, 0) \in T_{\tilde{y}} Y$. Differentiating (B.6) in s, s' at $(\tilde{x}, 0, 0)$ yields

$$\begin{gathered}\tfrac{\partial v}{\partial s}(\tilde{x}, 0, 0) - \tfrac{\partial v}{\partial s'}(\tilde{x}, 0, 0) = 0, \qquad \tfrac{\partial v}{\partial s}(\tilde{x}, 0, 0) = \tfrac{\partial u}{\partial s}(\tilde{x}, 0) = \hat{u}(\tilde{x}) \\ \text{and} \qquad \tfrac{\partial v}{\partial s'}(\tilde{x}, 0, 0) = \tfrac{\partial u'}{\partial s'}(\tilde{x}, 0) = \hat{u}'(\tilde{x}),\end{gathered}$$

so that $\hat{u}(\tilde{x}) = \hat{u}'(\tilde{x})$, for all $\tilde{x} \in X$. Thus $(U, u) \approx (U', u')$ forces $\hat{u} = \hat{u}'$ in $\Gamma^\infty(f^*(TY))$. Conversely one can show that $\hat{u} = \hat{u}'$ implies $(U, u) \approx (U', u')$. Also every $\hat{u} \in \Gamma^\infty(f^*(TY))$ comes from some (U, u) in (B.5). Hence $\approx$-equivalence classes $[U, u]$ are in 1-1 correspondence with $\hat{u} \in \Gamma^\infty(f^*(TY))$ by $[U, u] \mapsto \hat{u}$. So we can identify $\Gamma(\mathcal{T}_f Y)$ with $\Gamma^\infty(f^*(TY))$ when $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$.

B.4.2 Making $\Gamma(\mathcal{T}_f Y)$ into a $C^\infty(X)$-module

Section B.1.1 discussed the $\mathbb{R}$-algebra $C^\infty(X)$. We will give $\Gamma(\mathcal{T}_f Y)$ in §B.4.1 the structure of a $C^\infty(X)$-module.

Definition B.18. We continue in the situation of Definition B.16. To make $\Gamma(\mathcal{T}_f Y)$ into a $C^\infty(X)$-module we must define the product $a \cdot \alpha$ in $\Gamma(\mathcal{T}_f Y)$ for all $a \in C^\infty(X)$ and $\alpha \in \Gamma(\mathcal{T}_f Y)$, the sum $\alpha + \beta$ in $\Gamma(\mathcal{T}_f Y)$ for all $\alpha, \beta \in \Gamma(\mathcal{T}_f Y)$, and the zero element $0 \in \Gamma(\mathcal{T}_f Y)$, and verify they satisfy

$$\begin{gathered}\alpha + \beta = \beta + \alpha, \quad (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma), \\ 0_X \cdot \alpha = 0, \quad 1_X \cdot \alpha = \alpha, \quad a \cdot (b \cdot \alpha) = (a \cdot b) \cdot \alpha, \\ (a + b) \cdot \alpha = (a \cdot \alpha) + (b \cdot \alpha), \quad a \cdot (\alpha + \beta) = (a \cdot \alpha) + (a \cdot \beta),\end{gathered} \tag{B.7}$$

for all $a, b \in C^\infty(X)$ and $\alpha, \beta, \gamma \in \Gamma(\mathcal{T}_f Y)$, where $0_X, 1_X \in C^\infty(X)$ are the morphisms $0, 1 : X \to \mathbb{R}$.

To define $a \cdot \alpha$, let $a \in C^\infty(X)$ and $\alpha \in \Gamma(\mathcal{T}_f Y)$, and let (U, u) in (B.5) represent $\alpha = [U, u]$. Write $\tilde{\imath} : \tilde{U} \hookrightarrow X \times \mathbb{R}$ for the open submanifold with

$$\tilde{U}_{\rm top} = \bigl\{(x, s) \in X_{\rm top} \times \mathbb{R} : (x, a_{\rm top}(x)s) \in U_{\rm top}\bigr\}.$$

Form the commutative diagram in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd}
X \times \mathbb{R} \arrow[d, "\mathrm{id}_X \times (a \cdot \mathrm{id}_{\mathbb{R}})"] & \tilde{U} \arrow[l, hook', "\tilde{\imath}"] \arrow[d, "\star"'] \arrow[dr, "\tilde{u}"' near start] & X \arrow[l, "\star"', "\tilde{l}"] \arrow[d, "f"] \arrow[dl, "l" near start] \arrow[ll, bend right, "(\mathrm{id}_X, 0)"'] \\
X \times \mathbb{R} & U \arrow[l, hook', "i"] \arrow[r, "u"] & Y,
\end{tikzcd} \tag{B.8}$$

where morphisms labelled '$\star$' exist by Assumption 3.2(d), and $\mathrm{id}_X \times (a \cdot \mathrm{id}_{\mathbb{R}})$ maps $(x, s) \mapsto (x, a_{\rm top}(x)s)$ on $X_{\rm top} \times \mathbb{R}$. Then $\tilde{U}, \tilde{\imath}, \tilde{l}, \tilde{u}$ are a diagram of type (B.5). Define $a \cdot \alpha = [\tilde{U}, \tilde{u}] \in \Gamma(\mathcal{T}_f Y)$.

To show this is well defined, we must prove that if (U', u') is another representative for α, so that $(U, u) \approx (U', u')$, and $(\tilde{U}', \tilde{u}')$ is constructed from $a, (U', u')$ as in (B.8), then $(\tilde{U}, \tilde{u}) \approx (\tilde{U}', \tilde{u}')$, so that $[\tilde{U}, \tilde{u}] = [\tilde{U}', \tilde{u}']$. We do this by combining the data $j : V \hookrightarrow X \times \mathbb{R}^2$, $v : V \to Y$ satisfying (B.6) showing that $(U, u) \approx (U', u')$ with (B.8), now using $\mathrm{id}_X \times (a \cdot \mathrm{id}_{\mathbb{R}}) \times (a \cdot \mathrm{id}_{\mathbb{R}}) : X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$ in place of the left hand column of (B.8), to construct $\tilde{\jmath}, \tilde{V}, \tilde{v}$ showing that $(\tilde{U}, \tilde{u}) \approx (\tilde{U}', \tilde{u}')$. So $a \cdot \alpha$ is well defined.

To define $\alpha + \beta$, let $\alpha, \beta \in \Gamma(\mathcal{T}_f Y)$, and let $(U, u), (\hat{U}, \hat{u})$ in (B.5) represent $\alpha = [U, u]$ and $\beta = [\hat{U}, \hat{u}]$. Assumption 3.7(a) with $k = 2$ and $m_1 = m_2 = 1$ applied to $(U_1, u_1) = (U, u)$ and $(U_2, u_2) = (\hat{U}, \hat{u})$ gives an open $j : V \hookrightarrow X \times \mathbb{R}^2$ and $v : V \to Y$ such that $X_{\rm top} \times \{(0, 0)\} \subseteq V_{\rm top}$ and $v_{\rm top}(x, s, 0) = u_{\rm top}(x, s)$ for all (x, s) in $U_{\rm top}$ with $(x, s, 0)$ in $V_{\rm top}$ and $v_{\rm top}(x, 0, s) = \hat{u}_{\rm top}(x, s)$ for all (x, s) in $\hat{U}_{\rm top}$ with $(x, 0, s)$ in $V_{\rm top}$. Let $\check{\imath} : \check{U} \to X \times \mathbb{R}$ be the open submanifold with

$$\check{U}_{\rm top} = \bigl\{(x, s) \in X_{\rm top} \times \mathbb{R} : (x, s, s) \in V_{\rm top} \subseteq X_{\rm top} \times \mathbb{R}^2\bigr\}.$$

Form the commutative diagram in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd}
X \times \mathbb{R} \arrow[d, "\mathrm{id}_X \times (\mathrm{id}_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}})"] & \check{U} \arrow[l, hook', "\check{\imath}"] \arrow[d, "\star"'] \arrow[dr, "\check{u}"' near start] & X \arrow[l, "\star"', "\check{l}"] \arrow[d, "f"] \arrow[dl, "m" near start] \\
X \times \mathbb{R}^2 & V \arrow[l, hook', "j"] \arrow[r, "v"] & Y,
\end{tikzcd}$$

where morphisms labelled '$\star$' exist by Assumption 3.2(d), and $\mathrm{id}_X \times (\mathrm{id}_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}})$ maps $(x, s) \mapsto (x, s, s)$ on $X_{\rm top} \times \mathbb{R}$. Then $\check{U}, \check{\imath}, \check{l}, \check{u}$ are a diagram (B.5). Write $\alpha + \beta = [\check{U}, \check{u}]$ in $\Gamma(\mathcal{T}_f Y)$.

To show this is well defined, suppose $(U', u'), (\hat{U}', \hat{u}')$ are alternative representatives for α, β, so that $(U, u) \approx (U', u')$ and $(\hat{U}, \hat{u}) \approx (\hat{U}', \hat{u}')$, and use (V', v') to construct $(\check{U}', \check{u}')$ from $(U', u'), (\hat{U}', \hat{u}')$ as above. We must prove that

$(\check{U}, \check{u}) \approx (\check{U}', \check{u}')$. Let $\tilde{x} \in X_{\mathrm{top}}$, and let $j : V \hookrightarrow X \times \mathbb{R}^2$, $v : V \to Y$ satisfy (B.6) for $(U, u) \approx (U', u')$, and $\hat{\jmath} : \hat{V} \hookrightarrow X \times \mathbb{R}^2$, $\hat{v} : \hat{V} \to Y$ satisfy (B.6) for $(\hat{U}, \hat{u}) \approx (\hat{U}', \hat{u}')$. We will apply Assumption 3.7(a) five times to construct an open submanifold $k : W \hookrightarrow X \times \mathbb{R}^4$ with $(\tilde{x}, 0, 0, 0, 0) \in W_{\mathrm{top}}$ and a morphism $w : W \to Y$, such that for all $x \in X_{\mathrm{top}}$ and $q, r, s, t \in \mathbb{R}$ in the appropriate open sets we have

$$\begin{aligned} w_{\mathrm{top}}(x, q, 0, 0, 0) &= u_{\mathrm{top}}(x, q), & w_{\mathrm{top}}(x, 0, r, 0, 0) &= \hat{u}_{\mathrm{top}}(x, r), \\ w_{\mathrm{top}}(x, q, q, 0, 0) &= \check{u}_{\mathrm{top}}(x, q), & w_{\mathrm{top}}(x, 0, 0, s, 0) &= u'_{\mathrm{top}}(x, s), \\ w_{\mathrm{top}}(x, 0, 0, 0, t) &= \hat{u}'_{\mathrm{top}}(x, t), & w_{\mathrm{top}}(x, 0, 0, s, s) &= \check{u}'_{\mathrm{top}}(x, s), \\ w_{\mathrm{top}}(x, q, r, 0, 0) &= v_{\mathrm{top}}(x, q, r), & w_{\mathrm{top}}(x, 0, 0, s, t) &= \hat{v}_{\mathrm{top}}(x, s, t), \\ w_{\mathrm{top}}(x, q, 0, s, 0) &= v_{\mathrm{top}}(x, q, s), & w_{\mathrm{top}}(x, 0, r, 0, t) &= \hat{v}''_{\mathrm{top}}(x, r, t), \\ & & w_{\mathrm{top}}(x, q, r, -q, -r) &= f_{\mathrm{top}}(x). \end{aligned} \tag{B.9}$$

We do this in the following steps:

(a) Choose values of $w_{\mathrm{top}}(x, q, r, -q, t)$ to satisfy the second, fifth, tenth, and eleventh equations of (B.9), using Assumption 3.7(a) with $k = 2$, $n = 1$ and $X \times \mathbb{R}$ with variables $(x, x') \in X'_{\mathrm{top}} = X_{\mathrm{top}} \times \mathbb{R}$ in place of X, and variables $(x, q, r, -q, t) = (x, z_1, z_2 + x', -z_1, -x')$.

(b) Choose values of $w_{\mathrm{top}}(x, q, 0, s, t)$ to satisfy the first, fourth, fifth, sixth, eighth and ninth equations of (B.9), using Assumption 3.7(a) with $k = 2$, $n = 1$ and $X \times \mathbb{R}$ with variables $(x, x') \in X'_{\mathrm{top}} = X_{\mathrm{top}} \times \mathbb{R}$ in place of X, and variables $(x, q, 0, s, t) = (x, z_1, 0, x', z_2)$.

(c) Choose values of $w_{\mathrm{top}}(x, q, r, 0, t)$ to satisfy the first, second, third, fifth, seventh and tenth equations of (B.9), and with $w_{\mathrm{top}}(x, q, 0, 0, t)$ as already determined in (b), using Assumption 3.7(a) with $k = 3$, $n = 1$ and variables $(x, q, r, 0, t) = (x, z_1, z_2, 0, z_3)$.

(d) Choose values of $w_{\mathrm{top}}(x, q, r, s, 0)$ to satisfy the first–fourth, seventh and ninth equations of (B.9), and with $w_{\mathrm{top}}(x, q, r, -q, 0)$ as already determined in (a), using Assumption 3.7(a) with $k = 3$, $n = 1$ and variables $(x, q, r, s, 0) = (x, z_1 - z_3, z_2, z_3, 0)$.

(e) Choose values of $w_{\mathrm{top}}(x, q, r, s, t)$ agreeing with the choices made in (a)–(d), using Assumption 3.7(a) with $k = 4$, $n = 1$ and variables $(x, q, r, s, t) = (x, z_1 - z_3, z_2, z_3, z_4)$.

Write $\ddot{\jmath} : \ddot{V} \to X \times \mathbb{R}^2$ for the open submanifold with

$$\ddot{V}_{\mathrm{top}} = \bigl\{(x, q, s) \in X_{\mathrm{top}} \times \mathbb{R}^2 : (x, q, q, s, s) \in W_{\mathrm{top}} \subseteq X_{\mathrm{top}} \times \mathbb{R}^4\bigr\}.$$

Form the commutative diagram in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd} X \times \mathbb{R}^2 \arrow[d, "\mathrm{id}_X \times (\mathrm{id}_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}}) \times (\mathrm{id}_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}})"] & & \ddot{V} \arrow[ll, hook', "\ddot{\jmath}"'] \arrow[d, "\star"'] \arrow[drr, "\ddot{v}"] & & \\ X \times \mathbb{R}^4 & & W \arrow[ll, hook', "k"] \arrow[rr, "w"] & & Y, \end{tikzcd}$$

where the morphism '$\star$' exists by Assumption 3.2(d), and $\mathrm{id}_X \times (\mathrm{id}_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}}) \times (\mathrm{id}_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}})$ maps $(x, q, s) \mapsto (x, q, q, s, s)$ on $X_{\mathrm{top}} \times \mathbb{R}^2$. Then $\ddot{\jmath} : \ddot{V} \hookrightarrow X \times \mathbb{R}^2$ and $\ddot{v} : \ddot{V} \to Y$ satisfy (B.6) for $(\hat{U}, \hat{u}) \approx (\hat{U}', \hat{u}')$ at $\tilde{x} \in X_{\mathrm{top}}$, for all $\tilde{x}$. Hence $[\hat{U}, \hat{u}] = [\hat{U}', \hat{u}']$, and $\alpha + \beta$ is well defined.

Define $0 \in \Gamma(\mathcal{T}_f Y)$ to be $0 = [X \times \mathbb{R}, f \circ \pi_X]$, so that (B.5) becomes

$$\begin{tikzcd}
 & X \arrow[dl, "(\mathrm{id}_X,0)"'] \arrow[d, "l"] \arrow[dr, "f"] & \\
X \times \mathbb{R} & X \times \mathbb{R} \arrow[l, hook', "\mathrm{id}"] \arrow[r, "f\circ\pi_X"] & Y.
\end{tikzcd}$$

This defines all the data $\cdot, +, 0$ of the $C^\infty(X)$-module structure on $\Gamma(\mathcal{T}_f Y)$. It is now a long but straightforward calculation to show that the axioms (B.7) hold, and we leave this as an exercise for the reader.

B.4.3 Action of $v \in \Gamma(\mathcal{T}_f Y)$ as an f-derivation

If X is a classical manifold and $\alpha \in \Gamma^\infty(TX)$ is a vector field then α acts as a derivation $\Delta_\alpha : C^\infty(X) \to C^\infty(X)$ (and in fact as a C^∞-*derivation*, as in the author [65, §5.2]). We prove a relative version of this for $\dot{\mathbf{Man}}$.

Definition B.19. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$, and $\alpha \in \Gamma(\mathcal{T}_f Y)$. We will define a map $\Delta_\alpha : C^\infty(Y) \to C^\infty(X)$. Write $\alpha = [U, u]$ for (U, u) as in (B.5). Let $a \in C^\infty(Y)$, so that $a : Y \to \mathbb{R}$ and $a \circ u : U \to \mathbb{R}$ are morphisms in $\dot{\mathbf{Man}}$. Apply Assumption 3.5 to $f = a \circ u : U \to \mathbb{R}$. By (3.1)–(3.2), this gives a morphism $g : U \to \mathbb{R}$ in $\dot{\mathbf{Man}}$ such that

$$g_{\mathrm{top}}(x,t) = \begin{cases} t^{-1}[(a \circ u)_{\mathrm{top}}(x,t) - (a \circ u)_{\mathrm{top}}(x,0)], & t \neq 0, \\ \frac{\partial}{\partial t}(a \circ u)_{\mathrm{top}}(x,t), & t = 0, \end{cases} \tag{B.10}$$

and this determines g uniquely, by faithfulness in Assumption 3.2(a). Now define $\Delta_\alpha(a) = g \circ l : X \to \mathbb{R}$. Then $\Delta_\alpha(a) \in C^\infty(X)$, and (B.10) gives

$$\Delta_\alpha(a)_{\mathrm{top}}(x) = \tfrac{\partial}{\partial t}(a \circ u)_{\mathrm{top}}(x,t)|_{t=0} \qquad \text{for } x \in X_{\mathrm{top}}. \tag{B.11}$$

Let (U', u') be an alternative representative for α, and write $\Delta'_\alpha : C^\infty(Y) \to C^\infty(X)$ for the corresponding map. Then $(U, u) \approx (U', u')$, so by Definition B.16 for each $\tilde{x} \in X_{\mathrm{top}}$ there exist open $j : V \hookrightarrow X \times \mathbb{R}^2$ and $v : V \to Y$ satisfying (B.6). Then

$$\begin{aligned} \Delta_\alpha(a)_{\mathrm{top}}(\tilde{x}) &= \tfrac{\partial}{\partial s}(a \circ u)_{\mathrm{top}}(\tilde{x}, s)|_{s=0} = \tfrac{\partial}{\partial s}(a \circ v)_{\mathrm{top}}(\tilde{x}, s, 0)|_{s=0} \\ &= \tfrac{\partial}{\partial s'}(a \circ v)_{\mathrm{top}}(\tilde{x}, 0, s')|_{s'=0} = \tfrac{\partial}{\partial s'}(a \circ u')_{\mathrm{top}}(\tilde{x}, s')|_{s'=0} = \Delta'_\alpha(a)_{\mathrm{top}}(\tilde{x}), \end{aligned}$$

using (B.11) in the first and last steps, and differentiating (B.6) in s, s' at $s = s' = 0$ for the second–fourth. Hence $\Delta_\alpha = \Delta'_\alpha$, and Δ_α is well defined.

It is clear from (B.11) that $\Delta_\alpha : C^\infty(Y) \to C^\infty(X)$ is an $\mathbb{R}$-linear map. We will show in Proposition B.20 that it is both a derivation of $C^\infty(Y)$ as an $\mathbb{R}$-algebra, and a C^∞-derivation of $C^\infty(Y)$ as a C^∞-ring.

The next proposition follows easily from (B.11), the product and chain rules for differentiation, and Definition B.18.

Proposition B.20. *Work in the situation of Definition* B.19. *Then:*

(a) *Regard $f^*:C^\infty(Y)\to C^\infty(X)$ as a morphism of commutative $\mathbb{R}$-algebras as in* §B.1.1. *Then the $\mathbb{R}$-linear map $\Delta_\alpha:C^\infty(Y)\to C^\infty(X)$ satisfies*

$$\Delta_\alpha(a\cdot b)=f^*(a)\cdot\Delta_\alpha(b)+f^*(b)\cdot\Delta_\alpha(a)\quad\textit{for all } a,b\in C^\infty(Y).$$

*That is, Δ_α is a **relative derivation** for $f^*:C^\infty(Y)\to C^\infty(X)$.*

(b) *Regard $f^*:C^\infty(Y)\to C^\infty(X)$ as a morphism of C^∞-rings as in* §B.1.2, *and write the C^∞-ring operations on $C^\infty(X),C^\infty(Y)$ as Φ_g,Ψ_g respectively for smooth $g:\mathbb{R}^n\to\mathbb{R}$. Then $\Delta_\alpha:C^\infty(Y)\to C^\infty(X)$ satisfies*

$$\begin{aligned}\Delta_\alpha\bigl(\Psi_g(a_1,\dots,a_n)\bigr)&=\sum_{i=1}^n f^*\bigl(\Psi_{\frac{\partial g}{\partial x_i}}(a_1,\dots,a_n)\bigr)\cdot\Delta_\alpha(a_i)\\&=\sum_{i=1}^n \Phi_{\frac{\partial g}{\partial x_i}}(f^*(a_1),\dots,f^*(a_n))\cdot\Delta_\alpha(a_i)\end{aligned}\tag{B.12}$$

*for all $a_1,\dots,a_n\in C^\infty(Y)$. That is, Δ_α is a **relative C^∞-derivation** for $f^*:C^\infty(Y)\to C^\infty(X)$.*

(c) *If $\alpha,\beta\in\Gamma(\mathcal{T}_fY)$ then $\Delta_{\alpha+\beta}(a)=\Delta_\alpha(a)+\Delta_\beta(a)$ for all $a\in C^\infty(Y)$.*

(d) *If $a\in C^\infty(X)$ and $\alpha\in\Gamma(\mathcal{T}_fY)$ then $\Delta_{a\cdot\alpha}(b)=a\cdot\Delta_\alpha(b)$ for all $b\in C^\infty(Y)$.*

When $\dot{\mathbf{Man}}=\mathbf{Man}$, one can show that the map $\alpha\mapsto\Delta_\alpha$ is a 1-1 correspondence between elements of $\Gamma(\mathcal{T}_fY)$ and relative C^∞-derivations. But for general $\dot{\mathbf{Man}}$, it is not clear that $\alpha\mapsto\Delta_\alpha$ need be either injective or surjective.

B.4.4 Acting on modules $\Gamma(\mathcal{T}_fY)$ with morphisms in $\dot{\mathbf{Man}}$

Suppose $f:X\to Y$ and $g:Y\to Z$ are morphisms in $\dot{\mathbf{Man}}$. We will define natural morphisms $\Gamma(\mathcal{T}g):\Gamma(\mathcal{T}_fY)\to\Gamma(\mathcal{T}_{g\circ f}Z)$ and $f^*:\Gamma(\mathcal{T}_gZ)\to\Gamma(\mathcal{T}_{g\circ f}Z)$.

Definition B.21. Let $f:X\to Y$ and $g:Y\to Z$ be morphisms in $\dot{\mathbf{Man}}$. Sections B.4.1–B.4.2 define $C^\infty(X)$-modules $\Gamma(\mathcal{T}_fY)$ and $\Gamma(\mathcal{T}_{g\circ f}Z)$. Define a map $\Gamma(\mathcal{T}g):\Gamma(\mathcal{T}_fY)\to\Gamma(\mathcal{T}_{g\circ f}Z)$ by $\Gamma(\mathcal{T}g)([U,u])=[U,g\circ u]$. It is easy to check using §B.4.1–§B.4.2 that if $(U,u)\approx(U',u')$ then $(U,g\circ u)\approx(U',g\circ u')$, so that $\Gamma(\mathcal{T}g)$ is well-defined, and that it is a $C^\infty(X)$-module morphism.

For $[U,u] \in \Gamma(\mathcal{T}_g Z)$ defined by a pair (U,u) in a diagram (B.5) with Y,Z,g in place of X,Y,f, form the commutative diagram in $\dot{\mathbf{Man}}$:

$$
\begin{tikzcd}
 & X \arrow[dl, bend right, "{(\mathrm{id}_X,0)}"'] \arrow[d, "l'"'] \arrow[r, "f"] \arrow[drr, "g\circ f"] & Y \arrow[dr, bend left, "g"] \arrow[ddl, "{(\mathrm{id}_Y,0)}"'] \arrow[dd, "l"] & \\
X \times \mathbb{R} \arrow[dr, bend right, "f\times\mathrm{id}_{\mathbb{R}}"'] & U' \arrow[l, hook', "i'"'] \arrow[rr, "u'=u\circ m'"] \arrow[dr, "m'"] & & Z. \\
 & Y \times \mathbb{R} & U \arrow[l, hook', "i"'] \arrow[ur, bend right, "u"'] &
\end{tikzcd}
\tag{B.13}
$$

Here $i' : U' \hookrightarrow X \times \mathbb{R}$ is open with

$$U'_{\mathrm{top}} = \bigl\{(x,t) \in X_{\mathrm{top}} \times \mathbb{R} : (f_{\mathrm{top}}(x),t) \in U_{\mathrm{top}}\bigr\},$$

and unique l', m' exist making (B.13) commute by Assumption 3.2(d). Then U', i', l', u' in (B.13) are a diagram (B.5) for $g \circ f$, so that $[U',u'] \in \Gamma(\mathcal{T}_{g\circ f} Z)$. Define $f^*([U,u]) = [U',u']$.

To show that $[U',u']$ is independent of the choice of representative (U,u) for $[U,u]$, so that f^* is well defined, given another choice $(\hat{U},\hat{u})$ yielding $(\hat{U}',\hat{u}')$, as $(U,u) \approx (\hat{U},\hat{u})$ there exist V,v for each $\tilde{y} \in Y_{\mathrm{top}}$ satisfying (B.6) at $\tilde{y}$ over $g : Y \to Z$. Then for $\tilde{x} \in X_{\mathrm{top}}$ with $f_{\mathrm{top}}(\tilde{x}) = \tilde{y}$, we define V', v' satisfying (B.6) for $(U',u') \approx (\hat{U}',\hat{u}')$ at $\tilde{x}$ over $g \circ f : X \to Z$, by constructing V', v' from V, v in the same way that (B.13) generalizes (B.5). Hence $[U',u'] = [\hat{U}',\hat{u}']$, and $f^*([U,u])$ is well defined.

It is easy to check using §B.4.1–§B.4.2 that $f^*(\alpha+\beta) = f^*(\alpha) + f^*(\beta)$ and $f^*(a\cdot\alpha) = f^*(a)\cdot f^*(\alpha)$, for all $a \in C^\infty(Y)$ and $\alpha,\beta \in \Gamma(\mathcal{T}_g Z)$. That is, $f^* : \Gamma(\mathcal{T}_g Z) \to \Gamma(\mathcal{T}_{g\circ f} Z)$ is a module morphism relative to $f^* : C^\infty(Y) \to C^\infty(X)$.

If $e : W \to X$ is another morphism in $\dot{\mathbf{Man}}$, we see that

$$
\begin{aligned}
\Gamma(\mathcal{T}(g\circ f)) = \Gamma(\mathcal{T}g)\circ\Gamma(\mathcal{T}f) &: \Gamma(\mathcal{T}_e X) \longrightarrow \Gamma(\mathcal{T}_{g\circ f\circ e} Z),\\
(f\circ e)^* = e^*\circ f^* &: \Gamma(\mathcal{T}_g Z) \longrightarrow \Gamma(\mathcal{T}_{g\circ f\circ e} Z),\\
\Gamma(\mathcal{T}g)\circ e^* = e^*\circ\Gamma(\mathcal{T}g) &: \Gamma(\mathcal{T}_f Y) \longrightarrow \Gamma(\mathcal{T}_{g\circ f\circ e} Z).
\end{aligned}
\tag{B.14}
$$

Example B.22. If $f : X \to Y$ is a morphism in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$, we have $\Gamma(\mathcal{T}_f Y) \cong \Gamma^\infty(f^*(TY))$ as in Example B.17. For morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$, these isomorphisms identify

$$
\begin{aligned}
\Gamma(\mathcal{T}g) : \Gamma(\mathcal{T}_f Y) \to \Gamma(\mathcal{T}_{g\circ f} Z) \quad &\leftrightarrow \quad f^*(Tg)\circ : \Gamma^\infty(f^*(TY)) \to \Gamma^\infty((g\circ f)^*(TZ)),\\
f^* : \Gamma(\mathcal{T}_g Z) \to \Gamma(\mathcal{T}_{g\circ f} Z) \quad &\leftrightarrow \quad f^* : \Gamma^\infty(g^*(TZ)) \to \Gamma^\infty((g\circ f)^*(TZ)),
\end{aligned}
$$

where $Tg : TY \to g^*(TZ)$ is the derivative of g. This justifies the notation $\Gamma(\mathcal{T}g)$ and f^* in Definition B.21.

Lemma B.23. *Suppose $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\dot{\mathbf{Man}}$, with $g : Y \hookrightarrow Z$ an open submanifold. Then $\Gamma(\mathcal{T}g) : \Gamma(\mathcal{T}_f Y) \to \Gamma(\mathcal{T}_{g\circ f} Z)$ is an isomorphism of $C^\infty(X)$-modules.*

Proof. We will define an inverse map $I : \Gamma(\mathcal{T}_{g\circ f}Z) \to \Gamma(\mathcal{T}_f Y)$ for $\Gamma(\mathcal{T}g)$. Let $\alpha \in \Gamma(\mathcal{T}_{g\circ f}Z)$, and pick a representative (U, u) for $\alpha = [U, u]$, in a diagram (B.5). Let $i' : U' \hookrightarrow X \times \mathbb{R}$ be the open submanifold with

$$U'_{\mathrm{top}} = \bigl\{(x, t) \in X_{\mathrm{top}} \times \mathbb{R} : u_{\mathrm{top}}(x, t) \in Y_{\mathrm{top}} \subseteq Z_{\mathrm{top}}\bigr\}.$$

Then (B.5) extends to a commutative diagram

$$\begin{tikzcd}
 & & X \arrow[dll, "{(\mathrm{id}_X,0)}"'] \arrow[d, "l"] \arrow[drr, "g\circ f"] \arrow[ddl, "l'"'] \arrow[ddr, "f"] & & \\
X \times \mathbb{R} & & U \arrow[ll, hook', "i"'] \arrow[rr, "u"] & & Z, \\
 & U' \arrow[ul, hook, "i'"] \arrow[ur, hook, "j'"'] \arrow[rr, "u'"] & & Y \arrow[ur, hook, "g"'] &
\end{tikzcd} \tag{B.15}$$

where j', l', u' exist by Assumption 3.2(d) for the open submanifolds $i : U \hookrightarrow X \times \mathbb{R}$, $i' : U' \hookrightarrow X \times \mathbb{R}$ and $g : Y \hookrightarrow Z$ respectively. Then U', i', l', u' are a diagram (B.5) for $f : X \to Y$, so $[U', u'] \in \Gamma(\mathcal{T}_f Y)$. Define $I(\alpha) = [U', u']$.

A similar argument for V, v satisfying (B.6) shows $I(\alpha)$ is independent of the choice of (U, u), and so is well defined. To see that $\Gamma(\mathcal{T}g) \circ I = \mathrm{id}$, note that

$$\Gamma(\mathcal{T}g) \circ I(\alpha) = [U', g \circ u'] = [U', u \circ j'],$$

and use V, v in (B.6) with $v_{\mathrm{top}}(x, s, t) = u_{\mathrm{top}}(x, s + t)$ to show that $(U, u) \approx (U', u \circ j')$, so that $\Gamma(\mathcal{T}g) \circ I(\alpha) = [U, u] = \alpha$. To see that $I \circ \Gamma(\mathcal{T}g) = \mathrm{id}$, let $\beta = [U', u'] \in \Gamma(\mathcal{T}_f Y)$, so that $\Gamma(\mathcal{T}g)(\beta) = [U', g \circ u']$, and consider (B.15) with $U = U'$, $i = i'$, $l = l'$, $u = g \circ u'$ to see that $I \circ \Gamma(\mathcal{T}g)(\beta) = [U, u'] = \beta$. Therefore $\Gamma(\mathcal{T}g)$ is a bijection, and so an isomorphism of $C^\infty(X)$-modules. $\square$

B.4.5 The sheaves of $\mathcal{O}_X$-modules $\mathcal{T}X$ and $\mathcal{T}_f Y$

Next we define a sheaf of $\mathcal{O}_X$-modules $\mathcal{T}_f Y$ on X_{top}, with global sections $\mathcal{T}_f Y(X_{\mathrm{top}}) = \Gamma(\mathcal{T}_f Y)$. This justifies the notation $\Gamma(\mathcal{T}_f Y)$ in §B.4.1.

Definition B.24. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$. Section B.1.3 defines a sheaf of $\mathbb{R}$-algebras $\mathcal{O}_X$ on X_{top}. For each open submanifold $\chi' : X' \hookrightarrow X$ in $\dot{\mathbf{Man}}$, so that $X'_{\mathrm{top}} \subseteq X_{\mathrm{top}}$ is an open set and $f \circ \chi' : X' \to Y$ a morphism in $\dot{\mathbf{Man}}$, write $\mathcal{T}_f Y(X'_{\mathrm{top}}) = \Gamma(\mathcal{T}_{f\circ\chi'}Y)$ from Definition B.16, considered as a module over $\mathcal{O}_X(X'_{\mathrm{top}}) = C^\infty(X')$ as in Definition B.18. Note that when $\chi' : X' \hookrightarrow X$ is $\mathrm{id}_X : X \hookrightarrow X$ we have $\mathcal{T}_f Y(X_{\mathrm{top}}) = \Gamma(\mathcal{T}_f Y)$.

For each commutative triangle of open submanifolds in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd}
 & X' \arrow[dr, hook, "\chi'"] & \\
X'' \arrow[ur, hook, "\xi"] \arrow[rr, hook, "\chi''"] & & X,
\end{tikzcd} \tag{B.16}$$

using the notation of §B.4.4 define a map

$$\rho_{X'_{\mathrm{top}}X''_{\mathrm{top}}} = \xi^* : \mathcal{T}_f Y(X'_{\mathrm{top}}) = \Gamma(\mathcal{T}_{f\circ\chi'}Y) \longrightarrow \mathcal{T}_f Y(X''_{\mathrm{top}}) = \Gamma(\mathcal{T}_{f\circ\chi'\circ\xi}Y).$$

From §B.4.4, $\rho_{X'_{\rm top}X''_{\rm top}}$ intertwines the actions of $\mathcal{O}_X(X'_{\rm top}) = C^\infty(X')$ and $\mathcal{O}_X(X''_{\rm top}) = C^\infty(X'')$ on $\mathcal{T}_fY(X'_{\rm top}), \mathcal{T}_fY(X''_{\rm top})$ via the morphism $\rho_{X'_{\rm top}X''_{\rm top}} : \mathcal{O}_X(X'_{\rm top}) \to \mathcal{O}_X(X''_{\rm top})$ from §B.1.3.

Proposition B.25. *In Definition* B.24, *the data* $\mathcal{T}_fY(X'_{\rm top})$ *and* $\rho_{X'_{\rm top}X''_{\rm top}} : \mathcal{T}_fY(X'_{\rm top}) \to \mathcal{T}_fY(X''_{\rm top})$ *for all open* $X''_{\rm top} \subseteq X'_{\rm top} \subseteq X_{\rm top}$ *form a sheaf of* $\mathcal{O}_X$*-modules* $\mathcal{T}_fY$ *on* $X_{\rm top}$*, which we call the* ***tangent sheaf of*** f*. When* $Y = X$*,* $f = \mathrm{id}_X$*, we write* $\mathcal{T}X = \mathcal{T}_{\mathrm{id}_X}X$*, and call it the* ***tangent sheaf of*** X*.*

Proof. It is immediate from Definition B.24 and (B.14) that $\mathcal{T}_fY$ is a presheaf of $\mathcal{O}_X$-modules, that is, it satisfies Definition A.12(i)–(iii). Let $\chi' : X' \hookrightarrow X$ and $\chi''_a : X''_a \hookrightarrow X$ for $a \in A$ be open submanifolds with $\bigcup_{a\in A} X''_{a,\rm top} = X'_{\rm top}$, so that $\{X''_{a,\rm top} : a \in A\}$ is an open cover of $X'_{\rm top} \subseteq X_{\rm top}$. For each $a \in A$, as $X''_{a,\rm top} \subseteq X'_{\rm top} \subseteq X_{\rm top}$, Assumption 3.2(d) implies that there is a unique open submanifold $\xi_a : X''_a \hookrightarrow X'$ with $\chi''_a = \chi' \circ \xi_a$, as in (B.16).

For (iv), suppose $\alpha_1, \alpha_2 \in \mathcal{T}_fY(X'_{\rm top}) = \Gamma(\mathcal{T}_{f\circ\chi'}Y)$ with $\rho_{X'_{\rm top}X''_{a,\rm top}}(\alpha_1) = \rho_{X'_{\rm top}X''_{a,\rm top}}(\alpha_2)$ for all $a \in A$, so that $\xi_a^*(\alpha_1) = \xi_a^*(\alpha_2)$ in $\Gamma(\mathcal{T}_{f\circ\chi'\circ\xi_a}Y)$. Write $\alpha_c = [U_c, u_c]$ for $c = 1, 2$, where U_c, u_c live in a commutative diagram (B.5):

$$\begin{tikzcd} & X' \arrow[dl, "{(\mathrm{id}_{X'},0)}"'] \arrow[d, "l_c"] \arrow[dr, "f\circ\chi'"] & \\ X' \times \mathbb{R} & U_c \arrow[l, hook', "i_c"'] \arrow[r, "u_c"] & Y. \end{tikzcd}$$

From the definition of $\xi_a^*(\alpha_c)$ in §B.4.4, we see that if we define $h_{ac} : U_{ac} \hookrightarrow U_c$ to be the open submanifold with $U_{ac,\rm top} = U_{c,\rm top} \cap (X''_{a,\rm top} \times \mathbb{R}) \subseteq X'_{\rm top} \times \mathbb{R}$, then $\xi_a^*(\alpha_c) = [U_{ac}, u_c \circ h_{ac}]$. Hence $[U_{a1}, u_1 \circ h_{a1}] = [U_{a2}, u_2 \circ h_{a2}]$, so by Definition B.16, for each $\tilde{x} \in X''_{a,\rm top}$ there exist $j : V \hookrightarrow X''_a$ and $v : V \to Y$ satisfying (B.6). Then $\xi_a \circ j : V \hookrightarrow X'$ and $v : V \to Y$ satisfy (B.6) for $(U_1, u_1) \approx (U_2, u_2)$ at $\tilde{x} \in X'_{\rm top}$. As this holds for all $\tilde{x} \in X''_{a,\rm top}$, and $\bigcup_{a\in A} X''_{a,\rm top} = X'_{\rm top}$, we see that $\alpha_1 = [U_1, u_1] = [U_2, u_2] = \alpha_2$. Hence $\mathcal{T}_fY$ satisfies Definition A.12(iv).

For (v), suppose that $\alpha_a \in \mathcal{T}_fY(X''_{a,\rm top}) = \Gamma(\mathcal{T}_{f\circ\chi''_a}Y)$ for all $a \in A$ with

$$\rho_{X''_{a,\rm top}(X''_{a,\rm top}\cap X''_{b,\rm top})}(\alpha_a) = \rho_{X''_{b,\rm top}(X''_{a,\rm top}\cap X''_{b,\rm top})}(\alpha_b) \quad \text{for all } a, b \in A. \tag{B.17}$$

Write $\alpha_a = [U_a, u_a]$ for $a \in A$, where U_a, u_a live in a diagram (B.5):

$$\begin{tikzcd} & X''_a \arrow[dl, "{(\mathrm{id}_{X''_a},0)}"'] \arrow[d, "l_a"] \arrow[dr, "f\circ\chi''_a"] & \\ X''_a \times \mathbb{R} & U_a \arrow[l, hook', "i_a"'] \arrow[r, "u_a"] & Y. \end{tikzcd}$$

Let S_A be the set of all finite, nonempty subsets $B \subseteq A$. For each $B \in S_A$ write $\chi''_B : X''_B \hookrightarrow X'$ for the open submanifold with $X''_{B,\rm top} = \bigcap_{a\in B} X''_{a,\rm top}$. When $B = \{a\}$ we have $X''_{\{a\}} = X''_a$, $\chi''_{\{a\}} = \chi''_a$. If $C \subseteq B$ lie in S_A then there is a unique $\xi_{BC} : X''_B \hookrightarrow X''_C$ with $\chi''_B = \chi''_C \circ \xi_{BC}$ by Assumption 3.2(d).

For each $B \in S_A$ we will choose an open submanifold $k_B : W_B \hookrightarrow X''_B \times \prod_{b\in B} \mathbb{R}$ and a morphism $w_B : W_B \to Y$ in $\dot{\mathbf{Man}}$ with the properties:

(a) $X''_{B,\mathrm{top}} \times \{(0,\ldots,0)\} \subseteq W_{B,\mathrm{top}}$ for all $B \in S_A$.

(b) For $a \in A$ we have $W_{\{a\}} = U_a \hookrightarrow X''_a \times \mathbb{R} = X''_{\{a\}} \times \mathbb{R}$ and $w_{\{a\}} = u_a$.

(c) If $C \subsetneq B$ lie in S_A and $(x,(s_a)_{a\in C} \amalg (0)_{a\in B\setminus C}) \in W_{B,\mathrm{top}}$ then $(x,(s_a)_{a\in C})$ lies in $W_{C,\mathrm{top}}$ with $w_{C,\mathrm{top}}(x,(s_a)_{a\in C}) = w_{B,\mathrm{top}}(x,(s_a)_{a\in C} \amalg (0)_{a\in B\setminus C})$.

We do this by induction on $|B|$. For the first step, W_B, w_B are determined by (b) when $|B| = 1$, and (a) holds by definition of U_a, u_a. For the inductive step, suppose that $m \geqslant 1$ and we have chosen W_B, w_B for all $B \in S_A$ with $|B| \leqslant m$, such that (a),(c) hold whenever $|B| \leqslant m$. Let $B \in S_A$ with $|B| = m+1$, and write $B = \{a_1, \ldots, a_{m+1}\}$. Apply Assumption 3.7(a) with $k = m+1$, $n = 1$, and X''_B in place of X, taking $f_i : U_i \to Y$ to be the restriction of $w_{B\setminus\{a_i\}} : W_{B\setminus\{a_i\}} \to Y$ to the intersection of $W_{B\setminus\{a_i\}}$ with $X''_B \times \mathbb{R}^m$.

The compatibility condition between f_i, f_j in Assumption 3.7(a) follows from (c) above for $B \setminus \{a_i, a_j\} \subset B \setminus \{a_i\}$ and $B \setminus \{a_i, a_j\} \subset B \setminus \{a_j\}$. Therefore Assumption 3.7(a) gives W_B, w_B satisfying (a), and (c) when $C \subsetneq B$ with $|C| = m$. Then (c) for $|C| < m$ follows by taking $C \subsetneq B \setminus \{a_i\} \subsetneq B$. Hence by induction we can choose W_B, w_B satisfying (a)–(c) for all $B \in S_A$.

Now apply Proposition B.7 to choose a partition of unity $\{\eta_a : a \in A\}$ on X' subordinate to the open cover $\{X''_{a,\mathrm{top}} : a \in A\}$. Choose an open submanifold $i : U \hookrightarrow X' \times \mathbb{R}$ such that $X'_{\mathrm{top}} \times \{0\} \subseteq U_{\mathrm{top}}$ and if $(x,s) \in U_{\mathrm{top}}$ and $B = \bigl\{a \in A : x \in \operatorname{supp} \eta_{a,\mathrm{top}}\bigr\}$ then $(x,(\eta_{a,\mathrm{top}}(x)s)_{a\in B}) \in W_{B,\mathrm{top}}$. By (a) above and local finiteness of $\{\eta_a : a \in A\}$, this holds for any small enough open neighbourhood of $X'_{\mathrm{top}} \times \{0\}$ in $X' \times \mathbb{R}$.

We claim that there is a unique morphism $u : U \to Y$ in $\dot{\mathbf{Man}}$ such that for all $(x,s) \in U_{\mathrm{top}}$ with $B = \bigl\{a \in A : x \in \operatorname{supp} \eta_{a,\mathrm{top}}\bigr\}$ in S_A we have

$$u_{\mathrm{top}}(x,s) = w_{B,\mathrm{top}}(x,(\eta_{a,\mathrm{top}}(x)s)_{a\in B}). \tag{B.18}$$

To see this, note that as η_a for $a \in B$ and w_B are morphisms in $\dot{\mathbf{Man}}$, for each $B \in S_A$, equation (B.18) is the underlying continuous map of a morphism in $\dot{\mathbf{Man}}$ from an open submanifold of U to Y. Part (c) above implies that these continuous maps for $C \subseteq B$ agree on the overlap of their domains. If a point lies in the domain of the functions for $B, B' \in S_A$ then it lies in the domain for $B \cap B'$ by (c), and considering $B \cap B' \subseteq B$ and $B \cap B' \subseteq B'$ we see that the continuous maps for B, B' agree on the overlap of their domains. Hence by Assumption 3.3(a) there is a unique $u : U \to Y$ satisfying (B.18).

Now put $\alpha = [U,u] \in \mathcal{T}_f Y(X'_{\mathrm{top}}) = \Gamma(\mathcal{T}_{f\circ\chi'}Y)$. Fix $a \in A$, and let $\tilde{x} \in X''_{a,\mathrm{top}}$. Set $B = \bigl\{b \in A : \tilde{x} \in \operatorname{supp} \eta_{b,\mathrm{top}}\bigr\}$. Choose an open neighbourhood $R \hookrightarrow X''_a$ of $\tilde{x}$ in X''_a such that $R_{\mathrm{top}} \subseteq X''_{b,\mathrm{top}}$ for all $b \in B$, and $R_{\mathrm{top}} \cap \operatorname{supp} \eta_{c,\mathrm{top}} = \emptyset$ for all $c \in A \setminus B$. This is possible as $\operatorname{supp} \eta_{b,\mathrm{top}}$ is contained in $X''_{b,\mathrm{top}}$ and closed in

X'_{top}, and $\{\eta_a : a \in A\}$ is locally finite. We have

$$
\begin{aligned}
&\rho_{X''_{a,\mathrm{top}}R_{\mathrm{top}}} \circ \rho_{X'_{\mathrm{top}}X''_{a,\mathrm{top}}}(\alpha) = \rho_{X'_{\mathrm{top}}R_{\mathrm{top}}}(\alpha) = \textstyle\sum_{b\in B} \rho_{X''_{b,\mathrm{top}}R_{\mathrm{top}}}(\eta_b|_{X''_b} \cdot \alpha_b) \\
&= \textstyle\sum_{b\in B} \rho_{X'_{\mathrm{top}}R_{\mathrm{top}}}(\eta_b) \cdot \rho_{X''_{b,\mathrm{top}}R_{\mathrm{top}}}(\alpha_b) \\
&= \textstyle\sum_{b\in B} \rho_{X'_{\mathrm{top}}R_{\mathrm{top}}}(\eta_b) \cdot \rho_{(X''_{a,\mathrm{top}}\cap X''_{b,\mathrm{top}})R_{\mathrm{top}}} \circ \rho_{X''_{b,\mathrm{top}}(X''_{a,\mathrm{top}}\cap X''_{b,\mathrm{top}})}(\alpha_b) \\
&= \textstyle\sum_{b\in B} \rho_{X'_{\mathrm{top}}R_{\mathrm{top}}}(\eta_b) \cdot \rho_{(X''_{a,\mathrm{top}}\cap X''_{b,\mathrm{top}})R_{\mathrm{top}}} \circ \rho_{X''_{a,\mathrm{top}}(X''_{a,\mathrm{top}}\cap X''_{b,\mathrm{top}})}(\alpha_a) \\
&= \textstyle\sum_{b\in B} \rho_{X'_{\mathrm{top}}R_{\mathrm{top}}}(\eta_b) \cdot \rho_{X''_{a,\mathrm{top}}R_{\mathrm{top}}}(\alpha_a) = \rho_{X'_{\mathrm{top}}R_{\mathrm{top}}}\Bigl(\textstyle\sum_{b\in B} \eta_b\Bigr) \cdot \rho_{X''_{a,\mathrm{top}}R_{\mathrm{top}}}(\alpha_a) \\
&= \rho_{X'_{\mathrm{top}}R_{\mathrm{top}}}(1) \cdot \rho_{X''_{a,\mathrm{top}}R_{\mathrm{top}}}(\alpha_a) = \rho_{X''_{a,\mathrm{top}}R_{\mathrm{top}}}(\alpha_a).
\end{aligned}
\tag{B.19}
$$

Here the second step follows from comparing the definition (B.21) of $\alpha = [U, u]$ with the definitions of addition and multiplication by functions in $\Gamma(\mathcal{T}_{f|_R}Y)$ in §B.4.2, the fifth uses (B.17), the eighth holds as $\sum_{b\in B} \eta_b$ is 1 on R since $\{\eta_a : a \in A\}$ is a partition of unity with $R_{\mathrm{top}} \cap \operatorname{supp} \eta_{c,\mathrm{top}} = \emptyset$ for all $c \in A \setminus B$, and the other steps come from $\mathcal{T}_fY$ being a presheaf of $\mathcal{O}_X$-modules as above.

Since $X''_{a,\mathrm{top}}$ is covered by such open subsets $R_{\mathrm{top}} \subseteq X''_{a,\mathrm{top}}$, equation (B.19) and Definition A.12(iv) for $\mathcal{T}_fY$ (proved above) imply that $\rho_{X'_{\mathrm{top}}X''_{a,\mathrm{top}}}(\alpha) = \alpha_a$, for all $a \in A$. Therefore $\mathcal{T}_fY$ satisfies Definition A.12(v), and is a sheaf. $\square$

Here are some examples:

Example B.26. **(a)** When $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man}$, we have $\Gamma(\mathcal{T}_fY) \cong \Gamma^\infty(f^*(TY))$ as in Example B.17, and one can show that $\mathcal{T}_fY$ is canonically isomorphic to the sheaf of smooth sections of the vector bundle $f^*(TY) \to X$, so that $\mathcal{T}X$ is canonically isomorphic to the sheaf of smooth sections of $TX \to X$.

(b) When $\dot{\mathbf{M}}\mathbf{an}$ is one of the categories of manifolds with corners from Chapter 2:

$$\mathbf{Man^c_{in}}, \mathbf{Man^c_{st,in}}, \mathbf{Man^{gc}_{in}}, \mathbf{Man^{ac}_{in}}, \mathbf{Man^{ac}_{st,in}}, \mathbf{Man^{c,ac}_{in}}, \mathbf{Man^{c,ac}_{st,in}},$$

as in Example 3.8(ii), one can show that $\mathcal{T}_fY$ is the sheaf of smooth sections of the vector bundle $f^*({}^bTY) \to X$, so that $\mathcal{T}X$ is canonically isomorphic to the sheaf of smooth sections of the b-tangent bundle ${}^bTX \to X$.

(c) When $\dot{\mathbf{M}}\mathbf{an}$ is one of the categories of manifolds with corners from Chapter 2:

$$\mathbf{Man^c}, \mathbf{Man^c_{st}}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{ac}_{st}}, \mathbf{Man^{c,ac}}, \mathbf{Man^{c,ac}_{st}},$$

as in Example 3.8(ii), it turns out that $\mathcal{T}_fY$ is the sheaf of sections of the vector bundle of mixed rank $C(f)^*({}^bTC(Y))|_{C_0(X)} \to X$, using the corner functor $C(f) : C(X) \to C(Y)$ and the identification $X \cong C_0(X)$ from §2.2. If f is interior this reduces to $f^*({}^bTY) \to X$ as in **(b)**.

(d) When $\dot{\mathbf{M}}\mathbf{an} = \mathbf{Man^c_{we}}$ from §2.1, as in Example 3.8(ii), and $f : X \to Y$ in $\mathbf{Man^c_{we}}$ is weakly smooth but not smooth, in general $\mathcal{T}_fY$ is not even locally the sheaf of sections of a vector bundle on X.

B.4.6 Acting on sheaves $\mathcal{T}_f Y$ with morphisms in $\dot{\mathbf{Man}}$

We now lift the material of §B.4.4 from global sections $\Gamma(\mathcal{T}_f Y)$ to sheaves $\mathcal{T}_f Y$.

Definition B.27. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\dot{\mathbf{Man}}$. Define a morphism $\mathcal{T}g : \mathcal{T}_f Y \to \mathcal{T}_{g\circ f} Z$ of sheaves of $\mathcal{O}_X$-modules on $X_{\rm top}$ by, for each open submanifold $\chi' : X' \hookrightarrow X$ in $\dot{\mathbf{Man}}$,

$$\mathcal{T}g(X'_{\rm top}) = \Gamma(\mathcal{T}g) : \mathcal{T}_f Y(X'_{\rm top}) = \Gamma(\mathcal{T}_{f\circ\chi'} Y) \to \mathcal{T}_{g\circ f} Z(X'_{\rm top}) = \Gamma(\mathcal{T}_{g\circ f\circ\chi'} Z).$$

Using (B.14) we see that $\mathcal{T}g$ is a sheaf morphism.

On $Y_{\rm top}$ we have $\mathcal{T}_g Z$, a sheaf of $\mathcal{O}_Y$-modules, and $(f_{\rm top})_*(\mathcal{T}_{g\circ f} Z)$, a sheaf of $(f_{\rm top})_*(\mathcal{O}_X)$-modules. As in §B.1.3 we have a morphism $f_\sharp : \mathcal{O}_Y \to (f_{\rm top})_*(\mathcal{O}_X)$ of sheaves of $\mathbb{R}$-algebras or C^∞-rings on $Y_{\rm top}$. We will define a sheaf morphism $f_\flat : \mathcal{T}_g Z \to (f_{\rm top})_*(\mathcal{T}_{g\circ f} Z)$ on $Y_{\rm top}$ which is a module morphism under $f_\sharp$.

Let $\xi' : Y' \hookrightarrow Y$ be an open submanifold in $\dot{\mathbf{Man}}$, and let $\chi' : X' \hookrightarrow X$ be the open submanifold with $X'_{\rm top} = f_{\rm top}^{-1}(Y'_{\rm top}) \subseteq X_{\rm top}$. Then Assumption 3.2(d) gives a unique $f' : X' \to Y'$ with $\xi' \circ f' = f \circ \chi'$. Define

$$\begin{aligned} f_\flat(Y'_{\rm top}) = f'^* : \mathcal{T}_g Z(Y'_{\rm top}) = \Gamma(\mathcal{T}_{g\circ\xi'} Z) &\longrightarrow (f_{\rm top})_*(\mathcal{T}_{g\circ f} Z)(Y'_{\rm top}) \\ &= \mathcal{T}_{g\circ f} Z(X'_{\rm top}) = \Gamma(\mathcal{T}_{g\circ f\circ\chi'} Z) = \Gamma(\mathcal{T}_{g\circ\xi'\circ f'} Z). \end{aligned}$$

Using (B.14) we can prove that $f_\flat$ is a sheaf morphism. The module morphism property for $f_\flat$ follows from the corresponding property for f'^*.

Let $f^\flat : f_{\rm top}^{-1}(\mathcal{T}_g Z) \to \mathcal{T}_{g\circ f} Z$ on $X_{\rm top}$ be adjoint to $f_\flat : \mathcal{T}_g Z \to (f_{\rm top})_*(\mathcal{T}_{g\circ f} Z)$ under (A.18). Then $f_{\rm top}^{-1}(\mathcal{T}_g Z)$ is an $f_{\rm top}^{-1}(\mathcal{O}_Y)$-module, and $\mathcal{T}_{g\circ f} Z$ an $\mathcal{O}_X$-module, and $f^\flat$ is a module morphism under $f^\sharp : f_{\rm top}^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$.

If $e : W \to X$ is another morphism in $\dot{\mathbf{Man}}$, using (B.14) we can prove that

$$\begin{aligned} \mathcal{T}(g\circ f) = \mathcal{T}g \circ \mathcal{T}f : \mathcal{T}_e X &\longrightarrow \mathcal{T}_{g\circ f\circ e} Z, \\ (f\circ e)_\flat = (f_{\rm top})_*(e_\flat) \circ f_\flat : \mathcal{T}_g Z &\longrightarrow ((f\circ e)_{\rm top})_*(\mathcal{T}_{g\circ f\circ e} Z), \\ (e_{\rm top})_*(\mathcal{T}g) \circ e_\flat = e_\flat \circ \mathcal{T}g : \mathcal{T}_f Y &\longrightarrow (e_{\rm top})_*(\mathcal{T}_{g\circ f\circ e} Z). \end{aligned}$$

Using the adjoint property for $f_\flat, f^\flat$ above, the last two equations imply that

$$\begin{aligned} (f\circ e)^\flat = e^\flat \circ e_{\rm top}^{-1}(f^\flat) : (f\circ e)_{\rm top}^{-1}(\mathcal{T}_g Z) &\longrightarrow \mathcal{T}_{g\circ f\circ e} Z, \\ \mathcal{T}g \circ e^\flat = e^\flat \circ e_{\rm top}^{-1}(\mathcal{T}g) : e_{\rm top}^{-1}(\mathcal{T}_f Y) &\longrightarrow \mathcal{T}_{g\circ f\circ e} Z. \end{aligned}$$

Lemma B.23 implies:

Lemma B.28. *Suppose $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\dot{\mathbf{Man}}$, with $g : Y \hookrightarrow Z$ an open submanifold. Then $\mathcal{T}g : \mathcal{T}_f Y \to \mathcal{T}_{g\circ f} Z$ is an isomorphism of $\mathcal{O}_X$-modules.*

B.4.7 A pairing $\mu_X : \mathcal{T}X \times \mathcal{T}^*X \to \mathcal{O}_X$

Let $X \in \dot{\mathbf{Man}}$. In §B.3.1 we defined the cotangent sheaf $\mathcal{T}^*X$, and in §B.4.5 the tangent sheaf $\mathcal{T}X$, both $\mathcal{O}_X$-modules on $X_{\rm top}$. Note that in general neither is dual to the other. For example, when $\dot{\mathbf{Man}} = \mathbf{Man^c}$, as in Example B.12(b) $\mathcal{T}^*X$ is the sheaf of sections of the cotangent bundle $T^*X \to X$, and as in Example B.26(b),(c) $\mathcal{T}X$ is the sheaf of sections of the b-tangent bundle ${}^bTX \to X$, but $T^*X, {}^bTX$ are not dual vector bundles if $\partial X \neq \emptyset$. We defined $\mathcal{T}^*X$ using morphisms $X \to \mathbb{R}$ in $\dot{\mathbf{Man}}$, and $\mathcal{T}X$ using morphisms $X \times \mathbb{R} \to X$ in $\dot{\mathbf{Man}}$, so $\mathcal{T}X$ and $\mathcal{T}^*X$ depend on different data in $\dot{\mathbf{Man}}$.

We will define an $\mathcal{O}_X$-bilinear sheaf pairing $\mu_X : \mathcal{T}X \times \mathcal{T}^*X \to \mathcal{O}_X$ on $X_{\rm top}$, thought of as the pairing between vector fields and 1-forms on X. More generally, if $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$ we will define bilinear pairings $\mu_f : (f_{\rm top})_*(\mathcal{T}_fY) \times \mathcal{T}^*Y \to (f_{\rm top})_*(\mathcal{O}_X)$ on $Y_{\rm top}$, and $\mu^f : \mathcal{T}_fY \times f_{\rm top}^{-1}(\mathcal{T}^*Y) \to \mathcal{O}_X$ on $X_{\rm top}$.

Definition B.29. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$. Suppose $j : V \hookrightarrow Y$ is an open submanifold in $\dot{\mathbf{Man}}$, and let $i : U \hookrightarrow X$ be the open submanifold with $U_{\rm top} = f_{\rm top}^{-1}(V_{\rm top}) \subseteq X_{\rm top}$. Then Assumption 3.2(d) gives a unique morphism $f' : U \to V$ with $j \circ f' = f \circ i : U \to Y$.

From §B.1.3, §B.3.1 and §B.4.5 we have

$$\begin{aligned}&(f_{\rm top})_*(\mathcal{O}_X)(V_{\rm top}) = \mathcal{O}_X(U_{\rm top}) = C^\infty(U), \ \ \mathcal{P}\mathcal{T}^*Y(V_{\rm top}) = \Omega_{C^\infty(V)},\\ &(f_{\rm top})_*(\mathcal{T}_fY)(V_{\rm top}) = \mathcal{T}_fY(U_{\rm top}) = \Gamma(\mathcal{T}_{f\circ i}Y) = \Gamma(\mathcal{T}_{j\circ f'}Y) \cong \Gamma(\mathcal{T}_{f'}V),\end{aligned} \tag{B.20}$$

where for the last part $\Gamma(\mathcal{T}j) : \Gamma(\mathcal{T}_{f'}V) \to \Gamma(\mathcal{T}_{j\circ f'}Y)$ is an isomorphism by Lemma B.23. Identify $(f_{\rm top})_*(\mathcal{T}_fY)(V_{\rm top}) = \Gamma(\mathcal{T}_{f'}V)$ as in (B.20).

If $\alpha \in (f_{\rm top})_*(\mathcal{T}_fY)(V_{\rm top}) = \Gamma(\mathcal{T}_{f'}V)$ then §B.4.3 defines a relative C^∞-derivation $\Delta_\alpha : C^\infty(V) \to C^\infty(U)$ over $f' : U \hookrightarrow V$, satisfying (B.12). Regard $C^\infty(V)$ as a module over $C^\infty(U)$ using $f'^* : C^\infty(V) \to C^\infty(U)$. Then (B.12) implies that Δ_α is a C^∞-derivation as in (B.2), so the universal property of $\Omega_{C^\infty(V)}$ in Definition B.10 gives a unique $C^\infty(V)$-module morphism $\Gamma_\alpha : \Omega_{C^\infty(V)} \to C^\infty(U)$ with $\Delta_\alpha = \Gamma_\alpha \circ {\rm d}_{C^\infty(V)}$. Define

$$\begin{aligned}&\mathcal{P}\mu_f(V_{\rm top}) : (f_{\rm top})_*(\mathcal{T}_fY)(V_{\rm top}) \times \mathcal{P}\mathcal{T}^*Y(V_{\rm top}) \to (f_{\rm top})_*(\mathcal{O}_X)(V_{\rm top}),\\ &\mathcal{P}\mu_f(U_{\rm top}) : (\alpha, \beta) \mapsto \Gamma_\alpha(\beta).\end{aligned}$$

Then $\mathcal{P}\mu_f(U_{\rm top})$ is linear over $(f_{\rm top})_*(\mathcal{O}_X)(V_{\rm top}) = C^\infty(U)$ in α, since Δ_α is $C^\infty(U)$-linear in α by Proposition B.20(c),(d), and linear over $\mathcal{O}_Y(V_{\rm top}) = C^\infty(V)$ in β, via $f_\sharp(V_{\rm top})$ in (B.1).

It is easy to check that these maps $\mathcal{P}\mu_f(U_{\rm top})$ are compatible with restriction morphisms $\rho_{V_{\rm top}W_{\rm top}}$ for all open $W_{\rm top} \subseteq V_{\rm top} \subseteq Y_{\rm top}$. Thus, they define a bilinear pairing of presheaves $\mathcal{P}\mu_f : (f_{\rm top})_*(\mathcal{T}_fY) \times \mathcal{P}\mathcal{T}^*Y \to (f_{\rm top})_*(\mathcal{O}_X)$. So passing to the sheafification yields a bilinear pairing of sheaves

$$\mu_f : (f_{\rm top})_*(\mathcal{T}_fY) \times \mathcal{T}^*Y \longrightarrow (f_{\rm top})_*(\mathcal{O}_X).$$

Using the adjoint property of $(f_{\mathrm{top}})_*$ and f_{top}^{-1} as in (A.18), we can show that μ_f corresponds to a unique pairing

$$\mu^f : \mathcal{T}_f Y \times f_{\mathrm{top}}^{-1}(\mathcal{T}^* Y) \longrightarrow \mathcal{O}_X.$$

Here $\mu^f(\alpha,\beta)$ is $\mathcal{O}_X$-linear in α, but $f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)$-linear in β, using $f^\sharp : f_{\mathrm{top}}^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ from §B.1.3. To make μ^f $\mathcal{O}_X$-bilinear, we extend it to

$$\mu^f : \mathcal{T}_f Y \times (f_{\mathrm{top}}^{-1}(\mathcal{T}^* Y) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X) \longrightarrow \mathcal{O}_X,$$

or equivalently, to a morphism of $\mathcal{O}_X$-modules

$$\mu^f_* : \mathcal{T}_f Y \otimes_{\mathcal{O}_X} (f_{\mathrm{top}}^{-1}(\mathcal{T}^* Y) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X) \longrightarrow \mathcal{O}_X. \tag{B.21}$$

When $X = Y$ and $f = \mathrm{id}_X$, both μ^f, μ_f become an $\mathcal{O}_X$-bilinear pairing

$$\mu_X : \mathcal{T} X \times \mathcal{T}^* X \longrightarrow \mathcal{O}_X.$$

B.4.8 Morphisms $E \to \mathcal{T}_f Y$, $\mathcal{T}_f Y \to F$ for vector bundles $E, F \to X$

Definition B.30. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$, and $E, F \to X$ be vector bundles on X. Then §B.2.2 defines the $\mathcal{O}_X$-modules $\mathcal{E}, \mathcal{F}$ of sections of E, F, and §B.4.5 defines the $\mathcal{O}_X$-module $\mathcal{T}_f Y$. Define a *morphism* $\theta : E \to \mathcal{T}_f Y$ to be an $\mathcal{O}_X$-module morphism $\theta : \mathcal{E} \to \mathcal{T}_f Y$, and a *morphism* $\phi : \mathcal{T}_f Y \to F$ to be an $\mathcal{O}_X$-module morphism $\phi : \mathcal{T}_f Y \to \mathcal{F}$. That is, in our notation we will not distinguish between the vector bundles E, F and their sheaves of sections $\mathcal{E}, \mathcal{F}$.

By *composition* of such morphisms with each other, with morphisms of vector bundles, and with the $\mathcal{O}_X$-module morphisms in §B.4.6, we mean composition of $\mathcal{O}_X$-module morphisms, but identifying vector bundle morphisms $\mathrm{Hom}(E, F)$ with $\mathcal{O}_X$-module morphisms $\mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\mathcal{E}, \mathcal{F})$ as in §B.2.2. For example:

(a) If $\theta : E \to \mathcal{T}_f Y$ and $\phi : \mathcal{T}_f Y \to F$ are as above then $\phi \circ \theta : E \to F$ is the honest vector bundle morphism corresponding to $\phi \circ \theta : \mathcal{E} \to \mathcal{F}$.

(b) If $\theta : E \to \mathcal{T}_f Y$ is as above and $\lambda : D \to E$ is a vector bundle morphism as above we get a morphism $\theta \circ \lambda : D \to \mathcal{T}_f Y$.

(c) If $\theta : E \to \mathcal{T}_f Y$ is as above, $g : Y \to Z$ is a morphism in $\dot{\mathbf{Man}}$, and $\mathcal{T} g : \mathcal{T}_f Y \to \mathcal{T}_{g \circ f} Z$ is as in §B.4.6, we get a morphism $\mathcal{T} g \circ \theta : E \to \mathcal{T}_{g \circ f} Z$.

Example B.31. When $\dot{\mathbf{Man}} = \mathbf{Man}$, morphisms $\theta : E \to \mathcal{T}_f Y$, $\phi : \mathcal{T}_f Y \to F$ above are in natural 1-1 correspondence with vector bundle morphisms $\theta' : E \to f^*(TY)$, $\phi' : f^*(TY) \to F$ in the usual sense of differential geometry.

In Definition B.16 we wrote elements α of $\Gamma(\mathcal{T}_f Y)$ in terms of diagrams (B.5) in $\dot{\mathbf{Man}}$. We will now show that any morphism $\theta : E \to \mathcal{T}_f Y$ may be written in terms of a similar diagram.

Definition B.32. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$, and $\pi : E \to X$ be a vector bundle. Generalizing (B.5), consider commutative diagrams in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd}
& X \arrow[dl, "0_E"'] \arrow[d, "l"] \arrow[dr, "f"] & \\
E & V \arrow[l, hook', "j"'] \arrow[r, "v"] & Y,
\end{tikzcd} \tag{B.22}$$

where $0_E : X \to E$ is the zero section morphism as in §B.2.1, and $j : V \hookrightarrow E$ is an open submanifold with $0_{E,\mathrm{top}}(X_{\mathrm{top}}) \subseteq V_{\mathrm{top}} \subseteq E_{\mathrm{top}}$, and unique $l : X \to V$ with $j \circ l = 0_E$ exists by Assumption 3.2(d), and $v : V \to Y$ is a morphism in $\dot{\mathbf{Man}}$ with $v \circ l = f$. For brevity we write such a diagram as the pair (V, v).

Given such a pair (V, v) we will define a morphism $\theta_{V,v} : E \to \mathcal{T}_f Y$, in the sense of Definition B.30. Write $\mathcal{E}$ for the $\mathcal{O}_X$-module of sections of E. Let $\chi' : X' \hookrightarrow X$ be an open submanifold in $\dot{\mathbf{Man}}$, and set $E' = \chi'^*(E) = E|_{X'}$, so that $k : E' \hookrightarrow E$ is open in $\dot{\mathbf{Man}}$. We must define a $C^\infty(X')$-module morphism

$$\theta_{V,v}(X'_{\mathrm{top}}) : \mathcal{E}(X'_{\mathrm{top}}) = \Gamma^\infty(E') \longrightarrow \mathcal{T}_f Y(X'_{\mathrm{top}}) = \Gamma(\mathcal{T}_{f\circ\chi'} Y).$$

Suppose $e' \in \Gamma^\infty(E')$, so that $e' : X' \to E'$ with $\pi_{E'} \circ e' = \mathrm{id}_{X'}$. Then there is a unique morphism $\tilde{e}' : X' \times \mathbb{R} \to E'$ in $\dot{\mathbf{Man}}$ with $\tilde{e}'_{\mathrm{top}}(x, t) = t \cdot e'_{\mathrm{top}}(x) \in E'_{\mathrm{top}}$ for all $x \in X'_{\mathrm{top}}$ and $t \in \mathbb{R}$, where $t \cdot e'_{\mathrm{top}}(x)$ multiplies $e'_{\mathrm{top}}(x)$ in the vector space $E'_x \subseteq E'_{\mathrm{top}}$ by $t \in \mathbb{R}$. Let $i' : U' \hookrightarrow X' \times \mathbb{R}$ be the open submanifold with $U'_{\mathrm{top}} = \tilde{e}'^{-1}_{\mathrm{top}}(V_{\mathrm{top}})$. Consider the commutative diagram in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd}
& X' \arrow[dl, "(\mathrm{id}_{X'}{,}0)"'] \arrow[dr, "l'", "\star"'] \arrow[rr, hook, "\chi'"] \arrow[dd, "0_{E'}"] & & X \arrow[r, "f"] \arrow[dr, "l"] \arrow[dd, "0_E"'] & Y \\
X' \times \mathbb{R} \arrow[dr, "\tilde{e}'"'] & & U' \arrow[ll, hook', "i'"] \arrow[urr, "u'"] \arrow[rr, "m'", "\star"'] & & V, \arrow[u, "v"'] \arrow[dl, hook', "j"] \\
& E' \arrow[rr, hook, "k"] & & E &
\end{tikzcd}$$

where morphisms '$\hookrightarrow$' are open submanifolds, and morphisms '$\star$' exist by Assumption 3.2(d). Then $U', i', l', u' = v \circ m'$ are a diagram (B.5) for $f \circ \chi' : X' \to Y$, so $[U', u'] \in \Gamma(\mathcal{T}_{f\circ\chi'} Y)$ by Definition B.16. Define $\theta_{V,v}(X'_{\mathrm{top}})(e') = [U', u']$.

It is now straightforward to show using §B.4.2 and §B.4.5 that $\theta_{V,v}(X'_{\mathrm{top}})$ is a $C^\infty(X')$-module morphism, and that the maps $\theta_{V,v}(X'_{\mathrm{top}}), \theta_{V,v}(X''_{\mathrm{top}})$ for open $X''_{\mathrm{top}} \subseteq X'_{\mathrm{top}} \subseteq X_{\mathrm{top}}$ are compatible with restriction morphisms $\rho_{X'_{\mathrm{top}} X''_{\mathrm{top}}}$, so that $\theta_{V,v} : \mathcal{E} \to \mathcal{T}_f Y$ is an $\mathcal{O}_X$-module morphism.

Proposition B.33. *Let $f : X \to Y$ be a morphism in* $\dot{\mathbf{Man}}$, *and $\pi : E \to X$ be a vector bundle. Then every morphism $\theta : E \to \mathcal{T}_f Y$ in Definition* B.30 *is of the form $\theta = \theta_{V,v}$ in Definition* B.32 *for some diagram* (B.22).

Proof. Let X, Y, f, E, θ be as in the proposition. Write r for the rank of E and $\mathcal{E}$ for the $\mathcal{O}_X$-module of sections of E, so that $\theta : \mathcal{E} \to \mathcal{T}_f Y$ is an $\mathcal{O}_X$-module morphism. Choose an open cover $\{\chi_a : X'_a \hookrightarrow X\}$ such that $E_a := E|_{X'_a} = \chi_a^*(E)$

is a trivial vector bundle over X'_a for each $a \in A$, and choose an isomorphism $\Psi_a : E_a \to X'_a \times \mathbb{R}^r$ with the trivial vector bundle $X'_a \times \mathbb{R}^r \to X'_a$. Write $e_a^1, \ldots, e_a^r$ for the basis of sections of E_a identified by Ψ_a with the canonical basis of sections of $X'_a \times \mathbb{R}^r$. Then $e_a^k \in \mathcal{E}(X'_{a,\mathrm{top}}) = \Gamma^\infty(E|_{X'_a})$, so $\theta(X'_{a,\mathrm{top}})(e_a^k) \in \mathcal{T}_f Y(X'_{a,\mathrm{top}}) = \Gamma(\mathcal{T}_{f\circ\chi_a} Y)$. Choose a representative (U_a^k, u_a^k) for $\theta(X'_{a,\mathrm{top}})(e_a^k) = [U_a^k, u_a^k] \in \Gamma(\mathcal{T}_{f\circ\chi_a} Y)$ for all $a \in A$ and $k = 1, \ldots, r$, as in §B.4.1, so that U_a^k, u_a^k fit into a commutative diagram (B.5):

$$\begin{tikzcd} & X'_a \arrow[dl, "(\mathrm{id}_{X'_a},0)"'] \arrow[d, "l_a^k"] \arrow[dr, "f\circ\chi_a"] & \\ X'_a \times \mathbb{R} & U_a^k \arrow[l, hook', "i_a^k"'] \arrow[r, "u_a^k"] & Y. \end{tikzcd}$$

Apply Assumption 3.7(a) to construct a commutative diagram

$$\begin{tikzcd} & X'_a \arrow[dl, "(\mathrm{id}_{X'_a},0)"'] \arrow[d, "m_a"] \arrow[dr, "f\circ\chi_a"] & \\ E|_{X'_a} \cong X'_a \times \mathbb{R}^r & V_a \arrow[l, hook', "j_a"'] \arrow[r, "v_a"] & Y, \end{tikzcd}$$

such that $j_a : V_a \hookrightarrow X'_a \times \mathbb{R}^r$ is open, and if $(x, (0, \ldots, 0, s_k, 0, \ldots, 0)) \in V_{a,\mathrm{top}}$ with s_k the k^{th} coordinate in $\mathbb{R}^r$ then $(x, s_k) \in U^k_{a,\mathrm{top}}$ and $u^k_{a,\mathrm{top}}(x, s_k) = v_{a,\mathrm{top}}(x, (0, \ldots, 0, s_k, 0, \ldots, 0))$. Actually we apply Assumption 3.7(a) $2^r - r - 1$ times to choose $v_{a,\mathrm{top}}(x, (s_1, \ldots, s_r))$ with subsets of the $s_1, \ldots, s_r$ zero.

The next part of the proof follows that of part (v) of the sheaf property of $\mathcal{T}_f Y$ in Proposition B.25. Let S_A be the set of all finite, nonempty subsets $B \subseteq A$. For each $B \in S_A$ write $\chi_B : X'_B \hookrightarrow X$ for the open submanifold with $X'_{B,\mathrm{top}} = \bigcap_{a\in B} X'_{a,\mathrm{top}}$. When $B = \{a\}$ we have $X'_{\{a\}} = X'_a$, $\chi_{\{a\}} = \chi_a$. If $C \subseteq B$ lie in S_A then there is a unique $\xi_{BC} : X'_B \hookrightarrow X'_C$ with $\chi_B = \chi_C \circ \xi_{BC}$ by Assumption 3.2(d).

By the same proof as in the proof of Proposition B.25, using induction on $|B|$ and Assumption 3.7(a), for each $B \in S_A$ we choose an open submanifold $k_B : W_B \hookrightarrow \bigoplus_{b\in B} E|_{X'_B} \cong X'_B \times \prod_{b\in B} \mathbb{R}^r$ and a morphism $w_B : W_B \to Y$ with:

(a) $X'_{B,\mathrm{top}} \times \{(0, \ldots, 0)\} \subseteq W_{B,\mathrm{top}}$ for all $B \in S_A$.

(b) For $a \in A$ we have $W_{\{a\}} = V_a \hookrightarrow E|_{X'_a} \cong X'_{\{a\}} \times \mathbb{R}^r$ and $w_{\{a\}} = v_a$.

(c) If $C \subsetneq B$ lie in S_A and $(x, (\boldsymbol{s}_a)_{a\in C} \amalg (0)_{a\in B\setminus C}) \in W_{B,\mathrm{top}}$ then $(x, (\boldsymbol{s}_a)_{a\in C})$ lies in $W_{C,\mathrm{top}}$ with $w_{C,\mathrm{top}}(x, (\boldsymbol{s}_a)_{a\in C}) = w_{B,\mathrm{top}}(x, (\boldsymbol{s}_a)_{a\in C} \amalg (0)_{a\in B\setminus C})$.

Now apply Proposition B.7 to choose a partition of unity $\{\eta_a : a \in A\}$ on X' subordinate to the open cover $\{X'_{a,\mathrm{top}} : a \in A\}$. Choose an open submanifold $j : V \hookrightarrow E$ such that $0_{E,\mathrm{top}}(X_{\mathrm{top}}) \subseteq V_{\mathrm{top}}$ and if $e \in V_{\mathrm{top}} \subseteq E_{\mathrm{top}}$ with $\pi_{\mathrm{top}}(e) = x \in X_{\mathrm{top}}$ and $B = \{a \in A : x \in \operatorname{supp} \eta_{a,\mathrm{top}}\}$ then $(x, (\eta_{a,\mathrm{top}}(x)\pi_{\mathbb{R}^r} \circ \Psi_{a,\mathrm{top}}(e))_{a\in B}) \in W_{B,\mathrm{top}}$. By (a) above and local finiteness of $\{\eta_a : a \in A\}$, this holds for any small enough open neighbourhood of $0_{E,\mathrm{top}}(X_{\mathrm{top}})$ in E.

As for the construction of $u : U \to Y$ satisfying (B.18) in the proof of Proposition B.25, there is a unique morphism $v : V \to Y$ such that for all $e \in V_{\rm top}$ with $\pi_{\rm top}(e) = x \in X_{\rm top}$ and $B = \{a \in A : x \in \operatorname{supp} \eta_{a,{\rm top}}\}$ we have

$$v_{\rm top}(e) = w_{B,{\rm top}}(x, (\eta_{a,{\rm top}}(x) \cdot \pi_{\mathbb{R}^r} \circ \Psi_{a,{\rm top}}(e))_{a \in B}). \qquad \text{(B.23)}$$

Then $j : V \hookrightarrow E$ and $v : V \to Y$ fit into a diagram (B.22), and so give a morphism $\theta_{V,v} : E \to \mathcal{T}_f Y$ by Definition B.32. We will show that $\theta_{V,v} = \theta$.

Let $\tilde{x} \in X_{\rm top}$, and set $B = \{b \in A : \tilde{x} \in \operatorname{supp} \eta_{b,{\rm top}}\}$ in S_A. Choose an open neighbourhood $R \hookrightarrow X$ of $\tilde{x}$ in X such that $R_{\rm top} \subseteq X'_{b,{\rm top}}$ for all $b \in B$, and $R_{\rm top} \cap \operatorname{supp} \eta_{c,{\rm top}} = \emptyset$ for all $c \in A \setminus B$. This is possible as $\operatorname{supp} \eta_{b,{\rm top}}$ is contained in $X'_{b,{\rm top}}$ and closed in $X_{\rm top}$, and $\{\eta_a : a \in A\}$ is locally finite. Let $e \in \Gamma^\infty(E|_R)$. Then

$$\begin{aligned} \theta_{V,v}(R_{\rm top})(e) &= \sum_{a \in B} \eta_a|_R \cdot \theta_{V_a,v_a}(R_{\rm top})(\Psi_a|_R(e)) = \sum_{a \in B} \eta_a|_R \cdot \theta(R_{\rm top})(e) \\ &= 1 \cdot \theta(R_{\rm top})(e) = \theta(R_{\rm top})(e). \end{aligned} \qquad \text{(B.24)}$$

Here the first step follows from comparing the definition of $\theta_{V,v}$, equation (B.23), part (b) above, and the definitions of addition and multiplication by functions in $\Gamma(\mathcal{T}_{f|_R} Y)$ in §B.4.2. The second holds by definition of (V_a, v_a) above in terms of (U_a^k, u_a^k), where $\theta(X'_{a,{\rm top}})(e_a^k) = [U_a^k, u_a^k]$, and $e_a^1, \ldots, e_a^r$ are mapped by Ψ_a to the canonical basis of sections of $X'_a \times \mathbb{R}^r \to X'_a$. The third holds as $\sum_{b \in B} \eta_b$ is 1 on R since $\{\eta_a : a \in A\}$ is a partition of unity with $R_{\rm top} \cap \operatorname{supp} \eta_{c,{\rm top}} = \emptyset$ for all $c \in A \setminus B$.

Equation (B.24) shows that for any $\tilde{x} \in X_{\rm top}$ and any sufficiently small open neighbourhood $R_{\rm top}$ of $\tilde{x}$ in $X_{\rm top}$ we have $\theta_{V,v}(R_{\rm top}) = \theta(R_{\rm top}) : \mathcal{E}(R_{\rm top}) \to \mathcal{T}_f Y(R_{\rm top})$. Since $\theta_{V,v}, \theta$ are sheaf morphisms, this implies that $\theta_{V,v} = \theta$. $\square$

B.4.9 Notation for 'pullbacks' f^* by morphisms $f : X \to Y$

We will use the following notation for 'pullbacks' f^* by morphisms $f : X \to Y$.

Definition B.34. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$, and $E \to Y$ be a vector bundle on Y, and $\mathcal{E}$ the $\mathcal{O}_Y$-module of sections of E from §B.2.2. Then we can form the sheaf pullback $f_{\rm top}^{-1}(\mathcal{E})$ as in §A.5, which is a sheaf of modules over $f_{\rm top}^{-1}(\mathcal{O}_Y)$ on $X_{\rm top}$. In §B.1.3 we defined a morphism $f^\sharp : f_{\rm top}^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves of $\mathbb{R}$-algebras or C^∞-rings on $X_{\rm top}$. Thus we may form the $\mathcal{O}_X$-module $f_{\rm top}^{-1}(\mathcal{E}) \otimes_{f_{\rm top}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$ using $f^\sharp$.

We can also form the pullback vector bundle $f^*(E) \to X$ as in §B.2.1. The corresponding $\mathcal{O}_X$-module is canonically isomorphic to $f_{\rm top}^{-1}(\mathcal{E}) \otimes_{f_{\rm top}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, and we will identify it with $f_{\rm top}^{-1}(\mathcal{E}) \otimes_{f_{\rm top}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, and write it $f^*(\mathcal{E})$.

Let $F \to Y$ be another vector bundle, and $\theta : E \to F$ a vector bundle morphism, and $\tilde{\theta} : \mathcal{E} \to \mathcal{F}$ the corresponding $\mathcal{O}_Y$-module morphism. Then we may form the $\mathcal{O}_X$-module morphism

$$f^*(\tilde{\theta}) := f_{\mathrm{top}}^{-1}(\tilde{\theta}) \otimes \mathrm{id}_{\mathcal{O}_X} : f^*(\mathcal{E}) = f_{\mathrm{top}}^{-1}(\mathcal{E}) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \longrightarrow$$
$$f^*(\mathcal{F}) = f_{\mathrm{top}}^{-1}(\mathcal{F}) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X.$$

This is the $\mathcal{O}_X$-module morphism corresponding to the vector bundle morphism $f^*(\theta) : f^*(E) \to f^*(F)$ on X, as in §B.2.2.

Now let $g : Y \to Z$ be another morphism in $\dot{\mathbf{Man}}$, so we have an $\mathcal{O}_Y$-module $\mathcal{T}_g Z$ and an $\mathcal{O}_X$-module $\mathcal{T}_{g\circ f} Z$. We will often treat $\mathcal{T}_{g\circ f} Z$ *as if* it were the pullback $f^*(\mathcal{T}_g Z)$. This is an abuse of notation: for $f^\flat$ as in §B.4.6 and using $f^\sharp : f_{\mathrm{top}}^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$, we have an $\mathcal{O}_X$-module morphism

$$f^\flat \otimes \mathrm{id}_{\mathcal{O}_X} : f_{\mathrm{top}}^{-1}(\mathcal{T}_g Z) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \longrightarrow \mathcal{T}_{g\circ f} Z \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{T}_{g\circ f} Z. \tag{B.25}$$

It would be more consistent to write $f^*(\mathcal{T}_g Z) = f_{\mathrm{top}}^{-1}(\mathcal{T}_g Z) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$ (though we will not), but then $f^*(\mathcal{T}_g Z)$ and $\mathcal{T}_{g\circ f} Z$ would be different, as (B.25) need not be an isomorphism for general $\dot{\mathbf{Man}}$.

Suppose $E, \mathcal{E}$ are as above, and $\theta : E \to \mathcal{T}_g Z$ is a morphism (that is, $\theta : \mathcal{E} \to \mathcal{T}_g Z$ is an $\mathcal{O}_Y$-module morphism). Define a morphism $f^*(\theta) : f^*(E) \to \mathcal{T}_{g\circ f} Z$ by the commutative diagram of $\mathcal{O}_X$-modules

$$\begin{tikzcd}
f_{\mathrm{top}}^{-1}(\mathcal{E}) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \arrow[r, "f_{\mathrm{top}}^{-1}(\theta)\otimes \mathrm{id}_{\mathcal{O}_X}"'] \arrow[dr, bend right=10, "f^*(\theta)"'] & f_{\mathrm{top}}^{-1}(\mathcal{T}_g Z) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \arrow[d, "f^\flat \otimes \mathrm{id}_{\mathcal{O}_X}"'] \\
& \mathcal{T}_{g\circ f} Z \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{T}_{g\circ f} Z.
\end{tikzcd} \tag{B.26}$$

Here $f_{\mathrm{top}}^{-1}(\mathcal{E}) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$ is the $\mathcal{O}_X$-module corresponding to the vector bundle $f^*(E) \to X$, as above. Using this notation $f^*(\theta)$ we will avoid using the morphisms $f^\flat$ in Chapters 4–6.

Note that if $\phi : \mathcal{T}_g Z \to F$ is a morphism, we cannot define a pullback $f^*(\phi) : \mathcal{T}_{g\circ f} Z \to f^*(F)$, because the morphism (B.25) goes the wrong way.

Definition B.35. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$, and $F \to Y$ be a vector bundle, and $t \in \Gamma^\infty(F)$. Suppose ∇ is a connection on F, as in §B.3.2. Writing $\mathcal{F}$ for the $\mathcal{O}_Y$-module corresponding to F, we have $t \in \Gamma(\mathcal{F})$, so that $\nabla t \in \Gamma(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{T}^* Y)$. Define a morphism $f^*(\nabla t) : \mathcal{T}_f Y \to f^*(F)$, in the sense of §B.4.8, by the commutative diagram of $\mathcal{O}_X$-modules

$$\begin{tikzcd}
\mathcal{T}_f Y \arrow[r, "\otimes f_{\mathrm{top}}^{-1}(\nabla t)"'] \arrow[d, "f^*(\nabla t)"] & \mathcal{T}_f Y \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} (f_{\mathrm{top}}^{-1}(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{T}^* Y)) \arrow[d, "\cong"'] \\
f_{\mathrm{top}}^{-1}(\mathcal{F}) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X & f_{\mathrm{top}}^{-1}(\mathcal{F}) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} (\mathcal{T}_f Y \otimes_{\mathcal{O}_X} (f_{\mathrm{top}}^{-1}(\mathcal{T}^* Y) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X)), \arrow[l, "\mathrm{id} \otimes \mu_*^f"']
\end{tikzcd} \tag{B.27}$$

where μ_*^f is as in (B.21), and $f_{\mathrm{top}}^{-1}(\mathcal{F}) \otimes_{f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$ is the $\mathcal{O}_X$-module corresponding to $f^*(F) \to X$, as in Definition B.34.

B.5 The $O(s)$ and $O(s^2)$ notation

When $X \in \dot{\mathbf{Man}}$, and $E \to X$ is a vector bundle, and $s \in \Gamma^\infty(E)$, we now define several related uses of the notation '$O(s)$' and '$O(s^2)$'. This will be important in defining the (2-)categories of (m- and μ-)Kuranishi neighbourhoods in Chapters 4–6.

Definition B.36. Let X be an object in $\dot{\mathbf{Man}}$, and $E \to X$ be a vector bundle, and $s \in \Gamma^\infty(E)$ be a section. Then:

(i) If $F \to X$ is a vector bundle and $t_1, t_2 \in \Gamma^\infty(F)$, we write $t_2 = t_1 + O(s)$ if there exists a morphism $\alpha : E \to F$ such that $t_2 = t_1 + \alpha \circ s$ in $\Gamma^\infty(F)$.

Similarly, we write $t_2 = t_1 + O(s^2)$ if there exists $\beta : E \otimes E \to F$ such that $t_2 = t_1 + \beta \circ (s \otimes s)$ in $\Gamma^\infty(F)$. This implies that $t_2 = t_1 + O(s)$.

We can also apply this $O(s), O(s^2)$ notation to morphisms of vector bundles $\theta_1, \theta_2 : F \to G$, by regarding θ_1, θ_2 as sections of $F^* \otimes G$.

(ii) If $F \to X$ is a vector bundle, $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$, and $\Lambda_1, \Lambda_2 : F \to \mathcal{T}_f Y$ are morphisms as in §B.4.8, we write $\Lambda_2 = \Lambda_1 + O(s)$ if there exist open submanifolds $i : U \hookrightarrow X$ and $j : V \hookrightarrow E$ with $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}}$ and $0_{E,\mathrm{top}}(U_{\mathrm{top}}), s_{\mathrm{top}}(U_{\mathrm{top}}) \subseteq V_{\mathrm{top}}$, so that we have a commutative diagram in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd}
U \arrow[r, "k_1"] \arrow[d, hook, "i"] & V \arrow[d, hook, "j"] & U \arrow[l, "k_2"'] \arrow[d, hook, "i"] \\
X \arrow[r, "0_E"] \arrow[dr, "\mathrm{id}_X"'] & E \arrow[d, "\pi"] & X \arrow[l, "s"'] \arrow[dl, "\mathrm{id}_X"] \\
& X, &
\end{tikzcd} \qquad \text{(B.28)}$$

where the morphisms k_1, k_2 exist by Assumption 3.2(d). Also there should exist a morphism $\mathrm{M} : \pi^*(F)|_V \to \mathcal{T}_{f\circ\pi} Y|_V$ with $k_1^*(\mathrm{M}) = \Lambda_1|_U$ and $k_2^*(\mathrm{M}) = \Lambda_2|_U$ in morphisms $F|_U \to \mathcal{T}_f Y|_U$, where

$$k_a^*(\mathrm{M}) : k_a^* \circ \pi^*(F) = F|_U \longrightarrow \mathcal{T}_{f\circ\pi\circ k_a} Y = \mathcal{T}_f Y|_U$$

for $a = 1, 2$ are as in §B.4.8.

(iii) If $f, g : X \to Y$ are morphisms in $\dot{\mathbf{Man}}$, we write $g = f + O(s)$ if there is a diagram (B.28) as in (ii) with $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}}$, and a morphism $v : V \to Y$ in $\dot{\mathbf{Man}}$ with $v \circ k_1 = f|_U$ and $v \circ k_2 = g|_U$ in morphisms $U \to Y$ in $\dot{\mathbf{Man}}$.

(iv) Let $f, g : X \to Y$ with $g = f + O(s)$ be as in (iii), and $F \to X$, $G \to Y$ be vector bundles, and $\theta_1 : F \to f^*(G)$, $\theta_2 : F \to g^*(G)$ be morphisms. We wish to compare θ_1, θ_2, though they map to *different* vector bundles.

We write $\theta_2 = \theta_1 + O(s)$ if there is a diagram (B.28) with $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}}$ and a morphism $v : V \to Y$ with $v \circ k_1 = f|_U$ and $v \circ k_2 = g|_U$ as in (iii), and a morphism $\phi : \pi^*(F)|_V \to v^*(G)$ with $k_1^*(\phi) = \theta_1|_U$ and $k_2^*(\phi) = \theta_2|_U$, where $k_1^*(\phi), k_2^*(\phi)$ are as in §B.2.1.

(v) Let $f, g : X \to Y$ with $g = f + O(s)$ be as in (iii), and $F \to X$ be a vector bundle, and $\Lambda_1 : F \to \mathcal{T}_f Y$, $\Lambda_2 : F \to \mathcal{T}_g Y$ be morphisms, as in §B.4.8. We wish to compare Λ_1, Λ_2, though they map to *different* sheaves.

We write $\Lambda_2 = \Lambda_1 + O(s)$ if there is a diagram (B.28) with $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}}$ and a morphism $v : V \to Y$ with $v \circ k_1 = f|_U$ and $v \circ k_2 = g|_U$ as in (iii), and a morphism $\mathrm{M} : \pi^*(F)|_V \to \mathcal{T}_v Y$ with $k_1^*(\mathrm{M}) = \Lambda_1|_U$ and $k_2^*(\mathrm{M}) = \Lambda_2|_U$, where $k_1^*(\mathrm{M}), k_2^*(\mathrm{M})$ are as in §B.4.8.

(vi) Suppose $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$, and $F \to X$, $G \to Y$ are vector bundles, and $t \in \Gamma^\infty(G)$ with $f^*(t) = O(s)$ in the sense of (i), and $\Lambda : F \to \mathcal{T}_f Y$ is a morphism, as in §B.4.8, and $\theta : F \to f^*(G)$ is a vector bundle morphism, as in §B.2.1. We write $\theta = f^*(\mathrm{d}t) \circ \Lambda + O(s)$ if whenever ∇ is a connection on G we have $\theta = f^*(\nabla t) \circ \Lambda + O(s)$ in the sense of (i), where $f^*(\nabla t) : \mathcal{T}_f Y \to f^*(G)$ is as in §B.4.9, so that $f^*(\nabla t) \circ \Lambda : F \to f^*(G)$ is a vector bundle morphism as in §B.4.8.

Note that there exists a connection ∇ on G by Proposition B.14(a). If ∇, ∇' are two such connections then $\nabla' = \nabla + \Gamma$ for $\Gamma : \mathcal{G} \to \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{T}^* Y$ an $\mathcal{O}_Y$-module morphism, by Proposition B.14(b). Then

$$f^*(\nabla' t) \circ \Lambda = f^*(\nabla t) \circ \Lambda + [f_{\mathrm{top}}^{-1}(\Gamma) \circ \Lambda] \cdot f^*(t),$$

where $f_{\mathrm{top}}^{-1}(\Gamma) \circ \Lambda \in \Gamma^\infty(F^* \otimes f^*(G) \otimes f^*(G^*))$ is a natural section. Thus $f^*(\nabla' t) \circ \Lambda = f^*(\nabla t) \circ \Lambda + O(s)$, since $t = O(s)$. Hence the condition $\theta = f^*(\nabla t) \circ \Lambda + O(s)$ is independent of the choice of connection ∇ on G.

Note also that the '$f^*(\mathrm{d}t)$' in $\theta = f^*(\mathrm{d}t) \circ \Lambda + O(s)$ is just notation, intended to suggest this independence of the choice of ∇.

(vii) Let $f, g : X \to Y$ with $g = f + O(s)$ be as in (iii), and $\Lambda : E \to \mathcal{T}_f Y$ be a morphism in the sense of §B.4.8. We write $g = f + \Lambda \circ s + O(s^2)$ if there exists a commutative diagram in $\dot{\mathbf{Man}}$

$$\begin{tikzcd}
& Y & \\
U \arrow[r, "k_1"'] \arrow[d, hook, "i"] & V \arrow[u, "v"] \arrow[d, hook, "j"] & U \arrow[l, "k_2"] \arrow[d, hook, "i"] \\
X \arrow[r, "0_E"] \arrow[uur, bend left, "f"] & E & X, \arrow[l, "s"'] \arrow[uul, bend right, "g"']
\end{tikzcd} \tag{B.29}$$

with $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}}$, where morphisms i, j are open submanifolds, and morphisms k_1, k_2 exist by Assumption 3.2(d), and $\Lambda|_U = \theta_{V,v}$ as a morphism $E|_U \to \mathcal{T}_f Y|_U$, in the notation of §B.4.8.

Theorem 3.17, proved in §B.9, gives a long list of properties of the $O(s)$, $O(s^2)$ notation that we need for our theories of (m- and μ-)Kuranishi spaces.

Remark B.37. **(a)** When $\dot{\mathbf{Man}} = \mathbf{Man}$, and to some extent for general $\dot{\mathbf{Man}}$, we can interpret the $O(s)$ and $O(s^2)$ conditions in Definition B.36 in terms of

C^∞-algebraic geometry, as in §B.1.2 and [56, 65]. As in Proposition B.5 we can make $X \in \dot{\mathbf{Man}}$ into a C^∞-scheme $\underline{X} = (X_{\rm top}, \mathcal{O}_X)$. Given a vector bundle $E \to X$ and $s \in \Gamma^\infty(E)$, we have closed C^∞-subschemes $\underline{S}_1 \subseteq \underline{S}_2 \subseteq \underline{X}$, where $\underline{S}_1$ is defined by $s = 0$, and $\underline{S}_2$ by $s \otimes s = 0$.

The rough idea is that an equation on X holds up to $O(s)$ if when translated into C^∞-scheme language, the restriction of the equation to $\underline{S}_1 \subseteq \underline{X}$ holds exactly, and it holds up to $O(s^2)$ if its restriction to $\underline{S}_2 \subseteq \underline{X}$ holds exactly. For example, $t_2 = t_1 + O(s) \Leftrightarrow t_2|_{\underline{S}_1} = t_1|_{\underline{S}_1}$ and $t_2 = t_1 + O(s^2) \Leftrightarrow t_2|_{\underline{S}_2} = t_1|_{\underline{S}_2}$ in Definition B.36(i), for general $\dot{\mathbf{Man}}$.

Also morphisms $f, g : X \to Y$ in $\dot{\mathbf{Man}}$ translate to C^∞-scheme morphisms $\underline{f}, \underline{g} : \underline{X} \to \underline{Y}$. Then $g = f + O(s)$ implies that $\underline{g}|_{\underline{S}_1} = \underline{f}|_{\underline{S}_1}$ for general $\dot{\mathbf{Man}}$, and when $\dot{\mathbf{Man}} = \mathbf{Man}$ the two are equivalent. If we think of the $O(s), O(s^2)$ conditions as restriction to $\underline{S}_1, \underline{S}_2$ then much of Theorem 3.17 becomes obvious.

(b) In Definition B.36(i), we could instead have defined $t_2 = t_1 + O(s)$ in the style of (ii), using a diagram (B.28). One can prove using Assumption 3.5 that this would give an equivalent notion of when $t_2 = t_1 + O(s)$, and we implicitly show this in the second part of the proof of Theorem 3.17(f) in §B.9.

(c) We explain Definition B.36(vii). We have $\Lambda \circ s \in \Gamma(\mathcal{T}_f Y)$, where as in §B.4.1 elements of $\Gamma(\mathcal{T}_f Y)$ are defined using infinitesimal deformations of f amongst morphisms $X \to Y$ in $\dot{\mathbf{Man}}$. The equation '$g = f + \Lambda \circ s + O(s^2)$' means that $g = f + O(s)$, so that g is a small deformation of f near $s_{\rm top}^{-1}(0) \subseteq X_{\rm top}$, and to leading order near $s_{\rm top}^{-1}(0)$ is the infinitesimal deformation $\Lambda \circ s$ of f.

We could have generalized Definition B.36(vii) to define '$g = f + v + O(s^2)$' for any $v \in \Gamma(\mathcal{T}_f Y)$ with $v = O(s)$. It is not important that $v = \Lambda \circ s$ for some $\Lambda : E \to \mathcal{T}_f Y$, but we will only use the case $v = \Lambda \circ s$.

B.6 Discrete properties of morphisms in $\dot{\mathbf{Man}}$

Here is a condition for classes of morphisms in $\dot{\mathbf{Man}}$ to lift nicely to classes of (1-)morphisms in $\mathbf{m\dot{K}ur}, \boldsymbol{\mu}\mathbf{\dot{K}ur}, \mathbf{\dot{K}ur}$ in Chapters 4–6.

Definition B.38. Let $\boldsymbol{P}$ be a property of morphisms in $\dot{\mathbf{Man}}$, so that for any morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$, either f is $\boldsymbol{P}$, or f is not $\boldsymbol{P}$. For example, if $\dot{\mathbf{Man}}$ is $\mathbf{Man^c}$ from §2.1, then $\boldsymbol{P}$ could be interior, or b-normal.

We call $\boldsymbol{P}$ a *discrete* property of morphisms in $\dot{\mathbf{Man}}$ if:

(i) All diffeomorphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ are $\boldsymbol{P}$.

(ii) All open submanifolds $i : U \hookrightarrow X$ in $\dot{\mathbf{Man}}$ are $\boldsymbol{P}$.

(iii) If $f : X \to Y$ and $g : Y \to Z$ in $\dot{\mathbf{Man}}$ are $\boldsymbol{P}$ then $g \circ f : X \to Z$ is $\boldsymbol{P}$.

(iv) For a morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$ to be $\boldsymbol{P}$ is a *local property on* X, in the sense that if we can cover X by open submanifolds $i : U \hookrightarrow X$ such that $f \circ i : U \to Y$ is $\boldsymbol{P}$, then f is $\boldsymbol{P}$.

Some notation: if $f : X \to Y$ in $\dot{\mathbf{Man}}$ and $S \subseteq X_{\rm top}$ then we say that *f is $\boldsymbol{P}$ near S* if there exists an open submanifold $i : U \hookrightarrow X$ such that

$S \subseteq U_{\mathrm{top}} \subseteq X_{\mathrm{top}}$ and $f \circ i : U \to Y$ is $\boldsymbol{P}$. This is a well behaved notion as $\boldsymbol{P}$ is a local property, e.g. f is $\boldsymbol{P}$ if and only if f is $\boldsymbol{P}$ near each $x \in X_{\mathrm{top}}$.

(v) All morphisms in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$ are $\boldsymbol{P}$.

(vi) Suppose $f : X \times \mathbb{R} \to Y$ is a morphism in $\dot{\mathbf{Man}}$. If f is $\boldsymbol{P}$ near $X_{\mathrm{top}} \times \{0\}$ in $X_{\mathrm{top}} \times \mathbb{R}$, then f is $\boldsymbol{P}$.

(vii) Suppose $E \to X$ is a vector bundle in $\dot{\mathbf{Man}}$, and $s \in \Gamma^\infty(E)$, so that $s_{\mathrm{top}}^{-1}(0) \subseteq X_{\mathrm{top}}$, and $f, g : X \to Y$ are morphisms in $\dot{\mathbf{Man}}$ with $g = f + O(s)$ in the sense of Definition B.36(iii). Then f is $\boldsymbol{P}$ near $s_{\mathrm{top}}^{-1}(0)$ if and only if g is $\boldsymbol{P}$ near $s_{\mathrm{top}}^{-1}(0)$.

(viii) Suppose we are given a diagram in $\dot{\mathbf{Man}}$:

$$\begin{tikzcd}
U' \arrow[r, hook, "i'"] \arrow[dr, "f'"'] & U \arrow[r, hook, "i"] \arrow[dr, "f"'] & X \\
V' \arrow[r, hook, "j'"'] \arrow[ur, "g'"] & V \arrow[r, hook, "j"'] \arrow[ur, "g"] & Y,
\end{tikzcd}$$

where i, i', j, j' are open submanifolds in $\dot{\mathbf{Man}}$, and $f \circ i' = j \circ f' : U' \to Y$, $g \circ j' = i \circ g' : V' \to X$, and we are given points $x \in U'_{\mathrm{top}} \subseteq U_{\mathrm{top}} \subseteq X_{\mathrm{top}}$ and $y \in V'_{\mathrm{top}} \subseteq V_{\mathrm{top}} \subseteq Y_{\mathrm{top}}$ such that $f_{\mathrm{top}}(x) = y$ and $g_{\mathrm{top}}(y) = x$. Suppose too that there are vector bundles $E \to U'$ and $F \to V'$ and sections $s \in \Gamma^\infty(E)$, $t \in \Gamma^\infty(F)$ with $s(x) = t(y) = 0$, such that $g \circ f' = i \circ i' + O(s)$ on U' and $f \circ g' = j \circ j' + O(t)$ on V' in the sense of Definition B.36(iii). Then f, f' are $\boldsymbol{P}$ near x, and g, g' are $\boldsymbol{P}$ near y.

Example B.39. **(a)** When $\dot{\mathbf{Man}}$ is $\mathbf{Man^c}$ from §2.1, the following properties of morphisms in $\mathbf{Man^c}$ are discrete: interior, b-normal, strongly smooth, simple.

(b) When $\dot{\mathbf{Man}}$ is $\mathbf{Man^{gc}}$ from §2.4.1, the following properties of morphisms in $\mathbf{Man^{gc}}$ are discrete: interior, b-normal, simple.

(c) When $\dot{\mathbf{Man}}$ is $\mathbf{Man^{ac}}$ or $\mathbf{Man^{c,ac}}$ from §2.4.2, the following properties of morphisms in $\dot{\mathbf{Man}}$ are discrete: interior, b-normal, strongly a-smooth, simple.

B.7 Comparing different categories $\dot{\mathbf{Man}}$

To each category $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7, in Chapters 4–6 we will associate (2-)categories $\mathbf{m\dot{K}ur}, \boldsymbol{\mu}\mathbf{\dot{K}ur}, \mathbf{\dot{K}ur}$ of (m- and μ-)Kuranishi spaces. As in §3.2 there are many examples of such $\dot{\mathbf{Man}}$, such as $\dot{\mathbf{Man}} = \mathbf{Man}$ or $\mathbf{Man^c}$, and many functors between them, such as the inclusion $\mathbf{Man} \hookrightarrow \mathbf{Man^c}$.

Here is an important condition on functors between such categories $\dot{\mathbf{Man}}$:

Condition B.40. Let $\dot{\mathbf{Man}}, \ddot{\mathbf{Man}}$ satisfy Assumptions 3.1–3.7, and $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \to \ddot{\mathbf{Man}}$ be a functor in the commutative diagram

$$\begin{tikzcd} & \dot{\mathbf{Man}} \arrow[dr, "F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}"] \arrow[d, "F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}"] & \\ \mathbf{Man} \arrow[ur, "\subset"] \arrow[dr, "\subset"'] & & \mathbf{Top}, \\ & \ddot{\mathbf{Man}} \arrow[ur, "F_{\ddot{\mathbf{Man}}}^{\mathbf{Top}}"'] & \end{tikzcd} \tag{B.30}$$

where the functors $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}, F_{\ddot{\mathbf{Man}}}^{\mathbf{Top}}$ are as in Assumption 3.2, and the inclusions $\mathbf{Man} \hookrightarrow \dot{\mathbf{Man}}, \ddot{\mathbf{Man}}$ as in Assumption 3.4. We require $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ to take products, disjoint unions, and open submanifolds in $\dot{\mathbf{Man}}$ to products, disjoint unions, and open submanifolds in $\ddot{\mathbf{Man}}$, and to preserve dimensions.

Note that $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ must be faithful (injective on morphisms), as $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}}$ is.

Figure 3.1 on page I-47 gives a diagram of functors from Chapter 2 satisfying Condition B.40. In Chapters 4–6, when Condition B.40 holds, we will define natural (2)-functors

$$F_{\mathbf{m\dot{K}ur}}^{\mathbf{m\ddot{K}ur}} : \mathbf{m\dot{K}ur} \longrightarrow \mathbf{m\ddot{K}ur}, \;\; F_{\boldsymbol{\mu}\mathbf{\dot{K}ur}}^{\boldsymbol{\mu}\mathbf{\ddot{K}ur}} : \boldsymbol{\mu}\mathbf{\dot{K}ur} \longrightarrow \boldsymbol{\mu}\mathbf{\ddot{K}ur}, \;\; F_{\mathbf{\dot{K}ur}}^{\mathbf{\ddot{K}ur}} : \mathbf{\dot{K}ur} \longrightarrow \mathbf{\ddot{K}ur}$$

between the (2-)categories $\mathbf{m\dot{K}ur}, \boldsymbol{\mu}\mathbf{\dot{K}ur}, \mathbf{\dot{K}ur}$ and $\mathbf{m\ddot{K}ur}, \boldsymbol{\mu}\mathbf{\ddot{K}ur}, \mathbf{\ddot{K}ur}$ associated to $\dot{\mathbf{Man}}$ and $\ddot{\mathbf{Man}}$. To do this, we must relate the material of §B.1–§B.5 on differential geometry and the $O(s), O(s^2)$ notation in $\dot{\mathbf{Man}}$ and in $\ddot{\mathbf{Man}}$.

Definition B.41. Let Condition B.40 hold. We will use accents ‘ $\dot{}$ ’ and ‘ $\ddot{}$ ’ to denote objects associated to $\dot{\mathbf{Man}}$ and $\ddot{\mathbf{Man}}$, respectively. When something is independent of $\dot{\mathbf{Man}}$ or $\ddot{\mathbf{Man}}$ we omit the accent, so for instance we write X_{top} for the underlying topological space of $\dot{X} \in \dot{\mathbf{Man}}$.

Let $\dot{X}$ be an object in $\dot{\mathbf{Man}}$, and set $\ddot{X} = F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}(\dot{X})$. Then all the material of §B.1–§B.5 on $\dot{X}$ in $\dot{\mathbf{Man}}$ maps to corresponding material on $\ddot{X}$ in $\ddot{\mathbf{Man}}$ in a straightforward way. Where relevant we use $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ to denote the functors transforming structures on $\dot{X}$ to structures on $\ddot{X}$. In more detail:

(a) The commutative $\mathbb{R}$-algebra $C^\infty(\dot{X})$ in §B.1.1 is the set of morphisms $a : \dot{X} \to \mathbb{R}$ in $\dot{\mathbf{Man}}$. Applying $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ gives a map

$$F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : C^\infty(\dot{X}) \to C^\infty(\ddot{X}). \tag{B.31}$$

This is injective, as $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ is faithful, and an $\mathbb{R}$-algebra morphism, and a C^∞-ring morphism for the C^∞-ring structures in §B.1.2.

(b) Section B.1.3 defines the structure sheaves $\mathcal{O}_{\dot{X}}$ on X_{top} for $\dot{X} \in \dot{\mathbf{Man}}$, and $\mathcal{O}_{\ddot{X}}$ on X_{top} for $\ddot{X} \in \ddot{\mathbf{Man}}$. There is a natural morphism $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \mathcal{O}_{\dot{X}} \to \mathcal{O}_{\ddot{X}}$ of sheaves of $\mathbb{R}$-algebras or C^∞-rings on X_{top}, such that if $i : \dot{U} \hookrightarrow \dot{X}$ is an open submanifold in $\dot{\mathbf{Man}}$ then

$$F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}(U_{\mathrm{top}}) : \mathcal{O}_{\dot{X}}(U_{\mathrm{top}}) = C^\infty(\dot{U}) \longrightarrow \mathcal{O}_{\ddot{X}}(U_{\mathrm{top}}) = C^\infty(\ddot{U})$$

is the morphism (B.31) for $\dot U$.

(c) In §B.1.3–§B.2.2, $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ takes partitions of unity, vector bundles, sections, and $\mathcal{O}_{\dot X}$-modules of sections of vector bundles in $\dot{\mathbf{Man}}$, to their analogues in $\ddot{\mathbf{Man}}$, in the obvious way.

(d) In §B.3.1, we define the cotangent sheaf $\mathcal{T}^*\dot X$ as the sheafification of $\mathcal{PT}^*\dot X$, where if $i:\dot U\hookrightarrow\dot X$ is open in $\dot{\mathbf{Man}}$ then $\mathcal{PT}^*\dot X(U_{\rm top})=\Omega_{C^\infty(\dot U)}$.

Since $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}:C^\infty(\dot U)\to C^\infty(\ddot U)$ in (a) is a C^∞-ring morphism, Definition B.10 gives a module morphism

$$\mathcal{P}F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}(U_{\rm top}):=\Omega_{F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}}:\mathcal{PT}^*\dot X(U_{\rm top})=\Omega_{C^\infty(\dot U)}\to\mathcal{PT}^*\ddot X(U_{\rm top})=\Omega_{C^\infty(\ddot U)}.$$

These define a morphism $\mathcal{P}F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}:\mathcal{PT}^*\dot X\to\mathcal{PT}^*\ddot X$ of presheaves on $X_{\rm top}$. Sheafifying gives a morphism $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}:\mathcal{T}^*\dot X\to\mathcal{T}^*\ddot X$ of sheaves on $X_{\rm top}$, which is a module morphism under $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}:\mathcal{O}_{\dot X}\to\mathcal{O}_{\ddot X}$ from (b).

(e) Let $\dot E\to\dot X$ be a vector bundle, and $\dot{\mathcal{E}}$ the $\mathcal{O}_{\dot X}$-module of sections of $\dot E$ from §B.2.2, and $\dot\nabla:\dot{\mathcal{E}}\to\dot{\mathcal{E}}\otimes_{\mathcal{O}_{\dot X}}\mathcal{T}^*\dot X$ be a connection on $\dot E$, as in §B.3.2. Then one can show there is a unique connection $\ddot\nabla$ on $\ddot E$ such that the following diagram of morphisms on sheaves on $X_{\rm top}$ commutes:

$$\begin{tikzcd}
\dot{\mathcal{E}} \arrow[rr, "\dot\nabla"] \arrow[d, "F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}\text{ from (c)}"] && \dot{\mathcal{E}}\otimes_{\mathcal{O}_{\dot X}}\mathcal{T}^*\dot X \arrow[d, "F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}\text{ from (b),(c),(d)}"'] \\
\ddot{\mathcal{E}} \arrow[rr, "\ddot\nabla"] && \ddot{\mathcal{E}}\otimes_{\mathcal{O}_{\ddot X}}\mathcal{T}^*\ddot X.
\end{tikzcd}$$

(f) Let $\dot f:\dot X\to\dot Y$ be a morphism in $\dot{\mathbf{Man}}$, and $\ddot f:\ddot X\to\ddot Y$ its image in $\ddot{\mathbf{Man}}$ under $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$. Then §B.4.1–§B.4.2 define a $C^\infty(\dot X)$-module $\Gamma(\mathcal{T}_{\dot f}\dot Y)$. There is an obvious map

$$F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}:\Gamma(\mathcal{T}_{\dot f}\dot Y)\longrightarrow\Gamma(\mathcal{T}_{\ddot f}\ddot Y),\quad F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}:[\dot U,\dot u]\longmapsto[\ddot U,\ddot u].\tag{B.32}$$

To see this is well defined, note that in Definition B.16, if $(\dot U,\dot u)\approx(\dot U',\dot u')$ in $\dot{\mathbf{Man}}$ then $(\ddot U,\ddot u)\approx(\ddot U',\ddot u')$ in $\ddot{\mathbf{Man}}$, as $j,\dot V,\dot v$ in $\dot{\mathbf{Man}}$ satisfying (B.6) map to $\ddot\jmath,\ddot V,\ddot v$ in $\ddot{\mathbf{Man}}$ satisfying (B.6), so $[\ddot U,\ddot u]$ in (B.32) depends only on the equivalence class $[\dot U,\dot u]$.

Equation (B.32) is a module morphism under (B.31).

(g) Section B.4.5 defines the sheaves of $\mathcal{O}_{\dot X}$-modules $\mathcal{T}\dot X$ and $\mathcal{T}_{\dot f}\dot Y$. Using (B.32) we define sheaf morphisms $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}:\mathcal{T}\dot X\to\mathcal{T}\ddot X$ and $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}:\mathcal{T}_{\dot f}\dot Y\to\mathcal{T}_{\ddot f}\ddot Y$ which are module morphisms over $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}:\mathcal{O}_{\dot X}\to\mathcal{O}_{\ddot X}$ from (b).

(h) In §B.4.6–§B.4.9, $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ is compatible with the definitions and operations in the obvious way.

(i) In §B.5, $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ maps all the $O(\dot{s})$ and $O(\dot{s}^2)$ conditions in $\dot{\mathbf{Man}}$ from Definition B.36(i)–(vii) to the corresponding $O(\ddot{s})$ and $O(\ddot{s}^2)$ conditions in $\ddot{\mathbf{Man}}$, in the obvious way.

Remark B.42. The definitions of §B.1–§B.5 have been carefully designed so that material for $\dot{\mathbf{Man}}$ all transforms functorially to $\ddot{\mathbf{Man}}$ under $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ without problems, as in Definition B.41. It would have been easy, and more obvious, to write down definitions which lack this functorial behaviour.

Here is an example of this. Let $\dot{f} : \dot{X} \to \dot{Y}$ be a morphism in $\dot{\mathbf{Man}}$. In §B.4.3 we discussed relative (C^∞-)derivations $\dot{\Delta} : C^\infty(\dot{Y}) \to C^\infty(\dot{X})$. These are a natural notion of vector field over $\dot{f}$, and we could have defined $\Gamma(\mathcal{T}_{\dot{f}}\dot{Y})$ in §B.4.1 as a $C^\infty(\dot{X})$-module of such derivations. However, in the diagram

$$\begin{tikzcd}
C^\infty(\dot{Y}) \arrow[rr, "\dot{\Delta}"'] \arrow[d, "F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}"] & & C^\infty(\dot{X}) \arrow[d, "F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}"'] \\
C^\infty(\ddot{Y}) \arrow[rr, dotted, "\ddot{\Delta}"] & & C^\infty(\ddot{X}),
\end{tikzcd}$$

it is unclear whether a relative (C^∞-)derivation $\ddot{\Delta}$ must exist, or if it is unique. So defining $\mathcal{T}_{\dot{f}}\dot{Y}$ using (C^∞-)derivations would not be functorial under $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$.

For an inclusion of subcategories $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$ we can say more:

Proposition B.43. *Suppose $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}} : \dot{\mathbf{Man}} \hookrightarrow \ddot{\mathbf{Man}}$ is an inclusion of subcategories satisfying Condition* B.40, *and either:*

(a) *All objects of $\ddot{\mathbf{Man}}$ are objects of $\dot{\mathbf{Man}}$, and all morphisms $f : X \to \mathbb{R}$ in $\ddot{\mathbf{Man}}$ are morphisms in $\dot{\mathbf{Man}}$, and for a morphism $f : X \to Y$ in $\ddot{\mathbf{Man}}$ to lie in $\dot{\mathbf{Man}}$ is a* ***discrete*** *condition, as in Definition* B.38; *or*

(b) *$\dot{\mathbf{Man}}$ is a full subcategory of $\ddot{\mathbf{Man}}$ closed under isomorphisms in $\ddot{\mathbf{Man}}$.*

Then all the material of §B.1–§B.5 *for $\dot{\mathbf{Man}}$ is* ***exactly the same*** *if computed in $\dot{\mathbf{Man}}$ or $\ddot{\mathbf{Man}}$, and all the morphisms $F_{\dot{\mathbf{Man}}}^{\ddot{\mathbf{Man}}}$ in Definition* B.41 *are the identity maps. For example, if $f : X \to Y$ lies in $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$ then the relative tangent sheaves $(\mathcal{T}_fY)_{\dot{\mathbf{Man}}}, (\mathcal{T}_fY)_{\ddot{\mathbf{Man}}}$ on X_{top} from* §B.4 *computed in $\dot{\mathbf{Man}}$ and $\ddot{\mathbf{Man}}$ are not just canonically isomorphic, but actually the same sheaf.*

Proof. Suppose we start with an object X in $\dot{\mathbf{Man}}$, or a morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$, and then construct differential-geometric data in §B.1–§B.5 such as $C^\infty(X), \mathcal{O}_X, \mathcal{T}^*X, \mathcal{T}X$ or $\mathcal{T}_fY$, either in $\dot{\mathbf{Man}}$, or in $\ddot{\mathbf{Man}}$. The point of the proof is that when we do this in $\ddot{\mathbf{Man}}$, the constructions only ever involve objects and morphisms in $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$, so that the data $C^\infty(X), \mathcal{O}_X, \ldots, \mathcal{T}_fY$ are the same when computed in $\dot{\mathbf{Man}}$ or $\ddot{\mathbf{Man}}$.

Mostly this is straightforward to check, and we leave this to the reader. For example, for $X \in \dot{\mathbf{Man}}$ the C^∞-rings $C^\infty(X)_{\dot{\mathbf{Man}}}, C^\infty(X)_{\ddot{\mathbf{Man}}}$ are the sets of morphisms $f : X \to \mathbb{R}$ in $\dot{\mathbf{Man}}$ and in $\ddot{\mathbf{Man}}$. In case (a) these coincide by assumption, and in case (b) they coincide as $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$ is full. Then $\mathcal{O}_X, \mathcal{T}^*X$ are the same in $\dot{\mathbf{Man}}$ and $\ddot{\mathbf{Man}}$ as they are constructed from C^∞-rings $C^\infty(U)$ for open $i : U \hookrightarrow X$, which are the same in $\dot{\mathbf{Man}}$ and $\ddot{\mathbf{Man}}$.

We explain one subtle point concerning $\mathcal{T}_f Y$. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$, and consider the definition of $\Gamma(\mathcal{T}_f Y)$ in Definition B.16 in $\dot{\mathbf{Man}}$ and $\ddot{\mathbf{Man}}$. In case (a), for a diagram (B.5) in $\ddot{\mathbf{Man}}$, it is clear that the data $X, Y, X\times\mathbb{R}, f, i, (\mathrm{id}_X, 0)$ lie in $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$, but it is not obvious that $u : U \to Y$ lies in $\dot{\mathbf{Man}}$. However, we can prove this using Definition B.38.

Taking $E = U \times \mathbb{R} \to U$ to be the trivial line bundle and defining $s \in \Gamma^\infty(E)$ by $s(x,t) = ((x,t),t)$, we see from (B.5) that $u = f \circ \pi_X + O(s)$ in morphisms $U \to Y$ in $\ddot{\mathbf{Man}}$. But $f \circ \pi_X$ lies in $\dot{\mathbf{Man}}$, so u lies in $\dot{\mathbf{Man}}$ near $X_{\mathrm{top}} \times \{0\}$ in U_{top} by Definition B.38(vii). Then using Definition B.38(i)–(iv),(vi) and the assumption in Definition B.16 that U_{top} can be written as a union of subsets $X'_{\mathrm{top}} \times (-\epsilon, \epsilon)$ in $X_{\mathrm{top}} \times \mathbb{R}$ for $X'_{\mathrm{top}} \subseteq X_{\mathrm{top}}$ open and $\epsilon > 0$, we can deduce that $u : U \to Y$ lies in $\dot{\mathbf{Man}}$, so (B.5) is a diagram in $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$. Similarly, for $j : V \hookrightarrow X \times \mathbb{R}^2$, $v : V \to Y$ in $\ddot{\mathbf{Man}}$ satisfying (B.6) used to define the equivalence relation $\approx$ on pairs (U, u), making V smaller we can suppose that $V_{\mathrm{top}} = X'_{\mathrm{top}} \times (-\epsilon, \epsilon)^2$ for $\tilde{x} \in X'_{\mathrm{top}}$, and then V, j, v lie in $\dot{\mathbf{Man}} \subseteq \ddot{\mathbf{Man}}$, so that $\Gamma(\mathcal{T}_f Y)_{\dot{\mathbf{Man}}} = \Gamma(\mathcal{T}_f Y)_{\ddot{\mathbf{Man}}}$. $\square$

B.8 Differential geometry in $\dot{\mathbf{Man}}^{\mathbf{c}}$

Suppose $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumption 3.22 in §3.4. Then $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumptions 3.1–3.7, so §B.1–§B.5 applies in $\dot{\mathbf{Man}}^{\mathbf{c}}$. Section B.8.1 introduces new material for the corners case, such as morphisms $I_X^\diamond : \Pi_k^{-1}(\mathcal{T}X) \to \mathcal{T}C_k(X)$ analogous to those in (2.13). Section B.8.2 compares differential geometry in two categories $\dot{\mathbf{Man}}^{\mathbf{c}}, \ddot{\mathbf{Man}}^{\mathbf{c}}$, as in §B.7.

B.8.1 Action of the corner functor on tangent sheaves

In §4.6, for an m-Kuranishi space with corners $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur^c}$ we define the boundary $\partial\boldsymbol{X}$ and k-corners $C_k(\partial\boldsymbol{X})$, and we define the corner 2-functor $C : \mathbf{m\dot{K}ur^c} \to \mathbf{m\check{K}ur^c}$. To do this, for a manifold with corners X in $\dot{\mathbf{Man}}^{\mathbf{c}}$ with k-corner morphism $\Pi_k : C_k(X) \to X$ as in Assumption 3.22(d), we must lift differential geometry on X to differential geometry on $C_k(X)$.

Much of this follows by applying pullbacks in §B.1–§B.5 to Π_k. But we need one extra structure relating (relative) tangent sheaves on X and $C_k(X)$.

Definition B.44. Suppose $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, so that $C(f) : C(X) \to C(Y)$ and $\Pi : C(X) \to X$ are morphisms in $\check{\mathbf{Man}}^{\mathbf{c}}$. Then §B.4.5 defines the relative tangent sheaves $\mathcal{T}_f Y$ on X_{top} and $\mathcal{T}_{C(f)}C(Y)$ on $C(X)_{\mathrm{top}} = \coprod_{k\geqslant 0} C_k(X)_{\mathrm{top}}$, extending from $\dot{\mathbf{Man}}^{\mathbf{c}}$ to $\check{\mathbf{Man}}^{\mathbf{c}}$ in the obvious way.

We will define a morphism of sheaves on $C(X)_{\mathrm{top}}$:

$$I_f^\diamond : \Pi_{\mathrm{top}}^{-1}(\mathcal{T}_f Y) \longrightarrow \mathcal{T}_{C(f)}C(Y), \tag{B.33}$$

which is a module morphism under $\Pi^\sharp : \Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X) \to \mathcal{O}_{C(X)}$ from §B.1.3, where $\Pi_{\mathrm{top}}^{-1}(\mathcal{T}_f Y), \mathcal{T}_{C(f)}C(Y)$ are modules over $\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X), \mathcal{O}_{C(X)}$ respectively, as in §B.4.5. This does not follow from our previous constructions for $C(f), \Pi$, it is a new feature for manifolds with corners $\mathbf{\dot{M}an^c}$.

First we define an $\mathbb{R}$-linear map

$$\Gamma(I_{f,\diamond}) : \Gamma(\mathcal{T}_f Y) \longrightarrow \Gamma(\mathcal{T}_{C(f)}C(Y)). \tag{B.34}$$

Recall from §B.4.1 that $\Gamma(\mathcal{T}_f Y)$ is the set of $\approx$-equivalence classes $[U, u]$ of diagrams (B.5) in $\mathbf{\dot{M}an^c}$, where $\approx$ is defined using $j : V \hookrightarrow X \times \mathbb{R}^2$, $v : V \to Y$ in $\mathbf{\dot{M}an^c}$ satisfying (B.6). We have canonical isomorphisms

$$C(X \times \mathbb{R}) \cong C(X) \times C(\mathbb{R}) = C(X) \times C_0(\mathbb{R}) \cong C(X) \times \mathbb{R}, \tag{B.35}$$

where the first step comes from Assumption 3.22(h), the second from Assumption 3.22(e), and the third from $\Pi_0 : C_0(\mathbb{R}) \to \mathbb{R}$ an isomorphism in Assumption 3.22(d). Applying the corner functor $C : \mathbf{\dot{M}an^c} \to \mathbf{\check{\dot{M}}an^c}$ to (B.5) and making the identification (B.35) gives a commutative diagram in $\mathbf{\check{\dot{M}}an^c}$

$$\begin{tikzcd}
& C(X) \arrow[dl, "{(\mathrm{id}_{C(X)},0)}"'] \arrow[d, "C(l)"] \arrow[dr, "C(f)"] & \\
C(X) \times \mathbb{R} & C(U) \arrow[l, hook', "C(i)"'] \arrow[r, "C(u)"] & C(Y),
\end{tikzcd}$$

which is a diagram (B.5) for $C(f)$. Hence $[C(U), C(u)] \in \Gamma(\mathcal{T}_{C(f)}C(Y))$. Similarly, applying C to $j : V \hookrightarrow X \times \mathbb{R}^2$, $v : V \to Y$ satisfying (B.6) shows that if $(U, u) \approx (U', u')$ then $(C(U), C(u)) \approx (C(U'), C(u'))$, so the $\approx$-equivalence class $[C(U), C(u)]$ depends only on $[U, u]$. Define $\Gamma(I_{f,\diamond})$ in (B.34) by

$$\Gamma(I_{f,\diamond}) : [U, u] \longmapsto [C(U), C(u)].$$

Now $\Gamma(\mathcal{T}_f Y)$ is a module over $C^\infty(X)$ as in §B.4.2, and $\Gamma(\mathcal{T}_{C(f)}C(Y))$ a module over $C^\infty(C(X))$, and §B.1.1 defines a morphism $\Pi^* : C^\infty(X) \to C^\infty(C(X))$. If $a \in C^\infty(X)$, so that $a : X \to \mathbb{R}$ is a morphism in $\mathbf{\dot{M}an^c}$, then Assumption 3.22(g) implies that

$$\Pi^*(a) = a \circ \Pi = \Pi_0 \circ C(a) : C(X) \longrightarrow \mathbb{R} \qquad \text{in } \mathbf{\check{\dot{M}}an^c},$$

where $\Pi_0 : C_0(\mathbb{R}) \xrightarrow{\cong} \mathbb{R}$ is used in the identification (B.35). Using this we can easily show that (B.34) is a module morphism under $\Pi^* : C^\infty(X) \to C^\infty(C(X))$.

Suppose $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathbf{\dot{M}an^c}$. Then §B.4.4 defines morphisms $\Gamma(\mathcal{T}g) : \Gamma(\mathcal{T}_f Y) \to \Gamma(\mathcal{T}_{g\circ f} Z)$ and $f^* : \Gamma(\mathcal{T}_g Z) \to \Gamma(\mathcal{T}_{g\circ f} Z)$, and similarly for $\Gamma(\mathcal{T}\Pi), \Gamma(\mathcal{T}C(g))$ and $\Pi^*, C(f)^*$. By applying the corner functor C to the definitions we see that the following diagrams commute:

$$\begin{tikzcd}
\Gamma(\mathcal{T}_f Y) \arrow[rr, "\Gamma(I_{f,\diamond})"'] \arrow[d, "\Pi^*"] && \Gamma(\mathcal{T}_{C(f)}C(Y)) \arrow[d, "\Gamma(\mathcal{T}\Pi)"'] \\
\Gamma(\mathcal{T}_{f\circ\Pi}Y) \arrow[rr, equal] && \Gamma(\mathcal{T}_{\Pi\circ C(f)}Y),
\end{tikzcd} \tag{B.36}$$

$$\begin{tikzcd}
\Gamma(\mathcal{T}_f Y) \arrow[rr, "\Gamma(I_{f,\diamond})"'] \arrow[d, "\Gamma(\mathcal{T}g)"] && \Gamma(\mathcal{T}_{C(f)}C(Y)) \arrow[d, "\Gamma(\mathcal{T}C(g))"'] \\
\Gamma(\mathcal{T}_{g\circ f}Z) \arrow[r, "\Gamma(I_{g\circ f,\diamond})"] & \Gamma(\mathcal{T}_{C(g\circ f)}C(Z)) = \Gamma(\mathcal{T}_{C(g)\circ C(f)}C(Z)), &
\end{tikzcd} \tag{B.37}$$

$$\begin{tikzcd}
\Gamma(\mathcal{T}_g Z) \arrow[rr, "\Gamma(I_{g,\diamond})"'] \arrow[d, "f^*"] && \Gamma(\mathcal{T}_{C(g)}C(Z)) \arrow[d, "C(f)^*"'] \\
\Gamma(\mathcal{T}_{g\circ f}Z) \arrow[r, "\Gamma(I_{g\circ f,\diamond})"] & \Gamma(\mathcal{T}_{C(g\circ f)}C(Z)) = \Gamma(\mathcal{T}_{C(g)\circ C(f)}C(Z)). &
\end{tikzcd} \tag{B.38}$$

Let $i : X' \hookrightarrow X$ be an open submanifold in $\dot{\mathbf{Man}}^{\mathbf{c}}$, so that $C(i) : C(X') \hookrightarrow C(X)$ is an open submanifold in $\check{\dot{\mathbf{M}}}\mathbf{an}^{\mathbf{c}}$ by Assumption 3.22(j). Define

$$\begin{aligned}
I_{f,\diamond}(X'_{\rm top}) = \Gamma(I_{f\circ i,\diamond}) : (\mathcal{T}_f Y)(X'_{\rm top}) = \Gamma(\mathcal{T}_{f\circ i}Y) \\
\longrightarrow (\Pi_{\rm top})_*(\mathcal{T}_{C(f)}C(Y))(X'_{\rm top}) = \mathcal{T}_{C(f)}C(Y)(\Pi_{\rm top}^{-1}(X'_{\rm top})) \\
= \mathcal{T}_{C(f)}C(Y)(C(X')_{\rm top}) = \Gamma(\mathcal{T}_{C(f)\circ C(i)}C(Y)) = \Gamma(\mathcal{T}_{C(f\circ i)}C(Y)).
\end{aligned}$$

We claim that these $I_{f,\diamond}(X'_{\rm top}) : (\mathcal{T}_f Y)(X'_{\rm top}) \to (\Pi_{\rm top})_*(\mathcal{T}_{C(f)}C(Y))(X'_{\rm top})$ for all open $X'_{\rm top} \subseteq X_{\rm top}$ define a sheaf morphism

$$I_{f,\diamond} : \mathcal{T}_f Y \longrightarrow (\Pi_{\rm top})_*(\mathcal{T}_{C(f)}C(Y)) \tag{B.39}$$

on $X_{\rm top}$, as in §A.5. To prove this let $X''_{\rm top} \subseteq X'_{\rm top} \subseteq X_{\rm top}$ be open, corresponding to open submanifolds $i : X' \hookrightarrow X$, $j : X'' \hookrightarrow X'$, and use (B.38) with $j, f\circ i$ in place of f, g to show that $I_{f,\diamond}(X''_{\rm top}) \circ \rho_{X'_{\rm top}X''_{\rm top}} = \rho_{X'_{\rm top}X''_{\rm top}} \circ I_{f,\diamond}(X'_{\rm top})$. Here $\mathcal{T}_f Y, (\Pi_{\rm top})_*(\mathcal{T}_{C(f)}C(Y))$ are modules over $\mathcal{O}_X, (\Pi_{\rm top})_*(\mathcal{O}_{C(X)})$. As (B.34) is a module morphism under $\Pi^* : C^\infty(X) \to C^\infty(C(X))$, we see that $I_{f,\diamond}$ in (B.39) is a module morphism under $\Pi_\sharp : \mathcal{O}_X \to (\Pi_{\rm top})_*(\mathcal{O}_{C(X)})$ from §B.1.3.

Write $I_f^\diamond$ in (B.33) for the sheaf morphism on $C(X)_{\rm top}$ adjoint to $I_{f,\diamond}$ under (A.18). Since $I_{f,\diamond}$ is a module morphism under $\Pi_\sharp : \mathcal{O}_X \to (\Pi_{\rm top})_*(\mathcal{O}_{C(X)})$, and $\Pi^\sharp : \Pi_{\rm top}^{-1}(\mathcal{O}_X) \to \mathcal{O}_{C(X)}$ is adjoint to $\Pi_\sharp$ under (A.18) as in §B.1.3, we see that $I_f^\diamond$ is a module morphism under $\Pi^\sharp$.

If f is simple, so that $C(f) : C(X) \to C(Y)$ maps $C_k(X) \to C_k(Y)$ for $k \geqslant 0$ by Assumption 3.22(i), then $I_f^\diamond$ restricts to $I_f^\diamond : \Pi_{k,{\rm top}}^{-1}(\mathcal{T}_f Y) \to \mathcal{T}_{C_k(f)}C_k(Y)$ for each k. When $f = {\rm id}_X$, which is simple, with $\mathcal{T}X = \mathcal{T}_{{\rm id}_X}X$, we write $I_{{\rm id}_X}^\diamond$ as $I_X^\diamond : \Pi_{k,{\rm top}}^{-1}(\mathcal{T}X) \to \mathcal{T}C_k(X)$. This is an analogue of $I_X^\diamond : \Pi_k^*({}^bTX) \to {}^bT(C_k(X))$ in (2.13) for ordinary manifolds with corners $\mathbf{Man}^{\mathbf{c}}$.

Suppose $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$. Then by using (B.36)–(B.38) for all open subsets $X'_{\rm top} \subseteq X_{\rm top}$, $Y'_{\rm top} \subseteq Y_{\rm top}$, we can show that the following diagrams of sheaves on $X_{\rm top}$ and $Y_{\rm top}$ commute:

$$
\begin{tikzcd}
\mathcal{T}_f Y \arrow[r, "I_{f,\diamond}"'] \arrow[d, "\Pi_\flat"] & (\Pi_{\rm top})_*(\mathcal{T}_{C(f)}C(Y)) \arrow[d, "(\Pi_{\rm top})_*(\mathcal{T}\Pi)"'] \\
(\Pi_{\rm top})_*(\mathcal{T}_{f\circ\Pi}Y) \arrow[r, equal] & (\Pi_{\rm top})_*(\mathcal{T}_{\Pi\circ C(f)}Y),
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{T}_f Y \arrow[rr, "I_{f,\diamond}"'] \arrow[d, "\mathcal{T}g"] & & (\Pi_{\rm top})_*(\mathcal{T}_{C(f)}C(Y)) \arrow[d, "\mathcal{T}C(g)"'] \\
\mathcal{T}_{g\circ f} Z \arrow[r, "I_{g\circ f,\diamond}"] & (\Pi_{\rm top})_*(\mathcal{T}_{C(g\circ f)}C(Z)) \arrow[r, phantom, "="] & (\Pi_{\rm top})_*(\mathcal{T}_{C(g)\circ C(f)}C(Z)),
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{T}_g Z \arrow[r, "I_{g,\diamond}"'] \arrow[d, "f_\flat"] & (\Pi_{\rm top})_*(\mathcal{T}_{C(g)}C(Z)) \arrow[d, "(\Pi_{\rm top})_*(C(f)_\flat)"'] \\
(f_{\rm top})_*(\mathcal{T}_{g\circ f}Z) \arrow[r, "(f_{\rm top})_*(I_{g\circ f,\diamond})"] & \begin{array}{l}(f_{\rm top})_*\circ(\Pi_{\rm top})_*(\mathcal{T}_{C(g\circ f)}C(Z)) = \\ (\Pi_{\rm top})_*\circ(f_{\rm top})_*(\mathcal{T}_{C(g)\circ C(f)}C(Z)),\end{array}
\end{tikzcd}
$$

where $f_\flat, \mathcal{T}g$ are as in §B.4.6. Then using the adjoint property of $I_{f,\diamond}, f_\flat$ and $I_f^\diamond, f^\flat$ we deduce that the following diagrams of sheaves on $C(X)_{\rm top}$ commute:

$$
\begin{tikzcd}
\Pi_{\rm top}^{-1}(\mathcal{T}_f Y) \arrow[r, "I_f^\diamond"'] \arrow[d, "\Pi^\flat"] & \mathcal{T}_{C(f)}C(Y) \arrow[d, "\mathcal{T}\Pi"'] \\
\mathcal{T}_{f\circ\Pi}Y \arrow[r, equal] & \mathcal{T}_{\Pi\circ C(f)}Y
\end{tikzcd}
\tag{B.40}
$$

$$
\begin{tikzcd}
\Pi_{\rm top}^{-1}(\mathcal{T}_f Y) \arrow[rr, "I_f^\diamond"'] \arrow[d, "\Pi_{\rm top}^{-1}(\mathcal{T}g)"] & & \mathcal{T}_{C(f)}C(Y) \arrow[d, "\mathcal{T}C(g)"'] \\
\Pi_{\rm top}^{-1}(\mathcal{T}_{g\circ f}Z) \arrow[r, "I_{g\circ f}^\diamond"] & \mathcal{T}_{C(g\circ f)}C(Z) \arrow[r, phantom, "="] & \mathcal{T}_{C(g)\circ C(f)}C(Z),
\end{tikzcd}
\tag{B.41}
$$

$$
\begin{tikzcd}
\begin{array}{l}\Pi_{\rm top}^{-1}\circ f_{\rm top}^{-1}(\mathcal{T}_g Z) = \\ C(f)_{\rm top}^{-1}\circ\Pi_{\rm top}^{-1}(\mathcal{T}_g Z)\end{array} \arrow[rr, "C(f)_{\rm top}^{-1}(I_g^\diamond)"'] \arrow[d, "\Pi_{\rm top}^{-1}(f^\flat)"] & & C(f)_{\rm top}^{-1}(\mathcal{T}_{C(g)}C(Z)) \arrow[d, "C(f)^\flat"'] \\
\Pi_{\rm top}^{-1}(\mathcal{T}_{g\circ f}Z) \arrow[r, "I_{g\circ f}^\diamond"] & \mathcal{T}_{C(g\circ f)}C(Z) \arrow[r, phantom, "="] & \mathcal{T}_{C(g)\circ C(f)}C(Z).
\end{tikzcd}
\tag{B.42}
$$

We use these $I_f^\diamond$ to pull back morphisms $E \to \mathcal{T}_f Y$ by $\Pi : C(X) \to X$.

Definition B.45. Let $f : X \to Y$ be a morphism in $\dot{\mathbf{M}}\mathbf{an^c}$, and $E \to X$ be a vector bundle on X, and $\theta : E \to \mathcal{T}_f Y$ be a morphism on X in the sense of §B.4.8, so that $\theta : \mathcal{E} \to \mathcal{T}_f Y$ is an $\mathcal{O}_X$-module morphism, where $\mathcal{E}$ is the $\mathcal{O}_X$-module of sections of E as in §B.2.2.

Then we have a morphism $C(f) : C(X) \to C(Y)$ in $\check{\dot{\mathbf{M}}}\mathbf{an^c}$, and pulling back by $\Pi : C(X) \to X$ gives a vector bundle $\Pi^*(E) \to C(X)$. Define a morphism

$\Pi^\diamond(\theta):\Pi^*(E)\rightarrow\mathcal{T}_{C(f)}C(Y)$ on $C(X)$ by the commutative diagram

$$\begin{tikzcd}
\Pi^*(\mathcal{E}) \arrow[rr, equal] \arrow[d, "\Pi^\diamond(\theta)"] & & \Pi_{\mathrm{top}}^{-1}(\mathcal{E})\otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)}\mathcal{O}_{C(X)} \arrow[dd, "\Pi_{\mathrm{top}}^{-1}(\theta)\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"'] \\
\mathcal{T}_{C(f)}C(Y) \arrow[d, equal] & & \\
\mathcal{T}_{C(f)}C(Y)\otimes_{\mathcal{O}_{C(X)}}\mathcal{O}_{C(X)} & & \Pi_{\mathrm{top}}^{-1}(\mathcal{T}_fY)\otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)}\mathcal{O}_{C(X)}, \arrow[ll, "I_f^\diamond\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"']
\end{tikzcd} \tag{B.43}$$

where $\Pi^*(\mathcal{E})$ is the $\mathcal{O}_{C(X)}$-module of sections of $\Pi^*(E)\rightarrow C(X)$, and the bottom morphism in (B.43) is formed using the morphism $\Pi^\sharp:\Pi^{-1}(\mathcal{O}_X)\rightarrow\mathcal{O}_{C(X)}$ from §B.1.3, and is well defined as $I_f^\diamond$ is a module morphism over $\Pi^\sharp$.

In Definition B.32, given a diagram (B.22) involving $v:V\rightarrow Y$ for open $V\hookrightarrow E$ with $0_{E,\mathrm{top}}(X_{\mathrm{top}})\subseteq V_{\mathrm{top}}\subseteq E_{\mathrm{top}}$, we defined a morphism $\theta_{V,v}:E\rightarrow\mathcal{T}_fY$, and Proposition B.33 showed that every morphism $\theta:E\rightarrow\mathcal{T}_fY$ is of the form $\theta=\theta_{V,v}$ for some diagram (B.22). We can use this to interpret $\Pi^\diamond(\theta)$: applying $C:\dot{\mathbf{Man}}^{\mathbf{c}}\rightarrow\check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$ to (B.22) gives a diagram (B.22) for $C(f):C(X)\rightarrow C(Y)$ and $\Pi^*(E)\rightarrow C(X)$ in place of f,E. Hence $\theta_{C(V),C(v)}$ is a morphism $\Pi^*(E)\rightarrow\mathcal{T}_{C(f)}C(Y)$, and it is easy to see that

$$\Pi^\diamond(\theta_{V,v})=\theta_{C(V),C(v)}. \tag{B.44}$$

We think of $\Pi^\diamond(\theta)$ as a kind of pullback of θ by $\Pi:C(X)\rightarrow X$.

We write the restriction $\Pi^\diamond(\theta)|_{C_k(X)}$ for $k=0,1,\ldots$ as $\Pi_k^\diamond(\theta)$. Thus if $f:X\rightarrow Y$ is simple, so that $C(f)$ maps $C_k(X)\rightarrow C_k(Y)$ by Assumption 3.22(i), we have morphisms $\Pi_k^\diamond(\theta):\Pi_k^*(E)\rightarrow\mathcal{T}_{C_k(f)}C_k(Y)$ for $k=0,1,\ldots$.

Example B.46. Take $\dot{\mathbf{Man}}^{\mathbf{c}}=\mathbf{Man}^{\mathbf{c}}$, and let $f:X\rightarrow Y$ be an interior map in $\mathbf{Man}^{\mathbf{c}}$, and $E\rightarrow X$ be a vector bundle. Then $\mathcal{T}_fY$ is the sheaf of sections of $f^*({}^bTY)\rightarrow X$, as in Example B.26(b),(c), so morphisms $\theta:E\rightarrow\mathcal{T}_fY$ correspond to vector bundle morphisms $\tilde\theta:E\rightarrow f^*({}^bTY)$ on X. Then $\Pi^\diamond(\theta)$ corresponds to the composition of vector bundle morphisms on $C(X)$:

$$\Pi^*(E)\xrightarrow{\Pi^*(\tilde\theta)}\Pi^*\circ f^*({}^bTY)=C(f)^*\circ\Pi^*({}^bTY)\xrightarrow{C(f)^*(I_Y^\diamond)}C(f)^*({}^bTC(Y)),$$

where $I_Y^\diamond:\Pi^*({}^bTY)\rightarrow{}^bTC(Y)$ is as in (2.13).

Here are some properties of the morphisms $\Pi^\diamond(\theta)$:

Theorem B.47. (a) *Let $f:X\rightarrow Y$ be a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $E\rightarrow X$ be a vector bundle, and $\theta:E\rightarrow\mathcal{T}_fY$ be a morphism, in the sense of* §B.4.8. *Then the following diagram of sheaves on $C(X)_{\mathrm{top}}$ commutes:*

$$\begin{tikzcd}
\Pi^*(E) \arrow[r, "\Pi^\diamond(\theta)"'] \arrow[d, "\Pi^*(\theta)"] & \mathcal{T}_{C(f)}C(Y) \arrow[d, "\mathcal{T}\Pi"'] \\
\mathcal{T}_{f\circ\Pi}Y \arrow[r, equal] & \mathcal{T}_{\Pi\circ C(f)}Y,
\end{tikzcd}$$

where $\mathcal{T}\Pi$ and $\Pi^*(\theta)$ are defined in §B.4.6 *and* §B.4.9.

(b) *Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, $D, E \to X$ be vector bundles, $\lambda : D \to E$ a vector bundle morphism, and $\theta : E \to \mathcal{T}_f Y$ a morphism. Then*

$$\Pi^\diamond(\theta \circ \lambda) = \Pi^\diamond(\theta) \circ \Pi^*(\lambda) : \Pi^*(D) \longrightarrow \mathcal{T}_{C(f)}C(Y).$$

(c) *Let $f : X \to Y$, $g : Y \to Z$ be morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $E \to X$ be a vector bundle, and $\theta : E \to \mathcal{T}_f Y$ be a morphism. Then the following diagram of sheaves on $C(X)_{\mathrm{top}}$ commutes:*

$$\begin{tikzcd}
\Pi^*(E) \arrow[rr, "\Pi^\diamond(\theta)"'] \arrow[d, "\Pi^\diamond(\mathcal{T}g\circ\theta)"] & & \mathcal{T}_{C(f)}C(Y) \arrow[d, "\mathcal{T}C(g)"'] \\
\mathcal{T}_{C(g\circ f)}C(Z) \arrow[rr, equal] & & \mathcal{T}_{C(g)\circ C(f)}C(Z).
\end{tikzcd}$$

(d) *Let $f : X \to Y$, $g : Y \to Z$ be morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $F \to Y$ be a vector bundle, and $\phi : F \to \mathcal{T}_g Z$ be a morphism. Then*

$$\begin{aligned} C(f)^*(\Pi^\diamond(\phi)) = \Pi^\diamond(f^*(\phi)) : C(f)^* \circ \Pi^*(F) = \Pi^* \circ f^*(F) \\ \longrightarrow \mathcal{T}_{C(g)\circ C(f)}C(Z) = \mathcal{T}_{C(g\circ f)}C(Z). \end{aligned}$$

Proof. Part (a) can be proved by combining equations (B.26), (B.40) and (B.43). Part (b) follows from the commutative diagram

$$\begin{tikzcd}
\Pi^*(\mathcal{D}) \arrow[r, equal] \arrow[d, "\Pi^*(\lambda)"] \arrow[dd, bend right, "\Pi^\diamond(\theta\circ\lambda)"'] & \Pi_{\mathrm{top}}^{-1}(\mathcal{D}) \otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)} \mathcal{O}_{C(X)} \arrow[d, "\Pi_{\mathrm{top}}^{-1}(\lambda)\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"'] \arrow[ddd, bend left, "\Pi_{\mathrm{top}}^{-1}(\theta\circ\lambda)\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"] \\
\Pi^*(\mathcal{E}) \arrow[r, equal] \arrow[d, "\Pi^\diamond(\theta)"] & \Pi_{\mathrm{top}}^{-1}(\mathcal{E}) \otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)} \mathcal{O}_{C(X)} \arrow[dd, "\Pi_{\mathrm{top}}^{-1}(\theta)\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"'] \\
\mathcal{T}_{C(f)}C(Y) \arrow[d, equal] & \\
\mathcal{T}_{C(f)}C(Y) \otimes_{\mathcal{O}_{C(X)}} \mathcal{O}_{C(X)} & \Pi_{\mathrm{top}}^{-1}(\mathcal{T}_f Y) \otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)} \mathcal{O}_{C(X)}, \arrow[l, "I_f^\diamond\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"']
\end{tikzcd}$$

which combines equation (B.43) for θ and for $\theta \circ \lambda$.

Part (c) follows from the commutative diagram

$$\begin{tikzcd}
\Pi^*(\mathcal{E}) \arrow[r, equal] \arrow[d, "\Pi^\diamond(\theta)"] \arrow[dd, bend right, "\Pi^\diamond(\mathcal{T}g\circ\theta)"'] & \Pi_{\mathrm{top}}^{-1}(\mathcal{E}) \otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)} \mathcal{O}_{C(X)} \arrow[d, "\Pi_{\mathrm{top}}^{-1}(\theta)\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"'] \arrow[dd, bend left, "\Pi_{\mathrm{top}}^{-1}(\mathcal{T}g\circ\theta)\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"] \\
\mathcal{T}_{C(f)}C(Y) \otimes_{\mathcal{O}_{C(X)}} \mathcal{O}_{C(X)} \arrow[d, "\mathcal{T}C(g)"] & \Pi_{\mathrm{top}}^{-1}(\mathcal{T}_f Y) \otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)} \mathcal{O}_{C(X)} \arrow[l, "I_f^\diamond\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"'] \arrow[d, "\Pi_{\mathrm{top}}^{-1}(\mathcal{T}g)\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"'] \\
\mathcal{T}_{C(g\circ f)}C(Z) \otimes_{\mathcal{O}_{C(X)}} \mathcal{O}_{C(X)} & \Pi_{\mathrm{top}}^{-1}(\mathcal{T}_{g\circ f} Z) \otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)} \mathcal{O}_{C(X)}, \arrow[l, "I_{g\circ f}^\diamond\otimes\mathrm{id}_{\mathcal{O}_{C(X)}}"']
\end{tikzcd}$$

which combines (B.43) for θ and $\mathcal{T}g \circ \theta$, and (B.41) in the bottom square.

Part (d) follows from the commutative diagram

$$
\begin{tikzcd}[column sep=large]
\Pi^* \circ f^*(\mathcal{F}) \arrow[r, equal] \arrow[d, "\Pi^\diamond(\phi)"] \arrow[dd, bend right=40, "{\begin{array}{l} C(f)^*(\Pi^\diamond(\phi)) \\ =\Pi^\diamond(f^*(\phi)) \end{array}}"'] & \Pi_{\mathrm{top}}^{-1} \circ f_{\mathrm{top}}^{-1}(\mathcal{F}) \otimes_{\Pi_{\mathrm{top}}^{-1} \circ f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_{C(X)} \arrow[d, "\Pi_{\mathrm{top}}^{-1} \circ f_{\mathrm{top}}^{-1}(\phi) \otimes \mathrm{id}_{\mathcal{O}_{C(X)}}"'] \arrow[dd, bend left=40, "{\begin{array}{l} \Pi_{\mathrm{top}}^{-1}(f^*(\phi)) \\ \otimes \mathrm{id}_{\mathcal{O}_{C(X)}} \end{array}}"] \\
{\begin{array}{l} C(f)_{\mathrm{top}}^{-1}(\mathcal{T}_{C(g)} C(Z)) \\ \otimes_{C(f)_{\mathrm{top}}^{-1}(\mathcal{O}_{C(Y)})} \mathcal{O}_{C(X)} \end{array}} \arrow[d, "C(f)^\flat \otimes \mathrm{id}_{\mathcal{O}_{C(X)}}"] & {\begin{array}{l} \Pi_{\mathrm{top}}^{-1} \circ f_{\mathrm{top}}^{-1}(\mathcal{T}_g Z) \\ \otimes_{\Pi_{\mathrm{top}}^{-1} \circ f_{\mathrm{top}}^{-1}(\mathcal{O}_Y)} \mathcal{O}_{C(X)} \end{array}} \arrow[l, "{\begin{array}{l} C(f)_{\mathrm{top}}^{-1}(I_g^\diamond) \\ \otimes \mathrm{id}_{\mathcal{O}_{C(X)}} \end{array}}"'] \arrow[d, "\Pi_{\mathrm{top}}^{-1}(f^\flat) \otimes \mathrm{id}_{\mathcal{O}_{C(X)}}"'] \\
{\begin{array}{l} \mathcal{T}_{C(g \circ f)} C(Z) \\ \otimes_{\mathcal{O}_{C(X)}} \mathcal{O}_{C(X)} \end{array}} & {\begin{array}{l} \Pi_{\mathrm{top}}^{-1}(\mathcal{T}_{g \circ f} Z) \\ \otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)} \mathcal{O}_{C(X)}, \end{array}} \arrow[l, "I_{g \circ f}^\diamond \otimes \mathrm{id}_{\mathcal{O}_{C(X)}}"']
\end{tikzcd}
$$

which combines (B.43) for ϕ and $f^*(\phi)$, and (B.26) for $f^*(\phi)$ and $C(f)^*(\Pi^\diamond(\phi))$ in the right and left triangles, and (B.42) in the bottom square. $\square$

We show that all the $O(s)$ and $O(s^2)$ notation of Definition B.36(i)–(vii) on X pulls back under $\Pi : C(X) \to X$ to the corresponding $O(\Pi(s))$ and $O(\Pi(s)^2)$ notation on $C(X)$, using $\Pi^\diamond$ to pull back morphisms $\Lambda : E \to \mathcal{T}_f Y$.

Theorem B.48. *Let X be an object in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $E \to X$ be a vector bundle, and $s \in \Gamma^\infty(E)$ be a section. Then:*

- **(i)** *Suppose $F \to X$ is a vector bundle and $t_1, t_2 \in \Gamma^\infty(F)$ with $t_2 = t_1 + O(s)$ (or $t_2 = t_1 + O(s^2)$) on X as in Definition* B.36(i). *Then $\Pi^*(t_2) = \Pi^*(t_1) + O(\Pi^*(s))$ (or $\Pi^*(t_2) = \Pi^*(t_1) + O(\Pi^*(s)^2)$) on $C(X)$.*
- **(ii)** *Suppose $F \to X$ is a vector bundle, $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $\Lambda_1, \Lambda_2 : F \to \mathcal{T}_f Y$ are morphisms with $\Lambda_2 = \Lambda_1 + O(s)$ on X as in Definition* B.36(ii). *Then Definition* B.45 *gives morphisms $\Pi^\diamond(\Lambda_1), \Pi^\diamond(\Lambda_2) : \Pi^*(F) \to \mathcal{T}_{C(f)} C(Y)$ on $C(X)$, which satisfy $\Pi^\diamond(\Lambda_2) = \Pi^\diamond(\Lambda_1) + O(\Pi^*(s))$ on $C(X)$.*
- **(iii)** *Suppose $f, g : X \to Y$ are morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$ with $g = f + O(s)$ on X as in Definition* B.36(iii). *Then $C(g) = C(f) + O(\Pi^*(s))$ on $C(X)$.*
- **(iv)** *Suppose $f, g : X \to Y$ with $g = f + O(s)$ are in* **(iii)**, *and $F \to X$, $G \to Y$ are vector bundles, and $\theta_1 : F \to f^*(G)$, $\theta_2 : F \to g^*(G)$ are morphisms with $\theta_2 = \theta_1 + O(s)$ on X as in Definition* B.36(iv). *Then $\Pi^*(\theta_2) = \Pi^*(\theta_1) + O(\Pi^*(s))$ on $C(X)$.*
- **(v)** *Suppose $f, g : X \to Y$ with $g = f + O(s)$ are in* **(iii)**, *and $F \to X$ is a vector bundle, and $\Lambda_1 : F \to \mathcal{T}_f Y$, $\Lambda_2 : F \to \mathcal{T}_g Y$ are morphisms with $\Lambda_2 = \Lambda_1 + O(s)$ on X as in Definition* B.36(v). *Then $C(g) = C(f) + O(\Pi^*(s))$ on $C(X)$ by* **(iii)**, *and Definition* B.45 *gives morphisms $\Pi^\diamond(\Lambda_1) : \Pi^*(F) \to \mathcal{T}_{C(f)} C(Y)$, $\Pi^\diamond(\Lambda_2) : \Pi^*(F) \to \mathcal{T}_{C(g)} C(Y)$, which satisfy $\Pi^\diamond(\Lambda_2) = \Pi^\diamond(\Lambda_1) + O(\Pi^*(s))$ on $C(X)$.*

(vi) *Suppose $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and $F \to X$, $G \to Y$ are vector bundles, and $t \in \Gamma^\infty(G)$ with $f^*(t) = O(s)$, and $\Lambda : F \to \mathcal{T}_f Y$ is a morphism, and $\theta : F \to f^*(G)$ is a vector bundle morphism with $\theta = f^*(\mathrm{d}t) \circ \Lambda + O(s)$ on X as in Definition* B.36(vi). *Then $\Pi^*(\theta) = C(f)^*(\mathrm{d}\Pi^*(t)) \circ \Pi^\diamond(\Lambda) + O(\Pi^*(s))$ on $C(X)$.*

(vii) *Suppose $f, g : X \to Y$ with $g = f + O(s)$ are in* **(iii)**, *and $\Lambda : E \to \mathcal{T}_f Y$ is a morphism with $g = f + \Lambda \circ s + O(s^2)$ on X as in Definition* B.36(vii). *Then $C(g) = C(f) + \Pi^\diamond(\Lambda) \circ \Pi^*(s) + O(\Pi^*(s)^2)$ on $C(X)$.*

Proof. Part (i) is immediate on applying Π^* to Definition B.36(i).

For (ii), Definition B.36(ii) gives a diagram (B.28) with $s_{\mathrm{top}}^{-1}(0) \in U_{\mathrm{top}}$ and $\mathrm{M} : \pi^*(F)|_V \to \mathcal{T}_{f\circ\pi} Y$ with $k_1^*(\mathrm{M}) = \Lambda_1|_U$ and $k_2^*(\mathrm{M}) = \Lambda_2|_U$. Applying the corner functor C to (B.28) gives a diagram (B.28) for $\Pi^*(F)$ and $C(f) : C(X) \to C(Y)$, with $\Pi^*(s)_{\mathrm{top}}^{-1}(0) \subseteq C(U)_{\mathrm{top}}$. We have

$$\Pi^\diamond(\mathrm{M}) : C(\pi)^* \circ \Pi^*(F)|_{C(V)} \longrightarrow \mathcal{T}_{C(f)\circ C(\pi)} C(Y),$$

and $k_1^*(\mathrm{M}) = \Lambda_1|_U$, $k_2^*(\mathrm{M}) = \Lambda_2|_U$ and Theorem B.47(d) imply that

$$C(k_1)^* \circ \Pi^\diamond(\mathrm{M}) = \Pi^\diamond(\Lambda_1)|_{C(U)} \quad \text{and} \quad C(k_2)^* \circ \Pi^\diamond(\mathrm{M}) = \Pi^\diamond(\Lambda_2)|_{C(U)}.$$

Thus Definition B.36(ii) implies that $\Pi^\diamond(\Lambda_2) = \Pi^\diamond(\Lambda_1) + O(\Pi^*(s))$.

Parts (iii),(iv) are immediate on applying the corner functor C to Definition B.36(iii),(iv). Part (v) follows by a very similar argument to (ii).

For (vi), choose a connection ∇ on $G \to Y$, so that $\theta = f^*(\nabla t) \circ \Lambda + O(s)$ as in Definition B.36(i),(vi). Consider the diagram of sheaves on $C(X)$:

$$\begin{tikzcd}
& \mathcal{T}_{C(f)} C(Y) \arrow[d, "\mathcal{T}\Pi"] \arrow[dr, bend left, "C(f)^*(\nabla^\Pi \Pi^*(t))"] & \\
\begin{array}{l}\Pi^*(F) = \\ \Pi_{\mathrm{top}}^{-1}(F) \\ \otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)} \mathcal{O}_{C(X)}\end{array} \arrow[ur, bend left, "\Pi^\diamond(\Lambda)"] \arrow[r, "\Pi^*(\Lambda)"] \arrow[dr, bend right, "\Pi_{\mathrm{top}}^{-1}(\Lambda)\otimes \mathrm{id}_{\mathcal{O}_{C(X)}}"'] & \begin{array}{l}\mathcal{T}_{\Pi\circ C(f)} Y = \\ \mathcal{T}_{f\circ\Pi} Y \\ \otimes_{\mathcal{O}_{C(X)}} \mathcal{O}_{C(X)}\end{array} \arrow[r, "(f\circ\Pi)^*(\nabla t)"] & \begin{array}{l}(f\circ\Pi)^*(G) = \\ \Pi_{\mathrm{top}}^{-1}(f^*(G)) \\ \otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)} \mathcal{O}_{C(X)}\end{array} \\
& \Pi_{\mathrm{top}}^{-1}(\mathcal{T}_f Y) \otimes_{\Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X)} \mathcal{O}_{C(X)}. \arrow[u, "\Pi^\flat \otimes \mathrm{id}_{\mathcal{O}_{C(X)}}"] \arrow[ur, bend right, "\Pi_{\mathrm{top}}^{-1}(f^*(\nabla t))\otimes \mathrm{id}_{\mathcal{O}_{C(X)}}"'] &
\end{tikzcd} \tag{B.45}$$

Here the top left triangle commutes by Theorem B.47(a), and the bottom left by (B.26). We can show using the ideas of §B.3–§B.4 that there is a natural pullback connection $\nabla^\Pi = \Pi^*(\nabla)$ on $\Pi^*(G) \to C(Y)$ such that the top right triangle of (B.45) commutes, for any $t \in \Gamma^\infty(G)$.

We can prove from the definition of μ^f in §B.4.7 that the following commutes, as $\Pi : C(X) \to X$, $f : X \to Y$ are morphisms in $\check{\mathbf{Man}}^{\mathbf{c}}$:

$$\begin{tikzcd}
\Pi_{\mathrm{top}}^{-1}(\mathcal{T}_f Y) \times \Pi_{\mathrm{top}}^{-1} \circ f_{\mathrm{top}}^{-1}(\mathcal{T}^* Y) \arrow[r, "\Pi_{\mathrm{top}}^{-1}(\mu^f)"'] \arrow[d, "\Pi^\flat \times \mathrm{id}"] & \Pi_{\mathrm{top}}^{-1}(\mathcal{O}_X) \arrow[d, "\Pi^\sharp"'] \\
\mathcal{T}_{f\circ\Pi} Y \times (f \circ \Pi)_{\mathrm{top}}^{-1}(\mathcal{T}^* Y) \arrow[r, "\mu^{f\circ\Pi}"] & \mathcal{O}_{C(X)}.
\end{tikzcd} \tag{B.46}$$

Then comparing (B.27) for $(f \circ \Pi)^*(\nabla t)$ with the pullback of (B.27) for $f^*(\nabla t)$ by Π_{top}^{-1}, and using (B.46), we find the bottom right triangle in (B.45) commutes.

Therefore (B.45) commutes, so that

$$\begin{aligned}
C(f)^*(\nabla^\Pi \Pi^*(t)) \circ \Pi^\diamond(\Lambda) &= \Pi_{\mathrm{top}}^{-1}(f^*(\nabla t)) \otimes \mathrm{id}_{\mathcal{O}_{C(X)}} \circ \Pi_{\mathrm{top}}^{-1}(\Lambda) \otimes \mathrm{id}_{\mathcal{O}_{C(X)}} \\
&= \Pi^*(f^*(\nabla t) \circ \Lambda).
\end{aligned}$$

Since $\theta = f^*(\nabla t) \circ \Lambda + O(s)$ we have $\Pi^*(\theta) = \Pi^*(f^*(\nabla t) \circ \Lambda) + O(\Pi^*(s))$ by part (i), so $\Pi^*(\theta) = C(f)^*(\nabla^\Pi \Pi^*(t)) \circ \Pi^\diamond(\Lambda) + O(\Pi^*(s))$, proving part (vi).

For (vii), Definition B.36(vii) gives a diagram (B.29) with $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}}$ and $\Lambda|_U = \theta_{V,v}$. Applying the corner functor C to (B.29) gives a diagram (B.29) for $C(f), C(g) : C(X) \to C(Y)$, with $\Pi^*(s)_{\mathrm{top}}^{-1}(0) \subseteq C(U)_{\mathrm{top}}$, and (B.44) yields

$$\Pi^\diamond(\Lambda)|_{C(U)} = \Pi^\diamond(\Lambda|_U) = \Pi^\diamond(\theta_{V,v}) = \theta_{C(V),C(v)}.$$

Thus Definition B.36(vii) gives $C(g) = C(f) + \Pi^\diamond(\Lambda) \circ \Pi^*(s) + O(\Pi^*(s)^2)$. $\square$

B.8.2 Comparing different categories $\dot{\mathbf{Man}}^{\mathbf{c}}$

Condition B.40 in §B.7 gave a way to compare two categories $\dot{\mathbf{Man}}$, $\ddot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7. Here is the corners analogue. Figure 3.2 on page I-53 gives a diagram of functors from Chapter 2 satisfying Condition B.49.

Condition B.49. Let $\dot{\mathbf{Man}}^{\mathbf{c}}, \ddot{\mathbf{Man}}^{\mathbf{c}}$ satisfy Assumption 3.22, and $F_{\dot{\mathbf{Man}}^{\mathbf{c}}}^{\ddot{\mathbf{Man}}^{\mathbf{c}}} : \dot{\mathbf{Man}}^{\mathbf{c}} \to \ddot{\mathbf{Man}}^{\mathbf{c}}$ be a functor in the commutative diagram, as in (B.30):

$$\begin{tikzcd}
 & \dot{\mathbf{Man}}^{\mathbf{c}} \arrow[dr, "F_{\dot{\mathbf{Man}}^{\mathbf{c}}}^{\mathbf{Top}}"] \arrow[dd, "F_{\dot{\mathbf{Man}}^{\mathbf{c}}}^{\ddot{\mathbf{Man}}^{\mathbf{c}}}"] & \\
\mathbf{Man} \arrow[ur, "\subset"] \arrow[dr, "\subset"'] & & \mathbf{Top}. \\
 & \ddot{\mathbf{Man}}^{\mathbf{c}} \arrow[ur, "F_{\ddot{\mathbf{Man}}^{\mathbf{c}}}^{\mathbf{Top}}"'] &
\end{tikzcd}$$

We also require:

(i) $F_{\dot{\mathbf{Man}}^{\mathbf{c}}}^{\ddot{\mathbf{Man}}^{\mathbf{c}}}$ should take products, disjoint unions, open submanifolds, and simple maps in $\dot{\mathbf{Man}}^{\mathbf{c}}$ to products, disjoint unions, open submanifolds, and simple maps in $\ddot{\mathbf{Man}}^{\mathbf{c}}$, and preserve dimensions.

(ii) There are canonical isomorphisms $F_{\dot{\mathbf{Man}}^{\mathbf{c}}}^{\ddot{\mathbf{Man}}^{\mathbf{c}}}(C_k(X)) \cong C_k(F_{\dot{\mathbf{Man}}^{\mathbf{c}}}^{\ddot{\mathbf{Man}}^{\mathbf{c}}}(X))$ for all X in $\dot{\mathbf{Man}}^{\mathbf{c}}$ and $k \geqslant 0$, so $k = 1$ gives $F_{\dot{\mathbf{Man}}^{\mathbf{c}}}^{\ddot{\mathbf{Man}}^{\mathbf{c}}}(\partial X) \cong \partial(F_{\dot{\mathbf{Man}}^{\mathbf{c}}}^{\ddot{\mathbf{Man}}^{\mathbf{c}}}(X))$.

These isomorphisms commute with the projections $\Pi : C_k(X) \to X$ and $I_{k,l} : C_k(C_l(X)) \to C_{k+l}(X)$ in $\dot{\mathbf{M}}\mathbf{an^c}$ and $\ddot{\mathbf{M}}\mathbf{an^c}$, and induce a natural isomorphism $F_{\check{\dot{\mathbf{M}}}\mathbf{an^c}}^{\check{\ddot{\mathbf{M}}}\mathbf{an^c}} \circ C \Rightarrow C \circ F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}$ of functors $\dot{\mathbf{M}}\mathbf{an^c} \to \check{\ddot{\mathbf{M}}}\mathbf{an^c}$.

Remark B.50. Condition B.49 implies that $F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}} : \dot{\mathbf{M}}\mathbf{an^c} \to \ddot{\mathbf{M}}\mathbf{an^c}$ satisfies Condition B.40. Thus §B.7 applies, so that all the material of §B.1–§B.5 in $\dot{\mathbf{M}}\mathbf{an^c}$ maps functorially to its analogue in $\ddot{\mathbf{M}}\mathbf{an^c}$.

Because $F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}$ is compatible with the corner functors for $\dot{\mathbf{M}}\mathbf{an^c}, \ddot{\mathbf{M}}\mathbf{an^c}$ by Condition B.49(ii), these functorial maps from geometry in $\dot{\mathbf{M}}\mathbf{an^c}$ to geometry in $\ddot{\mathbf{M}}\mathbf{an^c}$ are also compatible with the material of §B.8.1. In more detail:

(a) Use the notation of Definition B.41, so that accents ‘ $\dot{}$ ’ and ‘ $\ddot{}$ ’ denote objects associated to $\dot{\mathbf{M}}\mathbf{an^c}$ and $\ddot{\mathbf{M}}\mathbf{an^c}$, respectively.

Suppose $\dot{f} : \dot{X} \to \dot{Y}$ is a morphism in $\dot{\mathbf{M}}\mathbf{an^c}$, so that $C(\dot{f}) : C(\dot{X}) \to C(\dot{Y})$ and $\dot{\Pi} : C(\dot{X}) \to \dot{X}$ are morphisms in $\check{\dot{\mathbf{M}}}\mathbf{an^c}$. We have relative tangent sheaves $\mathcal{T}_{\dot{f}}\dot{Y}$ on $X_{\rm top}$ and $\mathcal{T}_{C(\dot{f})}C(\dot{Y})$ on $C(X)_{\rm top}$, defined using differential geometry in $\dot{\mathbf{M}}\mathbf{an^c}$, and Definition B.44 defines a morphism $I_{\dot{f}}^{\diamond} : \Pi_{\rm top}^{-1}(\mathcal{T}_{\dot{f}}\dot{Y}) \to \mathcal{T}_{C(\dot{f})}C(\dot{Y})$ of sheaves on $C(X)_{\rm top}$.

Write $\ddot{f} : \ddot{X} \to \ddot{Y}$ for the image of $\dot{f} : \dot{X} \to \dot{Y}$ in $\ddot{\mathbf{M}}\mathbf{an^c}$. Then we have sheaves $\mathcal{T}_{\ddot{f}}\ddot{Y}$ on $X_{\rm top}$ and $\mathcal{T}_{C(\ddot{f})}C(\ddot{Y})$ on $C(X)_{\rm top}$ and a morphism $I_{\ddot{f}}^{\diamond} : \Pi_{\rm top}^{-1}(\mathcal{T}_{\ddot{f}}\ddot{Y}) \to \mathcal{T}_{C(\ddot{f})}C(\ddot{Y})$, defined using differential geometry in $\ddot{\mathbf{M}}\mathbf{an^c}$.

Definition B.41(g) gives sheaf morphisms $F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}} : \mathcal{T}_{\dot{f}}\dot{Y} \to \mathcal{T}_{\ddot{f}}\ddot{Y}$ and $F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}} : \mathcal{T}_{C(\dot{f})}C(\dot{Y}) \to \mathcal{T}_{C(\ddot{f})}C(\ddot{Y})$. Applying $F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}$ throughout Definition B.44 and using Condition B.49(ii), we see the following commutes:

$$\begin{tikzcd}
\Pi_{\rm top}^{-1}(\mathcal{T}_{\dot{f}}\dot{Y}) \arrow[rr, "I_{\dot{f}}^{\diamond}"'] \arrow[d, "\Pi_{\rm top}^{-1}(F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}})"] && \mathcal{T}_{C(\dot{f})}C(\dot{Y}) \arrow[d, "F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}"'] \\
\Pi_{\rm top}^{-1}(\mathcal{T}_{\ddot{f}}\ddot{Y}) \arrow[rr, "I_{\ddot{f}}^{\diamond}"] && \mathcal{T}_{C(\ddot{f})}C(\ddot{Y}).
\end{tikzcd}$$

(b) In a similar way to (a), if $\dot{f} : \dot{X} \to \dot{Y}$ is a morphism in $\dot{\mathbf{M}}\mathbf{an^c}$, and $\dot{E} \to \dot{X}$ is a vector bundle on $\dot{X}$, and $\dot{\theta} : \dot{E} \to \mathcal{T}_{\dot{f}}\dot{Y}$ is a morphism, then the following diagram of sheaves on $C(X)_{\rm top}$ commutes:

$$\begin{tikzcd}
\dot{\Pi}^*(\dot{E}) \arrow[rr, "\dot{\Pi}^{\diamond}(\dot{\theta})"'] \arrow[d, "F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}"] && \mathcal{T}_{C(\dot{f})}C(\dot{Y}) \arrow[d, "F_{\dot{\mathbf{M}}\mathbf{an^c}}^{\ddot{\mathbf{M}}\mathbf{an^c}}"'] \\
\ddot{\Pi}^*(\ddot{E}) \arrow[rr, "\ddot{\Pi}^{\diamond}(\ddot{\theta})"] && \mathcal{T}_{C(\ddot{f})}C(\ddot{Y}).
\end{tikzcd}$$

B.9 Proof of Theorem 3.17

We now prove Theorem 3.17(a)–(v). Though the theorem refers to the informal Definition 3.15, which summarizes Definition B.36, we use the precise notions from Definition B.36. Throughout this section, let X be an object in $\dot{\mathbf{Man}}$, and $\pi : E \to X$ be a vector bundle, and $s \in \Gamma^\infty(E)$ be a section.

Proof of Theorem 3.17(a), parts (i),(vi)

Let $F \to X$ be a vector bundle, $t_1, t_2 \in \Gamma^\infty(F)$, and $\{X_a : a \in A\}$ be a family of open submanifolds in X with $s_{\mathrm{top}}^{-1}(0) \subseteq \bigcup_{a\in A} X_{a,\mathrm{top}} \subseteq X_{\mathrm{top}}$, with $t_2|_{X_a} = t_1|_{X_a} + O(s)$ on X_a for $a \in A$. We will show that $t_2 = t_1 + O(s)$ on X.

Set $X_\infty = X \setminus s^{-1}(0)$, so that $\{X_a : a \in A\} \amalg \{X_\infty\}$ is an open cover of X. Choose a subordinate partition of unity $\{\eta_a : a \in A\} \amalg \{\eta_\infty\}$ on X, as in §B.1.4. As $t_2|_{X_a} = t_1|_{X_a} + O(s)$ there exists $\alpha_a : E|_{X_a} \to F|_{X_a}$ such that $t_2|_{X_a} = t_1|_{X_a} + \alpha_a \circ s|_{X_a}$ in $\Gamma^\infty(F|_{X_a})$ for $a \in A$, by Definition B.36(i). Since $s \neq 0$ on $X_\infty = X \setminus s^{-1}(0)$ there exists $\epsilon \in \Gamma^\infty(E^*|_{X_\infty})$ with $\epsilon \cdot (s|_{X_\infty}) = 1$. Define $\alpha : E \to F$ on X by $\alpha = \sum_{a\in A} \eta_a \cdot \alpha_a + \eta_\infty \cdot (t_2 - t_1) \otimes \epsilon$. It is easy to check that $t_2 = t_1 + \alpha \circ s$, so $t_2 = t_1 + O(s)$ on X. Thus the '$O(s)$' condition in Definition B.36(i) is local on $s_{\mathrm{top}}^{-1}(0) \subseteq X_{\mathrm{top}}$, as we have to prove.

The same method shows the '$O(s^2)$' condition in Definition B.36(i) is local on $s_{\mathrm{top}}^{-1}(0)$. Also Definition B.36(vi) is local on $s_{\mathrm{top}}^{-1}(0)$, as it is defined using (i).

Proof of Theorem 3.17(a), part (ii)

Let $F \to X$ be a vector bundle, $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$, and $\Lambda_1, \Lambda_2 : F \to \mathcal{T}_f Y$ be morphisms. Suppose $\{X_a : a \in A\}$ is a family of open submanifolds in X with $s_{\mathrm{top}}^{-1}(0) \subseteq \bigcup_{a\in A} X_{a,\mathrm{top}} \subseteq X_{\mathrm{top}}$, with $\Lambda_2|_{X_a} = \Lambda_1|_{X_a} + O(s)$ on X_a for $a \in A$. We will show that $\Lambda_2 = \Lambda_1 + O(s)$ on X.

As $\Lambda_2|_{X_a} = \Lambda_1|_{X_a} + O(s)$, by Definition B.36(ii), for each $a \in A$ there exists a commutative diagram (B.28) in $\dot{\mathbf{Man}}$, with $s_{\mathrm{top}}^{-1}(0) \cap X_{a,\mathrm{top}} \subseteq U_{a,\mathrm{top}} \subseteq X_{a,\mathrm{top}}$:

$$\begin{tikzcd}
U_a \arrow[r, "k_{1,a}"] \arrow[d, hook] & V_a \arrow[d, hook] & U_a \arrow[l, "k_{2,a}"'] \arrow[d, hook] \\
X_a \arrow[r, "0_E|_{X_a}"] \arrow[dr, hook] & E \arrow[d, "\pi"] & X_a \arrow[l, "s|_{X_a}"'] \arrow[dl, hook] \\
& X, &
\end{tikzcd} \tag{B.47}$$

where morphisms '$\hookrightarrow$' are open submanifolds, and there is a morphism $\mathrm{M}_a : \pi^*(F)|_{V_a} \to \mathcal{T}_{f\circ\pi} Y|_{V_a}$ with $k_{1,a}^*(\mathrm{M}_a) = \Lambda_1|_{U_a}$ and $k_{2,a}^*(\mathrm{M}_a) = \Lambda_2|_{U_a}$.

Let $U \hookrightarrow X$ and $V \hookrightarrow E$ be the open submanifolds with $U_{\mathrm{top}} = \bigcup_{a\in A} U_{a,\mathrm{top}}$ and $V_{\mathrm{top}} = \bigcup_{a\in A} V_{a,\mathrm{top}}$. Then $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}}$, since $s_{\mathrm{top}}^{-1}(0) \cap X_{a,\mathrm{top}} \subseteq U_{a,\mathrm{top}} \subseteq U_{\mathrm{top}}$ for $a \in A$ and $s_{\mathrm{top}}^{-1}(0) \subseteq \bigcup_{a\in A} X_{a,\mathrm{top}}$. By taking the union of (B.47) for $a \in A$, we see that U, V fit into a commutative diagram (B.28), including morphisms $k_1, k_2 : U \to V$ with $k_i|_{U_a} = k_{i,a}$ for $i = 1, 2$ and $a \in A$.

Now $\{V_a : a \in A\}$ is an open cover of V. Choose a subordinate partition of unity $\{\eta_a : a \in A\}$ on V. Define a morphism $\mathrm{M} : \pi^*(F)|_V \to \mathcal{T}_{f\circ\pi}Y|_V$ on V by $\mathrm{M} = \sum_{a\in A} \eta_a \cdot \mathrm{M}_a$. Here $\eta_a \cdot \mathrm{M}_a$ is initially defined only on $V_a \subseteq V$, but extends smoothly by zero to all of V as $\operatorname{supp} \eta_a \subseteq V_a$. For $i = 1, 2$ we have

$$k_i^*(\mathrm{M}) = k_i^*\Bigl(\sum_{a\in A} \eta_a \cdot \mathrm{M}_a\Bigr) = \sum_{a\in A} k_i^*(\eta_a) \cdot k_{i,a}^*(\mathrm{M}_a) = \sum_{a\in A} k_i^*(\eta_a) \cdot \Lambda_i|_{U_a} = \Lambda_i,$$

using $k_{i,a}^*(\mathrm{M}_a) = \Lambda_i|_{U_a}$ in the second step and $\sum_a \eta_a = 1$ in the third. Thus (B.28) and M imply that $\Lambda_2 = \Lambda_1 + O(s)$ on X, by Definition B.36(ii).

Proof of Theorem 3.17(a), parts (iii),(iv),(v),(vii)

Let $f, g : X \to Y$ be morphisms in $\dot{\mathbf{Man}}$, and $\{X_a : a \in A\}$ be a family of open submanifolds in X with $s_{\mathrm{top}}^{-1}(0) \subseteq \bigcup_{a\in A} X_{a,\mathrm{top}}$, such that $g|_{X_a} = f|_{X_a} + O(s)$ on X_a for $a \in A$. We will show that $g = f + O(s)$ on X.

By replacing each X_a by a subcover $\{X_{ab} : b \in B_a\}$ of X_a with $E|_{X_{ab}}$ trivial, we can suppose that $E|_{X_a}$ is trivial for all $a \in A$, and choose a trivialization $E|_{X_a} \cong X_a \times \mathbb{R}^r$, where $r = \operatorname{rank} E$.

Since $g|_{X_a} = f|_{X_a} + O(s)$ on X_a, by Definition B.36(iii) there exist a commutative diagram (B.47) and a morphism $v_a : V_a \to Y$ in $\dot{\mathbf{Man}}$ with $v_a \circ k_{1,a} = f|_{U_a}$ and $v_a \circ k_{2,a} = g|_{U_a}$ in morphisms $U_a \to Y$, for all $a \in A$.

The next part of the proof follows that of Propositions B.25 and B.33. Let S_A be the set of all finite, nonempty subsets $B \subseteq A$. For each $B \in S_A$ write $X_B \hookrightarrow X$ for the open submanifold with $X_{B,\mathrm{top}} = \bigcap_{a\in B} X_{a,\mathrm{top}}$. Let $X' \hookrightarrow X$ be the open submanifold with $X'_{\mathrm{top}} = \bigcup_{a\in A} X_{a,\mathrm{top}}$.

As in the proof of Proposition B.33, using induction on $|B|$ and Assumption 3.7(a), for each $B \in S_A$ we choose an open submanifold $k_B : W_B \hookrightarrow \bigoplus_{b\in B} E|_{X_B} \cong X_B \times \prod_{b\in B} \mathbb{R}^r$ and a morphism $v_B : W_B \to Y$ such that:

(a) $s_{\mathrm{top}}^{-1}(0) \times \{(0, \ldots, 0)\} \subseteq W_{B,\mathrm{top}}$ for all $B \in S_A$.

(b) For $a \in A$ we have $W_{\{a\}} = V_a \hookrightarrow E|_{X'_a} = X'_{\{a\}} \times \mathbb{R}^r$ and $v_{\{a\}} = v_a$.

(c) If $x \in X_{\mathrm{top}}$ and $t_b \in \mathbb{R}$ for $b \in B$ with $\sum_{b\in B} t_b = 1$ and $(x, (t_b \cdot s_{\mathrm{top}}(x))_{b\in B}) \in W_{B,\mathrm{top}}$ then $v_{B,\mathrm{top}}(x, (t_b \cdot s_{\mathrm{top}}(x))_{b\in B}) = g_{\mathrm{top}}(x)$.

(d) If $C \subsetneq B$ lie in S_A and $(x, (\boldsymbol{e}_a)_{a\in C} \amalg (0)_{a\in B\setminus C}) \in W_{B,\mathrm{top}}$ then $(x, (\boldsymbol{e}_a)_{a\in C})$ lies in $W_{C,\mathrm{top}}$ with $v_{C,\mathrm{top}}(x, (\boldsymbol{e}_a)_{a\in C}) = v_{B,\mathrm{top}}(x, (\boldsymbol{e}_a)_{a\in C} \amalg (0)_{a\in B\setminus C})$.

Here to prove part (c), which does not occur in the proof of Proposition B.33, we use $v_a \circ k_{2,a} = g|_{U_a}$ for $k_{2,a}$ as in (B.47) in the first step when $B = \{a\}$, and Assumption 3.7(b) in the inductive step.

Now apply Proposition B.7 to choose a partition of unity $\{\eta_a : a \in A\}$ on X' subordinate to the open cover $\{X_{a,\mathrm{top}} : a \in A\}$. Choose an open submanifold $j : V \hookrightarrow E$ such that $0_{E,\mathrm{top}}(s_{\mathrm{top}}^{-1}(0)) \subseteq V_{\mathrm{top}}$ and if $e \in V_{\mathrm{top}} \subseteq E_{\mathrm{top}}$ with $\pi_{\mathrm{top}}(e) = x \in X_{\mathrm{top}}$ and $B = \bigl\{a \in A : x \in \operatorname{supp} \eta_{a,\mathrm{top}}\bigr\}$ then $(x, (\eta_{a,\mathrm{top}}(x) \cdot e)_{a\in B}) \in W_{B,\mathrm{top}}$. By (a) above and local finiteness of $\{\eta_a : a \in A\}$, this holds for any small enough open neighbourhood of $0_{E,\mathrm{top}}(s_{\mathrm{top}}^{-1}(0))$ in E.

As in the proof of Proposition B.33, there is a unique morphism $v : V \to Y$ in $\dot{\mathbf{Man}}$ such that for all $e \in V_{\rm top}$ with $\pi_{\rm top}(e) = x \in X_{\rm top}$ and $B = \{a \in A : x \in \operatorname{supp} \eta_{a,\rm top}\}$ we have

$$v_{\rm top}(e) = v_{B,\rm top}(x, (\eta_{a,\rm top}(x) \cdot e)_{a\in B}). \tag{B.48}$$

Let $U \hookrightarrow X$ be the open submanifold with $U_{\rm top} = 0_{E,\rm top}^{-1}(V_{\rm top}) \cap s_{\rm top}^{-1}(V_{\rm top})$. Then $s_{\rm top}^{-1}(0) \subseteq U_{\rm top}$, as $0_{E,\rm top}(s_{\rm top}^{-1}(0)) \subseteq V_{\rm top}$. Then by Assumption 3.2(d) there are morphisms k_1, k_2 making a commutative diagram (B.28). For $x \in U_{\rm top}$ with $B = \{a \in A : x \in \operatorname{supp} \eta_{a,\rm top}\}$ we have

$$\begin{aligned}
(v \circ k_1)_{\rm top}(x) &= v_{\rm top} \circ 0_{E,\rm top}(x) = v_{B,\rm top}(x, (\eta_{a,\rm top}(x) \cdot 0)_{a\in B}) \\
&= v_{B,\rm top}(x, (0)_{a\in B}) = v_{\{b\},\rm top}(x, 0) = v_{b,\rm top}(0_{E,\rm top}(x)) = (f|_U)_{\rm top}(x), \\
(v \circ k_2)_{\rm top}(x) &= v_{\rm top} \circ s_{\rm top}(x) = v_{B,\rm top}(x, (\eta_{a,\rm top}(x) \cdot s_{\rm top}(x))_{a\in B}) = (g|_U)_{\rm top}(x),
\end{aligned}$$

where for both equations we use (B.28) and (B.48), for the first we pick $b \in B$ and use (b),(d) above with $C = \{b\}$ and $v_b \circ k_{1,b} = f|_{U_b}$, and for the second we use (c) above. As this holds for all $x \in U_{\rm top}$ we have $v \circ k_1 = f|_U$ and $v \circ k_2 = g|_U$. Thus $g = f + O(s)$ on X by Definition B.36(iii). Hence the '$O(s)$' condition in Definition B.36(iii) is local on $s_{\rm top}^{-1}(0) \subseteq X_{\rm top}$.

We prove locality of parts (iv),(v),(vii) by extensions of the proof above. For (iv),(v) we start with $\{X_a \hookrightarrow X : a \in A\}$ covering $s_{\rm top}^{-1}(0)$ in X, a diagram (B.47) and a morphism $v_a : V_a \to Y$ in $\dot{\mathbf{Man}}$ with $v_a \circ k_{1,a} = f|_{U_a}$ and $v_a \circ k_{2,a} = g|_{U_a}$ for all $a \in A$, as above, together with morphisms $\phi_a : \pi^*(F)|_{V_a} \to v_a^*(G)$ with $k_{i,a}^*(\phi) = \theta_i|_{U_a}$ for $a \in A$ and $i = 1, 2$ in case (iv), and morphisms $\mathrm{M}_a : \pi^*(F)|_{V_a} \to \mathcal{T}_{v_a}Y$ with $k_{i,a}^*(\mathrm{M}_a) = \Lambda_i|_{U_a}$ for $a \in A$ and $i = 1, 2$ in case (v).

Then we construct V, v, U, k_1, k_2 in a diagram (B.28) from the data X_a, U_a, V_a, v_a for $a \in A$ by an inductive argument as above. At the same time we construct a morphism $\phi : \pi^*(F)|_V \to v^*(G)$ with $k_i^*(\phi) = \theta_i|_U$ for $i = 1, 2$ in case (iv), and a morphism $\mathrm{M} : \pi^*(F)|_V \to \mathcal{T}_vY$ with $k_i^*(\mathrm{M}) = \Lambda_i|_U$ for $i = 1, 2$ in case (v). We do this by gluing together the ϕ_a (or the M_a) to make ϕ (or M) using the partition of unity $\{\eta_a : a \in A\}$, in a very similar way to the construction of v above. Therefore (iv),(v) are local on $s_{\rm top}^{-1}(0) \subseteq X_{\rm top}$.

To prove locality of (vii), given $\Lambda : E \to \mathcal{T}_fY$ and $\{X_a \hookrightarrow X : a \in A\}$ with $g|_{X_a} = f|_{X_a} + \Lambda \circ s|_{X_a} + O(s^2)$ on X_a for $a \in A$, we follow the proof of (iii) above constructing V, v, U with $s_{\rm top}^{-1}(0) \subseteq U_{\rm top}$ exactly, except that at the beginning we choose $v_a : V_a \to Y$ with $\Lambda|_{U_a} = \theta_{V_a,v_a}$ in the notation of §B.4.8, which is possible by Definition B.36(vii). The last part of the proof of Proposition B.33 then shows that $\Lambda|_U = \theta_{V,v}$, so $g = f + \Lambda \circ s + O(s^2)$ on X by Definition B.36(vii), and (vii) is local on $s_{\rm top}^{-1}(0) \subseteq X_{\rm top}$.

Proof of Theorem 3.17(b)

We will need the following lemma:

Lemma B.51. *In Definition* B.36(iv),(v), *the condition is independent of the choice of diagram* (B.28) *and morphism* $v : V \to Y$ *satisfying* (iii). *That is, if* (iv),(v) *hold for one choice of* (B.28), v, *then they hold for all possible choices.*

Proof. Let Definition B.36(iv) hold for U, V, k_1, k_2 as in (B.28) with $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}} \subseteq X_{\mathrm{top}}$ and $v : V \to Y$ with $v \circ k_1 = f|_U$, $v \circ k_2 = g|_U$ and $\phi : \pi^*(F)|_V \to v^*(G)$ with $k_1^*(\phi) = \theta_1|_U$ and $k_2^*(\phi) = \theta_2|_U$. Suppose that we are given an alternative diagram (B.28) involving $\tilde{U}, \tilde{V}, \tilde{k}_1, \tilde{k}_2$ with $s_{\mathrm{top}}^{-1}(0) \subseteq \tilde{U}_{\mathrm{top}} \subseteq X_{\mathrm{top}}$ and a morphism $\tilde{v} : \tilde{V} \to Y$ with $\tilde{v} \circ \tilde{k}_1 = f|_{\tilde{U}}$, $\tilde{v} \circ \tilde{k}_2 = g|_{\tilde{U}}$, as in (iii). We must construct a morphism $\tilde{\phi} : \pi^*(F)|_{\tilde{V}} \to \tilde{v}^*(G)$ with $\tilde{k}_1^*(\tilde{\phi}) = \theta_1|_{\tilde{U}}$ and $\tilde{k}_2^*(\tilde{\phi}) = \theta_2|_{\tilde{U}}$, so that (iv) also holds for the alternative choices (B.28) and $\tilde{v}$.

If we can prove such $\tilde{\phi}$ exist near any point $e \in \tilde{V}_{\mathrm{top}}$, then by taking an open cover of $\tilde{V}$ on which choices of $\tilde{\phi}$ exist, and combining them with a partition of unity, we see that such $\tilde{\phi}$ exists globally on $\tilde{V}$. The conditions on $\tilde{\phi}$ are only nontrivial near points $e = 0_{E,\mathrm{top}}(x') = s_{\mathrm{top}}(x')$ in $\tilde{V}_{\mathrm{top}}$ for $x' \in s_{\mathrm{top}}^{-1}(0) \subseteq X_{\mathrm{top}}$. We restrict to the preimages $U', V', \tilde{V}', \ldots$ in $U, V, \tilde{V}, \ldots$ of an open neighbourhood X' of x' in X with $E|_{X'}$ trivial, so that we may identify $E|_{X'} \cong X' \times \mathbb{R}^n$, and regard $s|_{X'}$ as a morphism $s' : X' \to \mathbb{R}^n$.

Then we have open $V', \tilde{V}' \hookrightarrow X' \times \mathbb{R}^n$ with $s_{\mathrm{top}}'^{-1}(0) \times \{0\} \subseteq V'_{\mathrm{top}}, \tilde{V}'_{\mathrm{top}}$ and morphisms $v' : V' \to Y$, $\tilde{v}' : \tilde{V}' \to Y$ with $v'_{\mathrm{top}}(x, 0) = f_{\mathrm{top}}(x)$, $v'_{\mathrm{top}}(x, s'_{\mathrm{top}}(x)) = g_{\mathrm{top}}(x)$, $\tilde{v}'_{\mathrm{top}}(x, 0) = f_{\mathrm{top}}(x)$ and $\tilde{v}'_{\mathrm{top}}(x, s'_{\mathrm{top}}(x)) = g_{\mathrm{top}}(x)$, for all $x \in X'_{\mathrm{top}}$ with the left hand sides defined. Assumption 3.7(b) now shows that there exist open $W' \hookrightarrow X' \times \mathbb{R}^n \times \mathbb{R}^n$ with $s_{\mathrm{top}}'^{-1}(0) \times \{(0,0)\} \subseteq W'_{\mathrm{top}}$ and a morphism $w' : W' \to Y$ with $w'_{\mathrm{top}}(x, \boldsymbol{z}, 0) = v'_{\mathrm{top}}(x, \boldsymbol{z})$ and $w'_{\mathrm{top}}(x, 0, \boldsymbol{z}) = \tilde{v}'_{\mathrm{top}}(x, \boldsymbol{z})$ and $w'_{\mathrm{top}}(x, t \cdot s_{\mathrm{top}}(x), (1-t) \cdot s_{\mathrm{top}}(x)) = g_{\mathrm{top}}(x)$ for all $x \in X'_{\mathrm{top}}$, $\boldsymbol{z} \in \mathbb{R}^n$ and $t \in \mathbb{R}$ for which both sides are defined.

We now choose a morphism $\psi : \pi_{X'}^*(F) \to w'^*(G)$ with $\psi|_{(x,\boldsymbol{z},0)} = \phi|_{(x,\boldsymbol{z})}$ for all $(x, \boldsymbol{z}) \in V'_{\mathrm{top}}$ with $(x, \boldsymbol{z}, 0) \in W'_{\mathrm{top}}$, and $\psi|_{(x, t\cdot s_{\mathrm{top}}(x), (1-t)\cdot s_{\mathrm{top}}(x))} = \theta_2(x)$ for all $x \in X_{\mathrm{top}}$ and $t \in \mathbb{R}$ with $(x, t \cdot s_{\mathrm{top}}(x), (1-t) \cdot s_{\mathrm{top}}(x)) \in W'_{\mathrm{top}}$. These two conditions are consistent at points $(x, s_{\mathrm{top}}(x), 0)$ as $k_2^*(\phi) = \theta_2|_U$. They prescribe ψ on cleanly-intersecting submanifolds of W', so making W' smaller if necessary, we can use Assumption 3.7(a) to show such ψ exists.

Let $\tilde{V}'' \hookrightarrow E|_{X'} \cong X' \times \mathbb{R}^n$ be the open submanifold with $\tilde{V}''_{\mathrm{top}} = \{(x, \boldsymbol{z}) : (x, 0, \boldsymbol{z}) \in W'_{\mathrm{top}}$ and $(x, \boldsymbol{z}) \in \tilde{V}'_{\mathrm{top}}\}$, and $\tilde{U}'' \hookrightarrow X'$ the open submanifold with $\tilde{U}''_{\mathrm{top}} = \{x \in \tilde{U}'_{\mathrm{top}} : (x, 0) \in \tilde{V}''_{\mathrm{top}}$ and $(x, s'_{\mathrm{top}}(x)) \in \tilde{V}''_{\mathrm{top}}\}$. Let $\tilde{l} : \tilde{V}'' \to W'$ be the morphism with $\tilde{l}(x, \boldsymbol{z}) = (x, 0, \boldsymbol{z})$ from Assumption 3.2(d). Define $\tilde{\phi}'' : \pi^*(F)|_{\tilde{V}''} \to \tilde{v}|_{\tilde{V}''}^*(G)$ by $\tilde{\phi}'' = \tilde{l}^*(\psi)$. Then $x' \in \tilde{U}''_{\mathrm{top}}$ and $e = 0_{E,\mathrm{top}}(x') \in \tilde{V}''_{\mathrm{top}}$, and $\tilde{k}_1|_{\tilde{U}''}^*(\tilde{\phi}'') = \theta_1|_{\tilde{U}''}$ and $\tilde{k}_2|_{\tilde{U}''}^*(\tilde{\phi}'') = \theta_2|_{\tilde{U}''}$. Hence $\tilde{\phi}$ satisfying the required conditions exists near e in $\tilde{V}_{\mathrm{top}}$ as required.

This proves the lemma for case (iv). The proof for (v) is similar, noting that we can use Assumption 3.7(a) and Proposition B.33 to show that morphisms $\mathrm{M} : \pi^*(F)|_V \to \mathcal{T}_v Y$ have the required extension properties at the point in the proof where we choose ψ. $\square$

It is now more-or-less immediate from the definitions that the conditions of Definition B.36(i),(ii),(iv)–(vi) are $C^\infty(X)$-linear. Here for (iv),(v) we must fix a diagram (B.28) and morphism $v : V \to Y$ satisfying (iii), and use these for all the different $O(s)$ conditions to be combined. This is possible by Lemma B.51.

For example, in (iv) suppose we have morphisms $\theta_1, \theta_1' : F \to f^*(G)$ and $\theta_2, \theta_2' : F \to g^*(G)$ with $\theta_2 = \theta_1 + O(s)$ and $\theta_2' = \theta_1' + O(s)$. Fix (B.28) and $v : V \to Y$ as above, so that (iv) gives $\phi, \phi' : \pi^*(F)|_V \to v^*(G)$ with $k_i^*(\phi) = \theta_i|_U$ and $k_i^*(\phi') = \theta_i'|_U$ for $i = 1, 2$. Then for $a, b \in C^\infty(X)$, considering

$$\pi|_V^*(a) \cdot \phi + \pi|_V^*(b) \cdot \phi' : \pi^*(F)|_V \longrightarrow v^*(G)$$

we see that $a\theta_2 + b\theta_2' = a\theta_1 + b\theta_1' + O(s)$, so (iv) is $C^\infty(X)$-linear.

Proof of Theorem 3.17(c)

It is clear from the definitions that the $O(s), O(s^2)$ conditions in Definition B.36(i) are equivalence relations. For (ii),(iii), reflexivity $\Lambda_1 = \Lambda_1 + O(s)$, $f = f + O(s)$ is easy (take $U = X$, $V = E$, $\mathrm{M} = 0$, $v = f \circ \pi$), and symmetry $\Lambda_2 = \Lambda_1 + O(s) \Rightarrow \Lambda_1 = \Lambda_2 + O(s)$, $g = f + O(s) \Rightarrow f = g + O(s)$ is also easy (apply the involution of E mapping $(x, e) \mapsto (x, s_{\rm top}(x) - e)$ on points to V, M, v). It remains to prove transitivity for (ii),(iii).

For (ii), let $F \to X$ be a vector bundle, $f : X \to Y$ a morphism, and $\Lambda_1, \Lambda_2, \Lambda_3 : F \to \mathcal{T}_f Y$ morphisms with $\Lambda_2 = \Lambda_1 + O(s)$ and $\Lambda_3 = \Lambda_2 + O(s)$. Then by Definition B.36(ii) there exist a diagram (B.28) including U, V, k_1, k_2 with $s_{\rm top}^{-1}(0) \subseteq U_{\rm top}$ and a morphism $\mathrm{M} : \pi^*(F)|_V \to \mathcal{T}_{f\circ\pi} Y|_V$ with $k_1^*(\mathrm{M}) = \Lambda_1|_U$ and $k_2^*(\mathrm{M}) = \Lambda_2|_U$. Also, there exist (B.28) including $\tilde U, \tilde V, \tilde k_1, \tilde k_2$ with $s_{\rm top}^{-1}(0) \subseteq \tilde U_{\rm top}$ and a morphism $\tilde{\mathrm{M}} : \pi^*(F)|_{\tilde V} \to \mathcal{T}_{f\circ\pi} Y|_{\tilde V}$ with $\tilde k_1^*(\tilde{\mathrm{M}}) = \Lambda_2|_{\tilde U}$ and $\tilde k_2^*(\tilde{\mathrm{M}}) = \Lambda_3|_{\tilde U}$. Then taking $\check U = U \cap \tilde U$, $\check V = V \cap \tilde V$, $\check k_i = k_i|_{\check U} = \tilde k_i|_{\check U}$ for $i = 1, 2$ and $\check{\mathrm{M}} = \mathrm{M}|_{\check U} + \tilde{\mathrm{M}}|_{\check U} - \pi^*(\Lambda_2)|_{\check U}$ we find that $\check k_1^*(\check{\mathrm{M}}) = \Lambda_1|_{\check U}$ and $\check k_2^*(\check{\mathrm{M}}) = \Lambda_3|_{\check U}$, so $\Lambda_3 = \Lambda_2 + O(s)$, and (ii) is an equivalence relation.

For (iii), suppose $f, g, h : X \to Y$ are morphisms in $\dot{\mathbf{Man}}$ with $g = f + O(s)$ and $h = g + O(s)$. Then there exist a diagram (B.28) including U, V, k_1, k_2 with $s_{\rm top}^{-1}(0) \subseteq U_{\rm top}$, and a morphism $v : V \to Y$ with $v \circ k_1 = f|_U$ and $v \circ k_2 = g|_U$. Also, there exist a diagram (B.28) including $\tilde U, \tilde V, \tilde k_1, \tilde k_2$ with $s_{\rm top}^{-1}(0) \subseteq \tilde U_{\rm top}$ and a morphism $\tilde v : \tilde V \to Y$ with $\tilde v \circ \tilde k_1 = g|_{\tilde U}$ and $\tilde v \circ \tilde k_2 = h|_{\tilde U}$.

We will prove that $h = f + O(s)$. By Theorem 3.17(a), proved above, it is enough to show that $h = f + O(s)$ near each point x' of $s_{\rm top}^{-1}(0) \subseteq X_{\rm top}$. We restrict to the preimages $U', V', \tilde U', \tilde V', \ldots$ in $U, V, \tilde U, \tilde V, \ldots$ of an open neighbourhood X' of x' in X with $E|_{X'}$ trivial, so that we may identify $E|_{X'} \cong X' \times \mathbb{R}^n$, and regard $s|_{X'}$ as a morphism $s' : X' \to \mathbb{R}^n$.

Then we have open $V', \tilde V' \hookrightarrow X' \times \mathbb{R}^n$ with $s_{\rm top}'^{-1}(0) \times \{0\} \subseteq V'_{\rm top}, \tilde V'_{\rm top}$ and morphisms $v' : V' \to Y$, $\tilde v' : \tilde V' \to Y$ with $v'_{\rm top}(x, 0) = f_{\rm top}(x)$, $v'_{\rm top}(x, s'_{\rm top}(x)) = g_{\rm top}(x)$, $\tilde v'_{\rm top}(x, 0) = g_{\rm top}(x)$ and $\tilde v'_{\rm top}(x, s'_{\rm top}(x)) = h_{\rm top}(x)$, for all $x \in X_{\rm top}$ with the left hand sides defined. Assumption 3.7(a) now gives open $W' \hookrightarrow X' \times \mathbb{R}^n \times \mathbb{R}^n$ with $s_{\rm top}'^{-1}(0) \times \{(0,0)\} \subseteq W'_{\rm top}$ and a morphism $w' : W' \to Y$ with

$$w'_{\rm top}(x, \boldsymbol{z}, 0) = v'_{\rm top}(x, \boldsymbol{z} + s'_{\rm top}(x)) \quad \text{and} \quad w'_{\rm top}(x, 0, \boldsymbol{z}) = \tilde v'_{\rm top}(x, \boldsymbol{z})$$

for all $x \in X_{\rm top}$, $\boldsymbol{z} \in \mathbb{R}^n$ for which both sides are defined. Here the $\boldsymbol{z} + s'_{\rm top}(x)$ means both equations prescribe $w'_{\rm top}(x, 0, 0) = g_{\rm top}(x)$, so they are consistent.

Now define $\check{V} \hookrightarrow E|_{X'} \cong X' \times \mathbb{R}^n$ to be the open submanifold with $\check{V}_{\rm top} = \{(x, \boldsymbol{z}) : (x, \boldsymbol{z} - s'_{\rm top}(x), \boldsymbol{z}) \in W'_{\rm top}\}$, and $\check{U} \hookrightarrow X'$ to be the open submanifold with $\check{U}_{\rm top} = \{x : (x, 0) \in \check{V}_{\rm top}$ and $(x, s'_{\rm top}(x)) \in \check{V}_{\rm top}\}$, and $\check{k}_1, \check{k}_2 : \check{U} \to \check{V}$, $\check{v} : \check{V} \to Y$ to be the morphisms with $\check{k}_{1,\rm top}(x) = (x, 0)$, $\check{k}_{2,\rm top}(x) = (x, s'_{\rm top}(x))$ and $\check{v}_{\rm top}(x, \boldsymbol{z}) = w'_{\rm top}(x, \boldsymbol{z} - s'_{\rm top}(x), \boldsymbol{z})$. Then

$$\check{v}_{\rm top} \circ \check{k}_{1,\rm top}(x) = \check{v}_{\rm top}(x, 0) = w'_{\rm top}(x, -s'_{\rm top}(x), 0) = v'_{\rm top}(x, 0) = f_{\rm top}(x),$$
$$\check{v}_{\rm top} \circ \check{k}_{2,\rm top}(x) = \check{v}_{\rm top}(x, s'_{\rm top}(x)) = w'_{\rm top}(x, 0, s'_{\rm top}(x)) = \tilde{v}'_{\rm top}(x, s'_{\rm top}(x)) = h_{\rm top}(x),$$

so $\check{v} \circ \check{k}_1 = f|_{\check{U}}$ and $\check{v} \circ \check{k}_2 = h|_{\check{U}}$, and $h = f + O(s)$ on X'. Hence $h = f + O(s)$ on X by Theorem 3.17(a), and (iii) is an equivalence relation.

Proof of Theorem 3.17(d)

This is a straightforward combination of the proofs that (i),(ii) are equivalence relations and (iii) is an equivalence relation in the proof of Theorem 3.17(c) above, and we leave it as an exercise.

Proof of Theorem 3.17(e), non Γ-equivariant case

As in the theorem, let $X_a \hookrightarrow X$ for $a \in A$ be open submanifolds with $s_{\rm top}^{-1}(0) \subseteq \bigcup_{a \in A} X_{a,\rm top}$. Write $X_{ab} \hookrightarrow X$ for the open submanifold with $X_{ab,\rm top} = X_{a,\rm top} \cap X_{b,\rm top}$ for $a, b \in A$. Suppose we are given morphisms $f_a : X_a \to Y$ in $\dot{\mathbf{Man}}$ for all $a \in A$ with $f_a|_{X_{ab}} = f_b|_{X_{ab}} + O(s)$ on X_{ab} for all $a, b \in A$. We must construct an open submanifold $X' \hookrightarrow X$ with $s_{\rm top}^{-1}(0) \subseteq X'_{\rm top}$ and a morphism $g : X' \to Y$ such that $g|_{X' \cap X_a} = f_a|_{X' \cap X_a} + O(s)$ for all $a \in A$.

Since $f_a|_{X_{ab}} = f_b|_{X_{ab}} + O(s)$ on X_{ab}, by Definition B.36(iii) there exists a diagram (B.28) including $U_{ab}, V_{ab}, k_{1,ab}, k_{2,ab}$ with $s_{\rm top}^{-1}(0) \cap X_{ab,\rm top} \subseteq U_{ab,\rm top} \subseteq X_{ab,\rm top}$ and a morphism $v_{ab} : V_{ab} \to Y$ with $v_{ab} \circ k_{1,ab} = f_a|_{U_{ab}}$ and $v_{ab} \circ k_{2,ab} = f_b|_{U_{ab}}$, for all $a, b \in A$. Making V_{ab} smaller, we can suppose that $0_{E,\rm top}(x) \in V_{ab,\rm top}$ if and only if $x \in U_{ab,\rm top}$, and $s_{\rm top}(x) \in V_{ab,\rm top}$ if and only if $x \in U_{ab,\rm top}$.

We will divide the proof into three steps:

(A) $A = \{1, 2\}$.

(B) $A = \mathbb{N}$ and $\{X_a : a \in \mathbb{N}\}$ is *locally finite*, i.e. each x in $X_{\rm top}$ has an open neighbourhood intersecting only finitely many $X_{a,\rm top}$.

(C) The general case.

We use the notation above in each step.

Step (A). Suppose $A = \{1, 2\}$. Let $\dot{X} \hookrightarrow X$ be the open submanifold with $\dot{X}_{\rm top} = X_{1,\rm top} \cup X_{2,\rm top}$. Then $s_{\rm top}^{-1}(0) \subseteq \bigcup_{a \in \{1,2\}} X_{a,\rm top} = \dot{X}_{\rm top}$. Choose a partition of unity $\{\eta_1, \eta_2\}$ on $\dot{X}$ subordinate to the open cover $\{X_1, X_2\}$. Let

$X' \hookrightarrow \dot{X}$ be the open submanifold with

$$\begin{aligned}X'_{\rm top} &= (X_{1,\rm top} \setminus \operatorname{supp} \eta_2) \amalg (X_{2,\rm top} \setminus \operatorname{supp} \eta_1) \\ &\quad \amalg \bigl\{x \in \operatorname{supp} \eta_1 \cap \operatorname{supp} \eta_2 : (x, \eta_{2,\rm top}(x) \cdot s_{\rm top}(x)) \in V_{12,\rm top}\bigr\}.\end{aligned}$$

Then $s_{\rm top}^{-1}(0) \subseteq X'_{\rm top}$. By Assumption 3.3(a) there is a unique $g : X' \to Y$ with

$$g_{\rm top}(x) = \begin{cases} f_{1,\rm top}(x), & x \in X_{1,\rm top} \setminus \operatorname{supp} \eta_2, \\ f_{2,\rm top}(x), & x \in X_{2,\rm top} \setminus \operatorname{supp} \eta_1, \\ v_{12,\rm top}(x, \eta_{2,\rm top}(x) \cdot s_{\rm top}(x)), & x \in X'_{\rm top} \cap U_{12,\rm top}. \end{cases}$$

This holds as the three possibilities for g are smooth maps on open subsets of X' covering X', which agree on the overlaps, since $v_{12,\rm top}(x, 0) = f_{1,\rm top}(x)$ and $v_{12,\rm top}(x, s_{\rm top}(x)) = f_{2,\rm top}(x)$.

To show that $g|_{X' \cap X_1} = f_1|_{X' \cap X_1} + O(s)$, define $V_1 \hookrightarrow E$, $U_1 \hookrightarrow X$ to be the open submanifolds and $v_1 : V_1 \to Y$, $k_{1,1}, k_{2,1} : U_1 \to V_1$ the morphisms with

$$\begin{aligned}V_{1,\rm top} &= \pi_{\rm top}^{-1}(X_{1,\rm top} \setminus \operatorname{supp} \eta_2) \\ &\quad \amalg \bigl\{(x, e) \in \pi_{\rm top}^{-1}(\operatorname{supp} \eta_2 \cap U_{12,\rm top}) : (x, \eta_{2,\rm top}(x) \cdot e) \in V_{12,\rm top}\bigr\}, \\ U_{1,\rm top} &= \bigl\{x \in X'_{\rm top} \cap X_{1,\rm top} : (x, 0) \in V_{1,\rm top} \text{ and } (x, s_{\rm top}(x)) \in V_{1,\rm top}\bigr\}, \\ v_{1,\rm top}(x, e) &= \begin{cases} f_{1,\rm top}(x), & x \in X_{1,\rm top} \setminus \operatorname{supp} \eta_2, \\ v_{12,\rm top}(x, \eta_{2,\rm top}(x) \cdot e), & (x, e) \in V_{1,\rm top} \cap \pi_{\rm top}^{-1}(U_{12,\rm top}), \end{cases} \\ k_{1,1,\rm top}(x) &= (x, 0), \qquad \text{and} \qquad k_{2,1,\rm top}(x) = (x, s_{\rm top}(x)).\end{aligned}$$

Again, the two possibilities for v_1 are smooth on an open cover of V_1, which agree on the overlap, since $v_{12,\rm top}(x, 0) = f_{1,\rm top}(x)$. Then $U_1, V_1, k_{1,1}, k_{2,1}$ form a diagram (B.28), and this and v_1 show that $g|_{X' \cap X_1} = f_1|_{X' \cap X_1} + O(s)$ by Definition B.36(iii). Similarly $g|_{X' \cap X_2} = f_2|_{X' \cap X_2} + O(s)$, proving step (A).

Step (B). Suppose $A = \mathbb{N}$ and $\{X_a : a \in \mathbb{N}\}$ is locally finite. By induction on $m = 1, 2, \ldots$ we will construct an open submanifold $X'_m \hookrightarrow X$ and a morphism $g_m : X'_m \to Y$ satisfying:

(i) $s_{\rm top}^{-1}(0) \cap \bigl(\bigcup_{a=1}^m X_{a,\rm top}\bigr) = s_{\rm top}^{-1}(0) \cap X'_{m,\rm top}$.

(ii) $g_m|_{X'_m \cap X_a} = f_a|_{X'_m \cap X_a} + O(s)$ for $a = 1, \ldots, m$.

(iii) If $m > 1$ and $x \in X_{\rm top} \setminus X_{m,\rm top}$ then $x \in X'_{m-1,\rm top}$ if and only if $x \in X'_{m,\rm top}$, and then $g_{m-1,\rm top}(x) = g_{m,\rm top}(x)$.

For the first step $m = 1$ we put $X'_1 = X_1$ and $g_1 = f_1$, and (i)–(iii) hold trivially. For the inductive step, suppose $m \geqslant 1$ and we have constructed $X'_1, g_1, \ldots, X'_m, g_m$ satisfying (i)–(iii). For each $a = 1, \ldots, m$ we have

$$\begin{aligned}g_m|_{X'_m \cap X_a \cap X_{m+1}} &= f_a|_{X'_m \cap X_a \cap X_{m+1}} + O(s) \\ &= f_{m+1}|_{X'_m \cap X_a \cap X_{m+1}} + O(s),\end{aligned} \tag{B.49}$$

using (ii) for g_m in the first step, and $f_a|_{X_{a(m+1)}} = f_{m+1}|_{X_{a(m+1)}} + O(s)$ and Theorem 3.17(c) in the second. Now (i) implies that

$$s_{\rm top}^{-1}(0) \cap (X'_m \cap X_{m+1})_{\rm top} \subseteq \bigcup_{a=1}^m (X'_m \cap X_a \cap X_{m+1})_{\rm top}.$$

Hence (B.49) and Theorem 3.17(a) imply that $g_m|_{X'_m \cap X_{m+1}} = f_a|_{X'_m \cap X_{m+1}} + O(s)$ on $X'_m \cap X_{m+1}$.

We now apply step (A) to combine $g_m : X'_m \to Y$ and $f_{m+1} : X_{m+1} \to Y$. This yields an open $X'_{m+1} \hookrightarrow X$ and a morphism $g_{m+1} : X'_{m+1} \to Y$ with

$$g_{m+1}|_{X'_{m+1} \cap X'_m} = g_m|_{X'_{m+1} \cap X'_m} + O(s), \tag{B.50}$$

$$g_{m+1}|_{X'_{m+1} \cap X_{m+1}} = f_{m+1}|_{X'_{m+1} \cap X_{m+1}} + O(s). \tag{B.51}$$

Parts (i),(iii) for X'_{m+1}, g_{m+1} are immediate from the construction. For (ii), the case $a = m+1$ for g_{m+1} is (B.51). For $a = 1, \ldots, m$ we have

$$g_{m+1}|_{X'_{m+1} \cap X'_m \cap X_a} = g_m|_{X'_{m+1} \cap X'_m \cap X_a} + O(s) = f_a|_{X'_{m+1} \cap X'_m \cap X_a} + O(s),$$

using (B.50), part (ii) for g_m, and Theorem 3.17(c) (proved above). Part (i) gives

$$s_{\rm top}^{-1}(0) \cap (X'_{m+1} \cap X'_m \cap X_a)_{\rm top} = s_{\rm top}^{-1}(0) \cap (X'_{m+1} \cap X_a)_{\rm top},$$

so Theorem 3.17(a) gives $g_{m+1}|_{X'_{m+1} \cap X_a} = g_m|_{X'_{m+1} \cap X_a} + O(s)$, proving (ii) for g_{m+1}. This completes the inductive step, so by induction we can choose X'_m, g_m satisfying (i)–(iii) for all $m = 1, 2, \ldots$.

We now claim that there are a unique open submanifold $X' \hookrightarrow X$ and morphism $g : X' \to Y$ with the property that $x \in X_{\rm top}$ lies in $X'_{\rm top}$ if and only if $x \in X'_{m,{\rm top}}$ for all $m \gg 0$ sufficiently large, and then $g_{\rm top}(x) = g_{m,{\rm top}}(x)$ for all $m \gg 0$ sufficiently large. To see this, write $X'_{\rm top}$ for the set of $x \in X_{\rm top}$ satisfying this condition. Fix $\tilde{x} \in X_{\rm top}$. Then local finiteness of $\{X_a : a \in \mathbb{N}\}$ means that $\tilde{x}$ has an open neighbourhood $U \hookrightarrow X$ in X such that $U_{\rm top} \cap X_{m,{\rm top}} = \emptyset$ for all $m \geqslant N$, for some $N \gg 0$.

Part (iii) implies that if $m \geqslant N$ and $x \in U_{\rm top}$ then $x \in X'_{m,{\rm top}}$ if and only if $x \in X'_{m+1,{\rm top}}$. Thus $U_{\rm top} \cap X'_{N,{\rm top}} = U_{\rm top} \cap X'_{N+1,{\rm top}} = U_{\rm top} \cap X'_{N+2,{\rm top}} = \cdots$, so that $U_{\rm top} \cap X'_{\rm top} = U_{\rm top} \cap X'_{N,{\rm top}}$, which is open. Hence we can cover $X_{\rm top}$ by open $U_{\rm top} \subseteq X_{\rm top}$ with $U_{\rm top} \cap X'_{\rm top}$ open, so $X'_{\rm top}$ is open in $X_{\rm top}$, and the open submanifold $X' \hookrightarrow X$ is well defined.

For $\tilde{x}, U, N$ as above, part (iii) also gives $g_{m,{\rm top}}(x) = g_{m+1,{\rm top}}(x)$ for any $x \in U_{\rm top} \cap X'_{\rm top}$ and $m \geqslant N$, so $g_{N,{\rm top}}(x) = g_{N+1,{\rm top}}(x) = g_{N+2,{\rm top}}(x) = \cdots$. Hence there is a unique map $g_{\rm top} : X'_{\rm top} \to Y_{\rm top}$ with $g_{\rm top}(x) = g_{m,{\rm top}}(x)$ for all $m \gg 0$ sufficiently large, where in U we have $g_{\rm top}|_{U_{\rm top} \cap X'_{\rm top}} = g_{N,{\rm top}}|_{U_{\rm top} \cap X'_{\rm top}}$. As $g_N|_{U \cap X'} : U \cap X' \to Y$ is a morphism in $\dot{\mathbf{Man}}$, and we can cover $X'_{\rm top}$ by such open $(U \cap X')_{\rm top}$, Assumption 3.3(a) implies that there is a unique morphism $g : X' \to Y$ in $\dot{\mathbf{Man}}$ with the prescribed $g_{\rm top}$.

Let $a \in \mathbb{N}$. Then as above we can cover $X' \cap X_a$ by open $U \hookrightarrow X' \cap X_a$ such that $U_{\rm top} \subseteq X'_{m,{\rm top}}$ and $g|_U = g_m|_U$ for $m \gg 0$, so that $m \geqslant a$. Then

$g|_U = g_m|_U = f_a|_U + O(s)$ by (ii), so Theorem 3.17(a) implies that $g|_{X'\cap X_a} = f_a|_{X'\cap X_a} + O(s)$, as we want. This completes step (B).

Step (C). Now consider the general case, with $\{X_a \hookrightarrow X : a \in A\}$ any open cover of X. Since $X_{\rm top}$ is Hausdorff, locally compact, and second countable by Assumption 3.2(b), it is also *paracompact* (i.e. every open cover has a locally finite refinement), and *Lindelöf* (i.e. every open cover has a countable subcover). So by paracompactness we can choose an open cover $\{\hat X_b \hookrightarrow X : b \in B\}$ of X which is locally finite, such that for all $b \in B$ there exists $a_b \in A$ with $X'_{b,\rm top} \subseteq X_{a_b,\rm top} \subseteq X_{\rm top}$. And by the Lindelöf property we can choose a countable subset $C \subseteq B$ such that $\{\hat X_c \hookrightarrow X : c \in C\}$ is still an open cover of X. Thus (adding extra empty $\hat X_c$ if C is finite) we can take $C = \mathbb{N}$.

For each $c \in \mathbb{N}$ set $\hat f_c = f_{a_c}|_{\hat X_c} : \hat X_c \to Y$. Then for all $c, d \in \mathbb{N}$ we have $\hat f_c|_{\hat X_{cd}} = \hat f_d|_{\hat X_{cd}} + O(s)$ since $f_{a_c}|_{X_{a_c a_d}} = f_{a_d}|_{X_{a_c a_d}} + O(s)$. Apply step (B) to $\{\hat X_c \hookrightarrow X : c \in \mathbb{N}\}$ and the $\hat f_c : \hat X_c \to Y$. This gives an open submanifold $X' \hookrightarrow X$ with $s_{\rm top}^{-1}(0) \subseteq X'_{\rm top}$ and a morphism $g : X' \to Y$ such that $g|_{X'\cap \hat X_c} = \hat f_c|_{X'\cap \hat X_c} + O(s)$ for all $c \in \mathbb{N}$. Let $a \in A$ and $c \in \mathbb{N}$. Then

$$g|_{X'\cap\hat X_c\cap X_a} = \hat f_c|_{X'\cap\hat X_c\cap X_a} + O(s) = f_{a_c}|_{X'\cap\hat X_c\cap X_a} + O(s) = f_a|_{X'\cap\hat X_c\cap X_a} + O(s),$$

using $f_{a_c}|_{X_{a_c a}} = f_a|_{X_{a_c a}} + O(s)$ and Theorem 3.17(c) (proved above). As this holds for all $c \in \mathbb{N}$ and the $\hat X_c$, $c \in \mathbb{N}$ cover $s_{\rm top}^{-1}(0)$, Theorem 3.17(a) implies that $g|_{X'\cap X_a} = f_a|_{X'\cap X_a} + O(s)$, as we want. This proves the first part of Theorem 3.17(e), without Γ-invariance/equivariance.

Proof of Theorem 3.17(e), the Γ-equivariant case

For the second part, we must show that if the initial data X, Y, $X_a \hookrightarrow X$, $f_a : X_a \to Y$ is invariant/equivariant under a finite group Γ, then we can choose X', g to be invariant/equivariant under Γ. To do this we must go through the whole proof above checking that each step can be done Γ-equivariantly. Most of this is easy or automatic – for example, when we choose the partition of unity $\{\eta_1, \eta_2\}$ in step (A), we can average η_1, η_2 over the Γ-action to make them Γ-invariant. But there is one point that needs a nontrivial proof.

Suppose as above we have X, Y, open $X_a \hookrightarrow X$ for $a \in A$, and morphisms $f_a : X_a \to Y$ with $f_a|_{X_{ab}} = f_b|_{X_{ab}} + O(s)$, and Γ acts on X, Y preserving the X_a, and the f_a are Γ-equivariant. Then by Definition B.36(iii) there exists a diagram (B.28) including $U_{ab}, V_{ab}, k_{1,ab}, k_{2,ab}$ with $s_{\rm top}^{-1}(0) \cap X_{ab,\rm top} \subseteq U_{ab,\rm top} \subseteq X_{ab,\rm top}$ and a morphism $v_{ab} : V_{ab} \to Y$ with $v_{ab} \circ k_{1,ab} = f_a|_{U_{ab}}$ and $v_{ab} \circ k_{2,ab} = f_b|_{U_{ab}}$, and these U_{ab}, V_{ab}, v_{ab} were used in the proof of step (A).

We can choose $U_{ab} \hookrightarrow X$ and $V_{ab} \hookrightarrow E$ to be Γ-invariant by replacing them by $\bigcap_{\gamma\in\Gamma} \gamma^{-1}(U_{ab})$ and $\bigcap_{\gamma\in\Gamma} \gamma^{-1}(V_{ab})$, and then $k_{1,ab}, k_{2,ab}$ are automatically Γ-equivariant. However, $v_{ab} : V_{ab} \to Y$ need not be Γ-equivariant.

We will show using Assumption 3.7(c) that given some choice of $U_{ab}, V_{ab}, k_{1,ab}, k_{2,ab}, v_{ab}$ that may not be Γ-invariant/equivariant, we can construct alternative choices $U'_{ab}, V'_{ab}, k'_{1,ab}, k'_{2,ab}, v'_{ab}$ which are Γ-invariant/equivariant.

First consider the case in which $E|_{X_{ab}}$ is trivial, with a Γ-equivariant trivialization $E|_{X_{ab}} \cong X_{ab} \times \mathbb{R}^n$, in which Γ acts linearly on the left on $\mathbb{R}^n$. Write $(\mathbb{R}^n)^{|\Gamma|}$ as $\bigoplus_{\gamma\in\Gamma}\mathbb{R}^n$, and elements of $(\mathbb{R}^n)^{|\Gamma|}$ as $(\boldsymbol{z}_\gamma)_{\gamma\in\Gamma}$ for $\boldsymbol{z}_\gamma \in \mathbb{R}^n$. Let Γ act linearly on $(\mathbb{R}^n)^{|\Gamma|}$, such that $\delta \in \Gamma$ acts in the given way on each copy of $\mathbb{R}^n$, but also δ permutes the indexing set Γ by right multiplication, so that

$$\delta : (\boldsymbol{z}_\gamma)_{\gamma\in\Gamma} \longmapsto (\delta \cdot \boldsymbol{z}_{\gamma\delta})_{\gamma\in\Gamma},$$

which gives a left action of Γ on $(\mathbb{R}^n)^{|\Gamma|}$.

We will use Assumption 3.7(c) to choose a Γ-invariant open submanifold $W_{ab} \hookrightarrow X_{ab} \times \bigoplus_{\gamma\in\Gamma}\mathbb{R}^n$ and a Γ-equivariant morphism $w_{ab} : W_{ab} \to Y$ such that

(i) $(s_{\rm top}^{-1}(0) \cap X_{ab,\rm top}) \times \{(0)_{\gamma\in\Gamma}\} \subseteq W_{ab,\rm top}$.

(ii) if $(x, \boldsymbol{z}) \in V_{ab,\rm top}$ and $\delta \in \Gamma$ with $\bigl(x, (\delta\cdot\boldsymbol{z})_\delta \amalg (0)_{\gamma\in\Gamma\setminus\{\delta\}}\bigr) \in W_{ab,\rm top}$ then $w_{ab,\rm top}\bigl(x, (\delta\cdot\boldsymbol{z})_\delta \amalg (0)_{\gamma\in\Gamma\setminus\{\delta\}}\bigr) = \delta \cdot v_{ab,\rm top}(x, \boldsymbol{z})$.

(iii) If $x \in X_{ab,\rm top}$ and $t_\gamma \in \mathbb{R}$ for $\gamma \in \Gamma$ with $\sum_{\gamma\in\Gamma} t_\gamma = 1$ and $\bigl(x, (t_\gamma \cdot s_{\rm top}(x))_{\gamma\in\Gamma}\bigr) \in W_{ab,\rm top}$ then $w_{ab,\rm top}\bigl(x, (t_\gamma \cdot s_{\rm top}(x))_{\gamma\in\Gamma}\bigr) = f_{b,\rm top}(x)$.

In fact we have to apply Assumption 3.7(c) finitely many times to choose $w_{ab,\rm top}\bigl(x, (\boldsymbol{z}_\gamma)_{\gamma\in B} \amalg (0)_{\gamma\in\Gamma\setminus B}\bigr)$ for all subsets $\emptyset \neq B \subseteq \Gamma$, by induction on increasing $|B| = 1, 2, \ldots, |\Gamma|$, following the proof of Proposition B.25 closely. When $B = \{\delta\}$ the values of $w_{ab,\rm top}$ are given by (ii). The condition that w_{ab} be Γ-equivariant means that the values of $w_{ab,\rm top}$ for $B \subseteq \Gamma$ determine the values for $B\delta$ for all $\delta \in \Gamma$, so we choose values of $w_{ab,\rm top}$ for one set B in each Γ-orbit of subsets $B' \subseteq \Gamma$. The values of $w_{ab,\rm top}$ for B must be chosen equivariant under $\mathrm{Stab}_\Gamma(B) = \{\delta \in \gamma : B\delta = B\}$, which is allowed by Assumption 3.7(c). Condition (iii) above comes from Assumption 3.7(b).

Now define $V'_{ab} \hookrightarrow E$, $U'_{ab} \hookrightarrow X$ to be the open submanifolds and $v'_{ab} : V'_{ab} \to Y$, $k'_{1,ab}, k'_{2,ab} : U'_{ab} \to V'_{ab}$ the morphisms defined on points by

$$\begin{aligned}
V'_{ab,\rm top} &= \bigl\{(x, \boldsymbol{z}) \in \bigl(E|_{X_{ab}}\bigr)_{\rm top} : \bigl(x, \bigl(\tfrac{1}{|\Gamma|}\boldsymbol{z}\bigr)_{\gamma\in\Gamma}\bigr) \in W_{ab,\rm top}\bigr\} \\
U'_{ab,\rm top} &= \bigl\{x \in X_{ab,\rm top} : (x, 0) \in V'_{ab,\rm top} \text{ and } (x, s_{\rm top}(x)) \in V'_{ab,\rm top}\bigr\}, \\
v'_{ab,\rm top}(x, \boldsymbol{z}) &= w_{ab,\rm top}\bigl(x, \bigl(\tfrac{1}{|\Gamma|}\boldsymbol{z}\bigr)_{\gamma\in\Gamma}\bigr), \\
k'_{1,ab,\rm top}(x) &= (x, 0), \qquad \text{and} \qquad k'_{2,ab,\rm top}(x) = (x, s_{\rm top}(x)).
\end{aligned}$$

Then we have $s_{\rm top}^{-1}(0) \cap X_{ab,\rm top} \subseteq U'_{ab,\rm top} \subseteq X_{ab,\rm top}$ and $v'_{ab} \circ k'_{1,ab} = f_a|_{U'_{ab}}$ and $v'_{ab} \circ k'_{2,ab} = f_b|_{U'_{ab}}$, as required to show that $f_a|_{X_{ab}} = f_b|_{X_{ab}} + O(s)$. Furthermore, as W_{ab} is Γ-invariant and w_{ab} is Γ-equivariant, we see that U'_{ab}, V'_{ab} are Γ-invariant and $v'_{ab}, k'_{1,ab}, k'_{2,ab}$ are Γ-equivariant, as we want.

Proof of Theorem 3.17(f)

Let $f, g : X \to Y$ be morphisms in $\dot{\mathbf{Man}}$ with $g = f + O(s)$, and $F \to X$, $G \to Y$ be vector bundles, and $\theta_1 : F \to f^*(G)$ be a morphism. We must show that there exists a morphism $\theta_2 : F \to g^*(G)$ with $\theta_2 = \theta_1 + O(s)$ as in Definition B.36(iv), and that such θ_2 are unique up to $O(s)$ as in Definition B.36(i).

First suppose $G \to Y$ is trivial, and choose a trivialization $G \cong Y \times \mathbb{R}^k$. Then $f^*(G)$ and $g^*(G)$ have induced trivializations $f^*(G) \cong X \times \mathbb{R}^k \cong g^*(G)$, giving an isomorphism $f^*(G) \cong g^*(G)$. Let $\theta_2 : F \to g^*(G)$ be the morphism identified with $\theta_1 : F \to f^*(G)$ by $f^*(G) \cong g^*(G)$. We claim that $\theta_2 = \theta_1 + O(s)$. To see this, let (B.28) and $v : V \to Y$ be as in Definition B.36(iii) for $g = f + O(s)$, and let $\phi : \pi^*(F)|_V \to v^*(G)$ be the morphism identified with $\pi^*(\theta_1)|_V : \pi^*(F)|_V \to (f \circ \pi)^*(G)|_V$ by the isomorphisms $v^*(G) \cong V \times \mathbb{R}^k \times (f \circ \pi)^*(G)|_V$. Then $k_1^*(\phi) = \theta_1|_U$ and $k_2^*(\phi) = \theta_2|_U$, so $\theta_2 = \theta_1 + O(s)$ by Definition B.36(iv).

Let $x \in s_{\rm top}^{-1}(0) \subseteq X_{\rm top}$, with $f_{\rm top}(x) = g_{\rm top}(x) = y \in Y_{\rm top}$. Choose an open neighbourhood $Y^y \hookrightarrow Y$ of y in Y with $G|_{Y^y}$ trivial. Let $X^x \hookrightarrow X$ be the open submanifold with $X^x_{\rm top} = f_{\rm top}^{-1}(Y^y_{\rm top}) \cap g_{\rm top}^{-1}(Y^y_{\rm top})$, so that $x \in X^x_{\rm top}$. Then we have morphisms $f|_{X^x}, g|_{X^x} : X^x \to Y^y$ with $g|_{X^x} = f|_{X^x} + O(s)$, and we have $\theta_1|_{X^x} : F|_{X^x} \to f|_{X^x}^*(G)$ with $G|_{Y^y}$ trivial. Hence from above there exists $\theta_2^x : F|_{X^x} \to g|_{X^x}^*(G)$ with $\theta_2^x = \theta_1|_{X^x} + O(s)$. Let $X^\infty \hookrightarrow X$ be the open submanifold with $X^\infty_{\rm top} = X_{\rm top} \setminus s_{\rm top}^{-1}(0)$. Set $\theta_2^\infty = 0 : F|_{X^\infty} \to g|_{X^\infty}^*(G)$. Then $\theta_2^\infty = \theta_1|_{X^\infty} + O(s)$, as $s \neq 0$ on X^∞.

Now $\{X^x : x \in s_{\rm top}^{-1}(0)\} \amalg \{X^\infty\}$ is an open cover of X. Choose a subordinate partition of unity $\{\eta^x : x \in s_{\rm top}^{-1}(0)\} \amalg \{\eta^\infty\}$ as in §B.1.4. Define $\theta_2 : F \to g^*(G)$ by $\theta_2 = \sum_{x \in s_{\rm top}^{-1}(0)} \eta^x \cdot \theta_2^x + \eta^\infty \cdot \theta^\infty$. Then using locality and $C^\infty(X)$-linearity in Theorem 3.17(a),(b) we see that $\theta_2 = \theta_1 + O(s)$ on X, as we have to prove.

Now suppose we have morphisms $\theta_2, \tilde\theta_2 : F \to g^*(G)$ with $\theta_2 = \theta_1 + O(s)$ and $\tilde\theta_2 = \theta_1 + O(s)$ as in Definition B.36(iv). We must show that $\tilde\theta_2 = \theta_2 + O(s)$ as in Definition B.36(i). By Theorem 3.17(a) it is enough to prove this locally near each $x \in s_{\rm top}^{-1}(0)$. So choose a small open neighbourhood X' of x. By Lemma B.51 we can use the same diagram (B.28) involving U, V, k_1, k_1 and morphism $v : V \to Y$ for verifying the conditions $\theta_2|_{X'} = \theta_1|_{X'} + O(s)$ and $\tilde\theta_2|_{X'} = \theta_1|_{X'} + O(s)$. Thus by Definition B.36(iv) there exist morphisms $\phi, \tilde\phi : \pi^*(F)|_V \to v^*(G)$ with $k_1^*(\phi) = k_1^*(\tilde\phi) = \theta_1|_U$, $k_2^*(\phi) = \theta_2|_U$ and $k_2^*(\tilde\phi) = \tilde\theta_2|_U$.

Making X', U', V' smaller we can suppose $E|_{X'}, F|_{X'}, f^*(G)|_{X'}, g^*(G)|_{X'}$, $v^*(G)$ are trivial, and choose isomorphisms $E|_{X'} \cong X' \times \mathbb{R}^n$, $F|_{X'} \cong X' \times \mathbb{R}^r$, $f^*(G)|_{X'} \cong X' \times \mathbb{R}^s \cong g^*(G)|_{X'}$, $v^*(G) \cong V \times \mathbb{R}^s$ which are compatible with $k_1^*(v^*(G)) = f^*(G)|_U$, $k_2^*(v^*(G)) = g^*(G)|_U$ for $U \subseteq X'$. Then we can interpret $s|_{X'}$ as a morphism $s' = (s'_1, \ldots, s'_n) : X' \to \mathbb{R}^n$ in $\dot{\mathbf{Man}}$, and $\theta_1|_{X'}, \theta_2|_{X'}, \tilde\theta_2|_{X'}$ as $\theta'_1, \theta'_2, \tilde\theta'_2 : X' \to (\mathbb{R}^r)^* \otimes \mathbb{R}^s$, and $\phi, \tilde\phi$ as $\phi', \tilde\phi' : V \to (\mathbb{R}^r)^* \otimes \mathbb{R}^s$.

We then have $V \hookrightarrow E|_{X'} \cong X' \times \mathbb{R}^n$ open, so writing points of $V_{\rm top}$ as $(x, \boldsymbol{z})$ for $x \in X'_{\rm top}$ and $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$, for all $x \in U_{\rm top} \subseteq X'_{\rm top}$ we have

$$\begin{aligned} \phi'_{\rm top}(x, 0) &= \theta'_{1,{\rm top}}(x), \quad \phi'_{\rm top}(x, s_{\rm top}(x)) = \theta'_{2,{\rm top}}(x), \\ \tilde\phi'_{\rm top}(x, 0) &= \theta'_{1,{\rm top}}(x), \quad \tilde\phi'_{\rm top}(x, s_{\rm top}(x)) = \tilde\theta'_{2,{\rm top}}(x). \end{aligned} \tag{B.52}$$

Applying Assumption 3.5 to $\tilde\phi' - \phi' : V \to (\mathbb{R}^r)^* \otimes \mathbb{R}^s$, we see that there exist morphisms $g_1, \ldots, g_n : V \to (\mathbb{R}^r)^* \otimes \mathbb{R}^s$ with

$$\tilde\phi'_{\rm top}(x, \boldsymbol{z}) - \phi'_{\rm top}(x, \boldsymbol{z}) = \textstyle\sum_{i=1}^n z_i \cdot g_{i,{\rm top}}(x, \boldsymbol{z}). \tag{B.53}$$

Define a vector bundle morphism $\alpha : E|_U \to F^*|_U \otimes g^*(G)|_U$ on points by

$$\alpha|_x : (e_1, \ldots, e_n) = \textstyle\sum_{i=1}^n e_i \cdot g_{i,\mathrm{top}}(x, s_{\mathrm{top}}(x)),$$

for $x \in U_{\mathrm{top}} \subseteq X'_{\mathrm{top}}$ and $(e_1, \ldots, e_n) \in E|_x \cong \mathbb{R}^n$, using the chosen trivializations. Then (B.52)–(B.53) imply that $\alpha \circ s = \tilde{\theta}_2|_U - \theta_2|_U$, so $\tilde{\theta}_2|_U = \theta_2|_U + O(s)$ on U as in Definition B.36(i). As $x \in U_{\mathrm{top}}$ and we can find such U for any $x \in s_{\mathrm{top}}^{-1}(0)$, Theorem 3.17(a) implies that $\tilde{\theta}_2 = \theta_2 + O(s)$, as we have to prove.

Proof of Theorem 3.17(g)

Let $f, g : X \to Y$ be morphisms with $g = f + O(s)$, and $F \to X$ be a vector bundle, and $\Lambda_1 : F \to \mathcal{T}_f Y$ be a morphism. We want to construct a morphism $\Lambda_2 : F \to \mathcal{T}_g Y$ with $\Lambda_2 = \Lambda_1 + O(s)$ as in Definition B.36(v), and show that such Λ_2 are unique up to $O(s)$ as in Definition B.36(ii).

As $g = f + O(s)$, by Definition B.36(iii) there is a commutative diagram (B.28) involving U, V, k_1, k_2 and a morphism $v : V \to Y$ with $v \circ k_1 = f|_U$ and $v \circ k_2 = g|_U$. By Proposition B.33 there exists a diagram (B.22)

$$\begin{tikzcd} & X \arrow[dl, "0_E"'] \arrow[d, "l"] \arrow[dr, "f"] & \\ F & W \arrow[l, hook', "j"'] \arrow[r, "w"] & Y, \end{tikzcd}$$

such that $\Lambda_1 = \theta_{W,w}$. Let $x \in s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}} \subseteq X_{\mathrm{top}}$, and choose an open neighbourhood $X^x \hookrightarrow U$ of x in U such that $E|_{X^x}, F|_{X^x}$ are trivial, and choose trivializations $E|_{X^x} \cong X^x \times \mathbb{R}^n$, $F|_{X^x} \cong X^x \times \mathbb{R}^r$.

We now use Assumption 3.7(a) with $k = 2$ to construct open $Z \hookrightarrow X^x \times \mathbb{R}^n \times \mathbb{R}^r$ with $X^x_{\mathrm{top}} \times \{(0,0)\} \subseteq Z_{\mathrm{top}}$ and a morphism $z : Z \to Y$ such that $z_{\mathrm{top}}(x', \boldsymbol{e}, 0) = v_{\mathrm{top}}(x', \boldsymbol{e})$ and $z_{\mathrm{top}}(x', 0, \boldsymbol{f}) = w_{\mathrm{top}}(x', \boldsymbol{f})$ for all $x' \in X^x_{\mathrm{top}}$, $\boldsymbol{e} \in \mathbb{R}^n$ and $\boldsymbol{f} \in \mathbb{R}^r$ for which both sides are defined. (Here to get $\mathbb{R}^n \oplus \mathbb{R}^n$ rather than $\mathbb{R}^n \oplus \mathbb{R}^r$, as in Assumption 3.7(a), we replace both n, r by $\max(n, r)$ and add an extra trivial factor of $\mathbb{R}^{|n-r|}$ to $E|_{X^x}$ or $F|_{X^x}$.)

Let $V' \hookrightarrow V$ and $U' \hookrightarrow X^x$ be the open submanifolds and $k'_1, k'_2 : U' \to V'$ the morphisms with

$$\begin{aligned} V'_{\mathrm{top}} &= \bigl\{(x', \boldsymbol{e}) : (x', \boldsymbol{e}, 0) \in Z_{\mathrm{top}},\ \ (x', \boldsymbol{e}) \in V_{\mathrm{top}}\bigr\}, \\ U'_{\mathrm{top}} &= \bigl\{x' : (x', 0) \in V'_{\mathrm{top}},\ \ (x', s_{\mathrm{top}}(x')) \in V'_{\mathrm{top}}\bigr\}, \\ k'_{1,\mathrm{top}}(x') = (x', 0), &\qquad k'_{2,\mathrm{top}}(x') = (x', s_{\mathrm{top}}(x')). \end{aligned}$$

Then $x \in U'_{\mathrm{top}}$. Define $\mathrm{M} : \pi^*(F)|_{V'} \to \mathcal{T}_v Y|_{V'}$ by $\mathrm{M} = \theta_{Z,z}$, in the notation of Definition B.32. Then $z_{\mathrm{top}}(x', 0, \boldsymbol{f}) = w_{\mathrm{top}}(x', \boldsymbol{f})$ with $\Lambda_1 = \theta_{W,w}$ and $k'_{1,\mathrm{top}}(x') = (x', 0)$ imply that $k_1'^*(\mathrm{M}) = \Lambda_1|_{U'}$. Define $\Lambda'_2 : F|_{U'} \to \mathcal{T}_g Y|_{U'}$ by $\Lambda'_2 = k_2'^*(\mathrm{M})$. Then Definition B.36(v) says that $\Lambda'_2 = \Lambda_1|_{U'} + O(s)$ on U'.

This shows that we can construct $\Lambda_2 : F \to \mathcal{T}_g Y$ with $\Lambda_2 = \Lambda_1 + O(s)$ locally near each x in $s_{\mathrm{top}}^{-1}(0)$. The proof can now be completed in a similar way to part (f).

Proof of Theorem 3.17(h)

Let $X, E, s, f, Y, F, G, t, \Lambda$ be as in Definition B.36(vi). By Proposition B.14(a) we may choose a connection ∇ on G. Then $\theta = f^*(\nabla t) \circ \Lambda : F \to f^*(G)$ is a vector bundle morphism as in §B.4.8, with $\theta = f^*(\mathrm{d}t) \circ \Lambda + O(s)$, so such θ exist as we want. Uniqueness of θ up to $O(s)$ in the sense of Definition B.36(i) is immediate from Definition B.36(vi) and Theorem 3.17(a).

Proof of Theorem 3.17(i)

Suppose $f, g : X \to Y$ are morphisms with $g = f + O(s)$. Then by Definition B.36(iii) there exists a diagram (B.28) involving U, V, k_1, k_2 with $s_{\rm top}^{-1}(0) \subseteq U_{\rm top} \subseteq X_{\rm top}$ and a morphism $v : V \to Y$ with $v \circ k_1 = f|_U$ and $v \circ k_2 = g|_U$. Then Definition B.32 gives $\theta_{V,v} : E|_U \to \mathcal{T}_f Y|_U$ with $g = f + \theta_{V,v} \circ s + O(s^2)$ on U. Let $W \hookrightarrow X$ be the open submanifold with $W_{\rm top} = X_{\rm top} \setminus s_{\rm top}^{-1}(0)$. Then $\{U, W\}$ is an open cover of X. Choose a subordinate partition of unity $\{\eta_U, \eta_W\}$ as in §B.1.4, and define $\Lambda = \eta_U \cdot \theta_{V,v} : E \to \mathcal{T}_f Y$. Then $g = f + \Lambda \circ s + O(s^2)$ on X, since near $s_{\rm top}^{-1}(0)$ in $X_{\rm top}$ we have $\Lambda = \theta_{V,v}$ with $g = f + \theta_{V,v} \circ s + O(s^2)$, and the condition is local near $s_{\rm top}^{-1}(0)$ by Theorem 3.17(a).

Proof of Theorem 3.17(j)

Let $f, g : X \to Y$ be morphisms in $\dot{\mathbf{Man}}$ with $g = f + O(s)$, and $\Lambda, \tilde\Lambda : E \to \mathcal{T}_f Y$ be morphisms with $g = f + \Lambda \circ s + O(s^2)$ as in Definition B.36(vii) and $\tilde\Lambda = \Lambda + O(s)$ as in Definition B.36(ii). We must prove that $g = f + \tilde\Lambda \circ s + O(s^2)$. By Theorem 3.17(a) it is enough to prove this near each $\check x$ in $s^{-1}(0) \subseteq X$.

So fix $\check x \in s^{-1}(0)$, and let $\check X$ be a small open neighbourhood of x in X on which $\check E = E|_{\check X}$ is trivial, and identify $\check E \cong \check X \times \mathbb{R}^n$. Write points of $\check E_{\rm top}$ as $(x, \boldsymbol{z})$ for $x \in \check X_{\rm top}$ and $\boldsymbol{z} \in \mathbb{R}^n$, and regard $\check s = s|_{\check X}$ as a morphism $\check s : \check X \to \mathbb{R}^n$. By Definition B.36(vii) there is a commutative diagram

$$\begin{tikzcd}
 & & Y & & \\
U^1 \arrow[rr, "k_1^1"] \arrow[d, hook] & & V^1 \arrow[u, "v^1"] \arrow[d, hook] & & U^1 \arrow[ll, "k_2^1"'] \arrow[d, hook] \\
\check X \arrow[rr, "\mathrm{id}_{\check X} \times 0"] \arrow[uurr, "f"] & & \check E = \check X \times \mathbb{R}^n & & \check X, \arrow[ll, "\mathrm{id}_{\check X} \times \check s"'] \arrow[uull, "g"']
\end{tikzcd} \tag{B.54}$$

with $\check s_{\rm top}^{-1}(0) \subseteq U^1_{\rm top}$ and $\Lambda|_{U^1} = \theta_{V^1, v^1}$, with morphisms '$\hookrightarrow$' open submanifolds.

By Definition B.36(ii) there is a commutative diagram

$$\begin{tikzcd}
U^2 \arrow[rr, "k_1^2"] \arrow[d, hook] & & V^2 \arrow[d, hook] & & U^2 \arrow[ll, "k_2^2"'] \arrow[d, hook] \\
\check X \arrow[rr, "\mathrm{id}_{\check X} \times 0"] & & \check E = \check X \times \mathbb{R}^n & & \check X, \arrow[ll, "\mathrm{id}_{\check X} \times \check s"']
\end{tikzcd}$$

with $\check s_{\rm top}^{-1}(0) \subseteq U^2_{\rm top}$ and a morphism $\mathrm{M} : \pi^*(\check E)|_{V^2} \to \mathcal{T}_{f \circ \pi} Y|_{V^2}$ with $k_1^{2*}(\mathrm{M}) = \Lambda|_{U^2}$ and $k_2^{2*}(\mathrm{M}) = \tilde\Lambda|_{U^2}$.

By Proposition B.33 there exists a diagram

$$
\begin{tikzcd}
& & V^2 \arrow[dll, "0_{\check E}"'] \arrow[d] \arrow[drr, "f\circ\pi\circ j'"] & & \\
\pi^*(\check E)|_{V^2}=V^2\times\mathbb{R}^n & & W^1 \arrow[ll, hook'] \arrow[rr, "w^1"] & & Y,
\end{tikzcd}
\tag{B.55}
$$

with $\mathrm{M}=\theta_{W^1,w^1}$. Define $V^3,V^4\hookrightarrow\check E=\check X\times\mathbb{R}^n$ to be the open submanifolds and $v^3:V^3\to Y$, $v^4:V^4\to Y$ the morphisms with

$$
\begin{aligned}
V^3_{\rm top}&=\bigl\{(x,\boldsymbol{z})\in\check X_{\rm top}\times\mathbb{R}^n:(x,0,\boldsymbol{z})\in W^1_{\rm top}\bigr\},\\
V^4_{\rm top}&=\bigl\{(x,\boldsymbol{z})\in\check X_{\rm top}\times\mathbb{R}^n:(x,\check s_{\rm top}(x),\boldsymbol{z})\in W^1_{\rm top}\bigr\},\\
v^3_{\rm top}(x,\boldsymbol{z})&=w^1_{\rm top}(x,0,\boldsymbol{z}),\quad v^4_{\rm top}(x,\boldsymbol{z})=w^1_{\rm top}(x,\check s_{\rm top}(x),\boldsymbol{z}).
\end{aligned}
\tag{B.56}
$$

Then $k_1^{2*}(\mathrm{M})=\Lambda|_{U^2}$ and $k_2^{2*}(\mathrm{M})=\tilde\Lambda|_{U^2}$ give $\Lambda|_{U^2}=\theta_{V^3,v^3}$ and $\tilde\Lambda|_{U^2}=\theta_{V^4,v^4}$.

Let $U^3\hookrightarrow\check X$ be the open submanifold with $U^3_{\rm top}=U^1_{\rm top}\cap U^2_{\rm top}$. Then $\theta_{V^1,v^1}|_{U^3}=\Lambda|_{U^3}=\theta_{V^3,v^3}|_{U^3}$. Therefore, extending Definition B.16, and making $\check X$ smaller if necessary, we can find an open submanifold $W^2\hookrightarrow\check X\times\mathbb{R}^n\times\mathbb{R}^n$ with $U^3_{\rm top}\times\{(0,0)\}\subseteq W^2_{\rm top}$ and a morphism $w^2:W^2\to Y$ with

$$
\begin{aligned}
w^2_{\rm top}(x,\boldsymbol{z}_1,0)=v^1_{\rm top}(x,\boldsymbol{z}_1),\qquad w^2_{\rm top}(x,0,\boldsymbol{z}_2)=v^3_{\rm top}(x,\boldsymbol{z}_2),\\
\text{and}\qquad w^2_{\rm top}(x,\boldsymbol{z}_1,-\boldsymbol{z}_1)=f_{\rm top}(x).
\end{aligned}
\tag{B.57}
$$

When $n=1$ the existence of W^2,w^2 follows from $\theta_{V^1,v^1}|_{U^3}=\theta_{V^3,v^3}|_{U^3}$ and Definition B.16, where $w^2_{\rm top}(x,\boldsymbol{z}_1,-\boldsymbol{z}_1)=f_{\rm top}(x)$ in (B.57) corresponds to $v_{\rm top}(x,s,-s)=f_{\rm top}(x)$ in (B.6). For $n>1$, we split $\theta_{V^1,v^1}|_{U^3},\theta_{V^3,v^3}|_{U^3}$ into n components in $\Gamma(\mathcal{T}_fY|_{U^3})$, each of which admits an extension to $W^2_i\hookrightarrow\check X\times\mathbb{R}\times\mathbb{R}$, $w^2_i:W^2_i\to Y$ as in (B.57) for $i=1,\ldots,n$, and then we use Assumption 3.7(a) repeatedly to construct W^2,w^2 in a similar way to the proof in Definition B.18 choosing $w_{\rm top}$ to satisfy (B.9).

Next we apply Assumption 3.7(a) with $k=3$ to choose open $Z\hookrightarrow\check X\times(\mathbb{R}^n)^3$ with $\check s^{-1}_{\rm top}(0)\times\{(0,0,0)\}\subseteq Z_{\rm top}$ and a morphism $z:Z\to Y$ with

$$
\begin{aligned}
z_{\rm top}(x,\boldsymbol{z}_1,\boldsymbol{z}_2,0)=f_{\rm top}(x),\qquad z_{\rm top}(x,\boldsymbol{z}_1,0,\boldsymbol{z}_3)=w^1_{\rm top}(x,\boldsymbol{z}_1,\boldsymbol{z}_3),\\
\text{and}\qquad z_{\rm top}(x,0,\boldsymbol{z}_2,\boldsymbol{z}_3)=w^2_{\rm top}(x,\boldsymbol{z}_2,\boldsymbol{z}_3-\boldsymbol{z}_2).
\end{aligned}
\tag{B.58}
$$

Here pairs of equations in (B.58) give the same values on intersections

$$
z_{\rm top}(x,\boldsymbol{z}_1,0,0)=f_{\rm top}(x),\;\; z_{\rm top}(x,0,\boldsymbol{z}_2)=f_{\rm top}(x),\;\; z_{\rm top}(x,0,0,\boldsymbol{z}_3)=v^3_{\rm top}(x,\boldsymbol{z}_3),
$$

by (B.54)–(B.57), so Assumption 3.7(a) applies.

Define $U^4\hookrightarrow\check X$, $V^5\hookrightarrow\check X\times\mathbb{R}^n$, $W^3\hookrightarrow\check X\times\mathbb{R}^n\times\mathbb{R}^n$ to be the open submanifolds and $v^5 : V^5 \to Y$, $w^3 : W^3 \to Y$ the morphisms with

$$\begin{aligned}
&U^4_{\rm top} = \bigl\{x \in \check X_{\rm top} : (x, s_{\rm top}(x), 0, 0) \in Z_{\rm top},\ (x, 0, s_{\rm top}(x), s_{\rm top}(x)) \in Z_{\rm top}\bigr\},\\
&V^5_{\rm top} = \bigl\{(x, \boldsymbol{z}_1) \in \check X_{\rm top} \times \mathbb{R}^n : (x, s_{\rm top}(x) - \boldsymbol{z}_1, \boldsymbol{z}_1, \boldsymbol{z}_1) \in Z_{\rm top}\bigr\},\\
&W^3_{\rm top} = \bigl\{(x, \boldsymbol{z}_1, \boldsymbol{z}_2) \in \check X_{\rm top} \times \mathbb{R}^n \times \mathbb{R}^n : (x, s_{\rm top}(x) - \boldsymbol{z}_1, \boldsymbol{z}_1, \boldsymbol{z}_1 + \boldsymbol{z}_2) \in Z_{\rm top}\bigr\},\\
&v^5_{\rm top}(x, \boldsymbol{z}_1) = z_{\rm top}(x, s_{\rm top}(x) - \boldsymbol{z}_1, \boldsymbol{z}_1, \boldsymbol{z}_1),\\
&w^3_{\rm top}(x, \boldsymbol{z}_1, \boldsymbol{z}_2) = z_{\rm top}(x, s_{\rm top}(x) - \boldsymbol{z}_1, \boldsymbol{z}_1, \boldsymbol{z}_1 + \boldsymbol{z}_2).
\end{aligned} \tag{B.59}$$

Then $s_{\rm top}^{-1}(0) \subseteq U^4_{\rm top}$. From (B.56), (B.58) and (B.59) we see that

$$w^3_{\rm top}(x, \boldsymbol{z}_1) = v^5_{\rm top}(x, \boldsymbol{z}_1),\ w^3_{\rm top}(x, 0, \boldsymbol{z}_2) = v^4_{\rm top}(x, \boldsymbol{z}_2),\ w^3_{\rm top}(x, \boldsymbol{z}_1, -\boldsymbol{z}_1) = f_{\rm top}(x).$$

Hence combining Definitions B.16, B.32 shows that $\theta_{V^5, v^5} = \theta_{V^4, v^4}|_{U^4}$. Now

$$\begin{aligned}
v^5_{\rm top}(x, \check s_{\rm top}(x)) &= z_{\rm top}(x, 0, \check s_{\rm top}(x), \check s_{\rm top}(x)) = w^2_{\rm top}(x, \check s_{\rm top}(x), 0)\\
&= v^1_{\rm top}(x, \check s_{\rm top}(x)) = g_{\rm top}(x)
\end{aligned}$$

for $x \in U^4_{\rm top}$, by (B.54), (B.57), (B.58), and (B.59). Thus Definition B.36(vii) with U^4, V^5, v^5 in (B.29) shows that $g|_{U^4} = f|_{U^4} + \theta_{V^5, v^5} \circ s + O(s^2)$. But from above $\tilde\Lambda|_{U^2} = \theta_{V^4, v^4}$ and $\theta_{V^5, v^5} = \theta_{V^4, v^4}|_{U^4}$. Therefore $g|_{U^4} = f|_{U^4} + \tilde\Lambda|_{U^4} \circ s + O(s^2)$ on U^4. Since $\check x \in U^4_{\rm top}$ and this holds for all $\check x \in s^{-1}(0)$, Theorem 3.17(a) implies that $g = f + \tilde\Lambda \circ s + O(s^2)$ on X, proving part (j).

Proof of Theorem 3.17(k)

Let $f, g : X \to Y$ be morphisms in $\dot{\mathbf{Man}}$ with $g = f + O(s)$, and $\Lambda : E \to \mathcal{T}_f Y$ be a morphism with $g = f + \Lambda \circ s + O(s^2)$. Theorem 3.17(g) gives $\tilde\Lambda : F \to \mathcal{T}_g Y$ with $\tilde\Lambda = \Lambda + O(s)$ as in Definition B.36(v), where $\tilde\Lambda$ is unique up to $O(s)$. We must show that $f = g + (-\tilde\Lambda) \circ s + O(s^2)$.

By Definition B.36(vii) there is a commutative diagram (B.29) involving U, V, k_1, k_2, v, with $s_{\rm top}^{-1}(0) \subseteq U_{\rm top}$ and $\Lambda|_U = \theta_{V,v}$. Define $V', V'' \hookrightarrow E$ to be the open submanifolds and $v' : V' \to Y$, $v'' : V'' \to Y$ the morphisms with

$$\begin{aligned}
V'_{\rm top} &= \bigl\{(x, e) \in E_{\rm top} : (x, s_{\rm top}(x) + e) \in V_{\rm top}\bigr\},\\
V''_{\rm top} &= \bigl\{(x, e) \in E_{\rm top} : (x, s_{\rm top}(x) - e) \in V_{\rm top}\bigr\},\\
v'_{\rm top}(x, e) &= v_{\rm top}(x, s_{\rm top}(x) + e), \quad v''_{\rm top}(x, e) = v_{\rm top}(x, s_{\rm top}(x) - e).
\end{aligned} \tag{B.60}$$

Then (B.29) implies that $0_{E,{\rm top}}(U_{\rm top}) \subseteq V'_{\rm top}, V''_{\rm top}$ and $v'_{\rm top}(x, 0) = v''_{\rm top}(x, 0) = g_{\rm top}(x)$ for $x \in U_{\rm top}$. Hence Definition B.32 defines morphisms

$$\theta_{V', v'} : E|_U \longrightarrow \mathcal{T}_g Y|_U, \qquad \theta_{V'', v''} : E|_U \longrightarrow \mathcal{T}_g Y|_U.$$

Since (B.60) gives $v''_{\rm top}(x, e) = v'_{\rm top}(x, -e)$ for all $(x, e) \in V''_{\rm top}$ we see from §B.4.2 that $\theta_{V'', v''} = -\theta_{V', v'}$. For $x \in U_{\rm top}$ we have $v''_{\rm top}(x, s_{\rm top}(x)) = v_{\rm top}(x, 0) =$

$f_{\rm top}(x)$ by (B.29) and (B.60). Hence $f|_U = g|_U + \theta_{V'',v''} \circ s + O(s^2)$ on U by Definition B.36(vii).

Writing $\pi : V \to X$ for the projection we have a vector bundle $\pi^*(E) \to V$. Write points of $\pi^*(E)$ as (x, e_1, e_2) where $\pi_{\rm top} : \pi^*(E)_{\rm top} \to V_{\rm top}$ maps $(x, e_1, e_2) \mapsto (x, e_1)$. Define $W \hookrightarrow \pi^*(E)$ to be the open submanifold and $w : W \to Y$ the morphism with

$$\begin{aligned} W_{\rm top} &= \bigl\{(x, e_1, e_2) \in \pi^*(E)_{\rm top} : (x, e_1 + e_2) \in V_{\rm top}\bigr\},\\ w_{\rm top}(x, e_1, e_2) &= v_{\rm top}(x, e_1 + e_2). \end{aligned}$$

Since $0_{\pi^*(E),{\rm top}}(V_{\rm top}) \subseteq W_{\rm top}$ with $w_{\rm top}(x, e_1, 0) = v_{\rm top}(x, e_1)$ for $(x, e_1) \in V_{\rm top}$, Definition B.32 defines a morphism $\theta_{W,w} : \pi^*(E) \longrightarrow \mathcal{T}_v Y$. As $k_1(x) = (x, 0)$ and $w_{\rm top}(x, 0, e) = v_{\rm top}(x, e)$ we have $k_1^*(\theta_{W,w}) = \theta_{V,v}|_U$. Since $k_2(x) = (x, s_{\rm top}(x))$ and $w_{\rm top}(x, s_{\rm top}(x), e) = v'_{\rm top}(x, e)$ we have $k_2^*(\theta_{W,w}) = \theta_{V',v'}$. Thus $\theta_{V',v'} = \theta_{V,v}|_U + O(s)$ by Definition B.36(ii).

We now have morphisms $\Lambda|_U, \theta_{V,v}|_U : E|_U \to \mathcal{T}_f Y|_U$ and $\tilde\Lambda|_U, \theta_{V',v'} : E|_U \to \mathcal{T}_g Y|_U$ with $\Lambda|_U = \theta_{V,v}|_U$ and $\tilde\Lambda|_U = \Lambda|_U + O(s)$, $\theta_{V',v'} = \theta_{V,v}|_U + O(s)$ as in Definition B.36(v). Thus uniqueness up to $O(s)$ in Theorem 3.17(g) shows that $\tilde\Lambda|_U = \theta_{V',v'} + O(s)$ as in Definition B.36(ii). Also $\theta_{V'',v''} = -\theta_{V',v'}$, so $\theta_{V'',v''} = -\tilde\Lambda|_U + O(s)$, and $f|_U = g|_U + \theta_{V'',v''} \circ s + O(s^2)$. Therefore Theorem 3.17(j) shows that $f|_U = g|_U + (-\tilde\Lambda|_U) \circ s + O(s^2)$. Since $s_{\rm top}^{-1}(0) \subseteq U_{\rm top}$, Theorem 3.17(a) now yields $f = g + (-\tilde\Lambda) \circ s + O(s^2)$, as we have to prove.

Proof of Theorem 3.17(l)

Let $f, g, h : X \to Y$ be morphisms in $\dot{\mathbf{Man}}$ with $g = f + O(s)$, $h = g + O(s)$ and $\Lambda_1 : E \to \mathcal{T}_f Y$, $\Lambda_2 : E \to \mathcal{T}_g Y$ be morphisms with $g = f + \Lambda_1 \circ s + O(s^2)$ and $h = g + \Lambda_2 \circ s + O(s^2)$. Theorem 3.17(g) gives $\tilde\Lambda_2 : E \to \mathcal{T}_f Y$ with $\tilde\Lambda_2 = \Lambda_2 + O(s)$ as in Definition B.36(v), unique up to $O(s)$. We must show that $h = f + (\Lambda_1 + \tilde\Lambda_2) \circ s + O(s^2)$.

Suppose first that $E \to X$ is trivial, and identify $E \cong X \times \mathbb{R}^n$. Write points of $E_{\rm top}$ as $(x, \boldsymbol{z})$ for $x \in X_{\rm top}$ and $\boldsymbol{z} \in \mathbb{R}^n$, and regard s as a morphism $X \to \mathbb{R}^n$. By Definition B.36(vii) there are commutative diagrams

$$\begin{tikzcd} & & Y & & \\ U^1 \arrow[rr, "k_1^1"'] \arrow[d, hook] & & V^1 \arrow[u, "v^1"] \arrow[d, hook] & & U^1 \arrow[ll, "k_2^1"] \arrow[d, hook] \\ X \arrow[rruu, "f"] \arrow[rr, "\mathrm{id}_X \times 0"] & & E = X \times \mathbb{R}^n & & X, \arrow[ll, "\mathrm{id}_X \times s"'] \arrow[lluu, "g"'] \end{tikzcd} \tag{B.61}$$

$$\begin{tikzcd} & & Y & & \\ U^2 \arrow[rr, "k_1^2"'] \arrow[d, hook] & & V^2 \arrow[u, "v^2"] \arrow[d, hook] & & U^2 \arrow[ll, "k_2^2"] \arrow[d, hook] \\ X \arrow[rruu, "g"] \arrow[rr, "\mathrm{id}_X \times 0"] & & E = X \times \mathbb{R}^n & & X, \arrow[ll, "\mathrm{id}_X \times s"'] \arrow[lluu, "h"'] \end{tikzcd} \tag{B.62}$$

with $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}}^1, U_{\mathrm{top}}^2$ and $\Lambda_1|_{U^1} = \theta_{V^1,v^1}$, $\Lambda_2|_{U^2} = \theta_{V^2,v^2}$.

Apply Assumption 3.7(a) with $k = 2$ to choose open $W^1 \hookrightarrow X \times \mathbb{R}^n \times \mathbb{R}^n$ with $s_{\mathrm{top}}^{-1}(0) \times \{(0,0)\} \subseteq W_{\mathrm{top}}^1$ and a morphism $w^1 : W^1 \to Y$ with

$$w_{\mathrm{top}}^1(x, \boldsymbol{z}_1, 0) = v_{\mathrm{top}}^1(x, \boldsymbol{z}_1 + s_{\mathrm{top}}(x)), \quad w_{\mathrm{top}}^1(x, 0, \boldsymbol{z}_2) = v_{\mathrm{top}}^2(x, \boldsymbol{z}_2). \tag{B.63}$$

Both equations have $w_{\mathrm{top}}^1(x, 0, 0) = g_{\mathrm{top}}(x)$ by (B.61)–(B.62), so Assumption 3.7(a) applies. Define open submanifolds $U^3 \hookrightarrow X$, $V^3, V^4, V^5 \hookrightarrow X \times \mathbb{R}^n$, $W^2 \hookrightarrow X \times \mathbb{R}^n \times \mathbb{R}^n$ and morphisms $v^3 : V^3 \to Y$, $v^4 : V^4 \to Y$, $v^5 : V^5 \to Y$, $w^2 : W^2 \to Y$, $k_1^3, k_2^3 : U^3 \to V^5$ with

$$\begin{aligned}
&U_{\mathrm{top}}^3 = \bigl\{x \in U_{\mathrm{top}}^1 \cap U_{\mathrm{top}}^2 : (x,0,0), (x, -s_{\mathrm{top}}(x), 0), (x, 0, s_{\mathrm{top}}(x)) \in W_{\mathrm{top}}^1\bigr\}\\
&V_{\mathrm{top}}^3 = \bigl\{(x, \boldsymbol{z}) \in X_{\mathrm{top}} \times \mathbb{R}^n : (x, -s_{\mathrm{top}}(x), \boldsymbol{z}) \in W_{\mathrm{top}}^1\bigr\},\\
&V_{\mathrm{top}}^4 = \bigl\{(x, \boldsymbol{z}) \in X_{\mathrm{top}} \times \mathbb{R}^n : (x, \boldsymbol{z} - s_{\mathrm{top}}(x), \boldsymbol{z}) \in W_{\mathrm{top}}^1\bigr\},\\
&V_{\mathrm{top}}^5 = \bigl\{(x, \boldsymbol{z}) \in X_{\mathrm{top}} \times \mathbb{R}^n : (x, \boldsymbol{z} - s_{\mathrm{top}}(x), 0) \in W_{\mathrm{top}}^1\bigr\},\\
&W_{\mathrm{top}}^2 = \bigl\{(x, \boldsymbol{z}_1, \boldsymbol{z}_2) \in X_{\mathrm{top}} \times \mathbb{R}^n \times \mathbb{R}^n : (x, \boldsymbol{z}_1 - s_{\mathrm{top}}(x), \boldsymbol{z}_2) \in W_{\mathrm{top}}^1\bigr\},\\
&v_{\mathrm{top}}^3(x, \boldsymbol{z}) = w_{\mathrm{top}}^1(x, -s_{\mathrm{top}}(x), \boldsymbol{z}), \quad v_{\mathrm{top}}^4(x, \boldsymbol{z}) = w_{\mathrm{top}}^1(x, \boldsymbol{z} - s_{\mathrm{top}}(x), \boldsymbol{z}),\\
&v_{\mathrm{top}}^5(x, \boldsymbol{z}) = w_{\mathrm{top}}^1(x, \boldsymbol{z} - s_{\mathrm{top}}(x), 0), \;\; w_{\mathrm{top}}^2(x, \boldsymbol{z}_1, \boldsymbol{z}_2) = w_{\mathrm{top}}^1(x, \boldsymbol{z}_1 - s_{\mathrm{top}}(x), \boldsymbol{z}_2),\\
&k_{1,\mathrm{top}}^3(x) = (x, 0) \quad \text{and} \quad k_{2,\mathrm{top}}^3(x) = (x, s_{\mathrm{top}}(x)).
\end{aligned} \tag{B.64}$$

Then (B.61)–(B.64) imply that

$$\begin{aligned}
v_{\mathrm{top}}^3(x, 0) = v_{\mathrm{top}}^4(x, 0) = w_{\mathrm{top}}^2(x, 0, 0) = f_{\mathrm{top}}(x), &\qquad w_{\mathrm{top}}^2(x, \boldsymbol{z}_1, 0) = v_{\mathrm{top}}^1(x, \boldsymbol{z}_1),\\
w_{\mathrm{top}}^2(x, 0, \boldsymbol{z}_2) = v_{\mathrm{top}}^3(x, \boldsymbol{z}_2), &\qquad w_{\mathrm{top}}^2(x, \boldsymbol{z}_1, \boldsymbol{z}_1) = v_{\mathrm{top}}^4(x, \boldsymbol{z}_1).
\end{aligned}$$

The first equation shows there are morphisms $\theta_{V^3,v^3}, \theta_{V^4,v^4} : E|_{U^3} \to \mathcal{T}_f Y|_{U^3}$, and the last three equations and the definition of addition in $\Gamma(\mathcal{T}_f Y)$ in §B.4.2 imply that $\theta_{V^4,v^4} = \theta_{V^1,v^1}|_{U^3} + \theta_{V^3,v^3}$. Also for $x \in U_{\mathrm{top}}^3$ we have

$$v_{\mathrm{top}}^4(x, s_{\mathrm{top}}(x)) = w_{\mathrm{top}}^1(x, 0, s_{\mathrm{top}}(x)) = v_{\mathrm{top}}^2(x, s_{\mathrm{top}}(x)) = h_{\mathrm{top}}(x)$$

by (B.62)–(B.64). Thus $h|_{U^3} = f|_{U^3} + \theta_{V^4,v^4} \circ s + O(s^2)$ by Definition B.36(vii).

Consider W^2 as an open set in the vector bundle $\pi : \pi^*(E) \to E$ acting on points by $\pi_{\mathrm{top}} : (x, \boldsymbol{z}_1, \boldsymbol{z}_2) \mapsto (z, \boldsymbol{z}_1)$. Then we have a morphism $\theta_{W^2,w^2} : \pi^*(E)|_{V^5} \to \mathcal{T}_{v^5} Y$. Since $k_{1,\mathrm{top}}^3(x) = (x, 0)$ with $w_{\mathrm{top}}^2(x, 0, \boldsymbol{z}_2) = v_{\mathrm{top}}^3(x, \boldsymbol{z}_2)$ we have $k_1^{3*}(\theta_{W^2,w^2}) = \theta_{V^3,v^3}|_{U^3}$, and as $k_{2,\mathrm{top}}^3(x) = (x, s_{\mathrm{top}}(x))$ with $w^2(x, s_{\mathrm{top}}(x), \boldsymbol{z}_2) = v_{\mathrm{top}}^2(x, \boldsymbol{z}_2)$ we have $k_2^{3*}(\theta_{W^2,w^2}) = \theta_{V^2,v^2}|_{U^3}$. Therefore $\theta_{V^3,v^3} = \theta_{V^2,v^2}|_{U^3} + O(s)$ by Definition B.36(ii).

We now have $\tilde{\Lambda}_2|_{U^3}, \theta_{V^3,v^3} : E|_{U^3} \to \mathcal{T}_f Y|_{U^3}$ and $\Lambda_2|_{U^3}, \theta_{V^2,v^2}|_{U^3} : E|_{U^3} \to \mathcal{T}_g Y|_{U^3}$ with $\Lambda_2|_{U^3} = \theta_{V^2,v^2}|_{U^3}$ and $\tilde{\Lambda}_2|_{U^3} = \Lambda|_{U^3} + O(s)$, $\theta_{V^3,v^3} = \theta_{V^2,v^2}|_{U^3} + O(s)$ as in Definition B.36(v). Thus uniqueness up to $O(s)$ in Theorem 3.17(g) shows that $\tilde{\Lambda}_2|_{U^3} = \theta_{V^3,v^3} + O(s)$ as in Definition B.36(ii). Also $\Lambda_1|_{U^3} = \theta_{V^1,v^1}|_{U^3}$ and $\theta_{V^4,v^4} = \theta_{V^1,v^1}|_{U^3} + \theta_{V^3,v^3}$ from above, so $\theta_{V^4,v^4} = \Lambda_1|_{U^3} + \tilde{\Lambda}_2|_{U^3} + O(s)$. But $h|_{U^3} = f|_{U^3} + \theta_{V^4,v^4} \circ s + O(s^2)$, so Theorem 3.17(j) shows that $h|_{U^3} =$

$f|_{U^3} + (\Lambda_1 + \tilde{\Lambda}_2)|_{U^3} \circ s + O(s^2)$. Since $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}}^3$, Theorem 3.17(a) now yields $h = f + (\Lambda_1 + \tilde{\Lambda}_2) \circ s + O(s^2)$.

This proves Theorem 3.17(l) when $E \to X$ is trivial. But $h = f + (\Lambda_1 + \tilde{\Lambda}_2) \circ s + O(s^2)$ is a local condition by Theorem 3.17(a), so by restricting to an open cover of subsets of X on which E is trivial, part (l) follows.

Proof of Theorem 3.17(m)

Let $f, g : X \to Y$ be morphisms with $g = f + O(s)$, and $\Lambda_1, \ldots, \Lambda_k : E \to \mathcal{T}_f Y$ be morphisms with $g = f + \Lambda_a \circ s + O(s^2)$ for $a = 1, \ldots, k$, and $\alpha_1, \ldots, \alpha_k \in C^\infty(X)$ with $\alpha_1 + \cdots + \alpha_k = 1$. We must show that $g = f + (\alpha_1 \cdot \Lambda_1 + \cdots + \alpha_k \cdot \Lambda_k) \circ s + O(s^2)$.

Suppose first that $E \to X$ is trivial, and identify $E \cong X \times \mathbb{R}^n$. Write points of E_{top} as $(x, \boldsymbol{z})$ for $x \in X_{\mathrm{top}}$ and $\boldsymbol{z} \in \mathbb{R}^n$, and regard s as a morphism $X \to \mathbb{R}^n$. By Definition B.36(vii), for $i = 1, \ldots, k$ there are commutative diagrams

$$\begin{tikzcd}
& & Y & & \\
U^i \arrow[rr, "k_1^i"'] \arrow[d, hook] & & V^i \arrow[u, "v^i"'] \arrow[d, hook] & & U^i \arrow[ll, "k_2^i"] \arrow[d, hook] \\
X \arrow[rr, "\mathrm{id}_X \times 0"] \arrow[uurr, bend left, "f"] & & E = X \times \mathbb{R}^n & & X, \arrow[ll, "\mathrm{id}_X \times s"'] \arrow[uull, bend right, "g"']
\end{tikzcd} \tag{B.65}$$

with $s_{\mathrm{top}}^{-1}(0) \subseteq U_{\mathrm{top}}^i$ and $\Lambda_i|_{U^i} = \theta_{V^i, v^i}$ for $i = 1, \ldots, k$.

Apply Assumption 3.7(b) to choose an open submanifold $W \hookrightarrow X \times (\mathbb{R}^n)^k$ and a morphism $w : W \to Y$ satisfying:

(i) $s_{\mathrm{top}}^{-1}(0) \times \{(0, \ldots, 0)\} \subseteq W_{\mathrm{top}} \subseteq X_{\mathrm{top}} \times (\mathbb{R}^n)^k$.

(ii) if $(x, (0, \ldots, 0, \boldsymbol{z}_i, 0, \ldots, 0)) \in W_{\mathrm{top}}$ with $\boldsymbol{z}_i$ in the i^{th} copy of $\mathbb{R}^n$ for $i = 1, \ldots, k$ then $(x, \boldsymbol{z}_i) \in V_{\mathrm{top}}^i$ and $v_{\mathrm{top}}^i(x, \boldsymbol{z}_i) = w_{\mathrm{top}}(x, (0, \ldots, 0, \boldsymbol{z}_i, 0, \ldots, 0))$.

(iii) If $x \in X_{\mathrm{top}}$ and $t_1, \ldots, t_k \in \mathbb{R}$ with $\sum_{i=1}^k t_i = 1$ and $(x, (t_1 \cdot s_{\mathrm{top}}(x), \ldots, t_k \cdot s_{\mathrm{top}}(x))) \in W_{\mathrm{top}}$ then $w_{\mathrm{top}}(x, (t_1 \cdot s_{\mathrm{top}}(x), \ldots, t_k \cdot s_{\mathrm{top}}(x))) = g_{\mathrm{top}}(x)$.

Actually we use Assumption 3.7(b) inductively $2^k - k - 1$ times to choose $w_{\mathrm{top}}(x, (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k))$ with subsets of the $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k$ zero, as for (a)–(d) in the proof of Theorem 3.17(a)(iii),(iv),(v),(vii) above.

Define open submanifolds $U' \hookrightarrow X$, $V' \hookrightarrow E = X \times \mathbb{R}^n$ and morphisms $v' : V' \to Y$, $k_1', k_2' : U' \to V'$ with

$$\begin{aligned}
&V'_{\mathrm{top}} = \big\{(x, \boldsymbol{z}) \in X_{\mathrm{top}} \times \mathbb{R}^n : (x, (\alpha_{1,\mathrm{top}}(x)\boldsymbol{z}, \ldots, \alpha_{k,\mathrm{top}}(x)\boldsymbol{z}) \in W_{\mathrm{top}}\big\},\\
&U'_{\mathrm{top}} = \big\{x \in X_{\mathrm{top}} : (x, 0) \in V'_{\mathrm{top}},\ \ (x, s_{\mathrm{top}}(x)) \in V'_{\mathrm{top}}\big\},\\
&v'_{\mathrm{top}}(x, \boldsymbol{z}) = w_{\mathrm{top}}(x, (x, (\alpha_{1,\mathrm{top}}(x)\boldsymbol{z}, \ldots, \alpha_{k,\mathrm{top}}(x)\boldsymbol{z})),\\
&k'_{1,\mathrm{top}}(x) = (x, 0) \quad \text{and} \quad k'_{2,\mathrm{top}}(x) = (x, s_{\mathrm{top}}(x)).
\end{aligned} \tag{B.66}$$

Then $v'_{\mathrm{top}}(x, 0) = w_{\mathrm{top}}(x, (0, \ldots, 0)) = v_{\mathrm{top}}^i(x, 0) = f_{\mathrm{top}}(x)$ for all $x \in U'_{\mathrm{top}}$ by (B.65)–(B.66) and (ii), so Definition B.32 gives $\theta_{V', v'} : E|_{U'} \to \mathcal{T}_f Y|_{U'}$. Also

$$v'_{\mathrm{top}}(x, s_{\mathrm{top}}(x)) = w_{\mathrm{top}}(x, (\alpha_{1,\mathrm{top}}(x) \cdot s_{\mathrm{top}}(x), \ldots, \alpha_{k,\mathrm{top}}(x) \cdot s_{\mathrm{top}}(x))) = g_{\mathrm{top}}(x)$$

for all $x \in U'_{\rm top}$ by (B.66), (iii) and $\sum_{i=1}^k \alpha_i = 1$, so $g|_{U'} = f|_{U'} + \theta_{V',v'} \circ s + O(s^2)$ by Definition B.36(vii). But comparing the definitions of W, w in (i)–(iii) above and the $C^\infty(X)$-module structure on $\Gamma(\mathcal{T}_f Y)$ in §B.4.2 we see that

$$\theta_{V',v'} = \textstyle\sum_{i=1}^k \alpha_i \cdot \theta_{V^i,v^i}|_{U'} = \sum_{i=1}^k \alpha_i \cdot \Lambda_i|_{U'}.$$

Hence $g|_{U'} = f|_{U'} + (\alpha_1 \cdot \Lambda_1 + \cdots + \alpha_k \cdot \Lambda_k)|_{U'} \circ s + O(s^2)$, so that $g = f + (\alpha_1 \cdot \Lambda_1 + \cdots + \alpha_k \cdot \Lambda_k) \circ s + O(s^2)$ by Theorem 3.17(a), as $s_{\rm top}^{-1}(0) \subseteq U'_{\rm top}$.

This proves Theorem 3.17(m) when $E \to X$ is trivial. But $g = f + (\alpha_1 \cdot \Lambda_1 + \cdots + \alpha_k \cdot \Lambda_k) \circ s + O(s^2)$ is a local condition by Theorem 3.17(a), so by restricting to an open cover of subsets of X on which E is trivial, part (m) follows.

Proofs of Theorem 3.17(n)–(v)

Theorem 3.17(n)–(v) all deal with pullbacks or pushforwards of the $O(s), O(s^2)$ conditions in Definition B.36 along a morphism $f : X \to Y$ or $g : Y \to Z$. Most of the proofs are pretty straightforward: we take a commutative diagram (etc.) that demonstrates the initial $O(s)$ or $O(s^2)$ condition, and pull back by f or compose with g, to get the commutative diagram (etc.) that demonstrates the final $O(s)$ or $O(s^2)$ condition. The most complex proof is for the second part of (p), so we explain this here, and leave the others as an exercise for the reader.

Suppose that $f : X \to Y$ and $g, h : Y \to Z$ are morphisms in $\dot{\mathbf{Man}}$, and $F \to Y$ is a vector bundle, and $t \in \Gamma^\infty(F)$, and $\theta : E \to f^*(F)$ is a morphism with $\theta \circ s = f^*(t) + O(s^2)$, and $\Lambda : F \to \mathcal{T}_g Z$ is a morphism with $h = g + \Lambda \circ t + O(t^2)$. We must show that $h \circ f = g \circ f + [f^*(\Lambda) \circ \theta] \circ s + O(s^2)$.

As $\theta \circ s = f^*(t) + O(s^2)$, by Definition B.36(i) there exists $\beta : E \otimes E \to f^*(F)$ such that $\theta \circ s = f^*(t) + \beta \circ (s \otimes s)$ in $\Gamma^\infty(f^*(F))$. Since $h = g + \Lambda \circ t + O(t^2)$, by Definition B.36(vii) there exists a commutative diagram in $\dot{\mathbf{Man}}$

$$\begin{tikzcd}
 & Z & \\
U \arrow[r, "k_1"'] \arrow[d, hook] & V \arrow[u, "v"] \arrow[d, hook] & U \arrow[l, "k_2"] \arrow[d, hook] \\
Y \arrow[r, "0_F"] \arrow[uur, bend left, "g"] & F & Y, \arrow[l, "t"'] \arrow[uul, bend right, "h"']
\end{tikzcd} \tag{B.67}$$

with $t_{\rm top}^{-1}(0) \subseteq U_{\rm top}$, and $\Lambda|_U = \theta_{V,v}$.

Define open submanifolds $U' \hookrightarrow X$, $V' \hookrightarrow E$ and morphisms $v' : V' \to Z$, $k'_1, k'_2 : U' \to V'$ with

$$\begin{aligned}
&V'_{\rm top} = \bigl\{(x,e) \in E_{\rm top} : (f_{\rm top}(x), \theta_{\rm top}|_x(e) - \beta_{\rm top}|_x(s_{\rm top}(x) \otimes e)) \in V_{\rm top}\bigr\}, \\
&U'_{\rm top} = \bigl\{x \in X_{\rm top} : (x,0) \in V'_{\rm top},\ \ (x, s_{\rm top}(x)) \in V'_{\rm top}\bigr\}, \\
&v'_{\rm top}(x,e) = v_{\rm top}(f_{\rm top}(x), \theta_{\rm top}|_x(e) - \beta_{\rm top}|_x(s_{\rm top}(x) \otimes e)), \\
&k'_{1,{\rm top}}(x) = (x,0) \quad \text{and} \quad k'_{2,{\rm top}}(x) = (x, s_{\rm top}(x)).
\end{aligned} \tag{B.68}$$

Then $s_{\rm top}^{-1}(0)\subseteq U'_{\rm top}$, as $f_{\rm top}(s_{\rm top}^{-1}(0))\subseteq t_{\rm top}^{-1}(0)$, and for $x\in U'_{\rm top}$ we have

$$v'_{\rm top}(x,0)=v_{\rm top}(f_{\rm top}(x),0)=g_{\rm top}\circ f_{\rm top}(x)=(g\circ f)_{\rm top}(x)$$

by (B.67)–(B.68), so Definition B.32 gives $\theta_{V',v'}:E|_{U'}\to\mathcal{T}_{g\circ f}Y|_{U'}$. Also

$$\begin{aligned}
v'_{\rm top}(x,s_{\rm top}(x))&=v_{\rm top}(f_{\rm top}(x),\theta_{\rm top}|_x(s_{\rm top}(x))-\beta_{\rm top}|_x(s_{\rm top}(x)\otimes s_{\rm top}(x)))\\
&=v_{\rm top}(f_{\rm top}(x),(\theta\circ s-\beta\cdot(s\otimes s))_{\rm top}|_x)=v_{\rm top}(f_{\rm top}(x),(f^*(t))_{\rm top}|_x)\\
&=v_{\rm top}(f_{\rm top}(x),t_{\rm top}(f_{\rm top}(x)))=h_{\rm top}\circ f_{\rm top}(x)=(h\circ f)_{\rm top}(x),
\end{aligned}$$

for $x\in U'_{\rm top}$ by (B.67)–(B.68) and $\theta\circ s=f^*(t)+\beta\circ(s\otimes s)$, so $h\circ f|_{U'}=g\circ f|_{U'}+\theta_{V',v'}\circ s+O(s^2)$ by Definition B.36(vii).

Now from the definition of pullbacks $f^*(\theta)$ in §B.4.9 we deduce that

$$\theta_{V',v'}=f^*(\theta_{V,v})\circ(\theta-\beta\cdot(s\otimes -))|^*_{U'}(\theta_{V,v})=f^*(\Lambda)\circ\theta|_{U'}-f^*(\Lambda)\circ[\beta\cdot(s\otimes -)]|_{U'},$$

as $\Lambda|_U=\theta_{V,v}$. Since the final term is linear in s we have $f^*(\Lambda)\circ\theta|_{U'}=\theta_{V',v'}+O(s)$. So $h\circ f|_{U'}=g\circ f|_{U'}+\theta_{V',v'}\circ s+O(s^2)$ and Theorem 3.17(j) imply that $h\circ f|_{U'}=g\circ f|_{U'}+f^*(\Lambda)\circ\theta|_{U'}\circ s+O(s^2)$, and then Theorem 3.17(a) and $s_{\rm top}^{-1}(0)\subseteq U'_{\rm top}$ give $h\circ f=g\circ f+f^*(\Lambda)\circ\theta\circ s+O(s^2)$. This proves the second part of Theorem 3.17(p).

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Glossary of notation, all volumes

Page references are in the form volume-page number. So, for example, II-57 means page 57 of volume II.

$\Gamma(\mathcal{E})$ global sections of a sheaf $\mathcal{E}$, I-230

$\Gamma^\infty(E)$ vector space of smooth sections of a vector bundle E, I-10, I-238

$\Omega_{\boldsymbol{X}} : K_{\partial \boldsymbol{X}} \to N_{\partial \boldsymbol{X}} \otimes i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}})$ isomorphism of canonical line bundles on boundary of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-67, II-76

$\Theta_{V,E,\Gamma,s,\psi} : (\det T^*V \otimes \det E)|_{s^{-1}(0)} \to \bar\psi^{-1}(K_{\boldsymbol{X}})$ isomorphism of line bundles from a Kuranishi neighbourhood (V, E, Γ, s, ψ) on a Kuranishi space $\boldsymbol{X}$, II-75

$\Theta_{V,E,s,\psi} : (\det T^*V \otimes \det E)|_{s^{-1}(0)} \to \psi^{-1}(K_{\boldsymbol{X}})$ isomorphism of line bundles from an m-Kuranishi neighbourhood (V, E, s, ψ) on an m-Kuranishi space $\boldsymbol{X}$, II-62

$\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}} : K_{\boldsymbol{W}} \to e^*(K_{\boldsymbol{X}}) \otimes f^*(K_{\boldsymbol{Y}}) \otimes (g \circ e)^*(K_{\boldsymbol{Z}})^*$ isomorphism of canonical bundles on w-transverse fibre product of (m-)Kuranishi spaces, II-96

$\alpha_{g,f,e} : (g \circ f) \circ e \Rightarrow g \circ (f \circ e)$ coherence 2-morphism in weak 2-category, I-224

$\beta_f : f \circ \mathrm{id}_X \Rightarrow f$ coherence 2-morphism in weak 2-category, I-224

$\delta_w^{\boldsymbol{g},\boldsymbol{h}} : T_z\boldsymbol{Z} \to O_w\boldsymbol{W}$ connecting morphism in w-transverse fibre product of (m-) Kuranishi spaces, II-92, II-116

$\gamma_f : \mathrm{id}_Y \circ f \Rightarrow f$ coherence 2-morphism in weak 2-category, I-224

$\gamma_f : N_{\partial X} \to (\partial f)^*(N_{\partial Y})$ isomorphism of normal line bundles of manifolds with corners, II-11

∇ connection on vector bundle $E \to X$ in $\dot{\mathbf{Man}}$, I-38, I-241

$C(X)$ corners $\coprod_{k=0}^{\dim X} C_k(X)$ of a manifold with corners X, I-8

$C(\boldsymbol{X})$ corners $\coprod_{k=0}^{\infty} C_k(\boldsymbol{X})$ of an (m or μ-)Kuranishi space $\boldsymbol{X}$, I-91, I-124, I-161

$C : \dot{\mathbf{K}}\mathbf{ur^c} \rightarrow \check{\dot{\mathbf{K}}}\mathbf{ur^c}$ corner 2-functor on Kuranishi spaces, I-161

$C : \mathbf{Man^c} \rightarrow \check{\mathbf{M}}\mathbf{an^c}$ corner functor on manifolds with corners, I-9

$C' : \mathbf{Man^c} \rightarrow \check{\mathbf{M}}\mathbf{an^c}$ second corner functor on manifolds with corners, I-9

$C : \mathbf{m}\dot{\mathbf{K}}\mathbf{ur^c} \rightarrow \mathbf{m}\check{\dot{\mathbf{K}}}\mathbf{ur^c}$ corner 2-functor on m-Kuranishi spaces, I-91

$C : \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c} \rightarrow \boldsymbol{\mu}\check{\dot{\mathbf{K}}}\mathbf{ur^c}$ corner functor on μ-Kuranishi spaces, I-124

$C : \dot{\mathbf{O}}\mathbf{rb^c} \rightarrow \check{\dot{\mathbf{O}}}\mathbf{rb^c}$ corner 2-functor on orbifolds with corners, I-178

$C^\infty(X)$ $\mathbb{R}$-algebra of smooth functions $X \rightarrow \mathbb{R}$ for a manifold X, I-10, I-233

$C_k(\boldsymbol{X})$ k-corners of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, I-81, I-123, I-157

$C_k(\mathfrak{X})$ k-corners of an orbifold with corners $\mathfrak{X}$, I-178

$C_k : \dot{\mathbf{K}}\mathbf{ur^c_{si}} \rightarrow \dot{\mathbf{K}}\mathbf{ur^c_{si}}$ k-corner 2-functor on Kuranishi spaces, I-161

$C_k : \mathbf{Man^c_{si}} \rightarrow \mathbf{Man^c_{si}}$ k-corner functor on manifolds with corners, I-9

$C_k : \mathbf{m}\dot{\mathbf{K}}\mathbf{ur^c_{si}} \rightarrow \mathbf{m}\dot{\mathbf{K}}\mathbf{ur^c_{si}}$ k-corner 2-functor on m-Kuranishi spaces, I-91

$C_k : \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}} \rightarrow \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}$ k-corner functor on μ-Kuranishi spaces, I-124

$C_k : \dot{\mathbf{O}}\mathbf{rb^c_{si}} \rightarrow \dot{\mathbf{O}}\mathbf{rb^c_{si}}$ k-corner 2-functor on orbifolds with corners, I-178

$\mathcal{C}^{\mathrm{op}}$ opposite category of category $\mathcal{C}$, I-221

$\mathbf{C^\infty Rings}$ category of C^∞-rings, I-234

$\mathbf{C^\infty Sch^{aff}}$ category of affine C^∞-schemes, I-37, I-236

$\partial : \dot{\mathbf{K}}\mathbf{ur^c_{si}} \rightarrow \dot{\mathbf{K}}\mathbf{ur^c_{si}}$ boundary 2-functor on Kuranishi spaces, I-161

$\partial : \mathbf{Man^c_{si}} \rightarrow \mathbf{Man^c_{si}}$ boundary functor on manifolds with corners, I-9

$\partial : \mathbf{m}\dot{\mathbf{K}}\mathbf{ur^c_{si}} \rightarrow \mathbf{m}\dot{\mathbf{K}}\mathbf{ur^c_{si}}$ boundary 2-functor on m-Kuranishi spaces, I-91

$\partial : \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}} \rightarrow \boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}$ boundary functor on μ-Kuranishi spaces, I-124

$\mathrm{depth}_X\, x$ the codimension k of the corner stratum $S^k(X)$ containing a point x in a manifold with corners X, I-6

$\mathbf{DerMan_{BN}}$ Borisov and Noel's ∞-category of derived manifolds, I-103

$\mathbf{DerMan_{Spi}}$ Spivak's ∞-category of derived manifolds, I-103

$\det(E^\bullet)$ determinant of a complex of vector spaces or vector bundles, II-52

$\mathrm{d}f : TX \rightarrow f^*(TY)$ derivative of a smooth map $f : X \rightarrow Y$, I-11

${}^b\mathrm{d}f : {}^bTX \rightarrow f^*({}^bTY)$ b-derivative of a smooth map $f : X \rightarrow Y$ of manifolds with corners, I-12

dMan 2-category of d-manifolds, a kind of derived manifold, I-103

$\partial \boldsymbol{X}$ boundary of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, I-86, I-124, I-160, I-161

$\partial \mathfrak{X}$ boundary of an orbifold with corners $\mathfrak{X}$, I-178

$f_{\rm top} : X_{\rm top} \to Y_{\rm top}$ underlying continuous map of morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$, I-31

GKN 2-category of global Kuranishi neighbourhoods over **Man**, I-142

$\mathbf{G\dot{K}N}$ 2-category of global Kuranishi neighbourhoods over $\dot{\mathbf{Man}}$, I-142

$\mathbf{GKN^c}$ 2-category of global Kuranishi neighbourhoods over manifolds with corners $\mathbf{Man^c}$, I-142

GmKN 2-category of global m-Kuranishi neighbourhoods over **Man**, I-59

$\mathbf{Gm\dot{K}N}$ 2-category of global m-Kuranishi neighbourhoods over $\dot{\mathbf{Man}}$, I-58

$\mathbf{GmKN^c}$ 2-category of global m-Kuranishi neighbourhoods over manifolds with corners $\mathbf{Man^c}$, I-59

$\mathbf{G\boldsymbol{\mu}KN}$ category of global μ-Kuranishi neighbourhoods over **Man**, I-111

$\mathbf{G\boldsymbol{\mu}\dot{K}N}$ category of global μ-Kuranishi neighbourhoods over $\dot{\mathbf{Man}}$, I-110

$\mathbf{G\boldsymbol{\mu}KN^c}$ category of global μ-Kuranishi neighbourhoods over manifolds with corners $\mathbf{Man^c}$, I-111

$G_x\boldsymbol{f} : G_x\boldsymbol{X} \to G_y\boldsymbol{Y}$ morphism of isotropy groups from 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\dot{\mathbf{K}}\mathbf{ur}$, I-168

$G_x\boldsymbol{X}$ isotropy group of a Kuranishi space $\boldsymbol{X}$ at a point $x \in \boldsymbol{X}$, I-166

$G_x\mathfrak{X}$ isotropy group of an orbifold $\mathfrak{X}$ at a point $x \in \mathfrak{X}$, I-176

$\mathrm{Ho}(\boldsymbol{\mathcal{C}})$ homotopy category of 2-category $\boldsymbol{\mathcal{C}}$, I-226

$I_f^\diamond : \Pi_{\rm top}^{-1}(\mathcal{T}_f Y) \to \mathcal{T}_{C(f)}C(Y)$ morphism of tangent sheaves in $\dot{\mathbf{Man}}^{\mathbf{c}}$, I-269

$I_X^\diamond : \Pi_k^*({}^bTX) \to {}^bT(C_k(X))$ natural morphism of b-tangent bundles over a manifold with corners X, I-12

$\boldsymbol{i_X} : \partial \boldsymbol{X} \to \boldsymbol{X}$ natural (1-)morphism of boundary of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, I-86, I-124, I-160

$I_X : {}^bTX \to TX$ natural morphism of (b-)tangent bundles of a manifold with corners X, I-11

$K_{\boldsymbol{f}} : f^*(K_{\boldsymbol{Y}}) \to K_{\boldsymbol{X}}$ isomorphism of canonical bundles from étale (1-)morphism of (m- or μ-)Kuranishi spaces $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, II-65

$\mathbf{KN}$ 2-category of Kuranishi neighbourhoods over manifolds $\mathbf{Man}$, I-142

$\dot{\mathbf{K}}\mathbf{N}$ 2-category of Kuranishi neighbourhoods over $\dot{\mathbf{M}}\mathbf{an}$, I-141

$\mathbf{KN^c}$ 2-category of Kuranishi neighbourhoods over manifolds with corners $\mathbf{Man^c}$, I-142

$\mathbf{KN}_S(X)$ 2-category of Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man}$, I-142

$\dot{\mathbf{K}}\mathbf{N}_S(X)$ 2-category of Kuranishi neighbourhoods over $S \subseteq X$ in $\dot{\mathbf{M}}\mathbf{an}$, I-142

$\mathbf{KN}^{\mathbf{c}}_S(X)$ 2-category of Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man^c}$, I-142

$\mathbf{Kur}$ 2-category of Kuranishi spaces over classical manifolds $\mathbf{Man}$, I-153

$\dot{\mathbf{K}}\mathbf{ur}$ 2-category of Kuranishi spaces over $\dot{\mathbf{M}}\mathbf{an}$, I-151

$\dot{\mathbf{K}}\mathbf{ur}_{\boldsymbol{P}}$ 2-category of Kuranishi spaces over $\dot{\mathbf{M}}\mathbf{an}$, and 1-morphisms with discrete property $\boldsymbol{P}$, I-154

$\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}$ 2-subcategory of Kuranishi spaces in $\dot{\mathbf{K}}\mathbf{ur}$ with all $G_x\boldsymbol{X} = \{1\}$, I-169

$\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{tr\Gamma}}$ 2-subcategory of Kuranishi spaces in $\dot{\mathbf{K}}\mathbf{ur}$ with all $\Gamma_i = \{1\}$, I-169

$\mathbf{Kur^{ac}}$ 2-category of Kuranishi spaces with a-corners, I-153

$\mathbf{Kur^c}$ 2-category of Kuranishi spaces with corners, I-153

$\check{\dot{\mathbf{K}}}\mathbf{ur^c}$ 2-category of Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, I-161

$\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}_{\boldsymbol{P}}$ 2-category of Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, and 1-morphisms which are $\boldsymbol{P}$, I-161

$\mathbf{Kur}^{\mathbf{c}}_{\mathbf{bn}}$ 2-category of Kuranishi spaces with corners, and b-normal 1-morphisms, I-154

$\mathbf{Kur}^{\mathbf{c}}_{\mathbf{in}}$ 2-category of Kuranishi spaces with corners, and interior 1-morphisms, I-154

$\mathbf{Kur}^{\mathbf{c}}_{\mathbf{si}}$ 2-category of Kuranishi spaces with corners, and simple 1-morphisms, I-154

$\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}_{\mathbf{si}}$ 2-category of Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, and simple 1-morphisms, I-161

$\mathbf{Kur}^{\mathbf{c}}_{\mathbf{st}}$ 2-category of Kuranishi spaces with corners, and strongly smooth 1-morphisms, I-154

$\mathbf{Kur}^{\mathbf{c}}_{\mathbf{st,bn}}$ 2-category of Kuranishi spaces with corners, and strongly smooth b-normal 1-morphisms, I-154

$\mathbf{Kur}^{\mathbf{c}}_{\mathbf{st,in}}$ 2-category of Kuranishi spaces with corners, and strongly smooth interior 1-morphisms, I-154

$\mathbf{Kur^c_{we}}$ 2-category of Kuranishi spaces with corners and weakly smooth 1-morphisms, I-153

$\dot{\mathbf{K}}\mathbf{ur^c}$ 2-category of Kuranishi spaces with corners associated to $\dot{\mathbf{M}}\mathbf{an^c}$, I-157

$\dot{\mathbf{K}}\mathbf{ur^c_{si}}$ 2-category of Kuranishi spaces with corners associated to $\dot{\mathbf{M}}\mathbf{an^c}$, and simple 1-morphisms, I-157

$\mathbf{Kur^{c,ac}}$ 2-category of Kuranishi spaces with corners and a-corners, I-153

$\mathbf{Kur^{c,ac}_{bn}}$ 2-category of Kuranishi spaces with corners and a-corners, and b-normal 1-morphisms, I-155

$\mathbf{Kur^{c,ac}_{in}}$ 2-category of Kuranishi spaces with corners and a-corners, and interior 1-morphisms, I-155

$\mathbf{Kur^{c,ac}_{si}}$ 2-category of Kuranishi spaces with corners and a-corners, and simple 1-morphisms, I-155

$\mathbf{Kur^{c,ac}_{st}}$ 2-category of Kuranishi spaces with corners and a-corners, and strongly a-smooth 1-morphisms, I-155

$\mathbf{Kur^{c,ac}_{st,bn}}$ 2-category of Kuranishi spaces with corners and a-corners, and strongly a-smooth b-normal 1-morphisms, I-155

$\mathbf{Kur^{c,ac}_{st,in}}$ 2-category of Kuranishi spaces with corners and a-corners, and strongly a-smooth interior 1-morphisms, I-155

$\mathbf{Kur^{gc}}$ 2-category of Kuranishi spaces with g-corners, I-153

$\mathbf{Kur^{gc}_{bn}}$ 2-category of Kuranishi spaces with g-corners, and b-normal 1-morphisms, I-155

$\mathbf{Kur^{gc}_{in}}$ 2-category of Kuranishi spaces with g-corners, and interior 1-morphisms, I-155

$\mathbf{Kur^{gc}_{si}}$ 2-category of Kuranishi spaces with g-corners, and simple 1-morphisms, I-155

K_X canonical bundle of a 'manifold' X in $\dot{\mathbf{M}}\mathbf{an}$, II-10

$K_{\boldsymbol{X}}$ canonical bundle of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-62, II-74

${}^bK_{\boldsymbol{X}}$ b-canonical bundle of an (m- or μ-)Kuranishi space with corners $\boldsymbol{X}$, II-66

$\mathbf{Man}$ category of classical manifolds, I-7

$\dot{\mathbf{M}}\mathbf{an}$ category of 'manifolds' satisfying Assumptions 3.1–3.7, I-31

$\ddot{\mathbf{M}}\mathbf{an}$ another category of 'manifolds' satisfying Assumptions 3.1–3.7, I-46

$\mathbf{Man^{ac}}$ category of manifolds with a-corners, I-18

$\mathbf{Man}^{\mathbf{ac}}_{\mathbf{bn}}$ category of manifolds with a-corners and b-normal maps, I-18

$\mathbf{Man}^{\mathbf{ac}}_{\mathbf{in}}$ category of manifolds with a-corners and interior maps, I-18

$\mathbf{Man}^{\mathbf{ac}}_{\mathbf{st}}$ category of manifolds with a-corners and strongly a-smooth maps, I-18

$\mathbf{Man}^{\mathbf{ac}}_{\mathbf{st,bn}}$ category of manifolds with a-corners and strongly a-smooth b-normal maps, I-18

$\mathbf{Man}^{\mathbf{ac}}_{\mathbf{st,in}}$ category of manifolds with a-corners and strongly a-smooth interior maps, I-18

$\mathbf{Man}^{\mathbf{b}}$ category of manifolds with boundary, I-7

$\mathbf{Man}^{\mathbf{b}}_{\mathbf{in}}$ category of manifolds with boundary and interior maps, I-7

$\mathbf{Man}^{\mathbf{b}}_{\mathbf{si}}$ category of manifolds with boundary and simple maps, I-7

$\mathbf{Man}^{\mathbf{c}}$ category of manifolds with corners, I-5

$\dot{\mathbf{M}}\mathbf{an}^{\mathbf{c}}$ category of ‘manifolds with corners’ satisfying Assumption 3.22, I-47

$\check{\dot{\mathbf{M}}}\mathbf{an}^{\mathbf{c}}$ category of ‘manifolds with corners’ of mixed dimension, I-48

$\check{\mathbf{M}}\mathbf{an}^{\mathbf{c}}$ category of manifolds with corners of mixed dimension, I-8

$\mathbf{Man}^{\mathbf{c}}_{\mathbf{bn}}$ category of manifolds with corners and b-normal maps, I-5

$\mathbf{Man}^{\mathbf{c}}_{\mathbf{in}}$ category of manifolds with corners and interior maps, I-5

$\check{\mathbf{M}}\mathbf{an}^{\mathbf{c}}_{\mathbf{in}}$ category of manifolds with corners of mixed dimension and interior maps, I-8

$\mathbf{Man}^{\mathbf{c}}_{\mathbf{si}}$ category of manifolds with corners and simple maps, I-5

$\check{\dot{\mathbf{M}}}\mathbf{an}^{\mathbf{c}}_{\mathbf{si}}$ category of ‘manifolds with corners’ of mixed dimension, and simple morphisms, I-48

$\mathbf{Man}^{\mathbf{c}}_{\mathbf{st}}$ category of manifolds with corners and strongly smooth maps, I-5

$\check{\mathbf{M}}\mathbf{an}^{\mathbf{c}}_{\mathbf{st}}$ category of manifolds with corners of mixed dimension and strongly smooth maps, I-8

$\mathbf{Man}^{\mathbf{c}}_{\mathbf{st,bn}}$ category of manifolds with corners and strongly smooth b-normal maps, I-5

$\mathbf{Man}^{\mathbf{c}}_{\mathbf{st,in}}$ category of manifolds with corners and strongly smooth interior maps, I-5

$\mathbf{Man}^{\mathbf{c}}_{\mathbf{we}}$ category of manifolds with corners and weakly smooth maps, I-5

$\mathbf{Man}^{\mathbf{c,ac}}$ category of manifolds with corners and a-corners, I-18

$\mathbf{Man^{c,ac}_{bn}}$ category of manifolds with corners and a-corners, and b-normal maps, I-19

$\mathbf{Man^{c,ac}_{in}}$ category of manifolds with corners and a-corners, and interior maps, I-18

$\mathbf{Man^{c,ac}_{si}}$ category of manifolds with corners and a-corners, and simple maps, I-19

$\mathbf{Man^{c,ac}_{in}}$ category of manifolds with corners and a-corners, and strongly a-smooth maps, I-19

$\mathbf{Man^{c,ac}_{st,bn}}$ category of manifolds with corners and a-corners, and strongly a-smooth b-normal maps, I-19

$\mathbf{Man^{c,ac}_{st,in}}$ category of manifolds with corners and a-corners, and strongly a-smooth interior maps, I-19

$\mathbf{Man^{gc}}$ category of manifolds with g-corners, I-16

$\mathbf{Man^{gc}_{in}}$ category of manifolds with g-corners and interior maps, I-16

$\mathbf{mKN}$ 2-category of m-Kuranishi neighbourhoods over manifolds $\mathbf{Man}$, I-59

$\mathbf{m\dot{K}N}$ 2-category of m-Kuranishi neighbourhoods over $\dot{\mathbf{M}}\mathbf{an}$, I-58

$\mathbf{mKN^c}$ 2-category of m-Kuranishi neighbourhoods over manifolds with corners $\mathbf{Man^c}$, I-59

$\mathbf{mKN}_S(X)$ 2-category of m-Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man}$, I-59

$\mathbf{m\dot{K}N}_S(X)$ 2-category of m-Kuranishi neighbourhoods over $S \subseteq X$ in $\dot{\mathbf{M}}\mathbf{an}$, I-58

$\mathbf{mKN}^{\mathbf{c}}_S(X)$ 2-category of m-Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man^c}$, I-59

$\mathbf{mKur}$ 2-category of m-Kuranishi spaces over classical manifolds $\mathbf{Man}$, I-72

$\mathbf{m\dot{K}ur}$ 2-category of m-Kuranishi spaces over $\dot{\mathbf{M}}\mathbf{an}$, I-72

$\mathbf{m\dot{K}ur}_{\boldsymbol{P}}$ 2-category of m-Kuranishi spaces over $\dot{\mathbf{M}}\mathbf{an}$, and 1-morphisms with discrete property $\boldsymbol{P}$, I-78

$\mathbf{mKur^{ac}}$ 2-category of m-Kuranishi spaces with a-corners, I-72

$\mathbf{mKur^{ac}_{bn}}$ 2-category of m-Kuranishi spaces with a-corners, and b-normal 1-morphisms, I-79

$\mathbf{mKur^{ac}_{in}}$ 2-category of m-Kuranishi spaces with a-corners, and interior 1-morphisms, I-79

$\mathbf{mKur^{ac}_{si}}$ 2-category of m-Kuranishi spaces with a-corners, and simple 1-morphisms, I-79

$\mathbf{mKur^{ac}_{st}}$ 2-category of m-Kuranishi spaces with a-corners, and strongly a-smooth 1-morphisms, I-79

$\mathbf{mKur^{ac}_{st,bn}}$ 2-category of m-Kuranishi spaces with a-corners, and strongly a-smooth b-normal 1-morphisms, I-79

$\mathbf{mKur^{ac}_{st,in}}$ 2-category of m-Kuranishi spaces with a-corners, and strongly a-smooth interior 1-morphisms, I-79

$\mathbf{mKur^{b}}$ 2-category of m-Kuranishi spaces with boundary, I-93

$\mathbf{mKur^{b}_{in}}$ 2-category of m-Kuranishi spaces with boundary, and interior 1-morphisms, I-93

$\mathbf{mKur^{b}_{si}}$ 2-category of m-Kuranishi spaces with boundary, and simple 1-morphisms, I-93

$\mathbf{mKur^{c}}$ 2-category of m-Kuranishi spaces with corners, I-72

$\mathbf{m\dot{\check{K}}ur^{c}}$ 2-category of m-Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, I-87

$\mathbf{m\dot{\check{K}}ur^{c}_{\boldsymbol{P}}}$ 2-category of m-Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, and 1-morphisms which are $\boldsymbol{P}$, I-91

$\mathbf{mKur^{c}_{bn}}$ 2-category of m-Kuranishi spaces with corners, and b-normal 1-morphisms, I-78

$\mathbf{mKur^{c}_{in}}$ 2-category of m-Kuranishi spaces with corners, and interior 1-morphisms, I-78

$\mathbf{mKur^{c}_{si}}$ 2-category of m-Kuranishi spaces with corners, and simple 1-morphisms, I-78

$\mathbf{m\dot{\check{K}}ur^{c}_{si}}$ 2-category of m-Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, and simple 1-morphisms, I-87

$\mathbf{mKur^{c}_{st}}$ 2-category of m-Kuranishi spaces with corners, and strongly smooth 1-morphisms, I-78

$\mathbf{mKur^{c}_{st,bn}}$ 2-category of m-Kuranishi spaces with corners, and strongly smooth b-normal 1-morphisms, I-78

$\mathbf{mKur^{c}_{st,in}}$ 2-category of m-Kuranishi spaces with corners, and strongly smooth interior 1-morphisms, I-78

$\mathbf{mKur^{c}_{we}}$ 2-category of m-Kuranishi spaces with corners and weakly smooth 1-morphisms, I-72

$\dot{\mathbf{mKur}}^{\mathbf{c}}$ 2-category of m-Kuranishi spaces with corners associated to $\dot{\mathbf{Man}}^{\mathbf{c}}$, I-81

$\mathbf{mKur}^{\mathbf{c,ac}}$ 2-category of m-Kuranishi spaces with corners and a-corners, I-72

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{bn}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and b-normal 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{in}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and interior 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{si}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and simple 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{st}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and strongly a-smooth 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{st,bn}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and strongly a-smooth b-normal 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{st,in}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and strongly a-smooth interior 1-morphisms, I-79

$\dot{\mathbf{mKur}}^{\mathbf{c}}_{\mathbf{si}}$ 2-category of m-Kuranishi spaces with corners associated to $\dot{\mathbf{Man}}^{\mathbf{c}}$, and simple 1-morphisms, I-81

$\mathbf{mKur}^{\mathbf{gc}}$ 2-category of m-Kuranishi spaces with g-corners, I-72

$\mathbf{mKur}^{\mathbf{gc}}_{\mathbf{bn}}$ 2-category of m-Kuranishi spaces with g-corners, and b-normal 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{gc}}_{\mathbf{in}}$ 2-category of m-Kuranishi spaces with g-corners, and interior 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{gc}}_{\mathbf{si}}$ 2-category of m-Kuranishi spaces with g-corners, and simple 1-morphisms, I-79

$\boldsymbol{\mu}\mathbf{KN}$ category of μ-Kuranishi neighbourhoods over manifolds $\mathbf{Man}$, I-111

$\boldsymbol{\mu}\dot{\mathbf{KN}}$ category of μ-Kuranishi neighbourhoods over $\dot{\mathbf{Man}}$, I-110

$\boldsymbol{\mu}\mathbf{KN}^{\mathbf{c}}$ category of μ-Kuranishi neighbourhoods over manifolds with corners $\mathbf{Man}^{\mathbf{c}}$, I-111

$\boldsymbol{\mu}\mathbf{KN}_S(X)$ category of μ-Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man}$, I-111

$\boldsymbol{\mu}\dot{\mathbf{KN}}_S(X)$ category of μ-Kuranishi neighbourhoods over $S \subseteq X$ in $\dot{\mathbf{Man}}$, I-110

$\boldsymbol{\mu}\mathbf{KN}^{\mathbf{c}}_S(X)$ category of μ-Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man}^{\mathbf{c}}$, I-111

$\boldsymbol{\mu}\mathbf{Kur}$ category of μ-Kuranishi spaces over classical manifolds $\mathbf{Man}$, I-117

$\boldsymbol{\mu}\dot{\mathbf{Kur}}$ category of μ-Kuranishi spaces over $\dot{\mathbf{Man}}$, I-116

$N_{C_k(X)}$ normal bundle of k-corners $C_k(X)$ in a manifold with corners X, I-12

${}^bN_{C_k(X)}$ b-normal bundle of k-corners $C_k(X)$ in a manifold with corners X, I-12

$N_{\partial X}$ normal line bundle of boundary ∂X in a manifold with corners X, I-12

$\tilde N_x f : \tilde N_x X \to \tilde N_y Y$ stratum normal map for manifolds with corners X, I-13

${}^b\tilde N_x f : {}^b\tilde N_x X \to {}^b\tilde N_y Y$ stratum b-normal map for morphism $f : X \to Y$ in $\mathbf{Man^c_{in}}$, I-14

$\tilde N_x X$ stratum normal space at x in a manifold with corners X, I-13

${}^b\tilde N_x X$ stratum b-normal space at x in a manifold with corners X, I-13

$\mathbf{Orb}_{\mathrm{CR}}$ Chen–Ruan's category of orbifolds, I-171

$\mathbf{Orb}_{C^\infty\mathrm{Sta}}$ 2-category of orbifolds as stacks on site $\mathbf{C^\infty Sch}$, I-172

$\mathbf{Orb}_{\mathrm{Kur}}$ 2-category of orbifolds as examples of Kuranishi spaces, I-175

$\mathbf{Orb}_{\mathrm{Le}}$ Lerman's 2-category of orbifolds, I-171

$\mathbf{Orb}_{\mathrm{ManSta}}$ 2-category of orbifolds as stacks on site $\mathbf{Man}$, I-171

$\mathbf{Orb}_{\mathrm{MP}}$ Moerdijk–Pronk's category of orbifolds, I-171

$\mathbf{Orb}_{\mathrm{Pr}}$ Pronk's 2-category of orbifolds, I-171

$\mathbf{Orb}_{\mathrm{ST}}$ Satake–Thurston's category of orbifolds, I-171

$\dot{\mathbf{O}}\mathbf{rb}$ 2-category of Kuranishi orbifolds associated to $\dot{\mathbf{M}}\mathbf{an}$, I-175

$\mathbf{Orb^{ac}}$ 2-category of orbifolds with a-corners, I-175

$\dot{\mathbf{O}}\mathbf{rb^c}$ 2-category of orbifolds with corners associated to $\dot{\mathbf{M}}\mathbf{an^c}$, I-178

$\mathbf{Orb^{c,ac}}$ 2-category of orbifolds with corners and a-corners, I-175

$\dot{\mathbf{O}}\mathbf{rb^c_{si}}$ 2-category of orbifolds with corners associated to $\dot{\mathbf{M}}\mathbf{an^c}$, and simple 1-morphisms, I-178

$\mathbf{Orb^c_{we}}$ 2-category of orbifolds with corners, and weakly smooth 1-morphisms, I-175

$\mathbf{Orb^c_{we}}$ 2-category of orbifolds with corners, I-175

$\mathbf{Orb^{eff}_{sur}}$ 2-category of effective orbifolds with 1-morphisms surjective on isotropy groups, I-35

$\mathbf{Orb^{gc}}$ 2-category of orbifolds with g-corners, I-175

$\mathcal{O}_X$ structure sheaf of object X in $\dot{\mathbf{M}}\mathbf{an}$, I-37, I-235

$O_x\boldsymbol{f}: O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$ obstruction map of (m- or μ-)Kuranishi spaces, II-17, II-21, II-22

${}^bO_x\boldsymbol{f}: {}^bO_x\boldsymbol{X} \to {}^bO_y\boldsymbol{Y}$ b-obstruction map of (m- or μ-)Kuranishi spaces with corners, II-19

$\tilde O_x\boldsymbol{f}: \tilde O_x\boldsymbol{X} \to \tilde O_y\boldsymbol{Y}$ stratum obstruction map of (m- or μ-)Kuranishi spaces with corners, II-19

$O_x\boldsymbol{X}$ obstruction space at x of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-16, II-21

$O_x^*\boldsymbol{X}$ coobstruction space at x of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-16, II-21

${}^bO_x\boldsymbol{X}$ b-obstruction space at x of an (m- or μ-)Kuranishi space with corners $\boldsymbol{X}$, II-19

$\tilde O_x\boldsymbol{X}$ stratum obstruction space at x of an (m- or μ-)Kuranishi space with corners $\boldsymbol{X}$, II-19

$\Phi_{ij}: (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ 1-morphism or coordinate change of Kuranishi neighbourhoods, I-136

$\Phi_{ij}: (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ 1-morphism or coordinate change of m-Kuranishi neighbourhoods, I-55

$[\Phi_{ij}]: (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ morphism or coordinate change of μ-Kuranishi neighbourhoods, I-109

$Q_xf: Q_xX \to Q_yY$ quasi-tangent map of morphism $f: X \to Y$ in $\dot{\mathbf{Man}}$, II-13

$Q_x\boldsymbol{f}: Q_x\boldsymbol{X} \to Q_y\boldsymbol{Y}$ quasi-tangent map of (m- or μ-)Kuranishi spaces, II-24, II-28

Q_xX quasi-tangent space at x of 'manifold' X in $\dot{\mathbf{Man}}$, II-13

$Q_x\boldsymbol{X}$ quasi-tangent space at x of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-24, II-28

$S^l(X)$ depth l stratum of a manifold with corners X, I-6

$Tf: TX \to TY$ derivative of a smooth map $f: X \to Y$, I-11

${}^bTf: {}^bTX \to {}^bTY$ b-derivative of an interior map $f: X \to Y$ of manifolds with corners, I-12

$\mathcal{T}_fY$ tangent sheaf of morphism $f: X \to Y$ in $\dot{\mathbf{Man}}$, I-38, I-251

$\mathcal{T}g: \mathcal{T}_fY \to \mathcal{T}_{g\circ f}Z$ morphism of tangent sheaves for $f: X \to Y$, $g: Y \to Z$ in $\dot{\mathbf{Man}}$, I-38, I-254

Top category of topological spaces, I-31

TX tangent bundle of a manifold X, I-11

T^*X cotangent bundle of a manifold X, I-11

$\mathcal{T}X$ tangent sheaf of 'manifold' X in $\dot{\mathbf{M}}\mathbf{an}$, I-38, I-251

$\mathcal{T}^*X$ cotangent sheaf of 'manifold' X in $\dot{\mathbf{M}}\mathbf{an}$, I-37, I-240

${}^bTX$ b-tangent bundle of a manifold with corners X, I-11

${}^bT^*X$ b-cotangent bundle of a manifold X, I-11

$T_xf : T_xX \to T_yY$ tangent map of morphism $f : X \to Y$ in $\dot{\mathbf{M}}\mathbf{an}$, II-4

${}^bT_xf : {}^bT_xX \to {}^bT_yY$ b-tangent map of interior map $f : X \to Y$ in $\mathbf{Man^c}$, I-12

$\tilde{T}_xf : \tilde{T}_xX \to \tilde{T}_yY$ stratum tangent map of morphism $f : X \to Y$ of manifolds with corners, II-4

$T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$ tangent map of (m- or μ-)Kuranishi spaces, II-17, II-21, II-22

${}^bT_x\boldsymbol{f} : {}^bT_x\boldsymbol{X} \to {}^bT_y\boldsymbol{Y}$ b-tangent map of (m- or μ-)Kuranishi spaces with corners, II-19

$\tilde{T}_x\boldsymbol{f} : \tilde{T}_x\boldsymbol{X} \to \tilde{T}_y\boldsymbol{Y}$ stratum tangent map of (m- or μ-)Kuranishi spaces with corners, II-19

T_xX tangent space at x of 'manifold' X in $\dot{\mathbf{M}}\mathbf{an}$, II-4

T_x^*X cotangent space at x of 'manifold' X in $\dot{\mathbf{M}}\mathbf{an}$, II-4

${}^bT_xX$ b-tangent space at x of a manifold with corners X, I-11

$\tilde{T}_xX$ stratum tangent space at x of a manifold with corners X, II-4

$T_x\boldsymbol{X}$ tangent space at x of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-16, II-21

$T_x^*\boldsymbol{X}$ cotangent space at x of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-16, II-21

${}^bT_x\boldsymbol{X}$ b-tangent space at x of an (m- or μ-)Kuranishi space with corners $\boldsymbol{X}$, II-19

$\tilde{T}_x\boldsymbol{X}$ stratum tangent space at x of an (m- or μ-)Kuranishi space with corners $\boldsymbol{X}$, II-19

(V, E, Γ, s) object in 2-category of global Kuranishi neighbourhoods $\mathbf{G\dot{K}N}$, I-142

(V, E, Γ, s, ψ) Kuranishi neighbourhood on topological space, I-135

(V, E, s) object in (2-)category of global m- or μ-Kuranishi neighbourhoods $\mathbf{Gm\dot{K}N}$ or $\mathbf{G\boldsymbol{\mu}\dot{K}N}$, I-58, I-110

(V, E, s, ψ) m- or μ-Kuranishi neighbourhood on topological space, I-55, I-109

X° interior of a manifold with corners X, I-6

$X_{\rm top}$ underlying topological space of object X in $\dot{\mathbf{M}}\mathbf{an}$, I-31

Index to all volumes

Page references are in the form volume-page number.