Kuranishi spaces and Symplectic Geometry

Volume II. Differential Geometry of (m-)Kuranishi spaces

Dominic Joyce

The Mathematical Institute,
Radcliffe Observatory Quarter,
Woodstock Road,
Oxford,
OX2 6GG,
U.K.
joyce@maths.ox.ac.uk

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Kuranishi spaces and Symplectic Geometry. Volume II.
Differential Geometry of (m-)Kuranishi spaces

Kuranishi spaces and Symplectic Geometry. Volume I.
Basic theory of (m-)Kuranishi spaces

Introduction to the series

On the foundations of Symplectic Geometry

Several important areas of Symplectic Geometry involve 'counting' moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves in a symplectic manifold (S, ω) satisfying some conditions, where J is an almost complex structure on S compatible with ω, and using the 'numbers of curves' to build some interesting theory, which is then shown to be independent of the choice of J. Areas of this type include Gromov–Witten theory [5, 30, 40, 46, 47, 51, 65, 67], Quantum Cohomology [46, 51], Lagrangian Floer cohomology [2, 12, 15, 20, 59, 70], Fukaya categories [9, 62, 64], Symplectic Field Theory [3, 7, 8], Contact Homology [6, 60], and Symplectic Cohomology [63].

Setting up the foundations of these areas, rigorously and in full generality, is a very long and difficult task, comparable to the work of Grothendieck and his school on the foundations of Algebraic Geometry, or the work of Lurie and Toën–Vezzosi on the foundations of Derived Algebraic Geometry. Any such foundational programme for Symplectic Geometry can be divided into five steps:

(i) We must define a suitable class of geometric structures $\mathcal{G}$ to put on the moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves we wish to 'count'. This must satisfy both (ii) and (iii) below.

(ii) Given a compact space X with geometric structure $\mathcal{G}$ and an 'orientation', we must define a 'virtual class' $[[X]_{\mathrm{virt}}]$ in some homology group, or a 'virtual chain' $[X]_{\mathrm{virt}}$ in the chains of the homology theory, which 'counts' X.

Actually, usually one studies a compact, oriented $\mathcal{G}$-space X with a 'smooth map' $f : X \to Y$ to a manifold Y, and defines $[[X]_{\mathrm{virt}}]$ or $[X]_{\mathrm{virt}}$ in a suitable (co)homology theory of Y, such as singular homology or de Rham cohomology. These virtual classes/(co)chains must satisfy a package of properties, including a deformation-invariance property.

(iii) We must prove that all the moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves that will be used in our theory have geometric structure $\mathcal{G}$, preferably in a natural way. Note that in order to make the moduli spaces $\bar{\mathcal{M}}$ compact (necessary for existence of virtual classes/chains), we have to include *singular* J-holomorphic curves in $\bar{\mathcal{M}}$. This makes construction of the $\mathcal{G}$-structure on $\bar{\mathcal{M}}$ significantly more difficult.

(iv) We combine (i)–(iii) to study the situation in Symplectic Geometry we are interested in, e.g. to define Lagrangian Floer cohomology $HF^*(L_1, L_2)$ for compact Lagrangians L_1, L_2 in a compact symplectic manifold (S, ω).

To do this we choose an almost complex structure J on (S, ω) and define a collection of moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves relevant to the problem. By (iii) these have structure $\mathcal{G}$, so by (ii) they have virtual classes/(co)chains $[\bar{\mathcal{M}}]_{\rm virt}$ in some (co)homology theory.

There will be geometric relationships between these moduli spaces – for instance, boundaries of moduli spaces may be written as sums of fibre products of other moduli spaces. By the package of properties in (ii), these geometric relationships should translate to algebraic relationships between the virtual classes/(co)chains, e.g. the boundaries of virtual cochains may be written as sums of cup products of other virtual cochains.

We use the virtual classes/(co)chains, and the algebraic identities they satisfy, and homological algebra, to build the theory we want – Quantum Cohomology, Lagrangian Floer Theory, and so on. We show the result is independent of the choice of almost complex structure J using the deformation-invariance properties of virtual classes/(co)chains.

(v) We apply our new machine to do something interesting in Symplectic Geometry, e.g. prove the Arnold Conjecture.

Many authors have worked on programmes of this type, since the introduction of J-holomorphic curve techniques into Symplectic Geometry by Gromov [32] in 1985. Oversimplifying somewhat, we can divide these approaches into three main groups, according to their answer to (i) above:

(A) **(Kuranishi-type spaces.)** In the work of Fukaya, Oh, Ohta and Ono [10–30], moduli spaces are given the structure of *Kuranishi spaces* (we will call their definition *FOOO Kuranishi spaces*).

Several other groups also work with Kuranishi-type spaces, including McDuff and Wehrheim [49, 50, 52–55], Pardon [60, 61], and the author in [42, 43] and this series.

(B) **(Polyfolds.)** In the work of Hofer, Wysocki and Zehnder [34–41], moduli spaces are given the structure of *polyfolds*.

(C) **(The rest of the world.)** One makes restrictive assumptions on the symplectic geometry – for instance, consider only noncompact, exact symplectic manifolds, and exact Lagrangians in them – takes J to be generic, and arranges that all the moduli spaces $\bar{\mathcal{M}}$ we are interested in are smooth manifolds (or possibly 'pseudomanifolds', manifolds with singularities in codimension 2). Then we form virtual classes/chains as for fundamental classes of manifolds. A good example of this approach is Seidel's construction [64] of Fukaya categories of Liouville domains.

We have not given complete references here, much important work is omitted.

Although Kuranishi-type spaces in (A), and polyfolds in (B), do exactly the same job, there is an important philosophical difference between them. Kuranishi spaces basically remember the minimal information needed to form virtual cycles/chains, and no more. Kuranishi spaces contain about the same amount of data as smooth manifolds, and include manifolds as examples.

In contrast, polyfolds remember the entire functional-analytic moduli problem, forgetting nothing. Any polyfold curve moduli space, even a moduli space of constant curves, is a hugely infinite-dimensional object, a vast amount of data.

Approach (C) makes one's life a lot simpler, but this comes at a cost. Firstly, one can only work in rather restricted situations, such as exact symplectic manifolds. And secondly, one must go through various contortions to ensure all the moduli spaces $\bar{\mathcal{M}}$ are manifolds, such as using domain-dependent almost complex structures, which are unnecessary in approaches (A),(B).

The aim and scope of the series, and its novel features

The aim of this series of books is to set up the foundations of these areas of Symplectic Geometry built using J-holomorphic curves following approach (A) above, using the author's own definition of Kuranishi space. We will do this starting from the beginning, rigorously, in detail, and as the author believes the subject ought to be done. The author hopes that in future, the series will provide a complete framework which symplectic geometers can refer to for theorems and proofs, and use large parts as a 'black box'.

The author currently plans four or more volumes, as follows:

Volume I. **Basic theory of (m-)Kuranishi spaces.** Definitions of the category $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ of μ-Kuranishi spaces, and the 2-categories $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ of m-Kuranishi spaces and $\dot{\mathbf{K}}\mathbf{ur}$ of Kuranishi spaces, over a category of 'manifolds' $\dot{\mathbf{M}}\mathbf{an}$ such as classical manifolds $\mathbf{Man}$ or manifolds with corners $\mathbf{Man^c}$. Boundaries, corners, and corner (2-)functors for (m- and μ-)Kuranishi spaces with corners. Relation to similar structures in the literature, including Fukaya–Oh–Ohta–Ono's Kuranishi spaces, and Hofer–Wysocki–Zehnder's polyfolds. 'Kuranishi moduli problems', our approach to putting Kuranishi structures on moduli spaces, canonical up to equivalence.

Volume II. **Differential Geometry of (m-)Kuranishi spaces.** Tangent and obstruction spaces for (m- and μ-)Kuranishi spaces. Canonical bundles and orientations. (W-)transversality, (w-)submersions, and existence of w-transverse fibre products in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ and $\dot{\mathbf{K}}\mathbf{ur}$. M-(co)homology of manifolds and orbifolds [44], virtual (co)chains and virtual (co)cycles for compact, oriented (m-)Kuranishi spaces in M-(co)homology. Orbifold strata of Kuranishi spaces. Bordism and cobordism for (m-)Kuranishi spaces.

Volume III. **Kuranishi structures on moduli spaces of J-holomorphic curves.** For very many moduli spaces of J-holomorphic curves $\bar{\mathcal{M}}$ of interest in Symplectic Geometry, including singular curves,

curves with Lagrangian boundary conditions, marked points, etc., we show that $\bar{\mathcal{M}}$ can be made into a Kuranishi space $\bar{\boldsymbol{\mathcal{M}}}$, uniquely up to equivalence in $\dot{\mathbf{K}}\mathbf{ur}$. We do this by a new method using 2-categories, similar to Grothendieck's representable functor approach to moduli spaces in Algebraic Geometry. We do the same for many other classes of moduli problems for nonlinear elliptic p.d.e.s, including gauge theory moduli spaces. Natural relations between moduli spaces, such as maps $F_i : \bar{\mathcal{M}}_{k+1} \to \bar{\mathcal{M}}_k$ forgetting a marked point, correspond to relations between the Kuranishi spaces, such as a 1-morphism $\boldsymbol{F}_i : \bar{\boldsymbol{\mathcal{M}}}_{k+1} \to \bar{\boldsymbol{\mathcal{M}}}_k$ in $\dot{\mathbf{K}}\mathbf{ur}$. We discuss orientations on Kuranishi moduli spaces.

Volumes IV– **Big theories in Symplectic Geometry.** To include Gromov–Witten invariants, Quantum Cohomology, Lagrangian Floer cohomology, and Fukaya categories.

For steps (i)–(v) above, (i)–(iii) will be tackled in volumes I–III respectively, and (iv)–(v) in volume IV onwards.

Readers familiar with the field will probably have noticed that our series sounds a lot like the work of Fukaya, Oh, Ohta and Ono [10–30], in particular, their 2009 two-volume book [15] on Lagrangian Floer cohomology. And it is very similar. On the large scale, and in a lot of the details, we have taken many ideas from Fukaya–Oh–Ohta–Ono, which the author acknowledges with thanks. Actually this is true of most foundational projects in this field: Fukaya, Oh, Ohta and Ono were the pioneers, and enormously creative, and subsequent authors have followed in their footsteps to a great extent.

However, there are features of our presentation that are genuinely new, and here we will highlight three:

(a) The use of *Derived Differential Geometry* in our Kuranishi space theory.

(b) The use of *M-(co)homology* to form virtual cycles and chains.

(c) The use of *'Kuranishi moduli problems'*, similar to Grothendieck's representable functor approach to moduli spaces in Algebraic Geometry, to prove moduli spaces of J-holomorphic curves have Kuranishi structures.

We discuss these in turn.

(a) Derived Differential Geometry

Derived Algebraic Geometry, developed by Lurie [48] and Toën–Vezzosi [68, 69], is the study of 'derived schemes' and 'derived stacks', enhanced versions of classical schemes and stacks with a richer geometric structure. They were introduced to study moduli spaces in Algebraic Geometry. Roughly, a classical moduli space $\mathcal{M}$ of objects E knows about the infinitesimal deformations of E, but not the obstructions to deformations. The corresponding derived moduli space $\boldsymbol{\mathcal{M}}$ remembers the deformations, obstructions, and higher obstructions.

Derived Algebraic Geometry has a less well-known cousin, Derived Differential Geometry, the study of 'derived' versions of smooth manifolds. Probably the first reference to Derived Differential Geometry is a short final paragraph in Lurie [48, §4.5]. Lurie's ideas were developed further in 2008 by his student David Spivak [66], who defined an ∞-category $\mathbf{DerMan_{Spi}}$ of 'derived manifolds'.

When I read Spivak's thesis [66], armed with a good knowledge of Fukaya–Oh–Ohta–Ono's Kuranishi space theory [15], I had a revelation:

Kuranishi spaces are really derived smooth orbifolds.

This should not be surprising, as derived schemes and Kuranishi spaces are both geometric structures designed to remember the obstructions in moduli problems.

This has important consequences for Symplectic Geometry: to understand Kuranishi spaces properly, we should use the insights and methods of Derived Algebraic Geometry. Fukaya–Oh–Ohta–Ono could not do this, as their Kuranishi spaces predate Derived Algebraic Geometry by several years. Since they lacked essential tools, their FOOO Kuranishi spaces are not really satisfactory as geometric spaces, though they are adequate for their applications. For example, they give no definition of morphism of FOOO Kuranishi spaces.

A very basic fact about Derived Algebraic Geometry is that it always happens in higher categories, usually ∞-categories. We have written our theory in terms of 2-categories, which are much simpler than ∞-categories. There are special features of our situation which mean that 2-categories are enough for our purposes. Firstly, the existence of partitions of unity in Differential Geometry means that structure sheaves are soft, and have no higher cohomology. Secondly, we are only interested in 'quasi-smooth' derived spaces, which have deformations and obstructions, but no higher obstructions. As we are studying Kuranishi spaces with deformations and obstructions – two levels of tangent directions – these spaces need to live in a higher category $\boldsymbol{\mathcal{C}}$ with at least two levels of morphism, 1- and 2-morphisms, so $\boldsymbol{\mathcal{C}}$ needs to be at least a 2-category.

Our Kuranishi spaces form a weak 2-category $\dot{\mathbf{K}}\mathbf{ur}$. One can take the homotopy category $\mathrm{Ho}(\dot{\mathbf{K}}\mathbf{ur})$ to get an ordinary category, but this loses important information. For example:

- 1-morphisms $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ in $\dot{\mathbf{K}}\mathbf{ur}$ are a 2-sheaf (stack) on $\boldsymbol{X}$, but morphisms $[\boldsymbol{f}]:\boldsymbol{X}\to\boldsymbol{Y}$ in $\mathrm{Ho}(\dot{\mathbf{K}}\mathbf{ur})$ are not a sheaf on $\boldsymbol{X}$, they are not 'local'. This is probably one reason why Fukaya et al. do not define morphisms for FOOO Kuranishi spaces, as higher category techniques would be needed.
- As in Chapter 11 of volume II, there is a good notion of (w-)transverse 1-morphisms $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ in $\dot{\mathbf{K}}\mathbf{ur}$, and (w-)transverse fibre products $\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exist in $\dot{\mathbf{K}}\mathbf{ur}$, characterized by a universal property involving the 2-morphisms in $\dot{\mathbf{K}}\mathbf{ur}$. In $\mathrm{Ho}(\dot{\mathbf{K}}\mathbf{ur})$ this universal property makes no sense, and (w-)transverse fibre products may not exist.

Derived Differential Geometry will be discussed in §4.8 of volume I.

(b) M-(co)homology and virtual cycles

In Fukaya–Oh–Ohta–Ono's Lagrangian Floer theory [15], a lot of extra complexity and hard work is due to the fact that their homology theory for forming virtual chains (singular homology) does not play nicely with FOOO Kuranishi spaces. For example, they deal with moduli spaces $\bar{\boldsymbol{\mathcal M}}_k(\alpha)$ of stable J-holomorphic discs Σ in (S,ω) with boundary in a Lagrangian L, with homology class $[\Sigma]=\alpha$ in $H_2(S,L;\mathbb{Z})$, and k boundary marked points. These satisfy boundary equations

$$\partial\bar{\boldsymbol{\mathcal M}}_k(\alpha)\simeq \textstyle\coprod_{\alpha=\beta+\gamma,\; k=i+j}\bar{\boldsymbol{\mathcal M}}_{i+1}(\beta)\times_{\mathbf{ev}_{i+1},L,\mathbf{ev}_{j+1}}\bar{\boldsymbol{\mathcal M}}_{j+1}(\gamma).$$

One would like to choose virtual chains $[\bar{\boldsymbol{\mathcal M}}_k(\alpha)]_{\rm virt}$ in homology satisfying

$$\partial[\bar{\boldsymbol{\mathcal M}}_k(\alpha)]_{\rm virt}=\textstyle\sum_{\alpha=\beta+\gamma,\; k=i+j}[\bar{\boldsymbol{\mathcal M}}_{i+1}(\beta)]_{\rm virt}\bullet_L[\bar{\boldsymbol{\mathcal M}}_{j+1}(\gamma)]_{\rm virt},$$

where $\bullet_L$ is a chain-level intersection product/cup product on the (co)homology of L. But singular homology has no chain-level intersection product.

In their later work [18, §12], [24], Fukaya et al. define virtual cochains in de Rham cohomology, which does have a cochain-level cup product. But there are disadvantages to this too, for example, one is forced to work in (co)homology over $\mathbb{R}$, rather than $\mathbb{Z}$ or $\mathbb{Q}$.

As in Chapter 12 of volume II, the author [44] defined new (co)homology theories $MH_*(X;R),MH^*(X;R)$ of manifolds and orbifolds X, called 'M-homology' and 'M-cohomology'. They satisfy the Eilenberg–Steenrod axioms, and so are canonically isomorphic to usual (co)homology $H_*(X;R),H^*(X;R)$, e.g. singular homology $H_*^{\rm si}(X;R)$. They are specially designed for forming virtual (co)chains for (m-)Kuranishi spaces, and have very good (co)chain-level properties.

In Chapter 13 of volume II we will explain how to form virtual (co)cycles and (co)chains for (m-)Kuranishi spaces in M-(co)homology. There is no need to perturb the (m-)Kuranishi space to do this. Our construction has a number of technical advantages over competing theories: we can make infinitely many compatible choices of virtual (co)chains, which can be made strictly compatible with relations between (m-)Kuranishi spaces, such as boundary formulae.

These technical advantages mean that applying our machinery to define some theory like Lagrangian Floer cohomology, Fukaya categories, or Symplectic Field Theory, will be significantly easier. Identities which only hold up to homotopy in the Fukaya–Oh–Ohta–Ono model, often hold on the nose in our version.

(c) Kuranishi moduli problems

The usual approaches to moduli spaces in Differential Geometry, and in Algebraic Geometry, are very different. In Differential Geometry, one defines a moduli space (e.g. of J-holomorphic curves, or instantons on a 4-manifold), initially as a set $\mathcal{M}$ of isomorphism classes of the objects of interest, and then adds extra structure: first a topology, and then an atlas of charts on $\mathcal{M}$ making the moduli space into a manifold or Kuranishi-type space. The individual charts are defined by writing the p.d.e. as a nonlinear Fredholm operator between Sobolev or Hölder spaces, and using the Implicit Function Theorem for Banach spaces.

In Algebraic Geometry, following Grothendieck, one begins by defining a functor F called the *moduli functor*, which encodes the behaviour of families of objects in the moduli problem. This might be of the form $F:(\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}}\to\mathbf{Sets}$

(to define a moduli $\mathbb{C}$-scheme) or $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ (to define a moduli $\mathbb{C}$-stack), where $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}, \mathbf{Sets}, \mathbf{Groupoids}$ are the categories of affine $\mathbb{C}$-schemes, and sets, and groupoids, and $(\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}}$ is the opposite category of $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$. Here if S is an affine $\mathbb{C}$-scheme then $F(S)$ is the set or groupoid of families of objects in the moduli problem over the base $\mathbb{C}$-scheme S.

We say that the moduli functor F is *representable* if there exists a $\mathbb{C}$-scheme $\mathcal{M}$ such that F is naturally isomorphic to $\mathrm{Hom}(-, \mathcal{M}) : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Sets}$, or an Artin $\mathbb{C}$-stack $\mathcal{M}$ such that F is naturally equivalent to $\mathbf{Hom}(-, \mathcal{M}) : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$. Then $\mathcal{M}$ is unique up to canonical isomorphism or canonical equivalence, and is called the *moduli scheme* or *moduli stack*.

As in Gomez [31, §2.1–§2.2], there are two equivalent ways to encode stacks, or moduli problems, as functors: either as a functor $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ as above, or as a *category fibred in groupoids* $G : \mathcal{C} \to \mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$, that is, a category $\mathcal{C}$ with a functor G to $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$ satisfying some lifting properties of morphisms in $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$ to morphisms in $\mathcal{C}$.

We introduce a new approach to constructing Kuranishi structures on Differential-Geometric moduli problems, including moduli of J-holomorphic curves, which is a 2-categorical analogue of the 'category fibred in groupoids' version of moduli functors in Algebraic Geometry. Our analogue of $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$ is the 2-category $\mathbf{G\dot{K}N}$ of *global Kuranishi neighbourhoods* (V, E, Γ, s), which are basically Kuranishi spaces $\boldsymbol{X}$ covered by a single chart (V, E, Γ, s, ψ).

We define a *Kuranishi moduli problem* (*KMP*) to be a 2-functor $F : \boldsymbol{\mathcal{C}} \to \mathbf{G\dot{K}N}$ satisfying some lifting properties, where $\boldsymbol{\mathcal{C}}$ is a 2-category. For example, if $\boldsymbol{\mathcal{M}} \in \mathbf{\dot{K}ur}$ is a Kuranishi space we can define a 2-category $\boldsymbol{\mathcal{C}}_{\boldsymbol{\mathcal{M}}}$ with objects $\big((V, E, \Gamma, s), \boldsymbol{f}\big)$ for $(V, E, \Gamma, s) \in \mathbf{G\dot{K}N}$ and $\boldsymbol{f} : (s^{-1}(0)/\Gamma, (V, E, \Gamma, s, \mathrm{id}_{s^{-1}(0)/\Gamma}))$ $\to \boldsymbol{\mathcal{M}}$ a 1-morphism, and a 2-functor $F_{\boldsymbol{\mathcal{M}}} : \boldsymbol{\mathcal{C}}_{\boldsymbol{\mathcal{M}}} \to \mathbf{G\dot{K}N}$ acting by $F_{\boldsymbol{\mathcal{M}}} : ((V, E, \Gamma, s), \boldsymbol{f}) \mapsto (V, E, \Gamma, s)$ on objects. A KMP $F : \boldsymbol{\mathcal{C}} \to \mathbf{G\dot{K}N}$ is called *representable* if it is equivalent in a certain sense to $F_{\boldsymbol{\mathcal{M}}} : \boldsymbol{\mathcal{C}}_{\boldsymbol{\mathcal{M}}} \to \mathbf{G\dot{K}N}$ for some $\boldsymbol{\mathcal{M}}$ in $\mathbf{\dot{K}ur}$, which is unique up to equivalence. Then Kuranishi moduli problems form a 2-category $\mathbf{\dot{K}MP}$, and the full 2-subcategory $\mathbf{\dot{K}MP^{re}}$ of representable KMP's is equivalent to $\mathbf{\dot{K}ur}$.

To construct a Kuranishi structure on some moduli space $\mathcal{M}$, e.g. a moduli space of J-holomorphic curves in some (S, ω), we carry out three steps:

1. Define a 2-category $\boldsymbol{\mathcal{C}}$ and 2-functor $F : \boldsymbol{\mathcal{C}} \to \mathbf{G\dot{K}N}$, where objects A in $\boldsymbol{\mathcal{C}}$ with $F(A) = (V, E, \Gamma, s)$ correspond to families of objects in the moduli problem over the base Kuranishi neighbourhood (V, E, Γ, s).
2. Prove that $F : \boldsymbol{\mathcal{C}} \to \mathbf{G\dot{K}N}$ is a Kuranishi moduli problem.
3. Prove that $F : \boldsymbol{\mathcal{C}} \to \mathbf{G\dot{K}N}$ is representable.

Here step (1) is usually fairly brief — far shorter than constructions of curve moduli spaces in [15, 30, 40], for instance. Step (2) is also short and uses standard arguments. The major effort is in (3). Step (3) has two parts: firstly we must show that a topological space $\mathcal{M}$ naturally associated to the KMP is Hausdorff and second countable (often we can quote this from the literature), and secondly we must prove that every point of $\mathcal{M}$ admits a Kuranishi neighbourhood with a certain universal property.

We compare our approach to moduli problems with other current approaches, such as those of Fukaya–Oh–Ohta–Ono or Hofer–Wysocki–Zehnder:

- Rival approaches are basically very long ad hoc constructions, the effort is in the definition itself. In our approach we have a short-ish definition, followed by a theorem (representability of the KMP) with a long proof.
- Rival approaches may involve making many arbitrary choices to construct the moduli space. In our approach the definition of the KMP is natural, with no arbitrary choices. If the KMP is representable, the corresponding Kuranishi space $\boldsymbol{\mathcal{M}}$ is unique up to canonical equivalence in $\dot{\mathbf{K}}\mathbf{ur}$.
- In our approach, morphisms between moduli spaces, e.g. forgetting a marked point, are usually easy and require almost no work to construct.

Kuranishi moduli problems are introduced in Chapter 8 of volume I, and volume III is dedicated to constructing Kuranishi structures on moduli spaces using the KMP method.

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Chapter 9

Introduction to volume II

In volume I of this series, given a category $\dot{\mathbf{Man}}$ of 'manifolds' satisfying some assumptions, such as classical manifolds $\mathbf{Man}$ or manifolds with corners $\mathbf{Man^c}$, we defined a corresponding category $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ of 'μ-Kuranishi spaces', and 2-categories $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ of 'm-Kuranishi spaces' and $\dot{\mathbf{K}}\mathbf{ur}$ of 'Kuranishi spaces'.

In this volume II, we study the differential geometry of these (m- and μ-) Kuranishi spaces, covering topics including tangent spaces $T_x\boldsymbol{X}$ and obstruction spaces $O_x\boldsymbol{X}$, canonical bundles $K_{\boldsymbol{X}}$ and orientations, (w-)submersions and (w-) transverse fibre products $\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ and $\dot{\mathbf{K}}\mathbf{ur}$, virtual chains and virtual cycles for compact, oriented (m-)Kuranishi spaces, orbifold strata of Kuranishi spaces, and (co)bordism of (m-)Kuranishi spaces.

We will be constantly referring to volume I. As it would take many pages to summarize the previous material we need, we have not tried to make this volume independent of volume I. So most readers will need a copy of volume I on hand to make sense of this book, unless they already know volume I well. The chapter numbering in this volume continues on from volume I, so all references to Chapters 1–8 and Appendices A, B are to volume I.

Chapter 10 defines and studies *tangent spaces* $T_x\boldsymbol{X}$ and *obstruction spaces* $O_x\boldsymbol{X}$ for (μ- or m-)Kuranishi spaces $\boldsymbol{X}$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur},\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur},\dot{\mathbf{K}}\mathbf{ur}$. These come from a suitable notion of tangent space T_xX in $\dot{\mathbf{Man}}$, where for categories of manifolds with corners $\mathbf{Man^c},\ldots$ there may be several versions $T_xX,{}^bT_xX,\tilde T_xX$, yielding different notions $T_x\boldsymbol{X},{}^bT_x\boldsymbol{X},\tilde T_x\boldsymbol{X}$, $O_x\boldsymbol{X},{}^bO_x\boldsymbol{X},\tilde O_x\boldsymbol{X}$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur},\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur},\dot{\mathbf{K}}\mathbf{ur}$. We also discuss applications, including orientations on (μ- and m-)Kuranishi spaces. Tangent and obstruction spaces are functorial under (1-)morphisms in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur},\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur},\dot{\mathbf{K}}\mathbf{ur}$, and are useful for stating conditions on 1-morphisms. For example, a 1-morphism $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ in $\mathbf{mKur}$ is étale (a local equivalence) if and only if $T_x\boldsymbol{f}:T_x\boldsymbol{X}\rightarrow T_y\boldsymbol{Y}$ and $O_x\boldsymbol{f}:O_x\boldsymbol{X}\rightarrow O_y\boldsymbol{Y}$ are isomorphisms for all $x\in\boldsymbol{X}$ with $\boldsymbol{f}(x)=y$ in $\boldsymbol{Y}$.

Chapter 11 studies transverse fibre products and submersions in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ and $\dot{\mathbf{K}}\mathbf{ur}$. Given suitable notions of when morphisms $g:X\rightarrow Z$, $h:Y\rightarrow Z$ in $\dot{\mathbf{Man}}$ are *transverse*, so that a fibre product $W=X\times_{g,Z,h}Y$ exists in $\dot{\mathbf{Man}}$ with $\dim W=\dim X+\dim Y-\dim Z$, or when $g:X\rightarrow Z$ is a *submersion*, so that g, h are transverse for any $h : Y \to Z$, we define notions of when 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ in $\mathbf{m\dot{K}ur}$ or $\mathbf{\dot{K}ur}$ are *w-transverse*, so that a 2-category fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in $\mathbf{m\dot{K}ur}$ or $\mathbf{\dot{K}ur}$ with $\operatorname{vdim} \boldsymbol{W} = \operatorname{vdim} \boldsymbol{X} + \operatorname{vdim} \boldsymbol{Y} - \operatorname{vdim} \boldsymbol{Z}$, or when $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ is a *w-submersion*, so that $\boldsymbol{g}, \boldsymbol{h}$ are w-transverse for any $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$.

For example, in Kuranishi spaces $\mathbf{Kur}$ over classical manifolds, 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are w-transverse if

$$O_x\boldsymbol{g} \oplus O_y\boldsymbol{Y} : O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y} \longrightarrow O_z\boldsymbol{Z}$$

is surjective for all $x \in \boldsymbol{X}$ and $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$, and then a fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in $\mathbf{Kur}$. This is automatic if $\boldsymbol{Z}$ is a manifold or orbifold, so that $O_z\boldsymbol{Z} = 0$ for all $z \in \boldsymbol{Z}$. Such fibre products will be important in applications in symplectic geometry.

In general, w-transverse fibre products do *not* exist in categories of μ-Kuranishi spaces $\boldsymbol{\mu}\mathbf{\dot{K}ur}$, nor in the homotopy categories $\operatorname{Ho}(\mathbf{m\dot{K}ur}), \operatorname{Ho}(\mathbf{\dot{K}ur})$. The 2-category structure on $\mathbf{m\dot{K}ur}$ and $\mathbf{\dot{K}ur}$ is essential for forming fibre products, as the universal property of such fibre products involves 2-morphisms. This is characteristic of 'derived' fibre products, and is an important reason for working in a 2-category or ∞-category when doing derived geometry.

Chapters 12–15 are not written yet, but will discuss virtual classes/chains for (m-)Kuranishi spaces using the author's theory of M-(co)homology [44], orbifold strata for Kuranishi spaces, and (co)bordism for (m-)Kuranishi spaces.

Chapter 10

Tangent and obstruction spaces

If X is a classical manifold then each $x \in X$ has a tangent space T_xX, and if $f : X \to Y$ is a smooth map there are functorial tangent maps $T_xf : T_xX \to T_yY$ for $x \in X$ with $f(x) = y \in Y$. For manifolds with corners $\mathbf{Man^c}, \mathbf{Man^{gc}}, \ldots$ there are (at least) two notions of tangent space $T_xX, {}^bT_xX$, as in §2.3.

For (m- or μ-)Kuranishi spaces $\boldsymbol{X}$, it turns out to be natural to define functorial *tangent spaces* $T_x\boldsymbol{X}$ and *obstruction spaces* $O_x\boldsymbol{X}$ for $x \in \boldsymbol{X}$. This chapter studies tangent and obstruction spaces, and applies them in several ways, for instance to define *orientations* on (m- or μ-)Kuranishi spaces $\boldsymbol{X}$.

10.1 Optional assumptions on tangent spaces

Suppose for the whole of this section that $\dot{\mathbf{M}}\mathbf{an}$ satisfies Assumptions 3.1–3.7. We now give optional assumptions on tangent spaces in $\dot{\mathbf{M}}\mathbf{an}$.

10.1.1 Tangent spaces

We ask that our 'manifolds' X have a notion of 'tangent space' T_xX satisfying many of the properties one expects. Note that we do *not* require $\dim T_xX = \dim X$, or that tangent spaces are the fibres of a vector bundle $TX \to X$, which are both false in some examples.

Assumption 10.1. (Tangent spaces.) (a) We are given a discrete property $\boldsymbol{A}$ of morphisms in $\dot{\mathbf{M}}\mathbf{an}$, in the sense of Definition 3.18, which may be trivial (i.e. all morphisms in $\dot{\mathbf{M}}\mathbf{an}$ may be $\boldsymbol{A}$), and should satisfy:

- **(i)** If $f : X \to Y$ is a morphism in $\dot{\mathbf{M}}\mathbf{an}$ with $Y \in \mathbf{Man}$, then f is $\boldsymbol{A}$.
- **(ii)** If $f : W \to Y$, $g : X \to Y$, $h : X \to Z$ are $\boldsymbol{A}$ morphisms in $\dot{\mathbf{M}}\mathbf{an}$ then the product $f \times h : W \times X \to Y \times Z$ and direct product $(g, h) : X \to Y \times Z$ from Assumption 3.1(e) are also $\boldsymbol{A}$.

  Projections $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ from products are $\boldsymbol{A}$.

(b) For all $X \in \dot{\mathbf{Man}}$ and $x \in X$, we are given a real vector space T_xX called the *tangent space of X at x*. For all $\boldsymbol{A}$ morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ and all $x \in X$ with $f(x) = y$ in Y, we are given a linear map $T_xf : T_xX \to T_yY$ called the *tangent map*. The dual vector space T_x^*X of T_xX is the *cotangent space*, and the dual linear map $T_x^*f : T_y^*Y \to T_x^*X$ of T_xf is the *cotangent map*. If $g : Y \to Z$ is another $\boldsymbol{A}$ morphism and $g(y) = z \in Z$ then $T_x(g \circ f) = T_yg \circ T_xf : T_xX \to T_zZ$. We have $T_x\mathrm{id}_X = \mathrm{id}_{T_xX} : T_xX \to T_xX$.

(c) For all $X, Y \in \dot{\mathbf{Man}}$ and $x \in X$, $y \in Y$ the morphism

$$T_{(x,y)}\pi_X \oplus T_{(x,y)}\pi_Y : T_{(x,y)}(X \times Y) \longrightarrow T_xX \oplus T_yY \tag{10.1}$$

is an isomorphism, where π_X, π_Y are $\boldsymbol{A}$ by **(a)(ii)**.

(d) If $i : U \hookrightarrow X$ is an open submanifold in $\dot{\mathbf{Man}}$ then $T_xi : T_xU \to T_xX$ is an isomorphism for all $x \in U \subseteq X$, so we may identify T_xU with T_xX.

(e) If $X \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}$ is a classical manifold and $x \in X$ then T_xX is (canonically isomorphic to) the usual tangent space T_xX of manifolds in differential geometry. If $f : X \to Y$ is a morphism in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}$, so that f is $\boldsymbol{A}$ by **(a)(i)**, and $x \in X$ with $f(x) = y \in Y$, then $T_xf : T_xX \to T_yY$ is the usual derivative of f at x in differential geometry.

Example 10.2. **(i)** If $\dot{\mathbf{Man}} = \mathbf{Man}$ then $\boldsymbol{A}$ must be trivial (i.e. all morphisms in $\mathbf{Man}$ are $\boldsymbol{A}$) by Assumption 10.1(a)(i), and T_xX, T_xf must be as usual in differential geometry by Assumption 10.1(e), and then Assumption 10.1 holds.

(ii) Let $\dot{\mathbf{Man}}$ be $\mathbf{Man^c}$ or $\mathbf{Man^c_{we}}$ from Chapter 2, and let $\boldsymbol{A}$ be trivial. Then as in §2.3, each $X \in \dot{\mathbf{Man}}$ has tangent spaces T_xX for all $x \in X$ and tangent maps $T_xf : T_xX \to T_yY$ for all morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ and $x \in X$ with $f(x) = y \in Y$, which satisfy Assumption 10.1.

(iii) Let $\dot{\mathbf{Man}}$ be one of $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$ from Chapter 2, and let $\boldsymbol{A}$ be *interior* maps in this category. Then as in §2.3–§2.4, each $X \in \dot{\mathbf{Man}}$ has b-tangent spaces ${}^bT_xX$ for all $x \in X$, and each interior morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$ has b-tangent maps ${}^bT_xf : {}^bT_xX \to {}^bT_yY$ for all $x \in X$ with $f(x) = y \in Y$, which satisfy Assumption 10.1.

(iv) Let $\dot{\mathbf{Man}}$ be one of $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$, and let $\boldsymbol{A}$ be trivial. Then as in §2.2, each $X \in \dot{\mathbf{Man}}$ with $\dim X = m$ has a depth stratification $X = \coprod_{k=0}^m S^k(X)$ with $S^k(X)$ a classical manifold of dimension $m - k$, and any morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$ preserves depth stratifications. (The latter does not hold for $\mathbf{Man^c_{we}}$, which we exclude).

For each $x \in S^k(X) \subseteq X$, define $\tilde{T}_xX = T_xS^k(X)$. We call this the *stratum tangent space* of X at x. If $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$ and $x \in S^k(X) \subseteq X$ with $f(x) = y \in S^l(Y) \subseteq Y$ then near $f|_{S^k(X)}$ is a smooth map of classical manifolds $S^k(X) \to S^l(Y)$ near x. Define

$$\tilde{T}_xf = T_x(f|_{S^k(X)}) : \tilde{T}_xX = T_xS^k(X) \longrightarrow \tilde{T}_yY = T_yS^l(Y).$$

Then these $\boldsymbol{A}, \tilde{T}_xX, \tilde{T}_xf$ satisfy Assumption 10.1.

(v) Let $\dot{\mathbf{Man}}$ satisfy Assumptions 3.1–3.7, and let $\boldsymbol{A}$ be trivial. Then as in §3.3.1(c) and §B.1.3, we define a functor $F_{\dot{\mathbf{Man}}}^{\mathbf{C^\infty Sch}}:\dot{\mathbf{Man}}\to\mathbf{C^\infty Sch^{aff}}$ to the category of affine C^∞-schemes. Now C^∞-schemes $\underline{X}=(X,\mathcal{O}_X)$ have a functorial notion of tangent space $T_x\underline{X}$ for $x\in X$, given by $T_x\underline{X}=(\Omega_{\underline{X},x}\otimes_{\mathcal{O}_{X,x}}\mathbb{R})^*$, where $\Omega_{\underline{X}}$ is the cotangent sheaf of $\underline{X}$ from [45, §5.6] (which we used in §B.4 to define $\mathcal{T}^*X$), and $\Omega_{\underline{X},x},\mathcal{O}_{X,x}$ are the stalks of $\Omega_{\underline{X}},\mathcal{O}_X$ at x.

Thus, for any $\dot{\mathbf{Man}}$ we can define $T_x^{C^\infty}X,T_x^{C^\infty}f$ satisfying Assumption 10.1 by applying $F_{\dot{\mathbf{Man}}}^{\mathbf{C^\infty Sch}}:\dot{\mathbf{Man}}\to\mathbf{C^\infty Sch^{aff}}$ and taking tangent spaces of C^∞-schemes. The result is canonically isomorphic to the tangent spaces T_xX in **(i)**,**(ii)** in those cases, but not isomorphic to ${}^bT_xX,\tilde T_xX$ in **(iii)**,**(iv)**.

Note that $\mathbf{Man^c}$ has three different tangent spaces satisfying Assumption 10.1 in **(ii)**–**(iv)**. Here is a way to compare different notions of tangent space:

Definition 10.3. Suppose we are given two notions of tangent space T_xX,T_xf for f with discrete property $\boldsymbol{A}$, and $T_x'X,T_x'f$ with discrete property $\boldsymbol{A}'$, both satisfying Assumption 10.1 in $\dot{\mathbf{Man}}$. A *natural transformation* $I:T\Rightarrow T'$ assigns a linear map $I_xX:T_xX\to T_x'X$ for all $X\in\dot{\mathbf{Man}}$ and $x\in X$, such that:

(i) If $f:X\to Y$ is a morphism in $\dot{\mathbf{Man}}$ which is both $\boldsymbol{A}$ and $\boldsymbol{A}'$, and $x\in X$ with $f(x)=y\in Y$, the following diagram commutes:

$$\begin{array}{ccc} T_xX & \xrightarrow{\quad T_xf\quad} & T_yY \\ \big\downarrow{\scriptstyle I_xX} & & \big\downarrow{\scriptstyle I_yY} \\ T_x'X & \xrightarrow{\quad T_x'f\quad} & T_y'Y. \end{array}$$

(ii) If $X\in\mathbf{Man}\subseteq\dot{\mathbf{Man}}$, so that $T_xX,T_x'X$ are both the usual tangent space T_xX by Assumption 10.1(e), then $I_xX=\mathrm{id}_{T_xX}$.

Example 10.4. **(a)** Let $\dot{\mathbf{Man}}=\mathbf{Man^c}$. Then Example 10.2(ii),(iii) define tangent spaces T_xX with $\boldsymbol{A}$ trivial, and ${}^bT_xX$ with $\boldsymbol{A}$ interior, satisfying Assumption 10.1. As in (2.10) in §2.3, there are natural maps $I_xX:{}^bT_xX\to T_xX$ satisfying Definition 10.3.

(b) When $\dot{\mathbf{Man}}=\mathbf{Man^c}$ there are injective maps $\iota_xX:\tilde T_xX\to T_xX$ in Example 10.2(ii),(iv), the inclusions $T_xS^k(X)\hookrightarrow T_xX$, satisfying Definition 10.3.

(c) Let $\dot{\mathbf{Man}}$ be one of $\mathbf{Man^c},\mathbf{Man^{gc}},\mathbf{Man^{ac}},\mathbf{Man^{c,ac}}$. Then there are natural surjective maps $\Pi_xX:{}^bT_xX\to\tilde T_xX$ in Example 10.2(iii),(iv) satisfying Definition 10.3.

We can also add a further assumption on dimensions of tangent spaces:

Assumption 10.5. Assumption 10.1 holds, and T_xX is finite-dimensional with $\dim T_xX=\dim X$ for all $X\in\dot{\mathbf{Man}}$ and $x\in X$.

This holds for Example 10.2(i)–(iii), but not for Example 10.2(iv)–(v).

To use Assumption 10.1, we will need the following notation:

Definition 10.6. Let Assumption 10.1 hold for $\dot{\mathbf{Man}}$, with discrete property $\boldsymbol{A}$ and data T_xX, T_xf. Suppose $\pi : E \to X$ is a vector bundle in $\dot{\mathbf{Man}}$, and $s \in \Gamma^\infty(E)$ be a section, and $x \in s^{-1}(0) \subseteq X$. We will define a linear map $\mathrm{d}_x s : T_xX \to E|_x$, where $E|_x$ is the fibre of E at x, which we think of as the derivative of s at x.

The section s, and the zero section 0_E, are both morphisms $X \to E$ in $\dot{\mathbf{Man}}$, with $s(x) = 0_E(x)$ as $x \in s^{-1}(0)$. Write $e = s(x) = 0_E(x)$. Then $\pi(e) = x$. Using Assumption 10.1(a) and Definition 3.18(iv) we can show that $s, 0_E, \pi$ are all $\boldsymbol{A}$. Hence Assumption 10.1 gives linear maps

$$T_xs : T_xX \longrightarrow T_eE, \quad T_x0_E : T_xX \longrightarrow T_eE, \quad T_e\pi : T_eE \longrightarrow T_xX,$$

with $T_e\pi \circ T_xs = T_e\pi \circ T_x0_E = \mathrm{id}_{T_xX}$ as $\pi \circ s = \pi \circ 0_E = \mathrm{id}_X$. By definition of vector bundles, there is an open neighbourhood U of x in X on which E is trivial, so $E|_U \cong U \times \mathbb{R}^k$ identifying $\pi|_U : E|_U \to U$ with $\pi_{\mathbb{R}^k} : U \times \mathbb{R}^k \to \mathbb{R}^k$. Thus from Assumption 10.1(c)–(e) we get a natural isomorphism

$$T_eE \cong T_xX \oplus \mathbb{R}^k \cong T_xX \oplus E|_x, \tag{10.2}$$

identifying $T_e\pi : T_eE \to T_xX$ with $\mathrm{id}_{T_xX} \oplus 0 : T_xX \oplus E|_x \to T_xX$, and $T_x0_E : T_xX \to T_eE$ with $\mathrm{id}_{T_xX} \oplus 0 : T_xX \to T_xX \oplus E|_x$. Write $\mathrm{d}_xs : T_xX \to E|_x$ for the composition of $T_xs : T_xX \longrightarrow T_eE$ with the projection $T_eE \to E|_x$ from (10.2). When $\dot{\mathbf{Man}} = \mathbf{Man}$, this $\mathrm{d}_xs : T_xX \to E|_x$ is $\nabla s|_x : T_xX \to E|_x$ for any connection ∇ on E, and is independent of the choice of ∇, as $s(x) = 0$.

10.1.2 Tangent spaces and differential geometry in $\dot{\mathbf{Man}}$

Suppose throughout this section that $\dot{\mathbf{Man}}$ satisfies Assumptions 3.1–3.7 and Assumption 10.1, so that we are given a discrete property $\boldsymbol{A}$ of morphisms in $\dot{\mathbf{Man}}$, and 'manifolds' V in $\dot{\mathbf{Man}}$ have tangent spaces T_xX for $x \in X$, and $\boldsymbol{A}$ morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ have functorial tangent maps $T_xf : T_xX \to T_yY$ for all $x \in X$ with $f(x) = y \in Y$. We will relate tangent spaces T_xX to (relative) tangent sheaves $\mathcal{T}X, \mathcal{T}_fY$ from §3.3.4 and §B.4.

Definition 10.7. Let $f : X \to Y$ be an $\boldsymbol{A}$ morphism in $\dot{\mathbf{Man}}$, and $\alpha \in \Gamma(\mathcal{T}_fY)$, and $x \in X$ with $f(x) = y \in Y$. We will define an element $\alpha|_x$ in T_yY.

By Definition B.16 we have $\alpha = [U, u]$ for $i : U \hookrightarrow X \times \mathbb{R}$ and $u : U \to Y$ in a diagram (B.5), with $u(x, 0) = y$. Using Definition B.38(iii),(viii) and that f is $\boldsymbol{A}$ we can show that u is $\boldsymbol{A}$ near $X \times \{0\}$. Thus we have linear maps

$$T_xX \oplus \mathbb{R} \xrightarrow[\cong]{} T_{(x,0)}(X \times \mathbb{R}) \xrightarrow[\cong]{(T_{(x,0)}i)^{-1}} T_{(x,0)}U \xrightarrow{T_{(x,0)}u} T_yY, \tag{10.3}$$

where the first two isomorphisms come from Assumption 10.1(c),(d),(e). Define $\alpha|_x$ to be the image of $(0, 1) \in T_xX \oplus \mathbb{R}$ under the composition of (10.3).

To show this is well defined, suppose also that $\alpha = [U', u']$ for U', u' in a diagram (B.5). Then $(U, u) \approx (U', u')$ in the notation of Definition B.16, so there exist open $j : V \hookrightarrow X \times \mathbb{R}^2$ and a morphism $v : V \to Y$ satisfying (B.6) with $\tilde{x} = x$. As for u we find that v is $\boldsymbol{A}$ near $(x, 0, 0)$, so as for (10.3) we have

$$T_xX \oplus \mathbb{R} \oplus \mathbb{R} \xrightarrow[\cong]{} T_{(x,0,0)}(X \times \mathbb{R}^2) \xrightarrow[\cong]{(T_{(x,0,0)}j)^{-1}} T_{(x,0,0)}V \xrightarrow{T_{(x,0,0)}v} T_yY.$$

The equations of (B.6) imply that

$$T_{(x,0,0)}v(w, s, 0) = T_{(x,0)}u(w, s), \qquad T_{(x,0,0)}v(w, 0, s') = (T_{(x,0)}u')(w, s'),$$
$$\text{and} \qquad T_{(x,0,0)}v(0, s, -s) = 0,$$

for $w \in T_xX$ and $s, s' \in \mathbb{R}$. Hence $T_{(x,0)}u(0, 1) = T_{(x,0)}u'(0, 1)$ by linearity of $T_{(x,0,0)}v$, so $\alpha|_x$ is independent of the choice of representative (U, u) for α, and is well defined.

From the definition of the $C^\infty(X)$-module structure on $\Gamma(\mathcal{T}_fY)$ in §B.4.2, we see that $\alpha \mapsto \alpha|_x$ is $\mathbb{R}$-linear, and satisfies $(a \cdot \alpha)|_x = a(x) \cdot (\alpha|_x)$ for all $a \in C^\infty(X)$ and $\alpha \in \Gamma(\mathcal{T}_fY)$.

Now let $E \to X$ be a vector bundle, and $\theta : E \to \mathcal{T}_fY$ be a morphism in the sense of §B.4.8. Then we have a map $\Gamma^\infty(E) \to T_yY$ taking $e \mapsto (\theta \circ e)|_x$ for all $e \in \Gamma^\infty(E)$, so that $\theta \circ e \in \Gamma(\mathcal{T}_fY)$. As this is $\mathbb{R}$-linear and satisfies $(\theta \circ (a \cdot e))|_x = a(x) \cdot (\theta \circ e)|_x$ for $a \in C^\infty(X)$ and $e \in \Gamma^\infty(E)$, the map $e \mapsto (\theta \circ e)|_x$ factors via $e|_x \in E|_x$. That is, there is a unique linear map $\theta|_x : E|_x \to T_yY$ with $(\theta \circ e)|_x = \theta|_x(e|_x)$ for all $e \in \Gamma^\infty(E)$.

Suppose $\theta : E \to \mathcal{T}_fY$ is of the form $\theta_{V,v}$ in the notation of Definition B.32 for some open $j : V \hookrightarrow E$ and $v : V \to Y$ in a diagram (B.22). Then v is $\boldsymbol{A}$ near $(x, 0)$ in V, and as for (10.3) we have linear maps

$$T_xX \oplus E|_x \xrightarrow[\cong]{} T_{(x,0)}E \xrightarrow[\cong]{(T_{(x,0)}j)^{-1}} T_{(x,0)}V \xrightarrow{T_{(x,0)}v} T_yY, \tag{10.4}$$

and we can show that $\theta|_x(e)$ is the image of $(0, e)$ under (10.4) for each $e \in E|_x$.

In the case when $\dot{\mathbf{Man}} = \mathbf{Man}$ and T_xX is the ordinary tangent space, $\mathcal{T}_fY$ is the sheaf of sections of $f^*(TY)$, so $\theta : E \to f^*(TY)$ is a vector bundle morphism on X, and $\theta|_x : E|_x \to f^*(TY)|_x = T_yY$ is just the fibre of the morphism at x.

The next proposition can be deduced from the definitions in a fairly straightforward way, using functoriality of tangent maps in Assumption 10.1(b), and writing θ using either (10.3) or (10.4). For example, in (a), if $\theta = \theta_{V,v}$ then $\mathcal{T}g \circ \theta = \theta_{V,g\circ v}$, and (a) follows from (10.4) and $T_{(x,0)}(g \circ v) = T_yg \circ T_{(x,0)}v$.

Proposition 10.8. (a) *Suppose $f : X \to Y$, $g : Y \to Z$ are $\boldsymbol{A}$ morphisms in $\dot{\mathbf{Man}}$, and $E \to X$ is a vector bundle, and $\theta : E \to \mathcal{T}_fY$ is a morphism, so that $\mathcal{T}g \circ \theta : E \to \mathcal{T}_{g\circ f}Z$ is a morphism as in* §3.3.4(c),(d) *and* §B.4.6, §B.4.8. *Then for all $x \in X$ with $f(x) = y \in Y$ and $g(y) = z \in Z$, we have*

$$T_yg \circ \theta|_x = (\mathcal{T}g \circ \theta)|_x : E|_x \longrightarrow T_zZ. \tag{10.5}$$

(b) *Suppose $f : X \to Y$, $g : Y \to Z$ are $\boldsymbol{A}$ morphisms in $\dot{\mathbf{Man}}$, and $F \to Y$ is a vector bundle, and $\theta : F \to \mathcal{T}_gZ$ is a morphism on Y, so that we have* *a morphism* $f^*(\theta) : f^*(F) \to \mathcal{T}_{g\circ f}Z$ *as in* §3.3.4(g) *and* §B.4.9*. Then for all* $x \in X$ *with* $f(x) = y \in Y$ *and* $g(y) = z \in Z$*, we have*

$$f^*(\theta)|_x = \theta|_y : f^*(F)|_x = F|_y \longrightarrow T_zZ. \tag{10.6}$$

(c) *Suppose* $f : X \to Y$ *is an* $\boldsymbol{A}$ *morphism in* $\dot{\mathbf{M}}\mathbf{an}$*, and* $E, F \to X$*,* $G \to Y$ *are vector bundles, and* $s \in \Gamma^\infty(E)$*,* $t \in \Gamma^\infty(G)$ *with* $f^*(t) = O(s)$*, and* $\Lambda : F \to \mathcal{T}_fY$ *is a morphism, and* $\theta : F \to f^*(G)$ *is a vector bundle morphism with* $\theta = f^*(\mathrm{d}t) \circ \Lambda + O(s)$ *in the sense of Definitions* 3.15(vi) *and* B.36(vi)*. Then for each* $x \in X$ *with* $s(x) = 0$ *and* $f(x) = y \in Y$*, we have*

$$\theta|_x = \mathrm{d}_y t \circ \Lambda|_x : E|_x \longrightarrow F|_y, \tag{10.7}$$

where $\mathrm{d}_y t$ *is as in Definition* 10.6*.*

(d) *Suppose* $f, g : X \to Y$ *are* $\boldsymbol{A}$ *morphisms in* $\dot{\mathbf{M}}\mathbf{an}$*, and* $E \to X$ *is a vector bundle, and* $s \in \Gamma^\infty(E)$*, and* $\Lambda : E \to \mathcal{T}_fY$ *be a morphism with* $g = f + \Lambda \circ s + O(s^2)$ *as in Definitions* 3.15(vii) *and* B.36(vii)*. Then for each* $x \in X$ *with* $s(x) = 0$*, so that* $f(x) = g(x) = y \in Y$*, we have*

$$T_xg = T_xf + \Lambda|_x \circ \mathrm{d}_x s : T_xX \longrightarrow T_yY. \tag{10.8}$$

10.1.3 Assumptions on $f : X \to \mathbb{R}^n$, and on local diffeomorphisms

Supposing Assumption 10.1 holds, we give some more assumptions on $\dot{\mathbf{M}}\mathbf{an}$, expressed in terms of tangent spaces T_xX. They will be used in §10.4–§10.5.

Assumption 10.9. Let Assumption 10.1 hold for $\dot{\mathbf{M}}\mathbf{an}$, giving notions of tangent space T_xX and tangent maps $T_xf : T_xX \to T_yY$ for $f : X \to Y$ in $\dot{\mathbf{M}}\mathbf{an}$ satisfying a discrete property $\boldsymbol{A}$.

Suppose $f : X \to \mathbb{R}^n$ is a morphism in $\dot{\mathbf{M}}\mathbf{an}$, so that f is $\boldsymbol{A}$ by Assumption 10.1(a)(i), and $x \in X$ such that $f(x) = 0$ and $T_xf : T_xX \to T_0\mathbb{R}^n = \mathbb{R}^n$ is surjective. Then there exists a commutative diagram in $\dot{\mathbf{M}}\mathbf{an}$:

$$\begin{array}{ccccc} x \in U & \xrightarrow[k]{\cong} & V \times W & \xrightarrow{\pi_W} & W \ni 0 \\ \big\downarrow{\scriptstyle i} & & & & \big\downarrow{\scriptstyle j} \\ X & & \xrightarrow{\quad f \quad} & & \mathbb{R}^n, \end{array} \tag{10.9}$$

where $i : U \hookrightarrow X$, $j : W \hookrightarrow \mathbb{R}^n$ are open submanifolds in $\dot{\mathbf{M}}\mathbf{an}$ with $x \in U \subseteq X$ and $0 \in W \subseteq \mathbb{R}^n$, and V is an object in $\dot{\mathbf{M}}\mathbf{an}$ with $\dim V = \dim X - n$, and $k : U \to V \times W$ is a diffeomorphism in $\dot{\mathbf{M}}\mathbf{an}$.

Suppose further that a finite group Γ acts on X fixing $x \in X$, and Γ acts linearly on $\mathbb{R}^n$, and $f : X \to \mathbb{R}^n$ is Γ-equivariant. Then we can choose U, W to be Γ-invariant, and V to have a Γ-action making (10.9) Γ-equivariant.

Example 10.10. **(a)** Assumption 10.9 holds for Example 10.2(i),(iii),(iv).

(b) As in Example 10.2(ii), let $\dot{\mathbf{Man}}$ be $\mathbf{Man^c}$ or $\mathbf{Man^c_{we}}$, and $\boldsymbol{A}$ be trivial, and T_xX, T_xf be as in §2.3. Then Assumption 10.9 *does not hold.* For example, let $f : X \to Y$ be the inclusion map $i : [0,\infty) \hookrightarrow \mathbb{R}$, and $x = 0 \in [0,\infty)$. Then $T_0 i : T_0[0,\infty) \to T_0\mathbb{R}$ is surjective, but no diagram (10.9) exists in $\dot{\mathbf{Man}}$.

Assumption 10.11. Let Assumption 10.1 hold for $\dot{\mathbf{Man}}$, giving notions of tangent space T_xX and tangent maps $T_xf : T_xX \to T_yY$ for $f : X \to Y$ in $\dot{\mathbf{Man}}$ satisfying a discrete property $\boldsymbol{A}$. We should be given another discrete property $\boldsymbol{B}$ of morphisms in $\dot{\mathbf{Man}}$, such that $\boldsymbol{B}$ implies $\boldsymbol{A}$.

Suppose $f : X \to Y$ is a $\boldsymbol{B}$ morphism in $\dot{\mathbf{Man}}$, and $x \in X$ with $f(x) = y$, and $T_xf : T_xX \to T_yY$ is an isomorphism. Then there should exist open submanifolds $i : U \hookrightarrow X$ and $j : V \hookrightarrow Y$ in $\dot{\mathbf{Man}}$ with $x \in U$ and $V = f(U) \subseteq Y$, so that there is a unique $f' : U \to V$ in $\dot{\mathbf{Man}}$ with $f \circ i = j \circ f'$ by Assumption 3.2(d), and $f' : U \to V$ should be a diffeomorphism in $\dot{\mathbf{Man}}$.

Example 10.12. **(i)** Let $\dot{\mathbf{Man}} = \mathbf{Man}$, and $\boldsymbol{A}$ be trivial, and T_xX, T_xf be as usual in differential geometry, so that Assumption 10.1 holds as in Example 10.2(i). Take $\boldsymbol{B}$ to be trivial. Then Assumption 10.11 holds.

(ii) Let $\dot{\mathbf{Man}} = \mathbf{Man^c}$ from Chapter 2, and $\boldsymbol{A}$ be trivial, and T_xX, T_xf be as in §2.3, so that Assumption 10.1 holds as in Example 10.2(ii). Take $\boldsymbol{B}$ to be simple morphisms. Then Assumption 10.11 holds. That is, if $f : X \to Y$ is a simple morphism in $\mathbf{Man^c}$ and $T_xf : T_xX \to T_yY$ is an isomorphism then f is a local diffeomorphism in $\mathbf{Man^c}$ near $x \in X$ and $y \in Y$.

Note that we do not allow $\dot{\mathbf{Man}} = \mathbf{Man^c_{we}}$ in this example, although Example 10.2(ii) includes $\mathbf{Man^c_{we}}$. One can show that the only discrete property $\boldsymbol{B}$ of morphisms in $\mathbf{Man^c_{we}}$ is $\boldsymbol{B}$ trivial, and Assumption 10.11 does not hold.

(iii) Let $\dot{\mathbf{Man}}$ be one of $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$ from Chapter 2, and $\boldsymbol{A}$ be interior maps, and consider b-tangent spaces ${}^bT_xX$ and b-tangent maps ${}^bT_xf : {}^bT_xX \to {}^bT_yY$ for interior f in $\dot{\mathbf{Man}}$ as in §2.3–§2.4, so that Assumption 10.1 holds as in Example 10.2(iii). Take $\boldsymbol{B}$ to be simple morphisms. Then $\boldsymbol{B}$ implies $\boldsymbol{A}$, as simple morphisms are interior, and Assumption 10.11 holds.

(iv) Let $\dot{\mathbf{Man}}$ be one of $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$ from Chapter 2, and $\boldsymbol{A}$ be trivial, and consider stratum tangent spaces $\tilde{T}_xX$ and stratum tangent maps $\tilde{T}_xf : \tilde{T}_xX \to \tilde{T}_yY$ as in Example 10.2(iv), so that Assumption 10.1 holds. Take $\boldsymbol{B}$ to be simple morphisms. Then Assumption 10.11 holds.

10.1.4 Assumptions on tangent bundles, and orientations

In the next assumption we suppose that tangent spaces T_xX in Assumption 10.1 are the fibres of a vector bundle $TX \to X$.

Assumption 10.13. (Tangent vector bundles.) **(a)** Let Assumption 10.1 hold for $\dot{\mathbf{Man}}$, with tangent spaces T_xX and discrete property $\boldsymbol{A}$. For each $X \in \dot{\mathbf{Man}}$ there is a natural vector bundle $\pi : TX \to X$ called the *tangent bundle*, of rank $\dim X$, whose fibre at each $x \in X$ is the tangent space T_xX.

The dual vector bundle of TX is called the *cotangent bundle* $T^*X \to X$, with fibres the cotangent spaces T_x^*X.

(b) If $f : X \to Y$ is an $\boldsymbol{A}$ morphism in $\dot{\mathbf{Man}}$ there is a natural vector bundle morphism $Tf : TX \to f^*(TY)$ on X, such that if $x \in X$ with $f(x) = y$ in Y then the fibre $Tf|_x$ of Tf at x is the tangent map $T_x f : T_x X \to T_y Y$.

The dual morphism is written $T^*f : f^*(T^*Y) \to T^*X$.

Using part (b) and §10.1.2 we can show that if $f : X \to Y$ is an $\boldsymbol{A}$ morphism in $\dot{\mathbf{Man}}$, and $E \to X$ is a vector bundle, and $\theta : E \to \mathcal{T}_f Y$ is a morphism, then there is a vector bundle morphism $\tilde\theta : E \to f^*(TY)$ on X whose fibre at $x \in X$ with $f(x) = y$ in Y is $\tilde\theta|_x = \theta|_x : E|_x \to T_y Y$ from Definition 10.7.

Example 10.14. As in Chapter 2, Assumption 10.13 holds for tangent spaces $T_x X$ in $\mathbf{Man}, \mathbf{Man^c}$ and $\mathbf{Man^c_{we}}$ from Example 10.2(i),(ii), and for b-tangent spaces ${}^bT_x X$ in $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$ from Example 10.2(iii). But it fails for stratum tangent spaces $\tilde T_x X$ in $\mathbf{Man^c}, \ldots, \mathbf{Man^{c,ac}}$ from Example 10.2(iv).

In §2.6 we discussed orientations on objects X in $\mathbf{Man}, \mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$, using the vector bundles $T^*X \to X$ or ${}^bT^*X \to X$. Under Assumption 10.13 we can make the same definitions in $\dot{\mathbf{Man}}$.

Definition 10.15. Let Assumption 10.13 hold for $\dot{\mathbf{Man}}$. An *orientation* o_X on an object X in $\dot{\mathbf{Man}}$ is an equivalence class $[\omega]$ of top-degree forms ω in $\Gamma^\infty(\Lambda^{\dim X} T^*X)$ with $\omega|_x \neq 0$ for all $x \in X$, where two such ω, ω' are equivalent if $\omega' = K \cdot \omega$ for $K : X \to (0, \infty)$ smooth. The *opposite orientation* is $-o_X = [-\omega]$. Then we call (X, o_X) an *oriented manifold*. Usually we just refer to X as an oriented manifold, and then we write $-X$ for X with the opposite orientation.

We will call the real line bundle $\Lambda^{\dim X} T^*X \to X$ the *canonical bundle* K_X of X. Then an orientation on X is an orientation on the fibres of K_X.

If $x \in X$ and $(v_1, \ldots, v_m)$ is a basis for $T_x X$, then we call $(v_1, \ldots, v_m)$ *oriented* if $\omega|_x \cdot v_1 \wedge \cdots \wedge v_m > 0$, and *anti-oriented* otherwise.

Let $f : X \to Y$ be a morphism in $\dot{\mathbf{Man}}$. A *coorientation* c_f on f is an orientation on the fibres of the line bundle $K_X \otimes f^*(K_Y^*)$ over X. That is, c_f is an equivalence class $[\gamma]$ of nonvanishing sections $\gamma \in \Gamma^\infty(K_X \otimes f^*(K_Y^*))$, where two such γ, γ' are equivalent if $\gamma' = K \cdot \gamma$ for $K : X \to (0, \infty)$ smooth. The *opposite coorientation* is $-c_f = [-\gamma]$. If Y is oriented then coorientations on f are equivalent to orientations on X. Orientations on X are equivalent to coorientations on $\pi : X \to *$, for $*$ the point in $\dot{\mathbf{Man}}$.

The reason we need Assumption 10.13 to define orientations, is that the vector bundle structure on $TX \to X$ gives us a notion of when orientations on $T_x X$ *vary continuously with* $x \in X$, which does not follow from Assumption 10.1 alone. We will use Convention 2.39 in $\dot{\mathbf{Man}}$ whenever it makes sense.

Here is an extension of Assumption 10.13 to manifolds with corners:

Assumption 10.16. Let Assumption 3.22 hold for $\dot{\mathbf{Man}}^{\mathbf{c}}$. Suppose Assumptions 10.1 and 10.13 hold for $\dot{\mathbf{Man}}^{\mathbf{c}}$, so that from Assumption 10.1 we have a discrete property $\boldsymbol{A}$ of morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and tangent spaces T_xX for objects X in $\dot{\mathbf{Man}}^{\mathbf{c}}$ which are fibres of the tangent bundle $TX \to X$, and tangent maps $T_xf : T_xX \to T_yY$ for $\boldsymbol{A}$ morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}^{\mathbf{c}}$, which are fibres of the vector bundle morphism $Tf : TX \to f^*(TY)$.

Assumption 3.22 includes a discrete property of morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$ called *simple maps*. We require that all simple maps are $\boldsymbol{A}$.

We require that either **(a)** or **(b)** holds for $\dot{\mathbf{Man}}^{\mathbf{c}}$, where:

(a) For each X in $\dot{\mathbf{Man}}^{\mathbf{c}}$, so that by Assumption 10.1(d) we have the boundary ∂X with morphism $i_X : \partial X \to X$, we are given a canonical exact sequence of vector bundles on ∂X:

$$0 \longrightarrow N_{\partial X} \xrightarrow{\alpha_X} i_X^*(TX) \xrightarrow{\beta_X} T(\partial X) \longrightarrow 0, \tag{10.10}$$

where $N_{\partial X}$ is a line bundle (rank 1 vector bundle) on ∂X, and there is natural orientation on the fibres of $N_{\partial X}$. If $f : X \to Y$ is simple in $\dot{\mathbf{Man}}^{\mathbf{c}}$, so that we have $\partial f : \partial X \to \partial Y$ with $i_Y \circ \partial f = f \circ i_X$ by Assumption 10.1(g),(i), then the following commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{\partial X} & \xrightarrow{\alpha_X} & i_X^*(TX) & \xrightarrow{\beta_X} & T(\partial X) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \gamma_f} & & \Big\downarrow{\scriptstyle i_X^*(Tf)} & & \Big\downarrow{\scriptstyle T(\partial f)} & & \\ 0 & \longrightarrow & (\partial f)^*(N_{\partial Y}) & \xrightarrow{(\partial f)^*(\alpha_Y)} & \begin{array}{c} i_X^*(f^*(TY)) \\ =(\partial f)^*(i_Y^*(TY)) \end{array} & \xrightarrow{(\partial f)^*(\beta_Y)} & (\partial f)^*(T(\partial Y)) & \longrightarrow & 0. \end{array} \tag{10.11}$$

Here a unique γ_f making (10.11) commute exists by exactness, and we require that γ_f should be an orientation-preserving isomorphism.

If $g : X \to Z$ is a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$ with $Z \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}^{\mathbf{c}}$, so that g and $g \circ i_X : \partial X \to Y$ are $\boldsymbol{A}$ by Assumption 10.1(a)(i) and $Tg, T(g \circ i_X)$ are defined by Assumption 10.11(b), we have

$$i_X^*(Tg) = T(g \circ i_X) \circ \beta_X : i_X^*(TX) \longrightarrow (g \circ i_X)^*(TZ). \tag{10.12}$$

(b) For each X in $\dot{\mathbf{Man}}^{\mathbf{c}}$ we have an exact sequence of vector bundles on ∂X:

$$0 \longrightarrow T(\partial X) \xrightarrow{\alpha_X} i_X^*(TX) \xrightarrow{\beta_X} N_{\partial X} \longrightarrow 0, \tag{10.13}$$

where $N_{\partial X}$ is a line bundle on ∂X, with a natural orientation on its fibres. If $f : X \to Y$ is simple in $\dot{\mathbf{Man}}^{\mathbf{c}}$, then the following commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(\partial X) & \xrightarrow{\alpha_X} & i_X^*(TX) & \xrightarrow{\beta_X} & N_{\partial X} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle T(\partial f)} & & \Big\downarrow{\scriptstyle i_X^*(Tf)} & & \Big\downarrow{\scriptstyle \gamma_f} & & \\ 0 & \longrightarrow & (\partial f)^*(T(\partial Y)) & \xrightarrow{(\partial f)^*(\alpha_Y)} & \begin{array}{c} i_X^*(f^*(TY)) \\ =(\partial f)^*(i_Y^*(TY)) \end{array} & \xrightarrow{(\partial f)^*(\beta_Y)} & (\partial f)^*(N_{\partial Y}) & \longrightarrow & 0. \end{array} \tag{10.14}$$

Here a unique γ_f making (10.14) commute exists by exactness, and we require that γ_f should be an orientation-preserving isomorphism.

If $g : X \to Z$ is a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$ with $Z \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}^{\mathbf{c}}$, then $g, g \circ i_X$ are $\boldsymbol{A}$, and in a similar way to (10.15) we have

$$T(g \circ i_X) = i_X^*(Tg) \circ \alpha_X : T(\partial X) \longrightarrow (g \circ i_X)^*(TZ). \tag{10.15}$$

In both cases we interpret $N_{\partial X}$ as the normal bundle of ∂X in X. Our convention is that $N_{\partial X}$ should be oriented by *outward-pointing* vectors.

Example 10.17. (i) Let $\dot{\mathbf{Man}}^{\mathbf{c}}$ be $\mathbf{Man}^{\mathbf{c}}, \mathbf{Man}^{\mathbf{gc}}, \mathbf{Man}^{\mathbf{ac}}$ or $\mathbf{Man}^{\mathbf{c,ac}}$ from Chapter 2, and $\boldsymbol{A}$ be interior maps, and use b-tangent spaces ${}^bT_xX$ and the b-tangent bundle ${}^bTX$ from §2.3. Then Assumption 10.16(a) holds, where (10.10) is equation (2.14) for $\mathbf{Man}^{\mathbf{c}}$ and $\mathbf{Man}^{\mathbf{gc}}$ (when ${}^bN_{\partial X} = \mathcal{O}_{\partial X}$ is naturally trivial), and (2.19) for $\mathbf{Man}^{\mathbf{ac}}$ and $\mathbf{Man}^{\mathbf{c,ac}}$ (when ${}^bN_{\partial X}$ is not naturally trivial).

(ii) Let $\dot{\mathbf{Man}}^{\mathbf{c}}$ be $\mathbf{Man}^{\mathbf{c}}$ from §2.1, and $\boldsymbol{A}$ be trivial, and use ordinary tangent spaces T_xX and the tangent bundle TX from §2.3. Then Assumption 10.16(b) holds, where (10.13) is equation (2.12).

As in Convention 2.39(c), from an orientation on a manifold with corners X in $\dot{\mathbf{Man}}^{\mathbf{c}}$, we can define an orientation on ∂X.

Definition 10.18. Work in the situation of Assumption 10.16, and let $X \in \dot{\mathbf{Man}}^{\mathbf{c}}$ with $\dim X = n$. In both cases (a),(b) we will define an isomorphism

$$\Omega_X : \Lambda^{n-1}T^*(\partial X) \longrightarrow N_{\partial X} \otimes i_X^*(\Lambda^n T^*X) \tag{10.16}$$

of line bundles on ∂X. In case (a), so that we have an exact sequence (10.10), if $U \subseteq \partial X$ is an open subset on which $T(\partial X), i_X^*(TX), N_{\partial X}$ are trivial, and $(c_1), (d_1, \ldots, d_n)$, and $(e_2, \ldots, e_n)$ are bases of sections of $N_{\partial X}|_U, i_X^*(TX)|_U$, $T(\partial X)|_U$ respectively with $\alpha_X(c_1) = d_1$ and $\beta_X(d_i) = e_i$ for $i = 2, \ldots, n$, and $(\delta_1, \ldots, \delta_n), (\epsilon_2, \ldots, \epsilon_n)$ are the bases of sections of $i_X^*(T^*X)|_U, T^*(\partial X)|_U$ dual to $(d_1, \ldots, d_n), (e_2, \ldots, e_n)$, then we define $\Omega_X|_U$ by

$$\Omega_X|_U : \epsilon_2 \wedge \cdots \wedge \epsilon_n \longmapsto c_1 \otimes (\delta_1 \wedge \cdots \wedge \delta_n). \tag{10.17}$$

It is easy to show that $\Omega_X|_U$ is independent of the choice of bases, and that such $\Omega_X|_U$ glue over open subsets $U \subseteq X$ covering X to give a unique global isomorphism Ω_X in (10.16).

In case (b), so that we instead have an exact sequence (10.13), we again define $\Omega_X|_U$ using bases $(c_1), \ldots, (\epsilon_2, \ldots, \epsilon_n)$, as above, but now we instead require that $\alpha_X(e_i) = d_i$ for $i = 2, \ldots, n$ and $\beta_X(d_1) = c_1$.

If X is oriented, then we have an orientation on the fibres of $\Lambda^n T^*X \to X$, and thus on the fibres of $i_X^*(\Lambda^n T^*X) \to \partial X$. But by Assumption 10.16(a),(b), we have an orientation on the fibres of $N_{\partial X} \to \partial X$. Tensoring these orientations together and pulling back by Ω_X in (10.16) gives an orientation on the fibres of $\Lambda^{n-1}T^*(\partial X) \to \partial X$, that is, an orientation on the manifold with corners ∂X.

Note that defining this orientation on ∂X involves an *orientation convention*, as in Convention 2.39, which in this case is the choice of how to write (10.17), together with the choice to orient $N_{\partial X}$ by outward-pointing vectors.

If X is oriented then by induction $\partial^k X$ is oriented for $k = 0, \ldots, \dim X$.

10.1.5 Quasi-tangent spaces

In Definition 2.16, for a manifold with corners X and $x \in X$ we defined stratum (b-)normal spaces $\tilde N_x X, {}^b\tilde N_x X$ and a commutative monoid $\tilde M_x X \subseteq {}^b\tilde N_x X$, which are functorial under (interior) morphisms in $\mathbf{Man^c}$. In §2.4.1 the ${}^b\tilde N_x X, \tilde M_x X$ are extended to manifolds with g-corners. We call these *quasi-tangent spaces*, as they behave rather like tangent spaces. Here is an assumption that will enable us to extend quasi-tangent spaces to (m- and μ-)Kuranishi spaces in §10.3.

Assumption 10.19. (Quasi-tangent spaces.) (a) We are given a category $\mathcal{Q}$ of some algebraic or geometric objects, which quasi-tangent spaces will take values in. Some examples of categories $\mathcal{Q}$ we are interested in are:

- **(i)** Finite-dimensional real vector spaces V and linear maps $\lambda : V \to V'$.
- **(ii)** Monoids M with $M \cong \mathbb{N}^k$ for $k \geqslant 0$, and monoid morphisms $\mu : M \to M'$.
- **(iii)** Toric monoids M, and monoid morphisms $\mu : M \to M'$.

We require that $\mathcal{Q}$ should have a terminal object, which we write as 0. Products $Q_1 \times Q_2$ of objects Q_1, Q_2 in $\mathcal{Q}$ (that is, fibre products $Q_1 \times_0 Q_2$) exist in $\mathcal{Q}$, with the usual universal property. We require that if $\{Q_i : i \in I\}$ is a set of objects in $\mathcal{Q}$, and $q_{ij} : Q_i \to Q_j$ are isomorphisms in $\mathcal{Q}$ for all $i, j \in I$ such that $q_{ik} = q_{jk} \circ q_{ij}$ for all $i, j, k \in I$, then there should exist a natural object $Q = [\coprod_{i \in I} Q_i]/\sim$ in $\mathcal{Q}$ with canonical isomorphisms $q_i : Q \to Q_i$ for $i \in I$ such that $q_j = q_{ij} \circ q_i$ for all $i, j \in I$. We think of Q as the quotient of the disjoint union $\coprod_{i \in I} Q_i$ (which may not be an object of $\mathcal{Q}$) by the equivalence relation $\sim$ induced by the q_{ij}.

(b) We are given a discrete property $\boldsymbol{C}$ of morphisms in $\dot{\mathbf{Man}}$, in the sense of Definition 3.18, which may be trivial (i.e. all morphisms in $\dot{\mathbf{Man}}$ may be $\boldsymbol{C}$), and should satisfy:

- **(i)** If $f : X \to Y$ is a morphism in $\dot{\mathbf{Man}}$ with $Y \in \mathbf{Man}$, then f is $\boldsymbol{C}$.
- **(ii)** If $f : W \to Y$, $g : X \to Y$, $h : X \to Z$ are $\boldsymbol{C}$ morphisms in $\dot{\mathbf{Man}}$ then the product $f \times h : W \times X \to Y \times Z$ and direct product $(g, h) : X \to Y \times Z$ from Assumption 3.1(e) are also $\boldsymbol{C}$.

  Projections $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ from products are $\boldsymbol{C}$.

(c) For all $X \in \dot{\mathbf{Man}}$ and $x \in X$, we are given an object $Q_x X$ in $\mathcal{Q}$ called the *quasi-tangent space* of X at x. For all $\boldsymbol{C}$ morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}$ and all $x \in X$ with $f(x) = y$ in Y, we are given a morphism $Q_x f : Q_x X \to Q_y Y$ in $\mathcal{Q}$ called the *quasi-tangent map*. These satisfy:

(i) If $f : X \to Y$, $g : Y \to Z$ are $\boldsymbol{C}$ morphisms in $\dot{\mathbf{Man}}$ and $x \in X$ with $f(x) = y$ in Y and $g(y) = z$ in Z then $Q_x(g \circ f) = Q_y g \circ Q_x f : Q_x X \to Q_z Z$. Also $Q_x \mathrm{id}_X = \mathrm{id}_{Q_x X} : Q_x X \to Q_x X$.

(ii) For all $X, Y \in \dot{\mathbf{Man}}$ and $x \in X$, $y \in Y$ the morphism

$$\big(Q_{(x,y)}\pi_X, Q_{(x,y)}\pi_Y\big) : Q_{(x,y)}(X \times Y) \longrightarrow Q_x X \times Q_y Y \qquad (10.18)$$

is an isomorphism in $\mathcal{Q}$, where π_X, π_Y are $\boldsymbol{C}$ by **(b)(ii)**.

(iii) If $i : U \hookrightarrow X$ is an open submanifold in $\dot{\mathbf{Man}}$ then $Q_x i : Q_x U \to Q_x X$ is an isomorphism for all $x \in U \subseteq X$, so we may identify $Q_x U$ with $Q_x X$.

(iv) If $X \in \mathbf{Man} \subseteq \dot{\mathbf{Man}}$ is a classical manifold and $x \in X$ then $Q_x X = 0$.

(v) Let X, Y be objects of $\dot{\mathbf{Man}}$, and $E \to X$ a vector bundle, and $s \in \Gamma^\infty(E)$ a section, and $f, g : X \to Y$ be $\boldsymbol{C}$ morphisms in $\dot{\mathbf{Man}}$ with $g = f + O(s)$ as in Definition 3.15(iii). Suppose $x \in s^{-1}(0) \subseteq X$, so that $f(x) = g(x) = y \in Y$. Then $Q_x f = Q_x g : Q_x X \to Q_y Y$.

Example 10.20. **(a)** Take $\dot{\mathbf{Man}}$ to be $\mathbf{Man^c}$ from §2.1, and $\boldsymbol{C}$ to be trivial (i.e. all morphisms in $\mathbf{Man^c}$ are $\boldsymbol{C}$), and $\mathcal{Q}$ to be the category of finite-dimensional real vector spaces. Definition 2.16 defines the stratum normal space $\tilde{N}_x X$, an object in $\mathcal{Q}$, for all $X \in \mathbf{Man^c}$ and $x \in X$, and a linear map $\tilde{N}_x f : \tilde{N}_x X \to \tilde{N}_y Y$, a morphism in $\mathcal{Q}$, for all morphisms $f : X \to Y$ in $\mathbf{Man^c}$ and $x \in X$ with $f(x) = y \in Y$. These satisfy Assumption 10.19.

(b) Take $\dot{\mathbf{Man}}$ to be $\mathbf{Man^c}$ from §2.1, and $\boldsymbol{C}$ to be interior morphisms, and $\mathcal{Q}$ to be the category of finite-dimensional real vector spaces. Definition 2.16 defines the stratum b-normal space ${}^b\tilde{N}_x X$, an object in $\mathcal{Q}$, for all $X \in \mathbf{Man^c}$ and $x \in X$, and a morphism ${}^b\tilde{N}_x f : {}^b\tilde{N}_x X \to {}^b\tilde{N}_y Y$ in $\mathcal{Q}$, for all interior morphisms $f : X \to Y$ in $\mathbf{Man^c}$ and $x \in X$ with $f(x) = y \in Y$. These satisfy Assumption 10.19.

(c) Take $\dot{\mathbf{Man}}$ to be $\mathbf{Man^c}$ from §2.1, and $\boldsymbol{C}$ to be interior morphisms, and $\mathcal{Q}$ to be the category of commutative monoids M with $M \cong \mathbb{N}^k$ for some $k \geqslant 0$. Definition 2.16 defines an object $\tilde{M}_x X$ in $\mathcal{Q}$ for all $X \in \mathbf{Man^c}$ and $x \in X$, and a morphism $\tilde{M}_x f : \tilde{M}_x X \to \tilde{M}_y Y$ in $\mathcal{Q}$, for all interior morphisms $f : X \to Y$ in $\mathbf{Man^c}$ and $x \in X$ with $f(x) = y \in Y$. These satisfy Assumption 10.19.

(d) Take $\dot{\mathbf{Man}}$ to be $\mathbf{Man^{gc}}$ from §2.4.1, and $\boldsymbol{C}$ to be interior morphisms, and $\mathcal{Q}$ to be the category of finite-dimensional real vector spaces. As in §2.4.1, the ${}^b\tilde{N}_x X$ and ${}^b\tilde{N}_x f : {}^b\tilde{N}_x X \to {}^b\tilde{N}_y Y$ in **(b)** are also defined for $X, Y \in \mathbf{Man^{gc}}$. These satisfy Assumption 10.19.

(e) Take $\dot{\mathbf{Man}}$ to be $\mathbf{Man^{gc}}$ from §2.4.1, and $\boldsymbol{C}$ to be interior morphisms, and $\mathcal{Q}$ to be the category of toric commutative monoids M. As in §2.4.1, the $\tilde{M}_x X$ and $\tilde{M}_x f : \tilde{M}_x X \to \tilde{M}_y Y$ in **(c)** are also defined for $X, Y \in \mathbf{Man^{gc}}$, though now $\tilde{M}_x X$ may be general toric monoids. These satisfy Assumption 10.19.

10.2 The definition of tangent and obstruction spaces

In this section we suppose $\dot{\mathbf{Man}}$ satisfies Assumption 10.1 in §10.1.1 throughout, so that we are given a discrete property $\boldsymbol{A}$ (possibly trivial) of morphisms in $\dot{\mathbf{Man}}$, and 'manifolds' V in $\dot{\mathbf{Man}}$ have tangent spaces T_vV for $v \in V$, and $\boldsymbol{A}$ morphisms $f : V \to W$ in $\dot{\mathbf{Man}}$ have functorial tangent maps $T_vf : T_vV \to T_wW$ for all $v \in V$ with $f(v) = w \in W$. For each (m- or μ-)Kuranishi space $\boldsymbol{X}$ we will define a *tangent space* $T_x\boldsymbol{X}$ and *obstruction space* $O_x\boldsymbol{X}$ for $x \in \boldsymbol{X}$, which behave functorially under $\boldsymbol{A}$ (1-)morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{m}\dot{\mathbf{Kur}}$, $\boldsymbol{\mu}\dot{\mathbf{Kur}}$, or $\dot{\mathbf{Kur}}$.

If we also suppose Assumption 10.5, which says that $\dim T_vV = \dim V$, then these satisfy $\dim T_x\boldsymbol{X} - \dim O_x\boldsymbol{X} = \mathrm{vdim}\,\boldsymbol{X}$.

10.2.1 Tangent and obstruction spaces for m-Kuranishi spaces

We define tangent and obstruction spaces $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ for m-Kuranishi spaces.

Definition 10.21. Let $\boldsymbol{X} = (X, \mathcal{K})$ be an m-Kuranishi space, with $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\ i,j\in I}, \Lambda_{ijk,\ i,j,k\in I}\big)$ and $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat\phi_{ij})$, $\Lambda_{ijk} = [\acute V_{ijk}, \hat\lambda_{ijk}]$ for all $i, j, k \in I$, as in Definition 4.14, and let $x \in \boldsymbol{X}$.

For each $i \in I$ with $x \in \mathrm{Im}\,\psi_i$, set $v_i = \psi_i^{-1}(x)$, and define real vector spaces K_i^x, C_i^x by the exact sequence

$$0 \longrightarrow K_i^x \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}_{v_i}s_i} E_i|_{v_i} \longrightarrow C_i^x \longrightarrow 0, \tag{10.19}$$

where $\mathrm{d}_{v_i}s_i$ is as in Definition 10.6, so that K_i^x, C_i^x are the kernel and cokernel of $\mathrm{d}_{v_i}s_i$. If Assumption 10.5 holds then Definition 4.14(b) gives

$$\dim K_i^x - \dim C_i^x = \dim T_{v_i}V_i - \dim E_i|_{v_i} = \dim V_i - \mathrm{rank}\,E_i = \mathrm{vdim}\,\boldsymbol{X}. \tag{10.20}$$

For $i, j \in I$ with $x \in \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$ we have $v_i \in V_{ij} \subseteq V_i$ with $\phi_{ij}(v_i) = v_j$ in V_j. Proposition 4.34(d) and Definition 4.33 imply that ϕ_{ij} is $\boldsymbol{A}$ near v_i, so $T_{v_i}\phi_{ij} : T_{v_i}V_i \to T_{v_j}V_j$ is defined. Thus we may form a diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i^x & \longrightarrow & T_{v_i}V_i & \xrightarrow{\mathrm{d}_{v_i}s_i} & E_i|_{v_i} & \longrightarrow & C_i^x & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \kappa^x_{\Phi_{ij}}} & & \downarrow{\scriptstyle T_{v_i}\phi_{ij}} & & \downarrow{\scriptstyle \hat\phi_{ij}|_{v_i}} & & \downarrow{\scriptstyle \gamma^x_{\Phi_{ij}}} & & \\ 0 & \longrightarrow & K_j^x & \longrightarrow & T_{v_j}V_j & \xrightarrow{\mathrm{d}_{v_j}s_j} & E_j|_{v_j} & \longrightarrow & C_j^x & \longrightarrow & 0. \end{array} \tag{10.21}$$

By differentiating Definition 4.2(d) at v_i we see the central square of (10.21) commutes, so by exactness there are unique linear $\kappa^x_{\Phi_{ij}}, \gamma^x_{\Phi_{ij}}$ making (10.21) commute.

If $i,j,k \in I$ with $x \in \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$ then we have a diagram

$$
\begin{array}{ccccccccc}
0 \longrightarrow & K_i^x & \longrightarrow & T_{v_i}V_i & \xrightarrow{\mathrm{d}_{v_i}s_i} & E_i|_{v_i} & \longrightarrow & C_i^x & \longrightarrow 0 \\
& \kappa^x_{\Phi_{ik}} \Big(\downarrow \kappa^x_{\Phi_{ij}} & & T_{v_i}\phi_{ik} \Big(\downarrow T_{v_i}\phi_{ij} & \hat\lambda_{ijk}|_{v_i} \swarrow & \hat\phi_{ij}|_{v_i} \downarrow \Big) \hat\phi_{ik}|_{v_i} & & \gamma^x_{\Phi_{ij}} \downarrow \Big) \gamma^x_{\Phi_{ik}} & \\
0 \longrightarrow & K_j^x & \longrightarrow & T_{v_j}V_k & \xrightarrow{\mathrm{d}_{v_j}s_j} & E_j|_{v_j} & \longrightarrow & C_j^x & \longrightarrow 0 \\
& \downarrow \kappa^x_{\Phi_{jk}} & & \downarrow T_{v_j}\phi_{jk} & & \hat\phi_{jk}|_{v_j} \downarrow & & \gamma^x_{\Phi_{jk}} \downarrow & \\
0 \longrightarrow & K_k^x & \longrightarrow & T_{v_k}V_k & \xrightarrow{\mathrm{d}_{v_k}s_k} & E_k|_{v_k} & \longrightarrow & C_k^x & \longrightarrow 0,
\end{array}
\tag{10.22}
$$

which combines (10.21) for i,j and j,k and i,k. Note that (10.22) may not commute: we can have $\phi_{ik} \neq \phi_{jk} \circ \phi_{ij}$ and $\hat\phi_{ik} \neq \phi_{ij}^*(\hat\phi_{jk}) \circ \hat\phi_{ij}$ near v_i in V_i, allowing

$$T_{v_i}\phi_{ik} \neq T_{v_j}\phi_{jk} \circ T_{v_i}\phi_{ij} \quad \text{and} \quad \hat\phi_{ik}|_{v_i} \neq \hat\phi_{jk}|_{v_j} \circ \hat\phi_{ij}|_{v_i}.$$

The 2-morphism $\Lambda_{ijk} = [\acute V_{ijk}, \hat\lambda_{ijk}] : \Phi_{jk} \circ \Phi_{ij} \Rightarrow \Phi_{ik}$ includes a morphism $\hat\lambda_{ijk} : E_i|_{\acute V_{ijk}} \to \mathcal{T}_{\phi_{jk}\circ\phi_{ij}} V_k|_{\acute V_{ijk}}$, where $v_i \in \acute V_{ijk} \subseteq V_i$. Thus as in §10.1.2, we have a linear map $\hat\lambda_{ijk}|_{v_i} : E_i|_{v_i} \to T_{v_k}V_k$, the arrow '$\dashrightarrow$' in (10.22). Applying (10.7)–(10.8) to equation (4.1) for Λ_{ijk} at v_i yields

$$
\begin{aligned}
T_{v_i}\phi_{ik} &= T_{v_j}\phi_{jk} \circ T_{v_i}\phi_{ij} + \hat\lambda_{ijk}|_{v_i} \circ \mathrm{d}_{v_i}s_i : T_{v_i}V_i \longrightarrow T_{v_k}V_k, \\
\hat\phi_{ik}|_{v_i} &= \hat\phi_{jk}|_{v_j} \circ \hat\phi_{ij}|_{v_i} + \mathrm{d}_{v_k}s_k \circ \hat\lambda_{ijk}|_{v_i} : E_i|_{v_i} \longrightarrow E_k|_{v_k}.
\end{aligned}
\tag{10.23}
$$

Comparing (10.22) and (10.23) and using exactness in the rows of (10.22), we deduce that

$$\kappa^x_{\Phi_{ik}} = \kappa^x_{\Phi_{jk}} \circ \kappa^x_{\Phi_{ij}} \quad \text{and} \quad \gamma^x_{\Phi_{ik}} = \gamma^x_{\Phi_{jk}} \circ \gamma^x_{\Phi_{ij}}. \tag{10.24}$$

When $k = i$ we have $\Phi_{ii} = \mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$ by Definition 4.14(f), so $\kappa^x_{\Phi_{ii}} = \mathrm{id}_{K_i^x}$, $\gamma^x_{\Phi_{ii}} = \mathrm{id}_{C_i^x}$, and from (10.24) we see that $\kappa^x_{\Phi_{ij}}, \gamma^x_{\Phi_{ij}}$ are isomorphisms, with inverses $\kappa^x_{\Phi_{ji}}, \gamma^x_{\Phi_{ji}}$.

Define the *tangent space* $T_x\boldsymbol{X}$ and *obstruction space* $O_x\boldsymbol{X}$ of $\boldsymbol{X}$ at x by

$$T_x\boldsymbol{X} = \textstyle\coprod_{i \in I : x \in \operatorname{Im}\psi_i} K_i^x / \approx \quad \text{and} \quad O_x\boldsymbol{X} = \textstyle\coprod_{i \in I : x \in \operatorname{Im}\psi_i} C_i^x / \asymp, \tag{10.25}$$

where $\approx$ is the equivalence relation $k_i \approx k_j$ if $k_i \in K_i^x$ and $k_j \in K_j^x$ with $\kappa^x_{\Phi_{ij}}(k_i) = k_j$, and $\asymp$ the equivalence relation $c_i \asymp c_j$ if $c_i \in C_i^x$ and $c_j \in C_j^x$ with $\gamma^x_{\Phi_{ij}}(c_i) = c_j$. Here (10.24) and $\kappa^x_{\Phi_{ij}}, \gamma^x_{\Phi_{ij}}$ isomorphisms with $\kappa^x_{\Phi_{ii}} = \mathrm{id}$, $\gamma^x_{\Phi_{ii}} = \mathrm{id}$ imply that $\approx, \asymp$ are equivalence relations. Then $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ are real vector spaces with canonical isomorphisms $T_x\boldsymbol{X} \cong K_i^x$ and $O_x\boldsymbol{X} \cong C_i^x$ for each $i \in I$ with $x \in \operatorname{Im}\psi_i$; the work above is just to make the definition of $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ independent of the choice of i.

If Assumption 10.5 holds then (10.20) gives

$$\dim T_x\boldsymbol{X} - \dim O_x\boldsymbol{X} = \operatorname{vdim}\boldsymbol{X}. \tag{10.26}$$

The dual vector spaces of $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ will be called the *cotangent space*, written $T_x^*\boldsymbol{X}$, and the *coobstruction space*, written $O_x^*\boldsymbol{X}$.

By (10.19), for any $i \in I$ with $x \in \operatorname{Im}\psi_i$ we have a canonical exact sequence

$$0 \longrightarrow T_x\boldsymbol{X} \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}_{v_i}s_i} E_i|_{v_i} \longrightarrow O_x\boldsymbol{X} \longrightarrow 0. \tag{10.27}$$

More generally, the argument above shows that if (V_a, E_a, s_a, ψ_a) is any m-Kuranishi neighbourhood on $\boldsymbol{X}$ in the sense of §4.7 with $x \in \operatorname{Im}\psi_a$, we have a canonical exact sequence analogous to (10.27).

Now let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism of m-Kuranishi spaces which is $\boldsymbol{A}$ in the sense of §4.5, with notation (4.6), (4.7), (4.9), and let $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$, so we have $T_x\boldsymbol{X}, O_x\boldsymbol{X},\ T_y\boldsymbol{Y}, O_y\boldsymbol{Y}$. Suppose $i \in I$ with $x \in \operatorname{Im}\chi_i$ and $j \in J$ with $y \in \operatorname{Im}\psi_j$, so we have a morphism $\boldsymbol{f}_{ij} = (U_{ij}, f_{ij}, \hat f_{ij})$ in $\boldsymbol{f}$, where f_{ij} is $\boldsymbol{A}$ near $\chi_i^{-1}(\operatorname{Im}\psi_j)$ by Definitions 4.33 and 4.35. As for (10.21), consider the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & T_x\boldsymbol{X} & \longrightarrow & T_{u_i}U_i & \xrightarrow{\mathrm{d}_{u_i}r_i} & D_i|_{u_i} & \longrightarrow & O_x\boldsymbol{X} & \longrightarrow 0 \\
& \Big\downarrow{\scriptstyle T_x\boldsymbol{f}} & & \Big\downarrow{\scriptstyle T_{u_i}f_{ij}} & & \Big\downarrow{\scriptstyle \hat f_{ij}|_{u_i}} & & \Big\downarrow{\scriptstyle O_x\boldsymbol{f}} & \\
0 \longrightarrow & T_y\boldsymbol{Y} & \longrightarrow & T_{v_j}V_j & \xrightarrow{\mathrm{d}_{v_j}s_j} & E_j|_{v_j} & \longrightarrow & O_y\boldsymbol{Y} & \longrightarrow 0,
\end{array} \tag{10.28}$$

where the rows are (10.27) for $\boldsymbol{X}, x, i$ and $\boldsymbol{Y}, y, j$ and so are exact. As for (10.21) the central square commutes, so there are unique linear maps $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$ and $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$ making (10.28) commute. A similar argument to the proof of (10.24) above shows that these $T_x\boldsymbol{f}, O_x\boldsymbol{f}$ are independent of the choices of $i \in I$ and $j \in J$, and so are well defined.

If (U_a, D_a, r_a, χ_a) and (V_b, E_b, s_b, ψ_b) are any m-Kuranishi neighbourhoods on $\boldsymbol{X}, \boldsymbol{Y}$ respectively in the sense of §4.7 with $x \in \operatorname{Im}\psi_a$, $y \in \operatorname{Im}\psi_b$, and $\boldsymbol{f}_{ab} = (U_{ab}, f_{ab}, \hat f_{ab})$ is the 1-morphism of m-Kuranishi neighbourhoods over $\boldsymbol{f}$ given by Theorem 4.56(b), then setting $u_a = \chi_a^{-1}(x)$, $v_b = \psi_b^{-1}(y)$, the argument of (10.28) shows that the following commutes, with exact rows:

$$\begin{array}{ccccccccc}
0 \longrightarrow & T_x\boldsymbol{X} & \longrightarrow & T_{u_a}U_a & \xrightarrow{\mathrm{d}_{u_a}r_a} & D_a|_{u_a} & \longrightarrow & O_x\boldsymbol{X} & \longrightarrow 0 \\
& \Big\downarrow{\scriptstyle T_x\boldsymbol{f}} & & \Big\downarrow{\scriptstyle T_{u_a}f_{ab}} & & \Big\downarrow{\scriptstyle \hat f_{ab}|_{u_a}} & & \Big\downarrow{\scriptstyle O_x\boldsymbol{f}} & \\
0 \longrightarrow & T_y\boldsymbol{Y} & \longrightarrow & T_{v_b}V_b & \xrightarrow{\mathrm{d}_{v_b}s_b} & E_b|_{v_b} & \longrightarrow & O_y\boldsymbol{Y} & \longrightarrow 0.
\end{array} \tag{10.29}$$

Suppose $\boldsymbol{e} : \boldsymbol{X} \to \boldsymbol{Y}$ is another 1-morphism of m-Kuranishi spaces, and $\boldsymbol{\eta} = (\boldsymbol{\eta}_{ij,\ i\in I,\ j\in J}) : \boldsymbol{e} \Rightarrow \boldsymbol{f}$ is a 2-morphism, so that $\boldsymbol{e}$ is $\boldsymbol{A}$ by Proposition 4.36(a). Then for x, y, i, j as above, consider the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & T_x\boldsymbol{X} & \longrightarrow & T_{u_i}U_i & \xrightarrow{\mathrm{d}_{u_i}r_i} & D_i|_{u_i} & \longrightarrow & O_x\boldsymbol{X} & \longrightarrow 0 \\
& {\scriptstyle T_x\boldsymbol{e}}\Big\downarrow\Big\downarrow{\scriptstyle T_x\boldsymbol{f}} & & {\scriptstyle T_{u_i}e_{ij}}\Big\downarrow\Big\downarrow{\scriptstyle T_{u_i}f_{ij}} & \swarrow{\scriptstyle \hat\eta_{ij}|_{v_i}} & {\scriptstyle \hat e_{ij}|_{u_i}}\Big\downarrow\Big\downarrow{\scriptstyle \hat f_{ij}|_{u_i}} & & {\scriptstyle O_x\boldsymbol{e}}\Big\downarrow\Big\downarrow{\scriptstyle O_x\boldsymbol{f}} & \\
0 \longrightarrow & T_y\boldsymbol{Y} & \longrightarrow & T_{v_j}V_j & \xrightarrow{\mathrm{d}_{v_j}s_j} & E_j|_{v_j} & \longrightarrow & O_y\boldsymbol{Y} & \longrightarrow 0.
\end{array} \tag{10.30}$$

As for (10.23), applying (10.7)–(10.8) to (4.1) for $\boldsymbol{\eta}_{ij} = [\acute V_{ij}, \hat\eta_{ij}]$ at v_i yields

$$\begin{aligned}
T_{u_i}f_{ij} &= T_{u_i}e_{ij} + \hat\eta_{ij}|_{v_i} \circ \mathrm{d}_{v_i}s_i : T_{v_i}V_i \longrightarrow T_{v_j}V_j, \\
\hat f_{ij}|_{u_i} &= \hat e_{ij}|_{u_i} + \mathrm{d}_{v_j}s_j \circ \hat\eta_{ij}|_{v_i} : E_i|_{v_i} \longrightarrow E_j|_{v_j}.
\end{aligned} \tag{10.31}$$

As for (10.24), combining (10.30) and (10.31) yields

$$T_x\boldsymbol{e} = T_x\boldsymbol{f} \quad \text{and} \quad O_x\boldsymbol{e} = O_x\boldsymbol{f}. \tag{10.32}$$

Thus, the maps $T_x\boldsymbol{f}, O_x\boldsymbol{f}$ depend only on the $\boldsymbol{A}$ morphism $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathrm{Ho}(\mathbf{m\dot{K}ur})$, and on $x \in \boldsymbol{X}$.

Now suppose $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ is another $\boldsymbol{A}$ 1-morphism of m-Kuranishi spaces and $\boldsymbol{g}(y) = z \in \boldsymbol{Z}$. In a similar way to (10.22), considering the diagram

$$
\begin{array}{ccccccccc}
0 \longrightarrow & T_x\boldsymbol{X} & \longrightarrow & T_{u_i}U_i & \xrightarrow{\;\mathrm{d}_{u_i}r_i\;} & D_i|_{u_i} & \longrightarrow & O_x\boldsymbol{X} & \longrightarrow 0 \\
{\scriptstyle T_x(\boldsymbol{g}\circ\boldsymbol{f})} & \downarrow {\scriptstyle T_x\boldsymbol{f}} & {\scriptstyle T_{v_i}(g\circ f)_{ik}} & \downarrow {\scriptstyle T_{u_i}f_{ij}} & {\scriptstyle \hat\theta^{\boldsymbol{g},\boldsymbol{f}}_{ijk}|_{v_i}} & \downarrow {\scriptstyle \hat f_{ij}|_{u_i}} & {\scriptstyle (\widehat{g\circ f})_{ik}|_{u_i}} & \downarrow {\scriptstyle O_x\boldsymbol{f}} & {\scriptstyle O_X(\boldsymbol{g}\circ\boldsymbol{f})} \\
0 \longrightarrow & T_y\boldsymbol{Y} & \longrightarrow & T_{v_j}V_k & \xrightarrow{\;\mathrm{d}_{v_j}s_j\;} & E_j|_{v_j} & \longrightarrow & O_y\boldsymbol{Y} & \longrightarrow 0 \\
 & \downarrow {\scriptstyle T_y\boldsymbol{g}} & & \downarrow {\scriptstyle T_{v_j}g_{jk}} & & \downarrow {\scriptstyle \hat g_{jk}|_{v_j}} & & \downarrow {\scriptstyle O_y\boldsymbol{g}} & \\
0 \longrightarrow & T_z\boldsymbol{Z} & \longrightarrow & T_{w_k}W_k & \xrightarrow{\;\mathrm{d}_{w_k}t_k\;} & F_k|_{v_k} & \longrightarrow & O_z\boldsymbol{Z} & \longrightarrow 0,
\end{array}
$$

applying (10.7)–(10.8) to (4.1) for $\Theta^{\boldsymbol{g},\boldsymbol{f}}_{ijk} = [\acute{V}^{\boldsymbol{g},\boldsymbol{f}}_{ijk}, \hat\theta^{\boldsymbol{g},\boldsymbol{f}}_{ijk}]$ in (4.24), we show that

$$
\begin{aligned}
T_x(\boldsymbol{g} \circ \boldsymbol{f}) &= T_y\boldsymbol{g} \circ T_x\boldsymbol{f} : T_x\boldsymbol{X} \longrightarrow T_z\boldsymbol{Z}, \\
O_x(\boldsymbol{g} \circ \boldsymbol{f}) &= O_y\boldsymbol{g} \circ O_x\boldsymbol{f} : O_x\boldsymbol{X} \longrightarrow O_z\boldsymbol{Z}.
\end{aligned} \tag{10.33}
$$

Also

$$
\begin{aligned}
T_x\mathbf{id}_{\boldsymbol{X}} &= \mathrm{id}_{T_x\boldsymbol{X}} : T_x\boldsymbol{X} \longrightarrow T_x\boldsymbol{X}, \\
O_x\mathbf{id}_{\boldsymbol{X}} &= \mathrm{id}_{O_x\boldsymbol{X}} : O_x\boldsymbol{X} \longrightarrow O_x\boldsymbol{X}.
\end{aligned} \tag{10.34}
$$

So tangent and obstruction spaces are functorial on the 2-category $\mathbf{m\dot{K}ur}_{\boldsymbol{A}}$.

Example 10.22. Let $\boldsymbol{X}, \boldsymbol{Y}$ be m-Kuranishi spaces, so that Example 4.31 defines the product m-Kuranishi space $\boldsymbol{X} \times \boldsymbol{Y}$. In Definition 10.21, using Assumption 10.1(c) it is easy to see that for all $(x, y) \in \boldsymbol{X}\times\boldsymbol{Y}$ we have canonical isomorphisms

$$T_{(x,y)}(\boldsymbol{X} \times \boldsymbol{Y}) \cong T_x\boldsymbol{X} \oplus T_y\boldsymbol{Y}, \qquad O_{(x,y)}(\boldsymbol{X} \times \boldsymbol{Y}) \cong O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y}. \tag{10.35}$$

Lemma 10.23. *In Definition* 10.21 *suppose* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is an equivalence in* $\mathbf{m\dot{K}ur}$, *so that* $\boldsymbol{f}$ *is* $\boldsymbol{A}$ *by Proposition* 4.36(c). *Then* $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$ *and* $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$ *are isomorphisms for all* $x \in \boldsymbol{X}$ *with* $\boldsymbol{f}(x) = y$ *in* $\boldsymbol{Y}$.

Proof. As $\boldsymbol{f}$ is an equivalence there exist an equivalence $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{X}$ and 2-morphisms $\boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \mathbf{id}_{\boldsymbol{X}}$ and $\boldsymbol{\zeta} : \boldsymbol{f} \circ \boldsymbol{g} \Rightarrow \mathbf{id}_{\boldsymbol{Y}}$. If $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$ then $\boldsymbol{g}(y) = x$. From (10.33), and (10.32) for $\boldsymbol{\eta}$, and (10.34), we see that

$$
\begin{aligned}
T_y\boldsymbol{g} \circ T_x\boldsymbol{f} &= T_x(\boldsymbol{g} \circ \boldsymbol{f}) = T_x\mathbf{id}_{\boldsymbol{X}} = \mathrm{id}_{T_x\boldsymbol{X}}, \\
O_y\boldsymbol{g} \circ O_x\boldsymbol{f} &= O_x(\boldsymbol{g} \circ \boldsymbol{f}) = O_x\mathbf{id}_{\boldsymbol{X}} = \mathrm{id}_{O_x\boldsymbol{X}}.
\end{aligned}
$$

Similarly $T_x\boldsymbol{f} \circ T_y\boldsymbol{g} = \mathrm{id}_{T_y\boldsymbol{Y}}$ and $O_x\boldsymbol{f} \circ O_y\boldsymbol{g} = \mathrm{id}_{O_y\boldsymbol{Y}}$. Thus $T_y\boldsymbol{g}, O_y\boldsymbol{g}$ are inverses for $T_x\boldsymbol{f}, O_x\boldsymbol{f}$, and $T_x\boldsymbol{f}, O_x\boldsymbol{f}$ are isomorphisms. $\square$

Remark 10.24. (a) Even when $\dot{\mathbf{Man}} = \mathbf{Man}$, in contrast to classical manifolds, $\dim T_x\boldsymbol{X}, \dim O_x\boldsymbol{X}$ may not be locally constant functions of $x \in \boldsymbol{X}$, but only upper semicontinuous, so $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ are not fibres of vector bundles on $\boldsymbol{X}$.

(b) In applications, tangent and obstruction spaces will often have the following interpretation. Suppose an m-Kuranishi space $\boldsymbol{X}$ is the moduli space of solutions of a nonlinear elliptic equation on a compact manifold, written as $\boldsymbol{X} \cong \Phi^{-1}(0)$ for $\Phi : \mathcal{V} \to \mathcal{E}$ a Fredholm section of a Banach vector bundle $\mathcal{E} \to \mathcal{V}$ over a Banach manifold $\mathcal{V}$. Then $\mathrm{d}_x\Phi : T_x\mathcal{V} \to \mathcal{E}_x$ is a linear Fredholm map of Banach spaces for $x \in \boldsymbol{X}$, and $T_x\boldsymbol{X} \cong \mathrm{Ker}(\mathrm{d}_x\Phi)$, $O_x\boldsymbol{X} \cong \mathrm{Coker}(\mathrm{d}_x\Phi)$, so that $\dim T_x\boldsymbol{X} - \dim O_x\boldsymbol{X} = \mathrm{vdim}\,\boldsymbol{X}$ is the Fredholm index $\mathrm{ind}(\mathrm{d}_x\Phi)$.

Combining Definition 10.21 and Example 10.2 yields:

Example 10.25. (i) In the 2-categories $\mathbf{mKur}, \mathbf{mKur^c}, \mathbf{mKur^c_{we}}$ from (4.37), we have notions of *tangent space* $T_x\boldsymbol{X}$ and *obstruction space* $O_x\boldsymbol{X}$ satisfying $\dim T_x\boldsymbol{X} - \dim O_x\boldsymbol{X} = \mathrm{vdim}\,\boldsymbol{X}$, based on the usual notion of tangent spaces T_xX when $\dot{\mathbf{Man}}$ is $\mathbf{Man}, \mathbf{Man^c}$ or $\mathbf{Man^c_{we}}$. For any 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{mKur}, \mathbf{mKur^c}, \mathbf{mKur^c_{we}}$ we have functorial *tangent maps* $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$ and *obstruction maps* $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$ for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$.

(ii) In the 2-categories $\mathbf{mKur^c}, \mathbf{mKur^{gc}}, \mathbf{mKur^{ac}}, \mathbf{mKur^{c,ac}}$ from (4.37), we have notions of *b-tangent space* ${}^bT_x\boldsymbol{X}$ and *b-obstruction space* ${}^bO_x\boldsymbol{X}$ satisfying $\dim {}^bT_x\boldsymbol{X} - \dim {}^bO_x\boldsymbol{X} = \mathrm{vdim}\,\boldsymbol{X}$, based on b-tangent spaces ${}^bT_xX$ from §2.3–§2.4 for the categories $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$. For any interior 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{mKur^c}, \ldots, \mathbf{mKur^{c,ac}}$ we have functorial *b-tangent maps* ${}^bT_x\boldsymbol{f} : {}^bT_x\boldsymbol{X} \to {}^bT_y\boldsymbol{Y}$ and *b-obstruction maps* ${}^bO_x\boldsymbol{f} : {}^bO_x\boldsymbol{X} \to {}^bO_y\boldsymbol{Y}$ for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$. Since ${}^bT_x\boldsymbol{f}, {}^bO_x\boldsymbol{f}$ are defined only for interior 1-morphisms $\boldsymbol{f}$, it is better to think of b-tangent and b-obstruction spaces ${}^bT_xX, {}^bO_x\boldsymbol{X}$ as attached to the 2-subcategories $\mathbf{mKur^c_{in}}, \mathbf{mKur^{gc}_{in}}, \mathbf{mKur^{ac}_{in}}$, $\mathbf{mKur^{c,ac}_{in}}$ from Definition 4.37.

(iii) In the 2-categories $\mathbf{mKur^c}, \mathbf{mKur^{gc}}, \mathbf{mKur^{ac}}, \mathbf{mKur^{c,ac}}$ from (4.37), we have notions of *stratum tangent space* $\tilde{T}_x\boldsymbol{X}$ and *stratum obstruction space* $\tilde{O}_x\boldsymbol{X}$, based on stratum tangent spaces $\tilde{T}_xX$ from Example 10.2(iv) for the categories $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$. They satisfy $\dim \tilde{T}_x\boldsymbol{X} - \dim \tilde{O}_x\boldsymbol{X} \leqslant \mathrm{vdim}\,\boldsymbol{X}$, but equality may not hold.

For any 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{mKur^c}, \mathbf{mKur^{gc}}, \mathbf{mKur^{ac}}, \mathbf{mKur^{c,ac}}$ we have functorial *stratum tangent maps* $\tilde{T}_x\boldsymbol{f} : \tilde{T}_x\boldsymbol{X} \to \tilde{T}_y\boldsymbol{Y}$ and *stratum obstruction maps* $\tilde{O}_x\boldsymbol{f} : \tilde{O}_x\boldsymbol{X} \to \tilde{O}_y\boldsymbol{Y}$ for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$.

(iv) For any $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7, the corresponding 2-category of m-Kuranishi spaces $\mathbf{m\dot{K}ur}$ has notions of C^∞*-tangent space* $T_x^{C^\infty}\boldsymbol{X}$ and C^∞*-obstruction space* $O_x^{C^\infty}\boldsymbol{X}$, functorial for all 1-morphisms in $\mathbf{m\dot{K}ur}$, based on tangent spaces of C^∞-schemes as in Example 10.2(v). They are canonically isomorphic to $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ in **(i)** in those cases.

Definition 10.26. Suppose we are given two notions of tangent space T_xX, T_xf with discrete property $\boldsymbol{A}$, and T'_xX, T'_xf with discrete property $\boldsymbol{A}'$, in $\dot{\mathbf{Man}}$ satisfying Assumption 10.1, and a natural transformation $I : T \Rightarrow T'$, as in

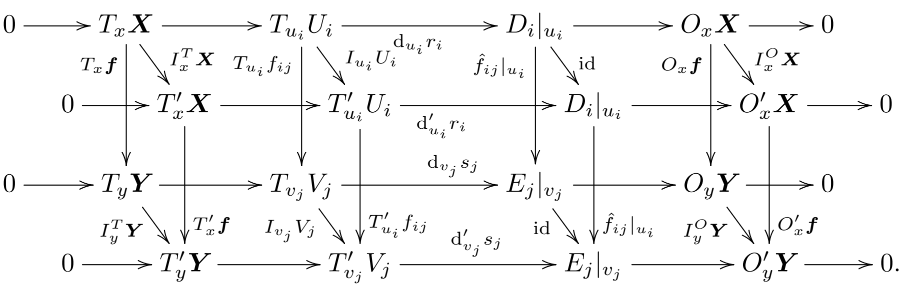
Definition 10.3. Then for each m-Kuranishi space $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$ and $x \in \boldsymbol{X}$, Definition 10.21 defines $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ and $T'_x\boldsymbol{X}, O'_x\boldsymbol{X}$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_x\boldsymbol{X} & \longrightarrow & T_{v_i}V_i & \xrightarrow{\mathrm{d}_{v_i}s_i} & E_i|_{v_i} & \longrightarrow & O_x\boldsymbol{X} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle I^T_x\boldsymbol{X}} & & \Big\downarrow{\scriptstyle I_{v_i}V_i} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle I^O_x\boldsymbol{X}} & & \\ 0 & \longrightarrow & T'_x\boldsymbol{X} & \longrightarrow & T'_{v_i}V_i & \xrightarrow{\mathrm{d}'_{v_i}s_i} & E_i|_{v_i} & \longrightarrow & O'_x\boldsymbol{X} & \longrightarrow & 0, \end{array} \tag{10.36}$$

where the rows are (10.27) for T, T', and are exact. Using Definitions 10.3 and 10.6 we can show that the central square of (10.36) commutes, so that by exactness there are unique linear maps $I^T_x\boldsymbol{X} : T_x\boldsymbol{X} \to T'_x\boldsymbol{X}$ and $I^O_x\boldsymbol{X} : O_x\boldsymbol{X} \to O'_x\boldsymbol{X}$ making (10.36) commute. One can show that these are independent of the choice of $i \in I$ as for (10.28).

Note that $I^O_x\boldsymbol{X}$ is always surjective. If $I_{v_i}V_i$ is injective then $I^T_x\boldsymbol{X}$ is injective. If $I_{v_i}V_i$ is surjective then $I^T_x\boldsymbol{X}$ is surjective and $I^O_x\boldsymbol{X}$ is an isomorphism.

Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism of m-Kuranishi spaces which is both $\boldsymbol{A}$ and $\boldsymbol{A}'$, with notation (4.6), (4.7), (4.9), let $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$, and consider the diagram



This combines (10.28) for T, T', and (10.36) for $\boldsymbol{X}, x$ and $\boldsymbol{Y}, y$. As the central cube commutes, by exactness the outer squares commute. That is, we have

$$I^T_y\boldsymbol{Y} \circ T_x\boldsymbol{f} = T'_x\boldsymbol{f} \circ I^T_x\boldsymbol{X} \quad \text{and} \quad I^O_y\boldsymbol{Y} \circ O_x\boldsymbol{f} = O'_x\boldsymbol{f} \circ I^O_x\boldsymbol{X}, \tag{10.37}$$

so the linear maps $I^T_x\boldsymbol{X}, I^O_x\boldsymbol{X}$ form natural transformations $I^T : T \Rightarrow T'$, $I^O : O \Rightarrow O'$ in $\mathbf{\dot{K}ur}$.

Combining Definition 10.26 and Examples 10.4 and 10.25 yields:

Example 10.27. **(a)** For $\boldsymbol{X}$ in $\mathbf{mKur^c}$ we have natural linear maps $I^T_x\boldsymbol{X} : {}^bT_x\boldsymbol{X} \to T_x\boldsymbol{X}$ and $I^O_x\boldsymbol{X} : {}^bO_x\boldsymbol{X} \to O_x\boldsymbol{X}$, for $T_x\boldsymbol{X}, O_x\boldsymbol{X}, {}^bT_x\boldsymbol{X}, {}^bO_X\boldsymbol{X}$ as in Example 10.25(i),(ii), where $I^O_x\boldsymbol{X}$ is always surjective.

(b) For $\boldsymbol{X}$ in $\mathbf{mKur^c}$ we have natural linear maps $\iota^T_x\boldsymbol{X} : \tilde T_x\boldsymbol{X} \to T_x\boldsymbol{X}$ and $\iota^O_x\boldsymbol{X} : \tilde O_x\boldsymbol{X} \to O_x\boldsymbol{X}$, for $T_x\boldsymbol{X}, O_x\boldsymbol{X}, \tilde T_x\boldsymbol{X}, \tilde O_X\boldsymbol{X}$ as in Example 10.25(i),(iii), where $\iota^T_x\boldsymbol{X}$ is always injective and $\iota^O_x\boldsymbol{X}$ is surjective.

(c) For $\boldsymbol{X}$ in any of $\mathbf{mKur^c}, \mathbf{mKur^{gc}}, \mathbf{mKur^{ac}}, \mathbf{mKur^{c,ac}}$, there are natural linear maps $\Pi^T_x\boldsymbol{X} : {}^bT_x\boldsymbol{X} \to \tilde T_x\boldsymbol{X}$ and $\Pi^O_x\boldsymbol{X} : {}^bO_x\boldsymbol{X} \to \tilde O_x\boldsymbol{X}$, for ${}^bT_x\boldsymbol{X}, {}^bO_x\boldsymbol{X}, \tilde T_x\boldsymbol{X}, \tilde O_X\boldsymbol{X}$ as in Example 10.25(ii),(iii), where $\Pi^T_x\boldsymbol{X}$ is always surjective and $\Pi^O_x\boldsymbol{X}$ is an isomorphism.

10.2.2 Tangent and obstruction spaces for μ-Kuranishi spaces

For μ-Kuranishi spaces in Chapter 5, by essentially exactly the same arguments as in §10.2.1, if $\dot{\mathbf{Man}}$ satisfies Assumption 10.1 with discrete property $\boldsymbol{A}$ then:

(a) For each μ-Kuranishi space $\boldsymbol{X}$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ and $x \in \boldsymbol{X}$ we can define the *tangent space* $T_x\boldsymbol{X}$ and *obstruction space* $O_x\boldsymbol{X}$, both real vector spaces.

(b) If Assumption 10.5 holds then $\dim T_x\boldsymbol{X} - \dim O_x\boldsymbol{X} = \operatorname{vdim}\boldsymbol{X}$.

(c) For each $\boldsymbol{A}$ morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ and $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$ we can define linear maps $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$ and $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$. These are functorial, that is, (10.33)–(10.34) hold.

(d) The analogues of Lemma 10.23, Examples 10.25, 10.27, Definition 10.26 hold.

10.2.3 Tangent and obstruction spaces for Kuranishi spaces

In §6.5, for a Kuranishi space $\boldsymbol{X}$ in $\dot{\mathbf{Kur}}$ and $x \in \boldsymbol{X}$ we defined a finite group $G_x\boldsymbol{X}$ called the *isotropy group*. It depends on arbitrary choices, and is natural up to isomorphism, but not up to canonical isomorphism.

Supposing Assumption 10.1 with discrete property $\boldsymbol{A}$, in §10.2.1, for an m-Kuranishi space $\boldsymbol{X}$, we defined a tangent space $T_x\boldsymbol{X}$ and an obstruction space $O_x\boldsymbol{X}$ for each $x \in \boldsymbol{X}$, which were unique up to canonical isomorphism and behaved functorially under $\boldsymbol{A}$ 1-morphisms and 2-morphisms of m-Kuranishi spaces. To define tangent and obstruction spaces for Kuranishi spaces, we must combine these two stories:

Definition 10.28. Let $\boldsymbol{X} = (X, \mathcal{K})$ be a Kuranishi space, with $\mathcal{K} = \big(I, (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\ i,j\in I}, \Lambda_{ijk,\ i,j,k\in I}\big)$, and let $x \in \boldsymbol{X}$.

In Definition 6.49 we defined the isotropy group $G_x\boldsymbol{X}$ by choosing $i \in I$ with $x \in \operatorname{Im}\psi_i$ and $v_i \in s_i^{-1}(0) \subseteq V_i$ with $\bar\psi_i(v_i) = x$, and setting $G_x\boldsymbol{X} = \operatorname{Stab}_{\Gamma_i}(v_i)$ as in (6.40). For these i, v_i, define the *tangent space* $T_x\boldsymbol{X}$ and *obstruction space* $O_x\boldsymbol{X}$ to be the kernel and cokernel of $\mathrm{d}_{v_i}s_i$, where $\mathrm{d}_{v_i}s_i$ is as in Definition 10.6, so that as in (10.27) we have an exact sequence

$$0 \longrightarrow T_x\boldsymbol{X} \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}_{v_i}s_i} E_i|_{v_i} \longrightarrow O_x\boldsymbol{X} \longrightarrow 0. \tag{10.38}$$

The actions of Γ_i on V_i, E_i induce linear actions of $G_x\boldsymbol{X}$ on $T_x\boldsymbol{X}, O_x\boldsymbol{X}$, by the commutative diagram for each $\gamma \in G_x\boldsymbol{X}$:

$$\begin{array}{ccccccccc} 0 \longrightarrow & T_x\boldsymbol{X} & \longrightarrow & T_{v_i}V_i & \xrightarrow{\mathrm{d}_{v_i}s_i} & E_i|_{v_i} & \longrightarrow & O_x\boldsymbol{X} & \longrightarrow 0 \\ & \downarrow{\scriptstyle \gamma\cdot} & & \downarrow{\scriptstyle T_{v_i}(\gamma\cdot)} & & \downarrow{\scriptstyle \gamma\cdot} & & \downarrow{\scriptstyle \gamma\cdot} & \\ 0 \longrightarrow & T_x\boldsymbol{X} & \longrightarrow & T_{v_i}V_i & \xrightarrow{\mathrm{d}_{v_i}s_i} & E_i|_{v_i} & \longrightarrow & O_x\boldsymbol{X} & \longrightarrow 0. \end{array}$$

This makes $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ into representations of $G_x\boldsymbol{X}$. The dual vector spaces of $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ are the *cotangent space* $T_x^*\boldsymbol{X}$ and the *coobstruction space* $O_x^*\boldsymbol{X}$.

If Assumption 10.5 holds then (10.38) implies that

$$\dim T_x\boldsymbol{X} - \dim O_x\boldsymbol{X} = \operatorname{vdim}\boldsymbol{X}. \tag{10.39}$$

Generalizing the discussion of Definition 6.49 on how $G_x\boldsymbol{X}$ depends on the choice of i, v_i, we can show that if $(G_x\boldsymbol{X}, T_x\boldsymbol{X}, O_x\boldsymbol{X})$ come from i, v_i, and $(G'_x\boldsymbol{X}, T'_x\boldsymbol{X}, O'_x\boldsymbol{X})$ come from alternative choices j, v_j, then by picking a point p in S_x in (6.41), we can define an isomorphism of triples

$$(I^G_x, I^T_x, I^O_x) : (G_x\boldsymbol{X}, T_x\boldsymbol{X}, O_x\boldsymbol{X}) \longrightarrow (G'_x\boldsymbol{X}, T'_x\boldsymbol{X}, O'_x\boldsymbol{X}).$$

If we instead picked $\tilde p \in S_x$ giving $(\tilde I^G_x, \tilde I^T_x, \tilde I^O_x)$, then there is a unique $\delta \in G'_x\boldsymbol{X}$ with $\delta \cdot p = \tilde p$, and we can show that $\tilde I^G_x(\gamma) = \delta I^G_x(\gamma)\delta^{-1}$, $\tilde I^T_x(v) = \delta \cdot I^T_x(v)$ and $\tilde I^O_x(w) = \delta \cdot I^O_x(w)$ for all $\gamma \in G_x\boldsymbol{X}$, $v \in T_x\boldsymbol{X}$, and $w \in O_x\boldsymbol{X}$. Such isomorphisms of triples behave as expected under compositions.

Now let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be an $\boldsymbol{A}$ 1-morphism in $\dot{\mathbf{K}}\mathbf{ur}$, with notation (6.15), (6.16), (6.18), and let $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$. As above we define $G_x\boldsymbol{X}, T_x\boldsymbol{X}, O_x\boldsymbol{X}$ using $i \in I$ and $u_i \in U_i$ with $\bar\chi_i(u_i) = x$, and $G_y\boldsymbol{Y}, T_y\boldsymbol{Y}, O_y\boldsymbol{Y}$ using $j \in J$ and $v_j \in V_j$ with $\bar\psi_j(v_j) = y$. By picking $p \in S_{x,\boldsymbol{f}}$ in (6.44), Definition 6.51 defines a group morphism $G_x\boldsymbol{f} : G_x\boldsymbol{X} \to G_y\boldsymbol{Y}$. As for (10.28), using the same p, define $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$, $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$ by the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & T_x\boldsymbol{X} & \longrightarrow & T_{u_i}U_i & \xrightarrow{\mathrm{d}_{u_i} r_i} & D_i|_{u_i} & \longrightarrow & O_x\boldsymbol{X} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle T_x\boldsymbol{f}} & & \Big\downarrow{\scriptstyle T_p f_{ij}\circ(T_p\pi_{ij})^{-1}} & & \Big\downarrow{\scriptstyle \hat f_{ij}|_p} & & \Big\downarrow{\scriptstyle O_x\boldsymbol{f}} & & \\
0 & \longrightarrow & T_y\boldsymbol{Y} & \longrightarrow & T_{v_j}V_j & \xrightarrow{\mathrm{d}_{v_j} s_j} & E_j|_{v_j} & \longrightarrow & O_y\boldsymbol{Y} & \longrightarrow & 0.
\end{array}$$

Then $T_x\boldsymbol{f}, O_x\boldsymbol{f}$ are $G_x\boldsymbol{f}$-equivariant linear maps.

Generalizing Definition 6.51, if $\tilde p \in S_{x,\boldsymbol{f}}$ is an alternative choice yielding $\tilde G_x\boldsymbol{f}, \tilde T_x\boldsymbol{f}, \tilde O_x\boldsymbol{f}$, there is a unique $\delta \in G_y\boldsymbol{Y}$ with $\delta \cdot p = \tilde p$, and then $\tilde G_x\boldsymbol{f}(\gamma) = \delta(G_x\boldsymbol{f}(\gamma))\delta^{-1}$, $\tilde T_x\boldsymbol{f}(v) = \delta \cdot T_x\boldsymbol{f}(v)$, $\tilde O_x\boldsymbol{f}(w) = \delta \cdot O_x\boldsymbol{f}(w)$ for all $\gamma \in G_x\boldsymbol{X}$, $v \in T_x\boldsymbol{X}$, and $w \in O_x\boldsymbol{X}$. That is, the triple $(G_x\boldsymbol{f}, T_x\boldsymbol{f}, O_x\boldsymbol{f})$ is canonical up to conjugation by an element of $G_y\boldsymbol{Y}$.

Continuing with the same notation, suppose $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ is another 1-morphism and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism in $\dot{\mathbf{K}}\mathbf{ur}$. Then $\boldsymbol{g}$ is $\boldsymbol{A}$ by Proposition 6.34(a), so as above we define $G_x\boldsymbol{g}, T_x\boldsymbol{g}, O_x\boldsymbol{g}$ by choosing $q \in S_{x,\boldsymbol{g}}$. As in Definition 6.51, if $\boldsymbol{\eta}_{ij}$ in $\boldsymbol{\eta}$ is represented by $(\dot P_{ij}, \eta_{ij}, \hat\eta_{ij})$, there is a unique element $G_x\boldsymbol{\eta} \in G_y\boldsymbol{Y}$ with $G_x\boldsymbol{\eta} \cdot \eta_{ij}(p) = q$. One can now check that

$$\begin{aligned}
&G_x\boldsymbol{g}(\gamma) = (G_x\boldsymbol{\eta})(G_x\boldsymbol{f}(\gamma))(G_x\boldsymbol{\eta})^{-1}, \quad T_x\boldsymbol{g}(v) = G_x\boldsymbol{\eta} \cdot T_x\boldsymbol{f}(v), \quad \text{and} \\
&O_x\boldsymbol{g}(w) = G_x\boldsymbol{\eta} \cdot O_x\boldsymbol{f}(w) \quad \text{for all } \gamma \in G_x\boldsymbol{X},\ v \in T_x\boldsymbol{X}, \text{ and } w \in O_x\boldsymbol{X}.
\end{aligned}$$

That is, $(G_x\boldsymbol{g}, T_x\boldsymbol{g}, O_x\boldsymbol{g})$ is conjugate to $(G_x\boldsymbol{f}, T_x\boldsymbol{f}, O_x\boldsymbol{f})$ under $G_x\boldsymbol{\eta} \in G_y\boldsymbol{Y}$, the same indeterminacy as in the definition of $(G_x\boldsymbol{f}, T_x\boldsymbol{f}, O_x\boldsymbol{f})$.

Suppose instead that $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ is another $\boldsymbol{A}$ 1-morphism of Kuranishi spaces and $\boldsymbol{g}(y) = z \in \boldsymbol{Z}$. Then as in Definition 6.51 there is a canonical element

$G_{x,\boldsymbol{g},\boldsymbol{f}} \in G_z\boldsymbol{Z}$ such that for all $\gamma \in G_x\boldsymbol{X}$, $v \in T_x\boldsymbol{X}$, $w \in O_x\boldsymbol{X}$ we have

$$\begin{aligned} G_x(\boldsymbol{g}\circ\boldsymbol{f})(\gamma) &= (G_{x,\boldsymbol{g},\boldsymbol{f}})((G_y\boldsymbol{g}\circ G_x\boldsymbol{f})(\gamma))(G_{x,\boldsymbol{g},\boldsymbol{f}})^{-1}, \\ T_x(\boldsymbol{g}\circ\boldsymbol{f})(v) &= G_{x,\boldsymbol{g},\boldsymbol{f}}\cdot(T_y\boldsymbol{g}\circ T_x\boldsymbol{f})(v), \\ O_x(\boldsymbol{g}\circ\boldsymbol{f})(w) &= G_{x,\boldsymbol{g},\boldsymbol{f}}\cdot(O_y\boldsymbol{g}\circ O_x\boldsymbol{f})(w). \end{aligned}$$

That is, $(G_x(\boldsymbol{g}\circ\boldsymbol{f}), T_x(\boldsymbol{g}\circ\boldsymbol{f}), O_x(\boldsymbol{g}\circ\boldsymbol{f}))$ is conjugate to $(G_y\boldsymbol{g}, T_y\boldsymbol{g}, O_y\boldsymbol{g})\circ(G_x\boldsymbol{f}, T_x\boldsymbol{f}, O_x\boldsymbol{f})$ under $G_{x,\boldsymbol{g},\boldsymbol{f}} \in G_z\boldsymbol{Z}$.

Remark 10.29. The definitions of $G_x\boldsymbol{X}, T_x\boldsymbol{X}, O_x\boldsymbol{X}, G_x\boldsymbol{f}, T_x\boldsymbol{f}, O_x\boldsymbol{f}$ above depend on arbitrary choices. We could use the Axiom of (Global) Choice as in Remark 4.21 to choose particular values for $G_x\boldsymbol{X}, \ldots, O_x\boldsymbol{f}$ for all $\boldsymbol{X}, x, \boldsymbol{f}$. But this is not really necessary, we can just bear the non-uniqueness in mind when working with them. All the definitions we make using $G_x\boldsymbol{X}, \ldots, O_x\boldsymbol{f}$ will be independent of the arbitrary choices in Definition 10.28.

The analogues of Lemma 10.23, Examples 10.25 and 10.27, and Definition 10.26 hold for our 2-categories of Kuranishi spaces.

10.3 Quasi-tangent spaces

In this section we suppose $\dot{\mathbf{Man}}$ satisfies Assumption 10.19 in §10.1.5 throughout, so that we are given a discrete property $\boldsymbol{C}$ (possibly trivial) of morphisms in $\dot{\mathbf{Man}}$, and 'manifolds' V in $\dot{\mathbf{Man}}$ have quasi-tangent spaces Q_vV for $v \in V$, which are objects in a category $\mathcal{Q}$, and $\boldsymbol{C}$ morphisms $f : V \to W$ in $\dot{\mathbf{Man}}$ have functorial quasi-tangent maps $Q_vf : Q_vV \to Q_wW$ for all $v \in V$ with $f(v) = w \in W$, which are morphisms in $\mathcal{Q}$.

For each (m- or μ-)Kuranishi space $\boldsymbol{X}$ we will define a *quasi-tangent space* $Q_x\boldsymbol{X}$ for $x \in \boldsymbol{X}$, with functorial morphisms $Q_x\boldsymbol{f} : Q_x\boldsymbol{X} \to Q_y\boldsymbol{Y}$ under $\boldsymbol{C}$ (1-)morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}, \boldsymbol{\mu}\mathbf{\dot{K}ur}$, or $\mathbf{\dot{K}ur}$. Unlike $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ in §10.2, there is no 'obstruction' version of $Q_x\boldsymbol{X}$. These $Q_x\boldsymbol{X}, Q_x\boldsymbol{f}$ are useful for imposing conditions on objects and (1-)morphisms in $\mathbf{m\dot{K}ur}, \boldsymbol{\mu}\mathbf{\dot{K}ur}$, and $\mathbf{\dot{K}ur}$, for instance in defining (w-)transversality and (w-)submersions in Chapter 11.

10.3.1 Quasi-tangent spaces for m-Kuranishi spaces

Here is the analogue of Definition 10.21 for quasi-tangent spaces:

Definition 10.30. Let $\boldsymbol{X} = (X, \mathcal{K})$ be an m-Kuranishi space, with $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\, i,j\in I}, \Lambda_{ijk,\, i,j,k\in I}\big)$ and $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat\phi_{ij})$, $\Lambda_{ijk} = [\acute V_{ijk}, \hat\lambda_{ijk}]$ for all $i, j, k \in I$, as in Definition 4.14, and let $x \in \boldsymbol{X}$.

For each $i \in I$ with $x \in \operatorname{Im}\psi_i$, set $v_i = \psi_i^{-1}(x)$ in $s_i^{-1}(0) \subseteq V_i$, so that we have an object $Q_{v_i}V_i$ in $\mathcal{Q}$ by Assumption 10.19(c). For $i, j \in I$ with $x \in \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ we have $v_i \in V_{ij} \subseteq V_i$ with $\phi_{ij} = v_j \in V_j$. Proposition 4.34(d) and Definition 4.33 imply that ϕ_{ij} is $\boldsymbol{C}$ near v_i, so $Q_{v_i}\phi_{ij} : Q_{v_i}V_i \to Q_{v_j}V_j$ is defined. When $j = i$ we have $\phi_{ii} = \mathrm{id}_{V_i}$, so $Q_{v_i}\phi_{ii} = \mathrm{id}_{Q_{v_i}V_i}$.

If $i,j,k\in I$ with $x\in \operatorname{Im}\psi_i\cap \operatorname{Im}\psi_j\cap \operatorname{Im}\psi_k$, Definition 4.3(b) for $\Lambda_{ijk}:\Phi_{jk}\circ\Phi_{ij}\Rightarrow\Phi_{ik}$ implies that $\phi_{ik}=\phi_{jk}\circ\phi_{ij}+O(s_i)$ near v_i, so

$$Q_{v_i}\phi_{ik}=Q_{v_j}\phi_{jk}\circ Q_{v_i}\phi_{ij}:Q_{v_i}V_i\longrightarrow Q_{v_j}V_j$$

by Assumption 10.19(c)(i),(v). Putting $k=i$ gives $Q_{v_j}\phi_{ji}\circ Q_{v_i}\phi_{ij}=\mathrm{id}_{Q_{v_i}V_i}$, and similarly $Q_{v_i}\phi_{ij}\circ Q_{v_j}\phi_{ji}=\mathrm{id}_{Q_{v_j}V_j}$, so $Q_{v_i}\phi_{ij}$ is an isomorphism. Hence by Assumption 10.19(a), we may define a natural object $Q_x\boldsymbol{X}$ in $\mathcal{Q}$ by

$$Q_x\boldsymbol{X}=\bigl[\textstyle\coprod_{i\in I:x\in\operatorname{Im}\psi_i}Q_{v_i}V_i\bigr]/\sim, \tag{10.40}$$

as in (10.25), where the equivalence relation $\sim$ is induced by the isomorphisms $Q_{v_i}\phi_{ij}:Q_{v_i}V_i\rightarrow Q_{v_j}V_j$, and there are canonical isomorphisms $Q_{x,i}:Q_x\boldsymbol{X}\rightarrow Q_{v_i}V_i$ in $\mathcal{Q}$ with $Q_{x,j}=Q_{v_i}\phi_{ij}\circ Q_{x,i}$ for all $i,j\in I$ with $x\in\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j$. We call $Q_x\boldsymbol{X}$ the *quasi-tangent space* of X at x.

More generally, if (V_a,E_a,s_a,ψ_a), $\Phi_{ai,\,i\in I}$, $\Lambda_{aij,\,i,j\in I}$ is any m-Kuranishi neighbourhood on $\boldsymbol{X}$ in the sense of §4.7 with $x\in\operatorname{Im}\psi_a$, and $v_a=\psi_a^{-1}(x)$, there is a canonical isomorphism $Q_{x,a}:Q_x\boldsymbol{X}\rightarrow Q_{v_a}V_a$ with $Q_{x,i}=Q_{v_a}\phi_{ai}\circ Q_{x,a}$ for all $i\in I$ with $x\in\operatorname{Im}\psi_i$.

Now let $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ be a 1-morphism of m-Kuranishi spaces which is $\boldsymbol{C}$ in the sense of §4.5, with notation (4.6), (4.7), (4.9), and let $x\in\boldsymbol{X}$ with $\boldsymbol{f}(x)=y$ in $\boldsymbol{Y}$, so we have objects $Q_x\boldsymbol{X},Q_y\boldsymbol{Y}$ in $\mathcal{Q}$. We claim that there is a unique morphism $Q_x\boldsymbol{f}:Q_x\boldsymbol{X}\rightarrow Q_y\boldsymbol{Y}$ in $\mathcal{Q}$, called the *quasi-tangent map*, such that the following diagram commutes:

$$\begin{array}{ccc} Q_x\boldsymbol{X} & \xrightarrow{\quad Q_x\boldsymbol{f}\quad} & Q_y\boldsymbol{Y} \\ {\scriptstyle Q_{x,i}}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle Q_{y,j}}\big\downarrow{\scriptstyle\cong} \\ Q_{u_i}U_i & \xrightarrow{\quad Q_{u_i}f_{ij}\quad} & Q_{v_j}V_j \end{array} \tag{10.41}$$

whenever $i\in I$ with $x\in\operatorname{Im}\chi_i$ and $u_i=\chi_i^{-1}(x)$, and $j\in J$ with $y\in\operatorname{Im}\psi_j$ and $v_j=\psi_j^{-1}(y)$. To see this, note that for fixed i,j there is a unique $Q_x\boldsymbol{f}$ making (10.41) commute. To show this $Q_x\boldsymbol{f}$ is independent of i,j, let i' be an alternative choice for i. From Definition 4.3(b) applied to the 2-morphism $\boldsymbol{F}_{ii'}^j:\boldsymbol{f}_{i'j}\circ\mathrm{T}_{ii'}\Rightarrow\boldsymbol{f}_{ij}$ in Definition 4.17(c), we see that $f_{i'j}\circ\tau_{ii'}=f_{ij}+O(r_i)$ near u_i in U_i, so $Q_{u_{i'}}f_{i'j}\circ Q_{u_i}\tau_{ii'}=Q_{u_i}f_{ij}$ by Assumption 10.19(c)(i),(v). Together with $Q_{x,i'}=Q_{u_i}\tau_{ii'}\circ Q_{x,i}$, this implies that $Q_x\boldsymbol{f}$ is unchanged by replacing i by i' in (10.41). Similarly, using $\boldsymbol{F}_i^{jj'}:\Upsilon_{jj'}\circ\boldsymbol{f}_{ij}\Rightarrow\boldsymbol{f}_{ij'}$ in Definition 4.17(d) we can show that $Q_x\boldsymbol{f}$ is unchanged by replacing j by an alternative choice j'.

More generally, if (U_a,D_a,r_a,χ_a), (V_b,E_b,s_b,ψ_b) are m-Kuranishi neighbourhoods on $\boldsymbol{X},\boldsymbol{Y}$ with $x\in\operatorname{Im}\chi_a$, $y\in\operatorname{Im}\psi_b$, and $\boldsymbol{f}_{ab}=(U_{ab},f_{ab},\hat f_{ab}):(U_a,D_a,r_a,\chi_a)\rightarrow(V_b,E_b,s_b,\psi_b)$ is a 1-morphism over $(S,\boldsymbol{f})$ for open $x\in S\subseteq\operatorname{Im}\chi_a\cap f^{-1}(\operatorname{Im}\psi_b)$ as in Theorem 4.56(b), then the following commutes:

$$\begin{array}{ccc} Q_x\boldsymbol{X} & \xrightarrow{\quad Q_x\boldsymbol{f}\quad} & Q_y\boldsymbol{Y} \\ {\scriptstyle Q_{x,a}}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle Q_{y,b}}\big\downarrow{\scriptstyle\cong} \\ Q_{u_a}U_a & \xrightarrow{\quad Q_{u_a}f_{ab}\quad} & Q_{v_b}V_b. \end{array} \tag{10.42}$$

Suppose $\boldsymbol{e} : \boldsymbol{X} \to \boldsymbol{Y}$ is another 1-morphism of m-Kuranishi spaces, and $\boldsymbol{\eta} = (\boldsymbol{\eta}_{ij,\, i\in I,\, j\in J}) : \boldsymbol{e} \Rightarrow \boldsymbol{f}$ is a 2-morphism, so that $\boldsymbol{e}$ is $\boldsymbol{C}$ by Proposition 4.36(a). Then for x, y, i, j, u_i, v_j as above, Definition 4.3(b) applied to the 2-morphism $\boldsymbol{\eta}_{ij} : \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{f}_{ij}$ shows that $f_{ij} = e_{ij} + O(r_i)$ near u_i in U_i, so $Q_{u_i} f_{ij} = Q_{u_i} e_{ij}$ by Assumption 10.19(c)(v). Thus comparing (10.41) for $\boldsymbol{e}, \boldsymbol{f}$ shows that $Q_x \boldsymbol{e} = Q_x \boldsymbol{f}$. Hence the morphisms $Q_x \boldsymbol{f}$ depend only on the $\boldsymbol{C}$ morphism $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathrm{Ho}(\mathbf{m\dot{K}ur})$, and on $x \in \boldsymbol{X}$.

Now suppose $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ is another $\boldsymbol{C}$ 1-morphism of m-Kuranishi spaces and $\boldsymbol{g}(y) = z \in \boldsymbol{Z}$ with notation (4.7)–(4.9), let $i \in I$, $j \in J$, $k \in K$ with $x \in \operatorname{Im} \chi_i$, $y \in \operatorname{Im} \psi_j$, $z \in \operatorname{Im} \omega_k$, and set $u_i = \chi_i^{-1}(x)$, $v_j = \psi_j^{-1}(y)$ and $v_k = \omega_k^{-1}(z)$. Then $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ is $\boldsymbol{C}$, and Definition 4.20 gives a 2-morphism $\Theta_{ijk}^{\boldsymbol{g},\boldsymbol{f}} : \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} \Rightarrow (\boldsymbol{g} \circ \boldsymbol{f})_{ik}$. Therefore $(g \circ f)_{ik} = g_{jk} \circ f_{ij} + O(r_i)$ near u_i, so Assumption 10.19(c)(i),(v) gives

$$Q_{u_i}(g \circ f)_{ik} = Q_{v_j} g_{jk} \circ Q_{u_i} f_{ij} : Q_{u_i} V_i \longrightarrow Q_{w_k} W_k.$$

Combining this with (10.41) for $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{g} \circ \boldsymbol{f}$ yields

$$Q_x(\boldsymbol{g} \circ \boldsymbol{f}) = Q_y \boldsymbol{g} \circ Q_x \boldsymbol{f}. \tag{10.43}$$

Also the definition of $\mathbf{id}_{\boldsymbol{X}}$ yields

$$Q_x \mathbf{id}_{\boldsymbol{X}} = \mathrm{id}_{Q_x \boldsymbol{X}} : Q_x \boldsymbol{X} \to Q_x \boldsymbol{X}. \tag{10.44}$$

So quasi-tangent spaces are functorial on the 2-category $\mathbf{m\dot{K}ur}_{\boldsymbol{C}}$.

As for Lemma 10.23, we can prove:

Lemma 10.31. *In Definition* 10.30 *suppose* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is an equivalence in* $\mathbf{m\dot{K}ur}$, *so that* $\boldsymbol{f}$ *is* $\boldsymbol{C}$ *by Proposition* 4.36(c). *Then* $Q_x \boldsymbol{f} : Q_x \boldsymbol{X} \to Q_y \boldsymbol{Y}$ *is an isomorphism in* $\mathcal{Q}$ *for all* $x \in \boldsymbol{X}$ *with* $\boldsymbol{f}(x) = y$ *in* $\boldsymbol{Y}$.

Combining Definition 10.30 and Example 10.20 yields:

Example 10.32. **(a)** In the 2-category $\mathbf{mKur^c}$ from (4.37), we have *stratum normal spaces* $\tilde{N}_x \boldsymbol{X}$ for all $\boldsymbol{X} \in \mathbf{mKur^c}$ and $x \in \boldsymbol{X}$, which are finite-dimensional real vector spaces, based on $\tilde{N}_v V$ in Definition 2.16 when $V \in \mathbf{Man^c}$ and $v \in V$. For any 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{mKur^c}$ we have functorial linear maps $\tilde{N}_x \boldsymbol{f} : \tilde{N}_x \boldsymbol{X} \to \tilde{N}_y \boldsymbol{Y}$ for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$.

(b) In the 2-category $\mathbf{mKur^c}$, we have *stratum b-normal spaces* ${}^b\tilde{N}_x \boldsymbol{X}$ for all $\boldsymbol{X}$ in $\mathbf{mKur^c}$ and $x \in \boldsymbol{X}$, which are finite-dimensional real vector spaces, based on ${}^b\tilde{N}_v V$ in Definition 2.16 when $V \in \mathbf{Man^c}$ and $v \in V$. For any interior 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{mKur^c}$ we have functorial linear maps ${}^b\tilde{N}_x \boldsymbol{f} : {}^b\tilde{N}_x \boldsymbol{X} \to {}^b\tilde{N}_y \boldsymbol{Y}$ for all x in $\boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$. We have $\dim \tilde{N}_x \boldsymbol{X} = \dim {}^b\tilde{N}_x \boldsymbol{X}$ for all $x, \boldsymbol{X}$, since $\dim \tilde{N}_v V = \dim {}^b\tilde{N}_v V$ for all $V \in \mathbf{Man^c}$ and $v \in V$. But in general there are no canonical isomorphisms $\tilde{N}_x \boldsymbol{X} \cong {}^b\tilde{N}_x \boldsymbol{X}$.

(c) In the 2-category $\mathbf{mKur^c}$, we have a commutative monoid $\tilde{M}_x \boldsymbol{X}$ for all $\boldsymbol{X}$ in $\mathbf{mKur^c}$ and $x \in \boldsymbol{X}$, with $\tilde{M}_x \boldsymbol{X} \cong \mathbb{N}^k$ for some $k \geqslant 0$, based on $\tilde{M}_v V$ in Definition

2.16 when $V \in \mathbf{Man^c}$ and $v \in V$. For any interior 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{mKur^c}$ we have functorial monoid morphisms $\tilde{M}_x\boldsymbol{f} : \tilde{M}_x\boldsymbol{X} \to \tilde{M}_y\boldsymbol{Y}$ for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$.

We have canonical isomorphisms ${}^b\tilde{N}_x\boldsymbol{X} \cong \tilde{M}_x\boldsymbol{X} \otimes_{\mathbb{N}} \mathbb{R}$ for all $x, \boldsymbol{X}$, as there are canonical isomorphisms ${}^b\tilde{N}_vV \cong \tilde{M}_vV \otimes_{\mathbb{N}} \mathbb{R}$, and these isomorphisms identify ${}^b\tilde{N}_x\boldsymbol{f} : {}^b\tilde{N}_x\boldsymbol{X} \to {}^b\tilde{N}_y\boldsymbol{Y}$ with $\tilde{M}_x\boldsymbol{f} \otimes \mathrm{id}_{\mathbb{R}} : \tilde{M}_x\boldsymbol{X} \otimes_{\mathbb{N}} \mathbb{R} \to \tilde{M}_y\boldsymbol{Y} \otimes_{\mathbb{N}} \mathbb{R}$.

(d) In the 2-category $\mathbf{mKur^{gc}}$ from (4.37), we have *stratum b-normal spaces* ${}^b\tilde{N}_x\boldsymbol{X}$ for all $\boldsymbol{X}$ in $\mathbf{mKur^{gc}}$ and $x \in \boldsymbol{X}$, based on ${}^b\tilde{N}_vV$ in §2.4.1 when $V \in \mathbf{Man^{gc}}$ and $v \in V$. For any interior 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{mKur^{gc}}$ we have functorial linear maps ${}^b\tilde{N}_x\boldsymbol{f} : {}^b\tilde{N}_x\boldsymbol{X} \to {}^b\tilde{N}_y\boldsymbol{Y}$ for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$. On $\mathbf{mKur^c} \subset \mathbf{mKur^{gc}}$ these agree with those in **(b)**.

(e) In the 2-category $\mathbf{mKur^{gc}}$, we have a toric commutative monoid $\tilde{M}_x\boldsymbol{X}$ for all $\boldsymbol{X}$ in $\mathbf{mKur^{gc}}$ and $x \in \boldsymbol{X}$, based on $\tilde{M}_vV$ in §2.4.1 when $V \in \mathbf{Man^{gc}}$ and $v \in V$. For any interior 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{mKur^{gc}}$ we have functorial monoid morphisms $\tilde{M}_x\boldsymbol{f} : \tilde{M}_x\boldsymbol{X} \to \tilde{M}_y\boldsymbol{Y}$ for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$. On $\mathbf{mKur^c} \subset \mathbf{mKur^{gc}}$ these agree with those in **(c)**.

We have canonical isomorphisms ${}^b\tilde{N}_x\boldsymbol{X} \cong \tilde{M}_x\boldsymbol{X} \otimes_{\mathbb{N}} \mathbb{R}$ for all $x, \boldsymbol{X}$, which identify ${}^b\tilde{N}_x\boldsymbol{f} : {}^b\tilde{N}_x\boldsymbol{X} \to {}^b\tilde{N}_y\boldsymbol{Y}$ with $\tilde{M}_x\boldsymbol{f} \otimes \mathrm{id}_{\mathbb{R}} : \tilde{M}_x\boldsymbol{X} \otimes_{\mathbb{N}} \mathbb{R} \to \tilde{M}_y\boldsymbol{Y} \otimes_{\mathbb{N}} \mathbb{R}$.

Quasi-tangent spaces are useful for stating conditions on objects and 1-morphisms in $\mathbf{m\dot{K}ur}$. For example:

- An object $\boldsymbol{X}$ in $\mathbf{mKur^{gc}}$ lies in $\mathbf{mKur^c} \subset \mathbf{mKur^{gc}}$ if and only if $\tilde{M}_xX \cong \mathbb{N}^k$ for all $x \in \boldsymbol{X}$, for $k \geqslant 0$ depending on x.
- An interior 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{mKur^c}$ or $\mathbf{mKur^{gc}}$ is simple if and only if $\tilde{M}_x\boldsymbol{f}$ is an isomorphism for all $x \in \boldsymbol{X}$.
- An interior 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{mKur^c}$ or $\mathbf{mKur^{gc}}$ is b-normal if and only if ${}^b\tilde{N}_x\boldsymbol{f}$ is surjective for all $x \in \boldsymbol{X}$.

Example 10.33. Let $\boldsymbol{X}$ be an object in $\mathbf{mKur^c}$, and $x \in \boldsymbol{X}$. Using the notation of Definitions 10.21 and 10.30, choose $i \in I$ with $x \in \operatorname{Im}\psi_i$, set $v_i = \psi_i^{-1}(x)$ in $s_i^{-1}(0) \subseteq V_i$, and consider the commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & \tilde{T}_{v_i}V_i & \xrightarrow{\tilde{\mathrm{d}}_{v_i}r_i} & E_i|_{v_i} & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \iota_{v_i}V_i} & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle 0} & & \\
\cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & T_{v_i}V_i & \xrightarrow{\mathrm{d}_{v_i}r_i} & E_i|_{v_i} & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \pi_{v_i}V_i} & & \downarrow & & \downarrow{\scriptstyle 0} & & \\
\cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & \tilde{N}_{v_i}V_i & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0. & &
\end{array} \qquad (10.45)$$

Here $T_{v_i}V_i, \tilde{T}_{v_i}V_i$ are as in Example 10.2(ii),(iv), and $\iota_{v_i}V_i$ is as in Example 10.4(b). The second column is (2.15) for V_i, v_i, which is exact, and the other columns are clearly exact. The rows of (10.45) are complexes. By equations (10.27), (10.40) and Examples 10.25(i),(iii) and 10.32(a), the first row has cohomology groups $\tilde{T}_x\boldsymbol{X}, \tilde{O}_x\boldsymbol{X}$, the second row $T_x\boldsymbol{X}, O_x\boldsymbol{X}$, and the third row $\tilde{N}_x\boldsymbol{X}, 0$.

Identifying (10.45) with equation (10.89), a standard piece of algebraic topology explained in Definition 10.69 below gives an exact sequence (10.90):

$$0 \longrightarrow \tilde{T}_x\boldsymbol{X} \xrightarrow{\iota_x^T\boldsymbol{X}} T_x\boldsymbol{X} \xrightarrow{\pi_x\boldsymbol{X}} \tilde{N}_x\boldsymbol{X} \xrightarrow{\delta_x\boldsymbol{X}} \tilde{O}_x\boldsymbol{X} \xrightarrow{\iota_x^O\boldsymbol{X}} O_x\boldsymbol{X} \longrightarrow 0. \tag{10.46}$$

Here $\iota_x^T\boldsymbol{X}, \iota_x^O\boldsymbol{X}$ are as in Example 10.27(b), and $\pi_x\boldsymbol{X}, \delta_x\boldsymbol{X}$ are natural linear maps, with $\delta_x\boldsymbol{X}$ a 'connecting morphism'. One can show as in Definitions 10.21 and 10.30 that $\pi_x\boldsymbol{X}, \delta_x\boldsymbol{X}$ are independent of the choice of $i \in I$.

Now let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in $\mathbf{mKur^c}$, and $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$. Then using equations (2.16), (10.28), (10.37), and (10.41), we can show that the following commutes, where $T_x\boldsymbol{f}, O_x\boldsymbol{f}, \tilde{T}_x\boldsymbol{f}, \tilde{O}_x\boldsymbol{f}$ are as in Example 10.25(i),(iii) and $\tilde{N}_x\boldsymbol{f}$ as in Example 10.32(a), and the rows are (10.46):

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \tilde{T}_x\boldsymbol{X} & \xrightarrow{\iota_x^T\boldsymbol{X}} & T_x\boldsymbol{X} & \xrightarrow{\pi_x\boldsymbol{X}} & \tilde{N}_x\boldsymbol{X} & \xrightarrow{\delta_x\boldsymbol{X}} & \tilde{O}_x\boldsymbol{X} & \xrightarrow{\iota_x^O\boldsymbol{X}} & O_x\boldsymbol{X} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{T}_x\boldsymbol{f}} & & \downarrow{\scriptstyle T_x\boldsymbol{f}} & & \downarrow{\scriptstyle \tilde{N}_x\boldsymbol{f}} & & \downarrow{\scriptstyle \tilde{O}_x\boldsymbol{f}} & & \downarrow{\scriptstyle O_x\boldsymbol{f}} & & \\
0 & \longrightarrow & \tilde{T}_y\boldsymbol{Y} & \xrightarrow{\iota_y^T\boldsymbol{Y}} & T_y\boldsymbol{Y} & \xrightarrow{\pi_y\boldsymbol{Y}} & \tilde{N}_y\boldsymbol{Y} & \xrightarrow{\delta_y\boldsymbol{Y}} & \tilde{O}_y\boldsymbol{Y} & \xrightarrow{\iota_y^O\boldsymbol{Y}} & O_y\boldsymbol{Y} & \longrightarrow & 0.
\end{array} \tag{10.47}$$

Example 10.34. Let $\boldsymbol{X}$ lie in $\mathbf{mKur^c}, \mathbf{mKur^{gc}}, \mathbf{mKur^{ac}}$ or $\mathbf{mKur^{c,ac}}$, and $x \in \boldsymbol{X}$. Then by a similar but simpler proof to Example 10.33 using (2.17) instead of (2.15), we find there is a natural exact sequence

$$0 \longrightarrow {}^b\tilde{N}_x\boldsymbol{X} \xrightarrow{{}^b\iota_x\boldsymbol{X}} {}^bT_x\boldsymbol{X} \xrightarrow{\Pi_x^T\boldsymbol{X}} \tilde{T}_x\boldsymbol{X} \longrightarrow 0, \tag{10.48}$$

where ${}^bT_x\boldsymbol{X}, \tilde{T}_x\boldsymbol{X}$ are as in Example 10.25(ii),(iii), and $\Pi_x^T\boldsymbol{X}$ as in Example 10.27(c), and ${}^b\tilde{N}_x\boldsymbol{X}$ as in Example 10.32(b). If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in $\mathbf{mKur^c}, \mathbf{mKur^{gc}}, \mathbf{mKur^{ac}}$ or $\mathbf{mKur^{c,ac}}$, and $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$ then as for (10.47) we have a commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}^b\tilde{N}_x\boldsymbol{X} & \xrightarrow{{}^b\iota_x\boldsymbol{X}} & {}^bT_x\boldsymbol{X} & \xrightarrow{\Pi_x^T\boldsymbol{X}} & \tilde{T}_x\boldsymbol{X} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle {}^b\tilde{N}_x\boldsymbol{f}} & & \downarrow{\scriptstyle {}^bT_x\boldsymbol{f}} & & \downarrow{\scriptstyle \tilde{T}_x\boldsymbol{f}} & & \\
0 & \longrightarrow & {}^b\tilde{N}_y\boldsymbol{Y} & \xrightarrow{{}^b\iota_y\boldsymbol{Y}} & {}^bT_y\boldsymbol{Y} & \xrightarrow{\Pi_y^T\boldsymbol{Y}} & \tilde{T}_y\boldsymbol{Y} & \longrightarrow & 0.
\end{array} \tag{10.49}$$

10.3.2 Quasi-tangent spaces for μ-Kuranishi spaces

For μ-Kuranishi spaces in Chapter 5, by essentially exactly the same arguments as in §10.3.1, if $\dot{\mathbf{Man}}$ satisfies Assumption 10.19 then:

(a) For each μ-Kuranishi space $\boldsymbol{X}$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ and $x \in \boldsymbol{X}$ we can define the *quasi-tangent space* $Q_x\boldsymbol{X}$, an object in $\mathcal{Q}$.

(b) For each $\boldsymbol{C}$ morphism $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ and $x\in\boldsymbol{X}$ with $\boldsymbol{f}(x)=y$ in $\boldsymbol{Y}$ we can define a morphism $Q_x\boldsymbol{f}:Q_x\boldsymbol{X}\to Q_y\boldsymbol{Y}$ in $\mathcal{Q}$. These are functorial, that is, (10.43)–(10.44) hold.

(c) The analogues of Lemma 10.31 and Examples 10.32–10.34 hold.

10.3.3 Quasi-tangent spaces for Kuranishi spaces

For quasi-tangent spaces of Kuranishi spaces, we combine the ideas of §10.3.1 and §10.2.3 in a straightforward way. The main points are these:

(a) Let $\boldsymbol{X}=(X,\mathcal{K})$ be a Kuranishi space, with $\mathcal{K}=\bigl(I,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\Phi_{ij,\,i,j\in I},\Lambda_{ijk,\,i,j,k\in I}\bigr)$, and let $x\in\boldsymbol{X}$. In Definition 6.49 we defined the isotropy group $G_x\boldsymbol{X}$ by choosing $i\in I$ with $x\in\operatorname{Im}\psi_i$ and $v_i\in s_i^{-1}(0)\subseteq V_i$ with $\bar\psi_i(v_i)=x$, and setting $G_x\boldsymbol{X}=\mathrm{Stab}_{\Gamma_i}(v_i)$ as in (6.40). For these i,v_i, we define the *quasi-tangent space* $Q_x\boldsymbol{X}$ in $\mathcal{Q}$ to be $Q_{v_i}V_i$.

(b) There is a natural action of $G_x\boldsymbol{X}$ on $Q_x\boldsymbol{X}$ by isomorphisms in $\mathcal{Q}$.

(c) $Q_x\boldsymbol{X}$ is independent of choices up to isomorphism in $\mathcal{Q}$, but not up to canonical isomorphism. Given two choices $Q_x\boldsymbol{X},Q'_x\boldsymbol{X}$, the isomorphism $Q_x\boldsymbol{X}\to Q'_x\boldsymbol{X}$ is natural only up to the action of $G_x\boldsymbol{X}$ on $Q'_x\boldsymbol{X}$.

(d) Let $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ be a $\boldsymbol{C}$ 1-morphism in $\dot{\mathbf{Kur}}$, with notation (6.15), (6.16), (6.18), and let $x\in\boldsymbol{X}$ with $y\in\boldsymbol{Y}$. By picking $p\in S_{x,\boldsymbol{f}}$ in (6.44), Definition 6.51 defines a group morphism $G_x\boldsymbol{f}:G_x\boldsymbol{X}\to G_y\boldsymbol{Y}$. Using the same p, define a morphism $Q_x\boldsymbol{f}:Q_x\boldsymbol{X}\to Q_y\boldsymbol{Y}$ in $\mathcal{Q}$ by the commutative diagram

$$\begin{array}{ccccc} Q_x\boldsymbol{X} & & \xrightarrow{\quad Q_x\boldsymbol{f}\quad} & & Q_y\boldsymbol{Y} \\ \| & & & & \| \\ Q_{u_i}U_i & \xleftarrow[\cong]{Q_p\pi_{ij}} & Q_pP_{ij} & \xrightarrow{Q_pf_{ij}} & Q_{v_j}V_j, \end{array}$$

where $Q_p\pi_{ij}$ is invertible as π_{ij} is étale. Then $Q_x\boldsymbol{f}$ is $G_x\boldsymbol{f}$-equivariant. It depends on the choice of p up to the action of $G_y\boldsymbol{Y}$ on $Q_y\boldsymbol{Y}$.

(e) Continuing from (d), suppose $\boldsymbol{e}:\boldsymbol{X}\to\boldsymbol{Y}$ is another 1-morphism and $\boldsymbol{\eta}:\boldsymbol{e}\Rightarrow\boldsymbol{f}$ a 2-morphism in $\dot{\mathbf{Kur}}$. Then $\boldsymbol{e}$ is $\boldsymbol{C}$ by Proposition 6.34(a). Definition 6.51 gives $G_x\boldsymbol{\eta}\in G_y\boldsymbol{Y}$, and we have $Q_x\boldsymbol{f}=G_x\boldsymbol{\eta}\cdot Q_x\boldsymbol{e}$.

(f) Continuing from (d), suppose $\boldsymbol{g}:\boldsymbol{Y}\to\boldsymbol{Z}$ is another $\boldsymbol{C}$ 1-morphism and $\boldsymbol{g}(y)=z\in\boldsymbol{Z}$. Then Definition 6.51 gives $G_{x,\boldsymbol{g},\boldsymbol{f}}\in G_z\boldsymbol{Z}$, and we have

$$Q_x(\boldsymbol{g}\circ\boldsymbol{f})=G_{x,\boldsymbol{g},\boldsymbol{f}}\cdot(Q_y\boldsymbol{g}\circ Q_x\boldsymbol{f}).$$

(f) The analogues of Lemma 10.31 and Examples 10.32–10.34 hold.

We leave the details to the reader.

10.4 Minimal (m-, μ-)Kuranishi neighbourhoods at $x \in \boldsymbol{X}$

In this section we suppose $\dot{\mathbf{Man}}$ satisfies Assumptions 10.1 and 10.9 in §10.1 throughout, so that we are given a discrete property $\boldsymbol{A}$ (possibly trivial) of morphisms in $\dot{\mathbf{Man}}$, and 'manifolds' V in $\dot{\mathbf{Man}}$ have tangent spaces T_vV for $v \in V$, and $\boldsymbol{A}$ morphisms $f : V \to W$ in $\dot{\mathbf{Man}}$ have functorial tangent maps $T_vf : T_vV \to T_wW$ for all $v \in V$ with $f(v) = w \in W$. For some results we also suppose Assumption 10.11.

We will use Assumption 10.9 to prove that if $\boldsymbol{X}$ is an m-Kuranishi space and $x \in X$ then we can find an m-Kuranishi neighbourhood (V, E, s, ψ) on $\boldsymbol{X}$ such that $x \in \operatorname{Im}\psi$ which is *minimal at* x in the sense that $\mathrm{d}_{\psi^{-1}(x)}s = 0$. Then we will use Assumption 10.11 to show that if (V', E', s', ψ') is another m-Kuranishi neighbourhood on $\boldsymbol{X}$ with $x \in \operatorname{Im}\psi'$ then (V', E', s', ψ') is locally isomorphic to (V, E, s, ψ) near x if (V', E', s', ψ') is minimal at x, and in general (V', E', s', ψ') is locally isomorphic to $(V \times \mathbb{R}^n, \pi^*(E) \oplus \mathbb{R}^n, \pi^*(s) \oplus \mathrm{id}_{\mathbb{R}^n}, \psi \circ \pi_V)$ near x.

We also generalize the results to μ-Kuranishi spaces, and to Kuranishi spaces, where a Kuranishi neighbourhood (V, E, Γ, s, ψ) on a Kuranishi space $\boldsymbol{X}$ is *minimal at* x if $x \in \operatorname{Im}\psi$, and $\Gamma \cong G_x\boldsymbol{X}$, so that $\bar{\psi}^{-1}(x)$ is a single point v in V fixed by Γ, and $\mathrm{d}_vs = 0$.

10.4.1 Minimal m-Kuranishi neighbourhoods at $x \in \boldsymbol{X}$

Definition 10.35. Let X be a topological space, and (V, E, s, ψ) be an m-Kuranishi neighbourhood on X in the sense of §4.1, and $x \in \operatorname{Im}\psi \subseteq X$. Set $v = \psi^{-1}(x) \in s^{-1}(0) \subseteq V$. Then Definition 10.6 defines a linear map of real vector spaces $\mathrm{d}_vs : T_vV \to E|_v$, the derivative of s at v, for T_vV as in Assumption 10.1(b). We say that (V, E, s, ψ) is *minimal at* x if $\mathrm{d}_vs = 0$.

Similarly, let $\boldsymbol{X} = (X, \mathcal{K})$ be an m-Kuranishi space in $\mathbf{m\dot{K}ur}$, and (V, E, s, ψ) be an m-Kuranishi neighbourhood on $\boldsymbol{X}$ in the sense of §4.7, and $x \in \operatorname{Im}\psi \subseteq X$ with $v = \psi^{-1}(x)$. Again we say that (V, E, s, ψ) is *minimal at* x if $\mathrm{d}_vs = 0$.

If (V, E, s, ψ) is an m-Kuranishi neighbourhood on $\boldsymbol{X}$ and $x \in \operatorname{Im}\psi$ with $v = \psi^{-1}(x)$ then as in (10.27) we have an exact sequence

$$0 \longrightarrow T_x\boldsymbol{X} \longrightarrow T_vV \xrightarrow{\mathrm{d}_vs} E|_v \longrightarrow O_x\boldsymbol{X} \longrightarrow 0.$$

Also $\operatorname{vdim}\boldsymbol{X} = \dim V - \operatorname{rank} E$. From these we easily deduce:

Lemma 10.36. *Let (V, E, s, ψ) be an m-Kuranishi neighbourhood on an m-Kuranishi space $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$, and $x \in \operatorname{Im}\psi$ with $v = \psi^{-1}(x) \in V$. Then*

$$\operatorname{rank} E \geqslant \dim O_x\boldsymbol{X} \qquad \text{and} \qquad \dim V \geqslant \operatorname{vdim}\boldsymbol{X} + \dim O_x\boldsymbol{X}, \qquad (10.50)$$

and (V, E, s, ψ) is minimal at x if and only if equality holds in (10.50).

If (V, E, s, ψ) is minimal at x there are natural isomorphisms $T_x\boldsymbol{X} \cong T_vV$ and $O_x\boldsymbol{X} \cong E|_v$.

We will be considering the question 'how many different m-Kuranishi neighbourhoods are there near x on an m-Kuranishi space $\boldsymbol{X}$?'. To answer this we need a notion of when two m-Kuranishi neighbourhoods on $\boldsymbol{X}$ are 'the same', which we call *strict isomorphism*.

Definition 10.37. Let $(V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)$ be m-Kuranishi neighbourhoods on a topological space X. A *strict isomorphism* $(\phi_{ij},\hat\phi_{ij}):(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ satisfies:

(a) $\phi_{ij}:V_i\rightarrow V_j$ is a diffeomorphism in $\dot{\mathbf{Man}}$.

(b) $\hat\phi_{ij}:E_i\rightarrow\phi_{ij}^*(E_j)$ is an isomorphism of vector bundles on V_i.

(c) $\hat\phi_{ij}(s_i)=\phi_{ij}^*(s_j)$ in $\Gamma^\infty(\phi_{ij}^*(E_j))$.

(d) $\psi_i=\psi_j\circ\phi_{ij}|_{s_i^{-1}(0)}:s_i^{-1}(0)\rightarrow X$, where $\phi_{ij}(s_i^{-1}(0))=s_j^{-1}(0)$ by (a)–(c).

Then $\Phi_{ij}=(V_i,\phi_{ij},\hat\phi_{ij}):(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ is a coordinate change over $\operatorname{Im}\psi_i=\operatorname{Im}\psi_j$.

If instead $(V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)$ are m-Kuranishi neighbourhoods on an m-Kuranishi space $\boldsymbol{X}$, we define strict isomorphisms as above, except that we also require Φ_{ij} to be one of the possible choices in Theorem 4.56(a).

We call m-Kuranishi neighbourhoods $(V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)$ on X or $\boldsymbol{X}$ *strictly isomorphic near* $S\subseteq\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\subseteq X$ if there exist open neighbourhoods U_i of $\psi_i^{-1}(S)$ in V_i and U_j of $\psi_j^{-1}(S)$ in V_j and a strict isomorphism

$$(\phi_{ij},\hat\phi_{ij}):(U_i,E_i|_{U_i},s_i|_{U_i},\psi_i|_{U_i})\longrightarrow(U_j,E_j|_{U_j},s_j|_{U_j},\psi_j|_{U_j}).$$

Given an m-Kuranishi neighbourhood (V,E,s,ψ) on X, we will construct a family $(V_{(n)},E_{(n)},s_{(n)},\psi_{(n)})$ for $n\in\mathbb{N}$ with $V_{(n)}=V\times\mathbb{R}^n$.

Definition 10.38. Let (V,E,s,ψ) be an m-Kuranishi neighbourhood on a topological space X, and let $n=0,1,\ldots$. Define an m-Kuranishi neighbourhood $(V_{(n)},E_{(n)},s_{(n)},\psi_{(n)})$ on X by

$$(V_{(n)},E_{(n)},s_{(n)},\psi_{(n)})=\bigl(V\times\mathbb{R}^n,\pi_V^*(E)\oplus\mathbb{R}^n,\pi_V^*(s)\oplus\mathrm{id}_{\mathbb{R}^n},\psi\circ\pi_V|_{s_{(n)}^{-1}(0)}\bigr).$$

In more detail, writing $\pi_V:V_{(n)}=V\times\mathbb{R}^n\rightarrow V$ for the projection, we define $E_{(n)}\rightarrow V_{(n)}$ to be the direct sum of $\pi_V^*(E)$ and the trivial vector bundle $\mathbb{R}^n$, so that $E_{(n)}=E\times\mathbb{R}^n\times\mathbb{R}^n$ as a manifold, and $\operatorname{rank}E_{(n)}=\operatorname{rank}E+n$, so that

$$\dim V_{(n)}-\operatorname{rank}E_{(n)}=(\dim V+n)-(\operatorname{rank}E+n)=\dim V-\operatorname{rank}E.$$

Writing points of E as (v,e) for $v\in V$ and $e\in E|_v$, and $s\in\Gamma^\infty(E)$ as mapping $v\mapsto(v,s(v))$ for $s(v)\in E|_v$, we may write points of $E_{(n)}$ as $(v,\boldsymbol{y},e,\boldsymbol{z})$ for $v\in V$, $e\in E|_v$ and $\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^n$, where $\pi:E_{(n)}\rightarrow V_{(n)}$ maps $\pi:(v,\boldsymbol{y},e,\boldsymbol{z})\mapsto(v,\boldsymbol{y})$. Then $s_{(n)}$ maps $s_{(n)}:(v,\boldsymbol{y})\mapsto(v,\boldsymbol{y},s(v),\boldsymbol{y})$. That is, the $\mathbb{R}^n$-component of $s_{(n)}$ in $E_{(n)}=\pi_V^*(E)\oplus\mathbb{R}^n$ maps $(v,\boldsymbol{y})\mapsto\boldsymbol{y}=\mathrm{id}_{\mathbb{R}^n}(\boldsymbol{y})$, so we write $s_{(n)}=\pi_V^*(s)\oplus\mathrm{id}_{\mathbb{R}^n}$.

Then $s_{(n)}^{-1}(0) = \{(v,0) : v \in s^{-1}(0)\} = s^{-1}(0) \times \{0\}$. Thus $\psi_{(n)} = \psi \circ \pi_V$ maps $(v,0) \mapsto \psi(v)$, and is a homeomorphism with $\operatorname{Im}\psi_{(n)} = \operatorname{Im}\psi \subseteq X$.

Define open submanifolds $V_{*(n)} \hookrightarrow V$, $V_{(n)*} \hookrightarrow V_{(n)}$ by $V_{*(n)} = V$ and $V_{(n)*} = V_{(n)}$, and morphisms $\phi_{*(n)} : V_{*(n)} \to V_{(n)}$, $\phi_{(n)*} : V_{(n)*} \to V$ by $\phi_{*(n)} = \mathrm{id}_V \times 0 : V_{*(n)} = V \to V_{(n)} = V \times \mathbb{R}^n$ and $\phi_{(n)*} = \pi_V : V_{(n)*} = V \times \mathbb{R}^n \to V$. Define vector bundle morphisms $\hat\phi_{*(n)} : E|_{V_{*(n)}} \to \phi_{*(n)}^*(E_{(n)})$, $\hat\phi_{(n)*} : E_{(n)}|_{V_{(n)*}} \to \phi_{(n)*}^*(E)$ by the commutative diagrams

$$
\begin{array}{ccc}
E|_{V_{*(n)}} & \xrightarrow[\hat\phi_{*(n)}]{} & \phi_{*(n)}^*(E_{(n)}) \\
\| & & \| \\
E & & (\mathrm{id}_V \times 0)^*(\pi_V^*(E) \oplus \mathbb{R}^n) \\
\big\downarrow{\scriptstyle \mathrm{id}_E \oplus 0} & & \| \\
E \oplus \mathbb{R}^n & = & (\mathrm{id}_V \times 0)^* \circ \pi_V^*(E) \oplus \mathbb{R}^n,
\end{array}
\qquad
\begin{array}{ccc}
E_{(n)}|_{V_{(n)*}} & \xrightarrow[\hat\phi_{(n)*}]{} & \phi_{(n)*}^*(E) \\
\| & & \| \\
E_{(n)} & & \\
\| & & \| \\
\pi_V^*(E) \oplus \mathbb{R}^n & \xrightarrow{\mathrm{id}_{\pi_V^*(E)} \oplus 0} & \pi_V^*(E).
\end{array}
$$

Then $\Phi_{*(n)} = (V_{*(n)}, \phi_{*(n)}, \hat\phi_{*(n)})$, $\Phi_{(n)*} = (V_{(n)*}, \phi_{(n)*}, \hat\phi_{(n)*})$ are 1-morphisms of m-Kuranishi neighbourhoods $\Phi_{*(n)} : (V, E, s, \psi) \to (V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})$ and $\Phi_{(n)*} : (V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)}) \to (V, E, s, \psi)$ on X over $S = \operatorname{Im}\psi = \operatorname{Im}\psi_{(n)}$.

Now $\phi_{*(n)} \circ \phi_{(n)*} = \mathrm{id}_V \times 0 : V \times \mathbb{R}^n \to V \times \mathbb{R}^n$. Thus we have isomorphisms

$$\mathcal{T}_{\phi_{*(n)} \circ \phi_{(n)*}} V_{(n)} = \mathcal{T}_{\mathrm{id}_V \times 0}(V \times \mathbb{R}^n) \cong \mathcal{T}_{\pi_V} V \oplus \mathcal{T}_0 \mathbb{R}^n \cong \mathcal{T}_{\pi_V} V \oplus \mathcal{O}_{V_{(n)}} \otimes \mathbb{R}^n.$$

Also $E_{(n)}|_{V_{(n)}} = \pi_V^*(E) \oplus \mathbb{R}^n$, so the sheaf of sections of $E_{(n)}|_{V_{(n)}}$ is isomorphic to $\pi_V^*(\mathcal{E}) \oplus \mathcal{O}_{V_{(n)}} \otimes_{\mathbb{R}} \mathbb{R}^n$, where $\mathcal{E}$ is the sheaf of sections of E. Define $\hat\lambda : E_{(n)}|_{V_{(n)}} \to \mathcal{T}_{\phi_{*(n)} \circ \phi_{(n)*}} V_{(n)}$ to be the $\mathcal{O}_{V_{(n)}}$-module morphism identified under these isomorphisms with

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix} : \begin{matrix} \pi_V^*(\mathcal{E}) \oplus \\ \mathcal{O}_{V_{(n)}} \otimes_{\mathbb{R}} \mathbb{R}^n \end{matrix} \longrightarrow \begin{matrix} \mathcal{T}_{\pi_V} V \oplus \\ \mathcal{O}_{V_{(n)}} \otimes_{\mathbb{R}} \mathbb{R}^n. \end{matrix}$$

We claim that $\Lambda = [V_{(n)}, \hat\lambda] : \Phi_{*(n)} \circ \Phi_{(n)*} \Rightarrow \mathrm{id}_{(V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})}$ is a 2-morphism of m-Kuranishi neighbourhoods over $\operatorname{Im}\psi = \operatorname{Im}\psi_{(n)}$. By Definition 4.3 we must show that

$$
\begin{aligned}
\mathrm{id}_V \times \mathrm{id}_{\mathbb{R}^n} &= \mathrm{id}_V \times 0 + \hat\lambda \circ s_{(n)} + O(s_{(n)}^2), \\
\begin{pmatrix} \mathrm{id}_{\pi^*(E)} & 0 \\ 0 & \mathrm{id}_{\mathbb{R}^n} \end{pmatrix} &= \begin{pmatrix} \mathrm{id}_{\pi^*(E)} \\ 0 \end{pmatrix} \begin{pmatrix} \mathrm{id}_{\pi^*(E)} & 0 \end{pmatrix} \\
&\quad + (\mathrm{id}_V \times 0)^*(\mathrm{d}s_{(n)}) \circ \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_{\mathbb{R}^n} \end{pmatrix} + O(s_{(n)}).
\end{aligned}
\tag{10.51}
$$

To prove these we must use the formal definitions in §B.3–§B.5. Define $w : E_{(n)} \to V_{(n)}$ to act by $w : (v, \boldsymbol{y}, e, \boldsymbol{z}) \mapsto (v, \boldsymbol{z})$ on points. Then $\hat\lambda = \theta_{E_{(n)}, w}$ in the notation of Definition B.32. Since

$$
\begin{aligned}
w \circ 0_{E_{(n)}}(v, \boldsymbol{y}) &= w(v, \boldsymbol{y}, 0, 0) = (v, 0) = (\mathrm{id}_V \times 0)(v, \boldsymbol{y}), \\
w \circ s_{(n)}(v, \boldsymbol{y}) &= w(v, \boldsymbol{y}, s(v), \boldsymbol{y}) = (v, \boldsymbol{y}) = (\mathrm{id}_V \times \mathrm{id}_{\mathbb{R}^n})(v, \boldsymbol{y}),
\end{aligned}
$$

Definition B.36(vii) implies the first equation of (10.51). Choose a connection ∇ on $E_{(n)} = \pi_V^*(E) \oplus \mathbb{R}^n$, in the sense of §B.3.2, which is the sum of a connection on $\pi_V^*(E)$ and the trivial connection on the trivial vector bundle $\mathbb{R}^n$. Then

$$(\mathrm{id}_V \times 0)^*(\nabla s_{(n)}) = \begin{pmatrix} \nabla_V s & \nabla_{\mathbb{R}^n} s \\ 0 & \mathrm{id} \end{pmatrix} : \begin{matrix} \mathcal{T}_{\pi_V} V \oplus \\ \mathcal{O}_{V_{(n)}} \otimes_{\mathbb{R}} \mathbb{R}^n \end{matrix} \longrightarrow \begin{matrix} \pi_V^*(\mathcal{E}) \oplus \\ \mathcal{O}_{V_{(n)}} \otimes_{\mathbb{R}} \mathbb{R}^n. \end{matrix}$$

The second equation of (10.51) then follows from Definition B.36(vi) and matrix multiplication. Hence $\Lambda : \Phi_{*(n)} \circ \Phi_{(n)*} \Rightarrow \mathrm{id}_{(V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})}$ is a 2-morphism over $\mathrm{Im}\,\psi$. From the definitions we see that $\Phi_{(n)*} \circ \Phi_{*(n)} = \mathrm{id}_{(V,E,s,\psi)}$, so $\mathrm{id}_{\mathrm{id}_{(V,E,s,\psi)}} : \Phi_{(n)*} \circ \Phi_{*(n)} \Rightarrow \mathrm{id}_{(V,E,s,\psi)}$ is a 2-morphism over $\mathrm{Im}\,\psi$. Therefore $\Phi_{*(n)}$ and $\Phi_{(n)*}$ are equivalences in the 2-category $\mathbf{m\dot{K}N}_{\mathrm{Im}\,\psi}(X)$, and are coordinate changes over $\mathrm{Im}\,\psi = \mathrm{Im}\,\psi_{(n)}$ by Definition 4.10.

Now let (V, E, s, ψ) be an m-Kuranishi neighbourhood on an m-Kuranishi space $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$, as in §4.7, with implicit extra data $\Phi_{*i,\ i \in I}$, $\Lambda_{*ij,\ i,j \in I}$, using the notation of Definition 4.49. For $n \geqslant 0$ and $i, j \in I$ define

$$\begin{aligned} \Phi_{(n)i} &= \Phi_{*i} \circ \Phi_{(n)*} : (V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)}) \longrightarrow (V_i, E_i, s_i, \psi_i), \\ \Lambda_{(n)ij} &= \Lambda_{*ij} * \mathrm{id}_{\Phi_{(n)*}} : \Phi_{ij} \circ \Phi_{(n)i} \Longrightarrow \Phi_{(n)j}. \end{aligned}$$

Then as $\Phi_{(n)*}$ is a coordinate change we see that $(V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})$ is also an m-Kuranishi neighbourhood on $\boldsymbol{X}$, with extra data $\Phi_{(n)i,\ i \in I}$, $\Lambda_{(n)ij,\ i,j \in I}$. Furthermore, it is easy to see that $\Phi_{*(n)} : (V, E, s, \psi) \to (V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})$ and $\Phi_{(n)*} : (V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)}) \to (V, E, s, \psi)$ are coordinate changes on $\boldsymbol{X}$ in the sense of Definition 4.51.

The next two propositions prove minimal m-Kuranishi neighbourhoods exist.

Proposition 10.39. *Suppose (V_i, E_i, s_i, ψ_i) is an m-Kuranishi neighbourhood on a topological space X, and $x \in \mathrm{Im}\,\psi_i \subseteq X$. Then there exists an m-Kuranishi neighbourhood (V, E, s, ψ) on X which is minimal at x, with $\mathrm{Im}\,\psi \subseteq \mathrm{Im}\,\psi_i \subseteq X$, and a coordinate change $\Phi_{*i} : (V, E, s, \psi) \to (V_i, E_i, s_i, \psi_i)$ over $S = \mathrm{Im}\,\psi$.*

Furthermore, (V_i, E_i, s_i, ψ_i) is strictly isomorphic to $(V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})$ near S in the sense of Definition 10.37, where $n = \dim V_i - \dim V \geqslant 0$ and $(V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})$ is constructed from (V, E, s, ψ) as in Definition 10.38, and this strict isomorphism locally identifies $\Phi_{*i} : (V, E, s, \psi) \to (V_i, E_i, s_i, \psi_i)$ with $\Phi_{*(n)} : (V, E, s, \psi) \to (V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})$ in Definition 10.38 near S.

Proof. Let $v_i = \psi_i^{-1}(x) \in s_i^{-1}(0) \subseteq V_i$. Then Definition 10.6 gives a linear map $\mathrm{d}_{v_i} s_i : T_{v_i} V_i \to E_i|_{v_i}$. Define n to be the dimension of the image of $\mathrm{d}_{v_i} s_i$ and $m = \mathrm{rank}\, E_i - n$, so that we may choose an isomorphism $E_i|_{v_i} \cong \mathbb{R}^m \oplus \mathbb{R}^n$ with $\mathrm{Im}\,\mathrm{d}_{v_i} s_i \cong \{0\} \oplus \mathbb{R}^n$. Choose an open neighbourhood V_i' of v_i in V_i with $E_i|_{V_i'}$ trivial, and choose a trivialization $E_i|_{V_i'} \cong V_i' \times (\mathbb{R}^m \oplus \mathbb{R}^n)$ which restricts to the chosen isomorphism $E_i|_{v_i} \cong \mathbb{R}^m \oplus \mathbb{R}^n$ at v_i. Then we may identify $s_i|_{V_i'}$ with $s_1 \oplus s_2$, where $s_1 : V_i' \to \mathbb{R}^m$, $s_2 : V_i' \to \mathbb{R}^n$ are morphisms in $\mathbf{\dot{M}an}$, and $\mathrm{d}_{v_i} s_i : T_{v_i} V_i \to E_i|_{v_i} \cong \mathbb{R}^m \oplus \mathbb{R}^n$ is identified with $T_{v_i} s_1 \oplus T_{v_i} s_2 : T_{v_i} V_i \to \mathbb{R}^m \oplus \mathbb{R}^n$. Hence $T_{v_i} s_1 = 0 : T_{v_i} V_i \to \mathbb{R}^m$, and $T_{v_i} s_2 : T_{v_i} V_i \to \mathbb{R}^n$ is surjective.

Apply Assumption 10.9 to $s_2 : V_i' \to \mathbb{R}^n$ at $v_i \in V_i'$, noting that s_2 is $\boldsymbol{A}$ by Assumption 10.1(a)(i). This gives open neighbourhoods U of v_i in V_i' and W of 0 in $\mathbb{R}^n$, an object V in $\dot{\mathbf{Man}}$ with $\dim V = \dim V_i - n$, and a diffeomorphism $\chi : U \to V \times W$ identifying $s_2|_U : U \to \mathbb{R}^n$ with $\pi_W : V \times W \to W \subseteq \mathbb{R}^n$.

We now have morphisms $s_1 \circ \chi^{-1} : V \times W \to \mathbb{R}^m$ and $s_2 \circ \chi^{-1} : V \times W \to \mathbb{R}^n$, where $0 \in W \subseteq \mathbb{R}^n$ is open, and $s_2 \circ \chi^{-1}$ maps $(v, \boldsymbol{w}) \mapsto \boldsymbol{w}$ for $v \in V$ and $\boldsymbol{w} = (w_1, \ldots, w_n) \in W$, since χ identifies $s_2|_U$ with π_W. Apply Assumption 3.5 to construct morphisms $g_j : V \times W \to \mathbb{R}^m$ for $j = 1, \ldots, n$ such that

$$s_1 \circ \chi^{-1}(v, (w_1, \ldots, w_n)) = s_1 \circ \chi^{-1}(v, (0, \ldots, 0)) + \sum_{j=1}^{n} w_j \cdot g_j(v, (w_1, \ldots, w_n))$$

for all $v \in V$ and $\boldsymbol{w} \in W$. Here $T_{v_i} s_1 = 0$ gives $g_j \circ \chi(v_i) = 0$ for $j = 1, \ldots, n$. Now we change the trivialization $E_i|_U \cong U \times (\mathbb{R}^m \oplus \mathbb{R}^n)$ by composing with the vector bundle isomorphism $U \times (\mathbb{R}^m \oplus \mathbb{R}^n) \to U \times (\mathbb{R}^m \oplus \mathbb{R}^n)$ acting by

$$(u, \boldsymbol{y}, \boldsymbol{z}) \longmapsto \bigl(u, \boldsymbol{y} - z_1 \cdot g_1 \circ \chi(u) - \cdots + z_n \cdot g_n \circ \chi(u), \boldsymbol{z}\bigr).$$

By definition of $g_1, \ldots, g_n$, at the point $u = \chi^{-1}(v, \boldsymbol{w})$ in U, this maps

$$s_1(u) \oplus s_2(u) = (s_1 \circ \chi^{-1})(v, \boldsymbol{w}) \oplus \boldsymbol{w} \longmapsto (s_1 \circ \chi^{-1})(v, 0) \oplus \boldsymbol{w}.$$

That is, changing s_1, s_2 along with the choice of trivialization, the effect is to leave s_2 unchanged, with $s_2 \circ \chi^{-1}(v, \boldsymbol{w}) = \boldsymbol{w}$, but to replace $s_1 \circ \chi^{-1}(v, \boldsymbol{w})$ by $s_1 \circ \chi^{-1}(v, 0)$, so that now $s_1 \circ \chi^{-1}(v, \boldsymbol{w})$ is independent of $\boldsymbol{w}$. As $g_j \circ \chi(v_i) = 0$, this replacement preserves the condition $\mathrm{d}_{v_i} s_1 = 0$. Write $\hat{\chi} : E_i|_U \to U \times (\mathbb{R}^m \oplus \mathbb{R}^n)$ for the new choice of trivialization.

Define $\pi : E \to V$ to be the trivial vector bundle $\pi_V : V \times \mathbb{R}^m \to V$, and define a section $s \in \Gamma^\infty(E)$, as a morphism $s : V \to E$, to be the composition

$$V \xrightarrow{(\mathrm{id}_V, 0)} V \times W \xrightarrow{(\pi_V, \chi^{-1})} V \times U \xrightarrow{\mathrm{id}_V \times s_1|_U} V \times \mathbb{R}^m = E.$$

Observe that the diffeomorphism $\chi : U \to V \times W$ identifies $U \cap s_i^{-1}(0)$ with

$$(s_1 \circ \chi^{-1})^{-1}(0) \cap (s_2 \circ \chi^{-1})^{-1}(0) = (s_1 \circ \chi^{-1})^{-1}(0) \cap (V \times \{0\}) = s^{-1}(0) \times \{0\}.$$

Hence defining $\psi : s^{-1}(0) \to X$ by $\psi = \psi_i \circ \chi^{-1} \circ (\mathrm{id}_{s^{-1}(0)}, 0)$, we see that ψ is a homeomorphism from $s^{-1}(0)$ to the open neighbourhood $\psi_i(U \cap s_i^{-1}(0))$ of x in $\mathrm{Im}\,\psi_i$. Therefore (V, E, s, ψ) is an m-Kuranishi neighbourhood on X, with $x \in \mathrm{Im}\,\psi \subseteq \mathrm{Im}\,\psi_i$. Also writing $v = \psi^{-1}(x) \in V$, then $\chi(v_i) = (v, 0)$, so $\mathrm{d}_v : T_v V \to E|_v$ is identified with the restriction of $T_{v_i} s_1 : T_{v_i} V_i \to \mathbb{R}^m$ to the subspace $T_v(\chi^{-1})[T_v V \oplus 0] \subseteq T_{v_i} V_i$. But $T_{v_i} s_1 = 0$, so $\mathrm{d}_v s = 0$, and (V, E, s, ψ) is minimal at x, as we have to prove.

Define a morphism $\phi_{*i} : V \to V_i$ and a vector bundle morphism $\hat{\phi}_{*i} : E \to \phi_{*i}^*(E_i)$ by the commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{\phi_{*i}} & V_i \\ \big\downarrow{\scriptstyle \mathrm{id}_V \times 0} & & \big\uparrow \\ V \times W & \xrightarrow{\chi^{-1}} & U, \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\hat{\phi}_{*i}} & \phi_{*i}^*(E_i) \\ \big\| & & \big\uparrow{\scriptstyle \cong} \\ V \times \mathbb{R}^m & \xrightarrow{\mathrm{id}_{V \times \mathbb{R}^m} \times 0} & V \times \mathbb{R}^m \times \mathbb{R}^n. \end{array}$$

Then $\Phi_{*i} = (V, \phi_{*i}, \hat{\phi}_{*i})$ is a 1-morphism of m-Kuranishi neighbourhoods $\Phi_{*i} : (V, E, s, \psi) \to (V_i, E_i, s_i, \psi_i)$ over $S = \operatorname{Im}\psi$, where Definition 4.2(d) holds as $\hat{\phi}_{*i}(s|_{V_{*i}}) = \phi_{*i}^*(s_i)$.

As $U \subseteq V_i$ is open, $(U, E_i|_U, s_i|_U, \psi_i|_U)$ is an m-Kuranishi neighbourhood on X. Also Definition 10.38 constructs $(V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})$ from $(V, E, s, \psi), n$ with $V_{(n)} = V \times \mathbb{R}^n$, so $V \times W \subseteq V_{(n)}$ is open, and we have an m-Kuranishi neighbourhood $(V \times W, E_{(n)}|_{V\times W}, s_{(n)}|_{V\times W}, \psi_{(n)}|_{V\times W})$ on X. From above we have isomorphisms $\chi : U \to V \times W$ and $\hat{\chi} : E_i|_U \to U \times \mathbb{R}^m \times \mathbb{R}^n = \chi^*(E_{(n)})$, since $E_{(n)} = V \times W \times \mathbb{R}^m \times \mathbb{R}^n$. We claim that

$$(\chi, \hat{\chi}) : (U, E_i|_U, s_i|_U, \psi_i|_U) \longrightarrow (V \times W, E_{(n)}|_{V\times W}, s_{(n)}|_{V\times W}, \psi_{(n)}|_{V\times W})$$

is a strict isomorphism. Here Definition 10.37(a),(b),(d) are immediate from the definitions, and (c) follows from $s_1 \circ \chi^{-1}(v, \boldsymbol{w}) = s_1 \circ \chi^{-1}(v, 0) = s(v)$ and $s_2 \circ \chi^{-1}(v, \boldsymbol{w}) = \boldsymbol{w} = \mathrm{id}_{\mathbb{R}^n}(\boldsymbol{w})$ above, and the definition of $s_{(n)}$. Thus (V_i, E_i, s_i, ψ_i) is strictly isomorphic to $(V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})$ near $S = \operatorname{Im}\psi$.

From the definitions we see that $\phi_{*(n)} = \chi \circ \phi_{*i}$ and $\hat{\phi}_{*(n)} = \hat{\chi} \circ \hat{\phi}_{*i}$, so $(\chi, \hat{\chi})$ locally identifies Φ_{*i} with $\Phi_{*(n)}$. By Definition 10.38, $\Phi_{*(n)}$ is a coordinate change, so Φ_{*i} is also a coordinate change. This completes the proof. □

Proposition 10.40. *Suppose* $\boldsymbol{X}$ *is an m-Kuranishi space in* $\mathbf{m\dot{K}ur}$ *and* $x \in \boldsymbol{X}$. *Then there exists an m-Kuranishi neighbourhood* (V, E, s, ψ) *on* $\boldsymbol{X}$, *in the sense of* §4.7, *which is minimal at* x.

Proof. Write $\boldsymbol{X} = (X, \mathcal{K})$ with $\mathcal{K} = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\, i,j\in I}, \Lambda_{ijk,\, i,j,k\in I}\bigr)$. Then there exists $h \in I$ with $x \in \operatorname{Im}\psi_h$. Proposition 10.39 constructs an m-Kuranishi neighbourhood (V, E, s, ψ) on the topological space X minimal at x with $x \in \operatorname{Im}\psi \subseteq \operatorname{Im}\psi_h \subseteq X$ and a coordinate change $\Phi'_{*h} : (V, E, s, \psi) \to (V_h, E_h, s_h, \psi_h)$. For all $i \in I$ set $\Phi_{*i} = \Phi_{hi} \circ \Phi'_{*h} : (V, E, s, \psi) \to (V_i, E_i, s_i, \psi_i)$, and for all $i, j \in I$ define

$$\Lambda_{*ij} = \Lambda_{hij} * \mathrm{id}_{\Phi'_{*h}} : \Phi_{ij} \circ \Phi_{*i} = \Phi_{ij} \circ \Phi_{hi} \circ \Phi'_{*h} \Longrightarrow \Phi_{hj} \circ \Phi'_{*h} = \Phi_{*j}.$$

Then (V, E, s, ψ) plus the data Φ_{*i}, Λ_{*ij} is an m-Kuranishi neighbourhood on the m-Kuranishi space $\boldsymbol{X}$ in the sense of Definition 4.49, since applying $- * \mathrm{id}_{\Phi'_{*h}}$ to (4.4) for $\mathcal{K}$ implies (4.57) for the Φ_{*i}, Λ_{*ij}. □

Remark 10.41. Definition 10.35 involves a choice of notion of tangent space T_vV for V in $\mathbf{\dot{M}an}$ in Assumption 10.1. As in Example 10.2, one category $\mathbf{\dot{M}an}$ can admit several different notions of tangent space, for example if $\mathbf{\dot{M}an}$ is $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}$ or $\mathbf{Man^{c,ac}}$ then both b-tangent spaces ${}^bT_vV$ and stratum tangent spaces $\tilde{T}_vV$ satisfy Assumptions 10.1 and 10.9.

Combining Lemma 10.36 and Proposition 10.40 we see that an m-Kuranishi neighbourhood (V, E, s, ψ) on $\boldsymbol{X}$ with $x \in \operatorname{Im}\psi$ is minimal at x if and only if $\dim V \leqslant \dim V'$ for all m-Kuranishi neighbourhoods (V', E', s', ψ') on $\boldsymbol{X}$ with $x \in \operatorname{Im}\psi'$. This characterization does not involve tangent spaces. Thus, whether or not (V, E, s, ψ) is minimal at x is *independent of the notion of tangent space* ${}^bT_vV, \tilde{T}_vV, \ldots$ *used to define minimality*, as long as there exists at least one notion of tangent space for $\mathbf{\dot{M}an}$ satisfying Assumptions 10.1 and 10.9.

10.4.2 Isomorphism of minimal m-Kuranishi neighbourhoods

In this section we also suppose Assumption 10.11, which was not needed in §10.4.1. We show that any two m-Kuranishi neighbourhoods minimal at $x \in X$ are strictly isomorphic near x, in the sense of Definition 10.37.

Proposition 10.42. *Let $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)$ be m-Kuranishi neighbourhoods on X which are both minimal at $x \in \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \subseteq X$, and $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat{\phi}_{ij}) : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ be a coordinate change over $x \in S \subseteq \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$. Then there exist open neighbourhoods U_i of $v_i = \psi_i^{-1}(x)$ in $V_{ij} \subseteq V_i$ and U_j of $v_j = \psi_j^{-1}(x)$ in V_j such that $\phi_{ij}|_{U_i} : U_i \to U_j$ is a diffeomorphism, and $\hat{\phi}_{ij}|_{U_i} : E_i|_{U_i} \to \phi_{ij}^*(E_j)|_{U_i}$ is an isomorphism.*

Furthermore there exists an isomorphism $\hat{\phi}'_{ij} : E_i|_{U_i} \to \phi_{ij}^*(E_j)|_{U_i}$ with $\hat{\phi}'_{ij} = \hat{\phi}_{ij}|_{U_i} + O(s_i)$ and $\hat{\phi}'_{ij}(s_i|_{U_i}) = \phi_{ij}^*(s_j)|_{U_i}$, so that

$$(\phi_{ij}|_{U_i}, \hat{\phi}'_{ij}) : (U_i, E_i|_{U_i}, s_i|_{U_i}, \psi_i|_{U_i}) \longrightarrow (U_j, E_j|_{U_j}, s_j|_{U_j}, \psi_j|_{U_j})$$

is a strict isomorphism of m-Kuranishi neighbourhoods over $T = \psi_i(U_i \cap s_i^{-1}(0))$. Also $[U_i, 0] : \Phi_{ij} \Rightarrow \Phi'_{ij} = (U_i, \phi_{ij}|_{U_i}, \hat{\phi}'_{ij})$ is a 2-morphism over T.

Proof. As in Definition 10.21 we have a commutative diagram (10.21) with exact rows, where $\kappa^x_{\Phi_{ij}}, \gamma^x_{\Phi_{ij}}$ are isomorphisms as Φ_{ij} is a coordinate change. But $\mathrm{d}_{v_i} s_i = \mathrm{d}_{v_j} s_j = 0$ as $(V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)$ are minimal at x. Hence (10.21) implies that $T_{v_i}\phi_{ij} : T_{v_i}V_i \to T_{v_j}V_j$ and $\hat{\phi}_{ij}|_{v_i} : E_i|_{v_i} \to E_j|_{v_j}$ are both isomorphisms. Also ϕ_{ij} is $\boldsymbol{B}$ near v_i by Proposition 4.34(d), for $\boldsymbol{B}$ the discrete property in Assumption 10.11. Hence as $T_{v_i}\phi_{ij}$ is an isomorphism, by Assumption 10.11 there exist open neighbourhoods U_i of v_i in V_{ij} and U_j of v_j in V_j such that $\phi_{ij}|_{U_i} : U_i \to U_j$ is a diffeomorphism in $\dot{\mathbf{Man}}$. Since $\hat{\phi}_{ij}|_{v_i} : E_i|_{v_i} \to E_j|_{v_j}$ is an isomorphism, $\hat{\phi}_{ij}$ is an isomorphism near v_i, so making U_i, U_j smaller we can suppose $\hat{\phi}_{ij}|_{U_i} : E_i|_{U_i} \to \phi_{ij}^*(E_j)|_{U_i}$ is an isomorphism.

We have $\hat{\phi}_{ij}(s_i|_{U_i}) = \phi_{ij}^*(s_j)|_{U_i} + O(s_i^2)$ by Definition 4.2(d), so by Definition 3.15(i) there exists $\alpha \in \Gamma^\infty(E_i^* \otimes E_i^* \otimes \phi_{ij}^*(E_j)|_{U_i})$ such that

$$\hat{\phi}_{ij}(s_i|_{U_i}) = \phi_{ij}^*(s_j)|_{U_i} + \alpha \cdot (s_i|_{U_i} \otimes s_i|_{U_i}).$$

Define a vector bundle morphism $\hat{\phi}'_{ij} : E_i|_{U_i} \to \phi_{ij}^*(E_j)|_{U_i}$ by

$$\hat{\phi}'_{ij}(e_i) = \hat{\phi}_{ij}|_{U_i}(e_i) - \alpha \cdot (e_i \otimes s_i|_{U_i})$$

for $e_i \in \Gamma^\infty(E_i|_{U_i})$. Clearly we have $\hat{\phi}'_{ij} = \hat{\phi}_{ij}|_{U_i} + O(s_i)$ and $\hat{\phi}'_{ij}(s_i|_{U_i}) = \phi_{ij}^*(s_j)|_{U_i}$, as in the proposition. Also $\hat{\phi}'_{ij}|_{v_i} = \hat{\phi}_{ij}|_{v_i}$ as $s_i|_{v_i} = 0$, and $\hat{\phi}_{ij}|_{v_i}$ is an isomorphism, so $\hat{\phi}'_{ij}$ is an isomorphism near v_i, and making U_i, U_j smaller we can suppose $\hat{\phi}'_{ij}$ is an isomorphism. The rest of the proposition is immediate. $\square$

Combining Proposition 10.42 with the material of §4.7 yields:

Proposition 10.43. *Let $\boldsymbol{X}$ be an m-Kuranishi space and $(V_a,E_a,s_a,\psi_a),(V_b,E_b,s_b,\psi_b)$ be m-Kuranishi neighbourhoods on $\boldsymbol{X}$ in the sense of §4.7 which are minimal at $x\in\boldsymbol{X}$ (these exist for any $x\in\boldsymbol{X}$ by Proposition 10.40). Theorem 4.56(a) gives a coordinate change $\Phi_{ab}=(V_{ab},\phi_{ab},\hat\phi_{ab}):(V_a,E_a,s_a,\psi_a)\to(V_b,E_b,s_b,\psi_b)$ on $\operatorname{Im}\psi_a\cap\operatorname{Im}\psi_b$, canonical up to 2-isomorphism.*

Then for small open neighbourhoods U_a of $\psi_a^{-1}(x)$ in $V_{ab}\subseteq V_a$ and U_b of $\psi_b^{-1}(x)$ in V_b, we may choose Φ_{ab} such that

$$(\phi_{ab}|_{U_a},\hat\phi_{ab}|_{U_a}):(U_a,E_a|_{U_a},s_a|_{U_a},\psi_a|_{U_a})\longrightarrow(U_b,E_b|_{U_b},s_b|_{U_b},\psi_b|_{U_b})$$

is a strict isomorphism of m-Kuranishi neighbourhoods on $\boldsymbol{X}$.

M-Kuranishi neighbourhoods (V_a,E_a,s_a,ψ_a) on $\boldsymbol{X}$ are classified up to strict isomorphism near x by $n=\dim V_a-\operatorname{vdim}\boldsymbol{X}-\dim O_x\boldsymbol{X}\in\mathbb{N}$.

Theorem 10.44. *Let $\boldsymbol{X}$ be an m-Kuranishi space in $\mathbf{m\dot{K}ur}$, and $x\in\boldsymbol{X}$, and (V,E,s,ψ) be an m-Kuranishi neighbourhood on $\boldsymbol{X}$ minimal at $x\in\boldsymbol{X}$, which exists by Proposition 10.40. Suppose (V_a,E_a,s_a,ψ_a) is any other m-Kuranishi neighbourhood on $\boldsymbol{X}$ with $x\in\operatorname{Im}\psi_a$. Then (V_a,E_a,s_a,ψ_a) is strictly isomorphic to $(V_{(n)},E_{(n)},s_{(n)},\psi_{(n)})$ near x in the sense of Definition 10.37, where*

$$n=\dim V_a-\dim V=\dim V_a-\operatorname{vdim}\boldsymbol{X}-\dim O_x\boldsymbol{X}\geqslant 0,\tag{10.52}$$

and $(V_{(n)},E_{(n)},s_{(n)},\psi_{(n)})$ is the m-Kuranishi neighbourhood on $\boldsymbol{X}$ constructed from $(V,E,s,\psi),n$ in Definition 10.38.

Proof. Let $\boldsymbol{X},x,(V,E,s,\psi),(V_a,E_a,s_a,\psi_a)$ be as in the theorem. Starting from (V_a,E_a,s_a,ψ_a), Propositions 10.39 and 10.40 construct an m-Kuranishi neighbourhood (V',E',s',ψ') on X or $\boldsymbol{X}$ which is minimal at x, such that $(V'_{(n)},E'_{(n)},s'_{(n)},\psi'_{(n)})$ is strictly isomorphic to (V_a,E_a,s_a,ψ_a) near x, by a strict isomorphism Ψ say, for $(V'_{(n)},E'_{(n)},s'_{(n)},\psi'_{(n)})$ constructed from (V',E',s',ψ') and $n=\dim V_a-\dim V'\geqslant 0$ in Definition 10.38. Then Proposition 10.43 shows that $(V,E,s,\psi),(V',E',s',\psi')$ are strictly isomorphic near x, by a strict isomorphism Ξ say, so $\dim V=\dim V'$, and (10.52) follows from (10.50).

Now consider the following diagram of coordinate changes of m-Kuranishi neighbourhoods on $\boldsymbol{X}$, defined near x, in the sense of Definition 4.51:

$$\begin{array}{ccccc}
(V,E,s,\psi) & \xleftarrow{\ \Phi_{(n)*}\ } & (V_{(n)},E_{(n)},s_{(n)},\psi_{(n)}) & & \\
\cong\downarrow\Xi & \Phi'_{*(n)}\circ\Xi\circ\Phi_{(n)*}\ \big\downarrow & \Rightarrow \quad \cong \big\downarrow \Xi_{(n)} & \searrow{\scriptstyle \Psi\circ\Xi_{(n)}}\ \cong & \\
(V',E',s',\psi') & \xrightarrow{\ \Phi'_{*(n)}\ } & (V'_{(n)},E'_{(n)},s'_{(n)},\psi'_{(n)}) & \xrightarrow[\cong]{\ \Psi\ } & (V_a,E_a,s_a,\psi_a).
\end{array}$$

Here arrows marked '$\cong$' are strict isomorphisms. The arrows '$\to$' exist from above and by Definition 10.38. Thus $\Phi'_{*(n)}\circ\Xi\circ\Phi_{(n)*}$ exists as a coordinate change on $\boldsymbol{X}$, by composition of coordinate changes in Definition 4.51.

Clearly Ξ induces a strict isomorphism $\Xi_{(n)}:(V_{(n)},E_{(n)},s_{(n)},\psi_{(n)})\to(V'_{(n)},E'_{(n)},s'_{(n)},\psi'_{(n)})$ near x, initially just as a coordinate change on X, not on

$\boldsymbol{X}$. However, there is a 2-morphism $\Phi'_{*(n)} \circ \Xi \circ \Phi_{(n)*} \Rightarrow \Xi_{(n)}$, constructed as for $\Lambda : \Phi_{*(n)} \circ \Phi_{(n)*} \Rightarrow \mathrm{id}_{(V_{(n)},E_{(n)},s_{(n)},\psi_{(n)})}$ in Definition 10.38. Therefore $\Xi_{(n)}$ is a coordinate change on $\boldsymbol{X}$, as $\Phi'_{*(n)} \circ \Xi \circ \Phi_{(n)*}$ is. Thus $\Psi \circ \Xi_{(n)}$: $(V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)}) \to (V_a, E_a, s_a, \psi_a)$ is a strict isomorphism of m-Kuranishi neighbourhoods on $\boldsymbol{X}$ near x, as required. □

As in Example 4.30, we say that an m-Kuranishi space $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$ *is a manifold* if $\boldsymbol{X} \simeq F_{\mathbf{\dot{M}an}}^{\mathbf{m\dot{K}ur}}(\tilde{X})$ in $\mathbf{m\dot{K}ur}$ for some $\tilde{X} \in \mathbf{\dot{M}an}$. We use Proposition 10.40 to give a criterion for this.

Theorem 10.45. *An m-Kuranishi space $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$ is a manifold, in the sense of Example* 4.30, *if and only if $O_x\boldsymbol{X} = 0$ for all $x \in \boldsymbol{X}$.*

Proof. The 'only if' part is obvious. For the 'if' part, suppose $\boldsymbol{X} = (X, \mathcal{K})$ lies in $\mathbf{m\dot{K}ur}$ with $O_x\boldsymbol{X} = 0$ for all $x \in X$. By Proposition 10.40, for each $x \in X$ we can choose an m-Kuranishi neighbourhood (V_x, E_x, s_x, ψ_x) on $\boldsymbol{X}$, as in §4.7, such that $x \in \operatorname{Im}\psi_x$ and (V_x, E_x, s_x, ψ_x) is minimal at x. But then $\operatorname{rank} E_x = \dim O_x\boldsymbol{X} = 0$ by Lemma 10.36, so $E_x = s_x = 0$. As the $\{\operatorname{Im}\psi_x : x \in X\}$ cover $\boldsymbol{X}$, Theorem 4.58 constructs $\boldsymbol{X}' = (X, \mathcal{K}')$ in $\mathbf{m\dot{K}ur}$ with $\mathcal{K}' = \big(X, (V_x, E_x, s_x, \psi_x)_{x\in X}, \Phi_{xy,\ x,y\in X}, \Lambda_{xyz,\ x,y,z\in X}\big)$ and $\boldsymbol{X} \simeq \boldsymbol{X}'$.

Since $E_x = s_x = 0$ for all $x \in X$, following the proof of Proposition 6.63 we can construct an object $\tilde{X}$ in $\mathbf{\dot{M}an}$ with topological space $\tilde{X} = X$ such that $F_{\mathbf{\dot{M}an}}^{\mathbf{m\dot{K}ur}}(\tilde{X}) \simeq \boldsymbol{X}'$, so that $F_{\mathbf{\dot{M}an}}^{\mathbf{m\dot{K}ur}}(\tilde{X}) \simeq \boldsymbol{X}$, and $\boldsymbol{X}$ is a manifold. □

All the results of §10.4.1–§10.4.2 apply in any 2-category $\mathbf{m\dot{K}ur}$ constructed from a category $\mathbf{\dot{M}an}$ satisfying Assumptions 3.1–3.7, 10.1, 10.9 and 10.11. By Examples 10.2, 10.10 and 10.12 and Definition 4.29, this includes the 2-categories

$$\mathbf{mKur}, \mathbf{mKur^c}, \mathbf{mKur^{gc}}, \mathbf{mKur^{ac}}, \mathbf{mKur^{c,ac}}. \tag{10.53}$$

10.4.3 Extension to μ-Kuranishi spaces

All of §10.4.1–§10.4.2 extends essentially immediately to μ-Kuranishi spaces. As in §5.2, μ-Kuranishi neighbourhoods are the same as m-Kuranishi neighbourhoods, and we call a μ-Kuranishi neighbourhood (V, E, s, ψ) on a topological space X (or on a μ-Kuranishi space $\boldsymbol{X}$) *minimal at* $x \in X$ if it is minimal at x as an m-Kuranishi neighbourhood. We leave the details to the reader.

10.4.4 Extension to Kuranishi spaces

Next we extend §10.4.1–§10.4.2 from m-Kuranishi spaces to Kuranishi spaces, by including finite groups Γ and isotropy groups $G_x\boldsymbol{X}$ throughout.

Here are the analogues of Definitions 10.35, 10.37 and 10.38.

Definition 10.46. Let (V, E, Γ, s, ψ) be a Kuranishi neighbourhood on a topological space X as in §6.1, and $x \in \operatorname{Im}\psi$. We call (V, E, Γ, s, ψ) *minimal at* x if

(a) $\bar{\psi}^{-1}(x)$ is a single point $\{v\}$ in V, and

(b) $\mathrm{d}_v s=0$, where v is as in (a) and $\mathrm{d}_v s: T_vV\rightarrow E|_v$ as in Definition 10.6.

Here $\bar\psi^{-1}(x)$ is a Γ-orbit in $s^{-1}(0)\subseteq V$, so (a) implies that v is fixed by Γ.

Similarly, let $\boldsymbol X=(X,\mathcal K)$ be a Kuranishi space in $\dot{\mathbf{Kur}}$, and (V,E,Γ,s,ψ) be a Kuranishi neighbourhood on $\boldsymbol X$ in the sense of §6.4, and $x\in\mathrm{Im}\,\psi\subseteq X$ with $v=\psi^{-1}(x)$. Again we call (V,E,Γ,s,ψ) *minimal at* x if (a),(b) hold. Then (a) implies that $G_x\boldsymbol X\cong\Gamma$, for $G_x\boldsymbol X$ the isotropy group of $\boldsymbol X$ from §6.5.

Definition 10.47. Let $\Phi_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ be a coordinate change of Kuranishi neighbourhoods on a topological space X. A *strict isomorphism* $(\sigma_{ij},\varphi_{ij},\hat\varphi_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ satisfies:

(a) $\sigma_{ij}:\Gamma_i\rightarrow\Gamma_j$ is an isomorphism of finite groups.

(b) $\varphi_{ij}:V_i\rightarrow V_j$ is a σ_{ij}-equivariant diffeomorphism in $\dot{\mathbf{Man}}$.

(c) $\hat\varphi_{ij}:E_i\rightarrow\phi_{ij}^*(E_j)$ is a σ_{ij}-equivariant vector bundle isomorphism on V_i.

(d) $\hat\varphi_{ij}(s_i)=\varphi_{ij}^*(s_j)$ in $\Gamma^\infty(\varphi_{ij}^*(E_j))$.

(e) $\bar\psi_i=\bar\psi_j\circ\varphi_{ij}|_{s_i^{-1}(0)}:s_i^{-1}(0)\rightarrow X$, where $\varphi_{ij}(s_i^{-1}(0))=s_j^{-1}(0)$ by (b)–(d).

Given a strict isomorphism $(\sigma_{ij},\varphi_{ij},\hat\varphi_{ij})$, we will define a coordinate change $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ over $\mathrm{Im}\,\psi_i=\mathrm{Im}\,\psi_j$. Set $P_{ij}=V_i\times\Gamma_j$, where $\Gamma_i\times\Gamma_j$ acts on P_{ij} by $(\gamma_i,\gamma_j):(v_i,\delta_j)\mapsto(\gamma_i\cdot v_i,\gamma_j\,\delta_j\,\sigma_{ij}(\gamma_i)^{-1})$. Define $\pi_{ij}:P_{ij}\rightarrow V_i$ by $\pi_{ij}:(v_i,\delta_j)\mapsto v_i$ and $\phi_{ij}:P_{ij}\rightarrow V_j$ by $\phi_{ij}:(v_i,\delta_j)\mapsto\delta_j\cdot\varphi_{ij}(v_i)$. Then π_{ij} is Γ_i-equivariant and Γ_j-invariant, and is a Γ_j-principal bundle, and ϕ_{ij} is Γ_i-invariant and Γ_j-equivariant.

At $(v_i,\delta_j)\in P_{ij}$, the morphism $\hat\phi_{ij}:\pi_{ij}^*(E_i)\rightarrow\phi_{ij}^*(E_j)$ must map $E_i|_{v_i}\rightarrow E_j|_{\delta_j\cdot\varphi_{ij}(v_i)}$. Let $\hat\phi_{ij}|_{(v_i,\delta_j)}$ be the composition of $\hat\varphi_{ij}|_{v_i}:E_i|_{v_i}\rightarrow E_j|_{\varphi_{ij}(v_i)}$ with the action of $\delta_j:E_j|_{\varphi_{ij}(v_i)}\rightarrow E_j|_{\delta_j\cdot\varphi_{ij}(v_i)}$. This defines $\hat\phi_{ij}$. It is now easy to show that $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$ is a 1-morphism $\Phi_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ over $\mathrm{Im}\,\psi_i$. Using the inverse of $(\sigma_{ij},\varphi_{ij},\hat\varphi_{ij})$ we construct a quasi-inverse Φ_{ji} for Φ_{ij} in the same way, so that Φ_{ij} is a coordinate change.

If instead $(V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)$ are Kuranishi neighbourhoods on a Kuranishi space $\boldsymbol X$, we define strict isomorphisms as above, except that we also require Φ_{ij} above to be one of the possible choices in Theorem 6.45(a).

We call Kuranishi neighbourhoods $(V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)$ on X or $\boldsymbol X$ *strictly isomorphic near* $S\subseteq\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j\subseteq X$ if there exist Γ_i- and Γ_j-invariant open neighbourhoods U_i of $\bar\psi_i^{-1}(S)$ in V_i and U_j of $\bar\psi_j^{-1}(S)$ in V_j, and a strict isomorphism

$$(\sigma_{ij},\varphi_{ij},\hat\varphi_{ij}):(U_i,E_i|_{U_i},\Gamma_i,s_i|_{U_i},\psi_i|_{U_i})\longrightarrow(U_j,E_j|_{U_j},\Gamma_j,s_j|_{U_j},\psi_j|_{U_j}).$$

Definition 10.48. Let (V,E,Γ,s,ψ) be a Kuranishi neighbourhood on a topological space X. Suppose we are given a finite group Δ, an injective morphism $\iota:\Gamma\hookrightarrow\Delta$, and a representation ρ of Γ on $\mathbb R^n$ for some $n=0,1,\ldots.$ We will define a Kuranishi neighbourhood $(V^{\Delta,\iota}_{(n),\rho},E^{\Delta,\iota}_{(n),\rho},\Delta,s^{\Delta,\iota}_{(n),\rho},\psi^{\Delta,\iota}_{(n),\rho})$ on X.

Define $V_{(n),\rho}^{\Delta,\iota} = (V \times \mathbb{R}^n \times \Delta)/\Gamma$, where Γ acts on $V \times \mathbb{R}^n \times \Delta$ by

$$\gamma : (v, \boldsymbol{y}, \delta) \longmapsto (\gamma \cdot v, \rho(\gamma)\boldsymbol{y}, \delta \cdot \iota(\gamma)^{-1}).$$

As the Γ-action is free and Γ is finite we can show using Assumptions 3.2(e) and 3.3(b) that the quotient $(V \times \mathbb{R}^n \times \Delta)/\Gamma$ exists in $\dot{\mathbf{Man}}$. Let Δ act on $V_{(n),\rho}^{\Delta,\iota}$ by

$$\delta' : (v, \boldsymbol{y}, \delta)\Gamma \longmapsto (v, \boldsymbol{y}, \delta' \cdot \delta)\Gamma.$$

Define $E_{(n),\rho}^{\Delta,\iota} = (E \times \mathbb{R}^n \times \mathbb{R}^n \times \Delta)/\Gamma$, where Γ acts on $E \times \mathbb{R}^n \times \mathbb{R}^n \times \Delta$ by

$$\gamma : \bigl((v,e), \boldsymbol{y}, \boldsymbol{z}, \delta\bigr) \longmapsto \bigl(\gamma \cdot (v,e), \rho(\gamma)\boldsymbol{y}, \rho(\gamma)\boldsymbol{z}, \delta \cdot \iota(\gamma)^{-1}\bigr).$$

Here we write points of E as (v,e) for $v \in V$ and $e \in E|_v$. The projection $\pi : E_{(n),\rho}^{\Delta,\iota} \to V_{(n),\rho}^{\Delta,\iota}$ making $E_{(n),\rho}^{\Delta,\iota}$ into a vector bundle acts by

$$\pi : \bigl((v,e), \boldsymbol{y}, \boldsymbol{z}, \delta\bigr)\Gamma \longmapsto (v, \boldsymbol{y}, \delta)\Gamma,$$

so that the fibre $E_{(n),\rho}^{\Delta,\iota}|_{(v,\boldsymbol{y},\delta)}$ is $E|_v \oplus \mathbb{R}^n \ni (e, \boldsymbol{z})$. Let Δ act on $E_{(n),\rho}^{\Delta,\iota}$ by

$$\delta' : \bigl((v,e), \boldsymbol{y}, \boldsymbol{z}, \delta\bigr)\Gamma \longmapsto \bigl((v,e), \boldsymbol{y}, \boldsymbol{z}, \delta' \cdot \delta\bigr)\Gamma.$$

Then π is Δ-equivariant. Define $s_{(n),\rho}^{\Delta,\iota} : V_{(n),\rho}^{\Delta,\iota} \to E_{(n),\rho}^{\Delta,\iota}$ by

$$s_{(n),\rho}^{\Delta,\iota} : (v, \boldsymbol{y}, \delta)\Gamma \longmapsto \bigl((v, s(v)), \boldsymbol{y}, \boldsymbol{y}, \delta\bigr)\Gamma,$$

where we write the action of $s : V \to E$ on points as $s : v \mapsto (v, s(v))$. Then $s_{(n),\rho}^{\Delta,\iota} \in \Gamma^\infty(E_{(n),\rho}^{\Delta,\iota})$ is Δ-equivariant. We have

$$(s_{(n),\rho}^{\Delta,\iota})^{-1}(0) = \bigl\{(v, \boldsymbol{y}, \delta)\Gamma \in V_{(n),\rho}^{\Delta,\iota} : s(v) = \boldsymbol{y} = 0\bigr\} = (s^{-1}(0) \times \{0\} \times \Delta)/\Gamma.$$

Hence we have a homeomorphism

$$I : (s_{(n),\rho}^{\Delta,\iota})^{-1}(0)/\Delta = [(s^{-1}(0) \times \{0\} \times \Delta)/\Gamma]/\Delta \longrightarrow s^{-1}(0)/\Gamma$$

mapping $I : [(v, 0, \delta)\Gamma]\Delta \mapsto v\Gamma$. Define $\psi_{(n),\rho}^{\Delta,\iota} = \psi \circ I : (s_{(n),\rho}^{\Delta,\iota})^{-1}(0)/\Delta \to X$. Then $\psi_{(n),\rho}^{\Delta,\iota}$ is a homeomorphism with the open set $\operatorname{Im}\psi_{(n),\rho}^{\Delta,\iota} = \operatorname{Im}\psi \subseteq X$. Thus $(V_{(n),\rho}^{\Delta,\iota}, E_{(n),\rho}^{\Delta,\iota}, \Delta, s_{(n),\rho}^{\Delta,\iota}, \psi_{(n),\rho}^{\Delta,\iota})$ is a Kuranishi neighbourhood on X.

Define a 1-morphism of Kuranishi neighbourhoods on X over $\operatorname{Im}\psi$

$$\Phi_{*(n)} = (P_{*(n)}, \pi_{*(n)}, \phi_{*(n)}, \hat{\phi}_{*(n)}) : (V, E, \Gamma, s, \psi) \longrightarrow (V_{(n),\rho}^{\Delta,\iota}, \ldots, \psi_{(n),\rho}^{\Delta,\iota})$$

by $P_{*(n)} = V \times \Delta$ with $\Gamma \times \Delta$-action $(\gamma, \delta') : (v, \delta) \mapsto (\gamma \cdot v, \delta' \cdot \delta \cdot \iota(\gamma)^{-1})$, and morphisms $\pi_{*(n)} : P_{*(n)} \to V$, $\phi_{*(n)} : P_{*(n)} \to V_{(n),\rho}^{\Delta,\iota}$, $\hat{\phi}_{*(n)} : \pi_{*(n)}^*(E) \to \phi_{*(n)}^*(E_{(n),\rho}^{\Delta,\iota})$ acting by

$$\pi_{*(n)} : (v, \delta) \longmapsto v, \quad \phi_{*(n)} : (v, \delta) \longmapsto (v, 0, \delta)\Gamma,$$
$$\hat{\phi}_{*(n)} : ((v, \delta), e) \longmapsto ((v, \delta), (e, 0)).$$

It is easy to check Definition 6.2 holds. Similarly define a 1-morphism

$$\Phi_{(n)*} = (P_{(n)*}, \pi_{(n)*}, \phi_{(n)*}, \hat{\phi}_{(n)*}) : (V^{\Delta,\iota}_{(n),\rho}, \ldots, \psi^{\Delta,\iota}_{(n),\rho}) \longrightarrow (V, E, \Gamma, s, \psi)$$

by $P_{(n)*} = V \times \mathbb{R}^n \times \Delta$ with $\Delta \times \Gamma$-action

$$(\delta', \gamma) : (v, \boldsymbol{y}, \delta) \longmapsto (\gamma \cdot v, \rho(\gamma)\boldsymbol{y}, \delta' \cdot \delta \cdot \iota(\gamma)^{-1}),$$

and $\pi_{(n)*} : P_{(n)*} \to V^{\Delta,\iota}_{(n),\rho}$, $\phi_{(n)*} : P_{(n)*} \to V$, $\hat{\phi}_{(n)*} : \pi^*_{(n)*}(E^{\Delta,\iota}_{(n),\rho}) \to \phi^*_{(n)*}(E)$ acting by

$$\pi_{(n)*} : (v, \boldsymbol{y}, \delta) \longmapsto (v, \boldsymbol{y}, \delta)\Gamma, \quad \phi_{(n)*} : (v, \boldsymbol{y}, \delta) \longmapsto v,$$
$$\hat{\phi}_{(n)*} : \bigl((v, \boldsymbol{y}, \delta), (e, \boldsymbol{z})\bigr) \longmapsto \bigl((v, \boldsymbol{y}, \delta), e\bigr).$$

As in Definition 10.38 but with extra contributions from finite groups Γ, Δ, we can define explicit 2-morphisms $\mathrm{K} : \Phi_{(n)*} \circ \Phi_{*(n)} \Rightarrow \mathrm{id}_{(V,E,\Gamma,s,\psi)}$ and $\Lambda : \Phi_{*(n)} \circ \Phi_{(n)*} \Rightarrow \mathrm{id}_{(V^{\Delta,\iota}_{(n),\rho}, \ldots, \psi^{\Delta,\iota}_{(n),\rho})}$ over $\mathrm{Im}\,\psi$, and we leave these as an exercise. Then K, Λ imply that $\Phi_{*(n)}, \Phi_{(n)*}$ are coordinate changes over $\mathrm{Im}\,\psi$.

Here is the analogue of Proposition 10.39:

Proposition 10.49. *Suppose $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ is a Kuranishi neighbourhood on a topological space X, and $x \in \mathrm{Im}\,\psi_i \subseteq X$. Then there exists a Kuranishi neighbourhood (V, E, Γ, s, ψ) on X which is minimal at x as in Definition* 10.46, *with $\mathrm{Im}\,\psi \subseteq \mathrm{Im}\,\psi_i \subseteq X$ and $\Gamma \subseteq \Gamma_i$ a subgroup, and a coordinate change $\Phi_{*i} : (V, E, \Gamma, s, \psi) \to (V_i, E_i, \Gamma_i, s_i, \psi_i)$ over $S = \mathrm{Im}\,\psi$.*

Furthermore, $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ is strictly isomorphic to $(V^{\Gamma_i,\iota}_{(n),\rho}, E^{\Gamma_i,\iota}_{(n),\rho}, \Gamma_i, s^{\Gamma_i,\iota}_{(n),\rho}, \psi^{\Gamma_i,\iota}_{(n),\rho})$ near S as in Definition 10.47, *where $n = \dim V_i - \dim V \geqslant 0$ and $(V^{\Gamma_i,\iota}_{(n),\rho}, \ldots, \psi^{\Gamma_i,\iota}_{(n),\rho})$ is constructed from (V, E, Γ, s, ψ) as in Definition* 10.48 *using the inclusion $\iota : \Gamma \hookrightarrow \Gamma_i$ and some representation ρ of Γ on $\mathbb{R}^n$, and this strict isomorphism locally identifies $\Phi_{*i} : (V, E, \Gamma, s, \psi) \to (V_i, E_i, \Gamma_i, s_i, \psi_i)$ with $\Phi_{*(n)} : (V, E, \Gamma, s, \psi) \to (V^{\Gamma_i,\iota}_{(n),\rho}, \ldots, \psi^{\Gamma_i,\iota}_{(n),\rho})$ in Definition* 10.48 *near S.*

Proof. Pick $v_i \in \bar{\psi}_i^{-1}(x) \subseteq s_i^{-1}(0) \subseteq V_i$, and define $\Gamma = \mathrm{Stab}_{\Gamma_i}(v_i) = \bigl\{\gamma \in \Gamma_i : \gamma(v_i) = v_i\bigr\}$, as a subgroup of Γ_i with inclusion $\iota : \Gamma \hookrightarrow \Gamma_i$. Then $\Gamma v_i = \bar{\psi}_i^{-1}(x)$ is $|\Gamma_i|/|\Gamma|$ points in V_i. Definition 10.6 gives a linear map $\mathrm{d}_{v_i} s_i : T_{v_i} V_i \to E_i|_{v_i}$. Here Γ acts linearly on $T_{v_i} V_i, E_i|_{v_i}$, and $\mathrm{d}_{v_i} s_i$ is Γ-equivariant. Define n to be the dimension of the image of $\mathrm{d}_{v_i} s_i$ and $m = \mathrm{rank}\, E_i - n$, so that we may choose a Γ-equivariant isomorphism $E_i|_{v_i} \cong \mathbb{R}^m \oplus \mathbb{R}^n$ with $\mathrm{Im}\,\mathrm{d}_{v_i} s_i \cong \{0\} \oplus \mathbb{R}^n$. Write ρ for the corresponding representation of Γ on $\mathbb{R}^n$.

Choose a Γ-invariant open neighbourhood V_i' of v_i in V_i with $E_i|_{V_i'}$ trivial, such that $(\delta \cdot V_i') \cap V_i = \emptyset$ for all $\delta \in \Gamma_i \setminus \Gamma$. Choose a Γ-equivariant trivialization $E_i|_{V_i'} \cong V_i' \times (\mathbb{R}^m \oplus \mathbb{R}^n)$ which restricts to the chosen isomorphism $E_i|_{v_i} \cong \mathbb{R}^m \oplus \mathbb{R}^n$ at v_i. Then we may identify $s_i|_{V_i'}$ with $s_1 \oplus s_2$, where $s_1 : V_i' \to \mathbb{R}^m$, $s_2 : V_i' \to \mathbb{R}^n$ are Γ-equivariant morphisms in $\dot{\mathbf{Man}}$, and $\mathrm{d}_{v_i} s_i : T_{v_i} V_i \to$

$E_i|_{v_i} \cong \mathbb{R}^m \oplus \mathbb{R}^n$ is identified with $T_{v_i}s_1 \oplus T_{v_i}s_2 : T_{v_i}V_i \to \mathbb{R}^m \oplus \mathbb{R}^n$. Hence $T_{v_i}s_1 = 0 : T_{v_i}V_i \to \mathbb{R}^m$, and $T_{v_i}s_2 : T_{v_i}V_i \to \mathbb{R}^n$ is surjective.

We now follow the proof of Proposition 10.39 to construct $v_i \in U \subseteq V_i'$, $\chi : U \xrightarrow{\cong} V \times W$, $\hat\chi : E_i|_U \to U \times (\mathbb{R}^m \oplus \mathbb{R}^n)$, $\pi : E \to V$, $s : V \to E$, and $v \in V$ with $\chi(v_i) = (v, 0)$ and $s(v) = \mathrm{d}_v s = 0$, but making everything Γ-invariant/equivariant, noting that Assumption 10.9 includes Γ-equivariance, and $(g_1, \ldots, g_n)$ can be made Γ-equivariant by averaging over the Γ-action. Define $\psi : s^{-1}(0)/\Gamma \to X$ by the commutative diagram

$$\begin{array}{ccccc} s^{-1}(0)/\Gamma & \xrightarrow[(\mathrm{id}_{s^{-1}(0)},0)/\Gamma]{} & [s^{-1}(0)\times\{0\}]/\Gamma & \xrightarrow[\chi|^{-1}_{U\cap s^{-1}(0)}/\Gamma]{} & (U\cap s^{-1}(0))/\Gamma \\ \downarrow{\scriptstyle\psi} & & & & \downarrow{\scriptstyle u\Gamma\mapsto u\Gamma_i} \\ X & & \xleftarrow{\qquad\psi_i\qquad} & & s^{-1}(0)/\Gamma_i. \end{array}$$

Here each arrow is a homeomorphism with an open subset, the top right as $\chi : U \to V \times W$ identifies $U \cap s_i^{-1}(0)$ with $s^{-1}(0) \times \{0\}$ and is Γ-equivariant, the right hand as U is Γ-invariant and $(\delta \cdot U) \cap U = \emptyset$ for $\delta \in \Gamma_i \setminus \Gamma$, and the bottom by Definition 6.1(e). Thus (V, E, Γ, s, ψ) is a Kuranishi neighbourhood on X with $x \in \mathrm{Im}\,\psi \subseteq \mathrm{Im}\,\psi_i \subseteq X$, and is minimal at x as in Definition 10.46. The rest of the proof is a straightforward generalization of that of Proposition 10.39. □

The next three results need Assumption 10.11. By modifying the proofs of Propositions 10.40, 10.42 and 10.43 and Theorems 10.44 and 10.45 to include finite groups, we can show:

Proposition 10.50. *Suppose $\boldsymbol{X}$ is a Kuranishi space in $\dot{\mathbf{K}}\mathbf{ur}$ and $x \in \boldsymbol{X}$. Then there exists a Kuranishi neighbourhood (V, E, Γ, s, ψ) on $\boldsymbol{X}$, as in §6.4, which is minimal at x as in Definition 10.46, with $\Gamma \cong G_x\boldsymbol{X}$. Any two Kuranishi neighbourhoods on $\boldsymbol{X}$ minimal at x are strictly isomorphic near x.*

Theorem 10.51. *Let $\boldsymbol{X}$ be a Kuranishi space in $\dot{\mathbf{K}}\mathbf{ur}$, and $x \in \boldsymbol{X}$, and (V, E, Γ, s, ψ) be a Kuranishi neighbourhood on $\boldsymbol{X}$ minimal at $x \in \boldsymbol{X}$, which exists by Proposition 10.50. Suppose $(V_a, E_a, \Gamma_a, s_a, \psi_a)$ is any other Kuranishi neighbourhood on $\boldsymbol{X}$ with $x \in \mathrm{Im}\,\psi_a$. Then $(V_a, E_a, \Gamma_a, s_a, \psi_a)$ is strictly isomorphic to $(V^{\Gamma_a,\iota}_{(n),\rho}, E^{\Gamma_a,\iota}_{(n),\rho}, \Gamma_a, s^{\Gamma_a,\iota}_{(n),\rho}, \psi^{\Gamma_a,\iota}_{(n),\rho})$ near x as in Definition 10.47, where*

$$n = \dim V_a - \dim V = \dim V_a - \mathrm{vdim}\,\boldsymbol{X} - \dim O_x\boldsymbol{X} \geqslant 0,$$

and $(V^{\Gamma_a,\iota}_{(n),\rho}, \ldots, \psi^{\Gamma_a,\iota}_{(n),\rho})$ is the Kuranishi neighbourhood on $\boldsymbol{X}$ constructed in Definition 10.48 from $(V, E, \Gamma, s, \psi), n$, an injective morphism $\iota : \Gamma \hookrightarrow \Gamma_a$, and some representation ρ of Γ on $\mathbb{R}^n$.

Theorem 10.52. *A Kuranishi space $\boldsymbol{X}$ in $\dot{\mathbf{K}}\mathbf{ur}$ is an orbifold, in the sense of Proposition 6.64, if and only if $O_x\boldsymbol{X} = 0$ for all $x \in \boldsymbol{X}$.*

The proof of Theorem 10.52 is simpler than that of Theorem 10.45, as we only need the analogue of the first part of the proof showing that $\boldsymbol{X} \simeq \boldsymbol{X}' = (X, \mathcal{K}')$ in $\dot{\mathbf{K}}\mathbf{ur}$ for $\mathcal{K}' = \big(I, (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \Lambda_{ijk,\ i,j,k\in I}\big)$ a Kuranishi structure with $E_i = s_i = 0$ for all $i \in I$. As for (10.53), the results of §10.4.4 above apply in the 2-categories

$$\mathbf{Kur}, \mathbf{Kur^c}, \mathbf{Kur^{gc}}, \mathbf{Kur^{ac}}, \mathbf{Kur^{c,ac}}.$$

10.5 Conditions for étale (1-)morphisms, equivalences, and coordinate changes

A (1-)morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}, \boldsymbol{\mu}\mathbf{\dot{K}ur}, \mathbf{\dot{K}ur}$ is called *étale* if it is locally an equivalence/isomorphism. We now prove necessary and sufficient conditions for (1-)morphisms $\boldsymbol{f}$ to be étale, and to be equivalences/isomorphisms, and for a (1-)morphism of (m- or μ-)Kuranishi neighbourhoods to be a coordinate change.

We suppose only that the category $\mathbf{\dot{M}an}$ used to define $\mathbf{m\dot{K}ur}, \boldsymbol{\mu}\mathbf{\dot{K}ur}, \mathbf{\dot{K}ur}$ satisfies Assumptions 3.1–3.7, and specify additional assumptions as needed.

10.5.1 Étale 1-morphisms, equivalences, and coordinate changes in $\mathbf{m\dot{K}ur}$

Definition 10.53. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in $\mathbf{m\dot{K}ur}$. We call $\boldsymbol{f}$ *étale* if it is a local equivalence. That is, $\boldsymbol{f}$ is étale if for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$ there exist open neighbourhoods $\boldsymbol{X}'$ of x in $\boldsymbol{X}$ and $\boldsymbol{Y}'$ of y in $\boldsymbol{Y}$ such that $\boldsymbol{f}(\boldsymbol{X}') \subseteq \boldsymbol{Y}'$, and $\boldsymbol{f}|_{\boldsymbol{X}'} : \boldsymbol{X}' \to \boldsymbol{Y}'$ is an equivalence in $\mathbf{m\dot{K}ur}$.

Theorem 10.54. *A* 1*-morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *in* $\mathbf{m\dot{K}ur}$ *is an equivalence if and only if* $\boldsymbol{f}$ *is étale and the underlying continuous map* $f : X \to Y$ *is a bijection.*

Proof. For the 'only if' part, let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be an equivalence. Then $\boldsymbol{f}$ is étale, as we can take $\boldsymbol{X}' = \boldsymbol{X}$, $\boldsymbol{Y}' = \boldsymbol{Y}$ in Definition 10.53, and $\boldsymbol{f}$ has a quasi-inverse $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{X}$ with $g = f^{-1} : Y \to X$, so that $f : X \to Y$ is a bijection.

For the 'if' part, suppose $\boldsymbol{f}$ is étale and $f : X \to Y$ is a bijection, and write $g = f^{-1} : Y \to X$ for the inverse map. As $\boldsymbol{f}$ is étale we can cover $\boldsymbol{X}, \boldsymbol{Y}$ by open $\boldsymbol{X}', \boldsymbol{Y}'$ such that $\boldsymbol{f}|_{\boldsymbol{X}'} : \boldsymbol{X}' \to \boldsymbol{Y}'$ is an equivalence, and then $g|_{Y'} : Y' \to X'$ is continuous. Thus g is continuous, and f, g are homeomorphisms.

Use notation (4.6), (4.7), (4.9) for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{f}$. Then for all $i \in I$ and $j \in J$ we have a 1-morphism $\boldsymbol{f}_{ij} : (U_i, D_i, r_i, \chi_i) \to (V_j, E_j, s_j, \psi_j)$ over (S, f) for $S = \operatorname{Im}\chi_i \cap f^{-1}(\operatorname{Im}\psi_j)$. Identifying X, Y using f, consider $\boldsymbol{f}_{ij}$ as a 1-morphism of m-Kuranishi neighbourhoods on X over S. Then $\boldsymbol{f}$ being étale means that $\boldsymbol{f}_{ij}$ is locally a coordinate change (i.e. locally an equivalence over id_X).

Theorem 4.13 says $\boldsymbol{\mathcal{E}qu}((U_i, D_i, r_i, \chi_i), (V_j, E_j, s_j, \psi_j))$ is a stack over S, so $\boldsymbol{f}_{ij}$ locally a coordinate change implies it is globally a coordinate change. Hence there exist a 1-morphism $\boldsymbol{g}_{ji} : (V_j, E_j, s_j, \psi_j) \to (U_i, D_i, r_i, \chi_i)$ and 2-morphisms $\boldsymbol{\iota}_{ij} : \boldsymbol{g}_{ji} \circ \boldsymbol{f}_{ij} \Rightarrow \mathrm{id}_{(U_i,D_i,r_i,\chi_i)}$, $\boldsymbol{\kappa}_{ji} : \boldsymbol{f}_{ij} \circ \boldsymbol{g}_{ji} \Rightarrow \mathrm{id}_{(V_j,E_j,s_j,\psi_j)}$ over S. By Proposition A.5 we choose these to satisfy $\boldsymbol{\kappa}_{ji} * \mathrm{id}_{\boldsymbol{f}_{ij}} = \mathrm{id}_{\boldsymbol{f}_{ij}} * \boldsymbol{\iota}_{ij}$ and $\boldsymbol{\iota}_{ij} * \mathrm{id}_{\boldsymbol{g}_{ji}} = \mathrm{id}_{\boldsymbol{g}_{ij}} * \boldsymbol{\kappa}_{ji}$. No longer identifying X, Y, we consider $\boldsymbol{g}_{ji}$ a 1-morphism over (T, g) for $T = \operatorname{Im}\psi_j \cap g^{-1}(\operatorname{Im}\chi_i)$, and $\boldsymbol{\iota}_{ij}, \boldsymbol{\kappa}_{ji}$ as 2-morphisms over S, T.

For all $j,j'\in J$ and $i,i'\in I$, define 2-morphisms $\boldsymbol{G}_{jj'}^{i}:\boldsymbol{g}_{j'i}\circ\Upsilon_{jj'}\Rightarrow\boldsymbol{g}_{ji}$, $\boldsymbol{G}_{j}^{ii'}:\mathrm{T}_{ii'}\circ\boldsymbol{g}_{ji}\Rightarrow\boldsymbol{g}_{ji'}$ by the commutative diagrams

$$
\begin{array}{ccccc}
\boldsymbol{g}_{j'i}\circ\Upsilon_{jj'} & = & \boldsymbol{g}_{j'i}\circ\Upsilon_{jj'}\circ\mathrm{id}_{(V_j,E_j,s_j,\psi_j)} & \xRightarrow[\mathrm{id}_{\boldsymbol{g}_{j'i}\circ\Upsilon_{jj'}}*\boldsymbol{\kappa}_{ji}^{-1}]{} & \boldsymbol{g}_{j'i}\circ\Upsilon_{jj'}\circ\boldsymbol{f}_{ij}\circ\boldsymbol{g}_{ji} \\
\Big\Downarrow{\scriptstyle \boldsymbol{G}_{jj'}^{i}} & & & & \Big\Downarrow{\scriptstyle \mathrm{id}_{\boldsymbol{g}_{j'i}}*\boldsymbol{F}_{i}^{jj'}*\mathrm{id}_{\boldsymbol{g}_{ji}}} \\
\boldsymbol{g}_{ji} & = & \mathrm{id}_{(U_i,D_i,r_i,\chi_i)}\circ\boldsymbol{g}_{ji} & \xLeftarrow{\boldsymbol{\iota}_{ij'}*\mathrm{id}_{\boldsymbol{g}_{ji}}} & \boldsymbol{g}_{j'i}\circ\boldsymbol{f}_{ij'}\circ\boldsymbol{g}_{ji},
\end{array}
\tag{10.54}
$$

$$
\begin{array}{ccccc}
\mathrm{T}_{ii'}\circ\boldsymbol{g}_{ji} & = & \mathrm{id}_{(U_i,D_i,r_i,\chi_i)}\circ\mathrm{T}_{ii'}\circ\boldsymbol{g}_{ji} & \xRightarrow[\boldsymbol{\iota}_{i'j}^{-1}*\mathrm{id}_{\mathrm{T}_{ii'}\circ\boldsymbol{g}_{ji}}]{} & \boldsymbol{g}_{ji'}\circ\boldsymbol{f}_{i'j}\circ\mathrm{T}_{ii'}\circ\boldsymbol{g}_{ji} \\
\Big\Downarrow{\scriptstyle \boldsymbol{G}_{j}^{ii'}} & & & & \Big\Downarrow{\scriptstyle \mathrm{id}_{\boldsymbol{g}_{ji'}}*\boldsymbol{F}_{ii'}^{j}*\mathrm{id}_{\boldsymbol{g}_{ji}}} \\
\boldsymbol{g}_{ji'} & = & \boldsymbol{g}_{ji'}\circ\mathrm{id}_{(V_j,E_j,s_j,\psi_j)} & \xLeftarrow{\mathrm{id}_{\boldsymbol{g}_{ji'}}*\boldsymbol{\kappa}_{ji}} & \boldsymbol{g}_{ji'}\circ\boldsymbol{f}_{ij}\circ\boldsymbol{g}_{ji}.
\end{array}
\tag{10.55}
$$

We now claim that as in (4.9),

$$
\boldsymbol{g}=\bigl(g,\boldsymbol{g}_{ji,\;j\in J,\;i\in I},\;\boldsymbol{G}_{jj',\;j,j'\in J}^{i,\;i\in I},\;\boldsymbol{G}_{j,\;j\in J}^{ii',\;i,i'\in I}\bigr)
$$

is a 1-morphism $\boldsymbol{g}:\boldsymbol{Y}\rightarrow\boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$. Definition 4.17(a)–(d) for $\boldsymbol{g}$ are immediate. Part (e) follows from (10.54)–(10.55) and (e) for $\boldsymbol{f}$ and $\boldsymbol{\iota}_{ij}*\mathrm{id}_{\boldsymbol{g}_{ji}}=\mathrm{id}_{\boldsymbol{g}_{ij}}*\boldsymbol{\kappa}_{ji}$. To prove (f), let $i\in I$ and $j,j',j''\in J$, and consider Figure 10.1. The small rectangle near the bottom commutes by Definition 4.17(h) for $\boldsymbol{f}$, the two parallel arrows on the right are equal as $\boldsymbol{\kappa}_{j'i}*\mathrm{id}_{\boldsymbol{f}_{ij'}}=\mathrm{id}_{\boldsymbol{f}_{ij'}}*\boldsymbol{\iota}_{ij'}$, three quadrilaterals commute by (10.54), and the rest of the diagram commutes by properties of 2-categories. Hence Figure 10.1 commutes, and the outside rectangle proves part (f) for $\boldsymbol{g}$. We can prove (g),(h) in a similar way. Thus $\boldsymbol{g}$ is a 1-morphism.

We claim that there are 2-morphisms $\boldsymbol{\eta}=(\boldsymbol{\eta}_{ii',\;i,i'\in I}):\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$ and $\boldsymbol{\zeta}=(\boldsymbol{\zeta}_{jj',\;j,j'\in J}):\boldsymbol{f}\circ\boldsymbol{g}\Rightarrow\mathbf{id}_{\boldsymbol{Y}}$ in $\mathbf{m\dot{K}ur}$, which are characterized uniquely by the property that for all $i,i'\in I$ and $j,j'\in J$, the following commute

$$
\begin{array}{ccccc}
\boldsymbol{g}_{ji'}\circ\boldsymbol{f}_{i'j}\circ\mathrm{T}_{ii'} & \xRightarrow[\mathrm{id}_{\boldsymbol{g}_{ji'}}*\boldsymbol{F}_{ii'}^{j}]{} & \boldsymbol{g}_{ji}\circ\boldsymbol{f}_{ij} & \xRightarrow[\Theta_{iji'}^{\boldsymbol{g},\boldsymbol{f}}]{} & (\boldsymbol{g}\circ\boldsymbol{f})_{ii'} \\
\Big\Downarrow{\scriptstyle \boldsymbol{\iota}_{i'j}*\mathrm{id}_{\mathrm{T}_{ii'}}} & & & & \Big\Downarrow{\scriptstyle \boldsymbol{\eta}_{ii'}} \\
\mathrm{id}_{(U_{i'},D_{i'},r_{i'},\chi_{i'})}\circ\mathrm{T}_{ii'} & = & \mathrm{T}_{ii'} & = & (\mathbf{id}_{\boldsymbol{X}})_{ii'},
\end{array}
\tag{10.56}
$$

$$
\begin{array}{ccccc}
\boldsymbol{f}_{ij'}\circ\boldsymbol{g}_{j'i}\circ\Upsilon_{jj'} & \xRightarrow[\mathrm{id}_{\boldsymbol{f}_{ij'}}*\boldsymbol{G}_{jj'}^{i}]{} & \boldsymbol{f}_{ij}\circ\boldsymbol{g}_{ji} & \xRightarrow[\Theta_{jij'}^{\boldsymbol{f},\boldsymbol{g}}]{} & (\boldsymbol{f}\circ\boldsymbol{g})_{jj'} \\
\Big\Downarrow{\scriptstyle \boldsymbol{\kappa}_{j'i}*\mathrm{id}_{\Upsilon_{jj'}}} & & & & \Big\Downarrow{\scriptstyle \boldsymbol{\zeta}_{jj'}} \\
\mathrm{id}_{(V_{j'},E_{j'},s_{j'},\psi_{j'})}\circ\Upsilon_{jj'} & = & \Upsilon_{jj'} & = & (\mathbf{id}_{\boldsymbol{Y}})_{jj'},
\end{array}
\tag{10.57}
$$

where $\Theta_{iji'}^{\boldsymbol{g},\boldsymbol{f}},\Theta_{jij'}^{\boldsymbol{f},\boldsymbol{g}}$ are as in Definition 4.20 for $\boldsymbol{g}\circ\boldsymbol{f},\boldsymbol{f}\circ\boldsymbol{g}$ in $\mathbf{m\dot{K}ur}$, and (10.56), (10.57) are in 2-morphisms of m-Kuranishi neighbourhoods over $S=\operatorname{Im}\chi_i\cap\operatorname{Im}\chi_{i'}\cap f^{-1}(\operatorname{Im}\psi_j)\subseteq X$ and $T=\operatorname{Im}\psi_j\cap\operatorname{Im}\psi_{j'}\cap g^{-1}(\operatorname{Im}\chi_i)\subseteq Y$.

To prove this for $\boldsymbol{\eta}$, first for $i,i'\in I$ and $j,j'\in J$ we show that (10.56) for i,i',j and for i,i',j' determine the same 2-morphism $\boldsymbol{\eta}_{ii'}$ on $\operatorname{Im}\chi_i\cap\operatorname{Im}\chi_{i'}\cap f^{-1}(\operatorname{Im}\psi_j\cap\operatorname{Im}\psi_{j'})$. Thus, as the $\operatorname{Im}\chi_i\cap\operatorname{Im}\chi_{i'}\cap f^{-1}(\operatorname{Im}\psi_j)$ for $j\in J$ cover $\operatorname{Im}\chi_i\cap\operatorname{Im}\chi_{i'}$, by the sheaf property of 2-morphisms in Theorem 4.13 there is

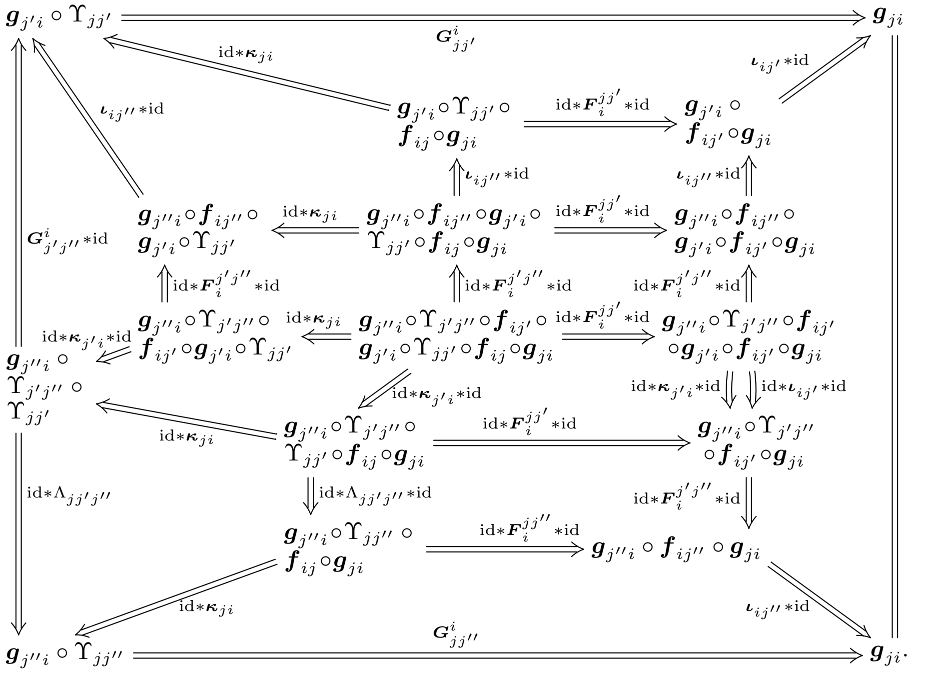


Figure 10.1: Proof of Definition 4.17(f) for $\boldsymbol{g}$

a unique 2-morphism $\boldsymbol{\eta}_{ii'}$ over $\operatorname{Im}\chi_i \cap \operatorname{Im}\chi_{i'}$ such that (10.56) commutes for all $j \in J$. Then we fix $j \in J$, and show these $\boldsymbol{\eta}_{ii'}$ satisfy the restrictions of Definition 4.18(a),(b) to the intersections of their domains with $f^{-1}(\operatorname{Im}\psi_j)$ using (10.54)–(10.56) and properties of the $\Theta^{\boldsymbol{g},\boldsymbol{f}}_{iji'}$ in Proposition 4.19. As $f^{-1}(\operatorname{Im}\psi_j)$ for $j \in J$ cover X, by the sheaf property of 2-morphisms this implies Definition 4.18(a),(b) for the $\boldsymbol{\eta}_{ii'}$, and $\boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \mathbf{id}_{\boldsymbol{X}}$ is a 2-morphism in $\mathbf{m\dot{K}ur}$. The proof for $\boldsymbol{\zeta}$ is the same. Hence $\boldsymbol{f}$ is an equivalence in $\mathbf{m\dot{K}ur}$, as we have to prove. □

Here is a necessary and sufficient condition for 1-morphisms in $\mathbf{m\dot{K}ur}$ to be étale. Combining it with Theorem 10.54 gives a necessary and sufficient condition for 1-morphisms to be equivalences.

Theorem 10.55. *Suppose the category* $\mathbf{\dot{M}an}$ *used to define* $\mathbf{m\dot{K}ur}$ *satisfies Assumptions* 3.1–3.7, 10.1, 10.9 *and* 10.11, *with tangent spaces written* T_uU *for* $U \in \mathbf{\dot{M}an}$, *and discrete properties* $\boldsymbol{A}, \boldsymbol{B}$, *where if* $f : U \to V$ *in* $\mathbf{\dot{M}an}$ *is* $\boldsymbol{A}$ *then tangent maps* $T_uf : T_uU \to T_vV$ *are defined, and if* f *is* $\boldsymbol{B}$ *(which implies* $\boldsymbol{A}$*) and* T_uf *is an isomorphism then* f *is a local diffeomorphism near* u.

Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *be a* 1*-morphism in* $\mathbf{m\dot{K}ur}$. *Then* $\boldsymbol{f}$ *is étale if and only if* $\boldsymbol{f}$ *is* $\boldsymbol{B}$ *and the linear maps* $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$, $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$ *from* §10.2.1 *are both isomorphisms for all* $x \in \boldsymbol{X}$ *with* $\boldsymbol{f}(x) = y$ *in* $\boldsymbol{Y}$.

The 'only if' part does not require Assumptions 10.9 *and* 10.11*.*

Proof. For the 'only if' part, suppose $\boldsymbol{f}$ is étale. Then for each $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$ there are open neighbourhoods $\boldsymbol{X}', \boldsymbol{Y}'$ of x, y in $\boldsymbol{X}, \boldsymbol{Y}$ with $\boldsymbol{f}|_{\boldsymbol{X}'} : \boldsymbol{X}' \to \boldsymbol{Y}'$ an equivalence. Thus $\boldsymbol{f}|_{\boldsymbol{X}'}$ is $\boldsymbol{A}$ and $\boldsymbol{B}$ by Proposition 4.36(c), and $T_x\boldsymbol{f}, O_x\boldsymbol{f}$ are isomorphisms by Lemma 10.23. As such $\boldsymbol{X}'$ cover $\boldsymbol{X}$, we see that $\boldsymbol{f}$ is locally $\boldsymbol{B}$, so it is $\boldsymbol{B}$ as this is a local condition by Definition 3.18(iv).

For the 'if' part, suppose $\boldsymbol{f}$ is $\boldsymbol{B}$ (which implies $\boldsymbol{f}$ is $\boldsymbol{A}$), and $T_x\boldsymbol{f}, O_x\boldsymbol{f}$ are isomorphisms for all $x \in \boldsymbol{X}$. Let $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$. By Proposition 10.40 we can choose m-Kuranishi neighbourhoods $(U_a, D_a, r_a, \chi_a), (V_b, E_b, s_b, \psi_b)$ on $\boldsymbol{X}, \boldsymbol{Y}$, as in §4.7, which are minimal at $x \in \operatorname{Im}\chi_a$ and $y \in \operatorname{Im}\psi_b$, as in §10.4.1. Making U_a smaller if necessary we can take $f(\operatorname{Im}\chi_a) \subseteq \operatorname{Im}\psi_b$. Theorem 4.56(b) now gives a 1-morphism $\boldsymbol{f}_{ab} = (U_{ab}, f_{ab}, \hat f_{ab}) : (U_a, D_a, r_a, \chi_a) \to (V_b, E_b, s_b, \psi_b)$ of m-Kuranishi neighbourhoods over $(\operatorname{Im}\chi_a, \boldsymbol{f})$ on $\boldsymbol{X}, \boldsymbol{Y}$, as in Definition 4.54.

Definition 4.2(d) says that $\hat f_{ab}(r_a) = f_{ab}^*(s_b) + O(r_a^2)$. By the argument in the proof of Proposition 10.42 we can choose $\hat f'_{ab} : D_a \to f_{ab}^*(E_b)$ with $\hat f'_{ab} = \hat f_{ab} + O(r_a)$ and $\hat f'_{ab}(r_a) = f_{ab}^*(s_b)$. Then replacing $\hat f_{ab}$ by $\hat f'_{ab}$, which is allowed in Theorem 4.56(b) as it does not change $\boldsymbol{f}_{ab}$ up to 2-isomorphism, we can suppose that $\hat f_{ab}(r_a) = f_{ab}^*(s_b)$.

Write $u_a = \chi_a^{-1}(x)$, $v_b = \psi_b^{-1}(y)$. Then (10.29) gives a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & T_x\boldsymbol{X} & \xrightarrow{\cong} & T_{u_a}U_a & \xrightarrow{\mathrm{d}_{u_a}r_a=0} & D_a|_{u_a} & \xrightarrow{\cong} & O_x\boldsymbol{X} & \longrightarrow & 0 \\
 & & \cong\big\downarrow T_x\boldsymbol{f} & & \big\downarrow T_{u_a}f_{ab} & & \big\downarrow \hat f_{ab}|_{u_a} & & \cong\big\downarrow O_x\boldsymbol{f} & & \\
0 & \longrightarrow & T_y\boldsymbol{Y} & \xrightarrow{\cong} & T_{v_b}V_b & \xrightarrow{\mathrm{d}_{v_b}s_b=0} & E_b|_{v_b} & \xrightarrow{\cong} & O_y\boldsymbol{Y} & \longrightarrow & 0,
\end{array}$$

with exact rows. By assumption $T_x\boldsymbol{f}, O_x\boldsymbol{f}$ are isomorphisms, and $\mathrm{d}_{u_a}r_a = \mathrm{d}_{v_b}s_b = 0$ as $(U_a, D_a, r_a, \chi_a), (V_b, E_b, s_b, \psi_b)$ are minimal at x, y, so the maps $T_x\boldsymbol{X} \to T_{u_a}U_a$, $D_a|_{u_a} \to O_x\boldsymbol{X}$, $T_y\boldsymbol{Y} \to T_{v_b}V_b$, $E_b|_{v_b} \to O_y\boldsymbol{Y}$ are isomorphisms. Hence $T_{u_a}f_{ab} : T_{u_a}U_a \to T_{v_b}V_b$ and $\hat f_{ab}|_{u_a} : D_a|_{u_a} \to E_b|_{v_b}$ are isomorphisms.

As $\boldsymbol{f}$ is $\boldsymbol{B}$, $\boldsymbol{f}_{ab}$ is $\boldsymbol{B}$, and f_{ab} is $\boldsymbol{B}$ near u_a. Since $T_{u_a}f_{ab} : T_{u_a}U_a \to T_{v_b}V_b$ is an isomorphism, Assumption 10.11 says that f_{ab} is a local diffeomorphism near u_a, so making U_a, U_{ab}, V_b smaller we can suppose $U_{ab} = U_a$ and $f_{ab} : U_a \to V_b$ is a diffeomorphism in $\dot{\mathbf{M}}\mathbf{an}$. Also $\hat f_{ab}|_{u_a} : D_a|_{u_a} \to E_b|_{v_b}$ an isomorphism implies that $\hat f_{ab} : D_a \to f_{ab}^*(E_b)$ is an isomorphism near u_a, so making U_a, U_{ab}, V_b smaller again we can suppose $\hat f_{ab}$ is an isomorphism.

Thus, we have a 1-morphism $\boldsymbol{f}_{ab} = (U_a, f_{ab}, \hat f_{ab}) : (U_a, D_a, r_a, \chi_a) \to (V_b, E_b, s_b, \psi_b)$ over $(\operatorname{Im}\chi_a, \boldsymbol{f})$ such that $f_{ab} : U_a \to V_b$ is a diffeomorphism and $\hat f_{ab} : D_a \to f_{ab}^*(E_b)$ is an isomorphism with $\hat f_{ab}(r_a) = f_{ab}^*(s_b)$. Let $\boldsymbol{X}' \subseteq \boldsymbol{X}$, $\boldsymbol{Y}' \subseteq \boldsymbol{Y}$ be the open neighbourhoods with topological spaces $X' = \operatorname{Im}\chi_a \subseteq X$, $Y' = \operatorname{Im}\psi_b \subseteq Y$. Then $f|_{X'} : X' \to Y'$ is a homeomorphism, as $f_{ab}|_{r_a^{-1}(0)} : r_a^{-1}(0) \to s_b^{-1}(0)$ is, so we can define $g = f|_{X'}^{-1} : Y' \to X'$, and then

$$\boldsymbol{g}_{ba} = \bigl(V_b, f_{ab}^{-1}, (f_{ab}^{-1})^*(\hat f_{ab}^{-1})\bigr) : (V_b, E_b, s_b, \psi_b) \longrightarrow (U_a, D_a, r_a, \chi_a)$$

is a 1-morphism of m-Kuranishi neighbourhoods over $(g, \operatorname{Im}\psi_b)$ which is a strict inverse for $\boldsymbol{f}_{ab}$, that is, $\boldsymbol{g}_{ba} \circ \boldsymbol{f}_{ab} = \mathrm{id}_{(U_a,D_a,r_a,\chi_a)}$, $\boldsymbol{f}_{ab} \circ \boldsymbol{g}_{ba} = \mathrm{id}_{(V_b,E_b,s_b,\psi_b)}$.

Clearly this implies that $\boldsymbol{f}|_{\boldsymbol{X}'}:\boldsymbol{X}'\to\boldsymbol{Y}'$ is an equivalence in $\mathbf{m\dot{K}ur}$. As we can find such open $x\in\boldsymbol{X}'\subseteq\boldsymbol{X}$, $y\in\boldsymbol{Y}'\subseteq\boldsymbol{Y}$ for all $x\in\boldsymbol{X}$ with $\boldsymbol{f}(x)=y$ in $\boldsymbol{Y}$, we see that $\boldsymbol{f}$ is étale, as we have to prove. □

We apply Theorems 10.54–10.55 to our examples of 2-categories $\mathbf{m\dot{K}ur}$:

Theorem 10.56. (a) *Work in the* 2*-category of m-Kuranishi spaces* $\mathbf{mKur}$ *constructed from* $\dot{\mathbf{Man}}=\mathbf{Man}$, *using ordinary tangent spaces* T_vV *for* $V\in\mathbf{Man}$. *Then a* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ *in* $\mathbf{mKur}$ *is étale if and only if* $T_x\boldsymbol{f}:T_x\boldsymbol{X}\to T_y\boldsymbol{Y}$, $O_x\boldsymbol{f}:O_x\boldsymbol{X}\to O_y\boldsymbol{Y}$ *are isomorphisms for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{f}(x)=y$ *in* $\boldsymbol{Y}$. *If this holds then* $\boldsymbol{f}$ *is an equivalence if and only if* $f:X\to Y$ *is a bijection.*

(b) *Work in the* 2*-category* $\mathbf{mKur^c}$ *constructed from* $\dot{\mathbf{Man}}=\mathbf{Man^c}$, *using ordinary tangent spaces* T_vV *for* $V\in\mathbf{Man^c}$. *Then a* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ *in* $\mathbf{mKur^c}$ *is étale if and only if* $\boldsymbol{f}$ *is simple and* $T_x\boldsymbol{f}:T_x\boldsymbol{X}\to T_y\boldsymbol{Y}$, $O_x\boldsymbol{f}:O_x\boldsymbol{X}\to O_y\boldsymbol{Y}$ *are isomorphisms for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{f}(x)=y$ *in* $\boldsymbol{Y}$. *If this holds then* $\boldsymbol{f}$ *is an equivalence if and only if* $f:X\to Y$ *is a bijection.*

(c) *Work in one of* $\mathbf{m\dot{K}ur}=\mathbf{mKur^c},\mathbf{mKur^{gc}},\mathbf{mKur^{ac}}$ *or* $\mathbf{mKur^{c,ac}}$ *constructed from* $\dot{\mathbf{Man}}=\mathbf{Man^c},\mathbf{Man^{gc}},\mathbf{Man^{ac}}$ *or* $\mathbf{Man^{c,ac}}$, *using b-tangent spaces* ${}^bT_vV$ *for* $V\in\dot{\mathbf{Man}}$, *as in* §2.3. *Then a* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ *in* $\mathbf{m\dot{K}ur}$ *is étale if and only if* $\boldsymbol{f}$ *is simple and* ${}^bT_x\boldsymbol{f}:{}^bT_x\boldsymbol{X}\to{}^bT_y\boldsymbol{Y}$, ${}^bO_x\boldsymbol{f}:{}^bO_x\boldsymbol{X}\to{}^bO_y\boldsymbol{Y}$ *are isomorphisms for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{f}(x)=y$ *in* $\boldsymbol{Y}$. *If this holds then* $\boldsymbol{f}$ *is an equivalence if and only if* $f:X\to Y$ *is a bijection.*

(d) *Work in one of* $\mathbf{m\dot{K}ur}=\mathbf{mKur^c},\mathbf{mKur^{gc}},\mathbf{mKur^{ac}}$ *or* $\mathbf{mKur^{c,ac}}$ *constructed from* $\dot{\mathbf{Man}}=\mathbf{Man^c},\mathbf{Man^{gc}},\mathbf{Man^{ac}}$ *or* $\mathbf{Man^{c,ac}}$, *using stratum tangent spaces* $\tilde{T}_vV$ *for* $V\in\dot{\mathbf{Man}}$, *as in Example* 10.2(iv)*. Then a* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ *in* $\mathbf{m\dot{K}ur}$ *is étale if and only if* $\boldsymbol{f}$ *is simple and* $\tilde{T}_x\boldsymbol{f}:\tilde{T}_x\boldsymbol{X}\to\tilde{T}_y\boldsymbol{Y}$, $\tilde{O}_x\boldsymbol{f}:\tilde{O}_x\boldsymbol{X}\to\tilde{O}_y\boldsymbol{Y}$ *are isomorphisms for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{f}(x)=y$ *in* $\boldsymbol{Y}$. *If this holds then* $\boldsymbol{f}$ *is an equivalence if and only if* $f:X\to Y$ *is a bijection.*

Proof. Parts (a),(c),(d) follow from Theorems 10.54–10.55 and Examples 10.2, 10.10 and 10.12. Part (b) does *not* follow directly from Theorems 10.54–10.55, since as in Example 10.10(b), Assumption 10.9 fails in $\dot{\mathbf{Man}}\mathbf{^c}$ for ordinary tangent spaces T_vV. Instead, we deduce (b) indirectly from (d). Suppose $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is simple and $x\in\boldsymbol{X}$ with $\boldsymbol{f}(x)=y$ in $\boldsymbol{Y}$. Then $\tilde{N}_x\boldsymbol{f}:\tilde{N}_x\boldsymbol{X}\to\tilde{N}_y\boldsymbol{Y}$ from Example 10.32(a) is an isomorphism as $\boldsymbol{f}$ is simple, so from equation (10.47) of Example 10.33 with exact rows we see that $T_x\boldsymbol{f},O_x\boldsymbol{f}$ are isomorphisms if and only if $\tilde{T}_x\boldsymbol{f},\tilde{O}_x\boldsymbol{f}$ are isomorpisms, and thus (b) follows from (d). □

Here is a criterion for when a 1-morphism of m-Kuranishi neighbourhoods is a coordinate change.

Theorem 10.57. *Suppose* $\dot{\mathbf{Man}}$ *satisfies Assumptions* 3.1–3.7, 10.1, 10.9 *and* 10.11, *with tangent spaces* T_vV *for* $V\in\dot{\mathbf{Man}}$, *and discrete properties* $\boldsymbol{A},\boldsymbol{B}$.

Let $\Phi_{ij}=(V_{ij},\phi_{ij},\hat{\phi}_{ij}):(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ *be a* 1*-morphism of m-Kuranishi neighbourhoods in* $\dot{\mathbf{Man}}$ *on a topological space* X *over an open*

$S \subseteq X$, as in §4.1, and suppose Φ_{ij} is $\boldsymbol{B}$. Let $x \in S$, and set $v_i = \psi_i^{-1}(x) \in V_i$ and $v_j = \psi_j^{-1}(x) \in V_j$. Consider the sequence of real vector spaces:

$$0 \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}_{v_i}s_i|_{v_i} \oplus T_{v_i}\phi_{ij}} E_i|_{v_i} \oplus T_{v_j}V_j \xrightarrow{-\hat\phi_{ij}|_{v_i} \oplus \mathrm{d}_{v_j}s_j} E_j|_{v_j} \longrightarrow 0. \quad (10.58)$$

Here $\mathrm{d}_{v_i}s_i, \mathrm{d}_{v_j}s_j$ are as in Definition 10.6, and differentiating Definition 4.2(d) at v_i implies that (10.58) is a complex. Then Φ_{ij} is a coordinate change over S in the sense of Definition 4.10 if and only if (10.58) is exact for all $x \in S$.

The 'only if' part does not require Assumptions 10.9 and 10.11.

Proof. We can regard Φ_{ij} as a 1-morphism $\Phi'_{ij} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$ between m-Kuranishi spaces $\boldsymbol{X}, \boldsymbol{Y}$ with only one m-Kuranishi neighbourhood, where the underlying continuous map of Φ'_{ij} is $\mathrm{id}_S : S \to S$. Then Φ_{ij} is a coordinate change if and only if Φ'_{ij} is an equivalence in $\mathbf{m\dot{K}ur}$, which holds if and only if Φ'_{ij} is étale by Theorem 10.54, as $\mathrm{id}_S : S \to S$ is a bijection.

Let $x \in S$, and set $v_i = \psi_i^{-1}(x) \in V_i$ and $v_j = \psi_j^{-1}(x) \in V_j$. As in (10.28) we have a commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & T_x\boldsymbol{X} & \longrightarrow & T_{v_i}V_i & \xrightarrow{\mathrm{d}_{v_i}s_i} & E_i|_{v_i} & \longrightarrow & O_x\boldsymbol{X} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T_x\Phi'_{ij}} & & \downarrow{\scriptstyle T_{v_i}\phi_{ij}} & & \downarrow{\scriptstyle \hat\phi_{ij}|_{v_i}} & & \downarrow{\scriptstyle O_x\Phi'_{ij}} & & \\
0 & \longrightarrow & T_x\boldsymbol{Y} & \longrightarrow & T_{v_j}V_j & \xrightarrow{\mathrm{d}_{v_j}s_j} & E_j|_{v_j} & \longrightarrow & O_x\boldsymbol{Y} & \longrightarrow & 0.
\end{array}$$

By elementary linear algebra we can show that (10.58) is exact if and only if $T_x\Phi'_{ij}$ and $O_x\Phi'_{ij}$ are isomorphisms. Thus (10.58) is exact for all $x \in S$ if and only if $T_x\Phi'_{ij}, O_x\Phi'_{ij}$ are isomorphisms for all $x \in S$, if and only if Φ'_{ij} is étale by Theorem 10.55, if and only if Φ_{ij} is a coordinate change. □

We apply Theorem 10.57 to our examples of 2-categories $\mathbf{m\dot{K}ur}$. Here as for Theorem 10.56, parts (a),(c),(d) follow from Theorem 10.57 and Examples 10.2, 10.10 and 10.12, and (b) can be deduced indirectly from (d), equation (10.47) of Example 10.33, and the proof of Theorem 10.57.

Theorem 10.58. *Working in a category $\mathbf{\dot{M}an}$ which we specify in* **(a)**–**(d)** *below, let $\Phi_{ij} = (V_{ij}, \phi_{ij}, \hat\phi_{ij}) : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ be a 1-morphism of m-Kuranishi neighbourhoods on a topological space X over an open $S \subseteq X$, and for each $x \in S$, set $v_i = \psi_i^{-1}(x) \in V_i$ and $v_j = \psi_j^{-1}(x) \in V_j$. Then:*

(a) *If $\mathbf{\dot{M}an} = \mathbf{Man}$ then Φ_{ij} is a coordinate change over S if and only if the following complex is exact for all $x \in S$:*

$$0 \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}_{v_i}s_i|_{v_i} \oplus T_{v_i}\phi_{ij}} E_i|_{v_i} \oplus T_{v_j}V_j \xrightarrow{-\hat\phi_{ij}|_{v_i} \oplus \mathrm{d}_{v_j}s_j} E_j|_{v_j} \longrightarrow 0. \quad (10.59)$$

(b) *If $\mathbf{\dot{M}an} = \mathbf{Man^c}$ then Φ_{ij} is a coordinate change over S if and only if ϕ_{ij} is simple near $\psi_i^{-1}(S)$ and (10.59) is exact for all $x \in S$.*

(c) *If* $\dot{\mathbf{Man}}$ *is one of* $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}$ *or* $\mathbf{Man^{c,ac}}$ *then* Φ_{ij} *is a coordinate change over* S *if and only if* ϕ_{ij} *is simple near* $\psi_i^{-1}(S)$ *and using b-tangent spaces from* §2.3, *the following is exact for all* $x \in S$*:*

$$0 \longrightarrow {}^bT_{v_i}V_i \xrightarrow{{}^b\mathrm{d}_{v_i}s_i|_{v_i}\oplus {}^bT_{v_i}\phi_{ij}} E_i|_{v_i}\oplus {}^bT_{v_j}V_j \xrightarrow{-\hat\phi_{ij}|_{v_i}\oplus {}^b\mathrm{d}_{v_j}s_j} E_j|_{v_j} \longrightarrow 0.$$

(d) *If* $\dot{\mathbf{Man}}$ *is one of* $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}$ *or* $\mathbf{Man^{c,ac}}$ *then* Φ_{ij} *is a coordinate change over* S *if and only if* ϕ_{ij} *is simple near* $\psi_i^{-1}(S)$ *and using stratum tangent spaces* $\tilde T_vV$ *from Example* 10.2(iv), *the following is exact for all* $x \in S$*:*

$$0 \longrightarrow \tilde T_{v_i}V_i \xrightarrow{\tilde{\mathrm{d}}_{v_i}s_i|_{v_i}\oplus \tilde T_{v_i}\phi_{ij}} E_i|_{v_i}\oplus \tilde T_{v_j}V_j \xrightarrow{-\hat\phi_{ij}|_{v_i}\oplus \tilde{\mathrm{d}}_{v_j}s_j} E_j|_{v_j} \longrightarrow 0.$$

10.5.2 Étale morphisms, isomorphisms, and coordinate changes in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$

All the material of §10.5.1 has analogues for μ-Kuranishi spaces $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ from Chapter 5. As $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ is an ordinary category, we replace equivalences in $\mathbf{m}\dot{\mathbf{Kur}}$ in §10.5.1 by isomorphisms in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$. So we define a morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ to be *étale* if it is a local isomorphism, that is, if for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$ there exist open neighbourhoods $\boldsymbol{X}'$ of x in $\boldsymbol{X}$ and $\boldsymbol{Y}'$ of y in $\boldsymbol{Y}$ such that $\boldsymbol{f}(\boldsymbol{X}') \subseteq \boldsymbol{Y}'$, and $\boldsymbol{f}|_{\boldsymbol{X}'} : \boldsymbol{X}' \to \boldsymbol{Y}'$ is an isomorphism in $\boldsymbol{\mu}\dot{\mathbf{Kur}}$.

The analogue of Theorem 10.54 for $\boldsymbol{\mu}\dot{\mathbf{Kur}}$ is much easier than the $\mathbf{m}\dot{\mathbf{Kur}}$ case in §10.5.1: it is a more-or-less immediate consequence of the sheaf property Theorem 5.10. The analogues of Theorems 10.55–10.58 have essentially the same proofs. We leave the details to the reader.

10.5.3 Étale 1-morphisms, equivalences, and coordinate changes in $\dot{\mathbf{Kur}}$

We now extend the material of §10.5.1 to Kuranishi spaces $\dot{\mathbf{Kur}}$ from Chapter 6. Our analogue of Definition 10.53 for Kuranishi spaces is just the same:

Definition 10.59. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in $\dot{\mathbf{Kur}}$. We call $\boldsymbol{f}$ *étale* if it is a local equivalence. That is, $\boldsymbol{f}$ is étale if for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$ there exist open neighbourhoods $\boldsymbol{X}'$ of x in $\boldsymbol{X}$ and $\boldsymbol{Y}'$ of y in $\boldsymbol{Y}$ such that $\boldsymbol{f}(\boldsymbol{X}') \subseteq \boldsymbol{Y}'$, and $\boldsymbol{f}|_{\boldsymbol{X}'} : \boldsymbol{X}' \to \boldsymbol{Y}'$ is an equivalence in $\dot{\mathbf{Kur}}$.

If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is étale and $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$ then $G_x\boldsymbol{f} : G_x\boldsymbol{X} \to G_y\boldsymbol{Y}$ from §6.5 is an isomorphism, since this holds for equivalences in $\dot{\mathbf{Kur}}$.

Remark 10.60. Our definition of étale is stronger than the usual definition of étale 1-morphisms of stacks in algebraic geometry, in which a 1-morphism $f : X \to Y$ is étale if it is representable and a local isomorphism *in the étale topology*, rather than the Zariski topology. With the algebro-geometric definition, which we do not use, $G_x\boldsymbol{f} : G_x\boldsymbol{X} \to G_y\boldsymbol{Y}$ need only be injective, not an isomorphism.

Here is the analogue of Theorem 10.54. It is proved in the same way, except that we ought to work in weak 2-categories rather than strict 2-categories, so in expressions like $\boldsymbol{g}_{j'i}\circ\boldsymbol{f}_{ij'}\circ\boldsymbol{g}_{ji}$ we have to insert brackets $(\boldsymbol{g}_{j'i}\circ\boldsymbol{f}_{ij'})\circ\boldsymbol{g}_{ji}$, and insert extra 2-morphisms $\boldsymbol{\alpha}_{*,*,*},\boldsymbol{\beta}_*,\boldsymbol{\gamma}_*$ from §6.1, which makes diagrams like Figure 10.1 grow unreasonably large. Since any weak 2-category can be strictified as in §A.3, the strict 2-category proof is guaranteed to extend.

Theorem 10.61. *A* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ *in* $\dot{\mathbf{K}}\mathbf{ur}$ *is an equivalence if and only if* $\boldsymbol{f}$ *is étale and the underlying continuous map* $f:X\rightarrow Y$ *is a bijection.*

Here is the analogue of Theorem 10.55. Its proof is a straightforward modification of that in §10.5.1 to include finite groups. We use Proposition 10.50 and Theorem 6.45(b) in place of Proposition 10.40 and Theorem 4.56(b) to obtain the 1-morphism $\boldsymbol{f}_{ab}:(U_a,D_a,\mathrm{B}_a,r_a,\chi_a)\rightarrow(V_b,E_b,\Gamma_b,s_b,\psi_b)$ over $(\mathrm{Im}\,\chi_a,\boldsymbol{f})$. As $(U_a,D_a,\mathrm{B}_a,r_a,\chi_a),(V_b,E_b,\Gamma_b,s_b,\psi_b)$ are minimal at x,y we have $\mathrm{B}_a\cong G_x\boldsymbol{X}$, $\Gamma_b\cong G_y\boldsymbol{Y}$, so $G_x\boldsymbol{f}:G_x\boldsymbol{X}\rightarrow G_y\boldsymbol{Y}$ an isomorphism implies that $\mathrm{B}_a\cong\Gamma_b$, which is used in the proof that we can modify $\boldsymbol{f}_{ab}$ to a strict isomorphism of Kuranishi neighbourhoods.

Theorem 10.62. *Suppose the category* $\dot{\mathbf{M}}\mathbf{an}$ *used to define* $\dot{\mathbf{K}}\mathbf{ur}$ *satisfies Assumptions* 3.1–3.7, 10.1, 10.9 *and* 10.11, *with tangent spaces written* T_uU *for* $U\in\dot{\mathbf{M}}\mathbf{an}$, *and discrete properties* $\boldsymbol{A},\boldsymbol{B}$, *where if* $f:U\rightarrow V$ *in* $\dot{\mathbf{M}}\mathbf{an}$ *is* $\boldsymbol{A}$ *then tangent maps* $T_uf:T_uU\rightarrow T_vV$ *are defined, and if* f *is* $\boldsymbol{B}$ *(which implies* $\boldsymbol{A}$*) and* T_uf *is an isomorphism then* f *is a local diffeomorphism near* u.

Let $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ *be a* 1*-morphism in* $\dot{\mathbf{K}}\mathbf{ur}$. *Then* $\boldsymbol{f}$ *is étale if and only if* $\boldsymbol{f}$ *is* $\boldsymbol{B}$ *and* $G_x\boldsymbol{f}:G_x\boldsymbol{X}\rightarrow G_y\boldsymbol{Y}$, $T_x\boldsymbol{f}:T_x\boldsymbol{X}\rightarrow T_y\boldsymbol{Y}$, $O_x\boldsymbol{f}:O_x\boldsymbol{X}\rightarrow O_y\boldsymbol{Y}$ *from* §6.5 *and* §10.2.3 *are isomorphisms for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{f}(x)=y$ *in* $\boldsymbol{Y}$.

The 'only if' part does not require Assumptions 10.9 *and* 10.11.

Here are the analogues of Theorem 10.56–10.58, all three proved in the same way, but using Theorems 10.61–10.62 in place of Theorems 10.54–10.55.

Theorem 10.63. (a) *Work in the* 2*-category of Kuranishi spaces* $\mathbf{Kur}$ *constructed from* $\dot{\mathbf{M}}\mathbf{an}=\mathbf{Man}$, *using ordinary tangent spaces* T_vV *for* $V\in\mathbf{Man}$. *Then a* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ *in* $\mathbf{Kur}$ *is étale if and only if* $G_x\boldsymbol{f}:G_x\boldsymbol{X}\rightarrow G_y\boldsymbol{Y}$, $T_x\boldsymbol{f}:T_x\boldsymbol{X}\rightarrow T_y\boldsymbol{Y}$, $O_x\boldsymbol{f}:O_x\boldsymbol{X}\rightarrow O_y\boldsymbol{Y}$ *are isomorphisms for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{f}(x)=y$ *in* $\boldsymbol{Y}$. *If this holds then* $\boldsymbol{f}$ *is an equivalence if and only if* $f:X\rightarrow Y$ *is a bijection.*

(b) *Work in the* 2*-category* $\mathbf{Kur^c}$ *constructed from* $\dot{\mathbf{M}}\mathbf{an}=\mathbf{Man^c}$, *using ordinary tangent spaces* T_vV *for* $V\in\mathbf{Man^c}$. *Then a* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ *in* $\mathbf{Kur^c}$ *is étale if and only if* $\boldsymbol{f}$ *is simple and* $G_x\boldsymbol{f}:G_x\boldsymbol{X}\rightarrow G_y\boldsymbol{Y}$, $T_x\boldsymbol{f}:T_x\boldsymbol{X}\rightarrow T_y\boldsymbol{Y}$, $O_x\boldsymbol{f}:O_x\boldsymbol{X}\rightarrow O_y\boldsymbol{Y}$ *are isomorphisms for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{f}(x)=y$ *in* $\boldsymbol{Y}$. *If this holds then* $\boldsymbol{f}$ *is an equivalence if and only if* $f:X\rightarrow Y$ *is a bijection.*

(c) *Work in one of* $\dot{\mathbf{K}}\mathbf{ur}=\mathbf{Kur^c},\mathbf{Kur^{gc}},\mathbf{Kur^{ac}}$ *or* $\mathbf{Kur^{c,ac}}$ *constructed from* $\dot{\mathbf{M}}\mathbf{an}=\mathbf{Man^c},\mathbf{Man^{gc}},\mathbf{Man^{ac}}$ *or* $\mathbf{Man^{c,ac}}$, *using b-tangent spaces* ${}^bT_vV$ *for* $V\in\dot{\mathbf{M}}\mathbf{an}$, *as in* §2.3. *Then a* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ *in* $\dot{\mathbf{K}}\mathbf{ur}$ *is étale if and only if* $\boldsymbol{f}$ *is simple and* $G_x\boldsymbol{f}:G_x\boldsymbol{X}\rightarrow G_y\boldsymbol{Y}$, ${}^bT_x\boldsymbol{f}:{}^bT_x\boldsymbol{X}\rightarrow{}^bT_y\boldsymbol{Y}$,

${}^bO_x\boldsymbol{f}:{}^bO_x\boldsymbol{X}\rightarrow{}^bO_y\boldsymbol{Y}$ *are isomorphisms for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{f}(x)=y$ *in* $\boldsymbol{Y}$. *If this holds then* $\boldsymbol{f}$ *is an equivalence if and only if* $f:X\rightarrow Y$ *is a bijection.*

(d) *Work in one of* $\dot{\mathbf{Kur}}=\mathbf{Kur^c},\mathbf{Kur^{gc}},\mathbf{Kur^{ac}}$ *or* $\mathbf{Kur^{c,ac}}$ *constructed from* $\dot{\mathbf{Man}}=\mathbf{Man^c},\mathbf{Man^{gc}},\mathbf{Man^{ac}}$ *or* $\mathbf{Man^{c,ac}}$, *using stratum tangent spaces* $\tilde T_vV$ *for* $V\in\dot{\mathbf{Man}}$, *as in Example* 10.2(iv). *Then a* 1*-morphism* $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ *in* $\dot{\mathbf{Kur}}$ *is étale if and only if* $\boldsymbol{f}$ *is simple and* $G_x\boldsymbol{f}:G_x\boldsymbol{X}\rightarrow G_y\boldsymbol{Y}$, $\tilde T_x\boldsymbol{f}:\tilde T_x\boldsymbol{X}\rightarrow\tilde T_y\boldsymbol{Y}$, $\tilde O_x\boldsymbol{f}:\tilde O_x\boldsymbol{X}\rightarrow\tilde O_y\boldsymbol{Y}$ *are isomorphisms for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{f}(x)=y$ *in* $\boldsymbol{Y}$. *If this holds then* $\boldsymbol{f}$ *is an equivalence if and only if* $f:X\rightarrow Y$ *is a bijection.*

Theorem 10.64. *Suppose* $\dot{\mathbf{Man}}$ *satisfies Assumptions* 3.1–3.7, 10.1, 10.9 *and* 10.11, *with tangent spaces* T_vV *for* $V\in\dot{\mathbf{Man}}$, *and discrete properties* $\boldsymbol{A},\boldsymbol{B}$.

Let $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ *be a* 1*-morphism of Kuranishi neighbourhoods over* $S\subseteq X$, *as in* §6.1, *and suppose* Φ_{ij} *is* $\boldsymbol{B}$. *Let* $p\in\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))\subseteq P_{ij}$, *and set* $v_i=\pi_{ij}(p)\in V_i$ *and* $v_j=\phi_{ij}(p)\in V_j$. *As in* (10.58), *consider the sequence of real vector spaces:*

$$0\longrightarrow T_{v_i}V_i\xrightarrow{\mathrm{d}_{v_i}s_i\oplus(T_p\phi_{ij}\circ(T_p\pi_{ij})^{-1})}E_i|_{v_i}\oplus T_{v_j}V_j\xrightarrow{-\hat\phi_{ij}|_p\oplus\mathrm{d}_{v_j}s_j}E_j|_{v_j}\longrightarrow 0. \quad (10.60)$$

Here $T_p\pi_{ij}:T_pP_{ij}\rightarrow T_{v_i}V_i$ *is invertible as* π_{ij} *is étale. Differentiating Definition* 6.2(e) *at* p *implies that* (10.60) *is a complex. Also consider the morphism of finite groups*

$$\begin{aligned}&\rho_p:\bigl\{(\gamma_i,\gamma_j)\in\Gamma_i\times\Gamma_j:(\gamma_i,\gamma_j)\cdot p=p\bigr\}\longrightarrow\bigl\{\gamma_j\in\Gamma_j:\gamma_j\cdot v_j=v_j\bigr\},\\&\rho_p:(\gamma_i,\gamma_j)\longmapsto\gamma_j.\end{aligned} \quad (10.61)$$

Then Φ_{ij} *is a coordinate change over* S, *in the sense of Definition* 6.11, *if and only if* (10.60) *is exact and* (10.61) *is an isomorphism for all* $p\in\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$.

The 'only if' part does not require Assumptions 10.9 *and* 10.11.

Theorem 10.65. *Working in a category* $\dot{\mathbf{Man}}$ *which we specify in* **(a)**–**(d)** *below, let* $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ *be a* 1*-morphism of Kuranishi neighbourhoods on a topological space* X *over an open subset* $S\subseteq X$. *Let* $p\in\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))\subseteq P_{ij}$, *set* $v_i=\pi_{ij}(p)\in V_i$ *and* $v_j=\phi_{ij}(p)\in V_j$, *and consider the morphism of finite groups*

$$\begin{aligned}&\rho_p:\bigl\{(\gamma_i,\gamma_j)\in\Gamma_i\times\Gamma_j:(\gamma_i,\gamma_j)\cdot p=p\bigr\}\longrightarrow\bigl\{\gamma_j\in\Gamma_j:\gamma_j\cdot v_j=v_j\bigr\},\\&\rho_p:(\gamma_i,\gamma_j)\longmapsto\gamma_j.\end{aligned} \quad (10.62)$$

Then:

(a) *If* $\dot{\mathbf{Man}}=\mathbf{Man}$ *then* Φ_{ij} *is a coordinate change over* S *if and only if for all* $p\in\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *equation* (10.62) *is an isomorphism, and the following is exact:*

$$0\longrightarrow T_{v_i}V_i\xrightarrow{\mathrm{d}_{v_i}s_i\oplus(T_p\phi_{ij}\circ(T_p\pi_{ij})^{-1})}E_i|_{v_i}\oplus T_{v_j}V_j\xrightarrow{-\hat\phi_{ij}|_p\oplus\mathrm{d}_{v_j}s_j}E_j|_{v_j}\longrightarrow 0. \quad (10.63)$$

(b) *If* $\dot{\mathbf{Man}}=\mathbf{Man^c}$ *then* Φ_{ij} *is a coordinate change over* S *if and only if* ϕ_{ij} *is simple near* $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *and for all* $p\in\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *equation* (10.62) *is an isomorphism and* (10.63) *is exact.*

(c) *If* $\dot{\mathbf{Man}}$ *is one of* $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}$ *or* $\mathbf{Man^{c,ac}}$ *then* Φ_{ij} *is a coordinate change over* S *if and only if* ϕ_{ij} *is simple near* $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *and using b-tangent spaces from* §2.3, *for all* $p \in \pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *equation* (10.62) *is an isomorphism and the following is exact:*

$$0 \longrightarrow {}^bT_{v_i}V_i \xrightarrow{{}^b\mathrm{d}_{v_i}s_i \oplus ({}^bT_p\phi_{ij}\circ({}^bT_p\pi_{ij})^{-1})} E_i|_{v_i} \oplus {}^bT_{v_j}V_j \xrightarrow{-\hat\phi_{ij}|_p \oplus {}^b\mathrm{d}_{v_j}s_j} E_j|_{v_j} \longrightarrow 0.$$

(d) *If* $\dot{\mathbf{Man}}$ *is one of* $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}$ *or* $\mathbf{Man^{c,ac}}$ *then* Φ_{ij} *is a coordinate change over* S *if and only if* ϕ_{ij} *is simple near* $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *and using stratum tangent spaces* $\tilde T_vV$ *from Example* 10.2(iv), *for all* $p \in \pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$, *equation* (10.62) *is an isomorphism and the following is exact:*

$$0 \longrightarrow \tilde T_{v_i}V_i \xrightarrow{\tilde{\mathrm{d}}_{v_i}s_i \oplus (\tilde T_p\phi_{ij}\circ(\tilde T_p\pi_{ij})^{-1})} E_i|_{v_i} \oplus \tilde T_{v_j}V_j \xrightarrow{-\hat\phi_{ij}|_p \oplus \tilde{\mathrm{d}}_{v_j}s_j} E_j|_{v_j} \longrightarrow 0.$$

Theorem 10.65(a)–(c) was quoted as Theorem 6.12 in volume I, and applied in Chapter 7 of volume I to show that FOOO coordinate changes and MW coordinate changes correspond to coordinate changes of Kuranishi neighbourhoods in our sense. This was important in the proofs in §7.5 that the geometric structures of Fukaya, Oh, Ohta and Ono [10–30], McDuff and Wehrheim [49, 50, 52–55], Yang [71–73], and Hofer, Wysocki and Zehnder [34–41], can all be mapped to our Kuranishi spaces.

10.6 Determinants of complexes

We now explain some homological algebra that will be needed in §10.7 to define canonical line bundles and orientations of (m-)Kuranishi spaces.

If E is a finite-dimensional real vector space the *determinant* is $\det E = \Lambda^{\dim E}E$, so that $\det E \cong \mathbb{R}$, and if F is another vector space with $\dim E = \dim F$ and $\alpha : E \to F$ is a linear map, we write $\det\alpha = \Lambda^{\dim E}\alpha : \det E \to \det F$. When $E = \mathbb{R}^n$ then $\det\alpha : \mathbb{R} \to \mathbb{R}$ is multiplication by the usual determinant of α as an $n \times n$ matrix. More generally, if $E \to X$ is a real vector bundle over a space X we write $\det E = \Lambda^{\mathrm{rank}\, E}E$, so that $\det E \to X$ is a real line bundle.

Our aim is to extend determinants $\det(E^\bullet)$ to finite-dimensional complexes $E^\bullet = \bigl(\cdots \to E^k \xrightarrow{\mathrm{d}^k} E^{k+1} \to \cdots\bigr)$ of vector spaces or vector bundles, and to relate $\det(E^\bullet)$ to $\det(H^*(E^\bullet))$. In §10.7, if (V, E, s, ψ) is an m-Kuranishi neighbourhood we will apply this to the complex $TV|_{s^{-1}(0)} \xrightarrow{\mathrm{d}s} E|_{s^{-1}(0)}$. Most of our results will only be used for length 2 complexes, but we prove the general case anyway. The subject involves many sign computations. Some of our orientation conventions — how to define orientations on (m-)Kuranishi spaces $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, and on products $\boldsymbol{X} \times \boldsymbol{Y}$ and fibre products $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ — are implicit in the choices of signs in equations such as (10.66), (10.69), and (10.93).

10.6.1 Determinants of complexes, and of their cohomology

If $E^\bullet = (E^*, \mathrm{d})$ is a bounded complex of finite-dimensional real vector spaces, we can form its *determinant* $\det(E^\bullet) = \bigotimes_{k\in\mathbb{Z}} (\Lambda^{\dim E^k} E^k)^{(-1)^k}$, a 1-dimensional real vector space. We now define an isomorphism $\Theta_{E^\bullet}$ between $\det(E^\bullet)$ and the determinant $\det(H^*(E^\bullet))$ of the cohomology of $E^\bullet$.

Definition 10.66. If E is a finite-dimensional real vector space we write $\det E = \Lambda^{\dim E} E$ for its top exterior power, so that $\det E$ is a 1-dimensional real vector space, with $\det E = \mathbb{R}$ if $E = 0$, and we write $(\det E)^{-1}$ for the dual vector space $(\det E)^*$. We also use the same notation if $E \to X$ is a vector bundle over some space X, so that $\det E = \Lambda^{\operatorname{rank} E} E$ is a real line bundle on X.

Suppose we are given a complex $E^\bullet$ of real vector spaces

$$\cdots \xrightarrow{\mathrm{d}^{k-2}} E^{k-1} \xrightarrow{\mathrm{d}^{k-1}} E^k \xrightarrow{\mathrm{d}^k} E^{k+1} \xrightarrow{\mathrm{d}^{k+1}} E^{k+2} \xrightarrow{\mathrm{d}^{k+2}} \cdots, \quad (10.64)$$

for $k \in \mathbb{Z}$, with $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$, where the E^k should be finite-dimensional with $E^k = 0$ for $|k| \gg 0$, say $E^k = 0$ unless $a \leqslant k \leqslant b$ for $a \leqslant b \in \mathbb{Z}$. Write $H^k(E^\bullet)$ for the k^{th} cohomology group of $E^\bullet$, so that $H^k(E^\bullet) = \operatorname{Ker} \mathrm{d}^k / \operatorname{Im} \mathrm{d}^{k-1}$ for $k \in \mathbb{Z}$. We will define an isomorphism

$$\Theta_{E^\bullet} : \bigotimes_{k=a}^b (\det E^k)^{(-1)^k} \longrightarrow \bigotimes_{k=a}^b (\det H^k(E^\bullet))^{(-1)^k}. \quad (10.65)$$

If $k < a$ or $k > b$ we have $E^k = H^k(E^\bullet) = 0$ and $\det E^k = \det H^k(E^\bullet) = \mathbb{R}$, and such terms do not change the tensor products in (10.65), so the left and right hand sides are independent of the choice of a, b with $E^k = 0$ unless $a \leqslant k \leqslant b$.

For each $k \in \mathbb{Z}$ define $m^k = \dim H^k(E^\bullet)$ and $n^k = \dim \operatorname{Im} \mathrm{d}^k$, so that $\dim E^k = n^{k-1} + m^k + n^k$. By induction on increasing k, choose bases $u_1^k, \ldots, u_{n^{k-1}}^k, v_1^k, \ldots, v_{m^k}^k, w_1^k, \ldots, w_{n^k}^k$ for E^k for each $k \in \mathbb{Z}$, such that $u_1^k, \ldots, u_{n^{k-1}}^k$ is a basis for $\operatorname{Im} \mathrm{d}^{k-1} \subseteq E^k$, and $u_1^k, \ldots, u_{n^{k-1}}^k, v_1^k, \ldots, v_{m^k}^k$ is a basis for $\operatorname{Ker} \mathrm{d}^k \subseteq E^k$, which forces $\mathrm{d}^k u_i^k = \mathrm{d}^k v_j^k = 0$ for all i, j, and $\mathrm{d}^k w_i^k = u_i^{k+1}$ for $i = 1, \ldots, n^k$. Then $[v_1^k], \ldots, [v_{m^k}^k]$ is a basis for $H^k(E^\bullet)$, where $[v_i^k]$ means $v_i^k + \operatorname{Im} \mathrm{d}^{k-1}$.

Define $\Theta_{E^\bullet}$ to be the unique isomorphism in (10.65) such that

$$\begin{aligned} \Theta_{E^\bullet} : \bigotimes_{k=a}^b \bigl(u_1^k \wedge \cdots \wedge u_{n^{k-1}}^k \wedge v_1^k \wedge \cdots \wedge v_{m^k}^k \wedge w_1^k \wedge \cdots \wedge w_{n^k}^k\bigr)^{(-1)^k} \longmapsto \\ \prod_{k=a}^b (-1)^{n^k(n^k+1)/2} \cdot \bigotimes_{k=a}^b \bigl([v_1^k] \wedge \cdots \wedge [v_{m^k}^k]\bigr)^{(-1)^k}. \end{aligned} \quad (10.66)$$

To show that this is independent of the choice of u_i^k, v_i^k, w_i^k, suppose $\tilde u_i^k, \tilde v_i^k, \tilde w_i^k$ are alternative choices. Then the two bases for E^k are related by a matrix

$$\begin{pmatrix} (\tilde u_i^k)_{i=1}^{n^{k-1}} \\ (\tilde v_i^k)_{i=1}^{m^k} \\ (\tilde w_i^k)_{i=1}^{n^k} \end{pmatrix} = \begin{pmatrix} A^k & 0 & 0 \\ * & B^k & 0 \\ * & * & C^k \end{pmatrix} \begin{pmatrix} (u_i^k)_{i=1}^{n^{k-1}} \\ (v_i^k)_{i=1}^{m^k} \\ (w_i^k)_{i=1}^{n^k} \end{pmatrix}$$

Here A^k, B^k, C^k are $n^{k-1} \times n^{k-1}$ and $m^k \times m^k$ and $n^k \times n^k$ real matrices, respectively, and the matrix has this lower triangular form as

$$\langle \tilde{u}_1^k, \ldots, \tilde{u}_{n^{k-1}}^k \rangle = \operatorname{Im} \mathrm{d}^{k-1} = \langle u_1^k, \ldots, u_{n^{k-1}}^k \rangle \qquad \text{and}$$
$$\langle \tilde{u}_1^k, \ldots, \tilde{u}_{n^{k-1}}^k, \tilde{v}_1^k, \ldots, \tilde{v}_{m^k}^k \rangle = \operatorname{Ker} \mathrm{d}^k = \langle u_1^k, \ldots, u_{n^{k-1}}^k, v_1^k, \ldots, v_{m^k}^k \rangle.$$

Also the two bases for $H^k(E^\bullet)$ are related by the matrix

$$\bigl([\tilde{v}_i^k]\bigr)_{i=1}^{m^k} = B^k \bigl([v_i^k]\bigr)_{i=1}^{m^k}.$$

Thus we see that

$$\begin{aligned}
&\tilde{u}_1^k \wedge \cdots \wedge \tilde{u}_{n^{k-1}}^k \wedge \tilde{v}_1^k \wedge \cdots \wedge \tilde{v}_{m^k}^k \wedge \tilde{w}_1^k \wedge \cdots \wedge \tilde{w}_{n^k}^k \\
&\quad = \det(A^k)\det(B^k)\det(C^k) \cdot u_1^k \wedge \cdots \wedge u_{n^{k-1}}^k \wedge v_1^k \wedge \cdots \wedge v_{m^k}^k \wedge w_1^k \wedge \cdots \wedge w_{n^k}^k, \\
&[\tilde{v}_1^k] \wedge \cdots \wedge [\tilde{v}_{m^k}^k] = \det(B^k) \cdot [v_1^k] \wedge \cdots \wedge [v_{m^k}^k].
\end{aligned}$$

Hence, if we change from the basis $u_1^k, \ldots, w_{n^k}^k$ of E^k to the basis $\tilde{u}_1^k, \ldots, \tilde{w}_{n^k}^k$ for all k, then the left hand side of (10.66) is multiplied by the factor

$$\textstyle\prod_{k=a}^b \bigl(\det(A^k)\det(B^k)\det(C^k)\bigr)^{(-1)^k}, \tag{10.67}$$

but the right hand side of (10.66) is multiplied by the apparently different factor

$$\textstyle\prod_{k=a}^b \bigl(\det(B^k)\bigr)^{(-1)^k}. \tag{10.68}$$

However, as $\mathrm{d}^k w_i^k = u_i^{k+1}$, $\mathrm{d}^k \tilde{w}_i^k = \tilde{u}_i^{k+1}$ we see that $C^k = A^{k+1}$, so that $\det(C^k) = \det(A^{k+1})$, and also $\det(A^a) = 1$ as $n^{a-1} = 0$ and $\det(C^b) = 1$ as $n^b = 0$. Therefore (10.67) and (10.68) are equal, so (10.66) is independent of the choice of bases $u_1^k, \ldots, w_{n^k}^k$ of E^k, and $\Theta_{E^\bullet}$ is well defined.

Suppose now that $E^\bullet$ in (10.64) is exact. Then $m^k = 0$ for all k, so as above we choose bases $u_1^k, \ldots, u_{n^{k-1}}^k, w_1^k, \ldots, w_{n^k}^k$ for E^k for each $k \in \mathbb{Z}$ with $\mathrm{d}^k u_i^k = 0$ and $\mathrm{d}^k w_i^k = u_i^{k+1}$ for all i, k. Define

$$\Psi_{E^\bullet} = \textstyle\bigotimes_{k=a}^b \bigl(u_1^k \wedge \cdots \wedge u_{n^{k-1}}^k \wedge w_1^k \wedge \cdots \wedge w_{n^k}^k\bigr)^{(-1)^k} \in \bigotimes_{k=a}^b (\det E^k)^{(-1)^k}. \tag{10.69}$$

This is independent of choices as above.

10.6.2 A continuity property of the isomorphisms $\Theta_{E^\bullet}$

We now prove a continuity property for the isomorphisms $\Theta_{E^\bullet}$ in §10.6.1. It will be used in §10.7.1 to define canonical line bundles $K_{\boldsymbol{X}}$ of m-Kuranishi spaces $\boldsymbol{X}$. Here (10.72) determines $\Xi_{\theta^\bullet}|_x$ for $x \in X$. The point is that these $\Xi_{\theta^\bullet}|_x$ *depend continuously on* $x \in X$, and so form an isomorphism of topological line bundles $\Xi_{\theta^\bullet}$ in (10.71). The sign $\prod_k (-1)^{n^k(n^k+1)/2}$ in (10.66) is needed to ensure this.

Proposition 10.67. *Suppose that X is a topological space, and we are given a commutative diagram of topological vector bundles and their morphisms on X:*

$$\begin{array}{ccccccccccc} \cdots \xrightarrow[\mathrm{d}^{k-2}]{} & E^{k-1} & \xrightarrow[\mathrm{d}^{k-1}]{} & E^k & \xrightarrow[\mathrm{d}^k]{} & E^{k+1} & \xrightarrow[\mathrm{d}^{k+1}]{} & E^{k+2} & \xrightarrow[\mathrm{d}^{k+2}]{} \cdots \\ & \downarrow \theta^{k-1} & & \downarrow \theta^k & & \downarrow \theta^{k+1} & & \downarrow \theta^{k+2} & \\ \cdots \xrightarrow{\check{\mathrm{d}}^{k-2}} & \check E^{k-1} & \xrightarrow{\check{\mathrm{d}}^{k-1}} & \check E^k & \xrightarrow{\check{\mathrm{d}}^k} & \check E^{k+1} & \xrightarrow{\check{\mathrm{d}}^{k+1}} & \check E^{k+2} & \xrightarrow{\check{\mathrm{d}}^{k+2}} \cdots, \end{array} \tag{10.70}$$

such that $\mathrm{d}^{k+1}\circ\mathrm{d}^k=\check{\mathrm{d}}^{k+1}\circ\check{\mathrm{d}}^k=0$ for all $k\in\mathbb{Z}$, and $E^k=\check E^k=0$ unless $a\leqslant k\leqslant b$ for $a\leqslant b$ in $\mathbb{Z}$. That is, $E^\bullet$, $\check E^\bullet$ are bounded complexes of topological vector bundles on X, and $\theta^\bullet:E^\bullet\rightarrow\check E^\bullet$ is a morphism of complexes.

For each $x\in X$ we have a morphism $\theta^\bullet|_x:E^\bullet|_x\rightarrow\check E^\bullet|_x$ of complexes of $\mathbb{R}$-vector spaces, which induces morphisms $H^k(\theta^\bullet|_x):H^k(E^\bullet|_x)\rightarrow H^k(\check E^\bullet|_x)$ on cohomology. Suppose $H^k(\theta^\bullet|_x)$ is an isomorphism for all $x\in X$ and $k\in\mathbb{Z}$. Then there exists a unique isomorphism of topological line bundles on X:

$$\Xi_{\theta^\bullet}:\textstyle\bigotimes_{k=a}^b(\det E^k)^{(-1)^k}\longrightarrow\bigotimes_{k=a}^b(\det\check E^k)^{(-1)^k} \tag{10.71}$$

such that for each $x\in X$, the following diagram of isomorphisms commutes

$$\begin{array}{ccc} \bigotimes_{k=a}^b(\det E^k)^{(-1)^k}|_x & \xrightarrow[\Xi_{\theta^\bullet}|_x]{} & \bigotimes_{k=a}^b(\det \check E^k)^{(-1)^k}|_x \\ \downarrow \Theta_{E^\bullet|_x} & & \Theta_{\check E^\bullet|_x}\downarrow \\ \bigotimes_{k=a}^b(\det H^k(E^\bullet|_x))^{(-1)^k} & \xrightarrow{\bigotimes_{k=a}^b(\det H^k(\theta^\bullet|_x))^{(-1)^k}} & \bigotimes_{k=a}^b(\det H^k(\check E^\bullet|_x))^{(-1)^k}, \end{array} \tag{10.72}$$

where $\Theta_{E^\bullet|_x},\Theta_{\check E^\bullet|_x}$ are as in Definition 10.66.

Proof. Fix $\tilde x\in X$, and set $\tilde m^k=\dim H^k(E^\bullet|_{\tilde x})=\dim H^k(\check E^\bullet|_{\tilde x})$, and $\tilde n^k=\dim\mathrm{Im}\,\mathrm{d}^k|_{\tilde x}$, and $\check{\tilde n}^k=\dim\mathrm{Im}\,\check{\mathrm{d}}^k|_{\tilde x}$. As in Definition 10.66, choose bases $\tilde u_1^k,\ldots,\tilde u^k_{\tilde n^{k-1}},\tilde v_1^k,\ldots,\tilde v^k_{\tilde m^k},\tilde w_1^k,\ldots,\tilde w^k_{\tilde n^k}$ for $E^k|_{\tilde x}$ and $\check{\tilde u}_1^k,\ldots,\check{\tilde u}^k_{\check{\tilde n}^{k-1}},\check{\tilde v}_1^k,\ldots,\check{\tilde v}^k_{\tilde m^k},\check{\tilde w}_1^k,\ldots,\check{\tilde w}^k_{\check{\tilde n}^k}$ for $\check E^k|_{\tilde x}$, such that $\mathrm{d}^k\tilde u_i^k=\mathrm{d}^k\tilde v_i^k=0$, $\mathrm{d}^k\tilde w_i^k=\tilde u_i^{k+1}$, $\check{\mathrm{d}}^k\check{\tilde u}_i^k=\check{\mathrm{d}}^k\check{\tilde v}_i^k=0$, and $\check{\mathrm{d}}^k\check{\tilde w}_i^k=\check{\tilde u}_i^{k+1}$ for all i,k. As $[\tilde v_1^k],\ldots,[\tilde v^k_{\tilde m^k}]$ is a basis for $H^k(E^\bullet|_{\tilde x})$, and $[\check{\tilde v}_1^k],\ldots,[\check{\tilde v}^k_{\tilde m^k}]$ is a basis for $H^k(\check E^\bullet|_{\tilde x})$, and $H^k(\theta^\bullet|_{\tilde x}):H^k(E^\bullet|_{\tilde x})\rightarrow H^k(\check E^\bullet|_{\tilde x})$ is an isomorphism, we can also choose the $\tilde v_i^k,\check{\tilde v}_i^k$ with $\theta^k|_{\tilde x}(\tilde v_i^k)=\check{\tilde v}_i^k$ for all i,k.

Now let $\tilde X$ be a small open neighbourhood of $\tilde x$ in X on which the $E^k,\check E^k$ are trivial for all k, and choose bases of sections $e_1^k,\ldots,e^k_{\tilde n^{k-1}},f_1^k,\ldots,f^k_{\tilde m^k},g_1^k,\ldots,g^k_{\tilde n^k}$ for $E^k|_{\tilde X}$ and $\check e_1^k,\ldots,\check e^k_{\check{\tilde n}^{k-1}},\check f_1^k,\ldots,\check f^k_{\tilde m^k},\check g_1^k,\ldots,\check g^k_{\check{\tilde n}^k}$ for $\check E^k|_{\tilde X}$, such that $e_i^k|_{\tilde x}=\tilde u_i^k$, $f_i^k|_{\tilde x}=\tilde v_i^k$, $g_i^k|_{\tilde x}=\tilde w_i^k$, $\check e_i^k|_{\tilde x}=\check{\tilde u}_i^k$, $\check f_i^k|_{\tilde x}=\check{\tilde v}_i^k$, and $\check g_i^k|_{\tilde x}=\check{\tilde w}_i^k$. Making $\tilde X$ smaller if necessary we can do this such that $\mathrm{d}^kg_i^k=e_i^{k+1}$ and $\check{\mathrm{d}}^k\check g_i^k=\check e_i^{k+1}$ for all i,k, as these hold for $\tilde u_i^k,\ldots,\check{\tilde w}_i^k$. Then $\mathrm{d}^ke_i^k=\check{\mathrm{d}}^k\check e_i^k=0$. Write

$$\mathrm{d}^kf_i^k=\textstyle\sum_{j=1}^{\tilde n^k}A_{ij}^{k+1}e_j^{k+1}+\sum_{j=1}^{\tilde m^{k+1}}B_{ij}^{k+1}f_j^{k+1}+\sum_{j=1}^{\tilde n^{k+1}}C_{ij}^{k+1}g_j^{k+1},$$

for $A_{ij}^{k+1},B_{ij}^{k+1},C_{ij}^{k+1}:\tilde X\rightarrow\mathbb{R}$ continuous and zero at x. Replacing f_i^k by $f_i^k-\sum_{i=1}^{n^k}A_{ij}^{k+1}g_j^k$ we can make $A_{ij}^{k+1}=0$ for all i,j,k. But then we have

$$0=\mathrm{d}^{k+1}\mathrm{d}^kf_i^k=\sum_{j=1}^{\tilde m^{k+1}}B_{ij}^{k+1}\Bigl(\sum_{l=1}^{\tilde m^{k+2}}B_{jl}^{k+2}f_l^{k+2}+\sum_{l=1}^{\tilde n^{k+2}}C_{jl}^{k+2}g_l^{k+2}\Bigr)+\sum_{j=1}^{\tilde n^{k+1}}C_{ij}^{k+1}e_j^{k+1},$$

so that $C_{ij}^{k+1}=0$ for all i,j,k. Thus we have

$$\mathrm{d}^k e_i^k=0,\qquad \mathrm{d}^k f_i^k=\textstyle\sum_{j=1}^{\tilde m^{k+1}} B_{ij}^{k+1} f_j^{k+1},\qquad \mathrm{d}^k g_i^k=e_i^{k+1}. \tag{10.73}$$

Replace $\check f_i^k$ by $\theta^k(f_j^k)$ for $i=1,\ldots,\tilde m^k$. Making $\tilde X$ smaller we can still suppose $\check e_1^k,\ldots,\check e^k_{\tilde{\check n}^{k-1}},\check f_1^k,\ldots,\check f^k_{\tilde m^k},\check g_1^k,\ldots,\check g^k_{\tilde{\check n}^k}$ is a basis of sections for $\check E^k|_{\tilde X}$, since this holds at x, and as $\check{\mathrm{d}}^k\circ\theta^k=\theta^{k+1}\circ\mathrm{d}^k$ we have

$$\check{\mathrm{d}}^k\check e_i^k=0,\qquad \check{\mathrm{d}}^k\check f_i^k=\textstyle\sum_{j=1}^{\tilde m^{k+1}} B_{ij}^{k+1}\check f_j^{k+1},\qquad \check{\mathrm{d}}^k\check g_i^k=\check e_i^{k+1}. \tag{10.74}$$

Now define an isomorphism of topological line bundles on $\tilde X$

$$\begin{aligned}
\Xi_{\theta^\bullet}|_{\tilde X}:&\textstyle\bigotimes_{k=a}^b(\det E^k)^{(-1)^k}|_{\tilde X}\longrightarrow\textstyle\bigotimes_{k=a}^b(\det \check E^k)^{(-1)^k}|_{\tilde X}\quad\text{by}\\
\Xi_{\theta^\bullet}|_{\tilde X}:&\textstyle\bigotimes_{k=a}^b\bigl(e_1^k\wedge\cdots\wedge e^k_{\tilde n^{k-1}}\wedge f_1^k\wedge\cdots\wedge f^k_{\tilde m^k}\wedge g_1^k\wedge\cdots\wedge g^k_{\tilde n^k}\bigr)^{(-1)^k}\longmapsto\\
&\textstyle\prod_{k=a}^b(-1)^{\tilde n^k(\tilde n^k+1)/2+\tilde{\check n}^k(\tilde{\check n}^k+1)/2}\,\cdot\\
&\textstyle\bigotimes_{k=a}^b\bigl(\check e_1^k\wedge\cdots\wedge\check e^k_{\tilde{\check n}^{k-1}}\wedge\check f_1^k\wedge\cdots\wedge\check f^k_{\tilde m^k}\wedge\check g_1^k\wedge\cdots\wedge\check g^k_{\tilde{\check n}^k}\bigr)^{(-1)^k}.
\end{aligned}\tag{10.75}$$

We claim that (10.72) commutes for $\Xi_{\theta^\bullet}|_{\tilde X}$ for all $x\in\tilde X$. To prove this, write

$$\begin{aligned}
E^k|_x&=\bigl\langle e_1^k|_x,\ldots,e^k_{\tilde n^{k-1}}|_x,f_1^k|_x,\ldots,f^k_{\tilde m^k}|_x,g_1^k|_x,\ldots,g^k_{\tilde n^k}|_x\bigr\rangle_{\mathbb R},\\
\check E^k|_x&=\bigl\langle \check e_1^k|_x,\ldots,\check e^k_{\tilde{\check n}^{k-1}}|_x,\check f_1^k|_x,\ldots,\check f^k_{\tilde m^k}|_x,\check g_1^k|_x,\ldots,\check g^k_{\tilde{\check n}^k}|_x\bigr\rangle_{\mathbb R},
\end{aligned}$$

and write $\mathrm{d}^k|_x:E^k|_x\to E^{k+1}|_x$ and $\check{\mathrm{d}}^k|_x:\check E^k|_x\to\check E^{k+1}|_x$ using (10.73)–(10.74). To define $\Theta_{E^\bullet|_x}$ in Definition 10.66 we choose bases $u_1^k,\ldots,u^k_{n^{k-1}},v_1^k,\ldots,v^k_{m^k}$, $w_1^k,\ldots,w^k_{n^k}$ for $E^k|_x$, where $n^k=\dim\operatorname{Im}\mathrm{d}^k|_x$. Since $\mathrm{d}^k|_x g_i^k|_x=e_i^{k+1}|_x$ for $i=1,\ldots,\tilde n^k$ we see that $n^k\geqslant\tilde n^k$, say $n^k=\tilde n^k+p^k$ for $p^k\geqslant 0$. Then $\tilde m^k=p^{k-1}+m^k+p^k$, since $n^{k-1}+m^k+n^k=\operatorname{rank}E^k=\tilde n^{k-1}+\tilde m^k+\tilde n^k$. We can also write $p^k=\operatorname{rank}\bigl(B_{ij}^{k+1}|_x\bigr)^{j=1,\ldots,\tilde m^{k+1}}_{i=1,\ldots,\tilde m^k}$. We choose the bases such that

$$\begin{gathered}
u_1^k,\ldots,u^k_{p^{k-1}}\in\bigl\langle f_1^k|_x,\ldots,f^k_{\tilde m^k}|_x\bigr\rangle_{\mathbb R},\quad u^k_{p^{k-1}+i}=e_i^k|_x,\ \ i=1,\ldots,\tilde n^{k-1},\\
v_1^k,\ldots,v^k_{m^k}\in\bigl\langle f_1^k|_x,\ldots,f^k_{\tilde m^k}|_x\bigr\rangle_{\mathbb R},\\
w_1^k,\ldots,w^k_{p^k}\in\bigl\langle f_1^k|_x,\ldots,f^k_{\tilde m^k}|_x\bigr\rangle_{\mathbb R},\quad w^k_{p^k+i}=g_i^k|_x,\ \ i=1,\ldots,\tilde n^k.
\end{gathered}\tag{10.76}$$

This is possible by (10.73). Let us write

$$u_1^k\wedge\cdots\wedge u^k_{p^{k-1}}\wedge v_1^k\wedge\cdots\wedge v^k_{m^k}\wedge w_1^k\wedge\cdots\wedge w^k_{p^k}=A^k\cdot f_1^k|_x\wedge\cdots\wedge f^k_{\tilde m^k}|_x \tag{10.77}$$

for $A^k\in\mathbb R\setminus\{0\}$, which holds as $u_1^k,\ldots,u^k_{p^{k-1}},v_1^k,\ldots,v^k_{m^k},w_1^k,\ldots,w^k_{p^k}$ is a basis for $\bigl\langle f_1^k|_x,\ldots,f^k_{\tilde m^k}|_x\bigr\rangle_{\mathbb R}$. Combining (10.76) and (10.77) gives

$$\begin{aligned}
&u_1^k\wedge\cdots\wedge u^k_{n^{k-1}}\wedge v_1^k\wedge\cdots\wedge v^k_{m^k}\wedge w_1^k\wedge\cdots\wedge w^k_{n^k}\\
&=(-1)^{p^{k-1}\tilde n^{k-1}}A^k\cdot e_1^k|_x\wedge\cdots\wedge e^k_{\tilde n^{k-1}}|_x\wedge f_1^k|_x\wedge\cdots\wedge f^k_{\tilde m^k}|_x\wedge g_1^k|_x\wedge\cdots\wedge g^k_{\tilde n^k}|_x.
\end{aligned}\tag{10.78}$$

Similarly, to define $\Theta_{\check E^\bullet|_x}$ in Definition 10.66, we choose bases $\check u^k_1,\ldots,\check u^k_{\check n^{k-1}}$, $\check v^k_1,\ldots,\check v^k_{m^k},\check w^k_1,\ldots,\check w^k_{\check n^k}$ for $\check E^k|_x$, where $\check n^k=\check{\tilde n}^k+p^k$, by

$$\begin{gathered}\check u^k_i=\theta^k(u^k_i),\;\; i=1,\ldots,p^{k-1},\quad \check u^k_{p^{k-1}+i}=\check e^k_i|_x,\;\; i=1,\ldots,\check{\tilde n}^{k-1},\\ \check v^k_i=\theta^k(v^k_i),\;\; i=1,\ldots,m^k,\\ \check w^k_i=\theta^k(w^k_i),\;\; i=1,\ldots,p^k,\quad \check w^k_{p^k+i}=\check g^k_i|_x,\;\; i=1,\ldots,\check{\tilde n}^k.\end{gathered}\tag{10.79}$$

This is possible by (10.73), (10.74), (10.76), (10.79) and $\check f^k_i=\theta^k(f^k_i)$. Applying θ^k to (10.77) yields

$$\check u^k_1\wedge\cdots\wedge\check u^k_{p^{k-1}}\wedge\check v^k_1\wedge\cdots\wedge\check v^k_{m^k}\wedge\check w^k_1\wedge\cdots\wedge\check w^k_{p^k}=A^k\cdot\check f^k_1|_x\wedge\cdots\wedge\check f^k_{\tilde m^k}|_x.\tag{10.80}$$

Combining (10.79) and (10.80) then gives

$$\begin{aligned}&\check u^k_1\wedge\cdots\wedge\check u^k_{\check n^{k-1}}\wedge\check v^k_1\wedge\cdots\wedge\check v^k_{m^k}\wedge\check w^k_1\wedge\cdots\wedge\check w^k_{\check n^k}\\ &=(-1)^{p^{k-1}\check{\tilde n}^{k-1}}A^k\cdot\check e^k_1|_x\wedge\cdots\wedge\check e^k_{\check{\tilde n}^{k-1}}|_x\wedge\check f^k_1|_x\wedge\cdots\wedge\check f^k_{\tilde m^k}|_x\wedge\check g^k_1|_x\wedge\cdots\wedge\check g^k_{\check{\tilde n}^k}|_x.\end{aligned}\tag{10.81}$$

To prove (10.72) commutes at $x\in\tilde X$, consider the diagram

$$\begin{array}{ccc}\begin{array}{l}\prod_{k=a}^b(-1)^{n^k(n^k+1)/2}\cdot\\ \bigotimes_{k=a}^b\bigl(u^k_1\wedge\cdots\wedge u^k_{n^{k-1}}\wedge v^k_1\wedge\\ \cdots\wedge v^k_{m^k}\wedge w^k_1\wedge\cdots\wedge w^k_{n^k}\bigr)^{(-1)^k}\\ =\prod_{k=a}^b(-1)^{n^k(n^k+1)/2}\cdot\\ \prod_{k=a}^b(-1)^{p^k\tilde n^k}A^k\cdot\\ \bigotimes_{k=a}^b\bigl(e^k_1|_x\wedge\cdots\wedge e^k_{\tilde n^{k-1}}|_x\\ \wedge f^k_1|_x\wedge\cdots\wedge f^k_{\tilde m^k}|_x\\ \wedge g^k_1|_x\wedge\cdots\wedge g^k_{\tilde n^k}|_x\bigr)^{(-1)^k}\end{array} & \xrightarrow{\Xi_{\theta^\bullet}|_x} & \begin{array}{l}\prod_{k=a}^b(-1)^{\check n^k(\check n^k+1)/2}\cdot\\ \bigotimes_{k=a}^b\bigl(\check u^k_1\wedge\cdots\wedge\check u^k_{\check n^{k-1}}\wedge\check v^k_1\wedge\\ \cdots\wedge\check v^k_{m^k}\wedge\check w^k_1\wedge\cdots\wedge\check w^k_{\check n^k}\bigr)^{(-1)^k}\\ =\prod_{k=a}^b(-1)^{\check n^k(\check n^k+1)/2}\cdot\\ \prod_{k=a}^b(-1)^{p^k\check{\tilde n}^k}A^k\cdot\\ \bigotimes_{k=a}^b\bigl(\check e^k_1|_x\wedge\cdots\wedge\check e^k_{\check{\tilde n}^{k-1}}|_x\\ \wedge\check f^k_1|_x\wedge\cdots\wedge\check f^k_{\tilde m^k}|_x\\ \wedge\check g^k_1|_x\wedge\cdots\wedge\check g^k_{\check{\tilde n}^k}|_x\bigr)^{(-1)^k}\end{array}\\ \Big\downarrow{\scriptstyle\Theta_{E^\bullet|_x}} & & \Big\downarrow{\scriptstyle\Theta_{\check E^\bullet|_x}}\\ \bigotimes_{k=a}^b\bigl([v^k_1]\wedge\cdots\wedge[v^k_{m^k}]\bigr)^{(-1)^k} & \xrightarrow{\bigotimes_{k=a}^b(\det H^k(\theta^\bullet|_x))^{(-1)^k}} & \bigotimes_{k=a}^b\bigl([\check v^k_1]\wedge\cdots\wedge[\check v^k_{m^k}]\bigr)^{(-1)^k}.\end{array}\tag{10.82}$$

Here the alternative expressions on the top left and top right come from (10.78) and (10.81). The left and right maps are $\Theta_{E^\bullet|_x},\Theta_{\check E^\bullet|_x}$ by (10.66), and the bottom map is $\bigotimes_k(\det H^k(\theta^\bullet|_x))^{(-1)^k}$ as $\theta^k(v^k_i)=\check v^k_i$. To see that the top map is $\Xi_{\theta^\bullet}|_x$ we use (10.75) and the sign identity

$$\begin{aligned}&\textstyle\prod_{k=a}^b(-1)^{n^k(n^k+1)/2}\cdot\prod_{k=a}^b(-1)^{p^k\tilde n^k}=\\ &\textstyle\prod_{k=a}^b(-1)^{\check n^k(\check n^k+1)/2}\cdot\prod_{k=a}^b(-1)^{p^k\check{\tilde n}^k}\cdot\prod_{k=a}^b(-1)^{\tilde n^k(\tilde n^k+1)/2+\check{\tilde n}^k(\check{\tilde n}^k+1)/2},\end{aligned}$$

which holds as $n^k=\tilde n^k+p^k$ and $\check n^k=\check{\tilde n}^k+p^k$.

Equation (10.82) shows that (10.72) commutes for all $x\in\tilde X$ for the isomorphism $\Xi_{\theta^\bullet}|_{\tilde X}$ defined in (10.75). We can cover X by such open $\tilde X\subseteq X$. Also (10.72) determines $\Xi_{\theta^\bullet}|_{\tilde X}$ at each $x\in\tilde X$, and so determines $\Xi_{\theta^\bullet}|_{\tilde X}$. Thus two such isomorphisms $\Xi_{\theta^\bullet}|_{\tilde X},\Xi_{\theta^\bullet}|_{\tilde X'}$ on open $\tilde X,\tilde X'\subseteq X$ must agree on the overlap $\tilde X\cap\tilde X'$. Hence these $\Xi_{\theta^\bullet}|_{\tilde X}$ glue to give a unique global isomorphism $\Xi_{\theta^\bullet}$ as in (10.71) such that (10.72) commutes for all $x\in X$, as we have to prove. $\square$

The proof of Proposition 10.67 also works if X is an object in $\dot{\mathbf{Man}}$, or some other kind of space, and (10.70)–(10.71) are diagrams in an appropriate category of vector bundles on X. We chose to use topological spaces and topological vector bundles as they are sufficient to define orientations in §10.7.

10.6.3 Determinants of direct sums of complexes

The next proposition will be used in §10.7 to define orientations of products $\boldsymbol{X}\times\boldsymbol{Y}$ of oriented (m-)Kuranishi spaces $\boldsymbol{X},\boldsymbol{Y}$.

Proposition 10.68. *Suppose $E^\bullet,F^\bullet$ are complexes of finite-dimensional real vector spaces with $E^k=F^k=0$ unless $a\leqslant k\leqslant b$ for $a\leqslant b\in\mathbb{Z}$. Then we have a complex $E^\bullet\oplus F^\bullet$ given by*

$$\cdots\longrightarrow \begin{array}{c}E^{k-1}\oplus\\ F^{k-1}\end{array}\xrightarrow{\begin{pmatrix}\mathrm{d}^{k-1} & 0\\ 0 & \mathrm{d}^{k-1}\end{pmatrix}}\begin{array}{c}E^{k}\oplus\\ F^{k}\end{array}\xrightarrow{\begin{pmatrix}\mathrm{d}^{k} & 0\\ 0 & \mathrm{d}^{k}\end{pmatrix}}\begin{array}{c}E^{k+1}\oplus\\ F^{k+1}\end{array}\longrightarrow\cdots. \tag{10.83}$$

Definition 10.66 *defines isomorphisms*

$$\begin{aligned}\Theta_{E^\bullet}&:\textstyle\bigotimes_{k=a}^b(\det E^k)^{(-1)^k}\longrightarrow\bigotimes_{k=a}^b(\det H^k(E^\bullet))^{(-1)^k},\\ \Theta_{F^\bullet}&:\textstyle\bigotimes_{k=a}^b(\det F^k)^{(-1)^k}\longrightarrow\bigotimes_{k=a}^b(\det H^k(F^\bullet))^{(-1)^k},\\ \Theta_{E^\bullet\oplus F^\bullet}&:\textstyle\bigotimes_{k=a}^b(\det(E^k\oplus F^k))^{(-1)^k}\longrightarrow\bigotimes_{k=a}^b(\det(H^k(E^\bullet)\oplus H^k(E^\bullet)))^{(-1)^k}.\end{aligned}$$

Define isomorphisms $I_{E^k,F^k}:\det(E^k\oplus F^k)\rightarrow\det E^k\otimes\det F^k$ such that if $e_1^k,\ldots,e_{M^k}^k$ and $f_1^k,\ldots,f_{N^k}^k$ are bases for E^k,F^k then

$$I_{E^k,F^k}:e_1^k\wedge\cdots\wedge e_{M^k}^k\wedge f_1^k\wedge\cdots\wedge f_{N^k}^k\longrightarrow\bigl(e_1^k\wedge\cdots\wedge e_{M^k}^k\bigr)\otimes\bigl(f_1^k\wedge\cdots\wedge f_{N^k}^k\bigr), \tag{10.84}$$

and similarly define $I_{H^k(E^\bullet),H^k(F^\bullet)}$. Then the following commutes:

$$\begin{array}{ccc}\textstyle\bigotimes_{k=a}^b(\det(E^k\oplus F^k))^{(-1)^k} & \xrightarrow[\Theta_{E^\bullet\oplus F^\bullet}]{} & \textstyle\bigotimes_{k=a}^b(\det(H^k(E^\bullet)\oplus H^k(F^\bullet)))^{(-1)^k}\\ \Big\downarrow{\scriptstyle \prod_{a\leqslant l<k\leqslant b}(-1)^{\dim E^k\dim F^l}\cdot\bigotimes_{k=a}^b(I_{E^k,F^k})^{(-1)^k}} & & {\scriptstyle \prod_{a\leqslant l<k\leqslant b}(-1)^{\dim H^k(E^\bullet)\dim H^l(F^\bullet)}\cdot\bigotimes_{k=a}^b(I_{H^k(E^\bullet),H^k(F^\bullet)})^{(-1)^k}}\Big\downarrow\\ \begin{array}{c}\bigotimes_{k=a}^b(\det E^k)^{(-1)^k}\otimes\\ \bigotimes_{k=a}^b(\det F^k)^{(-1)^k}\end{array} & \xrightarrow{\Theta_{E^\bullet}\otimes\Theta_{F^\bullet}} & \begin{array}{c}\bigotimes_{k=a}^b(\det H^k(E^\bullet))^{(-1)^k}\otimes\\ \bigotimes_{k=a}^b(\det H^k(F^\bullet))^{(-1)^k}.\end{array}\end{array} \tag{10.85}$$

Proof. As in Definition 10.66, choose bases $u_1^k,\ldots,u_{n^{k-1}}^k,v_1^k,\ldots,v_{m^k}^k,w_1^k,\ldots,$ $w_{n^k}^k$ for E^k for each $k\in\mathbb{Z}$, such that $\mathrm{d}^k u_i^k=\mathrm{d}^k v_i^k=0$ and $\mathrm{d}^k w_i^k=u_i^{k+1}$ for all i,k. And choose bases $\check u_1^k,\ldots,\check u_{\check n^{k-1}}^k,\check v_1^k,\ldots,\check v_{m^k}^k,\check w_1^k,\ldots,\check w_{\check n^k}^k$ for F^k such that $\mathrm{d}^k\check u_i^k=\mathrm{d}^k\check v_i^k=0$ and $\mathrm{d}^k\check w_i^k=\check u_i^{k+1}$ for all i,k. Then (10.66) gives

$$\begin{aligned}&\Theta_{E^\bullet}:\textstyle\bigotimes_{k=a}^b\bigl(u_1^k\wedge\cdots\wedge u_{n^{k-1}}^k\wedge v_1^k\wedge\cdots\wedge v_{m^k}^k\wedge w_1^k\wedge\cdots\wedge w_{n^k}^k\bigr)^{(-1)^k}\longmapsto\\ &\textstyle\prod_{k=a}^b(-1)^{n^k(n^k+1)/2}\cdot\bigotimes_{k=a}^b\bigl([v_1^k]\wedge\cdots\wedge[v_{m^k}^k]\bigr)^{(-1)^k},\end{aligned}\tag{10.86}$$

$$\begin{aligned}&\Theta_{F^\bullet}:\textstyle\bigotimes_{k=a}^b\bigl(\check u_1^k\wedge\cdots\wedge\check u_{\check n^{k-1}}^k\wedge\check v_1^k\wedge\cdots\wedge\check v_{\check m^k}^k\wedge\check w_1^k\wedge\cdots\wedge\check w_{\check n^k}^k\bigr)^{(-1)^k}\longmapsto\\ &\textstyle\prod_{k=a}^b(-1)^{\check n^k(\check n^k+1)/2}\cdot\bigotimes_{k=a}^b\bigl([\check v_1^k]\wedge\cdots\wedge[\check v_{\check m^k}^k]\bigr)^{(-1)^k},\end{aligned}\tag{10.87}$$

$$\begin{aligned}&\Theta_{E^\bullet\oplus F^\bullet}:\textstyle\bigotimes_{k=a}^b\bigl(u_1^k\wedge\cdots\wedge u_{n^{k-1}}^k\wedge\check u_1^k\wedge\cdots\wedge\check u_{\check n^{k-1}}^k\wedge v_1^k\wedge\cdots\wedge v_{m^k}^k\\ &\qquad\wedge\check v_1^k\wedge\cdots\wedge\check v_{\check m^k}^k\wedge w_1^k\wedge\cdots\wedge w_{n^k}^k\wedge\check w_1^k\wedge\cdots\wedge\check w_{\check n^k}^k\bigr)^{(-1)^k}\longmapsto\\ &\prod_{k=a}^b(-1)^{(n^k+\check n^k)(n^k+\check n^k+1)/2}\cdot\bigotimes_{k=a}^b\bigl([v_1^k]\wedge\cdots\wedge[v_{m^k}^k]\wedge[\check v_1^k]\wedge\cdots\wedge[\check v_{\check m^k}^k]\bigr)^{(-1)^k}.\end{aligned}\tag{10.88}$$

Equation (10.85) now follows from (10.84) and (10.86)–(10.88) by a computation with signs, where we use

$$\begin{aligned}&u_1^k\wedge\cdots\wedge u_{n^{k-1}}^k\wedge v_1^k\wedge\cdots\wedge v_{m^k}^k\wedge w_1^k\wedge\cdots\wedge w_{n^k}^k\wedge\check u_1^k\wedge\cdots\wedge\check u_{\check n^{k-1}}^k\wedge\check v_1^k\wedge\cdots\\ &\wedge\check v_{\check m^k}^k\wedge\check w_1^k\wedge\cdots\wedge\check w_{\check n^k}^k=(-1)^{n^k\check n^k+m^k\check n^{k-1}+\check m^k n^k+n^k\check n^{k-1}}\cdot u_1^k\wedge\cdots\wedge u_{n^{k-1}}^k\\ &\wedge\check u_1^k\wedge\cdots\wedge\check u_{\check n^{k-1}}^k\wedge v_1^k\wedge\cdots\wedge v_{m^k}^k\wedge\check v_1^k\wedge\cdots\wedge\check v_{\check m^k}^k\wedge w_1^k\wedge\cdots\wedge w_{n^k}^k\wedge\check w_1^k\wedge\cdots\wedge\check w_{\check n^k}^k\end{aligned}$$

to compare the left hand sides of (10.84) and (10.88). □

10.6.4 Determinants of short exact sequences of complexes

The next definition and proposition will be important in studying orientations on w-transverse fibre products in $\mathbf{m\dot{K}ur}$ or $\mathbf{\dot{K}ur}$ in Chapter 11. The definition is standard in (co)homology theory, as in Bredon [4, §IV.5] or Hatcher [33, §2.1].

Definition 10.69. Consider a commutative diagram of real vector spaces:

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \xrightarrow{\mathrm{d}^{k-2}} & E^{k-1} & \xrightarrow{\mathrm{d}^{k-1}} & E^k & \xrightarrow{\mathrm{d}^k} & E^{k+1} & \xrightarrow{\mathrm{d}^{k+1}} & E^{k+2} & \xrightarrow{\mathrm{d}^{k+2}} & \cdots\\ & & \downarrow{\scriptstyle\theta^{k-1}} & & \downarrow{\scriptstyle\theta^k} & & \downarrow{\scriptstyle\theta^{k+1}} & & \downarrow{\scriptstyle\theta^{k+2}} & & \\ \cdots & \xrightarrow{\mathrm{d}^{k-2}} & F^{k-1} & \xrightarrow{\mathrm{d}^{k-1}} & F^k & \xrightarrow{\mathrm{d}^k} & F^{k+1} & \xrightarrow{\mathrm{d}^{k+1}} & F^{k+2} & \xrightarrow{\mathrm{d}^{k+2}} & \cdots\\ & & \downarrow{\scriptstyle\psi^{k-1}} & & \downarrow{\scriptstyle\psi^k} & & \downarrow{\scriptstyle\psi^{k+1}} & & \downarrow{\scriptstyle\psi^{k+2}} & & \\ \cdots & \xrightarrow{\mathrm{d}^{k-2}} & G^{k-1} & \xrightarrow{\mathrm{d}^{k-1}} & G^k & \xrightarrow{\mathrm{d}^k} & G^{k+1} & \xrightarrow{\mathrm{d}^{k+1}} & G^{k+2} & \xrightarrow{\mathrm{d}^{k+2}} & \cdots\\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & 0, & & \end{array}\tag{10.89}$$

whose rows $E^\bullet,F^\bullet,G^\bullet$ are complexes, and whose columns are exact. Then $\theta^\bullet:E^\bullet\to F^\bullet$, $\psi^\bullet:F^\bullet\to G^\bullet$ are morphisms of complexes, and induce morphisms $H^k(\theta^\bullet):H^k(E^\bullet)\to H^k(F^\bullet)$, $H^k(\psi^\bullet):H^k(F^\bullet)\to H^k(G^\bullet)$ on cohomology.

We will define *connecting morphisms* $\delta^k_{\theta^\bullet,\psi^\bullet} : H^k(G^\bullet) \to H^{k+1}(E^\bullet)$. Let $\gamma \in H^k(G^\bullet)$, and write $\gamma = [g] = g + \operatorname{Im} \mathrm{d}^{k-1}$ for $g \in G^k$ with $\mathrm{d}^k(g) = 0$. Then $g = \psi^k(f)$ for some $f \in F^k$, by exactness of columns in (10.89), so $\mathrm{d}^k(f) \in F^{k+1}$. We have

$$\psi^{k+1}(\mathrm{d}^k f) = \mathrm{d}^k \circ \psi^k(f) = \mathrm{d}^k(g) = 0,$$

so $\mathrm{d}^k f = \theta^{k+1}(e)$ for some $e \in E^{k+1}$ by exactness of columns in (10.89). Then

$$\theta^{k+2} \circ \mathrm{d}^{k+1}(e) = \mathrm{d}^{k+1} \circ \theta^{k+1}(e) = \mathrm{d}^{k+1} \circ \mathrm{d}^k f = 0,$$

so $\mathrm{d}^{k+1}(e) = 0$ as θ^{k+2} is injective by exactness of columns in (10.89). Hence $[e] \in H^{k+1}(E^\bullet)$. Define $\delta^k_{\theta^\bullet,\psi^\bullet}(\gamma) = [e]$. A well known proof that can be found in Bredon [4, Th. IV.5.6] or Hatcher [33, Th. 2.16] shows that $\delta_{\theta^\bullet,\psi^\bullet}$ is well defined and linear, and the following sequence is exact

$$\cdots \to H^k(E^\bullet) \xrightarrow{H^k(\theta^\bullet)} H^k(F^\bullet) \xrightarrow{H^k(\psi^\bullet)} H^k(G^\bullet) \xrightarrow{\delta^k_{\theta^\bullet,\psi^\bullet}} H^{k+1}(E^\bullet) \to \cdots . \quad (10.90)$$

In the next proposition, note the similarity between the signs in (10.85) and (10.93). We can regard Proposition 10.68 as a special case of Proposition 10.70, with $0 \to E^\bullet \xrightarrow{\mathrm{id}\oplus 0} E^\bullet \oplus F^\bullet \xrightarrow{0\oplus\mathrm{id}} F^\bullet \to 0$ in place of equation (10.89).

Proposition 10.70. *Work in the situation of Definition* 10.69, *and suppose that* E^k, F^k, G^k *are finite-dimensional, and zero unless* $a \leqslant k \leqslant b$. *Then Definition* 10.66 *defines isomorphisms*

$$\begin{aligned}
\Theta_{E^\bullet} &: \textstyle\bigotimes_{k=a}^b (\det E^k)^{(-1)^k} \longrightarrow \bigotimes_{k=a}^b (\det H^k(E^\bullet))^{(-1)^k},\\
\Theta_{F^\bullet} &: \textstyle\bigotimes_{k=a}^b (\det F^k)^{(-1)^k} \longrightarrow \bigotimes_{k=a}^b (\det H^k(F^\bullet))^{(-1)^k},\\
\Theta_{G^\bullet} &: \textstyle\bigotimes_{k=a}^b (\det G^k)^{(-1)^k} \longrightarrow \bigotimes_{k=a}^b (\det H^k(G^\bullet))^{(-1)^k}.
\end{aligned} \quad (10.91)$$

Consider (10.90) *as an exact complex* $A^\bullet$ *with* $A^0 = H^0(E^\bullet)$, *and consider the* k^{th} *column of* (10.89) *as an exact complex* $B^\bullet_k$ *with* $B^0_k = E^k$. *Then* (10.69) *defines nonzero elements*

$$\begin{aligned}
\Psi_{A^\bullet} \in {} & \textstyle\bigotimes_{k=a}^b (\det H^k(E^\bullet))^{(-1)^k} \otimes \bigotimes_{k=a}^b (\det H^k(F^\bullet))^{(-1)^{k+1}}\\
& \textstyle\otimes \bigotimes_{k=a}^b (\det H^k(G^\bullet))^{(-1)^k},\\
\Psi_{B^\bullet_k} \in {} & (\det E^k) \otimes (\det F^k)^{-1} \otimes (\det G^k).
\end{aligned} \quad (10.92)$$

Then combining (10.91)–(10.92), *we have*

$$\begin{aligned}
&\prod_{a \leqslant l < k \leqslant b} (-1)^{\dim E^k \dim G^l} \cdot \bigl(\Theta_{E^\bullet} \otimes \Theta^{-1}_{F^\bullet} \otimes \Theta_{G^\bullet}\bigr)\bigl(\textstyle\bigotimes_{k=a}^b (\Psi_{B^\bullet_k})^{(-1)^k}\bigr)\\
&\qquad = \prod_{a \leqslant l < k \leqslant b} (-1)^{\dim H^k(E^\bullet) \dim H^l(G^\bullet)} \cdot \Psi_{A^\bullet}.
\end{aligned} \quad (10.93)$$

Proof. For $k \in \mathbb{Z}$, define

$$\begin{aligned} l^k &= \dim(\operatorname{Im} H^k(\theta^\bullet)), \;\; m^k = \dim(\operatorname{Im} H^k(\psi^\bullet)), \;\; n^k = \dim(\operatorname{Im} \delta^k_{\theta^\bullet,\psi^\bullet}), \\ p^k &= \dim(\operatorname{Im}(\mathrm{d}^k : E^k \to E^{k+1})), \;\; q^k = \dim(\operatorname{Im}(\mathrm{d}^k : G^k \to G^{k+1})). \end{aligned}$$

Then from (10.89) we deduce that

$$\begin{aligned} \dim E^k &= p^{k-1} + n^{k-1} + l^k + p^k, \\ \dim F^k &= p^{k-1} + n^{k-1} + q^{k-1} + l^k + m^k + p^k + n^k + q^k, \\ \dim G^k &= q^{k-1} + m^k + n^k + q^k, \quad \dim H^k(E^\bullet) = n^{k-1} + l^k, \\ \dim H^k(F^\bullet) &= l^k + m^k, \quad \text{and} \quad \dim H^k(G^\bullet) = m^k + n^k. \end{aligned} \tag{10.94}$$

For each $k \in \mathbb{Z}$, choose bases

$$\begin{aligned} &c_1^k, \ldots, c_{p^{k-1}}^k, b_1^k, \ldots, b_{n^{k-1}}^k, a_1^k, \ldots, a_{l^k}^k, d_1^k, \ldots, d_{p^k}^k \;\text{ for } E^k, \\ &\bar c_1^k, \ldots, \bar c_{p^{k-1}}^k, \bar b_1^k, \ldots, \bar b_{n^{k-1}}^k, g_1^k, \ldots, g_{q^{k-1}}^k, \bar a_1^k, \ldots, \bar a_{l^k}^k, \\ &e_1^k, \ldots, e_{m^k}^k, \bar d_1^k, \ldots, \bar d_{p^k}^k, f_1^k, \ldots, f_{n^k}^k, h_1^k, \ldots, h_{q^k}^k \;\text{ for } F^k, \\ &\bar g_1^k, \ldots, \bar g_{q^{k-1}}^k, \bar e_1^k, \ldots, \bar e_{m^k}^k, \bar f_1^k, \ldots, \bar f_{n^k}^k, \bar h_1^k, \ldots, \bar h_{q^k}^k \;\text{ for } G^k, \end{aligned}$$

such that d^k in $E^\bullet, F^\bullet, G^\bullet$ are given by

$$\begin{aligned} &\mathrm{d}^k(a_i^k) = 0, && \mathrm{d}^k(b_i^k) = 0, && \mathrm{d}^k(c_i^k) = 0, && \mathrm{d}^k(d_i^k) = c_i^{k+1}, \\ &\mathrm{d}^k(\bar a_i^k) = 0, && \mathrm{d}^k(e_i^k) = 0, && \mathrm{d}^k(\bar b_i^k) = 0, && \mathrm{d}^k(f_i^k) = \bar b_i^{k+1}, \\ &\mathrm{d}^k(\bar c_i^k) = 0, && \mathrm{d}^k(\bar d_i^k) = \bar c_i^{k+1}, && \mathrm{d}^k(g_i^k) = 0, && \mathrm{d}^k(h_i^k) = g_i^{k+1}, \\ &\mathrm{d}^k(\bar e_i^k) = 0, && \mathrm{d}^k(\bar f_i^k) = 0, && \mathrm{d}^k(\bar g_i^k) = 0, && \mathrm{d}^k(\bar h_i^k) = \bar g_i^{k+1}, \end{aligned}$$

and θ^k, ψ^k in (10.89) are given by

$$\begin{aligned} &\theta^k(a_i^k) = \bar a_i^k, && \theta^k(b_i^k) = \bar b_i^k, && \theta^k(c_i^k) = \bar c_i^k, && \theta^k(d_i^k) = \bar d_i^k, \\ &\psi^k(\bar a_i^k) = 0, && \psi^k(e_i^k) = \bar e_i^k, && \psi^k(\bar b_i^k) = 0, && \psi^k(f_i^k) = \bar f_i^k, \\ &\psi^k(\bar c_i^k) = 0, && \psi^k(\bar d_i^k) = 0, && \psi^k(g_i^k) = \bar g_i^k, && \psi^k(h_i^k) = \bar h_i^k. \end{aligned}$$

Then we have bases

$$\begin{aligned} &[b_1^k], \ldots, [b_{n^{k-1}}^k], [a_1^k], \ldots, [a_{l^k}^k] && \text{for } H^k(E^\bullet), \\ &[\bar a_1^k], \ldots, [\bar a_{l^k}^k], [e_1^k], \ldots, [e_{m^k}^k] && \text{for } H^k(F^\bullet), \\ &[\bar e_1^k], \ldots, [\bar e_{m^k}^k], [\bar f_1^k], \ldots, [\bar f_{n^k}^k] && \text{for } H^k(G^\bullet), \end{aligned}$$

where $H^k(\theta^\bullet), H^k(\psi^\bullet), \delta^k_{\theta^\bullet,\psi^\bullet}$ in (10.90) act by

$$\begin{aligned} &H^k(\theta^\bullet) : [a_i^k] \longmapsto [\bar a_i^k], \quad H^k(\theta^\bullet) : [b_i^k] \longmapsto 0, \quad H^k(\psi^\bullet) : [\bar a_i^k] \longmapsto 0, \\ &H^k(\psi^\bullet) : [e_i^k] \longmapsto [\bar e_i^k], \quad \delta^k_{\theta^\bullet,\psi^\bullet} : [\bar e_1^k] \longmapsto 0, \quad \delta^k_{\theta^\bullet,\psi^\bullet} : [\bar f_i^k] \longmapsto [b_i^{k+1}]. \end{aligned}$$

Definition 10.66 now implies that

$$
\begin{aligned}
\Psi_{A^\bullet} = {} & \textstyle\bigotimes_{k=a}^{b}\bigl([b_1^k]\wedge\cdots\wedge[b_{n^{k-1}}^k]\wedge[a_1^k]\wedge\cdots\wedge[a_{l^k}^k]\bigr)^{(-1)^k} \\
& \otimes \textstyle\bigotimes_{k=a}^{b}\bigl([\bar a_1^k]\wedge\cdots\wedge[\bar a_{l^k}^k]\wedge[e_1^k]\wedge\cdots\wedge[e_{m^k}^k]\bigr)^{(-1)^{k+1}} \\
& \otimes \textstyle\bigotimes_{k=a}^{b}\bigl([\bar e_1^k]\wedge\cdots\wedge[\bar e_{m^k}^k]\wedge[\bar f_1^k]\wedge\cdots\wedge[\bar f_{n^k}^k]\bigr)^{(-1)^k}, \qquad (10.95)\\
\Psi_{B_k^\bullet} = {} & (-1)^{q^{k-1}l^k+q^{k-1}p^k+m^kp^k}\cdot \\
& \bigl(c_1^k\wedge\cdots\wedge c_{p^{k-1}}^k\wedge b_1^k\wedge\cdots\wedge b_{n^{k-1}}^k\wedge a_1^k\wedge\cdots\wedge a_{l^k}^k\wedge d_1^k\wedge\cdots\wedge d_{p^k}^k\bigr) \\
& \otimes\bigl(\bar c_1^k\wedge\cdots\wedge\bar c_{p^{k-1}}^k\wedge\bar b_1^k\wedge\cdots\wedge\bar b_{n^{k-1}}^k\wedge g_1^k\wedge\cdots\wedge g_{q^{k-1}}^k\wedge\bar a_1^k\wedge\cdots\wedge\bar a_{l^k}^k \\
& \qquad \wedge e_1^k\wedge\cdots\wedge e_{m^k}^k\wedge\bar d_1^k\wedge\cdots\wedge\bar d_{p^k}^k\wedge f_1^k\wedge\cdots\wedge f_{n^k}^k\wedge h_1^k\wedge\cdots\wedge h_{q^k}^k\bigr)^{-1} \\
& \otimes\bigl(\bar g_1^k\wedge\cdots\wedge\bar g_{q^{k-1}}^k\wedge\bar e_1^k\wedge\cdots\wedge\bar e_{m^k}^k\wedge\bar f_1^k\wedge\cdots\wedge\bar f_{n^k}^k\wedge\bar h_1^k\wedge\cdots\wedge\bar h_{q^k}^k\bigr), \qquad (10.96)\\
\Theta_{E^\bullet}: {} & \bigotimes_{k=a}^{b}\bigl(c_1^k\wedge\cdots\wedge c_{p^{k-1}}^k\wedge b_1^k\wedge\cdots\wedge b_{n^{k-1}}^k\wedge a_1^k\wedge\cdots\wedge a_{l^k}^k\wedge d_1^k\wedge\cdots\wedge d_{p^k}^k\bigr)^{(-1)^k} \\
\longmapsto {} & \prod_{k=a}^{b}(-1)^{p^k(p^k+1)/2}\cdot\bigotimes_{k=a}^{b}\bigl([b_1^k]\wedge\cdots\wedge[b_{n^{k-1}}^k]\wedge[a_1^k]\wedge\cdots\wedge[a_{l^k}^k]\bigr)^{(-1)^k}, \qquad (10.97)\\
\Theta_{F^\bullet}: {} & \bigotimes_{k=a}^{b}\bigl(\bar c_1^k\wedge\cdots\wedge\bar c_{p^{k-1}}^k\wedge\bar b_1^k\wedge\cdots\wedge\bar b_{n^{k-1}}^k\wedge g_1^k\wedge\cdots\wedge g_{q^{k-1}}^k\wedge\bar a_1^k\wedge\cdots\wedge\bar a_{l^k}^k \\
& \qquad \wedge e_1^k\wedge\cdots\wedge e_{m^k}^k\wedge\bar d_1^k\wedge\cdots\wedge\bar d_{p^k}^k\wedge f_1^k\wedge\cdots\wedge f_{n^k}^k\wedge h_1^k\wedge\cdots\wedge h_{q^k}^k\bigr)^{(-1)^k} \\
\longmapsto {} & \prod_{k=a}^{b}(-1)^{\substack{(p^k+n^k+q^k)\cdot\\(p^k+n^k+q^k+1)/2}}\cdot\bigotimes_{k=a}^{b}\bigl([\bar a_1^k]\wedge\cdots\wedge[\bar a_{l^k}^k]\wedge[e_1^k]\wedge\cdots\wedge[e_{m^k}^k]\bigr)^{(-1)^k}, \qquad (10.98)\\
\Theta_{G^\bullet}: {} & \bigotimes_{k=a}^{b}\bigl(\bar g_1^k\wedge\cdots\wedge\bar g_{q^{k-1}}^k\wedge\bar e_1^k\wedge\cdots\wedge\bar e_{m^k}^k\wedge\bar f_1^k\wedge\cdots\wedge\bar f_{n^k}^k\wedge\bar h_1^k\wedge\cdots\wedge\bar h_{q^k}^k\bigr)^{(-1)^k} \\
\longmapsto {} & \prod_{k=a}^{b}(-1)^{q^k(q^k+1)/2}\cdot\bigotimes_{k=a}^{b}\bigl([\bar e_1^k]\wedge\cdots\wedge[\bar e_{m^k}^k]\wedge[\bar f_1^k]\wedge\cdots\wedge[\bar f_{n^k}^k]\bigr)^{(-1)^k}. \qquad (10.99)
\end{aligned}
$$

Here the sign in (10.96) is because, compared to the definition of $\Psi_{B_k^\bullet}$ in (10.69), we have reordered the basis elements for compatibility with (10.98). Equation (10.93) now follows from (10.94)–(10.99), after a computation with signs. □

10.7 Canonical line bundles and orientations

In this section we suppose throughout that $\dot{\mathbf{Man}}$ satisfies Assumptions 3.1–3.7, 10.1 and 10.13, so that objects X in $\dot{\mathbf{Man}}$ have functorial tangent spaces T_xX which are fibres of a tangent bundle $TX\rightarrow X$ of rank $\dim X$. The dual vector bundle is the cotangent bundle $T^*X\rightarrow X$. As in Definitions 2.38 and 10.15, its top exterior power $\Lambda^{\dim X}T^*X$ is the canonical bundle K_X of X, a real line bundle on X, and an orientation on X is an orientation on the fibres of K_X.

Our goal is to generalize this to (m- and μ-)Kuranishi spaces $\boldsymbol{X}$. In §10.7.1, for an m-Kuranishi space $\boldsymbol{X}=(X,\mathcal{K})$ in $\mathbf{m\dot{K}ur}$, we will define a topological real line bundle $K_{\boldsymbol{X}} \to X$, the *canonical bundle*, whose fibre at $x \in X$ is

$$K_{\boldsymbol{X}}|_x = \Lambda^{\dim T_x^*\boldsymbol{X}} T_x^*\boldsymbol{X} \otimes \Lambda^{\dim O_x\boldsymbol{X}} O_x\boldsymbol{X},$$

for $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ as in §10.2.1, using the material on determinants of complexes in §10.6. Then in §10.7.2 we define an orientation on $\boldsymbol{X}$ to be an orientation on the fibres of $K_{\boldsymbol{X}}$. Section 10.7.3 shows that if $\boldsymbol{X}$ is an oriented m-Kuranishi space with corners in $\mathbf{m\dot{K}ur^c}$, then there is a natural orientation on $\partial\boldsymbol{X}$, and hence on $\partial^k\boldsymbol{X}$ for $k = 1, 2, \ldots$. Sections 10.7.5–10.7.6 extend all this to μ-Kuranishi spaces and Kuranishi spaces.

The material of this section was inspired by Fukaya–Oh–Ohta–Ono's definition of orientations on FOOO Kuranishi spaces, as in Definition 7.8 and [15, Def. A1.17], [21, Def.s 3.1, 3.3, 3.5, & 3.10], and [30, Def. 5.8].

10.7.1 Canonical bundles of m-Kuranishi spaces

We now construct the *canonical bundle* $K_{\boldsymbol{X}} \to X$ of an m-Kuranishi space $\boldsymbol{X}$ in $\mathbf{mKur}$. Recall that we suppose $\mathbf{m\dot{K}ur}$ is constructed using $\mathbf{\dot{M}an}$ satisfying Assumptions 10.1 and 10.13, so that objects $V \in \mathbf{\dot{M}an}$ have tangent spaces T_vV which are the fibres of the tangent bundle $TV \to V$ with rank $\dim V$, and as in §10.2.1, $\boldsymbol{X}$ has tangent and obstruction spaces $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ for $x \in \boldsymbol{X}$.

Theorem 10.71. *Let $\boldsymbol{X} = (X, \mathcal{K})$ be an m-Kuranishi space in $\mathbf{m\dot{K}ur}$. Then there is a natural topological line bundle $\pi : K_{\boldsymbol{X}} \to X$ called the **canonical bundle** of $\boldsymbol{X}$, with fibres*

$$K_{\boldsymbol{X}}|_x = \det T_x^*\boldsymbol{X} \otimes \det O_x\boldsymbol{X} \tag{10.100}$$

for each $x \in X$, for $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ as in §10.2.1, with the property that if (V, E, s, ψ) is an m-Kuranishi neighbourhood on $\boldsymbol{X}$ in the sense of §4.7, then there is an isomorphism of topological real line bundles on $s^{-1}(0) \subseteq V$

$$\Theta_{V,E,s,\psi} : (\det T^*V \otimes \det E)|_{s^{-1}(0)} \longrightarrow \psi^{-1}(K_{\boldsymbol{X}}), \tag{10.101}$$

such that if $v \in s^{-1}(0) \subseteq V$ with $\psi(v) = x \in X$, so that as in (10.27) we have an exact sequence

$$0 \longrightarrow T_x\boldsymbol{X} \xrightarrow{\ \iota_x\ } T_vV \xrightarrow{\ \mathrm{d}_v s\ } E|_v \xrightarrow{\ \pi_x\ } O_x\boldsymbol{X} \longrightarrow 0, \tag{10.102}$$

and if $(c_1, \ldots, c_l), (d_1, \ldots, d_{l+m}), (e_1, \ldots, e_{m+n}), (f_1, \ldots, f_n)$ are bases for $T_x\boldsymbol{X}$, $T_vV, E|_v, O_x\boldsymbol{X}$ respectively with $\iota_x(c_i) = d_i$, $i = 1, \ldots, l$ and $\mathrm{d}_v s(d_{l+j}) = e_j$, $j = 1, \ldots, m$ and $\pi_x(e_{m+k}) = f_k$, $k = 1, \ldots, n$, and $(\gamma_1, \ldots, \gamma_l), (\delta_1, \ldots, \delta_{l+m})$ are dual bases to $(c_1, \ldots, c_l), (d_1, \ldots, d_{l+m})$ for $T_x^*\boldsymbol{X}, T_v^*V$, then

$$\begin{aligned} &\Theta_{V,E,s,\psi}|_v : \det T_v^*V \otimes \det E|_v \to \det T_x^*\boldsymbol{X} \otimes \det O_x\boldsymbol{X} \quad \textit{maps} \\ &\Theta_{V,E,s,\psi}|_v : (\delta_1 \wedge \cdots \wedge \delta_{l+m}) \otimes (e_1 \wedge \cdots \wedge e_{m+n}) \longmapsto \\ &\qquad (-1)^{m(m+1)/2} \cdot (\gamma_1 \wedge \cdots \wedge \gamma_l) \otimes (f_1 \wedge \cdots \wedge f_n). \end{aligned} \tag{10.103}$$

Proof. Just as a set, define $K_{\boldsymbol{X}}$ to be the disjoint union

$$K_{\boldsymbol{X}} = \textstyle\coprod_{x\in X}\bigl(\det T_x^*\boldsymbol{X}\otimes\det O_x\boldsymbol{X}\bigr),$$

and define $\pi : K_{\boldsymbol{X}} \to X$ to map $\pi : \det T_x^*\boldsymbol{X}\otimes\det O_x\boldsymbol{X} \mapsto x$, so that $K_{\boldsymbol{X}}|_x = \pi^{-1}(x)$ is as in (10.100) for $x \in X$. Define the structure of a 1-dimensional real vector space on $K_{\boldsymbol{X}}|_x$ for each $x \in X$ to be that coming from the right hand side of (10.100). To make $K_{\boldsymbol{X}}$ into a topological real line bundle, it remains to define a topology on the set $K_{\boldsymbol{X}}$, such that $\pi : K_{\boldsymbol{X}} \to X$ is a continuous map, and the usual local triviality condition for vector bundles holds.

Suppose (V, E, s, ψ) is an m-Kuranishi neighbourhood on $\boldsymbol{X}$. Consider the following complex $F^\bullet$ of topological real vector bundles on $s^{-1}(0) \subseteq V$:

$$\underset{\text{degree}}{\cdots} \xrightarrow{0} \underset{-3}{0} \xrightarrow{0} \underset{-2}{0} \xrightarrow{0} \underset{-1}{TV|_{s^{-1}(0)}} \xrightarrow{\mathrm{d}s} \underset{0}{E|_{s^{-1}(0)}} \xrightarrow{0} \underset{1}{0} \xrightarrow{0} \underset{2}{0} \xrightarrow{0} \cdots,$$

where $TV|_{s^{-1}(0)}$ is in degree -1 and $E|_{s^{-1}(0)}$ in degree 0, and $\mathrm{d}s$ is given by $\mathrm{d}s|_v = \mathrm{d}_v s$ for each $v \in s^{-1}(0)$, where $\mathrm{d}_v s$ is as in Definition 10.6. One can show that $\mathrm{d}_v s$ depends continuously on v, so that $\mathrm{d}s$ is a morphism of topological vector bundles.

Equation (10.102) shows that if $v \in s^{-1}(0)$ with $\psi(v) = x \in X$ then the cohomology of $F^\bullet|_v$ is $T_x\boldsymbol{X}$ in degree -1, and $O_x\boldsymbol{X}$ in degree 0, and 0 otherwise. Thus Definition 10.66 defines an isomorphism

$$\Theta_{F^\bullet|_v} : (\det T_vV)^{-1}\otimes(\det E|_v) \longrightarrow (\det T_x\boldsymbol{X})^{-1}\otimes(\det O_x\boldsymbol{X}).$$

Identifying $(\det T_vV)^{-1} = \det T_v^*V$ and $(\det T_x\boldsymbol{X})^{-1} = \det T_x^*\boldsymbol{X}$ and expanding Definition 10.66, we see that this $\Theta_{F^\bullet|_v}$ is exactly the map $\Theta_{V,E,s,\psi}|_v$ defined in (10.103). Thus, Definition 10.66 shows that $\Theta_{V,E,s,\psi}|_v$ is independent of choices of bases $(c_1, \ldots, c_l), \ldots, (f_1, \ldots, f_n)$.

Therefore we can define $\Theta_{V,E,s,\psi}$ in (10.101), just as a map of sets without yet considering topological line bundle structures, by taking $\Theta_{V,E,s,\psi}|_v$ for each $v \in s^{-1}(0)$ to be as in (10.103) for any choice of bases $(c_1, \ldots, c_l), \ldots, (f_1, \ldots, f_n)$. As $\psi : s^{-1}(0) \to \operatorname{Im}\psi$ is a homeomorphism, we can pushforward by ψ to obtain

$$\begin{aligned}\psi_*(\Theta_{V,E,s,\psi}) : \psi_*\bigl((\det T^*V\otimes\det E)|_{s^{-1}(0)}\bigr) \longrightarrow&\\ K_{\boldsymbol{X}}|_{\operatorname{Im}\psi} = \pi^{-1}(\operatorname{Im}\psi) \subseteq K_{\boldsymbol{X}},&\end{aligned} \tag{10.104}$$

which maps by $\Theta_{V,E,s,\psi}|_v$ over $x \in \operatorname{Im}\psi$ with $v = \psi^{-1}(x)$.

Now (10.104) is a bijection, with the left hand side a topological line bundle over $\operatorname{Im}\psi \subseteq X$. Hence there is a unique topology on $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi} = \pi^{-1}(\operatorname{Im}\psi) \subseteq K_{\boldsymbol{X}}$ making $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi} \to \operatorname{Im}\psi$ into a topological line bundle, such that (10.104) is an isomorphism of topological line bundles over $\operatorname{Im}\psi$.

Let (V', E', s', ψ') be another m-Kuranishi neighbourhood on $\boldsymbol{X}$, giving

$$\begin{aligned}\psi_*(\Theta_{V',E',s',\psi'}) : \psi'_*\bigl((\det T^*V'\otimes\det E')|_{s'^{-1}(0)}\bigr) \longrightarrow&\\ K_{\boldsymbol{X}}|_{\operatorname{Im}\psi'} = \pi^{-1}(\operatorname{Im}\psi') \subseteq K_{\boldsymbol{X}}.&\end{aligned} \tag{10.105}$$

So we have topologies on $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi}$ and $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi'}$ making (10.104)–(10.105) into isomorphisms of topological line bundles. We claim that these topologies agree on $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi\cap\operatorname{Im}\psi'}$. To prove this, note that Theorem 4.56(a) gives a coordinate change $\Phi=(\tilde V,\phi,\hat\phi):(V,E,s,\psi)\to(V',E',s',\psi')$ over $\operatorname{Im}\psi\cap\operatorname{Im}\psi'$ on $\boldsymbol{X}$, and consider the commutative diagram of topological vector bundles on $\tilde V\cap s^{-1}(0)$:

$$\begin{array}{ccccccccccc}
\cdots \xrightarrow{0} & 0 & \xrightarrow{0} & TV|_{\tilde V\cap s^{-1}(0)} & \xrightarrow{\mathrm{d}s} & E|_{\tilde V\cap s^{-1}(0)} & \xrightarrow{0} & 0 & \xrightarrow{0} \cdots \\
 & & & \big\downarrow{\scriptstyle T\phi|_{\tilde V\cap s^{-1}(0)}} & & \big\downarrow{\scriptstyle \hat\phi|_{\tilde V\cap s^{-1}(0)}} & & & \\
\cdots \xrightarrow{0} & 0 & \xrightarrow{0} & \phi^*(TV')|_{\tilde V\cap s^{-1}(0)} & \xrightarrow{\phi^*(\mathrm{d}s')} & \phi^*(E')|_{\tilde V\cap s^{-1}(0)} & \xrightarrow{0} & 0 & \xrightarrow{0} \cdots, \\
\text{degree} & -2 & & -1 & & 0 & & 1 &
\end{array} \tag{10.106}$$

where $T\phi|_{\tilde V\cap s^{-1}(0)}$ is defined by Assumption 10.13(b) since $\phi:\tilde V\to V'$ is $\boldsymbol{A}$ near $\tilde V\cap s^{-1}(0)$ by Proposition 4.34(d).

As in (10.70), regard the rows of (10.106) as complexes $F^\bullet,F'^\bullet$ of topological vector bundles, and the columns as a morphism of complexes $\theta^\bullet:F^\bullet\to F'^\bullet$. If $v\in\tilde V\cap s^{-1}(0)$ with $\phi(v)=v'\in s'^{-1}(0)$ and $\psi(v)=\psi'(v')=x\in\operatorname{Im}\psi\cap\operatorname{Im}\psi'$, then Definition 10.21 shows that $\theta^\bullet$ induces isomorphisms on cohomology groups of $F^\bullet,F'^\bullet$, and furthermore, under the identification of the cohomologies of $F^\bullet,F'^\bullet$ with $T_x\boldsymbol{X}$ in degree -1 and $O_x\boldsymbol{X}$ in degree 0, these isomorphisms are the identity maps on $T_x\boldsymbol{X},O_x\boldsymbol{X}$. Thus, Proposition 10.67 gives an isomorphism of topological line bundles on $\tilde V\cap s^{-1}(0)$:

$$\Xi_{\theta^\bullet}:(\det T^*V\otimes\det E)|_{\tilde V\cap s^{-1}(0)}\longrightarrow\phi^*(\det T^*V'\otimes\det E')|_{\cdots},$$

such that for all v,v',x as above, the following diagram (10.72) commutes

$$\begin{array}{ccc}
\det T_v^*V\otimes\det E|_v & \xrightarrow{\quad\Xi_{\theta^\bullet}|_v\quad} & \det T_{v'}^*V'\otimes\det E'|_{v'} \\
\big\downarrow{\scriptstyle \Theta_{V,E,s,\psi}|_v} & & \big\downarrow{\scriptstyle \Theta_{V',E',s',\psi'}|_{v'}} \\
(\det T_x\boldsymbol{X})^{-1}\otimes(\det O_x\boldsymbol{X}) & \xlongequal{\qquad} & (\det T_x\boldsymbol{X})^{-1}\otimes(\det O_x\boldsymbol{X}),
\end{array} \tag{10.107}$$

using the identifications of $\Theta_{F^\bullet|_v},\Theta_{F'^\bullet|_{v'}}$ with $\Theta_{V,E,s,\psi}|_v,\Theta_{V',E',s',\psi'}|_{v'}$ above.

Now $\psi_*(\Xi_{\theta^\bullet})$ is an isomorphism on $\operatorname{Im}\psi\cap\operatorname{Im}\psi'$ between the line bundles on the left hand sides of (10.104)–(10.105), and (10.107) for each $x\in\operatorname{Im}\psi\cap\operatorname{Im}\psi'$ shows that $\psi_*(\Xi_{\theta^\bullet})$ is compatible with (10.104)–(10.105). Thus, the topologies on $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi}$ and $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi'}$ from (10.104) and (10.105) agree on $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi\cap\operatorname{Im}\psi'}$, proving the claim.

Choose a family of m-Kuranishi neighbourhoods $\{(V_i,E_i,s_i,\psi_i):i\in I\}$ on $\boldsymbol{X}$ with $X=\bigcup_{i\in I}\operatorname{Im}\psi_i$ (for instance, those in the m-Kuranishi structure $\mathcal{K}$ on $\boldsymbol{X}=(X,\mathcal{K})$). Then we have topologies on $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi_i}$ for all $i\in I$ which agree on overlaps $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j}$ for all $i,j\in I$, so they glue to give a global topology on $K_{\boldsymbol{X}}$, which makes $\pi:K_{\boldsymbol{X}}\to X$ into a topological real line bundle. The compatibility between $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi}$ and $K_{\boldsymbol{X}}|_{\operatorname{Im}\psi'}$ on $\operatorname{Im}\psi\cap\operatorname{Im}\psi'$ above implies that this topology on $K_{\boldsymbol{X}}$ is independent of choices.

If (V,E,s,ψ) is any m-Kuranishi neighbourhood on $\boldsymbol{X}$, then by including (V,E,s,ψ) in the family $\{(V_i,E_i,s_i,\psi_i):i\in I\}$, by construction there is an isomorphism $\Theta_{V,E,s,\psi}$ in (10.101) with the properties required. □

Example 10.72. Using the notation of Example 4.30, let $X \in \dot{\mathbf{Man}}$, and let $\boldsymbol{X} = F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(X)$ be the corresponding m-Kuranishi space, so that $\boldsymbol{X}$ is covered by a single m-Kuranishi neighbourhood $(X, 0, 0, \mathrm{id}_X)$. Then $K_{\boldsymbol{X}}$ is canonically isomorphic to $K_X = \det T^*X \to X$, considered as a topological line bundle.

Canonical line bundles are functorial under étale 1-morphisms:

Proposition 10.73. *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be an étale 1-morphism in $\mathbf{m\dot{K}ur}$ as in* §10.5.1 *(for example, $\boldsymbol{f}$ could be an equivalence), so that Theorem* 10.71 *defines canonical bundles $K_{\boldsymbol{X}} \to X$, $K_{\boldsymbol{Y}} \to Y$. Then there is a natural isomorphism*

$$K_{\boldsymbol{f}} : f^*(K_{\boldsymbol{Y}}) \longrightarrow K_{\boldsymbol{X}} \tag{10.108}$$

of topological line bundles on X, such that for all $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$

$$\begin{aligned} &K_{\boldsymbol{f}}|_x = (\det T_x^*\boldsymbol{f}) \otimes (\det O_x\boldsymbol{f})^{-1} : \\ &\det T_y^*\boldsymbol{Y} \otimes \det O_y\boldsymbol{Y} \longrightarrow \det T_x^*\boldsymbol{X} \otimes \det O_x\boldsymbol{X}, \end{aligned} \tag{10.109}$$

where $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$, $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$ are as in §10.2.1 *and are isomorphisms by Theorem* 10.55, *and $T_x^*\boldsymbol{f} : T_y^*\boldsymbol{Y} \to T_x^*\boldsymbol{X}$ is dual to $T_x\boldsymbol{f}$.*

Proof. As a map of sets, $K_{\boldsymbol{f}}$ in (10.108) is determined uniquely by (10.109), and (10.109) is an isomorphism on the fibres at each $x \in X$. Thus, we need only show that this map $K_{\boldsymbol{f}}$ is continuous. Let $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y$ in $\boldsymbol{Y}$, and choose m-Kuranishi neighbourhoods (U_a, D_a, r_a, χ_a), (V_b, E_b, s_b, ψ_b) on $\boldsymbol{X}, \boldsymbol{Y}$ respectively with $x \in \mathrm{Im}\,\chi$ and $y \in \mathrm{Im}\,\psi$. Then Theorem 4.56(b) gives a 1-morphism $\boldsymbol{f}_{ab} = (U_{ab}, f_{ab}, \hat{f}_{ab}) : (U_a, D_a, r_a, \chi_a), (V_b, E_b, s_b, \psi_b)$ over $(\mathrm{Im}\,\chi_a \cap f^{-1}(\mathrm{Im}\,\psi_b), \boldsymbol{f})$.

By the argument in the proof of Theorem 10.71, but replacing (10.106) by

$$\begin{array}{ccccccccccc} \cdots \xrightarrow{0} & 0 & \xrightarrow{0} & TU_a|_{U_{ab}\cap r_a^{-1}(0)} & \xrightarrow{\mathrm{d}r_a} & D_a|_{U_{ab}\cap r_a^{-1}(0)} & \xrightarrow{0} & 0 & \xrightarrow{0} \cdots \\ & & & \Big\downarrow{\scriptstyle Tf_{ab}|_{U_{ab}\cap r_a^{-1}(0)}} & & \Big\downarrow{\scriptstyle \hat{f}_{ab}|_{U_{ab}\cap r_a^{-1}(0)}} & & & \\ \cdots \xrightarrow{0} & 0 & \xrightarrow{0} & f_{ab}^*(TV_b)|_{U_{ab}\cap r_a^{-1}(0)} & \xrightarrow{f_{ab}^*(\mathrm{d}s_b)} & f_{ab}^*(E_b)|_{U_{ab}\cap r_a^{-1}(0)} & \xrightarrow{0} & 0 & \xrightarrow{0} \cdots, \\ \text{degree} & -2 & & -1 & & 0 & & 1 & \end{array}$$

and noting that $T_x\boldsymbol{f}, O_x\boldsymbol{f}$ are isomorphisms, we obtain an isomorphism of topological line bundles on $U_{ab} \cap r_a^{-1}(0)$:

$$\Xi_{\theta^\bullet} : (\det T^*U_{ab} \otimes \det D_a)|_{U_{ab}\cap r_a^{-1}(0)} \longrightarrow f_{ab}^*(\det T^*V_b \otimes \det E_b)|_{\dots},$$

such that for all $u \in U_{ab} \cap r_a^{-1}(0)$ with $\chi_a(u) = x$ in $\boldsymbol{X}$, $f_{ab}(u) = v \in V_b$ and $\boldsymbol{f}(x) = \psi_b(v) = y$ in $\boldsymbol{Y}$ as above, as in (10.72) and (10.107) the following commutes:

$$\begin{array}{ccc} \det T_u^*U_{ab} \otimes \det D_a|_u & \xrightarrow{\quad \Xi_{\theta^\bullet}|_u \quad} & \det T_v^*V_b \otimes \det E_b|_v \\ \Big\downarrow{\scriptstyle \Theta_{U_a,D_a,r_a,\chi_a}|_u} & & \Big\downarrow{\scriptstyle \Theta_{V_b,E_b,s_b,\psi_b}|_v} \\ (\det T_x\boldsymbol{X})^{-1} \otimes (\det O_x\boldsymbol{X}) & \xleftarrow{\chi_a^*(K_{\boldsymbol{f}})|_u \,=\, K_{\boldsymbol{f}}|_x \text{ in (10.109)}} & (\det T_y\boldsymbol{Y})^{-1} \otimes (\det O_y\boldsymbol{Y}). \end{array} \tag{10.110}$$

As the top, left and right morphisms of (10.110) are restrictions to u of isomorphisms of topological line bundles $\Xi_{\theta^\bullet}, \Theta_{U_a,D_a,r_a,\chi_a}, \Theta_{V_b,E_b,s_b,\psi_b}$, it follows that $\chi_a^*(K_{\boldsymbol f})$ is an isomorphism of topological line bundles over $U_{ab} \cap r_a^{-1}(0)$, so that $K_{\boldsymbol f}$ is an isomorphism (and in particular is continuous) over $\mathrm{Im}\,\chi_a \cap f^{-1}(\mathrm{Im}\,\psi_b) \subseteq X$. Since we can cover X by such open $\mathrm{Im}\,\chi_a \cap f^{-1}(\mathrm{Im}\,\psi_b)$, this shows $K_{\boldsymbol f}$ in (10.108) is an isomorphism of topological line bundles. $\square$

By Examples 10.2 and 10.14, the results above apply when $\dot{\mathbf{mKur}}$ is one of

$$\mathbf{mKur}, \mathbf{mKur^c}, \mathbf{mKur^c_{we}}, \tag{10.111}$$

with $T_x\boldsymbol X, O_x\boldsymbol X$ and $K_{\boldsymbol X}$ defined using ordinary tangent spaces T_vV in $\mathbf{Man}$, $\mathbf{Man^c}, \mathbf{Man^c_{we}}$, and also when $\dot{\mathbf{mKur}}$ is one of

$$\mathbf{mKur^c}, \mathbf{mKur^{gc}}, \mathbf{mKur^{ac}}, \mathbf{mKur^{c,ac}}, \tag{10.112}$$

with ${}^bT_x\boldsymbol X, {}^bO_x\boldsymbol X, {}^bK_{\boldsymbol X}$ (using the obvious notation) defined using b-tangent spaces ${}^bT_vV$ in $\mathbf{Man^c}, \mathbf{Man^{gc}}, \mathbf{Man^{ac}}, \mathbf{Man^{c,ac}}$. Note that in $\mathbf{mKur^c}$ we have *two different notions of canonical bundle* $K_{\boldsymbol X}, {}^bK_{\boldsymbol X}$, defined using ordinary tangent bundles $TV \to V$ and b-tangent bundles ${}^bTV \to V$ in $\mathbf{Man^c}$. We will see in §10.7.2 that these yield equivalent notions of orientation on $\boldsymbol X$ in $\mathbf{mKur^c}$.

10.7.2 Orientations on m-Kuranishi spaces

Definition 10.74. Let $\boldsymbol X = (X, \mathcal K)$ be an m-Kuranishi space in $\dot{\mathbf{mKur}}$, so that Theorem 10.71 defines the canonical bundle $\pi : K_{\boldsymbol X} \to X$. An *orientation* $o_{\boldsymbol X}$ *on* $\boldsymbol X$ is an orientation on the fibres of $K_{\boldsymbol X}$.

That is, as in Definitions 2.38 and 10.15, an orientation $o_{\boldsymbol X}$ on $\boldsymbol X$ is an equivalence class $[\omega]$ of continuous sections $\omega \in \Gamma^0(K_{\boldsymbol X})$ with $\omega|_x \neq 0$ for all $x \in X$, where two such ω, ω' are equivalent if $\omega' = K \cdot \omega$ for $K : X \to (0,\infty)$ continuous. The *opposite orientation* is $-o_{\boldsymbol X} = [-\omega]$.

Then we call $(\boldsymbol X, o_{\boldsymbol X})$ an *oriented m-Kuranishi space*. Usually we suppress the orientation $o_{\boldsymbol X}$, and just refer to $\boldsymbol X$ as an oriented m-Kuranishi space, and then we write $-\boldsymbol X$ for $\boldsymbol X$ with the opposite orientation.

Proposition 10.73 implies that if $\boldsymbol f : \boldsymbol X \to \boldsymbol Y$ is an étale 1-morphism in $\dot{\mathbf{mKur}}$ then orientations $o_{\boldsymbol Y}$ on $\boldsymbol Y$ pull back to orientations $o_{\boldsymbol X} = \boldsymbol f^*(o_{\boldsymbol Y})$ on $\boldsymbol X$, where if $o_{\boldsymbol Y} = [\omega]$ then $o_{\boldsymbol X} = [K_{\boldsymbol f} \circ f^*(\omega)]$. If $\boldsymbol f$ is an equivalence, this defines a natural 1-1 correspondence between orientations on $\boldsymbol X$ and orientations on $\boldsymbol Y$.

Let $\boldsymbol f : \boldsymbol X \to \boldsymbol Y$ be a 1-morphism in $\dot{\mathbf{mKur}}$. A *coorientation* $c_{\boldsymbol f}$ on $\boldsymbol f$ is an orientation on the fibres of the line bundle $K_{\boldsymbol X} \otimes f^*(K_{\boldsymbol Y}^*)$ over X. That is, $c_{\boldsymbol f}$ is an equivalence class $[\gamma]$ of $\gamma \in \Gamma^0(K_{\boldsymbol X} \otimes f^*(K_{\boldsymbol Y}^*))$ with $\gamma|_x \neq 0$ for all $x \in X$, where two such γ, γ' are equivalent if $\gamma' = K \cdot \gamma$ for $K : X \to (0,\infty)$ continuous. The *opposite coorientation* is $-c_{\boldsymbol f} = [-\gamma]$. If $\boldsymbol Y$ is oriented then coorientations on $\boldsymbol f$ are equivalent to orientations on $\boldsymbol X$. Orientations on $\boldsymbol X$ are equivalent to coorientations on $\boldsymbol\pi : \boldsymbol X \to *$, for $*$ the point in $\dot{\mathbf{mKur}}$.

Remark 10.75. There are several equivalent ways to define orientations on m-Kuranishi spaces $\boldsymbol X = (X, \mathcal K)$ without first defining the canonical bundle $K_{\boldsymbol X}$.

Writing $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \Lambda_{ijk,\ i,j,k\in I}\big)$, an orientation on $\boldsymbol{X}$ is equivalent to the data of an orientation on the manifold E_i in $\dot{\mathbf{Man}}$ near $0_{E_i}(s_i^{-1}(0)) \subseteq E_i$, such that all the coordinate changes $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ are 'orientation-preserving' in a suitable sense.

The purpose of Definition 10.66 and Proposition 10.67 is to give us a good notion of when Φ_{ij} is orientation-preserving in the proof of Theorem 10.71. We do this using tangent spaces and tangent bundles, and implicitly we use the exact sequence (10.59) to compare orientations on (V_i, E_i, s_i, ψ_i) and (V_j, E_j, s_j, ψ_j).

It should still be possible to define orientations in $\mathbf{m\dot{K}ur}$ when the category $\dot{\mathbf{Man}}$ does not have tangent bundles $TV \to V$, but does have a well-behaved notion of orientation. To do this we would need an alternative way to define when Φ_{ij} is 'orientation-preserving', not involving tangent bundles.

As for (10.111)–(10.112), Definition 10.74 defines orientations on m-Kuranishi spaces $\boldsymbol{X}$ in the 2-categories $\mathbf{mKur}, \mathbf{mKur^c}, \mathbf{mKur^c_{we}}$, with $K_{\boldsymbol{X}}$ defined using tangent bundles $TV \to V$, and on $\boldsymbol{X}$ in the 2-categories $\mathbf{mKur^c}, \mathbf{mKur^{gc}}$, $\mathbf{mKur^{ac}}, \mathbf{mKur^{c,ac}}$, with ${}^bK_{\boldsymbol{X}}$ defined using b-tangent bundles ${}^bTV \to V$.

For $\boldsymbol{X} = (X, \mathcal{K})$ in $\mathbf{mKur^c}$, we have two canonical bundles $K_{\boldsymbol{X}}$ and ${}^bK_{\boldsymbol{X}}$, which are generally not canonically isomorphic. However, the notions of orientation on $\boldsymbol{X}$ defined using $K_{\boldsymbol{X}}$ and ${}^bK_{\boldsymbol{X}}$ are equivalent. This is because, as in §2.6, the notions of orientation on $E_i \in \mathbf{Man^c}$ defined using TE_i and ${}^bTE_i$ are equivalent, and as in Remark 10.75 an orientation on $\boldsymbol{X}$ is equivalent to local orientations on E_i in m-Kuranishi neighbourhoods (V_i, E_i, s_i, ψ_i) in $\mathcal{K}$.

Example 10.76. Using the notation of Example 4.30, let $X \in \dot{\mathbf{Man}}$, and let $\boldsymbol{X} = F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(X)$ be the corresponding m-Kuranishi space. Then combining Example 10.72 and Definitions 10.15 and 10.74 shows that orientations on X in $\dot{\mathbf{Man}}$, and on $\boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$, are equivalent.

10.7.3 Orienting boundaries of m-Kuranishi spaces with corners

Now suppose $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumptions 3.22 and 10.16, so that as in §4.6 we have a 2-category $\mathbf{m\dot{K}ur^c}$ of m-Kuranishi spaces with corners $\boldsymbol{X}$ which have boundaries $\partial\boldsymbol{X}$ and 1-morphisms $\boldsymbol{i_X} : \partial\boldsymbol{X} \to \boldsymbol{X}$ as in §4.6.1. Also $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumptions 10.1 and 10.13 by Assumption 10.16, so Theorem 10.71 defines canonical bundles $K_{\boldsymbol{X}} \to X$ and $K_{\partial\boldsymbol{X}} \to \partial X$. Our next theorem relates these. One should compare $\Omega_{\boldsymbol{X}}$ in (10.113) with Ω_X in (10.16) for $X \in \dot{\mathbf{Man}}^{\mathbf{c}}$.

Theorem 10.77. *Let $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfy Assumptions 3.22 and 10.16, and suppose $\boldsymbol{X}$ is an m-Kuranishi space with corners in $\mathbf{m\dot{K}ur^c}$. Then there is a natural isomorphism of topological line bundles on ∂X*

$$\Omega_{\boldsymbol{X}} : K_{\partial\boldsymbol{X}} \longrightarrow N_{\partial\boldsymbol{X}} \otimes i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}}), \tag{10.113}$$

where $N_{\partial\boldsymbol{X}}$ is a line bundle on ∂X, with a natural orientation on its fibres.

Suppose that (V_a, E_a, s_a, ψ_a) *is an m-Kuranishi neighbourhood on* $\boldsymbol{X}$, *as in* §4.7.1, *with* $\dim V_a = m_a$ *and* $\operatorname{rank} E_a = n_a$. *Then* §4.7.3 *defines an m-Kuranishi neighbourhood* $(V_{(1,a)}, E_{(1,a)}, s_{(1,a)}, \psi_{(1,a)})$ *on* $\partial\boldsymbol{X}$ *with* $V_{(1,a)} = \partial V_a$, $E_{(1,a)} = i_{V_a}^*(E_a)$, *and* $s_{(1,a)} = i_{V_a}^*(s_a)$. *Also Assumption* 10.16 *gives a (smooth) line bundle* $N_{\partial V_a} \to \partial V_a$, *with an orientation on its fibres. Then there is a natural isomorphism of topological line bundles on* $s_{(1,a)}^{-1}(0) \subseteq \partial V_a$

$$\Phi_{V_a,E_a,s_a,\psi_a} : N_{\partial V_a}|_{s_{(1,a)}^{-1}(0)} \longrightarrow \psi_{(1,a)}^{-1}(N_{\partial\boldsymbol{X}}), \tag{10.114}$$

which identifies the orientations on the fibres, such that the following commutes:

$$\begin{array}{ccc}
\begin{array}{l}(\det T^*(\partial V_a)\otimes \\ \det i_{V_a}^*(E_a))|_{s_{(1,a)}^{-1}(0)}\end{array} & \xrightarrow[\Omega_{V_a}\otimes \mathrm{id}_{\det i_{V_a}^*(E_a)|\cdots}]{} & \begin{array}{l}N_{\partial V_a}\otimes i_{V_a}^*(\det T^*V_a \\ \otimes \det E_a)|_{s_{(1,a)}^{-1}(0)}\end{array} \\
\Big\downarrow {\scriptstyle \Theta_{V_{(1,a)},E_{(1,a)},s_{(1,a)},\psi_{(1,a)}}} & & {\scriptstyle \Phi_{V_a,E_a,s_a,\psi_a}\otimes i_{V_a}|^*_{\cdots}(\Theta_{V_a,E_a,s_a,\psi_a})} \Big\downarrow \\
\psi_{(1,a)}^{-1}(K_{\partial\boldsymbol{X}}) & \xrightarrow{\Omega_{\boldsymbol{X}}} & \psi_{(1,a)}^{-1}(N_{\partial\boldsymbol{X}} \otimes i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}})),
\end{array} \tag{10.115}$$

where Ω_{V_a} *is as in* (10.16), *and* $\Theta_{V_a,E_a,s_a,\psi_a}, \Theta_{V_{(1,a)},E_{(1,a)},s_{(1,a)},\psi_{(1,a)}}$ *are as in* (10.101), *and* $\Omega_{\boldsymbol{X}}$ *is as in* (10.113), *and* $\Phi_{V_a,E_a,s_a,\psi_a}$ *is as in* (10.114).

Proof. Most of the theorem holds trivially, by definition. Define a topological line bundle $N_{\partial\boldsymbol{X}} \to \partial X$ by $N_{\partial\boldsymbol{X}} = K_{\partial\boldsymbol{X}} \otimes (i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}}))^*$, where $(i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}}))^*$ is the dual line bundle to $i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}})$, and define $\Omega_{\boldsymbol{X}}$ in (10.113) to be the inverse of

$$N_{\partial\boldsymbol{X}} \otimes i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}}) = K_{\partial\boldsymbol{X}} \otimes \big(i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}})\big)^* \otimes i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}}) \xrightarrow{\mathrm{id}\otimes \text{dual pairing}} K_{\partial\boldsymbol{X}}.$$

For the second part, since (10.115) is a diagram of isomorphisms of topological line bundles on $s_{(1,a)}^{-1}(0)$ with $\Phi_{V_a,E_a,s_a,\psi_a}$ the only undefined term, we define $\Phi_{V_a,E_a,s_a,\psi_a}$ to be the unique isomorphism in (10.114) such that (10.115) commutes.

We must construct an orientation on the fibres of $N_{\partial\boldsymbol{X}}$ such that (10.114) is orientation-preserving for all m-Kuranishi neighbourhoods (V_a, E_a, s_a, ψ_a) on $\boldsymbol{X}$. Since $\psi_{(1,a)} : s_{(1,a)}^{-1}(0) \to \operatorname{Im}\psi_{(1,a)}$ is a homeomorphism, there is a unique orientation on $N_{\partial\boldsymbol{X}}|_{\operatorname{Im}\psi_{(1,a)}}$ such that (10.114) is orientation-preserving. We will prove that for any two such (V_a, E_a, s_a, ψ_a), (V_b, E_b, s_b, ψ_b) on $\boldsymbol{X}$ we have

$$\begin{aligned}\Phi_{V_a,E_a,s_a,\psi_a}|_{V_{(1,a)(1,b)}\cap s_{(1,a)}^{-1}(0)} &= \partial\phi_{ab}|^*_{\cdots}(\Phi_{V_b,E_b,s_b,\psi_b}) \circ \gamma_{\phi_{ab}}|_{\cdots} : \\ N_{\partial V_a}|_{V_{(1,a)(1,b)}\cap s_{(1,a)}^{-1}(0)} &\longrightarrow \psi_{(1,a)}^{-1}(N_{\partial\boldsymbol{X}})|_{V_{(1,a)(1,b)}\cap s_{(1,a)}^{-1}(0)},\end{aligned} \tag{10.116}$$

where $\gamma_{\phi_{ab}} : N_{V_{ab}} \to \phi_{ab}^*(N_{V_b})$ is as in (10.11) or (10.14). As $\gamma_{\phi_{ab}}$ is orientation preserving by Assumption 10.16, equation (10.116) implies that the orientations on $N_{\partial\boldsymbol{X}}|_{\operatorname{Im}\psi_{(1,a)}}$ and $N_{\partial\boldsymbol{X}}|_{\operatorname{Im}\psi_{(1,b)}}$ agree on $\operatorname{Im}\psi_{(1,a)} \cap \operatorname{Im}\psi_{(1,b)}$. Because we can cover ∂X by such open $\operatorname{Im}\psi_{(1,a)} \subseteq \partial X$, there is a unique orientation on the fibres of $N_{\partial\boldsymbol{X}}$ with (10.114) orientation-preserving for all (V_a, E_a, s_a, ψ_a).

It remains to prove (10.116). Definition 4.60 constructs m-Kuranishi neighbourhoods $(V_{(1,a)}, E_{(1,a)}, s_{(1,a)}, \psi_{(1,a)})$, $(V_{(1,b)}, E_{(1,b)}, s_{(1,b)}, \psi_{(1,b)})$ on $\partial\boldsymbol{X}$ from $(V_a, E_a, s_a, \psi_a), (V_b, E_b, s_b, \psi_b)$. Theorem 4.56(a) gives a coordinate change

$$\Phi_{ab} = (V_{ab}, \phi_{ab}, \hat{\phi}_{ab}) : (V_a, E_a s_a, \psi_a) \longrightarrow (V_b, E_b, s_b, \psi_b)$$

over $\operatorname{Im}\psi_a \cap \operatorname{Im}\psi_b$ on $\boldsymbol{X}$. By Proposition 4.34(d), making V_{ab} smaller we can suppose $\phi_{ab} : V_{ab} \to V_b$ is simple, so $\partial\phi_{ab}$ is defined. Definition 4.61 constructs a coordinate change over $\operatorname{Im}\psi_{(1,a)} \cap \operatorname{Im}\psi_{(1,b)}$ on $\partial\boldsymbol{X}$

$$\begin{aligned}\Phi_{(1,a)(1,b)} = (V_{(1,a)(1,b)}, \phi_{(1,a)(1,b)}, \hat{\phi}_{(1,a)(1,b)}) : (V_{(1,a)}, E_{(1,a)}, s_{(1,a)}, \psi_{(1,a)})\\ \longrightarrow (V_{(1,b)}, E_{(1,b)}, s_{(1,b)}, \psi_{(1,b)}),\end{aligned}$$

with $V_{(1,a)(1,b)} = \partial V_{ab}$, $\phi_{(1,a)(1,b)} = \partial\phi_{ab}$, and $\hat{\phi}_{(1,a)(1,b)} = i_{V_{ab}}^*(\hat{\phi}_{ab})$.

Suppose Assumption 10.16(a) holds for $\dot{\mathbf{Man}}^{\mathbf{c}}$. Then by (10.11) we have a commutative diagram of vector bundles on $\partial V_{ab} \subseteq \partial V_a$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{\partial V_{ab}} & \xrightarrow{\alpha_{V_{ab}}} & i_{V_{ab}}^*(TV_{ab}) & \xrightarrow{\beta_{V_{ab}}} & T(\partial V_{ab}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_{\phi_{ab}}} & & \downarrow{\scriptstyle i_{V_{ab}}^*(T\phi_{ab})} & & \downarrow{\scriptstyle T(\partial\phi_{ab})} & & \\
0 & \longrightarrow & (\partial\phi_{ab})^*(N_{\partial V_b}) & \xrightarrow{(\partial\phi_{ab})^*(\alpha_{V_b})} & \begin{array}{c} i_{V_{ab}}^*(\phi_{ab}^*(TV_b)) \\ =(\partial\phi_{ab})^*(i_{V_b}^*(TV_b)) \end{array} & \xrightarrow{(\partial\phi_{ab})^*(\beta_{V_b})} & (\partial\phi_{ab})^*(T(\partial V_b)) & \longrightarrow & 0.
\end{array} \tag{10.117}$$

Let $v_a' \in V_{(1,a)(1,b)} \cap s_{(1,a)}^{-1}(0) \subseteq \partial V_{ab} \subseteq \partial V_a$, and set $v_a = i_{V_a}(v_a')$ in $V_{ab} \cap s_a^{-1}(0) \subseteq V_{ab} \subseteq V_a$, and $v_b' = \partial\phi_{ab}(v_a')$ in $V_{(1,b)} \cap s_{(1,b)}^{-1}(0) \subseteq \partial V_b$, and $v_b = i_{V_b}(v_b') = \phi_{ab}(v_a)$ in $s_b^{-1}(0) \subseteq V_b$, and $x' = \psi_{(1,a)}(v_a') = \psi_{(1,b)}(v_b')$ in ∂X, and $x = \psi_a(v_a) = \psi_b(v_b) = i_{\boldsymbol{X}}(x')$ in X. Set $m_a = \dim V_a$, $n_a = \operatorname{rank} E_a$, $m_b = \dim V_b$, $n_b = \operatorname{rank} E_b$, $m = \dim T_x X$ and $n = \dim O_x\boldsymbol{X}$. Then $m_a - n_a = m_b - n_b = m - n = \operatorname{vdim}\boldsymbol{X}$, so we have $m_a = m + p_a$, $n_a = n + p_a$, $m_b = m + p_b$, $n_b = n + p_b$ for $p_a, p_b \geqslant 0$.

As in (10.21) and (10.102) we have commutative diagrams

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & T_x\boldsymbol{X} & \xrightarrow{\iota_x^a} & T_{v_a}V_a & \xrightarrow{\mathrm{d}_{v_a}s_a} & E_a|_{v_a} & \xrightarrow{\pi_x^a} & O_x\boldsymbol{X} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle T_{v_a}\phi_{ab}} & & \downarrow{\scriptstyle \hat{\phi}_{ab}|_{v_a}} & & \| & & \\
0 & \longrightarrow & T_x\boldsymbol{X} & \xrightarrow{\iota_x^b} & T_{v_b}V_b & \xrightarrow{\mathrm{d}_{v_b}s_b} & E_b|_{v_b} & \xrightarrow{\pi_x^b} & O_x\boldsymbol{X} & \longrightarrow & 0,
\end{array} \tag{10.118}$$

$$\begin{array}{ccccccccccc}
0 & \rightarrow & T_{x'}(\partial\boldsymbol{X}) & \xrightarrow{\iota_{x'}^a} & T_{v_a'}(\partial V_a) & \xrightarrow{\mathrm{d}_{v_a'}s_{(1,a)}} & E_a|_{v_a} & \xrightarrow{\pi_{x'}^a} & O_{x'}(\partial\boldsymbol{X}) & \rightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle T_{v_a'}(\partial\phi_{ab})} & & \downarrow{\scriptstyle \hat{\phi}_{ab}|_{v_a}} & & \| & & \\
0 & \rightarrow & T_{x'}(\partial\boldsymbol{X}) & \xrightarrow{\iota_{x'}^b} & T_{v_b'}(\partial V_b) & \xrightarrow{\mathrm{d}_{v_b'}s_{(1,b)}} & E_b|_{v_b} & \xrightarrow{\pi_{x'}^b} & O_{x'}(\partial\boldsymbol{X}) & \rightarrow & 0,
\end{array} \tag{10.119}$$

with exact rows. Choose bases $(c_1, \ldots, c_m), (d_1^a, \ldots, d_{m+p_a}^a), (d_1^b, \ldots, d_{m+p_b}^b)$, $(e_1^a, \ldots, e_{p_a+n}^a), (e_1^b, \ldots, e_{p_b+n}^b), (f_1, \ldots, f_n)$ for $T_x\boldsymbol{X}, T_{v_a}V_a, E_a|_{v_a}, T_{v_b}V_b, E_b|_{v_b}$, $O_x\boldsymbol{X}$ respectively with

$$\begin{aligned}&\iota_x^a(c_i) = d_i^a,\ \iota_x^b(c_i) = d_i^b,\ i = 1, \ldots, m,\ \mathrm{d}_{v_a}s_a(d_{m+j}^a) = e_j^a,\ j = 1, \ldots, p_a,\\ &\mathrm{d}_{v_b}s_b(d_{m+j}^b) = e_j^b,\ j = 1, \ldots, p_b,\ \pi_x^a(e_{p_a+k}^a) = \pi_x^b(e_{p_b+k}^b) = f_k,\ k = 1, \ldots, n.\end{aligned} \tag{10.120}$$

Let $(\gamma_1,\ldots,\gamma_m),(\delta_1^a,\ldots,\delta_{m+p_a}^a),(\delta_1^b,\ldots,\delta_{m+p_b}^b)$ be the dual bases to $(c_1,\ldots,c_m),(d_1^a,\ldots,d_{m+p_a}^a),(d_1^b,\ldots,d_{m+p_b}^b)$. Then Theorem 10.71 gives

$$\begin{aligned}\Theta_{V_a,E_a,s_a,\psi_a}|_{v_a} : (\delta_1^a\wedge\cdots\wedge\delta_{m+p_a}^a)\otimes(e_1^a\wedge\cdots\wedge e_{p_a+n}^a)\longmapsto\\ (-1)^{p_a(p_a+1)/2}\cdot(\gamma_1\wedge\cdots\wedge\gamma_m)\otimes(f_1\wedge\cdots\wedge f_n),\end{aligned}\tag{10.121}$$

$$\begin{aligned}\Theta_{V_b,E_b,s_b,\psi_b}|_{v_b} : (\delta_1^b\wedge\cdots\wedge\delta_{m+p_b}^b)\otimes(e_1^b\wedge\cdots\wedge e_{p_b+n}^b)\longmapsto\\ (-1)^{p_b(p_b+1)/2}\cdot(\gamma_1\wedge\cdots\wedge\gamma_m)\otimes(f_1\wedge\cdots\wedge f_n).\end{aligned}\tag{10.122}$$

Now from (10.12) in Assumption 10.16(a) we can show that

$$\mathrm{d}_{v_a}s_a=\mathrm{d}_{v_a'}s_{(1,a)}\circ\beta_{V_{ab}}|_{v_a'}:T_{v_a}V_a\longrightarrow E_a|_{v_a}.$$

Exactness of the top line of (10.117) implies that

$$\begin{gathered}\mathrm{Im}(\mathrm{d}_{v_a'}s_{(1,a)})=\mathrm{Im}(\mathrm{d}_{v_a}s_a)=\bigl\langle e_1^a,\ldots,e_{p_a}^a\bigr\rangle_{\mathbb{R}},\\ \mathbb{R}\cong\mathrm{Im}(\alpha_{V_{ab}}|_{v_a'})\subseteq\mathrm{Ker}(\mathrm{d}_{v_a}s_a)=\bigl\langle d_1^a,\ldots,d_m^a\bigr\rangle_{\mathbb{R}}.\end{gathered}$$

Choose $(d_1^a,\ldots,d_{m+p_a}^a)$ with $\mathrm{Im}(\alpha_{V_{ab}}|_{v_a'})=\langle d_1^a\rangle_{\mathbb{R}}$. From (10.118) and $\iota_x^a(c_i)=d_i^a$, $\iota_x^b(c_i)=d_i^b$ we see that $T_{v_a}\phi_{ab}(d_i^a)=d_i^b$ for $i=1,\ldots,m$, so from (10.117) we deduce that $\mathrm{Im}(\alpha_{V_b}|_{v_b'})=\langle d_1^b\rangle_{\mathbb{R}}$. Thus there are unique $g_1^a\in N_{\partial V_{ab}}|_{v_a'}$ and $g_1^b\in N_{\partial V_b}|_{v_b'}$ with $\alpha_{V_{ab}}|_{v_a'}(g_1^a)=d_1^a$, $\alpha_{V_b}|_{v_b'}(g_1^b)=d_1^b$, and then $\gamma_{\phi_{ab}}|_{v_a'}(g_1^a)=g_2^a$. Set $d_i'^a=\beta_{V_{ab}}|_{v_a'}(d_i^a)$ for $i=2,\ldots,m+p_a$ and $d_i'^b=\beta_{V_b}|_{v_b'}(d_i^b)$ for $i=2,\ldots,m+p_b$. Then $(d_2'^a,\ldots,d_{m+p_a}'^a),(d_2'^b,\ldots,d_{m+p_b}'^b)$ are bases for $T_{v_a'}(\partial V_a),T_{v_b'}(\partial V_b)$, by exactness in the rows of (10.117). Let $(\delta_2'^a,\ldots,\delta_{m+p_a}'^a),(\delta_2'^b,\ldots,\delta_{m+p_b}'^b)$ be the dual bases for $T^*_{v_a'}(\partial V_a),T^*_{v_b'}(\partial V_b)$. Then Definition 10.18 gives

$$\Omega_{V_a}|_{v_a'}:\delta_2'^a\wedge\cdots\wedge\delta_{m+p_a}'^a\longmapsto g_1^a\otimes(\delta_1^a\wedge\cdots\wedge\delta_{m+p_a}^a),\tag{10.123}$$

$$\Omega_{V_b}|_{v_b'}:\delta_2'^b\wedge\cdots\wedge\delta_{m+p_b}'^b\longmapsto g_1^b\otimes(\delta_1^b\wedge\cdots\wedge\delta_{m+p_b}^b).\tag{10.124}$$

Using (10.118)–(10.120) we see there are unique bases $(c_2',\ldots,c_m'),(f_1',\ldots,f_n')$ for $T_{x'}(\partial\boldsymbol{X}),O_{x'}(\partial\boldsymbol{X})$ such that

$$\begin{gathered}\iota_{x'}^a(c_i')=d_i'^a,\;\;\iota_{x'}^b(c_i')=d_i'^b,\;\;i=2,\ldots,m,\\ \pi_{x'}^a(e_{p_a+k}^a)=f_k',\;\;\pi_{x'}^b(e_{p_b+k}^b)=f_k',\;\;k=1,\ldots,n.\end{gathered}$$

Let $(\gamma_2',\ldots,\gamma_m')$ be the dual basis to $(c_2',\ldots,c_m')$ for $T^*_{x'}(\partial\boldsymbol{X})$. Then as for (10.121)–(10.122), Theorem 10.71 gives

$$\begin{aligned}\Theta_{V_{(1,a)},E_{(1,a)},s_{(1,a)},\psi_{(1,a)}}|_{v_a'}:(\delta_2'^a\wedge\cdots\wedge\delta_{m+p_a}'^a)\otimes(e_1^a\wedge\cdots\wedge e_{p_a+n}^a)\\ \longmapsto(-1)^{p_a(p_a+1)/2}\cdot(\gamma_2'\wedge\cdots\wedge\gamma_m')\otimes(f_1'\wedge\cdots\wedge f_n'),\end{aligned}\tag{10.125}$$

$$\begin{aligned}\Theta_{V_{(1,b)},E_{(1,b)},s_{(1,b)},\psi_{(1,b)}}|_{v_b'}:(\delta_2'^b\wedge\cdots\wedge\delta_{m+p_b}'^b)\otimes(e_1^b\wedge\cdots\wedge e_{p_b+n}^b)\\ \longmapsto(-1)^{p_b(p_b+1)/2}\cdot(\gamma_2'\wedge\cdots\wedge\gamma_m')\otimes(f_1'\wedge\cdots\wedge f_n').\end{aligned}\tag{10.126}$$

From (10.115) and (10.121)–(10.126) we see that

$$\Phi_{V_a,E_a,s_a,\psi_a}|_{v'_a}(g_1^a) = \Phi_{V_b,E_b,s_b,\psi_b}|_{v'_b}(g_1^b) =$$
$$\bigl((\gamma_1 \wedge \cdots \wedge \gamma_m) \otimes (f_1 \wedge \cdots \wedge f_n)\bigr) \otimes \bigl((\gamma'_2 \wedge \cdots \wedge \gamma'_m) \otimes (f'_1 \wedge \cdots \wedge f'_n)\bigr)^{-1}.$$

This and $\gamma_{\phi_{ab}}|_{v'_a}(g_1^a) = g_1^b$ imply the restriction of (10.116) to v'_a, for any v'_a. Therefore (10.116) holds when $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumption 10.16(a). The proof for Assumption 10.16(b) is very similar, and we leave it to the reader. □

Example 10.78. Work in the 2-category $\mathbf{mKur^c}$ or $\mathbf{mKur^{gc}}$ of m-Kuranishi spaces with corners $\boldsymbol{X}$ defined using $\dot{\mathbf{Man}}^{\mathbf{c}} = \mathbf{Man^c}$ or $\mathbf{Man^{gc}}$ from Chapter 2, with (b-)canonical bundles ${}^bK_{\boldsymbol{X}}$ defined using b-tangent bundles ${}^bTV \to V$ from §2.3 for V in $\mathbf{Man^c}$ or $\mathbf{Man^{gc}}$. Then as in (2.14) and Example 10.17(i), the normal bundle $N_{\partial X}$ in (10.10) of Assumption 10.16(a) is naturally trivial, $N_{\partial X} = \mathcal{O}_{\partial X}$.

Thus, if $\boldsymbol{X}$ lies in $\mathbf{mKur^c}$ or $\mathbf{mKur^{gc}}$ then (10.114) in Theorem 10.77 implies that $N_{\partial \boldsymbol{X}}$ is naturally trivial on $\operatorname{Im}\psi_{(1,a)}$. As $\gamma_{\Phi_{ab}}$ in (10.117) respects the trivializations, they glue to a global natural trivialization $N_{\partial\boldsymbol{X}} \cong \mathcal{O}_{\partial X}$. Hence for $\boldsymbol{X}$ in $\mathbf{mKur^c}$ or $\mathbf{mKur^{gc}}$, we can replace (10.113) by a canonical isomorphism

$${}^b\Omega_{\boldsymbol{X}} : {}^bK_{\partial\boldsymbol{X}} \longrightarrow i_{\boldsymbol{X}}^*({}^bK_{\boldsymbol{X}}). \tag{10.127}$$

Here is the analogue of Definition 10.18:

Definition 10.79. Let $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfy Assumptions 3.22 and 10.16, and suppose $(\boldsymbol{X}, o_{\boldsymbol{X}})$ is an oriented m-Kuranishi space with corners in $\mathbf{m\dot{K}ur^c}$, as in §10.7.2. Then $o_{\boldsymbol{X}}$ is an orientation on the fibres of $K_{\boldsymbol{X}} \to X$, so $i_{\boldsymbol{X}}^*(o_{\boldsymbol{X}})$ is an orientation on the fibres of $i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}}) \to \partial X$. Theorem 10.77 gives a line bundle $N_{\partial\boldsymbol{X}} \to \partial X$ with an orientation $\nu_{\boldsymbol{X}}$ on its fibres, and an isomorphism $\Omega_{\boldsymbol{X}} : K_{\partial\boldsymbol{X}} \to N_{\partial\boldsymbol{X}} \otimes i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}})$. Thus there is a unique orientation $o_{\partial\boldsymbol{X}}$ on the fibres of $K_{\partial\boldsymbol{X}} \to \partial X$ identified by $\Omega_{\boldsymbol{X}}$ with $\nu_{\boldsymbol{X}} \otimes i_{\boldsymbol{X}}^*(o_{\boldsymbol{X}})$, and $o_{\partial\boldsymbol{X}}$ is an orientation on $\partial\boldsymbol{X}$.

In this way, if $\boldsymbol{X}$ is an oriented m-Kuranishi space with corners, then $\partial\boldsymbol{X}$ is oriented, and by induction $\partial^k\boldsymbol{X}$ is oriented for all $k = 0, 1, \ldots$. As for manifolds with corners in §2.6, the k-corners $C_k(\boldsymbol{X})$ for $k \geqslant 2$ need not be orientable.

10.7.4 Canonical bundles, orientations for products in $\mathbf{m\dot{K}ur}$

Products $\boldsymbol{X} \times \boldsymbol{Y}$ of m-Kuranishi spaces $\boldsymbol{X}, \boldsymbol{Y}$ were defined in Example 4.31. If $\boldsymbol{X}, \boldsymbol{Y}$ are oriented, the next theorem defines an orientation on $\boldsymbol{X} \times \boldsymbol{Y}$.

Theorem 10.80. *Let $\boldsymbol{X}, \boldsymbol{Y}$ be m-Kuranishi spaces in $\mathbf{m\dot{K}ur}$, so that Example* 4.31 *defines the product $\boldsymbol{X} \times \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$ with projections $\boldsymbol{\pi_X} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{X}$, $\boldsymbol{\pi_Y} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{Y}$, and Theorem* 10.71 *defines the canonical bundles $K_{\boldsymbol{X}}, K_{\boldsymbol{Y}}, K_{\boldsymbol{X}\times\boldsymbol{Y}}$ of $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{X} \times \boldsymbol{Y}$. There is a unique isomorphism of topological line bundles on $X \times Y$:*

$$\Upsilon_{\boldsymbol{X},\boldsymbol{Y}} : K_{\boldsymbol{X}\times\boldsymbol{Y}} \longrightarrow \pi_{\boldsymbol{X}}^*(K_{\boldsymbol{X}}) \otimes \pi_{\boldsymbol{Y}}{}^*(K_{\boldsymbol{Y}}), \tag{10.128}$$

such that if $x \in \boldsymbol{Y}$, $y \in \boldsymbol{Y}$ *and we identify* $T^*_{(x,y)}(\boldsymbol{X} \times \boldsymbol{Y}) = T^*_x\boldsymbol{X} \oplus T^*_y\boldsymbol{Y}$, $O_{(x,y)}(\boldsymbol{X} \times \boldsymbol{Y}) \cong O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y}$ *as in* (10.35), *and define isomorphisms*

$$\begin{aligned} I_{T^*_x\boldsymbol{X},T^*_y\boldsymbol{Y}} &: \det T^*_{(x,y)}(\boldsymbol{X} \times \boldsymbol{Y}) \longrightarrow \det(T^*_x\boldsymbol{X}) \otimes \det(T^*_y\boldsymbol{Y}), \\ I_{O_x\boldsymbol{X},O_y\boldsymbol{Y}} &: \det O_{(x,y)}(\boldsymbol{X} \times \boldsymbol{Y}) \longrightarrow \det(O_x\boldsymbol{X}) \otimes \det(O_y\boldsymbol{Y}) \end{aligned}$$

as in (10.84), *then*

$$\Upsilon_{\boldsymbol{X},\boldsymbol{Y}}|_{(x,y)} = (-1)^{\dim O_x\boldsymbol{X} \dim T_y\boldsymbol{Y}} \cdot I_{T^*_x\boldsymbol{X},T^*_y\boldsymbol{Y}} \otimes I_{O_x\boldsymbol{X},O_y\boldsymbol{Y}}. \tag{10.129}$$

Hence if $\boldsymbol{X}, \boldsymbol{Y}$ *are oriented there is a unique orientation on* $\boldsymbol{X} \times \boldsymbol{Y}$, *called the* ***product orientation***, *such that* (10.128) *is orientation-preserving.*

Proof. Equation (10.129) defines an isomorphism $\Upsilon_{\boldsymbol{X},\boldsymbol{Y}}|_{(x,y)} : K_{\boldsymbol{X}\times\boldsymbol{Y}}|_{(x,y)} \to \pi^*_{\boldsymbol{X}}(K_{\boldsymbol{X}}) \otimes \pi_{\boldsymbol{Y}}{}^*(K_{\boldsymbol{Y}})|_{(x,y)}$ for each $(x,y) \in X \times Y$. Thus there is a unique map of sets $\Upsilon_{\boldsymbol{X},\boldsymbol{Y}}$ in (10.128) which satisfies (10.129) for all $(x,y) \in X \times Y$. We must show that this map $\Upsilon_{\boldsymbol{X},\boldsymbol{Y}}$ is an isomorphism of topological line bundles. It is sufficient to do this locally near each (x,y) in $X \times Y$.

Fix $(x,y) \in X \times Y$, and let $(U_a, D_a, r_a, \chi_a), (V_b, E_b, s_b, \psi_b)$ be m-Kuranishi neighbourhoods on $\boldsymbol{X}, \boldsymbol{Y}$ with $x \in \operatorname{Im}\chi_a \subseteq X$, $y \in \operatorname{Im}\psi_b \subseteq Y$. Then as in Example 4.53 we have an m-Kuranishi neighbourhood

$$\bigl(U_a \times V_b, \pi^*_{U_a}(D_a) \oplus \pi^*_{V_b}(E_b), \pi^*_{U_a}(r_a) \oplus \pi^*_{V_b}(s_b), \chi_a \times \psi_b\bigr)$$

on $\boldsymbol{X} \times \boldsymbol{Y}$, with $(x,y) \in \operatorname{Im}(\chi_a \times \psi_b)$. Let $u = \chi_a^{-1}(x) \in r_a^{-1}(0) \subseteq U_a$, $v = \psi_b^{-1}(y) \in s_b^{-1}(0) \subseteq V_b$, so that as in Definition 10.6 we have linear maps $\mathrm{d}_u r_a : T_uU_a \to D_a|_u$ and $\mathrm{d}_v s_b : T_vV_b \to E_b|_v$.

As in the proof of Theorem 10.71, write $F^\bullet, G^\bullet$ for the complexes

$$\begin{array}{ccccccccccccccccc} \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & T_uU_a & \xrightarrow{\mathrm{d}_u r_a} & D_a|_u & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots, \\ \text{degree} & & -3 & & -2 & & -1 & & 0 & & 1 & & 2 & & \end{array}$$

$$\begin{array}{ccccccccccccccccc} \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & T_vV_b & \xrightarrow{\mathrm{d}s_b} & E_b|_v & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots. \\ \text{degree} & & -3 & & -2 & & -1 & & 0 & & 1 & & 2 & & \end{array}$$

Then Proposition 10.68 shows that the following commutes:

$$\begin{array}{ccc} \begin{array}{l}(\det(T_uU_a \oplus T_vV_b))^{-1} \\ \otimes \det(D_a|_u \oplus E_b|_v)\end{array} & \xrightarrow[\Theta_{F^\bullet \oplus G^\bullet}]{\hspace{3cm}} & K_{\boldsymbol{X}\times\boldsymbol{Y}}|_{(x,y)} \\ \Big\downarrow {\scriptstyle (-1)^{\operatorname{rank} D_a \dim V_b} \cdot \atop \scriptstyle I_{T^*_uU_a,T^*_vV_b} \otimes I_{D_a|_u,E_b|_v}} & & {\scriptstyle \Upsilon_{\boldsymbol{X},\boldsymbol{Y}}|_{(x,y)} = (-1)^{\dim O_x\boldsymbol{X} \dim T_y\boldsymbol{Y}} \cdot \atop \scriptstyle I_{T^*_x\boldsymbol{X},T^*_y\boldsymbol{Y}} \otimes I_{O_x\boldsymbol{X},O_y\boldsymbol{Y}}} \Big\downarrow \\ \begin{array}{l}((\det T_uU_a)^{-1} \otimes \det D_a|_u) \\ \otimes((\det T_vV_a)^{-1} \otimes \det E_b|_u)\end{array} & \xrightarrow{\Theta_{F^\bullet} \otimes \Theta_{G^\bullet}} & K_{\boldsymbol{X}}|_x \otimes K_{\boldsymbol{Y}}|_y. \end{array} \tag{10.130}$$

Now (10.130) is the fibre at $(x,y)\in r_a^{-1}(0)\times s_b^{-1}(0)$ of the commutative diagram of topological line bundles on $r_a^{-1}(0)\times s_b^{-1}(0)\subseteq U_a\times V_b$:

$$
\begin{array}{ccc}
\begin{array}{l}\det(T^*(U_a\times V_b)\otimes \\ \det((\pi_{U_a}^*(D_a)\oplus\pi_{V_b}^*(E_b)))|_{r_a^{-1}(0)\times s_b^{-1}(0)}\end{array} & \xrightarrow{\Theta_{U_a\times V_b,\cdots,\chi_a\times\psi_b}} & (\chi_a\times\psi_b)^{-1}(K_{\boldsymbol{X}\times\boldsymbol{Y}}) \\
\Big\downarrow {\scriptstyle \begin{array}{l}(-1)^{\operatorname{rank} D_a\dim V_b}\cdot \\ I_{T^*U_a,T^*V_b}\otimes I_{D_a,E_b}\end{array}} & & \Big\downarrow {\scriptstyle (\chi_a\times\psi_b)^{-1}(\Upsilon_{\boldsymbol{X},\boldsymbol{Y}})} \\
\begin{array}{l}\pi_{r_a^{-1}(0)}^*(\det T^*U_a\otimes\det D_a) \\ \otimes\pi_{s_b^{-1}(0)}^*(\det T^*V_a\otimes\det E_b)\end{array} & \xrightarrow{\begin{array}{l}\pi_{r_a^{-1}(0)}^*(\Theta_{U_a,D_a,r_a,\chi_a}) \\ \otimes\pi_{s_b^{-1}(0)}^*(\Theta_{V_b,E_b,s_b,\psi_b})\end{array}} & \begin{array}{l}(\chi_a\circ\pi_{r_a^{-1}(0)})^*(K_{\boldsymbol{X}}) \\ \otimes(\psi_b\circ\pi_{s_b^{-1}(0)})^*(K_{\boldsymbol{Y}}),\end{array}
\end{array}
\tag{10.131}
$$

where $\Theta_{U_a,D_a,r_a,\chi_a}$, $\Theta_{V_b,E_b,s_b,\psi_b}$ and $\Theta_{U_a\times V_b,\cdots,\chi_a\times\psi_b}$ are as in Theorem 10.71.

The top, bottom and left morphisms in (10.131) are isomorphisms of topological line bundles on $r_a^{-1}(0)\times s_b^{-1}(0)$. Hence the right hand morphism is an isomorphism, so $\Upsilon_{\boldsymbol{X},\boldsymbol{Y}}$ is an isomorphism on the open subset $\operatorname{Im}(\chi_a\times\psi_b)\subseteq X\times Y$, as $\chi_a\times\psi_b: r_a^{-1}(0)\times s_b^{-1}(0)\to\operatorname{Im}(\chi_a\times\psi_b)$ is a homeomorphism. Since we can cover $X\times Y$ by such open subsets $\operatorname{Im}(\chi_a\times\psi_b)$, we see that $\Upsilon_{\boldsymbol{X},\boldsymbol{Y}}$ is an isomorphism of topological line bundles, as we have to prove. □

The morphism $\Upsilon_{\boldsymbol{X},\boldsymbol{Y}}$ in (10.128), and hence the orientation on $\boldsymbol{X}\times\boldsymbol{Y}$ above, depend on our choice of *orientation conventions*, as in Convention 2.39, including various sign choices in §10.6–§10.7 and in (10.129). Different orientation conventions would change $\Upsilon_{\boldsymbol{X},\boldsymbol{Y}}$ and the orientation on $\boldsymbol{X}\times\boldsymbol{Y}$ by a sign depending on $\operatorname{vdim}\boldsymbol{X}$, $\operatorname{vdim}\boldsymbol{Y}$. If $\boldsymbol{X},\boldsymbol{Y}$ are manifolds then the orientation on $\boldsymbol{X}\times\boldsymbol{Y}$ agrees with that in Convention 2.39(a).

Proposition 10.81. *Suppose $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are oriented m-Kuranishi spaces. As in Example* 4.31, *products of m-Kuranishi spaces are commutative and associative up to canonical* 1*-isomorphism. When we include orientations,* (4.38) *becomes*

$$
\boldsymbol{X}\times\boldsymbol{Y}\cong(-1)^{\operatorname{vdim}\boldsymbol{X}\operatorname{vdim}\boldsymbol{Y}}\boldsymbol{Y}\times\boldsymbol{X},\qquad (\boldsymbol{X}\times\boldsymbol{Y})\times\boldsymbol{Z}\cong\boldsymbol{X}\times(\boldsymbol{Y}\times\boldsymbol{Z}). \tag{10.132}
$$

Proof. Let $x\in\boldsymbol{X}$ and $y\in\boldsymbol{Y}$, and consider the noncommutative diagram

$$
\begin{array}{ccc}
K_{\boldsymbol{X}\times\boldsymbol{Y}}|_{(x,y)} & \xrightarrow{\qquad\qquad\qquad} & K_{\boldsymbol{X}}|_x\otimes K_{\boldsymbol{Y}}|_y \\
 & {\scriptstyle \Upsilon_{\boldsymbol{X},\boldsymbol{Y}}|_{(x,y)}=(-1)^{\dim O_x\boldsymbol{X}\dim T_y\boldsymbol{Y}}\cdot I_{T_x^*\boldsymbol{X},T_y^*\boldsymbol{Y}}\otimes I_{O_x\boldsymbol{X},O_y\boldsymbol{Y}}} & \\
\Big\downarrow\cong & {\scriptstyle \Upsilon_{\boldsymbol{Y},\boldsymbol{X}}|_{(y,x)}=(-1)^{\dim O_y\boldsymbol{Y}\dim T_x\boldsymbol{X}}\cdot I_{T_y^*\boldsymbol{Y},T_x^*\boldsymbol{X}}\otimes I_{O_y\boldsymbol{Y},O_x\boldsymbol{X}}} & \cong\Big\downarrow \\
 & {\scriptstyle \cong(-1)^{\dim O_y\boldsymbol{Y}\dim T_x\boldsymbol{X}+\dim T_x\boldsymbol{X}\dim T_y\boldsymbol{Y}+\dim O_x\boldsymbol{X}\dim O_y\boldsymbol{Y}}\cdot} & \\
 & {\scriptstyle I_{T_x^*\boldsymbol{X},T_y^*\boldsymbol{Y}}\otimes I_{O_x\boldsymbol{X},O_y\boldsymbol{Y}}} & \\
K_{\boldsymbol{Y}\times\boldsymbol{X}}|_{(y,x)} & \xrightarrow{\qquad\qquad\qquad} & K_{\boldsymbol{Y}}|_y\otimes K_{\boldsymbol{X}}|_x.
\end{array}
\tag{10.133}
$$

Here the columns are the natural isomorphisms, and for the bottom morphism we use the fact that under the natural isomorphisms we have $I_{T_y^*\boldsymbol{Y},T_x^*\boldsymbol{X}}\cong$

$(-1)^{\dim T_x\boldsymbol{X}\dim T_y\boldsymbol{Y}} I_{T_x^*\boldsymbol{X},T_y^*\boldsymbol{Y}}$ and $I_{O_y\boldsymbol{Y},O_x\boldsymbol{X}} \cong (-1)^{\dim O_x\boldsymbol{X}\dim O_y\boldsymbol{Y}} I_{O_x\boldsymbol{X},O_y\boldsymbol{Y}}$. Thus, (10.133) fails to commute by an overall factor of

$$(-1)^{\dim O_x\boldsymbol{X}\dim T_y\boldsymbol{Y}} \cdot (-1)^{\dim O_y\boldsymbol{Y}\dim T_x\boldsymbol{X}+\dim T_x\boldsymbol{X}\dim T_y\boldsymbol{Y}+\dim O_x\boldsymbol{X}\dim O_y\boldsymbol{Y}}$$
$$= (-1)^{\operatorname{vdim}\boldsymbol{X}\operatorname{vdim}\boldsymbol{Y}},$$

since $\operatorname{vdim}\boldsymbol{X} = \dim T_x\boldsymbol{X} - \dim O_x\boldsymbol{X}$ and $\operatorname{vdim}\boldsymbol{Y} = \dim T_y\boldsymbol{Y} - \dim O_y\boldsymbol{Y}$ by (10.26). As this holds for all $(x,y)\in\boldsymbol{X}\times\boldsymbol{Y}$, the first equation of (10.132) follows, since $\Upsilon_{\boldsymbol{X},\boldsymbol{Y}}$ and $\Upsilon_{\boldsymbol{Y},\boldsymbol{X}}$ are used to define the orientations on $\boldsymbol{X}\times\boldsymbol{Y}$ and $\boldsymbol{Y}\times\boldsymbol{X}$. The second equation is easier, as the analogue of (10.133) does commute. □

10.7.5 Canonical bundles, orientations on μ-Kuranishi spaces

All the material of §10.7.1–§10.7.4 extends immediately to μ-Kuranishi spaces in Chapter 5, with no significant changes.

10.7.6 Canonical bundles, orientations on Kuranishi spaces

To extend §10.7.1–§10.7.4 to Kuranishi spaces in Chapter 6, there is one new issue. For a general Kuranishi space $\boldsymbol{X}$ in $\dot{\mathbf{Kur}}$, the naïve analogue of Theorem 10.71 is false, in that we may not be able to define a topological line bundle $\pi : K_{\boldsymbol{X}} \to X$ over X considered just as a topological space.

Really we should make X into a *Deligne–Mumford topological stack* (a kind of orbifold in topological spaces), as in Noohi [58], and then $\pi : K_{\boldsymbol{X}} \to X$ should be a line bundle in the sense of stacks or orbifolds. That is, X has finite isotropy groups G_xX for $x \in X$ as in §6.5, which may act nontrivially on the fibres $K_{\boldsymbol{X}}|_x$. The only possible nontrivial action is via $\{\pm 1\}$ acting on $\mathbb{R}$. Thus, as topological spaces, the fibres of $\pi : K_{\boldsymbol{X}} \to X$ may be either $\mathbb{R}$ or $\mathbb{R}/\{\pm 1\}$.

However, orientations on $\boldsymbol{X}$ only exist if G_xX acts trivially on $K_{\boldsymbol{X}}|_x$ for each $x \in X$, and then $K_{\boldsymbol{X}}$ does exist as a topological line bundle on X as a topological space. So we will restrict to this case, and not bother with topological stacks.

Definition 10.82. Let $\boldsymbol{X}$ be a Kuranishi space in $\dot{\mathbf{Kur}}$. Then as in §10.2.3, for each $x \in \boldsymbol{X}$ we have the isotropy group $G_x\boldsymbol{X}$, which acts linearly on the tangent and obstruction spaces $T_x\boldsymbol{X}, O_x\boldsymbol{X}$. We call $\boldsymbol{X}$ *locally orientable* if the induced action of $G_x\boldsymbol{X}$ on $\det T_x^*\boldsymbol{X} \otimes \det O_x\boldsymbol{X}$ is trivial for all $x \in \boldsymbol{X}$.

Here is the analogue of Theorem 10.71:

Theorem 10.83. *Let $\boldsymbol{X} = (X,\mathcal{K})$ be a locally orientable Kuranishi space in $\dot{\mathbf{Kur}}$. Then there is a natural topological line bundle $\pi : K_{\boldsymbol{X}} \to X$ called the **canonical bundle** of $\boldsymbol{X}$, with fibres for each $x \in X$ given by*

$$K_{\boldsymbol{X}}|_x = \det T_x^*\boldsymbol{X} \otimes \det O_x\boldsymbol{X}$$

for $T_x\boldsymbol{X}, O_x\boldsymbol{X}$ as in §10.2.3, with the property that if (V,E,Γ,s,ψ) is a Kuranishi neighbourhood on $\boldsymbol{X}$ in the sense of §6.4, then there is an isomorphism of topological real line bundles on $s^{-1}(0) \subseteq V$

$$\Theta_{V,E,\Gamma,s,\psi} : (\det T^*V \otimes \det E)|_{s^{-1}(0)} \longrightarrow \bar{\psi}^{-1}(K_{\boldsymbol{X}}), \tag{10.134}$$

such that if $v \in s^{-1}(0) \subseteq V$ *with* $\bar\psi(v) = x \in X$, *so that as in* (10.38) *we have an exact sequence*

$$0 \longrightarrow T_x\boldsymbol{X} \xrightarrow{\iota_x} T_vV \xrightarrow{\mathrm{d}_v s} E|_v \xrightarrow{\pi_x} O_x\boldsymbol{X} \longrightarrow 0,$$

and if $(c_1,\ldots,c_l),(d_1,\ldots,d_{l+m}),(e_1,\ldots,e_{m+n}),(f_1,\ldots,f_n)$ *are bases for* $T_x\boldsymbol{X}$, $T_vV, E|_v, O_x\boldsymbol{X}$ *respectively with* $\iota_x(c_i) = d_i$, $i = 1,\ldots,l$ *and* $\mathrm{d}_v s(d_{l+j}) = e_j$, $j = 1,\ldots,m$ *and* $\pi_x(e_{m+k}) = f_k$, $k = 1,\ldots,n$, *and* $(\gamma_1,\ldots,\gamma_l),(\delta_1,\ldots,\delta_{l+m})$ *are dual bases to* $(c_1,\ldots,c_l),(d_1,\ldots,d_{l+m})$ *for* $T_x^*\boldsymbol{X}, T_v^*V$, *then*

$$\begin{aligned}
&\Theta_{V,E,\Gamma,s,\psi}|_v : \det T_v^*V \otimes \det E|_v \to \det T_x^*\boldsymbol{X} \otimes \det O_x\boldsymbol{X} \quad \textit{maps} \\
&\Theta_{V,E,\Gamma,s,\psi}|_v : (\delta_1 \wedge \cdots \wedge \delta_{l+m}) \otimes (e_1 \wedge \cdots \wedge e_{m+n}) \longmapsto \\
&\qquad\qquad (-1)^{m(m+1)/2} \cdot (\gamma_1 \wedge \cdots \wedge \gamma_l) \otimes (f_1 \wedge \cdots \wedge f_n).
\end{aligned}$$

Proof. The proof is similar to that of Theorem 10.71, with one additional step: in the m-Kuranishi case, we make (10.104) by pushing $\Theta_{V,E,s,\psi}$ in (10.101) forward by the homeomorphism $\psi : s^{-1}(0) \to \operatorname{Im}\psi$. In the Kuranishi case, we have a Γ-equivariant $\Theta_{V,E,\Gamma,s,\psi}$ in (10.134) on $s^{-1}(0)$. Because of the locally orientable condition on $\boldsymbol{X}$, this pushes forward along the projection $s^{-1}(0) \to s^{-1}(0)/\Gamma$ to an isomorphism of topological line bundles on $s^{-1}(0)/\Gamma$, and this then pushes forward along the homeomorphism $\psi : s^{-1}(0)/\Gamma \to \operatorname{Im}\psi$ to give an analogue of (10.104). Also the analogue of (10.106) should take place on $\pi^{-1}(s^{-1}(0)) \subseteq P$ for $\Phi = (P, \pi, \phi, \hat\phi)$. We leave the details to the reader. □

The analogue of Proposition 10.73 holds for étale $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ between locally orientable Kuranishi spaces $\boldsymbol{X}, \boldsymbol{Y}$. Here is the analogue of Definition 10.74:

Definition 10.84. Let $\boldsymbol{X} = (X, \mathcal{K})$ be a locally orientable Kuranishi space in $\dot{\mathbf{K}}\mathbf{ur}$, so that Theorem 10.83 defines the canonical bundle $\pi : K_{\boldsymbol{X}} \to X$. An *orientation* $o_{\boldsymbol{X}}$ *on* $\boldsymbol{X}$ is an orientation on the fibres of $K_{\boldsymbol{X}}$. That is, $o_{\boldsymbol{X}}$ is an equivalence class $[\omega]$ of continuous sections $\omega \in \Gamma^0(K_{\boldsymbol{X}})$ with $\omega|_x \neq 0$ for all $x \in X$, where two such ω, ω' are equivalent if $\omega' = K \cdot \omega$ for $K : X \to (0,\infty)$ continuous. The *opposite orientation* is $-o_{\boldsymbol{X}} = [-\omega]$. Then we call $(\boldsymbol{X}, o_{\boldsymbol{X}})$ an *oriented Kuranishi space*. Usually we suppress $o_{\boldsymbol{X}}$, and just call $\boldsymbol{X}$ an oriented Kuranishi space, and then we write $-\boldsymbol{X}$ for $\boldsymbol{X}$ with the opposite orientation.

By the analogue of Proposition 10.73, if $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is an étale 1-morphism in $\dot{\mathbf{K}}\mathbf{ur}$ for $\boldsymbol{X}, \boldsymbol{Y}$ locally orientable then orientations $o_{\boldsymbol{Y}}$ on $\boldsymbol{Y}$ pull back to orientations $o_{\boldsymbol{X}} = \boldsymbol{f}^*(o_{\boldsymbol{Y}})$ on $\boldsymbol{X}$. If $\boldsymbol{f}$ is an equivalence, this defines a natural 1-1 correspondence between orientations on $\boldsymbol{X}$ and orientations on $\boldsymbol{Y}$.

Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in $\dot{\mathbf{K}}\mathbf{ur}$, with $\boldsymbol{X}, \boldsymbol{Y}$ locally orientable. A *coorientation* $c_{\boldsymbol{f}}$ on $\boldsymbol{f}$ is an orientation on the fibres of the line bundle $K_{\boldsymbol{X}} \otimes f^*(K_{\boldsymbol{Y}}^*)$ over X. That is, $c_{\boldsymbol{f}}$ is an equivalence class $[\gamma]$ of $\gamma \in \Gamma^0(K_{\boldsymbol{X}} \otimes f^*(K_{\boldsymbol{Y}}^*))$ with $\gamma|_x \neq 0$ for all $x \in X$, where two such γ, γ' are equivalent if $\gamma' = K \cdot \gamma$ for $K : X \to (0,\infty)$ continuous. The *opposite coorientation* is $-c_{\boldsymbol{f}} = [-\gamma]$. If $\boldsymbol{Y}$ is oriented then coorientations on $\boldsymbol{f}$ are equivalent to orientations on $\boldsymbol{X}$. Orientations on $\boldsymbol{X}$ are equivalent to coorientations on $\boldsymbol{\pi} : \boldsymbol{X} \to *$, for $*$ the point in $\dot{\mathbf{K}}\mathbf{ur}$.

The weak 2-functor $F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}:\mathbf{m\dot{K}ur}\hookrightarrow\mathbf{\dot{K}ur}$ from §6.2.4 identifies canonical bundles and orientations on an m-Kuranishi space $\boldsymbol{X}$ from §10.7.1–§10.7.2 with canonical bundles and orientations on the Kuranishi space $\boldsymbol{X}'=F_{\mathbf{m\dot{K}ur}}^{\mathbf{\dot{K}ur}}(\boldsymbol{X})$, which is automatically locally orientable as $G_x\boldsymbol{X}'=\{1\}$ for all $x\in\boldsymbol{X}'$.

Here are the analogues of Theorem 10.77 and Definition 10.79:

Theorem 10.85. *Let* $\mathbf{\dot{M}an^c}$ *satisfy Assumptions* 3.22 *and* 10.16, *and let* $\boldsymbol{X}$ *be a locally orientable Kuranishi space with corners in* $\mathbf{\dot{K}ur^c}$. *Then* $\partial\boldsymbol{X}$ *is locally orientable, and there is a natural isomorphism of topological line bundles on* ∂X

$$\Omega_{\boldsymbol{X}}:K_{\partial\boldsymbol{X}}\longrightarrow N_{\partial\boldsymbol{X}}\otimes i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}}),\tag{10.135}$$

where $N_{\partial\boldsymbol{X}}$ *is a line bundle on* ∂X, *with a natural orientation on its fibres.*

Suppose that $(V_a,E_a,\Gamma_a,s_a,\psi_a)$ *is a Kuranishi neighbourhood on* $\boldsymbol{X}$, *as in* §6.4, *with* $\dim V_a=m_a$ *and* $\operatorname{rank}E_a=n_a$. *Then as in* §6.4 *we have a Kuranishi neighbourhood* $(V_{(1,a)},E_{(1,a)},\Gamma_{(1,a)},s_{(1,a)},\psi_{(1,a)})$ *on* $\partial\boldsymbol{X}$ *with* $V_{(1,a)}=\partial V_a$, $E_{(1,a)}=i_{V_a}^*(E_a)$, $\Gamma_{(1,a)}=\Gamma_a$, *and* $s_{(1,a)}=i_{V_a}^*(s_a)$. *Also Assumption* 10.16 *gives a (smooth) line bundle* $N_{\partial V_a}\to\partial V_a$, *with an orientation on its fibres. Then there is a natural isomorphism of topological line bundles on* $s_{(1,a)}^{-1}(0)\subseteq\partial V_a$

$$\Phi_{V_a,E_a,\Gamma_a,s_a,\psi_a}:N_{\partial V_a}|_{s_{(1,a)}^{-1}(0)}\longrightarrow\bar\psi_{(1,a)}^{-1}(N_{\partial\boldsymbol{X}}),\tag{10.136}$$

which identifies the orientations on the fibres, such that the following commutes:

$$\begin{array}{ccc}
\begin{array}{l}(\det T^*\partial V_a\otimes\\ \det i_{V_a}^*(E_a))|_{s_{(1,a)}^{-1}(0)}\end{array} & \xrightarrow[\Omega_{V_a}\otimes\mathrm{id}_{\det i_{V_a}^*(E_a)|\cdots}]{} & \begin{array}{l}N_{\partial V_a}\otimes i_{V_a}^*(\det T^*V_a\\ \otimes\det E_a)|_{s(1,a)^{-1}(0)}\end{array}\\
\Big\downarrow{\scriptstyle\Theta_{V_{(1,a)},E_{(1,a)},s_{(1,a)},\psi_{(1,a)}}} & & {\scriptstyle\Phi_{V_a,E_a,s_a,\psi_a}\otimes i_{V_a}|^*_{\cdots}(\Theta_{V_a,E_a,s_a,\psi_a})}\Big\downarrow\\
\bar\psi_{(1,a)}^{-1}(K_{\partial\boldsymbol{X}}) & \xrightarrow{\Omega_{\boldsymbol{X}}} & \bar\psi_{(1,a)}^{-1}(N_{\partial\boldsymbol{X}}\otimes i_{\boldsymbol{X}}^*(K_{\boldsymbol{X}})),
\end{array}$$

where Ω_{V_a} *is as in* (10.16), *and* $\Theta_{V_a,E_a,\Gamma_a,s_a,\psi_a},\Theta_{V_{(1,a)},E_{(1,a)},\Gamma_{(1,a)},s_{(1,a)},\psi_{(1,a)}}$ *as in* (10.134), *and* $\Omega_{\boldsymbol{X}}$ *as in* (10.135), *and* $\Phi_{V_a,E_a,\Gamma_a,s_a,\psi_a}$ *as in* (10.136).

Proof. The proof is similar to that of Theorem 10.77, but with a few extra steps. Firstly, if in the situation of the theorem we have $v'_a\in s_{(1,a)}^{-1}(0)$ with $\bar\psi_{(1,a)}(v'_a)=x'\in\partial X$ and $v_a=i_{V_a}(v'_a)\in s_a^{-1}(0)$ and $i_{\boldsymbol{X}}(x')=\bar\psi_a(v_a)=x$ in $\boldsymbol{X}$, then as in the proof of Theorem 10.77 we can construct an isomorphism

$$\det T_{x'}^*(\partial\boldsymbol{X})\otimes\det O_{x'}(\partial\boldsymbol{X})\cong N_{\partial V_a}|_{v'_a}\otimes\det T_x^*\boldsymbol{X}\otimes\det O_x\boldsymbol{X},$$

which is equivariant under $G_{x'}(\partial\boldsymbol{X})\cong\mathrm{Stab}_{\Gamma_{(1,a)}}(v'_a)\subseteq\mathrm{Stab}_{\Gamma_a}(v_a)\cong G_x\boldsymbol{X}$. But $\mathrm{Stab}_{\Gamma_{(1,a)}}(v'_a)$ acts trivially on $N_{\partial V_a}|_{v'_a}$, as the action is defined using the γ_f in Assumption 10.16 which are orientation-preserving, and $G_x\boldsymbol{X}$ acts trivially on $\det T_x^*\boldsymbol{X}\otimes\det O_x\boldsymbol{X}$ as $\boldsymbol{X}$ is locally orientable. Hence $G_{x'}(\partial\boldsymbol{X})$ acts trivially on $\det T_{x'}^*(\partial\boldsymbol{X})\otimes\det O_{x'}(\partial\boldsymbol{X})$, so $\partial\boldsymbol{X}$ is locally orientable, as we have to prove.

Secondly, as the natural action of $\Gamma_{(1,a)}$ on $N_{\partial V_a}$ preserves orientations on the fibres, we can use $\Phi_{V_a,E_a,\Gamma_a,s_a,\psi_a}$ in (10.136) to induce a unique orientation on

$N_{\partial \boldsymbol{X}}|_{\operatorname{Im}\psi_{(1,a)}}$, as the orientation on $N_{\partial V_a}|_{s^{-1}_{(1,a)}(0)}$ descends through the quotient $s^{-1}_{(1,a)}(0) \to s^{-1}_{(1,a)}(0)/\Gamma_{(1,a)}$. We leave the details to the reader. $\square$

As in Example 10.78, working in $\mathbf{Kur^c}$ or $\mathbf{Kur^{gc}}$ with b-canonical bundles ${}^bK_{\boldsymbol{X}}$ in Theorem 10.85 defined using b-tangent bundles ${}^bTV \to V$ in $\mathbf{Man^c}$ or $\mathbf{Man^{gc}}$, the normal bundle $N_{\partial \boldsymbol{X}}$ in Theorem 10.85 is canonically trivial, $N_{\partial \boldsymbol{X}} \cong \mathcal{O}_{\partial X}$, so we can replace (10.135) by (10.127).

Definition 10.86. Let $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfy Assumptions 3.22 and 10.16, and suppose $(\boldsymbol{X}, o_{\boldsymbol{X}})$ is an oriented Kuranishi space with corners in $\dot{\mathbf{Kur}}^{\mathbf{c}}$. Then $\boldsymbol{X}$ is locally orientable by Definition 10.84 with canonical bundle $K_{\boldsymbol{X}} \to X$ from Theorem 10.83, and $o_{\boldsymbol{X}}$ is an orientation on the fibres of $K_{\boldsymbol{X}} \to X$. Theorem 10.85 shows that $\partial \boldsymbol{X}$ is locally orientable in $\dot{\mathbf{Kur}}^{\mathbf{c}}$, so that $K_{\partial \boldsymbol{X}} \to \partial X$ is defined, and gives a line bundle $N_{\partial \boldsymbol{X}} \to \partial X$ with an orientation $\nu_{\boldsymbol{X}}$ on its fibres, and an isomorphism $\Omega_{\boldsymbol{X}} : K_{\partial \boldsymbol{X}} \to N_{\partial \boldsymbol{X}} \otimes i^*_{\boldsymbol{X}}(K_{\boldsymbol{X}})$. Hence there is a unique orientation $o_{\partial \boldsymbol{X}}$ on the fibres of $K_{\partial \boldsymbol{X}} \to \partial X$ identified by $\Omega_{\boldsymbol{X}}$ with $\nu_{\boldsymbol{X}} \otimes i^*_{\boldsymbol{X}}(o_{\boldsymbol{X}})$, and $o_{\partial \boldsymbol{X}}$ is an orientation on $\partial \boldsymbol{X}$. Thus, if $\boldsymbol{X}$ is an oriented Kuranishi space with corners, then $\partial^k \boldsymbol{X}$ is naturally oriented for all $k = 0, 1, \ldots$.

The analogues of Theorem 10.80 and Proposition 10.81 hold for products $\boldsymbol{X} \times \boldsymbol{Y}$ of Kuranishi spaces $\boldsymbol{X} \times \boldsymbol{Y}$ defined as in Example 6.28, where we require $\boldsymbol{X}, \boldsymbol{Y}$ to be locally orientable, and then $\boldsymbol{X} \times \boldsymbol{Y}$ is also locally orientable, so that $K_{\boldsymbol{X}}, K_{\boldsymbol{Y}}, K_{\boldsymbol{X}\times\boldsymbol{Y}}$ exist. The proofs combine those of Theorems 10.80 and 10.83 and Proposition 10.81.

Chapter 11

Transverse fibre products and submersions

In the category of classical manifolds $\mathbf{Man}$, morphisms $g : X \to Z$, $h : Y \to Z$ are *transverse* if whenever $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$, then

$$T_x g \oplus T_y h : T_x X \oplus T_y Y \longrightarrow T_z Z$$

is surjective. If g, h are transverse then a fibre product $W = X \times_{g,Z,h} Y$ exists in the category $\mathbf{Man}$, as defined in §A.1, with $\dim W = \dim X + \dim Y - \dim Z$, in a Cartesian square in $\mathbf{Man}$:

$$\begin{array}{ccc} W & \xrightarrow{\quad f \quad} & Y \\ \downarrow{\scriptstyle e} & & {\scriptstyle h}\downarrow \\ X & \xrightarrow{\quad g \quad} & Z. \end{array}$$

Also $g : X \to Z$ is a *submersion* if $T_x g : T_x X \to T_z Z$ is surjective for all $x \in X$ with $g(x) = z \in Z$. If g is a submersion then g, h are transverse for any morphism $h : Y \to Z$ in $\mathbf{Man}$. Generalizations of all this to various categories $\mathbf{Man^c}, \mathbf{Man^c_{in}}, \mathbf{Man^{gc}}, \ldots$ of manifolds with (g-)corners were discussed in §2.5.

This chapter studies transversality, fibre products, and submersions for m-Kuranishi spaces and Kuranishi spaces. By 'fibre products' we mean 2-*category fibre products* in $\mathbf{m\dot{K}ur}$ and $\mathbf{\dot{K}ur}$ (or more generally in certain 2-subcategories $\mathbf{m\dot{K}ur}_{\boldsymbol{D}} \subseteq \mathbf{m\dot{K}ur}$ and $\mathbf{\dot{K}ur}_{\boldsymbol{D}} \subseteq \mathbf{\dot{K}ur}$), as defined in §A.4, which satisfy a complicated universal property involving 2-morphisms. Readers are advised to familiarize themselves with fibre products in both ordinary categories in §A.1, and in 2-categories in §A.4, before continuing.

As we explain in §11.4, these ideas do *not* extend nicely to the ordinary category of μ-Kuranishi spaces $\boldsymbol{\mu}\mathbf{\dot{K}ur} \simeq \mathrm{Ho}(\mathbf{m\dot{K}ur})$. The 2-category structure on $\mathbf{m\dot{K}ur}$ is essential for defining well-behaved transverse fibre products, and the universal property in $\mathbf{m\dot{K}ur}$ does not descend to $\mathrm{Ho}(\mathbf{m\dot{K}ur})$. We can still define a kind of 'transverse fibre product' in $\boldsymbol{\mu}\mathbf{\dot{K}ur}$, but it is not a category-theoretic fibre product, and it is not characterized by a universal property in $\boldsymbol{\mu}\mathbf{\dot{K}ur}$.

Optional assumptions on transversality and submersions in categories $\dot{\mathbf{M}}\mathbf{an}$, $\dot{\mathbf{M}}\mathbf{an}^{\mathbf{c}}$ are given in §11.1, extending those in Chapter 3. Section 11.2 discusses transverse fibre products in a general 2-category $\mathbf{m\dot{K}ur}$, and §11.3 works out these results in $\mathbf{mKur}, \mathbf{mKur}^{\mathbf{c}}_{\mathbf{st}}, \mathbf{mKur}^{\mathbf{gc}}$ and $\mathbf{mKur}^{\mathbf{c}}$. Section 11.4 considers fibre products of μ-Kuranishi spaces, and §11.5–§11.6 extend §11.2–§11.3 to Kuranishi spaces. Long proofs are postponed to §11.7–§11.11.

11.1 Optional assumptions on transverse fibre products

Suppose for the whole of this section that $\dot{\mathbf{M}}\mathbf{an}$ satisfies Assumptions 3.1–3.7. We now give optional assumptions on transversality and submersions in $\dot{\mathbf{M}}\mathbf{an}$.

11.1.1 'Transverse morphisms' and 'submersions' in $\dot{\mathbf{M}}\mathbf{an}$

Here is the basic assumption we will need to get a good notion of transverse fibre product in $\mathbf{m\dot{K}ur}, \dot{\mathbf{K}}\mathbf{ur}$ — part (b) will be essential in the proof of Theorem 11.17 in §11.2 on the existence of fibre products of w-transverse 1-morphisms of global m-Kuranishi neighbourhoods, which is the necessary local condition for existence of fibre products in $\mathbf{m\dot{K}ur}$. We write the assumption using choices of discrete properties $\boldsymbol{D}, \boldsymbol{E}$ to fit in with the results of §2.5.

Assumption 11.1. (Transverse fibre products.) (a) We are given discrete properties $\boldsymbol{D}, \boldsymbol{E}$ of morphisms in $\dot{\mathbf{M}}\mathbf{an}$, in the sense of Definition 3.18, where $\boldsymbol{D}$ implies $\boldsymbol{E}$. We require that the projections $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are $\boldsymbol{D}$ and $\boldsymbol{E}$ for all $X, Y \in \dot{\mathbf{M}}\mathbf{an}$. We write $\dot{\mathbf{M}}\mathbf{an}_{\boldsymbol{D}}$, $\dot{\mathbf{M}}\mathbf{an}_{\boldsymbol{E}}$ for the subcategories of $\dot{\mathbf{M}}\mathbf{an}$ with all objects, and only $\boldsymbol{D}$ and $\boldsymbol{E}$ morphisms.

(b) Let $g : X \to Z$ and $h : Y \to Z$ be morphisms in $\dot{\mathbf{M}}\mathbf{an}_{\boldsymbol{D}}$. We are given a notion of when g, h are *transverse*. This satisfies:

- **(i)** If g, h are transverse then a fibre product $W = X \times_{g,Z,h} Y$ exists in $\dot{\mathbf{M}}\mathbf{an}_{\boldsymbol{D}}$, as in Definition A.3, with $\dim W = \dim X + \dim Y - \dim Z$, in a Cartesian square in $\dot{\mathbf{M}}\mathbf{an}_{\boldsymbol{D}}$, so that e, f, g, h are $\boldsymbol{D}$ morphisms in $\dot{\mathbf{M}}\mathbf{an}$:

$$\begin{array}{ccc} W & \xrightarrow{\quad f \quad} & Y \\ \big\downarrow{\scriptstyle e} & & {\scriptstyle h}\big\downarrow \\ X & \xrightarrow{\quad g \quad} & Z. \end{array} \tag{11.1}$$

  Furthermore, (11.1) is also Cartesian in $\dot{\mathbf{M}}\mathbf{an}_{\boldsymbol{E}}$.

- **(ii)** In the situation of **(i)**, suppose $c : V \to X$, $d : V \to Y$ are morphisms in $\dot{\mathbf{M}}\mathbf{an}_{\boldsymbol{E}}$, and $E \to V$ is a vector bundle, and $s \in \Gamma^\infty(E)$ is a section, and $\mathrm{K} : E \to \mathcal{T}_{g\circ c}Z$ is a morphism, such that $h \circ d = g \circ c + \mathrm{K} \circ s + O(s^2)$ in the sense of Definition 3.15(vii). Then there exist an open neighbourhood V' of $s^{-1}(0)$ in V, and a morphism $b : V' \to W$ in $\dot{\mathbf{M}}\mathbf{an}_{\boldsymbol{E}}$, and morphisms $\Lambda : E|_{V'} \to \mathcal{T}_{e\circ b}X$, $\mathrm{M} : E|_{V'} \to \mathcal{T}_{f\circ b}Y$ with

$$c|_{V'} = e \circ b + \Lambda \circ s + O(s^2), \quad d|_{V'} = f \circ b + \mathrm{M} \circ s + O(s^2), \tag{11.2}$$

and if $\mathrm{K}' : E|_{V'} \to \mathcal{T}_{g\circ e\circ b}Z$ is a morphism with $\mathrm{K}|_{V'} = \mathrm{K}' + O(s)$ in the sense of Definition 3.15(v), which exists and is unique up to $O(s)$ by Theorem 3.17(g), as $g \circ c|_{V'} = g \circ e \circ b + O(s)$ by (11.2), then

$$\mathrm{K}' + \mathcal{T}g \circ \Lambda = \mathcal{T}h \circ \mathrm{M} + O(s) \tag{11.3}$$

in the sense of Definition 3.15(ii), where $\mathcal{T}g, \mathcal{T}h$ are as in §3.3.4(c).

(iii) In the situation of **(ii)**, suppose $\tilde{V}', \tilde{b}, \tilde{\Lambda}, \tilde{\mathrm{M}}$ are alternative choices for $V', b, \Lambda, \mathrm{M}$. Then there exists $\mathrm{N} : E|_{V'\cap\tilde{V}'} \to \mathcal{T}_b W|_{V'\cap\tilde{V}'}$ with

$$\tilde{b}|_{V'\cap\tilde{V}'} = b|_{V'\cap\tilde{V}'} + \mathrm{N} \circ s + O(s^2), \tag{11.4}$$

and if $\tilde{\Lambda}' : E|_{V'\cap\tilde{V}'} \to \mathcal{T}_{e\circ b}X|_{V'\cap\tilde{V}'}$, $\tilde{\mathrm{M}}' : E|_{V'\cap\tilde{V}'} \to \mathcal{T}_{f\circ b}Y|_{V'\cap\tilde{V}'}$ are morphisms with $\tilde{\Lambda}|_{V'\cap\tilde{V}'} = \tilde{\Lambda}' + O(s)$, $\tilde{\mathrm{M}}|_{V'\cap\tilde{V}'} = \tilde{\mathrm{M}}' + O(s)$, which exist and are unique up to $O(s)$ by Theorem 3.17(g), as $e\circ\tilde{b}|_{V'\cap\tilde{V}'} = e\circ b|_{V'\cap\tilde{V}'} + O(s)$ and $f \circ \tilde{b}|_{V'\cap\tilde{V}'} = f \circ b|_{V'\cap\tilde{V}'} + O(s)$ by (11.4), then

$$\Lambda|_{V'\cap\tilde{V}'} = \tilde{\Lambda}' + \mathcal{T}e \circ \mathrm{N} + O(s), \quad \mathrm{M}|_{V'\cap\tilde{V}'} = \tilde{\mathrm{M}}' + \mathcal{T}f \circ \mathrm{N} + O(s). \tag{11.5}$$

If $\check{\mathrm{N}} : E|_{V'\cap\tilde{V}'} \to \mathcal{T}_b W|_{V'\cap\tilde{V}'}$ satisfies (11.4)–(11.5) then $\check{\mathrm{N}} = \mathrm{N} + O(s)$.

(c) Let $g : X \to Z$ be a morphism in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$. We are given a notion of when g is a *submersion*. If g is a submersion and $h : Y \to Z$ is any morphism in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$, then g, h are transverse.

In fact any category $\dot{\mathbf{Man}}$ can be made to satisfy Assumption 11.1:

Example 11.2. Let $\dot{\mathbf{Man}}$ be any category satisfying Assumptions 3.1–3.7, and let $\boldsymbol{D}, \boldsymbol{E}$ be any discrete properties of morphisms in $\dot{\mathbf{Man}}$ satisfying Assumption 11.1(a) (for instance, $\boldsymbol{D}, \boldsymbol{E}$ could be trivial). Define morphisms $g : X \to Z$, $h : Y \to Z$ in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$ to be *transverse* if they satisfy Assumption 11.1(b). Define a $\boldsymbol{D}$ morphism $g : X \to Z$ to be a *submersion* if it satisfies Assumption 11.1(c). Then Assumption 11.1 holds, just by definition.

Let X, Y be any objects of $\dot{\mathbf{Man}}$, and $*$ be the point in $\dot{\mathbf{Man}}$, as in Assumption 3.1(c). Then the projections $\pi : X \to *$, $\pi : Y \to *$ satisfy Assumption 11.1(b), and so are transverse. Here in (b)(i) we take $W = X \times Y$, and in (b)(ii) we take $b = (c, d)$ and $\Lambda = \mathrm{M} = 0$. We will use this in discussing products of m-Kuranishi spaces in §11.2.3.

11.1.2 More assumptions on transversality and submersions

We now give six optional assumptions on transverse morphisms and submersions, which will imply similar properties for (m-)Kuranishi spaces. For the first, in Remark 2.37 we discuss when fibre products in $\mathbf{Man}, \mathbf{Man^c_{st}}, \ldots$ are also fibre products on the level of topological spaces.

Assumption 11.3. (Transverse fibre products are fibre products of topological spaces.) Suppose that Assumption 11.1 holds for $\dot{\mathbf{Man}}$, and in addition, the functor $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}} : \dot{\mathbf{Man}} \to \mathbf{Top}$ from Assumption 3.2 maps transverse fibre products in $\dot{\mathbf{Man}}$ to fibre products in $\mathbf{Top}$. That is, in the situation of Assumption 11.1(b)(i) we have a homeomorphism

$$(e, f) : W \longrightarrow \big\{(x, y) \in X \times Y : g(x) = h(y)\big\}.$$

Assumption 11.4. (Properties of submersions.) Suppose Assumption 11.1 holds for $\dot{\mathbf{Man}}$, and:

(a) If (11.1) is a Cartesian square in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$ with g a submersion, then f is a submersion.

(b) Products of submersions are submersions. That is, if $g : W \to Y$ and $h : X \to Z$ are submersions then $g \times h : W \times X \to Y \times Z$ is a submersion.

(c) The projection $\pi_X : X \times Y \to X$ is a submersion for all $X, Y \in \dot{\mathbf{Man}}$.

Assumption 11.5. (Tangent spaces of transverse fibre products.) Let $\dot{\mathbf{Man}}$ satisfy Assumption 10.1, with discrete property $\boldsymbol{A}$ and tangent spaces T_xX, and Assumption 11.1, with discrete properties $\boldsymbol{D}, \boldsymbol{E}$. Suppose that $\boldsymbol{D}$ implies $\boldsymbol{A}$, and whenever (11.1) is Cartesian in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$ with g, h transverse and $w \in W$ with $e(w) = x$ in X, $f(w) = y$ in Y and $g(x) = h(y) = z$ in Z, the following is an exact sequence of real vector spaces:

$$0 \longrightarrow T_wW \xrightarrow{T_we \oplus T_wf} T_xX \oplus T_yY \xrightarrow{T_xg \oplus -T_yh} T_zZ \longrightarrow 0.$$

Assumption 11.6. (Quasi-tangent spaces of transverse fibre products.) Let $\dot{\mathbf{Man}}$ satisfy Assumption 10.19, with discrete property $\boldsymbol{C}$ and quasi-tangent spaces Q_xX in a category $\mathcal{Q}$, and Assumption 11.1, with discrete properties $\boldsymbol{D}, \boldsymbol{E}$. Suppose that $\boldsymbol{D}$ implies $\boldsymbol{C}$, and whenever (11.1) is Cartesian in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$ with g, h transverse and $w \in W$ with $e(w) = x$ in X, $f(w) = y$ in Y and $g(x) = h(y) = z$ in Z, the following is Cartesian in $\mathcal{Q}$:

$$\begin{array}{ccc} Q_wW & \xrightarrow{Q_wf} & Q_yY \\ \big\downarrow{\scriptstyle Q_we} & & {\scriptstyle Q_yh}\big\downarrow \\ Q_xX & \xrightarrow{Q_xg} & Q_zZ. \end{array}$$

Assumption 11.7. (Compatibility with the corner functor.) Let $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfy Assumption 3.22 in §3.4, so that we have a corner functor $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}^{\mathbf{c}}$, and let Assumption 11.1 hold with $\dot{\mathbf{Man}}^{\mathbf{c}}$ in place of $\dot{\mathbf{Man}}$. Define transverse morphisms and submersions in $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}_{\boldsymbol{D}}$ in the obvious way: we call $g : \coprod_{l \geqslant 0} X_l \to \coprod_{n \geqslant 0} Z_n$ and $h : \coprod_{m \geqslant 0} Y_m \to \coprod_{n \geqslant 0} Z_n$ *transverse* in $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}_{\boldsymbol{D}}$

if $g|_{\cdots} : X_l \cap g^{-1}(Z_n) \to Z_n$ and $h|_{\cdots} : Y_m \cap h^{-1}(Z_n) \to Z_n$ are transverse in $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}}$ for all l, m, n, and similarly for submersions.

Suppose that C maps $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}} \to \check{\dot{\mathbf{Man}}}^{\mathbf{c}}_{\boldsymbol{D}}$ and $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{E}} \to \check{\dot{\mathbf{Man}}}^{\mathbf{c}}_{\boldsymbol{E}}$, and whenever (11.1) is a Cartesian square in $\dot{\mathbf{Man}}^{\mathbf{c}}$ with g, h transverse, then the following is Cartesian in $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}_{\boldsymbol{D}}$ and $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}_{\boldsymbol{E}}$, with $C(g), C(h)$ transverse in $\check{\dot{\mathbf{Man}}}^{\mathbf{c}}_{\boldsymbol{D}}$:

$$\begin{array}{ccc} C(W) & \xrightarrow[C(f)]{\qquad\qquad} & C(Y) \\ \Big\downarrow {\scriptstyle C(e)} & & {\scriptstyle C(h)} \Big\downarrow \\ C(X) & \xrightarrow{C(g)} & C(Z). \end{array}$$

Also, suppose that if g is a submersion then $C(g)$ is a submersion.

The next assumption is only nontrivial if $\boldsymbol{D} \neq \boldsymbol{E}$.

Assumption 11.8. (Fibre products with submersions in $\dot{\mathbf{Man}}_{\boldsymbol{E}}$.) Suppose that Assumption 11.1 holds for $\dot{\mathbf{Man}}$, and whenever $g : X \to Z$ is a submersion in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$, and $h : Y \to Z$ is any morphism in $\dot{\mathbf{Man}}_{\boldsymbol{E}}$ (not necessarily in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$), then a fibre product $W = X \times_{g,Z,h} Y$ exists in $\dot{\mathbf{Man}}_{\boldsymbol{E}}$, with $\dim W = \dim X + \dim Y - \dim Z$, in a Cartesian square (11.1) in $\dot{\mathbf{Man}}_{\boldsymbol{E}}$, and Assumption 11.1(b)(ii),(iii) hold for g, h. If Assumptions 11.3, 11.4(a) or 11.7 hold, then they also hold for fibre products $W = X \times_{g,Z,h} Y$ in $\dot{\mathbf{Man}}_{\boldsymbol{E}}$ with g a submersion.

11.1.3 Characterizing transversality and submersions

The next assumption gives necessary and sufficient conditions for when morphisms g, h in $\dot{\mathbf{Man}}^{\mathbf{c}}$ are transverse, or when g is a (strong) submersion, that extend nicely to (m-)Kuranishi spaces $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}, \dot{\mathbf{K}}\mathbf{ur}^{\mathbf{c}}$. The statement is complicated to allow these conditions to depend on several different things — maps of tangent spaces $T_x g, T_y h$, of quasi-tangent spaces $Q_x g, Q_y h$, and the corner maps $C(g), C(h)$ — since our examples in §2.5 depend on these.

We state it using $\dot{\mathbf{Man}}^{\mathbf{c}}$ in §3.4, so our conditions can involve the corner functor $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}^{\mathbf{c}}$. But as in Example 3.24(i), we can take $\dot{\mathbf{Man}}^{\mathbf{c}}$ to be any category $\dot{\mathbf{Man}}$ satisfying Assumptions 3.1–3.7 with $C_k(X) = \emptyset$ for all $X \in \dot{\mathbf{Man}}$ and $k > 0$, so the corners are not needed in all examples.

Assumption 11.9. Suppose $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumption 3.22 in §3.4, so that we have a corner functor $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}^{\mathbf{c}}$.

Suppose Assumption 10.1 holds for $\dot{\mathbf{Man}}^{\mathbf{c}}$, so we are given a discrete property $\boldsymbol{A}$ of morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and notions of *tangent space* $T_x X$ for X in $\dot{\mathbf{Man}}^{\mathbf{c}}$ and $x \in X$, and *tangent map* $T_x f : T_x X \to T_y Y$ for $\boldsymbol{A}$ morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}^{\mathbf{c}}$ and $x \in X$ with $f(x) = y$ in Y.

Suppose Assumption 10.19 holds for $\dot{\mathbf{Man}}^{\mathbf{c}}$, so we are given a category $\mathcal{Q}$, a discrete property $\boldsymbol{C}$ of morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and notions of *quasi-tangent space* $Q_x X$ in $\mathcal{Q}$ for X in $\dot{\mathbf{Man}}^{\mathbf{c}}$ and $x \in X$, and *quasi-tangent map* $Q_x f : Q_x X \to Q_y Y$

in $\mathcal{Q}$ for $\boldsymbol{C}$ morphisms $f : X \to Y$ in $\dot{\mathbf{Man}}^{\mathbf{c}}$ and $x \in X$ with $f(x) = y$ in Y. These may be trivial, i.e. $\mathcal{Q}$ could have one object and one morphism.

Suppose Assumption 11.1 holds for $\dot{\mathbf{Man}}^{\mathbf{c}}$, so we are given discrete properties $\boldsymbol{D}, \boldsymbol{E}$ of morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$, where $\boldsymbol{D}$ implies $\boldsymbol{E}$, and notions of *transverse morphisms* g, h and *submersions* g in $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}}$. We require that $\boldsymbol{D}$ implies $\boldsymbol{A}$ and $\boldsymbol{C}$, and:

(a) Let $g : X \to Z$ and $h : Y \to Z$ be morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}}$. Then g, h are transverse if and only if for all $x \in X$ and $y \in Y$ with $g(x) = h(y) = z$ in Z, the following linear map is surjective:

$$T_x g \oplus T_y h : T_x X \oplus T_y Y \longrightarrow T_z Z, \tag{11.6}$$

and an explicit condition (which may be trivial) holds, which we call 'condition $\boldsymbol{T}$', involving only (i)–(ii) below:

(i) Condition $\boldsymbol{T}$ may involve the quasi-tangent maps $Q_x g : Q_x X \to Q_z Z$ and $Q_x h : Q_y Y \to Q_z Z$ in $\mathcal{Q}$.

(ii) For all $j, k, l \geqslant 0$, condition $\boldsymbol{T}$ may involve the family of triples $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ for $\boldsymbol{x} \in C_j(X)$, $\boldsymbol{y} \in C_k(Y)$ with $\Pi_j(\boldsymbol{x}) = x$, $\Pi_k(\boldsymbol{y}) = y$, and $C(g)\boldsymbol{x} = C(h)\boldsymbol{y} = \boldsymbol{z}$ in $C_l(Z)$.

Condition $\boldsymbol{T}$ should only involve objects $Q_x X, \ldots$ in $\mathcal{Q}$ up to isomorphism, and subsets $\Pi_j^{-1}(x) \subseteq C_j(X), \ldots$ up to bijection.

(b) Taken together, the conditions in (a) are an *open condition* in x, y. That is, if both conditions hold for some x, y, z, then there are open neighbourhoods X' of x in X and Y' of y in Y such that both conditions also hold for all $x' \in X'$ and $y' \in Y'$ with $g(x') = h(y') = z' \in Z$.

(c) Suppose $g : X \to Z$, $h : Y \to Z$ are morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}}$ and $x \in X$, $y \in Y$ with $g(x) = h(y) = z \in Z$ are such that condition $\boldsymbol{T}$ holds, though (11.6) need not be surjective. Then there exist open $X' \hookrightarrow X \times \mathbb{R}^m$ and $Y' \hookrightarrow Y \times \mathbb{R}^n$ for $m, n \geqslant 0$ with $(x, 0) \in X'$ and $(y, 0) \in Y'$, and transverse morphisms $g' : X' \to Z$, $h' : Y' \to Z$ with $g'(\tilde{x}, 0) = g(\tilde{x})$, $h'(\tilde{y}, 0) = h(\tilde{y})$ for all $\tilde{x} \in X$, $\tilde{y} \in Y$ with $(\tilde{x}, 0) \in X'$ and $(\tilde{y}, 0) \in Y'$.

(d) Let $g : X \to Z$ be a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}}$. Then g is a submersion if and only if for all $x \in X$ with $g(x) = z$ in Z, the following is surjective:

$$T_x g : T_x X \longrightarrow T_z Z, \tag{11.7}$$

and an explicit condition (which may be trivial) holds, which we call 'condition $\boldsymbol{S}$', involving only (i)–(ii) below:

(i) Condition $\boldsymbol{S}$ may involve $Q_x g : Q_x X \to Q_z Z$.

(ii) For all $j, l \geqslant 0$, condition $\boldsymbol{S}$ may involve the family of pairs $(\boldsymbol{x}, \boldsymbol{z})$ where $\boldsymbol{x} \in C_j(X)$ with $\Pi_j(\boldsymbol{x}) = x$ and $C(g)\boldsymbol{x} = \boldsymbol{z}$ in $C_l(Z)$.

Condition $\boldsymbol{S}$ should only involve objects $Q_x X, \ldots$ in $\mathcal{Q}$ up to isomorphism, and subsets $\Pi_j^{-1}(x) \subseteq C_j(X), \ldots$ up to bijection.

(e) The conditions in (d) together are an open condition in $x \in X$.

(f) Suppose $g : X \to Z$ is a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}}$ and $x \in X$ with $g(x) = z$ in Z are such that condition $\boldsymbol{S}$ holds, though (11.7) need not be surjective. Then there exist open $X' \hookrightarrow X \times \mathbb{R}^m$ for $m \geqslant 0$ with $(x, 0) \in X'$ and a submersion $g' : X' \to Z$ with $g'(\tilde{x}, 0) = g(\tilde{x})$ for all $\tilde{x} \in X$ with $(\tilde{x}, 0) \in X'$.

(g) Suppose $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}}$ and $x \in X$ with $f(x) = y$ in Y and $g(y) = z$ in Z. If condition $\boldsymbol{S}$ holds for f at x, y and for g at y, z, then it holds for $g \circ f$ at x, z.

(h) Suppose $g : X \to Z$ is a morphism in $\dot{\mathbf{Man}}^{\mathbf{c}}$ with Z in $\mathbf{Man} \subseteq \dot{\mathbf{Man}}^{\mathbf{c}}$. Then g is $\boldsymbol{D}$, and condition $\boldsymbol{S}$ in (d) holds for all x, z.

11.1.4 Examples of categories satisfying the assumptions

Using the material of §2.5, we give several interesting examples in which Assumption 11.1 and various of Assumptions 11.3–11.9 hold:

Example 11.10. Take $\dot{\mathbf{Man}}$ to be the category of classical manifolds $\mathbf{Man}$, and $\boldsymbol{D}, \boldsymbol{E}$ to be trivial (i.e. all morphisms in $\mathbf{Man}$ are $\boldsymbol{D}$ and $\boldsymbol{E}$). As in Definition 2.21 in §2.5.1, define morphisms $g : X \to Z$, $h : Y \to Z$ in $\mathbf{Man}$ to be *transverse* if whenever $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$, then

$$T_x g \oplus T_y h : T_x X \oplus T_y Y \longrightarrow T_z Z$$

is surjective. Define $g : X \to Z$ to be a *submersion* if $T_x g : T_x X \to T_z Z$ is surjective for all $x \in X$ with $g(x) = z \in Z$. We claim that:

- Assumption 11.1 holds.
- Assumptions 11.3–11.5 hold.
- For Assumption 11.9, we take $\mathbf{Man}$ to be a category $\dot{\mathbf{Man}}^{\mathbf{c}}$ as in Example 3.24(i), with $C_k(X) = \emptyset$ for all $X \in \mathbf{Man}$ and $k > 0$. We take tangent spaces $T_x X$ to be as usual, and quasi-tangent spaces $Q_x X$ to be trivial, and conditions $\boldsymbol{T}$ and $\boldsymbol{S}$ are trivial. Then Assumption 11.9 holds.

Almost all the above is well known or obvious, but Assumption 11.1(b)(ii)–(iii) are new, so we prove them in Proposition 11.14 below.

Example 11.11. (a) Take $\dot{\mathbf{Man}}$ to be $\mathbf{Man}^{\mathbf{c}}$ from §2.1, and $\boldsymbol{D}$ to be strongly smooth morphisms, and $\boldsymbol{E}$ to be trivial, and define *s-transverse morphisms* and *s-submersions* in $\mathbf{Man}^{\mathbf{c}}_{\mathbf{st}}$ as in Definition 2.24 in §2.5.2. We claim that:

- Assumption 11.1 holds, where 'transverse' means s-transverse, and 'submersions' are s-submersions.
- Assumptions 11.3–11.4 hold.
- Assumption 11.5 holds for both ordinary tangent spaces $T_x X$ and stratum tangent spaces $\tilde{T}_x X$ in Example 10.2(ii),(iv).

- Assumption 11.6 holds for the stratum normal spaces $\tilde{N}_x X$ in Definition 2.16, as in Example 10.20(a).
- Assumption 11.8 holds, by Theorem 2.25(d).
- For Assumption 11.9, we take $\mathbf{Man^c}$ to be a category $\dot{\mathbf{M}}\mathbf{an^c}$ as in Example 3.24(a), with corner functor $C : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$ as in Definition 2.9. We take tangent spaces to be stratum tangent spaces $\tilde{T}_x X$, and quasi-tangent spaces to be stratum normal spaces $\tilde{N}_x X$. Condition $\boldsymbol{T}$ is that

$$\tilde{N}_x g \oplus \tilde{N}_y h : \tilde{N}_x X \oplus \tilde{N}_y Y \longrightarrow \tilde{N}_z Z \tag{11.8}$$

  is surjective. Condition $\boldsymbol{S}$ is that $\tilde{N}_x g : \tilde{N}_x X \to \tilde{N}_z Z$ is surjective. Then Assumption 11.9 holds.

Most of the above follows from §2.5.2, but Assumption 11.1(b)(ii)–(iii) are new, and we prove them in Proposition 11.14 below.

(b) We can also modify part **(a)** as follows. In Assumption 11.1 we take transversality in $\mathbf{Man^c_{st}}$ to be *t-transverse* morphisms in Definition 2.24. In Assumption 11.9, if $g : X \to Z$ and $h : Y \to Z$ are morphisms in $\mathbf{Man^c_{st}}$ and $x \in X$, $y \in Y$ with $g(x) = h(y) = z$ in Z, then the new condition $\boldsymbol{T}$ is that (11.8) is surjective, and for all $\boldsymbol{x} \in C_j(X)$ and $\boldsymbol{y} \in C_k(Y)$ with $\Pi_j(\boldsymbol{x}) = x$, $\Pi_k(\boldsymbol{y}) = y$, and $C(g)\boldsymbol{x} = C(h)\boldsymbol{y} = \boldsymbol{z}$ in $C_l(Z)$, we have $j + k \geqslant l$, and there is exactly one triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ with $j + k = l$.

Then Assumptions 11.1, 11.3–11.6 and 11.8–11.9 hold as in **(a)**, and in addition, Assumption 11.7 holds for both corner functors $C, C' : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$ in Definitions 2.9 and 2.11, by Theorem 2.25(b).

Example 11.12. **(a)** Take $\dot{\mathbf{M}}\mathbf{an}$ to be $\mathbf{Man^{gc}}$ from §2.4.1, and $\boldsymbol{D}, \boldsymbol{E}$ to be interior morphisms, and define *b-transverse morphisms* and *b-submersions* in $\mathbf{Man^{gc}_{in}}$ as in Definition 2.27 in §2.5.3. We claim that:

- Assumption 11.1 holds, where 'transverse' means b-transverse, and 'submersion' means b-submersion.
- Assumption 11.3 does *not* hold, as Example 2.35 shows.
- Assumption 11.4 holds.
- Assumption 11.5 holds for b-tangent spaces ${}^bT_x X$ in Example 10.2(iii).
- For Assumption 11.9, we take $\mathbf{Man^{gc}}$ to be a category $\dot{\mathbf{M}}\mathbf{an^c}$ as in Example 3.24(h). We take tangent spaces to be b-tangent spaces ${}^bT_x X$, and quasi-tangent spaces to be trivial. Conditions $\boldsymbol{T}$ and $\boldsymbol{S}$ are both trivial. Then Assumption 11.9 holds.

Most of the above follows from §2.5.3, and we prove Assumption 11.1(b)(ii)–(iii) in Proposition 11.14.

(b) Take $\dot{\mathbf{M}}\mathbf{an}$ to be $\mathbf{Man^{gc}}$ from §2.4.1, and $\boldsymbol{D}$ to be interior morphisms in $\mathbf{Man^{gc}}$, and $\boldsymbol{E}$ to be trivial, and define *c-transverse morphisms* and *b-fibrations* in $\mathbf{Man^{gc}_{in}}$ as in Definition 2.27 in §2.5.3. Then as in **(a)** we find that:

- Assumption 11.1 holds, where 'transverse' means c-transverse, and 'submersion' means b-fibration.
- Assumptions 11.3–11.4 hold.
- Assumption 11.5 holds for b-tangent spaces ${}^bT_xX$.
- Assumption 11.7 holds for the corner functor $C : \mathbf{Man^{gc}} \to \check{\mathbf{M}}\mathbf{an^{gc}}$ in §2.4.1, by Theorem 2.28(b).
- For Assumption 11.9, we take $\mathbf{Man^{gc}}$ to be a category $\dot{\mathbf{M}}\mathbf{an^{c}}$ as in Example 3.24(h), with corner functor $C : \mathbf{Man^{gc}} \to \check{\mathbf{M}}\mathbf{an^{gc}}$ as in §2.4.1. We take tangent spaces to be b-tangent spaces ${}^bT_xX$, and quasi-tangent spaces to be trivial.

  If $g : X \to Z$ and $h : Y \to Z$ are morphisms in $\mathbf{Man^{gc}_{in}}$ and $x \in X$, $y \in Y$ with $g(x) = h(y) = z$ in Z, condition $\boldsymbol{T}$ is that for all $\boldsymbol{x} \in C_j(X)$ and $\boldsymbol{y} \in C_k(Y)$ with $\Pi_j(\boldsymbol{x}) = x$, $\Pi_k(\boldsymbol{y}) = y$, and $C(g)\boldsymbol{x} = C(h)\boldsymbol{y} = \boldsymbol{z}$ in $C_l(Z)$, we have either $j + k > l$ or $j = k = l = 0$.

  If $g : X \to Z$ is a morphism in $\mathbf{Man^{gc}_{in}}$ and $x \in X$ with $g(x) = z \in Z$, condition $\boldsymbol{S}$ is that for all $\boldsymbol{x} \in C_j(X)$ with $\Pi_j(\boldsymbol{x}) = x$ and $C(g)\boldsymbol{x} = \boldsymbol{z}$ in $C_l(Z)$, we have $j \geqslant l$. Then Assumption 11.9 holds.

(c) We can also modify part **(b)** by instead taking 'submersions' to be *c-fibrations* in $\mathbf{Man^{gc}_{in}}$, as in Definition 2.27. In Assumption 11.9, if $g : X \to Z$ is a morphism in $\mathbf{Man^{gc}_{in}}$ and $x \in X$ with $g(x) = z \in Z$, the new condition $\boldsymbol{S}$ is that for all $\boldsymbol{x} \in C_j(X)$ with $\Pi_j(\boldsymbol{x}) = x$ and $C(g)\boldsymbol{x} = \boldsymbol{z}$ in $C_l(Z)$, we have $j \geqslant l$, and for each such $\boldsymbol{z}$ there is exactly one such $\boldsymbol{x}$ with $j = l$.

Then Assumptions 11.1, 11.3–11.5, 11.7 and 11.9 hold as in **(b)**, and in addition, Assumption 11.8 holds, by Theorem 2.28(e).

Example 11.13. **(a)** Take $\dot{\mathbf{M}}\mathbf{an}$ to be $\mathbf{Man^c}$ from §2.1, and $\boldsymbol{D}, \boldsymbol{E}$ to be interior morphisms, and define *sb-transverse morphisms* and *s-submersions* in $\mathbf{Man^c_{in}}$ by Definitions 2.24 and 2.31, as in §2.5.4. Then by restriction from $\mathbf{Man^{gc}_{in}}$ in Example 11.12(a), we see that:

- Assumption 11.1 holds, where 'transverse' means sb-transverse, and 'submersion' means s-submersion.
- Assumption 11.3 does *not* hold, as Example 2.35 shows.
- Assumption 11.4 holds.
- Assumption 11.5 holds for b-tangent spaces ${}^bT_xX$ in Example 10.2(iii).
- For Assumption 11.9, we take $\mathbf{Man^c}$ to be a category $\dot{\mathbf{M}}\mathbf{an^c}$ as in Example 3.24(a). We take tangent spaces to be b-tangent spaces ${}^bT_xX$, and quasi-tangent spaces to be monoids $\tilde M_xX$ as in Example 10.20(c). Condition $\boldsymbol{T}$ is that $\tilde M_xX \times_{\tilde M_xg,\tilde M_zZ,\tilde M_yh} \tilde M_yY \cong \mathbb{N}^n$ for $n \geqslant 0$, as in Definition 2.31. Condition $\boldsymbol{S}$ is that the monoid morphism $\tilde M_xg : \tilde M_xX \to \tilde M_zZ$ is isomorphic to a projection $\mathbb{N}^{m+n} \to \mathbb{N}^n$. Then Assumption 11.9 holds.

(b) Take $\dot{\mathbf{Man}}$ to be $\mathbf{Man^c}$ from §2.1, and $\boldsymbol{D}$ to be interior morphisms in $\mathbf{Man^c}$, and $\boldsymbol{E}$ to be trivial, and define *sc-transverse morphisms* and *s-submersions* in $\mathbf{Man^c_{in}}$ by Definitions 2.24 and 2.31, as in §2.5.4. Then by Example 11.11(a) and restriction from $\mathbf{Man^{gc}}$ in Example 11.12(b), we see that:

- Assumption 11.1 holds, where 'transverse' means sb-transverse, and 'submersion' means s-submersion.
- Assumptions 11.3–11.4 hold.
- Assumption 11.5 holds for b-tangent spaces ${}^bT_xX$.
- Assumption 11.6 holds for monoids $\tilde M_xX$.
- Assumption 11.7 holds for the corner functor $C : \mathbf{Man^c} \to \check{\mathbf{Man}}^{\mathbf{c}}$.
- Assumption 11.8 holds.
- For Assumption 11.9, we take $\mathbf{Man^c}$ to be a category $\dot{\mathbf{Man}}^{\mathbf{c}}$ as in Example 3.24(a), with corner functor $C : \mathbf{Man^c} \to \check{\mathbf{Man}}^{\mathbf{c}}$ as in §2.2. We take tangent spaces to be b-tangent spaces ${}^bT_xX$, and quasi-tangent spaces to be monoids $\tilde M_xX$. If $g : X \to Z$ and $h : Y \to Z$ are morphisms in $\mathbf{Man^c_{in}}$ and $x \in X$, $y \in Y$ with $g(x) = h(y) = z$ in Z, condition $\boldsymbol{T}$ is that $\tilde M_xX \times_{\tilde M_xg,\tilde M_zZ,\tilde M_yh} \tilde M_yY \cong \mathbb{N}^n$ for $n \geqslant 0$, and for all $\boldsymbol{x} \in C_j(X)$ and $\boldsymbol{y} \in C_k(Y)$ with $\Pi_j(\boldsymbol{x}) = x$, $\Pi_k(\boldsymbol{y}) = y$, and $C(g)\boldsymbol{x} = C(h)\boldsymbol{y} = \boldsymbol{z}$ in $C_l(Z)$, we have either $j + k > l$ or $j = k = l = 0$.

  If $g : X \to Z$ is a morphism in $\mathbf{Man^c_{in}}$ and $x \in X$ with $g(x) = z \in Z$, condition $\boldsymbol{S}$ is that $\tilde M_xg : \tilde M_xX \to \tilde M_zZ$ is isomorphic to a projection $\mathbb{N}^{m+n} \to \mathbb{N}^n$. Then Assumption 11.9 holds.

The next proposition will be proved in §11.7.

Proposition 11.14. *Examples* 11.10–11.13 *satisfy Assumption* 11.1(b)(ii),(iii).

11.2 Transverse fibre products and submersions in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$

We suppose throughout this section that the category $\dot{\mathbf{Man}}$ used to define $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ satisfies Assumptions 3.1–3.7 and 11.1, and will also specify additional assumptions as needed. Here Assumption 11.1 gives discrete properties $\boldsymbol{D}, \boldsymbol{E}$ of morphisms in $\dot{\mathbf{Man}}$, where $\boldsymbol{D}$ implies $\boldsymbol{E}$, defining subcategories $\dot{\mathbf{Man}}_{\boldsymbol{D}} \subseteq \dot{\mathbf{Man}}_{\boldsymbol{E}} \subseteq \dot{\mathbf{Man}}$ with all objects and only $\boldsymbol{D}, \boldsymbol{E}$ morphisms, and notions of when morphisms $g : X \to Z$, $h : Y \to Z$ in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$ are *transverse* (which implies that a fibre product $X \times_{g,Z,h} Y$ exists in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$, and is also a fibre product in $\dot{\mathbf{Man}}_{\boldsymbol{E}}$), and when $g : X \to Z$ is a *submersion* (which implies that if $h : Y \to Z$ is another morphism in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$ then g, h are transverse).

11.2.1 Fibre products of global m-Kuranishi neighbourhoods

We generalize transversality and submersions to 1-morphisms of m-Kuranishi neighbourhoods. We give both weak versions, 'w-transversality' and 'w-submersions', and strong versions, 'transversality' and 'submersions'.

Definition 11.15. Suppose $g : X \to Z$, $h : Y \to Z$ are continuous maps of topological spaces, and $(U_l, D_l, r_l, \chi_l), (V_m, E_m, s_m, \psi_m), (W_n, F_n, t_n, \omega_n)$ are m-Kuranishi neighbourhoods on X, Y, Z with $\operatorname{Im}\chi_l \subseteq g^{-1}(\operatorname{Im}\omega_n)$ and $\operatorname{Im}\psi_m \subseteq h^{-1}(\operatorname{Im}\omega_n)$, and

$$\begin{aligned} \boldsymbol{g}_{ln} &= (U_{ln}, g_{ln}, \hat{g}_{ln}) : (U_l, D_l, r_l, \chi_l) \longrightarrow (W_n, F_n, t_n, \omega_n), \\ \boldsymbol{h}_{mn} &= (V_{mn}, h_{mn}, \hat{h}_{mn}) : (V_m, E_m, s_m, \psi_m) \longrightarrow (W_n, F_n, t_n, \omega_n), \end{aligned}$$

are $\boldsymbol{D}$ 1-morphisms of m-Kuranishi neighbourhoods over $(\operatorname{Im}\chi_l, g), (\operatorname{Im}\psi_m, h)$.

We call $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ *weakly transverse*, or *w-transverse*, if there exist open neighbourhoods $\dot{U}_{ln}$ of $r_l^{-1}(0)$ in U_{ln}, and $\dot{V}_{mn}$ of $s_m^{-1}(0)$ in V_{mn}, such that:

(i) $g_{ln}|_{\dot{U}_{ln}} : \dot{U}_{ln} \to W_n$ and $h_{mn}|_{\dot{V}_{mn}} : \dot{V}_{mn} \to W_n$ are $\boldsymbol{D}$ morphisms in $\dot{\mathbf{Man}}$, which are transverse in the sense of Assumption 11.1(b); and

(ii) $\hat{g}_{ln}|_u \oplus \hat{h}_{mn}|_v : D_l|_u \oplus E_m|_v \to F_n|_w$ is surjective for all $u \in \dot{U}_{ln}$ and $v \in \dot{V}_{mn}$ with $g_{ln}(u) = h_{mn}(v) = w$ in W_n.

We call $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ *transverse* if they are w-transverse and in (ii) $\hat{g}_{ln}|_u \oplus \hat{h}_{mn}|_v$ is an isomorphism for all u, v.

We call $\boldsymbol{g}_{ln}$ a *weak submersion*, or a *w-submersion*, if there exists an open neighbourhood $\ddot{U}_{ln}$ of $r_l^{-1}(0)$ in U_{ln} such that:

(iii) $g_{ln}|_{\ddot{U}_{ln}} : \ddot{U}_{ln} \to W_n$ is a submersion in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$, as in Assumption 11.1(c).

(iv) $\hat{g}_{ln}|_u : D_l|_u \to F_n|_w$ is surjective for all $u \in \ddot{U}_{ln}$ with $g_{ln}(u) = w$ in W_n.

We call $\boldsymbol{g}_{ln}$ a *submersion* if it is a w-submersion and in (iv) $\hat{g}_{ln}|_u$ is an isomorphism for all u.

If $\boldsymbol{g}_{ln}$ is a w-submersion then $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are w-transverse for any $\boldsymbol{D}$ 1-morphism $\boldsymbol{h}_{mn} : (V_m, E_m, s_m, \psi_m) \to (W_n, F_n, t_n, \omega_n)$ over $(\operatorname{Im}\psi_m, h)$, by Assumption 11.1(c). Also if $\boldsymbol{g}_{ln}$ is a submersion then $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are transverse for any $\boldsymbol{D}$ 1-morphism $\boldsymbol{h}_{mn} : (V_m, E_m, s_m, \psi_m) \to (W_n, F_n, t_n, \omega_n)$ over $(\operatorname{Im}\psi_m, h)$ for which $E_m = 0$ is the zero vector bundle.

In Definition 4.8 we defined a strict 2-category $\mathbf{Gm\dot{K}N}$ of *global m-Kuranishi neighbourhoods*, where:

- Objects (V, E, s) in $\mathbf{Gm\dot{K}N}$ are a manifold V (object in $\dot{\mathbf{Man}}$), a vector bundle $E \to V$ and a section $s : V \to E$. Then $(V, E, s, \mathrm{id}_{s^{-1}(0)})$ is an m-Kuranishi neighbourhood on the topological space $s^{-1}(0) \subseteq V$, as in §4.1. They have *virtual dimension* $\operatorname{vdim}(V, E, s) = \dim V - \operatorname{rank} E$.

- 1-morphisms $\Phi_{ij}:(V_i,E_i,s_i)\to(V_j,E_j,s_j)$ in $\mathbf{Gm\dot{K}N}$ are triples $\Phi_{ij}=(V_{ij},\phi_{ij},\hat\phi_{ij})$ satisfying Definition 4.2(a)–(d) with $s_i^{-1}(0)$ in place of $\psi_i^{-1}(S)$. Then $\Phi_{ij}:(V_i,E_i,s_i,\mathrm{id}_{s_i^{-1}(0)})\to(V_j,E_j,s_j,\mathrm{id}_{s_j^{-1}(0)})$ is a 1-morphism of m-Kuranishi neighbourhoods over $\phi_{ij}|_{s_i^{-1}(0)}:s_i^{-1}(0)\to s_j^{-1}(0)$, as in §4.1.
- For 1-morphisms $\Phi_{ij},\Phi'_{ij}:(V_i,E_i,s_i)\to(V_j,E_j,s_j)$, a 2-morphism $\Lambda_{ij}:\Phi_{ij}\Rightarrow\Phi'_{ij}$ in $\mathbf{Gm\dot{K}N}$ is as in Definition 4.3, with $s_i^{-1}(0)$ in place of $\psi_i^{-1}(S)$.

We write $\mathbf{Gm\dot{K}N}_{\boldsymbol D}\subseteq\mathbf{Gm\dot{K}N}$ for the 2-subcategory with 1-morphisms Φ_{ij} which are $\boldsymbol D$, in the sense of Definition 4.33.

We will prove that w-transverse fibre products exist in $\mathbf{Gm\dot{K}N}_{\boldsymbol D}$:

Definition 11.16. Suppose we are given 1-morphisms in $\mathbf{Gm\dot{K}N}_{\boldsymbol D}$

$$\boldsymbol g_{ln}:(U_l,D_l,r_l)\longrightarrow(W_n,F_n,t_n),\quad \boldsymbol h_{mn}:(V_m,E_m,s_m)\longrightarrow(W_n,F_n,t_n),$$

which are w-transverse as in Definition 11.15. We will construct a fibre product

$$(T_k,C_k,q_k)=(U_l,D_l,r_l)\times_{\boldsymbol g_{ln},(W_n,F_n,t_n),\boldsymbol h_{mn}}(V_m,E_m,s_m) \tag{11.9}$$

in both $\mathbf{Gm\dot{K}N}_{\boldsymbol D}$ and $\mathbf{Gm\dot{K}N}_{\boldsymbol E}$.

Write $\boldsymbol g_{ln}=(U_{ln},g_{ln},\hat g_{ln})$ and $\boldsymbol h_{mn}=(V_{mn},h_{mn},\hat h_{mn})$. Then $\hat g_{ln}(r_l|_{U_{ln}})=g_{ln}^*(t_n)+O(r_l^2)$ by Definition 4.2(d), so Definition 3.15(i) gives $\epsilon:D_l\otimes D_l|_{U_{ln}}\to g_{ln}^*(F_n)$ with $\hat g_{ln}(r_l|_{U_{ln}})=g_{ln}^*(t_n)+\epsilon(r_l\otimes r_l|_{U_{ln}})$. Define $\hat g'_{ln}:D_l|_{U_{ln}}\to g_{ln}^*(F_n)$ by $\hat g'_{ln}(d)=\hat g_{ln}(d)-\epsilon(d\otimes r_l|_{U_{ln}})$. Replacing $\hat g_{ln}$ by $\hat g'_{ln}$, which does not change $\boldsymbol g_{ln}$ up to 2-isomorphism as $\hat g'_{ln}=\hat g_{ln}+O(r_l)$, we suppose that $\hat g_{ln}(r_l|_{U_{ln}})=g_{ln}^*(t_n)$, and similarly $\hat h_{mn}(s_m|_{V_{mn}})=h_{mn}^*(t_n)$. Making $\dot U_{ln},\dot V_{mn}$ smaller, we may suppose Definition 11.15(ii) still holds for the new $\hat g_{ln},\hat h_{mn}$.

For $\dot U_{ln},\dot V_{mn}$ as in Definition 11.15(i),(ii), define

$$T_k=\dot U_{ln}\times_{g_{ln}|_{\dot U_{ln}},W_n,h_{mn}|_{\dot V_{mn}}}\dot V_{mn}$$

to be the transverse fibre product in $\mathbf{\dot{M}an}_{\boldsymbol D}$ from Assumption 11.1(b), with projections $e_{kl}:T_k\to\dot U_{ln}\subseteq U_l$ and $f_{km}:T_k\to\dot V_{mn}\subseteq V_m$ in $\mathbf{\dot{M}an}_{\boldsymbol D}$. Then $g_{ln}\circ e_{kl}=h_{mn}\circ f_{km}$ and

$$\dim T_k=\dim U_l+\dim V_m-\dim W_n. \tag{11.10}$$

We have a morphism of vector bundles on T_k:

$$e_{kl}^*(\hat g_{ln})\oplus -f_{km}^*(\hat h_{mn}):e_{kl}^*(D_l)\oplus f_{km}^*(E_m)\longrightarrow e_{kl}^*(g_{ln}^*(F_n)). \tag{11.11}$$

If $t\in T_k$ with $e_{kl}(t)=u\in\dot U_{ln}$ and $f_{km}(t)=v\in\dot V_{mn}$ then $g_{ln}(u)=h_{mn}(v)=w\in W_n$ and the fibre of (11.11) at t is $\hat g_{ln}|_u\oplus-\hat h_{mn}|_v:D_l|_u\oplus E_m|_v\to F_n|_w$. So Definition 11.15(ii) implies that (11.11) is surjective. Define $C_k\to T_k$ to be the kernel of (11.11), as a vector subbundle of $e_{kl}^*(D_l)\oplus f_{km}^*(E_m)$ with

$$\operatorname{rank}C_k=\operatorname{rank}D_l+\operatorname{rank}E_m-\operatorname{rank}F_n. \tag{11.12}$$

Define vector bundle morphisms $\hat{e}_{kl} : C_k \to e_{kl}^*(D_l)$ and $\hat{f}_{km} : C_k \to f_{km}^*(D_l)$ to be the compositions of the inclusion $C_k \hookrightarrow e_{kl}^*(D_l) \oplus f_{km}^*(E_m)$ with the projections $e_{kl}^*(D_l) \oplus f_{km}^*(E_m) \to e_{kl}^*(D_l)$ and $e_{kl}^*(D_l) \oplus f_{km}^*(E_m) \to f_{km}^*(E_m)$. As C_k is the kernel of (11.11), noting the sign of $-f_{km}^*(\hat{h}_{mn})$ in (11.11), we have

$$e_{kl}^*(\hat{g}_{ln}) \circ \hat{e}_{kl} = f_{km}^*(\hat{h}_{mn}) \circ \hat{f}_{km} : C_k \longrightarrow e_{kl}^*(g_{ln}^*(F_n)) = f_{km}^*(h_{mn}^*(F_n)).$$

The section $e_{kl}^*(r_l) \oplus f_{km}^*(s_m)$ of $e_{kl}^*(D_l) \oplus f_{km}^*(E_m)$ over T_k satisfies

$$\begin{aligned}&\bigl(e_{kl}^*(\hat{g}_{ln}) \oplus -f_{km}^*(\hat{h}_{mn})\bigr)\bigl(e_{kl}^*(r_l) \oplus f_{km}^*(s_m)\bigr)\\ &\quad = e_{kl}^*(\hat{g}_{ln}(r_l)) - f_{km}^*(\hat{h}_{mn}(s_m)) = e_{kl}^* \circ g_{ln}^*(t_n) - f_{km}^* \circ h_{mn}^*(t_n) = 0,\end{aligned}$$

as $\hat{g}_{ln}(r_l|_{U_{ln}}) = g_{ln}^*(t_n)$ and $\hat{h}_{mn}(s_m|_{V_{mn}}) = h_{mn}^*(t_n)$. Thus $e_{kl}^*(r_l) \oplus f_{km}^*(s_m)$ lies in the kernel of (11.11), so it is a section of C_k. Define $q_k = e_{kl}^*(r_l) \oplus f_{km}^*(s_m)$ in $\Gamma^\infty(C_k)$. Then $\hat{e}_{kl}(q_k) = e_{kl}^*(r_l)$ and $\hat{f}_{km}(q_k) = f_{km}^*(s_m)$.

Then (T_k, C_k, q_k) is an object in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$. By (11.10) and (11.12) we have

$$\begin{aligned}\mathrm{vdim}(T_k, C_k, q_k) = \mathrm{vdim}(U_l, D_l, r_l) + \mathrm{vdim}(V_m, E_m, s_m)&\\ - \mathrm{vdim}(W_n, F_n, t_n).&\end{aligned} \tag{11.13}$$

Set $\boldsymbol{e}_{kl} = (T_k, e_{kl}, \hat{e}_{kl})$ and $\boldsymbol{f}_{km} = (T_k, f_{km}, \hat{f}_{km})$. Then $\boldsymbol{e}_{kl} : (T_k, C_k, q_k) \to (U_l, D_l, r_l)$ and $\boldsymbol{f}_{km} : (T_k, C_k, q_k) \to (V_m, E_m, s_m)$ are 1-morphisms in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$. Since $g_{ln} \circ e_{kl} = h_{mn} \circ f_{km}$ and $e_{kl}^*(\hat{g}_{ln}) \circ \hat{e}_{kl} = f_{km}^*(\hat{h}_{mn}) \circ \hat{f}_{km}$ we see that $\boldsymbol{g}_{ln} \circ \boldsymbol{e}_{kl} = \boldsymbol{h}_{mn} \circ \boldsymbol{f}_{km}$. Hence we have a 2-commutative diagram in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$:

$$\begin{array}{ccc} (T_k, C_k, q_k) & \xrightarrow[\quad \boldsymbol{f}_{km} \quad]{} & (V_m, E_m, s_m) \\ \Big\downarrow {\scriptstyle \boldsymbol{e}_{kl}} & \mathrm{id}_{\boldsymbol{g}_{ln} \circ \boldsymbol{e}_{kl}} \quad \Uparrow & {\scriptstyle \boldsymbol{h}_{mn}} \Big\downarrow \\ (U_l, D_l, r_l) & \xrightarrow{\quad \boldsymbol{g}_{ln} \quad} & (W_n, F_n, t_n). \end{array} \tag{11.14}$$

If $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are transverse, not just w-transverse, then (11.11) is an isomorphism, not just surjective, so C_k is the zero vector bundle, as it is the kernel of (11.11). Thus $(T_k, C_k, q_k) = (T_k, 0, 0)$ lies in the image of the obvious embedding $\mathbf{\dot{M}an_{\boldsymbol{D}}} \hookrightarrow \mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$.

The next theorem will be proved in §11.8.

Theorem 11.17. *In Definition* 11.16, *equation* (11.14) *is* 2*-Cartesian in both* $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$ *and* $\mathbf{Gm\dot{K}N_{\boldsymbol{E}}}$ *in the sense of Definition* A.11, *so that* (T_k, C_k, q_k) *is a fibre product in the* 2*-categories* $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}, \mathbf{Gm\dot{K}N_{\boldsymbol{E}}}$, *as in* (11.9).

11.2.2 (W-)transversality and fibre products in $\mathbf{m\dot{K}ur_{\boldsymbol{D}}}$

As in §4.5, for the discrete properties $\boldsymbol{D}, \boldsymbol{E}$ of morphisms in $\mathbf{\dot{M}an}$, we have a notion of when a 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$ is $\boldsymbol{D}$ or $\boldsymbol{E}$, and 2-subcategories $\mathbf{m\dot{K}ur_{\boldsymbol{D}}} \subseteq \mathbf{m\dot{K}ur_{\boldsymbol{E}}} \subseteq \mathbf{m\dot{K}ur}$ with only $\boldsymbol{D}$ or $\boldsymbol{E}$ 1-morphisms. We will define notions of (w-)transverse 1-morphisms and (w-)submersions in $\mathbf{m\dot{K}ur_{\boldsymbol{D}}}$.

Definition 11.18. Let $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ be 1-morphisms in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$. We call $\boldsymbol{g},\boldsymbol{h}$ or *w-transverse* (or *transverse*), if whenever $x\in\boldsymbol{X}$ and $y\in\boldsymbol{Y}$ with $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ in $\boldsymbol{Z}$, there exist m-Kuranishi neighbourhoods (U_l,D_l,r_l,χ_l), (V_m,E_m,s_m,ψ_m), (W_n,F_n,t_n,ω_n) on $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ as in §4.7 with $x\in\operatorname{Im}\chi_l\subseteq g^{-1}(\operatorname{Im}\omega_n)$, $y\in\operatorname{Im}\psi_m\subseteq h^{-1}(\operatorname{Im}\omega_n)$ and $z\in\operatorname{Im}\omega_n$, and 1-morphisms $\boldsymbol{g}_{ln}:(U_l,D_l,r_l,\chi_l)\rightarrow(W_n,F_n,t_n,\omega_n)$, $\boldsymbol{h}_{mn}:(V_m,E_m,s_m,\psi_m)\rightarrow(W_n,F_n,t_n,\omega_n)$ over $(\operatorname{Im}\chi_l,\boldsymbol{g})$ and $(\operatorname{Im}\psi_m,\boldsymbol{h})$, as in Definition 4.54, such that $\boldsymbol{g}_{ln},\boldsymbol{h}_{mn}$ are w-transverse (or transverse, respectively), as in Definition 11.16.

We call $\boldsymbol{g}$ a *w-submersion* (or a *submersion*), if whenever $x\in\boldsymbol{X}$ with $\boldsymbol{g}(x)=z\in\boldsymbol{Z}$, there exist m-Kuranishi neighbourhoods $(U_l,D_l,r_l,\chi_l),(W_n,F_n,t_n,\omega_n)$ on $\boldsymbol{X},\boldsymbol{Z}$ as in §4.7 with $x\in\operatorname{Im}\chi_l\subseteq g^{-1}(\operatorname{Im}\omega_n)$, $z\in\operatorname{Im}\omega_n$, and a 1-morphism $\boldsymbol{g}_{ln}:(U_l,D_l,r_l,\chi_l)\rightarrow(W_n,F_n,t_n,\omega_n)$ over $(\operatorname{Im}\chi_l,\boldsymbol{g})$, as in Definition 4.54, such that $\boldsymbol{g}_{ln}$ is a w-submersion (or a submersion, respectively), as in Definition 11.16.

Suppose $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$ is a w-submersion, and $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ is any $\boldsymbol{D}$ 1-morphism in $\mathbf{m\dot{K}ur}$. Let $x\in\boldsymbol{X}$ and $y\in\boldsymbol{Y}$ with $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ in $\boldsymbol{Z}$. As $\boldsymbol{g}$ is a w-submersion we can choose $\boldsymbol{g}_{ln}:(U_l,D_l,r_l,\chi_l)\rightarrow(W_n,F_n,t_n,\omega_n)$ with $x\in\operatorname{Im}\chi_l\subseteq g^{-1}(\operatorname{Im}\omega_n)$, $z\in\operatorname{Im}\omega_n$, and $\boldsymbol{g}_{ln}$ a w-submersion. Choose any m-Kuranishi neighbourhood (V_m,E_m,s_m,ψ_m) on $\boldsymbol{Y}$ with $y\in\operatorname{Im}\psi_m\subseteq h^{-1}(\operatorname{Im}\omega_n)$. Then Theorem 4.56(b) gives a $\boldsymbol{D}$ 1-morphism $\boldsymbol{h}_{mn}:(V_m,E_m,s_m,\psi_m)\rightarrow(W_n,F_n,t_n,\omega_n)$ over $(\operatorname{Im}\psi_m,\boldsymbol{h})$, and $\boldsymbol{g}_{ln},\boldsymbol{h}_{mn}$ are w-transverse as $\boldsymbol{g}_{ln}$ is a w-submersion. Hence $\boldsymbol{g},\boldsymbol{h}$ are w-transverse.

Similarly, suppose $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$ is a submersion, and $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ is a $\boldsymbol{D}$ 1-morphism in $\mathbf{m\dot{K}ur}$ such that $\boldsymbol{Y}$ is a manifold as in Example 4.30, that is, $\boldsymbol{Y}\simeq F_{\mathbf{\dot{M}an}}^{\mathbf{m\dot{K}ur}}(Y')$ for $Y'\in\mathbf{\dot{M}an}$. Then for $x\in\boldsymbol{X}$ and $y\in\boldsymbol{Y}$ with $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ in $\boldsymbol{Z}$ we can choose $\boldsymbol{g}_{ln},\boldsymbol{h}_{mn}$ as above with $\boldsymbol{g}_{ln}$ a submersion and $E_m=0$, so that $\boldsymbol{g}_{ln},\boldsymbol{h}_{mn}$ are transverse. Hence $\boldsymbol{g},\boldsymbol{h}$ are transverse.

The next important theorem will be proved in §11.9:

Theorem 11.19. *Let $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ be w-transverse 1-morphisms in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$. Then there exists a fibre product $\boldsymbol{W}=\boldsymbol{X}_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$, as in §A.4, with $\operatorname{vdim}\boldsymbol{W}=\operatorname{vdim}\boldsymbol{X}+\operatorname{vdim}\boldsymbol{Y}-\operatorname{vdim}\boldsymbol{Z}$, in a 2-Cartesian square:*

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow{\quad\boldsymbol{f}\quad} & \boldsymbol{Y} \\ \downarrow{\scriptstyle\boldsymbol{e}} & \boldsymbol{\eta}\Uparrow & \boldsymbol{h}\downarrow \\ \boldsymbol{X} & \xrightarrow{\quad\boldsymbol{g}\quad} & \boldsymbol{Z}. \end{array} \tag{11.15}$$

Equation (11.15) *is also 2-Cartesian in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$, so $\boldsymbol{W}$ is also a fibre product $\boldsymbol{X}_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$. Furthermore:*

(a) *If $\boldsymbol{g},\boldsymbol{h}$ are transverse then $\boldsymbol{W}$ is a manifold, as in Example 4.30. In particular, if $\boldsymbol{g}$ is a submersion and $\boldsymbol{Y}$ is a manifold, then $\boldsymbol{W}$ is a manifold.*

(b) *Suppose $(U_l,D_l,r_l,\chi_l),(V_m,E_m,s_m,\psi_m),(W_n,F_n,t_n,\omega_n)$ are m-Kuranishi neighbourhoods on $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$, as in §4.7, with $\operatorname{Im}\chi_l\subseteq g^{-1}(\operatorname{Im}\omega_n)$ and $\operatorname{Im}\psi_m\subseteq h^{-1}(\operatorname{Im}\omega_n)$, and $\boldsymbol{g}_{ln}:(U_l,D_l,r_l,\chi_l)\rightarrow(W_n,F_n,t_n,\omega_n)$, $\boldsymbol{h}_{mn}:(V_m,E_m,s_m,\psi_m)\rightarrow(W_n,F_n,t_n,\omega_n)$ are 1-morphisms of m-Kuranishi neighbourhoods on*

$\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ *over* $(\mathrm{Im}\,\chi_l,\boldsymbol{g})$ *and* $(\mathrm{Im}\,\psi_m,\boldsymbol{h})$, *as in* §4.7, *such that* $\boldsymbol{g}_{ln},\boldsymbol{h}_{mn}$ *are w-transverse, as in* §11.2.1. *Then there exist an m-Kuranishi neighbourhood* (T_k,C_k,q_k,φ_k) *on* $\boldsymbol{W}$ *with* $\mathrm{Im}\,\varphi_k = e^{-1}(\mathrm{Im}\,\chi_l)\cap f^{-1}(\mathrm{Im}\,\psi_m)\subseteq W$, *and* 1-*morphisms* $\boldsymbol{e}_{kl}:(T_k,C_k,q_k,\varphi_k)\to(U_l,D_l,r_l,\chi_l)$ *over* $(\mathrm{Im}\,\varphi_k,\boldsymbol{e})$ *and* $\boldsymbol{f}_{km}:(T_k,C_k,q_k,\varphi_k)\to(V_m,E_m,s_m,\psi_m)$ *over* $(\mathrm{Im}\,\varphi_k,\boldsymbol{f})$ *with* $\boldsymbol{g}_{ln}\circ\boldsymbol{e}_{kl}=\boldsymbol{h}_{mn}\circ\boldsymbol{f}_{km}$, *such that* (T_k,C_k,q_k) *and* $\boldsymbol{e}_{kl},\boldsymbol{f}_{km}$ *are constructed from* $(U_l,D_l,r_l),(V_m,E_m,s_m),(W_n,F_n,t_n)$ *and* $\boldsymbol{g}_{ln},\boldsymbol{h}_{mn}$ *exactly as in Definition* 11.16.

Also the unique 2-*morphism* $\boldsymbol{\eta}_{klmn}:\boldsymbol{g}_{ln}\circ\boldsymbol{e}_{kl}\Rightarrow\boldsymbol{h}_{mn}\circ\boldsymbol{f}_{km}$ *over* $(\mathrm{Im}\,\varphi_k,g\circ e)$ *constructed from* $\boldsymbol{\eta}:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ *in Theorem* 4.56(c) *is the identity.*

(c) *If* $\dot{\mathbf{Man}}$ *satisfies Assumption* 11.3 *then we can choose the topological space* W *in* $\boldsymbol{W}=(W,\mathcal{H})$ *to be* $W=\{(x,y)\in X\times Y: g(x)=h(y)\}$, *with* $e:W\to X$, $f:W\to Y$ *acting by* $e:(x,y)\mapsto x$ *and* $f:(x,y)\mapsto y$.

(d) *If* $\dot{\mathbf{Man}}$ *satisfies Assumption* 11.4(a) *and* (11.15) *is a* 2-*Cartesian square in* $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ *with* $\boldsymbol{g}$ *a w-submersion (or a submersion) then* $\boldsymbol{f}$ *is a w-submersion (or a submersion, respectively).*

(e) *If* $\dot{\mathbf{Man}}$ *satisfies Assumption* 10.1, *with tangent spaces* T_xX, *and satisfies Assumption* 11.5, *then using the notation of* §10.2, *whenever* (11.15) *is* 2-*Cartesian in* $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ *with* $\boldsymbol{g},\boldsymbol{h}$ *w-transverse and* $w\in\boldsymbol{W}$ *with* $\boldsymbol{e}(w)=x$ *in* $\boldsymbol{X}$, $\boldsymbol{f}(w)=y$ *in* $\boldsymbol{Y}$ *and* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$, *the following is an exact sequence:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_w\boldsymbol{W} & \xrightarrow{T_w\boldsymbol{e}\oplus T_w\boldsymbol{f}} & T_x\boldsymbol{X}\oplus T_y\boldsymbol{Y} & \xrightarrow{T_x\boldsymbol{g}\oplus -T_y\boldsymbol{h}} & T_z\boldsymbol{Z} \\ & & & & & & \big\downarrow{\scriptstyle \delta_w^{\boldsymbol{g},\boldsymbol{h}}} \\ 0 & \longleftarrow & O_z\boldsymbol{Z} & \xleftarrow{O_x\boldsymbol{g}\oplus -O_y\boldsymbol{h}} & O_x\boldsymbol{X}\oplus O_y\boldsymbol{Y} & \xleftarrow{O_w\boldsymbol{e}\oplus O_w\boldsymbol{f}} & O_w\boldsymbol{W}. \end{array} \qquad (11.16)$$

Here $\delta_w^{\boldsymbol{g},\boldsymbol{h}}:T_z\boldsymbol{Z}\to O_w\boldsymbol{W}$ *is a natural linear map defined as a connecting morphism, as in Definition* 10.69.

(f) *If* $\dot{\mathbf{Man}}$ *satisfies Assumption* 10.19, *with quasi-tangent spaces* Q_xX *in a category* $\mathcal{Q}$, *and satisfies Assumption* 11.6, *then whenever* (11.15) *is* 2-*Cartesian in* $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ *with* $\boldsymbol{g},\boldsymbol{h}$ *w-transverse and* $w\in\boldsymbol{W}$ *with* $\boldsymbol{e}(w)=x$ *in* $\boldsymbol{X}$, $\boldsymbol{f}(w)=y$ *in* $\boldsymbol{Y}$ *and* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$, *the following is Cartesian in* $\mathcal{Q}$*:*

$$\begin{array}{ccc} Q_w\boldsymbol{W} & \xrightarrow{Q_w\boldsymbol{f}} & Q_y\boldsymbol{Y} \\ \big\downarrow{\scriptstyle Q_w\boldsymbol{e}} & & \big\downarrow{\scriptstyle Q_y\boldsymbol{h}} \\ Q_x\boldsymbol{X} & \xrightarrow{Q_x\boldsymbol{g}} & Q_z\boldsymbol{Z}. \end{array} \qquad (11.17)$$

(g) *If* $\dot{\mathbf{Man}}^{\mathbf{c}}$ *satisfies Assumption* 3.22 *in* §3.4, *so that we have a corner functor* $C:\dot{\mathbf{Man}}^{\mathbf{c}}\to\check{\dot{\mathbf{Man}}}^{\mathbf{c}}$ *which extends to* $C:\mathbf{m\dot{K}ur}^{\mathbf{c}}\to\mathbf{m\check{\dot{K}}ur}^{\mathbf{c}}$ *as in* §4.6, *and Assumption* 11.1 *holds for* $\dot{\mathbf{Man}}^{\mathbf{c}}$, *and Assumption* 11.7 *holds, then whenever* (11.15) *is* 2-*Cartesian in* $\mathbf{m\dot{K}ur}^{\mathbf{c}}_{\boldsymbol{D}}$ *with* $\boldsymbol{g},\boldsymbol{h}$ *w-transverse (or transverse), then the following is* 2-*Cartesian in* $\mathbf{m\check{\dot{K}}ur}^{\mathbf{c}}_{\boldsymbol{D}}$ *and* $\mathbf{m\check{\dot{K}}ur}^{\mathbf{c}}_{\boldsymbol{E}}$, *with* $C(\boldsymbol{g}),C(\boldsymbol{h})$ *w-transverse (or transverse, respectively):*

$$\begin{array}{ccc} C(\boldsymbol{W}) & \xrightarrow{C(\boldsymbol{f})} & C(\boldsymbol{Y}) \\ \big\downarrow{\scriptstyle C(\boldsymbol{e})} & C(\boldsymbol{\eta})\Uparrow & \big\downarrow{\scriptstyle C(\boldsymbol{h})} \\ C(\boldsymbol{X}) & \xrightarrow{C(\boldsymbol{g})} & C(\boldsymbol{Z}). \end{array} \qquad (11.18)$$

Hence for $i \geqslant 0$ we have

$$C_i(\boldsymbol{W}) \simeq \coprod_{\substack{j,k,l\geqslant 0:\\ i=j+k-l}} \begin{aligned}&\bigl(C_j(\boldsymbol{X})\cap C(\boldsymbol{g})^{-1}(C_l(\boldsymbol{Z}))\bigr)\times_{C(\boldsymbol{g}),C_l(\boldsymbol{Z}),C(\boldsymbol{h})}\\ &\qquad\qquad\bigl(C_k(\boldsymbol{Y})\cap C(\boldsymbol{h})^{-1}(C_l(\boldsymbol{Z}))\bigr).\end{aligned} \tag{11.19}$$

When $i=1$, this computes the boundary $\partial\boldsymbol{W}$. In particular, if $\partial\boldsymbol{Z}=\emptyset$, so that $C_l(\boldsymbol{Z})=\emptyset$ for all $l>0$ by Assumption 3.22(f) *with $l=1$, we have*

$$\partial\boldsymbol{W}\simeq\bigl(\partial\boldsymbol{X}\times_{\boldsymbol{g}\circ\boldsymbol{i_X},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}\bigr)\amalg\bigl(\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}\circ\boldsymbol{i_Y}}\partial\boldsymbol{Y}\bigr). \tag{11.20}$$

Also, if $\boldsymbol{g}$ is a w-submersion (or a submersion), then $C(\boldsymbol{g})$ is a w-submersion (or a submersion, respectively).

(h) *If $\dot{\mathbf{Man}}$ satisfies Assumption* 11.8, *and $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ is a w-submersion in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$, and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ is any* 1*-morphism in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$ (not necessarily in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$), then a fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$, with* $\dim\boldsymbol{W}=\dim\boldsymbol{X}+\dim\boldsymbol{Y}-\dim\boldsymbol{Z}$, *in a* 2*-Cartesian square* (11.15) *in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$. The analogues of* **(a)**–**(d)** *and* **(g)** *hold for these fibre products.*

Example 11.20. Let $g:X\to Z$, $h:Y\to Z$ be transverse morphisms in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$, and let $W=X\times_{g,Z,h}Y$ in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$, with projections $e:W\to X$, $f:W\to Y$. Write $\boldsymbol{W},\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{e},\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}$ for the images of W,X,Y,Z,e,f,g,h in $\mathbf{m\dot{K}ur}$ under the 2-functor $F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}:\dot{\mathbf{Man}}\to\mathbf{m\dot{K}ur}$ from Example 4.30.

Then we have m-Kuranishi neighbourhoods $(W,0,0,\mathrm{id}_W)$ on $\boldsymbol{W}$, as in §4.7, and similarly for $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$. We have a 1-morphism $(W,e,0):(W,0,0,\mathrm{id}_W)\to(X,0,0,\mathrm{id}_X)$ over $(W,\boldsymbol{e})$, as in §4.7, and similarly for $\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}$.

These 1-morphisms $(X,g,0):(X,0,0,\mathrm{id}_X)\to(Z,0,0,\mathrm{id}_Z)$ and $(Y,h,0):(Y,0,0,\mathrm{id}_Y)\to(Z,0,0,\mathrm{id}_Z)$ are transverse as in Definition 11.15, where (i) holds as g,h are transverse in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$, and (ii) is trivial as D_l,E_m,F_n are zero. As these m-Kuranishi neighbourhoods cover $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$, we see that $\boldsymbol{g},\boldsymbol{h}$ are transverse by Definition 11.18, so a fibre product $\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Z}$ exists in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ by Theorem 11.19. We claim that this fibre product is $\boldsymbol{W}=F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}(W)$.

To see this, note that applying Definition 11.16 to the transverse $(X,g,0)$, $(Y,h,0)$ above yields $(T_k,C_k,q_k,\varphi_k)=(W,0,0,\mathrm{id}_W)$, so $(W,0,0,\mathrm{id}_W)$ is an m-Kuranishi neighbourhood on $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$ by Theorem 11.19(b), which covers $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$, and this forces $\boldsymbol{W}\simeq\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$. Thus, $F_{\dot{\mathbf{Man}}}^{\mathbf{m\dot{K}ur}}$ takes transverse fibre products in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$ and $\dot{\mathbf{Man}}_{\boldsymbol{E}}$ to transverse fibre products in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ and $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$.

11.2.3 Products of m-Kuranishi spaces

Let $\dot{\mathbf{Man}}$ be any category satisfying Assumptions 3.1–3.7. Apply Example 11.2 with $\boldsymbol{D},\boldsymbol{E}$ trivial to get notions of transverse morphisms and submersions in $\dot{\mathbf{Man}}$ satisfying Assumption 11.1. As in Example 11.2, for any $X,Y\in\dot{\mathbf{Man}}$ the projections $\pi:X\to *$ and $\pi:Y\to *$ are transverse in $\dot{\mathbf{Man}}$.

From Definitions 11.15 and 11.18 we see that for any $\boldsymbol{X},\boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$ the projections $\boldsymbol{\pi}:\boldsymbol{X}\to\boldsymbol{*}$, $\boldsymbol{\pi}:\boldsymbol{Y}\to\boldsymbol{*}$ are w-transverse, so a fibre product $\boldsymbol{X}\times_{\boldsymbol{*}}\boldsymbol{Y}$

exists in $\mathbf{m\dot{K}ur}$ by Theorem 11.19. Now a *product* in a category or 2-category is by definition a fibre product over the terminal object $*$. The fibre product property only determines $\boldsymbol{X}\times_*\boldsymbol{Y}$ up to canonical equivalence in $\mathbf{m\dot{K}ur}$. But from Theorem 11.19(b) we see that we can take $\boldsymbol{X}\times_*\boldsymbol{Y}$ and the 1-morphisms $\boldsymbol{e}:\boldsymbol{X}\times_*\boldsymbol{Y}\to\boldsymbol{X}$, $\boldsymbol{f}:\boldsymbol{X}\times_*\boldsymbol{Y}\to\boldsymbol{Y}$ to be the product $\boldsymbol{X}\times\boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$ in Example 4.31 and the projections $\boldsymbol{\pi_X}:\boldsymbol{X}\times\boldsymbol{Y}\to\boldsymbol{X}$, $\boldsymbol{\pi_Y}:\boldsymbol{X}\times\boldsymbol{Y}\to\boldsymbol{Y}$, which are uniquely defined.

This proves that the products $\boldsymbol{X}\times\boldsymbol{Y}$ defined in Example 4.31 have the universal property of products in the 2-category $\mathbf{m\dot{K}ur}$, that is, they are fibre products $\boldsymbol{X}\times_*\boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$. The existence of product m-Kuranishi neighbourhoods on $\boldsymbol{X}\times\boldsymbol{Y}$ in Example 4.53 follows from Theorem 11.19(b) with $W_n=*$.

As in Example 4.31, if $\boldsymbol{g}:\boldsymbol{W}\to\boldsymbol{Y}$, $\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Z}$ are 1-morphisms in $\mathbf{m\dot{K}ur}$ then we have a product 1-morphism $\boldsymbol{g}\times\boldsymbol{h}:\boldsymbol{W}\times\boldsymbol{X}\to\boldsymbol{Y}\times\boldsymbol{Z}$. Given 1-morphisms of m-Kuranishi neighbourhoods on $\boldsymbol{W},\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ over $\boldsymbol{g},\boldsymbol{h}$, we can write down a product 1-morphism of m-Kuranishi neighbourhoods on $\boldsymbol{W}\times\boldsymbol{X},\boldsymbol{Y}\times\boldsymbol{Z}$ over $\boldsymbol{g}\times\boldsymbol{h}$. Using these and Theorem 11.19(d) it is easy to prove:

Proposition 11.21. *Let* $\mathbf{\dot{M}an}$ *satisfy Assumptions* 11.1 *and* 11.4(b),(c). *Then products of w-submersions (or submersions) in* $\mathbf{m\dot{K}ur}$ *are w-submersions (or submersions, respectively). That is, if* $\boldsymbol{g}:\boldsymbol{W}\to\boldsymbol{Y}$ *and* $\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Z}$ *are (w-)submersions in* $\mathbf{m\dot{K}ur}$, *then* $\boldsymbol{g}\times\boldsymbol{h}:\boldsymbol{W}\times\boldsymbol{X}\to\boldsymbol{Y}\times\boldsymbol{Z}$ *is a (w-)submersion. Projections* $\boldsymbol{\pi_X}:\boldsymbol{X}\times\boldsymbol{Y}\to\boldsymbol{X}$, $\boldsymbol{\pi_Y}:\boldsymbol{X}\times\boldsymbol{Y}\to\boldsymbol{Y}$ *in* $\mathbf{m\dot{K}ur}$ *are w-submersions.*

11.2.4 Characterizing (w-)transversality and (w-)submersions

Assumption 11.9 in §11.1.3 gave necessary and sufficient conditions for morphisms g,h in $\mathbf{\dot{M}an^c}$ to be transverse, and for morphisms g to be submersions. The next theorem, proved in §11.10, extends these to conditions for 1-morphisms $\boldsymbol{g},\boldsymbol{h}$ in $\mathbf{m\dot{K}ur^c}$ to be (w-)transverse, and for 1-morphisms $\boldsymbol{g}$ to be (w-)submersions.

Theorem 11.22. *Let* $\mathbf{\dot{M}an^c}$ *satisfy Assumption* 3.22, *so that we have a corner functor* $C:\mathbf{\dot{M}an^c}\to\mathbf{\check{M}an^c}$, *and suppose Assumption* 11.9 *holds for* $\mathbf{\dot{M}an^c}$. *This requires that Assumption* 10.1 *holds, giving a notion of tangent spaces* T_xX *for* X *in* $\mathbf{\dot{M}an^c}$, *and that Assumption* 10.19 *holds, giving a notion of quasi-tangent spaces* Q_xX *in a category* $\mathcal{Q}$ *for* X *in* $\mathbf{\dot{M}an^c}$, *and that Assumption* 11.1 *holds, giving discrete properties* $\boldsymbol{D},\boldsymbol{E}$ *of morphisms in* $\mathbf{\dot{M}an^c}$ *and notions of transverse morphisms* g,h *and submersions* g *in* $\mathbf{\dot{M}an^c_{\boldsymbol{D}}}$.

As in §4.6, §10.2 *and* §10.3, *we define a* 2*-category* $\mathbf{m\dot{K}ur^c}$, *with a corner* 2*-functor* $\boldsymbol{C}:\mathbf{m\dot{K}ur^c}\to\mathbf{m\check{K}ur^c}$, *and notions of tangent, obstruction and quasi-tangent spaces* $T_x\boldsymbol{X},O_x\boldsymbol{X},Q_x\boldsymbol{X}$ *for* $\boldsymbol{X}$ *in* $\mathbf{m\dot{K}ur^c}$.

Now Assumption 11.9(a),(d) *involve a 'condition* $\boldsymbol{T}$*' on morphisms* $g:X\to Z$, $h:Y\to Z$ *in* $\mathbf{\dot{M}an^c_{\boldsymbol{D}}}$ *and points* $x\in X$, $y\in Y$ *with* $g(x)=h(y)=z\in Z$, *and a 'condition* $\boldsymbol{S}$*' on morphisms* $g:X\to Z$ *in* $\mathbf{\dot{M}an^c_{\boldsymbol{D}}}$ *and points* $x\in X$ *with* $g(x)=z\in Z$. *These conditions depend on the corner morphisms* $C(g),C(h)$ *and on quasi-tangent maps* Q_xg,Q_yh. *Observe that condition* $\boldsymbol{T}$ *also makes sense for* 1*-morphisms* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *in* $\mathbf{m\dot{K}ur^c_{\boldsymbol{D}}}$ *and* $x\in\boldsymbol{X}$, $y\in\boldsymbol{Y}$

with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ *in* $\boldsymbol{Z}$, *and condition* $\boldsymbol{S}$ *makes sense for* 1*-morphisms* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *in* $\mathbf{m\dot{K}ur^c_{\boldsymbol{D}}}$ *and* $x \in \boldsymbol{X}$ *with* $\boldsymbol{g}(x) = z \in \boldsymbol{Z}$. *Then:*

(a) *Let* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *be* 1*-morphisms in* $\mathbf{m\dot{K}ur^c_{\boldsymbol{D}}}$. *Then* $\boldsymbol{g}, \boldsymbol{h}$ *are w-transverse if and only if for all* $x \in \boldsymbol{X}$ *and* $y \in \boldsymbol{Y}$ *with* $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ *in* $\boldsymbol{Z}$, *condition* $\boldsymbol{T}$ *holds for* $\boldsymbol{g}, \boldsymbol{h}, x, y, z$, *and the following is surjective:*

$$O_x\boldsymbol{g} \oplus O_y\boldsymbol{h} : O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y} \longrightarrow O_z\boldsymbol{Z}. \tag{11.21}$$

If Assumption 10.9 *also holds for tangent spaces* T_xX *in* $\mathbf{\dot{M}an^c}$ *then* $\boldsymbol{g}, \boldsymbol{h}$ *are transverse if and only if for all* $x \in \boldsymbol{X}$ *and* $y \in \boldsymbol{Y}$ *with* $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ *in* $\boldsymbol{Z}$, *condition* $\boldsymbol{T}$ *holds for* $\boldsymbol{g}, \boldsymbol{h}, x, y, z$, *equation* (11.21) *is an isomorphism, and the following linear map is surjective:*

$$T_x\boldsymbol{g} \oplus T_y\boldsymbol{h} : T_x\boldsymbol{X} \oplus T_y\boldsymbol{Y} \longrightarrow T_z\boldsymbol{Z}. \tag{11.22}$$

(b) *Let* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *be a* 1*-morphism in* $\mathbf{m\dot{K}ur^c_{\boldsymbol{D}}}$. *Then* $\boldsymbol{g}$ *is a w-submersion if and only if for all* $x \in \boldsymbol{X}$ *with* $\boldsymbol{g}(x) = z$ *in* $\boldsymbol{Z}$, *condition* $\boldsymbol{S}$ *holds for* $\boldsymbol{g}, x, z$, *and the following linear map is surjective:*

$$O_x\boldsymbol{g} : O_x\boldsymbol{X} \longrightarrow O_z\boldsymbol{Z}. \tag{11.23}$$

If Assumption 10.9 *also holds then* $\boldsymbol{g}$ *is a submersion if and only if for all* $x \in \boldsymbol{X}$ *with* $\boldsymbol{g}(x) = z$ *in* $\boldsymbol{Z}$, *condition* $\boldsymbol{S}$ *holds for* $\boldsymbol{g}, x, z$, *equation* (11.23) *is an isomorphism, and the following is surjective:*

$$T_x\boldsymbol{g} : T_x\boldsymbol{X} \longrightarrow T_z\boldsymbol{Z}.$$

Combining Assumption 11.9(g) and Theorem 11.22(b) gives:

Corollary 11.23. *Let* $\mathbf{\dot{M}an^c}$ *satisfy Assumptions* 3.22 *and* 11.9. *Then compositions of w-submersions in* $\mathbf{m\dot{K}ur^c}$ *are w-submersions. If* $\mathbf{\dot{M}an^c}$ *also satisfies Assumption* 10.9 *then compositions of submersions in* $\mathbf{m\dot{K}ur^c}$ *are submersions.*

Combining Assumption 11.9(h) and Theorems 11.19(a) and 11.22(b) yields:

Corollary 11.24. *Let* $\mathbf{\dot{M}an^c}$ *satisfy Assumptions* 3.22 *and* 11.9, *so that Assumption* 11.1 *holds with discrete properties* $\boldsymbol{D}, \boldsymbol{E}$. *Suppose that* $\boldsymbol{Z}$ *is a classical manifold in* $\mathbf{m\dot{K}ur^c}$, *as in Example* 4.30. *Then any* 1*-morphism* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *in* $\mathbf{m\dot{K}ur^c}$ *is* $\boldsymbol{D}$ *and a w-submersion. Hence any* 1*-morphisms* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Z}$ *in* $\mathbf{m\dot{K}ur^c}$ *are w-transverse, and a fibre product* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *exists in* $\mathbf{m\dot{K}ur^c_{\boldsymbol{D}}}$, *and is also a fibre product in* $\mathbf{m\dot{K}ur^c_{\boldsymbol{E}}}$.

11.2.5 Orientations on w-transverse fibre products in $\mathbf{m\dot{K}ur}$

In this section we suppose throughout that $\dot{\mathbf{Man}}$ satisfies Assumptions 3.1–3.7, 10.1, 10.13, 11.1, and 11.5. Thus, objects X in $\dot{\mathbf{Man}}$ have tangent spaces T_xX which are fibres of a tangent bundle $TX \to X$ of rank $\dim X$, and these are used to define canonical bundles $K_{\boldsymbol{X}}$ and orientations on m-Kuranishi spaces $\boldsymbol{X}$ as in §10.7, and we can form w-transverse fibre products $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ as in Theorem 11.19.

Given orientations on $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, the next theorem defines an orientation on $\boldsymbol{W}$. It will be proved in §11.11. It is a generalization of Theorem 10.80 in §10.7.4 on orientations of products $\boldsymbol{X} \times \boldsymbol{Y}$, and reduces to this when $\boldsymbol{Z} = \boldsymbol{*}$, in which case $\Upsilon_{\boldsymbol{X},\boldsymbol{Y}}$ in Theorem 10.80 coincides with $\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{*}}$ below.

Theorem 11.25. *Suppose $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are w-transverse* 1-*morphisms in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$, so that a fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ by Theorem* 11.19, *in a* 2-*Cartesian square* (11.15). *Sections* 10.7.1–10.7.2 *define the canonical line bundles $K_{\boldsymbol{W}}, K_{\boldsymbol{X}}, K_{\boldsymbol{Y}}, K_{\boldsymbol{Z}}$ of $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, using tangent spaces and tangent bundles in $\dot{\mathbf{Man}}$ from Assumptions* 10.1 *and* 10.13, *and define orientations on $\boldsymbol{W}, \ldots, \boldsymbol{Z}$ to be orientations on the fibres of $K_{\boldsymbol{W}}, \ldots, K_{\boldsymbol{Z}}$.*

Then there is a unique isomorphism of topological line bundles on W:

$$\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}} : K_{\boldsymbol{W}} \longrightarrow e^*(K_{\boldsymbol{X}}) \otimes f^*(K_{\boldsymbol{Y}}) \otimes (g \circ e)^*(K_{\boldsymbol{Z}})^* \tag{11.24}$$

with the following property. Let $w \in \boldsymbol{W}$ with $\boldsymbol{e}(w) = x$ in $\boldsymbol{X}$, $\boldsymbol{f}(w) = y$ in $\boldsymbol{Y}$ and $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$. Then we can consider $\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}|_w$ as a nonzero element

$$\begin{aligned}\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}|_w &\in (K_{\boldsymbol{W}}|_w)^* \otimes K_{\boldsymbol{X}}|_x \otimes K_{\boldsymbol{Y}}|_y \otimes (K_{\boldsymbol{Z}}|_z)^* \\ &\cong (\det T_w^*\boldsymbol{W} \otimes \det O_w\boldsymbol{W})^{-1} \otimes \det T_x^*\boldsymbol{X} \otimes \det O_x\boldsymbol{X} \\ &\qquad \otimes \det T_y^*\boldsymbol{Y} \otimes \det O_y\boldsymbol{Y} \otimes (\det T_z^*\boldsymbol{Z} \otimes \det O_z\boldsymbol{Z})^{-1}.\end{aligned}$$

By Theorem 11.19(e) *we have an exact sequence*

$$\begin{array}{ccccccc} 0 \longrightarrow T_w\boldsymbol{W} & \xrightarrow[T_w\boldsymbol{e}\oplus T_w\boldsymbol{f}]{} & T_x\boldsymbol{X} \oplus T_y\boldsymbol{Y} & \xrightarrow[T_x\boldsymbol{g}\oplus -T_y\boldsymbol{h}]{} & T_z\boldsymbol{Z} \\ & & & & \downarrow{\scriptstyle \delta_w^{\boldsymbol{g},\boldsymbol{h}}} \\ 0 \longleftarrow O_z\boldsymbol{Z} & \xleftarrow{O_x\boldsymbol{g}\oplus -O_y\boldsymbol{h}} & O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y} & \xleftarrow{O_w\boldsymbol{e}\oplus O_w\boldsymbol{f}} & O_w\boldsymbol{W}. \end{array} \tag{11.25}$$

Consider (11.25) *as an exact complex $A^\bullet$ with $O_w\boldsymbol{W}$ in degree* 0, *so that* (10.69) *defines a nonzero element*

$$\begin{aligned}\Psi_{A^\bullet} \in{} & \det T_w^*\boldsymbol{W} \otimes (\det(T_x^*\boldsymbol{X} \oplus T_y^*\boldsymbol{Y}))^{-1} \otimes \det T_z^*\boldsymbol{Z} \\ & \otimes \det O_w\boldsymbol{W} \otimes (\det(O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y}))^{-1} \otimes \det O_z\boldsymbol{Z}.\end{aligned}$$

Then defining $I_{T_x^*\boldsymbol{X},T_y^*\boldsymbol{Y}}, I_{O_x\boldsymbol{X},O_y\boldsymbol{Y}}$ as in (10.84), *we have*

$$\begin{aligned}&\bigl(I_{T_x^*\boldsymbol{X},T_y^*\boldsymbol{Y}} \otimes I_{O_x\boldsymbol{X},O_y\boldsymbol{Y}}\bigr)(\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}|_w) \\ &\qquad = (-1)^{\dim O_w\boldsymbol{W} \dim T_z\boldsymbol{Z} + \dim O_x\boldsymbol{X} \dim T_y\boldsymbol{Y}} \cdot \Psi_{A^\bullet}^{-1}.\end{aligned} \tag{11.26}$$

Hence if $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are oriented there is a unique orientation on $\boldsymbol{W}$, called the ***fibre product orientation****, such that* (11.24) *is orientation-preserving.*

The morphism $\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}$ in (11.24), and hence the orientation on $\boldsymbol{W}$ above, depend on our choice of *orientation conventions*, as in Convention 2.39, including various sign choices in §10.6–§10.7 and in (11.26). Different orientation conventions would change $\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}$ and the orientation on $\boldsymbol{W}$ by a sign depending on $\mathrm{vdim}\,\boldsymbol{X},\mathrm{vdim}\,\boldsymbol{Y},\mathrm{vdim}\,\boldsymbol{Z}$. If $\boldsymbol{W},\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are manifolds then the orientation on $\boldsymbol{W}$ agrees with that in Convention 2.39(b).

Fibre products have natural commutativity and associativity properties, up to canonical equivalence in $\mathbf{m\dot{K}ur}$. For instance, for w-transverse $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ there is a natural equivalence $\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}\simeq\boldsymbol{Y}\times_{\boldsymbol{h},\boldsymbol{Z},\boldsymbol{g}}\boldsymbol{X}$. When we lift these to (multiple) fibre products of oriented m-Kuranishi spaces, the orientations on each side differ by some sign depending on the virtual dimensions of the factors. The next proposition, the m-Kuranishi analogue of Proposition 2.40, is a generalization of Proposition 10.81, and may be proved using the same method. Parts (b),(c) are the analogue of results by Fukaya et al. [15, Lem. 8.2.3(2),(3)] for FOOO Kuranishi spaces.

Proposition 11.26. *Suppose $\boldsymbol{V},\ldots,\boldsymbol{Z}$ are oriented m-Kuranishi spaces, and $\boldsymbol{e},\ldots,\boldsymbol{h}$ are* 1*-morphisms, and all fibre products below are w-transverse. Then the following canonical equivalences hold, in oriented m-Kuranishi spaces:*

(a) *For $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ we have*

$$\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}\simeq(-1)^{(\mathrm{vdim}\,\boldsymbol{X}-\mathrm{vdim}\,\boldsymbol{Z})(\mathrm{vdim}\,\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{Z})}\boldsymbol{Y}\times_{\boldsymbol{h},\boldsymbol{Z},\boldsymbol{g}}\boldsymbol{X}.$$

(b) *For $\boldsymbol{e}:\boldsymbol{V}\to\boldsymbol{Y}$, $\boldsymbol{f}:\boldsymbol{W}\to\boldsymbol{Y}$, $\boldsymbol{g}:\boldsymbol{W}\to\boldsymbol{Z}$, and $\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Z}$ we have*

$$\boldsymbol{V}\times_{\boldsymbol{e},\boldsymbol{Y},\boldsymbol{f}\circ\boldsymbol{\pi}_{\boldsymbol{W}}}\bigl(\boldsymbol{W}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{X}\bigr)\simeq\bigl(\boldsymbol{V}\times_{\boldsymbol{e},\boldsymbol{Y},\boldsymbol{f}}\boldsymbol{W}\bigr)\times_{\boldsymbol{g}\circ\boldsymbol{\pi}_{\boldsymbol{W}},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{X}.$$

(c) *For $\boldsymbol{e}:\boldsymbol{V}\to\boldsymbol{Y}$, $\boldsymbol{f}:\boldsymbol{V}\to\boldsymbol{Z}$, $\boldsymbol{g}:\boldsymbol{W}\to\boldsymbol{Y}$, and $\boldsymbol{h}:\boldsymbol{X}\to\boldsymbol{Z}$ we have*

$$\begin{aligned}&\boldsymbol{V}\times_{(\boldsymbol{e},\boldsymbol{f}),\boldsymbol{Y}\times\boldsymbol{Z},\boldsymbol{g}\times\boldsymbol{h}}(\boldsymbol{W}\times\boldsymbol{X})\simeq\\&\quad(-1)^{\mathrm{vdim}\,\boldsymbol{Z}(\mathrm{vdim}\,\boldsymbol{Y}+\mathrm{vdim}\,\boldsymbol{W})}(\boldsymbol{V}\times_{\boldsymbol{e},\boldsymbol{Y},\boldsymbol{g}}\boldsymbol{W})\times_{\boldsymbol{f}\circ\boldsymbol{\pi}_{\boldsymbol{V}},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{X}.\end{aligned}$$

By the same method we can also prove the following, the analogue of Fukaya et al. [15, Lem. 8.2.3(1)] for FOOO Kuranishi spaces:

Proposition 11.27. *Suppose $\mathbf{\dot{M}an^c}$ satisfies Assumptions* 3.22, 10.1, 10.13, 10.16, 11.1, *and* 11.5. *Let $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ be w-transverse* 1*-morphisms in $\mathbf{m\dot{K}ur^c}$ with $\partial\boldsymbol{Z}=\emptyset$, so that a fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in $\mathbf{m\dot{K}ur^c_{\boldsymbol{D}}}$ by Theorem* 11.19. *Suppose $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are oriented, so that $\boldsymbol{W}$ is oriented by Theorem* 11.25, *and $\partial\boldsymbol{W},\partial\boldsymbol{X},\partial\boldsymbol{Y},\partial\boldsymbol{Z}$ are oriented by Definition* 10.79. *Then as in* (11.20) *we have a canonical equivalence of oriented m-Kuranishi spaces:*

$$\partial\boldsymbol{W}\simeq\bigl(\partial\boldsymbol{X}\times_{\boldsymbol{g}\circ\boldsymbol{i}_{\boldsymbol{X}},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}\bigr)\amalg(-1)^{\mathrm{vdim}\,\boldsymbol{X}+\mathrm{vdim}\,\boldsymbol{Z}}\bigl(\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}\circ\boldsymbol{i}_{\boldsymbol{Y}}}\partial\boldsymbol{Y}\bigr).$$

11.3 Fibre products in $\mathbf{mKur}, \mathbf{mKur^c_{st}}, \mathbf{mKur^{gc}}, \mathbf{mKur^c}$

We now apply the results of §11.2 when $\dot{\mathbf{Man}}$ is $\mathbf{Man}, \mathbf{Man^c_{st}}, \mathbf{Man^{gc}}$ and $\mathbf{Man^c}$, using the material of §2.5 on transversality and submersions in these categories, and Examples 11.10–11.13 in §11.1.4.

11.3.1 Fibre products in mKur

Take $\dot{\mathbf{Man}}$ to be the category of classical manifolds $\mathbf{Man}$, with corresponding 2-category of m-Kuranishi spaces $\mathbf{mKur}$ as in Definition 4.29. We will use tangent spaces $T_x\boldsymbol{X}$ for $\boldsymbol{X}$ in $\mathbf{mKur}$ defined using ordinary tangent spaces T_vV in $\mathbf{Man}$, as in Example 10.25(i).

Definition 2.21 in §2.5.1 defines transverse morphisms and submersions in $\mathbf{Man}$, as usual in differential geometry. As in Example 11.10, these satisfy Assumption 11.1 with $\boldsymbol{D}, \boldsymbol{E}$ trivial, and Assumptions 11.3–11.5 and 11.9 also hold. So Definition 11.18 defines (w-)transverse 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ and (w-)submersions $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ in $\mathbf{mKur}$, in terms of the existence of covers of $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ by m-Kuranishi neighbourhoods on which we can represent $\boldsymbol{g}, \boldsymbol{h}$ in a special form. The next theorem summarizes Theorems 11.19, 11.22 and 11.25, Proposition 11.21, and Corollaries 11.23 and 11.24 in this case.

Theorem 11.28. **(a)** *Let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms in* $\mathbf{mKur}$. *Then $\boldsymbol{g}, \boldsymbol{h}$ are w-transverse if and only if for all $x \in \boldsymbol{X}$ and $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$, the following is surjective:*

$$O_x\boldsymbol{g} \oplus O_y\boldsymbol{h} : O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y} \longrightarrow O_z\boldsymbol{Z}. \tag{11.27}$$

This is automatic if $\boldsymbol{Z}$ is a manifold. Also $\boldsymbol{g}, \boldsymbol{h}$ are transverse if and only if for all x, y, z, equation (11.27) *is an isomorphism, and the following is surjective:*

$$T_x\boldsymbol{g} \oplus T_y\boldsymbol{h} : T_x\boldsymbol{X} \oplus T_y\boldsymbol{Y} \longrightarrow T_z\boldsymbol{Z}.$$

(b) *If $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are w-transverse in* $\mathbf{mKur}$ *then a fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in* $\mathbf{mKur}$, *in a 2-Cartesian square:*

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow{\quad \boldsymbol{f} \quad} & \boldsymbol{Y} \\ \downarrow \boldsymbol{e} & \boldsymbol{\eta}\Uparrow & \boldsymbol{h}\downarrow \\ \boldsymbol{X} & \xrightarrow{\quad \boldsymbol{g} \quad} & \boldsymbol{Z}. \end{array} \tag{11.28}$$

It has $\mathrm{vdim}\,\boldsymbol{W} = \mathrm{vdim}\,\boldsymbol{X} + \mathrm{vdim}\,\boldsymbol{Y} - \mathrm{vdim}\,\boldsymbol{Z}$, *and topological space* $W = \{(x,y) \in X \times Y : g(x) = h(y)\}$. *If $w \in \boldsymbol{W}$ with $\boldsymbol{e}(w) = x$ in $\boldsymbol{X}$, $\boldsymbol{f}(w) = y$ in $\boldsymbol{Y}$ and $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$, the following is an exact sequence:*

$$\begin{array}{ccccccc} 0 \longrightarrow & T_w\boldsymbol{W} & \xrightarrow{T_w\boldsymbol{e} \oplus T_w\boldsymbol{f}} & T_x\boldsymbol{X} \oplus T_y\boldsymbol{Y} & \xrightarrow{T_x\boldsymbol{g} \oplus -T_y\boldsymbol{h}} & T_z\boldsymbol{Z} & \\ & & & & & \downarrow \delta_w^{\boldsymbol{g},\boldsymbol{h}} & \\ 0 \longleftarrow & O_z\boldsymbol{Z} & \xleftarrow{O_x\boldsymbol{g} \oplus -O_y\boldsymbol{h}} & O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y} & \xleftarrow{O_w\boldsymbol{e} \oplus O_w\boldsymbol{f}} & O_w\boldsymbol{W}. & \end{array} \tag{11.29}$$

If $\boldsymbol{g},\boldsymbol{h}$ are transverse then $\boldsymbol{W}$ is a manifold.

(c) *In part* **(b)**, *using the theory of canonical bundles and orientations from* §10.7, *there is a natural isomorphism of topological line bundles on* W:

$$\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}:K_{\boldsymbol{W}}\longrightarrow e^*(K_{\boldsymbol{X}})\otimes f^*(K_{\boldsymbol{Y}})\otimes(g\circ e)^*(K_{\boldsymbol{Z}})^*. \tag{11.30}$$

Hence if $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are oriented there is a unique orientation on $\boldsymbol{W}$, called the ***fibre product orientation****, such that* (11.30) *is orientation-preserving. Proposition* 11.26 *holds for these fibre product orientations.*

(d) *Let $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ be a* 1*-morphism in* **mKur**. *Then $\boldsymbol{g}$ is a w-submersion if and only if $O_x\boldsymbol{g}:O_x\boldsymbol{X}\to O_z\boldsymbol{Z}$ is surjective for all $x\in\boldsymbol{X}$ with $\boldsymbol{g}(x)=z$ in $\boldsymbol{Z}$. Also $\boldsymbol{g}$ is a submersion if and only if $O_x\boldsymbol{g}:O_x\boldsymbol{X}\to O_z\boldsymbol{Z}$ is an isomorphism and $T_x\boldsymbol{g}:T_x\boldsymbol{X}\to T_z\boldsymbol{Z}$ is surjective for all x,z.*

(e) *If $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ are* 1*-morphisms in* **mKur** *with $\boldsymbol{g}$ a w-submersion then $\boldsymbol{g},\boldsymbol{h}$ are w-transverse. If $\boldsymbol{g}$ is a submersion and $\boldsymbol{Y}$ is a manifold then $\boldsymbol{g},\boldsymbol{h}$ are transverse.*

(f) *If* (11.28) *is* 2*-Cartesian in* **mKur** *with $\boldsymbol{g}$ a w-submersion (or a submersion) then $\boldsymbol{f}$ is a w-submersion (or a submersion).*

(g) *Compositions and products of (w-)submersions in* **mKur** *are (w-)submersions. Projections $\boldsymbol{\pi_X}:\boldsymbol{X}\times\boldsymbol{Y}\to\boldsymbol{X}$ in* **mKur** *are w-submersions.*

Example 11.29. Suppose $\boldsymbol{W}$ is an m-Kuranishi space covered by a single m-Kuranishi neighbourhood (V,E,s,ψ). Then we can write $\boldsymbol{W}$ as a w-transverse fibre product $\boldsymbol{W}\simeq\boldsymbol{V}\times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}}\boldsymbol{V}$ of manifolds in **mKur**, where $\boldsymbol{s},\boldsymbol{0}:\boldsymbol{V}\to\boldsymbol{E}$ are the images of the sections $s,0:V\to E$ under $F_{\mathbf{Man}}^{\mathbf{mKur}}:\mathbf{Man}\hookrightarrow\mathbf{mKur}$.

Example 11.30. Let $W\subseteq\mathbb{R}^n$ be any closed subset. By a lemma of Whitney's, we can write W as the zero set of a smooth function $g:\mathbb{R}^n\to\mathbb{R}$. Let $\boldsymbol{g}:\boldsymbol{\mathbb{R}^n}\to\boldsymbol{\mathbb{R}}$ and $\boldsymbol{0}:\boldsymbol{*}\to\boldsymbol{\mathbb{R}}$ be the images of $g:\mathbb{R}^n\to\mathbb{R}$ and $0:*\to\mathbb{R}$ under $F_{\mathbf{Man}}^{\mathbf{mKur}}:\mathbf{Man}\hookrightarrow\mathbf{mKur}$. Then $\boldsymbol{g},\boldsymbol{0}$ are w-transverse, so $\boldsymbol{W}=\boldsymbol{\mathbb{R}^n}\times_{\boldsymbol{g},\boldsymbol{\mathbb{R}},\boldsymbol{0}}\boldsymbol{*}$ is an m-Kuranishi space in **mKur**, with $\operatorname{vdim}\boldsymbol{W}=n-1$ and topological space W, by Theorem 11.28. This means that the topological spaces of m-Kuranishi spaces can be quite wild, fractals for example.

Example 11.31. Let $g:X\to Z$ and $h:Y\to Z$ be morphisms in **Man**, and $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ be their images under $F_{\mathbf{Man}}^{\mathbf{mKur}}$. Then $\boldsymbol{g},\boldsymbol{h}$ are w-transverse, so a fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in **mKur** by Theorem 11.28. In Example 11.20 we showed that if g,h are transverse in **Man**, so that a fibre product $W=X\times_{g,Z,h}Y$ exists in **Man**, then $\boldsymbol{W}\simeq F_{\mathbf{Man}}^{\mathbf{mKur}}(W)$.

If g,h are not transverse then the morphism $T_x\boldsymbol{g}\oplus-T_y\boldsymbol{h}:T_x\boldsymbol{X}\oplus T_y\boldsymbol{Y}\to T_z\boldsymbol{Z}$ in (11.29) is not surjective for some $w\in\boldsymbol{W}$, and then $O_w\boldsymbol{W}\neq 0$ by (11.29), so $\boldsymbol{W}$ is not a manifold. Hence, if a non-transverse fibre product $W=X\times_{g,Z,h}Y$ exists in **Man**, as in Example 2.23(ii)–(iv), then $\boldsymbol{W}\not\simeq F_{\mathbf{Man}}^{\mathbf{mKur}}(W)$.

11.3.2 Fibre products in $\mathbf{mKur^c_{st}}$ and $\mathbf{mKur^c}$

In §2.5.2, working in the subcategory $\mathbf{Man^c_{st}} \subset \mathbf{Man^c}$ from §2.1, we defined *s-transverse* and *t-transverse morphisms* and *s-submersions*. Example 11.11 explained how to fit these into the framework of Assumptions 11.1 and 11.3–11.9. The next theorem summarizes Theorems 11.19, 11.22 and 11.25, Proposition 11.21, and Corollaries 11.23 and 11.24 applied to Example 11.11. Equation (11.35) being exact is equivalent to (11.17) for the $\tilde N_x \boldsymbol{X}$ being Cartesian in real vector spaces.

Here $\mathbf{mKur^c_{st}} \subset \mathbf{mKur^c}$ are the 2-categories of m-Kuranishi spaces corresponding to $\mathbf{Man^c_{st}} \subset \mathbf{Man^c}$ as in Definition 4.29, the corner 2-functors $C, C' : \mathbf{mKur^c_{st}} \to \mathbf{m\check{K}ur^c_{st}}$ and $C, C' : \mathbf{mKur^c} \to \mathbf{m\check{K}ur^c}$ are as in Example 4.45, (stratum) tangent spaces $T_x\boldsymbol{X}, \tilde T_x\boldsymbol{X}$ are as in Example 10.25(i),(iii), and stratum normal spaces $\tilde N_x\boldsymbol{X}$ are as in Example 10.32(a).

We use the notation *ws-transverse*, *wt-transverse*, and *ws-submersions* for the notions of w-transverse and w-submersion in $\mathbf{mKur^c_{st}}$ corresponding to s- and t-transverse morphisms and s-submersions, and *s-transverse*, *t-transverse*, and *s-submersions* for the corresponding notions of transverse and submersion.

Theorem 11.32. (a) *Let* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *and* $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *be* 1*-morphisms in* $\mathbf{mKur^c_{st}}$*. Then* $\boldsymbol{g}, \boldsymbol{h}$ *are ws-transverse if and only if for all* $x \in \boldsymbol{X}$ *and* $y \in \boldsymbol{Y}$ *with* $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ *in* $\boldsymbol{Z}$*, the following linear maps are surjective:*

$$\tilde O_x\boldsymbol{g} \oplus \tilde O_y\boldsymbol{h} : \tilde O_x\boldsymbol{X} \oplus \tilde O_y\boldsymbol{Y} \longrightarrow \tilde O_z\boldsymbol{Z}, \tag{11.31}$$

$$\tilde N_x\boldsymbol{g} \oplus \tilde N_y\boldsymbol{h} : \tilde N_x\boldsymbol{X} \oplus \tilde N_y\boldsymbol{Y} \longrightarrow \tilde N_z\boldsymbol{Z}. \tag{11.32}$$

This is automatic if $\boldsymbol{Z}$ *is a classical manifold. Also* $\boldsymbol{g}, \boldsymbol{h}$ *are s-transverse if and only if for all* x, y, z*, equation* (11.31) *is an isomorphism, and* (11.32) *and the following are surjective:*

$$\tilde T_x\boldsymbol{g} \oplus \tilde T_y\boldsymbol{h} : \tilde T_x\boldsymbol{X} \oplus \tilde T_y\boldsymbol{Y} \longrightarrow \tilde T_z\boldsymbol{Z}. \tag{11.33}$$

Furthermore, $\boldsymbol{g}, \boldsymbol{h}$ *are wt-transverse (or t-transverse) if and only if they are ws-transverse (or s-transverse), and for all* x, y, z *as above, whenever* $\boldsymbol{x} \in C_j(\boldsymbol{X})$ *and* $\boldsymbol{y} \in C_k(\boldsymbol{Y})$ *with* $\boldsymbol{\Pi}_j(\boldsymbol{x}) = x$, $\boldsymbol{\Pi}_k(\boldsymbol{y}) = y$*, and* $C(\boldsymbol{g})\boldsymbol{x} = C(\boldsymbol{h})\boldsymbol{y} = \boldsymbol{z}$ *in* $C_l(\boldsymbol{Z})$*, we have* $j + k \geqslant l$*, and there is exactly one triple* $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ *with* $j + k = l$*.*

(b) *If* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *and* $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *are ws-transverse in* $\mathbf{mKur^c_{st}}$ *then a fibre product* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *exists in* $\mathbf{mKur^c_{st}}$*, in a* 2*-Cartesian square:*

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow[\qquad\boldsymbol{f}\qquad]{} & \boldsymbol{Y} \\ \downarrow{\scriptstyle \boldsymbol{e}} & \boldsymbol{\eta}\Uparrow & {\scriptstyle \boldsymbol{h}}\downarrow \\ \boldsymbol{X} & \xrightarrow{\qquad\boldsymbol{g}\qquad} & \boldsymbol{Z}. \end{array} \tag{11.34}$$

It has $\mathrm{vdim}\,\boldsymbol{W} = \mathrm{vdim}\,\boldsymbol{X} + \mathrm{vdim}\,\boldsymbol{Y} - \mathrm{vdim}\,\boldsymbol{Z}$*, and topological space* $W = \{(x, y) \in X \times Y : g(x) = h(y)\}$*. Equation* (11.34) *is also* 2*-Cartesian in* $\mathbf{mKur^c}$*.*

If $w \in \boldsymbol{W}$ with $\boldsymbol{e}(w) = x$ in $\boldsymbol{X}$, $\boldsymbol{f}(w) = y$ in $\boldsymbol{Y}$ and $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$, the following sequences are exact:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & T_w\boldsymbol{W} & \xrightarrow[T_w\boldsymbol{e}\oplus T_w\boldsymbol{f}]{} & T_x\boldsymbol{X}\oplus T_y\boldsymbol{Y} & \xrightarrow[T_x\boldsymbol{g}\oplus -T_y\boldsymbol{h}]{} & T_z\boldsymbol{Z} \\
 & & & & & & \big\downarrow{\scriptstyle \delta_w^{\boldsymbol{g},\boldsymbol{h}}} \\
0 & \longleftarrow & O_z\boldsymbol{Z} & \xleftarrow{O_x\boldsymbol{g}\oplus -O_y\boldsymbol{h}} & O_x\boldsymbol{X}\oplus O_y\boldsymbol{Y} & \xleftarrow{O_w\boldsymbol{e}\oplus O_w\boldsymbol{f}} & O_w\boldsymbol{W},
\end{array}
$$

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \tilde T_w\boldsymbol{W} & \xrightarrow[\tilde T_w\boldsymbol{e}\oplus \tilde T_w\boldsymbol{f}]{} & \tilde T_x\boldsymbol{X}\oplus \tilde T_y\boldsymbol{Y} & \xrightarrow[\tilde T_x\boldsymbol{g}\oplus -\tilde T_y\boldsymbol{h}]{} & \tilde T_z\boldsymbol{Z} \\
 & & & & & & \big\downarrow{\scriptstyle \tilde\delta_w^{\boldsymbol{g},\boldsymbol{h}}} \\
0 & \longleftarrow & \tilde O_z\boldsymbol{Z} & \xleftarrow{\tilde O_x\boldsymbol{g}\oplus -\tilde O_y\boldsymbol{h}} & \tilde O_x\boldsymbol{X}\oplus \tilde O_y\boldsymbol{Y} & \xleftarrow{\tilde O_w\boldsymbol{e}\oplus \tilde O_w\boldsymbol{f}} & \tilde O_w\boldsymbol{W},
\end{array}
$$

$$
0 \longrightarrow \tilde N_w\boldsymbol{W} \xrightarrow{\tilde N_w\boldsymbol{e}\oplus \tilde N_w\boldsymbol{f}} \tilde N_x\boldsymbol{X}\oplus \tilde N_y\boldsymbol{Y} \xrightarrow{\tilde N_x\boldsymbol{g}\oplus -\tilde N_y\boldsymbol{h}} \tilde N_z\boldsymbol{Z} \longrightarrow 0. \tag{11.35}
$$

If $\boldsymbol{g}, \boldsymbol{h}$ are s-transverse then $\boldsymbol{W}$ is a manifold.

(c) *In part* **(b)**, *if* (11.34) *is* 2*-Cartesian in* $\mathbf{mKur^c_{st}}$ *with $\boldsymbol{g}, \boldsymbol{h}$ wt-transverse (or t-transverse), then the following is* 2*-Cartesian in* $\mathbf{m\check{K}ur^c_{st}}$ *and* $\mathbf{m\check{K}ur^c}$, *with $C(\boldsymbol{g}), C(\boldsymbol{h})$ wt-transverse (or t-transverse, respectively):*

$$
\begin{array}{ccc}
C(\boldsymbol{W}) & \xrightarrow[C(\boldsymbol{f})]{} & C(\boldsymbol{Y}) \\
\big\downarrow{\scriptstyle C(\boldsymbol{e})} & C(\boldsymbol{\eta})\Uparrow & {\scriptstyle C(\boldsymbol{h})}\big\downarrow \\
C(\boldsymbol{X}) & \xrightarrow{C(\boldsymbol{g})} & C(\boldsymbol{Z}).
\end{array}
$$

Hence we have

$$
C_i(\boldsymbol{W}) \simeq \coprod_{\substack{j,k,l\geqslant 0:\\ i=j+k-l}} \begin{array}{l} \big(C_j(\boldsymbol{X}) \cap C(\boldsymbol{g})^{-1}(C_l(\boldsymbol{Z}))\big) \times_{C(\boldsymbol{g}),C_l(\boldsymbol{Z}),C(\boldsymbol{h})} \\ \qquad\qquad \big(C_k(\boldsymbol{Y}) \cap C(\boldsymbol{h})^{-1}(C_l(\boldsymbol{Z}))\big) \end{array}
$$

for $i \geqslant 0$. When $i = 1$, this computes the boundary $\partial\boldsymbol{W}$.

Also, if $\boldsymbol{g}$ is a ws-submersion (or an s-submersion), then $C(\boldsymbol{g})$ is a ws-submersion (or an s-submersion, respectively).

The analogue of the above also holds for $C' : \mathbf{mKur^c_{st}} \to \mathbf{m\check{K}ur^c_{st}}$.

(d) *In part* **(b)**, *using the theory of canonical bundles and orientations from* §10.7, *there is a natural isomorphism of topological line bundles on W:*

$$
\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}} : K_{\boldsymbol{W}} \longrightarrow e^*(K_{\boldsymbol{X}}) \otimes f^*(K_{\boldsymbol{Y}}) \otimes (g \circ e)^*(K_{\boldsymbol{Z}})^*. \tag{11.36}
$$

*Hence if $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ are oriented there is a unique orientation on $\boldsymbol{W}$, called the **fibre product orientation**, such that* (11.36) *is orientation-preserving. Propositions* 11.26 *and* 11.27 *hold for these fibre product orientations.*

(e) *Let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ be a* 1*-morphism in* $\mathbf{mKur^c_{st}}$. *Then $\boldsymbol{g}$ is a ws-submersion if and only if $\tilde O_x\boldsymbol{g} : \tilde O_x\boldsymbol{X} \to \tilde O_z\boldsymbol{Z}$ and $\tilde N_x\boldsymbol{g} : \tilde N_x\boldsymbol{X} \to \tilde N_z\boldsymbol{Z}$ are surjective for all $x \in \boldsymbol{X}$ with $\boldsymbol{g}(x) = z$ in $\boldsymbol{Z}$. Also $\boldsymbol{g}$ is an s-submersion if and only if $\tilde O_x\boldsymbol{g} : \tilde O_x\boldsymbol{X} \to \tilde O_z\boldsymbol{Z}$ is an isomorphism and $\tilde T_x\boldsymbol{g} : \tilde T_x\boldsymbol{X} \to \tilde T_z\boldsymbol{Z}$, $\tilde N_x\boldsymbol{g} : \tilde N_x\boldsymbol{X} \to \tilde N_z\boldsymbol{Z}$ are surjective for all x, z.*

(f) *If* $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ *are* 1*-morphisms in* $\mathbf{mKur^c_{st}}$ *with* $\boldsymbol{g}$ *a ws-submersion then* $\boldsymbol{g},\boldsymbol{h}$ *are ws-transverse and wt-transverse. If* $\boldsymbol{g}$ *is an s-submersion and* $\boldsymbol{Y}$ *is a manifold then* $\boldsymbol{g},\boldsymbol{h}$ *are s-transverse and t-transverse.*

(g) *If* (11.34) *is* 2*-Cartesian in* $\mathbf{mKur^c_{st}}$ *with* $\boldsymbol{g}$ *a ws-submersion (or an s-submersion) then* $\boldsymbol{f}$ *is a ws-submersion (or an s-submersion).*

(h) *Compositions and products of ws- or s-submersions in* $\mathbf{mKur^c_{st}}$ *are ws- or s-submersions. Projections* $\boldsymbol{\pi_X}:\boldsymbol{X}\times\boldsymbol{Y}\rightarrow\boldsymbol{X}$ *in* $\mathbf{mKur^c_{st}}$ *are ws-submersions.*

(i) *If* $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$ *is a ws-submersion in* $\mathbf{mKur^c_{st}}$*, and* $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ *is any* 1*-morphism in* $\mathbf{mKur^c}$ *(not necessarily in* $\mathbf{mKur^c_{st}}$*), then a fibre product* $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ *exists in* $\mathbf{mKur^c}$*, with* $\dim\boldsymbol{W}=\dim\boldsymbol{X}+\dim\boldsymbol{Y}-\dim\boldsymbol{Z}$*, in a* 2*-Cartesian square* (11.34) *in* $\mathbf{mKur^c}$*. It has topological space* $W=\{(x,y)\in X\times Y:g(x)=h(y)\}$*. The analogues of* **(c)**,**(g)** *hold for these fibre products. If* $\boldsymbol{g}$ *is an s-submersion and* $\boldsymbol{Y}$ *is a manifold then* $\boldsymbol{W}$ *is a manifold.*

Example 11.33. Define $X=Y=Z=[0,\infty)$ and $Z'=\mathbb{R}$, so that $Z\subset Z'$ is open. Define strongly smooth maps $g:X\rightarrow Z$, $h:Y\rightarrow Z$, $g':X\rightarrow Z'$ and $h':Y\rightarrow Z'$ by $g(x)=g'(x)=x$, $h(y)=h'(y)=y$. Let $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{Z}',\boldsymbol{g},\boldsymbol{h},\boldsymbol{g}',\boldsymbol{h}'$ be the images of X,Y,Z,Z',g,h,g',h' under $F_{\mathbf{Man^c_{st}}}^{\mathbf{mKur^c_{st}}}$.

Then $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$ are s-transverse. Also $\boldsymbol{g}':\boldsymbol{X}\rightarrow\boldsymbol{Z}'$, $\boldsymbol{h}':\boldsymbol{X}\rightarrow\boldsymbol{Z}'$ are ws-transverse, but are not s-transverse, as (11.33) for $\boldsymbol{g}',\boldsymbol{h}'$ is not surjective at $x=y=z=0$. Hence fibre products $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ and $\boldsymbol{W}'=\boldsymbol{X}\times_{\boldsymbol{g}',\boldsymbol{Z}',\boldsymbol{h}'}\boldsymbol{Y}$ exist in $\mathbf{mKur^c_{st}}$. Here $\boldsymbol{W}$ is $F_{\mathbf{Man^c_{st}}}^{\mathbf{mKur^c_{st}}}([0,\infty))$, but $\boldsymbol{W}'$ is not a manifold. We may cover $\boldsymbol{W}'$ by an m-Kuranishi neighbourhood (V,E,s,ψ), where $V=[0,\infty)^2$, and $E=[0,\infty)^2\times\mathbb{R}$ is the trivial vector bundle over V with fibre $\mathbb{R}$, and $s:V\rightarrow E$ maps $(x,y)\mapsto(x,y,x-y)$, and $\psi:(x,x)\mapsto x$.

Since $\boldsymbol{W}\not\simeq\boldsymbol{W}'$, this shows that the corners of $\boldsymbol{Z}$ can affect the fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\mathbf{mKur^c_{st}}$. This is not true for fibre products in $\mathbf{Man^c_{st}}$, where we have $X\times_{g,Z,h}Y\cong X\times_{g',Z',h'}Y$ when $Z\subset Z'$ and $g=g'$, $h=h'$.

11.3.3 Fibre products in $\mathbf{mKur^{gc}_{in}}$ and $\mathbf{mKur^{gc}}$

In §2.5.3, working in the subcategory $\mathbf{Man^{gc}_{in}}\subset\mathbf{Man^{gc}}$ from §2.4.1, we defined *b-transverse* and *c-transverse morphisms* and *b-submersions*, *b-fibrations*, and *c-fibrations*. Example 11.12 explained how to fit these into the framework of Assumptions 11.1 and 11.3–11.9. The next theorem summarizes Theorems 11.19, 11.22 and 11.25, Proposition 11.21, and Corollary 11.23 applied to Example 11.12.

Here $\mathbf{mKur^{gc}_{in}}\subset\mathbf{mKur^{gc}}$ are the 2-categories of m-Kuranishi spaces corresponding to $\mathbf{Man^{gc}_{in}}\subset\mathbf{Man^{gc}}$ as in Definition 4.29, the corner functor $C:\mathbf{mKur^{gc}}\rightarrow\mathbf{mKur^{gc}}$ is as in Example 4.45, and b-tangent spaces $T_x\boldsymbol{X}$ are as in Example 10.25(ii). We use the notation *wb-transverse*, *wc-transverse*, *wb-submersions*, *wb-fibrations*, *wc-fibrations* for the weak versions of b-transverse, …, c-fibrations in $\mathbf{mKur^{gc}_{in}}$ from Definition 11.18, and *b-transverse*, *c-transverse*, *b-submersions*, *b-fibrations*, and *c-fibrations* for the strong versions.

Theorem 11.34. **(a)** *Let* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *be* 1*-morphisms in* $\mathbf{mKur^{gc}_{in}}$*. Then* $\boldsymbol{g},\boldsymbol{h}$ *are wb-transverse if and only if for all* $x\in\boldsymbol{X}$ *and* $y\in\boldsymbol{Y}$ *with* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$*, the following linear map is surjective:*

$$ {}^bO_x\boldsymbol{g}\oplus {}^bO_y\boldsymbol{h}: {}^bO_x\boldsymbol{X}\oplus {}^bO_y\boldsymbol{Y}\longrightarrow {}^bO_z\boldsymbol{Z}. \tag{11.37}$$

This is automatic if $\boldsymbol{Z}$ *is a manifold. Also* $\boldsymbol{g},\boldsymbol{h}$ *are b-transverse if and only if for all* x,y,z*, equation* (11.37) *is an isomorphism, and the following is surjective:*

$$ {}^bT_x\boldsymbol{g}\oplus {}^bT_y\boldsymbol{h}: {}^bT_x\boldsymbol{X}\oplus {}^bT_y\boldsymbol{Y}\longrightarrow {}^bT_z\boldsymbol{Z}.$$

Furthermore, $\boldsymbol{g},\boldsymbol{h}$ *are wc-transverse (or c-transverse) if and only if they are wb-transverse (or b-transverse), and whenever* $\boldsymbol{x}\in C_j(\boldsymbol{X})$ *and* $\boldsymbol{y}\in C_k(\boldsymbol{Y})$ *with* $C(\boldsymbol{g})\boldsymbol{x}=C(\boldsymbol{h})\boldsymbol{y}=\boldsymbol{z}$ *in* $C_l(\boldsymbol{Z})$*, we have either* $j+k>l$*, or* $j=k=l=0$*.*

(b) *If* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *are wb-transverse in* $\mathbf{mKur^{gc}_{in}}$ *then a fibre product* $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ *exists in* $\mathbf{mKur^{gc}_{in}}$*, in a* 2*-Cartesian square:*

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow{\quad\boldsymbol{f}\quad} & \boldsymbol{Y} \\ \downarrow{\scriptstyle\boldsymbol{e}} & \boldsymbol{\eta}\Uparrow & {\scriptstyle\boldsymbol{h}}\downarrow \\ \boldsymbol{X} & \xrightarrow{\quad\boldsymbol{g}\quad} & \boldsymbol{Z}. \end{array} \tag{11.38}$$

It has $\operatorname{vdim}\boldsymbol{W}=\operatorname{vdim}\boldsymbol{X}+\operatorname{vdim}\boldsymbol{Y}-\operatorname{vdim}\boldsymbol{Z}$*. If* $w\in\boldsymbol{W}$ *with* $\boldsymbol{e}(w)=x$ *in* $\boldsymbol{X}$*,* $\boldsymbol{f}(w)=y$ *in* $\boldsymbol{Y}$ *and* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$*, the following sequence is exact:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}^bT_w\boldsymbol{W} & \xrightarrow{{}^bT_w\boldsymbol{e}\oplus {}^bT_w\boldsymbol{f}} & {}^bT_x\boldsymbol{X}\oplus {}^bT_y\boldsymbol{Y} & \xrightarrow{{}^bT_x\boldsymbol{g}\oplus - {}^bT_y\boldsymbol{h}} & {}^bT_z\boldsymbol{Z} \\ & & & & & & \downarrow {\scriptstyle {}^b\delta_w^{\boldsymbol{g},\boldsymbol{h}}} \\ 0 & \longleftarrow & {}^bO_z\boldsymbol{Z} & \xleftarrow{{}^bO_x\boldsymbol{g}\oplus - {}^bO_y\boldsymbol{h}} & {}^bO_x\boldsymbol{X}\oplus {}^bO_y\boldsymbol{Y} & \xleftarrow{{}^bO_w\boldsymbol{e}\oplus {}^bO_w\boldsymbol{f}} & {}^bO_w\boldsymbol{W}. \end{array}$$

If $\boldsymbol{g},\boldsymbol{h}$ *are b-transverse then* $\boldsymbol{W}$ *is a manifold.*

(c) *In* **(b)***, if* $\boldsymbol{g},\boldsymbol{h}$ *are wc-transverse then* $\boldsymbol{W}$ *has topological space* $W=\{(x,y)\in X\times Y: g(x)=h(y)\}$*, and* (11.38) *is also* 2*-Cartesian in* $\mathbf{mKur^{gc}}$*, and the following is* 2*-Cartesian in* $\mathbf{m\check{K}ur^{gc}_{in}}$ *and* $\mathbf{m\check{K}ur^{gc}}$*, with* $C(\boldsymbol{g}),C(\boldsymbol{h})$ *wc-transverse:*

$$\begin{array}{ccc} C(\boldsymbol{W}) & \xrightarrow{\quad C(\boldsymbol{f})\quad} & C(\boldsymbol{Y}) \\ \downarrow{\scriptstyle C(\boldsymbol{e})} & C(\boldsymbol{\eta})\Uparrow & {\scriptstyle C(\boldsymbol{h})}\downarrow \\ C(\boldsymbol{X}) & \xrightarrow{\quad C(\boldsymbol{g})\quad} & C(\boldsymbol{Z}). \end{array}$$

Hence we have

$$C_i(\boldsymbol{W})\simeq \coprod_{\substack{j,k,l\geqslant 0:\\ i=j+k-l}} \bigl(C_j(\boldsymbol{X})\cap C(\boldsymbol{g})^{-1}(C_l(\boldsymbol{Z}))\bigr)\times_{C(\boldsymbol{g}),C_l(\boldsymbol{Z}),C(\boldsymbol{h})} \bigl(C_k(\boldsymbol{Y})\cap C(\boldsymbol{h})^{-1}(C_l(\boldsymbol{Z}))\bigr)$$

for $i\geqslant 0$*. When* $i=1$*, this computes the boundary* $\partial\boldsymbol{W}$*.*

Also, if $\boldsymbol{g}$ *is a wb-fibration, or b-fibration, or wc-fibration, or c-fibration, then* $C(\boldsymbol{g})$ *is a wb-fibration, ..., or c-fibration, respectively.*

(d) *In part* **(b)**, *using the theory of b-canonical bundles and orientations from* §10.7, *there is a natural isomorphism of topological line bundles on W:*

$$ {}^b\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}} : {}^bK_{\boldsymbol{W}} \longrightarrow e^*({}^bK_{\boldsymbol{X}}) \otimes f^*({}^bK_{\boldsymbol{Y}}) \otimes (g\circ e)^*({}^bK_{\boldsymbol{Z}})^*. \tag{11.39} $$

*Hence if $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are oriented there is a unique orientation on $\boldsymbol{W}$, called the **fibre product orientation**, such that* (11.39) *is orientation-preserving. Propositions* 11.26 *and* 11.27 *hold for these fibre product orientations.*

(e) *Let $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ be a* 1*-morphism in* $\mathbf{mKur^{gc}_{in}}$*. Then $\boldsymbol{g}$ is a wb-submersion if and only if ${}^bO_x\boldsymbol{g}:{}^bO_x\boldsymbol{X}\to{}^bO_z\boldsymbol{Z}$ is surjective for all $x\in\boldsymbol{X}$ with $\boldsymbol{g}(x)=z$ in $\boldsymbol{Z}$. Also $\boldsymbol{g}$ is a b-submersion if and only if ${}^bO_x\boldsymbol{g}:{}^bO_x\boldsymbol{X}\to{}^bO_z\boldsymbol{Z}$ is an isomorphism and ${}^bT_x\boldsymbol{g}:{}^bT_x\boldsymbol{X}\to{}^bT_z\boldsymbol{Z}$ is surjective for all x,z.*

Furthermore $\boldsymbol{g}$ is a wb-fibration (or a b-fibration) if it is a wb-submersion (or b-submersion) and whenever there are $\boldsymbol{x},\boldsymbol{z}$ in $C_j(\boldsymbol{X}),C_l(\boldsymbol{Z})$ with $C(\boldsymbol{g})\boldsymbol{x}=\boldsymbol{z}$, we have $j\geqslant l$. And $\boldsymbol{g}$ is a wc-fibration (or a c-fibration) if it is a wb-fibration (or a b-fibration), and whenever $x\in\boldsymbol{X}$ and $\boldsymbol{z}\in C_l(\boldsymbol{Z})$ with $\boldsymbol{g}(x)=\boldsymbol{\Pi}_l(\boldsymbol{z})=z\in\boldsymbol{Z}$, then there is exactly one $\boldsymbol{x}\in C_l(\boldsymbol{X})$ with $\boldsymbol{\Pi}_l(\boldsymbol{x})=x$ and $C(\boldsymbol{g})\boldsymbol{x}=\boldsymbol{z}$.

(f) *If $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ are* 1*-morphisms in* $\mathbf{mKur^{gc}_{in}}$ *with $\boldsymbol{g}$ a wb-submersion (or wb-fibration) then $\boldsymbol{g},\boldsymbol{h}$ are wb-transverse (or wc-transverse, respectively). If $\boldsymbol{g}$ is a b-submersion (or b-fibration) and $\boldsymbol{Y}$ is a manifold then $\boldsymbol{g},\boldsymbol{h}$ are b-transverse (or c-transverse, respectively).*

(g) *If* (11.38) *is* 2*-Cartesian in* $\mathbf{mKur^{gc}_{in}}$ *with $\boldsymbol{g}$ a wb-submersion, b-submersion, wb-fibration, b-fibration, wc-fibration, or c-fibration, then $\boldsymbol{f}$ is a wb-submersion, ..., or c-fibration, respectively.*

(h) *Compositions and products of wb-submersions, b-submersions, wb-fibrations, b-fibrations, wc-fibrations, and c-fibrations, in* $\mathbf{mKur^{gc}_{in}}$ *are wb-submersions, ..., c-fibrations. Projections $\boldsymbol{\pi_X}:\boldsymbol{X}\times\boldsymbol{Y}\to\boldsymbol{X}$ in* $\mathbf{mKur^{gc}_{in}}$ *are wc-fibrations.*

(i) *If $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ is a wc-fibration in* $\mathbf{mKur^{gc}_{in}}$*, and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ is any* 1*-morphism in* $\mathbf{mKur^{gc}}$ *(not necessarily in* $\mathbf{mKur^{gc}_{in}}$*), then a fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in* $\mathbf{mKur^{gc}}$*, with* $\dim\boldsymbol{W}=\dim\boldsymbol{X}+\dim\boldsymbol{Y}-\dim\boldsymbol{Z}$*, in a* 2*-Cartesian square* (11.38) *in* $\mathbf{mKur^{gc}}$*. It has topological space $W=\{(x,y)\in X\times Y: g(x)=h(y)\}$. The analogues of* **(c)**,**(g)** *hold for these fibre products. If $\boldsymbol{g}$ is a c-fibration and $\boldsymbol{Y}$ is a manifold then $\boldsymbol{W}$ is a manifold.*

11.3.4 Fibre products in $\mathbf{mKur^{c}_{in}}$ and $\mathbf{mKur^{c}}$

In §2.5.4, working in the subcategory $\mathbf{Man^c_{in}}\subset\mathbf{Man^c}$ from §2.1, we defined *sb-transverse* and *sc-transverse morphisms.* Example 11.13 explained how to fit these into the framework of Assumptions 11.1 and 11.3–11.9, also using *s-submersions* from §2.5.2. The next theorem summarizes Theorems 11.19, 11.22 and 11.25 and Corollary 11.24 applied to Example 11.13.

Here $\mathbf{mKur^c_{in}}\subset\mathbf{mKur^c}$ are the 2-categories of m-Kuranishi spaces corresponding to $\mathbf{Man^c_{in}}\subset\mathbf{Man^c}$ as in Definition 4.29, the corner 2-functor $C:\mathbf{mKur^c}\to\mathbf{m\check{K}ur^c}$ is as in Example 4.45, b-tangent spaces ${}^bT_x\boldsymbol{X}$ are as in Example 10.25(ii), and monoids $\tilde M_x\boldsymbol{X}$ are as in Example 10.32(c).

We use the notation *wsb-transverse* and *wsc-transverse* for the notions of w-transverse in $\mathbf{mKur^c_{in}}$ corresponding to sb- and sc-transverse morphisms, and *sb-transverse*, *sc-transverse* for the notions of transverse. We omit some of the results on ws- and s-submersions, as they appeared already in Theorem 11.32.

Theorem 11.35. **(a)** *Let* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *be* 1*-morphisms in* $\mathbf{mKur^c_{in}}$. *Then* $\boldsymbol{g},\boldsymbol{h}$ *are wsb-transverse if and only if for all* $x\in\boldsymbol{X}$ *and* $y\in\boldsymbol{Y}$ *with* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$, *the following linear map is surjective:*

$$ {}^bO_x\boldsymbol{g}\oplus{}^bO_y\boldsymbol{h}:{}^bO_x\boldsymbol{X}\oplus{}^bO_y\boldsymbol{Y}\longrightarrow{}^bO_z\boldsymbol{Z}, \tag{11.40}$$

and we have an isomorphism of commutative monoids

$$ \tilde M_x\boldsymbol{X}\times_{\tilde M_x\boldsymbol{g},\tilde M_z\boldsymbol{Z},\tilde M_y\boldsymbol{h}}\tilde M_y\boldsymbol{Y}\cong\mathbb{N}^n\qquad\text{for } n\geqslant 0. \tag{11.41}$$

This is automatic if $\boldsymbol{Z}$ *is a classical manifold. Also* $\boldsymbol{g},\boldsymbol{h}$ *are sb-transverse if and only if for all* x,y,z, *equations* (11.40)–(11.41) *are isomorphisms, and the following is surjective:*

$$ {}^bT_x\boldsymbol{g}\oplus{}^bT_y\boldsymbol{h}:{}^bT_x\boldsymbol{X}\oplus{}^bT_y\boldsymbol{Y}\longrightarrow{}^bT_z\boldsymbol{Z}.$$

Furthermore, $\boldsymbol{g},\boldsymbol{h}$ *are wsc-transverse (or sc-transverse) if and only if they are wsb-transverse (or sb-transverse), and whenever* $\boldsymbol{x}\in C_j(\boldsymbol{X})$ *and* $\boldsymbol{y}\in C_k(\boldsymbol{Y})$ *with* $C(\boldsymbol{g})\boldsymbol{x}=C(\boldsymbol{h})\boldsymbol{y}=\boldsymbol{z}$ *in* $C_l(\boldsymbol{Z})$, *we have either* $j+k>l$, *or* $j=k=l=0$.

(b) *If* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *are wsb-transverse in* $\mathbf{mKur^c_{in}}$ *then a fibre product* $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ *exists in* $\mathbf{mKur^c_{in}}$, *in a* 2*-Cartesian square:*

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow{\quad\boldsymbol{f}\quad} & \boldsymbol{Y} \\ \downarrow{\scriptstyle\boldsymbol{e}} & \boldsymbol{\eta}\Uparrow & \boldsymbol{h}\downarrow \\ \boldsymbol{X} & \xrightarrow{\quad\boldsymbol{g}\quad} & \boldsymbol{Z}. \end{array} \tag{11.42}$$

It has $\mathrm{vdim}\,\boldsymbol{W}=\mathrm{vdim}\,\boldsymbol{X}+\mathrm{vdim}\,\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{Z}$. *If* $w\in\boldsymbol{W}$ *with* $\boldsymbol{e}(w)=x$ *in* $\boldsymbol{X}$, $\boldsymbol{f}(w)=y$ *in* $\boldsymbol{Y}$ *and* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$, *the following sequence is exact:*

$$\begin{array}{ccccccc} 0 \longrightarrow {}^bT_w\boldsymbol{W} & \xrightarrow{{}^bT_w\boldsymbol{e}\oplus{}^bT_w\boldsymbol{f}} & {}^bT_x\boldsymbol{X}\oplus{}^bT_y\boldsymbol{Y} & \xrightarrow{{}^bT_x\boldsymbol{g}\oplus-{}^bT_y\boldsymbol{h}} & {}^bT_z\boldsymbol{Z} \\ & & & & \downarrow{\scriptstyle {}^b\delta_w^{\boldsymbol{g},\boldsymbol{h}}} \\ 0 \longleftarrow {}^bO_z\boldsymbol{Z} & \xleftarrow{{}^bO_x\boldsymbol{g}\oplus-{}^bO_y\boldsymbol{h}} & {}^bO_x\boldsymbol{X}\oplus{}^bO_y\boldsymbol{Y} & \xleftarrow{{}^bO_w\boldsymbol{e}\oplus{}^bO_w\boldsymbol{f}} & {}^bO_w\boldsymbol{W}. \end{array}$$

If $\boldsymbol{g},\boldsymbol{h}$ *are sb-transverse then* $\boldsymbol{W}$ *is a manifold.*

(c) *In* **(b)**, *if* $\boldsymbol{g},\boldsymbol{h}$ *are wsc-transverse then* $\boldsymbol{W}$ *has topological space* $W=\{(x,y)\in X\times Y: g(x)=h(y)\}$, *and* (11.42) *is also* 2*-Cartesian in* $\mathbf{mKur^c}$, *and the following is* 2*-Cartesian in* $\mathbf{m\check{K}ur^c_{in}}$ *and* $\mathbf{m\check{K}ur^c}$, *with* $C(\boldsymbol{g}),C(\boldsymbol{h})$ *wsc-transverse:*

$$\begin{array}{ccc} C(\boldsymbol{W}) & \xrightarrow{\quad C(\boldsymbol{f})\quad} & C(\boldsymbol{Y}) \\ \downarrow{\scriptstyle C(\boldsymbol{e})} & C(\boldsymbol{\eta})\Uparrow & C(\boldsymbol{h})\downarrow \\ C(\boldsymbol{X}) & \xrightarrow{\quad C(\boldsymbol{g})\quad} & C(\boldsymbol{Z}). \end{array}$$

Hence we have

$$C_i(\boldsymbol{W}) \simeq \coprod_{\substack{j,k,l\geqslant 0:\\ i=j+k-l}} \bigl(C_j(\boldsymbol{X})\cap C(\boldsymbol{g})^{-1}(C_l(\boldsymbol{Z}))\bigr)\times_{C(\boldsymbol{g}),C_l(\boldsymbol{Z}),C(\boldsymbol{h})} \bigl(C_k(\boldsymbol{Y})\cap C(\boldsymbol{h})^{-1}(C_l(\boldsymbol{Z}))\bigr)$$

for $i \geqslant 0$. When $i=1$, this computes the boundary $\partial\boldsymbol{W}$.

Also, if $\boldsymbol{g}$ is a ws-submersion (or an s-submersion), then $C(\boldsymbol{g})$ is a ws-submersion (or an s-submersion, respectively).

(d) *In part* **(b)**, *using the theory of b-canonical bundles and orientations from §10.7, there is a natural isomorphism of topological line bundles on W:*

$$^b\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}} : {}^bK_{\boldsymbol{W}} \longrightarrow e^*({}^bK_{\boldsymbol{X}})\otimes f^*({}^bK_{\boldsymbol{Y}})\otimes (g\circ e)^*({}^bK_{\boldsymbol{Z}})^*. \tag{11.43}$$

Hence if $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are oriented there is a unique orientation on $\boldsymbol{W}$, called the ***fibre product orientation****, such that* (11.43) *is orientation-preserving.*

(e) *Let $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ be a* 1*-morphism in* $\mathbf{mKur^c_{in}}$. *Then $\boldsymbol{g}$ is a ws-submersion if and only if ${}^bO_x\boldsymbol{g} : {}^bO_x\boldsymbol{X}\to {}^bO_z\boldsymbol{Z}$ is surjective for all $x\in\boldsymbol{X}$ with $\boldsymbol{g}(x)=z$ in $\boldsymbol{Z}$, and the monoid morphism $\tilde M_x\boldsymbol{g} : \tilde M_x\boldsymbol{X}\to\tilde M_z\boldsymbol{Z}$ is isomorphic to a projection $\mathbb{N}^{m+n}\to\mathbb{N}^n$. Also $\boldsymbol{g}$ is an s-submersion if and only if ${}^bO_x\boldsymbol{g} : {}^bO_x\boldsymbol{X}\to {}^bO_z\boldsymbol{Z}$ is an isomorphism, and ${}^bT_x\boldsymbol{g} : {}^bT_x\boldsymbol{X}\to {}^bT_z\boldsymbol{Z}$ is surjective, and $\tilde M_x\boldsymbol{g}$ is isomorphic to a projection $\mathbb{N}^{m+n}\to\mathbb{N}^n$, for all x,z.*

(f) *If $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ are* 1*-morphisms in* $\mathbf{mKur^{gc}_{in}}$ *with $\boldsymbol{g}$ a ws-submersion then $\boldsymbol{g},\boldsymbol{h}$ are wsc-transverse. If $\boldsymbol{g}$ is an s-submersion and $\boldsymbol{Y}$ is a manifold then $\boldsymbol{g},\boldsymbol{h}$ are sc-transverse.*

11.4 Discussion of fibre products of μ-Kuranishi spaces

We now consider to what extent the results of §11.2–§11.3 may be extended to categories of μ-Kuranishi spaces $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ in Chapter 5. First consider an example:

Example 11.36. Let $X=Y=*$ be the point in **Man**, and $Z=\mathbb{R}^n$ for $n>0$, and $g:X\to Z$, $h:Y\to Z$ map $g:*\mapsto 0$ and $h:*\mapsto 0$. Then g,h are not transverse in **Man**, but a fibre product $W=X\times_{g,Z,h}Y$ exists in **Man**, with $W=*$. Note that $\dim W>\dim X+\dim Y-\dim Z$.

Write $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{g},\boldsymbol{h}$ for the images of X,Y,Z,g,h either in m-Kuranishi spaces **mKur** under $F_{\mathbf{Man}}^{\mathbf{mKur}}:\mathbf{Man}\to\mathbf{mKur}$ from Example 4.30, or in μ-Kuranishi spaces $\boldsymbol{\mu}\mathbf{Kur}$ under $F_{\mathbf{Man}}^{\mathbf{mKur}}:\mathbf{Man}\to\boldsymbol{\mu}\mathbf{Kur}$ from Example 5.16.

Then $\boldsymbol{g},\boldsymbol{h}$ are w-transverse in **mKur**, so a fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in the 2-category **mKur**, with $\operatorname{vdim}\boldsymbol{W}=-n$. It is a point with obstruction space $\mathbb{R}^n$, covered by an m-Kuranishi neighbourhood $(*,\mathbb{R}^n,0,\mathrm{id}_*)$.

As $\boldsymbol{X}=\boldsymbol{Y}=\boldsymbol{*}$ are the terminal object in the ordinary category $\boldsymbol{\mu}\mathbf{Kur}$, a fibre product $\tilde{\boldsymbol{W}}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ also exists in $\boldsymbol{\mu}\mathbf{Kur}$, but it is the point $\boldsymbol{*}$, as in **Man**, with $\operatorname{vdim}\tilde{\boldsymbol{W}}=0$, so $\operatorname{vdim}\tilde{\boldsymbol{W}}>\operatorname{vdim}\boldsymbol{X}+\operatorname{vdim}\boldsymbol{Y}-\operatorname{vdim}\boldsymbol{Z}$.

In this example, the fibre product $\tilde{\boldsymbol{W}}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\boldsymbol{\mu}\mathbf{Kur}$ is 'wrong', not the fibre product we want – it does not have the expected dimension, and is not locally described in μ-Kuranishi neighbourhoods by Definition 11.16.

As in Theorem 5.23 we have an equivalence $\mathrm{Ho}(\mathbf{mKur}) \simeq \boldsymbol{\mu}\mathbf{Kur}$. The moral is that the 2-category structure in $\mathbf{mKur}$ is crucial to get the 'correct' w-transverse fibre products, as the definition of 2-category fibre products in §A.4 involves the 2-morphisms in an essential way. Passing to the homotopy category $\mathrm{Ho}(\mathbf{mKur})$, or to $\boldsymbol{\mu}\mathbf{Kur}$, forgetting 2-morphisms, loses too much information for (w-)transverse fibre products to be well-behaved.

Our conclusion is that we should not study (w-)transverse fibre products in categories $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$, but we should work in the 2-categories $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ or $\dot{\mathbf{K}}\mathbf{ur}$ instead.

Despite this, there is nevertheless a sense in which well-behaved 'w-transverse fibre products' do exist in categories of μ-Kuranishi spaces $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$:

Definition 11.37. Suppose $\dot{\mathbf{M}}\mathbf{an}$ satisfies Assumptions 3.1–3.7 and 11.1, giving discrete properties $\boldsymbol{D},\boldsymbol{E}$ and notions of transverse morphisms and submersions. Let $\boldsymbol{g}':\boldsymbol{X}'\to\boldsymbol{Z}'$, $\boldsymbol{h}':\boldsymbol{Y}'\to\boldsymbol{Z}'$ be $\boldsymbol{D}$ morphisms in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$. As in §5.6.4 we can choose $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ with $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}(\boldsymbol{X})=\boldsymbol{X}'$, $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}(\boldsymbol{Y})=\boldsymbol{Y}'$, and $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}(\boldsymbol{Z})=\boldsymbol{Y}'$, and as in §5.6.3 we can choose 1-morphisms $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$, unique up to 2-isomorphism, such that $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}([\boldsymbol{g}])=\boldsymbol{g}'$ and $F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}([\boldsymbol{h}])=\boldsymbol{h}'$. Then $\boldsymbol{g},\boldsymbol{h}$ are $\boldsymbol{D}$. Define $\boldsymbol{g}',\boldsymbol{h}'$ to be *w-transverse* in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$ if $\boldsymbol{g},\boldsymbol{h}$ are w-transverse in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$. This is independent of choices.

If $\boldsymbol{g}',\boldsymbol{h}'$ are w-transverse then a fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$ by Theorem 11.17, with projections $\boldsymbol{e}:\boldsymbol{W}\to\boldsymbol{X}$, $\boldsymbol{f}:\boldsymbol{W}\to\boldsymbol{Y}$. Define $\boldsymbol{W}'=F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}(\boldsymbol{Y})$, $\boldsymbol{e}'=F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}([\boldsymbol{e}])$ and $\boldsymbol{f}'=F_{\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}}^{\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}}([\boldsymbol{f}])$. Then $\mathrm{vdim}\,\boldsymbol{W}'=\mathrm{vdim}\,\boldsymbol{X}'+\mathrm{vdim}\,\boldsymbol{Y}'-\mathrm{vdim}\,\boldsymbol{Z}'$, and we have a commutative square in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$:

$$\begin{array}{ccc} \boldsymbol{W}' & \xrightarrow{\quad\boldsymbol{f}'\quad} & \boldsymbol{Y}' \\ \Big\downarrow{\scriptstyle \boldsymbol{e}'} & & {\scriptstyle \boldsymbol{h}'}\Big\downarrow \\ \boldsymbol{X}' & \xrightarrow{\quad\boldsymbol{g}'\quad} & \boldsymbol{Z}'. \end{array} \tag{11.44}$$

In general (11.44) is *not Cartesian in* $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$, and $\boldsymbol{W}'$ is *not a fibre product* $\boldsymbol{X}'\times_{\boldsymbol{g}',\boldsymbol{Z}',\boldsymbol{h}'}\boldsymbol{Y}'$ *in* $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$, as Example 11.36 shows. But as $\boldsymbol{W}$ is unique up to canonical equivalence in $\mathbf{m}\dot{\mathbf{K}}\mathbf{ur}$, this $\boldsymbol{W}'$ is unique (that is, depends only on $\boldsymbol{X}',\boldsymbol{Y}',\boldsymbol{Z}',\boldsymbol{g}',\boldsymbol{h}'$) up to canonical isomorphism in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$.

By an abuse of notation, we could decide to call $\boldsymbol{W}'$ a 'w-transverse fibre product' in $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$, although it is not a fibre product in the category-theoretic sense. With this convention, the results of §11.2–§11.3 extend to μ-Kuranishi spaces in the obvious way. Such 'w-transverse fibre products' are an additional structure on $\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur}$. Fukaya, Oh, Ohta and Ono [15, §A1.2] define non-category-theoretic 'fibre products' $\boldsymbol{X}\times_Z\boldsymbol{Y}$ of FOOO Kuranishi spaces $\boldsymbol{X},\boldsymbol{Y}$ over manifolds Z in this sense, as in Definition 7.9.

11.5 Transverse fibre products and submersions in $\dot{\mathbf{K}}\mathbf{ur}$

Next we generalize §11.2–§11.3 to Kuranishi spaces $\dot{\mathbf{K}}\mathbf{ur}$. We suppose throughout this section that the category $\dot{\mathbf{M}}\mathbf{an}$ used to define $\dot{\mathbf{K}}\mathbf{ur}$ satisfies Assumptions

3.1–3.7 and 11.1, and will also specify additional assumptions as needed.

11.5.1 Transverse fibre products of orbifolds

Transverse fibre products of orbifolds are well understood, and are discussed by Adem, Leida and Ruan [1, Def. 1.41, Def. 2.7, Ex. 2.8], Chen and Ruan [5, p. 83], Moerdijk [56, §2.1 & §3.3], and Moerdijk and Pronk [57, §5]. Here are the analogues of Definition 2.21 and Theorem 2.22(a).

Definition 11.38. Write **Orb** for the 2-category of orbifolds, that is, for one of the equivalent 2-categories $\mathbf{Orb}_{\mathrm{Pr}}, \mathbf{Orb}_{\mathrm{Le}}, \mathbf{Orb}_{\mathrm{ManSta}}, \mathbf{Orb}_{C^\infty\mathrm{Sta}}, \mathbf{Orb}_{\mathrm{Kur}}$ in §6.6. Orbifolds $\mathfrak{X}$ have (weakly) functorial isotropy groups $G_x\mathfrak{X}$ and tangent spaces $T_x\mathfrak{X}$ for $x \in \mathfrak{X}$, as in §6.5 and §10.2. We call 1-morphisms $\mathfrak{g} : \mathfrak{X} \to \mathfrak{Z}$, $\mathfrak{h} : \mathfrak{Y} \to \mathfrak{Z}$ in **Orb** *transverse* if for all $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$ with $\mathfrak{g}(x) = \mathfrak{h}(y) = z \in \mathfrak{Z}$ and all $\gamma \in G_z\mathfrak{Z}$, the tangent morphism $T_x\mathfrak{g} \oplus (\gamma \cdot T_y\mathfrak{h}) : T_x\mathfrak{X} \oplus T_y\mathfrak{Y} \to T_z\mathfrak{Z}$ is surjective.

Theorem 11.39. *Suppose $\mathfrak{g} : \mathfrak{X} \to \mathfrak{Z}$ and $\mathfrak{h} : \mathfrak{Y} \to \mathfrak{Z}$ are transverse 1-morphisms in* **Orb**. *Then a fibre product $\mathfrak{W} = \mathfrak{X} \times_{\mathfrak{g},\mathfrak{Z},\mathfrak{h}} \mathfrak{Y}$ exists in the 2-category* **Orb**, *with $\dim \mathfrak{W} = \dim \mathfrak{X} + \dim \mathfrak{Y} - \dim \mathfrak{Z}$, in a 2-Cartesian square:*

$$\begin{array}{ccc} \mathfrak{W} & \xrightarrow{\quad\mathfrak{f}\quad} & \mathfrak{Y} \\ \downarrow{\scriptstyle\mathfrak{e}} & \eta\Uparrow & \downarrow{\scriptstyle\mathfrak{h}} \\ \mathfrak{X} & \xrightarrow{\quad\mathfrak{g}\quad} & \mathfrak{Z}. \end{array}$$

Just as a set, the underlying topological space may be written

$$W = \bigl\{(x,y,C) : x \in X,\ \ y \in Y,\ \ C \in G_x\mathfrak{g}(G_x\mathfrak{X})\backslash G_z\mathfrak{Z}/G_y\mathfrak{h}(G_y\mathfrak{Y})\bigr\}, \qquad (11.45)$$

where $\mathfrak{e}, \mathfrak{f}$ map $\mathfrak{e} : (x,y,C) \mapsto x$, $\mathfrak{f} : (x,y,C) \mapsto y$. The isotropy groups satisfy

$$G_{(x,y,C)}\mathfrak{W} \cong \bigl\{(\alpha,\beta) \in G_x\mathfrak{X} \times G_y\mathfrak{Y} : G_x\mathfrak{g}(\alpha)\,\gamma\,G_y\mathfrak{h}(\beta^{-1}) = \gamma\bigr\}$$

for fixed $\gamma \in C \subseteq G_z\mathfrak{Z}$.

Remark 11.40. (a) It is important that we work in a 2-*category* of orbifolds in Theorem 11.39. Transverse fibre products need not exist in the ordinary category Ho(**Orb**), and if they do exist they may be the 'wrong' fibre product.

(b) Note that we need not have $W \cong \bigl\{(x,y) \in X \times Y : \mathfrak{g}(x) = \mathfrak{h}(y)\bigr\}$ in Theorem 11.39, as either a set or a topological space. We discussed a similar phenomenon for fibre products in $\mathbf{Man^{gc}_{in}}, \mathbf{Man^{c}_{in}}$ in Remark 2.37, due to working in categories of interior maps. But the reasons here are different, and due to the 2-category structure. When we are working with spaces in a 2-category, points may have isotropy groups, and these isotropy groups modify the underlying sets/topological spaces of fibre products as in (11.45). There does not seem to be an easy description of the topology on (11.45) in terms of those on X, Y, Z.

(c) It may be surprising that we need $T_x\mathfrak{g} \oplus (\gamma \cdot T_y\mathfrak{h})$ to be surjective for all $\gamma \in G_z\mathfrak{Z}$ in Definition 11.38, rather than just requiring $T_x\mathfrak{g} \oplus T_y\mathfrak{h}$ to be surjective.

To see this is sensible, note that as in §10.2.3 the maps $T_x\mathfrak{g} : T_x\mathfrak{X} \to T_z\mathfrak{Z}$ and $T_y\mathfrak{h} : T_y\mathfrak{Y} \to T_z\mathfrak{Z}$ are defined using arbitrary choices, and are only canonical up to the actions $\gamma \cdot T_x\mathfrak{g}$, $\gamma \cdot T_x\mathfrak{h}$ of $\gamma \in G_z\mathfrak{Z}$. Also, surjectivity of $T_x\mathfrak{g} \oplus (\gamma \cdot T_y\mathfrak{h})$ is the transversality condition required at the point $(x, y, C) \in W$ in (11.45), where $C = G_x\mathfrak{g}(G_x\mathfrak{X})\,\gamma\, G_y\mathfrak{h}(G_y\mathfrak{Y})$.

11.5.2 Fibre products of global Kuranishi neighbourhoods

Here are the analogues of Definitions 11.15 and 11.16 and Theorem 11.17.

Definition 11.41. Suppose $g : X \to Z$, $h : Y \to Z$ are continuous maps of topological spaces, and $(U_l, D_l, \mathrm{B}_l, r_l, \chi_l), (V_m, E_m, \Gamma_m, s_m, \psi_m), (W_n, F_n, \Delta_n, t_n, \omega_n)$ are Kuranishi neighbourhoods on X, Y, Z with $\mathrm{Im}\,\chi_l \subseteq g^{-1}(\mathrm{Im}\,\omega_n)$ and $\mathrm{Im}\,\psi_m \subseteq h^{-1}(\mathrm{Im}\,\omega_n)$, and

$$\begin{aligned}\boldsymbol{g}_{ln} &= (P_{ln}, \pi_{ln}, g_{ln}, \hat{g}_{ln}) : (U_l, D_l, \mathrm{B}_l, r_l, \chi_l) \longrightarrow (W_n, F_n, \Delta_n, t_n, \omega_n),\\ \boldsymbol{h}_{mn} &= (P_{mn}, \pi_{mn}, h_{mn}, \hat{h}_{mn}) : (V_m, E_m, \Gamma_m, s_m, \psi_m) \longrightarrow (W_n, F_n, \Delta_n, t_n, \omega_n),\end{aligned}$$

are $\boldsymbol{D}$ 1-morphisms of Kuranishi neighbourhoods over $(\mathrm{Im}\,\chi_l, g)$, $(\mathrm{Im}\,\psi_m, h)$.

We call $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ *weakly transverse*, or *w-transverse*, if there exist open neighbourhoods $\dot{P}_{ln}, \dot{P}_{mn}$ of $\pi_{ln}^*(r_l)^{-1}(0)$ and $\pi_{mn}^*(s_m)^{-1}(0)$ in P_{ln}, P_{mn}, such that:

(i) $g_{ln}|_{\dot{P}_{ln}} : \dot{P}_{ln} \to W_n$ and $h_{mn}|_{\dot{P}_{mn}} : \dot{P}_{mn} \to W_n$ are $\boldsymbol{D}$ morphisms in $\dot{\mathbf{M}}\mathbf{an}$, which are transverse in the sense of Assumption 11.1(b).

(ii) $\hat{g}_{ln}|_p \oplus \hat{h}_{mn}|_q : D_l|_u \oplus E_m|_v \to F_n|_w$ is surjective for all $p \in \dot{P}_{ln}$ and $q \in \dot{P}_{mn}$ with $\pi_{ln}(p) = u \in U_l$, $\pi_{mn}(q) = v \in V_m$ and $g_{ln}(p) = h_{mn}(q) = w$ in W_n.

(iii) $\dot{P}_{ln}$ is invariant under $\mathrm{B}_l \times \Delta_n$, and $\dot{P}_{mn}$ is invariant under $\Gamma_m \times \Delta_n$.

We call $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ *transverse* if they are w-transverse and in (ii) $\hat{g}_{ln}|_p \oplus \hat{h}_{mn}|_q$ is an isomorphism for all p, q.

We call $\boldsymbol{g}_{ln}$ a *weak submersion*, or a *w-submersion*, if there exists a $\mathrm{B}_l \times \Delta_n$-invariant open neighbourhood $\ddot{P}_{ln}$ of $\pi_{ln}^*(r_l)^{-1}(0)$ in P_{ln} such that:

(iv) $g_{ln}|_{\ddot{P}_{ln}} : \ddot{P}_{ln} \to W_n$ is a submersion in $\dot{\mathbf{M}}\mathbf{an}_{\boldsymbol{D}}$, as in Assumption 11.1(c).

(v) $\hat{g}_{ln}|_p : D_l|_u \to F_n|_w$ is surjective for all $p \in \ddot{P}_{ln}$ with $\pi_{ln}(p) = u \in U_l$ and $g_{ln}(p) = w$ in W_n.

We call $\boldsymbol{g}_{ln}$ a *submersion* if it is a w-submersion and in (v) $\hat{g}_{ln}|_p$ is an isomorphism for all p.

If $\boldsymbol{g}_{ln}$ is a w-submersion then $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are w-transverse for any $\boldsymbol{D}$ 1-morphism $\boldsymbol{h}_{mn} : (V_m, E_m, \Gamma_m, s_m, \psi_m) \to (W_n, F_n, \Delta_n, t_n, \omega_n)$ over $(\mathrm{Im}\,\psi_m, h)$, by Assumption 11.1(c). Also if $\boldsymbol{g}_{ln}$ is a submersion then $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are transverse for any $\boldsymbol{D}$ 1-morphism $\boldsymbol{h}_{mn} : (V_m, E_m, \Gamma_m, s_m, \psi_m) \to (W_n, F_n, \Delta_n, t_n, \omega_n)$ over $(\mathrm{Im}\,\psi_m, h)$ for which $E_m = 0$ is the zero vector bundle.

In Definition 6.9 we defined a weak 2-category $\dot{\mathbf{GKN}}$ of *global Kuranishi neighbourhoods*, where:

- Objects (V, E, Γ, s) in $\dot{\mathbf{GKN}}$ are a manifold V (object in $\dot{\mathbf{Man}}$), a vector bundle $E \to V$, a finite group Γ acting on V, E preserving the structures, and a Γ-equivariant section $s : V \to E$. Then $(V, E, \Gamma, s, \mathrm{id}_{s^{-1}(0)/\Gamma})$ is a Kuranishi neighbourhood on the topological space $s^{-1}(0)/\Gamma$, as in §6.1. They have *virtual dimension* $\mathrm{vdim}(V, E, \Gamma, s) = \dim V - \mathrm{rank}\, E$.
- 1-morphisms $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i) \to (V_j, E_j, \Gamma_j, s_j)$ in $\dot{\mathbf{GKN}}$ are quadruples $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat{\phi}_{ij})$ satisfying Definition 6.2(a)–(e) with $s_i^{-1}(0)$ in place of $\bar{\psi}_i^{-1}(S)$. Then $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \mathrm{id}_{s_i^{-1}(0)/\Gamma_i}) \to (V_j, E_j, \Gamma_j, s_j, \mathrm{id}_{s_j^{-1}(0)/\Gamma_j})$ is a 1-morphism of Kuranishi neighbourhoods over the map $s_i^{-1}(0)/\Gamma_i \to s_j^{-1}(0)/\Gamma_j$ induced by ϕ_{ij}, π_{ij}, as in §6.1.
- For 1-morphisms $\Phi_{ij}, \Phi'_{ij} : (V_i, E_i, \Gamma_i, s_i) \to (V_j, E_j, \Gamma_j, s_j)$, a 2-morphism $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ in $\dot{\mathbf{GKN}}$ is as in Definition 6.4, with $s_i^{-1}(0)$ in place of $\bar{\psi}_i^{-1}(S)$.

We write $\dot{\mathbf{GKN}}_{\boldsymbol{D}} \subseteq \dot{\mathbf{GKN}}$ for the 2-subcategory with 1-morphisms Φ_{ij} which are $\boldsymbol{D}$, in the sense of Definition 6.31. The next (rather long) definition and theorem prove that w-transverse fibre products exist in $\dot{\mathbf{GKN}}_{\boldsymbol{D}}$.

Definition 11.42. Suppose we are given 1-morphisms in $\dot{\mathbf{GKN}}_{\boldsymbol{D}}$

$$\boldsymbol{g}_{ln} : (U_l, D_l, \mathrm{B}_l, r_l) \longrightarrow (W_n, F_n, \Delta_n, t_n),$$
$$\boldsymbol{h}_{mn} : (V_m, E_m, \Gamma_m, s_m) \longrightarrow (W_n, F_n, \Delta_n, t_n),$$

with $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ w-transverse in the sense of Definition 11.41. We will construct a fibre product

$$(T_k, C_k, \mathrm{A}_k, q_k) = (U_l, D_l, \mathrm{B}_l, r_l) \times_{\boldsymbol{g}_{ln}, (W_n, F_n, \Delta_n, t_n), \boldsymbol{h}_{mn}} (V_m, E_m, \Gamma_m, s_m) \quad (11.46)$$

in both $\dot{\mathbf{GKN}}_{\boldsymbol{D}}$ and $\dot{\mathbf{GKN}}_{\boldsymbol{E}}$.

Write $\boldsymbol{g}_{ln} = (P_{ln}, \pi_{ln}, g_{ln}, \hat{g}_{ln})$ and $\boldsymbol{h}_{mn} = (P_{mn}, \pi_{mn}, h_{mn}, \hat{h}_{mn})$. Then $\hat{g}_{ln}(\pi_{ln}^*(r_l)) = g_{ln}^*(t_n) + O(\pi_{ln}^*(r_l)^2)$ by Definition 6.2(e), so Definition 3.15(i) gives $\epsilon : \pi_{ln}^*(D_l) \otimes \pi_{ln}^*(D_l) \to g_{ln}^*(F_n)$ with $\hat{g}_{ln}(\pi_{ln}^*(r_l)) = g_{ln}^*(t_n) + \epsilon(\pi_{ln}^*(r_l) \otimes \pi_{ln}^*(r_l))$. By averaging over the $(\mathrm{B}_l \times \Delta_n)$-action we can suppose ϵ is $(\mathrm{B}_l \times \Delta_n)$-equivariant. Define $\hat{g}'_{ln} : \pi_{ln}^*(D_l) \to g_{ln}^*(F_n)$ by $\hat{g}'_{ln}(d) = \hat{g}_{ln}(d) - \epsilon(d \otimes \pi_{ln}^*(r_l))$. Replacing $\hat{g}_{ln}$ by $\hat{g}'_{ln}$, which does not change $\boldsymbol{g}_{ln}$ up to 2-isomorphism as $\hat{g}'_{ln} = \hat{g}_{ln} + O(\pi_{ln}^*(r_l))$, we may suppose that $\hat{g}_{ln}(\pi_{ln}^*(r_l)) = g_{ln}^*(t_n)$. Similarly we suppose that $\hat{h}_{mn}(\pi_{mn}^*(s_m)) = h_{mn}^*(t_n)$.

For $\dot{P}_{ln}, \dot{P}_{mn}$ as in Definition 11.41(i)–(iii), define

$$T_k = \dot{P}_{ln} \times_{g_{ln}|_{\dot{P}_{ln}}, W_n, h_{mn}|_{\dot{P}_{mn}}} \dot{P}_{mn} \quad (11.47)$$

to be the transverse fibre product in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$ from Assumption 11.1(b). Then

$$\dim T_k = \dim U_l + \dim V_m - \dim W_n, \quad (11.48)$$

as $\dim \dot P_{ln} = \dim U_l$, etc. Define a finite group $\mathrm{A}_k = \mathrm{B}_l \times \Gamma_m \times \Delta_n$. Since $g_{ln}|_{\dot P_{ln}}$ is B_l-invariant and Δ_n-equivariant, and $h_{mn}|_{\dot P_{mn}}$ is Γ_m-invariant and Δ_n-equivariant, A_k is a symmetry group of the fibre product (11.47), so there is a natural smooth action of A_k on T_k. If we can write points of T_k as (p, q) for $p \in \dot P_{ln}$, $q \in \dot P_{mn}$ with $g_{ln}(p) = h_{mn}(q) \in W_n$ then A_k acts on points by

$$(\beta, \gamma, \delta) : (p, q) \mapsto ((\beta, \delta) \cdot p, (\gamma, \delta) \cdot q),$$

noting that $g_{ln}((\beta, \delta) \cdot p) = \delta \cdot g_{ln}(p) = \delta \cdot h_{mn}(q) = h_{mn}((\gamma, \delta) \cdot q)$.

We have a morphism of vector bundles on T_k:

$$\begin{aligned} \pi^*_{\dot P_{ln}}(\hat g_{ln}) \oplus -\pi^*_{\dot P_{mn}}(\hat h_{mn}) : (\pi_{ln} \circ \pi_{\dot P_{ln}})^*(D_l) \oplus (\pi_{mn} \circ \pi_{\dot P_{mn}})^*(E_m) \\ \longrightarrow (g_{ln} \circ \pi_{\dot P_{ln}})^*(F_n). \end{aligned} \tag{11.49}$$

If $t \in T_k$ with $\pi_{\dot P_{ln}}(t) = p \in \dot P_{ln}$, $\pi_{\dot P_{mn}}(t) = q \in \dot P_{mn}$, $\pi_{ln}(p) = u \in U_{ln}$, $\pi_{mn}(q) = v \in V_{mn}$ and $g_{ln}(p) = h_{mn}(q) = w \in W_n$ then the fibre of (11.49) at t is $\hat g_{ln}|_p \oplus -\hat h_{mn}|_q : D_l|_u \oplus E_m|_v \to F_n|_w$. So Definition 11.41(ii) implies that (11.49) is surjective. Define $C_k \to T_k$ to be the kernel of (11.49), as a vector subbundle of $(\pi_{ln} \circ \pi_{\dot P_{ln}})^*(D_l) \oplus (\pi_{mn} \circ \pi_{\dot P_{mn}})^*(E_m)$ with

$$\operatorname{rank} C_k = \operatorname{rank} D_l + \operatorname{rank} E_m - \operatorname{rank} F_n. \tag{11.50}$$

Definition 6.2(d) for $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ says that $\hat g_{ln}$ is $(\mathrm{B}_l \times \Delta_n)$-equivariant and $\hat h_{ln}$ is $(\Gamma_m \times \Delta_n)$-equivariant. Including the trivial actions of Γ_m on D_l, F_n, and of B_l on E_m, F_n, means that $\hat g_{ln}, \hat h_{mn}$ are equivariant under $\mathrm{A}_k = \mathrm{B}_l \times \Gamma_m \times \Delta_n$. The pullbacks by $\pi_{\dot P_{ln}}, \pi_{\dot P_{mn}}$ are also A_k-equivariant, as $\pi_{\dot P_{ln}}, \pi_{\dot P_{mn}}$ are. So (11.49) is equivariant under the natural actions of A_k, and thus C_k has a natural A_k-action by restriction from the A_k-action on $(\pi_{ln} \circ \pi_{\dot P_{ln}})^*(D_l) \oplus (\pi_{mn} \circ \pi_{\dot P_{mn}})^*(E_m)$.

Write $\pi_{D_l} : C_k \to (\pi_{ln} \circ \pi_{\dot P_{ln}})^*(D_l)$, $\pi_{E_m} : C_k \to (\pi_{mn} \circ \pi_{\dot P_{mn}})^*(E_m)$ for the projections. Then as C_k is the kernel of (11.49) we have

$$\pi^*_{\dot P_{ln}}(\hat g_{ln}) \circ \pi_{D_l} = \pi^*_{\dot P_{mn}}(\hat h_{mn}) \circ \pi_{E_m} : C_k \longrightarrow (g_{ln} \circ \pi_{\dot P_{ln}})^*(F_n). \tag{11.51}$$

In sections of the left hand side of (11.49) over T_k, we have

$$\begin{aligned} &\big(\pi^*_{\dot P_{ln}}(\hat g_{ln}) \oplus -\pi^*_{\dot P_{mn}}(\hat h_{mn})\big)\big((\pi_{ln} \circ \pi_{\dot P_{ln}})^*(r_l) \oplus (\pi_{mn} \circ \pi_{\dot P_{mn}})^*(s_m)\big) \\ &\quad = \pi^*_{\dot P_{ln}} \circ \hat g_{ln} \circ \pi^*_{ln}(r_l) - \pi^*_{\dot P_{mn}} \circ \hat h_{mn} \circ \pi^*_{mn}(s_m) \\ &\quad = \pi^*_{\dot P_{ln}} \circ g^*_{ln}(t_n) - \pi^*_{\dot P_{mn}} \circ h^*_{mn}(t_n) = 0, \end{aligned}$$

as $\hat g_{ln}(\pi^*_{ln}(r_l)) = g^*_{ln}(t_n)$, $\hat h_{mn}(\pi^*_{mn}(s_m)) = h^*_{mn}(t_n)$, and $g_{ln} \circ \pi_{\dot P_{ln}} = h_{mn} \circ \pi_{\dot P_{mn}}$. Thus $(\pi_{ln} \circ \pi_{\dot P_{ln}})^*(r_l) \oplus (\pi_{mn} \circ \pi_{\dot P_{mn}})^*(s_m)$ lies in the kernel of (11.49), so it is a section of C_k. Write $q_k \in \Gamma^\infty(C_k)$ for this section. Then

$$\pi_{D_l}(q_k) = (\pi_{ln} \circ \pi_{\dot P_{ln}})^*(r_l) \quad \text{and} \quad \pi_{E_m}(q_k) = (\pi_{mn} \circ \pi_{\dot P_{mn}})^*(s_m). \tag{11.52}$$

Also q_k is A_k-equivariant, as $(\pi_{ln} \circ \pi_{\dot P_{ln}})^*(r_l)$ and $(\pi_{mn} \circ \pi_{\dot P_{mn}})^*(s_m)$ are.

Then $(T_k, C_k, \mathrm{A}_k, q_k)$ is an object in $\dot{\mathbf{GKN}}_{\boldsymbol{D}}$. By (11.48), (11.50) we have

$$\begin{aligned}&\mathrm{vdim}(T_k, C_k, \mathrm{A}_k, q_k) = \mathrm{vdim}(U_l, D_l, \mathrm{B}_l, r_l)\\&\quad + \mathrm{vdim}(V_m, E_m, \Gamma_m, s_m) - \mathrm{vdim}(W_n, F_n, \Delta_n, t_n).\end{aligned}$$

Define $P_{kl} = T_k \times \mathrm{B}_l$ and $P_{km} = T_k \times \Gamma_m$, as objects in $\dot{\mathbf{Man}}$. Define smooth actions of $\mathrm{A}_k \times \mathrm{B}_l$ on P_{kl}, and of $\mathrm{A}_k \times \Gamma_m$ on P_{km}, at the level of points by

$$\begin{aligned}((\beta, \gamma, \delta), \beta') &: (t, \beta'') \longmapsto ((\beta, \gamma, \delta) \cdot t, \beta'\beta''\beta^{-1}),\\((\beta, \gamma, \delta), \gamma') &: (t, \gamma'') \longmapsto ((\beta, \gamma, \delta) \cdot t, \gamma'\gamma''\gamma^{-1}).\end{aligned}$$

Define morphisms $\pi_{kl} = \pi_{T_k} : P_{kl} = T_k \times \mathrm{B}_l \to T_k$ and $\pi_{km} = \pi_{T_k} : P_{km} = T_k \times \Gamma_m \to T_k$ in $\dot{\mathbf{Man}}$. Then π_{kl} is an A_k-equivariant principal B_l-bundle over $T_{kl} = T_k$, and π_{km} an A_k-equivariant principal Γ_m-bundle over $T_{km} = T_k$.

Define morphisms $e_{kl} : P_{kl} \to U_l$ and $f_{km} : P_{km} \to V_m$ in $\dot{\mathbf{Man}}$ by

$$e_{kl}(t, \beta) = \beta \cdot \pi_{ln} \circ \pi_{\dot{P}_{ln}}(t), \qquad f_{km}(t, \gamma) = \gamma \cdot \pi_{lm} \circ \pi_{\dot{P}_{lm}}(t),$$

that is, $e_{kl}|_{T_k \times \{\beta\}} = \beta \cdot (\pi_{ln} \circ \pi_{\dot{P}_{ln}})$ and $\hat{f}_{km}|_{T_k \times \{\gamma\}} = \gamma \cdot (\pi_{lm} \circ \pi_{\dot{P}_{lm}})$ for $\beta \in \mathrm{B}_l$ and $\gamma \in \Gamma_m$. Then e_{kl} is A_k-invariant and B_l-equivariant, and f_{km} is A_k-invariant and Γ_m-equivariant. Also $e \circ \bar{\varphi}_k \circ \pi_{kl} = \bar{\chi}_l \circ e_{kl}$ on $\pi_{kl}^{-1}(q_k^{-1}(0)) \subseteq P_{kl}$ and $f \circ \bar{\varphi}_k \circ \pi_{km} = \bar{\psi}_m \circ f_{km}$ on $\pi_{km}^{-1}(q_k^{-1}(0)) \subseteq P_{km}$. And e_{kl}, f_{km} are $\boldsymbol{D}$, since $\pi_{\dot{P}_{ln}}, \pi_{\dot{P}_{lm}}$ are as (11.47) is a fibre product in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$, and $\beta \cdot \pi_{ln}$, $\gamma \cdot \pi_{ln}$ are étale.

Define morphisms $\hat{e}_{kl} : \pi_{kl}^*(C_k) \to e_{kl}^*(D_l)$ and $\hat{f}_{km} : \pi_{km}^*(C_k) \to f_{km}^*(E_m)$ by

$$\hat{e}_{kl}|_{T_k \times \{\beta\}} = (\pi_{ln} \circ \pi_{\dot{P}_{ln}})^*(\beta^\heartsuit) \circ \pi_{D_l}, \quad \hat{f}_{km}|_{T_k \times \{\gamma\}} = (\pi_{lm} \circ \pi_{\dot{P}_{lm}})^*(\gamma^\heartsuit) \circ \pi_{E_m}$$

for all $\beta \in \mathrm{B}_l$ and $\gamma \in \Gamma_m$, where $\beta^\heartsuit : D_l \to \beta^*(D_l)$ is the isomorphism from the lift of the B_l-action on U_l to D_l, with β^* the pullback by $\beta \cdot : U_l \to U_l$, and similarly for $\gamma^\heartsuit$. Then $\hat{e}_{kl}$ is $(\mathrm{A}_k \times \mathrm{B}_l)$-equivariant, and $\hat{f}_{km}$ is $(\mathrm{A}_k \times \Gamma_m)$-equivariant. We have

$$\begin{aligned}&\hat{e}_{kl}(\pi_{kl}^*(q_k))|_{T_k \times \{\beta\}} = (\pi_{ln} \circ \pi_{\dot{P}_{ln}})^*(\beta^\heartsuit) \circ \pi_{D_l}(\pi_{kl}^*(q_k))\\&\quad = (\pi_{ln} \circ \pi_{\dot{P}_{ln}})^*(\beta^\heartsuit) \circ (\pi_{ln} \circ \pi_{\dot{P}_{ln}})^*(r_l) = (\pi_{ln} \circ \pi_{\dot{P}_{ln}})^*(\beta^\heartsuit(r_l))\\&\quad = (\pi_{ln} \circ \pi_{\dot{P}_{ln}})^*(\beta^*(r_l)) = e_{kl}^*(r_l)|_{T_k \times \{\beta\}},\end{aligned}$$

using (11.52) in the second step and $\beta^\heartsuit(r_l) = \beta^*(r_l)$ as r_l is B_l-equivariant in the fourth. As this holds for all $\beta \in \mathrm{B}_l$ we see that $\hat{e}_{kl}(\pi_{kl}^*(q_k)) = e_{kl}^*(r_l)$, and similarly $\hat{f}_{km}(\pi_{km}^*(q_k)) = f_{km}^*(s_m)$.

Set $\boldsymbol{e}_{kl} = (P_{kl}, \pi_{kl}, e_{kl}, \hat{e}_{kl})$ and $\boldsymbol{f}_{km} = (P_{km}, \pi_{km}, f_{km}, \hat{f}_{km})$. Then $\boldsymbol{e}_{kl} : (T_k, C_k, \mathrm{A}_k, q_k) \to (U_l, D_l, \mathrm{B}_l, r_l)$ and $\boldsymbol{f}_{km} : (T_k, C_k, \mathrm{A}_k, q_k) \to (V_m, E_m, \Gamma_m, s_m)$ are 1-morphisms in $\dot{\mathbf{GKN}}_{\boldsymbol{D}}$, as we have verified Definition 6.2(a)–(e) for $\boldsymbol{e}_{kl}, \boldsymbol{f}_{km}$ above, and e_{kl}, f_{km} are $\boldsymbol{D}$.

Form the compositions $\boldsymbol{g}_{ln}\circ\boldsymbol{e}_{kl},\boldsymbol{h}_{mn}\circ\boldsymbol{f}_{kn}:(T_k,C_k,\mathrm{A}_k,q_k)\rightarrow(W_n,F_n,\Delta_n,t_n)$ using Definition 6.5, where we write

$$\boldsymbol{g}_{ln}\circ\boldsymbol{e}_{kl}=(P_{kln},\pi_{kln},a_{kln},\hat a_{kln}),\ \ \boldsymbol{h}_{mn}\circ\boldsymbol{f}_{km}=(P_{kmn},\pi_{kmn},b_{kmn},\hat b_{kmn}).$$

Then by Definition 6.5 we have

$$P_{kln}=\bigl(P_{kl}\times_{e_{kl},U_l,\pi_{ln}}P_{ln}\bigr)/\mathrm{B}_l=\bigl((T_k\times\mathrm{B}_l)\times_{e_{kl},U_l,\pi_{ln}}P_{ln}\bigr)/\mathrm{B}_l.$$

Define a morphism $\Phi_{kln}:T_k\times\Delta_n\rightarrow P_{kln}$ in $\dot{\mathbf{Man}}$ at the level of points by

$$\Phi_{kln}(t,\delta)=\bigl((t,1),\delta\cdot\pi_{\dot P_{ln}}(t)\bigr)\mathrm{B}_l.$$

We claim Φ_{kln} is a diffeomorphism. To see this, first note that the quotient B_l-action acts freely on the B_l factor in $T_k\times\mathrm{B}_l$, so we can restrict to $T_k\times\{1\}$ and omit the quotient, giving $P_{kln}\cong T_k\times_{\pi_{ln}\circ\pi_{\dot P_{ln}},U_l,\pi_{ln}}P_{ln}$. Then observe that if $(t,p)\in T_k\times_{U_l}P_{ln}$ then $\pi_{ln}[\pi_{\dot P_{ln}}(t)]=\pi_{ln}[u]$, but $\pi_{ln}:P_{ln}\rightarrow U_l$ is a principal Δ_n-bundle, so there exists a unique $\delta\in\Delta_n$ with $p=\delta\cdot\pi_{\dot P_{ln}}(t)$, and therefore $T_k\times\Delta_n\cong T_k\times_{U_l}P_{ln}$.

If we identify $P_{kln}=T_k\times\Delta_n$ using Φ_{kln}, then we find from Definition 6.5 that $\mathrm{A}_k\times\Delta_n$ acts on P_{kln} by

$$((\beta,\gamma,\delta),\delta'):(t,\delta'')\longmapsto((\beta,\gamma,\delta)\cdot t,\delta'\delta''\delta^{-1}),\tag{11.53}$$

and $\pi_{kln}:P_{kln}\rightarrow T_k$, $a_{kln}:P_{kln}\rightarrow W_n$, $\hat a_{kln}:\pi^*_{kln}(C_k)\rightarrow a^*_{kln}(F_n)$ act by

$$\pi_{kln}:(t,\delta)\longmapsto t,\qquad a_{kln}:(t,\delta)\longmapsto\delta\cdot g_{ln}\circ\pi_{\dot P_{ln}}(t),$$
$$\hat a_{kln}|_{(t,\delta)}=\hat g_{ln}|_{\delta\cdot\pi_{\dot P_{ln}}(t)}\circ\pi_{D_l}|_t=\delta^{\heartsuit}|_{g_{ln}\circ\pi_{\dot P_{ln}}(t)}\circ\hat g_{ln}|_{\pi_{\dot P_{ln}}(t)}\circ\pi_{D_l}|_t.$$

Similarly, there is a natural diffeomorphism $\Phi_{kmn}:T_k\times\Delta_n\rightarrow P_{kmn}$, and if we use it to identify $P_{kmn}=T_k\times\Delta_n$ then $\mathrm{A}_k\times\Delta_n$ acts on P_{kmn} as in (11.53), and $\pi_{kmn}:P_{kmn}\rightarrow T_k$, $b_{kmn}:P_{kmn}\rightarrow W_n$, $\hat b_{kmn}:\pi^*_{kmn}(C_k)\rightarrow b^*_{kmn}(F_n)$ act by

$$\pi_{kmn}:(t,\delta)\longmapsto t,\qquad b_{kmn}:(t,\delta)\longmapsto\delta\cdot h_{mn}\circ\pi_{\dot P_{mn}}(t),$$
$$\hat b_{kmn}|_{(t,\delta)}=\delta^{\heartsuit}|_{h_{mn}\circ\pi_{\dot P_{mn}}(t)}\circ\hat h_{mn}|_{\pi_{\dot P_{mn}}(t)}\circ\pi_{E_m}|_t.$$

Since $g_{ln}\circ\pi_{\dot P_{ln}}=h_{mn}\circ\pi_{\dot P_{mn}}$ by (11.47), and (11.51) holds, we see that these identifications $P_{kln}=T_k\times\Delta_n=P_{kmn}$ are $\mathrm{A}_k\times\Delta_n$-equivariant and identify $\pi_{kln},a_{kln},\hat a_{kln}$ with $\pi_{kmn},b_{kmn},\hat b_{kmn}$. That is, we have found a strict isomorphism between the 1-morphisms $\boldsymbol{g}_{ln}\circ\boldsymbol{e}_{kl},\boldsymbol{h}_{mn}\circ\boldsymbol{f}_{kn}$. It follows that

$$\boldsymbol{\eta}_{klmn}=[P_{kln},\Phi_{kmn}\circ\Phi_{kln}^{-1},0]:\boldsymbol{g}_{ln}\circ\boldsymbol{e}_{kl}\Longrightarrow\boldsymbol{h}_{mn}\circ\boldsymbol{f}_{kn}$$

is a 2-morphism in $\mathbf{G\dot{K}N}_{\boldsymbol{D}}$, and we have a 2-commutative diagram in $\mathbf{G\dot{K}N}_{\boldsymbol{D}}$:

$$\begin{array}{ccc}(T_k,C_k,\mathrm{A}_k,q_k) & \xrightarrow[\quad]{\boldsymbol{f}_{km}} & (V_m,E_m,\Gamma_m,s_m)\\ \Big\downarrow{\scriptstyle\boldsymbol{e}_{kl}} & \boldsymbol{\eta}_{klmn}\Uparrow & \Big\downarrow{\scriptstyle\boldsymbol{h}_{mn}}\\ (U_l,D_l,\mathrm{B}_l,r_l) & \xrightarrow{\boldsymbol{g}_{ln}} & (W_n,F_n,\Delta_n,t_n).\end{array}\tag{11.54}$$

If $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are transverse, not just w-transverse, then (11.49) is an isomorphism, not just surjective, so C_k is the zero vector bundle, as it is the kernel of (11.49). Thus $(T_k, C_k, \mathrm{A}_k, q_k,)$ is a quotient orbifold $[T_k/\mathrm{A}_k]$.

Theorem 11.43. *In Definition* 11.42, *equation* (11.54) *is* 2*-Cartesian in both* $\mathbf{G\dot{K}N}_{\boldsymbol{D}}$ *and* $\mathbf{G\dot{K}N}_{\boldsymbol{E}}$ *in the sense of Definition* A.11, *so that* $(T_k, C_k, \mathrm{A}_k, q_k)$ *is a fibre product in the* 2*-categories* $\mathbf{G\dot{K}N}_{\boldsymbol{D}}, \mathbf{G\dot{K}N}_{\boldsymbol{E}}$, *as in* (11.46).

The proof of Theorem 11.43 is the orbifold analogue of the proof of Theorem 11.17 in §11.8, and we leave it as a (long and rather dull) exercise for the reader.

11.5.3 (W-)transversality and fibre products in $\mathbf{\dot{K}ur}_{\boldsymbol{D}}$

Here are the analogues of Definition 11.18 and Theorem 11.19.

Definition 11.44. Let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms in $\mathbf{\dot{K}ur}_{\boldsymbol{D}}$. We call $\boldsymbol{g}, \boldsymbol{h}$ or *w-transverse* (or *transverse*), if whenever $x \in \boldsymbol{X}$ and $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$, there exist Kuranishi neighbourhoods $(U_l, D_l, \mathrm{B}_l, r_l, \chi_l)$, $(V_m, E_m, \Gamma_m, s_m, \psi_m)$, $(W_n, F_n, \Delta_n, t_n, \omega_n)$ on $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ as in §6.4 with $x \in \operatorname{Im}\chi_l \subseteq g^{-1}(\operatorname{Im}\omega_n)$, $y \in \operatorname{Im}\psi_m \subseteq h^{-1}(\operatorname{Im}\omega_n)$ and $z \in \operatorname{Im}\omega_n$, and 1-morphisms $\boldsymbol{g}_{ln} : (U_l, D_l, \mathrm{B}_l, r_l, \chi_l) \to (W_n, F_n, \Delta_n, t_n, \omega_n)$, $\boldsymbol{h}_{mn} : (V_m, E_m, \Gamma_m, s_m, \psi_m) \to (W_n, F_n, \Delta_n, t_n, \omega_n)$ over $(\operatorname{Im}\chi_l, \boldsymbol{g})$ and $(\operatorname{Im}\psi_m, \boldsymbol{h})$, as in Definition 6.44, such that $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are w-transverse (or transverse), as in Definition 11.42.

We call $\boldsymbol{g}$ a *w-submersion* (or a *submersion*), if whenever $x \in \boldsymbol{X}$ with $\boldsymbol{g}(x) = z \in \boldsymbol{Z}$, there exist Kuranishi neighbourhoods $(U_l, D_l, \mathrm{B}_l, r_l, \chi_l), (W_n, F_n, \Delta_n, t_n, \omega_n)$ on $\boldsymbol{X}, \boldsymbol{Z}$ as in §6.4 with $x \in \operatorname{Im}\chi_l \subseteq g^{-1}(\operatorname{Im}\omega_n)$, $z \in \operatorname{Im}\omega_n$, and a 1-morphism $\boldsymbol{g}_{ln} : (U_l, D_l, \mathrm{B}_l, r_l, \chi_l) \to (W_n, F_n, \Delta_n, t_n, \omega_n)$ over $(\operatorname{Im}\chi_l, \boldsymbol{g})$, as in Definition 6.44, such that $\boldsymbol{g}_{ln}$ is a w-submersion (or a submersion, respectively), as in Definition 11.42.

Suppose $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ is a w-submersion, and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ is any $\boldsymbol{D}$ 1-morphism in $\mathbf{\dot{K}ur}$. Let $x \in \boldsymbol{X}$ and $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$. As $\boldsymbol{g}$ is a w-submersion we can choose $\boldsymbol{g}_{ln} : (U_l, D_l, \mathrm{B}_l, r_l, \chi_l) \to (W_n, F_n, \Delta_n, t_n, \omega_n)$ with $x \in \operatorname{Im}\chi_l \subseteq g^{-1}(\operatorname{Im}\omega_n)$, $z \in \operatorname{Im}\omega_n$, and $\boldsymbol{g}_{ln}$ a w-submersion. Choose any Kuranishi neighbourhood $(V_m, E_m, \Gamma_m, s_m, \psi_m)$ on $\boldsymbol{Y}$ with $y \in \operatorname{Im}\psi_m \subseteq h^{-1}(\operatorname{Im}\omega_n)$. Then Theorem 6.45(b) gives a $\boldsymbol{D}$ 1-morphism $\boldsymbol{h}_{mn} : (V_m, E_m, \Gamma_m, s_m, \psi_m) \to (W_n, F_n, \Delta_n, t_n, \omega_n)$ over $(\operatorname{Im}\psi_m, \boldsymbol{h})$, and $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are w-transverse as $\boldsymbol{g}_{ln}$ is a w-submersion. Hence $\boldsymbol{g}, \boldsymbol{h}$ are w-transverse.

Similarly, suppose $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ is a submersion, and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ is a $\boldsymbol{D}$ 1-morphism in $\mathbf{\dot{K}ur}$ such that $\boldsymbol{Y}$ is an orbifold as in Proposition 6.64, that is, $\boldsymbol{Y} \simeq F_{\mathbf{\dot{O}rb}}^{\mathbf{\dot{K}ur}}(\mathfrak{Y})$ for $\mathfrak{Y} \in \mathbf{\dot{O}rb}$. Then for $x \in \boldsymbol{X}$ and $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$ we can choose $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ as above with $\boldsymbol{g}_{ln}$ a submersion and $E_m = 0$, so that $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are transverse. Hence $\boldsymbol{g}, \boldsymbol{h}$ are transverse.

Theorem 11.45. *Let* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *be w-transverse* 1*-morphisms in* $\mathbf{\dot{K}ur}_{\boldsymbol{D}}$. *Then there exists a fibre product* $\boldsymbol{W} = \boldsymbol{X}_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ *in* $\mathbf{\dot{K}ur}_{\boldsymbol{D}}$, *as in* §A.4, *with* $\mathrm{vdim}\,\boldsymbol{W}=\mathrm{vdim}\,\boldsymbol{X}+\mathrm{vdim}\,\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{Z}$, *in a* 2*-Cartesian square:*

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow[\boldsymbol{f}]{} & \boldsymbol{Y} \\ \downarrow \boldsymbol{e} & \boldsymbol{\eta}\Uparrow & \boldsymbol{h}\downarrow \\ \boldsymbol{X} & \xrightarrow{\boldsymbol{g}} & \boldsymbol{Z}. \end{array} \tag{11.55}$$

Equation (11.55) *is also* 2*-Cartesian in* $\dot{\mathbf{Kur}}_{\boldsymbol{E}}$, *so* $\boldsymbol{W}$ *is also a fibre product* $\boldsymbol{X}_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ *in* $\dot{\mathbf{Kur}}_{\boldsymbol{E}}$. *Furthermore:*

(a) *If* $\boldsymbol{g},\boldsymbol{h}$ *are transverse then* $\boldsymbol{W}$ *is an orbifold, as in Proposition* 6.64. *In particular, if* $\boldsymbol{g}$ *is a submersion and* $\boldsymbol{Y}$ *is an orbifold, then* $\boldsymbol{W}$ *is an orbifold.*

(b) *Suppose* $(U_l,D_l,\mathrm{B}_l,r_l,\chi_l)$, $(V_m,E_m,\Gamma_m,s_m,\psi_m)$, $(W_n,F_n,\Delta_n,t_n,\omega_n)$ *are Kuranishi neighbourhoods on* $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$, *as in* §6.4, *with* $\mathrm{Im}\,\chi_l\subseteq g^{-1}(\mathrm{Im}\,\omega_n)$ *and* $\mathrm{Im}\,\psi_m\subseteq h^{-1}(\mathrm{Im}\,\omega_n)$, *and* $\boldsymbol{g}_{ln}:(U_l,D_l,\mathrm{B}_l,r_l,\chi_l)\rightarrow(W_n,F_n,\Delta_n,t_n,\omega_n)$, $\boldsymbol{h}_{mn}:(V_m,E_m,\Gamma_m,s_m,\psi_m)\rightarrow(W_n,F_n,\Delta_n,t_n,\omega_n)$ *are* 1*-morphisms of Kuranishi neighbourhoods on* $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ *over* $(\mathrm{Im}\,\chi_l,\boldsymbol{g})$ *and* $(\mathrm{Im}\,\psi_m,\boldsymbol{h})$, *as in* §6.4, *such that* $\boldsymbol{g}_{ln},\boldsymbol{h}_{mn}$ *are w-transverse, as in* §11.5.2. *Then there exist a Kuranishi neighbourhood* $(T_k,C_k,\mathrm{A}_k,q_k,\varphi_k)$ *on* $\boldsymbol{W}$ *with* $\mathrm{Im}\,\varphi_k=e^{-1}(\mathrm{Im}\,\chi_l)\cap f^{-1}(\mathrm{Im}\,\psi_m)\subseteq W$, *and* 1*-morphisms* $\boldsymbol{e}_{kl}:(T_k,C_k,\mathrm{A}_k,q_k,\varphi_k)\rightarrow(U_l,D_l,\mathrm{B}_l,r_l,\chi_l)$ *over* $(\mathrm{Im}\,\varphi_k,\boldsymbol{e})$ *and* $\boldsymbol{f}_{km}:(T_k,C_k,\mathrm{A}_k,q_k,\varphi_k)\rightarrow(V_m,E_m,\Gamma_m,s_m,\psi_m)$ *over* $(\mathrm{Im}\,\varphi_k,\boldsymbol{f})$, *so that Theorem* 6.45(c) *gives a unique* 2*-morphism* $\boldsymbol{\eta}_{klmn}:\boldsymbol{g}_{ln}\circ\boldsymbol{e}_{kl}\Rightarrow\boldsymbol{h}_{mn}\circ\boldsymbol{f}_{km}$ *over* $(\mathrm{Im}\,\varphi_k,g\circ e)$ *constructed from* $\boldsymbol{\eta}:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$, *such that* T_k,C_k,A_k,q_k *and* $\boldsymbol{e}_{kl},\boldsymbol{f}_{km},\boldsymbol{\eta}_{klmn}$ *are constructed from* $(U_l,D_l,\mathrm{B}_l,r_l)$, (V_m,E_m,Γ_m,s_m), (W_n,F_n,Δ_n,t_n) *and* $\boldsymbol{g}_{ln},\boldsymbol{h}_{mn}$ *exactly as in Definition* 11.42.

(c) *If* $\dot{\mathbf{Man}}$ *satisfies Assumption* 11.3 *then just as a set, the underlying topological space* W *in* $\boldsymbol{W}=(W,\mathcal{H})$ *may be written*

$$W=\bigl\{(x,y,C):x\in X,\ \ y\in Y,\ \ C\in G_x\boldsymbol{g}(G_x\boldsymbol{X})\backslash G_z\boldsymbol{Z}/G_y\boldsymbol{h}(G_y\boldsymbol{Y})\bigr\}, \tag{11.56}$$

where $\boldsymbol{e},\boldsymbol{f}$ *map* $\boldsymbol{e}:(x,y,C)\mapsto x$, $\boldsymbol{f}:(x,y,C)\mapsto y$. *The isotropy groups satisfy*

$$G_{(x,y,C)}\boldsymbol{W}\cong\bigl\{(\alpha,\beta)\in G_x\boldsymbol{X}\times G_y\boldsymbol{Y}:G_x\boldsymbol{g}(\alpha)\,\gamma\,G_y\boldsymbol{h}(\beta^{-1})=\gamma\bigr\}$$

for fixed $\gamma\in C\subseteq G_z\boldsymbol{Z}$.

(d) *If* $\dot{\mathbf{Man}}$ *satisfies Assumption* 11.4(a) *and* (11.55) *is a* 2*-Cartesian square in* $\dot{\mathbf{Kur}}_{\boldsymbol{D}}$ *with* $\boldsymbol{g}$ *a w-submersion (or a submersion) then* $\boldsymbol{f}$ *is a w-submersion (or a submersion, respectively).*

(e) *If* $\dot{\mathbf{Man}}$ *satisfies Assumption* 10.1, *with tangent spaces* T_xX, *and satisfies Assumption* 11.5, *then using the notation of* §10.2, *whenever* (11.55) *is* 2*-Cartesian in* $\dot{\mathbf{Kur}}_{\boldsymbol{D}}$ *with* $\boldsymbol{g},\boldsymbol{h}$ *w-transverse and* $w\in\boldsymbol{W}$ *with* $\boldsymbol{e}(w)=x$ *in* $\boldsymbol{X}$, $\boldsymbol{f}(w)=y$ *in* $\boldsymbol{Y}$ *and* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$, *for some possible choices of* $T_w\boldsymbol{e},T_w\boldsymbol{f},T_x\boldsymbol{g},T_y\boldsymbol{h},O_w\boldsymbol{e},O_w\boldsymbol{f},O_x\boldsymbol{g},O_y\boldsymbol{h}$ *in Definition* 10.28 *depending on* w, *the following is an exact sequence:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_w\boldsymbol{W} & \xrightarrow[T_w\boldsymbol{e}\oplus T_w\boldsymbol{f}]{} & T_x\boldsymbol{X}\oplus T_y\boldsymbol{Y} & \xrightarrow[T_x\boldsymbol{g}\oplus -T_y\boldsymbol{h}]{} & T_z\boldsymbol{Z} \\ & & & & & & \downarrow \delta_w^{\boldsymbol{g},\boldsymbol{h}} \\ 0 & \longleftarrow & O_z\boldsymbol{Z} & \xleftarrow{O_x\boldsymbol{g}\oplus -O_y\boldsymbol{h}} & O_x\boldsymbol{X}\oplus O_y\boldsymbol{Y} & \xleftarrow{O_w\boldsymbol{e}\oplus O_w\boldsymbol{f}} & O_w\boldsymbol{W}. \end{array} \tag{11.57}$$

Here $\delta_w^{\boldsymbol{g},\boldsymbol{h}} : T_z\boldsymbol{Z} \to O_w\boldsymbol{W}$ *is a natural linear map defined as a connecting morphism, as in Definition* 10.69.

(f) *If* $\dot{\mathbf{M}}\mathbf{an}$ *satisfies Assumption* 10.19, *with quasi-tangent spaces* Q_xX *in a category* $\mathcal{Q}$, *and satisfies Assumption* 11.6, *then whenever* (11.55) *is* 2*-Cartesian in* $\dot{\mathbf{K}}\mathbf{ur}_{\boldsymbol{D}}$ *with* $\boldsymbol{g},\boldsymbol{h}$ *w-transverse and* $w \in \boldsymbol{W}$ *with* $\boldsymbol{e}(w) = x$ *in* $\boldsymbol{X}$, $\boldsymbol{f}(w) = y$ *in* $\boldsymbol{Y}$ *and* $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ *in* $\boldsymbol{Z}$, *the following is Cartesian in* $\mathcal{Q}$*:*

$$\begin{array}{ccc} Q_w\boldsymbol{W} & \xrightarrow{Q_w\boldsymbol{f}} & Q_y\boldsymbol{Y} \\ \downarrow{\scriptstyle Q_w\boldsymbol{e}} & & \downarrow{\scriptstyle Q_y\boldsymbol{h}} \\ Q_x\boldsymbol{X} & \xrightarrow{Q_x\boldsymbol{g}} & Q_z\boldsymbol{Z}. \end{array}$$

(g) *If* $\dot{\mathbf{M}}\mathbf{an^c}$ *satisfies Assumption* 3.22 *in* §3.4, *so that we have a corner functor* $C : \dot{\mathbf{M}}\mathbf{an^c} \to \check{\dot{\mathbf{M}}}\mathbf{an^c}$ *which extends to* $C : \dot{\mathbf{K}}\mathbf{ur^c} \to \check{\dot{\mathbf{K}}}\mathbf{ur^c}$ *as in* §6.3, *and Assumption* 11.1 *holds for* $\dot{\mathbf{M}}\mathbf{an^c}$, *and Assumption* 11.7 *holds, then whenever* (11.55) *is* 2*-Cartesian in* $\dot{\mathbf{K}}\mathbf{ur}_{\boldsymbol{D}}$ *with* $\boldsymbol{g},\boldsymbol{h}$ *w-transverse (or transverse), then the following is* 2*-Cartesian in* $\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}_{\boldsymbol{D}}$ *and* $\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}_{\boldsymbol{E}}$, *with* $C(\boldsymbol{g}), C(\boldsymbol{h})$ *w-transverse (or transverse, respectively):*

$$\begin{array}{ccc} C(\boldsymbol{W}) & \xrightarrow{C(\boldsymbol{f})} & C(\boldsymbol{Y}) \\ \downarrow{\scriptstyle C(\boldsymbol{e})} & C(\boldsymbol{\eta})\Uparrow & \downarrow{\scriptstyle C(\boldsymbol{h})} \\ C(\boldsymbol{X}) & \xrightarrow{C(\boldsymbol{g})} & C(\boldsymbol{Z}). \end{array}$$

Hence for $i \geqslant 0$ *we have*

$$C_i(\boldsymbol{W}) \simeq \coprod_{\substack{j,k,l\geqslant 0:\\ i=j+k-l}} \bigl(C_j(\boldsymbol{X}) \cap C(\boldsymbol{g})^{-1}(C_l(\boldsymbol{Z}))\bigr) \times_{C(\boldsymbol{g}),C_l(\boldsymbol{Z}),C(\boldsymbol{h})} \bigl(C_k(\boldsymbol{Y}) \cap C(\boldsymbol{h})^{-1}(C_l(\boldsymbol{Z}))\bigr).$$

When $i = 1$, *this computes the boundary* $\partial\boldsymbol{W}$. *In particular, if* $\partial\boldsymbol{Z} = \emptyset$, *so that* $C_l(\boldsymbol{Z}) = \emptyset$ *for all* $l > 0$ *by Assumption* 3.22(f) *with* $l = 1$, *we have*

$$\partial\boldsymbol{W} \simeq \bigl(\partial\boldsymbol{X} \times_{\boldsymbol{g}\circ\boldsymbol{i_X},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}\bigr) \amalg \bigl(\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}\circ\boldsymbol{i_Y}} \partial\boldsymbol{Y}\bigr).$$

Also, if $\boldsymbol{g}$ *is a w-submersion (or a submersion), then* $C(\boldsymbol{g})$ *is a w-submersion (or a submersion, respectively).*

(h) *If* $\dot{\mathbf{M}}\mathbf{an}$ *satisfies Assumption* 11.8, *and* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *is a w-submersion in* $\dot{\mathbf{K}}\mathbf{ur}_{\boldsymbol{D}}$, *and* $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *is any* 1*-morphism in* $\dot{\mathbf{K}}\mathbf{ur}_{\boldsymbol{E}}$ *(not necessarily in* $\dot{\mathbf{K}}\mathbf{ur}_{\boldsymbol{D}}$*), then a fibre product* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *exists in* $\dot{\mathbf{K}}\mathbf{ur}_{\boldsymbol{E}}$, *with* $\dim \boldsymbol{W} = \dim \boldsymbol{X} + \dim \boldsymbol{Y} - \dim \boldsymbol{Z}$, *in a* 2*-Cartesian square* (11.55) *in* $\dot{\mathbf{K}}\mathbf{ur}_{\boldsymbol{E}}$. *The analogues of* **(a)**–**(d)** *and* **(g)** *hold for these fibre products.*

The proof of Theorem 11.45 is the orbifold analogue of the proof of Theorem 11.19 in §11.9, and we again leave it as an exercise for the reader. Most of the proof requires only cosmetic changes. For the construction of the fibre product $\boldsymbol{W}$ we use Theorem 11.43 rather than Theorem 11.17, and we must include extra 2-morphisms $\alpha_{*,*,*}, \beta_*, \gamma_*$ from §6.1 as Kuranishi neighbourhoods form a weak rather than a strict 2-category, but otherwise the proof is the same.

Remark 11.46. Theorem 11.45(c) should be compared with Theorem 11.19(c) and Theorem 11.39. In Theorem 11.45(c) we do not describe the topological space W of $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ (as we did in Theorem 11.19(c)), but only the underlying set, which is the same as for orbifold fibre products in Theorem 11.39. As in Remark 11.40(b), the topological space does not have an easy description.

A good way to think about this is that just as an m-Kuranishi space $\boldsymbol{W}$ has an underlying topological space W, so a Kuranishi space $\boldsymbol{W}$ has an underlying *Deligne–Mumford topological stack* $\underline{W}$, a kind of orbifold version of topological spaces, as in Noohi [58]. Such stacks form a 2-category $\mathbf{Top_{DM}}$, and there is a weak 2-functor $F_{\dot{\mathbf{Kur}}}^{\mathbf{Top_{DM}}} : \dot{\mathbf{Kur}} \to \mathbf{Top_{DM}}$ mapping $\boldsymbol{W} \mapsto \underline{W}$.

If $\dot{\mathbf{Man}}$ satisfies Assumption 11.3, so that $F_{\dot{\mathbf{Man}}}^{\mathbf{Top}} : \dot{\mathbf{Man}} \to \mathbf{Top}$ takes transverse fibre products in $\mathbf{Man}$ to fibre products in $\mathbf{Top}$, then the 2-functor $F_{\dot{\mathbf{Kur}}}^{\mathbf{Top_{DM}}} : \dot{\mathbf{Kur}} \to \mathbf{Top_{DM}}$ takes w-transverse fibre products in $\dot{\mathbf{Kur}}$ to fibre products in $\mathbf{Top_{DM}}$. So in Theorem 11.45(c) we could say that $\underline{W} = \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y}$ is a fibre product of topological stacks.

All of §11.2.3–§11.2.5 can now be generalized to Kuranishi spaces, mostly with only cosmetic changes. Here is the analogue of Theorem 11.22. The important difference is that as for transversality for orbifolds in Definition 11.38, we must include the action of $\gamma \in G_z\boldsymbol{Z}$ on $Q_y\boldsymbol{h} : Q_y\boldsymbol{Y} \to Q_z\boldsymbol{Z}$ in 'condition $\boldsymbol{T}$', and on $O_y\boldsymbol{h} : O_y\boldsymbol{Y} \to O_z\boldsymbol{Z}$ and $T_y\boldsymbol{h} : T_y\boldsymbol{Y} \to T_z\boldsymbol{Z}$ in (11.58)–(11.59). This appears in the proof when we show the fibre product (11.47) is transverse in $\dot{\mathbf{Man}}$, as several points in (11.47) can lie over each (x, y, z) for $x \in \boldsymbol{X}$, $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$, and the transversality conditions at these points depend on $\gamma \in G_z\boldsymbol{Z}$.

Theorem 11.47. *Let $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfy Assumption* 3.22, *so that we have a corner functor $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}^{\mathbf{c}}$, and suppose Assumption* 11.9 *holds for $\dot{\mathbf{Man}}^{\mathbf{c}}$. This requires that Assumption* 10.1 *holds, giving a notion of tangent spaces T_xX for X in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and that Assumption* 10.19 *holds, giving a notion of quasi-tangent spaces Q_xX in a category $\mathcal{Q}$ for X in $\dot{\mathbf{Man}}^{\mathbf{c}}$, and that Assumption* 11.1 *holds, giving discrete properties $\boldsymbol{D}, \boldsymbol{E}$ of morphisms in $\dot{\mathbf{Man}}^{\mathbf{c}}$ and notions of transverse morphisms g, h and submersions g in $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}}$.*

As in §6.3, §10.2 *and* §10.3, *we define a 2-category $\dot{\mathbf{Kur}}^{\mathbf{c}}$, with a corner 2-functor $C : \dot{\mathbf{Kur}}^{\mathbf{c}} \to \check{\dot{\mathbf{Kur}}}^{\mathbf{c}}$, and notions of tangent, obstruction and quasi-tangent spaces $T_x\boldsymbol{X}, O_x\boldsymbol{X}, Q_x\boldsymbol{X}$ for $\boldsymbol{X}$ in $\dot{\mathbf{Kur}}^{\mathbf{c}}$.*

Now Assumption 11.9(a),(d) *involve a 'condition $\boldsymbol{T}$' on morphisms $g : X \to Z$, $h : Y \to Z$ in $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}}$ and points $x \in X$, $y \in Y$ with $g(x) = h(y) = z \in Z$, and a 'condition $\boldsymbol{S}$' on morphisms $g : X \to Z$ in $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}}$ and points $x \in X$ with $g(x) = z \in Z$. These conditions depend on the corner morphisms $C(g), C(h)$ and on quasi-tangent maps Q_xg, Q_yh. Then:*

(a) *Let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be* 1*-morphisms in $\dot{\mathbf{Kur}}^{\mathbf{c}}_{\boldsymbol{D}}$. Then $\boldsymbol{g}, \boldsymbol{h}$ are w-transverse if and only if for all $x \in \boldsymbol{X}$, $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$ and all $\gamma \in G_z\boldsymbol{Z}$, condition $\boldsymbol{T}$ holds for $\boldsymbol{g}, \boldsymbol{h}, x, y, z, \gamma$ using the morphisms $Q_x\boldsymbol{g} : Q_x\boldsymbol{X} \to Q_z\boldsymbol{Z}$ and $\gamma \cdot Q_x\boldsymbol{h} : Q_y\boldsymbol{Y} \to Q_z\boldsymbol{Z}$ in $\mathcal{Q}$ in Assumption*

11.9(a)(i), *where* $G_z\boldsymbol{Z}$ *acts on* $Q_z\boldsymbol{Z}$, *and the following is surjective:*

$$O_x\boldsymbol{g}\oplus(\gamma\cdot O_y\boldsymbol{h}):O_x\boldsymbol{X}\oplus O_y\boldsymbol{Y}\longrightarrow O_z\boldsymbol{Z}. \tag{11.58}$$

If Assumption 10.9 *also holds for tangent spaces* T_xX *in* $\dot{\mathbf{Man}}^{\mathbf{c}}$ *then* $\boldsymbol{g},\boldsymbol{h}$ *are transverse if and only if for all* $x\in\boldsymbol{X}$ *and* $y\in\boldsymbol{Y}$ *with* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$, *condition* $\boldsymbol{T}$ *holds for* $\boldsymbol{g},\boldsymbol{h},x,y,z,\gamma$ *as above, equation* (11.58) *is an isomorphism, and the following linear map is surjective:*

$$T_x\boldsymbol{g}\oplus(\gamma\cdot T_y\boldsymbol{h}):T_x\boldsymbol{X}\oplus T_y\boldsymbol{Y}\longrightarrow T_z\boldsymbol{Z}. \tag{11.59}$$

(b) *Let* $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$ *be a* 1*-morphism in* $\dot{\mathbf{Kur}}^{\mathbf{c}}_{\boldsymbol{D}}$. *Then* $\boldsymbol{g}$ *is a w-submersion if and only if for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{g}(x)=z$ *in* $\boldsymbol{Z}$, *condition* $\boldsymbol{S}$ *holds for* $\boldsymbol{g},x,z$, *and the following linear map is surjective:*

$$O_x\boldsymbol{g}:O_x\boldsymbol{X}\longrightarrow O_z\boldsymbol{Z}. \tag{11.60}$$

If Assumption 10.9 *also holds then* $\boldsymbol{g}$ *is a submersion if and only if for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{g}(x)=z$ *in* $\boldsymbol{Z}$, *condition* $\boldsymbol{S}$ *holds for* $\boldsymbol{g},x,z$, *equation* (11.60) *is an isomorphism, and the following is surjective:*

$$T_x\boldsymbol{g}:T_x\boldsymbol{X}\longrightarrow T_z\boldsymbol{Z}.$$

For the analogue of Theorem 11.25 we require $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ to be *locally orientable* Kuranishi spaces, as in §10.7.6, so that the canonical bundles $K_{\boldsymbol{X}},K_{\boldsymbol{Y}},K_{\boldsymbol{Z}}$ are defined as in Theorem 10.83. Then the w-transverse fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\dot{\mathbf{Kur}}_{\boldsymbol{D}}$ is also locally orientable, so that (11.24) makes sense.

Remark 11.48. We can relate Theorem 11.45(c),(e) and Theorem 11.47(a) as follows. Let $\dot{\mathbf{Man}}$ satisfy all the relevant assumptions, consider a w-transverse fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\dot{\mathbf{Kur}}$, and suppose $x\in\boldsymbol{X}$ and $y\in\boldsymbol{Y}$ with $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z\in\boldsymbol{Z}$. Defining the morphisms $G_x\boldsymbol{g}:G_x\boldsymbol{X}\rightarrow G_z\boldsymbol{Z}$ and $G_y\boldsymbol{h}:G_y\boldsymbol{Y}\rightarrow G_z\boldsymbol{Z}$ in §6.5 requires arbitrary choices. The same arbitrary choices are involved in the description (11.56) of W as a set, and in the linear maps $T_x\boldsymbol{g},O_x\boldsymbol{g},T_x\boldsymbol{h},O_x\boldsymbol{h}$ from §10.2.3 involved in (11.57)–(11.59).

If we take (11.56)–(11.59) all to be defined using the same arbitrary choices for $G_x\boldsymbol{g},G_y\boldsymbol{h}$, and we write $w\in W$ as (x,y,C) as in (11.56) with $\gamma\in C\subseteq G_z\boldsymbol{Z}$, then we may rewrite (11.57) as the exact sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow T_{(x,y,C)}\boldsymbol{W} & \longrightarrow & T_x\boldsymbol{X}\oplus T_y\boldsymbol{Y} & \xrightarrow{T_x\boldsymbol{g}\oplus(\gamma\cdot T_y\boldsymbol{h})} & T_z\boldsymbol{Z} \\ & & & & \downarrow \\ 0 \longleftarrow O_z\boldsymbol{Z} & \xleftarrow{O_x\boldsymbol{g}\oplus(\gamma\cdot O_y\boldsymbol{h})} & O_x\boldsymbol{X}\oplus O_y\boldsymbol{Y} & \longleftarrow & O_{(x,y,C)}\boldsymbol{W}. \end{array} \tag{11.61}$$

Thus we see that:

- We need (11.61) to be exact for all $C \in G_x\boldsymbol{g}(G_x\boldsymbol{X})\backslash G_z\boldsymbol{Z}/G_y\boldsymbol{h}(G_y\boldsymbol{Y})$, and hence for all $\gamma \in G_z\boldsymbol{Z}$. Thus it is necessary for $O_x\boldsymbol{g} \oplus (\gamma \cdot O_y\boldsymbol{h})$ to be surjective for all $\gamma \in G_z\boldsymbol{Z}$ for w-transverse $\boldsymbol{g}, \boldsymbol{h}$, as in Theorem 11.47(a).
- If $\boldsymbol{g}, \boldsymbol{h}$ are transverse then $\boldsymbol{W}$ is a manifold, and $O_{(x,y,C)}\boldsymbol{W} = 0$ for all (x, y, C). Thus by (11.61) it is necessary that $O_x\boldsymbol{g} \oplus (\gamma \cdot O_y\boldsymbol{h})$ is an isomorphism and $T_x\boldsymbol{g} \oplus (\gamma \cdot T_y\boldsymbol{h})$ is surjective for all $\gamma \in G_z\boldsymbol{Z}$ for transverse $\boldsymbol{g}, \boldsymbol{h}$, as in Theorem 11.47(a).

11.6 Fibre products in $\mathbf{Kur}, \mathbf{Kur}^{\mathbf{c}}_{\mathbf{st}}, \mathbf{Kur}^{\mathbf{gc}}$ and $\mathbf{Kur}^{\mathbf{c}}$

We now generalize §11.3 to Kuranishi spaces, using the material of §11.5.

11.6.1 Fibre products in Kur

As in §11.3.1, take $\dot{\mathbf{Man}}$ to be the category of classical manifolds **Man**, with corresponding 2-category of Kuranishi spaces **Kur** as in Definition 6.29. We will use tangent spaces $T_x\boldsymbol{X}$ for $\boldsymbol{X}$ in **Kur** defined using ordinary tangent spaces T_vV in **Man**. Definition 2.21 in §2.5.1 defines transverse morphisms and submersions in **Man**. As in Example 11.10, these satisfy Assumptions 11.1, 11.3–11.5 and 11.9. So Definition 11.44 defines (w-)transverse 1-morphisms and (w-)submersions in **Kur**. Here is the analogue of Theorem 11.28:

Theorem 11.49. **(a)** *Let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms in* **Kur**. *Then $\boldsymbol{g}, \boldsymbol{h}$ are w-transverse if and only if for all $x \in \boldsymbol{X}$, $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$ and all $\gamma \in G_z\boldsymbol{Z}$, the following is surjective:*

$$O_x\boldsymbol{g} \oplus (\gamma \cdot O_y\boldsymbol{h}) : O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y} \longrightarrow O_z\boldsymbol{Z}. \tag{11.62}$$

This is automatic if $\boldsymbol{Z}$ is an orbifold. Also $\boldsymbol{g}, \boldsymbol{h}$ are transverse if and only if for all x, y, z, γ, equation (11.62) *is an isomorphism, and the following is surjective:*

$$T_x\boldsymbol{g} \oplus (\gamma \cdot T_y\boldsymbol{h}) : T_x\boldsymbol{X} \oplus T_y\boldsymbol{Y} \longrightarrow T_z\boldsymbol{Z}.$$

(b) *If $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are w-transverse in* **Kur** *then a fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in* **Kur**, *in a* 2*-Cartesian square:*

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow{\quad \boldsymbol{f} \quad} & \boldsymbol{Y} \\ \downarrow{\scriptstyle \boldsymbol{e}} & \boldsymbol{\eta}\Uparrow & \downarrow{\scriptstyle \boldsymbol{h}} \\ \boldsymbol{X} & \xrightarrow{\quad \boldsymbol{g} \quad} & \boldsymbol{Z}. \end{array} \tag{11.63}$$

It has $\operatorname{vdim}\boldsymbol{W} = \operatorname{vdim}\boldsymbol{X} + \operatorname{vdim}\boldsymbol{Y} - \operatorname{vdim}\boldsymbol{Z}$. *Just as a set, the underlying topological space W in $\boldsymbol{W} = (W, \mathcal{H})$ may be written*

$$W = \bigl\{(x, y, C) : x \in X,\ \ y \in Y,\ \ C \in G_x\boldsymbol{g}(G_x\boldsymbol{X})\backslash G_z\boldsymbol{Z}/G_y\boldsymbol{h}(G_y\boldsymbol{Y})\bigr\},$$

where $\boldsymbol{e}, \boldsymbol{f}$ map $\boldsymbol{e} : (x, y, C) \mapsto x$, $\boldsymbol{f} : (x, y, C) \mapsto y$. The isotropy groups satisfy

$$G_{(x,y,C)}\boldsymbol{W} \cong \bigl\{(\alpha, \beta) \in G_x\boldsymbol{X} \times G_y\boldsymbol{Y} : G_x\boldsymbol{g}(\alpha)\,\gamma\,G_y\boldsymbol{h}(\beta^{-1}) = \gamma\bigr\}$$

for fixed $\gamma \in C \subseteq G_z\boldsymbol{Z}$*. If* $w \in \boldsymbol{W}$ *with* $\boldsymbol{e}(w) = x$ *in* $\boldsymbol{X}$*,* $\boldsymbol{f}(w) = y$ *in* $\boldsymbol{Y}$ *and* $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ *in* $\boldsymbol{Z}$*, for some possible choices of* $T_w\boldsymbol{e}, T_w\boldsymbol{f}, \ldots, O_y\boldsymbol{h}$ *in Definition* 10.28 *depending on* w*, the following is an exact sequence:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_w\boldsymbol{W} & \xrightarrow{T_w\boldsymbol{e}\oplus T_w\boldsymbol{f}} & T_x\boldsymbol{X} \oplus T_y\boldsymbol{Y} & \xrightarrow{T_x\boldsymbol{g}\oplus -T_y\boldsymbol{h}} & T_z\boldsymbol{Z} \\ & & & & & & \big\downarrow \delta_w^{\boldsymbol{g},\boldsymbol{h}} \\ 0 & \longleftarrow & O_z\boldsymbol{Z} & \xleftarrow{O_x\boldsymbol{g}\oplus -O_y\boldsymbol{h}} & O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y} & \xleftarrow{O_w\boldsymbol{e}\oplus O_w\boldsymbol{f}} & O_w\boldsymbol{W}. \end{array}$$

If $\boldsymbol{g}, \boldsymbol{h}$ *are transverse then* $\boldsymbol{W}$ *is an orbifold.*

(c) *In part* **(b)***, using the theory of canonical bundles and orientations from* §10.7.6*, suppose* $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ *are locally orientable. Then* $\boldsymbol{W}$ *is also locally orientable, and there is a natural isomorphism of topological line bundles on* W*:*

$$\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}} : K_{\boldsymbol{W}} \longrightarrow e^*(K_{\boldsymbol{X}}) \otimes f^*(K_{\boldsymbol{Y}}) \otimes (g \circ e)^*(K_{\boldsymbol{Z}})^*. \tag{11.64}$$

Hence if $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ *are oriented there is a unique orientation on* $\boldsymbol{W}$*, called the* ***fibre product orientation****, such that* (11.64) *is orientation-preserving. Proposition* 11.26 *holds for these fibre product orientations.*

(d) *Let* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *be a* 1*-morphism in* **Kur***. Then* $\boldsymbol{g}$ *is a w-submersion if and only if* $O_x\boldsymbol{g} : O_x\boldsymbol{X} \to O_z\boldsymbol{Z}$ *is surjective for all* $x \in \boldsymbol{X}$ *with* $\boldsymbol{g}(x) = z$ *in* $\boldsymbol{Z}$*. Also* $\boldsymbol{g}$ *is a submersion if and only if* $O_x\boldsymbol{g} : O_x\boldsymbol{X} \to O_z\boldsymbol{Z}$ *is an isomorphism and* $T_x\boldsymbol{g} : T_x\boldsymbol{X} \to T_z\boldsymbol{Z}$ *is surjective for all* x, z*.*

(e) *If* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *and* $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *are* 1*-morphisms in* **Kur** *with* $\boldsymbol{g}$ *a w-submersion then* $\boldsymbol{g}, \boldsymbol{h}$ *are w-transverse. If* $\boldsymbol{g}$ *is a submersion and* $\boldsymbol{Y}$ *is an orbifold then* $\boldsymbol{g}, \boldsymbol{h}$ *are transverse.*

(f) *If* (11.63) *is* 2*-Cartesian in* **Kur** *with* $\boldsymbol{g}$ *a w-submersion (or a submersion) then* $\boldsymbol{f}$ *is a w-submersion (or a submersion).*

(g) *Compositions and products of (w-)submersions in* **Kur** *are (w-)submersions. Projections* $\boldsymbol{\pi_X} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{X}$ *in* **Kur** *are w-submersions.*

11.6.2 Fibre products in $\mathbf{Kur^c_{st}}$ and $\mathbf{Kur^c}$

In §2.5.2, working in the subcategory $\mathbf{Man^c_{st}} \subset \mathbf{Man^c}$ from §2.1, we defined *s-transverse* and *t-transverse morphisms* and *s-submersions*. Example 11.11 explained how make these satisfy Assumptions 11.1 and x11.3–11.9.

The next theorem is the analogue of Theorem 11.32. Here $\mathbf{Kur^c_{st}} \subset \mathbf{Kur^c}$ are the 2-categories of Kuranishi spaces corresponding to $\mathbf{Man^c_{st}} \subset \mathbf{Man^c}$ as in Definition 6.29, the corner functors $C, C' : \mathbf{Kur^c_{st}} \to \mathbf{\check{K}ur^c_{st}}$ and $C, C' : \mathbf{Kur^c} \to \mathbf{\check{K}ur^c}$ are as in (6.36), (stratum) tangent spaces $T_x\boldsymbol{X}, \tilde{T}_x\boldsymbol{X}$ are as in Example 10.25(i),(iii), and stratum normal spaces $\tilde{N}_x\boldsymbol{X}$ are as in Example 10.32(a).

We use the notation *ws-transverse*, *wt-transverse*, and *ws-submersions* for the notions of w-transverse and w-submersion in $\mathbf{Kur^c_{st}}$ corresponding to s- and t-transverse morphisms and s-submersions, and *s-transverse*, *t-transverse*, and *s-submersions* for the corresponding notions of transverse and submersion.

Theorem 11.50. (a) *Let* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *be* 1*-morphisms in* $\mathbf{Kur^c_{st}}$. *Then* $\boldsymbol{g},\boldsymbol{h}$ *are ws-transverse if and only if for all* $x\in\boldsymbol{X}$, $y\in\boldsymbol{Y}$ *with* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$ *and all* $\gamma\in G_z\boldsymbol{Z}$, *the following linear maps are surjective:*

$$\tilde O_x\boldsymbol{g}\oplus(\gamma\cdot\tilde O_y\boldsymbol{h}):\tilde O_x\boldsymbol{X}\oplus\tilde O_y\boldsymbol{Y}\longrightarrow\tilde O_z\boldsymbol{Z},\tag{11.65}$$

$$\tilde N_x\boldsymbol{g}\oplus(\gamma\cdot\tilde N_y\boldsymbol{h}):\tilde N_x\boldsymbol{X}\oplus\tilde N_y\boldsymbol{Y}\longrightarrow\tilde N_z\boldsymbol{Z}.\tag{11.66}$$

This is automatic if $\boldsymbol{Z}$ *is a classical orbifold. Also* $\boldsymbol{g},\boldsymbol{h}$ *are s-transverse if and only if for all* x,y,z,γ, *equation* (11.65) *is an isomorphism, and* (11.66) *and the following are surjective:*

$$\tilde T_x\boldsymbol{g}\oplus(\gamma\cdot\tilde T_y\boldsymbol{h}):\tilde T_x\boldsymbol{X}\oplus\tilde T_y\boldsymbol{Y}\longrightarrow\tilde T_z\boldsymbol{Z}.$$

Furthermore, $\boldsymbol{g},\boldsymbol{h}$ *are wt-transverse (or t-transverse) if and only if they are ws-transverse (or s-transverse), and for all* x,y,z *as above, whenever* $\boldsymbol{x}\in C_j(\boldsymbol{X})$ *and* $\boldsymbol{y}\in C_k(\boldsymbol{Y})$ *with* $\boldsymbol{\Pi}_j(\boldsymbol{x})=x$, $\boldsymbol{\Pi}_k(\boldsymbol{y})=y$, *and* $C(\boldsymbol{g})\boldsymbol{x}=C(\boldsymbol{h})\boldsymbol{y}=\boldsymbol{z}$ *in* $C_l(\boldsymbol{Z})$, *we have* $j+k\geqslant l$, *and there is exactly one triple* $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ *with* $j+k=l$.

(b) *If* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *are ws-transverse in* $\mathbf{Kur^c_{st}}$ *then a fibre product* $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ *exists in* $\mathbf{Kur^c_{st}}$, *in a* 2*-Cartesian square:*

$$\begin{array}{ccc}\boldsymbol{W} & \xrightarrow[\quad\boldsymbol{f}\quad]{} & \boldsymbol{Y}\\ \downarrow{\scriptstyle\boldsymbol{e}} & \boldsymbol{\eta}\Uparrow & {\scriptstyle\boldsymbol{h}}\downarrow\\ \boldsymbol{X} & \xrightarrow{\quad\boldsymbol{g}\quad} & \boldsymbol{Z}.\end{array}\tag{11.67}$$

It has $\mathrm{vdim}\,\boldsymbol{W}=\mathrm{vdim}\,\boldsymbol{X}+\mathrm{vdim}\,\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{Z}$. *Just as a set, the underlying topological space* W *in* $\boldsymbol{W}=(W,\mathcal{H})$ *may be written*

$$W=\bigl\{(x,y,C):x\in X,\;\;y\in Y,\;\;C\in G_x\boldsymbol{g}(G_x\boldsymbol{X})\backslash G_z\boldsymbol{Z}/G_y\boldsymbol{h}(G_y\boldsymbol{Y})\bigr\},\tag{11.68}$$

where $\boldsymbol{e},\boldsymbol{f}$ *map* $\boldsymbol{e}:(x,y,C)\mapsto x$, $\boldsymbol{f}:(x,y,C)\mapsto y$. *The isotropy groups satisfy*

$$G_{(x,y,C)}\boldsymbol{W}\cong\bigl\{(\alpha,\beta)\in G_x\boldsymbol{X}\times G_y\boldsymbol{Y}:G_x\boldsymbol{g}(\alpha)\,\gamma\,G_y\boldsymbol{h}(\beta^{-1})=\gamma\bigr\}$$

for fixed $\gamma\in C\subseteq G_z\boldsymbol{Z}$. *Equation* (11.67) *is also* 2*-Cartesian in* $\mathbf{Kur^c}$.

If $w\in\boldsymbol{W}$ *with* $\boldsymbol{e}(w)=x$ *in* $\boldsymbol{X}$, $\boldsymbol{f}(w)=y$ *in* $\boldsymbol{Y}$ *and* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$, *for some possible choices of* $T_w\boldsymbol{e},\ldots,O_y\boldsymbol{h},\tilde T_w\boldsymbol{e},\ldots,\tilde O_y\boldsymbol{h},\tilde N_w\boldsymbol{e},\ldots,\tilde N_y\boldsymbol{h}$ *in Definition* 10.28 *and* §10.3.3 *depending on* w, *the following sequences are exact:*

$$\begin{array}{ccccccc}0 & \longrightarrow & T_w\boldsymbol{W} & \xrightarrow[T_w\boldsymbol{e}\oplus T_w\boldsymbol{f}]{} & T_x\boldsymbol{X}\oplus T_y\boldsymbol{Y} & \xrightarrow[T_x\boldsymbol{g}\oplus -T_y\boldsymbol{h}]{} & T_z\boldsymbol{Z}\\ &&&&&& \downarrow{\scriptstyle\delta_w^{\boldsymbol{g},\boldsymbol{h}}}\\ 0 & \longleftarrow & O_z\boldsymbol{Z} & \xleftarrow{O_x\boldsymbol{g}\oplus -O_y\boldsymbol{h}} & O_x\boldsymbol{X}\oplus O_y\boldsymbol{Y} & \xleftarrow{O_w\boldsymbol{e}\oplus O_w\boldsymbol{f}} & O_w\boldsymbol{W},\end{array}$$

$$\begin{array}{ccccccc}0 & \longrightarrow & \tilde T_w\boldsymbol{W} & \xrightarrow[\tilde T_w\boldsymbol{e}\oplus \tilde T_w\boldsymbol{f}]{} & \tilde T_x\boldsymbol{X}\oplus \tilde T_y\boldsymbol{Y} & \xrightarrow[\tilde T_x\boldsymbol{g}\oplus -\tilde T_y\boldsymbol{h}]{} & \tilde T_z\boldsymbol{Z}\\ &&&&&& \downarrow{\scriptstyle\tilde\delta_w^{\boldsymbol{g},\boldsymbol{h}}}\\ 0 & \longleftarrow & \tilde O_z\boldsymbol{Z} & \xleftarrow{\tilde O_x\boldsymbol{g}\oplus -\tilde O_y\boldsymbol{h}} & \tilde O_x\boldsymbol{X}\oplus \tilde O_y\boldsymbol{Y} & \xleftarrow{\tilde O_w\boldsymbol{e}\oplus \tilde O_w\boldsymbol{f}} & \tilde O_w\boldsymbol{W},\end{array}$$

$$0\longrightarrow\tilde N_w\boldsymbol{W}\xrightarrow{\tilde N_w\boldsymbol{e}\oplus\tilde N_w\boldsymbol{f}}\tilde N_x\boldsymbol{X}\oplus\tilde N_y\boldsymbol{Y}\xrightarrow{\tilde N_x\boldsymbol{g}\oplus-\tilde N_y\boldsymbol{h}}\tilde N_z\boldsymbol{Z}\longrightarrow 0.$$

If $\boldsymbol{g}, \boldsymbol{h}$ are s-transverse then $\boldsymbol{W}$ is an orbifold.

(c) *In part* **(b)**, *if* (11.67) *is* 2*-Cartesian in* $\mathbf{Kur^c_{st}}$ *with* $\boldsymbol{g}, \boldsymbol{h}$ *wt-transverse (or t-transverse), then the following is* 2*-Cartesian in* $\mathbf{\check{K}ur^c_{st}}$ *and* $\mathbf{\check{K}ur^c}$, *with* $C(\boldsymbol{g})$, $C(\boldsymbol{h})$ *wt-transverse (or t-transverse, respectively):*

$$\begin{array}{ccc} C(\boldsymbol{W}) & \xrightarrow[\qquad C(\boldsymbol{f}) \qquad]{} & C(\boldsymbol{Y}) \\ \Big\downarrow {\scriptstyle C(\boldsymbol{e})} & {\scriptstyle C(\boldsymbol{\eta})} \Uparrow & {\scriptstyle C(\boldsymbol{h})} \Big\downarrow \\ C(\boldsymbol{X}) & \xrightarrow{\qquad C(\boldsymbol{g}) \qquad} & C(\boldsymbol{Z}). \end{array}$$

Hence we have

$$C_i(\boldsymbol{W}) \simeq \coprod_{\substack{j,k,l \geqslant 0: \\ i=j+k-l}} \bigl(C_j(\boldsymbol{X}) \cap C(\boldsymbol{g})^{-1}(C_l(\boldsymbol{Z}))\bigr) \times_{C(\boldsymbol{g}), C_l(\boldsymbol{Z}), C(\boldsymbol{h})} \bigl(C_k(\boldsymbol{Y}) \cap C(\boldsymbol{h})^{-1}(C_l(\boldsymbol{Z}))\bigr)$$

for $i \geqslant 0$. *When* $i = 1$, *this computes the boundary* $\partial \boldsymbol{W}$.

Also, if $\boldsymbol{g}$ *is a ws-submersion (or an s-submersion), then* $C(\boldsymbol{g})$ *is a ws-submersion (or an s-submersion, respectively).*

The analogue of the above also holds for $C' : \mathbf{Kur^c_{st}} \rightarrow \mathbf{\check{K}ur^c_{st}}$.

(d) *In part* **(b)**, *using the theory of canonical bundles and orientations from* §10.7.6, *suppose* $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ *are locally orientable. Then* $\boldsymbol{W}$ *is also locally orientable, and there is a natural isomorphism of topological line bundles on* W:

$$\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}} : K_{\boldsymbol{W}} \longrightarrow e^*(K_{\boldsymbol{X}}) \otimes f^*(K_{\boldsymbol{Y}}) \otimes (g \circ e)^*(K_{\boldsymbol{Z}})^*. \tag{11.69}$$

Hence if $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ *are oriented there is a unique orientation on* $\boldsymbol{W}$, *called the* ***fibre product orientation***, *such that* (11.69) *is orientation-preserving. Propositions* 11.26 *and* 11.27 *hold for these fibre product orientations.*

(e) *Let* $\boldsymbol{g} : \boldsymbol{X} \rightarrow \boldsymbol{Z}$ *be a* 1*-morphism in* $\mathbf{Kur^c_{st}}$. *Then* $\boldsymbol{g}$ *is a ws-submersion if and only if* $\tilde{O}_x\boldsymbol{g} : \tilde{O}_x\boldsymbol{X} \rightarrow \tilde{O}_z\boldsymbol{Z}$ *and* $\tilde{N}_x\boldsymbol{g} : \tilde{N}_x\boldsymbol{X} \rightarrow \tilde{N}_z\boldsymbol{Z}$ *are surjective for all* $x \in \boldsymbol{X}$ *with* $\boldsymbol{g}(x) = z$ *in* $\boldsymbol{Z}$. *Also* $\boldsymbol{g}$ *is an s-submersion if and only if* $\tilde{O}_x\boldsymbol{g} : \tilde{O}_x\boldsymbol{X} \rightarrow \tilde{O}_z\boldsymbol{Z}$ *is an isomorphism and* $\tilde{T}_x\boldsymbol{g} : \tilde{T}_x\boldsymbol{X} \rightarrow \tilde{T}_z\boldsymbol{Z}$, $\tilde{N}_x\boldsymbol{g} : \tilde{N}_x\boldsymbol{X} \rightarrow \tilde{N}_z\boldsymbol{Z}$ *are surjective for all* x, z.

(f) *If* $\boldsymbol{g} : \boldsymbol{X} \rightarrow \boldsymbol{Z}$ *and* $\boldsymbol{h} : \boldsymbol{Y} \rightarrow \boldsymbol{Z}$ *are* 1*-morphisms in* $\mathbf{Kur^c_{st}}$ *with* $\boldsymbol{g}$ *a ws-submersion then* $\boldsymbol{g}, \boldsymbol{h}$ *are ws-transverse and wt-transverse. If* $\boldsymbol{g}$ *is an s-submersion and* $\boldsymbol{Y}$ *is an orbifold then* $\boldsymbol{g}, \boldsymbol{h}$ *are s-transverse and t-transverse.*

(g) *If* (11.67) *is* 2*-Cartesian in* $\mathbf{Kur^c_{st}}$ *with* $\boldsymbol{g}$ *a ws-submersion (or an s-submersion) then* $\boldsymbol{f}$ *is a ws-submersion (or an s-submersion).*

(h) *Compositions and products of ws- or s-submersions in* $\mathbf{Kur^c_{st}}$ *are ws- or s-submersions. Projections* $\boldsymbol{\pi_X} : \boldsymbol{X} \times \boldsymbol{Y} \rightarrow \boldsymbol{X}$ *in* $\mathbf{Kur^c_{st}}$ *are ws-submersions.*

(i) *If* $\boldsymbol{g} : \boldsymbol{X} \rightarrow \boldsymbol{Z}$ *is a ws-submersion in* $\mathbf{Kur^c_{st}}$, *and* $\boldsymbol{h} : \boldsymbol{Y} \rightarrow \boldsymbol{Z}$ *is any* 1*-morphism in* $\mathbf{Kur^c}$ *(not necessarily in* $\mathbf{Kur^c_{st}}$*), then a fibre product* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *exists in* $\mathbf{Kur^c}$, *with* $\dim \boldsymbol{W} = \dim \boldsymbol{X} + \dim \boldsymbol{Y} - \dim \boldsymbol{Z}$, *in a* 2*-Cartesian square* (11.67) *in* $\mathbf{Kur^c}$. *It has topological space* W *given as a set by* (11.68). *The analogues of* **(c)**,**(g)** *hold for these fibre products. If* $\boldsymbol{g}$ *is an s-submersion and* $\boldsymbol{Y}$ *is an orbifold then* $\boldsymbol{W}$ *is an orbifold.*

11.6.3 Fibre products in $\mathbf{Kur_{in}^{gc}}$ and $\mathbf{Kur^{gc}}$

In §2.5.3, working in $\mathbf{Man_{in}^{gc}} \subset \mathbf{Man^{gc}}$ from §2.4.1, we defined *b-transverse* and *c-transverse morphisms* and *b-submersions*, *b-fibrations*, and *c-fibrations*. Example 11.12 explained how to fit these into the framework of Assumptions 11.1 and 11.3–11.9. The next theorem is the analogue of Theorem 11.34.

Here $\mathbf{Kur_{in}^{gc}} \subset \mathbf{Kur^{gc}}$ are the 2-categories of Kuranishi spaces corresponding to $\mathbf{Man_{in}^{gc}} \subset \mathbf{Man^{gc}}$ as in Definition 6.29, the corner 2-functor $C : \mathbf{Kur^{gc}} \to \check{\mathbf{K}}\mathbf{ur^{gc}}$ is as in (6.36), and b-tangent spaces $T_x\boldsymbol{X}$ are as in Example 10.25(ii). We use the notation *wb-transverse*, *wc-transverse*, *wb-submersions*, *wb-fibrations*, *wc-fibrations* for the weak versions of b-transverse, ..., c-fibrations in $\mathbf{Kur_{in}^{gc}}$ from Definition 11.44, and *b-transverse*, *c-transverse*, *b-submersions*, *b-fibrations*, and *c-fibrations* for the strong versions.

Theorem 11.51. **(a)** *Let* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *and* $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *be* 1*-morphisms in* $\mathbf{Kur_{in}^{gc}}$. *Then* $\boldsymbol{g}, \boldsymbol{h}$ *are wb-transverse if and only if for all* $x \in \boldsymbol{X}$, $y \in \boldsymbol{Y}$ *with* $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ *in* $\boldsymbol{Z}$ *and all* $\gamma \in G_z\boldsymbol{Z}$, *the following linear map is surjective:*

$$ {}^bO_x\boldsymbol{g} \oplus (\gamma \cdot {}^bO_y\boldsymbol{h}) : {}^bO_x\boldsymbol{X} \oplus {}^bO_y\boldsymbol{Y} \longrightarrow {}^bO_z\boldsymbol{Z}. \tag{11.70}$$

This is automatic if $\boldsymbol{Z}$ *is an orbifold. Also* $\boldsymbol{g}, \boldsymbol{h}$ *are b-transverse if and only if for all* x, y, z, γ, *equation* (11.70) *is an isomorphism, and the following is surjective:*

$$ {}^bT_x\boldsymbol{g} \oplus (\gamma \cdot {}^bT_y\boldsymbol{h}) : {}^bT_x\boldsymbol{X} \oplus {}^bT_y\boldsymbol{Y} \longrightarrow {}^bT_z\boldsymbol{Z}.$$

Furthermore, $\boldsymbol{g}, \boldsymbol{h}$ *are wc-transverse (or c-transverse) if and only if they are wb-transverse (or b-transverse), and whenever* $\boldsymbol{x} \in C_j(\boldsymbol{X})$ *and* $\boldsymbol{y} \in C_k(\boldsymbol{Y})$ *with* $C(\boldsymbol{g})\boldsymbol{x} = C(\boldsymbol{h})\boldsymbol{y} = \boldsymbol{z}$ *in* $C_l(\boldsymbol{Z})$, *we have either* $j + k > l$, *or* $j = k = l = 0$.

(b) *If* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ *and* $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *are wb-transverse in* $\mathbf{Kur_{in}^{gc}}$ *then a fibre product* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *exists in* $\mathbf{Kur_{in}^{gc}}$, *in a 2-Cartesian square:*

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow{\quad \boldsymbol{f} \quad} & \boldsymbol{Y} \\ \downarrow \boldsymbol{e} & \boldsymbol{\eta} \Uparrow & \boldsymbol{h} \downarrow \\ \boldsymbol{X} & \xrightarrow{\quad \boldsymbol{g} \quad} & \boldsymbol{Z}. \end{array} \tag{11.71}$$

It has $\operatorname{vdim} \boldsymbol{W} = \operatorname{vdim} \boldsymbol{X} + \operatorname{vdim} \boldsymbol{Y} - \operatorname{vdim} \boldsymbol{Z}$. *If* $w \in \boldsymbol{W}$ *with* $\boldsymbol{e}(w) = x$ *in* $\boldsymbol{X}$, $\boldsymbol{f}(w) = y$ *in* $\boldsymbol{Y}$ *and* $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ *in* $\boldsymbol{Z}$, *for some possible choices of* ${}^bT_w\boldsymbol{e}, {}^bT_w\boldsymbol{f}, {}^bT_x\boldsymbol{g}, {}^bT_y\boldsymbol{h}, {}^bO_w\boldsymbol{e}, {}^bO_w\boldsymbol{f}, {}^bO_x\boldsymbol{g}, {}^bO_y\boldsymbol{h}$ *in Definition* 10.28 *depending on* w, *the following sequence is exact:*

$$\begin{array}{ccccccc} 0 \longrightarrow {}^bT_w\boldsymbol{W} & \xrightarrow{{}^bT_w\boldsymbol{e} \oplus {}^bT_w\boldsymbol{f}} & {}^bT_x\boldsymbol{X} \oplus {}^bT_y\boldsymbol{Y} & \xrightarrow{{}^bT_x\boldsymbol{g} \oplus - {}^bT_y\boldsymbol{h}} & {}^bT_z\boldsymbol{Z} \\ & & & & \downarrow {}^b\delta_w^{\boldsymbol{g},\boldsymbol{h}} \\ 0 \longleftarrow {}^bO_z\boldsymbol{Z} & \xleftarrow{{}^bO_x\boldsymbol{g} \oplus - {}^bO_y\boldsymbol{h}} & {}^bO_x\boldsymbol{X} \oplus {}^bO_y\boldsymbol{Y} & \xleftarrow{{}^bO_w\boldsymbol{e} \oplus {}^bO_w\boldsymbol{f}} & {}^bO_w\boldsymbol{W}. \end{array}$$

If $\boldsymbol{g}, \boldsymbol{h}$ *are b-transverse then* $\boldsymbol{W}$ *is an orbifold.*

(c) *In* **(b)**, *if* $\boldsymbol{g},\boldsymbol{h}$ *are wc-transverse then just as a set, the underlying topological space* W *in* $\boldsymbol{W}=(W,\mathcal{H})$ *may be written*

$$W=\bigl\{(x,y,C):x\in X,\;\; y\in Y,\;\; C\in G_x\boldsymbol{g}(G_x\boldsymbol{X})\backslash G_z\boldsymbol{Z}/G_y\boldsymbol{h}(G_y\boldsymbol{Y})\bigr\},\quad (11.72)$$

where $\boldsymbol{e},\boldsymbol{f}$ *map* $\boldsymbol{e}:(x,y,C)\mapsto x$, $\boldsymbol{f}:(x,y,C)\mapsto y$. *The isotropy groups satisfy*

$$G_{(x,y,C)}\boldsymbol{W}\cong\bigl\{(\alpha,\beta)\in G_x\boldsymbol{X}\times G_y\boldsymbol{Y}:G_x\boldsymbol{g}(\alpha)\,\gamma\,G_y\boldsymbol{h}(\beta^{-1})=\gamma\bigr\}$$

for fixed $\gamma\in C\subseteq G_z\boldsymbol{Z}$. *Also* (11.71) *is* 2*-Cartesian in* $\mathbf{Kur^{gc}}$, *and the following is* 2*-Cartesian in* $\check{\mathbf{K}}\mathbf{ur^{gc}_{in}}$ *and* $\check{\mathbf{K}}\mathbf{ur^{gc}}$, *with* $C(\boldsymbol{g}),C(\boldsymbol{h})$ *wc-transverse:*

$$\begin{array}{ccc} C(\boldsymbol{W}) & \xrightarrow[\quad C(\boldsymbol{f})\quad]{} & C(\boldsymbol{Y}) \\ \Big\downarrow{\scriptstyle C(\boldsymbol{e})} & C(\boldsymbol{\eta})\Uparrow & {\scriptstyle C(\boldsymbol{h})}\Big\downarrow \\ C(\boldsymbol{X}) & \xrightarrow{\quad C(\boldsymbol{g})\quad} & C(\boldsymbol{Z}). \end{array}$$

Hence we have

$$C_i(\boldsymbol{W})\simeq\coprod_{\substack{j,k,l\geqslant 0:\\ i=j+k-l}}\begin{array}{l}\bigl(C_j(\boldsymbol{X})\cap C(\boldsymbol{g})^{-1}(C_l(\boldsymbol{Z}))\bigr)\times_{C(\boldsymbol{g}),C_l(\boldsymbol{Z}),C(\boldsymbol{h})}\\ \qquad\qquad\bigl(C_k(\boldsymbol{Y})\cap C(\boldsymbol{h})^{-1}(C_l(\boldsymbol{Z}))\bigr)\end{array}$$

for $i\geqslant 0$. *When* $i=1$, *this computes the boundary* $\partial\boldsymbol{W}$.

Also, if $\boldsymbol{g}$ *is a wb-fibration, or b-fibration, or wc-fibration, or c-fibration, then* $C(\boldsymbol{g})$ *is a wb-fibration, ..., or c-fibration, respectively.*

(d) *In part* **(b)**, *using the theory of (b-)canonical bundles and orientations from* §10.7.6, *suppose* $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ *are locally orientable. Then* $\boldsymbol{W}$ *is also locally orientable, and there is a natural isomorphism of topological line bundles on* W:

$${}^b\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}:{}^bK_{\boldsymbol{W}}\longrightarrow e^*({}^bK_{\boldsymbol{X}})\otimes f^*({}^bK_{\boldsymbol{Y}})\otimes(g\circ e)^*({}^bK_{\boldsymbol{Z}})^*.\qquad (11.73)$$

Hence if $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ *are oriented there is a unique orientation on* $\boldsymbol{W}$, *called the* ***fibre product orientation***, *such that* (11.73) *is orientation-preserving. Propositions* 11.26 *and* 11.27 *hold for these fibre product orientations.*

(e) *Let* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *be a* 1*-morphism in* $\mathbf{Kur^{gc}_{in}}$. *Then* $\boldsymbol{g}$ *is a wb-submersion if and only if* ${}^bO_x\boldsymbol{g}:{}^bO_x\boldsymbol{X}\to{}^bO_z\boldsymbol{Z}$ *is surjective for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{g}(x)=z$ *in* $\boldsymbol{Z}$. *Also* $\boldsymbol{g}$ *is a b-submersion if and only if* ${}^bO_x\boldsymbol{g}:{}^bO_x\boldsymbol{X}\to{}^bO_z\boldsymbol{Z}$ *is an isomorphism and* ${}^bT_x\boldsymbol{g}:{}^bT_x\boldsymbol{X}\to{}^bT_z\boldsymbol{Z}$ *is surjective for all* x,z.

Furthermore $\boldsymbol{g}$ *is a wb-fibration (or a b-fibration) if it is a wb-submersion (or b-submersion) and whenever there are* $\boldsymbol{x},\boldsymbol{z}$ *in* $C_j(\boldsymbol{X}),C_l(\boldsymbol{Z})$ *with* $C(\boldsymbol{g})\boldsymbol{x}=\boldsymbol{z}$, *we have* $j\geqslant l$. *And* $\boldsymbol{g}$ *is a wc-fibration (or a c-fibration) if it is a wb-fibration (or a b-fibration), and whenever* $x\in\boldsymbol{X}$ *and* $\boldsymbol{z}\in C_l(\boldsymbol{Z})$ *with* $\boldsymbol{g}(x)=\boldsymbol{\Pi}_l(\boldsymbol{z})=z\in\boldsymbol{Z}$, *then there is exactly one* $\boldsymbol{x}\in C_l(\boldsymbol{X})$ *with* $\boldsymbol{\Pi}_l(\boldsymbol{x})=x$ *and* $C(\boldsymbol{g})\boldsymbol{x}=\boldsymbol{z}$.

(f) *If* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *are* 1*-morphisms in* $\mathbf{Kur^{gc}_{in}}$ *with* $\boldsymbol{g}$ *a wb-submersion (or wb-fibration) then* $\boldsymbol{g},\boldsymbol{h}$ *are wb-transverse (or wc-transverse, respectively). If* $\boldsymbol{g}$ *is a b-submersion (or b-fibration) and* $\boldsymbol{Y}$ *is an orbifold then* $\boldsymbol{g},\boldsymbol{h}$ *are b-transverse (or c-transverse, respectively).*

(g) *If* (11.71) *is* 2*-Cartesian in* $\mathbf{Kur^{gc}_{in}}$ *with* $\boldsymbol{g}$ *a wb-submersion, b-submersion, wb-fibration, b-fibration, wc-fibration, or c-fibration, then* $\boldsymbol{f}$ *is a wb-submersion, ..., or c-fibration, respectively.*

(h) *Compositions and products of wb-submersions, b-submersions, wb-fibrations, b-fibrations, wc-fibrations, and c-fibrations, in* $\mathbf{Kur^{gc}_{in}}$ *are wb-submersions, ..., c-fibrations. Projections* $\boldsymbol{\pi_X}:\boldsymbol{X}\times\boldsymbol{Y}\to\boldsymbol{X}$ *in* $\mathbf{Kur^{gc}_{in}}$ *are wc-fibrations.*

(i) *If* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *is a wc-fibration in* $\mathbf{Kur^{gc}_{in}}$, *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *is any* 1*-morphism in* $\mathbf{Kur^{gc}}$ *(not necessarily in* $\mathbf{Kur^{gc}_{in}}$*), then a fibre product* $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ *exists in* $\mathbf{Kur^{gc}}$, *with* $\dim\boldsymbol{W}=\dim\boldsymbol{X}+\dim\boldsymbol{Y}-\dim\boldsymbol{Z}$, *in a* 2*-Cartesian square* (11.71) *in* $\mathbf{Kur^{gc}}$. *It has topological space* W *given as a set by* (11.72)*. The analogues of* **(c)**,**(g)** *hold for these fibre products. If* $\boldsymbol{g}$ *is a c-fibration and* $\boldsymbol{Y}$ *is an orbifold then* $\boldsymbol{W}$ *is an orbifold.*

11.6.4 Fibre products in $\mathbf{Kur^{c}_{in}}$ and $\mathbf{Kur^{c}}$

In §2.5.4, working in the subcategory $\mathbf{Man^c_{in}}\subset\mathbf{Man^c}$ from §2.1, we defined *sb-transverse* and *sc-transverse morphisms.* Example 11.13 explained how to fit these into the framework of Assumptions 11.1 and 11.3–11.9, also using *s-submersions* from §2.5.2. The next theorem is the analogue of Theorem 11.35.

Here $\mathbf{Kur^c_{in}}\subset\mathbf{Kur^c}$ are the 2-categories of Kuranishi spaces corresponding to $\mathbf{Man^c_{in}}\subset\mathbf{Man^c}$ as in Definition 6.29, the corner 2-functor $C:\mathbf{Kur^c}\to\mathbf{\check{K}ur^c}$ is as in (6.36), b-tangent spaces ${}^bT_x\boldsymbol{X}$ are as in Example 10.25(ii), and monoids $\tilde M_x\boldsymbol{X}$ are as in Example 10.32(c). We use the notation *wsb-transverse* and *wsc-transverse* for the notions of w-transverse in $\mathbf{Kur^c_{in}}$ corresponding to sb- and sc-transverse morphisms, and *sb-transverse*, *sc-transverse* for the notions of transverse. Also ws-submersions and s-submersions are as in §11.6.2.

Theorem 11.52. (a) *Let* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *be* 1*-morphisms in* $\mathbf{Kur^c_{in}}$. *Then* $\boldsymbol{g},\boldsymbol{h}$ *are wsb-transverse if and only if for all* $x\in\boldsymbol{X}$, $y\in\boldsymbol{Y}$ *with* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$ *and all* $\gamma\in G_z\boldsymbol{Z}$, *the following linear map is surjective:*

$$ {}^bO_x\boldsymbol{g}\oplus(\gamma\cdot{}^bO_y\boldsymbol{h}):{}^bO_x\boldsymbol{X}\oplus{}^bO_y\boldsymbol{Y}\longrightarrow{}^bO_z\boldsymbol{Z}, \tag{11.74}$$

and we have an isomorphism of commutative monoids

$$\tilde M_x\boldsymbol{X}\times_{\tilde M_x\boldsymbol{g},\tilde M_z\boldsymbol{Z},(\gamma\cdot\tilde M_y\boldsymbol{h})}\tilde M_y\boldsymbol{Y}\cong\mathbb{N}^n\qquad\text{for } n\geqslant 0. \tag{11.75}$$

This is automatic if $\boldsymbol{Z}$ *is a classical orbifold. Also* $\boldsymbol{g},\boldsymbol{h}$ *are sb-transverse if and only if for all* x,y,z,γ, *equations* (11.74)–(11.75) *are isomorphisms, and the following is surjective:*

$${}^bT_x\boldsymbol{g}\oplus(\gamma\cdot{}^bT_y\boldsymbol{h}):{}^bT_x\boldsymbol{X}\oplus{}^bT_y\boldsymbol{Y}\longrightarrow{}^bT_z\boldsymbol{Z}.$$

Furthermore, $\boldsymbol{g},\boldsymbol{h}$ *are wsc-transverse (or sc-transverse) if and only if they are wsb-transverse (or sb-transverse), and whenever* $\boldsymbol{x}\in C_j(\boldsymbol{X})$ *and* $\boldsymbol{y}\in C_k(\boldsymbol{Y})$ *with* $C(\boldsymbol{g})\boldsymbol{x}=C(\boldsymbol{h})\boldsymbol{y}=\boldsymbol{z}$ *in* $C_l(\boldsymbol{Z})$, *we have either* $j+k>l$, *or* $j=k=l=0$.

(b) *If* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *are wsb-transverse in* $\mathbf{Kur^c_{in}}$ *then a fibre product* $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ *exists in* $\mathbf{Kur^c_{in}}$*, in a 2-Cartesian square:*

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow[\quad\boldsymbol{f}\quad]{} & \boldsymbol{Y} \\ \downarrow \boldsymbol{e} & \boldsymbol{\eta}\Uparrow & \boldsymbol{h}\downarrow \\ \boldsymbol{X} & \xrightarrow{\quad\boldsymbol{g}\quad} & \boldsymbol{Z}. \end{array} \tag{11.76}$$

It has $\operatorname{vdim}\boldsymbol{W}=\operatorname{vdim}\boldsymbol{X}+\operatorname{vdim}\boldsymbol{Y}-\operatorname{vdim}\boldsymbol{Z}$*. If* $w\in\boldsymbol{W}$ *with* $\boldsymbol{e}(w)=x$ *in* $\boldsymbol{X}$*,* $\boldsymbol{f}(w)=y$ *in* $\boldsymbol{Y}$ *and* $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ *in* $\boldsymbol{Z}$*, for some possible choices of* ${}^bT_w\boldsymbol{e},{}^bT_w\boldsymbol{f},{}^bT_x\boldsymbol{g},{}^bT_y\boldsymbol{h},{}^bO_w\boldsymbol{e},{}^bO_w\boldsymbol{f},{}^bO_x\boldsymbol{g},{}^bO_y\boldsymbol{h}$ *in Definition* 10.28 *depending on* w*, the following sequence is exact:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}^bT_w\boldsymbol{W} & \xrightarrow[{}^bT_w\boldsymbol{e}\oplus{}^bT_w\boldsymbol{f}]{} & {}^bT_x\boldsymbol{X}\oplus{}^bT_y\boldsymbol{Y} & \xrightarrow[{}^bT_x\boldsymbol{g}\oplus-{}^bT_y\boldsymbol{h}]{} & {}^bT_z\boldsymbol{Z} \\ & & & & & & \downarrow {}^b\delta_w^{\boldsymbol{g},\boldsymbol{h}} \\ 0 & \longleftarrow & {}^bO_z\boldsymbol{Z} & \xleftarrow{{}^bO_x\boldsymbol{g}\oplus-{}^bO_y\boldsymbol{h}} & {}^bO_x\boldsymbol{X}\oplus{}^bO_y\boldsymbol{Y} & \xleftarrow{{}^bO_w\boldsymbol{e}\oplus{}^bO_w\boldsymbol{f}} & {}^bO_w\boldsymbol{W}. \end{array}$$

If $\boldsymbol{g},\boldsymbol{h}$ *are sb-transverse then* $\boldsymbol{W}$ *is an orbifold.*

(c) *In* **(b)***, if* $\boldsymbol{g},\boldsymbol{h}$ *are wsc-transverse then just as a set, the underlying topological space* W *in* $\boldsymbol{W}=(W,\mathcal{H})$ *may be written*

$$W=\bigl\{(x,y,C):x\in X,\ \ y\in Y,\ \ C\in G_x\boldsymbol{g}(G_x\boldsymbol{X})\backslash G_z\boldsymbol{Z}/G_y\boldsymbol{h}(G_y\boldsymbol{Y})\bigr\},$$

where $\boldsymbol{e},\boldsymbol{f}$ *map* $\boldsymbol{e}:(x,y,C)\mapsto x$*,* $\boldsymbol{f}:(x,y,C)\mapsto y$*. The isotropy groups satisfy*

$$G_{(x,y,C)}\boldsymbol{W}\cong\bigl\{(\alpha,\beta)\in G_x\boldsymbol{X}\times G_y\boldsymbol{Y}:G_x\boldsymbol{g}(\alpha)\,\gamma\,G_y\boldsymbol{h}(\beta^{-1})=\gamma\bigr\}$$

for fixed $\gamma\in C\subseteq G_z\boldsymbol{Z}$*. Also* (11.76) *is 2-Cartesian in* $\mathbf{Kur^c}$*, and the following is 2-Cartesian in* $\check{\mathbf{K}}\mathbf{ur^c_{in}}$ *and* $\check{\mathbf{K}}\mathbf{ur^c}$*, with* $C(\boldsymbol{g}),C(\boldsymbol{h})$ *wsc-transverse:*

$$\begin{array}{ccc} C(\boldsymbol{W}) & \xrightarrow[\quad C(\boldsymbol{f})\quad]{} & C(\boldsymbol{Y}) \\ \downarrow C(\boldsymbol{e}) & C(\boldsymbol{\eta})\Uparrow & C(\boldsymbol{h})\downarrow \\ C(\boldsymbol{X}) & \xrightarrow{\quad C(\boldsymbol{g})\quad} & C(\boldsymbol{Z}). \end{array}$$

Hence we have

$$C_i(\boldsymbol{W})\simeq\coprod_{\substack{j,k,l\geqslant 0:\\ i=j+k-l}}\begin{array}[t]{l}\bigl(C_j(\boldsymbol{X})\cap C(\boldsymbol{g})^{-1}(C_l(\boldsymbol{Z}))\bigr)\times_{C(\boldsymbol{g}),C_l(\boldsymbol{Z}),C(\boldsymbol{h})}\\ \qquad\qquad\bigl(C_k(\boldsymbol{Y})\cap C(\boldsymbol{h})^{-1}(C_l(\boldsymbol{Z}))\bigr)\end{array}$$

for $i\geqslant 0$*. When* $i=1$*, this computes the boundary* $\partial\boldsymbol{W}$*.*

Also, if $\boldsymbol{g}$ *is a ws-submersion (or an s-submersion), then* $C(\boldsymbol{g})$ *is a ws-submersion (or an s-submersion, respectively).*

(d) *In part* **(b)***, using the theory of (b-)canonical bundles and orientations from* §10.7.6*, suppose* $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ *are locally orientable. Then* $\boldsymbol{W}$ *is also locally orientable, and there is a natural isomorphism of topological line bundles on* W*:*

$${}^b\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}:{}^bK_{\boldsymbol{W}}\longrightarrow e^*({}^bK_{\boldsymbol{X}})\otimes f^*({}^bK_{\boldsymbol{Y}})\otimes(g\circ e)^*({}^bK_{\boldsymbol{Z}})^*. \tag{11.77}$$

Hence if $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ *are oriented there is a unique orientation on* $\boldsymbol{W}$*, called the* ***fibre product orientation****, such that* (11.77) *is orientation-preserving.*

(e) *Let* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *be a* 1*-morphism in* $\mathbf{Kur^c_{in}}$*. Then* $\boldsymbol{g}$ *is a ws-submersion if and only if* ${}^bO_x\boldsymbol{g}:{}^bO_x\boldsymbol{X}\to{}^bO_z\boldsymbol{Z}$ *is surjective for all* $x\in\boldsymbol{X}$ *with* $\boldsymbol{g}(x)=z$ *in* $\boldsymbol{Z}$*, and the monoid morphism* $\tilde M_x\boldsymbol{g}:\tilde M_x\boldsymbol{X}\to\tilde M_z\boldsymbol{Z}$ *is isomorphic to a projection* $\mathbb{N}^{m+n}\to\mathbb{N}^n$*. Also* $\boldsymbol{g}$ *is an s-submersion if and only if* ${}^bO_x\boldsymbol{g}:{}^bO_x\boldsymbol{X}\to{}^bO_z\boldsymbol{Z}$ *is an isomorphism, and* ${}^bT_x\boldsymbol{g}:{}^bT_x\boldsymbol{X}\to{}^bT_z\boldsymbol{Z}$ *is surjective, and* $\tilde M_x\boldsymbol{g}$ *is isomorphic to a projection* $\mathbb{N}^{m+n}\to\mathbb{N}^n$*, for all* x,z*.*

(f) *If* $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ *and* $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ *are* 1*-morphisms in* $\mathbf{Kur^{gc}_{in}}$ *with* $\boldsymbol{g}$ *a ws-submersion then* $\boldsymbol{g},\boldsymbol{h}$ *are wsc-transverse. If* $\boldsymbol{g}$ *is an s-submersion and* $\boldsymbol{Y}$ *is an orbifold then* $\boldsymbol{g},\boldsymbol{h}$ *are sc-transverse.*

11.7 Proof of Proposition 11.14

11.7.1 The case of classical manifolds Man

First we prove the proposition for classical manifolds **Man** in Example 11.10. Let $g:X\to Z$, $h:Y\to Z$ be transverse morphisms in **Man**, with $W=X\times_{g,Z,h}Y$ in a Cartesian square (11.1). Write $\Delta_Z:Z\to Z\times Z$ for the diagonal map $\Delta_Z:z\mapsto(z,z)$. Then $\Delta_Z(Z)$ is an embedded submanifold of $Z\times Z$ with normal bundle $\nu_Z=\mathcal{T}Z\to Z$ in the exact sequence

$$0\longrightarrow\mathcal{T}Z\xrightarrow{\mathrm{id}\oplus\mathrm{id}}\mathcal{T}_{\Delta_Z}(Z\times Z)\cong\mathcal{T}Z\oplus\mathcal{T}Z\xrightarrow{\mathrm{id}\oplus-\mathrm{id}}\nu_Z=\mathcal{T}Z\longrightarrow0.\qquad(11.78)$$

Write points of the tangent bundle $\mathcal{T}Z$ as (z,u) for $z\in Z$ and $u\in T_zZ$. By a well known construction called a 'tubular neighbourhood', we may choose open neighbourhoods T_1 of the zero section in $\mathcal{T}Z\to Z$ and U_1 of $\Delta_Z(Z)$ in $Z\times Z$ and a diffeomorphism $\Phi_1:T_1\to U_1$ with $\Phi_1(z,0)=(z,z)$ for all $z\in Z$, such that the derivative of Φ_1 at the zero section $0(Z)$ induces the exact sequence (11.78). We may also choose T_1,U_1,Φ_1 so that $\Phi_1(z,u)=(z,z')$ for all $(z,u)\in T_1$. This and (11.78) imply that the derivative of Φ_1 at the zero section $0(Z)\subset T_1$ is

$$\mathcal{T}\Phi_1|_{0(Z)}=\begin{pmatrix}\mathrm{id} & 0\\ \mathrm{id} & -\mathrm{id}\end{pmatrix}:\mathcal{T}T_1|_{0(Z)}\cong\begin{matrix}\mathcal{T}Z\oplus\\ \mathcal{T}Z\end{matrix}\longrightarrow\mathcal{T}_{\Phi_1}U_1|_{0(Z)}\cong\begin{matrix}\mathcal{T}Z\oplus\\ \mathcal{T}Z.\end{matrix}\qquad(11.79)$$

The direct product $(e,f):W\to X\times Y$ embeds W as a submanifold in $X\times Y$, with normal bundle $\pi:\mathcal{T}_{g\circ e}Z\to W$ in the rightwards exact sequence

$$0\rightleftarrows\mathcal{T}W\underset{\gamma\oplus\delta}{\overset{\mathcal{T}e\oplus\mathcal{T}f}{\rightleftarrows}}\mathcal{T}_eX\oplus\mathcal{T}_fY\underset{\alpha\oplus\beta}{\overset{\mathcal{T}g\oplus-\mathcal{T}h}{\rightleftarrows}}\mathcal{T}_{g\circ e}Z\rightleftarrows0.\qquad(11.80)$$

Write points of $\mathcal{T}_{g\circ e}Z$ as (w,v) for $w\in W$ and $v\in T_{g\circ e(w)}Z$. Again, we can choose open neighbourhoods T_2 of the zero section in $\mathcal{T}_{g\circ e}Z$ and U_2 of $(e,f)(W)$ in $X\times Y$ and a diffeomorphism $\Phi_2:T_2\to U_2$ with $\Phi_2(w,0)=(e(w),f(w))$ for all $w\in W$, such that the derivative of Φ_2 at the zero section $0(W)$ induces the exact sequence (11.80). Making T_2, U_2 smaller we can suppose that $(g \times h)(U_2) \subseteq U_1$, so $\Psi := \Phi_1^{-1} \circ (g \times h) \circ \Phi_2$ is a well-defined smooth map $\Psi : T_2 \to T_1$.

We write the derivative of Φ_2 at the zero section $0(W) \subset T_2$ in the form

$$\mathcal{T}\Phi_2|_{0(W)} = \begin{pmatrix} \mathcal{T}e & \alpha \\ \mathcal{T}f & \beta \end{pmatrix} : \mathcal{T}T_2|_{0(W)} \cong \begin{matrix} \mathcal{T}W\oplus \\ \mathcal{T}_{g\circ e}Z \end{matrix} \longrightarrow \mathcal{T}_{\Phi_2}U_2|_{0(W)} \cong \begin{matrix} \mathcal{T}_eX\oplus \\ \mathcal{T}_fY. \end{matrix} \tag{11.81}$$

As the derivative of Φ_2 at $0(W)$ induces (11.80), we see that $\alpha \oplus \beta$ is a right inverse for $\mathcal{T}g \oplus -\mathcal{T}h$ in (11.80). This induces a unique splitting of (11.80). That is, there are unique morphisms γ, δ marked in (11.80) satisfying

$$\begin{aligned} &\mathcal{T}g \circ \alpha - \mathcal{T}h \circ \beta = \mathrm{id}_{\mathcal{T}_{g\circ e}Z}, \qquad && \gamma \circ \mathcal{T}e + \delta \circ \mathcal{T}f = \mathrm{id}_{\mathcal{T}W}, \\ &\alpha \circ \mathcal{T}g + \mathcal{T}e \circ \gamma = \mathrm{id}_{\mathcal{T}_eX}, && \mathcal{T}f \circ \delta - \beta \circ \mathcal{T}h = \mathrm{id}_{\mathcal{T}_fY}, \\ &\gamma\circ\alpha + \delta\circ\beta = 0, \quad \beta\circ\mathcal{T}g + \mathcal{T}f\circ\gamma = 0, && \mathcal{T}e\circ\delta - \alpha\circ\mathcal{T}h = 0. \end{aligned} \tag{11.82}$$

Combining the first equation of (11.82) with (11.79), (11.81), and $g \circ e = h \circ f$ yields

$$\begin{aligned} \mathcal{T}\Psi|_{0(W)} &= \mathcal{T}(\Phi_1^{-1} \circ (g \times h) \circ \Phi_2)|_{0(W)} = \begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} \begin{pmatrix} \mathcal{T}g & 0 \\ 0 & \mathcal{T}h \end{pmatrix} \begin{pmatrix} \mathcal{T}e & \alpha \\ \mathcal{T}f & \beta \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{T}(g\circ e) & \mathcal{T}g\circ\alpha \\ 0 & \mathrm{id}_{\mathcal{T}_{g\circ e}Z} \end{pmatrix} : \mathcal{T}T_2|_{0(W)} \cong \begin{matrix} \mathcal{T}W\oplus \\ \mathcal{T}_{g\circ e}Z \end{matrix} \to \mathcal{T}_\Psi T_1|_{0(Z)} \cong \begin{matrix} \mathcal{T}_{g\circ e}Z\oplus \\ \mathcal{T}_{g\circ e}Z. \end{matrix} \end{aligned} \tag{11.83}$$

Suppose as in Assumption 11.1(b)(ii) that $c : V \to X$, $d : V \to Y$ are morphisms in **Man**, and $E \to V$ is a vector bundle, and $s \in \Gamma^\infty(E)$ is a section, and $\mathrm{K} : E \to \mathcal{T}_{g\circ c}Z$ is a morphism, such that $h \circ d = g \circ c + \mathrm{K} \circ s + O(s^2)$.

Define $V' = \{v \in V : (c(v), d(v)) \in U_2\}$. If $v \in s^{-1}(0)$ then $h \circ d(v) = g \circ c(v)$ as $h \circ d = g \circ c + \mathrm{K} \circ s + O(s^2)$, so there is a unique $w \in W$ with $e(w) = c(v)$, $f(w) = d(v)$, so that $(c(v), d(v)) \in U_2$, and $v \in V'$. Hence V' is an open neighbourhood of $s^{-1}(0)$ in V. Define smooth maps $\Xi = \Phi_2^{-1} \circ (c, d)|_{V'} : V' \to T_2$ and $b = \pi \circ \Xi : V' \to W$, where $\pi : T_2 \to W$ is the restriction of $\pi : \mathcal{T}_{g\circ e}Z \to W$.

Define $t \in \Gamma^\infty(\mathcal{T}_{g\circ e\circ b}Z)$ by $\Xi(v) = (b(v), -t(v)) \in \mathcal{T}_{g\circ e}Z$ for $v \in V'$. Define $u \in \Gamma^\infty(\mathcal{T}_{g\circ c}Z|_{V'})$ by $\Psi \circ \Xi(v) = \Phi_1^{-1}(g \circ c(v), g \circ d(v)) = (g \circ c(v), -u(v))$ for $v \in V'$, noting that $\Phi_1(z, u) = (z, z')$ for $(z, u) \in T_1$. Combining $h \circ d = g \circ c + \mathrm{K} \circ s + O(s^2)$, $\Phi_1^{-1}(g \circ c(v), g \circ d(v)) = (g \circ c(v), -u(v))$ and (11.79) we see that

$$u = \mathrm{K} \circ s + O(s^2). \tag{11.84}$$

Now for $v \in V'$ we have

$$\begin{aligned} \Psi(b(v), 0) &= \Phi_1^{-1} \circ (g \times h)(e \circ b(v), f \circ b(v)) \\ &= \Phi_1^{-1}(g \circ e \circ b(v), g \circ e \circ b(v)) = (g \circ e \circ b(v), 0), \\ \Psi(b(v), -t(v)) &= \Phi_1^{-1} \circ (g \times h)(c(v), d(v)) \\ &= \Phi_1^{-1}(g \circ c(v), h \circ d(v)) = (g \circ c(v), -u(v)). \end{aligned}$$

Together with (11.83) these give

$$g \circ c = g \circ e \circ b + 0 \circ t + O(t^2), \quad u = t + O(t^2),$$

so inverting yields

$$g \circ e \circ b = g \circ c + 0 \circ u + O(u^2), \quad t = u + O(u^2). \tag{11.85}$$

Substituting (11.84) into the first equation of (11.85) gives $g \circ e \circ b = g \circ c + O(s)$. Thus by Theorem 3.17(g) there exists a morphism $\mathrm{K}' : E|_{V'} \to \mathcal{T}_{g\circ e\circ b}Z$ with $\mathrm{K}|_{V'} = \mathrm{K}' + O(s)$ in the sense of Definition 3.15(v), where K' is unique up to $O(s)$. Then substituting (11.84) into the second equation of (11.85) gives

$$t = \mathrm{K}' \circ s + O(s^2). \tag{11.86}$$

For $v \in V'$ we have

$$\Phi_2(b(v), 0) = (e \circ b(v), f \circ b(v)), \qquad \Phi_2(b(v), -t(v)) = (c(v), d(v)).$$

From these and (11.81) we see that

$$c|_{V'} = e \circ b + (-\alpha) \circ t + O(t^2), \qquad d|_{V'} = f \circ b + (-\beta) \circ t + O(t^2),$$

so substituting in (11.86) gives

$$c|_{V'} = e \circ b + \Lambda \circ s + O(s^2), \quad d|_{V'} = f \circ b + \mathrm{M} \circ s + O(s^2), \tag{11.87}$$

as in equation (11.2) in Assumption 11.1, where $\Lambda = -\alpha \circ \mathrm{K}'$ and $\mathrm{M} = -\beta \circ \mathrm{K}'$. Then composing the first equation of (11.82) on the right with K' gives

$$\mathrm{K}' + \mathcal{T}g \circ \Lambda = \mathcal{T}h \circ \mathrm{M} = \mathcal{T}h \circ \mathrm{M} + O(s), \tag{11.88}$$

which is equation (11.3). This proves Assumption 11.1(b)(ii) for $\dot{\mathbf{Man}} = \mathbf{Man}$.

Next suppose as in Assumption 11.1(b)(iii) that $\tilde V', \tilde b, \tilde\Lambda, \tilde{\mathrm{M}}, \tilde{\mathrm{K}}'$ are alternative choices for $V', b, \Lambda, \mathrm{M}, \mathrm{K}'$ above, so that $\tilde V'$ is an open neighbourhood of $s^{-1}(0)$ in V, and $\tilde b : \tilde V' \to W$ is a smooth map, and $\tilde\Lambda : E|_{\tilde V'} \to \mathcal{T}_{e\circ\tilde b}X$, $\tilde{\mathrm{M}} : E|_{\tilde V'} \to \mathcal{T}_{f\circ\tilde b}Y$ are morphisms with

$$c|_{\tilde V'} = e \circ \tilde b + \tilde\Lambda \circ s + O(s^2), \quad d|_{\tilde V'} = f \circ \tilde b + \tilde{\mathrm{M}} \circ s + O(s^2), \tag{11.89}$$

$$\tilde{\mathrm{K}}' + \mathcal{T}g \circ \tilde\Lambda = \mathcal{T}h \circ \tilde{\mathrm{M}} + O(s), \tag{11.90}$$

for $\tilde{\mathrm{K}}' : E|_{\tilde V'} \to \mathcal{T}_{g\circ e\circ\tilde b}Z$ a morphism with $\mathrm{K}|_{\tilde V'} = \tilde{\mathrm{K}}' + O(s)$.

By (11.87) and (11.89), in maps $V' \cap \tilde V' \to X \times Y$ we have

$$(c, d)|_{V'\cap\tilde V'} = (e, f) \circ b|_{V'\cap\tilde V'} + O(s), \;\; (c, d)|_{V'\cap\tilde V'} = (e, f) \circ \tilde b|_{V'\cap\tilde V'} + O(s),$$

so Theorem 3.17(c) implies that

$$(e, f) \circ \tilde b|_{V'\cap\tilde V'} = (e, f) \circ b|_{V'\cap\tilde V'} + O(s),$$

and thus $\tilde b|_{V'\cap\tilde V'} = b|_{V'\cap\tilde V'} + O(s)$, since (e, f) is an embedding. Hence by Theorem 3.17(g) there exist morphisms $\tilde\Lambda' : E|_{V'\cap\tilde V'} \to \mathcal{T}_{e\circ b}X|_{V'\cap\tilde V'}$, $\tilde{\mathrm{M}}' : E|_{V'\cap\tilde V'} \to \mathcal{T}_{f\circ b}Y|_{V'\cap\tilde V'}$ with $\tilde\Lambda|_{V'\cap\tilde V'} = \tilde\Lambda' + O(s)$, $\tilde{\mathrm{M}}|_{V'\cap\tilde V'} = \tilde{\mathrm{M}}' + O(s)$, and $\tilde\Lambda', \tilde{\mathrm{M}}'$ are unique up to $O(s)$. Equation (11.90) and $\mathrm{K}|_{V'} = \mathrm{K}' + O(s)$, $\mathrm{K}|_{\tilde V'} = \tilde{\mathrm{K}}' + O(s)$ now imply that

$$\mathrm{K}'|_{V'\cap\tilde V'} + \mathcal{T}g \circ \tilde\Lambda' = \mathcal{T}h \circ \tilde{\mathrm{M}}' + O(s). \tag{11.91}$$

Also (11.87), (11.89), $\tilde\Lambda|_{V'\cap\tilde V'} = \tilde\Lambda' + O(s)$, $\tilde{\mathrm{M}}|_{V'\cap\tilde V'} = \tilde{\mathrm{M}}' + O(s)$ and Theorem 3.17(k),(l) imply that

$$(e,f)\circ\tilde b|_{V'\cap\tilde V'} = (e,f)\circ b|_{V'\cap\tilde V'} + (\Lambda - \tilde\Lambda' \oplus \mathrm{M} - \tilde{\mathrm{M}}')\circ s + O(s^2). \tag{11.92}$$

Define $\mathrm{N} : E|_{V'\cap\tilde V'} \to \mathcal{T}_b W|_{V'\cap\tilde V'}$ by

$$\mathrm{N} = b^*(\gamma)\circ(\Lambda - \tilde\Lambda') + b^*(\delta)\circ(\mathrm{M} - \tilde{\mathrm{M}}'), \tag{11.93}$$

for γ,δ as in (11.80) and (11.82). Now in maps $V'\cap\tilde V' \to W$ we have

$$b|_{V'\cap\tilde V'} = \pi\circ\Phi_2^{-1}\circ(e,f)\circ b|_{V'\cap\tilde V'},\;\; \tilde b|_{V'\cap\tilde V'} = \pi\circ\Phi_2^{-1}\circ(e,f)\circ\tilde b|_{V'\cap\tilde V'}. \tag{11.94}$$

We have

$$\begin{aligned}
\tilde b|_{V'\cap\tilde V'} &= b|_{V'\cap\tilde V'} + [\mathcal{T}\pi\circ\mathcal{T}\Phi_2^{-1}\circ(\Lambda - \tilde\Lambda'\oplus\mathrm{M}-\tilde{\mathrm{M}}')]\circ s + O(s^2)\\
&= b|_{V'\cap\tilde V'} + \left[\begin{pmatrix}\mathrm{id}_{\mathcal{T}_bW} & 0\end{pmatrix} b^*\begin{pmatrix}\mathcal{T}e & \alpha\\ \mathcal{T}f & \beta\end{pmatrix}^{-1}\begin{pmatrix}\Lambda-\tilde\Lambda'\\ \mathrm{M}-\tilde{\mathrm{M}}'\end{pmatrix}\right]\circ s + O(s^2)\\
&= b|_{V'\cap\tilde V'} + \left[\begin{pmatrix}\mathrm{id}_{\mathcal{T}_bW} & 0\end{pmatrix} b^*\begin{pmatrix}\gamma & \delta\\ \mathcal{T}g & -\mathcal{T}h\end{pmatrix}\begin{pmatrix}\Lambda-\tilde\Lambda'\\ \mathrm{M}-\tilde{\mathrm{M}}'\end{pmatrix}\right]\circ s + O(s^2)\\
&= b|_{V'\cap\tilde V'} + \left[b^*(\gamma)\circ(\Lambda-\tilde\Lambda') + b^*(\delta)\circ(\mathrm{M}-\tilde{\mathrm{M}}')\right]\circ s + O(s^2)\\
&= b|_{V'\cap\tilde V'} + \mathrm{N}\circ s + O(s^2).
\end{aligned} \tag{11.95}$$

Here in the first step we use (11.92), (11.94), Theorem 3.17(k), and $\mathcal{T}(\pi\circ\Phi_2^{-1}) = \mathcal{T}\pi\circ\mathcal{T}\Phi_2^{-1}$. In the second we use (11.81), in the third we use (11.82) to invert the matrix explicitly, and in the fourth we use (11.93). This proves equation (11.4) in Assumption 11.1(b)(iii). Also we have

$$\begin{aligned}
\mathcal{T}e\circ\mathrm{N} &= \mathcal{T}e\circ b^*(\gamma)\circ(\Lambda-\tilde\Lambda') + \mathcal{T}e\circ b^*(\delta)\circ(\mathrm{M}-\tilde{\mathrm{M}}')\\
&= b^*(\mathcal{T}e\circ\gamma)\circ(\Lambda-\tilde\Lambda') + b^*(\mathcal{T}e\circ\delta)\circ(\mathrm{M}-\tilde{\mathrm{M}}')\\
&= b^*(\mathrm{id}_{\mathcal{T}_eX} - \alpha\circ\mathcal{T}g)\circ(\Lambda-\tilde\Lambda') + b^*(\alpha\circ\mathcal{T}h)\circ(\mathrm{M}-\tilde{\mathrm{M}}')\\
&= \Lambda-\tilde\Lambda' + b^*(\alpha)\circ\left[-\mathcal{T}g\circ(\Lambda-\tilde\Lambda') + \mathcal{T}h\circ(\mathrm{M}-\tilde{\mathrm{M}}')\right]\\
&= \Lambda-\tilde\Lambda' + b^*(\alpha)\circ\left[\mathrm{K}'|_{V'\cap\tilde V'} - \mathrm{K}'|_{V'\cap\tilde V'} + O(s)\right] = \Lambda-\tilde\Lambda' + O(s),
\end{aligned}$$

using (11.93) in the first step, (11.82) in the third, and (11.88), (11.91) in the fifth. This proves the first equation of (11.5), and the second equation is similar.

Suppose $\check{\mathrm{N}} : E|_{V'\cap\tilde V'} \to \mathcal{T}_bW|_{V'\cap\tilde V'}$ also satisfies (11.4)–(11.5). Subtracting the equations of (11.5) for $\mathrm{N}, \check{\mathrm{N}}$ gives

$$\mathcal{T}e\circ(\mathrm{N}-\check{\mathrm{N}}) = O(s), \qquad \mathcal{T}f\circ(\mathrm{N}-\check{\mathrm{N}}) = O(s).$$

Hence using (11.82) in the second step we have

$$\mathrm{N}-\check{\mathrm{N}} = \mathrm{id}_{\mathcal{T}W}\circ(\mathrm{N}-\check{\mathrm{N}}) = (\gamma\circ\mathcal{T}e + \delta\circ\mathcal{T}f)\circ(\mathrm{N}-\check{\mathrm{N}}) = O(s).$$

This completes Assumption 11.1(b)(iii) for $\dot{\mathbf{Man}} = \mathbf{Man}$ in Example 11.10.

11.7.2 The cases $\mathbf{Man^c_{in}}$ and $\mathbf{Man^{gc}_{in}}$

Next we explain how to modify the proof in §11.7.1 to work when both $\dot{\mathbf{Man}}_{\boldsymbol{D}}$ and $\dot{\mathbf{Man}}_{\boldsymbol{E}}$ are $\mathbf{Man^c_{in}}$ or $\mathbf{Man^{gc}_{in}}$, as in Examples 11.12(a) and 11.13(a). The difficulty is that the 'tubular neighbourhoods' $\Phi_1 : T_1 \to U_1$ and $\Phi_2 : T_2 \to U_2$ defined at the beginning of §11.7.1 may not exist.

To see the problem, consider $Z = [0,\infty)$. Then $\mathcal{T}Z = {}^bTZ \cong [0,\infty) \times \mathbb{R}$, where $(x,u) \in [0,\infty) \times \mathbb{R}$ represents $u \cdot x\frac{\partial}{\partial x} \in {}^bT_x[0,\infty)$, and $Z \times Z = [0,\infty)^2$ with $\Delta_Z(Z) = \bigl\{(x,x) : x \in [0,\infty)\bigr\} \subseteq [0,\infty)^2$. Thus $\mathcal{T}Z$ near the zero section $0(Z)$ is not diffeomorphic to $Z \times Z$ near $\Delta_Z(Z)$, as the corners are different at $(0,0) \in \mathcal{T}Z$ and $(0,0) \in Z \times Z$. So there do not exist open $0(Z) \subset T_1 \subseteq \mathcal{T}Z$ and $\Delta_Z(Z) \subset U_1 \subseteq Z \times Z$ and a diffeomorphism $\Phi_1 : T_1 \to U_1$.

Nonetheless, there is a construction which shares many of the important properties of tubular neighbourhoods in the corners case. We can choose open neighbourhoods T_1, T_2 of $0(Z), 0(W)$ in the vector bundles $\mathcal{T}Z = {}^bTZ \to Z$ and $\mathcal{T}_{g\circ e}Z = (g \circ e)^*({}^bTZ) \to W$, and interior maps $\Phi_1 : T_1 \to Z \times Z$, $\Phi_2 : T_2 \to X \times Y$, with the properties:

(a) $\Phi_1(z,0) = (z,z)$ and $\Phi_2(w,0) = (e(w), f(w))$ for all $z \in Z$ and $w \in W$.

(b) $\Phi_1(z,u) = (z,z')$ for all $(z,u) \in T_1$.

(c) ${}^b\mathrm{d}\Phi_1 : {}^bT(T_1) \to \Phi_1^*({}^bT(Z \times Z))$ and ${}^b\mathrm{d}\Phi_2 : {}^bT(T_2) \to \Phi_2^*({}^bT(X \times Y))$ are vector bundle isomorphisms.

(d) The derivatives ${}^b\mathrm{d}\Phi_1|_{0(Z)}$, ${}^b\mathrm{d}\Phi_2|_{0(W)}$ satisfy (11.79) and (11.81), where $\alpha \oplus \beta$ is a right inverse for $\mathcal{T}g \oplus -\mathcal{T}h$ in (11.80), so that (11.82) holds for some unique γ, δ.

(e) On the interiors, $\Phi_1|_{T_1^\circ} : T_1^\circ \to Z^\circ \times Z^\circ$ and $\Phi_2|_{T_2^\circ} : T_2^\circ \to X^\circ \times Y^\circ$ are diffeomorphisms with open subsets of their targets.

However, on $T_1 \setminus T_1^\circ$ and $T_2 \setminus T_2^\circ$, Φ_1, Φ_2 are generally *not injective*, and the images of Φ_1, Φ_2 are generally *not open* in $Z \times Z$ and $X \times Y$. So in particular, *the inverses* Φ_1^{-1} *and* Φ_2^{-1} *may not exist.*

(f) Although Φ_1^{-1}, Φ_2^{-1} may not exist, under some conditions on interior maps $a, b : V \to Z$ or $c : V \to X$, $d : V \to Y$, it may be automatic that $(a,b) : V \to Z \times Z$ factors via $\Phi_1 : T_1 \to Z \times Z$, or $(c,d) : V \to X \times Y$ factors via $\Phi_2 : T_2 \to X \times Y$. That is, there may exist unique interior $i : V \to T_1$ and $j : V \to T_2$ with $\Phi_1 \circ i = (a,b)$ and $\Phi_2 \circ j = (c,d)$. If Φ_1^{-1}, Φ_2^{-1} existed we would have $i = \Phi_1^{-1} \circ (a,b)$ and $j = \Phi_2^{-1} \circ (c,d)$. So we use factorization properties of this kind as a substitute for Φ_1^{-1}, Φ_2^{-1}.

For example, when $Z = [0,\infty)$ we can take $T_1 = \mathcal{T}Z = [0,\infty) \times \mathbb{R}$ and define $\Phi_1 : T_1 \to Z \times Z$ by $\Phi_1(x,u) = (x, e^{-u}x)$. Then $\Phi_1(z,u) = (z,z')$, as in (b). In the natural bases $x\frac{\partial}{\partial x}, \frac{\partial}{\partial u}$ for ${}^bT(\mathcal{T}Z)$ and $y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}$ for ${}^bT(Z \times Z)$, we see that $\mathcal{T}\Phi_1|_{0(Z)}$ maps $x\frac{\partial}{\partial x} \mapsto y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial u} \mapsto -z\frac{\partial}{\partial z}$, so $\mathcal{T}\Phi_1|_{0(Z)}$ has matrix $\bigl(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\bigr)$, and (11.79) holds as in (c). We have $\Phi_1(\{0\} \times \mathbb{R}) = \{(0,0)\}$, so Φ_1 is not injective, and the image $\Phi_1(T_1)$ is not open in $Z \times Z$, as in (e).

In the proof in §11.7.1, the problem is that we use Φ_1^{-1}, Φ_2^{-1} as follows:

(i) We define smooth $\Psi : T_2 \to T_1$ by $\Psi = \Phi_1^{-1} \circ (g \times h) \circ \Phi_2$.

(ii) We define smooth $\Xi : V' \to T_2$ by $\Xi = \Phi_2^{-1} \circ (c,d)|_{V'}$.

(iii) Equation (11.94) involves $\Phi_2^{-1} \circ (e,f)$.

(iv) Equations (11.83) and (11.95) involve $\mathcal{T}(\Phi_1^{-1})$ and $\mathcal{T}(\Phi_2^{-1})$.

Here (i)–(iii) are dealt with by the factorization property of Φ_1, Φ_2 in (f) above. For (i), if the open neighbourhood T_2 of $0(W)$ in $\mathcal{T}_{g\circ e}Z$ is small enough there is a unique interior map $\Psi : T_2 \to T_1$ with $\Phi_1 \circ \Psi = (g \times h) \circ \Phi_2$. For (ii), if V' is small enough there is a unique interior map $\Xi : V' \to T_2$ with $\Phi_2 \circ \Xi = (c,d)$. For (iii), $\Phi_2^{-1} \circ (e,f)$ is the zero section map $0 : W \to T_2 \subseteq \mathcal{T}_{g\circ e}Z$. For part (iv) we substitute $\mathcal{T}(\Phi_1^{-1}) = (\mathcal{T}\Phi_1)^{-1}$ and $\mathcal{T}(\Phi_2^{-1}) = (\mathcal{T}\Phi_2)^{-1}$, where $\mathcal{T}\Phi_1 = {}^b\mathrm{d}\Phi_1$ and $\mathcal{T}\Phi_2 = {}^b\mathrm{d}\Phi_2$ are vector bundle isomorphisms as in (c) above. With these modifications, the proof in §11.7.1 extends to work in $\mathbf{Man^c_{in}}$ and $\mathbf{Man^{gc}_{in}}$.

11.7.3 The cases $\mathbf{Man^c}$ and $\mathbf{Man^{gc}}$

Finally we modify the proofs in §11.7.1–§11.7.2 to work in the remaining cases of Examples 11.11–11.13, in which $\dot{\mathbf{Man}}_{\boldsymbol{E}}$ is $\mathbf{Man^c}$ or $\mathbf{Man^{gc}}$. In §11.7.2, it was important that we worked with *interior* maps, which are functorial for b-tangent bundles ${}^bTX$ in $\mathbf{Man^c_{in}}, \mathbf{Man^{gc}_{in}}$.

The new issues are that in the definition of the 'tubular neighbourhood' $\Phi_2 : T_2 \to X \times Y$ for $(e,f)(W) \subseteq X \times Y$, the map $(e,f) : W \to X \times Y$ may no longer be interior, which was essential in §11.7.2 to define Φ_2, T_2. Even if (e,f) is interior and Φ_2, T_2 in §11.7.2 are well defined, the maps $c : V \to X$, $d : V \to Y$ in Assumption 11.1(b)(ii) need not be interior, and if they are not, the lifting property of $(c,d) : V \to X \times Y$ in §11.7.2(f) may not hold, so that we cannot define $\Xi : V' \to T_2$ with $\Phi_2 \circ \Xi = (c,d)$ as in §11.7.1–§11.7.2.

Our solution is to use the corner functors $C : \mathbf{Man^c} \to \check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}}$, $C : \mathbf{Man^{gc}} \to \check{\mathbf{Man}}^{\mathbf{gc}}_{\mathbf{in}}$ from §2.2 and §2.4.1, which map to interior morphisms. Given a transverse Cartesian square (11.1) in $\mathbf{Man^c}$ or $\mathbf{Man^{gc}}$ in one of the remaining cases of Examples 11.11–11.13, we can consider the commutative diagram in $\check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}}$ or $\check{\mathbf{Man}}^{\mathbf{gc}}_{\mathbf{in}}$:

$$\begin{array}{ccc} C(W) & \xrightarrow{\quad C(f) \quad} & C(Y) \\ \downarrow{\scriptstyle C(e)} & & \downarrow{\scriptstyle C(h)} \\ C(X) & \xrightarrow{\quad C(g) \quad} & C(Z). \end{array} \tag{11.96}$$

We can show that in the cases we are interested in, (11.96) is *locally Cartesian and locally b-transverse on* $C(W)$. That is, if $\boldsymbol{w} \in C(W)$ with $C(e)\boldsymbol{w} = \boldsymbol{x} \in C(X)$, $C(f)\boldsymbol{w} = \boldsymbol{y} \in C(Y)$ and $C(g)\boldsymbol{x} = C(h)\boldsymbol{y} = \boldsymbol{z} \in C(Z)$, then $C(g), C(h)$ are b-transverse near $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ as in §2.5.3, and (11.96) is Cartesian near $\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $C(W), \ldots, C(Z)$. We do not claim (11.96) is Cartesian, nor that $C(g), C(h)$ are b-transverse, as these would be false in Example 2.26.

Thus $(C(e), C(f))$ embeds $C(W)$ as a submanifold of $C(X) \times C(Y)$, and the argument of §11.7.2 constructing 'tubular neighbourhoods' $\Phi_1 : T_1 \to Z \times Z$,

$\Phi_2 : T_2 \to X \times Y$ satisfying §11.7.2(a)–(f) works with $C(W), \ldots, C(h)$ in place of W, X, Y, Z, e, f, g, h, as $C(e), \ldots, C(h)$ are interior.

Now suppose as in Assumption 11.1(b)(ii) that $c : V \to X$, $d : V \to Y$ are morphisms in $\mathbf{Man^c}$ or $\mathbf{Man^{gc}}$, and $E \to V$ is a vector bundle, and $s \in \Gamma^\infty(E)$ is a section, and $\mathrm{K} : E \to \mathcal{T}_{g\circ c}Z$ is a morphism, such that $h \circ d = g \circ c + \mathrm{K} \circ s + O(s^2)$. Then we have a diagram in $\mathbf{\check{M}an^c_{in}}$ or $\mathbf{\check{M}an^{gc}_{in}}$:

$$\begin{array}{ccc} V \cong C_0(V) & \xrightarrow[C(d)|_{C_0(V)}]{} & C(Y) \\ \big\downarrow {\scriptstyle C(c)|_{C_0(V)}} & & {\scriptstyle C(h)} \big\downarrow \\ C(X) & \xrightarrow{C(g)} & C(Z). \end{array}$$

Under the isomorphism $V \cong C_0(V)$ there is a natural identification

$$\mathcal{T}_{g\circ c}Z \cong \mathcal{T}_{C(g)\circ C(c)|_{C_0(V)}}C(Z) \cong C(g \circ c)|^*_{C_0(V)}({}^bT(C(Z))).$$

Let $\check{\mathrm{K}} : E \to \mathcal{T}_{C(g)\circ C(c)|_{C_0(V)}}C(Z)$ correspond to K under this identification. Then we find that $C(h) \circ C(d)|_{C_0(V)} = C(g) \circ C(c)|_{C_0(V)} + \check{\mathrm{K}} \circ s + O(s^2)$. So we can repeat the argument of §11.7.1–§11.7.2 with $C_0(V), C(W), \ldots, C(Z)$, $C(c)|_{C_0(V)}, C(d)|_{C_0(V)}, C(e), \ldots, C(h), \check{\mathrm{K}}$ in place of $V, W, \ldots, Z, c, d, e, \ldots, h, \mathrm{K}$.

For Assumption 11.1(b)(ii) this constructs $\check{V}' \subseteq C_0(V)$, an interior morphism $\check{b} : C_0(V) \to C(W)$ and morphisms $\check{\Lambda} : E|_{V'} \to \mathcal{T}_{C(e)\circ\check{b}}C(X)$ and $\check{\mathrm{M}} : E|_{V'} \to \mathcal{T}_{C(f)\circ\check{b}}C(Y)$ with

$$C(c)|_{\check{V}'} = C(e) \circ \check{b} + \check{\Lambda} \circ s + O(s^2), \;\; C(d)|_{\check{V}'} = C(f) \circ \check{b} + \check{\mathrm{M}} \circ s + O(s^2). \quad (11.97)$$

Let $V' \subseteq V$ be identified with $\check{V}'$ under $V \cong C_0(V)$, let $b : V' \to W$ be identified with $\Pi \circ \check{b}$ under $V' \cong \check{V}'$, and let $\Lambda : E|_{V'} \to \mathcal{T}_{e\circ b}X$, $\mathrm{M} : E|_{V'} \to \mathcal{T}_{f\circ b}Y$ be identified with $\check{\Lambda}, \check{\mathrm{M}}$ as for $\mathrm{K} \cong \check{\mathrm{K}}$. Then (11.97) corresponds to (11.2). The rest of Assumption 11.1(b)(ii)–(iii) follow in the same way.

11.8 Proof of Theorem 11.17

Work in the situation of Definition 11.16. Since (11.14) is a 2-commutative square in $\mathbf{Gm\dot{K}N}_{\boldsymbol{D}}$, and $\mathbf{Gm\dot{K}N}_{\boldsymbol{D}} \subseteq \mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$ is an inclusion of 2-subcategories such that the 2-morphisms in $\mathbf{Gm\dot{K}N}_{\boldsymbol{D}}, \mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$ between given 1-morphisms in $\mathbf{Gm\dot{K}N}_{\boldsymbol{D}}$ coincide, if (11.14) is 2-Cartesian in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$ then it is 2-Cartesian in $\mathbf{Gm\dot{K}N}_{\boldsymbol{D}}$. Thus, we must verify the universal property of 2-category fibre products in Definition A.11 for (11.14) in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$.

Suppose we are given 1-morphisms in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$:

$$\boldsymbol{c}_{jl} : (S_j, B_j, p_j) \longrightarrow (U_l, D_l, r_l), \quad \boldsymbol{d}_{jm} : (S_j, B_j, p_j) \longrightarrow (V_m, E_m, s_m),$$

with $\boldsymbol{c}_{jl} = (S_{jl}, c_{jl}, \hat{c}_{jl})$ and $\boldsymbol{d}_{jm} = (S_{jm}, d_{jm}, \hat{d}_{jm})$, and let $\mathrm{K} = [\dot{S}_j, \hat{\kappa}] : \boldsymbol{g}_{ln} \circ \boldsymbol{c}_{jl} \Rightarrow \boldsymbol{h}_{mn} \circ \boldsymbol{d}_{jm}$ be a 2-morphism in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$. Then by Definition 4.3, $\dot{S}_j$ is an open neighbourhood of $p_j^{-1}(0)$ in $S_{jl} \cap S_{jm} \subseteq S_j$, and $\hat{\kappa} : B_j|_{\dot{S}_j} \to \mathcal{T}_{g_{ln}\circ c_{jl}} W_n|_{\dot{S}_j}$ is a morphism with

$$\begin{aligned} h_{mn} \circ d_{jm}|_{\dot{S}_j} &= g_{ln} \circ c_{jl}|_{\dot{S}_j} + \hat{\kappa} \circ p_j + O(p_j^2) \qquad \text{and} \\ d_{jm}^*(\hat{h}_{mn}) \circ \hat{d}_{jm}|_{\dot{S}_j} &= c_{jl}^*(\hat{g}_{ln}) \circ \hat{c}_{jl}|_{\dot{S}_j} + (g_{ln} \circ c_{jl})^*(\mathrm{d}t) \circ \hat{\kappa} + O(p_j). \end{aligned} \tag{11.98}$$

Assumption 11.1(b)(ii) now gives an open neighbourhood $\ddot{S}_j$ of $p_j^{-1}(0)$ in $\dot{S}_j$, a morphism $b_{jk} : \ddot{S}_j \to T_k$ in $\dot{\mathbf{Man}}_{\boldsymbol{E}}$, and morphisms $\hat{\lambda} : B_j|_{\ddot{S}_j} \to \mathcal{T}_{e_{kl}\circ b_{jk}} U_l$ and $\hat{\mu} : B_j|_{\ddot{S}_j} \to \mathcal{T}_{f_{km}\circ b_{jk}} V_m$ such that (11.2) becomes

$$c_{jl}|_{\ddot{S}_j} = e_{kl} \circ b_{jk} + \hat{\lambda} \circ p_j + O(p_j^2), \quad d_{jm}|_{\ddot{S}_j} = f_{km} \circ b_{jk} + \hat{\mu} \circ p_j + O(p_j^2). \tag{11.99}$$

Theorem 3.17(g) gives $\check{\kappa} : B_j|_{\ddot{S}_j} \to \mathcal{T}_{g_{ln}\circ e_{kl}\circ b_{jk}} W_n$ with $\check{\kappa} = \hat{\kappa}|_{\ddot{S}_j} + O(p_j)$, since $g_{ln} \circ c_{jl}|_{\ddot{S}_j} = g_{ln} \circ e_{kl} \circ b_{jk} + O(p_j)$ by (11.99), and then as in (11.3) we have

$$\check{\kappa} + \mathcal{T} g_{ln} \circ \hat{\lambda} = \mathcal{T} h_{mn} \circ \hat{\mu} + O(p_j). \tag{11.100}$$

Choose connections $\nabla^{D_l}, \nabla^{E_m}, \nabla^{F_n}$ on $D_l \to U_l$, $E_m \to V_m$, $F_n \to W_n$, as in §3.3.3 and §B.3.2, and write $\nabla^{g_{ln}^*(F_n)}, \nabla^{h_{mn}^*(F_n)}$ for the pullback connections from ∇^{F_n} on $g_{ln}^*(F_n) \to U_{ln}$, $h_{mn}^*(F_n) \to V_{mn}$. Then in morphisms $B_j|_{\ddot{S}_j} \to (g_{ln} \circ e_{kl} \circ b_{jk})^*(F_n)$ we have:

$$\begin{aligned} & b_{jk}^*\big[e_{kl}^*(\hat{g}_{ln}) \oplus - f_{km}^*(\hat{h}_{mn})\big] \circ \big[\big(\hat{c}_{jl}|_{\ddot{S}_j} - (e_{kl} \circ b_{jk})^*(\nabla^{D_l} r_l) \circ \hat{\lambda}\big) \\ & \qquad\qquad \oplus \big(\hat{d}_{jm}|_{\ddot{S}_j} - (f_{km} \circ b_{jk})^*(\nabla^{E_m} s_m) \circ \hat{\mu}\big)\big] \\ & = (e_{kl} \circ b_{jk})^*(\hat{g}_{ln}) \circ \hat{c}_{jl}|_{\ddot{S}_j} - (e_{kl} \circ b_{jk})^*(\hat{g}_{ln}) \circ (e_{kl} \circ b_{jk})^*(\nabla^{D_l} r_l) \circ \hat{\lambda} \\ & \quad - (f_{km} \circ b_{jk})^*(\hat{h}_{mn}) \circ \hat{d}_{jm}|_{\ddot{S}_j} + (f_{km} \circ b_{jk})^*(\hat{h}_{mn}) \circ (f_{km} \circ b_{jk})^*(\nabla^{E_m} s_m) \circ \hat{\mu} \\ & = c_{jl}^*(\hat{g}_{ln}) \circ \hat{c}_{jl}|_{\ddot{S}_j} - (e_{kl} \circ b_{jk})^*(\nabla^{g_{ln}^*(F_n)}(\hat{g}_{ln}(r_l))) \circ \hat{\lambda} \\ & \quad - d_{jm}^*(\hat{h}_{mn}) \circ \hat{d}_{jm}|_{\ddot{S}_j} + (f_{km} \circ b_{jk})^*(\nabla^{h_{mn}^*(F_n)}(\hat{h}_{mn}(s_m))) \circ \hat{\mu} + O(p_j) \\ & = c_{jl}^*(\hat{g}_{ln}) \circ \hat{c}_{jl}|_{\ddot{S}_j} - (e_{kl} \circ b_{jk})^*(\nabla^{g_{ln}^*(F_n)}(g_{ln}^*(t_n))) \circ \hat{\lambda} \\ & \quad - d_{jm}^*(\hat{h}_{mn}) \circ \hat{d}_{jm}|_{\ddot{S}_j} + (f_{km} \circ b_{jk})^*(\nabla^{h_{mn}^*(F_n)}(h_{mn}^*(t_n))) \circ \hat{\mu} + O(p_j) \\ & = c_{jl}^*(\hat{g}_{ln}) \circ \hat{c}_{jl}|_{\ddot{S}_j} - (g_{ln} \circ e_{kl} \circ b_{jk})^*(\nabla^{F_n} t_n) \circ \mathcal{T} g_{ln} \circ \hat{\lambda} \\ & \quad - d_{jm}^*(\hat{h}_{mn}) \circ \hat{d}_{jm}|_{\ddot{S}_j} + (h_{mn} \circ f_{km} \circ b_{jk})^*(\nabla^{F_n} t_n) \circ \mathcal{T} h_{mn} \circ \hat{\mu} + O(p_j) \\ & = c_{jl}^*(\hat{g}_{ln}) \circ \hat{c}_{jl}|_{\ddot{S}_j} - d_{jm}^*(\hat{h}_{mn}) \circ \hat{d}_{jm}|_{\ddot{S}_j} \\ & \quad + (g_{ln} \circ e_{kl} \circ b_{jk})^*(\nabla^{F_n} t_n) \circ \big[-\mathcal{T} g_{ln} \circ \hat{\lambda} + \mathcal{T} h_{mn} \circ \hat{\mu}\big] + O(p_j) \\ & = c_{jl}^*(\hat{g}_{ln}) \circ \hat{c}_{jl}|_{\ddot{S}_j} - d_{jm}^*(\hat{h}_{mn}) \circ \hat{d}_{jm}|_{\ddot{S}_j} + (g_{ln} \circ e_{kl} \circ b_{jk})^*(\nabla^{F_n} t_n) \circ \check{\kappa} + O(p_j) \\ & = c_{jl}^*(\hat{g}_{ln}) \circ \hat{c}_{jl}|_{\ddot{S}_j} - d_{jm}^*(\hat{h}_{mn}) \circ \hat{d}_{jm}|_{\ddot{S}_j} + (g_{ln} \circ c_{jl})^*(\nabla^{F_n} t_n) \circ \hat{\kappa}|_{\ddot{S}_j} + O(p_j) \\ & = 0 + O(p_j). \end{aligned} \tag{11.101}$$

Here the second step uses (11.99) and

$$\begin{aligned} \nabla^{g_{ln}^*(F_n)}(\hat{g}_{ln}(r_l)) &= \hat{g}_{ln} \circ \nabla^{D_l} r_l + O(r_l), \\ \nabla^{h_{mn}^*(F_n)}(\hat{h}_{mn}(s_m)) &= \hat{h}_{mn} \circ \nabla^{E_m} s_m + O(s_m). \end{aligned}$$

The third step uses $\hat{g}_{ln}(r_l|_{U_{ln}}) = g^*_{ln}(t_n)$ and $\hat{h}_{mn}(s_m|_{V_{mn}}) = h^*_{mn}(t_n)$. The fourth step uses

$$\begin{aligned}(e_{kl}\circ b_{jk})^*(\nabla^{g^*_{ln}(F_n)}(g^*_{ln}(t_n))) &= (g_{ln}\circ e_{kl}\circ b_{jk})^*(\nabla^{F_n}t_n)\circ\mathcal{T}g_{ln},\\ (f_{km}\circ b_{jk})^*(\nabla^{h^*_{mn}(F_n)}(h^*_{mn}(t_n))) &= (h_{mn}\circ f_{km}\circ b_{jk})^*(\nabla^{F_n}t_n)\circ\mathcal{T}h_{mn}.\end{aligned} \tag{11.102}$$

The fifth follows from $h_{mn}\circ f_{km} = g_{ln}\circ e_{kl}$, the sixth from (11.100), the seventh from (11.99) and $\check{\kappa} = \hat{\kappa}|_{\ddot{S}_j} + O(p_j)$, and the last from (11.98) and Definition 3.15(vi). This proves (11.101).

Now $b^*_{jk}(C_k)\to\ddot{S}_j$ is the kernel of the surjective vector bundle morphism

$$\begin{aligned}b^*_{jk}\bigl[e^*_{kl}(\hat{g}_{ln})\oplus -f^*_{km}(\hat{h}_{mn})\bigr] : (e_{kl}\circ b_{jk})^*(D_l)\oplus(f_{km}\circ b_{jk})^*(E_m)\\ \longrightarrow (g_{ln}\circ e_{kl}\circ b_{jk})^*(F_n),\end{aligned}$$

which occurs at the beginning of (11.101), and the inclusion of $b^*_{jk}(C_k)$ as the kernel is $b^*_{jk}(\hat{e}_{kl})\oplus b^*_{jk}(\hat{f}_{km})$. Since taking kernels of surjective vector bundle morphisms commutes with reducing modulo $O(p_j)$, equation (11.101) implies that there is a morphism $\hat{b}_{jk} : B_j|_{\ddot{S}_j}\to b^*_{jk}(C_k)$, unique up to $O(p_j)$, with

$$\begin{aligned}\bigl(b^*_{jk}(\hat{e}_{kl})\oplus b^*_{jk}(\hat{f}_{km})\bigr)(\hat{b}_{jk}) = \bigl(\hat{c}_{jl}|_{\ddot{S}_j} - (e_{kl}\circ b_{jk})^*(\nabla^{D_l}r_l)\circ\hat{\lambda}\bigr)\\ \oplus\bigl(\hat{d}_{jm}|_{\ddot{S}_j} - (f_{km}\circ b_{jk})^*(\nabla^{E_m}s_m)\circ\hat{\mu}\bigr) + O(p_j),\end{aligned} \tag{11.103}$$

which by Definition 3.15(vi) is equivalent to

$$\begin{aligned}\hat{c}_{jl}|_{\ddot{S}_j} &= b^*_{jk}(\hat{e}_{kl})\circ\hat{b}_{jk} + (e_{kl}\circ b_{jk})^*(\mathrm{d}r_l)\circ\hat{\lambda} + O(p_j),\\ \hat{d}_{jm}|_{\ddot{S}_j} &= b^*_{jk}(\hat{f}_{km})\circ\hat{b}_{jk} + (f_{km}\circ b_{jk})^*(\mathrm{d}s_m)\circ\hat{\mu} + O(p_j).\end{aligned} \tag{11.104}$$

We have

$$\begin{aligned}&\bigl(b^*_{jk}(\hat{e}_{kl})\oplus b^*_{jk}(\hat{f}_{km})\bigr)(\hat{b}_{jk}(p_j)) = \bigl(\hat{c}_{jl}(p_j)|_{\ddot{S}_j} - (e_{kl}\circ b_{jk})^*(\nabla^{D_l}r_l)\circ\hat{\lambda}\circ p_j\bigr)\\ &\quad\oplus\bigl(\hat{d}_{jm}(p_j)|_{\ddot{S}_j} - (f_{km}\circ b_{jk})^*(\nabla^{E_m}s_m)\circ\hat{\mu}\circ p_j\bigr)\\ &= \bigl(c^*_{jl}(r_l)|_{\ddot{S}_j} - (e_{kl}\circ b_{jk})^*(\nabla^{D_l}r_l)\circ\hat{\lambda}\circ p_j\bigr)\\ &\qquad\oplus\bigl(d^*_{jm}(s_m)|_{\ddot{S}_j} - (f_{km}\circ b_{jk})^*(\nabla^{E_m}s_m)\circ\hat{\mu}\circ p_j\bigr) + O(p_j^2)\\ &= \bigl(b^*_{jk}\circ e^*_{kl}(r_l)\bigr)\oplus\bigl(b^*_{jk}\circ f^*_{km}(s_m)\bigr) + O(p_j^2)\\ &= \bigl(b^*_{jk}(\hat{e}_{kl}(q_k))\bigr)\oplus\bigl(b^*_{jk}(\hat{f}_{km}(q_k))\bigr) + O(p_j^2)\\ &= \bigl(b^*_{jk}(\hat{e}_{kl})\oplus b^*_{jk}(\hat{f}_{km})\bigr)(b^*_{jk}(q_k)) + O(p_j^2),\end{aligned} \tag{11.105}$$

where the first step comes from (11.103), the second from Definition 4.2(d) for $\boldsymbol{c}_{jl}, \boldsymbol{d}_{jm}$, the third can be proved by pulling back r_l, s_m using the equations of (11.99), and the fourth follows from Definition 4.2(d) for $\boldsymbol{e}_{kl}, \boldsymbol{f}_{km}$.

As $b^*_{jk}(\hat{e}_{kl})\oplus b^*_{jk}(\hat{f}_{km})$ is injective, (11.105) shows that $\hat{b}_{jk}(p_j) = b^*_{jk}(q_k) + O(p_j^2)$. Thus $\boldsymbol{b}_{jk} = (\ddot{S}_j, b_{jk}, \hat{b}_{jk}) : (S_j, B_j, p_j)\to(T_k, C_k, q_k)$ is a 1-morphism in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$.

Definition 4.3 and equations (11.99) and (11.104) now give 2-morphisms

$$\Lambda = [\ddot{S}_j, \hat{\lambda}] : \boldsymbol{e}_{kl} \circ \boldsymbol{b}_{jk} \Longrightarrow \boldsymbol{c}_{jl},$$
$$\mathrm{M} = [\ddot{S}_j, \hat{\mu}] : \boldsymbol{f}_{km} \circ \boldsymbol{b}_{jk} \Longrightarrow \boldsymbol{d}_{jm},$$

in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$, and equation (11.100) is equivalent to the commutative diagram

$$\begin{array}{ccc} \boldsymbol{g}_{ln} \circ \boldsymbol{e}_{kl} \circ \boldsymbol{b}_{jk} & \xrightarrow[\mathrm{id}_{\boldsymbol{g}_{ln}\circ\boldsymbol{e}_{kl}} * \mathrm{id}_{\boldsymbol{b}_{jk}}]{} & \boldsymbol{h}_{mn} \circ \boldsymbol{f}_{km} \circ \boldsymbol{b}_{jk} \\ \Big\Downarrow{\scriptstyle \mathrm{id}_{\boldsymbol{g}_{ln}} * \Lambda} & & {\scriptstyle \mathrm{id}_{\boldsymbol{h}_{mn}} * \mathrm{M}}\Big\Downarrow \\ \boldsymbol{g}_{ln} \circ \boldsymbol{c}_{jl} & \xrightarrow{\mathrm{K}} & \boldsymbol{h}_{mn} \circ \boldsymbol{d}_{jm}, \end{array}$$

which is equation (A.16) for the 2-commutative square (11.14). This proves the first part of the universal property in Definition A.11.

For the second part, let $\boldsymbol{b}'_{jk} = (\ddot{S}'_j, b'_{jk}, \hat{b}'_{jk}) : (S_j, B_j, p_j) \to (T_k, C_k, q_k)$ be a 1-morphism in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$, and

$$\Lambda' = [\ddot{S}'_j, \hat{\lambda}'] : \boldsymbol{e}_{kl} \circ \boldsymbol{b}'_{jk} \Longrightarrow \boldsymbol{c}_{jl},$$
$$\mathrm{M}' = [\ddot{S}'_j, \hat{\mu}'] : \boldsymbol{f}_{km} \circ \boldsymbol{b}'_{jk} \Longrightarrow \boldsymbol{d}_{jm},$$

be 2-morphisms in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$, such that the following commutes

$$\begin{array}{ccc} \boldsymbol{g}_{ln} \circ \boldsymbol{e}_{kl} \circ \boldsymbol{b}'_{jk} & \xrightarrow[\mathrm{id}_{\boldsymbol{g}_{ln}\circ\boldsymbol{e}_{kl}} * \mathrm{id}_{\boldsymbol{b}'_{jk}}]{} & \boldsymbol{h}_{mn} \circ \boldsymbol{f}_{km} \circ \boldsymbol{b}'_{jk} \\ \Big\Downarrow{\scriptstyle \mathrm{id}_{\boldsymbol{g}_{ln}} * \Lambda'} & & {\scriptstyle \mathrm{id}_{\boldsymbol{h}_{mn}} * \mathrm{M}'}\Big\Downarrow \\ \boldsymbol{g}_{ln} \circ \boldsymbol{c}_{jl} & \xrightarrow{\mathrm{K}} & \boldsymbol{h}_{mn} \circ \boldsymbol{d}_{jm}, \end{array} \tag{11.106}$$

where making $\ddot{S}'_j$ smaller, we use the same open $p_j^{-1}(0) \subseteq \ddot{S}'_j \subseteq S_j$ in $\boldsymbol{b}'_{jk}, \Lambda', \mathrm{M}'$.

Then $b'_{jk} : \ddot{S}'_j \to T_k$ is a morphism in $\mathbf{\dot{M}an}_{\boldsymbol{E}}$, and $\hat{\lambda}' : B_j|_{\ddot{S}'_j} \to \mathcal{T}_{e_{kl}\circ b'_{jk}} U_l$ and $\hat{\mu}' : B_j|_{\ddot{S}'_j} \to \mathcal{T}_{f_{km}\circ b'_{jk}} V_m$ are morphisms, where by Definition 4.3(b)

$$\begin{aligned} c_{jl}|_{\ddot{S}'_j} &= e_{kl} \circ b'_{jk} + \hat{\lambda}' \circ p_j + O(p_j^2), \quad d_{jm}|_{\ddot{S}'_j} = f_{km} \circ b'_{jk} + \hat{\mu}' \circ p_j + O(p_j^2), \\ \hat{c}_{jl}|_{\ddot{S}'_j} &= b'^*_{jk}(\hat{e}_{kl}) \circ \hat{b}'_{jk} + (e_{kl} \circ b'_{jk})^*(\mathrm{d}r_l) \circ \hat{\lambda}' + O(p_j), \\ \hat{d}_{jm}|_{\ddot{S}'_j} &= b'^*_{jk}(\hat{f}_{km}) \circ \hat{b}'_{jk} + (f_{km} \circ b'_{jk})^*(\mathrm{d}s_m) \circ \hat{\mu}' + O(p_j), \end{aligned} \tag{11.107}$$

as in (11.99) and (11.104). Theorem 3.17(g) gives $\hat{\kappa}' : B_j|_{\ddot{S}'_j} \to \mathcal{T}_{g_{ln}\circ e_{kl}\circ b'_{jk}} W_n$ with $\hat{\kappa}' = \hat{\kappa}|_{\ddot{S}'_j} + O(p_j)$, since $g_{ln} \circ c_{jl}|_{\ddot{S}'_j} = g_{ln} \circ e_{kl} \circ b'_{jk} + O(p_j)$ by the first equation of (11.107), and then as in (11.100), equation (11.106) is equivalent to

$$\hat{\kappa}' + \mathcal{T} g_{ln} \circ \hat{\lambda}' = \mathcal{T} h_{mn} \circ \hat{\mu}' + O(p_j). \tag{11.108}$$

Applying Assumption 11.1(b)(iii) to the first line of (11.107), and (11.108), shows that there exists a morphism $\hat{\nu} : B_j|_{\ddot{S}_j \cap \ddot{S}'_j} \to \mathcal{T}_{b_{jk}} T_k|_{\ddot{S}_j \cap \ddot{S}'_j}$ with

$$b'_{jk}|_{\ddot{S}_j \cap \ddot{S}'_j} = b_{jk}|_{\ddot{S}_j \cap \ddot{S}'_j} + \hat{\nu} \circ p_j + O(p_j^2), \tag{11.109}$$

and if $\check\lambda' : B_j|_{\ddot S_j\cap\ddot S'_j} \to \mathcal{T}_{e_{kl}\circ b_{jk}}U_l|_{\ddot S_j\cap\ddot S'_j}$, $\check\mu' : B_j|_{\ddot S_j\cap\ddot S'_j} \to \mathcal{T}_{f_{km}\circ b_{jk}}V_m|_{\ddot S_j\cap\ddot S'_j}$ are morphisms with $\hat\lambda'|_{\ddot S_j\cap\ddot S'_j} = \check\lambda' + O(p_j)$, $\hat\mu'|_{\ddot S_j\cap\ddot S'_j} = \check\mu' + O(p_j)$, which exist and are unique up to $O(p_j)$ by Theorem 3.17(g), then

$$\hat\lambda|_{\ddot S_j\cap\ddot S'_j} = \check\lambda' + \mathcal{T}e_{kl}\circ\hat\nu + O(p_j), \quad \hat\mu|_{\ddot S_j\cap\ddot S'_j} = \check\mu' + \mathcal{T}f_{km}\circ\hat\nu + O(p_j). \quad (11.110)$$

Furthermore, $\hat\nu$ satisfying (11.109)–(11.110) is unique up to $O(p_j)$. Now

$$\begin{aligned}
&b'^*_{jk}(\hat e_{kl})\circ\hat b'_{jk}|_{\ddot S_j\cap\ddot S'_j} = \hat c_{jl}|_{\ddot S_j\cap\ddot S'_j} - (e_{kl}\circ b'_{jk})^*(\mathrm{d}r_l)\circ\hat\lambda'|_{\ddot S_j\cap\ddot S'_j} + O(p_j)\\
&= b^*_{jk}(\hat e_{kl})\circ\hat b_{jk}|_{\ddot S_j\cap\ddot S'_j} + (e_{kl}\circ b_{jk})^*(\mathrm{d}r_l)\circ\hat\lambda - (e_{kl}\circ b_{jk})^*(\mathrm{d}r_l)\circ\check\lambda' + O(p_j)\\
&= b^*_{jk}(\hat e_{kl})\circ\hat b_{jk}|_{\ddot S_j\cap\ddot S'_j} + (e_{kl}\circ b_{jk})^*(\nabla^{D_l}r_l)\circ\mathcal{T}e_{kl}\circ\hat\nu + O(p_j)\\
&= b^*_{jk}(\hat e_{kl})\circ\hat b_{jk}|_{\ddot S_j\cap\ddot S'_j} + b^*_{jk}(\nabla^{e^*_{kl}(D_l)}(e^*_{kl}(r_l))\circ\hat\nu + O(p_j)\\
&= b^*_{jk}(\hat e_{kl})\circ\hat b_{jk}|_{\ddot S_j\cap\ddot S'_j} + b^*_{jk}(\nabla^{e^*_{kl}(D_l)}(\hat e_{kl}(q_k))\circ\hat\nu + O(p_j)\\
&= b^*_{jk}(\hat e_{kl})\circ\big[\hat b_{jk}|_{\ddot S_j\cap\ddot S'_j} + b^*_{jk}(\nabla^{C_k}q_k)\circ\hat\nu\big] + O(p_j),
\end{aligned} \quad (11.111)$$

using the third equation of (11.107) in the first step, (11.104) and $e_{kl}\circ b_{jk}|_{\ddot S_j\cap\ddot S'_j} = e_{kl}\circ b'_{jk}|_{\ddot S_j\cap\ddot S'_j} + O(p_j)$ by (11.109) and $\hat\lambda'|_{\ddot S_j\cap\ddot S'_j} = \check\lambda' + O(p_j)$ in the second step, and (11.110) and choosing a connection ∇^{D_l} on $D_l \to U_l$ in the third.

In the fourth step of (11.111), as in (11.102) we use

$$(e_{kl}\circ b_{jk})^*(\nabla^{D_l}r_l)\circ\mathcal{T}e_{kl} = b^*_{jk}(\nabla^{e^*_{kl}(D_l)}(e^*_{kl}(r_l)) : \mathcal{T}_{b_{jk}}T_k|_{\ddot S_j\cap\ddot S'_j} \to (e_{kl}\circ b_{jk})^*(D_l),$$

where $\nabla^{e^*_{kl}(D_l)}$ is the pullback connection on $e^*_{kl}(D_l) \to T_k$ from ∇^{D_l}. The fifth step uses $\hat e_{kl}(q_k) = e^*_{kl}(r_l)$, and the sixth $\nabla^{e^*_{kl}(D_l)}(\hat e_{kl}(q_k)) = \hat e_{kl}\circ\nabla^{C_k}q_k + O(q_k)$ for ∇^{C_k} some connection on C_k, and $b^*_{jk}(q_k) = O(p_j)$. This proves (11.111). Similarly we have

$$b'^*_{jk}(\hat f_{km})\circ\hat b'_{jk}|_{\ddot S_j\cap\ddot S'_j} = b^*_{jk}(\hat f_{km})\circ\big[\hat b_{jk}|_{\ddot S_j\cap\ddot S'_j} + b^*_{jk}(\nabla^{C_k}q_k)\circ\hat\nu\big] + O(p_j). \quad (11.112)$$

Since $\hat e_{kl}\oplus\hat f_{km} : C_k \to e^*_{kl}(D_l)\oplus f^*_{km}(E_m)$ is injective, and $b'_{jk}|_{\ddot S_j\cap\ddot S'_j} = b_{jk}|_{\ddot S_j\cap\ddot S'_j} + O(p_j)$, equations (11.111)–(11.112) imply that as in (4.1),

$$\hat b'_{jk}|_{\ddot S_j\cap\ddot S'_j} = \hat b_{jk}|_{\ddot S_j\cap\ddot S'_j} + b^*_{jk}(\mathrm{d}q_k)\circ\hat\nu + O(p_j). \quad (11.113)$$

Equations (11.109) and (11.113) and $b = b'$ imply that

$$\mathrm{N} = [\ddot S_j\cap\ddot S'_j, \hat\nu] : \boldsymbol{b}_{jk} \Longrightarrow \boldsymbol{b}'_{jk}$$

is a 2-morphism in $\mathbf{Gm\dot{K}N_{\boldsymbol{E}}}$, and (11.110) is equivalent to

$$\Lambda = \Lambda'\odot(\mathrm{id}_{\boldsymbol{e}_{kl}} * \mathrm{N}) \quad\text{and}\quad \mathrm{M} = \mathrm{M}'\odot(\mathrm{id}_{\boldsymbol{f}_{km}} * \mathrm{N}).$$

That N is unique with these properties follows from the uniqueness of $\hat\nu$ satisfying (11.109)–(11.110) up to $O(p_j)$. This proves the second part of the universal property in Definition A.11, and completes the proof of Theorem 11.17.

11.9 Proof of Theorem 11.19

Suppose $\dot{\mathbf{Man}}$ satisfies Assumptions 3.1–3.7 and 11.1. Let $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ be 1-morphisms in $\mathbf{m\dot{K}ur}$, which will usually be w-transverse in $\mathbf{m\dot{K}ur_{\boldsymbol{D}}}$. The aim will be to construct a fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\mathbf{m\dot{K}ur_{\boldsymbol{D}}}$ or $\mathbf{m\dot{K}ur_{\boldsymbol{E}}}$, with projections $\boldsymbol{e}:\boldsymbol{W}\to\boldsymbol{X}$, $\boldsymbol{f}:\boldsymbol{W}\to\boldsymbol{Y}$ and a 2-morphism $\boldsymbol{\eta}:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ in a 2-Cartesian square (11.15). We will use notation (4.6)–(4.8) for $\boldsymbol{X}=(X,\mathcal{I})$, $\boldsymbol{Y}=(Y,\mathcal{J})$, $\boldsymbol{Z}=(Z,\mathcal{K})$, and our usual notation for $\boldsymbol{e},\ldots,\boldsymbol{h}$ and $\boldsymbol{\eta}$ as in (4.9) and Definition 4.18.

11.9.1 Constructing $\boldsymbol{W},\boldsymbol{e},\boldsymbol{f},\boldsymbol{\eta}$ when Assumption 11.3 holds

Let $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ be w-transverse 1-morphisms in $\mathbf{m\dot{K}ur}$. For simplicity, we first suppose that $\dot{\mathbf{Man}}$ also satisfies Assumption 11.3. Then as in Theorem 11.19(c) we will construct a fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\mathbf{m\dot{K}ur_{\boldsymbol{D}}}$ and $\mathbf{m\dot{K}ur_{\boldsymbol{E}}}$, with topological space $W=\bigl\{(x,y)\in X\times Y: g(x)=h(y)\bigr\}$, and continuous maps $e:W\to X$, $f:W\to Y$ acting by $e:(x,y)\mapsto x$ and $f:(x,y)\mapsto y$. The general case, which we tackle in §11.9.2, is more complicated, as we also have to construct W,e,f.

So let W,e,f be as above, and let $(x,y)\in W$ with $g(x)=h(y)=z$ in Z. Then by Definition 11.18 there exist m-Kuranishi neighbourhoods (U_l,D_l,r_l,χ_l), (V_m,E_m,s_m,ψ_m), (W_n,F_n,t_n,ω_n) on $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ as in §4.7 with $x\in\operatorname{Im}\chi_l\subseteq g^{-1}(\operatorname{Im}\omega_n)$, $y\in\operatorname{Im}\psi_m\subseteq h^{-1}(\operatorname{Im}\omega_n)$ and $z\in\operatorname{Im}\omega_n$, and 1-morphisms $\boldsymbol{g}_{ln}:(U_l,D_l,r_l,\chi_l)\to(W_n,F_n,t_n,\omega_n)$, $\boldsymbol{h}_{mn}:(V_m,E_m,s_m,\psi_m)\to(W_n,F_n,t_n,\omega_n)$ over $(\operatorname{Im}\chi_l,\boldsymbol{g})$ and $(\operatorname{Im}\psi_m,\boldsymbol{h})$, as in Definition 4.54, such that $\boldsymbol{g}_{ln},\boldsymbol{h}_{mn}$ are w-transverse as in Definition 11.16.

Apply Definition 11.16 and Theorem 11.17 to the 1-morphisms in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$

$$\boldsymbol{g}_{ln}:(U_l,D_l,r_l)\longrightarrow(W_n,F_n,t_n),\quad \boldsymbol{h}_{mn}:(V_m,E_m,s_m)\longrightarrow(W_n,F_n,t_n).$$

These construct a 2-Cartesian square (11.14) in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$ and $\mathbf{Gm\dot{K}N_{\boldsymbol{E}}}$. From (11.13) and Definition 4.14(b) for $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ we see that

$$\dim T_k-\operatorname{rank}C_k=\operatorname{vdim}\boldsymbol{X}+\operatorname{vdim}\boldsymbol{Y}-\operatorname{vdim}\boldsymbol{Z}.$$

Here by definition T_k is the transverse fibre product in $\dot{\mathbf{Man}}$:

$$T_k=\dot U_{ln}\times_{g_{ln}|_{\dot U_{ln}},W_n,h_{mn}|_{\dot V_{mn}}}\dot V_{mn},\tag{11.114}$$

for open $\dot U_{ln}\subseteq U_{ln}$, $\dot V_{mn}\subseteq V_{mn}$ satisfying Definition 11.15(i),(ii). As we suppose Assumption 11.3, by Assumption 3.2(e) we take T_k to have topological space

$$T_k=\bigl\{(u,v)\in\dot U_{ln}\times\dot V_{mn}:g_{ln}(u)=h_{mn}(v)\in W_n\bigr\},\tag{11.115}$$

and then $e_{kl}:T_k\to U_l$, $f_{km}:T_k\to V_m$ map $e_{kl}:(u,v)\mapsto u$, $f_{km}:(u,v)\mapsto v$.

Since $q_k=e_{kl}^*(r_l)\oplus f_{km}^*(s_m)$, we see that

$$q_k^{-1}(0)=\bigl\{(u,v)\in r_l^{-1}(0)\times s_m^{-1}(0):g_{ln}(u)=h_{mn}(v)\bigr\}.$$

Define $\varphi_k : q_k^{-1}(0) \to W$ by $\varphi_k(u,v) = (\chi_l(u), \psi_m(v))$. This is well defined as

$$g \circ \chi_l(u) = \omega_n \circ g_{ln}(u) = \omega_n \circ h_{mn}(v) = h \circ \psi_m(v),$$

using Definition 4.2(e) for $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$. As χ_l, ψ_m are homeomorphisms with their open images, φ_k is a homeomorphism with the open subset

$$\operatorname{Im}\varphi_k = \bigl\{(x,y)\in W : x\in \operatorname{Im}\chi_l,\ y\in\operatorname{Im}\psi_m\bigr\} = e^{-1}(\operatorname{Im}\chi_l)\cap f^{-1}(\operatorname{Im}\psi_m)\subseteq W.$$

Hence $(T_k, C_k, q_k, \varphi_k)$ is an m-Kuranishi neighbourhood on W. Since $e \circ \varphi_k = \chi_l \circ e_{kl}$ and $f \circ \varphi_k = \psi_m \circ f_{km}$ on $q_k^{-1}(0)$, $\boldsymbol{e}_{kl} : (T_k, C_k, q_k, \varphi_k) \to (U_l, D_l, r_l, \chi_l)$ is a 1-morphism over $(\operatorname{Im}\varphi_k, e)$ and $\boldsymbol{f}_{km} : (T_k, C_k, q_k, \varphi_k) \to (V_m, E_m, s_m, \psi_m)$ is a 1-morphism over $(\operatorname{Im}\varphi_k, f)$. Thus, generalizing (11.14) we have a 2-commutative diagram in $\mathbf{m\dot{K}N}_{\boldsymbol{D}}$ from Definition 4.8:

$$\begin{array}{ccc} (W, \operatorname{Im}\varphi_k, (T_k, C_k, q_k, \varphi_k)) & \xrightarrow[(f,\boldsymbol{f}_{km})]{} & (Y, \operatorname{Im}\psi_m, (V_m, E_m, s_m, \psi_m)) \\ \Big\downarrow{\scriptstyle (e,\boldsymbol{e}_{kl})} & \operatorname{id}\Uparrow & {\scriptstyle (h,\boldsymbol{h}_{mn})}\Big\downarrow \\ (X, \operatorname{Im}\chi_l, (U_l, D_l, r_l, \chi_l)) & \xrightarrow{(g,\boldsymbol{g}_{ln})} & (Z, \operatorname{Im}\omega_n, (W_n, F_n, t_n, \omega_n)). \end{array} \tag{11.116}$$

We can find such a diagram (11.116) with $(x,y) \in \operatorname{Im}\varphi_k \subseteq W$ for all (x,y) in W. Thus we can choose a family of such diagrams indexed by a in an indexing set A so that the subsets $\operatorname{Im}\varphi_k$ cover W. We change notation from subscripts k,l,m,n to subscripts $a, \dot a, \ddot a, \dddot a$, where $a \in A$, and $\dot a, \ddot a, \dddot a$ correspond to a, but have accents to help distinguish m-Kuranishi neighbourhoods on W, X, Y, Z. Thus, for $a \in A$ we have a family of 2-commutative diagrams in $\mathbf{m\dot{K}N}_{\boldsymbol{D}}$

$$\begin{array}{ccc} (W, \operatorname{Im}\varphi_a, (T_a, C_a, q_a, \varphi_a)) & \xrightarrow[(f,\boldsymbol{f}_{a\ddot a})]{} & (Y, \operatorname{Im}\psi_{\ddot a}, (V_{\ddot a}, E_{\ddot a}, s_{\ddot a}, \psi_{\ddot a})) \\ \Big\downarrow{\scriptstyle (e,\boldsymbol{e}_{a\dot a})} & \operatorname{id}\Uparrow & {\scriptstyle (h,\boldsymbol{h}_{\ddot a\dddot a})}\Big\downarrow \\ (X, \operatorname{Im}\chi_{\dot a}, (U_{\dot a}, D_{\dot a}, r_{\dot a}, \chi_{\dot a})) & \xrightarrow{(g,\boldsymbol{g}_{\dot a\dddot a})} & (Z, \operatorname{Im}\omega_{\dddot a}, (W_{\dddot a}, F_{\dddot a}, t_{\dddot a}, \omega_{\dddot a})), \end{array} \tag{11.117}$$

with $W = \bigcup_{a\in A} \operatorname{Im}\varphi_a$, such that as in (11.14) the following is 2-Cartesian in $\mathbf{Gm\dot{K}N}_{\boldsymbol{D}}$ and $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$:

$$\begin{array}{ccc} (T_a, C_a, q_a) & \xrightarrow[\boldsymbol{f}_{a\ddot a}]{} & (V_{\ddot a}, E_{\ddot a}, s_{\ddot a}) \\ \Big\downarrow{\scriptstyle \boldsymbol{e}_{a\dot a}} & \operatorname{id}\Uparrow & {\scriptstyle \boldsymbol{h}_{\ddot a\dddot a}}\Big\downarrow \\ (U_{\dot a}, D_{\dot a}, r_{\dot a}) & \xrightarrow{\boldsymbol{g}_{\dot a\dddot a}} & (W_{\dddot a}, F_{\dddot a}, t_{\dddot a}). \end{array} \tag{11.118}$$

Let $a, b \in A$. Then Theorem 4.56(a) gives coordinate changes

$$\begin{aligned} &\mathrm{T}_{\dot a\dot b} : (U_{\dot a}, D_{\dot a}, r_{\dot a}, \chi_{\dot a}) \longrightarrow (U_{\dot b}, D_{\dot b}, r_{\dot b}, \chi_{\dot b}) && \text{over } \operatorname{Im}\chi_{\dot a}\cap\operatorname{Im}\chi_{\dot b} \text{ on } \boldsymbol{X}, \\ &\Upsilon_{\ddot a\ddot b} : (V_{\ddot a}, E_{\ddot a}, s_{\ddot a}, \psi_{\ddot a}) \longrightarrow (V_{\ddot b}, E_{\ddot b}, s_{\ddot b}, \psi_{\ddot b}) && \text{over } \operatorname{Im}\psi_{\ddot a}\cap\operatorname{Im}\psi_{\ddot b} \text{ on } \boldsymbol{Y}, \\ &\Phi_{\dddot a\dddot b} : (W_{\dddot a}, F_{\dddot a}, t_{\dddot a}, \omega_{\dddot a}) \longrightarrow (W_{\dddot b}, F_{\dddot b}, t_{\dddot b}, \omega_{\dddot b}) && \text{over } \operatorname{Im}\omega_{\dddot a}\cap\operatorname{Im}\omega_{\dddot b} \text{ on } \boldsymbol{Z}, \end{aligned}$$

where we choose $\mathrm{T}_{\dot{a}\dot{a}}, \Upsilon_{\ddot{a}\ddot{a}}, \Phi_{\dddot{a}\dddot{a}}$ to be identities, and so Theorem 4.56(c) gives unique 2-morphisms

$$\begin{aligned} \boldsymbol{G}_{\dot{a}\dot{b}}^{\dddot{a}\dddot{b}} &: \boldsymbol{g}_{\dot{b}\dddot{b}} \circ \mathrm{T}_{\dot{a}\dot{b}} \Longrightarrow \Phi_{\dddot{a}\dddot{b}} \circ \boldsymbol{g}_{\dot{a}\dddot{a}} && \text{over } \operatorname{Im}\chi_{\dot{a}} \cap \operatorname{Im}\chi_{\dot{b}} \text{ on } \boldsymbol{X}, \\ \boldsymbol{H}_{\ddot{a}\ddot{b}}^{\dddot{a}\dddot{b}} &: \boldsymbol{h}_{\ddot{b}\dddot{b}} \circ \Upsilon_{\ddot{a}\ddot{b}} \Longrightarrow \Phi_{\dddot{a}\dddot{b}} \circ \boldsymbol{h}_{\ddot{a}\dddot{a}} && \text{over } \operatorname{Im}\psi_{\ddot{a}} \cap \operatorname{Im}\psi_{\ddot{b}} \text{ on } \boldsymbol{Y}, \end{aligned}$$

such that the analogue of (4.62) commutes. When $a = b$ these are identities, as $\mathrm{T}_{\dot{a}\dot{a}}, \Upsilon_{\ddot{a}\ddot{a}}, \Phi_{\dddot{a}\dddot{a}}$ are identities.

Writing $\mathrm{T}_{\dot{a}\dot{b}} = (U_{\dot{a}\dot{b}}, \tau_{\dot{a}\dot{b}}, \hat{\tau}_{\dot{a}\dot{b}})$ and $\Upsilon_{\ddot{a}\ddot{b}} = (V_{\ddot{a}\ddot{b}}, \upsilon_{\ddot{a}\ddot{b}}, \hat{\upsilon}_{\ddot{a}\ddot{b}})$, set $T_{ab} = e_{a\dot{a}}^{-1}(U_{\dot{a}\dot{b}}) \cap f_{a\ddot{a}}^{-1}(V_{\ddot{a}\ddot{b}})$. Then T_{ab} is an open neighbourhood of $\varphi_a^{-1}(\operatorname{Im}\varphi_a \cap \operatorname{Im}\varphi_b)$ in T_a. Consider the 1-morphisms in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$:

$$\begin{aligned} \mathrm{T}_{\dot{a}\dot{b}} \circ \boldsymbol{e}_{a\dot{a}}|_{T_{ab}} &: (T_{ab}, C_a|_{T_{ab}}, q_a|_{T_{ab}}) \longrightarrow (U_{\dot{b}}, D_{\dot{b}}, r_{\dot{b}}), \\ \Upsilon_{\ddot{a}\ddot{b}} \circ \boldsymbol{f}_{a\ddot{a}}|_{T_{ab}} &: (T_{ab}, C_a|_{T_{ab}}, q_a|_{T_{ab}}) \longrightarrow (V_{\ddot{b}}, E_{\ddot{b}}, s_{\ddot{b}}), \end{aligned}$$

and the 2-morphism

$$\big((\boldsymbol{H}_{\ddot{a}\ddot{b}}^{\dddot{a}\dddot{b}})^{-1} * \mathrm{id}_{\boldsymbol{f}_{a\ddot{a}}}\big) \odot \big(\boldsymbol{G}_{\dot{a}\dot{b}}^{\dddot{a}\dddot{b}} * \mathrm{id}_{\boldsymbol{e}_{a\dot{a}}}\big) : \boldsymbol{g}_{\dot{b}\dddot{b}} \circ \big[\mathrm{T}_{\dot{a}\dot{b}} \circ \boldsymbol{e}_{a\dot{a}}|_{T_{ab}}\big] \Longrightarrow \boldsymbol{h}_{\ddot{b}\dddot{b}} \circ \big[\Upsilon_{\ddot{a}\ddot{b}} \circ \boldsymbol{f}_{a\ddot{a}}|_{T_{ab}}\big],$$

noting that $\boldsymbol{g}_{\dot{a}\dddot{a}} \circ \boldsymbol{e}_{a\dot{a}} = \boldsymbol{h}_{\ddot{a}\dddot{a}} \circ \boldsymbol{f}_{a\ddot{a}}$ as in (11.118). Since (11.118) with b in place of a is 2-Cartesian in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$ by Theorem 11.17, the universal property in Definition A.11 gives a 1-morphism in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$, unique up to 2-isomorphism,

$$\Sigma_{ab} : (T_a, C_a, q_a)|_{T_{ab}} = (T_{ab}, C_a|_{T_{ab}}, q_a|_{T_{ab}}) \longrightarrow (T_b, C_b, q_b),$$

and 2-isomorphisms in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$

$$\boldsymbol{E}_{ab}^{\dot{a}\dot{b}} : \boldsymbol{e}_{b\dot{b}} \circ \Sigma_{ab} \Longrightarrow \mathrm{T}_{\dot{a}\dot{b}} \circ \boldsymbol{e}_{a\dot{a}}|_{T_{ab}}, \quad \boldsymbol{F}_{ab}^{\ddot{a}\ddot{b}} : \boldsymbol{f}_{b\ddot{b}} \circ \Sigma_{ab} \Longrightarrow \Upsilon_{\ddot{a}\ddot{b}} \circ \boldsymbol{f}_{a\ddot{a}}|_{T_{ab}}, \tag{11.119}$$

such that the following diagram of 2-isomorphisms commutes:

$$\begin{array}{ccc} \boldsymbol{g}_{\dot{b}\dddot{b}} \circ \boldsymbol{e}_{b\dot{b}} \circ \Sigma_{ab} & \xRightarrow{\quad\mathrm{id}\quad} & \boldsymbol{h}_{\ddot{b}\dddot{b}} \circ \boldsymbol{f}_{b\ddot{b}} \circ \Sigma_{ab} \\ \Big\Downarrow {\scriptstyle \mathrm{id}_{\boldsymbol{g}_{\dot{b}\dddot{b}}} * \boldsymbol{E}_{ab}^{\dot{a}\dot{b}}} & & {\scriptstyle \mathrm{id}_{\boldsymbol{h}_{\ddot{b}\dddot{b}}} * \boldsymbol{F}_{ab}^{\ddot{a}\ddot{b}}} \Big\Downarrow \\ \boldsymbol{g}_{\dot{b}\dddot{b}} \circ \mathrm{T}_{\dot{a}\dot{b}} \circ \boldsymbol{e}_{a\dot{a}}|_{T_{ab}} & \xRightarrow{((\boldsymbol{H}_{\ddot{a}\ddot{b}}^{\dddot{a}\dddot{b}})^{-1} * \mathrm{id}_{\boldsymbol{f}_{a\ddot{a}}}) \odot (\boldsymbol{G}_{\dot{a}\dot{b}}^{\dddot{a}\dddot{b}} * \mathrm{id}_{\boldsymbol{e}_{a\dot{a}}})} & \boldsymbol{h}_{\ddot{b}\dddot{b}} \circ \Upsilon_{\ddot{a}\ddot{b}} \circ \boldsymbol{f}_{a\ddot{a}}|_{T_{ab}}. \end{array} \tag{11.120}$$

As $\mathrm{T}_{\dot{a}\dot{a}}, \Upsilon_{\ddot{a}\ddot{a}}, \boldsymbol{G}_{\dot{a}\dot{a}}^{\dddot{a}\dddot{a}}, \boldsymbol{H}_{\ddot{a}\ddot{a}}^{\dddot{a}\dddot{a}}$ are identities, we can choose

$$\Sigma_{aa} = \mathrm{id}_{(T_a, C_a, q_a)}, \quad \boldsymbol{E}_{aa}^{\dot{a}\dot{a}} = \mathrm{id}_{\boldsymbol{e}_{a\dot{a}}}, \quad \text{and} \quad \boldsymbol{F}_{aa}^{\ddot{a}\ddot{a}} = \mathrm{id}_{\boldsymbol{f}_{a\ddot{a}}}. \tag{11.121}$$

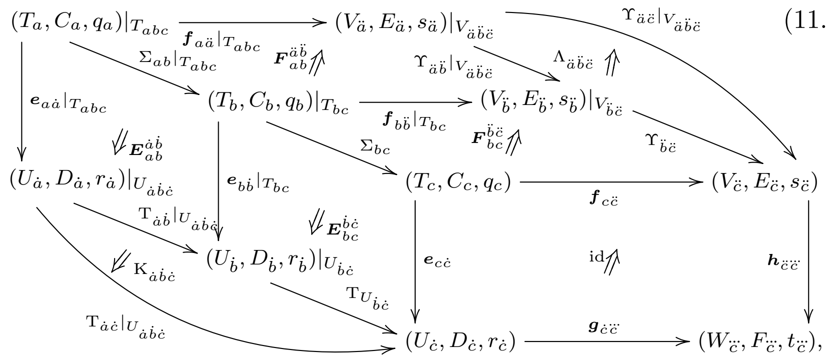
Now let $a, b, c \in A$. Then Theorem 4.56(c) gives unique 2-morphisms

$$\begin{aligned} \mathrm{K}_{\dot{a}\dot{b}\dot{c}} &: \mathrm{T}_{\dot{b}\dot{c}} \circ \mathrm{T}_{\dot{a}\dot{b}} \Longrightarrow \mathrm{T}_{\dot{a}\dot{c}} && \text{over } \operatorname{Im}\chi_{\dot{a}} \cap \operatorname{Im}\chi_{\dot{b}} \cap \operatorname{Im}\chi_{\dot{c}} \text{ on } \boldsymbol{X}, \\ \Lambda_{\ddot{a}\ddot{b}\ddot{c}} &: \Upsilon_{\ddot{b}\ddot{c}} \circ \Upsilon_{\ddot{a}\ddot{b}} \Longrightarrow \Upsilon_{\ddot{a}\ddot{c}} && \text{over } \operatorname{Im}\psi_{\ddot{a}} \cap \operatorname{Im}\psi_{\ddot{b}} \cap \operatorname{Im}\psi_{\ddot{c}} \text{ on } \boldsymbol{Y}, \end{aligned}$$

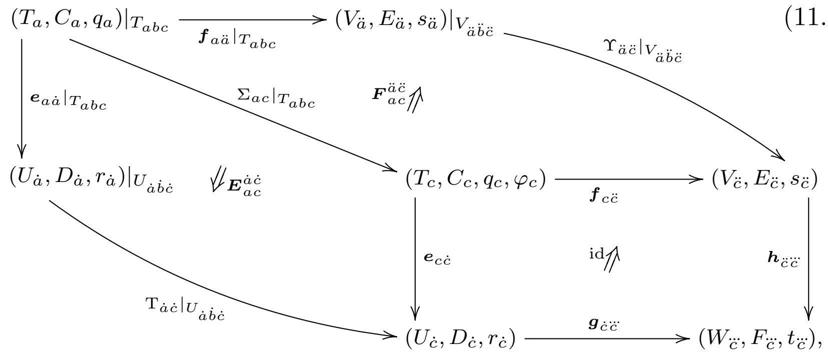
such that the analogue of (4.62) commutes. Using Theorem 4.56(d) we see that

$$\begin{aligned} \mathrm{K}_{\dot{a}\dot{c}\dot{d}} \odot (\mathrm{id}_{\mathrm{T}_{\dot{c}\dot{d}}} * \mathrm{K}_{\dot{a}\dot{b}\dot{c}}) &= \mathrm{K}_{\dot{a}\dot{b}\dot{d}} \odot (\mathrm{K}_{\dot{b}\dot{c}\dot{d}} * \mathrm{id}_{\mathrm{T}_{\dot{a}\dot{b}}}) : \mathrm{T}_{\dot{c}\dot{d}} \circ \mathrm{T}_{\dot{b}\dot{c}} \circ \mathrm{T}_{\dot{a}\dot{b}} \Longrightarrow \mathrm{T}_{\dot{a}\dot{d}}, \\ \Lambda_{\ddot{a}\ddot{c}\ddot{d}} \odot (\mathrm{id}_{\Upsilon_{\ddot{c}\ddot{d}}} * \Lambda_{\ddot{a}\ddot{b}\ddot{c}}) &= \Lambda_{\ddot{a}\ddot{b}\ddot{d}} \odot (\Lambda_{\ddot{b}\ddot{c}\ddot{d}} * \mathrm{id}_{\Upsilon_{\ddot{a}\ddot{b}}}) : \Upsilon_{\ddot{c}\ddot{d}} \circ \Upsilon_{\ddot{b}\ddot{c}} \circ \Upsilon_{\ddot{a}\ddot{b}} \Longrightarrow \Upsilon_{\ddot{a}\ddot{d}}. \end{aligned} \tag{11.122}$$

Compare the two 2-commutative diagrams:



(11.123)



(11.124)

where $T_{abc} = T_{ab} \cap T_{bc}$, and $U_{\dot a\dot b\dot c}, \ldots$ are defined in a similar way. By the last part of the universal property in Definition A.11 for (11.118) with c in place of a, there exists a unique 2-isomorphism $\mathrm{I}_{abc} : \Sigma_{bc} \circ \Sigma_{ab}|_{T_{abc}} \Rightarrow \Sigma_{ac}|_{T_{abc}}$, such that the following commute:

$$\begin{array}{ccccc}
\boldsymbol{e}_{c\dot c} \circ \Sigma_{bc} \circ \Sigma_{ab}|_{T_{abc}} & & \xRightarrow{\quad \mathrm{id}_{\boldsymbol{e}_{c\dot c}} * \mathrm{I}_{abc} \quad} & & \boldsymbol{e}_{c\dot c} \circ \Sigma_{ac}|_{T_{abc}} \\
\Big\Downarrow {\scriptstyle \boldsymbol{E}_{bc}^{\dot b\dot c} * \mathrm{id}_{\Sigma_{ab}}} & & & & {\scriptstyle \boldsymbol{E}_{ac}^{\dot a\dot c}} \Big\Downarrow \\
\mathrm{T}_{\dot b\dot c} \circ \boldsymbol{e}_{b\dot b} \circ \Sigma_{ab}|_{T_{abc}} & \xRightarrow{\mathrm{id}_{\mathrm{T}_{\dot b\dot c}} * \boldsymbol{E}_{ab}^{\dot a\dot b}} & \mathrm{T}_{\dot b\dot c} \circ \mathrm{T}_{\dot a\dot b} \circ \boldsymbol{e}_{a\dot a}|_{T_{abc}} & \xRightarrow{\mathrm{K}_{\dot a\dot b\dot c} * \mathrm{id}_{\boldsymbol{e}_{a\dot a}}} & \mathrm{T}_{\dot a\dot c} \circ \boldsymbol{e}_{a\dot a}|_{T_{abc}},
\end{array} \tag{11.125}$$

$$\begin{array}{ccccc}
\boldsymbol{f}_{c\ddot c} \circ \Sigma_{bc} \circ \Sigma_{ab}|_{T_{abc}} & & \xRightarrow{\quad \mathrm{id}_{\boldsymbol{f}_{c\ddot c}} * \mathrm{I}_{abc} \quad} & & \boldsymbol{f}_{c\ddot c} \circ \Sigma_{ac}|_{T_{abc}} \\
\Big\Downarrow {\scriptstyle \boldsymbol{F}_{bc}^{\ddot b\ddot c} * \mathrm{id}_{\Sigma_{ab}}} & & & & {\scriptstyle \boldsymbol{F}_{ac}^{\ddot a\ddot c}} \Big\Downarrow \\
\Upsilon_{\ddot b\ddot c} \circ \boldsymbol{f}_{b\ddot b} \circ \Sigma_{ab}|_{T_{abc}} & \xRightarrow{\mathrm{id}_{\Upsilon_{\ddot b\ddot c}} * \boldsymbol{F}_{ab}^{\ddot a\ddot b}} & \Upsilon_{\ddot b\ddot c} \circ \Upsilon_{\ddot a\ddot b} \circ \boldsymbol{f}_{a\ddot a}|_{T_{abc}} & \xRightarrow{\Lambda_{\ddot a\ddot b\ddot c} * \mathrm{id}_{\boldsymbol{f}_{a\ddot a}}} & \Upsilon_{\ddot a\ddot c} \circ \boldsymbol{f}_{a\ddot a}|_{T_{abc}}.
\end{array} \tag{11.126}$$

From (11.121) and (11.122) with $c = a$ we see that $\Sigma_{ba} \circ \Sigma_{ab} \cong \mathrm{id}_{(T_a, C_a, q_a, \varphi_a)}$, and similarly $\Sigma_{ab} \circ \Sigma_{ba} \cong \mathrm{id}_{(T_b, C_b, q_b, \varphi_b)}$. Hence $\Sigma_{ab} : (T_a, C_a, q_a, \varphi_a) \to (T_b, C_b, q_b, \varphi_b)$ is a coordinate change over $\mathrm{Im}\,\varphi_a \cap \mathrm{Im}\,\varphi_b$, with quasi-inverse Σ_{ba}. Also from (11.121) for a, b we can deduce that $\mathrm{I}_{aab} = \mathrm{I}_{abb} = \mathrm{id}_{\Sigma_{ab}}$.

Let $a, b, c, d \in A$, and consider the diagram of 2-morphisms over $\mathrm{Im}\,\varphi_a \cap$

$\operatorname{Im}\varphi_b \cap \operatorname{Im}\varphi_c \cap \operatorname{Im}\varphi_d$ on W:

$$
\begin{array}{ccccc}
\boldsymbol{e}_{d\dot d}\circ\Sigma_{cd}\circ\Sigma_{bc}\circ\Sigma_{ab} & & \xRightarrow{\mathrm{id}*\mathrm{I}_{abc}} & & \boldsymbol{e}_{d\dot d}\circ\Sigma_{cd}\circ\Sigma_{ac} \\
 & \searrow{\scriptstyle \boldsymbol{E}_{cd}^{\dot c\dot d}*\mathrm{id}} & & \swarrow{\scriptstyle \boldsymbol{E}_{cd}^{\dot c\dot d}*\mathrm{id}} & \\
 & \mathrm{T}_{\dot c\dot d}\circ\boldsymbol{e}_{c\dot c}\circ\Sigma_{bc}\circ\Sigma_{ab} & \xRightarrow{\mathrm{id}*\mathrm{I}_{abc}} & \mathrm{T}_{\dot c\dot d}\circ\boldsymbol{e}_{c\dot c}\circ\Sigma_{ac} & \\
{\scriptstyle \mathrm{id}*\mathrm{I}_{bcd}*\mathrm{id}}\Downarrow & \Downarrow{\scriptstyle \mathrm{id}*\boldsymbol{E}_{bc}^{\dot b\dot c}*\mathrm{id}} & & \Downarrow{\scriptstyle \mathrm{id}*\boldsymbol{E}_{ac}^{\dot a\dot c}} & \Downarrow{\scriptstyle \mathrm{id}*\mathrm{I}_{acd}} \\
 & \mathrm{T}_{\dot c\dot d}\circ\mathrm{T}_{\dot b\dot c}\circ\boldsymbol{e}_{b\dot b}\circ\Sigma_{ab} \xRightarrow{\mathrm{id}*\boldsymbol{E}_{ab}^{\dot a\dot b}} & \mathrm{T}_{\dot c\dot d}\circ\mathrm{T}_{\dot b\dot c}\circ\mathrm{T}_{\dot a\dot b}\circ\boldsymbol{e}_{a\dot a} & \xRightarrow{\mathrm{id}*\mathrm{K}_{\dot a\dot b\dot c}*\mathrm{id}} \mathrm{T}_{\dot c\dot d}\circ\mathrm{T}_{\dot a\dot c}\circ\boldsymbol{e}_{a\dot a} & \\
 & \Downarrow{\scriptstyle \mathrm{K}_{\dot b\dot c\dot d}*\mathrm{id}} & \Downarrow{\scriptstyle \mathrm{K}_{\dot b\dot c\dot d}*\mathrm{id}} & \Downarrow{\scriptstyle \mathrm{K}_{\dot a\dot c\dot d}*\mathrm{id}} & \\
 & \mathrm{T}_{\dot b\dot d}\circ\boldsymbol{e}_{b\dot b}\circ\Sigma_{ab} \xRightarrow{\mathrm{id}*\boldsymbol{E}_{ab}^{\dot a\dot b}} & \mathrm{T}_{\dot b\dot d}\circ\mathrm{T}_{\dot a\dot b}\circ\boldsymbol{e}_{a\dot a} & \xRightarrow{\mathrm{K}_{\dot a\dot b\dot d}*\mathrm{id}} \mathrm{T}_{\dot a\dot d}\circ\boldsymbol{e}_{a\dot a} & \\
 & \nearrow{\scriptstyle \boldsymbol{E}_{bd}^{\dot b\dot d}*\mathrm{id}} & & \nwarrow{\scriptstyle \boldsymbol{E}_{ad}^{\dot a\dot d}} & \\
\boldsymbol{e}_{d\dot d}\circ\Sigma_{bd}\circ\Sigma_{ab} & & \xRightarrow{\mathrm{id}*\mathrm{I}_{abd}} & & \boldsymbol{e}_{d\dot d}\circ\Sigma_{ad}.
\end{array}
\tag{11.127}
$$

Here four small quadrilaterals commute by (11.125), two commute by compatibility of vertical and horizontal composition, and one commutes by (11.122). So (11.127) commutes, implying that

$$\mathrm{id}_{\boldsymbol{e}_{d\dot d}} * \big(\mathrm{I}_{acd} \odot (\mathrm{id}_{\Sigma_{cd}} * \mathrm{I}_{abc})\big) = \mathrm{id}_{\boldsymbol{e}_{d\dot d}} * \big(\mathrm{I}_{abd} \odot (\mathrm{I}_{bcd} * \mathrm{id}_{\Sigma_{ab}})\big). \tag{11.128}$$

Similarly we can show that

$$\mathrm{id}_{\boldsymbol{f}_{d\ddot d}} * \big(\mathrm{I}_{acd} \odot (\mathrm{id}_{\Sigma_{cd}} * \mathrm{I}_{abc})\big) = \mathrm{id}_{\boldsymbol{e}_{d\dot d}} * \big(\mathrm{I}_{abd} \odot (\mathrm{I}_{bcd} * \mathrm{id}_{\Sigma_{ab}})\big). \tag{11.129}$$

By comparing two 2-commutative diagrams similar to (11.123)–(11.124) and using (11.122) and uniqueness of ϵ in Definition A.11 for the 2-Cartesian square (11.118) with d in place of a, we can use (11.128)–(11.129) to show that

$$\mathrm{I}_{acd} \odot (\mathrm{id}_{\Sigma_{cd}} * \mathrm{I}_{abc}) = \mathrm{I}_{abd} \odot (\mathrm{I}_{bcd} * \mathrm{id}_{\Sigma_{ab}}).$$

Now define $\boldsymbol{W} = (W, \mathcal{A})$, where $\mathcal{A} = \big(A, (T_a, C_a, q_a, \varphi_a)_{a\in A},\ \Sigma_{ab,\ a,b\in A},\ \mathrm{I}_{abc,\ a,b,c\in A}\big)$. Then W is Hausdorff and second countable as X, Y are, and we have already proved Definition 4.14(a)–(h) for $\mathcal{A}$ above, so that $\boldsymbol{W}$ is an m-Kuranishi space in $\mathbf{m\dot{K}ur}$ with $\operatorname{vdim}\boldsymbol{W} = \operatorname{vdim}\boldsymbol{X} + \operatorname{vdim}\boldsymbol{Y} - \operatorname{vdim}\boldsymbol{Z}$.

Define a 1-morphism $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$ in $\mathbf{m\dot{K}ur}$ by

$$\boldsymbol{e} = \big(e, \boldsymbol{e}_{ai,\ a\in A,\ i\in I},\ \boldsymbol{E}_{ab,\ a,b\in A}^{i,\ i\in I},\ \boldsymbol{E}_{a,\ a\in A}^{ij,\ i,j\in I}\big),$$

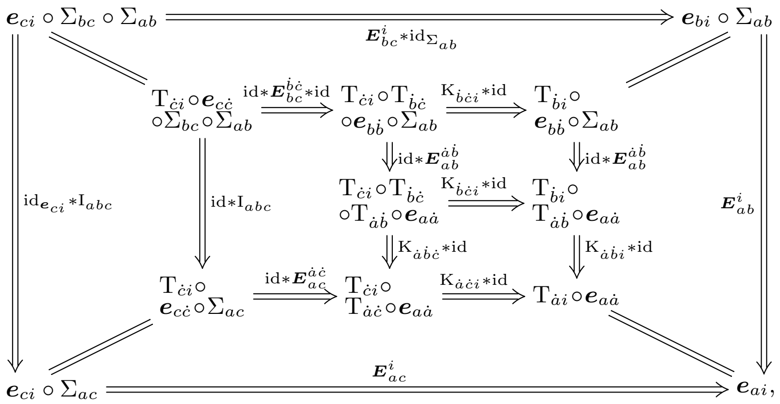
where $\boldsymbol{e}_{ai} = \mathrm{T}_{\dot a i} \circ \boldsymbol{e}_{a\dot a}$ and $\boldsymbol{E}_{ab}^{i}, \boldsymbol{E}_{a}^{ij}$ are defined by the 2-commutative diagrams

$$
\begin{array}{ccccc}
\boldsymbol{e}_{bi}\circ\Sigma_{ab} & & \xRightarrow{\quad\boldsymbol{E}_{ab}^{i}\quad} & & \boldsymbol{e}_{ai} \\
\| & & & & \| \\
\mathrm{T}_{\dot b i}\circ\boldsymbol{e}_{b\dot b}\circ\Sigma_{ab} & \xRightarrow{\mathrm{id}_{\mathrm{T}_{\dot b i}}*\boldsymbol{E}_{ab}^{\dot a\dot b}} & \mathrm{T}_{\dot b i}\circ\mathrm{T}_{\dot a\dot b}\circ\boldsymbol{e}_{a\dot a} & \xRightarrow{\mathrm{K}_{\dot a\dot b i}*\mathrm{id}_{\boldsymbol{e}_{a\dot a}}} & \mathrm{T}_{\dot a i}\circ\boldsymbol{e}_{a\dot a},
\end{array}
\tag{11.130}
$$

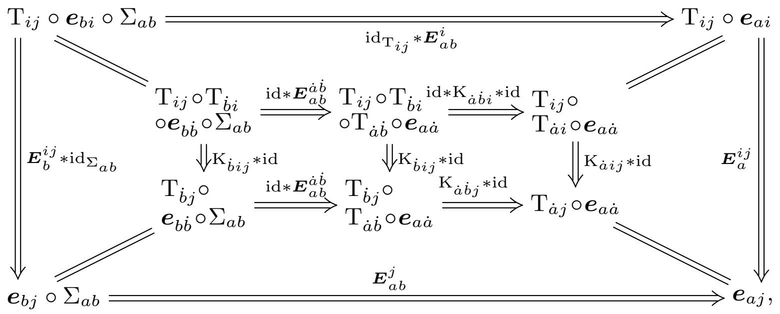
$$
\begin{array}{ccc}
\mathrm{T}_{ij}\circ\boldsymbol{e}_{ai} & \xRightarrow{\quad\boldsymbol{E}_{a}^{ij}\quad} & \boldsymbol{e}_{aj} \\
\| & & \| \\
\mathrm{T}_{ij}\circ\mathrm{T}_{\dot a i}\circ\boldsymbol{e}_{a\dot a} & \xRightarrow{\mathrm{K}_{\dot a ij}*\mathrm{id}_{\boldsymbol{e}_{a\dot a}}} & \mathrm{T}_{\dot a j}\circ\boldsymbol{e}_{a\dot a}.
\end{array}
\tag{11.131}
$$

Here $\boldsymbol{X}=(X,\mathcal{I})$ in (4.6), and $\mathrm{T}_{\dot{a}i}, \mathrm{K}_{\dot{a}ij}$ are the implicit data in the definition of the m-Kuranishi neighbourhood $(U_{\dot{a}}, D_{\dot{a}}, r_{\dot{a}}, \chi_{\dot{a}})$ on $\boldsymbol{X}$ in Definition 4.49, and the $\mathrm{K}_{\dot{a}\dot{b}i}$ are the implicit data in the definition of the coordinate change $\mathrm{T}_{\dot{a}\dot{b}} : (U_{\dot{a}}, D_{\dot{a}}, r_{\dot{a}}, \chi_{\dot{a}}) \to (U_{\dot{b}}, D_{\dot{b}}, r_{\dot{b}}, \chi_{\dot{b}})$ in Definition 4.51.

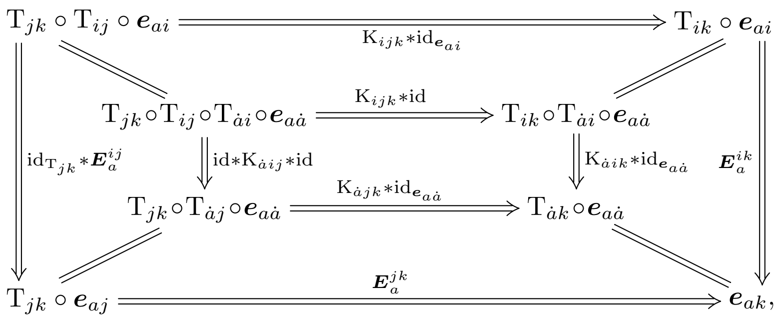
To show that $\boldsymbol{e}$ satisfies Definition 4.17, note that (a)–(d) are immediate, and (e) follows from $\Sigma_{aa}, \boldsymbol{E}_{aa}^{\dot{a}\dot{a}}, \mathrm{K}_{\dot{a}\dot{a}i}, \mathrm{K}_{\dot{a}ii}$ being identities, and (f)–(h) follow from the 2-commutative diagrams



(11.132)



(11.133)



(11.134)

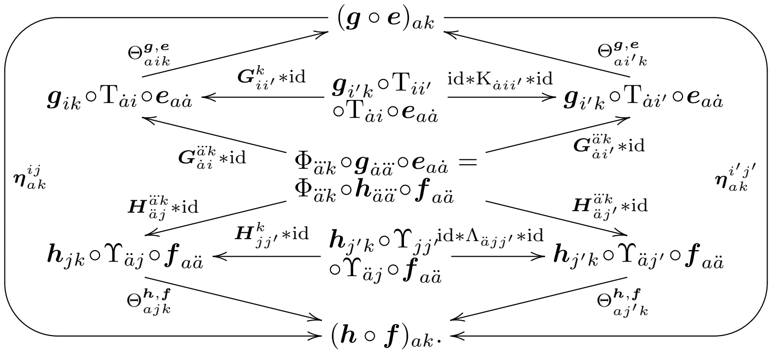
for all $a,b,c\in A$ and $i,j,k\in I$. Here (11.132) uses (4.62) for the 2-morphism $\mathrm{K}_{\dot{a}\dot{b}\dot{c}}$ constructed using Theorem 4.56(c), and (11.125), (11.130). Equation (11.133) uses (4.58) for the coordinate change $\mathrm{T}_{\dot{a}\dot{b}} : (U_{\dot{a}}, D_{\dot{a}}, r_{\dot{a}}, \chi_{\dot{a}}) \to (U_{\dot{b}}, D_{\dot{b}}, r_{\dot{b}}, \chi_{\dot{b}})$, and (11.130)–(11.131). Equation (11.134) uses (4.57) for the m-Kuranishi neighbourhood $(U_{\dot a}, D_{\dot a}, r_{\dot a}, \chi_{\dot a})$ on $\boldsymbol{X}$, and (11.131). All of (11.132)–(11.134) use compatibility of vertical and horizontal composition.

We define a 1-morphism $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}$ as for $\boldsymbol{e}$.

Definition 4.20 defines compositions $\boldsymbol{g}\circ\boldsymbol{e}, \boldsymbol{h}\circ\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$, with 2-morphisms of m-Kuranishi neighbourhoods $\Theta^{\boldsymbol{g},\boldsymbol{e}}_{aik}, \Theta^{\boldsymbol{h},\boldsymbol{f}}_{ajk}$ as in (4.24). We will define a 2-morphism $\boldsymbol{\eta} : \boldsymbol{g}\circ\boldsymbol{e} \Rightarrow \boldsymbol{h}\circ\boldsymbol{f}$ in $\mathbf{m\dot{K}ur}$, where $\boldsymbol{\eta} = \bigl(\boldsymbol{\eta}_{ak,\ a\in A,\ k\in K}\bigr)$. Let $a \in A$ and $k \in K$. We claim that there is a unique 2-morphism $\boldsymbol{\eta}_{ak} : (\boldsymbol{g}\circ\boldsymbol{e})_{ak} \Rightarrow (\boldsymbol{h}\circ\boldsymbol{f})_{ak}$ on $\operatorname{Im}\varphi_a \cap (g\circ e)^{-1}(\operatorname{Im}\omega_k)$ in W, such that for all $i \in I$ and $j \in J$, the following commutes on $\operatorname{Im}\varphi_a \cap e^{-1}(\operatorname{Im}\chi_i) \cap f^{-1}(\operatorname{Im}\psi_j) \cap (g\circ e)^{-1}(\operatorname{Im}\omega_k)$ in W:

$$\begin{array}{ccccccc}
\boldsymbol{g}_{ik}\circ\boldsymbol{e}_{ai} & \xRightarrow{\Theta^{\boldsymbol{g},\boldsymbol{e}}_{aik}} & (\boldsymbol{g}\circ\boldsymbol{e})_{ak} & \xRightarrow{\boldsymbol{\eta}_{ak}} & (\boldsymbol{h}\circ\boldsymbol{f})_{ak} & \xLeftarrow{\Theta^{\boldsymbol{h},\boldsymbol{f}}_{ajk}} & \boldsymbol{h}_{jk}\circ\boldsymbol{f}_{aj} \\
\| & & & & & & \| \\
\boldsymbol{g}_{ik}\circ\mathrm{T}_{\dot a i}\circ\boldsymbol{e}_{a\dot a} & \xLeftarrow{\boldsymbol{G}^{\ddot a k}_{\dot a i}*\mathrm{id}} & \Phi_{\ddot a k}\circ\boldsymbol{g}_{\dot a\ddot a}\circ\boldsymbol{e}_{a\dot a} & = & \Phi_{\ddot a k}\circ\boldsymbol{h}_{\ddot a\ddot a}\circ\boldsymbol{f}_{a\ddot a} & \xRightarrow{\boldsymbol{H}^{\ddot a k}_{\ddot a j}*\mathrm{id}} & \boldsymbol{h}_{jk}\circ\Upsilon_{\ddot a j}\circ\boldsymbol{f}_{a\ddot a}.
\end{array} \tag{11.135}$$

To prove the claim, write $\boldsymbol{\eta}^{ij}_{ak}$ for the 2-morphism $\boldsymbol{\eta}_{ak}$ which makes (11.135) commute. Let $i, i' \in I$ and $j, j' \in J$, and consider the diagram of 2-morphisms over $\operatorname{Im}\varphi_a \cap e^{-1}(\operatorname{Im}\chi_i \cap (\operatorname{Im}\chi_{i'}) \cap f^{-1}(\operatorname{Im}\psi_j \cap \operatorname{Im}\psi_{j'}) \cap (g\circ e)^{-1}(\operatorname{Im}\omega_k)$:



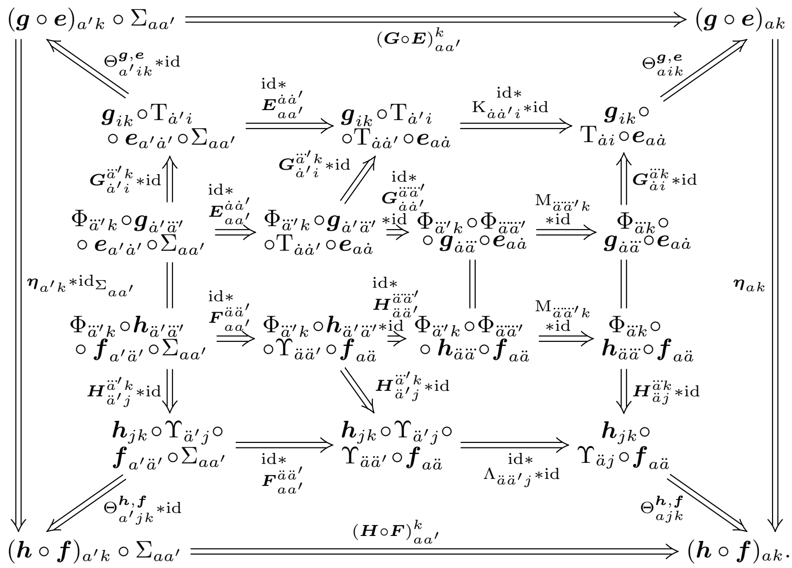
(11.136)

Here the outer pentagons commute by (11.135), the top and bottom quadrilaterals commute by (4.16) for $\boldsymbol{g}\circ\boldsymbol{e}$ and $\boldsymbol{h}\circ\boldsymbol{f}$, and the central two quadrilaterals commute by (4.59) for $\boldsymbol{g}_{\dot a\ddot a}$ and $\boldsymbol{h}_{\ddot a\ddot a}$. Thus (11.136) commutes, so $\boldsymbol{\eta}^{ij}_{ak} = \boldsymbol{\eta}^{i'j'}_{ak}$ on the intersection of their domains in W.

Now $\boldsymbol{\eta}^{ij}_{ak}$ is defined on $\operatorname{Im}\varphi_a \cap e^{-1}(\operatorname{Im}\chi_i) \cap f^{-1}(\operatorname{Im}\psi_j) \cap (g\circ e)^{-1}(\operatorname{Im}\omega_k)$, and for all $i \in I$ and $j \in J$ these form an open cover of the domain $\operatorname{Im}\varphi_a \cap (g\circ e)^{-1}(\operatorname{Im}\omega_k)$ of the 2-morphism $\boldsymbol{\eta}_{ak}$ that we want. So by the sheaf property of 2-morphisms of m-Kuranishi neighbourhoods in Theorem 4.13 and Definition A.17(iv), there is a unique 2-morphism $\boldsymbol{\eta}_{ak} : (\boldsymbol{g}\circ\boldsymbol{e})_{ak} \Rightarrow (\boldsymbol{h}\circ\boldsymbol{f})_{ak}$ over $\operatorname{Im}\varphi_a \cap (g\circ e)^{-1}(\operatorname{Im}\omega_k)$ such that $\boldsymbol{\eta}_{ak}|_{\operatorname{Im}\varphi_a\cap e^{-1}(\operatorname{Im}\chi_i)\cap f^{-1}(\operatorname{Im}\psi_j)\cap(g\circ e)^{-1}(\operatorname{Im}\omega_k)} = \boldsymbol{\eta}^{ij}_{ak}$ for all $i \in I$ and $j \in J$, so that (11.135) commutes, proving the claim.

To show $\boldsymbol{\eta} = \bigl(\boldsymbol{\eta}_{ak,\ a\in A,\ k\in K}\bigr) : \boldsymbol{g}\circ\boldsymbol{e} \Rightarrow \boldsymbol{h}\circ\boldsymbol{f}$ is a 2-morphism in $\mathbf{m\dot{K}ur}$, let

$a, a' \in A$, $i \in I$, $j \in J$ and $k \in K$, and consider the diagram of 2-morphisms



(11.137)

Here the left and right hexagons commute by (11.135), the top and bottom pentagons by (4.15) for $\boldsymbol{g} \circ \boldsymbol{e}, \boldsymbol{h} \circ \boldsymbol{f}$, the two centre left quadrilaterals by compatibility of vertical and horizontal composition, the centre left hexagon by (11.120), and the two centre right pentagons by (4.62) for $\boldsymbol{G}_{\dot{a}\dot{a}'}^{\dddot{a}\dddot{a}'}, \boldsymbol{H}_{\ddot{a}\ddot{a}'}^{\dddot{a}\dddot{a}'}$. Thus (11.137) commutes.

The outer rectangle of (11.137) proves the restriction of Definition 4.18(a) for $\boldsymbol{\eta}$ to the intersection of its domain with $e^{-1}(\operatorname{Im}\chi_i) \cap f^{-1}(\operatorname{Im}\psi_j)$. As these open subsets cover the domain, the sheaf property of 2-morphisms of m-Kuranishi neighbourhoods implies Definition 4.18(a) for $\boldsymbol{\eta}$. We prove Definition 4.18(b) in a similar way. Thus $\boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$ is a 2-morphism in $\mathbf{m\dot{K}ur}$, and we have constructed the 2-commutative diagram (11.15) in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$, in the case when Assumption 11.3 holds. We will show (11.15) is 2-Cartesian in §11.9.3.

11.9.2 Constructing $\boldsymbol{W}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{\eta}$ in the general case

Next we generalize the work of §11.9.1 to the case when Assumption 11.3 does not hold. Then in the first part of §11.9.1, we can no longer take $\boldsymbol{W}$ to have topological space $\{(x, y) \in X \times Y : g(x) = h(y)\}$ with $e : W \to X$, $f : W \to Y$ acting by $e : (x, y) \mapsto x$, $f : (x, y) \mapsto y$. Also for the fibre product T_k in $\mathbf{\dot{M}an}$ in (11.114), we cannot assume T_k has topological space (11.115).

We need to provide new definitions for W, e, f, and the continuous maps $\varphi_a : q_a^{-1}(0) \to W$ for $a \in A$. This is very similar to the definition of the topological space $C_k(X)$ and map $\Pi_k : C_k(X) \to X$ for $C_k(\boldsymbol{X}), \boldsymbol{\Pi}_k$ in Definition 4.39.

As in §11.9.1 we choose a family indexed by $a \in A$ of m-Kuranishi neighbourhoods $(U_{\dot{a}}, D_{\dot{a}}, r_{\dot{a}}, \chi_{\dot{a}})$, $(V_{\ddot{a}}, E_{\ddot{a}}, s_{\ddot{a}}, \psi_{\ddot{a}})$, $(W_{\dddot{a}}, F_{\dddot{a}}, t_{\dddot{a}}, \omega_{\dddot{a}})$ on $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ as in §4.7

with $\operatorname{Im}\chi_{\dot a} \subseteq g^{-1}(\operatorname{Im}\omega_{\dddot a})$, $\operatorname{Im}\psi_{\ddot a} \subseteq h^{-1}(\operatorname{Im}\omega_{\dddot a})$ and $\operatorname{Im}\omega_{\dddot a}$, and 1-morphisms $\boldsymbol{g}_{\dot a\dddot a} : (U_{\dot a}, D_{\dot a}, r_{\dot a}, \chi_{\dot a}) \to (W_{\dddot a}, F_{\dddot a}, t_{\dddot a}, \omega_{\dddot a})$, $\boldsymbol{h}_{\ddot a\dddot a} : (V_{\ddot a}, E_{\ddot a}, s_{\ddot a}, \psi_{\ddot a}) \to (W_{\dddot a}, F_{\dddot a}, t_{\dddot a}, \omega_{\dddot a})$ over $(\operatorname{Im}\chi_{\dot a}, \boldsymbol{g})$ and $(\operatorname{Im}\psi_{\ddot a}, \boldsymbol{h})$, as in Definition 4.54, such that $\boldsymbol{g}_{\dot a\dddot a}, \boldsymbol{h}_{\ddot a\dddot a}$ are w-transverse as in Definition 11.16, and

$$\big\{(x,y) \in X \times Y : g(x) = h(y)\big\} = \textstyle\bigcup_{a\in A}\big\{(x,y) \in \operatorname{Im}\chi_{\dot a} \times \operatorname{Im}\psi_{\ddot a} : g(x) = h(y)\big\}.$$

Applying Definition 11.16 and Theorem 11.17 to the w-transverse 1-morphisms $\boldsymbol{g}_{\dot a\dddot a}, \boldsymbol{h}_{\ddot a\dddot a}$ in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$ gives an object (T_a, C_a, q_a) in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$ in a 2-Cartesian square (11.118) in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$ and $\mathbf{Gm\dot{K}N_{\boldsymbol{E}}}$, for all $a \in A$.

Now follow §11.9.1 between (11.118) and (11.126). For all $a, b \in A$ this defines an open subset $T_{ab} \subseteq T_a$ and a 1-morphism $\Sigma_{ab} : (T_a, C_a, q_a)|_{T_{ab}} \to (T_b, C_b, q_b)$ in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$ with $\Sigma_{aa} = \mathrm{id}_{(T_a,C_a,q_a)}$, and for all $a, b, c \in A$ it defines an open subset $T_{abc} = T_{ab} \cap T_{bc} \subseteq T_a$ and a 2-morphism $\mathrm{I}_{abc} : \Sigma_{bc} \circ \Sigma_{ab}|_{T_{abc}} \Rightarrow \Sigma_{ac}|_{T_{abc}}$ in $\mathbf{Gm\dot{K}N_{\boldsymbol{D}}}$. None of this uses W, e, f, φ_a, which are not yet defined.

Definition 4.2(d) for Σ_{ab} shows we have a continuous map

$$\Sigma_{ab}|_{q_a^{-1}(0)\cap T_{ab}} : q_a^{-1}(0) \cap T_{ab} \longrightarrow q_b^{-1}(0), \quad a, b \in A. \tag{11.138}$$

Also $\Sigma_{aa} = \mathrm{id}_{(T_a,C_a,q_a)}$ and Definition 4.3 for I_{abc} imply that

$$\begin{aligned} &\Sigma_{aa}|_{q_a^{-1}(0)\cap T_{aa}} = \mathrm{id} : q_a^{-1}(0) \longrightarrow q_a^{-1}(0), \\ &\Sigma_{bc}|_{\cdots} \circ \Sigma_{ab}|_{\cdots} = \Sigma_{ac}|_{\cdots} : q_a^{-1}(0) \cap T_{ab} \cap T_{ac} \longrightarrow q_c^{-1}(0). \end{aligned} \tag{11.139}$$

Setting $c = a$ we see that $\Sigma_{ab}|_{q_a^{-1}(0)\cap T_{ab}} : q_a^{-1}(0) \cap T_{ab} \to q_b^{-1}(0) \cap T_{ba}$ is a homeomorphism, with inverse $\Sigma_{ba}|_{q_b^{-1}(0)\cap T_{ba}}$.

As for the definition of $C_k(X)$ in Definition 4.39, define a binary relation $\approx$ on $\coprod_{a\in A} q_a^{-1}(0)$ by $w_a \approx w_b$ if $a, b \in A$ and $w_a \in q_a^{-1}(0) \cap T_{ab}$ with $\Sigma_{ab}(w_a) = w_b$ in $q_b^{-1}(0)$. Then (11.138)–(11.139) imply that $\approx$ is an equivalence relation on $\coprod_{a\in A} q_a^{-1}(0)$. As in (4.49), define W to be the topological space

$$W = \big[\textstyle\coprod_{a\in A} q_a^{-1}(0)\big]\big/\approx,$$

with the quotient topology. For each $a \in A$ define $\varphi_a : q_a^{-1}(0) \to W$ by $\varphi_a : w_a \mapsto [w_a]$, where $[w_a]$ is the $\approx$-equivalence class of w_a.

Define $e : W \to X$ and $f : W \to Y$ by $e([w_a]) = \chi_{\dot a} \circ e_{a\dot a}(w_a)$ and $f([w_a]) = \psi_{\ddot a} \circ f_{a\ddot a}(w_a)$ for $a \in A$ and $w_a \in q_a^{-1}(0)$. To see that e is well defined, note that if $w_a \approx w_b$ as above, so that $\Sigma_{ab}(w_a) = w_b$, then

$$\chi_{\dot a} \circ e_{a\dot a}(w_a) = \chi_{\dot b} \circ \mathrm{T}_{\dot a\dot b} \circ e_{a\dot a}(w_a) = \chi_{\dot b} \circ e_{b\dot b} \circ \Sigma_{ab}(w_a) = \chi_{\dot b} \circ e_{b\dot b}(w_b),$$

using Definition 4.2(e) for the coordinate change $\mathrm{T}_{\dot a\dot b}$ on X in the first step, and the 2-morphism $\boldsymbol{E}_{ab}^{\dot a\dot b} : \boldsymbol{e}_{b\dot b} \circ \Sigma_{ab} \Rightarrow \mathrm{T}_{\dot a\dot b} \circ \boldsymbol{e}_{a\dot a}|_{T_{ab}}$ from (11.119) in the second. In the same way, f is well defined.

Very similar proofs to those in Definition 4.39 show that $\varphi_a : q_a^{-1}(0) \to W$ is a homeomorphism with an open set in W, so that $(T_a, C_a, q_a, \varphi_a)$ is an m-Kuranishi neighbourhood on W, and e, f are continuous with $\boldsymbol{e}_{a\dot a} : (T_a, C_a, q_a,$

$\varphi_a) \to (U_{\dot a}, D_{\dot a}, r_{\dot a}, \chi_{\dot a})$ a 1-morphism over $(\operatorname{Im}\varphi_a, e)$ and $\boldsymbol{f}_{a\ddot a} : (T_a, C_a, q_a, \varphi_a) \to (V_{\ddot a}, E_{\ddot a}, s_{\ddot a}, \psi_{\ddot a})$ a 1-morphism over $(\operatorname{Im}\varphi_a, f)$, and W is Hausdorff and second countable with $W = \bigcup_{a\in A} \operatorname{Im}\varphi_a$. Then the proofs in §11.9.1, but with these new W, e, f, φ_a, construct an m-Kuranishi space $\boldsymbol{W} = (W, \mathcal{A})$ and 1-morphisms $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ and a 2-morphism $\boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$ in $\mathbf{m\dot{K}ur}$.

11.9.3 Proving the universal property of the fibre product

We continue in the situation of §11.9.2. There, given w-transverse 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$, we constructed $\boldsymbol{W}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{\eta}$ in a 2-commutative square (11.15) in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$. We will now prove that (11.15) is 2-Cartesian in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$, by verifying the universal property in Definition A.11. This will also imply that (11.15) is 2-Cartesian in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$, as $\boldsymbol{D}$ implies $\boldsymbol{E}$.

Suppose we are given 1-morphisms $\boldsymbol{c} : \boldsymbol{V} \to \boldsymbol{X}$ and $\boldsymbol{d} : \boldsymbol{V} \to \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$ and a 2-morphism $\boldsymbol{\kappa} : \boldsymbol{g} \circ \boldsymbol{c} \Rightarrow \boldsymbol{h} \circ \boldsymbol{d}$. Write $\boldsymbol{V} = (V, \mathcal{L})$ with

$$\mathcal{L} = \bigl(L, (S_l, B_l, p_l, \upsilon_l)_{l\in L}, \mathrm{P}_{ll',\, l,l'\in L}, \mathrm{H}_{ll'l'',\, l,l',l''\in L}\bigr),$$

and use our usual notation for $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{\kappa}$. Our goal is to construct a 1-morphism $\boldsymbol{b} : \boldsymbol{V} \to \boldsymbol{W}$ in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$ and 2-morphisms $\boldsymbol{\zeta} : \boldsymbol{e} \circ \boldsymbol{b} \Rightarrow \boldsymbol{c}$, $\boldsymbol{\theta} : \boldsymbol{f} \circ \boldsymbol{b} \Rightarrow \boldsymbol{d}$ such that the following diagram (A.17) of 2-morphisms commutes:

$$\begin{array}{ccccc}
(\boldsymbol{g}\circ\boldsymbol{e})\circ\boldsymbol{b} & \xRightarrow{\;\boldsymbol{\eta}*\mathbf{id}_{\boldsymbol{b}}\;} & (\boldsymbol{h}\circ\boldsymbol{f})\circ\boldsymbol{b} & \xRightarrow{\;\boldsymbol{\alpha}_{\boldsymbol{h},\boldsymbol{f},\boldsymbol{b}}\;} & \boldsymbol{h}\circ(\boldsymbol{f}\circ\boldsymbol{b}) \\
\Big\Downarrow{\scriptstyle \boldsymbol{\alpha}_{\boldsymbol{g},\boldsymbol{e},\boldsymbol{b}}} & & & & \Big\Downarrow{\scriptstyle \mathbf{id}_{\boldsymbol{h}}*\boldsymbol{\theta}} \\
\boldsymbol{g}\circ(\boldsymbol{e}\circ\boldsymbol{b}) & \xRightarrow{\;\mathbf{id}_{\boldsymbol{g}}*\boldsymbol{\zeta}\;} & \boldsymbol{g}\circ\boldsymbol{c} & \xRightarrow{\;\boldsymbol{\kappa}\;} & \boldsymbol{h}\circ\boldsymbol{d}.
\end{array} \tag{11.140}$$

Let $a \in A$ and $l \in L$. Then $(U_{\dot a}, D_{\dot a}, r_{\dot a}, \chi_{\dot a})$ is an m-Kuranishi neighbourhood on $\boldsymbol{X}$, and $(S_l, B_l, p_l, \upsilon_l)$ is an m-Kuranishi neighbourhood on $\boldsymbol{V}$ as in Example 4.50. Thus Theorem 4.56(b) gives a 1-morphism $\boldsymbol{c}_{l\dot a} : (S_l, B_l, p_l, \upsilon_l) \to (U_{\dot a}, D_{\dot a}, r_{\dot a}, \chi_{\dot a})$ over $(\operatorname{Im}\upsilon_l \cap c^{-1}(\operatorname{Im}\chi_{\dot a}), \boldsymbol{c})$. Similarly we get a 1-morphism $\boldsymbol{d}_{l\ddot a} : (S_l, B_l, p_l, \upsilon_l) \to (V_{\ddot a}, E_{\ddot a}, s_{\ddot a}, \psi_{\ddot a})$ over $(\operatorname{Im}\upsilon_l \cap d^{-1}(\operatorname{Im}\psi_{\ddot a}), \boldsymbol{d})$. Composing gives $\boldsymbol{g}_{\dot a\dddot a} \circ \boldsymbol{c}_{l\dot a}$ over $\boldsymbol{g}\circ\boldsymbol{e}$ and $\boldsymbol{h}_{\ddot a\dddot a} \circ \boldsymbol{d}_{l\ddot a}$ over $\boldsymbol{h}\circ\boldsymbol{f}$. Hence Theorem 4.56(c) gives a unique 2-morphism $\boldsymbol{\kappa}_{l\dddot a} : \boldsymbol{g}_{\dot a\dddot a}\circ\boldsymbol{c}_{l\dot a} \Rightarrow \boldsymbol{h}_{\ddot a\dddot a}\circ\boldsymbol{d}_{l\ddot a}$ over $\operatorname{Im}\upsilon_l \cap c^{-1}(\operatorname{Im}\chi_{\dot a}) \cap d^{-1}(\operatorname{Im}\psi_{\ddot a})$ such that the analogue of (4.62) commutes.

Writing $\boldsymbol{c}_{l\dot a} = (S_{l\dot a}, c_{l\dot a}, \hat c_{l\dot a})$, $\boldsymbol{d}_{l\ddot a} = (S_{l\ddot a}, d_{l\ddot a}, \hat d_{l\ddot a})$ and setting $S_{la} = S_{l\dot a} \cap S_{l\ddot a}$, we now have a 2-commutative diagram in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$:

$$\begin{array}{ccc}
(S_l, B_l, p_l)|_{S_{la}} & \xrightarrow{\quad \boldsymbol{d}_{l\ddot a}|_{S_{la}} \quad} & (V_{\ddot a}, E_{\ddot a}, s_{\ddot a}) \\
\Big\downarrow{\scriptstyle \boldsymbol{c}_{l\dot a}|_{S_{la}}} & \boldsymbol{\kappa}_{l\dddot a}\Uparrow & \Big\downarrow{\scriptstyle \boldsymbol{h}_{\ddot a\dddot a}} \\
(U_{\dot a}, D_{\dot a}, r_{\dot a}) & \xrightarrow{\quad \boldsymbol{g}_{\dot a\dddot a} \quad} & (W_{\dddot a}, F_{\dddot a}, t_{\dddot a}).
\end{array}$$

The 2-Cartesian property of (11.118) in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$ gives a 1-morphism

$$\boldsymbol{b}_{la} : (S_l, B_l, p_l)|_{S_{la}} \longrightarrow (T_a, C_a, q_a),$$

and 2-morphisms

$$\boldsymbol{\zeta}_{la\dot a} : \boldsymbol{e}_{a\dot a} \circ \boldsymbol{b}_{la} \Longrightarrow \boldsymbol{c}_{l\dot a}|_{S_{la}}, \quad \boldsymbol{\theta}_{la\ddot a} : \boldsymbol{f}_{a\ddot a} \circ \boldsymbol{b}_{la} \Longrightarrow \boldsymbol{d}_{l\ddot a}|_{S_{la}}, \tag{11.141}$$

such that the following commutes

$$\begin{array}{ccc} \boldsymbol{g}_{\dot a\dddot a}\circ\boldsymbol{e}_{a\dot a}\circ\boldsymbol{b}_{la} & \xlongequal{\qquad\qquad} & \boldsymbol{h}_{\ddot a\dddot a}\circ\boldsymbol{f}_{a\ddot a}\circ\boldsymbol{b}_{la} \\ \Big\Downarrow{\scriptstyle \mathrm{id}_{\boldsymbol{g}_{\dot a\dddot a}}*\boldsymbol{\zeta}_{la\dot a}} & & {\scriptstyle \mathrm{id}_{\boldsymbol{h}_{\ddot a\dddot a}}*\boldsymbol{\theta}_{la\ddot a}}\Big\Downarrow \\ \boldsymbol{g}_{\dot a\dddot a}\circ\boldsymbol{c}_{l\dot a}\vert_{S_{la}} & \xRightarrow{\boldsymbol{\kappa}_{l\dddot a}} & \boldsymbol{h}_{\ddot a\dddot a}\circ\boldsymbol{d}_{l\ddot a}\vert_{S_{la}}. \end{array} \tag{11.142}$$

Now let $a\in A$ and $l,l'\in L$. Then we have 1-morphisms

$$\boldsymbol{b}_{la}\vert_{S_{la}\cap S_{ll'}},\boldsymbol{b}_{l'a}\circ\mathrm{P}_{ll'}\vert_{S_{la}\cap S_{ll'}}:(S_l,B_l,p_l)\vert_{S_{la}\cap S_{ll'}}\longrightarrow(T_a,C_a,q_a),$$

and 2-morphisms $\boldsymbol{\zeta}_{la\dot a},\boldsymbol{\theta}_{la\ddot a}$ in (11.141) such that (11.142) commutes, and

$$\begin{aligned} \boldsymbol{C}^{\dot a}_{ll'}\odot(\boldsymbol{\zeta}_{l'a\dot a}*\mathrm{id}_{\mathrm{P}_{ll'}}):\boldsymbol{e}_{a\dot a}\circ\boldsymbol{b}_{l'a}\circ\mathrm{P}_{ll'}\vert_{S_{la}\cap S_{ll'}}&\Longrightarrow\boldsymbol{c}_{l\dot a}\vert_{S_{la}\cap S_{ll'}},\\ \boldsymbol{D}^{\ddot a}_{ll'}\odot(\boldsymbol{\theta}_{l'a\ddot a}*\mathrm{id}_{\mathrm{P}_{ll'}}):\boldsymbol{f}_{a\ddot a}\circ\boldsymbol{b}_{l'a}\circ\mathrm{P}_{ll'}\vert_{S_{la}\cap S_{ll'}}&\Longrightarrow\boldsymbol{d}_{l\ddot a}\vert_{S_{la}\cap S_{ll'}}, \end{aligned}$$

for $\boldsymbol{C}^{\dot a}_{ll'}:\boldsymbol{c}_{l'\dot a}\circ\mathrm{P}_{ll'}\Rightarrow\boldsymbol{c}_{l\dot a}$ and $\boldsymbol{D}^{\ddot a}_{ll'}:\boldsymbol{d}_{l'\ddot a}\circ\mathrm{P}_{ll'}\Rightarrow\boldsymbol{d}_{l\ddot a}$ given by Theorem 4.56(c).

Using Theorem 4.56(c) we can show that the following commutes:

$$\begin{array}{ccc} \boldsymbol{g}_{\dot a\dddot a}\circ\boldsymbol{e}_{a\dot a}\circ\boldsymbol{b}_{l'a}\circ\mathrm{P}_{ll'}\vert_{S_{la}\cap S_{ll'}} & \xlongequal{\qquad\qquad} & \boldsymbol{h}_{\ddot a\dddot a}\circ\boldsymbol{f}_{a\ddot a}\circ\boldsymbol{b}_{l'a}\circ\mathrm{P}_{ll'}\vert_{S_{la}\cap S_{ll'}} \\ \Big\Downarrow{\scriptstyle \mathrm{id}_{\boldsymbol{g}_{\dot a\dddot a}}*(\boldsymbol{C}^{\dot a}_{ll'}\odot(\boldsymbol{\zeta}_{l'a\dot a}*\mathrm{id}_{\mathrm{P}_{ll'}}))} & & {\scriptstyle \mathrm{id}_{\boldsymbol{h}_{\ddot a\dddot a}}*(\boldsymbol{D}^{\ddot a}_{ll'}\odot(\boldsymbol{\theta}_{l'a\ddot a}*\mathrm{id}_{\mathrm{P}_{ll'}}))}\Big\Downarrow \\ \boldsymbol{g}_{\dot a\dddot a}\circ\boldsymbol{c}_{l\dot a}\vert_{S_{la}\cap S_{ll'}} & \xRightarrow{\boldsymbol{\kappa}_{l\dddot a}} & \boldsymbol{h}_{\ddot a\dddot a}\circ\boldsymbol{d}_{l\ddot a}\vert_{S_{la}\cap S_{ll'}}. \end{array}$$

Hence the second part of the universal property for the 2-Cartesian square (11.118) says that there is a unique 2-morphism in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$

$$\boldsymbol{B}^a_{ll'}:\boldsymbol{b}_{l'a}\circ\mathrm{P}_{ll'}\vert_{S_{la}\cap S_{ll'}}\Longrightarrow\boldsymbol{b}_{la}\vert_{S_{la}\cap S_{ll'}}$$

such that

$$\begin{aligned} \boldsymbol{C}^{\dot a}_{ll'}\odot(\boldsymbol{\zeta}_{l'a\dot a}*\mathrm{id}_{\mathrm{P}_{ll'}})&=\boldsymbol{\zeta}_{la\dot a}\odot(\mathrm{id}_{\boldsymbol{e}_{a\dot a}}*\boldsymbol{B}^a_{ll'}),\\ \boldsymbol{D}^{\ddot a}_{ll'}\odot(\boldsymbol{\theta}_{l'a\ddot a}*\mathrm{id}_{\mathrm{P}_{ll'}})&=\boldsymbol{\theta}_{la\ddot a}\odot(\mathrm{id}_{\boldsymbol{f}_{a\ddot a}}*\boldsymbol{B}^a_{ll'}). \end{aligned} \tag{11.143}$$

Note that the existence of $\boldsymbol{B}^a_{ll'}$ implies that

$$b_{la}\vert_{\mathrm{Im}\,\upsilon_l\cap\mathrm{Im}\,\upsilon_{l'}\cap c^{-1}(\mathrm{Im}\,\chi_{\dot a})\cap d^{-1}(\mathrm{Im}\,\psi_{\ddot a})}=b_{l'a}\vert_{\cdots}. \tag{11.144}$$

Next let $a,a'\in A$ and $l\in L$. A similar argument to the above yields a unique 2-morphism in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$

$$\boldsymbol{B}^{aa'}_l:\Sigma_{aa'}\circ\boldsymbol{b}_{la}\vert_{S_{la}\cap S_{la'}}\Rightarrow\boldsymbol{b}_{la'}\vert_{S_{la}\cap S_{la'}}$$

such that

$$\begin{aligned} \boldsymbol{C}^{\dot a\dot a'}_l\odot(\mathrm{id}_{\mathrm{T}_{\dot a\dot a'}}*\boldsymbol{\zeta}_{la\dot a})\odot(\boldsymbol{E}^{\dot a\dot a'}_{aa'}*\mathrm{id}_{\boldsymbol{b}_{la}})&=\boldsymbol{\zeta}_{la\dot a'}\odot(\mathrm{id}_{\boldsymbol{e}_{a'\dot a'}}*\boldsymbol{B}^{aa'}_l),\\ \boldsymbol{D}^{\ddot a\ddot a'}_l\odot(\mathrm{id}_{\Upsilon_{\ddot a\ddot a'}}*\boldsymbol{\theta}_{la\ddot a})\odot(\boldsymbol{F}^{\ddot a\ddot a'}_{aa'}*\mathrm{id}_{\boldsymbol{b}_{la}})&=\boldsymbol{\theta}_{la\ddot a'}\odot(\mathrm{id}_{\boldsymbol{f}_{a'\ddot a'}}*\boldsymbol{B}^{aa'}_l), \end{aligned}$$

where $\boldsymbol{C}_l^{\dot a\dot a'}:\mathrm{T}_{\dot a\dot a'}\circ\boldsymbol{c}_{l\dot a}\Rightarrow\boldsymbol{c}_{l\dot a'}$ and $\boldsymbol{D}_l^{\ddot a\ddot a'}:\Upsilon_{\ddot a\ddot a'}\circ\boldsymbol{d}_{l\ddot a}\Rightarrow\boldsymbol{d}_{l\ddot a'}$ are given by Theorem 4.56(c). Note that the existence of $\boldsymbol{B}_l^{aa'}$ implies that

$$b_{la}\vert_{\mathrm{Im}\,v_l\cap c^{-1}(\mathrm{Im}\,\chi_{\dot a}\cap\mathrm{Im}\,\chi_{\dot a'})\cap d^{-1}(\mathrm{Im}\,\psi_{\ddot a}\cap\mathrm{Im}\,\psi_{\ddot a'})}=b_{la'}\vert_{\cdots}.\tag{11.145}$$

As the domains of b_{la} for $a\in A$ and $l\in L$ cover V, equations (11.144) and (11.145) imply that there is a unique continuous map $b:V\to W$ with $b\vert_{\mathrm{Im}\,v_l\cap\mathrm{Im}\,v_{l'}\cap c^{-1}(\mathrm{Im}\,\chi_{\dot a})\cap d^{-1}(\mathrm{Im}\,\psi_{\ddot a})}=b_{la}$ for all $a\in A$ and $l\in L$. Define

$$\boldsymbol{b}=\bigl(b,\boldsymbol{b}_{la,\;l\in L,\;a\in A},\;\boldsymbol{B}^{a,\;a\in A}_{ll',\;l,l'\in L},\;\boldsymbol{B}^{aa',\;a,a'\in A}_{l,\;l\in L}\bigr).$$

We will show that $\boldsymbol{b}:\boldsymbol{V}\to\boldsymbol{W}$ is a 1-morphism in $\mathbf{m\dot{K}ur}$. Definition 4.17(a)–(d) are immediate. For (e), setting $l=l'$ we have $\boldsymbol{C}_{ll}^{\dot a}=\mathrm{id}=\boldsymbol{D}_{ll}^{\ddot a}$, so uniqueness of $\boldsymbol{B}_{ll}^a$ satisfying (11.143) gives $\boldsymbol{B}_{ll}^a=\mathrm{id}_{\boldsymbol{b}_{la}}$, and similarly $\boldsymbol{B}_l^{aa}=\mathrm{id}_{\boldsymbol{b}_{la}}$.

For (f), let $l,l',l''\in L$ and $a\in A$, and consider the diagram

$$\begin{array}{ccccc}
\boldsymbol{e}_{a\dot a}\circ\boldsymbol{b}_{l''a}\circ\mathrm{P}_{l'l''}\circ\mathrm{P}_{ll'} & & \xRightarrow{\mathrm{id}*\boldsymbol{B}^a_{l'l''}*\mathrm{id}} & & \boldsymbol{e}_{a\dot a}\circ\boldsymbol{b}_{l'a}\circ\mathrm{P}_{ll'} \\
 & \searrow{\scriptstyle\boldsymbol{\zeta}_{l''a\dot a}*\mathrm{id}} & & \swarrow{\scriptstyle\boldsymbol{\zeta}_{l'a\dot a}*\mathrm{id}} & \\
 & \boldsymbol{c}_{l''\dot a}\circ\mathrm{P}_{l'l''}\circ\mathrm{P}_{ll'} & \xRightarrow{\boldsymbol{C}^{\dot a}_{l'l''}*\mathrm{id}} & \boldsymbol{c}_{l'\dot a}\circ\mathrm{P}_{ll'} & \\
\Big\Downarrow{\scriptstyle\mathrm{id}*\mathrm{H}_{ll'l''}} & \Big\Downarrow{\scriptstyle\mathrm{id}*\mathrm{H}_{ll'l''}} & & \Big\Downarrow{\scriptstyle\boldsymbol{C}^{\dot a}_{ll'}} & \Big\Downarrow{\scriptstyle\mathrm{id}*\boldsymbol{B}^a_{ll'}} \\
 & \boldsymbol{c}_{l''\dot a}\circ\mathrm{P}_{ll''} & \xRightarrow{\boldsymbol{C}^{\dot a}_{ll''}} & \boldsymbol{c}_{l\dot a} & \\
 & \nearrow{\scriptstyle\boldsymbol{\zeta}_{l''a\dot a}*\mathrm{id}} & & \nwarrow{\scriptstyle\boldsymbol{\zeta}_{la\dot a}} & \\
\boldsymbol{e}_{a\dot a}\circ\boldsymbol{b}_{l''a}\circ\mathrm{P}_{ll''} & & \xRightarrow{\mathrm{id}*\boldsymbol{B}^a_{ll''}} & & \boldsymbol{e}_{a\dot a}\circ\boldsymbol{b}_{la}.
\end{array}\tag{11.146}$$

Here the top, bottom and right quadrilaterals commute by (11.143), the left by compatibility of vertical and horizontal composition, and the centre by Theorem 4.56(d). So (11.146) commutes, and so does the analogous diagram involving $\boldsymbol{f}_{a\ddot a},\boldsymbol{\theta}_{la\ddot a},\boldsymbol{D}^{\ddot a}_{ll'}$ in place of $\boldsymbol{e}_{a\dot a},\boldsymbol{\zeta}_{la\dot a},\boldsymbol{C}^{\dot a}_{ll'}$. Using these and uniqueness of $\boldsymbol{B}^a_{ll'}$ satisfying (11.143), we deduce that the following commutes:

$$\begin{array}{ccc}
\boldsymbol{b}_{l''a}\circ\mathrm{P}_{l'l''}\circ\mathrm{P}_{ll'} & \xRightarrow{\boldsymbol{B}^a_{l'l''}*\mathrm{id}_{\mathrm{P}_{ll'}}} & \boldsymbol{b}_{l'a}\circ\mathrm{P}_{ll'} \\
\Big\Downarrow{\scriptstyle\mathrm{id}_{\boldsymbol{b}_{l''a}}*\mathrm{H}_{ll'l''}} & & \Big\Downarrow{\scriptstyle\boldsymbol{B}^a_{ll'}} \\
\boldsymbol{b}_{l''a}\circ\mathrm{P}_{ll''} & \xRightarrow{\boldsymbol{B}^a_{ll''}} & \boldsymbol{b}_{la}.
\end{array}$$

This is Definition 4.17(f) for $\boldsymbol{b}$, and we prove (g),(h) in a similar way.

By the method used to construct $\boldsymbol{\eta}:\boldsymbol{g}\circ\boldsymbol{e}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$ in §11.9.1, we can show that there are unique 2-morphisms in $\mathbf{m\dot{K}ur}$

$$\boldsymbol{\zeta}=\bigl(\boldsymbol{\zeta}_{li,\;l\in L,\;i\in I}\bigr):\boldsymbol{e}\circ\boldsymbol{b}\Longrightarrow\boldsymbol{c},\quad\boldsymbol{\theta}=\bigl(\boldsymbol{\theta}_{lj,\;l\in L,\;j\in J}\bigr):\boldsymbol{f}\circ\boldsymbol{b}\Longrightarrow\boldsymbol{d},$$

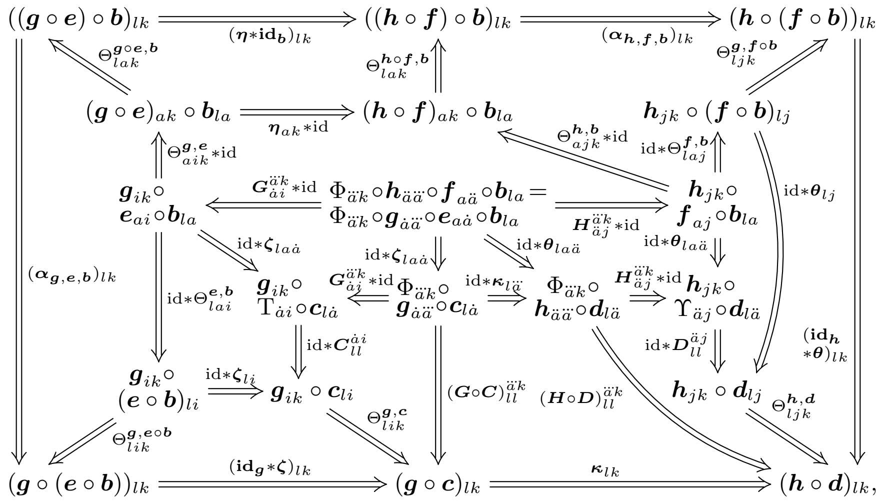
such that the following commute for all $l \in L$, $a \in A$, $i \in I$ and $j \in J$:

$$\begin{array}{ccccccc}
(\boldsymbol{e}\circ\boldsymbol{b})_{li} & & & \xRightarrow{\quad\boldsymbol{\zeta}_{li}\quad} & & & \boldsymbol{c}_{li} \\
\Uparrow \Theta^{\boldsymbol{e},\boldsymbol{b}}_{lai} & & & & & & \| \\
\boldsymbol{e}_{ai}\circ\boldsymbol{b}_{la} & = & \mathrm{T}_{\dot a i}\circ\boldsymbol{e}_{a\dot a}\circ\boldsymbol{b}_{la} & \xRightarrow{\mathrm{id}*\boldsymbol{\zeta}_{la\dot a}} & \mathrm{T}_{\dot a i}\circ\boldsymbol{c}_{l\dot a} & \xRightarrow{\boldsymbol{C}^{\dot a i}_{ll}} & \boldsymbol{c}_{li}\circ\mathrm{P}_{ll},
\end{array} \tag{11.147}$$

$$\begin{array}{ccccccc}
(\boldsymbol{f}\circ\boldsymbol{b})_{lj} & & & \xRightarrow{\quad\boldsymbol{\theta}_{lj}\quad} & & & \boldsymbol{d}_{lj} \\
\Uparrow \Theta^{\boldsymbol{f},\boldsymbol{b}}_{laj} & & & & & & \| \\
\boldsymbol{f}_{aj}\circ\boldsymbol{b}_{la} & = & \Upsilon_{\ddot a j}\circ\boldsymbol{f}_{a\ddot a}\circ\boldsymbol{b}_{la} & \xRightarrow{\mathrm{id}*\boldsymbol{\theta}_{la\ddot a}} & \Upsilon_{\ddot a j}\circ\boldsymbol{d}_{l\ddot a} & \xRightarrow{\boldsymbol{D}^{\ddot a j}_{ll}} & \boldsymbol{d}_{lj}\circ\mathrm{P}_{ll}.
\end{array} \tag{11.148}$$

Here $\Theta^{\boldsymbol{e},\boldsymbol{b}}_{lai}, \Theta^{\boldsymbol{f},\boldsymbol{b}}_{laj}$ are as in Definition 4.20 for $\boldsymbol{e}\circ\boldsymbol{b}, \boldsymbol{f}\circ\boldsymbol{b}$, and $\boldsymbol{C}^{\dot a i}_{ll} : \mathrm{T}_{\dot a i}\circ\boldsymbol{c}_{l\dot a} \Rightarrow \boldsymbol{c}_{li}\circ\mathrm{P}_{ll}$, $\boldsymbol{D}^{\ddot a j}_{ll} : \Upsilon_{\ddot a j}\circ\boldsymbol{d}_{l\ddot a} \Rightarrow \boldsymbol{d}_{lj}\circ\mathrm{P}_{ll}$ are as in Definition 4.54 for $\boldsymbol{c}_{l\dot a}, \boldsymbol{d}_{l\ddot a}$.

We now prove that (11.140) commutes by considering the diagram



(11.149)

for all $l \in L$, $a \in A$, $i \in I$, $j \in J$ and $k \in K$. Here the left and top right pentagons commute by (4.27), the top left, bottom left, and rightmost quadrilaterals by (4.30), the bottom right quadrilateral including $\boldsymbol{\kappa}_{lk}$ by (4.62) for $\boldsymbol{\kappa}_{l\ddot a}$, the quadrilaterals to left and right of this by (4.60), the bottom centre left quadrilateral and the right semicircle by (11.147)–(11.148), the centre triangle by (11.142), the two quadrilaterals to the left and right of this by compatibility of vertical and horizontal composition, and the top centre pentagon by (11.135).

Thus (11.149) commutes. The outside of (11.149) proves the restriction of the 'lk' component of (11.140) to the intersection of its domain with $b^{-1}(\mathrm{Im}\,\varphi_a) \cap c^{-1}(\mathrm{Im}\,\chi_i) \cap d^{-1}(\mathrm{Im}\,\psi_j)$. As these intersections for all $a \in A$, $i \in I$, $j \in J$ cover the whole domain, the sheaf property of 2-morphisms of m-Kuranishi neighbourhoods implies that (11.140) commutes. This proves the first part of the universal property in Definition A.11, the existence of $\boldsymbol{b}, \boldsymbol{\zeta}, \boldsymbol{\theta}$ satisfying (11.140).

For the second part, suppose $\tilde{\boldsymbol{b}} : \boldsymbol{V} \to \boldsymbol{W}$ is a 1-morphism in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$ and $\tilde{\boldsymbol{\zeta}} : \boldsymbol{e}\circ\tilde{\boldsymbol{b}} \Rightarrow \boldsymbol{c}$, $\tilde{\boldsymbol{\theta}} : \boldsymbol{f}\circ\tilde{\boldsymbol{b}} \Rightarrow \boldsymbol{d}$ are 2-morphisms such that the analogue of (11.140) commutes. Then $\tilde{\boldsymbol{b}}$ contains 1-morphisms $\tilde{\boldsymbol{b}}_{la}:(S_l,B_l,p_l,\upsilon_l)\to(T_a,C_a,q_a,\varphi_a)$, and running the construction of $\boldsymbol{\zeta},\boldsymbol{\theta}$ above in reverse, we find that as in (11.141) there are unique 2-morphisms $\tilde{\boldsymbol{\zeta}}_{la\dot a}:\boldsymbol{e}_{a\dot a}\circ\tilde{\boldsymbol{b}}_{la}\Rightarrow\boldsymbol{c}_{l\dot a}$, $\tilde{\boldsymbol{\theta}}_{la\ddot a}:\boldsymbol{f}_{a\ddot a}\circ\tilde{\boldsymbol{b}}_{la}\Rightarrow\boldsymbol{d}_{l\ddot a}$ such that the analogues of (11.147)–(11.148) commute for all $i\in I$ and $j\in J$:

$$
\begin{array}{ccccccc}
(\boldsymbol{e}\circ\tilde{\boldsymbol{b}})_{li} & & \xRightarrow{\qquad\qquad\qquad\tilde{\boldsymbol{\zeta}}_{li}\qquad\qquad\qquad} & & & & \boldsymbol{c}_{li} \\
\Uparrow\Theta^{\boldsymbol{e},\tilde{\boldsymbol{b}}}_{lai} & & & & & & \Vert \\
\boldsymbol{e}_{ai}\circ\tilde{\boldsymbol{b}}_{la} & = & \mathrm{T}_{\dot a i}\circ\boldsymbol{e}_{a\dot a}\circ\tilde{\boldsymbol{b}}_{la} & \xRightarrow{\mathrm{id}*\tilde{\boldsymbol{\zeta}}_{la\dot a}} & \mathrm{T}_{\dot a i}\circ\boldsymbol{c}_{l\dot a} & \xRightarrow{\boldsymbol{C}^{\dot a i}_{ll}} & \boldsymbol{c}_{li}\circ\mathrm{P}_{ll},
\end{array}
$$

$$
\begin{array}{ccccccc}
(\boldsymbol{f}\circ\tilde{\boldsymbol{b}})_{lj} & & \xRightarrow{\qquad\qquad\qquad\tilde{\boldsymbol{\theta}}_{lj}\qquad\qquad\qquad} & & & & \boldsymbol{d}_{lj} \\
\Uparrow\Theta^{\boldsymbol{f},\tilde{\boldsymbol{b}}}_{laj} & & & & & & \Vert \\
\boldsymbol{f}_{aj}\circ\tilde{\boldsymbol{b}}_{la} & = & \Upsilon_{\ddot a j}\circ\boldsymbol{f}_{a\ddot a}\circ\tilde{\boldsymbol{b}}_{la} & \xRightarrow{\mathrm{id}*\tilde{\boldsymbol{\theta}}_{la\ddot a}} & \Upsilon_{\ddot a j}\circ\boldsymbol{d}_{l\ddot a} & \xRightarrow{\boldsymbol{D}^{\ddot a j}_{ll}} & \boldsymbol{d}_{lj}\circ\mathrm{P}_{ll}.
\end{array}
$$

From the analogue of (11.140) we can use the analogue of (11.149) in reverse to prove that the analogue of (11.142) commutes:

$$
\begin{array}{ccc}
\boldsymbol{g}_{\dot a\ddot a}\circ\boldsymbol{e}_{a\dot a}\circ\tilde{\boldsymbol{b}}_{la} & = & \boldsymbol{h}_{\ddot a\ddot a}\circ\boldsymbol{f}_{a\ddot a}\circ\tilde{\boldsymbol{b}}_{la} \\
\Downarrow \mathrm{id}_{\boldsymbol{g}_{\dot a\ddot a}}*\tilde{\boldsymbol{\zeta}}_{la\dot a} & & \mathrm{id}_{\boldsymbol{h}_{\ddot a\ddot a}}*\tilde{\boldsymbol{\theta}}_{la\ddot a}\Downarrow \\
\boldsymbol{g}_{\dot a\ddot a}\circ\boldsymbol{c}_{l\dot a} & \xRightarrow{\boldsymbol{\kappa}_{l\ddot a}} & \boldsymbol{h}_{\ddot a\ddot a}\circ\boldsymbol{d}_{l\ddot a}.
\end{array}
$$

Then the second part of the universal property of the 2-Cartesian square (11.118) shows that there is a unique 2-isomorphism $\boldsymbol{\epsilon}_{la}:\boldsymbol{b}_{la}\Rightarrow\tilde{\boldsymbol{b}}_{la}$ with $\boldsymbol{\zeta}_{l\dot a}=\tilde{\boldsymbol{\zeta}}_{l\dot a}\odot(\mathrm{id}_{\boldsymbol{e}_{a\dot a}}*\boldsymbol{\epsilon}_{la})$ and $\boldsymbol{\theta}_{l\ddot a}=\tilde{\boldsymbol{\theta}}_{l\ddot a}\odot(\mathrm{id}_{\boldsymbol{f}_{a\ddot a}}*\boldsymbol{\epsilon}_{la})$. We can then check $\boldsymbol{\epsilon}=\big(\boldsymbol{\epsilon}_{la,\ l\in L,\ a\in A}\big):\boldsymbol{b}\Rightarrow\tilde{\boldsymbol{b}}$ is the unique 2-morphism with $\boldsymbol{\zeta}=\tilde{\boldsymbol{\zeta}}\odot(\mathbf{id}_{\boldsymbol{e}}*\boldsymbol{\epsilon})$ and $\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}\odot(\mathbf{id}_{\boldsymbol{f}}*\boldsymbol{\epsilon})$. This completes the proof that (11.15) is 2-Cartesian in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$, and hence in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$. We have now proved the first part of Theorem 11.19.

11.9.4 Proof of parts (a)–(h)

Finally we prove parts (a)–(h) of Theorem 11.19.

Part (a). Suppose $\boldsymbol{g},\boldsymbol{h}$ in §11.9.1–§11.9.3 are transverse, not just w-transverse. Then in §11.9.1–§11.9.2 we can choose the diagrams (11.117)–(11.118) for $a\in A$ with $\boldsymbol{g}_{\dot a\ddot a},\boldsymbol{h}_{\ddot a\ddot a}$ transverse, not just w-transverse. So as in Definition 11.16 we have $C_a=0$, as C_a is the kernel of (11.11), which is an isomorphism. Thus the m-Kuranishi structure on $\boldsymbol{W}$ has m-Kuranishi neighbourhoods (T_a,C_a,q_a,φ_a) with $C_a=q_a=0$ for all $a\in A$. Therefore $\boldsymbol{W}$ is a manifold as in the proof of Theorem 10.45.

Part (b). Suppose $(U_l,D_l,r_l,\chi_l),(V_m,E_m,s_m,\psi_m),(W_n,F_n,t_n,\omega_n),\boldsymbol{g}_{ln},\boldsymbol{h}_{mn}$ are as in Theorem 11.19(b), and $(T_k,C_k,q_k),\boldsymbol{e}_{kl},\boldsymbol{f}_{km}$ are constructed from them as in Definition 11.16. Then in §11.9.2, we can choose the diagram (11.117) for some $a\in A$ to be (11.116), so that $(T_a,C_a,q_a)=(T_k,C_k,q_k)$. Thus (T_a,C_a,q_a,φ_a) in the m-Kuranishi structure $\mathcal{A}$ of $\boldsymbol{W}=(W,\mathcal{A})$ in §11.9.1–§11.9.2 has $T_a=T_k$, $C_a=C_k$, and $q_a=q_k$, as in Theorem 11.19(b).

By Example 4.50, $(T_a, C_a, q_a, \varphi_a)$ is an m-Kuranishi neighbourhood on $\boldsymbol{W}$. The definitions of $\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{\eta}$ in §11.9.1–§11.9.2 then imply that $\boldsymbol{e}_{a\dot{a}} = \boldsymbol{e}_{kl}$ and $\boldsymbol{f}_{a\ddot{a}} = \boldsymbol{f}_{km}$ are 1-morphisms of m-Kuranishi neighbourhoods over $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ as in §4.7, and comparing (4.62) and (11.135) shows that the unique 2-morphism $\boldsymbol{\eta}_{a\dot{a}\ddot{a}\dddot{a}} = \boldsymbol{\eta}_{klmn} : \boldsymbol{g}_{ln} \circ \boldsymbol{e}_{kl} \Rightarrow \boldsymbol{h}_{mn} \circ \boldsymbol{f}_{km}$ constructed from $\boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$ in Theorem 4.56(b) is the identity, as in (11.116) and (11.117).

This proves part (b) in the special case that we choose to construct $\boldsymbol{W}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{\eta}$ in §11.9.1–§11.9.2 including the given data $(U_l, D_l, r_l, \chi_l), \ldots, \boldsymbol{h}_{mn}$. But any other possible choices of $\boldsymbol{W}', \boldsymbol{e}', \boldsymbol{f}', \boldsymbol{\eta}'$ in a 2-Cartesian square (11.15) are canonically equivalent to $\boldsymbol{W}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{\eta}$, by properties of fibre products, and we can use the canonical equivalence $\boldsymbol{i} : \boldsymbol{W} \to \boldsymbol{W}'$ and 2-morphisms $\boldsymbol{e}' \circ \boldsymbol{i} \Rightarrow \boldsymbol{e}$, $\boldsymbol{f}' \circ \boldsymbol{i} \Rightarrow \boldsymbol{f}$ to convert $(T_a, C_a, q_a, \varphi_a), \boldsymbol{e}_{a\dot{a}}, \boldsymbol{f}_{a\ddot{a}}$ to m-Kuranishi neighbourhoods and 1-morphisms over $\boldsymbol{W}', \boldsymbol{e}', \boldsymbol{f}'$ satisfying the required conditions.

Part (c). We have already proved (c) in §11.9.1 and §11.9.3, as in §11.9.1, when $\dot{\mathbf{Man}}$ satisfies Assumption 11.3 we constructed $\boldsymbol{W}, \boldsymbol{e}, \boldsymbol{f}$ with topological space $W = \{(x, y) \in X \times Y : g(x) = h(y)\}$, and maps $e : (x, y) \mapsto x$, $f : (x, y) \mapsto y$.

Part (d). Suppose $\dot{\mathbf{Man}}$ satisfies Assumption 11.4(a), and we are given a 2-Cartesian square (11.15) in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ with $\boldsymbol{g}$ a w-submersion, so that $\boldsymbol{g}, \boldsymbol{h}$ are w-transverse. Let $w \in \boldsymbol{W}$ with $\boldsymbol{e}(w) = x$ in $\boldsymbol{X}$ and $\boldsymbol{f}(w) = y$ in $\boldsymbol{Y}$. Then in (b) we can choose $\boldsymbol{g}_{ln} : (U_l, D_l, r_l, \chi_l) \to (W_n, F_n, t_n, \omega_n)$, $\boldsymbol{h}_{mn} : (V_m, E_m, s_m, \psi_m) \to (W_n, F_n, t_n, \omega_n)$ with $x \in \operatorname{Im} \chi_l$, $y \in \operatorname{Im} \psi_m$ and $\boldsymbol{g}_{ln}$ a w-submersion. So (b) gives $(T_k, C_k, q_k, \varphi_k), \boldsymbol{e}_{kl}, \boldsymbol{f}_{km}$ constructed as in Definition 11.16, and $w \in \operatorname{Im} \varphi_k$.

Then $g_{ln}|_{\dot{U}_{ln}} : \dot{U}_{ln} \to W_n$ is a submersion in the fibre product (11.114) for T_k by Definition 11.15(iii), so $f_{km} : T_k \to V_m$ is a submersion by Assumption 11.4(a). Also $\hat{g}_{ln}|_{\dot{U}_{ln}}$ is surjective by Definition 11.15(iv), which implies that $\hat{f}_{km} : C_k \to f_{km}^*(D_m)$ is surjective by the definition of $C_k, \hat{f}_{km}$ in Definition 11.16. Hence $\boldsymbol{f}_{km} = (T_k, f_{km}, \hat{f}_{km})$ is a w-submersion by Definition 11.15. As we can find such $\boldsymbol{f}_{km}$ over $(\operatorname{Im} \varphi_k, \boldsymbol{f})$ with $w \in \operatorname{Im} \varphi_k$ for all $w \in \boldsymbol{W}$, we see that $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{X}$ is a w-submersion by Definition 11.18.

Part (e). Suppose $\dot{\mathbf{Man}}$ satisfies Assumptions 10.1 and 11.5, and we are given a 2-Cartesian square (11.15) in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ with $\boldsymbol{g}, \boldsymbol{h}$ w-transverse. Let $w \in \boldsymbol{W}$ with $\boldsymbol{e}(w) = x$ in $\boldsymbol{X}$, $\boldsymbol{f}(w) = y$ in $\boldsymbol{Y}$, and $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$. Choose $(T_k, C_k, q_k, \varphi_k), \ldots, (W_n, F_n, t_n, \omega_n)$ and $\boldsymbol{e}_{kl}, \ldots, \boldsymbol{h}_{mn}$ as in (b) with $w \in \operatorname{Im} \varphi_k$, $x \in \operatorname{Im} \chi_l$, $y \in \operatorname{Im} \psi_m$ and $z \in \operatorname{Im} \omega_n$. Set $t_k = \varphi_k^{-1}(w)$, $u_l = \chi_l^{-1}(x)$,

$v_m = \psi_m^{-1}(y)$ and $w_n = \omega_n^{-1}(z)$, and consider the commutative diagram:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & T_{t_k}T_k & & \xrightarrow{\mathrm{d}_{t_k}q_k} & & C_k|_{t_k} & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \begin{pmatrix} T_{t_k}e_{kl} \\ T_{t_k}f_{km} \end{pmatrix}} & & & & \downarrow{\scriptstyle \begin{pmatrix} \hat e_{kl}|_{t_k} \\ \hat f_{km}|_{t_k} \end{pmatrix}} & & \downarrow{\scriptstyle 0} \\
\cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & \begin{array}{c} T_{u_l}U_l\oplus \\ T_{v_m}V_m \end{array} & & \xrightarrow{\begin{pmatrix} \mathrm{d}_{u_l}r_l & 0 \\ 0 & \mathrm{d}_{v_m}s_m \end{pmatrix}} & & \begin{array}{c} D_l|_{u_l}\oplus \\ E_m|_{v_m} \end{array} & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \begin{pmatrix} T_{u_l}g_{ln} & -T_{v_m}h_{mn} \end{pmatrix}} & & & & \downarrow{\scriptstyle \begin{pmatrix} \hat g_{ln}|_{u_l} & -\hat h_{mn}|_{v_m} \end{pmatrix}} & & \downarrow{\scriptstyle 0} \\
\cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & T_{w_n}W_n & & \xrightarrow{\mathrm{d}_{w_n}t_n} & & F_n|_{w_n} & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & & & 0 & & 0.
\end{array}
\tag{11.150}
$$

Here the second column is exact by Assumption 11.5 applied to the transverse fibre product (11.114) at t_k, and the third column is exact by Definition 11.16.

As in equation (10.27) of Definition 10.21, the cohomology groups of the first row of (11.150) at the second and third columns are $T_w\boldsymbol{W}$ and $O_w\boldsymbol{W}$, and similarly the second and third rows have cohomology $T_x\boldsymbol{X}\oplus T_y\boldsymbol{Y}, O_x\boldsymbol{X}\oplus O_y\boldsymbol{Y}$ and $T_z\boldsymbol{Z}, O_z\boldsymbol{Z}$.

In the setting of Definition 10.69, regard (11.150) as a diagram (10.89), a short exact sequence of complexes $E^\bullet, F^\bullet, G^\bullet$, the first, second and third rows of (11.150) respectively, with the third column of (11.150) in degree zero. Thus Definition 10.69 constructs a long exact sequence (10.90) from (11.150). This sequence is equation (11.16) in Theorem 11.19(d), as we want.

In more detail, our identification of the cohomology of the rows of (11.150) shows that the vector spaces in (10.90) are $0, T_w\boldsymbol{W}, T_x\boldsymbol{X}\oplus T_y\boldsymbol{Y},\ldots,O_z\boldsymbol{Z},0$ as in (11.16). Comparing Definitions 10.21 and 10.69 we see that the morphisms $H^k(\theta^\bullet), H^k(\psi^\bullet)$ in (10.90) for $k=-1,0$ are $T_w\boldsymbol{e}\oplus T_w\boldsymbol{f},\ldots,O_x\boldsymbol{g}\oplus -O_y\boldsymbol{h}$, as in (11.16). We define $\delta_w^{\boldsymbol{g},\boldsymbol{h}}$ in (11.16) to be the connecting morphism $\delta_{\theta^\bullet,\psi^\bullet}^{-1}$ in (10.90) from Definition 10.69. A proof similar to the definition of $T_x\boldsymbol{f}, O_x\boldsymbol{f}$ in Definition 10.21 shows $\delta_w^{\boldsymbol{g},\boldsymbol{h}}$ is independent of the choices of $(T_k,C_k,q_k,\varphi_k),\ldots,\boldsymbol{h}_{mn}$ above.

Part (f). Suppose $\dot{\mathbf{Man}}$ satisfies Assumptions 10.19 and 11.6, and we are given a 2-Cartesian square (11.15) in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ with $\boldsymbol{g},\boldsymbol{h}$ w-transverse. Let $w\in\boldsymbol{W}$ with $\boldsymbol{e}(w)=x$ in $\boldsymbol{X}$, $\boldsymbol{f}(w)=y$ in $\boldsymbol{Y}$, and $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ in $\boldsymbol{Z}$. Choose $(T_k,C_k,q_k,\varphi_k),\ldots,(W_n,F_n,t_n,\omega_n)$ and $\boldsymbol{e}_{kl},\ldots,\boldsymbol{h}_{mn}$ as in part (b) with $w\in\operatorname{Im}\varphi_k$, $x\in\operatorname{Im}\chi_l$, $y\in\operatorname{Im}\psi_m$ and $z\in\operatorname{Im}\omega_n$. Set $t_k=\varphi_k^{-1}(w)$, $u_l=\chi_l^{-1}(x)$, $v_m=\psi_m^{-1}(y)$ and $w_n=\omega_n^{-1}(z)$.

As the fibre product (11.114) is transverse, Assumption 11.6 says that

$$
\begin{array}{ccc}
Q_{t_k}T_k & \xrightarrow{Q_{t_k}e_{kl}} & Q_{v_m}V_m \\
\downarrow{\scriptstyle Q_{t_k}f_{km}} & & \downarrow{\scriptstyle Q_{v_m}h_{mn}} \\
Q_{u_l}U_l & \xrightarrow{Q_{u_l}g_{ln}} & Q_{w_n}W_n
\end{array}
\tag{11.151}
$$

is Cartesian in $\mathcal{Q}$. Now Definition 10.30 gives isomorphisms $Q_{w,k} : Q_w \boldsymbol{W} \to Q_{t_k} T_k, \ldots, Q_{z,n} : Q_z \boldsymbol{Z} \to Q_{w_n} W_n$ in $\mathcal{Q}$ such that (10.42) commutes for $\boldsymbol{e}_{kl}$, $\boldsymbol{f}_{km}, \boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$. Thus (11.151) is isomorphic in $\mathcal{Q}$ to the commutative square (11.17), so (11.17) is Cartesian in $\mathcal{Q}$, as we have to prove.

Part (g). Suppose $\dot{\mathbf{Man}}^{\mathbf{c}}$ satisfies Assumptions 3.22, 11.1, and 11.7, and we are given a 2-Cartesian square (11.15) in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ with $\boldsymbol{g}, \boldsymbol{h}$ w-transverse. Since $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$ maps $\dot{\mathbf{Man}}^{\mathbf{c}}_{\boldsymbol{D}} \to \check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}_{\boldsymbol{D}}$ by Assumption 11.7, the corner 2-functor $C : \mathbf{m\dot{K}ur}^{\mathbf{c}} \to \mathbf{m\check{\dot{K}}ur}^{\mathbf{c}}$ from §4.6 maps $\mathbf{m\dot{K}ur}^{\mathbf{c}}_{\boldsymbol{D}} \to \mathbf{m\check{\dot{K}}ur}^{\mathbf{c}}_{\boldsymbol{D}}$. Thus applying C to (11.15) shows (11.18) is a 2-commutative square in $\mathbf{m\check{\dot{K}}ur}^{\mathbf{c}}_{\boldsymbol{D}}$. We must show that $C(\boldsymbol{g}), C(\boldsymbol{h})$ are w-transverse, and (11.18) is 2-Cartesian.

Choose $(T_k, C_k, q_k, \varphi_k), \ldots, (W_n, F_n, t_n, \omega_n)$ and $\boldsymbol{e}_{kl}, \ldots, \boldsymbol{h}_{mn}$ as in part (b). Then Definitions 4.60 and 4.61 construct m-Kuranishi neighbourhoods $(T_{(a,k)}, C_{(a,k)}, q_{(a,k)}, \varphi_{(a,k)})$ on $C_a(\boldsymbol{W})$ for $a \geqslant 0$, and so on, and 1-morphisms $\boldsymbol{e}_{(a,k)(b,l)}, \ldots, \boldsymbol{h}_{(c,m)(d,n)}$ over $C(\boldsymbol{e}), \ldots, C(\boldsymbol{h})$ in a 2-commutative diagram in $\mathbf{\check{G}m\dot{K}N}^{\mathbf{c}}_{\boldsymbol{D}}$:

$$
\begin{array}{ccc}
\coprod_{a\geqslant 0}(T_{(a,k)}, C_{(a,k)}, q_{(a,k)}) & \xrightarrow{\coprod_{a,c\geqslant 0} \boldsymbol{f}_{(a,k)(c,m)}} & \coprod_{c\geqslant 0}(V_{(c,m)}, E_{(c,m)}, s_{(c,m)}) \\
\Big\downarrow {\scriptstyle \coprod_{a,b\geqslant 0} \boldsymbol{e}_{(a,k)(b,l)}} & \mathrm{id} \Uparrow & {\scriptstyle \coprod_{c,d\geqslant 0} \boldsymbol{h}_{(c,m)(d,n)}} \Big\downarrow \\
\coprod_{b\geqslant 0}(U_{(b,l)}, D_{(b,l)}, r_{(b,l)}) & \xrightarrow{\coprod_{b,d\geqslant 0} \boldsymbol{g}_{(b,l)(d,n)}} & \coprod_{d\geqslant 0}(W_{(d,n)}, F_{(d,n)}, t_{(d,n)}).
\end{array}
\tag{11.152}
$$

This is the result of applying the corner 2-functor to (11.14).

Applying $C : \dot{\mathbf{Man}}^{\mathbf{c}} \to \check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$ to the transverse fibre product (11.114) in $\dot{\mathbf{Man}}^{\mathbf{c}}$ and using Assumption 11.7 shows we have a fibre product in $\check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$

$$
C(T_k) = C(\dot{U}_{ln}) \times_{C(g_{ln}|_{\dot{U}_{ln}}), C(W_n), C(h_{mn}|_{\dot{V}_{mn}})} C(\dot{V}_{mn}), \tag{11.153}
$$

where $C(g_{ln}|_{\dot{U}_{ln}}), C(h_{mn}|_{\dot{V}_{mn}})$ are transverse in $\check{\dot{\mathbf{Man}}}{}^{\mathbf{c}}$. Note that the manifolds and smooth maps in (11.152) are the Cartesian square from (11.153).

Also, the vector bundles and linear maps in (11.152) are pullbacks of those in (11.14), so that $C_{(a,k)} = \Pi_a^*(C_k)$, $\hat{e}_{(a,k)(b,l)} = \Pi_a^*(\hat{e}_{kl})$, and so on. Therefore they satisfy the same surjectivity and exactness conditions as do those in (11.14). Thus Definition 11.15(i),(ii) for $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ imply Definition 11.15(i),(ii) for $\boldsymbol{g}_{(b,l)(d,n)}, \boldsymbol{h}_{(c,m)(d,n)}$, so $\boldsymbol{g}_{(b,l)(d,n)}, \boldsymbol{h}_{(c,m)(d,n)}$ are w-transverse for all $b, c, d \geqslant 0$, and the bottom and right 1-morphisms in (11.152) are w-transverse. As the domains of such $\boldsymbol{g}_{(b,l)(d,n)}, \boldsymbol{h}_{(c,m)(d,n)}$ cover $C(X) \times_{C(g), C(Z), C(h)} C(Y)$, we see that $C(\boldsymbol{g}), C(\boldsymbol{h})$ are w-transverse, as we want. The same proof shows that if $\boldsymbol{g}, \boldsymbol{h}$ are transverse then $C(\boldsymbol{g}), C(\boldsymbol{h})$ are transverse.

Given all this, equation (11.152) is built from the w-transverse 1-morphisms $\coprod_{b,d\geqslant 0} \boldsymbol{g}_{(b,l)(d,n)}$ and $\coprod_{c,d\geqslant 0} \boldsymbol{h}_{(c,m)(d,n)}$ in exactly the same way that equation (11.14) is built from the w-transverse 1-morphisms $\boldsymbol{g}_{ln}$ and $\boldsymbol{h}_{mn}$ in Definition 11.16. Therefore Theorem 11.17 shows that (11.152) is 2-Cartesian in $\mathbf{\check{G}m\dot{K}N}^{\mathbf{c}}_{\boldsymbol{D}}$ and $\mathbf{\check{G}m\dot{K}N}^{\mathbf{c}}_{\boldsymbol{E}}$.

In §11.9.3 we showed that when the 2-commutative square (11.15) can be covered by a family of diagrams (11.117)–(11.118) for $a \in A$ with (11.118) 2-Cartesian in $\mathbf{Gm\dot{K}N}_{\boldsymbol{D}}$ and $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}$, then (11.15) is 2-Cartesian in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$

and $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$. Since (11.18) can be covered by a family of diagrams (11.152) which are 2-Cartesian in $\mathbf{\check{G}m\dot{K}N}^{\mathbf{c}}_{\boldsymbol{D}}$ and $\mathbf{\check{G}m\dot{K}N}^{\mathbf{c}}_{\boldsymbol{E}}$, the same proof shows that (11.18) is 2-Cartesian in $\mathbf{m\check{\dot{K}}ur}^{\mathbf{c}}_{\boldsymbol{D}}$ and $\mathbf{m\check{\dot{K}}ur}^{\mathbf{c}}_{\boldsymbol{E}}$, as we want.

In the w-transverse 2-Cartesian square (11.18) in $\mathbf{m\check{\dot{K}}ur}^{\mathbf{c}}_{\boldsymbol{D}}$, suppose $w' \in C_i(\boldsymbol{W}) \subseteq C(\boldsymbol{W})$ with $C(\boldsymbol{e})w' = x'$ in $C_j(\boldsymbol{X})$, $C(\boldsymbol{f})w' = y'$ in $C_k(\boldsymbol{Y})$ and $C(\boldsymbol{g})x' = C(\boldsymbol{h})y' = z'$ in $C_l(\boldsymbol{Z})$. Locally near w' we have a w-transverse fibre product $C_i(\boldsymbol{W}) \simeq C_j(\boldsymbol{X}) \times_{C_l(\boldsymbol{Z})} C_k(\boldsymbol{Y})$, so the first part of Theorem 11.19 gives

$$\begin{aligned}\operatorname{vdim} \boldsymbol{W} - i = \operatorname{vdim} C_i(\boldsymbol{W}) &= \operatorname{vdim} C_j(\boldsymbol{X}) + \operatorname{vdim} C_k(\boldsymbol{Y}) - \operatorname{vdim} C_l(\boldsymbol{Z}) \\ &= \operatorname{vdim} \boldsymbol{X} - j + \operatorname{vdim} \boldsymbol{Y} - k - \operatorname{vdim} \boldsymbol{Z} + l.\end{aligned}$$

But also $\operatorname{vdim} \boldsymbol{W} = \operatorname{vdim} \boldsymbol{X} + \operatorname{vdim} \boldsymbol{Y} - \operatorname{vdim} \boldsymbol{Z}$, so that $i = j + k - l$. Therefore (11.18) being 2-Cartesian in $\mathbf{m\check{\dot{K}}ur}^{\mathbf{c}}_{\boldsymbol{D}}$ implies equation (11.19) holds in $\mathbf{m\dot{K}ur}^{\mathbf{c}}_{\boldsymbol{D}}$. When $i = 1$ and $\partial \boldsymbol{Z} = \emptyset$, in the union over j, k, l in (11.19) the only possibilities are $(j, k, l) = (1, 0, 0)$ and $(0, 1, 0)$, yielding equation (11.20).

Part (h). Suppose $\mathbf{\dot{M}an}$ satisfies Assumption 11.8, and $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ is a w-submersion in $\mathbf{m\dot{K}ur}_{\boldsymbol{D}}$, and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ is any morphism in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$. Then we can construct the fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ in $\mathbf{m\dot{K}ur}_{\boldsymbol{E}}$ by the method of §11.9.1–§11.9.3, but working in $\mathbf{Gm\dot{K}N}_{\boldsymbol{E}}, \mathbf{m\dot{K}ur}_{\boldsymbol{E}}$ rather than $\mathbf{Gm\dot{K}N}_{\boldsymbol{D}}, \mathbf{m\dot{K}ur}_{\boldsymbol{D}}$ throughout, and taking the $\boldsymbol{g}_{ln}, \boldsymbol{g}_{\dot{a}\ddot{a}}$ to be $\boldsymbol{D}$ w-submersions. The proofs of (a)–(d) and (g) above still work, with the obvious modifications.

This completes the proof of Theorem 11.19.

11.10 Proof of Theorem 11.22

11.10.1 Proof of Theorem 11.22(a)

Let $\mathbf{\dot{M}an}^{\mathbf{c}}$ satisfy Assumptions 3.22 and 11.9. Suppose $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are 1-morphisms in $\mathbf{m\dot{K}ur}^{\mathbf{c}}_{\boldsymbol{D}}$, and $x \in \boldsymbol{X}$, $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$.

For the first 'only if' part of (a), suppose $\boldsymbol{g}, \boldsymbol{h}$ are w-transverse. Then by Definition 11.18 there exist m-Kuranishi neighbourhoods (U_l, D_l, r_l, χ_l), (V_m, E_m, s_m, ψ_m), $(W_n, F_n, t_n, \omega_n)$ on $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ with $x \in \operatorname{Im} \chi_l \subseteq g^{-1}(\operatorname{Im} \omega_n)$, $y \in \operatorname{Im} \psi_m \subseteq h^{-1}(\operatorname{Im} \omega_n)$ and $z \in \operatorname{Im} \omega_n$, and 1-morphisms $\boldsymbol{g}_{ln} : (U_l, D_l, r_l, \chi_l) \to (W_n, F_n, t_n, \omega_n)$, $\boldsymbol{h}_{mn} : (V_m, E_m, s_m, \psi_m) \to (W_n, F_n, t_n, \omega_n)$ over $(\operatorname{Im} \chi_l, \boldsymbol{g})$ and $(\operatorname{Im} \psi_m, \boldsymbol{h})$, such that $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are w-transverse.

Write $u_l = \chi_l^{-1}(x) \in U_l$, $v_m = \psi_m^{-1}(y) \in V_m$ and $w_n = \omega_n^{-1}(z) \in W_n$. By (10.27)–(10.28) we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \begin{smallmatrix} T_x\boldsymbol{X} \oplus \\ T_y\boldsymbol{Y} \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} T_{u_l}U_l \oplus \\ T_{v_m}V_m \end{smallmatrix} & \xrightarrow{\mathrm{d}_{u_l} r_l \oplus \mathrm{d}_{v_m} s_m} & \begin{smallmatrix} D_l|_{u_l} \oplus \\ E_m|_{v_m} \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} O_x\boldsymbol{X} \oplus \\ O_y\boldsymbol{Y} \end{smallmatrix} & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle T_x\boldsymbol{g} \oplus T_y\boldsymbol{h}} & & \downarrow {\scriptstyle T_{u_l}g_{ln} \oplus T_{v_m}h_{mn}} & & \downarrow {\scriptstyle \hat{g}_{ln}|_{u_l} \oplus \hat{h}_{mn}|_{v_m}} & & \downarrow {\scriptstyle O_x\boldsymbol{g} \oplus O_y\boldsymbol{h}} & & \\ 0 & \longrightarrow & T_z\boldsymbol{Z} & \longrightarrow & T_{w_n}W_n & \xrightarrow{\mathrm{d}_{w_n} t_n} & F_n|_{w_n} & \longrightarrow & O_z\boldsymbol{Z} & \longrightarrow & 0. \end{array} \tag{11.154}$$

As $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are w-transverse, the third column of (11.154) is surjective by Definition 11.15(ii). Also $g_{ln} : U_{ln} \to W_n$ and $h_{mn} : V_{mn} \to W_n$ are transverse in $\dot{\mathbf{Man}}^{\mathbf{c}}$ near $u_l \in U_{ln}$ and $v_m \in V_{mn}$, so Assumption 11.9 says that the third column of (11.154) is surjective, and 'condition $\boldsymbol{T}$' holds for the data:

(i) The quasi-tangent maps $Q_{u_l} g_{ln} : Q_{u_l} U_l \to Q_{w_n} W_n$ and $Q_{v_m} h_{mn} : Q_{v_m} V_m \to Q_{w_n} W_n$ in $\mathcal{Q}$.

(ii) For all $i, j, k \geqslant 0$, the family of triples $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ for $\boldsymbol{u} \in C_i(U_l)$, $\boldsymbol{v} \in C_j(V_m)$ with $\Pi_i(\boldsymbol{u}) = u_l$, $\Pi_j(\boldsymbol{v}) = v_m$, and $C(g_{ln})\boldsymbol{u} = C(h_{mn})\boldsymbol{v} = \boldsymbol{w}$ in $C_k(W_n)$.

As the third column of (11.154) is surjective, the fourth column is surjective by exactness of rows, so (11.21) is surjective.

Definition 10.30 gives isomorphisms $Q_{x,l} : Q_x \boldsymbol{X} \to Q_{u_l} U_l$, etc., which identify $Q_x \boldsymbol{g} : Q_x \boldsymbol{X} \to Q_z \boldsymbol{Z}$ and $Q_y \boldsymbol{h} : Q_y \boldsymbol{Y} \to Q_z \boldsymbol{Z}$ with $Q_{u_l} g_{ln}, Q_{v_m} h_{mn}$ in (i) above. Also the maps $\chi_{(i,l)}, \psi_{(j,m)}, \omega_{(k,n)}$ from the definition of $C_i(\boldsymbol{X}), C_j(\boldsymbol{Y})$, $C_k(\boldsymbol{Z})$ in Definition 4.39 identify the sets in (ii) above with the corresponding sets from $C(\boldsymbol{g})|_{\cdots} : C_i(\boldsymbol{X}) \to C_k(\boldsymbol{Z})$, $C(\boldsymbol{h})|_{\cdots} : C_j(\boldsymbol{Y}) \to C_k(\boldsymbol{Z})$ over x, y, z. Hence condition $\boldsymbol{T}$ holding for (i),(ii) above implies that condition $\boldsymbol{T}$ holds for $\boldsymbol{g}, \boldsymbol{h}$ at x, y, z, noting the requirement in Assumption 11.9(a) that condition $\boldsymbol{T}$ only involves objects $Q_x X, \ldots$ in $\mathcal{Q}$ up to isomorphism, and subsets $\Pi_i^{-1}(x) \subseteq C_i(X), \ldots$ up to bijection. This proves the first 'only if' part of (a).

For the second 'only if' part of (a), suppose also that $\boldsymbol{g}, \boldsymbol{h}$ are transverse. Then condition $\boldsymbol{T}$ still holds for $\boldsymbol{g}, \boldsymbol{h}$ at x, y, z, and the third column of (11.154) is an isomorphism by Definition 11.15, and the second column is still surjective, so by exactness of rows the fourth column (which is (11.21)) is an isomorphism, and the first column (which is (11.22)) is surjective, as we have to prove.

For the first 'if' part of (a), suppose condition $\boldsymbol{T}$ holds for $\boldsymbol{g}, \boldsymbol{h}, x, y, z$ and (11.21) is surjective, for all x, y, z as above. Choose m-Kuranishi neighbourhoods (U_l, D_l, r_l, χ_l), (V_m, E_m, s_m, ψ_m), $(W_n, F_n, t_n, \omega_n)$ on $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ with $x \in \operatorname{Im}\chi_l \subseteq g^{-1}(\operatorname{Im}\omega_n)$, $y \in \operatorname{Im}\psi_m \subseteq h^{-1}(\operatorname{Im}\omega_n)$ and $z \in \operatorname{Im}\omega_n$. Theorem 4.56(b) gives 1-morphisms $\boldsymbol{g}_{ln} : (U_l, D_l, r_l, \chi_l) \to (W_n, F_n, t_n, \omega_n)$, $\boldsymbol{h}_{mn} : (V_m, E_m, s_m, \psi_m) \to (W_n, F_n, t_n, \omega_n)$ over $(\operatorname{Im}\chi_l, \boldsymbol{g})$ and $(\operatorname{Im}\psi_m, \boldsymbol{h})$.

Write $u_l = \chi_l^{-1}(x) \in U_l$, $v_m = \psi_m^{-1}(y) \in V_m$ and $w_n = \omega_n^{-1}(z) \in W_n$. As condition $\boldsymbol{T}$ holds for $\boldsymbol{g}, \boldsymbol{h}, x, y, z$, it holds for the data in (i),(ii) above, reversing the previous argument. Thus Assumption 11.9(c) says there exist open $(u_l, 0) \in U_{l'} \hookrightarrow U_{ln} \times \mathbb{R}^a$ and $(v_m, 0) \in V_{m'} \hookrightarrow V_{mn} \times \mathbb{R}^b$ for $a, b \geqslant 0$, and transverse morphisms $g_{l'n} : U_{l'} \to W_n$, $h_{m'n} : V_{m'} \to W_n$ with $g_{l'n}(u, 0) = g_{ln}(u)$, $h_{m'n}(v, 0) = h_{mn}(v)$ for all $u \in U_{ln}$, $v \in V_{mn}$ with $(u, 0) \in U_{l'}$ and $(v, 0) \in V_{m'}$.

As for $(V_{(n)}, E_{(n)}, s_{(n)}, \psi_{(n)})$ in Definition 10.38, define vector bundles $D_{l'} \to U_{l'}$, $E_{m'} \to V_{m'}$ by $D_{l'} = \pi_{U_l}^*(D_l) \oplus \mathbb{R}^a$, $E_{m'} = \pi_{V_m}^*(E_m) \oplus \mathbb{R}^b$. Define sections $r_{l'} = \pi_{U_l}^*(r_l) \oplus \mathrm{id}_{\mathbb{R}^a}$ in $\Gamma^\infty(D_{l'})$ and $s_{m'} = \pi_{V_m}^*(s_m) \oplus \mathrm{id}_{\mathbb{R}^b}$ in $\Gamma^\infty(E_{m'})$. Then $r_{l'}^{-1}(0) = (r_l^{-1}(0) \times \{0\}) \cap U_{l'}$ and $s_{m'}^{-1}(0) = (s_m^{-1}(0) \times \{0\}) \cap V_{m'}$. Define $\chi_{l'} : r_{l'}^{-1}(0) \to X$ by $\chi_{l'}(u, 0) = \chi_l(u)$, and $\psi_{m'} : s_{m'}^{-1}(0) \to Y$ by $\psi_{m'}(v, 0) = \psi_m(v)$. Then $(U_{l'}, D_{l'}, r_{l'}, \chi_{l'})$ and $(V_{m'}, E_{m'}, s_{m'}, \psi_{m'})$ are m-Kuranishi neighbourhoods on X, Y, with $x \in \operatorname{Im}\chi_{l'}$ and $y \in \operatorname{Im}\psi_{m'}$.

As for $\Phi_{(n)*}$ in Definition 10.38, we have coordinate changes

$$\mathrm{T}_{l'l} = (U_{l'}, \pi_{U_l}, \mathrm{id}_{\pi_{U_l}^*}(D_l) \oplus 0) : (U_{l'}, D_{l'}, r_{l'}, \chi_{l'}) \longrightarrow (U_l, D_l, r_l, \chi_l),$$
$$\Upsilon_{m'm} = (V_{m'}, \pi_{V_m}, \mathrm{id}_{\pi_{V_m}^*}(E_m) \oplus 0) : (V_{m'}, E_{m'}, s_{m'}, \psi_{m'}) \longrightarrow (V_m, E_m, s_m, \psi_m).$$

Using notation (4.6)–(4.8) for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ and defining $\mathrm{T}_{l'i} = \mathrm{T}_{li} \circ \mathrm{T}_{l'l}$, $\mathrm{K}_{l'ii'} = \mathrm{K}_{lii'} * \mathrm{id}_{\mathrm{T}_{l'l}}$, $\Upsilon_{m'j} = \Upsilon_{mj} \circ \Upsilon_{m'm}$, $\Lambda_{m'jj'} = \Lambda_{mjj'} * \mathrm{id}_{\Upsilon_{m'm}}$ for $i, i' \in I$ and $j, j' \in J$, where $\mathrm{T}_{li}, \mathrm{K}_{lii'}$ and $\Upsilon_{mj}, \Lambda_{mjj'}$ are the implicit extra data making $(U_l, D_l, r_l, \chi_l), (V_m, E_m, s_m, \psi_m)$ into m-Kuranishi neighbourhoods on $\boldsymbol{X}, \boldsymbol{Y}$ as in §4.7, then $\mathrm{T}_{l'i}, \mathrm{K}_{l'ii'}$ and $\Upsilon_{m'j}, \Lambda_{m'jj'}$ make $(U_{l'}, D_{l'}, r_{l'}, \chi_{l'})$ and $(V_{m'}, E_{m'}, s_{m'}, \psi_{m'})$ into m-Kuranishi neighbourhoods on $\boldsymbol{X}, \boldsymbol{Y}$. Similarly

$$\begin{aligned} \boldsymbol{g}_{ln} \circ \mathrm{T}_{l'l} = (U_{l'}, g_{ln} \circ \pi_{U_l}, \pi_{U_l}^*(\hat{g}_{ln}) \circ \pi_{\pi_{U_l}^*(D_l)} \oplus 0) :& \\ (U_{l'}, D_{l'}, r_{l'}, \chi_{l'}) \longrightarrow (W_n, F_n, t_n, \omega_n),& \\ \boldsymbol{h}_{mn} \circ \Upsilon_{m'm} = (V_{m'}, h_{mn} \circ \pi_{V_m}, \pi_{V_m}^*(\hat{h}_{mn}) \circ \pi_{\pi_{V_m}^*(E_m)} \oplus 0) :& \\ (V_{m'}, E_{m'}, s_{m'}, \psi_{m'}) \longrightarrow (W_n, F_n, t_n, \omega_n),& \end{aligned}$$

are 1-morphisms of m-Kuranishi neighbourhoods on $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ over $\boldsymbol{g}, \boldsymbol{h}$.

We have morphisms $g_{l'n} : U_{l'} \to W_n$ and $g_{ln} \circ \pi_{U_{ln}} : U_{l'} \to W_n$ in $\dot{\mathbf{Man}}^{\mathbf{c}}$. Define open $T \subseteq D_{l'}$ and a morphism $t : T \to W_n$ by

$$\begin{aligned} T = \big\{ ((u,(x_1,\ldots,x_a)),(d,(y_1,\ldots,y_a))) \in D_{l'} : (u,(y_1,\ldots,y_a)) \in U_{l'} \big\}, \\ t : ((u,(x_1,\ldots,x_a)),(d,(y_1,\ldots,y_a))) \longmapsto g'_{ln}(u,(y_1,\ldots,y_a)). \end{aligned}$$

Then whenever both sides are defined we have

$$\begin{aligned} t \circ 0_{D_{l'}}(u,(x_1,\ldots,x_a)) = g'_{ln}(u,(0,\ldots,0)) = g_{ln}(u) = g_{ln} \circ \pi_{U_l}(u,(x_1,\ldots,x_a)), \\ t \circ r_{l'}(u,(x_1,\ldots,x_a) = g'_{ln}(u,(x_1,\ldots,x_a)). \end{aligned}$$

Thus if we define $\hat{\eta} = \theta_{T,t} : D_{l'} \to \mathcal{T}_{g_{ln} \circ \pi_{U_l}} W_n$, using the notation of Definition B.32, then in the notation of Definitions 3.15(vii) and B.36(vii) we have

$$g_{l'n} = g_{ln} \circ \pi_{U_{ln}} + \hat{\eta} \circ r_{l'} + O(r_{l'})^2. \tag{11.155}$$

Equation (11.155) implies that $g_{l'n} = g_{ln} \circ \pi_{U_{ln}} + O(r_{l'})$. So by Theorem 3.17(g) there exists $\hat{\hat{g}}_{l'n} : D_{l'} \to g_{l'n}^*(F_n)$ with

$$\hat{\hat{g}}_{l'n} = (\hat{g}_{ln} \circ \pi_{\pi_{U_l}^*(D_l)} \oplus 0) + O(r_{l'}).$$

Define a vector bundle morphism $\hat{g}_{l'n} : D_{l'} \to g_{l'n}^*(F_n)$ by

$$\hat{g}_{l'n} = \hat{\hat{g}}_{l'n} + g_{l'n}^*(\nabla t_n) \circ \hat{\eta},$$

for ∇ some connection on $F_n \to W_n$. Then we have

$$\hat{g}_{l'n} = (\hat{g}_{ln} \circ \pi_{\pi_{U_l}^*(D_l)} \oplus 0) + g_{l'n}^*(\mathrm{d}t_n) \circ \hat{\eta} + O(r_{l'}), \tag{11.156}$$

in the sense of Definition 3.15(iv),(vi).

From Definitions 4.2 and 4.3 and (11.155)–(11.156) we can show that

$$\boldsymbol{g}_{l'n} = (U_{l'}, g_{l'n}, \hat{g}_{l'n}) : (U_{l'}, D_{l'}, r_{l'}, \chi_{l'}) \longrightarrow (W_n, F_n, t_n, \omega_n)$$

is a 1-morphism of m-Kuranishi neighbourhoods over $(\operatorname{Im}\chi_{l'}, g)$, and

$$\boldsymbol{\eta} = [U_{l'}, \hat{\eta}] : \boldsymbol{g}_{ln} \circ \mathrm{T}_{l'l} \Longrightarrow \boldsymbol{g}_{l'n}$$

is a 2-morphism. Then using §4.7.1, we can make $\boldsymbol{g}_{l'n}$ into a 1-morphism over $(\operatorname{Im}\chi_{l'}, \boldsymbol{g})$ in a unique way such that $\boldsymbol{\eta} : \boldsymbol{g}_{ln} \circ \mathrm{T}_{l'l} \Rightarrow \boldsymbol{g}_{l'n}$ is the unique 2-morphism given by Theorem 4.56(c). Similarly we construct

$$\boldsymbol{h}_{m'n} = (V_{m'}, h_{m'n}, \hat{h}_{m'n}) : (V_{m'}, E_{m'}, s_{m'}, \psi_{m'}) \longrightarrow (W_n, F_n, t_n, \omega_n)$$

over $(\operatorname{Im}\psi_{m'}, \boldsymbol{h})$, and a 2-morphism $\boldsymbol{\zeta} : \boldsymbol{h}_{mn} \circ \Upsilon_{m'm} \Rightarrow \boldsymbol{h}_{m'n}$.

Consider equation (11.154) for $\boldsymbol{g}_{l'n}, \boldsymbol{h}_{m'n}$ at $(u_l, 0) \in U_{l'}$, $(v_m, 0) \in V_{m'}$, $(w_n, 0) \in W_n$. Then the second column of (11.154) is surjective as $g_{l'n}, h_{m'n}$ are transverse, and the fourth column is surjective as (11.21) is surjective. Hence the third column is surjective by exactness. Thus Definition 11.15(ii) holds at $(u_l, 0), (v_m, 0)$, and this is an open condition. Also Definition 11.15(i) holds as $g_{l'n}, h_{m'n}$ are transverse. Thus making $U_{l'}, V_{m'}$ smaller, we can suppose $\boldsymbol{g}_{l'n}, \boldsymbol{h}_{m'n}$ are w-transverse. As we can find such $\boldsymbol{g}_{l'n}, \boldsymbol{h}_{m'n}$ with $x \in \operatorname{Im}\chi_{l'}$ and $y \in \operatorname{Im}\psi_{m'}$ for any x, y, z as above, $\boldsymbol{g}, \boldsymbol{h}$ are w-transverse by Definition 11.18. This proves the first 'if' part of (a).

For the second 'if' part, suppose that Assumption 10.9 holds for $\dot{\mathbf{Man}}^{\mathbf{c}}$, and for all $x \in \boldsymbol{X}$, $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$, condition $\boldsymbol{T}$ holds for $\boldsymbol{g}, \boldsymbol{h}, x, y, z$, (11.21) is an isomorphism, and (11.22) is surjective. For such x, y, z, we use Assumption 10.9 and Proposition 10.39 to choose m-Kuranishi neighbourhoods (U_l, D_l, r_l, χ_l), (V_m, E_m, s_m, ψ_m), $(W_n, F_n, t_n, \omega_n)$ on $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ which are minimal at $x \in \operatorname{Im}\chi_l \subseteq g^{-1}(\operatorname{Im}\omega_n)$, $y \in \operatorname{Im}\psi_m \subseteq h^{-1}(\operatorname{Im}\omega_n)$ and $z \in \operatorname{Im}\omega_n$. Theorem 4.56(b) gives 1-morphisms $\boldsymbol{g}_{ln} : (U_l, D_l, r_l, \chi_l) \to (W_n, F_n, t_n, \omega_n)$, $\boldsymbol{h}_{mn} : (V_m, E_m, s_m, \psi_m) \to (W_n, F_n, t_n, \omega_n)$ over $(\operatorname{Im}\chi_l, \boldsymbol{g})$ and $(\operatorname{Im}\psi_m, \boldsymbol{h})$.

Consider (11.154) for these $\boldsymbol{g}, \boldsymbol{h}$. Then the first column is (11.22), and so surjective, and the fourth column is (11.21), and so an isomorphism. But the middle morphisms $\mathrm{d}_{u_l} r_l, \mathrm{d}_{v_m} s_m, \mathrm{d}_{w_n} t_n$ are zero by minimality at x, y, z with $u_l = \chi_l^{-1}(x)$, $v_m = \psi_m^{-1}(y)$ and $w_n = \omega_n^{-1}(z)$. Hence by exactness the second column of (11.154) is surjective, and the third column is an isomorphism.

The argument for the first 'if' part shows that g_{ln}, h_{mn} satisfy condition $\boldsymbol{T}$ at u_l, v_m, w_n. This, surjectivity of the second column of (11.154), and Assumption 11.9(a),(b) imply that g_{ln}, h_{mn} are transverse near u_l, v_m. So making $U_{lm} \subseteq U_l$ and $V_{mn} \subseteq V_m$ smaller we can suppose g_{ln}, h_{mn} are transverse.

As the third column of (11.154) is an isomorphism, Definition 11.15(ii) holds at u_l, v_m, so making $U_{lm} \subseteq U_l$, $V_{mn} \subseteq V_m$ smaller again we can suppose Definition 11.15(ii) holds at all $u \in U_{ln}$, $v \in V_{mn}$ with $g_{ln}(u) = h_{mn}(v) \in W_n$. Then $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ are transverse. As we can find such $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ with $x \in \operatorname{Im}\chi_l$ and $y \in \operatorname{Im}\psi_m$ for any x, y, z as above, $\boldsymbol{g}, \boldsymbol{h}$ are transverse by Definition 11.18. This proves the second 'if' part, and completes Theorem 11.22(a).

11.10.2 Proof of Theorem 11.22(b)

We can prove part (b) in a very similar way to part (a) in §11.10.1. We work with $\boldsymbol{g}, x, z$ rather than $\boldsymbol{g}, \boldsymbol{h}, x, y, z$, and instead of (11.154) we use the equation

$$
\begin{CD}
0 @>>> T_x\boldsymbol{X} @>>> T_{u_l}U_l @>{\mathrm{d}_{u_l}r_l}>> D_l|_{u_l} @>>> O_x\boldsymbol{X} @>>> 0 \\
@. @VV{T_x\boldsymbol{g}}V @VV{T_{u_l}g_{ln}}V @V{\hat{g}_{ln}|_{u_l}}VV @VV{O_x\boldsymbol{g}}V \\
0 @>>> T_z\boldsymbol{Z} @>>> T_{w_n}W_n @>{\mathrm{d}_{w_n}t_n}>> F_n|_{w_n} @>>> O_z\boldsymbol{Z} @>>> 0.
\end{CD}
$$

We leave the details to the reader.

11.11 Proof of Theorem 11.25

Work in the situation of Theorem 11.25. Equation (11.26) defines an isomorphism $\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}|_w : K_{\boldsymbol{W}}|_w \longrightarrow e^*(K_{\boldsymbol{X}}) \otimes f^*(K_{\boldsymbol{Y}}) \otimes (g \circ e)^*(K_{\boldsymbol{Z}})^*|_w$ for each $w \in W$. Thus there is a unique map of sets $\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}$ in (11.24) which satisfies (11.26) for all $w \in W$. We must show that this map $\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}$ is an isomorphism of topological line bundles. It is sufficient to do this locally near each w in W.

Fix $w \in W$ with $e(w) = x$ in X, $f(w) = y$ in Y and $g(x) = h(y) = z$ in Z. Let (U_l, D_l, r_l, χ_l), (V_m, E_m, s_m, ψ_m), $(W_n, F_n, t_n, \omega_n)$ be m-Kuranishi neighbourhoods on $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, with $x \in \operatorname{Im}\chi_l \subseteq g^{-1}(\operatorname{Im}\omega_n)$, $y \in \operatorname{Im}\psi_m \subseteq h^{-1}(\operatorname{Im}\omega_n)$ and $z \in \operatorname{Im}\omega_n$, and let

$$
\begin{aligned}
\boldsymbol{g}_{ln} &= (U_{ln}, g_{ln}, \hat{g}_{ln}) : (U_l, D_l, r_l, \chi_l) \longrightarrow (W_n, F_n, t_n, \omega_n), \\
\boldsymbol{h}_{mn} &= (V_{mn}, h_{mn}, \hat{h}_{mn}) : (V_m, E_m, s_m, \psi_m) \longrightarrow (W_n, F_n, t_n, \omega_n),
\end{aligned}
$$

be w-transverse 1-morphisms over $(\operatorname{Im}\chi_l, \boldsymbol{g})$ and $(\operatorname{Im}\psi_m, \boldsymbol{h})$.

Theorem 11.19(b) now gives an m-Kuranishi neighbourhood $(T_k, C_k, q_k, \varphi_k)$ on $\boldsymbol{W}$ with $\operatorname{Im}\varphi_k = e^{-1}(\operatorname{Im}\chi_l) \cap f^{-1}(\operatorname{Im}\psi_m) \subseteq W$, so that $w \in \operatorname{Im}\varphi_k$, and 1-morphisms

$$
\begin{aligned}
\boldsymbol{e}_{kl} &= (T_k, e_{kl}, \hat{e}_{kl}) : (T_k, C_k, q_k, \varphi_k) \longrightarrow (U_l, D_l, r_l, \chi_l), \\
\boldsymbol{f}_{km} &= (T_k, f_{km}, \hat{f}_{km}) : (T_k, C_k, q_k, \varphi_k) \longrightarrow (V_m, E_m, s_m, \psi_m)
\end{aligned}
$$

over $(\operatorname{Im}\varphi_k, \boldsymbol{e})$ and $(\operatorname{Im}\varphi_k, \boldsymbol{f})$ with $\boldsymbol{g}_{ln} \circ \boldsymbol{e}_{kl} = \boldsymbol{h}_{mn} \circ \boldsymbol{f}_{km}$, such that T_k, C_k, q_k and $\boldsymbol{e}_{kl}, \boldsymbol{f}_{km}$ are constructed from $(U_l, D_l, r_l, \chi_l), (V_m, E_m, s_m, \psi_m), (W_n, F_n, t_n, \omega_n)$ and $\boldsymbol{g}_{ln}, \boldsymbol{h}_{mn}$ as in Definition 11.16. Thus

$$
T_k = \dot{U}_{ln} \times_{g_{ln}|_{\dot{U}_{ln}}, W_n, h_{mn}|_{\dot{V}_{mn}}} \dot{V}_{mn}
$$

is a transverse fibre product in $\dot{\mathbf{Man}}_{\boldsymbol{D}}$ for $\dot{U}_{ln} \subseteq U_{ln}$, $\dot{V}_{mn} \subseteq V_{mn}$ open.

Set $t_k = \varphi_k^{-1}(w)$, $u_l = \chi_l^{-1}(x)$, $v_m = \psi_m^{-1}(y)$ and $w_n = \omega_n^{-1}(z)$, and as in §11.9.4, consider the commutative diagram (11.150), with rows complexes and columns exact. In the setting of Definition 10.69, regard (11.150) as a diagram (10.89), a short exact sequence of complexes $E^\bullet, F^\bullet, G^\bullet$, the first, second and third rows of (11.150) respectively, with the third column of (11.150) in degree zero, so that the second and third columns of (11.150) become complexes $B^\bullet_{-1}$ and $B^\bullet_0$. Then (11.25) is the exact sequence (10.90) constructed from (11.150) in Definition 10.69, by the proof of Theorem 11.19(e), so Proposition 10.70 yields

$$\begin{aligned}(-1)^{\operatorname{rank} C_k \dim W_n}\cdot\bigl(\Theta_{E^\bullet}\otimes\Theta_{F^\bullet}^{-1}\otimes\Theta_{G^\bullet}\bigr)\bigl((\Psi_{B^\bullet_{-1}})^{-1}\otimes\Psi_{B^\bullet_0}\bigr)&\\ =(-1)^{\dim O_w\boldsymbol{W}\dim T_z\boldsymbol{Z}}\cdot\Psi_{A^\bullet}.&\end{aligned}\tag{11.157}$$

From Definition 10.66 and Theorem 10.71 we deduce that

$$\Theta_{T_k,C_k,q_k,\varphi_k}|_{t_k}=\Theta_{E^\bullet}:(\det T^*_{t_k}T_k\otimes\det C_k|_{t_k})\longrightarrow K_{\boldsymbol{X}}|_w,\tag{11.158}$$

$$\Theta_{W_n,F_n,t_n,\omega_n}|_{w_n}=\Theta_{G^\bullet}:(\det T^*_{w_n}W_n\otimes\det F_n|_{w_n})\longrightarrow K_{\boldsymbol{Z}}|_z.\tag{11.159}$$

Also $F^\bullet$ in (11.150) is the direct sum of two complexes coming from (U_l,D_l,r_l,χ_l) and (V_m,E_m,s_m,ψ_m). So Proposition 10.68 implies that the following commutes:

$$\begin{array}{ccc}\begin{array}{l}\det(T^*_{u_l}U_l\oplus T^*_{v_m}V_m)\otimes\\ \det(D_l|_{u_l}\oplus E_m|_{v_m})\end{array} & \xrightarrow{\Theta_{F^\bullet}} & \begin{array}{l}\det(T^*_x\boldsymbol{X}\oplus T^*_y\boldsymbol{Y})\otimes\\ \det(O_x\boldsymbol{X}\oplus O_y\boldsymbol{Y})\end{array}\\ \Big\downarrow{\scriptstyle (-1)^{\operatorname{rank} D_l\dim V_m}\cdot I_{T^*_{u_l}U_l,T^*_{v_m}V_m}\otimes I_{D_l|_{u_l},E_m|_{v_m}}} & & \Big\downarrow{\scriptstyle (-1)^{\dim O_x\boldsymbol{X}\dim T_y\boldsymbol{Y}}\cdot I_{T^*_x\boldsymbol{X},T^*_y\boldsymbol{Y}}\otimes I_{O_x\boldsymbol{X},O_y\boldsymbol{Y}}}\\ \begin{array}{l}(\det T^*_{u_l}U_l\otimes\det D_l|_{u_l})\otimes\\ (\det T^*_{v_m}V_m\otimes\det E_m|_{v_m})\end{array} & \xrightarrow{\Theta_{U_l,D_l,r_l,\chi_l}|_{u_l}\otimes\Theta_{V_m,E_m,s_m,\psi_m}|_{v_m}} & K_{\boldsymbol{X}}|_x\otimes K_{\boldsymbol{Y}}|_y.\end{array}\tag{11.160}$$

Combining equations (11.26) and (11.157)–(11.160) implies that

$$\begin{aligned}&(-1)^{\operatorname{rank} C_k\dim W_n+\operatorname{rank} D_l\dim V_m}\cdot\bigl(\Theta_{T_k,C_k,q_k,\varphi_k}|_{t_k}^{-1}\otimes\\ &\Theta_{U_l,D_l,r_l,\chi_l}|_{u_l}\otimes\Theta_{V_m,E_m,s_m,\psi_m}|_{v_m}\otimes\Theta_{W_n,F_n,t_n,\omega_n}|_{w_n}^{-1}\bigr)\\ &\circ\bigl(I_{T^*_{u_l}U_l,T^*_{v_m}V_m}\otimes I_{D_l|_{u_l},E_m|_{v_m}}\bigr)\bigl(\Psi_{B^\bullet_{-1}}\otimes(\Psi_{B^\bullet_0})^{-1}\bigr)=\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}|_w.\end{aligned}\tag{11.161}$$

Now (11.161) is the restriction to $t_k\in q_k^{-1}(0)$ of the equation

$$\begin{aligned}&(-1)^{\operatorname{rank} C_k\dim W_n+\operatorname{rank} D_l\dim V_m}\cdot\bigl(\Theta^{-1}_{T_k,C_k,q_k,\varphi_k}\otimes e_{kl}|^*_{q_k^{-1}(0)}(\Theta_{U_l,D_l,r_l,\chi_l})\\ &\otimes f_{km}|^*_{q_k^{-1}(0)}(\Theta_{V_m,E_m,s_m,\psi_m})\otimes(g_{ln}\circ e_{kl})|^*_{q_k^{-1}(0)}(\Theta^{-1}_{W_n,F_n,t_n,\omega_n})\bigr)\\ &\circ\bigl(I_{e^*_{kl}(T^*U_l),f^*_{km}(T^*V_m)}\otimes I_{e^*_{kl}(D_l),f^*_{km}(E_m)}\bigr)\big|_{q_k^{-1}(0)}\bigl(\Psi_{\tilde B^\bullet_{-1}}\otimes(\Psi_{\tilde B^\bullet_0})^{-1}\bigr)\\ &\qquad=\varphi_k^*(\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}),\end{aligned}\tag{11.162}$$

where $\tilde B^\bullet_{-1},\tilde B^\bullet_0$ are the complexes of topological vector bundles on $q_k^{-1}(0)$ whose fibres at t_k are the second and third columns of (11.150). Here $\Theta_{T_k,C_k,q_k,\varphi_k},\ldots,\Theta_{W_n,F_n,t_n,\omega_n}$ are isomorphisms of topological line bundles by Theorem 10.71, and $I_{e^*_{kl}(T^*U_l),f^*_{km}(T^*V_m)},I_{e^*_{kl}(D_l),f^*_{km}(E_m)}$ are also isomorphisms, and $\Psi_{\tilde B^\bullet_{-1}},\Psi_{\tilde B^\bullet_0}$ are nonvanishing continuous sections of topological line bundles.

Thus (11.162) implies that $\varphi_k^*(\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}})$ is a continuous, nonvanishing section of $\varphi_k^*\bigl((K_{\boldsymbol{W}})^*\otimes e^*(K_{\boldsymbol{X}})\otimes f^*(K_{\boldsymbol{Y}})\otimes(g\circ e)^*(K_{\boldsymbol{Z}})^*\bigr)$ on $q_k^{-1}(0)$. Therefore $\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}$

is a nonvanishing section of $(K_{\boldsymbol{W}})^* \otimes e^*(K_{\boldsymbol{X}}) \otimes f^*(K_{\boldsymbol{Y}}) \otimes (g \circ e)^*(K_{\boldsymbol{Z}})^*$, or equivalently an isomorphism $K_{\boldsymbol{W}} \to e^*(K_{\boldsymbol{X}}) \otimes f^*(K_{\boldsymbol{Y}}) \otimes (g \circ e)^*(K_{\boldsymbol{Z}})^*$, on the open subset $\operatorname{Im}\varphi_k \subseteq W$, as $\varphi_k : q_k^{-1}(0) \to \operatorname{Im}\varphi_k$ is a homeomorphism. Since we can cover W by such open subsets $\operatorname{Im}\varphi_k$, we see that $\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}$ is an isomorphism of topological line bundles, as we have to prove.

Chapter 12

M-homology and M-cohomology (Not written yet.)

Review of 'M-homology' and 'M-cohomology', which are new (co)homology theories $MH_*(X;R), MH^*(X;R)$ of manifolds and orbifolds X, due to the author [44]. They satisfy the Eilenberg–Steenrod axioms, and so are canonically isomorphic to usual (co)homology $H_*(X;R), H^*(X;R)$, e.g. singular homology $H_*^{\rm si}(X;R)$. They are specially designed for forming virtual (co)chains for (m-)Kuranishi spaces, and have very good (co)chain level properties.

Chapter 13

Virtual (co)cycles and (co)chains for (m-)Kuranishi spaces in M-(co)homology (Not written yet.)

We define an additional structure on an (m-)Kuranishi space with corners $\boldsymbol{X}$, and on 1-morphisms $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$, called a *vc-structure.* If $\boldsymbol{X}$ is a compact, oriented (m-)Kuranishi space with corners, Y is a classical manifold, and $\boldsymbol{f}:\boldsymbol{X}\to Y$ is a 1-morphism equipped with a vc-structure, we will define a *virtual chain* $[\boldsymbol{X}]_{\rm virt}$ in M-chains $MC_{{\rm vdim}\,\boldsymbol{X}}(Y;\mathbb{Z})$ (in the m-Kuranishi case) or $MC_{{\rm vdim}\,\boldsymbol{X}}(Y;\mathbb{Q})$ (in the Kuranishi case).

These vc-structures and virtual chains have lots of nice properties, which will be important in applications in symplectic geometry. If $\partial\boldsymbol{X}=\emptyset$ then $\partial[\boldsymbol{X}]_{\rm virt}=0$, so we have a homology class $[[\boldsymbol{X}]_{\rm virt}]$ in M-homology $MH_{{\rm vdim}\,\boldsymbol{X}}(Y;\mathbb{Z})$ or $MH_{{\rm vdim}\,\boldsymbol{X}}(Y;\mathbb{Q})$, the *virtual class.*

Such virtual chain and virtual cycle constructions are important in current approaches to symplectic geometry, such as the work of Fukaya–Oh–Ohta–Ono, Hofer–Wysocki–Zehnder and McDuff–Wehrheim discussed in §7.5 — see Remark 7.14 and Theorem 7.20. The point about our construction is that it will have very good technical properties, which will make defining theories such as Lagrangian Floer cohomology, Fukaya categories, and Symplectic Field Theory, much more convenient.

Chapter 14

Orbifold strata of Kuranishi spaces (Not written yet.)

Chapter 15

Bordism and cobordism for (m-)Kuranishi spaces (Not written yet.)

References for volume II

[1] A. Adem, J. Leida, and Y. Ruan, *Orbifolds and Stringy Topology*, vol. 171, Cambridge Tracts in Math., Cambridge University Press, Cambridge, 2007.

[2] M. Akaho and D. Joyce, *Immersed Lagrangian Floer theory*, J. Differential Geom. 86 (2010), 381–500. arXiv: `0803.0717`.

[3] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, *Compactness results in symplectic field theory*, Geom. Topol. 7 (2003), 799–888. arXiv: `math.SG/0308183`.

[4] G. E. Bredon, *Topology and Geometry*, vol. 139, Graduate Texts in Mathematics, Springer-Verlag, New York, 1993.

[5] W. Chen and Y. Ruan, *Orbifold Gromov–Witten theory*, in: *Orbifolds in mathematics and physics*, ed. by A. Adem, J. Morava, and Y. Ruan, vol. 310, Contemporary Mathematics, A.M.S., Providence, RI, 2002, 25–86. arXiv: `math.AG/0103156`.

[6] T. Ekholm, J. Etnyre, and M. Sullivan, *The contact homology of Legendrian submanifolds in* $\mathbb{R}^{2n+1}$, J. Differential Geom. 71 (2005), 177–305.

[7] Y. Eliashberg, *Symplectic field theory and its applications*, in: *International Congress of Mathematicians. Vol. I*, Eur. Math. Soc., Zürich, 2007, 217–246.

[8] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal. Special Volume, Part II (2000), 560–673. arXiv: `math.SG/0010059`.

[9] K. Fukaya, *Application of Floer homology of Lagrangian submanifolds to symplectic topology*, in: *Morse theoretic methods in nonlinear analysis and in symplectic topology*, vol. 217, NATO Sci. Ser. II Math. Phys. Chem., Springer, Dordrecht, 2006, 231–276.

[10] K. Fukaya, *Cyclic symmetry and adic convergence in Lagrangian Floer theory*, Kyoto J. Math. 50 (2010), 521–590. arXiv: `0907.4219`.

[11] K. Fukaya, *Counting pseudo-holomorphic discs in Calabi–Yau 3-fold*, Tohoku Math. J. 63 (2011), 697–727. arXiv: `0908.0148`.

[12] K. Fukaya, *Floer homology of Lagrangian submanifolds*, arXiv: `1106.4882`, 2013.

[13] K. Fukaya, *Lie groupoid, deformation of unstable curve, and construction of equivariant Kuranishi charts*, arXiv: 1701.02840, 2017.

[14] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Canonical models of filtered A_∞-algebras and Morse complexes*, in: *New perspectives and challenges in symplectic field theory*, vol. 49, CRM Proc. Lecture Notes, A.M.S., Providence, RI, 2009, 201–227. arXiv: 0812.1963.

[15] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer theory — anomaly and obstruction. Parts I & II.* Vol. 46.1 & 46.2, AMS/IP Studies in Advanced Mathematics, A.M.S./International Press, 2009.

[16] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Anchored Lagrangian submanifolds and their Floer theory*, in: *Mirror symmetry and tropical geometry*, vol. 527, Contemporary Mathematics, A.M.S., Providence, RI, 2010, 15–54. arXiv: 0907.2122.

[17] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Floer theory on compact toric manifolds I*, Duke Math. J. 151 (2010), 23–174. arXiv: 0802.1703.

[18] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Floer theory on compact toric manifolds II: bulk deformations*, Selecta Math 17 (2011), 609–711. arXiv: 0810.5654.

[19] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Spectral invariants with bulk, quasimorphisms and Lagrangian Floer theory*, arXiv: 1105.5123, 2011.

[20] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Floer theory on compact toric manifolds: survey*, in: vol. 17, Surv. Differ. Geom., Int. Press, Boston, MA, 2012, 229–298. arXiv: 1011.4044.

[21] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Technical details on Kuranishi structure and virtual fundamental chain*, arXiv: 1209.4410, 2012.

[22] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Displacement of polydisks and Lagrangian Floer theory*, arXiv: 1104.4267, 2013.

[23] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Floer theory over integers: spherically positive symplectic manifolds*, Pure Appl. Math. Q. 9 (2013), 189–289. arXiv: 1105.5124.

[24] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Kuranishi structure, Pseudo-holomorphic curve, and Virtual fundamental chain: Part 1*, arXiv: 1503.07631, 2015.

[25] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Exponential decay estimates and smoothness of the moduli space of pseudoholomorphic curves*, arXiv: 1603.07026, 2016.

[26] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Floer theory and mirror symmetry on compact toric manifolds*, Astérisque 376 (2016). arXiv: 1009.1648.

[27] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Shrinking good coordinate systems associated to Kuranishi structures*, J. Symplectic Geom. 14 (2016), 1295–1310. arXiv: 1405.1755.

[28] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Anti-symplectic involution and Floer cohomology*, Geom. Topol. 21 (2017), 1–106. arXiv: 0912.2646.

[29] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Kuranishi structure, Pseudo-holomorphic curve, and virtual fundamental chain: Part 2*, arXiv: 1704.01848, 2017.

[30] K. Fukaya and K. Ono, *Arnold Conjecture and Gromov–Witten invariant*, Topology 38 (1999), 933–1048.

[31] T. L. Gómez, *Algebraic stacks*, Proc. Indian Acad. Sci. Math. Sci. 111 (2001), 1–31. arXiv: math.AG/9911199.

[32] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), 307–347.

[33] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[34] H. Hofer, *Polyfolds and Fredholm Theory*, arXiv: 1412.4255, 2014.

[35] H. Hofer, K. Wysocki, and E. Zehnder, *A general Fredholm theory I: A splicing-based differential geometry*, J. Eur. Math. Soc. 9 (2007), 841–876. arXiv: math.FA/0612604.

[36] H. Hofer, K. Wysocki, and E. Zehnder, *Integration theory for zero sets of polyfold Fredholm sections*, arXiv: 0711.0781, 2007.

[37] H. Hofer, K. Wysocki, and E. Zehnder, *A general Fredholm theory II: Implicit function theorems*, Geom. Funct. Anal. 18 (2009), 206–293. arXiv: 0705.1310.

[38] H. Hofer, K. Wysocki, and E. Zehnder, *A general Fredholm theory III: Fredholm functors and polyfolds*, Geom. Topol. 13 (2009), 2279–2387. arXiv: 0810.0736.

[39] H. Hofer, K. Wysocki, and E. Zehnder, *Sc-smoothness, retractions and new models for smooth spaces*, Discrete Contin. Dyn. Syst. 28 (2010), 665–788. arXiv: 1002.3381.

[40] H. Hofer, K. Wysocki, and E. Zehnder, *Applications of polyfold theory I: the polyfolds of Gromov–Witten theory*, arXiv: 1107.2097, 2011.

[41] H. Hofer, K. Wysocki, and E. Zehnder, *Polyfold and Fredholm theory I: basic theory in M-polyfolds*, arXiv: 1407.3185, 2014.

[42] D. Joyce, *A new definition of Kuranishi space*, arXiv: 1409.6908, 2014.

[43] D. Joyce, *Kuranishi spaces as a 2-category*, arXiv: 1510.07444, 2015.

[44] D. Joyce, *Some new homology and cohomology theories of manifolds*, arXiv: 1509.05672, 2015.

[45] D. Joyce, *Algebraic Geometry over C^∞-rings*, to appear in Memoirs of the A.M.S., arXiv: 1001.0023, 2016.

[46] M. Kontsevich and Yu. Manin, *Gromov–Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. 164 (1994), 525–562. arXiv: hep-th/9402147.

[47] J. Li and G. Tian, *Comparison of algebraic and symplectic Gromov–Witten invariants*, Asian J. Math. 3 (1999), 689–728. arXiv: alg-geom/9712035.

[48] J. Lurie, *Derived Algebraic Geometry V: Structured spaces*, arXiv: 0905.0459, 2009.

[49] D. McDuff, *Notes on Kuranishi Atlases*, arXiv: 1411.4306, 2015.

[50] D. McDuff, *Strict orbifold atlases and weighted branched manifolds*, arXiv: 1506.05350, 2015.

[51] D. McDuff and D. Salamon, *J-holomorphic curves and quantum cohomology*, vol. 6, University Lecture Series, American Mathematical Society, Providence, RI, 1994.

[52] D. McDuff and K. Wehrheim, *Kuranishi atlases with trivial isotropy - the 2013 state of affairs*, arXiv: 1208.1340, 2013.

[53] D. McDuff and K. Wehrheim, *Smooth Kuranishi atlases with isotropy*, arXiv: 1508.01556, 2015.

[54] D. McDuff and K. Wehrheim, *The fundamental class of smooth Kuranishi atlases with trivial isotropy*, arXiv: 1508.01560, 2015.

[55] D. McDuff and K. Wehrheim, *The topology of Kuranishi atlases*, arXiv: 1508.01844, 2015.

[56] I. Moerdijk, *Orbifolds as groupoids: an introduction*, in: *Orbifolds in Mathematics and Physics*, ed. by A. Adem, J. Morava, and Y. Ruan, vol. 310, Contemporary Mathematics, A.M.S./International Press, Providence, RI, 2002, 205–222. arXiv: math.DG/0203100.

[57] I. Moerdijk and D. A. Pronk, *Orbifolds, sheaves and groupoids*, K-theory 12 (1997), 3–21.

[58] B. Noohi, *Foundations of topological stacks. I*, arXiv: math.AG/0503247, 2005.

[59] Y.-G. Oh and K. Fukaya, *Floer homology in symplectic geometry and in mirror symmetry*, in: *International Congress of Mathematicians. Vol. II*, Eur. Math. Soc., Zürich, 2006, 879–905. arXiv: math.SG/0601568.

[60] J. Pardon, *Contact homology and virtual fundamental cycles*, arXiv: 1508.03873, 2015.

[61] J. Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves*, Geom. Topol. 20 (2016), 779–1034. arXiv: 1309.2370.

[62] P. Seidel, *Fukaya categories and deformations*, in: *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, Higher Ed. Press, Beijing, 2002, 351–360. arXiv: math.SG/0206155.

[63] P. Seidel, *A biased view of symplectic cohomology*, in: *Current developments in mathematics, 2006*, Int. Press, Somerville, MA, 2008, 211–253. arXiv: 0704.2055.

[64] P. Seidel, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[65] B. Siebert, *Algebraic and symplectic Gromov–Witten invariants coincide*, Ann. Inst. Fourier (Grenoble) 49 (1999), 1743–1795. arXiv: math.AG/9804108.

[66] D. I. Spivak, *Derived smooth manifolds*, Duke Mathematical Journal 153 (2010), 55–128. arXiv: 0810.5174.

[67] M.F. Tehrani and K. Fukaya, *Gromov–Witten theory via Kuranishi structures*, arXiv: 1701.07821, 2017.

[68] B. Toën, *Higher and derived stacks: a global overview*, in: vol. 80 part 1, Proc. Symp. Pure Math., A.M.S., 2009, 435–487. arXiv: math.AG/0604504.

[69] B. Toën, *Derived Algebraic Geometry*, EMS Surveys in Mathematical Sciences 1 (2014), 153–240. arXiv: 1401.1044.

[70] K. Wehrheim and C. Woodward, *Quilted Floer cohomology*, Geom. Topol. 14 (2010), 833–902. arXiv: 0905.1370.

[71] D. Yang, *A choice-independent theory of Kuranishi structures and the polyfold–Kuranishi correspondence*, PhD thesis, New York University, 2014, URL: http://webusers.imj-prg.fr/~dingyu.yang/thesis.pdf.

[72] D. Yang, *The polyfold–Kuranishi correspondence I: A choice-independent theory of Kuranishi structures*, arXiv: 1402.7008, 2014.

[73] D. Yang, *Virtual harmony*, arXiv: 1510.06849, 2015.

Glossary of notation, all volumes

Page references are in the form volume-page number. So, for example, II-57 means page 57 of volume II.

$\Gamma(\mathcal{E})$ global sections of a sheaf $\mathcal{E}$, I-230

$\Gamma^\infty(E)$ vector space of smooth sections of a vector bundle E, I-10, I-238

$\Omega_{\boldsymbol{X}} : K_{\partial\boldsymbol{X}} \to N_{\partial\boldsymbol{X}} \otimes i^*_{\boldsymbol{X}}(K_{\boldsymbol{X}})$ isomorphism of canonical line bundles on boundary of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-67, II-76

$\Theta_{V,E,\Gamma,s,\psi} : (\det T^*V \otimes \det E)|_{s^{-1}(0)} \to \bar{\psi}^{-1}(K_{\boldsymbol{X}})$ isomorphism of line bundles from a Kuranishi neighbourhood (V,E,Γ,s,ψ) on a Kuranishi space $\boldsymbol{X}$, II-75

$\Theta_{V,E,s,\psi} : (\det T^*V \otimes \det E)|_{s^{-1}(0)} \to \psi^{-1}(K_{\boldsymbol{X}})$ isomorphism of line bundles from an m-Kuranishi neighbourhood (V,E,s,ψ) on an m-Kuranishi space $\boldsymbol{X}$, II-62

$\Upsilon_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}} : K_{\boldsymbol{W}} \to e^*(K_{\boldsymbol{X}}) \otimes f^*(K_{\boldsymbol{Y}}) \otimes (g \circ e)^*(K_{\boldsymbol{Z}})^*$ isomorphism of canonical bundles on w-transverse fibre product of (m-)Kuranishi spaces, II-96

$\alpha_{g,f,e} : (g \circ f) \circ e \Rightarrow g \circ (f \circ e)$ coherence 2-morphism in weak 2-category, I-224

$\beta_f : f \circ \mathrm{id}_X \Rightarrow f$ coherence 2-morphism in weak 2-category, I-224

$\delta_w^{\boldsymbol{g},\boldsymbol{h}} : T_z\boldsymbol{Z} \to O_w\boldsymbol{W}$ connecting morphism in w-transverse fibre product of (m-) Kuranishi spaces, II-92, II-116

$\gamma_f : \mathrm{id}_Y \circ f \Rightarrow f$ coherence 2-morphism in weak 2-category, I-224

$\gamma_f : N_{\partial X} \to (\partial f)^*(N_{\partial Y})$ isomorphism of normal line bundles of manifolds with corners, II-11

∇ connection on vector bundle $E \to X$ in $\dot{\mathbf{Man}}$, I-38, I-241

$C(X)$ corners $\coprod_{k=0}^{\dim X} C_k(X)$ of a manifold with corners X, I-8

$C(\boldsymbol{X})$ corners $\coprod_{k=0}^{\infty} C_k(\boldsymbol{X})$ of an (m or μ-)Kuranishi space $\boldsymbol{X}$, I-91, I-124, I-161

$C : \dot{\mathbf{Kur}}^{\mathbf{c}} \to \check{\dot{\mathbf{Kur}}}^{\mathbf{c}}$ corner 2-functor on Kuranishi spaces, I-161

$C : \mathbf{Man^c} \to \check{\mathbf{Man}}^{\mathbf{c}}$ corner functor on manifolds with corners, I-9

$C' : \mathbf{Man^c} \to \check{\mathbf{Man}}^{\mathbf{c}}$ second corner functor on manifolds with corners, I-9

$C : \mathbf{m}\dot{\mathbf{Kur}}^{\mathbf{c}} \to \mathbf{m}\check{\dot{\mathbf{Kur}}}^{\mathbf{c}}$ corner 2-functor on m-Kuranishi spaces, I-91

$C : \boldsymbol{\mu}\dot{\mathbf{Kur}}^{\mathbf{c}} \to \boldsymbol{\mu}\check{\dot{\mathbf{Kur}}}^{\mathbf{c}}$ corner functor on μ-Kuranishi spaces, I-124

$C : \dot{\mathbf{Orb}}^{\mathbf{c}} \to \check{\dot{\mathbf{Orb}}}^{\mathbf{c}}$ corner 2-functor on orbifolds with corners, I-178

$C^\infty(X)$ $\mathbb{R}$-algebra of smooth functions $X \to \mathbb{R}$ for a manifold X, I-10, I-233

$C_k(\boldsymbol{X})$ k-corners of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, I-81, I-123, I-157

$C_k(\mathfrak{X})$ k-corners of an orbifold with corners $\mathfrak{X}$, I-178

$C_k : \dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}} \to \dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}}$ k-corner 2-functor on Kuranishi spaces, I-161

$C_k : \mathbf{Man}^{\mathbf{c}}_{\mathbf{si}} \to \mathbf{Man}^{\mathbf{c}}_{\mathbf{si}}$ k-corner functor on manifolds with corners, I-9

$C_k : \mathbf{m}\dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}} \to \mathbf{m}\dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}}$ k-corner 2-functor on m-Kuranishi spaces, I-91

$C_k : \boldsymbol{\mu}\dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}} \to \boldsymbol{\mu}\dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}}$ k-corner functor on μ-Kuranishi spaces, I-124

$C_k : \dot{\mathbf{Orb}}^{\mathbf{c}}_{\mathbf{si}} \to \dot{\mathbf{Orb}}^{\mathbf{c}}_{\mathbf{si}}$ k-corner 2-functor on orbifolds with corners, I-178

$\mathcal{C}^{\mathrm{op}}$ opposite category of category $\mathcal{C}$, I-221

$\mathbf{C^\infty Rings}$ category of C^∞-rings, I-234

$\mathbf{C^\infty Sch^{aff}}$ category of affine C^∞-schemes, I-37, I-236

$\partial : \dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}} \to \dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}}$ boundary 2-functor on Kuranishi spaces, I-161

$\partial : \mathbf{Man}^{\mathbf{c}}_{\mathbf{si}} \to \mathbf{Man}^{\mathbf{c}}_{\mathbf{si}}$ boundary functor on manifolds with corners, I-9

$\partial : \mathbf{m}\dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}} \to \mathbf{m}\dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}}$ boundary 2-functor on m-Kuranishi spaces, I-91

$\partial : \boldsymbol{\mu}\dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}} \to \boldsymbol{\mu}\dot{\mathbf{Kur}}^{\mathbf{c}}_{\mathbf{si}}$ boundary functor on μ-Kuranishi spaces, I-124

$\mathrm{depth}_X\, x$ the codimension k of the corner stratum $S^k(X)$ containing a point x in a manifold with corners X, I-6

$\mathbf{DerMan_{BN}}$ Borisov and Noel's ∞-category of derived manifolds, I-103

$\mathbf{DerMan_{Spi}}$ Spivak's ∞-category of derived manifolds, I-103

$\det(E^\bullet)$ determinant of a complex of vector spaces or vector bundles, II-52

$\mathrm{d}f : TX \to f^*(TY)$ derivative of a smooth map $f : X \to Y$, I-11

${}^b\mathrm{d}f : {}^bTX \to f^*({}^bTY)$ b-derivative of a smooth map $f : X \to Y$ of manifolds with corners, I-12

$\mathbf{KN}$ 2-category of Kuranishi neighbourhoods over manifolds $\mathbf{Man}$, I-142

$\dot{\mathbf{K}}\mathbf{N}$ 2-category of Kuranishi neighbourhoods over $\dot{\mathbf{M}}\mathbf{an}$, I-141

$\mathbf{KN^c}$ 2-category of Kuranishi neighbourhoods over manifolds with corners $\mathbf{Man^c}$, I-142

$\mathbf{KN}_S(X)$ 2-category of Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man}$, I-142

$\dot{\mathbf{K}}\mathbf{N}_S(X)$ 2-category of Kuranishi neighbourhoods over $S \subseteq X$ in $\dot{\mathbf{M}}\mathbf{an}$, I-142

$\mathbf{KN}^{\mathbf{c}}_S(X)$ 2-category of Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man^c}$, I-142

$\mathbf{Kur}$ 2-category of Kuranishi spaces over classical manifolds $\mathbf{Man}$, I-153

$\dot{\mathbf{K}}\mathbf{ur}$ 2-category of Kuranishi spaces over $\dot{\mathbf{M}}\mathbf{an}$, I-151

$\dot{\mathbf{K}}\mathbf{ur}_{\boldsymbol{P}}$ 2-category of Kuranishi spaces over $\dot{\mathbf{M}}\mathbf{an}$, and 1-morphisms with discrete property $\boldsymbol{P}$, I-154

$\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{trG}}$ 2-subcategory of Kuranishi spaces in $\dot{\mathbf{K}}\mathbf{ur}$ with all $G_x\boldsymbol{X} = \{1\}$, I-169

$\dot{\mathbf{K}}\mathbf{ur}_{\mathbf{tr\Gamma}}$ 2-subcategory of Kuranishi spaces in $\dot{\mathbf{K}}\mathbf{ur}$ with all $\Gamma_i = \{1\}$, I-169

$\mathbf{Kur^{ac}}$ 2-category of Kuranishi spaces with a-corners, I-153

$\mathbf{Kur^c}$ 2-category of Kuranishi spaces with corners, I-153

$\check{\dot{\mathbf{K}}}\mathbf{ur^c}$ 2-category of Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, I-161

$\check{\dot{\mathbf{K}}}\mathbf{ur}^{\mathbf{c}}_{\boldsymbol{P}}$ 2-category of Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, and 1-morphisms which are $\boldsymbol{P}$, I-161

$\mathbf{Kur^c_{bn}}$ 2-category of Kuranishi spaces with corners, and b-normal 1-morphisms, I-154

$\mathbf{Kur^c_{in}}$ 2-category of Kuranishi spaces with corners, and interior 1-morphisms, I-154

$\mathbf{Kur^c_{si}}$ 2-category of Kuranishi spaces with corners, and simple 1-morphisms, I-154

$\check{\dot{\mathbf{K}}}\mathbf{ur^c_{si}}$ 2-category of Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, and simple 1-morphisms, I-161

$\mathbf{Kur^c_{st}}$ 2-category of Kuranishi spaces with corners, and strongly smooth 1-morphisms, I-154

$\mathbf{Kur^c_{st,bn}}$ 2-category of Kuranishi spaces with corners, and strongly smooth b-normal 1-morphisms, I-154

$\mathbf{Kur^c_{st,in}}$ 2-category of Kuranishi spaces with corners, and strongly smooth interior 1-morphisms, I-154

$\mathbf{Kur^{c}_{we}}$ 2-category of Kuranishi spaces with corners and weakly smooth 1-morphisms, I-153

$\dot{\mathbf{K}}\mathbf{ur^{c}}$ 2-category of Kuranishi spaces with corners associated to $\dot{\mathbf{M}}\mathbf{an^{c}}$, I-157

$\dot{\mathbf{K}}\mathbf{ur^{c}_{si}}$ 2-category of Kuranishi spaces with corners associated to $\dot{\mathbf{M}}\mathbf{an^{c}}$, and simple 1-morphisms, I-157

$\mathbf{Kur^{c,ac}}$ 2-category of Kuranishi spaces with corners and a-corners, I-153

$\mathbf{Kur^{c,ac}_{bn}}$ 2-category of Kuranishi spaces with corners and a-corners, and b-normal 1-morphisms, I-155

$\mathbf{Kur^{c,ac}_{in}}$ 2-category of Kuranishi spaces with corners and a-corners, and interior 1-morphisms, I-155

$\mathbf{Kur^{c,ac}_{si}}$ 2-category of Kuranishi spaces with corners and a-corners, and simple 1-morphisms, I-155

$\mathbf{Kur^{c,ac}_{st}}$ 2-category of Kuranishi spaces with corners and a-corners, and strongly a-smooth 1-morphisms, I-155

$\mathbf{Kur^{c,ac}_{st,bn}}$ 2-category of Kuranishi spaces with corners and a-corners, and strongly a-smooth b-normal 1-morphisms, I-155

$\mathbf{Kur^{c,ac}_{st,in}}$ 2-category of Kuranishi spaces with corners and a-corners, and strongly a-smooth interior 1-morphisms, I-155

$\mathbf{Kur^{gc}}$ 2-category of Kuranishi spaces with g-corners, I-153

$\mathbf{Kur^{gc}_{bn}}$ 2-category of Kuranishi spaces with g-corners, and b-normal 1-morphisms, I-155

$\mathbf{Kur^{gc}_{in}}$ 2-category of Kuranishi spaces with g-corners, and interior 1-morphisms, I-155

$\mathbf{Kur^{gc}_{si}}$ 2-category of Kuranishi spaces with g-corners, and simple 1-morphisms, I-155

K_X canonical bundle of a 'manifold' X in $\dot{\mathbf{M}}\mathbf{an}$, II-10

$K_{\boldsymbol{X}}$ canonical bundle of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-62, II-74

${}^bK_{\boldsymbol{X}}$ b-canonical bundle of an (m- or μ-)Kuranishi space with corners $\boldsymbol{X}$, II-66

$\mathbf{Man}$ category of classical manifolds, I-7

$\dot{\mathbf{M}}\mathbf{an}$ category of 'manifolds' satisfying Assumptions 3.1–3.7, I-31

$\ddot{\mathbf{M}}\mathbf{an}$ another category of 'manifolds' satisfying Assumptions 3.1–3.7, I-46

$\mathbf{Man^{ac}}$ category of manifolds with a-corners, I-18

$\mathbf{Man^{ac}_{bn}}$ category of manifolds with a-corners and b-normal maps, I-18

$\mathbf{Man^{ac}_{in}}$ category of manifolds with a-corners and interior maps, I-18

$\mathbf{Man^{ac}_{st}}$ category of manifolds with a-corners and strongly a-smooth maps, I-18

$\mathbf{Man^{ac}_{st,bn}}$ category of manifolds with a-corners and strongly a-smooth b-normal maps, I-18

$\mathbf{Man^{ac}_{st,in}}$ category of manifolds with a-corners and strongly a-smooth interior maps, I-18

$\mathbf{Man^{b}}$ category of manifolds with boundary, I-7

$\mathbf{Man^{b}_{in}}$ category of manifolds with boundary and interior maps, I-7

$\mathbf{Man^{b}_{si}}$ category of manifolds with boundary and simple maps, I-7

$\mathbf{Man^{c}}$ category of manifolds with corners, I-5

$\dot{\mathbf{M}}\mathbf{an^{c}}$ category of ‘manifolds with corners’ satisfying Assumption 3.22, I-47

$\check{\dot{\mathbf{M}}}\mathbf{an^{c}}$ category of ‘manifolds with corners’ of mixed dimension, I-48

$\check{\mathbf{M}}\mathbf{an^{c}}$ category of manifolds with corners of mixed dimension, I-8

$\mathbf{Man^{c}_{bn}}$ category of manifolds with corners and b-normal maps, I-5

$\mathbf{Man^{c}_{in}}$ category of manifolds with corners and interior maps, I-5

$\check{\mathbf{M}}\mathbf{an^{c}_{in}}$ category of manifolds with corners of mixed dimension and interior maps, I-8

$\mathbf{Man^{c}_{si}}$ category of manifolds with corners and simple maps, I-5

$\check{\dot{\mathbf{M}}}\mathbf{an^{c}_{si}}$ category of ‘manifolds with corners’ of mixed dimension, and simple morphisms, I-48

$\mathbf{Man^{c}_{st}}$ category of manifolds with corners and strongly smooth maps, I-5

$\check{\mathbf{M}}\mathbf{an^{c}_{st}}$ category of manifolds with corners of mixed dimension and strongly smooth maps, I-8

$\mathbf{Man^{c}_{st,bn}}$ category of manifolds with corners and strongly smooth b-normal maps, I-5

$\mathbf{Man^{c}_{st,in}}$ category of manifolds with corners and strongly smooth interior maps, I-5

$\mathbf{Man^{c}_{we}}$ category of manifolds with corners and weakly smooth maps, I-5

$\mathbf{Man^{c,ac}}$ category of manifolds with corners and a-corners, I-18

$\mathbf{Man^{c,ac}_{bn}}$ category of manifolds with corners and a-corners, and b-normal maps, I-19

$\mathbf{Man^{c,ac}_{in}}$ category of manifolds with corners and a-corners, and interior maps, I-18

$\mathbf{Man^{c,ac}_{si}}$ category of manifolds with corners and a-corners, and simple maps, I-19

$\mathbf{Man^{c,ac}_{in}}$ category of manifolds with corners and a-corners, and strongly a-smooth maps, I-19

$\mathbf{Man^{c,ac}_{st,bn}}$ category of manifolds with corners and a-corners, and strongly a-smooth b-normal maps, I-19

$\mathbf{Man^{c,ac}_{st,in}}$ category of manifolds with corners and a-corners, and strongly a-smooth interior maps, I-19

$\mathbf{Man^{gc}}$ category of manifolds with g-corners, I-16

$\mathbf{Man^{gc}_{in}}$ category of manifolds with g-corners and interior maps, I-16

$\mathbf{mKN}$ 2-category of m-Kuranishi neighbourhoods over manifolds $\mathbf{Man}$, I-59

$\mathbf{m\dot{K}N}$ 2-category of m-Kuranishi neighbourhoods over $\mathbf{\dot{M}an}$, I-58

$\mathbf{mKN^c}$ 2-category of m-Kuranishi neighbourhoods over manifolds with corners $\mathbf{Man^c}$, I-59

$\mathbf{mKN}_S(X)$ 2-category of m-Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man}$, I-59

$\mathbf{m\dot{K}N}_S(X)$ 2-category of m-Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{\dot{M}an}$, I-58

$\mathbf{mKN}^{\mathbf{c}}_S(X)$ 2-category of m-Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man^c}$, I-59

$\mathbf{mKur}$ 2-category of m-Kuranishi spaces over classical manifolds $\mathbf{Man}$, I-72

$\mathbf{m\dot{K}ur}$ 2-category of m-Kuranishi spaces over $\mathbf{\dot{M}an}$, I-72

$\mathbf{m\dot{K}ur}_{\boldsymbol{P}}$ 2-category of m-Kuranishi spaces over $\mathbf{\dot{M}an}$, and 1-morphisms with discrete property $\boldsymbol{P}$, I-78

$\mathbf{mKur^{ac}}$ 2-category of m-Kuranishi spaces with a-corners, I-72

$\mathbf{mKur^{ac}_{bn}}$ 2-category of m-Kuranishi spaces with a-corners, and b-normal 1-morphisms, I-79

$\mathbf{mKur^{ac}_{in}}$ 2-category of m-Kuranishi spaces with a-corners, and interior 1-morphisms, I-79

$\mathbf{mKur^{ac}_{si}}$ 2-category of m-Kuranishi spaces with a-corners, and simple 1-morphisms, I-79

$\mathbf{mKur^{ac}_{st}}$ 2-category of m-Kuranishi spaces with a-corners, and strongly a-smooth 1-morphisms, I-79

$\mathbf{mKur^{ac}_{st,bn}}$ 2-category of m-Kuranishi spaces with a-corners, and strongly a-smooth b-normal 1-morphisms, I-79

$\mathbf{mKur^{ac}_{st,in}}$ 2-category of m-Kuranishi spaces with a-corners, and strongly a-smooth interior 1-morphisms, I-79

$\mathbf{mKur^{b}}$ 2-category of m-Kuranishi spaces with boundary, I-93

$\mathbf{mKur^{b}_{in}}$ 2-category of m-Kuranishi spaces with boundary, and interior 1-morphisms, I-93

$\mathbf{mKur^{b}_{si}}$ 2-category of m-Kuranishi spaces with boundary, and simple 1-morphisms, I-93

$\mathbf{mKur^{c}}$ 2-category of m-Kuranishi spaces with corners, I-72

$\mathbf{m\check{\dot{K}}ur^{c}}$ 2-category of m-Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, I-87

$\mathbf{m\check{\dot{K}}ur^{c}_{\boldsymbol{P}}}$ 2-category of m-Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, and 1-morphisms which are $\boldsymbol{P}$, I-91

$\mathbf{mKur^{c}_{bn}}$ 2-category of m-Kuranishi spaces with corners, and b-normal 1-morphisms, I-78

$\mathbf{mKur^{c}_{in}}$ 2-category of m-Kuranishi spaces with corners, and interior 1-morphisms, I-78

$\mathbf{mKur^{c}_{si}}$ 2-category of m-Kuranishi spaces with corners, and simple 1-morphisms, I-78

$\mathbf{m\check{\dot{K}}ur^{c}_{si}}$ 2-category of m-Kuranishi spaces with corners over $\dot{\mathbf{M}}\mathbf{an^c}$ of mixed dimension, and simple 1-morphisms, I-87

$\mathbf{mKur^{c}_{st}}$ 2-category of m-Kuranishi spaces with corners, and strongly smooth 1-morphisms, I-78

$\mathbf{mKur^{c}_{st,bn}}$ 2-category of m-Kuranishi spaces with corners, and strongly smooth b-normal 1-morphisms, I-78

$\mathbf{mKur^{c}_{st,in}}$ 2-category of m-Kuranishi spaces with corners, and strongly smooth interior 1-morphisms, I-78

$\mathbf{mKur^{c}_{we}}$ 2-category of m-Kuranishi spaces with corners and weakly smooth 1-morphisms, I-72

$\dot{\mathbf{mKur}}^{\mathbf{c}}$ 2-category of m-Kuranishi spaces with corners associated to $\dot{\mathbf{Man}}^{\mathbf{c}}$, I-81

$\mathbf{mKur}^{\mathbf{c,ac}}$ 2-category of m-Kuranishi spaces with corners and a-corners, I-72

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{bn}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and b-normal 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{in}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and interior 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{si}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and simple 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{st}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and strongly a-smooth 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{st,bn}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and strongly a-smooth b-normal 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{c,ac}}_{\mathbf{st,in}}$ 2-category of m-Kuranishi spaces with corners and a-corners, and strongly a-smooth interior 1-morphisms, I-79

$\dot{\mathbf{mKur}}^{\mathbf{c}}_{\mathbf{si}}$ 2-category of m-Kuranishi spaces with corners associated to $\dot{\mathbf{Man}}^{\mathbf{c}}$, and simple 1-morphisms, I-81

$\mathbf{mKur}^{\mathbf{gc}}$ 2-category of m-Kuranishi spaces with g-corners, I-72

$\mathbf{mKur}^{\mathbf{gc}}_{\mathbf{bn}}$ 2-category of m-Kuranishi spaces with g-corners, and b-normal 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{gc}}_{\mathbf{in}}$ 2-category of m-Kuranishi spaces with g-corners, and interior 1-morphisms, I-79

$\mathbf{mKur}^{\mathbf{gc}}_{\mathbf{si}}$ 2-category of m-Kuranishi spaces with g-corners, and simple 1-morphisms, I-79

$\boldsymbol{\mu}\mathbf{KN}$ category of μ-Kuranishi neighbourhoods over manifolds **Man**, I-111

$\boldsymbol{\mu}\dot{\mathbf{KN}}$ category of μ-Kuranishi neighbourhoods over $\dot{\mathbf{Man}}$, I-110

$\boldsymbol{\mu}\mathbf{KN}^{\mathbf{c}}$ category of μ-Kuranishi neighbourhoods over manifolds with corners $\mathbf{Man}^{\mathbf{c}}$, I-111

$\boldsymbol{\mu}\mathbf{KN}_S(X)$ category of μ-Kuranishi neighbourhoods over $S \subseteq X$ in **Man**, I-111

$\boldsymbol{\mu}\dot{\mathbf{KN}}_S(X)$ category of μ-Kuranishi neighbourhoods over $S \subseteq X$ in $\dot{\mathbf{Man}}$, I-110

$\boldsymbol{\mu}\mathbf{KN}^{\mathbf{c}}_S(X)$ category of μ-Kuranishi neighbourhoods over $S \subseteq X$ in $\mathbf{Man}^{\mathbf{c}}$, I-111

$\boldsymbol{\mu}\mathbf{Kur}$ category of μ-Kuranishi spaces over classical manifolds **Man**, I-117

$\boldsymbol{\mu}\dot{\mathbf{Kur}}$ category of μ-Kuranishi spaces over $\dot{\mathbf{Man}}$, I-116

$\boldsymbol{\mu}\mathbf{Kur^c_{st}}$ category of μ-Kuranishi spaces with corners, and strongly smooth morphisms, I-119

$\boldsymbol{\mu}\mathbf{Kur^c_{st,bn}}$ category of μ-Kuranishi spaces with corners, and strongly smooth b-normal morphisms, I-119

$\boldsymbol{\mu}\mathbf{Kur^c_{st,in}}$ category of μ-Kuranishi spaces with corners, and strongly smooth interior morphisms, I-119

$\boldsymbol{\mu}\mathbf{Kur^c_{we}}$ category of μ-Kuranishi spaces with corners and weakly smooth morphisms, I-117

$\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c}$ category of μ-Kuranishi spaces with corners associated to $\dot{\mathbf{M}}\mathbf{an^c}$, I-122

$\boldsymbol{\mu}\mathbf{Kur^{c,ac}}$ category of μ-Kuranishi spaces with corners and a-corners, I-117

$\boldsymbol{\mu}\mathbf{Kur^{c,ac}_{bn}}$ category of μ-Kuranishi spaces with corners and a-corners, and b-normal morphisms, I-120

$\boldsymbol{\mu}\mathbf{Kur^{c,ac}_{in}}$ category of μ-Kuranishi spaces with corners and a-corners, and interior morphisms, I-120

$\boldsymbol{\mu}\mathbf{Kur^{c,ac}_{si}}$ category of μ-Kuranishi spaces with corners and a-corners, and simple morphisms, I-120

$\boldsymbol{\mu}\mathbf{Kur^{c,ac}_{st}}$ category of μ-Kuranishi spaces with corners and a-corners, and strongly a-smooth morphisms, I-120

$\boldsymbol{\mu}\mathbf{Kur^{c,ac}_{st,bn}}$ category of μ-Kuranishi spaces with corners and a-corners, and strongly a-smooth b-normal morphisms, I-120

$\boldsymbol{\mu}\mathbf{Kur^{c,ac}_{st,in}}$ category of μ-Kuranishi spaces with corners and a-corners, and strongly a-smooth interior morphisms, I-120

$\boldsymbol{\mu}\dot{\mathbf{K}}\mathbf{ur^c_{si}}$ category of μ-Kuranishi spaces with corners associated to $\dot{\mathbf{M}}\mathbf{an^c}$, and simple morphisms, I-122

$\boldsymbol{\mu}\mathbf{Kur^{gc}}$ category of μ-Kuranishi spaces with g-corners, I-117

$\boldsymbol{\mu}\mathbf{Kur^{gc}_{bn}}$ category of μ-Kuranishi spaces with g-corners, and b-normal morphisms, I-120

$\boldsymbol{\mu}\mathbf{Kur^{gc}_{in}}$ category of μ-Kuranishi spaces with g-corners, and interior morphisms, I-120

$\boldsymbol{\mu}\mathbf{Kur^{gc}_{si}}$ category of μ-Kuranishi spaces with g-corners, and simple morphisms, I-120

$\tilde M_x f : \tilde M_x X \to \tilde M_y Y$ monoid morphism for morphism $f : X \to Y$ in $\mathbf{Man^c_{in}}$, I-14

$\tilde M_x X$ monoid at a point x in a manifold with corners X, I-14

$N_{C_k(X)}$ normal bundle of k-corners $C_k(X)$ in a manifold with corners X, I-12

${}^bN_{C_k(X)}$ b-normal bundle of k-corners $C_k(X)$ in a manifold with corners X, I-12

$N_{\partial X}$ normal line bundle of boundary ∂X in a manifold with corners X, I-12

$\tilde N_x f : \tilde N_x X \to \tilde N_y Y$ stratum normal map for manifolds with corners X, I-13

${}^b\tilde N_x f : {}^b\tilde N_x X \to {}^b\tilde N_y Y$ stratum b-normal map for morphism $f : X \to Y$ in $\mathbf{Man^c_{in}}$, I-14

$\tilde N_x X$ stratum normal space at x in a manifold with corners X, I-13

${}^b\tilde N_x X$ stratum b-normal space at x in a manifold with corners X, I-13

$\mathbf{Orb}_{\rm CR}$ Chen–Ruan's category of orbifolds, I-171

$\mathbf{Orb}_{C^\infty{\rm Sta}}$ 2-category of orbifolds as stacks on site $\mathbf{C^\infty Sch}$, I-172

$\mathbf{Orb}_{\rm Kur}$ 2-category of orbifolds as examples of Kuranishi spaces, I-175

$\mathbf{Orb}_{\rm Le}$ Lerman's 2-category of orbifolds, I-171

$\mathbf{Orb}_{\rm ManSta}$ 2-category of orbifolds as stacks on site $\mathbf{Man}$, I-171

$\mathbf{Orb}_{\rm MP}$ Moerdijk–Pronk's category of orbifolds, I-171

$\mathbf{Orb}_{\rm Pr}$ Pronk's 2-category of orbifolds, I-171

$\mathbf{Orb}_{\rm ST}$ Satake–Thurston's category of orbifolds, I-171

$\dot{\mathbf{O}}\mathbf{rb}$ 2-category of Kuranishi orbifolds associated to $\dot{\mathbf{M}}\mathbf{an}$, I-175

$\mathbf{Orb^{ac}}$ 2-category of orbifolds with a-corners, I-175

$\dot{\mathbf{O}}\mathbf{rb^c}$ 2-category of orbifolds with corners associated to $\dot{\mathbf{M}}\mathbf{an^c}$, I-178

$\mathbf{Orb^{c,ac}}$ 2-category of orbifolds with corners and a-corners, I-175

$\dot{\mathbf{O}}\mathbf{rb^c_{si}}$ 2-category of orbifolds with corners associated to $\dot{\mathbf{M}}\mathbf{an^c}$, and simple 1-morphisms, I-178

$\mathbf{Orb^c_{we}}$ 2-category of orbifolds with corners, and weakly smooth 1-morphisms, I-175

$\mathbf{Orb^c_{we}}$ 2-category of orbifolds with corners, I-175

$\mathbf{Orb^{eff}_{sur}}$ 2-category of effective orbifolds with 1-morphisms surjective on isotropy groups, I-35

$\mathbf{Orb^{gc}}$ 2-category of orbifolds with g-corners, I-175

$\mathcal{O}_X$ structure sheaf of object X in $\dot{\mathbf{M}}\mathbf{an}$, I-37, I-235

$O_x\boldsymbol{f}:O_x\boldsymbol{X}\rightarrow O_y\boldsymbol{Y}$ obstruction map of (m- or μ-)Kuranishi spaces, II-17, II-21, II-22

${}^bO_x\boldsymbol{f}:{}^bO_x\boldsymbol{X}\rightarrow {}^bO_y\boldsymbol{Y}$ b-obstruction map of (m- or μ-)Kuranishi spaces with corners, II-19

$\tilde O_x\boldsymbol{f}:\tilde O_x\boldsymbol{X}\rightarrow \tilde O_y\boldsymbol{Y}$ stratum obstruction map of (m- or μ-)Kuranishi spaces with corners, II-19

$O_x\boldsymbol{X}$ obstruction space at x of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-16, II-21

$O_x^*\boldsymbol{X}$ coobstruction space at x of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-16, II-21

${}^bO_x\boldsymbol{X}$ b-obstruction space at x of an (m- or μ-)Kuranishi space with corners $\boldsymbol{X}$, II-19

$\tilde O_x\boldsymbol{X}$ stratum obstruction space at x of an (m- or μ-)Kuranishi space with corners $\boldsymbol{X}$, II-19

$\Phi_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ 1-morphism or coordinate change of Kuranishi neighbourhoods, I-136

$\Phi_{ij}:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ 1-morphism or coordinate change of m-Kuranishi neighbourhoods, I-55

$[\Phi_{ij}]:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ morphism or coordinate change of μ-Kuranishi neighbourhoods, I-109

$Q_xf:Q_xX\rightarrow Q_yY$ quasi-tangent map of morphism $f:X\rightarrow Y$ in $\dot{\mathbf{Man}}$, II-13

$Q_x\boldsymbol{f}:Q_x\boldsymbol{X}\rightarrow Q_y\boldsymbol{Y}$ quasi-tangent map of (m- or μ-)Kuranishi spaces, II-24, II-28

Q_xX quasi-tangent space at x of 'manifold' X in $\dot{\mathbf{Man}}$, II-13

$Q_x\boldsymbol{X}$ quasi-tangent space at x of an (m- or μ-)Kuranishi space $\boldsymbol{X}$, II-24, II-28

$S^l(X)$ depth l stratum of a manifold with corners X, I-6

$Tf:TX\rightarrow TY$ derivative of a smooth map $f:X\rightarrow Y$, I-11

${}^bTf:{}^bTX\rightarrow {}^bTY$ b-derivative of an interior map $f:X\rightarrow Y$ of manifolds with corners, I-12

$\mathcal{T}_fY$ tangent sheaf of morphism $f:X\rightarrow Y$ in $\dot{\mathbf{Man}}$, I-38, I-251

$\mathcal{T}g:\mathcal{T}_fY\rightarrow\mathcal{T}_{g\circ f}Z$ morphism of tangent sheaves for $f:X\rightarrow Y$, $g:Y\rightarrow Z$ in $\dot{\mathbf{Man}}$, I-38, I-254

Top category of topological spaces, I-31

TX tangent bundle of a manifold X, I-11

T^*X cotangent bundle of a manifold X, I-11

$\mathcal{T}X$ tangent sheaf of 'manifold' X in $\dot{\mathbf{Man}}$, I-38, I-251

$\mathcal{T}^*X$ cotangent sheaf of 'manifold' X in $\dot{\mathbf{Man}}$, I-37, I-240

${}^bTX$ b-tangent bundle of a manifold with corners X, I-11

${}^bT^*X$ b-cotangent bundle of a manifold X, I-11

$T_xf : T_xX \to T_yY$ tangent map of morphism $f : X \to Y$ in $\dot{\mathbf{Man}}$, II-4

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