Kuranishi (co)homology: a new tool in symplectic geometry. I. Overview

Dominic Joyce
Oxford University, UK

work in progress
based on
arXiv:0707.3572 v5, 10/08
summarized in
arXiv:0710.5634 v2, 10/08

These slides available at

www.maths.ox.ac.uk/∼joyce/talks.html

I.1. Motivation from Gromov–Witten theory

Let (M, ω) be a compact symplectic $2n$-manifold, J an almost complex structure compatible with ω, and $\beta \in H_2(M; \mathbb{Z})$. Then we can form moduli spaces $\bar{\mathcal{M}}_{g,m}(M, J, \beta)$ of iso. classes $[\Sigma, \vec{z}, u]$ of triples $(\Sigma, \vec{z}, u)$, where Σ is a genus g Riemann surface, possibly with nodal singularities, $\vec{z} = (z_1, \ldots, z_m)$ are distinct nonsingular marked points in Σ, and $u : \Sigma \to M$ is J-holomorphic with $u_*([\Sigma]) = \beta$ in $H_2(M; \mathbb{Z})$. Require $(\Sigma, \vec{z}, u)$ to be *stable*, i.e. $\mathrm{Aut}(\Sigma, \vec{z}, u)$ is finite.

Then $\bar{\mathcal{M}}_{g,m}(M,J,\beta)$ is a compact, Hausdorff topological space. If the deformation theory of $(\Sigma,\vec{z},u)$ is *unobstructed* for all $[\Sigma,\vec{z},u]$, expect $\bar{\mathcal{M}}_{g,m}(M,J,\beta)$ is an oriented orbifold without boundary of dimension

$$2\bigl(c_1(M)\cdot\beta+(n-3)(1-g)+m\bigr),$$

with smooth maps $\mathrm{ev}_1,\ldots,\mathrm{ev}_m:\bar{\mathcal{M}}_{g,m}(M,J,\beta)\rightarrow M$ by projection to $u(z_1),\ldots,u(z_m)$. Define *Gromov –Witten invariants*

$$GW_{g,m}(\beta;\alpha_1,\ldots,\alpha_m)=\int_{\bar{\mathcal{M}}_{g,m}(M,J,\beta)}\mathrm{ev}_1^*(\alpha_1)\cup\cdots\cup\mathrm{ev}_m^*(\alpha_m),$$

for $\alpha_1,\ldots,\alpha_m\in H^*(M)$ with $\deg\alpha_1+\cdots+\deg\alpha_m=\dim\bar{\mathcal{M}}_{g,m}(M,J,\beta)$.

If L is a Lagrangian in (M,ω) and $\beta \in H_2(M,L;\mathbb{Z})$ then we can form moduli spaces $\bar{\mathcal{M}}_{g,h,l,m}(M,L,J,\beta)$ of stable J-holomorphic curves Σ in M *with boundary* in L, with genus g, h boundary circles, l boundary and m interior marked points. If no obstructions then $\bar{\mathcal{M}}_{g,h,l,m}(M,L,J,\beta)$ would be a compact orbifold *with boundary and corners*.
The spaces $\bar{\mathcal{M}}_{0,1,k+1,0}(M,L,J,\beta)$ are used to define *Lagrangian Floer cohomology* of L. There should exist *open Gromov–Witten invariants* 'counting' curves in $\bar{\mathcal{M}}_{g,h,l,m}(M,L,J,\beta)$, under some conditions.

However, the $\bar{\mathcal{M}}_{g,m}(M,J,\beta)$ and $\bar{\mathcal{M}}_{g,h,l,m}(M,L,J,\beta)$ are usually obstructed. Even for generic J, always have branched covers of curves, curves with nodal singularities and constant components, etc., which cause obstructions. So have to deal with singularities. Often the actual dimension is larger than the expected ('virtual') dimension. One way to deal with this is to define a geometric structure called a *Kuranishi structure* on the moduli space, encoding the obstructions, making it into a *Kuranishi space*.

One then makes an abstract perturbation of the Kuranishi space (morally over $\mathbb{Q}$, not $\mathbb{Z}$) to turn it into a smooth manifold or orbifold of the expected dimension, and defines a *virtual chain* or *virtual cycle* in singular homology, which is then used to define G–W invariants, etc. For closed G–W invariants, Kuranishi spaces without boundaries, this works OK, pretty much. For Lagrangian Floer cohomology, open G–W invariants, Kuranishi spaces with boundaries and corners, it is horribly complex and technical, because of issues of compatibility of perturbations at the boundaries.

Government health warning:
Virtual cycle/chain constructions for curve moduli spaces can be habit-forming. Everyone invents their own.

Siebert 1996
Behrend and Fantechi 1997
Li and Tian 1998
Liu and Tian 1998
Fukaya and Ono 1999
Ruan 1999
McDuff 1999
Hofer et al. 2005 (polyfolds)
Fukaya, Oh, Ohta, Ono 2000,6
Lu and Tian 2006
Joyce 2007-8
...?

I.2. A new idea: Kuranishi (co)homology and Kuranishi (co)bordism

Here is my approach to virtual cycles. Let Y be an orbifold and R a $\mathbb{Q}$-algebra. I define *Kuranishi homology* $KH_*(Y;R)$, a homology theory of Y with coefficients in R, with chain complex $KC_*(Y;R)$ roughly speaking spanned by $[X,\boldsymbol{f}]$, for X a compact, oriented Kuranishi space with boundary and corners, and $\boldsymbol{f}: X \to Y$ a strongly smooth map. Actually, chains are spanned by $[X,\boldsymbol{f},\boldsymbol{G}]$ for $\boldsymbol{G}$ some extra *gauge-fixing data*. The boundary is

$$\partial : [X,\boldsymbol{f},\boldsymbol{G}] \mapsto [\partial X, \boldsymbol{f}|_{\partial X}, \boldsymbol{G}|_{\partial X}].$$

There is also a Poincaré dual theory of *Kuranishi cohomology* $KH^*(Y;R)$ with cochains $KC^*(Y;R)$ spanned by $[X, f, C]$, for f a strong submersion (weakly submersive), (X, f) co-oriented, and C co-gauge-fixing data.

The main result of the theory so far is that $KH_*(Y;R)$ is isomorphic to singular homology $H_*^{\mathrm{si}}(Y;R)$, and $KH^*(Y;R)$ to compactly-supported cohomology $H^*_{\mathrm{cs}}(Y;R)$.

I also define *effective Kuranishi* (*co*)-*homology* $KH_*^{\mathrm{ef}}(Y;R), KH^*_{\mathrm{ec}}(Y;R)$ for R a commutative ring, e.g. $R=\mathbb{Z}$, giving $H_*^{\mathrm{si}}(Y;R), H^*_{\mathrm{cs}}(Y;R)$. But this is less useful, as must restrict the K. spaces allowed in (co)chains.

As well as Kuranishi (co)homology, I define several different kinds of *Kuranishi (co)bordism*. The simplest is *Kuranishi bordism* $KB_*(Y;R)$ for Y an orbifold and R a commutative ring, spanned over R by $[X,\boldsymbol{f}]$ for X a compact oriented Kuranishi space *without boundary*, and $\boldsymbol{f}: X \rightarrow Y$ strongly smooth. We include the relation that if W is a compact oriented Kuranishi space with boundary but without corners, and $\boldsymbol{e}: W \rightarrow Y$ is strongly smooth, then $[\partial W, \boldsymbol{e}|_{\partial W}] = 0$ in $KB_*(Y;R)$.

Similarly, *Kuranishi cobordism* $KB^*(Y;R)$ is spanned over R by $[X, \boldsymbol{f}]$ for X a compact Kuranishi space without boundary, and $\boldsymbol{f} : X \to Y$ a cooriented strong submersion. *Almost complex Kuranishi (co)bordism* $KB_*^{\mathrm{ac}}, KB_{\mathrm{ca}}^*(Y;R)$ are variants spanned by $[X, (\boldsymbol{J}, \boldsymbol{K}), \boldsymbol{f}]$, for $X, \boldsymbol{f}$ as above and $(\boldsymbol{J}, \boldsymbol{K})$ an *almost complex structure* on X (similar to 'stably almost complex').

These (co)bordism theories are a natural context for closed Gromov–Witten theory, since $\bar{\mathcal{M}}_{g,m}(M, J, \beta)$ is oriented without boundary and has an almost complex structure.

So we can use Kuranishi (co)homology and (co)chains as a substitute for singular homology and chains, or for compactly-supported cohomology. This has huge advantages in Gromov–Witten theory.

In the closed case, $\big(\bar{\mathcal{M}}_{g,m}(M,J,\beta)$, $\mathbf{ev}_1\times\cdots\times\mathbf{ev}_m\big)$ defines a G–W type invariant directly in $KH_*(M^m;\mathbb{Q})$, without passing to virtual cycles. The isomorphism $KH_*(M^m;\mathbb{Q})\cong H^{\mathrm{si}}_*(M^m;\mathbb{Q})$ identifies these with Fukaya–Ono's G–W invariants.

I also have a conjectural approach to the Gopakumar-Vafa integrality conjecture using Kuranishi bordism and effective Kuranishi homology.

In the open case, for Lagrangian Floer cohomology, using Kuranishi chains instead of singular chains we can simplify [FOOO] significantly (joint with Manabu Akaho). We define a geometric A_∞-algebra on the Kuranishi cochains $\widehat{KC}^*(L;\Lambda_{\mathrm{nov}})$ over a Novikov ring, rather than building the A_∞ algebra by as an algebraic limit from a series of finite geometric approximations called $A_{n,K}$-algebras, on subcomplexes of the singular chains.

I also have a general theory of open Gromov–Witten invariants.

I.3. Kuranishi spaces

We follow Fukaya–Ono 1999, with modifications. Let X be a paracompact topological space and p lie in X. A *Kuranishi neighbourhood* (V_p, E_p, s_p, ψ_p) on X satisfies:

(i) V_p is a smooth orbifold, which may have boundary or corners;

(ii) $E_p \to V_p$ is an orbifold vector bundle, the *obstruction bundle*;

(iii) $s_p : V_p \to E_p$ is a smooth section, the *Kuranishi map*; and

(iv) ψ_p is a homeomorphism from $s_p^{-1}(0)$ to an open neighbourhood of p in X, where $s_p^{-1}(0) = \{v \in V_p : s_p(v) = 0\}$.

Let $(V_p,\ldots,\psi_p)$ and $(V_q,\ldots,\psi_q)$ be Kuranishi neighbourhoods of $p,q\in X$ with $\operatorname{Im}\psi_q\subseteq\operatorname{Im}\psi_p$. A *coordinate change* $(\phi_{pq},\hat\phi_{pq})$ from $(V_q,\ldots,\psi_q)$ to $(V_p,\ldots,\psi_p)$ satisfies:

(a) $\phi_{pq}:V_q\rightarrow V_p$ is a smooth embedding of orbifolds;

(b) $\hat\phi_{pq}:E_q\rightarrow\phi_{pq}^*(E_p)$ is an embedding of orbibundles over V_q;

(c) $\hat\phi_{pq}\circ s_q\equiv s_p\circ\phi_{pq}$;

(d) $\psi_q\equiv\psi_p\circ\phi_{pq}$; and

(e) $\mathrm{d}s_p$ induces an isomorphism on $s_q^{-1}(0)$ between the normal bundle of $\phi_{pq}(V_q)$ in V_p and $\phi_{pq}^*(E_p)/E_q$.

This forces $\dim V_p-\operatorname{rank}E_p=\dim V_q-\operatorname{rank}E_q$, but allows $\dim V_p>\dim V_q$, $\operatorname{rank}E_p>\operatorname{rank}E_q$.

A *Kuranishi structure* κ on X assigns *a germ of Kuranishi neighbourhoods* $(V_p,\ldots,\psi_p)$ for all $p\in X$, and a *germ of coordinate changes* $(\phi_{pq},\hat\phi_{pq})$ from $(V_p,\ldots,\psi_p)$ and $(V_q,\ldots,\psi_q)$ for all $p\in X$ and $q\in X$ close to p. These should satisfy that $\dim V_p-\operatorname{rank}E_p=\operatorname{vdim}X$ is independent of $p\in X$, and associativity of coordinate changes, $\phi_{pq}\circ\phi_{qr}=\phi_{pr}$, $\hat\phi_{pq}\circ\hat\phi_{qr}=\hat\phi_{pr}$. Then (X,κ) is a *Kuranishi space*.

One can define Kuranishi structures on moduli spaces of J-holomorphic curves (Fukaya et al., Liu,. . .).

Orbifolds Y are examples of Kuranishi spaces – take $(V_p, E_p, s_p, \psi_p) = (Y, Y, 0, \mathrm{id}_Y)$ for all $p \in Y$, where $Y \to Y$ is the zero vector bundle.

Many definitions for orbifolds generalize to Kuranishi spaces. For example, if X is a Kuranishi space and Y is an orbifold, a *strongly smooth map* $\boldsymbol{f} : X \to Y$ assigns smooth $f_p : V_p \to Y$ for $(V_p, \ldots, \psi_p)$ in the germ at p in X, with $f_p \circ \phi_{pq} \equiv f_q$ for coordinate changes. We call $\boldsymbol{f}$ a *strong submersion* if the f_p are submersions.

Can define *orientations* on Kuranishi spaces, and the *boundary* ∂X of a Kuranishi space. If $f_a : X_a \to Y$ for $a = 1, 2$ are strong submersions, then can define the *fibre product* $X_1 \times_Y X_2$ as a Kuranishi space.

A key fact for open Gromov–Witten theory: the boundary $\partial\bar{\mathcal{M}}_{g,h,l,m}(M, L, J, \beta)$ is isomorphic as a compact, oriented Kuranishi space to a disjoint union of fibre products $\bar{\mathcal{M}}_{g_1,h_1,l_1,m_1}(M, L, J, \beta_1) \times_L \bar{\mathcal{M}}_{g_2,h_2,l_2,m_2}(M, L, J, \beta_2)$, and similar pieces. Most of Lagrangian Floer cohomology and open G–W theory should really be a *purely algebraic* consequence of this.

I.4. J-holomorphic curves in symplectic geometry: on the foundations of the subject

Much of symplectic geometry involves using moduli spaces of J-holomorphic curves in (M,ω) to define some homological structure, and showing this is independent of the almost complex structure J, so the homological invariants depend only on (M,ω). This includes Gromov–Witten invariants (closed and open), Lagrangian Floer cohomology, contact homology, Symplectic Field Theory, Fukaya categories, ...

For all of these theories, we must solve four problems:

(a) Define a geometric structure which is the model structure for moduli spaces of J-holomorphic curves, e.g. Kuranishi spaces.

This structure must have analogues of differential geometry operations: orientations, smooth maps, submersions, fibre products.

(b) Show that the moduli spaces of J-holomorphic curves we wish to study carry the structure in (a), and that natural maps of these moduli spaces are 'smooth'. Prove identities between these structures.

(c) Use this geometric structure to construct *virtual cycles* or *virtual chains* for the moduli spaces, in some (co)chain complex. Translate relationships between moduli spaces into algebraic identities upon their virtual chains.
(d) Draw some interesting geometrical conclusions — for instance, define Gromov–Witten invariants or Lagrangian Floer cohomology, prove the Arnold Conjecture, and so on.

Kuranishi (co)homology and (co)-bordism are new tools for solving problem (c).

However, there may still be unresolved issues with problems (a) and (b). The foundations of the subject need some work. Hopefully resolved in final version of [FOOO]?

(a) What is the 'right' definition of Kuranishi space?
(b) Proof that moduli spaces of J-holomorphic curves are Kuranishi spaces: more detail needed, especially near singular curves, on smoothness of section s and coordinate changes $(\phi_{pq},\hat\phi_{pq})$ there, 'gluing', associativity of coordinate changes.