EXPLICIT CONSTRUCTION OF SELF-DUAL 4-MANIFOLDS

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1. Introduction. A self-dual metric or conformal structure on a 4-manifold M is a Riemannian metric g or conformal class $[g]$ for which the Weyl conformal curvature tensor W is self-dual; M is then called a self-dual 4-manifold. Compact self-dual 4-manifolds have been extensively studied and it is known [16] that very many compact 4-manifolds do admit families of self-dual metrics, but explicit examples of self-dual metrics that can be written down in coordinates are comparatively few. In this paper we shall provide a geometrical framework within which it is possible to construct self-dual structures by solving a linear rather than a nonlinear equation, and use it to construct some new explicit examples of compact self-dual 4-manifolds.

The two most basic compact self-dual 4-manifolds are the round metric on $\mathcal{S}^4$, which is conformally flat, and the Fubini-Study metric on $\mathbb{CP}^2$. Apart from these, the first examples of self-dual metrics on compact, simply-connected manifolds were Poon's family of self-dual metrics on $2\mathbb{CP}^2$ [13]. Then LeBrun, generalizing earlier work of Gibbons and Hawking [5], found that $U(1)$- invariant self-dual metrics can be constructed from solutions to a linear equation over hyperbolic 3-space $\mathcal{H}^3$. With this 'hyperbolic Ansatz' he wrote down a family of self-dual metrics on $n\mathbb{CP}^2$ for each n [9], which coincide with Poon's metrics when $n = 2$.

Using a similar argument, we shall study T^2- invariant self-dual metrics, and will show how to construct such metrics from solutions to a linear equation over the hyperbolic plane $\mathcal{H}^2$. Smooth actions of the torus T^2 on $n\mathbb{CP}^2$ can be classified up to equivariant diffeomorphisms [11], and for large n there are many possible actions. For every such T^2-action we shall construct a family of explicit self-dual metrics on $n\mathbb{CP}^2$, invariant under the action. One of these families for each n turns out to be a subfamily of LeBrun's metrics on $n\mathbb{CP}^2$, but the other families (for $n \geq 4$) are new. By the same technique we also write down explicit self-dual metrics on some other compact 4-manifolds with fundamental group $\mathbb{Z}$.

Chapter 1 studies 4-manifolds M equipped with a conformal structure $[g]$ and a connection ∇ preserving $[g]$. Conditions are given on the torsion and curvature of ∇ that imply $[g]$ is self-dual, but that do not require the torsion to vanish. Similar conditions are also given for $[g]$ to be conformal to an Einstein or Einstein-Weyl structure. The main inspiration for Chapter 1 is a formulation of the self-dual Einstein equations found by Ashtekar et al. [1].

Chapter 2 goes on to show how a conformal class $[g]$ and connection ∇ may be written down in such a way that the condition of Chapter 1 for $[g]$ to be self-dual is linear in a part of the data. The freedom in the torsion of ∇ is crucial here, because if the full torsion vanishes then the linear equation has only constant solutions. Since explicit solutions can often be found for linear equations, this provides a fruitful construction method for self-dual metrics. The result is linear constructions for self-dual metrics with one Killing vector or two commuting Killing vectors, and for self-dual hermitian metrics on a complex surface.

Chapter 3 applies the ideas to find locally T^2- invariant self-dual metrics on a T^2-bundle over a hyperbolic 2-manifold. The total space of the T^2- bundle can be realized as an open dense set in a compact 4-manifold M with a local T^2- action, and by looking for self-dual metrics on the T^2- bundle that extend to M, we find new self-dual metrics on $n\overline{\mathbb{CP}}^2$, the connected sum of n copies of $\overline{\mathbb{CP}}^2$, and on some compact 4-manifolds M with $\pi_1(M) = \mathbb{Z}$ and $b^-(M) = 0$. Chapters 2 and 3 rely heavily on the methods of LeBrun in [9].

1.1. Connections, curvature and conformal manifolds. Let M^n be a smooth manifold. A conformal structure $[g]$ on M is an equivalence class of Riemannian metrics on M, under the equivalence relation $g \sim \lambda g$ for all smooth, positive, real functions λ on M. Equivalently, a conformal structure is a $CO(n)$- structure on M, where $CO(n) = \mathbb{R}_+ \times SO(n)$ is the conformal group. Taking this point of view, we can construct a natural $\mathfrak{co}(n)$- bundle E on M associated to the adjoint representation of $CO(n)$ on $\mathfrak{co}(n)$; E is a subbundle of $TM \otimes T^*M$. Because $\mathfrak{co}(n)$ splits naturally as $\mathfrak{co}(n) = \mathbb{R} \oplus \mathfrak{so}(n)$, E splits naturally as $E = \mathbb{R} \oplus E_0$.

Let ∇ be a connection on M. The torsion $T = T(\nabla) \in TM \otimes \Lambda^2 T^*M$ of ∇ is a tensor on M, defined by $T(v,w) = \nabla_v w - \nabla_w v - [v,w]$ for all vector fields v, w on M, where $[v,w]$ is the Lie bracket of vector fields. The connection is *torsion-free* if T is zero. We say that ∇ preserves $[g]$ if for $g \in [g]$, there is a 1-form μ on M such that $\nabla g = \mu \otimes g$. Equivalently, we may say that ∇ is a $CO(n)$- connection. The curvature $R(\nabla)$ of a $CO(n)$- connection ∇ lies in $E \otimes \Lambda^2 T^*M$, as E is the $\mathfrak{co}(n)$- bundle associated to the $CO(n)$- structure. The splitting $E = \mathbb{R} \oplus E_0$ divides $R(\nabla)$ into an $\mathbb{R}$- component and an E_0- component, which we will write $R_0(\nabla)$.

Suppose ∇ is torsion-free and preserves $[g]$. We may form the curvature $R(\nabla) = R^i_{jkl}$ of ∇, and contract to get the Ricci curvature $\mathrm{Ric}(\nabla) = R^i_{jil}$. The contribution to $\mathrm{Ric}(\nabla)$ from the $\mathbb{R}$- component of $R(\nabla)$ is antisymmetric, and since ∇ is torsion-free, the contribution to $\mathrm{Ric}(\nabla)$ from the E_0 component $R_0(\nabla)$ of $R(\nabla)$ is symmetric, by the Bianchi identity.

DEFINITION 1.1.1. Let ∇ be a torsion-free connection on M preserving $[g]$. The pair $(\nabla, [g])$ is called an *Einstein-Weyl structure on* M if the symmetric part of $\mathrm{Ric}(\nabla)$ is λg for some $\lambda \in C^\infty(M)$ and $g \in [g]$.

Note that because the symmetric part of $\mathrm{Ric}(\nabla)$ is a contraction of $R_0(\nabla)$, this is a condition on $R_0(\nabla)$.

Now consider conformal structures in dimension 4. Let M be an oriented conformal 4-manifold with conformal structure $[g]$. Then $[g]$ and the orientation on M define an involution $*$ of $\Lambda^2 T^*M$ called the Hodge star. This splits $\Lambda^2 T^*M$ into $\Lambda^2 T^*M = \Lambda^+ \oplus \Lambda^-$, where Λ^+, Λ^- are the 3-dimensional subbundles of $\Lambda^2 T^*M$ of eigenvectors of $*$ associated to the eigenvalues $+1$ and -1 respectively. To make this explicit, fix some $g \in [g]$ and let $(\omega^1, \ldots, \omega^4)$ be a local quadruple of 1-forms on M that form an oriented orthonormal basis for T^*M w.r.t. g at each point. Then $\Lambda^\pm$ are given by

$$\begin{aligned} \Lambda^+ &= \langle \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4, \omega^1 \wedge \omega^3 + \omega^4 \wedge \omega^2, \omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3 \rangle, \\ \Lambda^- &= \langle \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4, \omega^1 \wedge \omega^3 - \omega^4 \wedge \omega^2, \omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3 \rangle. \end{aligned} \tag{1}$$

The Lie algebra $\mathfrak{co}(4)$ also splits naturally into $\mathfrak{co}(4) = \mathbb{R} \oplus \mathfrak{so}(3)^+ \oplus \mathfrak{so}(3)^-$, where

$\mathfrak{so}(3)^+, \mathfrak{so}(3)^-$ are the subalgebras of $\mathfrak{co}(4)$ that act trivially on Λ^-, Λ^+ respectively, and are isomorphic to $\mathfrak{so}(3)$. This gives a splitting $E = \mathbb{R} \oplus E^+ \oplus E^-$, and the subbundles E^+ and E^- associated to $\mathfrak{so}(3)^+$ and $\mathfrak{so}(3)^-$ are related to Λ^+ and Λ^- by $\Lambda^\pm \cong E^\pm \otimes K$, where K is a real line bundle, the square root of the bundle of volume forms.

If ∇ is a connection preserving $[g]$, then $R(\nabla)$ is a section of $E \otimes \Lambda^2 T^*M$. The splittings $E = \mathbb{R} \oplus E^+ \oplus E^-$ and $\Lambda^2 T^*M = \Lambda^+ \oplus \Lambda^-$ thus divide $R(\nabla)$ into six components. Also, because $\Lambda^\pm \cong E^\pm \otimes K$, the components $E^\pm \otimes \Lambda^\pm$ have the further splitting $E^\pm \otimes \Lambda^\pm = K \oplus S^2_0 E^\pm \otimes K \oplus \Lambda^2 E^\pm \otimes K$, since $E^\pm \otimes E^\pm$ has a canonical identity section and orthogonal bundle. So $R(\nabla)$ has a total of ten components. Suppose ∇ is torsion-free. Then curvature identities imply that the components in $\Lambda^2 E^\pm \otimes K$ are equal to the components in $\mathbb{R} \otimes \Lambda^\pm$, the components in $E^+ \otimes \Lambda^-$ and $E^- \otimes \Lambda^+$ are equal, and the trace components in $E^+ \otimes \Lambda^+$ and $E^- \otimes \Lambda^-$ are equal, so there are only six essentially different components.

The components in $\mathbb{R} \otimes \Lambda^\pm$ make up the $\mathbb{R}$- component of $R(\nabla)$, which is proportional to the curvature of the action of ∇ on the volume forms. If these components vanish then the connection is flat on the volume forms and there are local volume forms on M constant under ∇. This is the necessary and sufficient condition for a torsion-free connection ∇ preserving $[g]$ to be the Levi-Civita connection of some local $g \in [g]$. We have thus arrived at the decomposition of the Riemann tensor of a Riemannian 4-manifold, as given for instance in [2, p. 427]. Let $g \in [g]$, and let ∇ be the Levi-Civita connection of g. Then $R(\nabla)$ is the Riemann curvature of g, and the nonvanishing components are as follows:

(i) The components of $R(\nabla)$ in the trace parts of $E^\pm \otimes \Lambda^\pm$ are both proportional to the scalar curvature.

(ii) The component of $R(\nabla)$ in $S^2_0 E^+ \otimes K$ is called the self-dual Weyl curvature $W^+(g)$. It is an invariant of the conformal structure $[g]$.

(iii) The component of $R(\nabla)$ in $S^2_0 E^- \otimes K$ is called the anti-self-dual Weyl curvature $W^-(g)$. It is an invariant of the conformal structure $[g]$.

(iv) The components of $R(\nabla)$ in $E^+ \otimes \Lambda^-$ and $E^- \otimes \Lambda^+$ are equal under the isomorphism $E^+ \otimes \Lambda^- \cong E^- \otimes \Lambda^+$ induced by $\Lambda^\pm \cong E^\pm \otimes K$, and are the trace-free part of the Ricci curvature.

The decomposition is related to the definition of Einstein-Weyl structure above. As in part *(iv)*, if ∇ is a torsion-free connection preserving $[g]$, then the components of $R(\nabla)$ in $E^+ \otimes \Lambda^-$ and $E^- \otimes \Lambda^+$ are both equal to the trace-free, symmetric Ricci curvature of ∇. Thus by Definition 1.1.1, $(\nabla, [g])$ is an Einstein-Weyl structure if and only if these components vanish, and it is sufficient for one to vanish, say the component in $E^- \oplus \Lambda^+$. Also, from above a necessary and sufficient condition for ∇ to be the Levi-Civita connection of some local $g \in [g]$ is that the $\mathbb{R}$- component of $R(\nabla)$ should vanish, so this is a necessary and sufficient condition for an Einstein-Weyl manifold to be locally Einstein.

Let ∇ be a connection preserving $[g]$, possibly with torsion. Define $R^\pm(\nabla)$ to be the components of $R(\nabla)$ lying in $E^\pm \otimes \Lambda^2 T^*M$. Since $CO(4)/\{\pm 1\} \cong \mathbb{R}_+ \times SO(3)^+ \times SO(3)^-$, a $CO(4)$- connection is really made up of an $\mathbb{R}$- connection and two $SO(3)$- connections, and $R^\pm(\nabla)$ are the curvatures of these $SO(3)$- connections.

DEFINITION 1.1.2. Let ∇ be a $CO(4)$- connection preserving the conformal structure $[g]$ on M. The *anti-self-dual Weyl curvature of* ∇, $W^-(\nabla)$, is defined to be the component of $R^-(\nabla)$ in $S^2_0 E^- \otimes K$ in the decomposition above.

1.2. Self-dual conformal structures and connections with torsion. The results of this section, which are the foundation of the paper, were motivated by a formulation of the self-dual Einstein equations by Ashtekar et al. [1]. Section 1.3 will study Einstein metrics and Einstein-Weyl structures on 4-manifolds using the same ideas, and reprove some of the results of [1]. Let M be an oriented 4-manifold, $[g]$ a conformal structure on M, and ∇ a connection on M preserving $[g]$. Then by Definition 1.1.2, $W^-(\nabla)$ is a component of the curvature $R(\nabla)$ of ∇. Our first result shows that $W^-(\nabla)$ depends only on $[g]$ provided $T(\nabla)$ has a certain form.

THEOREM 1.2.1. *Let $[g]$ be a conformal class on M, and suppose $\nabla, \tilde{\nabla}$ are connections on M preserving $[g]$, with $\tilde{\nabla}$ torsion-free and $T(\nabla) = \tau + \kappa \wedge I_{TM}$, where $\tau \in \Gamma(TM \otimes \Lambda^+)$, I_{TM} is the identity section in $TM \otimes T^*M$, and κ is a 1-form. Then $W^-(\nabla) = W^-(\tilde{\nabla})$.*

Proof. The difference $\nabla - \tilde{\nabla}$ is a tensor, a section of $T^*M \otimes E$, and from §1.1 E splits as $E = \mathbb{R} \oplus E^+ \oplus E^-$. Let the first component of $\nabla - \tilde{\nabla}$ in the corresponding splitting of $T^*M \otimes E$ be λ; then λ is a 1-form. Let S be the component in $T^*M \otimes E^-$, and Q the sum of the components in $T^*M \otimes E^{\pm}$, so that S is a component of Q. Now R^- is the curvature of the $SO(3)^-$- connection forming part of the whole $CO(4)$- connection, and the difference between the $SO(3)^-$- components of ∇ and $\tilde{\nabla}$ is S. So by the formula for the curvature of a connection,

$$R^-(\nabla) = R^-(\tilde{\nabla}) + d_{\tilde{\nabla}} S + [S, S], \tag{2}$$

where the bracket $[\,,\,]$ is the Lie bracket in E^-, regarding it as a Lie algebra isomorphic to $\mathfrak{so}(3)$.

We shall show that the component in $S^2_0 E^- \otimes K$ of the extra terms on the right hand side of (2) is zero, so that taking $S^2_0 E^- \otimes K$- components gives $W^-(\nabla) = W^-(\tilde{\nabla})$, which is what we have to prove. To do this we must find an expression for S. Let $(\omega^1, \ldots, \omega^4)$ be a local quadruple of 1-forms on M that form an oriented orthonormal basis for T^*M at each point w.r.t. some $g \in [g]$, and let $(V_1, \ldots, V_4)$ be the dual basis of $(\omega^1, \ldots, \omega^4)$ for TM. We may write $E^- = \langle J_1, J_2, J_3 \rangle$, where

$$\begin{aligned} J_1 &= V_1 \otimes \omega^2 - V_2 \otimes \omega^1 - V_3 \otimes \omega^4 + V_4 \otimes \omega^3, \\ J_2 &= V_1 \otimes \omega^3 - V_3 \otimes \omega^1 - V_4 \otimes \omega^2 + V_2 \otimes \omega^4, \\ J_3 &= V_1 \otimes \omega^4 - V_4 \otimes \omega^1 - V_2 \otimes \omega^3 + V_3 \otimes \omega^2. \end{aligned} \tag{3}$$

Then the Lie bracket in E^- is $[J_1, J_2] = 2J_3$, $[J_2, J_3] = 2J_1$ and $[J_3, J_1] = 2J_2$.

In coordinates w.r.t. $\{V_j\}$ and $\{\omega^k\}$ we may write $\nabla - \tilde{\nabla} = \lambda_j \delta^i_k + Q^i_{jk}$, so antisymmetrizing over j, k gives the difference between the torsions:

$$\lambda_j \delta^i_k - \lambda_k \delta^i_j + Q^i_{jk} - Q^i_{kj} = \tau^i_{jk} + \kappa_j \delta^i_k - \kappa_k \delta^i_j. \tag{4}$$

But Q satisfies $Q^i_{jk} = -g^{ia} g_{kb} Q^b_{ja}$ for any $g \in [g]$. Using this symmetry it can be shown that if $Q^i_{jk} - Q^i_{kj} = U^i_{jk}$, then

$$2Q^i_{jk} = U^i_{jk} - g^{ia} g_{kb} U^b_{ja} - g^{ia} g_{jb} U^b_{ka}. \tag{5}$$

Rearranging (4) and applying (5) gives

$$2Q^i_{jk} = 2\delta^i_j\lambda_k - 2g^{ia}g_{jk}\lambda_a - 2\delta^i_j\kappa_k + 2g^{ia}g_{jk}\kappa_a + \tau^i_{jk} - g^{ia}g_{kb}\tau^b_{ja} - g^{ia}g_{jb}\tau^b_{ka}. \qquad (6)$$

Because S is the $T^*M \otimes E^-$ component of Q, this determines S in terms of κ, λ and τ. From (3) we calculate that $4S_j = -(J_\alpha)^k_i Q^i_{jk} \cdot J_\alpha$, where α is summed from 1 to 3. So by (6),

$$8S_j = -\big\{4(J_\alpha)^k_j\lambda_k - 4(J_\alpha)^k_j\kappa_k + 2(J_\alpha)^k_i\tau^i_{jk}\big\} \cdot J_\alpha, \qquad (7)$$

since $(J_\alpha)^k_i g^{ia}g_{jb}\tau^b_{ka} = 0$, as it is the contraction of an anti-self-dual term J_α and a self-dual term τ. But since τ^i_{jk} is self-dual in j, k, it is of type $(1,1)$ w.r.t. J_α (regarding this as an almost complex structure), and so $(J_\alpha)^a_j\tau^i_{ak} = -(J_\alpha)^a_k\tau^i_{ja}$. Substituting this into (7), we get

$$S = (J_1 \cdot \mu) \otimes J_1 + (J_2 \cdot \mu) \otimes J_2 + (J_3 \cdot \mu) \otimes J_3, \qquad (8)$$

where μ is the 1-form $\kappa/2 - \lambda/2 - \tau^i_{ik}\omega^k/4$, and $J_\alpha \cdot \mu$ is the contraction of the TM factor of J_α with $\mu \in T^*M$.

Evaluating the expression $[S,S]$ in (2) using (8) and the Lie bracket relations above, we get

$$[S,S] = 4\big\{J_1 \otimes (J_2 \cdot \mu) \wedge (J_3 \cdot \mu) + J_2 \otimes (J_3 \cdot \mu) \wedge (J_1 \cdot \mu) + J_3 \otimes (J_1 \cdot \mu) \wedge (J_2 \cdot \mu)\big\}. \qquad (9)$$

But the projection of $(J_2 \cdot \mu) \wedge (J_3 \cdot \mu)$ to Λ^- is $J_1 \otimes |\mu|^2 \in E^- \otimes K \cong \Lambda^-$, where we regard $|\mu|^2$ as a section of K. So the component of $[S,S]$ in $E^- \otimes E^- \otimes K$ is $4(J_1 \otimes J_1 + J_2 \otimes J_2 + J_3 \otimes J_3) \otimes |\mu|^2$, which is a multiple of the identity, and has no component in $S^2_0E^- \otimes K$.

Because $J_1 \otimes J_1 + J_2 \otimes J_2 + J_3 \otimes J_3$ is a tensor determined by the conformal class $[g]$ and the orientation, it follows that this tensor is constant with respect to $\tilde\nabla$, and we deduce from (8) that

$$\tilde\nabla S = J_\alpha \otimes (J_\alpha \cdot \tilde\nabla\mu), \qquad (10)$$

where α is summed from 1 to 3 and $(J_\alpha \cdot \tilde\nabla\mu)_{jk} = (J_\alpha)^i_k\tilde\nabla_j\mu_i$ in coordinates. Now $\tilde\nabla$ acting on S is the composition of the action of $\tilde\nabla$ on E^- with the action of $\tilde\nabla$ on 1-forms, and what we actually want is the action of $d_{\tilde\nabla}$ on S, which is the combination of $\tilde\nabla$ on E^- with the exterior derivative d on 1-forms. But because $\tilde\nabla$ is torsion-free, the antisymmetrization of the action of $\tilde\nabla$ on 1-forms is d, so $d_{\tilde\nabla}S$ is the antisymmetrization of $\tilde\nabla S$. So by (10),

$$d_{\tilde\nabla}S = J_\alpha \otimes (J_\alpha)^i_k(\tilde\nabla_j\mu_i)\omega^j \wedge \omega^k. \qquad (11)$$

It can be shown that the component of $d_{\tilde\nabla}S$ in $S^2E^- \otimes K$ is $(J_\alpha \otimes J_\alpha) \otimes g^{jk}\tilde\nabla_j\mu_k/2$, where $g^{jk}\tilde\nabla_j\mu_k$ is independent of the choice of $g \in [g]$ when regarded as a section of K. This is a multiple of the identity, and has no component in $S^2_0E^- \otimes K$. Therefore we have shown that the expression $d_{\tilde{\nabla}}S+[S,S]$ occurring in (2) has no component in $S^2_0E^-\otimes K$, so taking components in $S^2_0E^-\otimes K$ in (2) shows that $W^-(\nabla)=W^-(\tilde{\nabla})$. □

Using this Theorem, we can relate the curvature of ∇ to the conformal curvature of $[g]$.

LEMMA 1.2.2. *Let $[g]$ be a conformal class on M, and let ∇ be a connection on M preserving $[g]$ with $T(\nabla)=\tau+\kappa\wedge I_{TM}$ for some section τ of $TM\otimes\Lambda^+$ and some 1-form κ, where I_{TM} is the identity section in $TM\otimes T^*M$. Then $W^-(\nabla)$ is equal to the anti-self-dual Weyl curvature W^- of $[g]$, so that $[g]$ is self-dual if and only if $W^-(\nabla)=0$.*

Proof. Let $g\in[g]$, and let $\tilde{\nabla}$ be the Levi-Civita connection of g. Then $\tilde{\nabla}$ is torsion-free, and by the definition of $W^-(g)$ in §1.1, $W^-=W^-(\tilde{\nabla})$. But by Theorem 1.2.1, as $T(\nabla)=\tau+\kappa\wedge I_{TM}$, we have $W^-(\nabla)=W^-(\tilde{\nabla})$. Therefore $W^-(\nabla)=W^-$, and $[g]$ is self-dual if and only if $W^-(\nabla)=0$. □

The Lemma is a sufficient but not a necessary condition for $[g]$ to be self-dual, in the sense that if $[g]$ is a self-dual conformal class and ∇ preserves $[g]$, then it is not necessary for $W^-(\nabla)$ and $T^-(\nabla)$ to have the above form. Theorem 1.2.1 and Lemma 1.2.2 also imply the conformal invariance of the anti-self-dual Weyl curvature W^-. Let $T^-(\nabla)$ be the anti-self-dual part of $T(\nabla)$. Then putting $\kappa=0$ in Lemma 1.2.2 gives

COROLLARY 1.2.3. *Let $[g]$ be a conformal class on M, and ∇ be a connection on M preserving $[g]$. If $W^-(\nabla)=T^-(\nabla)=0$, then $[g]$ is self-dual.*

When the 1-form μ in the proof of Theorem 1.2.1 vanishes, the calculation shows that the $SO(3)^-$ part of the connections $\nabla,\tilde{\nabla}$ are equal. So we deduce

LEMMA 1.2.4. *Let $[g]$ be a conformal class on M, and let ∇ be a connection on M preserving $[g]$ with $T(\nabla)=\tau+\kappa\wedge I_{TM}$ for some section τ of $TM\otimes\Lambda^+$ and some 1-form κ. Then there exists a unique torsion-free connection $\tilde{\nabla}$ preserving $[g]$ with $SO(3)^-$-component equal to the $SO(3)^-$- component of ∇. This implies that $R^-(\nabla)=R^-(\tilde{\nabla})$.*

Proof. From the calculations in Theorem 1.2.1, we can see that there is a unique torsion-free connection $\tilde{\nabla}$ that preserves $[g]$ such that the first component of $\nabla-\tilde{\nabla}$ in the splitting $T^*M\otimes E=T^*M\oplus T^*M\otimes E^+\oplus T^*M\otimes E^-$ is some given 1-form λ. Choose λ to be the 1-form $\kappa-\tau^i_{ik}\omega^k/2$; this defines $\tilde{\nabla}$ uniquely. With this choice of λ, the 1-form μ in the proof of Theorem 1.2.1 is zero, so that $S=0$ by (8). But S is the difference between the $SO(3)^-$- components of ∇ and $\tilde{\nabla}$, so these components are equal. □

1.3. Einstein-Weyl structures, Einstein metrics and Ashtekar's equations. This section gives a different geometrical viewpoint on Ashtekar's formulation of the self-dual Einstein equations [1], which we hope will be a fruitful one. Recall from §1.1 that if ∇ is a connection on M^4 preserving a conformal class $[g]$, then $(\nabla,[g])$ is an Einstein-Weyl structure if and only if $R^-(\nabla)$ is anti-self-dual, and is locally Einstein if in addition the $\mathbb{R}$- component of $R(\nabla)$ is zero. Using Lemma 1.2.4 we shall prove a result analogous to Lemma 1.2.2 for Einstein-Weyl structures and Einstein metrics.

PROPOSITION 1.3.1. *Let $[g]$ be a conformal class on M, and ∇ be a connection on M preserving $[g]$. Suppose that $R^-(\nabla)$ is anti-self-dual and $T(\nabla)=\tau+\kappa\wedge I_{TM}$ for some section τ of $TM\otimes\Lambda^+$ and some 1-form κ. Then there exists a torsion-free connection $\tilde{\nabla}$ preserving $[g]$, such that $(\tilde{\nabla},[g])$ is an Einstein-Weyl structure.*

Suppose in addition that ∇ preserves a metric $g \in [g]$, and that $\tau^i_{ik}\omega^k - 2\kappa = df$ for some $f \in C^\infty(M)$. Then $e^f g$ is an Einstein metric on M.

Proof. By Lemma 1.2.4, there exists a unique connection $\tilde\nabla$ preserving $[g]$ that is torsion-free with $SO(3)^-$- component equal to that of ∇. Therefore $R^-(\tilde\nabla) = R^-(\nabla)$, which is anti-self-dual, so $(\tilde\nabla, [g])$ is an Einstein-Weyl structure.

Suppose now that ∇ preserves some $g \in [g]$ and that $\tau^i_{ik}\omega^k - 2\kappa = df$. The difference between the $\mathbb{R}$- components of ∇ and $\tilde\nabla$ is $\lambda = \kappa - \tau^i_{ik}\omega_k/2 = -df/2$, and as the $\mathbb{R}$ factor of $CO(4)$ acts to the power -2 on metrics it follows that $\tilde\nabla g = 2g \otimes \lambda = -g \otimes df$, and so $\tilde\nabla(e^f g) = 0$. Thus $\tilde\nabla$ is the Levi-Civita connection of $e^f g$, and $e^f g$ is Einstein. $\square$

When M is compact, the conditions for the Einstein case can be relaxed:

Lemma 1.3.2. *Let M be a compact 4-manifold, $[g]$ be a conformal class on M, and ∇ be a connection on M preserving $[g]$. Suppose $R^-(\nabla)$ lies in $S^2E^- \otimes K$ and $T(\nabla) = \tau + \kappa \wedge I_{TM}$ for some section τ of $TM \otimes \Lambda^+$ and some 1-form κ. Then locally there exist Einstein $g \in [g]$, and the obstruction to choosing a global Einstein $g \in [g]$ lies in $H^1(M, \mathbb{R})$. If in addition $R^-(\nabla)$ lies in $S^2_0E^- \otimes K$, then these metrics are Ricci-flat.*

Proof. Applying Proposition 1.3.1, we get a torsion-free connection $\tilde\nabla$ preserving $[g]$ with $R^-(\tilde\nabla) = R^-(\nabla)$. As $R^-(\nabla)$ lies in $S^2E^- \otimes K$, the component of $R^-(\tilde\nabla)$ in $\Lambda^2E^- \otimes K$ is zero. But from §1.1, this is equal to the Λ^-- part of the $\mathbb{R}$- component of $R(\tilde\nabla)$. Thus the $\mathbb{R}$- component of $R(\tilde\nabla)$ is a section of Λ^+. But the $\mathbb{R}$- component of $R(\tilde\nabla)$ is an exact 2-form, as it is the curvature of a topologically trivial real line bundle, so it is an exact, harmonic form on a compact Riemannian manifold, and is zero by Hodge theory.

Therefore the $\mathbb{R}$- component of $R(\tilde\nabla)$ is zero, and $\tilde\nabla$ is the Levi-Civita connection of local $g \in [g]$. As $R^-(\nabla)$ is anti-self-dual, these are Einstein, and if in addition $R^-(\nabla)$ lies in $S^2_0E^- \otimes K$ then the scalar curvature vanishes and they are Ricci-flat. The obstruction to patching them together to form a global Einstein metric lies in $H^1(M, \mathbb{R})$, as $\tilde\nabla$ induces a flat connection on the real line bundle of volume forms, and flat connections on a real line bundle are classified by $H^1(M, \mathbb{R})$; there exists a global volume form preserved by $\tilde\nabla$ if and only if this class in $H^1(M, \mathbb{R})$ is zero. $\square$

Applying Proposition 1.3.1 in the case that ∇ is flat, we recover Ashtekar's formulation of the self-dual Einstein equations [1], in the form of the Proposition on [10, p. 662].

Proposition 1.3.3 [1], [10]. *Let M be an oriented 4-manifold and let $V_1, \ldots, V_4$ be vector fields on M forming an oriented basis for TM at each point. Then $V_1, \ldots, V_4$ define a conformal structure $[g]$ on M. Suppose that $V_1, \ldots, V_4$ satisfy the three vector field equations*

$$[V_1, V_2] - [V_3, V_4] = 0, \qquad [V_1, V_3] - [V_4, V_2] = 0, \qquad [V_1, V_4] - [V_2, V_3] = 0. \tag{12}$$

Then $[g]$ is self-dual and there exists a torsion-free connection $\tilde\nabla$ preserving $[g]$ such that $\big(\tilde\nabla, [g]\big)$ is an Einstein-Weyl structure with zero scalar curvature. Suppose in addition that $V_1, \ldots, V_4$ preserve a volume form Ω on M. Then $\tilde\nabla$ is the Levi-Civita form of a self-dual, Ricci-flat metric $\tilde g \in [g]$.

Proof. There is a unique connection ∇ on M satisfying $\nabla V_j = 0$ for $j = 1, \ldots, 4$. Let g be the metric on M for which $V_1, \ldots, V_4$ are orthonormal and $[g]$ be the conformal structure of g. Then ∇ preserves g and $[g]$. The torsion T of ∇ is given by $T(V_i, V_j) = \nabla_{V_i} V_j - \nabla_{V_j} V_i - [V_i, V_j] = -[V_i, V_j]$. But as $V_1, \ldots, V_4$ are orthonormal, the anti-self-dual components of $T(\nabla)$ w.r.t. $[g]$ are $T(V_1, V_2) - T(V_3, V_4)$, $T(V_1, V_3) - T(V_4, V_2)$ and $T(V_1, V_4) - T(V_2, V_3)$. Therefore the condition $T^-(\nabla) = 0$ is the three equations (12).

Now since ∇ preserves $V_1, \ldots, V_4$, ∇ is flat and $R(\nabla) = 0$. Thus if (12) holds then $W^-(\nabla) = T^-(\nabla) = 0$ and by Corollary 1.2.3, $[g]$ is self-dual. Also the conditions of the first part of Proposition 1.3.1 hold, so there is a torsion-free connection $\tilde{\nabla}$ preserving $[g]$ such that $\big(\tilde{\nabla}, [g]\big)$ is an Einstein-Weyl structure. Because $R^-(\tilde{\nabla}) = R^-(\nabla) = 0$, the scalar curvature of $\tilde{\nabla}$ vanishes, which completes the first part of the Proposition.

Let $\omega^1, \ldots, \omega^4$ be the dual basis of 1-forms to $V_1, \ldots, V_4$. Write $T(\nabla)$ as $\tau^i_{jk} V_i \otimes \omega^j \otimes \omega^k$. Then an easy calculation shows that $\mathcal{L}_{V_k}(\omega^1 \wedge \ldots \wedge \omega^4) = -\tau^i_{ik} \omega^1 \wedge \ldots \wedge \omega^4$, where $\mathcal{L}$ is the Lie derivative. Suppose that $V_1, \ldots, V_4$ preserve a volume form $\Omega = e^f \omega^1 \wedge \ldots \wedge \omega^4$. Then $\mathcal{L}_{V_k} \Omega = 0$, which implies that $\tau^i_{ik} = V_k(f)$. As $\kappa = 0$ we have $\tau^i_{ik} \omega^k - 2\kappa = df$, and ∇ preserves g, so the second part of Proposition 1.3.1 applies and $\tilde{g} = e^f g$ is Einstein. But we know already that $[g]$ is self-dual and $\tilde{\nabla}$ has zero scalar curvature. Therefore $\tilde{g}$ is self-dual and Ricci-flat. $\square$

Vector field equations equivalent to (12) appear as [1, p. 633, eq. (2)], and in the form we have written them as [10, p. 662, eq. (8*a*)]. Another interpretation of (12) is this: the equations imply the vanishing of the Nijenhuis tensors of the almost complex structures J_1, J_2, J_3 defined by (3). Thus M has three anticommuting complex structures on M and so is a hypercomplex 4-manifold [15, §6], which are automatically self-dual and Einstein-Weyl. Note that in the second part of the Proposition, the volume form of the metric is $e^{2f} \omega^1 \wedge \ldots \wedge \omega^4$, which is not Ω, the volume form one might expect.

2. Constructions for self-dual metrics. In this chapter constructions will be given for self-dual metrics with one Killing vector or two commuting Killing vectors, and for self-dual Hermitian metrics on a complex surface with no symmetry assumption. The case of hypercomplex structures will be handled first in §2.1, as Proposition 1.3.3 (Ashtekar's equations) makes things simple. Then §2.2 and §2.3 generalize the results of §2.1 to ansatzes for general self-dual metrics with one and two Killing vectors respectively. The case of §2.2 has already been studied by Jones and Tod [6], and we reprove some of their results using the ideas of §1.2. Section 2.4 gives a classification result for T^2- invariant self-dual metrics compatible with a product structure $N \times T^2$, and shows how to find Kähler structures of the opposite orientation in the same conformal class. In §2.5 a curious construction for self-dual hermitian metrics is found.

2.1. Making hypercomplex structures using Ashtekar's equations. Ashtekar's equations (12) are bilinear in the vector fields $\{V_j\}$, and so are an example of the general problem of finding $u \in U$ with $Q(u, u) = 0$, where U, W are vector spaces, and $Q : U \times U \to W$ is a symmetric, bilinear map. Here is a simple method for reducing this bilinear problem to a linear problem: first find a starting solution u_0 in U satisfying $Q(u_0, u_0) = 0$. Choose a vector subspace Z of U satisfying $Q|_{Z \times Z} = 0$, and consider elements $u = u_0 + z$ of U, for $z \in Z$. Then $Q(u, u) = 2Q(u_0, z)$, so $Q(u, u) = 0$ if and only if $Q(u_0, z) = 0$, which is a linear restriction on z, regarding u_0 as fixed.

For many bilinear problems, the linear equations obtained in this way will be overdetermined and have no interesting solutions. But because (12) are first-order and involve only three linear combinations of the possible six $[V_i, V_j]$, the associated linear equation only prescribes about half of the first derivatives of z, so for well-chosen Z the linear equation will have nontrivial solutions.

Example 1. The Gibbons-Hawking Ansatz. Let M be $\mathbb{R}^4$ with the standard coordinates $(x_1, \ldots, x_4)$. We shall construct solutions of (12) with Killing vector $\partial/\partial x_1$. Let $f_1, \ldots, f_4$ be smooth real functions of x_2, x_3, x_4, and define $V_1 = f_1\partial/\partial x_1$ and $V_j = f_j\partial/\partial x_1 + \partial/\partial x_j$ for $j = 2, 3, 4$. Then (12) is equivalent to the three equations

$$\frac{\partial f_1}{\partial x_2} - \frac{\partial f_3}{\partial x_4} + \frac{\partial f_4}{\partial x_3} = 0, \qquad \frac{\partial f_1}{\partial x_3} - \frac{\partial f_4}{\partial x_2} + \frac{\partial f_2}{\partial x_4} = 0, \qquad \frac{\partial f_1}{\partial x_4} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = 0. \tag{13}$$

Thus to apply Proposition 1.3.3 to $V_1, \ldots, V_4$, (13) must hold and $(V_1, \ldots, V_4)$ must be an oriented basis for $T\mathbb{R}^4$ at each point. But the condition for this is simply $f_1 > 0$. So if $f_1 > 0$ and (13) holds, then Proposition 1.3.3 gives a hypercomplex structure on $\mathbb{R}^4$.

As $V_1, \ldots, V_4$ are independent of x_1, $\partial/\partial x_1$ is a Killing vector of the structure. When $f_1 = 1$ and $f_2 = f_3 = f_4 = 0$ we get the flat metric on $\mathbb{R}^4$. Working over $\mathbb{R}^3$ with coordinates (x_2, x_3, x_4), we may write $f_1 = U$, a function on $\mathbb{R}^3$, and regard $\omega = -(f_2, f_3, f_4)$ as a vector field on $\mathbb{R}^3$. Then (13) becomes the equation $\nabla U = \operatorname{curl}\omega$. This is [5, p. 431, eq. (3)], which is used by Gibbons and Hawking to construct all examples of self-dual, Ricci-flat metrics with a triholomorphic $U(1)$- symmetry.

Example 2. We shall now try the same trick for $\mathbb{R}^2$- invariant structures on $\mathbb{R}^4$ with commuting Killing vectors $\partial/\partial x_1$, $\partial/\partial x_2$. Let $e_1, \ldots, e_4$ and $f_1, \ldots, f_4$ be smooth real functions of x_3, x_4, and define $V_j = e_j\partial/\partial x_1 + f_j\partial/\partial x_2$ for $j = 1, 2$ and $V_j = e_j\partial/\partial x_1 + f_j\partial/\partial x_2 + \partial/\partial x_j$ for $j = 3, 4$. Then (12) is equivalent to the six equations

$$\begin{gathered} \frac{\partial e_3}{\partial x_4} - \frac{\partial e_4}{\partial x_3} = 0 \qquad \frac{\partial f_3}{\partial x_4} - \frac{\partial f_4}{\partial x_3} = 0, \\ \frac{\partial e_1}{\partial x_3} + \frac{\partial e_2}{\partial x_4} = 0, \qquad \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_4} = 0, \qquad \frac{\partial e_1}{\partial x_4} - \frac{\partial e_2}{\partial x_3} = 0, \qquad \frac{\partial f_1}{\partial x_4} - \frac{\partial f_2}{\partial x_3} = 0. \end{gathered} \tag{14}$$

The first of these shows that $e_3 = \partial\gamma/\partial x_3$ and $e_4 = \partial\gamma/\partial x_4$ for some function $\gamma(x_3, x_4)$, and by the coordinate change $(x_1, x_2, x_3, x_4) \mapsto (x_1 - \gamma, x_2, x_3, x_4)$ we eliminate the e_3, e_4 terms, and similarly the f_3, f_4 terms. So we may suppose after a coordinate change that $e_3 = e_4 = f_3 = f_4 = 0$.

The last four equations of (14) are four first-order linear equations in the four unknowns e_1, e_2, f_1, f_2. We can identify the equations: they are the Cauchy-Riemann equations for $e = e_1 - ie_2$ and $f = f_1 - if_2$ to be holomorphic functions of the complex variable $x_3 + ix_4$. Thus solutions of (14) are determined by a pair of holomorphic functions e, f on $\mathbb{C}$. The condition for this data to define an oriented basis of $T\mathbb{R}^4$ at each point is $e_1 f_2 - e_2 f_1 > 0$, which we can regard as $\operatorname{Im}(e\bar{f}) > 0$ or as a 2×2 determinant. If this condition and the last four equations of (14) both hold, then by Proposition 1.3.3 the data define a nonsingular hypercomplex structure on $\mathbb{R}^4$, preserved by an $\mathbb{R}^2$- action.

If $e = cf$ for some $c \in \mathbb{C}$, then there exists a coordinate change in x_3, x_4 taking $[g]$ to the flat conformal class on $\mathbb{R}^4$, but if e, f are not proportional we expect in general to get a nontrivial metric. Locally we can use this coordinate change to make e or f equal to 1, so that hypercomplex structures generated in this way really depend on only one holomorphic function.

Example 3. Here is another way to construct hypercomplex structures from holomorphic functions. Ashtekar's equations (12) may be written in a complex form:

$$[V_1 + iV_2, V_1 - iV_2] - [V_3 + iV_4, V_3 - iV_4] = 0, \qquad [V_1 + iV_2, V_3 - iV_4] = 0. \tag{15}$$

Let M be a complex surface, let (z_1, z_2) be local holomorphic coordinates, and define $V_1, \ldots, V_4$ locally by

$$V_1 + iV_2 = f_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2}, \qquad V_3 + iV_4 = f_3 \frac{\partial}{\partial z_1} + f_4 \frac{\partial}{\partial z_2}, \tag{16}$$

where f_j is a complex function on M. Substituting into (15), we find the equations are satisfied identically if $\partial f_j / \partial \overline{z}_k = 0$, i.e. if f_j is a holomorphic function w.r.t. the complex structure on M. So we can construct a hypercomplex structure on M locally out of four holomorphic functions $f_1, \ldots, f_4$, or globally out of two holomorphic vector fields.

The complex structure of M is of the opposite orientation to the new complex structures J_1, J_2, J_3 and commutes with them. If $u \cdot (V_1 + iV_2), u \cdot (V_3 + iV_4)$ commute for some nonzero holomorphic function u, then the metric is locally isomorphic to the flat metric on $\mathbb{R}^4$, but otherwise we in general expect a nontrivial metric. By a heuristic argument we can see that the hypercomplex structures really depend on one holomorphic function of two variables. The interesting thing about this construction is that it results in nontrivial structures that need have no Killing vectors, as compatibility with a complex structure does instead of an assumption of symmetry. The example will be generalized in §2.5.

2.2. Self-dual metrics with a Killing vector. In [6], Jones and Tod show that self-dual metrics with a Killing vector can be constructed from the solution of a linear equation (the generalized monopole equation) over a 3-manifold possessing an Einstein-Weyl structure. We shall now reprove this using Corollary 1.2.3, by generalizing Example 1 of §2.1. Let N be an oriented 3-manifold, M be $N \times \mathbb{R}$, and t be the projection $M \to \mathbb{R}$. Then dt is a 1-form on M, and $\partial / \partial t$ is a vector field which will be the Killing vector. Let γ be a smooth, positive function on N, ϕ be a 1-form on N, and h be a Riemannian metric on N. Define a metric g on M by $g = h + \gamma^{-2}(dt + \phi)^2$.

Let $H \subset TM$ be the subbundle of zeros of the 1-form $dt + \phi$, and let π be the projection from H to TN; π is an isomorphism on the fibres of H, and H is the orthogonal subspaces to $\partial/\partial t$. The metric h and the orientation on N together define the Hodge star $*_N$ of N, where $*_N : T^*N \to \Lambda^2 T^*N$ and $*_N : \Lambda^2 T^*N \to T^*N$. These lift to maps $\pi^*(*_N) : H^* \to \Lambda^2 H^*$ and $\pi^*(*_N) : \Lambda^2 H^* \to H^*$ on M. The splitting $T^*M = H^* \oplus \langle dt + \phi \rangle$ induces a splitting $\Lambda^2 T^*M = \Lambda^2 H^* \oplus H^* \wedge \langle dt + \phi \rangle$, and in this splitting, the Hodge star $*_M$ of g is

$$*_M \left(x, y \wedge \frac{dt + \phi}{\gamma} \right) = \left(\pi^*(*_N) y, \pi^*(*_N) x \wedge \frac{dt + \phi}{\gamma} \right), \tag{17}$$

where $x \in \Gamma(\Lambda^2 H^*)$ and $y \in \Gamma(H^*)$. This holds because $T^*M = H^* \oplus \langle dt + \phi \rangle$ is an orthogonal splitting w.r.t. g, and g agrees with $\pi^*(h)$ on H.

Let ∇^N be a torsion-free connection on N preserving the conformal class $[h]$ of h, such that $\nabla^N h = -2\mu \otimes h$ for some 1-form μ on N. Then the lift $\pi^*(\nabla^N)$ of ∇^N to H is a connection on H that preserves the conformal class $\pi^*\big([h]\big)$ on the fibres of H. Define a connection ∇ on M by $\nabla = \pi^*(\nabla^N) \oplus \nabla_\gamma$, where ∇_γ is the connection on the line bundle on M of vectors proportional to $\partial/\partial t$ defined by $\nabla_\gamma(\gamma\partial/\partial t) = \mu \otimes (\gamma\partial/\partial t)$. Then $\nabla g = -2\mu \otimes g$, so ∇ preserves the conformal class $[g]$.

To apply Corollary 1.2.3 to g and ∇ we need the torsion and curvature of ∇, which are given in the next two Lemmas. The proofs are straightforward calculations and are omitted, as they are similar to those of §2.3.

LEMMA 2.2.1. *In the situation above, $T^-(\nabla) = 0$ if and only if $d\gamma - \gamma\mu = -*_N d\phi$.*

LEMMA 2.2.2. *In the situation above, let $\zeta \in \Gamma(TN \otimes T^*N \otimes \Lambda^2 T^*N)$ be the section formed from the identity section of $\Lambda^2 TN \otimes \Lambda^2 T^*N$, viewed as a section of $TN \otimes TN \otimes \Lambda^2 T^*N$, by converting the second tensor factor to T^*N using the metric h of N. Let $R_0(\nabla^N)$ be the $SO(3)$- component of the curvature of the $CO(3)$- connection ∇^N. Then $W^-(\nabla) = 0$ if and only if $R_0(\nabla^N) = \sigma\zeta$ for some $\sigma \in C^\infty(N)$.*

Combining the two Lemmas and Corollary 1.2.3 gives

PROPOSITION 2.2.3. *In the situation above, suppose the data h, ϕ and γ used to construct g and the data ∇^N and μ used to construct ∇ satisfy $d\gamma - \gamma\mu = - *_N d\phi$ and $R_0(\nabla^N) = \sigma\zeta$ for some $\sigma \in C^\infty(N)$. Then g is self-dual.*

Proof. The connection ∇ preserves $[g]$, and by Lemmas 2.2.1 and 2.2.2, the conditions ensure that $W^-(\nabla) = T^-(\nabla) = 0$. Thus by Corollary 1.2.3, g is self-dual. □

We interpret the Proposition as follows. The component $R_0(\nabla^N)$ is the symmetric part of the Ricci curvature $\mathrm{Ric}(\nabla^N)$, and the projection of ζ to the space of Ricci tensors is h. So $R_0(\nabla^N) = \sigma\zeta$ if and only if the symmetric Ricci curvature of ∇^N is σh, which is the condition of §1.1 for $\big(\nabla^N, [h]\big)$ to be an Einstein-Weyl structure on N. The equation $d\gamma - \gamma\mu = - *_N d\phi$ is equivalent to the generalized monopole equation given by Jones and Tod [6, p. 567, eq. (2.14)]. This recovers their description of self-dual metrics with a Killing vector as solutions of a monopole equation over an Einstein-Weyl 3-manifold.

2.3. Self-dual metrics with two Killing vectors. We now give a similar construction for self-dual metrics on a product $M = N \times T^2$ for N a Riemann surface, following Example 2 of §2.1. The metrics are invariant under translations in T^2, so they have two commuting Killing vectors, and are compatible with the product structure in the sense that the tangent spaces of the factors are orthogonal. The analogue of the Einstein-Weyl 3-manifold of §2.2 is the 2-manifold N equipped with some geometric structure, and the analogue of the monopole equation is a linear equation on sections of a bundle.

Let N be an oriented 2-manifold, L be a vector bundle over N with oriented fibre $\mathbb{R}^2$, and ϕ be a section of $L^* \otimes \mathbb{R}^2$ over N that is invertible and orientation-preserving as a map $L \to \mathbb{R}^2$ at each point of N. Let $M = N \times T^2$. Then $TM = TN \oplus T(T^2)$ and $T^*M = T^*N \oplus T^*(T^2)$. Now the tangent space of T^2 is $\mathbb{R}^2$, so ϕ defines an identification between L and $T(T^2)$ and between L^* and $T^*(T^2)$, and with this identification we can write $TM = TN \oplus L$, $T^*M = T^*N \oplus L^*$. We shall make frequent use of this identification, to avoid writing the many factors of ϕ and ϕ^{-1} that would otherwise appear.

We shall also use the following convention: the Latin symbols a, b, c, d, e and the Greek symbols $\alpha, \beta, \gamma, \delta, \epsilon$ will be used as tensor indices, to indicate a tensor factor of TM or T^*M when they appear as a superscript or subscript, in the usual way. However, an index $a, \ldots, e$ means the tensor lies in TN or T^*N in that factor in the splittings $TM = TN \oplus L$, $T^*M = T^*N \oplus L^*$, and an index $\alpha, \ldots, \epsilon$ means the tensor lies in L or L^* in that factor. Moreover, when adding tensors, a and α will indicate the same tensor index, and so on. For instance, if V is a vector field on M we would write $v = v^a + v^\alpha$, where v^a, v^α are the components of v in TN and L, and if F is a tensor in $T^*N \otimes L^*$ then $F = F_{b\gamma}$, and $F_{b\gamma} - F_{c\beta}$ is the antisymmetrization of F and lies in $\Lambda^2 T^*M$.

Using this convention, let h_{ab} be a metric on N and $h_{\alpha\beta}$ a metric on the fibres of L over N. Define a metric g on M by $g = h_{ab} + h_{\alpha\beta}$, and let $[g]$ be the conformal class of g. The metrics $h_{ab}, h_{\alpha\beta}$ and the orientations of TN, L also define volume forms $\Omega_{ab}, \Omega_{\alpha\beta}$ and complex structures J^a_b, J^α_β on the fibres of TN and L respectively. Let μ be a 1-form on N, and let ∇^N, ∇^L be connections on TN, L over N such that ∇^N is torsion-free, and $\nabla^N h_{ab} = \mu \otimes h_{ab}$, $\nabla^L h_{\alpha\beta} = \mu \otimes h_{\alpha\beta}$. Then ∇^N, ∇^L lift to give a connection $\nabla^N \oplus \nabla^L$ on $TM = TN \oplus L$ over M, which satisfies $(\nabla^N \oplus \nabla^L) g = \mu \otimes g$, and therefore preserves $[g]$.

Let C be a section of $L^* \otimes TN \otimes L^*$ over N satisfying $C^a_{\beta\gamma} = C^a_{\gamma\beta}$, $C^a_{\beta\gamma} h^{\beta\gamma} = 0$ and $C^a_{\beta\gamma} = -J^a_d J^\epsilon_\beta C^d_{\epsilon\gamma}$. Then C is a section of a 2-dimensional subbundle of $L^* \otimes TN \otimes L^*$. From C we can form the tensors $h^{\alpha\delta} h_{ce} C^e_{\beta\delta}$ and $h^{\alpha\delta} h_{be} C^e_{\delta\gamma}$. We shall denote these $C^\alpha_{\beta c}$ and $C^\alpha_{b\gamma}$ respectively; as C is symmetric in the L^* indices there is no ambiguity here. Define a connection ∇ on M by

$$\nabla = \nabla^N \oplus \nabla^L + C^a_{\beta\gamma} - C^\alpha_{\beta c}. \tag{18}$$

The equation $C^a_{\beta\gamma} = -J^a_d J^\epsilon_\beta C^d_{\epsilon\gamma}$ implies that the last two terms in this equation are a section of $T^*N \otimes E^-$ in the notation of §1.1, so they modify only the $SO(3)^-$- component of the connection. Thus as $\nabla^N \oplus \nabla^L$ preserves $[g]$, ∇ does too.

LEMMA 2.3.1. *The torsions $T(\nabla^N \oplus \nabla^L)$, $T(\nabla)$ are given by*

$$T(\nabla^N \oplus \nabla^L) = -\nabla^L_b \phi_\gamma + \nabla^L_c \phi_\beta \quad \textit{and} \quad T(\nabla) = -\nabla^L_b \phi_\gamma + \nabla^L_c \phi_\beta + \phi_\alpha C^\alpha_{b\gamma} - \phi_\alpha C^\alpha_{c\beta}, \tag{19}$$

where the equations are in sections of $\mathbb{R}^2 \otimes T^*N \wedge L^*$.

Proof. Let ∇^0 be the trivial connection on T^2. Then $\nabla^N \oplus \nabla^0$ is a torsion-free connection on $M = N \otimes T^2$. A calculation shows that $\nabla^N \oplus \nabla^L = \nabla^N \oplus \nabla^0 - \nabla^L_b \phi_\gamma$, where the last term is a section of $T^*N \otimes \mathbb{R}^2 \otimes L^*$, but is regarded as a section of $T^*M \otimes TM \otimes T^*M$ and so can be added on to a connection. So the difference between the torsions of $\nabla^N \oplus \nabla^L$ and $\nabla^N \oplus \nabla^0$ is the antisymmetrization of $-\nabla^L_b \phi_\gamma$, which gives the first part of (19).

To get the torsion of ∇, by (18) we must antisymmetrize $C^a_{\beta\gamma} - C^\alpha_{\beta c}$ and add it to the torsion of $\nabla^N \oplus \nabla^L$. But $C^a_{\beta\gamma} = C^a_{\gamma\beta}$ so the first term makes no contribution, and the answer is the second part of (19). □

LEMMA 2.3.2. *In the notation of §1.1, the trace-free curvature $R_0(\nabla)$ is given by*

$$
\begin{aligned}
R_0(\nabla) = &- S_N J^a_b \Omega_{cd} - S_L J^\alpha_\beta \Omega_{cd} - \tfrac{1}{2}\|C\|^2 (J^\alpha_\beta \Omega_{\gamma\delta} - J^a_b \Omega_{\gamma\delta}) - (C^a_{\epsilon\beta} - C^\alpha_{\epsilon b})(C^\epsilon_{c\delta} - C^\epsilon_{d\gamma}) \\
&+ \nabla^{N,L}_c (C^a_{\delta\beta} - C^\alpha_{\delta b}) - \nabla^{N,L}_d (C^a_{\gamma\beta} - C^\alpha_{\gamma b}) + (C^a_{\epsilon\beta} - C^\alpha_{\epsilon b})\big(T(\nabla)^\epsilon_{c\delta} + T(\nabla)^\epsilon_{\gamma d}\big),
\end{aligned} \tag{20}
$$

where $\nabla^{N,L}$ is the connection on $L^* \otimes TN \otimes L^*$ induced by ∇^N and ∇^L and S_N, S_L are the scalar curvatures of the $U(1)$- components of $R(\nabla^N)$ and $R(\nabla^L)$.

Proof. Let $\nabla - \nabla^N \oplus \nabla^L = G^i_{jk}$. Then by the formula for the curvature of a connection

$$
R_0(\nabla)^i_{jkl} = R_0(\nabla^N \oplus \nabla^L)^i_{jkl} + D^{N,L}_k G^i_{lj} + G^i_{km} G^m_{lj} - G^i_{lm} G^m_{kj}, \tag{21}
$$

where $D^{N,L}$ is the combination of the exterior derivative d acting on the first T^*M factor of G, and $\nabla^N \oplus \nabla^L$ acting on the remaining tensor factor $TM \otimes T^*M$. It is more convenient to use $\nabla^{N,L}$ rather than $D^{N,L}$, using the formula

$$
D^{N,L}_k G^i_{lj} = \nabla^{N,L}_k G^i_{lj} - \nabla^{N,L}_l G^i_{kj} + G^i_{mj} T^m_{kl}, \tag{22}
$$

where $T = T(\nabla^N \oplus \nabla^L)$. Substituting into (21) and using (18) to give G^i_{jk} and Lemma 2.3.1 to give the form of T, we find that $R_0(\nabla)$ is

$$
\begin{aligned}
&R_0(\nabla^N \oplus \nabla^L) + \nabla^{N,L}_c (C^a_{\delta\beta} - C^\alpha_{\delta b}) - \nabla^{N,L}_d (C^a_{\gamma\beta} - C^\alpha_{\gamma b}) + (C^a_{\epsilon\beta} - C^\alpha_{\epsilon b})(T^\epsilon_{c\delta} + T^\epsilon_{\gamma d}) \\
&+ (C^a_{\gamma\epsilon} - C^\alpha_{\gamma e})(C^e_{\delta\beta} - C^\epsilon_{\delta b}) - (C^a_{\delta\epsilon} - C^\alpha_{\delta e})(C^e_{\gamma\beta} - C^\epsilon_{\gamma b}).
\end{aligned} \tag{23}
$$

Now $R_0(\nabla^N \oplus \nabla^L) = -S_N J^a_b \Omega_{cd} - S_L J^\alpha_\beta \Omega_{cd}$, and calculating in coordinates shows that the terms on the last line are equal to $-\frac{1}{2}\|C\|^2 (J^\alpha_\beta \Omega_{\gamma\delta} - J^a_b \Omega_{\gamma\delta})$. Also, by Lemma 2.3.1 we have $T = T(\nabla^N \oplus \nabla^L) = T(\nabla) - C^\alpha_{b\gamma} + C^\alpha_{c\beta}$. Substituting these things into (23) and rearranging gives (20). □

We can now prove the main result of this section, the promised construction for self-dual metrics.

THEOREM 2.3.3. *Suppose the following three conditions are satisfied:*
(i) $\nabla^L_a \phi_\beta + J^c_a J^\delta_\beta \nabla^L_c \phi_\delta = 2\phi_\gamma C^\gamma_{a\beta}$,
(ii) $S_N - S_L + 2\|C\|^2 = 0$, *and*
(iii) $J^e_b \nabla^{N,L}_e C^a_{\gamma\delta} = J^\epsilon_\gamma \nabla^{N,L}_b C^a_{\epsilon\delta}$.
Then the conformal class $[g]$ defined above on $M = N \times T^2$ is self-dual.

Proof. We shall apply Corollary 1.2.3, and to do this we need $W^-(\nabla) = T^-(\nabla) = 0$. The Hodge star $*$ of g acts on $\Lambda^2 T^*M$ by

$$
*\Omega_{ab} = \Omega_{\alpha\beta}, \qquad *\Omega_{\alpha\beta} = \Omega_{ab}, \qquad *(F_{a\beta} - F_{b\alpha}) = -J^c_a J^\delta_\beta F_{c\delta} + J^d_b J^\gamma_\alpha F_{d\gamma}. \tag{24}
$$

Thus by Lemma 2.3.1, $T^-(\nabla) = 0$ if and only if

$$
-\nabla^L_a \phi_\beta + \phi_\epsilon C^\epsilon_{a\beta} + J^c_a J^\delta_\beta (-\nabla^L_c \phi_\delta + \phi_\epsilon C^\epsilon_{c\delta}) = 0. \tag{25}
$$

But $C^\epsilon_{a\beta} = J^c_a J^\delta_\beta C^\epsilon_{c\delta}$ as $C^a_{\beta\gamma} = -J^a_d J^\epsilon_\beta C^d_{\epsilon\gamma}$, so (25) is condition (i).

Next we calculate the anti-self-dual component $R^-(\nabla)^-$ of $R^-(\nabla)$ from Lemma 2.3.2 and equation (24). As condition (i) ensures that $T^-(\nabla) = 0$, the terms in $T(\nabla)$ in (20) make no contribution to $R^-(\nabla)^-$. Also, condition (iii) is the necessary and sufficient condition for the terms in $\nabla^{N,L}C$ to make no contribution to $R^-(\nabla)^-$. Computing in coordinates we find that the contribution from the term $-(C^a_{\epsilon\beta} - C^\alpha_{\epsilon b})(C^\epsilon_{c\delta} - C^\epsilon_{d\gamma})$ is $\|C\|^2\Xi/4$, where

$$\Xi = \delta^a_d h_{\beta\gamma} - \delta^a_c h_{\beta\delta} + \delta^\alpha_\delta h_{bc} - \delta^\alpha_\gamma h_{bd} + J^a_c \Omega_{\beta\delta} - J^a_d \Omega_{\beta\gamma} + J^\alpha_\gamma \Omega_{bd} - J^\alpha_\delta \Omega_{bc}. \qquad (26)$$

Therefore if conditions (i) and (iii) hold we have

$$4R^-(\nabla)^- = -(S_N - S_L + \|C\|^2)(J^a_b - J^\alpha_\beta)(\Omega_{cd} - \Omega_{\gamma\delta}) + \|C\|^2\Xi. \qquad (27)$$

Now $R^-(\nabla)^-$ is a section of $S^2E^- \otimes K$, and $W^-(\nabla)$ is the projection of $R^-(\nabla)^-$ to $S^2_0E^- \otimes K$. Therefore $W^-(\nabla) = 0$ if and only if $R^-(\nabla)^-$ is proportional to the canonical section of $S^2E^- \otimes K$, which can be shown to be $(J^a_b - J^\alpha_\beta)(\Omega_{cd} - \Omega_{\gamma\delta}) + \Xi$. So by (27), $R^-(\nabla)^-$ is a multiple of the canonical section if and only if $S_N - S_L + \|C\|^2 = -\|C\|^2$, which is condition (ii). Thus if (i)-(iii) hold then $T^-(\nabla) = 0$ and $W^-(\nabla) = 0$, and by Corollary 1.2.3, $[g]$ is self-dual. □

2.4. Classifying T^2- invariant self-dual metrics on $N \times T^2$. Jones and Tod [6] show that all self-dual metrics with a non-vanishing Killing vector arise locally from Proposition 2.2.3. However, not all self-dual metrics with two commuting Killing vectors arise from Theorem 2.3.3; for instance, there are conformally flat metrics on $\mathcal{S}^3 \times \mathcal{S}^1$ that admit a T^2- action but have no compatible product structure $M = N \times T^2$, even locally. It will now be shown that the existence of such a product structure is sufficient for a T^2-invariant self-dual metric to come from Theorem 2.3.3.

LEMMA 2.4.1. *Suppose g is a T^2- invariant self-dual metric on $M = N \times T^2$ compatible with the product structure. Then g can be constructed using Theorem 2.3.3.*

Proof. It is trivial to define from g a metric h_{ab} on N, a line bundle L with metric $h_{\alpha\beta}$ and a nondegenerate, orientation-preserving section ϕ of $L^* \otimes \mathbb{R}^2$ such that $g = h_{ab} + h_{\alpha\beta}$, using ϕ to identify L and $T(T^2)$. Let μ be any 1-form on N and ∇^N be the unique torsion-free connection on N such that $\nabla^N h_{ab} = \mu \otimes h_{ab}$. Condition (i) of Theorem 2.3.3 gives a restriction on ∇^L and C, and it can be shown that ∇^L and C are uniquely defined by (i) and the condition $\nabla^L h_{\alpha\beta} = \mu \otimes h_{\alpha\beta}$. Let ∇ be the connection defined above on M. By Lemma 2.3.1, as condition (i) of Theorem 2.3.3 holds, $T^-(\nabla) = 0$. Since g is self-dual, Lemma 1.2.2 shows that $W^-(\nabla) = 0$. But the calculation of $W^-(\nabla)$ in Theorem 2.3.3 shows that conditions (ii) and (iii) of the Theorem also hold. Thus g can be constructed using Theorem 2.3.3. □

Let us regard the data $L, h_{ab}, h_{\alpha\beta}, C, \nabla^N$ and ∇^L of §2.3 as making up a single geometric structure on N, that must satisfy conditions (ii) and (iii) of Theorem 2.3.3. The remainder of this section classifies these geometric structures. We begin with an example which will be shown below to be in some sense generic. Let N be the hyperbolic plane

$\mathcal{H}^2$ with metric h_{ab} of scalar curvature $S_N = -1$ and ∇^N be the Levi-Civita connection of h_{ab}. Regarding TN as a complex line bundle over N, let L be the square-root of TN, i.e. a complex line bundle equipped with an identification $L \otimes_{\mathbb{C}} L \cong TN$. Then h_{ab} and ∇^N induce a metric $h_{\alpha\beta}$ and a connection ∇^L on L. The connection ∇^L preserves $h_{\alpha\beta}$, and its scalar curvature is $S_L = S_N/2 = -1/2$.

The complex identification $L \otimes_{\mathbb{C}} L \cong TN$ corresponds to a real identification $S^2_0 L \cong TN$. But therefore $TN \otimes S^2_0 L^* \cong TN \otimes T^*N$, and so the identity section in $TN \otimes T^*N$ gives rise to a natural section C of $TN \otimes S^2_0 L^*$; let C be normalized so that $\|C\|^2 = 1/4$ in the metric induced on $TN \otimes L^* \otimes L^*$ by h_{ab} and $h_{\alpha\beta}$. Then C satisfies the conditions $C^a_{\beta\gamma} = C^a_{\gamma\beta}$, $C^a_{\beta\gamma} h^{\beta\gamma} = 0$ and $C^a_{\beta\gamma} = -J^a_d J^\epsilon_\beta C^d_{\epsilon\gamma}$. Let ϕ be a section of $L^* \otimes \mathbb{R}^2$ over N, that is nondegenerate and orientation-preserving as a map $L \to \mathbb{R}^2$.

Lemma 2.4.2. *With the definitions above, if*

$$\nabla^L_a \phi_\beta + J^c_a J^\delta_\beta \nabla^L_c \phi_\delta = 2\phi_\gamma C^\gamma_{a\beta}, \tag{28}$$

then Theorem 2.3.3 defines a self-dual metric g on $M = \mathcal{H}^2 \times T^2$.

Proof. Condition (i) of the Theorem holds by (28), and conditions (ii) and (iii) hold since $S_N - S_L + 2\|C\|^2 = -1 + 1/2 + 2/4 = 0$, and $\nabla^{N,L} C = 0$ as C is the identity section. Therefore Theorem 2.3.3 gives a self-dual metric g on $\mathcal{H}^2 \times T^2$. □

Lemma 2.4.3. *Suppose g is a self-dual metric on $N \times T^2$ constructed using Theorem 2.3.3. If the data C of §2.3 is nowhere zero, then g is locally conformal to a metric constructed using Lemma 2.4.2.*

Proof. When C does not vanish, there is a unique conformal factor χ on N for which C satisfies $\|C\|^2 = 1/4$ w.r.t. χh_{ab} and $\chi h_{\alpha\beta}$. Define a new set of variables by $\tilde{C} = C$, $\tilde{\phi} = \phi$, $\tilde{h}_{ab} = \chi h_{ab}$, $\tilde{h}_{\alpha\beta} = \chi h_{\alpha\beta}$, and $\tilde{g} = \tilde{h}_{ab} + \tilde{h}_{\alpha\beta}$. Let $\tilde{\nabla}^N$ be the Levi-Civita connection of $\tilde{h}_{ab}$. It can easily be shown that there is a unique connection $\tilde{\nabla}^L$ on L for which $\tilde{\nabla}^L \tilde{h}_{\alpha\beta} = 0$ and condition (i) of Theorem 2.3.3 applies to the new variables. Let $\tilde{S}_N, \tilde{S}_L$ be the scalar curvatures of $\tilde{\nabla}^N$ and $\tilde{\nabla}^L$.

Since $\tilde{g}$ is conformal to g and so is self-dual and condition (i) of Theorem 2.3.3 applies to the new variables, conditions (ii) and (iii) apply as well, as in Lemma 2.4.1. Now C gives an identification between TN and $S^2_0 L$, and therefore identifies L and the square-root of TN as a complex line bundle. Computing in coordinates we find that condition (iii) of Theorem 2.3.3 can only hold if $\tilde{\nabla}^L$ is identified with the connection induced on the square-root of TN by $\tilde{\nabla}^N$. But this implies that $\tilde{S}_L = \tilde{S}_N/2$, and as $\|\tilde{C}\|^2 = 1/4$, condition (ii) of Theorem 2.3.3 gives that $\tilde{S}_N = -1$. So $\tilde{h}_{ab}$ has scalar curvature -1, and is locally isometric to the hyperbolic metric on $\mathcal{H}^2$. Thus we have shown that $\tilde{g}$ is constructed using Lemma 2.4.2. □

By the definition of C in §2.3, C is a section of a vector bundle with fibre $\mathbb{R}^2$ equipped with a complex structure and a connection preserving the complex structure. But this gives the vector bundle the structure of a holomorphic line bundle, and condition (iii) of Theorem 2.3.3 is that C should be a holomorphic section of this line bundle. So C behaves locally like a holomorphic function. Now a holomorphic function must either be identically zero, or have isolated zeros modelled on the zero of w^n at $w = 0$, for positive integers n. The case $C = 0$ is that of Example 2 in §2.1. Isolated zeros of C will be easier to understand after a diversion to look at anti-self-dual Kähler metrics.

PROPOSITION 2.4.4. *Let N be $\mathbb{C}$ with complex coordinate w and flat metric $h_{ab} = |dw|^2$, and let L be the trivial line bundle $\mathbb{C}$ over N with complex coordinate z and metric $h_{\alpha\beta} = |dz|^2$. Let ϕ be a nondegenerate, orientation-preserving section of $L^* \otimes \mathbb{R}^2$, written $\phi = dz \otimes \Phi + d\overline{z} \otimes \overline{\Phi}$ for Φ a section of $\mathbb{C}^2$ over N. Let $M = N \times T^2$, and as in §2.3 use ϕ to identify L with $T(T^2)$ in TM. Define an almost complex structure J on M by $J(\partial/\partial w) = \partial/\partial z$ and $J(\partial/\partial z) = -\partial/\partial w$, using this identification. Let ξ be a smooth, positive function on N and $\tilde{g}$ be the metric $\xi \det\phi \cdot (h_{ab} + h_{\alpha\beta})$ on M, where $\det\phi$ is a section of $\Lambda^2 L^* \otimes \Lambda^2\mathbb{R}^2$, which is identified with $\mathbb{R}$ to give a real function. Suppose that $\partial^2\xi/\partial w\partial\overline{w} = 0$ and*

$$2\xi\frac{\partial\Phi}{\partial\overline{w}} + \frac{\partial\xi}{\partial\overline{w}}\Phi + \frac{\partial\xi}{\partial w}\overline{\Phi} = 0. \tag{29}$$

Then J is an integrable complex structure and $\tilde{g}$ is anti-self-dual w.r.t. the complex orientation, and Kähler w.r.t. J.

Proof. As the orientation induced by J is the opposite to our usual orientation on $N\times T^2$, $\tilde{g}$ is anti-self-dual in the complex orientation if it is self-dual in our usual orientation. We shall apply Theorem 2.3.3 to show that $\tilde{g}$ is self-dual in the usual orientation, and in addition show that J is integrable and $\tilde{g}$ is Kähler. Let g be the metric $h_{ab} + h_{\alpha\beta}$ on M, which is conformal to $\tilde{g}$. Let ∇^N be the flat connection on N, and using complex notation, define a connection ∇^L on L and a section C of $L^* \otimes TN \otimes L^*$ by

$$\nabla^L\frac{\partial}{\partial z} = (Pdw - \overline{P}d\overline{w}) \otimes \frac{\partial}{\partial z} \qquad \text{and} \qquad C = Qdz \otimes \frac{\partial}{\partial w} \otimes dz + \overline{Q}d\overline{z} \otimes \frac{\partial}{\partial\overline{w}} \otimes d\overline{z}, \tag{30}$$

where P, Q are complex functions on N. Then ∇^L preserves $h_{\alpha\beta}$ and C satisfies the conditions in §2.3. Let ∇ be the connection on M defined in §2.3 using ∇^N, ∇^L and C.

Define $\tilde{\nabla}$ to be the Levi-Civita connection of $\tilde{g}$. The necessary and sufficient condition for J to be integrable and $\tilde{g}$ to be Kähler w.r.t. J is that $\tilde{\nabla}J = 0$. We shall apply the reasoning of the proof of Theorem 1.2.1 to the connections ∇, $\tilde{\nabla}$. Suppose that $T^-(\nabla) = 0$, which is condition (i) of Theorem 2.3.3 and by (30) works out to be

$$\frac{\partial\Phi}{\partial\overline{w}} + \overline{P}\Phi = Q\overline{\Phi}. \tag{31}$$

Then in the proof of Theorem 1.2.1 we have $\kappa = 0$, $T = \tau$ and $\lambda = -d\log(\xi\det\phi)/2$, so the 1-form μ is $d\log(\xi\det\phi)/4 - \tau^i_{ik}\omega^k/4$. Now $\tau = T(\nabla)$ is given by Lemma 2.3.1, and we find that τ^i_{ik} is the contraction of $\nabla^L\phi$ and ϕ^{-1}, which is easily shown to be equal to $d\log(\det\phi)$, so $\mu = (d\xi)/4\xi$. In complex notation, $\mu = Rdw + \overline{R}d\overline{w}$, where R is the complex function $(\partial\xi/\partial w)/4\xi$ on N.

From the proof of Theorem 1.2.1, the difference in the $SO(3)^-$- components of ∇ and $\tilde{\nabla}$ is S, where in complex notation, S is the real part of

$$-4Rdz \otimes \frac{\partial}{\partial z} \otimes dw + 4\overline{R}d\overline{z} \otimes \frac{\partial}{\partial w} \otimes dz + 2(Rdw - \overline{R}d\overline{w}) \otimes \left(\frac{\partial}{\partial z} \otimes dz - \frac{\partial}{\partial w} \otimes dw \right). \quad (32)$$

As J is a section of E^-, this implies that $\tilde{\nabla} J = 0$ if and only if $\nabla_j J^i_k = S^i_{jl} J^l_k - S^l_{jk} J^i_l$. But we can evaluate both sides of this equation using (30), (32) and the definition of J, and it turns out to hold if and only if $P = 2R$ and $Q = -2R$. Thus J is integrable and $\tilde{g}$ is Kähler w.r.t. J provided (31) holds with $P = 2R$ and $Q = -2R$. But substituting these values into (31) gives (29), so (29) is a sufficient condition for $(\tilde{g}, J)$ to be a Kähler structure on M.

It remains only to show that g is self-dual. We know already that condition (i) of Theorem 2.3.3 holds, so if (ii) and (iii) hold as well then the Theorem will show that g is self-dual, and the proof will be complete. From (30) we find that $S_L = -2\partial P/\partial\overline{w} - 2\partial\overline{P}/\partial w$ and $\|C\|^2 = 4|Q|^2$, and $S_N = 0$ as ∇^N is flat. Substituting in for P and Q, condition (ii) becomes $\partial^2\xi/\partial w\partial\overline{w} = 0$, which is assumed in the Proposition. A calculation shows that condition (iii) is equivalent to $\partial Q/\partial\overline{w} + 2\overline{P}Q = 0$, and substituting for P and Q, this also gives $\partial^2\xi/\partial w\partial\overline{w} = 0$. So conditions (ii) and (iii) of Theorem 2.3.3 hold, g is self-dual, and the Proposition is complete. □

Now a real function ξ on $N = \mathbb{C}$ satisfies $\partial^2\xi/\partial w\partial\overline{w} = 0$ if and only if it is the real part of a holomorphic function, which provides a great number of solutions. However, if $f : U \to V$ is a biholomorphism between two regions U, V of $\mathbb{C}$, then it is easy to see that the anti-self-dual Kähler structures constructed over V using some function ξ are the same as the anti-self-dual Kähler structures constructed over U using $\xi \circ f$. Thus if $\xi = \mathrm{Re}(e)$, where e is an invertible holomorphic function $V \to U$ with inverse f, then the structures constructed over V are the same as those constructed over U with the standard function $\xi = \mathrm{Re}(w)$. Locally, e is invertible if $\partial e/\partial w \neq 0$, and as C is proportional to $\partial e/\partial w$, this gives a similar statement to Lemma 2.4.3.

We can now provide a local model for self-dual metrics constructed using Theorem 2.3.3 in which C has an isolated zero in N. Consider the metrics constructed by Proposition 2.4.4 over a neighbourhood of 0 in N, using the function $\xi = \mathrm{Re}(1 + w^{n+1})$. This function is smooth and positive in the unit disc, and satisfies $\partial^2\xi/\partial w\partial\overline{w} = 0$, and by power series it can be shown that (29) admits many solutions. As C is proportional to $d\xi$, it has an isolated zero of order n at $w = 0$. Lemma 2.4.3 can easily be generalized to show that if g is a metric constructed using Theorem 2.4.3 for which C has an isolated zero of order n, then locally g can be constructed using Proposition 2.4.4 with function $\xi = \mathrm{Re}(1 + w^{n+1})$. Thus putting together all the results of this section, we have proved:

THEOREM 2.4.5. *Suppose g is a T^2- invariant self-dual metric on $M = N \times T^2$ compatible with the product structure. Then for every sufficiently small neighbourhood U of N, the restriction of g to $U \times T^2$ is constructed either using Example 2 of §2.1, or using Lemma 2.4.2, or using Proposition 2.4.4 in a neighbourhood of $w = 0$ with $\xi = \mathrm{Re}(1 + w^{n+1})$ for some positive integer n.*

This is a local classification of T^2- invariant self-dual metrics on $N \times T^2$. A paraphrase of the Theorem is that a T^2- invariant metric compatible with a product structure is locally constructed either over $\mathbb{R}^2$, or over $\mathcal{H}^2$, or over a branched cover of $\mathcal{H}^2$. For the inverse of $1 + w^{n+1}$ as a map $\mathbb{C} \to \mathbb{C}$ is a multifunction with a branch point of degree $n+1$ at 1, and using this one can regard the case of Proposition 2.4.4 with $\xi = \mathrm{Re}(1 + w^{n+1})$ as a construction over an $(n+1)$-fold branched cover of $\mathcal{H}^2$.

2.5. Self-dual hermitian metrics. Let M be a complex surface with complex structure J, and F a vector bundle over M with fibre $\mathbb{C}^2$. Let J' be the complex structure on the fibres of F, h a flat hermitian metric on the fibres, and ∇^F a connection on F preserving J' and h. Let ϕ be a section of $F^* \otimes TM$, that is invertible as a map $F \to TM$ and satisfies $\phi \circ J' = J \circ \phi$. Then using ϕ to identify F and TM, the metric h on the fibres of F defines a metric g on M, which is hermitian because h is hermitian and ϕ takes J' to J. Let ∇ be the connection on TM identified to ∇^F by ϕ. Then ∇ preserves g and J.

Now J splits 2-forms into the 2-forms of type $(1,1)$ and the 2-forms of type $(2,0) + (0,2)$, and as g is hermitian w.r.t. J, the forms of type $(2,0)+(0,2)$ are self-dual. We shall find conditions for $T = T(\nabla)$ and $R^-(\nabla)$ to be of type $(2,0) + (0,2)$, so that $T^-(\nabla) = W^-(\nabla) = 0$ and by Corollary 1.2.3, g is self-dual. A complex notation will be adopted throughout, by working in the complex vector bundles $F \otimes \mathbb{C}, TM \otimes \mathbb{C}$. In this notation, ϕ will be written $\phi = \Phi + \overline{\Phi}$, where Φ is the complex linear component of ϕ, and is a section of $(1 - iJ')F^* \otimes (1 - iJ)TM \subset F^* \otimes TM \otimes \mathbb{C}$.

The condition for T to be of type $(2,0) + (0,2)$ is that $T(x - iJx, y + iJy) = 0$ for all (real) sections x, y of TM. Let a, b be sections of $F \otimes \mathbb{C}$ of the form $a = a' - iJ'a'$, $b = b' - iJ'b'$, and v, w be the complex vector fields $v = \phi(a), w = \phi(b)$. Then

$$\begin{aligned} T(v, \overline{w}) &= \nabla_v \overline{w} - \nabla_{\overline{w}} v - [v, \overline{w}] \\ &= \overline{\Phi} \circ \nabla^F_v \overline{b} - \Phi \circ \nabla^F_{\overline{w}} a - [v, \overline{w}] \\ &= \overline{\Phi} \circ \partial^F_v \overline{b} - \Phi \circ \overline{\partial}^F_{\overline{w}} a - \partial_v\big(\overline{\Phi}(\overline{b})\big) + \overline{\partial}_{\overline{w}}\big(\Phi(a)\big) \\ &= -\big(\partial^F_v \overline{\Phi}\big)(b) + \big(\overline{\partial}^F_{\overline{w}} \Phi\big)(a). \end{aligned} \tag{33}$$

In this equation the operators $\overline{\partial}, \partial$ are the usual complex derivatives acting on sections of $(1-iJ)TM$ and $(1+iJ)TM$ respectively, the actions of $\overline{\partial}^F, \partial^F$ on $(1-iJ')F$ and $(1+iJ')F$ are the restrictions of the action of ∇^F, and the actions of $\overline{\partial}^F, \partial^F$ on $(1 - iJ')F^* \otimes (1 - iJ)TM$ and $(1 + iJ')F^* \otimes (1 + iJ)TM$ are the results of coupling the operators $\overline{\partial}, \partial$ and the operators $\overline{\partial}^F, \partial^F$ on $(1 - iJ')F^*$ and $(1 + iJ')F^*$.

From (33) we see that T is of type $(2,0) + (0,2)$ if $\overline{\partial}^F \Phi = 0$, as the complex conjugate of this equation gives $\partial^F \overline{\Phi} = 0$. This is a linear condition, which is surprising, and is the condition for Φ to be in some sense holomorphic. The solutions are only truly holomorphic if F has the structure of a holomorphic bundle over M compatible with ∇^F, in the sense that $\overline{\partial}^F$ is equal to the operator $\overline{\partial}$ defining the holomorphic structure. The condition for this is that $R(\nabla^F)$ should be of type $(1,1)$, and the holomorphic structure is then unique. Conversely, if F is a holomorphic bundle with a metric h in the fibres, there is a unique connection ∇^F preserving h compatible with the holomorphic structure.

Lemma 2.5.1. *Let M be a complex surface with hermitian metric g, and let ∇ be the unique connection on TM that preserves g and is compatible with the holomorphic structure of TM. Then $T(\nabla)$ is of type $(2,0) + (0,2)$ so that $T^-(\nabla) = 0$, and $W^-(\nabla) = W^-(g)$.*

Proof. Set $F = TM$, $h = g$ and $\nabla^F = \nabla$, and let ϕ be the identity map. Then as ϕ is holomorphic, $\overline{\partial}^F \Phi = 0$ and so from above, $T(\nabla)$ is of type $(2,0)+(0,2)$ and $T^-(\nabla) = 0$. Thus by Lemma 1.2.2, $W^-(\nabla) = W^-(g)$. $\square$

Suppose on the other hand that ∇^F has nontrivial curvature component of type $(2,0)+(0,2)$. Then F does not admit a compatible holomorphic structure, but in small neighbourhoods of M we expect the behaviour of solutions to $\overline{\partial}^F \Phi = 0$ to be very similar to the case ∇^F flat, and therefore that there will be many solutions to the equation, locally resembling quadruples of holomorphic functions. With these solutions we can make self-dual hermitian metrics, generalizing Example 3 of §2.1.

PROPOSITION 2.5.2. *Let M be a complex surface with complex structure J, and F a vector bundle over M with fibre $\mathbb{C}^2$, equipped with complex structure J', hermitian metric h, and connection ∇^F preserving h and J'. Let ϕ be an invertible section of $F^* \otimes TM$ satisfying $\phi \circ J' = J \circ \phi$, and let Φ be the complex linear component of ϕ in $F^* \otimes TM \otimes \mathbb{C}$. Define g to be the hermitian metric on M that is identified to h by ϕ. If $R^-(\nabla^F)$ is of type $(2,0)+(0,2)$ and Φ satisfies the linear equation $\overline{\partial}^F \Phi = 0$, then g is self-dual.*

Proof. Suppose $R^-(\nabla^F)$ is of type $(2,0)+(0,2)$ and $\overline{\partial}^F \Phi = 0$. Define the connection ∇ on M as above. Then ∇ preserves g, and from above $T^-(\nabla) = 0$ as $\overline{\partial}^F \Phi = 0$. Also, as $R^-(\nabla)$ is identified to $R^-(\nabla^F)$ by ϕ, $R^-(\nabla)$ is of type $(2,0)+(0,2)$ and so is self-dual, and thus $W^-(\nabla) = 0$. Therefore by Corollary 1.2.3, g is self-dual. $\square$

When ∇^F is flat we get the case of Example 3 of §2.1. Proposition 2.5.2 is similar to Proposition 2.2.3 and Theorem 2.3.3 in that it makes self-dual metrics using some fixed 'background' data M, F, h and ∇^F, and some data ϕ that satisfies a *linear* equation. The curious thing about this construction is that the metrics produced need not have Killing vectors. The Proposition is unlikely to yield new compact, self-dual, hermitian surfaces because a nontrivial compact self-dual 4-manifold M must have $b^+ - b^- > 0$, and this is a very restrictive condition on a compact surface.

3. Explicit self-dual metrics on compact 4-manifolds. In this chapter we shall construct compact self-dual 4-manifolds using Lemma 2.4.2. Firstly, some results of Orlik and Raymond on 4-manifolds with T^2- actions are summarized in §3.1. Then in §3.2, solutions will be found in coordinates to equation (28), which form the building blocks for many families of self-dual metrics on $\mathcal{H}^2 \times T^2$ constructed by the Proposition. Using the ideas of §3.1, it will be shown in §§3.3 and 3.4 that $\mathcal{H}^2 \times T^2$ or its quotient by $\mathbb{Z}$ can be included as an open dense set in compact 4-manifolds M in such a way that one of these self-dual metrics extends to a nonsingular, self-dual conformal structure on all of M. This gives a family of T^2- invariant self-dual metrics on $n\mathbb{CP}^2$ for each T^2- action on $n\mathbb{CP}^2$, and also self-dual metrics on some compact 4-manifolds M with $\pi_1(M) = \mathbb{Z}$ and $b^-(M) = 0$.

3.1. Compact 4-manifolds with a T^2- action. We begin by summarizing some results of Orlik and Raymond [11], that give a general picture of how T^2 can act smoothly on a compact 4-manifold M. As each orbit must have dimension 0,1 or 2, the stabilizer of each point is a closed subgroup of T^2 of dimension 0,1 or 2. For simplicity suppose all points of M have connected stabilizers, to avoid considering finite subgroups of T^2, and suppose T^2 does not act freely. Then the orbits of T^2 in M are of three types:

(a) single points fixed by the action of T^2,

(b) circles stabilized by some T^1 subgroup of T^2, and
(c) tori on which T^2 acts freely.

By the Theorem on [11, p. 537], the orbit space N of T^2 on M is a compact 2-manifold with nonempty boundary. Let N° be $N \setminus \partial N$, and suppose that N° has no compact components. Then the inverse image of N° in M is $N^\circ \times T^2$, which is the union of all orbits of type (c), one orbit for each point of N°. The subset $N^\circ \times T^2$ is open and dense in M, and the complement in M is the union of the orbits ∂N, which are of type (a) or (b).

Following [11, §§4, 5], all orbits of type (a) are isolated, and as M is compact there can be only finitely many. So ∂N is a compact 1-manifold, i.e. a union of circles, containing a finite number of points $p_1, \ldots, p_k$ that are orbits of type (a), and the rest are orbits of type (b). To each orbit of type (b) we associate a T^1 subgroup of T^2, the stabilizer of the orbit. Now the T^1 subgroups of T^2 are of the form $G(m,n) = \{(\psi,\chi) : m\psi + n\chi = 0,\ \psi,\chi \in T^1\}$, for m,n coprime integers. Because $G(m,n) = G(-m,-n)$, (m,n) and $(-m,-n)$ represent the same subgroup. Thus the set of T^1 subgroups of T^2 is discrete.

As the stabilizer of the orbit cannot change in a discontinuous fashion on $\partial N \setminus \{p_1, \ldots, p_k\}$, the T^1 stabilizer subgroup must be constant on each component of $\partial N \setminus \{p_1, \ldots, p_k\}$. By studying the torus action in the neighbourhood of a fixed point p_j, Orlik and Raymond find that if (m,n) and (m',n') are the pairs of coprime integers, defined up to sign, representing the T^1 stabilizer subgroups of T^2 for the components of $\partial N \setminus \{p_1, \ldots, p_k\}$ on either side of p_j, then $(m,n), (m',n')$ must satisfy $mn' - m'n = \pm 1$.

For M to be simply-connected, N must be simply-connected, so N is the closed disc Δ. Using the Poincaré disc picture of $\mathcal{H}^2$, we may regard $\mathcal{H}^2$ as the interior of Δ. The orientation of $\mathcal{H}^2$ induces an orientation on $\partial\Delta$, giving a notion of positive and negative direction to the tangent vectors of $\partial\Delta$. Let $k \ge 2$ and $p_1, \ldots, p_k$ be a set of distinct points in $\partial\Delta$, ordered so that p_{j+1} is the point adjacent to p_j in the positive direction on $\partial\Delta$, and p_1 is the point adjacent to p_k. Let $(m_1,n_1), \ldots, (m_k,n_k)$ be pairs of coprime integers satisfying $m_j n_{j+1} - m_{j+1} n_j = \pm 1$ for $1 \le j < k$ and $m_k n_1 - m_1 n_k = \pm 1$.

Then there is a compact 4-manifold M with an action of T^2, such that the orbit space of M is Δ, the union of the orbits of type (c) is $\mathcal{H}^2 \times T^2$, the orbits of type (a) are $p_1, \ldots, p_k$, the orbits in the interval (p_j, p_{j+1}) of $\partial\Delta$ have stabilizer $G(m_j,n_j)$, and the orbits in the interval (p_k, p_1) have stabilizer $G(m_k,n_k)$. Orlik and Raymond (first Theorem on [11, p. 547], and [11, §5]) show that this data defines a compact, simply-connected 4-manifold M with a torus action, that all compact, simply-connected 4-manifolds arise in this way, and that the possible manifolds are $\mathcal{S}^4$ and connected sums of the manifolds $\mathbb{CP}^2, \overline{\mathbb{CP}}^2$ and $\mathcal{S}^2 \times \mathcal{S}^2$.

We may also get some topological data on M from the integers m_j, n_j. The most fundamental topological invariants of a compact, simply-connected 4-manifold are $b^2(M), b^+(M)$ and $b^-(M)$, where $b^2(M)$ is the second Betti number of M and $b^\pm(M)$ are the dimensions of the spaces of closed self-dual and anti-self-dual 2-forms with respect to some metric on M, and $b^2(M) = b^+(M) + b^-(M)$. From Orlik and Raymond's classification, it can be deduced fairly easily that $b^2(M) = k - 2$.

The invariants $b^+(M), b^-(M)$ are slightly more difficult to pin down, but they can be defined as follows. Recall that the coprime integers (m_j,n_j) are defined only up to a sign. Let the sign be chosen such that $m_j \ge 0$ and $n_j > 0$ if $m_j = 0$; what matters here is to confine (m_j, n_j) to some $180°$ sector of $\mathbb{R}^2$. Then $b^{\pm}(M)$ are given by

$$b^-(M) - b^+(M) = m_k n_1 - m_1 n_k + \sum_{j=1}^{k-1} m_j n_{j+1} - m_{j+1} n_j. \tag{34}$$

In the particular case that $b^-(M) = 0$, M is $(k-2)\mathbb{CP}^2$, as out of the possible summands of M above, only $\mathbb{CP}^2$ makes no contribution to $b^-(M)$. We can use this to characterize all T^2- actions on $n\mathbb{CP}^2$, in Orlik and Raymond's form.

PROPOSITION 3.1.1. *Let n be a positive integer and $k = n + 2$. Then every smooth action of T^2 on $n\mathbb{CP}^2$ can be described in the manner above by coprime integers $(m_1, n_1), \ldots, (m_k, n_k)$ satisfying*
(i) $m_1 = 0$, $n_1 = 1$, $m_k = 1$ *and* $n_k = 0$,
(ii) $m_j > 0$ *for* $j > 1$, *and*
(iii) $m_j n_{j+1} - m_{j+1} n_j = -1$ *for* $1 \leq j < k$.
Conversely, every sequence $(m_1, n_1), \ldots, (m_k, n_k)$ of pairs of coprime integers satisfying (i)-(iii) gives rise to a T^2- action on $n\mathbb{CP}^2$.

Proof. Given a T^2- action on $n\mathbb{CP}^2$, by the results quoted above, there are pairs $(m_1, n_1), \ldots, (m_k, n_k)$ of coprime integers satisfying certain conditions. Because $k = 2 + b^2(M)$ and M is $n\mathbb{CP}^2$, $b^2(M) = n$ and $k = n + 2$ as in the Proposition. Since (m_j, n_j) is only defined up to sign, we may choose the sign of (m_k, n_k) such that $m_k n_1 - m_1 n_k = 1$. The group of orientation-preserving outer automorphisms of T^2 is $SL(2, \mathbb{Z})$, and by applying a suitable automorphism if necessary, we may suppose $(m_1, n_1) = (0, 1)$ and $(m_k, n_k) = (1, 0)$, which gives condition (i). Choose the signs of the remaining (m_j, n_j) by requiring that $m_j \geq 0$, and $n_j > 0$ if $m_j = 0$. Then (34) holds with these sign choices. As $b^-(M) = 0$, $b^+(M) = n$, and the right hand side of (34) is the sum of $n + 2$ terms each of which is ± 1, we deduce that one of $m_k n_1 - m_1 n_k$ and $m_j n_{j+1} - m_{j+1} n_j$ for $1 \leq j < k$ must be 1, and all the rest -1.

But we chose $m_k n_1 - m_1 n_k$ to be 1, so $m_j n_{j+1} - m_{j+1} n_j = -1$ for $1 \leq j < k$, which gives condition (iii). If $m_j = 0$ for some $j > 1$, then $n_j > 0$ and $m_{j-1} n_j = -1$, so $m_{j-1} < 0$, a contradiction. Thus $m_j > 0$ if $j > 1$, which is condition (ii). We have shown that the data $(m_1, n_1), \ldots, (m_k, n_k)$ satisfies (i)-(iii), proving the first part of the Proposition. For the second part, from Orlik and Raymond's results above, conditions (i)-(iii) ensure that the data does represent a compact, simply-connected manifold M with a T^2- action, and from above M has $b^+(M) = n$ and $b^-(M) = 0$. By the remark just before the Proposition, this means M is $n\mathbb{CP}^2$, which completes the second part of the Proposition. $\square$

It is an interesting combinatorial problem to compute the number of T^2- actions on $n\mathbb{CP}^2$ up to equivariant diffeomorphisms. We cannot do this precisely, but we can find the number of sets of data m_j, n_j satisfying the conditions of the Proposition.

PROPOSITION 3.1.2. *There are $(2n)!/n!(n+1)!$ sets of integers (m_j, n_j) satisfying the conditions of Proposition 3.1.1.*

Proof. Let $(m_1, n_1), \ldots, (m_k, n_k)$ satisfy the conditions of Proposition 3.1.1, and draw a graph in the plane as follows. Start with a convex k-gon, with vertices numbered $1, \ldots, k$ clockwise in order. Then for all pairs i, j such that $i, j \in \{1, \ldots, k\}$, $i+1 < j$, $(i, j) \neq (1, k)$ and $m_i n_j - m_j n_i = -1$, join vertices i and j by a straight edge. It can be shown using induction on k and the proof of the Theorem on [11, p. 553] that none of these added edges meet inside the k- gon, and that the graph is a division of the k-gon with numbered points into $k-2$ triangles.

Conversely, we claim that every subdivision of the numbered k-gon into triangles gives rise to a set of data (m_j, n_j) satisfying the conditions of Proposition 3.1.1. This set of data is explicitly constructed as follows: for $k = 3$, the (m_j, n_j) are in order $(0,1), (1,1), (1,0)$, and if $(m_1, n_1), \ldots, (m_k, n_k)$ are the data for some subdivision of the k-gon, then the data for the subdivision of the $(k+1)$- gon obtained by adding a triangle onto the edge joining vertices $j, j+1$ and adding 1 to the numbers of vertices $j+1, \ldots, k$ is obtained by inserting $(m_j + m_{j+1}, n_j + n_{j+1})$ into the sequence $(m_1, n_1), \ldots, (m_k, n_k)$ between (m_j, n_j) and (m_{j+1}, n_{j+1}).

This gives a 1-1 correspondence between sets of data $(m_1, n_1), \ldots, (m_k, n_k)$ satisfying the conditions of Proposition 3.1.1, and subdivisions into $k-2$ triangles of the k-gon with numbered vertices. The problem of counting such subdivisions is an old combinatorial problem going back to Euler, and Brown [3] gives a historical survey of it; I am indebted to an article by Jan Stevens for bringing this paper to my attention. Brown states that the number of such subdivisions is $(2k-4)!/(k-2)!(k-1)!$, and as $k = n+2$, this is $(2n)!/n!(n+1)!$, which is what we have to prove. $\square$

3.2. Self-dual metrics on torus bundles over the hyperbolic plane. Using the upper half-plane model for the hyperbolic plane, we shall put Lemma 2.4.2 in a more explicit form:

PROPOSITION 3.2.1. *Let (x_1, x_2) be coordinates on $\mathbb{R}^2$ and let N be the region $x_1 > 0$. Let ϕ_1, ϕ_2 be smooth functions $N \to \mathbb{R}^2$, and suppose they satisfy the three conditions*
(i) $\phi_1 \wedge \phi_2$ is a positive section of $\Lambda^2\mathbb{R}^2 \cong \mathbb{R}$,
(ii) $\partial\phi_1/\partial x_1 + \partial\phi_2/\partial x_2 = \phi_1/x_1$, and
(iii) $\partial\phi_1/\partial x_2 - \partial\phi_2/\partial x_1 = 0$.
Let ψ_1, ψ_2 be the unique maps $N \to (\mathbb{R}^2)^*$ for which $\phi_1 \otimes \psi_1 + \phi_2 \otimes \psi_2$ is the identity section in $\mathbb{R}^2 \otimes (\mathbb{R}^2)^*$. Let $M = N \times T^2$, and identify $T(T^2)$ with $\mathbb{R}^2$ in the usual way. Then the metric $g = (dx_1^2 + dx_2^2)/x_1^2 + \psi_1^2 + \psi_2^2$ is a self-dual metric on M.

Proof. Let N have metric $h_{ab} = (dx_1^2 + dx_2^2)/x_1^2$, define orthonormal vector fields on N by $V_1 = x_1\partial/\partial x_1$, $V_2 = x_1\partial/\partial x_2$, and let ω_1, ω_2 be the dual basis of 1-forms, so that $\omega_1 = dx_1/x_1, \omega_2 = dx_2/x_1$. The Levi-Civita connection of h_{ab} is given by $\nabla^N V_1 = -\omega_2 \otimes V_2$ and $\nabla^N V_2 = \omega_2 \otimes V_1$, and the curvature of ∇^N is

$$R(\nabla^N) = -(V_1 \otimes \omega_2 - V_2 \otimes \omega_1) \otimes \omega_1 \wedge \omega_2. \tag{35}$$

Thus the scalar curvature of h_{ab} is -1, and (N, h_{ab}) is the hyperbolic plane $\mathcal{H}^2$.

Let l_1, l_2 be orthonormal sections of L, the square root of the tangent bundle, such that the identification $S_0^2 L \cong TN$ is given by $l_1 \otimes l_1 - l_2 \otimes l_2 \cong V_1$ and $l_1 \otimes l_2 + l_2 \otimes l_1 \cong V_2$.

Let λ_1, λ_2 be the dual basis of l_1, l_2 for L^*. Then the metric $h_{\alpha\beta}$ is $\lambda_1^2+\lambda_2^2$, the connection ∇^L is given by $\nabla^L l_1 = -\omega_2 \otimes l_2/2$ and $\nabla^L l_2 = \omega_2 \otimes l_1/2$, and the canonical section C of $L^* \otimes TN \otimes L^*$ defined in §2.4 is given by

$$4C = \lambda_1 \otimes V_1 \otimes \lambda_1 - \lambda_2 \otimes V_1 \otimes \lambda_2 + \lambda_1 \otimes V_2 \otimes \lambda_2 + \lambda_2 \otimes V_2 \otimes \lambda_1, \tag{36}$$

which is normalized so that $\|C\|^2 = 1/4$ as required.

Let $\phi = \lambda_1 \otimes \phi_1 + \lambda_2 \otimes \phi_2$, so that ϕ is a section of $L^* \otimes \mathbb{R}^2$. The condition for ϕ to be nondegenerate and orientation-preserving is condition (i), and equation (28) of Lemma 2.4.2 is

$$\begin{aligned}
&(dx_1 \otimes \lambda_1 + dx_2 \otimes \lambda_2) \otimes \left(\frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2}\right) - (dx_1 \otimes \lambda_2 - dx_2 \otimes \lambda_1) \otimes \left(\frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1}\right)\\
&- \frac{1}{2}\big((\omega_1 \otimes \lambda_1 + \omega_2 \otimes \lambda_2) \otimes \phi_1 + (\omega_1 \otimes \lambda_2 - \omega_2 \otimes \lambda_1) \otimes \phi_2\big)\\
&= \frac{1}{2}\big((\omega_1 \otimes \lambda_1 + \omega_2 \otimes \lambda_2) \otimes \phi_1 - (\omega_1 \otimes \lambda_2 - \omega_2 \otimes \lambda_1) \otimes \phi_2\big).
\end{aligned} \tag{37}$$

But this simplifies to conditions (ii) and (iii). Thus if conditions (i)-(iii) hold, then Lemma 2.4.2 applies and gives a self-dual metric g on $M = N \times T^2$, which is $g = (dx_1^2 + dx_2^2)/x_1^2 + \psi_1^2 + \psi_2^2$ by the definition in §2.3. □

Conditions (ii) and (iii) are equivalent to equation (29) of Proposition 2.4.4 when $w = x_1 + ix_2$ and $\xi = x_1 = \mathrm{Re}(w)$. Thus every self-dual metric g constructed using Proposition 3.2.1 is conformal to a Kähler structure with the opposite orientation. In fact, there is a one-parameter family of Kähler structures (up to homothety) conformal to g, because the action of the hyperbolic isometry group $PSL(2,\mathbb{R})$ gives a family of identifications of $\mathcal{H}^2$ with the upper half-plane. Regarding $\mathcal{H}^2$ as a compact manifold with boundary $\mathcal{S}^1$, the Kähler structure depends on which point of the boundary is identified with ∞ in the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$.

The linear equations (ii), (iii) of the Proposition can be solved in $\mathbb{R}$ rather than $\mathbb{R}^2$, as the $\mathbb{R}^2$ tensor factor only enters in a trivial way. There is an obvious solution $\phi_1 = 0, \phi_2 = 1$ to conditions (ii) and (iii), which represents the section $\phi = \phi_j \otimes \lambda_j = \lambda_2$ of L^*. So define $f^{(\infty)}$ to be the section $-\lambda_2$ of L^*; then $f^{(\infty)}$ satisfies the equation

$$\nabla^L_a f_\beta + J^c_a J^\delta_\beta \nabla^L_c f_\delta = 2 f_\gamma C^\gamma_{a\beta} \tag{38}$$

on sections f of L^*, which is the analogue of (28). Now as (38) is a canonical equation over $\mathcal{H}^2$ it is invariant under orientation-preserving isometries of $\mathcal{H}^2$. Therefore we can generate a family of solutions to (38) from $f^{(\infty)}$ by applying isometries to it. The oriented isometry group of $\mathcal{H}^2$ is $PSL(2,\mathbb{R})$, which acts by

$$\tilde{x}_2 - i\tilde{x}_1 = \frac{a(x_2 - ix_1) + b}{c(x_2 - ix_1) + d}, \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), \tag{39}$$

where the centre $\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ of $SL(2,\mathbb{R})$ acts trivially. However, because of the square-root in the construction of L, in the lift to L^* of the action of $SL(2,\mathbb{R})$, $-\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ acts as a sign change in the fibres.

The upper triangular matrices in $SL(2,\mathbb{R})$ form a set with two connected components, and the matrices of the identity component fix $f^{(\infty)}$. Thus the images of $f^{(\infty)}$ under $SL(2,\mathbb{R})$ are a 1-parameter family, and a calculation with (39) shows the general member of this family to be

$$f^{(p)} = x_1\left(x_1^2+(x_2-p)^2\right)^{-1/2}\lambda_1 + (x_2-p)\left(x_1^2+(x_2-p)^2\right)^{-1/2}\lambda_2, \tag{40}$$

for $p\in\mathbb{R}$. Choosing the positive sign for the root and taking the limit $p\to\pm\infty$, the result is $\pm f^{(\infty)}$, so there is a sign change as p goes from $-\infty$ to $+\infty$.

Now $\mathcal{H}^2$ is diffeomorphic to an open disc, and by instead considering a closed disc we get a manifold with boundary $S^1=\mathbb{R}\cup\{\infty\}$. The solution (40) singles out the point $(x_1,x_2)=(0,p)$ on the boundary of $\mathcal{H}^2$, and is invariant (up to sign) under oriented isometries of $\mathcal{H}^2$ fixing this point. As the $f^{(p)}$ satisfy (38), if $p_1,\dots,p_k$ are elements of $\mathbb{R}\cup\{\infty\}$ and $v_1,\dots,v_k$ are elements of $\mathbb{R}^2$, then $\Sigma_{j=1}^k f^{(p_j)}\otimes v_j$ is a section of $L^*\otimes\mathbb{R}^2$ satisfying (28). In the next two sections we use solutions of this form to make compact self-dual 4-manifolds.

3.3. Torus-invariant self-dual metrics on $n\overline{\mathbb{CP}}^2$. To find examples of compact self-dual 4-manifolds, we shall take a compact 4-manifold M with a dense open set diffeomorphic to $\mathcal{H}^2\times T^2$ as in §3.1, and find a self-dual metric on $\mathcal{H}^2\times T^2$ using Lemma 2.4.2 that extends to a self-dual metric on all of M. The manifolds M for which this works turn out to be diffeomorphic to $n\overline{\mathbb{CP}}^2$. We begin with the conformally flat metric on $\mathcal{S}^4$, and use it as a local model for extending self-dual metrics on $\mathcal{H}^2\times T^2$ to M.

The four-sphere $\mathcal{S}^4$ is the 1-point compactification of $\mathbb{R}^4=\mathbb{R}^2\times\mathbb{R}^2$, and taking radial coordinates (r_j,y_j) in each factor, the round metric is conformal to $g_0=dr_1^2+r_1^2dy_1^2+dr_2^2+r_2^2dy_2^2$, where y_1,y_2 are coordinates of period 2π. Thus the metric on a dense open set of $\mathcal{S}^4$ is a T^2- invariant metric on a torus bundle over the quarter-plane $r_1>0, r_2>0$. To transform the quarter-plane to the half-plane, put $x_2-ix_1=(r_2-ir_1)^2$. Then in the coordinates (x_1,x_2,y_1,y_2), g_0 is

$$g_0=\frac{dx_1^2+dx_2^2}{4(x_1^2+x_2^2)^{1/2}}+\frac{1}{2}\bigl((x_1^2+x_2^2)^{1/2}-x_2\bigr)dy_1^2+\frac{1}{2}\bigl((x_1^2+x_2^2)^{1/2}+x_2\bigr)dy_2^2 \tag{41}$$

in the region $x_1>0$, so that g_0 is conformal to

$$g_0'=\frac{dx_1^2+dx_2^2}{x_1^2}+\frac{2(x_1^2+x_2^2)}{x_1^2}\Bigl\{\bigl(1-x_2(x_1^2+x_2^2)^{-1/2}\bigr)dy_1^2+\bigl(1+x_2(x_1^2+x_2^2)^{-1/2}\bigr)dy_2^2\Bigr\}. \tag{42}$$

In the notation of §3.2, define ϕ by

$$\phi=\tfrac{1}{2}f^{(0)}\otimes(1,-1)+\tfrac{1}{2}f^{(\infty)}\otimes(-1,-1). \tag{43}$$

Regarding ϕ as a map $L \to \mathbb{R}^2$, we calculate in coordinates that $\det \phi = x_1(x_1^2 + x_2^2)^{-1/2}/2 > 0$ so ϕ is nondegenerate and orientation-preserving, and ϕ satisfies (28) as it is a linear combination of the solutions of (28) found in §3.2. So Lemma 2.4.2 gives a self-dual metric on $N \times T^2$, and computing this metric in coordinates using Proposition 3.2.1 gives (42). Thus applying Lemma 2.4.2 to (43) yields an open dense set of $\mathcal{S}^4$ with the round metric, as a self-dual metric on $\mathcal{H}^2 \times T^2$. This provides a local model for compactifying self-dual metrics on $\mathcal{H}^2 \times T^2$ to get more general compact self-dual 4-manifolds.

Recall the decomposition of a 4-manifold M with a T^2- action into orbits of type (a), (b) and (c) given in §3.1. Here the two orbits of type (a) are at $(0,0)$ and ∞, the interval $x_1 = 0, x_2 < 0$ represents orbits of type (b) with stabilizer $G(0,1)$, and the interval $x_1 = 0, x_2 > 0$ represents orbits of type (b) with stabilizer $G(1,0)$. Observe from (40) and (43) that when $x_1 = 0$, ϕ is $\lambda_2 \otimes (0,1)$ when $x_2 < 0$ and $\lambda_2 \otimes (1,0)$ when $x_2 > 0$. Thus for the case of $\mathcal{S}^4$, at an orbit of type (b) with stabilizer $G(m,n)$, ϕ is equal to $\lambda_2 \otimes (m,n)$. This is true in general, and is what motivates the definition of ϕ in the next Theorem.

We shall now construct T^2- invariant self-dual metrics on $n\overline{\mathbb{CP}}^2$. Let n be a positive integer, k be $n+2$ and M be $n\overline{\mathbb{CP}}^2$ with some smooth T^2- action. Then by Proposition 3.1.1, M is described in Orlik and Raymond's form by a sequence $(m_1,n_1),\ldots,(m_k,n_k)$ of pairs of coprime integers satisfying conditions (i)-(iii) of Proposition 3.1.1.

THEOREM 3.3.1. *Let n, k, M and $(m_1,n_1),\ldots,(m_k,n_k)$ be as above. Let $p_1,\ldots,p_k$ be a strictly increasing sequence in $\mathbb{R}\cup\{\infty\}$, and define ϕ by*

$$\phi = \frac{f^{(p_k)} + f^{(p_1)}}{2} \otimes (m_k,n_k) + \sum_{j=1}^{k-1} \frac{f^{(p_j)} - f^{(p_{j+1})}}{2} \otimes (m_j,n_j). \tag{44}$$

Then ϕ is nondegenerate and orientation-preserving and satisfies equation (28) of Lemma 2.4.2, so the Proposition gives a self-dual metric g on $\mathcal{H}^2 \times T^2$. Regarding $\mathcal{H}^2 \times T^2$ as a dense open set of $n\overline{\mathbb{CP}}^2$, the metric extends to a nonsingular self-dual conformal structure on $n\overline{\mathbb{CP}}^2$. This construction gives a k-dimensional family of self-dual metrics on $n\overline{\mathbb{CP}}^2$ parametrized by $p_1,\ldots,p_k$, but because of the action of the hyperbolic isometry group $PSL(2,\mathbb{R})$ there are at most $n-1$ effective parameters.

Proof. The proof splits into two parts, the first part to prove that ϕ does give rise to a nonsingular self-dual metric on $\mathcal{H}^2 \times T^2$, and the second to show that this metric extends to $n\overline{\mathbb{CP}}^2$. For the first part, as ϕ is built up out of solutions to (28) it is only necessary to show it is nondegenerate and orientation-preserving, which we will do in two Lemmas.

LEMMA 3.3.2. *Let $c_1, c_2, c_3 \in \mathbb{R}$ with $c_1 < c_2 \leq c_3$. Then $\big(f^{(c_1)} - f^{(c_2)}\big) \wedge \big(f^{(c_3)} - f^{(\infty)}\big)$ is a negative section of $\Lambda^2 L^*$ over $\mathcal{H}^2$.*

Proof. Let $D = \big(f^{(c_1)} - f^{(c_2)}\big) \wedge \big(f^{(c_3)} - f^{(\infty)}\big)$, and let d_j be the function $d_j(x_1,x_2) = \big(x_1^2 + (x_2 - c_j)^2\big)^{1/2}$ for $j = 1,2,3$. Calculating in coordinates using the definition of $f^{(p)}$ in §3.2, we get

$$
\begin{aligned}
\frac{d_1d_2d_3D}{x_1} &= (d_2-d_1)\cdot(x_2-c_3+d_3)-(x_2-c_1)d_2+(x_2-c_2)d_1 \\
&= (d_2-d_1)d_3+d_1c_3-d_2c_3+c_1d_2-c_2d_1 \\
&= \bigl(d_3-(c_3-c_2)\bigr)(d_2-d_1)-d_2(c_2-c_1).
\end{aligned} \tag{45}
$$

But d_2, d_3 and c_3-c_2 are the lengths in the Euclidean metric of the three sides of the triangle in $\mathbb{R}^2$ with vertices $(x_1,x_2),(0,c_2),(0,c_3)$, and thus $\bigl|d_3-(c_3-c_2)\bigr|<d_2$ by the triangle inequality. Similarly d_1,d_2 and c_2-c_1 are the sides of a triangle, and so $|d_2-d_1|<c_2-c_1$. Multiplying these two inequalities together gives $\bigl(d_3-(c_3-c_2)\bigr)(d_2-d_1)<d_2(c_2-c_1)$, so by (45) $d_1d_2d_3D/x_1<0$, and thus $D<0$. $\square$

LEMMA 3.3.3. *The section ϕ of $L^*\otimes\mathbb{R}^2$ defined by (44) satisfies* $\det\phi>0$ *when $x_1>0$.*

Proof. Identifying $\Lambda^2\mathbb{R}^2$ with $\mathbb{R}$, from (44) we calculate that

$$
\det\phi=\frac{1}{4}\sum_{1\le j<l\le k}(m_jn_l-m_ln_j)D_{jl}, \tag{46}
$$

where $D_{jl}=\bigl(f^{(p_j)}-f^{(p_{j+1})}\bigr)\wedge\bigl(f^{(p_l)}-f^{(p_{l+1})}\bigr)$ for $l<k$, and $D_{jk}=\bigl(f^{(p_j)}-f^{(p_{j+1})}\bigr)\wedge\bigl(f^{(p_k)}+f^{(p_1)}\bigr)$, identifying Λ^2L^* with $\mathbb{R}$.

We shall show that $m_jn_l-m_ln_j<0$ for $1\le j<l\le k$, using conditions (i)-(iii) of Proposition 3.1.1. When $j=1$, $m_1n_l-m_ln_1=-m_l<0$ for $l>1$ by conditions (i) and (ii). When $j>1$, by conditions (ii) and (iii) we have $n_j/m_j>n_{j+1}/m_{j+1}$, so by induction $n_j/m_j>n_l/m_l$ when $1<j<l\le k$. Applying condition (ii) again gives $m_jn_l-m_ln_j<0$ in this case too. Thus $m_jn_l-m_ln_j<0$ for $1\le j<l\le k$. Therefore by (46) it is sufficient to prove that $D_{jl}<0$ whenever $1\le j<l\le k$, for then (46) is a sum of positive terms.

Now there exists an isometry of $\mathcal{H}^2$ taking $(0,p_j),(0,p_{j+1})$ and $(0,p_l)$ to $(0,c_1),(0,c_2)$ and $(0,c_3)$ for some $c_1,c_2,c_3\in\mathbb{R}$ with $c_1<c_2\le c_3$, and taking $(0,p_{l+1})$ to the point ∞. Under this isometry, D_{jl} is taken to the function D of Lemma 3.3.2. So $D_{jl}<0$ by Lemma 3.3.2, which completes the proof. $\square$

By Lemma 3.3.3, Lemma 2.4.2 applies to give a self-dual metric on $\mathcal{H}^2\times T^2$, viewed as a dense open set in M. It remains to show that the metric extends to a nonsingular metric on M. Similar compactification problems for S^1- invariant metrics have been dealt with in [5] and [9]. First we deal with compactifying orbits of type (b). Let K be the set $\bigl\{(x_1,x_2)\in\mathbb{R}^2 : x_1\ge 0\bigr\}\setminus\bigl\{(0,p_1),\ldots,(0,p_k)\bigr\}$. From the definitions (44) of ϕ and (40) of $f^{(p)}$, we can see that $\phi=x_1\lambda_1\otimes v_1+\lambda_2\otimes v_2$, where v_1,v_2 are continuous functions $K\to\mathbb{R}^2$ with $v_2=(m_j,n_j)+O(x_1^2)$ when $p_j<x_2<p_{j+1}$. Lemmas 3.3.2 and 3.3.3 ensure that $v_1\wedge v_2$ is a positive section of $\Lambda^2\mathbb{R}^2$ for $x_1>0$, and the proofs can easily be modified to show that this also holds on all of K.

But this is the condition for the matrix (v_1v_2) to have positive determinant, so the matrix is invertible, and there exist continuous functions $w_1,w_2 : K\to\bigl(\mathbb{R}^2\bigr)^*$ such that $v_1\otimes w_1+v_2\otimes w_2$ is the identity section in $\mathbb{R}^2\otimes\bigl(\mathbb{R}^2\bigr)^*$ over K. As v_1 is continuous and $v_2=(m_j,n_j)+O(x_1^2)$ for $x_1\ge 0$ and $p_j<x_2<p_{j+1}$, it follows that $w_1\circ(m_j,n_j)=O(x_1^2)$

and $w_2 \circ (m_j, n_j) = 1 + O(x_1^2)$ for $p_j < x_2 < p_{j+1}$. By Proposition 3.2.1, the metric g is $g = (dx_1^2 + dx_2^2)/x_1^2 + w_1^2/x_1^2 + w_2^2$, which is conformal to $g' = dx_1^2 + dx_2^2 + w_1^2 + x_1^2 w_2^2$. We claim that g' extends to a C^2 metric on $M \setminus \{(0, p_1), \ldots, (0, p_k), \infty\}$.

To justify this, recall from §3.1 that M has orbits of type (b) with stabilizer $G(m_j, n_j)$ at the points $x_1 = 0, p_j < x_2 < p_{j+1}$. Let m', n' be integers with $m_j n' - m' n_j = 1$, and y_1, y_2 be the standard coordinates on T^2 of period 2π. Define $z_1 = n_j y_1 - m_j y_2$ and $z_2 = n' y_1 - m' y_2$. Then (z_1, z_2) are also oriented coordinates of period 2π on T^2, and the relations $w_1 \circ (m_j, n_j) = O(x_1^2)$ and $w_2 \circ (m_j, n_j) = 1 + O(x_1^2)$ imply that $w_1 = s_1 dz_1 + O(x_1^2) dz_2$ and $w_2 = s_2 dz_1 + (1 + O(x_1^2)) dz_2$ for $x_1 > 0$ and $p_j < x_2 < p_{j+1}$, where $s_1 > 0$ as $\det(w_1 w_2) > 0$. Thus

$$g' = dx_1^2 + dx_2^2 + \big(s_1^2 + O(x_1^2)\big) dz_1^2 + \big(x_1^2 + O(x_1^4)\big) dz_2^2 + O(x_1^2) dz_1 dz_2. \tag{47}$$

The important terms here are $dx_1^2 + x_1^2 dz_2^2$. Since z_2 is a coordinate of period 2π and $x_1 \geq 0$, this is the flat metric in the plane, in radial coordinates. So if the z_2 coordinate is collapsed when $x_1 = 0$, the terms $dx_1^2 + x_1 dz_2^2$ extend to the added orbits in a nonsingular fashion. The group of translations in the z_2 coordinate is $G(m_j, n_j)$, so that the added points have stabilizer $G(m_j, n_j)$ as required. Examining the other terms in (47), it can be seen that g' extends to the added points to give a metric that is C^2 at the added points, which proves the claim we made above.

To extend the metric over the orbits of type (a) as well, we see from (41) that g' must be multiplied by an extra conformal factor that looks like $\big((x_1 - p_j)^2 + x_2^2\big)^{-1/2}$ near $(0, p_j)$. Once this conformal factor has been chosen, a similar analysis to the above shows that the rescaled metric extends to a C^2 metric at the orbits of type (a). The same treatment works for the orbit at ∞, which is either of type (a) or (b) depending on whether $p_k = \infty$ or not. So we conclude that the metric g produced from the (44) by Lemma 2.4.2 is conformal to a metric on M that is smooth on $\mathcal{H}^2 \times T^2$ and C^2 at the added points. But as the conformal structure is self-dual, local elliptic regularity implies that $[g]$ is also C^∞ at the added points, so it is a nonsingular, self-dual conformal structure on M.

This conformal structure depends on the increasing sequence of points $p_1, \ldots, p_k \in \mathbb{R} \cup \{\infty\}$, so we have made a k-dimensional family of self-dual metrics on $n\mathbb{CP}^2$. However, the hyperbolic isometry group $PSL(2, \mathbb{R})$ of dimension three acts on the sequences $(p_1, \ldots, p_k)$, and two sequences related by a hyperbolic isometry give the same self-dual conformal structure. As $k \geq 3$, $PSL(2, \mathbb{R})$ acts freely on increasing sequences $(p_1, \ldots, p_k)$ and the quotient family has dimension $k - 3 = n - 1$, finishing the proof of Theorem 3.3.1. □

By the remark after Proposition 3.2.1, the metric constructed above on $\mathcal{H}^2 \times T^2$ is conformal to a Kähler structure with the opposite orientation, using Proposition 2.4.4. Examining the Kähler structure reveals that it extends in a nonsingular fashion to all of M except the orbit at ∞. If the orbit at ∞ is of type (a) then the Kähler structure is asymptotically Euclidean close to the orbit at ∞, and if the orbit is of type (b) then the Kähler structure resembles the product Kähler structure on $\mathcal{S}^2 \times \mathcal{H}^2$ at ∞, which is conformally flat. Thus we have found complete, T^2- invariant, anti-self-dual Kähler structures on noncompact complex manifolds that have one noncompact end asymptotic to the standard Kähler structure on $\mathbb{C}^2$ or $\mathcal{S}^2 \times \mathcal{H}^2$. The underlying complex manifolds are n-fold blow-ups of $\mathbb{C}^2$ or $\mathcal{S}^2 \times \mathcal{H}^2$, and do not vary with $p_1, \ldots, p_k$.

Putting $(m_1, n_1) = (0, 1)$ and $(m_j, n_j) = (1, k - j)$ for $j = 2, \ldots, k$ gives data $\{(m_j, n_j)\}$ satisfying Proposition 3.1.1. It can be shown that the metrics constructed by Theorem 3.3.1 with this data are conformal to a subfamily of LeBrun's metrics on $n\mathbb{CP}^2$ [9], those that are made using points lying on a line in $\mathcal{H}^3$. It can also be shown that every T^2- invariant metric on $n\mathbb{CP}^2$ in LeBrun's family arises in this way, so that any metric constructed by Theorem 3.3.1 using a different T^2- action on $n\mathbb{CP}^2$ cannot be conformal to one of LeBrun's metrics.

We shall now make some remarks about the twistor spaces of the metrics of Theorem 3.3.1. There is a geometric structure called a *quaternionic structure* [15], which generalizes the notion of self-dual 4-manifold to $4n$ dimensions. In [7] the author defined a *quotient construction* for quaternionic manifolds. Starting with a quaternionic manifold of dimension $4n$ acted on by a Lie group G of dimension m, under certain conditions one can construct a quaternionic manifold of the lower dimension $4(n-m)$. When $n - m = 1$, this new quaternionic manifold is a self-dual 4-manifold.

Thus the quaternionic quotient construction gives a systematic method for finding examples of self-dual metrics. One advantage of this method is that the twistor spaces of the metrics are given explicitly by the construction. Now the most basic quaternionic manifold in dimension $4n + 4$ is $\mathbb{HP}^{n+1}$, and it is equipped with an action of $\mathrm{Sp}(n+2)$, which has maximal torus $U(1)^{n+2}$. By choosing $G = U(1)^n \subset U(1)^{n+2}$ and performing a quaternionic quotient, one produces self-dual 4-manifolds.

The metrics on $n\mathbb{CP}^2$ defined by Theorem 3.3.1 were originally found by the author in exactly this way. The data $(m_1, n_1), \ldots, (m_k, n_k)$ defines the subgroup $U(1)^n \subset U(1)^{n+2}$. In this picture, the twistor spaces of the metrics of Theorem 3.3.1 are given explicitly. Each twistor space Z appears as the quotient (in some sense) of a projective variety Q by $(\mathbb{C}^*)^n$, where Q is the intersection of n quadrics in $\mathbb{CP}^{2n+3}$. In particular, this shows that the twistor spaces of the metrics have algebraic dimension 3, i.e. are Moishezon.

Unfortunately, this picture of the twistor spaces, though simple in conception, is actually horribly messy to work out in detail, and is difficult to relate to the picture given in §3.3. For this reason we will not explain it here. If any reader can find a simpler construction for the twistor spaces based on the data of §3.3, the author would be interested to see it.

By a well-known argument involving a result of Poon [14], if the twistor space Z of a self-dual metric on a compact 4-manifold M is Moishezon, then M must be homeomorphic to $n\mathbb{CP}^2$, and so have $b^+(M) = n$ and $b^-(M) = 0$. But by Orlik and Raymond's results sketched in §3.1, if M is homeomorphic to $n\mathbb{CP}^2$ and has a T^2- action, then M is diffeomorphic to $n\mathbb{CP}^2$. Thus the Moishezon property of the twistor spaces, that we claimed above, gives a rather indirect way to see that any compact, self-dual 4-manifold constructed using the method of Theorem 3.3.1 must be diffeomorphic to $n\mathbb{CP}^2$.

CONJECTURE 3.3.4. *Let M be a compact, oriented, simply-connected 4-manifold, with a smooth T^2- action that has at least one fixed point. Suppose that M admits a T^2-invariant self-dual metric g. Then (M, g) is $\mathcal{S}^4$ with the round metric or $n\mathbb{CP}^2$ with one of the metrics constructed by Theorem 3.3.1.*

One direction from which this conjecture might be proved is algebraic geometry, as the T^2- action lifts to a holomorphic action of $(\mathbb{C}^*)^2$ on the twistor space Z, which must have strong consequences on the complex geometry of Z. Another approach might be to prove directly the existence of a compatible product structure, apply Theorem 2.4.5 to show that the metrics are constructed over $\mathcal{H}^2$ or a branched cover of it, and then to try and exclude the branched cover case.

3.4. Compact self-dual manifolds with fundamental group $\mathbb{Z}$. Self-dual metrics will now be constructed on compact self-dual 4-manifolds M with $\pi_1(M) = \mathbb{Z}$, using infinite sums of the solutions (40) of (28). The idea is this: let $\nu > 1$ be a fixed real number, let N be the set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$, and consider the action of $\mathbb{Z}$ on N given by $(x_1, x_2) \mapsto (\nu^n x_1, \nu^n x_2)$ for $n \in \mathbb{Z}$. The quotient $A = N/\mathbb{Z}$ is an annulus, with two S^1 boundary components. Moreover, the $\mathbb{Z}$- action preserves the hyperbolic metric $h_{ab} = (dx_1^2 + dx_2^2)/x_1^2$, so the annulus has a hyperbolic metric on it. We shall follow the method of §3.3 over A instead of over $\mathcal{H}^2$, and the rôle of the solutions $f^{(p)}$ picking out one boundary point of $\mathcal{H}^2$ will be taken by an infinite sum of the $f^{(p)}$ for $p = \nu^n p_0$ and $n \in \mathbb{Z}$.

There is one complication however: as $SL(2, \mathbb{Z})$ is the group of outer oriented automorphisms of T^2, oriented T^2- bundles over A are classified by homomorphisms $\mathbb{Z} \to SL(2, \mathbb{Z})$, up to conjugation by $SL(2, \mathbb{Z})$. So to find self-dual metrics on general T^2- bundles over A, we must find self-dual metrics on $\mathcal{H}^2 \times T^2$ that are invariant under the $\mathbb{Z}$- action

$$(x_1, x_2), \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \overset{n\in\mathbb{Z}}{\longmapsto} (\nu^n x_1, \nu^n x_2), R^n \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{48}$$

for R some fixed element of $SL(2, \mathbb{Z})$. Here (y_1, y_2) are the usual coordinates of period 2π on T^2. Let B be the T^2- bundle over A made by dividing $N \times T^2$ by the $\mathbb{Z}$- action (48). Now for nontrivial R, B has a natural local T^2- action, but not a global one. This means that the metrics we construct will usually have local T^2- actions but not global ones.

Generalizing the method of §3.3 to this situation involves us in three types of problem. Firstly, for the ideas to work at all, the infinite sums of solutions $f^{(p)}$ must converge to smooth solutions with the properties we expect. Secondly, the resulting section ϕ must be nondegenerate and orientation-preserving, and thirdly, there must be a compact manifold M in which B appears as a dense open set, and the self-dual metric on B must extend to a self-dual conformal structure on M. The second and third problems have been dealt with in §3.3 and need only minor modifications for this case. So the main work that remains to be done is to settle the question of convergence.

For the T^2- actions on $n\mathbb{CP}^2$ used in Theorem 3.3.1 there was a set of conditions defining the algebraic data already laid out in Proposition 3.1.1. The manifolds M with T^2-actions studied in this section are defined by algebraic data subject to similar conditions, and for convenience we collect these together here. In what follows we use the orientation of $\mathbb{R}^2$ to identify $\Lambda^2\mathbb{R}^2 \cong \mathbb{R}$, and $\mathbb{Z}^2$ is considered to be a subset of $\mathbb{R}^2$. The convention we use is that $(1, 0) \wedge (0, 1)$ is identified with $1 \in \mathbb{R}$.

DEFINITION 3.4.1. Let R be a matrix in $SL(2, \mathbb{Z})$ with distinct, positive eigenvalues r, r^{-1} with $r > 1$. Let X, Y be eigenvectors of R in $\mathbb{R}^2$ associated to r and r^{-1}, with signs chosen such that $X \wedge Y < 0$.
Let k, l be positive integers. Let $v_1, \ldots, v_k$ and $w_1, \ldots, w_l$ be nonzero, primitive elements of $\mathbb{Z}^2$ satisfying the following equations:

$$
\begin{array}{cccc}
v_j \wedge v_{j+1} = -1, & Rv_k \wedge v_1 = -1, & w_j \wedge w_{j+1} = -1, & w_l \wedge Rw_1 = -1, \\
v_j \wedge X > 0, & v_j \wedge Y < 0, & w_j \wedge Y > 0, & w_j \wedge X > 0.
\end{array} \tag{49}
$$

Extend the definition of v_j, w_j to all $j \in \mathbb{Z}$ by defining $v_{j+nk} = R^{-n}v_j$ and $w_{j+nl} = R^n w_j$. Let $\nu > 1$ be a real number. Let $p_1, \ldots, p_k, q_1, \ldots, q_l \in \mathbb{R}$ satisfy $p_1 < p_2 < \ldots < p_k < \nu^{-1}p_1 < 0$ and $0 < q_1 < q_2 < \ldots < q_l < \nu q_1$, and extend the definition of p_j, q_j to all $j \in \mathbb{Z}$ by defining $p_{j+nk} = \nu^{-n}p_j$ and $q_{j+nl} = \nu^n q_j$.

Using these definitions, the next Lemma defines a compact 4-manifold M using the ideas of §3.1. We omit the proof, which is a trivial verification except for the calculation of $b^{\pm}(M)$. This can be done by generalizing the expression given for $b^{\pm}(M)$ in §3.1 for the simply-connected case.

LEMMA 3.4.2. *With the definitions of Definition 3.4.1, let $\tilde{M}$ be the noncompact 4-manifold obtained from $\mathcal{H}^2 \times T^2$ by adding orbits of type (a) at $(x_1, x_2) = (0, p_j)$ and $(0, q_j)$ for all $j \in \mathbb{Z}$, adding orbits of type (b) with stabilizer $G(v_j)$ when $x_1 = 0$ and $x_2 \in (p_j, p_{j+1})$, and adding orbits of type (b) with stabilizer $G(w_j)$ when $x_1 = 0$ and $x_2 \in (q_j, q_{j+1})$, for all $j \in \mathbb{Z}$. Then $\tilde{M}$ has two noncompact ends, near $(x_1, x_2) = (0, 0)$ and ∞. The $\mathbb{Z}$- action (48) extends smoothly to $\tilde{M}$. Let M be the quotient of $\tilde{M}$ by (48). Then M is a smooth, Hausdorff, compact 4-manifold with $\pi_1(M) = \mathbb{Z}$, $b^+(M) = k + l$ and $b^-(M) = 0$, and B is a dense open set in M.*

PROPOSITION 3.4.3. *Make the definitions of Definition 3.4.1, and suppose that $r < \nu$. Define ϕ by*

$$
\phi(x_1, x_2) = \sum_{j=-\infty}^{\infty} \frac{f^{(p_j)} - f^{(p_{j+1})}}{2} \otimes v_j + \sum_{j=-\infty}^{\infty} \frac{f^{(q_j)} - f^{(q_{j+1})}}{2} \otimes w_j. \tag{50}
$$

These series converge converge uniformly in compact subsets of $\mathbb{R}^2$ not containing $(0, 0)$, $(0, p_j)$ or $(0, q_j)$, and so do all derivatives of the terms of the series in x_1, x_2. Therefore they converge to a smooth function ϕ away from these points, and give a self-dual conformal structure on the compact 4-manifold M of Lemma 3.4.2, using Lemma 2.4.2 and the ideas of Theorem 3.3.1. The conformal structure depends on the $k + l + 1$ parameters $p_1, \ldots, p_k, q_1, \ldots, q_l$ and ν, but there are at most $k + l$ effective parameters because multiplying p_j, q_j by a positive constant gives the same conformal structure.

Proof. We need only show that the first series in (50) is uniformly convergent, as the proof for the second series is the same. For j large and positive, $p_j = O(\nu^{-j/k})$, and on a compact subset of $\mathbb{R}^2$ not containing $(0, 0)$ or any $(0, p_l)$, $f^{(p_j)} - f^{(p_{j+1})}$ and all its derivatives are $O(\nu^{-j/k})$. By Definition 3.4.1, $v_j = O(r^{j/k})$ for large, positive j, and so the term in the series is $O(r^{j/k}\nu^{-j/k})$. But as $r < \nu$ by assumption, the series is uniformly convergent for j large and positive by comparison with a geometric series. Similarly, when j is large and negative, $f^{(p_j)} - f^{(p_{j+1})}$ and all its derivatives are $O(\nu^{j/k})$ in the compact subset, and $v_j = O(r^{-j/k})$, so the term in the series is $O(r^{-j/k}\nu^{j/k})$ which converges uniformly for j summed to $-\infty$, by comparison with a geometric series. This proves the first part of the Proposition.

So (50) converges uniformly and ϕ is well-defined. By properties of uniform convergence, ϕ is continuous, all its derivatives exist and are continuous, and the derivatives of ϕ are the sum of the derivatives of the terms in the series (50). But as the terms in the series were shown in §3.2 to satisfy (28), we deduce that ϕ satisfies (28). By uniform convergence we may write $\det \phi$ as an infinite sum of determinants of terms in the series (50). But from the conditions of Definition 3.4.1 and the proof of Lemma 3.3.3 it can be shown that each of these determinant terms is positive, so $\det \phi > 0$ as it is a convergent, infinite sum of positive terms, and thus ϕ is nondegenerate and orientation-preserving.

Therefore Lemma 2.4.2 applies to ϕ, giving a self-dual metric on $\mathcal{H}^2 \times T^2$. The definition of ϕ was chosen such that the metric should be invariant under the $\mathbb{Z}$- action (48), so dividing by $\mathbb{Z}$ we get a self-dual metric on B. But the T^2- bundle B is a dense open set in the compact 4-manifold M of Lemma 3.4.2, and the arguments in the proof of Theorem 3.3.1 show that this self-dual metric on B extends to a nonsingular self-dual conformal structure on M. The conformal structure depends on $p_1, \ldots, p_k, q_1, \ldots, q_l$ and ν. However, in the hyperbolic isometry group $PSL(2, \mathbb{R})$ there is a one-parameter subgroup of dilations fixing both 0 and ∞, which has the effect of multiplying p_j, q_j by positive constants, and this removes one parameter to give at most $k + l$ effective parameters. $\square$

In the cases I have examined, the manifold M of Lemma 3.4.2 has always been diffeomorphic to $\mathcal{S}^3 \times \mathcal{S}^1 \# (k + l)\overline{\mathbb{CP}}^2$, but I am not sure if this is always the case; a similar problem comes up in [12], where Orlik and Raymond were only able to give a stable classification of their manifolds. An interesting feature of Proposition 3.4.1 is the restriction $r < \nu$. The eigenvalue r is fixed by the local T^2- action on M, but ν is free to vary, and so we can consider what happens to the metrics of the Proposition as $\nu \to r$.

It turns out that when $\nu - r$ is positive and small, the section ϕ defined by (50) is large on all of the annulus A, except for a small neighbourhood of the boundary. Since the metric g is defined using the inverse of ϕ, this means that the T^2- orbits are small, and the Riemannian manifold M can be thought of as a fibre bundle over a 2-dimensional annulus with small fibres. Thus as $\nu \to r$, in a certain sense the metric on M collapses down to a metric on a 2-dimensional annulus. This is very similar to the collapses of Riemannian manifolds studied by Cheeger and Gromov [4].

The approach of this section should extend to manifolds with other fundamental groups. The method would be to find a discrete subgroup Γ of $PSL(2, \mathbb{R})$, such that $\mathcal{H}^2/\Gamma$ is a complete, nonsingular hyperbolic 2-manifold, with finitely many (and at least one) $\mathcal{S}^1$ boundary components. Then a T^2- bundle B is defined over $\mathcal{H}^2/\Gamma$ using a group homomorphism $\Gamma \to SL(2, \mathbb{Z})$, and self-dual metrics are defined on B by summing the solutions $f^{(p)}$ over Γ. The problem comes in showing that the sequences converge uniformly, under appropriate conditions. The next most obvious cases are the manifolds $2(\mathcal{S}^3 \times \mathcal{S}^1) \# n\overline{\mathbb{CP}}^2$, which have $\pi_1(M) = \mathbb{Z} * \mathbb{Z}$, and even for this fundamental group the convergence of series is quite nasty. Preliminary calculations indicate that the sequences ought to converge if all the closed geodesics on $\mathcal{H}^2/\Gamma$ are sufficiently long.

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