Compact 8-manifolds with holonomy $Spin(7)$

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1 Introduction

In Berger's classification [4] of the possible holonomy groups of a nonsymmetric, irreducible riemannian manifold, there are two special cases, the exceptional holonomy groups G_2 in 7 dimensions and $Spin(7)$ in 8 dimensions. Bryant [6] proved that such metrics exist locally, using the theory of exterior differential systems, and gave some explicit examples. Later, Bryant and Salamon [7] constructed explicit, *complete* metrics with holonomy G_2 and $Spin(7)$.

In two previous papers [12], [13], the author constructed many examples of *compact* riemannian 7-manifolds with holonomy G_2. This paper will construct examples of compact riemannian 8-manifolds with holonomy $Spin(7)$, using similar methods. We believe that these are the first examples known. Since metrics with holonomy $Spin(7)$ are ricci-flat, these are also new examples of compact, ricci-flat riemannian 8-manifolds.

Let M be an 8-manifold. A $Spin(7)$- structure on M can be encoded in a 4-form Ω on M, a special 4-form satisfying the condition that the stabilizer of Ω at each point should be isomorphic to $Spin(7)$. By an abuse of notation, we usually identify the $Spin(7)$- structure with its associated 4-form Ω. Since $Spin(7) \subset SO(8)$, the $Spin(7)$- structure also induces a riemannian metric g and an orientation on M.

It turns out that the holonomy group $\mathrm{Hol}(g)$ of g is a subgroup of $Spin(7)$, with $Spin(7)$- structure Ω, if and only if $d\Omega = 0$. The quantity $d\Omega$ is called the *torsion* of the $Spin(7)$- structure Ω, and Ω is called *torsion-free* if $d\Omega = 0$. Very briefly, the plan of the paper splits into four steps, as follows. Firstly, a compact 8-manifold M is given. Secondly, a $Spin(7)$- structure Ω on M is found, with *small torsion*, i.e. $d\Omega$ is small. Thirdly, Ω is deformed to a nearby $Spin(7)$-structure $\tilde{\Omega}$ with $d\tilde{\Omega} = 0$. Thus $\tilde{\Omega}$ is torsion-free, and if $\tilde{g}$ is the associated metric then $\mathrm{Hol}(\tilde{g}) \subset Spin(7)$. Fourthly, it is shown that $\mathrm{Hol}(\tilde{g})$ is $Spin(7)$, and not some proper subgroup.

The structure of the paper is designed around this division of the construction into four steps. There are six chapters. This first chapter is of introductory material. The second chapter has the full statements of the main results. In particular, §2.1 gives four theorems, called Theorems A-D, and a condition, called Condition E, which are the main tools used in the paper. Chapter 2 also contains discussion and informal explanation of the proofs, and some conclusions and questions for futher study. Then Chapters 3-6 provide the proofs of the results stated in Chap. 2. Each chapter is one of the four steps above. We will now describe these four steps in greater detail.

Step 1 Let T^8 be the 8-torus, and let $\hat{\Omega}$ be a flat $Spin(7)$- structure upon it. Suppose Γ is a finite group of isometries of T^8 preserving $\hat{\Omega}$. Then the quotient T^8/Γ is an orbifold, a compact 8-manifold with singularities, and carries a flat, torsion-free $Spin(7)$- structure $\hat{\Omega}$.

An 8-manifold M is defined by resolving the singularities of T^8/Γ. This M is a compact, nonsingular 8-manifold, with a continuous map $\pi : M \to T^8/\Gamma$ that is surjective, and is smooth and injective on the nonsingular part of T^8/Γ. We consider only orbifolds T^8/Γ with singularities of a certain very simple sort, and we resolve them in a standard way using something we call the Eguchi-Hanson space. This is done in Chap. 3.

Step 2 A constant $\theta > 0$ is given, and a smooth family $\{\Omega_t : 0 < t \leq \theta\}$ of $Spin(7)$- structures is explicitly defined on the compact 8-manifold M. The idea is that as $t \to 0$, Ω_t converges to the pull-back $\pi^*(\hat{\Omega})$ of the flat $Spin(7)$- structure $\hat{\Omega}$ on T^8/Γ, which is of course singular on M. Let g_t be the riemann metric associated to Ω_t.

We show that Condition E of §2.1 applies to the family $\{\Omega_t : 0 < t \leq \theta\}$. Condition E is a complex set of conditions on Ω_t and g_t, that gives bounds in terms of t for quantities such as $d\Omega_t$. Roughly speaking, the Condition says that $d\Omega_t = O(t^{9/2})$ in some sense, that the injectivity radius $\delta(g_t)$ should be at least $O(t)$, and that the Riemann curvature $R(g_t)$ should satisfy $\|R(g_t)\|_{C^0} = O(t^{-2})$. This is done in Chap. 4.

The point of this is that by choosing t very small, we can make $d\Omega_t$ as small as we like. If $d\Omega_t$ is small enough, it seems plausible that there might exist a $Spin(7)$- structure $\tilde{\Omega}_t$ close to Ω_t, for which $d\tilde{\Omega}_t = 0$.

Step 3 Two theorems, Theorems A and B, are proved in Chap. 5. Together these theorems show that if M is a compact 8-manifold and $\{\Omega_t : 0 < t \leq \theta\}$ is a family of $Spin(7)$- structures on M satisfying Condition E, then for all sufficiently small t, the $Spin(7)$- structure Ω_t can be deformed to a nearby $Spin(7)$- structure $\tilde{\Omega}_t$ with $\|\tilde{\Omega}_t - \Omega_t\|_{C^0} = O(t^{1/2})$, such that $d\tilde{\Omega}_t = 0$. Thus we show that there exist torsion-free $Spin(7)$- structures on M.

These theorems are results in analysis. We express $\tilde{\Omega}_t$ as the solution of a nonlinear elliptic equation, and attempt to converge to a solution of the equation by repeatedly solving a linear elliptic equation, to make a series of smaller and smaller corrections. Theorem A uses the machinery of elliptic operators, Sobolev and Hölder spaces, and elliptic regularity, to construct a smooth solution $\tilde{\Omega}_t$. Theorem B uses the curvature and injectivity radius estimates of Condition E to prove some facts needed by Theorem A.

Step 4 Chapter 6 proves two theorems, Theorems C and D. If a compact 8-manifold M has a torsion-free $Spin(7)$- structure Ω, then the holonomy group $\mathrm{Hol}(g)$ of the associated metric g is a subgroup of $Spin(7)$. But which subgroup? Theorem C shows that the subgroup $\mathrm{Hol}(g)$ is determined by topological invariants of M. In particular, if M is simply connected and satisfies $\hat{A}(M) = 1$, where $\hat{A}(M)$ is a given combination of the betti numbers, then g has holonomy $Spin(7)$. Combining this with Steps 1-3 we achieve our main result, the existence of metrics with holonomy $Spin(7)$ on compact 8-manifolds.

The proof of Theorem C relies on the fact that $\mathrm{Hol}(g)$ determines the constant spinors on M, and these determine the index of the positive Dirac operator on M. But by the Atiyah-Singer index theorem, this index is also given in terms of topological data. So $\mathrm{Hol}(g)$ determines a topological invariant of M, and conversely we may use this invariant to distinguish the different possibilities for $\mathrm{Hol}(g)$. Note that Theorem C is not original, but is already known.

Theorem D studies the deformation theory of torsion-free $Spin(7)$- structures. It shows that any torsion-free $Spin(7)$- structure on a compact, simply-connected 8-manifold is one of a smooth family of such structures, of a given dimension.

In Chap. 3, seven groups Γ are considered. It is a curious feature of the construction that the same orbifold T^8/Γ may admit many topologically distinct resolutions M, each with holonomy $Spin(7)$. Proposition 3.4.1 shows that these 7 examples generate at least 95 distinct, compact 8-manifolds. Thus the results of Chapters 3-6 prove the goal of this paper, stated in Theorem 2.1.2, which says that there are at least 95 distinct, simply-connected, compact 8-manifolds that have metrics with holonomy $Spin(7)$.

The remainder of this chapter has three sections, §1.1 which introduces the holonomy group $Spin(7)$, §1.2 on our analytic notation, and §1.3 on the Eguchi-Hanson space, which is a metric of holonomy $SU(2)$ on the noncompact manifold $T^*\mathbb{CP}^1$. For further introductory discussion and explanation of the results, the reader is advised to turn to Chap. 2.

1.1 The holonomy group $Spin(7)$

We begin with some necessary facts about the structure group $Spin(7)$, which can be found in [20, Chap. 12]. Let $\mathbb{R}^8$ be equipped with an orientation and its standard metric g, and let $\tau_1, \dots, \tau_8$ be an oriented orthonormal basis of $(\mathbb{R}^8)^*$.

Define a 4-form Ω_0 on $\mathbb{R}^8$ by

$$\begin{aligned}\Omega_0 = &\tau_1\wedge\tau_2\wedge\tau_5\wedge\tau_6 + \tau_1\wedge\tau_2\wedge\tau_7\wedge\tau_8 + \tau_3\wedge\tau_4\wedge\tau_5\wedge\tau_6 + \tau_3\wedge\tau_4\wedge\tau_7\wedge\tau_8 \\ &+ \tau_1\wedge\tau_3\wedge\tau_5\wedge\tau_7 - \tau_1\wedge\tau_3\wedge\tau_6\wedge\tau_8 - \tau_2\wedge\tau_4\wedge\tau_5\wedge\tau_7 + \tau_2\wedge\tau_4\wedge\tau_6\wedge\tau_8 \\ &- \tau_1\wedge\tau_4\wedge\tau_5\wedge\tau_8 - \tau_1\wedge\tau_4\wedge\tau_6\wedge\tau_7 - \tau_2\wedge\tau_3\wedge\tau_5\wedge\tau_8 - \tau_2\wedge\tau_3\wedge\tau_6\wedge\tau_7 \\ &+ \tau_1\wedge\tau_2\wedge\tau_3\wedge\tau_4 + \tau_5\wedge\tau_6\wedge\tau_7\wedge\tau_8.\end{aligned} \tag{1}$$

The subgroup of $GL(8,\mathbb{R})$ preserving Ω_0 is isomorphic to $Spin(7)$, the double cover of $SO(7)$, which is a compact, semisimple, 21-dimensional Lie group. It is a subgroup of $SO(8)$, so that the metric g can be reconstructed from Ω_0. The 4-form Ω_0 is self-dual, meaning that $*\Omega_0 = \Omega_0$, where $*$ is the Hodge star.

Let M be an 8-manifold, and define AM to be the subbundle of Λ^4T^*M of 4-forms that are identified with the 4-form Ω_0 of (1) by some isomorphism between $\mathbb{R}^8$ and T_mM. The fibre of AM is $GL(8,\mathbb{R})/Spin(7)$, so that AM is not a vector subbundle of Λ^4T^*M. Smooth sections of AM are called *admissible 4-forms*, and are equivalent to $Spin(7)$- structures on M. The fibre of AM has dimension 43, so that AM is of codimension 27 in Λ^4T^*M.

Let Ω be a smooth section of AM. Then Ω is a smooth 4-form on M, and defines a $Spin(7)$- structure on M. There is a 1-1 correspondence between $Spin(7)$- structures on M and sections Ω of AM, so by an abuse of notation we shall often identify a $Spin(7)$- structure with its associated 4-form Ω. A $Spin(7)$- structure Ω on M induces a natural metric g on M by the inclusion $Spin(7) \subset SO(8)$. Let ∇ be the Levi-Civita connection of g. The quantity $\nabla\Omega$ is called the *torsion* of the $Spin(7)$- structure, and Ω is called *torsion-free* if $\nabla\Omega = 0$. By [20, Lemma 12.4], $\nabla\Omega$ is determined by $d\Omega$, and $\nabla\Omega = 0$ if and only if $d\Omega = 0$.

The condition for g to have holonomy group $\mathrm{Hol}(g)$ contained in $Spin(7)$, and for Ω to be the associated $Spin(7)$- structure, is exactly that Ω should be torsion-free. Thus a torsion-free $Spin(7)$- structure Ω on M defines a metric g on M with $\mathrm{Hol}(g) \subset Spin(7)$. However, $\mathrm{Hol}(g)$ may be a proper subgroup of $Spin(7)$ rather than $Spin(7)$ itself. For compact manifolds M, Theorem C of §2.1 gives a test to determine $\mathrm{Hol}(g)$ using topological data from M. An important fact about the holonomy group $Spin(7)$ is that by [20, Lemma 12.6], if $\mathrm{Hol}(g) \subset Spin(7)$, then g is ricci-flat.

The action of $Spin(7)$ on $\mathbb{R}^8$ gives an action of $Spin(7)$ on $\Lambda^k(\mathbb{R}^8)^*$, which splits $\Lambda^k(\mathbb{R}^8)^*$ into an orthogonal direct sum of irreducible representations of $Spin(7)$. Suppose that M is an oriented 8-manifold with a $Spin(7)$- structure, so that M has a 4-form Ω and a metric g. Then in the same way, Λ^kT^*M splits into an orthogonal direct sum of subbundles with irreducible representations of $Spin(7)$ as fibres. The next result describes these splittings. We shall use the notation Λ^k_l for an irreducible representation of dimension l lying in Λ^kT^*M.

Proposition 1.1.1. *Let M be an oriented 8-manifold with $Spin(7)$- structure, giving a 4-form Ω and a metric g on M. Then Λ^kT^*M splits orthogonally into components as follows, where Λ^k_l is an irreducible representation of $Spin(7)$ of dimension l:*

(i) $\Lambda^1T^*M = \Lambda^1_8$, (ii) $\Lambda^2T^*M = \Lambda^2_7 \oplus \Lambda^2_{21}$, (iii) $\Lambda^3T^*M = \Lambda^3_8 \oplus \Lambda^3_{48}$,
(iv) $\Lambda^4T^*M = \Lambda^4_+T^*M \oplus \Lambda^4_-T^*M$, $\Lambda^4_+T^*M = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}$, $\Lambda^4_-T^*M = \Lambda^4_{35}$,
(v) $\Lambda^5T^*M = \Lambda^5_8 \oplus \Lambda^5_{48}$, (vi) $\Lambda^6T^*M = \Lambda^6_7 \oplus \Lambda^6_{21}$, (vii) $\Lambda^7T^*M = \Lambda^7_8$.

The Hodge star $*$ gives an isometry between Λ^k_l and Λ^{8-k}_l. In part (iv), $\Lambda^4_+T^*M$ and $\Lambda^4_-T^*M$ are the $+1$- and -1- eigenspaces of $*$ on Λ^4T^*M respectively.

Proof. Part (i) holds as $Spin(7)$ acts irreducibly on $\mathbb{R}^8$, and parts (ii), (iii) and (iv) are given in [20, Proposition 12.5]. Applying the Hodge star we deduce the splittings $(v), (vi)$ and (vii). □

Let the orthogonal projection from Λ^kT^*M to Λ^k_l be denoted π_l. Then, for instance, if $\xi \in \Gamma(\Lambda^2T^*M)$, $\xi = \pi_7(\xi) + \pi_{21}(\xi)$. This notation will be used throughout the paper. From above, the bundle AM is of codimension 27 in Λ^4T^*M. Clearly, the tangent space of the fibre of AM in Λ^4T^*M at Ω, depends on Ω alone. Using the splitting of Λ^4T^*M in Proposition 1.1.1, we see that $T_\Omega AM = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$, as this is the only natural vector subbundle with codimension 27.

1.2 Analytic preliminaries

Let M be a riemannian manifold with metric g, and V a vector bundle on M with metrics on the fibres and a connection ∇ preserving these metrics. In problems in analysis it is often useful to consider infinite-dimensional vector spaces of sections of V over M, and to equip these vector spaces with norms, making them into Banach spaces. In this paper you will meet four different types of Banach spaces of this sort, written $L^q_k(V)$, $L^q(V)$, $C^k(V)$ and $C^{k,\alpha}(V)$. We define them now, in order to fix our notation.

Let $q \geq 1$ be real and $k \geq 0$ be an integer. Define the *Sobolev space* $L^q_k(V)$ to be the set of locally integrable sections v of V that are k times weakly differentiable, and for which the *Sobolev norm*

$$\|v\|_{q,k} = \sum_{j=0}^{k} \left(\int_M |\nabla^j v|^q d\mu \right)^{1/q} \tag{2}$$

is finite. Here $d\mu$ is the volume form of g. The *Lesbesgue space* $L^q(V)$ is $L^q_0(V)$, and the *Lesbesgue norm* is $\|v\|_q = \|v\|_{q,0}$.

For integers $k \geq 0$, define the space $C^k(V)$ to be the space of continuous, bounded sections v of V that have k continuous, bounded derivatives, and define the norm $\|v\|_{C^k}$ by $\|v\|_{C^k} = \Sigma^k_{i=0} \sup_M |\nabla^i v|$. Let $\alpha \in (0,1)$, and let $k \geq 0$ be an integer. The *Hölder space* $C^{k,\alpha}(V)$ is the set of $v \in C^k(V)$ for which $\nabla^k v$ satisfies a Hölder condition of exponent α. The *Hölder norm* on $C^{k,\alpha}(V)$ is written $\|v\|_{C^{k,\alpha}}$, but we will not define it.

For the basic theory of Sobolev and Hölder spaces, we refer the reader to [2]. Later in the paper we shall make frequent use of two standard results in analysis, without explaining them or giving references. These are the Sobolev embedding theorem, and regularity results for elliptic operators. Embedding theorems are dealt with at length by Aubin in [2, §§2.3-2.9]. Elliptic regularity results can be found in [1], [2, §3.6] and [5, Theorems 27, 31, p. 463-4].

1.3 The Eguchi-Hanson space

A metric on an oriented 4-manifold with holonomy contained in $SU(2)$ is called a *hyperkähler structure* [20, p. 114]. Hyperkähler 4-manifolds have received a lot of attention, and much is known about different compact and noncompact examples. The simplest nontrivial example of a hyperkähler 4-manifold is the Eguchi-Hanson space [9], which is a complete hyperkähler metric on the noncompact 4-manifold $T^*\mathbb{CP}^1$. We will give this metric explicitly in coordinates, as it will be needed in Chap. 4.

Consider $\mathbb{C}^2$ with complex coordinates (z_1, z_2), acted upon by the involution $-1 : (z_1, z_2) \mapsto (-z_1, -z_2)$. Let X be the blow-up of $\mathbb{C}^2/\{\pm 1\}$ at the singular point, and let $\pi : X \to \mathbb{C}^2/\{\pm 1\}$ be the blow-down map. Then X is biholomorphic to $T^*\mathbb{CP}^1$, and has $\pi_1(X) = \{1\}$ and $H^2(X, \mathbb{R}) = \mathbb{R}$. The function $u = |z_1|^2 + |z_2|^2$ on $\mathbb{C}^2$ pushes down to $\mathbb{C}^2/\{\pm 1\}$ and so lifts to X. Let $t \geq 0$, and define a function f_t on X by

$$f_t = \sqrt{u^2 + t^4} + t^2 \log u - t^2 \log \left(\sqrt{u^2 + t^4} + t^2 \right). \tag{3}$$

This is the Kähler potential for the Eguchi-Hanson metric, and is taken from [15, p. 593]. Define a 2-form ω_t on X by $\omega_t = \frac{1}{2} i \partial \overline{\partial} f_t$. For $t > 0$, ω_t is the Kähler form of a Kähler metric h_t on X. This is the Eguchi-Hanson metric, and has holonomy $SU(2)$.

The dilation $(z_1, z_2) \mapsto (cz_1, cz_2)$ for some positive constant c induces an endomorphism of X, which takes h_t to $c^{-2} h_{ct}$. Thus the metrics h_t for $t > 0$ are all equivalent modulo diffeomorphisms and homotheties. Putting $t = 0$ gives $f = u$ in (3), so that h_0 is the pullback to X of the euclidean metric on $\mathbb{C}^2/\{\pm 1\}$. From (3) we can calculate the asymptotic behaviour of $h_t - h_0$. The result is

$$h_t = h_0 + t^4 A + B_t, \qquad \text{with } |A| = Cu^{-2} \text{ and } |B_t| = O(t^8 u^{-4}). \tag{4}$$

Here A and B_t are defined for $u > 0$, A is independent of t and homogeneous under dilations, C is a constant independent of t, and the norms $|\,.\,|$ are taken w.r.t. the metric h_0, which is nonsingular for $u > 0$. The estimate (4) shows that the Eguchi-Hanson metric is asymptotic to the flat metric h_0 at infinity. This behaviour is called *asymptotically locally euclidean*, or *ALE* for short.

2 Results and discussion

Section 2.1 gives the statements of the main results of the paper, and assuming these it is shown that there are many compact 8-manifolds that admit metrics with holonomy $Spin(7)$. The proofs of these results are deferred to Chapters 3-6. The remainder of the chapter discusses the results and their proofs, and enlarges on the explanation given in the introduction.

Section 2.2 recalls the Kummer construction for metrics with holonomy $SU(2)$ on the $K3$ surface, and uses this to motivate the idea of looking for metrics with holonomy $Spin(7)$ on a resolution of an orbifold T^8/Γ, and explains why the Eguchi-Hanson space of §1.3 should be involved in the desingularization. Then §2.3 discusses Theorems A and B and Condition E of §2.1, giving some reasons why the hypotheses of Condition E were chosen. Finally, §2.4 suggests some questions for further research.

2.1 The main results

The results of the paper hinge upon the following four theorems.

Theorem A. *Let $A_1, \ldots, A_4$ be positive constants. Then there exist positive constants κ, λ depending only on $A_1, \ldots, A_4$, such that whenever $0 < t \leq \kappa$, the following is true.*

Let M be a compact 8-manifold, and Ω a smooth section of AM on M. Suppose that ϕ is a smooth 4-form on M with $d\Omega + d\phi = 0$, and that these four conditions hold:

(i) $\|\phi\|_2 \leq A_1 t^{9/2}$,

(ii) $\|d\phi\|_{10} \leq A_2 t$,

(iii) if $\chi \in L_1^{10}(\Lambda_-^4)$ *then* $\|\nabla\chi\|_{10} \leq A_3\big(\|d\chi\|_{10} + t^{-21/5}\|\chi\|_2\big)$, *and*

(iv) if $\chi \in L_1^{10}(\Lambda_-^4)$ *then* $\chi \in C^0(\Lambda_-^4)$ *and* $\|\chi\|_{C^0} \leq A_4\big(t^{1/5}\|\nabla\chi\|_{10} + t^{-4}\|\chi\|_2\big)$.

Then there exists a smooth, closed section $\tilde{\Omega}$ of AM satisfying $\|\tilde{\Omega} - \Omega\|_{C^0} \leq \lambda t^{1/2}$. Thus $\tilde{\Omega}$ is a torsion-free $Spin(7)$- structure.

Theorem B. *Let ρ, K be positive constants. Then there exist positive constants A_3, A_4 depending only on ρ and K, such that the following is true.*

Let $t > 0$, and let M be a compact, oriented riemannian 8-manifold with metric g. Suppose that the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq \rho t$, and the Riemann curvature $R(g)$ of g satisfies $\|R(g)\|_{C^0} \leq K t^{-2}$. Then conditions (iii) and (iv) of Theorem A hold for the metric g on M.

Theorem C. *Suppose that M is a compact, simply-connected 8-manifold and that Ω is a torsion-free $Spin(7)$- structure on M, and let g be the associated metric. Then M is spin, and the volume form $\Omega \wedge \Omega$ gives a natural orientation on M. Define the $\hat{A}$- genus $\hat{A}(M)$ of M by*

$$24\hat{A}(M) = -1 + b^1 - b^2 + b^3 + b_+^4 - 2b_-^4, \tag{5}$$

where b^i are the betti numbers of M, and $b_\pm^4$ are the dimensions of the spaces of self-dual and anti-self-dual 4-forms in $H^4(M, \mathbb{R})$. Then $\hat{A}(M)$ is equal to 1, 2, 3 or 4, and the holonomy group $\mathrm{Hol}(g)$ of g is determined by $\hat{A}(M)$ as follows:

(*i*) $\mathrm{Hol}(g) = Spin(7)$ *if and only if* $\hat{A}(M) = 1$,

(*ii*) $\mathrm{Hol}(g) = SU(4)$ *if and only if* $\hat{A}(M) = 2$,

(*iii*) $\mathrm{Hol}(g) = Sp(2)$ *if and only if* $\hat{A}(M) = 3$, *and*

(*iv*) $\mathrm{Hol}(g) = SU(2) \times SU(2)$ *if and only if* $\hat{A}(M) = 4$.

Every compact, riemannian 8-manifold with holonomy group $Spin(7)$ *is simply-connected.*

Theorem D. *Let* M *be a simply-connected, compact 8-manifold admitting torsion-free* $Spin(7)$*- structures, let* $\mathcal{X}$ *be the set of torsion-free* $Spin(7)$*- structures on* M*, and let* $\mathcal{D}$ *be the group of diffeomorphisms of* M *isotopic to the identity. Then* $\mathcal{X}/\mathcal{D}$ *is a smooth manifold of dimension* $\hat{A}(M) + b^4_-(M)$.

The proofs of Theorems A and B will be given in Chap. 5, and of Theorems C and D in Chap. 6. Theorems A, B and D are mostly analysis, and Theorem C is mostly topology. In order to apply Theorems A-D, we will make use of the following condition.

Condition (E). Let M be a compact 8-manifold. Suppose there exists a constant $\theta \in (0,1)$, a family $\{\Omega_t : 0 < t \leq \theta\}$ of smooth $Spin(7)$- structures on M, and a family $\{\phi_t : 0 < t \leq \theta\}$ of smooth 4-forms on M. Let g_t be the metric on M induced by Ω_t. Let A_1, A_2, ρ and K be positive constants independent of t, and suppose that the following five properties hold for each $t \in (0, \theta]$.

(*i*) $d\Omega_t + d\phi_t = 0$,

(*ii*) $\|\phi_t\|_2 \leq A_1 t^{9/2}$,

(*iii*) $\|d\phi_t\|_{10} \leq A_2 t$,

(*iv*) the injectivity radius $\delta(g_t)$ of g_t satisfies $\delta(g_t) \geq \rho t$, and

(*v*) the Riemann curvature $R(g_t)$ of g_t satisfies $\left\|R(g_t)\right\|_{C^0} \leq K t^{-2}$.

Here all norms are taken w.r.t. the metric g_t on M.

To see how Condition E relates to Theorems A-D, we shall prove the next proposition.

Proposition 2.1.1. *Let* M *be a simply-connected, compact 8-manifold with* $\hat{A}(M) = 1$*, and suppose* M *satisfies Condition E. Then* M *admits metrics with holonomy* $Spin(7)$*, and the family of such metrics is a smooth manifold of dimension* $b^4_-(M) + 1$.

Proof. Apply Theorems A and B with the $Spin(7)$- structure Ω_t of Condition E in place of Ω, and the 4-form ϕ_t of Condition E in place of ϕ, for some $t \leq \theta$. Parts (*iv*) and (*v*) of Condition E show that the hypotheses of Theorem B hold, so by Theorem B, parts (*iii*) and (*iv*) of Theorem A hold for some A_3, A_4

depending on ρ and K. But parts (i)-(iii) of Condition E give the remaining hypotheses of Theorem A.

Thus the hypotheses of Theorem A hold for constants $A_1, \ldots, A_4$ independent of t, provided $0 < t \leq \theta$. Theorem A gives a constant $\kappa > 0$ depending only on $A_1, \ldots, A_4$. Suppose now that $0 < t \leq \min(\kappa, \theta)$. Then by Theorem A, there exists a torsion-free $Spin(7)$- structure $\tilde{\Omega}$ on M. Thus we have shown that for each t with $0 < t \leq \min(\kappa, \theta)$, there is a torsion-free $Spin(7)$- structure on M.

By assumption, M is simply-connected and $\hat{A}(M) = 1$. Therefore part (i) of Theorem C shows that if Ω is a torsion-free $Spin(7)$- structure on M and g the associated metric, then the holonomy group of g is $Spin(7)$. Finally, Theorem D shows that the family of such metrics is a smooth manifold of dimension $b^4_-(M) + 1$. This completes the proof. □

Our plan is this. We will construct, explicitly, a number of simply-connected, compact 8-manifolds M with $\hat{A}(M) = 1$. This is done in Chap. 3 by taking the quotient T^8/Γ of a torus T^8 with a flat $Spin(7)$- structure $\hat{\Omega}$, by a finite group of isometries Γ preserving $\hat{\Omega}$, and resolving the singularities of T^8/Γ in a special way to get M. Then in Chap. 4 we will write down on each M, explicitly in coordinates, two families $\{\Omega_t : 0 < t \leq \theta\}$ of $Spin(7)$- structures and $\{\phi_t : 0 < t \leq \theta\}$ of 4-forms, and verify that Condition E holds for these families.

Proposition 2.1.1 then shows that each of these 8-manifolds M has metrics with holonomy $Spin(7)$. The results of Chapters 3 and 4 are summarized in Proposition 3.4.1, which states that the examples in Chap. 3 include at least 95 topologically distinct 8-manifolds, and Theorem 4.2.5, which states that every example in Chap. 3 satisfies Condition E. Combining these two with Proposition 2.1.1 proves the following, which is the main goal of this paper.

Theorem 2.1.2. *There are at least 95 topologically distinct, compact, simply-connected 8-manifolds that admit metrics with holonomy $Spin(7)$.*

We believe these are the first known examples of compact riemannian 8-manifolds with holonomy $Spin(7)$.

2.2 Some motivation for the construction

In this section we try to explain why a resolution M of an orbifold T^8/Γ might carry metrics with holonomy $Spin(7)$, what sort of singularities in T^8/Γ we are able to resolve, and what the Eguchi-Hanson space of §1.3 is for. To do this we start by describing some mathematics that is already well known and understood: the Kummer construction for metrics with holonomy $SU(2)$ on the $K3$ surface.

The $K3$ surface is a compact complex surface, which admits a family of metrics with holonomy $SU(2)$, by Yau's solution of the Calabi conjecture. Although it is known that these metrics exist, the metrics are not known explicitly, and it is a difficult problem to describe what they are like. However, there is a method by Page [18] which gives an approximate description of some of the metrics, called the *Kummer construction*.

Let T^4 have coordinates $(x_1, \ldots, x_4)$, with $x_i \in \mathbb{R}/\mathbb{Z}$, and let -1 act on T^4 by $-1 : (x_1, \ldots, x_4) \mapsto (-x_1, \ldots, -x_4)$. Then $T^4/\{\pm 1\}$ is a singular 4-manifold with 16 singular points $x_i \in \{0, \frac{1}{2}\}$. Regarding T^4 as a complex manifold, let Y be the blow-up of $T^4/\{\pm 1\}$ at each singular point. Then Y is a compact, nonsingular complex surface called a $K3$ surface. The blow-ups are modelled on the Eguchi-Hanson space X of §1.3, so we may regard Y as the result of gluing 16 copies of X into the orbifold $T^4/\{\pm 1\}$.

There is a flat Kähler metric h on $T^4/\{\pm 1\}$, and X has a family $\{h_t : t > 0\}$ of metrics of holonomy $SU(2)$. It turns out that when t is small, there is a metric g_t on Y with holonomy $SU(2)$, such that g_t is close to the metric h_t near the resolution of each singular point, and g_t is close to h away from the singular points. Moreover, the smaller t is, the better the approximation of g_t with h_t and h. Both Topiwala [21] and LeBrun and Singer [16] have given proofs of the existence of metrics of holonomy $SU(2)$ on $K3$ using this idea. Their proofs show that g_t exists when t is sufficiently small.

Now let us move from 4-manifolds to 8-manifolds. Consider the products $T^4 \times (T^4/\{\pm 1\})$ and $(T^4/\{\pm 1\}) \times (T^4/\{\pm 1\})$, which we may write as $T^8/\mathbb{Z}_2$ and $T^8/\mathbb{Z}_2^2$. Using the resolution $K3$ of $T^4/\{\pm 1\}$, we see that $T^4 \times K3$ is a resolution of $T^8/\mathbb{Z}_2$, and admits metrics of holonomy $\{1\} \times SU(2)$, and $K3 \times K3$ is a resolution of $T^8/\mathbb{Z}_2^2$, with metrics of holonomy $SU(2) \times SU(2)$.

Both $\{1\} \times SU(2)$ and $SU(2) \times SU(2)$ are subgroups of $Spin(7)$. It follows that these 8-manifolds $T^4 \times K3$ and $K3 \times K3$ both have torsion-free $Spin(7)$-structures. Moreover, in view of the Kummer construction above, we may approximate some of the torsion-free $Spin(7)$- structures on $T^4 \times K3$ and $K3 \times K3$ by combining the flat $Spin(7)$- structure on $T^8/\mathbb{Z}_2$ or $T^8/\mathbb{Z}_2^2$ with the metrics h_t on the Eguchi-Hanson space X.

The single, motivating idea behind this paper is as follows. Above we have two examples of orbifolds $T^8/\mathbb{Z}_2$, $T^8/\mathbb{Z}_2^2$ that admit a resolution M, such that a flat $Spin(7)$- structure on the orbifold can be used to construct torsion-free $Spin(7)$- structures on M. Why not look for more general orbifolds T^8/Γ, with a flat $Spin(7)$- structure, such that all the singularities of T^8/Γ are modelled on the singularities of these two examples? By copying the examples we could define a resolution M, and it seems possible that this M might also admit torsion-free $Spin(7)$- structures. In good cases the associated metrics would have holonomy $Spin(7)$.

2.3 A discussion of the proof

The heart of the proof is Chapters 4 and 5. Chapter 4 proves that Condition E of §2.1 holds for the 8-manifolds M, and Chap. 5 proves Theorems A and B of §2.1, which show that if Condition E holds for some M, then M admits torsion-free $Spin(7)$- structures. In this section we hope to explain what the main issues in these proofs are, and why the particular hypotheses of Condition E were chosen.

First, some generalities. Let M be a compact 8-manifold, let Ω be a smooth $Spin(7)$- structure on M, and let g be the associated metric. The sort of result we want to prove says that if $d\Omega$ is sufficiently small, then there exists a new $Spin(7)$- structure $\tilde{\Omega}$, close to Ω, such that $d\tilde{\Omega} = 0$. Let us write $\eta = \tilde{\Omega} - \Omega$. Then $d\tilde{\Omega} = 0$ gives a p.d.e. for η.

Many proofs in analysis that construct a solution of a p.d.e., make use of an a priori bound for the solution. In this case, by an integration by parts we find that if $d\Omega + d\phi = 0$ for some smooth 4-form ϕ, and a solution η is small in C^0, then $\|\eta\|_2 \leq (1+\epsilon)\|\phi\|_2$ for some small $\epsilon > 0$. Therefore, let us suppose we have found a 4-form ϕ such that $d\Omega + d\phi = 0$ and $\|\phi\|_2$ is small. This gives an a priori bound for $\|\eta\|_2$.

What we really want, however, is a bound for $\|\eta\|_{C^0}$. We have a bound for $\|\eta\|_2$, and since $d\eta = -d\Omega$, any bound for $d\Omega$ gives a bound for $d\eta$. It turns out that using an elliptic regularity result and a Sobolev embedding result, we may prove that

$$\|\eta\|_{C^0} \leq C_1\|d\eta\|_q + C_2\|\eta\|_2. \tag{6}$$

Here $q > 8$, and C_1, C_2 are positive constants depending on q and g. Let us fix $q = 10$ for convenience; any $q > 8$ would do, provided it is not too large.

The important question is, how do the constants C_1 and C_2 depend on g? The dependence is through the injectivity radius and the curvature. Let $t > 0$, and suppose that the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq t$, and the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq t^{-2}$. Then it turns out, as in parts *(iii)* and *(iv)* of Theorem A, that $C_1 = O(t^{1/5})$ and $C_2 = O(t^{-4})$. From (6) we deduce that if $\|d\Omega\|_{10} \ll t^{-1/5}$, and $\|\phi\|_2 \ll t^4$, then η is small in C^0.

This estimate on $\|\eta\|_{C^0}$ is the important ingredient needed to make the proof work. Therefore, we are able to prove the following result. Suppose that ϕ is a 4-form with $d\Omega + d\phi = 0$, that $t > 0$, that $\delta(g) \geq t$ and $\|R(g)\|_{C^0} \leq t^{-2}$, and that $\|d\Omega\|_{10} \ll t^{-1/5}$ and $\|\phi\|_2 \ll t^4$. Then there exists a $Spin(7)$- structure $\tilde{\Omega}$ near to Ω with $d\tilde{\Omega} = 0$. This, approximately, is Theorems A and B combined. In brief, we can show that the torsion-free $Spin(7)$- structure $\tilde{\Omega}$ exists, provided the torsion $d\Omega$ is small compared to both the injectivity radius $\delta(g)$ and the Riemann curvature $\|R(g)\|_{C^0}$.

For the next stage of the argument, we require an orbifold T^8/Γ with a flat $Spin(7)$- structure $\hat{\Omega}$, that can be resolved using the Eguchi-Hanson space X of §1.3 to give a compact 8-manifold M. For simplicity, we will give our explanation for the example $T^8/\Gamma = T^4 \times (T^4/\{\pm 1\})$ and $M = T^4 \times K3$ in the last section. Let h be the euclidean metric on T^4 and $T^4/\{\pm 1\}$, and let h_t be the metric on X of §1.3.

We divide M into 3 regions. The first region contains the resolution of all the singular points, and is identified with 16 copies of $T^4 \times U$, where U is a subset of X. The second region is $T^4 \times V$, where V is a subset of $T^4/\{\pm 1\}$ containing none of the singular points. The third region is an 'annulus' that interpolates between the first and second regions. In Chap. 4 we will show how to define a family $\{\Omega_t : 0 < t \leq \theta\}$ of smooth $Spin(7)$- structures on M, with

g_t the metric associated to Ω_t.

On the first region $T^4 \times 16\,U$, g_t is $h + h_t$. On the second region $T^4 \times V$, g_t is $h + h$. On the third region, g_t interpolates smoothly between $h + h_t$ and $h + h$. On the first region, Ω_t comes from the flat structure on T^4 and the $SU(2)$- structure on X associated to h_t, and on the second region $\Omega_t = \hat{\Omega}$, the flat $Spin(7)$- structure. On the third region, Ω_t interpolates smoothly between the two. Thus $d\Omega_t = 0$ on the first and second regions, as the $SU(2)$- structure associated to h_t is torsion-free, but $d\Omega_t$ need not vanish on the third region, because of the 'errors' introduced by the interpolation.

How will $\delta(g_t)$ and $\|R(g_t)\|_{C^0}$ depend on t? Well, h_t is isometric to $t^2 h_1$, so $\delta(h_t) = O(t)$ and $\|R(h_t)\|_{C^0} = O(t^{-2})$. When t is small, h_t will make the dominant contributions to $\delta(g_t)$ and $\|R(g_t)\|_{C^0}$, so we deduce that $\delta(g_t) = O(t)$ and $\|R(g_t)\|_{C^0} = O(t^{-2})$ for small t. Suppose that we also define ϕ_t with $d\Omega_t + d\phi_t = 0$. From above, we want to know how $\|d\Omega_t\|_{10}$ and $\|\phi_t\|_2$ depend on t. Naturally, these depend on the precise definitions of Ω_t and ϕ_t.

The most naïve way to define Ω_t and g_t on the third region is simply to combine the values on the first and second regions using a partition of unity. In this case, $d\Omega_t$ is about the size of $h_t - h$. But from Eq. (4) of §1.3, $h_t - h = O(t^4)$. Because of this, it is easy to show that for Ω_t defined in this fashion, $\|d\Omega_t\|_{10} = O(t^4)$ for small t, and if ϕ_t is defined in a sensible way with $d\Omega_t + d\phi_t = 0$, then $\|\phi_t\|_2 = O(t^4)$ for small t.

Unfortunately, we are not yet in a position to apply the analytic result above. Putting Ω_t in place of Ω and ϕ_t in place of ϕ, we see that $\|d\Omega_t\|_{10} \ll t^{-1/5}$ holds for small t, but we know only that $\|\phi_t\|_2 = O(t^4)$, not that $\|\phi_t\|_2 \ll t^4$. Thus we are very close, but not quite able, to prove the result we want.

The solution the author has adopted to this problem is to define Ω_t and ϕ_t in a more sophisticated and careful manner, so that we achieve $\|\phi_t\|_2 = O(t^{9/2})$ for small t. Thus when t is small, we have $\|\phi_t\|_2 \ll t^4$ and the analytic result above applies. This is done by finding a way to cancel the terms of order t^4 introduced by the gluing.

The cost of this is that Chap. 4, in which Ω_t and ϕ_t are defined, is rather difficult to understand. Many auxiliary forms are introduced for no apparent reason, two forms X_t and Y_t are defined in §4.1, and the crucial equation $dX_t + dY_t = 0$ in §4.2 appears like a conjuring trick after some mystery cancellations. We apologize to the reader for this.

The only point of the complicated definitions in Chap. 4 is to obtain the power $t^{9/2}$ in part (ii) of Condition E of §2.1. If we just wanted the power t^4 then Chap. 4 would be a lot simpler, but then Theorems A and B would not apply. As it is we are forced to be devious, in order to cancel the terms of size t^4 that crop up when doing it the easy way. Note that this problem did not arise in the G_2 case of [12], and in that paper the naïve approach of joining two G_2- structures with a partition of unity was sufficient. The moral appears to be that seven dimensions are easier than eight.

2.4 Conclusions and questions

Although we have found at least 95 distinct 8-manifolds with holonomy $Spin(7)$, which seems a lot, all our examples are very similar to each other. This is in contrast to the G_2 case [12], [13], in which a wide variety of different orbifolds T^7/Γ were found that could be resolved to give 7-manifolds with holonomy G_2. I have tried hard to find more general orbifolds T^8/Γ that can be resolved with holonomy $Spin(7)$, using these elementary techniques, but I have not yet been successful.

One reason for this seems to be that many simple singularities of 8-orbifolds cannot be resolved with holonomy contained in $Spin(7)$. So, when considering orbifolds T^8/Γ, we often meet these unresolvable singularities. Here is an example. The simplest 8-orbifold singularity imaginable is the point 0 in $\mathbb{R}^8/\{\pm 1\}$. But there is no resolution of this singularity with a torsion-free, asymptotically locally euclidean $Spin(7)$- structure. In other words, this singularity cannot be resolved with holonomy $Spin(7)$.

Suppose, for a contradiction, that such a resolution of the singularity did exist. Consider the orbifold $T^8/\{\pm 1\}$. This has a flat $Spin(7)$- structure $\hat{\Omega}$, and 256 isolated singular points modelled on the singularity of $\mathbb{R}^8/\{\pm 1\}$. Let us resolve each singularity using the model resolution, to get a compact 8-manifold M. By the methods of this paper, M admits torsion-free $Spin(7)$- structures.

However, we can easily show that M is simply-connected and the betti numbers of M are $b^1 = 0$, $b^2 = 28 + 256a$, $b^3 = 256b$, $b^4_+ = 35 + 256c$, and $b^4_- = 35 + 256d$, where a, b, c, d are nonnegative integers. It follows that $24\hat{A}(M) = 256(b + c - a - 2d) - 64$, where $b + c - a - 2d$ is an integer. But from Theorem C we must have $\hat{A}(M) = 1, 2, 3$ or 4, which is a contradiction. Therefore the singularity of $\mathbb{R}^8/\{\pm 1\}$ cannot be resolved with holonomy $Spin(7)$.

The examples in this paper are probably only a small fraction of the 8-manifolds with holonomy $Spin(7)$ that actually exist. Clearly, it is desirable to find more examples and get a better idea of the 'geography' of such manifolds. I think that the next step may be to develop a theory of what sorts of orbifold singularity can be resolved with holonomy $Spin(7)$. I believe that there are many orbifolds T^8/Γ that can be resolved with holonomy $Spin(7)$, but I expect that in general these orbifolds will have nasty singular sets, with singular submanifolds intersecting in complicated ways.

Here are some problems that I feel are interesting, and may be worth investigating.

- What sorts of singularity of an orbifold T^8/Γ, can be resolved to give a compact 8-manifold M with holonomy contained in $Spin(7)$, following the method of this paper?
- Try to classify compact 8-manifolds M with holonomy $Spin(7)$ that arise from the desingularization of orbifolds T^8/Γ.
- All the compact 8-manifolds M of Chap. 3 satisfy the topological constraints $b^4_+(M) - b^4_-(M) = 64$ and $b^3(M) \equiv 0 \bmod 4$. Are there compact

8-manifolds M with holonomy $Spin(7)$ for which these constraints do not hold?

I conjecture that the answer to this question is Yes, and that these constraints are just features of the particular construction used in this paper.

- Any compact 8-manifold M with holonomy $Spin(7)$ must be spin and simply-connected and satisfy $\hat{A}(M) = 1$ and $b^4_+(M) \geq 1$. Can you prove any other interesting topological constraints on such M?
- Are there finitely or infinitely many distinct, compact 8-manifolds that admit metrics with holonomy $Spin(7)$?

  I conjecture that there are only finitely many. Certainly, one can show that only finitely many manifolds can arise by desingularizing orbifolds T^8/Γ.
- The Eguchi-Hanson space of §1.3 is the simplest in an infinite family of *asymptotically locally euclidean spaces* with holonomy $SU(2)$. Much is known about these ALE spaces, and more generally about ALE spaces with holonomy $SU(n)$. Do there exist ALE spaces with holonomy $Spin(7)$?

3 Construction of compact 8-manifolds

In this chapter we will write down a number of finite groups Γ acting on T^8, describe the singular set of the quotient T^8/Γ, and explain how to desingularize the orbifold T^8/Γ using the Eguchi-Hanson space X of §1.3, to get a compact 8-manifold M. The betti numbers and fundamental groups of the manifolds M are found. Results from Chapters 4-6 then show that these manifolds M admit metrics with holonomy $Spin(7)$.

Section 3.1 describes five sorts of singularity that can occur in an orbifold T^8/Γ. We explain how to resolve each singularity using the Eguchi-Hanson space, and how to calculate the betti numbers of the resulting 8-manifold. We encounter the phenomenon that one singularity may admit two different resolutions, both with holonomy $Spin(7)$. This greatly increases the number of different 8-manifolds M that result.

Sections 3.2 and 3.3 consider 7 related groups Γ acting on T^8, and construct resolutions M of T^8/Γ. Using later results it is shown that these manifolds possess metrics with holonomy $Spin(7)$. Finally, §3.4 examines the topology of the manifolds. Proposition 3.4.1 shows that at least 95 of them are topologically distinct.

3.1 Resolving singularities of orbifolds

We begin by defining some notation that will be used throughout Chapters 3 and 4. Let T^8 be the 8-dimensional torus, and let $\hat{\Omega}$ be a $Spin(7)$- structure on T^8 that is invariant under translations in T^8. Then $\hat{\Omega}$ is torsion-free, and defines a flat metric on T^8. Let Γ be a finite group of isometries of T^8 preserving $\hat{\Omega}$. The quotient T^8/Γ is a singular 8-manifold, and also a metric space.

If $x \in T^8$ and $x\Gamma$ is a singular point of T^8/Γ, then there is at least one nonidentity element $\nu \in \Gamma$ that fixes x. Conversely, if $\nu \in \Gamma$ is a nonidentity element and $x \in T^8$ is fixed by ν, then $x\Gamma$ is a singular point of T^8/Γ. In this case let us call $x\Gamma$ a ν- *singular point* of T^8/Γ. Define S_ν to be the set of ν-singular points in T^8/Γ, and S be the set of all singular points in T^8/Γ.

If M is a metric space, $N \subset M$ and $\zeta > 0$, define the ζ- *neighbourhood* of N in M to be the open set of points m in M a distance less than ζ from some point of N. Fix ζ to be some small positive constant (for the examples in this chapter, $\zeta = \frac{1}{9}$ will do). Define Z_ν to be the ζ- neighbourhood of S_ν in T^8/Γ, and Z to be the ζ- neighbourhood of S in T^8/Γ.

Let $\mathbb{R}^4$ have coordinates $y_1, \dots, y_4$ and metric $dy_1^2 + \cdots + dy_4^2$, and let $\mathbb{C}^2$ have complex coordinates z_1, z_2 and the euclidean metric $|dz_1|^2 + |dz_2|^2$. Then $\mathbb{R}^4$ and $\mathbb{C}^2$ are isometric, and we may identify them. Define B^4_ζ to be the open ball of radius ζ about 0 in $\mathbb{R}^4 \cong \mathbb{C}^2$. It will be useful to regard B^4_ζ as both a real and a complex space.

Let X denote the Eguchi-Hanson space of §1.3. Then X is equipped with a surjective map $\pi : X \to \mathbb{C}^2/\{\pm 1\}$. Define U be the inverse image of $B^4_\zeta/\{\pm 1\}$ in X. Then U is an open subset of X, and there is a map $\pi : U \to B^4_\zeta/\{\pm 1\}$ that resolves the singularity of $B^4_\zeta/\{\pm 1\}$ at 0. That is enough notation for the present.

We shall now consider five model 8-dimensional orbifolds, numbered (i)-(v). Each one exhibits a certain sort of singularity that we are interested in. In each case we will write down an orbifold T, we will define a subset $S \subset T$ which is part of the singular set of T, and we will define a 'resolution' R, which is an 8-manifold or orbifold together with a map $\pi : R \to T$ that resolves the singularities S.

In what follows, $(x_1, \dots, x_4)$ are coordinates on T^4, with each x_i in $\mathbb{R}/\mathbb{Z}$, $(y_1, \dots, y_4)$ are real coordinates on B^4_ζ, and (z_1, z_2) are complex coordinates on B^4_ζ. The standard action of -1 on T^4 is $-1 : (x_1, \dots, x_4) \mapsto (-x_1, \dots, -x_4)$, so that $T^4/\{\pm 1\}$ is a 4-orbifold. The standard action of -1 on B^4_ζ is $-1 : (y_1, \dots, y_4) \mapsto (-y_1, \dots, -y_4)$, or equivalently $-1 : (z_1, z_2) \mapsto (-z_1, -z_2)$, so that $B^4_\zeta/\{\pm 1\}$ has one singular point at 0.

(i) Our first model orbifold is $T = T^4 \times \big(B^4_\zeta/\{\pm 1\}\big)$. The singular set of this is $S = T^4 \times \{0\}$, which is one copy of T^4. The resolution of this is $R = T^4 \times U$. The map $\pi : U \to \big(B^4_\zeta/\{\pm 1\}\big)$ induces the resolving map $\pi : R \to T$.

(ii) The second model orbifold is $T = \big(T^4/\{\pm 1\}\big) \times \big(B^4_\zeta/\{\pm 1\}\big)$. The part of the singular set we are interested in is $S = \big(T^4/\{\pm 1\}\big) \times \{0\}$, one copy of $T^4/\{\pm 1\}$. The resolution is $R = \big(T^4/\{\pm 1\}\big) \times U$, with the obvious resolving map π.

(iii) The third model orbifold is $T = \big(B^4_\zeta/\{\pm 1\}\big) \times \big(B^4_\zeta/\{\pm 1\}\big)$. This time, the part of the singular set we are interested in is the single point $S = \{0 \times 0\}$. The resolution is $R = U \times U$, with the obvious map $\pi : R \to T$.

(iv) Define an isometric involution σ of $T^4 \times \big(B^4_\zeta/\{\pm 1\}\big)$ by

$$\sigma : (x_1, \ldots, x_4, y_1, \ldots, y_4) \longmapsto (\tfrac{1}{2} + x_1, x_2, -x_3, -x_4, y_1, y_2, -y_3, -y_4). \qquad (7)$$

Let $T = \big(T^4 \times (B^4_\zeta/\{\pm 1\})\big)/\langle\sigma\rangle$, and let S be the subset $S = \big(T^4 \times \{0\}\big)/\langle\sigma\rangle$, one copy of $T^4/\mathbb{Z}_2$, which is all of the singular set. Note that σ acts freely on T^4, as it changes the coordinate x_1. The resolution R is $R = (T^4 \times U)/\langle\sigma\rangle$, where σ acts on T^4 by $(x_1, \ldots, x_4) \mapsto (\frac{1}{2} + x_1, x_2, -x_3, -x_4)$, and σ acts on U by an isometric involution of U that projects to the action $(y_1, \ldots, y_4) \mapsto (y_1, y_2, -y_3, -y_4)$ on $B^4_\zeta/\{\pm 1\}$. The map $\pi : R \to T$ is the obvious one.

Now it turns out, as in [13, §3.1, Example 4], that there are two possible actions of σ on U. Let (z_1, z_2) be the complex coordinates on B^4_ζ inducing the complex structure on U. Consider the two involutions $(A) : (z_1, z_2) \mapsto (z_1, -z_2)$ and $(B) : (z_1, z_2) \mapsto (\overline{z}_1, \overline{z}_2)$. It is easy to see that (A) induces a holomorphic involution of U and (B) induces an antiholomorpic involution of U. Moreover, in appropriate real coordinates $(y_1, \ldots, y_4)$ on $B^4_\zeta/\{\pm 1\}$, both can be written as $(y_1, \ldots, y_4) \mapsto (y_1, y_2, -y_3, -y_4)$.

There is a rational curve $\mathbb{CP}^1 \subset U$, and its homology class generates $H_2(U, \mathbb{R})$. Since (A) is holomorphic and (B) antiholomorphic, it is clear that (A) preserves this homology class, but (B) changes its sign. Thus the actions (A) and (B) of σ are *topologically distinct*. They therefore give two different resolutions R, π of the orbifold T, which we will refer to as resolution (A) and resolution (B).

(v) The fifth orbifold is $T = \big(T^4/\{\pm 1\} \times B^4_\zeta/\{\pm 1\}\big)/\langle\sigma\rangle$, where σ has the action (7). Let S be the subset $S = \big(T^4/\{\pm 1\} \times \{0\}\big)/\langle\sigma\rangle$. Combining parts (ii) and (iv), we define R by $R = \big((T^4/\{\pm 1\}) \times U\big)/\langle\sigma\rangle$, and π in the obvious way. Again, the actions (A) and (B) of σ on U lead to two different resolutions R, π, which we will call resolutions (A) and (B).

The way we will use these model orbifolds and resolutions, is as follows. Suppose that the quotient T^8/Γ contains an open subset isometric to one of the model orbifolds T above. Then the subset of T^8/Γ identified with $S \subset T$ is called a *singularity of type* (i)-(v), as appropriate.

For instance, if T^8/Γ contains a copy of T^4 with a neighbourhood isometric to $T^4 \times \big(B^4_\zeta/\{\pm 1\}\big)$, we call this T^4 a *singularity of type* (i), and if T^8/Γ contains a point with a neighbourhood isometric to $(B^4_\zeta/\{\pm 1\}) \times (B^4_\zeta/\{\pm 1\})$, we call this point a *singularity of type* (iii).

Now suppose that T^8/Γ is a quotient for which the entire singular set is a union of singularities of types (i)-(v). We wish to define a compact, nonsingular 8-manifold M by resolving the singularities of T^8/Γ, such that the resolution of each singularity of type (i)-(v) is modelled on the 'resolution' R, π for that singularity, given above.

The basic idea is that for each singularity, we remove the subset T from T^8/Γ and replace it with the corresponding set R. The result should be a compact, nonsingular 8-manifold M, together with a map $\pi : M \to T^8/\Gamma$. In fact, this construction works, and the smooth 8-manifold M and the map $\pi : M \to T^8/\Gamma$ are well-defined. However, care must be taken in resolving the singularities of type (ii), (iii) and (v).

The problem is that singular points of type (iii) are not isolated, but instead they represent the transverse intersection of two submanifolds of singularities. Each of these submanifolds of singularities is a singularity of type (ii) or (v). Thus singularities of types (ii), (v) may intersect with other singularities of types (ii), (v), and the intersection points are intersections of type (iii). In defining the 8-manifold M, we must fit together the three resolutions at once. It is easy to see that this can be done, in only one way.

Having constructed the 8-manifold M, we would like to calculate its betti numbers and fundamental group. Here is a result for doing this.

Proposition 3.1.1. *Let M be the compact, nonsingular 8-manifold constructed above. Then the fundamental groups satisfy $\pi_1(M) \cong \pi_1(T^8/\Gamma)$. The betti numbers $b^k(M)$, $b^4_\pm(M)$ may be calculated by starting with the corresponding betti numbers of T^8/Γ and adding on a contribution for each of the singularities of type (i)-(v) in T^8/Γ. These contributions are as follows:*

Type (i) fixes b^1 and increases b^2 by 1, b^3 by 4, b^4_+ by 3 and b^4_- by 3.
Type (ii) fixes b^1 and b^3 and increases b^2 by 1, b^4_+ by 3 and b^4_- by 3.
Type (iii) fixes b^1, b^2, b^3 and b^4_- and increases b^4_+ by 1.
Type (iv) (A) fixes b^1 and increases b^2 by 1, b^3 by 2, b^4_+ by 1 and b^4_- by 1.
Type (iv) (B) fixes b^1 and b^2 and increases b^3 by 2, b^4_+ by 2 and b^4_- by 2.
Type (v) (A) fixes b^1 and b^3 and increases b^2 by 1, b^4_+ by 1 and b^4_- by 1.
Type (v) (B) fixes b^1, b^2 and b^3 and increases b^4_+ by 2 and b^4_- by 2.

Proof. Under the map $\pi : U \to B^4_\zeta/\{\pm 1\}$, the inverse image of 0 is $\mathcal{S}^\in$. Because $\mathcal{S}^\in$ is simply-connected, resolution of singularities using U does not change the fundamental group. It follows that $\pi_* : \pi_1(M) \to \pi_1(T^8/\Gamma)$ is an isomorphism, and so $\pi_1(M) \cong \pi_1(T^8/\Gamma)$.

It remains to calculate the betti numbers of M. This is easy to do using standard techniques in either homology or cohomology, and we will leave the details to the reader. One way to do it is to consider, for each singularity (i)-(v), the kernel of the natural map $\pi_* : H_*(R, \mathbb{R}) \to H_*(T, \mathbb{R})$ on the real homology. The dimension of the kernel in H_k gives the correction to b^k. Care must be taken with the overlaps in cases (ii), (iii) and (v), to avoid counting a homology class more than once. □

3.2 A family of 8-manifolds with holonomy $Spin(7)$

Let $(x_1, \ldots, x_8)$ be coordinates on $T^8 = \mathbb{R}^8/\mathbb{Z}^8$, where $x_i \in \mathbb{R}/\mathbb{Z}$. Define a section $\hat{\Omega}$ of $\Lambda^4 T^*(T^8)$ by Eq. (1) of §1.1, where τ_i is replaced by dx_i. Choose quadruples (c_1, c_2, c_5, c_6) and (d_1, d_3, d_5, d_7), where each c_i, d_i can take the value 0 or $\frac{1}{2}$. Let α, β, γ and δ be the involutions of T^8 defined by

$$\alpha\big((x_1, \ldots, x_8)\big) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8), \tag{8}$$

$$\beta\big((x_1, \ldots, x_8)\big) = (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8), \tag{9}$$

$$\gamma\big((x_1,\dots,x_8)\big) = (c_1 - x_1, c_2 - x_2, x_3, x_4, c_5 - x_5, c_6 - x_6, x_7, x_8), \tag{10}$$

$$\delta\big((x_1,\dots,x_8)\big) = (d_1 - x_1, x_2, d_3 - x_3, x_4, d_5 - x_5, x_6, d_7 - x_7, x_8). \tag{11}$$

By inspection, α, β, γ and δ preserve $\hat{\Omega}$, because of the careful choice of exactly which signs to change. It is easy to see that $\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 1$, and that $\alpha, \beta, \gamma, \delta$ all commute, since the constants c_i, d_i are 0 or $\frac{1}{2}$. Define Γ to be the group $\langle \alpha, \beta, \gamma, \delta \rangle$. Then $\Gamma \cong (\mathbb{Z}_2)^4$ is a group of automorphisms of T^8 preserving $\hat{\Omega}$.

Lemma 3.2.1. *Suppose that $(c_1, c_2) \neq (0,0)$, $\quad (c_5, c_6) \neq (0,0)$, $\quad (d_1, d_3) \neq (0,0)$, $\quad (d_5, d_7) \neq (0,0)$, and $(c_1, c_5) \neq (d_1, d_5)$. Then the fixed points of α, β, γ and δ are each 16 copies of T^4 in T^8, and the fixed points of $\alpha\beta$ are 256 points in T^8. These are the only nonidentity elements of Γ that have fixed points on T^8. The betti numbers of T^8/Γ are $b^1 = b^2 = b^3 = 0$ and $b^4_+ = b^4_- = 7$, and T^8/Γ is simply-connected.*

Proof. The fixed points of α are the points $(x_1, \dots, x_8)$ for which $x_1, \dots, x_4 \in \{0, \frac{1}{2}\}$. Clearly, these split up into 16 copies of T^4. In the same way, the fixed points of β, γ and δ are also 16 copies of T^4. The fixed points of $\alpha\beta$ are the points $(x_1, \dots, x_8)$ for which $x_1, \dots, x_8 \in \{0, \frac{1}{2}\}$, which is 256 points, as we have to prove.

Let us show that every other nonidentity element of Γ has no fixed points. First consider $\alpha\gamma$. It is given by

$$\alpha\gamma : (x_1, \dots, x_8) \longmapsto (x_1 - c_1, x_2 - c_2, -x_3, -x_4, c_5 - x_5, c_6 - x_6, x_7, x_8). \tag{12}$$

Since $(c_1, c_2) \neq (0,0)$, $\alpha\gamma$ acts as a nontrivial translation on the first two coordinates x_1, x_2. Thus $\alpha\gamma$ has no fixed points. Now consider $\alpha\gamma\delta$, which acts as

$$\begin{aligned}\alpha\gamma\delta : (x_1, \dots, x_8) &\longmapsto \\ &(d_1 - c_1 - x_1, x_2 - c_2, x_3 - d_3, -x_4, x_5 + c_5 - d_5, c_6 - x_6, d_7 - x_7, x_8).\end{aligned} \tag{13}$$

From coordinates x_2, x_3 and x_5, $\alpha\gamma\delta$ has fixed points only if $c_2 = d_3 = 0$ and $c_5 = d_5$. But if $c_2 = d_3 = 0$ then we must have $c_1 = d_1 = \frac{1}{2}$, since $(c_1, c_2) \neq (0,0) \neq (d_1, d_3)$. But $(c_1, c_5) \neq (d_1, d_5)$, so as $c_1 = d_1$ we must have $c_5 \neq d_5$. Therefore $c_2 = d_3 = 0$ and $c_5 = d_5$ cannot hold under the conditions of the lemma, and $\alpha\gamma\delta$ has no fixed points. Using similar arguments we can show that the other nonidentity elements also have no fixed points.

It remains to compute the betti numbers and fundamental group of T^8/Γ. Clearly, $H^k(T^8/\Gamma, \mathbb{R})$ is isomorphic to the set of constant k- forms on T^8 that are invariant under Γ. It is easy to show that for $0 < k < 8$, the only constant k- forms on T^8 invariant under Γ are the 14 elementary 4- forms in (1) that make up $\hat{\Omega}$. Then $b^1 = b^2 = b^3 = 0$ and $b^4_+ = b^4_- = 7$ follow very easily. It is also not difficult to show that T^8/Γ is simply-connected. $\square$

Now we will give four examples of these orbifolds T^8/Γ, and their resolutions.

Example 1. Put $(c_1, c_2, c_5, c_6) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(d_1, d_3, d_5, d_7) = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then Lemma 3.2.1 holds. Let us determine the singular set of T^8/Γ. From Lemma 3.2.1 we see that the singular set S of T^8/Γ is the union of $S_\alpha, S_\beta, S_\gamma$ and S_δ, with $S_{\alpha\beta} = S_\alpha \cap S_\beta$. Each of $S_\alpha, \ldots, S_\delta$ is the image in T^8/Γ of the 16 T^4 fixed by $\alpha, \ldots, \delta$.

Consider the action of $\langle \beta, \gamma, \delta \rangle$ on the set of 16 T^4 in T^8 fixed by α. It can be seen that β fixes this set, but $\langle \gamma, \delta \rangle$ acts freely upon it. On each T^4, β has the standard action of -1 on T^4. Thus dividing by β converts each T^4 to $T^4/\{\pm 1\}$, and dividing by $\langle \gamma, \delta \rangle$ identifies the 16 $T^4/\{\pm 1\}$ in 4 sets of 4. Therefore S_α is 4 copies of $T^4/\{\pm 1\}$.

The ζ- neighbourhood Z_α of S_α is 4 copies of $\bigl(T^4/\{\pm 1\}\bigr) \times \bigl(B^4_\zeta/\{\pm 1\}\bigr)$. But this is the orbifold given in §3.1 for a singularity of type (ii). It follows that S_α is composed of 4 singularities of type (ii). In the same way, it can be shown that S_β is 4 singularities of type (ii). Now $S_{\alpha\beta}$ is the image in T^8/Γ of the 256 points in T^8 fixed by $\alpha\beta$. These points are fixed by α and β, but $\langle \gamma, \delta \rangle$ acts freely upon them. Thus the 256 points are identified in 64 sets of 4 by $\langle \gamma, \delta \rangle$, and $S_{\alpha\beta}$ is 64 points. Each of these points is a singularity of type (iii).

It is easily shown that the action of $\langle \alpha, \beta, \delta \rangle$ on the set of 16 T^4 fixed by γ is free. Thus $\langle \alpha, \beta, \delta \rangle$ identifies the 16 T^4 in 2 sets of 8, and S_γ is 2 copies of T^4. The ζ- neighbourhood Z_γ is 2 copies of $T^4 \times \bigl(B^4_\zeta/\{\pm 1\}\bigr)$, which is the orbifold given in §3.1 for a singularity of type (i). So S_γ is 2 singularities of type (i). Similarly, S_δ is 2 singularities of type (i).

We have shown that in the notation of §3.1, the singular set S of T^8/Γ is 4 singularities of type (i), 8 singularities of type (ii), and 64 singularities of type (iii). Following the instructions in §3.1 we may resolve T^8/Γ to get an 8-manifold M. Combining Proposition 3.1.1 and Lemma 3.2.1 we compute the betti numbers and fundamental group of M, and we find that M is simply-connected and has betti numbers

$$b^2(M) = 12, \quad b^3(M) = 16, \quad b^4_+(M) = 107, \quad \text{and} \quad b^4_-(M) = 43. \tag{14}$$

From (14) we see that $\hat{A}(M) = 1$. (The formula for $\hat{A}(M)$ is given in Theorem C, §2.1.) Now Chap. 4 will show that Condition E of §2.1 applies to this manifold. Therefore Proposition 2.1.1 applies, and M admits a smooth, 44-dimensional family of metrics of holonomy $Spin(7)$.

Example 2. Here is a more complicated example. Put $(c_1, c_2, c_5, c_6) = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(d_1, d_3, d_5, d_7) = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, so that Lemma 3.2.1 holds. The difference with

Example 1 is that $\alpha\delta$ acts trivially on the set of 16 T^4 fixed by γ. Choose one of these T^4. The ζ- neighbourhood of this T^4 in T^8 is $T^4 \times B^4_\zeta$. Dividing by γ converts this to $T^4 \times \bigl(B^4_\zeta/\{\pm 1\}\bigr)$. Now it turns out that in a suitable coordinate system, the action of $\alpha\delta$ is the same as the action (7) of σ used in §3.1. So dividing by $\alpha\delta$ converts $T^4 \times \bigl(B^4_\zeta/\{\pm 1\}\bigr)$ into the model orbifold of type (iv) in §3.1.

Thus the action of $\langle\gamma, \alpha\delta\rangle$ produces 16 singularities of type (iv). But the group $\langle\alpha, \beta\rangle$ acts freely on these 16, identifying them in 4 groups of 4. It follows that S_γ is four singularities of type (iv) in the notation of §3.1, rather than 2 singularities of type (i). Therefore the singular set S of T^8/Γ consists of 2 type (i), 8 type (ii), 64 type (iii) and 4 type (iv) singularities.

As in Example 1, we may resolve T^8/Γ following the instructions in §3.1 for each singularity, to get a compact 8-manifold M. However, for each of the type (iv) singularities we have a choice of two different resolutions (A) and (B), giving 16 possibilities. Let us choose resolution (A) for j and resolution (B) for $4-j$ of the type (iv) singularities. Using Proposition 3.1.1 and Lemma 3.2.2, we find that M is simply-connected and has betti numbers

$$b^2(M)=10+j,\ b^3(M)=16,\ b^4_+(M)=109-j,\ b^4_-(M)=45-j, \quad j=0,\ldots,4, \tag{15}$$

so that of the 16 possibilities, at least 5 of the resulting manifolds are topologically distinct. From (15) we see that $\hat{A}(M)=1$, for all j. As in Example 1, Chap. 4 and Proposition 2.1.1 show that these manifolds have a smooth family of metrics with holonomy $Spin(7)$, of dimension $46-j$.

Example 3. Put $(c_1, c_2, c_5, c_6) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ and $(d_1, d_3, d_5, d_7) = (0, \frac{1}{2}, \frac{1}{2}, 0)$. In this case, the difference with Example 1 is that $\gamma\delta$ acts trivially on the set of 16 T^4 fixed by β. As in Example 2, it turns out that the action of $\gamma\delta$ may be identified with Eq. (7) of §3.1 in a suitable coordinate system. Because of this, S_β is 8 singularities of type (v) rather than 4 singularities of type (ii). The singular set S of T^8/Γ has 4 type (i), 4 type (ii), 64 type (iii) and 8 type (v) singularities. As in Examples 1 and 2, we may resolve T^8/Γ following the instructions in §3.1, to get a compact 8-manifold M.

From §3.1, for each of the 8 type (v) singularities we have a choice of two different resolutions (A) and (B), giving 256 possibilities. Let us choose resolution (A) for k and resolution (B) for $8-k$ of the type (v) singularities. Calculating from Proposition 3.1.1 and Lemma 3.2.1, we find that M has betti numbers

$$b^2(M)=8+k,\ b^3(M)=16,\ b^4_+(M)=111-k,\ b^4_-(M)=47-k, \quad k=0,\ldots,8. \tag{16}$$

As above, the manifolds are simply-connected, have $\hat{A}(M)=1$, and each carries a smooth family of metrics with holonomy $Spin(7)$, of dimension $48-k$.

Example 4. This example combines the features of Examples 2 and 3. Put $(c_1, c_2, c_5, c_6) = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(d_1, d_3, d_5, d_7) = (0, \frac{1}{2}, \frac{1}{2}, 0)$. Then $\alpha\delta$ acts trivially on the set of 16 T^4 fixed by γ, as in Example 2, and $\gamma\delta$ acts trivially on the set of 16 T^4 fixed by β, as in Example 3. Thus S_γ is 4 type (iv) and S_β is 8 type (v) singularities, and the singular set S of T^8/Γ has 2 type (i), 4 type (ii), 64 type (iii), 4 type (iv) and 8 type (v) singularities.

As in Examples 2 and 3, let us resolve T^8/Γ to give a simply-connected 8-manifold M, choosing the resolution (A) for j and (B) for $4-j$ of the type (iv) singularities, and choosing the resolution (A) for k and (B) for $8-k$ of the type (iv) singularities. Let $l=j+k$. The range of l is $0,\ldots,12$, as $j=0,\ldots,4$ and $k=0,\ldots,8$. Calculation shows that the betti numbers of M are

$$b^2(M)=6+l,\ b^3(M)=16,\ b^4_+(M)=113-l,\ b^4_-(M)=49-l,\quad l=0,\ldots,12. \tag{17}$$

The manifolds are simply-connected and have $\hat{A}(M)=1$, and each admits a smooth family of metrics with holonomy $Spin(7)$, of dimension $50-l$. More will be said about the topology of these manifolds in §3.4.

3.3 More examples

We shall now add a translation ϵ to the group $\langle\alpha,\beta,\gamma,\delta\rangle$ used in §3.2. Let the coordinates (x_i) on T^8 and the 4-form $\hat{\Omega}$ be as in §3.2. Choose quadruples (c_1,c_2,c_5,c_6) and (d_1,d_3,d_5,d_7), where each c_i,d_i can take the value 0 or $\frac{1}{2}$. Let α,β,γ and δ be the involutions of T^8 defined by equations (8)-(11) of §3.2, and let ϵ be the translation defined by

$$\epsilon\big((x_1,\ldots,x_8)\big)=(c_1+x_1,c_2+x_2,x_3,\tfrac{1}{2}+x_4,c_5+x_5,c_6+x_6,x_7,\tfrac{1}{2}+x_8). \tag{18}$$

By inspection, $\alpha,\ldots,\epsilon$ preserve $\hat{\Omega}$. It is easy to see that $\alpha^2=\beta^2=\gamma^2=\delta^2=\epsilon^2=1$, and that $\alpha,\ldots,\epsilon$ all commute, since the constants c_i,d_i are 0 or $\frac{1}{2}$. Define Γ to be the group $\langle\alpha,\beta,\gamma,\delta,\epsilon\rangle$. Then $\Gamma\cong(\mathbb{Z}_2)^5$ is a group of automorphisms of T^8 preserving $\hat{\Omega}$.

Using (8), (9), (10) and (18) we see that $\alpha\beta\gamma\epsilon$ acts on T^8 by

$$\alpha\beta\gamma\epsilon\big((x_1,\ldots,x_8)\big)=(x_1,x_2,-x_3,\tfrac{1}{2}-x_4,x_5,x_6,-x_7,\tfrac{1}{2}-x_8). \tag{19}$$

Because this fixes x_1,x_2,x_5 and x_6 we see that $\alpha\beta\gamma\epsilon$ fixes 16 copies of T^4 in T^8. This is the reason for the inclusion of the constants c_i in the definition (18) of ϵ.

Lemma 3.3.1. *Suppose that the hypotheses of Lemma 3.2.1 hold, and also that $(d_3,d_7)\neq(0,0)$. Then the fixed points of $\alpha,\beta,\gamma,\delta$ and $\alpha\beta\gamma\epsilon$ are each 16 copies of T^4 in T^8, and the fixed points of $\alpha\beta$ and $\alpha\beta\epsilon$ are each 256 points in T^8. These are the only nonidentity elements of Γ that have fixed points on T^8. The betti numbers of T^8/Γ are $b^1=b^2=b^3=0$ and $b^4_+=b^4_-=7$, and T^8/Γ is simply-connected.*

Proof. The proof is like that of Lemma 3.2.1, and is left as an exercise to the reader. The inclusion of c_1, c_2, c_5 and c_6 in (18) ensures that $\alpha\beta\gamma\epsilon$ has fixed points, and the addition of $\frac{1}{2}$ to x_4 and x_8 in (18) implies at once that 11 elements of Γ have no fixed points. $\square$

Here are three more examples. Examples 5 and 6 are based on Examples 1 and 4 of §3.2.

Example 5. Put $(c_1, c_2, c_5, c_6) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(d_1, d_3, d_5, d_7) = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, as in Example 1. Then Lemma 3.3.1 holds. Consider the singular set of T^8/Γ. It can be shown that $\langle\gamma, \delta, \epsilon\rangle$ acts freely on the set of 16 T^4 fixed by α, but β acts trivially on this set, and on each individual T^4, β acts as the involution -1. So S_α is 2 type (ii) singularities, in the notation of §3.1. In the same way, S_β, S_γ and $S_{\alpha\beta\gamma\epsilon}$ are each 2 type (ii) singularities.

The group $\langle\alpha, \beta, \gamma, \epsilon\rangle$ acts freely on the set of 16 T^4 fixed by δ, and therefore S_δ is 1 type (i) singularity. The group $\langle\gamma, \delta, \epsilon\rangle$ acts freely on the 256 fixed points of $\alpha\beta$, so that $S_{\alpha\beta}$ is 32 type (iii) singularities. Similarly, $S_{\alpha\beta\epsilon}$ is 32 type (iii) singularities. Thus the singular set S of T^8/Γ consists of 1 type (i), 8 type (ii), and 64 type (iii) singularities.

Following the instructions in §3.1, we construct a compact 8-manifold M by desingularizing T^8/Γ. Using Proposition 3.1.1 and Lemma 3.3.1 we may show that M is simply-connected, and has betti numbers

$$b^2(M) = 9, \quad b^3(M) = 4, \quad b^4_+(M) = 98, \quad \text{and} \quad b^4_-(M) = 34, \tag{20}$$

so that $\hat{A}(M) = 1$. (The formula for $\hat{A}(M)$ is given in Theorem C, §2.1.) Chapter 4 will show that Condition E of §2.1 holds for M, and thus Proposition 2.1.1 applies, and M admits a smooth, 35- dimensional family of metrics with holonomy $Spin(7)$.

Example 6. Set $(c_1, c_2, c_5, c_6) = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(d_1, d_3, d_5, d_7) = (0, \frac{1}{2}, \frac{1}{2}, 0)$, as in Example 4. The group $\langle\gamma, \delta, \epsilon\rangle$ acts freely on the set of 16 T^4 fixed by α, so S_α is 2 type (ii) singularities, and so is $S_{\alpha\beta\gamma\epsilon}$. But $\gamma\delta$ acts trivially on the set of 16 T^4 fixed by β, so S_β is 4 type (v) singularities.

Similarly, $\alpha\delta$ acts trivially on the set of 16 T^4 fixed by γ, and S_γ is 4 type (v) singularities. Also, $\beta\gamma\epsilon$ acts trivially on the set of 16 T^4 fixed by δ, so S_δ is 2 type (iv) singularities. The $\alpha\beta$- and $\alpha\beta\epsilon$- singular points together give 64 type (iii) singularities. Therefore S is 4 type (ii), 64 type (iii), 2 type (iv) and 8 type (v) singularities.

Let us resolve T^8/Γ to give an 8-manifold M, as in §3.1. Choose resolution (A) for j and (B) for $2-j$ of the type (iv) singularities, choose resolution (A) for k and (B) for $4-k$ of the type (v) singularities in S_β, and choose resolution (A) for l and (B) for $4-l$ of the type (v) singularities in S_γ. Let $m = j+k+l$. Computing with Proposition 3.1.1 and Lemma 3.3.1 shows that M is simply-connected with betti numbers

$$b^2(M)=4+m,\ b^3(M)=4,\ b^4_+(M)=103-m,\ b^4_-(M)=39-m, \quad m=0,\ldots,10. \tag{21}$$

As in the previous examples, M carries a smooth, $(40-m)$- dimensional family of metrics with holonomy $Spin(7)$. The topology of these manifolds will be discussed in §3.4.

Example 7. Put $(c_1,c_2,c_5,c_6) = (0,\frac{1}{2},0,\frac{1}{2})$ and $(d_1,d_3,d_5,d_7) = (\frac{1}{2},0,0,\frac{1}{2})$. Then Lemma 3.3.1 holds. This case is like Example 5, except that both $\alpha\gamma$ and ϵ act trivially on the set of 16 T^4 fixed by δ. On each individual T^4, ϵ is a translation, and so $T^4/\langle\epsilon\rangle$ is just T^4. Then $\alpha\gamma$ acts on this T^4 to produce a singularity of type (iv). Therefore S_δ is 4 type (iv) singularities, rather than 2 as we might have expected.

Thus S has 8 type (ii), 64 type (iii) and 4 type (iv) singularities. Let us resolve T^8/Γ to get M, using resolution (A) for j and (B) for $4-j$ of the type (iv) singularities. The result is a compact, simply-connected 8-manifold with betti numbers

$$b^2(M)=8+j,\ b^3(M)=8,\ b^4_+(M)=103-j,\ b^4_-(M)=39-j, \quad j=0,\ldots,4. \tag{22}$$

It has a smooth, $(40-j)$- dimensional family of metrics with holonomy $Spin(7)$.

3.4 Some topological considerations

Let us consider the topology of the compact 8-manifolds M constructed in Examples 1-7. Since they are all simply-connected, the betti numbers give 4 invariants b^2, b^3, b^4_+ and b^4_-. But the equation $\hat{A}(M) = 1$ gives the relation $b^3 + b^4_+ = b^2 + 2b^4_- + 25$, so only 3 of these invariants can be independent. Inspection of the betti numbers of the manifolds of Examples 1-7 reveals that the betti numbers of Example 4 include those of Examples 1-3, and the betti numbers of Example 6 include those of Example 5.

However, the 13 betti numbers of Example 4, the 11 betti numbers of Example 6 and the 5 betti numbers of Example 7 are all different, giving a total of 29 different sets of betti numbers. Now these 29 sets of betti numbers only fill out 2 of the remaining 3 parameters, because all of them satisfy the equation $b^4_+ - b^4_- = 64$. This is because there are always 64 singularities of type (iii) in our examples.

In addition to the betti numbers, we may also study the intersection product $\bigcap$ on the cohomology of M. Write $\bigcap_{i,j}$ for the product

$$\bigcap : H^i(M,\mathbb{R}) \times H^j(M,\mathbb{R}) \to H^{i+j}(M,\mathbb{R}). \tag{23}$$

Curiously, we find that $\bigcap_{2,3} = 0$ for all of the manifolds. Consider the map $\bigcap_{2,2}$ in the case of Example 4 of §3.2. It can be shown that $\dim(\mathrm{Im}\bigcap_{2,2})$ is equal to 3 when $k = 0$ and to $4k + 4$ when $k > 0$, where k is the integer used in Example 4. Thus k is a topological invariant of these manifolds.

However, $j+k=l$ is also a topological invariant, as it comes up in the betti numbers. Thus the manifolds of Example 4 include not just 13, but at least $5 \times 9 = 45$ distinct manifolds. Using similar arguments, we can show that in Example 6, both j and the unordered pair $\{k,l\}$ are topological invariants. There are 3 possibilities for j and 15 for $\{k,l\}$, so Example 6 includes not just 11, but $3 \times 15 = 45$ distinct manifolds. Thus we have proved the following.

Proposition 3.4.1. *Examples 4, 6 and 7 of this chapter together yield at least 95 topologically distinct, compact, simply-connected 8-manifolds.*

4 A proof that Condition E holds in our examples

We will now show that Condition E of §2.1 holds for all the manifolds M constructed in Examples 1-7 of Chap. 3. This will be done by explicitly constructing the 4-forms Ω_t, ϕ_t, and showing that they satisfy parts (i)-(v) of Condition E. Section 4.1 defines a number of auxiliary forms that we shall need, and §4.2 defines the families $\{\Omega_t : t \in (0,\theta]\}$ and $\{\phi_t : t \in (0,\theta]\}$ on M. The main result is given in Theorem 4.2.5.

In fact, we shall give the construction in full for the case of Example 1 of §3.2, and then indicate how to modify it for the other examples. Here is the notation that will be used in this chapter, up to the end of the proof of Theorem 4.2.4. Let Γ be the group of Example 1, and M the compact 8-manifold constructed there. Then there is a map $\pi : M \to T^8/\Gamma$, which resolves the singularities of T^8/Γ.

Whenever ν is used, we shall suppose that $\nu = \alpha, \beta, \gamma$ or δ. As in §3.1, S is the singular set of T^8/Γ, the union of the sets S_ν of ν- singular points. Also Z is the ζ- neighbourhood of S in T^8/Γ, and is the union of the sets Z_ν, the ζ-neighbourhoods of S_ν. Let P, Q and Q_ν be the open subsets of M defined by $P = M \setminus \pi^{-1}(S)$, $Q = \pi^{-1}(Z)$ and $Q_\nu = \pi^{-1}(Z_\nu)$. Then $M = P \cup Q$.

Because Example 1 has no singularities of type (iv) or (v), it can be seen that Z_ν decomposes isometrically as the product $Z_\nu \cong S_\nu \times B^4_\zeta/\{\pm 1\}$. This splitting lifts to give a splitting $Q_\nu \cong W_\nu \times U$, where W_ν is the desingularization of S_ν. Also, $Z_\alpha \cap Z_\beta$ is 64 copies of $\left(B^4_\zeta/\{\pm 1\}\right) \times \left(B^4_\zeta/\{\pm 1\}\right)$, so that $Q_\alpha \cap Q_\beta$ is 64 copies of $U \times U$.

From §3.1, the manifold U with its projection $\pi : U \to B^4_\zeta/\{\pm 1\}$ is in fact an open subset of the Eguchi-Hanson space X of §1.3, with its blowing-down map $\pi : X \to \mathbb{C}^2/\{\pm 1\}$. Now §1.3 defined a family of metrics h_t on X with holonomy $SU(2)$ for $t > 0$, and these restrict to give metrics on U. The metrics h_t on U for $t > 0$ desingularize the singular metric h_0 on U, which is the pull-back of the flat metric on $B^4_\zeta/\{\pm 1\}$.

4.1 Some auxiliary forms

We begin by defining some $Spin(7)$- structures $\hat{\Omega}$, $\Omega_{\nu,t}$ and $\Omega_{\alpha\beta,t}$ on various regions of M. Firstly, the flat $Spin(7)$- structure $\hat{\Omega}$ on T^8/Γ lifts to a $Spin(7)$- structure $\pi^*(\hat{\Omega})$ on P. By abuse of notation, we shall write $\hat{\Omega}$ in place of $\pi^*(\hat{\Omega})$. Both S_α, S_β have flat metrics with holonomy $\{\pm 1\}$, and S_γ, S_δ have flat metrics with holonomy $\{1\}$. Lifting the metric on S_ν to W_ν, we find that W_ν has a $\{\pm 1\}$- structure, which is singular in the cases $\nu = \alpha, \beta$.

The metrics h_t on U defined in §1.3 have holonomy $SU(2)$, and thus define a family of $SU(2)$- structures on U. Since $Q_\nu \cong W_\nu \times U$, it follows that Q_ν carries a family of $\{\pm 1\} \times SU(2)$- structures, which are singular for $\nu = \alpha$ and β. As $\{\pm 1\} \times SU(2)$ is a subgroup of $Spin(7)$, this induces a family of $Spin(7)$- structures on Q_ν.

Define $\Omega_{\nu,t}$ to be the torsion-free $Spin(7)$- structure on Q_ν defined using the metric h_t on U of §1.3. (Note that $\Omega_{\alpha,t}$ is singular on $\pi^*(S_\beta)$, and $\Omega_{\beta,t}$ is singular on $\pi^*(S_\alpha)$.) Because the inclusion $\{\pm 1\} \times SU(2) \subset Spin(7)$ is not unique, these $Spin(7)$- structures are not uniquely defined. When $t = 0$ we have $\Omega_{\nu,0}$, which is defined on $Q_\nu \cap P$. To define the family $\Omega_{\nu,t}$, we use the unique inclusion $\{\pm 1\} \times SU(2) \subset Spin(7)$ for which $\Omega_{\nu,0} = \hat{\Omega}$ on $Q_\nu \cap P$.

Now $Q_\alpha \cap Q_\beta$ is 64 copies of $U \times U$. On each of these copies, $\Omega_{\alpha,t}$ combines the singular $\{\pm 1\}$- structure on the first U factor with the Eguchi-Hanson $SU(2)$- structure on the second U factor, and $\Omega_{\beta,t}$ is the same but with the U factors reversed. Thus $\Omega_{\alpha,t}$ and $\Omega_{\beta,t}$ are different. Define a $Spin(7)$- structure $\Omega_{\alpha\beta,t}$ on $Q_\alpha \cap Q_\beta$ as follows. On each copy of $U \times U$, there is an $SU(2) \times SU(2)$- structure induced by the metric h_t on U with holonomy $SU(2)$, given in §1.3. Since $SU(2) \times SU(2) \subset Spin(7)$, the $SU(2) \times SU(2)$- structure induces a $Spin(7)$- structure $\Omega_{\alpha\beta,t}$ on $Q_\alpha \cap Q_\beta$. Here we use the unique inclusion $SU(2) \times SU(2) \subset Spin(7)$ for which $\Omega_{\alpha\beta,0} = \hat{\Omega}$ on $Q_\alpha \cap Q_\beta \cap P$.

Next, we will define a number of auxiliary forms for use in §4.2. Section 1.3 defined a function $u : X \to [0, \infty)$, and this restricts to U to give a function $u : U \to [0, \zeta^2)$. Let $u_\nu : Q_\nu \to [0, \zeta^2)$ be the lift of u to Q_ν in the splitting $Q_\nu \cong W_\nu \times U$. By the analysis which gave the estimate (4) in §1.3, it is easy to show that on $Q_\nu \cap P$, we have

$$\begin{gathered}\Omega_{\nu,t} = \hat{\Omega} + t^4 A_\nu + B_{\nu,t}, \text{ where } A_\nu = dC_\nu,\ B_{\nu,t} = dD_{\nu,t} \text{ and } |A_\nu| = O(u_\nu^{-2}), \\ |B_{\nu,t}| = O(t^8 u_\nu^{-4}),\ |C_\nu| = O(u_\nu^{-3/2}) \text{ and } |D_{\nu,t}| = O(t^8 u_\nu^{-7/2}).\end{gathered} \tag{24}$$

Here $A_\nu, B_{\nu,t}$ are closed 4-forms defined on $Q_\nu \cap P$, $C_\nu, D_{\nu,t}$ are 3-forms defined on $Q_\nu \cap P$, A_ν and C_ν are independent of t, A_ν lies in Λ^4_{35} w.r.t. the $Spin(7)$- structure $\hat{\Omega}$, and all norms $|\,.\,|$ above are taken w.r.t. the metric induced by $\hat{\Omega}$.

By analogy with (24) we find that on $Q_\alpha \cap Q_\beta \cap P$,

$$\begin{gathered}\Omega_{\alpha\beta,t} = \hat{\Omega} + t^4 A_\alpha + t^4 A_\beta + B_{\alpha,t} + B_{\beta,t} + B_{\alpha\beta,t}, \text{ where } B_{\alpha\beta,t} = dD_{\alpha\beta,t} \\ \text{and } |B_{\alpha\beta,t}| = O(t^8 u_\alpha^{-2} u_\beta^{-2}),\ |D_{\alpha\beta,t}| = O\big(t^8 u_\alpha^{-3/2} u_\beta^{-2} + t^8 u_\alpha^{-2} u_\beta^{-3/2}\big).\end{gathered} \tag{25}$$

Here $A_\alpha, A_\beta, B_{\alpha,t}$ and $B_{\beta,t}$ are as defined above, $B_{\alpha\beta,t}$ is a closed 4-form on $Q_\alpha \cap Q_\beta \cap P$, $D_{\alpha\beta,t}$ is a 3-form on $Q_\alpha \cap Q_\beta \cap P$, and norms $|\,.\,|$ are taken w.r.t. the metric induced by $\hat{\Omega}$.

Let $\tau : [0,\infty) \to [0,1]$ be a fixed, smooth, nonincreasing function with $\tau(r) = 1$ for $r \le \zeta^2/4$ and $\tau(r) = 0$ for $r \ge \zeta^2/2$. Define a 4-form w on T^8/Γ by

$$w = \pi_*\big(d\tau(u_\nu) \wedge C_\nu\big) \qquad \text{on } Z_\nu \text{ but not on } Z_\alpha \cap Z_\beta, \tag{26}$$

$$w = \pi_*\big(d\tau(u_\alpha) \wedge C_\alpha + d\tau(u_\beta) \wedge C_\beta\big) \qquad \text{on } Z_\alpha \cap Z_\beta, \tag{27}$$

and $w = 0$ elsewhere on T^8/Γ. Then w is a smooth 4-form on T^8/Γ. Since $\Lambda^4_{35} = \Lambda^4_-$, there is a 4-form x in $C^\infty(\Lambda^4_{35})$ with $dx = dw$. Thus $x - w$ is a closed 4-form, and it represents some class in $H^4(T^8/\Gamma, \mathbb{R})$.

Now it can be shown that every class in $H^4(T^8/\Gamma, \mathbb{R})$ is represented by a smooth, closed 4-form y, such that for each $\nu = \alpha, \beta, \gamma$ or δ, the restriction of y to $Z_\nu \cong S_\nu \times \big(B^4_\zeta/\{\pm 1\}\big)$ is of the form $a_\nu \omega_\nu + b_\nu \omega_\zeta$, where a_ν, b_ν are constants, ω_ν is the volume form of the euclidean metric on S_ν, and ω_ζ is the volume form of the euclidean metric on $B^4_\zeta/\{\pm 1\}$. Choose such a 4-form y representing the class $[x - w]$. Then $x - w - y$ is exact, so x can be written $x = w + y + dz$, where z is a smooth 3-form on T^8/Γ.

Define a 4-form E on P by $E = \pi^*(x)$. Define a 3-form F on $P \cap Q$ by $F = \pi^*(z)$. Both E and F are smooth and bounded w.r.t. the metrics on P, $P \cap Q$ induced by $\hat{\Omega}$, and E is a section of Λ^4_{35} in the splitting induced by $\hat{\Omega}$. Define a 4-form G on Q by $G = \pi^*(y)$. As $x = w + y + dz$ and $dy = 0$, E, F and G satisfy $dE = 0$ on $M \setminus Q$, $dG = 0$ on Q and

$$E = dF + G + d\tau(u_\nu) \wedge C_\nu \qquad \text{on } Q_\nu \text{ but not on } Q_\alpha \cap Q_\beta, \text{ and} \tag{28}$$

$$E = dF + G + d\tau(u_\alpha) \wedge C_\alpha + d\tau(u_\beta) \wedge C_\beta \qquad \text{on } Q_\alpha \cap Q_\beta. \tag{29}$$

On each Q_ν, $G = a_\nu \pi^*(\omega_\nu) + b_\nu \pi^*(\omega_\zeta)$. It happens that the 4-form $\pi^*(\omega_\nu)$ is also exactly equal to the volume form of the metric h_t of §1.3 on U, independently of t. Because of this, we see that

$$|G|^2 = a_\nu^2 + b_\nu^2 \quad \text{on } Q_\nu, \text{ where } |.| \text{ is the metric induced by } \hat{\Omega},\ \Omega_{\nu,t} \text{ or } \Omega_{\alpha\beta,t}, \text{ for each } t. \tag{30}$$

Thus G is bounded independently of t in all of the metrics we use on Q.

Finally, for $t \in (0, \frac{1}{2}]$, we define 4-forms X_t, Y_t on M as follows. On $M \setminus Q$ define

$$X_t = \hat{\Omega} + t^4 E \qquad \text{and} \qquad Y_t = 0. \tag{31}$$

On $Q_\nu \cap P$, except in $Q_\alpha \cap Q_\beta$, define

$$X_t = \hat{\Omega} + t^4\tau(u_\nu)A_\nu + \tau(u_\nu)B_{\nu,t} + t^4(1-\tau(u_\nu/t))E, \tag{32}$$

$$Y_t = d\tau(u_\nu)\wedge D_{\nu,t} - t^4 d\tau(u_\nu/t)\wedge F + t^4\tau(u_\nu/t)G. \tag{33}$$

On $Q_\alpha \cap Q_\beta \cap P$, define

$$\begin{aligned} X_t = \hat{\Omega} &+ \tau(u_\alpha)\{t^4 A_\alpha + B_{\alpha,t}\} + \tau(u_\beta)\{t^4 A_\beta + B_{\beta,t}\} \\ &+ \tau(u_\alpha)\tau(u_\beta)B_{\alpha\beta,t} + (1-\tau(u_\alpha/t))(1-\tau(u_\beta/t))t^4 E, \end{aligned} \tag{34}$$

$$\begin{aligned} Y_t =& d\tau(u_\alpha)\wedge\{t^4\tau(u_\beta/t)C_\alpha + D_{\alpha,t} + \tau(u_\beta)D_{\alpha\beta,t}\} \\ &+ d\tau(u_\beta)\wedge\{t^4\tau(u_\alpha/t)C_\beta + D_{\beta,t} + \tau(u_\alpha)D_{\alpha\beta,t}\} \\ &+ t^4 d\{(1-\tau(u_\alpha/t))(1-\tau(u_\beta/t))\}\wedge F \\ &+ t^4\big(\tau(u_\alpha/t) + \tau(u_\beta/t) - \tau(u_\alpha/t)\tau(u_\beta/t)\big)G. \end{aligned} \tag{35}$$

4.2 A family of $Spin(7)$- structures on M

It will presently be shown that the forms X_t and Y_t defined in the last section have three important properties. Firstly, X_t, Y_t are smooth on M and $dX_t + dY_t = 0$. Secondly, X_t is close to the subbundle AM at each point. And thirdly, Y_t is small on M, in a suitable sense. All the lengthy and obscure definitions of §4.1 were solely in order to achieve these three properties.

The purpose of X_t and Y_t is this. We will define Ω_t to be the smooth section of AM that is closest to X_t at each point, and ϕ_t to be $X_t + Y_t - \Omega_t$. Then $d\Omega_t + d\phi_t = 0$, since $dX_t + dY_t = 0$, and ϕ_t is the sum of $X_t - \Omega_t$ and Y_t, which are both small 4-forms in some sense. Because of these, Condition E of §2.1 will hold for Ω_t and ϕ_t.

Let all notation be as in §4.1, and make the convention that in this section C stands for *some positive constant independent of* t, but that each use of C can be a different constant. We begin with two lemmas.

Lemma 4.2.1. *The forms X_t and Y_t are smooth on M, and $dX_t + dY_t = 0$.*

Proof. It is clear that X_t, Y_t are smooth in P. From (24) and (32), we see that on $Q_\nu \cap P$, except in $Q_\alpha \cap Q_\beta$, we have

$$X_t = \Omega_{\nu,t} - \big(1-\tau(u_\nu)\big)\cdot\{t^4 A_\nu + B_{\nu,t}\} + t^4\big(1-\tau(u_\nu/t)\big)E. \tag{36}$$

So defining $X_t = \Omega_{\nu,t}$ in $Q_\nu \setminus P$ except in $Q_\alpha \cap Q_\beta$, X_t extends smoothly to Q_ν. From (25) and (34) we see that on $Q_\alpha \cap Q_\beta \cap P$, we have

$$\begin{aligned} X_t =& \Omega_{\alpha\beta,t} + (1-\tau(u_\alpha))(\Omega_{\beta,t} - \Omega_{\alpha\beta,t}) + (1-\tau(u_\beta))(\Omega_{\alpha,t} - \Omega_{\alpha\beta,t}) \\ &+ (1-\tau(u_\alpha))(1-\tau(u_\beta))B_{\alpha\beta,t} + t^4(1-\tau(u_\alpha/t))(1-\tau(u_\beta/t))E. \end{aligned} \tag{37}$$

Again, X_t extends smoothly to $Q_\alpha \cap Q_\beta$, and so X_t is smooth on M, as we have to prove. In a similar way, Y_t is smooth on M.

It remains to show that $dX_t + dY_t = 0$. Since we assume $t \le \frac{1}{2}$, the two regions $\zeta^2/4 < u_\nu < \zeta^2/2$ on which $\tau(u_\nu)$ changes, and $t\zeta^2/4 < u_\nu < t\zeta^2/2$ on which $\tau(u_\nu/t)$ changes, have no overlap. Now $dX_t = dY_t = 0$ on $M \setminus Q$, and calculating using (28), (29) and (32)-(35) shows that in Q_ν, but not in $Q_\alpha \cap Q_\beta$, we have

$$dX_t = -dY_t = d\tau(u_\nu) \wedge B_{\nu,t} - t^4 d\tau(u_\nu/t) \wedge E, \tag{38}$$

and in $Q_\alpha \cap Q_\beta$, we have

$$\begin{aligned} dX_t = -dY_t =& d\tau(u_\alpha) \wedge \bigl\{t^4\tau(u_\beta/t)A_\alpha + B_{\alpha,t} + \tau(u_\beta)B_{\alpha\beta,t}\bigr\} \\ &+ d\tau(u_\beta) \wedge \bigl\{t^4\tau(u_\alpha/t)A_\beta + B_{\beta,t} + \tau(u_\alpha)B_{\alpha\beta,t}\bigr\} \\ &+ t^4 d\bigl\{(1-\tau(u_\alpha/t))(1-\tau(u_\beta/t))\bigr\} \wedge E. \end{aligned} \tag{39}$$

Thus $dX_t + dY_t = 0$ on M, as we have to prove. $\square$

Equations (38) and (39) hold because, by design, various terms in dX_t and dY_t cancel out. For instance, the terms $t^4\tau(u_\nu)A_\nu$ and $t^4(1-\tau(u_\nu/t))E$ in (32) contribute cancelling terms $\pm t^4 d\tau(u_\nu) \wedge A_\nu$ to dX_t, as (28) shows that $dE = -d\tau(u_\nu) \wedge A_\nu$ in Q_ν. The next lemma shows that X_t is close to AM at each point. To see the point of dividing $X_t - \Omega$ into components χ and ξ, recall from §1.1 that $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$ is the tangent space of the fibres of AM at Ω.

Lemma 4.2.2. *At each point of M, we may write $X_t = \Omega + \chi + \xi$, where Ω is $\hat\Omega$, $\Omega_{\alpha\beta,t}$, or $\Omega_{\nu,t}$ for $\nu = \alpha, \beta, \gamma$ or δ. Here χ lies in $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$ w.r.t. Ω and satisfies $|\chi| \le Ct^4$ for small t, ξ lies in Λ^4_{27} and satisfies $|\xi| \le Ct^6$ for small t, and the metrics $|\,.\,|$ are those induced by Ω.*

Proof. On $M \setminus Q$ we take $\Omega = \hat\Omega$, $\chi = t^4 E$ and $\xi = 0$, by (31). Since from §4.1, E lies in Λ^4_{35} w.r.t. $\hat\Omega$ and is bounded on $M \setminus Q$, the lemma holds in this case.

On Q_ν, but not on $Q_\alpha \cap Q_\beta$, we take $\Omega = \Omega_{\nu,t}$. By (36) and the definition of τ, χ and ξ are zero unless $u_\nu \ge t\zeta^2/4$, and on this region $\Omega_{\nu,t} = \hat\Omega + O(t^2)$. Thus for small t, metrics calculated w.r.t. $\hat\Omega$ and $\Omega_{\nu,t}$ vary by no more than a fixed factor. Also, since E is bounded and lies in Λ^4_{35} w.r.t. $\hat\Omega$ on Q_ν, the component of E in Λ^4_{27} w.r.t. $\Omega_{\nu,t}$ is $O(t^2)$ on the region $u_\nu \ge t\zeta^2/4$. Using these two facts and the estimates (24), we prove easily that $|\chi| \le Ct^4$ and $|\xi| \le Ct^6$ on Q_ν for sufficiently small t.

On $Q_\alpha \cap Q_\beta$ we put $\Omega = \Omega_{\alpha\beta,t}$. Let us write $\Omega_{\beta,t} - \Omega = \chi' + \xi'$, where $\chi' \in \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$ and $\xi' \in \Lambda^4_{27}$. Since $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$ is the tangent space to AM at Ω, and $\Omega_{\beta,t} \in AM$, it follows that if $\Omega_{\beta,t} - \Omega$ is small, then $|\xi'| = O\bigl(|\chi'|^2\bigr)$. It can be shown that $|\Omega_{\beta,t} - \Omega| = O(t^4 u_\alpha^{-2})$ for small $t^4 u_\alpha^{-2}$ on $Q_\alpha \cap Q_\beta$. So if t is small and $u_\alpha \ge \zeta^2/4$, we have $|\Omega_{\beta,t} - \Omega| = O(t^4)$. We deduce that for small t, the term $(1-\tau(u_\alpha))(\Omega_{\beta,t} - \Omega_{\alpha\beta,t})$ in (37) contributes $O(t^4)$ to χ and $O(t^8)$ to ξ. In a similar way, it can be shown that the other terms in (37) contribute no more than $O(t^4)$ to χ and $O(t^6)$ to ξ. Thus on $Q_\alpha \cap Q_\beta$ we have $|\chi| \le Ct^4$ and $|\xi| \le Ct^6$ for small t, as we have to prove. □

We are now ready to define the $Spin(7)$- structure Ω_t. Choose a smooth metric g'_t on M that is C^1- close for small t to the metrics induced by $\hat{\Omega}$, $\Omega_{\nu,t}$ and $\Omega_{\alpha\beta,t}$ on their usual regions of M. Let Ω_t be the section of AM that is closest to X_t at each point of M, using the metric g'_t.

It is clear that provided X_t is sufficiently close to AM in the metric induced by g'_t, then Ω_t is well-defined and smooth. But by Lemma 4.2.2, when t is small, X_t is a distance no more than Ct^4 from AM w.r.t. the metric induced by Ω, and since g'_t was chosen close to this metric for small t, we see that Ω_t is well-defined and smooth for sufficiently small t.

Here are some estimates on $X_t - \Omega_t$ and Y_t.

Proposition 4.2.3. *There exists $\theta \in (0, \frac{1}{2}]$ such that whenever $0 < t \le \theta$, Ω_t is a smooth, well-defined section of AM. Let g_t be the metric on M induced by Ω_t. Then the following estimates hold for each $t \in (0, \theta]$, where all norms are w.r.t. the metric g_t.*

$$\begin{aligned} \|X_t - \Omega_t\|_{C^0} &\le Ct^6, & \big\|d(X_t - \Omega_t)\big\|_{C^0} &\le Ct^{7/2}, \\ \|Y_t\|_2 &\le Ct^{9/2}, & \|dY_t\|_{C^0} &\le Ct^{7/2}. \end{aligned} \tag{40}$$

Proof. From above, Ω_t is smooth and well-defined, when t is sufficiently small. Now Lemma 4.2.2 decomposes X_t as $X_t = \Omega + \chi + \xi$. As χ lies in the tangent space to AM and Ω_t is the closest point to X_t in AM, it follows that $|X_t - \Omega_t| = O\big(|\chi|^2 + |\xi|\big)$ when $|\chi|, |\xi|$ are small. Let g_t be the metric induced by Ω_t. Since g_t and the metric induced by Ω are close when t is small, the conclusions $|\chi| \le Ct^4$ and $|\xi| \le Ct^6$ of Lemma 4.2.2 also hold w.r.t. the metric g_t for small t, with increased constants C. Thus $\|X_t - \Omega_t\|_{C^0} \le Ct^6$ for t sufficiently small, as we have to prove.

In the notation of Lemma 4.2.2, it can be shown that $|\nabla(X_t - \Omega)| = O(t^{7/2})$ for small t on M, where ∇ and $|\,.\,|$ are taken w.r.t. the metric induced by Ω. Since $X_t - \Omega_t$ is a function of Ω and $X_t - \Omega$, it follows that $|\nabla(X_t - \Omega_t)| = O(t^{7/2})$ on M, where again ∇ and $|\,.\,|$ are w.r.t. Ω. So $|d(X_t - \Omega_t)| \le Ct^{7/2}$ w.r.t. Ω. But the metrics induced by Ω_t and Ω are close for small t, so that $\|d(X_t - \Omega_t)\|_{C^0} \le Ct^{7/2}$ also holds w.r.t. g_t for small t with increased constant C, as we have to prove.

The terms that make the largest contribution to $\|Y_t\|_2$ for small t are the term $-t^4 d\tau(u_\nu/t) \wedge F$ in (33) and the term $t^4 d\{(1-\tau(u_\alpha/t))(1-\tau(u_\beta/t))\} \wedge F$ in (35). To estimate these terms, observe that $|d\tau(u_\nu/t)|$ and $\big|d\{(1-\tau(u_\alpha/t))(1-\tau(u_\beta/t))\}\big|$ are both $O(t^{-1/2})$ and are both supported in a volume of order t^2.

Since F is bounded, it follows that both terms are $O(t^{7/2})$ and supported on a volume of order t^2, so their contributions to $\|Y_t\|_2$ are both of order $t^{9/2}$. It is not difficult to estimate the contribution of the remaining terms to $\|Y_t\|_2$, using (30) to estimate the terms in G, and all of them are smaller than $O(t^{9/2})$. Therefore $\|Y_t\|_2 \le Ct^{9/2}$ for small t, as we have to prove.

It is easy to see that each term in (38) and (39) is $O(t^{7/2})$ or smaller. Therefore $\|dY_t\|_{C^0} \le Ct^{7/2}$ for small t. We have shown that all the conclusions of the proposition hold when t is sufficiently small. Thus there exists $\theta \in (0,1)$ such that these conclusions hold whenever $0 < t \leq \theta$, and the proof is complete. □

At last, we are able to prove the following.

Theorem 4.2.4. *Condition E of §2.1 holds for the compact 8-manifold M of Example 1 in §3.2.*

Proof. Take θ and Ω_t to be as in Proposition 4.2.3, and define ϕ_t by $\phi_t = X_t + Y_t - \Omega_t$. Then ϕ_t is smooth and well-defined on M for $0 < t \leq \theta$. We must prove that parts (i)-(v) of Condition E hold for some positive constants A_1, A_2, ρ and K. Part (i) of Condition E, the equation $d\Omega_t + d\phi_t = 0$, follows immediately from the equation $dX_t + dY_t = 0$ of Lemma 4.2.1, as we have to prove.

For small t the volume of (M, g_t) is close to the volume of T^8/Γ, which is $1/16$. Thus $\mathrm{vol}(M, g_t) \leq 1$ for small t. It follows that

$$\|\phi_t\|_2 \leq \|X_t - \Omega_t\|_2 + \|Y_t\|_2 \leq \|X_t - \Omega_t\|_{C^0} + \|Y_t\|_2 \leq Ct^6 + Ct^{9/2}, \qquad (41)$$

using the estimates (40) of Proposition 4.2.3. So there is a constant $A_1 > 0$ independent of t such that $\|\phi_t\|_2 \leq A_1 t^{9/2}$ for $0 < t \leq \theta$, which is part (ii) of Condition E. Similarly, we have $\|d\phi_t\|_{10} \leq \|d(X_t - \Omega_t)\|_{C^0} + \|dY_t\|_{C^0} \leq Ct^{7/2}$ by (40), so that $\|d\phi_t\|_{10} \leq A_2 t$ for $0 < t \leq \theta$ and some $A_2 > 0$ independent of t, giving part (iii).

The metrics g_t are constructed out of the flat metric on T^8/Γ, and the metric h_t on the Eguchi-Hanson space, defined in §1.3. Because h_t is asymptotically flat, the injectivity radius and Riemann curvature satisfy $\delta(h_t) > 0$ and $\|R(h_t)\|_{C^0} < \infty$ even though the Eguchi-Hanson space is noncompact. But from §1.3, h_t and $c^{-2}h_{ct}$ are isometric, so conformal rescaling shows that $\delta(h_t) = Ct > 0$ and $\|R(h_t)\|_{C^0} = Ct^{-2}$ for $t > 0$.

It is easy to see that when t is small, it is the Eguchi-Hanson factors h_t in the metric g_t that give the smallest injectivity radius and the largest contribution to $|R(g_t)|$. It follows that $\delta(g_t)$ is at least $O(t)$ and $\|R(g_t)\|_{C^0}$ is $O(t^{-2})$ for t small. Thus parts (iv) and (v) hold for some constants $\rho, K > 0$ independent of t when t is small. Making θ smaller if necessary, we see that parts (i)-(v) of Condition E hold, which concludes the proof. □

The last theorem generalizes to give this, the main result of the chapter.

Theorem 4.2.5. *Condition E of §2.1 holds for all the compact 8-manifolds M constructed in Examples 1-7 of Chap. 3.*

Proof. All of the reasoning of this chapter, culminating in Theorem 4.2.4, actually applies to the other cases in Chap. 3 with only cosmetic changes. There are two main differences. Firstly, when M has singularities of type (iv) or (v) in the notation of §3.1, the formulae $Z_\nu \cong S_\nu \times \left(B^4_\zeta/\{\pm 1\}\right)$ and $Q_\nu \cong W_\nu \times U$ do not always hold. In this case there are natural double covers $\tilde{Z}_\nu$, $\tilde{Q}_\nu$ of Z_ν, Q_ν

that do split as products, so we may work instead on $\tilde{Z}_\nu$ and $\tilde{Q}_\nu$, and make everything $\mathbb{Z}_2$- invariant.

Secondly, for the examples of §3.3 we must allow $\nu \in \{\alpha, \beta, \gamma, \delta, \alpha\beta\gamma\epsilon\}$ instead of just $\{\alpha, \beta, \gamma, \delta\}$, and the special treatment given to the intersection $Q_\alpha \cap Q_\beta$ throughout must also be applied to the intersection $Q_\gamma \cap Q_{\alpha\beta\gamma\epsilon}$. Apart from these two changes the proofs for the other cases of Chap. 3 are almost the same. We leave it to the readers to verify this if they wish. □

5 Existence results for torsion-free $Spin(7)$- structures

In this chapter we will prove Theorems A and B of §2.1. Together these say that under certain conditions a $Spin(7)$- structure with small torsion on a compact 8-manifold may be deformed to a smooth $Spin(7)$- structure with zero torsion. The basic idea is straightforward – the condition for a $Spin(7)$- structure to be torsion-free is written as a nonlinear elliptic partial differential equation. Beginning with a $Spin(7)$- structure with small torsion, one finds a sequence of corrections by approximating the nonlinear equation by its linearization, and solving the linear equation. Finally, by estimating the correction terms in a suitable norm, it is shown that the sequence of corrections converges to a solution.

The difficulty arises because the $Spin(7)$- structure we start with is defined by desingularizing a singular space. To make the torsion small, we must use small Eguchi-Hanson spaces in the construction, and these give the manifold small injectivity radius and large curvature in some places. Now having small torsion is helpful, because it makes the corrections smaller, so the sequence tends to converge. But having small injectivity radius and large curvature are both unhelpful, as they make the corrections larger, so the sequence tends to diverge. Theorems A and B show that the first influence dominates the second, so that provided the desingularization is sufficiently small, there exists a nearby torsion-free $Spin(7)$- structure.

The problem of finding a torsion-free $Spin(7)$- structure on an 8-manifold M is diffeomorphism-invariant, and thus if $\tilde{\Omega}$ is a solution, then so are all the translates of $\tilde{\Omega}$ under diffeomorphisms of M, an infinite-dimensional family. In general it is easier to solve a problem that has a unique solution, than one with a huge family of solutions. Therefore, in proving Theorem A in §5.1 we shall write the p.d.e. for a torsion-free $Spin(7)$- structure $\tilde{\Omega}$ in a way that breaks the diffeomorphism-invariance, and has just one solution in each diffeomorphism class.

This will not be discussed in §5.1, but it is implicit in the design of the proof. Here is how it is done, for those who are curious. First, one fixes a smooth $Spin(7)$- structure Ω on M, with $d\Omega$ small. Then, one looks for smooth sections $\tilde{\Omega}$ of AM with $\tilde{\Omega} - \Omega$ small, satisfying $d\tilde{\Omega} = 0$. However, we consider only those sections $\tilde{\Omega}$ of AM such that $\tilde{\Omega} - \Omega$ is a section of the subbundle

$\Lambda^4_{27} \oplus \Lambda^4_{35}$ of $\Lambda^4 T^* M$, in the splitting induced by Ω in Proposition 1.1.1.

This is an 8-dimensional condition on $\tilde{\Omega}$ at each point, as the components of $\tilde{\Omega} - \Omega$ in Λ^4_1 and Λ^4_7 must vanish. But the Lie algebra of the diffeomorphism group is the vector fields on M, which also has 8 dimensions at each point. It can be shown that the condition $\tilde{\Omega} - \Omega \in C^\infty(\Lambda^4_{27} \oplus \Lambda^4_{35})$ is *transverse* to the infinite-dimensional action of the diffeomorphism group on $\tilde{\Omega}$, modulo finite-dimensional kernels and cokernels. In this way we achieve a 'gauge-fixing' of the diffeomorphism group, and formulate the problem so that the solution $\tilde{\Omega}$ is unique.

5.1 The proof of Theorem A

In this section we shall address the following question: suppose a compact 8-manifold M has a section Ω of AM defining a $Spin(7)$- structure on M, and also a small 4-form ϕ such that $\Omega + \phi$ is closed. Under what conditions is it possible to find a second section $\tilde{\Omega}$ of AM, such that $d\tilde{\Omega} = 0$, and $\tilde{\Omega} - \Omega$ is small? The problem is thus one of deforming a $Spin(7)$- structure Ω with small torsion to a structure $\tilde{\Omega}$ with zero torsion, which therefore yields a metric with holonomy contained in $Spin(7)$.

Theorem A. *Let $A_1, \ldots, A_4$ be positive constants. Then there exist positive constants κ, λ depending only on $A_1, \ldots, A_4$, such that whenever $0 < t \leq \kappa$, the following is true.*

Let M be a compact 8-manifold, and Ω a smooth section of AM on M. Suppose that ϕ is a smooth 4-form on M with $d\Omega + d\phi = 0$, and that these four conditions hold:

(i) $\|\phi\|_2 \leq A_1 t^{9/2}$,

(ii) $\|d\phi\|_{10} \leq A_2 t$,

(iii) if $\chi \in L^{10}_1(\Lambda^4_-)$ *then* $\|\nabla\chi\|_{10} \leq A_3\big(\|d\chi\|_{10} + t^{-21/5}\|\chi\|_2\big)$, *and*

(iv) if $\chi \in L^{10}_1(\Lambda^4_-)$ *then* $\chi \in C^0(\Lambda^4_-)$ *and* $\|\chi\|_{C^0} \leq A_4\big(t^{1/5}\|\nabla\chi\|_{10} + t^{-4}\|\chi\|_2\big)$.

Then there exists a smooth, closed section $\tilde{\Omega}$ of AM satisfying $\|\tilde{\Omega} - \Omega\|_{C^0} \leq \lambda t^{1/2}$. Thus $\tilde{\Omega}$ is a torsion-free $Spin(7)$- structure.

Proof. From §1.1, the tangent bundle of the fibres of AM at the section Ω is $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$. Therefore, if ξ is a section of $\Lambda^4 T^* M$ and $\|\xi - \Omega\|_{C^0}$ is small, ξ can be written uniquely as the sum of a section of AM, and a section of Λ^4_{27} that is small in C^0. In fact, we are only interested in ξ such that $\xi - \Omega$ is a section of Λ^4_{35}. To make the above precise, we shall prove the following lemma.

Lemma 5.1.1. *There are positive constants D_1, D_2, D_3 independent of M and Ω, such that there exists a unique smooth function Θ from the closed ball of radius D_1 in Λ^4_{35} to Λ^4_{27}, satisfying the following conditions:*

(i) $\Theta(0) = 0$,

(ii) $\Omega + u - \Theta(u) \in AM$,

(iii) $\bigl|\Theta(u) - \Theta(v)\bigr| \le D_2|u-v|\bigl(|u|+|v|\bigr)$,

(iv) $$\bigl|\nabla\bigl(\Theta(u)-\Theta(v)\bigr)\bigr| \le D_3\Bigl\{\bigl(|d\Omega||u-v| + |\nabla(u-v)|\bigr)\bigl(|u|+|v|\bigr) + |u-v|\bigl(|\nabla u| + |\nabla v|\bigr)\Bigr\},$$

where u,v are differentiable sections of Λ^4_{35} with $|u|,|v| \le 2D_1$.

Proof. Since $\Omega \in AM$ and the tangent space to AM at Ω is $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$, as above there exists a positive constant D_1 such that if $|u| \le D_1$ then $\Omega + u$ may be written uniquely as $\Omega + u = \Theta(u) + w$, where $\Theta(u) \in \Lambda^4_{27}$ and $w \in AM$. This defines a unique, smooth function Θ as in the lemma, for which properties (i) and (ii) hold trivially.

It remains to prove properties (iii) and (iv). Now Θ is smooth, and has zero first derivative in the fibres. It follows that Θ satisfies an estimate of the form (iii) for some constant D_2, and as this is a calculation at a point, D_2 is a universal constant, independent of M and Ω. To prove part (iv) we expand $\nabla\bigl(\Theta(u) - \Theta(v)\bigr)$ using the chain rule, into terms involving derivatives of u, v and terms involving derivatives of Θ. Since Θ is defined entirely using Ω, $\nabla\Theta$ is bounded pointwise by a multiple of $|\nabla\Omega|$. But from §1.1, $\nabla\Omega$ is determined by $d\Omega$, and so $|\nabla\Omega|$ is bounded by a multiple of $|d\Omega|$. Inequality (iv) then follows by calculation, for some positive constant D_3 independent of M and Ω. $\square$

The rest of Theorem A now follows from the following two propositions.

Proposition 5.1.2. *In the situation of Theorem A, there exist positive constants E_1, E_2 depending only on $A_1, \dots, A_4$, and a sequence $\{\eta_j\}_{j=0}^\infty$ in $L^{10}_2(\Lambda^4_{35})$ with $\eta_0 = 0$ satisfying the equation*

$$d\eta_{j+1} = d\phi + d\,\Theta(\eta_j) \tag{42}$$

for each $j \ge 0$, and the inequalities

(a) $\|\eta_j\|_2 \le 4A_1 t^{9/2}$, (d) $\|\eta_j - \eta_{j-1}\|_2 \le 4A_1 2^{-j} t^{9/2}$,

(b) $\|\nabla\eta_j\|_{10} \le E_1 t^{3/10}$, (e) $\bigl\|\nabla(\eta_j - \eta_{j-1})\bigr\|_{10} \le E_1 2^{-j} t^{3/10}$,

(c) $\|\eta_j\|_{C^0} \le E_2 t^{1/2}$, (f) $\|\eta_j - \eta_{j-1}\|_{C^0} \le E_2 2^{-j} t^{1/2}$.

Proposition 5.1.3. *In the situation of Proposition 5.1.2, the sequence $\{\eta_j\}_{j=0}^\infty$ converges in $L^{10}_1(\Lambda^4_{35})$. Let η be the limit of this sequence. Then η is smooth. Define $\tilde\Omega = \Omega + \eta - \Theta(\eta)$. Then $\tilde\Omega$ is a smooth section of AM satisfying $d\tilde\Omega = 0$ and $\|\tilde\Omega - \Omega\|_{C^0} \le \lambda t^{1/2}$, for some $\lambda > 0$ depending only on $A_1, \dots, A_4$.*

The proofs of the propositions will be given in order. At a number of points in the proofs we shall need t to be smaller than some positive constant defined in terms of $A_1, \dots, A_4$ and D_1, D_2, D_3. As a shorthand we shall simply say that this holds since $t \le \kappa$, and suppose without remark that κ has been chosen such that the relevant restriction holds. The reader may if she wishes go through the proof collecting the restrictions on the constant κ, and hence obtain an explicit expression for κ for which the theorem holds.

Proof of Proposition 5.1.2. The proof is by induction on j. We begin by proving existence for η_{k+1}, supposing that $\eta_0, \dots, \eta_k$ exist and satisfy (42) and (a)-(f) for $j \le k$. These imply that $\Theta(\eta_k)$ exists in $L^{10}_2(\Lambda^4_{27})$, since part (c) of Proposition 5.1.2 and the condition $t \le \kappa$ imply that $|\eta_k| \le D_1$, so that $\Theta(\eta_k)$ is well-defined by Lemma 5.1.1. Consider the operator $P : L^{10}_2(\Lambda^4_{35}) \to L^{10}(\Lambda^4_{35})$ defined by $P(\chi) = \pi_{35}(d^* d\chi)$. Since $\Lambda^4_{35} = \Lambda^4_-$ it is easy to see that P is the restriction of $\frac{1}{2}(d^*d + dd^*)$ to Λ^4_-. Thus P is elliptic and self-adjoint.

Define $\xi = \pi_{35}\bigl(d^*d\phi + d^*d\,\Theta(\eta_k)\bigr)$, so that $\xi \in L^{10}(\Lambda^4_{35})$. Let $\rho \in \operatorname{Ker} P$. Then $d\rho = 0$, so integrating by parts shows that the L^2- inner product $\langle \xi, \rho\rangle = 0$. As this holds for all $\rho \in \operatorname{Ker} P$, ξ is L^2- orthogonal to $\operatorname{Ker} P$. But P is self-adjoint, so $(\operatorname{Ker} P)^\perp = \operatorname{Im} P$, and $\xi \in \operatorname{Im} P$. Define η_{k+1} to be the unique element of $L^2(\Lambda^4_{35})$ for which $P(\eta_{k+1}) = \xi$ holds weakly, and $\eta_{k+1} - \pi_{35}(\phi)$ is L^2- orthogonal to $\operatorname{Ker} P$. Since P is elliptic, $\eta_{k+1} \in L^{10}_2(\Lambda^4_{35})$ by elliptic regularity, as $\xi \in L^{10}(\Lambda^4_{35})$.

Thus $\eta_{k+1} \in L^{10}_2(\Lambda^4_{35})$ exists and satisfies the equation

$$\pi_{35}(d^*d\eta_{k+1}) = \pi_{35}\bigl(d^*d\phi + d^*d\,\Theta(\eta_k)\bigr). \tag{43}$$

But (43) implies that $d^*d\bigl(\eta_{k+1} - \phi - \Theta(\eta_k)\bigr)$ is a d^*- exact 4-form lying in Λ^4_+. By Hodge theory any such form must be zero, and thus $d^*d\bigl(\eta_{k+1} - \phi - \Theta(\eta_k)\bigr) = 0$. Taking the L^2- inner product with $\eta_{k+1} - \phi - \Theta(\eta_k)$ and integrating by parts gives that $d\bigl(\eta_{k+1} - \phi - \Theta(\eta_k)\bigr) = 0$. Therefore (42) holds for $j = k$.

By Hodge theory, $H^4(M, \mathbb{R})$ splits as $H^4_+ \oplus H^4_-$, where $H^4_\pm$ are the subspaces of $H^4(M,\mathbb{R})$ represented by closed forms in $\Lambda^4_\pm$. Since $\Lambda^4_- = \Lambda^4_{35}$, $H^4_- \cong \operatorname{Ker} P$. But the definition of η_{k+1} ensures that $\eta_{k+1} - \phi - \Theta(\eta_k)$ is L^2- orthogonal to $\operatorname{Ker} P$. Now $\eta_{k+1} - \phi - \Theta(\eta_k)$ is closed, so that it represents a cohomology class in $H^4(M,\mathbb{R})$, and is L^2- orthogonal to $\operatorname{Ker} P$, which represents H^4_-. Therefore $\bigl[\eta_{k+1} - \phi - \Theta(\eta_k)\bigr] \in H^4_+$, so that

$$\bigl[\eta_{k+1} - \phi - \Theta(\eta_k)\bigr] \cup \bigl[\eta_{k+1} - \phi - \Theta(\eta_k)\bigr] \ge 0. \tag{44}$$

Let us write $\phi = \phi_+ + \phi_-$ where $\phi_\pm$ is the component of ϕ in $\Lambda^4_\pm$. Writing (44) out as an integral over M, and using the fact that $\eta_{k+1} \in \Lambda^4_-$ and $\Theta(\eta_k) \in \Lambda^4_+$, we find that

$$\int_M \Bigl\{ \bigl|\Theta(\eta_k) + \phi_+\bigr|^2 - \bigl|\eta_{k+1} - \phi_-\bigr|^2 \Bigr\} d\mu \ge 0, \tag{45}$$

so that $\bigl\|\eta_{k+1} - \phi_-\bigr\|_2 \le \bigl\|\Theta(\eta_k) + \phi_+\bigr\|_2$. Thus $\|\eta_{k+1}\|_2 \le 2\|\phi\|_2 + \bigl\|\Theta(\eta_k)\bigr\|_2$.

We have constructed by induction an element η_{k+1} in $L^{10}_2(\Lambda^4_{35})$ satisfying (42). In particular, setting $\eta_0 = 0$, we have constructed η_1. It remains to prove the inequalities (a)-(f). As $\eta_0 = 0$, inequalities (a)-(c) follow immediately from (d)-(f) respectively by induction on j. Therefore it is sufficient to establish the inequalities (d)-(f), and to do this by induction one must prove both the first step $j = 1$ and the inductive step. To make the logical structure of the proof easier to follow, the remainder will be split into two lemmas.

Lemma 5.1.4. *Parts (a) and (d) of Proposition 5.1.2 hold for $j = 1$. Suppose by induction that (42) and parts (a), (c) and (d) hold for $j \le k$. Then parts (a) and (d) hold for $j = k + 1$.*

Proof. From above, $\|\eta_{k+1}\|_2 \le 2\|\phi\|_2 + \big\|\Theta(\eta_k)\big\|_2$. Putting $k = 0$, substituting $\eta_0 = 0$ and $\Theta(0) = 0$ and applying part (i) of Theorem A gives $\|\eta_1\|_2 \le 2A_1 t^{9/2}$. This proves parts (a) and (d) of Proposition 5.1.2 for $j = 1$. Now suppose that (42) and parts (a), (c) and (d) hold for $j \le k$. Taking the difference of (42) for $j = k - 1, k$ gives $d(\eta_{k+1} - \eta_k) = d\big(\Theta(\eta_k) - \Theta(\eta_{k-1})\big)$. Applying the argument used above to prove that $\|\eta_{k+1}\|_2 \le 2\|\phi\|_2 + \big\|\Theta(\eta_k)\big\|_2$, we find that $\eta_{k+1} - \eta_k - \Theta(\eta_k) + \Theta(\eta_{k-1})$ is a closed form representing an element of H^4_+, so that $\big\|\eta_{k+1} - \eta_k\big\|_2 \le \big\|\Theta(\eta_k) - \Theta(\eta_{k-1})\big\|_2$. Thus part (iii) of Lemma 5.1.1 shows that

$$\big\|\eta_{k+1} - \eta_k\big\|_2 \le D_2\big\|\eta_k - \eta_{k-1}\big\|_2\big(\|\eta_k\|_{C^0} + \|\eta_{k-1}\|_{C^0}\big). \tag{46}$$

By part (c) of Proposition 5.1.2 for $j = k - 1, k$ and since $t \le \kappa$, we have $D_2\big(\|\eta_k\|_{C^0} + \|\eta_{k-1}\|_{C^0}\big) \le \frac{1}{2}$. Therefore part (d) for $j = k+1$ follows from (46) and part (d) for $j = k$, as we have to prove. Then part (a) for $j = k + 1$ follows by induction from part (d) for $j = 1, \ldots, k + 1$, and the lemma is complete. □

Lemma 5.1.5. *Parts (b), (c), (e) and (f) of Proposition 5.1.2 hold for $j = 1$. Suppose by induction that (42) and parts (a)-(f) hold for $j \le k$, and part (d) holds for $j = k + 1$. Then parts (b), (c), (e) and (f) hold for $j = k + 1$.*

Proof. By (42) we have $d\eta_1 = d\phi$. Applying parts (ii) and (iii) of Theorem A and part (d) of Proposition 5.1.2 for $j = 1$, we have

$$\|\nabla\eta_1\|_{10} \le A_3\big(A_2 t + 2A_1 t^{3/10}\big). \tag{47}$$

Since $t \le \kappa$, parts (b) and (e) of Proposition 5.1.2 hold for $j = 1$ with $E_1 = 6A_1A_3$. Using part (iv) of Theorem A and parts (d) and (e) of Proposition 5.1.2 for $j = 1$ we find that $\|\eta_1\|_{C^0} \le \frac{1}{2}A_4(E_1 + 4A_1)t^{1/2}$. Thus parts (c) and (f) of Proposition 5.1.2 hold for $j = 1$ with $E_2 = A_4(E_1 + 4A_1)$, as we have to prove.

Now suppose that (42) and parts (a)-(f) hold for $j \le k$, and part (d) holds for $j = k + 1$. Taking the difference of (42) for $j = k - 1, k$ we find that $d(\eta_{k+1} - \eta_k) = d\big(\Theta(\eta_k) - \Theta(\eta_{k-1})\big)$. Thus part (iii) of Theorem A and part (d) of Proposition 5.1.2 for $j = k + 1$ show that

$$\left\|\nabla(\eta_{k+1}-\eta_k)\right\|_{10} \le A_3\bigl(\left\|d(\Theta(\eta_k)-\Theta(\eta_{k-1}))\right\|_{10} + 4A_1 2^{-k-1}t^{3/10}\bigr). \quad (48)$$

Part (iv) of Lemma 5.1.1 shows that

$$\begin{aligned}\left\|d(\Theta(\eta_k)-\Theta(\eta_{k-1}))\right\|_{10} \le D_3\Bigl\{&\|\eta_k-\eta_{k-1}\|_{C^0}\bigl(\|\nabla\eta_k\|_{10}+\|\nabla\eta_{k-1}\|_{10}\bigr)\\ &+\bigl(\|d\phi\|_{10}\|\eta_k-\eta_{k-1}\|_{C^0}+\left\|\nabla(\eta_k-\eta_{k-1})\right\|_{10}\bigr)\bigl(\|\eta_k\|_{C^0}+\|\eta_{k-1}\|_{C^0}\bigr)\Bigr\}.\end{aligned} \quad (49)$$

Substituting part (ii) of Theorem A and parts (b), (c), (e) and (f) of Proposition 5.1.2 for $j = k-1, k$ into (49) we find that $\left\|d(\Theta(\eta_k)-\Theta(\eta_{k-1}))\right\|_{10} \le 2A_1 2^{-k-1}t^{3/10}$, as $t \le \kappa$. Thus (48) shows that $\left\|\nabla(\eta_{k+1}-\eta_k)\right\|_{10} \le E_1 2^{-k-1}t^{3/10}$, which proves part (e) of Proposition 5.1.2 for $j = k+1$. Part (f) for $j = k+1$ then follows immediately from parts (d) and (e) for $j = k+1$ and part (iv) of Theorem A. Parts (b) and (c) for $j = k+1$ follow by induction from parts (e) and (f) for $j = 1, \ldots, k+1$, and the lemma is finished. □

Lemmas 5.1.4 and 5.1.5 establish the first step and the inductive step of our marathon induction in j, and thus by induction in j, the sequence $\{\eta_j\}_{j=0}^{\infty}$ exists in $L_2^{10}(\Lambda^4_{35})$ and satisfies $\eta_0 = 0$, Eq. (42) and parts (a)-(f) of Proposition 5.1.2. This completes the proof of Proposition 5.1.2. □

Proof of Proposition 5.1.3. Parts (e) and (f) of Proposition 5.1.2 show that the sequence $\{\eta_j\}_{j=0}^{\infty}$ is convergent in $L_1^{10}(\Lambda^4_{35})$, by comparison with a geometric series. Let $\eta \in L_1^{10}(\Lambda^4_{35})$ be the limit of the sequence. Taking the limit in (42) shows that $d\eta = d\phi + d\,\Theta(\eta)$. Taking the limit in part (c) gives the estimate $\|\eta\|_{C^0} \le E_2 t^{1/2}$.

We will prove that η is smooth. There is a standard method called the 'bootstrap argument' for proving smoothness of solutions to a nonlinear differential equation with elliptic principal part. To use this method, we must write η as the solution of an elliptic equation. Let V be the vector bundle $\bigoplus_{i=0}^{8}\Lambda^i T^*M$ over M. Now the operator $d+d^* : C^\infty(V) \to C^\infty(V)$ is elliptic, and since $\Lambda^4_{35} = \Lambda^4_-$, $d^*\eta$ is given in terms of $d\eta$. First we write the equation $d\eta = d\phi + d\,\Theta(\eta)$ in the form

$$(d+d^*)\eta = d\phi + *d\phi + d\,\Theta(\eta) + *d\,\Theta(\eta), \quad (50)$$

where η and both sides of the equation are regarded as sections of V.

However, this equation is still unsuitable since both sides involve the first derivatives of η. Therefore we rewrite (50) in the form

$$(d+d^*)\eta + R(\eta, \nabla\eta) = d\phi + *d\phi + F(\eta). \quad (51)$$

Here $R(x,y)$ is a function depending on Ω and its derivatives, that is smooth in (x,y) and linear in y and satisfies $R(0,y) = 0$, $F(x)$ is a smooth function of x depending on Ω and its derivatives, and $F(\eta) - R(\eta,\nabla\eta) = d\,\Theta(\eta) + *d\,\Theta(\eta)$.

Since $d + d^*$ is elliptic and ellipticity is an open condition, we deduce that $d+d^*+R$ is also elliptic whenever η is sufficiently small in C^0. But from above, $\|\eta\|_{C^0} \le E_2 t^{1/2}$, so as $t \le \kappa$, $d + d^* + R$ is an elliptic operator of degree 1.

Since $\eta \in L^{10}_1(V)$, $\eta \in C^{0,1/5}(V)$ by the Sobolev embedding theorem. Thus the right hand side of (51) lies in $C^{0,1/5}(V)$. But the coefficients of $d + d^* + R$ are also in $C^{0,1/5}$, since they depend on η. Thus an elliptic regularity result [1, Theorem 10.7] applies to show that $\eta \in C^{1,1/5}(V)$.

Now suppose by induction that $\eta \in C^{k,1/5}(V)$ for some integer $k > 0$. Then the right hand side of (51) lies in $C^{k,1/5}(V)$, and the coefficients of $d + d^* + R$ lie in $C^{k,1/5}$. So by elliptic regularity, η lies in $C^{k+1,1/5}(V)$. Therefore by induction, $\eta \in C^{k,1/5}(V)$ for every integer $k > 0$, and so η is smooth, as we have to prove.

Define $\tilde\Omega = \Omega + \eta - \Theta(\eta)$. Then $\tilde\Omega$ is smooth, as η is smooth. Since $\|\eta\|_{C^0} \le E_2 t^{1/2}$ and $t \le \kappa$, we have $\|\eta\|_{C^0} \le D_1$, and so part (ii) of Lemma 5.1.1 applies to show that $\tilde\Omega$ is a section of AM, as we have to prove. The equation $d\tilde\Omega = 0$ follows from the equations $d\Omega + d\phi = 0$ of Theorem A and $d\eta = d\phi + d\,\Theta(\eta)$ above. Finally, the estimate $\|\tilde\Omega - \Omega\|_{C^0} \le \lambda t^{1/2}$, for some $\lambda > 0$ depending only on $A_1, \ldots, A_4$, follows easily from the inequality $\|\eta\|_{C^0} \le E_2 t^{1/2}$ above, the inequality $t \le \kappa$, and parts (i) and (iii) of Lemma 5.1.1. This completes the proofs of Proposition 5.1.3 and Theorem A. □

□

5.2 The proof of Theorem B

In this section we shall prove Theorem B of §2.1. We begin with two propositions.

Proposition 5.2.1. *Let B_1, B_2, B_3 be the balls of radii $1, 2, 3$ about 0 in $\mathbb{R}^8$, and let h be the euclidean metric on B_3. Then there exist positive constants F_1, F_2, F_3 such that if g is a riemannian metric on B_3 and $\|g - h\|_{C^{1,1/2}} \le F_1$, then the following is true.*

Let $\chi \in L^{10}_1(\Lambda^4_-)$ on B_3. Then χ satisfies the inequalities

$$\big\|\nabla\chi|_{B_2}\big\|_{10} \le F_2\big(\|d\chi\|_{10} + \|\chi\|_2\big), \tag{52}$$

$$\big\|\chi|_{B_2}\big\|_{C^0} \le F_3\big(\|\nabla\chi\|_{10} + \|\chi\|_2\big). \tag{53}$$

Here the bundle Λ^4_-, the connection ∇ and all norms are w.r.t. the metric g on B_3.

Proof. Estimates similar to (52) and (53) are well-known tools in analysis, so we will not give proofs. As $\chi \in \Lambda^4_-$, $d\chi$ and $d^*\chi$ are dual under $*$. Let $d + d^*$ be as in the proof of Proposition 5.1.3. It is an elliptic operator, so standard results apply, such as those in [1]. Inequality (52) is an interior estimate, a special case of an elliptic regularity result for $d + d^*$. To prove it, one may for instance modify the proof of [1, Theorem 10.3].

By the Sobolev embedding theorem, L_1^{10} is embedded in C^0. Thus inequality (53) is a Sobolev embedding result. Its special features are that $\|\chi\|_2$ rather than $\|\chi\|_{10}$ appears on the r.h.s., and the constant F_3 is independent of g. It can be shown by following the proof of [2, Lemma 2.22]. $\square$

Proposition 5.2.2. *Let ρ, K and F_1 be given, positive constants. Then there exists $L > 0$ depending on ρ, K and F_1 such that the following is true.*

Let $t > 0$ be a positive constant, and define $r = Lt$. Suppose (M, g) is a complete riemannian 8-manifold with injectivity radius $\delta(g)$ satisfying $\delta(g) \geq \rho t$, and Riemann curvature $R(g)$ satisfying $\|R(g)\|_{C^0} \leq Kt^{-2}$. Let ω be the volume of B_1. Then for each $m \in M$, the volume of the geodesic ball of radius r is at least $\frac{1}{2}\omega r^8$ and the volume of the geodesic ball of radius $4r$ is no more than $2\omega(4r)^8$. For each $m \in M$ there exists a smooth, injective map $\Psi_m : B_3 \to M$ such that $\Psi_m(0) = m$ and $\|r^{-2}\Psi_m^*(g) - h\|_{C^{1,1/2}} \leq F_1$, where the $C^{1,1/2}$- norm is taken w.r.t. the metric h on B_3. Moreover, $\Psi_m(B_2)$ contains the geodesic ball of radius r about m in M, and $\Psi_m(B_3)$ is contained in the geodesic ball of radius $4r$ about m in M.

Proof. It is easy to see that this result is in fact independent of t, since the result for $t = \hat{t}$ follows immediately from the result for $t = 1$ applied to the rescaled metric $\hat{g} = \hat{t}^{-2}g$. Therefore it suffices to prove the proposition for $t = 1$. What is required is systems of coordinates on open balls in M, in which the metric g is close to the euclidean metric h in the $C^{1,1/2}$- norm. These are provided by Jost and Karcher's theory of *harmonic coordinates* ([11], [10 p. 124]). Jost and Karcher show that if the injectivity radius is bounded below and the sectional curvature is bounded above, then there exist coordinate systems on all balls of a given radius, in which the metric is close to the euclidean metric in the $C^{1,\alpha}$- norm for any given $\alpha \in (0, 1)$.

We deduce that if $\delta(g) \geq \rho$ and $\|R(g)\|_{C^0} \leq K$ then for $L > 0$ sufficiently small and depending only on ρ, K and F_1, for each $m \in M$ there exists a coordinate system Ψ_m about m, which we may write as a map $\Psi_m : B_3 \to M$ with $\Psi_m(0) = m$, such that $\|L^{-2}\Psi_m^*(g) - h\|_{C^{1,1/2}} \leq F_1$, as we have to prove. Now the radius and volume of balls are controlled by the $C^{1,1/2}$- norm of the metric on the balls. Therefore if F_1 is small the volume and radius of the balls $\Psi_m(B_2)$, $\Psi_m(B_3)$ must be close to the volume and radius of B_2, B_3.

It follows that by making F_1 and L smaller if necessary, we can ensure that for each $m \in M$, the volume of the geodesic ball of radius L about m is at least $\frac{1}{2}\omega L^8$ and the volume of the geodesic ball of radius $4L$ about m is no more than $2\omega(4L)^8$, and that $\Psi_m(B_2)$ contains the geodesic ball of radius L about m and $\Psi_m(B_3)$ is contained in the geodesic ball of radius $4L$ about m. This completes the proposition. $\square$

Now we can prove Theorem B.

Theorem B. *Let ρ, K be positive constants. Then there exist positive constants A_3, A_4 depending only on ρ and K, such that the following is true.*

Let $t > 0$, and let M be a compact, oriented riemannian 8-manifold with metric g. Suppose that the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq \rho t$, and the Riemann curvature $R(g)$ of g satisfies $\|R(g)\|_{C^0} \leq Kt^{-2}$. Then conditions (iii) and (iv) of Theorem A hold for the metric g on M.

Proof. Let F_1 be the positive constant defined by Proposition 5.2.1, and apply Proposition 5.2.2. This constructs a constant $L > 0$ depending on ρ, K and F_1, such that for each $m \in M$ there exists a smooth, injective map $\Psi_m : B_3 \to M$ with $\Psi_m(0) = m$ and $\|r^{-2}\Psi_m^*(g) - h\|_{C^{1,1/2}} \leq F_1$, where $r = Lt$. Therefore Proposition 5.2.1 applies to the metric $r^{-2}\Psi_m^*(g)$ on B_3, to show that inequalities (52) and (53) hold for $\chi \in L_1^{10}(\Lambda_-^4)$ on B_3.

Now rescale the metric $r^{-2}\Psi_m^*(g)$ by a factor r^2 to give $\Psi_m^*(g)$, and regard (52) and (53) as inequalities for anti-self-dual 4-forms χ on M. We find that if $\chi \in L_1^{10}(\Lambda_-^4)$ on M, then

$$\left\|\nabla\chi|_{\Psi_m(B_2)}\right\|_{10} \leq F_2\Big\{\|d\chi|_{\Psi_m(B_3)}\|_{10} + r^{-21/5}\|\chi|_{\Psi_m(B_3)}\|_2\Big\}, \tag{54}$$

$$\left\|\chi|_{\Psi_m(B_2)}\right\|_{C^0} \leq F_3\Big\{r^{1/5}\left\|\nabla\chi|_{\Psi_m(B_3)}\right\|_{10} + r^{-4}\left\|\chi|_{\Psi_m(B_3)}\right\|_2\Big\}. \tag{55}$$

Raising (54) to the power 10 and manipulating yields the inequality

$$\int_{\Psi_m(B_2)} |\nabla\chi|^{10} d\mu \leq 2^9 F_2^{10} \int_{\Psi_m(B_3)} |d\chi|^{10} d\mu + 2^9 F_2^{10} r^{-42} \left(\int_{\Psi_m(B_3)} |\chi|^2 d\mu\right)^5. \tag{56}$$

Let $B_s(x)$ be the geodesic ball of radius s about x in M. Since $\Psi_m(B_2)$ contains the geodesic ball of radius r about m and $\Psi_m(B_3)$ is contained in the geodesic ball of radius $4r$ about m by Proposition 5.2.2, we may replace $\Psi_m(B_2)$ by $B_r(m)$ in the l.h.s. of (56) and $\Psi_m(B_3)$ by $B_{4r}(m)$ in the r.h.s. of (56). Integrating the result over M gives

$$\begin{aligned}\iint_{\substack{x,y\in M:\\ d(x,y)\leq r}} |\nabla\chi(x)|^{10} dx dy \leq 2^9 F_2^{10} &\iint_{\substack{x,y\in M:\\ d(x,y)\leq 4r}} |d\chi(x)|^{10} dx dy \\ &+ 2^9 F_2^{10} r^{-42} \int_{y\in M} \left(\int_{\substack{x\in M:\\ d(x,y)\leq 4r}} |\chi(x)|^2 dx\right)^5 dy.\end{aligned} \tag{57}$$

Here $d(x,y)$ is the geodesic distance between x and y. Exchanging the integrals in x and y shows that the l.h.s. of (57) is equal to $\int_M |\nabla\chi(x)|^{10}\,\mathrm{vol}(B_r(x))dx$. But by Proposition 5.2.1, $\mathrm{vol}(B_r(x)) \geq \frac{1}{2}\omega r^8$. Thus the l.h.s. of (57) is bounded below by $\frac{1}{2}\omega r^8 \int_M |\nabla\chi|^{10} dx$. Treating the r.h.s. of (57) in a similar way, we may show that

$$\begin{aligned}\tfrac{1}{2}\omega r^8 \int_M |\nabla\chi|^{10} dx \leq & 2^{26} F_2^{10} \omega r^8 \biggl\{ \int_M |d\chi|^{10} dx \\ & + r^{-42} \int_M |\chi|^2 dx \cdot \sup_{m\in M} \Bigl(\int_{B_{4r}(m)} |\chi(x)|^2 dx \Bigr)^4 \biggr\}.\end{aligned} \tag{58}$$

Cancelling factors and rearranging yields

$$\bigl\|\nabla\chi\bigr\|_{10}^{10} \leq 2^{27} F_2^{10} \Bigl\{ \bigl\|d\chi\bigr\|_{10}^{10} + r^{-42} \bigl\|\chi\bigr\|_2^{10} \Bigr\}. \tag{59}$$

Substituting $r = Lt$ into (59), raising the whole inequality to the power $1/10$ and manipulating shows that $\|\nabla\chi\|_{10} \leq A_3\bigl(\|d\chi\|_{10} + t^{-21/5}\|\chi\|_2\bigr)$, where $A_3 = 8F_2 \max\bigl(1, L^{-21/5}\bigr)$. This constant A_3 depends only on ρ and K. Thus part (iii) of Theorem A holds, as we have to prove.

Finally, taking the supremum of (55) over all $m \in M$ and substituting $r = Lt$ we find that

$$\|\chi\|_{C^0} \leq F_3\bigl(L^{1/5} t^{1/5} \|\nabla\chi\|_{10} + L^{-4} t^{-4} \|\chi\|_2\bigr), \tag{60}$$

for all $\chi \in L_1^{10}(\Lambda_-^4)$. Thus part (iv) of Theorem A holds with constant $A_4 = F_3 \max\bigl(L^{1/5}, L^{-4}\bigr)$, so that A_4 depends only on ρ and K, and the proof of Theorem B is complete. □

6 The topology of riemannian manifolds with holonomy $Spin(7)$

In this chapter we prove Theorems C and D of §2.1. Theorem C is proved in §6.1, and characterizes the holonomy group of a torsion-free $Spin(7)$- structure on a compact, simply-connected 8-manifold, using topological invariants of the manifold. The material of Theorem C is not new but is already common knowledge, and the author is grateful to Simon Salamon and McKenzie Wang for explaining it to him.

Section 6.2 proves Theorem D, which studies the moduli space of torsion-free $Spin(7)$- structures on a compact, simply-connected 8-manifold. This theorem is also not new, since Bryant and Harvey have an unpublished proof (announced in [6, p. 561]), and Wang has proved an infinitesimal version [23, Theorem 3.8]. But we believe this is the first published proof of the result.

6.1 The proof of Theorem C

Let M be a compact 8-manifold with a metric g with holonomy group contained in $Spin(7)$. Since $Spin(7)$ is simply-connected, any 8-manifold with a $Spin(7)$- structure is a spin manifold, so we may consider the spin bundle

$\Delta = \Delta_+ \oplus \Delta_-$ of M. Defined on the spin bundle is the elliptic Dirac operator $D : C^\infty(\Delta) \to C^\infty(\Delta)$, which is the sum of $D_+ : C^\infty(\Delta_+) \to C^\infty(\Delta_-)$ and $D_- : C^\infty(\Delta_-) \to C^\infty(\Delta_+)$. The *index* $\mathrm{ind}(D_+)$ of D_+ is the integer $\dim \operatorname{Ker} D_+ - \dim \operatorname{Coker} D_+$, and since D_- is the adjoint of D_+, this gives $\mathrm{ind}(D_+) = \dim \operatorname{Ker} D_+ - \dim \operatorname{Ker} D_-$. The Atiyah-Singer Index Theorem [3] gives a topological formula for the index of an elliptic operator, and by [3, Theorem 5.3], $\mathrm{ind}(D_+)$ is equal to the $\hat{A}$- genus of M, $\hat{A}(M)$, which for an 8-manifold is given in terms of the pontryagin classes $p_1(M)$, $p_2(M)$ by

$$45.2^7 \hat{A}(M) = 7p_1(M)^2 - 4p_2(M). \tag{61}$$

Spinors in $\operatorname{Ker} D$ are harmonic spinors. Since g has holonomy contained in $Spin(7)$, the scalar curvature of g is zero, and thus the Weitzenbock formula of Lichnerowicz [17, p. 8, eq. (7)] shows that any harmonic spinor is constant. So $\operatorname{Ker} D_\pm$ are exactly the spaces of constant positive and negative spinors. Now the constant spinors on M are determined by the holonomy group of g. Therefore the index $\mathrm{ind}(D_+)$ is determined by the holonomy group of g. To determine $\mathrm{ind}(D_+)$ for any given subgroup G of $Spin(7)$, one must look at the canonical representation of G on the fibres of Δ_+ and Δ_-. The kernels $\operatorname{Ker} D_\pm$ are the subbundles of $\Delta_\pm$ on which G acts trivially, so $\mathrm{ind}(D_+)$ is just the dimension of the G- invariant part of Δ_+ minus the dimension of the G- invariant part of Δ_-.

By following the reasoning of [19, §7], we shall give a formula for the $\hat{A}$- genus of M in terms of the betti numbers $b^i(M) = \dim H^i(M, \mathbb{R})$ and the positive and negative parts $b^4_\pm = \dim H^4_\pm(M, \mathbb{R})$. From [19, eq. (7.1)], the signature $b^4_+ - b^4_-$ of M is given by

$$7p_2(M) - p_1(M)^2 = 45(b^4_+ - b^4_-), \tag{62}$$

and by [19, eq. (7.3)], which applies to manifolds of holonomy $Spin(7)$ by the remark on [19, p. 166], the Euler characteristic of M is given by

$$4p_2(M) - p_1(M)^2 = 8(2 - 2b^1 + 2b^2 - 2b^3 + b^4). \tag{63}$$

Solving (62) and (63) for $p_1(M)^2$ and $p_2(M)$ and substituting into (61) gives

$$24\hat{A}(M) = -1 + b^1 - b^2 + b^3 + b^4_+ - 2b^4_-. \tag{64}$$

So the holonomy group of g determines the $\hat{A}$- genus of M, and the betti numbers of M must then satisfy (64). Conversely, given an 8-manifold M with holonomy contained in $Spin(7)$, we may calculate $\hat{A}(M)$ using (64), and this then restricts the possible holonomy groups of g.

Lemma 6.1.1. *Let M be a compact, simply-connected 8-manifold and g a metric on M with holonomy group contained in $Spin(7)$. Then the holonomy group of g is $Spin(7)$, $SU(4)$, $Sp(2)$ or $SU(2) \times SU(2)$.*

Proof. Since M is compact and simply-connected, the holonomy group $\mathrm{Hol}(g)$ must be a connected subgroup of $SO(8)$ that acts nontrivially on every nonzero vector in $\mathbb{R}^8$. Now Berger [4], [20] has given a classification of the possible connected holonomy groups of riemannian metrics. The list $Spin(7)$, $SU(4)$, $Sp(2)$, $SU(2) \times SU(2)$ can be shown to be the complete list of connected subgroups of $Spin(7)$ that are 8-dimensional holonomy groups in Berger's classification, and also act nontrivially on nonzero vectors. $\square$

Here is Theorem C.

Theorem C. *Suppose that M is a compact, simply-connected 8-manifold and that Ω is a torsion-free $Spin(7)$- structure on M, and let g be the associated metric. Then M is spin, and the volume form $\Omega \wedge \Omega$ gives a natural orientation on M. Define the $\hat{A}$- genus $\hat{A}(M)$ of M by*

$$24\hat{A}(M) = -1 + b^1 - b^2 + b^3 + b^4_+ - 2b^4_-, \tag{65}$$

where b^i are the betti numbers of M, and $b^4_\pm$ are the dimensions of the spaces of self-dual and anti-self-dual 4-forms in $H^4(M, \mathbb{R})$. Then $\hat{A}(M)$ is equal to $1, 2, 3$ or 4, and the holonomy group $\mathrm{Hol}(g)$ of g is determined by $\hat{A}(M)$ as follows:

(*i*) $\mathrm{Hol}(g) = Spin(7)$ *if and only if* $\hat{A}(M) = 1$,

(*ii*) $\mathrm{Hol}(g) = SU(4)$ *if and only if* $\hat{A}(M) = 2$,

(*iii*) $\mathrm{Hol}(g) = Sp(2)$ *if and only if* $\hat{A}(M) = 3$, *and*

(*iv*) $\mathrm{Hol}(g) = SU(2) \times SU(2)$ *if and only if* $\hat{A}(M) = 4$.

Every compact, riemannian 8-manifold with holonomy group $Spin(7)$ is simply-connected.

Proof. The Proposition of [22, p. 59] gives the dimensions of spaces of parallel spinors for different holonomy groups, and from this and the definition of $\hat{A}(M)$ we can deduce 'only if' parts of (i)-(iv). Now by Lemma 6.1.1 these are the only holonomy groups possible in these situation. Therefore $\hat{A}(M)$ is equal to 1,2,3 or 4, as we have to prove, and since the values of $\hat{A}(M)$ distinguish the four holonomy groups, the 'if' parts of (i)-(iv) also hold. This proves parts (i)-(iv).

It remains only to show that every compact, riemannian 8-manifold with holonomy group $Spin(7)$ is simply-connected. Let M be a compact riemannian 8-manifold with holonomy $Spin(7)$. Then M is ricci-flat and irreducible, so from the Cheeger-Gromoll splitting theorem [5, Cor. 6.67, p. 168] we deduce that M has finite fundamental group.

Let $\tilde{M}$ be the universal cover of M, and d the degree of the covering. Now, $\hat{A}(\tilde{M}) = 1$ by part (i), and $\hat{A}(M) = \hat{A}(\tilde{M})/d$, since the $\hat{A}$- genus is a characteristic class. However, since M has structure group $Spin(7)$, it is spin, and thus $\hat{A}(M)$ is an integer. It follows that the degree $d = 1$, and M is simply-connected, as we have to prove. $\square$

6.2 The proof of Theorem D

We shall now prove Theorem D.

Theorem D. *Let M be a simply-connected, compact 8-manifold admitting torsion-free $Spin(7)$- structures, let $\mathcal{X}$ be the set of torsion-free $Spin(7)$- structures on M, and let $\mathcal{D}$ be the group of diffeomorphisms of M isotopic to the identity. Then $\mathcal{X}/\mathcal{D}$ is a smooth manifold of dimension $\hat{A}(M) + b^4_-(M)$.*

Proof. This is a moduli space problem, and involves the quotient by an infinite-dimensional group. The standard approach to such a problem is to find a 'slice' for the action of the group. We shall use a version of Ebin's Slice Theorem [8, Theorem 7.4] for the space of riemannian metrics.

Theorem ([8]). *Let M be a compact manifold, let $\mathcal{D}$ be the group of diffeomorphisms of M isotopic to the identity, and let $\mathcal{R}$ be the set of riemannian metrics on M. Then $\mathcal{D}$ acts on $\mathcal{R}$. Let $g \in \mathcal{R}$, and let I_g be the subgroup of $\mathcal{D}$ fixing g.*

Define a linear subspace L_g of $C^\infty(S^2T^*M)$ by $h \in L_g$ if and only if $g^{ij}\nabla_i h_{jk} = 0$ in index notation, where ∇ is the Levi-Civita connection of g. Then there exists a subset S_g of $\mathcal{R}$ with the following three properties:

(i) *S_g is an open neighbourhood of g in L_g,*

(ii) *if $\alpha \in I_g$ then $\alpha \cdot S_g = S_g$, and*

(iii) *the natural projection from S_g/I_g to $\mathcal{R}/\mathcal{D}$ induces a homeomorphism between S_g/I_g and a neighbourhood of $g\mathcal{D}$ in $\mathcal{R}/\mathcal{D}$.*

The following Corollary immediately follows.

Corollary 6.2.1. *Let $\Pi : \mathcal{X} \to \mathcal{R}$ be the natural projection from a (torsion-free) $Spin(7)$- structure on M to the associated metric. Let $\Omega \in \mathcal{X}$, and let $g = \Pi(\Omega)$. Let S_g and I_g be as in the above theorem. Then $\Pi^*(S_g)/I_g$ is homeomorphic to a neighbourhood of $\Omega\mathcal{D}$ in $\mathcal{X}/\mathcal{D}$ under the natural projection.*

To prove Theorem D we shall study $\Pi^*(S_g)$. Let V be the vector space of smooth, exact 5-forms on M. Let $\Phi : C^\infty(AM) \to V$ be the restriction of the exterior derivative d to $C^\infty(AM)$. Then $\mathcal{X} = \operatorname{Ker} \oplus$. Let $W = C^\infty(T^*M)$. Define a map $\Psi : C^\infty(AM) \to W$ by $\Psi(\Omega') = g^{ij}\nabla_i g'_{jk}$ in coordinates, where $g' = \Pi(\Omega')$.

Now from the definition of L_g in Ebin's Slice Theorem, $\mathcal{X} \cap \operatorname{Ker} \ominus = \diamond^*(\mathcal{L}_\})$. So from part (i) of Ebin's Slice Theorem, we deduce that $\Pi^*(S_g)$ is an open neighbourhood of Ω in $\operatorname{Ker}\Phi \cap \operatorname{Ker}\Psi$. Thus we have shown that $\Pi^*(S_g)$ is an open neighbourhood of Ω in the kernel of the map

$$(\Phi, \Psi) : C^\infty(AM) \to V \oplus W. \tag{66}$$

The tangent space of $C^\infty(AM)$ at Ω may be written

$$T_\Omega C^\infty(AM) = C^\infty(\Lambda^4_1) \oplus C^\infty(\Lambda^4_7) \oplus C^\infty(\Lambda^4_{35}). \tag{67}$$

Using this splitting, we may write out the first derivatives $d\,\Phi, d\,\Psi$ of Φ and Ψ at Ω explicitly. Clearly, if (ξ_1, ξ_7, ξ_{35}) is a vector in $T_\Omega C^\infty(AM)$ then

$$d\,\Phi(\xi_1, \xi_7, \xi_{35}) = d\xi_1 + d\xi_7 + d\xi_{35} \in V. \tag{68}$$

Also, calculation shows that

$$d\,\Psi(\xi_1, \xi_7, \xi_{35}) = \pi_8(d\xi_1) - 7\pi_8(d\xi_{35}) \in C^\infty(\Lambda^5_8) \cong W. \tag{69}$$

Using this isomorphism $C^\infty(\Lambda^5_8) \cong W$, define $\Xi : C^\infty(AM) \to W$ by $\Xi(\Omega') = \Psi(\Omega') + 7\pi_8(\Phi(\Omega'))$. Then $\mathrm{Ker}\big((\Phi, \Psi)\big) = \mathrm{Ker}\big((\Phi, \Xi)\big)$, and from (68) and (69), the first derivative $d\,\Xi$ at Ω is given by

$$d\,\Xi(\xi_1, \xi_7, \xi_{35}) = 8\pi_8(d\xi_1) + 7\pi_8(d\xi_7) \in C^\infty(\Lambda^5_8) \cong W. \tag{70}$$

Now the spin bundles $\Delta_\pm$ of §6.1 satisfy $\Delta_+ \cong \Lambda^4_1 \oplus \Lambda^4_7$ and $\Delta_- \cong \Lambda^5_8$, and by scaling the factors Λ^4_1, Λ^4_7 correctly in the first isomorphism we can ensure that the Dirac operator D_+ is written

$$D_+(\xi_1, \xi_7) = 8\pi_8(d\xi_1) + 7\pi_8(d\xi_7) \tag{71}$$

in these representations of $\Delta_\pm$. So by (70), $d\,\Xi(\xi_1, \xi_7, \xi_{35}) = D_+(\xi_1, \xi_7)$. From §6.1, the kernel and cokernel of D_+ consist of constant sections. Since M is assumed to be simply-connected, there are no nonzero constant 1-forms on M, and thus the cokernel of D_+ is zero. And as the index of D_+ is $\hat{A}(M)$, the kernel of D_+ has dimension $\hat{A}(M)$.

We can now identify the image and kernel of $(d\,\Phi, d\,\Xi)$. Restricting to $C^\infty(\Lambda^4_{35})$, $d\,\Phi$ is surjective on V with kernel H^4_-, and $d\,\Xi$ is zero. Restricting to $C^\infty(\Lambda^4_1) \oplus C^\infty(\Lambda^4_7)$, $d\,\Xi$ is surjective on W with a kernel of constant sections of dimension $\hat{A}(M)$. Therefore $(d\,\Phi, d\,\Xi)$ is surjective on $V \oplus W$, and has kernel of dimension $\hat{A}(M) + b^4_-(M)$.

Next we shall apply an implicit function theorem, to show that the kernel of the map $(\Phi, \Xi) : C^\infty(A(M)) \to V \oplus W$ in a neighbourhood of Ω is a smooth manifold of dimension $\hat{A}(M) + b^4_-(M)$. There is an Implicit Function Theorem for Banach spaces [14, Theorem 2.1, p. 131]. But $C^\infty(AM), V$ and W do not have the structure of Banach spaces.

To get round this, one can work with $C^{k+1,\alpha}(AM)$ instead of $C^\infty(AM)$. Then V and W become Banach spaces with the $C^{k,\alpha}$- norms. Applying the Implicit Function Theorem for Banach spaces shows that the kernel of (Φ, Ξ) in any $C^{k+1,\alpha}(AM)$ is a smooth manifold of dimension $\hat{A}(M) + b^4_-(M)$. As this holds for all k, the kernel for each k must be smooth. Therefore the kernel of (Φ, Ξ) in a neighbourhood of Ω in $C^\infty(AM)$ is a smooth manifold of dimension $\hat{A}(M) + b^4_-(M)$.

Thus $\Pi^*(S_g)$ is a smooth manifold of dimension $\hat{A}(M) + b^4_-(M)$ near Ω. The calculations above identified the tangent space of $\Pi^*(S_g)$ at Ω, and it can be seen that the projection from this tangent space to $H^4(M,\mathbb{R})$, given by taking the cohomology class of the closed 4-form, is injective. So the map from $\Pi^*(S_g)$ to $H^4(M,\mathbb{R})$ is locally injective.

Now I_g acts trivially on $H^4(M,\mathbb{R})$, since each element of I_g is isotopic to the identity. Therefore I_g must act trivially on $\Pi^*(S_g)$ near Ω. Thus $\Pi^*(S_g)/I_g$ is just $\Pi^*(S_g)$ near Ω. From Corollary 6.2.1 we deduce that a neighbourhood of $\Omega\mathcal{D}$ in $\mathcal{X}/\mathcal{D}$ is homeomorphic to a neighbourhood in a smooth manifold of dimension $\hat{A}(M) + b^4_-(M)$. Since this holds for every point $\Omega\mathcal{D}$ in $\mathcal{X}/\mathcal{D}$, $\mathcal{X}/\mathcal{D}$ is a smooth manifold of dimension $\hat{A}(M) + b^4_-(M)$. This completes the proof of Theorem D. □

It is not difficult to modify the proof above for M not simply-connected. The dimension of the moduli space turns out to be $\hat{A}(M) + b^1(M) + b^4_-(M)$.

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