University of Oxford
Faculty of Mathematics and Natural Sciences

Department of Mathematics

Doctoral Thesis

Generalized Lagrangian mean curvature flow in almost Calabi–Yau manifolds

Submitted by Tapio Behrndt
born on 11th July 1983 in Berlin

Supervisor: Prof. Dominic Joyce

Oxford, 25th March 2011

Contents

1 Introduction

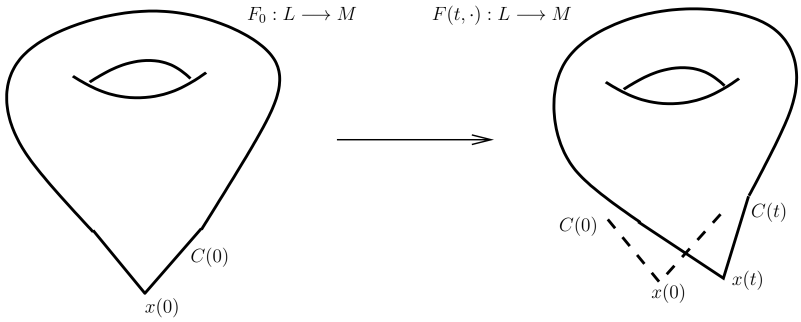
1.1 Motivation for this work

In a Calabi–Yau manifold M with holomorphic volume form Ω there is a distinguished class of submanifolds called special Lagrangian submanifolds. These are oriented Lagrangian submanifolds which are calibrated with respect to $\mathrm{Re}\,\Omega$. There has been growing interest in special Lagrangian submanifolds in the past decade since these are the key ingredient in the Strominger–Yau–Zaslow conjecture [53] which states mirror symmetry in terms of special Lagrangian torus fibrations.

Proving the existence of special Lagrangian submanifolds in a Calabi–Yau manifold is a hard problem. For instance Wolfson proved in [60] the existence of a K3-surface which has no special Lagrangian submanifolds. This shows how subtle the issue is. However, since special Lagrangian submanifolds are calibrated submanifolds, they are volume minimizers in their homology class. One possible approach to the study of the existence of special Lagrangian submanifolds is therefore through mean curvature flow, which is the negative gradient flow of the volume functional. The key observation here is due to Smoczyk [50] who proves that a compact Lagrangian submanifold in a Calabi–Yau manifold (or even in a Kähler–Einstein manifold) remains Lagrangian under the mean curvature flow. The naïve idea is therefore to take a Lagrangian submanifold in a Calabi–Yau manifold and to deform it under Lagrangian mean curvature flow to a special Lagrangian submanifold. The long-time convergence of the Lagrangian mean curvature flow to a special Lagrangian submanifold has so far only been verified in several special cases, see for instance Smoczyk, Wang [52] and Wang [59]. Also in [55] Thomas and Yau conjecture that for a given Lagrangian submanifold in a Calabi–Yau manifold, which satisfies a certain stability condition, the Lagrangian mean curvature flow exists for all time and converges to a special Lagrangian submanifold. In general however one expects that a Lagrangian submanifold will form a finite time singularity under the mean curvature flow. In fact, recently Neves [44] constructed examples of Lagrangian surfaces in two dimensional Calabi–Yau manifolds which develop a finite time singularity under the mean curvature flow. The appearance of finite time singularities in the Lagrangian mean curvature flow therefore seems to be unavoidable in general.

When a finite time singularity occurs there are two possibilities, depending on the kind of singularity, how the flow can be continued. The first possibility is as in Perelman's work [47] on the Ricci flow of three manifolds, where a surgery is performed before the singularity occurs and the flow is then continued. The other possibility to continue the Lagrangian mean curvature flow when a finite time singularity occurs is to evolve the singular Lagrangian submanifold by mean curvature flow in a specific class of singular Lagrangian submanifolds. In this work we study the latter possibility in the special case of isolated conical singularities.

The goal of this thesis is to study the (generalized) Lagrangian mean curvature flow of Lagrangian submanifolds in (almost) Calabi–Yau manifolds which have isolated conical singularities modelled on stable special Lagrangian cones. We show that for a given Lagrangian submanifold $F_0 : L \to M$ with isolated conical singularities modelled on stable special Lagrangian cones one can find for a short-time a solution $F(t,\cdot) : L \to M$, $0 \le t < T$, to the (generalized) Lagrangian mean curvature flow with initial condition $F_0 : L \to M$, by letting the conical singularities move around in M. The Lagrangian mean curvature flow of $F_0 : L \to M$ (here on the left) therefore looks after a short time like the surface on the right.



1.2 Summary of the thesis

This work is split into four parts and we give a short overview over each of these parts and also point out the new results we obtain.

The first part of this thesis consists of §2 and §3, where we discuss some standard theory of linear parabolic equations on domains of $\mathbb{R}^m$ and on compact Riemannian manifolds. We first introduce Hölder and Sobolev spaces on Riemannian manifolds and also the notion of parabolic Hölder and Sobolev spaces. Then we review some standard regularity theory for linear parabolic equations on domains. In §3.1 we explain the construction of the Friedrichs heat kernel on an arbitrary Riemannian manifold and then study existence and regularity of solutions to the Cauchy problem for the inhomogeneous heat equation on compact Riemannian manifolds. The material discussed in this first part is standard in geometric analysis and lays the foundation for the study of linear and nonlinear parabolic equations on Riemannian manifolds.

The second part is §4 and §5. In §4 we first review the necessary background material from Riemannian geometry and symplectic geometry. We then introduce almost Calabi–Yau manifolds, the generalized mean curvature vector, and discuss Lagrangian submanifold in almost Calabi–Yau manifolds. In §4.4 we study the deformation of Lagrangian submanifolds in cotangent bundles. In §5 we then introduce the generalized Lagrangian mean curvature flow in almost Calabi–Yau manifolds and present a new short time existence proof for the generalized Lagrangian mean curvature flow, when the initial Lagrangian submanifold is compact. Most of the material covered in §4 and §5 is well known and can be found in the literature. The definition of the generalized mean curvature vector field, however, and the method of proof of the short time existence of the generalized Lagrangian mean curvature flow appear to be new.

In the third part of this thesis, §6 and §7, we study the Laplace operator and the heat equation on Riemannian manifolds with conical singularities. We begin by introducing weighted Hölder and Sobolev spaces on Riemannian manifolds with conical singularities and by reviewing some standard results about the Laplace operator acting on weighted spaces. In §6.3 we then introduce the notion of discrete asymptotics on Riemannian manifolds with conical singularities and we define weighted spaces with discrete asymptotics. We then study the Laplace operator acting on weighted spaces with discrete asymptotics. In §7 we begin with the definition of weighted parabolic Hölder and Sobolev spaces with discrete asymptotics and then proceed to prove weighted Schauder and L^p-estimates for solutions of the inhomogeneous heat equation. We then discuss the asymptotics of the Friedrichs heat kernel on Riemannian manifolds with conical singularities following Mooers [41]. Finally in §7.4 and §7.5 we prove existence and maximal regularity of solutions to the Cauchy problem for the inhomogeneous heat equation, when the free term lies in a weighted parabolic Hölder or Sobolev space with discrete asymptotics. Our existence and regularity results generalize a result previously obtained by Coriasco, Schrohe, and Seiler [13, Thm. 6]. The results of this part of the thesis can also be found in the author's paper [7].

The fourth and final part of this work consists of §8 and §9, where we study the short time existence problem for the generalized Lagrangian mean curvature flow, when the initial Lagrangian submanifold has isolated conical singularities. In §8 we first introduce special Lagrangian cones and the notion of stability for special Lagrangian cones. Then we define Lagrangian submanifolds with isolated conical singularities and discuss several Lagrangian neighbourhood theorems for Lagrangian submanifolds with isolated conical singularities. In §9 we then study the generalized Lagrangian mean curvature flow with isolated conical singularities. The analysis of this problem turns out to be involved and very difficult from a technical point of view. The idea, however, is essentially the same as in the short time existence proof for the generalized Lagrangian mean curvature flow of compact Lagrangian submanifolds presented in §5. In §9.1 and §9.2 we first set up the short time existence problem and then in §9.3-§9.5 we prove short time existence of solutions. Finally in §9.6 we discuss some regularity theory of the flow. The results from §8 were already known through previous work of Joyce on special Lagrangian submanifolds with isolated conical singularities, see [24] and [25]. The material covered in §9 is new and the results of this section will also appear in the author's paper [6].

In §10, the final section of this thesis, we discuss in a purely formal way some open problems that are related to the material presented in this work. In §10.1 we first discuss differential operators of Laplace type and parabolic equations of Laplace type on compact Riemannian manifolds with conical singularities. The discussion essentially generalizes the results for the Laplace operator and the heat equation on compact Riemannian manifolds with conical singularities discussed in §6 and §7. Then in §10.2 and §10.3 we speculate about some further existence and regularity results for the generalized Lagrangian mean curvature flow with isolated conical singularities.

1.3 Acknowledgements

First I would like to thank my supervisor Dominic Joyce for suggesting the short time existence problem of the Lagrangian mean curvature flow with conical singularities to me and for helpful discussions about his work on special Lagrangian submanifolds. I am also grateful to Tom Ilmanen for his continuing support throughout my D.Phil. studies and for invitation to the ETH Zürich in Michaelmas Term 2009. In addition I would like to thank him for interesting and stimulating discussions. Further I wish to thank André Neves for several invitations to Imperial College London and for useful conversations, and my former diploma supervisor Helga Baum for her continuing support. Finally I would like to thank my girlfriend Judith for her patience and encouragement during the last two years.

My D.Phil. studies were supported by a Sloane Robinson Foundation Graduate Award 2008, 2009, and 2010 of the Lincoln College, a scholarship of the British Chamber of Commerce in Germany in 2008, and an EPSRC Research Studentship. I wish to thank all these institutions for their support.

2 Regularity theory for linear parabolic equations on domains

2.1 Hölder and Sobolev spaces on Riemannian manifolds

In this subsection we introduce Hölder and Sobolev spaces on Riemannian manifolds and also discuss the Sobolev Embedding Theorem and the Rellich–Kondrakov Theorem. Good references for the material presented in this section are Adams [1], Aubin [4, Ch. 2], Gilbarg and Trudinger [27, Ch. 7], and also Tartar [54].

We begin by introducing C^k-spaces and Hölder spaces. Let (M, g) be a Riemannian manifold. Throughout this thesis the term manifold means smooth manifold without boundary. For $k \in \mathbb{N}$ we denote by $C^k_{\text{loc}}(M)$ the space of k-times continuously differentiable functions $u : M \to \mathbb{R}$ and we set $C^\infty(M) = \bigcap_{k\in\mathbb{N}} C^k_{\text{loc}}(M)$, which is the space of smooth functions on M. We define the C^k-norm by

$$\|u\|_{C^k} = \sum_{j=0}^{k} \sup_{x\in M} |\nabla^j u(x)| \quad \text{for } u \in C^k_{\text{loc}}(M),$$

whenever it is finite, and we define the space $C^k(M)$ by

$$C^k(M) = \{u \in C^k_{\text{loc}}(M) \ : \ \|u\|_{C^k} < \infty\}.$$

Then $C^k(M)$ is a Banach space.

In the regularity theory for elliptic and parabolic partial differential equations it is more convenient to work with Hölder spaces than with C^k-spaces, since these turn out to have better regularity properties. Next we introduce Hölder spaces. Let $\alpha \in (0, 1)$ and T be a tensor field over M. Then we define a seminorm

$$[T]_\alpha = \sup_{\substack{x\neq y\in M \\ d_g(x,y)<\delta_g(x)}} \frac{|T(x) - T(y)|}{d_g(x, y)^\alpha},$$

whenever it is finite. Here $d_g(x, y)$ denotes the Riemannian distance of x and y with respect to g, and $\delta_g(x)$ denotes the injectivity radius of g at x. Moreover, $|T(x) - T(y)|$ is understood in the sense that we first take the parallel transport of $T(x)$ along the unique minimizing geodesic connecting x and y, and then compute the norm at the point y. We define the $C^{k,\alpha}$-norm by

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + [\nabla^k u]_\alpha \quad \text{for } u \in C^k_{\text{loc}}(M),$$

whenever it is finite. The number α is called the Hölder exponent. We denote by $C^{k,\alpha}_{\text{loc}}(M)$ the space of functions in $u \in C^k_{\text{loc}}(M)$ with finite $C^{k,\alpha}$-norm on every $N \subset\subset M$. Here $N \subset\subset M$ means that N is a smoothly embedded and open submanifold of M whose closure is compact in M, see §4.1 for the definition of embedded submanifold. We define the Hölder space $C^{k,\alpha}(M)$ by

$$C^{k,\alpha}(M) = \left\{u \in C^{k,\alpha}_{\text{loc}}(M) \ : \ \|u\|_{C^{k,\alpha}} < \infty\right\}.$$

Then $C^{k,\alpha}(M)$ is a Banach space.

The next class of function spaces we introduce are Sobolev spaces. In the definition of Sobolev spaces we will use of the notion of weak derivatives, which can be found in Gilbarg and Trudinger [27, Ch. 7, §3]. Let $k \in \mathbb{N}$ and $p \in [1, \infty)$. For a k-times weakly differentiable function $u : M \to \mathbb{R}$ we define the $W^{k,p}$-norm by

$$\|u\|_{W^{k,p}} = \left(\sum_{j=0}^{k} \int_M |\nabla^j u|^p \, \mathrm{d}V_g \right)^{1/p},$$

whenever it is finite. We denote by $W^{k,p}_{\mathrm{loc}}(M)$ the space of k-times weakly differentiable functions on M that have finite $W^{k,p}$-norm on every $N \subset\subset M$. The Sobolev space $W^{k,p}(M)$ is defined by

$$W^{k,p}(M) = \left\{ u \in W^{k,p}_{\mathrm{loc}}(M) \ : \ \|u\|_{W^{k,p}} < \infty \right\}.$$

Then $W^{k,p}(M)$ is a Banach space. If $k = 0$, then we write $L^p_{\mathrm{loc}}(M)$ and $L^p(M)$ instead of $W^{0,p}_{\mathrm{loc}}(M)$ and $W^{0,p}(M)$, respectively. Moreover, if $p = 2$ we can define a scalar product on $W^{k,2}(M)$ by

$$\langle u, v \rangle_{W^{k,2}} = \sum_{j=0}^{k} \int_M g(\nabla^j u, \nabla^j v) \, \mathrm{d}V_g \quad \text{for } u, v \in W^{k,2}(M). \tag{1}$$

Thus $W^{k,2}(M)$ is a Hilbert space.

An important tool in the existence and regularity theory for linear and nonlinear partial differential equations are the Sobolev Embedding Theorem and the Rellich–Kondrakov Theorem. The Sobolev Embedding Theorem gives embeddings between different Sobolev spaces and embeddings of Sobolev spaces into Hölder spaces.

Theorem 2.1 (**Sobolev Embedding Theorem**). *Let (M, g) be a compact m-dimensional Riemannian manifold. Let $k, l \in \mathbb{N}$, $p, q \in [1, \infty)$, and $\alpha \in (0, 1)$. Then the following hold.*

(i) *If $\frac{1}{p} \leq \frac{1}{q} + \frac{k-l}{m}$, then $W^{k,p}(M)$ embeds continuously into $W^{l,q}(M)$ by inclusion.*
(ii) *If $k - \frac{m}{p} \geq l + \alpha$, then $W^{k,p}(M)$ embeds continuously into $C^{l,\alpha}(M)$ by inclusion.*

Moreover (i) *and* (ii) *continue to hold when $M = \Omega$, where Ω is an open and bounded domain in $\mathbb{R}^m$.*

The proof of the Sobolev Embedding Theorem can be found in Gilbarg and Trudinger [27, Thm. 7.10] for domains in $\mathbb{R}^m$ and in Aubin [4, Thm. 2.20] for compact Riemannian manifolds.

The next theorem is the Rellich–Kondrakov Theorem, which states under what conditions the embeddings given by Sobolev Embedding Theorem are compact.

Theorem 2.2 (**Rellich–Kondrakov Theorem**). *Let (M, g) be a compact m-dimensional Riemannian manifold. Let $k, l \in \mathbb{N}$, $p, q \in [1, \infty)$, and $\alpha \in (0, 1)$. Then the following hold.*

(i) *If $\frac{1}{p} < \frac{1}{q} + \frac{k-l}{m}$, then the inclusion of $W^{k,p}(M)$ into $W^{l,q}(M)$ is compact.*
(ii) *If $k - \frac{m}{p} > l + \alpha$, then the inclusion of $W^{k,p}(M)$ into $C^{l,\alpha}(M)$ is compact.*

Moreover (i) *and* (ii) *continue to hold when* $M = \Omega$, *where* Ω *is an open and bounded domain in* $\mathbb{R}^m$.

A proof of the Rellich–Kondrakov Theorem for domains in $\mathbb{R}^m$ can be found in Gilbarg and Trudinger [27, Thm. 7.22] and for compact Riemannian manifolds in Aubin [4, Thm. 2.34].

2.2 Parabolic Hölder and Sobolev spaces

In this subsection we introduce parabolic Hölder and Sobolev spaces. These are Hölder and Sobolev spaces on $(0, T) \times M$, $T > 0$, where one derivative in the time direction compares to two derivatives in the spatial directions. By the time direction of $(0, T) \times M$ we mean the first variable while the spatial directions are in M. The reason for introducing these spaces is the heat operator maps between parabolic Hölder and Sobolev spaces. Good references on parabolic Hölder and Sobolev spaces are Krylov [30, Ch. 2, §2] and [31, Ch. 8, §5].

We first define C^k-spaces, Hölder spaces, and Sobolev spaces of maps $u : I \to X$, where $I \subset \mathbb{R}$ is an open and bounded interval and X is a Banach space. For $k \in \mathbb{N}$ we define $C^k_{\mathrm{loc}}(I; X)$ to be the space of k-times continuously differentiable maps $u : I \to X$. We define the C^k-norm by

$$\|u\|_{C^k} = \sum_{j=0}^{k} \sup_{t \in I} \|\partial_t^j u(t)\|_X \quad \text{for } u \in C^k_{\mathrm{loc}}(I; X),$$

whenever it is finite, and we define

$$C^k(I; X) = \left\{ u \in C^k_{\mathrm{loc}}(I; X) \ : \ \|u\|_{C^k} < \infty \right\}.$$

Moreover for $\alpha \in (0, 1)$ we define the $C^{k,\alpha}$-norm by

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + \sup_{t \neq s \in I} \frac{\|\partial_t^k u(t) - \partial_t^k u(s)\|_X}{|t - s|^\alpha} \quad \text{for } u \in C^k_{\mathrm{loc}}(I; X),$$

whenever it is finite. By $C^{k,\alpha}_{\mathrm{loc}}(I; X)$ we denote the space of maps $u \in C^k_{\mathrm{loc}}(I; X)$ with finite $C^{k,\alpha}$-norm on every $J \subset\subset I$, and we define

$$C^{k,\alpha}(I; X) = \left\{ u \in C^{k,\alpha}_{\mathrm{loc}}(I; X) \ : \ \|u\|_{C^{k,\alpha}} < \infty \right\}.$$

Then $C^k(I; X)$ and $C^{k,\alpha}(I; X)$ are both Banach spaces.

Next we define Sobolev spaces of maps $u : I \to X$. Let $k \in \mathbb{N}$ and $p \in [1, \infty)$. For a k-times weakly differentiable map $u : I \to X$ we define the $W^{k,p}$-norm by

$$\|u\|_{W^{k,p}} = \left(\sum_{j=0}^{k} \int_I \|\partial_t^j u(t)\|_X^p \, \mathrm{d}t \right)^{1/p},$$

whenever it is finite. The notion of weak derivatives of maps $u : I \to X$ with values in a Banach space can be found in Amann [2, Ch. III, §1.1]. We denote by $W^{k,p}_{\mathrm{loc}}(I; X)$ the space of k-times weakly differentiable maps $u : I \to X$ with finite $W^{k,p}$-norm on every $J \subset\subset I$, and we define

$$W^{k,p}(I; X) = \left\{ u \in W^{k,p}_{\mathrm{loc}}(I; X) \ : \ \|u\|_{W^{k,p}} < \infty \right\}.$$

Then $W^{k,p}(I;X)$ is a Banach space. If $k=0$, then we write $L^p_{\mathrm{loc}}(I;X)$ and $L^p(I;X)$ instead of $W^{0,p}_{\mathrm{loc}}(I;X)$ and $W^{0,p}(I;X)$, respectively.

An important result in the theory of linear and nonlinear parabolic equations is the so called Aubin–Dubinskiĭ Lemma. We will use the Aubin–Dubinskiĭ Lemma below in order to prove embedding results for parabolic Sobolev spaces.

Lemma 2.3 (**Aubin–Dubinskiĭ Lemma**). *Let $I \subset \mathbb{R}$ be an open and bounded interval, X, Y, Z Banach spaces, and $p \in (1,\infty)$. Assume that X embeds continuously into Y by inclusion, that the inclusion is compact, and that Y embeds continuously into Z by inclusion. Then the inclusion of $L^p(I;X) \cap W^{1,p}(I;Z)$ into $L^p(I;Y)$ is compact.*

The proof of the Aubin–Dubinskiĭ Lemma can be found in J.-P. Aubin [3] for the case when X and Z are reflexive and Dubinskiĭ [17] without the reflexivity assumption.

The next proposition is an important interpolation result for maps into Banach spaces. We will apply this result below in order to prove an interpolation result for parabolic Sobolev spaces.

Proposition 2.4. *Let $I \subset \mathbb{R}$ be an open and bounded interval, X, Y Banach spaces, and $p \in (1,\infty)$. Assume that X embeds continuously into Y by inclusion. Then $L^p(I;X) \cap W^{1,p}(I;Y)$ embeds continuously into $C^0(I;(X,Y)_{1/p,p})$ by inclusion. Here $(\cdot,\cdot)_{\theta,p}$ denotes the real interpolation method, see Amann* [2, Ch. I, §2.4].

The proof of Proposition 2.4 can be found in Amann [2, Ch. III, Thm. 4.10.2].

Next we define parabolic C^k-spaces and parabolic Hölder spaces. Let (M,g) be a Riemannian manifold and $k, l \in \mathbb{N}$ with $2k \le l$. We define

$$C^{k,l}(I \times M) = \bigcap_{j=0}^{k} C^j(I; C^{l-2j}(M)).$$

Then $C^{k,l}(I \times M)$ is a Banach space with norm given by

$$\|u\|_{C^{k,l}} = \sum_{i,j} \sup_{(t,x)\in I\times M} |\partial_t^i \nabla^j u(t,x)| \quad \text{for } u \in C^{k,l}(I \times M),$$

where the sum is taken over $i = 1,\dots,k$ and $j = 1,\dots,l$ with $2i + j \le l$. If $\alpha \in (0,1)$, then we define the parabolic Hölder space $C^{k,l,\alpha}(I \times M)$ by

$$C^{k,l,\alpha}(I \times M) = \bigcap_{j=0}^{k} C^{j,\alpha/2}(I; C^{l-2j}(M)) \cap C^j(I; C^{l-2j,\alpha}(M)).$$

Then $C^{k,l,\alpha}(I \times M)$ is a Banach spaces with norm given by

$$\|u\|_{C^{k,l,\alpha}} = \sum_{i,j} \Big\{ \sup_{(t,x)\in I\times M} |\partial_t^i \nabla^j u(t,x)| + \sup_{x\in M} [\partial_t^i \nabla^j u(\cdot,x)]_{\alpha/2} + \sup_{t\in I} [\partial_t^i \nabla^j u(t,\cdot)]_{\alpha} \Big\} \quad \text{for } u \in C^{k,l,\alpha}(I \times M),$$

where the sum is taken over $i = 1,\dots,k$ and $j = 1,\dots,l$ with $2i + j \le l$. Thus a function $u : I \times M \to \mathbb{R}$ lies in $C^{k,l,\alpha}(I \times M)$ if and only if all derivatives of the form $\partial_t^i \nabla^j u$ with $i \leq k$, $j \leq l$, and $2i + j \leq l$ exist and are Hölder continuous in time with Hölder exponent $\alpha/2$ and Hölder continuous on M with Hölder exponent α.

Finally we define parabolic Sobolev spaces. Let $k, l \in \mathbb{N}$ with $2k \leq l$, and $p \in [1, \infty)$. Then we define the parabolic Sobolev space $W^{k,l,p}(I \times M)$ by

$$W^{k,l,p}(I \times M) = \bigcap_{j=0}^{k} W^{j,p}(I; W^{l-2j,p}(M)).$$

Then $W^{k,l,p}(I \times M)$ is a Banach space with norm given by

$$\|u\|_{W^{k,l,p}} = \left(\sum_{i,j} \int_I \int_M |\partial_t^i \nabla^j u(t, \cdot)|^p \, \mathrm{d}V_g \, \mathrm{d}t \right)^{1/p} \qquad \text{for } u \in W^{k,l,p}(I \times M),$$

where the sum is taken over $i = 1, \ldots, k$ and $j = 1, \ldots, l$ with $2i + j \leq l$. Thus $W^{k,l,p}(I \times M)$ is the space of functions $u : (0, T) \times M \to \mathbb{R}$, such that all weak derivatives of the form $\partial_t^i \nabla^j u$ with $i \leq k$, $j \leq l$, and $2i + j \leq l$ lie in $W^{0,0,p}(I \times M)$. Note that $W^{0,0,p}(I \times M) = L^p(I \times M)$.

When defining parabolic Hölder and Sobolev spaces we assumed that $2k \leq l$, where k is the number of time derivatives and l is the number of spatial derivatives. These spaces can also be defined for arbitrary $k, l \in \mathbb{N}$, so the restriction is not necessary. We feel, however, that this restriction makes the definition of the spaces simpler and in our later applications we are allowed to choose l arbitrarily large anyway.

As a consequence of the Rellich–Kondrakov Theorem and the Aubin–Dubinskiĭ Lemma we obtain the following important embedding result for parabolic Sobolev spaces.

Proposition 2.5. *Let (M, g) be a compact Riemannian manifold, $I \subset \mathbb{R}$ an open and bounded interval, $k \in \mathbb{N}$ with $k \geq 2$, and $p \in (1, \infty)$. Then the inclusion of $W^{1,k,p}(I \times M)$ into $W^{0,k-1,p}(I \times M)$ is compact. The same result continues to hold if $M = \Omega$ is an open and bounded domain in $\mathbb{R}^m$.*

The next proposition gives an interpolation result for parabolic Sobolev spaces.

Proposition 2.6. *Let (M, g) be a compact m-dimensional Riemannian manifold, $I \subset \mathbb{R}$ an open and bounded interval, $k \in \mathbb{N}$ with $k \geq 2$, and $p \in (1, \infty)$. If $kp > 2 + m$, then $W^{1,k,p}(I \times M)$ embeds continuously into $C^{0,0}(I \times M)$ by inclusion. The same result continues to hold if $M = \Omega$ is an open and bounded domain in $\mathbb{R}^m$.*

Proof. From Proposition 2.4 it follows that $W^{1,k,p}(I \times M)$ embeds continuously into $C^0(I; (W^{k,p}(M), W^{k-2,p}(M))_{1/p,p})$, where $(\cdot, \cdot)_{1/p,p}$ is the real interpolation method. It can be shown that $(W^{k,p}(M), W^{k-2,p}(M))_{1/p,p} = W^{s,p}(M)$ with $s = k - \frac{2}{p}$, see for instance Tartar [54, Ch. 34]. Here $W^{s,p}(M)$ is a Sobolev space of fractional order, see Adams [1, Ch. VII, 7.36] and Tartar [54, Ch. 34]. The Sobolev Embedding Theorem continues to hold for Sobolev spaces of fractional order [1, Ch. VII, Thm 7.57], so $W^{s,p}(M)$ embeds continuously into $C^0(M)$ by inclusion provided $s - \frac{m}{p} > 0$, from which the proposition follows. $\square$

2.3 Linear parabolic equations on domains in $\mathbb{R}^m$

In this subsection we discuss a standard interior regularity result for weak solutions of linear parabolic equations on domains in $\mathbb{R}^m$ and also Schauder and L^p-estimates for solutions of linear parabolic equations. Useful reference about linear parabolic equations on domains in $\mathbb{R}^m$ are Friedman [18], Ladyžhenskaja, Solonnikov, and Ural'ceva [32], and Krylov [30] and [31].

Our discussion of weak solutions to linear parabolic equations follows Ladyžhenskaja et al. [32, Ch. III, §1]. Let $\Omega \subset \mathbb{R}^m$ be an open and bounded domain and $T > 0$. Let $a^{ij}, b^i, c : (0,T) \times \Omega \to \mathbb{R}$ be continuous with $a^{ij} = a^{ji}$ for $i, j = 1, \ldots, m$, and define a linear differential operator L acting on functions $u \in C^{1,2}((0,T) \times \Omega)$ by

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x^i}\left(a^{ij}(t,x)\frac{\partial u}{\partial x^j}\right) - b^i(t,x)\frac{\partial u}{\partial x^i} - c(t,x)u. \tag{2}$$

The functions a^{ij}, b^i, and c are called the coefficients of L. We assume that (2) is parabolic. This means that there exists a constant $\lambda > 0$, such that $\lambda^{-1}|\xi|^2 \le a^{ij}(t,x)\xi_i\xi_j \le \lambda|\xi|^2$ for $(t,x) \in (0,T) \times \Omega$ and $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$.

Next we define the notion of a weak solution to a linear parabolic differential equation of second order.

Definition 2.7. *Let $f : (0,T) \times \Omega \to \mathbb{R}$. A weak solution of $Lu = f$ is a function $u \in L^2((0,T); W^{1,2}(\Omega)) \cap C^0((0,T); L^2(\Omega))$ that satisfies*

$$-\int_0^T\int_\Omega u\frac{\partial \varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_\Omega a^{ij}\frac{\partial u}{\partial x^j}\frac{\partial \varphi}{\partial x^i} + b^i\frac{\partial u}{\partial x^i} + cu\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega f\varphi\,\mathrm{d}x\,\mathrm{d}t$$

for every $\varphi \in W^{1,2}((0,T); L^2(\Omega)) \cap L^2((0,T); W^{1,2}(\Omega))$ that vanishes on $\partial\Omega \times (0,T)$ and $\{0,T\} \times \Omega$. In $Lu = f$ the function f is called the free term.

In the next theorem the Hölder continuity of weak solutions to linear parabolic equations of second order is established, provided the coefficients and the free term are Hölder continuous. The theorem can be found in Ladyžhenskaja et al. [32, III, Thm. 12.1].

Theorem 2.8. *Let u be a weak solution of $Lu = f$. Let $k, l \in \mathbb{N}$ with $2k \le l$ and $\alpha \in (0,1)$. If the coefficients of L and f lie in $C^{k,l,\alpha}((0,T) \times \Omega)$, then $u \in C^{k+1,l+2,\alpha}(I \times \Omega')$ for every $I \subset\subset (0,T)$ and $\Omega' \subset\subset \Omega$. In particular, if the coefficients of L and f are smooth, then every weak solution of $Lu = f$ is smooth.*

The importance of Theorem 2.8 lies in the following fact. Often when one studies existence of solutions to a nonlinear parabolic differential equation $P(u) = 0$ one can apriori only show that solutions with low regularity exist. For instance one may find a function $u \in W^{1,k,p}((0,T) \times \Omega)$ for some $k \in \mathbb{N}$ with $k \ge 2$ and $p \in (1, \infty)$ that satisfies $P(u) = 0$. If u has sufficiently much regularity, then one can differentiate the equation $P(u) = 0$ and deduce that the derivatives of u are weak solutions to a linear parabolic differential equations with Hölder continuous coefficients and free term. But then Theorem 2.8 implies that the derivatives of u are Hölder continuous, so u itself is Hölder continuous as well. Often one is able to iterate this argument and to show that a solution to a nonlinear parabolic differential equation that has apriori only low regularity is in fact smooth. This procedure is often referred to as bootstrapping.

From now on let us assume that the coefficients of L are smooth on $(0,T)\times\Omega$. Then we have the following Schauder estimates for solutions of $Lu = f$.

Theorem 2.9. *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\alpha \in (0,1)$. Let $u \in C^{1,k,\alpha}((0,T)\times\Omega)$, $f \in C^{0,k-2,\alpha}((0,T)\times\Omega)$, and assume that $Lu = f$. Then for every $\Omega' \subset\subset \Omega$ there exists a constant $c > 0$ depending only on $\Omega', k, \alpha, \lambda$, and the $C^{0,k}$-norm of the coefficients of L on $(0,T)\times\Omega$, such that*

$$\|u\|_{C^{1,k,\alpha}} \leq c\left(\|f\|_{C^{0,k-2,\alpha}} + \|u\|_{C^{0,0}}\right),$$

where the norm on the left side is on $(0,T)\times\Omega'$ and the norm on the right side is on $(0,T)\times\Omega$.

The proof of the Schauder estimates can be found in Friedman [18, Ch. 3, §2] for instance.

Finally we state the L^p-estimates for second order linear parabolic equations, which can be found in Krylov [30, Ch. 5, §2, Thm. 5].

Theorem 2.10. *Let $k \in \mathbb{N}$ with $k \geq 2$ and $p \in (1,\infty)$. Let $u \in W^{1,2,p}((0,T)\times\Omega)$, $f \in W^{0,k-2,p}((0,T)\times\Omega)$, and assume that $Lu = f$. Then $u \in W^{1,k,p}((0,T)\times\Omega')$ for every $\Omega' \subset\subset \Omega$. Moreover for every $\Omega' \subset\subset \Omega$ there exists a constant $c > 0$ depending only on Ω', k, p, λ, and the $C^{0,k}$-norm of the coefficients of L on $(0,T)\times\Omega$, such that*

$$\|u\|_{W^{1,k,p}} \leq c\left(\|f\|_{W^{0,k-2,p}} + \|u\|_{W^{0,0,p}}\right),$$

where the norm on the left side is on $(0,T)\times\Omega'$ and the norm on the right side is on $(0,T)\times\Omega$.

3 The heat equation on compact Riemannian manifolds

3.1 The Friedrichs heat kernel on Riemannian manifolds

In this subsection we study the Friedrichs heat kernel on arbitrary Riemannian manifolds. The existence of the Friedrichs heat kernel follows from the spectral theorem for self-adjoint operators. Further we discuss the parametrix construction of Minakshisundaram and Pleijel [40] for the Friedrichs heat kernel. The construction of the Friedrichs heat kernel can be found in Davies [15, Ch. 5, §2]. This construction involves some advanced techniques from functional analysis including the Friedrichs extension, the spectral theorem for self-adjoint operators, and the functional calculus for self-adjoint operators, which can be found in Yosida [62, Ch. XI].

Let (M, g) be a Riemannian manifold and consider the Laplace operator acting as an unbounded operator

$$\Delta_g : C^\infty_{\mathrm{cs}}(M) \subset L^2(M) \to L^2(M), \tag{3}$$

where $C^\infty_{\mathrm{cs}}(M)$ is the space of smooth functions on M with compact support. Then (3) is symmetric and nonpositive, i.e. $\langle \Delta_g u, v\rangle_{L^2} = \langle u, \Delta_g v\rangle_{L^2}$ and $\langle \Delta_g u, u\rangle_{L^2} \leq 0$ for $u, v \in C^\infty_{\mathrm{cs}}(M)$. By Friedrichs' theorem [62, Ch. XI, §7, Thm. 2] there exists a closed and self-adjoint extension

$$\Delta_g : \mathrm{dom}(\Delta_g) \subset L^2(M) \to L^2(M), \tag{4}$$

called the Friedrichs extension. Since (4) is self-adjoint, the spectral theorem for self-adjoint operators [62, Ch. XI, §6, Thm. 1] shows that there exists a unique resolution of the identity $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ such that

$$\Delta_g = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E_\lambda. \tag{5}$$

Using the functional calculus for self-adjoint operators [62, Ch XI, §12] we can then define the Friedrichs heat semigroup $\{\exp(t\Delta_g)\}_{t>0}$ by

$$\exp(t\Delta_g) = \int_{-\infty}^{\infty} \exp(t\lambda) \, \mathrm{d}E_\lambda. \tag{6}$$

Then $\{\exp(t\Delta_g)\}_{t>0}$ is a semigroup of bounded operators on $L^2(M)$ that maps

$$\exp(t\Delta_g) : L^2(M) \to \bigcap_{j=0}^{\infty} \mathrm{dom}(\Delta_g^j) \tag{7}$$

for every $t > 0$. Moreover, if $\varphi \in L^2(M)$, then $u(t, \cdot) = \exp(t\Delta_g)\varphi$ is the unique solution to the Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(t, x) = \Delta_g u(t, x) && \text{for } (t, x) \in (0, T) \times M, \\ &u(0, x) = \varphi(x) && \text{for } x \in M \end{aligned}$$

with $u(t, \cdot) \in \mathrm{dom}(\Delta_g)$ for $t > 0$.

The next proposition shows that the action of the Friedrichs heat semigroup on $L^2(M)$ is given by an integral operator with a positive and symmetric integral kernel.

Proposition 3.1. *Let (M,g) be a Riemannian manifold and $\{\exp(t\Delta_g)\}_{t>0}$ the Friedrichs heat semigroup on (M,g). Then there exists a positive function $H \in C^\infty((0,\infty)\times M \times M)$, which is symmetric on $M \times M$, such that for every $\varphi \in L^2(M)$*

$$(\exp(t\Delta_g)\varphi)(x) = \int_M H(t,x,y)\varphi(y)\,\mathrm{d}V_g(y). \tag{8}$$

The function H is called the Friedrichs heat kernel on (M,g). In particular H satisfies

$$\begin{aligned} &\frac{\partial H}{\partial t}(t,x,y) = \Delta H(t,x,y) && \text{for } (t,x,y) \in (0,\infty)\times M \times M,\\ &H(0,x,y) = \delta_x(y) && \text{for } x,y \in M, \end{aligned}$$

where $\delta_x(y)$ is the delta distribution on (M,g).

The proof of Proposition 3.1 can be found in Cheeger and Yau [12, §1] and also in Davies [15, Thm 5.2.1].

On $\mathbb{R}^m$ an explicit formula for the heat kernel is known. In fact the Euclidean heat kernel is given by

$$H(t,x,y) = \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{9}$$

This formula can be easily derived by solving the Cauchy problem

$$\begin{aligned} &\frac{\partial H}{\partial t}(t,x,y) = \Delta H(t,x,y) && \text{for } (t,x) \in (0,\infty)\times \mathbb{R}^m,\\ &H(0,x,y) = \delta(x-y) && \text{for } x \in \mathbb{R}^m \end{aligned}$$

and fixed $y \in \mathbb{R}^m$ using the Fourier transform. Now if (M,g) is an m-dimensional Riemannian manifold and $x \in M$, then we can choose normal coordinates $(x_1,\ldots,x_m)$ at $x \in M$. Then the Riemannian metric g at x is the Euclidean metric on $\mathbb{R}^m$ and the Laplace operator Δ_g at x is the Laplace operator on $\mathbb{R}^m$. This suggests that (9) is at least locally a good approximation for the heat kernel on M. In fact, this is the statement of the theorem of Minakshisundaram and Pleijel [40, §1].

Theorem 3.2. *Let (M,g) be an m-dimensional Riemannian manifold and let H be the Friedrichs heat kernel on (M,g). Then near the diagonal in $M\times M$, H has an asymptotic expansion as $t \to 0$ of the form*

$$H(t,x,y) \sim \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{d_g(x,y)^2}{4t}\right) \sum_{j=0}^{\infty} a_j(x,y)t^j, \tag{10}$$

where $a_j \in C^\infty(M\times M)$ for $j \in \mathbb{N}$ and $a_0(x,x) = 1$ for $x \in M$.

A detailed discussion of Theorem 3.2 can also be found in Berger, Gauduchon, and Mazet [9, Ch. III, §E].

3.2 The Cauchy problem for the inhomogeneous heat equation

In this subsection we study the existence and regularity of solutions to the Cauchy problem for the inhomogeneous heat equation on compact Riemannian manifolds, i.e. for a given function $f : (0,T) \times M \to \mathbb{R}$ we search for a function $u : (0,T) \times M \to \mathbb{R}$ that extends continuously to $t = 0$ and satisfies

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \Delta_g u(t,x) + f(t,x) \quad && \text{for } (t,x) \in (0,T) \times M, \\ u(0,x) &= 0 && \text{for } x \in M. \end{aligned} \tag{11}$$

Here $T > 0$, (M, g) is a compact Riemannian manifold, and the function f is an element of a parabolic Hölder or Sobolev space. The results below follow from Theorem 3.2 and the standard regularity theory for the heat equation on $\mathbb{R}^m$ from Ladyženskaja et al. [32, IV,§1-§3]. Some results about linear parabolic equations on compact Riemannian manifolds can also be found in Aubin [4, Ch. 4, §4.2].

The following theorem is the main result about the existence and regularity of solutions to the Cauchy problem (11) in parabolic Hölder spaces.

Theorem 3.3. *Let (M, g) be a compact Riemannian manifold, $T > 0$, $k \in \mathbb{N}$ with $k \geq 2$, and $\alpha \in (0,1)$. Given $f \in C^{0,k-2,\alpha}((0,T) \times M)$, there exists a unique $u \in C^{1,k,\alpha}((0,T) \times M)$ solving the Cauchy problem* (11).

We only sketch the proof of Theorem 3.3, which is proved using Theorem 3.2 and the regularity theory for the heat equation on $\mathbb{R}^m$. Let H be the Friedrichs heat kernel on (M, g). For a given function $f : (0,T) \times M \to \mathbb{R}$ we define the convolution $H * f : (0,T) \times M \to \mathbb{R}$ of H and f by

$$(H * f)(t,x) = \int_0^t \int_M H(t-s,x,y) f(s,y) \, \mathrm{d}V_g(y) \, \mathrm{d}s, \tag{12}$$

whenever it is well defined. Denote $u = H * f$. Since H is a solution of the heat equation with initial condition the delta distribution, we have at least formally

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \left(\frac{\partial H}{\partial t} * f \right)(t,x) + \int_M H(0,x,y) f(t,y) \, \mathrm{d}V_g(y) \\ &= (\Delta_g H * f)(t,x) + f(t,x) = \Delta_g u(t,x) + f(t,x) \end{aligned} \tag{13}$$

for $(t,x) \in (0,T) \times M$. Moreover $u(0,x) = 0$ for $x \in M$, so at least formally u is a solution of the Cauchy problem (11). The problem is now to show that the computation in (13) is rigorous for certain functions f and that u possesses certain regularity. Let us consider the case when $f \in C^{0,0,\alpha}((0,T) \times M)$. Then we want to show that $u \in C^{1,2,\alpha}((0,T) \times M)$. Using Theorem 3.2 it is straightforward to check that $u \in C^{0,0}((0,T) \times M)$. Now consider the expression $\partial_t u$. We would like to switch differentiation and integration and we would like to write $\partial_t u = (\partial_t H) * f$. This, however, is not possible in general, since $\partial_t H$ is not locally integrable as we can see from Theorem 3.2. The trick to compute $\partial_t u$ is to use the Hölder continuity of f. Using Theorem 3.2 and the Hölder continuity of f it follows that $\partial_t H(t-s,x,y)(f(s,y) - f(s,x))$ is locally integrable. Moreover, since H is a solution to the heat equation, we have

$$\int_M \frac{\partial H}{\partial t}(t,x,y) \, \mathrm{d}V_g(y) = \int_M \Delta_g H(t,x,y) \, \mathrm{d}V_g(y) = 0$$

and thus we can write

$$\frac{\partial u}{\partial t}(t,x) = \int_0^t \int_M \frac{\partial H}{\partial t}(t-s,x,y)(f(s,y)-f(s,x))\,\mathrm{d}V_g(y)\,\mathrm{d}s + f(t,x).$$

It follows that u is once continuously differentiable in the time direction. In a similar way one can show that u is twice continuously differentiable in the spatial directions and hence $u \in C^{1,2}((0,T)\times M)$. In particular the formal computation in (13) is rigorous in this case and u is a solution of the Cauchy problem (11). It is straightforward to estimate the $C^{1,2,\alpha}$-norm of u in terms of H and the $C^{0,0,\alpha}$-norm of f, so that in fact $u \in C^{1,2,\alpha}((0,T)\times M)$.

The next theorem is an analogue of Theorem 3.3 for the case, when f lies in a parabolic Sobolev space.

Theorem 3.4. *Let (M,g) be a compact Riemannian manifold, $T>0$, $k\in\mathbb{N}$ with $k\geq 2$, and $p\in(1,\infty)$. Given $f\in W^{0,k-2,p}((0,T)\times M)$, there exists a unique $u\in W^{1,k,p}((0,T)\times M)$ solving the Cauchy problem* (11).

Theorem 3.4 is proved in a similar way as Theorem 3.3. For more details on the regularity theory for $u = H * f$ we refer the interested reader to Ladyženskaja et al. [32, IV,§1-§3].

For later applications we want to rephrase Theorem 3.4. Let (M,g) be a compact Riemannian manifold and let T,k,l, and p be as in Theorem 3.4. Denote

$$\tilde{W}^{1,k,p}((0,T)\times M) = \left\{u\in W^{1,k,p}((0,T)\times M) \ : \ u(0,\cdot)=0 \text{ on } M\right\}.$$

Note that if $u\in W^{1,k,p}((0,T)\times M)$, then u is uniformly Hölder continuous on $(0,T)$ by the Sobolev Embedding Theorem, and hence u extends continuously to $t=0$. In particular $u(0,\cdot): M\to\mathbb{R}$ is well defined. Then the statement of Theorem 3.4 is that

$$\frac{\partial}{\partial t} - \Delta_g : \tilde{W}^{1,k,p}((0,T)\times M) \longrightarrow W^{0,k-2,p}((0,T)\times M) \tag{14}$$

is a bijection. In particular the Open Mapping Theorem [33, XV, Thm. 1.3] implies that (14) is an isomorphism of Banach spaces. Theorem 3.3 can be rephrased in a similar way.

4 Background from Riemannian and symplectic geometry

4.1 Submanifolds in Riemannian manifolds

In this subsection we recall some standard notions from Riemannian submanifold geometry and also introduce the notion of calibrated submanifolds. For more on Riemannian submanifolds we refer the interested reader to Kobayashi and Nomizu [28, Ch. I, §1] and for an introduction to calibrated submanifolds to Harvey and Lawson [19] and Joyce [23, Ch. 4].

Let (M,g) be an m-dimensional Riemannian manifold and N a manifold of dimension n with $n \leq m$. An embedding of N into M is an injective C^1-map $F : N \to M$, such that the differential $\mathrm{d}F(x) : T_xN \to T_{F(x)}M$ is injective for every $x \in N$. The image $F(N)$ of an embedding $F : N \to M$ is then an n-dimensional C^1-submanifold of M. We say $F : N \to M$ is a C^k-embedding if $F : N \to M$ is an embedding and a C^k-map from N into M, and we say that $F : N \to M$ is a smooth embedding, if $F : N \to M$ is an embedding and a smooth map from N into M. If $F : N \to M$ is a C^k-embedding, then $F(N)$ is a C^k-submanifold of M. If $F : N \to M$ is a smooth embedding, then $F(N)$ is a smooth submanifold of M. Often we will refer to the embedding $F : N \to M$ as a submanifold of M.

A submanifold $F : N \to M$ defines an orthogonal decomposition of the vector bundle $F^*(TM)$ into $\mathrm{d}F(TN) \oplus \nu N$. The vector bundle νN over N is the normal bundle of $F : N \to M$. By $\pi_{\nu N}$ we will denote the orthogonal projection $F^*(TM) \to \nu N$ onto the normal bundle of $F : N \to M$.

The second fundamental form of a C^2-submanifold $F : N \to M$ is a section of the vector bundle $\odot^2 T^*N \otimes \nu N$ defined by $\mathrm{II}(X,Y) = \pi_{\nu N}(\nabla_{\mathrm{d}F(X)}\mathrm{d}F(Y))$ for $X, Y \in TN$. Here ∇ is the Levi-Civita connection of g. The mean curvature vector field of $F : N \to M$ is a section of νN defined by $H = \mathrm{tr}\ \mathrm{II}$, where the trace is taken with respect to the Riemannian metric $F^*(g)$ on N. A C^2-submanifold $F : N \to M$ is a minimal submanifold if the mean curvature vector field is zero. It can be shown that a compact C^2-submanifold $F : N \to M$ is minimal if and only if it is a critical point of the volume functional.

Next we define calibrated submanifolds. These are a special class of minimal submanifolds, which were introduced by Harvey and Lawson [19].

Definition 4.1. *Let (M,g) be a Riemannian manifold and let φ be a smooth and closed n-form on M. Then φ is a calibration if for every $x \in M$ and every oriented subspace $V \subset T_xM$ with $\dim V = n$ we have $\varphi(x)|_V \leq \mathrm{d}V_{g(x)|_V}$. Here $g(x)|_V$ is the Riemannian metric in T_xM restricted to the subspace V, and $\mathrm{d}V_{g(x)|_V}$ is defined using the orientation on V.*

If φ is a calibration on M, then an oriented n-dimensional submanifold $F : N \to M$ is calibrated with respect to φ if $F^*(\varphi) = \mathrm{d}V_{F^*(g)}$.

It is not difficult to show that compact calibrated submanifolds minimize volume among all submanifolds in their homology class, so that calibrated submanifolds are minimal submanifolds, see for instance [23, Prop. 4.1.4]. We will be interested in a particular class of calibrated submanifolds called special Lagrangian submanifolds, which we define in §4.3.

4.2 Symplectic manifolds and Lagrangian submanifolds

In this subsection we recall some basic definitions from symplectic geometry. A standard reference for symplectic geometry is McDuff and Salamon [36].

We begin with the definition of symplectic manifolds and Lagrangian submanifolds.

Definition 4.2. *A $2m$-dimensional symplectic manifold is a pair (M,ω), where M is a $2m$-dimensional manifold and ω is closed and non-degenerate two-form on M, i.e. $\mathrm{d}\omega = 0$ and $\omega^m(x) \neq 0$ for every $x \in M$. Let L be an m-dimensional manifold. A submanifold $F : L \to M$ of a symplectic manifold (M,ω) is a Lagrangian submanifold if $F^*(\omega) = 0$.*

The most elementary example of a symplectic manifold is $(\mathbb{C}^m, \omega')$, where $\omega' = \sum_{j=1}^m \mathrm{d}x_j \wedge \mathrm{d}y_j$, and $(x_1, \dots, y_m)$ are the usual real coordinates on $\mathbb{C}^m$. Denote by B_R the open ball of radius $R > 0$ about the origin in $\mathbb{C}^m$. Then in fact every symplectic manifold is locally isomorphic to (B_R, ω') for some small $R > 0$. This is the statement of Darboux' Theorem [36, Thm 3.15].

Theorem 4.3. *Let (M,ω) be a $2m$-dimensional symplectic manifold, $x \in M$, and let $A : \mathbb{C}^m \to T_xM$ be an isomorphism with $A^*(\omega) = \omega'$. Then there exists $R > 0$ and a smooth embedding $\Upsilon : B_R \to M$, such that $\Upsilon^*(\omega) = \omega'$, $\Upsilon(0) = x$, and $\mathrm{d}\Upsilon(0) = A$.*

An important example of a symplectic manifold is the cotangent bundle of a manifold. If M is an m-dimensional manifold, then the cotangent bundle T^*M of M is a $2m$-dimensional manifold and it has a canonical symplectic structure $\hat{\omega}$ defined as follows. Denote by $\pi : T^*M \to M$ the canonical projection and let $\hat{\lambda}$ be the one-form on T^*M defined by $\hat{\lambda}(\beta) = (\mathrm{d}\pi)^*(\beta)$ for $\beta \in T^*M$. Set $\hat{\omega} = -\mathrm{d}\hat{\lambda}$, then $\hat{\omega}$ is a symplectic structure on T^*M. For computations it is convenient to have an alternative description of $\hat{\omega}$ in local coordinates on T^*M. Let $(x_1, \dots, x_m)$ be local coordinates on M and extend these to local coordinates $(x_1, \dots, x_m, y_1, \dots, y_m)$ on T^*M, such that $(x_1, \dots, y_m)$ represents the one-form $y_1 \mathrm{d}x_1 + \cdots + y_m \mathrm{d}x_m$ in T_x^*M, where $x = (x_1, \dots, x_m)$. Then one can show that $\hat{\omega} = \sum_{j=1}^m \mathrm{d}x_j \wedge \mathrm{d}y_j$.

Next we define the notion of a Lagrangian neighbourhood of a Lagrangian submanifold in a symplectic manifold.

Definition 4.4. *Let (M,ω) be a symplectic manifold and $F : L \to M$ a Lagrangian submanifold of M. A Lagrangian neighbourhood for $F : L \to M$ is an embedding $\Phi_L : U_L \to M$ of an open neighbourhood U_L of the zero section in T^*L onto an open neighbourhood of $F(L)$ in M, such that $\Phi_L^*(\omega) = \hat{\omega}$ and $\Phi_L(x, 0) = F(x)$ for $x \in L$.*

Our later study of the generalized Lagrangian mean curvature flow of compact Lagrangian submanifolds is based on the existence of a Lagrangian neighbourhood for compact Lagrangian submanifolds, which is established in the next theorem.

Theorem 4.5 (Lagrangian Neighbourhood Theorem). *Let (M,ω) be a symplectic manifold and $F : L \to M$ a Lagrangian submanifold with L compact. Then there exists a Lagrangian neighbourhood $\Phi_L : U_L \to M$ for $F : L \to M$.*

A proof of the Lagrangian Neighbourhood Theorem for compact Lagrangian submanifolds can be found in McDuff and Salamon [36, Thm. 3.32].

4.3 Almost Calabi–Yau manifolds and Lagrangian submanifolds

In this subsection we introduce almost Calabi–Yau manifolds, which are the ambient spaces for the generalized Lagrangian mean curvature flow. We also introduce the generalized mean curvature vector field and discuss Lagrangian and special Lagrangian submanifolds in almost Calabi–Yau manifolds. The general theory about almost Calabi–Yau manifolds and special Lagrangian submanifolds in almost Calabi–Yau manifolds can be found in Joyce [23, Ch. 7 and 8]. In the discussion of the generalized mean curvature vector field we follow the author's paper [8], where also some additional material can be found.

We begin with the definition of almost Calabi–Yau manifolds following Joyce [23, Def. 8.4.3].

Definition 4.6. *An m-dimensional almost Calabi–Yau manifold is a quadruple (M, J, ω, Ω), where (M, J) is an m-dimensional complex manifold, ω is the Kähler form of a Kähler metric g on M, and Ω is a holomorphic volume form on M.*

Let (M, J, ω, Ω) be an m-dimensional almost Calabi–Yau manifold. The Ricci-form is the complex $(1,1)$-form given by $\rho(X, Y) = \mathrm{Ric}(JX, Y)$ for $X, Y \in TM$, where Ric is the Ricci-tensor of g. We define a function $\psi \in C^\infty(M)$ by

$$e^{2m\psi}\frac{\omega^m}{m!} = (-1)^{\frac{m(m-1)}{2}}\left(\frac{i}{2}\right)^m \Omega \wedge \bar{\Omega}. \qquad (15)$$

Then $|\Omega| = 2^{m/2}e^{m\psi}$, so that Ω is parallel if and only if ψ is constant. One can show that the Ricci-form of an almost Calabi–Yau manifold satisfies $\rho = dd^c \log|\Omega|$, see for instance Kobayashi and Nomizu [29, Ch. IX, §5]. Thus we find $\rho = m dd^c\psi$ and it follows that g is Ricci-flat if and only if ψ is constant. If $\psi \equiv 0$, then (M, J, ω, Ω) is a Calabi–Yau manifold [23, Ch. 8, §4].

Our motivation to work with almost Calabi–Yau manifolds and not only with Calabi–Yau manifolds is the following. The first nice feature of almost Calabi–Yau manifolds is that explicit almost Calabi–Yau metrics on compact manifolds are known, while there are no non-trivial Calabi–Yau metrics on compact manifolds explicitly known. For instance a quintic in $\mathbb{CP}^4$ equipped with the restriction of the Fubini–Study metric is an almost Calabi–Yau manifold. An even more important property of compact almost Calabi–Yau manifolds is that they appear in infinite dimensional families, while compact Calabi–Yau manifolds only appear in finite dimensional families due to the theorem of Tian [56] and Todorov [57], and Yau's proof of the Calabi conjecture [61]. In fact, recall that by the theorem of Tian and Todorov the moduli space $\mathcal{M}_{CY}$ of Calabi–Yau structures on a compact Calabi–Yau manifold is of dimension $h^{1,1}(M) + 2h^{n-1,1}(M) + 1$, where $h^{i,j}(M)$ are the Hodge numbers of M. In particular $\mathcal{M}_{CY}$ is finite dimensional. In the study of moduli spaces of J-holomorphic curves in symplectic manifolds it turns out that for a generic almost complex structure J the moduli space $\mathcal{M}_J$ of embedded J-holomorphic curves is a smooth manifold, while for a fixed almost complex structure J the space $\mathcal{M}_J$ can have singularities, see McDuff and Salamon [37] for details. The moduli space $\mathcal{M}_{ACY}$ of almost Calabi–Yau metrics is infinite dimensional and choosing a generic almost Calabi–Yau metric is therefore a more powerful thing to do than choosing a generic Calabi–Yau metric. We explain why this is of certain interest. It was proved by McLean [38] that if $F : L \to M$ is a special Lagrangian submanifold in a Calabi–Yau manifold, then the moduli space of compact special Lagrangian submanifolds $\mathcal{M}_{SL}$ is a smooth manifold of dimension $b^1(L)$, the first Betti number of L (see Definition 4.9 below for the notion of special Lagrangian submanifolds). An important question is whether it is possible to compactify $\mathcal{M}_{SL}$ in order to define invariants of Calabi–Yau manifolds by counting special Lagrangian submanifolds. One approach to this problem, due to Joyce, is to study the moduli space of special Lagrangian submanifolds with conical singularities in almost Calabi–Yau manifolds, see [26] for a survey of his results. In particular Joyce conjectures that for generic almost Calabi–Yau metrics the moduli space of special Lagrangian submanifolds with conical singularities is a smooth finite dimensional manifold.

The most important example of an (almost) Calabi–Yau manifold is $\mathbb{C}^m$ with its standard structure. Denote by $(x_1, \dots, x_m, y_1, \dots, y_m)$ the usual real coordinates on $\mathbb{C}^m$. We define a complex structure J', a non-degenerate two form ω', and a holomorphic volume form Ω' on $\mathbb{C}^m$ by

$$J'\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \quad \text{and } J'\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j} \quad \text{for } j = 1, \dots, m,$$
$$\omega' = \sum_{j=1}^{m} \mathrm{d}x_j \wedge \mathrm{d}y_j, \quad \Omega' = (\mathrm{d}x_1 + i\mathrm{d}y_1) \wedge \cdots \wedge (\mathrm{d}x_m + i\mathrm{d}y_m).$$

Then $(\mathbb{C}^m, J', \omega', \Omega')$ is an (almost) Calabi–Yau manifold and the corresponding Riemannian metric is the Euclidean metric $g' = \mathrm{d}x_1^2 + \cdots + \mathrm{d}y_m^2$.

Next we discuss Lagrangian submanifolds in almost Calabi–Yau manifolds. Thus let (M, J, ω, Ω) be an m-dimensional almost Calabi–Yau manifold and $F : L \to M$ a Lagrangian submanifold. We define a section α of the vector bundle $\mathrm{Hom}(\nu L, T^*L)$ by

$$\alpha(\xi) = \alpha_\xi = F^*(\xi \lrcorner\, \omega) \quad \text{for } \xi \in \nu L. \tag{16}$$

Since $F : L \to M$ is Lagrangian, α is an isomorphism in each fibre over L. Moreover, $\alpha^{-1}(\mathrm{d}u) = -J(\mathrm{d}F(\nabla u))$ for every $u \in C^1(L)$.

Assume that $F : L \to M$ is a C^2-Lagrangian submanifold and let H be the mean curvature vector field of $F : L \to M$. The one-form $\alpha_H = F^*(H \lrcorner\, \omega)$ on L is the mean curvature form of $F : L \to M$. It is true that $\mathrm{d}\alpha_H = F^*(\rho)$, where ρ is the Ricci-form, as first observed by Dazord [16]. Assume for the moment that (M, J, ω, Ω) is Calabi–Yau. Then $\rho \equiv 0$, as g is Ricci-flat. In particular α_H is closed and it follows from Cartan's formula that

$$F^*(\mathcal{L}_H \omega) = F^*(\mathrm{d}(H \lrcorner\, \omega)) = \mathrm{d}\alpha_H = 0.$$

Thus, if (M, J, ω, Ω) is Calabi–Yau, then the deformation of a Lagrangian submanifold in direction of the mean curvature vector field is an infinitesimal symplectic motion. Now if (M, J, ω, Ω) is an almost Calabi–Yau manifold, then the Ricci-form is given by $\rho = m\mathrm{d}\mathrm{d}^c\psi$. In particular $F^*(\mathcal{L}_H \omega) = mF^*(\mathrm{d}\mathrm{d}^c\psi)$ is nonzero in general. We therefore seek for a generalization of the mean curvature vector field with the property that the deformation of a Lagrangian submanifold in its direction is an infinitesimal symplectic motion. This leads to the definition of the generalized mean curvature vector field, which was introduced by the author in [8, §3].

Definition 4.7. *The generalized mean curvature vector field of $F : L \to M$ is the normal vector field $K = H - m\pi_{\nu L}(\nabla\psi)$, where H denotes the mean curvature vector field of $F : L \to M$. The one-form $\alpha_K = F^*(K \lrcorner\, \omega)$ is the generalized mean curvature form of $F : L \to M$.*

Note that if ψ is constant, then $K \equiv H$. Further observe that if $F : L \to M$ is Lagrangian, then we have

$$\begin{aligned} F^*(\mathcal{L}_K\omega) &= mF^*(\mathrm{dd}^c\psi) - mF^*(\mathrm{d}(\pi_{\nu L}(\nabla u)) \lrcorner\, \omega) \\ &= mF^*(\mathrm{dd}^c\psi) + mF^*(\mathrm{d}(\mathrm{d}\psi \circ J)) = 0. \end{aligned}$$

Thus if $F : L \to M$ is a Lagrangian submanifold in an almost Calabi–Yau manifold, then the deformation of $F : L \to M$ in direction of the generalized mean curvature vector field is an infinitesimal symplectic motion.

Next we define the Lagrangian angle of a Lagrangian submanifold. When $F : L \to M$ is a Lagrangian submanifold, then the Lagrangian angle is the map $\theta(F) : L \to \mathbb{R}/\pi\mathbb{Z}$ defined by

$$F^*(\Omega) = e^{i\theta(F)+mF^*(\psi)}\mathrm{d}V_{F^*(g)}. \tag{17}$$

Since $F : L \to M$ is a Lagrangian submanifold, $\theta(F)$ is in fact well defined, see for instance Harvey and Lawson [19, III.1]. In general $\theta(F) : L \to \mathbb{R}/\pi\mathbb{Z}$ cannot be lifted to a continuous function $\theta(F) : L \to \mathbb{R}$. However, $\mathrm{d}[\theta(F)]$ is a well defined closed one form on L, so it represents a cohomology class $\mu_F \in H^1(L, \mathbb{R})$ in the first de Rham cohomology group of L. Thus if $\mu_F = 0$, then $\theta(F) : L \to \mathbb{R}/\pi\mathbb{Z}$ can be lifted to a continuous function $\theta(F) : L \to \mathbb{R}$ and vice versa. The cohomology class μ_F is called the Maslov class of $F : L \to M$.

We now prove an important relation between the generalized mean curvature form of a Lagrangian submanifold $F : L \to M$ and the Lagrangian angle.

Proposition 4.8. *Let $F : L \to M$ be a Lagrangian submanifold in an almost Calabi–Yau manifold. Then the generalized mean curvature form of $F : L \to M$ satisfies $\alpha_K = -\mathrm{d}[\theta(F)]$.*

Proof. For every complex manifold (M, J) we have a natural decomposition of the bundle of complex m-forms given by

$$\Lambda^m T^*M \otimes \mathbb{C} = \bigoplus_{p+q=m} \Lambda^{p,q}T^*M. \tag{18}$$

See for instance [23, Ch. 5, §2] for a description of this decomposition. Since g is a Kähler metric, the complex structure J is parallel and therefore the decomposition (18) is invariant under the holonomy representation of g. Hence there exists a complex one-form η on M satisfying $\nabla\Omega = \eta \otimes \Omega$. Moreover, since Ω is holomorphic, η is in fact a one-form of type $(1, 0)$. Using $\Omega \wedge \bar{\Omega} = e^{2m\psi}\mathrm{d}V_g$ we find by computing $\nabla\left(\Omega \wedge \bar{\Omega}\right)$ that

$$(\eta + \bar{\eta}) \otimes \Omega \wedge \bar{\Omega} = 2m\mathrm{d}\psi \otimes \Omega \wedge \bar{\Omega}.$$

It follows that $\eta = 2m\partial\psi$ and thus $\nabla\Omega = 2m\partial\psi \otimes \Omega$. Following the computation of Thomas and Yau [55, Lem. 2.1] we find that

$$\nabla\Omega = (i\mathrm{d}[\theta(F)] + m\mathrm{d}\psi + i\alpha_H) \otimes \Omega$$

and hence $\alpha_H - m\mathrm{d}^c\psi = -\mathrm{d}[\theta(F)]$. By definition of the generalized mean curvature vector field $\alpha_H - m\mathrm{d}^c\psi = \alpha_K$, and hence it follows that $\alpha_K = -\mathrm{d}[\theta(F)]$ as we wanted to show. $\square$

Notice that as a consequence of Proposition 4.8, if $F : L \to M$ is a Lagrangian submanifold with zero Maslov class, then α_K is an exact one form and the deformation of $F : L \to M$ in direction of the generalized mean curvature vector field is an infinitesimal Hamiltonian motion.

Next we define a special class of Lagrangian submanifolds in almost Calabi–Yau manifolds called special Lagrangian submanifolds.

Definition 4.9. *Let $F : L \to M$ be a Lagrangian submanifold in an almost Calabi–Yau manifold (M, J, ω, Ω). Then $F : L \to M$ is a special Lagrangian submanifold with phase $e^{i\theta}$, $\theta \in \mathbb{R}$, if and only if*

$$F^*(\cos\theta \,\mathrm{Im}\,\Omega - \sin\theta \,\mathrm{Re}\,\Omega) = 0.$$

If $F : L \to M$ is a special Lagrangian submanifold with phase $e^{i\theta}$, then there is a unique orientation on L in which $F^*(\cos\theta \,\mathrm{Re}\,\Omega + \sin\theta \,\mathrm{Im}\,\Omega)$ is positive.

Note that a special Lagrangian submanifold $F : L \to M$ has zero Maslov-class, since $\theta(F)$ is constant on L and $\mathrm{d}[\theta(F)]$ represents μ_F by Proposition 4.8.

Definition 4.9 is not the usual definition of special Lagrangian submanifolds in terms of calibrations. Our definition is, however, equivalent to the definition of special Lagrangian submanifolds as a special class of calibrated submanifolds. In fact, if we define $\tilde{g}$ to be the conformally rescaled Riemannian metric on M given by $\tilde{g} = e^{2\psi}g$, then $\mathrm{Re}\,\Omega$ is a calibration on the Riemannian manifold $(M, \tilde{g})$ and we have the following alternative characterization of special Lagrangian submanifolds.

Proposition 4.10. *Let $F : L \to M$ be a Lagrangian submanifold in an almost Calabi–Yau manifold (M, J, ω, Ω). Then $F : L \to M$ is a special Lagrangian submanifold with phase $e^{i\theta}$, $\theta \in \mathbb{R}$, if and only if $F : L \to M$ is calibrated with respect to $\mathrm{Re}(e^{-i\theta}\Omega)$ for the metric $\tilde{g}$.*

Finally we mention that the definition of the generalized mean curvature vector field from Definition 4.7 can be extended to submanifolds in Kähler manifolds that are almost Einstein. Indeed, if (M, J, ω) is an m-dimensional Kähler manifold with Kähler metric g and ρ is the Ricci-form of g, then (M, J, ω) is said to be almost Einstein if there exists $\lambda \in \mathbb{R}$ and a function $\psi \in C^\infty(M)$ with $\rho = \lambda\omega + m\mathrm{dd}^c\psi$. Then if $F : L \to M$ is a C^2-submanifold, then the generalized mean curvature vector field of $F : L \to M$ is the normal vector field $K = H - m\pi_{\nu L}(\nabla\psi)$. More about the generalized mean curvature vector field can be found in the author's paper [8].

4.4 Lagrangian submanifolds in cotangent bundles

In this subsection we discuss Lagrangian submanifolds in cotangent bundles and their variations. This is of particular importance for the following reason. In §5.1 we will define the generalized Lagrangian mean curvature flow, which is a flow of Lagrangian submanifolds in an almost Calabi–Yau manifold. Later in §5.2, however, we will show that the generalized Lagrangian mean curvature flow can be seen as a flow of functions rather than of submanifolds. The differential of a function defines a Lagrangian submanifold of the cotangent bundle through its graph and therefore we first need to understand variations of Lagrangian submanifolds in cotangent bundles.

Let (M, J, ω, Ω) be an m-dimensional almost Calabi–Yau manifold and L an m-dimensional manifold. Let T^*L be the cotangent bundle of L and β a C^1-one-form on L. The graph of β is the submanifold

$$\hat{F} : L \longrightarrow T^*L, \quad \hat{F}(x) = (x, \beta(x)) \in T^*_x L \quad \text{for } x \in L. \tag{19}$$

We write Γ_β for $\hat{F}(L) = \{(x, \beta(x)) : x \in L\}$. As explained in §4.2, T^*L has a canonical symplectic structure $\hat{\omega}$. Then $\hat{F}^*(\hat{\omega}) = -\mathrm{d}\beta$, so that $\hat{F} : L \to T^*L$ is Lagrangian if and only if β is closed. In particular every function $u \in C^2(L)$ defines a Lagrangian submanifold $\hat{F} : L \to T^*L$ by $\hat{F}(x) = (x, \mathrm{d}u(x))$ for $x \in L$.

Now let $F : L \to M$ be a Lagrangian submanifold and assume that we are given a Lagrangian neighbourhood $\Phi_L : U_L \to M$ for $F : L \to M$. If β is a closed C^1-one-form on L with $\Gamma_\beta \subset U_L$, then we can define a submanifold by

$$\Phi_L \circ \beta : L \longrightarrow M, \quad (\Phi_L \circ \beta)(x) = \Phi_L(x, \beta(x)) \quad \text{for } x \in L.$$

Since $\Phi_L^*(\omega) = \hat{\omega}$, $\Phi_L \circ \beta : L \to M$ is a Lagrangian submanifold. Note that if L is compact, then, after reparametrizing by a diffeomorphism on L, every Lagrangian submanifold $\tilde{F} : L \to M$ that is C^1-close to $F : L \to M$ is given by $\Phi_L \circ \beta : L \to M$ for some closed one-form β on L.

When we study the generalized Lagrangian mean curvature flow as a flow of functions, we will study deformations of Lagrangian submanifolds of the form $\Phi_L \circ (\beta + s\eta) : L \to M$, for small $s \in \mathbb{R}$ and β, η are closed C^1-one-forms on L with $\Gamma_\beta \subset U_L$. The next important lemma gives a formula for the variation vector field of $\Phi_L \circ (\beta + s\eta) : L \to M$ along the submanifold $\Phi_L \circ \beta : L \to M$.

Lemma 4.11. *Let β, η be closed C^1-one-forms on L with $\Gamma_\beta \subset U_L$ and $\varepsilon > 0$ sufficiently small such that $\Gamma_{\beta + s\eta} \subset U_L$ for $s \in (-\varepsilon, \varepsilon)$. Then for every $s \in (-\varepsilon, \varepsilon)$, $\Phi_L \circ (\beta + s\eta) : L \to M$ is a Lagrangian submanifold and*

$$\frac{\mathrm{d}}{\mathrm{d}s} \Phi_L \circ (\beta + s\eta)\Big|_{s=0} = -\alpha^{-1}(\eta) + V(\eta),$$

where α is defined in §4.3 and $V(\eta) = \mathrm{d}(\Phi_L \circ \beta)(\hat{V}(\eta))$ is the tangential part of the variation vector field.

Proof. We choose local coordinates $(x_1, \ldots, x_m)$ on L and extend these to local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ on T^*L, such that $(x_1, \ldots, y_m)$ represents the one-form $y_1 \mathrm{d}x_1 + \cdots + y_m \mathrm{d}x_m$ in $T^*_x L$, where $x = (x_1, \ldots, x_m)$. Denote $\eta = \eta_i \mathrm{d}x_i$, where we sum over repeated indices. Then

$$\frac{\mathrm{d}}{\mathrm{d}s} \Phi_L(x, \beta(x) + s\eta(x))\Big|_{s=0} = \eta_i(x) \frac{\partial \Phi_L}{\partial y^i}(x, \beta(x)).$$

Here $\frac{\partial \Phi_L}{\partial y^i}$ is a section of the vector bundle $(\Phi_L \circ \beta)^*(TM)$. We need to compute the normal part of $\frac{\partial \Phi_L}{\partial y^i}$. Let $F = \Phi_L \circ \beta : L \to M$, then we have that

$$\begin{aligned} \pi_{\nu L}\left(\frac{\partial \Phi_L}{\partial y^i}\right) &= g^{ab} g\left(\frac{\partial \Phi_L}{\partial y^i}, J\left(\frac{\partial F}{\partial x^a}\right)\right) J\left(\frac{\partial F}{\partial x^b}\right) \\ &= g^{ab} \omega\left(\frac{\partial F}{\partial x^a}, \frac{\partial \Phi_L}{\partial y^i}\right) J\left(\frac{\partial F}{\partial x^b}\right). \end{aligned}$$

By definition $F = \Phi_L \circ \hat{F}$, where $\hat{F} : L \to T^*L$, $\hat{F}(x) = (x, \beta(x))$ for $x \in L$ is the graph of β. Denote by $(\hat{F}_1, \ldots, \hat{F}_{2m})$ the components of $\hat{F} : L \to T^*L$ in the coordinates $(x_1, \ldots, y_m)$ on T^*L. Then

$$\frac{\partial F}{\partial x^a} = \frac{\partial \Phi_L}{\partial x^a} + \frac{\partial \hat{F}_c}{\partial x^a} \frac{\partial \Phi_L}{\partial y^c}.$$

Since $\Phi_L^*(\omega) = \hat{\omega}$ and $\hat{\omega} = \sum_{j=1}^m \mathrm{d}x_j \wedge \mathrm{d}y_j$, we find that

$$\omega\left(\frac{\partial \Phi_L}{\partial x^a}, \frac{\partial \Phi_L}{\partial y^i}\right) = \hat{\omega}\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^i}\right) = \delta_{ai},$$

and similarly

$$\omega\left(\frac{\partial \Phi_L}{\partial y^c}, \frac{\partial \Phi_L}{\partial y^i}\right) = \hat{\omega}\left(\frac{\partial}{\partial y^c}, \frac{\partial}{\partial y^i}\right) = 0.$$

It follows that

$$\pi_{\nu L}\left(\frac{\partial \Phi_L}{\partial y^i}\right) = g^{ab}\delta_{ai} J\left(\frac{\partial F}{\partial x^b}\right) = g^{ib} J\left(\frac{\partial F}{\partial x^b}\right),$$

and hence

$$\pi_{\nu L}\left(\frac{\mathrm{d}}{\mathrm{d}s}\Phi_L \circ (\beta + s\eta)\big|_{s=0}\right) = g^{ib}\eta_i J\left(\frac{\partial F}{\partial x^b}\right) = -\alpha^{-1}(\eta).$$

□

5 Generalized Lagrangian mean curvature flow of compact Lagrangian submanifolds

5.1 Generalized Lagrangian mean curvature flow

In this subsection we introduce the generalized Lagrangian mean curvature flow in almost Calabi–Yau manifolds. For the definition of the generalized Lagrangian mean curvature flow we follow the author's paper [8]. General texts on the Lagrangian mean curvature flow are given by Wang [58] and with a special emphasis on the regularity theory of the Lagrangian mean curvature flow by Neves [43].

Throughout this subsection (M, J, ω, Ω) will be an m-dimensional almost Calabi–Yau manifold. We begin with the definition of the generalized Lagrangian mean curvature flow in almost Calabi–Yau manifolds.

Definition 5.1. *Let $F_0 : L \to M$ be a smooth Lagrangian submanifold in M. A smooth one-parameter family $\{F(t,\cdot)\}_{t\in(0,T)}$ of smooth Lagrangian submanifolds $F(t,\cdot) : L \to M$, which is continuous up to $t = 0$, is evolving by generalized Lagrangian mean curvature flow with initial condition $F_0 : L \to M$ if*

$$\begin{aligned} &\pi_{\nu L}\left(\frac{\partial F}{\partial t}\right)(t,x) = K(t,x) && \text{for } (t,x) \in (0,T)\times L,\\ &F(0,x) = F_0(x) && \text{for } x \in L. \end{aligned} \tag{20}$$

Here $K(t,\cdot)$ is the generalized mean curvature vector field of $F(t,\cdot) : L \to M$ for $t \in (0,T)$ as in Definition 4.7. If M is Calabi–Yau, then $\psi \equiv 0$ and $K \equiv H$. Then we say that $\{F(t,\cdot)\}_{t\in(0,T)}$ evolves by Lagrangian mean curvature flow.

The next theorem, proved in §5.2-§5.4, establishes the short time existence of the generalized Lagrangian mean curvature flow when L is compact.

Theorem 5.2. *Let $F_0 : L \to M$ be a smooth Lagrangian submanifold of an almost Calabi–Yau manifold with L compact. Then there exists $T > 0$ and a smooth one-parameter family $\{F(t,\cdot)\}_{t\in(0,T)}$ of smooth Lagrangian submanifolds $F(t,\cdot) : L \to M$, which is continuous up to $t = 0$ and evolves by generalized Lagrangian mean curvature flow with initial condition $F_0 : L \to M$.*

The system of partial differential equations (20) is, after reparametrizing by a family of diffeomorphisms on L, a quasilinear parabolic system. Hence, if L is compact, then it follows from the standard theory for parabolic equations on compact manifolds, see for instance Aubin [4, §4.2], that for every smooth submanifold $F_0 : L \to M$ there exists a smooth one-parameter family $\{F(t,\cdot)\}_{t\in(0,T)}$ of smooth submanifolds $F(t,\cdot) : L \to M$, which is continuous up to $t = 0$ and satisfies (20). Less obvious, however, is the fact that if $F_0 : L \to M$ is a Lagrangian submanifold, then $F(t,\cdot) : L \to M$ is a Lagrangian submanifold for every $t \in (0,T)$. The original proof of the fact that $F(t,\cdot) : L \to M$ is a Lagrangian submanifold for $t \in (0,T)$ uses long computations in local coordinates and the parabolic maximum principle. In §5.2 we will show how the generalized Lagrangian mean curvature flow can be integrated to a flow of functions on L rather than of embeddings of L into M. Using this interpretation of the generalized Lagrangian mean curvature flow we then present in §5.2-§5.4 a new proof of Theorem 5.2.

The idea of the Lagrangian mean curvature flow goes already back to Oh [45] in the early nineties. The existence of the Lagrangian mean curvature flow, however, was first proved by Smoczyk [50, Thm. 1.9] for the case when M is a Kähler–Einstein manifold. Recently there has been interest in generalizing the idea of the Lagrangian mean curvature flow. This led to the notion of generalized Lagrangian mean curvature flows first introduced by the author in [8], when M is a Kähler manifold that is almost Einstein, and later by Smoczyk and Wang [51], when M is an almost Kähler manifold that admits an Einstein connection.

The next proposition discusses another definition of the generalized Lagrangian mean curvature flow, which at least in the case when $F : L \to M$ is a compact Lagrangian submanifold, is equivalent to the previous one.

Proposition 5.3. *Let L be a compact manifold, $F_0 : L \to M$ a smooth Lagrangian submanifold, and $\{F(t,\cdot)\}_{(0,T)}$ a smooth one-parameter family of Lagrangian submanifolds $F(t,\cdot) : L \to M$, which is continuous up to $t = 0$ and evolves by generalized Lagrangian mean curvature flow with initial condition $F_0 : L \to M$. Then there exists a smooth one-parameter family $\{\varphi(t,\cdot)\}_{t\in(0,T)}$ of smooth diffeomorphisms of L, which is continuous up to $t = 0$, such that the following holds. The map $\varphi(0,\cdot) : L \to L$ is the identity on L and, if we define a one-parameter family $\{\tilde{F}(t,\cdot)\}_{t\in(0,T)}$ of Lagrangian submanifolds $\tilde{F}(t,\cdot) : L \to M$ by*

$$\tilde{F}(t,x) = F(t,\varphi(t,x)) \quad \textit{for } (t,x) \in (0,T) \times L, \tag{21}$$

then $\{\tilde{F}(t,\cdot)\}_{t\in(0,T)}$ is continuous up to $t = 0$ and satisfies

$$\begin{aligned} \frac{\partial \tilde{F}}{\partial t}(t,x) &= K(t,x) \quad \textit{for } (t,x) \in (0,T) \times L, \\ \tilde{F}(0,x) &= F_0(x) \qquad \textit{for } x \in L. \end{aligned} \tag{22}$$

Proof. We define $\{\varphi(t,\cdot)\}_{t\in[0,T)}$, such that $\varphi(0,\cdot) : L \to L$ is the identity on L and

$$\mathrm{d}[F(t,\cdot)]\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t,\cdot)\right) = -\pi_{\mathrm{d}F(t,\cdot)(TL)}\left(\frac{\partial F}{\partial t}(t,\cdot)\right) \quad \text{for } t \in (0,T). \tag{23}$$

The existence of $\{\varphi(t,\cdot)\}_{t\in[0,T)}$ is guaranteed by the Picard-Lindelöf Theorem [33, XIV, §3] and the compactness of L. It is then easy to see that $\{\tilde{F}(t,\cdot)\}_{t\in(0,T)}$ as defined in (21) is a solution of the Cauchy problem (22). □

Often (22) is used for the definition of the generalized Lagrangian mean curvature flow. Proposition 5.3 shows that (20) and (22) are equivalent up to tangential diffeomorphisms, provided L is compact. It is important to note, however, that in general (20) and (22) are not equivalent. For instance in the generalized Lagrangian mean curvature flow with isolated conical singularities, which we study in §9, we will find a solution to (20). The solution will then consist of Lagrangian submanifolds with isolated conical singularities and the singularities move around in the ambient space. In this case it is in general not possible to find a solution of (23) for a short time. Note anyway that if we are given solutions $\{F(t,\cdot)\}_{(0,T)}$ to the generalized Lagrangian mean curvature flow (20) and $\{\tilde{F}(t,\cdot)\}_{t\in(0,T)}$ to (22), then $F(t,L) = \tilde{F}(t,L)$ for $t \in (0,T)$. So $F(t,\cdot) : L \to M$ and $\tilde{F}(t,\cdot) : L \to M$ have the same image for each $t \in (0,T)$.

5.2 Integrating the generalized Lagrangian mean curvature flow

In this subsection we will show how the generalized Lagrangian mean curvature flow (20) can be integrated to a nonlinear differential equation on functions. In this way we get rid of the quasilinear system of parabolic differential equations given in (20). The price we have to pay, however, is that the new equation does depend in a nonlinear way on the second spatial derivatives of the function, i.e. is not quasilinear anymore.

Let (M, J, ω, Ω) be an m-dimensional almost Calabi–Yau manifold, L a compact m-dimensional manifold, and $F_0 : L \to M$ a Lagrangian submanifold. Then by the Lagrangian Neighbourhood Theorem, Theorem 4.5, there exists a Lagrangian neighbourhood $\Phi_L : U_L \to M$ for $F_0 : L \to M$. The main idea for integrating the generalized Lagrangian mean curvature flow lies in the observation that every Lagrangian submanifold $F : L \to M$ that is C^1-close to $F_0 : L \to M$ can be written as $F = \Phi_L \circ \beta : L \to M$ for a closed one-form β on L. Therefore if $\{F(t,\cdot)\}_{t\in(0,T)}$ evolves by generalized Lagrangian mean curvature flow with initial condition $F_0 : L \to M$, then $F(t,\cdot) = \Phi_L \circ \beta(t)$ for some smooth family $\{\beta(t)\}_{t\in(0,T)}$ of closed one-forms on L, that extends continuously to $t = 0$ with $\beta(0) = 0$. By Proposition 4.8, $\alpha_{K(t,\cdot)} = -\mathrm{d}[\theta(F(t,\cdot))]$ for $t \in (0,T)$. Therefore we expect that $\beta(t) = \mathrm{d}[u(t,\cdot)] + t\beta_0$ for $t \in (0,T)$, some function $u : (0,T) \times L \to \mathbb{R}$, that extends continuously to $t = 0$ with $u(0,\cdot) = 0$, and some representative $\beta_0 \in \mu_{F_0}$.

We now carry out these ideas in detail. Let $F_0 : L \to M$ be a Lagrangian submanifold with L compact and let $\Phi_L : U_L \to M$ be a Lagrangian neighbourhood for $F_0 : L \to M$, which exists by Theorem 4.5. Let μ_{F_0} be the Maslov class of $F_0 : L \to M$, and choose a smooth map $\alpha_0 : L \to \mathbb{R}/\pi\mathbb{Z}$ with $\mathrm{d}\alpha_0 \in \mu_{F_0}$. Denote $\beta_0 = \mathrm{d}\alpha_0$. Then there exists a smooth lift $\Theta(F_0) : L \to \mathbb{R}$ of the map $\theta(F_0) - \alpha_0 : L \to \mathbb{R}/\pi\mathbb{Z}$, and $\Theta(F_0)$ satisfies $\mathrm{d}[\Theta(F_0)] = \mathrm{d}[\theta(F_0)] - \beta_0$. Moreover, if $\{\eta(s)\}_{s\in(-\varepsilon,\varepsilon)}$, $\varepsilon > 0$, is a continuous family of closed one-forms defined on L with $\Gamma_{\eta(s)} \subset U_L$ for $s \in (-\varepsilon,\varepsilon)$ and $\eta(0) = 0$, then we can choose $\Theta(\Phi_L \circ \eta(s))$ to depend continuously on $s \in (-\varepsilon,\varepsilon)$.

We define a nonlinear differential operator P now as follows. Define a smooth one-parameter family $\{\beta(t)\}_{t\in(0,T)}$ of closed one-forms on L by $\beta(t) = t\beta_0$ for $t \in (0,T)$. Then $\{\beta(t)\}_{t\in(0,T)}$ extends continuously to $t = 0$ with $\beta(0) = 0$. Choose $T > 0$ small enough so that $\Gamma_{\beta(t)} \subset U_L$ for $(0,T)$. Then the domain of P is given by

$$\mathcal{D} = \big\{u \in C^\infty((0,T)\times L) \;:\; u \text{ extends continuously to } t = 0 \\ \text{and } \Gamma_{u(t,\cdot)+\beta(t)} \subset U_L \text{ for } t \in (0,T)\big\}$$

and we define

$$P : \mathcal{D} \to C^\infty((0,T)\times L), \quad P(u) = \frac{\partial u}{\partial t} - \Theta(\Phi_L \circ (\mathrm{d}u + \beta)).$$

If $u \in \mathcal{D}$, then $\Gamma_{u(t,\cdot)+\beta(t)} \subset U_L$ for every $t \in (0,T)$, and therefore the Lagrangian submanifold $\Phi_L \circ (\mathrm{d}[u(t,\cdot)] + \beta(t)) : L \to M$ is well defined for every $t \in (0,T)$. Hence $\Theta(\Phi_L \circ (\mathrm{d}[u(t,\cdot)] + \beta(t)))$ is well defined for every $t \in (0,T)$.

We consider the Cauchy problem

$$\begin{aligned} &P(u)(t,x) = 0 \quad \text{for } (t,x) \in (0,T) \times L, \\ &u(0,x) = 0 \qquad\;\; \text{for } x \in L. \end{aligned} \tag{24}$$

If we are given a solution $u \in \mathcal{D}$ of the Cauchy problem (24), then we obtain a solution to the generalized Lagrangian mean curvature flow.

Proposition 5.4. *Let $u \in \mathcal{D}$ be a solution of* (24)*. Define a one-parameter family $\{F(t,\cdot)\}_{t\in(0,T)}$ of submanifolds by*

$$F(t,\cdot) : L \longrightarrow M, \quad F(t,\cdot) = \Phi_L \circ (\mathrm{d}[u(t,\cdot)] + \beta(t)). \tag{25}$$

Then $\{F(t,\cdot)\}_{t\in(0,T)}$ is a smooth one-parameter family of smooth Lagrangian submanifolds, continuous up to $t = 0$, which evolves by generalized Lagrangian mean curvature flow with initial condition $F_0 : L \to M$.

Proof. Since $u(0,x) = 0$ for every $x \in L$ and $\beta(0) = 0$, it follows that $F(0,x) = F_0(x)$ for every $x \in L$. To show that $\{F(t,\cdot)\}_{t\in(0,T)}$ evolves by generalized Lagrangian mean curvature flow it suffices to show that $\alpha_{\frac{\partial F}{\partial t}} = \alpha_K$. Denote $X = \frac{\mathrm{d}}{\mathrm{d}t}\Phi_L \circ (\mathrm{d}u + \beta)$. Then X is a section of the vector bundle $(\Phi_L \circ (\mathrm{d}u+\beta))^*(TM)$. By Lemma 4.11 the normal part of X is equal to $-\alpha^{-1}(\mathrm{d}[\partial_t u] + \beta_0)$ and the tangential part is equal to $V(\mathrm{d}[\partial_t u] + \beta_0)$. Thus

$$\frac{\partial F}{\partial t} = -\alpha^{-1}\left(\mathrm{d}\left[\frac{\partial u}{\partial t}\right] + \beta_0\right) + V\left(\mathrm{d}\left[\frac{\partial u}{\partial t}\right] + \beta_0\right).$$

Since $\Phi_L \circ (\mathrm{d}u + \beta) : L \to M$ is a family of Lagrangian submanifolds,

$$(\Phi_L \circ (\mathrm{d}u + \beta))^*\left(V\left(\mathrm{d}\left[\frac{\partial u}{\partial t}\right] + \beta_0\right) \lrcorner\, \omega\right) = 0.$$

Thus we obtain

$$\alpha_{\frac{\partial F}{\partial t}} = -(\Phi_L \circ (\mathrm{d}u + \beta))^*\left(\alpha^{-1}\left(\mathrm{d}\left[\frac{\partial u}{\partial t}\right] + \beta_0\right) \lrcorner\, \omega\right) = -\mathrm{d}\left[\frac{\partial u}{\partial t}\right] - \beta_0.$$

Since $P(u) = 0$, $\partial_t u = \Theta(\Phi_L \circ (\mathrm{d}u + \beta))$ and hence

$$-\mathrm{d}\left[\frac{\partial u}{\partial t}\right] - \beta_0 = -\mathrm{d}[\Theta(F)] - \beta_0 = -\mathrm{d}[\theta(F)]$$

with $F = \Phi_L \circ (\mathrm{d}u + \beta)$. Thus $\alpha_{\frac{\partial F}{\partial t}} = -\mathrm{d}[\theta(F)]$. By Proposition 4.8 we have $\alpha_K = -\mathrm{d}[\theta(F)]$ and hence $\alpha_{\frac{\partial F}{\partial t}} = \alpha_K$, as we wanted to show. $\square$

We come to the important conclusion that the short time existence for the generalized Lagrangian mean curvature flow is equivalent to the short time existence of solutions to the Cauchy problem (24). Note that if $F_0 : L \to M$ has zero Maslov class, then we can choose $\beta = 0$ in Proposition 5.4.

5.3 Smoothness of P as a map between Banach manifolds

In this subsection we will show that $P : \mathcal{D} \to C^\infty((0,T) \times L)$ extends to a smooth map of certain Banach manifolds.

We first need to introduce some notation. For $k \in \mathbb{N}$ and $p \in (1,\infty)$ with $k - \frac{m}{p} > 2$ we define

$$\mathcal{D}^{k,p} = \left\{ u \in W^{1,k,p}((0,T) \times L) \; : \; \Gamma_{\mathrm{d}[u(t,\cdot)]+\beta(t)} \subset U_L \text{ for } t \in (0,T) \right\}.$$

If $u \in \mathcal{D}^{k,p}$, then $u(t,\cdot) \in C^2(L)$ for almost every $t \in (0,T)$ by the Sobolev Embedding Theorem as $k - \frac{m}{p} > 2$. In particular for almost every $t \in (0,T)$, $P(u)(t,\cdot) : L \to \mathbb{R}$ is well defined.

The goal of this section is to prove that $P : \mathcal{D}^{k,p} \to W^{0,k-2,p}((0,T) \times L)$ is smooth provided $k \in \mathbb{N}$ and $p \in (1,\infty)$ are sufficiently large. Note that if $u \in \mathcal{D}^{k,p}$, then only the first time derivative of u is guaranteed to be in L^p, whereas the spatial derivative of u up to order k lie in L^p. In order to prove that $P : \mathcal{D}^{k,p} \to W^{0,k-2,p}((0,T) \times L)$ is smooth we make use of the fact that we can take k to be arbitrarily large. On the other hand, when we prove short time existence of solutions of the Cauchy problem (24) in Proposition 5.8 below, we make use of the fact that u is only in $W^{1,p}$ in the time direction.

Let η be smooth and closed one-form on L with small C^0-norm. We define a function $F_\eta(x, \mathrm{d}u(x), \nabla \mathrm{d}u(x)) = \Theta(\Phi_L \circ (\eta + \mathrm{d}u))(x)$, where

$$F_\eta : \left\{ (x,y,z) \; : \; x \in L, \; y \in T^*_x L \text{ with } y + \eta \in U_L, \; z \in \otimes^2 T^*_x L \right\} \longrightarrow \mathbb{R}.$$

Then F_η is a smooth and nonlinear function on its domain, since Ω, g, ψ, and Φ_L are smooth. Furthermore we define a function Q_η on the domain of F_η by

$$Q_\eta(x,y,z) = F_\eta(x,y,z) - F_\eta(x,0,0) - (\partial_y F_\eta)(x,0,0) \cdot y - (\partial_z F_\eta)(x,0,0) \cdot z,$$

so Q_η is the remainder in the Taylor expansion of F_η to first order. In particular, by Taylor's Theorem [33, XIII, §6] we have for $a, b, c \geq 0$ and small $|y|, |z|$

$$(\nabla_x)^a (\partial_y)^b (\partial_z)^c Q_\eta(x,y,z) = O\left(|y|^{\max\{0,2-b\}} + |z|^{\max\{0,2-c\}} \right) \tag{26}$$

uniformly for $x \in L$, since L is compact. We begin by computing the linear terms in the Taylor expansion of F_η.

Lemma 5.5. *Let $u \in C^2(L)$ with $\Gamma_{\mathrm{d}u+\eta} \subset U_L$. Denote $\psi_\eta = (\Phi_L \circ \eta)^*(\psi)$ and $\theta_\eta = \theta(\Phi_L \circ \eta)$, which are smooth functions on L. Then*

$$(\partial_y F_\eta)(\cdot,0,0) \cdot \mathrm{d}u + (\partial_z F_\eta)(\cdot,0,0) \cdot \nabla \mathrm{d}u = \Delta u - m \mathrm{d}\psi_\eta(\nabla du) - \mathrm{d}\theta_\eta(\hat{V}(\mathrm{d}u)).$$

Here the Laplace operator and ∇ are computed using the Riemannian metric $(\Phi_L \circ \eta)^*(g)$ on L, and $\hat{V}(\mathrm{d}u)$ is defined as in Lemma 4.11.

Proof. Define $X = \frac{\mathrm{d}}{\mathrm{d}s} \Phi_L \circ (\eta + s\mathrm{d}u)|_{s=0}$. Then X is a section of the vector bundle $(\Phi_L \circ \eta)^*(TM)$. By Lemma 4.11 the normal part of X is equal to $-\alpha^{-1}(\mathrm{d}u) = J(\mathrm{d}(\Phi_L \circ \eta)(\nabla u))$ and the tangential part is equal to $V(\mathrm{d}u) = \mathrm{d}(\Phi_L \circ \eta)(\hat{V}(\mathrm{d}u))$ with $\hat{V}(\mathrm{d}u) \in TL$. Denote by θ_s the Lagrangian angle of $\Phi_L \circ (\eta + s\mathrm{d}u) : L \to M$. Further denote $\psi_s = (\Phi_L \circ (\eta + s\mathrm{d}u))^*(\psi)$ and $g_s = (\Phi_L \circ (\eta + s\mathrm{d}u))^*(g)$. By definition of the Lagrangian angle of $\Phi_L \circ (\eta + s\mathrm{d}u) : L \to M$ we then have

$$(\Phi_L \circ (\eta + s\mathrm{d}u))^*(\Omega) = e^{i\theta_s + m\psi_s} \mathrm{d}V_{g_s}. \tag{27}$$

Differentiating (27) on the left side with respect to s at $s=0$ gives

$$\frac{\mathrm{d}}{\mathrm{d}s}(\Phi_L\circ(\eta+s\mathrm{d}u))^*(\Omega)\Big|_{s=0}=(\Phi_L\circ\eta)^*(\mathcal{L}_X\Omega)=(\Phi_L\circ\eta)^*(\mathrm{d}(X\lrcorner\,\Omega))$$

by Cartan's formula, since Ω is closed. Now we decompose X into its tangential and normal part and we compute for the normal part

$$\begin{aligned}&(\Phi_L\circ\eta)^*(\mathrm{d}(J(\mathrm{d}(\Phi_L\circ\eta)(\nabla u))\lrcorner\,\Omega))\\&=i\cdot\mathrm{d}((\Phi_L\circ\eta)^*(\mathrm{d}(\Phi_L\circ\eta)(\nabla u)\lrcorner\,\Omega))=i\cdot\mathrm{d}(e^{i\theta_\eta+m\psi_\eta}\nabla u\lrcorner\,\mathrm{d}V_{g_\eta})\\&\quad=e^{i\theta_\eta+m\psi_\eta}\big\{i\mathrm{d}(\nabla u\lrcorner\,\mathrm{d}V_{g_\eta})-\mathrm{d}\theta_\eta\wedge(\nabla u\lrcorner\,\mathrm{d}V_{g_\eta})+im\mathrm{d}\psi_\eta\wedge(\nabla u\lrcorner\,\mathrm{d}V_{g_\eta})\big\}\\&\qquad=e^{i\theta_\eta+m\psi_\eta}\big\{i\Delta u-im\mathrm{d}\psi_\eta(\nabla u)+\mathrm{d}\theta_\eta(\nabla u)\big\}\mathrm{d}V_{g_\eta},\end{aligned}$$

where we use that Ω is holomorphic and (27). In a similar way we compute for the tangential part of X

$$\begin{aligned}&(\Phi_L\circ\eta)^*(\mathrm{d}(\mathrm{d}(\Phi_L\circ\eta)(\hat V(\mathrm{d}u))\lrcorner\,\Omega))=\mathrm{d}(e^{i\theta_\eta+m\psi_\eta}\hat V(\mathrm{d}u)\lrcorner\,\mathrm{d}V_{g_\eta})\\&=e^{i\theta_\eta+m\psi_\eta}\big\{i\mathrm{d}\theta_\eta\wedge(\hat V(\mathrm{d}u)\lrcorner\,\mathrm{d}V_{g_\eta})+m\mathrm{d}\psi_\eta\wedge(\hat V(\mathrm{d}u)\lrcorner\,\mathrm{d}V_{g_\eta})+\mathrm{d}(\hat V(\mathrm{d}u)\lrcorner\,\mathrm{d}V_{g_\eta})\big\}\\&\quad=e^{i\theta_\eta+m\psi_\eta}\big\{-i\mathrm{d}\theta_\eta(\hat V(\mathrm{d}u))-m\mathrm{d}\psi_\eta(\hat V(\mathrm{d}u))\big\}\mathrm{d}V_{g_\eta}+e^{i\theta_\eta+m\psi_\eta}\mathrm{d}(\hat V(\mathrm{d}u)\lrcorner\,\mathrm{d}V_{g_\eta}).\end{aligned}$$

Thus we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}(\Phi_L\circ(\eta+s\mathrm{d}u))^*(\Omega)\Big|_{s=0}&=e^{i\theta_\eta+m\psi_\eta}\big\{i\Delta u-im\mathrm{d}\psi_\eta(\nabla u)\\&\quad+\mathrm{d}\theta_\eta(\nabla u)-i\mathrm{d}\theta_\eta(\hat V(\mathrm{d}u))-m\mathrm{d}\psi_\eta(\hat V(\mathrm{d}u))\big\}\mathrm{d}V_{g_\eta}\\&\qquad+e^{i\theta_\eta+m\psi_\eta}\mathrm{d}(\hat V(\mathrm{d}u)\lrcorner\,\mathrm{d}V_{g_\eta}).\end{aligned}\tag{28}$$

Differentiating the left side in (27) with respect to s at $s=0$ we find that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}e^{i\theta_s+m\psi_s}\mathrm{d}V_{g_s}\Big|_{s=0}=\\&\qquad e^{i\theta_\eta+m\psi_\eta}\Big\{i\frac{\mathrm{d}\theta_s}{\mathrm{d}s}\Big|_{s=0}+m\mathrm{d}\psi_\eta(\hat V(\mathrm{d}u))-g(H_\eta,X)\Big\}\mathrm{d}V_{g_\eta},\end{aligned}\tag{29}$$

where H_η denotes the mean curvature vector field of $\Phi_L\circ\eta:L\to M$. Comparing (28) and (29) we conclude that

$$\frac{\mathrm{d}\theta_s}{\mathrm{d}s}\Big|_{s=0}=\Delta u-m\mathrm{d}\psi_\eta(\nabla u)-\mathrm{d}\theta_\eta(\hat V(\mathrm{d}u)),$$

from which the lemma follows. □

We can now prove the smoothness of $P:\mathcal{D}^{k,p}\to W^{0,k-2,p}((0,T)\times L)$ for sufficiently large $k\in\mathbb{N}$ and $p\in(1,\infty)$.

Proposition 5.6. *Let $k\in\mathbb{N}$, $p\in(1,\infty)$ such that $k\geq 6$ and $p>\max\{1,\frac{4+2m}{k-2}\}$. Then $P:\mathcal{D}^{k,p}\to W^{0,k-2,p}((0,T)\times L)$ is a smooth map of Banach manifolds.*

Proof. The first step is to show that $P:\mathcal{D}^{k,p}\to W^{0,k-2,p}((0,T)\times L)$ is well defined. Let $u\in\mathcal{D}^{k,p}$. Using Lemma 5.5 we can write

$$P(u)=\frac{\partial u}{\partial t}-\theta_\beta-\Delta u+m\mathrm{d}\psi_\beta(\nabla u)+\mathrm{d}\theta_\beta(\hat V(\mathrm{d}u))-Q_\beta(\cdot,\mathrm{d}u,\nabla\mathrm{d}u),\tag{30}$$

with $\psi_\beta = (\Phi_L \circ \beta)^*(\psi)$ and $\theta_\beta = \theta(\Phi_L \circ \beta)$. We show that each of the terms on the right side of (30) lies in $W^{0,k-2,p}((0,T) \times L)$. Recall that $\beta(t) = t\beta_0$ for $t \in (0,T)$. Since θ_β is a smooth function on $(0,T) \times L$, we have $\theta_\beta \in W^{0,k-2,p}((0,T) \times L)$. Moreover, from the definition of $\mathcal{D}^{k,p}$ it immediately follows that

$$\partial_t u, \Delta u, \mathrm{d}\psi_\beta(\nabla u), \mathrm{d}\theta_\beta(\hat{V}(\mathrm{d}u)) \in W^{0,k-2,p}((0,T) \times L).$$

It remains to show that $Q_\beta(\cdot, \mathrm{d}u, \nabla \mathrm{d}u)$ lies in $W^{0,k-2,p}((0,T) \times L)$. We first show that $Q_\beta(\cdot, \mathrm{d}u, \nabla \mathrm{d}u)$ lies in $W^{0,0,p}((0,T) \times L)$. By (26) we have

$$Q_\beta(\cdot, \mathrm{d}u, \nabla \mathrm{d}u) = O\left(|\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2\right).$$

Let us show that $|\nabla \mathrm{d}u|^2$ lies in $W^{0,0,p}((0,T)\times L)$. In a similar way one can show that $|\mathrm{d}u|^2$ lies in $W^{0,0,p}((0,T)\times L)$. Since $k \geq 4$, $|\nabla \mathrm{d}u|$ lies in $W^{1,k-2,p}((0,T)\times L)$. Since $p > \max\{1, \frac{2+m}{k-2}\}$, Proposition 2.6 implies that $W^{1,k-2,p}((0,T) \times L)$ embeds continuously into $C^{0,0}((0,T) \times L)$ by inclusion and hence $|\nabla \mathrm{d}u|^2 \in W^{0,0,p}((0,T) \times L)$. This shows that $Q_\beta(\cdot, \mathrm{d}u, \nabla \mathrm{d}u) \in W^{0,0,p}((0,T) \times L)$. For the derivatives of Q_β we have by the chain rule

$$\begin{aligned}\left|\nabla^j Q_\beta(x, \mathrm{d}u, \nabla \mathrm{d}u)\right| \leq j! \sum_{\substack{a,b,c\geq 0\\ a+b+c\leq j}} &\left|(\nabla_x)^a (\partial_y)^b (\partial_z)^c Q_\beta(x, \mathrm{d}u, \nabla \mathrm{d}u)\right| \\ \times \sum_{\substack{m_1,\ldots,m_b,n_1,\ldots,n_c\geq 1\\ a+m_1+\cdots+m_m+n_1+\cdots+n_c=j}} &\prod_{l=1}^{b} |\nabla^{m_l+1} u(x)| \prod_{l=1}^{c} |\nabla^{n_l+2} u(x)|\end{aligned} \tag{31}$$

for $j = 0, \ldots, k-2$. Using Proposition 2.6, the estimate (26), and the conditions on $k \in \mathbb{N}$ and $p \in (1,\infty)$ it is straightforward to show that each of the terms on the right side of (31) lies in $W^{0,0,p}((0,T) \times L)$ for $j = 0, \ldots, k-2$. Thus we finally obtain that $Q_\beta(\cdot, \mathrm{d}u, \nabla \mathrm{d}u) \in W^{0,k-2,p}((0,T) \times L)$ and hence $P : \mathcal{D}^{k,p} \to W^{0,k-2,p}((0,T) \times L)$ is well defined.

Next we show that $P : \mathcal{D}^{k,p} \to W^{0,k-2,p}((0,T) \times L)$ is a continuous map of Banach manifolds. Let $u, v \in \mathcal{D}^{k,p}$. Writing $P(u)$ and $P(v)$ as in (30) we find

$$\begin{aligned}\|P(u) - P(v)\|_{W^{0,k-2,p}} &= \|\Delta(u-v) + Q_\beta(\cdot, \mathrm{d}u, \nabla \mathrm{d}u) - Q_\beta(\cdot, \mathrm{d}v, \nabla \mathrm{d}v)\|_{W^{0,k-2,p}} \\ &\leq \|\Delta(u-v)\|_{W^{0,k-2,p}} + \|Q_\beta(\cdot, \mathrm{d}u, \nabla \mathrm{d}u) - Q_\beta(\cdot, \mathrm{d}v, \nabla \mathrm{d}v)\|_{W^{0,k-2,p}}\end{aligned}$$

Clearly $\|\Delta(u-v)\|_{W^{0,k-2,p}} \leq c\|u-v\|_{W^{1,k,p}}$ for some constant $c > 0$. Moreover, since Q_β is a smooth function on its domain, the second term can be estimated by the derivatives of Q_β and $\|u-v\|_{W^{1,k,p}}$ using the Mean Value Theorem [33, XIII, §4]. It follows that $P : \mathcal{D}^{k,p} \to W^{0,k-2,p}((0,T) \times L)$ is continuous. In a similar way one can show that $P : \mathcal{D}^{k,p} \to W^{0,k-2,p}((0,T) \times L)$ is in fact smooth. This completes the proof. □

5.4 Short time existence and regularity of solutions

In this subsection we first show that the Cauchy problem (24) has a solution $u \in \mathcal{D}^{k,p}$ for $k \in \mathbb{N}$ and $p \in (1,\infty)$ that satisfy the conditions of Proposition 5.6. We then improve the regularity of u and show that u is in fact smooth. Finally we give a proof of Theorem 5.2.

The first step in the short time existence proof is to show that the linearization of P at the initial condition is an isomorphism. From now on we assume that $k \in \mathbb{N}$ and $p \in (1,\infty)$ satisfy the conditions of Proposition 5.6, so that $P : \mathcal{D}^{k,p} \to W^{0,k-2,p}((0,T) \times L)$ is smooth by Proposition 5.6. Denote

$$\tilde{\mathcal{D}}^{k,p} = \{u \in \mathcal{D}^{k,p} \ : \ u(0,x) = 0 \text{ for } x \in L\}.$$

If $u \in \mathcal{D}^{k,p}$, then u is uniformly Hölder continuous on $(0,T)$ and therefore extends continuously to $t = 0$. Thus if $u \in \mathcal{D}^{k,p}$, then $u(0,\cdot) : L \to \mathbb{R}$ is well defined. Then a solution $u \in \mathcal{D}^{k,p}$ to the Cauchy problem (24) is the same as a function $u \in \tilde{\mathcal{D}}^{k,p}$ that solves the equation $P(u) = 0$.

Next we study the linearization of the operator $P : \mathcal{D}^{k,p} \to W^{0,k-2,p}((0,T) \times L)$ at the initial condition.

Proposition 5.7. *After making $T > 0$ smaller if necessary, the linearization of $P : \tilde{\mathcal{D}}^{k,p} \to W^{0,k-2,p}((0,T) \times L)$ at the initial condition*

$$\mathrm{d}P(0) : \tilde{W}^{1,k,p}((0,T) \times L) \longrightarrow W^{0,k-2,p}((0,T) \times L) \tag{32}$$

is an isomorphism of Banach spaces.

Proof. From Lemma 5.5 it follows that

$$\mathrm{d}P(0)(u) = \frac{\partial u}{\partial t} - \Delta u + m\mathrm{d}\psi_\beta(\nabla u) + \mathrm{d}\theta_\beta(\hat{V}(\mathrm{d}u)),$$

for $u \in \tilde{W}^{1,k,p}((0,T) \times L)$, where $\psi_\beta = (\Phi_L \circ \beta)^*(\psi)$, $\theta_\beta = \theta(\Phi_L \circ \beta)$, and the Laplace operator and ∇ are computed using the time dependent Riemannian metric $(\Phi_L \circ \beta)^*(g)$ on L. Define $Ku = m\mathrm{d}\psi_\beta(\nabla u) + \mathrm{d}\theta_\beta(\hat{V}(\mathrm{d}u))$ for $u \in \tilde{W}^{1,k,p}((0,T) \times L)$, so that $\mathrm{d}P(0) = \partial_t - \Delta + K$. Clearly K is a bounded operator from $\tilde{W}^{1,k,p}((0,T) \times L)$ into $W^{1,k-1,p}((0,T) \times L)$. By Proposition 2.5, the inclusion of $\tilde{W}^{1,k-1,p}((0,T) \times L)$ into $W^{0,k-2,p}((0,T) \times L)$ is compact and it follows that

$$K : \tilde{W}^{1,k,p}((0,T) \times L) \to W^{0,k-2,p}((0,T) \times L) \tag{33}$$

is a compact operator, as it is the composition of a bounded linear operator and a compact operator.

Let g_β be the time dependent Riemannian metric $(\Phi_L \circ \beta)^*(g)$ on L and $g_0 = F_0^*(g)$. Let Δ_0 be the Laplace operator on L computed using the Riemannian metric g_0. Then by Theorem 3.4 and the discussion following that theorem,

$$\frac{\partial}{\partial t} - \Delta_0 : \tilde{W}^{1,k,p}((0,T) \times L) \to W^{0,k-2,p}((0,T) \times L) \tag{34}$$

is an isomorphism. We show that

$$\frac{\partial}{\partial t} - \Delta : \tilde{W}^{1,k,p}((0,T) \times L) \to W^{0,k-2,p}((0,T) \times L) \tag{35}$$

is also an isomorphism, where the Laplace operator is computed using g_β. Let $u \in \tilde{W}^{1,k,p}((0,T) \times L)$. Then

$$\begin{aligned}
\|\Delta_0 u - \Delta u\|^p_{W^{0,k-2,p}} &= \int_0^T \|\Delta_0 u(t,\cdot) - \Delta u(t,\cdot)\|^p_{W^{k-2,p}} \,\mathrm{d}t \\
&\leq \sup_{t \in (0,T)} \|g_0 - g_{\beta(t)}\|_{C^{k-1}} \int_0^t \|\nabla^2 u\|^p_{W^{k-2,p}} \,\mathrm{d}t \leq \|g_0 - g_\beta\|_{C^{0,k-1}} \|u\|_{W^{1,k,p}}.
\end{aligned}$$

Since $\beta(t) = t\beta_0$ for $t \in (0,T)$, we can make $\|g_0 - g_\beta\|_{C^{0,k-1}}$ arbitrarily small by making $T > 0$ small. Hence (34) and (35) are arbitrarily close as bounded operators from $\tilde{W}^{1,k,p}((0,T)\times L)$ into $W^{0,k-2,p}((0,T)\times L)$. Since (34) is an isomorphism, it follows that for sufficiently small $T > 0$, (35) is an isomorphism. Since (35) is an isomorphism, it is a Fredholm operator of index zero. In particular, as (33) is a compact operator, it follows from the Fredholm alternative [33, XVII, §2] that (32) is a Fredholm operator of index zero. Thus in order to show that (32) is an isomorphism it suffices to show that (32) is injective.

Let $u \in \tilde{W}^{1,k,p}((0,T)\times L)$ be a solution of the Cauchy problem

$$\begin{aligned} &\mathrm{d}P(0)(u)(t,x) = 0 && \text{for } (t,x) \in (0,T)\times L,\\ &u(0,x) = 0 && \text{for } x \in L. \end{aligned}$$

Then u is a solution of a linear parabolic equation with smooth coefficients, and therefore u is smooth by Theorem 2.8. Moreover u extends continuously to $t = 0$. Denote $\varphi = \frac{1}{2}|u|^2$. Then $\varphi(0,\cdot) = 0$ on L and a short computation shows that $\mathrm{d}P(0)(\varphi) \leq 0$. The parabolic maximum principle, see for instance Friedman [18, Ch. 2, Thm. 1], implies that $\varphi \equiv 0$ and hence $u \equiv 0$. It follows that (32) is injective and therefore an isomorphism. □

We can now prove short time existence of solutions of the Cauchy problem (24). The strategy of the proof follows Aubin [4, Ch. 4, §4.2].

Proposition 5.8. *There exists $\tau > 0$ and $u \in \tilde{\mathcal{D}}^{k,p}$, such that $P(u) = 0$ on $(0,\tau)\times L$.*

Proof. By Proposition 5.7 the linearization of $P : \tilde{\mathcal{D}}^{k,p} \to W^{0,k-2,p}((0,T)\times L)$ at 0 is an isomorphism. Since $P : \tilde{\mathcal{D}}^{k,p} \to W^{0,k-2,p}((0,T)\times L)$ is smooth by Proposition 5.6, the Inverse Function Theorem for Banach manifolds [33, XIV, Thm. 1.2] shows that there exist open neighbourhoods V of 0 in $\tilde{\mathcal{D}}^{k,p}$ and W of $P(0)$ in $W^{0,k-2,p}((0,T)\times L)$, such that $P : V \to W$ is a smooth diffeomorphism. For $\tau \in (0,T)$ we define a function $w_\tau : (0,T)\times L \to \mathbb{R}$ by

$$w_\tau(t,x) = \begin{cases} 0 & \text{for } t < \tau,\ x \in L,\\ P(0)(t,x) & \text{for } t \geq \tau,\ x \in L. \end{cases}$$

Then $w_\tau \in W^{0,k-2,p}((0,T)\times L)$ for every $\tau > 0$ and we can make $w_\tau - P(0)$ arbitrarily small in $W^{0,k-2,p}((0,T)\times L)$ by making $\tau > 0$ small. In particular for sufficiently small $\tau > 0$, w_τ lies in W and there exists a unique $u \in V$ with $P(u) = w_\tau$. But then $P(u) = 0$ on $(0,\tau)\times L$ and $u(0,\cdot) = 0$ on L as we wanted to show. □

Since P is a nonlinear parabolic differential operator of second order, we can use the local regularity theory from §2.3 to show that a solution $u \in \mathcal{D}^{k,p}$ to $P(u) = 0$ is in fact smooth on $(0,T)\times L$.

Proposition 5.9. *Let $u \in \mathcal{D}^{k,p}$ with $P(u) = 0$. Then $u \in C^\infty((0,T)\times L)$.*

Proof. The proof follows from the local regularity theory for linear parabolic equations on domains and the bootstrapping method. We choose local coordinates $(x_1,\ldots,x_m) \in \Omega$ on L with $\Omega \subset \mathbb{R}^m$. Denote $u_j = \frac{\partial u}{\partial x^j}$ for $j = 1,\ldots,m$. Differentiating $P(u) = 0$ with respect to x_j we find that u_j is a solution of a linear parabolic differential equation of second order, with coefficients and free term being functions that depend smoothly on $\mathrm{d}u$ and $\nabla \mathrm{d}u$. Since $k \geq 6$ and $p \in (1, \infty)$ is sufficiently large, the Sobolev Embedding Theorem implies that the coefficients and the free term are Hölder continuous and lie in $C^{0,0,\alpha}((0,T) \times \Omega)$ for some $\alpha \in (0,1)$. But then Theorem 2.8 implies $u_j \in C^{1,2,\alpha}(I \times \Omega')$ for every $I \subset\subset (0,T)$, $\Omega' \subset\subset \Omega$, and $j = 1, \ldots, m$. Hence $u \in C^{1,3,\alpha}(I \times L)$ for every $I \subset\subset (0,T)$. Iterating this procedure then shows that $u \in C^{\infty}((0,T) \times L)$. □

We can now prove Theorem 5.2.

Proof of Theorem 5.2. We choose a Lagrangian neighbourhood $\Phi_L : U_L \to M$ for $F_0 : L \to M$. Let $k \in \mathbb{N}$ and $p \in (1, \infty)$ satisfy the assumptions from Proposition 5.6. Then by Proposition 5.8 there exists a solution $u \in \tilde{\mathcal{D}}^{k,p}$ of the Cauchy problem (24) on a short time interval $(0,T)$. Moreover, by Proposition 5.9, u is smooth on $(0,T) \times L$. We define a smooth one-parameter family $\{F(t, \cdot)\}_{t \in (0,T)}$ of smooth Lagrangian submanifolds $F(t, \cdot) : L \to M$, $t \in (0,T)$, by $F(t, \cdot) = \Phi_L \circ (\mathrm{d}[u(t, \cdot)] + \beta(t))$ for $t \in (0,T)$. Then by Proposition 5.4, $\{F(t, \cdot)\}_{t \in (0,T)}$ evolves by generalized Lagrangian mean curvature flow with initial condition $F_0 : L \to M$, as we wanted to show. □

6 The Laplace operator on Riemannian manifolds with conical singularities

6.1 Weighted Hölder and Sobolev spaces

We begin this subsection with the definition of Riemannian manifolds with conical singularities. We then proceed to define weighted Hölder and Sobolev spaces on Riemannian manifolds with conical singularities. Manifolds with ends and differential operators on manifolds with ends are extensively discussed in the literature, and we would like to mention the works of Lockhart and McOwen [34], Melrose [39], and Schulze [48] in particular.

We begin with the definition of manifolds with ends.

Definition 6.1. *Let M be an open and connected m-dimensional manifold with $m \geq 1$. Assume that we are given a compact m-dimensional submanifold $K \subset M$ with boundary, such that $M \backslash K$ has a finite number of pairwise disjoint, open, and connected components $S_1, \ldots, S_n$. Then M is a manifold with ends $S_1, \ldots, S_n$ if the following holds. There exist compact and connected $(m-1)$-dimensional manifolds $\Sigma_1, \ldots, \Sigma_n$, a constant $R > 0$, and diffeomorphisms $\phi_i : \Sigma_i \times (0, R) \to S_i$ for $i = 1, \ldots, n$. We say that $S_1, \ldots, S_n$ are the ends of M and that Σ_i is the link of S_i. Note that the boundary of K is diffeomorphic to $\bigsqcup_{i=1}^n \Sigma_i$.*

Next we define Riemannian cones.

Definition 6.2. *Let (Σ, h) be an $(m-1)$-dimensional compact and connected Riemannian manifold, $m \geq 1$. Let $C = (\Sigma \times (0, \infty)) \sqcup \{0\}$ and $C' = \Sigma \times (0, \infty)$ and write a general point in C and C' as (σ, r). Define a Riemannian metric $g = \mathrm{d}r^2 + r^2 h$ on C'. Then we say that (C, g) is the Riemannian cone over (Σ, h) with Riemannian cone metric g.*

Using the definition of manifolds with ends and of Riemannian cones we can now define compact Riemannian manifolds with conical singularities.

Definition 6.3. *Let (M, d) be a compact metric space, $x_1, \ldots, x_n$ distinct points in M, and denote $M' = M \backslash \{x_1, \ldots, x_n\}$. Suppose that M' is a smooth m-dimensional manifold, and assume that we are given $R > 0$, such that if we denote $S_i = \{x \in M \ : \ 0 < d(x, x_i) < R\}$ for $i = 1, \ldots, n$, then M' is a manifold with ends $S_1, \ldots, S_n$ as in Definition 6.1. Let Σ_i be the link of S_i for $i = 1, \ldots, n$. Assume that we are given a Riemannian metric g on M' that induces the metric d on M'. Let h_i be a Riemannian metric on Σ_i for $i = 1, \ldots, n$, and denote by (C_i, g_i) the Riemannian cone over (Σ_i, h_i) for $i = 1, \ldots, n$.*

Then we say that (M, g) is a compact m-dimensional Riemannian manifold with conical singularities at $x_1, \ldots, x_n$ modelled on $(C_1, g_1), \ldots, (C_n, g_n)$, if there exist diffeomorphisms $\phi_i : \Sigma_i \times (0, R) \to S_i$ for $i = 1, \ldots, n$, and $\mu_i \in \mathbb{R}$ with $\mu_i > 0$ for $i = 1, \ldots, n$, such that

$$\left|\nabla^k(\phi_i^*(g) - g_i)\right| = O(r^{\mu_i - k}) \quad \text{as } r \longrightarrow 0 \text{ for } k \in \mathbb{N} \tag{36}$$

and $i = 1, \ldots, n$. Here ∇ and $|\cdot|$ are computed using the Riemannian cone metric g_i on $\Sigma_i \times (0, R)$ for $i = 1, \ldots, n$.

In Definition 6.3 we have chosen $\mu_i > 0$ for $i = 1, \ldots, n$, since (36) then implies that the Riemannian metric $\phi_i^*(g)$ on $\Sigma_i \times (0, R)$ converges to the Riemannian cone metric g_i as $r \to 0$ for $i = 1, \ldots, n$. Thus on each end of M' the Riemannian metric g is in fact asymptotic to a Riemannian cone metric. Note in particular that if (M, g) is a compact Riemannian manifold and $x_1, \ldots, x_n$ are distinct points in M, then (M, g) is a compact Riemannian manifold with conical singularities at $x_1, \ldots, x_n$ with each singularity modelled on $\mathbb{R}^m$ with the Euclidean metric in polar coordinates.

Before we can define weighted function spaces on Riemannian manifolds with conical singularities we need to introduce the notion of a radius function.

Definition 6.4. *Let (M, g) be a compact Riemannian manifold with conical singularities as in Definition 6.3 and denote $K = M' \setminus \bigcup_{i=1}^n S_i$. A radius function on M' is a smooth function $\rho : M' \to (0, 1]$, such that $\rho \equiv 1$ on K and*

$$|\phi_i^*(\rho) - r| = O(r^{1+\varepsilon}) \quad \text{as } r \longrightarrow 0 \tag{37}$$

for some $\varepsilon > 0$. Here $|\cdot|$ is computed using the Riemannian cone metric g_i on $\Sigma_i \times (0, R)$ for $i = 1, \ldots, n$. A radius function always exists.

From now on let us fix a compact Riemannian manifold (M, g) with conical singularities as in Definition 6.3, and let us choose a radius function ρ on M'. If $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$, then we define a function $\rho^{\boldsymbol{\gamma}}$ on M' as follows. On S_i we set $\rho^{\boldsymbol{\gamma}} = \rho^{\gamma_i}$ for $i = 1, \ldots, n$ and $\rho^{\boldsymbol{\gamma}} \equiv 1$ on K. If $\boldsymbol{\gamma}, \boldsymbol{\mu} \in \mathbb{R}^n$, then we write $\boldsymbol{\gamma} \leq \boldsymbol{\mu}$ if $\gamma_i \leq \mu_i$ for $i = 1, \ldots, n$, and $\boldsymbol{\gamma} < \boldsymbol{\mu}$ if $\gamma_i < \mu_i$ for $i = 1, \ldots, n$. If $\boldsymbol{\gamma} \in \mathbb{R}^n$ and $a \in \mathbb{R}$, then we denote $\boldsymbol{\gamma} + a = (\gamma_1 + a, \ldots, \gamma_n + a) \in \mathbb{R}^n$.

We can now define weighted C^k-spaces on (M, g). Let $k \in \mathbb{N}$ and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Then the $C^k_{\boldsymbol{\gamma}}$-norm is defined by

$$\|u\|_{C^k_{\boldsymbol{\gamma}}} = \sum_{j=0}^{k} \sup_{x \in M'} |\rho(x)^{-\boldsymbol{\gamma}+j} \nabla^j u(x)| \quad \text{for } u \in C^k_{\text{loc}}(M'),$$

whenever it is finite. A different choice of radius function defines an equivalent norm. Note that a function $u \in C^k_{\text{loc}}(M')$ has finite $C^k_{\boldsymbol{\gamma}}$-norm if and only if $\nabla^j u$ grows at most like $\rho^{\boldsymbol{\gamma}-j}$ for $j = 0, \ldots, k$ as $\rho \to 0$. We define the weighted C^k-space $C^k_{\boldsymbol{\gamma}}(M')$ by

$$C^k_{\boldsymbol{\gamma}}(M') = \left\{ u \in C^k_{\text{loc}}(M') \;:\; \|u\|_{C^k_{\boldsymbol{\gamma}}} < \infty \right\}.$$

Then $C^k_{\boldsymbol{\gamma}}(M')$ is a Banach space. We set $C^\infty_{\boldsymbol{\gamma}}(M') = \bigcap_{k \in \mathbb{N}} C^k_{\boldsymbol{\gamma}}(M')$. The space $C^\infty_{\boldsymbol{\gamma}}(M')$ is in general not a Banach space.

Next we define weighted Hölder spaces on (M, g). Let $\alpha \in (0, 1)$ and T be a tensor field over M'. Then we define a seminorm by

$$[T]_{\alpha, \boldsymbol{\gamma}} = \sup_{\substack{x \neq y \in M' \\ d_g(x,y) < \delta_g(x)}} \left\{ \min\left\{\rho(x)^{-\boldsymbol{\gamma}}, \rho(y)^{-\boldsymbol{\gamma}}\right\} \frac{|T(x) - T(y)|}{d_g(x, y)^\alpha} \right\},$$

whenever it is finite, and we define the $C^{k,\alpha}_{\boldsymbol{\gamma}}$-norm by

$$\|u\|_{C^{k,\alpha}_{\boldsymbol{\gamma}}} = \|u\|_{C^k_{\boldsymbol{\gamma}}} + [\nabla^k u]_{\alpha, \boldsymbol{\gamma}-k} \quad \text{for } u \in C^{k,\alpha}_{\text{loc}}(M'),$$

whenever it is finite. A different choice of radius function defines an equivalent norm. The weighted Hölder space $C^{k,\alpha}_{\boldsymbol{\gamma}}(M')$ is defined by

$$C^{k,\alpha}_{\boldsymbol{\gamma}}(M') = \left\{ u \in C^{k,\alpha}_{\mathrm{loc}}(M') \;:\; \|u\|_{C^{k,\alpha}_{\boldsymbol{\gamma}}} < \infty \right\}.$$

Then $C^{k,\alpha}_{\boldsymbol{\gamma}}(M')$ is a Banach space.

The next proposition gives embeddings between different weighted Hölder spaces and states under which conditions these embeddings are compact.

Proposition 6.5. *Let (M, g) be a compact Riemannian manifold with conical singularities. Let $k, l \in \mathbb{N}$, $\alpha, \beta \in (0, 1)$, and $\boldsymbol{\gamma}, \boldsymbol{\delta} \in \mathbb{R}^n$. Then the following hold.*

(i) *If $l + \beta \le k + \alpha$ and $\boldsymbol{\delta} \le \boldsymbol{\gamma}$, then $C^{k,\alpha}_{\boldsymbol{\gamma}}(M')$ embeds continuously into $C^{l,\beta}_{\boldsymbol{\delta}}(M')$ by inclusion.*

(ii) *If $l + \beta < k + \alpha$ and $\boldsymbol{\delta} < \boldsymbol{\gamma}$, then the inclusion of $C^{k,\alpha}_{\boldsymbol{\gamma}}(M')$ into $C^{l,\beta}_{\boldsymbol{\delta}}(M')$ is compact.*

A proof of Proposition 6.5 can be found in Chaljub-Simon and Choquet-Bruhat [10, Lem. 2 and 3] for weighted Hölder spaces on asymptotically Euclidean manifolds. The proof for weighted Hölder spaces on compact Riemannian manifolds with conical singularities is a simple modification of the proof given by Chaljub-Simon and Choquet-Bruhat.

Finally we define weighted Sobolev spaces. Let $k \in \mathbb{N}$, $p \in [1, \infty)$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Then we define the $W^{k,p}_{\boldsymbol{\gamma}}$-norm by

$$\|u\|_{W^{k,p}_{\boldsymbol{\gamma}}} = \left(\sum_{j=0}^{k} \int_{M'} |\rho^{-\boldsymbol{\gamma}+j} \nabla^j u|^p \rho^{-m} \, \mathrm{d}V_g \right)^{1/p} \qquad \text{for } u \in W^{k,p}_{\mathrm{loc}}(M'),$$

whenever it is finite. A different choice of radius function defines an equivalent norm. We define the weighted Sobolev space $W^{k,p}_{\boldsymbol{\gamma}}(M')$ by

$$W^{k,p}_{\boldsymbol{\gamma}}(M') = \left\{ u \in W^{k,p}_{\mathrm{loc}}(M') \;:\; \|u\|_{W^{k,p}_{\boldsymbol{\gamma}}} < \infty \right\}.$$

Then $W^{k,p}_{\boldsymbol{\gamma}}(M')$ is a Banach space. If $k = 0$, then we write $L^p_{\boldsymbol{\gamma}}(M')$ instead of $W^{0,p}_{\boldsymbol{\gamma}}(M')$. Note that $L^p(M') = L^p_{-m/p}(M')$ and that $C^\infty_{\mathrm{cs}}(M')$, the space of smooth functions on M' with compact support, is dense in $W^{k,p}_{\boldsymbol{\gamma}}(M')$ for every $k \in \mathbb{N}$, $p \in [1, \infty)$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Moreover if $p = 2$, then we can define a scalar product by

$$\langle u, v \rangle_{W^{k,2}_{\boldsymbol{\gamma}}} = \sum_{j=0}^{k} \int_{M'} \rho^{-2\boldsymbol{\gamma}+2j} g(\nabla^j u, \nabla^j v) \rho^{-m} \, \mathrm{d}V_g \quad \text{for } u, v \in W^{k,2}_{\boldsymbol{\gamma}}(M').$$

Thus $W^{k,2}_{\boldsymbol{\gamma}}(M')$ is a Hilbert space.

The following proposition is easily verified using Hölder's inequality.

Proposition 6.6. *Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Then the scalar product on $L^2(M')$ given in (1) defines a dual pairing $L^p_{\boldsymbol{\gamma}}(M') \times L^q_{-m-\boldsymbol{\gamma}}(M') \longrightarrow \mathbb{R}$. Thus $L^p_{\boldsymbol{\gamma}}(M')$ and $L^q_{-m-\boldsymbol{\gamma}}(M')$ are Banach space duals of each other.*

The next theorem is a version of the Sobolev Embedding Theorem for weighted Hölder and Sobolev spaces.

Theorem 6.7. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3. Let $k,l \in \mathbb{N}$, $p,q \in [1,\infty)$, $\alpha \in (0,1)$, and $\boldsymbol{\gamma},\boldsymbol{\delta} \in \mathbb{R}^n$. Then the following hold.*

(i) *If $\frac{1}{p} \leq \frac{1}{q} + \frac{k-l}{m}$ and $\boldsymbol{\gamma} \geq \boldsymbol{\delta}$ then $W^{k,p}_{\boldsymbol{\gamma}}(M')$ embeds continuously into $W^{l,q}_{\boldsymbol{\delta}}(M')$ by inclusion.*

(ii) *If $k - \frac{m}{p} \geq l + \alpha$ and $\boldsymbol{\gamma} \geq \boldsymbol{\delta}$, then $W^{k,p}_{\boldsymbol{\gamma}}(M')$ embeds continuously into $C^{l,\alpha}_{\boldsymbol{\delta}}(M')$ by inclusion.*

The proof of the Sobolev Embedding Theorem for weighted spaces can be found in Bartnik [5, Thm. 1.2] for the case of asymptotically Euclidean manifolds. The proof of the Sobolev Embedding Theorem for weighted spaces on compact Riemannian manifolds with conical singularities is then a simple modification of Bartnik's proof.

The next theorem is the Rellich–Kondrakov Theorem for weighted Hölder and Sobolev spaces on compact Riemannian manifolds with conical singularities. For a proof of the Rellich–Kondrakov Theorem we again refer to Bartnik [5, Thm. 1.2, Lem 1.4].

Theorem 6.8. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3. Let $k,l \in \mathbb{N}$, $p,q \in [1,\infty)$, $\alpha \in (0,1)$, and let $\boldsymbol{\gamma},\boldsymbol{\delta} \in \mathbb{R}^n$. Then the following hold.*

(i) *If $\frac{1}{p} < \frac{1}{q} + \frac{k-l}{m}$ and $\boldsymbol{\gamma} > \boldsymbol{\delta}$, then the inclusion of $W^{k,p}_{\boldsymbol{\gamma}}(M')$ into $W^{l,q}_{\boldsymbol{\delta}}(M')$ is compact.*

(ii) *If $k - \frac{m}{p} > l + \alpha$ and $\boldsymbol{\gamma} > \boldsymbol{\delta}$, then the inclusion of $W^{k,p}_{\boldsymbol{\gamma}}(M')$ into $C^{l,\alpha}_{\boldsymbol{\delta}}(M')$ is compact.*

6.2 The Laplace operator on weighted spaces

In this subsection we discuss the Laplace operator on compact Riemannian manifolds with conical singularities acting on weighted Hölder and Sobolev spaces. There is a more general theory of elliptic cone differential operators, which generalizes the discussion of this subsection. The interested reader is referred to Lockhart and McOwen [34], Melrose [39], and Schulze [48]. Our discussion follows Lockhart and McOwen [34] and the presentation given by Joyce [24, §2].

Before we discuss the Laplace operator on compact Riemannian manifolds with conical singularities we recall the following standard theorem from geometric analysis.

Theorem 6.9. *Let (Σ,h) be a compact and connected Riemannian manifold. Then the spectrum $\sigma(\Delta_h)$ of the Laplace operator on Σ consists of eigenvalues only. The eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ form a decreasing sequence $0 = \lambda_0 > \lambda_1 \geq \ldots \to -\infty$ and every eigenspace is finite dimensional. Moreover there exists a complete orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}}$ of $L^2(\Sigma)$ consisting of smooth functions, such that $\Delta_h\varphi_j = \lambda_j\varphi_j$ for $j \in \mathbb{N}$.*

The proof of Theorem 6.9 can be found in Aubin [4, Thm. 4.2] or in Shubin [49, Ch. 1, Thm. 8.3] using pseudodifferential techniques.

Let (Σ, h) be a compact and connected $(m-1)$-dimensional Riemannian manifold, $m \geq 1$, and let (C, g) be the Riemannian cone over (Σ, h). A function $u : C' \to \mathbb{R}$ is said to be homogeneous of order α, if there exists a function $\varphi : \Sigma \to \mathbb{R}$, such that $u(\sigma, r) = r^\alpha \varphi(\sigma)$ for $(\sigma, r) \in C'$. The Laplace operator on C' is given by

$$\Delta_g u(\sigma, r) = \frac{\partial^2 u}{\partial r^2}(\sigma, r) + \frac{m-1}{r}\frac{\partial u}{\partial r}(\sigma, r) + \frac{1}{r^2}\Delta_h u(\sigma, r) \tag{38}$$

for $u \in C^2_{\mathrm{loc}}(C')$ and $(\sigma, r) \in C'$. Using (38) the following lemma is easily verified.

Lemma 6.10. *A homogeneous function $u(\sigma, r) = r^\alpha \varphi(\sigma)$ of order $\alpha \in \mathbb{R}$ on C' with $\varphi \in C^2(\Sigma)$ is harmonic if and only if $\Delta_h \varphi = -\alpha(\alpha + m - 2)\varphi$.*

Define

$$\mathcal{D}_\Sigma = \{\alpha \in \mathbb{R} \ : \ -\alpha(\alpha + m - 2) \in \sigma(\Delta_h)\}. \tag{39}$$

Then by Theorem 6.9, $\mathcal{D}_\Sigma$ is a discrete subset of $\mathbb{R}$ with no other accumulation points than $\pm\infty$. Moreover $\mathcal{D}_\Sigma \cap (2-m, 0) = \emptyset$, since Δ_h is non-positive. Finally from Lemma 6.10 it follows that $\mathcal{D}_\Sigma$ is the set of all $\alpha \in \mathbb{R}$ for which there exists a nonzero homogeneous harmonic function of order α on C. Define a function

$$m_\Sigma : \mathbb{R} \longrightarrow \mathbb{N}, \quad m_\Sigma(\alpha) = \dim\ker(\Delta_h + \alpha(\alpha + m - 2)).$$

Then $m_\Sigma(\alpha)$ is the multiplicity of the eigenvalue $-\alpha(\alpha + m - 2)$. Note that $\dim\ker(\Delta_h + \alpha(\alpha + m - 2))$ is finite for every $\alpha \in \mathbb{R}$ by Theorem 6.9 and that $m_\Sigma(\alpha) \neq 0$ if and only if $\alpha \in \mathcal{D}_\Sigma$. Finally we define a function $M_\Sigma : \mathbb{R} \to \mathbb{Z}$ by

$$M_\Sigma(\delta) = -\sum_{\alpha \in \mathcal{D}_\Sigma \cap (\delta, 0)} m_\Sigma(\alpha) \text{ if } \delta < 0, \ M_\Sigma(\delta) = \sum_{\alpha \in \mathcal{D}_\Sigma \cap [0, \delta)} m_\Sigma(\alpha) \text{ if } \delta \geq 0.$$

Then M_Σ is a monotone increasing function that is discontinuous exactly on $\mathcal{D}_\Sigma$. As $\mathcal{D}_\Sigma \cap (2-m, 0) = \emptyset$, we see that $M_\Sigma \equiv 0$ on $(2-m, 0)$. The set $\mathcal{D}_\Sigma$ and the function M_Σ play an important rôle in the Fredholm theory of the Laplace operator on compact Riemannian manifolds with conical singularities, see Theorem 6.12 below.

The next proposition gives the weighted Schauder and L^p-estimates for the Laplace operator on compact Riemannian manifolds with conical singularities.

Proposition 6.11. *Let (M, g) be a compact Riemannian manifold with conical singularities as in Definition 6.3 and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Let $u, f \in L^1_{\mathrm{loc}}(M')$ and assume that $\Delta_g u = f$ holds in the weak sense. Then the following hold.*

(i) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\alpha \in (0,1)$. If $f \in C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}(M')$ and $u \in C^0_{\boldsymbol{\gamma}}(M')$, then $u \in C^{k,\alpha}_{\boldsymbol{\gamma}}(M')$. Moreover there exists a constant $c > 0$ independent of u and f, such that*

$$\|u\|_{C^{k,\alpha}_{\boldsymbol{\gamma}}} \leq c\left(\|f\|_{C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}} + \|u\|_{C^0_{\boldsymbol{\gamma}}}\right). \tag{40}$$

(ii) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $p \in (1, \infty)$. If $f \in W^{k-2,p}_{\boldsymbol{\gamma}-2}(M')$ and $u \in L^p_{\boldsymbol{\gamma}}(M')$, then $u \in W^{k,p}_{\boldsymbol{\gamma}}(M')$. Moreover there exists a constant $c > 0$ independent of u and f, such that*

$$\|u\|_{W^{k,p}_{\boldsymbol{\gamma}}} \leq c\left(\|f\|_{W^{k-2,p}_{\boldsymbol{\gamma}-2}} + \|u\|_{L^p_{\boldsymbol{\gamma}}}\right). \tag{41}$$

The proof of Proposition 6.11 can be found in Marshall [35, Thm. 4.21].

Recall that a bounded linear operator $A : X \to Y$ between Banach spaces X and Y is a Fredholm operator, if it has finite dimensional kernel and its image is a closed subspace of Y of finite codimension. If $A : X \to Y$ is a Fredholm operator, then the Fredholm index of A is the integer $\operatorname{index} A = \dim\ker A - \dim\operatorname{coker} A$. The next theorem is the main Fredholm theorem for the Laplace operator on compact Riemannian manifolds with conical singularities.

Theorem 6.12. *Let (M, g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Then the following hold.*

(i) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\alpha \in (0, 1)$. Then*

$$\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma}}(M') \to C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}(M') \tag{42}$$

is a Fredholm operator if and only if $\gamma_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1, \ldots, n$. If $\gamma_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1, \ldots, n$, then the Fredholm index of (42) *is equal to $-\sum_{i=1}^n M_{\Sigma_i}(\gamma_i)$.*

(ii) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $p \in (1, \infty)$. Then*

$$\Delta_g : W^{k,p}_{\boldsymbol{\gamma}}(M') \to W^{k-2,p}_{\boldsymbol{\gamma}-2}(M') \tag{43}$$

is a Fredholm operator if and only if $\gamma_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1, \ldots, n$. If $\gamma_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1, \ldots, n$, then the Fredholm index of (43) *is equal to $-\sum_{i=1}^n M_{\Sigma_i}(\gamma_i)$.*

Furthermore the kernel of the operators (42) *and* (43) *is constant in $\boldsymbol{\gamma} \in \mathbb{R}^n$ on the connected components of $(\mathbb{R}\backslash\mathcal{D}_{\Sigma_1}) \times \cdots \times (\mathbb{R}\backslash\mathcal{D}_{\Sigma_n})$.*

The proof of Theorem 6.12 can be found in Lockhart and McOwen [34, Thm. 6.1] and in Marshall [35, Thm. 6.9]. In fact, Lockhart and McOwen prove the second part of Theorem 6.12 for the Laplace operator acting on weighted Sobolev spaces and Marshall deduces the first part of Theorem 6.12 for the Laplace operator acting on weighted Hölder spaces from the results of Lockhart and McOwen.

The following proposition is a simple consequence of Proposition 6.6 and Theorem 6.12.

Proposition 6.13. *Let (M, g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$. Let $k \in \mathbb{N}$ with $k \geq 2$, $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\gamma_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1, \ldots, n$. Then* (43) *is a Fredholm operator and its cokernel is isomorphic to the kernel of the operator $\Delta_g : W^{k,q}_{2-m-\boldsymbol{\gamma}}(M') \to W^{k-2,q}_{-m-\boldsymbol{\gamma}}(M')$.*

As before let (Σ, h) be a compact and connected $(m-1)$-dimensional Riemannian manifold, $m \geq 1$, and let (C, g) be the Riemannian cone over (Σ, h). Define

$$\mathcal{E}_\Sigma = \mathcal{D}_\Sigma \cup \{\beta \in \mathbb{R} \;:\; \beta = \alpha + 2k \text{ for } \alpha \in \mathcal{D}_\Sigma,\ k \in \mathbb{N} \text{ with } \alpha \geq 0 \text{ and } k \geq 1\}$$

and a function $n_\Sigma : \mathbb{R} \longrightarrow \mathbb{N}$ by

$$n_\Sigma(\beta) = m_\Sigma(\beta) + \sum_{k \geq 1,\ 2k \leq \beta} m_\Sigma(\beta - 2k).$$

Clearly if $\beta \notin \mathcal{E}_\Sigma$, then $n_\Sigma(\beta) = 0$. Also note that if $\beta < 2$, then $n_\Sigma(\beta) = m_\Sigma(\beta)$. Finally, if $\beta \in \mathcal{E}_\Sigma$, then n_Σ counts the multiplicity of the eigenvalues

$$-\beta(\beta + m - 2), -(\beta - 2)((\beta - 2) + m - 2), \ldots, -(\beta - 2k)((\beta - 2k) + m - 2)$$

for $2k \le \beta$. Finally we define a function $N_\Sigma : \mathbb{R} \longrightarrow \mathbb{N}$ by

$$N_\Sigma(\delta) = -\sum_{\beta \in \mathcal{D}_\Sigma \cap (\delta, 0)} n_\Sigma(\beta) \text{ if } \delta < 0, \; N_\Sigma(\delta) = \sum_{\beta \in \mathcal{D}_\Sigma \cap [0, \delta)} n_\Sigma(\beta) \text{ if } \delta \ge 0. \quad (44)$$

Then $N_\Sigma(\delta) = M_\Sigma(\delta)$ for $\delta \le 2$ and

$$M_\Sigma(\delta) = N_\Sigma(\delta) - N_\Sigma(\delta - 2) \quad \text{for } \delta \in \mathbb{R} \text{ with } \delta > 2. \quad (45)$$

The set $\mathcal{E}_\Sigma$ and the function N_Σ play a similar rôle in the study of the heat equation on compact Riemannian manifolds with conical singularities as $\mathcal{D}_\Sigma$ and M_Σ do in the study of the Laplace operator, see Theorem 7.10 and 7.13 below.

6.3 Weighted Hölder and Sobolev spaces with discrete asymptotics

In this subsection we first explain the construction of discrete asymptotics on Riemannian manifolds with conical singularities and then define weighted Hölder and Sobolev spaces with discrete asymptotics. The notion of discrete asymptotics in our specific setting appears to be new. There is however a strong similarity between our definition of discrete asymptotics and the index sets for polyhomogeneous conormal distributions considered by Melrose [39, Ch. 5, §10] and especially with the asymptotic types considered by Schulze [48, Ch. 2, §3].

We first explain our motivation for the introduction of discrete asymptotics. If (M, g) is a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \ge 3$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\gamma_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1, \ldots, n$, then for every $k \in \mathbb{N}$ with $k \ge 2$ and $p \in (1, \infty)$, $\Delta_g : W^{k,p}_{\boldsymbol{\gamma}}(M') \to W^{k-2,p}_{\boldsymbol{\gamma}-2}(M')$ is a Fredholm operator by Theorem 6.12. If $\boldsymbol{\gamma} > 0$, then it follows from Theorem 6.12 that the Fredholm index of $\Delta_g : W^{k,p}_{\boldsymbol{\gamma}}(M') \to W^{k-2,p}_{\boldsymbol{\gamma}-2}(M')$ is negative, so $\Delta_g : W^{k,p}_{\boldsymbol{\gamma}}(M') \to W^{k-2,p}_{\boldsymbol{\gamma}-2}(M')$ has a cokernel. The main idea behind our definition of discrete asymptotics is to enlarge the spaces $W^{k,p}_{\boldsymbol{\gamma}}(M')$ and $W^{k-2,p}_{\boldsymbol{\gamma}-2}(M')$ by finite dimensional spaces of functions that decay slower than $\rho^{\boldsymbol{\gamma}}$ and $\rho^{\boldsymbol{\gamma}-2}$, respectively, and that cancel the cokernel of $\Delta_g : W^{k,p}_{\boldsymbol{\gamma}}(M') \to W^{k-2,p}_{\boldsymbol{\gamma}-2}(M')$. More precisely our goal is to construct two finite dimensional spaces of functions V_1 and V_2 with $V_2 \subset V_1$ consisting of functions that decay slower than $\rho^{\boldsymbol{\gamma}}$ and $\rho^{\boldsymbol{\gamma}-2}$, respectively, such that the Laplace operator maps $W^{k,p}_{\boldsymbol{\gamma}}(M') \oplus V_1$ into $W^{k-2,p}_{\boldsymbol{\gamma}-2}(M') \oplus V_2$ and

$$\operatorname{index}\Big\{\Delta_g : W^{k,p}_{\boldsymbol{\gamma}}(M') \oplus V_1 \longrightarrow W^{k-2,p}_{\boldsymbol{\gamma}-2}(M') \oplus V_2\Big\} = 0 \quad (46)$$

We begin with the construction of the model space for the discrete asymptotics. Let (Σ, h) be a compact and connected $(m-1)$-dimensional Riemannian manifold, $m \ge 1$, and let (C, g) be the Riemannian cone over (Σ, h). For $\gamma \in \mathbb{R}$ we denote

$$H_\gamma(C') = \operatorname{span}\{u = r^\alpha \varphi \;:\; 0 \le \alpha < \gamma, \; \varphi \in C^\infty(\Sigma), \; u \text{ is harmonic}\},$$

which is the space of homogeneous harmonic functions of order α with $0 \leq \alpha < \gamma$. Then $\dim H_\gamma(C') = M_\Sigma(\gamma)$ for $\gamma \geq 2-m$, so $H_\gamma(C')$ is at least one dimensional for $\gamma > 0$. We define a finite dimensional vector space $V_{\mathsf{P}_\gamma}(C')$ by

$$V_{\mathsf{P}_\gamma}(C') = \operatorname{span}\left\{v = r^{2k}u \ : \ k \in \mathbb{N}, \ u = r^\alpha\varphi \in H_\gamma(C') \text{ and } \alpha + 2k < \gamma\right\}.$$

Note that the Laplace operator on C' maps $V_{\mathsf{P}_\gamma}(C') \to V_{\mathsf{P}_{\gamma-2}}(C')$ for every $\gamma \in \mathbb{R}$ as a consequence of (38). In particular the Laplace operator is nilpotent as a map $V_{\mathsf{P}_\gamma}(C') \to V_{\mathsf{P}_\gamma}(C')$. Also note that $\dim V_{\mathsf{P}_\gamma}(C') = N_\Sigma(\gamma)$ and $V_{\mathsf{P}_\gamma}(C') = H_\gamma(C')$ for $\gamma \leq 2$. The space $V_{\mathsf{P}_\gamma}(C')$ serves as the model space in the definition of discrete asymptotics on general Riemannian manifolds with conical singularities.

The definition of discrete asymptotics on general compact Riemannian manifolds with conical singularities is based on the following proposition.

Proposition 6.14. *Let (M, g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Then for every $\varepsilon > 0$ there exists a linear map*

$$\Psi_{\boldsymbol{\gamma}} : \bigoplus_{i=1}^n V_{\mathsf{P}_{\gamma_i}}(C'_i) \longrightarrow C^\infty(M'),$$

such that the following hold.

(i) *For every $v \in \bigoplus_{i=1}^n V_{\mathsf{P}_{\gamma_i}}(C'_i)$ with $v = (v_1, \dots, v_n)$ and $v_i = r^{\beta_i}\varphi_i$ where $\varphi_i \in C^\infty(\Sigma_i)$ for $i = 1, \dots, n$ we have*

$$|\nabla^k(\phi_i^*(\Psi_{\boldsymbol{\gamma}}(v)) - v_i)| = O(r^{\mu_i+\beta_i-\varepsilon-k}) \quad \text{as } r \longrightarrow 0 \text{ for } k \in \mathbb{N}$$

and $i = 1, \dots, n$.

(ii) *For every $v \in \bigoplus_{i=1}^n V_{\mathsf{P}_{\gamma_i}}(C'_i)$ with $v = (v_1, \dots, v_n)$ we have*

$$\Delta_g(\Psi_{\boldsymbol{\gamma}}(v)) - \sum_{i=0}^n \Psi_{\boldsymbol{\gamma}}(\Delta_{g_i}v_i) \in C^\infty_{\mathrm{cs}}(M').$$

Proof. Let $\varepsilon > 0$ be arbitrary. We define linear maps $\Psi_{\gamma_i} : V_{\mathsf{P}_{\gamma_i}}(C'_i) \to C^\infty(M')$ for $i = 1, \dots, n$, such that $\Psi_{\gamma_i} \equiv 0$ on $M' \backslash S_i$ and the following hold.

(a) For $v \in V_{\mathsf{P}_{\gamma_i}}(C'_i)$ with $v = r^{\beta_i}\varphi$ where $\varphi \in C^\infty(\Sigma)$ we have

$$|\nabla^k(\phi_i^*(\Psi_{\gamma_i}(v)) - v)| = O(r^{\mu_i+\beta_i-\varepsilon-k}) \quad \text{as } r \longrightarrow 0 \text{ for } k \in \mathbb{N}.$$

(b) For every $v \in V_{\mathsf{P}_{\gamma_i}}(C'_i)$ we have $\Delta_g(\Psi_{\gamma_i}(v)) - \Psi_{\gamma_i}(\Delta_{g_i}v) \in C^\infty_{\mathrm{cs}}(M')$.

The proposition then immediately follows by setting $\Psi_{\boldsymbol{\gamma}} = \Psi_{\gamma_1} \oplus \ldots \oplus \Psi_{\gamma_n}$.

Choose $R' > 0$ with $\frac{R}{2} < R' < R$ and denote $S'_i = \phi_i(\Sigma_i \times (0, R'))$ for $i = 1, \dots, n$. Pick any $i = 1 \dots, n$ and choose a function $\chi_i \in C^\infty(M')$ with $\chi_i \equiv 1$ on S'_i and $\chi_i \equiv 0$ on $M' \backslash S_i$. We first define $\Psi_{\gamma_i} : V_{\mathsf{P}_{\gamma_i}}(C'_i) \to C^\infty(M')$ for $v \in V_{\mathsf{P}_{\gamma_i}}(C'_i)$ with $\Delta_{g_i}v = 0$ and then proceed iteratively. Thus let $v \in V_{\mathsf{P}_{\gamma_i}}(C'_i)$ such that $\Delta_{g_i}v = 0$ and $v = r^{\beta_i}\varphi$ with $\varphi \in C^\infty(\Sigma)$. Denote $u = \chi_i\Delta_g((\phi_i^{-1})^*(v))$. Then u is a smooth function on M', since v and χ_i are smooth, is supported on S_i, and

$$|\nabla^k\phi_i^*(u)| = O(r^{\beta_i+\mu_i-2-k}) \quad \text{as } r \longrightarrow 0 \text{ for } k \in \mathbb{N}$$

by (36). Choose arbitrary $\beta_1,\ldots,\beta_{i-1},\beta_{i+1},\ldots,\beta_n\in\mathbb{R}$ and let $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_n)$. Since $\mathcal{D}_{\Sigma_j}\subset\mathbb{R}$ is discrete for $j=1,\ldots,n$ we can choose some arbitrary small $\varepsilon_1>0$ with $\varepsilon_1<\varepsilon$, such that $\beta_j+\mu_j-\varepsilon_1\notin\mathcal{D}_{\Sigma_j}$ for $j=1\ldots,n$. Then by Theorem 6.12

$$\Delta_g: W^{k,p}_{\boldsymbol{\beta}+\boldsymbol{\mu}-\varepsilon_1}(M')\longrightarrow W^{k-2,p}_{\boldsymbol{\beta}+\boldsymbol{\mu}-2-\varepsilon_1}(M')$$

is a Fredholm operator for every $k\in\mathbb{N}$ with $k\geq 2$ and $p\in(1,\infty)$. Since $C^\infty_{\rm cs}(M')$ is dense in $W^{k-2,p}_{\boldsymbol{\beta}+\boldsymbol{\mu}-2-\varepsilon_1}(M')$, we can choose a finite dimensional subspace $W\subset C^\infty_{\rm cs}(M')$ such that

$$W^{k-2,p}_{\boldsymbol{\beta}+\boldsymbol{\mu}-2-\varepsilon_1}(M')=\operatorname{im}\Big\{\Delta_g: W^{k,p}_{\boldsymbol{\beta}+\boldsymbol{\mu}-\varepsilon_1}(M')\longrightarrow W^{k-2,p}_{\boldsymbol{\beta}+\boldsymbol{\mu}-2-\varepsilon_1}(M')\Big\}\oplus W$$

for some $k\in\mathbb{N}$ with $k\geq 2$ and $p\in(1,\infty)$. Thus there exist unique $\tilde v\in W^{k,p}_{\boldsymbol{\beta}+\boldsymbol{\mu}-\varepsilon_1}(M')$ and $w\in W$, such that $u=\Delta_g\tilde v+w$. Moreover, since u and w are smooth, it follows from Theorem 6.7 and Proposition 6.11, (ii), that $\tilde v\in C^\infty_{\boldsymbol{\beta}+\boldsymbol{\mu}-\varepsilon_1}(M')$. We set $\Psi_{\gamma_i}(v)=\chi_i((\phi_i^{-1})^*(v)-\tilde v)$. Then it immediately follows from the construction of $\tilde v$ that $\Psi_{\gamma_i}(v)$ satisfies (a) and (b). In this way we define $\Psi_{\gamma_i}(v)$ for a basis of the subspace of $V_{\mathsf{P}_{\gamma_i}}(C_i')$ consisting of $v\in V_{\mathsf{P}_{\gamma_i}}(C_i')$ with $\Delta_{g_i}v=0$, and then extend Ψ_{γ_i} linearly to the whole subspace. This defines $\Psi_{\gamma_i}(v)$ for every $v\in V_{\mathsf{P}_{\gamma_i}(C_i')}$ with $\Delta_{g_i}v=0$.

The next step is to define $\Psi_{\gamma_i}(v')$ for $v'\in V_{\mathsf{P}_{\gamma_i}}(C_i')$ with $\Delta^2_{g_i}v'=0$. Thus let $v'\in V_{\mathsf{P}_{\gamma_i}}(C_i')$ with $\Delta^2_{g_i}v'=0$, such that $v'=r^{\beta_i'}\varphi'$ with $\varphi'\in C^\infty(\Sigma)$. Denote $v=\Delta_{g_i}v'$. Then $v\in V_{\mathsf{P}_{\gamma_i}}(C_i')$, $\Delta_{g_i}v=0$, and $v=r^{\beta_i}\varphi$ with $\beta_i=\beta_i'-2$ and $\varphi\in C^\infty(\Sigma)$. Furthermore $\Psi_{\gamma_i}(v)=\chi_i((\phi_i^{-1})^*(v)-\tilde v)$, where $\tilde v$ is defined above. Denote $h=\chi_i(\Delta_g((\phi_i^{-1})^*(v'))-(\phi_i^{-1})^*(\Delta_{g_i}v'))$, so that $\chi_i\Delta_g((\phi_i^{-1})^*(v'))=\chi_i(\phi_i^{-1})^*(v)+h$. Then h is a smooth function on M' that is supported on S_i and satisfies

$$|\nabla^k(\phi_i^{-1})^*(h)|=O(r^{\beta_i'+\mu_i-2-k})\quad\text{as } r\longrightarrow 0 \text{ for } k\in\mathbb{N}.$$

Define $f=\chi_i\tilde v+h$. Then f is a smooth function on M', since h, $\tilde v$, and χ_i are smooth. Furthermore $\chi_i\Delta_g v'=\Psi_{\gamma_i}(v)+f$. Since $\tilde v\in C^\infty_{\boldsymbol{\beta}+\boldsymbol{\mu}-\varepsilon_1}(M')$ and $\beta_i=\beta_i'-2$ we find that

$$|\nabla^k(\phi_i^{-1})^*(f)|=O(r^{\beta_i+\mu_i-\varepsilon_1-k})\quad\text{as } r\longrightarrow 0 \text{ for } k\in\mathbb{N}.$$

Now we proceed as before. Choose arbitrary $\beta_1',\ldots,\beta_{i-1}',\beta_{i+1}',\ldots,\beta_n'\in\mathbb{R}$ and denote $\boldsymbol{\beta}'=(\beta_1',\ldots,\beta_n')$. We can choose some arbitrary small $\varepsilon_2>0$ with $\varepsilon_1<\varepsilon_2<\varepsilon$, such that $\beta_j+\mu_j-\varepsilon_2\notin\mathcal{D}_{\Sigma_j}$ for $j=1,\ldots,n$. Then by Theorem 6.12 the operator

$$\Delta_g: W^{k,p}_{\boldsymbol{\beta}'+\boldsymbol{\mu}-\varepsilon_2}(M')\longrightarrow W^{k-2,p}_{\boldsymbol{\beta}'+\boldsymbol{\mu}-2-\varepsilon_2}(M')$$

is a Fredholm operator for every $k\in\mathbb{N}$ with $k\geq 2$ and $p\in(1,\infty)$. Thus we can choose a finite dimensional subspace $W\subset C^\infty_{\rm cs}(M)$, such that

$$W^{k-2,p}_{\boldsymbol{\beta}'+\boldsymbol{\mu}-2-\varepsilon_2}(M')=\operatorname{im}\Big\{\Delta_g: W^{k,p}_{\boldsymbol{\beta}+\boldsymbol{\mu}-\varepsilon_2}(M')\longrightarrow W^{k-2,p}_{\boldsymbol{\beta}+\boldsymbol{\mu}-2-\varepsilon_2}(M')\Big\}\oplus W$$

for some $k\in\mathbb{N}$ with $k\geq 2$ and $p\in(1,\infty)$. We choose unique $\tilde v'\in W^{k,p}_{\boldsymbol{\beta}'+\boldsymbol{\mu}-\varepsilon_2}(M')$ and $w\in W$, such that $f=\Delta_g\tilde v'+w$. Since f and w are smooth, Theorem 6.7 and Proposition 6.11 then imply that $\tilde v'\in C^\infty_{\boldsymbol{\beta}'+\boldsymbol{\mu}-\varepsilon_2}(M')$. We define

$\Psi_{\gamma_i}(v') = \chi_i((\phi_i^{-1})^*(v') - \tilde{v}')$. Then it is straightforward to check that $\Psi_{\gamma_i}(v')$ satisfies (a) and (b).

Finally, since $\Delta_{g_i} : V_{\mathsf{P}_{\gamma_i}}(C_i') \to V_{\mathsf{P}_{\gamma_i}}(C_i')$ is nilpotent, Ψ_{γ_i} can be defined for every $v \in V_{\mathsf{P}_{\gamma_i}}(C_i')$ by iteration. This completes the proof. $\square$

Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities, $m \geq 3$. Then using Proposition 6.14 we can define weighted C^k-spaces, Hölder spaces, and Sobolev spaces with discrete asymptotics as follows. For $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$ we define

$$C^k_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M') = C^k_{\boldsymbol{\gamma}}(M') \oplus \operatorname{im} \Psi_{\boldsymbol{\gamma}} \quad \text{and} \quad C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M') = C^{k,\alpha}_{\boldsymbol{\gamma}}(M') \oplus \operatorname{im} \Psi_{\boldsymbol{\gamma}}.$$

Both $C^k_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$ and $C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$ are Banach spaces, where the norm on the discrete asymptotics part is some finite dimensional norm. Finally if $p \in [1,\infty)$, then we define the weighted Sobolev space with discrete asymptotics $W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$ by

$$W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M') = W^{k,p}_{\boldsymbol{\gamma}}(M') \oplus \operatorname{im} \Psi_{\boldsymbol{\gamma}}.$$

Clearly $W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$ is a Banach space, where the norm on the discrete asymptotics part is some finite dimensional norm. Note that the discrete asymptotics are trivial if $\boldsymbol{\gamma} \leq 0$, so that in this case the weighted spaces with discrete asymptotics are simply weighted spaces as defined in §6.1.

6.4 The Laplace operator on weighted spaces with discrete asymptotics

In this subsection we discuss the Laplace operator acting on weighted Hölder and Sobolev spaces with discrete asymptotics and in particular we prove that (46) holds with our definition of weighted spaces with discrete asymptotics.

If (M,g) is a compact Riemannian manifold, then the Laplace operator defines an isomorphism of Banach spaces

$$\Delta_g : \left\{u \in C^{k,\alpha}(M) \ : \ \textstyle\int_M u \,\mathrm{d}V_g = 0\right\} \longrightarrow \left\{u \in C^{k-2,\alpha}(M) \ : \ \textstyle\int_M u \,\mathrm{d}V_g = 0\right\}$$

for every $k \in \mathbb{N}$ with $k \geq 2$ and $\alpha \in (0,1)$. In a similar way

$$\Delta_g : \left\{u \in W^{k,p}(M) \ : \ \textstyle\int_M u \,\mathrm{d}V_g = 0\right\} \longrightarrow \left\{u \in W^{k-2,p}(M) \ : \ \textstyle\int_M u \,\mathrm{d}V_g = 0\right\}$$

defines an isomorphism of Banach spaces for every $k \in \mathbb{N}$ with $k \geq 2$ and $p \in (1,\infty)$, see Aubin [4, Thm. 4.7]. Using the weighted Hölder and Sobolev spaces with discrete asymptotics we can now state a similar result for the Laplace operator on Riemannian manifolds with conical singularities.

Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\boldsymbol{\gamma} > 2-m$. For $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $p \in [1,\infty)$ we then define

$$C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 = \left\{u \in C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M') \ : \ \textstyle\int_{M'} u \,\mathrm{d}V_g = 0\right\},$$

and for $p \in [1,\infty)$

$$W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 = \left\{u \in W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M') \ : \ \textstyle\int_{M'} u \,\mathrm{d}V_g = 0\right\}.$$

From Proposition 6.14 it follows that for $k \in \mathbb{N}$ with $k \geq 2$ and $\alpha \in (0,1)$, $\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 \to C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-\mathbf{2}}}(M')_0$ and for $p \in [1,\infty)$, $\Delta_g : W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 \to W^{k-2,p}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-\mathbf{2}}}(M')_0$ are well defined linear operators. We then have the following result, which also verifies (46).

Proposition 6.15. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities, $m \geq 3$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\boldsymbol{\gamma} > 2-m$ and $\gamma_i \notin \mathcal{E}_{\Sigma_i}$ for $i = 1,\ldots,n$. Then the following hold.*

(i) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\alpha \in (0,1)$. Then*

$$\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 \to C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0 \tag{47}$$

is an isomorphism of Banach spaces.

(ii) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $p \in (1,\infty)$. Then*

$$\Delta_g : W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 \to W^{k-2,p}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0 \tag{48}$$

is an isomorphism of Banach spaces.

Proof. We demonstrate the proof of (i), the proof of (ii) goes similarly. Thus let $k \in \mathbb{N}$ with $k \geq 2$, $\alpha \in (0,1)$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\boldsymbol{\gamma} > 2-m$ and $\gamma_i \notin \mathcal{E}_{\Sigma_i}$ for $i = 1,\ldots,n$. Then by Theorem 6.12, $\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma}}(M')_0 \longrightarrow C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}(M')_0$ is a Fredholm operator and

$$\mathrm{index}\left\{\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma}}(M')_0 \longrightarrow C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}(M')_0\right\} = -\sum_{i=1}^n M_{\Sigma_i}(\gamma_i). \tag{49}$$

First we show that the operator $\Delta_g : C^{k+2,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 \to C^{k,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0$ has zero Fredholm index, i.e. that (46) holds. When we replace $C^{k,\alpha}_{\boldsymbol{\gamma}-2}(M')_0$ by $C^{k,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0$ in (49), then we enlarge the cokernel of the operator $\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma}}(M')_0 \longrightarrow C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}(M')_0$ by a finite dimensional space with dimension $\dim \mathrm{im}\, \Psi_{\boldsymbol{\gamma}-2}$. Thus we have

$$\begin{aligned}\mathrm{index}\left\{\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma}}(M')_0 \longrightarrow C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0\right\} &= \\ -\sum_{i=1}^n M_{\Sigma_i}(\gamma_i) - \dim \mathrm{im}\, \Psi_{\boldsymbol{\gamma}-2}&.\end{aligned} \tag{50}$$

Moreover, if we replace $C^{k,\alpha}_{\boldsymbol{\gamma}}(M')_0$ by the space $C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0$, then we enlarge the kernel and reduce the cokernel of the operator $\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma}}(M')_0 \longrightarrow C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0$. Thus it follows from (50) that

$$\begin{aligned}\mathrm{index}\left\{\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 \to C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0\right\} &= \\ -\sum_{i=1}^n M_{\Sigma_i}(\gamma_i) - \dim \mathrm{im}\, \Psi_{\boldsymbol{\gamma}-2} + \dim \mathrm{im}\, \Psi_{\boldsymbol{\gamma}}&\end{aligned} \tag{51}$$

By definition of $\Psi_{\boldsymbol{\gamma}}$ and $\Psi_{\boldsymbol{\gamma}-2}$ we have that

$$\dim \mathrm{im}\, \Psi_{\boldsymbol{\gamma}} = \sum_{i=1}^n N_{\Sigma_i}(\gamma_i) \quad \text{and} \quad \dim \mathrm{im}\, \Psi_{\boldsymbol{\gamma}-2} = \sum_{i=1}^n N_{\Sigma_i}(\gamma_i - 2),$$

where N_{Σ_i} is defined in (44) for $i = 1, \dots, n$. Using (45) we then conclude from (51) that

$$\operatorname{index}\left\{\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 \to C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0\right\} = 0.$$

Thus (46) holds and in order to show that $\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 \to C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0$ is a bijection it suffices to show that the kernel is trivial.

Let $u \in C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0$, such that $\Delta_g u = 0$, and let us first assume that $\boldsymbol{\gamma} > \frac{1}{2}(2-m)$. Then integration by parts gives

$$0 = \int_{M'} u \Delta_g u \, \mathrm{d}V_g = -\int_{M'} |\mathrm{d}u|^2 \, \mathrm{d}V_g$$

and hence $\mathrm{d}u = 0$. So u is constant on M', but $\int_{M'} u \, \mathrm{d}V_g = 0$, and hence $u \equiv 0$. Since $(2-m, 0)^n$ is a connected subset of $(\mathbb{R}^n \backslash \mathcal{D}_{\Sigma_1}) \times \cdots \times (\mathbb{R}^n \backslash \mathcal{D}_{\Sigma_n})$ that contains $(\frac{1}{2}(2-m), \dots, \frac{1}{2}(2-m))$, it follows from Theorem 6.12 that the kernel of $\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 \to C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0$ is trivial for every $\boldsymbol{\gamma} > 2-m$. Hence $\Delta_g : C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0 \to C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')_0$ is a bijection, and the Open Mapping Theorem [33, XV, Thm. 1.3] implies that this operator is an isomorphism of Banach spaces. □

The next proposition is a version of the Schauder and L^p-estimates for the Laplace operator acting on weighted spaces with discrete asymptotics.

Proposition 6.16. *Let (M, g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\gamma_i \notin \mathcal{E}_{\Sigma_i}$ for $i = 1, \dots, n$. Let $u, f \in L^1_{\mathrm{loc}}(M')$ and assume that $\Delta_g u = f$ holds in the weak sense. Then the following hold.*

(i) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\alpha \in (0,1)$. If $f \in C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')$ and $u \in C^0_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$, then $u \in C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$. Moreover there exists a constant $c > 0$ independent of u and f, such that*

$$\|u\|_{C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}} \leq c\left(\|f\|_{C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}} + \|u\|_{C^0_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}}\right). \tag{52}$$

(ii) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $p \in (1,\infty)$. If $f \in W^{k-2,p}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')$ and $u \in L^p_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$, then $u \in W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$. Moreover there exists a constant $c > 0$ independent of u and f, such that*

$$\|u\|_{W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}} \leq c\left(\|f\|_{W^{k-2,p}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}} + \|u\|_{L^p_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}}\right). \tag{53}$$

Proof. We demonstrate the proof of (i), the proof of (ii) works similarly. We can assume that $\boldsymbol{\gamma} > 0$, since otherwise the discrete asymptotics are trivial and we are in the situation of Proposition 6.11. Let $k \in \mathbb{N}$ with $k \geq 2$, $\alpha \in (0,1)$, and assume that $f \in C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}(M')$ and $u \in C^0_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$. Using that the discrete asymptotics are bounded functions on M' and the weighted Schauder estimates (40) we find that $u \in C^{k,\alpha}_{\mathbf{0}}(M')$. Hence $\Delta_g u = f$ and $\int_{M'} f \, \mathrm{d}V_g = 0$. Choose $\phi \in C^\infty_{\mathrm{cs}}(M')$ with $\int_{M'} \phi \, \mathrm{d}V_g = 1$ and write $u = u_0 + \lambda\phi$ with $u_0 \in C^{k,\alpha}_{\mathbf{0}}(M')_0$ and $\lambda \in \mathbb{R}$. Then Proposition 6.15, (i), implies $u_0 \in C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')_0$ and thus $u \in C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$ as we wanted to show.

It remains to prove the estimate (52). Write $u = u_1 + u_2$ with $u_1 \in C^{k,\alpha}_{\boldsymbol{\gamma}}(M')$, $u_2 \in \mathrm{im}\ \Psi_{\boldsymbol{\gamma}}$ and $f = f_1 + f_2$ with $f_1 \in C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}(M')$ and $f_2 \in \mathrm{im}\ \Psi_{\boldsymbol{\gamma}-2}$. Then

$$\Delta_g u_1 + \pi_{C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}}(\Delta_g u_2) = f_1 \quad \text{and} \quad \pi_{\mathrm{im}\ \Psi_{\boldsymbol{\gamma}-2}}(\Delta_g u_2) = f_2.$$

Using the weighted Schauder estimates and the continuity of the linear operator $\pi_{C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}} \circ \Delta_g : \mathrm{im}\ \Psi_{\boldsymbol{\gamma}} \to C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}(M')$ we find

$$\begin{aligned}
\|u_1\|_{C^{k,\alpha}_{\boldsymbol{\gamma}}} &\le c\left(\|f_1\|_{C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}} + \|\pi_{C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}}(\Delta_g u_2)\|_{C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}} + \|u_1\|_{C^0_{\boldsymbol{\gamma}}}\right)\\
&\le c\left(\|f_1\|_{C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}} + \|u_2\|_{\mathrm{im}\ \Psi_{\boldsymbol{\gamma}}} + \|u_1\|_{C^0_{\boldsymbol{\gamma}}}\right)\\
&= c\left(\|f_1\|_{C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}} + \|u\|_{C^0_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}}\right).
\end{aligned}$$

Lastly we estimate u_2 in terms of f. Choose some small $\varepsilon > 0$ such that $[\gamma_i - \varepsilon, \gamma_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ for $i = 1, \ldots, n$. Then by Theorem 6.12 and Proposition 6.13, $\Delta_g : W^{k,2}_{\boldsymbol{\gamma}-\varepsilon}(M') \to W^{k-2,2}_{\boldsymbol{\gamma}-\varepsilon-2}(M')$ is a Fredholm operator with cokernel being isomorphic to the kernel of $\Delta_g : W^{k,2}_{2-m-\boldsymbol{\gamma}+\varepsilon}(M) \to W^{k-2,2}_{-m-\boldsymbol{\gamma}+\varepsilon}(M')$. Since $u_1 \in C^{k,\alpha}_{\boldsymbol{\gamma}}(M')$, also $u_1 \in W^{k,p}_{\boldsymbol{\gamma}-\varepsilon}(M')$. Using integration by parts we therefore find that

$$\langle h, f_1\rangle_{L^2} = \langle h, \Delta_g u_1 + \pi_{C^{k-2,p}_{\boldsymbol{\gamma}-2}}(\Delta_g u_2)\rangle_{L^2} = \langle h, \pi_{C^{k-2,p}_{\boldsymbol{\gamma}-2}}(\Delta_g u_2)\rangle_{L^2}$$

for $h \in \ker\{\Delta_g : W^{k,2}_{2-m-\boldsymbol{\gamma}+\varepsilon}(M') \to W^{k-2,2}_{-m-\boldsymbol{\gamma}+\varepsilon}(M')\}$. Therefore f_1 determines $\sum_{i=1}^n M_{\Sigma_i}(\gamma_i)$ components of u_2. Moreover $\pi_{\mathrm{im}\ \Psi_{\boldsymbol{\gamma}-2}}(\Delta_g u_2) = f_2$, and hence f_2 determines $\sum_{i=1}^n N_{\Sigma_i}(\gamma_i - 2)$ different components of u_2. Thus by (45), f determines $\sum_{i=1}^n N_{\Sigma_i}(\gamma_i)$ components of u_2. Since $\dim \mathrm{im}\ \Psi_{\boldsymbol{\gamma}} = \sum_{i=1}^n N_{\Sigma_i}(\gamma_i)$, u_2 is uniquely determined by f. Hence $\|u_2\|_{\mathrm{im}\ \Psi_{\boldsymbol{\gamma}}} \le c\|f\|_{C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}}$. $\square$

7 The heat equation on Riemannian manifolds with conical singularities

7.1 Weighted parabolic Hölder and Sobolev spaces

In this subsection we define weighted parabolic Hölder and Sobolev spaces on Riemannian manifolds with conical singularities. These are parabolic Hölder and Sobolev spaces as defined in §2.2, with the only difference that we take the rate of decay of the functions into account. In fact, as for parabolic function spaces on general Riemannian manifolds in §2.2 we require in the definition of weighted parabolic function spaces on Riemannian manifolds with conical singularities that one spatial derivative compares to one time derivative. On a Riemannian manifold with conical singularities two spatial derivatives decrease the rate of decay of a function by two, and hence we need to require that each time derivative has to decrease the rate of decay of a function by two as well.

We begin with the definition of weighted parabolic C^k-spaces and Hölder spaces. Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, ρ a radius function on M', and $I \subset \mathbb{R}$ a bounded and open interval. For $k,l \in \mathbb{N}$ with $2k \leq l$ and $\boldsymbol{\gamma} \in \mathbb{R}^n$ we define

$$C^{k,l}_{\boldsymbol{\gamma}}(I \times M') = \bigcap_{j=0}^{k} C^j(I; C^{l-2j}_{\boldsymbol{\gamma}-2j}(M')).$$

Then $C^{k,l}_{\boldsymbol{\gamma}}(I \times M')$ is a Banach space with norm given by

$$\|u\|_{C^{k,l}_{\boldsymbol{\gamma}}} = \sum_{i,j} \sup_{(t,x)\in I\times M'} |\rho(x)^{-\boldsymbol{\gamma}+2i+j} \partial_t^i \nabla^j u(t,x)| \quad \text{for } u \in C^{k,l}_{\boldsymbol{\gamma}}(I \times M'),$$

where the sum is taken over $i = 1,\ldots,k$ and $j = 1,\ldots,l$ with $2i + j \leq l$. For $\alpha \in (0,1)$ we define the weighted parabolic Hölder space $C^{k,l,\alpha}_{\boldsymbol{\gamma}}(I \times M')$ by

$$C^{k,l,\alpha}_{\boldsymbol{\gamma}}(I \times M') = \bigcap_{j=0}^{k} C^{j,\alpha/2}(I; C^{l-2j}_{\boldsymbol{\gamma}-2j}(M')) \cap C^j(I; C^{l-2j,\alpha}_{\boldsymbol{\gamma}-2j}(M')).$$

The norm on $C^{k,l,\alpha}_{\boldsymbol{\gamma}}(I \times M')$ is given by

$$\begin{aligned}\|u\|_{C^{k,l,\alpha}_{\boldsymbol{\gamma}}} = \sum_{i,j} \Big\{ &\sup_{(t,x)\in I\times M'} |\rho(x)^{-\boldsymbol{\gamma}+2i+j} \partial_t^i \nabla^j u(t,x)| + \sup_{t\in I} [\partial_t^i \nabla^j u(t,\cdot)]_{\alpha,\boldsymbol{\gamma}-2i-j} \\ &+ \sup_{x\in M'} [\partial_t^i \nabla^j u(\cdot,x)]_{\alpha/2,\boldsymbol{\gamma}-2i-j} \Big\} \quad \text{for } u \in C^{k,l,\alpha}_{\boldsymbol{\gamma}}(I \times M'),\end{aligned}$$

where the sum is taken over $i = 1,\ldots,k$ and $j = 1,\ldots,l$ with $2i + j \leq l$. Then $C^{k,l,\alpha}_{\boldsymbol{\gamma}}(I \times M')$ is a Banach space.

Next we define weighted parabolic Sobolev spaces. Let $k,l \in \mathbb{N}$ with $2k \leq l$, $p \in [1,\infty)$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Then the weighted parabolic Sobolev space $W^{k,l,p}_{\boldsymbol{\gamma}}(I \times M')$ is given by

$$W^{k,l,p}_{\boldsymbol{\gamma}}(I \times M') = \bigcap_{j=0}^{k} W^{j,p}(I; W^{l-2j,p}_{\boldsymbol{\gamma}-2j}(M')).$$

Then $W^{k,l,p}_{\boldsymbol{\gamma}}(I\times M')$ is a Banach space with norm given by

$$\|u\|_{W^{k,l,p}_{\boldsymbol{\gamma}}}=\left(\sum_{i,j}\int_I\int_{M'}|\rho^{-\boldsymbol{\gamma}+2i+j}\partial_t^i\nabla^j u(t,\cdot)|^p\rho^{-m}\,\mathrm{d}V_g\,\mathrm{d}t\right)^{1/p}$$

for $u\in W^{k,l,p}_{\boldsymbol{\gamma}}(I\times M')$, where the sum is taken over $i=1,\ldots,k$ and $j=1,\ldots,l$ with $2i+j\leq l$.

Similar to Proposition 2.5 we have the following important embedding result for weighted parabolic Sobolev spaces.

Proposition 7.1. *Let (M,g) be a compact Riemannian manifold manifold with conical singularities as in Definition 6.3, $I\subset\mathbb{R}$ an open and bounded interval, $k\in\mathbb{N}$ with $k\geq 2$, $p\in(1,\infty)$, and $\boldsymbol{\gamma}\in\mathbb{R}^n$. Then $W^{1,k,p}_{\boldsymbol{\gamma}}(I\times M')$ embeds continuously into $W^{0,k-1,p}_{\boldsymbol{\gamma}-1}(I\times M')$ by inclusion and the inclusion is compact.*

The proof of Proposition 7.1 follows immediately from the Aubin–Dubinskiĭ Lemma and Theorem 6.8.

The next proposition is an interpolation result for weighted parabolic Sobolev spaces, which can be seen as a generalization of the interpolation result for parabolic Sobolev spaces from Proposition 2.6.

Proposition 7.2. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $I\subset\mathbb{R}$ an open and bounded interval, $k\in\mathbb{N}$ with $k\geq 2$, $p\in(1,\infty)$, and $\boldsymbol{\gamma}\in\mathbb{R}^n$. Let $\varepsilon>0$ and assume that $p>\frac{2}{\varepsilon}$ and $kp>2+m$. Then $W^{1,k,p}_{\boldsymbol{\gamma}}(I\times M')$ embeds continuously into $C^{0,0}_{\boldsymbol{\gamma}-\varepsilon}(I\times M')$ by inclusion.*

Proof. From Proposition 2.4 it follows that $W^{1,k,p}(I\times M')$ embeds continuously into $C^0(I;(W^{k,p}_{\boldsymbol{\gamma}}(M'),W^{k-2,p}_{\boldsymbol{\gamma}-2}(M'))_{1/p,p})$. From a result of Coriasco et al. [14, Lem. 5.4] it follows that $(W^{k,p}_{\boldsymbol{\gamma}}(M'),W^{k-2,p}_{\boldsymbol{\gamma}-2}(M'))_{1/p,p}$ embeds continuously into $W^{s,p}_{\boldsymbol{\gamma}-\varepsilon}(M')$ by inclusion for $s<k-\frac{2}{p}$, where $W^{s,p}_{\boldsymbol{\gamma}-\varepsilon}(M')$ is a weighted Sobolev space of fractional order. Theorem 6.7 continues to hold for weighted Sobolev spaces of fractional order and hence $W^{s,p}_{\boldsymbol{\gamma}-\varepsilon}(M')$ embeds continuously into $C^0_{\boldsymbol{\gamma}-\varepsilon}(M')$, from which the claim follows. □

Finally we define weighted parabolic spaces with discrete asymptotics. Thus if $m\geq 3$, then for $k,l\in\mathbb{N}$ with $2k\leq l$ we define the weighted parabolic C^k-space $C^{k,l}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(I\times M')$ with discrete asymptotics by

$$C^{k,l}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(I\times M')=\bigcap_{j=0}^{k}C^j(I;C^{l-2j}_{\boldsymbol{\gamma}-2j,\mathsf{P}_{\boldsymbol{\gamma}-2j}}(M')),$$

and if $\alpha\in(0,1)$, then we define the weighted parabolic Hölder space $C^{k,l,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(I\times M')$ with discrete asymptotics by

$$C^{k,l,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(I\times M')=\bigcap_{j=0}^{k}C^{j,\alpha/2}(I;C^{l-2j}_{\boldsymbol{\gamma}-2j,\mathsf{P}_{\boldsymbol{\gamma}-2j}}(M'))\cap C^j(I;C^{l-2j,\alpha}_{\boldsymbol{\gamma}-2j,\mathsf{P}_{\boldsymbol{\gamma}-2j}}(M')).$$

Then both $C^{k,l}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(I \times M')$ and $C^{k,l,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(I \times M')$ are Banach spaces. If $p \in [1,\infty)$, then we define the weighted parabolic Sobolev space $W^{k,l,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(I \times M')$ with discrete asymptotics by

$$W^{k,l,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(I \times M') = \bigcap_{j=0}^{k} W^{j,p}(I; W^{l-2j,p}_{\boldsymbol{\gamma}-2j,\mathsf{P}_{\boldsymbol{\gamma}-2j}}(M')).$$

Clearly $W^{k,l,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(I \times M')$ is a Banach space.

7.2 Weighted Schauder and L^p-estimates

In this subsection we prove weighted Schauder and L^p-estimates for solutions of the inhomogeneous heat equation on compact Riemannian manifolds with conical singularities.

The following proposition is a version of the Schauder estimates for solutions of the inhomogeneous heat equation on weighted parabolic Hölder spaces.

Proposition 7.3. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3. Let $T > 0$, $k \in \mathbb{N}$ with $k \geq 2$, $\alpha \in (0,1)$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Let $f \in C^{0,k-2,\alpha}_{\boldsymbol{\gamma}-2}((0,T)\times M')$ and $u \in C^{0,0}_{\boldsymbol{\gamma}}((0,T)\times M')$. Assume that $u \in C^{1,k,\alpha}((0,T)\times K)$ for every $K \subset\subset M'$ and $\partial_t u = \Delta_g u + f$. Then $u \in C^{1,k,\alpha}_{\boldsymbol{\gamma}}((0,T)\times M')$ and there exists a constant $c > 0$ independent of u and f, such that*

$$\|u\|_{C^{1,k,\alpha}_{\boldsymbol{\gamma}}} \leq c\left(\|f\|_{C^{0,k-2,\alpha}_{\boldsymbol{\gamma}-2}} + \|u\|_{C^{0,0}_{\boldsymbol{\gamma}}}\right). \tag{54}$$

Proof. Let $f \in C^{0,k-2,\alpha}_{\boldsymbol{\gamma}-2}((0,T)\times M')$ and assume that $u \in C^{0,0}_{\boldsymbol{\gamma}}((0,T)\times M')$ with $u \in C^{1,k,\alpha}((0,T)\times K)$ for every $K \subset\subset M'$. Then it follows from the Schauder estimates in Theorem 2.9 that for every $K, K' \subset\subset M'$ with $K' \subset\subset K$ there exists a constant $c > 0$, such that

$$\|u\|_{C^{1,k,\alpha}} \leq c\left(\|f\|_{C^{0,k-2,\alpha}} + \|u\|_{C^{0,0}}\right), \tag{55}$$

where the norm on the left side is over $(0,T)\times K'$ and the norm on the right side is over $(0,T)\times K$. Thus it remains to prove the Schauder estimate (54) on each end of M. Without loss of generality we can assume that $R \leq \sqrt{T}$. Then for $s \in (0,R)$ and $i = 1,\ldots,n$ we define

$$\delta_i^s : (\tfrac{1}{2},1)\times\Sigma_i\times(\tfrac{1}{2},1) \longrightarrow (0,T)\times\Sigma\times(0,R), \quad \delta_i^s(t,\sigma,r) = (s^2t,\sigma,sr).$$

Denote $u_i = \phi_i^*(u)$ and $f_i = \phi_i^*(f)$ for $i = 1,\ldots,n$ and define functions

$$u_i^s : (\tfrac{1}{2},1)\times\Sigma\times(\tfrac{1}{2},1) \to \mathbb{R}, \quad u_i^s = s^{-\gamma_i}(\delta_i^s)^*(u_i) \tag{56}$$

and functions

$$f_i^s : (\tfrac{1}{2},1)\times\Sigma\times(\tfrac{1}{2},1) \to \mathbb{R}, \quad f_i^s = s^{2-\gamma_i}(\delta_i^s)^*(f_i) \tag{57}$$

for $s \in (0,R)$ and $i = 1,\ldots,n$. Then there exists a constant $c > 0$, such that

$$\|u_i^s\|_{C^{0,0}}, \|f_i^s\|_{C^{0,k-2,\alpha}} \leq c \quad \text{on } (\tfrac{1}{2},1)\times\Sigma\times(\tfrac{1}{2},1) \tag{58}$$

for $s \in (0,R)$ and $i = 1,\ldots,n$. Using (38) and the definition of u_i^s and f_i^s in (56) and (57) we find that

$$\frac{\partial u_i^s}{\partial t} = \Delta_{g_i} u_i^s + L_i^s u_i^s + f_i^s \quad \text{on } (\tfrac{1}{2},1)\times\Sigma\times(\tfrac{1}{2},1)$$

for $i = 1,\ldots,n$, where L_i^s is a second order differential operator defined by

$$L_i^s v = s^2 \left\{ \Delta_{\phi_i^*(g)}((\delta_i^{1/s})^*(v)) - \Delta_{g_i}((\delta_i^{1/s})^*(v)) \right\} \circ \delta_i^s.$$

From (36) it follows that the coefficients of L_i^s and their derivatives converge to zero uniformly on compact subsets of $\Sigma_i \times (\frac{1}{2},1)$ as $s \to 0$. Using (58) and again the Schauder estimates from Theorem 2.9 it follows that there exists a constant $c > 0$, such that for every $s \in (0,\kappa)$, where $\kappa \in (0,R)$ is sufficiently small, and $i = 1,\ldots,n$ we have

$$\|u_i^s\|_{C^{1,k,\alpha}} \leq c \left(\|f_i^s\|_{C^{0,k-2,\alpha}} + \|u_i^s\|_{C^{0,0}} \right), \tag{59}$$

where the norm on the left side is on $(\frac{1}{2},1)\times\Sigma_i\times(\frac{2}{3},\frac{3}{4})$ and the norm on the right side is on $(\frac{1}{2},1)\times\Sigma_i\times(\frac{1}{2},1)$. Then it follows that $u \in C^{1,k,\alpha}_{\boldsymbol{\gamma}}((0,T)\times M')$ and (55) and (59) together imply (54). □

The next proposition gives the L^p-estimates for solutions of the inhomogeneous heat equation on weighted parabolic Sobolev spaces.

Proposition 7.4. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3. Let $T > 0$, $k \in \mathbb{N}$ with $k \geq 2$, $p \in (1,\infty)$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Let $f \in W^{0,k-2,p}_{\boldsymbol{\gamma}-2}((0,T)\times M')$ and $u \in W^{0,0,p}_{\boldsymbol{\gamma}}((0,T)\times M')$. Assume that $u \in W^{1,2}((0,T)\times K)$ for every $K \subset\subset M'$ and $\partial_t u = \Delta_g u + f$. Then $u \in W^{1,k,p}_{\boldsymbol{\gamma}}((0,T)\times M')$ and there exists a constant $c > 0$ independent of u and f, such that*

$$\|u\|_{W^{1,k,p}_{\boldsymbol{\gamma}}} \leq c \left(\|f\|_{W^{0,k-2,p}_{\boldsymbol{\gamma}-2}} + \|u\|_{W^{0,0,p}_{\boldsymbol{\gamma}}} \right). \tag{60}$$

Proposition 7.4 is proved in exactly the same way as Proposition 7.3.

7.3 Asymptotics of the Friedrichs heat kernel

In this subsection we study the asymptotic behaviour of the Friedrichs heat kernel on Riemannian manifolds with conical singularities. Our main reference is Mooers [41]. The heat kernel on Riemannian cones is also study by Cheeger [11] and on compact Riemannnian manifolds with conical singularities by Nagase [42] under the assumption that the Riemannian metric is isometric to a Riemannian cone metric near each singularity. Mooers' arguments are in principle the same as those given by Melrose in [39, Ch. 7], where the heat kernel on compact Riemannian manifolds with boundary is studied.

Before we begin our discussion of the asymptotics of the Friedrichs heat kernel on compact Riemannian manifolds with conical singularities let us consider the heat kernel on $\mathbb{R}^m$ as a motivating example. By introducing polar coordinates around the origin in $\mathbb{R}^m$ we can understand $\mathbb{R}^m$ as a Riemannian manifold with conical singularities at the origin. If $x, y \in \mathbb{R}^m \backslash \{0\}$ with $x = (\sigma, r), y = (\sigma', r') \in \mathcal{S}^{m-1} \times (0,\infty)$, then the distance of x and y is given by

$$|x-y|^2 \approx |r-r'|^2 + (r+r')^2 d_h(\sigma,\sigma')^2.$$

Here h is the standard Riemannian metric on $\mathcal{S}^{m-1}$. On $(0,\infty) \times (\mathbb{R}^m \backslash \{0\}) \times (\mathbb{R}^m \backslash \{0\})$ we then define functions $\rho_{\mathrm{bf}}, \rho_{\mathrm{tf}}, \rho_{\mathrm{lb}}, \rho_{\mathrm{rb}}$, and ρ_{tb} by

$$\begin{aligned}
\rho_{\mathrm{bf}}(t,x,y) &= \sqrt{t + r^2 + r'^2}, \quad \rho_{\mathrm{tf}}(t,x,y) = \frac{\sqrt{t + |x-y|^2}}{\sqrt{t + r^2 + r'^2}}, \\
\rho_{\mathrm{lb}}(t,x,y) &= \frac{r}{\sqrt{t + r^2 + r'^2}}, \quad \rho_{\mathrm{rb}}(t,x,y) = \frac{r'}{\sqrt{t + r^2 + r'^2}}, \\
\rho_{\mathrm{tb}}(t,x,y) &= \frac{\sqrt{t}}{\sqrt{t + |x-y|^2}}.
\end{aligned}$$

Then loosely speaking $\rho_{\mathrm{bf}}(t,x,y) = 0$ if and only if $t = 0$ and $r = r' = 0$, $\rho_{\mathrm{tf}}(t,x,y) = 0$ if and only if $t = 0$ and $x = y$, $\rho_{\mathrm{lb}}(t,x,y) = 0$ if and only if $r = 0$, $\rho_{\mathrm{rb}}(t,x,y) = 0$ if and only if $r' = 0$, and finally $\rho_{\mathrm{tb}}(t,x,y) = 0$ if and only if $t = 0$ and $x \neq y$. Now let us consider the Euclidean heat kernel H on $\mathbb{R}^m$ as given in (9). Then

$$H \sim \rho_{\mathrm{tf}}^{-m} \rho_{\mathrm{bf}}^{-m} \rho_{\mathrm{tb}}^{-m} \exp\left(-\frac{\rho_{\mathrm{bf}}^2 \rho_{\mathrm{tf}}^2 - \rho_{\mathrm{tb}}^2 \rho_{\mathrm{bf}}^2 \rho_{\mathrm{tf}}^2}{4 \rho_{\mathrm{tb}}^2 \rho_{\mathrm{bf}}^2 \rho_{\mathrm{tf}}^2} \right) = O\left(\rho_{\mathrm{tf}}^{-m} \rho_{\mathrm{bf}}^{-m} \rho_{\mathrm{tb}}^{\infty} \rho_{\mathrm{lb}}^0 \rho_{\mathrm{rb}}^0 \right).$$

Here $O(\rho_{\mathrm{tb}}^{\infty})$ means $O(\rho_{\mathrm{tb}}^k)$ for every $k \in \mathbb{N}$. It turns out that this is the leading order asymptotic behaviour of the Friedrichs heat kernel on general compact Riemannian manifolds with conical singularities.

Now let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$, and H the Friedrichs heat kernel on (M,g). The next proposition examines a simple property of the function $x \mapsto H(t,x,y)$ for fixed $t > 0$ and $y \in M'$.

Proposition 7.5. *Let $t > 0$ and $y \in M'$. Then the function $x \mapsto H(t,x,y)$ lies in $\bigcap_{\boldsymbol{\gamma} \in \mathbb{R}^n} C^{\infty}_{\boldsymbol{\gamma}, \mathsf{P}_{\boldsymbol{\gamma}}}(M')$.*

Proof. The Friedrichs heat semigroup $\{\exp(t\Delta_g)\}_{t>0}$ as defined by (5) and (6) is a semigroup of bounded operators on $L^2(M')$. Moreover for every $t > 0$, $\exp(t\Delta_g)$ maps $L^2(M')$ into $\bigcap_{j=0}^{\infty} \mathrm{dom}(\Delta_g^j)$ as in (7). It follows that for fixed $t > 0$ and $y \in M'$ the function $x \mapsto H(t,x,y)$ lies in $\bigcap_{j=0}^{\infty} \mathrm{dom}(\Delta_g^j)$. Using Proposition 6.15 and Theorem 6.7 we find that

$$\bigcap_{j=0}^{\infty} \mathrm{dom}(\Delta_g^j) = \bigcap_{\boldsymbol{\gamma} \in \mathbb{R}^n} C^{\infty}_{\boldsymbol{\gamma}, \mathsf{P}_{\boldsymbol{\gamma}}}(M')$$

from which the claim follows. □

Let H_i be the Friedrichs heat kernel on (C_i, g_i) for $i = 1, \ldots, n$, where $(C_1, g_1), \ldots, (C_n, g_n)$ are the model cones of (M,g) as in Definition 6.3. For $i = 1, \ldots, n$ and $s \in (0,\infty)$ define

$$\delta_i^s : (0,\infty) \times C_i \times C_i \longrightarrow (0,\infty) \times C_i \times C_i, \ \delta_i^s(t,\sigma,r,\sigma',r') = (s^2 t, \sigma, sr, \sigma', sr').$$

Then

$$(\delta_i^s)^*(t\Delta_{g_i}\varphi)(\sigma,r) = t\Delta_{g_i}(\delta_i^s)^*(\varphi)(\sigma,r) \quad \text{for } (\sigma,r) \in C_i \tag{61}$$

and $i = 1, \ldots, n$ and $\varphi \in \mathrm{dom}(\Delta_{g_i})$. Here $\mathrm{dom}(\Delta_{g_i})$ is the domain of the Friedrichs extension of the Laplace operator $\Delta_{g_i} : C^\infty_{\mathrm{cs}}(C_i) \subset L^2(C_i) \to L^2(C_i)$. Then (61) implies that

$$(\delta_i^s)^*(\exp(t\Delta_{g_i})\varphi) = \exp(t\Delta_{g_i})(\delta_i^s)^*(\varphi) \quad \text{for } \varphi \in \mathrm{dom}(\Delta_{g_i}) \tag{62}$$

and $i = 1, \ldots, n$. From (62) and Proposition 3.1 we then conclude that

$$(\delta_i^s)^*(H_i)(t, \sigma, r, \sigma', r') = s^{-m} H_i(t, r, \sigma, r', \sigma') \text{ for } (t, \sigma, r, \sigma', r') \in (0, \infty) \times C_i \times C_i$$

and $i = 1, \ldots, n$.

The next proposition describes the homogeneity of the Friedrichs heat kernel on (M, g) when g is isometric to the Riemannian cone metric g_i on each end of M' for $i = 1, \ldots, n$.

Proposition 7.6. *Let (M, g) be a Riemannian manifold with conical singularities as in Definition 6.3 and let $\tilde{g}$ be a Riemannian metric on M with $\phi_i^*(\tilde{g}) = g_i$ for $i = 1, \ldots, n$, i.e. the Riemannian metric is isometric to the Riemannian cone metric on each end of M'. Let $\tilde{H}$ be the Friedrichs heat kernel on $(M, \tilde{g})$. Then*

$$(\delta_i^s)^*(\phi_i^*(\tilde{H})) - s^{-m} H_i = O(s^\infty) \quad \textit{as } s \to 0$$

for $i = 1, \ldots, n$.

The proof of Proposition 7.6 can be found in Nagase [42, §5].

We now discuss parts of Mooers' parametrix construction for the Friedrichs heat kernel [41]. We explain this construction only in an informal way and the interested reader should consult Mooers' paper for a detailed description. In order to describe the asymptotics of the Friedrichs heat kernel it is convenient to introduce functions $\rho_{\mathrm{bf}}, \rho_{\mathrm{tf}}, \rho_{\mathrm{lb}}, \rho_{\mathrm{rb}}$, and ρ_{tb} on $(0, \infty) \times M' \times M'$ as follows. Let ρ be a radius function on M' and define

$$\begin{aligned}
\rho_{\mathrm{bf}}(t, x, y) &= \sqrt{t + \rho(x)^2 + \rho(y)^2}, \quad \rho_{\mathrm{tf}}(t, x, y) = \frac{\sqrt{t + d_g(x, y)^2}}{\sqrt{t + \rho(x)^2 + \rho(y)^2}}, \\
\rho_{\mathrm{lb}}(t, x, y) &= \frac{\rho(x)}{\sqrt{t + \rho(x)^2 + \rho(y)^2}}, \quad \rho_{\mathrm{rb}}(t, x, y) = \frac{\rho(y)}{\sqrt{t + \rho(x)^2 + \rho(y)^2}}, \\
\rho_{\mathrm{tb}}(t, x, y) &= \frac{\sqrt{t}}{\sqrt{t + d_g(x, y)^2}}.
\end{aligned}$$

Loosely speaking we have that $\rho_{\mathrm{bf}}(t, x, y) = 0$ if and only if $t = 0$ and $\rho(x) = \rho(y) = 0$, $\rho_{\mathrm{tf}}(t, x, y) = 0$ if and only if $t = 0$ and $x = y$, $\rho_{\mathrm{lb}}(t, x, y) = 0$ if and only if $\rho(x) = 0$, $\rho_{\mathrm{rb}}(t, x, y) = 0$ if and only if $\rho(y) = 0$, and finally $\rho_{\mathrm{tb}}(t, x, y) = 0$ if and only if $t = 0$ and $x \neq y$. In fact the functions $\rho_{\mathrm{bf}}, \rho_{\mathrm{tf}}, \rho_{\mathrm{lb}}, \rho_{\mathrm{rb}}$, and ρ_{tb} should be understood as boundary defining functions on the heat space of M, see Melrose [39, Ch. 7, §4].

From Theorem 3.2 we have a good understanding of the asymptotics of $H(t, x, y)$, when x and y lie in a compact region, so we only have to study the asymptotics of the heat kernel, when x and/or y are close to a singularity. The first step in the parametrix construction for the heat kernel is to find a rough parametrix H_0, i.e. a good first approximation, for H. The rough parametrix H_0 is constructed by gluing the heat kernels on the model cones of the conical singularities together with the heat kernel H. Since the Laplace operator on M' near each conical singularity is asymptotic to the Laplace operator on the model cone of the singularity, it follows that H_0 is a good first approximation for the heat kernel H and determines the leading order terms in the asymptotic expansion of H in terms of $\rho_{\mathrm{bf}}, \rho_{\mathrm{tf}}, \rho_{\mathrm{lb}}, \rho_{\mathrm{rb}}$, and ρ_{tb}. Using the discussion from above, we have a good understanding of the asymptotics of H_0, and, in fact, one can determine the expansion of H_0 in terms of the functions $\rho_{\mathrm{bf}}, \rho_{\mathrm{tf}}, \rho_{\mathrm{lb}}, \rho_{\mathrm{rb}}$, and ρ_{tb} and show that $H_0 \sim \rho_{\mathrm{tf}}^{-m}\rho_{\mathrm{bf}}^{-m}\rho_{\mathrm{tb}}^{\infty}\rho_{\mathrm{lb}}^{0}\rho_{\mathrm{rb}}^{0}$, see Mooers [41, Prop. 3.3]. (Note, however, that due to a mistake in [41, Lem. 3.2] the power -1 of the function ρ_{bf} in Mooers' result should be replaced by $-m$). What is left, is to solve away the error terms caused by the gluing procedure and the asymptoticness of the Laplace operator on M' to the Laplace operators on the model cones. This is done in Mooers [41, Prop. 3.4 – 3.8].

Of particular importance for us are the asymptotics of H when $\rho_{\mathrm{lb}}, \rho_{\mathrm{rb}} \to 0$, since this is where the discrete asymptotics come into play. Let $\boldsymbol{\gamma} \in \mathbb{R}^n$ and define $\boldsymbol{\gamma}^+, \boldsymbol{\gamma}^- \in \mathbb{R}^n$ by

$$\gamma_i^+ = \min\{\varepsilon \in \mathcal{E}_{\Sigma_i} \,:\, \varepsilon \geq \gamma_i\} \quad \text{and} \quad \gamma_i^- = \max\{\varepsilon \in \mathcal{E}_{\Sigma_i} \,:\, \varepsilon < \gamma_i\} \tag{63}$$

for $i = 1, \ldots, n$. For $\boldsymbol{\gamma} \in \mathbb{R}^n$ we choose a basis $\psi^1_{\boldsymbol{\gamma}}, \ldots, \psi^N_{\boldsymbol{\gamma}}$ for $\operatorname{im} \Psi_{\boldsymbol{\gamma}}$, where $N = \dim \operatorname{im} \Psi_{\boldsymbol{\gamma}}$. Recall from above that the function $x \mapsto H(t,x,y)$ lies in $\bigcap_{\boldsymbol{\gamma}\in\mathbb{R}^n} C^{\infty}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}(M')$ for fixed $t > 0$ and $y \in M'$. Now one can deduce from [41, Prop. 3.5] that there exist functions $H^1_{\boldsymbol{\gamma}}, \ldots, H^N_{\boldsymbol{\gamma}} \in C^\infty((0,\infty)\times M')$ that admit an asymptotic expansion of the form

$$H^j_{\boldsymbol{\gamma}} \sim \rho_{\mathrm{tf}}^{-m}\rho_{\mathrm{bf}}^{-m}\rho_{\mathrm{tb}}^{\infty}\rho_{\mathrm{rb}}^{-\boldsymbol{\gamma}^-} \quad \text{for } j = 1, \ldots, N, \tag{64}$$

and such that we have an asymptotic expansion of the form

$$H - \sum\nolimits_{j=1}^{N} \psi^j_{\boldsymbol{\gamma}} H^j_{\boldsymbol{\gamma}} \sim \rho_{\mathrm{tf}}^{-m}\rho_{\mathrm{bf}}^{-m}\rho_{\mathrm{tb}}^{\infty}\rho_{\mathrm{lb}}^{\boldsymbol{\gamma}^+}. \tag{65}$$

The time derivatives of H then admit a similar expansion and from (64) and (65) we then deduce the following result.

Theorem 7.7. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$, H the Friedrichs heat kernel on (M,g), and $\boldsymbol{\gamma} \in \mathbb{R}^n$. For $l \in \mathbb{N}$ choose a basis $\psi^1_{\boldsymbol{\gamma}-2l}, \ldots, \psi^{N_l}_{\boldsymbol{\gamma}-2l}$ for* $\operatorname{im} \Psi_{\boldsymbol{\gamma}-2l}$*, where $N_l = \dim \operatorname{im} \Psi_{\boldsymbol{\gamma}-2l}$. Then the following holds.*

For each $l \in \mathbb{N}$ there exist functions $H^1_{\boldsymbol{\gamma}-2l}, \ldots, H^{N_l}_{\boldsymbol{\gamma}-2l} \in C^\infty((0,\infty)\times M')$ and constants $c_l > 0$, such that for each $l \in \mathbb{N}$

$$|H^j_{\boldsymbol{\gamma}-2l}(t,y)| \leq c_l \cdot (t + \rho(y)^2)^{-\frac{m+(\boldsymbol{\gamma}-2l)^-}{2}} \quad \text{for } t > 0,\ y \in M'$$

and $j = 1, \ldots, N_l$, and

$$\begin{aligned}&\Big|\partial_t^l H(t,x,y) - \sum\nolimits_{j=1}^{N_l} \psi^j_{\boldsymbol{\gamma}-2l}(x) H^j_{\boldsymbol{\gamma}-2l}(t,y))\Big| \\ &\qquad\qquad \leq c_l \cdot (t + d_g(x,y)^2)^{-\frac{m+l}{2}} \left(\frac{\rho(x)^2}{\rho(x)^2+\rho(y)^2}\right)^{\frac{(\boldsymbol{\gamma}-2l)^+}{2}},\end{aligned}$$

for $t > 0$, and $x, y \in M'$. Here $\boldsymbol{\gamma}^+$ and $\boldsymbol{\gamma}^-$ are given in (63).

We should remark at this point that the wrong asymptotics of the Friedrichs heat kernel given by Mooers were also used by Jeffres and Loya in [22] in their study of the regularity of solutions of the heat equation on Riemannian manifolds with conical singularities.

7.4 The Cauchy problem for the inhomogeneous heat equation. I

In this subsection we prove existence and maximal regularity of solutions to the Cauchy problem for the inhomogeneous heat equation, when the free term lies in a weighted Hölder space with discrete asymptotics. There are only a few papers known to the author where parabolic equations on compact Riemannian manifolds with conical singularities are studied. We would like to mention the papers by Coriasco, Schrohe, and Seiler [13] and [14] in particular, since these initially motivated our study of the heat equation. Their approach to the study of linear parabolic equations on compact Riemannian manifolds with conical singularities is through semigroup theory, and therefore lasts heavily on techniques from functional analysis, while our approach using the heat kernel is more PDE style. In the special case of the heat equation our results, Theorems 7.10 and 7.13 below, generalize those obtained by Coriasco, Schrohe, and Seiler. We point out, however, that their results also apply to more general linear parabolic equations than just the heat equation.

Throughout this subsection (M, g) will be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$, ρ a radius function on M', and H will denote the Friedrichs heat kernel on (M, g). In the next two propositions we prove two elementary, though important, estimates for the convolution of H with powers of ρ.

Proposition 7.8. *Let $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\boldsymbol{\gamma} > 2 - m$ and $\gamma_i \notin \mathcal{E}_{\Sigma_i}$ for $i = 1, \ldots, n$. Then there exist a constants $c_l > 0$ for $l \in \mathbb{N}$, such that*

$$\left| \left(\left(\partial_t^l H - \sum\nolimits_{j=1}^{N_l} \psi^j_{\boldsymbol{\gamma}-2l} H^j_{\boldsymbol{\gamma}-2l} \right) * \rho^{\boldsymbol{\gamma}-2} \right)(t, x) \right| \leq c_l \cdot \rho(x)^{\boldsymbol{\gamma}-2l}$$

for every $t \in (0, \infty)$ and $x \in M'$. Here $\psi^j_{\boldsymbol{\gamma}-2l}$ and $H^j_{\boldsymbol{\gamma}-2l}$ are given by Theorem 7.7 for $j = 1, \ldots, N_l$.

Proof. We only consider the case $l = 0$, the general case is proved essentially in the same way. Denote

$$I(t, x) = \left(\left(H - \sum\nolimits_{j=1}^{N_0} \psi^j_{\boldsymbol{\gamma}} H^j_{\boldsymbol{\gamma}} \right) * \rho^{\boldsymbol{\gamma}-2} \right)(t, x)$$

for $t \in (0, \infty)$ and $x \in M'$. Using Theorem 7.7 we find that

$$\begin{aligned} |I(t, x)| &\leq \int_0^t \int_{M'} \left| H(s, x, y) - \sum\nolimits_{j=1}^{N_0} \psi^j_{\boldsymbol{\gamma}}(x) H^j_{\boldsymbol{\gamma}}(s, y) \right| \rho(y)^{\boldsymbol{\gamma}-2} \, \mathrm{d}V_g(y) \, \mathrm{d}s \\ &\leq c \cdot \rho(x)^{\boldsymbol{\gamma}^+} \int_{M'} \rho(y)^{\boldsymbol{\gamma}-2} (\rho(x)^2 + \rho(y)^2)^{-\frac{\boldsymbol{\gamma}^+}{2}} \int_0^t (s + d_g(x, y)^2)^{-\frac{m}{2}} \, \mathrm{d}s \, \mathrm{d}V_g(y), \end{aligned}$$

where $\boldsymbol{\gamma}^+$ is as in (63). Since $m \geq 3$, we can estimate the integral with respect to s by

$$\int_0^t (s + d_g(x,y)^2)^{-\frac{m}{2}} \, \mathrm{d}s \leq c \cdot d_g(x,y)^{2-m}$$

and thus obtain

$$|I(t,x)| \leq c \cdot \rho(x)^{\boldsymbol{\gamma}^+} \int_{M'} \rho(y)^{\boldsymbol{\gamma}-2} (\rho(x)^2 + \rho(y)^2)^{-\frac{\boldsymbol{\gamma}^+}{2}} d_g(x,y)^{2-m} \, \mathrm{d}V_g(y). \quad (66)$$

For the sake of simplicity we assume from now on that $\phi_i(g) = g_i$ for $i = 1, \ldots, n$. The general case then follows in a similar way because the error terms caused by the asymptotic condition (36) can be controlled by the estimates which we now prove. Let $R' > 0$ with $\frac{R}{2} < R' < R$ and assume that x lies in $S_i' = \phi_i(\Sigma_i \times (0, R'))$ for some $i = 1, \ldots, n$. The case $x \in M' \backslash S_i'$ is dealt with in a similar way. We now split the integral over M' in (66) into two integrals, one over S_i and the other one over $M' \backslash S_i$. We first study the integral over S_i.

If $y \in S_i$, then $d_g(x,y)^2 \geq c(r^2 + r'^2) d_h(\sigma, \sigma')^2$ for every $y \in S_i$ where $x = \phi_i(\sigma, r)$ and $y = \phi_i(\sigma', r')$. In particular $d_g(x,y)^2 \geq c(r^2 + r'^2) d_h(\sigma, \sigma')^2$ for every $y \in S_i$. Moreover we can assume that $\rho(x) = r$ and $\rho(y) = r'$. Using $\mathrm{d}V_{g_i}(\sigma', r') = r'^{m-1} \mathrm{d}r' \, \mathrm{d}V_{h_i}(\sigma')$ we thus obtain

$$\begin{aligned}
&\int_{S_i} \rho(y)^{\boldsymbol{\gamma}-2} (\rho(x)^2 + \rho(y)^2)^{-\frac{\boldsymbol{\gamma}^+}{2}} d_g(x,y)^{2-m} \, \mathrm{d}V_g(y) \\
&\qquad \leq c \int_0^R \int_{\Sigma_i} r'^{\gamma_i + m - 3} (r^2 + r'^2)^{1 - \frac{m + \gamma_i^+}{2}} d_h(\sigma, \sigma')^{2-m} \, \mathrm{d}V_h(\sigma') \, \mathrm{d}r' \\
&\qquad \leq c \int_0^R r'^{\gamma_i + m - 3} (r^2 + r'^2)^{1 - \frac{m + \gamma_i^+}{2}} \, \mathrm{d}r',
\end{aligned}$$

where in the last estimate we use that the integral with respect to σ' is finite, since $\dim \Sigma_i = m - 1$. With the change of variables $r' \mapsto \varrho = (\frac{r'}{r})^2$ we find

$$\int_0^R r'^{\gamma_i + m - 3} (r^2 + r'^2)^{1 - \frac{m + \gamma_i^+}{2}} \, \mathrm{d}r' \leq c \cdot r^{\gamma_i - \gamma_i^+} \int_0^\infty \varrho^{\frac{\gamma_i + m - 4}{2}} (1 + \varrho)^{1 - \frac{m + \gamma_i^+}{2}} \, \mathrm{d}\varrho.$$

Now the integral with respect to ϱ is finite if and only if $\frac{\gamma_i + m - 4}{2} > -1$ and $\frac{\gamma_i + m - 4}{2} + 1 - \frac{m + \gamma_i^+}{2} < -1$, which holds if and only if $2 - m < \gamma_i < \gamma_i^+$. Therefore we obtain that

$$\rho(x)^{\boldsymbol{\gamma}^+} \int_{S_i} \rho(y)^{\boldsymbol{\gamma}-2} (\rho(x)^2 + \rho(y)^2)^{-\frac{\boldsymbol{\gamma}^+}{2}} d_g(x,y)^{2-m} \, \mathrm{d}V_g(y) \leq c \rho(x)^{\boldsymbol{\gamma}}. \quad (67)$$

Now assume that $y \in M' \backslash S_i$. Then $d_g(x,y)$ is uniformly bounded from below, as $x \in S_i'$. Hence we can estimate $d_g(x,y)^2 \geq c(\rho(x)^2 + \rho(y)^2)$ uniformly for $y \in M' \backslash S_i$. From (66) we thus obtain

$$\begin{aligned}
&\int_{M' \backslash S_i} \rho(y)^{\boldsymbol{\gamma}-2} (\rho(x)^2 + \rho(y)^2)^{-\frac{\boldsymbol{\gamma}^+}{2}} d_g(x,y)^{2-m} \, \mathrm{d}V_g(y) \\
&\qquad \leq c \int_{M' \backslash S_i} \rho(y)^{\boldsymbol{\gamma}-2} (\rho(x)^2 + \rho(y)^2)^{1 - \frac{m + \boldsymbol{\gamma}^+}{2}} \, \mathrm{d}V_g(y).
\end{aligned}$$

Using the same estimates as before it is now straightforward to check that

$$\int_{M'\setminus S_i} \rho(y)^{\boldsymbol{\gamma}-2}(\rho(x)^2+\rho(y)^2)^{1-\frac{m+\boldsymbol{\gamma}^+}{2}}\,\mathrm{d}V_g(y) \le c\cdot\rho(x)^{\boldsymbol{\gamma}-\boldsymbol{\gamma}^+}.$$

We find that

$$\rho(x)^{\boldsymbol{\gamma}^+}\int_{M'\setminus S_i} \rho(y)^{\boldsymbol{\gamma}-2}(\rho(x)^2+\rho(y)^2)^{-\frac{\boldsymbol{\gamma}^+}{2}}d_g(x,y)^{2-m}\,\mathrm{d}V_g(y) \le c\rho(x)^{\boldsymbol{\gamma}}. \quad (68)$$

Finally from (66), (67), and (68) we conclude that $|I(t,x)| \le c\rho(x)^{\boldsymbol{\gamma}}$ for $t\in(0,\infty)$ and $x\in M'$, as we wanted to show. □

Proposition 7.9. *Let $\boldsymbol{\gamma}\in\mathbb{R}^n$ with $\gamma_i\notin\mathcal{E}_{\Sigma_i}$ for $i=1,\dots,n$. Then there exists a constant $c>0$, such that $\left|(H^j_{\boldsymbol{\gamma}-2l}*\rho^{\boldsymbol{\gamma}-2})(t)\right|\le c$ for every $t\in(0,\infty)$ and $j=1,\dots,N_l$, where $H^j_{\boldsymbol{\gamma}-2l}$ is as in Theorem 7.7.*

Proof. Again we only consider the case $l=0$. Fix some $j=1,\dots,N_0$ and denote $I(t)=(H^j_{\boldsymbol{\gamma}}*\rho^{\boldsymbol{\gamma}-2})(t)$ for $t\in(0,\infty)$. Using the estimates from Theorem 7.7 we obtain that

$$\begin{aligned}|I(t)| &\le \int_{M'}|H^j_{\boldsymbol{\gamma}}(s,y)|\rho(y)^{\boldsymbol{\gamma}-2}\,\mathrm{d}V_g(y)\\ &\le c\int_{M'}\rho(y)^{\boldsymbol{\gamma}-2}\int_0^t(s+\rho(y)^2)^{-\frac{m+\boldsymbol{\gamma}^-}{2}}\,\mathrm{d}s\,\mathrm{d}V_g(y),\end{aligned}$$

where $\boldsymbol{\gamma}^-$ is as in (63). We can estimate the integral with respect to s by

$$\int_0^t(s+\rho(y)^2)^{-\frac{m+\boldsymbol{\gamma}^-}{2}}\,\mathrm{d}s \le c\cdot\rho(y)^{2-m-\boldsymbol{\gamma}^-}$$

and hence we obtain

$$|I(t)| \le c\int_{M'}\rho(y)^{\boldsymbol{\gamma}-\boldsymbol{\gamma}^- -m}\,\mathrm{d}V_g(y). \quad (69)$$

Using that the Riemannian metric $\phi_i^*(g)$ is asymptotic to the Riemannian cone metric g_i on $\Sigma_i\times(0,R)$ and using that $\mathrm{d}V_{g_i}(\sigma,r)=r^{m-1}\mathrm{d}r\ \mathrm{d}V_{h_i}(\sigma)$ it follows that the integral in (69) is finite if and only if $\boldsymbol{\gamma}-\boldsymbol{\gamma}^- -m+(m-1)>-1$, which holds if and only if $\boldsymbol{\gamma}>\boldsymbol{\gamma}^-$. □

For $T>0$ and a given function $f:(0,T)\times M'\to\mathbb{R}$ we now consider the following Cauchy problem

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,x)=\Delta_g u(t,x)+f(t,x) && \text{for } (t,x)\in(0,T)\times M',\\ &u(0,x)=0 && \text{for } x\in M',\end{aligned} \quad (70)$$

i.e. we look for a function $u:(0,T)\times M'\to\mathbb{R}$ that extends continuously to $t=0$ and satisfies (70). Using Propositions 7.8 and 7.9 we are now able to prove existence and maximal regularity of solutions to (70), when f lies in a weighted Hölder space with discrete asymptotics.

Theorem 7.10. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$. Let $T > 0$, $k \in \mathbb{N}$ with $k \geq 2$, $\alpha \in (0,1)$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\boldsymbol{\gamma} > 2-m$ and $\gamma_i \notin \mathcal{E}_{\Sigma_i}$ for $i = 1,\dots,n$. Given $f \in C^{0,k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}((0,T)\times M')$, there exists a unique $u \in C^{1,k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T)\times M')$ solving the Cauchy problem* (70).

Proof. Let $f \in C^{0,k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}((0,T)\times M')$, then we can write $f = f_1 + f_2$ with

$$f_1 \in C^{0,k-2,\alpha}_{\boldsymbol{\gamma}-2}((0,T)\times M') \quad \text{and} \quad f_2 \in C^{0,\alpha/2}((0,T);\mathrm{im}\,\Psi_{\boldsymbol{\gamma}-2}).$$

Define $u = H * f$, $u_1 = H * f_1$, and $u_2 = H * f_2$, where H is the Friedrichs heat kernel and convolution is defined as in (12). Using Theorem 7.7 we can write

$$u_1(t,x) = \left(\left(H - \sum\nolimits_{j=1}^{N_0} \psi^j_{\boldsymbol{\gamma}} H^j_{\boldsymbol{\gamma}}\right) * f_1\right)(t,x) + \sum\nolimits_{j=1}^{N_0} \psi^j_{\boldsymbol{\gamma}}(x)(H^j_{\boldsymbol{\gamma}} * f_1)(t)$$

for $t \in (0,T)$ and $x \in M'$. Using $f_1 \in C^{0,k-2,\alpha}_{\boldsymbol{\gamma}-2}((0,T)\times M')$ and Proposition 7.8 we find that

$$\left|\left(\left(H - \sum\nolimits_{j=1}^{N_0} \psi^j_{\boldsymbol{\gamma}} H^j_{\boldsymbol{\gamma}}\right) * f_1\right)(t,x)\right| \leq c\|f_1\|_{C^{0,0}_{\boldsymbol{\gamma}-2}} \rho(x)^{\boldsymbol{\gamma}}.$$

Moreover, from Proposition 7.9 it follows that

$$|(H^j_{\boldsymbol{\gamma}} * f_1)(t)| \leq c\|f_1\|_{C^{0,0}_{\boldsymbol{\gamma}-2}}$$

for $j = 1,\dots,N_0$. Hence $u_1 \in C^{0,0}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T)\times M')$. In a similar way one can now show that in fact $u_1 \in C^{1,k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T)\times M')$.

Alternatively one can show that $u_1 \in C^{1,k,\alpha}((0,T)\times K)$ for every $K \subset\subset M'$. If $\boldsymbol{\gamma} < 0$, then the discrete asymptotics are trivial and the weighted Schauder estimates from Proposition 7.3 imply that $u_1 \in C^{1,k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T)\times M')$. If $\boldsymbol{\gamma} > 0$, then $u_1 \in C^{0,0}_{\mathbf{0}}((0,T)\times M')$, since the discrete asymptotics are bounded functions on M'. Therefore again the weighted Schauder estimates imply that in fact $u_1 \in C^{1,k,\alpha}_{\mathbf{0}}((0,T)\times M')$. But then using Proposition 6.15 and a simple iteration argument we conclude that $u_1 \in C^{1,k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T)\times M')$.

The same argument as before also shows that $u_2 = H*f_2$ lies in $C^{1,l,\alpha}_{\boldsymbol{\delta},\mathsf{P}_{\boldsymbol{\delta}}}((0,T)\times M')$ for every $l \in \mathbb{N}$ and $\boldsymbol{\delta} \in \mathbb{R}^n$. Hence $u \in C^{1,k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T)\times M')$ and u solves the Cauchy problem (70).

In order to show that u is the unique solution of the Cauchy problem (70) it suffices to show that if $u \in C^{1,k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T)\times M')$ solves the Cauchy problem (11) with $f \equiv 0$, then $u \equiv 0$. Thus let $u \in C^{1,k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T)\times M')$ be a solution of (11) with $f \equiv 0$ and assume first that $\boldsymbol{\gamma} > 1 - \frac{m}{2}$. Then for $t \in (0,T)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^2} = 2\langle \Delta_g u(t,\cdot), u(t,\cdot)\rangle_{L^2} = -2\|\nabla u(t,\cdot)\|^2_{L^2} \leq 0.$$

Since $u(0,\cdot) \equiv 0$, it follows that $u \equiv 0$. Now assume that $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $2-m < \boldsymbol{\gamma} \leq 1 - \frac{m}{2}$. Then it easily follows that $\int_{M'} u(t,x)\,\mathrm{d}V_g(x) = 0$ for $t \in (0,T)$. Using Proposition 6.15 we can define $u_1 = \Delta_g^{-1} u$. Then $u_1 \in C^{1,k+2,\alpha}_{\boldsymbol{\gamma}+2,\mathsf{P}_{\boldsymbol{\gamma}+2}}((0,T)\times$

M') and u_1 solves the Cauchy problem (11) with $f \equiv 0$. We can iterate this argument and define $u_l = \Delta_g^{-l} u$ for $l \in \mathbb{N}$ with $\boldsymbol{\gamma} + 2l > 1 - \frac{m}{2}$. Then $u_l \in C^{1,k+2l,\alpha}_{\boldsymbol{\gamma}+2l,\mathsf{P}_{\boldsymbol{\gamma}+2l}}((0,T) \times M')$ and u_l solves the Cauchy problem (11) with $f \equiv 0$. Then as above it follows that $u_l \equiv 0$ and hence $u \equiv 0$. This completes the proof of Theorem 7.10. □

7.5 The Cauchy problem for the inhomogeneous heat equation. II

In this subsection we prove existence and maximal regularity of solutions to the Cauchy problem for the inhomogeneous heat equation on compact Riemannian manifolds with conical singularities, when the free term lies in a weighted Sobolev space with discrete asymptotics.

We first recall Young's inequality which can be found in Krylov [30, Ch. 1, §8, Lem. 1] for instance. For $G, f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ the convolution of G and f is given by $(G * f)(x) = \int G(x-y) f(y) \, \mathrm{d}y$, whenever it is well defined. Then Young's inequality states that if $G \in L^1(\mathbb{R}^m)$ and $f \in L^p(\mathbb{R}^m)$ for $p \in [1, \infty)$, then the convolution $G * f$ lies in $L^p(\mathbb{R}^m)$ and $\|G * f\|_{L^p} \leq \|G\|_{L^1} \|f\|_{L^p}$.

We now prove a generalization of Young's inequality to Riemannian manifolds with conical singularities and weighted L^p-norms.

Proposition 7.11. *Let (M, g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $T > 0$, $p \in (1, \infty)$, and $\boldsymbol{\delta}, \boldsymbol{\varepsilon} \in \mathbb{R}^n$. Let $f \in W^{0,0,p}_{\boldsymbol{\varepsilon}}((0,T) \times M')$ and $G \in C^0_{\mathrm{loc}}(((0,T) \times (0,T) \times M' \times M') \backslash \Delta)$, where $\Delta = \{(t,t,x,x) \; : \; t \in (0,T), \; x \in M'\}$. Assume that*

$$\sup_{\substack{t \in (0,T) \\ x \in M'}} \rho(x)^{\boldsymbol{\alpha}_2} \|G(t, \cdot, x, \cdot)\|_{W^{0,0,1}_{-\boldsymbol{\beta}_2 - m}}, \quad \sup_{\substack{s \in (0,T) \\ y \in M'}} \rho(y)^{\boldsymbol{\beta}_1} \|G(\cdot, s, \cdot, y)\|_{W^{0,0,1}_{-\boldsymbol{\alpha}_1 + \boldsymbol{\delta} p}} < \infty$$

for some $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathbb{R}^n$ that satisfy

$$\frac{\boldsymbol{\alpha}_1}{p} + \boldsymbol{\alpha}_2 \left(1 - \frac{1}{p}\right) = 0 \quad \textit{and} \quad \frac{\boldsymbol{\beta}_1}{p} + \boldsymbol{\beta}_2 \left(1 - \frac{1}{p}\right) = \boldsymbol{\varepsilon} + \frac{m}{p}. \tag{71}$$

Then $G * f \in W^{0,0,p}_{\boldsymbol{\delta}}((0,T) \times M')$ and moreover

$$\|G * f\|_{W^{0,0,p}_{\boldsymbol{\delta}}} \leq \|f\|_{W^{0,0,p}_{\boldsymbol{\varepsilon}}} \sup_{\substack{t \in (0,T) \\ x \in M'}} \rho(x)^{\boldsymbol{\alpha}_2 (1 - \frac{1}{p})} \|G(t, \cdot, x, \cdot)\|^{1 - \frac{1}{p}}_{W^{0,0,1}_{-\boldsymbol{\beta}_2 - m}}$$
$$\times \sup_{\substack{s \in (0,T) \\ y \in M'}} \rho(y)^{\frac{\boldsymbol{\beta}_1}{p}} \|G(\cdot, s, \cdot, y)\|^{\frac{1}{p}}_{W^{0,0,1}_{-\boldsymbol{\alpha}_1 + \boldsymbol{\delta} p}}.$$

Proof. Without loss of generality we can assume that f and G are non-negative. We write

$$\begin{aligned} G(t,s,x,y) f(s,y) &= (G(t,s,x,y) f(s,y)^p)^{\frac{1}{p}} \, G(t,s,x,y)^{1 - \frac{1}{p}} \\ &= (G(t,s,x,y) f_{\boldsymbol{\varepsilon}}(s,y)^p)^{\frac{1}{p}} \, G(t,s,x,y)^{1 - \frac{1}{p}} \rho(y)^{\boldsymbol{\varepsilon} + \frac{m}{p}} \\ &= \left(\rho(x)^{\boldsymbol{\alpha}_1} \rho(y)^{\boldsymbol{\beta}_1} G(t,s,x,y) f_{\boldsymbol{\varepsilon}}(s,y)^p\right)^{\frac{1}{p}} \\ &\qquad \times \left(\rho(x)^{\boldsymbol{\alpha}_2} \rho(y)^{\boldsymbol{\beta}_2} G(t,s,x,y)\right)^{1 - \frac{1}{p}}, \end{aligned}$$

where $f_{\boldsymbol{\varepsilon}}(s,y) = \rho(y)^{-\boldsymbol{\varepsilon}-\frac{m}{p}} f(s,y)$ and $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathbb{R}^n$ satisfy (71). Using Hölder's inequality we find

$$|(G * f)(t,x)| \leq \left(\int_0^T \int_{M'} \rho(x)^{\boldsymbol{\alpha}_1} \rho(y)^{\boldsymbol{\beta}_1} G(t,s,x,y) f_{\boldsymbol{\varepsilon}}(s,y)^p \mathrm{d}V_g(y)\, \mathrm{d}s \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^T \int_{M'} \rho(x)^{\boldsymbol{\alpha}_2} \rho(y)^{\boldsymbol{\beta}_2} G(t,s,x,y)\, \mathrm{d}V_g(y)\, \mathrm{d}s \right)^{1-\frac{1}{p}}.$$

It follows that

$$\|G * f\|^p_{W^{0,0,p}_{\boldsymbol{\delta}}} \leq \int_0^T \int_{M'} \left\{ \int_0^T \int_{M'} \rho(x)^{\boldsymbol{\zeta}_1} \rho(y)^{\boldsymbol{\beta}_1} G(t,s,x,y) f_{\boldsymbol{\varepsilon}}(s,y)^p\, \mathrm{d}V_g(y)\, \mathrm{d}s \right.$$
$$\left. \times \left(\int_0^T \int_{M'} \rho(x)^{\boldsymbol{\alpha}_2} \rho(y)^{\boldsymbol{\beta}_2} G(t,s,x,y)\, \mathrm{d}V_g(y)\, \mathrm{d}s \right)^{p-1} \right\} \mathrm{d}V_g(x)\, \mathrm{d}t$$

with $\boldsymbol{\zeta}_1 = \boldsymbol{\alpha}_1 - \boldsymbol{\delta}p - m$. Observe that

$$\int_0^T \int_{M'} \rho(x)^{\boldsymbol{\alpha}_2} \rho(y)^{\boldsymbol{\beta}_2} G(t,s,x,y)\, \mathrm{d}V_g(y)\, \mathrm{d}s = \rho(x)^{\boldsymbol{\alpha}_2} \|G(t,\cdot,x,\cdot)\|_{W^{0,0,1}_{-\boldsymbol{\beta}_2-m}}.$$

Hence

$$\|G * f\|^p_{W^{0,0,p}_{\boldsymbol{\delta}}} \leq \left(\sup_{\substack{t \in (0,T) \\ x \in M'}} \rho(x)^{\boldsymbol{\alpha}_2} \|G(t,\cdot,x,\cdot)\|_{W^{0,0,1}_{-\boldsymbol{\beta}_2-m}} \right)^{p-1}$$
$$\times \int_0^T \int_{M'} \left\{ \int_0^T \int_{M} \rho(x)^{\boldsymbol{\zeta}_1} \rho(y)^{\boldsymbol{\beta}_1} G(t,s,x,y) f_{\boldsymbol{\varepsilon}}(s,y)^p\, \mathrm{d}V_g(y)\, \mathrm{d}s \right\} \mathrm{d}V_g(x)\, \mathrm{d}t.$$

Finally we have

$$\int_0^T \int_{M'} \left\{ \int_0^T \int_{M'} \rho(x)^{\boldsymbol{\zeta}_1} \rho(y)^{\boldsymbol{\beta}_1} G(t,s,x,y) f_{\boldsymbol{\varepsilon}}(s,y)^p\, \mathrm{d}V_g(y)\, \mathrm{d}s \right\} \mathrm{d}V_g(x)\, \mathrm{d}t$$
$$= \int_0^T \int_{M'} \left\{ \int_0^T \int_{M'} \rho(x)^{\boldsymbol{\zeta}_1} G(t,s,x,y)\, \mathrm{d}V_g(x)\, \mathrm{d}t \right\} \rho(y)^{\boldsymbol{\beta}_1} f_{\boldsymbol{\varepsilon}}(s,y)^p\, \mathrm{d}V_g(y)\, \mathrm{d}s$$
$$= \int_0^T \int_{M'} \|G(\cdot,s,\cdot,y)\|_{W^{0,0,1}_{-\boldsymbol{\alpha}_1+\boldsymbol{\delta}p}} \rho(y)^{\boldsymbol{\beta}_1} f_{\boldsymbol{\varepsilon}}(s,y)^p\, \mathrm{d}V_g(y)\, \mathrm{d}s$$
$$\leq \|f\|^p_{\boldsymbol{\varepsilon}} \sup_{\substack{s \in (0,T) \\ y \in M'}} \rho(y)^{\boldsymbol{\beta}_1} \|G(\cdot,s,\cdot,y)\|_{W^{0,0,1}_{-\boldsymbol{\alpha}_1+\boldsymbol{\delta}p}}$$

from which the claim follows. □

The next proposition is proved in a similar way to Proposition 7.11.

Proposition 7.12. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $T>0$, $p\in(1,\infty)$, and $\boldsymbol{\delta},\boldsymbol{\varepsilon}\in\mathbb{R}^n$. Let $f\in W^{0,0,p}_{\boldsymbol{\varepsilon}}((0,T)\times M')$ and $G\in C^0_{\mathrm{loc}}(((0,T)\times(0,T)\times M')\backslash\Delta)$, where $\Delta=\{(t,t,x)\ :\ t\in(0,T),\ x\in M'\}$. Assume that*

$$\sup_{t\in(0,T)}\|G(t,\cdot,\cdot)\|_{W^{0,0,1}_{-\boldsymbol{\alpha}_2-m}},\quad \sup_{\substack{s\in(0,T)\\ x\in M'}}\rho(x)^{\boldsymbol{\alpha}_1}\|G(\cdot,s,y)\|_{L^1}<\infty$$

for some $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2\in\mathbb{R}^n$ that satisfy

$$\frac{\boldsymbol{\alpha}_1}{p}+\boldsymbol{\alpha}_2\left(1-\frac{1}{p}\right)=\boldsymbol{\varepsilon}+\frac{m}{p}. \tag{72}$$

Then $G*f\in L^p((0,T))$ and moreover

$$\|G*f\|_{L^p}\leq\|f\|_{W^{0,0,p}_{\boldsymbol{\varepsilon}}}\sup_{t\in(0,T)}\|G(t,\cdot,\cdot)\|^{1-\frac{1}{p}}_{W^{0,0,1}_{-\boldsymbol{\alpha}_2-m}}\sup_{\substack{s\in(0,T)\\ x\in M'}}\rho(x)^{\frac{\boldsymbol{\alpha}_1}{p}}\|G(\cdot,s,x)\|^{\frac{1}{p}}_{L^1}.$$

For $T>0$ and a given function $f:(0,T)\times M'\to\mathbb{R}$ we now again consider the Cauchy problem (70). Using Propositions 7.11 and 7.12 we are now able to prove existence and maximal regularity of solutions to (70), when f lies in a weighted Sobolev space with discrete asymptotics.

Theorem 7.13. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m\geq 3$. Let $T>0$, $k\in\mathbb{N}$ with $k\geq 2$, $p\in(1,\infty)$, and $\boldsymbol{\gamma}\in\mathbb{R}^n$ with $\boldsymbol{\gamma}>2-m$ and $\gamma_i\notin\mathcal{E}_{\Sigma_i}$ for $i=1,\dots,n$. Given $f\in W^{0,k-2,p}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}((0,T)\times M')$, there exists a unique $u\in W^{1,k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T)\times M')$ solving the Cauchy problem* (70).

Proof. Let $f\in W^{0,k-2,p}_{\boldsymbol{\gamma}-2,\mathsf{P}_{\boldsymbol{\gamma}-2}}((0,T)\times M')$. Then we can write $f=f_1+f_2$ with

$$f_1\in W^{0,k-2,p}_{\boldsymbol{\gamma}-2}((0,T)\times M')\quad\text{and}\quad f_2\in L^p((0,T);\operatorname{im}\Psi_{\boldsymbol{\gamma}-2}).$$

Let H be the Friedrichs heat kernel on (M,g) and define $u=H*f$, $u_1=H*f_1$, and $u_2=H*f_2$, where convolution is defined as in (12).

The first step is to show that $u_1\in W^{0,0,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T)\times M)$. Using Theorem 7.7 we write

$$u_1(t,x)=\left(\left(H-\sum\nolimits_{j=1}^{N_0}\psi^j_{\boldsymbol{\gamma}}H^j_{\boldsymbol{\gamma}}\right)*f_1\right)(t,x)+\sum\nolimits_{j=1}^{N_0}\psi^j_{\boldsymbol{\gamma}}(x)(H^j_{\boldsymbol{\gamma}}*f_1)(t) \tag{73}$$

for $t\in(0,T)$ and $x\in M'$. We begin by showing that the first term on the right side of (73) lies in $W^{0,0,p}_{\boldsymbol{\gamma}}((0,T)\times M')$. Define $G\in C^0_{\mathrm{loc}}(((0,T)\times(0,T)\times M'\times M')\backslash\Delta)$ by

$$G(t,s,x,y)=H(|t-s|,x,y)-\sum\nolimits_{j=1}^{N_0}\psi^j_{\boldsymbol{\gamma}}(x)H^j_{\boldsymbol{\gamma}}(|t-s|,y).$$

Notice that

$$\left|\left(H-\sum\nolimits_{j=1}^{N_0}\psi^j_{\boldsymbol{\gamma}}H^j_{\boldsymbol{\gamma}}\right)*f_1(t,x)\right|\leq(|G|*|f_1|)(t,x).$$

We now apply Proposition 7.11 with $\boldsymbol{\delta} = \boldsymbol{\gamma}$ and $\boldsymbol{\varepsilon} = \boldsymbol{\gamma} - 2$. Then we have to show that that

$$\sup_{\substack{t\in(0,T)\\ x\in M'}} \rho(x)^{\boldsymbol{\alpha}_2} \|G(t,\cdot,x,\cdot)\|_{W^{0,0,1}_{-\boldsymbol{\beta}_2-m}}, \sup_{\substack{s\in(0,T)\\ y\in M'}} \rho(y)^{\boldsymbol{\beta}_1} \|G(\cdot,s,\cdot,y)\|_{W^{0,0,1}_{-\boldsymbol{\alpha}_1+\boldsymbol{\gamma}p}} < \infty,$$

where $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \in \mathbb{R}^n$ satisfy (71). Since $\boldsymbol{\gamma}^+ \geq 0$, where $\boldsymbol{\gamma}^+$ is as in (63), and $p > 1$, it suffices to prove that

$$\sup_{\substack{t\in(0,T)\\ x\in M'}} \rho(x)^{\boldsymbol{\alpha}_2} \|G(t,\cdot,x,\cdot)\|_{W^{0,0,1}_{-\boldsymbol{\beta}_2-m+\frac{\boldsymbol{\gamma}^+}{p-1}}}, \sup_{\substack{s\in(0,T)\\ y\in M'}} \rho(y)^{\boldsymbol{\beta}_1} \|G(\cdot,s,\cdot,y)\|_{W^{0,0,1}_{-\boldsymbol{\alpha}_1+\boldsymbol{\gamma}p}} < \infty. \tag{74}$$

We analyze the first term. Note that

$$\|G(t,\cdot,x,\cdot)\|_{W^{0,0,1}_{-\boldsymbol{\beta}_2-m+\frac{\boldsymbol{\gamma}^+}{p-1}}} = |G * \rho^{\boldsymbol{\beta}_2-\frac{\boldsymbol{\gamma}^+}{p-1}}|(t,x).$$

If $-m < \boldsymbol{\beta}_2 - \frac{\boldsymbol{\gamma}^+}{p-1} < \boldsymbol{\gamma}^+ - 2$, then by Proposition 7.8 there exists a constant $c > 0$, such that

$$|G * \rho^{\boldsymbol{\beta}_2-\frac{\boldsymbol{\gamma}^+}{p-1}}|(t,x) \leq c \cdot \rho(x)^{\boldsymbol{\beta}_2-\frac{\boldsymbol{\gamma}^+}{p-1}+2}.$$

Hence, if

$$\boldsymbol{\alpha}_2 + \boldsymbol{\beta}_2 - \frac{\boldsymbol{\gamma}^+}{p-1} + 2 \geq 0 \quad \text{and} \quad -m < \boldsymbol{\beta}_2 - \frac{\boldsymbol{\gamma}^+}{p-1} < \boldsymbol{\gamma}^+ - 2, \tag{75}$$

then the first term in (74) is finite. In a similar way we find that if

$$-m < \boldsymbol{\alpha}_1 - \boldsymbol{\gamma}p - m < 2 - \boldsymbol{\gamma}^+ \quad \text{and} \quad \boldsymbol{\beta}_1 + \boldsymbol{\alpha}_1 - \boldsymbol{\gamma}p - m + 2 \geq 0, \tag{76}$$

then the second term in (74) is finite. A straightforward computation now shows that (75) and (76) are equivalent to the existence of a $\boldsymbol{\beta} \in \mathbb{R}^n$ with

$$\frac{\boldsymbol{\gamma}^+}{p-1} - m < \boldsymbol{\beta} < \frac{\boldsymbol{\gamma}^+}{p-1} + \boldsymbol{\gamma}^+ - 2 \quad \text{and} \quad \frac{\boldsymbol{\gamma}p}{p-1} - 2 < \boldsymbol{\beta} < \frac{\boldsymbol{\gamma} + 2 - m + \boldsymbol{\gamma}^+}{p-1} - 2.$$

Such a $\boldsymbol{\beta}$ exists if and only if $m \geq 3$ and $2 - m < \boldsymbol{\gamma} < \boldsymbol{\gamma}^+$. It follows that

$$\left(H - \sum\nolimits_{j=1}^{N_0} \psi^j_{\boldsymbol{\gamma}} H^j_{\boldsymbol{\gamma}}\right) * f_1 \in W^{0,0,p}_{\boldsymbol{\gamma}}((0,T) \times M'). \tag{77}$$

The next step is to show that $H^j_{\boldsymbol{\gamma}} * f_1 \in L^p((0,T))$ for $j = 1, \ldots, N$. Fix some $j = 1, \ldots, N$ and define $G \in C^0_{\mathrm{loc}}(((0,T) \times (0,T) \times M') \backslash \Delta)$ by $G(t,s,x) = H^j_{\boldsymbol{\gamma}}(|t-s|, x)$. We now apply Proposition 7.12 with $\boldsymbol{\varepsilon} = \boldsymbol{\gamma} - 2$. Then it suffices to show that

$$\sup_{t\in(0,T)} \|G(t,\cdot,\cdot)\|_{W^{0,0,1}_{-\boldsymbol{\alpha}_2-m}}, \sup_{\substack{s\in(0,T)\\ x\in M'}} \rho(x)^{\boldsymbol{\alpha}_1} \|G(\cdot,s,y)\|_{L^1} < \infty \tag{78}$$

for some $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{R}^n$ that satisfy (72). Using Proposition 7.9 it follows that if

$$\boldsymbol{\alpha}_2 > \boldsymbol{\gamma}^- + 2 \quad \text{and} \quad \boldsymbol{\alpha}_1 + 2 - m - \boldsymbol{\gamma}^- \geq 0, \tag{79}$$

then the two terms in (78) are finite. A straightforward calculation shows that the conditions (72) and (79) are equivalent to $\boldsymbol{\gamma} > \boldsymbol{\gamma}^-$. Together with (77) we conclude that $u_1 \in W^{0,0,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T) \times M')$.

The same arguments as in the proof of Theorem 7.10 then show that in fact $u_1 \in W^{1,k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T) \times M')$ and $u_2 \in W^{1,l,p}_{\boldsymbol{\delta},\mathsf{P}_{\boldsymbol{\delta}}}((0,T) \times M')$ for every $l \in \mathbb{N}$ and $\boldsymbol{\delta} \in \mathbb{R}^n$. Hence $u \in W^{1,k,p}_{\boldsymbol{\gamma},\mathsf{P}_{\boldsymbol{\gamma}}}((0,T) \times M')$ as we wanted to show. Finally the uniqueness follows as in the proof of Theorem 7.10. $\square$

8 Lagrangian submanifolds with isolated conical singularities in almost Calabi–Yau manifolds

8.1 Special Lagrangian cones in $\mathbb{C}^m$

In this subsection we define special Lagrangian cones in $\mathbb{C}^m$ and introduce the notion of stability of special Lagrangian cones. Good references on special Lagrangian cones are Haskins [20] and Ohnita [46].

We begin with the definition of special Lagrangian cones in $\mathbb{C}^m$.

Definition 8.1. *Let $\iota_\Sigma : \Sigma \to \mathcal{S}^{2m-1}$ be a compact and connected $(m-1)$-dimensional submanifold of the $(2m-1)$-dimensional unit sphere $\mathcal{S}^{2m-1}$ in $\mathbb{R}^{2m}$. We identify Σ with its image $\iota_\Sigma(\Sigma) \subset \mathcal{S}^{2m-1}$. Define $\iota : \Sigma \times [0,\infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Denote $C = (\Sigma \times (0,\infty)) \sqcup \{0\}$, $C' = \Sigma \times (0,\infty)$ and identify C and C' with their images $\iota(C)$ and $\iota(C')$ under ι in $\mathbb{C}^m$. Then C is a special Lagrangian cone with phase $e^{i\theta}$, if ι restricted to $\Sigma \times (0,\infty)$ is a special Lagrangian submanifold of $\mathbb{C}^m$ with phase $e^{i\theta}$ in the sense of Definition 4.9.*

Let C be a special Lagrangian cone in $\mathbb{C}^m$. In §6.2 we discussed homogeneous harmonic functions on Riemannian cones. On a special Lagrangian cone there is a special class of homogeneous harmonic functions, namely those induced by the moment maps of the automorphism group of $(\mathbb{C}^m, J', \omega', \Omega')$.

The automorphism group of $(\mathbb{C}^m, \omega', g')$ is the Lie group $U(m) \ltimes \mathbb{C}^m$, where $\mathbb{C}^m$ acts by translations, and the automorphism group of $(\mathbb{C}^m, \omega', g', \Omega')$ is the Lie group $SU(m) \ltimes \mathbb{C}^m$. The Lie algebra of $U(m)$ is the space of skew-adjoint complex linear transformations, i.e.

$$\mathfrak{u}(m) = \left\{ A \in \mathfrak{gl}(m, \mathbb{C}) \ : \ A + \bar{A}^T = 0 \right\},$$

and the Lie algebra of $SU(m)$ is the space of the trace-free, skew-adjoint complex linear transformations, i.e.

$$\mathfrak{su}(m) = \{ A \in \mathfrak{u}(m) \ : \ \mathrm{tr}(A) = 0 \}.$$

Note that $\mathfrak{u}(m) = \mathfrak{su}(m) \oplus \mathfrak{u}(1)$.

Let $X = (A, v) \in \mathfrak{u}(m) \oplus \mathbb{C}^m$, with $A = (a_{ij})_{i,j=1,\ldots,m}$ and $v = (v_i)_{i=1,\ldots,m}$. Then X acts as a vector field on $\mathbb{C}^m$. Since $U(m) \ltimes \mathbb{C}^m$ preserves ω', $X \lrcorner\, \omega'$ is a closed one-form on $\mathbb{C}^m$ and thus there exists a unique smooth function $\mu_X : \mathbb{C}^m \to \mathbb{R}$, such that $\mathrm{d}\mu_X = X \lrcorner\, \omega'$ and $\mu_X(0) = 0$. Indeed, if $X = (A, v) \in \mathfrak{u}(m) \oplus \mathbb{C}^m$, then μ_X is given by

$$\mu_X = \frac{i}{2} \sum_{i,j=1}^{m} a_{ij} z_i \bar{z}_j + \frac{i}{2} \sum_{i=1}^{m} (v_i \bar{z}_i - \bar{v}_i z_i). \tag{80}$$

Moreover, since $a_{ij} = -\bar{a}_{ji}$ for $i, j = 1, \ldots, n$, we see that μ_X is a real quadratic polynomial. We call μ_X a moment map for X. For $X = (A, v, c) \in \mathfrak{u}(m) \oplus \mathbb{C}^m \oplus \mathbb{R}$ we define $\mu_X : \mathbb{C}^m \to \mathbb{R}$ by requiring that

$$\mathrm{d}\mu_X = X \lrcorner\, \omega' \quad \text{and} \quad \mu_X(0) = c. \tag{81}$$

A proof of the following proposition is given in Joyce [25, Prop. 3.5].

Proposition 8.2. *Let C be a special Lagrangian cone in $\mathbb{C}^m$ as in Definition 8.1 and let G be the maximal Lie subgroup of $SU(m)$ that preserves C. Then the following hold.*

(i) *Let $X \in \mathfrak{su}(m)$. Then $\iota^*(\mu_X)$ is a homogeneous harmonic function of order two on C'. Consequently the space of homogeneous harmonic functions of order two on C' is at least of dimension $m^2 - 1 - \dim G$.*

(ii) *Let $X \in \mathbb{C}^m$. Then $\iota^*(\mu_X)$ is a homogeneous harmonic function of order one on C'. Consequently the space of homogeneous harmonic functions of order one on C' is at least of dimension $2m$.*

Also note that if C is a special Lagrangian cone in $\mathbb{C}^m$ and $X \in \mathfrak{u}(1)$, then $\iota^*(\mu_X) = cr^2$ for some $c \in \mathbb{R}$.

Using Proposition 8.2 we can define the stability index of a special Lagrangian cone in $\mathbb{C}^m$ and the notion of stable special Lagrangian cones following Joyce [25, Def. 3.6].

Definition 8.3. *Let C be a special Lagrangian cone in $\mathbb{C}^m$ as in Definition 8.1 and let G be the maximal Lie subgroup of $SU(m)$ that preserves C. Then the stability index of C is the integer*

$$\text{s-index}(C) = M_\Sigma(2) - m^2 - 2m + \dim G,$$

where M_Σ is defined in §6.2. From Proposition 8.2 it follows that the stability index of a special Lagrangian cone is a non-negative integer. We say that a special Lagrangian cone C in $\mathbb{C}^m$ is stable if s-index$(C) = 0$.

Note that if C is a stable special Lagrangian cone as in Definition 8.1, then the only homogeneous harmonic functions on C' with rate α, where $0 \leq \alpha \leq 2$, are those induced by the $SU(m) \ltimes \mathbb{C}^m$-moment maps.

Examples of special Lagrangian cones can be found in Joyce [23, §8.3.2]. Examples of stable special Lagrangian cones, however, are hard to find and there are only a few examples known. The simplest example of a stable special Lagrangian cone is the Riemannian cone in $\mathbb{C}^3$ over T^2 with its standard metric. In this case C is given by

$$C = \left\{ \left(re^{i\phi_1}, re^{i\phi_2}, re^{i(\phi_1 - \phi_2)} \right) \;:\; r \in [0, \infty),\ \phi_1, \phi_2 \in [0, 2\pi) \right\} \subset \mathbb{C}^3$$

together with the Riemannian metric induced by the Euclidean metric on $\mathbb{C}^3$. Some other examples of stable special Lagrangian cones can be found in Ohnita's paper [46].

8.2 Lagrangian submanifolds with isolated conical singularities

In this subsection we define Lagrangian submanifolds with isolated conical singularities and prove a simple property of the Maslov class of Lagrangian submanifolds with isolated conical singularities. Related material about special Lagrangian submanifolds with conical singularities can be found in Joyce [24], [25], and [26] and also in Haskins and Pacini [21].

We now define Lagrangian submanifolds with isolated conical singularities in almost Calabi–Yau manifolds following Joyce [24, Def. 3.6].

Definition 8.4. *Let (M,J,ω,Ω) be an m-dimensional almost Calabi–Yau manifold and define $\psi\in C^\infty(M)$ as in (15). Let $x_1,\dots,x_n\in M$ be distinct points in M, $C_1,\dots,C_n$ special Lagrangian cones in $\mathbb{C}^m$ as in Definition 8.1 with embeddings $\iota_i:\Sigma_i\times(0,\infty)\to\mathbb{C}^m$ for $i=1,\dots,n$, and L an m-dimensional manifold with ends $S_1,\dots,S_n$ as in Definition 6.1. Then a Lagrangian submanifold $F:L\to M$ is a Lagrangian submanifold with isolated conical singularities at $x_1,\dots,x_n$ modelled on the special Lagrangian cones $C_1,\dots,C_n$, if the following holds.*

We are given isomorphisms $A_i:\mathbb{C}^m\to T_{x_i}M$ for $i=1,\dots,n$ with $A_i^*(\omega)=\omega'$ and $A_i^*(\Omega)=e^{i\theta_i+m\psi(x_i)}\Omega'$ for some $\theta_i\in\mathbb{R}$ and $i=1,\dots,n$. Then by Theorem 4.3 there exist $R>0$ and smooth embeddings $\Upsilon_i:B_R\to M$ with $\Upsilon_i(0)=x_i$, $\Upsilon_i^*(\omega)=\omega'$, and $\mathrm{d}\Upsilon_i(0)=A_i$ for $i=1,\dots,n$. Making $R>0$ smaller if necessary we can assume that $\Upsilon_1(B_R),\dots,\Upsilon_n(B_R)$ are pairwise disjoint in M. Then there should exist diffeomorphisms $\phi_i:\Sigma_i\times(0,R)\to S_i$ for $i=1,\dots,n$, such that $F\circ\phi_i$ maps $\Sigma_i\times(0,R)\to\Upsilon_i(B_R)$ for $i=1,\dots,n$, and there should exist $\nu_i\in(2,3)$ for $i=1,\dots,n$, such that

$$\left|\nabla^k(\Upsilon_i^{-1}\circ F\circ\phi_i-\iota_i)\right|=O(r^{\nu_i-1-k})\quad\text{as } r\longrightarrow 0 \text{ for } k\in\mathbb{N}. \tag{82}$$

Here ∇ and $|\cdot|$ are computed using the Riemannian cone metric $\iota_i^*(g')$ on $\Sigma_i\times(0,R)$. A Lagrangian submanifold $F:L\to M$ with isolated conical singularities modelled on special Lagrangian cones $C_1,\dots,C_n$ is said to have stable conical singularities, if $C_1,\dots,C_n$ are stable special Lagrangian cones in $\mathbb{C}^m$.

We have chosen $\nu_i\in(2,3)$ in Definition 8.4 for the following reasons. We need $\nu_i>2$ or otherwise (82) does not force the submanifold $F:L\to M$ to approach the cone $A_i(C_i)$ in $T_{x_i}M$ near x_i for $i=1,\dots,n$. Moreover $\nu_i<3$ guarantees that the definition is independent of the choice of Υ_i. Indeed, if we are given a different smooth embedding $\tilde\Upsilon_i:B_R\to M$ with $\tilde\Upsilon_i(0)=x_i$, $\tilde\Upsilon_i^*(\omega)=\omega'$, and $\mathrm{d}\tilde\Upsilon_i(0)=A_i$, then $\Upsilon_i-\tilde\Upsilon_i=O(r^2)$ on B_R by Taylor's Theorem. Therefore, since $\nu_i<3$, it follows that (82) holds with Υ_i replaced by $\tilde\Upsilon_i$.

If $F:L\to M$ is a Lagrangian submanifold with isolated conical singularities, then (82) implies that L together with the Riemannian metric $F^*(g)$ is a Riemannian manifold with conical singularities in the sense of Definition 6.3. In particular the analytical results from §6 apply to L together with the Riemannian metric $F^*(g)$.

The next proposition shows that the Maslov class of a Lagrangian submanifold with isolated conical singularities is an element of $H^1_{cs}(L,\mathbb{R})$, the first compactly supported de Rham cohomology group of L.

Proposition 8.5. *Let $F:L\to M$ be a Lagrangian submanifold with conical singularities as in Definition 8.4. Then the Maslov class μ_F of $F:L\to M$ may be defined as an element of $H^1_{cs}(L,\mathbb{R})$.*

Proof. Let η be a smooth and closed one-form on L and assume that there exists $\varepsilon>0$, such that $|\nabla^k\eta|=O(\rho^{-1+\varepsilon-k})$ for $k\in\mathbb{N}$. We first show that there exists a function $f\in C^\infty_\varepsilon(L)$, such that $\eta+\mathrm{d}f$ has compact support. Let $\eta_i=\phi_i^*(\eta)$ for $i=1,\dots,n$. Then $\eta_i(\sigma,r)=\eta_i^1(\sigma,r)\mathrm{d}r+\eta_i^2(\sigma,r)$ for $(\sigma,r)\in\Sigma_i\times(0,R)$ and $i=1,\dots,n$, where $\eta_i^1(\sigma,r)\in\mathbb{R}$ and $\eta_i^2(\sigma,r)\in T^*_\sigma\Sigma_i$ for $r\in(0,R)$ and

$i = 1, \dots, n$. Define functions

$$f_i(\sigma, r) = -\int_0^r \eta_i^1(\sigma, \varrho)\, \mathrm{d}\varrho \quad \text{for } (\sigma, r) \in \Sigma_i \times (0, R)$$

and $i = 1, \dots, n$. Since $|\eta_i| = O(r^{-1+\varepsilon})$ as $r \to 0$ it follows that f_i is well defined for $i = 1, \dots, n$. Moreover,

$$\mathrm{d}f_i(\sigma, r) = -\int_0^r \mathrm{d}_{\Sigma_i}\eta_i^1(\sigma, \varrho)\, \mathrm{d}\varrho - \eta_i^1(\sigma, r)\mathrm{d}r \quad \text{for } (\sigma, r) \in \Sigma_i \times (0, R)$$

and $i = 1, \dots, n$. Since η is closed we have $0 = \mathrm{d}\eta_i = \mathrm{d}_\Sigma \eta_i^1 \wedge \mathrm{d}r + \mathrm{d}_\Sigma \eta_i^2 - \frac{\partial \eta_i^2}{\partial r} \wedge \mathrm{d}r$ and thus $\mathrm{d}_\Sigma \eta_i^2 = 0$ and $\mathrm{d}_\Sigma \eta_i^1 = \frac{\partial \eta_i^2}{\partial r}$ for $i = 1, \dots, n$. Then it follows that $\mathrm{d}f_i = -\eta_i^2 - \eta_i^1 \mathrm{d}r$. Now we choose functions $\chi_1, \dots, \chi_n \in C^\infty(L)$, such that $\chi_i \equiv 1$ on $\phi_i(\Sigma_i \times (0, \frac{R}{2}))$ and $\chi_i \equiv 0$ on $L \setminus \bigcup_{i=1}^n \phi_i(\Sigma_i \times (0, R))$ and we define $f = \chi_1 f_1 + \dots + \chi_n f_n$. Then $f \in C_\varepsilon^\infty(M)$ and $\eta + \mathrm{d}f$ has compact support as we wanted to show.

By Proposition 4.8 the Maslov class of $F : L \to M$ is represented by α_K, the generalized mean curvature form of $F : L \to M$. Since $C_1, \dots, C_n$ are special Lagrangian cones, their mean curvature vector fields are zero. Then (82) implies that $|\nabla^k \alpha_K| = O(\rho^{\boldsymbol{\nu}-3-k})$ as $\rho \to 0$ for $k \in \mathbb{N}$ and hence $|\nabla^k \mathrm{d}\theta| = O(\rho^{\boldsymbol{\nu}-3-k})$ as $\rho \to 0$ for $k \in \mathbb{N}$. Since $\boldsymbol{\nu} - 2 > 0$, $|\nabla^k \mathrm{d}\theta| = O(\rho^{-1+\varepsilon-k})$ as $\rho \to 0$ for $k \in \mathbb{N}$ and some small $\varepsilon > 0$. Hence $\mu_F \in H^1_{cs}(L, \mathbb{R})$. □

8.3 Lagrangian neighbourhoods for Lagrangian submanifolds with isolated conical singularities

In this subsection we collect various Lagrangian neighbourhood theorems for Lagrangian submanifolds with isolated conical singularities proved by Joyce [24, §4].

Let C be a special Lagrangian cone in $\mathbb{C}^m$ as in Definition 8.1. Let $\sigma \in \Sigma$, $\tau \in T^*_\sigma\Sigma$ and $\varrho \in \mathbb{R}$. Then we denote by $(\sigma, r, \tau, \varrho)$ the point $\tau + \varrho \mathrm{d}r$ in $T^*_{(\sigma,r)}(\Sigma \times (0, \infty))$. For $s \in (0, \infty)$ we define

$$\delta^s : \Sigma \times (0, \infty) \to \Sigma \times (0, \infty), \quad \delta^s(\sigma, r) = (\sigma, sr). \tag{83}$$

Then δ^s induces an action of $(0, \infty)$ on $T^*(\Sigma \times (0, \infty))$ by $\delta^s_*(\sigma, r, \tau, \varrho) = (\sigma, sr, s^2\tau, s\varrho)$. Also observe that the canonical symplectic structure $\hat{\omega}$ on $T^*(\Sigma \times (0, \infty))$ satisfies $(\delta^s)^*(\hat{\omega}) = s^2\hat{\omega}$.

The following theorem is a Lagrangian neighbourhood theorem for special Lagrangian cones in $\mathbb{C}^m$. The proof can be found in Joyce [24, Thm 4.3].

Theorem 8.6. *Let C be a special Lagrangian cone in $\mathbb{C}^m$ as in Definition 8.1. Then there exists an open neighbourhood U_C of the zero section in $T^*(\Sigma \times (0, \infty))$ with $\delta^s_*(U_C) = U_C$ for $s \in (0, \infty)$ given by*

$$U_C = \left\{(\sigma, r, \tau, \varrho) \in T^*(\Sigma \times (0, \infty)) \ : \ |(\tau, \varrho)| < 2\zeta r\right\} \quad \textit{for some } \zeta > 0,$$

and there exists a Lagrangian neighbourhood $\Phi_C : U_C \to \mathbb{C}^m$ for $\iota : \Sigma \times (0, \infty) \to \mathbb{C}^m$, such that $s \cdot \Phi_C = \Phi_C \circ \delta^s$ for $s \in (0, \infty)$.

The next proposition is a result about the asymptotic behaviour of graphs of functions over special Lagrangian cones. The proof can be found in Joyce [24, Thm. 4.4].

Proposition 8.7. *Let C be a special Lagrangian cone in $\mathbb{C}^m$ as in Definition 8.1 and $\Phi_C : U_C \to \mathbb{C}^m$ a Lagrangian neighbourhood for $\iota : \Sigma \times (0,\infty) \to \mathbb{C}^m$ as in Theorem 8.6. Let $R > 0$, $\mu \in \mathbb{R}$, and $u \in C^k(\Sigma \times (0,R))$. Assume that*

$$|\nabla^j u| = O(r^{\mu-j}) \quad \text{as } r \longrightarrow 0 \text{ for } j = 0,\ldots,k$$

and $\Gamma_{\mathrm{d}u} \subset U_C$, where $\Gamma_{\mathrm{d}u} = \{(x, \mathrm{d}u(x)) \in T^*(\Sigma \times (0,R)) \ : \ x \in \Sigma \times (0,R)\}$ is the graph of $\mathrm{d}u$. Then

$$\left|\nabla^j(\Phi_C \circ \mathrm{d}u - \iota)\right| = O(r^{\mu-1-j}) \quad \text{as } r \longrightarrow 0 \text{ for } j = 0,\ldots,k-1.$$

Here ∇ and $|\cdot|$ are computed using the Riemannian cone metric $\iota^*(g')$ on $\Sigma \times (0,R)$.

The next theorem provides a special coordinate system for a Lagrangian submanifold with isolated conical singularities near each singular point. The proof can be found in Joyce [24, Thm. 4.4 & Lem. 4.5].

Theorem 8.8. *Let (M, J, ω, Ω) be an m-dimensional almost Calabi–Yau manifold and let $C_1,\ldots,C_n$ be special Lagrangian cones in $\mathbb{C}^m$ with embeddings $\iota_i : \Sigma_i \times (0,\infty) \to \mathbb{C}^m$ as in Definition 8.1 for $i = 1,\ldots,n$. Let $F : L \to M$ be a Lagrangian submanifold with isolated conical singularities at $x_1,\ldots,x_n \in M$ modelled on the special Lagrangian cones $C_1,\ldots,C_n$ as in Definition 8.4. For C_i we choose a Lagrangian neighbourhood $\Phi_{C_i} : U_{C_i} \to M$ as in Theorem 8.6 for $i = 1,\ldots,n$.*

After making $R > 0$ smaller if necessary, there exist unique functions $a_i : \Sigma_i \times (0,R) \to \mathbb{R}$ for $i = 1,\ldots,n$, such that $|\mathrm{d}a_i(\sigma,r)| < \zeta r$ for $(\sigma,r) \in \Sigma_i \times (0,R)$ and

$$\left|\nabla^k a_i\right| = O(r^{\nu_i-k}) \quad \text{as } r \longrightarrow 0 \text{ for } k \in \mathbb{N}$$

and $i = 1,\ldots,n$, where ∇ and $|\cdot|$ are computed using the Riemannian cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0,R)$ for $i = 1,\ldots,n$, such that the following holds.

The map $\Upsilon_i \circ \Phi_{C_i} \circ \mathrm{d}a_i : \Sigma_i \times (0,R) \to M$ is a diffeomorphism $\Sigma_i \times (0,R) \to F(S_i)$ for $i = 1,\ldots,n$, and if we define

$$\phi_i : \Sigma \times (0,R) \longrightarrow S_i, \quad \phi_i = F^{-1} \circ \Upsilon_i \circ \Phi_{C_i} \circ \mathrm{d}a_i$$

for $i = 1,\ldots,n$, then ϕ_i satisfies (82) for $i = 1,\ldots,n$.

Using the previous theorems we can now state a Lagrangian Neighbourhood Theorem for Lagrangian submanifolds with isolated conical singularities [24, Thm. 4.6].

Theorem 8.9. *Let (M, J, ω, Ω) be an m-dimensional almost Calabi–Yau manifold and let $C_1,\ldots,C_n$ be special Lagrangian cones in $\mathbb{C}^m$ with embeddings $\iota_i : \Sigma_i \times (0,\infty) \to \mathbb{C}^m$ for $i = 1,\ldots,n$ as in Definition 8.1. Let $F : L \to M$ be a Lagrangian submanifold with isolated conical singularities at $x_1,\ldots,x_n \in M$ modelled on the special Lagrangian cones $C_1,\ldots,C_n$ as in Definition 8.4. Finally let Φ_{C_i}, a_i, ϕ_i, and R be as in Theorem 8.8.*

Then there exists an open tubular neighbourhood U_L of the zero section in T^*L, such that

$$(\mathrm{d}\phi_i)^*(U_L) = \{(\sigma, r, \tau, \varrho) \in T^*(\Sigma_i \times (0,R)) \;:\; |(\tau, \varrho)| < \zeta r\}$$

for $i = 1, \ldots, n$ and a Lagrangian neighbourhood $\Phi_L : U_L \to M$ for $F : L \to M$, such that

$$(\Phi_L \circ \mathrm{d}\phi_i)(\sigma, r, \tau, \varrho) = (\Upsilon_i \circ \Phi_{C_i} \circ \mathrm{d}a_i)(\sigma, r, \tau, \varrho)$$

for every $(\sigma, r, \tau, \varrho) \in T^*(\Sigma_i \times (0,R))$ with $|(\tau, \varrho)| < \zeta r$.

8.4 Lagrangian neighbourhoods for families of Lagrangian submanifolds with isolated conical singularities

So far we have only discussed Lagrangian neighbourhoods for a single Lagrangian submanifold with isolated conical singularities. Later, when we prove short time existence of the generalized Lagrangian mean curvature flow for Lagrangian submanifolds with isolated conical singularities modelled on stable special Lagrangian cones, we allow the singularities to move around in the ambient space. Therefore we need to extend Theorem 8.9 to families of Lagrangian neighbourhoods for Lagrangian submanifolds with isolated conical singularities.

Let (M, J, ω, Ω) be an m-dimensional almost Calabi–Yau manifold, define $\psi \in C^\infty(M)$ as in (15), and let $F : L \to M$ be a Lagrangian submanifold with isolated conical singularities at $x_1, \ldots, x_n \in M$ modelled on special Lagrangian cones $C_1, \ldots, C_n$ as in Definition 8.4. We define a fibre bundle $\mathcal{A}$ over M by

$$\mathcal{A} = \big\{(x, A) \;:\; x \in M,\ A : \mathbb{C}^m \longrightarrow T_xM, \\ A^*(\omega) = \omega',\ A^*(\Omega) = e^{i\theta + m\psi(x)}\Omega' \text{ for some } \theta \in \mathbb{R}\big\}.$$

Then $B \in U(m)$ acts on $(x, A) \in \mathcal{A}_x$ by $B(x, A) = (x, A \circ B)$. This action of $U(m)$ is free and transitive on the fibres of $\mathcal{A}$ and thus $\mathcal{A}$ is a principal $U(m)$-bundle over M with $\dim \mathcal{A} = m^2 + 2m$.

Let G_i be the maximal Lie subgroup of $SU(m)$ that preserves C_i for $i = 1, \ldots, n$. If (x_i, A_i) and $(x_i, \hat{A}_i)$ lie in the same G_i-orbit, then they define equivalent choices for (x_i, A_i) in Definition 8.4. To avoid this let $\mathcal{E}_i$ be a small open ball of dimension $\dim \mathcal{A} - \dim G_i$ containing (x_i, A_i), which is transverse to the orbits of G_i for $i = 1, \ldots, n$. Then $G_i \cdot \mathcal{E}_i$ is open in $\mathcal{A}$. We set $\mathcal{E} = \mathcal{E}_1 \times \ldots \times \mathcal{E}_n$ and equip $\mathcal{E}$ with the Riemannian metric induced by the Riemannian metric on M. Then $\mathcal{E}$ parametrizes all nearby alternative choices for (x_i, A_i) in Definition 8.4. Note that $\dim \mathcal{E}_i = m^2 + 2m - \dim G_i$ for $i = 1, \ldots, n$ and $\dim \mathcal{E} = n(m^2 + 2m) - \sum_{i=1}^n \dim G_i$.

We now extend Theorem 8.9 to families $\{\Phi_L^e\}_{e \in \mathcal{E}}$ of Lagrangian neighbourhoods. Here for $e = (\hat{x}_1, \hat{A}_1, \ldots, \hat{x}_n, \hat{A}_n) \in \mathcal{E}$, $\Phi_L^e : U_L \to M$ is a Lagrangian neighbourhood for a Lagrangian submanifold with isolated conical singularities at $\hat{x}_1, \ldots, \hat{x}_n \in M$ and isomorphisms $\hat{A}_i : \mathbb{C}^m \to T_{\hat{x}_i}M$ for $i = 1, \ldots, n$ as in Definition 8.4. Such a theorem was proved by Joyce in [25, Thm. 5.2]. We will explain the proof in detail because later we have to make explicit use of the construction of the Lagrangian neighbourhoods. We begin with the following lemma.

Lemma 8.10. *Let $F : L \to M$ be a Lagrangian submanifold with isolated conical singularities as in Definition 8.4. Denote $e_0 = (x_1, A_1, \ldots, x_n, A_n)$ and define $\mathcal{E}$ as above. Then, after making $\mathcal{E}$ smaller if necessary, there exists a family $\{\Psi^e_M\}_{e\in\mathcal{E}}$ of smooth diffeomorphisms $\Psi^e_M : M \to M$, which depends smoothly on $e \in \mathcal{E}$, such that*

(i) *$\Psi^{e_0}_M$ is the identity on M,*
(ii) *Ψ^e_M is the identity on $M\backslash \bigcup_{i=1}^n \Upsilon_i(B_{R/2})$ for $e \in \mathcal{E}$,*
(iii) *$(\Psi^e_M)^*(\omega) = \omega$ for $e \in \mathcal{E}$,*
(iv) *$(\Psi^e_M \circ \Upsilon_i)(0) = \hat{x}_i$, $\mathrm{d}(\Psi^e_M \circ \Upsilon_i)(0) = \hat{A}_i$ for $i = 1, \ldots, n$ and $e \in \mathcal{E}$ with $e = (\hat{x}_1, \hat{A}_1, \ldots, \hat{x}_n, \hat{A}_n)$.*

Proof. We first construct families $\{\Psi^e_i\}_{e\in\mathcal{E}}$ of diffeomorphisms $\Psi^e_i : B_R \to B_R$ for $i = 1, \ldots, n$, which depend smoothly on $e \in \mathcal{E}$ and satisfy

(a) $\Psi^{e_0}_i$ is the identity on B_R for $i = 1, \ldots, n$,
(b) Ψ^e_i is the identity on $B_R\backslash B_{R/2}$ for $e \in \mathcal{E}$ and $i = 1, \ldots, n$,
(c) $(\Psi^e_i)^*(\omega') = \omega'$ for $e \in \mathcal{E}$ and $i = 1, \ldots, n$,
(d) $(\Upsilon_i \circ \Psi^e_i)(0) = \hat{x}_i$, $\mathrm{d}(\Upsilon_i \circ \Psi^e_i)(0) = \hat{A}_i$ for $e \in \mathcal{E}$ with $e = (\hat{x}_1, \hat{A}_1, \ldots, \hat{x}_n, \hat{A}_n)$ and $i = 1, \ldots, n$.

Let $e = (\hat{x}_1, \hat{A}_1, \ldots, \hat{x}_n, \hat{A}_n) \in \mathcal{E}$. Making $\mathcal{E}$ smaller if necessary we can assume that $\hat{x}_i \in \Upsilon_i(B_{R/4})$ for $i = 1, \ldots, n$. Denote $y_i = \Upsilon_i^{-1}(\hat{x}_i)$ for $i = 1, \ldots, n$ and define $B_i = (\mathrm{d}\Upsilon_i|_{y_i})^{-1} \circ \hat{A}_i$ for $i = 1, \ldots, n$. Since $\Upsilon_i^*(\omega) = \omega'$, $B_i \in Sp(2m, \mathbb{R})$ and so $(B_i, y_i) \in Sp(2m, \mathbb{R}) \ltimes \mathbb{R}^{2m}$. Here $Sp(2m) = \{A \in GL(2m, \mathbb{R}) : A^*(\omega') = \omega'\}$ is the automorphism group of $(\mathbb{R}^{2m}, \omega')$. Using standard techniques from symplectic geometry we can now define families $\{\Psi^e_i\}_{e\in\mathcal{E}}$ of diffeomorphisms $\Psi^e_i : B_R \to B_R$ for $i = 1, \ldots, n$, which depend smoothly on $e \in \mathcal{E}$, such that (a), (b), and (c) hold, and such that $\Psi^e_i = (B_i, y_i)$ on $B_{R/4}$ for $i = 1, \ldots, n$. But then by definition of (B_i, y_i) we see that (d) holds for $i = 1, \ldots, n$.

Now we define $\Psi^e_M : M \to M$ to be $\Upsilon_i \circ \Psi^e_i \circ \Upsilon_i^{-1}$ on $\Upsilon_i(B_R)$ for $i = 1, \ldots, n$ and the identity on $M\backslash \bigcup_{i=1}^n \Upsilon_i(B_R)$. This is clearly possible, since $\Upsilon_1(B_R), \ldots, \Upsilon_n(B_R)$ are pairwise disjoint in M. Since Ψ^e_i satisfies (a) – (d) for $i = 1, \ldots, n$, it follows that $\Psi^e_M : M \to M$ is a family of smooth diffeomorphisms of M, which depends smoothly on $e \in \mathcal{E}$ and satisfies (i) – (iv). □

Combining Theorem 8.9 and Lemma 8.10 we obtain the following Lagrangian neighbourhood theorem for families of Lagrangian submanifolds with isolated conical singularities.

Theorem 8.11. *Let (M, J, ω, Ω) be an m-dimensional almost Calabi–Yau manifold and $F : L \to M$ a Lagrangian submanifold with isolated conical singularities as in Definition 8.4. Let $\Phi_{C_i}, a_i, \phi_i, R, U_L$, and Φ_L for $i = 1, \ldots, n$ be as in Theorem 8.9. Denote $e_0 = (x_1, A_1, \ldots, x_n, A_n)$, let $\mathcal{E}$ be as above, and define $\Psi^e_M : M \to M$ as in Lemma 8.10.*

Define families of smooth embeddings $\{\Upsilon^e_i\}_{e\in\mathcal{E}}$, $\Upsilon^e_i : B_R \to M$ by $\Upsilon^e_i = \Psi^e_M \circ \Upsilon_i$ for $i = 1, \ldots, n$, and $\{\Phi^e_L\}_{e\in\mathcal{E}}$, $\Phi^e_L : U_L \to M$ by $\Phi^e_L = \Psi^e_M \circ \Phi_L$. Then $\{\Upsilon^e_i\}_{e\in\mathcal{E}}$ and $\{\Phi^e_L\}_{e\in\mathcal{E}}$ depend smoothly on $e \in \mathcal{E}$ for $i = 1, \ldots, n$, and

(i) *$\Upsilon^{e_0}_i = \Upsilon_i$, $(\Upsilon^e_i)^*(\omega) = \omega'$, $\Upsilon^e_i(0) = \hat{x}_i$, and $\mathrm{d}\Upsilon^e_i(0) = \hat{A}_i$ for every $e \in \mathcal{E}$ with $e = (\hat{x}_1, \hat{A}_1, \ldots, \hat{x}_n, \hat{A}_n)$,*

(ii) $\Phi_L^{e_0} = \Phi_L$, $(\Phi_L^e)^*(\omega) = \hat{\omega}$, *and* $\Phi_L^e \equiv \Phi_L$ *on* $\pi^{-1}(L \backslash \bigcup_{i=1}^n \phi_i(\Sigma_i \times (0, \frac{R}{2}))) \subset U_L$ *for every* $e \in \mathcal{E}$.

Moreover, for every $(\sigma, r, \tau, \varrho) \in T^*(\Sigma_i \times (0, R))$ *with* $|(\tau, \varrho)| < \zeta r$ *and* $e \in \mathcal{E}$ *we have*

$$(\Phi_L^e \circ \mathrm{d}\phi_i)(\sigma, r, \tau, \varrho) = (\Upsilon_i^e \circ \Phi_{C_i} \circ \mathrm{d}a_i)(\sigma, r, \tau, \varrho)$$

for $i = 1, \ldots, n$. *Finally for* $e \in \mathcal{E}$, $\Phi_L^e : U_L \to M$ *is a Lagrangian neighbourhood for a Lagrangian submanifold with isolated conical singularities at* $\hat{x}_1, \ldots, \hat{x}_n \in M$ *modelled on the special Lagrangian cones* $C_1, \ldots, C_n$ *with isomorphisms* $\hat{A}_i : \mathbb{C}^m \to T_{\hat{x}_i} M$ *for* $i = 1, \ldots, n$ *as in Definition 8.4.*

In §9 we will need an extension of Theorem 8.11. In fact, the manifold $\mathcal{E}$ corresponds to the rotations and translations of the model cones of the Lagrangian submanifold with isolated conical singularities. The rotations and translations of the model cones correspond to $SU(m) \ltimes \mathbb{C}^m$-moment maps. If $X \in \mathfrak{su}(m) \oplus \mathbb{C}^m$ and if μ_X is a moment map on $\mathbb{C}^m$, then $\mu_X + c$ with $c \in \mathbb{R}$ is another equivalent choice for a moment map of X, as we already explained in §8.1. For this reason we will now introduce a new manifold $\mathcal{F}$, that also allows us to vary this redundant parameter.

Let L be an m-dimensional manifold with ends $S_1, \ldots, S_n$ as in Definition 6.1. Choose $R' > 0$ such that $\frac{R}{2} < R' < R$ and denote $S_i' = \phi_i(\Sigma_i \times (0, R'))$ for $i = 1, \ldots, n$. We choose functions $q_i \in C^\infty(L)$ for $i = 1, \ldots, n$, such that $q_i \equiv 1$ on S_i' and $q_i \equiv 0$ on $M \backslash S_i$ for $i = 1, \ldots, n$. Let (M, J, ω, Ω) be an m-dimensional almost Calabi–Yau manifold and $F : L \to M$ a Lagrangian submanifold with isolated conical singularities as in Definition 8.4. Denote $e_0 = (x_1, A_1, \ldots, x_n, A_n)$ and let $\Phi_{C_i}, U_{C_i}, R, a_i, \phi_i, \Phi_L$, $U_L, \mathcal{E}, \Psi_M^e, \Upsilon_i^e$, and Φ_L^e for $i = 1, \ldots, n$ be as in Theorem 8.11.

Let $\mathcal{U}_1, \ldots, \mathcal{U}_n$ be open intervals in $\mathbb{R}$ containing 0 and let $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_n$. Define $\mathcal{F} = \mathcal{E} \times \mathcal{U}$ and $\mathcal{F}_i = \mathcal{E}_i \times \mathcal{U}_i$ for $i = 1, \ldots, n$. Denote by $f = (e, c)$ a general point in $\mathcal{F}$, where $e \in \mathcal{E}$ and $c = (c_1, \ldots, c_n) \in \mathcal{U}$. Let $f_0 = (e_0, 0)$. For $f \in \mathcal{F}$ we define families $\{\Upsilon_i^f\}_{f \in \mathcal{F}}$ of smooth embeddings $\Upsilon_i^f : B_R \to M$ simply by $\Upsilon_i^f \equiv \Upsilon_i^e$ for $i = 1, \ldots, n$. Denote

$$U_L' = \left\{ (x, \beta) \in U_L \ : \ \Gamma_{\beta + \sum_{i=1}^n c_i \mathrm{d}q_i} \in U_L \text{ for every } c \in \mathcal{U} \right\}.$$

Making $\mathcal{U}$ smaller if necessary we can ensure that U_L' is non-empty and contains the zero section in T^*L. Then we define a family $\{\Phi_L^f\}_{f \in \mathcal{F}}$ of smooth embeddings by

$$\Phi_L^f : U_L' \to M, \quad \Phi_L^f = \Phi_L^e \circ \sum_{i=1}^n c_i \mathrm{d}q_i. \tag{84}$$

Then $\{\Upsilon_i^f\}_{f \in \mathcal{F}}$ and $\{\Phi_L^f\}_{f \in \mathcal{F}}$ depend smoothly on $f \in \mathcal{F}$ for $i = 1, \ldots, n$. Moreover (i) in Theorem 8.11 continues to hold with $\Upsilon_i^{e_0}$ and Υ_i^e replaced by $\Upsilon_i^{f_0}$ and Υ_i^f, respectively, for $f \in \mathcal{F}$ and $i = 1, \ldots, n$. Furthermore, since $\mathrm{d}q_i$ is supported on $S_i \backslash S_i'$ for $i = 1, \ldots, n$, we have

$$(\Phi_L^f \circ \mathrm{d}\phi_i)(\sigma, r, \tau, \varrho) = (\Upsilon_i^f \circ \Phi_{C_i} \circ \mathrm{d}a_i)(\sigma, r, \tau, \varrho)$$

for every $(\sigma, r, \tau, \varrho) \in T^*(\Sigma_i \times (0, R'))$ with $|(\tau, \varrho)| < \zeta r$ for $i = 1, \ldots, n$. Finally, since $\Phi_L^e \equiv \Phi_L$ on $\pi^{-1}(L \backslash \bigcup_{i=1}^n \phi_i(\Sigma_i \times (0, \frac{R}{2})))$ by Theorem 8.11, we have

$\Phi_L^f \equiv \Phi_L$ on $\pi^{-1}(K) \subset T^*L$ for $f \in \mathcal{F}$, where $K = L \backslash \bigcup_{i=1}^n S_i$. Therefore (ii) in Theorem 8.11 continues to hold with $\Phi_L^{e_0}, \Phi_L^e$, and $\Sigma_i \times (0, \frac{R}{2})$ replaced by $\Phi_L^{f_0}, \Phi_L^f$, and $\Sigma_i \times (0, R)$, respectively.

9 Generalized Lagrangian mean curvature flow with isolated conical singularities

Throughout this section we fix the following data. Let (M, J, ω, Ω) an m-dimensional almost Calabi–Yau manifold, $m \geq 3$, with Riemannian metric g and define $\psi \in C^\infty(M)$ as in (15). Let L be an m-dimensional manifold with ends $S_1, \ldots, S_n$ as in Definition 6.1 and define R', S_i', and q_i for $i = 1, \ldots, n$ as in the end of §8.4. Denote $K = L \backslash \bigcup_{i=1}^n S_i$. Let $C_1, \ldots, C_n$ be stable special Lagrangian cones in $\mathbb{C}^m$ with embeddings $\iota_i : \Sigma_i \times (0, \infty) \to \mathbb{C}^m$ as in Definition 8.1 for $i = 1, \ldots, n$, and let $F_0 : L \to M$ be a Lagrangian submanifold with isolated conical singularities at $x_1, \ldots, x_n \in M$ modelled on the special Lagrangian cones $C_1, \ldots, C_n$. Later, from §9.5 on, we will also assume that the special Lagrangian cones are stable. Denote by G_i the maximal Lie subgroup of $SU(m)$ which preserves C_i for $i = 1, \ldots, n$. Choose identifications $A_i : \mathbb{C}^m \to T_{x_i}M$ and embeddings $\Upsilon_i : B_R \to M$ for $i = 1, \ldots, n$ and let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$ be as in Definition 8.4. Define $\mathcal{E}$ and $\mathcal{F}$ as in §8.4 and denote $e_0 = (x_1, A_1, \ldots, x_n, A_n) \in \mathcal{E}$ and $f_0 = (e_0, 0) \in \mathcal{F}$. Let $\Phi_{C_i}, U_{C_i}, R, a_i, \phi_i, \Phi_L, U_L, \mathcal{E}, \Psi_M^e, \Upsilon_i^e$, and Φ_L^e for $i = 1, \ldots, n$ be as in Theorem 8.11, and finally define Υ_i^f, Φ_L^f, and U_L' for $i = 1, \ldots, n$ as in the end of §8.4.

9.1 Deforming Lagrangian submanifolds with isolated conical singularities

In this subsection we study the deformations of the conical singularities of Lagrangian submanifolds of the form $\Phi_L^f \circ \mathrm{d}u : L \to M$, where $f \in \mathcal{F}$ and $u \in C^2(L)$ with $\Gamma_{\mathrm{d}u} \subset U_L'$. Let $v \in T_f\mathcal{F}$. Differentiating $\Phi_L^f \circ \mathrm{d}u$ with respect to f in the direction of v gives a C^1-section $\partial_v(\Phi_L^f \circ \mathrm{d}u)$ of the vector bundle $(\Phi_L^f \circ \mathrm{d}u)^*(TM)$, since $\Phi_L^f \circ \mathrm{d}u : L \to M$ is a C^1-submanifold. In the first part of this section we will show that $\partial_v(\Phi_L^f \circ \mathrm{d}u)$ can be extended to a smooth Hamiltonian vector field on M.

We begin by showing that the functions $q_1, \ldots, q_n$ on L can be extended to smooth functions on M.

Lemma 9.1. *There exist $\bar{q}_1, \ldots, \bar{q}_n \in C^\infty(M)$, such that $\bar{q}_i \equiv 0$ on $M \backslash \Upsilon_i(B_R)$, $\bar{q}_i \equiv 1$ on $\Upsilon_i(B_{R'})$ for $i = 1, \ldots, n$ and for every $u \in C^2(L)$ with $\Gamma_{\mathrm{d}u} \subset U_L'$ and $f \in \mathcal{F}$, $(\Phi_L^f \circ \mathrm{d}u)^*(\bar{q}_i) = q_i$ for $i = 1, \ldots, n$.*

Proof. We set $\bar{q}_i \equiv 1$ on $\Upsilon_i(B_{R'})$ and $\bar{q}_i \equiv 0$ on $M \backslash \Upsilon_i(B_R)$ for $i = 1, \ldots, n$. Moreover using that $\Phi_L : U_L \to M$ is an embedding we can define $\bar{q}_i$ on $\Phi_L(U_L \cap \pi^{-1}(S_i \backslash S_i'))$ by $\bar{q}_i(\Phi_L(x, \beta)) = q_i(x)$ for $x \in L$ and $\beta \in T_x^*L$ with $(x, \beta) \in U_L$ for $i = 1, \ldots, n$. Finally we extend $\bar{q}_i$ for $i = 1, \ldots, n$ to a smooth function on the whole of M. Using that $\Phi_L^e \equiv \Phi_L$ on $U_L' \cap \pi^{-1}(L \backslash \bigcup_{i=1}^n S_i')$ it follows that $\bar{q}_i$ satisfies $(\Phi_L^f \circ \mathrm{d}u)^*(\bar{q}_i) = q_i$ for every $u \in C^2(L)$ with $\Gamma_{\mathrm{d}u} \subset U_L'$ and $f \in \mathcal{F}$ for $i = 1, \ldots, n$. □

Define a smooth vector field $X_{\bar{q}_i}$ on M by $\mathrm{d}\bar{q}_i = X_{\bar{q}_i} \lrcorner\, \omega$ for $i = 1, \ldots, n$. Let $f \in \mathcal{F}$ and $v = (w, c) \in T_f\mathcal{F} = T_e\mathcal{E} \oplus \mathbb{R}^n$. Differentiating Ψ_M^e with respect to

$e \in \mathcal{E}$ in direction of $w \in T_e\mathcal{E}$ gives a smooth vector field $\partial_w \Psi^e_M$ on M. Define

$$X_f(v) = (\partial_w \Psi^e_M) \circ (\Psi^e_M)^{-1} - \sum_{i=1}^{n} c_i X_{\bar{q}_i}. \tag{85}$$

Then $X_f(v)$ is a smooth vector field M, which depends linearly on $v \in T_f\mathcal{F}$ and, since Ψ^e_M depends smoothly on $e \in \mathcal{E}$, $X_f(v)$ depends smoothly on $f \in \mathcal{F}$. In fact $X_f(v)$ is a Hamiltonian vector field as we will show in the next proposition.

Proposition 9.2. *Let $f \in \mathcal{F}$ and $v \in T_f\mathcal{F}$. Then there exists a unique smooth function $H_f(v)$ on M, which depends linearly on $v \in T_f\mathcal{F}$ and smoothly on $f \in \mathcal{F}$, such that $\mathrm{d}[H_f(v)] = X_f(v) \lrcorner\, \omega$ and $H_f(v) \equiv 0$ on $M \backslash \bigcup_{i=1}^n \Upsilon_i(B_R)$.*

Proof. By Lemma 8.10, Ψ^e_M is the identity on $M \backslash \bigcup_{i=1}^n \Upsilon_i(B_{R/2})$ for every $e \in \mathcal{E}$. Thus $X_f(v) = -\sum_{i=1}^n c_i X_{\bar{q}_i}$ on $M \backslash \bigcup_{i=1}^n \Upsilon_i(B_{R/2})$. Define $H_f(v)$ on $M \backslash \bigcup_{i=1}^n \Upsilon_i(B_{R/2})$ by $H_f(v) = -\sum_{i=1}^n c_i \bar{q}_i$. Then we have $H_f(v) \equiv 0$ on $M \backslash \bigcup_{i=1}^n \Upsilon_i(B_R)$, since $\bar{q}_i \equiv 0$ on $M \backslash \Upsilon_i(B_R)$ for $i = 1, \ldots, n$. Since $(\Psi^e_M)^*(\omega) = \omega$, it follows that $\mathcal{L}_{X_f(v)}\omega = 0$. Then by Cartan's formula $X_f(v) \lrcorner\, \omega$ is a closed one form on M. Since $\bigcup_{i=1}^n \Upsilon_i(B_{R'})$ is contractible, $X_f(v) \lrcorner\, \omega$ is exact on $\bigcup_{i=1}^n \Upsilon_i(B_{R'})$ and we can extend $H_f(v)$ to a smooth function on the whole of M such that $\mathrm{d}[H_f(v)] = X_f(v) \lrcorner\, \omega$ and $H_f(v)$ depends linearly on $v \in T_f\mathcal{F}$. $\square$

The next proposition shows that the Hamiltonian vector field $X_f(v)$ restricts to the variation field $\partial_v(\Phi^f_L \circ \mathrm{d}u)$.

Proposition 9.3. *Let $f \in \mathcal{F}$ and $v \in T_f\mathcal{F}$. Then for every $u \in C^2(L)$ with $\Gamma_{\mathrm{d}u} \subset U'_L$, we have*

$$(\Phi^f_L \circ \mathrm{d}u)^*(\mathrm{d}[H_f(v)]) = (\Phi^f_L \circ \mathrm{d}u)^*(\partial_v(\Phi^f_L \circ \mathrm{d}u) \lrcorner\, \omega).$$

Proof. Let $u \in C^2(L)$, $f \in \mathcal{F}$, and $v = (w, c) \in T_f\mathcal{F}$. By definition of $H_f(v)$ we have

$$(\Phi^f_L \circ \mathrm{d}u)^*(\mathrm{d}[H_f(v)]) = (\Phi^f_L \circ \mathrm{d}u)^*(X_f(v) \lrcorner\, \omega).$$

Denote $X = \frac{\mathrm{d}}{\mathrm{d}s}\Phi^f_L \circ (\mathrm{d}u + s\sum_{i=1}^n c_i \mathrm{d}q_i)|_{s=0}$. Then X is a section of the vector bundle $(\Phi^f_L \circ \mathrm{d}u)^*(TM)$ and by Lemma 4.11 the normal part of X is $-\sum_{i=1}^n c_i \alpha^{-1}(\mathrm{d}q_i)$ and the tangential part is $\sum_{i=1}^n c_i \mathrm{d}(\Phi^f_L \circ \mathrm{d}u)(\hat{V}(\mathrm{d}q_i))$. Using the definition of Φ^f_L from (84) we then find

$$\partial_v(\Phi^f_L \circ \mathrm{d}u) = \partial_w(\Phi^f_L \circ \mathrm{d}u) - \sum_{i=1}^{n} c_i \big\{\alpha^{-1}(\mathrm{d}q_i) - \mathrm{d}(\Phi^f \circ \mathrm{d}u)(\hat{V}(\mathrm{d}q_i))\big\}. \tag{86}$$

Using the definition of Φ^f_L again it follows that the pull back of the vector field $(\partial_w \Psi^e_M) \circ (\Psi^e_M)^{-1}$ with $\Phi^f_L \circ \mathrm{d}u : L \to M$ is equal to $\partial_w(\Phi^f_L \circ \mathrm{d}u)$. Hence

$$(\Phi^f_L \circ \mathrm{d}u)^*((\partial_w \Psi^e_M) \circ (\Psi^e_M)^{-1} \lrcorner\, \omega) = (\Phi^f_L \circ \mathrm{d}u)^*(\partial_w(\Phi^f_L \circ \mathrm{d}u) \lrcorner\, \omega). \tag{87}$$

By definition of α we clearly have

$$(\Phi^f_L \circ \mathrm{d}u)^*(\alpha^{-1}(\mathrm{d}q_i) \lrcorner\, \omega) = \mathrm{d}q_i \tag{88}$$

for $i = 1, \ldots, n$. Moreover, since $\Phi_L^f \circ \mathrm{d}u : L \to M$ is a Lagrangian submanifold, we find

$$(\Phi_L^f \circ \mathrm{d}u)^*(\mathrm{d}(\Phi^f \circ \mathrm{d}u)(\hat{V}(\mathrm{d}q_i)) \lrcorner\, \omega) = 0 \tag{89}$$

for $i = 1, \ldots, n$. The proposition now follows from (86)-(89) and the definition of $X_f(v)$ in (85). $\square$

Let $u \in C^k(L)$, $k > 1$, with $\Gamma_{\mathrm{d}u} \subset U_L'$ and $f \in \mathcal{F}$. Then $\Phi_L^f \circ \mathrm{d}u : L \to M$ is a C^{k-1}-submanifold of M. We define a linear map

$$\Xi_{(u,f)} : T_f\mathcal{F} \longrightarrow C^{k-1}_{\mathrm{loc}}(L), \quad \Xi_{(u,f)}(v) = (\Phi_L^f \circ \mathrm{d}u)^*(H_f(v)). \tag{90}$$

Then $\Xi_{(u,f)}(v) \equiv 0$ on K by Proposition 9.2 and

$$\mathrm{d}[\Xi_{(u,f)}(v)] = (\Phi_L^f \circ \mathrm{d}u)^*(\partial_v(\Phi_L^f \circ \mathrm{d}u) \lrcorner\, \omega)$$

by Proposition 9.3. We show that $\Xi_{(u,f)}(v)$ is asymptotic to the pull back of a $U(m) \ltimes \mathbb{C}^m$-moment map on each end of L.

Proposition 9.4. *Let $\boldsymbol{\mu} \in \mathbb{R}^n$ with $2 < \boldsymbol{\mu} \le \boldsymbol{\nu}$, $k > 1$, and $u \in C^k_{\boldsymbol{\mu}}(L)$ with $\Gamma_{\mathrm{d}u} \subset U_L'$. Let $f = (e, c) \in \mathcal{F}$, denote $f_i = (\hat{x}_i, \hat{A}_i, c_i)$ for $i = 1, \ldots, n$, and let $v = (v_1, \ldots, v_n) \in T_{f_1}\mathcal{F}_1 \oplus \cdots \oplus T_{f_n}\mathcal{F}_n$, where $\mathcal{F}_i = \mathcal{E}_i \times \mathcal{U}_i$ for $i = 1, \ldots, n$ as in the end of §8.4. Then the following hold.*

(i) *If $v_i \in T_{\hat{A}_i}\mathcal{A}_{\hat{x}_i}$, then there exists a unique $X_i \in \mathfrak{u}(m) \oplus \mathbb{R}$, such that*

$$\left|\nabla^j(\phi_i^*(\Xi_{(u,f)}(v)) - \iota_i^*(\mu_{X_i}))\right| = O(r^{\mu_i - j}) \quad \textit{as } r \to 0$$

for $j = 0, \ldots, k-1$.

(ii) *If $v_i \in T_{\hat{x}_i}M$, then there exists a unique $X_i \in \mathbb{C}^m \oplus \mathbb{R}$, such that*

$$\left|\nabla^j(\phi_i^*(\Xi_{(u,f)}(v)) - \iota_i^*(\mu_{X_i}))\right| = O(r^{\mu_i - 1 - j}) \quad \textit{as } r \to 0$$

for $j = 0, \ldots, k-1$.

Moreover for every $u \in C^k(L)$ with $\Gamma_{\mathrm{d}u} \subset U_L'$ and $f \in \mathcal{F}$ the map (90) is a monomorphism and $\dim \mathrm{im}\, \Xi_{(u,f)} = n(m^2 + 2m + 1) - \sum_{i=1}^n \dim G_i$.

Proof. We demonstrate the proof of (i). Choose some $i = 1, \ldots, n$ and assume that $v_i \in T_{\hat{A}_i}\mathcal{A}_{\hat{x}_i}$. Pushing the vector field $X_f(v)$ forward with $(\mathrm{d}\Upsilon_i^f)^{-1}$ gives a smooth vector field X_i on B_R. Then by construction of $X_f(v)$ and Υ_i^f we find that $X_i = \mathrm{d}\Psi_i^e(\partial_{v_i}\Psi_i^e)$, where Ψ_i^e is as in the proof of Lemma 8.10. On $B_{R/4}$ we have $\Psi_i^e = \hat{A}_i$ and thus $X_i = \hat{A}_i \circ v_i$. In particular $X_i \in \mathrm{u}(m)$. Using $(\Upsilon_i^f)^*(\omega) = \omega'$ we can write

$$\mathrm{d}[\Xi_{(u,f)}(v)] = (\iota_i \circ \phi_i^{-1})^*(\mathrm{d}\mu_{X_i}) + ((\Upsilon_i^f)^{-1} \circ (\Phi_L^f \circ \mathrm{d}u) - \iota_i \circ \phi_i^{-1})^*(X_i \lrcorner\, \omega'),$$

where μ_{X_i} is defined as in (80). Then using $X_i \lrcorner\, \omega' = O(r)$ and (82) we find that

$$\left|\nabla^j(\phi_i^*(\mathrm{d}[\Xi_{(u,f)}(v)]) - \iota_i^*(\mathrm{d}\mu_{X_i}))\right| = O(r^{\mu_i - 1 - j}) \quad \text{as } r \to 0$$

for $j = 0, \ldots, k-2$. Hence by adding a constant to X_i as in (80), if necessary, we conclude that

$$\left|\nabla^j(\phi_i^*(\Xi_{(u,f)}(v)) - \iota_i^*(\mu_{X_i})\right| = O(r^{\mu_i - 1 - j}) \quad \text{as } r \to 0$$

for $j = 0, \ldots, k-1$. This proves (i), and (ii) is proved in a similar way.

The injectivity of the map (90) follows directly from its definition. Moreover, since $\dim \mathcal{F} = n(m^2 + 2m + 1) - \sum_{i=1}^{n} \dim G_i$, it follows that the image of $T_f\mathcal{F}$ under $\Xi_{(u,f)}$ has the same dimension. □

Note in particular that if $2 < \boldsymbol{\mu} \leq \boldsymbol{\nu}$ with $(2, \mu_i] \cap \mathcal{E}_{\Sigma_i} = \emptyset$ for $i = 1, \ldots, n$ and the special Lagrangian cones $C_1, \ldots, C_n$ are stable, then $\dim \operatorname{im} \Xi_{(0,f_0)} = \dim \operatorname{im} \Psi_{\boldsymbol{\mu}}$. Here $\Psi_{\boldsymbol{\mu}}$ is defined as in Proposition 6.14 and/or Proposition 10.3.

9.2 Integrating the generalized Lagrangian mean curvature flow with isolated conical singularities

As in §5.2 we show in this subsection how the generalized Lagrangian mean curvature flow of Lagrangian submanifold with isolated conical singularities can be written as a nonlinear equation of a function u on L and the parameter $f \in \mathcal{F}$. The only major difference to the approach in §5.2 is that we have to build in the parameter $f \in \mathcal{F}$ into our equation, in order to be able to translate and rotate the cones in M.

Since $F_0 : L \to M$ is a Lagrangian submanifold with isolated conical singularities modelled on stable special Lagrangian cones, it follows from Proposition 8.5 that the Maslov class μ_{F_0} of $F_0 : L \to M$ is an element of $H^1_{\mathrm{cs}}(L, \mathbb{R})$. Choose a smooth map $\alpha_0 : L \to \mathbb{R}/\pi\mathbb{Z}$ with $\mathrm{d}\alpha_0 \in \mu_{F_0}$. Since L retracts onto K, we can choose α_0 to be supported on K. We denote $\beta_0 = \mathrm{d}\alpha_0$ which is a smooth one-form on L, that is supported on K and represents the Maslov class of $F_0 : L \to M$. As in §5.2 we can then choose a smooth lift $\Theta(F_0) : L \to \mathbb{R}$ of $\theta(F_0) - \alpha_0 : L \to \mathbb{R}/\pi\mathbb{Z}$. Then $\mathrm{d}[\Theta(F_0)] = \mathrm{d}[\theta(F_0)] - \beta_0$. Finally, if $\{\eta(s)\}_{s \in (-\varepsilon,\varepsilon)}$, $\varepsilon > 0$, is a continuous family of closed one-forms defined on L with $\Gamma_{\eta(s)} \subset U_L$ for $s \in (-\varepsilon, \varepsilon)$ and $\eta(0) = 0$, then we may choose $\Theta(\Phi_L \circ \eta(s))$ to depend continuously on $s \in (-\varepsilon, \varepsilon)$.

We now define an operator P as follows. We then define a smooth one-parameter family of closed one forms $\{\beta(t)\}_{t \in (0,T)}$ by $\beta(t) = t\beta_0$. Then $\{\beta(t)\}_{t \in (0,T)}$ extends continuously to $t = 0$ with $\beta(0) = 0$. We define the domain of P is given by

$$\begin{aligned}\mathcal{D} = \{(u, f) \,:\, & u \in C^\infty((0,T) \times L),\ f \in C^\infty((0,T); \mathcal{F}),\ u \text{ and } f \text{ extend} \\ & \text{continuously to } t = 0, \text{ and } \Gamma_{\mathrm{d}[u(t,\cdot)] + \beta(t)} \subset U'_L \text{ for } t \in (0,T)\}.\end{aligned}$$

The operator $P : \mathcal{D} \to C^\infty((0,T) \times L)$ is then defined by

$$P(u, f) = \frac{\partial u}{\partial t} - \Theta(\Phi^f_L \circ (\mathrm{d}u + \beta)) - \Xi_{(u,f)}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right).$$

In the remainder we will study the following Cauchy problem

$$\begin{aligned} & P(u, f)(t, x) = 0 && \text{for } (t, x) \in (0, T) \times L, \\ & u(0, x) = 0 && \text{for } x \in L, \\ & f(0) = f_0. && \end{aligned} \tag{91}$$

If we are given a solution $(u, f) \in \mathcal{D}$ of (91), then we obtain a solution to the generalized Lagrangian mean curvature flow with initial condition $F_0 : L \to M$.

Proposition 9.5. *Suppose $(u, f) \in \mathcal{D}$ is a solution of the Cauchy problem* (91)*. Define a one-parameter family $\{F(t,\cdot)\}_{t\in(0,T)}$ of submanifolds by*

$$F(t,\cdot) : L \longrightarrow M, \quad F(t,\cdot) = \Phi_L^{f(t)} \circ (\mathrm{d}[u(t,\cdot)] + \beta(t)). \tag{92}$$

Then $\{F(t,\cdot)\}_{t\in(0,T)}$ is a smooth one-parameter family of smooth Lagrangian submanifolds, continuous up to $t = 0$, which evolves by generalized Lagrangian mean curvature flow with initial condition $F_0 : L \to M$.

Proof. Let $(u, f) \in \mathcal{D}$ be a solution of the Cauchy problem (91). Since u and f extend continuously to $t = 0$ with $u(0,\cdot) = 0$ on L and $f(0) = f_0$, $\{F(t,\cdot)\}_{t\in(0,T)}$ is a smooth one-parameter family of Lagrangian submanifolds, continuous up to $t = 0$, with

$$F(0, x) = \Phi_L^{f(0)}(x, \mathrm{d}[u(0,\cdot)](x) + \beta(0)) = \Phi_L^{f_0}(x, 0) = F_0(x)$$

for $x \in L$. Thus it remains to show that $\{F(t,\cdot)\}_{t\in(0,T)}$ evolves by generalized Lagrangian mean curvature flow. We show that $\alpha_{\frac{\partial F}{\partial t}} = \alpha_K$. Denote $X = \frac{\mathrm{d}}{\mathrm{d}t}\Phi_L^f \circ (\mathrm{d}u+\beta)$. Then X is a section of the vector bundle $(\Phi_L^f \circ (\mathrm{d}u+\beta))^*(TM)$ and using Lemma 4.11 it follows that

$$X = \partial_{\frac{\mathrm{d}f}{\mathrm{d}t}} \left(\Phi_L^f \circ (\mathrm{d}u + \beta)\right) - \alpha^{-1} \left(\mathrm{d}\left[\frac{\partial u}{\partial t}\right] + \beta_0\right) + V\left(\mathrm{d}\left[\frac{\partial u}{\partial t}\right] + \beta_0\right).$$

Since $\Phi_L^f \circ (\mathrm{d}u + \beta) : L \to M$ is a Lagrangian submanifold,

$$(\Phi_L^f \circ (\mathrm{d}u + \beta))^* \left(V\left(\mathrm{d}\left[\frac{\partial u}{\partial t}\right] + \beta_0\right) \lrcorner\, \omega\right) = 0.$$

Moreover, using $P(u, f) = 0$ and the definition of $\Theta(\Phi_L^f \circ (\mathrm{d}u + \beta))$ we find

$$\mathrm{d}\left[\frac{\partial u}{\partial t}\right] + \beta_0 = \mathrm{d}[\theta(\Phi_L^f \circ (\mathrm{d}u + \beta))] + \mathrm{d}\left[\Xi_{(u,f)}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\right].$$

and hence we obtain

$$\begin{aligned}\alpha_{\frac{\partial F}{\partial t}} = {} & (\Phi_L^f \circ (\mathrm{d}u + \beta))^*(\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ (\mathrm{d}u + \beta)) \lrcorner\, \omega) \\ & -\mathrm{d}\left[\Xi_{(u,f)}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\right] - \mathrm{d}[\theta(\Phi_L^f \circ (\mathrm{d}u + \beta))].\end{aligned}$$

Since $\Xi_{(u,f)} \equiv 0$ on K and $\beta(t)$ is supported on K for every $t \in (0, T)$, we have

$$(\Phi_L^f \circ (\mathrm{d}u + \beta))^*(\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ (\mathrm{d}u + \beta)) = \mathrm{d}\left[\Xi_{(u,f)}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\right]$$

and thus $\alpha_{\frac{\partial F}{\partial t}} = -\mathrm{d}[\theta(F)]$. By Proposition 4.8 the generalized mean curvature form satisfies $\alpha_K = -\mathrm{d}[\theta(F)]$, from which the claim follows. $\square$

9.3 Smoothness of P as a map between Banach manifolds

The goal of this subsection is to prove that $P : \mathcal{D} \to C^\infty((0,T)\times L)$ extends to a smooth map between certain Banach manifolds. The main difference to §5.3 is that we now also have to study the regularity of the map P on the ends of L.

We first extend the domain of the operator P. The manifold $\mathcal{F}$ embeds into $\mathbb{R}^s$ for some sufficiently large $s \in \mathbb{N}$. Let $p \in (1,\infty)$ and $f \in W^{1,p}((0,T);\mathbb{R}^s)$. Then $f : (0,T) \to \mathcal{F}$ is continuous by the Sobolev Embedding Theorem and the condition $f(t) \in \mathcal{F}$ makes sense for every $t \in (0,T)$. We define the Banach manifold $W^{1,p}((0,T);\mathcal{F})$ by

$$W^{1,p}((0,T);\mathcal{F}) = \left\{ f \in W^{1,p}((0,T);\mathbb{R}^s) \ : \ f(t) \in \mathcal{F} \text{ for } t \in (0,T) \right\}.$$

Let $k \in \mathbb{N}$, $p \in (1,\infty)$ with $k - \frac{m}{p} > 2$, and $\boldsymbol{\mu} \in \mathbb{R}^n$ with $2 < \boldsymbol{\mu} < 3$. For $T > 0$ small enough, such that $\Gamma_{\beta(t)} \subset U'_L$ for $t \in (0,T)$, we define

$$\begin{aligned}\mathcal{D}^{k,p}_{\boldsymbol{\mu}} = \big\{ (u,f) \ : \ u \in W^{1,k,p}_{\boldsymbol{\mu}}((0,T)\times L),\ f \in W^{1,p}((0,T);\mathcal{F}),\\ \text{such that } \Gamma_{\mathrm{d}[u(t,.)]+\beta(t)} \subset U'_L \text{ for } t \in (0,T)\big\}.\end{aligned}$$

Let $(u,f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$. Since $k - \frac{m}{p} > 2$, it follows from the Sobolev Embedding Theorem that $\Phi_L^{f(t)} \circ (\mathrm{d}[u(t,\cdot)] + \beta(t)) : L \to M$ is a C^1-Lagrangian submanifold for almost every $t \in (0,T)$. In particular $\Theta(\Phi_L^{f(t)} \circ (\mathrm{d}[u(t,\cdot)] + \beta(t)))$ is well defined for almost every $t \in (0,T)$ and P acts on $\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$.

In order to define the target space for P acting on $\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ we have to introduce a weighted parabolic Sobolev space with discrete asymptotics. First we define a weighted Sobolev space $W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L)$ with discrete asymptotics by

$$W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L) = W^{k-2,p}_{\boldsymbol{\mu}-2}(L) \oplus \mathrm{span}\{q_1,\ldots,q_n\}.$$

Note that this is a natural definition for a weighted Sobolev space with discrete asymptotics in this case, since the q_i's are constant on each end of L and $\boldsymbol{\mu} > 2$. Further we define the weighted parabolic Sobolev space $W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ with discrete asymptotics by

$$W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L) = L^p((0,T);W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L)).$$

The main result of this subsection is that $P : \mathcal{D}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu}-2}((0,T)\times L)$ is a smooth map provided $k \in \mathbb{N}$ and $p \in (1,\infty)$ are sufficiently large and $\boldsymbol{\mu} < \boldsymbol{\nu}$. The idea of the proof follows Proposition 5.6, the new difficulty, however, is that we have to deal with the regularity of the operator P on the ends of L.

Notice that for $(u,f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ we only require u and f to have one time derivative that lies in L^p, whereas we are allowed to choose $k \in \mathbb{N}$ and $p \in (1,\infty)$ arbitrary large. By choosing $k \in \mathbb{N}$ and $p \in (1,\infty)$ sufficiently large we are guaranteed that P maps $\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ smoothly into another Banach manifold. We require u and f only to have one time derivative in L^p to make an argument in the short time existence proof in Proposition 9.13 work.

Let $\eta \in C^\infty(T^*L)$ be a closed one-form with $\Gamma_\eta \subset U'_L$, which is supported on K, and define $F_\eta(f,x,\mathrm{d}u(x),\nabla\mathrm{d}u(x)) = \Theta(\Phi^f_L \circ (\mathrm{d}u + \eta))(x)$, where

$$\begin{aligned}F_\eta : \big\{ (f,x,y,z) \ : \ f \in \mathcal{F},\ x \in L,\\ y \in T^*_xL \text{ with } y + \eta(x) \in U'_L,\ z \in \otimes^2 T^*_xL \big\} \longrightarrow \mathbb{R}.\end{aligned}$$

Then F_η is a smooth and nonlinear function on its domain, since Ω, g, ψ, and Φ_L^f are smooth and Φ_L^f depends smoothly on $f \in \mathcal{F}$. Furthermore we define a function Q_η on the domain of F_η by

$$\begin{aligned} Q_\eta(f,x,y,z) = F_\eta(f,x,y,z) - F_\eta(f_0,x,0,0) \\ - (\partial_y F_\eta)(f_0,x,0,0)\cdot y - (\partial_z F_\eta)(f_0,x,0,0)\cdot z. \end{aligned} \tag{93}$$

Since F_η is smooth, Q_η is a smooth and nonlinear function on its domain.

Next define $F_i(\sigma,r,\mathrm{d}u_i(\sigma,r),\nabla\mathrm{d}u_i(\sigma,r)) = \theta(\Phi_{C_i}\circ\mathrm{d}u_i)(\sigma,r)$, where

$$\begin{aligned} F_i : \big\{(\sigma,r,y_i,z_i)\ :\ (\sigma,r)\in\Sigma_i\times(0,R), \\ y_i \in T^*_{(\sigma,r)}(\Sigma_i\times(0,R))\cap U_{C_i},\ z_i\in\otimes^2 T^*_{(\sigma,r)}(\Sigma_i\times(0,R))\big\} \longrightarrow \mathbb{R} \end{aligned}$$

for $i=1,\dots,n$. Then $F_i(\sigma,r,y,z)$ is a smooth and nonlinear function on its domain, since Ω', g', and Φ_{C_i} are smooth for $i=1,\dots,n$. Furthermore we define functions Q_i on the domain of F_i by

$$\begin{aligned} Q_i(\sigma,r,y,z) = F_i(\sigma,r,y,z) - F_i(\sigma,r,0,0) \\ - (\partial_y F_i)(\sigma,r,0,0)\cdot y - (\partial_z F_i)(\sigma,r,0,0)\cdot z. \end{aligned} \tag{94}$$

for $i=1,\dots,n$. Then Q_i is a smooth nonlinear function on its domain for $i=1,\dots,n$, since F_i is smooth for $i=1,\dots,n$. Let $u_i\in C^2(\Sigma_i\times(0,R))$ with $\Gamma_{\mathrm{d}u_i}\subset U_{C_i}$ for $i=1,\dots,n$. Then it follows from Lemma 5.5 that

$$(\partial_y F_i)(\sigma,r,0,0)\cdot\mathrm{d}u_i(\sigma,r) + (\partial_z F_i)(\sigma,r,0,0)\cdot\nabla\mathrm{d}u_i(\sigma,r) = \Delta u_i(\sigma,r) \tag{95}$$

for $(\sigma,r)\in\Sigma_i\times(0,R)$ and $i=1,\dots,n$. Moreover, since C_i is a special Lagrangian cone, C_i has constant phase $e^{i\theta_i}$ and we can write

$$\begin{aligned} F_i(\sigma,r,\mathrm{d}u_i(\sigma,r),\nabla\mathrm{d}u_i(\sigma,r)) = \\ \theta_i + \Delta u_i(\sigma,r) + Q_i(\sigma,r,\mathrm{d}u_i(\sigma,r),\nabla\mathrm{d}u_i(\sigma,r)) \end{aligned} \tag{96}$$

for $(\sigma,r)\in\Sigma_i\times(0,R)$ and $i=1,\dots,n$.

Next we prove some estimates for the functions Q_i.

Lemma 9.6. *For $a,b,c\geq 0$ and small $r^{-1}|y|,|z|$ we have*

$$(\nabla_x)^a(\partial_y)^b(\partial_z)^c Q_i(x,y,z) = O\left(r^{-a-\max\{2,b\}}|y|^{\max\{0,2-b\}} + r^{-a}|z|^{\max\{0,2-c\}}\right)$$

uniformly for $x=(\sigma,r)\in\Sigma_i\times(0,R)$ and $i=1,\dots,n$.

Proof. Since F_i is smooth on its domain, Taylor's Theorem implies that for $a,b,c\geq 0$ and small $|y|,|z|$,

$$(\nabla_x)^a(\partial_y)^b(\partial_z)^c Q_i(x,y,z) = O\left(|y|^{\max\{0,2-b\}} + |z|^{\max\{0,2-c\}}\right) \tag{97}$$

for fixed $x=(\sigma,r)\in\Sigma_i\times(0,R)$ and $i=1,\dots,n$.

For $s\in(0,1)$ and $i=1,\dots,n$ define $\delta_i^s:\Sigma_i\times(0,R)\longrightarrow\Sigma_i\times(0,R)$ by $\delta_i^s(\sigma,r)=(\sigma,sr)$. Using the invariance of the Lagrangian angle under dilations and $(\delta_i^s)^*(\Phi_{C_i}) = s\cdot\Phi_{C_i}$, we find $\theta(\Phi_{C_i}\circ\mathrm{d}u_i) = \theta(\Phi_{C_i}\circ\mathrm{d}u_i^s)\circ\delta_i^s$, where

$u_i^s = s^2 u_i \circ (\delta_i^s)^{-1}$. Since $\Delta u_i = (\Delta u_i^s) \circ \delta_i^s$, it follows from the definition of Q_i that

$$Q_i(\sigma, r, \mathrm{d}u_i(\sigma, t), \nabla \mathrm{d}u_i(\sigma, r)) = Q_i(\sigma, sr, \mathrm{d}u_i^s(\sigma, sr), \nabla \mathrm{d}u_i^s(\sigma, sr)) \tag{98}$$

for $s \in (0,1)$ and $(\sigma, r) \in \Sigma_i \times (0, R)$. Since $|\mathrm{d}u_i^s(\sigma, sr)| = s^2 |\mathrm{d}u_i(\sigma, r)|$ and $|\nabla \mathrm{d}u_i^s(\sigma, sr)| = |\nabla \mathrm{d}u_i(\sigma, r)|$ it follows from (97), (98), and the compactness of Σ_i that for small $r^{-1}|\mathrm{d}u_i(\sigma, r)|$ and $|\nabla \mathrm{d}u_i(\sigma, r)|$,

$$Q_i(\sigma, r, \mathrm{d}u_i(\sigma, r), \nabla \mathrm{d}u_i(\sigma, r)) = O\left(r^{-2}|\mathrm{d}u_i(\sigma, r)|^2 + |\nabla \mathrm{d}u_i(\sigma, r)|^2\right)$$

uniformly for $(\sigma, r) \in \Sigma_i \times (0, R)$. The derivatives of Q_i are then estimated in a similar way. This completes the proof of the lemma. $\square$

Let $a \in C^\infty(L)$ be a smooth function on L that satisfies $\phi_i^*(a) = a_i$ for $i = 1, \ldots, n$. Here a_i is as defined in Theorem 8.11. Then we define functions R_i on the domain of F_i by

$$R_i(f, \sigma, r, y_i, z_i) = F_\eta(f, x, y - \mathrm{d}a(x), z - \nabla \mathrm{d}a(x)) - F_i(\sigma, r, y_i, z_i), \tag{99}$$

for $i = 1, \ldots, n$, where $x = \phi_i(\sigma, r), y_i = \phi_i^*(y)$, and $z_i = \phi_i^*(z)$. Then R_i is smooth on its domain for $i = 1, \ldots, n$, since F_η and F_i are smooth for $i = 1, \ldots, n$.

Lemma 9.7. *For $a, b, c \geq 0$ we have*

$$(\nabla_x)^a (\partial_y)^b (\partial_z)^c R_i(f, \sigma, r, y, z) = O(r^{1-a-b})$$

uniformly for $x = (\sigma, r) \in \Sigma_i \times (0, R)$ and $i = 1, \ldots, n$.

Proof. From Theorem 8.11 we obtain

$$R_i(f, \sigma, r, \mathrm{d}u_i(\sigma, r), \nabla \mathrm{d}u_i(\sigma, r)) = \theta(\Upsilon_i^f \circ \Phi_{C_i} \circ \mathrm{d}u_i)(\sigma, r) - \theta(\Phi_{C_i} \circ \mathrm{d}u_i)(\sigma, r)$$

for $i = 1, \ldots, n$. Denote $g_i^1 = (\Upsilon_i^f \circ \Phi_{C_i} \circ \mathrm{d}u_i)^*(g')$ and $g_i^2 = (\Phi_{C_i} \circ \mathrm{d}u_i)^*(g')$ for $i = 1, \ldots, n$. Since $\mathrm{d}\Upsilon_i^f|_0 = \hat{A}_i$, we have by Taylor's Theorem

$$\mathrm{d}V_{g_i^1} - e^{-i\theta_i - m\psi(\hat{x}_i)} \mathrm{d}V_{g_i^2} = O(r)$$

for $i = 1, \ldots, n$. Moreover $(\Upsilon_i^f)^*(\Omega)|_0 = e^{i\theta_i + m\psi(\hat{x}_i)}\Omega'$ and hence

$$e^{-i\theta_i - m\psi(\hat{x}_i)} (\Upsilon_i^f)^*(\Omega') - \Omega' = O(r)$$

for $i = 1, \ldots, n$ by Taylor's Theorem. Using the definition of the Lagrangian angle we conclude that $R_i(f, \sigma, r, y, z) = O(r)$ for $i = 1, \ldots, n$. The derivatives of R_i are then estimated in a similar way using Lemma 9.6. $\square$

Using Lemmas 9.6 and 9.7 we can prove the following estimates for the function Q_η as defined in (93).

Lemma 9.8. *For $a, b, c \geq 0$ and small $\rho^{-1}(x)|y|$, $|z|$, and $d(f, f_0)$ we have*

$$(\nabla_x)^a (\partial_y)^b (\partial_z)^c Q_\eta(f, x, y, z) =$$
$$O\left(\rho(x)^{-a-\max\{2,b\}}|y|^{\max\{0,2-b\}} + \rho(x)^{-a}|z|^{\max\{0,2-c\}} + \rho(x)^{1-a-b} d(f, f_0)\right),$$

uniformly for $x \in L$. Here $d(f, f_0)$ denotes the distance of f to f_0 in $\mathcal{F}$.

Proof. By Taylor's Theorem we have

$$(\nabla_x)^a(\partial_y)^b(\partial_z)^c Q_\eta(f,x,y,z) = O\left(|y|^{\max\{0,2-b\}} + |z|^{\max\{0,2-b\}} + d(f,f_0)\right)$$

uniformly for x in compact subsets of L and for small $|y|, |z|$, and $d(f,f_0)$. Let $x \in S_i$ with $x = \phi_i(\sigma,r)$, $(\sigma,r) \in \Sigma_i \times (0,R)$ for some $i = 1,\dots,n$. Then using (93), (94), and (99) we can write

$$\begin{aligned}
&Q_\eta(f,x,y,z) = Q_i(\sigma,r,y_i + \mathrm{d}a_i(\sigma,r), z_i + \nabla\mathrm{d}a_i(\sigma,r)) \\
&- Q_i(\sigma,r,\mathrm{d}a_i(\sigma,r),\nabla\mathrm{d}a_i(\sigma,r)) - (\partial_y Q_i)(\sigma,r,\mathrm{d}a_i(\sigma,r),\nabla\mathrm{d}a_i(\sigma,r))\cdot y_i \\
&- (\partial_z Q_i)(\sigma,r,\mathrm{d}a_i(\sigma,r),\nabla\mathrm{d}a_i(\sigma,r))\cdot z_i + R_i(f,\sigma,r,y_i+\mathrm{d}a_i(\sigma,r),z_i+\nabla\mathrm{d}a_i(\sigma,r)) \\
&- R_i(f,\sigma,r,\mathrm{d}a_i(\sigma,r),\nabla\mathrm{d}a_i(\sigma,r)) - (\partial_y R_i)(f,\sigma,r,\mathrm{d}a_i(\sigma,r),\nabla\mathrm{d}a_i(\sigma,r))\cdot y_i \\
&\qquad - (\partial_z R_i)(f,\sigma,r,\mathrm{d}a_i(\sigma,r),\nabla\mathrm{d}a_i(\sigma,r))\cdot z_i.
\end{aligned}$$

The lemma now follows from Taylor's Theorem and Lemma 9.6 and 9.7. □

Finally let us define a function

$$S : \left\{(f,v,x,y) \ : \ f \in \mathcal{F},\ v \in T_f\mathcal{F},\ x \in L,\ y \in T_x^*L \cap U_L\right\} \longrightarrow \mathbb{R}$$

by $S(f,v,x,\mathrm{d}u(x)) = \Xi_{(u,f)}(v)(x)$, where $\Xi_{(u,f)}(v)$ is defined in (90). Since Φ_L^f is smooth and depends smoothly on $f \in \mathcal{F}$, S is a smooth and nonlinear function on its domain. By definition of P we can now write

$$P(u,f) = \frac{\partial u}{\partial t} - F_\beta(f,\cdot,\mathrm{d}u,\nabla\mathrm{d}u) - S(f,\tfrac{\mathrm{d}f}{\mathrm{d}t},\cdot,\mathrm{d}u), \qquad (100)$$

where $\{\beta(t)\}_{t\in(0,T)}$ is given by $\beta(t) = t\beta_0$ and we now use F_η with $\eta = \beta(t)$ for $t \in (0,T)$.

We can now prove the smoothness of $P : \mathcal{D}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ for sufficiently large $k \in \mathbb{N}$ and $p \in (1,\infty)$.

Proposition 9.9. *Let $\boldsymbol{\mu} \in \mathbb{R}^n$ with $2 < \boldsymbol{\mu} < \boldsymbol{\nu}$, and define $\varepsilon \in \mathbb{R}$ by $\varepsilon = \min_{i=1,\dots,n} \frac{\mu_i - 2}{4}$. Let $k \in \mathbb{N}$ and $p \in (1,\infty)$ with $k \geq 6$ and $p > \max\{m, \frac{4+2m}{k-2}, \frac{2}{\varepsilon}\}$. Then $P : \mathcal{D}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ is a smooth map of Banach manifolds.*

Proof. We first show that $P : \mathcal{D}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ is well defined, i.e. that for $(u,f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ we have $P(u,f) \in W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$. We show that each of the terms on the right side of (100) lies in $W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$. Clearly we have $\partial_t u \in W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ by definition of $\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$. In order to show that $F_\beta(f,\cdot,\mathrm{d}u,\nabla\mathrm{d}u) \in W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ we first expand F_β by

$$\begin{aligned}
F_\beta(f,\cdot,\mathrm{d}u,\nabla\mathrm{d}u) = F_\beta(f_0,\cdot,0,0) + \Delta u - m\mathrm{d}\psi_\beta(\nabla u)& \\
- \mathrm{d}\theta_\beta(\hat{V}(\mathrm{d}u)) + Q_\beta(f,\cdot,\mathrm{d}u,\nabla\mathrm{d}u).&
\end{aligned} \qquad (101)$$

We show separately that each of the terms on the right side of (101) lies in $W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$. It is clear from the definition of $\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ that Δu and $\mathrm{d}\psi_\beta(\nabla u)$ both lie in $W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$. Next we show that $F_\beta(f_0,\cdot,0,0)$ lies in $W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$. We choose functions $\chi_i \in C^\infty(L)$ for $i = 1,\dots,n$, such that $\chi_i \equiv 1$ on $\phi_i(\Sigma_i \times (0, \frac{R}{2}))$ and $\chi_i \equiv 0$ on $L \backslash S_i$. It is clear that $F_\beta(f_0, \cdot, 0, 0) \in W^{0,k-2,p}((0,T) \times K')$ for every $K' \subset\subset L$. Using (96) and (99) we can write

$$F_\beta(f_0, x, 0, 0) = \theta_i + \Delta a_i(\sigma, r) + Q_i(\sigma, r, \mathrm{d}a_i(\sigma, r), \nabla \mathrm{d}a_i(\sigma, r)) \\ + R_i(f_0, \sigma, r, \mathrm{d}a_i(\sigma, r), \nabla \mathrm{d}a_i(\sigma, r))$$

for $x \in S_i$ with $x = \phi_i(\sigma, r)$, where θ_i is the Lagrangian angle of C_i. Clearly

$$\chi_i(\phi_i^{-1})^*(\theta_i) \in W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T) \times L) \tag{102}$$

for $i = 1, \ldots, n$. Since $a \in C^\infty_{\boldsymbol{\nu}}(L)$ and $\boldsymbol{\mu} < \boldsymbol{\nu}$, it follows that

$$\chi_i(\phi_i^{-1})^*(\Delta a_i) \in W^{0,k-2,p}_{\boldsymbol{\mu}-2}((0,T) \times L) \tag{103}$$

for $i = 1, \ldots, n$. Since $\boldsymbol{\mu} - 2 < 1$, it follows from Lemma 9.7 that

$$\chi_i(\phi_i^{-1})^*(R_i(f_0, \cdot, \cdot, \mathrm{d}a_i, \nabla \mathrm{d}a_i)) \in W^{0,k-2,p}_{\boldsymbol{\mu}-2}((0,T) \times L). \tag{104}$$

Finally, by Lemma 9.6

$$Q_i(\sigma, r, \mathrm{d}a_i(\sigma, r), \nabla \mathrm{d}a_i(\sigma, r)) = O\left(r^{-2}|\mathrm{d}a_i(\sigma, r)|^2 + |\nabla \mathrm{d}a_i(\sigma, r)|^2\right)$$

uniformly in $(\sigma, r) \in \Sigma_i \times (0, R)$. Since $a \in C^\infty_{\boldsymbol{\nu}}(L)$, $r^{-2}|\mathrm{d}a_i(\sigma, r)|^2 = O(r^{2\nu_i - 4})$ and $|\nabla \mathrm{d}a_i(\sigma, r)|^2 = O(r^{2\nu_i - 4})$ as $r \to 0$ for $i = 1, \ldots, n$. Since $2\boldsymbol{\nu} - 4 > \boldsymbol{\mu} - 2$, it follows that

$$\chi_i(\phi_i^{-1})^*(Q_i(\cdot, \cdot, \mathrm{d}a_i, \nabla \mathrm{d}a_i)) \in W^{0,0,p}_{\boldsymbol{\mu}-2}((0,T) \times L).$$

The spatial derivatives of Q_i up to order $k - 2$ can now be estimated in the same way as in the proof of Proposition 5.6 and we thus obtain that

$$\chi_i(\phi_i^{-1})^*(Q_i(\cdot, \cdot, \mathrm{d}a_i, \nabla \mathrm{d}a_i)) \in W^{0,k-2,p}_{\boldsymbol{\mu}-2}((0,T) \times L). \tag{105}$$

From (102)-(105) it follows that $F_\beta(f, \cdot, 0, 0) \in W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T) \times L)$ and we also conclude that $\mathrm{d}\theta_\beta(\hat{V}(\mathrm{d}u))$ lies in $W^{0,k-2,p}_{\boldsymbol{\mu}-2}((0,T) \times L)$.

It remains to show that $Q_\beta(f, \cdot, \mathrm{d}u, \nabla \mathrm{d}u) \in W^{0,k-2,p}_{\boldsymbol{\mu}-2}((0,T) \times L)$. From Lemma 9.8 we have the estimate

$$Q_\beta(f, \cdot, \mathrm{d}u, \nabla \mathrm{d}u) = O\left(\rho^{-2}|\mathrm{d}u|^2 + |\nabla \mathrm{d}u|^2 + \rho \cdot d(f, f_0)\right)$$

uniformly on L. Since $k \geq 4$, $|\nabla \mathrm{d}u| \in W^{1,k-2,p}_{\boldsymbol{\mu}-2}((0,T) \times L)$, and since $p > \frac{2}{\varepsilon}$, Proposition 7.2 implies that $|\nabla \mathrm{d}u|^2 \in C^{0,0}_{\boldsymbol{\mu}-2-\varepsilon}((0,T) \times L)$. Thus $|\nabla \mathrm{d}u|^2 \in C^{0,0}_{2\boldsymbol{\mu}-4-2\varepsilon}((0,T) \times L)$. Since $2\boldsymbol{\mu} - 4 - 2\varepsilon > \boldsymbol{\mu} - 2$, $C^{0,0}_{2\boldsymbol{\mu}-4-2\varepsilon}((0,T) \times L)$ embeds continuously into $W^{0,0,p}_{\boldsymbol{\mu}-2}((0,T) \times L)$ by inclusion and hence $|\nabla \mathrm{d}u|^2 \in W^{0,0,p}_{\boldsymbol{\mu}-2}((0,T) \times L)$. In a similar way one can show that $\rho^{-2}|\mathrm{d}u|^2 \in W^{0,0,p}_{\boldsymbol{\mu}-2}((0,T) \times L)$. For $t \in (0,T)$ we have $\rho \cdot d(f(t), f_0(t)) \in C^0_{\mathbf{1}}(L)$ and since $\boldsymbol{\mu} - 2 < 1$, $\rho \cdot d(f(t), f_0(t)) \in L^p_{\boldsymbol{\mu}-2}(L)$. Then it follows from the Mean Value Theorem [33, XIII, §4] that $\rho \cdot d(f, f_0) \in W^{0,0,p}_{\boldsymbol{\mu}-2}((0,T) \times L)$. Thus $Q_\beta(f, \cdot, \mathrm{d}u, \nabla \mathrm{d}u)$ lies in $W^{0,0,p}_{\boldsymbol{\mu}-2}((0,T) \times L)$ and using the same arguments as in the proof of Proposition 5.6 and as above one can show that in fact $Q_\beta(f, \cdot, \mathrm{d}u, \nabla \mathrm{d}u) \in W^{0,k-2,p}_{\boldsymbol{\mu}-2}((0,T) \times L)$. Hence each of the terms on the right side of (101) lies in $W^{0,k-2,p}_{\boldsymbol{\mu}-2}((0,T)\times L)$ and therefore $F_{t\beta}(f,\cdot,\mathrm{d}u,\nabla\mathrm{d}u)\in W^{0,k-2,p}_{\boldsymbol{\mu}-2}((0,T)\times L)$.

Finally we show that $S(f,\frac{\mathrm{d}f}{\mathrm{d}t},\cdot,\mathrm{d}u)\in W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$. Since $p>m$, Theorem 6.7 implies that $u(t,\cdot)\in C^{k-1}_{\boldsymbol{\mu}}(L)$ for almost every $t\in(0,T)$. Then by Proposition 9.4, $S(f(t),\frac{\mathrm{d}f}{\mathrm{d}t}(t),\cdot,\mathrm{d}[u(t,\cdot)])$ lies in $C^{k-2}_{\mathbf{1}}(L)\oplus\mathrm{span}\{q_1,\dots,q_n\}$ for almost every $t\in(0,T)$. Since $\boldsymbol{\mu}-2<1$, $C^{k-2}_{\mathbf{1}}(L)$ embeds continuously into $W^{k-2,p}_{\boldsymbol{\mu}-2}(L)$ by inclusion and hence $S(f(t),\frac{\mathrm{d}f}{\mathrm{d}t}(t),\cdot,\mathrm{d}[u(t,\cdot)])\in W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L)$ for almost every $t\in(0,T)$. In particular, since S is smooth on its domain, the Mean Value Theorem implies that $S(f,\frac{\mathrm{d}f}{\mathrm{d}t},\cdot,\mathrm{d}u)\in W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$.

Thus we finally conclude that $P(u,f)\in W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ for every $(u,f)\in\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ and therefore the map $P:\mathcal{D}^{k,p}_{\boldsymbol{\mu}}\to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ is well defined. The smoothness of $P:\mathcal{D}^{k,p}_{\boldsymbol{\mu}}\to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ now follows from smoothness of Q_β and S and the same arguments as in Proposition 5.6. $\square$

9.4 The linearization of P and structure of the equation

In this subsection we compute the linearization of the operator $P:\mathcal{D}^{k,p}_{\boldsymbol{\mu}}\to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$. But first we need to take a closer look at the operator P. From now on we also assume that the special Lagrangian cones $C_1,\dots,C_n$ are stable, and we choose $\boldsymbol{\mu}\in\mathbb{R}^n$ with $2<\boldsymbol{\mu}<\boldsymbol{\nu}$, such that $(2,\mu_i]\cap\mathcal{E}_{\Sigma_i}=\emptyset$. We also fix $k\in\mathbb{N}$ and $p\in(1,\infty)$ that satisfy the conditions of Proposition 9.9, so that $P:\mathcal{D}^{k,p}_{\boldsymbol{\mu}}\to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ is a smooth map of Banach manifolds.

Let $(u,f)\in\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$. Then it follows from (90) that $t\mapsto\Xi_{(u(t,\cdot),f(t))}$ is a section of the vector bundle $f^*(\mathrm{Hom}(T\mathcal{F},C^1_{\mathrm{loc}}(L)))$ over the manifold $(0,T)$. Define

$$V_{\mathsf{P}_{(u,f)}}(L)=\mathrm{im}\,\big\{\Xi_{(u,f)}:f^*(T\mathcal{F})\longrightarrow C^1_{\mathrm{loc}}(L)\big\}. \tag{106}$$

Then $V_{\mathsf{P}_{(u,f)}}(L)$ is a finite dimensional vector bundle over $(0,T)$ with fibre dimension $\dim\mathcal{F}$. Also note that if u is smooth, then each fibre of $V_{\mathsf{P}_{(u,f)}}(L)$ consists of smooth functions on L. From Proposition 9.4 it follows that for every $(u,f)\in\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$, $V_{\mathsf{P}_{(u,f)}}(L)$ has zero intersection with $L^p_{\boldsymbol{\mu}}(L)$ in each fibre over $(0,T)$. Hence we can define

$$W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}(L)=W^{k,p}_{\boldsymbol{\mu}}(L)\oplus V_{\mathsf{P}_{(u,f)}}(L).$$

Then $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}(L)$ is a Banach bundle over the Banach manifold $\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ with fibres being weighted Sobolev spaces with discrete asymptotics. If u and f are constant in time, so for instance at the initial condition $u=0$ and $f=f_0$, then $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}(L)$ is simply a weighted Sobolev space with discrete asymptotics.

In §6.3 we also defined the function space $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}(L)$. Observe that the discrete asymptotic parts of the Banach spaces $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}(L)$ and $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}(L)$ are isomorphic for every $t\in(0,T)$, since $(2,\mu_i]\cap\mathcal{E}_{\Sigma_i}=\emptyset$ for $i=1,\dots,n$ and $C_1,\dots,C_n$ are stable special Lagrangian cones. Note carefully, however, that the discrete asymptotics of the spaces $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}(L)$ and $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}(L)$ are in general not the same. The discrete asymptotics of the space $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}(L)$ are defined using the harmonic functions on the cones $C_1,\dots,C_n$ and are in fact functions that are harmonic on each end of L. On the other hand the discrete asymptotics of the space $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}(L)$ are defined by the map $\Xi_{(u,f)}$, which is defined by the deformations of the Lagrangian neighbourhoods $\{\Phi^f_L\}_{f\in\mathcal{F}}$ in $f\in\mathcal{F}$. The deformations of $\{\Phi^f_L\}_{f\in\mathcal{F}}$ in $f\in\mathcal{F}$ correspond to moment maps in $\mathbb{C}^m$ and these restrict to homogeneous harmonic functions on $C_1,\ldots,C_n$, and by Proposition 9.4, $\Xi_{(u,f)}$ is asymptotic to one of these moment maps. Therefore the discrete asymptotics of the space $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}(L)$ are in general only harmonic in an asymptotic sense near each conical singularity. For this reason we need to impose an extra condition on the generalized mean curvature form of the initial Lagrangian submanifold $F_0 : L \to \mathbb{R}$, when we prove short time existence in §9.5.

Following §7.1 we can now define the weighted parabolic Sobolev space $W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}((0,T)\times L)$ with discrete asymptotics by

$$W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}((0,T)\times L) = L^p((0,T);W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}(L)) \cap W^{1,p}((0,T);W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L)).$$

Consider the linearization of the operator $P : \mathcal{D}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ at some $(u,f)\in\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$, which is a linear operator

$$\mathrm{d}P(u,f) : W^{1,k,p}_{\boldsymbol{\mu}}((0,T)\times L)\oplus W^{1,p}((0,T);f^*(T\mathcal{F})) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L).$$

Using $\Xi_{(u,f)}$ to identify $f^*(T\mathcal{F})$ with $V_{\mathsf{P}_{(u,f)}}(L)$, we can understand the linearization of $P : \mathcal{D}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ at (u,f) as a linear operator

$$\mathrm{d}P(u,f) : W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}((0,T)\times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L). \tag{107}$$

Let $(u,f)\in\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$, $w\in T_f\mathcal{F}$, and denote $\eta = \mathrm{d}u+\beta$. Then $\partial_w(\Phi^f_L\circ\eta)$ is a section of the vector bundle $(\Phi^f_L\circ\eta)^*(TM)$. From Proposition 9.3 it follows that the normal part of $\partial_w(\Phi^f_L\circ\eta)$ is equal to $-J(\mathrm{d}(\Phi^f_L\circ\eta)(\nabla\Xi_{(u,f)}(w)))$. Moreover we define $\hat W(\Xi_{(u,f)}(w))\in TL$ by requiring that $-\mathrm{d}(\Phi^f_L\circ\eta)(\hat W(\Xi_{(u,f)}(w)))$ is equal to the tangential part of $\partial_w(\Phi^f_L\circ\eta)$. Thus

$$\partial_w(\Phi^f_L\circ\eta) = -J(\mathrm{d}(\Phi^f_L\circ\eta)(\nabla\Xi_{(u,f)}(w))) - \mathrm{d}(\Phi^f_L\circ\eta)(\hat W(\Xi_{(u,f)}(w))).$$

In the next proposition we compute an explicit formula for the operator (107).

Proposition 9.10. *Let $(u,f)\in\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ and $v-\Xi_{(u,f)}(w)\in W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}((0,T)\times L)$, where $v\in W^{1,k,p}_{\boldsymbol{\mu}}((0,T)\times L)$ and $w\in W^{1,p}((0,T);f^*(T\mathcal{F}))$. Then*

$$\begin{aligned}\mathrm{d}P(u,f)(v,\Xi_{(u,f)}(w)) &= \frac{\partial}{\partial t}(v-\Xi_{(u,f)}(w)) - \Delta(v-\Xi_{(u,f)}(w))\\ &+ m\mathrm{d}\psi_{\mathrm{d}u+\beta}(\nabla(v-\Xi_{(u,f)}(w))) + \mathrm{d}\theta_{\mathrm{d}u+\beta}(\hat V(\mathrm{d}v)-\hat W(\Xi_{(u,f)}(w)))\\ &- \mathrm{d}[\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t})](\hat V(\mathrm{d}v)-\hat W(\Xi_{(u,f)}(w))) + \mathrm{d}[\partial_t u](\hat W(\Xi_{(u,f)}(w)))\\ &+ \mathrm{d}[\Xi_{(u,f)}(w)](\hat V(\mathrm{d}[\partial_t u]) - \hat W(\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t})) - \mathrm{d}v(\hat W(\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t}))),\end{aligned} \tag{108}$$

Here $\psi_{\mathrm{d}u+\beta} = (\Phi^f_L\circ(\mathrm{d}u+\beta))^*(\psi)$, $\theta_{\mathrm{d}u+\beta} = \theta(\Phi^f_L\circ(\mathrm{d}u+\beta))$, and the Laplace operator and ∇ are computed using the time dependent Riemannian metric $(\Phi^f_L\circ$

$(\mathrm{d}u+\beta))^*(g)$ *on* L*. In particular at* $(0, f_0) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ *we have*

$$
\begin{aligned}
\mathrm{d}P(0,f_0)(v,\Xi_{(u,f)}(w)) &= \frac{\partial}{\partial t}(v-\Xi_{(0,f_0)}(w)) - \Delta(v-\Xi_{(0,f_0)}(w)) \\
&\quad + m\mathrm{d}\psi_\beta(\nabla(v-\Xi_{(0,f_0)}(w))) + \mathrm{d}\theta_\beta(\hat{V}(\mathrm{d}v) - \hat{W}(\Xi_{(0,f_0)}(w))).
\end{aligned} \tag{109}
$$

Proof. Denote $\eta = \mathrm{d}u+\beta$. Recall that β is supported on K, so that $\eta = \mathrm{d}u$ on each end of L. We first compute $\mathrm{d}P(u,f)(v,0)$. From Lemma 5.5 we obtain that

$$
\frac{\mathrm{d}}{\mathrm{d}s}\Theta(\Phi_L^f \circ (\eta + s\mathrm{d}v))\Big|_{s=0} = \Delta v - m\mathrm{d}\psi_\eta(\nabla v) - \mathrm{d}\theta_\eta(\hat{V}(\mathrm{d}v)).
$$

Moreover using Lemma 4.11 we find that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Xi_{(u+sv,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t})\Big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}H_f(\tfrac{\mathrm{d}f}{\mathrm{d}t}) \circ (\Phi_L^f \circ (\eta + s\mathrm{d}v))\Big|_{s=0} \\
&= \mathrm{d}[H_f(\tfrac{\mathrm{d}f}{\mathrm{d}t})](-\alpha^{-1}(\mathrm{d}v) + V(\mathrm{d}v)) \\
&= \mathrm{d}v(\hat{W}(\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t}))) + \mathrm{d}[\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t})](\hat{V}(\mathrm{d}v)).
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\mathrm{d}P(u,f)(v,0) = \frac{\partial v}{\partial t} &- \Delta v + m\mathrm{d}\psi_\eta(\nabla v) + \mathrm{d}\theta_\eta(\hat{V}(\mathrm{d}v)) \\
&- \mathrm{d}v(\hat{W}(\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t}))) - \mathrm{d}[\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t})](\hat{V}(\mathrm{d}v)).
\end{aligned}
$$

The next step is to compute $\mathrm{d}P(u,f)(0,w)$. Let $\{f_s\}_{s\in(-\varepsilon,\varepsilon)}$, $\varepsilon > 0$, be a smooth curve in $\mathcal{F}$ with $f_0 = f$ and $\frac{\mathrm{d}f_s}{\mathrm{d}s}\big|_{s=0} = w$. The normal part of the vector field $\partial_w(\Phi_L^f \circ \eta)$ along $\Phi_L^f \circ \eta : L \to M$ is $-J(\mathrm{d}(\Phi_L^f \circ \eta)(\nabla \Xi_{(u,f)}(w)))$ and the tangential part is equal to $-\mathrm{d}(\Phi_L^f \circ \eta)(\hat{W}(\Xi_{(u,f)}(w)))$. Then the same computation as in Lemma 5.5 shows that

$$
\frac{\mathrm{d}}{\mathrm{d}s}\Theta(\Phi_L^{f_s} \circ \eta)\Big|_{s=0} = -\Delta \Xi_{(u,f)}(w) + m\mathrm{d}\psi_\eta(\nabla \Xi_{(u,f)}(w)) + \mathrm{d}\theta_\eta(\hat{W}(\Xi_{(u,f)}(w))).
$$

Moreover, recalling the definition of H_f from Proposition 9.2 and of $\Xi_{(u,f)}$ from (90), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Xi_{(u,f_s)}(\tfrac{\mathrm{d}f_s}{\mathrm{d}t})\Big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}H_{f_s}(\tfrac{\mathrm{d}f_s}{\mathrm{d}t}) \circ (\Phi_L^{f_s} \circ \eta)\Big|_{s=0} \\
&= \Xi_{(u,f)}(\tfrac{\mathrm{d}w}{\mathrm{d}t}) + \partial_w H_f(\tfrac{\mathrm{d}f}{\mathrm{d}t}) \circ (\Phi_L^f \circ \eta) + \mathrm{d}[H_f(\tfrac{\mathrm{d}f}{\mathrm{d}t})](\partial_w(\Phi_L^f \circ \eta)),
\end{aligned}
$$

and in a similar way we find

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Xi_{(u,f)}(w) = \Xi_{(u,f)}(\tfrac{\mathrm{d}w}{\mathrm{d}t}) &+ \partial_{\frac{\mathrm{d}f}{\mathrm{d}t}} H_f(w) \circ (\Phi_L^f \circ \eta) + \mathrm{d}[H_f(w)](\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ \eta)) \\
&+ \mathrm{d}[\partial_t u](\hat{W}(\Xi_{(u,f)}(w))) + \mathrm{d}[\Xi_{(u,f)}(w)](\hat{V}(\mathrm{d}[\partial_t u])).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Xi_{(u,f_s)}(\tfrac{\mathrm{d}f_s}{\mathrm{d}t})\Big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\Xi_{(u,f)}(w) + \partial_w H_f(\tfrac{\mathrm{d}f}{\mathrm{d}t}) \circ (\Phi_L^f \circ \eta) \\
&\quad - \partial_{\frac{\mathrm{d}f}{\mathrm{d}t}} H_f(w) \circ (\Phi_L^f \circ \eta) + \mathrm{d}[H_f(\tfrac{\mathrm{d}f}{\mathrm{d}t})](\partial_w(\Phi_L^f \circ \eta)) \\
&\quad - \mathrm{d}[H_f(w)](\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ \eta)) - \mathrm{d}[\partial_t u](\hat{W}(\Xi_{(u,f)}(w))) \\
&\quad - \mathrm{d}[\Xi_{(u,f)}(w)](\hat{V}(\mathrm{d}[\partial_t u])).
\end{aligned}
$$

Next we compute the term $\partial_w H_f(\frac{\mathrm{d}f}{\mathrm{d}t}) - \partial_{\frac{\mathrm{d}f}{\mathrm{d}t}} H_f(w)$. Let $\{\tilde{f}_s\}_{s\in(-\varepsilon,\varepsilon)}$ be a smooth curve in $\mathcal{F}$ with $\tilde{f}_0 = f$ and $\frac{\mathrm{d}\tilde{f}_s}{\mathrm{d}s}\big|_{s=0} = \frac{\mathrm{d}f}{\mathrm{d}t}$. We extend $w \in T_f\mathcal{F}$ to a parallel vector field w_s along the curve $s \mapsto \tilde{f}_s$ in $\mathcal{F}$ and we extend $\frac{\mathrm{d}f}{\mathrm{d}t} \in T_f\mathcal{F}$ to a parallel vector field $(\frac{\mathrm{d}f}{\mathrm{d}t})_s$ along the curve $s \mapsto f_s$ in $\mathcal{F}$. Then

$$\partial_w H_f(\tfrac{\mathrm{d}f}{\mathrm{d}t}) \circ (\Phi_L^f \circ \eta) = \frac{\mathrm{d}}{\mathrm{d}s} H_{f_s}((\tfrac{\mathrm{d}f}{\mathrm{d}t})_s) \circ (\Phi_L^{f_s} \circ \eta)\Big|_{s=0} - \mathrm{d}[H_f(\tfrac{\mathrm{d}f}{\mathrm{d}t})](\partial_w(\Phi_L^f \circ \eta)),$$

$$\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}} H_f(w) \circ (\Phi_L^f \circ \eta)) = \frac{\mathrm{d}}{\mathrm{d}s} H_{\tilde{f}_s}(w_s) \circ (\Phi_L^{\tilde{f}_s} \circ \eta))\Big|_{s=0} - \mathrm{d}[H_f(w)](\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ \eta)).$$

Thus in order to find $\partial_w H_f(\frac{\mathrm{d}f}{\mathrm{d}t}) - \partial_{\frac{\mathrm{d}f}{\mathrm{d}t}} H_f(w)$ we need to compute the term $\frac{\mathrm{d}}{\mathrm{d}s} H_{f_s}(\frac{\mathrm{d}f_s}{\mathrm{d}t}) - H_{\tilde{f}_s}(w_s)\big|_{s=0}$. We have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}(\Phi_L^{f_s} \circ \eta)^*(\mathrm{d}[H_{f_s}((\tfrac{\mathrm{d}f}{\mathrm{d}t})_s)]) - (\Phi_L^{\tilde{f}_s} \circ \eta)^*(\mathrm{d}[H_{\tilde{f}_s}(w_s)])\Big|_{s=0} \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}s}(\Phi_L^{f_s} \circ \eta)^* \left(\partial_{(\frac{\mathrm{d}f}{\mathrm{d}t})_s}(\Phi_L^{f_s} \circ \eta) \lrcorner\, \omega\right) - (\Phi_L^{\tilde{f}_s} \circ \eta)^* \left(\partial_{w_s}(\Phi_L^{\tilde{f}_s} \circ \eta) \lrcorner\, \omega\right)\Big|_{s=0} \\
&\quad = (\Phi_L^f \circ \eta)^* \left(\mathcal{L}_{\partial_w(\Phi_L^f \circ \eta)}\left(\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ \eta) \lrcorner\, \omega\right) - \mathcal{L}_{\frac{\mathrm{d}f}{\mathrm{d}t}}\left(\partial_{\partial_w(\Phi_L^f \circ \eta)}(\Phi_L^f \circ \eta) \lrcorner\, \omega\right)\right) \\
&\qquad + (\Phi_L^f \circ \eta)^* \left(\frac{\mathrm{d}}{\mathrm{d}s}\partial_{(\frac{\mathrm{d}f}{\mathrm{d}t})_s}(\Phi_L^{f_s} \circ \eta) \lrcorner\, \omega - \partial_{w_s}(\Phi_L^{\tilde{f}_s} \circ \eta) \lrcorner\, \omega\Big|_{s=0}\right) \\
&\quad = (\Phi_L^f \circ \eta)^*(2 \cdot \mathrm{d}[\omega(\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ \eta), \partial_w(\Phi_L^f \circ \eta))] + [\partial_w(\Phi_L^f \circ \eta), \partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ \eta)] \lrcorner\, \omega) \\
&\quad = (\Phi_L^f \circ \eta)^*(\mathrm{d}[\omega(\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ \eta), \partial_w(\Phi_L^f \circ \eta))]).
\end{aligned}$$

Here $[\cdot,\cdot]$ denotes the Lie bracket, and in the last step we use that for two Hamiltonian vector fields X, Y, $[X,Y] \lrcorner\, \omega = \mathrm{d}[\omega(X,Y)]$ holds. Decomposing $\partial_w(\Phi_L^f \circ \eta)$ and $\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ \eta)$ into their tangential and normal parts and using that $\Phi_L^f \circ \eta : L \to M$ is Lagrangian we obtain

$$\begin{aligned}
&(\Phi_L^f \circ \eta)^*(\omega(\partial_{\frac{\mathrm{d}f}{\mathrm{d}t}}(\Phi_L^f \circ \eta), \partial_w(\Phi_L^f \circ \eta))) \\
&\qquad = \mathrm{d}[\Xi_{(u,f)}(w)](\hat{W}(\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t}))) - \mathrm{d}[\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t})](\hat{W}(\Xi_{(u,f)}(w))).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Xi_{(u,f_s)}(\tfrac{\mathrm{d}f_s}{\mathrm{d}t})\Big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\Xi_{(u,f)}(w) - \mathrm{d}[\partial_t u](\hat{W}(\Xi_{(u,f)}(w))) \\
&\quad - \mathrm{d}[\Xi_{(u,f)}(w)](\hat{V}(\mathrm{d}[\partial_t u])) + \mathrm{d}[\Xi_{(u,f)}(w)](\hat{W}(\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t}))) \\
&\quad - \mathrm{d}[\Xi_{(u,f)}(\tfrac{\mathrm{d}f}{\mathrm{d}t})](\hat{W}(\Xi_{(u,f)}(w))),
\end{aligned}$$

and the proposition follows. □

9.5 Short time existence with low regularity

In this subsection we show that there exists $(u, f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$, which solves the Cauchy problem (91). The regularity of u and f are then improved in §9.6.

If we make an extra assumption of the generalized mean curvature form of the initial Lagrangian submanifold $F_0 : L \to M$, then we can prove the following important lemma. In §10.2 we will discuss whether it is possible to drop this extra condition on the generalized mean curvature form.

Lemma 9.11. *Assume that the special Lagrangian cones $C_1,\dots,C_n$ are stable in the sense of Definition 8.3, and choose $\boldsymbol{\mu}\in\mathbb{R}^n$ with $2<\boldsymbol{\mu}<\boldsymbol{\nu}$ small enough such that $(2,\mu_i]\cap\mathcal{E}_{\Sigma_i}=\emptyset$ for $i=1,\dots,n$. Moreover, assume that there exists $\boldsymbol{\varepsilon}\in\mathbb{R}^n$ with $\boldsymbol{\varepsilon}>0$, such that $|\alpha_K|=O(\rho^{\boldsymbol{\varepsilon}})$ as $\rho\to 0$, where $\alpha_K=-\mathrm{d}[\theta(F_0)]$ is the generalized mean curvature form of the initial Lagrangian submanifold $F_0:L\to M$. Then $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}(L)=W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}(L)$.*

Proof. By Proposition 9.10 the linearization of $P:\mathcal{D}^{k,p}_{\boldsymbol{\mu}}\to W^{0,k-2,p}_{\boldsymbol{\mu},\mathsf{Q}}((0,T)\times L)$ at the initial condition $(0,f_0)\in\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ is a map

$$\mathrm{d}P(0,f_0):W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T)\times L)\longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L) \tag{110}$$

and is given by

$$\begin{aligned}\mathrm{d}P(0,f_0)(v,\Xi_{(0,f_0)}(w))&=\frac{\partial}{\partial t}(v-\Xi_{(0,f_0)}(w))-\Delta(v-\Xi_{(0,f_0)}(w))\\&\quad+m\mathrm{d}\psi_\beta(\nabla(v-\Xi_{(0,f_0)}(w)))+\mathrm{d}\theta_\beta(\hat V(\mathrm{d}v)-\hat W(\Xi_{(0,f_0)}(w))),\end{aligned}\tag{111}$$

where the Laplace operator is computed using the time dependent Riemannian metric $(\Phi^{f_0}_L\circ\beta)^*(g)$, and ψ_β and θ_β are defined by $\psi_\beta=(\Phi^{f_0}_L\circ\beta)^*(\psi)$ and $\theta_\beta=\theta(\Phi^{f_0}_L\circ\beta)$, respectively. Recall that $\{\beta(t)\}_{t\in(0,T)}$ is given by $\beta(t)=t\beta_0$, and β_0 is supported on K. In particular note that the Riemannian metric $(\Phi^{f_0}_L\circ\beta)^*(g)$ is constant on each end of L. Also notice carefully that even after identifying $T_{f_0}\mathcal{F}$ with a space of discrete asymptotics on L, the operator $\mathrm{d}P(0,f_0)$ is not a differential operator on the sum $v-\Xi_{(0,f_0)}(w)$.

Define a linear operator A on $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}(L)$ by

$$\begin{aligned}A(v,\Xi_{(0,f_0)}(w))&=\Delta(v-\Xi_{(0,f_0)}(w))\\&\quad-m\mathrm{d}\psi_\beta(\nabla(v-\Xi_{(0,f_0)}(w)))-\mathrm{d}\theta_\beta(\hat V(\mathrm{d}v)-\hat W(\Xi_{(0,f_0)}(w)))\end{aligned}$$

so that we can write $\mathrm{d}P(0,f_0)=\partial_t-A$. Clearly ∂_t is a bounded linear operator $W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T)\times L)\to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$, and therefore A is also a bounded linear operator between these spaces. In particular it follows that for each fixed $t\in(0,T)$, A is a bounded linear operator $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}(L)\to W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L)$. Define linear operators K_1 and K_2 on $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}(L)$ by

$$K_1(v,\Xi_{(0,f_0)}(w))=-m\mathrm{d}\psi_0(\nabla(v-\Xi_{(0,f_0)}(w))) \tag{112}$$

and

$$K_2(v,\Xi_{(0,f_0)}(w))=-\mathrm{d}\theta_0(\hat V(\mathrm{d}v)-\hat W(w)) \tag{113}$$

for $v\in W^{1,k,p}_{\boldsymbol{\mu}}((0,T)\times L)$ and $w\in W^{1,p}((0,T);T_{f_0}\mathcal{F})$. We show that both, K_1 and K_2 define bounded linear operators

$$K_1,K_2:W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}(L)\longrightarrow W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L). \tag{114}$$

We begin with K_1. Since ψ is a smooth function on M, $\mathrm{d}\psi_\beta$ is a smooth and bounded one-form on L. It follows that $K_1:W^{k,p}_{\boldsymbol{\mu}}(L)\longrightarrow W^{k-1,p}_{\boldsymbol{\mu}-1}(L)$ is a well defined, bounded linear operator, so in particular $K_1:W^{k,p}_{\boldsymbol{\mu}}(L)\to$

$W^{k-2,p}_{\boldsymbol{\mu}-2}(L)$ is a well defined bounded linear operator. It remains to study the action of K_1 on the discrete asymptotics part. By Proposition 9.4 we can write $\Xi_{(0,f_0)}(w) = (\iota_i \circ \phi_i^{-1})^*(\mu_{X_i}) + O(\rho^{\nu_i - 1})$ on each end of L, for some moment map μ_{X_i}. Using Taylor's Theorem to expand μ_{X_i}, it is then straightforward to show that $K_1 : V_{\mathsf{P}_{(0,f_0)}}(L) \to W^{k-2,p}_{\boldsymbol{\nu}-2,\mathsf{Q}}(L)$ is a well defined, bounded linear operator. In particular, since $\boldsymbol{\nu} > \boldsymbol{\mu}$, $K_1 : V_{\mathsf{P}_{(0,f_0)}}(L) \to W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L)$ is a well defined, bounded linear operator.

Next we show that K_2 in (114) is a well defined, bounded operator. We now use the extra assumption on the generalized mean curvature form of the initial Lagrangian submanifold $F_0 : L \to M$. Thus assume that $|\alpha_K| = O(\rho^{\boldsymbol{\varepsilon}})$ for some $\boldsymbol{\varepsilon} \in \mathbb{R}^n$ with $\boldsymbol{\varepsilon} > 0$. Using $|\alpha_K| = O(\rho^{\boldsymbol{\varepsilon}})$ it follows that $K_2 : W^{k,p}_{\boldsymbol{\mu}}(L) \to W^{k-2,p}_{\boldsymbol{\mu}-1+\boldsymbol{\varepsilon}}(L)$ is a well defined, bounded linear operator and hence $K_2 : W^{k,p}_{\boldsymbol{\mu}}(L) \to W^{k-2,p}_{\boldsymbol{\mu}-2}(L)$ is a well defined, bounded linear operator. Note that here it would be sufficient to assume that $\boldsymbol{\varepsilon} > \boldsymbol{\mu} - 2$ (which holds anyway, even without the extra assumption on the generalized mean curvature form). It remains to study the action of K_2 on the discrete asymptotics part, and this is where we have to use $\boldsymbol{\varepsilon} > 0$. The vector field $\hat{W}(\Xi_{(0,f_0)}(w))$ on L is defined by restricting a smooth vector field on M to the submanifold $F_0 : L \to M$. In particular $\hat{W}(w)$ is a smooth and bounded vector field in L. The same arguments as before and the extra assumption on the generalized mean curvature form then imply that $K_2 : V_{\mathsf{P}_{(0,f_0)}}(L) \to W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L)$ is a well defined, bounded linear operator.

Now we have proved that both K_1 and K_2 are bounded linear operators $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}(L) \to W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L)$. Since A is also a bounded linear operator between these spaces it follows that the Laplace operator $\Delta : W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}(L) \to W^{k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}(L)$ is a bounded linear operator. Since $(2,\mu_i] \cap \mathcal{E}_{\Sigma_i} = \emptyset$ for $i = 1, \dots, n$ and the special Lagrangian cones $C_1, \dots, C_n$ are stable the discrete asymptotics parts of $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}(L)$ and $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}(L)$ are isomorphic, and it follows from Proposition 6.15 that $W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}(L) = W^{k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}(L)$. □

The rest of the short time existence proof now follows the same strategy as in §5.4. We define

$$\tilde{\mathcal{D}}^{k,p}_{\boldsymbol{\mu}} = \big\{(u,f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}} \ : \ u(0,\cdot) = 0 \text{ on } L,\ f(0) = f_0\big\}.$$

Recall that if $(u,f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$, then u and f extend continuously to $t = 0$, since they are uniformly Hölder continuous on $(0,T)$. Moreover observe that $(u,f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ is a solution of the Cauchy problem (91) if and only if $(u,f) \in \tilde{\mathcal{D}}^{k,p}_{\boldsymbol{\mu}}$ and $P(u,f) = 0$. We define

$$\tilde{W}^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}((0,T) \times L) = \big\{v \in W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(u,f)}}((0,T) \times L) \ : \ v(0,\cdot) = 0 \text{ on } L\big\}.$$

In the next proposition we show that the linearization of the nonlinear operator $P : \tilde{\mathcal{D}}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T) \times L)$ at the initial condition $(0, f_0)$ is an isomorphism.

Proposition 9.12. *Under the assumptions of Lemma 9.11 the linear operator*

$$\mathrm{d}P(0,f_0) : \tilde{W}^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T) \times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T) \times L) \qquad (115)$$

is an isomorphism of Banach spaces for $T > 0$ sufficiently small.

Proof. By Proposition 9.10 the linearization of $P : \tilde{\mathcal{D}}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu},\mathsf{Q}}((0,T)\times L)$ at the initial condition $(0, f_0) \in \tilde{\mathcal{D}}^{k,p}_{\boldsymbol{\mu}}$ is given by

$$\begin{aligned} \mathrm{d}P(0,f_0)(v,\Xi_{(0,f_0)}(w)) &= \frac{\partial}{\partial t}(v - \Xi_{(0,f_0)}(w)) - \Delta(v - \Xi_{(0,f_0)}(w)) \\ &\quad + m\mathrm{d}\psi_\beta(\nabla(v - \Xi_{(0,f_0)}(w))) + \mathrm{d}\theta_\beta(\hat{V}(\mathrm{d}v) - \hat{W}(\Xi_{(0,f_0)}(w))), \end{aligned} \tag{116}$$

where again the Laplace operator is computed using the time dependent Riemannian metric $(\Phi^{f_0}_L \circ \beta)^*(g)$, and ψ_β and θ_β are given by $\psi_\beta = (\Phi^{f_0}_L \circ \beta)^*(\psi)$ and $\theta_\beta = \theta(\Phi^{f_0}_L \circ \beta)$, respectively.

We define two linear operators K_1 and K_2 as in (112) and (113). As proved in Lemma 9.11, K_1 and K_2 then define bounded linear operators

$$K_1, K_2 : W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T)\times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L). \tag{117}$$

Write $\mathrm{d}P(0,f_0) = \partial_t - \Delta + K_1 + K_2$. From Lemma 9.11 it follows that $W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T)\times L) = W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}((0,T)\times L)$. Furthermore we clearly have that $W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L) = W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{P}_{\boldsymbol{\mu}-2}}((0,T)\times L)$ and hence

$$\mathrm{d}P(0,f_0) : W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}((0,T)\times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{P}_{\boldsymbol{\mu}-2}}((0,T)\times L)$$

is a well defined, bounded linear operator. Since

$$K_1, K_2 : W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}((0,T)\times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{P}_{\boldsymbol{\mu}-2}}((0,T)\times L)$$

are well defined, bounded linear operators, it follows in particular that

$$\frac{\partial}{\partial t} - \Delta : W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}((0,T)\times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{P}_{\boldsymbol{\mu}-2}}((0,T)\times L) \tag{118}$$

is a well defined, bounded linear operator.

Next we show that (118) is an isomorphism. Define a Riemannian metric g_0 on L by $g_0 = (\Phi^{f_0}_L \circ \beta_0)^*(g)$ and consider the linear operator

$$\frac{\partial}{\partial t} - \Delta_0 : \tilde{W}^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}((0,T)\times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{P}_{\boldsymbol{\mu}-2}}((0,T)\times L), \tag{119}$$

where the Laplace operator is now defined using the Riemannian metric g_0. By Theorem 7.13 the linear operator (119) is an isomorphism of Banach spaces. Since (118) and (119) map between the same spaces, the same argument as in the proof of Proposition 5.7 implies that for $T > 0$ sufficiently small, the operator (118) is also an isomorphism.

Now we are almost done. In fact, since (118) is an isomorphism the same arguments as in the proof of Proposition 5.7 then imply that for $T > 0$ the operator (115) is an isomorphism. In fact, by choosing $T > 0$ small we can make the operators $\mathrm{d}P(0,f_0)$ and $\partial_t - \Delta$ to be arbitrary close in the operator norm $\tilde{W}^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}((0,T)\times L) \to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{P}_{\boldsymbol{\mu}-2}}((0,T)\times L)$. Since $\partial_t - \Delta$ is an isomorphism between these spaces, it follows that for $T > 0$ sufficiently small the operator (115) is an isomorphism. □

We can now prove short time existence of solutions with low regularity to the Cauchy problem (91) as in §5.4.

Proposition 9.13. *Let $\boldsymbol{\mu} \in \mathbb{R}^n$ with $2 < \boldsymbol{\mu} < 3$ and $(2, \mu_i] \cap \mathcal{E}_{\Sigma_i} = \emptyset$ for $i = 1, \ldots, n$. Then under the assumptions of Lemma 9.11 there exists $\tau > 0$ and $(u, f) \in \tilde{\mathcal{D}}^{k,p}_{\boldsymbol{\mu}}$, such that $P(u, f) = 0$ on the time interval $(0, \tau)$.*

Proof. By Proposition 9.12,

$$\mathrm{d}P(0, f_0) : \tilde{W}^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T) \times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T) \times L) \tag{120}$$

is an isomorphism of Banach spaces. Since $P : \tilde{\mathcal{D}}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T) \times M)$ is smooth by Proposition 9.9, the Inverse Function Theorem for Banach manifolds [33, XIV, Thm. 1.2] shows that there exist open neighbourhoods $V \subset \tilde{\mathcal{D}}^{k,p}_{\boldsymbol{\mu}}$ of $(0, f_0)$ and $W \subset W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T) \times L)$ of $P(0, f_0)$, such that $P : V \to W$ is a smooth diffeomorphism. For $\tau \in (0,T)$ we define a function w_τ on $(0,T) \times L$ by

$$w_\tau(t,x) = \begin{cases} 0 & \text{for } t < \tau \text{ and } x \in L, \\ P(0, f_0)(t,x) & \text{for } t \geq \tau \text{ and } x \in L. \end{cases}$$

Then $w_\tau \in W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T) \times L)$ for every $\tau \in (0,T)$. In particular we can make $w_\tau - P(0, f_0)$ arbitrary small in $W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T) \times L)$ by making $\tau > 0$ small. Thus for $\tau > 0$ sufficiently small we have $w_\tau \in W$ and there exists $(u, f) \in V$ with $P(u, f) = w_\tau$. But then $P(u, f) = 0$ on $(0, \tau)$ as we wanted to show. □

9.6 Regularity of solutions and short time existence of the flow

In this subsection we study the regularity of solutions to $P(u, f) = 0$. We define the functions $F_\beta, Q_\beta, F_i, Q_i$, and R_i for $i = 1, \ldots, n$ as in §9.3. We fix $\boldsymbol{\mu} \in \mathbb{R}^n$ with $2 < \boldsymbol{\mu} < \boldsymbol{\nu}$ and $(2, \mu_i] \cap \mathcal{E}_{\Sigma_i} = \emptyset$ for $i = 1, \ldots, n$, and we fix $k \in \mathbb{N}$ and $p \in (1, \infty)$ as in Proposition 9.9.

We begin with the study of the interior regularity of solutions to $P(u, f) = 0$.

Lemma 9.14. *Let $(u, f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ be a solution of $P(u, f) = 0$. Then $u(t, \cdot) \in C^\infty(L)$ for every $t \in (0,T)$.*

Proof. Let $(u, f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ be a solution of $P(u, f) = 0$. Then

$$0 = \frac{\partial u}{\partial t} - F_\beta(f, \cdot, \mathrm{d}u, \nabla \mathrm{d}u) - S(f, \tfrac{\mathrm{d}f}{\mathrm{d}t}, \cdot, \mathrm{d}u).$$

Since F_β and S are smooth functions on their domains, and S depends on the first derivative of u only, the same method as in the proof of Proposition 5.9 implies that $u(t, \cdot)$ is smooth on L for every $t \in (0,T)$. Note at this point however that we cannot conclude that u is smooth in the time direction, since S depends on $\frac{\mathrm{d}f}{\mathrm{d}t}$. □

Now we study the regularity of solutions to the Cauchy problem (91) on the ends of L. We define functions

$$S_i(f, v, \sigma, r, \mathrm{d}u_i(\sigma, r)) = S(f, v, \phi_i(\sigma, r), (\phi_i)_*(\mathrm{d}u_i(\sigma, r))),$$

where

$$S_i : \big\{(f, v, \sigma, r, y_i) \ : \ f \in \mathcal{F}, \ v \in T_f\mathcal{F}, \\ (\sigma, r) \in \Sigma_i \times (0, R), \ (\phi_i)_*(y_i) \in U'_L\big\} \longrightarrow \mathbb{R}$$

for $i = 1, \dots, n$. First we improve the regularity of the derivatives of the solutions $(u, f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ to $P(u, f) = 0$ on each end of L.

Lemma 9.15. *Let $(u, f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ be a solution of $P(u, f) = 0$ and let $a \in C^\infty_{\boldsymbol{\nu}}(L)$ with $\phi_i^*(a) = a_i$ for $i = 1, \dots, n$. Assume that $u + a \in W^{1,2,p}_{\boldsymbol{\gamma}}((0, T) \times L)$ for some $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\boldsymbol{\gamma} > 2$. Then $u + a \in W^{1,l,p}_{\boldsymbol{\gamma}}((0, T) \times L))$ for every $l \in \mathbb{N}$.*

Proof. By Lemma 9.14, $u \in W^{1,l,p}((0, T) \times K)$ for every compact $K \subset L$ and hence $u + a \in W^{1,l,p}((0, T) \times K)$ for every $K \subset\subset L$. Thus we only have to improve the regularity of $u + a$ on each end of L. Denote $u_i = \phi_i^*(u)$ for $i = 1, \dots, n$ and define $v_i = u_i + a_i$ and $w_i = \partial_r v_i$ for $i = 1, \dots, n$. Since $P(u, f) = 0$ and $\partial_t a_i = 0$ for $i = 1, \dots, n$, it follows that v_i satisfies

$$\begin{aligned}\frac{\partial v_i}{\partial t}(\sigma, r) = \theta_i + \Delta v_i(\sigma, r) &+ Q_i(\sigma, r, \mathrm{d}v_i(\sigma, r), \nabla \mathrm{d}v_i(\sigma, r)) \\ &+ R_i(f, \sigma, r, \mathrm{d}v_i(\sigma, r), \nabla \mathrm{d}v_i(\sigma, r)) + S_i(f, \tfrac{\mathrm{d}f}{\mathrm{d}t}, \sigma, r, \mathrm{d}v_i(\sigma, r))\end{aligned} \tag{121}$$

for $(\sigma, r) \in \Sigma_i \times (0, R)$ and $i = 1, \dots, n$. Let $\kappa > 0$ be sufficiently small. Then for $s \in (0, \kappa)$ and $i = 1, \dots, n$ we define

$$\delta_i^s : (\tfrac{1}{2}, 1) \times \Sigma_i \times (\tfrac{1}{2}, 1) \to (0, T) \times \Sigma_i \times (0, R), \quad \delta_i^s(t, \sigma, r) = (s^2 t, \sigma, sr).$$

For $s \in (0, \kappa)$ and $i = 1, \dots, n$ we define functions

$$v_i^s : (\tfrac{1}{2}, 1) \times \Sigma_i \times (\tfrac{1}{2}, 1), 1) \to \mathbb{R}, \quad v_i^s = s^{-\gamma_i}(\delta_i^s)^*(v_i)$$

for $i = 1, \dots, n$, and further we define

$$w_i^s : (\tfrac{1}{2}, 1) \times \Sigma_i \times (\tfrac{1}{2}, 1), 1) \to \mathbb{R}, \quad w_i^s = s^{1-\gamma_i}(\delta_i^s)^*(w_i)$$

for $i = 1, \dots, n$. Then there exist constants $c_i > 0$ for $i = 1, \dots, n$ independent of $s \in (0, \kappa)$, such that $\|v_i^s\|_{W^{1,2,p}} \le c_i$ and $\|w_i^s\|_{W^{0,1,p}} \le c_i$ on $(\frac{1}{2}, 1) \times \Sigma_i \times (\frac{1}{2}, 1)$ for $s \in (0, \kappa)$ and $i = 1, \dots, n$.

Differentiating (121) on both sides with respect to r shows that w_i^s satisfies

$$\frac{\partial w_i^s}{\partial t}(t, \sigma, r) = (L + K_i^s) w_i^s(t, \sigma, r) + f_i^s(t, \sigma, r) \tag{122}$$

for $(t, \sigma, r) \in (\frac{1}{2}, 1) \times \Sigma_i \times (\frac{1}{2}, 1)$ and $i = 1, \dots, n$. Here L is a second order differential operator that is given by

$$L w_i^s(\sigma, r) = \Delta w_i^s(\sigma, r) - (m-2) r^{-2} w_i^s(\sigma, r)$$

for $(\sigma, r) \in \Sigma_i \times (0, R)$ and $i = 1, \dots, n$, $f_i^s : (\frac{1}{2}, 1) \times \Sigma_i \times (\frac{1}{2}, 1) \to \mathbb{R}$ is defined by

$$\begin{aligned}f_i^s(t, \sigma, r) = &- 2r^{-3}(\Delta_{h_i} v_i^s)(t, \sigma, r) + s^{3-\gamma_i}(\partial_r S_i)(t, \sigma, sr, s^{\gamma_i}(\delta_i^s)_*(\mathrm{d}v_i^s)) \\ &+ s^{3-\gamma_i}(\partial_r Q_i)(\sigma, sr, s^{\gamma_i}(\delta_i^s)_*(\mathrm{d}v_i^s), s^{\gamma_i}(\delta_i^s)_*(\nabla \mathrm{d}v_i^s)) \\ &+ s^{3-\gamma_i}(\partial_r R_i)(f, \sigma, sr, s^{\gamma_i}(\delta_i^s)_*(\mathrm{d}v_i^s), s^{\gamma_i}(\delta_i^s)_*(\nabla \mathrm{d}v_i^s)) \\ &+ s^{3-\gamma_i}(\partial_r S_i)(f, \tfrac{\mathrm{d}f}{\mathrm{d}t}, \sigma, sr, s^{\gamma_i}(\delta_i^s)_*(\mathrm{d}v_i^s)),\end{aligned}$$

for $i = 1, \dots, n$, and K_i^s is a linear second order differential operator defined by

$$\begin{aligned}
K_i^s w_i^s(\sigma, r) = &s^2 (\partial_z Q_i)(\sigma, sr, s^{\gamma_i}(\delta_i^s)_*(\mathrm{d}v_i^s), s^{\gamma_i}(\delta_i^s)_*(\nabla \mathrm{d}v_i^s)) \cdot (\delta_i^s)_*(\nabla \mathrm{d}w_i^s) \\
&+ s^2 (\partial_z R_i)(f, \sigma, sr, s^{\gamma_i}(\delta_i^s)_*(\mathrm{d}v_i^s), s^{\gamma_i}(\delta_i^s)_*(\nabla \mathrm{d}v_i^s)) \cdot (\delta_i^s)_*(\nabla \mathrm{d}w_i^s) \\
&+ s^2 (\partial_y Q_i)(\sigma, sr, s^{\gamma_i}(\delta_i^s)_*(\mathrm{d}v_i^s), s^{\gamma_i}(\delta_i^s)_*(\nabla \mathrm{d}v_i)) \cdot (\delta_i^s)_*(\mathrm{d}w_i^s) \\
&+ s^2 (\partial_y R_i)(f, \sigma, sr, s^{\gamma_i}(\delta_i^s)_*(\mathrm{d}v_i^s), s^{\gamma_i}(\delta_i^s)_*(\nabla \mathrm{d}v_i^s)) \cdot (\delta_i^s)_*(\mathrm{d}w_i^s) \\
&+ s^2 (\partial_y S_i)(f, \tfrac{\mathrm{d}f}{\mathrm{d}t}, \sigma, sr, s^{\gamma_i}(\delta_i^s)_*(\mathrm{d}v_i^s)) \cdot (\delta_i^s)_*(\mathrm{d}w_i^s)
\end{aligned}$$

for $i = 1, \dots, n$. By Lemma 9.6 and 9.7, $\|f_i^s\|_{W^{0,0,p}}$ is uniformly bounded on compact subsets of $(\frac{1}{2}, 1) \times \Sigma_i \times (\frac{1}{2}, 1)$ as $s \to 0$ for $i = 1, \dots, n$. Moreover, using Lemmas 9.6 and 9.7 again we see that the coefficients of the differential operator K_i^s and their derivatives converge uniformly to zero on compact subsets of $(\frac{1}{2}, 1) \times \Sigma_i \times (\frac{1}{2}, 1)$ as $s \to 0$ for $i = 1, \dots, n$. Thus the L^p-estimates from Theorem 2.10 show that there exist constants $c_i' > 0$ for $i = 1, \dots, n$ independent of $s \in (0, \kappa)$, such that $\|w_i^s\|_{W^{0,2,p}} \leq c_i'$ on $(\frac{2}{3}, \frac{3}{4}) \times \Sigma_i \times (\frac{2}{3}, \frac{3}{4})$ for $i = 1, \dots, n$, and so $\|v_i^s\|_{W^{0,3,p}} \leq c_i'$ on $(\frac{2}{3}, \frac{3}{4}) \times \Sigma_i \times (\frac{2}{3}, \frac{3}{4})$ for $i = 1, \dots, n$. Hence

$$u + a \in W_{\boldsymbol{\gamma}}^{0,3,p}((0,T) \times L) \cap W_{\boldsymbol{\gamma}}^{1,2,p}((0,T) \times L).$$

Iterating this argument then shows that $u + a \in W_{\boldsymbol{\gamma}}^{1,l,p}((0,T) \times L))$ for every $l \in \mathbb{N}$, as we wanted to show. $\square$

Finally we show that the rate of decay of the function $u + a$ improves for positive time. Loosely speaking $\Phi_L^{f(t)} \circ (\mathrm{d}[u(t,\cdot)] + \beta(t))$ can be written near the conical singularities as the graph of $\mathrm{d}[u(t,\cdot)] + \mathrm{d}a$, and therefore we expect that the rate of decay of $u + a$, and not of u, improves for positive time. Another way to state this is to say that the special Lagrangian cones are attractors for the generalized Lagrangian mean curvature flow with conical singularities.

Lemma 9.16. *Let $(u, f) \in \tilde{\mathcal{D}}_{\boldsymbol{\mu}}^{k,p}$ be a solution of the Cauchy problem* (91) *and let $a \in C_{\boldsymbol{\nu}}^\infty(L)$ with $\phi_i^*(a) = a_i$ for $i = 1, \dots, n$. Then $u + a \in W_{\boldsymbol{\gamma}}^{1,2,p}(I \times L)$ for every $I \subset\subset (0,T)$ and for every $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $2 < \boldsymbol{\gamma} < 3$ and $(2, \gamma_i] \cap \mathcal{E}_{\Sigma_i} = \emptyset$ for $i = 1, \dots, n$.*

Proof. Denote $v = u + a$. Since $\boldsymbol{\mu} < \boldsymbol{\nu}$, we have $u + a \in W_{\boldsymbol{\mu}}^{1,k,p}((0,T) \times L)$. Denote $u_i = \phi_i^*(u)$ and define $v_i = u_i + a_i$ for $i = 1, \dots, n$. Then v_i satisfies

$$\begin{aligned}
\frac{\partial v_i}{\partial t}(t, \sigma, r) &= \Delta v_i(t, \sigma, r) + h_i(t, \sigma, r) && \text{for } (t, \sigma, r) \in (0,T) \times \Sigma_i \times (0, R), \\
v_i(0, \sigma, r) &= a_i(\sigma, r) && \text{for } (\sigma, r) \in \Sigma_i \times (0, R)
\end{aligned} \tag{123}$$

and $i = 1, \dots, n$, where $h_i : (0,T) \times \Sigma_i \times (0,R) \to \mathbb{R}$ is given by

$$\begin{aligned}
h_i(t, \sigma, r) = &\theta_i + Q_i(\sigma, r, \mathrm{d}v_i(\sigma, r), \nabla \mathrm{d}v_i(\sigma, r))(t) \\
&+ S_i(f, \tfrac{\mathrm{d}f}{\mathrm{d}t}, \sigma, r, \mathrm{d}v_i(\sigma, r))(t) + R_i(f, \sigma, r, \mathrm{d}v_i(\sigma, r), \nabla \mathrm{d}v_i(\sigma, r))(t)
\end{aligned}$$

for $i = 1, \dots, n$ and S_i as defined above.

Now choose some Riemannian metric $\tilde{g}$ on L with $\phi_i^*(g_i) = \tilde{g}$ for $i = 1, \dots, n$. Let $h \in W^{0,k-2,p}((0,T) \times L)$ with $\phi_i^*(h) = h_i$ for $i = 1, \dots, n$. Then it follows from Lemma 9.6 and 9.7 that $h \in W^{0,k-2,p}_{2\boldsymbol{\mu}-4}((0,T)\times L)$. Moreover, since v_i satisfies (123) for $i = 1,\ldots,n$, we find that

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= \Delta_{\tilde g} v(t,x) + h(t,x) + r(t,x) \quad &&\text{for } (t,x) \in (0,T)\times L, \\ v(0,x) &= a(x) &&\text{for } x \in L, \end{aligned} \tag{124}$$

where $r \in W^{0,k-2,p}((0,T)\times L)$ is supported on $(0,T)\times (L\backslash\bigcup_{i=1}^n S_i)$. Let $\tilde H$ be the Friedrichs heat kernel on $(L,\tilde g)$. Then by uniqueness of solutions to the heat equation we find that v is given by

$$v(t,x) = (\tilde H * (h+r))(t,x) + (\exp(t\Delta_{\tilde g})a)(x) \quad \text{for } (t,x) \in (0,T)\times L. \tag{125}$$

Since $h \in W^{0,k-2,p}_{2\boldsymbol{\mu}-4}((0,T)\times L)$ it follows as in Theorem 7.10 that $\tilde H * (h+r)$ lies in $W^{1,k,p}_{2\boldsymbol{\mu}-2,\mathsf{P}_{2\boldsymbol{\mu}-2}}((0,T)\times L)$. Moreover from Proposition 7.5 it follows that the second term on the right side of (125) lies in $C^\infty_{\mathsf{P}_{\boldsymbol{\delta}}}(L)$ for every $t \in (0,T)$ and $\boldsymbol{\delta} \in \mathbb{R}$. Hence

$$v \in W^{1,k,p}_{\boldsymbol{\mu}}(I\times L) \cap W^{1,k,p}_{2\boldsymbol{\mu}-2,\mathsf{P}_{2\boldsymbol{\mu}-2}}(I\times L)$$

for every $I \subset\subset (0,T)$. In particular, if $(2, 2\mu_i - 2] \cap \mathcal{E}_{\Sigma_i} = \emptyset$ for $i = 1,\ldots,n$, then it follows that $v \in W^{1,k,p}_{2\boldsymbol{\mu}-2}(I\times L)$ for every $I \subset\subset (0,T)$. Iterating this procedure we find that $v \in W^{1,k,p}_{\boldsymbol{\gamma}}(I\times L)$ for every $I \subset\subset (0,T)$ and every $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $2 < \gamma < 3$ and $(2,\gamma_i] \cap \mathcal{E}_{\Sigma_i} = \emptyset$ for $i = 1,\ldots,n$ and the lemma follows. □

The next proposition summarizes the previous three regularity results and shows that the generalized Lagrangian mean curvature flow we obtain is smooth in time.

Proposition 9.17. *Let $(u,f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ be a solution of the Cauchy problem (91). Then f defines $W^{1,p}$-one-parameter families $\{x_i(t)\}_{t\in(0,T)}$ of points in M for $i = 1,\ldots,n$ and of isomorphisms $\{A_i(t)\}_{t\in(0,T)}$ for $i = 1,\ldots,n$ with $A_i(t) \in \mathcal{A}_{x_i(t)}$ for $i = 1,\ldots,n$. Finally define a one-parameter family $\{F(t,\cdot)\}_{t\in(0,T)}$ of Lagrangian submanifolds as in (92). Then $\{F(t,\cdot)\}_{t\in(0,T)}$ is a $W^{1,p}$-one-parameter family of smooth Lagrangian submanifolds with isolated conical singularities modelled on $C_1,\ldots,C_n$. For $t \in (0,T)$ the Lagrangian submanifold $F(t,\cdot) : L \to M$ has conical singularities at $x_1(t),\ldots,x_n(t)$ and isomorphisms $A_i(t) \in \mathcal{A}_{x_i(t)}$ for $i = 1,\ldots,n$ as in Definition 8.4. Moreover for every $t \in (0,T)$, $F(t,\cdot) : L \to M$ satisfies (82) for every $\boldsymbol{\gamma} \in \mathbb{R}$ with $2 < \boldsymbol{\gamma} < 3$ and $(2,\gamma_i] \cap \mathcal{E}_{\Sigma_i} = \emptyset$ for $i = 1,\ldots,n$.*

The proof of Proposition 9.17 follows immediately from Proposition 8.7, Theorem 8.11, and Lemmas 9.14-9.16.

We are finally ready to prove our main theorem about the short time existence of the generalized Lagrangian mean curvature flow when the initial condition is a Lagrangian submanifold with isolated stable conical singularities.

Theorem 9.18. *Let (M,J,ω,Ω) be an m-dimensional almost Calabi–Yau manifold, $m \geq 3$, $C_1,\ldots,C_n$ stable special Lagrangian cones in $\mathbb{C}^m$, and $F_0 : L \to M$ a Lagrangian submanifold with isolated conical singularities at $x_1,\ldots,x_n$, modelled on the stable special Lagrangian cones $C_1,\ldots,C_n$ as in Definition 8.4, and assume that the generalized mean curvature form α_K of $F_0 : L \to M$ satisfies $|\alpha_K| = O(\rho^{\boldsymbol{\varepsilon}})$ for some $\boldsymbol{\varepsilon} \in \mathbb{R}^n$ with $\boldsymbol{\varepsilon} > 0$. Then there exists $T > 0$, Hölder* *continuous one-parameter families of points $\{x_i(t)\}_{t\in(0,T)}$ in M for $i=1,\dots,n$, continuous up to $t=0$, with $x_i(0)=x_i$ for $i=1,\dots,n$, and Hölder continuous one-parameter families $\{A_i(t)\}_{t\in(0,T)}$ of isomorphisms $A_i(t)\in\mathcal{A}_{x_i(t)}$ for $i=1,\dots,n$, continuous up to $t=0$, with $A_i(0)=A_i$ for $i=1,\dots,n$, such that the following holds.*

There exists a Hölder continuous one-parameter family $\{F(t,\cdot)\}_{t\in(0,T)}$ of smooth Lagrangian submanifolds $F(t,\cdot):L\to M$, continuous up to $t=0$, with isolated conical singularities at $x_1(t),\dots,x_n(t)$ modelled on the special Lagrangian cones $C_1,\dots,C_n$ and with isomorphisms $A_1(t),\dots,A_n(t)$, $A_i(t):\mathbb{C}^m\to T_{x_i(t)}M$ for $i=1,\dots,n$ as in Definition 8.4, which evolves by generalized Lagrangian mean curvature flow with initial condition $F_0:L\to M$. Moreover, for every $t\in(0,T)$ the Lagrangian submanifold $F(t,\cdot):L\to M$ satisfies (82) *for every $\boldsymbol{\gamma}\in\mathbb{R}^n$ with $\gamma_i\in(2,3)$ and $(2,\gamma_i]\cap\mathcal{E}_{\Sigma_i}=\emptyset$ for $i=1,\dots,n$. Finally $\{F(t,\cdot)\}_{t\in(0,T)}$ is the unique solution to the generalized Lagrangian mean curvature flow in this particular class of Lagrangian submanifolds.*

Proof. Let $F_0:L\to M$ be as in the theorem and $f_0=(x_1,A_1,0,\dots,x_n,A_n,0)$. Define $\mathcal{F}$ as in §8.4 and choose Lagrangian neighbourhoods $\{\Phi_L^f\}_{f\in\mathcal{F}}$ as in §8.4. Let $\boldsymbol{\mu}\in\mathbb{R}^n$ with $2<\boldsymbol{\mu}<\boldsymbol{\nu}$ and let $k\in\mathbb{N}$ and $p\in(1,\infty)$ be sufficiently large. Then by Proposition 9.13 there exists a solution $(u,f)\in\mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ of the Cauchy problem (91). Then f defines $W^{1,p}$-one-parameter families $\{x_i(t)\}_{t\in(0,T)}$ of points in M for $i=1,\dots,n$ and of isomorphisms $\{A_i(t)\}_{t\in(0,T)}$ for $i=1,\dots,n$ with $A_i(t)\in\mathcal{A}_{x_i(t)}$ for $i=1,\dots,n$. In particular, by the Sobolev Embedding Theorem the families $\{x_i(t)\}_{t\in(0,T)}$ and $\{A_i(t)\}_{t\in(0,T)}$ for $i=1,\dots,n$ are uniformly Hölder continuous with respect to $t\in(0,T)$, and therefore $\{x_i(t)\}_{t\in(0,T)}$ and $\{A_i(t)\}_{t\in(0,T)}$ extend continuously to $t=0$. We define a $W^{1,p}$-one-parameter family of Lagrangian submanifolds $\{F(t,\cdot)\}_{t\in(0,T)}$ as in (92). Then $\{F(t,\cdot)\}_{t\in(0,T)}$ is a Hölder continuous one parameter family of Lagrangian submanifolds. By Proposition 9.17, $F(t,\cdot):L\to M$ is a smooth Lagrangian submanifold with isolated conical singularities at $x_1(t),\dots,x_n(t)\in M$ modelled on the stable special Lagrangian cones $C_1,\dots,C_n$ with isomorphisms $A_i(t):\mathbb{C}^m\to T_{x_i(t)}M$ for $i=1,\dots,n$ as in Definition 8.4. Furthermore by Lemma 9.17, for every $t\in(0,T)$ the Lagrangian submanifold $F(t,\cdot):L\to M$ satisfies (82) for every $\boldsymbol{\gamma}\in\mathbb{R}^n$ with $\gamma_i\in(2,3)$ and $(2,\gamma_i]\cap\mathcal{E}_{\Sigma_i}=\emptyset$ for $i=1,\dots,n$. Finally Proposition 9.5 shows that $\{F(t,\cdot)\}_{t\in(0,T)}$ evolves by generalized Lagrangian mean curvature flow with initial condition $F_0:L\to M$. The uniqueness of solutions to the Cauchy problem (91) follows immediately from Proposition 9.12 and the Inverse Function Theorem and implies the uniqueness of $\{F(t,\cdot)\}_{t\in(0,T)}$ in this particular class of Lagrangian submanifolds. □

10 Open problems related to the thesis

In this section we discuss some open problems that are related to the material presented in this work. We emphasize again that this discussion will be purely formal and at many stages we do not attempt to give rigorous mathematical proofs of the results, which we claim to be true.

10.1 Parabolic equations of Laplace type on compact Riemannian manifolds with conical singularities

In this subsection we study differential operators of Laplace type and parabolic equations of Laplace type. Differential operators of Laplace type are differential operators of second order on compact Riemannian manifolds with conical singularities that differ from the Laplace operator only by a lower order term that is well behaved in a certain sense that we will make precise below. Our discussion of parabolic equations of Laplace type follows the same ideas as in §7.3 and we explain how a fundamental solution to parabolic equations of Laplace type can be constructed. The reader is also referred to Melrose [39, Ch. 7, §], where the fundamental solution for general second order parabolic equations on Riemannian manifolds with boundary is constructed. Finally we present two existence and maximal regularity results for parabolic equations of Laplace type which generalize the results obtained in §7.4 and §7.5.

We begin with the definition of differential operators of Laplace type on Riemannian manifolds with conical singularities.

Definition 10.1. *Let (M,g) be a compact Riemannian manifold with conical singularities as in Definition 6.3, ρ a radius functions on M', and L a linear second order differential operator on M'. Then L is said to be a differential operator of Laplace type if the following holds. There exist $\boldsymbol{\delta} \in \mathbb{R}^n$ with $\boldsymbol{\delta} > 0$, a smooth vector field X on M' with $|\nabla^j X| = O(\rho^{\boldsymbol{\delta}-1-j})$ as $\rho \to 0$ for $j \in \mathbb{N}$, and a function $b \in C^\infty_{\boldsymbol{\delta}-2}(M')$, such that*

$$Lu = \Delta_g u + g(X, \nabla u) + b \cdot u \quad \text{for } u \in C^2_{\mathrm{loc}}(M'). \tag{126}$$

The reason we call these operators of Laplace type is that to leading order these operators coincide with the Laplace operator. In fact, if L is a differential operator of Laplace type, then the principal symbol of L is equal to the principal symbol of the Laplace operator, see Shubin [49, §5.1] for the definition of the principal symbol of a differential operator. Thus away from the conical singularities L and the Laplace operator coincide to leading order. Moreover, close to the singularity the Laplace operator dominates the lower order terms of L, essentially because the coefficients of the lower order terms of L decay sufficiently fast near the singularity. A way to state this in terms of symbols of differential operators is to say that L and the Laplace operator have the same indicial operator in the sense of Melrose [39, Ch. 4, §15], or the same conormal symbol in the sense of Schulze [48, Ch. 2, §1.2]. Therefore the Laplace operator and L agree to leading order also near the singularities, and we expect that the Laplace operator and L, and also the corresponding parabolic equations, have a similar theory.

The Fredholm theory for differential operators of Laplace type follows immediately from Proposition 6.12, the Fredholm alternative, and the observation that the lower order terms of a differential operator of Laplace type define a compact operator between the weighted spaces under consideration. For instance the lower order terms define a compact operator $W^{k,p}_{\boldsymbol{\gamma}}(M') \to W^{k-2,p}_{\boldsymbol{\gamma}-2}(M')$.

Proposition 10.2. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$, L a differential operator of Laplace type, and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Then the following hold.*

(i) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\alpha \in (0,1)$. Then*

$$L : C^{k,\alpha}_{\boldsymbol{\gamma}}(M') \to C^{k-2,\alpha}_{\boldsymbol{\gamma}-2}(M') \tag{127}$$

is a Fredholm operator if and only if $\gamma_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1,\dots,n$, where $\mathcal{D}_{\Sigma_i}$ is defined in (39)*. If $\gamma_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1,\dots,n$, then the Fredholm index of* (127) *is equal to $-\sum_{i=1}^n M_{\Sigma_i}(\gamma_i)$.*

(ii) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $p \in (1,\infty)$. Then*

$$L : W^{k,p}_{\boldsymbol{\gamma}}(M') \to W^{k-2,p}_{\boldsymbol{\gamma}-2}(M') \tag{128}$$

is a Fredholm operator if and only if $\gamma_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1,\dots,n$. If $\gamma_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1,\dots,n$, then the Fredholm index of (128) *is equal to $-\sum_{i=1}^n M_{\Sigma_i}(\gamma_i)$.*

Once we know the Fredholm theory for differential operators of Laplace type it is straightforward to construct discrete asymptotics for these operators and to define weighted spaces with discrete asymptotics. In fact we have the following analogue of Proposition 6.14.

Proposition 10.3. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$, L a differential operator of Laplace type, and $\boldsymbol{\gamma} \in \mathbb{R}^n$. Then for every $\varepsilon > 0$ there exists a linear map*

$$\Psi^L_{\boldsymbol{\gamma}} : \bigoplus_{i=1}^n V_{\mathsf{P}_{\gamma_i}}(C_i) \longrightarrow C^\infty(M'),$$

such that the following hold.

(i) *For every $v \in \bigoplus_{i=1}^n V_{\mathsf{P}_{\gamma_i}}(C_i)$ with $v = (v_1,\dots,v_n)$ and $v_i = r^{\beta_i}\varphi_i$ where $\varphi_i \in C^\infty(\Sigma)$ for $i = 1,\dots,n$ we have*

$$|\nabla^k(\phi_i^*(\Psi^L_{\boldsymbol{\gamma}}(v)) - v_i)| = O(r^{\mu_i+\beta_i-\varepsilon-k}) \quad \textit{as } r \longrightarrow 0 \textit{ for } k \in \mathbb{N}$$

and $i = 1,\dots,n$.

(ii) *For every $v \in \bigoplus_{i=1}^n V_{\mathsf{P}_{\gamma_i}}(C_i)$ with $v = (v_1,\dots,v_n)$ we have*

$$L(\Psi^L_{\boldsymbol{\gamma}}(v)) - \sum_{i=0}^n \Psi^L_{\boldsymbol{\gamma}}(\Delta_{g_i} v_i) \in C^\infty_{\mathrm{cs}}(M').$$

Moreover, if $\boldsymbol{\delta} > 1$, then we may take $\Psi^L_{\boldsymbol{\gamma}} = \Psi_{\boldsymbol{\gamma}}$, where $\Psi_{\boldsymbol{\gamma}}$ is defined in Proposition 6.14.

Note that since L and the Laplace operator coincide to leading order near each conical singularities, we use the same model space in the construction of the discrete asymptotics for L in Proposition 10.3 as in the definition of the discrete asymptotics for the Laplace operator in Proposition 6.14.

We can now proceed to define weighted C^k-spaces, Hölder spaces, and Sobolev spaces with discrete asymptotics. In fact, for $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $\boldsymbol{\gamma} \in \mathbb{R}^n$ we define

$$C^k_{\boldsymbol{\gamma},\mathsf{P}^L_{\boldsymbol{\gamma}}}(M') = C^k_{\boldsymbol{\gamma}}(M') \oplus \operatorname{im}\,\Psi^L_{\boldsymbol{\gamma}} \quad \text{and} \quad C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}^L_{\boldsymbol{\gamma}}}(M') = C^{k,\alpha}_{\boldsymbol{\gamma}}(M') \oplus \operatorname{im}\,\Psi^L_{\boldsymbol{\gamma}}$$

and finally for $p \in [1,\infty)$ we define

$$W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}^L_{\boldsymbol{\gamma}}}(M') = W^{k,p}_{\boldsymbol{\gamma}}(M') \oplus \operatorname{im}\,\Psi^L_{\boldsymbol{\gamma}}.$$

Then $C^k_{\boldsymbol{\gamma},\mathsf{P}^L_{\boldsymbol{\gamma}}}(M')$, $C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}^L_{\boldsymbol{\gamma}}}(M')$, and $W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}^L_{\boldsymbol{\gamma}}}(M')$ are Banach spaces, where we choose some finite dimensional norm on the discrete asymptotics part.

Following the proof of Proposition 6.15 we find the following result.

Proposition 10.4. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities, $m \geq 3$, L a differential operator of Laplace type as in* (126)*, and $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\boldsymbol{\gamma} > 2-m$ and $\gamma_i \notin \mathcal{E}_{\Sigma_i}$. Then the following hold.*

(i) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\alpha \in (0,1)$. Then*

$$L : C^{k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}^L_{\boldsymbol{\gamma}}}(M') \longrightarrow C^{k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}^L_{\boldsymbol{\gamma}-2}}(M') \tag{129}$$

is a Fredholm operator of index zero.

(ii) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $p \in (1,\infty)$. Then*

$$L : W^{k,p}_{\boldsymbol{\gamma},\mathsf{P}^L_{\boldsymbol{\gamma}}}(M') \longrightarrow W^{k-2,p}_{\boldsymbol{\gamma}-2,\mathsf{P}^L_{\boldsymbol{\gamma}-2}}(M') \tag{130}$$

is a Fredholm operator of index zero.

We now begin our discussion of parabolic equations of Laplace type. The next theorem establishes the existence of a fundamental solution to parabolic equations of Laplace type and also states that the fundamental solution satisfies similar estimates as the Friedrichs heat kernel in Theorem 7.7.

Theorem 10.5. *Let (M,g) be a compact m-dimensional Riemannian manifold with conical singularities as in Definition 6.3, $m \geq 3$, and L a differential operator of Laplace type. Then there exists $H \in C^\infty((0,\infty) \times M' \times M')$ solving the Cauchy problem*

$$\begin{aligned} &\frac{\partial H}{\partial t}(t,x,y) = LH(t,x,y), && \text{for } (t,x,y) \in (0,T) \times M' \times M', \\ &H(0,x,y) = \delta_x(y), && \text{for } x \in M'. \end{aligned} \tag{131}$$

Moreover the following holds.

Let $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\gamma_i \notin \mathcal{E}_{\Sigma_i}$. For $l \in \mathbb{N}$ choose a basis $\psi^1_{\boldsymbol{\gamma}-2l},\ldots,\psi^{N_l}_{\boldsymbol{\gamma}-2l}$ for the vector space $\operatorname{im}\,\Psi^L_{\boldsymbol{\gamma}}$*, where $N_l = \dim \operatorname{im}\,\Psi^L_{\boldsymbol{\mu}}$. Then for each $l \in \mathbb{N}$ there exist functions $H^1_{\boldsymbol{\gamma}-2l},\ldots,H^{N_l}_{\boldsymbol{\gamma}-2l} \in C^\infty((0,\infty)\times M')$ and constants $c_l > 0$, such that for every $l \in \mathbb{N}$*

$$|H^j_{\boldsymbol{\gamma}-2l}(t,y)| \leq c(t+\rho(y)^2)^{-\frac{m+(\boldsymbol{\gamma}-2l)^-}{2}} \quad \text{for } t>0,\ y \in M'$$

and $j = 1, \dots, N_l$, *and*

$$\Big|\partial_t^l H(t,x,y) - \sum_{j=1}^{N_l} \psi^j_{\boldsymbol{\gamma}-2l}(x) H^j_{\boldsymbol{\gamma}-2l}(t,y))\Big| \\ \leq c_l (t + d_g(x,y)^2)^{-\frac{m+2l}{2}} \left(\frac{\rho(x)^2}{\rho(x)^2+\rho(y)^2}\right)^{\frac{(\boldsymbol{\gamma}-2l)^+}{2}}$$

for $t \in (0,\infty)$ *and* $x, y \in M'$.

The proof of Theorem 10.5 should more or less follow the same ideas as in the construction of the Friedrichs heat kernel discussed in §7.3. By Theorem 7.7 we are given the Friedrichs heat kernel H_0, which is a fundamental solution for the heat equation and satisfies a bunch of estimates. The fact that L and the Laplace operator coincide to leading order implies that H_0 satisfies

$$\begin{aligned} &\frac{\partial H_0}{\partial t}(t,x,y) = LH_0(t,x,y) + R_0(t,x,y), && \text{for } (t,x,y) \in (0,T) \times M' \times M', \\ &H_0(0,x,y) = \delta_x(y), && \text{for } x \in M', \end{aligned}$$

where $R_0 \in C^\infty((0,\infty) \times M' \times M')$ is a lower order remainder in a pseudodifferential calculus that can be defined on the heat space of M. Using the same ideas as in [41, Prop. 3.4-3.8] and the discrete asymptotics given by Proposition 10.3 one can construct a function $H \in C^\infty((0,\infty) \times M' \times M')$ that is a fundamental solution of (131) and satisfies the estimates in Theorem 10.5.

Once Theorem 10.5 is known it is straightforward to generalize the results from §7.4 and §7.5 to the Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = Lu(t,x) + f(t,x) && \text{for } (t,x) \in (0,T) \times M, \\ &u(0,x) = 0 && \text{for } x \in M. \end{aligned} \tag{132}$$

The next theorem gives an existence and maximal regularity result for solutions of (132) in the case when the free term lies in a weighted parabolic Hölder space with discrete asymptotics.

Theorem 10.6. *Let* (M,g) *be a compact* m*-dimensional Riemannian manifold with conical singularities,* $m \geq 3$, L *a differential operator of Laplace type as in* (126), $T > 0$, $k \in \mathbb{N}$ *with* $k \geq 2$, $\alpha \in (0,1)$, *and* $\boldsymbol{\gamma} \in \mathbb{R}^n$ *with* $\boldsymbol{\gamma} > 2 - m$ *and* $\gamma_i \notin \mathcal{E}_{\Sigma_i}$. *Given* $f \in C^{0,k-2,\alpha}_{\boldsymbol{\gamma}-2,\mathsf{P}^L_{\boldsymbol{\gamma}-2}}((0,T) \times M')$, *there exists a unique* $u \in C^{1,k,\alpha}_{\boldsymbol{\gamma},\mathsf{P}^L_{\boldsymbol{\gamma}}}((0,T) \times M')$ *solving the Cauchy problem* (132).

Theorem 10.6 is proved in exactly the same way as Theorem 7.10. Note, however, that the discrete asymptotics are now defined using Proposition 10.3.

In a similar way we obtain the following theorem when the free term lies in a weighted parabolic Sobolev space with discrete asymptotics.

Theorem 10.7. *Let* (M,g) *be a compact* m*-dimensional Riemannian manifold with conical singularities,* $m \geq 3$, L *a differential operator of Laplace type as in* (126), $T > 0$, $k \in \mathbb{N}$ *with* $k \geq 2$, $p \in (1,\infty)$, *and* $\boldsymbol{\gamma} \in \mathbb{R}^n$ *with* $\boldsymbol{\gamma} > 2 - m$ *and* $\gamma_i \notin \mathcal{E}_{\Sigma_i}$. *Given* $f \in W^{0,k-2,p}_{\boldsymbol{\gamma}-2,\mathsf{P}^L_{\boldsymbol{\gamma}-2}}((0,T) \times M')$, *there exists a unique* $u \in W^{1,k,p}_{\boldsymbol{\gamma},\mathsf{P}^L_{\boldsymbol{\gamma}}}((0,T) \times M')$ *solving the Cauchy problem* (132).

10.2 Short time existence of the generalized Lagrangian mean curvature flow with isolated conical singularities

In this subsection we discuss why we think that it should be possible to drop the extra assumption on the generalized mean curvature form in Proposition 9.12. The problems discussed in this and the next subsection mainly deal with the fact that we do not fully understand the operator $P : \tilde{\mathcal{D}}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ and the rôle of the discrete asymptotics in this geometric context.

The result we want to speculate about is a generalization of Proposition 9.12. In fact we believe that Proposition 9.12 can be replaced by the following result.

Proposition 10.8. *Choose $\boldsymbol{\mu} \in \mathbb{R}^n$ with $2 < \boldsymbol{\mu} < 3$ and $(2,\mu_i] \cap \mathcal{E}_{\Sigma_i} = \emptyset$ for $i = 1,\dots,n$. Then for $T > 0$ sufficiently small, the linear operator*

$$\mathrm{d}P(0,f_0) : \tilde{W}^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T)\times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L) \tag{133}$$

is an isomorphism of Banach spaces.

In Proposition 10.8 we also used the extra assumption that the generalized mean curvature form α_K of the initial Lagrangian submanifold $F_0 : L \to M$ satisfies $|\alpha_K| = O(\rho^{\boldsymbol{\varepsilon}})$ for some $\boldsymbol{\varepsilon} \in \mathbb{R}^n$ with $\boldsymbol{\varepsilon} > 0$. We then used this condition in Lemma 9.11 to show that the discrete asymptotics defined by $\Xi_{(0,f_0)}$ are the same as the ones defined by Proposition 6.14. Then we were able to use our regularity results for the heat equation from Theorem 7.13 to show that the linearization of P at the initial condition is an isomorphism.

Recall from Proposition 9.10 that the linearization of the operator $P : \tilde{\mathcal{D}}^{k,p}_{\boldsymbol{\mu}} \to W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L)$ at the initial condition $(0,f_0)$ is an operator

$$\mathrm{d}P(0,f_0) : \tilde{W}^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T)\times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L) \tag{134}$$

given by

$$\begin{aligned} \mathrm{d}P(0,f_0)(v,\Xi_{(0,f_0)}(w)) = {} & \frac{\partial}{\partial t}(v - \Xi_{(0,f_0)}(w)) - \Delta(v - \Xi_{(0,f_0)}(w)) \\ & + m\mathrm{d}\psi_\beta(\nabla(v - \Xi_{(0,f_0)}(w))) + \mathrm{d}\theta_\beta(\hat{V}(\mathrm{d}v) - \hat{W}(\Xi_{(0,f_0)}(w))), \end{aligned} \tag{135}$$

for $v \in W^{1,k,p}_{\boldsymbol{\mu}}((0,T)\times L)$ and $w \in W^{1,p}((0,T);T_{f_0}\mathcal{F})$. It the proof of Proposition 10.8 we made use of the fact that the spaces $W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T)\times L)$ and $W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}((0,T)\times L)$ are the same, as proved in Lemma 9.11. This made it possible to compare the geometric operator in (134) and (135) together with the geometric discrete asymptotics and the heat operator with the discrete asymptotics given by Proposition 6.14. In general it will not be true that the spaces $W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T)\times L)$ and $W^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{\boldsymbol{\mu}}}((0,T)\times L)$ are the same. However, using Theorem 10.7 we can now argue as follows. Let us define operators

$$H, K : \tilde{W}^{1,k,p}_{\boldsymbol{\mu},\mathsf{P}_{(0,f_0)}}((0,T)\times L) \longrightarrow W^{0,k-2,p}_{\boldsymbol{\mu}-2,\mathsf{Q}}((0,T)\times L) \tag{136}$$

by

$$\begin{aligned} H(v,\Xi_{(0,f_0)}(w)) = {} & \frac{\partial}{\partial t}(v - \Xi_{(0,f_0)}(w)) - \Delta(v - \Xi_{(0,f_0)}(w)) \\ & + m\mathrm{d}\psi_\beta(\nabla(v - \Xi_{(0,f_0)}(w))) + \mathrm{d}\theta_\beta(\hat{W}(v - \Xi_{(0,f_0)}(w))) \end{aligned} \tag{137}$$

and $K = \mathrm{d}\theta_\beta(\hat{V}(\mathrm{d}v) - \hat{W}(v))$. Note that H and K in (136) are well defined and that $\mathrm{d}P(0, f_0)(v, \Xi_{(0,f_0)}(w)) = H + K$. It is not hard to see that H is a parabolic operator of Laplace type with discrete asymptotics defined by $\Xi_{(0,f_0)}$. Thus Theorem 10.7 implies that H in (136) is an isomorphism for $T > 0$ sufficiently small. The same techniques used in the proofs of Propositions 5.7 and 9.12 then imply that (134) is an isomorphism.

From here on it is now straightforward to prove Proposition 10.8 and then the short time existence of the generalized Lagrangian mean curvature flow with isolated conical singularities without the extra assumption on the generalized mean curvature form of the initial Lagrangian submanifold.

10.3 Regularity theory for the generalized Lagrangian mean curvature flow with isolated conical singularities

In this subsection we speculate on some further regularity theory for the generalized Lagrangian mean curvature flow with isolated conical singularities.

The first regularity problem we want to discuss is the time regularity of solutions $(u, f) \in \mathcal{D}^{k,p}_{\boldsymbol{\mu}}$ to $P(u, f) = 0$. In fact we would like to show that the functions $u : (0, T) \times L \to \mathbb{R}$ are smooth in the time variable and that the family $\{f(t)\}_{t\in(0,T)} \subset \mathcal{F}$ is a smooth one-parameter family. Then it would follow that the one-parameter family $\{F(t, \cdot)\}_{t\in(0,T)}$ of smooth Lagrangian submanifolds from Theorem 9.18 is in fact a smooth one-parameter family of smooth Lagrangian submanifolds. In particular it would follow that the translations and rotations of the model cones of the Lagrangian submanifolds are smooth. The problem in proving such a result come again from the structure of the equation $P(u, f) = 0$, because we do not really understand the structure of this equation and the discrete asymptotics involved in this problem.

Using the speculation from §10.1, §10.2, and this subsection we finish this thesis with the following conjecture about the short time existence for the generalized Lagrangian mean curvature flow with isolated conical singularities.

Conjecture 10.9. *Let (M, J, ω, Ω) be an m-dimensional almost Calabi–Yau manifold, $m \geq 3$, $C_1, \ldots, C_n$ stable special Lagrangian cones in $\mathbb{C}^m$, and $F_0 : L \to M$ a Lagrangian submanifold with isolated conical singularities at $x_1, \ldots, x_n$, modelled on the stable special Lagrangian cones $C_1, \ldots, C_n$ as in Definition 8.4. Then there exists $T > 0$, smooth one-parameter families of points $\{x_i(t)\}_{t\in(0,T)}$ in M for $i = 1, \ldots, n$, continuous up to $t = 0$, with $x_i(0) = x_i$ for $i = 1, \ldots, n$, and smooth one-parameter families $\{A_i(t)\}_{t\in(0,T)}$ of isomorphisms $A_i(t) \in \mathcal{A}_{x_i(t)}$ for $i = 1, \ldots, n$, continuous up to $t = 0$, with $A_i(0) = A_i$ for $i = 1, \ldots, n$, such that the following holds.*

There exists a smooth one-parameter family $\{F(t, \cdot)\}_{t\in(0,T)}$ of smooth Lagrangian submanifolds $F(t, \cdot) : L \to M$, continuous up to $t = 0$, with isolated conical singularities at $x_1(t), \ldots, x_n(t)$ modelled on the special Lagrangian cones $C_1, \ldots, C_n$ and with isomorphisms $A_1(t), \ldots, A_n(t)$, $A_i(t) : \mathbb{C}^m \to T_{x_i(t)}M$ for $i = 1, \ldots, n$ as in Definition 8.4, which evolves by generalized Lagrangian mean curvature flow with initial condition $F_0 : L \to M$. Moreover, for every $t \in (0, T)$ the Lagrangian submanifold $F(t, \cdot) : L \to M$ satisfies (82) *for every $\boldsymbol{\gamma} \in \mathbb{R}^n$ with $\gamma_i \in (2, 3)$ and $(2, \gamma_i] \cap \mathcal{E}_{\Sigma_i} = \emptyset$ for $i = 1, \ldots, n$. Finally $\{F(t, \cdot)\}_{t\in(0,T)}$ is the unique solution to the generalized Lagrangian mean curvature flow in this particular class of Lagrangian submanifolds.*

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