Derived symplectic structures in generalized Donaldson–Thomas theory and categorification

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In Loving Memory of My Beloved Father

Abstract

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This thesis presents a series of results obtained in [13, 18, 19, 23–25, 87]. In [19], we prove a Darboux theorem for derived schemes with symplectic forms of degree $k < 0$, in the sense of [142]. We use this to show that the classical scheme $X = t_0(\boldsymbol{X})$ has the structure of an *algebraic d-critical locus*, in the sense of Joyce [87]. Then, if (X, s) is an *oriented d-critical locus*, we prove in [18] that there is a natural perverse sheaf $P^\bullet_{X,s}$ on X, and in [25], we construct a natural motive $MF_{X,s}$, in a certain quotient ring $\bar{\mathcal{M}}^{\hat{\mu}}_X$ of the $\hat{\mu}$-equivariant motivic Grothendieck ring $\mathcal{M}^{\hat{\mu}}_X$, and used in Kontsevich and Soibelman's theory of motivic Donaldson–Thomas invariants [102]. In [13], we obtain similar results for k-shifted symplectic derived Artin stacks.

We apply this theory to *categorifying* Donaldson–Thomas invariants of Calabi–Yau 3-folds, and to categorifying Lagrangian intersections in a complex symplectic manifold using perverse sheaves, and to prove the existence of natural motives on moduli schemes of coherent sheaves on a Calabi–Yau 3-fold equipped with 'orientation data', as required in Kontsevich and Soibelman's motivic Donaldson–Thomas theory [102], and on intersections $L \cap M$ of oriented Lagrangians L, M in an algebraic symplectic manifold (S, ω). In [23] we show that if (S, ω) is a complex symplectic manifold, and L, M are complex Lagrangians in S, then the intersection $X = L \cap M$, as a complex analytic subspace of S, extends naturally to a complex analytic d-critical locus (X, s) in the sense of Joyce [87]. If the canonical bundles K_L, K_M have square roots $K_L^{1/2}, K_M^{1/2}$ then (X, s) is oriented, and we provide a direct construction of a perverse sheaf $P^\bullet_{L,M}$ on X, which coincides with the one constructed in [18].

In [24] we have a more in depth investigation in *generalized Donaldson–Thomas invariants* $\bar{DT}^\alpha(\tau)$ defined by Joyce and Song [85]. We propose a new algebraic method to extend the theory to algebraically closed fields $\mathbb{K}$ of characteristic zero, rather than $\mathbb{K} = \mathbb{C}$, and we conjecture the extension of generalized Donaldson–Thomas theory to compactly supported coherent sheaves on noncompact quasi-projective Calabi–Yau 3-folds, and to complexes of coherent sheaves on Calabi–Yau 3-folds.

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Contents

Introduction

In the following we will summarize some motivations and background which permit to allocate our problem and state the main results. After that, we outline the contents of the thesis.

Notations and conventions

Throughout $\mathbb{K}$ will be an algebraically closed field with $\operatorname{char}\mathbb{K} = 0$. Classical $\mathbb{K}$-schemes and Artin $\mathbb{K}$-stacks will be written $W, X, Y, Z, \ldots$, and derived $\mathbb{K}$-schemes and derived Artin $\mathbb{K}$-stacks in bold as $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \ldots$. Basic references for $\mathbb{K}$-schemes are Hartshorne [65], for Artin $\mathbb{K}$-stacks Laumon and Moret-Bailly [109], and for derived $\mathbb{K}$-schemes and derived Artin $\mathbb{K}$-stacks Toën and Vezzosi [142,171–175]. All (classical) $\mathbb{K}$-schemes and Artin $\mathbb{K}$-stacks X are assumed separated and locally of finite type. All derived $\mathbb{K}$-schemes and derived $\mathbb{K}$-stacks $\boldsymbol{X}$ are assumed to be locally finitely presented. We write $\mathrm{Sch}_{\mathbb{K}}$ for the category of $\mathbb{K}$-schemes, $\mathrm{Art}_{\mathbb{K}}$ for the 2-category of Artin $\mathbb{K}$-stacks, $\mathbf{dSch}_{\mathbb{K}}$ for the ∞-category of derived $\mathbb{K}$-schemes, and $\mathbf{dArt}_{\mathbb{K}}$ for the ∞-category of derived Artin $\mathbb{K}$-stacks, and $t_0 : \mathbf{dSch}_{\mathbb{K}} \to \mathrm{Sch}_{\mathbb{K}}$, $t_0 : \mathbf{dArt}_{\mathbb{K}} \to \mathrm{Art}_{\mathbb{K}}$ for the classical truncation functors. Finally, all complex analytic spaces Hausdorff and locally of finite type.

D-critical loci

In §2, we will introduce the theory of d-critical loci from [87]. Recall that to say that a scheme X has an *obstruction theory* means, very roughly speaking, that one is endowed with a complex of vector bundles encoding informations on the deformations and obstructions spaces of X. When this obstruction theory is *symmetric*, Behrend [5] proved that X can be described as the zero locus of an almost closed 1-form. Schemes with symmetric obstruction theories are the basis of Joyce and Song's theory of Donaldson–Thomas invariants of Calabi–Yau 3-folds [85]. In the attempt to resolve the questions about *categorification* in Donaldson–Thomas theory, that is, to produce a natural graded $\mathbb{Q}$-vector space thought of as some kind of generalized cohomology of the moduli space, whose graded dimension is the virtual counting of the moduli space itself, the author and her collaborators tried for some time to construct perverse sheaves, and motivic Milnor fibres, from a scheme with symmetric obstruction theory, but failed. Moreover, more recently, Pandharipande and Thomas [140] gave examples of schemes X with symmetric obstruction theories with X not locally isomorphic to a critical locus. This was the major signal that almost closed 1-forms were not enough to resolve the questions.

Later, Pantev, Toën, Vaquié and Vezzosi [142,179] defined a new notion of *derived critical locus*. It is set in the context of Toën and Vezzosi's theory of derived algebraic geometry [173–175], and consists of a quasi-smooth derived scheme $\boldsymbol{X}$ equipped with a -1-shifted symplectic structure ω. Examples of -1-shifted symplectic derived schemes are the critical locus $\mathrm{Crit}(f)$ of a regular function $f : U \to \mathbb{A}^1$ on a smooth $\mathbb{K}$-scheme U, or the intersection $L \cap M$ of smooth Lagrangians L, M in an algebraic symplectic manifold (S, ω), or the moduli scheme $\mathcal{M}$ of stable coherent sheaves on a Calabi–Yau 3-fold. Behrend's schemes with symmetric obstruction theories [5] can be interpreted as 'semiclassical truncation' of -1-shifted symplectic derived schemes. If $(\boldsymbol{X},\omega)$ is a -1-shifted symplectic derived scheme in the sense of Pantev et al. [142], then the classical scheme $X = t_0(\boldsymbol{X})$ has a symmetric obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ with $\mathcal{E}^\bullet = i^*(\mathbb{L}_{\boldsymbol{X}})$ and $\theta = i^*(\omega_0)$, where $i : X \hookrightarrow \boldsymbol{X}$ is the inclusion.

Very recently, and inspired by issues coming from the failed attempts to use schemes with symmetric obstruction theory for applications especially in Donaldson–Thomas theory from one hand, and from the other motivated by searching for a theory much more simpler than the derived algebraic geometry, Joyce defined a new class of geometric objects called *d-critical loci* (X, s), which are classical schemes X with an extra (classical, not derived) geometric structure s that records information on how X may locally be written as a classical critical locus $\mathrm{Crit}(H)$ of a regular function $H : U \to \mathbb{A}^1$ on a smooth scheme U. They are much simpler than -1-shifted symplectic derived schemes, and are entirely 'classical', in the sense that they are defined up to isomorphism in an ordinary category using classical algebraic geometry in the style of Hartshorne [65], rather than being defined up to equivalence in an ∞-category using homotopy theory and derived algebraic geometry as in [173–175].

There is a (non-full) truncation functor from -1-shifted symplectic derived schemes to algebraic d-critical loci. Given an algebraic d-critical locus (X, s), then Zariski locally on X we can construct both a -1-shifted symplectic derived scheme $(\boldsymbol{X},\omega)$, and a symmetric obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$, uniquely up to equivalence, but we cannot combine these local models to make $(\boldsymbol{X},\omega)$ or $\mathcal{E}^\bullet, \phi$ globally on X because of difficulties with gluing 'derived' objects on open covers. Schemes with symmetric obstruction theories can contain strictly less information than algebraic d-critical loci. On the other hand, schemes with (symmetric) obstruction theories can contain global, nonlocal information which is forgotten by algebraic d-critical loci. It turned out that the theory of d-critical loci, and its extension to d-critical stacks, has applications to generalizations of (motivic) Donaldson–Thomas theory of Calabi–Yau 3-folds, as in [85, 102, 104, 167] and to *categorification* of Donaldson–Thomas invariants, and hence for constructing *cohomological Hall algebras*, following Kontsevich and Soibelman [104].

Darboux theorem for derived symplectic schemes

In §1.1-§1.2 and §3, we present results in [19]. In the context of Toën and Vezzosi's theory of derived algebraic geometry [171–175], Pantev, Toën, Vaquié and Vezzosi [142,179] defined a notion of *k-shifted symplectic structure* ω on a derived scheme or stack $\boldsymbol{X}$, for $k \in \mathbb{Z}$. If $\boldsymbol{X}$ is a derived scheme and ω a 0-shifted symplectic structure, then $\boldsymbol{X} = X$ is a smooth classical scheme and $\omega \in H^0(\Lambda^2 T^* X)$ a classical symplectic structure on X. Pantev et al. [142] introduced a notion of Lagrangian $\boldsymbol{i} : \boldsymbol{L} \to \boldsymbol{X}$ in a k-shifted symplectic derived stack $(\boldsymbol{X},\omega)$, and showed that the fibre product $\boldsymbol{L} \times_{\boldsymbol{X}} \boldsymbol{M}$ of Lagrangians $\boldsymbol{i} : \boldsymbol{L} \to \boldsymbol{X}$, $\boldsymbol{j} : \boldsymbol{M} \to \boldsymbol{X}$ is $(k-1)$-shifted symplectic. Thus, (derived) intersections $L \cap M$ of Lagrangians L, M in a classical algebraic symplectic manifold (S,ω) are -1-shifted symplectic. They also proved that if Y is a Calabi–Yau m-fold then the derived moduli stacks $\boldsymbol{\mathcal{M}}$ of (complexes of) coherent sheaves on Y carry a natural $(2-m)$-shifted symplectic structure.

In [19], in the case $k = -1$, we prove that a -1-shifted symplectic derived scheme $(\boldsymbol{X},\tilde{\omega})$ is Zariski locally equivalent to the derived critical locus $\mathbf{Crit}(H)$ of a regular function $H : U \to \mathbb{A}^1$ on a smooth scheme U and that the underlying classical scheme $X = t_0(\boldsymbol{X})$ extends naturally to an algebraic d-critical locus (X, s), that is, as above, we define a truncation functor from -1-shifted symplectic derived schemes to algebraic d-critical loci.

More generally, we actually proved that if $\boldsymbol{X}$ is a derived scheme and $\tilde{\omega}$ a k-shifted symplectic structure on $\boldsymbol{X}$ for $k < 0$ with $k \not\equiv 2 \mod 4$, then $(\boldsymbol{X},\tilde{\omega})$ is Zariski locally equivalent to $(\mathbf{Spec}\, A, \omega)$, for $\mathbf{Spec}\, A$ an affine derived scheme in which the cdga A is smooth in degree zero and quasi-free in negative degrees, and has Darboux-like coordinates x^i_j, y^{k-i}_j with respect to which the symplectic form $\omega = \sum_{i,j} \mathrm{d}_{dR} y^{k-i}_j \mathrm{d}_{dR} x^i_j$ is standard, and in which the differential in A is given by a Poisson bracket with a Hamiltonian function H of degree $k+1$. When $k<0$ with $k \equiv 2 \mod 4$ we give two statements, one Zariski local in $\boldsymbol{X}$ in which the symplectic form ω on $\mathbf{Spec}\, A$ is standard except for the part in the degree $k/2$ variables, which depends on some functions q_i, and one étale local in $\boldsymbol{X}$ in which ω is entirely standard. Here is [19, Thm. 5.18]:

Theorem *Let $\boldsymbol{X}$ be a derived $\mathbb{K}$-scheme with k-shifted symplectic form $\tilde\omega$ for $k<0$, and $x \in \boldsymbol{X}$. Then there exists a standard form cdga A over $\mathbb{K}$ which is minimal at $p \in \operatorname{Spec} H^0(A)$ in the sense of [19, §4], a k-shifted symplectic form ω on $\mathbf{Spec}\, A$, and a morphism $\boldsymbol{f} : \boldsymbol{U} = \mathbf{Spec}\, A \to \boldsymbol{X}$ with $\boldsymbol{f}(p) = x$ and $\boldsymbol{f}^*(\tilde\omega) \sim \omega$, such that if k is odd or divisible by 4, then $\boldsymbol{f}$ is a Zariski open inclusion, and A, ω are in Darboux form, and if $k \equiv 2 \mod 4$, then $\boldsymbol{f}$ is étale, and A, ω are in strong Darboux form, as in [19, §5].*

Let Y be a Calabi–Yau m-fold over $\mathbb{K}$, that is, a smooth projective $\mathbb{K}$-scheme with $H^i(\mathcal{O}_Y) = \mathbb{K}$ for $i = 0, m$ and $H^i(\mathcal{O}_Y) = 0$ for $0 < i < m$. Suppose $\mathcal{M}$ is a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves in $\operatorname{coh}(Y)$, where we call $F \in \operatorname{coh}(Y)$ *simple* if $\operatorname{Hom}(F,F) = \mathbb{K}$. As we will discuss in §3, there is a corresponding derived moduli scheme $\boldsymbol{\mathcal{M}}$ with $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$, and $\boldsymbol{\mathcal{M}}$ has a $(2-m)-$shifted symplectic structure ω, so we deduce that $(\boldsymbol{\mathcal{M}}, \omega)$ is Zariski locally modelled on $(\mathbf{Spec}\, A, \omega)$, and $\mathcal{M}$ is Zariski locally modelled on $\operatorname{Spec} H^0(A)$. In the case $m = 3$, so that $k = -1$, we get [19, Cor. 5.19]:

Corollary *Suppose Y is a Calabi–Yau 3-fold over a field $\mathbb{K}$, and $\mathcal{M}$ is a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves on Y. Then for each $[F] \in \mathcal{M}$, there exist a smooth $\mathbb{K}$-scheme U with $\dim U = \dim \operatorname{Ext}^1(F,F)$, a regular function $f : U \to \mathbb{A}^1$, and an isomorphism from $\operatorname{Crit}(f) \subseteq U$ to a Zariski open neighbourhood of $[F]$ in $\mathcal{M}$.*

Here $\dim U = \dim \operatorname{Ext}^1(F,F)$ comes from A minimal at p and $\boldsymbol{f}(p) = [F]$. This is a new result in Donaldson–Thomas theory. When $\mathbb{K} = \mathbb{C}$ and $\mathcal{M}$ is a moduli space of simple coherent sheaves on Y, using gauge theory and transcendental complex methods, Joyce and Song [85, Th. 5.4] prove that the underlying complex analytic space $\mathcal{M}^{\mathrm{an}}$ of $\mathcal{M}$ is locally of the form $\operatorname{Crit}(f)$ for U a complex manifold and $f : U \to \mathbb{C}$ a holomorphic function. Behrend and Getzler announced the analogue of [85, Th. 5.4] for moduli of complexes in $D^b \operatorname{coh}(Y)$, but the proof has not yet appeared. Over general $\mathbb{K}$, as in Kontsevich and Soibelman [102, §3.3] the formal neighbourhood $\hat{\mathcal{M}}_{[F]}$ of $\mathcal{M}$ at any $[F] \in \mathcal{M}$ is isomorphic to the critical locus $\operatorname{Crit}(\hat f)$ of a formal power series $\hat f$ on $\operatorname{Ext}^1(F,F)$ with only cubic and higher terms.

In [19, Cor. 5.20] we studied the case $m = 4$, so that $k = -2$, and we deduce a local description of Calabi–Yau 4-fold moduli schemes:

Corollary *Suppose Y is a Calabi–Yau 4-fold over a field $\mathbb{K}$, and $\mathcal{M}$ is a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves on Y. Then for each $[F] \in \mathcal{M}$, there exist a smooth $\mathbb{K}$-scheme U with $\dim U = \dim \operatorname{Ext}^1(F,F)$, a vector bundle $E \to U$ with $\operatorname{rank} E = \dim \operatorname{Ext}^2(F,F)$, a nondegenerate quadratic form Q on E, a section $s \in H^0(E)$ with $Q(s,s) = 0$, and an isomorphism from $s^{-1}(0) \subseteq U$ to a Zariski open neighbourhood of $[F]$ in $\mathcal{M}$.*

If (S, ω) is an algebraic symplectic manifold over $\mathbb{K}$, that is, a 0-shifted symplectic derived $\mathbb{K}$-scheme in the language of [142], and $L, M \subseteq S$ are Lagrangians, then Pantev et al. [142, Th. 2.10] show that the derived intersection $\boldsymbol{X} = L \times_S M$ has a -1-shifted symplectic structure. Here is [19, Cor. 5.21]:

Corollary *Suppose (S, ω) is an algebraic symplectic manifold, and L, M are algebraic Lagrangian submanifolds in S. Then the intersection $X = L \cap M$, as a classical $\mathbb{K}$-subscheme of S, is Zariski locally modelled on the critical locus $\operatorname{Crit}(f)$ of a regular function $f : U \to \mathbb{A}^1$ on a smooth $\mathbb{K}$-scheme U.*

In real or complex symplectic geometry, it is easy to prove analogues of that using Darboux' Theorem or the Lagrangian Neighbourhood Theorem. However, these do not hold for algebraic symplectic manifolds, so it is not obvious how to prove the above result using classical techniques.

Here are [19, Thm. 6.6 & Cor. 6.7]:

Theorem *Suppose* $(\boldsymbol{X},\tilde{\omega})$ *is a* -1*-shifted symplectic derived* $\mathbb{K}$*-scheme, and let* $X=t_0(\boldsymbol{X})$ *be the associated classical* $\mathbb{K}$*-scheme of* $\boldsymbol{X}$*. Then* X *extends uniquely to an algebraic d-critical locus* (X,s)*, with the property that whenever* $(\mathbf{Spec}\,A,\omega)$ *is a* -1*-shifted symplectic derived* $\mathbb{K}$*-scheme in Darboux form with Hamiltonian* $H\in A(0)$*, as in [19, Ex.s 5.8 & 5.15], and* $\boldsymbol{f}:\mathbf{Spec}\,A\to\boldsymbol{X}$ *is an equivalence in* $\mathbf{dSch}_{\mathbb{K}}$ *with a Zariski open derived* $\mathbb{K}$*-subscheme* $\boldsymbol{R}\subseteq\boldsymbol{X}$ *with* $\boldsymbol{f}^*(\tilde{\omega})\sim\omega$*, writing* $U=\operatorname{Spec}A(0)$*,* $R=t_0(\boldsymbol{R})$*,* $f=t_0(\boldsymbol{f})$ *so that* $H:U\to\mathbb{A}^1$ *is regular and* $f:\operatorname{Crit}(H)\to R$ *is an isomorphism, for* $\operatorname{Crit}(H)\subseteq U$ *the classical critical locus of* H*, then* (R,U,H,f^{-1}) *is a critical chart on* (X,s)*. The canonical bundle* $K_{X,s}$ *from Theorem* 2.1.6 *is naturally isomorphic to the determinant line bundle* $\det(\mathbb{L}_{\boldsymbol{X}})|_{X^{\rm red}}$ *of the cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$ *of* $\boldsymbol{X}$*.*

We can think of the above result as defining a *truncation functor*

$$\begin{aligned} F:\bigl\{&\text{category of }-1\text{-shifted symplectic derived }\mathbb{K}\text{-schemes }(\boldsymbol{X},\omega)\bigr\}\\ &\longrightarrow\bigl\{\text{category of algebraic d-critical loci }(X,s)\text{ over }\mathbb{K}\bigr\},\end{aligned}$$

where the morphisms $\boldsymbol{f}:(\boldsymbol{X},\omega)\to(\boldsymbol{Y},\omega')$ in the first line are (homotopy classes of) étale maps $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ with $\boldsymbol{f}^*(\omega')\sim\omega$, and the morphisms $f:(X,s)\to(Y,t)$ in the second line are étale maps $f:X\to Y$ with $f^*(t)=s$. In [87, Ex. 2.17] Joyce gives an example of -1-shifted symplectic derived schemes $(\boldsymbol{X},\omega),(\boldsymbol{Y},\omega')$, both global critical loci, such that $\boldsymbol{X},\boldsymbol{Y}$ are not equivalent as derived $\mathbb{K}$-schemes, but their truncations $F(\boldsymbol{X},\omega),F(\boldsymbol{Y},\omega')$ are isomorphic as algebraic d-critical loci. Thus, the functor F is not full.

Suppose again Y is a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves in $\operatorname{coh}(Y)$. Then Thomas [167] defined a natural *perfect obstruction theory* $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ on $\mathcal{M}$ in the sense of Behrend and Fantechi [6], and Behrend [5] showed that $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ can be made into a *symmetric obstruction theory*. Now in derived algebraic geometry $\mathcal{M}=t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli $\mathbb{K}$-scheme, and the obstruction theory $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ from [167] is $\mathbb{L}_{t_0}:\mathbb{L}_{\boldsymbol{\mathcal{M}}}|_{\mathcal{M}}\to\mathbb{L}_{\mathcal{M}}$. Pantev et al. [142, §2.1] prove $\boldsymbol{\mathcal{M}}$ has a -1-shifted symplectic structure ω, and the symmetric structure on $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ from [5] is $\omega^0|_{\mathcal{M}}$. So we have [19, Cor. 6.7]:

Corollary *Suppose* Y *is a Calabi–Yau* 3*-fold over* $\mathbb{K}$*, and* $\mathcal{M}$ *is a classical moduli* $\mathbb{K}$*-scheme of simple coherent sheaves in* $\operatorname{coh}(Y)$ *with perfect obstruction theory* $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ *as in Thomas* [167]*. Then* $\mathcal{M}$ *extends naturally to an algebraic d-critical locus* $(\mathcal{M},s)$*. The canonical bundle* $K_{\mathcal{M},s}$ *from Theorem* 2.1.6 *is naturally isomorphic to* $\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$*.*

If (S,ω) is an algebraic symplectic manifold over $\mathbb{K}$ and $L,M\subseteq S$ are Lagrangians, then Pantev et al. [142, Th. 2.10] show that the derived intersection $\boldsymbol{X}=L\times_S M$ has a -1-shifted symplectic structure. If $X=t_0(\boldsymbol{X})$ then $\mathbb{L}_{\boldsymbol{X}}|_X\simeq[T^*S|_X\to T^*L|_X\oplus T^*M|_X]$ with $T^*S|_X$ in degree -1 and $T^*L|_X\oplus T^*M|_X$ in degree zero. Hence $\det(\mathbb{L}_{\boldsymbol{X}}|_X)\cong K_S|_X^{-1}\otimes K_L|_X\otimes K_M|_X\cong K_L|_X\otimes K_M|_X$, since $K_S\cong\mathcal{O}_S$. So we obtain [19, Cor. 6.8]:

Corollary *Suppose* (S,ω) *is an algebraic symplectic manifold over* $\mathbb{K}$*, and* L,M *are algebraic Lagrangians in* S*. Then the intersection* $X=L\cap M$*, as a* $\mathbb{K}$*-subscheme of* S*, extends naturally to an algebraic d-critical locus* (X,s)*. The canonical bundle* $K_{X,s}$ *from Theorem* 2.1.6 *is isomorphic to* $K_L|_{X^{\rm red}}\otimes K_M|_{X^{\rm red}}$*.*

Symmetries and stabilization for sheaves of vanishing cycles

In §1.3 we introduce some background material on perverse sheaves, which are used in §4 to present results obtain in [18]. Let U be a smooth $\mathbb{C}$-scheme and $f : U \to \mathbb{C}$ a regular function, and write $X = \mathrm{Crit}(f)$, as a $\mathbb{C}$-subscheme of U. Then following [18], one can define the *perverse sheaf of vanishing cycles* $\mathcal{PV}^\bullet_{U,f}$ on X. Formally, $X = \coprod_{c \in f(X)} X_c$, where $X_c \subseteq X$ is the open and closed $\mathbb{C}$-subscheme of points $x \in X$ with $f(x) = c$, and $\mathcal{PV}^\bullet_{U,f}|_{X_c} = \phi^p_{f-c}(A_U[\dim U])|_{X_c}$ for each $c \in f(X)$, where $A_U[\dim U]$ is the constant perverse sheaf on U over a base ring A, and $\phi^p_{f-c} : \mathrm{Perv}(U) \to \mathrm{Perv}(f^{-1}(c))$ is the vanishing cycle functor for $f - c : U \to \mathbb{C}$. In [18] we prove some results on $\mathcal{PV}^\bullet_{U,f}$.

Let U, f, X be as above, and write X^{red} for the reduced $\mathbb{C}$-subscheme of X. Suppose $\Phi : U \to U$ is an isomorphism with $f \circ \Phi = f$ and $\Phi|_X = \mathrm{id}_X$. Then Φ induces a natural isomorphism $\Phi_* : \mathcal{PV}^\bullet_{U,f} \to \mathcal{PV}^\bullet_{U,f}$. In [18, Thm. 3.1] we prove that $\mathrm{d}\Phi|_{TU|_{X^{\mathrm{red}}}} : TU|_{X^{\mathrm{red}}} \to TU|_{X^{\mathrm{red}}}$ has determinant $\det\bigl(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr) : X^{\mathrm{red}} \to \mathbb{C} \setminus \{0\}$ which is a locally constant map $X^{\mathrm{red}} \to \{\pm 1\}$, and $\Phi_* : \mathcal{PV}^\bullet_{U,f} \to \mathcal{PV}^\bullet_{U,f}$ is multiplication by $\det\bigl(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr)$.

Let U, f, X be as above, and write $I_X \subseteq \mathcal{O}_U$ for the sheaf of ideals of regular functions $U \to \mathbb{C}$ vanishing on X. For each $k = 1, 2, \ldots$, write $X^{(k)}$ for the k^{th} *order thickening of* X *in* U, that is, $X^{(k)}$ is the closed $\mathbb{C}$-subscheme of U defined by the vanishing of the sheaf of ideals I^k_X in $\mathcal{O}_U$. Write $f^{(k)} := f|_{X^{(k)}} : X^{(k)} \to \mathbb{C}$. In [18, Thm. 4.1] we prove that the perverse sheaf $\mathcal{PV}^\bullet_{U,f}$ depends only on the third-order thickenings $(X^{(3)}, f^{(3)})$ up to canonical isomorphism. In fact, étale locally, $\mathcal{PV}^\bullet_{U,f}$ depends only on $(X^{(2)}, f^{(2)})$ up to non-canonical isomorphism, with isomorphisms natural up to sign.

Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Let $\Phi : U \hookrightarrow V$ be a closed embedding of $\mathbb{C}$-schemes with $f = g \circ \Phi : U \to \mathbb{C}$, and suppose $\Phi|_X : X \to Y$ is an isomorphism. Then [18, Thm. 5.4] constructs a natural isomorphism of perverse sheaves on X:

$$\Theta_\Phi : \mathcal{PV}^\bullet_{U,f} \longrightarrow \Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi,$$

where $\pi_\Phi : P_\Phi \to X$ is a certain principal $\mathbb{Z}/2\mathbb{Z}$-bundle on X. Writing N_{UV} for the normal bundle of U in V, then the Hessian $\mathrm{Hess}\, g$ induces a nondegenerate quadratic form q_{UV} on $N_{UV}|_X$, and P_Φ parametrizes square roots of $\det(q_{UV}) : K_U^2|_X \to \Phi|_X^*(K_V^2)$. Moreover, Θ_Φ are functorial in a suitable sense under compositions of embeddings $\Phi : U \hookrightarrow V$, $\Psi : V \hookrightarrow W$. The theorem is proved by showing that étale locally there exist equivalences $V \simeq U \times \mathbb{C}^n$ identifying $\Phi(U)$ with $U \times \{0\}$ and $g : V \to \mathbb{C}$ with $f \boxplus z_1^2 + \cdots + z_n^2 : U \times \mathbb{C}^n \to \mathbb{C}$, and applying étale local isomorphisms of perverse sheaves

$$\mathcal{PV}^\bullet_{U,f} \cong \mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong \mathcal{PV}^\bullet_{U \times \mathbb{C}^n, f \boxplus z_1^2+\cdots+z_n^2} \cong \mathcal{PV}^\bullet_{V,g},$$

using $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong A_{\{0\}}$ in the first step, and the Thom–Sebastiani Theorem for perverse sheaves in the second. Passing from $f : U \to \mathbb{C}$ to $g = f \boxplus z_1^2 + \cdots + z_n^2 : U \times \mathbb{C}^n \to \mathbb{C}$ is an important idea in singularity theory, and it is known as *stabilization*, and f and g are called *stably equivalent*. So, this result concerns the behaviour of perverse sheaves of vanishing cycles under stabilization.

We use this result in the proof of [18, Thm. 6.9], where we prove that if (X, s) is an algebraic d-critical locus over $\mathbb{C}$ with an 'orientation', then we may define a natural perverse sheaf $P^\bullet_{X,s}$ on X, such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ then $P^\bullet_{X,s}$ is locally modelled on $\mathcal{PV}^\bullet_{U,f}$. Note that although we have explained our results only for $\mathbb{C}$-schemes and perverse sheaves upon them, the proofs are quite general and work in several contexts.

These results have exciting applications in the categorification of Donaldson–Thomas theory on Calabi–Yau 3-folds, and in defining a new kind of 'Fukaya category' of complex Lagrangians in complex symplectic manifold, as explained in [18, Cor. 6.10, 6.11 & 6.12]:

Corollary *Let* $(\boldsymbol{X},\omega)$ *be a* -1*-shifted symplectic derived scheme over* $\mathbb{C}$ *in the sense of Pantev et al.* [142], *and* $X=t_0(\boldsymbol{X})$ *the associated classical* $\mathbb{C}$*-scheme. Suppose we are given a square root* $\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}$ *for* $\det(\mathbb{L}_{\boldsymbol{X}})|_X$*. Then we may define* $P^\bullet_{\boldsymbol{X},\omega}\in\mathrm{Perv}(X)$*, uniquely up to canonical isomorphism, and isomorphisms* $\Sigma_{\boldsymbol{X},\omega}:P^\bullet_{\boldsymbol{X},\omega}\to\mathbb{D}_X(P^\bullet_{\boldsymbol{X},\omega})$, $\mathrm{T}_{\boldsymbol{X},\omega}:P^\bullet_{\boldsymbol{X},\omega}\to P^\bullet_{\boldsymbol{X},\omega}$*. The same applies for* $\mathscr{D}$*-modules and mixed Hodge modules on* X*, and for* l*-adic perverse sheaves and* $\mathscr{D}$*-modules on* X *if* $\boldsymbol{X}$ *is over* $\mathbb{K}$ *with* $\mathrm{char}\,\mathbb{K}=0$.

Corollary *Let* Y *be a Calabi–Yau* 3*-fold over* $\mathbb{C}$*, and* $\mathcal{M}$ *a classical moduli* $\mathbb{K}$*-scheme of simple coherent sheaves in* $\mathrm{coh}(Y)$ *with natural (symmetric) obstruction theory* $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ *as in Behrend* [5], *Thomas* [167]*. Suppose we are given a square root* $\det(\mathcal{E}^\bullet)^{1/2}$ *for* $\det(\mathcal{E}^\bullet)$*. Then we may define* $P^\bullet_{\mathcal{M}}\in\mathrm{Perv}(\mathcal{M})$*, uniquely up to canonical isomorphism, and isomorphisms* $\Sigma_{\mathcal{M}}:P^\bullet_{\mathcal{M}}\to\mathbb{D}_{\mathcal{M}}(P^\bullet_{\mathcal{M}})$, $\mathrm{T}_{\mathcal{M}}:P^\bullet_{\mathcal{M}}\to P^\bullet_{\mathcal{M}}$*. The same applies for* $\mathscr{D}$*-modules and mixed Hodge modules on* $\mathcal{M}$*, and for* l*-adic perverse sheaves and* $\mathscr{D}$*-modules on* $\mathcal{M}$ *if* $Y,\mathcal{M}$ *are over* $\mathbb{K}$ *with* $\mathrm{char}\,\mathbb{K}=0$.

Corollary *Let* (S,ω) *be a complex symplectic manifold and* L,M *complex Lagrangian submanifolds in* S*, and write* $X=L\cap M$*, as a complex analytic subspace of* S*. Suppose we are given square roots* $K_L^{1/2},K_M^{1/2}$ *for* K_L,K_M*. Then we may define* $P^\bullet_{L,M}\in\mathrm{Perv}(X)$*, uniquely up to canonical isomorphism, and isomorphisms* $\Sigma_{L,M}:P^\bullet_{L,M}\to\mathbb{D}_X(P^\bullet_{L,M})$, $\mathrm{T}_{L,M}:P^\bullet_{L,M}\to P^\bullet_{L,M}$*. The same applies for* $\mathscr{D}$*-modules and mixed Hodge modules on* X*.*

The above is relevant to the *categorification* of Donaldson–Thomas theory. As in [5, §1.2], the perverse sheaf $P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}$ constructed on the Donaldson–Thomas moduli space $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ of stable sheaves has pointwise Euler characteristic $\chi\bigl(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\bigr)=\nu$. This implies that when A is a field, say $A=\mathbb{Q}$, the (compactly-supported) hypercohomologies $\mathbb{H}^*\bigl(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\bigr),\mathbb{H}^*_{\mathrm{cs}}\bigl(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\bigr)$ satisfy

$$\sum_{k\in\mathbb{Z}}(-1)^k\dim\mathbb{H}^k\bigl(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\bigr)=\sum_{k\in\mathbb{Z}}(-1)^k\dim\mathbb{H}^k_{\mathrm{cs}}\bigl(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\bigr)=\chi\bigl(\mathcal{M}^\alpha_{\mathrm{st}}(\tau),\nu\bigr)=DT^\alpha(\tau),\qquad(0.0.1)$$

where $\mathbb{H}^k\bigl(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\bigr)\cong\mathbb{H}^{-k}_{\mathrm{cs}}\bigl(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\bigr)^*$ by Verdier duality. That is, we have produced a natural graded $\mathbb{Q}$-vector space $\mathbb{H}^*\bigl(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\bigr)$, thought of as some kind of generalized cohomology of $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$, whose graded dimension is $DT^\alpha(\tau)$. This gives a new interpretation of the Donaldson–Thomas invariant $DT^\alpha(\tau)$. In fact, as discussed at length in [166, §3], the first natural "refinement" or "quantization" direction of a Donaldson–Thomas invariant $DT^\alpha(\tau)\in\mathbb{Z}$ is not the Poincaré polynomial of this cohomology, but its weight polynomial $w\bigl(\mathbb{H}^*(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}),t\bigr)\in\mathbb{Z}\bigl[t^{\pm\frac{1}{2}}\bigr]$, defined using the mixed Hodge structure on the cohomology of the mixed Hodge module version of $P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}$, which exists assuming that $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ is projective. This is related to work by other authors. The idea of categorifying Donaldson–Thomas invariants using perverse sheaves or $\mathscr{D}$-modules is probably first due to Behrend [5], and for Hilbert schemes $\mathrm{Hilb}^n(Y)$ of a Calabi–Yau 3-fold Y is discussed by Dimca and Szendrői [35] and Behrend, Bryan and Szendrői [9, §3.4], using mixed Hodge modules. Our result answers a question of Joyce and Song [85, Question 5.7(a)].

As in [85,102] representations of *quivers with superpotentials* (Q,W) give 3-Calabi–Yau triangulated categories, and one can define Donaldson–Thomas type invariants $DT^\alpha_{Q,W}(\tau)$ 'counting' such representations, which are simple algebraic 'toy models' for Donaldson–Thomas invariants of Calabi–Yau 3-folds. Kontsevich and Soibelman [104] explain how to categorify these quiver invariants $DT^\alpha_{Q,W}(\tau)$, and define an associative multiplication on the categorification to make a *Cohomological Hall Algebra*. Our work [18] was strongly motivated by the aim of extending [104] to define Cohomological Hall Algebras for Calabi–Yau 3-folds. We point out also that the square root $\det(\mathcal{E}^\bullet)^{1/2}$ corresponds roughly to *orientation data* in the work of Kontsevich and Soibelman [102, §5], [104].

Finally, we cite Kiem and Li [99] who have recently proved an analogue of Corollary 4.4.3 by complex analytic methods, beginning from Joyce and Song's result [85, Th. 5.4], proved using gauge theory, that $\mathcal{M}_{\rm st}^\alpha(\tau)$ is locally isomorphic to $\mathrm{Crit}(f)$ as a complex analytic space, for V a complex manifold and $f:V\to\mathbb{C}$ holomorphic.

Motivic vanishing cycles and critical loci

In §1.4 we introduce some background material on motives, which are used in §5 to present results obtain in [25]. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, U a smooth $\mathbb{K}$-scheme, $f:U\to\mathbb{A}^1$ a regular function, and $U_0=f^{-1}(0)$, $X=\mathrm{Crit}(f)$ as closed $\mathbb{K}$-subschemes of U. Following Denef and Loeser [31, 32] and Looijenga [120], in [25] we define the *motivic nearby cycle* $MF_{U,f}^{\rm mot}$ in the monodromic Grothendieck group $K_0^{\hat\mu}(U_0)$ of $\hat\mu$-equivariant motives on U_0, and the *motivic vanishing cycle* $MF_{U,f}^{\mathrm{mot},\phi}$ in the ring $\mathcal{M}_X^{\hat\mu}=K_0^{\hat\mu}(X)[\mathbb{L}^{-1}]$ with Tate motive $\mathbb{L}=[\mathbb{A}^1]$ inverted. Here $MF_{U,f}^{\rm mot}$ is the motivic analogue of the constructible complex of nearby cycles $\psi_f(\mathbb{Q}_U)\in\mathrm{Perv}(U_0)$ in [18], and $MF_{U,f}^{\mathrm{mot},\phi}$ the motivic analogue of the perverse sheaf of vanishing cycles $\mathcal{PV}_{U,f}^\bullet=\phi_f(\mathbb{Q}_U[\dim U-1])\in\mathrm{Perv}(X)$ in [18] (at least when $X\subseteq U_0$). The fibre $MF_{U,f}^{\rm mot}(x)$ of $MF_{U,f}^{\rm mot}$ at $x\in U_0$ is the *motivic Milnor fibre* of f at x from [31, 32, 120], the algebraic analogue of the Milnor fibre $MF_f(x)$ at x of a holomorphic function $f:U\to\mathbb{C}$ on a complex manifold U.

In [25] we study $MF_{U,f}^{\rm mot}$. In [25, Thm. 4.2], we show that $MF_{U,f}^{\mathrm{mot},\phi}\in\mathcal{M}_X^{\hat\mu}$ depends only on the third-order thickenings $U^{(3)},f^{(3)}$ of U,f at X, where $\mathcal{O}_{U^{(3)}}=\mathcal{O}_U/I_X^3$, for $I_X\subseteq\mathcal{O}_U$ the ideal of functions $U\to\mathbb{A}^1$ vanishing on X, and $f^{(3)}=f|_{U^{(3)}}$. We also show by example that $U^{(2)},f^{(2)}$ do not determine $MF_{U,f}^{\mathrm{mot},\phi}$.

Then, in [25, §3], we define a natural motive $\Upsilon(P)\in\mathcal{M}_X^{\hat\mu}$ for each principal $\mathbb{Z}_2$-bundle $P\to X$. As in Denef and Loeser [32] and Looijenga [120], there is a (non-obvious) commutative, associative multiplication $\odot$ on $\mathcal{M}_X^{\hat\mu}$ which appears in the motivic Thom–Sebastiani Theorem [31,32,120]. Then we define a new ring of motives $\bar{\mathcal{M}}_Y^{\hat\mu}$ for each $\mathbb{K}$-scheme Y to be the quotient of $\bigl(\mathcal{M}_Y^{\hat\mu},\odot\bigr)$ by the ideal generated by pushforwards $\phi_*\bigl(\Upsilon(P\otimes_{\mathbb{Z}_2}Q)-\Upsilon(P)\odot\Upsilon(Q)\bigr)$ for all $\mathbb{K}$-scheme morphisms $\phi:X\to Y$ and principal $\mathbb{Z}_2$-bundles $P,Q\to X$, and then $\Upsilon(P\otimes_{\mathbb{Z}_2}Q)=\Upsilon(P)\odot\Upsilon(Q)$ holds in $\bar{\mathcal{M}}_X^{\hat\mu}$. Note that Kontsevich and Soibelman in [102, §4.5] defined the motivic rings $\bar{\mathcal{M}}^\mu(X)$ in which their motivic Donaldson–Thomas invariants take values, imposing a relation which implies that the motivic vanishing cycle $MF_{E,q}^{\mathrm{mot},\phi}$ of a nondegenerate quadratic form q on a vector bundle $E\to U$ depends only on the triple $\bigl(\mathrm{rank}\,E,\Lambda^{\rm top}E,\det q\bigr)$. This implies our relation $\Upsilon(P\otimes_{\mathbb{Z}_2}Q)=\Upsilon(P)\odot\Upsilon(Q)$. So Kontsevich and Soibelman's ring $\bar{\mathcal{M}}^\mu(X)$ is a quotient of $\bar{\mathcal{M}}_X^{\hat\mu}$.

In [25, Thm. 4.4] we prove that if U,V are smooth $\mathbb{K}$-schemes, $f:U\to\mathbb{A}^1$, $g:V\to\mathbb{A}^1$ are regular, $X=\mathrm{Crit}(f)$, $Y=\mathrm{Crit}(g)$, and $\Phi:U\to V$ is an embedding with $f=g\circ\Phi$ and $\Phi|_X:X\to Y$ an isomorphism, then $\Phi|_X^*\bigl(MF_{V,g}^{\mathrm{mot},\phi}\bigr)=MF_{U,f}^{\mathrm{mot},\phi}\odot\Upsilon(P_\Phi)$ in $\bar{\mathcal{M}}_X^{\hat\mu}$, for $P_\Phi\to X$ a principal $\mathbb{Z}_2$-bundle parametrizing orientations of the nondegenerate quadratic form $\mathrm{Hess}\,g$ on $N_{UV}|_X$, with $N_{UV}\to U$ the normal bundle of $\Phi(U)$ in V. The analogous result [18, Thm. 5.4] for perverse sheaves of vanishing cycles $\mathcal{PV}_{U,f}^\bullet$, as above, says that $\Phi|_X^*(\mathcal{PV}_{V,g}^\bullet)\cong\mathcal{PV}_{U,f}^\bullet\otimes_{\mathbb{Z}_2}P_\Phi$.

For these U,V,f,g,Φ, [87, Prop. 2.23] shows that étale locally on V we have equivalences $V\sim U\times\mathbb{A}^n$ identifying $g\sim f\boxplus z_1^2+\cdots+z_n^2$ and $\Phi\sim\mathrm{id}_U\times 0$. So if we could work étale locally, we would have

$$\begin{aligned}\Phi|_X^*\bigl(MF_{V,g}^{\mathrm{mot},\phi}\bigr)&\sim(\mathrm{id}_X\times 0)^*\bigl(MF_{U\times\mathbb{A}^n,f\boxplus z_1^2+\cdots+z_n^2}^{\mathrm{mot},\phi}\bigr)\\&=MF_{U,f}^{\mathrm{mot},\phi}\boxdot MF_{\mathbb{A}^n,z_1^2+\cdots+z_n^2}^{\mathrm{mot},\phi}=MF_{U,f}^{\mathrm{mot},\phi}\boxdot 1_{\{0\}}=MF_{U,f}^{\mathrm{mot},\phi},\end{aligned}$$

using the motivic Thom–Sebastiani theorem in the second step. However, for motives we must work Zariski locally, so we need a more complicated proof involving the (étale locally trivial) correction factor $\Upsilon(P_\Phi)$. In singularity theory, passing from f to $f \boxplus z_1^2 + \cdots + z_n^2$ is known as *stabilization*, our result is about the behaviour of motivic vanishing cycles under stabilization.

Finally, we use that to prove [25, Thm. 5.10], which roughly says that if (X, s) is an algebraic d-critical locus over $\mathbb{K}$ with an 'orientation', then we may define a natural motive $MF_{X,s}$ in $\bar{\mathcal{M}}_X^{\hat{\mu}}$, such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$ then $MF_{X,s}$ is locally modelled on $MF_{U,f}^{\mathrm{mot},\phi} \odot \Upsilon(P)$, where $P \to X$ is a principal $\mathbb{Z}_2$-bundle relating the 'orientations' on (X, s) and $\mathrm{Crit}(f)$.

The following are [25, Cor. 5.12, 5.13 & 5.14]:

Corollary *Let $(\boldsymbol{X}, \omega)$ be a -1-shifted symplectic derived scheme over $\mathbb{K}$ in the sense of Pantev et al.* [142], *and $X = t_0(\boldsymbol{X})$ the associated classical $\mathbb{K}$-scheme, assumed of finite type. Suppose we are given a square root $\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}$ for $\det(\mathbb{L}_{\boldsymbol{X}})|_X$. Then we may define a natural motive $MF_{\boldsymbol{X},\omega} \in \bar{\mathcal{M}}_X^{\hat{\mu}}$.*

Corollary *Suppose Y is a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ is a finite type moduli $\mathbb{K}$-scheme of simple coherent sheaves in $\mathrm{coh}(Y)$, with obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ as in Thomas* [167]. *Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$. Then we may define a natural motive $MF_{\mathcal{M}} \in \bar{\mathcal{M}}_{\mathcal{M}}^{\hat{\mu}}$.*

Kontsevich and Soibelman define a motive over $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$, by associating a formal power series to each (not necessarily closed) point, and taking its motivic Milnor fibre. The question of how these formal power series and motivic Milnor fibres vary in families over the base $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$ is not really addressed in [102]. Our result answers this question, showing that Zariski locally in $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$ we can take the formal power series and motivic Milnor fibres to all come from a regular function $f : U \to \mathbb{A}^1$ on a smooth $\mathbb{K}$-scheme U. As before, the square root $\det(\mathcal{E}^\bullet)^{1/2}$ required in Corollary 5.3.3 corresponds roughly to *orientation data* in Kontsevich and Soibelman [102, §5], [104].

Corollary *Let (S, ω) be an algebraic symplectic manifold and L, M finite type algebraic Lagrangian submanifolds in S, and write $X = L \cap M$, as a subscheme of S. Suppose we are given square roots $K_L^{1/2}, K_M^{1/2}$ for K_L, K_M. Then we may define a natural motive $MF_{L,M} \in \bar{\mathcal{M}}_X^{\hat{\mu}}$.*

Generalization to symplectic derived stacks

In §6 we describe results obtained in [13], where we extend the results of [19], [18], [25] from $\mathbb{K}$-schemes to Artin $\mathbb{K}$-stacks, using the notion of d-critical stack from [87]. Here is [13, Thm. 2.10]:

Theorem *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$ a k-shifted symplectic derived Artin $\mathbb{K}$-stack as in* [142] *for $k < 0$, and $p \in \boldsymbol{X}(\mathbb{K})$ be a $\mathbb{K}$-point of $\boldsymbol{X}$. Then we can construct the following data:*

- **(a)** *Affine derived $\mathbb{K}$-schemes $\boldsymbol{U} = \mathbf{Spec}\, A$, $\boldsymbol{V} = \mathbf{Spec}\, B$, where A, B are commutative differential graded $\mathbb{K}$-algebras (cdgas) in degrees $\leqslant 0$, of an explicit 'standard form' defined in* §3.
- **(b)** *A morphism of derived stacks $\boldsymbol{\varphi} : \boldsymbol{U} = \mathbf{Spec}\, A \to \boldsymbol{X}$ which is smooth of the minimal possible relative dimension $n = \dim H^1(\mathbb{L}_{\boldsymbol{X}}|_p)$.*
- **(c)** *An inclusion $\iota : B \hookrightarrow A$ of B as a dg-subalgebra of A, so that $\boldsymbol{i} = \mathbf{Spec}\, \iota : \boldsymbol{U} \to \boldsymbol{V}$ is a morphism of derived $\mathbb{K}$-schemes. On classical schemes, $i = t_0(\boldsymbol{i}) : U = t_0(\boldsymbol{U}) \to V = t_0(\boldsymbol{V})$ is an isomorphism.*
- **(d)** *A $\mathbb{K}$-point $\tilde{p} \in \mathrm{Spec}\, H^0(A)$ with $\boldsymbol{\varphi}(\tilde{p}) = p$, such that the 'standard form' cdgas A, B have the minimal possible numbers of generators $\dim H^j(\mathbb{L}_{\boldsymbol{U}}|_{\tilde{p}}), \dim H^j(\mathbb{L}_{\boldsymbol{V}}|_{\boldsymbol{i}(\tilde{p})})$ in each degree $j = 0, -1, \ldots, k, k-1$.*

(e) *An equivalence of relative (co)tangent complexes* $\mathbb{L}_{\boldsymbol{U}/\boldsymbol{V}} \simeq \mathbb{T}_{\boldsymbol{U}/\boldsymbol{X}}[1-k]$. *Hence* $\mathbb{L}_{\boldsymbol{U}/\boldsymbol{V}}$ *is a vector bundle of rank* n *in degree* $k-1$.

(f) *A* k*-shifted symplectic structure* $\omega_B = (\omega_B^0, 0, \ldots)$ *on* $\boldsymbol{V} = \mathbf{Spec}\, B$ *which is in 'Darboux form' in the sense of* [19, §5] *and* §3, *with* $\boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}}) \sim \boldsymbol{i}^*(\omega_B)$ *in* k*-shifted closed* 2*-forms on* $\boldsymbol{U}$.

For example, if $k = -2d-1$ *for* $d = 0, 1, \ldots$ *then the 'standard form' and 'Darboux form' conditions above mean the following. The degree* 0 *part* B^0 *of* B *is a smooth* $\mathbb{K}$*-algebra of dimension* m_0, *and we are given* $x_1^0, \ldots, x_{m_0}^0 \in B^0$ *such that* $(x_1^0, \ldots, x_{m_0}^0)$ *are étale coordinates on all of* $V(0) = \operatorname{Spec} B^0$. *As a graded commutative algebra,* B *is freely generated over* B^0 *by variables*

$$\begin{array}{ll} x_1^{-i}, \ldots, x_{m_i}^{-i} & \text{in degree } -i \text{ for } i = 1, \ldots, d, \\ y_1^{i-2d-1}, \ldots, y_{m_i}^{i-2d-1} & \text{in degree } i-2d-1 \text{ for } i = 0, 1, \ldots, d. \end{array}$$

We have $\omega_B^0 = \sum_{i=0}^{d} \sum_{j=1}^{m_i} \mathrm{d}_{dR} y_j^{i-2d-1} \, \mathrm{d}_{dR} x_j^{-i}$ *in* $(\Lambda^2 \Omega_B^1)^{-2d-1}$. *The differential* d *on the cdga* B *is* $\mathrm{d}b = \{H, b\}$ *for* $b \in B$, *where* $\{\,,\,\} : B \times B \to B$ *is the Poisson bracket defined using the inverse of* ω_B^0, *and* $H \in B^{-2d}$ *is a Hamiltonian function satisfying the classical master equation* $\{H, H\} = 0$. *Also* $B \subset A$, *and* A *is freely generated as a graded commutative algebra over* B *by additional variables* $w_1^{-2d-2}, \ldots, w_n^{-2d-2}$ *in degree* $-2d-2$.

Theorem above says that given a k-shifted derived Artin stack $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$ for $k < 0$, near each $p \in \boldsymbol{X}$ we can find a smooth atlas $\boldsymbol{\varphi} : \boldsymbol{U} \to \boldsymbol{X}$ with $\boldsymbol{U} = \mathbf{Spec}\, A$ an affine derived scheme, such that $(\boldsymbol{U}, \boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}}))$ is in a standard 'Darboux form'. Although $(\boldsymbol{U}, \boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}}))$ is not k-shifted symplectic, as $\boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}})$ is not nondegenerate, we can build from $(\boldsymbol{U}, \boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}}))$ in a natural way a 'Darboux form' k-shifted symplectic derived scheme$(\boldsymbol{V}, \omega_B)$, which is equivalent to $(\boldsymbol{U}, \boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}}))$ except in degree $k-1$.

The following are [13, Cor. 2.11 & Cor. 2.12]:

Corollary *Let* $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$ *be a* -1*-shifted symplectic derived Artin* $\mathbb{K}$*-stack, and* $X = t_0(\boldsymbol{X})$ *the corresponding classical Artin* $\mathbb{K}$*-stack. Then for each* $p \in X$ *there exist a smooth* $\mathbb{K}$*-scheme* U *with dimension* $\dim H^0(\mathbb{L}_X|_p)$, *a point* $t \in U$, *a regular function* $f : U \to \mathbb{A}^1$ *with* $\mathrm{d}_{dR} f|_t = 0$, *so that* $T := \operatorname{Crit}(f) \subseteq U$ *is a closed* $\mathbb{K}$*-subscheme with* $t \in T$, *and a morphism* $\varphi : T \to X$ *which is smooth of relative dimension* $\dim H^1(\mathbb{L}_X|_p)$, *with* $\varphi(t) = p$. *We may take* $f|_{T^{\mathrm{red}}} = 0$.

Thus, the underlying classical stack X of a -1-shifted symplectic derived stack $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$ admits an atlas consisting of critical loci of regular functions on smooth schemes.

Now let Y be a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or complexes $F^\bullet$ in $D^b \operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$. Then $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$, for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli stack. The (open) condition $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$ is needed to make $\boldsymbol{\mathcal{M}}$ 1-geometric and 1-truncated (that is, a derived Artin stack, in our terminology); without it, $\mathcal{M}, \boldsymbol{\mathcal{M}}$ would be a higher derived stack. Pantev et al. [142, §2.1] prove $\boldsymbol{\mathcal{M}}$ has a -1-shifted symplectic structure $\omega_{\boldsymbol{\mathcal{M}}}$. Applying the above Corollary and using $H^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}}|_{[F]}) \cong \operatorname{Ext}^{1-i}(F, F)^*$ yields a new result on classical 3-Calabi–Yau moduli stacks, the statement of which involves no derived geometry:

Corollary *Suppose* Y *is a Calabi–Yau* 3*-fold over* $\mathbb{K}$, *and* $\mathcal{M}$ *a classical moduli* $\mathbb{K}$*-stack of coherent sheaves* F, *or more generally of complexes* $F^\bullet$ *in* $D^b \operatorname{coh}(Y)$ *with* $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$. *Then for each* $[F] \in \mathcal{M}$, *there exist a smooth* $\mathbb{K}$*-scheme* U *with* $\dim U = \dim \operatorname{Ext}^1(F, F)$, *a point* $u \in U$, *a regular function* $f : U \to \mathbb{A}^1$ *with* $\mathrm{d}_{dR} f|_u = 0$, *and a morphism* $\varphi : \operatorname{Crit}(f) \to \mathcal{M}$ *which is smooth of relative dimension* $\dim \operatorname{Hom}(F, F)$, *with* $\varphi(u) = [F]$.

This is an analogue of [19, Cor. 5.19]. When $\mathbb{K} = \mathbb{C}$, a related result for coherent sheaves only, with U a complex manifold and f a holomorphic function, was proved by Joyce and Song [85, Th. 5.5] using gauge theory and transcendental complex methods.

Here is [13, Thm. 3.18], a stack version of [19, Thm. 6.6]:

Theorem *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$ a -1-shifted symplectic derived Artin $\mathbb{K}$-stack, and $X = t_0(\boldsymbol{X})$ the corresponding classical Artin $\mathbb{K}$-stack. Then there exists a unique d-critical structure $s \in H^0(\mathcal{S}_X^0)$ on X, making (X, s) into a d-critical stack, with the following properties:*

(a) *Let U, $f : U \to \mathbb{A}^1$, $T = \mathrm{Crit}(f)$ and $\varphi : T \to X$ be as in Corollary 6.1.5, with $f|_{T^{\mathrm{red}}} = 0$. There is a unique $s_T \in H^0(\mathcal{S}_T^0)$ on T with $\iota_{T,U}(s_T) = i^{-1}(f) + I_{T,U}^2$, and (T, s_T) is an algebraic d-critical locus. Then $s(T, \varphi) = s_T$ in $H^0(\mathcal{S}_T^0)$.*

(b) *The canonical bundle $K_{X,s}$ of (X, s) from Theorem 2.2.6 is naturally isomorphic to the restriction $\det(\mathbb{L}_{\boldsymbol{X}})|_{X^{\mathrm{red}}}$ to $X^{\mathrm{red}} \subseteq X \subseteq \boldsymbol{X}$ of the determinant line bundle $\det(\mathbb{L}_{\boldsymbol{X}})$ of the cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ of $\boldsymbol{X}$.*

We can think about it as defining a *truncation functor*

$$\begin{aligned} F : \big\{ &\infty\text{-category of } -1\text{-shifted symplectic derived Artin } \mathbb{K}\text{-stacks } (\boldsymbol{X}, \omega_{\boldsymbol{X}}) \big\} \\ &\longrightarrow \big\{ 2\text{-category of d-critical stacks } (X, s) \text{ over } \mathbb{K} \big\}. \end{aligned}$$

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-stack of coherent sheaves in $\mathrm{coh}(Y)$, or complexes of coherent sheaves in $D^b \mathrm{coh}(Y)$. There is a natural obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ on $\mathcal{M}$, where $\mathcal{E}^\bullet \in D_{\mathrm{qcoh}}(\mathcal{M})$ is perfect in the interval $[-2, 1]$, and $h^i(\mathcal{E}^\bullet)|_F \cong \mathrm{Ext}^{1-i}(F, F)^*$ for each $\mathbb{K}$-point $F \in \mathcal{M}$, regarding F as an object in $\mathrm{coh}(Y)$ or $D^b \mathrm{coh}(Y)$. Now in derived algebraic geometry $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli $\mathbb{K}$-stack, and $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is $\mathbb{L}_{t_0} : \mathbb{L}_{\boldsymbol{\mathcal{M}}}|_{\mathcal{M}} \to \mathbb{L}_{\mathcal{M}}$. Pantev et al. [142, §2.1] prove $\boldsymbol{\mathcal{M}}$ has a -1-shifted symplectic structure ω. Thus we obtain [13, Cor. 3.19]:

Corollary *Suppose Y is a Calabi–Yau 3-fold over $\mathbb{K}$ of characteristic zero, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-stack of coherent sheaves F in $\mathrm{coh}(Y)$, or complexes of coherent sheaves $F^\bullet$ in $D^b \mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, with obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Then $\mathcal{M}$ extends naturally to an algebraic d-critical locus $(\mathcal{M}, s)$. The canonical bundle $K_{\mathcal{M},s}$ from Theorem 2.2.6 is naturally isomorphic to $\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\mathrm{red}}}$.*

Here is [13, Cor. 4.13], the stack version of [18, Cor. 6.10]:

Corollary *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $(\boldsymbol{X}, \omega)$ a -1-shifted symplectic derived Artin $\mathbb{K}$-stack, and $X = t_0(\boldsymbol{X})$ the associated classical Artin $\mathbb{K}$-stack. Suppose we are given a square root $\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}$. Then working in l-adic perverse sheaves on stacks [13, §4] we may define a perverse sheaf $\check{P}^\bullet_{\boldsymbol{X},\omega}$ on X uniquely up to canonical isomorphism, and Verdier duality and monodromy isomorphisms $\check{\Sigma}_{\boldsymbol{X},\omega} : \check{P}^\bullet_{\boldsymbol{X},\omega} \to \mathbb{D}_X(\check{P}^\bullet_{\boldsymbol{X},\omega})$ and $\check{\mathrm{T}}_{\boldsymbol{X},\omega} : \check{P}^\bullet_{\boldsymbol{X},\omega} \to \check{P}^\bullet_{\boldsymbol{X},\omega}$. These are characterized by the fact that given a diagram $\boldsymbol{U} = \mathbf{Crit}(f : U \to \mathbb{A}^1) \xleftarrow{\ \boldsymbol{i}\ } \boldsymbol{V} \xrightarrow{\ \boldsymbol{\varphi}\ } \boldsymbol{X}$ such that U is a smooth $\mathbb{K}$-scheme, $\boldsymbol{\varphi}$ smooth of dimension n, $\mathbb{L}_{\boldsymbol{V}/\boldsymbol{U}} \simeq \mathbb{T}_{\boldsymbol{V}/\boldsymbol{X}}[2]$, $\boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}}) \sim \boldsymbol{i}^*(\omega_{\boldsymbol{U}})$ for $\omega_{\boldsymbol{U}}$ the natural -1-shifted symplectic structure on $\boldsymbol{U} = \mathbf{Crit}(f : U \to \mathbb{A}^1)$, and $\varphi^*(\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}) \cong i^*(K_U) \otimes \Lambda^n \mathbb{T}_{\boldsymbol{V}/\boldsymbol{X}}$, then $\varphi^*(\check{P}^\bullet_{\boldsymbol{X},\omega})[n]$, $\varphi^*(\check{\Sigma}^\bullet_{\boldsymbol{X},\omega})[n]$, $\varphi^*(\check{\mathrm{T}}^\bullet_{\boldsymbol{X},\omega})[n]$ are canonically isomorphic to $i^*(\mathcal{PV}_{U,f})$, $i^*(\sigma_{U,f})$, $i^*(\tau_{U,f})$, for $\mathcal{PV}_{U,f}, \sigma_{U,f}, \tau_{U,f}$ as in [13] . The same applies in the other theories of perverse sheaves and $\mathscr{D}$-modules on stacks.*

Here is [13, Cor. 4.14], the stack version of [18, Cor. 6.11]:

Corollary *Let Y be a Calabi–Yau 3-fold over an algebraically closed field $\mathbb{K}$ of characteristic zero, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-stack of coherent sheaves F in $\mathrm{coh}(Y)$, or of complexes $F^\bullet$ in $D^b \mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, with obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$. Then working in l-adic perverse sheaves on stacks [13, §4], we may define a natural perverse sheaf $\check{P}^\bullet_{\mathcal{M}} \in \mathrm{Perv}(\mathcal{M})$, and Verdier duality and monodromy isomorphisms $\check{\Sigma}_{\mathcal{M}} : \check{P}^\bullet_{\mathcal{M}} \to \mathbb{D}_{\mathcal{M}}(\check{P}^\bullet_{\mathcal{M}})$ and $\check{\mathrm{T}}_{\mathcal{M}} : \check{P}^\bullet_{\mathcal{M}} \to \check{P}^\bullet_{\mathcal{M}}$. The pointwise Euler characteristic of $\check{P}^\bullet_{\mathcal{M}}$ is the Behrend function $\nu_{\mathcal{M}}$ of $\mathcal{M}$ from Joyce and Song* [85, §4], *so that $\check{P}^\bullet_{\mathcal{M}}$ is in effect a categorification of the Donaldson–Thomas theory of $\mathcal{M}$. The same applies in the other theories of perverse sheaves and $\mathscr{D}$-modules on stacks.*

Here is [13, Cor. 5.16], the stack version of [25, Cor. 5.12]:

Corollary *Let $(\boldsymbol{X}, \omega)$ be a -1-shifted symplectic derived Artin $\mathbb{K}$-stack in the sense of Pantev et al.* [142], *and $X = t_0(\boldsymbol{X})$ the associated classical Artin $\mathbb{K}$-stack, assumed of finite type and locally a global quotient. Suppose we are given a square root $\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}$ for $\det(\mathbb{L}_{\boldsymbol{X}})|_X$. Then we may define a natural motive $MF_{\boldsymbol{X},\omega} \in \overline{\mathcal{M}}^{\mathrm{st},\hat{\mu}}_X$, which is characterized by the fact that given a diagram*

$$\boldsymbol{U} = \mathbf{Crit}(f : U \to \mathbb{A}^1) \xleftarrow{\;\;\boldsymbol{i}\;\;} \boldsymbol{V} \xrightarrow{\;\;\boldsymbol{\varphi}\;\;} \boldsymbol{X}$$

such that U is a smooth $\mathbb{K}$-scheme, $\boldsymbol{\varphi}$ is smooth of dimension n, $\mathbb{L}_{\boldsymbol{V}/\boldsymbol{U}} \simeq \mathbb{T}_{\boldsymbol{V}/\boldsymbol{X}}[2]$, $\boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}}) \sim \boldsymbol{i}^*(\omega_{\boldsymbol{U}})$ for $\omega_{\boldsymbol{U}}$ the natural -1-shifted symplectic structure on $\boldsymbol{U} = \mathbf{Crit}(f : U \to \mathbb{A}^1)$, and $\varphi^*(\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}) \cong i^*(K_U) \otimes \Lambda^n \mathbb{T}_{\boldsymbol{V}/\boldsymbol{X}}$, then $\varphi^*(MF_{\boldsymbol{X},\omega}) = \mathbb{L}^{n/2} \odot i^*(MF^{\mathrm{mot},\phi}_{U,f})$ in $\overline{\mathcal{M}}^{\mathrm{st},\hat{\mu}}_V$.

Here is [13, Cor. 5.17], the stack version of [25, Cor. 5.13]:

Corollary *Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ a finite type classical moduli $\mathbb{K}$-stack of coherent sheaves in $\mathrm{coh}(Y)$, with natural obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$. Then we may define a natural motive $MF_{\mathcal{M}} \in \overline{\mathcal{M}}^{\mathrm{st},\hat{\mu}}_{\mathcal{M}}$.*

This is relevant to Kontsevich and Soibelman's theory of *motivic Donaldson–Thomas invariants* [102]. Again, our square root $\det(\mathcal{E}^\bullet)^{1/2}$ roughly coincides with their *orientation data* [102, §5]. In [102, §6.2], given a finite type moduli stack $\mathcal{M}$ of coherent sheaves on a Calabi–Yau 3-fold Y with orientation data, they define a motive $\int_{\mathcal{M}} 1$ in a ring D^μ isomorphic to our $\overline{\mathcal{M}}^{\mathrm{st},\hat{\mu}}_{\mathbb{K}}$. We expect this should agree with $\pi_*(MF_{\mathcal{M}})$ in our notation, with $\pi : \mathcal{M} \to \operatorname{Spec} \mathbb{K}$ the projection. This $\int_{\mathcal{M}} 1$ is roughly the motivic Donaldson–Thomas invariant of $\mathcal{M}$. Their construction involves expressing $\mathcal{M}$ near each point in terms of the critical locus of a formal power series. Kontsevich and Soibelman's constructions were partly conjectural, and our results may fill some gaps in their theory.

Donaldson–Thomas theory

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A *Calabi–Yau* 3*-fold* is a smooth projective 3-fold X over $\mathbb{C}$ or $\mathbb{K}$, with trivial canonical bundle K_X and $H^1(\mathcal{O}_X) = 0$. Fix a very ample line bundle $\mathcal{O}_X(1)$ on X, and let τ be Gieseker stability on the abelian category of coherent sheaves $\mathrm{coh}(X)$ on X with respect to $\mathcal{O}_X(1)$. If E is a coherent sheaf on X then the class $[E] \in K^{\mathrm{num}}(\mathrm{coh}(X))$ is in effect the Chern character $\mathrm{ch}(E)$ of E in the Chow ring $A^*(X)_{\mathbb{Q}}$ as in [46]. For a class α in the numerical Grothendieck group $K^{\mathrm{num}}(\mathrm{coh}(X))$, write $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau), \mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ for the coarse moduli schemes of τ-(semi)stable sheaves E with class $[E] = \alpha$. Then $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau)$ is a projective $\mathbb{C}$ or $\mathbb{K}$-scheme whose points correspond to S-equivalence classes of τ-semistable sheaves, and $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ is an open subscheme of $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau)$ whose points correspond to isomorphism classes of τ-stable sheaves. Write $\mathfrak{M}$ for the moduli stack of coherent sheaves E on X. It is an Artin $\mathbb{C}$ or

$\mathbb{K}$-stack, locally of finite type and has affine geometric stabilizers. For $\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))$, write $\mathfrak{M}^\alpha$ for the open and closed substack of E with $[E] = \alpha$ in $K^{\mathrm{num}}(\mathrm{coh}(X))$. Write $\mathfrak{M}^\alpha_{\mathrm{ss}}(\tau), \mathfrak{M}^\alpha_{\mathrm{st}}(\tau)$ for the substacks of τ-(semi)stable sheaves E in class $[E] = \alpha$, which are finite type open substacks of $\mathfrak{M}^\alpha$.

In 1998, Thomas [167], following his proposal with Donaldson [36], motivates a *holomorphic Casson invariant* and defines the *Donaldson–Thomas invariants* $DT^\alpha(\tau)$ which are integers 'counting' τ-stable coherent sheaves with Chern character α on a Calabi–Yau 3-fold X over $\mathbb{K}$, where τ denotes Gieseker stability for some ample line bundle on X. Mathematically, and in 'modern' terms, he found that $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ is endowed with a symmetric obstruction theory and defined

$$DT^\alpha(\tau) \quad = \int_{[\mathcal{M}^\alpha_{\mathrm{st}}(\tau)]^{\mathrm{vir}}} 1$$

which is mathematical reflection of the heuristic that views $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ as the critical locus of the *holomorphic Chern-Simons functional* and the shadow of a more deeper 'derived' geometry. A crucial result is that the invariants are unchanged under deformations of the underlying geometry of X. Finally we remark that the conventional definition of Thomas [167] works only for classes α containing no strictly τ-semistable sheaves and this permits to work just with schemes rather than stacks as the stable moduli scheme itself already encodes all the information about the Ext groups, and thus about the tangent-obstruction complex of the moduli functor.

In 2005, Behrend [5] proved a *virtual Gauss–Bonnet theorem* which in particular yields that Donaldson–Thomas type invariants can be written as a weighted Euler characteristic

$$DT^\alpha(\tau) = \chi\big(\mathcal{M}^\alpha_{\mathrm{st}}(\tau), \nu_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big)$$

of the stable moduli scheme $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ by a constructible function $\nu_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}$, as a consequence known in literature as the *Behrend function*. It depends only on the scheme structure of $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$, and it is convenient to think about it as a multiplicity function. An important moral is that it is better to 'count' points in a moduli scheme by the weighted Euler characteristic rather than the unweighted one as it often gives answers unchanged under deformations of the underlying geometry. It is worth to point out that this equation is local, and 'motivic', and makes sense even for non-proper finite type $\mathbb{K}$-schemes. Anyway, using this formula to generalize the classical picture by defining the Donaldson–Thomas invariants as $\chi\big(\mathcal{M}^\alpha_{\mathrm{ss}}(\tau), \nu_{\mathcal{M}^\alpha_{\mathrm{ss}}(\tau)}\big)$ when $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau) \neq \mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ is not a good idea as in the case there are strictly τ-semistable sheaves, the moduli scheme $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau)$ is no more a good model and suggest that schemes are no more 'enough' to extend the theory. The crucial work by Behrend [5] suggests that Donaldson–Thomas invariants can be written as motivic invariants, like those studied by Joyce in [75–80], and so it raises the possibility that one can extend the results of [75–80] to Donaldson–Thomas invariants by including Behrend functions as weights.

Thus, in 2005, Joyce and Song [85] proposed a theory of *generalized Donaldson–Thomas invariants* $\bar{DT}^\alpha(\tau)$. They are rational numbers which 'count' both τ-stable and τ-semistable coherent sheaves with Chern character α on a compact Calabi–Yau 3-fold X over $\mathbb{C}$; strictly τ-semistable sheaves must be counted with complicated rational weights. The $\bar{DT}^\alpha(\tau)$ are defined for all classes α, and are equal to $DT^\alpha(\tau)$ when it is defined. They are unchanged under deformations of X, and transform by a wall-crossing formula under change of stability condition τ. The theory is valid also for compactly supported coherent sheaves on *compactly embeddable* noncompact Calabi–Yau 3-folds in the complex analytic topology. To prove all this they study the local structure of the moduli stack $\mathfrak{M}$ of coherent sheaves on X. They first show that $\mathfrak{M}$ is Zariski locally isomorphic to the moduli stack $\mathfrak{Vect}$ of algebraic vector bundles on X. Then they use *gauge theory* on complex vector bundles and transcendental complex analytic methods to show that an atlas for $\mathfrak{M}$ may be written locally in the complex analytic topology as $\mathrm{Crit}(f)$ for $f : U \to \mathbb{C}$ a holomorphic function on a complex manifold U. They use this to deduce identities on the Behrend function $\nu_{\mathfrak{M}}$ through the Milnor fibre description of Behrend functions. These identities

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{\bar{\chi}([E_1],[E_2])} \nu_{\mathfrak{M}}(E_1)\nu_{\mathfrak{M}}(E_2),$$

$$\int_{\substack{[\lambda]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)):\\ \lambda \Leftrightarrow 0\to E_1\to F\to E_2\to 0}} \nu_{\mathfrak{M}}(F)\,\mathrm{d}\chi \quad - \quad \int_{\substack{[\mu]\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2)):\\ \mu \Leftrightarrow 0\to E_2\to D\to E_1\to 0}} \nu_{\mathfrak{M}}(D)\,\mathrm{d}\chi \quad = \quad (e_{21} - e_{12})\ \ \nu_{\mathfrak{M}}(E_1 \oplus E_2),$$

where $e_{21} = \dim \mathrm{Ext}^1(E_2, E_1)$ and $e_{12} = \dim \mathrm{Ext}^1(E_1, E_2)$ for $E_1, E_2 \in \mathrm{coh}(X)$, are crucial for the whole program of Joyce and Song, which is based on the idea that Behrend's approach should be integrated with Joyce's theory [75–80]. As the proof uses gauge theory and transcendental methods, it works only over $\mathbb{C}$ and forces them to put constraints on the Calabi–Yau 3-fold they can define generalized Donaldson–Thomas invariants for. Finally, in [85, §4.5], when $\mathbb{K} = \mathbb{C}$, the Chern character embeds $K^{\mathrm{num}}(\mathrm{coh}(X))$ in $H^{\mathrm{even}}(X;\mathbb{Q})$, and the Voisin Hodge conjecture result [182] for Calabi–Yau over $\mathbb{C}$ completely characterize its image. They use this to show $K^{\mathrm{num}}(\mathrm{coh}(X))$ is unchanged under deformations of X. This is important for the $\bar{DT}^{\alpha}(\tau)$ with $\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))$ to be invariant under deformations of X even to make sense.

In 2008 and 2010, with two subsequent papers [102, 104], Kontsevich and Soibelman also studied generalizations of Donaldson–Thomas invariants, both in the direction of motivic and categorified Donaldson–Thomas invariants. In [102], they proposed a very general version of the theory, which, very roughly speaking, can be outlined saying that, supposing for the sake of simplicity that $\mathcal{M}^{\alpha}_{\mathrm{st}}(\tau) = \mathcal{M}^{\alpha}_{\mathrm{ss}}(\tau)$, their oversimplified idea is to define *motivic Donaldson–Thomas invariants* $DT^{\alpha}_{\mathrm{mot}} = \Upsilon(\mathcal{M}^{\alpha}_{\mathrm{st}}(\tau), \nu_{\mathrm{mot}})$, where ν_{mot} is a complicated constructible function which we can refer to as the *motivic Behrend function* for a general motivic invariant Υ. Their construction is closely related to Joyce and Song's construction, even if they work in a more general context: they consider derived categories of coherent sheaves, Bridgeland stability conditions, and general motivic invariants, whereas Joyce and Song work with abelian categories of coherent sheaves, Gieseker stability, and the Euler characteristic. However, the price to work in a more general context is that most results depend on conjectures (motivic Behrend function identities, existence of orientation data, absence of poles). In particular, Kontsevich and Soibelman's parallel passages of Joyce and Song's proof of the Behrend function identities [102, §4.4 & §6.3] work over a field $\mathbb{K}$ of characteristic zero, and say that the formal completion $\hat{\mathfrak{M}}_{[E]}$ of $\mathfrak{M}$ at $[E]$ can be written in terms of $\mathrm{Crit}(f)$ for f a formal power series on $\mathrm{Ext}^1(E, E)$, with no convergence criteria. Their analogue [102, Conj. 4], concerns the *motivic Milnor fibre* of the formal power series f. So the Behrend function identities are related to a conjecture of Kontsevich and Soibelman [102, Conj. 4] and its application in [102, §6.3], and could probably be deduced from it. Anyway, Joyce and Song's approach [85] is not wholly algebro-geometric – it uses gauge theory, and transcendental complex analytic geometry methods. Therefore this method will not suffice to prove the parallel conjectures in Kontsevich and Soibelman [102, Conj. 4], which are supposed to hold for general fields $\mathbb{K}$ as well as $\mathbb{C}$, and for general motivic invariants of algebraic $\mathbb{K}$-schemes as well as for the topological Euler characteristic. Recently, in 2012, Le Quy Thuong [112] provided a proof for this conjecture using some deep high technology results from motivic integration.

Following Joyce and Song's proposal, and using the machinery in [19, 87], we provide in §7 an extension of the theory of generalized Donaldson–Thomas invariants in [85] to algebraically closed fields $\mathbb{K}$ of characteristic zero. Our argument provides the algebraic analogue of [85, Thm 5.5], [85, Thm 5.11] and [85, Cor. 5.28] which are enough to extend [85] at least for compact Calabi–Yau 3-folds. Unfortunately, to extend the whole project to complexes of sheaves and to compactly supported sheaves on a noncompact quasi-projective Calabi–Yau 3-fold, we would need other results also from derived algebraic geometry which we do not have at the present. We hope to come back on this point in a future work.

We will show that an atlas for $\mathfrak{M}$ near $[E] \in \mathfrak{M}(\mathbb{K})$ may be written locally in the étale topology as the zero locus $\mathrm{d}f^{-1}(0)$ for a G-invariant regular function f defined on a étale neighborhood of $0 \in U(\mathbb{K})$ in the affine $\mathbb{K}$-space $\mathrm{Ext}^1(E, E)$, where G is a maximal torus of $\mathrm{Aut}(E)$.

Based on this picture, we give an algebraic proof of the Behrend function identities. We point out that our approach is actually valid much more generally for any stack which is locally a global quotient, and we actually do not use any particular properties of coherent sheaves on Calabi–Yau 3-folds. In the past, the author tried a picture in which the moduli stack of coherent sheaves was locally described as a zero locus of an algebraic almost closed 1-form in the sense of [5], which turned out later to be a wrong direction to follow.

Finally, we will study the deformation invariance properties of $\bar{DT}^{\alpha}(\tau)$ under changes of the underlying geometry of X, characterizing a globally constant lattice containing the image through the Chern character of $K^{\mathrm{num}}(\mathrm{coh}(X))$ and in which classes α vary.

The implications are quite exciting and far-reaching. Our algebraic method could lead to the extension of generalized Donaldson–Thomas theory to the derived categorical context. The plan to extend from abelian to derived categories the theory of Joyce and Song [85] starts by reinterpreting the series of papers by Joyce [75–82] in this new general setup. We expect that a well-behaved theory of invariants counting τ-semistable objects in triangulated categories in the style of Joyce's theory exists, and we hope to come back to it in a future work.

Categorifying complex Lagrangian intersections

Let (S, ω) be a complex symplectic manifold, i.e., a complex manifold S endowed with a closed non-degenerate holomorphic 2-form $\omega \in \Omega^2_S$. Denote the complex dimension of S by $2n$. A complex submanifold $M \subset S$ is *Lagrangian* if the restriction of ω to a 2-form on M vanishes and $\dim M = n$. Let $X = L \cap M$ be the intersection as a complex analytic space. Then X carries a canonical symmetric obstruction theory $\varphi : E^\bullet \to \mathbb{L}_X$ in the sense of [6], which can be represented by the complex $E^\bullet \simeq [T^*S|_X \to T^*L|_X \oplus T^*M|_X]$ with $T^*S|_X$ in degree -1 and $T^*L|_X \oplus T^*M|_X$ in degree zero. Hence $\det(E^\bullet) \cong K_L|_X \otimes K_M|_X$. Inspired by [102, §5.2] in primis and then by [18, §2.4] and close to [87, §5.2], we will say that if we are given square roots $K_L^{1/2}, K_M^{1/2}$ for K_L, K_M, then X has *orientation data*. In this case we will also say that L, M are *oriented* Lagrangians, see Remark 8.1.4.

We start from well known facts from complex symplectic geometry. It is well established that every complex symplectic manifold S is locally isomorphic to the cotangent bundle T^*N of a complex manifold N. The fibres of the induced vector bundle structure on S are Lagrangian submanifolds, so complex analytically locally defining on S a foliation by Lagrangian submanifolds, i.e., a *polarization*. The data of a polarization for us will be used as a way to describe locally in the complex analytic topology the Lagrangian intersection X as a critical locus $X \cong \mathrm{Crit}(f : U \to \mathbb{C})$, where f is a holomorphic function on a complex manifold U. One moral of this approach is that every polarization defines a set of data for X which we will call a *chart*, by analogy with critical charts defined by [87, §2.1], and thus the choice of a family of polarizations on a complex symplectic manifold provides a family of charts which will be useful to defining some geometric structures on them and consequently get a global object on X by *gluing*. This will become more clear later. In conclusion, on each chart defined by the choice of a polarization, there is naturally associated a perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$.

Now, a natural problem to investigate is the following. Given analytic open $R_i, R_j \subseteq X$ with isomorphisms $R_i \cong \mathrm{Crit}(f_i)$, $R_j \cong \mathrm{Crit}(f_j)$ for holomorphic $f_i : U_i \to \mathbb{C}$ and $f_j : U_j \to \mathbb{C}$, we have to understand whether the perverse sheaves $\mathcal{P}^\bullet_{R_i} = \mathcal{PV}^\bullet_{U_i,f_i}$ on R_i and $\mathcal{P}^\bullet_{R_j} = \mathcal{PV}^\bullet_{U_j,f_j}$ on R_j are isomorphic over $R_i \cap R_j$, and if so, whether the isomorphism is canonical, for only then can we hope to glue the $\mathcal{P}^\bullet_{R_i}$ for $i \in I$ to make $\mathcal{P}^\bullet_{L,M}$. Studying these issues led to this project.

Our approach was inspired by a work of Behrend and Fantechi [8]. They also investigated Lagrangian intersections in complex symplectic manifolds, but their project is probably more ambitious, as they show the existence of deeply interesting structures carried by the intersection. Unfortunately, their construction has some crucial mistakes. Our project started exactly with the aim to fix them and develop then an independent theory. In the meantime, the author worked with other collaborators on the large project [18,19,25] discussed also above, involving Lagrangian intersections too, but our methods here want to be self contained and independent from that. In particular, the analogue of our theorem below for algebraic symplectic manifolds and algebraic manifolds follows from [18, 19, 142], but the complex analytic case is not available in [19, 142].

In §8.2 we will state and prove the following result:

Theorem *Let (S,ω) be a complex symplectic manifold and L, M oriented complex Lagrangian submanifolds in S, and write $X = L\cap M$, as a complex analytic subspace of S. Then we may define $P^\bullet_{L,M}\in \mathrm{Perv}(X)$, uniquely up to canonical isomorphism, and isomorphisms $\Sigma_{L,M} : P^\bullet_{L,M} \to \mathbb{D}_X(P^\bullet_{L,M})$, $\mathrm{T}_{L,M} : P^\bullet_{L,M} \to P^\bullet_{L,M}$, respectively the Verdier duality and the monodromy isomorphisms. These $P^\bullet_{L,M}\in \mathrm{Perv}(X), \Sigma_{L,M}, \mathrm{T}_{L,M}$ are characterized by the following property.*

Given a choice of local Darboux coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ in the sense of Definition 8.1.1 such that L is locally identified in coordinates with the graph $\Gamma_{\mathrm{d}f(x_1,\ldots,x_n)}$ of $\mathrm{d}f$ for f a holomorphic function defined locally on an open $U\subset\mathbb{C}^n$, and M is locally identified in coordinates with the graph $\Gamma_{\mathrm{d}g(x_1,\ldots,x_n)}$ of $\mathrm{d}g$ for g a holomorphic function defined locally on U, and the orientations $K_L^{1/2}, K_M^{1/2}$ are the trivial square roots of $K_L\cong\langle \mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_n\rangle\cong K_M$, then $P^\bullet_{L,M}\cong \mathcal{PV}^\bullet_{U,g-f}$, where $\mathcal{PV}^\bullet_{U,g-f}$ is the perverse sheaf of vanishing cycles of $g-f$, and $\Sigma_{L,M}$ and $\mathrm{T}_{L,M}$ are respectively the Verdier duality $\sigma_{U,g-f}$ and the monodromy $\tau_{U,g-f}$ introduced in §1.3. The same applies for $\mathscr{D}$-modules and mixed Hodge modules on X.

Here is a sketch of the method of proof, given in detail in §8.2.1–8.2.3.

Given (S,ω) a complex symplectic manifold we want to construct a global perverse sheaf $P^\bullet_{L,M}\in\mathrm{Perv}(X)$, by gluing together local data coming from choices of polarizations by isomorphisms. We consider an open cover $\{S_i\}_{i\in I}$ of S and polarizations $\pi_i : S_i\to E_i$, always assumed to be transverse to both the Lagrangians L and M. We use the following method:

(i) For each polarization $\pi_i : S_i \to E_i$ transverse to both the Lagrangian submanifolds L and M, we define a perverse sheaf of vanishing cycle $\mathcal{PV}^\bullet_{f_i}$, naturally defined on the chart induced by the choice of a polarization. and a principal $\mathbb{Z}_2$-bundle Q_{f_i}, which roughly speaking parametrizes isomorphisms $K_L^{1/2}\cong K_M^{1/2}$ compatible with π_i.

(ii) For two such polarizations E_i and E_j, transverse to each other, and to both the Lagrangians, we have a way to define two perverse sheaves of vanishing cycles, $\mathcal{PV}^\bullet_{f_i}$ and $\mathcal{PV}^\bullet_{f_j}$, again with principal $\mathbb{Z}_2$-bundles, each of them parametrizing choices of square roots of the canonical bundles of $L\cong\Gamma_{\mathrm{d}f_i}$ and $M\cong\Gamma_{\mathrm{d}f_j}$. In this case we find an isomorphism Ψ_{ij} on double overlap $S_i\cap S_j$ between $\mathcal{PV}^\bullet_{f_i}\otimes_{\mathbb{Z}_2} Q_{f_i}$ and $\mathcal{PV}^\bullet_{f_j}\otimes_{\mathbb{Z}_2} Q_{f_j}$.

(iii) For four such polarizations E_i, E_j, E_k and E_l with E_i not necessarily transverse to E_k, we obtain equality between $\Psi_{ij}\circ\Psi_{jk}$ and $\Psi_{il}\circ\Psi_{lk}$ on $S_i\cap S_j\cap S_k\cap S_l$.

As perverse sheaves form a stack, there exists $P^\bullet_{L,M}$ on X, unique up to canonical isomorphism, with $P^\bullet_{L,M}|_{S_i}\cong\mathcal{PV}^\bullet_{f_i}\otimes_{\mathbb{Z}_2} Q_{f_i}$, for all $i\in I$.

Our perverse sheaf $P^\bullet_{L,M}$ categorifies Lagrangian intersection numbers, in the sense that the constructible function

$$p\to\sum_i(-1)^i\dim_{\mathbb{C}}\mathbb{H}^i_{\{p\}}(X,P^\bullet_{L,M}),$$

is equal to the well known Behrend function ν_X in [5] by construction, using the expression of the Behrend function of a critical locus in terms of the Milnor fibre, as in [5], and so

$$\chi(X,\nu_X)=\sum_i(-1)^i\dim_{\mathbb{C}}\mathbb{H}^i(X,P^\bullet_{L,M}).$$

This resolves a long-standing question in the categorification of Lagrangian intersection number, and it may have exciting far reaching consequences in symplectic geometry and topological field theory. In [89], Kapustin and Rozansky study boundary conditions and defects in a three-dimensional topological sigma-model with a complex symplectic target space, the Rozansky-Witten model. They conjecture the existence of an interesting 2-category, the 2-category of boundary conditions. Their toy model for symplectic manifold is a cotangent bundle of some manifold. In this case, this category is related to the category of matrix factorizations [139]. Thus, we strongly believe that constructing a sheaf of $\mathbb{Z}_2$-periodic triangulated categories on Lagrangian intersection would yield an answer to their conjecture. In the language of categorification, this would give a second categorification of the intersection numbers, the first being given by the hypercohomology of the perverse sheaf constructed in the present work. Also, this construction should be compatible with the Gerstenhaber and Batalin–Vilkovisky structures in the sense of [4, Conj. 1.3.1].

Our second main result proved in §8.3 constitutes another bridge between our work and [18, 19, 87]. Pantev et al. [142] show that *derived* intersections $L\cap M$ of algebraic Lagrangians L, M in an algebraic symplectic manifold (S,ω) have -1-shifted symplectic structures, so that Theorem 6.6 in [18] gives them the structure of algebraic d-critical loci in the sense of [87]. Our second main result shows a complex analytic version of this, which is not available from [19, 142], that is, the *classical* intersection $L\cap M$ of complex Lagrangians L, M in a complex symplectic manifold (S,ω) has the structure of an (oriented) complex analytic d-critical locus.

Theorem *Suppose (S,ω) is a complex symplectic manifold, and L, M are (oriented) complex Lagrangian submanifolds in S. Then the intersection $X=L\cap M$, as a complex analytic subspace of S, extends naturally to a (oriented) complex analytic d-critical locus (X,s). The canonical bundle $K_{X,s}$ in the sense of [87, §2.4] is naturally isomorphic to $K_L|_{X^{\mathrm{red}}}\otimes K_M|_{X^{\mathrm{red}}}$.*

It would be interesting to prove an analogous version of this also for a class of 'derived Lagrangians' in (S,ω). Some of the authors of [18] are working on defining a 'Fukaya category' of (derived) complex Lagrangians in a complex symplectic manifold, using $\mathbb{H}^*(P^\bullet_{L,M})$ as morphisms.

Chapter 1

Background material

This chapter contains the basic well established material needed to state our results in the following chapters. Expert readers can skip it.

1.1 Commutative differential graded algebras

In this section we introduce general definitions and conventions from classical algebraic geometry. We call it *classical* to distinguish it from the most recent theory of *derived* algebraic geometry. We start by reviewing some definitions and notations from [19, §2].

Definition 1.1.1. A *commutative graded algebra* A over $\mathbb{K}$ concentrated in non-positive degrees is an algebra A with a decomposition $A = \bigoplus_{i\leqslant 0} A^i$ and an associative product $m : A^i \otimes A^j \to A^{i+j}$ satisfying $fg = (-1)^{|f||g|}gf$ for homogeneous elements $f, g \in A$.

Define a *derivation of degree* k from A to a graded module M to be a $\mathbb{K}$-linear map $\delta : A \to M$ that is homogeneous of degree k and satisfies $\delta(fg) = \delta(f)g + (-1)^{k|f|}f\delta(g)$. There is a universal derivation into a module of *Kähler differentials* Ω^1_A, which can be constructed as I/I^2 for $I = \mathrm{Ker}(m : A \otimes A \to A)$. The universal derivation $\delta : A \to \Omega^1_A$ is then computed as $\delta(a) = a \otimes 1 - 1 \otimes a \in I/I^2$.

In the particular case when $M = A$ one sometimes refers to a derivation $X : A \to A$ of degree k as a *vector field of degree* k. Define the *graded Lie bracket* of two homogeneous vector fields X, Y by $[X, Y] := XY - (-1)^{|X||Y|}YX$. On any commutative graded algebra, there is a canonical degree 0 *Euler vector field* E which acts on a homogeneous element $f \in A$ via $E(f) = |f|f$.

Define the *de Rham algebra* of A to be the free commutative graded algebra over A on the graded module $\Omega^1_A[1]$:

$$\mathrm{DR}(A) := \mathrm{Sym}_A(\Omega^1_A[1]) = \bigoplus_p \Lambda^p\Omega^1_A[p]. \tag{1.1.1}$$

We endow $\mathrm{DR}(A)$ with the de Rham operator d_{dR}, which is the unique square-zero derivation of degree -1 on the commutative graded algebra $\mathrm{DR}(A)$ such that for $f \in A$, $\mathrm{d}_{dR}(f) = \delta(f)[1] \in \Omega^1_A[1]$. Thus $\mathrm{d}_{dR}(fg) = \mathrm{d}_{dR}(f)g + (-1)^{|f|}f\,\mathrm{d}_{dR}(g)$ for $f, g \in A$ and $\mathrm{d}_{dR}(\alpha\cdot\beta) = \mathrm{d}_{dR}(\alpha)\beta + (-1)^{|\alpha|}\alpha\mathrm{d}_{dR}(\beta)$ for any two $\alpha, \beta \in \mathrm{DR}(A)$. The de Rham algebra $\mathrm{DR}(A)$ has two gradings, one induced by the grading on A and on the module $\Omega^1[1]$ and the other given by p in the decomposition $\mathrm{DR}(A) := \mathrm{Sym}_A(\Omega^1_A[1]) \cong \bigoplus_p \Lambda^p\Omega^1_A[p]$. We shall refer to the first grading as *degree* and the second grading as *weight*. Thus the de Rham operator d_{dR} has degree -1 and weight $+1$.

Definition 1.1.2. Given a homogeneous vector field X on A of degree $|X|$, the *contraction operator* ι_X on $\mathrm{DR}(A)$ is defined to be the unique derivation of degree $|X| + 1$ such that $\iota_X f = 0$

and $\iota_X \mathrm{d}_{dR}(f) = X(f)$ for all $f \in A$. We define the *Lie derivative* L_X along a vector field X by

$$L_X = [\iota_X, \mathrm{d}_{dR}] = \iota_X \mathrm{d}_{dR} - (-1)^{|X|+1} \mathrm{d}_{dR} \iota_X = \iota_X \mathrm{d}_{dR} + (-1)^{|X|} \mathrm{d}_{dR} \iota_X.$$

It is a derivation of $\mathrm{DR}(A)$ of degree $|X|$. In particular, the Lie derivative along E is of degree 0.

Given $f \in A$, and a homogeneous form $\alpha \in \Lambda^p \Omega^1_A[p]$, we have

$$L_E f = \iota_E \mathrm{d}_{dR} f = E(f) = |f| f, \quad L_E \mathrm{d}_{dR} f = |f| \mathrm{d}_{dR} f, \quad L_E \alpha = \iota_E \mathrm{d}_{dR} \alpha + \mathrm{d}_{dR} \iota_E \alpha = (|\alpha| + p)\alpha.$$

Note that a de Rham closed form α can fail to be exact only if it lives on $A^0 \subset A$.

Given X, Y homogeneous vector fields on A, we have the following equalities of derivations:

$$[\mathrm{d}_{dR}, L_X] = 0, \quad [\iota_X, \iota_Y] = 0, \quad [L_X, \iota_Y] = \iota_{[X,Y]}, \quad [L_X, L_Y] = L_{[X,Y]} \quad \text{on } \mathrm{DR}(A).$$

Definition 1.1.3. A *commutative differential graded algebra* or *cdga* (A, d) is a commutative graded algebra A over $\mathbb{K}$, endowed with a square-zero derivation d of degree 1. Usually we write A rather than (A, d), leaving d implicit. Note that the cohomology $H^*(A)$ of A with respect to the differential d is a commutative graded algebra.

Definition 1.1.4. Let (A, d) be a cdga. Then as in Definition 1.1.1, to the underlying commutative graded algebra A we associate the module of Kähler differentials Ω^1_A with universal degree 0 derivation $\delta : A \to \Omega^1_A$, and the de Rham algebra $\mathrm{DR}(A) = \mathrm{Sym}_A\big(\Omega^1_A[1]\big)$ in (1.1.1), with degree -1 de Rham differential $\mathrm{d}_{dR} : \mathrm{DR}(A) \to \mathrm{DR}(A)$. The differential d on A induces a unique differential on Ω^1_A, also denoted d, satisfying $\mathrm{d} \circ \delta = \delta \circ \mathrm{d} : A \to \Omega^1_A$, and making (Ω^1_A, d) into a dg-module. Moreover, d on $\Omega^1_A[1]$ anti-commutes with the de Rham operator $\mathrm{d}_{dR} : A \to \Omega^1[1]$. We extend the differential d uniquely to all of $\mathrm{DR}(A)$ by requiring it to be a derivation of degree 1 with respect to the multiplication on $\mathrm{DR}(A)$. We will basically work with (A, d) for which the Kähler differentials (Ω^1_A, d) give a model for the *cotangent complex* $\mathbb{L}_{(A,\mathrm{d})}$ of (A, d), and we will basically identify $\mathbb{L}_{(A,\mathrm{d})}$ and (Ω^1_A, d).

Similarly, given a map $A \to B$ of cdgas, we can define the *relative Kähler differentials* $\Omega^1_{B/A}$, and when the map $A \to B$ is nice enough (for example, B is obtained from A adding free generators of some degree and imposing a differential, as in [19, Ex. 2.8]), then the relative Kähler differentials give a model for the *relative cotangent complex* $\mathbb{L}_{B/A}$.

1.2 Derived algebraic geometry

We give a brief sketch on Toën and Vezzosi's derived algebraic geometry [171–175], and Pantev, Toën, Vaquié and Vezzosi's theory of k-shifted symplectic structures on derived schemes and stacks [142, 179], which is central to our program. Following our principal reference [142], we prefer to use the Toën–Vezzosi version, instead of the Lurie one [121, 122]. Our slogan will be that *a derived* $\mathbb{K}$*-scheme is a geometric space locally modelled on* **Spec** A *for* A *a cdga over* $\mathbb{K}$, just as a classical $\mathbb{K}$-scheme is a space locally modelled on $\mathrm{Spec}\, A$ for A a commutative $\mathbb{K}$-algebra. We refer to [13, 19] for more details.

1.2.1 Derived schemes and stacks

Fix an algebraically closed base field $\mathbb{K}$, of characteristic zero. Toën and Vezzosi define the ∞-category $\mathbf{dSt}_{\mathbb{K}}$ of *derived* $\mathbb{K}$*-stacks* (or D^-*-stacks*) [175, Def. 2.2.2.14], [173, Def. 4.2]. All derived $\mathbb{K}$-stacks $\boldsymbol{X}$ in this paper are assumed to be *locally finitely presented.* There is a *spectrum functor*

$$\mathbf{Spec} : \{\text{commutative differential graded } \mathbb{K}\text{-algebras, degrees} \leqslant 0\} \longrightarrow \mathbf{dSt}_{\mathbb{K}}\,.$$

All cdgas will be in degrees $\leqslant 0$. A derived $\mathbb{K}$-stack $\boldsymbol{X}$ is called an *affine derived $\mathbb{K}$-scheme* if $\boldsymbol{X}$ is equivalent in $\mathbf{dSt}_{\mathbb{K}}$ to $\mathbf{Spec}\, A$ for some cdga A over $\mathbb{K}$. As in [173, §4.2], a derived $\mathbb{K}$-stack $\boldsymbol{X}$ is called a *derived $\mathbb{K}$-scheme* if it may be covered by Zariski open $\boldsymbol{Y} \subseteq \boldsymbol{X}$ with $\boldsymbol{Y}$ an affine derived $\mathbb{K}$-scheme. Write $\mathbf{dSch}_{\mathbb{K}}$ for the full ∞-subcategory of derived $\mathbb{K}$-schemes in $\mathbf{dSt}_{\mathbb{K}}$.

We call a derived $\mathbb{K}$-stack $\boldsymbol{X}$ a *derived Artin $\mathbb{K}$-stack* if it is *m-geometric* for some m [175, Def. 1.3.3.1] and the underlying classical stack is 1-truncated (that is, just a stack, not a higher stack). Any such $\boldsymbol{X}$ admits a smooth surjective morphism $\boldsymbol{\varphi} : \boldsymbol{U} \to \boldsymbol{X}$, an *atlas*, with $\boldsymbol{U}$ a derived $\mathbb{K}$-scheme. Write $\mathbf{dArt}_{\mathbb{K}}$ for the full ∞-subcategory of derived Artin $\mathbb{K}$-stacks in $\mathbf{dSt}_{\mathbb{K}}$. Then $\mathbf{dSch}_{\mathbb{K}} \subset \mathbf{dArt}_{\mathbb{K}} \subset \mathbf{dSt}_{\mathbb{K}}$.

Write $\mathrm{Sch}_{\mathbb{K}}$ for the category of $\mathbb{K}$-schemes X, and $\mathrm{Art}_{\mathbb{K}}$ for the 2-category of Artin $\mathbb{K}$-stacks X. By an abuse of notation we regard $\mathrm{Sch}_{\mathbb{K}}$ as a discrete 2-subcategory of $\mathrm{Art}_{\mathbb{K}}$, so that $\mathrm{Sch}_{\mathbb{K}} \subset \mathrm{Art}_{\mathbb{K}}$. As in [175, Prop. 2.1.2.1], there is an *inclusion functor* $i : \mathrm{Art}_{\mathbb{K}} \to \mathbf{dArt}_{\mathbb{K}}$ mapping $\mathrm{Sch}_{\mathbb{K}} \to \mathbf{dSch}_{\mathbb{K}}$, and a *classical truncation functor* $t_0 : \mathbf{dArt}_{\mathbb{K}} \to \mathrm{Art}_{\mathbb{K}}$ mapping $\mathbf{dSch}_{\mathbb{K}} \to \mathrm{Sch}_{\mathbb{K}}$.

A derived Artin $\mathbb{K}$-stack $\boldsymbol{X}$ has a *cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$ of finite cohomological amplitude $[-m, 1]$ and a dual *tangent complex* $\mathbb{T}_{\boldsymbol{X}}$ [175, §1.4], [173, §4.2.4–§4.2.5] in a stable ∞-category $L_{\mathrm{qcoh}}(\boldsymbol{X})$ defined in [173, §3.1.7, §4.2.4]. When X is a classical scheme or stack, then the homotopy category of $L_{\mathrm{qcoh}}(\boldsymbol{X})$ is nothing but the triangulated category $D_{\mathrm{qcoh}}(X)$. These have the usual properties of (co)tangent complexes. For instance, if $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a morphism in $\mathbf{dArt}_{\mathbb{K}}$ there is a distinguished triangle

$$\boldsymbol{f}^*(\mathbb{L}_{\boldsymbol{Y}}) \xrightarrow{\mathbb{L}_{\boldsymbol{f}}} \mathbb{L}_{\boldsymbol{X}} \longrightarrow \mathbb{L}_{\boldsymbol{X}/\boldsymbol{Y}} \longrightarrow \boldsymbol{f}^*(\mathbb{L}_{\boldsymbol{Y}})[1], \tag{1.2.1}$$

where $\mathbb{L}_{\boldsymbol{X}/\boldsymbol{Y}}$ is the *relative cotangent complex* of $\boldsymbol{f}$. Here $\boldsymbol{f}$ is smooth of relative dimension n if and only if $\mathbb{L}_{\boldsymbol{X}/\boldsymbol{Y}}$ is locally free of rank n, and $\boldsymbol{f}$ is étale if and only if $\mathbb{L}_{\boldsymbol{X}/\boldsymbol{Y}} = 0$. See [19, §3.3] for a complete list of properties of cotangent complexes used in our results.

Now suppose A is a cdga over $\mathbb{K}$, and $\boldsymbol{X}$ a derived $\mathbb{K}$-scheme with $\boldsymbol{X} \simeq \mathbf{Spec}\, A$ in $\mathbf{dSch}_{\mathbb{K}}$. Then we have an equivalence of triangulated categories $L_{\mathrm{qcoh}}(\boldsymbol{X}) \simeq D(\mathrm{mod}\, A)$, where $D(\mathrm{mod}\, A)$ is the derived category of dg-modules over A. This equivalence identifies cotangent complexes $\mathbb{L}_{\boldsymbol{X}} \simeq \mathbb{L}_A$. If A is of standard form as §3.1 the Kähler differentials Ω^1_A are a model for $\mathbb{L}_A$ in $D(\mathrm{mod}\, A)$, and in §3 we will give a simple explicit description of Ω^1_A. Thus, if $\boldsymbol{X}$ is a derived $\mathbb{K}$-scheme with $\boldsymbol{X} \simeq \mathbf{Spec}\, A$ for A a standard form cdga, we can understand $\mathbb{L}_{\boldsymbol{X}}$ well. We will use this to do computations with *k-shifted p-forms* and *k-shifted closed p-forms* on $\boldsymbol{X}$, as in §1.2.2.

1.2.2 Shifted symplectic structures

Let $\boldsymbol{X}$ be a derived stack. Pantev, Toën, Vaquié and Vezzosi [142] defined *k-shifted p-forms*, *k-shifted closed p-forms*, and *k-shifted symplectic structures* on $\boldsymbol{X}$, for $k \in \mathbb{Z}$ and $p \geqslant 0$. One first defines these notions on derived affine schemes and then defines the general notions by smooth descent. Since our main theorems are statements about the local structure of derived schemes and stacks endowed with shifted symplectic forms, it suffices for us to describe the affine case. The basic idea is this:

(a) Define the exterior powers $\Lambda^p \mathbb{L}_{\boldsymbol{X}}$ in $L_{\mathrm{qcoh}}(\boldsymbol{X})$ for $p = 0, 1, \ldots$. Regard $\Lambda^p \mathbb{L}_{\boldsymbol{X}}$ as a complex, with differential d:

$$\cdots \xrightarrow{\mathrm{d}} (\Lambda^p \mathbb{L}_{\boldsymbol{X}})^{k-1} \xrightarrow{\mathrm{d}} (\Lambda^p \mathbb{L}_{\boldsymbol{X}})^{k} \xrightarrow{\mathrm{d}} (\Lambda^p \mathbb{L}_{\boldsymbol{X}})^{k+1} \xrightarrow{\mathrm{d}} \cdots .$$

Then a *k-shifted p-form*, or *p-form of degree k*, is an element ω^0 of $(\Lambda^p \mathbb{L}_{\boldsymbol{X}})^k$ with $\mathrm{d}\omega^0 = 0$. Mostly we are interested in the cohomology class $[\omega^0] \in H^k(\Lambda^p \mathbb{L}_{\boldsymbol{X}})$.

(b) There are *de Rham differentials* $\mathrm{d}_{dR} : \Lambda^p \mathbb{L}_{\boldsymbol{X}} \to \Lambda^{p+1} \mathbb{L}_{\boldsymbol{X}}$ with $\mathrm{d}_{dR} \circ \mathrm{d}_{dR} = \mathrm{d} \circ \mathrm{d}_{dR} + \mathrm{d}_{dR} \circ \mathrm{d} = 0$. Then a *$k$-shifted closed p-form*, or *closed p-form of degree k*, is a sequence $\omega =$

$(\omega^0, \omega^1, \omega^2, \ldots)$ with ω^i in $(\Lambda^{p+i}\mathbb{L}_{\boldsymbol{X}})^{k-i}$ for $i \geqslant 0$, satisfying $\mathrm{d}\omega^0 = 0$ and $\mathrm{d}_{dR}\omega^i + \mathrm{d}\omega^{i+1} = 0$ for $i = 0, 1, \ldots$. That is, $\omega = (\omega^0, \omega^1, \omega^2, \ldots)$ is a k-cycle in the negative cyclic complex

$$\Big(\big(\prod_{i=0}^{\infty}(\Lambda^{p+i}\mathbb{L}_{\boldsymbol{X}})^{k-i}\big)_{k\in\mathbb{Z}}, \mathrm{d} + \mathrm{d}_{dR}\Big).$$

Mostly we are interested in the cohomology class $[\omega] = [\omega^0, \omega^1, \ldots]$ in the cohomology of this complex. We will write $\omega \sim \omega'$ if ω, ω' are k-shifted closed p-forms with the same cohomology class $[\omega] = [\omega']$. There is a map $(\omega^0, \omega^1, \omega^2, \ldots) \mapsto \omega^0$ from k-shifted closed p-forms to k-shifted p-forms.

(c) A 2-form ω^0 of degree k on $\boldsymbol{X}$ induces a morphism $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$ in $L_{\mathrm{qcoh}}(\boldsymbol{X})$. We call ω^0 *nondegenerate* if $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$ is an equivalence. A closed 2-form ω of degree k on $\boldsymbol{X}$ for $k \in \mathbb{Z}$ is called a *k-shifted symplectic structure* if the corresponding 2-form $\omega^0 = \pi(\omega)$ is nondegenerate.

The families (simplicial sets) of p-forms and of closed p-forms of degree k on $\boldsymbol{X}$ are written $\mathcal{A}^p_{\mathbb{K}}(\boldsymbol{X}, k)$ and $\mathcal{A}^{p,\mathrm{cl}}_{\mathbb{K}}(\boldsymbol{X}, k)$, respectively. There is a morphism $\pi : \mathcal{A}^{p,\mathrm{cl}}_{\mathbb{K}}(\boldsymbol{X}, k) \to \mathcal{A}^p_{\mathbb{K}}(\boldsymbol{X}, k)$, which is in general neither injective nor surjective. In [142, Def. 1.7], Pantev et al. define a simplicial set $\mathcal{A}^p_{\mathbb{K}}(\boldsymbol{X}, k)$ of *p-forms of degree* $k \in \mathbb{Z}$ on the derived $\mathbb{K}$-scheme $\boldsymbol{X} = \operatorname{Spec} A$ by

$$\mathcal{A}^p_{\boldsymbol{X}}(Y, k) = \big|\Lambda^p\mathbb{L}_A[k]\big|. \tag{1.2.2}$$

As explained above, in our case we may take $\Lambda^p\mathbb{L}_A = \Lambda^p\Omega^1_A$. Thus (1.2.2) yields

$$\pi_0\big(\mathcal{A}^p_{\mathbb{K}}(\boldsymbol{X}, k)\big) \cong H^k\big(\Lambda^p\Omega^1_A, \mathrm{d}\big) = H^{k-p}\big(\Lambda^p\Omega^1_A[p], \mathrm{d}\big). \tag{1.2.3}$$

So, (connected components of the simplicial set of) p-forms of degree k on $\boldsymbol{X}$ are just k-cohomology classes of the complex $\big(\Lambda^p\Omega^1_A, \mathrm{d}\big)$. We prefer to deal with explicit representatives, rather than cohomology classes. The definition of the simplicial set $\mathcal{A}^{p,\mathrm{cl}}_{\mathbb{K}}(\boldsymbol{X}, k)$ of *closed p-forms of degree* $k \in \mathbb{Z}$ on $\boldsymbol{X} = \mathbf{Spec}\, A$ in Pantev et al. [142, Def. 1.7] yields

$$\mathcal{A}^{p,\mathrm{cl}}_{\mathbb{K}}(\boldsymbol{X}, k) = \big|\prod_{i\geqslant 0}\Lambda^{p+i}\,\mathbb{L}_A[k-i]\big|. \tag{1.2.4}$$

In our case we may take $\Lambda^{p+i}\,\mathbb{L}_A[k-i] = \Lambda^{p+i}\Omega^1_A[k-i]$. Thus, as for (1.2.2)–(1.2.3), equation (1.2.4) implies that

$$\pi_0\big(\mathcal{A}^{p,\mathrm{cl}}_{\mathbb{K}}(\boldsymbol{X}, k)\big) \cong H^0\big(\prod_{i\geqslant 0}\Lambda^{p+i}\,\Omega^1_A[k-i], \mathrm{d} + \mathrm{d}_{dR}\big) = H^k\big(\prod_{i\geqslant 0}\Lambda^{p+i}\Omega^1_A[-i], \mathrm{d} + \mathrm{d}_{dR}\big).$$

If a derived $\mathbb{K}$-scheme $\boldsymbol{X}$ has a 0-shifted symplectic structure then $\boldsymbol{X}$ is a smooth $\mathbb{K}$-scheme X with a classical symplectic structure. Pantev et al. [142] construct k-shifted symplectic structures on several classes of derived moduli stacks. If Y is a Calabi–Yau m-fold and $\boldsymbol{\mathcal{M}}$ a derived moduli stack of coherent sheaves or perfect complexes on Y, then $\boldsymbol{\mathcal{M}}$ has a $(2-m)$-shifted symplectic structure. We are particularly interested in the case $m = 3$, so $k = -1$.

1.3 Perverse sheaves on schemes and stacks

Next, we review the theory of perverse sheaves on schemes §1.3.1 and stacks §1.3.2. Perverse sheaves, and the related theories of $\mathscr{D}$-modules and mixed Hodge modules, make sense in several contexts, both algebraic and complex analytic: perverse sheaves on $\mathbb{C}$-schemes [12, 34] and on complex analytic spaces [34] with coefficients in a ring A (usually $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$), $\mathscr{D}$-modules on

$\mathbb{C}$-schemes [16] and on complex analytic spaces [15, 155], perverse sheaves on $\mathbb{K}$-schemes with coefficients in $\mathbb{Z}/l^n\mathbb{Z}$, $\mathbb{Z}_l$, $\mathbb{Q}_l$, or $\bar{\mathbb{Q}}_l$ for $l \neq \operatorname{char}\mathbb{K} \neq 2$ a prime [12], $\mathscr{D}$-modules on $\mathbb{K}$-schemes for $\mathbb{K}$ an algebraically closed field [15], mixed Hodge modules on $\mathbb{C}$-schemes, and on complex analytic spaces [152,154]. Perverse sheaves are easiest to define, and have the nicest properties, for schemes X over $\mathbb{C}$, since then one can make use of the complex analytic topology. We follow [13, §2], that is why we decided to focus more on those for this brief introduction on the subject. A good introductory reference on perverse sheaves on $\mathbb{C}$-schemes and complex analytic spaces is Dimca [34]. Three other books are Kashiwara and Schapira [90], Schürmann [157], and Hotta, Tanisaki and Takeuchi [67]. Massey [126] and Rietsch [146] are surveys on perverse sheaves, and Beilinson, Bernstein and Deligne [12] is an important primary source, who cover both $\mathbb{Q}$-perverse sheaves on $\mathbb{C}$-schemes, and $\mathbb{Q}_l$-perverse sheaves on $\mathbb{K}$-schemes. An introduction to perverse sheaves on schemes and to related theories mentioned above, suited to our purposes, can be found in [18, §2] and [19, §4].

1.3.1 Perverse sheaves on $\mathbb{C}$-schemes and $\mathbb{K}$-schemes

Definition 1.3.1. Let X be a $\mathbb{C}$-scheme (always assumed separated and of finite type) and A a well-behaved commutative base ring, usually $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$. Write X^{an} for the set of $\mathbb{C}$-points of X with the complex analytic topology. Consider sheaves of A-modules $\mathcal{S}$ on X^{an}. A sheaf $\mathcal{S}$ is called *constructible* if all the stalks $\mathcal{S}_x$ for $x \in X^{\mathrm{an}}$ are finite type A-modules, and there is a locally finite stratification $X^{\mathrm{an}} = \coprod_{j\in J} X_j^{\mathrm{an}}$ of X^{an}, where $X_j \subseteq X$ for $j \in J$ are $\mathbb{C}$-subschemes of X and $X_j^{\mathrm{an}} \subseteq X^{\mathrm{an}}$ the corresponding subsets of $\mathbb{C}$-points, such that $\mathcal{S}|_{X_j^{\mathrm{an}}}$ is an A-local system for all $j \in J$.

Write $D(X)$ for the derived category of complexes $\mathcal{C}^\bullet$ of sheaves of A-modules on X^{an}. Write $D^b_c(X)$ for the full subcategory of bounded complexes $\mathcal{C}^\bullet$ in $D(X)$ whose cohomology sheaves $\mathcal{H}^m(\mathcal{C}^\bullet)$ are constructible for all $m \in \mathbb{Z}$. Then $D(X), D^b_c(X)$ are triangulated categories. An example of a constructible complex on X is the *constant sheaf* A_X on X with fibre A at each point.

Grothendieck's "six operations on sheaves" $f^*, f^!, Rf_*, Rf_!, \mathcal{RH}om, \overset{L}{\otimes}$ act on $D(X)$ preserving the subcategory $D^b_c(X)$. There is a functor $\mathbb{D}_X : D^b_c(X) \to D^b_c(X)^{\mathrm{op}}$ with $\mathbb{D}_X \circ \mathbb{D}_X \cong \mathrm{id} : D^b_c(X) \to D^b_c(X)$, called *Verdier duality.* It reverses shifts, that is, $\mathbb{D}_X\big(\mathcal{C}^\bullet[k]\big) = \big(\mathbb{D}_X(\mathcal{C}^\bullet)\big)[-k]$ for $\mathcal{C}^\bullet$ in $D^b_c(X)$ and $k \in \mathbb{Z}$.

For each $x \in X^{\mathrm{an}}$, let $i_x : * \to X$ map $i_x : * \mapsto x$. If $\mathcal{C}^\bullet \in D^b_c(X)$, then the *support* $\operatorname{supp}^m \mathcal{C}^\bullet$ and *cosupport* $\operatorname{cosupp}^m \mathcal{C}^\bullet$ of $\mathcal{H}^m(\mathcal{C}^\bullet)$ for $m \in \mathbb{Z}$ are

$$\operatorname{supp}^m \mathcal{C}^\bullet = \overline{\big\{x \in X^{\mathrm{an}} : \mathcal{H}^m(i_x^*(\mathcal{C}^\bullet)) \neq 0\big\}}, \quad \operatorname{cosupp}^m \mathcal{C}^\bullet = \overline{\big\{x \in X^{\mathrm{an}} : \mathcal{H}^m(i_x^!(\mathcal{C}^\bullet)) \neq 0\big\}},$$

where $\overline{\{\cdots\}}$ means the closure in X^{an}. We call $\mathcal{C}^\bullet$ *perverse*, or a *perverse sheaf*, if $\dim_{\mathbb{C}} \operatorname{supp}^{-m} \mathcal{C}^\bullet \leqslant m$ and $\dim_{\mathbb{C}} \operatorname{cosupp}^m \mathcal{C}^\bullet \leqslant m$ for all $m \in \mathbb{Z}$. Write $\operatorname{Perv}(X)$ for the full subcategory of perverse sheaves in $D^b_c(X)$. Then $\operatorname{Perv}(X)$ is an abelian category, the heart of a t-structure on $D^b_c(X)$.

Next we recall Definition 4.2 from [13], where we extend Definition 1.3.1 to $\mathbb{K}$-schemes X over fields $\mathbb{K} \neq \mathbb{C}$. Then the complex analytic topology is not available, and the best we can do is the étale topology. Finding good definitions of $D(X), D^b_c(X), \operatorname{Perv}(X)$ turns out to depend strongly on the base ring A, so we temporarily include A in our notation, writing $D(X,A), D^b_c(X,A), \operatorname{Perv}(X,A)$. The primary source is Beilinson, Bernstein and Deligne [12], and useful references are Ekedahl [42], Freitag and Kiehl [43], and Kiehl and Weissauer [96].

Definition 1.3.2. Let $\mathbb{K}$ be an algebraically closed field with $\operatorname{char}\mathbb{K} \neq 2$, and X a $\mathbb{K}$-scheme (always assumed separated and of finite type). Then:

(a) If A is a commutative ring with finite characteristic $\operatorname{char} A > 0$ coprime to $\operatorname{char} \mathbb{K}$, then we can define $D(X, A)$ to be the derived category of sheaves of A-modules on X in the étale topology, and $D^b_c(X, A)$ to be the full subcategory of bounded complexes with constructible cohomology. This works in particular for $A = \mathbb{Z}/l^n\mathbb{Z}$, with l a prime coprime to $\operatorname{char} \mathbb{K}$.

(b) Let l be a prime coprime to $\operatorname{char} \mathbb{K}$. The ring of *$l$-adic integers* $\mathbb{Z}_l$ are $\mathbb{Z}_l = \varprojlim_n \mathbb{Z}/l^n\mathbb{Z}$. It has characteristic zero. We define $D^b_c(X, \mathbb{Z}_l) = \varprojlim_n D^b_c(X, \mathbb{Z}/l^n\mathbb{Z})$, for $D^b_c(X, \mathbb{Z}/l^n\mathbb{Z})$ as in (a). Objects of $D^b_c(X, \mathbb{Z}_l)$ are projective systems of $\mathbb{Z}/l^n\mathbb{Z}$-sheaves on X in the étale topology.

(c) The *l-adic rationals* $\mathbb{Q}_l$ is the field of fractions of $\mathbb{Z}_l$. We define $D^b_c(X, \mathbb{Q}_l) = D^b_c(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. That is, objects $\mathcal{P}^\bullet, \mathcal{Q}^\bullet$ of $D^b_c(X, \mathbb{Q}_l)$ are objects of $D^b_c(X, \mathbb{Z}_l)$, and $\operatorname{Hom}_{D^b_c(X,\mathbb{Q}_l)}(\mathcal{P}^\bullet, \mathcal{Q}^\bullet) = \operatorname{Hom}_{D^b_c(X,\mathbb{Z}_l)}(\mathcal{P}^\bullet, \mathcal{Q}^\bullet) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$.

(d) The algebraic closure $\bar{\mathbb{Q}}_l$ of $\mathbb{Q}_l$ is noncanonically isomorphic to $\mathbb{C}$. We define $D^b_c(X, \bar{\mathbb{Q}}_l) = \varinjlim_E D^b_c(X, E)$, where the limit is over finite field extensions E of $\mathbb{Q}_l$ in $\bar{\mathbb{Q}}_l$.

As in [12, 42, 43, 96], in each case the same package of properties as perverse sheaves over $\mathbb{C}$-schemes has been developed, including Grothendieck's six operations $f^*, f^!, Rf_*, Rf_!, \mathcal{RH}om, \overset{L}{\otimes}$ and Verdier duality $\mathbb{D}_X$, and an abelian category of perverse sheaves $\operatorname{Perv}(X, A) \subset D^b_c(X, A)$ which is the heart of a t-structure. We will refer to case (a) as *perverse sheaves with finite coefficients*, and cases (b)–(d) as *perverse sheaves with l-adic coefficients*.

The rest of this section works for perverse sheaves over $\mathbb{C}$-schemes and $\mathbb{K}$-schemes, with coefficients in A, and we will not distinguish the two; by 'X is a scheme' we mean either X is a $\mathbb{C}$-scheme or X is a $\mathbb{K}$-scheme.

Definition 1.3.3. Let U be a smooth scheme and $f : U \to \mathbb{A}^1$ a regular function, and write U_0 for the subscheme $f^{-1}(0) \subseteq U$. Then we can define the (*shifted*) *nearby cycle functor* $\psi^p_f : D^b_c(U) \to D^b_c(U_0)$ and the (*shifted*) *vanishing cycle functor* $\phi^p_f : D^b_c(U) \to D^b_c(U_0)$. Both map $\operatorname{Perv}(U) \to \operatorname{Perv}(U_0)$. The shift $A_U[\dim U]$ of the constant sheaf A_U is perverse, so $\phi^p_f(A_U[\dim U]) \in \operatorname{Perv}(U_0)$.

Write $X = \operatorname{Crit}(f)$. Then $f|_{X^{\rm red}}$ is locally constant on X, so we have a decomposition $X = \coprod_{c \in f(X)} X_c$, where $X_c \subseteq X$ is the open and closed subscheme of points $p \in X$ with $f(p) = c$. It turns out that $\phi^p_f(A_U[\dim U])$ is supported on $X_0 \subseteq X \subseteq U$. Define the *perverse sheaf of vanishing cycles* $\mathcal{PV}^\bullet_{U,f}$ *of* U, f in $\operatorname{Perv}(X)$ or $\operatorname{Perv}(U)$ to be $\mathcal{PV}^\bullet_{U,f} = \bigoplus_{c \in f(X)} \phi^p_{f-c}(A_U[\dim U])|_{X_c}$.

Using an isomorphism $\mathbb{D}_U(A_U) \cong A_U[2\dim U]$ and a compatibility between ϕ^p_f and $\mathbb{D}_U, \mathbb{D}_{U_0}$, in [18, §2.4] we define a canonical *Verdier duality isomorphism* $\sigma_{U,f} : \mathcal{PV}^\bullet_{U,f} \xrightarrow{\cong} \mathbb{D}_X(\mathcal{PV}^\bullet_{U,f})$.

There are monodromy natural transformations $M_{U,f} : \psi^p_f \Rightarrow \psi^p_f$ and $M_{U,f} : \phi^p_f \Rightarrow \phi^p_f$, and using these in [18, §2.4] we define the *twisted monodromy operator* $\tau_{U,f} : \mathcal{PV}^\bullet_{U,f} \xrightarrow{\cong} \mathcal{PV}^\bullet_{U,f}$.

Here are some results connecting perverse sheaves and smooth morphisms [13, Prop.s 4.4 & 4.6, Thm. 4.5]. Theorem 1.3.5 (proved in [12, Th. 3.2.4], see also [106, §2.3]) is crucial in our program [13, 18], and it is the reason why perverse sheaves extend to Artin stacks. In [18, Thm. 2.6], we give a version of Theorem 1.3.5 in the étale topology. The analogue for $D^b_c(X)$ or $D(X)$ rather than $\operatorname{Perv}(X)$ is false. One moral is that perverse sheaves behave like sheaves, rather than like complexes.

Proposition 1.3.4. *Let $\Phi : X \to Y$ be a scheme morphism smooth of relative dimension d. Then the (exceptional) inverse image functors $\Phi^*, \Phi^! : D^b_c(Y) \to D^b_c(X)$ satisfy $\Phi^*[d] \cong \Phi^![-d]$, where $\Phi^*[d], \Phi^![-d]$ are $\Phi^*, \Phi^!$ shifted by $\pm d$. Furthermore $\Phi^*[d], \Phi^![-d]$ map* $\operatorname{Perv}(Y) \to \operatorname{Perv}(X)$.

Theorem 1.3.5. *Let X be a scheme. Then perverse sheaves on X form a **stack** (a kind of sheaf of categories) on X **in the smooth topology**. Explicitly, this means the following. Let $\{u_i : U_i \to X\}_{i\in I}$ be a **smooth open cover** for X, so that $u_i : U_i \to X$ is a scheme morphism smooth of relative dimension d_i for $i \in I$, with $\coprod_i u_i$ surjective. Write $U_{ij} = U_i \times_{u_i,X,u_j} U_j$ for $i, j \in I$ with projections*

$$\pi^i_{ij} : U_{ij} \longrightarrow U_i, \quad \pi^j_{ij} : U_{ij} \longrightarrow U_j, \quad u_{ij} = u_i \circ \pi^i_{ij} = u_j \circ \pi^j_{ij} : U_{ij} \longrightarrow X.$$

Similarly, write $U_{ijk} = U_i \times_X U_j \times_X U_k$ for $i, j, k \in I$ with projections

$$\pi^{ij}_{ijk} : U_{ijk} \longrightarrow U_{ij}, \quad \pi^{ik}_{ijk} : U_{ijk} \longrightarrow U_{ik}, \quad \pi^{jk}_{ijk} : U_{ijk} \longrightarrow U_{jk},$$
$$\pi^i_{ijk} : U_{ijk} \to U_i, \ \pi^j_{ijk} : U_{ijk} \to U_j, \ \pi^k_{ijk} : U_{ijk} \to U_k, \ u_{ijk} : U_{ijk} \to X,$$

so that $\pi^i_{ijk} = \pi^i_{ij} \circ \pi^{ij}_{ijk}$, $u_{ijk} = u_{ij} \circ \pi^{ij}_{ijk} = u_i \circ \pi^i_{ijk}$, and so on. All these morphisms $u_i, \pi^i_{ij}, \ldots, u_{ijk}$ are smooth of known relative dimensions, so $u_i^*[d_i] \cong u_i^![-d_i]$ maps $\mathrm{Perv}(X) \to \mathrm{Perv}(U_i)$ by Proposition 1.3.4, and similarly for $\pi^i_{ij}, \ldots, u_{ijk}$. With this notation:

(i) *Suppose $\mathcal{P}^\bullet, \mathcal{Q}^\bullet \in \mathrm{Perv}(X)$, and we are given $\alpha_i : u_i^*[d_i](\mathcal{P}^\bullet) \to u_i^*[d_i](\mathcal{Q}^\bullet)$ in $\mathrm{Perv}(U_i)$ for all $i \in I$ such that for all $i, j \in I$ we have*

$$(\pi^i_{ij})^*[d_j](\alpha_i) = (\pi^j_{ij})^*[d_i](\alpha_j) : u_{ij}^*[d_i + d_j](\mathcal{P}^\bullet) \longrightarrow u_{ij}^*[d_i + d_j](\mathcal{Q}^\bullet).$$

Then there is a unique $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ with $\alpha_i = u_i^*[d_i](\alpha)$ for all $i \in I$.

(ii) *Suppose we are given $\mathcal{P}_i^\bullet \in \mathrm{Perv}(U_i)$ for all $i \in I$ and isomorphisms $\alpha_{ij} : (\pi^i_{ij})^*[d_j](\mathcal{P}_i^\bullet) \to (\pi^j_{ij})^*[d_i](\mathcal{P}_j^\bullet)$ in $\mathrm{Perv}(U_{ij})$ for all $i, j \in I$ with $\alpha_{ii} = \mathrm{id}$ and*

$$(\pi^{jk}_{ijk})^*[d_i](\alpha_{jk}) \circ (\pi^{ij}_{ijk})^*[d_k](\alpha_{ij}) = (\pi^{ik}_{ijk})^*[d_j](\alpha_{ik}) : (\pi^i_{ijk})^*[d_j + d_k](\mathcal{P}_i) \longrightarrow (\pi^k_{ijk})^*[d_i + d_j](\mathcal{P}_k)$$

in $\mathrm{Perv}(U_{ijk})$ for all $i, j, k \in I$. Then there exists $\mathcal{P}^\bullet$ in $\mathrm{Perv}(X)$, unique up to canonical isomorphism, with isomorphisms $\beta_i : u_i^*(\mathcal{P}^\bullet) \to \mathcal{P}_i^\bullet$ for each $i \in I$, satisfying $\alpha_{ij} \circ (\pi^i_{ij})^*(\beta_i) = (\pi^j_{ij})^*(\beta_j) : u_{ij}^*(\mathcal{P}^\bullet) \to (\pi^j_{ij})^*(\mathcal{P}_j^\bullet)$ for all $i, j \in I$.

Proposition 1.3.6. *Let $\Phi : U \to V$ be a scheme morphism smooth of relative dimension d and $g : V \to \mathbb{A}^1$ be regular, and set $f = g \circ \Phi : U \to \mathbb{A}^1$. Then*

- **(a)** *There are natural isomorphisms of functors $\mathrm{Perv}(V) \to \mathrm{Perv}(U_0)$:*

$$\Phi_0^*[d] \circ \psi_g^p \cong \psi_f^p \circ \Phi^*[d] \quad \textit{and} \quad \Phi_0^*[d] \circ \phi_g^p \cong \phi_f^p \circ \Phi^*[d], \tag{1.3.1}$$

  where $U_0 = f^{-1}(0) \subseteq U$, $V_0 = g^{-1}(0) \subseteq V$ and $\Phi_0 = \Phi|_{U_0} : U_0 \to V_0$.

- **(b)** *Write $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$, so that $\Phi|_X : X \to Y$ is smooth of dimension d. Then there is a canonical isomorphism*

$$\Xi_\Phi : \Phi|_X^*[d](\mathcal{PV}^\bullet_{V,g}) \xrightarrow{\cong} \mathcal{PV}^\bullet_{U,f} \quad \textit{in } \mathrm{Perv}(X), \tag{1.3.2}$$

  which identifies $\Phi|_X^*[d](\sigma_{V,g}), \Phi|_X^*[d](\tau_{V,g})$ with $\sigma_{U,f}, \tau_{U,f}$.

We recall [18, Def. 2.8]. If $P \to X$ is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle on a $\mathbb{C}$-scheme X, and $\mathcal{Q}^\bullet \in \mathrm{Perv}(X)$, we will define a perverse sheaf $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P$ as follows:

Definition 1.3.7. Let X be a $\mathbb{C}$-scheme. A *principal* $\mathbb{Z}/2\mathbb{Z}$*-bundle* $P \to X$ is a proper, surjective, étale morphism of $\mathbb{C}$-schemes $\pi : P \to X$ together with a free involution $\sigma : P \to P$, such that the orbits of $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$ are the fibres of π. We will use the ideas of *isomorphism* of principal bundles $\iota : P \to P'$, *section* $s : X \to P$, *tensor product* $P \otimes_{\mathbb{Z}/2\mathbb{Z}} P'$, and *pullback* $f^*(P) \to W$ under a $\mathbb{C}$-scheme morphism $f : W \to X$, all of which are defined in the obvious ways. Let $P \to X$ be a principal $\mathbb{Z}/2\mathbb{Z}$-bundle. Write $\mathcal{L}_P \in D^b_c(X)$ for the rank one A-local system on X induced from P by the nontrivial representation of $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$ on A. It is characterized by $\pi_*(A_P) \cong A_X \oplus \mathcal{L}_P$. For each $\mathcal{Q}^\bullet \in D^b_c(X)$, write $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P \in D^b_c(X)$ for $\mathcal{Q}^\bullet \overset{L}{\otimes} \mathcal{L}_P$, and call it $\mathcal{Q}^\bullet$ *twisted by* P. If $\mathcal{Q}^\bullet$ is perverse then $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P$ is perverse.

There is a 'Thom–Sebastiani Theorem for perverse sheaves', due to Massey [125] and Schürmann [157, Cor. 1.3.4]. Following [18, Thm. 2.12], applied to $\mathcal{PV}^\bullet_{U,f}$, it yields:

Theorem 1.3.8. *Let* U, V *be smooth* $\mathbb{C}$*-schemes and* $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ *be regular, so that* $f \boxplus g : U \times V \to \mathbb{C}$ *is regular with* $(f \boxplus g)(u, v) := f(u) + g(v)$*. Set* $X = \mathrm{Crit}(f)$ *and* $Y = \mathrm{Crit}(g)$ *as* $\mathbb{C}$*-subschemes of* U, V*, so that* $\mathrm{Crit}(f \boxplus g) = X \times Y$*. Then there is a natural isomorphism*

$$\mathcal{TS}_{U,f,V,g} : \mathcal{PV}^\bullet_{U\times V, f\boxplus g} \longrightarrow \mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{V,g} \tag{1.3.3}$$

in $\mathrm{Perv}(X \times Y)$*, such that the following diagrams commute:*

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U\times V,f\boxplus g} \arrow[rr, "\sigma_{U\times V,f\boxplus g}"'] \arrow[d, "\mathcal{TS}_{U,f,V,g}"] & & \mathbb{D}_{X\times Y}(\mathcal{PV}^\bullet_{U\times V,f\boxplus g}) \\
\mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{V,g} \arrow[r, "\sigma_{U,f} \overset{L}{\boxtimes} \sigma_{V,g}"] & \mathbb{D}_X(\mathcal{PV}^\bullet_{U,f}) \overset{L}{\boxtimes} \mathbb{D}_Y(\mathcal{PV}^\bullet_{V,g}) \arrow[r, "\cong"] & \mathbb{D}_{X\times Y}\bigl(\mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{V,g}\bigr), \arrow[u, "\mathbb{D}_{X\times Y}(\mathcal{TS}_{U,f,V,g})"']
\end{tikzcd} \tag{1.3.4}$$

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U\times V,f\boxplus g} \arrow[r, "\tau_{U\times V,f\boxplus g}"'] \arrow[d, "\mathcal{TS}_{U,f,V,g}"] & \mathcal{PV}^\bullet_{U\times V,f\boxplus g} \arrow[d, "\mathcal{TS}_{U,f,V,g}"'] \\
\mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{V,g} \arrow[r, "\tau_{U,f} \overset{L}{\boxtimes} \tau_{V,g}"] & \mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{V,g}.
\end{tikzcd} \tag{1.3.5}$$

1.3.2 Perverse sheaves on stacks

Note that because of Proposition 1.3.4 and Theorem 1.3.5, any of the theories of perverse sheaves on $\mathbb{C}$-schemes or $\mathbb{K}$-schemes discussed in §1.3.1 can be extended to Artin $\mathbb{C}$-stacks or Artin $\mathbb{K}$-stacks X in a naïve way, using the philosophy discussed in §2.2 and [87, §2.7] of defining sheaves on X in terms of sheaves on schemes T for smooth $t : T \to X$, in particular Proposition 2.2.1. This is discussed in [13, §4.3]. We first recall [13, Def. 4.9]:

Definition 1.3.9. Fix one of the theories of perverse sheaves on $\mathbb{K}$-schemes discussed in §1.3.1, over an allowed base ring A, where we include the special case $\mathbb{K} = \mathbb{C}$ and A is general as in [34]. Let X be an Artin $\mathbb{K}$-stack, always assumed locally of finite type. We will explain how to define an abelian category $\mathrm{Perv}_{\mathrm{naï}}(X)$ of *naïve perverse sheaves* on X:

(A) Define an object $\mathcal{P}$ of $\mathrm{Perv}_{\mathrm{naï}}(X)$ to assign

(a) For each $\mathbb{K}$-scheme T and smooth 1-morphism $t : T \to X$, a perverse sheaf $\mathcal{P}(T, t) \in \mathrm{Perv}(T)$ on T in our chosen $\mathbb{K}$-scheme perverse sheaf theory.

(b) For each 2-commutative diagram in $\mathrm{Art}_{\mathbb{K}}$:

$$
\begin{tikzcd}
 & U \arrow[dr, "u"] & \\
T \arrow[ur, "\phi"] \arrow[rr, "t"'] \arrow[rr, phantom, shift left=3, "\eta\Uparrow"] & & X,
\end{tikzcd}
\tag{1.3.6}
$$

where T, U are $\mathbb{K}$-schemes and ϕ, t, u are smooth with ϕ of dimension d, an isomorphism $\mathcal{P}(\phi,\eta) : \phi^*[d](\mathcal{P}(U,u)) \to \mathcal{P}(T,t)$ in $\mathrm{Perv}(T)$.

This data must satisfy the following condition:

(i) For each 2-commutative diagram in $\mathrm{Art}_{\mathbb{K}}$:

$$
\begin{tikzcd}
 & V \arrow[dr, "v"] & \\
U \arrow[ur, "\psi"] \arrow[rr, "u"] \arrow[rr, phantom, shift left=3, "\zeta\Uparrow"] \arrow[rr, phantom, shift right=2, "\eta\Uparrow"] & & X, \\
T \arrow[u, "\phi"] \arrow[urr, bend right=10, "t"'] & &
\end{tikzcd}
$$

with T, U, V $\mathbb{K}$-schemes and ϕ, ψ, t, u, v smooth with ϕ, ψ of dimensions d, e, we must have

$$
\begin{aligned}
\mathcal{P}\bigl(\psi\circ\phi, (\zeta * \mathrm{id}_\phi)\odot\eta\bigr) &= \mathcal{P}(\phi,\eta)\circ\phi^*[d](\mathcal{P}(\psi,\zeta)) \quad \text{as morphisms} \\
(\psi\circ\phi)^*[d+e](\mathcal{P}(V,v)) &= \phi*[d]\circ\psi^*[e](\mathcal{P}(V,v)) \longrightarrow \mathcal{P}(T,t).
\end{aligned}
$$

(B) Morphisms $\alpha : \mathcal{P} \to \mathcal{Q}$ of $\mathrm{Perv}_{\text{naï}}(X)$ comprise a morphism $\alpha(T,t) : \mathcal{P}(T,t) \to \mathcal{Q}(T,t)$ in $\mathrm{Perv}(T)$ for all smooth 1-morphisms $t : T \to X$ from a scheme T, such that for each diagram (1.3.6) in (b) the following commutes:

$$
\begin{tikzcd}
\phi^*[d](\mathcal{P}(U,u)) \arrow[r, "\mathcal{P}(\phi,\eta)"'] \arrow[d, "\phi^*[d](\alpha(U,u))"] & \mathcal{P}(T,t) \arrow[d, "\alpha(T,t)"'] \\
\phi^*[d](\mathcal{Q}(U,u)) \arrow[r, "\mathcal{Q}(\phi,\eta)"] & \mathcal{Q}(T,t).
\end{tikzcd}
$$

(C) Composition of morphisms $\mathcal{P} \xrightarrow{\alpha} \mathcal{Q} \xrightarrow{\beta} \mathcal{R}$ in $\mathrm{Perv}_{\text{naï}}(X)$ is $(\beta\circ\alpha)(T,t) = \beta(T,t)\circ\alpha(T,t)$. Identity morphisms $\mathrm{id}_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$ are $\mathrm{id}_{\mathcal{P}}(T,t) = \mathrm{id}_{\mathcal{P}(T,t)}$.

We can also define a category of *naïve $\mathscr{D}$-modules* on X in the same way.

However, for a satisfactory theory of perverse sheaves on Artin stacks, we want more: we would like the category $\mathrm{Perv}(X)$ of perverse sheaves on X to be the heart of a t-structure on a triangulated category $D^b_c(X)$ of 'constructible complexes', which may not be equivalent to $D^b\,\mathrm{Perv}(X)$, and we would like Grothendieck's "six operations on sheaves" $f^*, f^!, Rf_*, Rf_!, \mathcal{RH}om, \overset{L}{\otimes}$, and Verdier duality operators $\mathbb{D}_X$, to act on these ambient categories $D^b_c(X)$. Other than pullbacks $f^*, f^!$ by smooth 1-morphisms $f : X \to Y$ and operators $\mathbb{D}_X$, none of this is obvious using the definition of perverse sheaves $\mathrm{Perv}_{\text{naï}}(X)$ above. Thus, the main issue in developing a good theories of perverse sheaves on Artin stacks X is not defining the categories $\mathrm{Perv}(X)$ or $\mathrm{Perv}_{\text{naï}}(X)$ themselves, but defining the categories $D^b_c(X)$ and the six operations $f^*, \ldots, \overset{L}{\otimes}$ upon them, and then defining a perverse t-structure on $D^b_c(X)$ with heart $\mathrm{Perv}(X)$. If (a)–(c) of Definition 1.3.9 hold for these $D^b_c(X), \mathrm{Perv}(X)$, it will then be automatic [108, §7] that $\mathrm{Perv}(X) \simeq \mathrm{Perv}_{\text{naï}}(X)$ for $\mathrm{Perv}_{\text{naï}}(X)$ as in Definition 1.3.9. In [13, §4.4], we summarize Laszlo and Olsson's theory [106–108] of perverse sheaves on Artin $\mathbb{K}$-stacks, with finite or l-adic coefficient ring A. Then in [13, §4.5] we outline a theory of perverse sheaves on Artin $\mathbb{C}$-stacks over general base rings A, using the methods of [106–108]. There, we outline a way of extending Dimca's theory [34] of perverse sheaves on

$\mathbb{C}$-schemes X, which uses the complex analytic topology on the underlying set of $\mathbb{C}$-points $X_{\rm an}$ and works over general coefficient rings A, to Artin $\mathbb{C}$-stacks. The key is to work with sheaves on a suitable site:

Definition 1.3.10. Let X be an Artin $\mathbb{C}$-stack (always assumed locally of finite type). Define the *lisse-analytic site* $\text{Lis-an}(X)$ of X as follows. The underlying category of $\text{Lis-an}(X)$ has objects triples (P,T,t) where $t:T\to X$ is a smooth 1-morphism from a $\mathbb{C}$-scheme T, and $P\subseteq T_{\rm an}$ is an open subset in the complex analytic topology of the set $T_{\rm an}$ of $\mathbb{C}$-points of T. A morphism $(\phi,\eta):(P,T,t)\to(Q,U,u)$ in the underlying category is a morphism of $\mathbb{C}$-schemes $\phi:T\to U$ with $\phi_{\rm an}(P)\subseteq Q\subseteq U_{\rm an}$ with a 2-morphism of Artin $\mathbb{C}$-stacks $\eta:u\Rightarrow t\circ\phi$. Composition of morphisms $(P,T,t)\xrightarrow{(\phi,\eta)}(Q,U,u)\xrightarrow{(\psi,\zeta)}(R,V,v)$ is $(\psi,\zeta)\circ(\phi,\eta):=\bigl(\psi\circ\phi,(\zeta*\mathrm{id}_\phi)\odot\eta\bigr)$. The coverings of an object (P,T,t) in the Grothendieck topology on $\text{Lis-an}(X)$ are those collections of morphisms $\bigl\{(\phi_i,\eta_i):(P_i,T_i,t_i)\to(P,T,t)\bigr\}_{i\in I}$ such that $\phi_i:T_i\to T$ is étale with $(\phi_i)_{\rm an}|_{P_i}:P_i\to(\phi_i)_{\rm an}(P_i)$ a homeomorphism for $i\in I$, and $\bigl\{(\phi_i)_{\rm an}(P_i):i\in I\bigr\}$ is an open cover of P.

To build our theory of constructible complexes $D^b_c(X,A)$ with six operations and perverse sheaves $\mathrm{Perv}(X,A)$ over a general commutative ring A, we now follow the method of Laszlo and Olsson [106,108] for finite coefficients, but using sheaves on the lisse-analytic site $\text{Lis-an}(X)$ rather than on the lisse-étale site $\text{Lis-ét}(X)$. Since cohomology in the lisse-analytic topology yields the answer one wants over even general rings A, their programme works without imposing finiteness conditions on A. Laszlo and Olsson remark [106, p. 1] that their method applies to other situations such as complex analytic stacks.

If X is a $\mathbb{C}$-scheme, then the categories of sheaves of sets, A-modules, … on the lisse-analytic site $\text{Lis-an}(X)$ are equivalent to the categories of ordinary sheaves of sets, A-modules, … on $X_{\rm an}$ with the complex analytic topology. Therefore, if X is a $\mathbb{C}$-scheme then these definitions of $D^b_c(X),\mathrm{Perv}(X)$ for X an Artin $\mathbb{C}$-stack are equivalent to those in Dimca [34] for X a $\mathbb{C}$-scheme.

The conclusion is that one can extend Dimca's theory of constructible complexes $D^b_c(X)$ and perverse sheaves $\mathrm{Perv}(X)$ on $\mathbb{C}$-schemes to Artin $\mathbb{C}$-stacks, the six operations f^*, $f^!$, Rf_*, $Rf_!$, $\mathcal{RH}om$, $\overset{L}{\otimes}$ are defined on $D^b_c(X)$, the usual package of properties hold including the stack analogues of Proposition 1.3.4 and Theorem 1.3.5, and $\mathrm{Perv}(X)$ is equivalent to the category $\mathrm{Perv}_{\text{naï}}(X)$ in [13, §4.3].

1.4 Motives on schemes and stacks

Here we introduce some background material on motives. Our notation follows [13,25].

1.4.1 Rings of motives on $\mathbb{K}$-schemes

We begin by defining rings of motives $K_0(\mathrm{Sch}_X),\mathcal{M}_X,K_0^{\hat\mu}(\mathrm{Sch}_X),\mathcal{M}_X^{\hat\mu}$ for a $\mathbb{K}$-scheme X. Some references are Denef and Loeser [30–33], Looijenga [120], and Joyce [86].

Definition 1.4.1. Let X be a $\mathbb{K}$-scheme (always assumed of finite type). Consider pairs (R,ρ), where R is a $\mathbb{K}$-scheme and $\rho:R\to X$ is a morphism. Call two pairs (R,ρ), (R',ρ') *equivalent* if there is an isomorphism $\iota:R\to R'$ with $\rho=\rho'\circ\iota$. Write $[R,\rho]$ for the equivalence class of (R,ρ). If (R,ρ) is a pair and S is a closed $\mathbb{K}$-subscheme of R then $(S,\rho|_S)$, $(R\setminus S,\rho|_{R\setminus S})$ are pairs of the same kind. Define the *Grothendieck ring* $K_0(\mathrm{Sch}_X)$ *of the category* Sch_X *of* $\mathbb{K}$*-schemes over* X to be the abelian group generated by equivalence classes $[R,\rho]$, with the relation that for each closed $\mathbb{K}$-subscheme S of R we have

$$[R,\rho]=[S,\rho|_S]+[R\setminus S,\rho|_{R\setminus S}]. \tag{1.4.1}$$

Define a product '$\cdot$' on $K_0(\mathrm{Sch}_X)$ by

$$[R,\rho]\cdot[S,\sigma]=[R\times_{\rho,X,\sigma}S,\rho\circ\pi_R]. \tag{1.4.2}$$

This is compatible with (1.4.1), and extends to a biadditive, commutative, associative product $\cdot : K_0(\mathrm{Sch}_X)\times K_0(\mathrm{Sch}_X)\to K_0(\mathrm{Sch}_X)$. It makes $K_0(\mathrm{Sch}_X)$ into a commutative ring, with identity $1_X=[X,\mathrm{id}_X]$. Define $\mathbb{L}=[\mathbb{A}^1\times X,\pi_X]$ in $K_0(\mathrm{Sch}_X)$. We denote by

$$\mathcal{M}_X=K_0(\mathrm{Sch}_X)[\mathbb{L}^{-1}] \tag{1.4.3}$$

the ring obtained from $K_0(\mathrm{Sch}_X)$ by inverting $\mathbb{L}$. When $X=\operatorname{Spec}\mathbb{K}$ we write $K_0(\mathrm{Sch}_{\mathbb{K}}),\mathcal{M}_{\mathbb{K}}$ instead of $K_0(\mathrm{Sch}_X),\mathcal{M}_X$.

The *external tensor products* $\boxtimes : K_0(\mathrm{Sch}_X)\times K_0(\mathrm{Sch}_Y)\to K_0(\mathrm{Sch}_{X\times Y})$ and $\boxtimes:\mathcal{M}_X\times\mathcal{M}_Y\to\mathcal{M}_{X\times Y}$ are

$$\Bigl(\sum_{i\in I}c_i[R_i,\rho_i]\Bigr)\boxtimes\Bigl(\sum_{j\in J}d_j[S_j,\sigma_j]\Bigr)=\sum_{i\in I,\,j\in J}c_id_j[R_i\times S_j,\rho_i\times\sigma_j], \tag{1.4.4}$$

for finite I,J. They are biadditive, commutative, and associative. Taking $Y=\operatorname{Spec}\mathbb{K}$ and using $X\times\operatorname{Spec}\mathbb{K}\cong X$, we see that $\boxtimes$ makes $K_0(\mathrm{Sch}_X),\mathcal{M}_X$ into modules over $K_0(\mathrm{Sch}_{\mathbb{K}}),\mathcal{M}_{\mathbb{K}}$.

Let $\phi:X\to Y$ be a morphism of $\mathbb{K}$-schemes. Define the *pushforwards* $\phi_*:K_0(\mathrm{Sch}_X)\to K_0(\mathrm{Sch}_Y)$ and $\phi_*:\mathcal{M}_X\to\mathcal{M}_Y$ by

$$\phi_*:\sum_{i=1}^n c_i[R_i,\rho_i]\longmapsto\sum_{i=1}^n c_i[R_i,\phi\circ\rho_i]. \tag{1.4.5}$$

This intertwines the relation (1.4.1), and so is well-defined.

Define *pullbacks* $\phi^*:K_0(\mathrm{Sch}_Y)\to K_0(\mathrm{Sch}_X)$ and $\phi^*:\mathcal{M}_Y\to\mathcal{M}_X$ by

$$\phi^*:\sum_{i=1}^n c_i[R_i,\rho_i]\longmapsto\sum_{i=1}^n c_i[R_i\times_{\rho_i,Y,\phi}X,\pi_X]. \tag{1.4.6}$$

Pushforwards and pullbacks have the obvious functoriality properties. As in [86, Th. 3.5], pushforwards and pullbacks commute in Cartesian squares, that is, if

$$\begin{tikzcd} W \arrow[r, "\eta"] \arrow[d, "\theta"] & Y \arrow[d, "\psi"] \\ X \arrow[r, "\phi"] & Z \end{tikzcd} \quad \text{is a Cartesian square in the category } \mathrm{Sch}_{\mathbb{K}} \text{ then the following commutes:} \quad \begin{tikzcd} \mathcal{M}_W \arrow[r, "\eta_*"] & \mathcal{M}_Y \\ \mathcal{M}_X \arrow[r, "\phi_*"] \arrow[u, "\theta^*"] & \mathcal{M}_Z, \arrow[u, "\psi^*"] \end{tikzcd} \tag{1.4.7}$$

and the analogue holds for $K_0(\mathrm{Sch}_W),\ldots,K_0(\mathrm{Sch}_Z)$.

Definition 1.4.2. For $n=1,2,\ldots$, write μ_n for the group of all n^{th} roots of unity in $\mathbb{K}$, which is assumed algebraically closed of characteristic zero, so that $\mu_n\cong\mathbb{Z}_n$. Then μ_n is the $\mathbb{K}$-scheme $\operatorname{Spec}(\mathbb{K}[x]/(x^n-1))$. The μ_n form a projective system, with respect to the maps $\mu_{nd}\to\mu_n$ mapping $x\mapsto x^d$ for all $d,n=1,2,\ldots$. Define the group $\hat\mu$ to be the projective limit of the μ_n. Note that $\hat\mu$ is not a $\mathbb{K}$-scheme, but is a pro-scheme.

Let R be a $\mathbb{K}$-scheme. A *good* μ_n*-action* on R is a group action $r_n:\mu_n\times R\to R$ such that such that each orbit is contained in an open affine subscheme of R and $\rho\circ r_n(\gamma)\cong\rho$ for all $\gamma\in\mu_n$. A *good* $\hat\mu$*-action on* R is a group action $\hat r:\hat\mu\times R\to R$ which factors through a good μ_n-action, for some n. We will write $\hat\iota:\hat\mu\times R\to R$ for the trivial $\hat\mu$-action on R, which is automatically good.

Consider triples $(R,\rho,\hat r)$, where R is a $\mathbb{K}$-scheme, $\rho:R\to X$ a morphism, and $\hat r:\hat\mu\times R\to R$ a good $\hat\mu$-action on R. Call two such triples $(R,\rho,\hat r),(R',\rho',\hat r')$ *equivalent* if there exists a $\hat\mu$-equivariant isomorphism $\iota:R\to R'$ with $\rho=\rho'\circ\iota$. Write $[R,\rho,\hat r]$ for the equivalence class of $(R,\rho,\hat r)$.

The *monodromic Grothendieck group* $K_0^{\hat\mu}(\mathrm{Sch}_X)$ is the abelian group generated by such equivalence classes $[R,\rho,\hat r]$, with the relations:

(i) $[R,\rho,\hat r]=[S,\rho|_S,\hat r|_S]+[R\setminus S,\rho|_{R\setminus S},\hat r|_{R\setminus S}]$ for each closed $\hat\mu$-invariant $\mathbb{K}$-subscheme S of R;

(ii) given $[R_1,\rho_1,\hat r_1],[R_2,\rho_2,\hat r_2]$ with $\pi:R_2\to R_1$ a $\hat\mu$-equivariant vector bundle of rank d over R_1 and $\rho_2=\rho_1\circ\pi$, then $[R_2,\rho_2]=[R_1\times\mathbb{A}^d,\rho_1\circ\pi,\hat r_1\times\hat\iota]$.

There is a natural biadditive product '$\cdot$' on $K_0^{\hat\mu}(\mathrm{Sch}_X)$ given by

$$[R,\rho,\hat r]\cdot[S,\sigma,\hat s]=[R\times_{\rho,X,\sigma}S,\rho\circ\pi_R,\hat r\times\hat s],\tag{1.4.8}$$

making $K_0^{\hat\mu}(\mathrm{Sch}_X)$ into a commutative ring, with identity $1_X=[X,\mathrm{id}_X,\hat\iota]$.

Define $\mathbb{L}=[\mathbb{A}^1\times X,\pi_X,\hat\iota]$ in $K_0^{\hat\mu}(\mathrm{Sch}_X)$. We denote by

$$\mathcal{M}_X^{\hat\mu}=K_0^{\hat\mu}(\mathrm{Sch}_X)[\mathbb{L}^{-1}]$$

the ring obtained from $K_0^{\hat\mu}(\mathrm{Sch}_X)$ by inverting $\mathbb{L}$. When $X=\operatorname{Spec}\mathbb{K}$ we write $K_0^{\hat\mu}(\mathrm{Sch}_{\mathbb{K}}),\mathcal{M}_{\mathbb{K}}^{\hat\mu}$ instead of $K_0^{\hat\mu}(\mathrm{Sch}_X),\mathcal{M}_X^{\hat\mu}$.

The *external tensor products* $\boxtimes:K_0^{\hat\mu}(\mathrm{Sch}_X)\times K_0^{\hat\mu}(\mathrm{Sch}_Y)\to K_0^{\hat\mu}(\mathrm{Sch}_{X\times Y})$ and $\boxtimes:\mathcal{M}_X^{\hat\mu}\times\mathcal{M}_Y^{\hat\mu}\to\mathcal{M}_{X\times Y}^{\hat\mu}$ are

$$\Bigl(\sum_{i\in I}c_i[R_i,\rho_i,\hat r_i]\Bigr)\boxtimes\Bigl(\sum_{j\in J}d_j[S_j,\sigma_j,\hat s_j]\Bigr)=\sum_{i\in I,\,j\in J}c_id_j[R_i\times S_j,\rho_i\times\sigma_j,\hat r_i\times\hat s_j],\tag{1.4.9}$$

for finite I,J. Pushforwards ϕ_* and pullbacks ϕ^* are defined for $K_0^{\hat\mu}(\mathrm{Sch}_X),\mathcal{M}_X^{\hat\mu}$ in the obvious way, and the analogue of (1.4.7) holds.

There are natural morphisms of commutative rings

$$\begin{aligned}i_X:K_0(\mathrm{Sch}_X)&\longrightarrow K_0^{\hat\mu}(\mathrm{Sch}_X),&\quad i_X:\mathcal{M}_X&\longrightarrow\mathcal{M}_X^{\hat\mu},\\ \Pi_X:K_0^{\hat\mu}(\mathrm{Sch}_X)&\longrightarrow K_0(\mathrm{Sch}_X),&\quad \Pi_X:\mathcal{M}_X^{\hat\mu}&\longrightarrow\mathcal{M}_X,\end{aligned}\tag{1.4.10}$$

given by $i_X:[R,\rho]\mapsto[R,\rho,\hat\iota]$ and $\Pi_X:[R,\rho,\hat r]\mapsto[R,\rho]$.

Following Looijenga [120, §7] and Denef and Loeser [32, §5], we introduce a second multiplication '$\odot$' on $K_0^{\hat\mu}(\mathrm{Sch}_X),\mathcal{M}_X^{\hat\mu}$ (written '$*$' in [32, 120]).

Definition 1.4.3. Let X be a $\mathbb{K}$-scheme and $[R,\rho,\hat r],[S,\sigma,\hat s]$ be generators of $K_0^{\hat\mu}(\mathrm{Sch}_X)$. Then there exists $n\geqslant 1$ such that the $\hat\mu$-actions $\hat r,\hat s$ on R,S factor through μ_n-actions r_n,s_n. Define J_n to be the Fermat curve $J_n=\bigl\{(t,u)\in(\mathbb{A}^1\setminus\{0\})^2:t^n+u^n=1\bigr\}$. Let $\mu_n\times\mu_n$ act on $J_n\times(R\times_X S)$ by $(\alpha,\alpha')\cdot\bigl((t,u),(v,w)\bigr)=\bigl((\alpha\cdot t,\alpha'\cdot u),(r_n(\alpha)(v),s_n(\alpha')(w))\bigr)$. Write $J_n(R,S)=(J_n\times(R\times_X S))/(\mu_n\times\mu_n)$ for the quotient $\mathbb{K}$-scheme, and define a μ_n-action υ_n on $J_n(R,S)$ by $\upsilon_n(\alpha)\bigl((t,u),v,w\bigr)(\mu_n\times\mu_n)=\bigl((\alpha\cdot t,\alpha\cdot u),v,w\bigr)(\mu_n\times\mu_n)$. Let $\hat\upsilon$ be the induced good $\hat\mu$-action on $J_n(R,S)$, and set

$$[R,\rho,\hat r]\odot[S,\sigma,\hat s]=(\mathbb{L}-1)\cdot\bigl[(R\times_X S)/\mu_n,\hat\iota\bigr]-\bigl[J_n(R,S),\hat\upsilon\bigr]\tag{1.4.11}$$

in $K_0^{\hat\mu}(\mathrm{Sch}_X)$ and $\mathcal{M}_X^{\hat\mu}$. This turns out to be independent of n, and defines commutative, associative products $\odot$ on $K_0^{\hat\mu}(\mathrm{Sch}_X)$ and $\mathcal{M}_X^{\hat\mu}$.

Let X, Y be $\mathbb{K}$-schemes. As for Definitions 1.4.1 and 1.4.2, we define products

$$\boxdot : K_0^{\hat\mu}(\mathrm{Sch}_X)\times K_0^{\hat\mu}(\mathrm{Sch}_Y)\to K_0^{\hat\mu}(\mathrm{Sch}_{X\times Y}),\ \ \boxdot : \mathcal{M}_X^{\hat\mu}\times\mathcal{M}_X^{\hat\mu}\to\mathcal{M}_{X\times Y}^{\hat\mu}$$

by following the definition above for $[R,\rho,\hat r]\in K_0^{\hat\mu}(\mathrm{Sch}_X)$, $[S,\sigma,\hat s]\in K_0^{\hat\mu}(\mathrm{Sch}_Y)$, but taking products $R\times S$ rather than fibre products $R\times_X S$. These $\boxdot$ are also commutative and associative in the appropriate sense. Taking $Y=\operatorname{Spec}\mathbb{K}$ and using $X\times\operatorname{Spec}\mathbb{K}\cong X$, we see that $\boxdot$ makes $K_0^{\hat\mu}(\mathrm{Sch}_X),\mathcal{M}_X^{\hat\mu}$ into modules over $K_0^{\hat\mu}(\mathrm{Sch}_{\mathbb{K}}),\mathcal{M}_{\mathbb{K}}^{\hat\mu}$. For generators $[R,\rho,\hat r]$ and $[S,\sigma,\hat\iota]=i_X([S,\sigma])$ in $K_0^{\hat\mu}(\mathrm{Sch}_X)$ or $\mathcal{M}_X^{\hat\mu}$ where $[S,\sigma,\hat\iota]$ has trivial $\hat\mu$-action $\hat\iota$, one can show that $[R,\rho,\hat r]\odot[S,\sigma,\hat\iota]=[R,\rho,\hat r]\cdot[S,\sigma,\hat\iota]$. Thus i_X is a ring morphism $\bigl(K_0(\mathrm{Sch}_X),\cdot\bigr)\to\bigl(K_0^{\hat\mu}(\mathrm{Sch}_X),\odot\bigr)$ and $\bigl(\mathcal{M}_X,\cdot\bigr)\to\bigl(\mathcal{M}_X^{\hat\mu},\odot\bigr)$. However, Π_X is not a ring morphism $\bigl(K_0^{\hat\mu}(\mathrm{Sch}_X),\odot\bigr)\to\bigl(K_0(\mathrm{Sch}_X),\cdot\bigr)$ or $\bigl(\mathcal{M}_X^{\hat\mu},\odot\bigr)\to\bigl(\mathcal{M}_X,\cdot\bigr)$. Since $\mathbb{L}=[\mathbb{A}^1\times X,\pi_X,\hat\iota]$ this implies that $M\cdot\mathbb{L}=M\odot\mathbb{L}$ for all M in $K_0^{\hat\mu}(\mathrm{Sch}_X),\mathcal{M}_X^{\hat\mu}$.

Definition 1.4.4. Define the element $\mathbb{L}^{1/2}$ in $K_0^{\hat\mu}(\mathrm{Sch}_X)$ and $\mathcal{M}_X^{\hat\mu}$ by

$$\mathbb{L}^{1/2}=[X,\mathrm{id}_X,\hat\iota]-[X\times\mu_2,\hat r], \tag{1.4.12}$$

where $[X,\mathrm{id}_X,\hat\iota]$ with trivial $\hat\mu$-action $\hat\iota$ is the identity 1_X in $K_0^{\hat\mu}(\mathrm{Sch}_X),\mathcal{M}_X^{\hat\mu}$, and $X\times\mu_2=X\times\{1,-1\}$ is two copies of X with nontrivial $\hat\mu$-action $\hat r$ induced by the left action of μ_2 on itself, exchanging the two copies of X. Applying (1.4.11) with $n=2$, we can show that $\mathbb{L}^{1/2}\odot\mathbb{L}^{1/2}=\mathbb{L}$. Thus, $\mathbb{L}^{1/2}$ in (1.4.12) is a square root for $\mathbb{L}$ in the rings $\bigl(K_0^{\hat\mu}(\mathrm{Sch}_X),\odot\bigr),\bigl(\mathcal{M}_X^{\hat\mu},\odot\bigr)$. Note that $\mathbb{L}^{1/2}\cdot\mathbb{L}^{1/2}\neq\mathbb{L}$.

Equivalently, we could have defined

$$\mathbb{L}_X^{1/2}=[X,\mathrm{id}_X,\hat\iota]\boxdot\mathbb{L}_{\mathbb{K}}^{1/2}\in K_0^{\hat\mu}(\mathrm{Sch}_X), \tag{1.4.13}$$

where $\mathbb{L}_{\mathbb{K}}^{1/2}\in K_0^{\hat\mu}(\mathrm{Sch}_{\mathbb{K}})$. We can now define unique elements $\mathbb{L}^{n/2}$ in $K_0^{\hat\mu}(\mathrm{Sch}_X)$ for all $n=0,1,2,\ldots$ and $\mathbb{L}^{n/2}$ in $\mathcal{M}_X^{\hat\mu}$ for all $n\in\mathbb{Z}$ in the obvious way, such that $\mathbb{L}^{m/2}\odot\mathbb{L}^{n/2}=\mathbb{L}^{(m+n)/2}$ for all $m,n\geqslant 0$ or $m,n\in\mathbb{Z}$.

Next, following [25, §2.5], which was motivated by ideas in Kontsevich and Soibelman [102, §4.5], we define principal $\mathbb{Z}/2\mathbb{Z}$-bundles $P\to X$, associated motives $\Upsilon(P)$, and a quotient ring of motives $\bar{\mathcal{M}}_X^{\hat\mu}$ in which $\Upsilon(P\otimes_{\mathbb{Z}/2\mathbb{Z}}Q)=\Upsilon(P)\odot\Upsilon(Q)$ for all P,Q.

Definition 1.4.5. Let X be a $\mathbb{K}$-scheme. A *principal* $\mathbb{Z}/2\mathbb{Z}$*-bundle* $P\to X$ is a proper, surjective, étale morphism of $\mathbb{K}$-schemes $\pi:P\to X$ together with a free involution $\sigma:P\to P$, such that the orbits of $\mathbb{Z}/2\mathbb{Z}=\{1,\sigma\}$ are the fibres of π. The *trivial* $\mathbb{Z}/2\mathbb{Z}$*-bundle* is $\pi_X:X\times\mathbb{Z}/2\mathbb{Z}\to X$. We will use the ideas of *isomorphism* of principal bundles $\iota:P\to Q$, *section* $s:X\to P$, *tensor product* $P\otimes_{\mathbb{Z}/2\mathbb{Z}}Q$, and *pullback* $f^*(P)\to Y$ under a 1-morphism of stacks $f:Y\to X$, all of which are defined in the obvious ways. Write $(\mathbb{Z}/2\mathbb{Z})(X)$ for the abelian group of isomorphism classes $[P]$ of principal $\mathbb{Z}/2\mathbb{Z}$-bundles $P\to X$, with multiplication $[P]\cdot[Q]=[P\otimes_{\mathbb{Z}/2\mathbb{Z}}Q]$ and identity $[X\times\mathbb{Z}/2\mathbb{Z}]$. Since $P\otimes_{\mathbb{Z}/2\mathbb{Z}}P\cong X\times\mathbb{Z}/2\mathbb{Z}$ for each $P\to X$, each element of $(\mathbb{Z}/2\mathbb{Z})(X)$ is self-inverse, and has order 1 or 2. If $\pi:P\to X$ is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle over X, define a motive

$$\Upsilon(P)=\mathbb{L}^{-1/2}\odot\bigl([X,\mathrm{id},\hat\iota]-[P,\pi,\hat r]\bigr)\in\mathcal{M}_X^{\hat\mu},$$

where $\hat r$ is the $\hat\mu$-action on P induced by the μ_2-action on P from the principal $\mathbb{Z}/2\mathbb{Z}$-bundle structure, as $\mu_2\cong\mathbb{Z}/2\mathbb{Z}$. If $P=X\times\mathbb{Z}/2\mathbb{Z}$ is the trivial $\mathbb{Z}/2\mathbb{Z}$-bundle then

$$\Upsilon(X\times\mathbb{Z}/2\mathbb{Z})=\mathbb{L}^{-1/2}\odot\bigl([X,\mathrm{id},\hat\iota]-[X\times\mathbb{Z}/2\mathbb{Z},\pi,\hat r]\bigr)=\mathbb{L}^{-1/2}\odot\mathbb{L}^{1/2}\odot[X,\mathrm{id},\hat\iota]=[X,\mathrm{id},\hat\iota],$$

using (1.4.12). Note that $[X, \mathrm{id}, \hat{\imath}]$ is the identity in the ring $\mathcal{M}_X^{\hat{\mu}}$. As $\Upsilon(P)$ only depends on P up to isomorphism, Υ factors via $(\mathbb{Z}/2\mathbb{Z})(X)$, and we may consider Υ as a map $(\mathbb{Z}/2\mathbb{Z})(X) \to \mathcal{M}_X^{\hat{\mu}}$.

For our applications, we want $\Upsilon : (\mathbb{Z}/2\mathbb{Z})(X) \to \mathcal{M}_X^{\hat{\mu}}$ to be a group morphism with respect to the multiplication $\odot$ on $\mathcal{M}_X^{\hat{\mu}}$, but we cannot prove that it is. Our solution is to pass to a quotient ring $\bar{\mathcal{M}}_X^{\hat{\mu}}$ of $\mathcal{M}_X^{\hat{\mu}}$ such that the induced map $\Upsilon : (\mathbb{Z}/2\mathbb{Z})(X) \to \bar{\mathcal{M}}_X^{\hat{\mu}}$ is a group morphism. If we simply defined $\bar{\mathcal{M}}_X^{\hat{\mu}}$ to be the quotient ring of $\mathcal{M}_X^{\hat{\mu}}$ by the relations $\Upsilon(P \otimes_{\mathbb{Z}/2\mathbb{Z}} Q) - \Upsilon(P) \odot \Upsilon(Q) = 0$ for all $[P], [Q]$ in $(\mathbb{Z}/2\mathbb{Z})(X)$ then pushforwards $\phi_* : \bar{\mathcal{M}}_X^{\hat{\mu}} \to \bar{\mathcal{M}}_Y^{\hat{\mu}}$ would not be defined for general 1-morphisms $\phi : X \to Y$. So we impose a more complicated relation.

For each $\mathbb{K}$-scheme Y, define $I_Y^{\hat{\mu}}$ to be the ideal in the commutative ring $\big(\mathcal{M}_Y^{\hat{\mu}}, \odot\big)$ generated by elements $\phi_*\big(\Upsilon(P \otimes_{\mathbb{Z}/2\mathbb{Z}} Q) - \Upsilon(P) \odot \Upsilon(Q)\big)$ for all $\mathbb{K}$-scheme morphisms $\phi : X \to Y$ and principal $\mathbb{Z}/2\mathbb{Z}$-bundles $P, Q \to X$, and define $\bar{\mathcal{M}}_Y^{\hat{\mu}} = \mathcal{M}_Y^{\hat{\mu}} / I_Y^{\hat{\mu}}$ to be the quotient, as a commutative ring with multiplication '$\odot$', with projection $\Pi_Y^{\hat{\mu}} : \mathcal{M}_Y^{\hat{\mu}} \to \bar{\mathcal{M}}_Y^{\hat{\mu}}$. Kontsevich and Soibelman [102, §4.5] introduce a relation in their motivic rings which has a similar effect.

Note that in $\bar{\mathcal{M}}_Y^{\hat{\mu}}$ we do not have the second multiplication '$\cdot$', since we do not require $I_Y^{\hat{\mu}}$ to be an ideal in $\big(\mathcal{M}_Y^{\hat{\mu}}, \cdot\big)$. Also $\boxtimes$ and $\Pi_Y : \mathcal{M}_Y^{\hat{\mu}} \to \mathcal{M}_Y$ on $\mathcal{M}_Y^{\hat{\mu}}$ do not descend to $\bar{\mathcal{M}}_Y^{\hat{\mu}}$. Apart from this, all the structures on $\mathcal{M}_Y^{\hat{\mu}}$ above descend to $\bar{\mathcal{M}}_Y^{\hat{\mu}}$: operations $\odot, \boxdot$, pushforwards ϕ_* and pullbacks ϕ^*, and elements $\mathbb{L}, \mathbb{L}^{1/2}, \Upsilon(P)$. By definition, $\bar{\mathcal{M}}_X^{\hat{\mu}}$ has the property that

$$\Upsilon(P \otimes_{\mathbb{Z}/2\mathbb{Z}} Q) = \Upsilon(P) \odot \Upsilon(Q) \quad \text{in } \bar{\mathcal{M}}_X^{\hat{\mu}} \tag{1.4.14}$$

for all principal $\mathbb{Z}/2\mathbb{Z}$-bundles $P, Q \to X$.

Following Denef and Loeser [32], we define motivic nearby cycles, motivic Milnor fibres, and motivic vanishing cycles:

Definition 1.4.6. Let U be a smooth $\mathbb{K}$-scheme and $f : U \to \mathbb{A}^1$ a regular function, and set $U_0 = f^{-1}(0) \subseteq U$. Then Denef and Loeser [32, §3.5] and Looijenga [120, §5] define the *motivic nearby cycle* of f, an element $MF_{U,f}^{\mathrm{mot}}$ of $\mathcal{M}_{U_0}^{\hat{\mu}}$ or $\bar{\mathcal{M}}_{U_0}^{\hat{\mu}}$. It has an intrinsic definition using arc spaces and the motivic zeta function, which we will not explain, but we will give a formula [32, §3.3], [120, §5] for $MF_{U,f}^{\mathrm{mot}}$ involving choosing a resolution of f.

If $f = 0$ then $MF_{U,f}^{\mathrm{mot}} = 0$, so suppose f is not constant. By Hironaka's Theorem [66] we can choose a *resolution* $(\tilde{U}, \pi)$ of f. That is, $\tilde{U}$ is a smooth $\mathbb{K}$-scheme and $\pi : \tilde{U} \to U$ a proper morphism, such that $\pi|_{\tilde{U} \setminus \pi^{-1}(U_0)} : \tilde{U} \setminus \pi^{-1}(U_0) \to U \setminus U_0$ is an isomorphism, and $\pi^{-1}(U_0)^{\mathrm{red}}$ has only normal crossings as a $\mathbb{K}$-subscheme of $\tilde{U}$. Write E_i, $i \in J$ for the irreducible components of $\pi^{-1}(U_0)$. For each $i \in J$, denote by N_i the multiplicity of E_i in the divisor of $f \circ \pi$ on $\tilde{U}$, and by $\nu_i - 1$ the multiplicity of E_i in the divisor of $\pi^*(\mathrm{d}x)$, where $\mathrm{d}x$ is a local non vanishing volume form at any point of $\pi(E_i)$. For $I \subset J$, we consider the smooth $\mathbb{K}$-scheme $E_I^\circ = \big(\bigcap_{i \in I} E_i\big) \setminus \big(\bigcup_{j \in J \setminus I} E_j\big)$.

Let $m_I = \gcd(N_i)_{i \in I}$. We introduce an unramified Galois cover $\tilde{E}_I^\circ$ of E_I°, with Galois group μ_{m_I}, as follows. Let $\tilde{U}'$ be an affine Zariski open subset of $\tilde{U}$, such that, on $\tilde{U}'$, $f \circ \pi = uv^{m_I}$, with $u : \tilde{U}' \to \mathbb{A}^1 \setminus \{0\}$ and $v : \tilde{U}' \to \mathbb{A}^1$. Then the restriction of $\tilde{E}_I^\circ$ above $E_I^\circ \cap \tilde{U}'$, denoted by $\tilde{E}_I^\circ \cap \tilde{U}'$, is defined as

$$\tilde{E}_I^\circ \cap \tilde{U}' = \big\{(z, w) \in \mathbb{A}^1 \times (E_I^\circ \cap \tilde{U}') : z^{m_I} = u(w)^{-1}\big\}.$$

Gluing together the covers $\tilde{E}_I^\circ \cap \tilde{U}'$ in the obvious way, we obtain the cover $\tilde{E}_I^\circ$ of E_I° which has a natural μ_{m_I}-action ρ_I, obtained by multiplying the z-coordinate by elements of μ_{m_I}. This μ_{m_I}-action on $\tilde{E}_I^\circ$ induces a $\hat{\mu}$-action $\hat{\rho}_I$ on $\tilde{E}_I^\circ$. Then

$$MF_{U,f}^{\mathrm{mot}} = \sum_{\emptyset \neq I \subseteq J} (1 - \mathbb{L})^{|I|-1} \big[\tilde{E}_I^\circ, \pi_{U_0}, \hat{\rho}_I\big] \quad \text{in } \mathcal{M}_{U_0}^{\hat{\mu}}. \tag{1.4.15}$$

It is independent of the choice of resolution $(\tilde{U},\pi)$. The fibre $MF^{\rm mot}_{U,f}|_x$ at each $x\in U_0$ is called the *motivic Milnor fibre* of f at x.

Now let $X=\mathrm{Crit}(f)\subseteq U$, as a closed $\mathbb{K}$-subscheme of U. Since f is constant on the reduced scheme $X^{\rm red}$, $f(X)$ is finite, and we may write $X=\coprod_{c\in f(X)} X_c$, where $X_c\subseteq X$ is the open and closed $\mathbb{K}$-subscheme with $X_c^{\rm red}=f|_{X^{\rm red}}^{-1}(c)$. Consider the restriction $MF^{\rm mot}_{U,f}|_{U_0\setminus X_0}$ in $\mathcal{M}^{\hat\mu}_{U_0\setminus X_0}$ or $\overline{\mathcal{M}}^{\hat\mu}_{U_0\setminus X_0}$. We can choose $(\tilde{U},\pi)$ above with $\pi|_{\tilde U\setminus\pi^{-1}(X_0)}:\tilde U\setminus\pi^{-1}(X_0)\to U\setminus X_0$ an isomorphism. Write $D_1,\ldots,D_k$ for the irreducible components of $\pi^{-1}(U_0\setminus X_0)\cong U_0\setminus X_0$. They are disjoint as $\pi^{-1}(U_0\setminus X_0)$ is nonsingular. The closures $\overline{D}_1,\ldots,\overline{D}_k$ (which need not be disjoint) are among the divisors E_i, so we write $\overline{D}_a=E_{i_a}$ for $a=1,\ldots,k$, with $\{i_1,\ldots,i_k\}\subseteq I$. Clearly $N_{i_a}=\nu_{i_a}=1$ for $a=1,\ldots,k$. Then in (1.4.15) the only nonzero contributions to $MF^{\rm mot}_{U,f}|_{U_0\setminus X_0}$ are from $I=\{i_a\}$ for $a=1,\ldots,k$, with $\tilde{E}^\circ_{\{i_a\}}\cong E^\circ_{\{i_a\}}\cong D_a$, and the $\hat\mu$-action on $\tilde{E}^\circ_{\{i_a\}}$ is trivial as it factors through the action of $\mu_1=\{1\}$. Hence

$$MF^{\rm mot}_{U,f}|_{U_0\setminus X_0}=\sum_{a=1}^k\big[\tilde{E}^\circ_{\{i_a\}},\pi_{U_0\setminus X_0},\hat\iota\big]=\sum_{a=1}^k\big[D_a,\pi_{U_0\setminus X_0},\hat\iota\big]=\big[U_0\setminus X_0,\mathrm{id}_{U_0\setminus X_0},\hat\iota\big].$$

Therefore $[U_0,\mathrm{id}_{U_0},\hat\iota]-MF^{\rm mot}_{U,f}$ is supported on $X_0\subseteq U_0$, and by restricting to X_0 we regard it as an element of $\mathcal{M}^{\hat\mu}_{X_0}$ or $\overline{\mathcal{M}}^{\hat\mu}_{X_0}$. Define the *motivic vanishing cycle* $MF^{{\rm mot},\phi}_{U,f}$ of f in $\mathcal{M}^{\hat\mu}_X$ or $\overline{\mathcal{M}}^{\hat\mu}_X$ by

$$MF^{{\rm mot},\phi}_{U,f}\big|_{X_c}=\mathbb{L}^{-\dim U/2}\odot\big([U_c,\mathrm{id}_{U_c},\hat\iota]-MF^{\rm mot}_{U,f-c}\big)\big|_{X_c}\tag{1.4.16}$$

for each $c\in f(X)$, where $\odot$ and $\mathbb{L}^{-\dim U/2}$ are as in Definitions 1.4.3 and 1.4.4.

Here is the motivic Thom–Sebastiani Theorem of Denef–Loeser and Looijenga [31, 32, 120], stated using the notation of [25, §2.4].

Theorem 1.4.7. *Let U,V be smooth $\mathbb{K}$-schemes, $f:U\to\mathbb{A}^1$, $g:V\to\mathbb{A}^1$ regular functions, and $X=\mathrm{Crit}(f)$, $Y=\mathrm{Crit}(g)$. Write $f\boxplus g:U\times V\to\mathbb{A}^1$ for the regular function mapping $f\boxplus g:(u,v)\mapsto f(u)+g(v)$. Then $MF^{{\rm mot},\phi}_{U\times V,f\boxplus g}=MF^{{\rm mot},\phi}_{U,f}\boxdot MF^{{\rm mot},\phi}_{V,g}$ in $\mathcal{M}^{\hat\mu}_{X\times Y}$.*

Example 1.4.8. Define $f:\mathbb{A}^n\to\mathbb{A}^1$ by $f(z_1,\ldots,z_n)=z_1^2+\cdots+z_n^2$ for $n\geqslant 1$. Then using Theorem 1.4.7, induction on n, and

$$MF^{{\rm mot},\phi}_{\mathbb{A}^1,z^2}=\mathbb{L}^{-1/2}\odot\big(1-[\mu_2,\hat\rho]\big)=\mathbb{L}^{-1/2}\odot\mathbb{L}^{1/2}=1,$$

shows that

$$MF^{{\rm mot},\phi}_{\mathbb{A}^n,z_1^2+\cdots+z_n^2}=MF^{{\rm mot},\phi}_{\mathbb{A}^1,z^2}\boxdot\cdots\boxdot MF^{{\rm mot},\phi}_{\mathbb{A}^1,z^2}=1\boxdot\cdots\boxdot 1=1.\tag{1.4.17}$$

If V is a finite-dimensional $\mathbb{K}$-vector space and q a nondegenerate quadratic form on V, then $(V,q)\cong(\mathbb{A}^n,z_1^2+\cdots+z_n^2)$ for $n=\dim V$, so $MF^{{\rm mot},\phi}_{V,q}=1$. That is the purpose of the factors $\mathbb{L}^{-\dim U/2}$.

In [25, Thm. 2.16], reported below, we show how motivic vanishing cycles change under stabilization by a nondegenerate quadratic form. The term $\mathbb{L}^{\dim U/2}\odot MF^{{\rm mot},\phi}_{E,q}$ in (1.4.18) may be regarded as the *relative motivic vanishing cycle* of (E,q) relative to U.

Theorem 1.4.9. *Let U be a smooth $\mathbb{K}$-scheme, $\pi:E\to U$ a vector bundle over U, $f:U\to\mathbb{A}^1$ a regular function, q a nondegenerate quadratic form on E, and $X=\mathrm{Crit}(f)$. Regard (the total space of) E as a smooth $\mathbb{K}$-scheme and $q,f\circ\pi:E\to\mathbb{A}^1$ as regular functions on E, so that*

$f\circ\pi+q:E\to\mathbb{A}^1$ is also a regular function. Identify U with the zero section in E, so that $X\subseteq U\subseteq E$, and we have $\mathcal{M}^{\hat\mu}_X\subseteq\mathcal{M}^{\hat\mu}_U\subseteq\mathcal{M}^{\hat\mu}_E$. Then in $\mathcal{M}^{\hat\mu}_E$ we have

$$MF^{\mathrm{mot},\phi}_{E,f\circ\pi+q}=MF^{\mathrm{mot},\phi}_{U,f}\odot\big(\mathbb{L}^{\dim U/2}\odot MF^{\mathrm{mot},\phi}_{E,q}\big). \tag{1.4.18}$$

Theorem 2.20 in [25] gives an expression (1.4.19) for motivic vanishing cycles $MF^{\mathrm{mot},\phi}_{E,q}$ of nondegenerate quadratic forms on vector bundles. The proof uses (1.4.14), and so holds only in $\bar{\mathcal{M}}^{\hat\mu}_U$ rather than in $\mathcal{M}^{\hat\mu}_U$. Note that $\mathbb{L}^{\dim U/2}\odot MF^{\mathrm{mot},\phi}_{E,q}$ in (1.4.19) also occurs in equation (1.4.18) of Theorem 1.4.9.

Theorem 1.4.10. *Let U be a smooth $\mathbb{K}$-scheme, $E\to U$ a vector bundle of rank r, and $q\in H^0(S^2E^*)$ a nondegenerate quadratic form on the fibres of E. Regard $q:E\to\mathbb{A}^1$ as a regular function on the total space of E, which is a nondegenerate homogeneous quadratic polynomial on each fibre E_u of E, so that $\mathrm{Crit}(q)\subseteq E$ is the zero section of E, which we identify with U.*

Then $\Lambda^rE\to U$ is a line bundle, and the determinant $\det(q)$ is a nonvanishing section of $(\Lambda^rE^*)^{\otimes^2}$, or equivalently an isomorphism $(\Lambda^rE)\otimes_{\mathcal{O}_U}(\Lambda^rE)\to\mathcal{O}_U$. Thus there is a principal $\mathbb{Z}_2$-bundle $P\to U$, unique up to isomorphism, corresponding to $\big(\Lambda^rE,\det(q)\big)$ under the 1-1 correspondence [25, Rem. 2.18]. We have

$$\Upsilon(P)=\mathbb{L}^{\dim U/2}\odot MF^{\mathrm{mot},\phi}_{E,q}\quad\text{in }\bar{\mathcal{M}}^{\hat\mu}_U. \tag{1.4.19}$$

Theorem 1.4.10 is more-or-less equivalent to material in Kontsevich and Soibelman [102, §5.1]. It implies that $MF^{\mathrm{mot},\phi}_{E,q}$ depends only on $U,r,\Lambda^rE,\det(q)$, which is important in their definition of motivic Donaldson–Thomas invariants. As for our $\bar{\mathcal{M}}^{\hat\mu}_X$, Kontsevich and Soibelman [102, §4.5] also introduce an extra relation in their ring of motives to make the analogue of Theorem 1.4.10 true. We defined the ring $\bar{\mathcal{M}}^{\hat\mu}_Y$ by imposing the pushforward ϕ_* of relation (1.4.14) in $\mathcal{M}^{\hat\mu}_X$ under all morphisms $\phi:X\to Y$. We may rewrite (1.4.19) as

$$\Upsilon(P)=MF^{\mathrm{mot},\phi,\mathrm{rel}}_{E\to U,q}, \tag{1.4.20}$$

for $P\to U$ the principal $\mathbb{Z}_2$-bundle corresponding to $\big(\Lambda^rE,\det(q)\big)$ in the sense of [25, Rem. 2.18]. We may regard (1.4.20) as a relation in $\mathcal{M}^{\hat\mu}_U$, which is equivalent to (1.4.14). Thus, an alternative definition of the rings $\bar{\mathcal{M}}^{\hat\mu}_Y$, closer in spirit to Kontsevich and Soibelman [102, §4.5 & §5.1], is to impose the relation in $\mathcal{M}^{\hat\mu}_Y$ that for all $\mathbb{K}$-scheme morphisms $\phi:U\to Y$, rank r vector bundles $E\to U$, and nondegenerate quadratic forms $q\in H^0(S^2E^*)$, the pushforward $\phi_*\big(MF^{\mathrm{mot},\phi,\mathrm{rel}}_{E\to U,q}\big)$ depends only on U,ϕ and $\big(\Lambda^rE,\det(q)\big)$. See [25, Rem. 2.21] for a more detailed discussion.

1.4.2 Motives on stacks

We now generalize the material of §1.4.1 to Artin stacks following [13, §5]. Our definitions are new, but very similar to work by Joyce [86] on 'stack functions', and Kontsevich and Soibelman [104, §4.1–§4.2]. As in [86], we restrict our attention to Artin $\mathbb{K}$-stacks X (always assumed of finite type) with *affine geometric stabilizers*. Later we will restrict further, to stacks which are *locally a global quotient*.

Definition 1.4.11. An Artin $\mathbb{K}$-stack X has *affine geometric stabilizers* if the stabilizer group $\mathrm{Iso}_X(x)$ is an affine algebraic group for all points $x\in X$.

An Artin $\mathbb{K}$-stack X is *locally a global quotient* if we may cover X by Zariski open $\mathbb{K}$-substacks $Y \subseteq X$ equivalent to global quotients $[S/\operatorname{GL}(n,\mathbb{K})]$, where S is a $\mathbb{K}$-scheme with a $\operatorname{GL}(n,\mathbb{K})$-action. If X is locally a global quotient then it has affine geometric stabilizers, since the stabilizer groups of $[S/\operatorname{GL}(n,\mathbb{K})]$ are closed $\mathbb{K}$-subgroups of $\operatorname{GL}(n,\mathbb{K})$, and so are affine. The authors do not know any example of an Artin $\mathbb{K}$-stack with affine geometric stabilizers which is not locally a global quotient.

Deligne–Mumford stacks have affine geometric stabilizers, and are locally a global quotient if their stabilizers are generically trivial. If $\mathcal{M}$ is a moduli stack of coherent sheaves F on a projective scheme Y, then using Quot-schemes one can show that $\mathcal{M}$ is locally a global quotient. If $\mathcal{M}$ is a moduli stack of complexes $F^\bullet$ in $D^b\operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet,F^\bullet)=0$ then $\mathcal{M}$ has affine geometric stabilizers, since $\operatorname{Iso}_{\mathcal{M}}(F^\bullet)$ is the invertible elements in the finite-dimensional algebra $\operatorname{Hom}(F^\bullet,F^\bullet)$, and so is affine. We require affine geometric stabilizers to use a result of Kresch [105, Prop. 3.5.9]:

Proposition 1.4.12 (Kresch)**.** *Let X be a (finite type) Artin $\mathbb{K}$-stack with affine geometric stabilizers. Then X admits a stratification $X=\coprod_{i\in I}X_i$, for I a finite set and $X_i\subseteq X$ a locally closed $\mathbb{K}$-substack, such that X_i is equivalent to a global quotient stack $[S_i/\operatorname{GL}(n_i,\mathbb{K})]$ for each $i\in I$, where S_i is a (finite type) $\mathbb{K}$-scheme with an action of $\operatorname{GL}(n_i,\mathbb{K})$. Conversely, any Artin $\mathbb{K}$-stack X admitting such a stratification has affine geometric stabilizers.*

For the rest of this paper, all Artin $\mathbb{K}$-stacks X are assumed to have affine geometric stabilizers. Here are [13, Def.s 5.11-5.13], the analogues of Definitions 1.4.1 and 1.4.2:

Definition 1.4.13. Let X be an Artin $\mathbb{K}$-stack (always assumed to be of finite type, with affine geometric stabilizers). Consider pairs (R,ρ), where R is a $\mathbb{K}$-scheme and $\rho:R\to X$ a 1-morphism. Call two pairs (R,ρ), (R',ρ') *equivalent* if there exists an isomorphism $\iota:R\to R'$ such that $\rho'\circ\iota$ and ρ are 2-isomorphic 1-morphisms $R\to X$. Write $[R,\rho]$ for the equivalence class of (R,ρ). Define the Grothendieck ring $K_0(\mathrm{Sch}_X)$ of the category of $\mathbb{K}$-schemes over X to be the abelian group generated by equivalence classes $[R,\rho]$, such that as for (1.4.1) for each closed $\mathbb{K}$-subscheme S of R we have

$$[R,\rho]=[S,\rho|_S]+[R\setminus S,\rho|_{R\setminus S}].$$

When $X=\operatorname{Spec}\mathbb{K}$ we write $K_0(\mathrm{Sch}_{\mathbb{K}})$ instead of $K_0(\mathrm{Sch}_X)$. Define a biadditive, commutative, associative product '$\cdot$' on $K_0(\mathrm{Sch}_X)$ as in (1.4.2). It makes $K_0(\mathrm{Sch}_X)$ into a commutative ring, in general without identity. If X is a $\mathbb{K}$-scheme $K_0(\mathrm{Sch}_X)$ is as in Definition 1.4.1, with identity $[X,\mathrm{id}_X]$. For Artin $\mathbb{K}$-stacks X,Y, define a biadditive, commutative, associative *external tensor product* $\boxtimes:K_0(\mathrm{Sch}_X)\times K_0(\mathrm{Sch}_Y)\to K_0(\mathrm{Sch}_{X\times Y})$ by (1.4.4). Taking $Y=\operatorname{Spec}\mathbb{K}$ we see that $\boxtimes$ makes $K_0(\mathrm{Sch}_X)$ into a module over $K_0(\mathrm{Sch}_{\mathbb{K}})$.

Next we will define a stack analogue $\mathcal{M}_X^{\mathrm{stk}}$ of the motivic ring $\mathcal{M}_X$ of (1.4.3) for $\mathbb{K}$-schemes X. Since we have no identity in $K_0(\mathrm{Sch}_X)$ if X is not a scheme, and we have not defined a Tate motive $\mathbb{L}$ in $K_0(\mathrm{Sch}_X)$, the analogue of (1.4.3) does not make sense. Instead, we use the $K_0(\mathrm{Sch}_{\mathbb{K}})$-module structure, and define

$$\mathcal{M}_X^{\mathrm{stk}}=K_0(\mathrm{Sch}_X)\otimes_{K_0(\mathrm{Sch}_{\mathbb{K}})}K_0(\mathrm{Sch}_{\mathbb{K}})\big[\mathbb{L}^{-1},(\mathbb{L}^k-1)^{-1},\ k=1,2,\ldots\big], \tag{1.4.21}$$

where $\mathbb{L}\in K_0(\mathrm{Sch}_{\mathbb{K}})$ is as in Definition 1.4.1. The product '$\cdot$' descends to $\mathcal{M}_X^{\mathrm{stk}}$. When $X=\operatorname{Spec}\mathbb{K}$ we write $\mathcal{M}_{\mathbb{K}}^{\mathrm{stk}}$ instead of $\mathcal{M}_X^{\mathrm{stk}}$. Note that for X a $\mathbb{K}$-scheme, $\mathcal{M}_X^{\mathrm{stk}}$ is not isomorphic to $\mathcal{M}_X$ in (1.4.3), since we invert $\mathbb{L}^k-1$ in $\mathcal{M}_X^{\mathrm{stk}}$ but not in $\mathcal{M}_X$, but there is a natural projection $\mathcal{M}_X\to\mathcal{M}_X^{\mathrm{stk}}$. The reason we invert $\mathbb{L}^k-1$ as well as $\mathbb{L}$ is that the motive of $\operatorname{GL}(n,\mathbb{K})$ in $\mathcal{M}_{\mathbb{K}}$ is

$$[\operatorname{GL}(n,\mathbb{K})]:=[\operatorname{GL}(n,\mathbb{K}),\pi_{\operatorname{Spec}\mathbb{K}}]=\mathbb{L}^{n(n-1)/2}\prod_{k=1}^n(\mathbb{L}^k-1),$$

so that $[\mathrm{GL}(n,\mathbb{K})]$ is invertible in $\mathcal{M}_{\mathbb{K}}^{\mathrm{stk}}$.

Let X be an Artin $\mathbb{K}$-stack (as usual of finite type, with affine geometric stabilizers). Then Proposition 1.4.12 gives a finite stratification $X=\coprod_{i\in I}X_i$ with $X_i\simeq[S_i/\mathrm{GL}(n_i,\mathbb{K})]$. Write $\pi_i:S_i\to X$ for the composition of 1-morphisms $S_i\to[S_i/\mathrm{GL}(n_i,\mathbb{K})]\xrightarrow{\sim}X_i\hookrightarrow X$. Define elements $1_X,\mathbb{L}\in\mathcal{M}_X^{\mathrm{stk}}$ by

$$1_X=\sum_{i\in I}[\mathrm{GL}(n_i,\mathbb{K})]^{-1}\boxtimes[S_i,\pi_i],\qquad \mathbb{L}=\sum_{i\in I}[\mathrm{GL}(n_i,\mathbb{K})]^{-1}\boxtimes[\mathbb{A}^1\times S_i,\pi_i\circ\pi_{S_i}],\tag{1.4.22}$$

where $[\mathrm{GL}(n_i,\mathbb{K})]^{-1}\in\mathcal{M}_{\mathbb{K}}^{\mathrm{stk}}$ exists as above. It is easy to verify that these $1_X,\mathbb{L}$ are independent of the choice of I,X_i,S_i,n_i, and that 1_X is the identity in $(\mathcal{M}_X^{\mathrm{stk}},\cdot)$, see [13, §5] for a proof.

Let $\phi:X\to Y$ be a 1-morphism of Artin $\mathbb{K}$-stacks. Define the *pushforwards* $\phi_*:K_0(\mathrm{Sch}_X)\to K_0(\mathrm{Sch}_Y)$ and $\phi_*:\mathcal{M}_X^{\mathrm{stk}}\to\mathcal{M}_Y^{\mathrm{stk}}$ by (1.4.5). If ϕ is representable we may also define *pullbacks* $\phi^*:K_0(\mathrm{Sch}_Y)\to K_0(\mathrm{Sch}_X)$ and $\phi^*:\mathcal{M}_Y^{\mathrm{stk}}\to\mathcal{M}_X^{\mathrm{stk}}$ by (1.4.6). (Here ϕ is *representable in* $\mathbb{K}$*-schemes* if $X\times_{\phi,Y,u}U$ is a $\mathbb{K}$-scheme for all $u:U\to Y$ with U a $\mathbb{K}$-scheme.) But if ϕ is not representable then $R_i\times_{\rho_i,Y,\phi}X$ in (1.4.6) may not be a $\mathbb{K}$-scheme, so (1.4.6) does not make sense.

However, for general 1-morphisms $\phi:X\to Y$ we can still define a pullback morphism $\phi^*:\mathcal{M}_Y^{\mathrm{stk}}\to\mathcal{M}_X^{\mathrm{stk}}$ as follows. Proposition 1.4.12 gives a finite stratification $X=\coprod_{i\in I}X_i$ with $X_i\simeq[S_i/\mathrm{GL}(n_i,\mathbb{K})]$. Let $\pi_i:S_i\to X$ be as above, and define a group morphism $\phi^*:\mathcal{M}_Y^{\mathrm{stk}}\to\mathcal{M}_X^{\mathrm{stk}}$ by

$$\phi^*:\sum_{j=1}^n c_j[R_j,\rho_j]\longmapsto\sum_{j=1}^n c_j\sum_{i\in I}[\mathrm{GL}(n_i,\mathbb{K})]^{-1}\boxtimes[R_j\times_{\rho_j,Y,\phi\circ\pi_i}S_i,\pi_X].\tag{1.4.23}$$

If ϕ is representable in $\mathbb{K}$-schemes, this is the result of multiplying (1.4.6) by equation (1.4.22) for 1_X, and so the two definitions of ϕ^* agree. One can show that ϕ^* is independent of the choice of I,X_i,S_i,n_i, and that pullbacks ϕ^* have the usual functoriality properties. As in [86, Th. 3.5], the analogue of (1.4.7) holds for 2-Cartesian squares in Artin $\mathbb{K}$-stacks.

Definition 1.4.14. Let X be an Artin $\mathbb{K}$-stack. Consider triples $(R,\rho,\hat r)$, where R is a $\mathbb{K}$-scheme, $\rho:R\to X$ a 1-morphism, and $\hat r:\hat\mu\times R\to R$ a good $\hat\mu$-action on R, in the sense of Definition 1.4.2. Call two such triples $(R,\rho,\hat r),(R',\rho',\hat r')$ *equivalent* if there exists a $\hat\mu$-equivariant isomorphism $\iota:R\to R'$ and a 2-isomorphism $\rho\cong\rho'\circ\iota$. Write $[R,\rho,\hat r]$ for the equivalence class of $(R,\rho,\hat r)$.

The *monodromic Grothendieck group* $K_0^{\hat\mu}(\mathrm{Sch}_X)$ is the abelian group generated by such equivalence classes $[R,\rho,\hat r]$, with relations (i),(ii) as in Definition 1.4.2, except that we require a 2-isomorphism $\rho_2\cong\rho_1\circ\pi$ rather than equality $\rho_2=\rho_1\circ\pi$ in (ii). Define a biadditive, commutative, associative product '$\cdot$' on $K_0^{\hat\mu}(\mathrm{Sch}_X)$ as in (1.4.8). As for $K_0(\mathrm{Sch}_X)$ in Definition 1.4.13, this makes $K_0^{\hat\mu}(\mathrm{Sch}_X)$ into a commutative ring, in general without identity. If X is a $\mathbb{K}$-scheme $K_0^{\hat\mu}(\mathrm{Sch}_X)$ is as in Definition 1.4.2, with identity $[X,\mathrm{id}_X,\hat\iota]$. For Artin $\mathbb{K}$-stacks X,Y, define a biadditive, commutative, associative *external tensor product* $\boxtimes:K_0^{\hat\mu}(\mathrm{Sch}_X)\times K_0^{\hat\mu}(\mathrm{Sch}_Y)\to K_0^{\hat\mu}(\mathrm{Sch}_{X\times Y})$ by (1.4.9). Taking $Y=\operatorname{Spec}\mathbb{K}$, this makes $K_0^{\hat\mu}(\mathrm{Sch}_X)$ into a module over $K_0^{\hat\mu}(\mathrm{Sch}_{\mathbb{K}})$.

As for (1.4.21), using the $K_0^{\hat\mu}(\mathrm{Sch}_{\mathbb{K}})$-module structure on $K_0^{\hat\mu}(\mathrm{Sch}_X)$ define

$$\mathcal{M}_X^{\mathrm{st},\hat\mu}=K_0^{\hat\mu}(\mathrm{Sch}_X)\otimes_{K_0^{\hat\mu}(\mathrm{Sch}_{\mathbb{K}})}K_0^{\hat\mu}(\mathrm{Sch}_{\mathbb{K}})\big[\mathbb{L}^{-1},(\mathbb{L}^k-1)^{-1},\ k=1,2,\ldots\big].$$

The product '$\cdot$' descends to $\mathcal{M}_X^{\mathrm{st},\hat\mu}$. When $X=\operatorname{Spec}\mathbb{K}$ we write $\mathcal{M}_{\mathbb{K}}^{\mathrm{st},\hat\mu}$ instead of $\mathcal{M}_X^{\mathrm{st},\hat\mu}$. Using the data X_i,S_i,n_i of Proposition 1.4.12, as in (1.4.22) define elements $1_X,\mathbb{L}\in\mathcal{M}_X^{\mathrm{st},\hat\mu}$ by

$$1_X=\sum_{i\in I}[\mathrm{GL}(n_i,\mathbb{K})]^{-1}\boxtimes[S_i,\pi_i,\hat\iota],\ \mathbb{L}=\sum_{i\in I}[\mathrm{GL}(n_i,\mathbb{K})]^{-1}\boxtimes[\mathbb{A}^1\times S_i,\pi_i\circ\pi_{S_i},\hat\iota].\tag{1.4.24}$$

These are independent of choices, and 1_X is the identity in $\mathcal{M}_X^{\mathrm{st},\hat{\mu}}$.

Let $\phi : X \to Y$ be a 1-morphism of Artin $\mathbb{K}$-stacks. Define the *pushforwards* $\phi_* : K_0^{\hat{\mu}}(\mathrm{Sch}_X) \to K_0^{\hat{\mu}}(\mathrm{Sch}_Y)$ and $\phi_* : \mathcal{M}_X^{\mathrm{st},\hat{\mu}} \to \mathcal{M}_Y^{\mathrm{st},\hat{\mu}}$ by the analogue of (1.4.5). If ϕ is representable in $\mathbb{K}$-schemes we may also define *pullbacks* $\phi^* : K_0^{\hat{\mu}}(\mathrm{Sch}_Y) \to K_0^{\hat{\mu}}(\mathrm{Sch}_X)$ and $\phi^* : \mathcal{M}_Y^{\mathrm{st},\hat{\mu}} \to \mathcal{M}_X^{\mathrm{st},\hat{\mu}}$ by the analogue of (1.4.6). If ϕ is not representable in $\mathbb{K}$-schemes, we can still define $\phi^* : \mathcal{M}_Y^{\mathrm{st},\hat{\mu}} \to \mathcal{M}_X^{\mathrm{st},\hat{\mu}}$ by the analogue of (1.4.23). Pushforwards and pullbacks have the usual functoriality properties, and the analogue of (1.4.7) holds for 2-Cartesian squares in Artin $\mathbb{K}$-stacks.

As for (1.4.10), there are natural morphisms of commutative rings

$$\begin{aligned} i_X &: K_0(\mathrm{Sch}_X) \longrightarrow K_0^{\hat{\mu}}(\mathrm{Sch}_X), & i_X &: \mathcal{M}_X^{\mathrm{stk}} \longrightarrow \mathcal{M}_X^{\mathrm{st},\hat{\mu}}, \\ \Pi_X &: K_0^{\hat{\mu}}(\mathrm{Sch}_X) \longrightarrow K_0(\mathrm{Sch}_X), & \Pi_X &: \mathcal{M}_X^{\mathrm{st},\hat{\mu}} \longrightarrow \mathcal{M}_X^{\mathrm{stk}}, \end{aligned}$$

given by $i_X : [R,\rho] \mapsto [R,\rho,\hat{\iota}]$ and $\Pi_X : [R,\rho,\hat{r}] \mapsto [R,\rho]$. If X is a $\mathbb{K}$-scheme, there is a natural projection $\mathcal{M}_X^{\hat{\mu}} \to \mathcal{M}_X^{\mathrm{st},\hat{\mu}}$.

The analogue of Definition 1.4.3, defining another associative, commutative product '$\odot$' on $K_0^{\hat{\mu}}(\mathrm{Sch}_X)$ and $\mathcal{M}_X^{\mathrm{st},\hat{\mu}}$ and an external version '$\boxdot$', works essentially without change. For the analogue of Definition 1.4.4, following (1.4.13) we define $\mathbb{L}^{1/2}$ in $\mathcal{M}_X^{\mathrm{st},\hat{\mu}}$ only by $\mathbb{L}^{1/2} = 1_X \boxdot \mathbb{L}_{\mathbb{K}}^{1/2} \in \mathcal{M}_X^{\mathrm{st},\hat{\mu}}$, where 1_X is as in (1.4.24), and $\mathbb{L}_{\mathbb{K}}^{1/2} \in \mathcal{M}_{\mathbb{K}}^{\mathrm{st},\hat{\mu}}$ as in (1.4.12). Then $\mathbb{L}^{1/2} \odot \mathbb{L}^{1/2} = \mathbb{L}$ in $\mathcal{M}_X^{\mathrm{st},\hat{\mu}}$, and we can define $\mathbb{L}^{n/2}$ in $\mathcal{M}_X^{\mathrm{st},\hat{\mu}}$ for all $n \in \mathbb{Z}$ in the obvious way. Here is [25, Def. 5.13] the stack analogue of Definition 1.4.5:

Definition 1.4.15. For each Artin $\mathbb{K}$-stack Y, define $I_Y^{\mathrm{st},\hat{\mu}}$ to be the ideal in the commutative ring $\big(\mathcal{M}_Y^{\mathrm{st},\hat{\mu}}, \odot\big)$ generated by elements $\phi_*\big(\Upsilon^{\mathrm{stk}}(P \otimes_{\mathbb{Z}/2\mathbb{Z}} Q) - \Upsilon(P)^{\mathrm{stk}} \odot \Upsilon^{\mathrm{stk}}(Q)\big)$ for all 1-morphisms $\phi : X \to Y$ with X a $\mathbb{K}$-scheme and principal $\mathbb{Z}/2\mathbb{Z}$-bundles $P, Q \to X$, where $\Upsilon^{\mathrm{stk}}(P)$, $\Upsilon^{\mathrm{stk}}(Q)$, $\Upsilon^{\mathrm{stk}}(P \otimes_{\mathbb{Z}/2\mathbb{Z}} Q)$ are the images in $\mathcal{M}_X^{\mathrm{st},\hat{\mu}}$ of the elements $\Upsilon(P), \Upsilon(Q), \Upsilon(P \otimes_{\mathbb{Z}/2\mathbb{Z}} Q)$ in $\mathcal{M}_X^{\hat{\mu}}$ from Definition 1.4.5. Define $\bar{\mathcal{M}}_Y^{\mathrm{st},\hat{\mu}} = \mathcal{M}_Y^{\mathrm{st},\hat{\mu}} / I_Y^{\mathrm{st},\hat{\mu}}$ to be the quotient, as a commutative ring with multiplication '$\odot$', with projection $\Pi_Y^{\hat{\mu}} : \mathcal{M}_Y^{\hat{\mu}} \to \bar{\mathcal{M}}_Y^{\hat{\mu}}$. The second multiplication '$\cdot$', external product $\boxtimes$, and projection $\Pi_Y : \mathcal{M}_Y^{\mathrm{st},\hat{\mu}} \to \mathcal{M}_Y^{\mathrm{stk}}$ on $\mathcal{M}_Y^{\mathrm{st},\hat{\mu}}$ do not descend to $\bar{\mathcal{M}}_Y^{\mathrm{st},\hat{\mu}}$. The other structures $\odot, \boxdot, 1_Y, \mathbb{L}, \phi_*, \phi^*, i_Y, \mathbb{L}^{1/2}$ do descend to $\bar{\mathcal{M}}_Y^{\mathrm{st},\hat{\mu}}$.

If X is a $\mathbb{K}$-scheme, we have a natural projection $\bar{\mathcal{M}}_X^{\hat{\mu}} \to \bar{\mathcal{M}}_X^{\mathrm{st},\hat{\mu}}$. So in particular, the motives $MF_{X,s} \in \bar{\mathcal{M}}_X^{\hat{\mu}}$ in Theorem 5.3.1 also make sense in $\bar{\mathcal{M}}_X^{\mathrm{st},\hat{\mu}}$. We will use this in Theorem 6.4.2.

Chapter 2

D-critical loci

We summarize the theory of d-critical schemes and stacks introduced by Joyce [87]. There are two versions of the theory, complex analytic and algebraic d-critical loci, sometimes we give results for both the versions simultaneously, otherwise just briefly indicate the differences between the two, referring to [87] for details. We will need this material for the following chapters.

2.1 D-critical schemes

Let X be a complex analytic space or a $\mathbb{K}$-scheme. Then [87, Th. 2.1 & Prop. 2.3] associates a natural sheaf $\mathcal{S}_X$ to X, such that, very briefly, sections of $\mathcal{S}_X$ parametrize different ways of writing X as $\mathrm{Crit}(f)$ for U a complex manifold or smooth $\mathbb{K}$-scheme and $f : U \to \mathbb{C}$ holomorphic or $f : U \to \mathbb{A}^1$ regular. Let us state it for $\mathbb{K}$-schemes. The natural sheaf of $\mathbb{K}$-algebras $\mathcal{S}_X$ on X in either the Zariski or étale topologies, has the following properties:

(a) Suppose $R \subseteq X$ is Zariski open, U is a smooth $\mathbb{K}$-scheme, and $i : R \hookrightarrow U$ a closed embedding. Define an ideal $I_{R,U} \subseteq i^{-1}(\mathcal{O}_U)$ by the exact sequence $0 \longrightarrow I_{R,U} \longrightarrow i^{-1}(\mathcal{O}_U) \xrightarrow{i^\sharp} \mathcal{O}_X|_R \longrightarrow 0$, where $\mathcal{O}_X, \mathcal{O}_U$ are the sheaves of regular functions on X, U. Then there is an exact sequence on R, where $\mathrm{d} : f + I_{R,U}^2 \mapsto \mathrm{d}f + I_{R,U} \cdot i^{-1}(T^*U)$

$$0 \longrightarrow \mathcal{S}_X|_R \xrightarrow{\iota_{R,U}} \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \xrightarrow{\mathrm{d}} \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)}.$$

(b) Let $R \subseteq S \subseteq X$ be Zariski open, U, V be smooth $\mathbb{K}$-schemes, $i : R \hookrightarrow U$, $j : S \hookrightarrow V$ closed embeddings, and $\Phi : U \to V$ a morphism with $\Phi \circ i = j|_R : R \to V$. Then the following diagram of sheaves on R commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}_X|_R \arrow[r, "\iota_{S,V}|_R"] \arrow[d, "\mathrm{id}"] & \frac{j^{-1}(\mathcal{O}_V)}{I_{S,V}^2}\Big|_R \arrow[r, "\mathrm{d}"] \arrow[d, "i^{-1}(\Phi^\sharp)"] & \frac{j^{-1}(T^*V)}{I_{S,V} \cdot j^{-1}(T^*V)}\Big|_R \arrow[d, "i^{-1}(\mathrm{d}\Phi)"] \\
0 \arrow[r] & \mathcal{S}_X|_R \arrow[r, "\iota_{R,U}"] & \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \arrow[r, "\mathrm{d}"] & \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)}.
\end{tikzcd} \tag{2.1.1}$$

(c) There is a natural decomposition $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{K}_X$, where $\mathbb{K}_X$ is the constant sheaf on X with fibre $\mathbb{K}$, and $\mathcal{S}_X^0 \subset \mathcal{S}_X$ is the kernel of the composition $\mathcal{S}_X \longrightarrow \mathcal{O}_X \xrightarrow{i_X^\sharp} \mathcal{O}_{X^{\mathrm{red}}}$, with $i_X : X^{\mathrm{red}} \hookrightarrow X$ the reduced $\mathbb{K}$-subscheme of X.

(d) Let $\phi : X \to Y$ be a morphism of $\mathbb{K}$-schemes. Then there is a unique morphism $\phi^\star : \phi^{-1}(\mathcal{S}_Y) \to \mathcal{S}_X$ of sheaves of $\mathbb{K}$-algebras on X, which maps $\phi^{-1}(\mathcal{S}_Y^0) \to \mathcal{S}_X^0$, such that if $R \subseteq X$, $S \subseteq Y$ are Zariski open with $\phi(R) \subseteq S$, U, V are smooth schemes, $i : R \hookrightarrow U$, $j : S \hookrightarrow V$ are closed embeddings, and $\Phi : U \to V$ is a morphism with $\Phi \circ i = j \circ \phi|_R : R \to V$, then as for (2.1.1) the following diagram of sheaves on R commutes:

$$\begin{tikzcd} 0 \arrow[r] & \phi^{-1}(\mathcal{S}_Y)|_R \arrow[r, "\phi^{-1}(\iota_{S,V})|_R"'] \arrow[d, "\phi^\star|_R"] & \frac{\phi^{-1}\circ j^{-1}(\mathcal{O}_V)|_R}{\phi^{-1}(I_{S,V}^2)|_R} \arrow[r, "\phi^{-1}(\mathrm{d})"'] \arrow[d, "i^{-1}(\Phi^\sharp)"] & \frac{\phi^{-1}(j^{-1}(T^*V))|_R}{\phi^{-1}(I_{S,V}\cdot j^{-1}(T^*V))|_R} \arrow[d, "i^{-1}(\mathrm{d}\Phi)"'] \\ 0 \arrow[r] & \mathcal{S}_X|_R \arrow[r, "\iota_{R,U}"] & \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \arrow[r, "\mathrm{d}"] & \frac{i^{-1}(T^*U)}{I_{R,U}\cdot i^{-1}(T^*U)}. \end{tikzcd} \tag{2.1.2}$$

(e) If $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ are smooth morphisms of $\mathbb{K}$-schemes, then $(\psi\circ\phi)^\star = \phi^\star \circ \phi^{-1}(\psi^\star) : (\psi\circ\phi)^{-1}(\mathcal{S}_Z) = \phi^{-1}\circ\psi^{-1}(\mathcal{S}_Z) \longrightarrow \mathcal{S}_X$. If $\phi : X \to Y$ is $\mathrm{id}_X : X \to X$ then $\mathrm{id}_X^\star = \mathrm{id}_{\mathcal{S}_X} : \mathrm{id}_X^{-1}(\mathcal{S}_X) = \mathcal{S}_X \to \mathcal{S}_X$.

Remark 2.1.1. Suppose we have U a complex manifold, $f : U \to \mathbb{C}$ an holomorphic function, and $X = \mathrm{Crit}(f)$, as a closed complex analytic subspace of U. Write $i : X \hookrightarrow U$ for the inclusion, and $I_{X,U} \subseteq i^{-1}(\mathcal{O}_U)$ for the sheaf of ideals vanishing on $X \subseteq U$. Then we obtain a natural section $s \in H^0(\mathcal{S}_X)$. Essentially $s = f + I_{\mathrm{d}f}^2$, where $I_{\mathrm{d}f} \subseteq \mathcal{O}_U$ is the ideal generated by $\mathrm{d}f$. Note that $f|_X = f + I_{\mathrm{d}f}$, so s determines $f|_X$. Basically, s remembers all of the information about f which makes sense intrinsically on X, rather than on the ambient space U.

Following [87, Def. 2.5] we define algebraic d-critical loci:

Definition 2.1.2. An (*algebraic*) *d-critical locus* over a field $\mathbb{K}$ is a pair (X, s), where X is a $\mathbb{K}$-scheme and $s \in H^0(\mathcal{S}_X^0)$, such that for each $x \in X$, there exists a Zariski open neighbourhood R of x in X, a smooth $\mathbb{K}$-scheme U, a regular function $f : U \to \mathbb{A}^1 = \mathbb{K}$, and a closed embedding $i : R \hookrightarrow U$, such that $i(R) = \mathrm{Crit}(f)$ as $\mathbb{K}$-subschemes of U, and $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$. We call the quadruple (R, U, f, i) a *critical chart* on (X, s). If $U' \subseteq U$ is a Zariski open, and $R' = i^{-1}(U') \subseteq R$, $i' = i|_{R'} : R' \hookrightarrow U'$, and $f' = f|_{U'}$, then (R', U', f', i') is a critical chart on (X, s), and we call it a *subchart* of (R, U, f, i), and we write $(R', U', f', i') \subseteq (R, U, f, i)$.

Let $(R, U, f, i), (S, V, g, j)$ be critical charts on (X, s), with $R \subseteq S \subseteq X$. An *embedding* of (R, U, f, i) in (S, V, g, j) is a locally closed embedding $\Phi : U \hookrightarrow V$ such that $\Phi \circ i = j|_R$ and $f = g \circ \Phi$. As a shorthand we write $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$. If $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ and $\Psi : (S, V, g, j) \hookrightarrow (T, W, h, k)$ are embeddings, then $\Psi \circ \Phi : (R, U, i, e) \hookrightarrow (T, W, h, k)$ is also an embedding.

A *morphism* $\phi : (X, s) \to (Y, t)$ of d-critical loci $(X, s), (Y, t)$ is a $\mathbb{K}$-scheme morphism $\phi : X \to Y$ with $\phi^\star(t) = s$. This makes d-critical loci into a category.

Remark 2.1.3. (a) For (X, s) to be a (complex analytic or algebraic) d-critical locus places strong local restrictions on the singularities of X. For example, Behrend [5] notes that if X has reduced local complete intersection singularities then locally it cannot be the zeroes of an almost closed 1-form on a smooth space, and hence not locally a critical locus, and Pandharipande and Thomas [140] give examples which are zeroes of almost closed 1-forms, but are not locally critical loci.

(b) If $X = \mathrm{Crit}(f)$ for holomorphic $f : U \to \mathbb{C}$, then $f|_{X^{\mathrm{red}}}$ is locally constant, and we can write $f = f^0 + c$ uniquely near X in U for $f^0 : U \to \mathbb{C}$ holomorphic with $\mathrm{Crit}(f^0) = X = \mathrm{Crit}(f)$, $f^0|_{X^{\mathrm{red}}} = 0$, and $c : U \to \mathbb{C}$ locally constant with $c|_{X^{\mathrm{red}}} = f|_{X^{\mathrm{red}}}$. Defining d-critical loci using

$s \in H^0(\mathcal{S}_X^0)$ corresponds to remembering only the function f^0 near X in U, and forgetting the locally constant function $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{C}$.

(c) It is natural to ask what is the relation between d-critical loci and schemes with symmetric obstruction theories. In [87, ex. 2.16], Joyce shows a case in which the algebraic d-critical locus remembers more information, locally, than the symmetric obstruction theory. In [87, ex. 2.17], Joyce shows that the (symmetric) obstruction theory remembers global, non-local information which is forgotten by the algebraic d-critical locus.

(e) One could think about critical charts as Kuranishi neighbourhoods on a topological space, and embeddings as analogous to coordinate changes between Kuranishi neighbourhoods.

Here are [87, Prop.s 2.8, 2.30, Th.s 2.20, 2.28, Def. 2.31, Rem 2.32 & Cor. 2.33]:

Proposition 2.1.4. *Let $\phi : X \to Y$ be a smooth morphism of $\mathbb{K}$-schemes. Suppose $t \in H^0(\mathcal{S}_Y^0)$, and set $s := \phi^\star(t) \in H^0(\mathcal{S}_X^0)$. If (Y,t) is a d-critical locus, then (X,s) is a d-critical locus, and $\phi : (X,s) \to (Y,t)$ is a morphism of d-critical loci. Conversely, if also $\phi : X \to Y$ is surjective, then (X,s) a d-critical locus implies (Y,t) is a d-critical locus.*

Theorem 2.1.5. *Suppose (X,s) is an algebraic d-critical locus, and $(R,U,f,i),(S,V,g,j)$ are critical charts on (X,s). Then for each $x \in R \cap S \subseteq X$ there exist subcharts $(R',U',f',i') \subseteq (R,U,f,i)$, $(S',V',g',j') \subseteq (S,V,g,j)$ with $x \in R' \cap S' \subseteq X$, a critical chart (T,W,h,k) on (X,s), and embeddings $\Phi : (R',U',f',i') \hookrightarrow (T,W,h,k)$, $\Psi : (S',V',g',j') \hookrightarrow (T,W,h,k)$.*

Theorem 2.1.6. *Let (X,s) be an algebraic d-critical locus, and $X^{\mathrm{red}} \subseteq X$ the associated reduced $\mathbb{K}$-subscheme. Then there exists a line bundle $K_{X,s}$ on X^{red} which we call the **canonical bundle** of (X,s), which is natural up to canonical isomorphism, and is characterized by the following properties:*

(a) *For each $x \in X^{\mathrm{red}}$, there is a canonical isomorphism*

$$\kappa_x : K_{X,s}|_x \xrightarrow{\cong} \left(\Lambda^{\mathrm{top}} T_x^* X\right)^{\otimes 2}, \tag{2.1.3}$$

where $T_x X$ is the Zariski tangent space of X at x.

(b) *If (R,U,f,i) is a critical chart on (X,s), there is a natural isomorphism*

$$\iota_{R,U,f,i} : K_{X,s}|_{R^{\mathrm{red}}} \longrightarrow i^*\left(K_U^{\otimes 2}\right)|_{R^{\mathrm{red}}}, \tag{2.1.4}$$

where $K_U = \Lambda^{\dim U} T^* U$ is the canonical bundle of U in the usual sense.

(c) *In the situation of* **(b)**, *let $x \in R$. Then we have an exact sequence*

$$0 \longrightarrow T_x X \xrightarrow{\mathrm{d}i|_x} T_{i(x)} U \xrightarrow{\mathrm{Hess}_{i(x)} f} T_{i(x)}^* U \xrightarrow{\mathrm{d}i|_x^*} T_x^* X \longrightarrow 0, \tag{2.1.5}$$

and the following diagram commutes:

$$\begin{tikzcd} K_{X,s}|_x \arrow[rr, "\kappa_x"] \arrow[drr, "\iota_{R,U,f,i}|_x"'] & & \left(\Lambda^{\mathrm{top}} T_x^* X\right)^{\otimes 2} \arrow[d, "\alpha_{x,R,U,f,i}"] \\ & & K_U|_{i(x)}^{\otimes 2}, \end{tikzcd}$$

where $\alpha_{x,R,U,f,i}$ is induced by taking top exterior powers in (2.1.5).

Proposition 2.1.7. *Suppose $\phi:(X,s)\to(Y,t)$ is a morphism of d-critical loci with $\phi:X\to Y$ smooth, as in Proposition* 2.1.4*. The* ***relative cotangent bundle*** $T^*_{X/Y}$ *is a vector bundle of mixed rank on X in the exact sequence of coherent sheaves on X:*

$$0 \longrightarrow \phi^*(T^*Y) \xrightarrow{\mathrm{d}\phi^*} T^*X \longrightarrow T^*_{X/Y} \longrightarrow 0. \tag{2.1.6}$$

There is a natural isomorphism of line bundles on X^{red}:

$$\Upsilon_\phi : \phi|^*_{X^{\mathrm{red}}}(K_{Y,t}) \otimes \left(\Lambda^{\mathrm{top}} T^*_{X/Y}\right)\big|^{\otimes^2}_{X^{\mathrm{red}}} \xrightarrow{\cong} K_{X,s}, \tag{2.1.7}$$

such that for each $x\in X^{\mathrm{red}}$ the following diagram of isomorphisms commutes:

$$\begin{tikzcd}
K_{Y,t}|_{\phi(x)} \otimes \left(\Lambda^{\mathrm{top}} T^*_{X/Y}|_x\right)^{\otimes^2} \arrow[r, "\Upsilon_\phi|_x"'] \arrow[d, "\kappa_{\phi(x)}\otimes \mathrm{id}"] & K_{X,s}|_x \arrow[d, "\kappa_x"'] \\
\left(\Lambda^{\mathrm{top}} T^*_{\phi(x)}Y\right)^{\otimes^2} \otimes \left(\Lambda^{\mathrm{top}} T^*_{X/Y}|_x\right)^{\otimes^2} \arrow[r, "\upsilon_x^{\otimes^2}"] & \left(\Lambda^{\mathrm{top}} T^*_x X\right)^{\otimes^2},
\end{tikzcd} \tag{2.1.8}$$

where $\kappa_x,\kappa_{\phi(x)}$ are as in (2.1.3)*, and $\upsilon_x:\Lambda^{\mathrm{top}}T^*_{\phi(x)}Y\otimes\Lambda^{\mathrm{top}}T^*_{X/Y}|_x\to\Lambda^{\mathrm{top}}T^*_xX$ is obtained by restricting* (2.1.6) *to x and taking top exterior powers.*

Definition 2.1.8. Let (X,s) be an algebraic d-critical locus, and $K_{X,s}$ its canonical bundle from Theorem 2.1.6. An *orientation* on (X,s) is a choice of square root line bundle $K^{1/2}_{X,s}$ for $K_{X,s}$ on X^{red}. That is, an orientation is a line bundle L on X^{red}, together with an isomorphism $L^{\otimes^2}=L\otimes L\cong K_{X,s}$. A d-critical locus with an orientation will be called an *oriented d-critical locus.*

Remark 2.1.9. In view of equation (2.1.3), one might hope to define a canonical orientation $K^{1/2}_{X,s}$ for a d-critical locus (X,s) by $K^{1/2}_{X,s}\big|_x=\Lambda^{\mathrm{top}}T^*_xX$ for $x\in X^{\mathrm{red}}$. However, *this does not work*, as the spaces $\Lambda^{\mathrm{top}}T^*_xX$ do not vary continuously with $x\in X^{\mathrm{red}}$ if X is not smooth. In [87, Ex. 2.39], Joyce shows that d-critical loci need not admit orientations.

In the situation of Proposition 2.1.7, the factor $(\Lambda^{\mathrm{top}}T^*_{X/Y})|^{\otimes^2}_{X^{\mathrm{red}}}$ in (2.1.7) has a natural square root $(\Lambda^{\mathrm{top}}T^*_{X/Y})|_{X^{\mathrm{red}}}$. Thus we deduce:

Corollary 2.1.10. *Let $\phi:(X,s)\to(Y,t)$ be a morphism of d-critical loci with $\phi:X\to Y$ smooth. Then each orientation $K^{1/2}_{Y,t}$ for (Y,t) lifts to a natural orientation $K^{1/2}_{X,s}=\phi|^*_{X^{\mathrm{red}}}(K^{1/2}_{Y,t})\otimes(\Lambda^{\mathrm{top}}T^*_{X/Y})|_{X^{\mathrm{red}}}$ for (X,s).*

Remark 2.1.11. There is also an interpretation of orientations in terms of principal $\mathbb{Z}_2$-bundles. The line bundle $K_{X,s}$ in Theorem 2.1.6 is characterized uniquely up to isomorphism equivalently by part (ii) and the following property based on Definition 4.3.2 which we will discuss in §4: let $\Phi:(R,U,f,i)\hookrightarrow(S,V,g,j)$ be an embedding of critical charts on (X,s), and let J_Φ be as in Definition 4.3.2. Then

$$\iota_{S,V,g,j}|_{R^{\mathrm{red}}} = J_\Phi\circ\iota_{R,U,f,i} : K_{X,s}|_{R^{\mathrm{red}}} \longrightarrow j^*\big(K_V^{\otimes^2}\big)\big|_{R^{\mathrm{red}}}. \tag{2.1.9}$$

Using this equivalent characterization of canonical bundles, we can express orientations in terms of *principal $\mathbb{Z}_2$-bundles.*

Proposition 2.1.12. *Let (X,s) be a d-critical locus. Then Definition 4.3.2 induces an isomorphism between isomorphism classes of orientations $K^{1/2}_{X,s}$ on (X,s), and isomorphism classes of the following collections of data:*

(a) *For each critical chart* (R, U, f, i) *on* (X, s)*, a choice of principal* $\mathbb{Z}_2$*-bundle* $\pi_{R,U,f,i} : Q_{R,U,f,i} \to R$ *on* R*, and*

(b) *For each embedding of critical charts* $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$*, a choice of isomorphism* $\Lambda_\Phi : Q_{S,V,g,j}|_R \to P_\Phi \otimes_{\mathbb{Z}_2} Q_{R,U,f,i}$ *as in Definition 4.3.2,*

such that [87, eq 2.38] commutes for all embeddings $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$*,* $\Psi : (S, V, g, j) \hookrightarrow (T, W, h, k)$*, where* $P_\Phi, P_\Psi, P_{\Psi\circ\Phi}, \Xi_{\Psi,\Phi}$ *are as in the first part of Definition 4.3.2.*

Let $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on a d-critical locus (X, s). Define N_{UV}, q_{UV} as in Theorem 4.3.1, and $\pi_\Phi : P_\Phi \to R$ as in Definition 4.3.2. Then an alternative interpretation of P_Φ is as *the principal* $\mathbb{Z}_2$*-bundle of orientations of the nondegenerate quadratic form* q_{UV} *on the vector bundle* $i^*(N_{UV})$ *over* R. Thus, Proposition 2.1.12 shows that an orientation $K_{X,s}^{1/2}$ on (X, s) is equivalent to giving principal $\mathbb{Z}_2$-bundles $Q_{R,U,f,i} \to R$ for each chart (R, U, f, i) on (X, s), such that $Q_{R,U,f,i}$ and $Q_{S,V,g,j}|_R$ differ by the principal $\mathbb{Z}_2$-bundle of orientations of q_{UV} for each embedding $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$. This is why the term *orientation* has been chosen for $K_{X,s}^{1/2}$. It is closely relation to the notion of *orientation data* in Kontsevich and Soibelman [102, §5].

2.2 D-critical stacks

In [87, §2.7–§2.8] Joyce extends the material of §2.1 from $\mathbb{K}$-schemes to Artin $\mathbb{K}$-stacks. He works in the context of the theory of *sheaves on Artin stacks* by Laumon and Moret-Bailly [109], so for the reader's convenience we recall the following:

Proposition 2.2.1 (Laumon and Moret-Bailly [109])**.** *Let* X *be an Artin* $\mathbb{K}$*-stack. The category of sheaves of sets on* X *in the lisse-étale topology is equivalent to the category* $\mathrm{Sh}(X)$ *defined as follows:*

(A) *Objects* $\mathcal{A}$ *of* $\mathrm{Sh}(X)$ *comprise the following data:*

(a) *For each* $\mathbb{K}$*-scheme* T *and smooth* 1*-morphism* $t : T \to X$ *in* $\mathrm{Art}_\mathbb{K}$*, we are given a sheaf of sets* $\mathcal{A}(T, t)$ *on* T*, in the étale topology.*

(b) *For each* 2*-commutative diagram in* $\mathrm{Art}_\mathbb{K}$*:*

$$\begin{tikzcd} & U \arrow[dr, "u"] \arrow[d, phantom, "\eta\Uparrow"] & \\ T \arrow[ur, "\phi"] \arrow[rr, "t"'] & {} & X, \end{tikzcd} \tag{2.2.1}$$

where T, U *are schemes and* $t : T \to X$*,* $u : U \to X$ *are smooth* 1*-morphisms in* $\mathrm{Art}_\mathbb{K}$*, we are given a morphism* $\mathcal{A}(\phi, \eta) : \phi^{-1}(\mathcal{A}(U, u)) \to \mathcal{A}(T, t)$ *of étale sheaves of sets on* T*.*

This data must satisfy the following conditions:

(i) *If* $\phi : T \to U$ *in* **(b)** *is étale, then* $\mathcal{A}(\phi, \eta)$ *is an isomorphism.*

(ii) *For each* 2*-commutative diagram in* $\mathrm{Art}_\mathbb{K}$*:*

$$\begin{tikzcd} & V \arrow[dr, "v"] \arrow[d, phantom, "\zeta\Uparrow"] & \\ U \arrow[ur, "\psi"] \arrow[rr, "u"'] & {} \arrow[d, phantom, "\eta\Uparrow"] & X, \\ T \arrow[u, "\phi"] \arrow[urr, bend right, "t"'] & {} & \end{tikzcd}$$

with T, U, V schemes and t, u, v smooth, we must have

$$\begin{aligned}\mathcal{A}\big(\psi\circ\phi,(\zeta*\mathrm{id}_\phi)\odot\eta\big)=\mathcal{A}(\phi,\eta)\circ\phi^{-1}(\mathcal{A}(\psi,\zeta))\quad \textit{as morphisms}\\ (\psi\circ\phi)^{-1}(\mathcal{A}(V,v))=\phi^{-1}\circ\psi^{-1}(\mathcal{A}(V,v))\longrightarrow\mathcal{A}(T,t).\end{aligned}$$

(B) *Morphisms $\alpha : \mathcal{A}\to\mathcal{B}$ of $\mathrm{Sh}(X)$ comprise a morphism $\alpha(T,t) : \mathcal{A}(T,t)\to\mathcal{B}(T,t)$ of étale sheaves of sets on a scheme T for all smooth 1-morphisms $t : T\to X$, such that for each diagram* (2.2.1) *in* **(b)** *the following commutes:*

$$\begin{tikzcd}
\phi^{-1}(\mathcal{A}(U,u)) \arrow[rr, "\mathcal{A}(\phi,\eta)"'] \arrow[d, "\phi^{-1}(\alpha(U,u))"] && \mathcal{A}(T,t) \arrow[d, "\alpha(T,t)"'] \\
\phi^{-1}(\mathcal{B}(U,u)) \arrow[rr, "\mathcal{B}(\phi,\eta)"] && \mathcal{B}(T,t).
\end{tikzcd}$$

(C) *Composition of morphisms $\mathcal{A}\xrightarrow{\alpha}\mathcal{B}\xrightarrow{\beta}\mathcal{C}$ in $\mathrm{Sh}(X)$ is $(\beta\circ\alpha)(T,t)=\beta(T,t)\circ\alpha(T,t)$. Identity morphisms $\mathrm{id}_\mathcal{A} : \mathcal{A}\to\mathcal{A}$ are $\mathrm{id}_\mathcal{A}(T,t)=\mathrm{id}_{\mathcal{A}(T,t)}$.*

The analogue of all the above also holds for (étale) sheaves of $\mathbb{K}$-vector spaces, sheaves of $\mathbb{K}$-algebras, and so on, in place of (étale) sheaves of sets. Furthermore, the analogue of all the above holds for quasi-coherent sheaves, (or coherent sheaves, or vector bundles, or line bundles) on X, where in **(a)** *$\mathcal{A}(T,t)$ becomes a quasi-coherent sheaf (or coherent sheaf, or vector bundle, or line bundle) on T, in* **(b)** *we replace $\phi^{-1}(\mathcal{A}(U,u))$ by the pullback $\phi^*(\mathcal{A}(U,u))$ of quasi-coherent sheaves (etc.), and $\mathcal{A}(\phi,\eta),\alpha(T,t)$ become morphisms of quasi-coherent sheaves (etc.) on T.*

We can also describe ***global sections*** *of sheaves on Artin $\mathbb{K}$-stacks in the above framework: a global section $s\in H^0(\mathcal{A})$ of $\mathcal{A}$ in part* **(A)** *assigns a global section $s(T,t)\in H^0(\mathcal{A}(T,t))$ of $\mathcal{A}(T,t)$ on T for all smooth $t : T\to X$ from a scheme T, such that $\mathcal{A}(\phi,\eta)^*(s(U,u))=s(T,t)$ in $H^0(\mathcal{A}(T,t))$ for all 2-commutative diagrams* (2.2.1) *with t,u smooth.*

In [87, Cor. 2.52] Joyce generalizes the sheaves $\mathcal{S}_X,\mathcal{S}_X^0$ in §2.1 to Artin $\mathbb{K}$-stacks:

Proposition 2.2.2. *Let X be an Artin $\mathbb{K}$-stack, and write $\mathrm{Sh}(X)_{\mathbb{K}\text{-alg}}$ and $\mathrm{Sh}(X)_{\mathbb{K}\text{-vect}}$ for the categories of sheaves of $\mathbb{K}$-algebras and $\mathbb{K}$-vector spaces on X defined in Proposition* 2.2.1*. Then:*

- **(a)** *We may define canonical objects $\mathcal{S}_X$ in both $\mathrm{Sh}(X)_{\mathbb{K}\text{-alg}}$ and $\mathrm{Sh}(X)_{\mathbb{K}\text{-vect}}$ by $\mathcal{S}_X(T,t):=\mathcal{S}_T$ for all smooth morphisms $t : T\to X$ for $T\in\mathrm{Sch}_\mathbb{K}$, for $\mathcal{S}_T$ as in* §2.1 *taken to be a sheaf of $\mathbb{K}$-algebras (or $\mathbb{K}$-vector spaces) on T in the étale topology, and $\mathcal{S}_X(\phi,\eta):=\phi^\star :$ $\phi^{-1}(\mathcal{S}_X(U,u))=\phi^{-1}(\mathcal{S}_U)\to\mathcal{S}_T=\mathcal{S}_X(T,t)$ for all 2-commutative diagrams* (2.2.1) *in $\mathrm{Art}_\mathbb{K}$ with t,u smooth, where $\phi^\star$ is as in* §2.1*.*
- **(b)** *There is a natural decomposition $\mathcal{S}_X=\mathbb{K}_X\oplus\mathcal{S}_X^0$ in $\mathrm{Sh}(X)_{\mathbb{K}\text{-vect}}$ induced by the splitting $\mathcal{S}_X(T,t)=\mathcal{S}_T=\mathbb{K}_T\oplus\mathcal{S}_T^0$ in* §2.1*, where $\mathbb{K}_X$ is a sheaf of $\mathbb{K}$-subalgebras of $\mathcal{S}_X$ in $\mathrm{Sh}(X)_{\mathbb{K}\text{-alg}}$, and $\mathcal{S}_X^0$ a sheaf of ideals in $\mathcal{S}_X$.*

Here [87, Def. 2.53] is the generalization of Definition 2.1.2 to Artin stacks.

Definition 2.2.3. A *d-critical stack* (X,s) is an Artin $\mathbb{K}$-stack X and a global section $s\in H^0(\mathcal{S}_X^0)$, where $\mathcal{S}_X^0$ is as in Proposition 2.2.2, such that $\big(T,s(T,t)\big)$ is an algebraic d-critical locus in the sense of Definition 2.1.2 for all smooth morphisms $t : T\to X$ with $T\in\mathrm{Sch}_\mathbb{K}$.

Here is a convenient way to understand d-critical stacks (X,s) in terms of d-critical structures on an atlas $t : T\to X$ for X from [87, Prop. 2.54].

Proposition 2.2.4. *Suppose we are given a 2-commutative diagram in* $\mathrm{Art}_{\mathbb{K}}$*:*

$$\begin{tikzcd} U \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\eta\Uparrow"] & T \arrow[d, "t"'] \\ T \arrow[r, "t"] & X, \end{tikzcd} \tag{2.2.2}$$

where X is an Artin $\mathbb{K}$-stack, T, U are $\mathbb{K}$-schemes, t, π_1, π_2 are smooth 1-morphisms, $t : T \to X$ is surjective, and the 1-morphism $U \to T \times_{t,X,t} T$ induced by (2.2.2) *is surjective. For instance, this happens if $U \rightrightarrows T$ is a groupoid in $\mathbb{K}$-schemes, and $X = [U \rightrightarrows T]$ the associated groupoid stack. Then:*

(i) *Let $\mathcal{S}_X$ be as in Proposition* 2.2.2*, and $\mathcal{S}_T, \mathcal{S}_U$ be as in* §2*, regarded as sheaves on T, U in the étale topology, and define $\pi_i^\star : \pi_i^{-1}(\mathcal{S}_T) \to \mathcal{S}_U$ as in* §**??** *for $i = 1, 2$. Consider the map $t^* : H^0(\mathcal{S}_X) \to H^0(\mathcal{S}_T)$ mapping $t^* : s \mapsto s(T, t)$. This is injective, and induces a bijection*

$$t^* : H^0(\mathcal{S}_X) \xrightarrow{\cong} \{ s' \in H^0(\mathcal{S}_T) : \pi_1^\star(s') = \pi_2^\star(s') \text{ in } H^0(\mathcal{S}_U) \}. \tag{2.2.3}$$

The analogue holds for $\mathcal{S}_X^0, \mathcal{S}_T^0, \mathcal{S}_U^0$.

(ii) *Suppose $s \in H^0(\mathcal{S}_X^0)$, so that $t^*(s) \in H^0(\mathcal{S}_T^0)$ with $\pi_1^\star \circ t^*(s) = \pi_2^\star \circ t^*(s)$. Then (X, s) is a d-critical stack if and only if $\big(T, t^*(s)\big)$ is an algebraic d-critical locus, and then $\big(U, \pi_1^\star \circ t^*(s)\big)$ is also an algebraic d-critical locus.*

Next, we report [87, Ex. 2.55], where Joyce considers quotient stacks $X = [T/G]$.

Example 2.2.5. Suppose an algebraic $\mathbb{K}$-group G acts on a $\mathbb{K}$-scheme T with action $\mu : G \times T \to T$, and write X for the quotient Artin $\mathbb{K}$-stack $[T/G]$. Then as in (2.2.2) there is a natural 2-Cartesian diagram

$$\begin{tikzcd} G \times T \arrow[r, "\mu"'] \arrow[d, "\pi_T"] \arrow[dr, phantom, "\eta\Uparrow"] & T \arrow[d, "t"'] \\ T \arrow[r, "t"] & X = [T/G], \end{tikzcd}$$

where $t : T \to X$ is a smooth atlas for X. If $s' \in H^0(\mathcal{S}_T^0)$ then $\pi_1^\star(s') = \pi_2^\star(s')$ in (2.2.3) becomes $\pi_T^\star(s') = \mu^\star(s')$ on $G \times T$, that is, s' is G-invariant. Hence, Proposition 2.2.4 shows that d-critical structures s on $X = [T/G]$ are in 1-1 correspondence with G-invariant d-critical structures s' on T.

Here [87, Th. 2.56] is an analogue of Theorem 2.1.6.

Theorem 2.2.6. *Let (X, s) be a d-critical stack. Using the description of quasi-coherent sheaves on X^{red} in Proposition* 2.2.1 *there is a line bundle $K_{X,s}$ on the reduced $\mathbb{K}$-substack X^{red} of X called the* ***canonical bundle*** *of (X, s), unique up to canonical isomorphism, such that:*

(a) *For each point $x \in X^{\mathrm{red}} \subseteq X$ we have a canonical isomorphism*

$$\kappa_x : K_{X,s}|_x \xrightarrow{\cong} \big(\Lambda^{\mathrm{top}} T_x^* X\big)^{\otimes^2} \otimes \big(\Lambda^{\mathrm{top}} \mathfrak{Iso}_x(X)\big)^{\otimes^2}, \tag{2.2.4}$$

where $T_x^* X$ is the Zariski cotangent space of X at x, and $\mathfrak{Iso}_x(X)$ the Lie algebra of the isotropy group (stabilizer group) $\mathrm{Iso}_x(X)$ of X at x.

(b) *If T is a $\mathbb{K}$-scheme and $t : T \to X$ a smooth 1-morphism, so that $t^{\mathrm{red}} : T^{\mathrm{red}} \to X^{\mathrm{red}}$ is also smooth, then there is a natural isomorphism of line bundles on T^{red}:*

$$\Gamma_{T,t} : K_{X,s}(T^{\mathrm{red}}, t^{\mathrm{red}}) \xrightarrow{\cong} K_{T,s(T,t)} \otimes \big(\Lambda^{\mathrm{top}} T_{T/X}^*\big)\big|_{T^{\mathrm{red}}}^{\otimes^{-2}}. \tag{2.2.5}$$

Here $\big(T, s(T, t)\big)$ is an algebraic d-critical locus by Definition 2.2.3*, and $K_{T,s(T,t)} \to T^{\mathrm{red}}$ is its canonical bundle from Theorem* 2.1.6*.*

(c) *If $t : T \to X$ is a smooth 1-morphism, we have a distinguished triangle in $D_{\mathrm{qcoh}}(T)$:*

$$t^*(\mathbb{L}_X) \xrightarrow{\mathbb{L}_t} \mathbb{L}_T \longrightarrow T^*_{T/X} \longrightarrow t^*(\mathbb{L}_X)[1], \tag{2.2.6}$$

where $\mathbb{L}_T, \mathbb{L}_X$ are the cotangent complexes of T, X, and $T^*_{T/X}$ the relative cotangent bundle of $t : T \to X$, a vector bundle of mixed rank on T. Let $p \in T^{\mathrm{red}} \subseteq T$, so that $t(p) := t \circ p \in X$. Taking the long exact cohomology sequence of (2.2.6) *and restricting to $p \in T$ gives an exact sequence*

$$0 \longrightarrow T^*_{t(p)}X \longrightarrow T^*_pT \longrightarrow T^*_{T/X}|_p \longrightarrow \mathfrak{Iso}_{t(p)}(X)^* \longrightarrow 0. \tag{2.2.7}$$

Then the following diagram commutes:

$$\begin{tikzcd}
K_{X,s}|_{t(p)} \arrow[r, equal] \arrow[d, "\kappa_{t(p)}"] & K_{X,s}(T^{\mathrm{red}}, t^{\mathrm{red}})|_p \arrow[r, "\Gamma_{T,t}|_p"'] & K_{T,s(T,t)}|_p \otimes \big(\Lambda^{\mathrm{top}} T^*_{T/X}\big)\big|_p^{\otimes^{-2}} \arrow[d, "\kappa_p \otimes \mathrm{id}"'] \\
\big(\Lambda^{\mathrm{top}} T^*_{t(p)}X\big)^{\otimes^2} \otimes \big(\Lambda^{\mathrm{top}} \mathfrak{Iso}_{t(p)}(X)\big)^{\otimes^2} \arrow[rr, "\alpha_p^2"] & & \big(\Lambda^{\mathrm{top}} T^*_pT\big)^{\otimes^2} \otimes \big(\Lambda^{\mathrm{top}} T^*_{T/X}\big)\big|_p^{\otimes^{-2}},
\end{tikzcd}$$

where $\kappa_p, \kappa_{t(p)}, \Gamma_{T,t}$ are as in (2.1.3), (2.2.4) *and* (2.2.5), *respectively, and $\alpha_p : \Lambda^{\mathrm{top}} T^*_{t(p)}X \otimes \Lambda^{\mathrm{top}} \mathfrak{Iso}_{t(p)}(X) \xrightarrow{\cong} \Lambda^{\mathrm{top}} T^*_pT \otimes \Lambda^{\mathrm{top}} T^*_{T/X}|_p^{-1}$ is induced by taking top exterior powers in* (2.2.7).

Here [87, Def. 2.57] is the analogue of Definition 2.1.8:

Definition 2.2.7. Let (X, s) be a d-critical stack, and $K_{X,s}$ its canonical bundle from Theorem 2.2.6. An *orientation* on (X, s) is a choice of square root line bundle $K^{1/2}_{X,s}$ for $K_{X,s}$ on X^{red}. That is, an orientation is a line bundle L on X^{red}, together with an isomorphism $L^{\otimes^2} = L \otimes L \cong K_{X,s}$. A d-critical stack with an orientation will be called an *oriented d-critical stack.*

Let (X, s) be an oriented d-critical stack. Then for each smooth $t : T \to X$ we have a square root $K^{1/2}_{X,s}(T^{\mathrm{red}}, t^{\mathrm{red}})$. Thus by (2.2.5), $K^{1/2}_{X,s}(T^{\mathrm{red}}, t^{\mathrm{red}}) \otimes (\Lambda^{\mathrm{top}} \mathbb{L}_{T/X})|_{T^{\mathrm{red}}}$ is a square root for $K_{T,s(T,t)}$. This proves [87, Lem. 2.58]:

Lemma 2.2.8. *Let (X, s) be a d-critical stack. Then an orientation $K^{1/2}_{X,s}$ for (X, s) determines a canonical orientation $K^{1/2}_{T,s(T,t)}$ for the algebraic d-critical locus $\big(T, s(T, t)\big)$, for all smooth $t : T \to X$ with T a $\mathbb{K}$-scheme.*

2.3 Equivariant d-critical loci

Here we summarizes some results about group actions on algebraic d-critical loci from [87].

Definition 2.3.1. Let (X, s) be an algebraic d-critical locus over $\mathbb{K}$, and $\mu : G \times X \to X$ an action of an algebraic $\mathbb{K}$-group G on the $\mathbb{K}$-scheme X. We also write the action as $\mu(\gamma) : X \to X$ for $\gamma \in G$. We say that (X, s) is *G-invariant* if $\mu(\gamma)^\star(s) = s$ for all $\gamma \in G$, or equivalently, if $\mu^\star(s) = \pi_X^\star(s)$ in $H^0(\mathcal{S}^0_{G\times X})$, where $\pi_X : G \times X \to X$ is the projection.

Let $\chi : G \to \mathbb{G}_m$ be a morphism of algebraic $\mathbb{K}$-groups, that is, a character of G, where $\mathbb{G}_m = \mathbb{K} \setminus \{0\}$ is the multiplicative group. We say that (X, s) is *G-equivariant, with character χ*, if $\mu(\gamma)^\star(s) = \chi(\gamma) \cdot s$ for all $\gamma \in G$, or equivalently, if $\mu^\star(s) = (\chi \circ \pi_G) \cdot (\pi_X^\star(s))$ in $H^0(\mathcal{S}^0_{G\times X})$, where $H^0(\mathcal{O}_G) \ni \chi$ acts on $H^0(\mathcal{S}^0_{G\times X})$ by multiplication, as G is a smooth $\mathbb{K}$-scheme.

Suppose (X,s) is G-invariant or G-equivariant, with $\chi = 1$ in the G-invariant case. We call a critical chart (R,U,f,i) on (X,s) with a G-action $\rho : G \times U \to U$ a *G-equivariant critical chart* if $R \subseteq X$ is a G-invariant open subscheme, and $i : R \hookrightarrow U$, $f : U \to \mathbb{A}^1$ are equivariant with respect to the actions $\mu|_{G\times R}, \rho, \chi$ of G on $R, U, \mathbb{A}^1$, respectively.

We call a subchart $(R',U',f',i') \subseteq (R,U,f,i)$ a *G-equivariant subchart* if $R' \subseteq R$ and $U' \subseteq U$ are G-invariant open subschemes. Then $(R',U',f',i'), \rho'$ is a G-equivariant critical chart, where $\rho' = \rho|_{G\times U'}$.

Note that X may not be covered by G-equivariant critical charts without extra assumptions on X, G. We will restrict to the case when G is a torus, with a 'good' action on X:

Definition 2.3.2. Let X be a $\mathbb{K}$-scheme, G an algebraic $\mathbb{K}$-torus, and $\mu : G \times X \to X$ an action of G on X. We call μ a *good action* if X admits a Zariski open cover by G-invariant affine open $\mathbb{K}$-subschemes $U \subseteq X$.

A torus-equivariant d-critical locus (X,s) admits an open cover by equivariant critical charts if and only if the torus action is good:

Proposition 2.3.3. *Let (X,s) be an algebraic d-critical locus which is invariant or equivariant under the action $\mu : G \times X \to X$ of an algebraic torus G.*
(a) *If μ is good then for all $x \in X$ there exists a G-equivariant critical chart $(R,U,f,i), \rho$ on (X,s) with $x \in R$, and we may take* $\dim U = \dim T_x X$.
(b) *Conversely, if for all $x \in X$ there exists a G-equivariant critical chart $(R,U,f,i), \rho$ on (X,s) with $x \in R$, then μ is good.*

Chapter 3

A Darboux theorem for symplectic derived schemes

This chapter is based on [19]. In general, we will not go into details and proofs of results, for which we refer to [19]. We will need this material for the following chapters.

3.1 'Standard form' affine derived schemes

The next definition summarizes [19, Ex. 2.8, Def. 2.9 & Def. 2.13]. A useful review of [19] is provided in [13, §2.3–2.4], which is our main reference.

Definition 3.1.1. We will explain how to inductively construct a sequence of commutative differential graded algebras (cdgas) $A(0), A(1), \ldots, A(n) = A$ over $\mathbb{K}$ with $A(0)$ a smooth $\mathbb{K}$-algebra and $A(k)$ having underlying commutative graded algebra free over $A(0)$ on generators of degrees $-1, \ldots, -k$. We will call A a *standard form* cdga. We will write $\boldsymbol{U}(i) = \mathbf{Spec}\, A(i)$ for $i = 0, \ldots, n$ and $\boldsymbol{U} = \boldsymbol{U}(n) = \mathbf{Spec}\, A$ for the corresponding affine derived $\mathbb{K}$-schemes, where $\boldsymbol{U}(0) = U(0)$ is a smooth classical $\mathbb{K}$-scheme, which contains $\operatorname{Spec} H^0(A)$ as a closed $\mathbb{K}$-subscheme.

Begin with a commutative algebra $A(0)$ smooth over $\mathbb{K}$. Choose a free $A(0)$-module M^{-1} of finite rank together with a map $\pi^{-1} : M^{-1} \to A(0)$. Define a cdga $A(1)$ whose underlying commutative graded algebra is free over $A(0)$ with generators given by M^{-1} in degree -1 and with differential d determined by the map $\pi^{-1} : M^{-1} \to A(0)$. By construction, we have $H^0(A(1)) = A(0)/I$, where the ideal $I \subseteq A(0)$ is the image of the map $\pi^{-1} : M^{-1} \to A(0)$. Note that $A(1)$ fits in a homotopy pushout diagram of cdgas

$$\begin{tikzcd} \mathrm{Sym}_{A(0)}(M^{-1}) \arrow[r, "0_*"'] \arrow[d, "\pi_*^{-1}"] & A(0) \arrow[d] \\ A(0) \arrow[r, "f^{-1}"] & A(1), \end{tikzcd}$$

with morphisms $\pi_*^{-1}, 0_*$ induced by $\pi^{-1}, 0 : M^{-1} \to A(0)$. Write $f^{-1} : A(0) \to A(1)$ for the resulting map of algebras.

Next, choose a free $A(1)$-module M^{-2} of finite rank together with a map $\pi^{-2} : M^{-2}[1] \to A(1)$. Define a cdga $A(2)$ whose underlying commutative graded algebra is free over $A(1)$ with generators given by M^{-2} in degree -2 and with differential d determined by the map $\pi^{-2} : M^{-2}[1] \to A(1)$. Write f^{-2} for the resulting map of algebras $A(1) \to A(2)$. As the underlying commutative graded algebra of $A(1)$ was free over $A(0)$ on generators of degree -1, the underlying commutative graded algebra of $A(2)$ is free over $A(0)$ on generators of degrees $-1, -2$. Since $A(2)$ is obtained from

$A(1)$ by adding generators in degree -2, we have $H^0(A(1)) \cong H^0(A(2)) \cong A(0)/I$. Note that $A(2)$ fits in a homotopy pushout diagram of cdgas

$$\begin{tikzcd} \mathrm{Sym}_{A(1)}(M^{-2}[1]) \arrow[r, "0_*"'] \arrow[d, "\pi_*^{-2}"] & A(1) \arrow[d] \\ A(1) \arrow[r, "f^{-2}"] & A(2), \end{tikzcd}$$

with morphisms $\pi_*^{-2}, 0_*$ induced by $\pi^{-2}, 0 : M^{-2}[1] \to A(1)$.

Continuing in this manner inductively, we define a cdga $A(n) = A$ with $A^0 = A(0)$ and $H^0(A) = A(0)/I$, whose underlying commutative graded algebra is free over $A(0)$ on generators of degrees $-1, \ldots, -n$. We call any cdga A constructed in this way a *standard form* cdga.

If A is of standard form, we will call a cdga A' a *localization* of A if $A' = A \otimes_{A^0} A^0[f^{-1}]$ for $f \in A^0$, that is, A' is obtained by inverting f in A. Then A' is also of standard form, with $A'^0 \cong A^0[f^{-1}]$. If $p \in \operatorname{Spec} H^0(A)$ with $f(p) \neq 0$, we call A' a *localization of A around p*.

Let A be a standard form cdga. We call A *minimal* at $p \in \operatorname{Spec} H^0(A)$ if for all $k = 1, \ldots, n$ the compositions $H^{-k}\big(\mathbb{L}_{A(k)/A(k-1)}\big) \longrightarrow H^{1-k}\big(\mathbb{L}_{A(k-1)}\big) \longrightarrow H^{1-k}\big(\mathbb{L}_{A(k-1)/A(k-2)}\big)$ in the cotangent complexes restricted to $\operatorname{Spec} H^0(A)$ vanish at p. For more on this point, see [19, Prop. 2.12].

Here are [19, Th.s 4.1 & 4.2]. They say that any derived scheme $\boldsymbol{X}$ is locally modelled on $\mathbf{Spec}\, A$ for a (minimal) standard form cdga A, and give us a way to compare two such local models $\boldsymbol{f} : \mathbf{Spec}\, A \hookrightarrow \boldsymbol{X}$, $\boldsymbol{g} : \mathbf{Spec}\, B \hookrightarrow \boldsymbol{X}$.

Theorem 3.1.2. *Let $\boldsymbol{X}$ be a derived $\mathbb{K}$-scheme, and $x \in \boldsymbol{X}$. Then there exist a standard form cdga A over $\mathbb{K}$ which is minimal at a point $p \in \operatorname{Spec} H^0(A)$, in the sense of Definition* 3.1.1, *and a morphism $\boldsymbol{f} : \boldsymbol{U} = \mathbf{Spec}\, A \to \boldsymbol{X}$ in $\mathbf{dSch}_{\mathbb{K}}$ which is a Zariski open inclusion with $\boldsymbol{f}(p) = x$.*

We think of $A, \boldsymbol{f}$ in Theorem 3.1.2 as like a coordinate system on $\boldsymbol{X}$ near x. As well as being able to choose coordinates near any point, we want to be able to compare different coordinate systems on their overlaps. That is, given local equivalences $\boldsymbol{f} : \mathbf{Spec}\, A \to \boldsymbol{X}$, $\boldsymbol{g} : \mathbf{Spec}\, B \to \boldsymbol{X}$, we would like to compare the cdgas A, B on the overlap of their images in $\boldsymbol{X}$. For general A, B we cannot (even locally) find a cdga morphism $\alpha : B \to A$ with $\boldsymbol{f} \simeq \boldsymbol{g} \circ \mathbf{Spec}\, \alpha$. However, the next theorem, which is [19, Thm. 4.2] shows we can find a third cdga C and open inclusions $\alpha : A \to C$, $\beta : B \to C$ with $\boldsymbol{f} \circ \mathbf{Spec}\, \alpha \simeq \boldsymbol{g} \circ \mathbf{Spec}\, \beta$. This will be important in the proof of Theorem 3.3.1.

Theorem 3.1.3. *Let $\boldsymbol{X}$ be a derived $\mathbb{K}$-scheme, A, B be standard form cdgas over* $\mathbb{K}$, *and $\boldsymbol{f} : \mathbf{Spec}\, A \to \boldsymbol{X}$, $\boldsymbol{g} : \mathbf{Spec}\, B \to \boldsymbol{X}$ be Zariski open inclusions in $\mathbf{dSch}_{\mathbb{K}}$. Suppose $p \in \operatorname{Spec} H^0(A)$ and $q \in \operatorname{Spec} H^0(B)$ with $\boldsymbol{f}(p) = \boldsymbol{g}(q)$ in $\boldsymbol{X}$. Then there exist a standard form cdga C over $\mathbb{K}$ which is minimal at r in $\operatorname{Spec} H^0(C)$ and morphisms of cdgas $\alpha : A \to C$, $\beta : B \to C$ which are Zariski open inclusions, such that $\mathbf{Spec}\, \alpha : r \mapsto p$, $\mathbf{Spec}\, \beta : r \mapsto q$, and $\boldsymbol{f} \circ \mathbf{Spec}\, \alpha \simeq \boldsymbol{g} \circ \mathbf{Spec}\, \beta$ as morphisms $\mathbf{Spec}\, C \to \boldsymbol{X}$ in $\mathbf{dSch}_{\mathbb{K}}$. If instead $\boldsymbol{f}, \boldsymbol{g}$ are étale rather than Zariski open inclusions, the same holds with α, β étale rather than Zariski open inclusions.*

One important advantage of working with derived schemes $\boldsymbol{U} = \mathbf{Spec}\, A$ for A a standard form cdga, is that the cotangent complex $\mathbb{L}_{\boldsymbol{U}}$ and its exterior powers $\Lambda^p \mathbb{L}_{\boldsymbol{U}}$ can be written simply and explicitly in terms of A. As in [19, §2, §3.3] the differential-graded module of *Kähler differentials* Ω^1_A is a model for $\mathbb{L}_{\boldsymbol{U}}$. If $U(0) = \operatorname{Spec} A^0$ admits global étale coordinates $(x_1^0, \ldots, x_{m_0}^0)$, then Ω^1_A is a finitely-generated free A-module, generated by $\mathrm{d}_{dR} x_1^{-i}, \ldots, \mathrm{d}_{dR} x_{m_i}^{-i}$ in degree $-i$ for $i = 0, \ldots, n$, where $x_1^{-i}, \ldots, x_{m_i}^{-i}$ are $A(i-1)$-bases for the free finite rank $A(i-1)$-modules M^{-i} for $i = 1, \ldots, n$, in the notation of Definition 3.1.1. Because of this, on $\boldsymbol{U} = \mathbf{Spec}\, A$, the k-shifted (closed) p-forms from [142] discussed in §1.2.2 can be written down explicitly in coordinates. Here is [19, Prop. 5.7].

Part (a) implies that for a k-shifted symplectic form $\omega = (\omega^0, \omega^1, \omega^2, \ldots)$ on a standard form $\boldsymbol{U} = \mathbf{Spec}\, A$, up to equivalence we may take $\omega^1 = \omega^2 = \cdots = 0$, which simplifies calculations a lot. Let us note here that the proof of [19, Prop. 5.7] uses the interpretation of shifted symplectic forms as representing classes in negative cyclic homology. As this is a quite technical part, we omit it here, referring to [19] for details.

Proposition 3.1.4. **(a)** *Let $\omega = (\omega^0, \omega^1, \omega^2, \ldots)$ be a closed 2-form of degree $k < 0$ on $\boldsymbol{U} = \mathbf{Spec}\, A$, for A a standard form cdga over $\mathbb{K}$. Then there exist $\Phi \in A^{k+1}$ and $\phi \in (\Omega^1_A)^k$ such that $\mathrm{d}\Phi = 0$ in A^{k+2} and $\mathrm{d}_{dR}\Phi + \mathrm{d}\phi = 0$ in $(\Omega^1_A)^{k+1}$ and $\omega \sim (\mathrm{d}_{dR}\phi, 0, 0, \ldots)$.*

(b) *In the case $k = -1$ in* **(a)** *we have $\Phi \in A^0 = A(0)$, so we can consider the restriction $\Phi|_{U^{\mathrm{red}}}$ of Φ to the reduced $\mathbb{K}$-subscheme U^{red} of $U = t_0(\boldsymbol{U}) = \mathrm{Spec}\, H^0(A)$. Then $\Phi|_{U^{\mathrm{red}}}$ is locally constant on U^{red}, and we may choose (Φ, ϕ) in* **(a)** *such that $\Phi|_{U^{\mathrm{red}}} = 0$.*

(c) *Suppose (Φ, ϕ) and (Φ', ϕ') are alternative choices in part* **(a)** *for fixed $\omega, k, \boldsymbol{U}, A$, where if $k = -1$ we suppose $\Phi|_{U^{\mathrm{red}}} = 0 = \Phi'|_{U^{\mathrm{red}}}$ as in* **(b)**. *Then there exist $\Psi \in A^k$ and $\psi \in (\Omega^1_A)^{k-1}$ with $\Phi - \Phi' = \mathrm{d}\Psi$ and $\phi - \phi' = \mathrm{d}_{dR}\Psi + \mathrm{d}\psi$.*

3.2 'Darboux form' shifted symplectic derived schemes

The next definition summarizes [19, Ex.s 5.8–5.10].

Definition 3.2.1. Fix $d = 0, 1, \ldots$. We will explain how to define a class of explicit standard form cdgas $(A, \mathrm{d}) = A(n)$ for $n = 2d + 1$ with a very simple, explicit k-shifted symplectic form $\omega = (\omega^0, 0, 0, \ldots)$ on $\boldsymbol{U} = \mathbf{Spec}\, A$ for $k = -2d - 1$. We will say that A, ω are in *Darboux form.*

First choose a smooth $\mathbb{K}$-algebra $A(0)$ of dimension m_0. Localizing $A(0)$ if necessary, we may assume that there exist $x^0_1, \ldots, x^0_{m_0} \in A(0)$ such that $\mathrm{d}_{dR}x^0_1, \ldots, \mathrm{d}_{dR}x^0_{m_0}$ form a basis of $\Omega^1_{A(0)}$ over $A(0)$. Geometrically, $U(0) = \mathrm{Spec}\, A(0)$ is a smooth $\mathbb{K}$-scheme of dimension m_0, and $(x^0_1, \ldots, x^0_{m_0}) : U(0) \to \mathbb{A}^{m_0}$ are global étale coordinates on $U(0)$. Next, choose $m_1, \ldots, m_d \in \mathbb{N} = \{0, 1, \ldots\}$. Define A as a commutative graded algebra to be the free algebra over $A(0)$ generated by variables

$$\begin{aligned} &x^{-i}_1, \ldots, x^{-i}_{m_i} && \text{in degree } -i \text{ for } i = 1, \ldots, d, \text{ and} \\ &y^{i-2d-1}_1, \ldots, y^{i-2d-1}_{m_i} && \text{in degree } i - 2d - 1 \text{ for } i = 0, 1, \ldots, d. \end{aligned} \tag{3.2.1}$$

So the upper index i in x^i_j, y^i_j always indicates the degree. We will define the differential d in the cdga (A, d) later. The spaces $(\Lambda^p\Omega^1_A)^k$ and the de Rham differential d_{dR} upon them depend only on the commutative graded algebra A, not on the (not yet defined) differential d. Note that Ω^1_A is the free A-module with basis $\mathrm{d}_{dR}x^{-i}_j, \mathrm{d}_{dR}y^{i-2d-1}_j$ for $i = 0, \ldots, d$ and $j = 1, \ldots, m_i$. Define

$$\omega^0 = \sum_{i=0}^{d} \sum_{j=1}^{m_i} \mathrm{d}_{dR}y^{i-2d-1}_j \, \mathrm{d}_{dR}x^{-i}_j \qquad \text{in } (\Lambda^2\Omega^1_A)^{-2d-1}. \tag{3.2.2}$$

Then $\mathrm{d}_{dR}\omega^0 = 0$ in $(\Lambda^3\Omega^1_A)^{-2d-1}$. Now choose H in A^{-2d}, which we will call the *Hamiltonian*, and which we require to satisfy the *classical master equation*

$$\sum_{i=1}^{d} \sum_{j=1}^{m_i} \frac{\partial H}{\partial x^{-i}_j} \, \frac{\partial H}{\partial y^{i-2d-1}_j} = 0 \qquad \text{in } A^{1-2d}. \tag{3.2.3}$$

The classical master equation can be expressed invariantly as $\{H, H\} = 0$, where $\{,\}$ is a certain shifted Poisson bracket. For more on this, consult [19, §5.7]. Note that (3.2.3) is trivial when

$d=0$, so that $k=-1$, as $A^1=0$. Define the differential d on A by $\mathrm{d}=0$ on $A(0)$, and

$$\mathrm{d}x_j^{-i}=\frac{\partial H}{\partial y_j^{i-2d-1}},\quad \mathrm{d}y_j^{i-2d-1}=\frac{\partial H}{\partial x_j^{-i}},\quad \begin{array}{l} i=0,\ldots,d,\\ j=1,\ldots,m_i.\end{array} \tag{3.2.4}$$

Then $\mathrm{d}\circ\mathrm{d}=0$, and (A,d) is a standard form cdga $A=A(n)$ as in Definition 3.1.1 for $n=2d+1$, defined using free modules $M^{-i}=\langle x_1^{-i},\ldots,x_{m_i}^{-i}\rangle_{A(i-1)}$ for $i=1,\ldots,d$ and $M^{i-2d-1}=\langle y_1^{i-2d-1},\ldots,y_{m_i}^{i-2d-1}\rangle_{A(2d-i)}$ for $i=0,\ldots,d$. Then $\omega=(\omega^0,0,0,\ldots)$ is a k-shifted symplectic structure on $\boldsymbol{U}=\mathbf{Spec}\,A$ for $k=-2d-1$. Define $\Phi\in A^{-2d}$ and $\phi\in(\Omega_A^1)^{-2d-1}$ by $\Phi=-\frac{1}{2d+1}H$ and

$$\phi=\frac{1}{2d+1}\sum_{i=0}^{d}\sum_{j=1}^{m_i}\bigl[(2d+1-i)y_j^{i-2d-1}\,\mathrm{d}_{dR}x_j^{-i}+i\,x_j^{-i}\,\mathrm{d}_{dR}y_j^{i-2d-1}\bigr]. \tag{3.2.5}$$

Then $\mathrm{d}\Phi=0$, $\mathrm{d}_{dR}\Phi+\mathrm{d}\phi=0$, and $\omega^0=\mathrm{d}_{dR}\phi$, as in Proposition 3.1.4(a). We say that A,ω are in *Darboux form* for $k=-2d-1$.

In [19, Ex.s 5.9 & 5.10] we give similar Darboux forms for $k=-4d$ and $k=-4d-2$ with $d=0,1,2,\ldots$. We will not give all the details. In brief, when $k=-4d$, rather than (3.2.1), A is freely generated over $A(0)$ by the variables

$$\begin{array}{ll} x_1^{-i},\ldots,x_{m_i}^{-i} & \text{in degree } -i \text{ for } i=1,\ldots,2d-1,\\ x_1^{-2d},\ldots,x_{m_{2d}}^{-2d},y_1^{-2d},\ldots,y_{m_{2d}}^{-2d} & \text{in degree } -2d, \text{ and}\\ y_1^{i-4d},\ldots,y_{m_i}^{i-4d} & \text{in degree } i-4d \text{ for } i=0,1,\ldots,2d-1,\end{array}$$

and $\omega^0\in(\Lambda^2\Omega_A^1)^{-4d}$ with $\mathrm{d}_{dR}\omega^0=0$ is given by

$$\omega^0=\sum_{i=0}^{2d}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{i-4d}\,\mathrm{d}_{dR}x_j^{-i}\qquad\text{in }(\Lambda^2\Omega_A^1)^{-4d},$$

and d on A is defined as in (3.2.4) using $H\in A^{1-4d}$ satisfying the analogue of (3.2.3). We then say that $A,\boldsymbol{U}=\mathbf{Spec}\,A,\omega$ are in *Darboux form* for $k=-4d$.

Similarly, when $k=-4d-2$, A is freely generated over $A(0)$ by the variables

$$\begin{array}{ll} x_1^{-i},\ldots,x_{m_i}^{-i} & \text{in degree } -i \text{ for } i=1,\ldots,2d,\\ z_1^{-2d-1},\ldots,z_{m_{2d+1}}^{-2d-1} & \text{in degree } -2d-1, \text{ and}\\ y_1^{i-4d-2},\ldots,y_{m_i}^{i-4d-2} & \text{in degree } i-4d-2 \text{ for } i=0,1,\ldots,2d,\end{array}$$

and $\omega^0\in(\Lambda^2\Omega_A^1)^{-4d-2}$ with $\mathrm{d}_{dR}\omega^0=0$ is given by

$$\omega^0=\sum_{i=0}^{2d}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{i-4d-2}\,\mathrm{d}_{dR}x_j^{-i}+\sum_{j=1}^{m_{2d+1}}\mathrm{d}_{dR}z_j^{-2d-1}\,\mathrm{d}_{dR}z_j^{-2d-1},$$

and d is defined as in (3.2.4) using $H\in A^{-4d-1}$ satisfying the classical master equation

$$\sum_{i=1}^{2d}\sum_{j=1}^{m_i}\frac{\partial H}{\partial x_j^{-i}}\,\frac{\partial H}{\partial y_j^{i-4d-2}}+\frac{1}{4}\sum_{j=1}^{m_{2d+1}}\left(\frac{\partial H}{\partial z_j^{-2d-1}}\right)^2=0\qquad\text{in }A^{-4d}.$$

We then say that A,ω are in *strong Darboux form* for $k=-4d-2$. There is also a *weak Darboux form* [19, Ex. 5.12] in this case, which we will not discuss.

Here is [19, Th. 5.18], the main result of [19]. We consider it to be a shifted symplectic analogue of Darboux' Theorem, as it shows that we can choose 'coordinate systems' on a k-shifted symplectic derived scheme $(\boldsymbol{X},\omega)$ in which ω assumes a standard form. Bouaziz and Grojnowski [17] also independently prove a similar theorem.

Theorem 3.2.2. *Let $\boldsymbol{X}$ be a derived $\mathbb{K}$-scheme with k-shifted symplectic form $\tilde\omega$ for $k<0$, and $x\in\boldsymbol{X}$. Then there exists a standard form cdga A over $\mathbb{K}$ which is minimal at $p\in\operatorname{Spec}H^0(A)$, a k-shifted symplectic form ω on $\mathbf{Spec}\,A$, and a morphism $\boldsymbol{f}:\boldsymbol{U}=\mathbf{Spec}\,A\to\boldsymbol{X}$ with $\boldsymbol{f}(p)=x$ and $\boldsymbol{f}^*(\tilde\omega)\sim\omega$, such that:*

- **(i)** *If k is odd or divisible by 4, then $\boldsymbol{f}$ is a Zariski open inclusion, and A,ω are in Darboux form, as in Definition* 3.2.1*.*
- **(ii)** *If $k\equiv 2 \mod 4$, then $\boldsymbol{f}$ is étale, and A,ω are in strong Darboux form, as in Definition* 3.2.1*.*

Let Y be a Calabi–Yau m-fold over $\mathbb{K}$ for $m\geqslant 3$, that is, a smooth projective $\mathbb{K}$-scheme with $H^i(\mathcal{O}_Y)=\mathbb{K}$ for $i=0,m$ and $H^i(\mathcal{O}_Y)=0$ for $0<i<m$. Suppose $\mathcal{M}$ is a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves in $\operatorname{coh}(Y)$, where we call $F\in\operatorname{coh}(Y)$ *simple* if $\operatorname{Hom}(F,F)=\mathbb{K}$. We point out that our moduli scheme $\mathcal{M}$ of simple coherent sheaves is based on Inaba's definition [74, Dfn 0.1] of the moduli functor which allows an equivalence relation tensoring by a line bundle over the base.

There is a corresponding definition of a derived moduli functor of coherent sheaves, also including tensoring by a line bundle over the base in the definition of the image simplicial set, yielding a derived enhancement $\boldsymbol{\mathcal{M}}$ of $\mathcal{M}$ with $\mathcal{M}=t_0(\boldsymbol{\mathcal{M}})$. Initially we know that $\boldsymbol{\mathcal{M}}$ is a derived stack, but since the classical truncation $t_0(\boldsymbol{\mathcal{M}})$ is a classical scheme, it follows that $\boldsymbol{\mathcal{M}}$ is a derived scheme.

Pantev et al. [142] do not including tensoring by a line bundle over the base in their definition, so that they consider the corresponding classical moduli stack $\mathcal{M}'$ of simple coherent sheaves, which is a classical Artin stack in which each point has isotropy group $\mathbb{G}_m$. Naively one might guess that the relationship between $\mathcal{M}$ and $\mathcal{M}'$ is that $\mathcal{M}'=[\mathcal{M}/\mathbb{G}_m]$ as a quotient stack, with a projection $\mathcal{M}\to\mathcal{M}'$, which is a $\mathbb{G}_m$-principal bundle, but this may be wrong. If we instead defined a moduli space $\widetilde{\mathcal{M}}$ of simple sheaves with fixed determinant, then (at least for torsion-free coherent sheaves, thus of positive rank) we would indeed get a projection $\widetilde{\mathcal{M}}\to\mathcal{M}'$ which is a $\mathbb{G}_m$-principal bundle. However, in general $\widetilde{\mathcal{M}}$ would not be a scheme, but a Deligne-Mumford stack, with stabilizer group $\mathbb{Z}_n$ at $[E]$ for $n=\operatorname{rank}E$.

In fact, the relationship between $\mathcal{M}$ and $\mathcal{M}'$ is more-or-less the opposite of this naive guess. There is a projection $\pi:\mathcal{M}'\to\mathcal{M}$ which is a $\mathbb{G}_m$-gerbe, with fibre $[*/\mathbb{G}_m]$ at each point. One can regard $[*/\mathbb{G}_m]$ as a kind of stacky algebraic group (a 2-group), which acts on $\mathcal{M}'$, and then $\mathcal{M}=[\mathcal{M}'/[*/\mathbb{G}_m]]$, and $\pi:\mathcal{M}'\to\mathcal{M}$ is a '$[*/\mathbb{G}_m]$ - principal bundle'. For torsion-free sheaves, the composition $\widetilde{\mathcal{M}}\to\mathcal{M}'\to\mathcal{M}$ is étale, a $\mathbb{Z}_n$-principal bundle for sheaves of rank n.

Now we consider the derived picture. Let us denote the derived enhancement of $\mathcal{M}'$ by $\boldsymbol{\mathcal{M}'}$, with $t_0(\boldsymbol{\mathcal{M}'})=\mathcal{M}'$. Pantev et al. [142, §2.1] prove $\boldsymbol{\mathcal{M}'}$ has a $(2-m)$-shifted symplectic structure. We can deduce that also $\boldsymbol{\mathcal{M}}$ has a $(2-m)$-shifted symplectic structure, as follows. Joyce and Song [85, Thm. 5.4] prove that for any finite type moduli stack of coherent sheaves U on a Calabi-Yau m-fold Y (their definition requires Y smooth and projective with $H^i(\mathcal{O}_Y)=0$ for $0<i<m$, as assumed above), then by applying a finite number of Seidel-Thomas twists by $\mathcal{O}_Y(-n)$ for $n\gg 0$ and a shift $[-m]$, we can identify U with a moduli stack of vector bundles E with rank $E\gg 0$. Seidel–Thomas twists and shifts commute with forming derived moduli stacks, with or without tensoring by a line bundle over the base. (Note that Seidel–Thomas twists and shifts do not commute with fixing determinants, though).

Thus, to show that the derived moduli scheme $\boldsymbol{\mathcal{M}}$ has a $(2-m)$-shifted symplectic structure, it is sufficient (at least for $\boldsymbol{\mathcal{M}}$ of finite type and $H^i(\mathcal{O}_Y)=0$ for $0<i<m$) to consider the case when $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{M}}'$ are derived moduli schemes/stacks of vector bundles of fixed rank n. In this case $\boldsymbol{\mathcal{M}}'$ is an open derived substack of the mapping stack $\mathbf{Map}(Y,[*/\mathrm{GL}(n,\mathbb{K})])$, which is the derived moduli stack $\mathfrak{Vect}_Y'^{n}$ of rank n vector bundles on Y. The $(2-m)$-shifted symplectic structure on $\boldsymbol{\mathcal{M}}'$ and $\mathbf{Map}(Y,[*/\mathrm{GL}(n,\mathbb{K})])$ is then induced as in [142, §2.1] from the Calabi–Yau m-fold structure on Y and the 2-shifted symplectic structure on $[*/\mathrm{GL}(n,\mathbb{K})]$, which in turn comes from the natural choice of a $\mathrm{GL}(n,\mathbb{K})$-invariant nondegenerate quadratic form on the Lie algebra $\mathfrak{gl}(n,\mathbb{K})$.

Now $\boldsymbol{\mathcal{M}}$ is in effect a derived moduli scheme of projective vector bundles, so that $\boldsymbol{\mathcal{M}}$ is a derived open substack of $\mathbf{Map}(Y,[*/\mathrm{PGL}(n,\mathbb{K})])$. Thus, $\boldsymbol{\mathcal{M}}$ has a $(2-m)$-shifted symplectic structure as in [142, §2.1] induced from the 2-shifted symplectic structure on $[*/\mathrm{PGL}(n,\mathbb{K})]$, which comes from the natural $\mathrm{PGL}(n,\mathbb{K})$-invariant nondegenerate quadratic form on the Lie algebra $\mathfrak{pgl}(n,\mathbb{K})$ (the Killing form; any $\mathrm{PGL}(n,\mathbb{K})$-invariant quadratic form on $\mathfrak{pgl}(n,\mathbb{K})$ is a multiple of the Killing form). The projection $\boldsymbol{\mathcal{M}}'\to\boldsymbol{\mathcal{M}}$ is the restriction of the projection $\mathbf{Map}(Y,[*/\mathrm{GL}(n,\mathbb{K})])\to\mathbf{Map}(Y,[*/\mathrm{PGL}(n,\mathbb{K})])$ induced by composition with the morphism $[*/\mathrm{GL}(n,\mathbb{K})]\to[*/\mathrm{PGL}(n,\mathbb{K})]$ coming from the algebraic group morphism $\mathrm{GL}(n,\mathbb{K})\to\mathrm{PGL}(n,\mathbb{K})$.

If we instead define a derived Deligne–Mumford moduli stack $\widetilde{\boldsymbol{\mathcal{M}}}$ with $\widetilde{\mathcal{M}}=t_0(\widetilde{\boldsymbol{\mathcal{M}}})$ by fixing determinants of vector bundles, then $\widetilde{\boldsymbol{\mathcal{M}}}$ is open in $\mathbf{Map}(Y,[*/\mathrm{SL}(n,\mathbb{K})])$, and the maps $\widetilde{\boldsymbol{\mathcal{M}}}\to\boldsymbol{\mathcal{M}}'\to\boldsymbol{\mathcal{M}}$ above come from composition with the Artin stack morphisms $[*/\mathrm{SL}(n,\mathbb{K})]\to[*/\mathrm{GL}(n,\mathbb{K})]\to[*/\mathrm{PGL}(n,\mathbb{K})]$ induced by the algebraic group morphisms $\mathrm{SL}(n,\mathbb{K})\to\mathrm{GL}(n,\mathbb{K})\to\mathrm{PGL}(n,\mathbb{K})$.

We said above that at the classical level,we can regard $\mathcal{M}$ as a quotient $[\mathcal{M}'/[*/\mathbb{G}_m]]$ by the stacky algebraic group $[*/\mathbb{G}_m]$. At the derived level, we can regard the $(2-m)$-shifted symplectic derived scheme $(\boldsymbol{\mathcal{M}},\omega)$ as a shifted symplectic quotient of the $(2-m)$-shifted symplectic derived stack $(\boldsymbol{\mathcal{M}}',\omega')$ by $[*/\mathbb{G}_m]$, using the derived shifted symplectic quotient construction of Safronov [151]. Taking the zero set of the moment map modifies $\boldsymbol{\mathcal{M}}'$ in degree $1-m<0$, which is not visible at the level of the classical truncations $\mathcal{M}$, $\mathcal{M}'$.

We have shown that the classical moduli scheme $\mathcal{M}$ can be enhanced to a derived moduli scheme $\boldsymbol{\mathcal{M}}$ with $\mathcal{M}=t_0(\boldsymbol{\mathcal{M}})$, and $\boldsymbol{\mathcal{M}}$ carries a $(2-m)$-shifted symplectic structure ω. Theorem 3.2.2 gives Zariski local models $(\mathbf{Spec}\,A,\omega)$ for $(\boldsymbol{\mathcal{M}},\omega)$ in Darboux form, with $\boldsymbol{\mathcal{M}}$ Zariski locally modelled on $\mathrm{Spec}\,H^0(A)$. In the case $m=3$, so that $k=-1$, we deduce [19, Cor. 5.19]:

Corollary 3.2.3. *Suppose Y is a Calabi–Yau 3-fold over a field $\mathbb{K}$, and $\mathcal{M}$ is a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves on Y. Then for each $[F]\in\mathcal{M}$, there exist a smooth $\mathbb{K}$-scheme U with $\dim U=\dim\mathrm{Ext}^1(F,F)$, a regular function $f:U\to\mathbb{A}^1$, and an isomorphism from $\mathrm{Crit}(f)\subseteq U$ to a Zariski open neighbourhood of $[F]$ in $\mathcal{M}$.*

Here $\dim U=\dim\mathrm{Ext}^1(F,F)$ comes from A minimal at p and $\boldsymbol{f}(p)=[F]$ in Theorem 3.2.2. As we will discuss in details in §7, related results are important in Donaldson–Thomas theory [85, 102, 104]. When $\mathbb{K}=\mathbb{C}$ and $\mathcal{M}$ is a moduli space of simple coherent sheaves on Y, using gauge theory and transcendental complex methods, Joyce and Song [85, Th. 5.4] prove that the underlying complex analytic space $\mathcal{M}^{\mathrm{an}}$ of $\mathcal{M}$ is locally of the form $\mathrm{Crit}(f)$ for U a complex manifold and $f:U\to\mathbb{C}$ a holomorphic function. Behrend and Getzler announced the analogue of [85, Th. 5.4] for moduli of complexes in $D^b\,\mathrm{coh}(Y)$, but the proof has not yet appeared. Over general $\mathbb{K}$, as in Kontsevich and Soibelman [102, §3.3] the formal neighbourhood $\hat{\mathcal{M}}_{[F]}$ of $\mathcal{M}$ at any $[F]\in\mathcal{M}$ is isomorphic to the critical locus $\mathrm{Crit}(\hat{f})$ of a formal power series $\hat{f}$ on $\mathrm{Ext}^1(F,F)$ with only cubic and higher terms.

In the case $m = 4$, so that $k = -2$, from Example 5.16 in [19] we deduce in [19, Cor. 5.20] a local description of Calabi–Yau 4-fold moduli schemes:

Corollary 3.2.4. *Suppose Y is a Calabi–Yau 4-fold over a field $\mathbb{K}$, and $\mathcal{M}$ is a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves on Y. Then for each $[F] \in \mathcal{M}$, there exist a smooth $\mathbb{K}$-scheme U with $\dim U = \dim \mathrm{Ext}^1(F,F)$, a vector bundle $E \to U$ with $\operatorname{rank} E = \dim \mathrm{Ext}^2(F,F)$, a nondegenerate quadratic form Q on E, a section $s \in H^0(E)$ with $Q(s,s) = 0$, and an isomorphism from $s^{-1}(0) \subseteq U$ to a Zariski open neighbourhood of $[F]$ in $\mathcal{M}$.*

If (S,ω) is an algebraic symplectic manifold over $\mathbb{K}$ (that is, a 0-shifted symplectic derived $\mathbb{K}$-scheme in the language of [142]) and $L, M \subseteq S$ are Lagrangians, then Pantev et al. [142, Th. 2.10] show that the derived intersection $\boldsymbol{X} = L \times_S M$ has a -1-shifted symplectic structure. So Theorem 3.2.2 imply [19, Cor. 5.21]:

Corollary 3.2.5. *Suppose (S,ω) is an algebraic symplectic manifold, and L, M are algebraic Lagrangian submanifolds in S. Then the intersection $X = L \cap M$, as a classical $\mathbb{K}$-subscheme of S, is Zariski locally modelled on the critical locus $\mathrm{Crit}(f)$ of a regular function $f : U \to \mathbb{A}^1$ on a smooth $\mathbb{K}$-scheme U.*

We remark that in real or complex symplectic geometry, it is easy to prove analogues of Corollary 3.2.5 using Darboux' Theorem or the Lagrangian Neighbourhood Theorem. However, these do not hold for algebraic symplectic manifolds, so it is not obvious how to prove Corollary 3.2.5 using classical techniques.

In [19, §5.7] we give an alternative point of view of Theorem 3.2.2 using a certain Hamiltonian construction from the mathematical physics literature [27, §4]. We view the differential d on A as being a *cohomological vector field* Q, which is to say a square-zero, degree 1 derivation of the algebra A. Using the symplectic form ω, one defines Hamiltonian vector fields together with a graded Poisson bracket, and shows that the cohomological vector field Q has a Hamiltonian function $H = k\Phi$ satisfying the classical master equation $\{H,H\} = 0$, and that the action of Q on an element $f \in A$ is given by the Poisson bracket $\{H, f\}$. Here is [19, Def. 5.22]:

Definition 3.2.6. Let (A, d) be a standard form cdga. Recall that a 2-form $\omega^0 : \mathbb{T}_A \to \mathbb{L}_A[k]$ is non-degenerate if it is a quasi-isomorphism. We say that a 2-form is *strictly non-degenerate* if it is an isomorphism between the underlying graded modules of $\mathbb{T}_A$ and $\mathbb{L}_A[k]$ obtained by forgetting the differentials. As in [19, Prop. 5.24], one can easily prove that given (A, d) a standard form cdga minimal at p with non-degenerate 2-form $\omega^0 : \mathbb{T}_A \to \mathbb{L}_A[k]$ of degree $k < 0$, then ω^0 is strictly non-degenerate in a neighbourhood of $p \in \mathrm{Spec}(A^0)$.

Definition 3.2.7. Given a homogeneous element $H \in A$, the associated *Hamiltonian vector field* X_H is uniquely defined by the requirement that $\iota_{X_H}\omega^0 = \mathrm{d}_{dR}H$. Given homogeneous $f, g \in A$, define the *Poisson bracket* $\{f,g\} = (-1)^{|f|-k-1}X_f(g)$. See also [19, Def. 5.29].

The following lemma singles out the condition to impose on a Hamiltonian H to get a useful cohomological vector field [19, Lem. 5.31]:

Lemma 3.2.8. *Let A be a graded commutative algebra with (strictly) non-degenerate 2-form ω^0 of degree $k < 0$ together with the induced Poisson bracket $\{\,,\,\}$ of degree $-k$. For a Hamiltonian $H \in A$ of degree $k+1$, the Hamiltonian vector field X_H gives a derivation of A of degree 1, and for each $f \in A$ we have the identity $X_H(f) = \{H, f\}$. The vector field X_H is square-zero ('cohomological') with non-trivial $H^0(A)$ if and only if H satisfies the* ***classical master equation:*** *$\{H,H\} = 0$. Defining a differential d on A by $\mathrm{d}f = X_H(f) = \{H, f\}$, we have that $\mathrm{d}\omega^0 = 0$, so that $\omega = (\omega^0, 0, 0, \ldots)$ gives a k-shifted symplectic form on the cdga (A, d).*

Next, [19, Def. 5.33 & Thm. 5.34]:

Definition 3.2.9. A standard form cdga (A, d) with k-shifted symplectic form $\omega = (\omega^0, 0, \dots)$ is said to be in *Hamiltonian form* if the 2-form ω^0 is strictly non-degenerate and the differential d on A is equal as a derivation to a Hamiltonian vector field X_H for some function $H \in A$ of degree $k+1$.

Theorem 3.2.10. *A derived $\mathbb{K}$-scheme X with symplectic form $\tilde{\omega}$ of degree $k < 0$ is Zariski locally of Hamiltonian form.*

3.3 −1-shifted symplectic derived schemes and d-critical loci

Here we review [19, §6]. The following is [19, Thm. 6.6]:

Theorem 3.3.1. *Suppose $(\boldsymbol{X}, \tilde{\omega})$ is a −1-shifted symplectic derived $\mathbb{K}$-scheme, and let $X = t_0(\boldsymbol{X})$ be the associated classical $\mathbb{K}$-scheme of $\boldsymbol{X}$. Then X extends uniquely to an algebraic d-critical locus (X, s), with the property that whenever $(\mathbf{Spec}\, A, \omega)$ is a −1-shifted symplectic derived $\mathbb{K}$-scheme in Darboux form with Hamiltonian $H \in A(0)$, as in [19, Ex.s 5.8 & 5.15] and explained in Definition 3.2.1, and $\boldsymbol{f} : \mathbf{Spec}\, A \to \boldsymbol{X}$ is an equivalence in $\mathbf{dSch}_{\mathbb{K}}$ with a Zariski open derived $\mathbb{K}$-subscheme $\boldsymbol{R} \subseteq \boldsymbol{X}$ with $\boldsymbol{f}^*(\tilde{\omega}) \sim \omega$, writing $U = \operatorname{Spec} A(0)$, $R = t_0(\boldsymbol{R})$, $f = t_0(\boldsymbol{f})$ so that $H : U \to \mathbb{A}^1$ is regular and $f : \operatorname{Crit}(H) \to R$ is an isomorphism, for $\operatorname{Crit}(H) \subseteq U$ the classical critical locus of H, then (R, U, H, f^{-1}) is a critical chart on (X, s).*

The canonical bundle $K_{X,s}$ from Theorem 2.1.6 is naturally isomorphic to the determinant line bundle $\det(\mathbb{L}_{\boldsymbol{X}})|_{X^{\mathrm{red}}}$ of the cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ of $\boldsymbol{X}$.

We can think of Theorem 3.3.1 as defining a *truncation functor*

$$\begin{aligned} F : \{&\text{category of −1-shifted symplectic derived } \mathbb{K}\text{-schemes } (\boldsymbol{X}, \omega)\} \\ &\longrightarrow \{\text{category of algebraic d-critical loci } (X, s) \text{ over } \mathbb{K}\}, \end{aligned} \tag{3.3.1}$$

where the morphisms $\boldsymbol{f} : (\boldsymbol{X}, \omega) \to (\boldsymbol{Y}, \omega')$ in the first line are (homotopy classes of) étale maps $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ with $\boldsymbol{f}^*(\omega') \sim \omega$, and the morphisms $f : (X, s) \to (Y, t)$ in the second line are étale maps $f : X \to Y$ with $f^*(t) = s$. In [87, Ex. 2.17] we give an example of −1-shifted symplectic derived schemes $(\boldsymbol{X}, \omega), (\boldsymbol{Y}, \omega')$, both global critical loci, such that $\boldsymbol{X}, \boldsymbol{Y}$ are not equivalent as derived $\mathbb{K}$-schemes, but their truncations $F(\boldsymbol{X}, \omega), F(\boldsymbol{Y}, \omega')$ are isomorphic as algebraic d-critical loci. Thus, the functor F in (3.3.1) is not full. We briefly recall proof of Theorem 3.3.1, which is in [19, §6.3].

Proof. Let $(\boldsymbol{X}, \tilde{\omega})$ be a −1-shifted symplectic derived $\mathbb{K}$-scheme, and $X = t_0(\boldsymbol{X})$. For the first part of Theorem 3.3.1, we must construct a section $s \in H^0(\mathcal{S}_X^0)$ such that (X, s) is a d-critical locus, and if $A, \omega, H, \boldsymbol{f}, \boldsymbol{R}, U, R, f$ are as in Theorem 3.3.1, then (R, U, H, f^{-1}) is a critical chart on (X, s). The condition that (R, U, H, f^{-1}) is a critical chart determines $s|_R$ uniquely.

Theorem 3.2.2(i) implies that for any $x \in \boldsymbol{X}$, we can find such $A, \omega, H, \boldsymbol{f}, \boldsymbol{R}, U, R, f$ with $x \in R \subseteq X$. So the condition in Theorem 3.3.1 determines $s|_R$ for Zariski open $R \subseteq X$ in an open cover of X. Thus $s \in H^0(\mathcal{S}_X^0)$ satisfying the conditions of the theorem is unique if it exists, and it exists if and only if the prescribed values $s|_R, s|_S$ agree on overlaps $R \cap S$ between open sets $R, S \subseteq X$. So suppose $A, \omega, H, \boldsymbol{f}, \boldsymbol{R}, U, R, f$ and $B, \check{\omega}, \check{H}, \boldsymbol{g}, \boldsymbol{S}, V, S, g$ are two choices above. Write s_R and s_S for the sections of $\mathcal{S}_X^0$ on $R, S \subseteq X$ determined by the critical charts (R, U, H, f^{-1}) and $(S, V, \check{H}, g^{-1})$, so that by the theory of d-critical loci in §2

$$\iota_{R,U}(s_R) = (f^{-1})^{-1}(H) + I_{R,U}^2, \qquad \iota_{S,V}(s_S) = (g^{-1})^{-1}(\check{H}) + I_{S,V}^2. \tag{3.3.2}$$

We must show that $s_R|_{R\cap S} = s_S|_{R\cap S}$. Let $x \in R\cap S$, so that $x = f(p) = g(q)$ for unique $p \in \operatorname{Crit} H \subseteq U$ and $q \in \operatorname{Crit}(\check H) \subseteq V$. Then using the method of Theorem 3.1.3 constructs a standard form cdga C minimal at $r \in \operatorname{Spec} H^0(C)$, and Zariski open inclusions $\alpha : A \to C$, $\beta : B \to C$ with $\boldsymbol{f} \circ \mathbf{Spec}\,\alpha \simeq \boldsymbol{g} \circ \mathbf{Spec}\,\beta$, such that the smooth $\mathbb{K}$-scheme $W = \operatorname{Spec} C(0)$ and $\mathbb{K}$-scheme morphisms $a = \operatorname{Spec}\alpha(0) : W \to U$, $b = \operatorname{Spec}\beta(0) : W \to V$ satisfy by [19, eq. 5.37]

$$a^*(H) - b^*(\check H) \in \bigl(\mathrm{d}C^{-1}\bigr)^2 = \bigl(a^*(\mathrm{d}_{dR}H)\bigr)^2 = \bigl(b^*(\mathrm{d}_{dR}\check H)\bigr)^2 \subset C(0). \tag{3.3.3}$$

Write $Z = \operatorname{Spec} H^0(C)$, regarded as a closed $\mathbb{K}$-subscheme of $W = \operatorname{Spec} C(0)$. Then $f \circ a|_Z = g\circ b|_Z : Z \to X$ is an isomorphism with a Zariski open $\mathbb{K}$-subscheme $T \subseteq R\cap S \subseteq X$ with $x \in T$. Define $s_T \in H^0\bigl(\mathcal{S}^0_X|_T\bigr)$ by

$$\iota_{T,W}(s_T) = \bigl((f\circ a|_Z)^{-1}\bigr)^{-1}(a^*(H)) + I^2_{T,W} = \bigl((g\circ b|_Z)^{-1}\bigr)^{-1}(b^*(\check H)) + I^2_{T,W}, \tag{3.3.4}$$

using the notation of §2.1 for the embedding $(f\circ a|_Z)^{-1} = (g\circ b|_Z)^{-1} : T \hookrightarrow W$ of T in the smooth $\mathbb{K}$-scheme W, where the two expressions on the right hand side of (3.3.4) are equal by (3.3.3), since $I_{T,W} = \bigl((f\circ a|_Z)^{-1}\bigr)^{-1}\bigl((a^*(\mathrm{d}_{dR}H))\bigr) = \bigl((g\circ b|_Z)^{-1}\bigr)^{-1}\bigl((b^*(\mathrm{d}_{dR}\check H))\bigr)$. We now have

$$\begin{aligned}\iota_{T,W}(s_R|_T) &= \bigl((f\circ a|_Z)^{-1}\bigr)^{-1}(a_\#)\circ \iota_{R,U}|_T(s_R|_T) = \bigl((f\circ a|_Z)^{-1}\bigr)^{-1}(a_\#)\circ \bigl((f^{-1})^{-1}(H) + I^2_{R,U}\bigr)\big|_T \\ &= \bigl((f\circ a|_Z)^{-1}\bigr)^{-1}(a^*(H)) + I^2_{T,W} = \iota_{T,W}(s_T),\end{aligned}$$

using (2.1.1) with $T, W, (f\circ a|_Z)^{-1}, R, U, f^{-1}, a$ in place of R, U, i, S, V, j, Φ in the first step, (3.3.2) in the second, and (3.3.4) in the fourth. Hence $s_R|_T = s_T$, as $\iota_{T,W}$ is injective in §2.1. Similarly $s_S|_T = s_T$, so $s_R|_T = s_S|_T$. As we can cover $R\cap S$ by such open $x \in T \subseteq R\cap S$, this implies that $s_R|_{R\cap S} = s_S|_{R\cap S}$, and the first part of Theorem 3.3.1 follows.

For the second part of the theorem, let $A, \omega, H, \boldsymbol{f}, \boldsymbol{R}, U, R, f$ be as in Theorem 3.3.1, so that (R, U, H, f^{-1}) is a critical chart on (X, s), and write $Y = \operatorname{Spec} H^0(A) \subseteq U$, so that $f : Y \to R$ is an isomorphism. Then (2.1.4) gives a natural isomorphism

$$\iota_{R,U,H,f^{-1}} : K_{X,s}|_{R^{\mathrm{red}}} \longrightarrow (f^{-1})^*\bigl(K_U^{\otimes^2}\bigr)|_{R^{\mathrm{red}}}. \tag{3.3.5}$$

Also $\mathbb{L}_{\boldsymbol{f}} : \boldsymbol{f}^*(\mathbb{L}_{\boldsymbol{X}}) \to \mathbb{L}_A \simeq \Omega^1_A$ is a quasi-isomorphism as $\boldsymbol{f}$ is a Zariski open inclusion. Hence $\det(\mathbb{L}_{\boldsymbol{f}})|_{Y^{\mathrm{red}}} : f^*(\det(\mathbb{L}_{\boldsymbol{X}})|_{R^{\mathrm{red}}}) \to \det(\Omega^1_A)|_{Y^{\mathrm{red}}}$ is an isomorphism, so pulling back by $f^{-1}|_{R^{\mathrm{red}}}$ gives an isomorphism

$$(f^{-1}|_{R^{\mathrm{red}}})^*\bigl(\det(\mathbb{L}_{\boldsymbol{f}})|_{Y^{\mathrm{red}}}\bigr) : \det(\mathbb{L}_{\boldsymbol{X}})|_{R^{\mathrm{red}}} \longrightarrow (f^{-1}|_{R^{\mathrm{red}}})^*\bigl(\det(\Omega^1_A)|_{Y^{\mathrm{red}}}\bigr). \tag{3.3.6}$$

Now by the theory of obstruction theories as in [5], we have a natural isomorphism

$$\Omega^1_A|_{Y^{\mathrm{red}}} \cong \bigl[TU|_{Y^{\mathrm{red}}} \xrightarrow{\partial^2 H|_{Y^{\mathrm{red}}}} T^*U|_{Y^{\mathrm{red}}}\bigr],$$

with $TU|_{Y^{\mathrm{red}}}$ in degree -1 and $T^*U|_{Y^{\mathrm{red}}}$ in degree 0. Thus we have a natural isomorphism

$$\det(\Omega^1_A)|_{Y^{\mathrm{red}}} \cong K_U^{\otimes^2}|_{Y^{\mathrm{red}}}. \tag{3.3.7}$$

Combining (8.3.1)–(3.3.7) gives a natural isomorphism

$$K_{X,s}|_{R^{\mathrm{red}}} \longrightarrow \det(\mathbb{L}_{\boldsymbol{X}})|_{R^{\mathrm{red}}}, \tag{3.3.8}$$

for each critical chart (R, U, H, f^{-1}) constructed from $A, \omega, H, \boldsymbol{f}, \boldsymbol{R}$ as above. Combining Theorem 3.1.3 on comparing the charts (R, U, H, f^{-1}) with Definition 4.3.2 defining the isomorphism J_Φ in (2.1.9), one can show that the canonical isomorphisms (3.3.8) on $R^{\mathrm{red}}, S^{\mathrm{red}}$ from two such charts (R, U, H, f^{-1}) and $(S, V, \check H, g^{-1})$ are equal on the overlap $(R\cap S)^{\mathrm{red}}$. Therefore the isomorphisms (3.3.8) glue to give a global canonical isomorphism $K_{X,s} \cong \det(\mathbb{L}_{\boldsymbol{X}})|_{X^{\mathrm{red}}}$. This completes the proof of Theorem 3.3.1. □

Suppose Y is a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves in $\mathrm{coh}(Y)$. Then Thomas [167] defined a natural *perfect obstruction theory* $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ on $\mathcal{M}$ in the sense of Behrend and Fantechi [6], and Behrend [5] showed that $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ can be made into a *symmetric obstruction theory*. Now in derived algebraic geometry $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli $\mathbb{K}$-scheme, and the obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ from [70, 167] is $\mathbb{L}_{t_0} : \mathbb{L}_{\boldsymbol{\mathcal{M}}}|_{\mathcal{M}} \to \mathbb{L}_{\mathcal{M}}$. As previously discussed, we can deduce from Pantev et al. [142, §2.1] that $\boldsymbol{\mathcal{M}}$ has a −1-shifted symplectic structure ω, and the symmetric structure on $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ from [5] is $\omega^0|_{\mathcal{M}}$. So as for Corollary 3.2.3, Theorem 3.3.1 implies [19, Cor. 6.7]:

Corollary 3.3.2. *Suppose Y is a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ is a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves in* $\mathrm{coh}(Y)$ *with perfect obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ as in Thomas* [167]*. Then $\mathcal{M}$ extends naturally to an algebraic d-critical locus $(\mathcal{M}, s)$. The canonical bundle $K_{\mathcal{M},s}$ from Theorem* 2.1.6 *is naturally isomorphic to* $\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\mathrm{red}}}$.

If (S, ω) is an algebraic symplectic manifold over $\mathbb{K}$ and $L, M \subseteq S$ are Lagrangians, then Pantev et al. [142, Th. 2.10] show that the derived intersection $\boldsymbol{X} = L \times_S M$ has a −1-shifted symplectic structure. If $X = t_0(\boldsymbol{X})$ then $\mathbb{L}_{\boldsymbol{X}}|_X \simeq [T^*S|_X \to T^*L|_X \oplus T^*M|_X]$ with $T^*S|_X$ in degree −1 and $T^*L|_X \oplus T^*M|_X$ in degree zero. Hence $\det(\mathbb{L}_{\boldsymbol{X}}|_X) \cong K_S|_X^{-1} \otimes K_L|_X \otimes K_M|_X \cong K_L|_X \otimes K_M|_X$, since $K_S \cong \mathcal{O}_S$. So as for Corollary 3.2.5, Theorem 3.3.1 implies [19, Cor. 6.8]:

Corollary 3.3.3. *Suppose (S, ω) is an algebraic symplectic manifold over $\mathbb{K}$, and L, M are algebraic Lagrangians in S. Then the intersection $X = L \cap M$, as a $\mathbb{K}$-subscheme of S, extends naturally to an algebraic d-critical locus (X, s). The canonical bundle $K_{X,s}$ from Theorem* 2.1.6 *is isomorphic to $K_L|_{X^{\mathrm{red}}} \otimes K_M|_{X^{\mathrm{red}}}$.*

The author [23, §3] proves a complex analytic analogue of Corollary 3.3.3, explained in §8. In §4 and §5, we will use these results explaining how these ideas turn out to be crucial in the program of the author and her collaborators in [18] and [25].

Chapter 4

Symmetries and stabilization for sheaves of vanishing cycles

This chapter is based on [18], in which we study sheaves of vanishing cycles introduced in §1.3. In general, we will not go into details and proofs of results, for which we refer to [18].

4.1 Action of symmetries on vanishing cycles

We recall [18, Def. 2.14], which basically introduces some notation for pullbacks of $\mathcal{PV}^{\bullet}_{V,g}$ by étale morphisms. We use notation from §1.3.

Definition 4.1.1. Let U, V be smooth $\mathbb{C}$-schemes, $\Phi : U \to V$ an étale morphism, and $g : V \to \mathbb{C}$ a regular function. Write $f = g \circ \Phi : U \to \mathbb{C}$, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Then $\Phi|_X : X \to Y$ is étale. Define an isomorphism

$$\mathcal{PV}_{\Phi} : \mathcal{PV}^{\bullet}_{U,f} \longrightarrow \Phi|_X^*\big(\mathcal{PV}^{\bullet}_{V,g}\big) \quad \text{in } \mathrm{Perv}(X) \tag{4.1.1}$$

by the commutative diagram for each $c \in f(X) \subseteq g(Y)$:

$$\begin{tikzcd} \mathcal{PV}^{\bullet}_{U,f}|_{X_c} = \phi^p_{f-c}(A_U[\dim U])|_{X_c} \arrow[r, "\alpha"'] \arrow[d, "\mathcal{PV}_{\Phi}|_{X_c}"] & \phi^p_{f-c} \circ \Phi^*(A_V[\dim V]))|_{X_c} \arrow[d, "\beta"'] \\ \Phi|_{X_c}^*\big(\mathcal{PV}^{\bullet}_{V,g}\big) \arrow[r, equal] & \Phi_0^* \circ \phi^p_{g-c} \circ (A_V[\dim V]))|_{X_c}. \end{tikzcd} \tag{4.1.2}$$

If $U = V$, $f = g$ and $\Phi = \mathrm{id}_U$ then $\mathcal{PV}_{\mathrm{id}_U} = \mathrm{id}_{\mathcal{PV}^{\bullet}_{U,f}}$, and the isomorphisms $\mathcal{PV}_{\Phi}$ are functorial.

We recall [18, Thm. 3.1, Cor. 3.2]. Proof of them can be found in [18, §3.1-3.2].

Theorem 4.1.2. *Let U, V be smooth $\mathbb{C}$-schemes, $\Phi, \Psi : U \to V$ étale morphisms, and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ regular functions with $g \circ \Phi = f = g \circ \Psi$. Write $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V, so that $\Phi|_X, \Psi|_X : X \to Y$ are étale morphisms. Suppose $\Phi|_X = \Psi|_X$. Then:*

(a) *As Φ, Ψ are étale, $\mathrm{d}\Phi : TU \to \Phi^*(TV)$, $\mathrm{d}\Psi : TU \to \Psi^*(TV)$ are isomorphisms of vector bundles. Restricting to the reduced $\mathbb{C}$-subscheme X^{red} of X, and using $\Phi|_{X^{\mathrm{red}}} = \Psi|_{X^{\mathrm{red}}}$ as $\Phi|_X = \Psi|_X$, gives isomorphisms $\mathrm{d}\Phi|_{X^{\mathrm{red}}}, \mathrm{d}\Psi|_{X^{\mathrm{red}}} : TU|_{X^{\mathrm{red}}} \longrightarrow \Phi|_{X^{\mathrm{red}}}^*(TV)$, and thus $\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}} : TU|_{X^{\mathrm{red}}} \longrightarrow TU|_{X^{\mathrm{red}}}$. Hence $\det\big(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) : X^{\mathrm{red}} \to \mathbb{C} \setminus \{0\}$ is a regular function. Then $\det\big(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\big)$ is a locally constant map $X^{\mathrm{red}} \to \{\pm 1\} \subset \mathbb{C} \setminus \{0\}$.*

(b) *Definition* 4.1.1 *defines isomorphisms* $\mathcal{PV}_\Phi, \mathcal{PV}_\Psi : \mathcal{PV}^\bullet_{U,f} \to \Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big)$ *in* $\mathrm{Perv}(X)$. *These are related by*

$$\mathcal{PV}_\Phi = \det\big(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) \cdot \mathcal{PV}_\Psi, \tag{4.1.3}$$

regarding $\det\big(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) : X \to \{\pm 1\}$ *as a locally constant map of topological spaces, where* X, X^{red} *have the same topological space.*

The analogues of these results also hold for $\mathscr{D}$*-modules and mixed Hodge modules on* $\mathbb{C}$*-schemes, for* l*-adic perverse sheaves and* $\mathscr{D}$*-modules on* $\mathbb{K}$*-schemes, and (with* Φ, Ψ *local biholomorphisms and* f, g *analytic functions) for perverse sheaves,* $\mathscr{D}$*-modules and mixed Hodge modules on complex analytic spaces.*

By taking $U = V$, $f = g$, Φ an isomorphism and $\Psi = \mathrm{id}_U$, we deduce a result on the action of symmetries on perverse sheaves of vanishing cycles:

Corollary 4.1.3. *Let* U *be a smooth* $\mathbb{C}$*-scheme,* $\Phi : U \to U$ *an isomorphism, and* $f : U \to \mathbb{C}$ *be regular with* $f \circ \Phi = f$. *Write* $X = \mathrm{Crit}(f)$ *as a* $\mathbb{C}$*-subscheme of* U *and* X^{red} *for its reduced* $\mathbb{C}$*-subscheme, and suppose* $\Phi|_X = \mathrm{id}_X$. *Then* $\det\big(\mathrm{d}\Phi|_{X^{\mathrm{red}}} : TU|_{X^{\mathrm{red}}} \to TU|_{X^{\mathrm{red}}}\big)$ *is a locally constant map* $X^{\mathrm{red}} \to \{\pm 1\}$, *and* $\mathcal{PV}_\Phi : \mathcal{PV}^\bullet_{U,f} \xrightarrow{\cong} \mathcal{PV}^\bullet_{U,f}$ *in* $\mathrm{Perv}(X)$ *from Definition* 4.1.1 *is multiplication by* $\det\big(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) = \pm 1$.

Here is a crucial example following [18, Ex.s 2.13, 2.15, 3.3].

Example 4.1.4. Define $f : \mathbb{C}^n \to \mathbb{C}$ by $f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$ for $n > 1$. Then $\mathrm{Crit}(f) = \{0\}$, so $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} = \phi_f^p(A_{\mathbb{C}^n}[n])|_{\{0\}}$ is a perverse sheaf on the point $\{0\}$. Following Dimca [34, Prop. 4.2.2, Ex. 4.2.3 & Ex. 4.2.6], we find that there is a canonical isomorphism

$$\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong H^{n-1}\big(MF_f(0); A\big) \otimes_A A_{\{0\}}, \tag{4.1.4}$$

where $MF_f(0)$ is the *Milnor fibre* of f at 0, as in [34, p. 103]. Since $f(z) = z_1^2 + \cdots + z_n^2$ is homogeneous, we see that

$$MF_f(0) \cong \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n : f(z_1, \ldots, z_n) = 1\big\} \cong T^*\mathcal{S}^{n-1},$$

so that $H^{n-1}\big(MF_f(0); A\big) \cong H^{n-1}\big(\mathcal{S}^{n-1}; A\big) \cong A$. Therefore we have

$$\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong A_{\{0\}}. \tag{4.1.5}$$

This isomorphism (4.1.5) is *natural up to sign* (unless the base ring A has characteristic 2, in which case (4.1.5) is natural), as it depends on the choice of isomorphism $H^{n-1}(\mathcal{S}^{n-1}, A) \cong A$, which corresponds to an orientation for $\mathcal{S}^{n-1}$. This uncertainty of signs will be important in §4.3.

We can also use Milnor fibres to compute the monodromy operator on $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2}$. There is a monodromy map $\mu_f : MF_f(0) \to MF_f(0)$, natural up to isotopy, which is the monodromy in the Milnor fibration of f at 0. Under the identification $MF_f(0) \cong T^*\mathcal{S}^{n-1}$ we may take μ_f to be the map $\mathrm{d}(-1) : T^*\mathcal{S}^{n-1} \to T^*\mathcal{S}^{n-1}$ induced by $-1 : \mathcal{S}^{n-1} \to \mathcal{S}^{n-1}$ mapping $-1 : (x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_n)$. This multiplies orientations on $\mathcal{S}^{n-1}$ by $(-1)^n$. Thus, $\mu_{f*} : H^{n-1}(\mathcal{S}^{n-1}, A) \to H^{n-1}(\mathcal{S}^{n-1}, A)$ multiplies by $(-1)^n$.

By [34, Prop. 4.2.2], equation (4.1.4) identifies the action of the monodromy operator $M_{\mathbb{C}^n,f}|_{\{0\}}$ on $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2}$ with the action of μ_{f*} on $H^{n-1}(\mathcal{S}^{n-1}, A)$. So $M_{\mathbb{C}^n,f}|_{\{0\}}$ is multiplication by $(-1)^n$. Combining this with the sign change $(-1)^{\dim U}$ in [18, §2.4] for $U = \mathbb{C}^n$ shows that the twisted monodromy is

$$\tau_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} = \mathrm{id} : \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \longrightarrow \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2}. \tag{4.1.6}$$

Equations (4.1.5)–(4.1.6) also hold for $n=0,1$, though (4.1.4) does not.

In Definition 4.1.1, set $U=V=\mathbb{C}^n$ and $f(z_1,\ldots,z_n)=g(z_1,\ldots,z_n)=z_1^2+\cdots+z_n^2$, so that $X=Y=\{0\}\subset\mathbb{C}^n$. Let $M\in\mathrm{O}(n,\mathbb{C})$ be an orthogonal matrix, so that $M:\mathbb{C}^n\to\mathbb{C}^n$ is an isomorphism with $f=g\circ M$ and $M|_{\{0\}}=\mathrm{id}_{\{0\}}$. As $M|_Y=\mathrm{id}_Y$, Definition 4.1.1 defines an isomorphism

$$\mathcal{PV}_M:\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}\longrightarrow\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}. \tag{4.1.7}$$

Equation (4.1.4) describes $\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}$ in terms of $MF_f(0)\cong T^*\mathcal{S}^{n-1}$. Now $M|_{MF_f(0)}:MF_f(0)\to MF_f(0)$ multiplies orientations on $\mathcal{S}^{n-1}$ by $\det M$, so

$$(M|_{MF_f(0)})_*:H^{n-1}\bigl(MF_f(0);A\bigr)\to H^{n-1}\bigl(MF_f(0);A\bigr)$$

is multiplication by $\det M$. Thus (4.1.4) implies that $\mathcal{PV}_M$ in (4.1.7) is multiplication by $\det M=\pm1$. Let $\Phi,\Psi\in\mathrm{O}(n,\mathbb{C})$ be orthogonal matrices, so that $\det\Phi,\det\Psi\in\{\pm1\}$ and $\Phi,\Psi:\mathbb{C}^n\to\mathbb{C}^n$ are isomorphisms with $f=g\circ\Phi=g\circ\Phi$ and $\Phi|_{\{0\}}=\Psi|_{\{0\}}=\mathrm{id}_{\{0\}}$. In Theorem 4.1.2(a) we have $\mathrm{d}\Psi|_{X^{\rm red}}^{-1}\circ\mathrm{d}\Phi|_{X^{\rm red}}=\Psi^{-1}\circ\Phi:\mathbb{C}^n\longrightarrow\mathbb{C}^n$, so that $\det\bigl(\mathrm{d}\Psi|_{X^{\rm red}}^{-1}\circ\mathrm{d}\Phi|_{X^{\rm red}}\bigr)=\det\Psi^{-1}\det\Phi=\pm1$. For Theorem 4.1.2(b), we get that $\mathcal{PV}_\Phi,\mathcal{PV}_\Psi:A_{\{0\}}\to A_{\{0\}}$ are multiplication by $\det\Phi,\det\Psi$, so $\mathcal{PV}_\Phi=(\det\Psi^{-1}\det\Phi)\cdot\mathcal{PV}_\Psi$, as in (4.1.3).

4.2 Dependence of $\mathcal{PV}^\bullet_{U,f}$ on f

Here we present an independent result obtained in [18], but not necessary for the rest of the paper. We will use the following notation from [18, §4].

Definition 4.2.1. Let U be a smooth $\mathbb{C}$-scheme, $f:U\to\mathbb{C}$ a regular function, and $X=\mathrm{Crit}(f)$ as a closed $\mathbb{C}$-subscheme of U. Write $I_X\subseteq\mathcal{O}_U$ for the sheaf of ideals of regular functions $U\to\mathbb{C}$ vanishing on X, so that $I_X=I_{\mathrm{d}f}$. For each $k=1,2,\ldots$, write $X^{(k)}$ for the $k^{\rm th}$ *order thickening of X in U*, that is, $X^{(k)}$ is the closed $\mathbb{C}$-subscheme of U defined by the vanishing of the sheaf of ideals I_X^k in $\mathcal{O}_U$. Also write $X^{\rm red}$ for the reduced $\mathbb{C}$-subscheme of U, and $X^{(\infty)}$ or $\hat U$ for the formal completion of U along X. Then we have a chain of inclusions of closed $\mathbb{C}$-subschemes of U

$$X^{\rm red}\subseteq X=X^{(1)}\subseteq X^{(2)}\subseteq X^{(3)}\subseteq\cdots\subseteq X^{(\infty)}=\hat U\subseteq U, \tag{4.2.1}$$

although technically $X^{(\infty)}=\hat U$ is not a scheme, but a formal scheme. Write $f^{(k)}:=f|_{X^{(k)}}:X^{(k)}\to\mathbb{C}$, and $f^{\rm red}:=f|_{X^{\rm red}}:X^{\rm red}\to\mathbb{C}$, and $f^{(\infty)}$ or $\hat f:=f|_{\hat U}:\hat U\to\mathbb{C}$, so that $f^{(k)},f^{\rm red}$ are regular functions on the $\mathbb{C}$-schemes $X^{(k)},X^{\rm red}$, and $f^{(\infty)}=\hat f$ a formal function on the formal $\mathbb{C}$-scheme $X^{(\infty)}=\hat U$. Note that $f^{\rm red}:X^{\rm red}\to\mathbb{C}$ is locally constant, since $X=\mathrm{Crit}(f)$. We also use the same notation for complex analytic spaces and $\mathbb{K}$-schemes.

Now we can ask: how much of the sequence (4.2.1) does $\mathcal{PV}^\bullet_{U,f}$ depend on? That is, is $\mathcal{PV}^\bullet_{U,f}$ (canonically?) determined by $(X^{\rm red},f^{\rm red})$, or by $(X^{(k)},f^{(k)})$ for some $k=1,2,\ldots$, or by $(\hat U,\hat f)$, as well as by (U,f)? The next theorem [18, Thm. 4.2] shows that $\mathcal{PV}^\bullet_{U,f}$ is determined up to canonical isomorphism by $(X^{(3)},f^{(3)})$, and hence a fortiori also by $(X^{(k)},f^{(k)})$ for $k>3$ and by $(\hat U,\hat f)$:

Theorem 4.2.2. *Let U,V be smooth $\mathbb{C}$-schemes, $f:U\to\mathbb{C}$, $g:V\to\mathbb{C}$ be regular functions, and $X=\mathrm{Crit}(f)$, $Y=\mathrm{Crit}(g)$ as closed $\mathbb{C}$-subschemes of U,V, so that we have perverse sheaves $\mathcal{PV}^\bullet_{U,f},\mathcal{PV}^\bullet_{V,g}$ on X,Y. Define $X^{(3)},f^{(3)}$ and $Y^{(3)},g^{(3)}$ as in Definition 4.2.1, and suppose $\Phi:X^{(3)}\to Y^{(3)}$ is an isomorphism with $g^{(3)}\circ\Phi=f^{(3)}$, so that $\Phi|_X:X\to Y\subseteq Y^{(3)}$ is an isomorphism. Then there is a canonical isomorphism in* $\mathrm{Perv}(X)$

$$\Omega_\Phi:\mathcal{PV}^\bullet_{U,f}\longrightarrow\Phi|_X^*(\mathcal{PV}^\bullet_{V,g}), \tag{4.2.2}$$

which is characterized by the property that if T is a smooth $\mathbb{C}$-scheme and $\pi_U : T \to U$, $\pi_V : T \to V$ are étale morphisms with $e := f \circ \pi_U = g \circ \pi_V : T \to \mathbb{C}$, so that $\pi_U|_Q : Q \to X$, $\pi_V|_Q : Q \to Y$ are étale for $Q := \mathrm{Crit}(e)$, and $\Phi \circ \pi_U|_{Q^{(2)}} = \pi_V|_{Q^{(2)}} : Q^{(2)} \to Y^{(2)}$, then

$$\pi_U|_Q^*(\Omega_\Phi) \circ \mathcal{PV}_{\pi_U} = \mathcal{PV}_{\pi_V} : \mathcal{PV}^\bullet_{T,e} \longrightarrow \pi_V|_Q^*(\mathcal{PV}^\bullet_{U,f}). \tag{4.2.3}$$

Also the following commute, where $\sigma_{U,f}, \sigma_{V,g}, \tau_{U,f}, \tau_{V,g}$ are as in §1.3.1:

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f} \arrow[rr, "\sigma_{U,f}"'] \arrow[d, "\Omega_\Phi"] & & \mathbb{D}_X(\mathcal{PV}^\bullet_{U,f}) \\
\Phi|_X^*(\mathcal{PV}^\bullet_{V,g}) \arrow[r, "\Phi|_X^*(\sigma_{V,g})"] & \Phi|_X^*\big(\mathbb{D}_Y(\mathcal{PV}^\bullet_{V,g})\big) \arrow[r, "\cong"] & \mathbb{D}_X\big(\Phi|_X^*(\mathcal{PV}^\bullet_{V,g})\big), \arrow[u, "\mathbb{D}_X(\Omega_\Phi)"']
\end{tikzcd} \tag{4.2.4}$$

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f} \arrow[r, "\tau_{U,f}"'] \arrow[d, "\Omega_\Phi"] & \mathcal{PV}^\bullet_{U,f} \arrow[d, "\Omega_\Phi"'] \\
\Phi|_X^*(\mathcal{PV}^\bullet_{V,g}) \arrow[r, "\Phi|_X^*(\tau_{V,g})"] & \Phi|_X^*(\mathcal{PV}^\bullet_{V,g}).
\end{tikzcd} \tag{4.2.5}$$

If there exists an étale morphism $\Xi : U \to V$ with $g \circ \Xi = f : U \to \mathbb{C}$ and $\Xi|_{X^{(3)}} = \Phi : X^{(3)} \to Y^{(3)}$ then $\Omega_\Phi = \mathcal{PV}_\Xi$, for $\mathcal{PV}_\Xi$ as in (4.1.1).

If W is another smooth $\mathbb{C}$-scheme, $h : W \to \mathbb{C}$ is regular, $Z = \mathrm{Crit}(h)$, and $\Psi : Y^{(3)} \to Z^{(3)}$ is an isomorphism with $h^{(3)} \circ \Psi = g^{(3)}$ then

$$\Omega_{\Psi\circ\Phi} = \Phi|_X^*(\Omega_\Psi) \circ \Omega_\Phi : \mathcal{PV}^\bullet_{U,f} \longrightarrow (\Psi \circ \Phi)|_X^*(\mathcal{PV}^\bullet_{W,h}). \tag{4.2.6}$$

If $U = V$, $f = g$, $X = Y$ and $\Phi = \mathrm{id}_{X^{(3)}}$ then $\Omega_{\mathrm{id}_{X^{(3)}}} = \mathrm{id}_{\mathcal{PV}^\bullet_{U,f}}$.

The analogues of all the above also hold with appropriate modifications for $\mathscr{D}$-modules on $\mathbb{C}$-schemes, for perverse sheaves and $\mathscr{D}$-modules on complex analytic spaces, for l-adic perverse sheaves and $\mathscr{D}$-modules on $\mathbb{K}$-schemes, and for mixed Hodge modules on $\mathbb{C}$-schemes and complex analytic spaces.

See [18, Rem. 4.5] for a discussion about it, and [18, §4.1-4.3] for the proof.

4.3 Stabilizing perverse sheaves of vanishing cycles

To set up notation for Theorem 4.3.4 below, we need the following theorem, which is proved in Joyce [87, Prop.s 2.22, 2.23 & 2.25]. There, it is stated in terms of critical charts as in §2, but here we just need a simplified version.

Theorem 4.3.1 (Joyce [87])**.** *Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Let $\Phi : U \hookrightarrow V$ be a closed embedding of $\mathbb{C}$-schemes with $f = g \circ \Phi : U \to \mathbb{C}$, and suppose $\Phi|_X : X \to V \supseteq Y$ is an isomorphism $\Phi|_X : X \to Y$. Then:*

(i) *For each $x \in X \subseteq U$ there exist smooth $\mathbb{C}$-schemes U', V', a point $x' \in U'$ and morphisms $\iota : U' \to U$, $\jmath : V' \to V$, $\Phi' : U' \to V'$, $\alpha : V' \to U$ and $\beta : V' \to \mathbb{C}^n$, where $n = \dim V - \dim U$, such that $\iota(x') = x$, and $\iota, \jmath$ and $\alpha \times \beta : V \to U \times \mathbb{C}^n$ are étale, and the following diagram commutes*

$$\begin{tikzcd}
U \arrow[d, "\Phi"] & U' \arrow[l, "\iota"'] \arrow[r, "\iota"] \arrow[d, "\Phi'"] & U \arrow[d, "\mathrm{id}_U \times 0"'] \\
V & V' \arrow[l, "\jmath"'] \arrow[r, "\alpha\times\beta"] & U \times \mathbb{C}^n,
\end{tikzcd} \tag{4.3.1}$$

and $g \circ \jmath = f \circ \alpha + (z_1^2 + \cdots + z_n^2) \circ \beta : V' \to \mathbb{C}$. Thus, setting $f' := f \circ \iota : U' \to \mathbb{C}$, $g' := g \circ \jmath : V' \to \mathbb{C}$, $X' := \mathrm{Crit}(f') \subseteq U'$, and $Y' := \mathrm{Crit}(g') \subseteq V'$, then $f' = g' \circ \Phi' : U' \to \mathbb{C}$, *and* $\Phi'|_{X'} : X' \to Y'$, $\iota|_{X'} : X' \to X$, $\jmath|_{Y'} : Y' \to Y$, $\alpha|_{Y'} : Y' \to X$ *are étale. We also require that* $\Phi \circ \alpha|_{Y'} = \jmath|_{Y'} : Y' \to Y$.

(ii) *Write* N_{UV} *for the normal bundle of* $\Phi(U)$ *in* V, *regarded as an algebraic vector bundle on* U *in the exact sequence of vector bundles on* U:

$$0 \longrightarrow TU \xrightarrow{\mathrm{d}\Phi} \Phi^*(TU) \xrightarrow{\Pi_{UV}} N_{UV} \longrightarrow 0. \tag{4.3.2}$$

Then there exists a unique $q_{UV} \in H^0(S^2 N_{UV}^*|_X)$ *which is a nondegenerate quadratic form on* $N_{UV}|_X$, *such that whenever* $U', V', \iota, \jmath, \Phi', \beta, n, X'$ *are as in* **(i)**, *writing* $\langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_{U'}$ *for the trivial vector bundle on* U' *with basis* $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$, *there is a natural isomorphism* $\hat\beta : \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_{U'} \to \iota^*(N_{UV}^*)$ *making the following diagram commute:*

$$\begin{tikzcd}
\iota^*(N_{UV}^*) \arrow[rr, "\iota^*(\Pi_{UV}^*)"'] & & \iota^* \circ \Phi^*(T^*V) = \Phi'^* \circ \jmath^*(T^*V) \arrow[d, "\Phi'^*(\mathrm{d}\jmath^*)"'] \\
\langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_{U'} = \Phi'^* \circ \beta^*(T_0^*\mathbb{C}^n) \arrow[u, dotted, "\hat\beta"'] \arrow[rr, "\Phi'^*(\mathrm{d}\beta^*)"] & & \Phi'^*(T^*V'),
\end{tikzcd} \tag{4.3.3}$$

$$\text{and} \qquad \iota|_{X'}^*(q_{UV}) = (S^2\hat\beta)|_{X'}(\mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n). \tag{4.3.4}$$

(iii) *Now suppose* W *is another smooth* $\mathbb{C}$*-scheme,* $h : W \to \mathbb{C}$ *is regular,* $Z = \mathrm{Crit}(h)$ *as a* $\mathbb{C}$*-subscheme of* W, *and* $\Psi : V \hookrightarrow W$ *is a closed embedding of* $\mathbb{C}$*-schemes with* $g = h \circ \Psi : V \to \mathbb{C}$ *and* $\Psi|_Y : Y \to Z$ *an isomorphism. Define* N_{VW}, q_{VW} *and* N_{UW}, q_{UW} *using* $\Psi : V \hookrightarrow W$ *and* $\Psi \circ \Phi : U \hookrightarrow W$ *as in* **(ii)** *above. Then there are unique morphisms* $\gamma_{UVW}, \delta_{UVW}$ *which make the following diagram of vector bundles on* U *commute, with straight lines exact:*

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & 0 \arrow[dr] & & TU \arrow[dl, "\mathrm{d}\Phi"'] \arrow[ddl, "\mathrm{d}(\Psi\circ\Phi)"] & 0 \arrow[l] \\
 & & & \Phi^*(TV) \arrow[dll, "\Pi_{UV}"'] \arrow[d, "\Phi^*(\mathrm{d}\Psi)"] & & \\
0 \arrow[r] & N_{UV} \arrow[dl] \arrow[dr, dotted, "\gamma_{UVW}"'] & & (\Psi\circ\Phi)^*(TW) \arrow[dl, "\Pi_{UW}"] \arrow[ddr, "\Phi^*(\Pi_{VW})"] & & \\
0 & & N_{UW} \arrow[dl] \arrow[drr, dotted, "\delta_{UVW}"'] & & & \\
 & 0 & & & \Phi^*(N_{VW}) \arrow[r] \arrow[d] & 0 \\
 & & & & 0. &
\end{tikzcd} \tag{4.3.5}$$

Restricting to X *gives an exact sequence of vector bundles:*

$$0 \longrightarrow N_{UV}|_X \xrightarrow{\gamma_{UVW}|_X} N_{UW}|_X \xrightarrow{\delta_{UVW}|_X} \Phi|_X^*(N_{VW}) \longrightarrow 0. \tag{4.3.6}$$

Then there is a natural isomorphism of vector bundles on X

$$N_{UW}|_X \cong N_{UV}|_X \oplus \Phi|_X^*(N_{VW}), \tag{4.3.7}$$

compatible with the exact sequence (4.3.6), *which identifies*

$$\begin{aligned} q_{UW} &\cong q_{UV} \oplus \Phi|_X^*(q_{VW}) \oplus 0 \qquad \textit{under the splitting} \\ S^2 N_{UW}|_X^* &\cong S^2 N_{UV}|_X^* \oplus \Phi|_X^*\bigl(S^2 N_{VW}^*|_Y\bigr) \oplus \bigl(N_{UV}^*|_X \otimes \Phi|_X^*(N_{VW}^*)\bigr). \end{aligned} \tag{4.3.8}$$

(iv) *Parts* **(i)–(iii)** *also hold for* $\mathbb{K}$*-schemes over an algebraically closed field* $\mathbb{K}$ *with* $\mathrm{char}\,\mathbb{K} \neq 2$, *rather than* $\mathbb{C}$*-schemes.*

(v) *Analogues of* **(i)–(iii)** *hold for complex analytic spaces, replacing the smooth $\mathbb{C}$-schemes U, V, W by complex manifolds, the regular functions f, g, h by holomorphic functions, the $\mathbb{C}$-schemes X, Y, Z by complex analytic spaces, the étale open sets $\iota : U' \to U$, $\jmath : V' \to V$ by complex analytic open sets $U' \subseteq U$, $V' \subseteq V$, and with $\alpha \times \beta : V' \to U \times \mathbb{C}^n$ a biholomorphism with a complex analytic open neighbourhood of $(x, 0)$ in $U \times \mathbb{C}^n$.*

Following [87, Def.s 2.26 & 2.34], we define:

Definition 4.3.2. Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Suppose $\Phi : U \hookrightarrow V$ is a closed embedding of $\mathbb{C}$-schemes with $f = g \circ \Phi : U \to \mathbb{C}$ and $\Phi|_X : X \to Y$ an isomorphism. Then Theorem 4.3.1(ii) defines the normal bundle N_{UV} of U in V, a vector bundle on U of rank $n = \dim V - \dim U$, and a nondegenerate quadratic form $q_{UV} \in H^0(S^2 N_{UV}^*|_X)$. Taking top exterior powers in the dual of (4.3.2) gives an isomorphism of line bundles on U

$$\rho_{UV} : K_U \otimes \Lambda^n N_{UV}^* \xrightarrow{\cong} \Phi^*(K_V),$$

where K_U, K_V are the canonical bundles of U, V. Write X^{red} for the reduced $\mathbb{C}$-subscheme of X. As q_{UV} is a nondegenerate quadratic form on $N_{VW}|_X$, its determinant $\det(q_{VW})$ is a nonzero section of $(\Lambda^n N_{VW}^*)|_X^{\otimes^2}$. Define an isomorphism of line bundles on X^{red}:

$$J_\Phi = \rho_{UV}^{\otimes^2} \circ \bigl(\mathrm{id}_{K_U^2|_{X^{\mathrm{red}}}} \otimes \det(q_{UV})|_{X^{\mathrm{red}}}\bigr) : K_U^{\otimes^2}\big|_{X^{\mathrm{red}}} \xrightarrow{\cong} \Phi|_{X^{\mathrm{red}}}^*\bigl(K_V^{\otimes^2}\bigr). \tag{4.3.9}$$

Since principal $\mathbb{Z}/2\mathbb{Z}$-bundles $\pi : P \to X$ in the sense of Definition 1.3.7 are an (étale or complex analytic) topological notion, and X^{red} and X have the same topological space (even in the étale or complex analytic topology), principal $\mathbb{Z}/2\mathbb{Z}$-bundles on X^{red} and on X are equivalent. Define $\pi_\Phi : P_\Phi \to X$ to be the principal $\mathbb{Z}/2\mathbb{Z}$-bundle which parametrizes square roots of J_Φ on X^{red}. That is, (étale or complex analytic) local sections $s_\alpha : X \to P_\Phi$ of P_Φ correspond to local isomorphisms $\alpha : K_U|_{X^{\mathrm{red}}} \to \Phi|_{X^{\mathrm{red}}}^*(K_V)$ on X^{red} with $\alpha \otimes \alpha = J_\Phi$.

Now suppose W is another smooth $\mathbb{C}$-scheme, $h : W \to \mathbb{C}$ is regular, $Z = \mathrm{Crit}(h)$ as a $\mathbb{C}$-subscheme of W, and $\Psi : V \hookrightarrow W$ is a closed embedding of $\mathbb{C}$-schemes with $g = h \circ \Psi : V \to \mathbb{C}$ and $\Psi|_Y : Y \to Z$ an isomorphism. Then Theorem 4.3.1(iii) applies, and from (4.3.7)–(4.3.8) we can deduce that

$$J_{\Psi\circ\Phi} = \Phi|_{X^{\mathrm{red}}}^*(J_\Psi) \circ J_\Phi : K_U^{\otimes^2}\big|_{X^{\mathrm{red}}} \xrightarrow{\cong} (\Psi \circ \Phi)|_{X^{\mathrm{red}}}^*\bigl(K_W^{\otimes^2}\bigr) = \Phi|_{X^{\mathrm{red}}}^*\bigl[\Psi|_{Y^{\mathrm{red}}}^*\bigl(K_W^{\otimes^2}\bigr)\bigr]. \tag{4.3.10}$$

For the principal $\mathbb{Z}/2\mathbb{Z}$-bundles $\pi_\Phi : P_\Phi \to X$, $\pi_\Psi : P_\Psi \to Y$, $\pi_{\Psi\circ\Phi} : P_{\Psi\circ\Phi} \to X$, equation (4.3.10) implies that there is a canonical isomorphism

$$\Xi_{\Psi,\Phi} : P_{\Psi\circ\Phi} \xrightarrow{\cong} \Phi|_X^*(P_\Psi) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi. \tag{4.3.11}$$

It is also easy to see that these $\Xi_{\Psi,\Phi}$ have an associativity property under triple compositions, that is, given another smooth $\mathbb{C}$-scheme T, regular $e : T \to \mathbb{C}$ with $Q := \mathrm{Crit}(e)$, and $\Upsilon : T \hookrightarrow U$ a closed embedding with $e = f \circ \Upsilon : T \to \mathbb{C}$ and $\Upsilon|_Q : Q \to X$ an isomorphism, then

$$\begin{aligned}\bigl(\mathrm{id}_{(\Phi\circ\Upsilon)|_Q^*(P_\Psi)} \otimes \Xi_{\Phi,\Upsilon}\bigr) \circ \Xi_{\Psi,\Phi\circ\Upsilon} = \bigl(\Upsilon|_Q^*(\Xi_{\Psi,\Phi}) \otimes \mathrm{id}_{P_\Upsilon}\bigr) \circ \Xi_{\Psi\circ\Phi,\Upsilon} :\\ P_{\Psi\circ\Phi\circ\Upsilon} \longrightarrow (\Phi \circ \Upsilon)|_Q^*(P_\Psi) \otimes_{\mathbb{Z}/2\mathbb{Z}} \Upsilon|_Q^*(P_\Phi) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Upsilon.\end{aligned} \tag{4.3.12}$$

Analogues of all the above also work for $\mathbb{K}$-schemes over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}\,\mathbb{K} \neq 2$, as in Theorem 4.3.1(iv), and for complex manifolds and complex analytic spaces, as in Theorem 4.3.1(v).

The reason for restricting to $X^{\rm red}$ above is the following [87, Prop. 2.27], whose proof uses the fact that $X^{\rm red}$ is reduced in an essential way.

Lemma 4.3.3. *In Definition* 4.3.2, *the isomorphism* J_Φ *in* (4.3.9) *and the principal* $\mathbb{Z}/2\mathbb{Z}$*-bundle* $\pi_\Phi:P_\Phi\to X$ *depend only on* U,V,X,Y,f,g *and* $\Phi|_X:X\to Y$*. That is, they do not depend on* $\Phi:U\to V$ *apart from* $\Phi|_X:X\to Y$*.*

Using the notation of Definition 4.3.2, we can state [18, Thm. 5.4].

Theorem 4.3.4. **(a)** *Let* U,V *be smooth* $\mathbb{C}$*-schemes,* $f:U\to\mathbb{C}$, $g:V\to\mathbb{C}$ *be regular, and* $X=\mathrm{Crit}(f)$, $Y=\mathrm{Crit}(g)$ *as* $\mathbb{C}$*-subschemes of* U,V*. Let* $\Phi:U\hookrightarrow V$ *be a closed embedding of* $\mathbb{C}$*-schemes with* $f=g\circ\Phi:U\to\mathbb{C}$, *and suppose* $\Phi|_X:X\to V\supseteq Y$ *is an isomorphism* $\Phi|_X:X\to Y$*. Then there is a natural isomorphism of perverse sheaves on* X*:*

$$\Theta_\Phi:\mathcal{PV}^\bullet_{U,f}\longrightarrow\Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi,\tag{4.3.13}$$

where $\mathcal{PV}^\bullet_{U,f},\mathcal{PV}^\bullet_{V,g}$ *are the perverse sheaves of vanishing cycles from* §1.3.1, *and* P_Φ *the principal* $\mathbb{Z}/2\mathbb{Z}$*-bundle from Definition* 4.3.2, *and if* $\mathcal{Q}^\bullet$ *is a perverse sheaf on* X *then* $\mathcal{Q}^\bullet\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi$ *is as in Definition* 1.3.7*. Also the following diagrams commute, where* $\sigma_{U,f},\sigma_{V,g},\tau_{U,f},\tau_{V,g}$ *are as in* §1.3.1

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f} \arrow[r, "\Theta_\Phi"'] \arrow[d, "\sigma_{U,f}"] & \Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi \arrow[r, "\Phi|_X^*(\sigma_{V,g})\otimes\mathrm{id}"'] & \Phi|_X^*\big(\mathbb{D}_Y(\mathcal{PV}^\bullet_{V,g})\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi \arrow[d, "\cong"'] \\
\mathbb{D}_X(\mathcal{PV}^\bullet_{U,f}) & & \mathbb{D}_X\big(\Phi|_X^*(\mathcal{PV}^\bullet_{V,g})\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi\big), \arrow[ll, "\mathbb{D}_X(\Theta_\Phi)"']
\end{tikzcd}\tag{4.3.14}$$

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f} \arrow[r, "\Theta_\Phi"'] \arrow[d, "\tau_{U,f}"] & \Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi \arrow[d, "\Phi|_X^*(\tau_{V,g})\otimes\mathrm{id}"'] \\
\mathcal{PV}^\bullet_{U,f} \arrow[r, "\Theta_\Phi"] & \Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi.
\end{tikzcd}\tag{4.3.15}$$

If $U=V$, $f=g$, $\Phi=\mathrm{id}_U$ *then* $\pi_\Phi:P_\Phi\to X$ *is trivial, and* Θ_Φ *corresponds to* $\mathrm{id}_{\mathcal{PV}^\bullet_{U,f}}$ *under the natural isomorphism* $\mathrm{id}_X^*(\mathcal{PV}^\bullet_{U,f})\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi\cong\mathcal{PV}^\bullet_{U,f}$*.*
(b) *The isomorphism* Θ_Φ *in* (4.3.13) *depends only on* U,V,X,Y,f,g *and* $\Phi|_X:X\to Y$*. That is, if* $\tilde\Phi:U\to V$ *is an alternative choice for* Φ *with* $\Phi|_X=\tilde\Phi|_X:X\to Y$, *then* $\Theta_\Phi=\Theta_{\tilde\Phi}$, *noting that* $P_\Phi=P_{\tilde\Phi}$ *by Lemma* 4.3.3*.*

(c) *Now suppose* W *is another smooth* $\mathbb{C}$*-scheme,* $h:W\to\mathbb{C}$ *is regular,* $Z=\mathrm{Crit}(h)$, *and* $\Psi:V\hookrightarrow W$ *is a closed embedding with* $g=h\circ\Psi:V\to\mathbb{C}$ *and* $\Psi|_Y:Y\to Z$ *an isomorphism. Then Definition* 4.3.2 *defines principal* $\mathbb{Z}/2\mathbb{Z}$*-bundles* $\pi_\Phi:P_\Phi\to X$, $\pi_\Psi:P_\Psi\to Y$, $\pi_{\Psi\circ\Phi}:P_{\Psi\circ\Phi}\to X$ *and an isomorphism* $\Xi_{\Psi,\Phi}$ *in* (4.3.11), *and part* **(a)** *defines isomorphisms of perverse sheaves* $\Theta_\Phi,\Theta_{\Psi\circ\Phi}$ *on* X *and* Θ_Ψ *on* Y*. Then the following commutes in* $\mathrm{Perv}(X)$*:*

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f} \arrow[rr, "\Theta_{\Psi\circ\Phi}"'] \arrow[d, "\Theta_\Phi"] & & (\Psi\circ\Phi)|_X^*\big(\mathcal{PV}^\bullet_{W,h}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_{\Psi\circ\Phi} \arrow[d, "\mathrm{id}\otimes\Xi_{\Psi,\Phi}"'] \\
\Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi \arrow[r, "\Phi|_X^*(\Theta_\Psi)\otimes\mathrm{id}"] & \Phi|_X^*\circ\Psi|_Y^*\big(\mathcal{PV}^\bullet_{W,h}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}\Phi|_X^*(P_\Psi)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi. &
\end{tikzcd}\tag{4.3.16}$$

(d) *The analogues of* **(a)**–**(c)** *also hold for* $\mathscr{D}$*-modules on* $\mathbb{C}$*-schemes, for perverse sheaves and* $\mathscr{D}$*-modules on complex analytic spaces, for* l*-adic perverse sheaves and* $\mathscr{D}$*-modules on* $\mathbb{K}$*-schemes, and for mixed Hodge modules on* $\mathbb{C}$*-schemes and complex analytic spaces.*

Theorem 4.3.4 has a long proof in [18, §5], which uses crucially material in §4.1 on the action of symmetries on vanishing cycles.

4.4 Perverse sheaves on oriented d-critical loci

We state [18, Thm. 6.9]. It is proved in [18, §6.3]. We decided to do not repeat the proof here, as we use a similar technique in §8.

Theorem 4.4.1. *Let (X, s) be an oriented algebraic d-critical locus over $\mathbb{C}$, with orientation $K_{X,s}^{1/2}$. Then for any well-behaved base ring A, such as $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$, there exists a perverse sheaf $P_{X,s}^{\bullet}$ in* $\mathrm{Perv}(X)$ *over A, which is natural up to canonical isomorphism, and Verdier duality and monodromy isomorphisms*

$$\Sigma_{X,s} : P_{X,s}^{\bullet} \longrightarrow \mathbb{D}_X\big(P_{X,s}^{\bullet}\big), \qquad \mathrm{T}_{X,s} : P_{X,s}^{\bullet} \longrightarrow P_{X,s}^{\bullet}, \tag{4.4.1}$$

which are characterized by the following properties:

(i) *If (R, U, f, i) is a critical chart on (X, s), there is a natural isomorphism*

$$\omega_{R,U,f,i} : P_{X,s}^{\bullet}|_R \longrightarrow i^*\big(\mathcal{PV}_{U,f}^{\bullet}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}, \tag{4.4.2}$$

where $\pi_{R,U,f,i} : Q_{R,U,f,i} \to R$ is the principal $\mathbb{Z}/2\mathbb{Z}$-bundle parametrizing local isomorphisms $\alpha : K_{X,s}^{1/2} \to i^*(K_U)|_{R^{\mathrm{red}}}$ with $\alpha \otimes \alpha = \iota_{R,U,f,i}$, for $\iota_{R,U,f,i}$ as in (2.1.4)*. Furthermore the following commute in* $\mathrm{Perv}(R)$*:*

$$\begin{tikzcd}
P_{X,s}^{\bullet}|_R \arrow[r, "\omega_{R,U,f,i}"'] \arrow[d, "\Sigma_{X,s}|_R"] & i^*\big(\mathcal{PV}_{U,f}^{\bullet}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \arrow[d, "i^*(\sigma_{U,f})\otimes \mathrm{id}_{Q_{R,U,f,i}}"'] \\
\mathbb{D}_R\big(P_{X,s}^{\bullet}|_R\big) & i^*\big(\mathbb{D}_{\mathrm{Crit}(f)}(\mathcal{PV}_{U,f}^{\bullet})\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \cong \mathbb{D}_R\big(i^*(\mathcal{PV}_{U,f}^{\bullet}) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}\big), \arrow[l, "\mathbb{D}_R(\omega_{R,U,f,i})"']
\end{tikzcd} \tag{4.4.3}$$

$$\begin{tikzcd}
P_{X,s}^{\bullet}|_R \arrow[r, "\omega_{R,U,f,i}"'] \arrow[d, "\mathrm{T}_{X,s}|_R"] & i^*\big(\mathcal{PV}_{U,f}^{\bullet}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \arrow[d, "i^*(\tau_{U,f})\otimes \mathrm{id}_{Q_{R,U,f,i}}"'] \\
P_{X,s}^{\bullet}|_R \arrow[r, "\omega_{R,U,f,i}"] & i^*\big(\mathcal{PV}_{U,f}^{\bullet}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}.
\end{tikzcd} \tag{4.4.4}$$

(ii) *Let $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on (X, s). Then there is a natural isomorphism of principal $\mathbb{Z}/2\mathbb{Z}$-bundles*

$$\Lambda_\Phi : Q_{S,V,g,j}|_R \xrightarrow{\cong} i^*(P_\Phi) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \tag{4.4.5}$$

on R, for P_Φ as in Definition 4.3.2*, defined as follows: local isomorphisms*

$$\alpha : K_{X,s}^{1/2}|_{R^{\mathrm{red}}} \longrightarrow i^*(K_U)|_{R^{\mathrm{red}}},\ \beta : K_{X,s}^{1/2}|_{R^{\mathrm{red}}} \longrightarrow j^*(K_V)|_{R^{\mathrm{red}}},\ \gamma : i^*(K_U)|_{R^{\mathrm{red}}} \longrightarrow j^*(K_V)|_{R^{\mathrm{red}}}$$

with $\alpha \otimes \alpha = \iota_{R,U,f,i}$, $\beta \otimes \beta = \iota_{S,V,g,j}|_{R^{\mathrm{red}}}$, $\gamma \otimes \gamma = i|_{R^{\mathrm{red}}}^*(J_\Phi)$ correspond to local sections $s_\alpha : R \to Q_{R,U,f,i}$, $s_\beta : R \to Q_{S,V,g,j}|_R$, $s_\gamma : R \to i^*(P_\Phi)$. Equation (2.1.9) *shows that $\beta = \gamma \circ \alpha$ is a possible solution for β, and we define Λ_Φ in* (8.2.7) *such that $\Lambda_\Phi(s_\beta) = s_\gamma \otimes_{\mathbb{Z}/2\mathbb{Z}} s_\alpha$ if and only if $\beta = \gamma \circ \alpha$. Then the following diagram commutes in* $\mathrm{Perv}(R)$*, for Θ_Φ as in* (4.3.13)*:*

$$\begin{tikzcd}
P_{X,s}^{\bullet}|_R \arrow[r, "\omega_{R,U,f,i}"'] \arrow[d, "\omega_{S,V,g,j}|_R"] & i^*\big(\mathcal{PV}_{U,f}^{\bullet}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \arrow[d, "i^*(\Theta_\Phi)\otimes \mathrm{id}_{Q_{R,U,f,i}}"'] \\
j^*\big(\mathcal{PV}_{V,g}^{\bullet}\big)|_R \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{S,V,g,j}|_R \arrow[r, "\mathrm{id}_{j^*(\mathcal{PV}_{V,g}^{\bullet})}\otimes\Lambda_\Phi"] & i^*\big(\Phi^*(\mathcal{PV}_{V,g}^{\bullet}) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}.
\end{tikzcd} \tag{4.4.6}$$

The analogues of all the above also hold for $\mathscr{D}$-modules on oriented algebraic d-critical loci over $\mathbb{C}$, for perverse sheaves and $\mathscr{D}$-modules on oriented complex analytic d-critical loci, for l-adic perverse sheaves and $\mathscr{D}$-modules on oriented algebraic d-critical loci over $\mathbb{K}$, and for mixed Hodge modules on oriented algebraic d-critical loci over $\mathbb{C}$ and oriented complex analytic d-critical loci.

From Theorem 3.3.1 and Corollaries 3.3.2 and 3.3.3 we deduce [18, Cor.s 6.10, 6.11 & 6.12]:

Corollary 4.4.2. *Let $(\boldsymbol{X},\omega)$ be a -1-shifted symplectic derived scheme over $\mathbb{C}$ in the sense of Pantev et al.* [142], *and $X = t_0(\boldsymbol{X})$ the associated classical $\mathbb{C}$-scheme. Suppose we are given a square root $\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}$ for $\det(\mathbb{L}_{\boldsymbol{X}})|_X$. Then we may define $P^\bullet_{\boldsymbol{X},\omega} \in \mathrm{Perv}(X)$, uniquely up to canonical isomorphism, and isomorphisms $\Sigma_{\boldsymbol{X},\omega} : P^\bullet_{\boldsymbol{X},\omega} \to \mathbb{D}_X(P^\bullet_{\boldsymbol{X},\omega})$, $\mathrm{T}_{\boldsymbol{X},\omega} : P^\bullet_{\boldsymbol{X},\omega} \to P^\bullet_{\boldsymbol{X},\omega}$. The same applies for $\mathscr{D}$-modules and mixed Hodge modules on X, and for l-adic perverse sheaves and $\mathscr{D}$-modules on X if $\boldsymbol{X}$ is over $\mathbb{K}$ with $\mathrm{char}\,\mathbb{K} = 0$.*

Corollary 4.4.3. *Let Y be a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves in $\mathrm{coh}(Y)$ with natural (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ as in Behrend* [5], *Thomas* [167]. *Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$. Then we may define $P^\bullet_{\mathcal{M}} \in \mathrm{Perv}(\mathcal{M})$, uniquely up to canonical isomorphism, and isomorphisms $\Sigma_{\mathcal{M}} : P^\bullet_{\mathcal{M}} \to \mathbb{D}_{\mathcal{M}}(P^\bullet_{\mathcal{M}})$, $\mathrm{T}_{\mathcal{M}} : P^\bullet_{\mathcal{M}} \to P^\bullet_{\mathcal{M}}$. The same applies for $\mathscr{D}$-modules and mixed Hodge modules on $\mathcal{M}$, and for l-adic perverse sheaves and $\mathscr{D}$-modules on $\mathcal{M}$ if $Y, \mathcal{M}$ are over $\mathbb{K}$ with $\mathrm{char}\,\mathbb{K} = 0$.*

Corollary 4.4.4. *Let (S,ω) be a complex symplectic manifold and L, M complex Lagrangian submanifolds in S, and write $X = L \cap M$, as a complex analytic subspace of S. Suppose we are given square roots $K_L^{1/2}, K_M^{1/2}$ for K_L, K_M. Then we may define $P^\bullet_{L,M} \in \mathrm{Perv}(X)$, uniquely up to canonical isomorphism, and isomorphisms $\Sigma_{L,M} : P^\bullet_{L,M} \to \mathbb{D}_X(P^\bullet_{L,M})$, $\mathrm{T}_{L,M} : P^\bullet_{L,M} \to P^\bullet_{L,M}$. The same applies for $\mathscr{D}$-modules and mixed Hodge modules on X.*

We will go back in details to Donaldson–Thomas theory and Lagrangian intersections respectively in §7 and §8.

Chapter 5

On motivic vanishing cycles of critical loci

This chapter is based on [25], in which we study motivic vanishing cycles introduced in §1.4. We keep the same notation as in §1.4. In general, we will not go into details and proofs of results, for which we refer to [25].

5.1 Dependence of $MF_{U,f}^{\mathrm{mot},\phi}$ on f

As done in §4.2, we can ask: how much of the sequence $X^{\mathrm{red}} \subseteq X = X^{(1)} \subseteq X^{(2)} \subseteq X^{(3)} \subseteq \cdots \subseteq X^{(\infty)} = \hat{U} \subseteq U$ of closed $\mathbb{K}$-subschemes of U, does $MF_{U,f}^{\mathrm{mot},\phi}$ depend on? That is, is $MF_{U,f}^{\mathrm{mot},\phi}$ determined by $(X^{\mathrm{red}}, f^{\mathrm{red}})$, or by $(X^{(k)}, f^{(k)})$ for some $k = 1, 2, \ldots$, or by $(\hat{U}, \hat{f})$, as well as by (U, f)? In [25, Thm. 3.1] we show that $MF_{U,f}^{\mathrm{mot},\phi}$ is determined by $(X^{(3)}, f^{(3)})$, and hence a fortiori also by $(X^{(k)}, f^{(k)})$ for $k > 3$ and by $(\hat{U}, \hat{f})$. Again, this is not necessary for the sequel, but we include it for completeness and because we think it can be useful for other applications.

Theorem 5.1.1. *Let U, V be smooth $\mathbb{K}$-schemes, $f : U \to \mathbb{A}^1$, $g : V \to \mathbb{A}^1$ be regular functions, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as closed $\mathbb{K}$-subschemes of U, V, so that the motivic vanishing cycles $MF_{U,f}^{\mathrm{mot},\phi}$, $MF_{V,g}^{\mathrm{mot},\phi}$ are defined on X, Y. Define $X^{(3)}, f^{(3)}$ and $Y^{(3)}, g^{(3)}$ as in Definition* 4.2.1, *and suppose $\Phi : X^{(3)} \to Y^{(3)}$ is an isomorphism with $g^{(3)} \circ \Phi = f^{(3)}$, so that $\Phi|_X : X \to Y \subseteq Y^{(3)}$ is an isomorphism. Then*

$$MF_{U,f}^{\mathrm{mot},\phi} = \Phi|_X^*\big(MF_{V,g}^{\mathrm{mot},\phi}\big) \quad \text{in } \mathcal{M}_X^{\hat{\mu}} \text{ and } \overline{\mathcal{M}}_X^{\hat{\mu}}. \tag{5.1.1}$$

Remark 5.1.2. We can define motivic vanishing cycles $MF_{\hat{U},\hat{f}}^{\mathrm{mot},\phi}$ for a class of formal functions $\hat{f}$ on formal schemes $\hat{U}$ using Theorem 5.1.1. Let U be a smooth $\mathbb{K}$-scheme, $X \subseteq U$ a closed $\mathbb{K}$-subscheme, and $\hat{U}$ the formal completion of U along X. Suppose $f : \hat{U} \to \mathbb{A}^1$ is a formal function with $\mathrm{Crit}(f) = X \subseteq \hat{U}$. Then there is a unique $MF_{\hat{U},\hat{f}}^{\mathrm{mot},\phi}$ in $\mathcal{M}_X^{\hat{\mu}}$ or $\overline{\mathcal{M}}_X^{\hat{\mu}}$ with the property that if $U' \subseteq U$ is Zariski open with $X' = X \cap U'$ and $g : U' \to \mathbb{A}^1$ is regular with $g + I_{X'}^3 = \hat{f}|_{\hat{U}'} + I_X^3$ in $H^0(\mathcal{O}_{U'}/I_{X'}^3)$ then $MF_{\hat{U},\hat{f}}^{\mathrm{mot},\phi}|_{X'} = MF_{U',g}^{\mathrm{mot},\phi}|_{X'}$. Theorem 5.1.1 shows that $MF_{U',g}^{\mathrm{mot},\phi}|_{X'}$ is independent of the choice of g with $g + I_{X'}^3 = \hat{f}|_{\hat{U}'} + I_X^3$, so $MF_{\hat{U},\hat{f}}^{\mathrm{mot},\phi}$ is well-defined. Motivic Milnor fibres for formal functions were also defined by Nicaise and Sebag [135] in the context of formal geometry.

In [25, Ex. 3.4], we show that Theorem 5.1.1 with $X^{(2)}$, $Y^{(2)}$, $f^{(2)}$, $g^{(2)}$ in place of $X^{(3)}$, $Y^{(3)}$, $f^{(3)}$, $g^{(3)}$ is false, so we cannot do better than $(X^{(3)}, f^{(3)})$ in Theorem 5.1.1.

Taking $U = V$, $X = Y$ and $\Phi = \mathrm{id}_{X^{(3)}}$ we get an obvious corollary of Theorem 5.1.1:

Theorem 5.1.3. *Let U be a smooth $\mathbb{K}$-scheme and $f,g : U \to \mathbb{A}^1$ regular functions. Suppose $X := \mathrm{Crit}(f) = \mathrm{Crit}(g)$ and $f^{(3)} = g^{(3)}$, that is, $f + I_X^3 = g + I_X^3$ in $H^0(\mathcal{O}_U/I_X^3)$, where $I_X \subseteq \mathcal{O}_U$ is the ideal of regular functions vanishing on X. Then $MF^{\mathrm{mot},\phi}_{U,f} = MF^{\mathrm{mot},\phi}_{U,g}$ in $\mathcal{M}^{\hat\mu}_X$ and $\bar{\mathcal{M}}^{\hat\mu}_X$.*

5.2 Stabilizing motivic vanishing cycles

To set up notation for Theorem 5.2.2 below, we need the following theorem, which is proved in Joyce [87, Prop.s 2.24 & 2.25(c)].

Theorem 5.2.1 (Joyce [87])**.** *Let U,V be smooth $\mathbb{K}$-schemes, $f : U \to \mathbb{A}^1$, $g : V \to \mathbb{A}^1$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{K}$-subschemes of U,V. Let $\Phi : U \hookrightarrow V$ be a closed embedding of $\mathbb{K}$-schemes with $f = g \circ \Phi : U \to \mathbb{A}^1$, and suppose $\Phi|_X : X \to V \supseteq Y$ is an isomorphism $\Phi|_X : X \to Y$. Then:*

(i) *For each $x \in X \subseteq U$ there exist a Zariski open neighbourhood U' of x in U, a smooth $\mathbb{K}$-scheme V', and morphisms $\jmath : V' \to V$, $\Phi' : U' \to V'$, $\alpha : V' \to U'$, $\beta : V' \to \mathbb{A}^n$, and $q_1,\ldots,q_n : U' \to \mathbb{A}^1\setminus\{0\}$, where $n = \dim V - \dim U$, such that $\jmath : V' \to V$ and $\alpha\times\beta : V' \to U'\times\mathbb{A}^n$ are étale, $\Phi|_{U'} = \jmath\circ\Phi'$, $\alpha\circ\Phi' = \mathrm{id}_{U'}$, $\beta\circ\Phi' = 0$, and*

$$g\circ\jmath = f\circ\alpha + (q_1\circ\alpha)\cdot(z_1^2\circ\beta) + \cdots + (q_n\circ\alpha)\cdot(z_n^2\circ\beta). \tag{5.2.1}$$

Thus, setting $f' = f|_{U'}$, $g' = g\circ\jmath$, $X' = \mathrm{Crit}(f') = X\cap U'$, and $Y' = \mathrm{Crit}(g')$, then $f' = g'\circ\Phi'$, and $\Phi'|_{X'} : X' \to Y'$, $\jmath|_{Y'} : Y' \to Y$, $\alpha|_{Y'} : Y' \to X$ are étale. We require that $\Phi\circ\alpha|_{Y'} = \jmath|_{Y'} : Y' \to Y$.

(ii) *Write N_{UV} for the normal bundle of $\Phi(U)$ in V, regarded as a vector bundle on U in the exact sequence*

$$0 \longrightarrow TU \xrightarrow{\mathrm{d}\Phi} \Phi^*(TV) \xrightarrow{\Pi_{UV}} N_{UV} \longrightarrow 0, \tag{5.2.2}$$

so that $N_{UV}|_X$ is a vector bundle on X. Then there exists a unique $q_{UV} \in H^0\big(S^2N_{UV}|_X^*\big)$ which is a nondegenerate quadratic form on $N_{UV}|_X$, such that whenever $x,U',V',\jmath,\Phi',\alpha,\beta,n,q_a$ are as in **(i)***, then there is an isomorphism $\hat\beta : \langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{U'} \to N_{UV}^*|_{U'}$ making the following commute:*

$$\begin{tikzcd}
N_{UV}^*|_{U'} \arrow[rr, "\Pi_{UV}^*|_{U'}"'] & & \Phi|_{U'}^*(T^*V) = \Phi'^*\circ\jmath^*(T^*V) \arrow[d, "\Phi'^*(\mathrm{d}\jmath^*)"'] \\
\langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{U'} = \Phi'^*\circ\beta^*(T_0^*\mathbb{A}^n) \arrow[u, dotted, "\hat\beta"'] \arrow[rr, "\Phi'^*(\mathrm{d}\beta^*)"] & & \Phi'^*(T^*V'),
\end{tikzcd}$$

and if $X' = X\cap U'$, then $q_{UV}|_{X'} = \big[q_1\cdot S^2\hat\beta(\mathrm{d}z_1\otimes\mathrm{d}z_1) + \cdots + q_n\cdot S^2\hat\beta(\mathrm{d}z_n\otimes\mathrm{d}z_n)\big]\big|_{X'}$.

Here, we state [25, Thm. 4.4] and we recall briefly its proof.

Theorem 5.2.2. *Let U,V be smooth $\mathbb{K}$-schemes, $f : U \to \mathbb{A}^1$, $g : V \to \mathbb{A}^1$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{K}$-subschemes of U,V. Let $\Phi : U \hookrightarrow V$ be an embedding of $\mathbb{K}$-schemes with $f = g\circ\Phi : U \to \mathbb{A}^1$, and suppose $\Phi|_X : X \to V \supseteq Y$ is an isomorphism $\Phi|_X : X \to Y$. Then*

$$\Phi|_X^*\big(MF^{\mathrm{mot},\phi}_{V,g}\big) = MF^{\mathrm{mot},\phi}_{U,f}\odot\Upsilon(P_\Phi) \quad \text{in } \bar{\mathcal{M}}^{\hat\mu}_X, \tag{5.2.3}$$

where P_Φ is as in Definition 4.3.2 and Υ is defined in §1.4.1.

Proof. Suppose U,V,f,g,X,Y,Φ are as in Theorem 5.2.2, and use the notation N_{UV},q_{UV} from Theorem 5.2.1(ii) and J_Φ,P_Φ from Definition 4.3.2. Let $x,U',V',\jmath,\Phi',\alpha,\beta,q_1,\ldots,q_n,f',g',X',Y'$

be as in Theorem 5.2.1(i). Then in $\bar{\mathcal{M}}^{\hat{\mu}}_{X'}$ we have

$$\begin{aligned}
\Phi|_X^*\big(MF^{\mathrm{mot},\phi}_{V,g}\big)\big|_{X'} &= \Phi'|_X^* \circ \jmath|_{Y'}^*\big(MF^{\mathrm{mot},\phi}_{V,g}\big) = \Phi'|_X^*\big(MF^{\mathrm{mot},\phi}_{V',g'}\big)\\
&= \Phi'|_X^* \circ (\alpha\times\beta)^*\big(MF^{\mathrm{mot},\phi}_{U'\times\mathbb{A}^n, f\circ\pi_{U'}+\Sigma_{i=1}^n(q_i\circ\pi_{U'})\cdot(z_i^2\circ\pi_{\mathbb{A}^n})}\big)\\
&= (\mathrm{id}_{X'}\times 0)^*\big(MF^{\mathrm{mot},\phi}_{U',f'}\odot \mathbb{L}^{\dim U/2}\odot MF^{\mathrm{mot},\phi}_{U'\times\mathbb{A}^n,\Sigma_{i=1}^n(q_i\circ\pi_{U'})\cdot(z_i^2\circ\pi_{\mathbb{A}^n})}\big)\\
&= (\mathrm{id}_{X'}\times 0)^*\big(MF^{\mathrm{mot},\phi}_{U',f'}\odot \Upsilon(P_{q_1\cdots q_n})\big) = MF^{\mathrm{mot},\phi}_{U',f'}\odot\Upsilon\big(P_{q_1\cdots q_n}|_{X'}\big)\\
&= MF^{\mathrm{mot},\phi}_{U',f'}\odot\Upsilon\big(P_\Phi|_{X'}\big) = \big(MF^{\mathrm{mot},\phi}_{U,f}\odot\Upsilon(P_\Phi)\big)\big|_{X'},
\end{aligned} \tag{5.2.4}$$

using $\Phi|_{U'} = \jmath\circ\Phi'$ in the first step, $\jmath : V'\to V$ étale with $g' = g\circ\jmath$ in the second, $\alpha\times\beta : V'\to U'\times\mathbb{A}^n$ étale and (5.2.1) in the third, and $\alpha\circ\Phi' = \mathrm{id}_{U'}$, $\beta\circ\Phi' = 0$, and Theorem 1.4.9 in the fourth.

In the fifth step of (5.2.4), we apply Theorem 1.4.10 to the vector bundle $U'\times\mathbb{A}^n\to U'$ and nondegenerate quadratic form $\sum_{i=1}^n (q_i\circ\pi_{U'})\cdot(z_i^2\circ\pi_{\mathbb{A}^n})$, and we write $P_{q_1\cdots q_n}\to U'$ for the principal $\mathbb{Z}_2$-bundle corresponding to $\big(\mathcal{O}_{U'}, q_1\cdots q_n\big)$ under correspondence 2.19 in [25]. The sixth step uses $MF^{\mathrm{mot},\phi}_{U',f'}$ supported on $X'\cong X'\times\{0\}$, the seventh that $P_{q_1\cdots q_n}|_{X'}\cong P_\Phi|_{X'}$ since Theorem 5.2.1(ii) implies an identification between $q_1\cdots q_n$ and $\det(q_{UV})$ on X' and $P_{q_1\cdots q_n}|_{X'}, P_\Phi|_{X'}$ parametrize square roots of $q_1\cdots q_n$ and $\det(q_{UV})$ on X', and the eighth that $U'\subseteq U$ is open with $f' = f|_{U'}$. Equation (5.2.4) proves the restriction of (5.2.3) to the Zariski open set $X'\subseteq X$. Since we can cover X by such open X', Theorem 5.2.2 follows. □

5.3 Motivic vanishing cycles on d-critical loci

Here is [25, Thm. 5.10]:

Theorem 5.3.1. *Let (X,s) be a finite type algebraic d-critical locus with a choice of orientation $K^{1/2}_{X,s}$. There exists a unique motive $MF_{X,s}\in\bar{\mathcal{M}}^{\hat{\mu}}_X$ with the property that if (R,U,f,i) is a critical chart on (X,s), then*

$$MF_{X,s}|_R = i^*\big(MF^{\mathrm{mot},\phi}_{U,f}\big)\odot\Upsilon(Q_{R,U,f,i}) \quad \text{in } \bar{\mathcal{M}}^{\hat{\mu}}_R, \tag{5.3.1}$$

where $Q_{R,U,f,i}\to R$ is the principal $\mathbb{Z}_2$-bundle parametrizing local isomorphisms $\alpha : K^{1/2}_{X,s}|_{R^{\mathrm{red}}}\to i^*(K_U)|_{R^{\mathrm{red}}}$ with $\alpha\otimes\alpha = \iota_{R,U,f,i}$, for $\iota_{R,U,f,i}$ as in (2.1.4).

Proof. Let (X,s) be an algebraic d-critical locus with orientation $K^{1/2}_{X,s}$. We must construct $MF_{X,s}\in\bar{\mathcal{M}}^{\hat{\mu}}_X$ satisfying (5.3.1) for each critical chart (R,U,f,i). Since such $R\subseteq X$ form a Zariski open cover of X, and (5.3.1) determines $MF_{X,s}|_R$, there exists a unique $MF_{X,s}$ satisfying (5.3.1) for all (R,U,f,i) if and only if the prescribed values $MF_{X,s}|_R$ agree on overlaps between critical charts. That is, we must prove that if (R,U,f,i) and (S,V,g,j) are critical charts, then

$$\big[i^*\big(MF^{\mathrm{mot},\phi}_{U,f}\big)\odot\Upsilon(Q_{R,U,f,i})\big]\big|_{R\cap S} = \big[j^*\big(MF^{\mathrm{mot},\phi}_{V,g}\big)\odot\Upsilon(Q_{S,V,g,j})\big]\big|_{R\cap S}. \tag{5.3.2}$$

Fix $x\in R\cap S\subseteq X$, and let $(R',U',f',i'),(S',V',g',j'),(T,W,h,k),\Phi,\Psi$ be as in Theorem 2.1.5. Then as in Theorem 4.4.1 there is a natural isomorphism of principal $\mathbb{Z}_2$-bundles on R'

$$\Lambda_\Phi : Q_{T,W,h,k}|_{R'} \xrightarrow{\cong} i|_{R'}^*(P_\Phi)\otimes_{\mathbb{Z}_2} Q_{R,U,f,i}|_{R'}, \tag{5.3.3}$$

for $P_\Phi \to \mathrm{Crit}(f')$ the principal $\mathbb{Z}_2$-bundle of orientations of $\big(N_{U'W}|_{\mathrm{Crit}(f')}, q_{U'W}\big)$. We now have

$$\begin{aligned}
&\big[k^*\big(MF^{\mathrm{mot},\phi}_{W,h}\big) \odot \Upsilon(Q_{T,W,h,k})\big]\big|_{R'} = i'^*\big[\Phi|^*_{\mathrm{Crit}(f')}\big(MF^{\mathrm{mot},\phi}_{W,h}\big)\big] \odot \Upsilon(Q_{T,W,h,k})|_{R'} \\
&\quad = i'^*\big[MF^{\mathrm{mot},\phi}_{U',f'} \odot \Upsilon(P_\Phi)\big] \odot \Upsilon(Q_{T,W,h,k})|_{R'} = i|^*_{R'}\big[MF^{\mathrm{mot},\phi}_{U,f}\big] \odot \Upsilon\big(i|^*_{R'}(P_\Phi)\big) \odot \Upsilon\big(Q_{T,W,h,k}|_{R'}\big) \\
&\quad = i|^*_{R'}\big[MF^{\mathrm{mot},\phi}_{U,f}\big] \odot \Upsilon\big(i|^*_{R'}(P_\Phi) \otimes_{\mathbb{Z}_2} Q_{T,W,h,k}|_{R'}\big) = i|^*_{R'}\big[MF^{\mathrm{mot},\phi}_{U,f}\big] \odot \Upsilon\big(Q_{R,U,f,i}|_{R'}\big) \\
&\quad = \big[i^*\big(MF^{\mathrm{mot},\phi}_{U,f}\big) \odot \Upsilon(Q_{R,U,f,i})\big]\big|_{R'},
\end{aligned} \tag{5.3.4}$$

using $\Phi|_{\mathrm{Crit}(f')} \circ i' = k|_{R'}$ in the first step, Theorem 5.2.2 for $\Phi : (U', f') \to (W, h)$ in the second, $U' \subseteq U$, $f' = f|_{U'}$ and functoriality of Υ in the third, (1.4.14) in the fourth, and (5.3.3) in the fifth.

Similarly, from $\Psi : (S', V', g', j') \hookrightarrow (T, W, h, k)$ we obtain

$$\big[k^*\big(MF^{\mathrm{mot},\phi}_{W,h}\big) \odot \Upsilon(Q_{T,W,h,k})\big]\big|_{S'} = \big[j^*\big(MF^{\mathrm{mot},\phi}_{V,g}\big) \odot \Upsilon(Q_{S,V,g,j})\big]\big|_{S'}. \tag{5.3.5}$$

Combining the restrictions of (5.3.4)–(5.3.5) to $R' \cap S'$ proves the restriction of (5.3.2) to $R' \cap S'$. Since we can cover $R \cap S$ by such Zariski open $R' \cap S' \subseteq R \cap S$, this proves (5.3.2), and hence Theorem 5.3.1. $\square$

From Theorem 3.3.1 and Corollaries 3.3.2 and 3.3.3 we deduce [25, Cor.s 5.12, 5.13 & 5.14]:

Corollary 5.3.2. *Let $(\boldsymbol{X}, \omega)$ be a -1-shifted symplectic derived scheme over $\mathbb{K}$ in the sense of* [142], *and $X = t_0(\boldsymbol{X})$ the associated classical $\mathbb{K}$-scheme, assumed of finite type. Suppose we are given a square root $\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}$ for $\det(\mathbb{L}_{\boldsymbol{X}})|_X$. Then we may define a natural motive $MF_{\boldsymbol{X},\omega} \in \bar{\mathcal{M}}^{\hat\mu}_X$.*

Corollary 5.3.3. *Suppose Y is a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ is a finite type moduli $\mathbb{K}$-scheme of simple coherent sheaves in $\mathrm{coh}(Y)$ with obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ as in Thomas* [167]. *Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$. Then we may define a natural motive $MF_{\mathcal{M}} \in \bar{\mathcal{M}}^{\hat\mu}_{\mathcal{M}}$.*

Corollary 5.3.4. *Let (S, ω) be an algebraic symplectic manifold and L, M finite type algebraic Lagrangian submanifolds in S, and write $X = L \cap M$, as a subscheme of S. Suppose we are given square roots $K_L^{1/2}, K_M^{1/2}$ for K_L, K_M. Then we may define a natural motive $MF_{L,M} \in \bar{\mathcal{M}}^{\hat\mu}_X$.*

Corollary 5.3.3 has applications to Donaldson–Thomas theory, discussed in §7.

We conclude saying that recently Maulik proved a torus localization formula for the motives $MF_{X,s}$ of Theorem 5.3.1, [133]. See [25, §5.3] for a brief discussion about it.

Chapter 6

A Darboux Theorem for shifted symplectic structures on derived Artin stacks, with applications

This chapter extend results of the previous chapters §3, §4 and §5 to Artin $\mathbb{K}$-stacks, and it is based on [13]. As usual, we will not go into details and proofs of results, for which we refer to [13].

6.1 'Darboux form' atlases for shifted symplectic stacks

We follow [13, §2.5-2.7]. We start by generalizing Definition 3.1.1 and Theorems 3.1.2–3.1.3 to derived Artin stacks [13, Def. 2.7, Thm.s 2.8 & 2.9]. The proofs can be found in [13, §2.5].

Definition 6.1.1. Let $\boldsymbol{X}$ be a derived Artin $\mathbb{K}$-stack, and p a point of $\boldsymbol{X}$. By this we mean a morphism $p : \operatorname{Spec}\mathbb{K} \to \boldsymbol{X}$; we may also call p a $\mathbb{K}$-point of $\boldsymbol{X}$. A *standard form open neighbourhood* $(A, \boldsymbol{\varphi}, \tilde{p})$ of p, in the smooth topology, means a standard form cdga A over $\mathbb{K}$ in the sense of Definition 3.1.1, so that $\boldsymbol{U} = \mathbf{Spec}\, A$ is an affine derived $\mathbb{K}$-scheme, and a morphism $\boldsymbol{\varphi} : \boldsymbol{U} \to \boldsymbol{X}$ which is smooth of some relative dimension $n \geqslant 0$, and a $\mathbb{K}$-point $\tilde{p}$ in $\boldsymbol{U}$ with $p = \boldsymbol{\varphi}(\tilde{p})$, that is, there is an equivalence of morphisms $p \simeq \boldsymbol{\varphi} \circ \tilde{p} : \operatorname{Spec}\mathbb{K} \to \boldsymbol{X}$. If we do not specify $p, \tilde{p}$, we just call $(A, \boldsymbol{\varphi})$ a *standard form open neighbourhood* in $\boldsymbol{X}$. For such $\boldsymbol{X}, p, (A, \boldsymbol{\varphi}, \tilde{p}), n$, as for (1.2.1) we have the standard fibre sequence

$$\boldsymbol{\varphi}^*(\mathbb{L}_{\boldsymbol{X}}) \xrightarrow{\mathbb{L}_{\varphi}} \mathbb{L}_{\boldsymbol{U}} \longrightarrow \mathbb{L}_{\boldsymbol{U}/\boldsymbol{X}} \longrightarrow \boldsymbol{\varphi}^*(\mathbb{L}_{\boldsymbol{X}})[1], \tag{6.1.1}$$

where $\mathbb{L}_{\boldsymbol{U}/\boldsymbol{X}}$ is locally free of rank n. Restricting (6.1.1) to $\tilde{p}$ and taking cohomology, we have the following:

(a) There are isomorphisms $H^i\big(\mathbb{L}_{\boldsymbol{X}}|_p\big) \cong H^i\big(\mathbb{L}_{\boldsymbol{U}}|_{\tilde{p}}\big)$ for $i < 0$.

(b) Since $\boldsymbol{U}$ is not stacky, $H^1\big(\mathbb{L}_{\boldsymbol{U}}|_{\tilde{p}}\big) = 0$ and so there is an exact sequence of $\mathbb{K}$-vector spaces

$$0 \longrightarrow H^0\big(\mathbb{L}_{\boldsymbol{X}}|_p\big) \longrightarrow H^0\big(\mathbb{L}_{\boldsymbol{U}}|_{\tilde{p}}\big) \longrightarrow H^0\big(\mathbb{L}_{\boldsymbol{U}/\boldsymbol{X}}|_{\tilde{p}}\big) \longrightarrow H^1\big(\mathbb{L}_{\boldsymbol{X}}|_p\big) \longrightarrow 0,$$

where $H^0\big(\mathbb{L}_{\boldsymbol{U}/\boldsymbol{X}}|_{\tilde{p}}\big) \cong \mathbb{K}^n$. Therefore $n \geqslant \dim H^1\big(\mathbb{L}_{\boldsymbol{X}}|_p\big)$. Note that $H^1\big(\mathbb{L}_{\boldsymbol{X}}|_p\big) \cong \mathfrak{Iso}_{\boldsymbol{X}}(p)^*$, where $\mathfrak{Iso}_{\boldsymbol{X}}(p)$ is the Lie algebra of the isotropy group $\mathrm{Iso}_{\boldsymbol{X}}(p)$ of $\boldsymbol{X}$ at p, which is an algebraic $\mathbb{K}$-group. In particular, the minimal possible relative dimension $n = \operatorname{rank}(\mathbb{L}_{\boldsymbol{U}/\boldsymbol{X}})$ of a neighbourhood $\boldsymbol{\varphi} : \boldsymbol{U} \to \boldsymbol{X}$ of p is $n = \dim H^1\big(\mathbb{L}_{\boldsymbol{X}}|_p\big)$.

(c) If $\boldsymbol{\varphi}$ is smooth of minimal relative dimension $n=\dim H^1(\mathbb{L}_{\boldsymbol{X}}|_p)$, then

$$H^0(\mathbb{L}_{\boldsymbol{X}}|_p)\cong H^0(\mathbb{L}_{\boldsymbol{U}}|_{\tilde p})\quad\text{and}\quad H^0(\mathbb{L}_{\boldsymbol{U}/\boldsymbol{X}}|_{\tilde p})\cong H^1(\mathbb{L}_{\boldsymbol{X}}|_p). \tag{6.1.2}$$

We call a standard form open neighbourhood $(A,\boldsymbol{\varphi},\tilde p)$ *minimal at* p if A is minimal at $\tilde p$ in the sense of Definition 3.1.1 and $n=\dim H^1(\mathbb{L}_{\boldsymbol{X}}|_p)$. Then parts (a),(c) imply that $A(0)$ is smooth of dimension $m_0=\dim H^0(\mathbb{L}_{\boldsymbol{X}}|_p)$, and A has $m_i=\dim H^{-i}(\mathbb{L}_{\boldsymbol{X}}|_p)$ generators in degree $-i$ for $i=1,2,\ldots$.

Theorem 6.1.2. *Let $\boldsymbol{X}$ be a derived Artin $\mathbb{K}$-stack, and p a point of $\boldsymbol{X}$. Then there exists a minimal standard form open neighbourhood $(A,\boldsymbol{\varphi},\tilde p)$ of p, in the sense of Definition* 6.1.1.

Theorem 6.1.3. *Let $\boldsymbol{X}$ be a derived Artin $\mathbb{K}$-stack and $(A,\boldsymbol{\varphi}),(B,\boldsymbol{\psi})$ standard form open neighbourhoods in $\boldsymbol{X}$, and write $\boldsymbol{U}=\mathbf{Spec}\,A$, $\boldsymbol{V}=\mathbf{Spec}\,B$. Then for each $p\in\boldsymbol{U}\times_{\boldsymbol{X}}\boldsymbol{V}$ there exist a standard form cdga C over $\mathbb{K}$ minimal at $q\in\boldsymbol{W}=\mathbf{Spec}\,C$, an étale morphism $\boldsymbol{i}:\boldsymbol{W}\to\boldsymbol{U}\times_{\boldsymbol{X}}\boldsymbol{V}$ with $\boldsymbol{i}(q)=p$, and cdga morphisms $\alpha:A\to C$, $\beta:B\to C$ with $\boldsymbol{\pi_U}\circ\boldsymbol{i}\simeq\mathbf{Spec}\,\alpha:\boldsymbol{W}\to\boldsymbol{U}$ and $\boldsymbol{\pi_V}\circ\boldsymbol{i}\simeq\mathbf{Spec}\,\beta:\boldsymbol{W}\to\boldsymbol{V}$.*

Here is [13, Thm. 2.10], a stack analogue of Theorem 3.2.2. Note that (a)(i)–(v) are modelled closely on the first part of Definition 3.2.1, and equations (6.1.3)–(6.1.7) are analogues of or identical to (3.2.1)–(3.2.5). The proof can be found in [13, §2.6].

Theorem 6.1.4. **(a)** *Let $(\boldsymbol{X},\omega_{\boldsymbol{X}})$ be a k-shifted symplectic derived Artin $\mathbb{K}$-stack, where $k=-2d-1$ for $d=0,1,2,\ldots$, and $p\in\boldsymbol{X}$. Then we can construct a minimal standard form open neighbourhood $(A,\boldsymbol{\varphi}:\boldsymbol{U}\to\boldsymbol{X},\tilde p)$ of p in the sense of Definition* 6.1.1, *and a k-shifted closed 2-form $\omega=(\omega^0,0,\ldots)$ on $\boldsymbol{U}=\mathbf{Spec}\,A$ for $\omega^0\in(\Lambda^2\Omega^1_A)^k$, such that $\boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}})\sim\omega$ in k-shifted closed 2-forms on $\boldsymbol{U}=\mathbf{Spec}\,A$. Furthermore, A,ω are in a standard ‘Darboux form’, a modified version of Definition* 3.2.1, *as follows:*

(i) *The degree 0 part A^0 of A is a smooth $\mathbb{K}$-algebra of dimension m_0, and we are given $x^0_1,\ldots,x^0_{m_0}\in A^0$ such that $\mathrm{d}_{dR}x^0_1,\ldots,\mathrm{d}_{dR}x^0_{m_0}$ form a basis of $\Omega^1_{A^0}$ over A^0.*

(ii) *As a graded commutative algebra, A is freely generated over A^0 by variables*

$$\begin{aligned}
&x^{-i}_1,\ldots,x^{-i}_{m_i} && \text{in degree } -i \text{ for } i=1,\ldots,d,\\
&y^{i-2d-1}_1,\ldots,y^{i-2d-1}_{m_i} && \text{in degree } i-2d-1 \text{ for } i=0,1,\ldots,d,\\
&w^{-2d-2}_1,\ldots,w^{-2d-2}_n && \text{in degree } -2d-2,
\end{aligned} \tag{6.1.3}$$

for $m_0,\ldots,m_d\geqslant 0$ with m_0 as in **(i)**, *and $n=\dim H^1(\mathbb{L}_{\boldsymbol{X}}|_p)$ the relative dimension of $\boldsymbol{\varphi}$. The upper index i in w^i_j,x^i_j,y^i_j is the degree. Then*

$$\omega^0=\sum_{i=0}^d\sum_{j=1}^{m_i}\mathrm{d}_{dR}y^{i-2d-1}_j\,\mathrm{d}_{dR}x^{-i}_j\qquad\text{in } (\Lambda^2\Omega^1_A)^{-2d-1}. \tag{6.1.4}$$

(iii) *We are given H in A^{-2d}, called the* ***Hamiltonian****, which satisfies the* ***classical master equation***

$$\sum_{i=1}^d\sum_{j=1}^{m_i}\frac{\partial H}{\partial x^{-i}_j}\,\frac{\partial H}{\partial y^{i-2d-1}_j}=0\qquad\text{in } A^{1-2d}. \tag{6.1.5}$$

The differential d on A satisfies $\mathrm{d}=0$ on A^0, and

$$\mathrm{d}x^{-i}_j=\frac{\partial H}{\partial y^{i-2d-1}_j},\quad \mathrm{d}y^{i-2d-1}_j=\frac{\partial H}{\partial x^{-i}_j},\quad \begin{array}{l} i=0,\ldots,d,\\ j=1,\ldots,m_i.\end{array} \tag{6.1.6}$$

Note that (6.1.6) ***does not specify*** $\mathrm{d}w^{-2d-2}_j$ *for $j=1,\ldots,n$, and so* ***does not completely determine*** d *on A.*

(iv) *Define* $\Phi \in A^{-2d}$ *and* $\phi \in (\Omega^1_A)^{-2d-1}$ *by* $\Phi = -\frac{1}{2d+1} H$ *and*

$$\phi = \frac{1}{2d+1}\sum_{i=0}^{d}\sum_{j=1}^{m_i}\big[(2d+1-i)y_j^{i-2d-1}\,\mathrm{d}_{dR}x_j^{-i} + i\,x_j^{-i}\,\mathrm{d}_{dR}y_j^{i-2d-1}\big]. \tag{6.1.7}$$

Then $\mathrm{d}\Phi = 0$, $\mathrm{d}_{dR}\Phi + \mathrm{d}\phi = 0$, *and* $\omega^0 = \mathrm{d}_{dR}\phi$.

(v) *Minimality of* $(A, \boldsymbol{\varphi}, \tilde p)$ *means that* $\mathrm{d}w_j^{-2d-2}|_{\tilde p} = 0$ *for* $j = 1, \ldots, n$ *and*

$$\mathrm{d}x_j^{-i}\big|_{\tilde p} = \frac{\partial H}{\partial y_j^{i-2d-1}}\bigg|_{\tilde p} = 0 = \mathrm{d}y_j^{i-2d-1}\big|_{\tilde p} = \frac{\partial H}{\partial x_j^{-i}}\bigg|_{\tilde p}, \quad \begin{array}{l} i = 0, \ldots, d, \\ j = 1, \ldots, m_i. \end{array}$$

(b) *In part* **(a)**, *let* B *be the graded subalgebra of* A *generated by* A^0 *and the variables* x_j^i, y_j^i *in* **(ii)** *for all* i, j, *with inclusion* $\iota : B \hookrightarrow A$. *Then* B *is closed under* d, *and so is a dg-subalgebra of* A. *For degree reasons* H, Φ *above cannot depend on the* w_j^{-2d-2}, *so* $H, \Phi \in B$. *Also the data* ω, ω^0, ϕ *in* $\Omega^1_A, \Lambda^2\Omega^1_A$ *above are the images under* ι *of* $\omega_B, \omega_B^0, \phi_B$ *in* $\Omega^1_B, \Lambda^2\Omega^1_B$. *Then* ω_B *is a* k*-shifted symplectic structure on* $\boldsymbol{V} = \mathbf{Spec}\, B$, *and* B, ω_B *is in Darboux form as in Definition* 3.2.1, *and* B *is minimal at* $\tilde p$ *as in Definition* 3.1.1.

Geometrically, we have a diagram of morphisms in $\mathbf{dArt}_{\mathbb{K}}$:

$$\boldsymbol{V} = \mathbf{Spec}\, B \xleftarrow{\ \boldsymbol{i} = \mathbf{Spec}\,\iota\ } \boldsymbol{U} = \mathbf{Spec}\, A \xrightarrow{\ \boldsymbol{\varphi}\ } \boldsymbol{X},$$

where $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$, $(\boldsymbol{V}, \omega_B)$ *are* k*-shifted symplectic, with* $\boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}}) \sim \boldsymbol{i}^*(\omega_B)$ *in* k*-shifted closed* 2*-forms on* $\boldsymbol{U}$. *We can think of* $\boldsymbol{\varphi} : \boldsymbol{U} \to \boldsymbol{X}$ *as a 'submersion', and* $\boldsymbol{i} : \boldsymbol{U} \hookrightarrow \boldsymbol{V}$ *as an embedding of* $\boldsymbol{U}$ *as a derived subscheme of* $\boldsymbol{V}$. *On classical schemes,* $i = t_0(\boldsymbol{i}) : U = t_0(\boldsymbol{U}) \to V = t_0(\boldsymbol{V})$ *is an isomorphism. There is a natural equivalence of relative (co)tangent complexes*

$$\mathbb{L}_{\boldsymbol{U}/\boldsymbol{V}} \simeq \mathbb{T}_{\boldsymbol{U}/\boldsymbol{X}}[1-k]. \tag{6.1.8}$$

(c) *The obvious analogues of* **(a)**,**(b)** *also hold if* $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$ *is a* k*-shifted symplectic derived Artin* $\mathbb{K}$*-stack for* $k < 0$ *with* $k \equiv 0 \mod 4$ *or* $k \equiv 2 \mod 4$. *In each case, the algebra* A *is the corresponding algebra from Definition* 3.2.1, *modified by adding generators* $w_1^{k-1}, \ldots, w_n^{k-1}$ *in degree* $k-1$.

In the case $k = -1$, as in [19, Ex. 5.15] the classical $\mathbb{K}$-schemes $U \cong V$ in Theorem 6.1.4(a),(b) are isomorphic to $\mathrm{Crit}\big(H : U(0) \to \mathbb{A}^1\big)$. So changing notation from $U(0), H, \tilde p$ to U, f, u, using $H^i\big(\mathbb{L}_X|_p\big) \cong H^i\big(\mathbb{L}_{\boldsymbol{X}}|_p\big)$ for $X = t_0(\boldsymbol{X})$ and $i = 0, 1$, and applying Proposition 3.1.4(b) to get $f|_{T^{\mathrm{red}}} = 0$, we deduce [13, Cor. 2.11]:

Corollary 6.1.5. *Let* $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$ *be a* -1*-shifted symplectic derived Artin* $\mathbb{K}$*-stack, and* $X = t_0(\boldsymbol{X})$ *the corresponding classical Artin* $\mathbb{K}$*-stack. Then for each* $p \in X$ *there exist a smooth* $\mathbb{K}$*-scheme* U *with dimension* $\dim H^0\big(\mathbb{L}_X|_p\big)$, *a point* $t \in U$, *a regular function* $f : U \to \mathbb{A}^1$ *with* $\mathrm{d}_{dR}f|_t = 0$, *so that* $T := \mathrm{Crit}(f) \subseteq U$ *is a closed* $\mathbb{K}$*-subscheme with* $t \in T$, *and a morphism* $\varphi : T \to X$ *which is smooth of relative dimension* $\dim H^1\big(\mathbb{L}_X|_p\big)$, *with* $\varphi(t) = p$. *We may take* $f|_{T^{\mathrm{red}}} = 0$.

Here the derived critical locus $\mathbf{Crit}(f : U \to \mathbb{A}^1)$, *as a* -1*-shifted symplectic derived scheme, agrees with* $(\boldsymbol{V}, \omega_B)$ *in Theorem* 6.1.4, *and* $\varphi : T \to X$ *corresponds to* $t_0(\boldsymbol{\varphi}) \circ t_0(\boldsymbol{i})^{-1}$ *in Theorem* 6.1.4.

Thus, the underlying classical stack X of a -1-shifted symplectic derived stack $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$ admits an atlas consisting of critical loci of regular functions on smooth schemes.

Now let Y be a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or complexes $F^\bullet$ in $D^b\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$. Then $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$, for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli stack. The (open) condition $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$ is needed to make $\boldsymbol{\mathcal{M}}$ 1-truncated (that is, a derived Artin stack, in our terminology), and so make $\mathcal{M} =$

$t_0(\boldsymbol{\mathcal{M}})$ an ordinary, and not higher, stack. Pantev et al. [142, §2.1] prove $\boldsymbol{\mathcal{M}}$ has a -1-shifted symplectic structure $\omega_{\boldsymbol{\mathcal{M}}}$. Applying Corollary 6.1.5 and using $H^i\big(\mathbb{L}_{\boldsymbol{\mathcal{M}}}|_{[F]}\big) \cong \mathrm{Ext}^{1-i}(F,F)^*$ yields a new result on classical 3-Calabi–Yau moduli stacks, the statement of which involves no derived geometry [13, Cor. 2.12]:

Corollary 6.1.6. *Suppose Y is a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-stack of coherent sheaves F, or more generally of complexes $F^\bullet$ in $D^b\,\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet,F^\bullet)=0$. Then for each $[F]\in\mathcal{M}$, there exist a smooth $\mathbb{K}$-scheme U with $\dim U=\dim \mathrm{Ext}^1(F,F)$, a point $u\in U$, a regular function $f:U\to\mathbb{A}^1$ with $\mathrm{d}_{dR}f|_u=0$, and a morphism $\varphi:\mathrm{Crit}(f)\to\mathcal{M}$ which is smooth of relative dimension $\dim \mathrm{Hom}(F,F)$, with $\varphi(u)=[F]$.*

This is an analogue of Corollary 3.2.3. When $\mathbb{K}=\mathbb{C}$, a related result for coherent sheaves only, with U a complex manifold and f a holomorphic function, was proved by Joyce and Song [85, Th. 5.5] using gauge theory and transcendental complex methods. This will be important in §7.

Finally, we state [13, Thm. 2.13], on comparing Darboux form atlases on overlaps, as in §3.

Proposition 6.1.7. *Let $(\boldsymbol{X},\omega_{\boldsymbol{X}})$ be a -1-shifted symplectic derived Artin $\mathbb{K}$-stack, and $X=t_0(\boldsymbol{X})$ the corresponding classical Artin $\mathbb{K}$-stack. Suppose $U,f:U\to\mathbb{A}^1$, $\varphi:T=\mathrm{Crit}(f)\to X$ and $U',f':U'\to\mathbb{A}^1$, $\varphi':T'=\mathrm{Crit}(f')\to X$ are two choices of the data constructed in Corollary 6.1.5 for points $p,p'\in\boldsymbol{X}$, with $f|_{T^{\mathrm{red}}}=0=f'|_{T'^{\mathrm{red}}}$. Let $q\in T\times_{\varphi,X,\varphi'}T'$. Then there exist a smooth $\mathbb{K}$-scheme V, a closed $\mathbb{K}$-subscheme $R\subseteq V$, a point $r\in R$, and morphisms $\theta:V\to U$, $\theta':V\to U'$ with $\theta(R)\subseteq T$, $\theta'(R)\subseteq T'$ such that the following diagram 2-commutes (homotopy commutes) in $\mathrm{Art}_{\mathbb{K}}$:*

$$\begin{tikzcd}
V \arrow[rr, "\theta'"] \arrow[dd, "\theta"] & & U' \arrow[r, "f'"] & \mathbb{A}^1 \\
& R \arrow[ul, hook', "\mathrm{inc}"'] \arrow[rr, "\theta'|_R"] \arrow[dd, "\theta|_R"] \arrow[ddrr, phantom, "\eta\,\Uparrow"] & & T' \arrow[ul, hook', "\mathrm{inc}"] \arrow[dd, "\varphi'"] \\
U \arrow[d, "f"] & & & \\
\mathbb{A}^1 & T \arrow[ul, hook', "\mathrm{inc}"] \arrow[rr, "\varphi"] & & X,
\end{tikzcd}$$

and the induced morphism $R\to T\times_X T'$ is étale and maps $r\mapsto q$. Furthermore $f\circ\theta-f'\circ\theta'\in I_{R,V}^2$, where $I_{R,V}\subseteq\mathcal{O}_V$ is the ideal of functions vanishing on $R\subseteq V$.

6.2 A truncation functor to d-critical stacks

Here is [13, Thm. 3.18], a stack version of Theorem 3.3.1.

Theorem 6.2.1. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $(\boldsymbol{X},\omega_{\boldsymbol{X}})$ a -1-shifted symplectic derived Artin $\mathbb{K}$-stack, and $X=t_0(\boldsymbol{X})$ the corresponding classical Artin $\mathbb{K}$-stack. Then there exists a unique d-critical structure $s\in H^0(\mathcal{S}_X^0)$ on X, making (X,s) into a d-critical stack, with the following properties:*

(a) *Let U, $f:U\to\mathbb{A}^1$, $T=\mathrm{Crit}(f)$ and $\varphi:T\to X$ be as in Corollary 6.1.5, with $f|_{T^{\mathrm{red}}}=0$. There is a unique $s_T\in H^0(\mathcal{S}_T^0)$ on T with $\iota_{T,U}(s_T)=i^{-1}(f)+I_{T,U}^2$, and (T,s_T) is an algebraic d-critical locus. Then $s(T,\varphi)=s_T$ in $H^0(\mathcal{S}_T^0)$.*

(b) *The canonical bundle $K_{X,s}$ of (X,s) from Theorem 2.2.6 is naturally isomorphic to the restriction $\det(\mathbb{L}_{\boldsymbol{X}})|_{X^{\mathrm{red}}}$ to $X^{\mathrm{red}}\subseteq X\subseteq\boldsymbol{X}$ of the determinant line bundle $\det(\mathbb{L}_{\boldsymbol{X}})$ of the cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ of $\boldsymbol{X}$.*

We can think of Theorem 6.2.1 as defining a *truncation functor*

$$\begin{aligned} F : \big\{&\infty\text{-category of } -1\text{-shifted symplectic derived Artin } \mathbb{K}\text{-stacks } (\boldsymbol{X}, \omega_{\boldsymbol{X}})\big\} \\ &\longrightarrow \big\{2\text{-category of d-critical stacks } (X, s) \text{ over } \mathbb{K}\big\}. \end{aligned}$$

The following will be important in §7. Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-stack of coherent sheaves in $\mathrm{coh}(Y)$, or complexes of coherent sheaves in $D^b\,\mathrm{coh}(Y)$. There is a natural obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ on $\mathcal{M}$, where $\mathcal{E}^\bullet \in D_{\mathrm{qcoh}}(\mathcal{M})$ is perfect in the interval $[-2, 1]$, and $h^i(\mathcal{E}^\bullet)|_F \cong \mathrm{Ext}^{1-i}(F, F)^*$ for each $\mathbb{K}$-point $F \in \mathcal{M}$, regarding F as an object in $\mathrm{coh}(Y)$ or $D^b\,\mathrm{coh}(Y)$. Now in derived algebraic geometry $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli $\mathbb{K}$-stack, and $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is $\mathbb{L}_{t_0} : \mathbb{L}_{\boldsymbol{\mathcal{M}}}|_{\mathcal{M}} \to \mathbb{L}_{\mathcal{M}}$. Pantev et al. [142, §2.1] prove $\boldsymbol{\mathcal{M}}$ has a -1-shifted symplectic structure ω. Thus Theorem 6.2.1 implies [13, Cor. 3.19]:

Corollary 6.2.2. *Suppose Y is a Calabi–Yau 3-fold over $\mathbb{K}$ of characteristic zero, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-stack of coherent sheaves F in $\mathrm{coh}(Y)$, or complexes of coherent sheaves $F^\bullet$ in $D^b\,\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, with obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Then $\mathcal{M}$ extends naturally to an algebraic d-critical locus $(\mathcal{M}, s)$. The canonical bundle $K_{\mathcal{M},s}$ from Theorem 2.2.6 is naturally isomorphic to $\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\mathrm{red}}}$.*

6.3 Perverse sheaves on d-critical stacks

We state [13, Prop. 4.8] on the behavior of the perverse sheaves $P^\bullet_{X,s}$ of Theorem 4.4.1 under smooth pullback, which will be the main ingredient in the proof of our main result of the section [13, Thm. 4.12], Theorem 6.3.2 below. It is proved in [13, §4.2].

Proposition 6.3.1. **(a)** *Let $\phi : (X, s) \to (Y, t)$ be a morphism of algebraic d-critical loci over $\mathbb{C}$, in the sense of §2 and suppose $\phi : X \to Y$ is smooth of relative dimension d. Let $K^{1/2}_{Y,t}$ be an orientation for (Y, t), so that Corollary 2.1.10 defines an induced orientation $K^{1/2}_{X,s}$ for (X, s). Theorem 4.4.1 defines perverse sheaves $P^\bullet_{X,s}, P^\bullet_{Y,t}$ on X, Y. Then there is a natural isomorphism*

$$\Delta_\phi : \phi^*[d](P^\bullet_{Y,t}) \xrightarrow{\cong} P^\bullet_{X,s} \quad \textit{in } \mathrm{Perv}(X) \tag{6.3.1}$$

which is characterized by the property that if $(R, U, f, i), (S, V, g, j)$ are critical charts on $(X, s), (Y, t)$ with $\phi(R) \subseteq S$ and $\Phi : U \to V$ is smooth of relative dimension d with $f = g \circ \Phi$ and $\Phi \circ i = j \circ \phi$, then the following commutes

$$\begin{tikzcd} \phi|_R^*[d](P^\bullet_{Y,t}) \arrow[r, "\phi|_R^*[d](\omega_{S,V,g,j})"'] \arrow[d, "\Delta_\phi|_R"] & \phi|_R^*[d]\big(j^*\big(\mathcal{PV}^\bullet_{V,g}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{S,V,g,j}\big) \arrow[d, "i^*(\Xi_\Phi)\otimes\alpha_\Phi"'] \\ P^\bullet_{X,s}|_R \arrow[r, "\omega_{R,U,f,i}"] & i^*\big(\mathcal{PV}^\bullet_{U,f}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}, \end{tikzcd} \tag{6.3.2}$$

where Ξ_Φ is as in (1.3.2) and $\alpha_\Phi : \phi|_R^*[d](Q_{S,V,g,j}) \to Q_{R,U,f,i}$ is the natural isomorphism. Also Δ_ϕ identifies $\phi^*[d](\Sigma_{Y,t}), \phi^*[d](\mathrm{T}_{Y,t})$ with $\Sigma_{X,s}, \mathrm{T}_{X,s}$.

(b) *If $\psi : (Y, t) \to (Z, u)$ is another morphism of algebraic d-critical loci over $\mathbb{C}$ smooth of relative dimension e, then*

$$\Delta_{\psi\circ\phi} = \Delta_\phi \circ \phi^*[d](\Delta_\psi) : (\psi \circ \phi)^*[d + e](P^\bullet_{Z,u}) \xrightarrow{\cong} P^\bullet_{X,s}. \tag{6.3.3}$$

(c) *Analogues of* **(a)**,**(b)** *hold for algebraic d-critical loci (X, s) over general fields $\mathbb{K}$ in the settings of l-adic perverse sheaves and of $\mathscr{D}$-modules, and for algebraic d-critical loci (X, s) over $\mathbb{C}$ in the setting of mixed Hodge modules.*

Here is [13, Thm. 4.12, Cor.s 4.13 & 4.14], the analogue of Theorem 4.4.1 and Corollaries 4.4.2-4.4.3.

Theorem 6.3.2. *Let (X,s) be an oriented d-critical stack over $\mathbb{K}$ (allowing $\mathbb{K}=\mathbb{C}$) with orientation $K_{X,s}^{1/2}$. Fix a theory of perverse sheaves on $\mathbb{K}$-schemes from §1.3, and let $\mathrm{Perv}_{\mathrm{naï}}(X)$ be the corresponding category of naïve perverse sheaves on X. Then we may define $\mathcal{P}_{X,s}\in\mathrm{Perv}_{\mathrm{naï}}(X)$ and Verdier duality and monodromy isomorphisms*

$$\Sigma_{X,s}:\mathcal{P}_{X,s}\longrightarrow\mathbb{D}_X(\mathcal{P}_{X,s}),\qquad \mathrm{T}_{X,s}:\mathcal{P}_{X,s}\longrightarrow\mathcal{P}_{X,s},$$

as follows:

- **(a)** *If $t:T\to X$ is smooth with T a $\mathbb{K}$-scheme, so that $(T,s(T,t))$ is an algebraic d-critical locus with natural orientation $K_{T,s(T,t)}^{1/2}$ as in Lemma 2.2.8, then $\mathcal{P}_{X,s}(T,t)=P^\bullet_{T,s(T,t)}$ in $\mathrm{Perv}(T)$, where $P^\bullet_{T,s(T,t)}$ is the perverse sheaf on the oriented algebraic d-critical locus $(T,s(T,t))$ over $\mathbb{K}$ given by Theorem 4.4.1. Also $\Sigma_{X,s}(T,t)=\Sigma_{T,s(T,t)}$ and $\mathrm{T}_{X,s}(T,t)=\mathrm{T}_{T,s(T,t)}$.*
- **(b)** *For each 2-commutative diagram in $\mathrm{Art}_{\mathbb{K}}$*

$$\begin{tikzcd} & U \arrow[dr, "u"] & \\ T \arrow[ur, "\phi"] \arrow[rr, "t"'] \arrow[rr, phantom, shift left=3ex, "\eta\Uparrow"] & & X \end{tikzcd}$$

with T,U $\mathbb{K}$-schemes and ϕ,t,u smooth with ϕ of dimension d, we have

$$\mathcal{P}_{X,s}(\phi,\eta)=\Delta_\phi:\phi^*[d](\mathcal{P}_{X,s}(U,u))=\phi^*[d](P^\bullet_{U,s(U,u)})\longrightarrow\mathcal{P}_{X,s}(T,t)=P^\bullet_{T,s(T,t)},$$

where Δ_ϕ is as in Proposition 6.3.1.

If we work with perverse sheaves on $\mathbb{K}$-schemes in the sense of [12] over a base ring A with either $\operatorname{char}A>0$ coprime to $\operatorname{char}\mathbb{K}$, or $A=\mathbb{Z}_l,\mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$ with l coprime to $\operatorname{char}\mathbb{K}$, then $\mathrm{Perv}_{\mathrm{naï}}(X)\simeq\mathrm{Perv}(X)$ as in [13, §4.4] and §1.3, where $\mathrm{Perv}(X)\subset D^b_c(X)$ is the category of perverse sheaves on X over A defined by Laszlo and Olsson [106–108]. Thus $\mathcal{P}_{X,s}$ corresponds to $\check{P}^\bullet_{X,s}\in\mathrm{Perv}(X)$ unique up to canonical isomorphism, and $\Sigma_{X,s},\mathrm{T}_{X,s}$ correspond to isomorphisms

$$\check{\Sigma}_{X,s}:\check{P}^\bullet_{X,s}\longrightarrow\mathbb{D}_X(\check{P}^\bullet_{X,s}),\quad \check{\mathrm{T}}_{X,s}:\check{P}^\bullet_{X,s}\longrightarrow\check{P}^\bullet_{X,s}\quad\text{in }\mathrm{Perv}(X).$$

The analogue will also hold in any other theory of perverse sheaves or $\mathscr{D}$-modules on schemes and Artin stacks with the package of properties discussed in §1.3 including the six operations $f^*,f^!,Rf_*,Rf_!,\mathcal{RH}om,\overset{L}{\otimes}$, Verdier duality $\mathbb{D}_X$, and descent in the smooth topology as in Theorem 1.3.5.

Proof. Proposition 6.3.1(b) implies that the data $\mathcal{P}_{X,s}(T,t),\mathcal{P}_{X,s}(\phi,\eta)$ in (a),(b) satisfy Definition 1.3.9(A)(i). Thus $\mathcal{P}_{X,s}$ is an object of $\mathrm{Perv}_{\mathrm{naï}}(X)$. Similarly, the last part of Proposition 6.3.1(a) implies that $\Sigma_{X,s},\mathrm{T}_{X,s}$ are morphisms in $\mathrm{Perv}_{\mathrm{naï}}(X)$. The last part is immediate from the discussion of [13, §4.3-4.5] and briefly recalled in §1.3. □

Combining Theorems 6.1.4, 6.2.1 and 6.3.2 and Corollary 6.2.2 yields:

Corollary 6.3.3. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $(\boldsymbol{X},\omega)$ a -1-shifted symplectic derived Artin $\mathbb{K}$-stack, and $X=t_0(\boldsymbol{X})$ the associated classical Artin $\mathbb{K}$-stack. Suppose we are given a square root $\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}$. Then working in l-adic perverse sheaves on* *stacks* [106–108], *we may define a perverse sheaf* $\check{P}^\bullet_{\boldsymbol{X},\omega}$ *on* X *uniquely up to canonical isomorphism, and Verdier duality and monodromy isomorphisms* $\check{\Sigma}_{\boldsymbol{X},\omega} : \check{P}^\bullet_{\boldsymbol{X},\omega} \to \mathbb{D}_X(\check{P}^\bullet_{\boldsymbol{X},\omega})$ *and* $\check{\mathrm{T}}_{\boldsymbol{X},\omega} : \check{P}^\bullet_{\boldsymbol{X},\omega} \to \check{P}^\bullet_{\boldsymbol{X},\omega}$. *These are characterized by the fact that given a diagram*

$$\boldsymbol{U} = \mathbf{Crit}(f : U \to \mathbb{A}^1) \xleftarrow{\ \boldsymbol{i}\ } \boldsymbol{V} \xrightarrow{\ \boldsymbol{\varphi}\ } \boldsymbol{X}$$

such that U *is a smooth* $\mathbb{K}$*-scheme,* $\boldsymbol{\varphi}$ *smooth of dimension* n, $\mathbb{L}_{\boldsymbol{V}/\boldsymbol{U}} \simeq \mathbb{T}_{\boldsymbol{V}/\boldsymbol{X}}[2]$, $\boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}}) \sim \boldsymbol{i}^*(\omega_{\boldsymbol{U}})$ *for* $\omega_{\boldsymbol{U}}$ *the natural* -1*-shifted symplectic structure on* $\boldsymbol{U} = \mathbf{Crit}(f : U \to \mathbb{A}^1)$, *and* $\varphi^*(\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}) \cong i^*(K_U) \otimes \Lambda^n \mathbb{T}_{\boldsymbol{V}/\boldsymbol{X}}$, *then* $\varphi^*(\check{P}^\bullet_{\boldsymbol{X},\omega})[n]$, $\varphi^*(\check{\Sigma}^\bullet_{\boldsymbol{X},\omega})[n]$, $\varphi^*(\check{\mathrm{T}}^\bullet_{\boldsymbol{X},\omega})[n]$ *are canonically isomorphic to* $i^*(\mathcal{PV}_{U,f})$, $i^*(\sigma_{U,f})$, $i^*(\tau_{U,f})$, *for* $\mathcal{PV}_{U,f}, \sigma_{U,f}, \tau_{U,f}$ *as in 1.3.1. The same applies in the other theories of perverse sheaves and* $\mathscr{D}$*-modules on stacks.*

The following will be discussed also in §7:

Corollary 6.3.4. *Let* Y *be a Calabi–Yau* 3*-fold over an algebraically closed field* $\mathbb{K}$ *of characteristic zero, and* $\mathcal{M}$ *a classical moduli* $\mathbb{K}$*-stack of coherent sheaves* F *in* $\mathrm{coh}(Y)$, *or of complexes* $F^\bullet$ *in* $D^b\,\mathrm{coh}(Y)$ *with* $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, *with obstruction theory* $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. *Suppose we are given a square root* $\det(\mathcal{E}^\bullet)^{1/2}$. *Then working in* l*-adic perverse sheaves on stacks* [106–108], *we may define a natural perverse sheaf* $\check{P}^\bullet_{\mathcal{M}} \in \mathrm{Perv}(\mathcal{M})$, *and Verdier duality and monodromy isomorphisms* $\check{\Sigma}_{\mathcal{M}} : \check{P}^\bullet_{\mathcal{M}} \to \mathbb{D}_{\mathcal{M}}(\check{P}^\bullet_{\mathcal{M}})$ *and* $\check{\mathrm{T}}_{\mathcal{M}} : \check{P}^\bullet_{\mathcal{M}} \to \check{P}^\bullet_{\mathcal{M}}$. *The pointwise Euler characteristic of* $\check{P}^\bullet_{\mathcal{M}}$ *is the Behrend function* $\nu_{\mathcal{M}}$ *of* $\mathcal{M}$ *from Joyce and Song* [85, §4], *so that* $\check{P}^\bullet_{\mathcal{M}}$ *is in effect a categorification of the Donaldson–Thomas theory of* $\mathcal{M}$. *The same applies in the other theories of perverse sheaves and* $\mathscr{D}$*-modules on stacks.*

For completeness, we conclude reporting an example [13, Ex. 4.15]:

Example 6.3.5. Suppose an algebraic $\mathbb{K}$-group G acts on a $\mathbb{K}$-scheme T with action $\mu : G \times T \to T$, and write X for the quotient Artin $\mathbb{K}$-stack $[T/G]$, and $t : T \to [T/G]$ for the natural quotient 1-morphism. As in Example 2.2.5, there is a 1-1 correspondence between d-critical structures s on $X = [T/G]$ and G-invariant d-critical structures s' on T, such that $s' = s(T,t)$. Also, from Lemma 2.2.8 we see that there is a 1-1 correspondence between orientations $K^{1/2}_{X,s}$ for (X,s), and G-invariant orientations $K^{1/2}_{T,s'}$ for (T,s'), given by $K^{1/2}_{T,s'} = K^{1/2}_{X,s}(T^{\mathrm{red}}, t^{\mathrm{red}}) \otimes (\Lambda^{\mathrm{top}}\mathbb{L}_{T/X})|_{T^{\mathrm{red}}}$. Choose such $s, s', K^{1/2}_{X,s}, K^{1/2}_{T,s'}$, so that Theorems 4.4.1 and 6.3.2 give perverse sheaves $P^\bullet_{T,s'}, \check{P}^\bullet_{X,s}$ on T, X.

We would like to relate the hypercohomologies $\mathbb{H}^*(T, P^\bullet_{T,s'})$ and $\mathbb{H}^*(X, \check{P}^\bullet_{X,s})$. We have that $t^*(\check{P}^\bullet_{X,s})[\dim G] \cong P^\bullet_{T,s'}$ and thus

$$R^q t_* P^\bullet_{T,s'} \cong R^q t_* t^*(\check{P}^\bullet_{X,s})[\dim G] \cong \check{P}^\bullet_{X,s} \otimes_{A_X} R^q t_*(A_T)[\dim G],$$

where A_T is the constant sheaf on T with fibre the base ring A. Therefore, the Leray–Serre spectral sequence for the fibration $t : T \to X$ with fibre G, twisted by $\check{P}^\bullet_{X,s}$, can be interpreted as a spectral sequence

$$E^{\bullet,\bullet} \Longrightarrow \mathbb{H}^\bullet(T, P^\bullet_{T,s'}) \quad \text{with} \quad E_2^{p,q} = \mathbb{H}^p\big(X, \check{P}^\bullet_{X,s} \otimes_{A_X} R^q t_*(A_T)[\dim G]\big),$$

where $R^q t_*(A_T)[\dim G]$ is locally constant on X with fibre $H^{q-\dim G}(G, A)$.

We also have a projection $\pi : X = [T/G] \to [*/G]$ for $* = \operatorname{Spec}\mathbb{K}$ with fibre T. The Leray–Serre spectral sequence for π gives a spectral sequence

$$E^{\bullet,\bullet} \Longrightarrow \mathbb{H}^\bullet(X, \check{P}^\bullet_{X,s}) \quad \text{with} \quad E_2^{p,q} = \mathbb{H}^p\big([*/G], \mathbb{H}^{q+\dim G}(T, P^\bullet_{T,s'})\big).$$

If G is finite we can consider the $\mathbb{H}^*(T, P^\bullet_{T,s'})$ as G-modules and $\mathbb{H}^*([*/G], -)$ as group cohomology $H^*_{\mathrm{grp}}(G, -)$, giving a spectral sequence

$$H^p_{\mathrm{grp}}\bigl(G, \mathbb{H}^q(T, P^\bullet_{T,s'})\bigr) \Longrightarrow \mathbb{H}^{p+q}(X, \check{P}^\bullet_{X,s}).$$

6.4 Motives on d-critical stacks

We start by recalling [13, Prop. 5.8], a result on smooth pullbacks and pushforwards of the motives $MF_{X,s}$ of Theorem 5.3.1, a motivic analogue of Proposition 6.3.1(a).

Proposition 6.4.1. *Let $\phi : (X, s) \to (Y, t)$ be a morphism of (finite type) algebraic d-critical loci in the sense of §2 and suppose $\phi : X \to Y$ is smooth of relative dimension n. Let $K^{1/2}_{Y,t}$ be an orientation for (Y, t), so that Corollary* 2.1.10 *defines an induced orientation $K^{1/2}_{X,s}$ for (X, s). Theorem 5.3.1 now defines motives $MF_{X,s}, MF_{Y,t}$ on X, Y. These are related by*

$$\phi^*\bigl(MF_{Y,t}\bigr) = \mathbb{L}^{n/2} \odot MF_{X,s} \in \bar{\mathcal{M}}^{\hat{\mu}}_X, \tag{6.4.1}$$

$$\phi_*\bigl(MF_{X,s}\bigr) = \mathbb{L}^{-n/2} \odot MF_{Y,t} \odot [X, \phi, \hat{\iota}] \in \bar{\mathcal{M}}^{\hat{\mu}}_Y. \tag{6.4.2}$$

Proof. If $x \in X$ with $\phi(x) = y \in Y$ then the proof of Proposition 2.1.4 in [87] shows we may choose critical charts $(R, U, f, i), (S, V, g, j)$ on $(X, s), (Y, t)$ with $x \in R$, $y \in \phi(R) \subseteq S$ of minimal dimensions $\dim U = \dim T_x X$, $\dim V = \dim T_y Y$, and $\Phi : U \to V$ smooth of relative dimension n with $f = g \circ \Phi$ and $\Phi \circ i = j \circ \phi$. Let $\pi : \tilde{V} \to V$ be an embedded resolution of singularities of g. Then $\tilde{U} := U \times_{\Phi,V,\pi} \tilde{V}$ is an embedded resolution of singularities of f, since Φ is smooth and $f = g \circ \Phi$. As in Definition 1.4.6, let F_i for $i \in J$ be the irreducible components of $\pi^{-1}(V_0)$, so that $\pi^{-1}(V_0) = \bigcup_{i \in J} F_i$, with multiplicities N_i in the divisor of $g \circ \pi$ on $\tilde{V}$, and $\nu_i - 1$ in the divisor of $\pi^*(\mathrm{d}x)$, and define $F^\circ_I = \bigl(\bigcap_{i \in I} F_i\bigr) \setminus \bigl(\bigcup_{j \in J \setminus I} F_j\bigr)$ and covers $\tilde{F}^\circ_I \to F^\circ_I$ for all $I \subseteq J$. Define $E_i = U \times_{\Phi,V,\pi|_{F_i}} F_i \subset \pi^{-1}(U_0) \subset \tilde{U}$. Then $\pi^{-1}(U_0) = \bigcup_{i \in J} E_i$. The E_i need not be irreducible, or nonempty, but this is not important. Neglecting this, we can treat the E_i, $i \in J$ as the components for $(\tilde{U}, \pi)$ in Definition 1.4.6, and then they have the same multiplicities N_i, ν_i as the F_i for $(\tilde{V}, \pi)$, and the $E^\circ_I, \tilde{E}^\circ_I$ for $I \subseteq J$ defined in Definition 1.4.6 satisfy $E^\circ_I \cong U \times_V F^\circ_I$ and $\tilde{E}^\circ_I \cong U \times_V \tilde{F}^\circ_I$. Thus we have

$$\begin{aligned} MF^{\mathrm{mot}}_{U,f} &= \sum_{\emptyset \neq I \subseteq J} (1 - \mathbb{L})^{|I|-1} \bigl[\tilde{E}^\circ_I, \pi_{U_0}, \hat{\rho}_I\bigr] = \sum_{\emptyset \neq I \subseteq J} (1 - \mathbb{L})^{|I|-1} \bigl[\tilde{F}^\circ_I \times_{\pi_{V_0}, V_0, \Phi|_{U_0}} U_0, \pi_{U_0}, \hat{\rho}_I\bigr] \\ &= \Phi|^*_{U_0}\Bigl[\sum_{\emptyset \neq I \subseteq J} (1 - \mathbb{L})^{|I|-1} \bigl[\tilde{F}^\circ_I, \pi_{V_0}, \hat{\rho}_I\bigr]\Bigr] = \Phi|^*_{U_0}\bigl(MF^{\mathrm{mot}}_{V,g}\bigr). \end{aligned}$$

So from (1.4.16) we deduce that

$$\Phi|^*_{\mathrm{Crit}(f)}(MF^{\mathrm{mot},\phi}_{V,g}) = \mathbb{L}^{n/2} \odot MF^{\mathrm{mot},\phi}_{U,f}, \tag{6.4.3}$$

using $\Phi|^*_{U_c}\bigl([V_c, \mathrm{id}_{V_c}, \hat{\iota}]\bigr) = [U_c, \mathrm{id}_{U_c}, \hat{\iota}]$, where the factor $\mathbb{L}^{n/2}$ is to convert the factor $\mathbb{L}^{-\dim U/2}$ in $MF^{\mathrm{mot},\phi}_{U,f}$ to the factor $\mathbb{L}^{-\dim V/2}$ in $MF^{\mathrm{mot},\phi}_{V,g}$. Combining (6.4.3) with Theorem 5.3.1 for $(X, s), (R, U, f, i)$ and the pullback of Theorem 5.3.1 for $(Y, t), (S, V, g, j)$ by $\phi|_R : R \to S$, and noting that $\phi^* \circ j^* = i^* \circ \Phi|^*_{\mathrm{Crit}(f)}$ since $j \circ \phi = \Phi \circ i$, we deduce the restriction of (6.4.1) to $R \subseteq X$. As we can cover X by such open R, this proves (6.4.1). Equation (6.4.2) follows by applying ϕ_* and noting that $\phi_* \circ \phi^*(M) = M \odot [X, \phi, \hat{\iota}]$ for all $\phi : X \to Y$ and $M \in \bar{\mathcal{M}}^{\hat{\mu}}_Y$. $\square$

Here is [13, Thm. 5.14], the analogue of Theorem 5.3.1.

Theorem 6.4.2. *Let* (X, s) *be an oriented d-critical stack, with orientation* $K_{X,s}^{1/2}$*, where* X *is assumed of finite type and locally a global quotient. Then there exists a unique motive* $MF_{X,s} \in \bar{\mathcal{M}}_X^{\mathrm{st},\hat{\mu}}$ *such that if* T *is a finite type* $\mathbb{K}$*-scheme and* $t : T \to X$ *is smooth of relative dimension* n*, so that* $(T, s(T, t))$ *is an algebraic d-critical locus over* $\mathbb{K}$ *with natural orientation* $K_{T,s(T,t)}^{1/2}$ *as in Lemma* 2.2.8*, then*

$$t^*\big(MF_{X,s}\big) = \mathbb{L}^{n/2} \odot MF_{T,s(T,t)} \quad \text{in } \bar{\mathcal{M}}_T^{\mathrm{st},\hat{\mu}}, \tag{6.4.4}$$

where $MF_{T,s(T,t)} \in \bar{\mathcal{M}}_T^{\mathrm{st},\hat{\mu}}$ *is as in Theorem* 5.3.1*, projected from* $\bar{\mathcal{M}}_T^{\hat{\mu}}$ *to* $\bar{\mathcal{M}}_T^{\mathrm{st},\hat{\mu}}$*, and* $t^* : \bar{\mathcal{M}}_X^{\mathrm{st},\hat{\mu}} \to \bar{\mathcal{M}}_T^{\mathrm{st},\hat{\mu}}$ *is the pullback.*

We refer to [13, Rem. 5.15] for a discussion about how to relax the assumptions in Theorem 6.4.2 that X is of *finite type*, and *locally a global quotient*. Combining Theorems 6.1.4, 6.2.1, 6.4.2 and Corollary 6.2.2, and noting as in §7 that moduli stacks of coherent sheaves are locally global quotients, yields [13, Cor.s 5.16 & 5.17], the analogue of Corollaries 5.3.2-5.3.3:

Corollary 6.4.3. *Let* $(\boldsymbol{X}, \omega)$ *be a* -1*-shifted symplectic derived Artin* $\mathbb{K}$*-stack in the sense of Pantev et al.* [142]*, and* $X = t_0(\boldsymbol{X})$ *the associated classical Artin* $\mathbb{K}$*-stack, assumed of finite type and locally a global quotient. Suppose we are given a square root* $\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}$ *for* $\det(\mathbb{L}_{\boldsymbol{X}})|_X$*. Then we may define a natural motive* $MF_{\boldsymbol{X},\omega} \in \bar{\mathcal{M}}_X^{\mathrm{st},\hat{\mu}}$*, which is characterized by the fact that given a diagram* $\boldsymbol{U} = \mathbf{Crit}(f : U \to \mathbb{A}^1) \xleftarrow{\boldsymbol{i}} \boldsymbol{V} \xrightarrow{\boldsymbol{\varphi}} \boldsymbol{X}$ *such that* U *is a smooth* $\mathbb{K}$*-scheme,* $\boldsymbol{\varphi}$ *is smooth of dimension* n*,* $\mathbb{L}_{\boldsymbol{V}/\boldsymbol{U}} \simeq \mathbb{T}_{\boldsymbol{V}/\boldsymbol{X}}[2]$*,* $\boldsymbol{\varphi}^*(\omega_{\boldsymbol{X}}) \sim \boldsymbol{i}^*(\omega_{\boldsymbol{U}})$ *for* $\omega_{\boldsymbol{U}}$ *the natural* -1*-shifted symplectic structure on* $\boldsymbol{U} = \mathbf{Crit}(f : U \to \mathbb{A}^1)$*, and* $\varphi^*(\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}) \cong i^*(K_U) \otimes \Lambda^n \mathbb{T}_{\boldsymbol{V}/\boldsymbol{X}}$*, then* $\varphi^*(MF_{\boldsymbol{X},\omega}) = \mathbb{L}^{n/2} \odot i^*(MF_{U,f}^{\mathrm{mot},\phi})$ *in* $\bar{\mathcal{M}}_V^{\mathrm{st},\hat{\mu}}$*.*

Corollary 6.4.4. *Let* Y *be a Calabi–Yau* 3*-fold over* $\mathbb{K}$*, and* $\mathcal{M}$ *a finite type classical moduli* $\mathbb{K}$*-stack of coherent sheaves in* $\mathrm{coh}(Y)$*, with natural obstruction theory* $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$*. Suppose we are given a square root* $\det(\mathcal{E}^\bullet)^{1/2}$ *for* $\det(\mathcal{E}^\bullet)$*. Then we may define a natural motive* $MF_{\mathcal{M}} \in \bar{\mathcal{M}}_{\mathcal{M}}^{\mathrm{st},\hat{\mu}}$*.*

Corollary 6.4.4 is relevant to Kontsevich and Soibelman's theory of *motivic Donaldson–Thomas invariants* [102]. Our square root $\det(\mathcal{E}^\bullet)^{1/2}$ roughly coincides with their *orientation data* [102, §5]. In [102, §6.2], given a finite type moduli stack $\mathcal{M}$ of coherent sheaves on a Calabi–Yau 3-fold Y with orientation data, they define a motive $\int_{\mathcal{M}} 1$ in a ring D^μ isomorphic to our $\bar{\mathcal{M}}_{\mathbb{K}}^{\mathrm{st},\hat{\mu}}$. We expect this should agree with $\pi_*(MF_{\mathcal{M}})$ in our notation, with $\pi : \mathcal{M} \to \operatorname{Spec} \mathbb{K}$ the projection. This $\int_{\mathcal{M}} 1$ is roughly the motivic Donaldson–Thomas invariant of $\mathcal{M}$. Their construction involves expressing $\mathcal{M}$ near each point in terms of the critical locus of a formal power series. Kontsevich and Soibelman's constructions were partly conjectural, and our results may fill some gaps in their theory. See also §7. We will not give the proof of Theorem 6.4.2 here, which can be found in [13, §5.5]. We conclude with [13, Ex. 5.18]:

Example 6.4.5. As in [86, Def. 2.1], an algebraic $\mathbb{K}$-group G is called *special* if every étale locally trivial principal G-bundle over a $\mathbb{K}$-scheme is Zariski locally trivial. Any special $\mathbb{K}$-group can be embedded as a closed $\mathbb{K}$-subgroup $G \subseteq \mathrm{GL}(n, \mathbb{K})$, and then $\mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})/G$ is a Zariski locally trivial principal G-bundle, so taking motives in $\mathcal{M}_{\mathbb{K}}^{\mathrm{stk}}$ gives $[\mathrm{GL}(n, \mathbb{K})] = [G] \cdot [\mathrm{GL}(n, \mathbb{K})/G]$. Hence $[G]$ is invertible in $\mathcal{M}_{\mathbb{K}}^{\mathrm{stk}}$, with $[G]^{-1} = [\mathrm{GL}(n, \mathbb{K})/G] \cdot [\mathrm{GL}(n, \mathbb{K})]^{-1}$.

Some examples of special $\mathbb{K}$-groups are $\mathbb{G}_m, \mathrm{GL}(n, \mathbb{K}), \mathrm{SL}(n, \mathbb{K}), \mathrm{Sp}(2n, \mathbb{K})$, and the group of invertible elements $A^\times$ of any finite-dimensional $\mathbb{K}$-algebra A. Products of special groups are special. Special $\mathbb{K}$-groups are always affine and connected, so nontrivial finite groups are not special.

Suppose a special $\mathbb{K}$-group G of dimension n acts on a finite type, oriented algebraic d-critical locus (T,s') over $\mathbb{K}$ preserving $s'\in H^0(\mathcal{S}_T^0)$ and the orientation $K_{T,s'}^{1/2}$. Write $X=[T/G]$ for the quotient stack and $t:T\to X$ for the projection. Then s' descends to a unique d-critical structure s on X with $s'=s(T,t)$ as in Example 2.2.5, and using Theorem 2.2.6 we also find that the orientation $K_{T,s'}^{1/2}$ descends to a unique orientation $K_{X,s}^{1/2}$ on the d-critical stack (X,s) with $K_{X,s}^{1/2}(T^{\mathrm{red}},t^{\mathrm{red}})\cong K_{T,s'}^{1/2}\otimes\bigl(\Lambda^{\mathrm{top}}T_{T/X}^*\bigr)\big|_{T^{\mathrm{red}}}^{\otimes^{-1}}$. Theorem 6.4.2 gives $MF_{X,s}\in\bar{\mathcal{M}}_X^{\mathrm{st},\hat\mu}$ with $t^*\bigl(MF_{X,s}\bigr)=\mathbb{L}^{n/2}\odot MF_{T,s'}$ in $\bar{\mathcal{M}}_T^{\mathrm{st},\hat\mu}$. Applying t_* and using $t_*\circ t^*(M)=[T,t,\hat\imath]\odot M$ for $M\in\bar{\mathcal{M}}_X^{\mathrm{st},\hat\mu}$ gives

$$MF_{X,s}\odot[T,t,\hat\imath]=\mathbb{L}^{n/2}\odot t_*(MF_{T,s'}).\tag{6.4.5}$$

Now $t:T\to X$ is a principal G-bundle, and so Zariski locally trivial as G is special. Therefore $[T,t,\hat\imath]=[G,\hat\imath]\boxdot 1_X$, where $[G,\hat\imath]=i_{\mathbb{K}}([G])\in\mathcal{M}_{\mathbb{K}}^{\mathrm{st},\hat\mu}$. As $[G]$ is invertible, so is $[G,\hat\imath]$. Thus multiplying (6.4.5) by $[G,\hat\imath]^{-1}$ gives $MF_{X,s}=[G,\hat\imath]^{-1}\boxdot\bigl(\mathbb{L}^{n/2}\odot t_*(MF_{T,s'})\bigr)$.

Chapter 7

Generalizations of Donaldson–Thomas theory

Generalized Donaldson–Thomas invariants $\bar{DT}^{\alpha}(\tau)$ defined by Joyce and Song [85] are rational numbers which 'count' both τ-stable and τ-semistable coherent sheaves with Chern character α on a Calabi–Yau 3-fold X, where τ denotes Gieseker stability for some ample line bundle on X. The $\bar{DT}^{\alpha}(\tau)$ are defined for all classes α, and are equal to the classical $DT^{\alpha}(\tau)$ defined by Thomas [167] when it is defined. They are unchanged under deformations of X, and transform by a wall-crossing formula under change of stability condition τ. Joyce and Song use gauge theory and transcendental complex analytic methods, so that their theory of generalized Donaldson–Thomas invariants is valid only in the complex case. This also forces them to put constraints on the Calabi–Yau 3-fold they can define generalized Donaldson–Thomas invariants for. We will review their theory in §7.1.

We will propose a new algebraic method extending the theory to algebraically closed fields $\mathbb{K}$ of characteristic zero, and partly to triangulated categories and for non necessarily compact Calabi–Yau 3-folds under some hypothesis.

We will use results discussed in §2–§6 to describe the local structure of the moduli stack $\mathfrak{M}$ of (complexes of) coherent sheaves on X, showing that an atlas for $\mathfrak{M}$ carries the structure of a $\mathrm{GL}(n,\mathbb{K})$-invariant d-critical locus in the sense of [87] and thus it may be written locally as the zero locus of a regular function defined on an étale neighborhood in the tangent space of $\mathfrak{M}$ and use this to deduce identities on the Behrend function $\nu_{\mathfrak{M}}$.

Moreover, when $\mathbb{K}=\mathbb{C}$, [85, Thm. 4.9] uses the integral Hodge conjecture result by Voisin for Calabi–Yau 3-folds over $\mathbb{C}$ to show that the numerical Grothendieck group $K^{\mathrm{num}}(\mathrm{coh}(X))$ is unchanged under deformations of X. This is important for the results that $\bar{DT}^{\alpha}(\tau)$ for $\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))$ are invariant under deformations of X, even to make sense. We will provide an algebraic proof of that result, characterizing the numerical Grothendieck group of a Calabi–Yau 3-fold in terms of a globally constant lattice described using the Picard scheme.

7.1 Donaldson–Thomas theory: background material

This section should be conceived as background picture in which next sections should be allocated. The competent reader can skip directly to §7.2.

7.1.1 Obstruction theories and Donaldson–Thomas type invariants

This section will briefly recall material from [6], [114] and then [167] which provide both important notions used in the sequel and a hopefully interesting picture of Donaldson–Thomas theory.

Obstruction theories

Suppose that X is a subscheme of a smooth scheme M, cut out by a section s of a rank r vector bundle $E \to M$. Then the *expected dimension*, or virtual dimension, of X is $n-r$, the dimension it would have if the section s was transverse. If it is not transverse, one wants to take a correct $(n-r)$-cycle on X. As the section s induces a cone in $E_{|X}$, one may then intersect this cone with the zero section of X inside E to get a cycle of expected dimension on X. The key observation is that one works entirely on X and not in the ambient scheme M. The deformation theory of the moduli problem is often endowed with the infinitesimal version of $s : M \to E$ on X, namely the linearization of s, yielding the exact sequence $0 \longrightarrow TX \longrightarrow TM_{|X} \xrightarrow{\mathrm{d}s} E_{|X} \longrightarrow Ob \longrightarrow 0$, for some cokernel Ob, which in the moduli problem becomes the *obstruction sheaf*.

Moduli spaces in algebraic geometry often have an expected dimension at each point, which is a lower bound for the dimension at that point. Sometimes it may not coincide with the actual dimension of the moduli space and sometimes it is not possible to get a space of the expected dimension. When one has a moduli space X one obtains *numerical invariants* by integrating certain cohomology classes over the virtual moduli cycle, a class of the expected dimension in its Chow ring.

One example is the moduli space of torsion-free, semi-stable vector bundles on a surface which yields the *Donaldson theory* and which provides a set of differential invariants of 4-manifolds. Another one is the moduli space of stable maps from curves of genus g to a fixed projective variety which yields the *Gromov–Witten invariants*, a kind of generalization of the classical enumerative invariant which counts the number of algebraic curves with appropriate constraints in a variety. In both cases, these invariants are intersection theories on the moduli spaces, respectively, of vector bundles over the surfaces, and of stable maps from curves to a variety. The fundamental class is the core of an intersection theory. However, for Gromov–Witten invariants for example, one cannot take the fundamental class of the whole moduli space directly. The virtual moduli cycle, roughly speaking, plays the role of the fundamental class in an appropriate "good" intersection theory.

A nice picture to start with is the following situation: when the expected dimension does not coincide with the actual dimension of the moduli space, one may view this as if the moduli space is a subspace of an 'ambient' space cut out by a set of 'equations' whose vanishing loci do not meet transversely. Such a situation is well understood in the following setting described in the Introduction of [114]: let X, Y and W be smooth varieties, $X, Y \to W$ and let $Z = X \times_W Y$. Then $[X] \cdot [Y]$, the intersection of the cycle $[X]$ and $[Y]$, is a cycle in A_*W of dimension $\dim X + \dim Y - \dim W$. When $\dim Z = \dim X + \dim Y - \dim W$, then $[Z] = [X] \cdot [Y]$. Otherwise, $[Z]$ may not be $[X] \cdot [Y]$. The *excess intersection theory* gives that one can find a cycle in A_*Z so that it is $[X] \cdot [Y]$. One may view this cycle as the virtual cycle of Z representing $[X] \cdot [Y]$. Following Fulton–MacPherson's normal cone construction (in [46–48]), this cycle is the image of the cycle of the normal cone to Z in X, denoted by $C_{Z/X}$, under the Gysin homomorphism $s^* : A_*(C_{Y/W} \times_Y Z) \to A_*Z$, where $s : Z \to C_{Y/W} \times_Y Z$ is the zero section. This theory does not apply directly to moduli schemes, since, except for some isolated cases, it is impossible to find pairs $X \to W$ and $Y \to W$ for smooth X, Y and W so that $X \times_W Y$ is the moduli space and $[X] \cdot [Y]$ so defined is the virtual moduli cycle one needs.

Behrend and Fantechi [6] and Li and Tian [114] give two different approaches to deal with this. Very briefly, the strategy to Li and Tian's approach in [114] is that rather than trying to find an embedding of the moduli space into some ambient space, they will construct a cone in a vector bundle directly, say $C \subset V$, over the moduli space and then define the virtual moduli cycle to be $s^*[C]$, where s is the zero section of V. The pair $C \subset V$ will be constructed based on a choice of the *tangent-obstruction complex* of the moduli functor. The construction commutes with Gysin maps and carries a good invariance property.

In [6] Behrend and Fantechi introduce the notion of *cone stacks* over a scheme X (or more generally for Deligne–Mumford stacks). These are Artin stacks which are locally the quotient of a cone by a vector bundle acting on it. They call a cone *abelian* if it is defined as $\operatorname{Spec}\operatorname{Sym}\mathscr{F}$, where $\mathscr{F}$ is a coherent sheaf on X. Every cone is contained as a closed subcone in a minimal abelian one, which is called its *abelian hull*. The notions of being abelian and of abelian hull generalize immediately to cone stacks. Then, for a complex $E^\bullet$ in the derived category $D(X)$ of quasicoherent sheaves on X which satisfies some suitable assumptions (denoted by $(*)$, see Definition 7.1.1), there is an associated abelian cone stack $h^1/h^0((E^\bullet)^\vee)$. In particular the *cotangent complex* $L_X^\bullet$ of X constructed by Illusie [72] (a helpful review is given in Illusie [73, §1]) satisfies condition $(*)$, so one can define the abelian cone stack $\mathfrak{N}_X := h^1/h^0((L_X^\bullet)^\vee)$, the *intrinsic normal sheaf.* More directly, $\mathfrak{N}_X$ is constructed as follows: étale locally on X, embed an open set U of X in a smooth scheme W, and take the stack quotient of the *normal sheaf* (viewed as abelian cone) $N_{U/W}$ by the natural action of $TW_{|_U}$. One can glue these abelian cone stacks together to get $\mathfrak{N}_X$. The *intrinsic normal cone* $\mathfrak{C}_X$ is the closed subcone stack of $\mathfrak{N}_X$ defined by replacing $N_{U/W}$ by the *normal cone* $C_{U/W}$ in the previous construction. In particular, the intrinsic normal sheaf $\mathfrak{N}_X$ of X carries the obstructions for deformations of affine X-schemes. With this motivation, they introduce the notion of *obstruction theory* for X. To say that a scheme X has an *obstruction theory* means, very roughly speaking, that one is endowed with a complex of vector bundles encoding informations on the deformations and obstructions spaces of X. That is, this is an object $E^\bullet$ in the derived category together with a morphism $E^\bullet \to L_X^\bullet$, satisfying Condition $(*)$ and such that the induced map $\mathfrak{N}_X \to h^1/h^0((E^\bullet)^\vee)$ is a closed immersion. One denotes the sheaf $h^1(E^{\bullet\vee})$ by Ob, the obstruction sheaf of the obstruction theory. It contains the obstructions to the smoothness of X. When an obstruction theory $E^\bullet$ is *perfect*, $\mathfrak{E} = h^1/h^0((E^\bullet)^\vee)$ is a vector bundle stack. Once an obstruction theory is given, with the additional technical assumption that it admits a global resolution, one can define a virtual fundamental class of the expected dimension: one has a vector bundle stack $\mathfrak{E}$ with a closed subcone stack $\mathfrak{C}_X$, and to define the virtual fundamental class of X with respect to $E^\bullet$ one simply intersects $\mathfrak{C}_{\mathfrak{X}}$ with the zero section of $\mathfrak{E}$. To get round of the problem of dealing with Chow groups for Artin stacks, Behrend and Fantechi choose to assume that $E^\bullet$ is globally given by a homomorphism of vector bundles $F^{-1} \to F^0$. Then $\mathfrak{C}_X$ gives rise to a cone C in $F_1 = F^{-1\vee}$ and one intersects C with the zero section of F_1 (see [105] for a statement without this assumption).

So, recall the following definitions from Behrend and Fantechi [5–7]:

Definition 7.1.1. Let $D(Y)$ be the derived category of quasicoherent sheaves on a $\mathbb{K}$-scheme Y.

(a) A complex $E^\bullet \in D(Y)$ is *perfect of perfect amplitude contained in* $[a,b]$, if étale locally on Y, $E^\bullet$ is quasi-isomorphic to a complex of locally free sheaves of finite rank in degrees $a, a+1, \ldots, b$.

(b) A complex $E^\bullet \in D(Y)$ *satisfies condition* $(*)$ if

(i) $h^i(E^\bullet) = 0$ for all $i > 0$,

(ii) $h^i(E^\bullet)$ is coherent for $i = 0, -1$.

(c) An *obstruction theory* for Y is a morphism $\varphi : E^\bullet \to L_Y$ in $D(Y)$, where $L_Y = L_{Y/\operatorname{Spec}\mathbb{K}}$ is the cotangent complex of Y, and E satisfies condition $(*)$, and $h^0(\varphi)$ is an isomorphism, and $h^{-1}(\varphi)$ is an epimorphism.

(d) An obstruction theory $\varphi : E^\bullet \to L_Y$ is called *perfect* if $E^\bullet$ is perfect of perfect amplitude contained in $[-1,0]$.

(e) A perfect obstruction theory $\varphi : E^\bullet \to L_Y$ on Y is called *symmetric* if there exists an isomorphism $\vartheta : E^\bullet \to E^{\bullet\vee}[1]$, such that $\vartheta^\vee[1] = \vartheta$. Here $E^{\bullet\vee} = R\mathcal{H}om\,(E^\bullet, \mathcal{O}_Y)$ is the *dual* of $E^\bullet$, and $\vartheta^\vee$ the dual morphism of ϑ.

(f) If moreover Y is a scheme with a G-action, where G is an algebraic group, an *equivariant* perfect obstruction theory is a morphism $E^\bullet \to L_Y$ in the category $D(Y)^G$, which is a perfect obstruction theory as a morphism in $D(Y)$ (this definition is originally due to Graber–Pandharipande [57]). Here $D(Y)^G$ denotes the derived category of the abelian category of G-equivariant quasicoherent $\mathcal{O}_Y$-modules.

(g) A *symmetric equivariant* obstruction theory (or an *equivariant symmetric* obstruction theory) is a pair $(E^\bullet \to L_Y, E^\bullet \to E^{\bullet\vee}[1])$ of morphisms in the category $D(Y)^G$, such that $E^\bullet \to L_Y$ is an equivariant perfect obstruction theory and $\vartheta : E^\bullet \to E^{\bullet\vee}[1]$ is an isomorphism satisfying $\vartheta^\vee[1] = \vartheta$ in $D(Y)^G$. Note that this is more than requiring that the obstruction theory be equivariant and symmetric, separately, as said in [7].

If instead $Y \xrightarrow{\psi} U$ is a morphism of $\mathbb{K}$-schemes, so Y is a U-scheme, we define *relative* perfect obstruction theories $\phi : E^\bullet \to L_{Y/U}$ in the obvious way.

Behrend and Fantechi [6, Th. 4.5] prove the following theorem, which both explains the term obstruction theory and provides a criterion for verification in practice:

Theorem 7.1.2. *The following conditions are equivalent for $E^\bullet \in D(Y)$ satisfying condition $(*)$.*

(a) *The morphism $\phi : E^\bullet \to L_Y$ is an obstruction theory.*

(b) *Suppose that we are given a square-zero extension $\overline{T}$ of T with ideal sheaf J, with $T, \overline{T}$ affine, and a morphism $g : T \to Y$. The morphism ϕ induces an element $\phi^*(\omega(g)) \in \mathrm{Ext}^1(g^*E^\bullet, J)$ from $\omega(g) \in \mathrm{Ext}^1(g^*L_Y, J)$ by composition. Then $\phi^*(\omega(g))$ vanishes if and only if there exists an extension $\overline{g}$ of g. If it vanishes, then the set of extensions form a torsor under* $\mathrm{Hom}(g^*E^\bullet, J)$.

Some examples can be found in [7]: Lagrangian intersections, sheaves on Calabi–Yau 3-folds, stable maps to Calabi–Yau 3-folds. Next section will concentrate on Donaldson–Thomas obstruction theory as in [167].

Donaldson–Thomas invariants of Calabi–Yau 3-folds

Donaldson–Thomas invariants $DT^\alpha(\tau)$ are the virtual counts of stable sheaves on Calabi–Yau 3-folds X. They were defined by Richard Thomas [167], following a proposal of Donaldson and Thomas [36, §3], from the idea of defining an *holomorphic analogue* of the classical Casson invariant.

More precisely, mathematically, Donaldson–Thomas invariants are constructed as follows. Deformation theory gives rise to a perfect obstruction theory [6] (or a tangent-obstruction complex in the language of [114]) on the moduli space of stable sheaves $\mathcal{M}^\alpha_{\rm st}(\tau)$. Recall that Thomas supposes $\mathcal{M}^\alpha_{\rm st}(\tau) = \mathcal{M}^\alpha_{\rm ss}(\tau)$, that is, there are no strictly semistable sheaves E in class α, which implies the properness of $\mathcal{M}^\alpha_{\rm st}(\tau)$. As Thomas points out in [167], the obstruction sheaf is equal to $\Omega_{\mathcal{M}^\alpha_{\rm st}(\tau)}$, the sheaf of Kähler differentials, and hence the tangents $T_{\mathcal{M}^\alpha_{\rm st}(\tau)}$ are dual to the obstructions. This expresses a certain symmetry of the obstruction theory on $\mathcal{M}^\alpha_{\rm st}(\tau)$ and is a mathematical reflection of the heuristic that views $\mathcal{M}^\alpha_{\rm st}(\tau)$ as the critical locus of a holomorphic functional (the holomorphic Chern-Simons functional). Associated to the perfect obstruction theory is the virtual fundamental class, an element of the Chow group $A_*(\mathcal{M}^\alpha_{\rm st}(\tau))$ of algebraic cycles modulo rational equivalence on $\mathcal{M}^\alpha_{\rm st}(\tau)$. One implication of the symmetry of the obstruction theory is the fact that the virtual fundamental class $[\mathcal{M}_{\rm st}^\alpha(\tau)]^{\rm vir}$ is of degree zero. It can hence be integrated over the proper space of stable sheaves to an integer, the Donaldson–Thomas invariant or 'virtual count' of $\mathcal{M}_{\rm st}^\alpha(\tau)$

$$DT^\alpha(\tau) \quad = \int_{[\mathcal{M}_{\rm st}^\alpha(\tau)]^{\rm vir}} 1. \tag{7.1.1}$$

In fact Thomas did not define invariants $DT^\alpha(\tau)$ counting sheaves with fixed class $\alpha \in K^{\rm num}(\text{coh}(X))$, but coarser invariants $DT^P(\tau)$ counting sheaves with fixed Hilbert polynomial $P(t) \in \mathbb{Q}[t]$. Thus

$$\mathcal{M}_{\rm ss}^P(\tau) \;=\; \coprod_{\alpha:P_\alpha=P} \mathcal{M}_{\rm ss}^\alpha(\tau) \quad \rightsquigarrow \quad DT^P(\tau) \;=\; \sum_{\alpha\in K^{\rm num}(\text{coh}(X)):P_\alpha=P} DT^\alpha(\tau),$$

is the relationship with Joyce and Song's version $DT^\alpha(\tau)$ reviewed in §7.1.3, where the r.h.s. has only finitely many nonzero terms in the sum. Here, Thomas' main result [167, §3]:

Theorem 7.1.3. *For each Hilbert polynomial $P(t)$, the invariant $DT^P(\tau)$ is unchanged by continuous deformations of the underlying Calabi–Yau 3-fold X over $\mathbb{K}$.*

The same proof shows that $DT^\alpha(\tau)$ for $\alpha \in K^{\rm num}(\text{coh}(X))$ is deformation-invariant, *provided* it is known that the group $K^{\rm num}(\text{coh}(X))$ is deformation-invariant, so that this statement makes sense. This issue is discussed in [85, §4.5]. There, it is shown that when $\mathbb{K} = \mathbb{C}$ one can describe $K^{\rm num}(\text{coh}(X))$ in terms of cohomology groups $H^*(X;\mathbb{Z})$, $H^*(X;\mathbb{Q})$, so that $K^{\rm num}(\text{coh}(X))$ is manifestly deformation-invariant, and therefore $DT^\alpha(\tau)$ is also deformation-invariant. Theorem [85, Thm. 4.19] crucially uses the integral Hodge conjecture result by [182] for Calabi–Yau 3-folds over $\mathbb{C}$. In [85, Rmk 4.20(e)], Joyce and Song propose to extend that description over an algebraically closed base field $\mathbb{K}$ of characteristic zero by replacing $H^*(X;\mathbb{Q})$ by the *algebraic de Rham cohomology* $H^*_{\rm dR}(X)$ of Hartshorne [64]. For X a smooth projective $\mathbb{K}$-scheme, $H^*_{\rm dR}(X)$ is a finite-dimensional vector space over $\mathbb{K}$. There is a Chern character map $\text{ch} : K^{\rm num}(\text{coh}(X)) \hookrightarrow H^{\rm even}_{\rm dR}(X)$. In [64, §4], Hartshorne considers how $H^*_{\rm dR}(X_t)$ varies in families $X_t : t \in T$, and defines a *Gauss–Manin connection*, which makes sense of $H^*_{\rm dR}(X_t)$ being locally constant in t. In §7.2.3 we will use another idea to characterize the numerical Grothendieck group of a Calabi–Yau 3-fold in terms of a globally constant lattice described using the Picard scheme.

Next section will introduce the Behrend function and the work done by Behrend in [5], which has been crucial for the development of Donaldson–Thomas theory.

7.1.2 Microlocal geometry and the Behrend function

This section briefly explains Behrend's approach [5] to Donaldson–Thomas invariants as Euler characteristics of moduli schemes weighted by the Behrend function. It was introduced by Behrend [5] for finite type $\mathbb{C}$-schemes X; in [85, §4.1] it has been generalized to Artin $\mathbb{K}$-stacks. Behrend functions are also defined for complex analytic spaces $X_{\rm an}$, and the Behrend function of a $\mathbb{C}$-scheme X coincides with that of the underlying complex analytic space $X_{\rm an}$. The theory is also valid for $\mathbb{K}$-schemes acted on by a reductive linear algebraic group. A good reference for this section, other than the original paper by Behrend [5], are [85, §4] and [138] for the equivariant version. We point out here that a detailed discussion about the Behrend function can be found also in [26].

Microlocal approach to the Behrend function

In [5], Behrend suggests a microlocal approach to the problem. The first part of the discussion describes how the Behrend function is defined while the second part, although not detailed and not directly involved in the rest of the paper, aim to give a more complete picture.

The definition of the Behrend function. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and X a finite type $\mathbb{K}$-scheme. Suppose $X \hookrightarrow M$ is an embedding of X as a closed subscheme of a smooth $\mathbb{K}$-scheme M. Then one has a commutative diagram

$$\begin{tikzcd} Z_*(X) \arrow[r, "\mathrm{Eu}", "\cong"'] \arrow[dr, "c_0^M"'] & \mathrm{CF}_{\mathbb{Z}}(X) \arrow[r, "\mathrm{Ch}", "\cong"'] \arrow[d, "c_0^{SM}"] & \mathcal{L}_X(M) \arrow[dl, "0^!"] \\ & A_0(X) & \end{tikzcd} \tag{7.1.2}$$

where the two horizontal arrows are isomorphisms. Here $Z_*(X)$ denotes the group of *algebraic cycles* on X, as in Fulton [46], and $\mathrm{CF}_{\mathbb{Z}}(X)$ the group of $\mathbb{Z}$-valued constructible functions on X in the sense of [75]. The *local Euler obstruction* is a group isomorphism $\mathrm{Eu} : Z_*(X) \to \mathrm{CF}_{\mathbb{Z}}(X)$. The local Euler obstruction was first defined by MacPherson [124] to solve the problem of existence of covariantly functorial Chern classes, answering thus a Deligne–Grothendieck conjecture when $\mathbb{K} = \mathbb{C}$, using complex analysis, but Gonzalez–Sprinberg [56] provides an alternative algebraic definition which works over any algebraically closed field $\mathbb{K}$ of characteristic zero. It is the obstruction to extending a certain section of the tautological bundle on the *Nash blowup*. More precisely, if V is a prime cycle on X, the constructible function $\mathrm{Eu}(V)$ is given by

$$\mathrm{Eu}(V) : x \longmapsto \int_{\mu^{-1}(x)} c(\tilde{T}) \cap s(\mu^{-1}(x), \tilde{V}),$$

where $\mu : \tilde{V} \to V$ is the Nash blowup of V, $\tilde{T}$ the dual of the universal quotient bundle, c the total Chern class and s the Segre class of the normal cone to a closed immersion. Kennedy [95, Lem. 4] proves that $\mathrm{Eu}(V)$ is constructible.

As pointed out in the next section, it is worth observing that independently, at about the same time, Kashiwara proved an *index theorem* over $\mathbb{C}$ for a holonomic $\mathcal{D}$-module relating its local Euler characteristic and the local Euler obstruction with respect to an appropriate stratification (see [54] for details). It coincides with the one defined above and this is equivalent to saying that the diagram (7.1.4) below commutes.

Observe that this part of the diagram exists without the embedding into M and is sufficient to give the definition of the Behrend function as follow. Let $C_{X/M}$ be the *normal cone* of X in M, as in [46, p.73], and $\pi : C_{X/M} \to X$ the projection. As in [5, §1.1], define a cycle $\mathfrak{C}_{X/M} \in Z_*(X)$ by

$$\mathfrak{C}_{X/M} = \sum_{C'} (-1)^{\dim \pi(C')} \mathrm{mult}(C') \pi(C'),$$

where the sum is over all irreducible components C' of $C_{X/M}$. It turns out that $\mathfrak{C}_{X/M}$ depends only on X, and not on the embedding $X \hookrightarrow M$. Behrend [5, Prop. 1.1] proves that given a finite type $\mathbb{K}$-scheme X, there exists a unique cycle $\mathfrak{C}_X \in Z_*(X)$, such that for any étale map $\varphi : U \to X$ for a $\mathbb{K}$-scheme U and any closed embedding $U \hookrightarrow M$ into a smooth $\mathbb{K}$-scheme M, one has $\varphi^*(\mathfrak{C}_X) = \mathfrak{C}_{U/M}$ in $Z_*(U)$. If X is a subscheme of a smooth M one takes $U = X$ and get $\mathfrak{C}_X = \mathfrak{C}_{X/M}$. Behrend calls $\mathfrak{C}_X$ the *signed support of the intrinsic normal cone*, or the *distinguished cycle* of X. For each finite type $\mathbb{K}$-scheme X, define the *Behrend function* ν_X in $\mathrm{CF}_{\mathbb{Z}}(X)$ by $\nu_X = \mathrm{Eu}(\mathfrak{C}_X)$, as in Behrend [5, §1.2].

For completeness, the section now describes the other side of the diagram (7.1.2), which yields another possible way to define the Behrend function. Write $\mathcal{L}_X(M)$ for the free abelian group generated by closed, irreducible, reduced, conical Lagrangian, $\mathbb{K}$-subvariety in Ω_M lying over cycles contained in X. The isomorphism $\mathrm{Ch} : \mathrm{CF}_{\mathbb{Z}}(X) \to \mathcal{L}_X(M)$ maps a constructible function to its characteristic cycle, which is a conic Lagrangian cycle on Ω_M supported inside X defined in the following way. Consider the commutative diagram of group isomorphisms that fits in the diagram (7.1.2):

$$\begin{tikzcd} Z_*(M) \arrow[r, "\mathrm{Eu}"] \arrow[rr, bend right, "L"'] & \mathrm{CF}_{\mathbb{Z}}(M) \arrow[r, "\mathrm{Ch}"] & \mathcal{L}(M). \end{tikzcd} \qquad (7.1.3)$$

Here $L : Z_*(M) \to \mathcal{L}(M)$ is defined on any prime cycle V by $L : V \to (-1)^{\dim(V)}\ell(V)$, where $\ell(V)$ is the closure of the conormal bundle of any nonsingular dense open subset of V. Then Eu, L are isomorphisms, and the *characteristic cycle map* $\mathrm{Ch} : \mathrm{CF}_{\mathbb{Z}}(M) \to \mathcal{L}(M) \subset Z_{\dim M}(\Omega_M)$ is defined to be the unique isomorphism making (7.1.3) commute. In the complex case Ginsburg [54] describes the inverse of this map as *intersection multiplicity* between two conical Lagrangian cycles. This formula is crucial in [5, §4.3], where Behrend gives an expression for the Behrend function in terms of linking numbers, which has a validity also in the case it is not known if a scheme admitting a symmetric obstruction theory can locally be written as the critical locus of a regular function on a smooth scheme (Theorem 7.1.10). See also [46, Ex. 19.2.4].

The maps to $A_0(X)$ are the degree zero *Chern-Mather class*, the degree zero *Schwartz-MacPherson Chern class*, and the intersection with the zero section, respectively. The Mather class is a homomorphism $c^M : Z_*(X) \to A_*(X)$, whose definition is a globalization of the construction of the local Euler obstruction. One has $c^M(V) = \mu_*\big(c(\widetilde{T}) \cap [\widetilde{V}]\big)$, for a prime cycle V of degree p on X with the same notation as above. For a the expression in terms of normal cones, see for example [148, §1]. Applying c^M to the cycle $\mathfrak{C}_X$, one obtains the *Aluffi class* $\alpha_X = c^M(\mathfrak{C}_X) \in A_*(X)$ defined in [1]. If X is smooth, its Aluffi class equals $\alpha_X = c(\Omega_X) \cap [X]$.

Now given a symmetric obstruction theory on X, the *cone of curvilinear obstructions* $cv \hookrightarrow ob = \Omega_X$, pulls back to a cone in $\Omega_{M_{|X}}$ via the epimorphism $\Omega_{M_{|X}} \to \Omega_X$. Via the embedding $\Omega_{M_{|X}} \hookrightarrow \Omega_M$ one obtains a conic subscheme $C \hookrightarrow \Omega_M$, the *obstruction cone* for the embedding $X \hookrightarrow M$. Behrend proves that the virtual fundamental class is $[X]^{\mathrm{vir}} = 0^![C]$. The key fact is that C is *Lagrangian*. Because of this, there exists a unique constructible function ν_X on X such that $\mathrm{Ch}(\nu_X) = [C]$ and $c_0^{SM}(\nu_X) = [X]^{\mathrm{vir}}$. Then Theorem 7.1.7 below follows as an application of MacPherson's theorem [124] (or equivalently from the microlocal index theorem of Kashiwara [90]), which one can think of as a kind of generalization of the *Gauss–Bonnet theorem* to singular schemes. See Theorem 7.1.7 below for its validity over $\mathbb{K}$. The cycle $\mathfrak{C}_X$ such that $\mathrm{Eu}(\mathfrak{C}_X) = \nu_X$ is as defined above, the (signed) support of the intrinsic normal cone of X. The Aluffi class $\alpha_X = c^M(\mathfrak{C}_X) = c^{SM}(\nu_X)$ has thus the property that its degree zero component is the virtual fundamental class of any symmetric obstruction theory on X.

In the case $\mathbb{K} = \mathbb{C}$, using MacPherson's complex analytic definition of the local Euler obstruction [124], the definition of ν_X makes sense in the framework of complex analytic geometry, and so Behrend functions can be defined for complex analytic spaces X_{an}. Thus, as in [85, Prop. 4.2] one has that if X is a finite type $\mathbb{K}$-scheme, then the Behrend function ν_X is a well-defined $\mathbb{Z}$-valued constructible function on X, in the Zariski topology. If Y is a complex analytic space then the Behrend function ν_Y is a well-defined $\mathbb{Z}$-valued locally constructible function on Y, in the analytic topology. Finally, if X is a finite type $\mathbb{C}$-scheme, with underlying complex analytic space X_{an}, then the algebraic Behrend function ν_X and the analytic Behrend function $\nu_{X_{\mathrm{an}}}$ coincide. In particular, ν_X depends only on the complex analytic space X_{an} underlying X, locally in the analytic topology. Finally, the definition of Behrend functions is valid over $\mathbb{K}$-schemes, algebraic $\mathbb{K}$-spaces and Artin $\mathbb{K}$-stacks, locally of finite type (see [85, Prop. 4.4]).

Categorifying the theory. We now relate the theory of Behrend functions to the categorification program which was one of the main application of the whole program [13, 18, 19, 25, 87].

For this paragraph, restrict to $\mathbb{K} = \mathbb{C}$ for simplicity. There exists a sophisticated modern theory of linear partial differential equations on a smooth complex algebraic variety X, sometimes called *microlocal analysis*, because it involves analysis on the cotangent bundle T^*X; this yields a theory which is invariant with respect to the action of the whole group of canonical transformation of T^*X while the usual theory is only invariant under the subgroup induced by diffeomorphism of X. It is sometimes called $\mathcal{D}$*-module theory,* because it involves sheaves of modules $\mathcal{M}$ over the sheaf of rings of holomorphic linear partial differential operators of finite order $\mathcal{D} = \mathcal{D}_X$; these rings are noncommutative, left and right Noetherian, and have finite global homological dimension. It is also sometimes called *algebraic analysis* because it involves such algebraic constructions as $\mathrm{Ext}^i_{\mathcal{D}}(\mathcal{M},\mathcal{N})$. The theory as it is known today grew out of the work done in the 1960s by the school of Mikio Sato in Japan. During the 1970's, one of the central themes in $\mathcal{D}$-module theory was David Hilbert's twenty-first problem, now called the *Riemann-Hilbert problem.* A generalization of it may be stated as the problem to solve the *Riemann-Hilbert correspondence,* which, roughly speaking, describes the nature of the correspondence between a system of differential equations and its solutions. A comprehensive reference is the book of Kashiwara and Shapira [90], while an interesting eclectic vision on the subject is provided by Ginsburg [54]. One has the following commutative diagram:

$$\begin{tikzcd} \text{(perverse) constructible sheaves} \arrow[d, "\chi"'] & \text{(regular) holonomic modules} \arrow[l, "\sim"', "DR"] \arrow[d, "SS"] \\ \text{constructible functions} \arrow[r, "\mathrm{Ch}", "\sim"'] & \text{Lagrangian cycles in } T^*X. \end{tikzcd} \tag{7.1.4}$$

Recall that here SS denotes the *characteristic cycle map* which to a $\mathcal{D}$-module $\mathcal{M}$ associates its *characteristic cycle.* It is the formal linear combination of irreducible components of the *characteristic variety* (the support of the graded sheaf $\mathrm{gr}\mathcal{M}$ associated to $\mathcal{M}$) counted with their multiplicities. It looks like

$$SS(\mathcal{M}) = \sum m_\alpha(\mathcal{M}) \cdot \overline{T^*_{X_\alpha} X}$$

for a stratification $\{X_\alpha\}$ of X, where $m_\alpha(\mathcal{M})$ are positive integers and $\overline{T^*_{X_\alpha} X}$ is the closure of the conormal bundle $T^*_{X_\alpha}X$. Each component of the characteristic variety has dimension at least $\dim(X)$. A $\mathcal{D}$-module $\mathcal{M}$ is called *holonomic* if its characteristic variety is pure of dimension $\dim(X)$. To have also *regular singularities* means, very roughly speaking, that the system is determined by its principal symbol.

So, to a holonomic system it has been associated an object of microlocal nature, the characteristic cycle. On the other side, the Riemann-Hilbert correspondence associates to an holonomic system $\mathcal{M}$ its *De Rham complex,*

$$\mathrm{DR}(\mathcal{M}):\ 0 \longrightarrow \Omega^0(\mathcal{M}) \xrightarrow{\ d\ } \Omega^1(\mathcal{M}) \xrightarrow{\ d\ } \dots \xrightarrow{\ d\ } \Omega^{\dim(X)}(\mathcal{M}) \xrightarrow{\ d\ } 0,$$

where $\Omega^p(\mathcal{M})$ is the sheaf of $\mathcal{M}$-valued p-forms on X and d is the differential defined by Cartan formula. As an object in the derived category it can be expressed as

$$\mathrm{DR}(\mathcal{M}) = \mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}\,(\mathcal{O}_X, \mathcal{M})[\dim(X)]. \tag{7.1.5}$$

If $\mathcal{M}$ is holonomic, $\mathrm{DR}(\mathcal{M})$ is constructible and determines $\mathcal{M}$ provided that the latter has regular singularities.

Now, given a constructible sheaf $\mathcal{C}^\bullet$ there is associated a constructible function on X: define a map $\chi_X : \mathrm{Obj}\,(D^b_{\mathrm{Con}}(X)) \to \mathrm{CF}^{\mathrm{an}}_{\mathbb{Z}}(X)$ by taking Euler characteristics of the cohomology of stalks of complexes, given by

$$\chi_X(\mathcal{C}^\bullet) : x \longmapsto \sum_{k\in\mathbb{Z}} (-1)^k \dim \mathcal{H}^k(\mathcal{C}^\bullet)_x. \tag{7.1.6}$$

Since distinguished triangles in $D^b_{\mathrm{Con}}(X)$ give long exact sequences on cohomology of stalks $\mathcal{H}^k(-)_x$, this χ_X is additive over distinguished triangles, and so descends to a group morphism $\chi_X : K_0(D^b_{\mathrm{Con}}(X)) \to \mathrm{CF}^{\mathrm{an}}_{\mathbb{Z}}(X)$. These maps $\chi_X : \mathrm{Obj}(D^b_{\mathrm{Con}}(X)) \to \mathrm{CF}^{\mathrm{an}}_{\mathbb{Z}}(X)$ and $\chi_X : K_0(D^b_{\mathrm{Con}}(X)) \to \mathrm{CF}^{\mathrm{an}}_{\mathbb{Z}}(X)$ are surjective, since $\mathrm{CF}^{\mathrm{an}}_{\mathbb{Z}}(X)$ is spanned by the characteristic functions of closed analytic cycles Y in X, and each such Y lifts to a perverse sheaf in $D^b_{\mathrm{Con}}(X)$. In category-theoretic terms, $X \mapsto D^b_{\mathrm{Con}}(X)$ is a functor D^b_{Con} from complex analytic spaces to triangulated categories, and $X \mapsto \mathrm{CF}^{\mathrm{an}}_{\mathbb{Z}}(X)$ is a functor $\mathrm{CF}^{\mathrm{an}}_{\mathbb{Z}}$ from complex analytic spaces to abelian groups, and $X \mapsto \chi_X$ is a natural transformation χ from D^b_{Con} to $\mathrm{CF}^{\mathrm{an}}_{\mathbb{Z}}$.

Thus, if $\mathcal{M}$ is a regular holonomic $\mathcal{D}$-module on X, then $\nu_X = \chi_X(\mathrm{DR}(\mathcal{M}))$, in the notation of (7.1.5) and (7.1.6).

In the case X is the critical scheme of a regular function f on a smooth scheme M, Behrend [5] gives the following expression for the Behrend function due to Parusiński and Pragacz [143]. This formula has been crucial in [85]. For the definition of the *Milnor fibres* for holomorphic functions on complex analytic spaces and the a review on *vanishing cycles* a survey paper on the subject is Massey [126], and three books are Kashiwara and Schapira [90], Dimca [34], and Schürmann [159]. Over the field $\mathbb{C}$, Saito's theory of *mixed Hodge modules* [152] provides a generalization of the theory of perverse sheaves with more structure, which may also be a context in which to generalize Donaldson–Thomas theory.

Theorem 7.1.4. *Let U be a complex manifold of dimension n, and $f : U \to \mathbb{C}$ a holomorphic function, and define X to be the complex analytic space* $\mathrm{Crit}(f)$ *contained in $U_0 = f^{-1}(\{0\})$. Then the Behrend function ν_X of X is given by*

$$\nu_X(x) = (-1)^{\dim U}\big(1 - \chi(MF_f(x))\big) \qquad \text{for } x \in X. \tag{7.1.7}$$

Moreover, the perverse sheaf of vanishing cycles $\phi_f(\underline{\mathbb{Q}}[n-1])$ on U_0 is supported on X, and

$$\chi_{U_0}\big(\phi_f(\underline{\mathbb{Q}}[n-1])\big)(x) = \begin{cases} \nu_X(x), & x \in X, \\ 0, & x \in U_0 \setminus X, \end{cases} \tag{7.1.8}$$

where ν_X is the Behrend function of the complex analytic space X.

Thus, if X is the Donaldson–Thomas moduli space of stable sheaves, one can, heuristically, think of ν_X as the *Euler characteristic of the perverse sheaf of vanishing cycles of the holomorphic Chern-Simons functional.* This is naturally related to Corollary 4.4.3 in §4 and to the important relation (0.0.1) discussed in the Introduction.

The Behrend function and its characterization

Here we will point out some important remarks and properties of the Behrend function.

Behrend function as a multiplicity function in the weighted Euler characteristic. It is worth to report here [85, §1.2] which provides a good way to think of Behrend functions as *multiplicity functions.* If X is a finite type $\mathbb{C}$-scheme then the Euler characteristic $\chi(X)$ 'counts' points without multiplicity, so that each point of $X(\mathbb{C})$ contributes 1 to $\chi(X)$. If X^{red} is the underlying reduced $\mathbb{C}$-scheme then $X^{\mathrm{red}}(\mathbb{C}) = X(\mathbb{C})$, so $\chi(X^{\mathrm{red}}) = \chi(X)$, and $\chi(X)$ does not see non-reduced behaviour in X. However, the *weighted Euler characteristic* $\chi(X, \nu_X)$ 'counts' each $x \in X(\mathbb{C})$ weighted by its multiplicity $\nu_X(x)$. The Behrend function ν_X detects non-reduced behaviour, so in general $\chi(X, \nu_X) \neq \chi(X^{\mathrm{red}}, \nu_{X^{\mathrm{red}}})$. For example, let X be the k-fold point $\mathrm{Spec}\big(\mathbb{C}[z]/(z^k)\big)$ for $k \geqslant 1$. Then $X(\mathbb{C})$ is a single point x with $\nu_X(x) = k$, so $\chi(X) = 1 = \chi(X^{\mathrm{red}}, \nu_{X^{\mathrm{red}}})$, but $\chi(X, \nu_X) = k$.

An important moral of [5] is that (at least in moduli problems with symmetric obstruction theories, such as Donaldson–Thomas theory) it is better to 'count' points in a moduli scheme $\mathcal{M}$ by the weighted Euler characteristic $\chi(\mathcal{M},\nu_{\mathcal{M}})$ than by the unweighted Euler characteristic $\chi(\mathcal{M})$. One reason is that $\chi(\mathcal{M},\nu_{\mathcal{M}})$ often gives answers unchanged under deformations of the underlying geometry, but $\chi(\mathcal{M})$ does not. For example, consider the family of $\mathbb{C}$-schemes $X_t = \operatorname{Spec}\bigl(\mathbb{C}[z]/(z^2-t^2)\bigr)$ for $t\in\mathbb{C}$. Then X_t is two reduced points $\pm t$ for $t\neq 0$, and a double point when $t=0$. So as above we find that $\chi(X_t,\nu_{X_t})=2$ for all t, which is deformation-invariant, but $\chi(X_t)$ is 2 for $t\neq 0$ and 1 for $t=0$, which is not deformation-invariant.

Properties of the Behrend function. Here are some important properties of Behrend functions. They are proved by Behrend [5, §1.2 & Prop. 1.5] when $\mathbb{K}=\mathbb{C}$, but his proof is valid for general $\mathbb{K}$.

Theorem 7.1.5. *Let X,Y be Artin $\mathbb{K}$-stacks locally of finite type. Then:*

(i) *If X is smooth of dimension n then $\nu_X\equiv(-1)^n$.*

(ii) *If $\varphi: X\to Y$ is smooth with relative dimension n then $\nu_X\equiv(-1)^n\varphi^*(\nu_Y)$.*

(iii) *$\nu_{X\times Y}\equiv\nu_X\boxdot\nu_Y$, where $(\nu_X\boxdot\nu_Y)(x,y)=\nu_X(x)\nu_Y(y)$.*

Let us recall [85, Thm 4.11]. It is stated using the Milnor fibre, but its proof works algebraically over $\mathbb{K}$.

Theorem 7.1.6. *Let U be a smooth $\mathbb{K}$-variety, $f:U\to\mathbb{A}^1_{\mathbb{K}}$ a regular function over U, and V a smooth $\mathbb{K}$-subvariety of U, and $v\in V\cap\operatorname{Crit}(f)$. Define $\tilde U$ to be the blowup of U along V, with blowup map $\pi:\tilde U\to U$, and set $\tilde f=f\circ\pi:\tilde U\to\mathbb{A}^1_{\mathbb{K}}$. Then $\pi^{-1}(v)=\mathbb{P}(T_vU/T_vV)$ is contained in $\operatorname{Crit}(\tilde f)$, and*

$$\nu_{\operatorname{Crit}(f)}(v)\quad=\int_{w\in\mathbb{P}(T_vU/T_vV)}\nu_{\operatorname{Crit}(\tilde f)}(w)\,\mathrm{d}\chi\quad+\quad(-1)^{\dim U-\dim V}\bigl(1-\dim U+\dim V\bigr)\nu_{\operatorname{Crit}(f|_V)}(v),$$

where $w\mapsto\nu_{\operatorname{Crit}(f)}(w)$ is a constructible function on $\mathbb{P}(T_vU/T_vV)$, and the integral is the Euler characteristic of $\mathbb{P}(T_vU/T_vV)$ weighted by this.

One can see the next result as a kind of *virtual Gauss–Bonnet formula*. It is crucial for Donaldson–Thomas theory. It is proved by Behrend [5, Th. 4.18] when $\mathbb{K}=\mathbb{C}$, but his proof is valid for general $\mathbb{K}$. It depends crucially on [5, Prop. 1.12] which again depend on an application of MacPherson's theorem [124] over $\mathbb{C}$ but valid over general $\mathbb{K}$ thanks to Kennedy [95] and the definition of the Euler characteristic over algebraically closed field $\mathbb{K}$ of characteristic zero given by Joyce [75]. See also an independent construction of the Schwartz–MacPherson Chern class given by Aluffi [2].

Theorem 7.1.7. *Let X a proper $\mathbb{K}$-scheme with a symmetric obstruction theory, and $[X]^{\mathrm{vir}}\in A_0(X)$ the corresponding virtual class. Then*

$$\int_{[X]^{\mathrm{vir}}}1=\chi(X,\nu_X)\in\mathbb{Z},$$

where $\chi(X,\nu_X)=\int_{X(\mathbb{K})}\nu_X\,\mathrm{d}\chi$ is the Euler characteristic of X weighted by the Behrend function ν_X of X. In particular, $\int_{[X]^{\mathrm{vir}}}1$ depends only on the $\mathbb{K}$-scheme structure of X, not on the choice of symmetric obstruction theory.

Theorem 7.1.7 implies that $DT^\alpha(\tau)$ in (7.1.1) is given by

$$DT^\alpha(\tau) = \chi\big(\mathcal{M}^\alpha_{\rm st}(\tau), \nu_{\mathcal{M}^\alpha_{\rm st}(\tau)}\big). \tag{7.1.9}$$

There is a big difference between the two equations (7.1.1) and (7.1.9) defining Donaldson–Thomas invariants. Equation (7.1.1) is non-local, and non-motivic, and makes sense only if $\mathcal{M}^\alpha_{\rm st}(\tau)$ is a proper $\mathbb{K}$-scheme. But (7.1.9) is local, and (in a sense) motivic, and makes sense for arbitrary finite type $\mathbb{K}$-schemes $\mathcal{M}^\alpha_{\rm st}(\tau)$. In fact, one could take (7.1.9) to be the definition of Donaldson–Thomas invariants even when $\mathcal{M}^\alpha_{\rm ss}(\tau) \neq \mathcal{M}^\alpha_{\rm st}(\tau)$, but in [85, §6.5] Joyce and Song argued that this is not a good idea, as then $DT^\alpha(\tau)$ would not be unchanged under deformations of X. In [85, §6.5] Joyce and Song say:

> 'Equation (7.1.9) was the inspiration for this book. It shows that Donaldson–Thomas invariants $DT^\alpha(\tau)$ can be written as *motivic* invariants, like those studied in [77–81], and so it raises the possibility that we can extend the results of [77–81] to Donaldson–Thomas invariants by including Behrend functions as weights.'

Almost closed 1-forms. In [140] Pandharipande and Thomas give a counterexample to the idea that every scheme admitting a symmetric obstruction theory can locally be written as the critical locus of a regular function on a smooth scheme. This limits the usefulness of the above formula for $\nu_X(x)$ in terms of the Milnor fibre. Here is the more general approach due to Behrend [5], which the author tried to use to give a strictly algebraic proof on the Behrend function identities, but later this proof turned out to be not completely correct.

Definition 7.1.8. Let $\mathbb{K}$ be an algebraically closed field, and M a smooth $\mathbb{K}$-scheme. Let ω be an algebraic 1-form on M, that is, $\omega \in H^0(T^*M)$. Call ω *almost closed* if $\mathrm{d}\omega$ is a section of $I_\omega \cdot \Lambda^2 T^*M$, where I_ω is the ideal sheaf of the zero locus $\omega^{-1}(0)$ of ω. Equivalently, $\mathrm{d}\omega|_{\omega^{-1}(0)}$ is zero as a section of $\Lambda^2 T^*M|_{\omega^{-1}(0)}$. In (étale) local coordinates $(z_1, \ldots, z_n)$ on M, if $\omega = f_1 \mathrm{d}z_1 + \cdots + f_n \mathrm{d}z_n$, then ω is almost closed provided $\frac{\partial f_j}{\partial z_k} \equiv \frac{\partial f_k}{\partial z_j} \mod (f_1, \ldots, f_n)$.

Let M be a smooth Deligne–Mumford stack and ω an almost closed 1-form on M with zero locus $X = Z(\omega)$. It is a general principle, that a section of a vector bundle defines a perfect obstruction theory for the zero locus of the section. This obstruction theory is given by

$$\begin{tikzcd} {[T_{M_{|X}}} \arrow[r, "d\circ\omega^\vee"] \arrow[d, "\omega^\vee"'] & {\Omega_{M_{|X}}]} \arrow[d, "1"] \\ {[I/I^2} \arrow[r, "\mathrm{d}"] & {\Omega_{M_{|X}}]} \end{tikzcd} \tag{7.1.10}$$

This obstruction theory is symmetric, in a canonical way, because under the assumption that ω is almost closed one has that $d \circ \omega^\vee$ is self-dual, as a homomorphism of vector bundles over X.

Behrend [5, Prop. 3.14] proves a kind of converse of that, by a proof valid for general $\mathbb{K}$, which says that, at least locally, every symmetric obstruction theory is given in this way by an almost closed 1-form.

Proposition 7.1.9. *Let $\mathbb{K}$ be an algebraically closed field, and X a $\mathbb{K}$-scheme with a symmetric obstruction theory. Then X may be covered by Zariski open sets $Y \subseteq X$ such that there exists a smooth $\mathbb{K}$-scheme M, an almost closed 1-form ω on M, and an isomorphism of $\mathbb{K}$-schemes $Y \cong \omega^{-1}(0)$.*

Restricting to $\mathbb{K} = \mathbb{C}$, Behrend [5, Prop. 4.22] gives an expression for the Behrend function of the zero locus of an almost closed 1-form as a *linking number*. It is possible to use it to give an algebraic proof of the first Behrend identity over $\mathbb{C}$.

Proposition 7.1.10. *Let M be a smooth scheme and ω an almost closed 1-form on M, and let $Y = \omega^{-1}(0)$ be the scheme-theoretic zero locus of ω. Fix p a closed point in Y, choose étale coordinates $(x_1, \ldots, x_n)$ on M around p with $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ the associated canonical coordinates for T^*M. Write $\omega = \sum_{i=1}^{n} f_i \mathrm{d}x_i$ in these coordinates. One can identify T^*M near p with $\mathbb{C}^{2n}$. Then for all $\eta \in \mathbb{C}$ and $\epsilon \in \mathbb{R}$ with $0 < |\eta| \ll \epsilon \ll 1$ one has*

$$\nu_Y(p) = L_{\mathcal{S}_\epsilon}\big(\Gamma_{\eta^{-1}\omega} \cap \mathcal{S}_\epsilon, \Delta \cap \mathcal{S}_\epsilon\big), \tag{7.1.11}$$

where

- $\mathcal{S}_\epsilon = \big\{(x_1, \ldots, p_n) \in \mathbb{C}^{2n} : |x_1|^2 + \cdots + |p_n|^2 = \epsilon^2\big\}$ *is the sphere of radius ϵ in $\mathbb{C}^{2n}$,*
- $\Gamma_{\eta^{-1}\omega}$ *is the graph of $\eta^{-1}\omega$ regarded locally as a complex submanifold of $\mathbb{C}^{2n}$ of real dimension $2n$ oriented so that $M \longrightarrow \Omega_M$ is orientation preserving and defined by the equations $\{\eta p_i = f_i(x)\}$,*
- $\Delta = \big\{(x_1, \ldots, p_n) \in \mathbb{C}^{2n} : p_j = \bar{x}_j,\ j = 1, \ldots, n\big\}$, *i.e. the image of the smooth map $M \longrightarrow \Omega_M$ given by the section $\mathrm{d}\varrho$ of Ω_M, with*

  $$\varrho = \sum_i x_i \bar{x}_i + \sum_i p_i \bar{p}_i$$

  the square of the distance function defined on Ω_M by the choice of coordinates of real dimension $2n$,
- $L_{\mathcal{S}_\epsilon}(\ ,\ )$ *is the linking number of two disjoint, closed, oriented $(n-1)$-submanifolds in $\mathcal{S}_\epsilon$.*

We remark here that Δ is not a complex submanifold, but only a real submanifold. Thus, there are no good generalizations of Δ to other fields $\mathbb{K}$.

7.1.3 Generalizations of Donaldson–Thomas theory

Next it will be briefly reviewed how the theory of generalized Donaldson–Thomas invariants has been developed, starting from the series of papers [75–81] about constructible functions, stack functions, Ringel–Hall algebras, counting invariants for Calabi–Yau 3-folds, and wall-crossing and then summarizing the main results in [85] including the definition of generalized Donaldson–Thomas invariants $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$, their deformation-invariance, and wall-crossing formulae under change of stability condition τ. In the sequel, there are two paragraphs on statements and a sketch of proofs of the theorems [85, Thm 5.5] and [85, Thm 5.11] on which this paper is concentrated. We conclude with a brief and rough remark on Kontsevich and Soibelman's parallel approach to Donaldson–Thomas theory [102], focusing more on analogies and differences with Joyce and Song's construction [85] rather than going into a detailed exposition.

Brief sketch of background from [75–81]

Here it will be recalled a few important ideas from [75–81]. They deal with *Artin stacks* rather than coarse moduli schemes, as in [167]. Let X be a Calabi–Yau 3-fold over $\mathbb{C}$, and write $\mathfrak{M}$ for the moduli stack of all coherent sheaves E on X. It is an Artin $\mathbb{C}$-stack.

The ring of *stack functions* $\mathrm{SF}(\mathfrak{M})$ in [76] is basically the Grothendieck group $K_0(\mathrm{Sta}_{\mathfrak{M}})$ of the 2-category $\mathrm{Sta}_{\mathfrak{M}}$ of stacks over $\mathfrak{M}$. That is, $\mathrm{SF}(\mathfrak{M})$ is generated by isomorphism classes $[(\mathfrak{R},\rho)]$ of representable 1-morphisms $\rho:\mathfrak{R}\to\mathfrak{M}$ for $\mathfrak{R}$ a finite type Artin $\mathbb{C}$-stack, with the relation

$$[(\mathfrak{R},\rho)]=[(\mathfrak{S},\rho|_{\mathfrak{S}})]+[(\mathfrak{R}\setminus\mathfrak{S},\rho|_{\mathfrak{R}\setminus\mathfrak{S}})]$$

when $\mathfrak{S}$ is a closed $\mathbb{C}$-substack of $\mathfrak{R}$. In [76] Joyce studies different kinds of stack function spaces with other choices of generators and relations, and operations on these spaces. These include projections $\Pi_n^{\mathrm{vi}}:\mathrm{SF}(\mathfrak{M})\to\mathrm{SF}(\mathfrak{M})$ to stack functions of *virtual rank* n, which act on $[(\mathfrak{R},\rho)]$ by modifying $\mathfrak{R}$ depending on its stabilizer groups.

In [78, §5.2] he defines a *Ringel–Hall* type algebra $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$ of stack functions *with algebra stabilizers* on $\mathfrak{M}$, with an associative, non-commutative multiplication $*$ and in [78, §5.2] he defines a Lie subalgebra $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$ of stack functions *supported on virtual indecomposables*. In [78, §6.5] he defines an explicit Lie algebra $L(X)$ to be the $\mathbb{Q}$-vector space with basis of symbols λ^α for $\alpha\in K^{\mathrm{num}}(\mathrm{coh}(X))$, with Lie bracket

$$[\lambda^\alpha,\lambda^\beta]=\bar{\chi}(\alpha,\beta)\lambda^{\alpha+\beta}, \tag{7.1.12}$$

for $\alpha,\beta\in K^{\mathrm{num}}(\mathrm{coh}(X))$, where $\bar{\chi}(\,,\,)$ is the *Euler form* on $K^{\mathrm{num}}(\mathrm{coh}(X))$ defined as follows:

$$\bar{\chi}([E],[F])=\sum_{i\geq 0}(-1)^i\dim\mathrm{Ext}^i(E,F) \tag{7.1.13}$$

for all $E,F\in\mathrm{coh}(X)$. As X is a Calabi–Yau 3-fold, $\bar{\chi}$ is antisymmetric, so (7.1.12) satisfies the Jacobi identity and makes $L(X)$ into an infinite-dimensional Lie algebra over $\mathbb{Q}$.

Then in [78, §6.6] Joyce defines a *Lie algebra morphism* $\Psi:\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})\to L(X)$, which, roughly speaking, is of the form

$$\Psi(f)\quad=\sum_{\alpha\in K^{\mathrm{num}}(\mathrm{coh}(X))}\chi^{\mathrm{stk}}\big(f|_{\mathfrak{M}^\alpha}\big)\lambda^\alpha, \tag{7.1.14}$$

where $f=\sum_{i=1}^m c_i[(\mathfrak{R}_i,\rho_i)]$ is a stack function on M, and $\mathfrak{M}^\alpha$ is the substack in $\mathfrak{M}$ of sheaves E with class α, and χ^{stk} is a kind of stack-theoretic Euler characteristic. But in fact the definition of Ψ, and the proof that Ψ is a Lie algebra morphism, are highly nontrivial, and use many ideas from [75, 76, 78], including those of 'virtual rank' and 'virtual indecomposable'. The problem is that the obvious definition of χ^{stk} usually involves dividing by zero, so defining (7.1.14) in a way that makes sense is quite subtle. The proof that Ψ is a Lie algebra morphism uses *Serre duality* and the assumption that X is a Calabi–Yau 3-fold.

Now let τ be a stability condition on $\mathrm{coh}(X)$, such as Gieseker stability. Then one has open, finite type substacks $\mathfrak{M}_{\mathrm{ss}}^\alpha(\tau),\mathfrak{M}_{\mathrm{st}}^\alpha(\tau)$ in $\mathfrak{M}$ of τ-(semi)stable sheaves E in class α, for all $\alpha\in K^{\mathrm{num}}(\mathrm{coh}(X))$. Write $\bar{\delta}_{\mathrm{ss}}^\alpha(\tau)$ for the characteristic function of $\mathfrak{M}_{\mathrm{ss}}^\alpha(\tau)$, in the sense of stack functions [76]. Then $\bar{\delta}_{\mathrm{ss}}^\alpha(\tau)\in\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$. In [79, §8], Joyce defines elements $\bar{\epsilon}^\alpha(\tau)$ in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$ by

$$\bar{\epsilon}^\alpha(\tau)\quad=\sum_{\substack{n\geqslant 1,\ \alpha_1,\dots,\alpha_n\in K^{\mathrm{num}}(\mathrm{coh}(X)):\\ \alpha_1+\cdots+\alpha_n=\alpha,\ \tau(\alpha_i)=\tau(\alpha),\ \text{all } i}}\frac{(-1)^{n-1}}{n}\ \bar{\delta}_{\mathrm{ss}}^{\alpha_1}(\tau)*\bar{\delta}_{\mathrm{ss}}^{\alpha_2}(\tau)*\cdots*\bar{\delta}_{\mathrm{ss}}^{\alpha_n}(\tau), \tag{7.1.15}$$

where $*$ is the Ringel–Hall multiplication in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$. Then [79, Thm. 8.7] shows that $\bar{\epsilon}^\alpha(\tau)$ lies in the Lie subalgebra $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$, a nontrivial result. Thus one can apply the Lie algebra morphism Ψ to $\bar{\epsilon}^\alpha(\tau)$. In [80, §6.6] he defines invariants $J^\alpha(\tau)\in\mathbb{Q}$ for all $\alpha\in K^{\mathrm{num}}(\mathrm{coh}(X))$ by

$$\Psi\big(\bar{\epsilon}^\alpha(\tau)\big)=J^\alpha(\tau)\lambda^\alpha. \tag{7.1.16}$$

These $J^\alpha(\tau)$ are rational numbers 'counting' τ-semistable sheaves E in class α. When $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau) = \mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ then $J^\alpha(\tau) = \chi(\mathcal{M}_{\mathrm{st}}^\alpha(\tau))$, that is, $J^\alpha(\tau)$ is the naïve Euler characteristic of the moduli space $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$. This is *not* weighted by the Behrend function $\nu_{\mathcal{M}_{\mathrm{st}}^\alpha(\tau)}$, and so in general does not coincide with the Donaldson–Thomas invariant $DT^\alpha(\tau)$ in (7.1.12). As the $J^\alpha(\tau)$ do not include Behrend functions, they do not count semistable sheaves with multiplicity, and so they will not in general be unchanged under deformations of the underlying Calabi–Yau 3-fold, as Donaldson–Thomas invariants are. However, the $J^\alpha(\tau)$ do have very good properties under change of stability condition. In [80] Joyce shows that if $\tau, \tilde\tau$ are two stability conditions on $\mathrm{coh}(X)$, then it is possible to write $\bar\epsilon^\alpha(\tilde\tau)$ in terms of a (complicated) explicit formula involving the $\bar\epsilon^\beta(\tau)$ for $\beta \in K^{\mathrm{num}}(\mathrm{coh}(X))$ and the Lie bracket in $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$. Applying the Lie algebra morphism Ψ shows that $J^\alpha(\tilde\tau)\lambda^\alpha$ may be written in terms of the $J^\beta(\tau)\lambda^\beta$ and the Lie bracket in $L(X)$, and hence [80, Thm. 6.28] yields an explicit transformation law for the $J^\alpha(\tau)$ under change of stability condition. In [81] he shows how to encode invariants $J^\alpha(\tau)$ satisfying a transformation law in generating functions on a complex manifold of stability conditions, which are both holomorphic and continuous, despite the discontinuous wall-crossing behaviour of the $J^\alpha(\tau)$.

Summary of the main results from [85]

The basic idea behind the project developed in [85] is that the Behrend function $\nu_{\mathfrak{M}}$ of the moduli stack $\mathfrak{M}$ of coherent sheaves in X should be inserted as a weight in the programme of [75–81] summarized in §7.1.3. Thus one will obtain weighted versions $\tilde\Psi$ of the Lie algebra morphism Ψ of (7.1.14), and $\bar{D}T^\alpha(\tau)$ of the counting invariant $J^\alpha(\tau) \in \mathbb{Q}$ in (7.1.16). Here is how this is worked out in [85].

Joyce and Song define a modification $\tilde L(X)$ of the Lie algebra $L(X)$ above, the $\mathbb{Q}$-vector space with basis of symbols $\tilde\lambda^\alpha$ for $\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))$, with Lie bracket

$$[\tilde\lambda^\alpha, \tilde\lambda^\beta] = (-1)^{\bar\chi(\alpha,\beta)}\bar\chi(\alpha,\beta)\tilde\lambda^{\alpha+\beta},$$

which is (7.1.14) with a sign change. Then they define a *Lie algebra morphism* $\tilde\Psi : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \to \tilde L(X)$. Roughly speaking this is of the form

$$\tilde\Psi(f) \quad = \sum_{\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))} \chi^{\mathrm{stk}}\big(f|_{\mathfrak{M}^\alpha}, \nu_{\mathfrak{M}}\big)\tilde\lambda^\alpha, \tag{7.1.17}$$

that is, in (7.1.14) we replace the stack-theoretic Euler characteristic χ^{stk} with a stack-theoretic Euler characteristic weighted by the Behrend function $\nu_{\mathfrak{M}}$. The proof that $\tilde\Psi$ is a Lie algebra morphism combines the proof in [78] that Ψ is a Lie algebra morphism with the two *Behrend function identities* (7.1.18)–(7.1.19) proved in [85, thm. 5.11] and reported below. Proving (7.1.18)–(7.1.19) requires a deep understanding of the local structure of the moduli stack $\mathfrak{M}$, which is of interest in itself. First they show using a composition of *Seidel–Thomas twists* by $\mathcal{O}_X(-n)$ for $n \gg 0$ that $\mathfrak{M}$ is locally 1-isomorphic to the moduli stack $\mathfrak{Vect}$ of vector bundles on X. Then they prove that near $[E] \in \mathfrak{Vect}(\mathbb{C})$, an atlas for $\mathfrak{Vect}$ can be written locally in the complex analytic topology in the form $\mathrm{Crit}(f)$ for $f : U \to \mathbb{C}$ a holomorphic function on an open set U in $\mathrm{Ext}^1(E,E)$. These U, f are *not algebraic*, they are constructed using gauge theory on the complex vector bundle E over X and transcendental methods. Finally, they deduce (7.1.18)–(7.1.19) using the Milnor fibre expression (7.1.7) for Behrend functions applied to these U, f.

Before going on with the review of Joyce and Song's program, it is worth to stop for a while on some details about [85, Thm 5.5] and [85, Thm 5.11], the statements of the theorems and how they prove it.

Gauge theory and transcendental complex analytic geometry from [85]. In [85, Thm. 5.5] Joyce and Song give a local characterization of an atlas for the moduli stack $\mathfrak{M}$ as the critical points of a holomorphic function on a complex manifold. The statement and a sketch of its proof are reported below. Some background references are Kobayashi [101, §VII.3], Lübke and Teleman [118, §4.1 & §4.3], Friedman and Morgan [44, §4.1–§4.2] and Miyajima [129].

Theorem 7.1.11. *Let X be a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathfrak{M}$ the moduli stack of coherent sheaves on X. Suppose E is a coherent sheaf on X, so that $[E] \in \mathfrak{M}(\mathbb{C})$. Let G be a maximal reductive subgroup in $\mathrm{Aut}(E)$, and $G^{\mathbb{C}}$ its complexification. Then $G^{\mathbb{C}}$ is an algebraic $\mathbb{C}$-subgroup of $\mathrm{Aut}(E)$, a maximal reductive subgroup, and $G^{\mathbb{C}} = \mathrm{Aut}(E)$ if and only if $\mathrm{Aut}(E)$ is reductive. There exists a quasiprojective $\mathbb{C}$-scheme S, an action of $G^{\mathbb{C}}$ on S, a point $s \in S(\mathbb{C})$ fixed by $G^{\mathbb{C}}$, and a 1-morphism of Artin $\mathbb{C}$-stacks $\Phi : [S/G^{\mathbb{C}}] \to \mathfrak{M}$, which is smooth of relative dimension $\dim \mathrm{Aut}(E) - \dim G^{\mathbb{C}}$, where $[S/G^{\mathbb{C}}]$ is the quotient stack, such that $\Phi(s\, G^{\mathbb{C}}) = [E]$, the induced morphism on stabilizer groups $\Phi_* : \mathrm{Iso}_{[S/G^{\mathbb{C}}]}(s\, G^{\mathbb{C}}) \to \mathrm{Iso}_{\mathfrak{M}}([E])$ is the natural morphism $G^{\mathbb{C}} \hookrightarrow \mathrm{Aut}(E) \cong \mathrm{Iso}_{\mathfrak{M}}([E])$, and $\mathrm{d}\Phi|_{s\, G^{\mathbb{C}}} : T_s S \cong T_{s\, G^{\mathbb{C}}}[S/G^{\mathbb{C}}] \to T_{[E]}\mathfrak{M} \cong \mathrm{Ext}^1(E,E)$ is an isomorphism. Furthermore, S parametrizes a formally versal family $(S, \mathcal{D})$ of coherent sheaves on X, equivariant under the action of $G^{\mathbb{C}}$ on S, with fibre $\mathcal{D}_s \cong E$ at s. If $\mathrm{Aut}(E)$ is reductive then Φ is étale.*

Write S_{an} for the complex analytic space underlying the $\mathbb{C}$-scheme S. Then there exists an open neighbourhood U of 0 in $\mathrm{Ext}^1(E,E)$ in the analytic topology, a holomorphic function $f : U \to \mathbb{C}$ with $f(0) = \mathrm{d}f|_0 = 0$, an open neighbourhood V of s in S_{an}, and an isomorphism of complex analytic spaces $\Xi : \mathrm{Crit}(f) \to V$, such that $\Xi(0) = s$ and $\mathrm{d}\Xi|_0 : T_0\, \mathrm{Crit}(f) \to T_s V$ is the inverse of $\mathrm{d}\Phi|_{s\, G^{\mathbb{C}}} : T_s S \to \mathrm{Ext}^1(E,E)$. Moreover we can choose U, f, V to be $G^{\mathbb{C}}$-invariant, and Ξ to be $G^{\mathbb{C}}$-equivariant.

In [85], Theorem 7.1.11 gives Joyce and Song the possibility to use the Milnor fibre formula (7.1.7) for the Behrend function of $\mathrm{Crit}(f)$ to study the Behrend function $\nu_{\mathfrak{M}}$, crucially used in proving Behrend identities. The proof of Theorem 7.1.11 comes in two parts. First it is shown in [85, §8] that $\mathfrak{M}$ near $[E]$ is locally isomorphic, as an Artin $\mathbb{C}$-stack, to the moduli stack $\mathfrak{Vect}$ of *algebraic vector bundles* on X near $[E']$ for some vector bundle $E' \to X$. The proof uses algebraic geometry, and is valid for X a Calabi–Yau m-fold for any $m > 0$ over any algebraically closed field $\mathbb{K}$. The local morphism $\mathfrak{M} \to \mathfrak{Vect}$ is the composition of shifts and m *Seidel–Thomas twists* by $\mathcal{O}_X(-n)$ for $n \gg 0$. Thus, it is enough to prove Theorem 7.1.11 with $\mathfrak{Vect}$ in place of $\mathfrak{M}$. This is done in [85, §9] using gauge theory on vector bundles over X. An interesting motivation for this approach could be found in [36, §3] and [167, §2]. Let $E \to X$ be a fixed complex (not holomorphic) vector bundle over X. Write $\mathscr{A}$ for the infinite-dimensional affine space of smooth *semiconnections* ($\bar\partial$-operators) on E, and $\mathscr{G}$ for the infinite-dimensional Lie group of *smooth gauge transformations* of E. Then $\mathscr{G}$ acts on $\mathscr{A}$, and $\mathscr{B} = \mathscr{A}/\mathscr{G}$ is the space of gauge-equivalence classes of semiconnections on E. Fix $\bar\partial_E$ in $\mathscr{A}$ coming from a holomorphic vector bundle structure on E. Then points in $\mathscr{A}$ are of the form $\bar\partial_E + A$ for $A \in C^\infty\big(\mathrm{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,1}T^*X\big)$, and $\bar\partial_E + A$ makes E into a holomorphic vector bundle if $F_A^{0,2} = \bar\partial_E A + A \wedge A$ is zero in $C^\infty\big(\mathrm{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2}T^*X\big)$. Thus, the moduli space (stack) of holomorphic vector bundle structures on E is isomorphic to $\{\bar\partial_E + A \in \mathscr{A} : F_A^{0,2} = 0\}/\mathscr{G}$. In [167], it is observed that when X is a Calabi–Yau 3-fold, there is a natural holomorphic function $CS : \mathscr{A} \to \mathbb{C}$ called the *holomorphic Chern–Simons functional*, invariant under $\mathscr{G}$ up to addition of constants, such that $\{\bar\partial_E + A \in \mathscr{A} : F_A^{0,2} = 0\}$ is the critical locus of CS. Thus, $\mathfrak{Vect}$ is (informally) locally the critical points of a holomorphic function CS on an infinite-dimensional complex stack $\mathscr{B} = \mathscr{A}/\mathscr{G}$. To prove Theorem 7.1.11 Joyce and Song show that one can find a finite-dimensional complex submanifold U in $\mathscr{A}$ and a finite-dimensional complex Lie subgroup $G^{\mathbb{C}}$ in $\mathscr{G}$ preserving U such that the theorem holds with $f = CS|_U$. These U, f are *not algebraic*, they are constructed using gauge theory on the complex vector bundle E over X and transcendental methods.

The Behrend function identities from [85]. In [85, Thm. 5.11] Behrend function identities are proven: they are the crucial step to define the Lie algebra morphism $\tilde{\Psi}$ below and then the generalized Donaldson–Thomas invariants:

Theorem 7.1.12. *Let X be a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathfrak{M}$ the moduli stack of coherent sheaves on X. The Behrend function $\nu_{\mathfrak{M}} : \mathfrak{M}(\mathbb{C}) \to \mathbb{Z}$ is a natural locally constructible function on $\mathfrak{M}$. For all $E_1, E_2 \in \mathrm{coh}(X)$, it satisfies:*

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{\bar{\chi}([E_1],[E_2])}\nu_{\mathfrak{M}}(E_1)\nu_{\mathfrak{M}}(E_2), \tag{7.1.18}$$

$$\int_{\substack{[\lambda]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)):\\ \lambda\Leftrightarrow 0\to E_1\to F\to E_2\to 0}} \nu_{\mathfrak{M}}(F)\,\mathrm{d}\chi \quad - \int_{\substack{[\mu]\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2)):\\ \mu\Leftrightarrow 0\to E_2\to D\to E_1\to 0}} \nu_{\mathfrak{M}}(D)\,\mathrm{d}\chi \quad = \quad (e_{21}-e_{12})\ \nu_{\mathfrak{M}}(E_1\oplus E_2), \tag{7.1.19}$$

where $e_{21} = \dim \mathrm{Ext}^1(E_2, E_1)$ and $e_{12} = \dim \mathrm{Ext}^1(E_1, E_2)$ for $E_1, E_2 \in \mathrm{coh}(X)$. Here $\bar{\chi}([E_1], [E_2])$ in (7.1.18) is the Euler form as in (7.1.13), and in (7.1.19) the correspondence between $[\lambda] \in \mathbb{P}(\mathrm{Ext}^1(E_2, E_1))$ and $F \in \mathrm{coh}(X)$ is that $[\lambda] \in \mathbb{P}(\mathrm{Ext}^1(E_2, E_1))$ lifts to some $0 \neq \lambda \in \mathrm{Ext}^1(E_2, E_1)$, which corresponds to a short exact sequence $0 \to E_1 \to F \to E_2 \to 0$ in $\mathrm{coh}(X)$ in the usual way. The function $[\lambda] \mapsto \nu_{\mathfrak{M}}(F)$ is a constructible function $\mathbb{P}(\mathrm{Ext}^1(E_2, E_1)) \to \mathbb{Z}$, and the integrals in (7.1.19) are integrals of constructible functions using the Euler characteristic as measure.

Joyce and Song prove Theorem 7.1.12 using Theorem 7.1.11 and the Milnor fibre description of Behrend functions from §7.1.3. They apply Theorem 7.1.11 to $E = E_1 \oplus E_2$, and take the maximal reductive subgroup G of $\mathrm{Aut}(E)$ to contain the subgroup $\{\mathrm{id}_{E_1} + \lambda \mathrm{id}_{E_2} : \lambda \in \mathrm{U}(1)\}$, so that $G^{\mathbb{C}}$ contains $\{\mathrm{id}_{E_1} + \lambda \mathrm{id}_{E_2} : \lambda \in \mathbb{G}_m\}$. Equations (7.1.18) and (7.1.19) are proved by a kind of localization using this $\mathbb{G}_m$-action on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$. More precisely, Theorem 7.1.11 gives an atlas for $\mathfrak{M}$ near E as $\mathrm{Crit}(f)$ near 0, where f is a holomorphic function defined near 0 on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ and f is invariant under the action of $T = \{\mathrm{id}_{E_1} + \lambda \mathrm{id}_{E_2} : \lambda \in \mathrm{U}(1)\}$ on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ by conjugation. The fixed points of T on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ are $\mathrm{Ext}^1(E_1, E_1) \oplus \mathrm{Ext}^1(E_2, E_2)$ and heuristically one can says that the restriction of f to these fixed points is $f_1 + f_2$, where f_j is defined near 0 in $\mathrm{Ext}^1(E_j, E_j)$ and $\mathrm{Crit}(f_j)$ is an atlas for $\mathfrak{M}$ near E_j. The Milnor fibre $MF_f(0)$ is invariant under T, so by localization one has

$$\chi(MF_f(0)) = \chi(MF_f(0)^T) = \chi(MF_{f_1+f_2}(0)).$$

A product property of Behrend functions, which may be seen as a kind of *Thom-Sebastiani theorem*, gives

$$1 - \chi(MF_{f_1+f_2}(0)) = (1 - \chi(MF_{f_1}(0)))(1 - \chi(MF_{f_2}(0))).$$

Then the identity (7.1.18) follows from Theorem 7.1.4:

$$\nu_{\mathfrak{M}}(E) = (-1)^{\dim \mathrm{Ext}^1(E,E) - \dim \mathrm{Hom}(E,E)}(1 - \chi(MF_f(0))),$$

and the analogues for E_1 and E_2. Equation (7.1.19) uses a more involved argument to do with the Milnor fibres of f at non-fixed points of the $U(1)$-action. The proof of Theorem 7.1.12 uses gauge theory, and transcendental complex analytic geometry methods, and is valid only over $\mathbb{K} = \mathbb{C}$. However, as pointed out in [85, Question 5.12], Theorem 7.1.12 makes sense as a statement in algebraic geometry, for Calabi–Yau 3-folds over $\mathbb{K}$.

In [85, §5], Joyce and Song then define *generalized Donaldson–Thomas invariants* $\bar{DT}^{\alpha}(\tau) \in \mathbb{Q}$ by

$$\tilde{\Psi}\big(\bar{\epsilon}^{\alpha}(\tau)\big) = -\bar{DT}^{\alpha}(\tau)\tilde{\lambda}^{\alpha}, \tag{7.1.20}$$

as in (7.1.16). When $\mathcal{M}^\alpha_{\rm ss}(\tau)=\mathcal{M}^\alpha_{\rm st}(\tau)$ then $\bar\epsilon^\alpha(\tau)=\bar\delta^\alpha_{\rm ss}(\tau)$, and (7.1.17) gives

$$\tilde\Psi\big(\bar\epsilon^\alpha(\tau)\big)=\chi^{\rm stk}\big(\mathfrak{M}^\alpha_{\rm st}(\tau),\nu_{\mathfrak{M}^\alpha_{\rm st}(\tau)}\big)\tilde\lambda^\alpha. \tag{7.1.21}$$

The projection $\pi:\mathfrak{M}^\alpha_{\rm st}(\tau)\to\mathcal{M}^\alpha_{\rm st}(\tau)$ from the moduli stack to the coarse moduli scheme is smooth of dimension -1, so $\nu_{\mathfrak{M}^\alpha_{\rm st}(\tau)}=-\pi^*(\nu_{\mathcal{M}^\alpha_{\rm st}(\tau)})$ by (ii) in §7.1.2, and comparing (7.1.9), (7.1.20), (7.1.21) shows that $\bar{DT}^\alpha(\tau)=DT^\alpha(\tau)$. But the new invariants $\bar{DT}^\alpha(\tau)$ are also defined for α with $\mathcal{M}^\alpha_{\rm ss}(\tau)\neq\mathcal{M}^\alpha_{\rm st}(\tau)$, when conventional Donaldson–Thomas invariants $DT^\alpha(\tau)$ are not defined.

Thanks to Theorem 7.1.11 and Theorem 7.1.12, $\tilde\Psi$ is a Lie algebra morphism [85, §5.3], thus the change of stability condition formula for the $\bar\epsilon^\alpha(\tau)$ in [80] implies a formula for the elements $-\bar{DT}^\alpha(\tau)\tilde\lambda^\alpha$ in $\tilde L(X)$, and thus a transformation law for the invariants $\bar{DT}^\alpha(\tau)$, using combinatorial coefficients.

To study the new invariants $\bar{DT}^\alpha(\tau)$, it is helpful to introduce another family of invariants $PI^{\alpha,n}(\tau')$, similar to Pandharipande–Thomas invariants [140]. Let $n\gg 0$ be fixed. A *stable pair* is a nonzero morphism $s:\mathcal{O}_X(-n)\to E$ in $\mathrm{coh}(X)$ such that E is τ-semistable, and if $\mathrm{Im}\, s\subset E'\subset E$ with $E'\neq E$ then $\tau([E'])<\tau([E])$. For $\alpha\in K^{\rm num}(\mathrm{coh}(X))$ and $n\gg 0$, the moduli space $\mathcal{M}^{\alpha,n}_{\rm stp}(\tau')$ of stable pairs $s:\mathcal{O}_X(-n)\to X$ with $[E]=\alpha$ is a fine moduli scheme, which is proper and has a symmetric obstruction theory. Joyce and Song define

$$PI^{\alpha,n}(\tau') \quad = \int_{[\mathcal{M}^{\alpha,n}_{\rm stp}(\tau')]^{\rm vir}} 1 \quad = \quad \chi\big(\mathcal{M}^{\alpha,n}_{\rm stp}(\tau'),\nu_{\mathcal{M}^{\alpha,n}_{\rm stp}(\tau')}\big)\in\mathbb{Z}, \tag{7.1.22}$$

where the second equality follows from Theorem 7.1.7. By a similar proof to that for Donaldson–Thomas invariants in [167], Joyce and Song find that $PI^{\alpha,n}(\tau')$ is unchanged under deformations of the underlying Calabi–Yau 3-fold X. By a wall-crossing proof similar to that for $\bar{DT}^\alpha(\tau)$, they show that $PI^{\alpha,n}(\tau')$ can be written in terms of the $\bar{DT}^\beta(\tau)$. As $PI^{\alpha,n}(\tau')$ is deformation-invariant, one deduces from this relation by induction on $\mathrm{rank}\,\alpha$ with $\dim\alpha$ fixed that $\bar{DT}^\alpha(\tau)$ is also deformation-invariant.

The pair invariants $PI^{\alpha,n}(\tau')$ are a useful tool for computing the $\bar{DT}^\alpha(\tau)$ in examples in [85, §6]. The method is to describe the moduli spaces $\mathcal{M}^{\alpha,n}_{\rm stp}(\tau')$ explicitly, and then use (7.1.22) to compute $PI^{\alpha,n}(\tau')$, and their relation with $\bar{DT}^\alpha(\tau)$ to deduce the values of $\bar{DT}^\alpha(\tau)$. Their point of view is that the $\bar{DT}^\alpha(\tau)$ are of primary interest, and the $PI^{\alpha,n}(\tau')$ are secondary invariants, of less interest in themselves.

Motivic Donaldson–Thomas invariants: Kontsevich and Soibelman's approach from [102]. Kontsevich and Soibelman in [102] also studied generalizations of Donaldson–Thomas invariants. They work in a more general context but their results are in great part based on conjectures. They consider derived categories of coherent sheaves, Bridgeland stability conditions [20], and general motivic invariants, whereas Joyce and Song work with abelian categories of coherent sheaves, Gieseker stability, and the Euler characteristic. Kontsevich and Soibelman's motivic functions in the equivariant setting [102, §4.2], motivic Hall algebra [102, §6.1], motivic quantum torus [102, §6.2] and their algebra morphism to define Donaldson–Thomas invariants [102, Thm. 8] all have an analogue in Joyce and Song's program.

It is worth to note here some points (see [85, §1.6] for the entire discussion).

(a) Joyce was probably the first to approach Donaldson–Thomas type invariants in an abstract categorical setting. He developed the technique of motivic stack functions and understood the relevance of motives to the counting problem [75–80]. The main limitation of his approach was due to the fact that he worked with abelian rather than triangulated categories.

For many applications, especially to physics, one needs triangulated categories. The more recent theory of Joyce and Song [85] fixes some of these gaps and fits well with the general philosophy of [102] (and actually Joyce and Song use some ideas from Kontsevich and Soibelman). They deal with concrete examples of categories (e.g. the category of coherent sheaves) and construct numerical invariants via Behrend approach. It is difficult to prove that they are in fact invariants of triangulated categories which is manifest in [102].

(b) Kontsevich and Soibelman write their wall-crossing formulae in terms of products in a pronilpotent Lie group while Joyce and Song's formulae are written in terms of combinatorial coefficients.

(c) Equations (7.1.18)–(7.1.19) are related to a conjecture of Kontsevich and Soibelman [102, Conj. 4] and its application in [102, §6.3], and could probably be deduced from it. Joyce and Song got the idea of proving (7.1.18)–(7.1.19) by localization using the $\mathbb{G}_m$-action on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ from [102]. However, Kontsevich and Soibelman approach [102, Conj. 4] via formal power series and non-Archimedean geometry. Their analogue concerns the 'motivic Milnor fibre' of the formal power series f. Instead, in Theorem 7.1.11 Joyce and Song in effect first prove that they can choose the formal power series to be convergent, and then use ordinary differential geometry and Milnor fibres.

(d) While Joyce's series of papers [75–80] develops the difficult idea of 'virtual rank' and 'virtual indecomposables', Kontsevich and Soibelman have no analogue of these. They come up against the problem (specialization from virtual Poincaré polynomial to Euler characteristic) this technology was designed to solve in the 'absence of poles conjecture' [102, §7].

Section 7.3 proposes new ideas for further research also in the direction of Kontsevich and Soibelman's paper [102].

7.2 The main results

We will prove and use the algebraic analogue of Theorem 7.1.11, which we can state as follows:

Theorem 7.2.1. *Let X be a Calabi–Yau 3-fold over $\mathbb{K}$, and write $\mathfrak{M}$ for the moduli stack of coherent sheaves on X. Then for each $[E] \in \mathfrak{M}(\mathbb{K})$, there exists a smooth affine $\mathbb{K}$-scheme U, a point $p \in \mathrm{U}(\mathbb{K})$, an étale morphism $u : U \to \mathrm{Ext}^1(E,E)$ with $u(p) = 0$, a regular function $f : U \to \mathbb{A}^1$ with $f|_p = \partial f|_p = 0$, and a 1-morphism $\xi : \mathrm{Crit}(f) \to \mathfrak{M}$ smooth of relative dimension $\dim \mathrm{Aut}(E)$, with $\xi(p) = [E] \in \mathfrak{M}(\mathbb{K})$, such that if $\iota : \mathrm{Ext}^1(E,E) \to T_{[E]}\mathfrak{M}$ is the natural isomorphism, then $\mathrm{d}\xi|_p = \iota \circ \mathrm{d}u|_p : T_pU \to T_{[E]}\mathfrak{M}$. Moreover, let G be a maximal algebraic torus in $\mathrm{Aut}(E)$, acting on $\mathrm{Ext}^1(E,E)$ by $\gamma : \epsilon \mapsto \gamma \circ \epsilon \circ \gamma^{-1}$. Then we can choose U, p, u, f, ξ and a G-action on U such that u is G-equivariant and p, f are G-invariant, so that $\mathrm{Crit}(f)$ is G-invariant, and $\xi : \mathrm{Crit}(f) \to \mathfrak{M}$ factors through the projection $\mathrm{Crit}(f) \to [\mathrm{Crit}(f)/G]$.*

Note that you can regard $u : U \to \mathrm{Ext}^1(E,E)$ as an étale open neighbourhood of 0 in $\mathrm{Ext}^1(E,E)$. Theorem 7.2.1 will be proved in §7.2.1, using §2. Next, we will use this to prove the algebraic analogue of Theorem 7.1.12:

Theorem 7.2.2. *Let X be a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathfrak{M}$ the moduli stack of coherent sheaves on X. The Behrend function $\nu_{\mathfrak{M}} : \mathfrak{M}(\mathbb{K}) \to \mathbb{Z}$ is a natural locally constructible function on $\mathfrak{M}$. For all $E_1, E_2 \in \mathrm{coh}(X)$, it satisfies:*

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{\bar{\chi}([E_1],[E_2])} \nu_{\mathfrak{M}}(E_1)\nu_{\mathfrak{M}}(E_2), \tag{7.2.1}$$

$$\int_{\substack{[\lambda]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)):\\ \lambda\Leftrightarrow 0\to E_1\to F\to E_2\to 0}} \nu_{\mathfrak{M}}(F)\,\mathrm{d}\chi \;-\; \int_{\substack{[\mu]\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2)):\\ \mu\Leftrightarrow 0\to E_2\to D\to E_1\to 0}} \nu_{\mathfrak{M}}(D)\,\mathrm{d}\chi \;=\; (e_{21}-e_{12})\;\nu_{\mathfrak{M}}(E_1\oplus E_2), \tag{7.2.2}$$

where $e_{21} = \dim \mathrm{Ext}^1(E_2, E_1)$ *and* $e_{12} = \dim \mathrm{Ext}^1(E_1, E_2)$ *for* $E_1, E_2 \in \mathrm{coh}(X)$*. Here* $\bar{\chi}([E_1], [E_2])$ *in* (7.2.1) *is the Euler form as in* (7.1.13)*, and in* (7.2.2) *the correspondence between* $[\lambda] \in \mathbb{P}(\mathrm{Ext}^1(E_2, E_1))$ *and* $F \in \mathrm{coh}(X)$ *is that* $[\lambda] \in \mathbb{P}(\mathrm{Ext}^1(E_2, E_1))$ *lifts to some* $0 \neq \lambda \in \mathrm{Ext}^1(E_2, E_1)$*, which corresponds to a short exact sequence* $0 \to E_1 \to F \to E_2 \to 0$ *in* $\mathrm{coh}(X)$ *in the usual way. The function* $[\lambda] \mapsto \nu_{\mathfrak{M}}(F)$ *is a constructible function* $\mathbb{P}(\mathrm{Ext}^1(E_2, E_1)) \to \mathbb{Z}$*, and the integrals in* (7.2.2) *are integrals of constructible functions using the Euler characteristic as measure.*

As in §7.1.3, the identities (7.2.1)–(7.2.2) are crucial for the whole program in [85], and will be proved in §7.2.2.

In the next theorem, the condition that $\mathrm{Ext}^{<0}(E^\bullet, E^\bullet) = 0$ is necessary for $\tilde{\mathfrak{M}}$ to be an Artin stack, rather than a higher stack. Note that this condition is automatically satisfied by complexes $E^\bullet$ which are semistable in any stability condition, for example Bridgeland stability conditions [20]. Therefore to prove wall-crossing formulae for Donaldson-Thomas invariants in the derived category $D^b \mathrm{coh}(X)$ under change of stability condition by the "dominant stability condition" method of [78–81, 90], it is enough to know the Behrend function identities (7.2.1)–(7.2.2) for complexes $E^\bullet$ with $\mathrm{Ext}^{<0}(E^\bullet, E^\bullet) = 0$, and we do not need to deal with complexes $E^\bullet$ with $\mathrm{Ext}^{<0}(E^\bullet, E^\bullet) \neq 0$, or with higher stacks.

Theorem 7.2.3. *Let* X *be a Calabi–Yau 3-fold over* $\mathbb{K}$*, and write* $\tilde{\mathfrak{M}}$ *for the moduli stack of complexes* $E^\bullet$ *in* $D^b \mathrm{coh}(X)$ *with* $\mathrm{Ext}^{<0}(E^\bullet, E^\bullet) = 0$*. This is an Artin stack by [70]. Let* $[E^\bullet] \in \tilde{\mathfrak{M}}(\mathbb{K})$*, and suppose that a Zariski open neighbourhood of* $[E^\bullet]$ *in* $\tilde{\mathfrak{M}}(\mathbb{K})$ *is equivalent to a global quotient* $[S/\mathrm{GL}(n, \mathbb{K})]$ *for* S *a* $\mathbb{K}$*-scheme with a* $\mathrm{GL}(n, \mathbb{K})$*-action. Then the analogues of Theorems 7.2.1 and 7.2.2 hold with* $\tilde{\mathfrak{M}}, E^\bullet$ *in place of* $\mathfrak{M}, E$*.*

The condition on $\tilde{\mathfrak{M}}$ that it should be *locally a global quotient*, is known for the moduli stack of coherent sheaves $\mathfrak{M}$ using Quot schemes. A proof of that can be found in [85, §9.3], where Joyce and Song uses the standard method for constructing coarse moduli schemes of semistable coherent sheaves in Huybrechts and Lehn [71], adapting it for Artin stacks, and an argument similar to parts of that of Luna's Etale Slice Theorem [119, §III]. However, this is not known for the moduli stack of complexes. The author expects Theorem 7.2.3 to hold without this technical assumption, but currently can't prove it.

The proof of Theorem 7.2.3 is the same as the proof of Theorem 7.2.2, substituting sheaves with complexes of sheaves, and accordingly making the obvious modifications.

Finally, in §7.2.3 we will characterize the numerical Grothendieck group of a Calabi–Yau 3-fold in terms of a deformation invariant lattice described using the Picard group. First of all, using existence results, and smoothness and properness properties of the relative Picard scheme in a family of Calabi–Yau 3-folds, one proves that the Picard groups form a local system. Actually, it is a local system with finite monodromy, so it can be made trivial after passing to a finite étale cover of the base scheme, as formulated in the analogue of [85, Thm. 4.21], which studies the monodromy of the Picard scheme instead of the numerical Grothendieck group in a family. Then, Theorem 7.2.4, a substitute for [85, Thm. 4.19], which does not need the integral Hodge conjecture result by Voisin [182] for Calabi–Yau 3-folds over $\mathbb{C}$ and which is valid over $\mathbb{K}$, characterizes the numerical Grothendieck group of a Calabi–Yau 3-fold in terms of a globally constant lattice described using the Picard scheme:

Theorem 7.2.4. *Let* X *be a Calabi–Yau* 3*-fold over* $\mathbb{K}$ *with* $H^1(\mathcal{O}_X)=0$*. Define*

$$\begin{aligned}\Lambda_X = \big\{&(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \text{ where } \lambda_0, \lambda_3 \in \mathbb{Q},\ \lambda_1 \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q},\ \lambda_2 \in \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Q}) \text{ such that}\\ &\lambda_0 \in \mathbb{Z},\ \lambda_1 \in \mathrm{Pic}(X)/\textit{torsion},\ \lambda_2 - \tfrac{1}{2}\lambda_1^2 \in \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z}),\ \lambda_3 + \tfrac{1}{12}\lambda_1 c_2(TX) \in \mathbb{Z}\big\},\end{aligned}$$

where λ_1^2 *is defined as the map* $\alpha \in \mathrm{Pic}(X) \to \frac{1}{2}c_1(\lambda_1)\cdot c_1(\lambda_1)\cdot c_1(\alpha) \in A^3(X)_{\mathbb{Q}} \cong \mathbb{Q}$, *and* $\frac{1}{12}\lambda_1 c_2(TX)$ *is defined as* $\frac{1}{12}c_1(\lambda_1)\cdot c_2(TX) \in A^3(X)_{\mathbb{Q}} \cong \mathbb{Q}$. *Then for any family of Calabi-Yau 3-folds* $\pi : \mathcal{X} \to S$ *over a connected base* S *with* $X = \pi^{-1}(s_0)$, *the lattices* Λ_{X_s} *form a local system of abelian groups over* S *with fibre* Λ_X. *Furthermore, the monodromy of this system lies in a finite subgroup of* $\mathrm{Aut}(\Lambda_X)$, *so after passing to an étale cover* $\tilde{S} \to S$ *of* S, *we can take the local system to be trivial, and coherently identify* $\Lambda_{X_{\tilde{s}}} \cong \Lambda_X$ *for all* $\tilde{s} \in \tilde{S}$. *Finally, the Chern character gives an injective morphism* $\mathrm{ch} : K^{\mathrm{num}}(\mathrm{coh}(X)) \hookrightarrow \Lambda_X$.

Following [85], this yields

Theorem 7.2.5. *The generalized Donaldson–Thomas invariants* $\bar{DT}^\alpha(\tau)$ *over* $\mathbb{K}$ *for* $\alpha \in \Lambda_X$ *are unchanged under deformations of the underlying Calabi–Yau 3-fold X, by which we mean the following: let* $\mathfrak{X} \xrightarrow{\varphi} T$ *a smooth projective morphism of algebraic* $\mathbb{K}$*-varieties* X, T, *with* T *connected. Let* $\mathcal{O}_{\mathfrak{X}}(1)$ *be a relative very ample line bundle for* $\mathfrak{X} \xrightarrow{\varphi} T$. *For each* $t \in T(\mathbb{K})$, *write* X_t *for the fibre* $X \times_{\varphi,T,t} \mathrm{Spec}\,\mathbb{K}$ *of* φ *over* t, *and* $\mathcal{O}_{X_t}(1)$ *for* $\mathcal{O}_{\mathfrak{X}}(1)|_{X_t}$. *Suppose that* X_t *is a smooth Calabi–Yau 3-fold over* $\mathbb{K}$ *for all* $t \in T(\mathbb{K})$, *with* $H^1(\mathcal{O}_{X_t}) = 0$. *Then the generalized Donaldson–Thomas invariants* $\bar{DT}^\alpha(\tau)_t$ *are independent of* $t \in T(\mathbb{K})$.

More precisely, the isomorphism $\Lambda_{X_t} = \Lambda_X$ is canonical up to action of a finite group Γ, the monodromy on T, and $DT^\alpha(\tau)_t$ are independent of the action of Γ on α, so whichever identification $\Lambda_{X_t} = \Lambda_X$ is chosen, it is still true $DT^\alpha(\tau)_t$ independent of t.

Now, recall that in [85] Joyce and Song used the assumption that the base field is the field of complex numbers $\mathbb{K} = \mathbb{C}$ for the Calabi–Yau 3-fold X in three main ways:

- **(a)** Theorem 7.1.11 in §7.1.3 is proved using gauge theory and transcendental complex analytic methods, and work only over $\mathbb{K} = \mathbb{C}$. It is used to prove the Behrend function identities (7.1.18)–(7.1.19), which are vital for much of their results, including the wall crossing formula for the $\bar{DT}^\alpha(\tau)$, and the relation between $PI^{\alpha,n}(\tau'), \bar{DT}^\alpha(\tau)$.
- **(b)** In [85, §4.5], when $\mathbb{K} = \mathbb{C}$ the Chern character embeds $K^{\mathrm{num}}(\mathrm{coh}(X))$ in $H^{\mathrm{even}}(X;\mathbb{Q})$, and they use this to show $K^{\mathrm{num}}(\mathrm{coh}(X))$ is unchanged under deformations of X. This is important for the results that $\bar{DT}^\alpha(\tau)$ and $PI^{\alpha,n}(\tau')$ for $\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))$ are invariant under deformations of X even to make sense.
- **(c)** Their notion of 'compactly embeddable' noncompact Calabi-Yau 3-folds in [85, §6.7] is complex analytic and does not make sense for general $\mathbb{K}$. This constrains the noncompact Calabi–Yau 3-folds they can define generalized Donaldson–Thomas invariants for.

Now Theorem 7.2.1 and Theorem 7.2.2 extend the results in **(a)** over algebraically closed field $\mathbb{K}$ of characteristic zero. As noted in [75], constructible functions methods fail for $\mathbb{K}$ of positive characteristic. Because of this, the alternative descriptions (7.1.9) and (7.1.22), for $DT^\alpha(\tau)$ and $PI^{\alpha,n}(\tau')$ as weighted Euler characteristics, and the definition of $\bar{DT}^\alpha(\tau)$ in §7.1.3, cannot work in positive characteristic, so working over an algebraically closed field of characteristic zero is about as general as is reasonable.

The point **(a)** above has consequences also on **(c)**, because Joyce and Song only need the notion of 'compactly embeddable' as their complex analytic proof of (7.1.18)–(7.1.19) requires X compact. Unfortunately the given algebraic version of (7.1.18)–(7.1.19) in Theorem 7.2.2 uses results from derived algebraic geometry, and the author does not know if they apply also for compactly supported sheaves on a noncompact X. We can prove a version of that under some technical assumptions, as stated in §7.3. Observe, also, that in the noncompact case you cannot expect to have the deformation invariance property unless in some particular cases in which the moduli space is proper. The extension of **(b)** to $\mathbb{K}$ is given in Section 7.2.3, which yields Theorem 7.2.5, thanks to which it is possible to extend [85, Cor. 5.28] about the deformation invariance of the generalized Donaldson–Thomas invariants in the compact case to algebraically closed fields $\mathbb{K}$ of characteristic zero. Thus, this proves our main theorem:

Theorem 7.2.6. *The theory of generalized Donaldson–Thomas invariants defined in [85] is valid over algebraically closed fields of characteristic zero.*

Next, we will respectively prove Theorems 7.2.1, 7.2.2 and 7.2.4 in §7.2.1, §7.2.2 and §7.2.3.

7.2.1 Local description of the Donaldson–Thomas moduli space

Let us fix a moduli stack $\mathfrak{M}$ which is locally a global quotient. In particular, $\mathfrak{M}$ can be the moduli stack of coherent sheaves over a Calabi-Yau 3-fold X, so that the theory exposed in §2 and §6 applies.

The first step in order to proving Theorem 7.2.1 is to show the existence of a quasiprojective $\mathbb{K}$-scheme S, an action of G on S, a point $x \in S(\mathbb{K})$ fixed by G, and a 1-morphism of Artin $\mathbb{K}$-stacks $\xi : [S/G] \to \mathfrak{M}$, which is smooth of relative dimension $\dim \mathrm{Aut}(E) - \dim G$, where $[S/G]$ is the quotient stack, such that $\xi(x\,G) = [E]$, the induced morphism on stabilizer groups $\xi_* : \mathrm{Iso}_{[S/G]}(x\,G) \to \mathrm{Iso}_{\mathfrak{M}}([E])$ is the natural morphism $G \hookrightarrow \mathrm{Aut}(E) \cong \mathrm{Iso}_{\mathfrak{M}}([E])$, and $\mathrm{d}\xi|_{x\,G} : T_xS \cong T_{x\,G}[S/G] \to T_{[E]}\mathfrak{M} \cong \mathrm{Ext}^1(E,E)$ is an isomorphism.

As $\mathfrak{M}$ is locally a global quotient, let's say $\mathfrak{M}$ is locally $[Q/H]$ with $H = \mathrm{GL}(n,\mathbb{K})$, and a $\mathbb{K}$-scheme Q which is H-invariant, so that the projection $[Q/H] \to \mathfrak{M}$ is a 1-isomorphism with an open $\mathbb{K}$-substack $\mathfrak{Q}$ of $\mathfrak{M}$. This 1-isomorphism identifies the stabilizer groups $\mathrm{Iso}_{\mathfrak{M}}([E]) = \mathrm{Aut}(E)$ and $\mathrm{Iso}_{[Q/H]}(xH) = \mathrm{Stab}_H(x)$, and the Zariski tangent spaces $T_{[E]}\mathfrak{M} \cong \mathrm{Ext}^1(E,E)$ and $T_{xH}[Q/H] \cong T_xQ/T_x(xH)$, so one has natural isomorphisms $\mathrm{Aut}(E) \cong \mathrm{Stab}_H(x)$ and $\mathrm{Ext}^1(E,E) \cong T_xQ/T_x(xH)$, and G is identified as a subgroup of H.

To obtain the 1-morphism with the required properties, following [85, §9.3] and Luna's Etale Slice Theorem [119, §III], we obtain an atlas S as a G-invariant, locally closed $\mathbb{K}$-subscheme in Q with $x \in S(\mathbb{K})$, such that $T_xQ = T_xS \oplus T_x(xH)$, and the morphism $\mu : S \times H \to Q$ induced by the inclusion $S \hookrightarrow Q$ and the H-action on Q is smooth of relative dimension $\dim \mathrm{Aut}(E)$. Here $x \in Q(\mathbb{K})$ project to the point xH in $\mathfrak{Q}(\mathbb{K})$ identified with $[E] \in \mathfrak{M}(\mathbb{K})$ under the 1-isomorphism $\mathfrak{Q} \cong [Q/H]$ and G, a $\mathbb{K}$-subgroup of the $\mathbb{K}$-group H, is as in the statement of Theorem 7.2.1, that is, a maximal torus in $\mathrm{Aut}(E)$. Since S is invariant under the $\mathbb{K}$-subgroup G of the $\mathbb{K}$-group H acting on Q, the inclusion $i : S \hookrightarrow Q$ induces a representable 1-morphism of quotient stacks $i_* : [S/G] \to [Q/H]$. In [85], Joyce and Song found that i_* is smooth of relative dimension $\dim \mathrm{Aut}(E) - \dim G$. Combining the 1-morphism $i_* : [S/G] \to [Q/H]$, the 1-isomorphism $\mathfrak{Q} \cong [Q/H]$, and the open inclusion $\mathfrak{Q} \hookrightarrow \mathfrak{M}$, yields a 1-morphism $\xi : [S/G] \to \mathfrak{M}$, as required for Theorem 7.2.1. This ξ is smooth of relative dimension $\dim \mathrm{Aut}(E) - \dim G$, as i_* is. The conditions that $\xi(x\,G) = [E]$ and that $\xi_* : \mathrm{Iso}_{[S/G]}(x\,G) \to \mathrm{Iso}_{\mathfrak{M}}([E])$ is the natural $G \hookrightarrow \mathrm{Aut}(E) \cong \mathrm{Iso}_{\mathfrak{M}}([E])$ in Theorem 7.1.11 are immediate from the construction. That $\mathrm{d}\xi|_{x\,G} : T_xS \cong T_{x\,G}[S/G] \to T_{[E]}\mathfrak{M} \cong \mathrm{Ext}^1(E,E)$ is an isomorphism follows from $T_{[E]}\mathfrak{M} \cong T_{xH}[Q/H] \cong T_xQ/T_x(xH)$ and $T_xQ = T_xS \oplus T_x(xH)$.

In conclusion, we can summarize as follows: given a point $[E] \in \mathfrak{M}(\mathbb{K})$, that is an equivalence class of a (complex of) coherent sheaves, we will denote by G a maximal torus in $\mathrm{Aut}(E)$. As $\mathfrak{M}$ is locally a global quotient, there exists an atlas S, which is a scheme over $\mathbb{K}$, and a smooth morphism $\pi : S \to \mathfrak{M}$, with π smooth of relative dimension $\dim G$. If $x \in S$ is the point corresponding to $E \in \mathfrak{M}(\mathbb{K})$, then π smooth of $\dim G$ means that π has *minimal dimension* near E, that is $T_xS = \mathrm{Ext}^1(E,E)$. Moreover, the atlas S is endowed with a G-action, so that π descends to a morphism $[S/G] \to \mathfrak{M}$. Note next that the maximal torus G acts on S preserving x. By replacing S by a G-equivariant étale open neighbourhood S' of x, we can suppose S is affine. Thus the atlas S in the sense of Corollary 6.1.5 and Theorem 6.2.1 for the moduli stack $\mathfrak{M}$ carries a d-critical locus structure (S, s_S) which is G-equivariant in the sense of §2.3.

Using Proposition 2.3.3, there exists a G-invariant critical chart (R,U,f,i) in the sense of §2 for (S,s) with x in R, and $\dim U$ is minimal so that $T_{i(x)}U = T_xR = \mathrm{Ext}^1(E,E)$. Making U smaller if necessary, we can choose G-equivariant étale coordinates $U \to \mathbb{A}^n \cong \mathrm{Ext}^1(E,E)$ near $i(x)$, sending $i(x)$ to 0, and with $T_{i(x)}U = \mathrm{Ext}^1(E,E)$ the given identification. Then we can regard $U \to \mathrm{Ext}^1(E,E)$ as a G-equivariant étale open neighbourhood of 0 in $\mathrm{Ext}^1(E,E)$, which concludes the proof of Theorem 7.2.1.

7.2.2 Behrend function identities

Now we are ready to prove Theorem 7.2.2. Let X be a Calabi–Yau 3-fold over an algebraically closed field $\mathbb{K}$ of characteristic zero, $\mathfrak{M}$ the moduli stack of coherent sheaves on X, and E_1, E_2 be coherent sheaves on X. Set $E = E_1 \oplus E_2$. Using the splitting

$$\mathrm{Ext}^1(E,E) = \mathrm{Ext}^1(E_1,E_1) \oplus \mathrm{Ext}^1(E_2,E_2) \oplus \mathrm{Ext}^1(E_1,E_2) \oplus \mathrm{Ext}^1(E_2,E_1), \tag{7.2.3}$$

write elements of $\mathrm{Ext}^1(E,E)$ as $(\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21})$ with $\epsilon_{ij} \in \mathrm{Ext}^1(E_i,E_j)$. For simplicity, we will write $e_{ij} = \dim \mathrm{Ext}^1(E_i,E_j)$. Choose a maximal torus G of $\mathrm{Aut}(E)$ which contains the subgroup $T = \{\mathrm{id}_{E_1} + \lambda \mathrm{id}_{E_2} : \lambda \in \mathbb{G}_m\}$, which acts on $\mathrm{Ext}^1(E,E)$ by

$$\lambda : (\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21}) \mapsto (\epsilon_{11}, \epsilon_{22}, \lambda^{-1}\epsilon_{12}, \lambda\epsilon_{21}). \tag{7.2.4}$$

Apply Theorem 7.2.1 with these E and G. This gives a G-equivariant étale morphism $u : U \to \mathrm{Ext}^1(E,E)$ with U a smooth affine G-invariant $\mathbb{K}$-scheme, and $u(p) = 0$, for $p \in U(\mathbb{K})$, a G-invariant regular function $f : U \to \mathbb{A}^1_{\mathbb{K}}$ on U with $f|_p = \partial f|_p = 0$, an open neighbourhood V of s in S, and a 1-morphism $\xi : \mathrm{Crit}(f) \to \mathfrak{M}$ smooth of relative dimension $\dim \mathrm{Aut}(E)$, with $\xi(p) = [E] \in \mathfrak{M}(\mathbb{K})$ and $\mathrm{d}\xi|_p : T_p(\mathrm{Crit}(f)) = \mathrm{Ext}^1(E,E) \to T_{[E]}\mathfrak{M}$ the natural isomorphism. Then the Behrend function $\nu_{\mathfrak{M}}$ at $[E] = [E_1 \oplus E_2]$ satisfies

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{\dim \mathrm{Aut}(E)} \nu_{\mathrm{Crit}(f)}(0), \tag{7.2.5}$$

where one uses that ξ is smooth of relative dimension $\dim \mathrm{Aut}(E)$, and Theorem 7.1.5 to say that

$$\nu_{\mathrm{Crit}(f)} = (-1)^{\dim(\mathrm{Aut}(E))} \xi^*(\nu_{\mathfrak{M}}).$$

On the other hand, the last part of the proof of (7.2.1) in [85, Section 10.1] uses algebraic methods and gives

$$\nu_{\mathfrak{M}}(E_1)\nu_{\mathfrak{M}}(E_2) = \nu_{\mathfrak{M}\times\mathfrak{M}}(E_1,E_2) = (-1)^{\dim \mathrm{Aut}(E_1) + \dim \mathrm{Aut}(E_2)} \nu_{\mathrm{Crit}(f^G)}(0), \tag{7.2.6}$$

where $\nu_{Crit(f^G)}(0) = \nu_{\mathrm{Crit}(f)^G}(0) = \nu_{\mathrm{Crit}(f|_{U \cap \mathrm{Ext}^1(E,E)^G})}(0)$ and U is as in Theorem 7.2.1 and $\mathrm{Ext}^1(E,E)^G$ denotes the fixed point locus of $\mathrm{Ext}^1(E,E)$ for the G-action. Thus what actually remains to prove in order to establish identity (7.2.1) is

$$\nu_{\mathrm{Crit}(f)}(0) = (-1)^{\dim \mathrm{Ext}^1(E_1,E_2) + \dim \mathrm{Ext}^1(E_2,E_1)} \nu_{\mathrm{Crit}(f^G)}(0). \tag{7.2.7}$$

This is a generalization of a result in [7] over $\mathbb{C}$ in the case of an isolated $\mathbb{C}^*$-fixed point. Combining equations (7.2.5), (7.2.6) and (7.2.7) and sorting out the signs as in [85, Section 10.1] proves equation (7.2.1). Equation (7.2.7) will be crucial also for the proof of the second Behrend identity (7.2.2).

Let us start by recalling an easy result similar to [85, Prop. 10.1], but now in the étale topology. Let $u : U \to \mathrm{Ext}^1(E,E)$ be the étale map as in §7.2.1, and $p \in U$ such that $u(p) = 0$. We will consider points $(0,0,\epsilon_{12},0), (0,0,0,\epsilon_{21}) \in \mathrm{Ext}^1(E,E)$ basically like points in U. This is because we consider a unique lift $\alpha(e_{12})$ of $(0,0,\epsilon_{12},0)\in \mathrm{Ext}^1(E,E)$ to U, such that $u(\alpha(e_{12}))=(0,0,e_{12},0)$ and $\lim\limits_{\lambda\to 0}\lambda.\alpha(e_{12})=p$, using that $\lim\limits_{\lambda\to 0}(0,0,\lambda^{-1}\epsilon_{12},0)=(0,0,0,0)$. In the sequel we will also treat points of the blow-up of $u^{-1}(\epsilon_{12}=0)$ in U like points of the blow-up of $e_{12}=0$ in $\mathrm{Ext}^1(E,E)$, and so on.

So we can state the following result, for the proof of which we cite [85, Prop.10.1], with appropriate obvious modifications, working in the étale topology.

Proposition 7.2.7. *Let $\epsilon_{12}\in \mathrm{Ext}^1(E_1,E_2)$ and $\epsilon_{21}\in \mathrm{Ext}^1(E_2,E_1)$. Then*

- **(i)** $(0,0,\epsilon_{12},0),(0,0,0,\epsilon_{21})\in \mathrm{Crit}(f)\subseteq U\subseteq \mathrm{Ext}^1(E,E)$, *and* $(0,0,\epsilon_{12},0),(0,0,0,\epsilon_{21})\in V\subseteq S(\mathbb{K})\subseteq \mathrm{Ext}^1(E,E)$;
- **(ii)** ξ *maps* $(0,0,\epsilon_{12},0)\mapsto(0,0,\epsilon_{12},0)$ *and* $(0,0,0,\epsilon_{21})\mapsto(0,0,0,\epsilon_{21})$; *and*
- **(iii)** *the induced morphism on closed points* $[S/\mathrm{Aut}(E)](\mathbb{K})\to\mathfrak{M}(\mathbb{K})$ *maps* $[(0,0,0,\epsilon_{21})]\mapsto[F]$ *and* $[(0,0,\epsilon_{12},0)]\mapsto[F']$, *where the exact sequences* $0\to E_1\to F\to E_2\to 0$ *and* $0\to E_2\to F'\to E_1\to 0$ *in* $\mathrm{coh}(X)$ *correspond to* $\epsilon_{21}\in\mathrm{Ext}^1(E_2,E_1)$ *and* $\epsilon_{12}\in\mathrm{Ext}^1(E_1,E_2)$, *respectively.*

Now use the idea in [85, §10.2]. Set $U'=\big\{(\epsilon_{11},\epsilon_{22},\epsilon_{12},\epsilon_{21})\in U:\epsilon_{21}\neq 0\big\}$, an open set in U, (this is using our informal notation of writing points of the étale open set $U\hookrightarrow \mathrm{Ext}^1(E,E)$ as if they were points of $\mathrm{Ext}^1(E,E)$. Formally we should write $U'=\{p\in U:u(p)=(\epsilon_{11},\epsilon_{22},\epsilon_{12},\epsilon_{21}),\epsilon_{21}\neq 0\}$.) Write V' for the submanifold of $(\epsilon_{11},\epsilon_{22},\epsilon_{12},\epsilon_{21})\in U'$ with $\epsilon_{12}=0$. Let $\tilde U'$ be the blowup of U' along V', with projection $\pi':\tilde U'\to U'$. With our convention on the notation, points of $\tilde U'$ may be written $(\epsilon_{11},\epsilon_{22},[\epsilon_{12}],\lambda\epsilon_{12},\epsilon_{21})$, where $[\epsilon_{12}]\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))$, and $\lambda\in\mathbb{K}$, and $\epsilon_{21}\neq 0$. Write $f'=f|_{U'}$ and $\tilde f'=f'\circ\pi'$. Then applying Theorem 7.1.6 to $U',V',f',\tilde U',\pi',\tilde f'$ at the point $(0,0,0,\epsilon_{21})\in U'$ gives

$$
\begin{aligned}
\nu_{\mathrm{Crit}(f)}(0,0,0,\epsilon_{21}) \quad &= \int\limits_{[\epsilon_{12}]\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))} \nu_{\mathrm{Crit}(\tilde f')}(0,0,[\epsilon_{12}],0,\epsilon_{21})\,\mathrm{d}\chi+ \\
&\qquad (-1)^{e_{12}}\big(1-e_{12}\big)\nu_{\mathrm{Crit}(f|_{V'})}(0,0,0,\epsilon_{21}).
\end{aligned}
\tag{7.2.8}
$$

Here $\nu_{\mathrm{Crit}(f)}(0,0,0,\epsilon_{21})$ is independent of the choice of ϵ_{21} representing $[\epsilon_{21}]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))$, and is a constructible function of $[\epsilon_{21}]$, so the integrals in (7.2.8) are well-defined. Note that $\nu_{\mathrm{Crit}(f)}$ and the other Behrend functions in the sequel are nonzero just on the zero loci of the corresponding functions, so here and in the sequel the integrals over the whole $\mathbb{P}(\mathrm{Ext}^1(\ldots))$ actually are just over the points that lie in these zero loci. Adopt this convention for the whole section.

Similarly consider the analogous situation exchanging the role of ϵ_{12} and ϵ_{21}. Set $U''=\big\{(\epsilon_{11},\epsilon_{22},\epsilon_{12},\epsilon_{21})\in U:\epsilon_{12}\neq 0\big\}$, an open set in U, and write $V''=\big\{(\epsilon_{11},\epsilon_{22},\epsilon_{12},\epsilon_{21})\in U'':\epsilon_{21}=0\big\}$. Let $\tilde U''$ be the blowup of U'' along V'', with projection $\pi'':\tilde U''\to U''$. Points of $\tilde U''$ may be written $(\epsilon_{11},\epsilon_{22},\epsilon_{12},[\epsilon_{21}],\lambda\epsilon_{21})$, where $[\epsilon_{21}]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))$, and $\lambda\in\mathbb{K}$, and $\epsilon_{12}\neq 0$. Write $f''=f|_{U''}$ and $\tilde f''=f''\circ\pi''$. Similarly to the previous situation, we can apply Theorem 7.1.6 to $U'',V'',f'',\tilde U'',\pi'',\tilde f''$ at the point $(0,0,\epsilon_{12},0)\in U''$ which gives

$$
\begin{aligned}
\nu_{\mathrm{Crit}(f)}(0,0,\epsilon_{12},0) \quad &= \int\limits_{[\epsilon_{21}]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))} \nu_{\mathrm{Crit}(\tilde f'')}(0,0,\epsilon_{12},0,[\epsilon_{21}])\,\mathrm{d}\chi+ \\
&\qquad (-1)^{e_{21}}\big(1-e_{21}\big)\nu_{\mathrm{Crit}(f|_{V''})}(0,0,\epsilon_{12},0).
\end{aligned}
\tag{7.2.9}
$$

Let $L_{12}\to\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))$ and $L_{21}\to\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))$ be the tautological line bundles, so that the fibre of L_{12} over a point $[\epsilon_{12}]$ in $\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))$ is the 1-dimensional subspace $\{\lambda\,\epsilon_{12}:\lambda\in\mathbb{K}\}$

in $\mathrm{Ext}^1(E_1, E_2)$. Consider the fibre product

$$\begin{tikzcd} Z \arrow[r, "\text{étale}"] \arrow[d] & \mathrm{Ext}^1(E_1,E_1)\times \mathrm{Ext}^1(E_2,E_2)\times (L_{12}\oplus L_{21}) \arrow[d] \\ U \arrow[r, "\text{étale}"] & \mathrm{Ext}^1(E,E) \end{tikzcd}$$

where the horizontal maps are étale morphisms. Informally, this defines $Z \subseteq \mathrm{Ext}^1(E_1, E_1) \times \mathrm{Ext}^1(E_2, E_2) \times (L_{12} \oplus L_{21})$ to be the étale open subset of points $\big(\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], \lambda_1\, \epsilon_{12}, [\epsilon_{21}], \lambda_2\, \epsilon_{21}\big)$ for $\lambda_i \in \mathbb{K}$, for which $(\epsilon_{21}, \epsilon_{22}, \lambda_1\, \epsilon_{12}, \lambda_2\, \epsilon_{21})$ lies in U. In other words, $\mathrm{Ext}^1(E_1, E_1)\times \mathrm{Ext}^1(E_2, E_2)\times (L_{12} \oplus L_{21})$ is obtained from $\mathrm{Ext}^1(E, E)$ by two commuting blow-ups to $\mathrm{Ext}^1(E, E)$ at the transversely intersecting submanifolds $\{e_{12} = 0\}$ and $\{e_{21} = 0\}$ and Z is obtained doing two commuting blow-ups to U at the transversely intersecting submanifolds in U pulling back $\{e_{12} = 0\}$ and $\{e_{21} = 0\}$. Observe that Z contains both $\tilde{U}'$ and $\tilde{U}''$, which respectively have subspaces $\mathrm{Crit}(\tilde{f}')$ and $\mathrm{Crit}(\tilde{f}'')$.

We claim that there is an étale open set $W \hookrightarrow \mathrm{Ext}^1(E_1, E_1) \times \mathrm{Ext}^1(E_2, E_2) \times (L_{12} \otimes L_{21})$ fitting into a Cartesian square:

$$\begin{tikzcd} Z \arrow[r, "\text{étale}"] \arrow[d, "\Pi"'] & \mathrm{Ext}^1(E_1,E_1)\times \mathrm{Ext}^1(E_2,E_2)\times (L_{12}\oplus L_{21}) \arrow[d, "\Pi'"] \\ W \arrow[r, "\text{étale}"] & \mathrm{Ext}^1(E_1,E_1)\times \mathrm{Ext}^1(E_2,E_2)\times (L_{12}\otimes L_{21}) \end{tikzcd} \tag{7.2.10}$$

with Π a $\mathbb{G}_m$-invariant morphism. Note that this does not define W as a fibre product, or characterize W by a universal property: rather, it says Z is a fibre product involving W.

To see that such a W exists, first consider the morphism $L_{12} \oplus L_{21} \to L_{12} \otimes L_{21}$, which is the right hand column of (7.2.10), omitting constant factors. This is a morphism of fibre bundles over $\mathbb{P}(\mathrm{Ext}^1(E_1, E_2)) \times \mathbb{P}(\mathrm{Ext}^1(E_2, E_1))$, with each morphism of fibres modelled on the morphism $\mathbb{A}^2 \to \mathbb{A}^1$ mapping $(x, y) \mapsto xy$. The group $\mathbb{G}_m$ acts on $L_{12} \oplus L_{21}$ preserving the bundle structure, on the fibres $\mathbb{A}^2$ as $\lambda : (x, y) \mapsto (\lambda^{-1}x, \lambda y)$, and the map $L_{12} \oplus L_{21} \to L_{12} \otimes L_{21}$ is $\mathbb{G}_m$-invariant.

Now this map $\mathbb{A}^2 \to \mathbb{A}^1$ mapping $(x, y) \mapsto xy$ is in fact a GIT quotient by the given $\mathbb{G}_m$ action, using the trivial linearization in which all of $\mathbb{A}^2$ is GIT-semistable. Restricting to the open subschemes $\{x \neq 0\}$ or $\{y \neq 0\}$ in $\mathbb{A}^2$, the $\mathbb{G}_m$-action becomes free, and $\{x \neq 0\} \to \mathbb{A}^1$, $\{y \neq 0\} \to \mathbb{A}^1$ are principal $\mathbb{G}_m$-bundles.

Similarly, it is an easy exercise in GIT to show that the morphism $L_{12} \oplus L_{21} \to L_{12} \otimes L_{21}$ is a GIT quotient by $\mathbb{G}_m$, using the trivial linearization in which all of $L_{12} \oplus L_{21}$ is GIT semistable, and restricting to the open subschemes $\{e_{12} \neq 0\}$ and $\{e_{21} \neq 0\}$ in $L_{12} \oplus L_{21}$ gives principal $\mathbb{G}_m$-bundles.

From all this, we see that we can define $\Pi : Z \to W$ to be the GIT quotient of Z by $\mathbb{G}_m$, with the trivial linearization in which all of Z is GIT semistable, and then W exists and fits into a Cartesian square (7.2.10) with étale horizontal morphisms. Also, the restriction of Π to the open subsets $\tilde{U}'$ and $\tilde{U}''$ in Z, corresponding to $\{e_{21} \neq 0\}$ and $\{e_{12} \neq 0\}$, are principal $\mathbb{G}_m$-bundles. So in particular, the restrictions of Π to $\tilde{U}'$ and $\tilde{U}''$ are smooth.

Here is the crucial point: $\mathrm{Crit}(\tilde{f}') \subset \tilde{U}'$ and $\mathrm{Crit}(\tilde{f}'') \subset \tilde{U}''$ are $\mathbb{G}_m$-invariant subschemes, where $\tilde{f}'$ and $\tilde{f}''$ are the restriction respectively on $\tilde{U}'$ and $\tilde{U}''$ of the G-invariant function $\tilde{f}$ on Z, pullback of the G-invariant function f on U through the projection $Z \to U$. Now, as $\tilde{f}$ is $\mathbb{G}_m$ invariant, it pushes down to the GIT quotient W of Z by $\mathbb{G}_m$. So there exists a function $\hat{f} : W \to \mathbb{A}^1_{\mathbb{K}}$ so that $\tilde{f} = \hat{f} \circ \Pi$. Define the scheme Q to be $\mathrm{Crit}(\hat{f})$. As the projection Π is smooth on $\tilde{U}'$ and $\tilde{U}''$, we get that $\mathrm{Crit}(\tilde{f}') = \Pi^{-1}(Q) \cap \tilde{U}'$ and $\mathrm{Crit}(\tilde{f}'') = \Pi^{-1}(Q) \cap \tilde{U}''$ and both

$\Pi : \mathrm{Crit}(\tilde{f}') \to Q$ and $\Pi : \mathrm{Crit}(\tilde{f}'') \to Q$ are smooth of relative dimension 1. Thus Theorem 7.1.5 yields that $\nu_{\mathrm{Crit}(\tilde{f}')} = -\Pi^*(\nu_Q)$ and $\nu_{\mathrm{Crit}(\tilde{f}'')} = -\Pi^*(\nu_Q)$ and then

$$\nu_{\mathrm{Crit}(\tilde{f}')}(0,0,[\epsilon_{12}],0,\epsilon_{21}) = -\nu_Q(0,0,[\epsilon_{12}],[\epsilon_{21}],0) = \nu_{\mathrm{Crit}(\tilde{f}'')}(0,0,\epsilon_{12},0,[\epsilon_{21}]), \tag{7.2.11}$$

where the sign comes from the fact that the map Π is smooth of relative dimension 1. Moreover observe that

$$\nu_{\mathrm{Crit}(f|_{V'})}(0,0,0,\epsilon_{21}) = (-1)^{e_{21}}\nu_{\mathrm{Crit}(f)^G}(0,0,0,0). \tag{7.2.12}$$

This is because the G-invariance of f imply that its values on $(\epsilon_{11},\epsilon_{22},0,\epsilon_{21})$ and $(\epsilon_{11},\epsilon_{22},0,0)$ are the same and the projection $\mathrm{Crit}(f|_{V'}) \to \mathrm{Crit}(f|_{U^G})$ is smooth of relative dimension e_{21}. For the same reason, one has

$$\nu_{\mathrm{Crit}(f|_{V''})}(0,0,\epsilon_{12},0) = (-1)^{e_{12}}\nu_{\mathrm{Crit}(f)^G}(0,0,0,0). \tag{7.2.13}$$

Now, substitute equations (7.2.11), (7.2.12) and (7.2.13) into (7.2.8) and (7.2.9). One gets

$$\begin{aligned}\nu_{\mathrm{Crit}(f)}(0,0,0,\epsilon_{21}) &= -\int_{[\epsilon_{12}]\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))} \nu_Q(0,0,[\epsilon_{12}],[\epsilon_{21}],0)\,\mathrm{d}\chi + \\ &\qquad (-1)^{e_{12}+e_{21}}\big(1-e_{12}\big)\nu_{\mathrm{Crit}(f)^G}(0,0,0,0),\end{aligned} \tag{7.2.14}$$

$$\begin{aligned}\nu_{\mathrm{Crit}(f)}(0,0,\epsilon_{12},0) &= -\int_{[\epsilon_{21}]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))} \nu_Q(0,0,[\epsilon_{12}],[\epsilon_{21}],0)\,\mathrm{d}\chi + \\ &\qquad (-1)^{e_{12}+e_{21}}\big(1-e_{21}\big)\nu_{\mathrm{Crit}(f)^G}(0,0,0,0).\end{aligned} \tag{7.2.15}$$

Since $\chi\big(\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))\big) = e_{21}$ and $\chi\big(\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))\big) = e_{12}$, integrating (7.2.14) over $[\epsilon_{21}] \in \mathbb{P}(\mathrm{Ext}^1(E_2,E_1))$, and (7.2.15) over $[\epsilon_{12}] \in \mathbb{P}(\mathrm{Ext}^1(E_1,E_2))$, yields respectively

$$\begin{aligned}\int_{[\epsilon_{21}]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))} \nu_{\mathrm{Crit}(f)}(0,0,0,\epsilon_{21})\,\mathrm{d}\chi &= -\int_{([\epsilon_{12}],[\epsilon_{21}])\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))\times\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))} \nu_Q(0,0,[\epsilon_{12}],[\epsilon_{21}],0)\,\mathrm{d}\chi \\ &\quad + (-1)^{e_{12}+e_{21}}\big(1-e_{12}\big)e_{21}\nu_{\mathrm{Crit}(f)^G}(0),\end{aligned} \tag{7.2.16}$$

$$\begin{aligned}\int_{[\epsilon_{12}]\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))} \nu_{\mathrm{Crit}(f)}(0,0,\epsilon_{12},0)\,\mathrm{d}\chi &= -\int_{([\epsilon_{12}],[\epsilon_{21}])\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))\times\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))} \nu_Q(0,0,[\epsilon_{12}],[\epsilon_{21}],0)\,\mathrm{d}\chi \\ &\quad + (-1)^{e_{12}+e_{21}}\big(1-e_{21}\big)e_{12}\nu_{\mathrm{Crit}(f)^G}(0),\end{aligned} \tag{7.2.17}$$

Subtracting (7.2.16) from (7.2.17), gives

$$\begin{aligned}\int_{[\epsilon_{21}]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))} \nu_{\mathrm{Crit}(f)}(0,0,0,\epsilon_{21})\,\mathrm{d}\chi - \int_{[\epsilon_{12}]\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))} \nu_{\mathrm{Crit}(f)}(0,0,\epsilon_{12},0)\,\mathrm{d}\chi = \\ (-1)^{e_{12}+e_{21}}\big(e_{21}-e_{12}\big)\nu_{\mathrm{Crit}(f)^G}(0).\end{aligned} \tag{7.2.18}$$

Consider equation (7.2.18) applied substituting $\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)\oplus\mathbb{K})$ to $\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))$. This adds one dimension to $\mathrm{Ext}^1(E,E)$. Denote $\tilde{\tilde{f}}$ the lift of f to $\mathrm{Ext}^1(E,E)\oplus\mathbb{K}$. In this case equation (7.2.18) becomes

$$\begin{aligned}\int_{[\epsilon_{21}]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)\oplus\mathbb{K})} \nu_{\mathrm{Crit}(\tilde{\tilde{f}})}(0,0,0,\epsilon_{21}\oplus\lambda)\,\mathrm{d}\chi - \int_{[\epsilon_{12}]\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))} \nu_{\mathrm{Crit}(\tilde{\tilde{f}})}(0,0,\epsilon_{12},0)\,\mathrm{d}\chi = \\ (-1)^{1+e_{12}+e_{21}}\big(1+e_{21}-e_{12}\big)\nu_{\mathrm{Crit}(\tilde{\tilde{f}})^G}(0),\end{aligned} \tag{7.2.19}$$

Now, observe that $\nu_{\mathrm{Crit}(f)} = -\nu_{\mathrm{Crit}(\tilde{\tilde{f}})}$ from Theorem 7.1.5 and $\nu_{\mathrm{Crit}(\tilde{\tilde{f}})^G}(0) = \nu_{\mathrm{Crit}(f)^G}(0)$ as $(\mathrm{Ext}^1(E,E)\oplus\mathbb{K})^G = \mathrm{Ext}^1(E,E)^G \oplus 0$ and the map $\mathrm{Crit}(\tilde{\tilde{f}})^G \to \mathrm{Crit}(f)^G$ is étale. Thus

$$-\int_{[\epsilon_{21}]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))} \nu_{\mathrm{Crit}(f)}(0,0,0,\epsilon_{21})\,\mathrm{d}\chi \;-\; \nu_{\mathrm{Crit}(f)}(0,0,0,0) \;+ \int_{[\epsilon_{12}]\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2))} \nu_{\mathrm{Crit}(f)}(0,0,\epsilon_{12},0)\,\mathrm{d}\chi = (-1)^{1+e_{12}+e_{21}}\big(1+e_{21}-e_{12}\big)\nu_{\mathrm{Crit}(f)^G}(0). \tag{7.2.20}$$

Here, $\nu_{\mathrm{Crit}(f)}(0)$ on the l.h.s. comes from the fact that the $\mathbb{G}_m$-action over $\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)\oplus\mathbb{K})$ fixes $\mathbb{P}(\mathrm{Ext}^1(E_2,E_1))$ and $[0,1]$; the free orbits of the $\mathbb{G}_m$-action contribute zero to the weighted Euler characteristic. Then one uses that $\nu_{\mathrm{Crit}(\tilde{\tilde{f}})}$ valued over $[0,1]$ is equal to $-\nu_{\mathrm{Crit}(f)}(0)$. Adding (7.2.18) and (7.2.20) yields (7.2.7), which concludes the proof of identity (7.2.1).

The conclusion of the proof of identity (7.2.2) is now easy. Let $0 \neq \epsilon_{21} \in \mathrm{Ext}^1(E_2,E_1)$ correspond to the short exact sequence $0 \to E_1 \to F \to E_2 \to 0$ in $\mathrm{coh}(X)$. Then

$$\nu_{\mathfrak{M}}(F) = (-1)^{\dim \mathrm{Aut}(E)}\nu_{\mathrm{Crit}(f)}(0,0,0,\epsilon_{21}) \tag{7.2.21}$$

using $\xi_* : [(0,0,0,\epsilon_{21})] \mapsto [F]$ from Proposition 7.2.7 and ξ smooth of relative dimension $\dim(\mathrm{Aut}(E))$ and properties of Behrend function in Theorem 7.1.5. Substituting (7.2.21) and its analogue for D in the place of F into (7.2.2), using equation (7.2.5) and identity (7.2.7) to substitute for $\nu_{\mathfrak{M}}(E_1\oplus E_2)$, and cancelling factors of $(-1)^{\dim \mathrm{Aut}(E)}$, one gets that (7.2.2) is equivalent to (7.2.18), which concludes the proof.

7.2.3 Deformation invariance issue

Thomas' original definition (7.1.1) of $DT^\alpha(\tau)$, and Joyce and Song's definition (7.1.22) of the pair invariants $PI^{\alpha,n}(\tau')$, are both valid over $\mathbb{K}$. Joyce and Song suggest to solve problem (b) in §7.2 to work in [85, Rmk 4.20 (e)], replacing $H^*(X;\mathbb{Q})$ by the *algebraic de Rham cohomology* $H^*_{\mathrm{dR}}(X)$ of Hartshorne [64]. Here we suggest another argument which is based on the theory of *Picard schemes* by Grothendieck [62, 63]. Other references are [3, 100]. Even if our argument will not prove that the numerical Grothendieck groups are deformation invariant, as this last fact depend deeply on the integral Hodge conjecture type result [182] which we are not able to prove in this more general context, we will however find a deformation invariant lattice Λ_{X_t} containing its image through the Chern character map and define $\bar{DT}^\alpha(\tau)_t$ for $\alpha \in \Lambda_{X_t}$ which will be deformation invariant. For the whole section we will work in the étale topology.

To prove deformation-invariance we need to work not with a single Calabi–Yau 3-fold X over $\mathbb{K}$, but with a *family* of Calabi–Yau 3-folds $\mathfrak{X} \xrightarrow{\varphi} T$ over a base $\mathbb{K}$-scheme T. Taking $T = \operatorname{Spec}\mathbb{K}$ recovers the case of one Calabi–Yau 3-fold. Here are our assumptions and notation for such families. Let $\mathfrak{X} \xrightarrow{\varphi} T$ be a smooth projective morphism of algebraic $\mathbb{K}$-varieties X, T, with T connected. Let $\mathcal{O}_{\mathfrak{X}}(1)$ be a relative very ample line bundle for $\mathfrak{X} \xrightarrow{\varphi} T$. For each $t \in T(\mathbb{K})$, write X_t for the fibre $X \times_{\varphi,T,t} \operatorname{Spec}\mathbb{K}$ of φ over t, and $\mathcal{O}_{X_t}(1)$ for $\mathcal{O}_{\mathfrak{X}}(1)|_{X_t}$. Suppose that X_t is a smooth Calabi–Yau 3-fold over $\mathbb{K}$ for all $t \in T(\mathbb{K})$, with $H^1(\mathcal{O}_{X_t}) = 0$.

There are some important existence theorems which refine the original Grothendieck's theorem [62, Thm. 3.1]. In [3, Thm. 7.3], Artin proves that given $f : X \to S$ a flat, proper, and finitely presented map of algebraic spaces cohomologically flat in dimension zero, then the relative Picard scheme $\mathrm{Pic}_{X/S}$ exists as an algebraic space which is locally of finite presentation over S. Its fibres are the Picard schemes $\mathrm{Pic}(X_s)$ of the fibres. They form a family whose total space is $\mathrm{Pic}_{X/S}$. In [63, Prop. 2.10] Grothendieck shows that if $H^2(X_s, \mathcal{O}_{X_s}) = 0$ for some $s \in S$, there exists a neighborhood U of s such that the scheme $\mathrm{Pic}_{X/S|_U}$ is smooth, and in this case $\dim(\mathrm{Pic}(X_s)) = \dim(H^1(X_s, \mathcal{O}_{X_s}))$.

In our case, $\mathrm{Pic}_{\mathfrak{X}/T}$ exists and is smooth with 0-dimensional fibres which are the Picard schemes $\mathrm{Pic}(X_t)$. These results yield that the Picard schemes $\mathrm{Pic}(X_t)$ for $t \in T(\mathbb{K})$ are canonically isomorphic *locally* in $T(\mathbb{K})$ in the étale topology. Observe that at the moment we don't have canonical isomorphisms $\mathrm{Pic}(X_t) \cong \mathrm{Pic}(X)$ for all $t \in T(\mathbb{K})$ (this would be canonically isomorphic *globally* in $T(\mathbb{K})$). Instead, we mean that the groups $\mathrm{Pic}(X_t)$ for $t \in T(\mathbb{K})$ form a *local system of abelian groups* over $T(\mathbb{K})$, with fibre $\mathrm{Pic}(X)$.

When $\mathbb{K} = \mathbb{C}$, Joyce and Song proved [85, §4] that $K^{\mathrm{num}}(\mathrm{coh}(X_t))$ form a local system of abelian groups over $T(\mathbb{K})$, with fibre $K^{\mathrm{num}}(\mathrm{coh}(X))$. This means that in simply-connected regions of $T(\mathbb{C})$ in the complex analytic topology the $K^{\mathrm{num}}(\mathrm{coh}(X_t))$ are all canonically isomorphic, and isomorphic to $K(\mathrm{coh}(X))$. But around loops in $T(\mathbb{C})$, this isomorphism with $K(\mathrm{coh}(X))$ can change by *monodromy*, by an automorphism $\mu : K(\mathrm{coh}(X)) \to K(\mathrm{coh}(X))$ of $K(\mathrm{coh}(X))$. In [85, Thm 4.21] they showed that the group of such monodromies μ is finite, and so it is possible to make it trivial by passing to a finite cover $\tilde{T}$ of T. If they worked instead with invariants $PI^{P,n}(\tau')$ counting pairs $s : \mathcal{O}_X(-n) \to E$ in which E has fixed Hilbert polynomial P, rather than fixed class $\alpha \in K^{\mathrm{num}}(\mathrm{coh}(X))$, as in Thomas' original definition of Donaldson–Thomas invariants [167], then they could drop the assumption on $K^{\mathrm{num}}(\mathrm{coh}(X_t))$ in Theorem [85, Thm. 5.25].

Similarly, we now study monodromy phenomena for $\mathrm{Pic}(X_t)$ in families of smooth $\mathbb{K}$-schemes $\mathfrak{X} \to T$ in the étale topology following the idea of [85, Thm. 4.21]. We find that we can always eliminate such monodromy by passing to a finite cover $\tilde{T}$ of T. This is crucial to prove deformation-invariance of the $\bar{DT}^{\alpha}(\tau), PI^{\alpha,n}(\tau')$ in [85, §12].

Theorem 7.2.8. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $\varphi : \mathfrak{X} \to T$ a smooth projective morphism of $\mathbb{K}$-schemes with T connected, and $\mathcal{O}_{\mathfrak{X}}(1)$ a relative very ample line bundle on $\mathfrak{X}$, so that for each $t \in T(\mathbb{K})$, the fibre X_t of φ is a smooth projective $\mathbb{K}$-scheme with very ample line bundle $\mathcal{O}_{X_t}(1)$. Suppose the Picard schemes* $\mathrm{Pic}(X_t)$ *are locally constant in $T(\mathbb{K})$, so that $t \mapsto \mathrm{Pic}(X_t)$ is a local system of abelian groups on T. Fix a base point $s \in T(\mathbb{K})$, and let Γ be the monodromy group of* $\mathrm{Pic}(X_s)$. *Then Γ is a finite group. There exists a finite étale cover $\pi : \tilde{T} \to T$ of degree $|\Gamma|$, with $\tilde{T}$ a connected $\mathbb{K}$-scheme, such that writing $\tilde{\mathfrak{X}} = \mathfrak{X} \times_T \tilde{T}$ and $\tilde{\varphi} : \tilde{\mathfrak{X}} \to \tilde{T}$ for the natural projection, with fibre $\tilde{X}_{\tilde{t}}$ at $\tilde{t} \in \tilde{T}(\mathbb{K})$, then* $\mathrm{Pic}(\tilde{X}_{\tilde{t}})$ *for all $\tilde{t} \in \tilde{T}(\mathbb{K})$ are all globally canonically isomorphic to* $\mathrm{Pic}(X_s)$. *That is, the local system $\tilde{t} \mapsto \mathrm{Pic}(\tilde{X}_{\tilde{t}})$ on $\tilde{T}$ is trivial in the étale topology.*

Proof. As $\mathrm{Pic}(X_s)$ is finitely generated, one can choose classes $[L_1], \dots, [L_k] \in \mathrm{Pic}(X_s)$ as generators. Let $P_1, \dots, P_k$ be the Hilbert polynomials respectively of $[L_1], \dots, [L_k]$ with respect to $\mathcal{O}_{X_s}(1)$. Let $\gamma \in \Gamma$, and consider the images $\gamma \cdot [L_i] \in \mathrm{Pic}(X_s)$ for $i = 1, \dots, k$. As we assume $\mathcal{O}_{\mathfrak{X}}(1)$ is globally defined on T and does not change under monodromy, it follows that the Hilbert polynomials $P_1, \dots, P_k$ do not change under monodromy. Hence $\gamma \cdot [L_i]$ has Hilbert polynomial P_i. Again one uses properness to show that the set $\mathrm{Pic}^{P_i}(X_s)$ composed by isomorphism classes of line bundles in $\mathrm{Pic}(X_s)$ with Hilbert polynomial P_i for some $i = 1, \dots, k$ is a finite set, that is, every P_i is the Hilbert polynomial of only finitely many classes $[R_1], \dots, [R_{n_i}]$ in $\mathrm{Pic}(X_s)$. It follows that for each $\gamma \in \Gamma$ we have $\gamma \cdot [L_i] \in \{[R_1], \dots, [R_{n_i}]\}$. So there are at most $n_1 \cdots n_k$ possibilities for $(\gamma \cdot [L_1], \dots, \gamma \cdot [L_k])$. But $(\gamma \cdot [L_1], \dots, \gamma \cdot [L_k])$ determines γ as $[L_1], \dots, [L_k]$ generate $\mathrm{Pic}(X_s)$. Hence $|\Gamma| \leqslant n_1 \cdots n_k$, and Γ is finite.

We can now construct an étale cover $\pi : \tilde{T} \to T$ which is a principal Γ-bundle, and so has degree $|\Gamma|$, such that the $\mathbb{K}$-points of $\tilde{T}$ are pairs (t, ι) where $t \in T(\mathbb{K})$ and $\iota : \mathrm{Pic}(X_t) \to \mathrm{Pic}(X_s)$ is an isomorphism from the properness and smoothness argument above, and Γ acts freely on $\tilde{T}(\mathbb{K})$ by $\gamma : (t, \iota) \mapsto (t, \gamma \circ \iota)$, so that the Γ-orbits correspond to points $t \in T(\mathbb{K})$. Then for $\tilde{t} = (t, \iota)$ we have $\tilde{X}_{\tilde{t}} = X_t$, with canonical isomorphism $\iota : \mathrm{Pic}(\tilde{X}_{\tilde{t}}) \to \mathrm{Pic}(X_s)$. ☐

So the conclusion is that from properness and smoothness argument, $\mathrm{Pic}(X_t)$ are canonically isomorphic locally in $T(\mathbb{K})$. But by Theorem 7.2.8, one can pass to a finite cover $\tilde{T}$ of T, so that the $\mathrm{Pic}(\tilde{X}_{\tilde{t}})$ are canonically isomorphic globally in $\tilde{T}(\mathbb{K})$. So, replacing $\mathfrak{X}, T$ by $\tilde{\mathfrak{X}}, \tilde{T}$, we will assume from here that the Picard schemes $\mathrm{Pic}(X_t)$ for $t \in T(\mathbb{K})$ are all canonically isomorphic globally in $T(\mathbb{K})$, and we write $\mathrm{Pic}(X)$ for this group $\mathrm{Pic}(X_t)$ up to canonical isomorphism.

In Theorem [85, Thm. 4.19] Joyce and Song showed that when $\mathbb{K} = \mathbb{C}$ and $H^1(\mathcal{O}_X) = 0$ the numerical Grothendieck group $K^{\mathrm{num}}(\mathrm{coh}(X))$ is unchanged under small deformations of X up to canonical isomorphism. As we said, here we will not prove this result. So, the idea is to construct a globally constant lattice Λ_X using the globally constancy of the Picard schemes such that there exist an inclusion $K^{\mathrm{num}}(\mathrm{coh}(X)) \hookrightarrow \Lambda_X$. It could happen that the image of the numerical Grothendieck group varies with t as it has to do with the integral Hodge conjecture as in [85, Thm. 4.19], but this does not affect the deformation invariance of $\bar{DT}^\alpha(\tau)$ as for them to be deformation invariant is enough to find a deformation invariant lattice in which the classes α vary. Next, we describe such lattice Λ_X and explain how the numerical Grothendieck group $K^{\mathrm{num}}(\mathrm{coh}(X))$ is contained in it. Our idea follows [85, Thm. 4.19].

Let X be a Calabi–Yau 3-fold over $\mathbb{K}$, with $H^1(\mathcal{O}_X) = 0$ and consider the *Chern character*, as in Hartshorne [65]: for each $E \in \mathrm{coh}(X)$ we have the rank $r(E) \in A^0(X) \cong \mathbb{Z}$, and the Chern classes $c_i(E) \in A^i(X)$ for $i = 1, 2, 3$. It is useful to organize these into the Chern character $\mathrm{ch}(E)$ in $A^*(X)_{\mathbb{Q}}$, where $\mathrm{ch}(E) = \mathrm{ch}_0(E) + \mathrm{ch}_1(E) + \mathrm{ch}_2(E) + \mathrm{ch}_3(E)$ with $\mathrm{ch}_i(E) \in A^i(X)_{\mathbb{Q}}$:

$$\begin{gathered}\mathrm{ch}_0(E) = r(E), \quad \mathrm{ch}_1(E) = c_1(E), \quad \mathrm{ch}_2(E) = \tfrac{1}{2}\big(c_1(E)^2 - 2c_2(E)\big), \\ \mathrm{ch}_3(E) = \tfrac{1}{6}\big(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)\big).\end{gathered} \tag{7.2.22}$$

By the Hirzebruch–Riemann–Roch Theorem [65, Th. A.4.1], the Euler form on coherent sheaves E, F is given in terms of their Chern characters by

$$\bar{\chi}\big([E],[F]\big) = \deg\big(\mathrm{ch}(E)^\vee \cdot \mathrm{ch}(F) \cdot \mathrm{td}(TX)\big)_3, \tag{7.2.23}$$

where $(\cdot)_3$ denotes the component of degree 3 in $A^*(X)_{\mathbb{Q}}$ and where $\mathrm{td}(TX)$ is the *Todd class* of TX, which is $1+\frac{1}{12}c_2(TX)$ as X is a Calabi–Yau 3-fold, and $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)^\vee = (\lambda_0, -\lambda_1, \lambda_2, -\lambda_3)$, writing $(\lambda_0, \ldots, \lambda_3) \in A^*(X)$ with $\lambda_i \in A^i(X)$. Define:

$$\begin{aligned}\Lambda_X = \big\{&(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \text{ where } \lambda_0, \lambda_3 \in \mathbb{Q},\ \lambda_1 \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q},\ \lambda_2 \in \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Q}) \text{ such that} \\ &\lambda_0 \in \mathbb{Z},\ \lambda_1 \in \mathrm{Pic}(X)/\text{torsion},\ \lambda_2 - \tfrac{1}{2}\lambda_1^2 \in \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z}),\ \lambda_3 + \tfrac{1}{12}\lambda_1 c_2(TX) \in \mathbb{Z}\big\},\end{aligned}$$

where λ_1^2 is defined as the map $\alpha \in \mathrm{Pic}(X) \to \frac{1}{2}c_1(\lambda_1) \cdot c_1(\lambda_1) \cdot c_1(\alpha) \in A^3(X)_{\mathbb{Q}} \cong \mathbb{Q}$, and $\frac{1}{12}\lambda_1 c_2(TX)$ is defined as $\frac{1}{12}c_1(\lambda_1) \cdot c_2(TX) \in A^3(X)_{\mathbb{Q}} \cong \mathbb{Q}$. Theorem 7.2.4 states that Λ_X is deformation invariant and the Chern character gives an injective morphism $\mathrm{ch} : K^{\mathrm{num}}(\mathrm{coh}(X)) \hookrightarrow \Lambda_X$. The proof of Theorem 7.2.4 is straightforward:

Proof. The proof follows exactly as in [85, Thm. 4.19] and the fact the Picard scheme $\mathrm{Pic}(X)$ is globally constant in families from the argument above yields that the lattice Λ_X is deformation invariant. Moreover, the proof that $\mathrm{ch}\big(K^{\mathrm{num}}(\mathrm{coh}(X))\big) \subseteq \Lambda_X$ is again as in [85, Thm. 4.19]. Observe that we do not prove that $\mathrm{ch}\big(K^{\mathrm{num}}(\mathrm{coh}(X))\big) = \Lambda_X$, fact which uses Voisin's Hodge conjecture proof for Calabi–Yau 3-folds over $\mathbb{C}$ [182]. □

Question 7.2.9. *Does Voisin's result [182] work over $\mathbb{K}$ in terms of* $\mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z})$*?*

This concludes the discussion of problem (b) in §7.2 and yields the deformation-invariance of $DT^\alpha(\tau)$, $PI^{\alpha,n}(\tau')$ over $\mathbb{K}$.

7.3 Implications and conjectures

In this section we sketch some exciting implications of the theory and propose new ideas for further research. One proposal is in the direction of extending Donaldson–Thomas invariants to compactly supported coherent sheaves on noncompact quasi-projective Calabi–Yau 3-folds. A second idea is in the derived categorical framework trying to establish a theory of generalized Donaldson–Thomas invariants for objects in the derived category of coherent sheaves. Here we expose the problems and illustrate some possible approaches when known.

7.3.1 Noncompact Calabi–Yau 3-folds

We start by recalling the following definition from [85, Def. 6.27]:

Definition 7.3.1. Let X be a noncompact Calabi-Yau 3-fold over $\mathbb{C}$. We call X compactly embeddable if whenever $K \subset X$ is a compact subset, in the analytic topology, there exists an open neighbourhood U of K in X in the analytic topology, a compact Calabi-Yau 3-fold Y over $\mathbb{C}$ with $H^1(\mathcal{O}_Y) = 0$, an open subset V of Y in the analytic topology, and an isomorphism of complex manifolds $\varphi : U \to V$.

Joyce and Song only need the notion of '*compactly embeddable*' as their complex analytic proof of (7.1.18)–(7.1.19) recalled in §7.1.3, requires X compact; but unfortunately the given algebraic version of (7.1.18)–(7.1.19) in Theorem 7.2.2 uses results from derived algebraic geometry [142, 171–175], and the author does not know if they apply also for compactly supported sheaves on a noncompact X.

More precisely, in [142] it is shown that if X is a projective Calabi-Yau m-fold then the derived moduli stack $\mathfrak{M}_{\mathrm{Perf}(X)}$ of perfect complexes of coherent sheaves on X is $(2-m)$-shifted symplectic. It is not obvious that if X is a quasi-projective Calabi-Yau m-fold, possibly noncompact, then the derived moduli stack $\mathfrak{M}_{\mathrm{Perf}_{cs}(X)}$ of perfect complexes on X with compactly-supported cohomology is also $(2-m)$-shifted symplectic.

At the present, we can state the following result. We thank Bertrand Toën for explaining this to us.

Theorem 7.3.2. *Suppose Z is smooth projective of dimension m, and $s \in H^0(K_Z^{-1})$, and $X \subset Z$ is Zariski open with s nonvanishing on X, so that X is a (generally non compact) quasi-projective Calabi-Yau m-fold. Then the derived moduli stack $\mathfrak{M}_{\mathrm{Perf}_{cs}}(X)$ of compactly-supported coherent sheaves on X, or of perfect complexes on X with compactly-supported cohomology, is $(2-m)$-shifted symplectic.*

Proof. Let Z be smooth and projective of dimension m, and s be any section of K_Z^{-1}. Let Y be the derived scheme of zeros of s and $X = Z \setminus Y$. Then, Y is equipped with a canonical O-orientation in the sense of [142] of dimension $m-1$, so $\mathfrak{M}_{\mathrm{Perf}}(Y)$ is $(3-m)$-symplectic, even if Y is not smooth. The restriction map $\mathfrak{M}_{\mathrm{Perf}}(Z) \to \mathfrak{M}_{\mathrm{Perf}}(Y)$ is moreover Lagrangian. The map $* \to \mathfrak{M}_{\mathrm{Perf}}(Y)$, corresponding to the zero object is étale, and thus its pull-back provides a Lagrangian map $\mathfrak{M}_{\mathrm{Perf}_{cs}}(X) \to *$, or, equivalently, a $(2-m)$-symplectic structure on $\mathfrak{M}_{\mathrm{Perf}_{cs}}(X)$. Now if X' is open in X, then $\mathfrak{M}_{\mathrm{Perf}_{cs}}(X') \to \mathfrak{M}_{\mathrm{Perf}_{cs}}(X)$ is an open immersion, so $\mathfrak{M}_{\mathrm{Perf}_{cs}}(X')$ is also $(2-m)$-symplectic. □

We remark the following:

(a) We point out that the condition of Theorem 7.3.2 is similar to the compactly-embeddable condition in [85, Def. 6.27], but more general, as we do not require Z to be a Calabi-Yau.

(*b*) Observe that in the non-compact case we cannot expect to have the deformation invariance property of Donaldson–Thomas invariants, except in some particular cases in which the moduli space is proper.

(*c*) Note that we need the noncompact Calabi–Yau to be quasi-projective in order to have a quasi projective Quot scheme [136, Thm. 6.3].

We conclude the section with the following:

Conjecture 7.3.3. The theory of generalized Donaldson–Thomas invariants defined in [85] is valid over algebraically closed fields of characteristic zero for compactly supported coherent sheaves on noncompact quasi-projective Calabi–Yau 3-folds. In this last case, one can define $\bar{DT}^\alpha(\tau)$ and prove the wall–crossing formulae and the relation with $PI^{\alpha,n}(\tau')$ is still valid, while one loses the deformation invariance property and the properness of moduli spaces.

7.3.2 Derived categorical framework

Our algebraic method could lead to the extension of generalized Donaldson–Thomas theory to the derived categorical context. The plan to extend the theory of Joyce and Song [85] from abelian to derived categories starts by reinterpreting the series of papers by Joyce [75–82] in this new general setup. In particular:

(*a*) Defining configurations in triangulated categories $\mathcal{T}$ requires to replace the exact sequences by distinguished triangles.

(*b*) Constructing moduli stacks of objects and configurations in $\mathcal{T}$. Again, the theory of derived algebraic geometry [142, 171–175] can give us a satisfactory answer.

(*c*) Defining stability conditions on triangulated categories can be approached using Bridgeland's results [20], and its extension by Gorodentscev et al. [51], which combines Bridgeland's idea with Rudakov's definition for abelian categories [147]. Since Joyce's stability conditions [77] are based on Rudakov, the modifications should be straightforward.

(*d*) The 'nonfunctoriality of the cone' in triangulated categories causes that the triangulated category versions of some operations on configurations are defined up to isomorphism, but not canonically, which yields that corresponding diagrams may be commutative, but not Cartesian as in the abelian case. In particular, one loses the associativity of the Ringel-Hall algebra of stack functions, which is a crucial object in Joyce and Song framework. We expect that derived Hall algebra approach of Toën [172] resolves this issue. See also [117].

The list above does not represent a big difficulty. The main issues actually are: proving existence of Bridgeland stability conditions (or other type) on the derived category; proving that semistable moduli schemes and stacks are finite type (permissible), and proving that two stability conditions can be joined by a path of permissible stability conditions.

Theorem 7.2.3 is just one of the steps in developing this program. The author thus expects that a well-behaved theory of invariants counting τ-semistable objects in triangulated categories in the style of Joyce's theory exists, that is, Theorem 7.2.6 should be valid also in the derived categorical context:

Conjecture 7.3.4. The theory of generalized Donaldson–Thomas invariants defined in [85] is valid for complexes of coherent sheaves on Calabi-Yau 3-folds over algebraically closed fields of characteristic zero.

Chapter 8

Categorifying complex Lagrangian intersections

This chapter begins with a section of background material on basic notions in symplectic geometry. In §8.2, we state and prove our first main result on the construction of a canonical global perverse sheaf on complex Lagrangian intersections. In §8.3 we prove our second main result on the d-critical locus structure carried by Lagrangian intersections. Finally, the last section sketches some implications of the theory and proposes new ideas for further research.

Throughout we will work in the complex analytic topology over $\mathbb{C}$. We will denote by (S,ω) a complex symplectic manifold endowed with a symplectic form ω, and its Lagrangian submanifolds will be always assumed to be nonsingular. Note that all complex analytic spaces in this paper are locally of finite type, which is necessary for the existence of embeddings $i : X \hookrightarrow U$ for U a complex manifold. Fix a well-behaved commutative base ring A (where 'well-behaved' means that we need assumptions on A such as A is regular noetherian, of finite global dimension or finite Krull dimension, a principal ideal domain, or a Dedekind domain, at various points in the theory), to study sheaves of A-modules. For some results A must be a field. Usually we take $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$.

8.1 Lagrangian intersections in complex symplectic manifolds

We will start with a basic definition to fix the notation:

Definition 8.1.1. Let (S,ω) be a *symplectic manifold*, i.e., a complex manifold S endowed with a closed non-degenerate holomorphic 2-form $\omega \in \Omega^2_S$. Denote the complex dimension of S by $2n$.

A complex submanifold $M \subset S$ is *Lagrangian* if the restriction of the symplectic form ω on S to a 2-form on M vanishes and $\dim M = n$.

Holomorphic coordinates, $x_1, \ldots, x_n, y_1, \ldots, y_n$ on an open subset $S' \subset S$ in the complex analytic topology, are called *Darboux coordinates* if $\omega = \sum_{i=1}^{n} \mathrm{d}y_i \wedge \mathrm{d}x_i$.

Definition 8.1.2. Given an n-dimensional manifold N, let us denote by $S = T^*N$ its cotangent bundle. For any chosen point $p \in U \subset N$, for U an open subset of N containing x, let us denote by $(x_1, \ldots, x_n)$ a set of coordinates. Then for any $x \in U$, the differentials $(\mathrm{d}x_1)_x, \ldots, (\mathrm{d}x_n)_x$ form a basis of $T^*_x N$.

Namely, if $y \in T^*_x N$ then $y = \sum_{i=1}^{n} y_i (\mathrm{d}x_i)_x$ for some complex coefficients $y_1, \ldots, y_n$. This induces a set of coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on T^*U, so a coordinate chart for T^*N, induced by

$(x_1, \ldots, x_n)$ on U. It is well known that transition functions on the overlaps are holomorphic and this gives the structure of a complex manifold of dimension $2n$ to T^*N.

Next, one can define a 2-form on T^*U by $\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$. It is easy to check that the definition is coordinate-independent. Define the 1-form $\alpha = \sum_{i=1}^{n} y_i \wedge \mathrm{d}x_i$. Clearly $\omega = -\mathrm{d}\alpha$, and α is intrinsically defined. The 1-form α is called in literature the *Liouville form*, and the 2-form ω is the *canonical symplectic form*.

Next, we will review symmetric obstruction theories on Lagrangian intersections from [6], and we state a crucial definition for our program.

Let (S, ω) be a complex symplectic manifold as above, and $L, M \subseteq S$ be Lagrangian submanifolds. Let $X = L \cap M$ be the intersection as a complex analytic space. Then X carries a canonical symmetric obstruction theory $\varphi : E^\bullet \to \mathbb{L}_X$ in the sense of [6], which can be represented by the complex $E^\bullet \simeq [T^*S|_X \to T^*L|_X \oplus T^*M|_X]$ with $T^*S|_X$ in degree -1 and $T^*L|_X \oplus T^*M|_X$ in degree zero. Hence

$$\det(E^\bullet) \cong K_S|_X^{-1} \otimes K_L|_X \otimes K_M|_X \cong K_L|_X \otimes K_M|_X, \tag{8.1.1}$$

since $K_S \cong \mathcal{O}_S$. This motivates the following:

Definition 8.1.3. We define an *orientation* of a complex Lagrangian submanifold L to be a choice of square root line bundle $K_L^{1/2}$ for K_L.

Remark 8.1.4. The previous definition is inspired by [18] and close to 'orientation data' in Kontsevich and Soibelman [102]. We point out that *spin structure* could have been a better choice of name than orientation, but we use orientations for consistency with [18, 19, 25, 87]. Also, for *real* Lagrangians, a square root $K_L^{1/2}$ induces an orientation on L in the usual sense.

Now, we recall well known established results in complex symplectic geometry which will be used to prove our main results. We start with the *complex Darboux theorem.*

Theorem 8.1.5. *Let (S, ω) be a complex symplectic manifold. Then locally in the complex analytic topology around a point $p \in S$ is always possible to choose holomorphic Darboux coordinates.*

So, basically, every symplectic manifold S is locally isomorphic to the cotangent bundle T^*N of a manifold N. The fibres of the induced vector bundle structure on S are Lagrangian submanifolds, so complex analytically locally defining on S a foliation by Lagrangian submanifolds, i.e., a *polarization*:

Definition 8.1.6. A *polarization* of a symplectic manifold (S, ω) is a holomorphic Lagrangian fibration $\pi : S' \to E$, where $S' \subseteq S$ is open.

Note that it is always possible to choose locally near a point $p \in S$ in the complex analytic topology Darboux coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and compatible coordinates x_i on E such that π can be identified with the projection $(x_1, \ldots, x_n, y_1, \ldots, y_n) \to (x_1, \ldots, x_n)$.

We will usually consider polarizations which are *transverse* to the Lagrangians whose intersection we wish to study. If L, M are complex Lagrangian submanifolds in (S, ω), and we consider the projection

$$\pi : (x_1, \ldots, x_n, y_1, \ldots, y_n) \to (x_1, \ldots, x_n)$$

defining local coordinates on L, then we can always assume to choose such coordinates x_i, y_i transverse to L, M at a point, and transverse to other coordinate systems too. Very briefly, this is because, if the projection $\pi : (x_1, \ldots, x_n, y_1, \ldots, y_n) \to (x_1, \ldots, x_n)$ is not transverse, we can change coordinates $(\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_1, \ldots, \tilde{y}_n)$ by a linear transformation by a generic matrix $M \in \mathrm{Sp}(2n, \mathbb{C})$. This matrix preserves the symplectic form, so $\omega = \sum_i \mathrm{d}\tilde{x}_i \wedge \mathrm{d}\tilde{y}_i$, and the generic Lagrangian plane $< \frac{\partial}{\partial \tilde{y}_1}, \ldots, \frac{\partial}{\partial \tilde{y}_n} >$ intersects transversely T_pL and T_pM for $p \in S$. Thus $\pi : (\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_1, \ldots, \tilde{y}_n) \to (\tilde{x}_1, \ldots, \tilde{x}_n)$ is a polarization transverse to L, M near p, and so π is a transverse polarization for L, M on S', for an open neighbourhood S' of p in S. In conclusion, we are using the projection π as a polarization, and we assume that the leaves are transverse to the two Lagrangians.

Recall now the *Lagrangian neighbourhood theorem*:

Theorem 8.1.7. *If $M \subset (S, \omega)$ is a complex Lagrangian submanifold, then there exists a complex analytic neighbourhood $V \subset M$ of a point $p \in M$ isomorphic as a complex symplectic manifold to a neighbourhood U of p in T^*M, and M is identified with the zero section in T^*M.*

Note that (S, ω) need not be isomorphic to T^*M in a neighbourhood of M, but just in a neighbourhood of a point $p \in M$. So, we may identify locally M with N and, by making L smaller if necessary, we have the following picture:

Lemma 8.1.8. *Choose locally near a point $p \in S$ in the complex analytic topology Darboux coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and compatible coordinates x_i on E such that $\pi : S \to E$ can be identified with the projection $(x_1, \ldots, x_n, y_1, \ldots, y_n) \to (x_1, \ldots, x_n)$. Now, given a polarization $(x_1, \ldots, x_n, y_1, \ldots, y_n) \to (x_1, \ldots, x_n)$ defining local coordinates on L, then L is given by*

$$\left\{ \left(x_1, \ldots, x_n, \frac{\mathrm{d}f}{\mathrm{d}x_1}, \ldots, \frac{\mathrm{d}f}{\mathrm{d}x_n} \right) : \ x_1, \ldots, x_n \in M' \right\}$$

for a holomorphic function $f(x_1, \ldots, x_n)$ defined locally on an open $M' \subset M \subset S$, where M is the Lagrangian submanifold identified with the zero section, and the polarization π with projection $T^*M \to M$.

So, in conclusion, if $\pi : S' \to E$ is a polarization, and M a Lagrangian submanifold with $\pi : M \to E$ transverse near x in M, then locally there is a unique isomorphism $S' \cong T^*M$ identifying M with zero section and π with projection $T^*M \to M$. Then any other Lagrangian L in S transverse to π is locally described by the graph of $\mathrm{d}f$, for a holomorphic function f locally defined on M. It is now straightforward to deduce that, as M is graph of 0, and L is graph of $\mathrm{d}f$, then the intersection $X = L \cap M$ is the critical locus $(\mathrm{d}f)^{-1}(0)$.

We can summarize in this way. Let (S, ω) be a complex symplectic manifold, and $L, M \subseteq S$ be Lagrangian submanifolds. Let $X = L \cap M$ be the intersection as a complex analytic space. Then X is complex analytically locally modelled on the zero locus of the 1-form $\mathrm{d}f$, that is on the critical locus $\mathrm{Crit}(f : U \to \mathbb{C})$ for a holomorphic function f on a smooth manifold U. So, X carries a natural perverse sheaf of vanishing cycles $\mathcal{PV}^{\bullet}_{U,f}$ in the notation of §1.3, and a natural problem to investigate is the following. Given open $R_i, R_j \subseteq S$ with isomorphisms $R_i \cong \mathrm{Crit}(f_i)$, $R_j \cong \mathrm{Crit}(f_j)$ for holomorphic $f_i : U_i \to \mathbb{C}$ and $f_j : U_j \to \mathbb{C}$, we have to understand whether the perverse sheaves $\mathcal{P}^{\bullet}_{R_i} = \mathcal{PV}^{\bullet}_{U_i,f_i}$ on R_i and $\mathcal{P}^{\bullet}_{R_j} = \mathcal{PV}^{\bullet}_{U_j,f_j}$ on R_j are isomorphic over $R_i \cap R_j$, and if so, whether the isomorphism is canonical, for only then can we hope to glue the $\mathcal{P}^{\bullet}_{R_i}$ for $i \in I$ to make $\mathcal{P}^{\bullet}_{L,M}$. We will develop this program in §8.2.

8.2 Canonical perverse sheaves on Lagrangian intersections

We can state our main result.

Theorem 8.2.1. *Let (S,ω) be a complex symplectic manifold and L, M oriented complex Lagrangian submanifolds in S, and write $X = L \cap M$, as a complex analytic subspace of S. Then we may define $P^\bullet_{L,M} \in \mathrm{Perv}(X)$, uniquely up to canonical isomorphism, and isomorphisms*

$$\Sigma_{L,M} : P^\bullet_{L,M} \to \mathbb{D}_X(P^\bullet_{L,M}), \qquad \mathrm{T}_{L,M} : P^\bullet_{L,M} \to P^\bullet_{L,M}, \tag{8.2.1}$$

respectively the Verdier duality and the monodromy isomorphisms. These $P^\bullet_{L,M} \in \mathrm{Perv}(X), \Sigma_{L,M}$, $\mathrm{T}_{L,M}$ are locally characterized by the following property.

Given a choice of local Darboux coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ in the sense of Definition 8.1.1 such that L is locally identified in coordinates with the graph $\Gamma_{\mathrm{d}f(x_1,\dots,x_n)}$ of $\mathrm{d}f$ for f a holomorphic function defined locally on an open $U \subset \mathbb{C}^n$, and M is locally identified in coordinates with the graph $\Gamma_{\mathrm{d}g(x_1,\dots,x_n)}$ of $\mathrm{d}g$ for g a holomorphic function defined locally on U, and the orientations $K_L^{1/2}, K_M^{1/2}$ are the trivial square roots of $K_L \cong \langle \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \rangle \cong K_M$, then there is a canonical isomorphism $P^\bullet_{L,M} \cong \mathcal{PV}^\bullet_{U,g-f}$, where $\mathcal{PV}^\bullet_{U,g-f}$ is the perverse sheaf of vanishing cycles of $g-f$, and $\Sigma_{L,M}$ and $\mathrm{T}_{L,M}$ are respectively the Verdier duality $\sigma_{U,g-f}$ and the monodromy $\tau_{U,g-f}$ introduced in §1.3. The same applies for $\mathscr{D}$-modules and mixed Hodge modules on X.

A convenient way to express this is in terms of *charts*, by which we mean a set of data locally defined by the choice of a polarization $\pi : S \to E$. Charts are analogous to *critical charts* defined by [87, §2], as in §2. We will show in §8.3 that they are actually critical charts and they define the structure of a d-critical locus on the Lagrangian intersection, but for this section we will not use it.

We explained in §8.1 that the local choice of a polarization on (S,ω) yields a local description of the Lagrangian intersection as a critical locus $P \cong \mathrm{Crit}(f)$ for a closed embedding $i : P \hookrightarrow U$, P open in X, and a holomorphic function $f : U \subset L \to \mathbb{C}$, where U is an open submanifold of L. We have a local symplectic identification $S \cong T^*U \subseteq T^*L$, which identifies $L \subset S$ with the zero section in T^*L, and $M \subset S$ with the graph $\Gamma_{\mathrm{d}f}$ of $\mathrm{d}f$, and $\pi : S \to E \cong L$ with the projection $T^*L \to L$. So, for each polarization $\pi : S \to E$ we have naturally induced a set of data (P, U, f, i), which we will call an *L-chart*. We will also consider *M-charts*, namely charts coming from polarizations that identify the other Lagrangian M with the zero section, that is, charts like (Q, V, g, j) where $Q \cong \mathrm{Crit}(g)$ for a closed embedding $j : Q \hookrightarrow V$, Q open in X, and a holomorphic function $g : V \subset M \to \mathbb{C}$, where V is an open submanifold of M. We have a local symplectic identification $S \cong T^*V \subseteq T^*M$, which identifies $M \subset S$ with the zero section in T^*M, and $L \subset S$ with the graph $\Gamma_{\mathrm{d}g}$ of $\mathrm{d}g$, and $\pi : S \to E \cong M$ with the projection $T^*M \to M$. We will use also *LM-charts*.

Using this general technique, let us fix the following notation we will use for the rest of the paper. We will consider mainly three kinds of *charts*, where by charts we basically mean a set of data associated to a choice of one or two polarizations for our complex symplectic manifold:

(*a*) *L-charts* (P, U, f, i) are induced by a polarizations $\pi : S' \to E$ transverse to L, M with $S' \subset S$ open, $P \subset X$ open, and $U \subset L$ open, and $f : U \to \mathbb{C}$ holomorphic, and $i : P \hookrightarrow U \subset L$ the inclusion, with $i : P \to \mathrm{Crit}(f)$ an isomorphism, so that we have local identifications

- $(S,\omega) = T^*U$;
- $L =$ zero section;
- $M = \Gamma_{\mathrm{d}f}$;
- $E = L = U$;
- $\pi : S \to E$ with $\pi : T^*U \to U$.

(*b*) *M-charts* (Q, V, g, j) are induced by a polarization $\tilde{\pi} : \tilde{S} \to F$ transverse to L, M, with $\tilde{S} \subset S$ open, $Q \subset X$ open, and $V \subset M$ open, and $g : V \to \mathbb{C}$ holomorphic, and $j : Q \hookrightarrow V \subset M$ the inclusion, with $j : Q \to \mathrm{Crit}(g)$ an isomorphism, so that we have local identifications

- $(S, -\omega) = T^*V$;
- $M =$ zero section;
- $L = \Gamma_{\mathrm{d}g}$;
- $F = M = V$;
- $\tilde{\pi} : S \to F$ with $\pi : T^*V \to V$.

(*c*) *LM-charts* (R, W, h, k) are induced by polarizations $\pi : S' \to E$, $\tilde{\pi} : \tilde{S} \to F$ transverse to L, M and to each other on $\hat{S} = S' \cap \tilde{S}$. We have $W \subset L \times M$ open, $h : W \to \mathbb{C}$ holomorphic, $k : R \to W \subset L \times M$ the diagonal map, with $k : R \to \mathrm{Crit}(h)$ an isomorphism, and local identifications

- $(S', \omega) \times (\tilde{S}, -\omega) = T^*W$;
- $L \times M =$ zero section;
- diagonal $\Delta_S = \Gamma_{\mathrm{d}h}$;
- $E \times F = L \times M = W$;
- $\pi \times \tilde{\pi} : \hat{S} \to E \times F$ with $\pi : T^*W \to W$.

Note that $R = P \cap Q$. These three kinds of charts will be related by Proposition 8.2.2, which will give an embedding from an open subset of an L-chart (P, U, f, i) into an LM-chart (R, W, h, k), and similarly an embedding from an open subset of an M-chart (Q, V, g, j) into (R, W, h, k).

Moreover, we will explain in §8.2.1 that the choice of a polarization $\pi : S \to E$ naturally induces local biholomorphisms $L \xrightarrow{\cong} M$ or $M \xrightarrow{\cong} L$, and thus isomorphisms

$$\Theta : K_L|_X \xrightarrow{\cong} K_M|_X \quad \text{or} \quad \Xi : K_M|_X \xrightarrow{\cong} K_L|_X \tag{8.2.2}$$

between the canonical bundles of the Lagrangian submanifolds. We define $\pi_{P,U,f,i} : Q_{P,U,f,i} \to P$ to be the principal $\mathbb{Z}_2$-bundle parametrizing local isomorphisms

$$\vartheta : K_L^{1/2}|_X \xrightarrow{\cong} K_M^{1/2}|_X \quad \text{or} \quad \xi : K_M^{1/2}|_X \xrightarrow{\cong} K_L^{1/2}|_X \tag{8.2.3}$$

such that $\vartheta \otimes \vartheta = \Theta$ or $\xi \otimes \xi = \Xi$.

Also, on each L-chart, M-chart, LM-chart, we have a natural perverse sheaf of vanishing cycles associated to the local description of the Lagrangian intersection as a critical locus. So we get a perverse sheaf of vanishing cycles $i^*(\mathcal{PV}^\bullet_{U,f})$ on P, $j^*(\mathcal{PV}^\bullet_{V,g})$ on Q, and $k^*(\mathcal{PV}^\bullet_{W,h})$ on R. These perverse sheaves together with principal $\mathbb{Z}_2$-bundles parametrizing square roots of isomorphisms (8.2.2) are the objects we want to glue.

Then $P^\bullet_{L,M} \in \mathrm{Perv}(X)$ is characterized by the following properties:

(i) If (P, U, f, i) is a an L-chart, M-chart, or LM-chart, there is a natural isomorphism

$$\omega_{P,U,f,i} : P^\bullet_{L,M}|_P \longrightarrow i^*\big(\mathcal{PV}^\bullet_{U,f}\big) \otimes_{\mathbb{Z}_2} Q_{P,U,f,i}, . \tag{8.2.4}$$

Furthermore the following commute in $\mathrm{Perv}(P)$:

$$
\begin{tikzcd}
P^\bullet_{L,M}|_P \arrow[r, "\omega_{P,U,f,i}"'] \arrow[d, "\Sigma_{L,M}|_P"] & i^*\big(\mathcal{PV}^\bullet_{U,f}\big)\otimes_{\mathbb{Z}_2} Q_{P,U,f,i} \arrow[d, "i^*(\sigma_{U,f})\otimes\mathrm{id}_{Q_{P,U,f,i}}"'] \\
\mathbb{D}_P\big(P^\bullet_{L,M}|_P\big) & i^*\big(\mathbb{D}_{\mathrm{Crit}(f)}(\mathcal{PV}^\bullet_{U,f})\big)\otimes_{\mathbb{Z}_2} Q_{P,U,f,i} \cong \mathbb{D}_P\big(i^*(\mathcal{PV}^\bullet_{U,f})\otimes_{\mathbb{Z}_2} Q_{P,U,f,i}\big), \arrow[l, "\mathbb{D}_P(\omega_{P,U,f,i})"']
\end{tikzcd}
\tag{8.2.5}
$$

$$
\begin{tikzcd}
P^\bullet_{L,M}|_P \arrow[r, "\omega_{P,U,f,i}"'] \arrow[d, "\mathrm{T}_{L,M}|_P"] & i^*\big(\mathcal{PV}^\bullet_{U,f}\big)\otimes_{\mathbb{Z}_2} Q_{P,U,f,i} \arrow[d, "i^*(\tau_{U,f})\otimes\mathrm{id}_{Q_{P,U,f,i}}"'] \\
P^\bullet_{L,M}|_R \arrow[r, "\omega_{P,U,f,i}"] & i^*\big(\mathcal{PV}^\bullet_{U,f}\big)\otimes_{\mathbb{Z}_2} Q_{P,U,f,i}.
\end{tikzcd}
\tag{8.2.6}
$$

(ii) Let $\pi : S' \to E$ and $\tilde\pi : \tilde S \to F$ be polarizations transverse to L, M, and transverse to each other on $S' \cap \tilde S$. Then from π we get an L-chart (P, U, f, i), from $\tilde\pi$ we get an M-chart (Q, V, g, j), and from π and $\tilde\pi$ together we get an LM-chart (R, W, h, k). Write (P', U', f', i') for the L-chart determined by $\pi|_{S'\cap\tilde S} : S' \cap \tilde S \to E$, and (Q', V', g', i') for the M-chart determined by $\tilde\pi|_{S'\cap\tilde S} : S'\cap\tilde S \to F$. Then $P' \subseteq P$, $U' \subseteq U$ are open and $f' = f|_{U'}$, $i' = i|_{P'}$, so (P', U', f', i') is a *subchart* of (P, U, f, i) in the sense of §2. We write this as $(P', U', f', i') \subseteq (P, U, f, i)$. Similarly, $(Q', V', g', j') \subseteq (Q, V, g, j)$. Also $P' = Q' = R = X \cap S' \cap \tilde S$.

In this situation, Proposition 8.2.2 will show that there exist closed embeddings $\Phi : U' \hookrightarrow W$ and $\Psi : V' \hookrightarrow W$ such that so that $h \circ \Phi = f : U' \to \mathbb{C}$ and $h \circ \Psi = g : V' \to \mathbb{C}$. Moreover $\mathrm{Crit}(f) \cong \mathrm{Crit}(h) \cong \mathrm{Crit}(g)$ as complex analytic spaces, and f, h and g, h are pairs of *stably equivalent* functions. Inspired by [87, Def. 2.18], we will say that $\Phi : (P', U', f', i') \hookrightarrow (R, W, h, k)$ is an embedding of charts if Φ is a locally closed embedding $U' \hookrightarrow W$ of complex manifolds such that $\Phi \circ i' = k|_{P'} : P' \to W$ and $f = h \circ \Phi : U' \to \mathbb{C}$. As a shorthand we write $\Phi : (P', U', f', i') \hookrightarrow (R, W, h, k)$ to mean Φ is an embedding of (P', U', f', i') in (R, W, h, k). In brief, Proposition 8.2.2 in §8.2.1 will define two embeddings of charts $\Phi : (P', U', f', i') \hookrightarrow (R, W, h, k)$ and $\Psi : (Q', V', g', j') \hookrightarrow (R, W, h, k)$.

Given the embedding of charts $\Phi : (P', U', f', i') \hookrightarrow (R, W, h, k)$, there is a natural isomorphism of principal $\mathbb{Z}_2$-bundles

$$\Lambda_\Phi : Q_{R,W,h,k}|_{P'} \xrightarrow{\cong} i^*(P_\Phi) \otimes_{\mathbb{Z}_2} Q_{P',U',f',i'} \tag{8.2.7}$$

on P', for P_Φ defined as follows: local isomorphisms

$$\begin{gathered}\alpha : K_X^{1/2}|_{P'^{\mathrm{red}}} \to i^*(K_{U'})|_{P'^{\mathrm{red}}},\ \beta : K_X^{1/2}|_{P'^{\mathrm{red}}} \to j'^*(K_W)|_{P'^{\mathrm{red}}},\\ \gamma : i'^*(K_{U'})|_{P'^{\mathrm{red}}} \to j^*(K_W)|_{P'^{\mathrm{red}}}\end{gathered} \tag{8.2.8}$$

with $\alpha\otimes\alpha = \iota_{P',U',f',i'}$, $\beta\otimes\beta = \iota_{R,W,h,k}|_{P'^{\mathrm{red}}}$, $\gamma\otimes\gamma = i|^*_{P'^{\mathrm{red}}}(J_\Phi)$ correspond to local sections $s_\alpha : P' \to Q_{P',U',f',i'}$, $s_\beta : P' \to Q_{R,W,h,k}|_{P'}$, $s_\gamma : P' \to i'^*(P_\Phi)$, for J_Φ as in Definition 4.3.2, and for isomorphisms $\iota_{R,W,h,k} : K_X \to i^*(K_W^{\otimes 2})|_{P'^{\mathrm{red}}}$ induced by the polarization $E_1 \times E_2$.

Then for each embedding of charts, the following diagram commutes in $\mathrm{Perv}(P')$, for Θ_Φ as in (4.3.13):

$$
\begin{tikzcd}
P^\bullet_{L,M}|_{P'} \arrow[r, "\omega_{P',U',f',i'}"'] \arrow[d, "\omega_{R,W,h,k}|_{P'}"] & i^*\big(\mathcal{PV}^\bullet_{U',f'}\big)\otimes_{\mathbb{Z}_2} Q_{P',U',f',i'} \arrow[d, "i'^*(\Theta_\Phi)\otimes\mathrm{id}_{Q_{P',U',f',i'}}"'] \\
j^*\big(\mathcal{PV}^\bullet_{W,h}\big)|_{P'}\otimes_{\mathbb{Z}_2} Q_{R,W,h,k}|_{P'} \arrow[r, "\mathrm{id}_{j^*(\mathcal{PV}^\bullet_{W,h})}\otimes\Lambda_\Phi"] & i^*\big(\Phi|_X^*(\mathcal{PV}^\bullet_{W,h})\otimes_{\mathbb{Z}_2} P_\Phi\big)\otimes_{\mathbb{Z}_2} Q_{P',U',f',i'}.
\end{tikzcd}
\tag{8.2.9}
$$

We will have an analogous commutative diagram induced by Ψ on $\mathrm{Perv}(Q')$:

$$
\begin{tikzcd}
P^{\bullet}_{L,M}|_{Q'} \arrow[rr, "\omega_{Q',V',g',j'}"'] \arrow[d, "\omega_{R,W,h,k}|_{Q'}"] & & i^*\big(\mathcal{PV}^{\bullet}_{V,g}\big) \otimes_{\mathbb{Z}_2} Q_{Q',V',g',j'} \arrow[d, "j'^*(\Theta_\Psi)\otimes \mathrm{id}_{Q_{Q',V',g',j'}}"'] \\
j'^*\big(\mathcal{PV}^{\bullet}_{W,h}\big)|_{Q'} \otimes_{\mathbb{Z}_2} Q_{R,W,h,k}|_{Q'} \arrow[rr, "\mathrm{id}_{j'^*(\mathcal{PV}^{\bullet}_{W,h})}\otimes\Lambda_\Psi"] & & i^*\big(\Psi|_X^*(\mathcal{PV}^{\bullet}_{W,h}) \otimes_{\mathbb{Z}_2} P_\Psi\big) \otimes_{\mathbb{Z}_2} Q_{Q',V',g',j'}.
\end{tikzcd}
\tag{8.2.10}
$$

Using Theorem 4.3.4, we get isomorphisms

$$
\begin{aligned}
\alpha &: (i'^*(\mathcal{PV}^{\bullet}_{U',f'}) \otimes Q_{P',U',f',i'})|_R \cong (k^*(\mathcal{PV}^{\bullet}_{W,h}) \otimes Q_{R,W,h,k}),\\
\beta &: (j'^*(\mathcal{PV}^{\bullet}_{V',g'}) \otimes Q_{Q',V',g',j'})|_R \cong (k^*(\mathcal{PV}^{\bullet}_{W,h}) \otimes Q_{R,W,h,k}).
\end{aligned}
$$

Combining these, we get an isomorphism

$$
\beta^{-1}\circ\alpha : (i'^*(\mathcal{PV}^{\bullet}_{U',f'}) \otimes Q_{P',U',f',i'})|_R \cong (j'^*(\mathcal{PV}^{\bullet}_{V',g'}) \otimes Q_{Q',V',g',j'})|_R, \tag{8.2.11}
$$

that is, an isomorphism of perverse sheaves from L-charts and M-charts in $\mathrm{Perv}(P' \cap Q')$. Later, in §8.2.2 we will involve also two other polarizations for an associativity result. More precisely, following notation of §8.2.2 we want that if we have two L-charts (P_1, U_1, f_1, i_1) and (P_3, U_3, f_3, i_3) and two M-charts (Q_2, V_2, g_2, j_2) and (Q_4, V_4, g_4, j_4) then

$$
\begin{aligned}
&\alpha_{32}|_Y^{-1}\circ\beta_{32}|_Y\circ\beta_{12}|_Y^{-1}\circ\alpha_{12}|_Y = \alpha_{34}|_Y^{-1}\circ\beta_{34}|_Y\circ\beta_{14}|_Y^{-1}\circ\alpha_{14}|_Y :\\
&\qquad\big(\mathcal{PV}^{\bullet}_{U_1,f_1}\otimes_{\mathbb{Z}_2} Q_{P_1,U_1,f_1,i_1}\big)|_Y \longrightarrow \big(\mathcal{PV}^{\bullet}_{U_3,f_3}\otimes_{\mathbb{Z}_2} Q_{P_3,U_3,f_3,i_3}\big)|_Y.
\end{aligned}
\tag{8.2.12}
$$

Theorem 8.2.1 will be proved in §8.2.1–§8.2.3. In §8.2.3, we will provide a descent argument, which is the most technical part of the paper. We outline here our method of the proof.

Let $\{U_a\}_{a\in I}$ be an analytic open cover for $X = L\cap M$, induced by polarizations $\pi_a : S_a \to E_a$ for $a \in I$, transverse to both L and M, and write $U_{ab} = U_a \cap U_b$ for $a, b \in I$. Similarly, write $U_{abc} = U_a\cap U_b\cap U_c$ for $a, b, c \in I$. Define $\mathcal{P}_a$ to be $i_a^*(\mathcal{PV}^{\bullet}_{U_a,f_a})\otimes_{\mathbb{Z}_2} Q_{P_a,U_a,f_a,i_a}$ from the discussion above, and isomorphisms $\gamma_{ab} : \mathcal{P}^{\bullet}_a|_{U_{ab}} \to \mathcal{P}^{\bullet}_b|_{U_{ab}}$ in $\mathrm{Perv}(U_{ab})$ for all $a, b \in I$ with $\beta_{aa} = \mathrm{id}$ and

$$
\gamma_{bc}|_{U_{abc}}\circ\gamma_{ab}|_{U_{abc}} = \gamma_{ac}|_{U_{abc}} : \mathcal{P}_a|_{U_{abc}} \longrightarrow \mathcal{P}_c|_{U_{abc}}
$$

in $\mathrm{Perv}(U_{abc})$ for all $a, b, c \in I$.

The construction is independent of the choice of $\{U_a\}_{a\in I}$ above. Then by Theorem 1.3.5, there exists $\mathcal{P}^{\bullet}$ in $\mathrm{Perv}(X)$, unique up to canonical isomorphism, with isomorphisms $\omega_a : \mathcal{P}^{\bullet}|_{U_a} \to \mathcal{P}^{\bullet}_a$ for each $a \in I$, satisfying $\gamma_{ab}\circ\omega_a|_{U_{ab}} = \omega_b|_{U_{ab}} : \mathcal{P}^{\bullet}|_{U_{ab}} \to \mathcal{P}^{\bullet}_b|_{U_{ab}}$ for all $a, b \in I$, which concludes the proof of Theorem 8.2.1. We will carry out this program in §8.2.1–§8.2.3.

Theorem 8.2.1 resolves a long-standing question in the categorification of Lagrangian intersection number: our perverse sheaf $P^{\bullet}_{L,M}$ categorifies Lagrangian intersection numbers, in the sense that the constructible function

$$
p \to \sum_i (-1)^{i-2n}\dim_{\mathbb{C}}\mathbb{H}^i_{\{p\}}(X, P^{\bullet}_{L,M}),
$$

is equal to the well known Behrend function ν_X in [5] by construction, using the expression of the Behrend function of a critical locus in terms of the Milnor fibre, as in [5] and recalled in §7.1, and so

$$
\chi(X,\nu_X) = \sum_i (-1)^{i-2n}\dim_A \mathbb{H}^i(X, P^{\bullet}_{L,M}), \tag{8.2.13}
$$

for a base ring A, which in this case is a field. If the intersection X is compact, then $[L]\cap[M]$ is given by (8.2.13), where $[L], [M]$ are the homology classes of L and M in S.

Theorem 8.2.1 and its proof provide thus a direct construction of the perverse sheaf defined in Corollary 4.4.4 in §4. Moreover, our construction may have exciting far reaching applications in symplectic geometry and topological field theory, as discussed in §8.4.

8.2.1 Canonical isomorphism of perverse sheaves on double overlaps

Given a complex symplectic manifold (S,ω) and Lagrangian submanifolds L, M in S, define X to be their intersection as a complex analytic space. Using results in §8.1, locally in the complex analytic topology near a point $x \in X$, we can choose an open set $S' \subset S$ and a polarization transverse to both L and M such that $S' \cong T^*L$ and $M \cong \Gamma_{\mathrm{d}f}$ so that $X = \mathrm{Crit}(f)$ for a holomorphic function f defined on $U = L \cap S'$. Thus we get a perverse sheaf of vanishing cycle $\mathcal{PV}^\bullet_{U,f}$. In this section we will investigate how two such local descriptions are related.

Consider $\pi_1,\pi_2 : S_1,S_2 \to E_1,E_2$ two polarizations transverse to each other and both transverse to both L and M. Choose open neighbourhoods U_1 of $X\cap S_1$ in $L\cap S_1$ with $\pi_1(U_1) \subset \pi_1(M\cap S_1)$ and V_2 of $X\cap S_2$ in $M\cap S_2$ with $\pi_2(V_2) \subset \pi_2(L\cap S_2)$. Then we get respectively the local identifications

$$\begin{gathered}U_1 \cong \pi_1(U_1) \subset E_1,\ S_1 \supset \pi_1^{-1}(\pi_1(U_1)) \cong T^*U_1,\ L\cap\pi_1^{-1}(\pi_1(U_1)) \cong \Gamma_0,\\ M\cap\pi_1^{-1}(\pi_1(U_1)) \cong \Gamma_{\mathrm{d}f_1},\ f_1 : U_1 \to \mathbb{C},\end{gathered}$$

$$\begin{gathered}V_2 \cong \pi_2(V_2) \subset E_2,\ \bar{S}_2 \supset \pi_2^{-1}(\pi_2(V_2)) \cong T^*V_2,\ M\cap\pi_2^{-1}(\pi_2(V_2)) \cong \Gamma_0,\\ L\cap\pi_2^{-1}(\pi_2(V_2)) \cong \Gamma_{\mathrm{d}g_2},\ g_2 : V_2 \to \mathbb{C}.\end{gathered}$$

Choose an open neighbourhood W_{12} of $\{(x,x) : x \in X\cap S_1\cap S_2\}$ in $U_1\times V_2$ with $(\pi_1\times\pi_2)(W_{12}) \subset (\pi_1\times\pi_2)(\Delta_S\cap(S_1\times S_2))$. Choose open neighbourhoods U_1' of $X\cap S_1\cap S_2$ in U_1 with $\{(l,\pi_2|_M^{-1}\circ\pi_2(l)) : l\in U_1'\} \subset W_{12}$ and V_2' of $X\cap S_1\cap S_2$ in V_2 with $\{(\pi_1|_L^{-1}\circ\pi_1(m)) : m\in V_2'\} \subset W_{12}$. Then we have:

Proposition 8.2.2. *In the situation above, starting from polarizations $\pi_1,\pi_2 : S_1,S_2 \to E_1,E_2$ and defining $f_1 : U_1 \to \mathbb{C}$ using π_1 and $g_2 : V_2 \to \mathbb{C}$ using π_2 and using the given notation, there exists locally a holomorphic function $h_{12} : W_{12} \to \mathbb{C}$ such that the following diagram*

$$\begin{tikzcd} U_1' \arrow[r, "\Phi_{12}=\mathrm{id}_{U_1'}\times\pi_1|_{V_2'}^{-1}"] \arrow[dr, "f_1|_{U_1'}"'] & W_{12} \arrow[d, "h_{12}"] & V_2' \arrow[l, "\Psi_{12}=\pi_2|_{U_1'}^{-1}\times\mathrm{id}_{V_2'}"'] \arrow[dl, "g_2|_{V_2'}"] \\ & \mathbb{C} & \end{tikzcd} \tag{8.2.14}$$

is commutative, that is

$$h_{12}(l,\pi_1|_{V_2'}^{-1}(l)) = f_1(l), \quad \text{and} \quad h_{12}(\pi_2|_{U_1'}^{-1}(m),m) = g_2(m), \tag{8.2.15}$$

for every $l\in U_1'$, $m\in V_2'$. Moreover, $\Phi_{12} = \mathrm{id}_{U_1'}\times\pi_1|_{V_2'}^{-1}$ and $\Psi_{12} = \pi_2|_{U_1'}^{-1}\times\mathrm{id}_{V_2'}$ induce isomorphisms $\mathrm{Crit}(h_{12}) \cong \mathrm{Crit}(f_1) \cong \mathrm{Crit}(g_2)$ as complex analytic spaces locally in the complex analytic topology. In particular, from Theorem 4.3.1, we can choose $(z_1,\ldots,z_n)$ coordinates normal to $(\mathrm{id}_L\times\pi_1|_{V_2'}^{-1})(U_1')$ in W_{12}, and $(w_1,\ldots,w_n)$ coordinates normal to $(\pi_2|_{U_1'}^{-1}\times\mathrm{id}_{V_2'})(V_2')$ in W_{12}, under which we can write $h_{12} \cong f_1 \boxplus z_1^2+\ldots+z_n^2$ and $h_{12} \cong g_2 \boxplus w_1^2+\ldots+w_n^2$. Following §4, we will say that f_1 and g_2 are both stably equivalent to h_{12}.

Proof. Consider the product symplectic manifold $(S\times\bar{S},\omega\oplus(-\omega))$, where $\bar{S}$ denotes the symplectic manifold S corresponding to the symplectic form with the opposite sign. In $S\times\bar{S}$ consider the Lagrangian submanifolds $N_1:=L\times M$ and $N_2:=\Delta_S$, the diagonal. As explained in §8.1, identify locally $(S\times\bar{S},\omega\oplus-\omega)$ with $(T^*(L\times M),\omega_{L\times M})$, where $\omega_{L\times M}$ is the symplectic form on $T^*(L\times M)$, and thus $\pi_1\times\pi_2$ is identified with the projection $\pi:T^*(L\times M)\to L\times M$, that is N_1 with the zero section, and N_2 with the graph $\Gamma_{\mathrm{d}h_{12}}$ for a holomorphic function $h_{12}:L\times M\to\mathbb{C}$ normalized by $h_{12}|_{(\pi_1\times\pi_2)((L\times M)\cap\Delta_S)}=0$. Consider the submanifold $P:=S\times M\subset S\times\bar{S}$ and intersect the Lagrangians N_1 and N_2 with this submanifold, yielding respectively $N_1\cap P=N_1$ and $N_2\cap P=\Delta_M$, which both lie in $S\times M$. Observe that $\omega\oplus(-\omega)|_P=p_1^{-1}\omega$, where $p_i:S\times S\to S$ are the projections to the i-th factor. Consider the following diagram of inclusions and projections in $S\times\bar{S}$ and S:

$$\begin{tikzcd}
 & N_1\cap P=N_1 \arrow[dr,"\subset"] \arrow[dd,"p_1"] & & \\
N_2\cap P=\Delta_M \arrow[rr,"\subset"] \arrow[dd,"p_1"] & & P=S\times M \arrow[r,"\subset"] \arrow[dd,"p_1"] & S\times\bar{S} \arrow[dd,"p_1"] \\
 & L \arrow[dr,"\subset"] & & \\
M \arrow[rr,"\subset"] & & S \arrow[r,equal] & S.
\end{tikzcd} \tag{8.2.16}$$

Under the local symplectomorphisms $S\cong T^*U_1$ and $S\times\bar{S}\cong T^*(U_1\times V_2)$, equation (8.2.16) is identified with the diagram:

$$\begin{tikzcd}
 & z(U_1)\times z(V_2) \arrow[dr,"\subset"] \arrow[dd,"\pi_{T^*U_1}"] & & \\
\Gamma_{\mathrm{d}h_{12}}|_{(\mathrm{id}_{U_1}\times\pi_1|_{V_2'}^{-1})(U_1)} \arrow[rr,"\subset"] \arrow[dd,"\pi_{T^*U_1}"] & & T^*U_1\times z(V_2) \arrow[r,"\subset"] \arrow[dd,"\pi_{T^*U_1}"] & T^*(U_1\times V_2) \arrow[dd,"\pi_{T^*U_1}"] \\
 & z(U_1) \arrow[dr,"\subset"] & & \\
\Gamma_{\mathrm{d}f_1} \arrow[rr,"\subset"] & & T^*U_1 \arrow[r,equal] & T^*U_1.
\end{tikzcd} \tag{8.2.17}$$

Here $z:U_1\to T^*U_1$, $z:V_2\to T^*V_2$ are the zero section maps. To see that $N_2\cap P=\Delta_{V_2}$ is identified with $\Gamma_{\mathrm{d}h_{12}}|_{(\mathrm{id}_{U_1}\times\pi_1|_{V_2'}^{-1})(U_1)}$, note that N_2 is identified with $\Gamma_{\mathrm{d}h_{12}}$, and

$$(\pi_1\times\pi_2)(\Delta_{V_2})=(\pi_1|_{V_2'}\times\mathrm{id}_{V_2})(V_2)=(\mathrm{id}_{U_1}\times\pi_1|_{V_2'}^{-1})(U_1)\subset U_1\times V_2,$$

so that Δ_{V_2} is identified with a subset of $T^*(U_1\times V_2)|_{(\mathrm{id}_{U_1}\times\pi_1|_{V_2'}^{-1})(U_1)}\subset T^*(U_1\times V_2)$.

Equation (8.2.17) shows that $\pi_{T^*U_1}$ maps

$$\Gamma_{\mathrm{d}h_{12}}|_{(\mathrm{id}_{U_1}\times\pi_1|_{V_2'}^{-1})(U_1)}\to\Gamma_{\mathrm{d}f_1}.$$

Writing points of T^*U_1 as (x,α) for $x\in U_1$ and $\alpha\in T_x^*U_1$, and points of T^*V_2 as (y,β) for $y\in V_2$ and $\beta\in T_x^*V_2$, we have

$$\begin{aligned}\Gamma_{\mathrm{d}f_1}&=\big\{\big(x,\mathrm{d}f_1(x)\big):x\in U_1\big\},\\ \Gamma_{\mathrm{d}h_{12}}|_{(\mathrm{id}_{U_1}\times\pi_1|_{V_2'}^{-1})(U_1)}&=\big\{\big(x,\mathrm{d}_xh_{12}(x,\pi_1|_{V_2'}^{-1}(x)),\pi_1|_{V_2'}^{-1}(x),0\big):x\in U_1\big\},\end{aligned}$$

where the final term $\beta=0$ as $\Gamma_{\mathrm{d}h_{12}}|_{(\mathrm{id}_{U_1}\times\pi_1|_{V_2'}^{-1})(U_1)}\subset T^*U_1\times z(V_2)$. The projection $\pi_{T^*U_1}$: $T^*U_1\times T^*V_2\to T^*V_2$ maps $(x,\alpha,y,\beta)\mapsto(x,\alpha)$. So from (8.2.17) we see that $\mathrm{d}_xh_{12}(x,\pi_1|_{V_2'}^{-1}(x))=\mathrm{d}_xf_1(x)$ for $x\in U_1$, that is, $\mathrm{d}\big(h_{12}\circ(\mathrm{id}_{U_1}\times\pi_1|_{V_2'}^{-1})\big)=\mathrm{d}f_1$ in 1-forms on U_1. Therefore $h_{12}\circ\big(\mathrm{id}_{U_1}\times\pi_1|_{V_2'}^{-1})\big)=f_1+c$ for some $c\in\mathbb{R}$. But f_1 and h_{12} are normalized by $f_1|_{U_1\cap V_2}=0$ and $h_{12}|_{N_1\cap N_2}=0$, so as $p_1(N_1 \cap N_2) \subset U_1 \cap V_2$ we see that $c = 0$. Hence $h_{12} \circ \big(\mathrm{id}_{U_1} \times \pi_1|_{V_2'}^{-1}\big) = f_1$, and the left hand triangle of (8.2.14) commutes.

Using an analogous argument replacing (8.2.16)–(8.2.17) by the equations:

$$\begin{tikzcd}
& N_1 \cap Q = N_1 \arrow[dd, "p_2"] \arrow[dr, "\subset"] & & \\
N_2 \cap Q = \Delta_L \arrow[rr, "\subset"'] \arrow[dd, "p_2"] & & Q := L \times S \arrow[r, "\subset"'] \arrow[dd, "p_2"] & S \times \bar{S} \arrow[dd, "p_2"'] \\
& M \arrow[dr, "\subset"] & & \\
L \arrow[rr, "\subset"] & & \bar{S} \arrow[r, equal] & \bar{S},
\end{tikzcd}$$

$$\begin{tikzcd}
& z(U_1) \times z(V_2) \arrow[dd, "\pi_{T^*V_2}"] \arrow[dr, "\subset"] & & \\
\Gamma_{\mathrm{d}h_{12}}|_{(\pi_2|_{U_1'}^{-1} \times \mathrm{id}_{V_2})(V_2)} \arrow[rr, "\subset"'] \arrow[dd, "\pi_{T^*V_2}"] & & z(U_1) \times T^*V_2 \arrow[r, "\subset"'] \arrow[dd, "\pi_{T^*V_2}"] & T^*(U_1 \times V_2) \arrow[dd, "\pi_{T^*V_2}"'] \\
& z(U_1) \arrow[dr, "\subset"] & & \\
\Gamma_{\mathrm{d}g_2} \arrow[rr, "\subset"] & & T^*V_2 \arrow[r, equal] & T^*V_2.
\end{tikzcd}$$

we see that the right hand triangle of (8.2.14) commutes.

Finally, the last part of Proposition 8.2.2 follows directly from Theorem 4.3.1(i). □

Note that the local biholomorphisms $\pi_1|_{V_2'}^{-1}$, $\pi_2|_{U_1'}^{-1}$ coming from polarizations π_1, π_2, induce isomorphisms (8.2.2) between the canonical bundles of the Lagrangian submanifolds. In terms of charts, we have an L-chart (P_1, U_1, f_1, i_1), an M-chart (Q_2, V_2, g_2, j_2) and an LM-chart $(R_{12}, W_{12}, h_{12}, k_{12})$ induced by E_1, E_2, $E_1 \times E_2$ respectively, where $P_1 = X \cap U_1$, $Q_2 = X \cap V_2$, $R_{12} = \{x \in X : (x, x) \in W_{12}\}$. Let us denote the corresponding principal $\mathbb{Z}_2$-bundles Q_{P_1,U_1,f_1,i_1}, Q_{Q_2,V_2,g_2,j_2} and $Q_{R_{12},W_{12},h_{12},k_{12}}$ parametrizing square roots of these isomorphisms of canonical bundles as explained in the introduction of §8.2.

Note that Proposition 8.2.2 defined two embeddings $\Phi_{12} : U_1' \hookrightarrow W_{12}$ and $\Psi_{12} : V_2' \hookrightarrow W_{12}$ which satisfy all the properties of Definition 4.3.2, giving embeddings of charts $\Phi_{12} : (P_1', U_1', f_1', i_1') \hookrightarrow (R_{12}, W_{12}, h_{12}, k_{12})$ and $\Psi_{12} : (Q_2', V_2', g_2', j_2') \hookrightarrow (R_{12}, W_{12}, h_{12}, k_{12})$, where (P_1', U_1', f_1', i_1') is a subchart of (P_1, U_1, f_1, i_1), and (Q_2', V_2', g_2', j_2') is a subchart of (Q_2, V_2, g_2, j_2) with $P_1' = Q_2' = R_{12}$.

Thus Definition 4.3.2 gives isomorphisms of line bundles on P^{red}:

$$J_{\Phi_{12}} : K_{U_1}^{\otimes^2}\big|_{P_1'^{\mathrm{red}}} \xrightarrow{\cong} \Phi_{12}|^*_{P_1'^{\mathrm{red}}}\big(K_{W_{12}}^{\otimes^2}\big), \tag{8.2.18}$$

induced by $\Phi_{12} : (P_1', U_1', f_1', i_1') \hookrightarrow (R_{12}, W_{12}, h_{12}, k_{12})$, and

$$J_{\Psi_{12}} : K_{V_2}^{\otimes^2}\big|_{Q_2'^{\mathrm{red}}} \xrightarrow{\cong} \Psi_{12}|^*_{Q_2'^{\mathrm{red}}}\big(K_{W_{12}}^{\otimes^2}\big), \tag{8.2.19}$$

induced by $\Psi_{12} : (Q_2', V_2', g_2', j_2') \hookrightarrow (R_{12}, W_{12}, h_{12}, k_{12})$.

Following Definition 4.3.2, define $\pi_{\Phi_{12}} : P_{\Phi_{12}} \to P_1'$, $\pi_{\Psi_{12}} : P_{\Psi_{12}} \to Q_2'$ to be the principal $\mathbb{Z}_2$-bundles parametrizing square roots of $J_{\Phi_{12}}, J_{\Psi_{12}}$ on R_{12}^{red}. Then we naturally get isomorphisms of principal $\mathbb{Z}_2$-bundles Λ_Φ and Λ_Ψ

$$\Lambda_{\Phi_{12}} : Q_{R_{12},W_{12},h_{12},k_{12}} \xrightarrow{\cong} P_{\Phi_{12}} \otimes_{\mathbb{Z}_2} Q_{P_1,U_1,f_1,i_1}|_{R_{12}}, \tag{8.2.20}$$

$$\Lambda_{\Psi_{12}} : Q_{R_{12},W_{12},h_{12},k_{12}} \xrightarrow{\cong} P_{\Psi_{12}} \otimes_{\mathbb{Z}_2} Q_{Q_2,V_2,g_2,j_2}|_{R_{12}}. \tag{8.2.21}$$

Thus, we can apply Theorem 4.3.4, which yields natural isomorphisms of perverse sheaves

$$\Theta_{\Phi_{12}} : \mathcal{PV}^{\bullet}_{U_1',f_1'} \longrightarrow \Phi_{12}|^*_{P_1'}\big(\mathcal{PV}^{\bullet}_{W_{12},h_{12}}\big) \otimes_{\mathbb{Z}_2} P_{\Phi_{12}}, \tag{8.2.22}$$

$$\Theta_\Psi : \mathcal{PV}^\bullet_{V_2',g_2'} \longrightarrow \Psi_{12}|^*_{Q_2'}\big(\mathcal{PV}^\bullet_{W_{12},h_{12}}\big) \otimes_{\mathbb{Z}_2} P_{\Psi_{12}}, \tag{8.2.23}$$

where $\mathcal{PV}^\bullet_{U_1',f_1'}, \mathcal{PV}^\bullet_{V_2',g_2'}, \mathcal{PV}^\bullet_{W_{12},h_{12}}$ are the perverse sheaves of vanishing cycles from §1.3, and if $\mathcal{Q}^\bullet$ is a perverse sheaf on X then $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}_2} P_{\Phi_{12}}$ is as in Definition 1.3.7. Also diagrams (4.3.14) and (4.3.15) commute. Now, combining the isomorphisms (8.2.20)–(8.2.23) we get isomorphisms

$$\alpha_{12} = \Theta_{\Phi_{12}} \otimes \Lambda^{-1}_{\Phi_{12}} : \big(\mathcal{PV}^\bullet_{U_1,f_1} \otimes_{\mathbb{Z}_2} Q_{P_1,U_1,f_1,i_1}\big)|_{R_{12}} \longrightarrow \mathcal{PV}^\bullet_{W_{12},h_{12}} \otimes_{\mathbb{Z}_2} Q_{R_{12},W_{12},h_{12},k_{12}}, \tag{8.2.24}$$

$$\beta_{12} = \Theta_{\Psi_{12}} \otimes \Lambda^{-1}_{\Psi_{12}} : \big(\mathcal{PV}^\bullet_{V_2,g_2} \otimes_{\mathbb{Z}_2} Q_{Q_2,V_2,g_2,j_2}\big)|_{R_{12}} \longrightarrow \mathcal{PV}^\bullet_{W_{12},h_{12}} \otimes_{\mathbb{Z}_2} Q_{R_{12},W_{12},h_{12},k_{12}}, \tag{8.2.25}$$

$$\beta^{-1}_{12} \circ \alpha_{12} : \big(\mathcal{PV}^\bullet_{U_1,f_1} \otimes_{\mathbb{Z}_2} Q_{P_1,U_1,f_1,i_1}\big)|_{R_{12}} \longrightarrow \big(\mathcal{PV}^\bullet_{V_2,g_2} \otimes_{\mathbb{Z}_2} Q_{Q_2,V_2,g_2,j_2}\big)|_{R_{12}}. \tag{8.2.26}$$

8.2.2 Comparing perverse sheaves of vanishing cycles associated to polarizations

Given a complex symplectic manifold (S,ω) and L, M Lagrangian submanifolds in S, define X to be their intersection. From §8.1, locally in the complex analytic topology near a point $x \in X$, we can choose an open set $S' \subset S$ and we can choose a polarization transverse to both L and M such that $S' \cong T^*L$ and $M \cong \Gamma_{\mathrm{d}f}$ so that $X \cap S' = \mathrm{Crit}(f)$ for a holomorphic function f defined on $U \subseteq L \cap S'$. Thus we get a perverse sheaf of vanishing cycle $\mathcal{PV}^\bullet_{U,f}$. In §8.2.1 we compared perverse sheaves of vanishing cycles associated to two transverse polarizations. In this section we will investigate about how they behave if we consider four polarizations, pairwise transverse in a 4-cycle. This result will be used in §8.2.3 to prove Theorem 8.2.1.

We choose four polarizations $\pi_i : S \to E_i$ for $i = 1, \dots, 4$ all transverse to each other except perhaps for the pairs E_1, E_3 and E_2, E_4. Define L-charts $(P_1,U_1,f_1,i_1), (P_3,U_3,f_3,i_3)$ from π_1, π_3 and M-charts $(Q_2,V_2,g_2,j_2), (Q_4,V_4,g_4,j_4)$ from π_2, π_4, as in the beginning of §8.2. Define LM-charts $(R_{12},W_{12},h_{12},k_{12})$ from π_1,π_2, $(R_{32},W_{32},h_{32},k_{32})$ from π_3,π_2, $(R_{34},W_{34},h_{34},k_{34})$ from π_3,π_4, $(R_{14},W_{14},h_{14},k_{14})$ from π_1,π_4 as in §8.2.1, with embeddings of charts $\Phi_{12},\Psi_{12},\dots,\Phi_{14},\Psi_{14}$ from subcharts of $(P_a,U_a,f_a,i_a),(Q_b,V_b,g_b,j_b)$ to $(R_{ab},W_{ab},h_{ab},k_{ab})$.

Similarly to Proposition 8.2.2, we have the following result:

Proposition 8.2.3. *Given a complex symplectic manifold (S,ω) and L,M Lagrangian submanifolds in S, define X to be their intersection. Suppose we are given four polarizations $\pi_1,\dots,\pi_4$, and choose data (P_a,U_a,f_a,i_a) for $a = 1,3$, (Q_b,V_b,g_b,j_b) for $a = 2,4$ and $(R_{ab},W_{ab},h_{ab},k_{ab}),\Phi_{ab},\Psi_{ab}$ for $ab = 12,32,34,14$ as above.*

Then there exist an open set Z in $U_1 \times V_2 \times U_3 \times V_4$, a holomorphic function $F : Z \to \mathbb{C}$, and open neighbourhoods U_a' of $X \cap S_1 \cap S_2 \cap S_3 \cap S_4$ in U_a, and V_b' of $X \cap S_1 \cap S_2 \cap S_3 \cap S_4$ in V_b, and $W_{ab'}$ of $\big\{(x,x) : x \in X \cap S_1 \cap S_2 \cap S_3 \cap S_4\big\}$ in W_{ab} for all $a = 1,3$, $b = 2,4$ such that the *following diagram commutes:*

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
U_1' \arrow[rr, "\Phi_{12}|_{U_1'}=\mathrm{id}_{U_1}\times\pi_1|_{V_2}^{-1}"'] \arrow[d, "\Phi_{14}|_{U_1'}=\pi_1|_{V_4}^{-1}\times\mathrm{id}_{U_1}"'] \arrow[drr, "\mathrm{id}_{U_1}\times\pi_1|_M^{-1}\times(\pi_3|_M\circ\pi_1|_M^{-1})\times\pi_1|_M^{-1}"] \arrow[dddrrr, "f_1|_{U_1'}"] & & W_{12}' \arrow[d, "\pi_{U_1}\times\pi_{V_2}\times(\pi_3|_M\circ\pi_{V_2})\times\pi_{V_2}"'] \arrow[dddr, "h_{12}|_{W_{12}'}"] & & V_2' \arrow[ll, "\Psi_{12}|_{V_2'}=\pi_2|_L^{-1}\times id_{V_2'}"'] \arrow[d, "\Psi_{32}|_{V_2'}=id_{V_2'}\times\pi_2|_L^{-1}"] \arrow[dll, "\pi_2|_L^{-1}\times\mathrm{id}_{\tilde M}\times\pi_3|_{\tilde M}\times\mathrm{id}_{V_2'}"'] \arrow[dddl, "g_2|_{V_2'}"] \\
W_{14}' \arrow[rr, "\pi_{U_1}\times(\pi_2|_L\circ(\pi_4|_L\times\pi_1|_M)^{-1})\times(\pi_3|_M\circ(\pi_4|_L\times\pi_1|_M)^{-1})\times\pi_{V_4'}"] \arrow[ddrrr, "h_{14}|_{W_{14}'}"] & & Z \arrow[ddr, "F"] & & W_{32}' \arrow[ll, "\pi_{U_3'}\times(\pi_2|_L\circ\pi_{U_3'})\times(\pi_3|_M\circ\pi_{V_2'})\times\pi_{V_2'}"'] \arrow[ddl, "h_{32}|_{W_{32}'}"'] \\
V_4' \arrow[u, "\Psi_{14}|_{V_4'}=id_{V_4'}\times\pi_4|_L^{-1}"] \arrow[rr, "\Psi_{34}|_{V_4'}=\pi_4|_L^{-1}\times id_{V_4'}"'] \arrow[urr, "\pi_4|_L^{-1}\times(\pi_2|_L\circ\pi_4|_L^{-1})\times\pi_4|_L^{-1}\times\mathrm{id}_{V_4'}"] \arrow[drrr, "g_4|_{V_4'}"'] & & W_{34}' \arrow[u, "\pi_{U_3'}\times(\pi_2|_L\circ\pi_{U_3'})\times\pi_{U_3'}\times\pi_{V_4'}"'] \arrow[dr, "h_{34}|_{W_{34}'}"'] & & U_3' \arrow[ll, "\Phi_{34}|_{U_3'}=id_{U_3'}\times\pi_3|_M^{-1}"] \arrow[u, "\Phi_{32}|_{U_3'}=\pi_3|_M^{-1}\times id_{U_3'}"'] \arrow[ull, "\mathrm{id}_{U_3'}\times\pi_2|_L\times\mathrm{id}_{U_3'}\times\pi_3|_M^{-1}"] \arrow[dl, "f_3|_{U_3'}"] \\
& & & \mathbb{C}. &
\end{tikzcd}
\tag{8.2.27}
$$

Moreover, locally in the complex analytic topology $\mathrm{Crit}(F) \cong \mathrm{Crit}(h_{ij}) \cong \mathrm{Crit}(f_i) \cong \mathrm{Crit}(g_j)$ *as complex analytic spaces for all* i, j. *In particular, we can choose appropriate coordinate systems under which we can write* F *as the sum of functions* h_{ij} *or* f_i *or* g_j *and non degenerate quadratic forms, that is they are all stably equivalent to each other in the sense of Theorem 4.3.1.*

Proof. In equation (8.2.27) there are three kinds of small triangles:

(i) Eight triangles with vertices U_a', W_{ab}', Z or V_b', W_{ab}', Z.

(ii) Eight triangles with vertices $U_a', W_{ab}', \mathbb{C}$ or $V_b', W_{ab}', \mathbb{C}$.

(iii) Four triangles with vertices $W_{ab}', Z, \mathbb{C}$.

To show that (8.2.27) commutes, we must show all these triangles commute. For the triangles of type (i) we can just check this by hand in an elementary way. The triangles of type (ii) commute by Proposition 8.2.2 applied to π_1, π_2 or π_3, π_2 or π_3, π_4 or π_1, π_4. This leaves the triangles of type (iii), which we will show commute by a similar proof to Proposition 8.2.2.

Consider the product symplectic manifold $(S \times \bar{S} \times S \times \bar{S}, \omega \oplus -\omega \oplus \omega \oplus -\omega)$, where $\bar{S}$ denotes the symplectic manifold S corresponding to the symplectic form with the opposite sign. Write $p_i : S \times S \times S \times S \to S$ for the projection to the i-th factor. In $S \times \bar{S} \times S \times \bar{S}$ consider the Lagrangian submanifolds $N_1 := L \times \Delta_S \times M$ and $N_2 := \Delta_S \times \Delta_S$. Identify it with $T^*(L \times M \times L \times M)$, and thus $\pi_1 \times \pi_2 \times \pi_3 \times \pi_4$ with $\pi : T^*(L \times M \times L \times M) \to L \times M \times L \times M$, that is N_1 with the zero section, and $N_2 := \Gamma_{\mathrm{d}F}$ for a holomorphic function $F : Z \to \mathbb{C}$ for open $Z \subseteq L \times M \times L \times M$ normalized by $F|_{(\pi_1 \times \pi_2 \times \pi_3 \times \pi_4)(N_1 \cap N_2)} = 0$. Consider the submanifolds

$$P_{12} := S \times S \times \Delta_S^{34}, \quad P_{32} := L \times S \times S \times M, \quad P_{34} := \Delta_S^{12} \times S \times S, \quad P_{14} := S \times \Delta_S^{32} \times S.$$

In the same style as the proof of Proposition 8.2.14, intersect the Lagrangians with these submanifolds. We can either identify N_1 with the zero section and $N_2 = \Gamma_{\mathrm{d}F}$, or $N_1 = \Gamma_{-\mathrm{d}F}$ and N_2 with the zero section. We will use both the options. Let us start with the submanifold

P_{12}, for which we use the second identification. Consider the following diagram of inclusions and projections in $S \times \bar{S} \times S \times \bar{S}$ and $S \times \bar{S}$:

$$\begin{tikzcd}
& N_1 \cap P_{12} = L \times \Delta_M^{234} \arrow[dd, "p_1\times p_2"] \arrow[dr, "\subset"] & & \\
N_2 \cap P_{12} = N_2 \arrow[rr, "\subset"'] \arrow[dd, "p_1\times p_2"] & & P_{12} = S \times S \times \Delta_S^{34} \arrow[r, "\subset"'] \arrow[dd, "p_1\times p_2"] & S \times \bar{S} \times S \times \bar{S} \arrow[dd, "p_1\times p_2"'] \\
& L \times M \arrow[dr, "\subset"] & & \\
\Delta_S \arrow[rr, "\subset"] & & S \times \bar{S} \arrow[r, equal] & S \times \bar{S}.
\end{tikzcd} \tag{8.2.28}$$

Under the local symplectomorphisms $S\times\bar{S} \cong T^*(L\times M)$ and $S\times\bar{S}\times S\times\bar{S} \cong T^*(L\times M\times L\times M)$, equation (8.2.28) is identified with the diagram:

$$\begin{tikzcd}
& \Gamma_{-\mathrm{d}F}|_{(\pi_L\times\pi_M\times(\pi_3|_M\circ\pi_M)\times\pi_M)(W'_{12})} \arrow[dd, "\pi_{12}"] \arrow[dr, "\subset"] & & \\
z(L)\times z(M)\times z(L)\times z(M) \arrow[rr, "\subset"'] \arrow[dd, "\pi_{12}"] & & z(L)\times z(M)\times T^*L\times T^*M \arrow[r, "\subset"'] \arrow[dd, "\pi_{12}"] & T^*L\times T^*M\times T^*L\times T^*M \arrow[dd, "\pi_{12}"'] \\
& \Gamma_{-\mathrm{d}h_{12}} \arrow[dr, "\subset"] & & \\
z(L)\times z(M) \arrow[rr, "\subset"] & & T^*L\times T^*M \arrow[r, equal] & T^*L\times T^*M.
\end{tikzcd} \tag{8.2.29}$$

Here $z : L \to T^*L$, $M \to T^*M$ are the zero section maps. To understand this, note that $\pi_1\times\pi_2\times\pi_2\times\pi_4$ maps $N_1\cap P_{12} = L\times\Delta_M^{234}$ to the submanifold $(\pi_L\times\pi_M\times(\pi_3|_M\circ\pi_M)\times\pi_M)(L\times M)$ in $L\times M\times L\times M$. Our identification $S\times\bar{S}\times S\times\bar{S} \cong T^*(L\times M\times L\times M)$ maps $N_1 \mapsto \Gamma_{-\mathrm{d}F}$. Hence the top term $N_1\cap P_{12}$ in (8.2.28) is identified with the top term $\Gamma_{-\mathrm{d}F}|_{(\pi_L\times\pi_M\times(\pi_3|_M\circ\pi_M)\times\pi_M)(W'_{12})}$ in (8.2.29). As for (8.2.16)–(8.2.17), we see from (8.2.28)–(8.2.29) that the triangle of type (iii) in (8.2.27) with vertices the top centre $L\times M$, and $L\times M\times L\times M$, and $\mathbb{C}$, commutes.

Similarly, taking intersections with the submanifold P_{14} gives a diagram analogous to (8.2.28):

$$\begin{tikzcd}
& N_1 \cap P_{14} = N_1 \arrow[dd, "p_1\times p_4"] \arrow[dr, "\subset"] & & \\
N_2 \cap P_{14} = \Delta_S^{1234} \arrow[rr, "\subset"'] \arrow[dd, "p_1\times p_4"] & & P_{14} = S \times \Delta_S^{23} \times S \arrow[r, "\subset"'] \arrow[dd, "p_1\times p_4"] & S \times \bar{S} \times S \times \bar{S} \arrow[dd, "p_1\times p_4"'] \\
& L \times M \arrow[dr, "\subset"] & & \\
\Delta_S \arrow[rr, "\subset"] & & S \times \bar{S} \arrow[r, equal] & S \times \bar{S}.
\end{tikzcd}$$

Using the first identification, this is identified with the diagram

$$\begin{tikzcd}
& z(L)\times z(M)\times z(L)\times z(M) \arrow[dd, "\pi_{14}"] \arrow[dr, "\subset"] & & \\
\begin{array}{c}\Gamma_{\mathrm{d}F}|_{(\pi_L\times(\pi_2|_L\circ(\pi_4|_L\times\pi_1|_M)^{-1})\times} \\ {}_{(\pi_3|_M\circ(\pi_4|_L\times\pi_1|_M)^{-1})\times\pi_M)(W'_{14})}\end{array} \arrow[rr, "\subset"'] \arrow[dd, "\pi_{14}"] & & z(L)\times T^*M\times T^*L\times z(M) \arrow[r, "\subset"'] \arrow[dd, "\pi_{14}"] & T^*(L\times M\times L\times M) \arrow[dd, "\pi_{14}"'] \\
& z(L)\times z(M) \arrow[dr, "\subset"] & & \\
\Gamma_{\mathrm{d}h_{14}} \arrow[rr, "\subset"] & & T^*(L\times M) \arrow[r, equal] & T^*(L\times M).
\end{tikzcd}$$

Here $\pi_1\times\pi_2\times\pi_2\times\pi_4$ maps $N_2\cap P_{14} = \Delta_S^{1234}$ to $(\pi_L\times(\pi_2|_L\circ(\pi_4|_L\times\pi_1|_M)^{-1})\times(\pi_3|_M\circ(\pi_4|_L\times\pi_1|_M)^{-1})\times\pi_M)(L\times M)$, and we identify N_2 with $\Gamma_{\mathrm{d}F}$, which is how we get the first term on the middle line. From this we see that the triangle of type (iii) in (8.2.27) with vertices the left hand $L\times M$, and $L\times M\times L\times M$, and $\mathbb{C}$, commutes. The remaining two type (iii) triangles can be shown to commute in a similar way. Hence (8.2.27) commutes. Finally, the last part of Proposition 8.2.3 follows directly from Theorem 4.3.1(i). □

In the situation of Proposition 8.2.3, set $Y = X \cap S_1 \cap S_2 \cap S_3 \cap S_4$. Then following the reasoning of (8.2.18)–(8.2.26) which defined the isomorphisms of perverse sheaves α_{12}, β_{12} in (8.2.24)–(8.2.25), from (8.2.27) we get a commutative diagram of isomorphisms of perverse sheaves:

$$
\begin{tikzcd}
(\mathcal{PV}^\bullet_{U_1,f_1} \otimes_{\mathbb{Z}_2} Q_{P_1,U_1,f_1,i_1})|_Y \arrow[r, "\alpha_{12}|_Y"'] \arrow[d, "\alpha_{14}|_Y"] \arrow[dr] & (\mathcal{PV}^\bullet_{W_{12},h_{12}} \otimes_{\mathbb{Z}_2} Q_{R_{12},W_{12},h_{12},k_{12}})|_Y \arrow[d] & (\mathcal{PV}^\bullet_{V_2,g_2} \otimes_{\mathbb{Z}_2} Q_{Q_2,V_2,g_2,j_2})|_Y \arrow[l, "\beta_{12}|_Y"] \arrow[d, "\beta_{32}|_Y"'] \arrow[dl] \\
(\mathcal{PV}^\bullet_{W_{14},h_{14}} \otimes_{\mathbb{Z}_2} Q_{R_{14},W_{14},h_{14},k_{14}})|_Y \arrow[r] & \mathcal{PV}^\bullet_{Z,F} \otimes_{\mathbb{Z}_2} Q_{Z,F} & (\mathcal{PV}^\bullet_{W_{32},h_{32}} \otimes_{\mathbb{Z}_2} Q_{R_{32},W_{32},h_{32},k_{12}})|_Y \arrow[l] \\
(\mathcal{PV}^\bullet_{V_4,g_4} \otimes_{\mathbb{Z}_2} Q_{Q_4,V_4,g_4,j_4})|_Y \arrow[r, "\beta_{34}|_Y"] \arrow[u, "\beta_{14}|_Y"] \arrow[ur] & (\mathcal{PV}^\bullet_{W_{34},h_{34}} \otimes_{\mathbb{Z}_2} Q_{R_{34},W_{34},h_{34},k_{34}})|_Y \arrow[u] & (\mathcal{PV}^\bullet_{U_3,f_3} \otimes_{\mathbb{Z}_2} Q_{P_3,U_3,f_3,i_3})|_Y \arrow[l, "\alpha_{34}|_Y"'] \arrow[u, "\alpha_{32}|_Y"'] \arrow[ul]
\end{tikzcd}
\tag{8.2.30}
$$

Since (8.2.30) commutes, we deduce that

$$
\begin{aligned}
\alpha_{32}|_Y^{-1} \circ \beta_{32}|_Y \circ \beta_{12}|_Y^{-1} \circ \alpha_{12}|_Y = \alpha_{34}|_Y^{-1} \circ \beta_{34}|_Y \circ \beta_{14}|_Y^{-1} \circ \alpha_{14}|_Y : \\
(\mathcal{PV}^\bullet_{U_1,f_1} \otimes_{\mathbb{Z}_2} Q_{P_1,U_1,f_1,i_1})|_Y \longrightarrow (\mathcal{PV}^\bullet_{U_3,f_3} \otimes_{\mathbb{Z}_2} Q_{P_3,U_3,f_3,i_3})|_Y.
\end{aligned}
\tag{8.2.31}
$$

Equation (8.2.31) tells us something important. Suppose we start with polarizations $\pi_1 : S_1 \to E_1$ and $\pi_3 : S_3 \to E_3$ transverse to L, M, and use them to define L-charts (P_1, U_1, f_1, i_1) and (P_3, U_3, f_3, i_3), and hence perverse sheaves $\mathcal{PV}^\bullet_{U_1,f_1} \otimes_{\mathbb{Z}_2} Q_{P_1,U_1,f_1,i_1}$ on $P_1 = X \cap S_1$ and $\mathcal{PV}^\bullet_{U_3,f_3} \otimes_{\mathbb{Z}_2} Q_{P_3,U_3,f_3,i_3}$ on $P_3 = X \cap S_3$. We wish to relate these perverse sheaves on the overlap $X \cap S_1 \cap S_3$. To do this, we choose another polarization $\pi_2 : S_2 \to E_2$ transverse to L, M, π_1, π_3, and define an M-chart (Q_2, V_2, g_2, j_2) and LM-charts $(R_{12}, W_{12}, h_{12}, k_{12})$ and $(R_{32}, W_{32}, h_{32}, k_{32})$. Then as in (8.2.26), $\alpha_{32}^{-1} \circ \beta_{32} \circ \beta_{12}^{-1} \circ \alpha_{12}$ provides the isomorphism $\mathcal{PV}^\bullet_{U_1,f_1} \otimes_{\mathbb{Z}_2} Q_{P_1,U_1,f_1,i_1} \cong \mathcal{PV}^\bullet_{U_3,f_3} \otimes_{\mathbb{Z}_2} Q_{P_3,U_3,f_3,i_3}$ we want on $X \cap S_1 \cap S_2 \cap S_3$. Equation (8.2.31) shows that this isomorphism is independent of the choice of polarization $\pi_2 : S_2 \to E_2$.

8.2.3 Descent for perverse sheaves

To conclude the proof of Theorem 8.2.1, we use Theorem 1.3.5, so in particular a descent argument to glue and get a global perverse sheaf. In this section we adopt the point of view of charts induced by polarizations. This proof follows similar ideas to [18, §6.3].

Let (S, ω) be a complex symplectic manifold and L, M complex Lagrangian submanifolds in S, and write $X = L \cap M$, as a complex analytic subspace of S. Suppose we are given square roots $K_L^{1/2}, K_M^{1/2}$ for K_L, K_M. We may choose a family of polarizations $\pi_a : S_a \to E_a$ which defines a family $\{(P_a, U_a, f_a, i_a) : a \in A\}$ of L-charts (P_a, U_a, f_a, i_a) on X such that $\{P_a : a \in A\}$ is an analytic open cover of the analytic space X, so that $P_a \cong \mathrm{Crit}(f_a)$ for holomorphic functions $f_a : U_a \to \mathbb{C}$, and U_a complex manifolds (Lagrangians), and $i_a : P_a \hookrightarrow U_a$ closed embeddings.

Then for each $a \in A$ we have a perverse sheaf

$$
i_a^*(\mathcal{PV}^\bullet_{U_a,f_a}) \otimes_{\mathbb{Z}_2} Q_{P_a,U_a,f_a,i_a} \in \mathrm{Perv}(P_a),
\tag{8.2.32}
$$

for Q_{P_a,U_a,f_a,i_a} the principal $\mathbb{Z}_2$ bundle defined in §8.2 point (i) parametrizing choices of square roots of canonical bundles $K_L^{1/2} \xrightarrow{\cong} K_M^{1/2}$ which square to isomorphisms (8.2.2). As explained already in the introduction of §8.2, the idea of the proof is to use Theorem 1.3.5(ii) to glue the perverse sheaves (8.2.32) on the analytic open cover $\{P_a : a \in A\}$ to get a global perverse sheaf $P^\bullet_{L,M}$ on X.

We already know from Proposition 8.2.2 that, given an L-chart (P, U, f, i) and an M-chart (Q, V, g, j) we have the isomorphism (8.2.11), which we recall here:

$$\beta^{-1} \circ \alpha : (i^*(\mathcal{PV}^\bullet_{U,f}) \otimes Q_{P,U,f,i})|_P \xrightarrow{\cong} (j^*(\mathcal{PV}^\bullet_{V,g}) \otimes Q_{Q,V,g,j})|_P,$$

that is, an isomorphism of perverse sheaves from L-charts and M-charts in $\operatorname{Perv}(P \cap Q)$.

Now, to develop our program, we have to show that if (P_a, U_a, f_a, i_a) and (P_b, U_b, f_b, i_b) are L-charts, then we have a canonical isomorphism

$$\delta_{ab} : (i_a^*(\mathcal{PV}^\bullet_{U_a,f_a}) \otimes Q_{P_a,U_a,f_a,i_a})|_{P_a \cap P_b} \xrightarrow{\cong} (i_b^*(\mathcal{PV}^\bullet_{U_b,f_b}) \otimes Q_{P_b,U_b,f_b,i_b})|_{P_a \cap P_b} \tag{8.2.33}$$

with the property that for any M-chart (Q, V, g, j) coming from $\tilde\pi : \tilde S \to F$ transverse to π_a and π_b, we have

$$\delta_{ab}|_{P_a \cap P_b \cap Q} = \alpha^{-1}_{U_b,f_b,W',h'}|_{P_a \cap P_b \cap Q} \circ \beta_{W',h',V,g}|_{P_a \cap P_b \cap Q} \circ \beta^{-1}_{W,h,V,g}|_{P_a \cap P_b \cap Q} \circ \alpha_{U_a,f_a,W,h}|_{P_a \cap P_b \cap Q}. \tag{8.2.34}$$

To prove this, we first use Proposition 8.2.3, which provides an associativity result as in (8.2.12) or (8.2.31). In particular, it shows that if (Q', V', g', j') is another such M-chart, then

$$\begin{aligned}&\alpha^{-1}_{U_b,f_b,W',h'}|_{P_a \cap P_b \cap Q \cap Q'} \circ \beta_{W',h',V,g}|_{P_a \cap P_b \cap Q \cap Q'} \circ \beta^{-1}_{W,h,V,g}|_{P_a \cap P_b \cap Q \cap Q'} \circ \alpha_{U_a,f_a,W,h}|_{P_a \cap P_b \cap Q \cap Q'} = \\ &\alpha^{-1}_{U_b,f_b,W'',h''}|_{P_a \cap P_b \cap Q \cap Q'} \circ \beta_{W'',h'',V',g'}|_{P_a \cap P_b \cap Q \cap Q'} \circ \beta^{-1}_{W''',h''',V',g'}|_{P_a \cap P_b \cap Q \cap Q'} \circ \alpha_{U_a,f_a,W''',h'''}|_{P_a \cap P_b \cap Q \cap Q'}.\end{aligned}$$

Fix two charts (P_a, U_a, f_a, i_a) and (P_b, U_b, f_b, i_b), and choose a family $\{(Q_c, V_c, g_c, j_c) : c \in I\}$ of M-charts (Q_c, V_c, g_c, j_c) on X transverse to both (P_a, U_a, f_a, i_a) and (P_b, U_b, f_b, i_b), such that $\{Q_c : c \in I\}$ is an analytic open cover of $P_a \cap P_b$. Then, we can use the sheaf property of morphisms of perverse sheaves in the sense of Theorem 1.3.5, to get δ_{ab} as in (8.2.34) by gluing

$$\alpha^{-1}_{U_b,f_b,W',h'}|_{P_a \cap P_b \cap Q} \circ \beta_{W',h',V,g}|_{P_a \cap P_b \cap Q} \circ \beta^{-1}_{W,h,V,g}|_{P_a \cap P_b \cap Q} \circ \alpha_{U_a,f_a,W,h}|_{P_a \cap P_b \cap Q}$$

on the open cover $\{Q_c : c \in I\}$. Also, δ_{ab} satisfy (8.2.34) for all (Q, V, g, j), and this is independent of choice of (Q, V, g, j). This is because we can run the construction above with the family $\{(P_a, U_a, f_a, i_a) : a \in A\} \amalg \{(Q, V, g, j)\}$, yielding the same result.

Moreover, on $P_a \cap P_b \cap P_c$ we have $\delta_{bc} \circ \delta_{ab} = \delta_{ac}$. This is because, given locally a polarization $\tilde\pi : \tilde S \to F$ transverse to all of π_a, π_b, π_c, then on $P_a \cap P_b \cap P_c \cap Q$, we can easily check that

$$\begin{aligned}\delta_{bc} \circ \delta_{ab}|_{P_a \cap P_b \cap P_c \cap Q} =& (\alpha^{-1}_{U_c,f_c,W'',h''} \circ \beta_{W'',h'',V,g} \circ \beta^{-1}_{W',h',V,g} \circ \alpha_{U_b,f_b,W',h'}) \circ \\ & (\alpha^{-1}_{U_b,f_b,W',h'} \circ \beta^{-1}_{W',h',V,g} \circ \beta^{-1}_{W,h,V,g} \circ \alpha_{U_a,f_a,W,h}) = \\ & (\alpha^{-1}_{U_c,f_c,W'',h''} \circ \beta_{W'',h'',V,g} \circ \beta^{-1}_{W,h,V,g} \circ \alpha_{U_a,f_a,W,h}) = \delta_{ac}|_{P_a \cap P_b \cap P_c \cap Q}.\end{aligned}$$

As we can cover $P_a \cap P_b \cap P_c$ by such open $P_a \cap P_b \cap P_c \cap Q$, we deduce that $\delta_{bc} \circ \delta_{ab} = \delta_{ac}$ by the sheaf property of morphisms of perverse sheaves in the sense of Theorem 1.3.5.

In conclusion, we have an open cover of X by L-charts (P_a, U_a, f_a, i_a), and isomorphisms (8.2.33), satisfying $\delta_{bc} \circ \delta_{ab} = \delta_{ac}$. So by stack property of perverse sheaves in the sense of Theorem 1.3.5(ii), we get that there exists $P^\bullet_{L,M}$ in $\operatorname{Perv}(X)$, unique up to canonical isomorphism, with isomorphisms

$$\omega_{P_a,U_a,f_a,i_a} : P^\bullet_{L,M}|_{P_a} \xrightarrow{\cong} i_a^*(\mathcal{PV}^\bullet_{U_a,f_a}) \otimes_{\mathbb{Z}_2} Q_{P_a,U_a,f_a,i_a}$$

as in (8.2.4) for each $a \in A$, with $\delta_{ab} \circ \omega_{P_a,U_a,f_a,i_a}|_{P_a \cap P_b} = \omega_{P_b,U_b,f_b,i_b}|_{P_a \cap P_b}$ for all $a, b \in A$. Also, (8.2.5)–(8.2.6) with (P_a, U_a, f_a, i_a) in place of (P, U, f, i) define isomorphisms $\Sigma_{L,M}|_{P_a}$, $\mathrm{T}_{L,M}|_{P_a}$

for each $a \in A$. The prescribed values for $\Sigma_{L,M}|_{P_a}, \mathrm{T}_{L,M}|_{P_a}$ and $\Sigma_{L,M}|_{P_b}, \mathrm{T}_{L,M}|_{P_b}$ agree when restricted to $P_a \cap P_b$ for all $a, b \in A$. Hence, Theorem 1.3.5(i) gives unique isomorphisms $\Sigma_{L,M}, \mathrm{T}_{L,M}$ in (8.2.1) such that (8.2.5)–(8.2.6) commute with (P_a, U_a, f_a, i_a) in place of (P, U, f, i) for all $a \in A$.

Also, the whole construction is independent of the choice of the family of L-charts and polarizations. This is because we can suppose $\{(P_a, U_a, f_a, i_a) : a \in A\}$ and $\{(\tilde{P}_a, \tilde{U}_a, \tilde{f}_a, \tilde{\imath}_a) : a \in \tilde{A}\}$ are alternative choices above, yielding $P^{\bullet}_{L,M}, \Sigma_{L,M}, \mathrm{T}_{L,M}$ and $\tilde{P}^{\bullet}_{L,M}, \tilde{\Sigma}_{L,M}, \tilde{\mathrm{T}}_{L,M}$. Then applying the same construction to the family $\{(P_a, U_a, f_a, i_a) : a \in A\} \amalg \{(\tilde{P}_a, \tilde{U}_a, \tilde{f}_a, \tilde{\imath}_a) : a \in \tilde{A}\}$ to get $\hat{P}^{\bullet}_{L,M}$, we have canonical isomorphisms $P^{\bullet}_{L,M} \cong \hat{P}^{\bullet}_{L,M} \cong \tilde{P}^{\bullet}_{L,M}$, which identify $\Sigma_{L,M}, \mathrm{T}_{L,M}$ with $\tilde{\Sigma}_{L,M}, \tilde{\mathrm{T}}_{L,M}$. Thus $P^{\bullet}_{L,M}, \Sigma_{L,M}, \mathrm{T}_{L,M}$ are independent of choices up to canonical isomorphism.

8.3 Analytic d-critical locus structure on complex Lagrangian intersections

Pantev et al. [142] show that derived intersections $L \cap M$ of algebraic Lagrangians L, M in an algebraic symplectic manifold (S, ω) have (-1)–shifted symplectic structures, so that Theorem 6.6 in [19], discussed also in §3, gives them the structure of algebraic d-critical loci. Here, we will prove a complex analytic version of this. The result of this section, which is the complex analytic version of Corollary 3.3.3 in §3, is Theorem 8.3.1, which states that the Lagrangian intersection $L \cap M$ of (oriented) complex Lagrangians L, M has the structure of an (oriented) complex analytic d-critical locus. Notice at this point that we could have then used [18, Thm 6.9] to define a perverse sheaf $P^{\bullet}_{L,M}$ on $L \cap M$, instead of going through Theorem 8.2.1 in §8.2, but we wanted to provide a clear and direct proof about how to glue perverse sheaves on complex Lagrangian intersections in a complex analytic setup, and using only classical and symplectic geometry. Note also that we cannot prove Theorem 8.3.1 by going via [142], as they do not do a complex analytic version.

Here is the result of the section.

Theorem 8.3.1. *Suppose (S, ω) is a complex symplectic manifold, and L, M are (oriented) complex Lagrangian submanifolds in S. Then the intersection $X = L \cap M$, as a complex analytic subspace of S, extends naturally to a (oriented) complex analytic d-critical locus (X, s). The canonical bundle $K_{X,s}$ is naturally isomorphic to $K_L|_{X^{\mathrm{red}}} \otimes K_M|_{X^{\mathrm{red}}}$.*

Proof. Let (S, ω) be a complex symplectic manifold, and $L, M \subset S$ two complex Lagrangian submanifolds of S. Given the complex analytic space $X = L \cap M$, we must construct a section $s \in H^0(S^0_X)$ such that (X, s) is a complex analytic d-critical locus. We use notation from §8.2, and in particular the notions of L-chart, M-chart, and LM-chart.

We claim that there is a unique d-critical structure s on X, such that

1. every L-chart (P, U, f, i) from a polarization $\pi_1 : S_1 \to E_1$ transverse to L, M is a critical chart on (X, s);

2. every M-chart (Q, V, g, j) from a polarization $\pi_2 : S_2 \to E_2$ transverse to L, M is a critical chart on (X, s).

where L-charts and M-charts are defined using transverse polarizations. To show this we note that the L-chart (P, U, f, i) determines a d-critical structure s_P on P, and similarly the M-chart (Q, V, g, j) determines a d-critical structure s_Q on Q.

Next, for given L-charts and M-charts, we use the LM-charts (R, W, h, k) and Proposition 8.2.2 in §8.2 to show that $s_P|_{P \cap Q} = s_Q|_{P \cap Q}$.

Then, we choose a locally finite cover of L-charts (P_a, U_a, f_a, i_a) for $a \in A$ covering X, from polarizations transverse to L, M. We choose M-charts (Q_b, U_b, f_b, i_b) for $b \in B$ covering X, from polarizations transverse to L, M and all polarizations used to define the (P_a, U_a, f_a, i_a). Then we get: $s_{P_a}|_{P_a \cap Q_b} = s_{Q_b}|_{P_a \cap Q_b}$ for all a, b. Hence $s_{P_a}|_{P_a \cap P_{a'} \cap Q_b} = s_{P_{a'}}|_{P_a \cap P_{a'} \cap Q_b}$ for all $a, a' \in A$, $b \in B$. As the Q_b cover X, we have $s_{P_a}|_{P_a \cap P_{a'}} = s_{P_{a'}}|_{P_a \cap P_{a'}}$, for all $a, a' \in A$.

So there exists a unique section s with $s|_{P_a} = s_{P_a}$, for all $a \in A$, as S^0_X is a sheaf. Finally, following the same technique of §8.2.3, the construction is independence of choices.

For the second part of the theorem, let (P, U, f, i), be a critical chart on (X, s). Then Theorem 2.1.6(i) gives a natural isomorphism

$$\iota_{P,U,f,i} : K_{X,s}|_{P^{\mathrm{red}}} \longrightarrow i^*\big(K_U^{\otimes^2}\big)|_{P^{\mathrm{red}}}. \tag{8.3.1}$$

Using (8.2.2), note that $K_U^2 \cong K_L \otimes K_M$, as the polarization π identifies both L, M with U locally, giving isomorphisms $K_U|_X \cong K_L|_X \cong K_M|_X$. Now comparing with (8.1.1), we get $K_{X,s}|_{P^{\mathrm{red}}} \cong \det(\mathbb{L}_X)|_{P^{\mathrm{red}}}$ for each (P, U, f, i), critical chart on (X, s). Comparing two critical charts, one can show that the canonical isomorphisms constructed above from two such charts are equal on the overlap. Therefore the isomorphisms glue to give a global canonical isomorphism $K_{X,s} \cong \det(\mathbb{L}_X)|_{X^{\mathrm{red}}}$. This completes the proof of Theorem 8.3.1. □

Note that we did not use $LMLM$ charts and Proposition 8.2.3 in §8.2.2. That is because we are constructing a section s of a sheaf, (effectively, a morphism in a category), rather than a (perverse) sheaf (an object in a category), so basically we only have to go up to double overlaps, not triple overlaps.

8.4 Relation with other works and further research

In this section we briefly discuss related work in the literature, and outline some ideas for future investigation.

The work of Behrend and Fantechi [8] The main inspiration for the present work was a result by Behrend and Fantechi [8] in 2006. Their project aims to construct and study Gerstenhaber and Batalin–Vilkovisky structures on Lagrangian intersections. They consider a pair L, M, of complex Lagrangian submanifolds in a complex symplectic manifold (S, ω), and they show that one can equip the graded algebra $\mathcal{T}or_{-i}^{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ with a Gerstenhaber bracket, and the graded sheaf $\mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ with a Batalin–Vilkovisky type differential. The approach is the same as our approach, and in fact we were inspired by that: it is based on the holomorphic version of the Darboux theorem, that is, any holomorphic symplectic manifold is locally isomorphic to a cotangent bundle, thus reducing the case of a general Lagrangian intersection to the special case where one of the two Lagrangian is identified with the zero section of the cotangent bundle of the symplectic manifold, and the second one is the graph of a holomorphic function locally defined on the first Lagrangian.

In particular, Behrend and Fantechi [8, Th.s 4.3 & 5.2] claim to construct canonical $\mathbb{C}$-linear differentials

$$\mathrm{d} : \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \longrightarrow \mathcal{E}xt^{i+1}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$$

with $\mathrm{d}^2 = 0$, such that $\big(\mathcal{E}xt^*_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M), \mathrm{d}\big)$ is a constructible complex, called the *virtual de Rham complex* of the Lagrangian intersection X. Conjecturally, $(\mathcal{E}^\bullet, \mathrm{d})$ categorifies Lagrangian intersection numbers, in the sense that the constructible function

$$p \to \sum_i (-1)^{i-\dim(S)} \dim_{\mathbb{C}} \mathbb{H}^i_{\{p\}}(X, (\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M), \mathrm{d})),$$

of fiberwise Euler characteristic of $(\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M),\mathrm{d})$ is equal to the well known Behrend function ν_X in [5], and so

$$\chi(X,\nu_X)=\sum_i(-1)^{i-\dim(S)}\dim_{\mathbb{C}}\mathbb{H}^i(X,(\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M),\mathrm{d})).$$

Their main theorem [8, Th. 4.3] claims that the locally defined de Rham differentials coming from the picture given by the holomorphic Darboux theorem, do not depend on the choices of coordinates involved in the choice of a polarization of S, so that they can claim the existence of a global canonical differential. Unfortunately, there is a mistake in the proof. To fix this one should instead work with $\mathcal{E}xt^*_{\mathcal{O}_S}(K_L^{1/2},K_M^{1/2})$ for square roots $K_L^{1/2},K_M^{1/2}$ as in §8.2. Also the relation between their virtual de Rham complex and vanishing cycles relies on a conjecture of Kapranov [88, Rmk. 2.12(b)], which later turned out to be true just over the ring of Laurent series - see Sabbah [148, Th. 1.1] (deformation–quantization setting, see discussion below).

The work of Kashiwara and Schapira [91] Kashiwara and Schapira [92] develop a theory of *deformation quantization modules*, or *DQ-modules*, on a complex symplectic manifold (S,ω), which roughly may be regarded as symplectic versions of $\mathscr{D}$-modules. *Holonomic* DQ-modules $\mathcal{D}^\bullet$ are supported on (possibly singular) complex Lagrangians L in S. If L is a smooth, closed, complex Lagrangian in S and $K_L^{1/2}$ a square root of K_L, D'Agnolo and Schapira [29] show that there exists a simple holonomic DQ-module $\mathcal{D}^\bullet$ supported on L.

If $\mathcal{D}^\bullet,\mathcal{E}^\bullet$ are simple holonomic DQ-modules on S supported on smooth Lagrangians L,M, then Kashiwara and Schapira [91] show that $R\mathscr{H}om(\mathcal{D}^\bullet,\mathcal{E}^\bullet)[n]$ is a perverse sheaf on S over the field $\mathbb{C}((\hbar))$, supported on $X=L\cap M$. Pierre Schapira explained to the authors of [18] how to prove that $R\mathscr{H}om(\mathcal{D}^\bullet,\mathcal{E}^\bullet)[n]\cong P^\bullet_{L,M}$, when $P^\bullet_{L,M}$ is defined over the base ring $A=\mathbb{C}((\hbar))$.

The work of Baranovsky and Ginzburg [4] Apart from the mistake in the proof, Behrend and Fantechi's work [8] gives a new important understanding of a rich structure on Lagrangian intersection, investigated also by Baranovsky and Ginzburg [4], who obtained analogous results for any pair of smooth coisotropic submanifolds L,M of arbitrary smooth Poisson algebraic varieties S considering first order deformations of the structure sheaf $\mathcal{O}_S$ to a sheaf of non-commutative algebras and of the structure sheaves $\mathcal{O}_L$ and $\mathcal{O}_M$ to sheaves of right and left modules over the deformed algebra. The construction is canonically defined and it is independent of the choices of deformations involved. The proof of their main result, Theorem 4.3.1 in [4], shows that sometimes the Gerstenhaber and Batalin–Vilkovisky structures on Tor or Ext are well-defined globally. In their construction, this is the case, for instance, whenever in the setting of the proof of [4, Thm 4.3.1], some cocycles are defined globally.

The work of Kapustin and Rozansky [89] In [89], Kapustin and Rozansky study boundary conditions and defects in a three-dimensional topological sigma-model with a complex symplectic target space, the Rozansky-Witten model. It turns out that this model has a deep relation with the problem of deformation quantization of the derived category of coherent sheaves on a complex manifold, regarded as a symmetric monoidal category, and in particular with categorified algebraic geometry in the sense of [14, 176]. Namely, in the case when the target space of the Rozansky-Witten model has the form of the cotangent bundle T^*Y, where Y is a complex manifold, the 2-category of boundary conditions is very similar to the 2-category of derived categorical sheaves on Y. More precisely, given a complex symplectic manifold (S,ω), Kapustin and Rozansky conjecture the existence of an interesting 2-category, with objects complex Lagrangians L with $K_L^{1/2}$, such that $\mathrm{Hom}(L,M)$ is a $\mathbb{Z}_2$-periodic triangulated category, and if $L\cap M$ is locally modelled on

$\mathrm{Crit}(f : U \to \mathbb{C})$ for $f : U \to \mathbb{C}$ is a holomorphic function on a manifold U, then $\mathrm{Hom}(L, M)$ is locally modelled on the matrix factorization category $MF(U, f)$ as in [139].

Matrix factorization and second categorification It would be interesting to construct a sheaf of $\mathbb{Z}_2$-periodic triangulated categories on Lagrangian intersection, which, in the language of categorification, would yield a second categorification of the intersection numbers, the first being given by the hypercohomology of the perverse sheaf constructed in the present work.

Also, this construction should be compatible with the Gerstenhaber and Batalin–Vilkovisky structures in the sense of [4, Conj. 1.3.1].

Fukaya category for derived Lagrangian and d-critical loci It would be interesting to extend Theorem 8.3.1 to a class of 'derived Lagrangians' in (S, ω).

Given a pair L, M, of derived complex Lagrangian submanifolds in the sense of [142] in a complex symplectic manifold (S, ω), with $\dim_{\mathbb{C}} S = 2n$, Joyce conjectures that there should be some kind of approximate comparison $\mathbb{H}^k(P^\bullet_{L,M}) \approx HF^{k+n}(L, M)$, where $HF^*(L, M)$ is the *Lagrangian Floer cohomology* of Fukaya, Oh, Ohta and Ono [45]. Some of the authors of [18] are working on defining a 'Fukaya category' of (derived) complex Lagrangians in a complex symplectic manifold, using $\mathbb{H}^*(P^\bullet_{L,M})$ as morphisms. See [18] for a more detailed discussion.

Bibliography

[1] P. Aluffi, *Weighted Chern-Mather classes and Milnor classes of hypersurfaces*, in Singularities–Sapporo 1998, volume 29 of Adv. Stud. Pure Math., pages 1–20. Kinokuniya, Tokyo, 2000.

[2] P. Aluffi, *Limits of Chow groups, and a new construction of Chern–Schwartz–MacPherson classes,* Pure Appl. Math. Q. 2 (2006), no. 4, part 2, 915–941.

[3] M. Artin, *Algebraization of formal moduli: I*, in Global Analysis, papers in honor of K. Kodaira, Princeton U. Press, 1969, pp. 21–71.

[4] V. Baranovsky and V. Ginzburg, *Gerstenhaber–Batalin–Vilkoviski structures on coisotropic intersections*, arXiv:0907.0037.

[5] K. Behrend, *Donaldson–Thomas type invariants via microlocal geometry*, Ann. of Math. 170 (2009), 1307–1338. math.AG/0507523.

[6] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. 128 (1997), 45–88. alg-geom/9601010.

[7] K. Behrend and B. Fantechi, *Symmetric obstruction theories and Hilbert schemes of points on threefolds*, Algebra and Number Theory 2 (2008), 313–345, math.AG/0512556.

[8] K. Behrend and B. Fantechi, *Gerstenhaber and Batalin–Vilkovisky structures on Lagrangian intersections*, pages 1–47 in *Algebra, arithmetic, and geometry*, Progr. Math. 269, Birkhäuser, Boston, MA, 2009.

[9] K. Behrend, J. Bryan and B. Szendrői, *Motivic degree zero Donaldson–Thomas invariants*, Invent. Math. 192 (2013), 111–160. arXiv:0909.5088.

[10] K. Behrend and E. Getzler, *On holomorphic Chern-Simons functional*, preprint.

[11] K. Behrend and Z. Hua, *Derived moduli space of complexes of coherent sheaves*, preprint.

[12] A.A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers,* Astérisque 100, 1982.

[13] O. Ben-Bassat, C. Brav, V. Bussi, and D. Joyce, *A 'Darboux Theorem' for shifted symplectic structures on derived Artin stacks, with applications*, arXiv:1312.0090, 2013.

[14] D. Ben-Zvi, J. Francis and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry,* arXiv:0805.0157.

[15] J.-E. Björk, *Analytic $\mathscr{D}$-modules and applications*, Kluwer, Dordrecht, 1993.

[16] A. Borel et al., *Algebraic $\mathscr{D}$-modules*, Perspectives in Mathematics 2, Academic Press, 1987.

[17] E. Bouaziz and I. Grojnowski, *A d-shifted Darboux theorem*, arXiv:1309.2197, 2013.

[18] C. Brav, V. Bussi, D. Dupont, D. Joyce, and B. Szendrői, *Symmetries and stabilization for sheaves of vanishing cycles*, arXiv:1211.3259, 2012.

[19] C. Brav, V. Bussi and D. Joyce, *A Darboux theorem for derived schemes with shifted symplectic structure*, arXiv:1305.6302, 2013.

[20] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. 166 (2007), 317–345. math.AG/0212237.

[21] T. Bridgeland, *Hall algebras and curve-counting invariants*, arXiv:1002.4374, 2010.

[22] T. Bridgeland, *An introduction to motivic Hall algebras*, arXiv:1002.4372, 2010.

[23] V. Bussi, *Categorification of Lagrangian intersections on complex symplectic manifolds using perverse sheaves of vanishing cycles*, arXiv:1404.1329.

[24] V. Bussi, *Generalized Donaldson–Thomas theory over fields* $\mathbb{K} \neq \mathbb{C}$, arXiv:1403.2403.

[25] V. Bussi, D. Joyce and S. Meinhardt, *On motivic vanishing cycles of critical loci*, arXiv:1305.6428.

[26] V. Bussi and S. Yokura, *Naive motivic Donaldson–Thomas type Hirzebruch classes and some problems*, arXiv:1306.4725.

[27] A.S. Cattaneo and F. Schätz, *Introduction to supergeometry*, Rev. Math. Phys. 23 (2011), 669–690. arXiv:1011.3401.

[28] H-L. Chang and J. Li, *Semi-Perfect Obstruction theory and DT Invariants of Derived Objects*, arXiv:1105.3261, 2011.

[29] A. D'Agnolo and P. Schapira, *Quantization of complex Lagrangian submanifolds*, Advances in Math. 213 (2007), 358–379. math.AG/0506064.

[30] J. Denef and F. Loeser, *Motivic Igusa zeta functions*, J. Algebraic Geom. 7 (1998), 505–537. math.AG/9803040.

[31] J. Denef and F. Loeser, *Motivic exponential integrals and a motivic Thom–Sebastiani Theorem*, Duke Math. J. 99 (1999), 285–309. math.AG/9803048.

[32] J. Denef and F. Loeser, *Geometry on arc spaces of algebraic varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 327–348, Progr. Math. 201, Birkhäuser, Basel, 2001. math.AG/0006050.

[33] J. Denef and F. Loeser, *Lefschetz numbers of iterates of the monodromy and truncated arcs*, Topology 41 (2002), 1031–1040. math.AG/0001105.

[34] A. Dimca, *Sheaves in Topology*, Universitext, Springer-Verlag, Berlin, 2004.

[35] A. Dimca and B. Szendrői, *The Milnor fibre of the Pfaffian and the Hilbert scheme of four points on* $\mathbb{C}^3$, Math. Res. Lett. 16 (2009) 1037–1055. arXiv:0904.2419.

[36] S.K. Donaldson and R.P. Thomas, *Gauge Theory in Higher Dimensions*, Chapter 3 in S.A. Huggett, L.J. Mason, K.P. Tod, S.T. Tsou and N.M.J. Woodhouse, editors, *The Geometric Universe*, Oxford University Press, Oxford, 1998.

[37] A. Dubson, *Classes caractéristiques des variétés singuliéres,* C. R. Acad. Sci. Paris Sér. A–B 287 (1978), no. 4, A237–A240.

[38] A. Dubson, *Formule pour l'indice des complexes constructibles et D-modules holonomes*, C. R. Acad. Sci. Paris 298 (1984), 113–116.

[39] A. Dubson, *Calcul des invariants numériques des singularités et applications*, preprint, Bonn 1981.

[40] D. Edidin, W. Graham, *Equivariant intersection theory,* Invent. Math. 131 (1998), no. 3, 595–634.

[41] D. Eisenbud, *Commutative algebra*, Graduate Texts in Math. 150, Springer-Verlag, New York, 1995.

[42] T. Ekedahl, *On the adic formalism*, pages 197–218 in *The Grothendieck Festschrift, vol. II*, Progr. Math. 87, Birkhäuser, Boston, 1990.

[43] E. Freitag and R. Kiehl, *Etale cohomology and the Weil Conjecture*, Ergeb. der Math. und ihrer Grenzgebiete 13, Springer-Verlag, 1988.

[44] R. Friedman and J.W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergeb. der Math. und ihrer Grenzgebiete 27, Springer-Verlag, 1994.

[45] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory — anomaly and obstruction*, Parts I & II. AMS/IP Studies in Advanced Mathematics, 46.1 & 46.2, A.M.S./International Press, 2009.

[46] W. Fulton, *Intersection Theory*, Ergeb. Math. und ihrer Grenzgebiete 2, Springer-Verlag, Berlin, 1984.

[47] W. Fulton and R. MacPherson, *Intersecting cycles on an algebraic variety*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976) ed. P. Holm, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 179–197.

[48] W. Fulton and R. MacPherson, *Defining algebraic intersections*, in Algebraic Geometry, Proceedings, Tromso, Norway, 1977, Springer Lecture Notes 687 (1978), 1–30.

[49] W. Fulton and R. MacPherson, *Categorical Framework for the Study of Singular Spaces* , Memoirs of the American Mathematical Society 243, 1981.

[50] T. Gaffney and H. Hauser, *Characterizing singularities of varieties and of mappings*, Invent. math. 81 (1985), 427–447.

[51] A. Gorodentscev, S. Kuleshov and A. Rudakov, *Stability data and t-structures on a triangulated category*, math.AG/0312442, 2003.

[52] S.I. Gelfand and Y.I. Manin, *Methods of Homological Algebra*, second edition, Springer-Verlag, Berlin, 2003.

[53] W.D. Gillam, *Deformation of quotients on a product*, available at www.math.brown.edu

[54] V. Ginsburg, *Characteristic varieties and vanishing cycles*, Invent. Math., 84; p.327–402, 1986.

[55] T.L. Gómez, *Algebraic stacks*, Proc. Indian Acad. Sci. Math. Sci. 111 (2001), 1–31, math.AG/9911199.

[56] G. Gonzalez–Sprinberg, *L'obstruction locale d'Euler et le théoréme de MacPherson*, Astérisque 82–83 (1981), 7–32.

[57] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math., 135(2): 487–518, 1999.

[58] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[59] A. Grothendieck, *Elements de Géométrie Algébrique*, part I Publ. Math. IHES 4 (1960), part II Publ. Math. IHES 8 (1961), part III Publ. Math. IHES 11 (1960) and 17 (1963), and part IV Publ. Math. IHES 20 (1964), 24 (1965), 28 (1966) and 32 (1967).

[60] A. Grothendieck, *Revêtements Étales et Groupe Fondamental (SGA1)*, Springer Lecture Notes 224, Springer-Verlag, Berlin, 1971.

[61] A. Grothendieck, *Techniques de construction et théorémes d'existence en géométrie algébrique IV: Les schémas de Hilbert*, Séminaire Bourbaki, 1960/61, no. 221.

[62] A. Grothendieck, *Technique de descente et théorémes d'existence en géométrie algébrique V: Les schémas de Picard. Théorémes d'existence*, Séminaire Bourbaki, 1961/62, no. 232.

[63] A. Grothendieck, *Technique de descente et théorémes d'existence en géométrie algébrique VI: Les schémas de Picard. Propriétés générales*, Séminaire Bourbaki, 1961/62, no. 236.

[64] R. Hartshorne, *On the de Rham cohomology of algebraic varieties*, Publ. Math. I.H.E.S. 45 (1975), 5–99.

[65] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. 52, Springer, New York, 1977.

[66] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero I, II*, Ann. Math. 79 (1964), 109–203 and 205–326.

[67] R. Hotta, T. Tanisaki and K. Takeuchi, *$\mathscr{D}$-modules, perverse sheaves, and representation theory*, Progr. Math. 236, Birkhäuser, Boston, MA, 2008.

[68] Z. Hua, *Chern–Simons functions on toric Calabi–Yau threefolds and Donaldson–Thomas theory*, arXiv:1103.1921, 2011

[69] D. Huybrechts, *Fourier–Mukai transforms in Algebraic Geometry*, Oxford University Press, Oxford, 2006.

[70] D. Huybrechts and R.P. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes*, Math. Ann. 346 (2010), 545–569. arXiv:0805.3527.

[71] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Math. E31, Vieweg, Braunschweig/Wiesbaden, 1997.

[72] L. Illusie, *Complexe cotangent et déformations. I*, Springer Lecture Notes in Math. 239, Springer-Verlag, Berlin, 1971.

[73] L. Illusie, *Cotangent complex and deformations of torsors and group schemes*, pages 159–189 in Springer Lecture Notes in Math. 274, Springer-Verlag, Berlin, 1972.

[74] M. Inaba, *Toward a definition of moduli of complexes of coherent sheaves on a projective scheme*, J. Math. Kyoto Univ. (JMKYAZ) 42-2 (2002), 317–329.

[75] D. Joyce, *Constructible functions on Artin stacks*, J. L.M.S. 74 (2006), 583–606, math.AG/0403305.

[76] D. Joyce, *Motivic invariants of Artin stacks and 'stack functions'*, Quart. J. Math. 58 (2007), 345–392, math.AG/0509722.

[77] D. Joyce, *Configurations in abelian categories. I. Basic properties and moduli stacks*, Adv. Math. 203 (2006), 194–255, math.AG/0312190.

[78] D. Joyce, *Configurations in abelian categories. II. Ringel–Hall algebras*, Adv. Math. 210 (2007), 635–706, math.AG/0503029.

[79] D. Joyce, *Configurations in abelian categories. III. Stability conditions and identities*, Adv. Math. 215 (2007), 153–219, math.AG/0410267.

[80] D. Joyce, *Configurations in abelian categories. IV. Invariants and changing stability conditions*, Adv. Math. 217 (2008), 125–204, math.AG/0410268.

[81] D. Joyce, *Holomorphic generating functions for invariants counting coherent sheaves on Calabi–Yau 3-folds*, Geometry and Topology 11 (2007), 667–725, hep-th/0607039.

[82] D. Joyce, *Generalized Donaldson–Thomas invariants*, arXiv:0910.0105, 2009.

[83] D. Joyce *Donaldson–Thomas theory: introduction and open problems*, October 2011, available at http://people.maths.ox.ac.uk/joyce/MICAtalk.pdf

[84] D. Joyce, *A classical model for derived critical loci*, arXiv:1304.4508, 2013.

[85] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, Mem. Amer. Math. Soc. 217 (2012), no. 1020. arXiv:0810.5645.

[86] D. Joyce, *Motivic invariants of Artin stacks and 'stack functions'*, Quart. J. Math. 58 (2007), 345–392. math.AG/0509722.

[87] D. Joyce, *A classical model for derived critical loci*, arXiv:1304.4508, 2013.

[88] M. Kapranov, *DG-modules over de Rham algebra*, Algebraic Geometry (Chicago, 1989), Lecture Notes in Math., vol. 1479, Springer, Berlin, 1991, p. 57–86.

[89] A. Kapustin and L. Rozansky, *Three-dimensional topological field theory and symplectic algebraic geometry II* (2009). Preprint arXiv:0909.3643.

[90] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Math. Wiss. 292, Springer-Verlag, Berlin, 1990.

[91] M. Kashiwara and P. Schapira, *Constructibility and duality for simple modules on symplectic manifolds*, Amer. J. Math. 130 (2008), 207–237. math.QA/0512047.

[92] M. Kashiwara and P. Schapira, *Deformation quantization modules*, Astérisque 345, 2012.

[93] N.M. Katz and G. Laumon, *Transformation de Fourier et majoration des sommes exponentielles,* Publ. Math. I.H.E.S. 62 (1985), 145–202.

[94] G. Kemper, *Separating invariants*, J. Symbolic Comput. 44 (2009), no. 9, 1212–1222, https://www.ricam.oeaw.ac.at/mega2007 /electronic/37.pdf

[95] G. Kennedy, *MacPherson's Chern classes of singular algebraic varieties*, Comm. Algebra 18 (1990), 2821–2839.

[96] R. Kiehl and R. Weissauer, *Weil Conjectures, perverse sheaves and l'adic Fourier transform*, Springer-Verlag, 2001.

[97] Y. H. Kiem and J. Li. *Localizing virtual cycles by cosections*, Preprint, arXiv:1007.3085.

[98] Y. H. Kiem and J. Li, *A wall crossing formula of Donaldson–Thomas invariants without Chern-Simons functional*, arXiv:0905.4770.

[99] Y.-H. Kiem and J. Li, *Categorification of Donaldson–Thomas invariants via perverse sheaves*, arXiv:1212.6444, 2012.

[100] S. L. Kleiman, *The Picard scheme*, In Fundamental algebraic geometry, volume 123 of Math. Surveys Monogr., pages 235–321. Amer. Math. Soc., Providence, RI, 2005, http://arxiv.org/pdf/math/0504020v1.

[101] S. Kobayashi, *Differential geometry of complex vector bundles*, Publ. Math. Soc. Japan 15, Princeton University Press, Princeton, NJ, 1987.

[102] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, Mirror symmetry and tropical geometry, 55–89, Contemp. Math., 527, Amer. Math. Soc., Providence, RI, 2010, arXiv:0811.2435, 2008.

[103] M. Kontsevich and Y. Soibelman, *Motivic Donaldson–Thomas invariants: summary of results*, arXiv:0910.4315, 2009.

[104] M. Kontsevich and Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants*, Commun. Number Theory Phys. 5 (2011), 231–352. arXiv:1006.2706.

[105] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. 138 (1999), 495–536. math.AG/9810166.

[106] Y. Laszlo and M. Olsson, *The six operations for sheaves on Artin stacks. I* Publ. Math. I.H.E.S. 107 (2008), 109–168. math.AG/0512097.

[107] Y. Laszlo and M. Olsson, *The six operations for sheaves on Artin stacks. II*, Publ. Math. I.H.E.S. 107 (2008), 169–210. math.AG/0603680.

[108] Y. Laszlo and M. Olsson, *Perverse t-structure on Artin stacks*, Math. Z. 261 (2009), 737–748. math.AG/0606175.

[109] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergeb. der Math. und ihrer Grenzgebiete 39, Springer-Verlag, Berlin, 2000.

[110] D.T. Le and B. Teissier, *Variétés polaires et classes de Chern des variétés singuliéres*, Ann. of Math. 1981.

[111] M. Levine and R. Pandharipande, *Algebraic cobordism revisited*, math.AG/0605196.

[112] T. Le Quy, *Proofs of the integral identity conjecture over algebraically closed fields*, arXiv:1206.5334.

[113] J. Li, *Zero dimensional Donaldson–Thomas invariants of threefolds*, Geometry and Topology 10 (2006), 2117–2171, math.AG/0604490.

[114] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, Journal of the American Mathematical Society 11 (1998), 119–174. math.AG/9602007.

[115] W.P. Li, Z. Qin, *Donaldson–Thomas invariants of certain Calabi–Yau 3-folds*, arXiv1002.4080, 2010.

[116] M. Lieblich, *Moduli of complexes on a proper morphism*, J. of Algebraic Geometry 15 (2006) 175-206.

[117] P. Lowrey, *The moduli stack and motivic Hall algebra for the bounded derived category*, http://arxiv.org/pdf/1110.5117.pdf.

[118] M. Lübke and A. Teleman, *The Kobayashi–Hitchin correspondence*, World Scientific, Singapore, 1995.

[119] D. Luna, *Slices étales*, Bull. Soc. Math. France, Mémoire 33 (1973), 81–105.

[120] E. Looijenga, *Motivic measures*, Séminaire Bourbaki, Vol. 1999/2000. Astérisque 276 (2002), 267–297. math.AG/0006220.

[121] J. Lurie, *Derived Algebraic Geometry*, PhD thesis, M.I.T., 2004. Available at www.math.harvard.edu/∼lurie/papers/DAG.pdf.

[122] J. Lurie, *Higher algebra*, available at http://www.math.harvard.edu/lurie, 2011.

[123] S. Mac Lane, *Categories for the working mathematician*, Graduate texts in mathematics 5, Springer, 1998.

[124] R.D. MacPherson, *Chern classes for singular algebraic varieties*, Ann. Math. 100 (1974), 423–432.

[125] D. Massey, *The Sebastiani–Thom isomorphism in the derived category*, Compositio Math. 125 (2001), 353–362. math.AG/9908101.

[126] D.B. Massey, *Notes on perverse sheaves and vanishing cycles*, arXiv:math/9908107, 1999.

[127] D. Massey, *Natural commuting of vanishing cycles and the Verdier dual*, arXiv:0908.2799, 2009.

[128] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[129] K. Miyajima, *Kuranishi family of vector bundles and algebraic description of Einstein–Hermitian connections*, Publ. RIMS, Kyoto Univ. 25 (1989), 301–320.

[130] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson–Thomas theory. I*, Compos. Math. 142 (2006), 1263–1285. math.AG/0312059.

[131] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson–Thomas theory. II*, Compos. Math. 142 (2006), 1286–1304. math.AG/0406092.

[132] D. Maulik, R. Pandharipande, R.P. Thomas, *Curves on K3 surfaces and modular forms*, J.Topology p.937–996, (2010), math.AG/1001.2719.

[133] D. Maulik, *Motivic residues and Donaldson–Thomas theory*, in preparation, 2013.

[134] V. Navarro Aznar, *Sur les multiplicités de Schubert locales des faisceaux algébriques cohérents*, Compositio Mathematica, tome 48, no. 3 (1983), p. 311–326.

[135] J. Nicaise and J. Sebag, *Motivic Serre invariants, ramification, and the analytic Milnor fibre*, Invent. Math. 168 (2007), 133–173.

[136] N. Nitsure, *Construction of Hilbert and Quot schemes*, Fundamental algebraic geometry, 105–137, Math. Surveys Monogr., 123, Amer. Math. Soc., Providence, RI, 2005.

[137] N. Nitsure, *Deformation Theory for Vector Bundles*, Moduli spaces and vector bundles LMS lecture note series 359 (2009) in honour of Peter Newstead.

[138] T. Ohmoto, *Equivariant Chern classes of singular algebraic varieties with group actions,* Math. Proc. Cambridge Philos. Soc. 140 (2006), no. 1, 115–134.

[139] D.O. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Tr. Mat. Inst. Steklova 246 (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 240262

[140] R. Pandharipande and R.P. Thomas, *Almost closed* 1*-forms*, arXiv:1204.3958, 2012.

[141] R. Pandharipande and R.P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. 178 (2009), 407–447. arXiv:0707.2348.

[142] T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. I.H.E.S. 117 (2013), 271–328. arXiv:1111.3209.

[143] A. Parusiński and P. Pragacz, *Characteristic classes of hypersurfaces and characteristic cycles*, J. Alg. Geom. 10 (2001), 63–79. math.AG/9801102.

[144] C.P. Ramanujam, *On a geometric interpretation of multiplicity*, Inventiones Mathematicae, Volume 22, Number 1, 63–67.

[145] S. Raskin, *The cotangent stack*, seminar note in www.math.harvard.edu

[146] K. Rietsch, *An introduction to perverse sheaves*, pages 391–429 in V. Dlab and C.M. Ringel, editors, *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, Fields Inst. Commun. 40, A.M.S., Providence, RI, 2004. math.RT/0307349.

[147] A. Rudakov, *Stability for an Abelian Category*, Journal of Algebra 197 (1997), 231–245.

[148] C. Sabbah, *Quelques remarques sur la géometrie des espaces conormaux*, Asterisque 130 (1985), 161–192.

[149] C. Sabbah, *Espaces conormaux bivariants*, Thése, Université Paris VII, 1986.

[150] C. Sabbah, *On a twisted de Rham complex, II*, arXiv:1012.3818, 2010.

[151] P. Safronov, *Quasi-Hamiltonian reduction via classical Chern-Simons theory* , arXiv:1311.6429, 2013.

[152] M. Saito, *Modules de Hodge polarisables*, Publ. RIMS 24 (1988), 849–995.

[153] M. Saito, *Duality for vanishing cycle functors*, Publ. RIMS 25 (1989), 889–921.

[154] M. Saito, *Mixed Hodge Modules*, Publ. RIMS 26 (1990), 221–333.

[155] M. Saito, *$\mathscr{D}$-modules on analytic spaces*, Publ. RIMS 27 (1991), 291–332.

[156] M. Saito, *Thom–Sebastiani Theorem for Hodge Modules*, preprint, 2010.

[157] J. Schürmann, *Topology of singular spaces and constructible sheaves*, Monografie Matematyczne 63, Birkhäuser, Basel, 2003.

[158] P. Schapira, *Cycles Lagrangiens, fonctions constructibles et applications*, Seminaire EDP, Publ. Ecole Polytechnique (1988/89)

[159] J. Schürmann, *Lectures on characteristic classes of constructible functions. Notes by Piotr Pragacz and Andrzej Weber*, Trends Math., Topics in cohomological studies of algebraic varieties, 175–201, Birkhäuser, Basel, 2005.

[160] J. Schürmann, *A general intersection formula for Lagrangian cycles*, Compositio Math. 140 (1004) 1037–1052

[161] J. Schürmann and S. Yokura, *A survey of characteristic classes of singular spaces*, Singularity theory, 865–952, World Sci. Publ., Hackensack, NJ, 2007. arXiv:math/0511175.

[162] P. Seidel and R.P. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. 108 (2001), 37–108. math.AG/0001043.

[163] J.P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier 6 (1956), 1–42.

[164] P. Sosna, *Linearisations of triangulated categories with respect to finite group actions,* arXiv:1108.2144v1.

[165] J. Stoppa and R.P. Thomas, *Hilbert schemes and stable pairs: GIT and derived category wall crossings*, arXiv:0903.1444, 2009.

[166] B. Szendroi, *Nekrasov's partition function and refined Donaldson–Thomas theory: the rank one case*, SIGMA (2012) 088, 16pp. arXiv:1210.5181.

[167] R.P. Thomas, *A holomorphic Casson invariant for Calabi–Yau* 3*-folds, and bundles on* $K3$ *fibrations*, J. Diff. Geom. 54 (2000), 367–438, math.AG/9806111.

[168] Y. Toda, *Curve counting theories via stable objects I. DT/PT correspondence*, arXiv:0902.4371, 2009.

[169] Y. Toda, *Curve counting theories via stable objects II: DT/ncDT flop formula*, arXiv:0909.5129, 2011.

[170] B. Toën, *Higher and derived stacks: a global overview*, pages 435–487 in *Algebraic Geometry — Seattle 2005*, Proc. Symp. Pure Math. 80, Part 1, A.M.S., Providence, RI, 2009. math.AG/0604504.

[171] B. Toën, *Moduli of objects in dg-categories*, arXiv:math/0503269,2007.

[172] B. Toën, *Derived Hall Algebras*, arXiv:math/0501343, 2005.

[173] B. Toën, *Higher and derived stacks: a global overview*, pages 435–487 in *Algebraic Geometry — Seattle 2005*, Proc. Symp. Pure Math. 80, Part 1, A.M.S., Providence, RI, 2009. math.AG/0604504.

[174] B. Toën and G. Vezzosi, *Homotopical Algebraic Geometry II: Geometric Stacks and Applications*, Mem. Amer. Math. Soc. 193 (2008), no. 902. math.AG/0404373.

[175] B. Toën and G. Vezzosi, *From HAG to DAG: derived moduli stacks*, pages 173–216 in *Axiomatic, enriched and motivic homotopy theory*, NATO Sci. Ser. II Math. Phys. Chem., 131, Kluwer, Dordrecht, 2004. math.AG/0210407.

[176] B. Toën and C. Vezzosi, *A note on Chern character, loop spaces and derived algebraic geometry,* arXiv:0804.1274.

[177] B. Totaro, *The Chow ring of a classifying space,* Algebraic K-theory (Seattle, WA, 1997), 249–281, Proc. Sympos. Pure Math., 67, Amer. Math. Soc., Providence, RI, 1999.

[178] J.-L. Verdier, *Spécialisation des classes de Chern*, Exp. 7 in *Caractéristique d'Euler–Poincaré*, Seminaire E.N.S. 1978-9, Astérisque 82–83 (1981).

[179] G. Vezzosi, *Derived critical loci I – Basics*, arXiv:1109.5213, 2011.

[180] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math., 97, 613–670, 1989.

[181] V. Voevodsky, *Triangulated categories of motives over a field*, pages 188–238 in *Cycles, Transfers and Motivic Homology Theories*, Ann. of Math. Studies 143, 2000.

[182] C. Voisin, *On integral Hodge classes on uniruled or Calabi–Yau threefolds*, pages 43–73 in S. Mukai, Y. Miyaoka, S. Mori, A. Moriwaki and I. Nakamura, editors, *Moduli Spaces and Arithmetic Geometry (Kyoto 2004)*, Advanced Studies in Pure Math. 45, Math. Soc. Japan, Tokyo, 2006. math.AG/0412279.

[183] S. Yokura, *Bivariant Theories of Constructible Functions and Grothendieck Transformations*, Topology and Its Applications, Vol. 123, (2002), pp. 283–296.

[184] Z. Zhang, *Moduli spaces of sheaves on K3 surfaces and symplectic stacks,* arXiv:1111.6294v1.

[185] J. Zhou, *Classes de Chern en théorie bivariante*, in Thése, Université Aix-Marseille II (1995).