Calabi–Yau and Special Lagrangian 3-folds with conical singularities and their desingularizations

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Abstract

In this thesis we study the desingularizations of Calabi–Yau and special Lagrangian (SL) 3-folds with conical singularities. Let $(M_0, J_0, \omega_0, \Omega_0)$ be a Calabi–Yau 3-fold with conical singularities x_i for $i = 1, \ldots, n$ modelled on Calabi–Yau cones $(V_i, J_{V_i}, \omega_{V_i}, \Omega_{V_i})$. Suppose $(Y_i, J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i})$ is an Asymptotically Conical (AC) Calabi–Yau 3-fold modelled on the Calabi–Yau cone $(V_i, J_{V_i}, \omega_{V_i}, \Omega_{V_i})$ for $i = 1, \ldots, n$.

For the Calabi–Yau desingularization, we first rescale each Y_i by a small $t > 0$ and then glue into M_0 at x_i. This gives a family of *nearly Calabi–Yau* 3-folds M_t, and when t is sufficiently small, we show by applying Joyce's existence result for torsion-free G_2-structures to $S^1 \times M_t$ that the nearly Calabi–Yau structures can be deformed to genuine Calabi–Yau structures, and hence obtaining a desingularization of M_0. We first treat the case $\lambda_i < -3$ and proceed to the obstructed case $\lambda_i = -3$ where λ_i denotes the *rate* at which the AC Calabi–Yau 3-fold Y_i converges to the cone V_i. The principal analytic tool we use in the obstructed case is the theory of weighted Sobolev spaces from Lockhart and McOwen.

Our result on the case $\lambda_i < -3$ can be applied to desingularizing Calabi–Yau 3-orbifolds with isolated singularities which enable us to describe what the Calabi–Yau metrics locally look like on crepant resolutions of orbifolds. When $\lambda_i = -3$ our result gives a desingularization of Calabi–Yau 3-folds with ordinary double points, which is an analytic version of Friedman's result giving necessary and sufficient conditions for smoothing ordinary double points. Our approach in both cases uses the metrics on the Calabi–Yau 3-folds and is analytic, rather than the complex structure and being complex algebraic.

For the special Lagrangian desingularizations, suppose N_0 is an SL 3-fold in M_0 with conical singularities at the same points x_i modelled on SL cones C_i in V_i, and suppose L_i is an AC SL 3-fold in Y_i modelled on an SL cone C_i. We then simultaneously desingularize M_0 and N_0 by gluing in rescaled Y_i and L_i at each x_i. The construction is achieved by applying Joyce's analytic result on deforming Lagrangian submanifolds to nearby special Lagrangian submanifolds. As an application, we take two examples for M_0, namely the orbifold $T^6/\mathbb{Z}_3$ and a quintic 3-fold. We construct some singular SL 3-folds N_0 in M_0 and AC SL 3-folds L_i in the corresponding Y_i, and glue them together to obtain examples of nonsingular SL 3-folds in the desingularized Calabi–Yau 3-folds.

Contents

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Chapter 1

Introduction

This thesis is devoted to the study of a class of singular Calabi–Yau and special Lagrangian 3-folds. A Calabi–Yau manifold is a Kähler manifold (M, J, ω) with a covariant constant holomorphic volume form Ω satisfying $\omega^m/m! = (-1)^{m(m-1)/2}\,(i/2)^m\,\Omega\wedge\bar{\Omega}$, where m is the complex dimension of M. The canonical line bundle K_M of a Calabi–Yau manifold M is always trivial, so that the first Chern class $c_1(M)$ of M must vanish, and the vanishing of $c_1(M)$ is necessary for having a Ricci-flat metric. To prove that it is also sufficient is difficult, and this problem was first considered by Calabi in a more general context. He raised the famous conjecture on whether any representative of $c_1(M)$ can be the Ricci-form of some Kähler metric. Calabi showed that if such a Kähler metric exists, then it must be unique. Yau then provided the proof that such a metric always exists if M is compact.

From the holonomy point of view, a Calabi–Yau manifold is precisely a Riemannian manifold (M, g) with holonomy group $\mathrm{Hol}(g)$ contained in $\mathrm{SU}(m)$. For Calabi–Yau m-folds with holonomy group $\mathrm{SU}(m)$, the Hodge numbers $h^{p,q}$ satisfy: $h^{0,0} = h^{m,0} = 1$ and $h^{p,0} = 0$ for $0 < p < m$. If we collect results on the Hodge numbers of Calabi–Yau manifolds for the case $m = 3$, we find that the only independent Hodge numbers are $h^{1,1}$ and $h^{2,1}$, and the Euler numbers of Calabi–Yau 3-folds are then $\chi = 2(h^{1,1} - h^{2,1})$.

In the most popular version of string theory, a branch of theoretical physics, the space we live in looks locally like a product of 4-dimensional Minkowski space and a compact Calabi–Yau 3-fold M. The Calabi–Yau condition on M is necessary because of supersymmetry. *Mirror symmetry*, which had been known to both physicists and mathematicians for some time, is a phenomenon in string theory that there are mirror pairs of Calabi–Yau 3-folds M and $\check{M}$ which are physically equivalent. Much efforts have been put on explaining mirror symmetry in terms of mathematical contents. One feature of a mirror pair is the interchange of Hodge numbers, i.e. $h^{p,q}(M) = h^{3-p,q}(\check{M})$. This in particular means that the two non-trivial Hodge numbers $h^{1,1}$ and $h^{2,1}$ are interchanged between M and $\check{M}$, i.e.

$$h^{1,1}(M) \;=\; h^{2,1}(\check{M}) \quad \text{and} \quad h^{2,1}(M) \;=\; h^{1,1}(\check{M}),$$

and that $\chi(M) = -\chi(\check{M})$. In fact, mirror symmetry is a much stronger statement than the mere existence of mirror pairs of Calabi–Yau manifolds and the exchange of Hodge numbers.

It involves an idea that, roughly speaking, there is a natural isomorphism which identifies the complex moduli of $\check{M}$ and the complexified Kähler moduli of M, and vice versa. If such a mirror map existed, then information on holomorphic invariants of one Calabi–Yau 3-fold M could yield information on symplectic invariants of the mirror $\check{M}$, and vice versa. One of the very first examples constructed by physicists is the study of the mirror family to *quintic 3-folds*, see [12]. Take a one-parameter family of quintics

$$M_\psi = \left\{ [z_0, z_1, z_2, z_3, z_4] \in \mathbb{CP}^4 : \sum_j z_j^5 - 5\psi \prod_j z_j \ = \ 0 \right\}$$

which is a family of Calabi–Yau 3-folds with $h^{1,1}(M_\psi) = 1$ and $h^{2,1}(M_\psi) = 101$. Consider an action of $\{(a_0, \ldots, a_4) \in (\mathbb{Z}_5)^5 : \prod_j a_j = 1\} \cong (\mathbb{Z}_5)^4$ given by rescaling of z_j by fifth roots of unity. Dividing by the diagonal $\mathbb{Z}_5$ projective stabilizer we get a $(\mathbb{Z}_5)^3$-action on M_ψ. Then the mirror of the quintic is given by crepant resolutions $\check{M}_\psi$ of the quotient $M_\psi/(\mathbb{Z}_5)^3$ with $h^{1,1}(\check{M}_\psi) = 101$ and $h^{2,1}(\check{M}_\psi) = 1$.

Singularities and their desingularizations are important for understanding the mirror quintic. There are many different types of singularities and ways of desingularizing them. One kind of frequently appearing singular Calabi–Yau manifolds is known as *conifolds* (see [11]), which are smooth Calabi–Yau manifolds apart from a number of isolated conical singularities. Conifolds correspond to the points where the moduli spaces of Calabi–Yau manifolds meet. The neighbourhood of a singular point of a conifold X can be described by a complex quadric in $\mathbb{C}^4$,

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0,$$

which is known in the mathematical literature as an *ordinary double point* or *node*. The quadric is in fact topologically a cone over $S^2 \times S^3$. There are two different ways of repairing the singularities in a conifold. The first is by *deformation*, where the quadric is deformed to

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \epsilon$$

for some nonzero $\epsilon \in \mathbb{C}$. The deformed conifold $\tilde{X}$ is a smooth manifold, each singular point having been replaced by an S^3. As this smoothing of the singularity results from changing the polynomial, it corresponds to the desingularization arising from deforming the complex structure. Another way to remove the singularities on a conifold is by making a *small resolution* of X. This yields a smooth manifold $\hat{X}$ in which each singular point is replaced by an $S^2 \cong \mathbb{CP}^1$. The process of varying a complex structure from a smooth Calabi–Yau manifold $\tilde{X}$ so that a conifold singularity appears, and then resolving that conifold so that a new $S^2 \cong \mathbb{CP}^1$ appears is called a *conifold transition*. Thus conifold singularities provide a transition between topologically distinct Calabi–Yau's $\tilde{X}$ and $\hat{X}$.

These issues provide the primary motivation for the first half of this thesis: the study of Calabi–Yau 3-folds with conical singularities and their desingularizations, including the smoothings of Calabi–Yau 3-folds with ordinary double points. A novel feature of our approach is that it uses the metrics on singular Calabi–Yau 3-folds and is analytic, rather than just the complex structure, and being complex algebraic. Thus we provide new analytic proofs of results previously known in complex algebraic setting.

Special Lagrangian (SL) submanifolds in Calabi–Yau manifolds play an important role in the explanation of Mirror symmetry. They are examples of *calibrated submanifolds*, appearing in Harvey and Lawson [22], which generalizes the concept of volume-minimizing property of complex submanifolds of Kähler manifolds. Let (M, J, ω, Ω) be a Calabi–Yau manifold of complex dimension m. Then $\mathrm{Re}(\Omega)$ is a calibrated form whose calibrated submanifolds are real m-dimensional special Lagrangian submanifolds (SL m-folds).

There has been extensive research in the mathematics literature on special Lagrangian and other calibrated submanifolds. Moreover, a lot of SL m-folds in $\mathbb{C}^m$ have been constructed explicitly using various techniques. For example, Joyce [29] constructed SL m-folds by using large symmetry groups. In particular, a large family of cohomogeneity one SL m-folds are constructed using moment map techniques. Moreover, he constructed SL cones in $\mathbb{C}^m$, providing local models for conical singularities in SL m-folds in general Calabi–Yau m-folds. Haskins [23] focused on dimension three and explored examples of SL cones in $\mathbb{C}^3$. In [42] McLean studied the deformation theory for calibrated submanifolds. In particular, he showed that local deformations of a smooth compact SL m-fold L in a Calabi–Yau m-fold (M, J, ω, Ω) are always unobstructed and the moduli space $\mathcal{M}_L$ is a smooth manifold of dimension $b_1(L)$, the first Betti number of L.

Special Lagrangian submanifolds attracted much interest in connection with the *SYZ conjecture* proposed by Strominger, Yau and Zaslow [49] in 1996, which explains Mirror symmetry between Calabi–Yau 3-folds. The precise formulation of the conjecture has not yet been worked out. Roughly speaking, it states that given a mirror pair $(M, \check{M})$ of Calabi–Yau 3-folds there should be SL T^3-fibrations $f, \check{f}$ degenerating over a common discriminant locus $\Delta \subset B$ and such that for each $b \in B \setminus \Delta$, the fibres $F_b = f^{-1}(b)$ and $\check{F}_b = \check{f}^{-1}(b)$ are nonsingular SL 3-tori T^3 in M and $\check{M}$ which are in some sense dual to one another. Much progress has been made on it, in particular, Joyce [30] tried a local geometric approach to the conjecture, and suggested that the final form of the SYZ conjecture should be an asymptotic statement about 1-parameter families of Calabi–Yau 3-folds approaching the *large complex structure limit*.

A part of the conjecture asserts that the mirror $\check{M}$ of a Calabi–Yau 3-fold M can be obtained by some suitable compactification of the dual of the SL T^3-fibration on M. Therefore to find a compactification and understand the relations with the Mirror symmetry one should understand the singularities of the moduli space of SL m-folds.

Perhaps the simplest singularities to understand are isolated singularities modelled on SL cones. Joyce has developed a comprehensive programme on the desingularization of SL m-folds with conical singularities in (almost) Calabi–Yau manifolds and their deformation theory in his recent series of papers [31]-[35]. The SL m-folds with conical singularities are desingularized by gluing in at the singular points some nonsingular SL m-folds in $\mathbb{C}^m$ which are asymptotic to SL cones at infinity. Furthermore, Joyce [28] proposed to define an invariant of Calabi–Yau 3-folds, analogous to Gromov-Witten invariants in symplectic geometry, by a weighted count of SL homology 3-spheres in a given homology class. Understanding the singularities and the compactifications of the moduli space of SL m-folds will also be important for this programme.

The second half of the thesis will then be devoted to the study of a simple kind of singular SL

3-folds living in singular Calabi–Yau 3-folds with singularities at the same points. We shall simultaneously desingularize the singular Calabi–Yau and SL 3-folds. This can be compared with the work by Joyce [33] where singular SL 3-folds are desingularized in nonsingular Calabi–Yau 3-folds.

A guide to the chapters

The objects of our study are Calabi–Yau and special Lagrangian 3-folds with a kind of singular points known as conical singularities. Our aim is to develop an analytic desingularization theory for both Calabi–Yau and SL 3-folds with conical singularities. With this goal in mind, we are concerned mostly with the following two constructions in this thesis:

(1) Desingularizing Calabi–Yau 3-folds M_0 with conical singularities x_i for $i = 1, \ldots, n$ modelled on some Calabi–Yau cones V_i by gluing in Asymptotically Conical (AC) Calabi–Yau 3-folds Y_i;

(2) Desingularizing SL 3-folds N_0 (in the Calabi–Yau 3-fold M_0) with conical singularities at the same points x_i modelled on SL cones C_i (in the Calabi–Yau cones V_i) by gluing in AC SL 3-folds L_i (in the AC Calabi–Yau 3-folds Y_i).

This thesis is organized as follows. In Chapter 2 we describe some standard material from Calabi–Yau manifolds, special Lagrangian geometry and give an introduction to analysis on compact manifolds. We define Calabi–Yau manifolds and look at some of their properties in §2.1.3. Section 2.1.4 gives some examples of Calabi–Yau manifolds, with emphasis on the crepant resolutions of quotient singularities in Example 2.6, and the cotangent bundles of spheres in Example 2.7. Section 2.2 introduces special Lagrangian geometry, in which we begin with some basic concepts in Symplectic geometry. The study of SL m-folds in $\mathbb{C}^m$ and their constructions will be given in §2.2.3 and 2.2.4. We then study SL m-folds in Calabi–Yau m-folds in §2.2.5, including McLean's deformation result and a discussion on fixed point sets of antiholomorphic involutions. Finally in §2.3, we give some necessary notations and concepts in analysis for our later chapters, involving the Sobolev Embedding Theorem and an elliptic regularity result.

In Chapter 3 we study Calabi–Yau desingularizations mentioned in (1). A Calabi–Yau 3-fold $(M_0, J_0, \omega_0, \Omega_0)$ with conical singularities is a Calabi–Yau 3-fold with a finite number of distinct singular points x_i for $i = 1, \ldots, n$, such that near each singular point x_i, M_0 looks like some Calabi–Yau cone V_i, and all the structures J_0, ω_0, Ω_0 on M_0 converge to the cone structures $J_{V_i}, \omega_{V_i}, \Omega_{V_i}$ with some *rate* ν and with all their derivatives. We shall assume the existence of such kind of singular Calabi–Yau 3-folds throughout the thesis, in other words, we assume that there are singular Calabi–Yau metrics on some complex manifolds with conical singularities. Asymptotically Conical (AC) Calabi–Yau 3-folds Y_i are nonsingular Calabi–Yau 3-folds which have a similar definition to that of Calabi–Yau 3-folds with conical singularities, so that all the structures $J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i}$ approach the cone structures $J_{V_i}, \omega_{V_i}, \Omega_{V_i}$ at infinity with some rate λ_i and with all their derivatives. We then apply a homothety to each Y_i, in other words, rescale Y_i by a small $t > 0$, and glue them into M_0 at each x_i. This gives a family of *nearly Calabi–Yau* 3-folds M_t. The point is then to prove that for sufficiently small $t > 0$, the nearly Calabi–Yau structures on M_t can be perturbed or deformed to genuine Calabi–Yau structures. We show this step by inducing G_2-structures on the real 7-dimensional manifolds $S^1 \times M_t$ from the nearly Calabi–Yau structures on M_t. The induced G_2-structures have *small torsion* for sufficiently small $t > 0$, and we then apply Joyce's existence result [26, Thm.11.6.1] for torsion-free G_2-structures to show that the induced G_2-structures can actually be deformed to have zero torsion. We can now pull back the torsion-free G_2-structures to the 6-folds M_t to obtain genuine Calabi–Yau structures.

Section 3.1 provides some background material on SU(3)- and G_2-structures on real 6- and 7-folds respectively. These concepts are useful throughout the chapter. In §3.2 we define nearly Calabi–Yau structures on some real 6-folds. We start by introducing nearly Calabi–Yau structures in §3.2.1. Then we induce G_2-structures from them on 7-folds $S^1 \times M$ in §3.2.2. We go on in §3.2.3 to prove our main result on the existence of Calabi–Yau structures on some 6-folds. As the proof requires results from Joyce's proof on the existence of torsion-free G_2-structures, we include them there and modify some of the conditions to fit into our situation. The analytic result we obtain in §3.2.3 will be important to the constructions of Calabi–Yau desingularization later. Section 3.3 studies Calabi–Yau cones V_i, Calabi–Yau 3-folds M_0 with conical singularities and AC Calabi–Yau 3-folds Y_i. We give definitions and examples, and prove a Darboux type theorem for both M_0 and Y_i. In §3.4, we prove our desingularization result for an easier case $\lambda_i < -3$. We first construct a family of nearly Calabi–Yau 3-folds M_t by gluing Y_i into M_0, and then we show that the nearly Calabi–Yau structures have "small enough torsion" when t is sufficiently small so that our analytic result applies, thus obtaining genuine Calabi–Yau structures on M_t. Finally in §3.4.4 we give an application of our result and provide an example of desingularizing the Calabi–Yau 3-orbifold $T^6/\mathbb{Z}_3$. The appropriate AC Calabi–Yau 3-folds are given by the canonical line bundle $K_{\mathbb{CP}^2}$ over $\mathbb{CP}^2$. The resulting desingularization we obtained is in fact the crepant resolution, the blow-up the singular points, in which each singular point is replaced by a copy of $\mathbb{CP}^2$. An application of our result gives a description of what the Calabi–Yau metric, whose existence is proved by Yau [52], looks like on the crepant resolution of the orbifold $T^6/\mathbb{Z}_3$.

Chapter 4 is an extension or a generalization of Chapter 3. During the gluing process, we may encounter a kind of cohomological obstruction to defining a 3-form Ω_t on M_t which interpolates between the 3-form Ω_0 on M_0 and the scaled 3-form $t^3\Omega_{Y_i}$ on Y_i if the AC Calabi–Yau 3-fold Y_i has rate $\lambda_i = -3$. We then extend our desingularization theorem to the more complicated situation when $\lambda_i = -3$. Allowing $\lambda_i = -3$ also causes analytical difficulties in the way that the 3-form Ω_t will contribute an error of size which is too large for some of the hypotheses of our desingularization theorem to hold.

In order to extend our result to the case $\lambda_i = -3$, we replace the 3-form Ω_0 on M_0 by $\Omega_0 + \chi$, where χ is a closed (2,1)-form on M_0 such that $\omega_0 \wedge \chi = 0$ and χ is asymptotic to some given closed (2,1)-form ξ_i with order $O(r^{-3})$ on V_i. The advantage of adding such a (2,1)-form χ to Ω_0 is that it corresponds to a change of Calabi–Yau structure up to first order, that is, the "error" from being a Calabi–Yau 3-form is of order $O(|\chi|^2)$, and this will have the effect of squaring the size of the original error term, which will then be small enough to apply our desingularization result.

The machinery we use to construct such a (2,1)-form χ is the analytic theory of weighted

Sobolev spaces on manifolds with ends developed by Lockhart and McOwen [39].

We begin in §4.1 by establishing the necessary notations. Section 4.2 studies the analytic theory for weighted Sobolev spaces due to Lockhart and McOwen [39]. In §4.3, we construct our desired (2,1)-form χ on M_0. We then glue Y_i's into M_0, constructing the nearly Calabi–Yau structures in §4.4. Section 4.5 gives the main result on Calabi–Yau desingularization when $\lambda_i = -3$ which generalizes the result in Chapter 3. Finally in §4.6 we focus on a kind of singular Calabi–Yau 3-fold where the singularities are known as *ordinary double points*. We shall assume the existence of such kind of singular Calabi–Yau manfolds for the methods developed to apply. The desingularization of this kind of Calabi–Yau 3-folds belongs to the case $\lambda_i = -3$, and we apply our main result to repair ordinary double points. We conclude by showing that our result is in some way equivalent to Friedman's result [16], giving necessary and sufficient conditions for smoothing ordinary double points.

In Chapters 5 and 6, we focus on the SL desingularizations mentioned in (2). SL 3-folds inside the above Calabi–Yau 3-folds M_0, V_i and Y_i are studied. An SL 3-fold N_0 with conical singularities at x_i is basically a singular SL 3-fold in M_0 that approaches an SL cone C_i near each x_i. The way we define N_0 is to express it near x_i as a graph of some *exact* 1-form da_i on C_i with decay rate $\mu > 0$. To be more precise, N_0 is locally the image of the graph of da_i under the embedding Ψ_{C_i} mapping from a Lagrangian neighbourhood of C_i to the Calabi–Yau cone V_i. Next we define AC SL 3-folds L_i, which are nonsingular SL 3-folds in the AC Calabi–Yau 3-folds Y_i asymptotic to SL cones C_i at infinity. Similar to the definition of N_0, we define L_i at infinity to be the image of the graph of some exact 1-form db_i on C_i with rate κ_i under a Lagrangian neighbourhood embedding. During the Calabi–Yau desingularization, the AC Calabi–Yau 3-folds Y_i's are glued into M_0 at x_i, thus we can pick AC SL 3-folds L_i in Y_i, so that they can be glued into N_0 at x_i. This yields a 1-parameter family of compact nonsingular 3-folds N_t in the nearly Calabi–Yau 3-folds M_t. We construct N_t to be Lagrangian.

After deforming the nearly Calabi–Yau structures to genuine Calabi–Yau structures on M_t, the nonsingular 3-fold N_t is still Lagrangian after applying a diffeomorphism of M_t close to the identity. Our next step is to deform N_t to SL 3-folds $\hat{N}_t$ for small enough t. In a recent paper [33], Joyce proved an analytic existence result for SL m-folds, showing that under certain conditions a compact nonsingular Lagrangian m-fold in a Calabi–Yau m-fold which is close to being special Lagrangian can be deformed to a nearby SL m-fold. Our second objective, the special Lagrangian desingularizations, can then be achieved by adapting Joyce's result.

In order to illustrate our desingularization theorem, we include some examples of SL 3-folds in the Calabi–Yau 3-folds we discussed before. We construct N_0 in the Calabi–Yau 3-orbifold $T^6/\mathbb{Z}_3$ and L_i in the corresponding AC Calabi–Yau 3-fold $K_{\mathbb{CP}^2}$. In particular, we show that an SL T^3 in $T^6/\mathbb{Z}_3$ with one conical singularity is desingularized by gluing in the real canonical line bundle $K_{\mathbb{RP}^2}$ in $K_{\mathbb{CP}^2}$, yielding a $T^3 \# \mathbb{R}^3$. We also construct N_0 in some quintic Calabi–Yau 3-fold which contains ordinary double points, and L_i in the corresponding AC Calabi–Yau 3-fold T^*S^3. An example is given by taking N_0 as the fixed point set of some antiholomorphic isometric involution, and L_i as some SO(3)-invariant $\mathbb{R}^3$'s or $S^2 \times \mathbb{R}$'s or S^3's in T^*S^3.

Section 5.1 discusses SL cones and the Lagrangian Neighbourhood Theorem on them. We give definitions of SL m-folds with conical singularities in §5.2 and AC SL m-folds in §5.3. Finally in §5.4, examples of AC SL m-folds are constructed in two kinds of AC Calabi–Yau m-folds. Section §5.4.1 gives SL m-folds as fixed points of antiholomorphic isometric involution σ, T^{m-1}-invariant SL m-folds, and SO(m)-invariant SL m-folds in $K_{\mathbb{CP}^2}$. In §5.4.2, we provide fixed point sets examples in T^*S^m, together with T^2-invariant and SO(3)-invariant examples in T^*S^3.

In Chapter 6, we desingularize SL 3-folds with conical singularities, and produce some new SL 3-folds in crepant resolutions of $T^6/\mathbb{Z}_3$ and some quintics. Section 6.1 states Joyce's SL desingularization result. We construct a family of Lagrangian 3-folds N_t in §6.2. In order to apply Joyce's result, we need to have estimates of various norms of some 3-forms restricted on N_t. We use analysis to obtain all the estimates we need in §6.3. We prove the main theorem on desingularizing SL 3-folds with conical singularities in §6.4. In the last section, §6.5, we construct examples by gluing different AC SL 3-folds we obtained in Chapter 5 into some SL 3-folds with conical singularities in the Calabi–Yau 3-orbifold $T^6/\mathbb{Z}_3$ and in some quintic 3-folds.

Chapter 2

Background material

2.1 Calabi–Yau manifolds

In this thesis we shall approach the subject of Calabi–Yau manifolds very much from the point of view of differential geometry, rather than algebraic geometry. Here we would like to introduce them via Kähler geometry and holonomy groups. We begin with a very brief description of the necessary concepts from holonomy groups and a discussion on the Calabi conjecture. Then we go on to the definition and general properties of Calabi–Yau manifolds. Finally, we will explore some examples of both compact and non-compact Calabi–Yau manifolds.

2.1.1 Brief review on Holonomy groups

We introduce the notion of holonomy groups in this section. For further details and discussions about holonomy groups, we refer the reader to [3, Chapter 10] and [26, Chapters 2 and 3].

Let (M, g) be a Riemannian manifold of dimension n with Levi-Civita connection ∇. Given $p \in M$, define the *holonomy group* $\mathrm{Hol}_p(g)$ of g at p to be the group of linear automorphisms of T_pM obtained from parallel transports defined by ∇ around loops based at p. If M is connected, the holonomy group is independent of basepoint because if we join any two points $p, q \in M$ by a piecewise smooth curve γ in M, then we have a group isomorphism $\mathrm{Hol}_p(g) \cong \mathrm{Hol}_q(g)$ given by $x \mapsto P_\gamma \circ x \circ P_\gamma^{-1}$ where P_γ denotes the parallel transport map along γ. Hence we shall often drop the subscripts p and write the holonomy group as $\mathrm{Hol}(g)$.

For all Riemannian metrics g, $\mathrm{Hol}(g)$ is a Lie subgroup of $\mathrm{O}(n)$. One important result about holonomy groups is that the covariant constant (parallel) tensors on the manifold are invariant under the holonomy group $\mathrm{Hol}(g)$, and conversely a tensor at a point which is invariant under $\mathrm{Hol}(g)$ can be extended to a unique covariant constant tensor on the manifold. This is based on the fact that the covariant constant tensors are invariant under parallel transport and so they are entirely determined by $\mathrm{Hol}(g)$. In 1955, Berger [2] gave a list of all possible subgroups of $\mathrm{O}(n)$ that can be the holonomy groups of a Riemannian manifold (M, g) under certain assumptions on M and g :

Theorem 2.1 *If (M, g) is a simply-connected Riemannian manifold of dimension n, and g is irreducible and non-symmetric, then exactly one of the following seven cases holds.*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,

(ii) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$ *in* $\mathrm{SO}(2m)$,

(iii) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *in* $\mathrm{SO}(2m)$,

(iv) $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *in* $\mathrm{SO}(4m)$,

(v) $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\,\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4m)$,

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $\mathrm{SO}(7)$, *or*

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *in* $\mathrm{SO}(8)$.

Our focus will be on type (iii) as, by the next definition, they are holonomy groups for Calabi–Yau metrics. Note that Kähler metrics g on a complex manifold have holonomy groups $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$, and this property can also be used to define Kähler metrics. Metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)$ are called *hyperkähler* metrics.

2.1.2 The Calabi conjecture

Let (M, J, ω) be a compact Kähler manifold with Kähler metric g and Ricci form η. We know that η is a real, closed (1,1)-form representing $c_1(M)$ in $H^2(M, \mathbb{R})$. Calabi raised the famous conjecture about prescribing the Ricci curvature on (M, J, ω), namely, given a real, closed (1,1)-form φ representing $c_1(M)$, can we find a Kähler metric h on M whose Ricci form is φ? This is known as the Calabi conjecture and it was solved by Yau [52] in 1976. We state it as the following :

Theorem 2.2 *Suppose (M, J, ω) is a compact Kähler manifold with Kähler metric g. If φ is a closed real (1,1)-form representing $c_1(M)$, then there exists a unique Kähler metric h, with Kähler form ω', on M such that $[\omega] = [\omega'] \in H^2(M, \mathbb{R})$, i.e. g and h are in the same Kähler class, and the Ricci form of h is φ.*

One of its applications is that when $c_1(M) = 0$, it implies there exists a unique Ricci-flat Kähler metric in each Kähler class of M. *Calabi–Yau manifolds*, which we shall discuss in the next section, are Ricci-flat, thus an important consequence of the Calabi conjecture is the existence of large families of Calabi–Yau manifolds. Note that Yau's proof of the Calabi conjecture is based on existence theorems for solutions of non-linear partial differential equations, but it does not provide a way to write down the Ricci-flat metric explicitly. In the non-compact case the situation is better in this respect. Examples are the Calabi metrics [10] and the Eguchi-Hanson metrics [15] on the crepant resolutions of quotient singularities, and the Stenzel metrics [48] on cotangent bundles of rank one symmetric spaces. They will play an important role in this thesis, providing examples of a kind of Calabi–Yau manifolds known as *Asymptotically Conical (AC) Calabi–Yau manifolds.*

2.1.3 Basic definitions and properties of Calabi–Yau manifolds

For some further reading on Calabi–Yau manifolds, we refer the reader to [26, Chapter 6] and [25, Chapter 6].

Definition 2.3 A *Calabi–Yau m-fold* is a Kähler manifold (M, J, ω) of complex dimension m with a covariant constant holomorphic volume form Ω satisfying

$$\omega^m/m! = (-1)^{m(m-1)/2}\,(i/2)^m\,\Omega \wedge \bar{\Omega}. \tag{2.1}$$

Then we say that (J, ω, Ω) constitutes a *Calabi–Yau structure* on M and we shall denote a Calabi–Yau manifold as a quadruple (M, J, ω, Ω).

We should note here that there are several inequivalent definitions of Calabi–Yau manifolds in use in the literature. For example, one may define Calabi–Yau manifolds to be compact Kähler manifolds with vanishing first Chern class. We will explain more on $c_1(M) = 0$ later this section.

Clearly the holomorphic volume form Ω is unique up to a change of phase $\Omega \mapsto e^{i\theta}\Omega$ from the normalization formula (2.1), and the constant factor there is chosen so that the real part $\mathrm{Re}(\Omega)$ of Ω is a *calibration* on M. (We shall discuss calibrations in §2.2.2)

Another equivalent way of defining a Calabi–Yau m-fold is to require that the Riemannian $2m$-fold (M, g) has holonomy group $\mathrm{Hol}(g)$ contained in $\mathrm{SU}(m)$. This can be seen using the result about holonomy groups mentioned in §2.1.1. Suppose there is a covariant constant holomorphic volume form Ω on M, then for each $p \in M$, the holonomy group $\mathrm{Hol}(g)$ must preserve Ω_p on $\mathbb{C}^m$, where we identify the tangent space T_pM with $\mathbb{C}^m$. But the subgroup of $\mathrm{O}(2m)$ preserving Ω_p is $\mathrm{SU}(m)$, so $\mathrm{Hol}(g)$ must be contained in $\mathrm{SU}(m)$. Conversely, if we define a holomorphic $(m,0)$-form θ on $\mathbb{C}^m$ by $\theta = dz_1 \wedge \cdots \wedge dz_m$, then, since it is preserved by $\mathrm{SU}(m)$ and thus by $\mathrm{Hol}(g)$, θ can be extended to a covariant constant tensor Ω on M and Ω can be written in the form $dz_1 \wedge \cdots \wedge dz_m$ at each p of M.

In complex dimension 2, the holonomy groups of Calabi–Yau manifolds are contained in $\mathrm{SU}(2)$, which is isomorphic to $\mathrm{Sp}(1)$. Thus all Calabi–Yau 2-folds are *hyperkähler* and have the whole 2-sphere S^2 of integrable complex structures. By the classification theory of compact complex surfaces, it turns out that all compact Calabi–Yau 2-folds are either $K3$ *surfaces* or 4-tori T^4. Hence they are well understood, and we shall normally focus on Calabi–Yau m-folds for

$m \geq 3$.

We see that a Calabi–Yau m-fold (M, J, ω, Ω) admits a covariant constant holomorphic $(m, 0)$-form Ω which is analogous to the holomorphic volume form $dz_1 \wedge \cdots \wedge dz_m$ on $\mathbb{C}^m$. It is a section of the *canonical bundle* $K_M = \bigwedge^{m,0} T^*M$ of M. Thus the canonical bundle K_M admits a nowhere vanishing holomorphic section Ω and such a section exists if and only if K_M is trivial. Then the connection on K_M induced from the Kähler metric g must be flat. By the fact that the curvature of this connection is the Ricci-form of g, we conclude that the Ricci curvature of g vanishes. Consequently, every Calabi–Yau m-fold is Ricci-flat.

Recall from §2.1.2 that an important application of the Calabi conjecture is the construction of compact Calabi–Yau m-folds. Here is how it works. Suppose (M, J) is a compact complex manifold admitting Kähler metrics with $c_1(M) = 0$, then by Yau's proof of the Calabi conjecture, every Kähler class on M contains a unique Ricci-flat metric g. Thus (M, J, ω) is a compact Ricci-flat Kähler manifold, where ω is the Kähler form of g. Assume further that K_M is trivial, i.e. there exists a nowhere vanishing holomorphic section Ω on M. Then by applying the "Bochner argument" on compact Ricci-flat Kähler manifolds, one can show that any $(p, 0)$-form on M is closed if and only if it is covariant constant. It is easy to see that Ω is closed and thus is covariant constant. Therefore, up to a change of phase, we can make (M, J, ω) into a Calabi–Yau m-fold (M, J, ω, Ω).

Now suppose (M, J, ω, Ω) is a compact Calabi–Yau m-fold with $\mathrm{Hol}(g) = \mathrm{SU}(m)$. It follows by using general facts about compact Ricci-flat manifolds that (M, J, ω, Ω) has finite fundamental group $\pi_1(M)$. Also, by using the "Bochner argument" mentioned before, one can prove that on (M, J, ω, Ω), the $(p, 0)$-th *Dolbeault cohomology group* $H^{p,0}_{\bar\partial}(M)$ is isomorphic to the vector space of covariant constant $(p, 0)$-forms on M. It can be shown that this vector space is equal to $\mathbb{C}$ when $p = 0$ or m, and 0 otherwise. Therefore on a Calabi–Yau m-fold (M, J, ω, Ω) with $\mathrm{Hol}(g) = \mathrm{SU}(m)$, its *Hodge numbers* $h^{p,q}$ (the dimensions of the (p, q)-th Dolbeault cohomology groups) satisfy: $h^{0,0} = h^{m,0} = 1$ and $h^{p,0} = 0$ for $p \neq 0, m$. For the case $m = 3$, $h^{0,0} = h^{3,0} = 1$ and $h^{1,0} = h^{2,0} = 0$, together with the symmetry properties of a Hodge diamond, i.e. $h^{p,q} = h^{q,p}$ and $h^{p,q} = h^{m-p,m-q}$ (the *Serre duality*, $m = 3$ in this case), we end up with the following :

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

2.1.4 Examples of Calabi–Yau m-folds

We collect some examples of Calabi–Yau manifolds in this section. Examples 2.4 and 2.5 are taken from [26, §6.7], Example 2.6 from [10, p.284-5] and [26, Example 8.2.5], and Example 2.7

from [48, §7].

Examples 2.4 In this example we consider complex hypersurfaces in $\mathbb{CP}^m$ which are expressed as the zero set of a homogeneous polynomial. Let X be a nonsingular hypersurface of degree d in $\mathbb{CP}^m$, i.e.

$$X = \{[z_0, \dots, z_m] \in \mathbb{CP}^m : f(z_0, \dots, z_m) = 0\}$$

where $f(z_0, \dots, z_m)$ is a nonzero homogeneous polynomial of degree d. One can show by the *adjunction formula* ([20, p.147]) that the canonical bundle K_X is trivial, and therefore $c_1(X) = 0$, if and only if $d = m+1$. Since X is compact and Kähler, thus if the degree of X is $m+1$, then by Yau's proof of the Calabi conjecture, X admits a Ricci-flat metric. Therefore, the holomorphic section Ω of X is covariant constant, and X can be made into a Calabi–Yau manifold. Consequently, any nonsingular hypersurface of degree $m+1$ in $\mathbb{CP}^m$ is a Calabi–Yau $(m-1)$-fold. Thus we have found a simple way of constructing Calabi–Yau m-folds. For $m = 2$, we have cubic curves in $\mathbb{CP}^2$, which are 2-tori T^2, and for $m = 3$, we have quartic surfaces in $\mathbb{CP}^3$, which are $K3$ surfaces. As we are mainly interested in Calabi–Yau 3-folds in this thesis, we shall discuss more on quintic hypersurfaces in $\mathbb{CP}^4$ later.

Examples 2.5 Suppose $X \subseteq \mathbb{CP}^m$ is a *complete intersection* of hypersurfaces, i.e. $X = H_1 \cap \cdots \cap H_k$ where $H_1, \dots, H_k$ are hypersurfaces in $\mathbb{CP}^m$ which intersect transversely along X, so that $\dim X = m - k$. In a similar way to the previous example, one can show that X is a Calabi–Yau $(m-k)$-fold if and only if $d_1 + \cdots + d_k = m + 1$ where $d_1, \dots, d_k$ are the degrees of $H_1, \dots, H_k$. Now if $d_j = 1$ for some j, then X can be regarded as the intersection of $k-1$ hypersurfaces in $\mathbb{CP}^{m-1}$ omitting H_j, so we can assume $d_j \geq 2$ for all j. This construction yields a finite number of topologically distinct Calabi–Yau m-folds for each m.

Examples 2.6 (Crepant resolutions of quotient singularities)

We now want to consider the *crepant resolution* X of some quotient singularities $\mathbb{C}^m/G$, where G is a finite subgroup of $\mathrm{SU}(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$ (some references are given by Joyce [26, §6.4-§6.6] and Roan [45]). A *quotient singularity* $\mathbb{C}^m/G$ is a singular complex manifold obtained by taking the quotient of $\mathbb{C}^m$ by the G-action, where G is a nontrivial finite subgroup of $\mathrm{GL}(m, \mathbb{C})$. For each $p \in \mathbb{C}^m$, $G \cdot p$ represents a G-orbit of p in $\mathbb{C}^m$ and it is a singular point in $\mathbb{C}^m/G$ if the stabilizer $\mathrm{Stab}(p) = \{\gamma \in G : \gamma \cdot p = p\}$ of p is nontrivial. Hence 0 is always a singular point of $\mathbb{C}^m/G$ and it will be the unique singular point if G acts freely on $\mathbb{C}^m \setminus \{0\}$. Also, geometric structures can be "pushed down" from $\mathbb{C}^m$ to the nonsingular part of $\mathbb{C}^m/G$ if and only if they are G-invariant.

Now suppose G is a finite subgroup of $\mathrm{SU}(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$. Let X be a nonsingular complex manifold. Then adopting an analytic definition, rather than an algebraic one, (X, π) is called a *resolution* of $\mathbb{C}^m/G$ if $\pi : X \to \mathbb{C}^m/G$ is a proper holomorphic map which is surjective and is a biholomorphism between $X \setminus \pi^{-1}(0)$ and $\mathbb{C}^m/G \setminus \{0\}$. Often $\pi^{-1}(0)$ will be a compact submanifold of X, or a finite union of submanifolds. Thus we can repair the singularities through the process of resolution by replacing each singular point by a submanifold. One technique to construct a resolution is to apply a *blow-up.* A resolution (X,π) of $\mathbb{C}^m/G$ with $c_1(X)=0$ is called a *crepant resolution.* Crepant resolutions of $\mathbb{C}^m/G$ for $m=2,3$ exist for every finite subgroup of $\mathrm{SU}(m)$ (see for instance Roan [45, Thm. 1] for $m=3$) and are well understood. In particular, they are unique for $m=2$.

On a crepant resolution (X,π) of $\mathbb{C}^m/G$ where G is some finite subgroup of $\mathrm{SU}(m)$ acting freely on $\mathbb{C}^m\setminus\{0\}$, one can define an *Asymptotically Locally Euclidean (ALE) Kähler metric* on it, making it an ALE Kähler manifold (for an introduction to ALE metrics, see [26, §8.1-§8.2]). Recall that one of the applications of the Calabi conjecture is the existence of a unique Ricci-flat Kähler metric in each Kähler class on a compact Kähler manifold whose first Chern class vanishes. Joyce [27, Thm. 3.3] proved a result analogous to this for ALE Kähler manifolds (X,π), asserting that in each Kähler class of ALE Kähler metrics on X there exists a unique Ricci-flat ALE Kähler metric g satisfying certain asymptotic conditions. Moreover, he showed ([27, Thm. 3.4]) that such a Ricci-flat ALE Kähler metric g has holonomy $\mathrm{SU}(m)$.

An explicit example of a Ricci-flat ALE Kähler manifold asymptotic to $\mathbb{C}^m/\mathbb{Z}_m$ for $m\geq 3$ is given by Calabi [10, p. 284-5]. Let $\zeta=e^{2\pi i/m}$ be the m-th root of unity, and define an action generated by ζ on $\mathbb{C}^m$ by

$$\zeta^k\cdot(z_1,\dots,z_m)\;=\;(\zeta^k z_1,\dots,\zeta^k z_m)$$

for $0\leq k\leq m-1$. The group $\mathbb{Z}_m=\{1,\zeta,\dots,\zeta^{m-1}\}$ is a subgroup of $\mathrm{SU}(m)$, as $\zeta^m=1$, and it acts freely on $\mathbb{C}^m\setminus\{0\}$. Then the quotient $\mathbb{C}^m/\mathbb{Z}_m$ has a resolution of singularities (X,π) given by a *blow-up* of $\mathbb{C}^m/\mathbb{Z}_m$ at 0. In fact, X is the total space of the canonical bundle over $\mathbb{CP}^{m-1}$ with $\pi^{-1}(0)\cong\mathbb{CP}^{m-1}$ and it is a crepant resolution of $\mathbb{C}^m/\mathbb{Z}_m$. Now define $f:\mathbb{C}^m/\mathbb{Z}_m\setminus\{0\}\longrightarrow\mathbb{R}$ by

$$f=\sqrt[m]{r^{2m}+1}+\frac{1}{m}\sum_{j=0}^{m-1}\zeta^j\log\Big(\sqrt[m]{r^{2m}+1}-\zeta^j\Big),\tag{2.1}$$

where r is the radius function on $\mathbb{C}^m/\mathbb{Z}_m$. Then f is a well-defined, smooth real function on $\mathbb{C}^m/\mathbb{Z}_m\setminus\{0\}$, so it defines a Kähler metric g on $X\setminus\pi^{-1}(0)$ which has f as the Kähler potential. Calabi [10] shows that g can be extended to all of X and is a Ricci-flat metric with $\mathrm{Hol}(g)=\mathrm{SU}(m)$. Moreover, from the explicit form of f, we see that $f=r^2+O(r^{2-2m})$ for large r, which implies that g is an ALE Kähler metric on X. It follows that with Calabi's metric, the crepant resolution X of $\mathbb{C}^m/\mathbb{Z}_m$, or equivalently the total space $K_{\mathbb{CP}^{m-1}}$ of the canonical line bundle over $\mathbb{CP}^{m-1}$, is a Calabi–Yau m-fold asymptotic to $\mathbb{C}^m/\mathbb{Z}_m$.

Examples 2.7 (Cotangent bundles of spheres)

In [48], Stenzel constructed complete, Ricci-flat Kähler metrics on the cotangent bundle T^*S^m of the sphere S^m, and more generally on the "complexification" of compact rank one globally symmetric spaces, making them into Calabi–Yau m-folds. His approach was to use the large symmetry group of these manifolds to reduce the problem to solving an ordinary differential equations. For the case $m=2$, Stenzel's metric coincides with the Eguchi-Hanson metric [15].

The Calabi–Yau 3-fold T^*S^3, which is known in the physics literature as the deformed conifold, is of particular interest to us.

Let us realize the cotangent bundle T^*S^m as follows:

$$T^*S^m = \big\{(v,\xi) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} : |v| = 1,\ \langle v,\xi\rangle = 0\big\}.$$

According to [50], we can map the cotangent bundle T^*S^m diffeomorphically to the affine quadric

$$Q_1 = \Big\{(z_1, \dots, z_{m+1}) \in \mathbb{C}^{m+1} : \sum_{j=1}^{m+1} z_j^2 = 1\Big\}$$

via the identification $T^*S^m \longrightarrow\ Q_1$ given by

$$(v,\xi) \longmapsto\ v\cosh|\xi|\ +\ i\,\frac{\xi}{|\xi|}\sinh|\xi|$$

so that the standard symplectic form on $\mathbb{C}^{m+1}$ restricted to Q_1 is identified with the canonical symplectic form on T^*S^m.

We now briefly describe the Kähler potential of the Stenzel metric on Q_1 [48, §7], and thus on T^*S^m using the above identification. Let $(z_1, \dots, z_{m+1})$ be coordinates on $\mathbb{C}^{m+1}$, and $r^2 = |z_1|^2 + \cdots + |z_{m+1}|^2$. Then Stenzel's Ricci-flat metric on Q_1 is given by

$$g_{Q_1}\ = \sum_{j,k=1}^{m+1} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} f(r^2)\, dz_j\, d\bar{z}_k$$

where f is a smooth real function of r^2 satisfying the differential equation

$$\frac{d}{dw}\Big(\big(\frac{df}{dw}\big)^m\Big)\ =\ mc(\sinh w)^{m-1},$$

where $w = \cosh^{-1}(r^2)$ and c is some positive constant. We shall come back to this again and focus on $m = 3$ in Chapter 4 when we study the desingularization of *Calabi–Yau 3-folds with ordinary double points.*

2.2 Special Lagrangian geometry

We begin in §2.2.1 with some background from Symplectic geometry. The standard Darboux Theorem and the Lagrangian Neighbourhood Theorem will be discussed. Section 2.2.2 provides an introduction to calibrated geometry. In §2.2.3 and 2.2.4 we discuss special Lagrangian geometry in $\mathbb{C}^m$ and construct some explicit examples. Finally, §2.2.5 is devoted to the study of special Lagrangian submanifolds in Calabi–Yau manifolds, and we will give a result on deformations of compact special Lagrangian submanifolds.

2.2.1 Some basic concepts in Symplectic geometry

We shall recall some elementary definitions and results in symplectic geometry. A basic reference is McDuff and Salamon [41].

Let M be a smooth manifold of even dimension $2m$. A *symplectic structure* on M is a closed nondegenerate 2-form ω. Endowed with the symplectic structure ω, the smooth manifold M is a *symplectic manifold*, and we shall write it as (M, ω). If (M, ω) and $(\tilde{M}, \tilde{\omega})$ are symplectic manifolds, a diffeomorphism $F : M \longrightarrow \tilde{M}$ satisfying $F^*\tilde{\omega} = \omega$ is called a *symplectomorphism*.

An important example of a symplectic manifold is the cotangent bundle of any smooth n-dimensional manifold L, which carries a canonical symplectic structure ω_{can} that we now describe. Recall that the standard coordinates of $(p, \alpha) \in T^*L$ are defined to be $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, where $(x_1, \ldots, x_n)$ denotes the coordinate representation of p, and $(y_1, \ldots, y_n)$ denotes the cotangent coordinate representation of α on the fibre T_p^*L so that $\alpha = \sum_{j=1}^n y_j dx_j$. The symplectic form is then given by $\omega_{\text{can}} = -d\alpha = \sum_{j=1}^n dx_j \wedge dy_j$.

The nondegeneracy condition on the 2-form ω means that there is a canonical isomorphism between the tangent and cotangent bundle via $TM \longrightarrow T^*M : X \longmapsto \iota(X)\omega$. A smooth vector field X on M is said to be *symplectic* if $\iota(X)\omega$ is a closed 1-form.

Recall that given a smooth vector field X on a compact manifold M, for any $p \in M$, one can generate a family of diffeomorphisms ψ_t, the flow of X, by

$$\frac{d}{dt}\,\psi_t(p) \;=\; X|_{\psi_t(p)}, \qquad \psi_0(p) = p.$$

Using Cartan's formula and the fact that ω is a closed 2-form on the symplectic manifold (M, ω), we see that the vector field X is symplectic if and only if $\mathcal{L}_X\omega = 0$, i.e. ω is invariant under the flow of X, or equivalently, ψ_t is a symplectomorphism for each t.

The next theorem, known as *Darboux's Theorem* [41, Thm. 3.15], is one of the most fundamental results in the theory of symplectic structures. Basically it states that every symplectic manifold (M, ω) is locally symplectomorphic to the standard symplectic form $(\mathbb{R}^{2m}, \hat{\omega})$. In other words, the prototype of a local piece of a $2m$-dimensional symplectic manifold is $(\mathbb{R}^{2m}, \hat{\omega})$.

Theorem 2.8 (Darboux's Theorem) *Let (M, ω) be a $2m$-dimensional symplectic manifold. For any $p \in M$, there exist local coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ centered at p in which $\omega = \sum_{j=1}^m dx_j \wedge dy_j$.*

Any coordinates with this property are called *Darboux coordinates* or *canonical coordinates*.

A submanifold L in (M, ω) is called *Lagrangian* if $\omega|_L = 0$ and $\dim L = m$. The real m-dimensional space $\{(x_1, \ldots, x_m, 0, \ldots, 0) : x_j \in \mathbb{R}\} \cong \mathbb{R}^m$ is certainly a Lagrangian submanifold of $\mathbb{C}^m$. There is a natural way of manufacturing Lagrangian submanifolds in $\mathbb{C}^m$. Suppose f is a smooth real function on $\mathbb{R}^m$, then the graph of df,

$$\left\{ \left(x_1, \ldots, x_m, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m} \right) \,:\, x_j \in \mathbb{R} \right\},$$

is Lagrangian. More generally, if α is a smooth 1-form on a smooth manifold L, and if $\Gamma(\alpha)$ denotes the graph of α in the cotangent bundle T^*L with its canonical symplectic structure ω_{can}, one can show that $\omega_{\text{can}}|_{\Gamma(\alpha)} = -\pi^*(d\alpha)$, where $\pi : \Gamma(\alpha) \longrightarrow L$ is the natural projection, and therefore the graph $\Gamma(\alpha)$ of α in T^*L is Lagrangian if and only if α is closed. By taking $\alpha = 0$, we see that the zero section of T^*L is a Lagrangian submanifold.

Next we discuss the *Lagrangian Neighbourhood Theorem* [41, Thm. 3.33], which shows that any compact Lagrangian submanifold L is locally symplectomorphic to the zero section in T^*L.

Theorem 2.9 *Let (M, ω) be a symplectic manifold and $L \subset M$ a compact Lagrangian submanifold. Then there exists an open tubular neighbourhood U of the zero section in T^*L, and an embedding $\Psi : U \longrightarrow M$ such that*

$$\Psi^*(\omega) = \omega_{\text{can}}, \qquad \Psi|_L = \text{Id},$$

where ω_{can} is the canonical symplectic form on T^*L.

The proof is based on the fact that the normal bundle $\nu(L)$ of L in M is isomorphic to the cotangent bundle T^*L. Indeed, as $\omega(u, v) = g(u, Jv)$ for any $u, v \in TM$, where J is an almost complex structure and g the Hermitian metric on M, and as $\omega|_L = 0$, it suggests that J induces an isomorphism between TL and $\nu(L)$. Together with the isomorphism between T^*L and TL provided by the metric g, we thus obtain $\nu(L) \cong T^*L$.

Now we try to look at the Lagrangian submanifolds that are "close" to L in M. Suppose $\tilde{L}$ is a submanifold that is C^1-close to L in M. Then we may consider it as the image of the graph of a small section of $\nu(L)$ under the exponential map. The argument before suggests that $\tilde{L}$ can also be considered as the image $\Psi(\Gamma(\alpha))$ of the graph $\Gamma(\alpha)$ of a C^1-small 1-form α on L. If $\tilde{L}$ is also a Lagrangian submanifold in M, then

$$0 \;=\; \Psi^*(\omega|_{\tilde{L}}) \;=\; \omega_{\text{can}}|_{\Gamma(\alpha)} \;=\; -\pi^*(d\alpha).$$

Therefore we have established a 1-1 correspondence between Lagrangian submanifolds $\tilde{L}$ close to L in M and small closed 1-forms on L.

In the final part of this section, we discuss *moment maps* for actions of Lie groups on a symplectic manifold. These maps are an important tool for studying Lagrangian and special Lagrangian submanifolds.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. An action of a Lie group G on a manifold M is a smooth group homomorphism

$$\begin{aligned} \psi \;:\; G &\longrightarrow \text{Diff}\,(M) \\ g &\longmapsto \psi_g, \end{aligned}$$

where Diff (M) is the group of diffeomorphisms of M. Let (M,ω) be a symplectic manifold, and G a compact Lie group with an action $\psi : G \longrightarrow \mathrm{Diff}\,(M)$. The action ψ is *symplectic* if ψ_g is a symplectomorphism for every $g \in G$, in other words, $\psi \in \mathrm{Symp}\,(M,\omega)$, where Symp (M,ω) is the group of symplectomorphisms of (M,ω).

The symplectic action of G on (M,ω) induces a linear map from the Lie algebra $\mathfrak{g}$ of G to the space of vector fields:

$$\begin{aligned} \phi \ : \ \mathfrak{g} &\longrightarrow C^\infty(TM) \\ x &\longmapsto \phi(x) = \frac{d}{dt}\,\psi_{\exp(tx)}\Big|_{t=0}, \end{aligned}$$

where $\exp : \mathfrak{g} \longrightarrow G$ is the exponential map. (For instance, when $G = S^1$, we identify $\mathfrak{g}$ with $\mathbb{R}$, and the exponential map $\exp : \mathbb{R} \longrightarrow S^1$ becomes $\theta \longmapsto e^{2\pi i\theta}$.) Recall that the flow of a vector field is a symplectomorphism if and only if the vector field is symplectic. It follows that $\phi(x)$ is symplectic for every $x \in \mathfrak{g}$, and so the 1-form $\iota(\phi(x))\omega$ is closed for each x.

Let G be a compact Lie group with a symplectic action $\psi : G \longrightarrow \mathrm{Symp}\,(M,\omega)$, and $\mathfrak{g}$ the Lie algebra of G with dual vector space $\mathfrak{g}^*$.

Definition 2.10 A *moment map* for the G-action on (M,ω) is a smooth map $\mu : M \to \mathfrak{g}^*$ satisfying

(i) $\iota(\phi(x))\,\omega = \langle x, d\mu\rangle$ for any $x \in \mathfrak{g}$, where $\langle\cdot,\cdot\rangle$ denotes the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$,

(ii) μ is *equivariant* with respect to the G-action ψ_g on M and the coadjoint G-action Ad^* on $\mathfrak{g}^*$, i.e. the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} M & \xrightarrow{\ \mu\ } & \mathfrak{g}^* \\ {\scriptstyle\psi_g}\downarrow & & \downarrow{\scriptstyle\mathrm{Ad}^*_g} \\ M & \xrightarrow[\ \mu\ ]{} & \mathfrak{g}^* \end{array}$$

Note that μ is determined up to an additive constant by (i). A first example is given by the angular momentum in $\mathbb{R}^3$, in which $G = \mathrm{SO}(3)$ acts diagonally on $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ with its standard symplectic structure (see for instance [41, p.165]). This case is indeed the origin of the term moment map.

Define the centre $Z(\mathfrak{g}^*)$ to be the subspace of $\mathfrak{g}^*$ fixed by the coadjoint G-action, i.e.

$$Z(\mathfrak{g}^*) = \{c \in \mathfrak{g}^* : \mathrm{Ad}^*_g\,(c) = c \ \text{ for all } g \in G\}.$$

Using the equivariance $\mu(\psi_g(p)) = \mathrm{Ad}^*_g\,(\mu(p))$ from the definition, one can easily see that the level set $\mu^{-1}(c)$ in M is G-invariant if and only if $c \in Z(\mathfrak{g}^*)$.

Now consider $M = \mathbb{C}^m$ with the standard symplectic form $\hat\omega$, and the group $\mathrm{U}(m) \ltimes \mathbb{C}^m$ acting on $\mathbb{C}^m$ by : $z \mapsto Az + b$ for $A \in \mathrm{U}(m)$ and $b \in \mathbb{C}^m$. It is the group of automorphisms of $\mathbb{C}^m$ that preserves $\hat g$, $\hat\omega$ and so it takes Lagrangian m-folds to Lagrangian m-folds. Suppose G is a connected Lie subgroup of $\mathrm{U}(m) \ltimes \mathbb{C}^m$ with Lie algebra $\mathfrak{g}$. Then G preserves the symplectic form $\hat{\omega}$ and hence the G-action is symplectic. We state here a useful result [29, Prop. 4.2] that relates G-invariant Lagrangian m-folds and level sets of the moment map of G:

Proposition 2.11 *Let G be a connected Lie subgroup of $\mathrm{U}(m) \ltimes \mathbb{C}^m$ with Lie algebra $\mathfrak{g}$ and L a connected G-invariant Lagrangian m-fold in $\mathbb{C}^m$, then G admits a moment map $\mu : \mathbb{C}^m \to \mathfrak{g}^*$, and $L \subseteq \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

Here is why the moment map μ is constant on L. Since L is invariant under the G-action, the vector fields $\phi(x) \in C^\infty(T\mathbb{C}^m)$ are tangent to L for all $x \in \mathfrak{g}$. Let v be a vector tangent to L. As L is Lagrangian, $\hat{\omega}|_L = 0$, we have $0 = \hat{\omega}(\phi(x), v) = \langle x, d\mu(v) \rangle$ for any $x \in \mathfrak{g}$. Thus the differential of the moment map $d\mu$ is zero on L, and the result follows. Consequently, G-invariant Lagrangian m-folds in $\mathbb{C}^m$ lie in level sets of the moment map μ of G.

Later in §2.2.4, we shall apply the technique of moment maps to construct special Lagrangian submanifolds in $\mathbb{C}^m$ and in Calabi–Yau manifolds.

2.2.2 Brief review on calibrated geometries

The concept of calibrated geometry was first introduced by Harvey and Lawson [22] in 1982. Calibrated submanifolds, in particular special Lagrangian submanifolds, are believed to play an important role in mirror symmetry, and hence they have recently received a lot of attention. The simplest examples of calibrated submanifolds are complex submanifolds of Kähler manifolds. This suggests that complex submanifolds can be placed in the more general context of calibrated submanifolds.

Let (M, g) be a Riemannian manifold, and φ a closed k-form on M. φ is called a *calibration* if for each oriented tangent k-plane V in T_xM, $\varphi|_V \leq \mathrm{vol}(V)$ for all $x \in M$. Here $\mathrm{vol}(V)$ denotes the volume form on V induced by the Riemannian metric g. An oriented k-dimensional submanifold N is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}(T_xN)$ for all $x \in N$, i.e. φ restricts to the volume form of the induced metric on N. It follows from Stokes' theorem that any compact calibrated submanifold N has least volume in its homology class. Indeed, if N' is another compact oriented k-dimensional submanifold of M, with $[N'] = [N] \in H_k(M, \mathbb{R})$, then

$$\mathrm{Vol}(N) = \int_{x \in N} \mathrm{vol}(T_xN) = \int_{x \in N} \varphi|_{T_xN} = \int_{x \in N'} \varphi|_{T_xN'} \leq \int_{x \in N'} \mathrm{vol}(T_xN') = \mathrm{Vol}(N').$$

Classical examples of calibrated submanifolds are given by complex submanifolds of Kähler manifolds. Suppose (M, J, g) is a Kähler manifold and ω is the Kähler form of g. Set $\varphi_k = \frac{1}{k!}\,\omega^k$ for some $1 \leq k \leq m = \dim_{\mathbb{C}} M$. Since ω is closed on M, so φ_k is a closed $2k$-form. Furthermore, by *Wirtinger's Inequality*, $\varphi_k|_V \leq \mathrm{vol}(V)$ for any oriented tangent real $2k$-plane V, with equality if and only if V is the tangent complex k-plane. As a result, φ_k is a calibration and if N is any oriented real $2k$-dimensional submanifold of M calibrated by φ_k, then it is a complex k-dimensional submanifold. It follows that compact complex submanifolds of a Kähler manifold are volume-minimizing in their homology class.

Another class of examples of calibrated geometries given by Harvey and Lawson [22] is that of special Lagrangian geometry. The flat version of the geometry exists on $\mathbb{C}^m$ and gives a distinguished class of minimal real m-dimensional submanifolds of $\mathbb{C}^m$. These submanifolds can also be defined in a wider class of complex manifolds - Calabi–Yau manifolds. The calibration is given by $\mathrm{Re}\,(e^{i\theta}\Omega)$ where Ω is the holomorphic $(m,0)$-form on the Calabi–Yau manifold, and these special Lagrangian calibrations are one of the main objects of the thesis. Note that there is a S^1 family of these calibrations on each Calabi–Yau manifold, and since Calabi–Yau manifolds are Kähler, they possess the Kähler calibration as well.

There are also two natural classes of calibrated submanifolds in *G_2-manifolds*. A G_2-manifold M is 7-manifold equipped with a *torsion-free* G_2-structure (φ, g) where φ is a closed 3-form and g its induced metric. More discussion on G_2-structures will be given in §3.1. *Associative 3-folds* and *coassociative 4-folds* in M are 3- and 4-submanifolds with calibrations given by φ and $*\varphi$ respectively. Here $*$ is the Hodge star of the metric g.

2.2.3 Special Lagrangian submanifolds in $\mathbb{C}^m$

We start by defining special Lagrangian submanifolds in the flat complex Euclidean space $\mathbb{C}^m$. Take standard complex coordinates $z_1 = x_1 + iy_1, \ldots, z_m = x_m + iy_m$ on $\mathbb{C}^m$. Let $\hat{g}$ be the Euclidean metric, $\hat{\omega}$ the Kähler form of $\hat{g}$ and $\hat{\Omega} = dz_1 \wedge \cdots \wedge dz_m$ the volume form on $\mathbb{C}^m$. The real part of $\hat{\Omega}$, $\mathrm{Re}(\hat{\Omega})$, and the imaginary part, $\mathrm{Im}(\hat{\Omega})$, are closed real m-forms. Harvey and Lawson [22, III. 1] showed that $\mathrm{Re}(\hat{\Omega})$ is indeed a calibration, thus we can define :

Definition 2.12 Let L be an oriented real m-dimensional submanifold in $\mathbb{C}^m$. L is called a *special Lagrangian submanifold* (*SL m-fold*) of $\mathbb{C}^m$ if it is calibrated by $\mathrm{Re}(\hat{\Omega})$ and we call $\mathrm{Re}(\hat{\Omega})$ the *special Lagrangian calibration*. More generally, we say L is an *SL m-fold with phase $e^{i\theta}$* for $\theta \in [0, 2\pi)$ if it is calibrated by $\mathrm{Re}(e^{-i\theta}\,\hat{\Omega})$.

We will usually work with $\theta = 0$, i.e. SL m-folds with phase 1, and when we discuss SL m-folds without specifying a phase, it means that we are talking about SL m-folds with phase 1.

Note that every special Lagrangian submanifold L in $\mathbb{C}^m$ is non-compact. For if L is compact, then by Stokes' theorem, we know that $\int_L \eta = 0$ for any exact m-form η on L. Now $\hat{\Omega}$ is a real closed m-form on $\mathbb{C}^m$ which can be written as $d(z_1\, dz_2 \wedge \cdots \wedge dz_m)$. Thus $\hat{\Omega}$ is exact and $\mathrm{Re}(\hat{\Omega})|_L$ is an exact m-form on L and so $\int_L \mathrm{Re}(\hat{\Omega})|_L = 0$. But L is calibrated by $\mathrm{Re}(\hat{\Omega})$, giving $\int_L \mathrm{Re}(\hat{\Omega})|_L = \int_L \mathrm{vol}(L) = \mathrm{Vol}(L) \neq 0$, which is a contradiction. Thus compact special Lagrangian submanifolds only exist in general Calabi–Yau manifolds.

Define the *special Lagrangian m-planes* to be those oriented real m-planes in $\mathbb{C}^m$ calibrated by $\mathrm{Re}(\hat{\Omega})$ and denote the set of all special Lagrangian m-planes by *SLag*. Now consider a real vector subspace $U = \{(x_1, \ldots, x_m, 0, \ldots, 0) : x_j \in \mathbb{R}\} \subset \mathbb{R}^{2m}$ equipped with the standard orientation. The $\mathrm{SU}(m)$-orbit of U in the Grassmannian $Gr(\mathbb{R}^m, \mathbb{R}^{2m})$ is the set $\{V = A \cdot U :$

$A \in \mathrm{SU}(m)\}$. Clearly, $\mathrm{SU}(m)$ acts transitively on it. Also, the stabilizer $\mathrm{Stab}(U)$ of U is $\mathrm{SO}(m)$. Thus the $\mathrm{SU}(m)$-orbit of U has the structure of a homogeneous manifold which is isomorphic to $\mathrm{SU}(m)/\mathrm{SO}(m)$ and has real dimension $\frac{1}{2}(m^2 + m - 2)$. Note that U is calibrated by $\mathrm{Re}(\hat{\Omega})$ and since $\mathrm{SU}(m)$ preserves g_0 and $\hat{\Omega}$, so $A \cdot U$ is also calibrated by $\mathrm{Re}(\hat{\Omega})$ for all $A \in \mathrm{SU}(m)$. On the other hand, any oriented real m-plane $V \in \mathbb{C}^m$ calibrated by $\mathrm{Re}(\hat{\Omega})$, i.e. $V \in \mathit{SLag}$, is of the form $V = A \cdot U$ for some $A \in \mathrm{SU}(m)$. As a result, we have the following identifications :

$$\mathit{SLag} \quad \cong \quad \mathrm{SU}(m)\text{-orbit of } U \text{ in } Gr(\mathbb{R}^m, \mathbb{R}^{2m}) \quad \cong \quad \mathrm{SU}(m)/\mathrm{SO}(m).$$

Observe that the Kähler form $\hat{\omega}$ and the imaginary part $\mathrm{Im}(\hat{\Omega})$ of $\hat{\Omega}$ restrict to zero on U. By the fact that $\mathrm{SU}(m)$ preserves $\hat{\omega}$ and $\hat{\Omega}$ and acts transitively on SLag, we have $\hat{\omega}|_V = \mathrm{Im}(\hat{\Omega})|_V = 0$ for all $V \in \mathit{SLag}$. Consequently, if L is an SL m-fold in $\mathbb{C}^m$, then $\hat{\omega}|_L = \mathrm{Im}(\hat{\Omega})|_L = 0$. It turns out that the vanishing of these two real forms gives a necessary and sufficient condition for a submanifold in $\mathbb{C}^m$ being special Lagrangian [22, III. Cor. 1.11], and can be regarded as an alternative characterization of the special Lagrangian condition:

Proposition 2.13 *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into a special Lagrangian submanifold of $\mathbb{C}^m$ if and only if $\hat{\omega}|_L = \mathrm{Im}(\hat{\Omega})|_L = 0$.*

We shall use this as the *definition* of SL m-folds most of the time. Recall from §2.2.1 that a real m-dimensional submanifold L of a real $2m$-dimensional symplectic manifold (M, ω) is Lagrangian if ω restricts to zero on L. Thus by Proposition 2.13, special Lagrangian submanifolds are Lagrangian submanifolds in $\mathbb{C}^m$ satisfying an extra condition $\mathrm{Im}(\hat{\Omega})|_L = 0$. The next result shows that special Lagrangian submanifolds with some phase $e^{i\theta}$ are minimal Lagrangian submanifolds [22, Prop. 2.17]:

Proposition 2.14 *A connected Lagrangian submanifold L in $\mathbb{C}^m$ is minimal (mean curvature $H = 0$) if and only if L is special Lagrangian with phase $e^{i\theta}$ for some $\theta \in [0, 2\pi)$.*

In the case $m = 1$, one can see that $U \cong \mathbb{R}$ is the only special Lagrangian plane in $\mathbb{C}$, so this case is trivial. For $m = 2$, we know that $\mathbb{C}^2$ carries three distinct complex structures I, J and K satisfying $I^2 = J^2 = K^2 = -\mathrm{Id}$ and $K = IJ$. It turns out that, by using Proposition 2.13 for instance, a real surface L in $\mathbb{C}^2$ is special Lagrangian with respect to one complex structure if and only if it is holomorphic with respect to another one of these complex structures. Thus special Lagrangian geometry is equivalent to complex geometry in this case, and is well understood.

Examples 2.15 (Special Lagrangian Graphs) We have seen that if $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a smooth function, then the graph $\Gamma(df)$ of the 1-form df is a Lagrangian submanifold of $\mathbb{C}^m$. We shall discuss here the additional condition for $\Gamma(df)$ to be special Lagrangian.

Denote by $\mathrm{Hess}(f)$ the Hessian matrix of f, namely the $m \times m$ matrix

$$\mathrm{Hess}(f) = \left(\frac{\partial^2 f}{\partial x_i \, \partial x_j} \right)_{1 \le i,j \le m}.$$

The tangent space to $\Gamma(df)$ at the point $(x_1, \dots, x_m, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m})$ is the image of $\mathbb{R}^m$ under the linear map $\mathrm{Id} + i\,\mathrm{Hess}(f)(x_1, \dots, x_m)$. Hence the graph $\Gamma(df)$ is special Lagrangian if and only if

$$\mathrm{Im}\left(\det_{\mathbb{C}}\left(\mathrm{Id} + i\,\mathrm{Hess}(f)\right)\right) = 0 \ \text{ on } \mathbb{C}^m.$$

For $m = 1$, the differential equation is $f''(x) = 0$, and so $y = f'(x) =$ constant, which just says the horizontal real lines are special Lagrangian. For $m = 2$, the equation becomes $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$, and hence $\Gamma(df)$ is a special Lagrangian 2-fold if and only if f is harmonic. Note that the condition is linear. This is, however, no longer the case starting from dimension 3. The function f has to satisfy the above nonlinear second order elliptic partial differential equation for $m \geq 3$. When $m = 3$, the condition is $\Delta f = \det \mathrm{Hess}(f)$, where Δ is the Laplacian on $\mathbb{R}^3$. Observe that the first order term in this partial differential equation is Δf, so the *linearization* of this equation is $\Delta f = 0$. This should be compared with McLean's Theorem (Theorem 2.24 below) on the deformation theory of compact special Lagrangian submanifolds in general Calabi–Yau manifolds.

2.2.4 Constructions of SL m-folds in $\mathbb{C}^m$

In this section we describe some ways of constructing SL m-folds in $\mathbb{C}^m$. These yield many explicit examples, including some singular SL m-folds, for later chapters where we will consider these examples in Calabi–Yau m-folds.

(i) Construction by moment maps

We shall give here a construction of SL m-folds as level sets of m moment maps for some G-action on $\mathbb{C}^m$. With the discussion in §2.2.1, we assume that G is a connected Lie subgroup of $\mathrm{U}(m) \ltimes \mathbb{C}^m$, so that the G-action is symplectic. Proposition 2.11 shows that G-invariant Lagrangian m-folds in $\mathbb{C}^m$ lie in the level sets of the moment map of G. If we instead consider the group $\mathrm{SU}(m) \ltimes \mathbb{C}^m$, which is the group of automorphisms of $\mathbb{C}^m$ preserving the structures $\hat{g}, \hat{\omega}$ and $\hat{\Omega}$, we get the result for SL m-folds, that is, G-invariant SL m-folds contained in the level set of the moment map of G. But so far we have only been using the Kähler form $\hat{\omega}$ and the moment map associated to it. As SL m-folds are characterized by the vanishing of the two forms $\hat{\omega}$ and $\mathrm{Im}(\hat{\Omega})$, one may ask whether there is an analogue of moment maps associated to $\hat{\omega}$, i.e. a kind of "generalized moment map" associated to $\mathrm{Im}(\hat{\Omega})$ such that these maps are constants on SL m-folds. First we need the following lemma:

Lemma 2.16 *Let G be a connected Lie subgroup of $\mathrm{SU}(m) \ltimes \mathbb{C}^m$ with Lie algebra $\mathfrak{g}$. Define a natural product $\lambda_k : \bigwedge^k \mathfrak{g} \to \bigwedge^{k-1} \mathfrak{g}$ by*

$$\lambda_k(x_1 \wedge \dots \wedge x_k) := \sum_{1 \leq i < j \leq k} (-1)^{i+j-1}\,[x_i, x_j] \wedge x_1 \wedge \dots \wedge \widehat{x_i} \wedge \dots \wedge \widehat{x_j} \wedge \dots \wedge x_k$$

where $[\,,]$ denotes the Lie bracket on $\mathfrak{g}$ and $x_1, \dots, x_k$ are elements in $\mathfrak{g}$. Let $\phi : \mathfrak{g} \longrightarrow C^\infty(T\mathbb{C}^m)$ be the natural action of $\mathfrak{g}$ on $\mathbb{C}^m$ by vector fields, and it induces a linear map $\bigwedge^k \mathfrak{g} \longrightarrow C^\infty(\bigwedge^k T\mathbb{C}^m)$. Then for any $x_1, \dots, x_k \in \mathfrak{g}$,

$$d\left(\iota\left(\phi(x_1) \wedge \dots \wedge \phi(x_k)\right) \mathrm{Im}(\hat{\Omega})\right) = \iota\left(\phi(\lambda_k(x_1 \wedge \dots \wedge x_k))\right) \mathrm{Im}(\hat{\Omega}).$$

We can show this by induction on k, and the proof can be found in [13, Lem. 4.3]. Now suppose $0 \neq \xi \in \bigwedge^{m-1} \mathfrak{g}$. If $\lambda_{m-1}(\xi) = 0$, then by Lemma 2.16, we have

$$d(\,\iota(\phi(\xi))\mathrm{Im}(\hat{\Omega})) = \,\iota(\phi(\lambda_{m-1}(\xi)))\mathrm{Im}(\hat{\Omega}) = 0.$$

Thus $\iota(\phi(\xi))\mathrm{Im}(\hat{\Omega})$ is a closed 1-form on $\mathbb{C}^m$, which is therefore equal to $d\eta$ for some smooth function $\eta : \mathbb{C}^m \to \mathbb{R}$. If L is any connected G-invariant SL m-fold in $\mathbb{C}^m$, then for each $p \in L$, $\phi(\xi)|_p \in \bigwedge^{m-1} T_pL$ and so $d\eta|_{T_pL} = 0$, as $\mathrm{Im}(\hat{\Omega})|_{T_pL} = 0$. Hence η is constant on L. Consequently, the function η suits our requirement and can then be viewed as a "generalized moment map" associated to $\mathrm{Im}(\hat{\Omega})$.

For any connected Lie group G, if $\xi \in \bigwedge^k \mathfrak{g}$ is invariant under the induced $\mathrm{Ad}(G)$-action on $\bigwedge^k \mathfrak{g}$, i.e. the Lie derivative $\mathcal{L}_v\xi$ of ξ vanishes for any $v \in \mathfrak{g}$, then $\lambda_k(\xi) = 0$. This is because if we set $D\xi(v) = \mathcal{L}_v\xi$, then $D\xi$ gives a linear map from $\mathfrak{g}$ to $\bigwedge^k \mathfrak{g}$, and so is an element of $\mathfrak{g}^* \otimes \bigwedge^k \mathfrak{g}$. Now one can show that $\lambda_k(\xi) = \pi(D\xi)$ where $\pi : \mathfrak{g}^* \otimes \bigwedge^k \mathfrak{g} \to \bigwedge^{k-1} \mathfrak{g}$ is a contraction map that contracts $\mathfrak{g}^*$ with the first factor in $\bigwedge^k \mathfrak{g}$, and hence $\mathcal{L}_v\xi = 0$ for any $v \in \mathfrak{g}$ implies $\lambda_k(\xi) = 0$.

Suppose that $\dim G = m - 1$ and that there is an element $\xi \in \bigwedge^{m-1} \mathfrak{g}$ which is invariant under the induced $\mathrm{Ad}(G)$-action on $\bigwedge^{m-1} \mathfrak{g}$. As the consequence of Lemma 4.3, the "generalized moment map" η exists and L must lie in the level set of η and hence it lies in the level set of m functions :

$$\{z \in \mathbb{C}^m : \, \mu(z) = c\,, \; \eta(z) = c'\}$$

for some $c \in Z(\mathfrak{g}^*)$ and $c' \in \mathbb{R}$.

Now we describe a method of constructing SL m-folds using moment map techniques:

Proposition 2.17 *Let G be a connected $(m-1)$-dimensional Lie subgroup of $\mathrm{SU}(m) \ltimes \mathbb{C}^m$ with Lie algebra $\mathfrak{g}$. Suppose $0 \neq \xi \in \bigwedge^{m-1} \mathfrak{g}$ is invariant under $\mathrm{Ad}(G)$. Identify the dual $\mathfrak{g}^*$ of the Lie algebra with $\mathbb{R}^{m-1}$. Suppose G admits a moment map $\mu = (\mu_1, \ldots, \mu_{m-1})$ and let η be the "generalized moment map" of G. For each $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$ with $(c_1, \ldots, c_{m-1}) \in Z(\mathfrak{g}^*)$, define*

$$L_c = \{z \in \mathbb{C}^m : \, \mu_1(z) = c_1 \;, \, \ldots, \; \mu_{m-1}(z) = c_{m-1} \;, \; \eta(z) = c_m\}.$$

Then L_c is an SL m-fold in $\mathbb{C}^m$ wherever it is nonsingular.

The proof can be found in [13, Prop. 4.4]. Goldstein [19] proved a similar result where he considered non-compact Calabi–Yau manifolds with a Hamiltonian structure-preserving torus action. He showed if M is a non-compact Calabi–Yau m-fold with $H^1(M) = 0$, and suppose there is a Hamiltonian structure-preserving $(m-1)$-torus action on M, then for a generic point $(c_1, \ldots, c_m) \in \mathbb{R}^m$, the level set $\{\mu \equiv (c_1, \ldots, c_{m-1}),\, \eta \equiv c_m\}$ is an SL m-fold in M.

Examples 2.18 This example is given by Harvey and Lawson [22, III.3.A]. Let G be the subgroup $T^{m-1} = \{(e^{i\theta_1}, \ldots, e^{i\theta_{m-1}}) : \theta_j \in \mathbb{R}\}$ acting on $\mathbb{C}^m$ by :

$$(e^{i\theta_1}, \ldots, e^{i\theta_{m-1}}) \cdot (z_1, \ldots, z_m) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_{m-1}} z_{m-1},\, e^{-i(\theta_1 + \cdots + \theta_{m-1})} z_m).$$

Then G lies in $\mathrm{SU}(m)$. The moment map for this G-action is $\mu = (\mu_1, \dots, \mu_{m-1})$ where $\mu_j(z_1, \dots, z_m) = |z_j|^2 - |z_m|^2$. Since G is abelian, the coadjoint action of G is trivial and hence $Z(\mathfrak{g}^*) = \mathfrak{g}^* = \mathbb{R}^{m-1}$. Moreover, λ_{m-1} is a zero map and so the "generalized moment map" η exists. One can show that η is some constant multiple of $z_1 \cdots z_m + (-1)^m \bar{z}_1 \cdots \bar{z}_m$ and so we can take $\eta(z_1, \dots, z_m) = \mathrm{Re}(z_1 \cdots z_m)$ if m is even and $\mathrm{Im}(z_1 \cdots z_m)$ if m is odd. Then for each $c = (c_1, \dots, c_m) \in \mathbb{R}^m$, define

$$\begin{aligned} L_c := \{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_j|^2 - |z_m|^2 = c_j \text{ for } j = 1, \dots, m-1, \text{ and} \\ \mathrm{Re}(z_1 \cdots z_m) = c_m \text{ if } m \text{ is even}, \ \mathrm{Im}(z_1 \cdots z_m) = c_m \text{ if } m \text{ is odd}\}. \end{aligned}$$

By Proposition 2.17, L_c is an SL m-fold in $\mathbb{C}^m$ wherever it is nonsingular. One can show that if $c_m \neq 0$, then L_c is nonsingular and diffeomorphic to $T^{m-1} \times \mathbb{R}$. When $c_m = 0$, it may be nonsingular or have various kinds of singularity, depending on the values of $c_1, \dots, c_{m-1}$. We are particularly interested in SL 3-folds, so let us take $m = 3$. Then L_c has an isolated singular point at the origin if $c = (0, 0, 0)$, and is topologically the union of two copies of $T^1 \times \mathbb{C}$ intersecting at T^1 (where the singularities are located) if $c = (r, 0, 0), (0, r, 0)$ or $(-r, -r, 0)$ for $r > 0$. All other L_c's are nonsingular and diffeomorphic to $T^2 \times \mathbb{R}$.

Examples 2.19 This example is taken from Marshall [40] which gives an application of Proposition 2.17 to a non-abelian Lie group G that is a subgroup of $\mathrm{SU}(4)$ isomorphic to $\mathrm{SU}(2)$, and it acts on $\mathbb{C}^4$ by the usual matrix multiplication. One can take a basis for the Lie algebra $\mathfrak{g}$ of G to be the matrices:

$$e_1 = \begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -3i \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 & i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & 2i & 0 \\ 0 & 2i & 0 & i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}.$$

We have the following relations :

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1 \quad [e_3, e_1] = 2e_2.$$

Take $\xi = e_1 \wedge e_2 \wedge e_3 \in \bigwedge^3 \mathfrak{g}$, then ξ is $\mathrm{Ad}(G)$-invariant and so $\lambda_3(\xi) = 0$. The moment map for this G-action is given by $\mu = (\mu_1, \mu_2, \mu_3)$ where

$$\begin{aligned} \mu_1(z_1, z_2, z_3, z_4) &= \sqrt{3}\,\mathrm{Re}(z_1\bar{z}_2 + z_3\bar{z}_4) + 2\mathrm{Re}(z_2\bar{z}_3), \\ \mu_2(z_1, z_2, z_3, z_4) &= \sqrt{3}\,\mathrm{Im}(z_1\bar{z}_2 + z_3\bar{z}_4) + 2\mathrm{Im}(z_2\bar{z}_3), \\ \mu_3(z_1, z_2, z_3, z_4) &= 3|z_1|^2 + |z_2|^2 - |z_3|^2 - 3|z_4|^2 \end{aligned}$$

and the generalized moment map η is given by

$$\eta(z_1, z_2, z_3, z_4) = \mathrm{Im}\big(2\sqrt{3}(z_1 z_3^2 + z_2^3 z_4) - 9 z_1 z_2 z_3 z_4 + \frac{9}{2}\, z_1^2 z_4^2 - \frac{3}{2}\, z_2^2 z_3^2\big).$$

Since $Z(\mathfrak{g}^*) = Z(\mathfrak{su}(2)^*) = \{0\}$, then for each $c \in \mathbb{R}$,

$$L_c = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \mu \equiv 0, \ \eta \equiv c\}$$

is a G-invariant SL 4-fold in $\mathbb{C}^4$.

(ii) Cohomogeneity one examples

We will discuss a kind of G-invariant SL m-fold L with cohomogeneity one, i.e. the G-orbits are of codimension one in L, and construct examples of cohomogeneity one SL m-folds in $\mathbb{C}^m$. This technique of using a large symmetry group can reduce the problem of solving partial differential equations to ordinary differential equations.

Suppose G is a connected Lie subgroup of $\mathrm{SU}(m) \ltimes \mathbb{C}^m$ with Lie algebra $\mathfrak{g}$ and moment map $\mu : \mathbb{C}^m \longrightarrow \mathfrak{g}^*$. Let $\mathcal{O}$ be a G-orbit in $\mathbb{C}^m$ with $\dim \mathcal{O} = m-1$ and $\mathcal{O} \subseteq \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$. Joyce [29, Thm. 4.5] showed that there exists a locally unique, G-invariant SL m-fold L in $\mathbb{C}^m$ containing $\mathcal{O}$. Also, $L \subseteq \mu^{-1}(c)$ and L is locally diffeomorphic to $(-\epsilon, \epsilon) \times \mathcal{O}$ for some $\epsilon > 0$. His construction is to solve an ordinary differential equation in a 1-parameter family of $(m-1)$-dimensional G-orbits in $\mathbb{C}^m$. The strategy of constructing cohomogeneity one G-invariant SL m-folds in $\mathbb{C}^m$ is to first choose a suitable Lie subgroup G in $\mathrm{SU}(m) \ltimes \mathbb{C}^m$ with moment map μ, and then work out the types of G-orbit $\mathcal{O}$ in $\mu^{-1}(c)$ for $c \in Z(\mathfrak{g}^*)$, and see if any have dimension $m-1$. We can then construct L by solving a first-order ordinary differential equation in $(m-1)$-dimensional G-orbits. Details can be found in [29, Thm. 3.3 & 4.5].

Here is another example taken from Harvey and Lawson [22, III.3.B]:

Examples 2.20 Let G be the subgroup $\mathrm{SO}(3)$ of $\mathrm{SU}(3)$. Then its Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ consists of skew-symmetric 3×3 real matrices. One can show that the moment map μ of G is

$$\mu(z_1, z_2, z_3) = (\mathrm{Im}(z_1 \bar{z}_2),\ \mathrm{Im}(z_2 \bar{z}_3),\ \mathrm{Im}(z_3 \bar{z}_1)).$$

Now, since $Z(\mathfrak{g}^*) = \{0\}$, we have to find G-orbits in $\mu^{-1}(0)$. It can be shown that $\mu^{-1}(0)$ is equal to $\{(\lambda x_1, \lambda x_2, \lambda x_3) : \lambda \in \mathbb{C},\ x_1, x_2, x_3 \in \mathbb{R}\}$. Then we normalize x_1, x_2, x_3 so that $x_1^2 + x_2^2 + x_3^2 = 1$. Hence the G-orbits in $\mu^{-1}(0)$ are of the form $\mathcal{O}_\lambda = \{(\lambda x_1, \lambda x_2, \lambda x_3) : x_j \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 = 1\}$ for each $\lambda \in \mathbb{C}$. Then $\dim \mathcal{O}_\lambda = 2$ for $\lambda \neq 0$, which is of the type cohomogeneity one. Now we set up a first order ordinary differential equation on G-orbits $\mathcal{O}_\lambda$. Let P be the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Define $\phi_t : P \to \mathbb{C}^3$ by $\phi_t(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3)$ where $\lambda = \lambda(t) \in \mathbb{C}$. Then calculation shows that the ordinary differential equation is given by

$$\frac{d\lambda}{dt} = \bar{\lambda}^2.$$

One can see that since $d(\lambda^3)/dt = 3(\lambda^2)\, d\lambda/dt = 3|\lambda|^4$, so it is real, and $d(\mathrm{Im}(\lambda^3))/dt = 0$. Consequently, for each $c \in \mathbb{R}$,

$$L_c := \{(\lambda x_1, \lambda x_2, \lambda x_3) : x_j \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 = 1,\ \lambda \in \mathbb{C},\ \mathrm{Im}(\lambda^3) = c\ \}$$

is an SL 3-fold in $\mathbb{C}^3$. When $c = 0$, it is a union of three copies of $\mathbb{R}^3$ intersecting at 0, and when $c \neq 0$, it is nonsingular and is diffeomorphic to three disjoint unions of $S^2 \times \mathbb{R}$.

This example can be generalized to higher dimensional cases. For $G = \mathrm{SO}(m)$ in $\mathrm{SU}(m)$, and for each $c \in \mathbb{R}$, then

$$L_c := \{(\lambda x_1, \ldots, \lambda x_m) : x_j \in \mathbb{R},\ x_1^2 + \cdots + x_m^2 = 1,\ \lambda \in \mathbb{C},\ \mathrm{Im}(\lambda^m) = c\ \}$$

is an SL m-fold in $\mathbb{C}^m$. When $c = 0$, it is a union of m copies of $\mathbb{R}^m$ intersecting at 0, and when $c \neq 0$ and m is even (resp. odd), it is nonsingular and is diffeomorphic to $m/2$ (resp. m) disjoint unions of $S^{m-1} \times \mathbb{R}$. It is not hard to see that if we restrict $\arg(\lambda)$ to the range $(0, \pi/m)$, we get one copy of such $S^{m-1} \times \mathbb{R}$, converging to $\mathbb{R}^m \cup e^{i\pi/m}\mathbb{R}^m$. This illustrates the following result on *SL cones* and *Asymptotically Conical (AC) SL m-folds* in $\mathbb{C}^m$, proved by Haskins [23, Thm. A] and Joyce [29, Thm. 6.4]:

Theorem 2.21 *Let C be a closed SL cone in $\mathbb{C}^m$ with isolated singular point 0, and write $\Sigma = \{z \in C : |z| = 1\}$. For each $c > 0$, define*

$$\begin{aligned} L_c &= \{\lambda z : z \in \Sigma,\ \lambda \in \mathbb{C},\ \mathrm{Im}(\lambda^m) = c,\ \arg(\lambda) \in (0, \pi/m)\,\} \\ &= \{(c\sin(m\theta))^{-1/m} e^{i\theta} z\, :\, z \in \Sigma,\ \theta \in (0, \pi/m)\,\}. \end{aligned}$$

Then L_c is an immersed AC SL m-fold in $\mathbb{C}^m$ diffeomorphic to $\Sigma \times \mathbb{R}$, and asymptotic to $C \cup e^{i\pi/m}C$.

Roughly speaking, a SL cone C is an SL m-fold in $\mathbb{C}^m$ invariant under dilations, and an AC SL m-fold L in $\mathbb{C}^m$ is a nonsingular SL m-fold which converges to some SL cone C at infinity. We will discuss these submanifolds in detail later in Chapter 5. Note that for sufficiently small θ, we have $(c\sin(m\theta))^{-1/m}\sin\theta \approx (mc)^{-1}r^{1-m}$ where r is given by $(c\sin(m\theta))^{-1/m}$ from the theorem, and we see that one end of L_c is approaching the cone $\mathbb{R}^m$ with "rate" $1 - m$ when $r \to \infty$. We shall define precisely what we mean by "rate" in Chapter 5. The same happens on the other end as well. Thus Theorem 2.21 associates to any SL cone C a 1-parameter family of AC SL m-folds L_c, asymptotic to the union of two SL cones $C \cup e^{i\pi/m}C$ with rate $1 - m$. We recover Example 2.20 if we take $C = \mathbb{R}^m$ and $\Sigma = S^{m-1}$.

2.2.5 SL m-folds in Calabi–Yau m-folds

We now extend the notion of special Lagrangian submanifolds from $\mathbb{C}^m$ to Calabi–Yau manifolds. In order to define SL m-folds, one needs an analogue of the constant holomorphic $(m, 0)$-form $dz_1 \wedge \cdots \wedge dz_m$ on $\mathbb{C}^m$. Recall that there is such a constant holomorphic $(m, 0)$-form on a Calabi–Yau m-fold, therefore the concept of special Lagrangian geometry can be generalized to the Calabi–Yau setting. Now suppose (M, J, ω, Ω) is a Calabi–Yau m-fold. The normalization condition assures that at each point, ω and Ω can be written as the standard Kähler form $\hat{\omega}$ and holomorphic volume form $\hat{\Omega}$ on $\mathbb{C}^m$. Then $\mathrm{Re}(\Omega)$, and also $\mathrm{Re}(e^{i\theta}\Omega)$ for $\theta \in [0, 2\pi)$, are calibrations and we can define SL m-folds in Calabi–Yau m-folds in a similar way:

Definition 2.22 Let (M, J, ω, Ω) be a Calabi–Yau m-fold and L an oriented real m-dimensional submanifold in M. L is a *special Lagrangian submanifold* (*SL m-fold*) *with phase* $e^{i\theta}$ if it is calibrated by $\mathrm{Re}(e^{-i\theta}\Omega)$.

Thus each Calabi–Yau m-fold naturally comes with an S^1-family of special Lagrangian submanifolds. By Proposition 2.13, we can also take the following alternative definition of SL m-folds:

Proposition 2.23 *Let (M, J, ω, Ω) be a Calabi–Yau m-fold and L a real m-dimensional submanifold in M. Then L admits an orientation making it into an SL m-fold with phase $e^{i\theta}$ in M if and only if $\omega|_L = \mathrm{Im}(e^{-i\theta}\Omega)|_L = 0$.*

Suppose L is a compact SL m-fold in (M, J, ω, Ω) with phase $e^{i\theta}$, then

$$[\Omega]\cdot[L] \ = \int_L \Omega = e^{i\theta}\int_L e^{-i\theta}\Omega = e^{i\theta}\int_L \mathrm{Re}(e^{-i\theta}\Omega) + ie^{i\theta}\int_L \mathrm{Im}(e^{-i\theta}\Omega) = e^{i\theta}\mathrm{Vol}(L),$$

where $[\Omega] \in H^m(M, \mathbb{C})$ and $[L] \in H_m(M, \mathbb{Z})$. Here we used the fact that $\mathrm{Re}(e^{-i\theta}\Omega)$ is a calibration and $\mathrm{Im}(e^{-i\theta}\Omega)$ vanishes on L. Thus we see that the homology class $[L]$ determines the phase $e^{i\theta}$ of L. If we only study SL m-folds in a fixed homology class in M, we can rescale the phase $\Omega \mapsto e^{i\theta}\Omega$, so that we can always suppose L has phase 1.

In [42], McLean studied the deformation theory for calibrated submanifolds. In particular, he proved the following result for SL m-folds in Calabi–Yau m-folds [42, Thm 3.6]:

Theorem 2.24 *Let L be a compact SL m-fold in a Calabi–Yau m-fold (M, J, ω, Ω). Then the moduli space $\mathcal{M}_L$ of special Lagrangian deformations of L is a smooth manifold of dimension $b^1(L)$, the first Betti number of L.*

Thus if L is a compact SL m-fold with $H^1(L, \mathbb{R}) = 0$, for example, if L is a homology m-sphere for $m \geq 2$, then $\mathcal{M}_L$ has dimension 0, and so L admits no nontrivial special Lagrangian deformations in M.

Here we briefly sketch the proof of Theorem 2.24. Let U be an open tubular neighbourhood of L in T^*L, and $\Psi : U \longrightarrow M$ the Lagrangian neighbourhood embedding of Theorem 2.9 with $\Psi^*(\omega) = \omega_{\mathrm{can}}$ and $\Psi|_L = \mathrm{Id}$. Suppose $\tilde{L}$ is a submanifold of M which is C^1-close to L in M, and write $C^\infty(U) = \{\alpha \in C^\infty(T^*N) : \Gamma(\alpha) \subset U\}$. The argument after Theorem 2.9 shows that $\tilde{L}$ can be written as the image $\Psi(\Gamma(\alpha))$ of the graph $\Gamma(\alpha)$ of some C^1-small 1-form $\alpha \in C^\infty(U)$. Moreover, $\tilde{L} = \Psi(\Gamma(\alpha))$ is Lagrangian if and only if $d\alpha = 0$, as we have discussed. Let $\pi : U \longrightarrow L$ be the natural projection. Then one can show that $\tilde{L} = \Psi(\Gamma(\alpha))$ is an SL m-fold in M if and only if $d\alpha = 0$ and $\pi_*(\Psi^*(\mathrm{Im}(\Omega))|_{\Gamma(\alpha)}) = 0$.

Since $\pi_*(\Psi^*(\mathrm{Im}(\Omega))|_{\Gamma(\alpha)})$ is an m-form on L, it is a multiple of the volume form dV of the induced metric $g|_L$ on L. Define a function $F : C^\infty(U) \longrightarrow C^\infty(L)$ by $\pi_*(\Psi^*(\mathrm{Im}(\Omega))|_{\Gamma(\alpha)}) = F(\alpha)\,dV$. Then $\tilde{L}$ is special Lagrangian if and only if $d\alpha = 0 = F(\alpha)$. Calculation shows (see Prop. 2.10 in [32]) that the linearization $dF|_0(\alpha)$ of F is given by $d^*\alpha$, where $*$ is the Hodge star of $g|_L$. Therefore the first order special Lagrangian deformations correspond to closed and coclosed 1-forms, or equivalently, harmonic 1-forms on L. Consider a function G mapping $\alpha \longmapsto (d\alpha, F(\alpha)\,dV)$ between appropriate Banach spaces of differential forms, then one can show that G actually maps to some Banach spaces of exact 2-forms and exact m-forms, and can rewrite the problem such that the linearization of G is surjective. Hence the Banach space Implicit Function Theorem implies that $G^{-1}(0,0)$ is a manifold with tangent space at 0 isomorphic to the vector space of harmonic 1-forms, and elliptic regularity shows that 1-forms in $G^{-1}(0,0)$ are smooth.

As a result, the local moduli space has the same dimension as the de Rham cohomology groups $H^1(L)$ and $H^{m-1}(L)$ (by Poincaré duality). Note that the product $H^1(L) \times H^{m-1}(L)$ has a natural symplectic structure that makes it into a symplectic manifold. Using this observation, Hitchin [24] proved that the local moduli space $\mathcal{M}_L$ of the SL m-fold L can be immersed in the symplectic manifold $H^1(L) \times H^{m-1}(L)$ as a Lagrangian submanifold.

There is a natural way of manufacturing SL m-folds in some Calabi–Yau m-folds (M, J, ω, Ω) endowed with a *real structure*, which is an *antiholomorphic isometric involution* $\sigma : M \longrightarrow M$ on M satisfying $\sigma^2 = \mathrm{Id}$, $\sigma^*(J) = -J$, $\sigma^*(g) = g$, $\sigma^*(\omega) = -\omega$ and $\sigma^*(\Omega) = \bar{\Omega}$. This involution generalizes the concept of the usual complex conjugation in $\mathbb{C}^m$. The following result shows that the fixed point set of σ on M is a real m-dimensional submanifold, and is indeed special Lagrangian (cf. [26, §11.9 Method 2]):

Proposition 2.25 *Suppose (M, J, ω, Ω) is a Calabi–Yau m-fold possessing an antiholomorphic isometric involution $\sigma : M \longrightarrow M$, then the fixed point set L of σ is an SL m-fold of M.*

Proof. Choose any holomorphic chart $(U_\alpha, \varphi_\alpha)$ on M, then the pullback $\varphi_\alpha^*\sigma$ of σ is a real structure on $\mathbb{C}^m$, whose fixed point set is diffeomorphic to $\mathbb{R}^m$. Hence the fixed point set L of σ on each chart U_α is diffeomorphic to $\mathbb{R}^m$, and is therefore a real m-fold. To see L is special Lagrangian, note that for each point $p \in L$, $\sigma(p) = p$ and $(\sigma_*)_p : T_pM \longrightarrow T_pM$ with $(\sigma_*)_p^2 = \mathrm{Id}$. Decompose the real $2m$-dimensional vector space T_pM into the $(+1)$-eigenspace U and (-1)-eigenspace V. Clearly, U is just the tangent space T_pL of L at p. It follows that

$$-\omega(u,v) = \sigma^*\omega(u,v) = \omega((\sigma_*)_p\, u, (\sigma_*)_p\, v) = \omega(u,v)$$

for any $u, v \in T_pL$, thus $\omega|_L = 0$. Similarly $\mathrm{Im}(\Omega)$ restricts to zero on L, and thus L is special Lagrangian. □

We shall see some examples of SL m-folds constructed by this method in Chapter 5.

2.3 Analysis on compact manifolds

In this section we are going to provide basic material on analysis on compact manifolds. Some good references are given by Aubin [1, Chapters 2,3], Besse [3, Appendix] and Joyce [26, Chapter 1].

We shall refer to spaces L^p, L^p_k, C^k and $C^{k,\alpha}$ as the Banach spaces defined in [26, §1.2]. Let (M, g) be a Riemannian m-fold. It is usual to consider these Banach spaces as vector spaces of functions on M. Here we will consider vector spaces of sections of vector bundles over M, in particular the bundles $\Lambda^r T^*M$ of r-forms for $0 \leq r \leq m$. Let $E \to M$ be a vector bundle over M with a fibre metric and a compatible connection ∇_E.

Definition 2.26 Let (M, g) be a Riemannian m-fold. Denote by dV the volume form of the Riemannian metric g. For $p \geq 1$, we define the *Lebesgue space* $L^p(E)$ to be the set of sections ξ of E over M that are locally integrable and such that the norm

$$\|\xi\|_{L^p} = \left(\int_M |\xi|^p dV\right)^{1/p}$$

is finite. For $k \geq 0$, we define the *Sobolev space* $L^p_k(E)$ to be the set of sections ξ of E over M that are k times weakly differentiable and for which the norm

$$\|\xi\|_{L^p_k} = \left(\sum_{j=0}^{k} \int_M |\nabla^j_E \xi|^p dV\right)^{1/p}$$

is finite. Then $L^p_k(E)$ is a Banach space for each $p \geq 1$, and in addition, $L^2_k(E)$ is a Hilbert space.

Next we introduce the C^k spaces and the Hölder spaces $C^{k,\alpha}$.

Definition 2.27 Let (M, g) be a Riemannian m-fold. For $k \geq 0$, we define $C^k(E)$ to be the space of sections ξ of E over M with k continuous derivatives such that the norm

$$\|\xi\|_{C^k} = \sum_{j=0}^{k} \sup_M |\nabla^j_E \xi|$$

is finite. Let $d(x, y)$ be the distance computed w.r.t. g and let $\alpha \in (0, 1)$. A section η of E over M is *Hölder continuous with exponent* α if

$$[\eta]_\alpha = \sup_{\substack{x \neq y \in M \\ d(x,y) < \delta(g)}} \frac{|\eta_x - \eta_y|}{d(x, y)^\alpha}$$

is finite. Here $\delta(g)$ is the injectivity radius of the metric g. The quantity $|\eta_x - \eta_y|$ can be understood by using the identification between the fibres over x and y obtained from parallel translation along a geodesic from x to y of length $d(x, y)$. Then the *Hölder space* $C^{k,\alpha}(E)$ is the set of $\xi \in C^k(E)$ such that $\nabla^k_E \xi$ is Hölder continuous with exponent α and the norm

$$\|\xi\|_{C^{k,\alpha}} = \|\xi\|_{C^k} + [\nabla^k_E \xi]_\alpha$$

is finite.

The relationships between these L^p_k and $C^{k,\alpha}$ spaces can be seen from the Sobolev Embedding theorem on Riemannian manifolds. The proof can be found in [1, Chapter 2].

Theorem 2.28 (Sobolev Embedding Theorem) *Let (M, g) be a compact Riemannian m-fold. Suppose $k \geq l \geq 0$ are integers, $p, q \geq 1$ and $\alpha \in (0, 1)$. Then*

(a) *If* $\frac{1}{p} \leq \frac{1}{q} + \frac{k-l}{m}$, *then* $L^p_k(E) \hookrightarrow L^q_l(E)$ *is a continuous inclusion.*

(b) *If* $\frac{1}{p} \leq \frac{k-l-\alpha}{m}$, *then* $L^p_k(E) \hookrightarrow C^{l,\alpha}(E)$ *is a continuous inclusion.*

We shall need a "weighted" version of the Sobolev Embedding Theorem in Chapter 4 to deal with some kinds of noncompact manifolds.

Next we discuss elliptic operators on a compact Riemannian m-fold (M,g). For an introduction, see [3, Appendix D-G] and [26, §1.3]. Let $E, F \to M$ be vector bundles over M, and let $P : C^\infty(E) \longrightarrow C^\infty(F)$ be a smooth, linear differential operator of order $l \geq 1$. In index notation, denote by $A^{i_1 \cdots i_l}$ the leading coefficients, i.e. the coefficients of the highest order derivative, of P. Then for any 1-form $\theta \in T^*M$, $A^{i_1 \cdots i_l}\theta_{i_1} \ldots \theta_{i_l}$ takes values in $E^* \otimes F$, i.e. a linear map from E to F. Define $\sigma_\theta(P) = A^{i_1 \cdots i_l}\theta_{i_1} \ldots \theta_{i_l}$ for any 1-form θ. Let $\sigma(P)$ be the map $\sigma(P) : T^*M \times E \longrightarrow F$ which assigns each (θ, ξ) to the value $\sigma_\theta(P)(\xi)$. Then $\sigma(P)$ is called the principal symbol of P, and we say P is an elliptic operator if for each nonzero $\theta \in T^*M$, the linear map $\sigma_\theta(P)$ is invertible.

Now consider P acts by $P : L^p_{k+l}(E) \longrightarrow L^p_k(F)$ and $P : C^{k+l,\alpha}(E) \longrightarrow C^{k,\alpha}(F)$. On a compact Riemannian m-fold (M,g), for $p \geq 1$ and $k \geq 0$, we have the smooth, linear elliptic, self-adjoint operator

$$d + d^* : L^p_{k+1}\big(\textstyle\bigoplus_{j=0}^m \Lambda^j T^*M\big) \longrightarrow L^p_k\big(\textstyle\bigoplus_{j=0}^m \Lambda^j T^*M\big)$$

of order 1, and the Laplacian

$$\Delta = dd^* + d^*d = (d+d^*)^2 : L^p_{k+2}\big(\textstyle\bigoplus_{j=0}^m \Lambda^j T^*M\big) \longrightarrow L^p_k\big(\textstyle\bigoplus_{j=0}^m \Lambda^j T^*M\big),$$

which is a linear elliptic, self-adjoint operator of order 2. Similar definitions hold for Hölder spaces.

Elliptic operators are important in analysis on compact manifolds, as they have the following nice regularity property:

Theorem 2.29 *Let (M,g) be a compact Riemannian m-fold, $E, F \longrightarrow M$ be vector bundles over M, and P a smooth, linear elliptic operator of order $l \geq 1$. Let $p \geq 1$ and $k \geq 0$ be an integer. Suppose $\xi \in L^1(E)$ and $\eta \in L^1(F)$ with $P\xi = \eta$ holds weakly. If $\eta \in L^p_k(F)$, then $\xi \in L^p_{k+l}(E)$, and*

$$\|\xi\|_{L^p_{k+l}} \leq C\big(\|\eta\|_{L^p_k} + \|\xi\|_{L^1}\big),$$

for some $C > 0$ independent of ξ, η.

For a proof, see Morrey [43, Thm. 6.4.8]. The analogous result holds for Hölder spaces, that is, if $\eta \in C^{k,\alpha}(F)$, then $\xi \in C^{k+l,\alpha}(E)$, and

$$\|\xi\|_{C^{k+l,\alpha}} \leq C\big(\|\eta\|_{C^{k,\alpha}} + \|\xi\|_{C^0}\big),$$

for some $C > 0$ independent of ξ, η. These estimates are called the L^p *estimates* and *Schauder estimates* respectively. One of the reasons for introducing the more complicated Sobolev and

Hölder spaces is that the C^l spaces do not have this kind of regularity property. If $\xi \in L^1(E)$ with $P\xi = 0$, then by Theorem 2.29 we have $\xi \in L^p_k(E)$ for all $k \geq 0$, which implies $\xi \in C^\infty(E)$. As a result, the kernel Ker$P$ of P is independent of p and k, and is a subspace of $C^\infty(E)$. In fact, it is also of finite dimensions ([26, Thm. 1.5.1]).

If one restricts ξ so that it is L^2-orthogonal to Ker$P$, then we obtain ([26, Prop. 1.5.2]):

Proposition 2.30 *Let (M, g) be a compact Riemannian m-fold, $E, F \longrightarrow M$ be vector bundles over M, and P a smooth, linear elliptic operator of order $l \geq 1$. Let $p \geq 1$ and $k \geq 0$ be an integer. If $\xi \in L^p_{k+l}(E)$ and $\xi \perp \mathrm{Ker}P$, then there is a constant $D > 0$, independent of ξ, such that*

$$\|\xi\|_{L^p_{k+l}} \leq D\|P\xi\|_{L^p_k}.$$

Similarly, if $\alpha \in (0, 1)$ and $k \geq 0$ is an integer, and if $\xi \in C^{k+l,\alpha}(E)$ and $\xi \perp \mathrm{Ker}P$, then there is a constant $D > 0$, independent of ξ, such that

$$\|\xi\|_{C^{k+l,\alpha}} \leq D\|P\xi\|_{C^{k,\alpha}}.$$

We shall finish this section by giving an existence result for the equation $P\xi = \eta$. Recall that if X, Y are finite-dimensional inner product spaces and $P : X \longrightarrow Y$ is a linear map, one can solve the equation $Px = y$ if and only if $y \perp \mathrm{Ker}\, P^*$, where P^* is the dual of the linear map, and we have the orthogonal decomposition $Y = P(X) \oplus \mathrm{Ker}\, P^*$. We will see in the next theorem that the same criteria is needed for linear elliptic operators, and hence the existence theory for a linear elliptic operator on a compact manifold is very similar to the finite-dimensional case. Suppose $P : C^\infty(E) \longrightarrow C^\infty(F)$ is a smooth, linear elliptic operator of order $l \geq 1$, then the *formal adjoint* $P^* : C^\infty(F) \longrightarrow C^\infty(E)$ of P is the unique linear elliptic operator of order l with $\langle P\xi, \eta\rangle_{L^2(F)} = \langle \xi, P^*\eta\rangle_{L^2(E)}$ whenever $\xi \in C^\infty(E), \eta \in C^\infty(F)$.

Theorem 2.31 *Let (M, g) be a compact Riemannian m-fold, $E, F \longrightarrow M$ be vector bundles over M, and P a smooth, linear elliptic operator of order $l \geq 1$. Let $k \geq 0$ be an integer, $p \geq 1$ and $\alpha \in (0, 1)$. Then the maps $P : L^p_{k+l}(E) \longrightarrow L^p_k(F)$ and $P : C^{k+l,\alpha}(E) \longrightarrow C^{k,\alpha}(F)$ have closed images. If $\eta \in L^p_k(F)$, then there is a solution $\xi \in L^p_{k+l}(E)$ of $P\xi = \eta$ if and only if $\eta \perp \mathrm{Ker}\, P^*$. Similarly, if $\eta \in C^{k,\alpha}(F)$, then there is a solution $\xi \in C^{k+l,\alpha}(E)$ of $P\xi = \eta$ if and only if $\eta \perp \mathrm{Ker}\, P^*$. In both cases, the solution ξ is unique if $\xi \perp \mathrm{Ker}\, P$.*

The proof can be found in [26, Thm. 1.5.3]. Thus we have from Theorem 2.31 the L^2-orthogonal decompositions:

$$L^p_k(F) = P(L^p_{k+l}(E)) \oplus \mathrm{Ker}P^* \quad \text{and} \quad C^{k,\alpha}(F) = P(C^{k+l,\alpha}(E)) \oplus \mathrm{Ker}P^*.$$

Chapter 3

Desingularizations of Calabi–Yau 3-folds with conical singularities

In this chapter we study *Calabi–Yau 3-folds* M_0 *with conical singularities* at a finite number of points $x_1, \ldots, x_n$ modelled on *Calabi–Yau cones* $V_1, \ldots, V_n$. Throughout this thesis we shall assume the existence of such kind of manifolds M_0, or in order words, we shall assume that there exist singular Calabi–Yau metrics on some compact complex manifolds with conical singularities. We then construct desingularizations of M_0, obtaining a 1-parameter family of compact, nonsingular Calabi–Yau 3-folds which has M_0 as the limit. We shall achieve this by first choosing *Asymptotically Conical Calabi–Yau 3-folds* Y_i, for $i = 1, \ldots, n$, where Y_i converges to the Calabi–Yau cone V_i at infinity, and then gluing Y_i into M_0 at x_i after applying a homothety. We thus obtain a 1-parameter family of *nearly Calabi–Yau 3-folds* M_t depending on a small real variable t. For sufficiently small $t > 0$, we show that the nearly Calabi–Yau structures on M_t can be deformed to genuine Calabi–Yau structures, and therefore obtaining the desingularizations of M_0. Our result can be applied to resolving orbifold singularities and hence provides a quantitative description of the Calabi–Yau metrics on the crepant resolutions.

We begin in §3.1 by giving some background material for this chapter. Section 3.2 introduces nearly Calabi–Yau structures on 6-dimensional manifolds and the induced G_2-structures on 7-dimensional manifolds. We also prove the existence result for genuine Calabi–Yau structures, using Joyce's existence result for torsion-free G_2-structures [26, Thm. 11.6.1] with some modifications. Then we define in §3.3 the main objects of this chapter, namely the *Calabi–Yau cones, Calabi–Yau m-folds with conical singularities* and *Asymptotically Conical Calabi–Yau m-folds,* and provide some examples. Finally we show in §3.4 the construction of the desingularization in the simplest case where there are no obstructions. We shall then give an application of our result in §3.4.4 which involves desingularizing Calabi–Yau 3-orbifolds with isolated singularities. Note that in the orbifold case, the existence of singular Calabi–Yau metrics is known. We can then describe what the Calabi–Yau metrics on the crepant resolution of the orbifold locally look like. Our approach in this chapter is analytic and metrics on singular Calabi–Yau 3-folds are considered, rather than just their complex structures. In this way, we provide an analytic way to study results previously known in algebraic geometry.

3.1 Background material on SU(3)- and G_2-structures

In this section we provide some background on Calabi–Yau 3-folds, SU(3)-structures on 6-folds and G_2-structures on 7-folds. They will play an essential role in our construction of desingularizations of compact Calabi–Yau 3-folds with conical singularities. Let us begin with studying Calabi–Yau 3-folds. Some useful introductory references on Calabi–Yau manifolds are [21] and [26, Chapter 6].

Definition 3.1 A *Calabi–Yau 3-fold* is a Kähler manifold (M, J, g) of complex dimension 3 with a covariant constant holomorphic volume form Ω such that it satisfies $\omega^3 = \frac{3i}{4}\,\Omega \wedge \bar{\Omega}$ where ω is the Kähler form of g. We say that (J, ω, Ω) constitutes a *Calabi–Yau structure* on M and write a Calabi–Yau manifold as a quadruple (M, J, ω, Ω).

Thus for each $x \in M$, there is an isomorphism between T_xM and $\mathbb{C}^3$ that identifies g_x, ω_x and Ω_x with the flat metric g_0, the real 2-form ω_0 and the complex 3-form Ω_0 on $\mathbb{C}^3$ given by

$$g_0 = |dz_1|^2 + |dz_2|^2 + |dz_3|^2, \quad \omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$$
$$\text{and} \quad \Omega_0 = dz_1 \wedge dz_2 \wedge dz_3,$$

where (z_1, z_2, z_3) are coordinates on $\mathbb{C}^3$. Calabi–Yau manifolds are automatically Ricci-flat, and one can use Yau's solution of the Calabi conjecture ([52] and [26, Chapter 5]) to show the existence of families of Calabi–Yau manifolds. Another equivalent way of defining a Calabi–Yau 3-fold is to require that the Riemannian 6-fold (M, g) has holonomy group $\mathrm{Hol}(g)$ contained in SU(3). One can then show that M admits a holomorphic volume form satisfying the normalization formula.

We shall now consider SU(3)-structures on 6-folds and G_2-structures on 7-folds and relate them to Calabi–Yau structures. An SU(3)-structure on a real 6-fold M is a principal subbundle of the frame bundle of M, with fibre SU(3). Each SU(3)-structure gives rise to an almost complex structure J, a real 2-form ω and a complex 3-form Ω with the properties that

1. ω is of type (1,1) w.r.t. J and is positive,
2. Ω is of type (3,0) w.r.t. J and is nonvanishing, and
3. $\omega^3 = \frac{3i}{4}\,\Omega \wedge \bar{\Omega}$.

We will refer to (J, ω, Ω) as an SU(3)-structure. If in addition $d\omega = 0$ and $d\Omega = 0$, then J is integrable and Ω is a holomorphic (3,0)-form, and the closedness of ω implies the associated Hermitian metric g is Kähler. Thus $d\omega$ and $d\Omega$ can be thought of as the *torsion* of the SU(3)-structure, and when they both vanish the SU(3)-structure is *torsion-free*. Note that property (3) implies that the holomorphic (3,0)-form Ω has constant length, so it is covariant constant. Therefore we have

Proposition 3.2 *Let M be a real 6-fold and (J, ω, Ω) an SU(3)-structure on M. Let g be the Hermitian metric with Hermitian form ω. Then the following are equivalent:*

(i) $d\omega = 0$ and $d\Omega = 0$ *on M,*

(ii) (J, ω, Ω) *is torsion-free,*

(iii) (J, ω, Ω) *gives a Calabi–Yau structure on M, and*

(iv) $\mathrm{Hol}(g) \subseteq \mathrm{SU}(3)$.

Now we discuss G_2-structures in 7-folds. The books by Salamon [46, §11-§12] and Joyce [26, Chapter 10] are good introductions to G_2. In the theory of Riemannian holonomy groups, one of the exceptional cases in Berger's classification [2] is given by G_2 in 7 dimensions. Bryant and Salamon [8] found explicit, complete metrics with holonomy G_2 on noncompact manifolds, and Joyce [26] constructed examples of compact 7-folds with holonomy G_2. The exceptional Lie group G_2 is the subgroup of GL(7,$\mathbb{R}$) preserving the 3-form

$$\begin{aligned}\varphi_0 = {} & dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_4 \wedge dx_5 + dx_1 \wedge dx_6 \wedge dx_7 + dx_2 \wedge dx_4 \wedge dx_6 \\ & - dx_2 \wedge dx_5 \wedge dx_7 - dx_3 \wedge dx_4 \wedge dx_7 - dx_3 \wedge dx_5 \wedge dx_6\end{aligned}$$

on $\mathbb{R}^7$ with coordinates $(x_1, \ldots, x_7)$. This group also preserves the metric

$$g_0 = dx_1^2 + \cdots + dx_7^2,$$

the 4-form

$$\begin{aligned}*\varphi_0 = {} & dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7 + dx_2 \wedge dx_3 \wedge dx_6 \wedge dx_7 + dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \\ & + dx_1 \wedge dx_3 \wedge dx_5 \wedge dx_7 - dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_6 - dx_1 \wedge dx_2 \wedge dx_5 \wedge dx_6 \\ & - dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_7,\end{aligned}$$

and the orientation on $\mathbb{R}^7$. Let X be an oriented 7-fold. We say that a 3-form φ (a 4-form ψ) on X is *positive* if for each $p \in X$, there exists an oriented isomorphism between T_pX and $\mathbb{R}^7$ identifying φ and the 3-form φ_0 (the 4-form $*\varphi_0$).

A G_2-structure on a 7-fold X is a principal subbundle of the oriented frame bundle of X, with fibre G_2. Thus there is a 1-1 correspondence between positive 3-forms and G_2-structures on X. Moreover, to any positive 3-form on X one can associate a unique positive 4-form $*\varphi$ and metric g, such that φ, $*\varphi$ and g are identified with φ_0, $*\varphi_0$ and g_0 under an isomorphism between T_pX and $\mathbb{R}^7$, for each $p \in M$. We shall refer to (φ, g) as a G_2-structure. Suppose (φ, g) is a G_2-structure on X, and ∇ is the Levi-Civita connection of g. We call $\nabla\varphi$ the *torsion* of the G_2-structure (φ, g), and if $\nabla\varphi = 0$, then the G_2-structure is *torsion-free*. Here is a result from [26, Prop. 10.1.3]:

Proposition 3.3 *Let X be a real 7-fold and (φ, g) a G_2-structure on X. Then the following are equivalent:*

(i) (φ, g) *is torsion-free,*

(ii) $\nabla\varphi = 0$ *on* X*, where* ∇ *is the Levi-Civita connection of* g*,*

(iii) $d\varphi = d^*\varphi = 0$ *on* X*, and*

(iv) $\mathrm{Hol}(g) \subseteq G_2$*, and* φ *is the induced 3-form.*

Now if (M, J, ω, Ω) is a Calabi–Yau 3-fold with the Calabi–Yau metric g_M, then by Proposition 3.2, (J, ω, Ω) gives a torsion-free SU(3)-structure on M, and $\mathrm{Hol}(g_M) \subseteq \mathrm{SU}(3)$. By considering SU(3) as a subgroup of G_2, the 7-fold $S^1 \times M$ has a torsion-free G_2-structure, which is constructed by the following result [26, Prop. 11.1.2]:

Proposition 3.4 *Suppose* (J, ω, Ω) *is a torsion-free SU(3)-structure on a 6-fold* M*. Let* s *be a coordinate on* S^1*. Define a metric* g *and a 3-form* φ *on* $S^1 \times M$ *by*

$$g = ds^2 + g_M \quad \textit{and} \quad \varphi = ds \wedge \omega + \mathrm{Re}(\Omega).$$

Then (φ, g) *is a torsion-free* G_2*-structure on* $S^1 \times M$*, and*

$$*\varphi = \frac{1}{2}\,\omega \wedge \omega - ds \wedge \mathrm{Im}(\Omega).$$

3.2 Nearly Calabi–Yau structures

This section introduces the notion of a *nearly Calabi–Yau structure* on an oriented 6-fold M. We begin in §3.2.1 by giving the definition of a nearly Calabi–Yau structure (ω, Ω) on M, and showing that if M admits a genuine Calabi–Yau structure, then any real closed 2-form ω and complex closed 3-form Ω on M which are sufficiently close to the genuine Calabi–Yau structure gives a nearly Calabi–Yau structure. Section 3.2.2 constructs G_2-structures on the 7-fold $S^1 \times M$. Finally, we give the main result of the section, the existence of genuine Calabi–Yau structures on M, in §3.2.3. It is based on the existence result for torsion-free G_2-structures on compact 7-folds by Joyce [26, Thm. 11.6.1].

3.2.1 Introduction to nearly Calabi–Yau structures

Let M be an oriented 6-fold. A nearly Calabi–Yau structure on M consists of a real closed 2-form ω, and a complex closed 3-form Ω on M. Basically, the idea of a nearly Calabi–Yau structure (ω, Ω) is that it corresponds to an SU(3)-structure with "small torsion", and hence approximates a genuine Calabi–Yau structure, which is equivalent to a torsion-free SU(3)-structure. So let us start with generating an SU(3)-structure on M from (ω, Ω).

First we write $\Omega = \theta_1 + i\theta_2$, so θ_1 and θ_2 are both real closed 3-forms. Suppose θ_1 has stabilizer $\mathrm{SL}(3, \mathbb{C}) \subset \mathrm{GL}_+(6, \mathbb{R})$ at each $p \in M$, then the orbit of θ_1 in $\bigwedge^3 T_p^*M$ under the action of $\mathrm{GL}_+(6, \mathbb{R})$ is $\mathrm{GL}_+(6, \mathbb{R})/\mathrm{SL}(3, \mathbb{C})$. For each $p \in M$, define $\bigwedge^3_+ T_p^*M$ to be the subset of

3-forms $\theta \in \bigwedge^3 T_p^* M$ for which there exists an oriented isomorphism between T_pM and $\mathbb{R}^6 \cong \mathbb{C}^3$ identifying θ and the 3-form $\mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3)$ where (z_1, z_2, z_3) are coordinates on $\mathbb{C}^3$. Then $\bigwedge^3_+ T_p^* M \cong \mathrm{GL}_+(6,\mathbb{R})/\mathrm{SL}(3,\mathbb{C})$, as $\mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3)$ has stabilizer $\mathrm{SL}(3,\mathbb{C})$. Then $\theta_1|_p$ lies in $\bigwedge^3_+ T_p^* M$ for each $p \in M$. It is easy to see that dim $\bigwedge^3_+ T_p^* M =$ dim $\mathrm{GL}_+(6,\mathbb{R})/\mathrm{SL}(3,\mathbb{C}) =$ dim $\bigwedge^3 T_p^* M = 20$, so $\bigwedge^3_+ T_p^* M$ is an open subset of $\bigwedge^3 T_p^* M$ for each $p \in M$. Therefore any 3-form on M which is sufficiently close to a 3-form in $\bigwedge^3_+ T_p^* M$ still lies in $\bigwedge^3_+ T_p^* M$, or equivalently, has stabilizer $\mathrm{SL}(3,\mathbb{C})$ at each point on M.

The oriented frame bundle F_+ of M is the bundle over M whose fibre at $p \in M$ is the set of oriented isomorphisms between T_pM and $\mathbb{R}^6$. Let P be the subset of F_+ consisting of oriented isomorphisms between T_pM and $\mathbb{R}^6$ which identify θ_1 at p and $\mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3)$. It is well-defined as we have assumed that $\theta_1|_p \in \bigwedge^3_+ T_p^* M$. Thus θ_1 defines a principal subbundle P of the oriented frame bundle F_+, with fibre $\mathrm{SL}(3,\mathbb{C})$, that is, an $\mathrm{SL}(3,\mathbb{C})$-structure on M. As $\mathrm{SL}(3,\mathbb{C})$ acts on $\mathbb{R}^6 \cong \mathbb{C}^3$ preserving the complex structure J_0 on $\mathbb{C}^3$, we obtain a unique almost complex structure J' on M.

Note that the forms ω, Ω are not necessarily of type (1,1) and (3,0) with respect to J' respectively. We then denote by $\omega^{(1,1)}$ the (1,1)-component of ω with respect to J' and define a 3-form θ_2' on M by $\theta_2'(u,v,w) := \theta_1(J' u, v, w)$ for all $u, v, w \in TM$, or in index notation, $(\theta_2')_{abc} = (J')_a^d\,(\theta_1)_{dbc}$. Suppose that $\omega^{(1,1)}$ is a positive (1,1)-form, that is, $\omega^{(1,1)}(v, J'v) > 0$ for any nonzero $v \in TM$. Write $\Omega' = \theta_1 + i\theta_2'$, then Ω' is a (3,0)-form with respect to J'. In general, θ_2' will not be a closed 3-form, unless J' is integrable.

We want $(J', \omega^{(1,1)}, \Omega')$ to be an SU(3)-structure, but the problem with this is the usual normalization formula defining a Calabi–Yau manifold may not hold for $\omega^{(1,1)}$ and Ω', that is, $(\omega^{(1,1)})^3 \neq \frac{3i}{4}\,\Omega' \wedge \bar{\Omega}'\ (= \frac{3}{2}\,\theta_1 \wedge \theta_2')$ in general. We then define a smooth function $f : M \to (0, \infty)$ by

$$(\omega^{(1,1)})^3 = f \cdot \frac{3}{2}\,\theta_1 \wedge \theta_2'. \tag{3.1}$$

Consequently, if we rescale $\omega^{(1,1)}$ by setting $\omega' = f^{-\frac{1}{3}}\,\omega^{(1,1)}$, we have $\omega'^3 = \frac{3}{2}\,\theta_1 \wedge \theta_2'$. Given that $\omega^{(1,1)}$, and hence ω', is positive, then one can determine a Hermitian metric g_M on M from ω' and J' by $g_M(u,v) = \omega'(u, J'v)$ for all $u, v \in TM$.

Now we are ready to give the definition of a nearly Calabi–Yau structure on M:

Definition 3.5 Let M be an oriented 6-fold, and let ω be a real closed 2-form, and $\Omega = \theta_1 + i\theta_2$ a complex closed 3-form on M. Let $\epsilon_0 \in (0,1]$ be a fixed small constant, to be chosen later in Lemma 3.7. Then (ω, Ω) constitutes a *nearly Calabi–Yau structure* on M if

(i) the real closed 3-form θ_1 has stabilizer $\mathrm{SL}(3,\mathbb{C})$ at each point of M, or equivalently, it lies in $\bigwedge^3_+ T_p^* M$ for each $p \in M$.
Then we can associate a unique almost complex structure J' and a unique real 3-form θ_2' such that $\Omega' = \theta_1 + i\theta_2'$ is a (3,0)-form with respect to J'.

(ii) the (1,1)-component $\omega^{(1,1)}$ of ω with respect to J' is positive.

Then we can associate a Hermitian metric g_M on M from ω' and J', where $\omega' = f^{-\frac{1}{3}}\,\omega^{(1,1)}$ is the rescaled (1,1)-part of ω and f is defined by (3.1).

(iii) the following inequalities hold for some ϵ with $0 < \epsilon \leq \epsilon_0$:

$$|\theta_2 - \theta_2'|_{g_M} < \epsilon\,, \tag{3.2}$$

$$|\omega^{(2,0)}|_{g_M} < \epsilon\,, \text{and} \tag{3.3}$$

$$\left|\omega^3 - \frac{3}{2}\,\theta_1 \wedge \theta_2\right|_{g_M} < \epsilon \tag{3.4}$$

where the norms $|\cdot|_{g_M}$ are measured by the metric g_M.

If (ω, Ω) is a nearly Calabi–Yau structure on M, one can show that the function f defined in (3.1) satisfies

$$|f-1| < C_0\epsilon, \tag{3.5}$$

for some constant $C_0 > 0$, i.e. f is approximately equal to 1 for sufficiently small ϵ, as we would expect.

The next result shows that if M admits a genuine Calabi–Yau structure, then any real closed 2-form ω and complex closed 3-form Ω on M which are sufficiently close to the genuine Calabi–Yau structure gives a nearly Calabi–Yau structure.

Proposition 3.6 *There exist constants $\epsilon_1, C, C' > 0$ such that whenever $0 < \epsilon \leq \epsilon_1$, the following is true.*

Let M be an oriented 6-fold. Suppose $(\tilde{J}, \tilde{\omega}, \tilde{\Omega})$ is a Calabi–Yau structure with Calabi–Yau metric $\tilde{g}$, ω a real closed 2-form, and $\Omega = \theta_1 + i\theta_2$ a complex closed 3-form on M, satisfying

$$|\tilde{\omega} - \omega|_{\tilde{g}} < \epsilon \quad \textit{and} \quad |\tilde{\Omega} - \Omega|_{\tilde{g}} < \epsilon, \tag{3.6}$$

then (ω, Ω) is a nearly Calabi–Yau structure on M with metric g_M satisfying

$$|\tilde{g} - g_M|_{\tilde{g}} < C\epsilon \quad \textit{and} \quad |\tilde{g}^{-1} - g_M^{-1}|_{\tilde{g}} < C'\epsilon, \tag{3.7}$$

Proof. From (3.6) we have $|\mathrm{Re}(\tilde{\Omega}) - \theta_1|_{\tilde{g}} < \epsilon$, which means that if we choose ϵ_1 to be sufficiently small, then θ_1 has stabilizer SL(3,$\mathbb{C}$) since the stabilizer condition is an open condition as we mentioned before. So we can associate a unique almost complex structure J', with $|\tilde{J} - J'|_{\tilde{g}} < C_1\epsilon$ for some constant $C_1 > 0$, and a unique real 3-form θ_2' such that $\Omega' = \theta_1 + i\theta_2'$ is a (3,0)-form with respect to J'.

One can deduce from (3.6) and $|\tilde{J} - J'|_{\tilde{g}} < C_1\epsilon$ that $|\tilde{\omega} - \omega^{(1,1)}|_{\tilde{g}} < C_2\epsilon$ for some $C_2 > 0$. Make ϵ_1 smaller if necessary, then $\omega^{(1,1)}$ is a positive (1,1)-form with respect to J' since the positivity is also an open condition. Then we can define a metric g_M by $g_M(u,v) = \omega'(u, J'v)$ for all $u, v \in TM$, where $\omega' = f^{-\frac{1}{3}}\,\omega^{(1,1)}$ and f is defined by (3.1). Now we show that f is close to 1. In fact,

$$
\begin{aligned}
|(f-1)\omega'^3|_{\tilde g} &= |(\omega^{(1,1)})^3 - \omega'^3|_{\tilde g} \\
&\le |(\omega^{(1,1)})^3 - \tilde\omega^3|_{\tilde g} + \left|\tilde\omega^3 - \frac{3}{2}\mathrm{Re}(\tilde\Omega)\wedge\mathrm{Im}(\tilde\Omega)\right|_{\tilde g} + \left|\frac{3}{2}\mathrm{Re}(\tilde\Omega)\wedge\mathrm{Im}(\tilde\Omega) - \frac{3}{2}\theta_1\wedge\theta_2'\right|_{\tilde g}.
\end{aligned}
\tag{3.8}
$$

Since $|\tilde\omega - \omega^{(1,1)}|_{\tilde g} < C_2\epsilon$, the first term of right hand side of (3.8) is of size $O(\epsilon)$. The second term vanishes as $(\tilde\omega, \tilde\Omega)$ is a Calabi–Yau structure. From $|\mathrm{Re}(\tilde\Omega) - \theta_1|_{\tilde g} < \epsilon$, and $|\tilde J - J'|_{\tilde g} < C_1\epsilon$, we have $|\mathrm{Im}(\tilde\Omega) - \theta_2'|_{\tilde g} < C_3\epsilon$ for some $C_3 > 0$ and hence the third term also has size $O(\epsilon)$. Summing up, we have $f-1$ is of size $O(\epsilon)$.

Using the fact that $|\tilde\omega - \omega^{(1,1)}|_{\tilde g} < C_2\epsilon$ and $|f-1| = O(\epsilon)$ we can show $|\tilde\omega - \omega'|_{\tilde g} < C_4\epsilon$ for some $C_4 > 0$. Together with $|\tilde J - J'|_{\tilde g} < C_1\epsilon$, we obtain first part of (3.7), that is, $|\tilde g - g_M|_{\tilde g} < C\epsilon$ for some $C > 0$. If ϵ is small enough such that $C\epsilon < \frac{1}{2}$, then one can deduce that $|\tilde g^{-1} - g_M^{-1}|_{\tilde g} < C'\epsilon$ for some $C' > 0$. This implies that $\tilde g$ and g_M are uniformly equivalent metrics, and hence norms of any tensor on M taken with respect to $\tilde g$ and with respect to g_M differ by a bounded factor.

It remains to check (3.2)-(3.4) of Definition 3.5. But it is not hard to get bounds for (3.2)-(3.4) in terms of ϵ with respect to the metric $\tilde g$, and so by making ϵ smaller and using the equivalence of the metrics, we obtain (3.2)-(3.4). $\square$

3.2.2 G_2-structures on $S^1 \times M$

Let (ω, Ω) be a nearly Calabi–Yau structure on M. From §3.2.1 we know that (J', ω', Ω') gives an SU(3)-structure with metric g_M on M. In this section, we would like to discuss G_2-structures on the 7-fold $S^1 \times M$, which is essential for the main result in next section. Let s be a coordinate on S^1. Now define a 3-form φ' and a metric g' on $S^1 \times M$ by

$$\varphi' = ds\wedge\omega' + \theta_1 \quad\text{and}\quad g' = ds^2 + g_M. \tag{3.9}$$

It turns out that (φ', g') defines a G_2-structure (with torsion) on $S^1 \times M$. The associated 4-form $*_{g'}\varphi'$ on $S^1 \times M$ is then given by

$$*_{g'}\varphi' = \frac{1}{2}\,\omega'\wedge\omega' - ds\wedge\theta_2'. \tag{3.10}$$

Also, we can construct another 3-form φ and 4-form χ on $S^1 \times M$ by

$$\varphi = ds\wedge\omega + \theta_1 \quad\text{and}\quad \chi = \frac{1}{2}\,\omega\wedge\omega - ds\wedge\theta_2. \tag{3.11}$$

The next lemma shows that the forms in (3.11) are close to the G_2-forms φ' and $*_{g'}\varphi'$ if we take ϵ_0 in the definition of nearly Calabi–Yau manifolds to be sufficiently small.

Lemma 3.7 *There exist constants C_1, C_2, C_3 and $C_4 > 0$ such that if ϵ_0 in Definition 3.5 is chosen sufficiently small, then the following is true.*

Let (φ', g') be the G_2-structure given by (3.9), $*_{g'}\varphi'$ the associated 4-form given by (3.10), φ the 3-form and χ the 4-form given by (3.11) on $S^1 \times M$. Then

$$|\varphi - \varphi'|_{g'} < C_1\epsilon \tag{3.12}$$

where $\epsilon \in (0, \epsilon_0]$ is the small constant in (iii) of Definition 3.5. Hence φ is also a positive 3-form on $S^1 \times M$, and it defines another G_2-structure (φ, g). Moreover, the associated metric g and the 4-form $*_g\varphi$ satisfy

$$|g - g'|_{g'} < C_2\epsilon, \quad |g^{-1} - g'^{-1}|_{g'} < C_3\,\epsilon \ \text{ and} \tag{3.13}$$

$$|*_g\varphi - \chi|_{g'} < C_4\,\epsilon. \tag{3.14}$$

Proof. From (3.9) and (3.11), we have $|\varphi-\varphi'|_{g'} = |ds\wedge(\omega-\omega')|_{g'} = |\omega-\omega'|_{g_M} < C_1\,\epsilon$ for some constant $C_1 > 0$, where we used (3.3) and (3.5). Now we choose ϵ_0 in Definition 3.5 such that $C_1\,\epsilon_0$ is small enough and φ is a positive 3-form on $S^1 \times M$. Then we can associate a metric g from φ. Using the fact that such associations are continuous we have $|g - g'|_{g'} < C_2\,\epsilon$ for some $C_2 > 0$. Also, by using the same argument as in Proposition 3.6, we obtain $|g^{-1} - g'^{-1}|_{g'} < C_3\,\epsilon$ for some $C_3 > 0$, which shows (3.13). For (3.14), we have

$$\begin{aligned} |*_g\varphi - \chi|_{g'} &\le |*_g\varphi - *_{g'}\varphi'|_{g'} + |*_{g'}\varphi' - \chi|_{g'} \\ &\le |(*_g - *_{g'})\varphi|_{g'} + |*_{g'}\,(\varphi-\varphi')|_{g'} + |*_{g'}\varphi' - \chi|_{g'}. \end{aligned} \tag{3.15}$$

The first term of right hand side has the same size as $|g - g'|_{g'}$, which is bounded in (3.13). The second term is just $|\varphi - \varphi'|_{g'}$ since $*_{g'}$ is an isometry with respect to g', and so it is bounded in (3.12). For the last term, we have

$$|*_{g'}\varphi' - \chi|_{g'} = \left|\frac{1}{2}\,(\omega'\wedge\omega' - \omega\wedge\omega) - ds\wedge(\theta_2' - \theta_2)\right|_{g'}$$

from (3.10) and (3.11). Using $|\omega - \omega'|_{g_M} < C_1\,\epsilon$ and (3.2), we can show that this term has size $O(\epsilon)$. Summing up together, we get $|*_g\varphi - \chi|_{g'} < C_4\,\epsilon$ for some constant $C_4 > 0$. □

3.2.3 An existence result for Calabi–Yau structures on M

In the last part of this section we present our main result which shows that when ϵ_0 is sufficiently small, the nearly Calabi–Yau structure on M can be deformed to a genuine Calabi–Yau structure. The proof is based on an existence result for torsion-free G_2-structures given by Joyce [26, Thm. 11.6.1], which shows using analysis that any G_2-structure on a compact 7-fold with sufficiently small torsion can be deformed to a nearby torsion-free G_2-structure. We shall adopt a slightly modified version of this result, which improves the bounds of various norms to fit into our situation.

Our problem is clearly a 6-dimensional one, but we jump into 7-dimensions to consider G_2-structures. Apparently this seems to complicate the problem, but there is an advantage that we already have the analytic existence theorem from Joyce, which is the key result that makes the entire thing work. Another advantage is that while the Calabi–Yau structure involves J, ω, Ω, the G_2-structure is packaged in a single 3-form φ such that small perturbations of φ still give a G_2-structure. This makes the proofs simpler in 7-dimensions.

We refer to the spaces L^q, L^q_k, C^k and $C^{k,\alpha}$ as the Banach spaces defined in Chapter 2. Let us begin by stating Joyce's result, with improvements to powers of t:

Theorem 3.8 *Let $\kappa > 0$ and $D_1, D_2, D_3 > 0$ be constants. Then there exist constants $\epsilon \in (0, 1]$ and $K > 0$ such that whenever $0 < t \le \epsilon$, the following is true.*

Let X be compact 7-fold, and (φ, g) a G_2-structure on X with $d\varphi = 0$. Suppose ψ is a smooth 3-form on X with $d^*\psi = d^*\varphi$, and

(i) $\|\psi\|_{L^2} \le D_1 t^{\frac{7}{2}+\kappa}$, $\|\psi\|_{C^0} \le D_1 t^{\kappa}$ *and* $\|d^*\psi\|_{L^{14}} \le D_1 t^{-\frac{1}{2}+\kappa}$,

(ii) *the injectivity radius $\delta(g)$ satisfies $\delta(g) \ge D_2 t$, and*

(iii) *the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \le D_3 t^{-2}$.*

Then there exists a smooth, torsion-free G_2-structure $(\tilde{\varphi}, \tilde{g})$ on X such that $\|\tilde{\varphi} - \varphi\|_{C^0} \le K t^{\kappa}$ and $[\tilde{\varphi}] = [\varphi]$ in $H^3(X, \mathbb{R})$.

The proof of it depends upon the following two results. We state them here and then we will improve the powers of t so that Theorem 3.8 can be modified to fit into our situation for the 7-fold $S^1 \times M$.

Theorem 3.9 *Let $D_2, D_3, t > 0$ be constants, and suppose (X, g) is a complete Riemannian 7-fold, whose injectivity radius $\delta(g)$ and Riemann curvature $R(g)$ satisfy $\delta(g) \ge D_2 t$ and $\|R(g)\|_{C^0} \le D_3 t^{-2}$. Then there exist $K_1, K_2 > 0$ depending only on D_2 and D_3, such that if $\chi \in L^{14}_1(\Lambda^3 T^* X) \cap L^2(\Lambda^3 T^* X)$ then*

$$\|\nabla\chi\|_{L^{14}} \le K_1 \left(\|d\chi\|_{L^{14}} + \|d^*\chi\|_{L^{14}} + t^{-4}\|\chi\|_{L^2}\right)$$
$$\text{and} \quad \|\chi\|_{C^0} \le K_2 \left(t^{\frac{1}{2}}\|\nabla\chi\|_{L^{14}} + t^{-\frac{7}{2}}\|\chi\|_{L^2}\right).$$

The second result is:

Theorem 3.10 *Let $\kappa > 0$ and $D_1, K_1, K_2 > 0$ be constants. Then there exist constants $\epsilon \in (0, 1]$, K_3 and $K > 0$ such that whenever $0 < t \le \epsilon$, the following is true.*

Let X be a compact 7-fold, and (φ, g) a G_2-structure on X with $d\varphi = 0$. Suppose ψ is a smooth 3-form on X with $d^*\psi = d^*\varphi$, and

(i) $\|\psi\|_{L^2} \le D_1 t^{\frac{7}{2}+\kappa}$, $\|\psi\|_{C^0} \le D_1 t^{\kappa}$ *and* $\|d^*\psi\|_{L^{14}} \le D_1 t^{-\frac{1}{2}+\kappa}$,

(ii) *if* $\chi \in L^{14}_1(\Lambda^3 T^* X)$ *then* $\|\nabla\chi\|_{L^{14}} \le K_1 \left(\|d\chi\|_{L^{14}} + \|d^*\chi\|_{L^{14}} + t^{-4}\|\chi\|_{L^2}\right)$,

(iii) *if* $\chi \in L^{14}_1(\Lambda^3 T^* X)$ *then* $\|\chi\|_{C^0} \le K_2 \left(t^{\frac{1}{2}}\|\nabla\chi\|_{L^{14}} + t^{-\frac{7}{2}}\|\chi\|_{L^2}\right)$.

Let ϵ_1 be as in Definition 10.3.3, and F as in Proposition 10.3.5 of Joyce [26]. Denote by π_1 the orthogonal projection from $\Lambda^3 T^*X$ to the 1-dimensional component of the decomposition into irreducible representation of G_2. Then there exist sequences $\{\eta_j\}_{j=0}^{\infty}$ in $L_2^{14}(\Lambda^2 T^*X)$ and $\{f_j\}_{j=0}^{\infty}$ in $L_1^{14}(X)$ with $\eta_0 = f_0 = 0$, satisfying the equations

$$(dd^* + d^*d)\eta_j = d^*\psi + d^*(f_{j-1}\psi) + *dF(d\eta_{j-1}) \text{ and } f_j\varphi = \frac{7}{3}\pi_1(d\eta_j)$$

for each $j > 0$, and the inequalities

(a) $\|d\eta_j\|_{L^2} \leq 2D_1 t^{\frac{7}{2}+\kappa}$, (d) $\|d\eta_j - d\eta_{j-1}\|_{L^2} \leq 2D_1 2^{-j} t^{\frac{7}{2}+\kappa}$,

(b) $\|\nabla d\eta_j\|_{L^{14}} \leq K_3 t^{-\frac{1}{2}+\kappa}$, (e) $\|\nabla(d\eta_j - d\eta_{j-1})\|_{L^{14}} \leq K_3 2^{-j} t^{-\frac{1}{2}+\kappa}$,

(c) $\|d\eta_j\|_{C^0} \leq K t^{\kappa} \leq \epsilon_1$ *and* (f) $\|d\eta_j - d\eta_{j-1}\|_{C^0} \leq K 2^{-j} t^{\kappa}$.

We shall first modify Theorem 3.9 by considering the 6-dimensional version of those analytic estimates. In Theorem 3.9, the first inequality is derived from an elliptic regularity estimate for the operator $d + d^*$ on 3-forms on X. The second inequality follows from the Sobolev Embedding Theorem, which states that L_k^q embeds in $C^{l,\alpha}$ if $\frac{1}{q} \leq \frac{k-l-\alpha}{n}$ where n is the dimension of the underlying Riemannian manifold. For the 7-dimensional case, we have L_1^{14} embeds in $C^{0,1/2}$ which then embeds in C^0, whereas in 6 dimensions, we have L_1^{12} embeds in $C^{0,1/2}$. We can use this to show

Theorem 3.11 *Let $D_2, D_3, t > 0$ be constants, and suppose (M, g) is a complete Riemannian 6-fold, whose injectivity radius $\delta(g)$ and Riemann curvature $R(g)$ satisfy $\delta(g) \geq D_2 t$ and $\|R(g)\|_{C^0} \leq D_3 t^{-2}$. Then there exist $K_1, K_2 > 0$ depending only on D_2 and D_3, such that if $\chi \in L_1^{12}(\Lambda^3 T^*M) \cap L^2(\Lambda^3 T^*M)$ then*

$$\|\nabla\chi\|_{L^{12}} \leq K_1 \left(\|d\chi\|_{L^{12}} + \|d^*\chi\|_{L^{12}} + t^{-\frac{7}{2}}\|\chi\|_{L^2}\right)$$

$$\text{and} \quad \|\chi\|_{C^0} \leq K_2 \left(t^{\frac{1}{2}}\|\nabla\chi\|_{L^{12}} + t^{-3}\|\chi\|_{L^2}\right).$$

The proof of it is similar to [26, Thm. G1, p.298]. We can first prove the case for $t = 1$, and the case for general $t > 0$ follows by conformal rescaling: apply the $t = 1$ case to the metric $t^{-2}g$. The factors of t compensate for powers of t which the norms scaled by in replacing g by $t^{-2}g$.

By considering S^1-invariant forms and S^1-invariant G_2-structures on the 7-fold $S^1 \times M$, we can use the Sobolev Embedding Theorem in 6 dimensions, rather than in 7 dimensions. This has an advantage that the powers of t and the inequalities are calculated in 6 dimensions, though the norms are computed on the 7-fold $S^1 \times M$. Note that the length of S^1 is fixed, i.e. independent of t while the metric on M is rescaled by t. Here is the modified version of Theorem 3.10:

Theorem 3.12 *Let $\kappa > 0$ and $D_1, K_1, K_2 > 0$ be constants. Then there exist constants $\epsilon \in (0, 1]$, K_3 and $K > 0$ such that whenever $0 < t \leq \epsilon$, the following is true.*

Let M be a compact 6-fold, and (φ, g) an S^1-invariant G_2-structure on $S^1 \times M$ with $d\varphi = 0$. Suppose ψ is an S^1-invariant smooth 3-form on the 7-fold $S^1 \times M$ with $d^*\psi = d^*\varphi$, and

(i) $\|\psi\|_{L^2} \leq D_1 t^{3+\kappa}$, $\|\psi\|_{C^0} \leq D_1 t^{\kappa}$ *and* $\|d^*\psi\|_{L^{12}} \leq D_1 t^{-\frac{1}{2}+\kappa}$,

(ii) *if* $\chi \in L^{12}_1(\Lambda^3 T^*(S^1 \times M))$ *is* S^1*-invariant, then* $\|\nabla\chi\|_{L^{12}} \leq K_1\,(\|d\chi\|_{L^{12}} + \|d^*\chi\|_{L^{12}} + t^{-\frac{7}{2}}\|\chi\|_{L^2})$,

(iii) *if* $\chi \in L^{12}_1(\Lambda^3 T^*(S^1 \times M))$ *is* S^1*-invariant, then* $\|\chi\|_{C^0} \leq K_2\,(t^{\frac{1}{2}}\|\nabla\chi\|_{L^{12}} + t^{-3}\|\chi\|_{L^2})$.

With the same notations as in Theorem 3.10, there exist sequences $\{\eta_j\}_{j=0}^{\infty}$ *in* $L^{12}_2(\Lambda^2 T^*(S^1 \times M))$ *and* $\{f_j\}_{j=0}^{\infty}$ *in* $L^{12}_1(S^1 \times M)$ *with* η_j, f_j *being all* S^1*-invariant and* $\eta_0 = f_0 = 0$*, satisfying the equations*

$$(dd^* + d^*d)\eta_j = d^*\psi + d^*(f_{j-1}\psi) + *\,dF(d\eta_{j-1}) \text{ and } f_j\varphi = \frac{7}{3}\pi_1(d\eta_j)$$

for each $j > 0$*, and the inequalities*

(a) $\|d\eta_j\|_{L^2} \leq 2D_1 t^{3+\kappa}$,
(b) $\|\nabla d\eta_j\|_{L^{12}} \leq K_3 t^{-\frac{1}{2}+\kappa}$,
(c) $\|d\eta_j\|_{C^0} \leq K t^{\kappa} \leq \epsilon_1$ *and*
(d) $\|d\eta_j - d\eta_{j-1}\|_{L^2} \leq 2D_1 2^{-j} t^{3+\kappa}$,
(e) $\|\nabla(d\eta_j - d\eta_{j-1})\|_{L^{12}} \leq K_3 2^{-j} t^{-\frac{1}{2}+\kappa}$,
(f) $\|d\eta_j - d\eta_{j-1}\|_{C^0} \leq K 2^{-j} t^{\kappa}$.

Here ∇ *and* $\|\cdot\|$ *are computed using* g *on* $S^1 \times M$.

Thus Theorem 3.12 is essentially an S^1-invariant version of Theorem 3.10. We are now ready to state the modified version of Theorem 3.8, to be used to obtain our main result.

Theorem 3.13 *Let* $\kappa > 0$ *and* $D_1, D_2, D_3 > 0$ *be constants. Then there exist constants* $\epsilon \in (0, 1]$ *and* $K > 0$ *such that whenever* $0 < t \leq \epsilon$*, the following is true.*

Let M *be a compact 6-fold, and* (φ, g) *an* S^1*-invariant* G_2*-structure on* $S^1 \times M$ *with* $d\varphi = 0$*. Suppose* ψ *is an* S^1*-invariant smooth 3-form on* $S^1 \times M$ *with* $d^*\psi = d^*\varphi$*, and*

(i) $\|\psi\|_{L^2} \leq D_1 t^{3+\kappa}$, $\|\psi\|_{C^0} \leq D_1 t^{\kappa}$ *and* $\|d^*\psi\|_{L^{12}} \leq D_1 t^{-\frac{1}{2}+\kappa}$,

(ii) *the injectivity radius* $\delta(g)$ *satisfies* $\delta(g) \geq D_2 t$*, and*

(iii) *the Riemann curvature* $R(g)$ *satisfies* $\|R(g)\|_{C^0} \leq D_3 t^{-2}$.

Then there exists a smooth, torsion-free S^1*-invariant* G_2*-structure* $(\tilde{\varphi}, \tilde{g})$ *on* $S^1 \times M$ *such that* $\|\tilde{\varphi} - \varphi\|_{C^0} \leq K t^{\kappa}$ *and* $[\tilde{\varphi}] = [\varphi]$ *in* $H^3(S^1 \times M, \mathbb{R})$.

Theorem 3.13 follows from Theorems 3.11 and 3.12 as on [26, p.296-297]. Note that since all the data η_j, f_j, etc. in Joyce's proof are S^1-invariant, so the limit of sequence is still S^1-invariant as it is unique, and hence the limiting G_2-structure obtained is also S^1-invariant.

In the remaining part of this section, we shall derive an existence result for genuine Calabi–Yau structures. Our strategy is the following. We start with the nearly Calabi–Yau structure (ω, Ω) on M, then from §3.2.2 one can induce a G_2-structure (φ, g) on $S^1 \times M$. It can then be shown in the following theorem that, under appropriate hypotheses on the nearly Calabi–Yau structure (ω, Ω), the induced G_2-structure satisfies all the conditions in Theorem 3.13, and therefore can be deformed to have zero torsion. Finally, we pull back this torsion-free G_2-structure to obtain a genuine Calabi–Yau structure on M.

Theorem 3.14 *Let $\kappa > 0$ and $E_1, E_2, E_3, E_4 > 0$ be constants. Then there exist constants $\epsilon \in (0, 1]$ and $K > 0$ such that whenever $0 < t \leq \epsilon$, the following is true.*

Let M be a compact 6-fold, and (ω, Ω) a nearly Calabi–Yau structure on M. Let ω', g_M and θ_2' be as in §3.2.1. Suppose

(i) $\|\omega - \omega'\|_{L^2} \leq E_1 t^{3+\kappa}$, $\|\omega - \omega'\|_{C^0} \leq E_1 t^{\kappa}$, $\|\theta_2 - \theta_2'\|_{L^2} \leq E_1 t^{3+\kappa}$, *and* $\|\theta_2 - \theta_2'\|_{C^0} \leq E_1 t^{\kappa}$,

(ii) $\|\nabla(\omega - \omega')\|_{L^{12}} \leq E_1 t^{-\frac{1}{2}+\kappa}$, $\|\nabla\omega\|_{L^{12}} \leq E_1 t^{-\frac{1}{2}+\kappa}$, *and* $\|\nabla\theta_1\|_{L^{12}} \leq E_1 t^{-\frac{1}{2}+\kappa}$,

(iii) $\|\nabla(\omega - \omega')\|_{C^0} \leq E_2 t^{\kappa-1}$, *and* $\|\nabla^2(\omega - \omega')\|_{C^0} \leq E_2 t^{\kappa-2}$,

(iv) *the injectivity radius $\delta(g_M)$ satisfies $\delta(g_M) \geq E_3 t$, and*

(v) *the Riemann curvature $R(g_M)$ satisfies $\|R(g_M)\|_{C^0} \leq E_4 t^{-2}$.*

Then there exists a Calabi–Yau structure $(\tilde{J}, \tilde{\omega}, \tilde{\Omega})$ on M such that $\|\tilde{\omega} - \omega\|_{C^0} \leq K t^{\kappa}$ and $\|\tilde{\Omega} - \Omega\|_{C^0} \leq K t^{\kappa}$. Moreover, if $H^1(M, \mathbb{R}) = 0$, then the cohomology classes satisfy $[\mathrm{Re}(\Omega)] = [\mathrm{Re}(\tilde{\Omega})] \in H^3(M, \mathbb{R})$ and $[\omega] = c\,[\tilde{\omega}] \in H^2(M, \mathbb{R})$ for some $c > 0$. Here the connection ∇ and all norms are computed with respect to g_M.

Proof. Let φ be the 3-form on $S^1 \times M$ given by (3.11). Then Lemma 3.7 shows that (φ, g) is a G_2-structure on $S^1 \times M$, with $d\varphi = 0$ as $d\omega = d\theta_1 = 0$. Define a 3-form $\psi = \varphi - *_g\chi$ on $S^1 \times M$, where χ is the 4-form given by (3.11). Then $d^*\psi = d^*\varphi - d^*(*_g\chi) = d^*\varphi + *_g d\chi = d^*\varphi$, since $d^* *_g = - *_g d$ on 3-forms and $d\chi = 0$. Now

$$|\psi|_g = |*_g\psi|_g = |*_g\varphi - \chi|_g \leq C\,|*_g\varphi - \chi|_{g'}$$

where $C > 0$ is some constant relating norms w.r.t. the uniformly equivalent metrics g and $g' = ds^2 + g_M$. From (3.15), one can show that $|*_g\varphi - \chi|_{g'} \leq C_1(|\omega - \omega'|^2_{g_M} + |\omega - \omega'|_{g_M} + |\theta_2 - \theta_2'|_{g_M})$ for some $C_1 > 0$, and hence

$$|\psi|_g \leq C_2\,(\,|\omega - \omega'|^2_{g_M} + |\omega - \omega'|_{g_M} + |\theta_2 - \theta_2'|_{g_M})$$

for some $C_2 > 0$. Consequently, we have

$$\begin{aligned} &\|\psi\|_{C^0} \leq C_2\,(\,\|\omega - \omega'\|^2_{C^0} + \|\omega - \omega'\|_{C^0} + \|\theta_2 - \theta_2'\|_{C^0}), \quad \text{and} \\ &\|\psi\|_{L^2} \leq C_3\,(\,\|\omega - \omega'\|_{C^0} \cdot \|\omega - \omega'\|_{L^2} + \|\omega - \omega'\|_{L^2} + \|\theta_2 - \theta_2'\|_{L^2}) \end{aligned}$$

for some $C_3 > 0$, which then imply

$$\|\psi\|_{C^0} \leq C_4 t^{\kappa} \quad \text{and} \quad \|\psi\|_{L^2} \leq C_4 t^{3+\kappa} \quad \text{for some } C_4 > 0, \tag{3.16}$$

where we have used condition (i) and $t \leq \epsilon \leq 1$. This verifies the first two inequalities of (i) in Theorem 3.13, and we now proceed to the last one. Denote by ∇^g and $\nabla^{g'}$ the connections computed using g and g' respectively. Since $d^*\psi = d^*\varphi$, we get

$$|d^*\psi|_g = |d^*\varphi|_g \leq |\nabla^g\varphi|_g \leq C\,|\nabla^g\varphi|_{g'}. \tag{3.17}$$

Denote by A the difference of the two torsion-free connections ∇^g and $\nabla^{g'}$. Thus A transforms as a tensor and it satisfies $A^k_{ij} = A^k_{ji}$. We need the following proposition to obtain the bound for $|d^*\psi|_g$.

Proposition 3.15 *In the situation above, we have*

$$|\nabla^{g'}\varphi|_{g'} \leq |\nabla\omega|_{g_M} + |\nabla\theta_1|_{g_M}, \tag{3.18}$$

$$|\nabla^{g'}\varphi'|_{g'} \leq |\nabla(\omega-\omega')|_{g_M} + |\nabla\omega|_{g_M} + |\nabla\theta_1|_{g_M}, \tag{3.19}$$

$$|A|_{g'} \leq \frac{3}{2}\,|g^{-1}|_{g'}\cdot|\nabla^{g'}g|_{g'}, \tag{3.20}$$

$$\text{and}\quad |\nabla^{g'}g|_{g'} \leq B_1\,|\nabla^{g'}(\varphi-\varphi')|_{g'} + B_2\,|\varphi-\varphi'|_{g'}\cdot\big(|\nabla^{g'}\varphi|_{g'} + |\nabla^{g'}\varphi'|_{g'}\big) \tag{3.21}$$

for some $B_1, B_2 > 0$ depending on a small upper bound for $|\varphi-\varphi'|_{g'}$.

Proof. For the first one, note that

$$\begin{aligned}|\nabla^{g'}\varphi|_{g'} &= |\nabla^{g'}(ds\wedge\omega+\theta_1)|_{g'}\\ &\leq |\nabla^{g'}ds|_{g'}\cdot|\omega|_{g'} + |ds|_{g'}\cdot|\nabla^{g'}\omega|_{g'} + |\nabla^{g'}\theta_1|_{g'}.\end{aligned}$$

Since ds is a constant 1-form with length 1 w.r.t. the metric g', equation (3.18) follows. The second inequality follows easily from the first one. For (3.20), we have from the definition of the tensor A,

$$A^k_{ij} = \Gamma^k_{ij} - \Gamma'^k_{ij}$$

where Γ^k_{ij} and Γ'^k_{ij} are the Christoffel symbols of the Levi-Civita connections ∇^g and $\nabla^{g'}$ respectively. Consider now the term $\nabla^{g'}g$, and expressing it in index notation,

$$\nabla^{g'}_a g_{bc} = \partial_a g_{bc} - \Gamma'^d_{ab}g_{dc} - \Gamma'^d_{ac}g_{bd}.$$

Then

$$\begin{aligned}\Gamma^k_{ij} &= \frac{1}{2}\,g^{kl}\big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\big)\\ &= \frac{1}{2}g^{kl}\big[\big(\nabla^{g'}_i g_{jl} + \Gamma'^m_{ij}g_{ml} + \Gamma'^m_{il}g_{jm}\big) + \big(\nabla^{g'}_j g_{il} + \Gamma'^m_{ji}g_{ml} + \Gamma'^m_{jl}g_{im}\big)\\ &\qquad - \big(\nabla^{g'}_l g_{ij} + \Gamma'^m_{li}g_{mj} + \Gamma'^m_{lj}g_{im}\big)\big]\\ &= \frac{1}{2}\,g^{kl}\big(\nabla^{g'}_i g_{jl} + \nabla^{g'}_j g_{il} - \nabla^{g'}_l g_{ij} + 2\,\Gamma'^m_{ij}g_{ml}\big)\end{aligned}$$

by the fact that $\nabla^{g'}$ is torsion-free. Hence,

$$\begin{aligned}A^k_{ij} &= \Gamma^k_{ij} - \Gamma'^k_{ij}\\ &= \frac{1}{2}\,g^{kl}\big(\nabla^{g'}_i g_{jl} + \nabla^{g'}_j g_{il} - \nabla^{g'}_l g_{ij}\big),\end{aligned}$$

which gives rise to (3.20).

To verify the last inequality, first let F be the smooth function that maps each positive 3-form to its associated metric. Then $F(\varphi) = g$ and $F(\varphi') = g'$. As $F(\varphi)$ depends pointwise on φ, we can write $F(\varphi)(x) = F(x, \varphi(x))$ for all $x \in S^1 \times M$ and $\varphi(x)$ in the vector space $\bigwedge^3 T_x^*(S^1 \times M)$. We may then take partial derivative in the $\varphi(x)$ direction without using a connection, and write ∂ for the partial derivative in this direction. Now,

$$\begin{aligned}
|\nabla^{g'} g|_{g'} &= |\nabla^{g'}(g - g')|_{g'} \\
&= |\nabla^{g'}(F(\varphi) - F(\varphi'))|_{g'} \\
&= \Big| \int_0^1 \frac{d}{dr} \nabla^{g'} \big(F(x, \varphi'(x) + r(\varphi(x) - \varphi'(x)))\big)\, dr \Big|_{g'} \\
&= \Big| \int_0^1 \nabla^{g'} \big(\frac{d}{dr} F(x, \varphi'(x) + r(\varphi(x) - \varphi'(x)))\big)\, dr \Big|_{g'} \\
&= \Big| \int_0^1 \nabla^{g'} \big(\partial F(x, \varphi'(x) + r(\varphi(x) - \varphi'(x))) \cdot (\varphi(x) - \varphi'(x))\big)\, dr \Big|_{g'} \\
&= \Big| \int_0^1 \big[(\nabla^{g'} \partial F)(x, \varphi'(x) + r(\varphi(x) - \varphi'(x))) + \partial^2 F(x, \varphi'(x) + r(\varphi(x) - \varphi'(x))) \\
&\qquad \cdot \nabla^{g'}(\varphi'(x) + r(\varphi(x) - \varphi'(x)))\big] \cdot (\varphi(x) - \varphi'(x)) + \partial F(x, \varphi'(x) + r(\varphi(x) - \varphi'(x))) \\
&\qquad \cdot \nabla^{g'}(\varphi(x) - \varphi'(x))\, dr \Big|_{g'}.
\end{aligned}$$

It can be shown, by using the fact that continuous functions are bounded over compact spaces, that for any ϕ which is close enough to φ', we have

$$|\partial F(x, \phi(x))|_{g'} \leq B_1 \quad \text{and} \quad |\partial^2 F(x, \phi(x))|_{g'} \leq 2B_2$$

for some constants $B_1, B_2 > 0$, and as this is a calculation at a point, B_1, B_2 are constants depend only on a small upper bound for $|\varphi - \varphi'|_{g'}$. Moreover, if we choose geodesic normal coordinates at x, then the Christoffel symbols Γ'^k_{ij} of $\nabla^{g'}$ vanish at x, so $\nabla^{g'}$ reduces to the usual partial differentiation at x and it follows that

$$\nabla^{g'} \partial F(x, \varphi'(x) + r(\varphi(x) - \varphi'(x))) = 0$$

since F, and hence ∂F is invariant under translation along the directions of coordinate vectors. Consequently,

$$\begin{aligned}
|\nabla^{g'} g|_{g'} &\leq B_1 |\nabla^{g'}(\varphi(x) - \varphi'(x))|_{g'} \\
&\qquad + 2\, B_2\, |\varphi(x) - \varphi'(x)|_{g'} \cdot \int_0^1 |\nabla^{g'}(\varphi'(x) + r(\varphi(x) - \varphi'(x)))|_{g'}\, dr \\
&\leq B_1 |\nabla^{g'}(\varphi(x) - \varphi'(x))|_{g'} + B_2\, |\varphi(x) - \varphi'(x)|_{g'} \cdot \big(|\nabla^{g'} \varphi|_{g'} + |\nabla^{g'} \varphi'|_{g'}\big)
\end{aligned}$$

and this finishes the proof of Proposition 3.15. $\square$

Applying the above estimates to (3.17) shows that

$$\begin{aligned}
|d^*\psi|_g &\le C|\nabla^g\varphi|_{g'} \\
&\le C|\nabla^{g'}\varphi|_{g'} + C|A|_{g'}\cdot|\varphi|_{g'} \\
&\le C|\nabla\omega|_{g_M} + C|\nabla\theta_1|_{g_M} + \frac{3}{2}C\ |g^{-1}|_{g'}\cdot|\nabla^{g'}g|_{g'}\cdot|\varphi|_{g'} \quad \text{by (3.18) and (3.20)} \\
&\le C|\nabla\omega|_{g_M} + C|\nabla\theta_1|_{g_M} + C_5\big(B_1\,|\nabla^{g'}(\varphi-\varphi')|_{g'} + B_2\,|\varphi-\varphi'|_{g'}\cdot(|\nabla^{g'}\varphi|_{g'} + |\nabla^{g'}\varphi'|_{g'})\big) \\
&\qquad \text{by (3.21) and the fact that } g^{-1} \text{ and } \varphi \text{ are bounded by some constants w.r.t. } g' \\
&\le C|\nabla\omega|_{g_M} + C|\nabla\theta_1|_{g_M} + C_6|\nabla(\omega-\omega')|_{g_M} + C_7|\omega-\omega'|_{g_M}\cdot(|\nabla(\omega-\omega')|_{g_M} + 2|\nabla\omega|_{g_M} \\
&\qquad + 2|\nabla\theta_1|_{g_M}) \quad \text{by (3.19)},
\end{aligned}$$

where $C_5, C_6, C_7 > 0$ are some constants. From the second inequality of (i), we have $\|\omega-\omega'\|_{C^0} \le E_1 t^\kappa \le E_1$ as $t \le \epsilon \le 1$. Combining with condition (ii) shows that

$$\begin{aligned}
\|d^*\psi\|_{L^{12}} &\le C\|\nabla\omega\|_{L^{12}} + C\|\nabla\theta_1\|_{L^{12}} + C_6\|\nabla(\omega-\omega')\|_{L^{12}} \\
&\qquad + C_7\|\omega-\omega'\|_{C^0}\cdot(\|\nabla(\omega-\omega')\|_{L^{12}} + 2\|\nabla\omega\|_{L^{12}} + 2\|\nabla\theta_1\|_{L^{12}}) \\
&\le (CE_1 + CE_1 + C_6E_1 + C_7E_1\,(E_1 + 2E_1 + 2E_1))t^{-\frac{1}{2}+\kappa}.
\end{aligned}$$

Thus, together with (3.16), we have verified condition (i) of Theorem 3.13.

Given (iv) and (v), the injectivity radius and the Riemann curvature of $g' = ds^2 + g_M$ satisfy $\delta(g') \ge \min(\mu t, \pi) = \mu t$ for small t, and $\|R(g')\|_{C^0} \le \nu t^{-2}$ for some μ, $\nu > 0$. This is equivalent to saying that $\delta(t^{-2}g') \ge \mu$ and $\|R(t^{-2}g')\|_{C^0} \le \nu$. To verify (ii) and (iii) of Theorem 3.13, it is enough to show that the metrics $t^{-2}g$ and $t^{-2}g'$ are C^2-close w.r.t. $t^{-2}g'$, since this would imply $\delta(t^{-2}g) \ge \tilde\mu$ and $\|R(t^{-2}g)\|_{C^0} \le \tilde\nu$ for some $\tilde\mu, \tilde\nu > 0$, which is what we need. But the second inequality of condition (i) and condition (iii) in the hypotheses ensure that $\|g-g'\|_{C^0}, t\|\nabla(g-g')\|_{C^0}$ and $t^2\|\nabla^2(g-g')\|_{C^0}$ are all of size $O(t^\kappa)$, where the connection ∇ and all norms are computed using g'. It follows that $\|t^{-2}g - t^{-2}g'\|_{C^0}$, $\|\nabla(t^{-2}g - t^{-2}g')\|_{C^0}$ and $\|\nabla^2(t^{-2}g - t^{-2}g')\|_{C^0}$ are all of the same size $O(t^\kappa)$, where the connection ∇ and all norms are computed with respect to $t^{-2}g'$ this time. Hence $t^{-2}g$ is C^2-close to $t^{-2}g'$ w.r.t. $t^{-2}g'$.

Therefore Theorem 3.13 gives a torsion-free G_2-structure $(\tilde\varphi, \tilde g)$ on $S^1 \times M$. It remains to construct a Calabi–Yau structure on M from $(\tilde\varphi, \tilde g)$. Denote $\frac{\partial}{\partial s}$ by the Killing vector w.r.t. g such that $\iota\left(\frac{\partial}{\partial s}\right) ds = 1$. Then $\frac{\partial}{\partial s}$ is also a Killing vector w.r.t. $\tilde g$ since $(\tilde\varphi, \tilde g)$ is S^1-invariant. Using the fact that Killing vectors on a torsion-free compact G_2-manifold are covariant constant (this can be shown by applying Lemma 10.2.5 of [26] and the Bochner argument on the 2-form $\iota(v)\tilde\varphi$ where v is a Killing vector), we have $\nabla^{\tilde g}\frac{\partial}{\partial s} = 0$, and hence $\left|\frac{\partial}{\partial s}\right|_{\tilde g}$ equals to some constant c. Define a 1-form $d\tilde s$ on $S^1 \times M$ by $(d\tilde s)_a = \frac{1}{c}\,\tilde g_{ab}\big(\frac{\partial}{\partial s}\big)^b$. Then $d\tilde s$ is closed, of unit length w.r.t. $\tilde g$, and $\iota\left(\frac{\partial}{\partial s}\right) d\tilde s = c$, and we may write $d\tilde s = c\,ds + \alpha'$ for some closed 1-form α' on $S^1 \times M$ with $\iota\left(\frac{\partial}{\partial s}\right)\alpha' = 0$. The 1-form α' is thus the pullback of some closed 1-form α on M via the projection map $\pi : S^1 \times M \longrightarrow M$, i.e. $\alpha' = \pi^*(\alpha)$. Since by assumption $H^1(M, \mathbb{R}) = 0$, α, and hence α', is exact. Therefore we have $[d\tilde s] = c\,[ds]$.

Using the fact that $\frac{\partial}{\partial s}$ is a Killing vector and $\tilde\varphi$ is a closed 3-form, we have

$$d\Big(\iota\Big(\frac{\partial}{\partial s}\Big)\tilde\varphi\Big) = \mathcal{L}_{\frac{\partial}{\partial s}}\tilde\varphi = 0,$$

so $\iota\left(\frac{\partial}{\partial s}\right)\tilde{\varphi}$, and similarly $\iota\left(\frac{\partial}{\partial s}\right)(*_{\tilde{g}}\tilde{\varphi})$ are both S^1-invariant closed forms on $S^1 \times M$. Then we can define a closed 2-form $\tilde{\omega}$ and closed 3-forms $\tilde{\theta_1}$ and $\tilde{\theta_2}$ on M by

$$\tilde{\omega}_x = \frac{1}{c}\left(\iota\left(\frac{\partial}{\partial s}\right)\tilde{\varphi}\right)_{(s,x)}\Big|_{T_xM}, \quad (\tilde{\theta_1})_x = \tilde{\varphi}_{(s,x)} - (d\tilde{s}\wedge\tilde{\omega})_{(s,x)}|_{T_xM},$$

$$\text{and} \quad (\tilde{\theta_2})_x = -\frac{1}{c}\left(\iota\left(\frac{\partial}{\partial s}\right)(*_{\tilde{g}}\,\tilde{\varphi})\right)_{(s,x)}\Big|_{T_xM}$$

for each $x \in M$ and any $s \in S^1$. Identify $T_{(s,x)}(S^1 \times M)$ with $\mathbb{R}^7 \cong \mathbb{R}\oplus\mathbb{C}^3$ such that T_xM is identified with $\mathbb{C}^3$, $(\tilde{\varphi},\tilde{g})$ with the flat G_2-structure (φ_0, g_0), $\frac{1}{c}(\frac{\partial}{\partial s})$ with $\frac{\partial}{\partial x_1}$, and $d\tilde{s}$ with dx_1 where x_1 is the coordinate on $\mathbb{R}$. Then calculation shows that at each $x \in M$, $(\tilde{J},\tilde{\omega},\tilde{\Omega})$ can be identified with the standard Calabi–Yau structure (J_0,ω_0,Ω_0) on $\mathbb{C}^3$, where $\tilde{\Omega} = \tilde{\theta_1} + i\tilde{\theta_2}$, and $\tilde{J}$ is the associated complex structure. It follows that $(\tilde{J},\tilde{\omega},\tilde{\Omega})$ gives a Calabi–Yau structure on M with $\tilde{\varphi} = d\tilde{s}\wedge\tilde{\omega} + \tilde{\theta_1}$ and $*_{\tilde{g}}\tilde{\varphi} = \frac{1}{2}\tilde{\omega}\wedge\tilde{\omega} - d\tilde{s}\wedge\tilde{\theta_2}$ on $S^1 \times M$.

It is not hard to show $\|\tilde{\omega}-\omega\|_{C^0} \leq Kt^{\kappa}$ and, by making K larger if necessary, $\|\tilde{\Omega}-\Omega\|_{C^0} \leq Kt^{\kappa}$ provided that $\|\tilde{\varphi}-\varphi\|_{C^0} \leq Kt^{\kappa}$, which is a consequence from Theorem 3.13. Moreover, as $[\tilde{\varphi}] = [\varphi]$ and $[d\tilde{s}] = c\,[ds]$, it follows that $[\omega] = c\,[\tilde{\omega}]$ and $[\tilde{\theta_1}] = [\theta_1]$. This completes the proof of Theorem 3.14. □

Remarks

1. In general we can't guarantee $[\mathrm{Im}(\tilde{\Omega})] = [\mathrm{Im}(\Omega)]$ since, roughly speaking, $[\mathrm{Im}(\tilde{\Omega})]$ is locally determined by $[\mathrm{Re}(\tilde{\Omega})]$, whereas $[\mathrm{Im}(\Omega)]$ is free to change slightly, as long as the inequality (3.2) is satisfied. Hence $[\mathrm{Im}(\Omega)]$ is independent of $[\mathrm{Re}(\Omega)]$ and it follows that $[\mathrm{Im}(\tilde{\Omega})]$ can't possibly be determined by $[\mathrm{Im}(\Omega)]$.

2. If $H^1(M,\mathbb{R}) \neq 0$, then α' may not be exact, and we have to modify the cohomological formula for $[\mathrm{Re}(\tilde{\Omega})]$ to
$$[\mathrm{Re}(\tilde{\Omega})] = [\mathrm{Re}(\Omega)] - [\alpha]\cup[\tilde{\omega}].$$

3. There is an alternative way of obtaining the Calabi–Yau structure on M from the holonomy point of view. Since $(\tilde{\varphi},\tilde{g})$ is torsion-free, $\mathrm{Hol}(\tilde{g}) \subseteq G_2$. Moreover, $\mathrm{Hol}(\tilde{g})$ fixes the vector $\frac{\partial}{\partial s}$ as $\nabla^{\tilde{g}}\frac{\partial}{\partial s} = 0$. It turns out that $\mathrm{Hol}(\tilde{g})$ actually lies in SU(3) and hence the torsion-free G_2-structure $(\tilde{\varphi},\tilde{g})$ must come from a Calabi–Yau structure on M.

3.3 Calabi–Yau cones, Calabi–Yau manifolds with conical singularities and Asymptotically Conical Calabi–Yau manifolds

In this section we define Calabi–Yau cones, Calabi–Yau manifolds with conical singularities and Asymptotically Conical Calabi–Yau manifolds. We will give some examples and provide results analogous to the usual Darboux Theorem on symplectic manifolds for the Calabi–Yau manifolds with conical singularities and Asymptotically Conical Calabi–Yau manifolds. The conical singularities in Calabi–Yau 3-folds will be desingularized in section 3.4 by using the existence result obtained in section 3.2.

3.3.1 Preliminaries on Calabi–Yau cones

We will give our definition of Calabi–Yau cones and provide several examples in this section. Let us first consider the $\mathbb{C}^m$ case. Write $\mathbb{C}^m$ as $S^{2m-1}\times(0,\infty)\cup\{0\}$, a cone over the $(2m-1)$-dimensional sphere. Let r be a coordinate on $(0,\infty)$. Then the standard metric $\hat{g}$, Kähler form $\hat{\omega}$ and holomorphic volume form $\hat{\Omega}$ on $\mathbb{C}^m$ can be written as

$$\hat{g}=r^2\hat{g}|_{S^{2m-1}}+dr^2,\quad \hat{\omega}=r^2\hat{\omega}|_{S^{2m-1}}+rdr\wedge\alpha$$
$$\text{and}\quad \hat{\Omega}=r^m\hat{\Omega}|_{S^{2m-1}}+r^{m-1}dr\wedge\beta,$$

where α is a real 1-form and β a complex $(m-1)$-form on S^{2m-1}. Hence they scale as

$$\hat{g}|_{S^{2m-1}\times\{r\}}=r^2\hat{g}|_{S^{2m-1}},\quad \hat{\omega}|_{S^{2m-1}\times\{r\}}=r^2\hat{\omega}|_{S^{2m-1}}$$
$$\text{and}\quad \hat{\Omega}|_{S^{2m-1}\times\{r\}}=r^m\hat{\Omega}|_{S^{2m-1}}.$$

Motivated by this standard case, we give our definition of a *Calabi–Yau cone*:

Definition 3.16 Let Γ be a compact $(2m-1)$-dimensional smooth manifold, and let $V=\{0\}\cup V'$ where $V'=\Gamma\times(0,\infty)$. Write points on V' as (γ,r). V is called a *Calabi–Yau cone* if V' is a Calabi–Yau m-fold with a Calabi–Yau structure (J_V,ω_V,Ω_V) and its associated Calabi–Yau metric g_V satisfying

$$\begin{aligned} g_V=&r^2g_V|_{\Gamma\times\{1\}}+dr^2,\quad \omega_V=r^2\omega_V|_{\Gamma\times\{1\}}+rdr\wedge\alpha\\ &\text{and}\quad \Omega_V=r^m\Omega_V|_{\Gamma\times\{1\}}+r^{m-1}dr\wedge\beta.\end{aligned}\tag{3.22}$$

Here we identify Γ with $\Gamma\times\{1\}$, and α is a real 1-form and β a complex $(m-1)$-form on Γ.

We remark here that in Sasaki-Einstein geometry, a Riemannian manifold (M,g) of dimension $(2m-1)$ is Sasaki-Einstein if and only if the cone over M with metric r^2g+dr^2 is Calabi–Yau, i.e. a Calabi–Yau cone. Thus in our case, V is a Calabi–Yau cone is equivalent to Γ being Sasaki-Einstein. There has been considerable interest recently in Sasaki-Einstein geometry due to a new construction of an infinite family of explicit Sasaki-Einstein metrics on five dimensions, particularly on $S^2 \times S^3$ [18]. Much work has been done by Boyer and Galicki on Sasaki-Einstein and 3-Sasakian geometry, see for examples [5] and [6].

Let X be the radial vector field on V such that $X_{(\gamma,r)} = \frac{1}{2}r\frac{\partial}{\partial r}$ for any $(\gamma, r) \in \Gamma \times (0, \infty)$. Then $r^2\alpha = 2\iota(X)\omega_V$ and $r^m\beta = 2\iota(X)\Omega_V$. Moreover,

$$\begin{aligned}\mathcal{L}_X\omega_V &= d(\iota(X)\omega_V) \quad \text{as } d\omega_V = 0 \\ &= \frac{1}{2}d(r^2\alpha) \\ &= \frac{1}{2}r^2 d\alpha + r dr \wedge \alpha.\end{aligned}$$

It can be shown that $d\alpha = 2\omega_V|_\Gamma$ by using $d\omega_V = 0$ and the formula for ω_V in (3.22). Therefore we have $\mathcal{L}_X\omega_V = \omega_V$. The flow of X thus expands the Kähler form ω_V exponentially and X is then a Liouville vector field, which is a kind of vector field satisfying $\mathcal{L}_X\omega = \omega$ on a symplectic manifold (M, ω). In a similar way, we can show $\mathcal{L}_X\Omega_V = \frac{m}{2}\Omega_V$ and $\mathcal{L}_X g_V = g_V$. It follows that $\mathcal{L}_X J_V = 0$, and hence X is a holomorphic vector field. In particular, the 1-form α defines a contact form on Γ, which makes Γ a contact $(2m-1)$-fold.

The tangent space $T_{(\gamma,r)}V$ decomposes as $T_{(\gamma,r)}V = T_\gamma\Gamma \oplus < X_{(\gamma,r)} >_{\mathbb{R}}$ for any $(\gamma, r) \in \Gamma \times (0, \infty)$. Note that $Z := J_V X$ is a vector field on Γ, and it is complete as Γ is compact. Now $\iota(Z)\omega_V$ is a 1-form such that $\iota(X)(\iota(Z)\omega_V) = g_V(X, X) = \frac{1}{4}r^2$ and $\iota(Z)\omega_V\big|_{\Gamma\times\{r\}} = 0$, hence we can write $\iota(Z)\omega_V = \frac{1}{2}r dr$. It follows that

$$\mathcal{L}_Z\omega_V = d(\iota(Z)\omega_V) = d(r dr) = 0.$$

For the holomorphic volume form, we use the fact that if Ω is a holomorphic $(m, 0)$-form and v a holomorphic vector field, then $\mathcal{L}_{Jv}\Omega = i\mathcal{L}_v\Omega$ where J is the complex structure. Now $Z = J_V X$ is a holomorphic vector field, this gives $\mathcal{L}_Z\Omega_V = im\Omega_V$.

Now we define a *complex dilation* on the Calabi–Yau cone V. The flow of Z generates the diffeomorphism $\exp(\theta Z)$ on Γ for each $\theta \in \mathbb{R}$. Thus for each $\theta \in \mathbb{R}$ and $t > 0$, we can define a *complex dilation* ψ on V which is given by $\psi(0) = 0$ and $\psi(\gamma, r) = (\exp(\theta Z)(\gamma), tr)$.

Lemma 3.17 *Let $\psi : V \longrightarrow V$ be the complex dilation defined above. Then $\psi^*(g_V) = t^2 g_V$, $\psi^*(\omega_V) = t^2\omega_V$ and $\psi^*(\Omega_V) = t^m e^{im\theta}\,\Omega_V$.*

Proof. It follows from $\mathcal{L}_Z\omega_V = 0$ that $\exp(\theta Z)^*(\omega_V) = \omega_V$ and hence $\psi^*(\omega_V) = t^2\omega_V$ by the scaling of t. The formula for the metric g_V follows similarly. For the holomorphic $(m, 0)$-form Ω_V, observe that

$$\frac{d}{d\theta}\exp(\theta Z)^*(\Omega_V)\Big|_{\theta=0} = \mathcal{L}_Z\Omega_V = im\,\Omega_V,$$

and this means $\exp(\theta Z)^*(\Omega_V) = e^{im\theta}\,\Omega_V$. Thus together with the scaling of t, we have $\psi^*(\Omega_V) = t^m e^{im\theta}\,\Omega_V$. □

In the situation of our standard example $\mathbb{C}^m$, the complex dilation is given by complex multiplication $\psi : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ sending z to ψz, where $\psi = te^{i\theta} \in \mathbb{C}$. It is easy to see the above properties for the standard structures $\hat{g}$, $\hat{\omega}$ and $\hat{\Omega}$.

Examples 3.18 A trivial example is given by $\mathbb{C}^m$, a cone on S^{2m-1}. Some nontrivial examples can be constructed as follows: Let G be a finite subgroup of $\mathrm{SU}(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$, then the quotient singularity $\mathbb{C}^m/G$ is a Calabi–Yau cone. An example of this type is given by the $\mathbb{Z}_m$-action described in Example 2.6: define an action generated by ζ on $\mathbb{C}^m$ by

$$\zeta^k \cdot (z_1, \ldots, z_m) \;=\; (\zeta^k z_1, \ldots, \zeta^k z_m)$$

where $\zeta = e^{2\pi i/m}$ and $0 \le k \le m-1$. Note that $\zeta^m = 1$, so $\mathbb{Z}_m = \{1, \zeta, \ldots, \zeta^{m-1}\}$ is a subgroup of $\mathrm{SU}(m)$ and acts freely on $\mathbb{C}^m \setminus \{0\}$. Then $\mathbb{C}^m/\mathbb{Z}_m$ is a Calabi–Yau cone.

Examples 3.19 Consider the cone V defined by the quadric $\sum_{j=1}^{m+1} z_j^2 = 0$ on $\mathbb{C}^{m+1}$. As we have mentioned in Chapter 1, the singularity at the origin is known as an *ordinary double point*, or a *node*. It can be shown that the link Γ of V is an S^{m-1}-bundle over S^m. We are particularly interested in the case $m = 3$. Then Γ has the topology of $S^2 \times S^3$, and hence V is topologically a cone on $S^2 \times S^3$ since any S^2-bundle over S^3 is trivial. Candelas and Ossa [11] constructed a Calabi–Yau metric on V, thus making it a Calabi–Yau cone. Under the correspondence between Sasaki-Einstein metrics on the link and Calabi–Yau metrics on the cone, the existence of a homogeneous Sasaki-Einstein metrics on $S^2 \times S^3$ even dates back to the work of Tanno [51]. One can also describe V as follows. Consider the blow-up $\widetilde{\mathbb{C}}^4$ of $\mathbb{C}^4$ at origin. It introduces an exceptional divisor $\mathbb{CP}^3$, and the blow-up $\widetilde{V}$ of the cone V inside $\widetilde{\mathbb{C}}^4$ meets this $\mathbb{CP}^3$ at $S \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. The exceptional divisor $\mathbb{CP}^3$ corresponds to the zero section of the line bundle L given by $\widetilde{\mathbb{C}}^4 \longrightarrow \mathbb{CP}^3$, and so its normal bundle is isomorphic to L. Hence the normal bundle $\mathcal{O}(-1,-1)$ over $\mathbb{CP}^1 \times \mathbb{CP}^1$ is isomorphic to the line bundle $\widetilde{V} \longrightarrow S$. This gives us the following isomorphisms:

$$V \setminus \{0\} \;\cong\; \widetilde{V} \setminus S \;\cong\; \mathcal{O}(-1,-1) \setminus (\mathbb{CP}^1 \times \mathbb{CP}^1).$$

Examples 3.20 Suppose S is Kähler-Einstein with positive scalar curvature, Calabi [10, p.284-5] constructed a 1-parameter family of Calabi–Yau metrics g_t for $t \ge 0$ on the canonical line bundle K_S. When $t > 0$, g_t is a nonsingular complete metric on K_S and when $t = 0$, g_0 degenerates on S and thus gives a cone metric on $K_S \setminus S$, which then makes $K_S \setminus S$ a Calabi–Yau cone with S "collapsed" to the vertex of the cone.

(i) One of the standard examples of Kähler-Einstein manifolds with positive scalar curvature is given by the complex surface $S \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. Calabi's construction thus applies to it and yields a Calabi–Yau metric on $K_S = \mathcal{O}(-2,-2) \longrightarrow S$. Note that $\mathcal{O}(-1,-1)$ is a double cover of $\mathcal{O}(-2,-2)$ away from the zero section S, so we have the following relation between the cone $K_S \setminus S$ and the cone V described in Example 3.19:

$$K_S \setminus S \;\cong\; (V \setminus \{0\})/\mathbb{Z}_2.$$

(ii) Boyer, Galicki and Kollár [7] constructed Kähler-Einstein metrics on some compact orbifolds, particularly on orbifolds of the form given by the quotient $(L \setminus \{0\})/\mathbb{C}^*$, where $L = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : \sum_{j=1}^m z_j^{a_j} = 0\}$ for some positive integers a_j satisfying certain conditions. The set L is a hypersurface in $\mathbb{C}^m$, and $\mathbb{C}^*$ acts naturally on $\mathbb{C}^m$ by $\lambda : (z_1, \ldots, z_m) \mapsto (\lambda^{a_1} z_1, \ldots, \lambda^{a_m} z_m)$. It follows from Calabi's construction, in the category of orbifolds, that $L \setminus \{0\}$ is a Calabi–Yau cone.

3.3.2 Calabi–Yau m-folds with conical singularities

We define Calabi–Yau m-folds with conical singularities in this section. As we have discussed before, we shall always assume the existence of Calabi–Yau metrics on such kind of singular manifolds in this thesis. A class of Calabi–Yau m-folds with conical singularities are given by orbifolds, in which case the existence of such singular Calabi–Yau metrics is known (see [26, Thm. 6.5.6]). We shall see an example in §3.4.4. After the definition of Calabi–Yau m-folds with conical singularities, we show that there exist coordinate systems that can trivialize the symplectic forms of Calabi–Yau m-folds with conical singularities.

Definition 3.21 Let $(M_0, J_0, \omega_0, \Omega_0)$ be a singular Calabi–Yau m-fold with isolated singularities $x_1, \ldots, x_n \in M_0$, and no other singularities. We say that M_0 is a *Calabi–Yau m-fold with conical singularities* x_i for $i = 1, \ldots, n$ with rate $\nu > 0$ modelled on Calabi–Yau cones $(V_i, J_{V_i}, \omega_{V_i}, \Omega_{V_i})$ if there exist a small $\epsilon > 0$, a small open neighbourhood S_i of x_i in M_0, and a diffeomorphism $\Phi_i : \Gamma_i \times (0, \epsilon) \longrightarrow S_i \setminus \{x_i\}$ for each i such that

$$|\nabla^k(\Phi_i^*(\omega_0) - \omega_{V_i})|_{g_{V_i}} = O(r^{\nu-k}), \text{ and} \tag{3.23}$$

$$|\nabla^k(\Phi_i^*(\Omega_0) - \Omega_{V_i})|_{g_{V_i}} = O(r^{\nu-k}) \quad \text{as } r \to 0 \text{ and for all } k \geq 0. \tag{3.24}$$

Here Γ_i is the link of V_i, and ∇, $|\cdot|_{g_{V_i}}$ are computed using the cone metric g_{V_i}.

Note that the asymptotic conditions on g_0 and J_0 follow from those on ω_0 and Ω_0, namely,

$$|\nabla^k(\Phi_i^*(g_0) - g_{V_i})|_{g_{V_i}} = O(r^{\nu-k}), \text{ and} \tag{3.25}$$

$$|\nabla^k(\Phi_i^*(J_0) - J_{V_i})|_{g_{V_i}} = O(r^{\nu-k}) \quad \text{as } r \to 0 \text{ and for all } k \geq 0, \tag{3.26}$$

and so it is enough to just assume asymptotic conditions on ω_0 and Ω_0.

We will usually assume that M_0 is compact. The point of the definition is that M_0 is locally modelled on $\Gamma_i \times (0, \epsilon)$ near x_i, and as $r \to 0$, all the structures g_0, J_0, ω_0 and Ω_0 on M_0 converge to the cone structures $g_{V_i}, J_{V_i}, \omega_{V_i}$ and Ω_{V_i} with rate ν and with all their derivatives.

Two diffeomorphisms, or two coordinate systems, Φ_i and Φ_i' are equivalent if and only if the following relation holds:

$$|\nabla^k(\Phi_i - \Phi_i')|_{g_{V_i}} = O(r^{\nu+1-k}) \quad \text{as } r \to 0 \text{ and for all } k \geq 0.$$

Here we interpret the difference between Φ_i and Φ_i' using local coordinates on the image $S_i \setminus \{x_i\}$. Thus if Φ_i and Φ_i' are equivalent, we have

$$\begin{aligned} |\nabla^k(\Phi_i^*(\omega_0) - \omega_{V_i})|_{g_{V_i}} &\leq |\nabla^k(\Phi_i^*(\omega_0) - (\Phi_i')^*(\omega_0))|_{g_{V_i}} + |\nabla^k((\Phi_i')^*(\omega_0) - \omega_{V_i})|_{g_{V_i}} \\ &= O(r^{\nu-k}) + |\nabla^k((\Phi_i')^*(\omega_0) - \omega_{V_i})|_{g_{V_i}} \end{aligned}$$

and we see that Φ_i satisfies (3.23) (and similarly (3.24)) if and only if Φ_i' does.

The 2-forms $\Phi_i^*(\omega_0)$ and ω_{V_i} are closed on $\Gamma_i \times (0, \epsilon)$ and so $\Phi_i^*(\omega_0) - \omega_{V_i}$ represents a cohomology class in $H^2(\Gamma_i \times (0,\epsilon), \mathbb{R}) \cong H^2(\Gamma_i, \mathbb{R})$. Similarly, $\Phi_i^*(\Omega_0) - \Omega_{V_i}$ represents a cohomology class in $H^m(\Gamma_i \times (0,\epsilon), \mathbb{C}) \cong H^m(\Gamma_i, \mathbb{C})$. It turns out that in the conical singularity case, these two classes $[\Phi_i^*(\omega_0) - \omega_{V_i}]$ and $[\Phi_i^*(\Omega_0) - \Omega_{V_i}]$ are automatically zero:

Lemma 3.22 *Let $(M_0, J_0, \omega_0, \Omega_0)$ be a compact Calabi–Yau m-fold with conical singularities x_i for $i = 1, \ldots, n$ with rate $\nu > 0$ modelled on Calabi–Yau cones $(V_i, J_{V_i}, \omega_{V_i}, \Omega_{V_i})$. Then $[\Phi_i^*(\omega_0) - \omega_{V_i}] = 0$ in $H^2(\Gamma_i, \mathbb{R})$ and $[\Phi_i^*(\Omega_0) - \Omega_{V_i}] = 0$ in $H^m(\Gamma_i, \mathbb{C})$.*

Proof. Suppose Σ_i is a 2-cycle in Γ_i for $i = 1, \ldots, n$. Then

$$\begin{aligned} \int_{\Sigma_i \times \{r\}} \left(\Phi_i^*(\omega_0) - \omega_{V_i}\right) &= \mathrm{vol}(\Sigma_i) \cdot O(r^\nu) \quad \text{by (3.23)} \\ &= O(r^{\nu+2}) \end{aligned}$$

Hence $\nu > 0$ implies the above integral approaches 0 as $r \to 0$. Then

$$[\Phi_i^*(\omega_0) - \omega_{V_i}] \cdot [\Sigma_i] = 0$$

for any 2-cycle Σ_i, and hence $[\Phi_i^*(\omega_0) - \omega_{V_i}] = 0 \in H^2(\Gamma_i \times (0,\epsilon), \mathbb{R}) \cong H^2(\Gamma_i, \mathbb{R})$. The case for $[\Phi_i^*(\Omega_0) - \Omega_{V_i}]$ follows similarly by considering m-cycles in Γ_i. □

One may ask whether the symplectic form ω_0 on S_i near x_i in M_0 can actually be symplectomorphic to the cone form ω_{V_i} near the origin, rather than just having an asymptotic relation in (3.23). Theorem 3.24 below shows that this is indeed the case for Calabi–Yau m-folds with conical singularities, and can be regarded as an analogue of the usual Darboux theorem on symplectic manifolds. Before that, we need the following lemma:

Lemma 3.23 *Let $X_{t,i}$ be a smooth family of vector fields on $\Gamma_i \times (0, \epsilon)$ for $t \in [0,1]$ and $i = 1, \ldots, n$ with $|X_{t,i}|_{g_{V_i}} = O(r^\delta)$ for some $\delta > 1$. Then there exist an $\epsilon' \in (0, \epsilon)$ and a family of smooth maps $\psi_{t,i} : \Gamma_i \times (0, \epsilon') \longrightarrow \Gamma_i \times (0, \epsilon)$ such that $\psi_{t,i}$ is a diffeomorphism with its image and satisfies*

$$\frac{d}{dt}\psi_{t,i}(\gamma, r) = X_{t,i}|_{\psi_{t,i}(\gamma,r)} \in T_{\psi_{t,i}(\gamma,r)}(\Gamma_i \times (0, \epsilon')), \qquad \psi_{0,i}(\gamma, r) = (\gamma, r)$$

for all $(\gamma, r) \in \Gamma_i \times (0, \epsilon')$ and all $t \in [0,1]$, $i = 1, \ldots, n$.

This may be proved using the method of [41, p. 93-95]. Roughly speaking, the condition that $|X_{t,i}|_{g_{V_i}} = O(r^\delta)$ for some $\delta > 1$ prevents the r-coordinate of $\psi_{t,i}(\gamma, r)$ from going to 0 or ϵ for small r and for any $t \in [0,1]$, and therefore $\psi_{t,i}(\gamma, r)$ exists for all $t \in [0,1]$.

Theorem 3.24 *Let $(M_0, J_0, \omega_0, \Omega_0)$ be a compact Calabi–Yau m-fold with conical singularities x_i for $i = 1, \ldots, n$ with rate $\nu > 0$ modelled on Calabi–Yau cones $(V_i, J_{V_i}, \omega_{V_i}, \Omega_{V_i})$. Then there exist an $\epsilon' > 0$, an open neighbourhood S'_i of x_i and a diffeomorphism $\Phi'_i : \Gamma_i \times (0, \epsilon') \longrightarrow S'_i \setminus \{x_i\}$ for each i such that $(\Phi'_i)^*(\omega_0) = \omega_{V_i}$ and $|\nabla^k((\Phi'_i)^*(\Omega_0) - \Omega_{V_i})|_{g_{V_i}} = O(r^{\nu-k})$ for all $k \geq 0$.*

Proof. By the definition of a Calabi–Yau m-fold with conical singularities, there is an $\epsilon > 0$ and a diffeomorphism $\Phi_i : \Gamma_i \times (0, \epsilon) \longrightarrow S_i \setminus \{x_i\}$ for each i such that $|\nabla^k(\Phi_i^*(\omega_0) - \omega_{V_i})|_{g_{V_i}} = O(r^{\nu-k})$ as $r \to 0$ and for all $k \geq 0$. Write $\omega^i = \Phi_i^*(\omega_0)$. Thus we have two symplectic forms, namely the cone form ω_{V_i} and the asymptotic cone form ω^i on $\Gamma_i \times (0, \epsilon)$. Following Moser's proof of the usual Darboux Theorem [41, p.93], we construct a 1-parameter family of closed 2-forms

$$\omega^{t,i} = \omega_{V_i} + t(\omega^i - \omega_{V_i}) \quad \text{for } t \in [0,1] \tag{3.27}$$

on $\Gamma_i \times (0, \epsilon)$. Make ϵ smaller if necessary so that $\omega^{t,i}$ is symplectic for all $t \in [0,1]$. Then

$$\frac{d}{dt}\omega^{t,i} = \omega^i - \omega_{V_i}. \tag{3.28}$$

By Lemma 3.22, $[\omega^i - \omega_{V_i}] = 0$ in $H^2(\Gamma_i, \mathbb{R})$ and hence $\omega^i - \omega_{V_i}$ is exact. Suppose $\eta^i = \omega^i - \omega_{V_i}$, and write η^i as $\eta_0^i(\gamma, r) + \eta_1^i(\gamma, r) \wedge dr$, where $\eta_0^i(\gamma, r) \in \Lambda^2 T_\gamma^* \Gamma_i$ and $\eta_1^i(\gamma, r) \in \Lambda^1 T_\gamma^* \Gamma_i$. Then η^i is an exact 2-form such that

$$\frac{d}{dt}\omega^{t,i} = \eta^i.$$

Now we want to choose a 1-form σ^i such that $\eta^i = d\sigma^i$. Define

$$\sigma^i(\gamma, r) = -\int_0^r \eta_1^i(\gamma, s) ds. \tag{3.29}$$

By the fact that $|\eta^i|_{g_{V_i}} = |\omega^i - \omega_{V_i}|_{g_{V_i}} = O(r^\nu)$ as $r \to 0$, we have $|\eta_0^i|_{g_{V_i}} = O(r^\nu) = |\eta_1^i|_{g_{V_i}}$ as $r \to 0$. For each fixed r, $\eta_1^i(\gamma, r)$ is a 1-form on $\Gamma_i \cong \Gamma_i \times \{1\}$, and so $|\eta_1^i(\gamma, r)|_{g_{V_i}|_{\Gamma_i \times \{1\}}} = r|\eta_{1,i}(\gamma, r)|_{g_{V_i}} = O(r^{\nu+1})$. Then σ^i is well-defined since η_1^i is of size $O(r^{\nu+1})$ w.r.t. the fixed metric $g_{V_i}|_{\Gamma_i \times \{1\}}$ where $\nu + 1 > -1$ as $\nu > 0$. It follows by integration that $\sigma^i(\gamma, r)$ is of size $O(r^{\nu+2})$ w.r.t. the fixed metric $g_{V_i}|_{\Gamma_i \times \{1\}}$ and of size $O(r^{\nu+1})$ w.r.t. the cone metric g_{V_i}.

Since $d\eta^i = 0$, we have $d_{\Gamma_i}\eta_0^i = 0$ and

$$\frac{\partial \eta_0^i}{\partial r} + d_{\Gamma_i}\eta_1^i = 0, \tag{3.30}$$

where d_{Γ_i} denotes the exterior differentiation in the γ direction. Therefore,

$$\begin{aligned} d\sigma^i(\gamma, r) &= -\int_0^r d_{\Gamma_i}(\eta_1^i(\gamma, s))ds - dr \wedge \frac{\partial}{\partial r}\Big(\int_0^r \eta_1^i(\gamma, s)ds\Big) \\ &= \int_0^r \frac{\partial \eta_0^i}{\partial s}(\gamma, s)ds - dr \wedge \eta_1^i(\gamma, r) \quad \text{by (3.30).} \end{aligned}$$

Applying the argument before, the 2-form η_0^i is of size $O(r^{\nu+2})$ w.r.t. the fixed metric $g_{V_i}|_{\Gamma_i\times\{1\}}$. Thus $|\eta_0^i(\gamma,s)|_{g_{V_i}|_{\Gamma_i\times\{1\}}}\to 0$ as $s\to 0$. This gives $d\sigma^i(\gamma,r)=\eta_0^i(\gamma,r)+\eta_1^i(\gamma,r)\wedge dr=\eta^i(\gamma,r)$. Therefore, we obtain a 1-form σ^i on $\Gamma_i\times(0,\epsilon)$ such that

$$\frac{d}{dt}\omega^{t,i}=d\sigma^i.$$

Now from the definition of $\sigma^i(\gamma,r)$, it can be shown that

$$|\nabla^k\sigma^i(\gamma,r)|_{g_{V_i}|_{\Gamma_i\times\{1\}}}\leq\; k\,|\nabla^{k-1}\eta_1^i(\gamma,r)|_{g_{V_i}|_{\Gamma_i\times\{1\}}}+\int_0^r|\nabla^k\eta_1^i(\gamma,s)|_{g_{V_i}|_{\Gamma_i\times\{1\}}}ds.$$

Moreover, the $(0,k+1)$-tensor $\nabla^k\eta_1^i$ satisfies

$$|\nabla^k\eta_1^i(\gamma,s)|_{g_{V_i}|_{\Gamma_i\times\{1\}}}=s^{k+1}\,|\nabla^k\eta_1^i(\gamma,s)|_{g_{V_i}}=s^{k+1}O(s^{\nu-k})=O(s^{\nu+1})$$

as $|\eta_1^i(\gamma,s)|_{g_{V_i}}=O(s^\nu)$. Thus the integral in the right hand side of the above inequality converges. Note that we have the same condition for $\sigma^i(\gamma,r)$ and its k-th derivative $\nabla^k\sigma^i(\gamma,r)$ to be well-defined, namely $\nu+1>-1$, which holds automatically in our case where $\nu>0$. We then deduce that $|\nabla^k\sigma^i(\gamma,r)|_{g_{V_i}|_{\Gamma_i\times\{1\}}}=O(r^{\nu+2})$ and hence

$$|\nabla^k\sigma^i(\gamma,r)|_{g_{V_i}}=O(r^{\nu-k+1})\quad\text{as } r\to 0 \text{ and for all } k\geq 0. \tag{3.31}$$

Now define a family of vector fields $X_{t,i}$ via

$$\sigma^i+\iota(X_{t,i})\omega^{t,i}=0.$$

Then we have

$$|\nabla^k X_{t,i}|_{g_{V_i}}=O(r^{\nu-k+1})\quad\text{as } r\to 0 \text{ and for all } k\geq 0. \tag{3.32}$$

Lemma 3.23 thus yields a family of diffeomorphisms $\psi_{t,i}$ on V_i such that $\psi_{t,i}^*(\omega^{t,i})=\omega^{0,i}$. In particular, we have constructed $\psi_{1,i}:\Gamma_i\times(0,\epsilon')\longrightarrow\Gamma_i\times(0,\epsilon)$ for some $\epsilon'\in(0,\epsilon)$ which is a diffeomorphism with its image satisfying

$$\psi_{1,i}^*(\omega)=\psi_{1,i}^*(\omega^{1,i})=\omega^{0,i}=\omega_{V_i}.$$

Write $\Phi_i'=\Phi_i\circ\psi_{1,i}$ and $S_i'=\Phi_i\circ\psi_{1,i}(\Gamma_i\times(0,\epsilon'))$, then $\Phi_i':\Gamma_i\times(0,\epsilon')\longrightarrow S_i'\setminus\{x_i\}$ is a diffeomorphism such that

$$(\Phi_i')^*(\omega_0)=\psi_{1,i}^*(\Phi_i^*(\omega_0))=\psi_{1,i}^*(\omega^i)=\omega_{V_i},$$

as required.

From (3.32) we have $|\nabla^k\psi_{t,i}|_{g_{V_i}}=O(r^{\nu-k+1})$ for all $k\geq 0$ since, roughly speaking, $\psi_{t,i}=\mathrm{Id}+\int_0^t X_{s,i}ds$ to the first order. It doesn't exactly make sense as $\psi_{t,i}$ and Id map to different points on V_i. But we could express them in terms of local coordinates $(x_1,\dots,x_{2m-1},r)$ on $\Gamma_i\times(0,\epsilon')$. Let $\psi_{t,i}^j(x_1,\dots,x_{2m-1},r)$ be the j-th component function of $\psi_{t,i}$ for $j=1,\dots,2m$. Then $\partial^k(\psi_{t,i}^j(x_1,\dots,x_{2m-1},r)-x_j)=O(r^{\nu-k+1})$ for $j=1,\dots,2m-1$ and for all $k\geq 0$, and $\partial^k(\psi_{t,i}^{2m}(x_1,\dots,x_{2m-1},r)-r)=O(r^{\nu-k+1})$ for all $k\geq 0$ where ∂ denotes the usual partial differentiation at the point $(x_1, \ldots, x_{2m-1}, r)$. It follows that $\partial^k(\psi_{t,i} - \mathrm{Id})(x_1, \ldots, x_{2m-1}, r) = O(r^{\nu-k+1})$ for all $k \geq 0$. Consequently we have

$$|\partial^k(\psi_{t,i} - \mathrm{Id})^*(\Phi_i^*(\Omega_0))|_{g_{V_i}} = O(r^{\nu-k})$$

at the point $(x_1, \ldots, x_{2m-1}, r)$. As a result, we have at each point on $\Gamma_i \times (0, \epsilon')$

$$\begin{aligned}
\left|\nabla^k((\Phi_i')^*(\Omega_0) - \Omega_{V_i})\right|_{g_{V_i}} &= \left|\nabla^k(\psi_{1,i}^*(\Phi_i^*(\Omega_0)) - \Omega_{V_i})\right|_{g_{V_i}} \\
&\leq \left|\partial^k(\psi_1^*(\Phi_i^*(\Omega_0)) - \Phi_i^*(\Omega_0))\right|_{g_{V_i}} \\
&\quad + \left|\partial^k(\Phi_i^*(\Omega_0) - \Omega_{V_i})\right|_{g_{V_i}} \\
&= O(r^{\nu-k}) + O(r^{\nu-k}) = O(r^{\nu-k})
\end{aligned}$$

for all $k \geq 0$. This completes the proof. □

3.3.3 Asymptotically Conical Calabi–Yau m-folds

In the last part we study Asymptotically Conical (AC) Calabi–Yau m-folds. We shall provide some examples and give an analogue of Theorem 3.24 for AC Calabi–Yau m-folds.

Definition 3.25 Let $(V, J_V, \omega_V, \Omega_V)$ be a Calabi–Yau cone of complex dimension m with link Γ. Let $(Y, J_Y, \omega_Y, \Omega_Y)$ be a complete, nonsingular Calabi–Yau m-fold. Then Y is an *Asymptotically Conical (AC) Calabi–Yau m-fold* with rate $\lambda < 0$ modelled on $(V, J_V, \omega_V, \Omega_V)$ if there exist a compact subset $K \subset Y$, and a diffeomorphism $\Upsilon : \Gamma \times (R, \infty) \longrightarrow Y \setminus K$ for some $R > 0$ such that

$$|\nabla^k(\Upsilon^*(\omega_Y) - \omega_V)|_{g_V} = O(r^{\lambda-k}), \text{ and} \tag{3.33}$$
$$|\nabla^k(\Upsilon^*(\Omega_Y) - \Omega_V)|_{g_V} = O(r^{\lambda-k}) \text{ as } r \to \infty \text{ and for all } k \geq 0. \tag{3.34}$$

Here ∇ and $|\cdot|$ are computed using the cone metric g_V.

Similar asymptotic conditions on g_Y and J_Y can be deduced from (3.33) and (3.34). The coordinates Υ and Υ' are equivalent if and only if $|\nabla^k(\Upsilon - \Upsilon')|_{g_V} = O(r^{\lambda+1-k})$ as $r \to \infty$ and for all $k \geq 0$.

Remark If Y is an AC Calabi–Yau m-fold which is not a $\mathbb{C}^m$, then Y can only have one end, or equivalently, the link Γ is connected. One can show this by using the *Cheeger-Gromoll splitting theorem* (see for example [3, §6.G]). Suppose Y has more than one end. As Y is complete and Ricci-flat, Cheeger-Gromoll splitting theorem tells us that we can always split off a line so that Y is isometric to a product $N \times \mathbb{C}$, where $\mathbb{C}$ carries the Euclidean metric. Now, either N is a flat $\mathbb{C}^{m-1}$, in which case $Y = \mathbb{C}^m$, or N has nonzero curvature at some $p \in N$, in which case the curvature of $N \times \mathbb{C}$ is of order $O(1)$ as we go to infinity in $\{p\} \times \mathbb{C}$. But then this contradicts the AC condition which requires the curvature to decay at $O(r^{-2})$. Therefore Y cannot have more than one end, and so from now on, we shall always take the link Γ to be a *compact, connected*

(2m − 1)-dimensional Sasaki-Einstein manifold.

Unlike the conical singularity case, $[\Upsilon^*(\omega_Y) - \omega_V]$ and $[\Upsilon^*(\Omega_Y) - \Omega_V]$ need not be zero cohomology classes. Here are some conditions:

Lemma 3.26 *Let $(Y, J_Y, \omega_Y, \Omega_Y)$ be an AC Calabi–Yau m-fold with rate $\lambda < 0$ modelled on Calabi–Yau cones $(V, J_V, \omega_V, \Omega_V)$. If $\lambda < -2$ or $H^2(\Gamma, \mathbb{R}) = 0$, then $[\Upsilon^*(\omega_Y) - \omega_V] = 0$. If $\lambda < -m$ or $H^m(\Gamma, \mathbb{C}) = 0$, then $[\Upsilon^*(\Omega_Y) - \Omega_V] = 0$.*

The proof of it is similar to that of Lemma 3.22, except we now have $O(r^{\lambda+2})$ for the integral of the difference of symplectic forms and $O(r^{\lambda+m})$ for the holomorphic $(m, 0)$-forms. Hence if $\lambda < -2$, the integral approaches 0 as $r \to \infty$, which implies $[\Upsilon^*(\omega_Y) - \omega_V] = 0$. The same argument applies to the holomorphic $(m, 0)$-forms.

We shall normally consider the case $\lambda < -2$, so that $\Upsilon^*(\omega_Y) - \omega_V$ is always exact. Moreover, when $\lambda < -2$, the proof for the analogue of Darboux Theorem works similarly to that for the conical singularities case. It is not clear whether the theorem holds for $\lambda \geq -2$ and $[\Upsilon^*(\omega_Y) - \omega_V] = 0$ or not.

Theorem 3.27 *Let $(Y, J_Y, \omega_Y, \Omega_Y)$ be an AC Calabi–Yau m-fold with rate $\lambda < -2$ modelled on Calabi–Yau cones $(V, J_V, \omega_V, \Omega_V)$. Then there exist a $R' > 0$ and a diffeomorphism $\Upsilon' : \Gamma \times (R', \infty) \longrightarrow Y \setminus K$ such that $(\Upsilon')^*(\omega_Y) = \omega_V$ and $|\nabla^k((\Upsilon')^*(\Omega_Y) - \Omega_V)|_{g_V} = O(r^{\lambda-k})$ for all $k \geq 0$.*

One can prove it in the same way as the proof of Theorem 3.24. The condition on the rate λ is essential for this proof to work. Since $|\eta_1|_{g_V} = O(r^\lambda)$, we need $\lambda < -2$ to construct the 1-form σ. Moreover, in proving Theorem 3.24, we encountered the norm of $\eta_0(\gamma, r)$ w.r.t. the fixed metric: $|\eta_0(\gamma, r)|_{g_V|_{\Gamma\times\{1\}}} = r^2|\eta_0(\gamma, r)|_{g_V}$, which is equal to $O(r^{\lambda+2})$ in this case. Therefore we need $\lambda < -2$ in order to have $\eta_0 \to 0$ as $r \to \infty$.

Examples 3.28 Let G be a finite subgroup of $\mathrm{SU}(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$, and (X, π) a crepant resolution of the Calabi–Yau cone $V = \mathbb{C}^m/G$ given in Example 3.18. Then in each Kähler class of ALE Kähler metrics on X there is a unique ALE Ricci-flat Kähler metric (see Joyce [26], Chapter 8) and X is then an AC Calabi–Yau m-fold asymptotic to the cone $\mathbb{C}^m/G$. In this case, it follows from [26, Thm. 8.2.3] that the rate λ is $-2m$.

If we take $G = \mathbb{Z}_m$ acting on $\mathbb{C}^m$ as in Example 3.18, then a crepant resolution is given by the blow-up of $\mathbb{C}^m/\mathbb{Z}_m$ at 0, which is also the total space of the canonical line bundle $K_{\mathbb{CP}^{m-1}}$ over $\mathbb{CP}^{m-1}$. An explicit ALE Ricci-flat Kähler metric is given in [10, p.284-5] and also in [26, Example 8.2.5]. We have seen this in Example 2.6.

Examples 3.29 Consider the Calabi–Yau cone $V = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}$ described in Example 3.19. As we have discussed in Chapter 1, manifolds with such kind of singular points are known as conifolds, and there are two different ways of repairing the singularities, corresponding to two kinds of AC Calabi–Yau manifolds. The first one is called the *small resolution* of V, given by

$$\widetilde{V} = \big\{((z_1,\dots,z_4),[w_1,w_2]) \in \mathbb{C}^4 \times \mathbb{CP}^1 : z_1w_2 = z_4w_1,\ z_3w_2 = z_2w_1\big\}.$$

It is essentially isomorphic to the normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ over $\mathbb{CP}^1$ with fibre $\mathbb{C}^2$, and is also isomorphic to V away from the origin where it is replaced by the whole $\mathbb{CP}^1$. Note that one can obtain a second small resolution by swapping z_3 and z_4 in $\widetilde{V}$. Candelas and de la Ossa [11, p.258] constructed Calabi–Yau metrics on $\widetilde{V}$, and it is an AC Calabi–Yau 3-fold with rate -2.

The other is known as the *deformation* or *smoothing*, where V is deformed to $Q_\epsilon = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = \epsilon\}$ with ϵ a nonzero constant. This has the effect of replacing the node by an S^3. We have seen this in Example 2.7, in which we mentioned that the cotangent bundle T^*S^3 of S^3 can be identified with Q_1, or Q_ϵ, and there is a symplectomorphism which identifies the standard symplectic form on $\mathbb{C}^4$ restricted to Q_ϵ and the canonical symplectic form on the cotangent bundle T^*S^3 of S^3. More importantly, Stenzel [48, p.161] constructed a Calabi–Yau metric on Q_ϵ whose Kähler potential has to satisfy a certain ordinary differential equation. Thus Q_ϵ, or equivalently T^*S^3, is an AC Calabi–Yau 3-fold (with rate $\lambda = -3$). More details will be given in §4.6 of Chapter 4.

Examples 3.30 Calabi [10, p.284-5] constructed a 1-parameter family of AC Calabi–Yau metrics on the canonical bundle K_S of any Kähler-Einstein $(m-1)$-fold S with positive scalar curvature, so that K_S is an AC Calabi–Yau m-fold modelled on the Calabi–Yau cone $K_S \setminus S$ with rate $\lambda = -2m$.

For the case in Example 3.20 (i), the $\mathcal{O}(-2,-2)$-bundle is AC Calabi–Yau asymptotic to the cone $\mathcal{O}(-2,-2)\setminus(\mathbb{CP}^1\times\mathbb{CP}^1)$ with rate -6.

Note that if we take $S = \mathbb{CP}^{m-1}$, then we recover the case in Example 3.28 with $G = \mathbb{Z}_m$.

3.4 Calabi–Yau desingularizations

This section studies desingularizations of a compact Calabi–Yau 3-fold M_0 with conical singularities using an AC Calabi–Yau 3-fold Y_i with rate λ_i for $i = 1,\dots,n$. We shall only treat the simplest case here, in which $\lambda_i < -3$ so that $\Upsilon^*_{t,i}(t^3\Omega_{Y_i}) - \Omega_{V_i}$ is exact by Lemma 3.26. We explicitly construct a 1-parameter family of diffeomorphic, nonsingular compact 6-folds M_t for small t in §3.4.1. Then in §3.4.2 we construct a real closed 2-form ω_t and a complex closed 3-form Ω_t on M_t and show that they give nearly Calabi–Yau structures on M_t for small enough t. Section 3.4.3 contains the main result of this chapter, in which we show that the nearly Calabi–Yau structure (ω_t,Ω_t) on M_t can be deformed to a genuine Calabi–Yau structure $(\tilde\omega_t,\tilde\Omega_t)$ for small t by applying Theorem 3.14. Finally in §3.4.4, we apply our result to some examples studied before. We shall also discuss the case when $\lambda_i = -3$.

3.4.1 Construction of M_t

Let $(M_0, J_0, \omega_0, \Omega_0)$ be a compact Calabi–Yau 3-fold with conical singularities x_i with rate ν modelled on Calabi–Yau cones $(V_i, J_{V_i}, \omega_{V_i}, \Omega_{V_i})$ for $i = 1, \ldots, n$. By Theorem 3.24, there exists an $\epsilon > 0$, a small open neighbourhood S_i of x_i in M_0 and a diffeomorphism $\Phi_i : \Gamma_i \times (0, \epsilon) \longrightarrow S_i \setminus \{x_i\}$ for each i such that $\Phi_i^*(\omega_0) = \omega_{V_i}$.

Let $(Y_i, J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i})$ be an AC Calabi–Yau 3-fold with rate $\lambda_i < -3$ modelled on the same Calabi–Yau cone V_i. Theorem 3.27 shows that there is a diffeomorphism $\Upsilon_i : \Gamma_i \times (R, \infty) \longrightarrow Y_i \setminus K_i$ for some $R > 0$ such that

$$\Upsilon_i^*(\omega_{Y_i}) = \omega_{V_i} \quad \text{and} \quad |\nabla^k(\Upsilon_i^*(\Omega_{Y_i}) - \Omega_{V_i})|_{g_{V_i}} = O(r^{\lambda_i - k})$$

as $r \to \infty$ for all $k \geq 0$. We then apply a homothety to Y_i such that

$$(Y_i, J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i}) \longmapsto (Y_i, J_{Y_i}, t^2\omega_{Y_i}, t^3\Omega_{Y_i}).$$

Then $(Y_i, J_{Y_i}, t^2\omega_{Y_i}, t^3\Omega_{Y_i})$ is also an AC Calabi–Yau 3-fold, with the diffeomorphism $\Upsilon_{t,i} : \Gamma_i \times (tR, \infty) \longrightarrow Y_i \setminus K_i$ given by

$$\Upsilon_{t,i}(\gamma, r) = \Upsilon_i(\gamma, t^{-1}r).$$

Our goal is to desingularize $(M_0, J_0, \omega_0, \Omega_0)$ by gluing $(Y_i, J_{Y_i}, t^2\omega_{Y_i}, t^3\Omega_{Y_i})$ in at x_i to produce a family of compact nonsingular Calabi–Yau 3-folds.

Fix $\alpha \in (0, 1)$ and let $t > 0$ be small enough that $tR < t^\alpha < 2t^\alpha < \epsilon$. Define

$$P_{t,i} = K_i \cup \Upsilon_{t,i}(\Gamma_i \times (tR, 2t^\alpha)) \subset Y_i \quad \text{and}$$
$$Q_t = M_0 \setminus \bigcup_{i=1}^{n} \Phi_i(\Gamma_i \times (0, t^\alpha)) \subset M_0$$

The diffeomorphism $\Phi_i \circ \Upsilon_{t,i}^{-1}$ identifies $\Upsilon_{t,i}(\Gamma_i \times (t^\alpha, 2t^\alpha)) \subset P_{t,i}$ and $\Phi_i(\Gamma_i \times (t^\alpha, 2t^\alpha)) \subset Q_t$, and we define the intersection $P_{t,i} \cap Q_t$ to be the region $\Upsilon_{t,i}(\Gamma_i \times (t^\alpha, 2t^\alpha)) \cong \Phi_i(\Gamma_i \times (t^\alpha, 2t^\alpha)) \cong \Gamma_i \times (t^\alpha, 2t^\alpha)$. Define M_t to be the quotient space of the union $(\bigcup_{i=1}^n P_{t,i}) \cup Q_t$ under the equivalence relation identifying the two annuli $\Upsilon_{t,i}(\Gamma_i \times (t^\alpha, 2t^\alpha))$ and $\Phi_i(\Gamma_i \times (t^\alpha, 2t^\alpha))$ for $i = 1, \ldots, n$. Then M_t is a smooth nonsingular compact 6-fold for each t.

3.4.2 Nearly Calabi–Yau structures (ω_t, Ω_t): the case $\lambda_i < -3$

In this section we construct on M_t a real closed 2-form ω_t and a complex closed 3-form Ω_t, and show they together give nearly Calabi–Yau structures on M_t for small enough t. Define

$$\omega_t = \begin{cases} \omega_0 \quad \text{on} \;\; Q_t, \\ t^2\omega_{Y_i} \quad \text{on} \;\; P_{t,i} \text{ for } i = 1, \ldots, n. \end{cases}$$

This is well-defined as $\Phi_i^*(\omega_0) = \omega_{V_i} = \Upsilon_{t,i}^*(t^2\omega_{Y_i})$ on each intersection $P_{t,i} \cap Q_t$ by Theorem 3.24 and 3.27. Thus ω_t gives a symplectic form on M_t.

Let $F : \mathbb{R} \longrightarrow [0,1]$ be a smooth, increasing function with $F(s) = 0$ for $s \leq 1$ and $F(s) = 1$ for $s \geq 2$. Then for $r \in (tR, \epsilon)$, $F(t^{-\alpha}r) = 0$ for $tR < r \leq t^\alpha$ and $F(t^{-\alpha}r) = 1$ for $2t^\alpha \leq r < \epsilon$. We now define a complex 3-form on M_t. From (3.24), we have $|\Phi_i^*(\Omega_0) - \Omega_{V_i}|_{g_{V_i}} = O(r^\nu)$. As $\nu > 0$, it follows that $\Phi_i^*(\Omega_0) - \Omega_{V_i}$ is exact, and we can write

$$\Phi_i^*(\Omega_0) = \Omega_{V_i} + dA_i \tag{3.35}$$

for some complex 2-form $A_i(\gamma, r)$ on $\Gamma_i \times (0, \epsilon)$ satisfying

$$|\nabla^k A_i(\gamma, r)|_{g_{V_i}} = O(r^{\nu+1-k}) \quad \text{as } r \to 0 \ \text{ for all } k \geq 0. \tag{3.36}$$

The case $k = 0$ follows by defining A_i by integration as in Theorem 3.24. Similarly, as we have assumed $\lambda_i < -3$ to simplify the problem, the 3-form $\Upsilon_i^*(\Omega_{Y_i}) - \Omega_{V_i}$ is exact by Lemma 3.26 and we can write

$$\Upsilon_i^*(\Omega_{Y_i}) = \Omega_{V_i} + dB_i$$

for some complex 2-form $B_i(\gamma, r)$ on $\Gamma_i \times (R, \infty)$ satisfying

$$|\nabla^k B_i(\gamma, r)|_{g_{V_i}} = O(r^{\lambda_i+1-k}) \quad \text{as } r \to \infty \text{ and for all } k \geq 0.$$

Then we apply a homothety to Y_i and rescale the forms to get $B_i(\gamma, t^{-1}r)$ on $\Gamma_i \times (tR, \infty)$ such that

$$\Upsilon_{t,i}^*(t^3\Omega_{Y_i}) = \Omega_{V_i} + t^3 dB_i(\gamma, t^{-1}r) \tag{3.37}$$

and

$$|\nabla^k B_i(\gamma, t^{-1}r)|_{g_{V_i}} = O(t^{-\lambda_i-3} r^{\lambda_i+1-k}) \quad \text{for } r > tR \text{ and for all } k \geq 0. \tag{3.38}$$

Define a smooth, complex closed 3-form Ω_t on M_t by

$$\Omega_t = \begin{cases} \Omega_0 \quad \text{on } \ Q_t \setminus \big[(\bigcup_{i=1}^n P_{t,i}) \cap Q_t\big], \\ \Omega_{V_i} + d\big[F(t^{-\alpha}r)A_i(\gamma, r) + t^3(1 - F(t^{-\alpha}r))B_i(\gamma, t^{-1}r)\big] \ \text{ on } P_{t,i} \cap Q_t, \ \text{ for } i = 1, \ldots, n, \\ t^3\Omega_{Y_i} \ \text{ on } \ P_{t,i} \setminus (P_{t,i} \cap Q_t) \text{ for } i = 1, \ldots, n. \end{cases} \tag{3.39}$$

Note that when $2t^\alpha \leq r < \epsilon$ we have $F(t^{-\alpha}r) = 1$ so that $\Omega_t = \Phi_i^*(\Omega_0)$ by (3.35), and when $tR < r \leq t^\alpha$ we have $F(t^{-\alpha}r) = 0$, so that $\Omega_t = \Upsilon_{t,i}^*(t^3\Omega_{Y_i})$ by (3.37). Therefore, Ω_t interpolates between $\Phi_i^*(\Omega_0)$ near $r = \epsilon$ and $\Upsilon_{t,i}^*(t^3\Omega_{Y_i})$ near $r = tR$.

Recall that if we get a real closed 2-form ω and a complex 3-form Ω which are sufficiently close to the Kähler form $\tilde{\omega}$ and the holomorphic volume form $\tilde{\Omega}$ of a Calabi–Yau structure respectively, then Proposition 3.6 tells us that (ω, Ω) gives a nearly Calabi–Yau structure on the manifold. Making use of this idea, we have

Proposition 3.31 *Let M_t, ω_t and Ω_t be defined as above. Then (ω_t, Ω_t) gives a nearly Calabi–Yau structure on M_t for sufficiently small t.*

Proof. We only have to prove the statement on each $P_{t,i} \cap Q_t$, as $(M_t, \omega_t, \Omega_t)$ is Calabi–Yau on $P_{t,i} \setminus (P_{t,i} \cap Q_t)$ and on $Q_t \setminus \big[(\bigcup_{i=1}^n P_{t,i}) \cap Q_t$, and hence is nearly Calabi-Yau. We prove it by applying Proposition 3.6, that is, we show on each $P_{t,i} \cap Q_t$ that (ω_t, Ω_t) is sufficiently close to the genuine Calabi–Yau structure $(\omega_{V_i}, \Omega_{V_i})$ coming from the Calabi–Yau cone V_i for small t. We choose to compare with $(\omega_{V_i}, \Omega_{V_i})$ rather than either of the Calabi–Yau structures (ω_0, Ω_0) and $(t^2\omega_{Y_i}, t^3\Omega_{Y_i})$ on $P_{t,i} \cap Q_t$ since we have already got bounds on norms for various forms w.r.t. the cone metric g_{V_i}. Now $\omega_t = \omega_{V_i}$ on $P_{t,i} \cap Q_t$, while

$$\Omega_t - \Omega_{V_i} = d\big[F(t^{-\alpha}r)A_i(\gamma, r) + t^3(1 - F(t^{-\alpha}r))B_i(\gamma, t^{-1}r)\big] \quad \text{on} \quad P_{t,i} \cap Q_t$$

by (3.39). Calculation shows that

$$|(\Omega_t - \Omega_{V_i})(\gamma, r)|_{g_{V_i}} = O(t^{-\lambda_i(1-\alpha)}) + O(t^{\alpha\nu}) \quad \text{for } r \in (t^\alpha, 2t^\alpha), \tag{3.40}$$

and hence $|\Omega_t - \Omega_{V_i}|_{g_{V_i}} \leq C_0 t^\gamma$ where $C_0 > 0$ is some constant and $\gamma = \min(-\lambda_i(1-\alpha),\ \alpha\nu)$. Hence Proposition 3.6 applies with $\epsilon = C_0 t^\gamma$ if t is small enough such that $C_0 t^\gamma \leq \epsilon_1$, and so (ω_t, Ω_t) gives a nearly Calabi–Yau structure on $P_{t,i} \cap Q_t$. This completes the proof. $\square$

Therefore we can associate an almost complex structure J_t and a real 3-form $\theta'_{2,t}$ such that $\Omega'_t := \mathrm{Re}(\Omega_t) + i\theta'_{2,t}$ is a (3,0)-form w.r.t. J_t. Moreover, we have the 2-form ω'_t, which is the rescaled (1,1)-part of ω_t w.r.t. J_t, and the associated metric g_t on M_t. Following similar arguments to Proposition 3.6, we conclude that $|g_t - g_{V_i}|_{g_{V_i}} = O(t^{-\lambda_i(1-\alpha)}) + O(t^{\alpha\nu}) = |g_t^{-1} - g_V^{-1}|_{g_{V_i}}$.

3.4.3 The main result

We are now ready to prove our first main result in this thesis: the desingularization of compact Calabi–Yau 3-folds M_0 with conical singularities in the simplest case $\lambda_i < -3$, assuming the existence of singular Calabi–Yau metrics on them. We prove it using Theorem 3.14, the analytic existence result for genuine Calabi–Yau structures.

Theorem 3.32 *Suppose $(M_0, J_0, \omega_0, \Omega_0)$ is a compact Calabi–Yau 3-fold with conical singularities x_i with rate $\nu > 0$ modelled on Calabi–Yau cones $(V_i, J_{V_i}, \omega_{V_i}, \Omega_{V_i})$ for $i = 1, \ldots, n$. Let $(Y_i, J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i})$ be an AC Calabi–Yau 3-fold with rate $\lambda_i < -3$ modelled on the same Calabi–Yau cone V_i. Define a family $(M_t, \omega_t, \Omega_t)$ of nonsingular compact nearly Calabi–Yau 3-folds, with the associated metrics g_t as in §3.4.1 and §3.4.2.*

Then M_t admits a Calabi–Yau structure $(\tilde{J}_t, \tilde{\omega}_t, \tilde{\Omega}_t)$ such that $\|\tilde{\omega}_t - \omega_t\|_{C^0} \leq Kt^\kappa$ and $\|\tilde{\Omega}_t - \Omega_t\|_{C^0} \leq Kt^\kappa$ for some $\kappa, K > 0$ and for sufficiently small t. The cohomology classes satisfy $[\mathrm{Re}(\Omega_t)] = [\mathrm{Re}(\tilde{\Omega}_t)] \in H^3(M_t, \mathbb{R})$ and $[\omega_t] = c_t\,[\tilde{\omega}_t] \in H^2(M_t, \mathbb{R})$ for some $c_t > 0$. Here all norms are computed with respect to g_t.

Proof. First we estimate the norms of $\omega_t - \omega'_t$ and $\mathrm{Im}(\Omega_t) - \theta'_{2,t} = \mathrm{Im}(\Omega_t) - \mathrm{Im}(\Omega'_t)$ on each $P_{t,i} \cap Q_t$, as in part (i) of Theorem 3.14. Since ω'_t depends on $\mathrm{Re}(\Omega_t)$ and ω_t $(= \omega_{V_i}$ on $P_{t,i} \cap Q_t)$

on $\mathrm{Re}(\Omega_{V_i})$, it follows that

$$\begin{aligned}|\omega_t-\omega_t'|_{g_t} &\le C_1|\omega_t-\omega_t'|_{g_{V_i}} \le C_2|\mathrm{Re}(\Omega_t)-\mathrm{Re}(\Omega_{V_i})|_{g_{V_i}} \le C_2|\Omega_t-\Omega_{V_i}|_{g_{V_i}}\\ &= O(t^{-\lambda_i(1-\alpha)})+O(t^{\alpha\nu})\end{aligned}$$

for some constants $C_1, C_2>0$ and hence

$$\|\omega_t-\omega_t'\|_{C^0} = O(t^{-\lambda_i(1-\alpha)})+O(t^{\alpha\nu}). \tag{3.41}$$

From the fact that $\mathrm{vol}(P_{t,i}\cap Q_t)=O(t^{6\alpha})$, we have

$$\|\omega_t-\omega_t'\|_{L^2} = O(t^{6\alpha/2})\cdot\|\omega_t-\omega_t'\|_{C^0} = O(t^{3\alpha-\lambda_i(1-\alpha)})+O(t^{3\alpha+\alpha\nu}). \tag{3.42}$$

Furthermore,

$$\begin{aligned}|\mathrm{Im}(\Omega_t)-\mathrm{Im}(\Omega_t')|_{g_t} &\le C_3\,|\mathrm{Im}(\Omega_t)-\mathrm{Im}(\Omega_t')|_{g_{V_i}}\\ &\le C_3\,|\mathrm{Im}(\Omega_t)-\mathrm{Im}(\Omega_{V_i})|_{g_{V_i}}+C_3\,|\mathrm{Im}(\Omega_{V_i})-\mathrm{Im}(\Omega_t')|_{g_{V_i}}\\ &\le C_3\,|\Omega_t-\Omega_{V_i}|_{g_{V_i}}+C_4\,|\mathrm{Re}(\Omega_{V_i})-\mathrm{Re}(\Omega_t)|_{g_{V_i}}\\ &= O(t^{-\lambda_i(1-\alpha)})+O(t^{\alpha\nu})\end{aligned}$$

for some constants $C_3, C_4>0$, as $\mathrm{Im}(\Omega_{V_i})$ is determined by $\mathrm{Re}(\Omega_{V_i})$ and $\mathrm{Im}(\Omega_t)'$ by $\mathrm{Re}(\Omega_t)$. Therefore,

$$\|\mathrm{Im}(\Omega_t)-\mathrm{Im}(\Omega_t')\|_{C^0} = O(t^{-\lambda_i(1-\alpha)})+O(t^{\alpha\nu}) \tag{3.43}$$

and

$$\|\mathrm{Im}(\Omega_t)-\mathrm{Im}(\Omega_t')\|_{L^2} = O(t^{3\alpha-\lambda_i(1-\alpha)})+O(t^{3\alpha+\alpha\nu}). \tag{3.44}$$

It can be deduced from (3.39) and (3.40) that $|\nabla^{g_t}(\Omega_t-\Omega_{V_i})|_{g_t}=O(t^{-\lambda_i(1-\alpha)-\alpha})+O(t^{\alpha\nu-\alpha})$ and $|(\nabla^{g_t})^2(\Omega_t-\Omega_{V_i})|_{g_t}=O(t^{-\lambda_i(1-\alpha)-2\alpha})+O(t^{\alpha\nu-2\alpha})$, which imply the equalities

$$\|\nabla^{g_t}(\omega_t-\omega_t')\|_{C^0}=O(t^{-\lambda_i(1-\alpha)-\alpha})+O(t^{\alpha\nu-\alpha}) \tag{3.45}$$

and

$$\|(\nabla^{g_t})^2(\omega_t-\omega_t')\|_{C^0}=O(t^{-\lambda_i(1-\alpha)-2\alpha})+O(t^{\alpha\nu-2\alpha}). \tag{3.46}$$

Then the L^{12}-norm satisfies

$$\begin{aligned}\|\nabla^{g_t}(\omega_t-\omega_t')\|_{L^{12}} &= O(t^{6\alpha/12})\cdot\|\nabla^{g_t}(\omega_t-\omega_t')\|_{C^0}\\ &= O(t^{-\frac{\alpha}{2}-\lambda_i(1-\alpha)})+O(t^{-\frac{\alpha}{2}+\alpha\nu}).\end{aligned} \tag{3.47}$$

Finally, we estimate the L^{12}-norms of $\nabla^{g_t}\omega_t$ and $\nabla^{g_t}\mathrm{Re}(\Omega_t)$. Note that

$$|\nabla^{g_t}\omega_t|_{g_t} \le C_5\,|(\nabla^{g_t}-\nabla^{g_{V_i}})\omega_t\,|_{g_{V_i}}+C_5\,|\nabla^{g_{V_i}}\omega_t|_{g_{V_i}} = C_5\,|(\nabla^{g_t}-\nabla^{g_{V_i}})\omega_t\,|_{g_{V_i}}$$

for some constant $C_5>0$, as $\omega_t=\omega_{V_i}$ on $P_{t,i}\cap Q_t$ and $\nabla^{g_{V_i}}\omega_{V_i}=0$. Then

$$\begin{aligned}|\nabla^{g_t}\omega_t|_{g_t} &\le C_5\,|(\nabla^{g_t}-\nabla^{g_{V_i}})|_{g_{V_i}}\cdot|\omega_t|_{g_{V_i}}\\ &\le \frac{3}{2}C_5\,|g_t^{-1}|_{g_{V_i}}\cdot|\nabla^{g_{V_i}}g_t|_{g_{V_i}}\cdot|\omega_t|_{g_{V_i}} \quad \text{by (3.20)}\\ &= C_6|\nabla^{g_{V_i}}(g_t-g_{V_i})|_{g_{V_i}} \quad \text{as } \nabla^{g_{V_i}}g_{V_i}=0.\end{aligned}$$

Here C_6 is an upper bound for $\frac{3}{2}\,C_5\,|g_t^{-1}|_{g_{V_i}} \cdot |\omega_t|_{g_{V_i}}$ which is independent of t. It follows that

$$|\nabla^{g_t}\omega_t|_{g_t} \;=\; O(t^{-\lambda_i(1-\alpha)-\alpha}) + O(t^{\alpha\nu-\alpha}),$$

and consequently

$$\|\nabla^{g_t}\omega_t\|_{L^{12}} \;=\; O(t^{-\frac{\alpha}{2}-\lambda_i(1-\alpha)}) + O(t^{-\frac{\alpha}{2}+\alpha\nu}). \tag{3.48}$$

A similar argument shows

$$\|\nabla^{g_t}\mathrm{Re}(\Omega_t)\|_{L^{12}} \;=\; O(t^{-\frac{\alpha}{2}-\lambda_i(1-\alpha)}) + O(t^{-\frac{\alpha}{2}+\alpha\nu}). \tag{3.49}$$

Now for parts (i) to (iii) of Theorem 3.14 to hold, we need:

$$\begin{cases} -\lambda_i(1-\alpha) \geq \kappa, \quad \alpha\nu \geq \kappa \quad \text{from (3.41) and (3.43)}, \\ 3\alpha - \lambda_i(1-\alpha) \geq 3+\kappa, \quad 3\alpha + \alpha\nu \geq 3+\kappa \quad \text{from (3.42) and (3.44)}, \\ -\frac{\alpha}{2} - \lambda_i(1-\alpha) \geq -\frac{1}{2}+\kappa, \quad -\frac{\alpha}{2}+\alpha\nu \geq -\frac{1}{2}+\kappa \quad \text{from (3.47), (3.48) and (3.49)}, \\ -\lambda_i(1-\alpha) - \alpha \geq \kappa - 1, \quad \alpha\nu - \alpha \geq \kappa - 1 \quad \text{from (3.45)}, \\ -\lambda_i(1-\alpha) - 2\alpha \geq \kappa - 2, \quad \alpha\nu - 2\alpha \geq \kappa - 2 \quad \text{from (3.46)}. \end{cases}$$

Observe that the second set of inequalities imply all the others, as $\alpha \leq 1$. Therefore, calculations using these two inequalities show that there exist solutions $\alpha \in (0,1)$ and $\kappa > 0$ for any $\nu > 0$ and $\lambda_i < -3$. For example, we could take

$$\alpha = \frac{1}{2}\left(\frac{6+\nu}{3+\nu}\right) \in (0,1) \quad \text{and} \quad \kappa = \min\Big((1-\alpha)(-3-\lambda_1), \ldots, (1-\alpha)(-3-\lambda_n), \frac{\nu}{2}\Big) > 0.$$

For parts (iv) and (v) of Theorem 3.14, note that under the homothety $g_{Y_i} \mapsto t^2 g_{Y_i}$ on the AC Calabi–Yau 3-fold Y_i we have $\delta(t^2 g_{Y_i}) = t\,\delta(g_{Y_i})$ and $\|R(t^2 g_{Y_i})\|_{C^0} = t^{-2}\|R(g_{Y_i})\|_{C^0}$. Moreover, the dominant contributions to $\delta(g_t)$ and $\|R(g_t)\|_{C^0}$ for small t come from $\delta(t^2 g_{Y_i})$ and $\|R(t^2 g_{Y_i})\|_{C^0}$ which are proportional to t and t^{-2}. Thus there exist constants $E_3, E_4 > 0$ such that (iv), (v) of Theorem 3.14 hold for sufficiently small t. Hence by Theorem 3.14, M_t admits a Calabi–Yau structure $(\tilde{J}_t, \tilde{\omega}_t, \tilde{\Omega}_t)$ such that $\|\tilde{\omega}_t - \omega_t\|_{C^0} \leq Kt^{\kappa}$ and $\|\tilde{\Omega}_t - \Omega_t\|_{C^0} \leq Kt^{\kappa}$ for some $\kappa, K > 0$ and for sufficiently small t.

Finally, the cohomology condition in Theorem 3.14 holds automatically here. This can be seen by applying the following lemma:

Lemma 3.33 *Let $(Y, J_Y, \omega_Y, \Omega_Y)$ be an AC Calabi–Yau m-fold with rate $\lambda < 0$ modelled on a Calabi–Yau cone $(V, J_V, \omega_V, \Omega_V)$ with link Γ. Then Y has holonomy $\{1\}$, or $G \ltimes \mathrm{Sp}(m/2)$ for some finite group G and m even, or $\mathrm{SU}(m)$.*

Remark A related argument in the ALE case is given by Joyce [26, Thm. 8.2.4] in which he shows that $\mathrm{Hol}(g_Y) = \mathrm{SU}(m)$ when Y is the crepant resolution of the Calabi–Yau cone $\mathbb{C}^m/G$ for a finite subgroup G of $\mathrm{SU}(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$.

Proof of Lemma 3.33. First of all, we show that the universal cover $\tilde{Y}$ of Y is also AC, and then we exclude the reducible holonomy groups of $\tilde{Y}$ except $\{1\}$ and use Berger's holonomy classification on $\tilde{Y}$ to show the lemma. We have the compact subset $K \subset Y$, and a diffeomorphism $\Upsilon : \Gamma \times (R, \infty) \longrightarrow Y \setminus K$. Since Γ is a compact manifold with positive curvature Einstein metric, its fundamental group $\pi_1(\Gamma)$ is finite (see [3, cor. 6.67]). Denote by π the covering map $\pi : \tilde{Y} \longrightarrow Y$, and $\tilde{K}$ the preimage $\pi^{-1}(K)$ in $\tilde{Y}$. Then $\tilde{Y} \setminus \tilde{K}$ is some number (possibly infinite) of copies of $\Gamma' \times (R, \infty)$, where Γ' is some connected cover of Γ. But Γ' must be a finite cover of Γ, say a k-fold cover, as Γ has finite fundamental group, so it is compact, and $\Gamma' \times (R, \infty)$ is an AC end. The remark after Definition 3.25 now shows that $\tilde{Y}$ can have only one end, so $\tilde{Y}$ is AC asymptotic to $\Gamma' \times (R, \infty)$, and is a finite k-fold cover of Y.

Now the universal cover $\tilde{Y}$ is AC. If the holonomy of $\tilde{Y}$ is reducible, then either it is $\{1\}$ in which case $\tilde{Y}$ is a $\mathbb{C}^m$, or $\tilde{Y}$ is isometric to a product. We claim that in this case $\tilde{Y}$ can be writen as $\tilde{Y} \cong W \times X$ where W has nontrivial holonomy and X is noncompact. To see this, since $\tilde{Y}$ is not a $\mathbb{C}^m$, so one of W and X has nontrivial holonomy, and since $\tilde{Y}$ is noncompact, so one of W and X is noncompact. By considering the cases (a) $W = \mathbb{C}^k$ for some $1 \leq k \leq m-1$; (b) W has nontrivial holonomy and is compact; (c) W has nontrivial holonomy and is noncompact, and same for X, it is not hard to see that, swapping W and X if necessary, the claim is true.

We now take $w \in W$ with nonzero curvature R_w, which is possible as W has nontrivial holonomy. Consider $\{w\} \times X$ in $\tilde{Y}$, then the curvature of of $W \times X$ is of order $O(1)$ as we go to infinity in $\{w\} \times X$ since X is noncompact, but this contradicts the AC condition of $\tilde{Y}$ which requires the curvature to decay at $O(r^{-2})$. Therefore we have excluded reducible holonomy groups for $\tilde{Y}$ except $\{1\}$, and hence by Berger's classification, the possibilities for holonomy of $\tilde{Y}$ are $\mathrm{SU}(m)$, $\mathrm{Sp}(m/2)$ and $\{1\}$. Then the holonomy of our AC Calabi–Yau m-fold Y is either $\mathrm{SU}(m)$, $G \ltimes \mathrm{Sp}(m/2)$ for G a quotient group of the finite covering group of $\pi : \tilde{Y} \longrightarrow Y$, or $\{1\}$. This completes the proof of Lemma 3.33. □

Now, back to the proof of Theorem 3.32, $\mathrm{Hol}(g_{Y_i})$ lies inside the holonomy group $\mathrm{Hol}(g_t)$, as g_t equals g_{Y_i} on appropriate region of M_t. Thus Lemma 3.33 implies that $\mathrm{Hol}(g_t) = \mathrm{SU}(3)$ as well. Since $\tilde{g}_t$ converges to g_t as $t \to 0$, and since holonomy groups are semicontinuous under limits, i.e. $\mathrm{Hol}\Big(\lim_{t\to 0} g_t\Big) \subseteq \lim_{t \to 0} \mathrm{Hol}(g_t)$, therefore $\mathrm{Hol}(\tilde{g}_t)$ must be the whole $\mathrm{SU}(3)$. The first cohomology group $H^1(M_t, \mathbb{R})$ therefore vanishes for each sufficiently small t, and the theorem now follows from Theorem 3.14. □

3.4.4 Conclusions

We conclude by applying the above result to some examples given in §3.3. First consider the situation in Example 3.28 and take $m = 3$. Then the crepant resolution X of the Calabi–Yau cone $\mathbb{C}^3/G$ is an AC Calabi–Yau 3-fold with rate -6. Thus Theorem 3.32 applies and we can desingularize any compact Calabi–Yau 3-fold M_0 with conical singularities modelled on $\mathbb{C}^3/G$, or equivalently, any Calabi–Yau 3-orbifold with isolated singularities, by the gluing process. Note that in general we have to assume the existence of singular Calabi–Yau metrics on manifolds with conical singularities, but in the orbifold case, there is a result asserting the existence of Calabi–Yau metrics: if M is a compact Kähler orbifold with $c_1(M) = 0$, then there is a unique Ricci-flat Kähler metric in each Kähler class on M (see for instance [26, Thm. 6.5.6]).

If we take $G = \mathbb{Z}_3$, a standard example of compact Calabi–Yau 3-orbifold with isolated singularities is given in [26, Example 6.6.3]. Define a lattice Λ in $\mathbb{C}^3$ by

$$\Lambda = \mathbb{Z}^3 \oplus \zeta\mathbb{Z}^3 = \{(a_1 + b_1\zeta,\ a_2 + b_2\zeta,\ a_3 + b_3\zeta) : a_j, b_j \in \mathbb{Z}\}$$

where $\zeta = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{2\pi i/3}$ denotes the cube root of unity. Let T^6 be the quotient $\mathbb{C}^3/\Lambda$, with a flat Calabi–Yau structure (J, ω, Ω). Write points on T^6 as $(z_1, z_2, z_3) + \Lambda$ for $(z_1, z_2, z_3) \in \mathbb{C}^3$. We can also regard T^6 as the product of three T^2's where each T^2 is the quotient $\mathbb{C}/(\mathbb{Z} \oplus \zeta\mathbb{Z})$.

Define an action generated by ζ on T^6 by

$$\zeta \,:\, (z_1, z_2, z_3) + \Lambda \longmapsto (\zeta z_1, \zeta z_2, \zeta z_3) + \Lambda.$$

This ζ-action is well-defined, as $\zeta \cdot \Lambda = \Lambda$. The group $\mathbb{Z}_3 = \{1, \zeta, \zeta^2\}$ is a finite group of automorphisms of T^6, preserving the flat Calabi–Yau structure on it. Thus the toroidal orbifold $T^6/\mathbb{Z}_3$ is a Calabi–Yau 3-orbifold, which can also be expressed as $\mathbb{C}^3/A$, where A is the group generated rotations by ζ and translations in Λ. Write points on $T^6/\mathbb{Z}_3$ as $\mathbb{Z}_3 \cdot (z_1, z_2, z_3) + \Lambda$.

In each T^2 there are three fixed points of ζ located at 0, $\frac{1}{2} + \frac{i}{2\sqrt{3}}$, $\frac{i}{\sqrt{3}}$. The element $\zeta^2 = \zeta^{-1}$ clearly has the same fixed points. Altogether the orbifold $T^6/\mathbb{Z}_3$ has then 27 isolated singularities. Note that $\frac{1}{2} + \frac{i}{2\sqrt{3}} = \frac{2i}{\sqrt{3}} - \zeta$, so we write the 27 fixed points on T^6 as

$$\Big\{(c_1, c_2, c_3) + \Lambda : c_1, c_2, c_3 \in \big\{0, \frac{i}{\sqrt{3}}, \frac{2i}{\sqrt{3}}\big\}\Big\}.$$

Now these singular points are locally modelled on the Calabi–Yau cone $\mathbb{C}^3/\mathbb{Z}_3$, thus making the orbifold $T^6/\mathbb{Z}_3$ a Calabi–Yau 3-fold with conical singularities. Applying Theorem 3.32, we can desingularize $T^6/\mathbb{Z}_3$ by gluing in AC Calabi–Yau 3-folds $K_{\mathbb{CP}^2}$ (with rate -6) at the singular points, obtaining a Calabi–Yau desingularization of $T^6/\mathbb{Z}_3$.

Now the *Schlessinger Rigidity Theorem* [47] states that if G is a finite subgroup of GL(m,$\mathbb{C}$) and the singularities of $\mathbb{C}^m/G$ are all of codimension at least three, then $\mathbb{C}^m/G$ is rigid, i.e. it admits no nontrivial deformations. It can then be shown by using this rigidity theorem that if we desingularize a Calabi–Yau 3-orbifold with isolated singularities modelled on $\mathbb{C}^3/\mathbb{Z}_3$ by gluing, we shall obtain a crepant resolution of the original orbifold.

On the crepant resolution the existence of Calabi–Yau metrics is guaranteed by Yau's solution to the Calabi conjecture [52]. However, it does not provide a way to write down the Calabi–Yau metrics explicitly, and so in general we do not know much about what the Calabi–Yau metrics are like. But in the orbifold case, our result tells a bit more by giving a quantitative description of these Calabi–Yau metrics, showing that these metrics locally look like the metrics obtained by gluing the orbifold metrics and the ALE metrics on the crepant resolution of $\mathbb{C}^3/G$.

Our result can also be applied to desingularize compact Calabi–Yau 3-folds with conical singularities modelled on the Calabi–Yau cone $\mathcal{O}(-2,-2) \setminus (\mathbb{CP}^1 \times \mathbb{CP}^1)$ by gluing in the AC Calabi–Yau 3-fold $\mathcal{O}(-2,-2)$-bundle with rate -6. Thus we could resolve a kind of singularity which is not of orbifold type.

Theorem 3.32 deals with the simplest case $\lambda_i < -3$, and we shall extend it in Chapter 4 by including the case $\lambda_i = -3$, so that the result can be applicable to a larger class of AC Calabi–Yau 3-folds.

Chapter 4

Desingularizations of Calabi–Yau 3-folds with conical singularities: The obstructed case

In the last chapter we have developed an analytic tool to desingularize Calabi–Yau 3-folds with conical singularities. There we used AC Calabi–Yau 3-folds Y_i with rate $\lambda_i < -3$, and hence $[\Upsilon^*_{t,i}(t^3\Omega_{Y_i}) - \Omega_{V_i}] = 0$ by Lemma 3.26. This chapter extends Theorem 3.32 to a more complicated situation, in which we relax the assumption $\lambda_i < -3$ to allow $\lambda_i = -3$ and $[\Upsilon^*_{t,i}(t^3\Omega_{Y_i}) - \Omega_{V_i}] \neq 0$ in $H^3(\Gamma_i, \mathbb{C})$. But the cohomology class $[\Phi^*(\Omega_0) - \Omega_{V_i}]$ is always zero by Lemma 3.22, so there are *cohomological obstructions* to defining the closed 3-form Ω_t which interpolates between $\Phi^*(\Omega_0)$ and $\Upsilon^*_t(t^3\Omega_{Y_i})$. Thus allowing $\lambda_i = -3$ introduces global problems to our gluing method. Moreover, since we have now $|\Upsilon^*_t(t^3\Omega_{Y_i}) - \Omega_{V_i}|_{g_{V_i}} = O(r^{-3})$, our definition of Ω_t will contribute $O(t^{3(1-\alpha)}) + O(t^{\alpha\nu})$ to the error, which is too large for parts (i)-(iii) of Theorem 3.14 to hold.

The method we use here is to replace the holomorphic (3,0)-form Ω_0 on M_0 by $\Omega_0 + t^3\chi$, where χ is some closed and coclosed (2,1)-form with appropriate asymptotic behaviour, and $[\Phi^*_i(\chi)] = [\Upsilon^*_{t,i}(t^3\Omega_Y)]$ on $H^3(\Gamma_i, \mathbb{C})$. We shall show in the following that such χ exists and it cancels out the $O(t^{-3}r^3)$ terms such that Theorem 3.14 can handle the size of the error introduced.

As mentioned in the last chapter, we shall restrict ourselves to the case $\lambda_i < -2$ so that $\Upsilon^*(\omega_{Y_i}) - \omega_{V_i}$ is always exact. Here we only make the improvement from $\lambda_i < -3$ to $\lambda \leq -3$ since the case $\lambda \in (-3, -2)$ will introduce extra terms which contribute errors that we cannot cope with in our estimates later. This seems not a big step forward. However, there are some examples of AC Calabi–Yau 3-folds with rate -3 and $[\Upsilon^*_{t,i}(t^3\Omega_{Y_i}) - \Omega_{V_i}] \neq 0$, so that our result can be applicable to a larger class of AC Calabi–Yau 3-folds.

We begin in §4.1 by setting up notations and giving a brief account of why a (2,1)-form is needed for our construction. Section 4.2 provides a detailed discussion on the analytic theory of Weighted Sobolev spaces due to Lockhart and McOwen [39]. In §4.3, we apply materials in §4.2 to construct our desired (2,1)-form on M_0. We then glue Y_i's into M_0, constructing the nearly

Calabi–Yau structures in §4.4, as in Chapter 3. Section 4.5 gives the main result on Calabi–Yau desingularization when $\lambda_i = -3$ which generalizes the result in Chapter 3.

In contrast to Chapter 3, where the desingularization result is local, there are global topological conditions for the desingularization to be possible, which can relate different singular points.

In the last section we focus on some singular Calabi–Yau 3-folds where the singularities are known as the *ordinary double points*. Unlike the orbifold case we discussed in last chapter, we shall assume the existence of singular Calabi–Yau metrics on compact complex 3-folds with ordinary double points, as we do not currently know any existence result of Calabi–Yau metrics on such kind of manifolds. The desingularization of Calabi–Yau 3-folds with ordinary double points belongs to the case $\lambda_i = -3$, as the AC Calabi–Yau 3-folds Q_ϵ have rate -3. After constructing a nice coordinate system on Q_ϵ, we apply our main result to repair ordinary double points. We conclude by showing that our result is in some way equivalent to Friedman's result [16] on smoothing ordinary double points, hence providing another analytic proof of a known result in algebraic geometry.

4.1 The geometric set-up

We shall consider the following situation. Let $(M_0, J_0, \omega_0, \Omega_0)$ be a compact Calabi–Yau 3-fold with finitely many conical singularities at $x_1, \dots, x_n$ with same rate $\nu > 0$ modelled on Calabi–Yau cones $V_1, \dots, V_n$. Write $M_0' = M_0 \setminus \{x_1, \dots, x_n\}$. For simplicity we suppose M_0' is connected. Denote by $\Gamma_1, \dots, \Gamma_n$ the links of $V_1, \dots, V_n$. As was discussed in the remark after Definition 3.25, we take $\Gamma_1, \dots, \Gamma_n$ to be connected. For $i = 1, \dots, n$ and some small $\epsilon > 0$, there should exist an open neighbourhood S_i of x_i such that the closures $\bar{S}_1, \dots, \bar{S}_n$ are disjoint in M_0. By Theorem 3.24, there should exist a diffeomorphism $\Phi_i : \Gamma_i \times (0, \epsilon) \longrightarrow S_i \setminus \{x_i\}$ such that

$$\Phi_i^*(\omega_0) = \omega_{V_i} \text{ and } |\nabla^k(\Phi_i^*(\Omega_0) - \Omega_{V_i})|_{g_{V_i}} = O(r^{\nu - k}) \tag{4.1}$$

for $i = 1, \dots, n$ and all $k \geq 0$.

Let $(Y_i, J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i})$ be AC Calabi–Yau 3-folds with rates $\lambda_i = -3$ modelled on the same cones V_i. Then there should exist a compact subset $K_i \subset Y_i$ and, by Theorem 3.27, a diffeomorphism $\Upsilon_i : \Gamma_i \times (R, \infty) \longrightarrow Y_i \setminus K_i$ for some $R > 0$ such that

$$\Upsilon_i^*(\omega_{Y_i}) = \omega_{V_i} \text{ and } |\nabla^k(\Upsilon_i^*(\Omega_{Y_i}) - \Omega_{V_i})|_{g_{V_i}} = O(r^{\lambda_i - k})$$

for $i = 1, \dots, n$ and all $k \geq 0$. Suppose there is a closed *homogeneous* (2,1)-form ξ_i of order -3 on V_i with $\omega_{V_i} \wedge \xi_i = 0$, and a diffeomorphism Υ_i such that

$$\Upsilon_i^*(\omega_{Y_i}) = \omega_{V_i} \text{ and } |\nabla^k(\Upsilon_i^*(\Omega_{Y_i}) - \Omega_{V_i} - \xi_i)|_{g_{V_i}} = O(r^{\lambda_i' - k}) \tag{4.2}$$

for $i = 1, \dots, n$, $\lambda_i' < -3$ and all $k \geq 0$. We shall see the meaning of a homogeneous k-form of order α in the beginning of §4.2.2. The point of this condition is to give a nice coordinate system on Y_i so that $(\omega_{Y_i}, \Omega_{Y_i})$ can be written as $(\omega_{V_i}, \Omega_{V_i} + \xi_i + O(r^{\lambda_i'}))$ on V_i. Note that the term $O(r^{\lambda_i'})$ for $\lambda_i' < -3$ is an exact 3-form as it decays faster than $O(r^{-3})$, so we have $[\xi_i] = [\Upsilon_i^*(\Omega_{Y_i}) - \Omega_{V_i}] \in H^3(\Gamma_i, \mathbb{C})$, and $\bigoplus_{i=1}^n [\xi_i] \in \bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})$. We shall assume the existence of such a ξ_i throughout the chapter, and will justify this when we apply our result to desingularize Calabi–Yau 3-folds with ordinary double points in the last section.

We would like to construct a closed (2,1)-form χ with the properties that $\omega_0 \wedge \chi = 0$ and $|\Phi_i^*(\chi) - \xi_i|_{g_{V_i}} = O(r^{-3+\delta})$ for some small $\delta > 0$. The reason for using such a (2,1)-form is that the change of the Calabi–Yau structure $(\omega_0, \Omega_0) \longmapsto (\omega_0, \Omega_0 + \chi)$ is a deformation of Calabi–Yau structures to first order. Here is a way to see this: Suppose (J, ω, Ω) is a Calabi–Yau structure and (J', ω, Ω') is a nearby Calabi–Yau structure with the same Kähler form ω. From the fact that the tangent space of the set of (3,0)-forms w.r.t. some complex structures is the space of (3,0)-forms and (2,1)-forms, we have $\Omega' = \Omega +$ (3,0)-piece + (2,1)-piece to first order. We can just write $\Omega' = \Omega + \chi$ for a (2,1)-form χ if there is no rescaling of Ω involved. Since (J, ω, Ω) and (J', ω, Ω') are Calabi–Yau structures, so $\omega \wedge \Omega = 0$ and $\omega \wedge \Omega' = 0$. Then $\omega \wedge (\Omega + \chi) = 0$, and it follows that $\omega \wedge \chi = 0$, which is why we want the (2,1)-form χ to satisfy $\omega_0 \wedge \chi = 0$.

The advantage of adding a trace-free (2,1)-form χ to Ω_0 is that it introduces a torsion to $(\omega_0, \Omega_0 + \chi)$ of order $O(|\chi|^2)$, rather than $O(|\chi|)$. It will turn out in §4.5 that the effect of having the term $O(|\chi|^2)$ will change a $O(t^3 r^{-3})$ term to a $O(t^6 r^{-6})$ term which will be a small enough error to apply our result.

Observe from (4.2) that $(J_{V_i}, \omega_{V_i}, \Omega_{V_i})$ and $(J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i})$ are two Calabi–Yau structures "close" to each other for large r, with $\Upsilon_i^*(\omega_{Y_i}) = \omega_{V_i}$ and $\Upsilon_i^*(\Omega_{Y_i}) = \Omega_{V_i} + \xi_i + O(r^{\lambda_i'})$. The argument before then shows that $(\omega_{V_i}, \Omega_{V_i}) \longmapsto (\omega_{V_i}, \Omega_{V_i} + \xi_i)$ is a change of Calabi–Yau structures to first order, which implies that ξ_i is of type (3,0) and (2,1) with $\omega_{V_i} \wedge \xi_i = 0$. Thus (4.2) is consistent with what we have assumed on ξ_i.

To construct the (2,1)-form χ we have to do some analysis on M_0', which takes up sections §4.2 and §4.3. We will be first showing that there exists a closed complex 3-form χ' on M_0' with the prescribed asymptotic behaviour under the condition that $\bigoplus_{i=1}^n [\xi_i]$ lies in a certain subspace of $\bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})$. We then project χ' to its "trace-free" (2,1)-component (see §4.3) to obtain our desired χ.

4.2 Analysis on Calabi–Yau 3-folds with conical singularities

The principal analytical tool we shall be using to construct the (2,1)-form χ is the theory of weighted Sobolev spaces on manifolds with ends due to Lockhart and McOwen [39], particularly the Fredholm properties of the linear elliptic operators $d+d^*$ and dd^*+d^*d on differential forms on the noncompact Calabi–Yau 3-fold M_0'.

It is known that on compact manifolds, elliptic operators such as $d+d^*$ and dd^*+d^*d have good regularity properties and they are Fredholm maps between appropriate Sobolev spaces, which are therefore useful tools in problems involving elliptic operators on compact manifolds. Now as M_0' is noncompact, the Sobolev spaces do not have such kind of properties, and it suggests that these spaces are not good choices of Banach spaces for studying elliptic operators on M_0'. Instead, it turns out to be helpful to introduce the concept of weighted Sobolev spaces.

4.2.1 Weighted Sobolev spaces

As the central object in this chapter is a Calabi–Yau manifold, it is enough for us to consider even dimensional manifolds in the following sections.

Definition 4.1 Let (M,g) be a compact Riemannian $2m$-fold with finitely many conical singularities at $x_1,\dots,x_n$. That is, there are Riemannian cones $V_i \cong \Gamma_i \times (0,\infty)\cup\{0\}$ for $i=1,\dots,n$ with Γ_i (identified with $\Gamma_i\times\{1\}$) compact, small open neighbourhoods S_i of x_i in M, and diffeomorphisms $\Phi_i:\Gamma_i\times(0,\epsilon)\longrightarrow S_i\setminus\{x_i\}$ such that

$$|\nabla^k(\Phi_i^*(g)-g_{V_i})|_{g_{V_i}} = O(r^{\nu-k}) \quad \text{as } r\to 0 \text{ and for all } k\geq 0$$

for some rate $\nu>0$ for all i, where r is a coordinate on $(0,\epsilon)$, $g_{V_i}=r^2 g_{V_i}|_{\Gamma_i\times\{1\}}+dr^2$ is the cone metric on V_i and ∇ is the Levi-Civita connection of g_{V_i}.

Write $M'=M\setminus\{x_1,\dots,x_n\}$. Define a *radius function* ρ on M' to be a smooth function $\rho: M'\longrightarrow(0,1]$ such that $\Phi_i^*(\rho)=r$ on $\Gamma_i\times(0,\frac{1}{2}\epsilon)$ for $i=1,\dots,n$ and $\rho\equiv 1$ on $M\setminus\bigcup_{i=1}^n S_i$. Essentially it measures the distance to the singular points. Radius functions always exist.

For $\beta\in\mathbb{R}$, the function ρ^β is well-defined and smooth on M', and equals to $\rho(y)^\beta$ for $y\in S_i\setminus\{x_i\}$, $i=1,\dots,n$, and 1 for $y\in M\setminus\bigcup_{i=1}^n S_i$. Note that in our case, we will only consider the same power β of ρ for each i.

Now we are going to define the weighted Sobolev spaces of complex k-forms on M'. Let $\Lambda^k_{\mathbb{C}}T^*M'$ be the vector bundle of complex k-forms on M', equipped with the metric g and the Levi-Civita connection ∇. For $p\geq 1$, $l\geq 0$ and $\beta\in\mathbb{R}$, we define the *weighted Sobolev space* $L^p_{l,\beta}(\Lambda^k_{\mathbb{C}}T^*M')$ to be the set of complex k-forms η on M' that are locally integrable and l times weakly differentiable, and for which the norm

$$\|\eta\|_{L^p_{l,\beta}} = \left(\sum_{j=0}^{l} \int_{M'} |\rho^{-\beta+j} \nabla^j \eta|^p \, \rho^{-2m} dV \right)^{1/p} \tag{4.3}$$

is finite. Then $L^p_{l,\beta}(\Lambda^k_{\mathbb{C}} T^* M')$ is a Banach space, and $L^2_{l,\beta}(\Lambda^k_{\mathbb{C}} T^* M')$ a Hilbert space. Note that the norm is defined in a way similar to the usual Sobolev norm, with an addition of a weight $\rho^{-\beta p + jp - 2m}$. The idea of this is an element η in $L^p_{l,\beta}(\Lambda^k_{\mathbb{C}} T^* M')$ is a L^p_l k-form on M' which decays at most like ρ^β near x_i as $\rho \to 0$, and thus the index $\beta \in \mathbb{R}$ can be interpreted as an *order of growth*. Moreover, $\nabla^j \eta$ decays at most like $\rho^{\beta - j}$ near x_i for $j = 1, \ldots, l$. As a vector space of forms, $L^p_{l,\beta}(\Lambda^k_{\mathbb{C}} T^* M')$ is independent of choice of radius function ρ, and all choices of ρ give equivalent norms.

It is often important to know for which rate β the $L^p_{0,\beta}$-norm equals the standard L^p-norm of forms on M. Note that from (4.3), the rate we need is $\beta = -2m/p$, and therefore

$$L^p_{0,-2m/p}(\Lambda^k_{\mathbb{C}} T^* M') = L^p(\Lambda^k_{\mathbb{C}} T^* M'). \tag{4.4}$$

We shall need the analogue of the Sobolev Embedding Theorem (Theorem 2.28) for weighted Sobolev spaces, which is adapted from [39, Lem. 7.2]:

Theorem 4.2 (Weighted Sobolev Embedding Theorem) *In the situation above, suppose $l \geq n \geq 0$ are integers, $p, q > 1$ and $\beta, \gamma \in \mathbb{R}$. If $\frac{1}{p} \leq \frac{1}{q} + \frac{l-n}{2m}$ and either*

(i) $p \leq q$ with $\beta \geq \gamma$, or

(ii) $q < p$ with $\beta > \gamma$,

then

$$L^p_{l,\beta}(\Lambda^k_{\mathbb{C}} T^* M') \hookrightarrow L^q_{n,\gamma}(\Lambda^k_{\mathbb{C}} T^* M')$$

is a continuous inclusion.

We can also define weighted Sobolev spaces on AC Riemannian $2m$-folds in a sense analogous to Definition 4.1. However, we shall only treat it very briefly here since our main focus will be on the case for conical singularities and we shall only need the analysis for AC Calabi–Yau 3-folds in Theorem 6.10 later.

Let (Y, g) be a complete, nonsingular Riemannian $2m$-fold. Then Y is *Asymptotically Conical (AC)* with rate $\lambda < 0$ if there is a Riemannian cone $V \cong \Gamma \times (0, \infty) \cup \{0\}$ of dimension $2m$ with Γ compact and connected, a compact subset $K \subset Y$, and a diffeomorphism $\Upsilon : \Gamma \times (R, \infty) \longrightarrow Y \setminus K$ for some $R > 0$ such that

$$|\nabla^k(\Upsilon^*(g) - g_V)|_{g_V} = O(r^{\lambda - k}) \quad \text{as } r \to \infty \text{ and for all } k \geq 0,$$

where r is the coordinate on (R, ∞), $g_V = r^2 g_V|_{\Gamma \times \{1\}} + dr^2$ is the cone metric on V and ∇ is the Levi-Civita connection of g_V. Define a radius function ρ on Y to be a smooth function

$\rho : Y \longrightarrow [R, \infty)$ such that $\rho \equiv R$ on K and $\Upsilon^*(\rho) = r$ on $\Gamma \times (2R, \infty)$. Let $\Lambda^k_{\mathbb{C}} T^* Y$ be the vector bundle of complex k-forms on Y, equipped with the metric g and the Levi-Civita connection ∇. For $p \geq 1$, $l \geq 0$ and $\beta \in \mathbb{R}$, we define the *weighted Sobolev space* $L^p_{l,\beta}(\Lambda^k_{\mathbb{C}} T^* Y)$ to be the set of complex k-forms η on Y that are locally integrable and l times weakly differentiable, and for which the norm

$$\|\eta\|_{L^p_{l,\beta}} = \left(\sum_{j=0}^{l} \int_Y |\rho^{-\beta+j} \nabla^j \eta|^p \, \rho^{-2m} dV \right)^{1/p}$$

is finite. Then $L^p_{l,\beta}(\Lambda^k_{\mathbb{C}} T^* Y)$ is a Banach space, and $L^2_{l,\beta}(\Lambda^k_{\mathbb{C}} T^* Y)$ a Hilbert space.

Generally speaking, the theory of weighted Sobolev spaces on AC manifolds is very similar to that on manifolds with conical singularities, except that for some cases like the embedding theorems, we have to reverse the directions of the inequalites involving the rates, e.g. we need $\beta \leq \gamma$, $\beta < \gamma$ for the AC version of Theorem 4.2.

4.2.2 $d + d^*$ and $dd^* + d^*d$ on manifolds with conical singularities

Next we discuss the analysis of the elliptic operators $d + d^*$ and $dd^* + d^* d$ on manifolds with conical singularities. In the situation of §4.2.1, suppose M is a compact complex manifold of real dimension $2m$ with conical singularities, we are interested in studying the maps

$$\begin{aligned} (d+d^*)^p_{l+1,\beta} \quad &: \quad L^p_{l+1,\beta}(\Lambda^*_{\mathbb{C}} T^* M') \longrightarrow L^p_{l,\beta-1}(\Lambda^*_{\mathbb{C}} T^* M') \quad \text{and} \\ \Delta^p_{l+2,\beta} = (dd^* + d^* d)^p_{l+2,\beta} \quad &: \quad L^p_{l+2,\beta}(\Lambda^*_{\mathbb{C}} T^* M') \longrightarrow L^p_{l,\beta-2}(\Lambda^*_{\mathbb{C}} T^* M'). \end{aligned}$$

for $p > 1$, $l \geq 0$ and $\beta \in \mathbb{R}$. Here we denote by $\Lambda^*_{\mathbb{C}} T^* M'$ the direct sum of spaces $\Lambda^k_{\mathbb{C}} T^* M'$ for $k = 0, \ldots, 2m$.

To begin with, we shall study the operators

$$\begin{aligned} d + d^*_{V_i} \quad &: \quad C^\infty(\Lambda^*_{\mathbb{C}} T^* V_i') \longrightarrow C^\infty(\Lambda^*_{\mathbb{C}} T^* V_i') \quad \text{and} \\ \Delta_{V_i} = dd^*_{V_i} + d^*_{V_i} d \quad &: \quad C^\infty(\Lambda^*_{\mathbb{C}} T^* V_i') \longrightarrow C^\infty(\Lambda^*_{\mathbb{C}} T^* V_i') \end{aligned}$$

on the Riemannian cone V_i' for each i.

For $\alpha \in \mathbb{R}$, $k = 0, \ldots, 2m$ and $i = 1, \ldots, n$, we say that a k-form η^i_k on the Riemannian cone V_i is *homogeneous of order* α if

$$\eta^i_k = r^{\alpha+k} \gamma^i_k + r^{\alpha+k-1} dr \wedge \delta^i_{k-1}$$

for some k-form γ^i_k and $(k-1)$-form δ^i_{k-1} on Γ_i. We set $\gamma^i_{2m} = 0$ and $\delta^i_{-1} = 0$ for all i. It follows that

$$|\eta^i_k|_{g_{V_i}} = O(r^\alpha)$$

since

$$|\gamma^i_k|_{g_{V_i}} = O(r^{-k}) \quad \text{and} \quad |\delta^i_{k-1}|_{g_{V_i}} = O(r^{-k+1}).$$

We remark here that this definition is different from the usual sense of a k-form being homogeneous of degree p (meaning that the Lie derivative by $r\frac{\partial}{\partial r}$ is multiplication by p). A k-form being homogeneous of order α in our sense is in fact homogeneous of degree $\alpha+k$ in the usual sense. Write $\eta^i = \sum_{k=0}^{2m} \eta_k^i \in C^\infty(\Lambda_{\mathbb{C}}^* T^* V_i')$. Then η^i lies in the kernel $\mathrm{Ker}(d+d_{V_i}^*)$ of the operator $d+d_{V_i}^*$ if and only if $d\eta_k^i + d_{V_i}^*\eta_{k+2}^i = 0$ for $k=0,\ldots,2m$. On the other hand, η^i lies in $\mathrm{Ker}(\Delta_{V_i})$ if and only if $\Delta_{V_i}\eta_k^i = 0$ for $k=0,\ldots,2m$, as Δ_{V_i} takes k-forms to k-forms for $k=0,\ldots,2m$. We now give a more explicit description of the kernels of these two operators for homogeneous forms of order α:

Proposition 4.3 *Let $\eta_k^i = r^{\alpha+k}\gamma_k^i + r^{\alpha+k-1}dr\wedge\delta_{k-1}^i$ be a homogeneous k-form of order α on the Riemannian cone $V_i' = \Gamma_i \times (0,\infty)$ for $k=0,\ldots,2m$, $i=1,\ldots,n$, and for $\gamma_k^i \in C^\infty(\Lambda_{\mathbb{C}}^k T^*\Gamma_i)$ and $\delta_{k-1}^i \in C^\infty(\Lambda_{\mathbb{C}}^{k-1} T^*\Gamma_i)$. Write $\eta^i = \sum_{k=0}^{2m}\eta_k^i \in C^\infty(\Lambda_{\mathbb{C}}^* T^* V_i')$. Then*

(i) $(d+d_{V_i}^*)\eta^i = 0$ *if and only if*

$$\begin{aligned} d\gamma_k^i + d_{\Gamma_i}^*\gamma_{k+2}^i &= (\alpha-k+2m-2)\delta_{k+1}^i, \text{ and} \\ d\delta_{k-1}^i + d_{\Gamma_i}^*\delta_{k+1}^i &= (\alpha+k)\gamma_k^i \quad \text{for } k=0,\ldots,2m. \end{aligned} \tag{4.5}$$

(ii) $\Delta_{V_i}\eta^i = 0$ *if and only if*

$$\begin{aligned} \Delta_{\Gamma_i}\gamma_k^i &= (\alpha+k)(\alpha-k+2m-2)\gamma_k^i + 2d\delta_{k-1}^i, \text{ and} \\ \Delta_{\Gamma_i}\delta_{k-1}^i &= (\alpha-k+2m)(\alpha+k-2)\delta_{k-1}^i + 2d_{\Gamma_i}^*\gamma_k^i \\ &\text{for } k=0,\ldots,2m. \end{aligned} \tag{4.6}$$

Here $d_{\Gamma_i}^*$ and $\Delta_{\Gamma_i} = dd_{\Gamma_i}^* + d_{\Gamma_i}^* d$ are computed using the metric $g_{V_i}|_{\Gamma_i\times\{1\}}$.

Proof. Direct computation shows:

$$d\eta_k^i = (\alpha+k)r^{\alpha+k-1}dr\wedge\gamma_k^i + r^{\alpha+k}d\gamma_k^i - r^{\alpha+k-1}dr\wedge d\delta_{k-1}^i \tag{4.7}$$

$$\begin{aligned} *_{V_i}d\eta_k^i = {}& (\alpha+k)r^{2m+\alpha-k-2} *_{\Gamma_i}\gamma_k^i + (-1)^{k+1}r^{2m+\alpha-k-3}dr\wedge *_{\Gamma_i}d\gamma_k^i \\ &- r^{2m+\alpha-k-2} *_{\Gamma_i} d\delta_{k-1}^i \end{aligned}$$

$$\begin{aligned} d*_{V_i}d\eta_k^i = {}& (\alpha+k)r^{2m+\alpha-k-2}d *_{\Gamma_i}\gamma_k^i \\ &+ (\alpha+k)(2m+\alpha-k-2)r^{2m+\alpha-k-3}dr\wedge *_{\Gamma_i}\gamma_k^i \\ &+ (-1)^k r^{2m+\alpha-k-3}dr\wedge d *_{\Gamma_i} d\gamma_k^i - r^{2m+\alpha-k-2}d *_{\Gamma_i} d\delta_{k-1}^i \\ &- (2m+\alpha-k-2)r^{2m+\alpha-k-3}dr\wedge *_{\Gamma_i} d\delta_{k-1}^i \end{aligned}$$

$$\begin{aligned} *_{V_i}d*_{V_i}d\eta_k^i = {}& (-1)^{2m-k}(\alpha+k)r^{\alpha+k-3}dr\wedge *_{\Gamma_i}d *_{\Gamma_i}\gamma_k^i \\ &+ (-1)^{k(2m-1-k)}(\alpha+k)(2m+\alpha-k-2)r^{\alpha+k-2}\gamma_k^i \\ &+ (-1)^k r^{\alpha+k-2} *_{\Gamma_i} d *_{\Gamma_i} d\gamma_k^i \\ &+ (-1)^{2m-1-k}r^{\alpha+k-3}dr\wedge *_{\Gamma_i}d *_{\Gamma_i} d\delta_{k-1}^i \\ &- (-1)^{k(2m-1-k)}(2m+\alpha-k-2)r^{\alpha+k-2}d\delta_{k-1}^i. \end{aligned}$$

Hence with $d^*_{\Gamma_i} = (-1)^k *_{\Gamma_i} d*_{\Gamma_i}$ on k-forms and $d^*_{V_i} = - *_{V_i} d*_{V_i}$ (since $\dim_{\mathbb{R}} \Gamma_i = 2m-1$ and $\dim_{\mathbb{R}} V_i = 2m$)

$$\begin{aligned} d^*_{V_i} d\eta^i_k = &- (\alpha+k) r^{\alpha+k-3} dr \wedge d^*_{\Gamma_i}\gamma^i_k \\ &- (\alpha+k)(2m+\alpha-k-2) r^{\alpha+k-2}\gamma^i_k + r^{\alpha+k-2} d^*_{\Gamma_i} d\gamma^i_k \\ &+ r^{\alpha+k-3} dr \wedge d^*_{\Gamma_i} d\delta^i_{k-1} + (2m+\alpha-k-2) r^{\alpha+k-2} d\delta^i_{k-1}. \end{aligned} \tag{4.8}$$

Analogously

$$*_{V_i}\eta^i_k = (-1)^k r^{2m+\alpha-k-1} dr \wedge *_{\Gamma_i}\gamma^i_k + r^{2m+\alpha-k} *_{\Gamma_i} \delta^i_{k-1}$$

$$\begin{aligned} d*_{V_i}\eta^i_k = &(-1)^{k+1} r^{2m+\alpha-k-1} dr \wedge d *_{\Gamma_i}\gamma^i_k \\ &+ (2m+\alpha-k) r^{2m+\alpha-k-1} dr \wedge *_{\Gamma_i}\delta^i_{k-1} + r^{2m+\alpha-k} d *_{\Gamma_i} \delta^i_{k-1} \end{aligned}$$

$$\begin{aligned} *_{V_i} d*_{V_i}\eta^i_k = &(-1)^{k+1} r^{\alpha+k-2} *_{\Gamma_i} d *_{\Gamma_i}\gamma^i_k \\ &+ (-1)^{(k-1)(2m-k)}(2m+\alpha-k) r^{\alpha+k-2}\delta^i_{k-1} \\ &+ (-1)^{2m-k+1} r^{\alpha+k-3} dr \wedge *_{\Gamma_i} d *_{\Gamma_i} \delta^i_{k-1}. \end{aligned}$$

Thus

$$\begin{aligned} d^*_{V_i}\eta^i_k = &r^{\alpha+k-2} d^*_{\Gamma_i}\gamma^i_k - (2m+\alpha-k) r^{\alpha+k-2}\delta^i_{k-1} \\ &- r^{\alpha+k-3} dr \wedge d^*_{\Gamma_i}\delta^i_{k-1}, \end{aligned} \tag{4.9}$$

and hence

$$\begin{aligned} dd^*_{V_i}\eta^i_k = &(\alpha+k-2) r^{\alpha+k-3} dr \wedge d^*_{\Gamma_i}\gamma^i_k + r^{\alpha+k-2} dd^*_{\Gamma_i}\gamma^i_k \\ &- (2m+\alpha-k)(\alpha+k-2) r^{\alpha+k-3} dr \wedge \delta^i_{k-1} \\ &- (2m+\alpha-k) r^{\alpha+k-2} d\delta^i_{k-1} + r^{\alpha+k-3} dr \wedge dd^*_{\Gamma_i}\delta^i_{k-1}. \end{aligned} \tag{4.10}$$

Now replace k by $k+2$ in (4.9). Together with (4.7), these yield (i) of the proposition as $(d + d^*_{V_i})\eta^i = 0$ if and only if $d\eta^i_k + d^*_{V_i}\eta^i_{k+2} = 0$ for $k = 0, \ldots, 2m$. Equations (4.8) and (4.10) yield

$$\begin{aligned} \Delta_{V_i}\eta^i_k &= d^*_{V_i} d\eta^i_k + dd^*_{V_i}\eta^i_k \\ &= r^{\alpha+k-2}\big(\Delta_{\Gamma_i}\gamma^i_k - (\alpha+k)(2m+\alpha-k-2)\gamma^i_k - 2d\delta^i_{k-1}\big) \\ &\quad + r^{\alpha+k-3} dr \wedge \big(\Delta_{\Gamma_i}\delta^i_{k-1} - (2m+\alpha-k)(\alpha+k-2)\delta^i_{k-1} - 2d^*_{\Gamma_i}\gamma^i_k\big). \end{aligned} \tag{4.11}$$

Hence (ii) follows from (4.11), as $\Delta_{V_i}\eta^i = 0$ if and only if $\Delta_{V_i}\eta^i_k = 0$ for $k = 0, \ldots, 2m$. □

Recall that an operator between Banach spaces is *Fredholm* if it has finite-dimensional kernel and cokernel. Write

$$\Lambda^{even}_{\mathbb{C}} T^*M' = \bigoplus_{j=0}^{m} \Lambda^{2j}_{\mathbb{C}} T^*M' \text{ and } \Lambda^{odd}_{\mathbb{C}} T^*M' = \bigoplus_{j=0}^{m-1} \Lambda^{2j+1}_{\mathbb{C}} T^*M'.$$

We are going to study the operators

$$(d+d^*)^p_{l+1,\beta} : L^p_{l+1,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M') \longrightarrow L^p_{l,\beta-1}(\Lambda^{odd}_{\mathbb{C}} T^*M')$$

and

$$\Delta^p_{l+2,\beta} : L^p_{l+2,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M') \longrightarrow L^p_{l,\beta-2}(\Lambda^{even}_{\mathbb{C}} T^*M')$$

for $p > 1$, $l \geq 0$ and $\beta \in \mathbb{R}$, and give a result which shows that they are Fredholm under certain conditions on β.

The corresponding operators on the cone V_i are:

$$\begin{aligned} d + d^*_{V_i} &: C^\infty(\Lambda^{even}_{\mathbb{C}} T^*V_i') \longrightarrow C^\infty(\Lambda^{odd}_{\mathbb{C}} T^*V_i') \quad \text{and} \\ \Delta_{V_i} &: C^\infty(\Lambda^{even}_{\mathbb{C}} T^*V_i') \longrightarrow C^\infty(\Lambda^{even}_{\mathbb{C}} T^*V_i'). \end{aligned}$$

Note that the kernel of $d+d^*_{V_i}$ on all forms splits into kernels on even forms and odd forms. Thus for $\eta^i = \sum_{j=0}^m \eta^i_{2j} \in C^\infty(\Lambda^{even}_{\mathbb{C}} T^*V_i')$, (4.5) still holds for even k. It is easy to see that (4.6) is also true when $\eta^i = \sum_{j=0}^m \eta^i_{2j}$ and $k = 2j$, as the kernel of Δ_{V_i} on all forms splits into kernels on each k-form.

Before stating the result, we define:

Definition 4.4 In the situation of §4.2.1 and §4.2.2, define

$$\begin{aligned} \mathcal{D}_{d+d^*_{V_i}} = \{\, \alpha \in \mathbb{R} : & \text{ there exist } \eta^i_{2j} \text{ for } j = 0, \ldots, m, \text{ not all zero, such that} \\ & (4.5) \text{ holds for } k = 2j\}, \end{aligned} \tag{4.12}$$

and for $k = 0, \ldots, 2m$, define

$$\mathcal{D}_{\Delta^k_{V_i}} = \{\, \alpha \in \mathbb{R} : \text{there exist a nonzero } \eta^i_k \text{ such that (4.6) holds}\}, \tag{4.13}$$

where we denote by $\Delta^k_{V_i}$ the Laplacian of k-forms on V_i.

We shall sometimes refer to these sets as the exceptional sets of the corresponding operators. Effectively $\mathcal{D}_{d+d^*_{V_i}}$ is the set of $\alpha \in \mathbb{R}$ for which there exist homogeneous $2j$-forms η^i_{2j}, for $j = 0, \ldots, m$ not all zero, of order α on V_i' satisfying $d\eta^i_{2j} + d^*_{V_i}\eta^i_{2j+2} = 0$. On the other hand, $\mathcal{D}_{\Delta^k_{V_i}}$ is the set of $\alpha \in \mathbb{R}$ for which there exists a nonzero homogeneous harmonic k-form η^i_k of order α on V_i'. By the property that the kernel of the Laplacian Δ_{V_i} on even forms is graded into the kernels of $\Delta^k_{V_i}$ for even k, we have

$$\mathcal{D}_{\Delta_{V_i}} = \bigcup_{j=0}^m \mathcal{D}_{\Delta^{2j}_{V_i}}.$$

Then Lockhart and McOwen show [39, Thm. 1.1]:

Theorem 4.5 *In the situation above, $\mathcal{D}_{d+d^*_{V_i}}$ and $\mathcal{D}_{\Delta_{V_i}}$ are discrete subsets of $\mathbb{R}$. Moreover, for $p > 1$, $l \geq 0$ and $\beta \in \mathbb{R}$, the map*

$$(d + d^*)^p_{l+1,\beta} : L^p_{l+1,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M') \longrightarrow L^p_{l,\beta-1}(\Lambda^{odd}_{\mathbb{C}} T^*M')$$

is Fredholm if and only if $\beta \in \mathbb{R} \setminus \mathcal{D}_{d+d^*_{V_i}}$ for $i = 1, \ldots, n$. Similarly the map

$$\Delta^p_{l+2,\beta} : L^p_{l+2,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M') \longrightarrow L^p_{l,\beta-2}(\Lambda^{even}_{\mathbb{C}} T^*M')$$

is Fredholm if and only if $\beta \in \mathbb{R} \setminus \mathcal{D}_{\Delta_{V_i}}$ for $i = 1, \ldots, n$.

We shall require an elliptic regularity result for weighted Sobolev spaces analogous to Theorem 2.29, which is taken from Lockhart [38, Thm. 3.7]:

Theorem 4.6 *In the situation above, suppose $p > 1$, $l \geq 0$ and $\beta \in \mathbb{R}$. If $\eta \in L^p_{0,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M')$ lies in L^p_1 locally, and $\xi \in L^p_{l,\beta-1}(\Lambda^{odd}_{\mathbb{C}} T^*M')$ with $(d+d^*)^p_{l+1,\beta}\eta = \xi$, then $\eta \in L^p_{l+1,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M')$ and $\|\eta\|_{L^p_{l+1,\beta}} \leq C\,(\|\eta\|_{L^p_{0,\beta}} + \|\xi\|_{L^p_{l,\beta-1}})$ for some $C > 0$ independent of η and ξ.*

An analogous result also holds for $\Delta^p_{l+2,\beta}$. We shall now look at how the kernels of the two operators depend on p, l and β. As $\beta \in \mathbb{R} \setminus \mathcal{D}_{d+d^*_{V_i}}$ for $i = 1, \ldots, n$, $(d+d^*)^p_{l+1,\beta}$ is Fredholm, and $\mathrm{Ker}((d+d^*)^p_{l+1,\beta})$ is finite-dimensional. Let $\eta \in \mathrm{Ker}((d+d^*)^p_{l+1,\beta})$, then $\eta \in L^p_{l+1,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M')$ and $(d+d^*)\eta = 0$. By using Theorem 4.6 with $\xi = 0$, we have $\eta \in \mathrm{Ker}((d+d^*)^p_{l+1,\beta})$ for all $l \geq 0$.

Lockhart and McOwen show [39, Lem. 7.3] that the kernel $\mathrm{Ker}((d+d^*)^p_{l+1,\beta})$ is independent of $p > 1$ for $n = 1$, where n is the number of singular points of M_0. The case for $n > 1$ is generalized in [39, §8]. Moreover, they show [39, Lem. 7.1] that if $[\beta, \beta+\epsilon] \subset \mathbb{R} \setminus \mathcal{D}_{d+d^*_{V_i}}$, then $\mathrm{Ker}((d+d^*)^p_{l+1,\beta}) = \mathrm{Ker}((d+d^*)^p_{l+1,\beta+\epsilon})$. This proves:

Theorem 4.7 *For $\beta \in \mathbb{R} \setminus \mathcal{D}_{d+d^*_{V_i}}$ for $i = 1, \ldots, n$, the kernel $\mathrm{Ker}((d+d^*)^p_{l+1,\beta})$ is independent of $p > 1$, and $l \geq 0$, and is invariant under small changes of β, i.e. $\mathrm{Ker}((d+d^*)^p_{l+1,\beta}) = \mathrm{Ker}((d+d^*)^p_{l+1,\beta+\epsilon})$ for $[\beta, \beta+\epsilon] \subset \mathbb{R} \setminus \mathcal{D}_{d+d^*_{V_i}}$. An analogous result also holds for $\mathrm{Ker}(\Delta^p_{l+2,\beta})$.*

Now let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta \in \mathbb{R}$, define a map

$$\langle\, , \,\rangle : L^p_{0,\beta}(\Lambda^k_{\mathbb{C}} T^*M') \times L^q_{0,-\beta-2m}(\Lambda^{2m-k}_{\mathbb{C}} T^*M') \longrightarrow \mathbb{R}$$

for all $k = 0, \ldots, 2m$ by

$$\langle \eta, \xi \rangle = \int_{M'} \eta \wedge \xi.$$

Then (4.3) gives $\eta\rho^{-\beta-2m/p} \in L^p(\Lambda^k_{\mathbb{C}} T^*M')$, and by using $\frac{1}{p} + \frac{1}{q} = 1$, we have $\xi\rho^{\beta+2m/p} \in L^q(\Lambda^{2m-k}_{\mathbb{C}} T^*M')$. Here L^p and L^q denotes the usual Lebesgue spaces. It can then be deduced, by applying the Hölder's inequality, that $\langle\, , \,\rangle$ is well-defined and continuous, and defines a pairing between $L^p_{0,\beta}(\Lambda^k_{\mathbb{C}} T^*M')$ and $L^q_{0,-\beta-2m}(\Lambda^{2m-k}_{\mathbb{C}} T^*M')$ so that they are dual Banach spaces.

Note that this pairing between certain weighted Sobolev spaces of complex k and $(2m-k)$-forms induces pairings between weighted Sobolev spaces of even forms and between weighted Sobolev spaces of odd forms, thus we have maps

$$L^p_{0,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M') \times L^q_{0,-\beta-2m}(\Lambda^{even}_{\mathbb{C}} T^*M') \longrightarrow \mathbb{R}$$

and

$$L^p_{0,\beta}(\Lambda^{odd}_{\mathbb{C}} T^*M') \times L^q_{0,-\beta-2m}(\Lambda^{odd}_{\mathbb{C}} T^*M') \longrightarrow \mathbb{R}.$$

These maps will also be written as $\langle\, , \,\rangle$.

Following similar arguments as in [31, Lem. 2.13], it can be shown that if $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $k, l \geq 0$ and $\beta \in \mathbb{R}$, then for all $\eta \in L^p_{k+1,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M')$ and $\xi \in L^q_{l+1,-\beta-2m+1}(\Lambda^{odd}_{\mathbb{C}} T^*M')$, integration by parts is valid, so that we have

$$\langle\, (d+d^*)\eta, \xi\,\rangle = \langle\, \eta, (d+d^*)\xi\,\rangle\,.$$

Note that the left side is the pairing between odd forms whereas the right side is that between even forms, as the operator

$$(d+d^*)^q_{l+1,-\beta-2m+1} : L^q_{l+1,-\beta-2m+1}(\Lambda^{odd}_{\mathbb{C}} T^*M') \longrightarrow L^q_{l,-\beta-2m}(\Lambda^{even}_{\mathbb{C}} T^*M')$$

on the right side is the adjoint, or the dual operator, of

$$(d+d^*)^p_{k+1,\beta} : L^p_{k+1,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M') \longrightarrow L^p_{k,\beta-1}(\Lambda^{odd}_{\mathbb{C}} T^*M').$$

From now on, we shall denote by $d + d^*_{ev}$ and $d + d^*_{od}$ the operator $d + d^*$ on even and odd forms on M' respectively, and by $(d + d^*_{V_i})_{ev}$ and $(d + d^*_{V_i})_{od}$ the operator $d + d^*_{V_i}$ on even and odd forms on V_i' respectively.

Likewise, for all $\eta \in L^p_{k+2,\beta}(\Lambda^{even}_{\mathbb{C}} T^*M')$ and $\xi \in L^q_{l+2,-\beta-2m+2}(\Lambda^{even}_{\mathbb{C}} T^*M')$, integration by parts is valid and we have

$$\langle\, \Delta\eta, \xi\,\rangle = \langle\, \eta, \Delta\xi\,\rangle\,.$$

Using the idea of the proof of [31, Thm. 2.14], we can now describe the cokernels of the operators $(d + d^*_{ev})^p_{k+1,\beta}$ and $\Delta^p_{k+2,\beta}$:

Theorem 4.8 *In the situation above, let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $k, l \geq 0$, we have*

(i) *For all $\beta \in \mathbb{R} \setminus \mathcal{D}_{(d+d^*_{V_i})_{ev}}$ for $i = 1, \ldots, n$, $\eta \in L^p_{k,\beta-1}(\Lambda^{odd}_{\mathbb{C}} T^*M')$ lies in the image of $(d + d^*_{ev})^p_{k+1,\beta}$ if and only if $\langle\, \eta, \xi\,\rangle = 0$ for all $\xi \in \mathrm{Ker}((d + d^*_{od})^q_{l+1,-\beta-2m+1})$. Hence we have the isomorphism*

$$\mathrm{Coker}((d + d^*_{ev})^p_{k+1,\beta}) \;\cong\; \mathrm{Ker}((d + d^*_{od})^q_{l+1,-\beta-2m+1})^*. \tag{4.14}$$

(ii) *For all $\beta \in \mathbb{R} \setminus \mathcal{D}_{\Delta_{V_i}}$ for $i = 1, \ldots, n$, $\eta \in L^p_{k,\beta-2}(\Lambda^{even}_{\mathbb{C}} T^*M')$ lies in the image of $\Delta^p_{k+2,\beta}$ if and only if $\langle\, \eta, \xi\rangle = 0$ for all $\xi \in \mathrm{Ker}((d+d^*)^q_{l+2,-\beta-2m+2})$. Hence we have the isomorphism*

$$\mathrm{Coker}(\Delta^p_{k+2,\beta}) \;\cong\; \mathrm{Ker}(\Delta^q_{l+2,-\beta-2m+2})^*. \tag{4.15}$$

The index $\mathrm{ind}(A)$ of a Fredholm operator A is defined by $\mathrm{ind}(A) = \dim\, \mathrm{Ker}(A) - \dim\, \mathrm{Coker}(A)$. When $\beta \in \mathbb{R} \setminus \mathcal{D}_{(d+d^*_{V_i})_{ev}}$ for all i, we see from (4.14) that

$$\mathrm{ind}((d + d^*_{ev})^p_{k+1,\beta}) = \dim\, \mathrm{Ker}((d + d^*_{ev})^p_{k+1,\beta}) - \dim\, \mathrm{Ker}((d + d^*_{od})^q_{l+1,-\beta-2m+1}), \tag{4.16}$$

and when $\beta \in \mathbb{R} \setminus \mathcal{D}_{\Delta_{V_i}}$ for all i, (4.15) gives

$$\mathrm{ind}(\Delta^p_{k+2,\beta}) = \dim\, \mathrm{Ker}(\Delta^p_{k+2,\beta}) - \dim\, \mathrm{Ker}(\Delta^q_{l+2,-\beta-2m+2}). \tag{4.17}$$

Definition 4.9 Let $\alpha \in \mathcal{D}_{(d+d^*_{V_i})_{ev}}$. Define $d^i_1(\alpha)$ to be the dimension of the vector space of solutions of $(d+d^*_{V_i})_{ev}(\mu^i)=0$ of the form

$$\mu^i(\gamma,r)=\sum_{s=0}^{t}\sum_{j=0}^{m}(\log\, r)^s\eta^i_{2j,s} \tag{4.18}$$

where $\eta^i_{2j,s}=r^{\alpha+2j}\gamma^i_{2j,s}+r^{\alpha+2j-1}dr\wedge\delta^i_{2j-1,s}$ is a homogeneous $2j$-form of order α on the Riemannian cone $V'_i=\Gamma_i\times(0,\infty)$ for $i=1,\ldots,n$, $s=0,\ldots,t$ and for $\gamma^i_{2j,s}\in C^\infty(\Lambda^{2j}_{\mathbb{C}}\,T^*\Gamma_i)$ and $\delta^i_{2j-1,s}\in C^\infty(\Lambda^{2j-1}_{\mathbb{C}}\,T^*\Gamma_i)$. Thus $\mu^i(\gamma,r)$ is a polynomial in $\log r$ of degree t with coefficients in $C^\infty(\Lambda^{even}_{\mathbb{C}}\,T^*\Gamma_i)\oplus C^\infty(\Lambda^{odd}_{\mathbb{C}}\,T^*\Gamma_i)$.

We can also define $d^i_2(\alpha)$ for Δ_{V_i} in a similar way. But we will not need this quantity, so we shall not discuss it in details.

Lockhart and McOwen show [39, Thm. 1.2] that:

Theorem 4.10 *In the situation above, let $p>1, k\geq 0$, and $\beta_1,\beta_2\in\mathbb{R}\setminus\mathcal{D}_{(d+d^*_{V_i})_{ev}}$ for $i=1,\ldots,n$ with $\beta_1\leq\beta_2$. Then*

$$\mathrm{ind}((d+d^*_{ev})^p_{k+1,\beta_1})-\mathrm{ind}((d+d^*_{ev})^p_{k+1,\beta_2})=\sum_{i=1}^{n}\sum_{\alpha\in\mathcal{D}_{(d+d^*_{V_i})_{ev}}\cap(\beta_1,\beta_2)}d^i_1(\alpha). \tag{4.19}$$

4.2.3 $d+d^*$ and dd^*+d^*d on Calabi–Yau 3-folds with conical singularities

Now we restrict to the situation when $M=M_0$ is a Calabi–Yau 3-fold with isolated conical singularities at $x_1,\ldots,x_n$ modelled on Calabi–Yau cones $V_1,\ldots,V_n$. We have $m=3$ in this case. Also,

$$\begin{aligned}\Lambda^{even}_{\mathbb{C}}T^*M'_0&=\Lambda^0_{\mathbb{C}}T^*M'_0\oplus\Lambda^2_{\mathbb{C}}T^*M'_0\oplus\Lambda^4_{\mathbb{C}}T^*M'_0\oplus\Lambda^6_{\mathbb{C}}T^*M'_0\\ \text{and}\quad\Lambda^{odd}_{\mathbb{C}}T^*M'_0&=\Lambda^1_{\mathbb{C}}T^*M'_0\oplus\Lambda^3_{\mathbb{C}}T^*M'_0\oplus\Lambda^5_{\mathbb{C}}T^*M'_0.\end{aligned}$$

We see from (4.4) that for $k=0,\ldots,6$,

$$L^p_{0,-6/p}(\Lambda^k_{\mathbb{C}}T^*M'_0)=L^p(\Lambda^k_{\mathbb{C}}T^*M'_0)$$

and in particular,

$$L^2_{0,-3}(\Lambda^k_{\mathbb{C}}T^*M'_0)=L^2(\Lambda^k_{\mathbb{C}}T^*M'_0). \tag{4.20}$$

First of all we study the kernels of the operators

$$(d+d^*_{ev})^p_{k+2,\beta}:L^p_{k+2,\beta}(\Lambda^{even}_{\mathbb{C}}T^*M'_0)\longrightarrow L^p_{k+1,\beta-1}(\Lambda^{odd}_{\mathbb{C}}T^*M'_0)$$

and

$$\Delta^p_{k+2,\beta}:L^p_{k+2,\beta}(\Lambda^{even}_{\mathbb{C}}T^*M'_0)\longrightarrow L^p_{k,\beta-2}(\Lambda^{even}_{\mathbb{C}}T^*M'_0).$$

Lemma 4.11 *For $p > 1$, $k \geq 0$ and $\beta \in \mathbb{R}$, we have*

$$\operatorname{Ker}((d+d^*_{ev})^p_{k+2,\beta}) \subseteq \operatorname{Ker}(\Delta^p_{k+2,\beta}),$$

and equality holds if $\beta \geq -2$.

Proof. The inclusion follows from the fact that $\Delta = dd^* + d^*d = (d+d^*)^2$ on any space of twice differentiable forms. Suppose now $\beta \geq -2$ and $\chi = \chi_0 + \chi_2 + \chi_4 + \chi_6 \in \operatorname{Ker}(\Delta^p_{k+2,\beta})$. Then, as $\operatorname{Ker}(\Delta^p_{k+2,\beta})$ is graded, $\chi_{2j} \in \operatorname{Ker}(\Delta^p_{k+2,\beta})$ for $j = 0, \ldots, 3$. Since the kernel $\operatorname{Ker}(\Delta^p_{k+2,\beta})$ is independent of $p > 1$ by Theorem 4.7, we have $\chi_{2j} \in \operatorname{Ker}(\Delta^2_{k+2,\beta})$. Thus χ_{2j} lies in $L^2_{k+2,\beta}(\Lambda^{2j}_{\mathbb{C}} T^* M_0')$, and hence $d\chi_{2j} \in L^2_{k+1,\beta-1}(\Lambda^{2j+1}_{\mathbb{C}} T^* M_0')$ and $d^*\chi_{2j} \in L^2_{k+1,\beta-1}(\Lambda^{2j-1}_{\mathbb{C}} T^* M_0')$. Now using the Weighted Sobolev Embedding Theorem (Theorem 4.2 (i)), we have

$$d\chi_{2j}, d^*\chi_{2j} \in L^2_{k+1,\beta-1} \hookrightarrow L^2_{0,-3} = L^2$$

for $\beta \geq -2$. Consequently, integration by parts gives

$$\|d\chi_{2j}\|^2_{L^2} + \|d^*\chi_{2j}\|^2_{L^2} \;=\; \langle d\chi_{2j}, d\chi_{2j}\rangle_{L^2} + \langle d^*\chi_{2j}, d^*\chi_{2j}\rangle_{L^2} \;=\; \langle \chi_{2j}, \Delta\chi_{2j}\rangle_{L^2} \;=\; 0,$$

which implies $d\chi_{2j} = d^*\chi_{2j} = 0$. It follows that χ_{2j} and hence χ lies in $\operatorname{Ker}((d+d^*_{ev})^p_{k+2,\beta})$. □

Lemma 4.11 is certainly true when we restrict to k-forms, that is, for $p > 1$, $l \geq 0$ and $\beta \in \mathbb{R}$, we have

$$\operatorname{Ker}\left((d+d^*)^p_{l+2,\beta}|_{\Lambda^k_{\mathbb{C}}}\right) \subseteq \operatorname{Ker}\left(\Delta^p_{l+2,\beta}|_{\Lambda^k_{\mathbb{C}}}\right),$$

and equality holds for $\beta \geq -2$.

Proposition 4.12 *(i) There are no nonzero homogeneous harmonic functions (and 6-forms) of order α on V_i' for $i = 1, \ldots, n$ and $\alpha \in (-4, 0)$. Hence we have*

$$\mathcal{D}_{\Delta^0_{V_i}} \cap (-4,0) \;=\; \mathcal{D}_{\Delta^6_{V_i}} \cap (-4,0) \;=\; \emptyset. \tag{4.21}$$

(ii) There are no nonzero homogeneous harmonic 1-forms (and 5-forms) of order α on V_i' for $i = 1, \ldots, n$ and $\alpha \in (-3, -1)$. Hence we have

$$\mathcal{D}_{\Delta^1_{V_i}} \cap (-3,-1) \;=\; \mathcal{D}_{\Delta^5_{V_i}} \cap (-3,-1) \;=\; \emptyset. \tag{4.22}$$

Proof. We shall apply Proposition 4.3 to deduce this result. Suppose $\eta^i_0 = r^\alpha \gamma^i_0$ is a nonzero homogeneous harmonic function of order $\alpha \in (-4, 0)$. Then (4.6) gives

$$\Delta_{\Gamma_i} \gamma^i_0 = \alpha(\alpha+4)\gamma^i_0$$

for $0 \neq \gamma^i_0 \in C^\infty(\Gamma_i)$, which means $\alpha(\alpha+4)$ is an eigenvalue of Δ_{Γ_i}. As $\alpha \in (-4, 0)$ this contradicts the fact that eigenvalues of Δ_{Γ_i} are nonnegative.

For (ii), we suppose that $\eta^i_1 = r^{\alpha+1}\gamma^i_1 + r^\alpha dr \wedge \delta^i_0$ is a nonzero homogeneous harmonic 1-form of order $\alpha \in (-3, -1)$. Thus γ^i_1 and δ^i_0 are not both zero. Again, (4.6) gives

$$\Delta_{\Gamma_i} \gamma^i_1 = (\alpha+1)(\alpha+3)\gamma^i_1 + 2d\delta^i_0, \text{ and} \tag{4.23}$$

$$\Delta_{\Gamma_i} \delta^i_0 = (\alpha+5)(\alpha-1)\delta^i_0 + 2d^*_{\Gamma_i}\gamma^i_1. \tag{4.24}$$

Apply d to both sides of (4.23), we get

$$\Delta_{\Gamma_i} d\gamma_1^i = (\alpha+1)(\alpha+3)d\gamma_1^i.$$

Since $\alpha \in (-3,-1)$, $(\alpha+1)(\alpha+3)$ is always negative, and hence $d\gamma_1^i = 0$. Now (4.23) becomes

$$dd^*_{\Gamma_i}\gamma_1^i = (\alpha+1)(\alpha+3)\gamma_1^i + 2d\delta_0^i. \tag{4.25}$$

Apply d to both sides of (4.24), then

$$\Delta_{\Gamma_i} d\delta_0^i = (\alpha+5)(\alpha-1)d\delta_0^i + 2dd^*_{\Gamma_i}\gamma_1^i,$$

and by (4.25) it becomes

$$\Delta_{\Gamma_i} d\delta_0^i - (\alpha+5)(\alpha-1)d\delta_0^i = 2(\alpha+1)(\alpha+3)\gamma_1^i + 4d\delta_0^i.$$

It follows that

$$\gamma_1^i = d\left(\frac{\Delta_{\Gamma_i}\delta_0^i - ((\alpha+5)(\alpha-1)+4)\delta_0^i}{2(\alpha+1)(\alpha+3)}\right),$$

thus γ_1^i is an exact 1-form, and we define $f^i = \frac{\Delta_{\Gamma_i}\delta_0^i - ((\alpha+5)(\alpha-1)+4)\delta_0^i}{2(\alpha+1)(\alpha+3)}$. It is well-defined as $\alpha \in (-3,-1)$. Now (4.23) becomes

$$\Delta_{\Gamma_i} df^i = (\alpha+1)(\alpha+3)df^i + 2d\delta_0^i.$$

Integrating this we obtain

$$\Delta_{\Gamma_i} f^i = (\alpha+1)(\alpha+3)(f^i + c) + 2\delta_0^i, \tag{4.26}$$

where c is an arbitrary constant. Also, (4.24) becomes

$$\Delta_{\Gamma_i}\delta_0^i = (\alpha+5)(\alpha-1)\delta_0^i + 2\Delta_{\Gamma_i} f^i. \tag{4.27}$$

We now claim that there exist $A, \lambda \in \mathbb{R}$ such that

$$\Delta_{\Gamma_i}(f^i + c + \lambda\delta_0^i) = A(f^i + c + \lambda\delta_0^i) \tag{4.28}$$

holds. Indeed, (4.26) and (4.27) give

$$\begin{aligned}
&\Delta_{\Gamma_i}(f^i + c + \lambda\delta_0^i)\\
&= (\alpha+1)(\alpha+3)(f^i + c) + 2\delta_0^i + \lambda(\alpha+5)(\alpha-1)\delta_0^i + 2\lambda\Delta_{\Gamma_i} f^i\\
&= (\alpha+1)(\alpha+3)(1+2\lambda)(f^i + c) + (2 + 4\lambda + \lambda(\alpha+5)(\alpha-1))\delta_0^i.
\end{aligned}$$

Setting $A = (\alpha+1)(\alpha+3)(1+2\lambda)$ and $A\lambda = 2 + 4\lambda + \lambda(\alpha+5)(\alpha-1)$, we obtain a quadratic equation upon A:

$$A^2 + (6 - 2(\alpha+2)^2)A + (\alpha-1)(\alpha+1)(\alpha+3)(\alpha+5) = 0.$$

Solving it we get $A = (\alpha+1)(\alpha+5)$ or $(\alpha-1)(\alpha+3)$, which respectively gives $\lambda = \frac{1}{\alpha+3}$ or $\frac{-1}{\alpha+1}$. Hence for $\alpha \in (-3,-1)$, A is always negative whereas λ is always positive.

We have shown there exist $A < 0$ and $\lambda > 0$ such that (4.28) holds for $\alpha \in (-3,-1)$, which then implies $f^i + c + \lambda\delta_0^i = 0$. Putting $f^i = -\lambda\delta_0^i - c$ into (4.27), we get

$$(1+2\lambda)\Delta_{\Gamma_i}\delta_0^i = (\alpha+5)(\alpha-1)\delta_0^i.$$

Since $\lambda > 0$, $(\alpha+5)(\alpha-1)/(1+2\lambda)$ is negative and so $\delta_0^i = 0$. But then $\gamma_1^i = df^i = 0$, which is a contradiction. This completes the proof. $\square$

Now Theorem 4.7 and Proposition 4.12 prove that $\operatorname{Ker}(\Delta^p_{k+2,\beta}|_{\Lambda^0_{\mathbb{C}}})$ and $\operatorname{Ker}(\Delta^p_{k+2,\beta}|_{\Lambda^6_{\mathbb{C}}})$ are independent of β for $\beta \in (-4,0)$, whereas $\operatorname{Ker}(\Delta^p_{k+2,\beta}|_{\Lambda^1_{\mathbb{C}}})$ and $\operatorname{Ker}(\Delta^p_{k+2,\beta}|_{\Lambda^5_{\mathbb{C}}})$ are independent of β for $\beta \in (-3,-1)$. Together with Lemma 4.11, we obtain

Proposition 4.13 *For $p > 1$, $k \geq 0$ and $\beta \in \mathbb{R}$, we have:*

(i) If $\chi \in \operatorname{Ker}(\Delta^p_{k+2,\beta}|_{\Lambda^0_{\mathbb{C}}})$ or $\operatorname{Ker}(\Delta^p_{k+2,\beta}|_{\Lambda^6_{\mathbb{C}}})$ and $\beta > -4$, then χ is covariant constant.

(ii) If $\chi \in \operatorname{Ker}(\Delta^p_{k+2,\beta}|_{\Lambda^1_{\mathbb{C}}})$ or $\operatorname{Ker}(\Delta^p_{k+2,\beta}|_{\Lambda^5_{\mathbb{C}}})$ and $\beta > -3$, then χ is closed and coclosed.

In our main construction of the 3-form χ on M_0', we will come across the kernel $\operatorname{Ker}((d+d^*_{ev})^q_{l+2,-2-\delta})$ of $d+d^*_{ev}$ for some small $\delta > 0$. The reason is basically that we need to solve an equation involving the elliptic operator $d+d^*_{od}$ with rate $-3+\delta$, which means we need to study its cokernel and hence the kernel of $d+d^*_{ev}$ at rate $-2-\delta$ by Theorem 4.8.

Observe from Proposition 4.3 that if $b^2(\Gamma_i) > 0$, then there exist nonzero homogeneous even forms of order -2 in the kernel of $(d+d^*_{V_i})_{ev}$ (by taking $\eta_0^i = \eta_4^i = \eta_6^i = 0$ and $\eta_2^i = \gamma_2^i$ for some nonzero closed and coclosed 2-form γ_2^i on Γ_i), so $-2 \in \mathcal{D}_{(d+d^*_{V_i})_{ev}}$. Now choose $\delta > 0$ small enough such that $\mathcal{D}_{(d+d^*_{V_i})_{ev}} \cap [-2-\delta, -2+\delta] \subseteq \{-2\}$ for $i = 1,\ldots,n$. Here we include the case when $\mathcal{D}_{(d+d^*_{V_i})_{ev}} \cap [-2-\delta,-2+\delta] = \emptyset$ for some i, so that the situation when $b^2(\Gamma_i) = 0$ is allowed. By the fact that $(d+d^*_{ev})^q_{l+2,\beta}$ and $(d+d^*_{od})^p_{k+2,-\beta-5}$ are dual operators for $p,q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, $k,l \geq 0$ and $\beta \in \mathbb{R}$, they are Fredholm for the same rate β, and hence we have

$$\mathcal{D}_{(d+d^*_{V_i})_{od}} = \{-\beta-5 : \beta \in \mathcal{D}_{(d+d^*_{V_i})_{ev}}\}.$$

It follows that $\mathcal{D}_{(d+d^*_{V_i})_{od}}$ is the reflection of $\mathcal{D}_{(d+d^*_{V_i})_{ev}}$ at $-5/2$. Furthermore, we have

$$\mathcal{D}_{(d+d^*_{V_i})_{ev}} \cap [-2-\delta,-2+\delta] \subseteq \{-2\} \iff \mathcal{D}_{(d+d^*_{V_i})_{od}} \cap [-3-\delta,-3+\delta] \subseteq \{-3\}$$

for $i = 1,\ldots,n$, and if $b^2(\Gamma_i) = b^3(\Gamma_i) > 0$ for some i, then $-2 \in \mathcal{D}_{(d+d^*_{V_i})_{ev}}$, or equivalently, $-3 \in \mathcal{D}_{(d+d^*_{V_i})_{od}}$. We shall see later that the other direction is also true.

In [39, §5], Lockhart and McOwen show that the kernels of some elliptic operators on manifolds with conical singularities have asymptotic expansions in terms of homogeneous solutions. More precisely, suppose A is a certain kind of elliptic operator on a manifold with a conical singularity with rate ν and $\mathcal{D}_A$ denotes its exceptional set. If $\beta_1,\beta_2 \in \mathbb{R}\setminus\mathcal{D}_A$ with $\beta_1 \leq \beta_2$, then any $u \in \operatorname{Ker}(A^p_{k,\beta_1})$ can be written as

$$\sum_{s=0}^{t}\sum_{j=1}^{N}(\log r)^s u_{j,s} + O(r^{\min(m+\nu,\beta_2)})$$

on the asymptotic ends, where $u_{j,s}$ denotes the homogeneous solution of order $\alpha_j \in (\beta_1,\beta_2)$ for $j = 1,\ldots,N$ and m is the minimum of α_j.

In our case, we have chosen $\delta > 0$ so that $\mathcal{D}_{(d+d^*_{V_i})_{ev}} \cap [-2-\delta, -2+\delta] \subseteq \{-2\}$ for $i = 1, \ldots, n$. We also choose δ such that $0 < \delta < \nu$, where ν is the rate of the conical singularities of M'_0. This implies that any χ in the kernel $\mathrm{Ker}\left((d + d^*_{ev})^q_{l+2,-2-\delta}\right)$ of $d + d^*_{ev}$ can be written as $\Phi^*_i(\chi) = \mu^i + O(r^{-2+\delta})$, where $\mu^i(\gamma, r)$ is of the form (4.18) for $\alpha = -2$, on $\Gamma_i \times (0, \epsilon)$ for $i = 1, \ldots, n$. We are going to show that $\mu^i(\gamma, r)$ is actually of degree 0, i.e. $t = 0$, and therefore

$$\Phi^*_i(\chi) = \eta^i + O(r^{-2+\delta})$$

where $\eta^i = \eta^i_0 + \eta^i_2 + \eta^i_4 + \eta^i_6 \in C^\infty(\Lambda^{even}_{\mathbb{C}} T^* V'_i)$ of order -2. Let us first describe the kernel of $(d + d^*_{V_i})_{ev}$ for homogeneous even forms of order -2.

Lemma 4.14 *Let $\mathcal{K}_i$ be the vector space of solutions of $(d + d^*_{V_i})_{ev}(\eta^i) = 0$ for homogeneous forms $\eta^i = \eta^i_0 + \eta^i_2 + \eta^i_4 + \eta^i_6 \in C^\infty(\Lambda^{even}_{\mathbb{C}} T^* V'_i)$ of order -2. Then*

$$\mathcal{K}_i = \{\gamma^i_2 + r dr \wedge \delta^i_3 \in C^\infty(\Lambda^2_{\mathbb{C}} T^* V'_i \oplus \Lambda^4_{\mathbb{C}} T^* V'_i) : d\gamma^i_2 = d^*_{\Gamma_i}\gamma^i_2 = 0 \text{ and } d\delta^i_3 = d^*_{\Gamma_i}\delta^i_3 = 0\}.$$

Proof. Putting $\alpha = -2$ and $m = 3$ in (4.5), it becomes

$$\begin{aligned} d\gamma^i_0 + d^*_{\Gamma_i}\gamma^i_2 &= 2\delta^i_1, & d\gamma^i_2 + d^*_{\Gamma_i}\gamma^i_4 &= 0, & d\gamma^i_4 &= -2\delta^i_5, \\ d^*_{\Gamma_i}\delta^i_1 &= -2\gamma^i_0, & d\delta^i_1 + d^*_{\Gamma_i}\delta^i_3 &= 0, & d\delta^i_3 + d^*_{\Gamma_i}\delta^i_5 &= 2\gamma^i_4. \end{aligned} \tag{4.29}$$

Since dA and $d^*_{\Gamma_i}B$ are mutually orthogonal, the middle two equations imply

$$d\gamma^i_2 = d^*_{\Gamma_i}\gamma^i_4 = d\delta^i_1 = d^*_{\Gamma_i}\delta^i_3 = 0.$$

As $-2 \in (-4, 0)$, by (i) of Proposition 4.12, we have $\eta^i_0 = \eta^i_6 = 0$, and hence $\gamma^i_0 = \delta^i_5 = 0$. Now δ^i_1 is coexact from the first equation of (4.29) and $d\delta^i_1 = 0$, we have

$$2\left\langle \delta^i_1, \delta^i_1 \right\rangle_{L^2} = \left\langle \delta^i_1, d^*_{\Gamma_i}\gamma^i_2 \right\rangle_{L^2} = \left\langle d\delta^i_1, \gamma^i_2 \right\rangle_{L^2} = 0.$$

Thus $\delta^i_1 = 0$, and hence $d^*_{\Gamma_i}\gamma^i_2 = 0$. Similarly we have $\gamma^i_4 = 0$, so that $d\delta^i_3 = 0$. The result follows. □

Observe that the kernel $\mathcal{K}_i$ splits into two components which correspond, after some calculation, respectively to the kernel of $d + d^*_{V_i}$ for homogeneous 2-forms and the kernel for homogeneous 4-forms of order -2. Using the fact that Γ_i is compact, we have

Corollary 4.15 *The map*

$$\mathcal{K}_i \longrightarrow H^2(\Gamma_i, \mathbb{C}) \oplus H^3(\Gamma_i, \mathbb{C})$$

given by $\gamma^i_2 + r dr \wedge \delta^i_3 \mapsto ([\gamma^i_2], [\delta^i_3])$ is an isomorphism, and hence $\dim \mathcal{K}_i = 2\, b^2(\Gamma_i)$.

If $-2 \in \mathcal{D}_{(d+d^*_{V_i})_{ev}}$, then $\mathcal{K}_i \neq 0$, and so $b^2(\Gamma_i) = b^3(\Gamma_i) > 0$. Thus we have shown that for $i = 1, \ldots, n$,

$$-2 \in \mathcal{D}_{(d+d^*_{V_i})_{ev}} \iff -3 \in \mathcal{D}_{(d+d^*_{V_i})_{od}} \iff b^2(\Gamma_i) = b^3(\Gamma_i) > 0.$$

Proposition 4.16 *Suppose $\mu^i(\gamma, r)$ is a solution of $(d+d^*_{V_i})_{ev}(\mu^i) = 0$ of the form (4.18). Then for $\alpha = -2$, $\mu^i(\gamma, r)$ is of degree $t = 0$.*

Proof. In our case $\mu^i(\gamma, r)$ can be written as

$$\mu^i(\gamma, r) = \sum_{s=0}^{t} (\log\, r)^s (\eta^i_{0,s} + \eta^i_{2,s} + \eta^i_{4,s} + \eta^i_{6,s}).$$

Comparing the leading coefficient we then deduce from Lemma 4.14 that $\gamma^i_{0,t} = \gamma^i_{4,t} = \delta^i_{1,t} = \delta^i_{5,t} = 0$, and $\gamma^i_{2,t} + rdr \wedge \delta^i_{3,t} \in \mathcal{K}_i$, so that $d\gamma^i_{2,t} = d^*_{\Gamma_i}\gamma^i_{2,t} = 0$ and $d\delta^i_{3,t} = d^*_{\Gamma_i}\delta^i_{3,t} = 0$. Now $(d + d^*_{V_i})_{ev}(\mu^i) = 0$ gives

$$\sum_{s=0}^{t} d\big((\log r)^s \eta^i_{2j,s}\big) + d^*\big((\log r)^s \eta^i_{2j+2,s}\big) = 0$$

for $j = 0, 1, 2$. Lets focus on the equation for $j = 1$, which is equivalent to

$$\begin{aligned}\sum_{s=0}^{t} (\log r)^{s-1}\big(&r^{-1} s dr \wedge \gamma^i_{2,s} + (\log r) d\gamma^i_{2,s} - r^{-1}(\log r) dr \wedge d\delta^i_{1,s} \\ &+ (\log r) d^*_{\Gamma_i}\gamma^i_{4,s} - s\delta^i_{3,s} - r^{-1}(\log r) dr \wedge d^*_{\Gamma_i}\delta^i_{3,s}\big) = 0,\end{aligned}$$

Suppose $t \geq 1$, then comparing coefficients of $(\log r)^{t-1}$ gives

$$t\,\gamma^i_{2,t} = d\delta^i_{1,t-1} + d^*_{\Gamma_i}\delta^i_{3,t-1}, \quad t\,\delta^i_{3,t} = d\gamma^i_{2,t-1} + d^*_{\Gamma_i}\gamma^i_{4,t-1}. \tag{4.30}$$

Since $\gamma^i_{2,t}$ and $\delta^i_{3,t}$ are both closed and coclosed, and hence harmonic, it then follows from the orthogonality of the Hodge decomposition on compact manifolds that $\gamma^i_{2,t} = \delta^i_{3,t} = 0$. This contradicts the fact that $\mu^i(\gamma, r)$ is a polynomial of degree t with leading coefficient $\gamma^i_{2,t} + rdr \wedge \delta^i_{3,t}$. This completes the proof. □

We can now say something about the change of index of $d + d^*_{ev}$ at rate -2. Recall that the quantity $d^i_1(\alpha)$ in Definition 4.9 is the dimension of the vector space of solutions of $(d + d^*_{V_i})_{ev}(\mu^i) = 0$ of the form (4.18). Then Corollary 4.15 and Proposition 4.16 imply

$$d^i_1(-2) = \dim \mathcal{K}_i = 2b^2(\Gamma_i)$$

and it follows from (4.19) that we have

Corollary 4.17 *Suppose $p > 1$, $k \geq 0$ and $0 < \delta < \nu$ with $\mathcal{D}_{(d+d^*_{V_i})_{ev}} \cap [-2-\delta, -2+\delta] \subseteq \{-2\}$ for $i = 1, \ldots, n$. Then*

$$\mathrm{ind}((d + d^*_{ev})^p_{k+2,-2-\delta}) - \mathrm{ind}((d + d^*_{ev})^p_{k+2,-2+\delta}) = 2\sum_{i=1}^{n} b^2(\Gamma_i). \tag{4.31}$$

Before presenting the main theorem (Theorem 4.19) in this section, we need the following result, Proposition 4.18, which can be deduced from Lockhart's example [38, Example (0.15) & (0.16)]. For later purpose, we also include the result for the AC case.

Proposition 4.18 *(i) Let $(M_0, J_0, \omega_0, \Omega_0)$ be a compact Calabi–Yau 3-fold with finitely many conical singularities $x_1, \ldots, x_n$ with rate $\nu > 0$ modelled on Calabi–Yau cones $V_1, \ldots, V_n$. Then the vector space of closed and coclosed k-forms in $L^2(\Lambda^k_{\mathbb{C}} T^* M_0')$ is isomorphic under the map $\eta \mapsto [\eta]$ with $H^k(M_0', \mathbb{C})$ for $k < 3$, and with the image of $H^k_{cs}(M_0', \mathbb{C})$ in $H^k(M_0', \mathbb{C})$ for $k = 3$.*

(ii) Let $(Y, J_Y, \omega_Y, \Omega_Y)$ be an AC Calabi–Yau 3-fold with rate $\lambda < 0$ modelled on a Calabi–Yau cone V. Then the vector space of closed and coclosed k-forms in $L^2(\Lambda^k_{\mathbb{C}} T^* Y)$ is isomorphic under the map $\eta \mapsto [\eta]$ with $H^k(Y, \mathbb{C})$ for $k > 3$, and with the image of $H^k_{cs}(Y, \mathbb{C})$ in $H^k(Y, \mathbb{C})$ for $k = 3$.

Proof. Part (i) follows from Lockhart's Example (0.16) in [38]. From our definition, we have $\Phi_i^*(g_0) = g_{V_i} + O(r^\nu)$, which can be written as $\Phi_i^*(g_0) = e^{-2z}(g_{V_i}|_{\Gamma_i \times \{1\}} + dz^2) + O(e^{-z\nu})$ for $z = -\log r$. Then g_0 is *admissible* in Lockhart's sense [38, §2], $\rho := -z$ is decreasing and satisfies the inequality $\rho < -[(1+\delta)/2] \log z$ on $\Gamma_i \times [1, \infty)$ for some $\delta > 0$. Taking our $\bigcup_{i=1}^n S_i \setminus \{x_i\}$ to be Lockhart's M_∞, our $M_0 \setminus \bigcup_{i=1}^n S_i$ to be his M_0 and our $\coprod_{i=1}^n \Gamma_i$ to be his ∂M_0, these fit into the situation in Lockhart's Example (0.16), and the result follows by taking his n to be 6.

For the AC case, the proof follows from Lockhart's Example (0.15), and is similar to the argument in the conical singularities case except that we take $z = \log r$ and $\rho = z$, so that ρ is an increasing function of z this time, which is the reason for the inequality of k being reversed in the AC case. □

Theorem 4.19 *Suppose $p > 1$, $k \geq 0$ and $0 < \delta < \nu$ with $\mathcal{D}_{(d+d^*_{V_i})_{ev}} \cap [-2-\delta, -2+\delta] \subseteq \{-2\}$ for $i = 1, \ldots, n$. If χ lies in the kernel $\mathrm{Ker}\,((d+d^*_{ev})^p_{k+2,-2-\delta})$ of $d + d^*_{ev}$, then we have*

$$\Phi_i^*(\chi) = \gamma_2^i + r dr \wedge \delta_3^i + O(r^{-2+\delta}) \quad on \ \ \Gamma_i \times (0, \epsilon),$$

where $\gamma_2^i + r dr \wedge \delta_3^i \in \mathcal{K}_i$. Furthermore, the kernel $\mathrm{Ker}\,((d+d^*_{ev})^p_{k+2,-2-\delta})$ is graded so that it is a direct sum of vector spaces of closed and coclosed 0-forms, 2-forms, 4-forms and 6-forms.

Proof. Applying [39, §5] and the argument just before Lemma 4.14, we have $\Phi_i^*(\chi) = \mu^i + O(r^{-2+\delta})$. The first part of the proposition then follows from Lemma 4.14 and Proposition 4.16. Write $\chi = \chi_0 + \chi_2 + \chi_4 + \chi_6 \in \mathrm{Ker}\,((d+d^*_{ev})^p_{k+2,-2-\delta})$. Then $\Phi_i^*(\chi_0) = O(r^{-2+\delta})$, $\Phi_i^*(\chi_2) = \gamma_2^i + O(r^{-2+\delta})$, $\Phi_i^*(\chi_4) = r dr \wedge \delta_3^i + O(r^{-2+\delta})$ and $\Phi_i^*(\chi_6) = O(r^{-2+\delta})$. Moreover, χ satisfies

$$d\chi_0 + d^*\chi_2 = 0, \;\; d\chi_2 + d^*\chi_4 = 0 \;\; \text{and} \;\; d\chi_4 + d^*\chi_6 = 0. \tag{4.32}$$

Now $d\chi_2$ is a 3-form which is exact, and hence closed, and is coclosed by the second equation above. Also, $\Phi_i^*(d\chi_2) = d\gamma_2^i + O(r^{-3+\delta}) = O(r^{-3+\delta})$. Since $\mathcal{D}_{(d+d^*_{V_i})_{ev}} \cap [-2-\delta, -2+\delta] \subseteq \{-2\}$ for $i = 1, \ldots, n$, we have $-2-\delta \in \mathbb{R} \setminus \mathcal{D}_{(d+d^*_{V_i})_{ev}}$. As discussed earlier, this implies that $-3+\delta \in \mathbb{R} \setminus \mathcal{D}_{(d+d^*_{V_i})_{od}}$. Using Theorem 4.7 we have $d\chi_2 \in \mathrm{Ker}\,((d+d^*_{od})^2_{k+2,-3+\delta}|_{\Lambda^3_{\mathbb{C}}})$.

Now from Proposition 4.18, the vector space of closed and coclosed 3-forms in $L^2(\Lambda^3_{\mathbb{C}} T^* M_0')$ is isomorphic to the image of $H^3_{cs}(M_0', \mathbb{C})$ in $H^3(M_0', \mathbb{C})$. By (4.20), the vector space of closed and coclosed 3-forms in $L^2(\Lambda^3_{\mathbb{C}} T^* M_0')$ is just $\mathrm{Ker}\,((d+d^*_{od})^2_{k+2,-3}|_{\Lambda^3_{\mathbb{C}}})$. This implies that we have an injective map from $\mathrm{Ker}\,((d+d^*_{od})^2_{k+2,-3}|_{\Lambda^3_{\mathbb{C}}})$ to $H^3(M'_0,\mathbb{C})$. Together with the natural inclusion from $\mathrm{Ker}\,((d+d^*_{od})^2_{k+2,-3+\delta}|_{\Lambda^3_{\mathbb{C}}})$ to $\mathrm{Ker}\,((d+d^*_{od})^2_{k+2,-3}|_{\Lambda^3_{\mathbb{C}}})$, the 3-form $d\chi_2$ being exact implies that $d\chi_2=0$. This gives $d^*\chi_4=0$ as well.

Observe that $\chi\in\mathrm{Ker}\,((d+d^*_{ev})^p_{k+2,-2-\delta})\subseteq\mathrm{Ker}\,(\Delta^p_{k+2,-2-\delta})$, and since $\mathrm{Ker}\,(\Delta^p_{k+2,-2-\delta})$ is graded, we have $\chi_0,\chi_6\in\mathrm{Ker}\,(\Delta^p_{k+2,-2-\delta})$. Using (i) of Proposition 4.13, both χ_0 and χ_6 are constants. It follows that $d\chi_0=0$ and $d^*\chi_6=0$, and hence $d^*\chi_2=d\chi_4=0$ from (4.32). This proves the theorem. □

We would like to finish this section by studying some algebraic topology on the compact Calabi–Yau 3-fold M_0 with conical singularities. Consider M'_0 as the interior of a compact manifold $\bar{M}'_0$ with boundary $\partial\bar{M}'_0$ the disjoint union $\coprod_{i=1}^n\Gamma_i$. We have the usual de Rham cohomology groups $H^k(M'_0,\mathbb{C})$ and $H^k(\Gamma_i,\mathbb{C})$, and the compactly-supported de Rham cohomology groups $H^k_{cs}(M'_0,\mathbb{C})$. Let $b^k(M'_0)$, $b^k(\Gamma_i)$ and $b^k_{cs}(M'_0)$ be the corresponding Betti numbers. Note that by Poincaré duality we have $H^k_{cs}(M'_0,\mathbb{C})\cong H^{6-k}(M'_0,\mathbb{C})^*$ and $H^k(\Gamma_i,\mathbb{C})\cong H^{5-k}(\Gamma_i,\mathbb{C})^*$, which give $b^k_{cs}(M'_0)=b^{6-k}(M'_0)$ and $b^k(\Gamma_i)=b^{5-k}(\Gamma_i)$. We are going to build a long exact sequence of cohomology groups $H^k(M'_0,\mathbb{C})$, $H^k(\Gamma_i,\mathbb{C})$ and $H^k_{cs}(M'_0,\mathbb{C})$. Clearly, we have the natural maps

$$\phi_k:H^k_{cs}(M'_0,\mathbb{C})\longrightarrow H^k(M'_0,\mathbb{C})$$

given by $\phi_k([\chi])=[\chi]$. For $r\in(0,\epsilon)$, let $\iota^i_r:\Gamma_i\longrightarrow\Gamma_i\times(0,\epsilon)$ be the inclusion $\gamma\mapsto(\gamma,r)$. Thus the maps $\Phi_i\circ\iota^i_r:\Gamma_i\longrightarrow S_i\setminus\{x_i\}$ give embeddings from Γ_i to M'_0, and they induce the pull-back maps

$$\rho_k:H^k(M'_0,\mathbb{C})\longrightarrow\textstyle\bigoplus_{i=1}^n H^k(\Gamma_i,\mathbb{C})$$

defined by $\rho_k([\chi])=\bigoplus_{i=1}^n[(\Phi_i\circ\iota^i_r)^*\chi]$. Finally, let F be a smooth function on M'_0 such that $F=1$ on $\Phi_i\big(\Gamma_i\times(0,\frac{1}{2}\epsilon)\big)$ for all $i=1,\dots,n$ and $F=0$ on $M'_0\setminus\bigcup_{i=1}^n S_i$. Then we can define the boundary maps

$$\partial_k:\textstyle\bigoplus_{i=1}^n H^k(\Gamma_i,\mathbb{C})\longrightarrow H^{k+1}_{cs}(M'_0,\mathbb{C})$$

by $\partial_k(\bigoplus_{i=1}^n[\chi_i])=[d(F\pi_1^*\chi_1+\cdots+F\pi_n^*\chi_n)]$, where $\pi_i:\Gamma_i\times(0,\epsilon)\longrightarrow\Gamma_i$ is the projection map. Using the natural long exact sequence for relative homology and the Poincaré duality isomorphisms, we obtain the following long exact sequence:

$$\cdots\rightarrow H^k_{cs}(M'_0,\mathbb{C})\xrightarrow{\phi_k}H^k(M'_0,\mathbb{C})\xrightarrow{\rho_k}\textstyle\bigoplus_{i=1}^n H^k(\Gamma_i,\mathbb{C})\xrightarrow{\partial_k}H^{k+1}_{cs}(M'_0,\mathbb{C})\rightarrow\cdots \tag{4.33}$$

for $k=0,\dots,6$. For simplicity we suppose that M'_0 has no compact connected components so that $H^0_{cs}(M'_0,\mathbb{C})=H^6(M'_0,\mathbb{C})=0$.

Consider now the dual vector spaces $\bigoplus_{i=1}^n H^2(\Gamma_i,\mathbb{C})$ and $\bigoplus_{i=1}^n H^3(\Gamma_i,\mathbb{C})$, and the pull back maps $\rho_2:H^2(M'_0,\mathbb{C})\longrightarrow\bigoplus_{i=1}^n H^2(\Gamma_i,\mathbb{C})$ and $\rho_3:H^3(M'_0,\mathbb{C})\longrightarrow\bigoplus_{i=1}^n H^3(\Gamma_i,\mathbb{C})$. The following result shows the subspaces $\rho_2(H^2(M'_0,\mathbb{C}))$ in $\bigoplus_{i=1}^n H^2(\Gamma_i,\mathbb{C})$ and $\rho_3(H^3(M'_0,\mathbb{C}))$ in $\bigoplus_{i=1}^n H^3(\Gamma_i,\mathbb{C})$ are annihilators of each other:

Proposition 4.20 *Let $[\alpha]\in H^2(M'_0,\mathbb{C})$ and $[\beta]\in H^3(M'_0,\mathbb{C})$. If $\rho_2([\alpha])=\bigoplus_{i=1}^n[\alpha_i]$ and $\rho_3([\beta])=\bigoplus_{i=1}^n[\beta_i]$, then*

$$(\textstyle\bigoplus_{i=1}^n[\alpha_i])\cup(\bigoplus_{i=1}^n[\beta_i])=0,$$

where $\cup : \bigoplus_{i=1}^n H^2(\Gamma_i, \mathbb{C}) \times \bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C}) \longrightarrow \mathbb{C}$ *denotes the cup product. Moreover,*

$$\dim \rho_2(H^2(M_0', \mathbb{C})) + \dim \rho_3(H^3(M_0', \mathbb{C})) = \sum_{i=1}^n b^2(\Gamma_i).$$

Hence $\rho_2(H^2(M_0', \mathbb{C}))$ *and* $\rho_3(H^3(M_0', \mathbb{C}))$ *are annihilators of each other, and in particular, if* $\bigoplus_{i=1}^n [\beta_i]$ *lies in* $\bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})$*, then*

$$\begin{aligned} \bigoplus_{i=1}^n [\beta_i] \in \rho_3(H^3(M_0', \mathbb{C})) \Longleftrightarrow & (\bigoplus_{i=1}^n [\alpha_i]) \cup (\bigoplus_{i=1}^n [\beta_i]) = 0 \\ & \textit{for all } \bigoplus_{i=1}^n [\alpha_i] \in \rho_2(H^2(M_0', \mathbb{C})). \end{aligned}$$

Proof.

$$\begin{aligned} \bigoplus_{i=1}^n [\alpha_i] \cup \bigoplus_{i=1}^n [\beta_i] &= \sum_{i=1}^n [\alpha_i] \cup [\beta_i] \\ &= \sum_{i=1}^n \int_{\Gamma_i} \alpha_i \wedge \beta_i \\ &= \int_{\partial \bar{M}_0'} \alpha \wedge \beta \quad \text{since } \bigoplus_{i=1}^n [\alpha_i] \text{ and } \bigoplus_{i=1}^n [\beta_i] \text{ lie in the} \\ &\qquad\qquad\qquad \text{image of } \rho_2 \text{ and } \rho_3 \text{ respectively.} \\ &= \int_{\bar{M}_0'} d(\alpha \wedge \beta) \quad \text{by Stokes' Theorem} \\ &= 0 \quad \text{as both } \alpha \text{ and } \beta \text{ are closed.} \end{aligned}$$

Note that $\rho_2(H^2(M_0', \mathbb{C})) \cong \operatorname{Ker} \partial_2$ by exactness of (4.33), and $\rho_3(H^3(M_0', \mathbb{C})) \cong \partial_2(\bigoplus_{i=1}^n H^2(\Gamma_i, \mathbb{C}))$ as $H^3_{cs}(M_0', \mathbb{C}) \cong H^3(M_0', \mathbb{C})^*$ and $\bigoplus_{i=1}^n H^2(\Gamma_i, \mathbb{C}) \cong \bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})^*$ by Poincaré duality. It follows that

$$\begin{aligned} & \dim \rho_2(H^2(M_0', \mathbb{C})) + \dim \rho_3(H^3(M_0', \mathbb{C})) \\ = & \dim \operatorname{Ker} \partial_2 + \dim \partial_2(\bigoplus_{i=1}^n H^2(\Gamma_i, \mathbb{C})) \\ = & \dim \bigoplus_{i=1}^n H^2(\Gamma_i, \mathbb{C}) \\ = & \sum_{i=1}^n b^2(\Gamma_i). \end{aligned}$$

This completes the proof. □

4.3 Construction of χ

As was mentioned in §4.1, what we need for our desingularization is a closed (2,1)-form χ with $\omega_0 \wedge \chi = 0$ and prescribed asymptotic behaviour $\Phi_i^*(\chi) = \xi_i + O(r^{-3+\delta})$ on each $\Gamma_i \times (0, \epsilon)$.

Obviously $\bigoplus_{i=1}^n [\xi_i] \in \bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})$ lies in $\rho_3(H^3(M_0', \mathbb{C}))$ is a necessary condition for such a 3-form χ to exist. It is because if χ exists, then $d\chi = 0$ and $[\chi]$ is well-defined in $H^3(M_0', \mathbb{C})$. The asymptotic condition $\Phi_i^*(\chi) = \xi_i + O(r^{-3+\delta})$ would imply $\rho_3([\chi]) = \bigoplus_{i=1}^n [\xi_i]$, and hence

$\bigoplus_{i=1}^{n}[\xi_i]$ lies in the image of ρ_3.

In this section we prove that this is also a sufficient condition for the existence of such a complex 3-form χ on M_0'. We shall apply the theory in §4.2 and show that the image of ρ_2 in $\bigoplus_{i=1}^{n} H^2(\Gamma_i,\mathbb{C})$ causes obstruction to solving a certain elliptic equation. The existence of χ is equivalent to the existence of solution to this elliptic equation. Thus if $\bigoplus_{i=1}^{n}[\xi_i]$ lies in $\rho_3(H^3(M_0',\mathbb{C}))$, then Proposition 4.20 tells us that it annihilates the image of ρ_2 and so the elliptic equation is indeed unobstructed, which means we can solve for χ.

Let us first fix our attention on the complex 3-forms $\Lambda^3_{\mathbb{C}}T^*M_0'$ on M_0'. It has complex dimension 20. Using the complex structure J_0 it can be decomposed according to the type:

$$\Lambda^3_{\mathbb{C}}T^*M_0' = \Lambda^{3,0}\oplus\Lambda^{2,1}\oplus\Lambda^{1,2}\oplus\Lambda^{0,3}.$$

The Hodge star $*$, acts as a complex linear operator, maps $\Lambda^3_{\mathbb{C}}T^*M_0'$ to itself with $*^2=-1$. We can then decompose $\Lambda^3_{\mathbb{C}}T^*M_0'$ into parts:

$$\Lambda^3_{\mathbb{C}}T^*M_0' = \Lambda^3_+\oplus\Lambda^3_-,$$

where $\Lambda^3_\pm$ denotes the $\pm i$ eigenspaces of $*$ on $\Lambda^3_{\mathbb{C}}T^*M_0'$. Note that complex conjugation gives an isomorphism between Λ^3_+ and Λ^3_-, thus

$$\dim_{\mathbb{C}}\Lambda^3_+ = \dim_{\mathbb{C}}\Lambda^3_- = \frac{1}{2}\dim_{\mathbb{C}}\Lambda^3_{\mathbb{C}}T^*M_0' = 10.$$

Given any $\varphi\in\Lambda^3_{\mathbb{C}}T^*M_0'$ we can write $\varphi=\varphi_++\varphi_-$ where $\varphi_+=\frac{1}{2}(\varphi-i*\varphi)\in\Lambda^3_+$ and $\varphi_-=\frac{1}{2}(\varphi+i*\varphi)\in\Lambda^3_-$.

Using the Kähler form ω_0 we can split $\Lambda^{2,1}$ and $\Lambda^{1,2}$ into:

$$\Lambda^{2,1}=\Lambda^{2,1}_0\oplus(\omega_0\wedge\Lambda^{1,0}) \text{ and } \Lambda^{1,2}=\Lambda^{1,2}_0\oplus(\omega_0\wedge\Lambda^{0,1}),$$

where $\Lambda^{2,1}_0$ and $\Lambda^{1,2}_0$ denote respectively the kernels of the maps $\Lambda^{2,1}\longrightarrow\Lambda^{3,2}$ and $\Lambda^{1,2}\longrightarrow\Lambda^{2,3}$, both given by $\alpha\longmapsto\omega_0\wedge\alpha$. We shall sometimes call these spaces the trace-free components. Certainly every form in $\Lambda^{3,0}$ and $\Lambda^{0,3}$ is also trace-free. Here is an algebraic relation:

Lemma 4.21 *The $(+i)$-eigenspace of the Hodge star $*$ on $\Lambda^3_{\mathbb{C}}T^*M_0'$ is*

$$\Lambda^3_+=\Lambda^{2,1}_0\oplus(\omega_0\wedge\Lambda^{0,1})\oplus\Lambda^{0,3}$$

and the $(-i)$-eigenspace is

$$\Lambda^3_-=\Lambda^{3,0}\oplus(\omega_0\wedge\Lambda^{1,0})\oplus\Lambda^{1,2}_0.$$

We shall see this by direct checking from the model space $\mathbb{C}^3$: Let (z_1,z_2,z_3) be complex coordinates on $\mathbb{C}^3$. The trace-free (2,1)-forms are spanned by

$$dz_a\wedge dz_b\wedge d\bar{z}_c \quad\text{and}\quad \frac{1}{\sqrt{2}}\big(dz_a\wedge dz_b\wedge d\bar{z}_b-dz_a\wedge dz_c\wedge d\bar{z}_c\big),$$

the forms in $\omega_0\wedge\Lambda^{0,1}$ by

$$\frac{i}{\sqrt{2}}\big(dz_a\wedge d\bar{z}_a\wedge d\bar{z}_c+dz_b\wedge d\bar{z}_b\wedge d\bar{z}_c\big),$$

and the $\Lambda^{0,3}$-forms by

$$d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$$

for a, b, c distinct. Together they give rise to a space of complex dimension 10, which is the same as Λ^3_+. As $*$ is complex linear, we have

$$\alpha \wedge \overline{*\alpha} = (\alpha, \alpha) \cdot \mathrm{vol}$$

where $(\cdot, \cdot)$ is the pointwise inner product, and vol is the volume form $i\, dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$. Putting α to be the basis elements and since they are of unit length, we check that they all satisfy $*\alpha = i\, \alpha$.

Before moving on to the construction of χ, we pause to look at the harmonic 1-forms in $\mathrm{Ker}\,(\Delta^p_{k+2,\beta}|_{\Lambda^1_{\mathbb{C}}})$ for $\beta \geq -3$ through (covariant) constant 1-forms on M_0':

Proposition 4.22 *For $p > 1$, $k \geq 0$ and $\beta \in \mathbb{R}$, we have* $\mathrm{Ker}\,(\Delta^p_{k+2,\beta}|_{\Lambda^1_{\mathbb{C}}}) = 0$ *for $\beta \geq -3$. Hence $H^1(M_0', \mathbb{C}) = 0$.*

Proof. Take $\alpha \in \mathrm{Ker}\,(\Delta^p_{k+2,-3}|_{\Lambda^1_{\mathbb{C}}})$. Using the *upper semi-continuous* property of the kernel $\mathrm{Ker}\,(\Delta^2_{k+2,-3}|_{\Lambda^1_{\mathbb{C}}})$, we have

$$\mathrm{Ker}\,(\Delta^2_{k+2,-3}|_{\Lambda^1_{\mathbb{C}}}) = \mathrm{Ker}\,(\Delta^2_{k+2,-3+\delta}|_{\Lambda^1_{\mathbb{C}}})$$

for small enough $\delta > 0$. The kernel is upper semi-continuous at rate -3 because when $-3 \in \mathcal{D}_{\Lambda^1_{V_i}}$, all elements added to the kernel of $\Delta|_{\Lambda^1_{\mathbb{C}}}$ at rate -3 look like $O(r^{-3}(\log r)^t)$ by the argument before Lemma 4.14. However, the $L^2_{0,-3}$-norm for these things does not exist, which means that they do not lie in $L^2_{0,-3}$, and hence not in $\mathrm{Ker}\,(\Delta^2_{k+2,-3}|_{\Lambda^1_{\mathbb{C}}})$. We then conclude that $\mathrm{Ker}\,(\Delta^2_{k+2,\mu}|_{\Lambda^1_{\mathbb{C}}}) \cong \mathrm{Ker}\,(\Delta^2_{k+2,-3}|_{\Lambda^1_{\mathbb{C}}})$ for $-3 < \mu \leq -3 + \delta$. Together with Theorem 4.7 we have $\alpha \in \mathrm{Ker}\,(\Delta^p_{k+2,-3}|_{\Lambda^1_{\mathbb{C}}}) = \mathrm{Ker}\,(\Delta^2_{k+2,-3+\delta}|_{\Lambda^1_{\mathbb{C}}})$.

Similar to the argument in showing Proposition 4.13, we have $\alpha \in \mathrm{Ker}\,(\Delta^2_{k+2,-1-\mu}|_{\Lambda^1_{\mathbb{C}}})$ for small $\mu > 0$, which then imply $\nabla\alpha \in L^2_{k+1,-2-\mu} \hookrightarrow L^2_{0,-3} = L^2$. Now the Weitzenbock formula for 1-forms shows that $\Delta^p_{k+2,-3}\,\alpha = \nabla^*\nabla\alpha$ since g_0 is Ricci-flat. As a result, integration by parts gives

$$\|\nabla\alpha\|^2_{L^2} = \langle\nabla\alpha, \nabla\alpha\rangle_{L^2} = \langle\nabla^*\nabla\alpha, \alpha\rangle_{L^2} = \left\langle \Delta^p_{k+2,-3}\,\alpha, \alpha \right\rangle_{L^2} = 0,$$

which is valid as we have $\nabla\alpha \in L^2$. Hence $\nabla\alpha = 0$, and α is therefore a constant 1-form on M_0'. Thus we have shown that every 1-form in the kernel $\mathrm{Ker}\,(\Delta^p_{k+2,-3}|_{\Lambda^1_{\mathbb{C}}})$ is constant.

Suppose there is a nonzero constant 1-form on M_0'. Then there exist constant vector fields v_1, v_2 on M_0'. Define an integrable distribution $\langle v_1, v_2\rangle$ on M_0, and we see that M_0 is locally a product $N \times \mathbb{R}^2$, with product metric. As the flow along a constant vector field is an isometry, so it takes singular points to singular points, and hence the leaf of foliation passing through the singular point x_i is all singular, which is a contradiction as the singularities are nonisolated in M_0. Thus there are no nonzero constant 1-forms on M_0'. Combining this with the previous paragraph, we obtain $\mathrm{Ker}\,(\Delta^p_{k+2,-3}|_{\Lambda^1_{\mathbb{C}}}) = 0$.

Applying Proposition 4.18 on 1-forms, the vector space of closed and coclosed 1-forms in $L^2_{k+2,-3}(\Lambda^1_{\mathbb{C}} T^* M'_0)$ is isomorphic to $H^1(M'_0, \mathbb{C})$. We then deduce

$$H^1(M'_0, \mathbb{C}) \cong \mathrm{Ker}\,((d + d^*_{od})^2_{k+2,-3}|_{\Lambda^1_{\mathbb{C}}}) \subseteq \mathrm{Ker}\,(\Delta^p_{k+2,-3}|_{\Lambda^1_{\mathbb{C}}}) = 0.$$

This completes the proof. □

Using the upper semi-continuity and Proposition 4.13 (ii), we know that the 1-forms in $\mathrm{Ker}\,(\Delta^p_{k+2,\beta}|_{\Lambda^1_{\mathbb{C}}})$ for $\beta \geq -3$ are closed and coclosed. Proposition 4.22 is thus a stronger result, showing that all of them are zero by using the property that Calabi–Yau 3-folds with conical singularities do not support constant 1-forms.

A similar result holds for 5-forms, as $\mathrm{Ker}\,(\Delta^p_{k+2,\beta}|_{\Lambda^1_{\mathbb{C}}}) \cong \mathrm{Ker}\,(\Delta^p_{k+2,\beta}|_{\Lambda^5_{\mathbb{C}}})$. Furthermore, we have $H^5_{cs}(M'_0, \mathbb{C}) \cong H^1(M'_0, \mathbb{C}) = 0$.

Now consider the elliptic operator

$$\begin{aligned} d_+ + d^* :\ C^\infty(\Lambda^3_+ \oplus \Lambda^5_{\mathbb{C}}) &\longrightarrow C^\infty(\Lambda^4_{\mathbb{C}} \oplus \Lambda^6_{\mathbb{C}}) \\ (\varphi_3, \varphi_5) &\longmapsto (d\varphi_3 + d^*\varphi_5, d\varphi_5). \end{aligned}$$

Here d_+ is the restriction of d to the forms in Λ^3_+. We shall apply the theory in §4.2 to study the operator

$$(d_+ + d^*)^p_{k+2,-3+\delta} :\ L^p_{k+2,-3+\delta}(\Lambda^3_+ \oplus \Lambda^5_{\mathbb{C}}) \longrightarrow L^p_{k+1,-4+\delta}(\Lambda^4_{\mathbb{C}} \oplus \Lambda^6_{\mathbb{C}}). \tag{4.34}$$

Define $V \subseteq H^3(M'_0, \mathbb{C})$ to be the image of $\phi_3 : H^3_{cs}(M'_0, \mathbb{C}) \longrightarrow H^3(M'_0, \mathbb{C})$. Its dimension can be calculated using the long exact sequence (4.33): As we have assumed that M'_0, Γ_i's are connected and M'_0 has no compact connected components, then $H^0(M'_0, \mathbb{C}) = \mathbb{C}$, $\bigoplus_{i=1}^n H^0(\Gamma_i, \mathbb{C}) = \mathbb{C}^n$ and $H^0_{cs}(M'_0, \mathbb{C}) = 0$. Combining these with $H^1(M'_0, \mathbb{C}) = 0$ from Proposition 4.22 gives a short exact sequence in the beginning of (4.33). Hence taking alternating sums of dimensions shows that

$$\begin{aligned} \dim V &= b^3_{cs}(M'_0) - \sum_{i=1}^n b^2(\Gamma_i) + b^2(M'_0) - b^2_{cs}(M'_0) + \sum_{i=1}^n b^1(\Gamma_i) \\ &= b^2(M'_0) + b^3(M'_0) - b^4(M'_0) + \sum_{i=1}^n b^1(\Gamma_i) - \sum_{i=1}^n b^2(\Gamma_i). \end{aligned}$$

Proposition 4.18 shows that V is isomorphic to $\mathrm{Ker}\,((d + d^*_{od})^2_{k+2,-3}|_{\Lambda^3_{\mathbb{C}}})$. Suppose $V = V_+ \oplus V_-$ is the decomposition of V into $\pm i$ eigenspaces of $*$. Then $\dim V_+ = \dim V_- = \frac{1}{2} \dim V$, as V_+ is isomorphic to V_- under complex conjugation. We now identify the kernel of $(d_+ + d^*)^p_{k+2,-3+\delta}$ and V_+, and hence one can interpret $\mathrm{Ker}\,((d_+ + d^*)^p_{k+2,-3+\delta})$ as a "half" of $\mathrm{Ker}\,((d + d^*_{od})^2_{k+2,-3}|_{\Lambda^3_{\mathbb{C}}})$.

Theorem 4.23 *Suppose $p > 1$, $k \geq 0$ and $0 < \delta < \nu$ with $\mathcal{D}_{(d+d^*_{V_i})_{od}} \cap [-3-\delta, -3+\delta] \subseteq \{-3\}$. Then the kernel $\mathrm{Ker}\,((d_+ + d^*)^p_{k+2,-3+\delta})$ is a vector space of closed Λ^3_+-forms. The map $\mathrm{Ker}\,((d_+ + d^*)^p_{k+2,-3+\delta}) \longrightarrow H^3(M'_0, \mathbb{C})$ given by $\chi \mapsto [\chi]$ induces an isomorphism of $\mathrm{Ker}\,((d_+ + d^*)^p_{k+2,-3+\delta})$ with V_+ defined above. Hence $\dim \mathrm{Ker}\,((d_+ + d^*)^p_{k+2,-3+\delta}) = \dim V_+$.*

Proof. Suppose $(\varphi_3, \varphi_5) \in \operatorname{Ker}((d_+ + d^*)^p_{k+2,-3+\delta})$. Then we have $d\varphi_3 + d^*\varphi_5 = 0$ and $d\varphi_5 = 0$. Applying $*$ to $d\varphi_3 + d^*\varphi_5 = 0$ and using the fact that $*\varphi_3 = i\varphi_3$, we get $d^*\varphi_3 + i\, d(*\varphi_5) = 0$. Therefore $(d + d^*_{od})(i * \varphi_5 + \varphi_3 + \varphi_5) = 0$, that is, the mixed form $i * \varphi_5 + \varphi_3 + \varphi_5$ lies in the kernel of $(d + d^*_{od})^p_{k+2,-3+\delta}$. It follows that $i * \varphi_5 + \varphi_3 + \varphi_5 \in \operatorname{Ker}(\Delta^p_{k+2,-3+\delta})$, and so $i*\varphi_5 \in \operatorname{Ker}(\Delta^p_{k+2,-3+\delta}|_{\Lambda^1_{\mathbb{C}}})$. Proposition 4.22 then gives $i*\varphi_5$, and hence $\varphi_5 = 0$. Consequently the kernel $\operatorname{Ker}((d_+ + d^*)^p_{k+2,-3+\delta})$ is a vector space of closed Λ^3_+-forms.

By Theorem 4.7 and the upper semi-continuity of the kernel (see the proof of Proposition 4.22), we have $\operatorname{Ker}((d_+ + d^*)^p_{k+2,-3+\delta}) \cong \operatorname{Ker}((d_+ + d^*)^2_{k+2,-3+\delta}) \cong \operatorname{Ker}((d_+ + d^*)^2_{k+2,-3})$. It follows that $\chi \mapsto [\chi]$ induces an isomorphism of $\operatorname{Ker}((d_+ + d^*)^p_{k+2,-3+\delta})$ with closed L^2-forms in Λ^3_+ at rate -3, which is just the space V_+. This proves the theorem. □

Consider now the dual operator

$$\begin{aligned} d^*_+ + d:\ C^\infty(\Lambda^4_{\mathbb{C}} \oplus \Lambda^6_{\mathbb{C}}) &\longrightarrow C^\infty(\Lambda^3_+ \oplus \Lambda^5_{\mathbb{C}}) \\ (\varphi_4, \varphi_6) &\longmapsto (d^*_+\varphi_4, d\varphi_4 + d^*\varphi_6), \end{aligned}$$

where d^*_+ is the projection of d^* to Λ^3_+-forms.

The following result identifies the kernel and cokernel of $d_+ + d^*$ for small $\delta > 0$:

Theorem 4.24 *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $k, l \geq 0$ and $0 < \delta < \nu$ with $\mathcal{D}_{d+d^*_{V_i}} \cap [-3-\delta, -3+\delta] \subseteq \{-3\}$. Then the operator $d_+ + d^*$ in (4.34) is Fredholm with*

$$\operatorname{Coker}((d_+ + d^*)^p_{k+2,-3+\delta}) \cong \operatorname{Ker}((d^*_+ + d)^q_{l+2,-2-\delta})^*.$$

Moreover, $\operatorname{Ker}((d^*_+ + d)^q_{l+2,-2-\delta})$ *is a direct sum of vector spaces of closed and coclosed 4-forms and constant 6-forms.*

Proof. The first part follows from §4.2. To prove the last part, we use a similar technique as in the proof of Theorem 4.23. Suppose $(\varphi_4, \varphi_6) \in \operatorname{Ker}((d^*_+ + d)^q_{l+2,-2-\delta})$, so that $d^*_+\varphi_4 = 0$ and $d\varphi_4 + d^*\varphi_6 = 0$. The first one implies $d^*\varphi_4 + id(*\varphi_4) = 0$. Therefore we have $(d + d^*_{ev})(i* \varphi_6 + i* \varphi_4 + \varphi_4 + \varphi_6) = 0$, which means the mixed form $i* \varphi_6 + i* \varphi_4 + \varphi_4 + \varphi_6$ lies in $\operatorname{Ker}((d + d^*_{ev})^q_{l+2,-2-\delta})$. Hence $\operatorname{Ker}((d^*_+ + d)^q_{l+2,-2-\delta}) \subseteq \operatorname{Ker}((d + d^*_{ev})^q_{l+2,-2-\delta})$. The result then follows from Theorem 4.19 showing that $\operatorname{Ker}((d + d^*_{ev})^q_{l+2,-2-\delta})$ is graded. □

After so much analytic background and preparation, we finally come to the construction of our desired 3-form on M'_0. Take any smooth complex 3-form χ', not necessarily closed or coclosed, in $L^p_{k+2,-3+\delta}(\Lambda^3_+)$ for some $p > 1, k \geq 0$ and $0 < \delta < \nu$ with $\mathcal{D}_{(d+d^*_{V_i})_{od}} \cap [-3-\delta, -3+\delta] \subseteq \{-3\}$, such that

$$|\nabla^j(\Phi^*_i(\chi') - \xi_i)|_{g_{V_i}} = O(r^{-3+\delta-j}) \quad \text{for all } j \geq 0 \tag{4.35}$$

for $i = 1, \ldots, n$, where ∇ is the Levi-Civita connection of the cone metric g_{V_i}.

Write $\chi = \chi' + \varphi$ for some complex 3-form φ on M'_0. We shall try and solve the following elliptic equation:

$$(d_+ + d^*)(\varphi, \zeta) = (-d\chi', 0),$$

that is, $d\varphi + d^*\zeta = -d\chi'$ and $d\zeta = 0$. From (4.35) we have $\Phi_i^*(d\chi') = O(r^{-4+\delta})$. Then by an elliptic regularity result (Theorem 4.6 with $l = k+1, \beta = -3+\delta$), if (φ, ζ) exists, it lies in $L^p_{k+2,-3+\delta}(\Lambda^3_+ \oplus \Lambda^5_{\mathbb{C}})$. Thus we hope to solve

$$(d_+ + d^*)^p_{k+2,-3+\delta}(\varphi, \zeta) = (-d\chi', 0). \tag{4.36}$$

A general fact about the existence of solution to the equation (4.36) is that a solution (φ, ζ) exists in $L^p_{k+2,-3+\delta}(\Lambda^3_+ \oplus \Lambda^5_{\mathbb{C}})$ if and only if $(-d\chi', 0)$ is L^2-orthogonal to the cokernel $\operatorname{Coker}((d_+ + d^*)^p_{k+2,-3+\delta})$, or equivalently to the kernel $\operatorname{Ker}((d_+^* + d)^q_{l+2,-2-\delta})$ by Theorem 4.24. We shall prove that under the condition that $\bigoplus_{i=1}^n [\xi_i] \in \bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})$ lies in the image of ρ_3, then $(-d\chi', 0)$ is automatically orthogonal to $\operatorname{Ker}((d_+^* + d)^q_{l+2,-2-\delta})$, and hence the equation is indeed unobstructed and so is solvable. Furthermore, we shall show that $\zeta = 0$, and hence $d\varphi = -d\chi'$, which then implies $d\chi = 0$. Since $\chi', \varphi \in \Lambda^3_+$, we have $\chi \in \Lambda^3_+$ as well. Here is our main result in this section:

Theorem 4.25 *Suppose $\bigoplus_{i=1}^n [\xi_i] \in \bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})$ lies in $\rho_3(H^3(M_0', \mathbb{C}))$. Then there exists a complex 3-form χ on M_0' satisfying*

(i) $\chi \in \Lambda^3_+$, and $d\chi = 0$,

(ii) $|\nabla^j(\Phi_i^*(\chi) - \xi_i)|_{g_{V_i}} = O(r^{-3+\delta-j})$ for $i = 1, \dots, n$ and all $j \geq 0$.

Proof. The discussion above suggests that the desired χ exists if we can solve (4.36) for $(\varphi, \zeta) \in L^p_{k+2,-3+\delta}(\Lambda^3_+ \oplus \Lambda^5_{\mathbb{C}})$.

Take any $(\alpha, \beta) \in \operatorname{Ker}((d_+^* + d)^q_{l+2,-2-\delta})$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $l \geq 0$. Equation (4.36) is solvable if and only if

$$\langle (d\chi', 0), (\alpha, \beta) \rangle_{L^2} = 0.$$

Theorem 4.24 tells us that α is a closed and coclosed 4-form, then

$$\begin{aligned}
\langle (d\chi', 0), (\alpha, \beta) \rangle_{L^2} &= \lim_{r \to 0} \int_{K_r} d\chi' \wedge *\alpha \,+\, 0 \wedge *\beta \quad \text{where } K_r := \{x \in M_0' : \rho(x) \geq r\} \\
&= \lim_{r \to 0} \int_{K_r} d(\chi' \wedge *\alpha) \quad \text{as } \alpha \text{ is coclosed} \\
&= \lim_{r \to 0} \int_{\partial K_r} \chi' \wedge *\alpha \quad \text{by Stokes' Theorem.}
\end{aligned}$$

Recall that $\Phi_i^*(\chi') = \xi_i + O(r^{-3+\delta})$ on $\Gamma_i \times (0, \epsilon)$. As α lies in the 4-form part in $\operatorname{Ker}((d_+^* + d)^q_{l+2,-2-\delta}) \subseteq \operatorname{Ker}((d + d_{ev}^*)^q_{l+2,-2-\delta})$, Theorem 4.19 shows that $\Phi_i^*(\alpha) = rdr \wedge \delta_3^i + O(r^{-2+\delta})$ where $\delta_3^i \in C^\infty(\Lambda^3_{\mathbb{C}} T^*\Gamma_i)$ with $d\delta_3^i = d^*_{\Gamma_i}\delta_3^i = 0$. We then have $\Phi_i^*(*\alpha) = *_{\Gamma_i}\delta_3^i + O(r^{-2+\delta})$ on $\Gamma_i \times (0, \epsilon)$. Using the fact that $\xi_i = O(r^{-3})$ and $*_{\Gamma_i}\delta_3^i = O(r^{-2})$ we obtain

$$\begin{aligned}
\langle (d\chi', 0), (\alpha, \beta) \rangle_{L^2} &= \lim_{r \to 0} \sum_{i=1}^n \int_{\Gamma_i \times \{r\}} (\xi_i \wedge *_{\Gamma_i}\delta_3^i \,+\, O(r^{-5+\delta})) \\
&= \lim_{r \to 0} \sum_{i=1}^n [\xi_i] \cup [*_{\Gamma_i}\delta_3^i] \,+\, O(r^{-5+\delta}) \cdot O(r^5) \quad \text{by the definition of cup} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{product and using } \operatorname{vol}(\Gamma_i \times \{r\}) = O(r^5) \\
&= \sum_{i=1}^n [\xi_i] \cup [*_{\Gamma_i}\delta_3^i]
\end{aligned} \tag{4.37}$$

as $\delta > 0$ and the cohomology classes are independent of r. From the hypothesis, we have $\bigoplus_{i=1}^{n}[\xi_i] \in \rho_3(H^3(M_0',\mathbb{C}))$. Thus by Proposition 4.20, $(\bigoplus_{i=1}^{n}[\xi_i]) \cup (\bigoplus_{i=1}^{n}[\theta_i]) = 0$, or $\sum_{i=1}^{n}[\xi_i] \cup [\theta_i] = 0$, for all $\bigoplus_{i=1}^{n}[\theta_i] \in \rho_2(H^2(M_0',\mathbb{C}))$. Since $\rho_2([*\alpha]) = \bigoplus_{i=1}^{n}[*_{\Gamma_i}\delta_3^i]$ from $\Phi_i^*(*\alpha) = *_{\Gamma_i}\delta_3^i + O(r^{-2+\delta})$, so $\bigoplus_{i=1}^{n}[*_{\Gamma_i}\delta_3^i]$ lies in the image of ρ_2 and hence $\sum_{i=1}^{n}[\xi_i] \cup [*_{\Gamma_i}\delta_3^i] = 0$. Thus from (4.37) we have shown that $\langle (d\chi', 0), (\alpha,\beta)\rangle_{L^2} = 0$ for any $(\alpha,\beta) \in \operatorname{Ker}((d_+^* + d)^q_{l+2,-2-\delta})$. It follows that a solution (φ,ζ) to (4.36) exists.

As $\chi', \varphi \in \Lambda_+^3$, $\chi = \chi' + \varphi$ also lies in Λ_+^3. Since $d\varphi + d^*\zeta = -d\chi'$, we have $dd^*\zeta = 0$. Together with $d\zeta = 0$, we see that $\zeta \in \operatorname{Ker}(\Delta^p_{k+2,-3+\delta}|_{\Lambda^5_{\mathbb{C}}})$, which is then equal to zero by the remark after Proposition 4.22. This gives $d\chi = d\chi' + d\varphi = 0$. Moreover, using the asymptotic condition (4.35) for χ' and the fact that φ is a smooth 3-form decaying at rate $O(r^{-3+\delta})$ near x_i, we obtain (ii) of the theorem. This completes the proof. $\square$

The space $\operatorname{Ker}((d_+^* + d)^q_{l+2,-2-\delta})$ is in effect the obstruction space to the existence of the 3-form χ, as it is the obstruction space to solving (4.36). However, we can overcome all the obstructions by fixing $\bigoplus_{i=1}^{n}[\xi_i] \in \rho_3(H^3(M_0',\mathbb{C}))$, which cause (4.37) to vanish automatically. As we explained before, this condition is clearly necessary for the existence of χ, thus we have shown that the necessary condition is also sufficient.

Proposition 4.26 *The complex 3-form χ in Theorem 4.25 can be projected to the $\Lambda_0^{2,1}$-component $\chi_0^{(2,1)}$ of χ, satisfying*

(i) $d\chi_0^{(2,1)} = 0$ and hence $d^*\chi_0^{(2,1)} = 0$, and

(ii) $\Phi_i^*(\chi_0^{(2,1)}) = \xi_i + dC_i$,

where C_i is a complex 2-form on $\Gamma_i \times (0,\epsilon)$ and $|\nabla^j C_i|_{g_{V_i}} = O(r^{-2+\delta-j})$ for $i = 1,\dots,n$ and all $j \geq 0$.

Proof. As $\chi \in \Lambda_+^3$, the algebraic relation from Lemma 4.21 gives

$$\chi = \chi_0^{(2,1)} + (\omega_0 \wedge \theta^{(0,1)}) + \chi^{(0,3)}. \tag{4.38}$$

Since χ is closed and coclosed, it is harmonic and thus all its components in (4.38) are also harmonic. In particular, $\theta^{(0,1)}$ is harmonic. Note that $\omega_0 \wedge \chi = \omega_0 \wedge \omega_0 \wedge \theta^{(0,1)}$ as both $\chi_0^{(2,1)}$ and $\chi^{(0,3)}$ are trace-free. Using Theorem 4.25 (ii) we have $\Phi_i^*(\chi) = \xi_i + O(r^{-3+\delta})$. It follows that $\omega_0 \wedge \omega_0 \wedge \theta^{(0,1)} = O(r^{-3+\delta})$ as $\xi_i \wedge \omega_{V_i} = 0$. Hence $\theta^{(0,1)} \in \operatorname{Ker}(\Delta^p_{k+2,-3+\delta}|_{\Lambda^1_{\mathbb{C}}})$, which then gives $\theta^{(0,1)} = 0$ by applying the Weitzenbock formula for 1-forms as in the proof of Proposition 4.22. This kills the bit $\omega_0 \wedge \theta^{(0,1)}$ in χ and (4.38) implies

$$0 = d\chi = d\chi_0^{(2,1)} + d\chi^{(0,3)}.$$

The components in $d\chi$ are of type (3,1) and (2,2) coming from $d\chi_0^{(2,1)}$ and of type (1,3) coming from $d\chi^{(0,3)}$. Consequently, we have $d\chi_0^{(2,1)} = d\chi^{(0,3)} = 0$. This proves (i) as $\chi_0^{(2,1)} \in \Lambda_+^3$.

To show (ii), let $\pi : \Lambda_{\mathbb{C}}^3 \longrightarrow \Lambda_0^{2,1}$ be the projection of complex 3-forms on M_0' to their $\Lambda_0^{2,1}$-components, using structures on M_0', and let π_{V_i} be the similar projection on V_i, using structures on V_i. Hence we have $\pi(\chi) = \chi_0^{(2,1)}$, and $\pi_{V_i}(\xi_i) = \xi_i$, as ξ_i is by assumption a trace-free (2,1)-form on V_i. Since the Calabi–Yau structures on M_0 and V_i agree up to order $O(r^\nu)$ by definition, we have $\Phi_i^* \circ \pi = \pi_{V_i} \circ \Phi_i^* + O(r^{-3+\nu})$. It follows that

$$\begin{aligned}\Phi_i^*(\chi_0^{(2,1)}) = \Phi_i^*(\pi(\chi)) = \pi_{V_i}(\Phi_i^*(\chi)) + O(r^{-3+\nu}) &= \pi_{V_i}(\xi_i) + O(r^{-3+\delta}) + O(r^{-3+\nu})\\ &= \xi_i + O(r^{-3+\delta}) \quad \text{as } \delta < \nu.\end{aligned}$$

The error term is an exact 3-form as it decays faster than $O(r^{-3})$, and we may now define C_i by integration as in Theorem 3.24. so that $\Phi_i^*(\chi_0^{(2,1)}) = \xi_i + dC_i$. The size of the derivatives of C_i can be deduced from the result of integration, as we have seen in Theorem 3.24. This proves the proposition. □

4.4 Nearly Calabi–Yau structures (ω_t, Ω_t): the case $\lambda_i = -3$

This section constructs a 1-parameter family of smooth, nonsingular compact 6-folds M_t, as in §3.4.1. We then construct nearly Calabi–Yau structures (ω_t, Ω_t) on M_t for small enough t, in which we include a correction term to the definition of Ω_t in §3.4.2. We use the notation in §4.1, §4.2 and §4.3. Suppose that $\bigoplus_{i=1}^n [\xi_i] \in \bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})$ lies in $\rho_3(H^3(M_0', \mathbb{C}))$, so that $\chi_0^{(2,1)}$ exists on M_0' by Proposition 4.26. We shall use η in place of $\chi_0^{(2,1)}$ from now on. For $i = 1, \ldots, n$ apply a homothety to Y_i as before. Then $(Y_i, J_{Y_i}, t^2\omega_{Y_i}, t^3\Omega_{Y_i})$ is also an AC Calabi–Yau 3-fold, with the diffeomorphism $\Upsilon_{t,i} : \Gamma_i \times (tR, \infty) \longrightarrow Y_i \setminus K_i$ given by $\Upsilon_{t,i}(\gamma, r) = \Upsilon_i(\gamma, t^{-1}r)$. The nonsingular 6-fold M_t will be constructed exactly in the same way as in §3.4.1. We now define the nearly Calabi–Yau structures (ω_t, Ω_t) on M_t. The 2-form ω_t will have the same definition as in §3.4.2. As discussed in §3.4.2, we can write

$$\Phi_i^*(\Omega_0) = \Omega_{V_i} + dA_i \tag{4.39}$$

for $i = 1, \ldots, n$ and some complex 2-form $A_i(\gamma, r)$ on $\Gamma_i \times (0, \epsilon)$ satisfying

$$|\nabla^k A_i(\gamma, r)|_{g_{V_i}} = O(r^{\nu+1-k}) \quad \text{as } r \to 0 \ \text{ for all } k \geq 0. \tag{4.40}$$

Then by Proposition 4.26 (ii), we have

$$\Phi_i^*(\Omega_0 + t^3\eta) = \Omega_{V_i} + t^3\xi_i + d(A_i + t^3C_i) \tag{4.41}$$

for $i = 1, \ldots, n$. As for the AC Calabi–Yau 3-folds, we have from §4.1

$$\Upsilon_{t,i}^*(t^3\Omega_{Y_i}) = \Omega_{V_i} + t^3\xi_i + t^3 dB_i \tag{4.42}$$

for $i = 1, \ldots, n$ and some complex 2-form $B_i(\gamma, t^{-1}r)$ on $\Gamma_i \times (tR, \infty)$ satisfying

$$\begin{aligned}|\nabla^k B_i(\gamma, t^{-1}r)|_{g_{V_i}} = O(t^{-\lambda_i'-3} r^{\lambda_i'+1-k}) \quad &\text{for } r > tR, \ \lambda_i' < -3\\ &\text{and all } k \geq 0.\end{aligned} \tag{4.43}$$

Let $F : \mathbb{R} \longrightarrow [0, 1]$ be the smooth function introduced in §3.4.2. We now define a smooth complex closed 3-form Ω_t on M_t by

$$\Omega_t = \begin{cases} \Omega_0 + t^3\eta \quad \text{on} \;\; Q_t \setminus \big[\big(\bigcup_{i=1}^n P_{t,i}\big) \cap Q_t\big], \\ \Omega_{V_i} + t^3\xi_i + d\big[F(t^{-\alpha}r)A_i(\gamma,r) + t^3F(t^{-\alpha}r)C_i(\gamma,r) \\ \quad +t^3(1-F(t^{-\alpha}r))B_i(\gamma,t^{-1}r)\big] \\ \quad \text{on } P_{t,i}\cap Q_t \;\; \text{for } i=1,\ldots,n, \\ t^3\Omega_{Y_i} \;\; \text{on} \;\; P_{t,i}\setminus(P_{t,i}\cap Q_t) \;\; \text{for } i=1,\ldots,n. \end{cases} \tag{4.44}$$

Then when $2t^\alpha \le r < \epsilon$ we have $F(t^{-\alpha}r) = 1$ so that $\Omega_t = \Phi_i^*(\Omega_0 + t^3\eta)$ for each i by (4.41), and when $tR < r \le t^\alpha$ we have $F(t^{-\alpha}r) = 0$ so that $\Omega_t = \Upsilon_{t,i}^*(t^3\Omega_{Y_i})$ by (4.42). Thus Ω_t interpolates between $\Phi_i^*(\Omega_0 + t^3\eta)$ near $r = \epsilon$ and $\Upsilon_{t,i}^*(t^3\Omega_{Y_i})$ near $r = tR$.

Let M_t, ω_t and Ω_t be defined as above. For $i = 1,\ldots,n$, (M_t,ω_t,Ω_t) is just $(Y_i, t^2\omega_{Y_i}, t^3\Omega_{Y_i})$ on $P_{t,i}\setminus(P_{t,i}\cap Q_t)$. On each $P_{t,i}\cap Q_t$, we have $\omega_t = \omega_{V_i}$ and

$$\begin{aligned}\Omega_t - \Omega_{V_i} = t^3\xi_i + d\big[&F(t^{-\alpha}r)A_i(\gamma,r) + t^3F(t^{-\alpha}r)C_i(\gamma,r) \\ &+ t^3(1-F(t^{-\alpha}r))B_i(\gamma,t^{-1}r)\big]\end{aligned}$$

by (4.44). Then Proposition 4.26, (4.40) and (4.43) gives

$$\begin{aligned}|\Omega_t - \Omega_{V_i}|_{g_{V_i}} &= O(t^{3(1-\alpha)}) + O(t^{\alpha\nu}) + O(t^{3(1-\alpha)+\alpha\delta}) + O(t^{-\lambda_i'(1-\alpha)}) \\ &= O(t^{\alpha\nu}) + O(t^{3(1-\alpha)}) \quad \text{for } r\in(t^\alpha,2t^\alpha)\end{aligned} \tag{4.45}$$

since both $O(t^{3(1-\alpha)+\alpha\delta})$ and $O(t^{-\lambda_i'(1-\alpha)})$ absorb in the term $O(t^{3(1-\alpha)})$. Hence $|\Omega_t-\Omega_{V_i}|_{g_{V_i}} \le K_0t^\kappa$ where $K_0>0$ is some constant and $\kappa = \min(\alpha\nu, 3(1-\alpha))$. On $Q_t\setminus\big[\big(\bigcup_{i=1}^n P_{t,i}\big)\cap Q_t\big]$, we have

$$|\Omega_t - \Omega_0|_{g_0} = |t^3\eta|_{g_0} = O(t^{3(1-\alpha)}), \tag{4.46}$$

thus $|\Omega_t-\Omega_0|_{g_0} \le K_1t^{3(1-\alpha)}$ where $K_1>0$ is some constant. Consequently, (ω_t,Ω_t) is sufficiently close to the genuine Calabi–Yau structures on $Q_t\setminus\big[\big(\bigcup_{i=1}^n P_{t,i}\big)\cap Q_t\big]$ and $P_{t,i}\cap Q_t$ and is equal to $(t^2\omega_{Y_i}, t^3\Omega_{Y_i})$ on $P_{t,i}\setminus(P_{t,i}\cap Q_t)$. Thus by Proposition 3.6, we have proved:

Proposition 4.27 *Let M_t, ω_t and Ω_t be defined as above. Then (ω_t,Ω_t) gives a nearly Calabi–Yau structure on M_t for sufficiently small t.*

Similar to the unobstructed case in Chapter 3, we can then associate an almost complex structure J_t and a real 3-form $\theta_{2,t}'$ such that $\Omega_t' = \mathrm{Re}(\Omega_t) + i\theta_{2,t}'$ is a (3,0)-form w.r.t. J_t. Moreover, we have the 2-form ω_t', which is the rescaled (1,1)-part of ω_t w.r.t. J_t, and the associated metric g_t on M_t. It follows from Proposition 3.6 that $|g_t - g_{V_i}|_{g_{V_i}} = O(t^{\alpha\nu}) + O(t^{3(1-\alpha)}) = |g_t^{-1} - g_{V_i}^{-1}|_{g_{V_i}}$ on $P_{t,i}\cap Q_t$, whereas $|g_t - g_{V_i}|_{g_{V_i}} = O(t^{3(1-\alpha)}) = |g_t^{-1} - g_{V_i}^{-1}|_{g_{V_i}}$ on $Q_t\setminus\big[\big(\bigcup_{i=1}^n P_{t,i}\big)\cap Q_t\big]$.

4.5 The main result when $\lambda_i = -3$

Here is our main result on desingularizing compact Calabi–Yau 3-folds M_0 with finitely many conical singularities, the analogue of Theorem 3.32, but allowing $\lambda_i = -3$ and $[\Upsilon_i^*(\Omega_{Y_i})-\Omega_{V_i}] \neq 0$ in $H^3(\Gamma_i, \mathbb{C})$.

Theorem 4.28 *Suppose $(M_0, J_0, \omega_0, \Omega_0)$ is a compact Calabi–Yau 3-fold with finitely many conical singularities $x_1, \ldots, x_n$ with rate $\nu > 0$ modelled on Calabi–Yau cones $V_1, \ldots, V_n$. Let $(Y_1, J_{Y_1}, \omega_{Y_1}, \Omega_{Y_1})$, $\ldots$, $(Y_n, J_{Y_n}, \omega_{Y_n}, \Omega_{Y_n})$ be AC Calabi–Yau 3-folds with rates -3 modelled on the same Calabi–Yau cones $V_1, \ldots, V_n$.*

Suppose that there is a closed, homogeneous, trace-free (2,1)-form ξ_i of order -3 on V_i such that (4.2) holds for $i = 1, \ldots, n$, and that $\bigoplus_{i=1}^n [\xi_i] \in \bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})$ lies in $\rho_3(H^3(M_0', \mathbb{C}))$.

Define a family $(M_t, \omega_t, \Omega_t)$ of nonsingular compact nearly Calabi–Yau 3-folds with the associated metrics g_t as in §4.4.

Then M_t admits a Calabi–Yau structure $(\tilde{J}_t, \tilde{\omega}_t, \tilde{\Omega}_t)$ such that $\|\tilde{\omega}_t - \omega_t\|_{C^0} \leq Kt^\kappa$ and $\|\tilde{\Omega}_t - \Omega_t\|_{C^0} \leq Kt^\kappa$ for some $\kappa, K > 0$ and for sufficiently small t. The cohomology classes satisfy $[\mathrm{Re}(\Omega_t)] = [\mathrm{Re}(\tilde{\Omega}_t)] \in H^3(M_t, \mathbb{R})$ and $[\omega_t] = c_t\,[\tilde{\omega}_t] \in H^2(M_t, \mathbb{R})$ for some $c_t > 0$. Here all norms are computed with respect to g_t.

Proof. The scheme of the proof follows exactly in the same manner as in Theorem 3.32 for the unobstructed case where $\lambda_i < -3$. Thus we only have to estimate various norms of forms on the regions $P_{t,i} \cap Q_t$ and $Q_t \setminus \big[\big(\bigcup_{i=1}^n P_{t,i}\big) \cap Q_t\big]$, so as to verify (i)-(iii) of Theorem 3.14. On the annuli $P_{t,i} \cap Q_t$ for $i = 1, \ldots, n$ we have

$$|\Omega_t - (\Omega_{V_i} + t^3\xi_i)|_{g_{V_i}} = O(t^{\alpha\nu}) + O(t^{-\lambda_i'(1-\alpha)})$$

by (4.44) where $\lambda_i' < -3$. Now fix $x \in V_i'$ and consider the space of all (ω_x, Ω_x) where ω is a real 2-form and Ω a complex 3-form on V_i'. Let S_x be the subspace of (ω_x, Ω_x) which are SU(3)-structures at x, that is, there is an isomorphism between T_xV_i' and $\mathbb{C}^3$ such that ω_x is identified with $\hat{\omega}$ in $\mathbb{C}^3$ and Ω_x with $\hat{\Omega}$ in $\mathbb{C}^3$ (cf. the discussion after Definition 3.1). Since ξ_i is a trace-free (2,1)-form on V_i, changing Ω_{V_i} by ξ_i will give a deformation of an SU(3)-structures to first order, and so there is an SU(3)-structure $(\omega_{V_i}, \Omega_{V_i}')$ on V_i, which is the projection of $(\omega_{V_i}, \Omega_{V_i} + t^3\xi_i)$ onto the subspace S_x for all $x \in V_i'$, such that

$$\begin{aligned} |(\Omega_{V_i} + t^3\xi_i) - \Omega_{V_i}'|_{g_{V_i}} &= O(|t^3\xi_i|_{g_{V_i}}^2) \\ &= O(t^{6(1-\alpha)}) \quad \text{for } r \in (t^\alpha, 2t^\alpha). \end{aligned}$$

Therefore,

$$\begin{aligned} |\Omega_t - \Omega_{V_i}'|_{g_{V_i}} &\leq |\Omega_t - (\Omega_{V_i} + t^3\xi_i)|_{g_{V_i}} + |(\Omega_{V_i} + t^3\xi_i) - \Omega_{V_i}'|_{g_{V_i}} \\ &= O(t^{\alpha\nu}) + O(t^{-\lambda_i'(1-\alpha)}) + O(t^{6(1-\alpha)}). \end{aligned}$$

Following the argument from the proof of Theorem 3.32, the crucial things to look at are the L^2-norms:

$$\|\omega_t - \omega_t'\|_{L^2} = O(t^{3\alpha+\alpha\nu}) + O(t^{3\alpha-\lambda_i'(1-\alpha)}) + O(t^{6-3\alpha}) = \|\mathrm{Im}(\Omega_t) - \mathrm{Im}(\Omega_t')\|_{L^2}.$$

Although there is an extra term $O(t^{6-3\alpha})$ in the estimate, we can still show that there exist solutions $\alpha \in (0,1)$ and $\kappa > 0$ to the inequalities

$$3\alpha + \alpha\nu \geq 3 + \kappa, \quad 3\alpha - \lambda_i'(1-\alpha) \geq 3 + \kappa \quad \text{and} \quad 6 - 3\alpha \geq 3 + \kappa$$

for any $\nu > 0$ and $\lambda_i' < -3$, since the extra inequality holds automatically for any $\alpha \in (0,1)$. Hence on the annuli $P_{t,i} \cap Q_t$, we show that (i)-(iii) of Theorem 3.14 hold for sufficiently small t, and it remains to estimate the norms on the region $Q_t \setminus \left[\left(\bigcup_{i=1}^n P_{t,i}\right) \cap Q_t\right]$.

The hypothesis ensure the existence of η on M_0'. Using the fact that η is of type (2,1), $\Omega_t = \Omega_0 + t^3\eta$ on $Q_t \setminus \left[\left(\bigcup_{i=1}^n P_{t,i}\right) \cap Q_t\right]$ is then a change of (3,0)-form up to first order. Since $\Omega_t' = \mathrm{Re}(\Omega_t) + i\theta_{2,t}'$ is of type (3,0) w.r.t. J_t, we have $|\Omega_t - \Omega_t'|_{g_t} = O(|t^3\eta|_{g_t}^2)$, which gives $|\mathrm{Im}(\Omega_t) - \mathrm{Im}(\Omega_t')|_{g_t} = O(|t^3\eta|_{g_t}^2)$. For $r \in (2t^\alpha, \epsilon)$,

$$|\mathrm{Im}(\Omega_t) - \mathrm{Im}(\Omega_t')|_{g_t} = O(t^{6(1-\alpha)}). \tag{4.47}$$

Now note that

$$\begin{aligned}
\mathrm{Re}(\Omega_t) \wedge \mathrm{Im}(\Omega_t') &= \mathrm{Re}(\Omega_t) \wedge \mathrm{Im}(\Omega_t) + \mathrm{Re}(\Omega_t) \wedge (\mathrm{Im}(\Omega_t') - \mathrm{Im}(\Omega_t)) \\
&= (\mathrm{Re}(\Omega_0 + t^3\eta) \wedge \mathrm{Im}(\Omega_0 + t^3\eta)) + O(t^{6(1-\alpha)}) \quad \text{by (4.47)} \\
&= \mathrm{Re}(\Omega_0) \wedge \mathrm{Im}(\Omega_0) \; + \; \mathrm{Re}(t^3\eta) \wedge \mathrm{Im}(t^3\eta) \; + \; O(t^{6(1-\alpha)})
\end{aligned}$$

as both wedge products $\mathrm{Re}(\Omega_0) \wedge \mathrm{Im}(t^3\eta)$ and $\mathrm{Im}(\Omega_0) \wedge \mathrm{Re}(t^3\eta)$ cannot give (3,3)-forms. Hence we have

$$\mathrm{Re}(\Omega_t) \wedge \mathrm{Im}(\Omega_t') \; = \; \mathrm{Re}(\Omega_0) \wedge \mathrm{Im}(\Omega_0) \; + \; O(t^{6(1-\alpha)}). \tag{4.48}$$

For the difference between ω_t and ω_t', note that $\omega_t = \omega_0$ on $Q_t \setminus \left[\left(\bigcup_{i=1}^n P_{t,i}\right) \cap Q_t\right]$, and hence $\omega_t - \omega_t^{(1,1)}$ w.r.t. J_t is essentially $\omega_0^{(0,2)}$, which is isomorphic to $\omega_0 \wedge \Omega_t'$. Now,

$$\begin{aligned}
\omega_0 \wedge \Omega_t' &= \omega_0 \wedge (\Omega_t + i(\mathrm{Im}(\Omega_t') - \mathrm{Im}(\Omega_t))) \\
&= \omega_0 \wedge (\Omega_0 + t^3\eta + O(t^{6(1-\alpha)}) \quad \text{by (4.47)} \\
&= O(t^{6(1-\alpha)}) \quad \text{since } \omega_0 \wedge \Omega_0 = 0 = \omega_0 \wedge \eta
\end{aligned}$$

Hence

$$\omega_t^{(1,1)} = \omega_0 + O(t^{6(1-\alpha)}),$$

and

$$(\omega_t^{(1,1)})^3 = \omega_0^3 + O(t^{6(1-\alpha)})$$

By comparing this to (4.48), we see that the function f for rescaling is equal to $1 + O(t^{6(1-\alpha)})$. Thus $\omega_t' = f^{-\frac{1}{3}}\omega_t^{(1,1)} = \omega_0 + O(t^{6(1-\alpha)})$, and we have

$$|\omega_t - \omega_t'|_{g_t} = O(t^{6(1-\alpha)}). \tag{4.49}$$

Once again by focussing on the L^2-norms, we obtain

$$\|\omega_t - \omega_t'\|_{L^2} = O(t^{6-3\alpha}) = \|\mathrm{Im}(\Omega_t) - \mathrm{Im}(\Omega_t')\|_{L^2},$$

which have been handled before. As a consequence, (i)-(iii) of Theorem 3.14 hold on $Q_t \setminus \left[\left(\bigcup_{i=1}^n P_{t,i}\right) \cap Q_t\right]$ for sufficiently small t. The theorem now follows from Theorem 3.14, arguing in the same way as in the proof of Theorem 3.32. □

Remark Theorem 4.28 can be regarded as a generalization of Theorem 3.32. If an AC Calabi–Yau 3-fold $(Y_i, J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i})$ has rate $\lambda_i < -3$, it also has rate -3, in which case we take $\xi_i = 0$ and the conditions for ξ_i in Theorem 4.28 are then satisfied automatically. Hence Theorem 3.32 is a special case of Theorem 4.28.

4.6 Desingularizations of Calabi–Yau 3-folds with ordinary double points

We now apply the theory of the previous sections to desingularize the simple kind of singularities known as *ordinary double points*, or *nodes* of Calabi–Yau 3-folds. We remark here that, unlike the orbifold case, we need to assume the existence of singular Calabi–Yau metrics on compact complex 3-folds with ordinary double points for the methods developed in the thesis to apply, as we do not yet have any existence result for Calabi–Yau metrics on such kind of manifolds. As we shall see in the following, the AC Calabi–Yau 3-fold we use for gluing is the cotangent bundle T^*S^3 of S^3. Our goal in this section is to construct a nice coordinate system on T^*S^3 so that the conditions of (4.2) holds, and hence our main result is applicable to desingularizing Calabi–Yau 3-folds with ordinary double points. Some reading on ordinary double points are Friedman [16], Joyce [26, Example 6.3.4] and Reid [44].

Definition 4.29 Let V be the cone in $\mathbb{C}^4$ defined by the complex quadric

$$\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\},$$

which is smooth apart from the origin. The singularity at the origin is called an *ordinary double point*, or a *node*. It is a kind of double point as both $F = z_1^2 + z_2^2 + z_3^2 + z_4^2$ and dF vanish at the origin, but the hessian $\mathrm{Hess}\, F(0)$ is nondegenerate. We have seen this in Example 3.19. By making a linear change of coordinates: $x = z_1 + iz_2$, $y = z_1 - iz_2$, $z = z_3 + iz_4$ and $w = -z_3 - iz_4$, we can write the quadric in the form

$$\{(x, y, z, w) \in \mathbb{C}^4 : xy = zw\}.$$

Stenzel constructed [48] Ricci-flat metrics on V (in fact on any complex m-dimensional quadric), thus making V a Calabi–Yau cone. It can be shown that the link Γ has the topology of $S^2 \times S^3$, and hence V is topologically a cone on $S^2 \times S^3$. One can also describe V as follows. Consider the blow-up $\widetilde{\mathbb{C}}^4$ of $\mathbb{C}^4$ at origin. It introduces an exceptional divisor $\mathbb{CP}^3$, and the blow-up $\widetilde{V}$ of the cone V inside $\widetilde{\mathbb{C}}^4$ meets this $\mathbb{CP}^3$ at $S \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. The exceptional divisor $\mathbb{CP}^3$ corresponds to the zero section of the line bundle L given by $\widetilde{\mathbb{C}}^4 \longrightarrow \mathbb{CP}^3$, and so its normal bundle is isomorphic to L. Hence the normal bundle $\mathcal{O}(-1,-1)$ over $\mathbb{CP}^1 \times \mathbb{CP}^1$ is isomorphic to the line bundle $\widetilde{V} \longrightarrow S$. This gives us the following isomorphisms:

$$V \setminus \{0\} \;\cong\; \widetilde{V} \setminus S \;\cong\; \mathcal{O}(-1,-1) \setminus (\mathbb{CP}^1 \times \mathbb{CP}^1).$$

Let $V_1, \ldots, V_n$ be Calabi–Yau cones defined by the complex quadric in $\mathbb{C}^4$. A *Calabi–Yau 3-fold* $(M_0, J_0, \omega_0, \Omega_0)$ *with ordinary double points* $x_1, \ldots, x_n$ is a Calabi–Yau 3-fold with conical singularities at $x_1, \ldots, x_n$ with rate $\nu > 0$ modelled on the Calabi–Yau cones $V_1, \ldots, V_n$.

There are two different ways of repairing the ordinary double points, as we have discussed briefly in first part of Chapter 1. The first is by making a *small resolution*, in which the singular points are replaced by rational curves $\mathbb{CP}^1$. To see this explicitly, we define $\widetilde{V}_i^+ \subset \mathbb{C}^4 \times \mathbb{CP}^1$ for $i = 1, \ldots, n$ by

$$\widetilde{V}_i^+ = \big\{((x,y,z,w),[u_1,u_2]) \in \mathbb{C}^4 \times \mathbb{CP}^1 : xu_2 = wu_1,\ zu_2 = yu_1\big\}$$

and define $\pi_i^+ : \widetilde{V}_i^+ \longrightarrow \mathbb{C}^4$ to be the projection to the first factor. Since u_1, u_2 are not both zero, the two equations $xu_2 = wu_1$ and $zu_2 = yu_1$ imply $xy = zw$ which is the defining equation of the alternative form of V_i. This means π_i^+ maps $\widetilde{V}_i^+$ to V_i. Moreover, π_i^+ is an isomorphism except at 0, and so $\widetilde{V}_i^+$ is isomorphic to V_i away from the origin, and replaces the origin by $(\pi_i^+)^{-1}(0) = \mathbb{CP}^1$. Essentially $\widetilde{V}_i^+$ is the normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{CP}^1$ with fibre $\mathbb{C}^2$. Thus we have

$$V_i \setminus \{0\} \;\cong\; \widetilde{V}_i^+ \setminus (\pi_i^+)^{-1}(0) \;\cong\; \mathcal{O}(-1) \oplus \mathcal{O}(-1) \setminus \mathbb{CP}^1.$$

$\widetilde{V}_i^+$ is called a small resolution of V_i. Note that one can obtain a second small resolution $\widetilde{V}_i^-$ by swapping z and w (which preserves $xy = zw$, i.e. preserves V_i) in $\widetilde{V}_i^+$. These two small resolutions are related by a process in algebraic geometry called a *flop*.

Candelas and de la Ossa [11, p.258] constructed Ricci-flat metrics on $\widetilde{V}_i^\pm$, and in our notation, these metrics satisfy $\Upsilon_i^*(g_{Y_i}) - g_{V_i} = O(r^{-2})$ for $Y_i = \widetilde{V}_i^\pm$. Therefore the small resolutions $\widetilde{V}_i^\pm$ are AC Calabi–Yau 3-folds with rate -2, and this is a rate out of reach by our techniques developed in this chapter.

Another way of desingularizing the ordinary double points is by *deformation* or *smoothing* (cf. Example 2.7 and Example 3.29), where V_i is deformed to

$$Q_\epsilon = \{(z_1,z_2,z_3,z_4) \in \mathbb{C}^4 \,:\, z_1^2 + z_2^2 + z_3^2 + z_4^2 = \epsilon\}$$

for some nonzero $\epsilon \in \mathbb{C}$. This has the effect of replacing the singularity by a 3-sphere S^3. We now relate the deformation to the cotangent bundle of the 3-sphere. By identifying the cotangent bundle with the tangent bundle via the round metric, we can realize T^*S^3 as follows:

$$T^*S^3 = \{(v,\xi) \in \mathbb{R}^4 \times \mathbb{R}^4 \,:\, |v| = 1,\ \langle v, \xi\rangle = 0\}.$$

It has a canonical symplectic structure, and following [50], we can map the cotangent bundle T^*S^3 diffeomorphically to the deformation Q_ϵ by

$$\begin{aligned} F_\epsilon \,:\, T^*S^3 &\longrightarrow\; Q_\epsilon \\ (v,\xi) &\mapsto\; \sqrt{\epsilon}\, v \cosh|\xi| \,+\, i\sqrt{\epsilon}\, \frac{\xi}{|\xi|} \sinh|\xi|, \end{aligned}$$

so that the standard symplectic form on $\mathbb{C}^4$ restricted to Q_ϵ is identified with the canonical symplectic form on T^*S^3. We now fix our attention on $\epsilon = 1$.

Consider the action of the group SO(4) on Q_1 given by the usual matrix multiplication:

$$(z_1, z_2, z_3, z_4)^T \longmapsto A \cdot (z_1, z_2, z_3, z_4)^T \qquad \text{for } A \in \mathrm{SO}(4),$$

and on T^*S^3 given by

$$(v, \xi) \longmapsto (Av, A\xi) \qquad \text{for } A \in \mathrm{SO}(4).$$

Observe that F_1 maps T^*S^3 to Q_1 equivariantly with respect to these SO(4)-actions. On T^*S^3, the action is transitive on each level set $|\xi| = c > 0$. Take $v = (1, 0, 0, 0)^T$ and $\xi = (0, c, 0, 0)^T$, the stabilizer at (v, ξ) is of the form

$$\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \qquad \text{where } I \text{ is the } 2 \times 2 \text{ identity matrix and } B \in \mathrm{SO}(2).$$

Thus T^*S^3 or Q_1 admits an SO(4)-action with generic orbit SO(4)/SO(2), which implies that the action is of cohomogeneity one, that is, each generic orbit has real codimension one in T^*S^3 or Q_1. As the zero section S^3 of T^*S^3 corresponds to the case when $\xi = 0$, we have the fibration

$$0 \longrightarrow \mathrm{SO}(4)/\mathrm{SO}(2) \longrightarrow T^*S^3 \setminus S^3 \longrightarrow (0, \infty) \longrightarrow 0. \tag{4.50}$$

As a complex hypersurface of $\mathbb{C}^4$, Q_1 inherits a complex structure. With respect to this complex structure, Stenzel [48, p.161] constructed an SO(4)-invariant Ricci-flat metric on Q_1, and hence on the cotangent bundle T^*S^3. Denote by ω_{Q_1} the Kähler form corresponding to Stenzel's Ricci-flat metric, and Ω_{Q_1} the holomorphic (3,0)-form which can be computed explicitly by the relation:

$$\frac{1}{2}\, d(z_1^2 + z_2^2 + z_3^2 + z_4^2 - 1) \wedge \Omega_{Q_1} = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4.$$

Thus on $\{z_1 \neq 0\}$, we have

$$\Omega_{Q_1} = \frac{1}{z_1}\, dz_2 \wedge dz_3 \wedge dz_4 \Big|_{Q_1}. \tag{4.51}$$

Note that Stenzel's Kähler form ω_{Q_1} is the unique SO(4)-invariant Kähler form (see [48], Lemma 5) normalized w.r.t. Ω_{Q_1}, that is, $(J_{Q_1}, \omega_{Q_1}, \Omega_{Q_1})$ is a Calabi–Yau structure on Q_1, where J_{Q_1} is the complex structure on Q_1 inherited from the standard complex structure on $\mathbb{C}^4$.

We hope to construct a special section of the fibration (4.50), i.e. a map $s : (0, \infty) \longrightarrow Q_1 \setminus S^3$ such that $s(x)$ is a unique point in the codimension one orbit SO(4)/SO(2) corresponding to $x \in (0, \infty)$. Then we obtain a natural identification

$$\mathrm{SO}(4)/\mathrm{SO}(2) \times (0, \infty) \;\cong\; Q_1 \setminus S^3$$

given by $(A \cdot \mathrm{SO}(2),\, x) \longmapsto A \cdot s(x)$ for $A \in \mathrm{SO}(4)$, which is well-defined as the stabilizer at $s(x)$ is also SO(2).

For this purpose, we are going to construct a distinguished real curve in $T^*S^3 \cong Q_1$ which corresponds to the fixed point set of some automorphisms on Q_1. The section s can then be defined so that its image in Q_1 is exactly the real curve. Here is an example of a suitable automorphism σ:

$$(z_1, z_2, z_3, z_4) \longmapsto (\bar{z}_1, -\bar{z}_2, \bar{z}_4, -\bar{z}_3).$$

Clearly, $\sigma^4 = \mathrm{Id}$, and hence σ generates a group $\langle\sigma\rangle$ of automorphisms of Q_1 isomorphic to $\mathbb{Z}_4$. Since σ is defined by conjugation, we have $\sigma_*(J_{Q_1}) = -J_{Q_1}$. From (4.51), we obtain $\sigma_*(\Omega_{Q_1}) = -\bar{\Omega}_{Q_1}$. With respect to the complex structure $-\sigma_*(J_{Q_1}) = J_{Q_1}$, $-\sigma_*(\omega_{Q_1})$ is also an SO(4)-invariant Ricci-flat Kähler form, which is therefore equal to ω_{Q_1} by uniqueness of SO(4)-invariant Kähler forms normalized w.r.t. Ω_{Q_1}. Hence σ preserves Stenzel's Calabi–Yau structure on $T^*S^3 \cong Q_1$ up to a sign, which is the main reason of using such an automorphism to make a $\mathbb{Z}_4$-action. Together with the SO(4)-action on Q_1 described before, we now have a $\mathbb{Z}_4 \ltimes \mathrm{SO}(4)$-action on Q_1, where the $\mathbb{Z}_4$-action interacts nicely with the SO(4)-action in the sense that $\sigma \circ A \circ \sigma^{-1} \in \mathrm{SO}(4)$ for each $A \in \mathrm{SO}(4)$, and that it commutes with the action of SO(2). We will discuss this again later.

The fixed locus of σ is given by $\{(x_1, ix_2, 0, 0) : x_1, x_2 \in \mathbb{R},\ x_1^2 - x_2^2 = 1\}$. Choose a parametrization by $x_1 = \cosh x$ and $x_2 = \sinh x$ for $x \in \mathbb{R}$. Define $s : (0,\infty) \longrightarrow Q_1 \setminus S^3$ by $s(x) = (\cosh x, i \sinh x, 0, 0)$. Now we fix an identification $\varphi : \mathrm{SO}(4)/\mathrm{SO}(2) \times (0,\infty) \longrightarrow Q_1 \setminus S^3$ given by

$$\varphi(A \cdot \mathrm{SO}(2),\, x) = A \cdot (\cosh x, i \sinh x, 0, 0) \qquad \text{for } A \in \mathrm{SO}(4) \text{ and } x \in (0,\infty).$$

Denote by X_{ij} the 4×4-matrix with ij-th entry $= 1$, ji-th entry $= -1$ and all other entries zero. Then $\{e_1 = X_{12},\ e_2 = X_{13},\ e_3 = X_{14},\ e_4 = X_{23},\ e_5 = X_{24},\ e_6 = X_{34}\}$ gives a basis for the Lie algebra $\mathfrak{so}(4)$. Project it to $\mathfrak{so}(4)/\mathfrak{so}(2)$, then $\{e_1, e_2, e_3, e_4, e_5\}$ forms a basis. The differential $d\varphi$ at $(\mathrm{Id} \cdot \mathrm{SO}(2), x)$ then identifies the tangent space $T_{(\cosh x, i \sinh x, 0,0)} Q_1$ with the vector space $\mathfrak{so}(4)/\mathfrak{so}(2) \times \mathbb{R}$ by:

$$\begin{aligned}
e_1 &\longmapsto \Big(i \sinh x \frac{\partial}{\partial z_1} - \cosh x \frac{\partial}{\partial z_2}\Big) + \Big(-i \sinh x \frac{\partial}{\partial \bar{z}_1} - \cosh x \frac{\partial}{\partial \bar{z}_2}\Big)\\
e_2 &\longmapsto -\cosh x \frac{\partial}{\partial z_3} - \cosh x \frac{\partial}{\partial \bar{z}_3}\\
e_3 &\longmapsto -\cosh x \frac{\partial}{\partial z_4} - \cosh x \frac{\partial}{\partial \bar{z}_4}\\
e_4 &\longmapsto -i \sinh x \frac{\partial}{\partial z_3} + i \sinh x \frac{\partial}{\partial \bar{z}_3}\\
e_5 &\longmapsto -i \sinh x \frac{\partial}{\partial z_4} + i \sinh x \frac{\partial}{\partial \bar{z}_4}\\
\frac{\partial}{\partial x} &\longmapsto \Big(\sinh x \frac{\partial}{\partial z_1} + i \cosh x \frac{\partial}{\partial z_2}\Big) + \Big(\sinh x \frac{\partial}{\partial \bar{z}_1} - i \cosh x \frac{\partial}{\partial \bar{z}_2}\Big).
\end{aligned} \tag{4.52}$$

We are interested in looking at all the closed homogeneous 2-forms on $\mathrm{SO}(4)/\mathrm{SO}(2) \times (0,\infty)$ with $\sigma = -1$, as the Kähler form on the cone and on Q_1 are of this type. This leads us to look at the SO(4)-invariant 2-forms on SO(4)/SO(2), which is equivalent to considering the Ad(SO(2))-invariant 2-forms on $\mathfrak{so}(4)/\mathfrak{so}(2)$. Regard the tangent space $T_{(\cosh x, i \sinh x, 0,0)} Q_1$ as a $\mathbb{C}^3$ subspace of $\mathbb{C}^4$ given by $\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (\cosh x)\, z_1 + i\,(\sinh x)\, z_2 = 0\}$. Write $w_1 = z_2$, $w_2 = z_3$ and $w_3 = z_4$ so that the $\mathbb{C}^3$ with coordinates (w_1, w_2, w_3) parametrizes $T_{(\cosh x, i \sinh x, 0, 0)} Q_1$ by $(w_1, w_2, w_3) \mapsto (-i\,(\tanh x)\, w_1, w_1, w_2, w_3)$.

The action of SO(2) can be written as:

$$(w_1, w_2, w_3) \longmapsto (w_1,\, \cos\theta w_2 + \sin\theta w_3,\, -\sin\theta w_2 + \cos\theta w_3) \quad \text{for } \theta \in \mathbb{R}.$$

Calculation shows that the invariant 2-forms are given by

$$dw_1 \wedge d\bar{w}_1,\ dw_2 \wedge d\bar{w}_2 + dw_3 \wedge d\bar{w}_3,\ \mathrm{Re}\,(dw_2 \wedge dw_3),\ \mathrm{Im}\,(dw_2 \wedge dw_3) \text{ and } \mathrm{Re}\,(dw_2 \wedge d\bar{w}_3).$$

Thus the space of SO(2)-invariant 2-forms on $\mathbb{C}^3$ has real dimension 5. For the $\mathbb{Z}_4$-action σ, we have

$$(w_1, w_2, w_3) \longmapsto (-\bar{w}_1, \bar{w}_3, -\bar{w}_2).$$

Note that this action commutes with the above SO(2)-action. Among the SO(2)-invariant 2-forms, σ acts as -1 on $dw_1 \wedge d\bar{w}_1$, $dw_2 \wedge d\bar{w}_2 + dw_3 \wedge d\bar{w}_3$ and $\mathrm{Im}\,(dw_2 \wedge dw_3)$, and as 1 on $\mathrm{Re}\,(dw_2 \wedge dw_3)$ and $\mathrm{Re}\,(dw_2 \wedge d\bar{w}_3)$.

We now proceed to work out the complex structure on $\mathfrak{so}(4)/\mathfrak{so}(2) \times \mathbb{R}$ which corresponds to the complex structure on $T_{(\cosh x, i \sinh x, 0,0)} Q_1$, i.e. multiplication by i on $\mathbb{C}^3$. Then we can write the above SO(2)- and $\mathbb{Z}_4$-action in terms of $e_1, \dots, e_5, \partial/\partial x$, and obtain the invariant 2-forms on $\mathfrak{so}(4)/\mathfrak{so}(2) \times \mathbb{R}$ by pulling back by φ. In view of (4.52), the corresponding complex structure J on $\mathfrak{so}(4)/\mathfrak{so}(2) \times \mathbb{R}$ is given by

$$\begin{aligned} J : e_1 &\longmapsto -\frac{\partial}{\partial x}, & e_2 &\longmapsto (\coth x)\, e_4, & e_3 &\longmapsto (\coth x)\, e_5, \\ e_4 &\longmapsto -(\tanh x)\, e_2, & e_5 &\longmapsto -(\tanh x)\, e_3, & \frac{\partial}{\partial x} &\longmapsto e_1. \end{aligned} \tag{4.53}$$

Denote by $\omega^1, \dots, \omega^6, dx$ the dual basis of $e_1, \dots, e_6, \partial/\partial x$. Then $\{\omega^1, \dots, \omega^5\}$ is a basis for $(\mathfrak{so}(4)/\mathfrak{so}(2))^*$. Calculation shows that

$$\begin{aligned} \varphi^*(dz_1) &= \sinh x \cdot (dx + i\omega^1), & \varphi^*(dz_2) &= i \cosh x \cdot (dx + i\omega^1), \\ \varphi^*(dz_3) &= -\cosh x\, \omega^2 - i \sinh x\, \omega^4, & \varphi^*(dz_4) &= -\cosh x\, \omega^3 - i \sinh x\, \omega^5 \end{aligned} \tag{4.54}$$

and the SO(2)-invariant 2-forms with $\sigma = -1$ on $\mathfrak{so}(4)/\mathfrak{so}(2) \times \mathbb{R}$ are given by

$$\begin{aligned} \varphi^*(\tfrac{i}{2}\, dw_1 \wedge d\bar{w}_1) &= \cosh^2 x \cdot dx \wedge \omega^1, \\ \varphi^*(\tfrac{i}{2}\,(dw_2 \wedge d\bar{w}_2 + dw_3 \wedge d\bar{w}_3)) &= \cosh x \cdot \sinh x \cdot (\omega^2 \wedge \omega^4 + \omega^3 \wedge \omega^5), \\ \varphi^*(\mathrm{Im}\,(dw_2 \wedge dw_3)) &= \cosh x \cdot \sinh x \cdot (\omega^2 \wedge \omega^5 - \omega^3 \wedge \omega^4). \end{aligned}$$

Write $\alpha = \omega^1$, $\beta = \omega^2 \wedge \omega^4 + \omega^3 \wedge \omega^5$ and $\gamma = \omega^2 \wedge \omega^5 - \omega^3 \wedge \omega^4$. A general SO(4)-invariant 2-form with $\sigma = -1$ can then be written as

$$f_1(x)\, dx \wedge \alpha + f_2(x)\, \beta + f_3(x)\, \gamma$$

for some smooth real functions f_1, f_2 and f_3.

On the Lie algebra $\mathfrak{so}(4)$, the Lie brackets for the basis elements satisfy $[X_{ab}, X_{ac}] = -X_{bc}$, and $[X_{ab}, X_{cd}] = 0$ if a, b, c, d are distinct. Using these relations yields all the structural constants C^k_{ij} of the Lie bracket, and by applying the Maurer-Cartan equation $d\omega^k = -\sum_{i<j} C^k_{ij}\,\omega^i \wedge \omega^j$, we obtain

$$\begin{array}{lll} d\omega^1 = \omega^2 \wedge \omega^4 + \omega^3 \wedge \omega^5, & d\omega^2 = -\omega^1 \wedge \omega^4 + \omega^3 \wedge \omega^6, & d\omega^3 = -\omega^1 \wedge \omega^5 - \omega^2 \wedge \omega^6, \\ d\omega^4 = \omega^1 \wedge \omega^2 + \omega^5 \wedge \omega^6, & d\omega^5 = \omega^1 \wedge \omega^3 - \omega^4 \wedge \omega^6, & d\omega^6 = \omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^5. \end{array} \tag{4.55}$$

It follows that $\beta = d\omega^1 = d\alpha$ is an exact 2-form, and

$$\begin{aligned} d\gamma &= d\omega^2 \wedge \omega^5 - \omega^2 \wedge d\omega^5 - d\omega^3 \wedge \omega^4 + \omega^3 \wedge d\omega^4 \\ &= -\omega^1 \wedge \omega^4 \wedge \omega^5 + \omega^3 \wedge \omega^6 \wedge \omega^5 - \omega^2 \wedge \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^4 \wedge \omega^6 + \omega^1 \wedge \omega^5 \wedge \omega^4 \\ &\quad + \omega^2 \wedge \omega^6 \wedge \omega^4 + \omega^3 \wedge \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^5 \wedge \omega^6 \\ &= 2\,\omega^1 \wedge (\omega^2 \wedge \omega^3 - \omega^4 \wedge \omega^5) \neq 0. \end{aligned}$$

Suppose $f_1(x)\,dx \wedge \alpha + f_2(x)\,\beta + f_3(x)\,\gamma$ is a closed 2-form. For the remainder of this section, we shall use a prime to indicate differentiation w.r.t. x. Then

$$\begin{aligned} 0 &= d(f_1(x)\,dx \wedge \alpha + f_2(x)\,\beta + f_3(x)\,\gamma) \\ &= -f_1(x)\,dx \wedge d\alpha + f_2'(x)\,dx \wedge \beta + f_2(x)\,d\beta + f_3'(x)\,dx \wedge \gamma + f_3(x)\,d\gamma \\ &= (-f_1(x) + f_2'(x))\,dx \wedge d\alpha + f_3'(x)\,dx \wedge \gamma + f_3(x)\,d\gamma. \end{aligned}$$

This amounts to $f_3(x) = 0$, as $d\gamma \neq 0$. It follows that there is no γ component in a closed 2-form, and $f_2'(x) = f_1(x)$. We thus obtain

Lemma 4.30 *The* $\mathrm{SO}(4)$*-invariant homogeneous closed 2-forms on* $\mathrm{SO}(4)/\mathrm{SO}(2) \times (0, \infty)$ *with the* $\mathbb{Z}_4$*-action* $\sigma = -1$ *are of the form*

$$u'(x)\,dx \wedge \alpha + u(x)\,\beta \tag{4.56}$$

where u *is some smooth real function.*

On Q_1 Stenzel's Kähler form is given by $\omega_{Q_1} = d(J_{Q_1} df(w))$ where $w = \cosh^{-1}(|z_1|^2 + \cdots + |z_4|^2) = \cosh^{-1}(|\cosh x|^2 + |i\sinh x|^2) = 2x$, and $f(w)$ is the Kähler potential satisfying the differential equation

$$\frac{d}{dw}\Big(\big(\frac{df}{dw}\big)^3\Big) = 3c\,(\sinh w)^2 \tag{4.57}$$

for some constant c to be fixed later. With the initial condition $f'(0) = 0$ the derivative of f is given by

$$\frac{df}{dw} = \Big(\frac{3c}{4}\Big(\frac{e^{2w}}{2} - 2w - \frac{e^{-2w}}{2}\Big)\Big)^{1/3}$$

or in terms of our coordinate x,

$$\frac{1}{2}\,f'(2x) = \Big(\frac{3c}{4}\Big(\frac{e^{4x}}{2} - 4x - \frac{e^{-4x}}{2}\Big)\Big)^{1/3}. \tag{4.58}$$

Expressing ω_{Q_1} in the form of (4.56) we have

$$\omega_{Q_1} = \frac{d^2 f}{dw^2}\, dw \wedge \alpha + \frac{df}{dw}\beta$$

or

$$\omega_{Q_1} = \frac{1}{2}\,(f''(2x)\, dx \wedge \alpha + f'(2x)\beta). \tag{4.59}$$

Next we proceed to compute the pullback of the holomorphic (3,0)-form Ω_{Q_1} on $\mathrm{SO}(4)/\mathrm{SO}(2)\times(0,\infty)$. At the point $(\cosh x, i\sinh x, 0, 0)$, (4.51) and (4.54) yield

$$\begin{aligned}
\varphi^*(\Omega_{Q_1}) &= \frac{1}{\cosh x}\big(i\cosh x\cdot(dx+i\omega^1)\wedge(-\cosh x\,\omega^2 - i\sinh x\,\omega^4)\wedge(-\cosh x\,\omega^3 - i\sinh x\,\omega^5)\big)\\
&= i\,(\cosh^2 x\cdot dx\wedge\omega^2\wedge\omega^3 - \sinh^2 x\cdot dx\wedge\omega^4\wedge\omega^5 - \cosh x\cdot\sinh x\cdot\omega^1\wedge\omega^2\wedge\omega^5\\
&\quad - \cosh x\cdot\sinh x\cdot\omega^1\wedge\omega^4\wedge\omega^3 + i\cosh^2 x\cdot\omega^1\wedge\omega^2\wedge\omega^3 - i\sinh^2 x\cdot\omega^1\wedge\omega^4\wedge\omega^5\\
&\quad + i\cosh x\cdot\sinh x\cdot dx\wedge\omega^4\wedge\omega^3 + i\cosh x\cdot\sinh x\cdot dx\wedge\omega^2\wedge\omega^5).
\end{aligned}$$

Now we can determine the value of the constant c in (4.57) by requiring the normalization formula holds, i.e.

$$\omega_{Q_1}^3 = \frac{3}{2}\,\mathrm{Re}\,\Omega_{Q_1}\wedge\mathrm{Im}\,\Omega_{Q_1}.$$

The above expression for Ω_{Q_1} gives

$$\frac{3}{2}\,\mathrm{Re}\,\Omega_{Q_1}\wedge\mathrm{Im}\,\Omega_{Q_1} = -6\cosh^2 x\cdot\sinh^2 x\cdot(dx\wedge\omega^1\wedge\cdots\wedge\omega^5).$$

whereas from (4.59),

$$\omega_{Q_1}^3 = -\frac{3}{4}\,f''(2x)(f'(2x))^2(dx\wedge\omega^1\wedge\cdots\wedge\omega^5).$$

Equation (4.57) implies

$$f''(2x)(f'(2x))^2 = 64\,c\cosh^2 x\cdot\sinh^2 x,$$

which then gives $c = 1/8$.

Putting $\cosh^2 x = \frac{1}{4}\,(e^{2x}+2+e^{-2x})$, $\sinh^2 x = \frac{1}{4}\,(e^{2x}-2+e^{-2x})$ and $\cosh x\cdot\sinh x = \frac{1}{4}\,(e^{2x}-e^{-2x})$ into the formula for Ω_{Q_1} yields

$$\begin{aligned}
\varphi^*(\Omega_{Q_1}) &= \frac{i}{4}\,e^{2x}\,(dx\wedge\omega^2\wedge\omega^3 - dx\wedge\omega^4\wedge\omega^5 - \omega^1\wedge\omega^2\wedge\omega^5 - \omega^1\wedge\omega^4\wedge\omega^3\\
&\qquad + i\omega^1\wedge\omega^2\wedge\omega^3 - i\omega^1\wedge\omega^4\wedge\omega^5 + i dx\wedge\omega^4\wedge\omega^3 + i dx\wedge\omega^2\wedge\omega^5)\\
&\quad + \frac{i}{2}\,(dx\wedge\omega^2\wedge\omega^3 + dx\wedge\omega^4\wedge\omega^5 + i\omega^1\wedge\omega^2\wedge\omega^3 + i\omega^1\wedge\omega^4\wedge\omega^5) + O(e^{-2x})\\
&= \frac{i}{4}\,e^{2x}\,(dx+i\omega^1)\wedge(\omega^2+i\omega^4)\wedge(\omega^3+i\omega^5) + \frac{i}{2}\,(dx+i\omega^1)\wedge(\omega^2\wedge\omega^3+\omega^4\wedge\omega^5)\\
&\quad + O(e^{-2x}).
\end{aligned} \tag{4.60}$$

Consider now the Calabi–Yau structure $(J_{Q_0}, \omega_{Q_0}, \Omega_{Q_0})$ on the cone Q_0. Instead of using $\cosh w = |z_1|^2+\cdots+|z_4|^2$, we use $\frac{1}{2}\,e^{\tilde{w}} = |z_1|^2+\cdots+|z_4|^2$ for Q_0 case. Then $\tilde{w} =$

$\log(2(|z_1|^2+\cdots+|z_4|^2))$. Again, we choose a parametrization of the fixed locus of σ on Q_0 given by $(\frac{1}{2}e^{\tilde{x}}, \frac{i}{2}e^{\tilde{x}}, 0, 0)$ for $\tilde{x} \in (0,\infty)$. Hence $\tilde{w} = \log(2(|\frac{1}{2}e^{\tilde{x}}|^2 + |\frac{i}{2}e^{\tilde{x}}|^2)) = 2\tilde{x}$.

Let $\phi : \mathrm{SO}(4)/\mathrm{SO}(2) \times (0,\infty) \longrightarrow Q_0$ be a coordinate map for Q_0 defined by

$$\phi(A\cdot \mathrm{SO}(2),\, \tilde{x}) = A\cdot\big(\frac{1}{2}e^{\tilde{x}},\, \frac{i}{2}e^{\tilde{x}},\, 0,\, 0\big) \quad \text{for } A \in \mathrm{SO}(4) \text{ and } \tilde{x} \in (0,\infty).$$

From this, we are going to get a new radial coordinate $r \in (0,\infty)$ representing the radial distance on Q_0, and write the Calabi–Yau structure in terms of r.

The same analysis for Q_0 instead of Q_1 gives

$$\begin{aligned} \omega_{Q_0} &= \left(\frac{3}{64}\right)^{1/3} e^{4\tilde{x}/3}\big(\frac{4}{3}\,d\tilde{x}\wedge\alpha+\beta\big), \\ \text{and } \phi^*(\Omega_{Q_0}) &= \frac{i}{4}\,e^{2\tilde{x}}\,(d\tilde{x}\wedge\omega^2\wedge\omega^3 - d\tilde{x}\wedge\omega^4\wedge\omega^5 - \omega^1\wedge\omega^2\wedge\omega^5 - \omega^1\wedge\omega^4\wedge\omega^3 \\ &\qquad + i\omega^1\wedge\omega^2\wedge\omega^3 - i\omega^1\wedge\omega^4\wedge\omega^5 + i d\tilde{x}\wedge\omega^4\wedge\omega^3 + i d\tilde{x}\wedge\omega^2\wedge\omega^5) \end{aligned} \tag{4.61}$$

where we have used $c = 1/8$.

On the other hand, in terms of the radial coordinate r, the complex 3-form Ω_{Q_0} on the cone Q_0 with link $\mathrm{SO}(4)/\mathrm{SO}(2)$ (or $S^2\times S^3$) can be written as

$$\phi^*(\Omega_{Q_0}) = r^3\cdot(\text{3-form on } \mathrm{SO}(4)/\mathrm{SO}(2)) + r^2\,dr\wedge(\text{2-form on } \mathrm{SO}(4)/\mathrm{SO}(2)).$$

Comparing this with (4.61), we obtain

$$r^3 \;=\; k\,e^{2\tilde{x}}$$

for some constant k to be determined later. It follows that

$$d\tilde{x} \;=\; \frac{3}{2r}\,dr, \qquad \text{or} \qquad \frac{\partial}{\partial\tilde{x}} \;=\; \frac{2r}{3}\,\frac{\partial}{\partial r}.$$

As Q_1 approaches the cone Q_0 asymptotically when $x\to\infty$, and for large x, we have $x\approx\tilde{x}$. So from (4.53), the complex structure $\tilde{J}$ on $\mathfrak{so}(4)/\mathfrak{so}(2)\times\mathbb{R}$ corresponding to J_{Q_0} is given by

$$e_1 \longmapsto -\frac{\partial}{\partial\tilde{x}}, \qquad e_2\longmapsto e_4, \qquad e_3\longmapsto e_5, \qquad e_4\longmapsto -e_2, \qquad e_5\longmapsto -e_3, \qquad \frac{\partial}{\partial\tilde{x}}\longmapsto e_1$$

since $\tanh x$, $\coth x\to 1$ as $x\to\infty$. With respect to this complex structure, the vectors e_1 and $\frac{\partial}{\partial r}$, using the relation $\frac{\partial}{\partial\tilde{x}} = \frac{2r}{3}\frac{\partial}{\partial r}$, transform as

$$e_1\longmapsto -\frac{2r}{3}\,\frac{\partial}{\partial r}, \qquad \frac{\partial}{\partial r}\longmapsto \frac{3}{2r}\,e_1.$$

By definition of the cone metric g_{Q_0} and the radial coordinate r, we have

$$\begin{aligned} g_{Q_0}\big(\frac{\partial}{\partial r},\,\frac{\partial}{\partial r}\big) \;=\; 1 \;&\Longrightarrow\; \omega_{Q_0}\big(\frac{\partial}{\partial r},\,\tilde{J}\frac{\partial}{\partial r}\big) \;=\; 1 \\ &\Longrightarrow\; \omega_{Q_0}\big(\frac{\partial}{\partial r},\,e_1\big) \;=\; \frac{2r}{3}. \end{aligned}$$

Substituting $r^3 = k\, e^{2\tilde{x}}$ and $d\tilde{x} = \frac{3}{2r}\, dr$ into ω_{Q_0} in (4.61) gives

$$\omega_{Q_0} = \left(\frac{3}{64}\right)^{1/3} k^{-2/3}\,(2r\, dr \wedge \alpha + r^2\, \beta).$$

The value for k can then be computed by applying $\frac{\partial}{\partial r}$ and e_1 to ω_{Q_0}:

$$\left(\frac{3}{64}\right)^{1/3} k^{-2/3} \cdot 2r = \frac{2r}{3},$$

giving $k = 9/8$, and ω_{Q_0}, in terms of r, is given by

$$\omega_{Q_0} = \frac{1}{3}\,(2r\, dr \wedge \alpha + r^2\, \beta), \tag{4.62}$$

which also satisfies Lemma 4.30 for $u(r) = \frac{r^2}{3}$.

Thus in terms of the radial coordinate r, the coordinate map $\phi : \mathrm{SO}(4)/\mathrm{SO}(2) \times (0, \infty) \longrightarrow Q_0$ can now be written as:

$$\phi(A \cdot \mathrm{SO}(2),\, r) = A \cdot \big(\frac{\sqrt{2}}{3}\, r^{3/2},\, \frac{i\sqrt{2}}{3}\, r^{3/2},\, 0,\, 0\big) \quad \text{for } A \in \mathrm{SO}(4) \text{ and } r \in (0, \infty),$$

and from (4.61) we have

$$\begin{aligned}
\phi^*(\Omega_{Q_0}) = \frac{2i}{9}\, r^3\, \big(&\frac{3}{2r}\, dr \wedge \omega^2 \wedge \omega^3 - \frac{3}{2r}\, dr \wedge \omega^4 \wedge \omega^5 - \omega^1 \wedge \omega^2 \wedge \omega^5 - \omega^1 \wedge \omega^4 \wedge \omega^3 \\
&+ i\omega^1 \wedge \omega^2 \wedge \omega^3 - i\omega^1 \wedge \omega^4 \wedge \omega^5 + \frac{3i}{2r}\, dr \wedge \omega^4 \wedge \omega^3 + \frac{3i}{2r}\, dr \wedge \omega^2 \wedge \omega^5\big) \\
= \frac{2i}{9}\, r^3\, \big(&\frac{3}{2r} dr + i\omega^1\big) \wedge (\omega^2 + i\omega^4) \wedge (\omega^3 + i\omega^5).
\end{aligned} \tag{4.63}$$

It is easy to check that the normalization formula $\omega_{Q_0}^3 = \frac{3}{2}\, \mathrm{Re}\,(\Omega_{Q_0}) \wedge \mathrm{Im}\,(\Omega_{Q_0})$ holds.

The relation between the radial coordinate $r \in (0, \infty)$ for the cone Q_0 and the coordinate $x \in (0, \infty)$ for Q_1 can be obtained by equating ω_{Q_1} in (4.59) and ω_{Q_0} in (4.62) so that the map $\Upsilon : Q_0 \longrightarrow Q_1$ we hope to construct preserves the symplectic forms. Explicitly, this is given by

$$\frac{1}{3}\, r^2 = \frac{1}{2}\, f'(2x),$$

which is equivalent to the relation

$$r = \left(\frac{81}{32}\, \Big(\frac{e^{4x}}{2} - 4x - \frac{e^{-4x}}{2}\Big)\right)^{1/6}. \tag{4.64}$$

This implies

$$r = \left(\frac{9}{8}\right)^{1/3} e^{2x/3}\, \big(1 + O(x\, e^{-4x})\big)$$

and hence

$$e^{2x} = \frac{8}{9}\, r^3\, \big(1 + O(r^{-6} \log r)\big).$$

Putting this into (4.60), and using the fact that for large x, $dx \approx d\tilde{x} = \frac{3}{2r} dr$, we have

$$\begin{aligned}
\varphi^*(\Omega_{Q_1}) =& \frac{2i}{9}\, r^3\, \big(1 + O(r^{-6} \log r)\big)\, \big(\frac{3}{2r} dr + i\omega^1\big) \wedge (\omega^2 + i\omega^4) \wedge (\omega^3 + i\omega^5) + \frac{i}{2}\, \big(\frac{3}{2r} dr + i\omega^1\big) \\
&\wedge (\omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^5) + O(r^{-3}) \\
=& \phi^*(\Omega_{Q_0}) + O(r^{-3} \log r) + \frac{i}{2}\, \big(\frac{3}{2r}\, dr + i\omega^1\big) \wedge (\omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^5) + O(r^{-3}) \quad \text{by (4.63).}
\end{aligned}$$

We remark here that both $O(r^{-3}\log r)$ and $O(r^{-3})$ are measured w.r.t. the cylinder metric, and notice that the term $O(r^{-3})$ can be absorbed in $O(r^{-3}\log r)$. Denote by ξ the 3-form $\frac{i}{2}\,(\frac{3}{2r}dr + i\omega^1) \wedge (\omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^5)$, thus we have

$$\varphi^*(\Omega_{Q_1}) = \phi^*(\Omega_{Q_0}) + \xi + O(r^{-3}\log r). \tag{4.65}$$

Measuring (4.65) w.r.t. the cone metric g_{Q_0}, the error terms ξ and $O(r^{-3}\log r)$ will then have size $O(r^{-3})$ and $O(r^{-6}\log r)$ respectively. This suggests that ξ is the 3-form we need.

We can finally define the map $\Upsilon : Q_0 \longrightarrow Q_1$ so that the following diagram commutes:

$$\begin{tikzcd} \mathrm{SO}(4)/\mathrm{SO}(2) \times (0,\infty) \arrow[r, "(A\cdot\mathrm{SO}(2),\, r)\mapsto(A\cdot\mathrm{SO}(2),\, x)"] \arrow[d, "\phi"'] & \mathrm{SO}(4)/\mathrm{SO}(2) \times (0,\infty) \arrow[d, "\varphi"] \\ Q_0 \arrow[r, "\Upsilon"'] & Q_1 \end{tikzcd}$$

The map in the upper arrow is the change of variable from r to x on $(0,\infty)$ given by the inverse of (4.64). We have therefore constructed a nice coordinate system such that

$$\Upsilon^*(\omega_{Q_1}) = \omega_{Q_0} \quad \text{and} \quad \Upsilon^*(\Omega_{Q_1}) = \Omega_{Q_0} + \xi + O(r^{-6}\log r)$$

computed using the cone metric g_{Q_0}. It is not hard to see that, for the holomorphic (3,0)-forms, the equation for derivatives also holds, i.e.

$$\nabla^k(\Upsilon^*(\Omega_{Q_1}) - \Omega_{Q_0} - \xi) = O(r^{-6-k}\log r) \quad \text{for all } k \geq 0$$

where ∇ is the Levi-Civita connection of g_{Q_0}. In order to show the conditions of (4.2) holds, we still need to check if ξ is a closed, trace-free (2,1)-form. Indeed,

$$\begin{aligned} d\xi &= \frac{i}{2}\,((i\,d\omega^1) \wedge (\omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^5) - (\frac{3}{2r}dr + i\omega^1) \wedge d(\omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^5)) \\ &= \frac{i}{2}\,(i\,(\omega^2 \wedge \omega^4 + \omega^3 \wedge \omega^5) \wedge (\omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^5) - (\frac{3}{2r}dr + i\omega^1) \wedge d(d\omega^6)) \quad \text{from (4.55)} \\ &= 0, \end{aligned}$$

which shows ξ is closed. Moreover, it is of type (2,1) with respect to the complex structure J_{Q_0} on the cone Q_0. From the complex structure on $\mathfrak{so}(4)/\mathfrak{so}(2) \times \mathbb{R}$ corresponding to J_{Q_0}, which is the limit of the complex structure in (4.53), we have

$$\omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^5 = \omega^2 \wedge \tilde{J}\omega^5 - \tilde{J}\omega^2 \wedge \omega^5,$$

which is of type (1,1), as it is $\tilde{J}$-invariant, and

$$\frac{3}{2r}dr + i\omega^1 = \tilde{J}\omega^1 + i\omega^1,$$

which is of type (1,0). Hence $\xi = \frac{i}{2}\,(\frac{3}{2r}dr + i\omega^1) \wedge (\omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^5)$ is a (2,1)-form. Finally, it is easy to see that ξ is trace-free by using Lemma 4.30, as $(dr \wedge \alpha) \wedge \xi = 0$ and $\beta \wedge \xi = 0$.

Therefore we have just illustrated that there is a closed, trace-free (2,1)-form ξ with order $O(r^{-3})$ w.r.t. the cone metric, and a diffeomorphism Υ such that (4.2) holds. The cohomology class $[\xi]$ can be expressed as $[\xi] = [\Upsilon^*(\Omega_{Q_1}) - \Omega_{Q_0}] = [\frac{-1}{2}\omega^1 \wedge (\omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^5)]$, which is a nonzero fixed class in $H^3(S^2 \times S^3, \mathbb{C})$. Thus Q_1 is an AC Calabi–Yau 3-fold with rate -3. By transferring our nice coordinates from Q_1 to Q_{ϵ_i}, where $\epsilon_i \in \mathbb{C} \setminus \{0\}$, and $i = 1, \dots, n$, we can then use Q_{ϵ_i}, or the cotangent bundle $(T^*S^3)_i$, as the AC Calabi–Yau 3-fold $(Y_i, J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i})$ to desingularize Calabi–Yau 3-folds with ordinary double points $x_1, \dots, x_n$. The transformation can be achieved by using the map $f_{\epsilon_i} : Q_{\epsilon_i} \longrightarrow Q_1$ given by $(z_1, \dots, z_4) \mapsto (\epsilon_i^{-1/2} z_1, \dots, \epsilon_i^{-1/2} z_4)$. From (4.51), we have

$$\Omega_{Q_{\epsilon_i}} = \epsilon_i\, f^*_{\epsilon_i}(\Omega_{Q_1}).$$

Note that f_{ϵ_i} maps Q_0 to Q_0 and $\Omega_{Q_0} = \epsilon_i\, f^*_{\epsilon_i}(\Omega_{Q_0})$. Define $\Upsilon_{\epsilon_i} = f^{-1}_{\epsilon_i} \circ \Upsilon \circ f_{\epsilon_i} : Q_0 \longrightarrow Q_{\epsilon_i}$. It follows that

$$\Upsilon^*_{\epsilon_i}(\Omega_{Q_{\epsilon_i}}) = \Omega_{Q_0} + \epsilon_i\, \xi + O(r^{-6} \log r)$$

where we have used $f^*_{\epsilon_i}(\xi) = \xi$, as ξ is invariant under scaling of r. The equation for derivatives of the above formula also holds. Define $\xi_{\epsilon_i} = \epsilon_i\, \xi$. Then the cohomology class $[\xi_{\epsilon_i}] \in H^3((S^2 \times S^3)_i, \mathbb{C})$ equals $\epsilon_i\, [\xi]$, which is nonzero as both ϵ_i and $[\xi] \neq 0$.

Our main result in §4.5 can then be applied to desingularize Calabi–Yau 3-folds with ordinary double points, assuming the existence of these singular Calabi–Yau manifolds:

Theorem 4.31 *Let $(M_0, J_0, \omega_0, \Omega_0)$ be a Calabi–Yau 3-fold with ordinary double points $x_1, \dots, x_n$, that is, a Calabi–Yau 3-fold with conical singularities $x_1, \dots, x_n$ with some rate $\nu > 0$ modelled on the Calabi–Yau cones $V_1, \dots, V_n$ defined by the complex quadric in $\mathbb{C}^4$. If there exist $\epsilon_i \in \mathbb{C} \setminus \{0\}$ for $i = 1, \dots, n$ such that $\bigoplus_{i=1}^n [\xi_{\epsilon_i}] \in \bigoplus_{i=1}^n H^3((S^2 \times S^3)_i, \mathbb{C})$ lies in $\rho_3(H^3(M_0', \mathbb{C}))$, then there exists a smooth family M_t of nonsingular compact Calabi–Yau 3-folds M_t for sufficiently small $t > 0$ constructed by gluing the AC Calabi–Yau 3-fold Q_{ϵ_i}, which is the cotangent bundle $(T^*S^3)_i$, into M_0 at x_i for $i = 1, \dots, n$.*

We remark here that we can choose such ϵ_i for $i = 1, \dots, n$ if and only if $\rho_3(H^3(M_0', \mathbb{C}))$ contains an element with every component in $\bigoplus_{i=1}^n H^3((S^2 \times S^3)_i, \mathbb{C})$ nonzero. In particular, if the cohomology class $[(S^3)_i]$ of $(S^3)_i$ vanishes for some i, then we can't choose ϵ_i nonzero.

We finish this chapter by relating our theorem to an algebraic geometry result on smoothing of singular Calabi–Yau 3-folds with ordinary double points given by Friedman [17, Cor. 8.8]. Suppose Y is a small resolution of the singular Calabi–Yau 3-fold M_0 with ordinary double points at $x_1, \dots, x_n$, and suppose $C_i \subset Y$ is the S^2 introduced at each x_i. Friedman proved that M_0 admits a smoothing if there is a linear relation between the homology classes $[C_i]$ of the $C_i \cong S^2$, namely

$$\sum_{i=1}^n \alpha_i\, [C_i] = 0 \quad \text{in } H_2(Y, \mathbb{C}) \quad \text{with all } \alpha_i \neq 0.$$

We claim that our condition on the cohomology classes $[\xi_{\epsilon_i}]$ in Theorem 4.31 is equivalent to Friedman's condition on the homology classes $[C_i]$.

As we have shown, $\bigoplus_{i=1}^n [\xi_{\epsilon_i}] \in \rho_3(H^3(M_0', \mathbb{C}))$ implies there exists a closed 3-form χ in M_0' such that $\Phi_i^*(\chi) = \xi_{\epsilon_i} + O(r^{-3+\delta})$. Thus for each i, $\int_{(S^3)_i} \chi = [\xi_{\epsilon_i}] \cdot [S^3]$, which is nonzero as

$[\xi_{\epsilon_i}] \neq 0$ in $H^3((S^2 \times S^3)_i, \mathbb{C})$. Now consider a part of the exact sequence of the pair $(Y, Y \setminus \bigcup_{i=1}^n C_i)$:

$$\cdots \longrightarrow H^3(Y \setminus \textstyle\bigcup_{i=1}^n C_i,\, \mathbb{C}) \longrightarrow H^4(Y,\, Y \setminus \bigcup_{i=1}^n C_i,\, \mathbb{C}) \longrightarrow H^4(Y, \mathbb{C}) \longrightarrow \cdots$$

Note that $H^3(M_0', \mathbb{C}) \cong H^3(Y \setminus \bigcup_{i=1}^n C_i, \mathbb{C})$, and using the *Thom isomorphism* (see for example [4, p. 63]), we have $H^4(Y,\, Y \setminus \bigcup_{i=1}^n C_i,\, \mathbb{C}) \cong H^0(\bigcup_{i=1}^n C_i, \mathbb{C})$, which is isomorphic to $H_2(\bigcup_{i=1}^n C_i, \mathbb{C})$ ($\cong \bigoplus_i \mathbb{C}$) by Poincaré duality. It then maps to $H^4(Y, \mathbb{C}) \cong H_2(Y, \mathbb{C})$ by inclusion. The exact sequence now becomes:

$$\cdots \longrightarrow H^3(M_0', \mathbb{C}) \longrightarrow H_2(\textstyle\bigcup_{i=1}^n C_i, \mathbb{C}) \longrightarrow H_2(Y, \mathbb{C}) \longrightarrow \cdots$$

with $[\chi] \in H^3(M_0', \mathbb{C})$ maps to $\sum_{i=1}^n \big(\int_{(S^3)_i} \chi\big)\,[C_i]$, and thus maps to 0 by exactness. This gives Friedman's condition on the homology classes $[C_i]$, as $\int_{(S^3)_i} \chi$ is nonzero for all i. Thus our cohomology condition implies Friedman's homology relation, and it is not hard to see the inverse is also true. It follows that our desingularization result in this chapter provides an analytic way, rather than the existing algebro-geometric way, of repairing the ordinary double points.

Friedman's result, together with Yau's solution to the Calabi conjecture, gives a unique Calabi–Yau metric on the deformation or smoothing of M_0. Our result then describes explicitly what this Calabi–Yau metric looks like, showing that it is in fact obtained by gluing the Calabi–Yau metric on the singular Calabi–Yau 3-fold M_0 with ordinary double points and the Calabi–Yau metrics on each $(T^*S^3)_i$.

Chapter 5

SL m-folds with conical singularities and AC SL m-folds

In Chapters 3 and 4 we studied the desingularization of Calabi–Yau 3-folds with conical singularities. The objects or data we had were Calabi–Yau cones $V_1, \ldots, V_n$, a Calabi–Yau 3-fold M_0 with conical singularities $x_1, \ldots, x_n$ modelled on $V_1, \ldots, V_n$, and AC Calabi–Yau 3-folds $Y_1, \ldots, Y_n$, modelled on the same cones $V_1, \ldots, V_n$. This chapter is devoted to studying special Lagrangian 3-folds inside the above objects. The extra data in this chapter will be SL cones $C_1, \ldots, C_n$ in $V_1, \ldots, V_n$, a singular SL 3-fold N_0 in M_0 with conical singularities at the same points $x_1, \ldots, x_n$ and modelled on SL cones $C_1, \ldots, C_n$, and AC SL 3-folds $L_1, \ldots, L_n$ in $Y_1, \ldots, Y_n$ modelled on same cones $C_1, \ldots, C_n$.

We shall introduce the notion of SL cones in §5.1, SL m-folds with conical singularities in §5.2 and AC SL m-folds in §5.3. Finally, we give some examples of AC SL m-folds in AC Calabi–Yau m-folds given by the canonical line bundle $K_{\mathbb{CP}^{m-1}}$ over $\mathbb{CP}^{m-1}$ and the cotangent bundle T^*S^m of spheres.

5.1 Special Lagrangian cones and their Lagrangian Neighbourhoods

We shall carry on using the notations in §4.1 for the ambient Calabi–Yau setting, and set up new notations for the special Lagrangian submanifolds inside based on them. In this section we consider special Lagrangian cones in Calabi–Yau cones.

Definition 5.1 For $i = 1, \ldots, n$, let C_i be an SL m-fold, which is closed and nonsingular except at 0, in the Calabi–Yau cone $(V_i, J_{V_i}, \omega_{V_i}, \Omega_{V_i})$. Recall that $V_i' = V_i \setminus \{0\}$ can be written as $\Gamma_i \times (0, \infty)$ where Γ_i is a compact, connected $(2m-1)$-dimensional smooth manifold. Then C_i is an *SL cone* in V_i if $C_i' = C_i \setminus \{0\}$ can be written as $\Sigma_i \times (0, \infty)$ for some compact, nonsingular

$(m-1)$-dimensional submanifold Σ_i of Γ_i. Let g_{C_i} be the restriction of the Calabi–Yau cone metric g_{V_i} to C_i.

We call Σ_i the link of the SL cone C_i. Here we can allow Σ_i to be disconnected, or equivalently C_i' disconnected, though we have assumed for simplicity Γ_i is connected.

Recall from §3.3.1 that Γ_i is a contact $(2m-1)$-fold with a contact 1-form α_i defined by $r^2\alpha_i = \iota(X_i)\omega_{V_i}$ where X_i is the radial vector field on V_i. The fact that the SL cone $C_i' = \Sigma_i \times (0,\infty)$ is Lagrangian in the Calabi–Yau cone $V_i' = \Gamma_i \times (0,\infty)$ implies Σ_i is a $(m-1)$-dimensional Legendrian submanifold in Γ_i. In §3.3.1, there is also a complex dilation $\lambda_{t,\theta} : V_i \longrightarrow V_i$, given by $\lambda_{t,\theta}(\gamma, r) = (\exp(\theta Z_i)(\gamma), tr)$ for $\theta \in \mathbb{R}$ and $t > 0$, where $Z_i = J_{V_i}X_i$. When $\theta = 0$, $\lambda_{t,0}$ gives a "real dilation" $(\gamma, r) \mapsto (\gamma, tr)$, and the cone C_i is therefore invariant under this real dilation, i.e. $C_i = \lambda_{t,0}(C_i)$ for all $t > 0$. We note that $\lambda_{t,0}(C_i)$ is still special Lagrangian in V_i since

$$0 = \lambda_{t,0}^*(\omega_{V_i}|_{C_i}) = t^2\omega_{V_i}|_{\lambda_{t,0}(C_i)}, \quad \text{and} \quad 0 = \lambda_{t,0}^*(\mathrm{Im}(\Omega_{V_i})|_{C_i}) = t^m\mathrm{Im}(\Omega_{V_i})|_{\lambda_{t,0}(C_i)}.$$

Let $\iota_i : \Sigma_i \times (0,\infty) \longrightarrow \Gamma_i \times (0,\infty)$ be the inclusion map given by $\iota_i(\sigma, r) = (\sigma, r)$. We identify $\Sigma_i \cong \Sigma_i \times \{1\}$. Let $(\sigma, r) \in \Sigma_i \times (0,\infty)$. A 1-form on $\Sigma_i \times (0,\infty)$ at the point (σ, r) can be expressed as $\eta + c\,dr$, where $\eta \in T_\sigma^*\Sigma_i$ and $c \in \mathbb{R}$. Use (σ, r, η, c) to denote a point in $T^*_{(\sigma,r)}(\Sigma_i \times (0,\infty))$. Identify $\Sigma_i \times (0,\infty)$ as the zero section $\{(\sigma, r, \eta, c) : \eta = c = 0\}$ of the cotangent bundle $T^*(\Sigma_i \times (0,\infty))$. Define a dilation action on $T^*(\Sigma_i \times (0,\infty))$ by

$$t \cdot (\sigma, r, \eta, c) = (\sigma, tr, t^2\,\eta, tc), \tag{5.1}$$

where $t \in \mathbb{R}_+$. This t-action restricts to the usual dilation on the cone $\Sigma_i \times (0,\infty)$, and the pullback of the canonical symplectic form ω_{can} on $T^*(\Sigma_i \times (0,\infty))$ by t satisfies $t^*(\omega_{\mathrm{can}}) = t^2\omega_{\mathrm{can}}$.

As we have seen in Theorem 2.9, the Lagrangian Neighbourhood Theorem gives that any compact Lagrangian submanifold N in a symplectic manifold looks locally like the zero section in T^*N. We are going to extend the Lagrangian Neighbourhood Theorem to special Lagrangian cones C_i in the Calabi–Yau cones V_i:

Theorem 5.2 *With the above notations, there exist an open tubular neighbourhood U_{C_i} of the zero section $\Sigma_i \times (0,\infty)$ in $T^*(\Sigma_i \times (0,\infty))$, which is invariant under the t-action, and an embedding $\Psi_{C_i} : U_{C_i} \longrightarrow V_i' \cong \Gamma_i \times (0,\infty)$ such that*

$$\Psi_{C_i}|_{\Sigma_i \times (0,\infty)} = \iota_i, \quad \Psi_{C_i}^*(\omega_{V_i}) = \omega_{\mathrm{can}} \quad \text{and} \quad \Psi_{C_i} \circ t = \lambda_{t,0} \circ \Psi_{C_i}$$

for $t \in \mathbb{R}_+$, where t acts on U_{C_i} as in (5.1), and $\lambda_{t,0}$ is the dilation on the Calabi–Yau cone V_i.

Theorem 5.2 can be proved by arguing in the same way as in the proof of [31, Thm. 4.3], which applies [31, Thm. 4.2], a version of a result of Weinstein on Lagrangian foliations. Here is the rough idea. In order to use Theorem 4.2 of [31], we need a smooth family $L_{(\sigma,r)}$ of non-compact Lagrangian submanifolds in $\Gamma_i \times (0,\infty)$ containing the point (σ, r) and transverse to $\Sigma_i \times (0,\infty)$ at (σ, r), i.e. $T_{(\sigma,r)}L_{(\sigma,r)} \cap T_{(\sigma,r)}(\Sigma_i \times (0,\infty)) = \{0\}$, with $L_{(\sigma,tr)} = \lambda_{t,0}\,(L_{(\sigma,r)})$.

Since $\Sigma_i \cong \Sigma_i \times \{1\}$ is compact, it is possible to choose such a family for $r = 1$, i.e. we can choose noncompact Lagrangian submanifolds $L_{(\sigma,1)}$ for all $\sigma \in \Sigma_i$, transverse to $\Sigma_i \times (0,\infty)$ at $(\sigma, 1)$. Define $L_{(\sigma,r)} = \lambda_{r,0}\left(L_{(\sigma,1)}\right)$, then the family $\{L_{(\sigma,r)} : (\sigma, r) \in \Sigma_i \times (0,\infty)\}$ is what we need.

As we shall see, the Lagrangian neighbourhoods for C_i, together with the Lagrangian neighbourhoods for N_0 (SL 3-fold with conical singularities) and L_i (AC SL 3-fold), are useful in the analytic result in Theorem 6.1 as we will glue all these neighbourhoods together.

5.2 SL m-folds with conical singularities

After defining SL cones in Calabi–Yau cones, we now define SL m-folds N_0 with conical singularities. Essentially, N_0 is an SL m-fold in some Calabi–Yau m-fold M_0 with conical singularities at $x_1, \dots, x_n$, and it is asymptotic at $x_1, \dots, x_n$ to the SL cones $C_1, \dots, C_n$ in the Calabi–Yau cones $V_1, \dots, V_n$. The idea is to define N_0 as a graph of some 1-form ζ_i on C_i near x_i for $i = 1, \dots, n$. The fact that N_0 is Lagrangian implies ζ_i is a closed 1-form. Moreover, for N_0 to approach C_i near x_i, ζ_i should decay at least like $O(r)$ which then implies ζ_i is in fact exact. As a result, we are able to express N_0 as a graph of some exact 1-form da_i.

We write $(M_0, J_0, \omega_0, \Omega_0)$ for a Calabi–Yau m-fold with conical singularities at $x_1, \dots, x_n$ with rate $\nu > 0$, modelled on Calabi–Yau cones $V_1, \dots, V_n$ with diffeomorphism $\Phi_i : \Gamma_i \times (0, \epsilon) \longrightarrow S_i \setminus \{x_i\}$ such that

$$\Phi_i^*(\omega_0) = \omega_{V_i} \text{ and } |\nabla^k(\Phi_i^*(\Omega_0) - \Omega_{V_i})|_{g_{V_i}} = O(r^{\nu-k}) \text{ as } r \to 0$$

for $i = 1, \dots, n$ and all $k \geq 0$. Here is the definition of SL m-folds with conical singularities:

Definition 5.3 Let N_0 be a singular SL m-fold in M_0, with singularities at $x_1, \dots, x_n$ and no other singularities. Then N_0 is an *SL m-fold with conical singularities at* $x_1, \dots, x_n$ with rate $\mu \in (1, \nu + 1)$, modelled on SL cones $C_1, \dots, C_n$, if there exist an open neighbourhood $T_i \subset S_i$ of x_i in N_0, and a smooth function a_i on $\Sigma_i \times (0, \epsilon')$ for $i = 1, \dots, n$ and $\epsilon' < \epsilon$, satisfying

$$|\nabla^k a_i|_{g_{C_i}} = O(r^{\mu+1-k}) \quad \text{as } r \to 0 \text{ and for all } k \geq 0, \tag{5.2}$$

computing ∇ and $|\cdot|_{g_{C_i}}$ using the cone metric g_{C_i}, such that

$$T_i \setminus \{x_i\} = \Phi_i \circ \Psi_{C_i}(\Gamma(d\,a_i)) \tag{5.3}$$

where $\Psi_{C_i} : U_{C_i} \longrightarrow V_i'$ is the embedding from the Lagrangian neighbourhood U_{C_i} to the Calabi–Yau cone V_i', and $\Gamma(d\,a_i)$ is the graph of the 1-form $d\,a_i$. From (5.2), $d\,a_i$ has rate $O(r^{\mu})$ as $r \to 0$, which shows it is a small 1-form, and by making ϵ' smaller if necessary, the graph $\Gamma(d\,a_i)$ of $d\,a_i$ lies in U_{C_i}.

We require the rate μ to be greater than 1 to ensure N_0 approaches the cone C_i near x_i. For the upper bound, we choose μ to be less than $\nu + 1$ so that whether N_0 is an SL m-fold with conical singularities with rate μ is independent of the choice of the diffeomorphisms or coordinates Φ_i amongst equivalent coordinates. Recall that two coordinates Φ_i and Φ'_i are equivalent if and only if the following relation holds

$$|\nabla^k(\Phi_i - \Phi'_i)|_{g_{V_i}} = O(r^{\nu+1-k}) \quad \text{as } r \to 0 \text{ and for all } k \geq 0.$$

Here we interpret the difference between Φ_i and Φ'_i using local coordinates on the image $S_i \setminus \{x_i\}$. From (5.3), $T_i \setminus \{x_i\}$ is written as $\Phi_i \circ \Psi_{C_i}(\Gamma(d\,a_i))$. If a'_i is another smooth function such that $T_i \setminus \{x_i\} = \Phi'_i \circ \Psi_{C_i}(\Gamma(d\,a'_i))$ under the coordinate Φ'_i, then

$$\begin{aligned} |\nabla^{k+1} a'_i|_{g_{C_i}} &= |\nabla^k(\Phi_i - \Phi'_i)|_{g_{V_i}} + |\nabla^{k+1} a_i|_{g_{C_i}} \\ &= O(r^{\nu+1-k}) + O(r^{\mu-k}) \quad \text{from (5.2)} \\ &= O(r^{\mu-k}) \quad \text{provided } \mu \leq \nu + 1 \end{aligned}$$

for $k \geq 0$. Integration then gives $|a'_i|_{g_{C_i}} = O(r^{\mu+1})$. Hence (5.2) for a_i is equivalent to (5.2) for a'_i, and the definition of SL m-folds with conical singularities with rate μ is therefore independent of the choice of Φ_i.

Now we give a result on the construction of Lagrangian neighbourhoods for N_0, compatible with the Lagrangian neighbourhoods for C_i in Theorem 5.2. The proof is similar to that of Theorem 4.6 of [31].

Theorem 5.4 *With the above notations, there exists an open tubular neighbourhood U_{N_0} of the zero section N_0 in T^*N_0 such that*

$$d\Phi_i|_{U_{C_i} \cap T^*(\Sigma_i \times (0,\epsilon'))}\,(U_{C_i} \cap T^*(\Sigma_i \times (0,\epsilon'))) \;=\; T^*(T_i \setminus \{x_i\}) \cap U_{N_0} \quad \textit{for } i = 1, \ldots, n,$$

and there exists an embedding $\Psi_{N_0} : U_{N_0} \longrightarrow M_0$ with $\Psi_{N_0}|_{N_0} = Id$, $\Psi^*_{N_0}(\omega_0) = \omega_{T^*N_0}$, where Id is the identity map on N_0 and $\omega_{T^*N_0}$ the canonical symplectic structure on T^*N_0, such that

$$\Psi_{N_0} \circ d\Phi_i|_{U_{C_i} \cap T^*(\Sigma_i \times (0,\epsilon'))}(\sigma, r, \eta, c) \;=\; \Phi_i \circ \Psi_{C_i}(\sigma, r, \eta + da_i^1(\sigma, r), c + da_i^2(\sigma, r))$$

for $i = 1, \ldots, n$, $(\sigma, r, \eta, c) \in U_{C_i} \cap T^*(\Sigma_i \times (0, \epsilon'))$, and for $da_i(\sigma, r) = da_i^1(\sigma, r) + da_i^2(\sigma, r)dr$ with $da_i^1(\sigma, r) \in T^*_\sigma \Sigma_i$ and $da_i^2(\sigma, r) \in \mathbb{R}$.

Let us focus on the zero section $\{\eta = c = 0\} \cong \Sigma_i \times (0, \epsilon')$. The first part of the theorem gives

$$d\Phi_i|_{U_{C_i} \cap T^*(\Sigma_i \times (0,\epsilon'))}(\sigma, r, 0, 0) \;=\; T_i \setminus \{x_i\}.$$

This is consistent with the second part of the theorem, as from the left hand side we have

$$\Psi_{N_0} \circ d\Phi_i|_{U_{C_i} \cap T^*(\Sigma_i \times (0,\epsilon'))}(\sigma, r, 0, 0) \;=\; \Psi_{N_0}(T_i \setminus \{x_i\}) \;=\; T_i \setminus \{x_i\},$$

using $\Psi_{N_0}|_{N_0} = \mathrm{Id}$, and the right hand side gives

$$\Phi_i \circ \Psi_{C_i}(\sigma, r, da_i^1(\sigma, r), da_i^2(\sigma, r)) \;=\; \Phi_i \circ \Psi_{C_i}(\Gamma(da_i)) \;=\; T_i \setminus \{x_i\}$$

by (5.3).

5.3 AC SL m-folds

We proceed to define AC SL m-folds L_i in AC Calabi–Yau m-folds Y_i for $i = 1, \ldots, n$. Similar to the conical singularities case, we want to define L_i as a graph of some closed 1-form χ_i near infinity. The condition for L_i to converge to C_i at infinity is that χ_i has size $O(r^{\kappa_i})$ for large r and $\kappa_i < 1$.

Let $(Y_i, J_{Y_i}, \omega_{Y_i}, \Omega_{Y_i})$ be an AC Calabi–Yau m-fold with rate $\lambda_i \leq -m$ modelled on Calabi–Yau cones V_i with diffeomorphism $\Upsilon_i : \Gamma_i \times (R, \infty) \longrightarrow Y_i \setminus K_i$ such that

$$\Upsilon_i^*(\omega_{Y_i}) = \omega_{V_i} \text{ and } |\nabla^k(\Upsilon_i^*(\Omega_{Y_i}) - \Omega_{V_i})|_{g_{V_i}} = O(r^{\lambda_i - k}) \text{ as } r \to \infty \text{ and for all } k \geq 0.$$

Definition 5.5 Let L_i be a nonsingular SL m-fold in Y_i for $i = 1, \ldots, n$. Then L_i is an *AC SL 3-fold* with rate $\kappa_i \in (\lambda_i + 1, 1)$, modelled on SL cones C_i if there exist a compact subset $H_i \subset L_i$ and a smooth closed 1-form χ_i on $\Sigma_i \times (R', \infty)$ for $R' > R$, satisfying

$$|\nabla^k \chi_i|_{g_{C_i}} = O(r^{\kappa_i - k}) \quad \text{as } r \to \infty \text{ and for all } k \geq 0, \tag{5.4}$$

computing ∇ and $|\cdot|_{g_{C_i}}$ using the cone metric g_{C_i}, such that

$$L_i \setminus H_i = \Upsilon_i \circ \Psi_{C_i}(\Gamma(\chi_i)). \tag{5.5}$$

Equation (5.4) implies χ_i has rate $O(r^{\kappa_i})$ as $r \to \infty$, and by making R' larger if necessary, the graph $\Gamma(\chi_i)$ of χ_i lies in U_{C_i}.

Analogous to the upper bound for the rate μ in the conical singularities case, we require $\kappa_i > \lambda_i + 1$ so that the definition of AC SL m-folds with rate κ_i does not depend on the choice of the coordinate Υ_i.

Here we work with closed 1-forms with rate $\kappa_i < 1$ in the definition. However, assuming χ_i to be closed is not enough for our purpose, as we hope to express L_i as a graph of an exact 1-form near infinity. The reason for that is if the 1-form χ_i was not exact, we shall come across global topological obstructions (cf. the obstructed case in [34], or the $\lambda_i = -3$ case for Calabi–Yau 3-folds in Chapter 4) which we do not want to deal with.

Note that if we assume $\kappa_i < -1$, then χ_i is automatically exact, and it can be written as $d\,b_i$ for some smooth function b_i on $\Sigma_i \times (R', \infty)$. We can construct b_i by integration: Write $\chi_i(\sigma, r) = \chi_i^1(\sigma, r) + \chi_i^2(\sigma, r)\, dr$ where $\chi_i^1(\sigma, r) \in T_\sigma^*\Sigma_i$ and $\chi_i^2(\sigma, r) \in \mathbb{R}$. Similar to the argument we used before in defining the 1-form in Theorem 3.24, we define $b_i(\sigma, r) = \int_r^\infty \chi_i^2(\sigma, s) ds$, which is well-defined if $\kappa_i < -1$, and it satisfies $b_i(\sigma, r) = O(r^{\kappa_i + 1})$. Then $\chi_i(\sigma, r) = d\,b_i(\sigma, r)$. It follows that assuming $\kappa_i < -1$ will suit our purpose, and in fact we shall see in Chapter 6 that we need to assume $\kappa_i < -3/2$ to apply Theorem 6.1 (for the case $m = 3$). Thus from now on, we adjust the rate κ_i of AC SL m-folds to be less than -1, so that $\kappa_i \in (\lambda_i + 1, -1)$, and equations (5.4) and (5.5) in the definition become respectively

$$|\nabla^k b_i|_{g_{C_i}} = O(r^{\kappa_i + 1 - k}) \quad \text{as } r \to \infty \text{ and for all } k \geq 0, \tag{5.6}$$

and

$$L_i \setminus H_i = \Upsilon_i \circ \Psi_{C_i}(\Gamma(d\,b_i)). \tag{5.7}$$

In the last part of this section, we give the Lagrangian Neighbourhood Theorem for AC SL m-folds L_i (compare to [31, Thm. 7.5]), which is an analogue of Theorem 5.4:

Theorem 5.6 *With the above notations, for $i = 1, \ldots, n$, there exists an open tubular neighbourhood U_{L_i} of the zero section L_i in T^*L_i such that*

$$d\Upsilon_i|_{U_{C_i}\cap T^*(\Sigma_i\times(R',\infty))}\,(U_{C_i} \cap T^*(\Sigma_i \times (R',\infty))) \;=\; T^*(L_i \setminus H_i) \cap U_{L_i},$$

and there exists an embedding $\Psi_{L_i} : U_{L_i} \longrightarrow Y_i$ with $\Psi_{L_i}|_{L_i} = Id$, $\Psi^*_{L_i}(\omega_{Y_i}) = \omega_{T^*L_i}$, where Id is the identity map on L_i and $\omega_{T^*L_i}$ the canonical symplectic structure on T^*L_i, such that

$$\Psi_{L_i} \circ d\Upsilon_i|_{U_{C_i}\cap T^*(\Sigma_i\times(R',\infty))}(\sigma, r, \eta, c) \;=\; \Upsilon_i \circ \Psi_{C_i}(\sigma, r, \eta + db^1_i(\sigma, r), c + db^2_i(\sigma, r))$$

for $(\sigma, r, \eta, c) \in U_{C_i} \cap T^*(\Sigma_i \times (R',\infty))$ and for $db_i(\sigma, r) = db^1_i(\sigma, r) + db^2_i(\sigma, r)dr$ with $db^1_i(\sigma, r) \in T^*_\sigma\Sigma_i$ and $db^2_i(\sigma, r) \in \mathbb{R}$.

5.4 Some examples of AC SL m-folds

This section gives some examples of AC SL m-folds in the following AC Calabi–Yau m-folds: (i) the crepant resolution of the Calabi–Yau cone $\mathbb{C}^m/\mathbb{Z}_m$, or equivalently, the total space of the canonical line bundle $K_{\mathbb{CP}^{m-1}}$ over $\mathbb{CP}^{m-1}$ endowed with Calabi's metric, as described in Example 3.28; (ii) the cotangent bundle T^*S^m of S^m ($\cong$ the complex quadric Q_ϵ) endowed with Stenzel's metric, as described in Example 3.29 and §4.6. We shall basically use methods described in §2.2.4 of Chapter 2, and generalize the constructions to AC Calabi–Yau m-folds.

5.4.1 AC SL m-folds in $K_{\mathbb{CP}^{m-1}}$

Examples 5.7 Our first example of an AC SL m-fold will be given by the fixed point set of an antiholomorphic isometric involution (Proposition 2.25) on the AC Calabi–Yau m-fold $K_{\mathbb{CP}^{m-1}}$ (Example 3.28). Consider the usual complex conjugation $\sigma_0 : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ having $\mathbb{R}^m$ as its fixed point set. Since $\sigma_0 \circ \gamma \circ \sigma_0^{-1} = \gamma^{-1}$ for any $\gamma \in \mathbb{Z}_m$, σ_0 induces an involution on $\mathbb{C}^m/\mathbb{Z}_m$, and then lifts to σ on the crepant resolution of $\mathbb{C}^m/\mathbb{Z}_m$, or $K_{\mathbb{CP}^{m-1}}$. This map σ is actually an antiholomorphic isometric involution on $K_{\mathbb{CP}^{m-1}}$: since σ is induced by the complex conjugation, it satisfies $\sigma^2 = \mathrm{Id}$ and $\sigma^*(J) = -J$ where J is the complex structure on $K_{\mathbb{CP}^{m-1}}$. We know that $\sigma^*(r) = r$, where r is the radius function on $\mathbb{C}^m/\mathbb{Z}_m \setminus \{0\}$, and hence $\sigma^*(f) = f$ as the Kähler potential f defined by Calabi is a function of r^2. It follows that $\sigma^*(g) = g$ and hence $\sigma^*(\omega) = -\omega$. Furthermore, $\sigma^*(\Omega) = \bar{\Omega}$ on $K_{\mathbb{CP}^{m-1}}$ since the holomorphic volume form on $K_{\mathbb{CP}^{m-1}}$ is just the same as that on $\mathbb{C}^m$. Consequently, $\sigma : K_{\mathbb{CP}^{m-1}} \longrightarrow K_{\mathbb{CP}^{m-1}}$ is an antiholomorphic isometric involution on $K_{\mathbb{CP}^{m-1}}$, and the fixed point set of σ is an SL m-fold of $K_{\mathbb{CP}^{m-1}}$.

Next we investigate the fixed point set of σ in $K_{\mathbb{CP}^{m-1}}$, which is just the union of the fixed points set L_0 of $\sigma_0 : \mathbb{C}^m/\mathbb{Z}_m \longrightarrow \mathbb{C}^m/\mathbb{Z}_m$ with the origin removed and the fixed points set of $\sigma|_{\mathbb{CP}^{m-1}} : \mathbb{CP}^{m-1} \longrightarrow \mathbb{CP}^{m-1}$ in $\mathbb{CP}^{m-1}$. Now a point $\mathbb{Z}_m \cdot (z_1, \ldots, z_m)$ in $\mathbb{C}^m/\mathbb{Z}_m$ is fixed by σ_0 whenever $(\bar{z}_1, \ldots, \bar{z}_m) = (z_1\, e^{2\pi ik/m}, \ldots, z_m\, e^{2\pi ik/m})$ for some $0 \leq k \leq m-1$. It follows that $z_j = r_j\, e^{\pi ik/m}$ where $r_j \in \mathbb{R}$, $j = 1, \ldots, m$ and hence the fixed point set L_0 in $\mathbb{C}^m/\mathbb{Z}_m$ is given by

$$L_0 = \{\mathbb{Z}_m \cdot (r_1\, e^{\pi ik/m}, \ldots, r_m\, e^{\pi ik/m}) : r_j \in \mathbb{R},\ k = 0, 1, \ldots, m-1\}.$$

Observe that $\mathbb{Z}_m \cdot (r_1\, e^{\pi ik/m}, \ldots, r_m\, e^{\pi ik/m})$ is equal to $\mathbb{Z}_m \cdot (r_1, \ldots, r_m)$ for k even and $\mathbb{Z}_m \cdot (r_1\, e^{\pi i/m}, \ldots, r_m\, e^{\pi i/m})$ for k odd. Thus L_0 has two components,

$$L_0 = (\mathbb{Z}_m \cdot \mathbb{R}^m)/\mathbb{Z}_m \cup e^{\pi i/m}\, ((\mathbb{Z}_m \cdot \mathbb{R}^m)/\mathbb{Z}_m). \tag{5.8}$$

(i) When m is even, then $-1 \in \mathbb{Z}_m$, and $\mathbb{Z}_2$ is the subgroup of $\mathbb{Z}_m$ fixing $\mathbb{R}^m$. In this case, L_0 is topologically a union of two copies of $\mathbb{R}^m/\mathbb{Z}_2$, i.e. two cones on $\mathbb{RP}^{m-1}$ meeting at 0;

(ii) When m is odd, then $\frac{m-1}{2} \in \mathbb{Z}$, and hence

$$\begin{aligned} e^{\pi i/m}\, \mathbb{Z}_m \cdot (r_1, \ldots, r_m) &= e^{\pi i/m}\, e^{\frac{2\pi i}{m}(\frac{m-1}{2})}\, \mathbb{Z}_m \cdot (r_1, \ldots, r_m) \\ &= e^{\pi i}\, \mathbb{Z}_m \cdot (r_1, \ldots, r_m) \\ &= \mathbb{Z}_m \cdot (-r_1, \ldots, -r_m). \end{aligned}$$

It follows from (5.8) that $L_0 = (\mathbb{Z}_m \cdot \mathbb{R}^m)/\mathbb{Z}_m$, which is topologically a copy of $\mathbb{R}^m$, or equivalently, a cone on S^{m-1}.

Together with the fixed point set of $\sigma|_{\mathbb{CP}^{m-1}} : \mathbb{CP}^{m-1} \longrightarrow \mathbb{CP}^{m-1}$, which is just $\mathbb{RP}^{m-1}$, this yields the fixed point set L of σ, i.e. an AC SL m-fold in $K_{\mathbb{CP}^{m-1}}$:

(i) When m is even, L is homeomorphic to $\mathbb{RP}^{m-1} \times \mathbb{R}$;

(ii) When m is odd, L is homeomorphic to $S^{m-1} \times (0, \infty) \cup \mathbb{RP}^{m-1}$. We may regard L as the quotient $(S^{m-1} \times \mathbb{R})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts freely on $S^{m-1} \times \mathbb{R}$.

In both cases, L is the canonical line bundle $K_{\mathbb{RP}^{m-1}}$ over $\mathbb{RP}^{m-1}$. In fact, $K_{\mathbb{RP}^{m-1}}$ is trivial (nontrivial) if m is even (odd), as $\mathbb{RP}^{m-1}$ is oriented (non-oriented).

We are particularly interested in the case $m = 3$. From the above analysis we obtain an AC SL 3-fold $K_{\mathbb{RP}^2}$ as the fixed point set of some antiholomorphic isometric involution in the AC Calabi–Yau 3-fold $K_{\mathbb{CP}^2}$. Observe that $K_{\mathbb{RP}^2}$ admits a double cover which is diffeomorphic to $S^2 \times \mathbb{R}$. We remark that this AC SL 3-fold $K_{\mathbb{RP}^2}$ has rate "$\kappa = -\infty$", since there will be coordinates in which $K_{\mathbb{RP}^2}$ is a cone. As we have discussed before in the definition of AC SL m-folds, we require $1 + \text{rate for } K_{\mathbb{CP}^2} < \kappa < -1$. Thus we could say the rate for $K_{\mathbb{RP}^2}$ is any $\kappa \in (-5, -1)$, as $K_{\mathbb{CP}^2}$ has rate -6.

Examples 5.8 Next we describe an example of a T^{m-1}-invariant AC SL m-fold in $K_{\mathbb{CP}^{m-1}}$, as given in [13, §5.1.1]. The idea of the construction is to use the method of moment maps, similar to Proposition 2.17. The T^{m-1}-action on $K_{\mathbb{CP}^{m-1}}$ is just the T^{m-1}-action on $K_{\mathbb{CP}^{m-1}} \setminus \mathbb{CP}^{m-1} \cong \mathbb{C}^m/\mathbb{Z}_m \setminus \{0\}$ given by

$$(e^{i\theta_1}, \ldots, e^{i\theta_m}) \cdot (z_1, \ldots, z_m) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_m} z_m)$$

where $\theta_1 + \cdots + \theta_m = 0$, and the T^{m-1}-action on $\mathbb{CP}^{m-1}$. According to the calculation in [13, §5.1.1], the moment map μ for the T^{m-1}-action on $K_{\mathbb{CP}^{m-1}}$ is given by:

$$\mu(z_1, \ldots, z_m) = f'(r^2)\left(|z_1|^2 - \frac{1}{m}\,r^2,\ \ldots, |z_m|^2 - \frac{1}{m}\,r^2\right)$$

on the bit $K_{\mathbb{CP}^{m-1}} \setminus \mathbb{CP}^{m-1} \cong \mathbb{C}^m/\mathbb{Z}_m \setminus \{0\}$. Here f is the Kähler potential defined by Calabi. On $\mathbb{CP}^{m-1}$, we have

$$\mu([w_1, \ldots, w_m]) = \left(|w_1|^2 - \frac{1}{m},\ \ldots, |w_m|^2 - \frac{1}{m}\right)$$

where $w_1, \ldots, w_m$ are normalized such that $|w_1|^2 + \cdots + |w_m|^2 = 1$. Up to a constant factor, this is just the same as the moment map of T^{m-1} acting on $\mathbb{CP}^{m-1}$ with the Fubini-Study metric. The image of $\mathbb{CP}^{m-1}$ under the moment map μ is

$$\mu(\mathbb{CP}^{m-1}) \cong \left\{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1 + \cdots + x_m = 0,\ \frac{-1}{m} \le x_j \le \frac{m-1}{m}\ \forall\, j\right\}.$$

This is a simplex in $\mathbb{R}^m$, and in fact the convex hull of m vertices given by $(\frac{m-1}{m}, \frac{-1}{m}, \ldots, \frac{-1}{m})$, $(\frac{-1}{m}, \frac{m-1}{m}, \ldots, \frac{-1}{m}), \ldots, (\frac{-1}{m}, \frac{-1}{m}, \ldots, \frac{m-1}{m})$, corresponding to the image of points $p_1, \ldots, p_m$ under μ, where $p_j = [0, \ldots, 0, \overset{j\text{ th}}{1}, 0, \ldots, 0]$. Moreover, these p_j's are fixed points of the T^{m-1}-action on $\mathbb{CP}^{m-1}$. This situation illustrates a well-known *convexity theorem* of Atiyah, Guillemin and Sternberg (see for instance McDuff and Salamon [41], Thm. 5.47, p.180).

It follows that if $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$ with $c_1 + \cdots + c_m = 0$ does not lie in the convex hull of the m points $\mu(p_1), \ldots, \mu(p_m)$, then $\mu^{-1}(c) \cap \mathbb{CP}^{m-1} = \emptyset$. On the other hand, for each c in the $(m-1)$-simplex $\mu(\mathbb{CP}^{m-1})$, $\mu^{-1}(c)|_{\mathbb{CP}^{m-1}}$ is a T^{m-1}-orbit in $\mathbb{CP}^{m-1}$. It turns out that if c lies on a k-dimensional face of the simplex for some $k = 0, 1, \ldots, m-1$, $\mu^{-1}(c)|_{\mathbb{CP}^{m-1}}$ will be a k-torus T^k in $\mathbb{CP}^{m-1}$. Note that a 0-torus is just a point corresponding to one of the p_j's which are the fixed points of the T^{m-1}-action on $\mathbb{CP}^{m-1}$.

The "generalized moment map" η of the T^{m-1}-action on $K_{\mathbb{CP}^{m-1}}$ is basically the same as that of the T^{m-1}-action on $\mathbb{C}^m$, because the metric on $K_{\mathbb{CP}^{m-1}}$ constructed by Calabi has the same holomorphic volume form as that on $\mathbb{C}^m$. Thus from Example 2.18, η is given by $\mathrm{Re}(z_1 \cdots z_m)$ if m is even and $\mathrm{Im}(z_1 \cdots z_m)$ if m is odd. Note that $\eta \equiv 0$ on $\mathbb{CP}^{m-1}$.

We can now write down the SL m-fold in $K_{\mathbb{CP}^{m-1}}$ as the level sets of μ and η. Let $c = (c_1, \ldots, c_m, c')$ where $c_1, \ldots, c_m, c' \in \mathbb{R}$ and $c_1 + \cdots + c_m = 0$. Define

$$\begin{aligned} L_c = \Big\{&(z_1, \ldots, z_m) \in \mathbb{C}^m \setminus \{0\} : f'(r^2)\left(|z_j|^2 - \frac{1}{m}\,r^2\right) = c_j\,,\ \text{for } j = 1, \ldots, m, \\ &\text{and } \mathrm{Re}(z_1 \cdots z_m) = c' \text{ if } m \text{ is even},\ \ \mathrm{Im}(z_1 \cdots z_m) = c' \text{ if } m \text{ is odd}\Big\}. \end{aligned}$$

Then L_c is a T^{m-1}-invariant SL m-fold in $\mathbb{C}^m \setminus \{0\}$ with respect to Calabi's metric. Since L_c is invariant under the $\mathbb{Z}_m$-action on $\mathbb{C}^m \setminus \{0\}$, it follows that the quotient $L_c/\mathbb{Z}_m$ is a T^{m-1}-invariant SL m-fold in $\mathbb{C}^m/\mathbb{Z}_m \setminus \{0\}$, and converges to the cone

$$\begin{aligned} \Big\{\mathbb{Z}_m \cdot (z_1, \ldots, z_m) : |z_1|^2 = \cdots = |z_m|^2 = \frac{1}{m}\,r^2,\ &\text{and } \mathrm{Re}(z_1 \cdots z_m) = c' \text{ if } m \text{ is even}, \\ &\mathrm{Im}(z_1 \cdots z_m) = c' \text{ if } m \text{ is odd}\Big\} \end{aligned}$$

in $\mathbb{C}^m/\mathbb{Z}_m$.

Now suppose $c' \neq 0$, then $L_c/\mathbb{Z}_m$ will not intersect $\mathbb{CP}^{m-1}$ in $K_{\mathbb{CP}^{m-1}}$, as $\eta \equiv 0$ on $\mathbb{CP}^{m-1}$. Consequently, $L_c/\mathbb{Z}_m$ is a T^{m-1}-invariant SL m-fold in $K_{\mathbb{CP}^{m-1}}$ for any $c_1, \ldots, c_m, c' \in \mathbb{R}$ with $c_1 + \cdots + c_m = 0$ and $c' \neq 0$.

For the case $c' = 0$, if $(c_1, \ldots, c_m)$ does not lie in the convex hull of the m points $\mu(p_1), \ldots, \mu(p_m)$ in $\mathbb{R}^{m-1}$, then $L_c/\mathbb{Z}_m$ is a T^{m-1}-invariant SL m-fold in $K_{\mathbb{CP}^{m-1}}$. If the point $(c_1, \ldots, c_m)$ is on a k-dimensional face of the simplex for some $k = 0, 1, \ldots, m-1$, then we have to include a k-torus T^k in $\mathbb{CP}^{m-1}$, and hence in this case we obtain a T^{m-1}-invariant SL m-fold $L_c/\mathbb{Z}_m \cup T^k$ in $K_{\mathbb{CP}^{m-1}}$. Note that the singular behaviour for these SL m-folds $L_c/\mathbb{Z}_m \cup T^k$ is analogous to the case in Example 2.18.

The case $m = 3$ gives examples of T^2-invariant SL 3-folds in $K_{\mathbb{CP}^2}$. If $k = 0$, then $L_c/\mathbb{Z}_3 \cup \{p_j\}$, for $j = 1, 2, 3$, has an isolated singular point at $p_j \in \mathbb{CP}^2$. If $k = 1$, then the whole T^1 is where the singularities located in $L_c/\mathbb{Z}_3 \cup T^1$, and for $k = 2$, we have a nonsingular SL 3-fold $L_c/\mathbb{Z}_3 \cup T^2$ in $K_{\mathbb{CP}^2}$.

Examples 5.9 This example constructs AC SL 3-folds in $K_{\mathbb{CP}^2}$ invariant under the standard SO(3)-action. From the calculation of the moment map of the U(m)-action on $\mathbb{C}^m$ w.r.t. Calabi's metric in [13, §5.1.1], it can be seen that the moment map $\mu : K_{\mathbb{CP}^2} \longrightarrow \mathfrak{so}(3)^* \cong \mathbb{R}^3$ of the SO(3)-action on $K_{\mathbb{CP}^2}$ is given by

$$\mu = f'(r^2)\,(\mathrm{Im}(z_1\bar{z}_2), \mathrm{Im}(z_2\bar{z}_3), \mathrm{Im}(z_3\bar{z}_1)).$$

Again, f denotes the Kähler potential defined by Calabi. Since $Z(\mathfrak{so}(3)^*) = \{0\}$, it follows that any SO(3)-invariant SL 3-fold in $K_{\mathbb{CP}^2}$ must lie in the level set $\mu^{-1}(0)$. Using the same construction as in Example 2.20 of Chapter 2, we obtain a family of SL 3-folds L_c in $K_{\mathbb{CP}^2}$ which has the same form as in Example 2.20, since the level sets of both moment maps coincide. Thus for $c \neq 0$, L_c is an AC SL 3-fold in $K_{\mathbb{CP}^2}$ with Calabi's metric and is diffeomorphic to $S^2 \times \mathbb{R}$. As the cone $(\mathbb{Z}_3 \cdot \mathbb{R}^3)/\mathbb{Z}_3$ is identified with the cone $e^{i\pi/3} \cdot (\mathbb{Z}_3 \cdot \mathbb{R}^3)/\mathbb{Z}_3$ in $\mathbb{C}^3/\mathbb{Z}_3$, L_c converges to two copies of $(\mathbb{Z}_3 \cdot \mathbb{R}^3)/\mathbb{Z}_3$ in $\mathbb{C}^3/\mathbb{Z}_3$ from Theorem 2.21. We have seen that the rate for the SO(3)-invariant SL 3-folds in $\mathbb{C}^3$ is -2, thus L_c also has rate -2. With the discussion in Example 5.7 the fixed point set $K_{\mathbb{RP}^2}$ admits a double cover diffeomorphic to $S^2 \times \mathbb{R}$, which is just one possible L_c, and hence we have obtained a family of deformations of a double cover of Example 5.7 for $m = 3$. One can generalize this example to higher dimensions and obtain SO(m)-invariant AC SL m-folds in $K_{\mathbb{CP}^{m-1}}$.

5.4.2 AC SL m-folds in T^*S^m

Examples 5.10 As we have seen in Example 2.7 and §4.6, the cotangent bundle T^*S^m can be described as an AC Calabi–Yau m-fold, which can also be viewed as the complex quadric $Q_\epsilon = \{\sum_j z_j^2 = \epsilon\}$ for $0 \neq \epsilon \in \mathbb{C}$. We first use the fixed point set of an antiholomorphic isometric involution to construct SL m-folds in Q_ϵ. Let $\sigma : \mathbb{C}^{m+1} \longrightarrow \mathbb{C}^{m+1}$ be the usual complex conjugation. Suppose $\epsilon \in \mathbb{R}$ with $\epsilon > 0$, then σ maps Q_ϵ to Q_ϵ, and it is an antiholomorphic isometric involution on Q_ϵ with respect to Stenzel's metric. In this case, the fixed point set L is the real points of Q_ϵ, which corresponds to the zero section S^m of the cotangent bundle T^*S^m, and therefore S^m is an SL m-fold in T^*S^m. More generally, if we take $\sigma_k : Q_\epsilon \longrightarrow Q_\epsilon$ with

$$\sigma_k(z_1, \ldots, z_{m+1}) \;=\; (\bar{z}_1, \ldots, \bar{z}_k, -\bar{z}_{k+1}, \ldots, -\bar{z}_{m+1})$$

for $0 \le k \le m+1$ and write $z_j = x_j + iy_j$ for $j = 1, \ldots, m+1$, then the fixed point set L_k for σ_k is given by

$$L_k = \{(x_1, \ldots, x_k, iy_{k+1}, \ldots, iy_{m+1}) \in \mathbb{C}^{m+1} : \; x_1^2 + \cdots + x_k^2 - y_{k+1}^2 - \cdots - y_{m+1}^2 = \epsilon\}.$$

If $\epsilon > 0$, then $L_0 = \emptyset$ and for $k \ge 1$, L_k is topologically an $S^{k-1} \times \mathbb{R}^{m-k+1}$ in T^*S^m and it follows that $S^{k-1} \times \mathbb{R}^{m-k+1}$ is an SL m-fold with phase i^{m-k+1} in T^*S^m for each $0 \le k \le m$. In fact this agrees with a standard result in symplectic geometry that given a submanifold N in a manifold M, the annihilator $TN^\perp$ of TN is Lagrangian in T^*M with its canonical symplectic structure. Here those $S^{k-1} \times \mathbb{R}^{m-k+1}$'s are exactly the annihilators $(TS^{k-1})^\perp$ of TS^{k-1} in T^*S^m. Similar result holds for $\epsilon < 0$ where L_k now corresponds to $S^{m-k} \times \mathbb{R}^k$ for $0 \le k \le m$ and $L_{m+1} = \emptyset$.

The case $m = 3$ yields four topologically distinct SL 3-folds S^3, $S^2 \times \mathbb{R}$, $S^1 \times \mathbb{R}^2$ and $S^0 \times \mathbb{R}^3$ (or two disjoint copies of $\mathbb{R}^3$) in $Q_\epsilon \cong T^*S^3$. Recall that the complex quadric Q_ϵ is the deformation of the singular Calabi–Yau cone Q_0, a cone on $S^2 \times S^3$. As $\epsilon \to 0$, the fixed point sets in Q_ϵ converges to fixed point sets in Q_0 with different topologies:
(i) the 3-sphere S^3 collapses to a single point in Q_0;
(ii) the $S^2 \times \mathbb{R}$ converges to two cones on S^2, i.e. two $\mathbb{R}^3$ intersecting at 0;
(iii) the $S^1 \times \mathbb{R}^2$ converges to one cone on $S^1 \times S^1 \cong T^2$; and
(iv) the $S^0 \times \mathbb{R}^3$ converges to two $\mathbb{R}^3$ intersecting at 0.

Examples 5.11 We describe here an example of T^2-invariant SL 3-fold (§5.2.2 of [13]) in Q_ϵ constructed by the method of moment maps. Take T^2 as a subgroup of SO(4) defined by :

$$\left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & 0 & -\sin\theta_2 & \cos\theta_2 \end{pmatrix} : \theta_1, \, \theta_2 \in \mathbb{R} \right\}.$$

The associated vector fields for the two basis elements x_1, x_2 of the Lie algebra of T^2 are :

$$\phi(x_1) = z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_2}$$
$$\phi(x_2) = z_4 \frac{\partial}{\partial z_3} - z_3 \frac{\partial}{\partial z_4} + \bar{z}_4 \frac{\partial}{\partial \bar{z}_3} - \bar{z}_3 \frac{\partial}{\partial \bar{z}_4}.$$

The moment map $\mu : Q_\epsilon \to \mathbb{R}^2$ for this T^2-action is given by

$$\mu(z_1, z_2, z_3, z_4) = f'(r^2) \left(\mathrm{Im}(z_2\bar{z}_1), \, \mathrm{Im}(z_4\bar{z}_3)\right),$$

where f is the Kähler potential defined by Stenzel.

Let Ω_{Q_ϵ} be the holomorphic volume form on Q_ϵ, as given in (4.51). Then by solving the equation $\iota(\phi(x_1)\wedge\phi(x_2))\,\Omega_{Q_\epsilon} = d\alpha$ for α, we obtain the generalized moment map $\eta = \mathrm{Im}(\alpha)$. Recall that $dz_1\wedge dz_2\wedge dz_3\wedge dz_4 = (z_1dz_1 + z_2dz_2 + z_3dz_3 + z_4dz_4)\wedge\Omega_{Q_\epsilon}$, thus taking interior products with $\phi(x_1)\wedge\phi(x_2)$ on both sides gives

$$\begin{aligned}\iota(\phi(x_1)\wedge\phi(x_2))\,(dz_1\wedge dz_2\wedge dz_3\wedge dz_4) &= (z_1dz_1+\cdots+z_4dz_4)\wedge(\iota(\phi(x_1)\wedge\phi(x_2))\,\Omega_{Q_\epsilon})\\ &= (z_1dz_1+z_2dz_2+z_3dz_3+z_4dz_4)\wedge d\alpha\\ &= -d\big(\alpha\,(z_1dz_1+z_2dz_2+z_3dz_3+z_4dz_4)\big).\end{aligned}$$

Here we have used the fact that $z_1^2+z_2^2+z_3^2+z_4^2$ is T^2-invariant, and so $\iota(\phi(x_j))(z_1dz_1+z_2dz_2+z_3dz_3+z_4dz_4) = 0$ for $j = 1,2$. After some brief calculations, we obtain $\alpha = z_1^2+z_2^2-z_3^2-z_4^2$ up to a constant multiple, and hence the generalized moment map $\eta : Q_\epsilon \to \mathbb{R}$ is given by

$$\eta = \mathrm{Im}(z_1^2+z_2^2-z_3^2-z_4^2).$$

Consequently, for each $c = (c_1,c_2,c_3)\in\mathbb{R}^3$,

$$\begin{aligned}L_c = \{(z_1,z_2,z_3,z_4)\in Q_\epsilon :\ & f'(r^2)\,\mathrm{Im}(z_2\bar z_1) = c_1,\ f'(r^2)\,\mathrm{Im}(z_4\bar z_3) = c_2,\\ & \mathrm{Im}(z_1^2+z_2^2-z_3^2-z_4^2) = c_3\}\end{aligned}$$

is a T^2-invariant SL 3-fold in Q_ϵ. As $\epsilon\to 0$, L_c converges to the following T^2-invariant cone C in Q_0:

$$C \;=\; \{(z_1,z_2,z_3,z_4)\in Q_0 :\ \mathrm{Im}(z_2\bar z_1) = 0,\ \mathrm{Im}(z_4\bar z_3) = 0,\ \mathrm{Im}(z_1^2+z_2^2-z_3^2-z_4^2) = 0\}.$$

As we have $z_1^2+z_2^2+z_3^2+z_4^2 = 0$ on Q_0, the last equation defining C thus becomes $\mathrm{Im}(z_1^2+z_2^2) = 0 = \mathrm{Im}(z_3^2+z_4^2)$. It can then be shown that the cone C can actually be written as the union of

$$\{(x_1,x_2,iy_3,iy_4) : x_j,y_j\in\mathbb{R},\ x_1^2+x_2^2-y_3^2-y_4^2 = 0\}$$

$$\text{and}\quad \{(iy_1,iy_2,x_3,x_4) : x_j,y_j\in\mathbb{R},\ -y_1^2-y_2^2+x_3^2+x_4^2 = 0\},$$

i.e. T^2-invariant cones coming from the fixed point sets of some antiholomorphic isometric involution on Q_0. Note also that the fixed point set L_2 for σ_2 in Example 5.10 is a special case of this example, as it corresponds to the case when $c_1 = c_2 = c_3 = 0$.

Examples 5.12 The last example we give here is on SO(3)-invariant AC SL 3-folds in Q_ϵ for $0\neq\epsilon\in\mathbb{C}$. Take an SO(3) subgroup of SO(4) with matrices of the form

$$\{\begin{pmatrix} A & 0\\ 0 & 1\end{pmatrix},\ A\in\mathrm{SO}(3)\}\,.$$

We shall apply the construction by the cohomogeneity one method described in Example 2.20, we first calculate the moment map $\mu : Q_\epsilon \longrightarrow \mathfrak{so}(3)^*\cong\mathbb{R}^3$ of this SO(3)-action, which is given by

$$(z_1,z_2,z_3,z_4)\longmapsto f'(r^2)\,(\mathrm{Im}(z_1\bar z_2),\,\mathrm{Im}(z_2\bar z_3),\,\mathrm{Im}(z_3\bar z_1))$$

where f is the Kähler potential defined by Stenzel. Following exactly the same technique in Example 2.20, it can be shown that the SO(3)-orbits in $\mu^{-1}(0)$ can be written as

$$\{(\sqrt{\epsilon-\lambda^2}\,x_1,\ \sqrt{\epsilon-\lambda^2}\,x_2,\ \sqrt{\epsilon-\lambda^2}\,x_3,\lambda)\ :\ x_j\in\mathbb{R},\ x_1^2+x_2^2+x_3^2 = 1\}$$

for each $\lambda \in \mathbb{C}$, and the differential equation in λ becomes

$$\operatorname{Im}\left(\frac{d\lambda}{dt}\sqrt{\epsilon-\lambda^2}\right)=0. \tag{5.9}$$

Integrating (5.9) and we get

$$P(\lambda)=\frac{\epsilon}{2}\arcsin\left(\frac{\lambda}{\sqrt{\epsilon}}\right)+\frac{1}{2}\lambda\sqrt{\epsilon-\lambda^2}+\text{constant}.$$

Hence we have obtained a family of SO(3)-invariant SL 3-folds

$$L_c=\left\{(\sqrt{\epsilon-\lambda^2}\,x_1,\sqrt{\epsilon-\lambda^2}\,x_2,\sqrt{\epsilon-\lambda^2}\,x_3,\lambda)\,:\,x_j\in\mathbb{R},\ x_1^2+x_2^2+x_3^2=1,\ \operatorname{Im}(P(\lambda))=c\right\}$$

for $c \in \mathbb{R}$. In what follows, we shall focus on the flow lines of $P(\lambda)$ given by (5.9) and describe the topology of L_c. Let us first consider the case when $\epsilon = 1$. Then we have two singular points at ± 1, and there is a flow line along the segment $[-1, 1]$ where our SL 3-fold L_c corresponds to an S^3.

When λ is close to 1, write $\lambda = 1 + w$, then for w small enough, we have

$$\operatorname{Im}\left(\frac{d\lambda}{dt}\sqrt{1-\lambda^2}\right)\approx\operatorname{Im}\left(\frac{dw}{dt}\sqrt{-2w}\right).$$

Calculation shows that this yields three flow lines ending at 1, one of which corresponds to the line segment $[-1, 1]$, and the other two, A_1 and A_2, look like half lines $\{\arg(z) = \pi/3\}$ and $\{\arg(z) = 5\pi/3\}$ near the point 1 respectively. Similarly, for λ close to -1, the flow lines are $[-1, 1]$, A_3 and A_4, where A_3, A_4 look like half lines $\{\arg(z) = 2\pi/3\}$ and $\{\arg(z) = 4\pi/3\}$ near the point -1 respectively. Consequently, we have four different SL cones on S^2, i.e. four different SL $\mathbb{R}^3$'s represented by A_1, A_2, A_3 and A_4, in which the two $\mathbb{R}^3$'s corresponding to A_1 and A_2 intersect at the point $(0, 0, 0, 1)$ on Q_1, and the two $\mathbb{R}^3$'s corresponding to A_3 and A_4 intersect at the point $(0, 0, 0, -1)$ on Q_1.

For large $|\lambda|$, (5.9) for $\epsilon = 1$ becomes:

$$\operatorname{Im}\left(\frac{d\lambda}{dt}\sqrt{1-\lambda^2}\right)\approx\operatorname{Im}\left(\frac{d\lambda}{dt}(i\lambda)\right),$$

which implies $\operatorname{Re}(\lambda^2)$ = constant. It follows that A_1, A_2, A_3 and A_4 are asymptotic to the lines $\{\arg(z) = \pi/4\}$, $\{\arg(z) = 7\pi/4\}$, $\{\arg(z) = 3\pi/4\}$ and $\{\arg(z) = 5\pi/4\}$ respectively. Thus the four $\mathbb{R}^3$'s represented by A_1, A_2, A_3, A_4 are AC SL 3-folds in Q_1 asymptotic respectively to the cones C_0, C_3, C_1, C_2 in Q_0 given by:

$$C_k\ =\ \{i^k e^{i\pi/4}(ix_1,ix_2,ix_3,x_4): x_j\in\mathbb{R},\ x_4>0,\ x_1^2+x_2^2+x_3^2-x_4^2=0\}\quad\text{for } k=0,1,2,3.$$

All flow lines other than $[-1, 1]$ and the A_j's correspond to $S^2 \times \mathbb{R}$, and we have four different kinds of such of them, flowing between the regions bounded by $A_1 \cup A_2$, $A_1 \cup [-1, 1] \cup A_3$, $A_3 \cup A_4$ and $A_4 \cup [-1, 1] \cup A_2$. These yields four families of AC SL 3-folds L_c, homeomorphic to $S^2 \times \mathbb{R}$ and asymptotic to the cones $C_0 \cup C_1$, $C_1 \cup C_2$, $C_2 \cup C_3$ and $C_3 \cup C_0$. Summarizing the above results, we have obtained the following topologically distinct families of SO(3)-invariant SL 3-folds L_c in Q_1:

(i) a 3-sphere S^3;

(ii) 4 different AC SL $\mathbb{R}^3$'s, converging to 4 SL cones C_j's in Q_0; and

(iii) 4 different kinds of AC SL $S^2 \times \mathbb{R}$, converging to unions of cones $C_0 \cup C_1$, $C_1 \cup C_2$, $C_2 \cup C_3$ and $C_3 \cup C_0$.

After interpreting the case $\epsilon = 1$, we proceed to discuss the situation for $\epsilon \neq 1$ with $\mathrm{Re}(\sqrt{\epsilon}) > 0$ and $\mathrm{Im}(\sqrt{\epsilon}) > 0$. The singular points for (5.9) are now $\pm\sqrt{\epsilon}$. Unlike the $\epsilon = 1$ case, there is *no* flow line along the line segment $[-\sqrt{\epsilon}, \sqrt{\epsilon}]$. As a result, we have *no* special Lagrangian 3-spheres S^3 (with phase 1) for this case.

When λ is close to $\sqrt{\epsilon}$, we also have three flow lines B_1, B_2, B_3 ending at $\sqrt{\epsilon}$, converging to three half lines near $\sqrt{\epsilon}$ with slopes depending on $\arg(\sqrt{\epsilon})$. A similar situation appears near the point $-\sqrt{\epsilon}$, where we have three flow lines B_4, B_5, B_6 ending at $-\sqrt{\epsilon}$. Thus there are six different special Lagrangian $\mathbb{R}^3$'s B_1, B_2, B_3, B_4, B_5 and B_6, where the first three intersect at the point $(0, 0, 0, \sqrt{\epsilon})$ in Q_ϵ, while the last three intersect at the point $(0, 0, 0, -\sqrt{\epsilon})$ in Q_ϵ. Now for large $|\lambda|$, the asymptotic behaviour of the flow lines is similar to the case $\epsilon = 1$ before. We fix B_1 to be the flow line ending at $\sqrt{\epsilon}$ and converging to the line $\{\arg(z) = \pi/4\}$ at infinity, B_2 the flow line converging to $\{\arg(z) = 7\pi/4\}$ and B_3 the flow line converging to $\{\arg(z) = 3\pi/4\}$. Thus B_1, B_2 and B_3 are AC SL 3-folds in Q_ϵ, asymptotic respectively to the cones C_0, C_3, C_1 in Q_0. On the other hand, we fix B_4 to be the flow line ending at $-\sqrt{\epsilon}$ and converging to line $\{\arg(z) = 5\pi/4\}$ at infinity, B_5 the flow line converging to $\{\arg(z) = 3\pi/4\}$ and B_6 the flow line converging to $\{\arg(z) = 7\pi/4\}$. Then B_4, B_5 and B_6 are AC SL 3-folds in Q_ϵ, asymptotic respectively to the cones C_2, C_1, C_3 in Q_0. We note that although there are six different $\mathbb{R}^3$'s converging to six cones at infinity, the pair $\{B_2, B_6\}$ shares the same asymptotic cone C_3, and $\{B_3, B_5\}$ the cone C_1.

Apart from the B_j's, all other flow lines represent $S^2 \times \mathbb{R}$, flowing between the regions bounded by $B_1 \cup B_2$, $B_1 \cup B_3$, $B_4 \cup B_5$, $B_4 \cup B_6$ and $B_2 \cup B_3 \cup B_5 \cup B_6$, and so we have five different kinds of AC SL 3-folds $S^2 \times \mathbb{R}$. To summarize, when $\epsilon \neq 1$ with $\mathrm{Re}(\sqrt{\epsilon}) > 0$ and $\mathrm{Im}(\sqrt{\epsilon}) > 0$, we have the following two topologically distinct families of SO(3)-invariant SL 3-folds in Q_ϵ:

(i) 6 different AC SL $\mathbb{R}^3$'s, converging to 4 SL cones C_j's in Q_0; and

(ii) 5 different kinds of AC SL $S^2 \times \mathbb{R}$, converging to 5 unions of cones $C_0 \cup C_1$, $C_1 \cup C_2$, $C_2 \cup C_3$, $C_3 \cup C_0$ and $C_1 \cup C_3$.

Let us finish by stating the situation when $\epsilon \neq 1$ with $\mathrm{Re}(\sqrt{\epsilon}) > 0$ and $\mathrm{Im}(\sqrt{\epsilon}) < 0$. The whole picture of the flow lines will be a kind of rotation of the previous one by i. Again, we have three flow lines B'_1, B'_2, B'_3 ending at $\sqrt{\epsilon}$, converge to $\{\arg(z) = 7\pi/4\}$, $\{\arg(z) = 5\pi/4\}$ and $\{\arg(z) = \pi/4\}$ respectively. The other three B'_4, B'_5, B'_6 ending at $-\sqrt{\epsilon}$, converge to $\{\arg(z) = 3\pi/4\}$, $\{\arg(z) = \pi/4\}$ and $\{\arg(z) = 5\pi/4\}$ respectively. Hence the six B'_j's are AC SL $\mathbb{R}^3$'s in Q_ϵ, with B'_1 converges to the cone C_3, B'_2, B'_6 to C_2, B'_4 to C_1 and B'_3, B'_5 to C_0. Similarly, we have five kinds of AC SL $S^2 \times \mathbb{R}$, flowing between the regions bounded by $B'_1 \cup B'_2$, $B'_4 \cup B'_6$, $B'_4 \cup B'_5$, $B'_1 \cup B'_3$ and $B'_2 \cup B'_3 \cup B'_5 \cup B'_6$.

Chapter 6

Desingularizations of SL 3-folds with conical singularities

After introducing SL m-folds with conical singularities and AC SL m-folds in the corresponding Calabi–Yau m-folds, we are going to study simultaneous desingularization of Calabi–Yau and SL 3-folds. Suppose N_0 is an SL 3-fold in M_0 with conical singularities at the same points x_i for $i = 1, \ldots, n$, and L_i an AC SL 3-fold in Y_i. When we glue in a rescaled Y_i to M_0 at each x_i, we also glue in a rescaled L_i to N_0 at each x_i. The idea of the desingularization is to construct a family of nonsingular 3-folds N_t in the family of nonsingular nearly Calabi–Yau 3-folds $(M_t, \omega_t, \Omega_t)$ so that N_t is *Lagrangian* in $(M_t, \omega_t, \Omega_t)$. Our result on Calabi–Yau desingularization then gives genuine Calabi–Yau 3-folds $(M_t, \tilde{J}_t, \tilde{\omega}_t, \tilde{\Omega}_t)$. Choose a suitable coordinate/diffeomorphism ψ_t on M_t so that $\omega_t = \psi_t^*(c_t \tilde{\omega}_t)$. Then the pullback $(\hat{J}_t, \hat{\omega}_t, \hat{\Omega}_t)$ of $(\tilde{J}_t, \tilde{\omega}_t, \tilde{\Omega}_t)$ under ψ_t is also a genuine Calabi–Yau structure on M_t, and N_t is Lagrangian in the Calabi–Yau 3-folds $(M_t, \hat{J}_t, \hat{\omega}_t, \hat{\Omega}_t)$ as well.

The analytic result we need in this chapter is adapted from Joyce [33, Thm. 5.3], in which he shows that when t is sufficiently small, we can deform the Lagrangian m-fold N_t to a compact nonsingular SL m-fold. The hypotheses in Joyce's theorem involves estimates of various kinds of norms of $\mathrm{Im}\,(\hat{\Omega}_t)|_{N_t}$. This suggest us to compute the term $\mathrm{Im}\,(\hat{\Omega}_t)$ restricting on different regions of N_t.

We begin in §6.1 by establishing necessary notations and discussing Joyce's analytic result. In §6.2 we construct Lagrangian 3-folds N_t by gluing in L_i to N_0 at each x_i. Then in §6.3 we compute the estimates of the size of $\mathrm{Im}\,(\hat{\Omega}_t)|_{N_t}$. We divide the whole computation into three components, as given in equation (6.4). Using the concept of *local injectivity radius*, together with a kind of *isoperimetric inequality* and the elliptic regularity result, we finally verify all the conditions, and so we are able to prove a result on SL desingularizations in §6.4. In the last section, §6.5, we illustrate our main result by taking two examples of Calabi–Yau 3-folds with conical singularities, namely the orbifold $T^6/\mathbb{Z}_3$ and some quintic 3-folds, and perform the desingularization simultaneously for both Calabi–Yau and SL 3-folds with conical singularities. We use AC SL 3-folds from §5.4 for gluing, thus obtaining various kinds of nonsingular SL 3-folds in the nonsingular Calabi–Yau 3-folds.

Results related to the desingularization of SL m-folds can also be found, for instance, in Butscher [9], Joyce [33], [34], [35], and Lee [37]. Butscher shows existence of SL connected sums of two compact SL m-folds (in $\mathbb{C}^m$ with boundary) at one point by gluing in Lawlor necks (for definition, see [36]). Joyce proves a desingularization result of SL m-folds with conical singularities in nonsingular *almost* Calabi–Yau m-folds by gluing in AC SL m-folds in $\mathbb{C}^m$. He also studies desingularizations in families of almost Calabi–Yau m-folds. Lee considers a compact, connected, immersed SL m-fold in a Calabi–Yau m-fold, whose self-intersection points satisfy an angle criterion. She uses Lawlor necks for gluing at the singular points.

6.1 Joyce's desingularization theory

Joyce has developed a comprehensive desingularization theory of SL m-folds with isolated conical singularities in Calabi–Yau m-folds (and more generally in *almost* Calabi–Yau m-folds). His approach is to glue in appropriate AC SL m-folds in $\mathbb{C}^m$ which are asymptotic to some SL cones, thus obtaining a 1-parameter family of compact nonsingular Lagrangian m-folds. Then he proves using analysis that the Lagrangian m-folds which are close to being special Lagrangian can actually be deformed to SL m-folds in the Calabi–Yau m-fold. The whole programme on SL m-folds with isolated conical singularities is given in the series of his papers [31, 32, 33, 34, 35].

We shall now fix $m = 3$ to fit into our situation for desingularizing SL 3-folds in Calabi–Yau 3-folds.

Before stating Joyce's analytic result, we need to establish the necessary notations, which can be found in Definition 5.2 of [33]. However we shall only consider the following particular case of his definition:

Let (M, J, ω, Ω) be a Calabi–Yau 3-fold, with Calabi–Yau metric g. Since we only deal with Calabi–Yau manifolds, not the more general class of almost Calabi–Yau manifolds, we can take the smooth function ψ in Definition 5.2 of [33] to be 1 on M, so that condition (ii) of Theorem 5.3 in [33] becomes trivial in our case.

Suppose $N \subset M$ is a Lagrangian 3-fold. Restricting the Calabi–Yau metric g on N, we obtain a metric $h = g|_N$, with volume form dV. The holomorphic (3,0)-form Ω restricts to a 3-form $\Omega|_N$ on N. As N is Lagrangian, we can write

$$\Omega|_N = e^{i\theta}\, dV$$

for some phase function θ on N, which equals zero if and only if N is special Lagrangian (with phase 1). Suppose $[\mathrm{Im}(\Omega)|_N] = 0$ in $H^3(N, \mathbb{R})$, or equivalently,

$$\int_N \mathrm{Im}(\Omega) = \int_N \sin\theta\, dV = 0.$$

Clearly, this is a necessary condition for N to be special Lagrangian in M. Note that this can be satisfied by choosing the phase of Ω appropriately, which means this is actually a fairly mild restriction.

In (iii) of Theorem 5.3 in [33], we need a 3-form β and a connection $\acute{\nabla}$ on T^*N ($\hat{\nabla}$ in the original notation). For $r > 0$, define $\mathcal{B}_r \subset T^*N$ to be $\mathcal{B}_r = \{\alpha \in C^\infty(T^*N) : |\alpha|_{h^{-1}} < r\}$, where $|\cdot|_{h^{-1}}$ is computed using the metric h^{-1}. The Levi-Civita connection ∇ of h on TN induces a splitting $T\mathcal{B}_r = H \oplus V$, where $H \cong TN$ and $V \cong T^*N$ are the horizontal and vertical subbundles of T^*N. Define $\acute{h}$ on $\mathcal{B}_r$ such that H and V are orthogonal w.r.t. $\acute{h}$, and $\acute{h}|_H = \pi^*(h)$, $\acute{h}|_V = \pi^*(h^{-1})$ where $\pi : \mathcal{B}_r \to N$ denotes the natural projection. Let $\acute{\nabla}$ be the connection on $T\mathcal{B}_r$ given by lift of ∇ on N in the horizontal directions H, and by partial differentiation in the vertical directions V. Since N is Lagrangian in M, for small $r > 0$, the Lagrangian Neighbourhood Theorem gives an embedding $\Psi : \mathcal{B}_r \to M$ such that $\Psi^*(\omega) = \omega_{\mathrm{can}}$ and $\Psi|_N = \mathrm{Id}$, where ω_{can} is the natural symplectic structure on $\mathcal{B}_r \subset T^*N$. Finally, we define a 3-form β on $\mathcal{B}_r$ by $\beta = \Psi^*(\mathrm{Im}\,(\Omega))$, the pullback of the imaginary part of the holomorphic (3,0)-form Ω on M by the Lagrangian embedding.

The finite dimensional vector space W in Definition 5.2 of [33] has to do with the number of connected components of $N_0 \setminus \{x_1, \ldots, x_n\}$ where N_0 is a SL 3-fold with conical singularities at $x_1, \ldots, x_n$. We will make the nonsingular Lagrangian 3-fold N_t (which is our N defined above) by gluing AC SL 3-folds $L_1, \ldots, L_n$ into N_0. If $N_0 \setminus \{x_1, \ldots, x_n\}$ is not connected, then each of the L_i's is connected but may contain more than one end, so that N_t consists of several components of N_0 joined by "small necks" from L_i's. The vector space W will then be a space of functions which is approximately constant on each component of $N_0 \setminus \{x_1, \ldots, x_n\}$ and changes on small necks. The dimension $\dim W$ will be the number of connected component of $N_0 \setminus \{x_1, \ldots, x_n\}$. For the sake of simplicity, we only study the case when $N_0 \setminus \{x_1, \ldots, x_n\}$ is connected, which means we can take $W = \langle 1 \rangle$, the space of constant functions. As a result, condition (vii) of Theorem 5.3 of [33] is trivial and we can drop it entirely. The natural projection $\pi_W : L^2(N) \to W$ is now given by

$$\pi_W(v) = \mathrm{vol}(N)^{-1} \int_N v\, dV$$

for $W = \langle 1 \rangle$, and we have $\pi_W(v) = 0 \iff \int_N v\, dV = 0$. It follows that the last inequality of Theorem 5.3 (i) in [33] holds automatically, as we have assumed $\int_N \sin\theta\, dV = 0$. Moreover, we can replace $\pi_W(v) = 0$ by $\int_N v\, dV = 0$ in (vi) of the theorem to leave W out of our definition.

We are now ready to state the following analytic existence result for SL 3-folds, adapted from [33, Thm. 5.3]:

Theorem 6.1 *Let $\kappa' > 1$ and $A_1, \ldots, A_6 > 0$. Then there exist $\epsilon, K > 0$ depending only on $\kappa', A_1, \ldots, A_6$ such that the following holds.*

Refer to the notation in §6.1. Suppose $0 < t \le \epsilon$ and $r = A_1 t$, and

(i) $\|\sin\theta\|_{L^{6/5}} \le A_2 t^{\kappa'+3/2}$, $\|\sin\theta\|_{C^0} \le A_2 t^{\kappa'-1}$ *and* $\|d\sin\theta\|_{L^6} \le A_2 t^{\kappa'-3/2}$.

(ii) $\|\acute{\nabla}^k \beta\|_{C^0} \le A_3 t^{-k}$ *for $k = 0, 1, 2$ and 3.*

(iii) *The injectivity radius $\delta(h)$ satisfies $\delta(h) \ge A_4 t$.*

(iv) *The Riemann curvature* $R(h)$ *satisfies* $\|R(h)\|_{C^0} \leq A_5 t^{-2}$.

(v) *If* $v \in L^2_1(N)$ *with* $\int_N v\, dV = 0$, *then* $v \in L^6(N)$, *and* $\|v\|_{L^6} \leq A_6 \|dv\|_{L^2}$.

Here all norms are computed using the metric h *on* N *in* (i), (iv) *and* (v), *and the metric* $\acute{h}$ *on* $\mathcal{B}_{A_1 t}$ *in* (ii). *Then there exists* $f \in C^\infty(N)$ *with* $\int_N f\, dV = 0$, *such that* $\|df\|_{C^0} \leq K t^{\kappa'} < A_1 t$ *and* $\hat{N} = \Psi_*(\Gamma(df))$ *is an immersed special Lagrangian 3-fold in* (M, J, ω, Ω).

For a small 1-form $\alpha \in C^\infty(T^*N)$, $\Psi_*(\Gamma(\alpha))$ is a Lagrangian submanifold in M if α is closed, and it is a special Lagrangian submanifold if α also satisfies $d^*(\cos\theta\, \alpha) = \sin\theta + Q(\alpha)$ where Q is a smooth map with $Q(\alpha) = O(|\alpha|^2 + |\nabla\alpha|^2)$ (see [33, Lemma 5.7]). Now if $f \in C^\infty(N)$ is a small function in $C^1(N)$, then df is a small 1-form, and $\Psi_*(\Gamma(df))$ is special Lagrangian if and only if $d^*(\cos\theta\, df) = \sin\theta + Q(df)$. Thus the function f in the theorem is basically the solution to the above nonlinear elliptic equation. Joyce [33, §5.5] solved this equation by constructing inductively a sequence $(f_k)_{k=0}^\infty$ in $C^\infty(N)$ with $f_0 = 0$ and $\int_N f_k\, dV = 0$ satisfying $d^*(\cos\theta df_k) = \sin\theta + Q(df_{k-1})$ for $k \geq 1$. Then he showed that this sequence converges in some Sobolev space to f which satisfies the nonlinear elliptic equation and is smooth by elliptic regularity.

6.2 Construction of N_t

In this section we are going to define a 1-parameter family of compact nonsingular Lagrangian 3-folds N_t in the nearly Calabi–Yau 3-folds M_t for small $t > 0$. Recall that we desingularize the Calabi–Yau 3-fold M_0 with conical singularities by first applying a homothety to each AC Calabi–Yau 3-fold Y_i, and then gluing it into M_0 at x_i for $i = 1, \dots, n$ to make the nonsingular M_t's. For the special Lagrangians inside these Calabi–Yau's, we then desingularize N_0 (inside $(M_0, J_0, \omega_0, \Omega_0)$) by gluing L_i (inside $(Y_i, J_{Y_i}, t^2\omega_{Y_i}, t^3\Omega_{Y_i})$) into N_0 at x_i for each i. Note that after applying the homothety to Y_i, equations (5.6) and (5.7) now become

$$|\nabla^k b_i(\sigma, t^{-1}r)|_{g_{C_i}} = O(t^{-\kappa_i - 1} r^{\kappa_i + 1 - k}) \quad \text{as } r \to \infty \text{ and for all } k \geq 0, \tag{6.1}$$

and

$$L_i \setminus H_i = \Upsilon_{t,i} \circ \Psi_{C_i}(\Gamma(t^2 d\, b_i(\sigma, t^{-1}r))) \tag{6.2}$$

where the diffeomorphism $\Upsilon_{t,i} : \Gamma_i \times (tR, \infty) \longrightarrow Y_i \setminus K_i$ is given by $\Upsilon_{t,i}(\gamma, r) = \Upsilon_i(\gamma, t^{-1}r)$.

Now for $i = 1, \dots, n$, $\alpha \in (0,1)$ and small enough $t > 0$ with $tR < tR' < t^\alpha < 2t^\alpha < \epsilon' < \epsilon$, define a smooth function $u_{t,i}$ on $\Sigma_i \times (tR', \epsilon')$ by

$$u_{t,i}(\sigma, r) = F(t^{-\alpha}r)\, a_i(\sigma, r) + t^2(1 - F(t^{-\alpha}r))\, b_i(\sigma, t^{-1}r). \tag{6.3}$$

Again, F is the smooth, increasing function we used before in defining Ω_t in Calabi–Yau desingularizations. Thus we have $F(t^{-\alpha}r) = 1$ when $2t^\alpha \leq r < \epsilon'$, in which case $u_{t,i}(\sigma, r) = a_i(\sigma, r)$, and $F(t^{-\alpha}r) = 0$ when $tR' \leq r < t^\alpha$, in which case $u_{t,i}(\sigma, r) = t^2 b_i(\sigma, t^{-1}r)$.

Here we are going to use the same $\alpha \in (0,1)$ as in the previous Calabi–Yau 3-fold desingularizations, and we shall require α to satisfy certain inequalities for this SL desingularization as well.

Define N_t to be the union of $(N_0 \setminus \bigcup_{i=1}^n T_i)$, $\bigcup_{i=1}^n \Psi_{C_i}(\Gamma(du_{t,i}))$, and $\bigcup_{i=1}^n H_i$. Basically, we construct N_t in the way that for $r > \epsilon'$, N_t is just $(N_0 \setminus \bigcup_{i=1}^n T_i) \subset N_0$. For $tR' < r < \epsilon'$, N_t is diffeomorphic to the union of the graphs $\Gamma(du_{t,i})$ of the 1-forms $du_{t,i}$ which in fact interpolate between the graphs $\Gamma(da_i)$ of da_i, i.e. part of N_0, for $2t^\alpha \le r < \epsilon'$ and the graphs $\Gamma(t^2 db_i)$ of $t^2 db_i$, i.e. part of L_i, for $tR' \le r < t^\alpha$. Finally for $r < tR'$, N_t is the union of $H_i \subset L_i$.

Under our construction the boundary of each H_i $(\cong \Sigma_i)$ joins smoothly onto $\bigcup_{i=1}^n \Psi_{C_i}(\Gamma(du_{t,i}))$ at the $\Sigma_i \times \{tR'\}$ end, and the boundary of $(N_0 \setminus \bigcup_{i=1}^n T_i)$, which is the disjoint union of the Σ_i, joins smoothly onto $\bigcup_{i=1}^n \Psi_{C_i}(\Gamma(du_{t,i}))$ at the $\Sigma_i \times \{\epsilon'\}$ end. Thus N_t is a compact smooth manifold without boundary. More importantly, N_t is in fact a Lagrangian submanifold:

Proposition 6.2 *N_t is a Lagrangian 3-fold in the nearly Calabi–Yau 3-fold $(M_t, \omega_t, \Omega_t)$ for sufficiently small $t > 0$.*

Proof. We shall look at the symplectic form ω_t restricts to different regions of N_t. Since N_t coincides with N_0, which is Lagrangian in M_0, in the component $N_0 \setminus \bigcup_{i=1}^n T_i$, and ω_t equals ω_0 on this part, we see that $\omega_t \equiv 0$ on $N_0 \setminus \bigcup_{i=1}^n T_i$. In the same way, as N_t coincides with the union of L_i in the component $\bigcup_{i=1}^n H_i$, and ω_t is now $t^2 \omega_{Y_i}$ on each L_i, thus we have $\omega_t \equiv 0$ on $\bigcup_{i=1}^n H_i$, as L_i is Lagrangian in Y_i. For the middle part, N_t is given by $\bigcup_{i=1}^n \Psi_{C_i}(\Gamma(du_{t,i}))$. As ω_t is equal to ω_{V_i} on each $\Psi_{C_i}(\Gamma(du_{t,i}))$, and $\Psi_{C_i}^*(\omega_{V_i}) = \omega_{\mathrm{can}}$ from Theorem 5.2, where ω_{can} is the canonical symplectic form on $T^*(\Sigma_i \times (0,\infty))$, we get $\omega_t \equiv 0$ on $\bigcup_{i=1}^n \Psi_{C_i}(\Gamma(du_{t,i}))$ as $\Gamma(du_{t,i})$ is the graph of a closed 1-form $du_{t,i}$ on $\Sigma_i \times (tR', \epsilon')$. It follows that $\omega_t|_{N_t} = 0$, and N_t is then Lagrangian in M_t. □

Now we deform the underlying nearly Calabi–Yau structure on M_t to a genuine Calabi–Yau structure $(\tilde{J}_t, \tilde{\omega}_t, \tilde{\Omega}_t)$ for small t by applying Theorem 4.28. As shown in the theorem, we have the relation $[\omega_t] = c_t\,[\tilde{\omega}_t] \in H^2(M_t, \mathbb{R})$ for some $c_t > 0$ between the cohomology classes of the Kähler forms. Thus ω_t and $c_t\,\tilde{\omega}_t$ are in the same cohomology class. Using Moser's type argument there is a diffeomorphism $\psi_t : M_t \longrightarrow M_t$ on M_t satisfying $\psi_t^*(c_t\,\tilde{\omega}_t) = \omega_t$. Write $\hat{\omega}_t = \psi_t^*(\tilde{\omega}_t)$, $\hat{J}_t = \psi_t^*(\tilde{J}_t)$, $\hat{g}_t = \psi_t^*(\tilde{g}_t)$ and $\hat{\Omega}_t = \psi_t^*(\tilde{\Omega}_t)$ under the new coordinates. The fact that ω_t and $c_t\,\tilde{\omega}_t$ are close for small $t > 0$ means that the diffeomorphism ψ_t is close to identity, which then implies that the complex 3-forms are also close under the new coordinates, i.e. $\hat{\Omega}_t \approx \Omega_t$. We shall evaluate this difference in the next section.

As we have arranged $\omega_t = c_t\,\hat{\omega}_t$ by applying a diffeomorphism ψ_t, it follows that $\hat{\omega}_t|_{N_t} = 0$ since $c_t > 0$, and so we obtain:

Proposition 6.3 *N_t is a Lagrangian 3-fold in the Calabi–Yau 3-fold $(M_t, \hat{J}_t, \hat{\omega}_t, \hat{\Omega}_t)$ for sufficiently small $t > 0$.*

6.3 Estimates of $\mathrm{Im}(\hat{\Omega}_t)|_{N_t}$

We have constructed a family of compact, nonsingular Lagrangian 3-folds N_t by gluing L_i into N_0 at each x_i, and our next step is to apply Theorem 6.1 to deform N_t to a special Lagrangian 3-fold $\hat{N}_t$ in the Calabi–Yau 3-fold $(M_t, \hat{J}_t, \hat{\omega}_t, \hat{\Omega}_t)$ for small enough $t > 0$. This leads us to consider the estimates of various norms of $\mathrm{Im}(\hat{\Omega}_t)|_{N_t}$ for (i) of Theorem 6.1.

Let $h_t, \tilde{h}_t$ and $\hat{h}_t$ be the restrictions of $g_t, \tilde{g}_t$ and $\hat{g}_t$ to N_t respectively. In view of Theorem 4.28, or Theorem 3.32, the metrics $g_t, \tilde{g}_t$, and hence $h_t, \tilde{h}_t$, are uniformly equivalent in t, so norms of any tensor on N_t measuring with respect to $h_t, \tilde{h}_t$ only differ by a bounded factor independent of t. We shall see later the uniform equivalence between $\hat{h}_t$ and the other two.

Here is the basic estimate, computing with respect to $\hat{h}_t$:

$$\left|\mathrm{Im}(\hat{\Omega}_t)|_{N_t}\right|_{\hat{h}_t} \le \left|(\hat{\Omega}_t - \tilde{\Omega}_t)|_{N_t}\right|_{\hat{h}_t} + \left|(\tilde{\Omega}_t - \Omega_t)|_{N_t}\right|_{\hat{h}_t} + \left|\mathrm{Im}(\Omega_t)|_{N_t}\right|_{\hat{h}_t}. \tag{6.4}$$

We hope to arrange for this error to be small enough that we can deform N_t to an SL 3-fold by using the analytic result in Theorem 6.1. The first term $(\hat{\Omega}_t - \tilde{\Omega}_t)|_{N_t}$ in the right side of (6.4) is basically the error coming from changing coordinates on M_t, which can be estimated by considering the diffeomorphism ψ_t from Moser's argument. The second term $(\tilde{\Omega}_t - \Omega_t)|_{N_t}$ is the error arising from deforming the nearly Calabi–Yau structure (ω_t, Ω_t) to the genuine Calabi–Yau structure $(\tilde{J}_t, \tilde{\omega}_t, \tilde{\Omega}_t)$ on M_t. We already have the C^0-estimates from Theorem 3.32 or Theorem 4.28, but for part (i) of Theorem 6.1 to hold we need to improve and get a better control of this term. We shall devote most of the section to achieving this. For the final term $\mathrm{Im}(\Omega_t)|_{N_t}$, we can estimate it by restricting Ω_t to different regions of N_t.

Let us first evaluate the last term $\mathrm{Im}(\Omega_t)$ on various components of N_t. We first adopt the definition of the complex 3-form Ω_t from (3.39) when $\lambda_i < -3$, and we will treat the case $\lambda_i = -3$ afterwards. From (3.39) we have

$$\Omega_t = \begin{cases} \Omega_0 \quad \text{on} \ \ Q_t \setminus [(\bigcup_{i=1}^n P_{t,i}) \cap Q_t], \\ \Omega_{V_i} + d\big[F(t^{-\alpha} r) A_i(\gamma, r) + t^3(1 - F(t^{-\alpha} r)) B_i(\gamma, t^{-1} r)\big] \ \ \text{on} \ P_{t,i} \cap Q_t, \text{ for } i = 1, \dots, n, \\ t^3 \Omega_{Y_i} \quad \text{on} \ \ P_{t,i} \setminus (P_{t,i} \cap Q_t), \text{ for } i = 1, \dots, n. \end{cases}$$

On $Q_t \setminus [(\bigcup_{i=1}^n P_{t,i}) \cap Q_t]$, Ω_t is given by Ω_0, and N_t is the union of $N_0 \setminus \bigcup_{i=1}^n T_i$ and $\bigcup_{i=1}^n \Psi_{C_i}(\Gamma(da_i))$, which is a part of N_0. Thus $\mathrm{Im}(\Omega_t) = 0$ on this region of N_t, as N_0 is special Lagrangian in M_0. Similarly, on $P_{t,i} \setminus (P_{t,i} \cap Q_t)$ for each i, Ω_t is given by $t^3 \Omega_{Y_i}$, and N_t is the union of $\bigcup_{i=1}^n H_i$ and $\bigcup_{i=1}^n \Psi_{C_i}(\Gamma(t^2 db_i))$ which lies in L_i. It follows that $\mathrm{Im}(\Omega_t) = 0$ on this region of N_t, as L_i is special Lagrangian in Y_i. For the annuli region, we have

$$\mathrm{Im}(\Omega_t)|_{N_t} = \mathrm{Im}\big(\Omega_{V_i} + d\big[F(t^{-\alpha} r) A_i(\gamma, r) + t^3(1 - F(t^{-\alpha} r)) B_i(\gamma, t^{-1} r)\big]\big)\big|_{\Psi_{C_i}(\Gamma(du_{t,i}))}. \tag{6.5}$$

Consider the term $\mathrm{Im}(\Omega_{V_i})|_{\Psi_{C_i}(\Gamma(du_{t,i}))}$. Regard C_i as the zero section in $T^*(\Sigma_i \times (0, \infty))$. Then the difference between $\mathrm{Im}(\Omega_{V_i})|_{\Psi_{C_i}(\Gamma(du_{t,i}))}$ and $\mathrm{Im}(\Omega_{V_i})|_{C_i}$ is given by

$$O(\,|\acute{\nabla} \Omega_{V_i}|_{g_{V_i}} \cdot |du_{t,i}|_{g_{C_i}}\,) + O(\,|\Omega_{V_i}|_{g_{V_i}} \cdot |\nabla du_{t,i}|_{g_{C_i}}\,),$$

where $\acute{\nabla}$ denotes a connection on $T^*(\Sigma_i \times (0,\infty))$ (see §6.1 for the construction of $\acute{\nabla}$ on T^*N), and ∇ the connection on C_i computed using the metric g_{C_i}. Roughly speaking, the first term is coming from moving base points, whereas the second term from changing tangent spaces. Now we have $|\acute{\nabla}\Omega_{V_i}|_{g_{V_i}} = O(t^{-\alpha})$ on the annulus, and $|\Omega_{V_i}|_{g_{V_i}}$ is a constant. Together with the fact that C_i is special Lagrangian in V_i, i.e. $\mathrm{Im}(\Omega_{V_i})|_{C_i} = 0$, we obtain the size for $\mathrm{Im}(\Omega_{V_i})|_{\Psi_{C_i}(\Gamma(du_{t,i}))}$:

$$\big|\,\mathrm{Im}(\Omega_{V_i})|_{\Psi_{C_i}(\Gamma(du_{t,i}))}\,\big|_{g_{C_i}} = O(\,t^{-\alpha}|du_{t,i}|_{g_{C_i}}\,) + O(\,|\nabla du_{t,i}|_{g_{C_i}}\,) \tag{6.6}$$

for $r \in (t^\alpha, 2t^\alpha)$. Using (5.2) and (6.1), and the definition of $u_{t,i}$ in (6.3) we get

$$\begin{aligned} |du_{t,i}|_{g_{C_i}} &= O(t^{\mu\alpha}) + O(t^{1-\kappa_i(1-\alpha)}), \quad \text{and} \\ |\nabla du_{t,i}|_{g_{C_i}} &= O(t^{(\mu-1)\alpha}) + O(t^{(1-\kappa_i)(1-\alpha)}) \quad \text{for } r \in (t^\alpha, 2t^\alpha). \end{aligned} \tag{6.7}$$

Putting (6.7) into (6.6), and using the estimates for A_i and B_i from (3.36) and (3.38), we compute the size for (6.5):

$$\begin{aligned} \big|\,\mathrm{Im}(\Omega_t)|_{N_t}\,\big|_{g_{C_i}} &= O(t^{(\mu-1)\alpha}) + O(t^{(1-\kappa_i)(1-\alpha)}) + O(t^{\alpha\nu}) + O(t^{-\lambda_i(1-\alpha)}) \\ &= O(t^{(\mu-1)\alpha}) + O(t^{(1-\kappa_i)(1-\alpha)}) \quad \text{for } r \in (t^\alpha, 2t^\alpha). \end{aligned} \tag{6.8}$$

The term $O(t^{\alpha\nu})$ is absorbed into $O(t^{(\mu-1)\alpha})$ as we have chosen $\mu < \nu + 1$ in the definition of SL 3-folds with conical singularities, and similarly the term $O(t^{-\lambda_i(1-\alpha)})$ is absorbed into $O(t^{(1-\kappa_i)(1-\alpha)})$ as $\kappa_i > \lambda_i + 1$ in the definition of AC SL 3-folds.

Summing up all these, and using the uniform equivalence between metrics g_{C_i} and h_t (follows from that between g_{V_i} and g_t), we see that

Proposition 6.4 *In the situation above, the error term $\mathrm{Im}(\Omega_t)|_{N_t}$, for the case $\lambda_i < -3$, satisfies*

$$\big|\,\mathrm{Im}(\Omega_t)|_{N_t}\,\big|_{h_t} = \begin{cases} 0 \quad \textit{on} \;\; N_t \cap \big(Q_t \setminus [(\bigcup_{i=1}^n P_{t,i}) \cap Q_t]\big), \\ O(t^{(\mu-1)\alpha}) + O(t^{(1-\kappa_i)(1-\alpha)}) \quad \textit{on} \;\; N_t \cap \big(P_{t,i} \cap Q_t\big), \;\; \textit{for} \;\; i = 1,\ldots,n, \\ 0 \quad \textit{on} \;\; N_t \cap \big(P_{t,i} \setminus (P_{t,i} \cap Q_t)\big), \;\; \textit{for} \;\; i = 1,\ldots,n. \end{cases}$$

Now we briefly sketch the case $\lambda_i = -3$. Note from (4.44) that on $Q_t \setminus [(\bigcup_{i=1}^n P_{t,i}) \cap Q_t]$ the extra term $t^3\eta$ contributes $O(t^3)$ on $M_0 \setminus \bigcup_{i=1}^n S_i$, and $O(t^3 r^{-3})$ on $\Gamma_i \times (2t^\alpha, \epsilon)$ to the error, whereas on the annulus $P_{t,i} \cap Q_t$, the extra terms $t^3\xi_i$ and $t^3 F(t^{-\alpha}r)C_i(\gamma, r)$ contribute $O(t^{3(1-\alpha)})$ and $O(t^{3(1-\alpha)+\alpha\delta})$ respectively. Calculation then shows that

Proposition 6.5 *In the situation above, the error term $\mathrm{Im}(\Omega_t)|_{N_t}$, for the case $\lambda_i = -3$, satisfies*

$$\big|\,\mathrm{Im}(\Omega_t)|_{N_t}\,\big|_{h_t} = \begin{cases} O(t^3) \quad \textit{on} \;\; N_t \cap \big(M_0 \setminus \bigcup_{i=1}^n S_i\big), \\ O(t^3 r^{-3}) \quad \textit{on} \;\; N_t \cap \big(\Gamma_i \times (2t^\alpha, \epsilon)\big), \;\; \textit{for} \;\; i = 1,\ldots,n, \\ O(t^{3(1-\alpha)}) + O(t^{(\mu-1)\alpha}) + O(t^{(1-\kappa_i)(1-\alpha)}) \quad \textit{on} \;\; N_t \cap \big(P_{t,i} \cap Q_t\big), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for} \;\; i = 1,\ldots,n, \\ 0 \quad \textit{on} \;\; N_t \cap \big(P_{t,i} \setminus (P_{t,i} \cap Q_t)\big), \;\; \textit{for} \;\; i = 1,\ldots,n. \end{cases}$$

Next we estimate the term $(\tilde{\Omega}_t - \Omega_t)|_{N_t}$ in (6.4), which comes from deforming the nearly Calabi–Yau structure to the genuine Calabi–Yau structure on M_t. From Theorem 3.32 or Theorem 4.28, we have the C^0-estimates for $\tilde{\Omega}_t - \Omega_t$ on the whole M_t given by $\|\tilde{\Omega}_t - \Omega_t\|_{C^0} = O(t^\kappa)$ for some $\kappa > 0$. This term then contributes $O(t^\kappa)$ to the basic estimate (6.4), which in turn contributes $O(t^\kappa)$ to different norms of $\sin\theta$ in Theorem 6.1. But for the $L^{6/5}$-estimate to hold, one needs $\kappa \geq \kappa' + 3/2 > 5/2$ as $\kappa' > 1$. Since we have no a priori control of $\kappa > 0$, this will put a strong restriction on κ, and in turn on the rates ν and λ_i as well.

To resolve this problem we are going to improve the global L^2-estimate for $\tilde{\Omega}_t - \Omega_t$ to C^0-estimates locally by applying modified versions of Theorem 3.11 and Theorem 3.12 (see also Theorems G1 and G2 in [26, §11.6] for 7 dimensions). Note that if we look back on the construction of $\tilde{\Omega}_t$ in Theorem 3.14, the size of $\tilde{\Omega}_t - \Omega_t$ is of the same order as the size of $d\eta_t = \tilde{\varphi}_t - \varphi_t$, in other words, the error introduced when deforming the nearly Calabi–Yau structure to the genuine Calabi–Yau structure on the 6-fold M_t is essentially the same as that introduced when deforming the G_2-structure to the torsion-free G_2-structure on the 7-fold $S^1 \times M_t$. It suggests that in order to get a better control of the C^0-norm of $\tilde{\Omega}_t - \Omega_t$ on M_t, one could consider improving the C^0-norm of $d\eta_t$ on $S^1 \times M_t$.

In [26, §11.6] Joyce proved an existence result for torsion-free G_2-structures by constructing the 2-form η upon solving a nonlinear elliptic p.d.e. (equation (11.33) of [26]) in η. His method of solving the p.d.e. is to inductively construct sequences of 2-forms $\{\eta_j\}_{j=0}^\infty$ and functions $\{f_j\}_{j=0}^\infty$ with $\eta_0 = f_0 = 0$, and then he showed that these sequences converge in some Sobolev spaces to limits η and f which satisfy the p.d.e. The C^0-estimate of $d\eta$ is derived from the C^0-estimates of the sequence elements $d\eta_j$. So to improve $\|d\eta\|_{C^0}$, we need to improve Theorems G1 and G2 in [26], or more appropriately, the 6-dimensional version of them, i.e. Theorems 3.11 and 3.12 in Chapter 3, in our situation.

Here is the modified version of Theorem 3.11:

Theorem 6.6 *Let $D_2, D_3 > 0$ be constants, and suppose (M, g) is a complete Riemannian 6-fold with a continuous function ρ having the following properties:*

(1) the injectivity radius of geodesics $\delta(g)_x$ of (M, g) starting at x satisfies $\delta(g)_x \geq D_2\rho(x)$,

(2) the Riemann curvature $R(g)$ satisfies $|R(g)|_g \leq D_3\rho^{-2}$ on M, and

(3) for all $x \in M$, we have $1/2\,\rho(x) \leq \rho \leq 2\rho(x)$ on balls $B_{D_2\rho(x)}(x)$ of radius $D_2\rho(x)$ about x.

Then there exist $K_1, K_2 > 0$ depending only on D_2 and D_3, such that if $\chi \in L^{12}_1(\Lambda^3T^*M) \cap L^2(\Lambda^3T^*M)$ then

$$\|\rho^{7/2}\,\nabla\chi\|_{L^{12}} \leq K_1\,(\,\|\rho^{7/2}\,d\chi\|_{L^{12}} + \|\rho^{7/2}\,d^*\chi\|_{L^{12}} + \|\chi\|_{L^2}\,)$$

$$\textit{and} \quad \|\rho^3\,\chi\|_{C^0} \leq K_2\,(\,\|\rho^{7/2}\nabla\chi\|_{L^{12}} + \|\chi\|_{L^2}\,).$$

We shall call ρ a *local injectivity radius function* on M. Condition (3) ensures that ρ does not change quickly, and we may treat it as constant on $B_{D_2\rho(x)}(x)$. Moreover, (1) and (3) imply

$\delta(g)_y \geq 1/2\, D_2\rho(x)$ for all $y \in B_{D_2\rho(x)}(x)$, whereas (2) and (3) give $|R(g)|_g \leq 4D_3\rho(x)^{-2}$ on $B_{D_2\rho(x)}(x)$. The right hand sides of these inequalities are just constants, so that we get control of injectivity radius and Riemann curvature on balls about x with radius at most $D_2\rho(x)$, which can then be compared with Euclidean balls.

As in Theorem 3.11, we can prove Theorem 6.6 using the same method of proof as that of Theorem G1 in [26, p. 298], but we now use balls of radius $L\rho(x)$, where $0 < L < D_2$, about x instead of Lt in the proof of Theorem G1. Note that it is important to have the constants D_2, D_3 in the theorem independent of t, so that we have K_1 and K_2 are independent of t as well.

Next we give the improved result for Theorem 3.12:

Theorem 6.7 *Let $\kappa > 0$ and $D_1, D_2, D_3, D_4, K_1, K_2 > 0$ be constants. Then there exist constants $\epsilon \in (0,1]$, K_3 and $K > 0$ such that whenever $0 < t \leq \epsilon$, the following is true.*

Let M be a compact 6-fold, with metric g_M and a local injectivity radius function ρ satisfying (1), (2) and (3) in Theorem 6.6. Suppose ρ also satisfies $\rho \geq D_4 t > 0$ on M. Let (φ, g) be an S^1-invariant G_2-structure on $S^1 \times M$ with $d\varphi = 0$. Suppose ψ is an S^1-invariant smooth 3-form on the 7-fold $S^1 \times M$ with $d^*\psi = d^*\varphi$, and

(i) $\|\psi\|_{L^2} \leq D_1 t^{3+\kappa}$, $\|\rho^3\, \psi\|_{C^0} \leq D_1 t^{3+\kappa}$ *and* $\|\rho^{7/2}\, d^*\psi\|_{L^{12}} \leq D_1 t^{3+\kappa}$,

(ii) *if* $\chi \in L^{12}_1(\Lambda^3 T^*(S^1 \times M))$ *is S^1-invariant, then* $\|\rho^{7/2}\, \nabla\chi\|_{L^{12}} \leq K_1\, (\|\rho^{7/2}\, d\chi\|_{L^{12}} + \|\rho^{7/2}\, d^*\chi\|_{L^{12}} + \|\chi\|_{L^2})$,

(iii) *if* $\chi \in L^{12}_1(\Lambda^3 T^*(S^1 \times M))$ *is S^1-invariant, then* $\|\rho^3\, \chi\|_{C^0} \leq K_2\, (\|\rho^{7/2}\nabla\chi\|_{L^{12}} + \|\chi\|_{L^2})$.

With the same notation as in Theorem 3.10, there exist sequences $\{\eta_j\}_{j=0}^\infty$ in $L^{12}_2(\Lambda^2 T^*(S^1 \times M))$ and $\{f_j\}_{j=0}^\infty$ in $L^{12}_1(S^1 \times M)$ with η_j, f_j being all S^1-invariant and $\eta_0 = f_0 = 0$, satisfying the equations

$$(dd^* + d^*d)\eta_j = d^*\psi + d^*(f_{j-1}\psi) + *\, dF(d\eta_{j-1}) \text{ and } f_j\varphi = \frac{7}{3}\pi_1(d\eta_j)$$

for each $j > 0$, and the inequalities

(a) $\|d\eta_j\|_{L^2} \leq 2D_1 t^{3+\kappa}$,
(b) $\|\rho^{7/2}\, \nabla d\eta_j\|_{L^{12}} \leq K_3 t^{3+\kappa}$,
(c) $\|\rho^3\, d\eta_j\|_{C^0} \leq K t^{3+\kappa}$ *and*
(d) $\|d\eta_j - d\eta_{j-1}\|_{L^2} \leq 2D_1 2^{-j} t^{3+\kappa}$,
(e) $\|\rho^{7/2}\, \nabla(d\eta_j - d\eta_{j-1})\|_{L^{12}} \leq K_3 2^{-j} t^{3+\kappa}$,
(f) $\|\rho^3\, (d\eta_j - d\eta_{j-1})\|_{C^0} \leq K 2^{-j} t^{3+\kappa}$.

Here ∇ and $\|\cdot\|$ are computed using g on $S^1 \times M$.

We can prove Theorem 6.7 by applying Theorem 6.6 in place of Theorem 3.11, and then follow the same arguments in the proof of Theorem G2 [26, p. 299]. The only extra issue here is that we need a lower bound for ρ: $\rho \geq D_4 t > 0$ on M. The inequality in part (c) implies $|d\eta_j|_g \leq K t^{3+\kappa}\rho^{-3} \leq K D_4^{-3} t^\kappa$ if $\rho \geq D_4 t$. Thus assuming $\rho \geq D_4 t$ on M gives $|d\eta_j|_g \leq \epsilon_1$ if t is sufficiently small, where ϵ_1 is the small positive constant defined in Definition 10.3.3 in [26]. The lower bound for ρ therefore ensures $d\eta_j$ is small enough for each j which is needed to apply

Proposition 10.3.5 in [26]. We also make a remark here about the difference between (a), (d) and (b), (c), (e), (f) in the theorem: (a) and (d) are global estimates on the whole manifold, as we get $\|d\eta_j\|_{L^2}^2$ and $\|d\eta_j - d\eta_{j-1}\|_{L^2}^2$ from the elliptic equation in the theorem by integration by parts. On the other hand, (b), (c), (e) and (f) are local estimates on small balls, and so we are allowed to insert powers of ρ.

Let us now return to our Calabi–Yau 3-fold M_t. Define a function ρ_t on M_t by

$$\rho_t = \begin{cases} \epsilon & \text{on } \ M_0 \setminus \bigcup_{i=1}^n S_i, \\ r & \text{on } \ \Gamma_i \times (tR, \epsilon), \ \text{ for } \ i = 1, \ldots, n, \\ tR & \text{on } \ K_i \subset Y_i, \ \text{ for } \ i = 1, \ldots, n. \end{cases} \tag{6.9}$$

We claim that ρ_t is a local injectivity radius function on M_t. To see properties (1) and (2), recall that the way we construct g_t on M_t is, on $M_0 \setminus \bigcup_{i=1}^n S_i$ it is equal to g_0, on the annulus $\Gamma_i \times (tR, \epsilon)$ it is g_{V_i}, and on $K_i \subset Y_i$ it is $t^2 g_{Y_i}$. It follows that for $x \in M_0 \setminus \bigcup_{i=1}^n S_i$, we have $\delta(g_t)_x = \delta(g_0)_x \geq C_1$ and $|R(g_t)|_{g_t} = |R(g_0)|_{g_0} \leq C_2$ for some constant $C_1, C_2 > 0$, as the metric here is independent of t. For $x \in \Gamma_i \times (tR, \epsilon)$, we have $\delta(g_t)_x = \delta(g_{V_i})_x \geq C_3 r$ and $|R(g_t)|_{g_t} = |R(g_{V_i})|_{g_{V_i}} \leq C_4 r^{-2}$ for some constant $C_3, C_4 > 0$, as the length scale for the cone metric is given by r. Finally for $x \in K_i \subset Y_i$, we have $\delta(g_t)_x = \delta(t^2 g_{Y_i})_x \geq C_5 t$ and $|R(g_t)|_{g_t} = |R(t^2 g_{Y_i})|_{t^2 g_{Y_i}} \leq C_6 t^{-2}$ for some constant $C_5, C_6 > 0$, as the length scale for the metric $t^2 g_{Y_i}$ is given by t. Thus from the explicit definition of ρ_t in (6.9), (1) and (2) hold with $D_2 = \min(C_1 \epsilon^{-1}, C_3, C_5 R^{-1})$ and $D_3 = \max(C_2 \epsilon^2, C_4, C_6 R^2)$.

Condition (3) holds with small enough $D_2 << 1/2$, thus by making D_2 smaller if necessary, ρ_t satisfies (3) as well. Therefore Theorem 6.6 applies to (M_t, g_t) and ρ_t. Let $D_4 = R$, then $\rho_t \geq D_4 t$ on M_t, and we thus have a lower bound for ρ_t. It follows that Theorem 6.7 also applies to (M_t, g_t) and ρ_t.

As proved in Theorem G2, the sequence $\{\eta_j\}$ converges to η in some Sobolev space of $\Lambda^2 T^*(S^1 \times M)$. From part (c) of Theorem 6.7, we deduce that $\|\rho^3 \, d\eta\|_{C^0} \leq K t^{3+\kappa}$, that is, $|d\eta|_g = O(t^{3+\kappa} \rho^{-3})$. Thus on our 7-fold $S^1 \times M_t$, we have $|d\eta_t|_{g_{\varphi_t}} = O(t^{3+\kappa} \rho_t^{-3})$, where the norm is measured by the metric g_{φ_t} associated to the G_2 3-form $\varphi_t = ds \wedge \omega_t + \mathrm{Re}(\Omega_t)$. Using (6.9) and the fact that the metrics g_{φ_t} and $ds^2 + g_t$ on $S^1 \times M_t$ are uniformly equivalent (see Lemma 3.7), we obtain for $(s, x) \in S^1 \times M_t$,

$$\big| (d\eta_t)_{(s,x)}|_{T_x M_t} \big|_{g_t} = \begin{cases} O(t^{3+\kappa}) & \text{for } x \in M_0 \setminus \bigcup_{i=1}^n S_i, \\ O(t^{3+\kappa} r^{-3}) & \text{for } x \in \Gamma_i \times (tR, \epsilon), \ \text{ for } \ i = 1, \ldots, n, \\ O(t^{\kappa}) & \text{for } x \in K_i \subset Y_i, \ \text{ for } \ i = 1, \ldots, n, \end{cases} \tag{6.10}$$

which then implies the improved C^0-estimate of $\tilde{\Omega}_t - \Omega_t$ given by:

Proposition 6.8 *In the situation above, the error term $(\tilde{\Omega}_t - \Omega_t)|_{N_t}$ satisfies*

$$\big|\,(\tilde{\Omega}_t - \Omega_t)|_{N_t}\,\big|_{h_t} = \begin{cases} O(t^{3+\kappa}) & \textit{on } \ N_t \cap (M_0 \setminus \bigcup_{i=1}^n S_i), \\ O(t^{3+\kappa} r^{-3}) & \textit{on } \ N_t \cap (\Gamma_i \times (tR, \epsilon)), \ \textit{for } \ i = 1, \ldots, n, \\ O(t^{\kappa}) & \textit{on } \ N_t \cap K_i, \ \textit{for } \ i = 1, \ldots, n. \end{cases}$$

To finish the basic estimate, it remains to compute the error term $(\hat{\Omega}_t - \tilde{\Omega}_t)|_{N_t}$. This term arises from changing the coordinates on M_t by applying the diffeomorphism ψ_t obtained from Moser's argument. We claim that this term is of the same order as the term $(\tilde{\Omega}_t - \Omega_t)|_{N_t}$.

Recall that we use Moser's argument to construct the diffeomorphism $\psi_t : M_t \longrightarrow M_t$ so that $\psi_t^*(c_t \tilde{\omega}_t) = \omega_t$, and we write $\hat{\Omega}_t = \psi_t^*(\tilde{\Omega}_t)$. Thus the difference between $\hat{\Omega}_t$ and $\tilde{\Omega}_t$ is essentially given by the term "$\partial(\psi_t - \mathrm{Id})$", where Id denotes the identity map on M_t. Here what we mean by the difference between ψ_t and Id can be interpreted in terms of local coordinates on M_t (similar arguments appeared in the proof of Theorem 3.24), and we use ∂ to denote the usual partial differentiation.

Now as $c_t \tilde{\omega}_t$ and ω_t are cohomologous, we write $c_t \tilde{\omega}_t - \omega_t = d\sigma_t$ for some smooth 1-form σ_t. Note that $c_t \tilde{\omega}_t - \omega_t = \iota(\frac{\partial}{\partial s})(\tilde{\varphi}_t - \varphi_t) = \iota(\frac{\partial}{\partial s}) d\eta_t$ is just a component of $d\eta_t$, and so $|d\sigma_t|_{g_t}$ is given by (6.10). We claim that we can choose a small 1-form σ_t uniquely on M_t, so that Moser's argument defines "small" vector fields X_t (see also in the proof of Theorem 3.24), and then constructs "small" diffeomorphisms ψ_t by representing them as the flow of X_t on M_t. Our technique is to adopt a kind of *isoperimetric inequality* which is similar to the one in (v) of Theorem 6.1, but we are working with 1-forms on the real 6-folds M_0' and Y_i. Using the notation on the weighted Sobolev spaces in Chapter 4, we have the following:

Proposition 6.9 *There exists a constant $C_1 > 0$ such that $\|\sigma\|_{L^3} \leq C_1(\|d\sigma\|_{L^2} + \|d^*\sigma\|_{L^2})$ for all $\sigma \in L^2_{1,-2}(\Lambda^1_{\mathbb{C}} T^* M_0')$.*

Proof. By the Weighted Sobolev Embedding theorem (Theorem 4.2 (i)), $L^2_{1,-2}(\Lambda^1_{\mathbb{C}})$ embeds in $L^3_{0,-2}(\Lambda^1_{\mathbb{C}})$ in 6 dimensions. From (4.4), we have $L^3(\Lambda^1_{\mathbb{C}}) = L^3_{0,-2}(\Lambda^1_{\mathbb{C}})$. Then $\|\sigma\|_{L^3} \leq D_1 \|\sigma\|_{L^2_{1,-2}}$ for some constant $D_1 > 0$.

Now $(d + d^*_{od})^2_{1,-2} : L^2_{1,-2}(\Lambda^{odd}_{\mathbb{C}}) \longrightarrow L^2_{0,-3}(\Lambda^{even}_{\mathbb{C}})$ is an elliptic operator, so we can apply elliptic regularity (Theorem 4.6) which gives $\|\sigma\|_{L^2_{1,-2}} \leq D_2(\|(d + d^*)\sigma\|_{L^2_{0,-3}} + \|\sigma\|_{L^2_{0,-2}})$ for some constant $D_2 > 0$ independent of σ. If σ is also L^2-orthogonal to the kernel $\mathrm{Ker}((d + d^*_{od})^2_{1,-2})$, then one can drop the term $\|\sigma\|_{L^2_{0,-2}}$ on right hand side by increasing D_2, and thus obtaining $\|\sigma\|_{L^2_{1,-2}} \leq D_2 \|(d + d^*)\sigma\|_{L^2_{0,-3}}$ (analogous result in Proposition 2.30). By (4.20), we have $L^2 = L^2_{0,-3}$, and so $\|\sigma\|_{L^2_{1,-2}} \leq D_2(\|d\sigma\|_{L^2} + \|d^*\sigma\|_{L^2})$. Moreover, the kernel $\mathrm{Ker}((d + d^*_{od})^2_{0,-3}|_{\Lambda^1_{\mathbb{C}}})$ is exactly the vector space of closed and coclosed 1-forms in $L^2(\Lambda^1_{\mathbb{C}})$, which is isomorphic to $H^1(M_0', \mathbb{C})$ by Proposition 4.18. But $H^1(M_0', \mathbb{C}) = 0$ from Proposition 4.22, so σ is always L^2-orthogonal to $\mathrm{Ker}((d + d^*_{od})^2_{1,-2})$, and hence $\|\sigma\|_{L^2_{1,-2}} \leq D_2(\|d\sigma\|_{L^2} + \|d^*\sigma\|_{L^2})$ holds. Combining this with $\|\sigma\|_{L^3} \leq D_1 \|\sigma\|_{L^2_{1,-2}}$, the result follows with $C_1 = D_1 D_2$. □

A similar result holds on AC Calabi–Yau 3-folds Y_i:

Proposition 6.10 *There exists a constant $C_2 > 0$ such that $\|\sigma\|_{L^3} \le C_2(\|d\sigma\|_{L^2} + \|d^*\sigma\|_{L^2})$ for all $\sigma \in L^2_{1,-2}(\Lambda^1_{\mathbb{C}}T^*Y_i)$, for $i = 1, \dots, n$.*

The proof follows very closely the proof of Proposition 6.9, but this time we apply a result from Dodziuk [14, Cor. 1, p.24], which tells us that there are no nonzero L^2-harmonic 1-forms on any complete, oriented, Riemannian manifold with nonnegative Ricci curvature and infinite volume, on our AC Calabi–Yau 3-folds Y_i. This implies that there are no nonzero closed and coclosed 1-forms in $L^2(\Lambda^1_{\mathbb{C}}T^*Y_i)$, and hence σ is always L^2-orthogonal to the kernel of $d + d^*$. We remark here that the inequality in Proposition 6.10 is invariant under homotheties, which means the inequality also holds on $(Y_i, J_{Y_i}, t^2\omega_{Y_i}, t^3\Omega_{Y_i})$ with the same constant. Now take $C = \max(C_1, C_2)$, we have $\|\sigma\|_{L^3} \le C(\|d\sigma\|_{L^2} + \|d^*\sigma\|_{L^2})$ for 1-forms σ on M'_0 and Y_i for $i = 1, \dots, n$.

We now proceed to "glue" together the inequalities on M'_0 and Y_i to obtain an inequality for 1-forms on M_t for small $t > 0$. Choose $u, v > 0$ with $v < u < \alpha$ such that $tR < t^\alpha < 2t^\alpha < t^u < t^v < \epsilon$ for small $t > 0$. Let $H : (0, \infty) \longrightarrow [0, 1]$ be a smooth decreasing function so that $H(s) = 1$ for $s \in (0, v]$, and $H(s) = 0$ for $s \in [u, \infty)$. Define a function $G_t : M_t \longrightarrow [0, 1]$ by $G_t(x) = 1$ for $x \in M_0 \setminus \bigcup_{i=1}^n S_i$, $G_t(x) = H(\log r/\log t)$ for $x \in \Gamma_i \times (tR, \epsilon)$, $i = 1, \dots, n$, and $G_t(x) = 0$ for $x \in K_i \subset Y_i$, $i = 1, \dots, n$. Observe that $G_t \equiv 0$ on K_i and $\Gamma_i \times (tR, t^u)$ for $i = 1, \dots, n$, and $G_t \equiv 1$ on $\Gamma_i \times (t^v, \epsilon)$ and $M_0 \setminus \bigcup_{i=1}^n S_i$ for $i = 1, \dots, n$.

Let σ_t be a smooth 1-form on M_t with $d\sigma_t = c_t\tilde{\omega}_t - \omega_t$. Since $H^1(M_t, \mathbb{C}) = 0$ from a general fact on compact Calabi–Yau manifolds, so σ_t is automatically orthogonal to the space of closed and coclosed 1-forms on M_t, and it follows that we can choose σ_t uniquely by requiring $d^*\sigma_t = 0$, where d^* is computed using the metric g_t.

Write $\sigma_t = G_t\sigma_t + (1 - G_t)\sigma_t$. We can regard $G_t\sigma_t$ as a compactly-supported 1-form on M'_0, and $(1 - G_t)\sigma_t$ a sum of compactly-supported 1-forms on Y_i. Applying Proposition 6.9 to $G_t\sigma_t$, using the metric g_0 on the support of G_t and putting the constant $C = \max(C_1, C_2)$ gives

$$\begin{aligned}
\|G_t\sigma_t\|_{L^3} &\le C\big(\|d(G_t\sigma_t)\|_{L^2} + \|d^*(G_t\sigma_t)\|_{L^2}\big)\\
&\le C\big(\|G_t\, d\sigma_t\|_{L^2} + \|dG_t \wedge \sigma_t\|_{L^2} + \|dG_t\|_{L^6} \cdot \|\sigma_t\|_{L^3} + \|G_t\, d^*\sigma_t\|_{L^2}\big)\\
&\le C\big(\|G_t\, d\sigma_t\|_{L^2} + 2\|dG_t\|_{L^6} \cdot \|\sigma_t\|_{L^3}\big),
\end{aligned}$$

where we have used $d^*\sigma_t = 0$ and Hölder's inequality. The same inequality holds with the metric g_t, as it coincides with g_0 on the support of G_t. For the 1-form $(1 - G_t)\sigma_t$, since the support of $1 - G_t$ in Y_i is $K_i \cup (\Gamma_i \times (tR, t^v))$ for $i = 1, \dots, n$, we apply Proposition 6.10, using the constant C, and get

$$\begin{aligned}
&\|(1 - G_t)\sigma_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^3}\\
&\le C\big(\|d((1 - G_t)\sigma_t)|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^2} + \|d^*((1 - G_t)\sigma_t)|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^2}\big)\\
&\le C\big(\|(1 - G_t)d\sigma_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^2} + 2\|dG_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^6} \cdot \|\sigma_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^3}\\
&\quad + \|(1 - G_t)d^*\sigma_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^2}\big)
\end{aligned}$$

using the metric $t^2g_{Y_i}$. As the metric $t^2g_{Y_i}$ coincides with g_t for $r \le t^\alpha$, and is close to it for $t^\alpha \le r \le t^v$, $d^*\sigma_t$ equals zero for $r \le t^\alpha$, and is small for $t^\alpha \le r \le t^v$, computed using $t^2g_{Y_i}$.

Thus by increasing C, we have

$$\begin{aligned}\|(1-G_t)\sigma_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^3} \;\le\; & C\big(\,\|((1-G_t)d\sigma_t)|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^2}\\ &+\,2\|dG_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^6}\cdot\|\sigma_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^3}\big),\end{aligned}$$

computed using g_t. It follows that

$$\begin{aligned}\|(1-G_t)\sigma_t\|_{L^3} \;\le\; & \sum_{i=1}^n \|(1-G_t)\sigma_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^3}\\ \le\; & C\sum_{i=1}^n\big(\,\|((1-G_t)d\sigma_t)|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^2}\\ &\qquad+\,2\|dG_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^6}\cdot\|\sigma_t|_{K_i\cup(\Gamma_i\times(tR,t^v))}\|_{L^3}\big)\\ \le\; & C\sqrt{n}\,\big(\,\|(1-G_t)d\sigma_t\|_{L^2}\,+\,2\|dG_t\|_{L^6}\cdot\|\sigma_t\|_{L^3}\big),\end{aligned}$$

where we used the inequality of arithmetic and geometric means on the last row. Consequently we have

$$\begin{aligned}\|\sigma_t\|_{L^3} &\le \|G_t\sigma_t\|_{L^3}\;+\;\|(1-G_t)\,\sigma_t\|_{L^3}\\ &\le C\big(\,\|G_t\,d\sigma_t\|_{L^2}+2\|dG_t\|_{L^6}\cdot\|\sigma_t\|_{L^3}\big)+C\sqrt{n}\,\big(\|(1-G_t)\,d\sigma_t\|_{L^2}+2\|dG_t\|_{L^6}\cdot\|\sigma_t\|_{L^3}\big),\end{aligned}$$

which implies

$$\begin{aligned}\big(1-2C(1+\sqrt{n})\,\|dG_t\|_{L^6}\big)\cdot\|\sigma_t\|_{L^3} \;\le\; & C\|G_t\,d\sigma_t\|_{L^2}\;+\;C\sqrt{n}\,\|(1-G_t)\,d\sigma_t\|_{L^2}\\ \le\; & C(1+\sqrt{n})\,\|d\sigma_t\|_{L^2}\end{aligned}$$

as $\|G_t\,d\sigma_t\|_{L^2}$, $\|(1-G_t)\,d\sigma_t\|_{L^2} \;\le\; \|d\sigma_t\|_{L^2}$. Calculation shows that the L^6-norm of dG_t is given by $O(|\log t|^{-5/6})$, which tends to zero as $t\to 0$. Thus for sufficiently small $t>0$, we can make

$$2C(1+\sqrt{n})\,\|dG_t\|_{L^6}\le 1/2.$$

Therefore

$$\|\sigma_t\|_{L^3}\;\le\;2C(1+\sqrt{n})\,\|d\sigma_t\|_{L^2},$$

and hence we have proved:

Theorem 6.11 *Suppose σ_t is a smooth 1-form on M_t with $d\sigma_t=c_t\tilde{\omega}_t-\omega_t$ and $d^*\sigma_t=0$. Then there exists a constant $K>0$, independent of t, such that*

$$\|\sigma_t\|_{L^3}\;\le\;K\,\|d\sigma_t\|_{L^2}$$

for sufficiently small $t>0$.

As we have seen earlier, $|d\sigma_t|_{g_t}$ is of the same order as $|(d\eta_t)_{(s,x)}|_{T_xM_t}|_{g_t}$, and so is given by (6.10), i.e. $|d\sigma_t|_{g_t}=O(t^{3+\kappa}\rho_t^{-3})$ on balls of radius $O(\rho_t(x))$ about $x\in M_t$, where ρ_t is given in (6.9). Thus we have estimates of $(d+d^*)\sigma_t=d\sigma_t$ and all derivatives, given by $|\nabla^l d\sigma_t|_{g_t}=O(t^{3+\kappa}\rho_t^{-3-l})$ for $l\ge 0$. Moreover, we have the global estimate $\|d\sigma_t\|_{L^2}=O(t^{3+\kappa})$ on the whole M_t as in Theorem 6.7, which implies $\|\sigma_t\|_{L^3}=O(t^{3+\kappa})$ from Theorem 6.11. Now using similar arguments to the proof of Theorem 6.7, the elliptic regularity for the operator $d+d^*$ and the global estimate $\|\sigma_t\|_{L^3}=O(t^{3+\kappa})$ give

$$|\nabla^l\sigma_t|_{g_t}=O(t^{3+\kappa}\rho_t^{-2-l})$$

for $l \geq 0$. It follows that we have the same estimates for $|\nabla^l X_t|_{g_t}$, and hence for $|\nabla^l(\psi_t - \mathrm{Id})|_{g_t}$ which can be interpreted using local coordinates. Now this diffeomorphism estimate for $l = 1$ is sufficient to prove our expected size for the term $(\hat{\Omega}_t - \tilde{\Omega}_t)|_{N_t}$:

Proposition 6.12 *In the situation above, the error term $(\hat{\Omega}_t - \tilde{\Omega}_t)|_{N_t}$ satisfies*

$$\big|(\hat{\Omega}_t - \tilde{\Omega}_t)|_{N_t}\big|_{h_t} = \begin{cases} O(t^{3+\kappa}) & \text{on } N_t \cap (M_0 \setminus \bigcup_{i=1}^n S_i), \\ O(t^{3+\kappa} r^{-3}) & \text{on } N_t \cap (\Gamma_i \times (tR, \epsilon)), \text{ for } i = 1, \ldots, n, \\ O(t^\kappa) & \text{on } N_t \cap K_i, \text{ for } i = 1, \ldots, n. \end{cases}$$

Before proceeding to combining the errors to get the basic estimate in (6.4), let us return to the issue on the uniform equivalence between the metrics $\hat{h}_t$ and h_t. We already got the size for $(\tilde{\Omega}_t - \Omega_t)|_{N_t}$ and $(\tilde{\omega}_t - \omega_t)|_{N_t}$, both have same order. The size for $(\hat{\Omega}_t - \tilde{\Omega}_t)|_{N_t}$ and $(\hat{\omega}_t - \tilde{\omega}_t)|_{N_t}$ are essentially given by the "difference" between the diffeomorphism ψ_t and the identity, and we have shown that it is of the same order as the size for $(\tilde{\Omega}_t - \Omega_t)|_{N_t}$ or $(\tilde{\omega}_t - \omega_t)|_{N_t}$. It follows that the size for $(\hat{\Omega}_t - \Omega_t)|_{N_t}$ and $(\hat{\omega}_t - \omega_t)|_{N_t}$ has the same order as $(\tilde{\Omega}_t - \Omega_t)|_{N_t}$ or $(\tilde{\omega}_t - \omega_t)|_{N_t}$. This implies the metrics $\hat{h}_t$ and h_t are uniformly equivalent in t, and therefore Propositions 6.4, 6.5, 6.8 and 6.12 also hold for $\hat{h}_t$.

We summarize the above estimates from Propositions 6.4, 6.5, 6.8 and 6.12 in the following table, measuring w.r.t $\hat{h}_t$:

	$(\hat{\Omega}_t - \tilde{\Omega}_t)\vert_{N_t}$	$(\tilde{\Omega}_t - \Omega_t)\vert_{N_t}$	$\mathrm{Im}(\Omega_t)\vert_{N_t}$ $(\lambda_i < -3)$	$\mathrm{Im}(\Omega_t)\vert_{N_t}$ $(\lambda_i = -3)$
$N_t \cap (M_0 \setminus \bigcup_{i=1}^n S_i)$	$O(t^{3+\kappa})$	$O(t^{3+\kappa})$	0	$O(t^3)$
$N_t \cap (\Gamma_i \times (2t^\alpha, \epsilon))$	$O(t^{3+\kappa} r^{-3})$	$O(t^{3+\kappa} r^{-3})$	0	$O(t^3 r^{-3})$
$N_t \cap (\Gamma_i \times (t^\alpha, 2t^\alpha))$	$O(t^{3(1-\alpha)+\kappa})$	$O(t^{3(1-\alpha)+\kappa})$	$O(t^{(\mu-1)\alpha})$ $+O(t^{(1-\kappa_i)(1-\alpha)})$	$O(t^{3(1-\alpha)}) + O(t^{(\mu-1)\alpha})$ $+O(t^{(1-\kappa_i)(1-\alpha)})$
$N_t \cap (\Gamma_i \times (tR, t^\alpha))$	$O(t^{3+\kappa} r^{-3})$	$O(t^{3+\kappa} r^{-3})$	0	0
$N_t \cap K_i$	$O(t^\kappa)$	$O(t^\kappa)$	0	0

Table 6.1 The estimate (6.4) on different regions of N_t

As in §6.1 we may write $\hat{\Omega}_t|_{N_t} = e^{i\theta_t} dV_t$ for some phase function $e^{i\theta_t}$ on N_t. Here dV_t is the volume form induced by the metric $\hat{h}_t$. Then $\mathrm{Im}(\hat{\Omega}_t)|_{N_t} = \sin\theta_t \, dV_t$. We see from the table that, for the case $\lambda_i < -3$,

$$|\sin\theta_t|_{\hat{h}_t} = \begin{cases} O(t^{3+\kappa}) & \text{on } N_t \cap (M_0 \setminus \bigcup_{i=1}^n S_i), \\ O(t^{3+\kappa} r^{-3}) & \text{on } N_t \cap (\Gamma_i \times (2t^\alpha, \epsilon)), \\ O(t^{3(1-\alpha)+\kappa}) + O(t^{(\mu-1)\alpha}) + O(t^{(1-\kappa_i)(1-\alpha)}) & \text{on } N_t \cap (\Gamma_i \times (t^\alpha, 2t^\alpha)), \\ O(t^{3+\kappa} r^{-3}) & \text{on } N_t \cap (\Gamma_i \times (tR, t^\alpha)), \\ O(t^\kappa) & \text{on } N_t \cap K_i, \end{cases} \tag{6.11}$$

for all $i = 1, \ldots, n$. For the case $\lambda_i = -3$, we have

$$|\sin\theta_t|_{\hat{h}_t} = \begin{cases} O(t^3) & \text{on } N_t \cap (M_0 \setminus \bigcup_{i=1}^n S_i), \\ O(t^3 r^{-3}) & \text{on } N_t \cap (\Gamma_i \times (2t^\alpha, \epsilon)), \\ O(t^{3(1-\alpha)}) + O(t^{(\mu-1)\alpha}) + O(t^{(1-\kappa_i)(1-\alpha)}) & \text{on } N_t \cap (\Gamma_i \times (t^\alpha, 2t^\alpha)), \\ O(t^{3+\kappa} r^{-3}) & \text{on } N_t \cap (\Gamma_i \times (tR, t^\alpha)), \\ O(t^\kappa) & \text{on } N_t \cap K_i, \end{cases} \tag{6.12}$$

for all $i = 1, \ldots, n$. Note that on each region the error from (6.12) dominates that from (6.11), so (i) of Theorem 6.1 holds when $\lambda_i = -3$ implies it also holds when $\lambda_i < -3$. Thus it is enough for us to consider (6.12) only.

We also need to estimate the derivative $d\sin\theta_t$. Using similar arguments it can be deduced that (when $\lambda_i = -3$)

$$|d\sin\theta_t|_{\hat{h}_t} = \begin{cases} O(t^3) & \text{on } N_t \cap (M_0 \setminus \bigcup_{i=1}^n S_i), \\ O(t^3 r^{-4}) & \text{on } N_t \cap (\Gamma_i \times (2t^\alpha, \epsilon)), \\ O(t^{3(1-\alpha)-\alpha}) + O(t^{(\mu-1)\alpha-\alpha}) + O(t^{(1-\kappa_i)(1-\alpha)-\alpha}) & \text{on } N_t \cap (\Gamma_i \times (t^\alpha, 2t^\alpha)), \\ O(t^{3+\kappa} r^{-4}) & \text{on } N_t \cap (\Gamma_i \times (tR, t^\alpha)), \\ O(t^{\kappa-1}) & \text{on } N_t \cap K_i, \end{cases} \tag{6.13}$$

for all $i = 1, \ldots, n$. Here we used equations (5.2) and (6.1) to obtain the bound on $N_t \cap (\Gamma_i \times (t^\alpha, 2t^\alpha))$.

As in part (i) of Theorem 6.1, we need bounds for $\|\sin\theta_t\|_{L^{6/5}}$, $\|\sin\theta_t\|_{C^0}$ and $\|d\sin\theta_t\|_{L^6}$, computing norms w.r.t. $\hat{h}_t$. Since $\mathrm{vol}(N_t \cap K_i) = O(t^3)$ and $\mathrm{vol}(N_t \cap (\Gamma_i \times (t^\alpha, 2t^\alpha))) = O(t^{3\alpha})$, and $\mathrm{vol}(N_t \cap (M_0 \setminus \bigcup_{i=1}^n S_i)) = O(1)$, it follows that

$$\begin{aligned} \|\sin\theta_t\|_{L^{6/5}} &= O(1)^{5/6} \cdot O(t^3) + O\left(\sum_{i=1}^n \mathrm{vol}(\Sigma_i)^{5/6} \left(\int_{2t^\alpha}^{\epsilon} (t^3 r^{-3})^{6/5} r^2 dr\right)^{5/6}\right) \\ &\quad + O(t^{3\alpha})^{5/6} \cdot \big(O(t^{3(1-\alpha)}) + O(t^{(\mu-1)\alpha}) + \sum_{i=1}^n O(t^{(1-\kappa_i)(1-\alpha)})\big) \\ &\quad + O\left(\sum_{i=1}^n \mathrm{vol}(\Sigma_i)^{5/6} \left(\int_{tR}^{t^\alpha} (t^{3+\kappa} r^{-3})^{6/5} r^2 dr\right)^{5/6}\right) + O(t^3)^{5/6} \cdot O(t^\kappa) \\ &= O(t^{3(1-\alpha)+5\alpha/2}) + O(t^{(\mu-1)\alpha+5\alpha/2}) + \sum_{i=1}^n O(t^{(1-\kappa_i)(1-\alpha)+5\alpha/2}) + O(t^{\kappa+5/2}) \end{aligned} \tag{6.14}$$

using (6.12). Similarly, we have

$$\begin{aligned}\|\sin\theta_t\|_{C^0} &= O(t^3) + O(t^3(2t^\alpha)^{-3}) + O(t^{3(1-\alpha)}) + O(t^{(\mu-1)\alpha}) + \sum_{i=1}^n O(t^{(1-\kappa_i)(1-\alpha)}) \\ &\quad + O(t^{3+\kappa}(tR)^{-3}) + O(t^\kappa) \\ &= O(t^{3(1-\alpha)}) + O(t^{(\mu-1)\alpha}) + \sum_{i=1}^n O(t^{(1-\kappa_i)(1-\alpha)}) + O(t^\kappa). \end{aligned} \tag{6.15}$$

Using the estimate (6.13) for the derivative, we have

$$\begin{aligned}\|d\sin\theta_t\|_{L^6} &= O(1)^{1/6}\cdot O(t^3) + O\left(\sum_{i=1}^n \mathrm{vol}(\Sigma_i)^{1/6}\left(\int_{2t^\alpha}^{\epsilon}(t^3r^{-4})^6\,r^2dr\right)^{1/6}\right) \\ &\quad + O(t^{3\alpha})^{1/6}\cdot\big(O(t^{3(1-\alpha)-\alpha}) + O(t^{(\mu-1)\alpha-\alpha}) + \sum_{i=1}^n O(t^{(1-\kappa_i)(1-\alpha)-\alpha})\big) \\ &\quad + O\left(\sum_{i=1}^n \mathrm{vol}(\Sigma_i)^{1/6}\left(\int_{tR}^{t^\alpha}(t^{3+\kappa}r^{-4})^6\,r^2dr\right)^{1/6}\right) + O(t^3)^{1/6}\cdot O(t^{\kappa-1}) \\ &= O(t^{3(1-\alpha)-\alpha/2}) + O(t^{(\mu-1)\alpha-\alpha/2}) + \sum_{i=1}^n O(t^{(1-\kappa_i)(1-\alpha)-\alpha/2}) + O(t^{\kappa-1/2}). \end{aligned} \tag{6.16}$$

Now for part (i) of Theorem 6.1 to hold, we need:

$$\begin{cases}\kappa+5/2\ge\kappa'+3/2, & 3(1-\alpha)+5\alpha/2\ge\kappa'+3/2,\\ (\mu-1)\alpha+5\alpha/2\ge\kappa'+3/2, \quad \text{and} & (1-\kappa_i)(1-\alpha)+5\alpha/2\ge\kappa'+3/2\end{cases} \tag{6.17}$$

from (6.14),

$$\begin{cases}\kappa\ge\kappa'-1, & 3(1-\alpha)\ge\kappa'-1,\\ (\mu-1)\alpha\ge\kappa'-1, \quad \text{and} & (1-\kappa_i)(1-\alpha)\ge\kappa'-1\end{cases} \tag{6.18}$$

from (6.15), and

$$\begin{cases}\kappa-1/2\ge\kappa'-3/2, & 3(1-\alpha)-\alpha/2\ge\kappa'-3/2,\\ (\mu-1)\alpha-\alpha/2\ge\kappa'-3/2, \quad \text{and} & (1-\kappa_i)(1-\alpha)-\alpha/2\ge\kappa'-3/2\end{cases} \tag{6.19}$$

from (6.16).

Calculations show that given $\kappa > 0$, $\mu > 1$ and $\kappa_i < -3/2$, we can choose $\kappa' > 1$ close to 1, and $\alpha \in (0,1)$ close to 1, such that (6.17), (6.18) and (6.19) hold. Here is the place where we need to assume the rate κ_i of AC SL 3-folds L_i to be less than $-3/2$. As we have fixed the rate λ_i of the AC Calabi–Yau 3-fold Y_i to satisfy $\lambda_i \le -3$, and also we require $\kappa_i > \lambda_i + 1$ in Definition 5.5, so that assuming $\kappa_i < -3/2$ is still possible.

Therefore, we have shown that there exist $\kappa' > 1$ and $A_2 > 0$ such that $\|\sin\theta_t\|_{L^{6/5}} \le A_2 t^{\kappa'+3/2}$, $\|\sin\theta_t\|_{C^0} \le A_2 t^{\kappa'-1}$ and $\|d\sin\theta_t\|_{L^6} \le A_2 t^{\kappa'-3/2}$ for sufficiently small $t > 0$, i.e. (i) of Theorem 6.1 holds for N_t.

6.4 Desingularizations of N_0

This section gives the main result of the chapter, the desingularizations of SL 3-folds N_0 with conical singularities. The proof of it is based on an analytic existence theorem for SL 3-folds, Theorem 6.1, which is adapted from Joyce's result [33, Thm. 5.3]. We have already verified part (i) of Theorem 6.1 in §6.3, and it remains to check (ii) to (v) hold for the Lagrangian 3-folds N_t we constructed.

Theorem 6.13 *Suppose $(M_0, J_0, \omega_0, \Omega_0)$ is a compact Calabi–Yau 3-fold with finitely many conical singularities at $x_1, \ldots, x_n$ with rate $\nu > 0$ modelled on Calabi–Yau cones $V_1, \ldots, V_n$. Let $(Y_1, J_{Y_1}, \omega_{Y_1}, \Omega_{Y_1}), \ldots, (Y_n, J_{Y_n}, \omega_{Y_n}, \Omega_{Y_n})$ be AC Calabi–Yau 3-folds with rates $\lambda_1, \ldots, \lambda_n \leq -3$ modelled on the same Calabi–Yau cones $V_1, \ldots, V_n$.*

Suppose that there is a closed, homogeneous, trace-free (2,1)-form ξ_i of order -3 on V_i such that (4.2) holds for $i = 1, \ldots, n$, and that $\bigoplus_{i=1}^n [\xi_i] \in \bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})$ lies in $\rho_3(H^3(M_0', \mathbb{C}))$, where ρ_3 denotes the natural pull-back map $H^3(M_0', \mathbb{C}) \longrightarrow \bigoplus_{i=1}^n H^3(\Gamma_i, \mathbb{C})$.

Then Theorem 4.28 gives a family of Calabi–Yau 3-folds $(M_t, \tilde{J}_t, \tilde{\omega}_t, \tilde{\Omega}_t)$ for sufficiently small $t > 0$. Apply the diffeomorphism $\psi_t : M_t \longrightarrow M_t$ on M_t to get Calabi–Yau structures $(\hat{J}_t, \hat{\omega}_t, \hat{\Omega}_t)$, as in §6.2.

Let N_0 be a compact SL 3-fold in M_0 with the same conical singularities at $x_1, \ldots, x_n$ with rate $\mu \in (1, \nu+1)$ modelled on SL cones $C_1, \ldots, C_n$. Suppose $N_0 \setminus \{x_1, \ldots, x_n\}$ is connected. Let $L_1, \ldots, L_n$ be AC SL 3-folds in $Y_1, \ldots, Y_n$ with rates $\kappa_1 \in (\lambda_1+1, -3/2), \ldots, \kappa_n \in (\lambda_n+1, -3/2)$ modelled on the same SL cones $C_1, \ldots, C_n$.

Then there exists a family of compact nonsingular SL 3-folds $\hat{N}_t$ in $(M_t, \hat{J}_t, \hat{\omega}_t, \hat{\Omega}_t)$ for sufficiently small t, such that $\hat{N}_t$ is constructed by deforming the Lagrangian 3-fold N_t which is made by gluing L_i into N_0 at x_i for $i = 1, \ldots, n$.

Proof. First of all, we have to check that N_t satisfies the conditions in §6.1. Let us start with evaluating the integral $\int_{N_t} \mathrm{Im}(\hat{\Omega}_t)$. Calculation using (6.11) shows that

$$\int_{N_t} \mathrm{Im}(\hat{\Omega}_t) \;=\; O(t^{3+\tau})$$

for some $\tau \in (0, \kappa)$ when $\lambda_i < -3$. For the case $\lambda_i = -3$, we compute

$$\int_{N_t} \mathrm{Im}(\hat{\Omega}_t) \;=\; O(t^3 \log t),$$

using (6.12). As mentioned in §6.1, we can rescale the phase for $\hat{\Omega}_t$ by $\hat{\Omega}_t \mapsto e^{i\zeta_t} \hat{\Omega}_t$ such that $\int_{N_t} \mathrm{Im}(e^{i\zeta_t} \hat{\Omega}_t) = 0$. Thus we have $\sin\theta_t \mapsto \sin(\theta_t + \zeta_t) \approx \sin(\theta_t) + \zeta_t$, and the size for the term ζ_t is approximately given by the ratio between $\int_{N_t} \mathrm{Im}(\hat{\Omega}_t)$ and $\int_{N_t} \mathrm{Re}(\hat{\Omega}_t)$. Now since $\int_{N_t} \mathrm{Re}(\hat{\Omega}_t) \approx \mathrm{vol}(N_t) = O(1)$, the correction term ζ_t essentially contributes $O(t^{3+\tau})$ when

$\lambda_i < -3$, and $O(t^3 \log t)$ when $\lambda_i = -3$ to $\sin(\theta_t)$. As we have shown that $\|\sin(\theta_t)\|_{L^{6/5}} = O(t^{\kappa'+3/2})$ for some $\kappa' > 1$, then $\|\sin(\theta_t + \zeta_t)\|_{L^{6/5}} = O(t^{\kappa'+3/2}) + O(t^{3+\tau})$ when $\lambda_i < -3$ and $\|\sin(\theta_t + \zeta_t)\|_{L^{6/5}} = O(t^{\kappa'+3/2}) + O(t^3 \log t)$ when $\lambda_i = -3$. But for both cases the term $O(t^{\kappa'+3/2})$ will be dominant if κ' is close to 1. As a result, the rescaling of phases does not affect the $L^{6/5}$-estimates in (i) of Theorem 6.1 at all. For the terms $\|\sin(\theta_t)\|_{C^0}$ and $\|d\sin(\theta_t)\|_{L^6}$, calculation shows that the rescaling of phases does not affect the estimates as well.

As we have assumed $N_0 \setminus \{x_1, \ldots, x_n\}$ is connected, we can take the finite dimensional vector space W to be the space of constant functions, i.e. $W = \langle 1 \rangle$, as in §6.1.

Under our construction, N, h and $\acute{h}$ in §6.1 are replaced by N_t, $\hat{h}_t$ and $\acute{\hat{h}}_t$ respectively, and we thus need to show (i), (iii), (iv) and (v) hold using the metric $\hat{h}_t$ on N_t, and (ii) holds using the metric $\acute{\hat{h}}_t$ on T^*N_t. Basically the proof for (iii) and (iv) using the metric h_t and for (ii) using the metric $\acute{h}_t$ can be found in [33, Thm. 6.8], and the proof for (v) using the metric h_t is given in [33, Thm. 6.12]. Thus our approach to showing (ii)-(v) in Theorem 6.1 is to apply Theorems 6.8 and 6.12 in [33] together with the uniform equivalence between the metrics h_t and $\hat{h}_t$.

We have shown in §6.3 that given $\kappa > 0$, $\mu > 1$ and $\kappa_i < -3/2$ for $i = 1, \ldots, n$, there exists $\kappa' > 1$ and $A_2 > 0$ such that (i) of Theorem 6.1 holds for sufficiently small $t > 0$, measuring w.r.t. the metric $\hat{h}_t$.

For part (v), Theorem 6.12 in [33] shows that there exists $A_6 > 0$ such that (v) holds using the metric h_t. Note that the assumption on the connectedness of $N_0 \setminus \{x_1, \ldots, x_n\}$ is used here. The fact that $v \in L^6(N_t)$ follows from $L^2_1(N_t) \hookrightarrow L^6(N_t)$ by the Sobolev Embedding Theorem (Theorem 2.28). The idea of proving the inequality for the metric h_t on N_t for small t is to combine the Sobolev embedding inequalities on $N_0 \setminus \{x_1, \ldots, x_n\}$ and L_i. Now as h_t and $\hat{h}_t$ are uniformly equivalent metrics, so (v) is true for h_t if and only if it is true for $\hat{h}_t$. As a result, by making A_6 larger if necessary, (v) holds with the metric $\hat{h}_t$.

To deduce (iii) and (iv) for $\hat{h}_t$, we first apply Theorem 6.8 in [33] to show they are true for h_t for some $A_4, A_5 > 0$. The idea of which is to consider the behaviour of the metric h_t for small t. Since h_t is $t^2 g_{Y_i}|_{L_i}$ on H_i and on $\Psi_{C_i}(\Gamma(du_{t,i}))$ near $\Sigma_i \times \{tR'\}$ for each i, we have $\delta(t^2 g_{Y_i}|_{L_i}) = t\delta(g_{Y_i}|_{L_i})$ and $\|R(t^2 g_{Y_i}|_{L_i})\|_{C^0} = t^{-2}\|R(g_{Y_i}|_{L_i})\|_{C^0}$. For small $t > 0$, the dominant contributions to $\delta(h_t)$ and $\|R(h_t)\|_{C^0}$ come from $\delta(t^2 g_{Y_i}|_{L_i})$ and $\|R(t^2 g_{Y_i}|_{L_i})\|_{C^0}$ for some i, and hence we have $\delta(h_t) = O(t)$ and $\|R(h_t)\|_{C^0} = O(t^{-2})$. Now we prove (iii) and (iv) also hold, increasing A_4, A_5 if necessary, for $\hat{h}_t$ by showing the metrics $t^{-2}\hat{h}_t$ and $t^{-2}h_t$ are C^2-close w.r.t. $t^{-2}h_t$ (compare to the similar argument in the proof of Theorem 3.14). From the estimate we know using elliptic regularity on balls of radius $O(t)$, we have $|(d\eta_t)_{(s,x)}|_{T_x M_t}|_{g_t} = O(t^{\kappa})$ for $(s, x) \in S^1 \times M_t$. Here we do not need to use the improved estimate for $d\eta_t$ as in §6.3. Then we have $|(\nabla^{g_t})^l\,(d\eta_t)_{(s,x)}|_{T_x M_t}|_{g_t} = O(t^{\kappa - l})$ for $l \geq 0$, where ∇^{g_t} denotes the Levi-Civita connection of g_t. This implies

$$|(\nabla^{g_t})^l(\tilde{\Omega}_t - \Omega_t)|_{g_t} \;=\; O(t^{\kappa-l}) \;=\; |(\nabla^{g_t})^l(\tilde{\omega}_t - \omega_t)|_{g_t} \quad \text{for } l \geq 0,$$

and from Moser's argument in §6.3, we also have

$$|(\nabla^{g_t})^l(\hat{\Omega}_t - \tilde{\Omega}_t)|_{g_t} \;=\; O(t^{\kappa-l}) \;=\; |(\nabla^{g_t})^l(\hat{\omega}_t - \tilde{\omega}_t)|_{g_t} \quad \text{for } l \geq 0.$$

Putting together implies

$$|(\nabla^{g_t})^l(\hat{g}_t - g_t)|_{g_t} \;=\; O(t^{\kappa-l}) \quad \text{for } l \geq 0. \tag{6.20}$$

Denote by ∇^{h_t} the Levi-Civita connection of $h_t = g_t|_{N_t}$. Then we have

$$\nabla^{g_t}(\hat{g}_t - g_t)|_{N_t} \;=\; \nabla^{h_t}(\hat{h}_t - h_t) \;+\; \text{bilinear terms in } (\hat{g}_t - g_t)|_{N_t} \text{ and } T,$$

where T is the second fundamental form of N_t in M_t w.r.t. g_t. The largest contribution to $|T|_{h_t}$ comes from the second fundamental form T_{L_i} of L_i in Y_i w.r.t. $t^2 g_{Y_i}$. As $|T_{L_i}|_{g_{Y_i}|_{L_i}}$ is bounded on L_i, conformal rescaling then shows $|T_{L_i}|_{t^2 g_{Y_i}|_{L_i}} = O(t^{-1})$. Thus we have $|T|_{h_t} = O(t^{-1})$, and more generally $|\,(\nabla^{h_t})^l T\,|_{h_t} = O(t^{-l-1})$ for $l \geq 0$. The estimate for the l^{th} derivative of $\hat{h}_t - h_t$ then follows from the relation

$$\begin{aligned}\big|(\nabla^{g_t})^l(\hat{g}_t - g_t)|_{N_t}\big|_{h_t} \;=\;& \Big|(\nabla^{h_t})^l(\hat{h}_t - h_t)\Big|_{h_t} + O\Big(\sum |\,(\nabla^{h_t})^j T\,|_{h_t}\cdot\big|(\nabla^{g_t})^{l-j-1}(\hat{g}_t - g_t)|_{N_t}\big|_{h_t}\Big)\\ &+\; O\Big(\sum |\,(\nabla^{h_t})^{j_1} T\,|_{h_t}\cdot|\,(\nabla^{h_t})^{j_2} T\,|_{h_t}\cdot\big|(\nabla^{g_t})^{l-j_1-j_2-2}(\hat{g}_t - g_t)|_{N_t}\big|_{h_t}\Big)\\ &+\cdots+\; O\big(|T|_{h_t}\cdots|T|_{h_t}\cdot(\hat{g}_t - g_t)|_{N_t}|_{h_t}\big).\end{aligned}$$

Note that the terms $O(\cdot)$ all have size $O(t^{\kappa-l})$, and therefore by (6.20) we see that

$$\Big|(\nabla^{h_t})^l(\hat{h}_t - h_t)\Big|_{h_t} \;=\; O(t^{\kappa-l}) \quad \text{for } l \geq 0. \tag{6.21}$$

In particular, (6.21) shows $|\hat{h}_t - h_t|_{h_t}$, $t\,|\nabla^{h_t}(\hat{h}_t - h_t)|_{h_t}$ and $t^2\,|(\nabla^{h_t})^2(\hat{h}_t - h_t)|_{h_t}$ are all of size $O(t^\kappa)$. It follows that $|t^{-2}\hat{h}_t - t^{-2}h_t|_{t^{-2}h_t}$, $|\nabla^{t^{-2}h_t}(t^{-2}\hat{h}_t - t^{-2}h_t)|_{t^{-2}h_t}$ and $|(\nabla^{t^{-2}h_t})^2(t^{-2}\hat{h}_t - t^{-2}h_t)|_{t^{-2}h_t}$ are all of the same size $O(t^\kappa)$, where $\nabla^{t^{-2}h_t}$ and $|\cdot|_{t^{-2}h_t}$ are computed using $t^{-2}h_t$. Therefore the metrics $t^{-2}\hat{h}_t$ and $t^{-2}h_t$ are C^2-close w.r.t. $t^{-2}h_t$ for small t, and hence (iii) and (iv) are true for $\hat{h}_t$ as well.

We remain to show (ii), using the metric $\acute{\hat{h}}_t$ and the connection $\nabla^{\acute{\hat{h}}_t}$ on T^*N_t. Here we recall the construction of $\acute{\hat{h}}_t$ and $\nabla^{\acute{\hat{h}}_t}$, as in §6.1. Write $T(T^*N_t) = H_t \oplus V_t$, where $H_t \cong TN_t$ and $V_t \cong T^*N_t$ are the horizontal and vertical subbundles w.r.t. $\nabla^{\hat{h}_t}$, and define $\acute{\hat{h}}_t|_{H_t} = \hat{h}_t$ and $\acute{\hat{h}}_t|_{V_t} = \hat{h}_t^{-1}$. The connection $\nabla^{\acute{\hat{h}}_t}$ is given by the lift of the Levi-Civita connection $\nabla^{\hat{h}_t}$ of $\hat{h}_t$ in H_t, and by partial differentiation in V_t. Following the steps in Definition 6.7 in [33], we define Lagrangian neighbourhoods U_{N_t}, Ψ_{N_t} for N_t by gluing together the Lagrangian neighbourhoods U_{N_0}, Ψ_{N_0} for N_0 from Theorem 5.4and Lagrangian neighbourhoods U_{L_i}, Ψ_{L_i} for L_i from Theorem 5.6. The neighbourhood U_{N_t} is an open tubular neighbourhood of N_t in T^*N_t, and $\Psi_{N_t} : U_{N_t} \longrightarrow M_t$ is an embedding with $\Psi_{N_t}|_{N_t} = \mathrm{Id}$ and $\Psi^*_{N_t}(\omega_t) = \omega_{T^*N_t}$ where $\omega_{T^*N_t}$ is the canonical symplectic structure on T^*N_t. Recall that we have $c_t\hat{\omega}_t = \omega_t$, so $\Psi^*_{N_t}(c_t\hat{\omega}_t) = \omega_{T^*N_t}$. Now we define 3-forms β_t and $\hat{\beta}_t$ by $\beta_t = \Psi^*_{N_t}(\mathrm{Im}(\Omega_t))$ and $\hat{\beta}_t = \Psi^*_{N_t}(\mathrm{Im}(c_t^{3/2}\hat{\Omega}_t))$.

Using arguments in Theorem 6.8 in [33], we have $\|\,(\nabla^{\acute{h}_t})^l\beta_t\,\|_{C^0} \leq A_3\,t^{-l}$ for $l = 0,1,2,3$ on $\mathcal{B}_{A_1 t} \subset U_{N_t}$ for some $A_1, A_3 > 0$, where the norm is measuring w.r.t. $\acute{h}_t$. To prove (ii) in our case, we try to get from estimates on β_t to estimates on $\hat{\beta}_t$. Note that $\hat{\beta}_t$ and β_t are C^l-close w.r.t. $t^2 g_t$. This follows from $|(\nabla^{g_t})^l(\hat{\Omega}_t - \Omega_t)|_{g_t} \;=\; O(t^{\kappa-l}) \;=\; |(\nabla^{g_t})^l(\hat{\omega}_t - \omega_t)|_{g_t}$ for $l \geq 0$, which we have discussed earlier. Combining the C^{l+1}-closeness of $t^{-2}\hat{h}_t$ and $t^{-2}h_t$ from (6.21), we get a similar estimate for $\hat{\beta}_t$, using the metric $\acute{\hat{h}}_t$. Thus making A_1 smaller and A_3 larger if necessary, we obtain $\|\,(\nabla^{\acute{\hat{h}}_t})^l\hat{\beta}_t\,\|_{C^0} \leq A_3\,t^{-l}$ for $l = 0,1,2,3$ on $\mathcal{B}_{A_1 t} \subset U_{N_t}$, where the norm is computed using $\acute{\hat{h}}_t$.

The theorem now follows from Theorem 6.1 which shows that for sufficiently small $t > 0$ we can deform N_t to a nearby special Lagrangian 3-fold $\hat{N}_t = (\Psi_{N_t})_*(\Gamma(df_t))$ for some $f_t \in C^\infty(N_t)$ with $\int_{N_t} f_t dV_t = 0$ and $\|df_t\|_{C^0} \leq Kt^{\kappa'} \leq A_1 t$. This completes the proof of Theorem 6.13. □

6.5 Applications of the desingularization theory

We conclude with applying the results of §6.4 to two cases where the ambient Calabi–Yau 3-folds M_0 are taken to be (i) the Calabi–Yau 3-orbifold $T^6/\mathbb{Z}_3$, as described in §3.4.4, and (ii) the Calabi–Yau 3-fold with ordinary double points given by some explicit quintic 3-fold. In most cases, we shall take the SL 3-folds N_0 with conical singularities as the fixed point set of some antiholomorphic involutions on those Calabi–Yau 3-folds, whereas the AC SL 3-folds L_i will be taken from examples in §5.4 inside the corresponding AC Calabi–Yau 3-folds.

Example 6.14 Take the Calabi–Yau 3-fold M_0 with conical singularities to be the Calabi–Yau 3-orbifold $T^6/\mathbb{Z}_3$ given in [26, Example 6.6.3] and also in §3.4.4. Applying our desingularization result in Theorem 3.32, we can desingularize $T^6/\mathbb{Z}_3$ by gluing in AC Calabi–Yau 3-folds $K_{\mathbb{CP}^2}$ at the singular points, obtaining the crepant resolution of $T^6/\mathbb{Z}_3$. We shall use the notations in §3.4.4.

Now we produce examples of SL 3-folds N_0 with conical singularities in $T^6/\mathbb{Z}_3$ by using the fixed point set of an antiholomorphic isometric involution (see Proposition 2.25), a well-known way of producing special Lagrangians in Calabi–Yau manifolds. Recall that an antiholomorphic isometric involution of a Calabi–Yau manifold (M, J, ω, Ω) is a diffeomorphism $\sigma : M \longrightarrow M$ such that $\sigma^2 = \mathrm{Id}$, $\sigma^*(J) = -J$, $\sigma^*(\omega) = -\omega$, $\sigma^*(\Omega) = \bar{\Omega}$ and $\sigma^*(g) = g$, where g is the associated Calabi–Yau metric.

Let $\sigma_0 : T^6 \longrightarrow T^6$ be the complex conjugation given by

$$\sigma_0 : (z_1, z_2, z_3) + \Lambda \longmapsto (\bar{z}_1, \bar{z}_2, \bar{z}_3) + \Lambda,$$

which is well-defined as $\bar{\Lambda} = \Lambda$. The fixed points of σ_0 satisfy $z_j = \bar{z}_j + a_j + b_j\zeta$ for some $a_j, b_j \in \mathbb{Z}$. Write $z_j = x_j + y_j\zeta$ for $x_j, y_j \in \mathbb{R}$. It follows that $z_j = x_j + a_j\zeta$ and $b_j = 2a_j$, and hence the fixed point set of σ_0 is given by

$$\{(x_1, x_2, x_3) + \Lambda : x_j \in \mathbb{R}\},$$

and is then topologically a T^3.

Since $\sigma_0 \cdot \zeta \cdot \sigma_0^{-1} = \zeta^{-1}$, the map σ_0 on T^6 induces a conjugation σ on $T^6/\mathbb{Z}_3$ which is given by

$$\sigma : \mathbb{Z}_3 \cdot (z_1, z_2, z_3) + \Lambda \longmapsto \mathbb{Z}_3 \cdot (\bar{z}_1, \bar{z}_2, \bar{z}_3) + \Lambda.$$

Observe that σ_0 swaps ζT^3 and $\zeta^2 T^3$, and fixes T^3 as above. But in the orbifold level, T^3, ζT^3 and $\zeta^2 T^3$ are the same. Moreover, it is not hard to see that the map $T^6 \longrightarrow T^6/\mathbb{Z}_3$ is injective when restricted to $T^3 \subset T^6$, which means the image of T^3 in $T^6/\mathbb{Z}_3$ is homeomorphic to T^3. As a result, the fixed point set of σ is topologically a T^3, and is given by $(\mathbb{Z}_3 \cdot T^3)/\mathbb{Z}_3$, which is then our SL 3-fold N_0 in $M_0 = T^6/\mathbb{Z}_3$.

It is worth knowing how σ acts on those 27 orbifold singular points, and see how many of them are being fixed by σ, which will then be the singular points of the SL 3-fold N_0. It turns out that σ only fixes the point $\mathbb{Z}_3 \cdot (0,0,0) + \Lambda$, i.e. 0 in $T^6/\mathbb{Z}_3$, and swaps the other 26 points in pairs, for example, $\mathbb{Z}_3 \cdot (0, \frac{i}{\sqrt{3}}, \frac{2i}{\sqrt{3}}) + \Lambda \longleftrightarrow \mathbb{Z}_3 \cdot (0, \frac{2i}{\sqrt{3}}, \frac{i}{\sqrt{3}}) + \Lambda$. This means that we have constructed an SL 3-fold N_0, which is topologically a T^3, with one singular point at 0 in $T^6/\mathbb{Z}_3$, modelled on the SL cone $(\mathbb{Z}_3 \cdot \mathbb{R}^3)/\mathbb{Z}_3$ in $\mathbb{C}^3/\mathbb{Z}_3$.

To desingularize this N_0 we glue in at the singular point some appropriate pieces of AC SL 3-folds in the AC Calabi–Yau 3-folds $K_{\mathbb{CP}^2}$, the canonical bundle over $\mathbb{CP}^2$. As we have discussed in §3.4.4, the Calabi–Yau desingularization we get is the crepant resolution of orbifold $T^6/\mathbb{Z}_3$, and so the nonsingular SL 3-folds we constructed will sit inside this crepant resolution. Our first example of an AC SL 3-fold will be taken from Example 5.7 in which the real line bundle $K_{\mathbb{RP}^2}$ over $\mathbb{RP}^2$ is constructed as the fixed point set of an antiholomorphic isometric involution. By gluing this $K_{\mathbb{RP}^2}$ into $N_0 = (\mathbb{Z}_3 \cdot T^3)/\mathbb{Z}_3$ at the singular point, we obtain a nonsingular SL 3-fold in the crepant resolution of $M_0 = T^6/\mathbb{Z}_3$. Topologically, what we obtain will be a real blow-up of T^3 at a point, i.e. replacing a point by an $\mathbb{RP}^2$, which can also be interpreted as a $T^3 \# \mathbb{RP}^3$. As we have discussed in Example 5.7, the AC SL 3-fold $K_{\mathbb{RP}^2}$ has rate $\kappa = -\infty$, and in order to fit into our desingularization theorem, we could choose the rate for $K_{\mathbb{RP}^2}$ to be any $\kappa \in (-5, -3/2)$.

The next example of AC SL 3-folds in $K_{\mathbb{CP}^2}$ is given by Example 5.9. There we have constructed a family of SO(3)-invariant SL 3-folds L_c diffeomorphic to $S^2 \times \mathbb{R}$ which converges to two copies of the cone $((\mathbb{Z}_3 \cdot \mathbb{R}^3)/\mathbb{Z}_3)$ in $\mathbb{C}^3/\mathbb{Z}_3$. If we now take N_0 to be a connected double cover of T^3 so that we have two singular points in the same place in $M_0 = T^6/\mathbb{Z}_3$ (and $N_0 \setminus \{0\}$ is connected), we can desingularize this N_0 by doing the connected sum with an $S^2 \times \mathbb{R}$, obtaining an SL 3-fold which is homeomorphic to a $T^3 \# (S^1 \times S^2)$ in the crepant resolution of $T^6/\mathbb{Z}_3$. We mentioned in Example 5.9 that one possible L_c will be a double cover of $K_{\mathbb{RP}^2}$, which means the family of nonsingular SL 3-folds we constructed here will be deformations of a double cover of the SL 3-fold $T^3 \# \mathbb{RP}^3$ in the first example. Notice also that each L_c has rate -2, as discussed in Example 5.9, and so our desingularization theorem works.

Here is an alternative way of producing SL 3-folds in the crepant resolution of $T^6/\mathbb{Z}_3$. Split T^6 as $T^6 = T^4 \times T^2$. Let X be an SL 2-fold (but not a T^2) in T^4, or equivalently, a complex curve w.r.t. another complex structure on T^4. Take N_0 to be the image of $X \times S^1$ in $T^6/\mathbb{Z}_3$, and suppose N_0 passes through a singular point in $T^6/\mathbb{Z}_3$. By gluing in $K_{\mathbb{RP}^2}$ at the singular point, we get a real blow-up of N_0 at a point, homeomorphic to $N_0 \# \mathbb{RP}^3$. In these examples, we obtain SL 3-folds not from fixed points of an involution on the crepant resolution of $T^6/\mathbb{Z}_3$, as N_0 does not come from one.

Example 6.15 Let us take M_0 to be a quintic 3-fold (taken from [21, §18.2]) given by

$$M_0 = \big\{[z_0, z_1, z_2, z_3, z_4] \in \mathbb{CP}^4 : f(z_0, z_1, z_2, z_3, z_4) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4 = 0\big\}.$$

We see that M_0 is a hypersurface of $\mathbb{CP}^4$ defined by a homogeneous quintic polynomial f, and it has trivial canonical bundle by the adjunction formula, as we have seen in Example 2.4. Now assume the existence of a singular Calabi–Yau metric on M_0, so that M_0 is a Calabi–Yau 3-fold with conical singularities given by the ordinary double points, and hence our desingularization result can be applied to M_0.

By looking at the Jacobian of f, the singular points are given by $[\zeta^{a_0}, \zeta^{a_1}, \zeta^{a_2}, \zeta^{a_3}, \zeta^{a_4}]$ where $\zeta = e^{2\pi i/5}$ and $a_j \in \mathbb{Z}_5$ with $\sum a_j = 0$. Thus there are 125 singular points in total. To see why they are ordinary double points, let us first focus on the point $[1, 1, 1, 1, 1]$ and investigate the neighbourhood of it in M_0. Take $z_0 = 1$, and write $z_j = 1 + w_j$ for $j = 1, \dots, 4$. Then $f(z_0, z_1, z_2, z_3, z_4) = 1 + (1 + w_1)^5 + \cdots + (1 + w_4)^5 - 5(1 + w_1)\cdots(1 + w_4)$. We see that the constant and linear terms vanish, and the quadratic term is given by

$$q(w_1, w_2, w_3, w_4) = 10(w_1^2 + w_2^2 + w_3^2 + w_4^2) - 5(w_1w_2 + w_1w_3 + w_1w_4 + w_2w_3 + w_2w_4 + w_3w_4).$$

Thus M_0 is locally modelled on

$$\{(w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : q(w_1, w_2, w_3, w_4) = 0\}$$

near the point $[1, 1, 1, 1, 1]$. In matrix representation, q is given by

$$\begin{pmatrix} 10 & -5/2 & -5/2 & -5/2 \\ -5/2 & 10 & -5/2 & -5/2 \\ -5/2 & -5/2 & 10 & -5/2 \\ -5/2 & -5/2 & -5/2 & 10 \end{pmatrix}$$

and the eigenvalues are given by: 5/2, 25/2, 25/2, 25/2. It follows that under an appropriate linear change of coordinates, M_0 is locally modelled on

$$\{(w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0\},$$

i.e. the cone Q_0 described in Example 3.19 and §4.6, near $[1, 1, 1, 1, 1]$. Hence the point $[1, 1, 1, 1, 1]$ is an ordinary double point for M_0. Similar arguments show that all the singular points $[\zeta^{a_0}, \zeta^{a_1}, \zeta^{a_2}, \zeta^{a_3}, \zeta^{a_4}]$ are actually ordinary double points.

Consider the following 1-parameter family of quintic 3-folds:

$$M_\psi = \big\{[z_0, z_1, z_2, z_3, z_4] \in \mathbb{CP}^4 : z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0z_1z_2z_3z_4 = 0\big\}.$$

This family is well-studied, and a lot of work has been done on constructing "mirror Calabi–Yau 3-folds" of M_ψ, see [12]. It can be shown that when $\psi^5 = 1$, M_ψ is singular with 125 ordinary double points (our M_0 belongs to this sub-family). When $\psi^5 \neq 1$, M_ψ is nonsingular, with $h^{1,1}(M_\psi) = 1$ and $h^{2,1}(M_\psi) = 101$. Thus there are 101 families of complex deformations of M_0, and so we can deform M_0 smoothly to obtain a family of nonsingular Calabi–Yau 3-folds M_ψ with $\psi = 1 + \epsilon$ for some small nonzero $\epsilon \in \mathbb{C}$. In fact, M_ψ is what we shall get by gluing in some

Q_{ϵ_i} to each singular point x_i at M_0, and so M_ψ is a Calabi–Yau desingularization obtained in Chapter 4. In addition, $\psi = \infty$ corresponds to the singular variety $\{z_0 z_1 z_2 z_3 z_4 = 0\}$, which is the union of five $\mathbb{CP}^3$'s ($\{z_j = 0\}$).

Consider the following involution on M_0:

$$\sigma : [z_0, z_1, z_2, z_3, z_4] \longmapsto [\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_4, \bar{z}_3].$$

The fixed point set of σ is given by

$$\mathrm{Fix}(\sigma) = \{[x_0, x_1, x_2, x_3 + ix_4, x_3 - ix_4] \in \mathbb{CP}^4 : x_j \in \mathbb{R},\ f(x_0, x_1, x_2, x_3 + ix_4, x_3 - ix_4) = 0\}.$$

Among the singular points on M_0, five of which are fixed by σ: $p_1 = [1,1,1,1,1]$, $p_2 = [1,1,1,\zeta,\zeta^4]$, $p_3 = [1,1,1,\zeta^2,\zeta^3]$, $p_4 = [1,1,1,\zeta^3,\zeta^2]$, and $p_5 = [1,1,1,\zeta^4,\zeta]$. As before, we look at a neighbourhood of the point $p_1 = [1,1,1,1,1]$ by taking $z_0 = 1$ and $z_j = 1 + w_j$ for $j = 1, \ldots, 4$. Thus near p_1, the fixed point set $\mathrm{Fix}(\sigma)$ of σ is locally modelled on

$$\{(x_1, x_2, x_3 + ix_4, x_3 - ix_4) : q(x_1, x_2, x_3 + ix_4, x_3 - ix_4) = 0\},$$

where $q(x_1, x_2, x_3 + ix_4, x_3 - ix_4)$

$$= (x_1, x_2, x_3 + ix_4, x_3 - ix_4) \cdot \begin{pmatrix} 10 & -5/2 & -5/2 & -5/2 \\ -5/2 & 10 & -5/2 & -5/2 \\ -5/2 & -5/2 & 10 & -5/2 \\ -5/2 & -5/2 & -5/2 & 10 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 + ix_4 \\ x_3 - ix_4 \end{pmatrix}$$

$$= (x_1, x_2, x_3, x_4) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{pmatrix} \cdot \begin{pmatrix} 10 & -5/2 & -5/2 & -5/2 \\ -5/2 & 10 & -5/2 & -5/2 \\ -5/2 & -5/2 & 10 & -5/2 \\ -5/2 & -5/2 & -5/2 & 10 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= (x_1, x_2, x_3, x_4) \cdot \begin{pmatrix} 10 & -5/2 & -5 & 0 \\ -5/2 & 10 & -5 & 0 \\ -5 & -5 & 15 & 0 \\ 0 & 0 & 0 & -25 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

This matrix has three positive and one negative eigenvalues, and hence by performing a linear change of coordinates, the fixed point set of σ is locally modelled on

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0\} \tag{6.22}$$

near p_1, i.e. two cones on S^2 (or two copies of $\mathbb{R}^3$) meeting at 0. This is also the case for each of the other four singular points $p_2, \ldots, p_5$.

Let us look at the fixed point set $\mathrm{Fix}(\sigma)$ of σ restricted to the subspaces $\{z_0 = z_1 = z_2\}$, that is, we consider

$$\begin{aligned} U = \{&[1,1,1,x+iy,x-iy] : x,y \in \mathbb{R},\ 2x^5 - 20x^3y^2 + 10xy^4 - 5x^2 - 5y^2 + 3 = 0\} \\ &\cup \{[0,0,0,x+iy,x-iy] : x,y \in \mathbb{R},\ y \neq 0,\ 2x^5 - 20x^3y^2 + 10xy^4 = 0\}. \end{aligned}$$

Calculation shows that the second bit of U consists of five points in $\mathbb{CP}^4$: $a = [0,0,0,i,-i]$, $b = [0,0,0,1+i\sqrt{5+2\sqrt{5}},\, 1-i\sqrt{5+2\sqrt{5}}]$, $c = [0,0,0,1-i\sqrt{5+2\sqrt{5}},\, 1+i\sqrt{5+2\sqrt{5}}]$, $d = [0,0,0,1+i\sqrt{5-2\sqrt{5}},\, 1-i\sqrt{5-2\sqrt{5}}]$ and $e = [0,0,0,1-i\sqrt{5-2\sqrt{5}},\, 1+i\sqrt{5-2\sqrt{5}}]$. For the first bit, we illustrate it on $\mathbb{R}^2$:

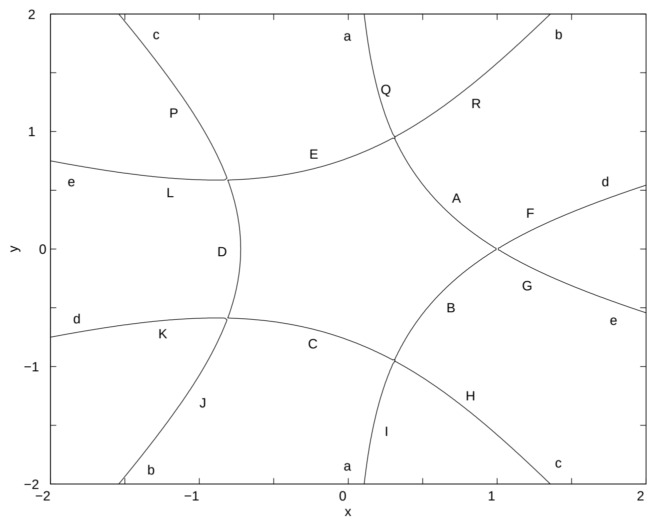

Figure 6.1: Sketch of $2x^5 - 20x^3y^2 + 10xy^4 - 5x^2 - 5y^2 + 3 = 0$ on $\mathbb{R}^2$

The above figure gives a representation of U on $\mathbb{R}^2$, with five asymptotic lines $\{x = 0\}$, $\{y = \sqrt{1+\frac{2\sqrt{5}}{5}}\,x\}$, $\{y = -\sqrt{1+\frac{2\sqrt{5}}{5}}\,x\}$, $\{y = \sqrt{1-\frac{2\sqrt{5}}{5}}\,x\}$ and $\{y = -\sqrt{1-\frac{2\sqrt{5}}{5}}\,x\}$ corresponding to the five points a,b,c,d,e in $\mathbb{CP}^4$ respectively. We see that the graph is connected and contains all five ordinary double points $p_1,\ldots,p_5$, and so does U.

Now we take N_0 as the connected component of $\mathrm{Fix}(\sigma)$ containing $p_1 = [1,1,1,1,1]$. Then $U \subset N_0$, and so N_0 contains $p_1,\ldots,p_5$ as well. Thus N_0 is a singular SL 3-fold with conical singularities $p_1,\ldots,p_5$ modelled on cones of the form (6.22). As before, we need to know whether $N_0' = N_0 \setminus \{p_1,\ldots,p_5\}$ is connected or not in order to apply our desingularization theorem. Observe from Figure 6.1 that the graph with $p_1,\ldots,p_5$ removed has 15 connected components, but we have to include the five points a,b,c,d,e "at infinity", which implies $U \setminus \{p_1,\ldots,p_5\}$ has 10 connected components. Near each singular point, N_0 is modelled on the cone given by (6.22), which is a two-sided cone on S^2 $\big(\{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1^2+x_2^2+x_3^2-x_4^2 = 0,\ x_4 > 0\} \cup \{0\} \cup \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1^2+x_2^2+x_3^2-x_4^2 = 0,\ x_4 < 0\}\big)$ such that the sign of x_4 determines the side of the cone. Thus the point A is actually connected to F in N_0' as they lie in the same side of the two-sided cone at p_1. We now show by using an informal argument how to connect the 10 components of U in N_0'. From Figure 6.1, the point A is connected to F, as we mentioned above, then F is connected to K though the infinity point c, and K is connected to D as they lie in the same side of the two-sided cone at p_4, and so on. We can then trace a path:

$$A \to F \xrightarrow{d} K \to D \to L \xrightarrow{e} G \to B \to H \xrightarrow{c} P \to E \to Q \xrightarrow{a} I \to C \to J \xrightarrow{b} R \to A.$$

We see that the 10 components of $U \setminus \{p_1,\ldots,p_5\}$ are connected in N_0', and hence N_0' is connected, which is what we want.

To desingularize N_0 we take SO(3)-invariant AC SL 3-folds in Q_ϵ in Example 5.12 and glue them into N_0 at the singular points $p_1, \dots, p_5$. Recall in that example all the AC SL 3-folds we constructed are asymptotic to the (one-sided) cones C_0, C_1, C_2, C_3, or to unions of cones $C_k \cup C_l$ in Q_0. The local model of N_0 near each singular point is given by the two-sided cone of the form (6.22). Thus at each p_j, we can regard the cone as either the union $C_0 \cup C_2$ or $C_1 \cup C_3$. Let us fix it to be $C_0 \cup C_2$ at each p_j.

(i) $\mathrm{Im}(\sqrt{\epsilon}) > 0$, $\mathrm{Re}(\sqrt{\epsilon}) > 0$

In this case, we have only one option for desingularizing N_0. We glue in at each p_j two disjoint union of AC SL $\mathbb{R}^3$'s B_1, B_4, ending at the points $(0, 0, 0, \sqrt{\epsilon})$, $(0, 0, 0, -\sqrt{\epsilon})$ in Q_ϵ and asymptotic to C_0, C_2 respectively.

(ii) $\mathrm{Im}(\sqrt{\epsilon}) = 0$, $\mathrm{Re}(\sqrt{\epsilon}) > 0$

We specify this to the case $\epsilon = 1$. Again, we have only one option for desingularizing N_0, which is given by gluing in at each p_j two disjoint union of AC SL $\mathbb{R}^3$'s A_1, A_4, ending at the points $(0, 0, 0, 1)$, $(0, 0, 0, -1)$ in Q_1 and asymptotic to C_0, C_2 respectively.

(iii) $\mathrm{Im}(\sqrt{\epsilon}) < 0$, $\mathrm{Re}(\sqrt{\epsilon}) > 0$

This time we have five varieties of choice for the desingularization. The options are: (1) $B_2' \cup B_3'$, intersecting at $(0, 0, 0, \sqrt{\epsilon})$; (2) $B_5' \cup B_6'$, intersecting at $(0, 0, 0, -\sqrt{\epsilon})$; (3) disjoint union of B_2' and B_5'; (4) disjoint union of B_3' and B_6'; and (5) a family of AC SL $S^2 \times \mathbb{R}$'s given by the flow lines between the region bounded by B_2', B_3', B_5' and B_6'. Thus at each p_j, we have four different choices of AC SL $\mathbb{R}^3 \cup \mathbb{R}^3$'s corresponding to (1)-(4) to glue in, or we can choose one out of a family of AC SL $S^2 \times \mathbb{R}$'s given in (5) to desingularize the p_j's in N_0. As a result, we get different possibilities for the topology of $\hat{N}_t$, and it would be interesting to consider how the classes of SL 3-folds $\hat{N}_t$ we construct vary as ϵ goes round the loop.

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