Moduli Spaces of Sheaves on Toric Varieties

Martijn Kool
Lincoln College
Hilary Term 2010

Thesis submitted for the degree of Doctor of Philosophy at the University of Oxford

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Abstract

Extending work of Klyachko and Perling, we develop a combinatorial description of pure equivariant sheaves of any dimension on an arbitrary nonsingular toric variety X. Using GIT, this allows us to construct explicit moduli spaces of pure equivariant sheaves on X corepresenting natural moduli functors (similar to work of Payne in the case of equivariant vector bundles). The action of the algebraic torus on X lifts to the moduli space of all Gieseker stable sheaves on X and we express its fixed point locus explicitly in terms of moduli spaces of pure equivariant sheaves on X. One of the problems arising is to find an equivariant line bundle on the side of the GIT problem, which precisely recovers Gieseker stability. In the case of torsion free equivariant sheaves, we can always construct such equivariant line bundles. As a by-product, we get a combinatorial description of the fixed point locus of the moduli space of μ-stable reflexive sheaves on X.

As an application, we study generating functions of Euler characteristics of moduli spaces of μ-stable torsion free sheaves on X, where X is in addition a surface. We obtain a general expression for such generating functions in terms of Euler characteristics of moduli spaces of stable configurations of linear subspaces. The expression holds for any choice of X, ample divisor, rank and first Chern class. It can be further simplified in examples, which allows us to compute some new and known generating functions (due to Göttsche, Klyachko and Yoshioka). In general, these generating functions depend on choice of stability condition, enabling us to study wall-crossing phenomena and relate to work of Göttsche and Joyce. As another application, we compute some Euler characteristics of moduli spaces of μ-stable pure dimension 1 sheaves on $\mathbb{P}^2$. These can be seen as genus zero Gopakumar–Vafa invariants of $K_{\mathbb{P}^2}$ by a general conjecture of Katz.

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Preface

One of the great successes of algebraic geometry is its capability of constructing moduli spaces of geometric objects. Whenever one studies a specific type of geometric objects, such as certain varieties, schemes, morphisms, vector bundles, sheaves et cetera, one considers the set of isomorphism classes of such geometric objects and one would like to put a "natural" geometric structure on this set. Such a problem is called a moduli problem and a solution to it is called a moduli space of the type of geometric objects under consideration. "Natural" means one should have a notion of families of the kind of geometric objects under consideration and the geometric structure should be compatible with this notion. This leads to the elegant concepts of fine and coarse moduli spaces. Grothendieck's Quot scheme provides a powerful tool for constructing fine moduli spaces for certain moduli problems. Unfortunately, many moduli problems do not have fine moduli spaces. However, using in addition tools from geometric invariant theory (GIT), one can often construct coarse moduli spaces for a moduli problem. These techniques have become classical in algebraic geometry. The moduli spaces constructed by these methods are schemes. A more modern viewpoint is to construct moduli spaces as stacks. In this case, the moduli spaces remember the automorphism groups of the geometric objects under consideration. Moduli spaces of geometric objects tend to be unwieldy. A moduli space is said to satisfy Murphy's Law when any singularity type of finite type over $\mathbb{Z}$ appears on it. As Vakil has shown [Vak], many known moduli spaces satisfy Murphy's Law, even for moduli problems of perfectly normal types of geometric objects such as nonsingular curves in projective space, smooth surfaces with very ample canonical bundle and Gieseker stable sheaves. Our main interest is in the moduli space of Gieseker stable sheaves.

Another great feature of algebraic geometry is its capability of dealing with explicit examples. Important results in modern day high-energy physics, in particular string theory, have been established by means of explicit computations in algebraic geometry. String theory is currently the most promising attempt to unite general relativity and the standard model. Quantising the fundamental building blocks of string theory, i.e. strings as opposed to point particles, requires a spacetime with more than the four conventional dimensions. In one popular version, anomalies of the superconformal algebra cancel in the case spacetime is the product of a conventional four-dimensional spacetime and a six-dimensional space having the structure of a Calabi–Yau threefold. String theory has led to many discoveries about Calabi–Yau threefolds such as Mirror Symmetry. Mirror Symmetry suggests that for (many) Calabi–Yau threefolds X, there should exist a mirror Calabi–Yau threefold X° such that X and X° have the same superconformal field theory. Hence, geometrically very different Calabi–Yau threefolds can give rise to the same physical reality. Mathematically, this implies that for (many) Calabi–Yau threefolds X, there should exist a mirror Calabi–Yau threefold X° such that their Hodge diamonds are related by $h^{p,q}(X) = h^{3-p,q}(X^\circ)$. Evidence for this consequence has been established by considering large numbers of examples of Calabi–Yau threefolds coming from constructions in toric geometry [CK, Ch. 4]. In fact, toric geometry has provided an important laboratory for testing many deep predictions of string theory in specific examples.

The easiest example of an algebraic variety is affine space. Projective space can be seen as a particular way of gluing several affine spaces together. Toric geometry in some sense generalises this procedure. One starts with a finite amount of combinatorial data, i.e. a fan in a lattice, and this data gives you a recipe for gluing several copies of affine space together (in the nonsingular case). The resulting varieties are called toric varieties and come equipped with the regular action of an algebraic torus. Many topological and geometric properties of toric varieties are determined in a purely combinatorial way by their fan and lattice. This makes toric varieties very combinatorial objects suitable for explicit computations. Although the collection of toric varieties should not be considered representative of the collection of all algebraic varieties, e.g. toric varieties are always rational, they provide a perfect testing ground for studying complicated questions in algebraic geometry. In addition, their explicit nature makes them attractive to physicists, who often want to do explicit computations.

The central objects of study of this thesis are moduli spaces of Gieseker stable sheaves on nonsingular projective toric varieties. Along the way we will also study moduli spaces of μ-stable reflexive sheaves on nonsingular projective toric varieties. The regular action of the algebraic torus lifts to these moduli spaces and one of our main goals is to give a combinatorial description of the fixed point loci of these moduli spaces. Extending work of Klyachko [Kly1], [Kly2], [Kly3], [Kly4] and Perling [Per1], [Per2], we develop a combinatorial description of pure equivariant sheaves on an arbitrary nonsingular toric variety X. Using GIT and this combinatorial description, we construct explicit coarse moduli spaces of pure equivariant sheaves on X corepresenting natural moduli functors (similar to techniques used in the case of equivariant vector bundles by Payne [Pay]). We show how the fixed point locus of the moduli space of all Gieseker stable sheaves on X can be expressed in terms of the explicit moduli spaces of pure equivariant sheaves on X. One of the problems arising is to find an equivariant line bundle on the side of the GIT problem, which precisely recovers Gieseker stability. In the case of torsion free equivariant sheaves, we construct ample equivariant line bundles with this property. As a by-product, we construct particularly simple ample equivariant line bundles recovering μ-stability for reflexive equivariant sheaves and give a combinatorial description of the fixed point locus of the moduli space of μ-stable reflexive sheaves on X. These form the topics of the first chapter[1].

Explicit knowledge about moduli spaces of geometric objects can be important for computing invariants associated to the moduli space. An interesting example is to com-

[1]Most of the material of chapter 1 can be found in [Koo1].

pute motivic invariants such as the virtual Hodge polynomial, the virtual Poincaré polynomial or the Euler characteristic of a moduli space. These invariants give information about the topology of the moduli space, although they are in general not invariant under deformations of the underlying algebraic variety on which the geometric objects live. Another important example are the generalised Donaldson–Thomas invariants and BPS invariants of a Calabi–Yau threefold X as recently introduced by Joyce and Song [JS]. They are constructed using moduli spaces of Gieseker or μ-semistable sheaves on X (realised as Artin stacks). Generalised Donaldson–Thomas invariants and BPS invariants are invariant under deformations of X and therefore provide deep geometric invariants relevant to both mathematics and physics. In string theory, B-branes on X corresponding to BPS states are described by (semi)stable coherent sheaves on X. Intuitively, BPS invariants of X count the number of such states on X and therefore are conjectured to be integers. This is known as the Integrality Conjecture for BPS invariants. In general, moduli spaces of Gieseker semistable sheaves depend on a choice of ample line bundle on the underlying algebraic variety on which the sheaves live. I.e. they depend on a choice of stability condition. As a consequence, motivic and geometric invariants associated to them also depend on this choice of stability condition. Variations of the stability condition leads to nice wall-crossing formulae. Generating functions of motivic and geometric invariants are often expected and sometimes known to have interesting modular properties. These modular properties are predicted by string theory. For example, roughly, the S-Duality Conjecture predicts certain generating functions of Euler characteristics of moduli spaces of sheaves on surfaces to be modular forms.

Localisation is an important method for computing motivic and geometric invariants in the case the underlying algebraic variety on which the geometric objects live is toric. Morally, one should be able to use the torus action to reduce the computation to a problem involving the fixed point locus. For example, the Euler characteristic of a quasi-projective variety with regular action of an algebraic torus is just the Euler characteristic of the fixed point locus. As an application of the combinatorial description of fixed point loci of moduli spaces of sheaves on toric varieties derived in chapter 1, we study generating functions of Euler characteristics of moduli spaces of μ-stable torsion free sheaves on nonsingular complete toric surfaces. We express such a generating function in terms of Euler characteristics of moduli spaces of stable configurations of linear subspaces. The expression holds for any choice of nonsingular complete toric surface, ample divisor, rank and first Chern class. The expression can be further simplified in examples. In the rank 1 case, we recover a well-known result derived for general nonsingular projective surfaces by Göttsche [Got1]. In the rank 2 case on the projective plane $\mathbb{P}^2$, we compare our result to work of Klyachko [Kly4] and Yoshioka [Yos]. In the rank 2 case on $\mathbb{P}^1 \times \mathbb{P}^1$ or any Hirzebruch surface $\mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 1}$), we find a formula with explicit dependence on choice of stability condition, which allows us to study wall-crossing phenomena. We compare our expression to results by Göttsche [Got2] and Joyce [Joy2] and perform various consistency checks. In the rank 3 case on the projective plane $\mathbb{P}^2$, we obtain an explicit but complicated expression, which allows for numerical computations. The general combinatorial description of fixed point loci of moduli spaces of sheaves on toric varieties of chapter 1 is more widely applicable. We will discuss examples of computations of generating functions of Euler characteristics of moduli spaces of pure dimension 1 sheaves on the projective plane $\mathbb{P}^2$. These Euler characteristics can be seen as genus zero Gopakumar–Vafa invariants of the canonical bundle $K_{\mathbb{P}^2}$ by a general conjecture of Katz [Kat]. Our examples are consistent with Katz' Conjecture. These form the topics of the second chapter[2].

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[2]Most of the material of this chapter can be found in [Koo2].

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Chapter 1

Fixed Point Loci of Moduli Spaces of Sheaves on Toric Varieties

Vakil has shown that the moduli space of Gieseker stable sheaves satisfies Murphy's Law, meaning every singularity type of finite type over $\mathbb{Z}$ appears on the moduli space [Vak]. Hence the moduli space $\mathcal{M}_P^s$ of Gieseker stable sheaves with Hilbert polynomial P on a projective variety X with ample line bundle $\mathcal{O}_X(1)$ can become very complicated. Now assume X is a nonsingular projective toric variety with torus T. We can lift the action of T on X to an action of T on the moduli space $\mathcal{M}_P^s$. One of the goals of this chapter is to find a combinatorial description of the fixed point locus $(\mathcal{M}_P^s)^T$ using techniques of toric geometry.

Klyachko has given a combinatorial description of equivariant vector bundles and, more generally, reflexive equivariant and torsion free equivariant sheaves on a nonsingular toric variety [Kly1], [Kly2], [Kly3], [Kly4]. This description gives a relatively easy way to compute Chern characters and sheaf cohomology of such sheaves. Klyachko's work has been reconsidered and extended by Knutson and Sharpe in [KS1], [KS2]. They sketch how his combinatorial description can be used to construct moduli spaces of equivariant vector bundles and reflexive equivariant sheaves. Perling has given a general description of equivariant quasi-coherent sheaves on toric varieties in [Per1], [Per2]. He gives a detailed study of the moduli space of rank 2 equivariant vector bundles on nonsingular toric surfaces in [Per3]. A systematic construction of the moduli spaces of equivariant vector bundles on toric varieties has been given by Payne [Pay]. He considers families of equivariant vector bundles on toric varieties and shows the moduli space of rank 3 equivariant vector bundles on toric varieties satisfies Murphy's Law.

In the current chapter, we will present a combinatorial description of pure equivariant sheaves on nonsingular toric varieties (Theorem 1.1.12), generalising the known combinatorial description of torsion free equivariant sheaves due to Klyachko [Kly4]. Using this combinatorial description, we construct coarse moduli spaces of pure equivariant sheaves on nonsingular projective toric varieties (Theorem 1.2.13), corepresenting natural moduli functors. For this, we develop an explicit description of families of pure equivariant sheaves on nonsingular projective toric varieties (Theorem 1.2.9), analogous to Payne's description in the case of families of equivariant vector bundles [Pay]. The moduli spaces of pure equivariant sheaves on nonsingular projective toric varieties are constructed using GIT. It is important to note that these moduli spaces are explicit and combinatorial in nature, which makes them suitable for computations. We are interested in the case where GIT stability coincides with Gieseker stability, which is the natural notion of stability for coherent sheaves. Consequently, we would like the existence of an equivariant line bundle in the GIT problem, which precisely recovers Gieseker stability. In the case of reflexive equivariant sheaves and μ-stability, some aspects of this issue are discussed in [KS1], [KS2] and [Kly4]. We construct ample equivariant line bundles matching GIT and Gieseker stability for torsion free equivariant sheaves in general (Theorem 1.2.22). Subsequently, we consider the moduli space $\mathcal{M}_P^s$ of all Gieseker stable sheaves with (arbitrary) fixed Hilbert polynomial P on a nonsingular projective toric variety X with torus T and ample line bundle $\mathcal{O}_X(1)$. We lift the action of the torus T to $\mathcal{M}_P^s$, describe the closed points of the fixed point locus $(\mathcal{M}_P^s)^T$ and study the difference between invariant and equivariant simple sheaves. We study deformation theoretic aspects of equivariant sheaves and describe the fixed point locus $(\mathcal{M}_P^s)^T$ in terms of moduli spaces of pure equivariant sheaves on X.

Theorem 1.0.1 (Corollary 1.3.10)**.** *Let X be a nonsingular projective toric variety. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and let P be a choice of Hilbert polynomial of degree* $\dim(X)$*. Then there is a canonical isomorphism*

$$(\mathcal{M}_P^s)^T \cong \coprod_{\vec{\chi} \in (\mathcal{X}_P^0)^{gf}} \mathcal{M}_{\vec{\chi}}^{0,s}.$$

Here the right hand side of the equation is a disjoint union of moduli spaces of torsion free equivariant sheaves on X. It is important to note that the moduli spaces on the right hand side are explicit and combinatorial in nature and their construction is very different from the construction of $\mathcal{M}_P^s$, which makes use of Quot schemes and requires boundedness results [HL, Ch. 1–4]. The theorem gives us a combinatorial description of $(\mathcal{M}_P^s)^T$. Explicit knowledge of $(\mathcal{M}_P^s)^T$ is useful for computing invariants associated to $\mathcal{M}_P^s$, e.g. the Euler characteristic of $\mathcal{M}_P^s$, using localisation techniques. We exploit these ideas in the next chapter in the case X is a nonsingular complete toric surface to obtain expressions for generating functions of Euler characteristics of moduli spaces of μ-stable torsion free sheaves on X. These computations can be used to study wall-crossing phenomena, i.e. study the dependence of these generating functions on choice of ample line bundle $\mathcal{O}_X(1)$ on X. We will mention some of these results in this chapter without further details. Most of the formulation and proof of the above theorem holds similarly for P of any degree. The only complication arising in the general case is to find an equivariant line bundle in the GIT problem, which precisely reproduces Gieseker stability. Currently, we can only achieve this in full generality for P of degree $\dim(X)$, i.e. for torsion free sheaves, though we will develop the rest of the theory for arbitrary P (Theorem 1.3.9). As a by-product, we will construct moduli spaces of μ-stable reflexive equivariant sheaves on nonsingular projective toric varieties (Theorem 1.3.14) and express the fixed point loci of moduli spaces of μ-stable reflexive sheaves on nonsingular projective toric varieties in terms of them (Theorem 1.3.15). In the case of reflexive equivariant sheaves, we will construct particularly simple ample equivariant line bundles in the GIT problem, which precisely recover μ-stability.

1.1 Pure Equivariant Sheaves on Toric Varieties

In this section, we will give a combinatorial description of pure equivariant sheaves on nonsingular toric varieties. After recalling the notion of an equivariant and a pure sheaf, we will give the combinatorial description in the affine case. Subsequently, we will pass to the general case. Our main tool will be Perling's notion of σ-families. In order to avoid cumbersome notation, we will first treat the case of irreducible support in detail and discuss the general case at the end.

We recall the notion of a G-equivariant sheaf.

Definition 1.1.1. Let G be an affine algebraic group acting regularly on a scheme[1] X of finite type over k. Denote the group action by $\sigma : G \times X \longrightarrow X$, denote projection to the second factor by $p_2 : G \times X \longrightarrow X$ and denote multiplication on G by $\mu : G \times G \longrightarrow G$. Moreover, denote projection to the last two factors by $p_{23} : G \times G \times X \longrightarrow G \times X$. Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules on X. A *G-equivariant structure* on $\mathcal{E}$ is an isomorphism $\Phi : \sigma^*\mathcal{E} \longrightarrow p_2^*\mathcal{E}$ such that

$$(\mu \times 1_X)^*\Phi = p_{23}^*\Phi \circ (1_G \times \sigma)^*\Phi.$$

This equation is called *the cocycle condition.* A sheaf of $\mathcal{O}_X$-modules endowed with a G-equivariant structure is called a *G-equivariant sheaf.* A *G-equivariant morphism* from a G-equivariant sheaf $(\mathcal{E}, \Phi)$ to a G-equivariant sheaf $(\mathcal{F}, \Psi)$ is a morphism $\theta : \mathcal{E} \longrightarrow \mathcal{F}$ of sheaves of $\mathcal{O}_X$-modules such that $p_2^*\theta \circ \Phi = \Psi \circ \sigma^*\theta$. We denote the k-vector space of G-equivariant morphisms from $(\mathcal{E}, \Phi)$ to $(\mathcal{F}, \Psi)$ by $G\text{-Hom}(\mathcal{E}, \mathcal{F})$. $\oslash$

Using the above definition, we can form the k-linear additive category of G-equivariant sheaves which we will denote by $\mathrm{Mod}^G(X)$. Similarly, one can construct the categories of G-equivariant (quasi-)coherent sheaves $\mathrm{Qco}^G(X)$ and $\mathrm{Coh}^G(X)$. These are abelian categories and $\mathrm{Qco}^G(X)$ has enough injectives [Toh, Ch. V].

[1]In this chapter, all schemes will be schemes over k an algebraically closed field of characteristic 0, unless stated otherwise.

Now let X be a toric variety, so $G = T$ is the algebraic torus[2]. Denote the fan by Δ, the character group by $M = X(T)$ and the group of one-parameter subgroups by N (so $M = N^\vee$ and we have a natural pairing between the two lattices $\langle \cdot, \cdot \rangle : M \times N \longrightarrow \mathbb{Z}$). The elements σ of Δ are in bijective correspondence with the invariant affine open subsets U_σ of X. In particular, for a strongly convex rational polyhedral cone $\sigma \in \Delta$ (which lies in the lattice N) we have $U_\sigma = \mathrm{Spec}(k[S_\sigma])$, where $k[S_\sigma]$ is the semigroup algebra associated to the semigroup S_σ defined by

$$\begin{aligned} &S_\sigma = \sigma^\vee \cap M, \\ &\sigma^\vee = \{u \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}. \end{aligned}$$

We will denote the element of $k[M]$ corresponding to $m \in M$ by $\chi(m)$ and write the group operation on $k[M]$ multiplicatively, so $\chi(m)\chi(m') = \chi(m + m')$. We obtain the following M-graded k-algebras

$$\Gamma(U_\sigma, \mathcal{O}_X) = \bigoplus_{m \in S_\sigma} k\chi(m) \subset \bigoplus_{m \in M} k\chi(m) = \Gamma(T, \mathcal{O}_X). \tag{1.1}$$

There is a regular action of T on $\Gamma(U_\sigma, \mathcal{O}_X)$. For $t \in T$ a closed point and $f : U_\sigma \longrightarrow k$ a regular function, one defines

$$(t \cdot f)(x) = f(t \cdot x).$$

The regular action of T on U_σ induces a decomposition into weight spaces (Complete Reducibility Theorem [Per1, Thm. 2.30]). This decomposition coincides precisely with the decomposition in equation (1.1). More generally, if $(\mathcal{E}, \Phi)$ is an equivariant quasi-coherent sheaf on X, there is a natural regular action of T on $\Gamma(U_\sigma, \mathcal{E})$ [Per1, Subsect. 2.2.2, Ch. 4]. This action can be described as follows. For any closed point $t \in T$, let $i_t : X \longrightarrow T \times X$ be the inclusion and define $\Phi_t = i_t^* \Phi : t^*\mathcal{E} \longrightarrow \mathcal{E}$. From the cocycle condition, we obtain $\Phi_{st} = \Phi_t \circ t^*\Phi_s$ for all closed points $s, t \in T$ (see Definition 1.1.1). Also, for $f \in \Gamma(U_\sigma, \mathcal{E})$

[2] When dealing with toric geometry, we use the notation of the standard reference [Ful].

we have a canonically lifted section $t^*f \in \Gamma(U_\sigma, t^*\mathcal{E})$, which allows us to define

$$t \cdot f = \Phi_t(t^*f).$$

Again, we get a decomposition into weight spaces [Per1, Thm. 2.30]

$$\Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{m \in M} \Gamma(U_\sigma, \mathcal{E})_m.$$

In particular, for $\mathcal{E} = \mathcal{O}_X$, we obtain $\Gamma(U_\sigma, \mathcal{O}_X)_m = k\chi(m)$ if $m \in S_\sigma$ and $\Gamma(U_\sigma, \mathcal{O}_X)_m = 0$ otherwise. It is not difficult to deduce from the previous discussion that the functor $\Gamma(U_\sigma, -)$ induces an equivalence between the category of equivariant quasi-coherent (resp. coherent) sheaves on U_σ and the category of M-graded (resp. finitely generated M-graded) S_σ-modules [Per1, Prop. 2.31].

Before we proceed to use the previous notions to give Perling's characterisation of equivariant quasi-coherent sheaves on affine toric varieties in terms of σ-families, we remind the reader of the notion of a pure sheaf.

Definition 1.1.2. Let $\mathcal{E} \neq 0$ be a coherent sheaf on a scheme X of finite type over k. The sheaf $\mathcal{E}$ is said to be *pure* of dimension d if $\dim(\mathcal{F}) = d$ for any coherent subsheaf $0 \neq \mathcal{F} \subset \mathcal{E}$. Here the *dimension* of a coherent sheaf $\mathcal{F}$ is defined to be the dimension of the support $\mathrm{Supp}(\mathcal{F})$ of the coherent sheaf $\mathcal{F}$. In the case X is in addition integral, we also refer to a pure sheaf on X of dimension $\dim(X)$ as a *torsion free* sheaf on X. $\oslash$

For future purposes, we state the following easy results.

Proposition 1.1.3. *Let X be a scheme of finite type over k, $\{U_i\}$ an open cover of X and $\mathcal{E} \neq 0$ a coherent sheaf on X. Then $\mathcal{E}$ is pure of dimension d if and only if for each i the restriction $\mathcal{E}|_{U_i}$ is zero or pure of dimension d.*

Proof. The "if" part is trivial. Assume $\mathcal{E} \neq 0$ is pure of dimension d but there is a coherent subsheaf $0 \neq \mathcal{F} \subset \mathcal{E}|_{U_i}$ having dimension $< d$ for some i. Let $Z = \overline{\mathrm{Supp}(\mathcal{F})}$ (where the bar denotes closure in X) and consider the coherent subsheaf $\mathcal{E}_Z \subset \mathcal{E}$ defined by $\mathcal{E}_Z(U) = \ker(\mathcal{E}(U) \longrightarrow \mathcal{E}(U \setminus Z))$ for all open subsets $U \subset X$. This sheaf is nonzero because $0 \neq \mathcal{F} \subset \mathcal{E}_Z|_{U_i}$ yet $\mathrm{Supp}(\mathcal{E}_Z) \subset Z$ so $\dim(\mathcal{E}_Z) < d$, contradicting purity. $\square$

Proposition 1.1.4. *Let X, Y be schemes of finite type over k and let X be reduced. Denote by $p_2 : X \times Y \longrightarrow Y$ projection to the second component. Let $\mathcal{E}$ be a coherent sheaf on $X \times Y$, $\mathcal{F}$ a coherent sheaf on Y and $\Phi, \Psi : \mathcal{E} \longrightarrow p_2^* \mathcal{F}$ morphisms. For any closed point $x \in X$, let $i_x : Y \longrightarrow X \times Y$ be the induced morphism. If $i_x^* \Phi = i_x^* \Psi$ for all closed points $x \in X$, then $\Phi = \Psi$.*

Proof. Using open affine covers, it is enough to prove the case $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$, where R, S are finitely generated k-algebras and R has no nilpotent elements. Consider the finitely generated $R \otimes_k S$-module $E = \Gamma(X, \mathcal{E})$ and the finitely generated S-module $F = \Gamma(Y, \mathcal{F})$. Let $\Phi, \Psi : E \longrightarrow F \otimes_k R$ be the induced morphisms [Har1, Prop. II.5.2]. Let $e \in E$ and let $\xi = \Phi(e) - \Psi(e)$. We need to prove $\xi = 0$. But we know that for any maximal ideal $\mathfrak{m} \subset R$, the induced morphism

$$F \otimes_k R \longrightarrow F \otimes_k R/\mathfrak{m} \cong F,$$

maps ξ to zero [Har1, Prop. II.5.2]. Since R has no nilpotent elements, the intersection of all its maximal ideals is zero $\bigcap_{\mathfrak{m} \subset R} \mathfrak{m} = \sqrt{(0)} = (0)$ ([AM, Prop. 1.8], [Eis, Thm. 4.19]), hence $\xi = 0$. $\square$

Proposition 1.1.5. *Let X be a scheme of finite type over k, G an affine algebraic group acting regularly on X and $\mathcal{E} \neq 0$ a G-equivariant coherent sheaf on X. Then $\mathcal{E}$ is pure of dimension d if and only if all its nontrivial G-equivariant coherent subsheaves have dimension d.*

Proof. There is a unique filtration

$$0 \subset T_0(\mathcal{E}) \subset \cdots \subset T_d(\mathcal{E}) = \mathcal{E},$$

where $T_i(\mathcal{E})$ is the maximal coherent subsheaf of $\mathcal{E}$ of dimension $\leq i$. This filtration is called the torsion filtration of $\mathcal{E}$ [HL, Sect. 1.1]. We claim each $T_i(\mathcal{E})$ is an equivariant coherent subsheaf of $\mathcal{E}$, i.e. the morphism

$$\sigma^*(T_i(\mathcal{E})) \hookrightarrow \sigma^*(\mathcal{E}) \xrightarrow{\Phi} p_2^*(\mathcal{E}),$$

factors through $p_2^*(T_i(\mathcal{E}))$. This would imply the proposition. By definition of $T_i(\mathcal{E})$, the morphism

$$g^*(T_i(\mathcal{E})) \hookrightarrow g^*(\mathcal{E}) \xrightarrow{i_g^*\Phi} \mathcal{E},$$

factors through $T_i(\mathcal{E})$ for any closed point $g \in G$. The result now follows from Proposition 1.1.4 applied to the morphisms

$$\sigma^*(T_i(\mathcal{E})) \hookrightarrow \sigma^*(\mathcal{E}) \xrightarrow{\Phi} p_2^*(\mathcal{E}) \longrightarrow p_2^*(\mathcal{E}/T_i(\mathcal{E})),$$
$$\sigma^*(T_i(\mathcal{E})) \xrightarrow{0} p_2^*(\mathcal{E}/T_i(\mathcal{E})).$$

□

Let U_σ be an affine toric variety defined by a cone σ in a lattice N. We have already seen that $\Gamma(U_\sigma, -)$ induces an equivalence between the category of equivariant quasi-coherent sheaves on U_σ and the category of M-graded $k[S_\sigma]$-modules. The latter category can be conveniently reformulated using Perling's notion of a σ-family [Per1, Def. 4.2].

Definition 1.1.6. For $m, m' \in M$, $m \leq_\sigma m'$ means $m' - m \in S_\sigma$. A *σ-family* consists of the following data: a family of k-vector spaces $\{E_m^\sigma\}_{m \in M}$ and k-linear maps $\chi_{m,m'}^\sigma : E_m^\sigma \longrightarrow E_{m'}^\sigma$ for all $m \leq_\sigma m'$, such that $\chi_{m,m}^\sigma = 1$ and $\chi_{m,m''}^\sigma = \chi_{m',m''}^\sigma \circ \chi_{m,m'}^\sigma$ for all $m \leq_\sigma m' \leq_\sigma m''$. A *morphism of σ-families* $\hat{\phi}^\sigma : \hat{E}^\sigma \longrightarrow \hat{F}^\sigma$ is a family of k-linear maps $\{\phi_m : E_m^\sigma \longrightarrow F_m^\sigma\}_{m \in M}$, such that $\phi_{m'}^\sigma \circ (\chi_E)_{m,m'}^\sigma = (\chi_F)_{m,m'}^\sigma \circ \phi_m^\sigma$ for all $m \leq_\sigma m'$. ⊘

Let $(\mathcal{E}, \Phi)$ be an equivariant quasi-coherent sheaf on U_σ. Denote the corresponding M-graded $k[S_\sigma]$-module by $E^\sigma = \bigoplus_{m \in M} E_m^\sigma$. This gives us a σ-family $\{E_m^\sigma\}_{m \in M}$ by taking

$$\chi_{m,m'}^\sigma : E_m^\sigma \longrightarrow E_{m'}^\sigma, \ \chi_{m,m'}^\sigma(e) = \chi(m' - m)e,$$

for all $m \leq_\sigma m'$. This establishes an equivalence between the category of equivariant quasi-coherent sheaves on U_σ and the category of σ-families [Per1, Thm. 4.5].

Recall that an affine toric variety U_σ defined by a cone σ of dimension s in a lattice N of rank r is nonsingular if and only if σ is generated by part of a $\mathbb{Z}$-basis for N. Assume this is the case, then $U_\sigma \cong k^s \times (k^*)^{r-s}$. Let $\sigma(1) = \{\rho_1, \ldots, \rho_s\}$ be the rays (i.e. 1-dimensional faces) of σ. Let $n(\rho_i)$ be the first integral lattice point on the ray ρ_i. Then $(n(\rho_1), \ldots, n(\rho_s))$ is part of a $\mathbb{Z}$-basis for N. Let $(m(\rho_1), \ldots, m(\rho_s))$ be the corresponding part of a dual basis for M. The cosets $([m(\rho_1)], \ldots, [m(\rho_s)])$ form a $\mathbb{Z}$-basis for $M/S_\sigma^\perp$. Here $S_\sigma^\perp$ denotes the subgroup $S_\sigma^\perp = \sigma^\perp \cap M$, where $\sigma^\perp = \{u \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle u, v \rangle = 0 \text{ for all } v \in \sigma\}$. We obtain $M/S_\sigma^\perp \cong \mathbb{Z}^s$. Let $\hat{E}^\sigma$ be a σ-family. We can repackage the data in $\hat{E}^\sigma$ somewhat more efficiently as follows. First of all, note that for all $m' - m \in S_\sigma^\perp$, the k-linear map $\chi^\sigma_{m,m'} : E^\sigma_m \longrightarrow E^\sigma_{m'}$ is an isomorphism, so we might just as well restrict attention to σ-families having $\chi^\sigma_{m,m'} = 1$ (and hence $E^\sigma_m = E^\sigma_{m'}$) for all $m' - m \in S_\sigma^\perp$. We can then rewrite for any $\lambda_1, \ldots, \lambda_s \in \mathbb{Z}$

$$
\begin{aligned}
&E^\sigma(\lambda_1, \ldots, \lambda_s) = E^\sigma_m, \text{ where } m = \sum_{i=1}^s \lambda_i m(\rho_i),\\
&\chi_1^\sigma(\lambda_1, \ldots, \lambda_s) : E^\sigma(\lambda_1, \ldots, \lambda_s) \longrightarrow E^\sigma(\lambda_1 + 1, \lambda_2, \ldots, \lambda_s),\\
&\chi_1^\sigma(\lambda_1, \ldots, \lambda_s) = \chi^\sigma_{m,m'}, \text{ where } m = \sum_{i=1}^s \lambda_i m(\rho_i),\ m' = m(\rho_1) + m,\\
&\ldots
\end{aligned}
$$

When we would like to suppress the domain, we also denote these maps somewhat sloppily by $x_1 \cdot = \chi_1^\sigma(\lambda_1, \ldots, \lambda_s)$, ..., $x_s \cdot = \chi_s^\sigma(\lambda_1, \ldots, \lambda_s)$. These k-linear maps satisfy $x_i x_j = x_j x_i$ for all $i, j = 1, \ldots, s$. The equivalence between the category of equivariant quasi-coherent sheaves on U_σ and the category of σ-families restricts to an equivalence between the full subcategories of equivariant coherent sheaves on U_σ and the category of finite σ-families (see [Per1, Def. 4.10, Prop. 4.11]). A finite σ-family is a σ-family $\hat{E}^\sigma$ such that all $E^\sigma(\lambda_1, \ldots, \lambda_s)$ are finite-dimensional k-vector spaces, there are $A_1, \ldots, A_s \in \mathbb{Z}$ such that $E^\sigma(\lambda_1, \ldots, \lambda_s) = 0$ unless $\lambda_1 \geq A_1$, ..., $\lambda_s \geq A_s$ and there are only finitely many $(\Lambda_1, \ldots, \Lambda_s) \in \mathbb{Z}^s$ such that

$$
\begin{aligned}
&E^{\sigma}(\Lambda_1, \ldots, \Lambda_s) \\
&\neq \operatorname{span}_k \left\{ x_1^{\Lambda_1-\lambda_1} \cdots x_s^{\Lambda_s-\lambda_s} e \;\middle|\; e \in E^{\sigma}(\lambda_1, \ldots, \lambda_s) \text{ with } \Lambda_i - \lambda_i \geq 0, \text{ not all } 0 \right\}.
\end{aligned}
$$

1.1.1 Combinatorial Descriptions in the Case of Irreducible Support

Going from affine toric varieties to general toric varieties, Perling introduces the notion of Δ-families [Per1, Sect. 4.2], which are basically collections of σ-families, for all cones σ in the fan Δ, satisfying certain compatibility conditions. We will not use this notion. Instead, we will first study pure equivariant sheaves on nonsingular affine toric varieties and then use gluing to go to general toric varieties. In order to avoid heavy notation, we will restrict to the case of irreducible support and defer the general case to the next subsection. Recall that for a toric variety X defined by a fan Δ in a lattice N, there is a bijective correspondence between the elements of Δ and the invariant closed (irreducible) subvarieties of X [Ful, Sect. 3.1]. The correspondence associates to a cone $\sigma \in \Delta$ the invariant closed subvariety $V(\sigma) \subset X$, which is defined to be the closure in X of the unique orbit of minimal dimension in U_σ. If $\dim(\sigma) = s$, then $\operatorname{codim}(V(\sigma)) = s$.

Proposition 1.1.7. *Let U_σ be a nonsingular affine toric variety defined by a cone[3] σ in a lattice N of rank r. Let $\mathcal{E} \neq 0$ be an equivariant coherent sheaf on U_σ with irreducible support. Then $\operatorname{Supp}(\mathcal{E}) = V(\tau)$, for some $\tau \prec \sigma$. Now fix $\tau \prec \sigma$, let $(\rho_1, \ldots, \rho_r)$ be the rays of σ and $(\rho_1, \ldots, \rho_s) \subset (\rho_1, \ldots, \rho_r)$ the rays of τ. Then $\operatorname{Supp}(\mathcal{E}) = V(\tau)$ if and only if there are integers $B_1, \ldots, B_s$ such that $E^{\sigma}(\lambda_1, \ldots, \lambda_r) = 0$ unless $\lambda_1 \leq B_1$, …, $\lambda_s \leq B_s$, but for each $\lambda_i \neq \lambda_1, \ldots, \lambda_s$ there is no such upper bound.*

Proof. Note that $V(\tau)$ is defined by the prime ideal $\mathcal{I}_\tau = \langle \chi(m(\rho_1)), \ldots, \chi(m(\rho_s)) \rangle$. Define the open subset $U = U_\sigma \setminus V(\tau) = D(\chi(m(\rho_1))) \cup \cdots \cup D(\chi(m(\rho_s)))$, where $D(\chi(m(\rho_i)))$ is the set of all prime ideals not containing $\chi(m(\rho_i))$. The open subset

[3] From now on, in this setting we will always assume $\dim(\sigma) = r$, so $U_\sigma \cong \mathbb{A}^r$.

$D(\chi(m(\rho_i))) = \mathrm{Spec}(k[S_\sigma][\chi(-m(\rho_i))])$. Clearly [Har1, Prop. II.5.2]

$$\begin{aligned}\mathrm{Supp}(\mathcal{E}) \subset V(\tau) &\Longleftrightarrow \mathcal{E}|_{D(\chi(m(\rho_1)))} = \cdots = \mathcal{E}|_{D(\chi(m(\rho_s)))} = 0 \\ &\Longleftrightarrow \Gamma(U_\sigma, \mathcal{E}) \otimes_{k[S_\sigma]} k[S_\sigma][\chi(-m(\rho_1))] = 0 \\ &\qquad \ldots \\ &\qquad \Gamma(U_\sigma, \mathcal{E}) \otimes_{k[S_\sigma]} k[S_\sigma][\chi(-m(\rho_s))] = 0.\end{aligned}$$

Since $\Gamma(U_\sigma, \mathcal{E})$ is finitely generated, we in fact have

$$\begin{aligned}\mathrm{Supp}(\mathcal{E}) \subset V(\tau) \Longleftrightarrow &\exists \kappa_1, \ldots, \kappa_s \in \mathbb{Z}_{>0} \\ &\chi(m(\rho_1))^{\kappa_1}\Gamma(U_\sigma, \mathcal{E}) = \cdots = \chi(m_{\rho_s})^{\kappa_s}\Gamma(U_\sigma, \mathcal{E}) = 0.\end{aligned}$$

The proof now easily follows from the fact that the σ-family corresponding to $\mathcal{E}$ is finite. $\square$

Proposition 1.1.8. *Let U_σ be a nonsingular affine toric variety defined by a cone σ in a lattice N of rank r. Let $\tau \prec \sigma$, let $(\rho_1, \ldots, \rho_r)$ be the rays of σ and $(\rho_1, \ldots, \rho_s) \subset (\rho_1, \ldots, \rho_r)$ the rays of τ. Then the category of pure equivariant sheaves $\mathcal{E}$ on U_σ with support $V(\tau)$ is equivalent to the category of σ-families $\hat{E}^\sigma$ having the following properties:*

(i) *There are integers $A_1 \leq B_1, \ldots, A_s \leq B_s, A_{s+1}, \ldots, A_r$ such that $E^\sigma(\lambda_1, \ldots \lambda_r) = 0$ unless $A_1 \leq \lambda_1 \leq B_1$, …, $A_s \leq \lambda_s \leq B_s$, $A_{s+1} \leq \lambda_{s+1}$, …, $A_r \leq \lambda_r$.*

(ii) *For all integers $A_1 \leq \Lambda_1 \leq B_1$, …, $A_s \leq \Lambda_s \leq B_s$, there is a finite dimensional k-vector space $E^\sigma(\Lambda_1, \ldots, \Lambda_s, \infty, \ldots, \infty)$ (not all of them zero) satisfying the following properties. All vector spaces $E^\sigma(\Lambda_1, \ldots, \Lambda_s, \lambda_{s+1}, \ldots, \lambda_r)$ are subspaces of $E^\sigma(\Lambda_1, \ldots, \Lambda_s, \infty, \ldots, \infty)$ and the maps $x_{s+1}, \ldots, x_r$ are inclusions. Moreover, there are integers $\lambda_{s+1}, \ldots, \lambda_r$ such that we have $E^\sigma(\Lambda_1, \ldots, \Lambda_s, \lambda_{s+1}, \ldots, \lambda_r) = E^\sigma(\Lambda_1, \ldots, \Lambda_s, \infty, \ldots, \infty)$.*

Proof. Let $\mathcal{E}$ be a pure equivariant sheaf with support $V(\tau)$ and corresponding σ-family $\hat{E}^\sigma$. Then (i) follows from Proposition 1.1.7. For (ii), it is enough to prove $x_{s+1}, \ldots, x_r$

are injective (the rest follows from the fact that $\hat{E}^\sigma$ is finite). Assume one of them is not injective, say, without loss of generality, x_{s+1}. I.e.

$$\chi^\sigma_{s+1}(\Lambda_1, \ldots, \Lambda_s, \Lambda_{s+1}, \Lambda_{s+2}, \ldots, \Lambda_r) :$$
$$E^\sigma(\Lambda_1, \ldots, \Lambda_s, \Lambda_{s+1}, \Lambda_{s+2}, \ldots, \Lambda_r) \longrightarrow E^\sigma(\Lambda_1, \ldots, \Lambda_s, \Lambda_{s+1} + 1, \Lambda_{s+2}, \ldots, \Lambda_r),$$

is not injective where $A_1 \leq \Lambda_1 \leq B_1$, ..., $A_s \leq \Lambda_s \leq B_s$, $\Lambda_{s+1} \geq A_{s+1}$, ..., $\Lambda_r \geq A_r$. Define

$$F^\sigma(\Lambda_1, \ldots, \Lambda_s, \Lambda_{s+1}, \Lambda_{s+2}, \ldots, \Lambda_r) = \ker\, \chi^\sigma_{s+1}(\Lambda_1, \ldots, \Lambda_s, \Lambda_{s+1}, \Lambda_{s+2}, \ldots, \Lambda_r) \neq 0,$$
$$F^\sigma(\Lambda_1 + k_1, \ldots, \Lambda_s + k_s, \Lambda_{s+1}, \Lambda_{s+2} + k_{s+2}, \ldots, \Lambda_r + k_r)$$
$$= \left(\prod_{i \neq s+1} x_i^{k_i}\right)(F^\sigma(\Lambda_1, \ldots, \Lambda_s, \Lambda_{s+1}, \ldots, \Lambda_r)),\ \forall k_1, \ldots, k_s, k_{s+2}, \ldots, k_r \in \mathbb{Z}_{\geq 0},$$
$$F^\sigma(\lambda_1, \ldots, \lambda_r) = 0, \text{ otherwise.}$$

Clearly, $\hat{F}^\sigma$ defines a nontrivial equivariant coherent subsheaf of $\mathcal{E}$, hence its support has to be $V(\tau)$. By construction, there is an integer B_{s+1} such that $F^\sigma(\lambda_1, \ldots, \lambda_r) = 0$ outside some region $\lambda_1 \leq B_1$, ..., $\lambda_s \leq B_s$, $\lambda_{s+1} \leq B_{s+1}$, which contradicts Proposition 1.1.7.

Conversely, let $\mathcal{E}$ be an equivariant quasi-coherent sheaf with corresponding σ-family $\hat{E}^\Delta$ as in (i), (ii). It is easy to see that $\mathcal{E}$ is coherent and $\mathrm{Supp}(\mathcal{E}) \subset V(\tau)$ (see also proof of Proposition 1.1.7). It is enough to show that any nontrivial equivariant coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ has support $V(\tau)$ by Proposition 1.1.5. Suppose there is a $0 \neq \mathcal{F} \subset \mathcal{E}$ equivariant coherent subsheaf with support $V(\nu_1) \cup \cdots \cup V(\nu_p) \subsetneq V(\tau)$. Then τ is a proper face of each ν_i. Let $\rho^{(\nu_i)}$ be a ray of ν_i, that is not a ray of τ. Then

$$\mathcal{F}|_{D\left(\chi\left(m\left(\rho^{(\nu_1)}\right)\right)\cdots\chi\left(m\left(\rho^{(\nu_p)}\right)\right)\right)} = 0.$$

As a consequence, there is a $\kappa \in \mathbb{Z}_{>0}$ such that

$$\chi(m(\rho_{s+1}))^\kappa \cdots \chi(m(\rho_r))^\kappa \Gamma(U_\sigma, \mathcal{F}) = 0.$$

Let $\hat{F}^\sigma$ be the σ-family corresponding to $\mathcal{F}$. We obtain that there is a $B \in \mathbb{Z}$ such that $F^\sigma(\lambda_1, \ldots, \lambda_s, \mu, \ldots, \mu) = 0$ for all $\lambda_1, \ldots, \lambda_s \in \mathbb{Z}$ and $\mu > B$. But by injectivity of $x_{s+1}, \ldots, x_r$, we obtain $\hat{F}^\sigma = 0$, so $\mathcal{F} = 0$, which is a contradiction. $\square$

In order to generalise the result of the previous proposition to arbitrary nonsingular toric varieties, we need the following proposition for gluing purposes.

Proposition 1.1.9. *Let U_σ be a nonsingular affine toric variety defined by a cone σ in a lattice N of rank r. Let $\mathcal{E}$ be a pure equivariant sheaf on U_σ with support $V(\tau)$ where $\tau \prec \sigma$. Let $(\rho_1, \ldots, \rho_r)$ be the rays of σ and $(\rho_1, \ldots, \rho_s) \subset (\rho_1, \ldots, \rho_r)$ the rays of τ. Let $\nu \prec \sigma$ be a proper face and consider the equivariant coherent sheaf $\mathcal{E}|_{U_\nu}$. Then the ν-family corresponding to $\mathcal{E}|_{U_\nu}$ is described in terms of the σ-family corresponding to $\mathcal{E}$ as follows:*

(i) *Assume τ is not a face of ν. Then $\mathcal{E}|_{U_\nu} = 0$.*

(ii) *Assume $\tau \prec \nu$. Let $(\rho_1, \ldots, \rho_s, \rho_{s+1}, \ldots, \rho_{s+t}) \subset (\rho_1, \ldots, \rho_r)$ be the rays of ν. Then for all $\lambda_1, \ldots, \lambda_{s+t} \in \mathbb{Z}$ we have*

$$E^\nu(\lambda_1, \ldots, \lambda_{s+t}) = E^\sigma(\lambda_1, \ldots, \lambda_{s+t}, \infty, \ldots, \infty),$$
$$\chi_i^\nu(\lambda_1, \ldots, \lambda_{s+t}) = \chi_i^\sigma(\lambda_1, \ldots, \lambda_{s+t}, \infty, \ldots, \infty), \ \forall i = 1, \ldots, s+t.$$

Proof. There is an integral element $m_\nu \in$ relative interior$(\nu^\perp \cap \sigma^\vee)$, such that $S_\nu = S_\sigma + \mathbb{Z}_{\geq 0}(-m_\nu)$ (e.g. [Per1, Thm. 3.14]). Let $\rho_{i_1}, \ldots, \rho_{i_p}$ be the rays of ν and let $\rho_{j_1}, \ldots, \rho_{j_q}$ be all the other rays (so $p + q = r$). Then

$$m_\nu = \sum_{k=1}^{q} \gamma_k m(\rho_{j_k}),$$

where all $\gamma_k > 0$ integers. We obtain [Har1, Prop. II.5.2]

$$\begin{aligned} \Gamma(U_\nu, \mathcal{E}|_{U_\nu}) &\cong \Gamma(U_\sigma, \mathcal{E}) \otimes_{k[S_\sigma]} k[S_\nu] \\ &= \Gamma(U_\sigma, \mathcal{E}) \otimes_{k[S_\sigma]} k[S_\sigma][\chi(-m(\rho_{j_1}))^{\gamma_{j_1}}, \ldots, \chi(-m(\rho_{j_q}))^{\gamma_{j_q}}]. \end{aligned} \tag{1.2}$$

Case 1: τ is not a face of ν. Trivial because $V(\tau) \cap U_\nu = \varnothing$. We can also see this case algebraically as follows. There is a ray ρ_l of τ which is not a ray of ν. Hence ρ_l is equal to one of $\rho_{j_1}, \ldots, \rho_{j_q}$, say without loss of generality $\rho_l = \rho_{j_1} = \rho_1$. There is a $\kappa > 0$ such that $\chi(m(\rho_1))^\kappa \Gamma(U_\sigma, \mathcal{E}|_{U_\sigma}) = 0$ by Proposition 1.1.7. From equation (1.2), we deduce $\Gamma(U_\nu, \mathcal{E}|_{U_\nu}) = 0$.

Case 2: $\tau \prec \nu$. In this case, we can number the rays $\rho_{i_1}, \ldots, \rho_{i_p}$ of ν as follows $(\rho_1, \ldots, \rho_s, \rho_{s+1}, \ldots, \rho_{s+t})$. Assume $\mathcal{E}$ is described by a σ-family $\hat{E}^\sigma$ as in Proposition 1.1.8. Note that $\Gamma(U_\sigma, \mathcal{E}) \otimes_{k[S_\sigma]} k[S_\nu]$ has a natural M-grading [Per1, Sect. 2.5]. In particular, for a fixed $m \in M$, the elements of degree m are finite sums of expressions of the form $e \otimes \chi(m'')$, where $e \in E^\sigma_{m'}$, $m' \in M$, $m'' \in S_\nu$ such that $m' + m'' = m$. Now fix $m = \sum_{i=1}^r \lambda_i m(\rho_i) \in M$, $m' = \sum_{i=1}^r \alpha_i m(\rho_i) \in M$ and $m'' \in S_\nu$, so $m'' = \sum_{i=1}^r \beta_i m(\rho_i) - u \sum_{i=s+t+1}^r \gamma_i m(\rho_i)$ with $\beta_1, \ldots, \beta_r, u \geq 0$. Assume $m = m' + m''$ and consider the element $e \otimes \chi(m'')$ with $e \in E^\sigma_{m'}$. We can now rewrite $e \otimes \chi(m'') = e' \otimes \chi(m''')$, where

$$
\begin{aligned}
e' &= \chi\left(\sum_{i=1}^r \beta_i m(\rho_i)\right) \cdot e \in E^\sigma(\lambda_1, \ldots, \lambda_{s+t}, \alpha_{s+t+1} + \beta_{s+t+1}, \ldots, \alpha_r + \beta_r), \\
\chi(m''') &= \chi\left(-u \sum_{i=s+t+1}^r \gamma_i m(\rho_i)\right).
\end{aligned}
$$

For $v > 0$ large enough

$$
\begin{aligned}
&e' \otimes \chi(m''') = \chi\left(v \sum_{i=s+t+1}^r \gamma_i m(\rho_i)\right) \cdot e' \otimes \chi\left(-(u+v) \sum_{i=s+t+1}^r \gamma_i m(\rho_i)\right), \\
&\text{where } \chi\left(v \sum_{i=s+t+1}^r \gamma_i m(\rho_i)\right) \cdot e' \in E^\sigma(\lambda_1, \ldots, \lambda_{s+t}, \infty, \ldots, \infty).
\end{aligned}
$$

From these remarks, one easily deduces the assertion. $\square$

As a special case of the above proposition we get the following result. If we take $\nu = \tau$, then we obtain that for all integers $\lambda_1, \ldots, \lambda_r$

$$
E^\sigma(\lambda_1, \ldots, \lambda_r) \subset E^\sigma(\lambda_1, \ldots, \lambda_s, \infty, \ldots, \infty) = E^\tau(\lambda_1, \ldots, \lambda_s).
$$

We obtain that all $E^{\sigma}(\lambda_1, \ldots, \lambda_r)$ are subspaces of $E^{\tau}(\lambda_1, \ldots, \lambda_s)$.

Combining Propositions 1.1.8 and 1.1.9, we obtain a combinatorial description of pure equivariant sheaves with irreducible support on nonsingular toric varieties.

Theorem 1.1.10. *Let X be a nonsingular toric variety with fan*[4] *Δ in a lattice N of rank r. Let $\tau \in \Delta$ and consider the invariant closed subvariety $V(\tau)$. It is covered by U_σ, where $\sigma \in \Delta$ has dimension r and $\tau \prec \sigma$. Denote these cones by $\sigma_1, \ldots, \sigma_l$. For each $i = 1, \ldots, l$, let $\left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ be the rays of σ_i and let $\left(\rho_1^{(i)}, \ldots, \rho_s^{(i)}\right) \subset \left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ be the rays of τ. The category of pure equivariant sheaves on X with support $V(\tau)$ is equivalent to the category $\mathcal{C}^\tau$, which can be described as follows. An object $\hat{E}^\Delta$ of $\mathcal{C}^\tau$ consists of the following data:*

(i) *For each $i = 1, \ldots, l$ we have a σ_i-family $\hat{E}^{\sigma_i}$ as described in Proposition 1.1.8.*

(ii) *Let $i, j = 1, \ldots, l$. Let $\left\{\rho_{i_1}^{(i)}, \ldots, \rho_{i_p}^{(i)}\right\} \subset \left\{\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right\}$ resp. $\left\{\rho_{j_1}^{(j)}, \ldots, \rho_{j_p}^{(j)}\right\} \subset \left\{\rho_1^{(j)}, \ldots, \rho_r^{(j)}\right\}$ be the rays of $\sigma_i \cap \sigma_j$ in σ_i respectively σ_j, labeled in such a way that $\rho_{i_k}^{(i)} = \rho_{j_k}^{(j)}$ for all $k = 1, \ldots, p$. Now let $\lambda_1^{(i)}, \ldots, \lambda_r^{(i)} \in \mathbb{Z} \cup \{\infty\}$, $\lambda_1^{(j)}, \ldots, \lambda_r^{(j)} \in \mathbb{Z} \cup \{\infty\}$ be such that $\lambda_{i_k}^{(i)} = \lambda_{j_k}^{(j)} \in \mathbb{Z}$ for all $k = 1, \ldots, p$ and $\lambda_n^{(i)} = \lambda_n^{(j)} = \infty$ otherwise. Then*

$$
\begin{aligned}
E^{\sigma_i}\left(\sum_{k=1}^{r} \lambda_k^{(i)} m\left(\rho_k^{(i)}\right)\right) &= E^{\sigma_j}\left(\sum_{k=1}^{r} \lambda_k^{(j)} m\left(\rho_k^{(j)}\right)\right), \\
\chi_n^{\sigma_i}\left(\sum_{k=1}^{r} \lambda_k^{(i)} m\left(\rho_k^{(i)}\right)\right) &= \chi_n^{\sigma_j}\left(\sum_{k=1}^{r} \lambda_k^{(j)} m\left(\rho_k^{(j)}\right)\right), \ \forall n = 1, \ldots, r.
\end{aligned}
$$

The morphisms of $\mathcal{C}^\tau$ are described as follows. If $\hat{E}^\Delta$, $\hat{F}^\Delta$ are two objects, then a morphism $\hat{\phi}^\Delta : \hat{E}^\Delta \longrightarrow \hat{F}^\Delta$ is a collection of morphisms of σ-families $\{\hat{\phi}^{\sigma_i} : \hat{E}^{\sigma_i} \longrightarrow \hat{F}^{\sigma_i}\}_{i=1,\ldots,l}$ such that for all i, j as in (ii) *one has*

$$
\phi^{\sigma_i}\left(\sum_{k=1}^{r} \lambda_k^{(i)} m\left(\rho_k^{(i)}\right)\right) = \phi^{\sigma_j}\left(\sum_{k=1}^{r} \lambda_k^{(j)} m\left(\rho_k^{(j)}\right)\right).
$$

[4]From now on, in this setting we will always assume every cone of Δ is contained in a cone of dimension r. Therefore, we can cover X by copies of $\mathbb{A}^r$.

Proof. Note that $V(\tau)$ is covered by the star of τ, i.e. the cones $\sigma \in \Delta$ such that $\tau \prec \sigma$ [Ful, Sect. 3.1]. Let $\sigma_1, \ldots, \sigma_l \in \Delta$ be the cones of maximal dimension in the star of τ. Let $\mathcal{E}$ be a pure equivariant sheaf on X with support $V(\tau)$. Then $\mathcal{E}|_{U_{\sigma_i}}$ is a pure equivariant sheaf on U_{σ_i} with support $V(\tau) \cap U_{\sigma_i}$ for all $i = 1, \ldots, l$ (using Proposition 1.1.3). Using Proposition 1.1.8, we get a σ_i-family $\hat{E}^{\sigma_i}$ for all $i = 1, \ldots, l$ (this gives (i) of the theorem). Using Proposition 1.1.9, we see that these σ-families have to glue as in (ii) (up to isomorphism). $\square$

In the above theorem, we will refer to the category $\mathcal{C}^\tau$ as the category of pure Δ-families with support $V(\tau)$. If we take $\tau = 0$ to be the apex in this theorem, we obtain the known combinatorial description of torsion free equivariant sheaves on nonsingular toric varieties initially due to Klyachko [Kly4] and also discussed by Knutson and Sharpe [KS1, Sect. 4.5] and Perling [Per1, Subsect. 4.4.2]. The theorem generalises this description. In the case $\tau = 0$ is the apex, we will refer to the category $\mathcal{C}^0$ as the category of torsion free Δ-families. In the above theorem, denote by $\mathcal{C}^{\tau,fr}$ the full subcategory of $\mathcal{C}^\tau$ consisting of those elements having all limiting vector spaces $E^{\sigma_i}(\Lambda_1, \ldots, \Lambda_s, \infty, \ldots, \infty)$ equal to $k^{\oplus r}$ for some r. We refer to $\mathcal{C}^{\tau,fr}$ as the category of framed pure Δ-families with support $V(\tau)$. This notion does not make much sense now because $\mathcal{C}^{\tau,fr}$ is equivalent to $\mathcal{C}^\tau$, but framing will become relevant when looking at families.

1.1.2 Combinatorial Descriptions in the General Case

The results of the previous subsection generalise in a straightforward way to the case of general –not necessarily irreducible– support. Since the proofs will require no essentially new ideas, we will just discuss the results.

Let us first discuss the generalisation of Proposition 1.1.7. Let U_σ be a nonsingular affine toric variety defined by a cone σ in a lattice N of rank r. Let $\mathcal{E} \neq 0$ be an equivariant coherent sheaf on U_σ. Then $\mathrm{Supp}(\mathcal{E}) = V(\tau_1) \cup \cdots \cup V(\tau_a)$ for some faces $\tau_1, \ldots, \tau_a \prec \sigma$. Now fix faces $\tau_1, \ldots, \tau_a \prec \sigma$, let $(\rho_1, \ldots, \rho_r)$ be the rays of σ and let $\left(\rho_1^{(\alpha)}, \ldots, \rho_{s_\alpha}^{(\alpha)}\right) \subset (\rho_1, \ldots, \rho_r)$ be the rays of τ_α for all $\alpha = 1, \ldots, a$. Assume $\tau_\alpha \not\prec \tau_\beta$ for all $\alpha, \beta = 1, \ldots, a$ with $\alpha \neq \beta$. Then $\mathrm{Supp}(\mathcal{E}) = V(\tau_1) \cup \cdots \cup V(\tau_a)$ if and only if the following property holds:

$E^\sigma(\lambda_1, \ldots, \lambda_r) = 0$ unless $(\lambda_1, \ldots, \lambda_r) \in \mathcal{R}$ where $\mathcal{R} \subset N$ is defined by inequalities as follows: there are integers $A_1, \ldots, A_r$ and integers $B_1^{(\alpha)}, \ldots, B_{s_\alpha}^{(\alpha)}$ for each $\alpha = 1, \ldots, a$ such that the region $\mathcal{R}$ is defined by

$$
\begin{aligned}
&[A_1 \leq \lambda_1 \wedge \cdots \wedge A_r \leq \lambda_r)] \\
&\wedge [(\lambda_1^{(1)} \leq B_1^{(1)} \wedge \cdots \wedge \lambda_{s_1}^{(1)} \leq B_{s_1}^{(1)}) \vee \cdots \vee (\lambda_1^{(a)} \leq B_1^{(a)} \wedge \cdots \wedge \lambda_{s_a}^{(a)} \leq B_{s_a}^{(a)})],
\end{aligned}
$$

moreover, there is no region $\mathcal{R}'$ of such a form with more upper bounds contained in $\mathcal{R}$ with the same property. Here $\lambda_i^{(j)}$ corresponds to the coordinate associated to the ray $\rho_i^{(j)}$ defined above.

Note that if we assume in addition that $\mathcal{E}$ is pure, then all the $V(\tau_\alpha)$ have the same dimension so $s_1 = \cdots = s_a = s$. If $\dim(\mathcal{E}) = d$, then $\mathrm{Supp}(\mathcal{E}) = V(\tau_1) \cup \cdots \cup V(\tau_a)$, where $\tau_1, \ldots, \tau_a$ are some faces of σ of dimension $s = r - d$. One possible support would be taking $\tau_1, \ldots, \tau_a$ all faces of σ of dimension $s = r - d$. In this case, the region $\mathcal{R}$ will be a disjoint union of the following form

$$
\begin{aligned}
&\{[A_1, B_1] \times \cdots \times [A_s, B_s] \times (B_{s+1}, \infty) \times \cdots \times (B_r, \infty)\} \\
&\sqcup \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.3)\\
&\sqcup \{(B_1, \infty) \times \cdots \times (B_{r-s}, \infty) \times [A_{r-s+1}, B_{r-s+1}] \times \cdots \times [A_r, B_r]\} \\
&\sqcup \{[A_1, B_1] \times \cdots \times [A_{s+1}, B_{s+1}] \times (B_{s+2}, \infty) \times \cdots \times (B_r, \infty)\} \\
&\sqcup \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.4)\\
&\sqcup \{(B_1, \infty) \times \cdots \times (B_{r-s-1}, \infty) \times [A_{r-s}, B_{r-s}] \times \cdots \times [A_r, B_r]\} \\
&\sqcup \cdots \\
&\sqcup \{[A_1, B_1] \times \cdots \times [A_r, B_r]\}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.5)
\end{aligned}
$$

for some integers $A_1, \ldots, A_r$ and $B_1, \ldots, B_r$. Here (1.3) is a disjoint union of $\binom{r}{s}$ regions with s upper bounds. Denote these regions of $\mathcal{R}$ by $\mathcal{R}_\mu^s$, where $\mu = 1, \ldots, \binom{r}{s}$. Here (1.4) is a disjoint union of $\binom{r}{s+1}$ regions with $s+1$ upper bounds. Denote these regions of $\mathcal{R}$ by $\mathcal{R}_\mu^{s+1}$, where $\mu = 1, \ldots, \binom{r}{s+1}$. Et cetera. Finally, (1.5) is a disjoint union of $\binom{r}{r} = 1$ regions with r upper bounds (i.e. only $[A_1, B_1] \times \cdots \times [A_r, B_r]$). Denote this region of $\mathcal{R}$ by $\mathcal{R}_\mu^r$. We will use this notation later on. Using the techniques of the previous subsection, one easily proves the following proposition.

Proposition 1.1.11. *Let U_σ be a nonsingular affine toric variety defined by a cone σ in a lattice N of rank r. Let $\tau_1, \ldots, \tau_a \prec \sigma$ be all faces of dimension s. Then the category of pure equivariant sheaves $\mathcal{E}$ on U_σ with support $V(\tau_1) \cup \cdots \cup V(\tau_a)$ is equivalent to the category of σ-families $\hat{E}^\sigma$ satisfying the following properties:*

(i) *There are integers $A_1, \ldots, A_r$, $B_1, \ldots, B_r$ such that $E^\sigma(\lambda_1, \ldots, \lambda_r) = 0$ unless $(\lambda_1, \ldots, \lambda_r) \in \mathcal{R}$, where the region $\mathcal{R}$ is as above.*

(ii) *Any region $\mathcal{R}_\mu^i = [A_1, B_1] \times \cdots \times [A_i, B_i] \times (B_{i+1}, \infty) \times \cdots \times (B_r, \infty)$ of $\mathcal{R}$ satisfies the following properties*[5]*. Firstly, for any integers $A_1 \leq \Lambda_1 \leq B_1$, …, $A_i \leq \Lambda_i \leq B_i$ there is a finite-dimensional k-vector space $E^\sigma(\Lambda_1, \ldots, \Lambda_i, \infty, \ldots, \infty)$, such that $E^\sigma(\Lambda_1, \ldots, \Lambda_i, \lambda_{i+1}, \ldots, \lambda_r) = E^\sigma(\Lambda_1, \ldots, \Lambda_i, \infty, \ldots, \infty)$ for some integers $\lambda_{i+1} > B_{i+1}, \ldots, \lambda_r > B_r$. Moreover, if $\mathcal{R}_\mu^i$ is one of the regions $\mathcal{R}_\mu^s$, not all $E^\sigma(\Lambda_1, \ldots, \Lambda_i, \infty, \ldots, \infty)$ are zero. Secondly, $\chi_{i+1}^\sigma(\vec{\lambda}), \ldots, \chi_r^\sigma(\vec{\lambda})$ are inclusions for all $\vec{\lambda} \in \mathcal{R}_\mu^i$. Finally, if $j_1, \ldots, j_{s+1} \in \{1, \ldots, i\}$ are distinct, then for any $\vec{\lambda} \in \mathcal{R}_\mu^i$ the following k-linear map is injective*

$$\begin{aligned} E^\sigma(\vec{\lambda}) \hookrightarrow\ & E^\sigma(\lambda_1, \ldots, \lambda_{j_1-1}, B_{j_1}+1, \lambda_{j_1+1}, \ldots, \lambda_r \\ & \oplus \cdots \\ & \oplus E^\sigma(\lambda_1, \ldots, \lambda_{j_{s+1}-1}, B_{j_{s+1}}+1, \lambda_{j_{s+1}+1}, \ldots, \lambda_r), \end{aligned} \tag{1.6}$$

[5]Without loss of generality, we denote this region in such a way that the i upper bounds occur in the first i intervals. The general case is clear.

$$
\begin{aligned}
&\left(\chi^{\sigma}_{j_1}(\lambda_1,\ldots,\lambda_{j_1-1},B_{j_1},\lambda_{j_1+1},\ldots,\lambda_r)\circ\cdots\right.\\
&\quad\left.\circ\chi^{\sigma}_{j_1}(\lambda_1,\ldots,\lambda_{j_1-1},\lambda_{j_1},\lambda_{j_1+1},\ldots,\lambda_r)\right)\\
&\oplus\cdots\\
&\oplus\left(\chi^{\sigma}_{j_{s+1}}(\lambda_1,\ldots,\lambda_{j_{s+1}-1},B_{j_{s+1}},\lambda_{j_{s+1}+1},\ldots,\lambda_r)\circ\cdots\right.\\
&\quad\left.\circ\chi^{\sigma}_{j_{s+1}}(\lambda_1,\ldots,\lambda_{j_{s+1}-1},\lambda_{j_{s+1}},\lambda_{j_{s+1}+1},\ldots,\lambda_r)\right).
\end{aligned}
$$

Note that in this proposition, the only essentially new type of condition compared to Proposition 1.1.8 is condition (1.6). We obtain the following theorem.

Theorem 1.1.12. *Let X be a nonsingular toric variety with fan Δ in a lattice N of rank r. Let $\sigma_1,\ldots,\sigma_l$ be all cones of Δ of dimension r. Denote the rays of σ_i by $\left(\rho^{(i)}_1,\ldots,\rho^{(i)}_r\right)$ for all $i=1,\ldots,l$. Let $\tau_1,\ldots,\tau_a$ be all cones of Δ of dimension s. The category of pure equivariant sheaves on X with support $V(\tau_1)\cup\cdots\cup V(\tau_a)$ is equivalent to the category $\mathcal{C}^{\tau_1,\ldots,\tau_a}$, which can be described as follows. An object $\hat{E}^{\Delta}$ of $\mathcal{C}^{\tau_1,\ldots,\tau_a}$ consists of the following data:*

(i) *For each $i=1,\ldots,l$ we have a σ_i-family $\hat{E}^{\sigma_i}$ as described in Proposition 1.1.11.*

(ii) *Let $i,j=1,\ldots,l$. Let $\left\{\rho^{(i)}_{i_1},\ldots,\rho^{(i)}_{i_p}\right\}\subset\left\{\rho^{(i)}_1,\ldots,\rho^{(i)}_r\right\}$ resp. $\left\{\rho^{(j)}_{j_1},\ldots,\rho^{(j)}_{j_p}\right\}\subset\left\{\rho^{(j)}_1,\ldots,\rho^{(j)}_r\right\}$ be the rays of $\sigma_i\cap\sigma_j$ in σ_i respectively σ_j, labeled in such a way that $\rho^{(i)}_{i_k}=\rho^{(j)}_{j_k}$ for all $k=1,\ldots,p$. Now let $\lambda^{(i)}_1,\ldots,\lambda^{(i)}_r\in\mathbb{Z}\cup\{\infty\}$, $\lambda^{(j)}_1,\ldots,\lambda^{(j)}_r\in\mathbb{Z}\cup\{\infty\}$ be such that $\lambda^{(i)}_{i_k}=\lambda^{(j)}_{j_k}\in\mathbb{Z}$ for all $k=1,\ldots,p$ and $\lambda^{(i)}_n=\lambda^{(j)}_n=\infty$ otherwise. Then*

$$
\begin{aligned}
E^{\sigma_i}\left(\sum_{k=1}^{r}\lambda^{(i)}_k m\left(\rho^{(i)}_k\right)\right)&=E^{\sigma_j}\left(\sum_{k=1}^{r}\lambda^{(j)}_k m\left(\rho^{(j)}_k\right)\right),\\
\chi^{\sigma_i}_n\left(\sum_{k=1}^{r}\lambda^{(i)}_k m\left(\rho^{(i)}_k\right)\right)&=\chi^{\sigma_j}_n\left(\sum_{k=1}^{r}\lambda^{(j)}_k m\left(\rho^{(j)}_k\right)\right),\ \forall n=1,\ldots,r.
\end{aligned}
$$

The morphisms of $\mathcal{C}^{\tau_1,\ldots,\tau_a}$ are described as follows. If $\hat{E}^{\Delta}$, $\hat{F}^{\Delta}$ are two objects, then a morphism $\hat{\phi}^{\Delta}:\hat{E}^{\Delta}\longrightarrow\hat{F}^{\Delta}$ is a collection of morphisms of σ-families $\{\hat{\phi}^{\sigma_i}:\hat{E}^{\sigma_i}\longrightarrow$

$\hat{F}^{\sigma_i}\}_{i=1,\ldots,l}$ *such that for all* i, j *as in* (ii) *one has*

$$\phi^{\sigma_i}\left(\sum_{k=1}^{r} \lambda_k^{(i)} m\left(\rho_k^{(i)}\right)\right) = \phi^{\sigma_j}\left(\sum_{k=1}^{r} \lambda_k^{(j)} m\left(\rho_k^{(j)}\right)\right).$$

Although we only described the "maximally reducible" case in Proposition 1.1.11 and Theorem 1.1.12, the reader will have no difficulty writing down the case of arbitrary reducible support. We refrain from doing this since the notation will become too cumbersome, whereas the ideas are the same.

1.2 Moduli Spaces of Equivariant Sheaves on Toric Varieties

In this section, we discuss how the combinatorial description of pure equivariant sheaves on nonsingular toric varieties of Theorems 1.1.10 and 1.1.12 can be used to define a moduli problem and a coarse moduli space of such sheaves using GIT. We will start by defining the relevant moduli functors and studying families. Subsequently, we will perform GIT quotients and show we have obtained coarse moduli spaces. Again, for notational convenience, we will first discuss the case of irreducible support and discuss the general case only briefly afterwards. The GIT construction gives rise to various notions of GIT stability depending on a choice of equivariant line bundle. In order to recover geometric results, we need an equivariant line bundle which precisely recovers Gieseker stability. We will construct such (ample) equivariant line bundles for torsion free equivariant sheaves in general. As a by-product, for reflexive equivariant sheaves, we can always construct particularly simple ample equivariant line bundles matching GIT stability and μ-stability (subsection 1.3.4).

1.2.1 Moduli Functors

We start by defining some topological data.

Definition 1.2.1. Let X be a nonsingular toric variety and use notation as in Theorem 1.1.10. Recall that $\sigma_1, \ldots, \sigma_l$ are the cones of maximal dimension having τ as a face. Let $\mathcal{E}$ be a pure equivariant sheaf on X with support $V(\tau)$. The *characteristic function* $\vec{\chi}_{\mathcal{E}}$ of $\mathcal{E}$ is defined to be the map

$$\vec{\chi}_{\mathcal{E}} : M \longrightarrow \mathbb{Z}^l,$$
$$\vec{\chi}_{\mathcal{E}}(m) = (\chi_{\mathcal{E}}^{\sigma_1}(m), \ldots, \chi_{\mathcal{E}}^{\sigma_l}(m)) = (\dim_k(E_m^{\sigma_1}), \ldots, \dim_k(E_m^{\sigma_l})).$$

We denote the set of all characteristic functions of pure equivariant sheaves on X with support $V(\tau)$ by $\mathcal{X}^\tau$. $\oslash$

Assume X is a nonsingular projective toric variety. Let $\mathcal{O}_X(1)$ be an ample line bundle on X, so we can speak of Gieseker (semi)stable sheaves on X [HL, Def. 1.2.4]. Let $\vec{\chi} \in \mathcal{X}^\tau$. We will be interested in moduli problems of Gieseker (semi)stable pure equivariant sheaves on X with support $V(\tau)$ and characteristic function $\vec{\chi}$. This means we need to define moduli functors, i.e. we need an appropriate notion of a family. Let Sch/k be the category of k-schemes of finite type. Let S be a k-scheme of finite type and, for any $s \in S$, define the natural morphism $\iota_s : \mathrm{Spec}(k(s)) \longrightarrow S$, where $k(s)$ is the residue field of s. We define an equivariant S-flat family to be an equivariant coherent sheaf $\mathcal{F}$ on $X \times S$ (S with trivial torus action), which is flat w.r.t. the projection $p_S : X \times S \longrightarrow S$. Such a family $\mathcal{F}$ is said to be Gieseker semistable with support $V(\tau)$ and characteristic function $\vec{\chi}$, if $\mathcal{F}_s = (1_X \times \iota_s)^*\mathcal{F}$ is Gieseker semistable with support $V(\tau) \times \mathrm{Spec}(k(s))$ and characteristic function $\vec{\chi}$ for all $s \in S$. Two such families $\mathcal{F}_1, \mathcal{F}_2$ are said to be equivalent if there is a line bundle $L \in \mathrm{Pic}(S)$ and an equivariant isomorphism $\mathcal{F}_1 \cong \mathcal{F}_2 \otimes p_S^* L$, where L is being considered as an equivariant sheaf on S with trivial equivariant structure. Denote the set of Gieseker semistable equivariant S-flat families with support $V(\tau)$ and characteristic function $\vec{\chi}$ modulo equivalence by $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau, ss}(S)$. We obtain a moduli functor

$$\underline{\mathcal{M}}_{\vec{\chi}}^{\tau,ss} : (Sch/k)^o \longrightarrow Sets,$$
$$S \mapsto \underline{\mathcal{M}}_{\vec{\chi}}^{\tau,ss}(S),$$
$$(f : S' \longrightarrow S) \mapsto \underline{\mathcal{M}}_{\vec{\chi}}^{\tau,ss}(f) = f^* : \underline{\mathcal{M}}_{\vec{\chi}}^{\tau,ss}(S) \longrightarrow \underline{\mathcal{M}}_{\vec{\chi}}^{\tau,ss}(S').$$

Similarly, we obtain a moduli problem and a moduli functor $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau,s}$ in the geometrically Gieseker stable case. Also note that we could have defined alternative moduli functors $\underline{\mathcal{M}}_{\vec{\chi}}^{\prime\tau,ss}$, $\underline{\mathcal{M}}_{\vec{\chi}}^{\prime\tau,s}$ by using just equivariant isomorphism as the equivalence relation instead of the one above. We start with the following proposition.

Proposition 1.2.2. *Let X be a nonsingular toric variety. Let S be a connected k-scheme of finite type and let $\mathcal{F}$ be an equivariant S-flat family. Then the characteristic functions of the fibres $\vec{\chi}_{\mathcal{F}_s}$ are constant on S.*

Proof. Let σ be a cone of the fan Δ. Let $V = \mathrm{Spec}(A) \subset S$ be an affine open subset[6]. It is enough to prove that for all $m \in M$

$$\chi_{\mathcal{F}_s}^{\sigma}(m) = \dim_{k(s)} \Gamma(U_\sigma \times k(s), \mathcal{F}_s|_{U_\sigma \times k(s)})_m,$$

is constant for all $s \in V$. Note that the equivariant coherent sheaf $\mathcal{F}|_{U_\sigma \times V}$ corresponds to a finitely generated M-graded $k[S_\sigma] \otimes_k A$-module

$$\Gamma(U_\sigma \times V, \mathcal{F}) = \bigoplus_{m \in M} F_m^{\sigma},$$

where all F_m^{σ} are in fact finitely generated A-modules, so they correspond to coherent sheaves $\mathcal{F}_m^{\sigma}$ on V. Since $\mathcal{F}$ is S-flat, each $\mathcal{F}_m^{\sigma}$ is a locally free sheaf of some finite rank $r(m)$ [Har1, Prop. III.9.2]. Fix $s \in V$ and consider the natural morphism $A \longrightarrow k(s)$,

[6] Note that from now on, for R a commutative ring, we often sloppily write R instead of $\mathrm{Spec}(R)$, when no confusion is likely to arise.

then

$$\Gamma(U_\sigma \times k(s), \mathcal{F}_s|_{U_\sigma \times k(s)}) \cong \bigoplus_{m \in M} F_m^\sigma \otimes_A k(s) \cong \bigoplus_{m \in M} k(s)^{\oplus r(m)}.$$

Consequently, $\chi^\sigma_{\mathcal{F}_s}(m) = r(m)$ for all $m \in M$. $\square$

1.2.2 Families

The question now arises to what extent the various moduli functors defined in the previous subsection are corepresentable. In order to answer this question, we will give a combinatorial description of a family. Payne has studied a similar problem in [Pay] for equivariant vector bundles on toric varieties.

We start with some straight-forward generalisations of the theory in section 1.1.

Definition 1.2.3. Let U_σ be an affine toric variety defined by a cone σ in a lattice N. Let S be a k-scheme of finite type. A *σ-family over S* consists of the following data: a family of quasi-coherent sheaves $\{\mathcal{F}^\sigma_m\}_{m\in M}$ on S and morphisms $\chi^\sigma_{m,m'} : \mathcal{F}^\sigma_m \longrightarrow \mathcal{F}^\sigma_{m'}$ for all $m \leq_\sigma m'$, such that $\chi^\sigma_{m,m} = 1$ and $\chi^\sigma_{m,m''} = \chi^\sigma_{m',m''} \circ \chi^\sigma_{m,m'}$ for all $m \leq_\sigma m' \leq_\sigma m''$. A *morphism $\hat{\phi}^\sigma : \hat{\mathcal{F}}^\sigma \longrightarrow \hat{\mathcal{G}}^\sigma$ of σ-families over S* is a family of morphisms $\{\phi_m : \mathcal{F}^\sigma_m \longrightarrow \mathcal{G}^\sigma_m\}_{m\in M}$, such that $\phi^\sigma_{m'} \circ (\chi_\mathcal{F})^\sigma_{m,m'} = (\chi_\mathcal{G})^\sigma_{m,m'} \circ \phi^\sigma_m$ for all $m \leq_\sigma m'$. $\oslash$

Proposition 1.2.4. *Let U_σ be a nonsingular affine toric variety defined by a cone σ and let S be a k-scheme of finite type. The category of equivariant quasi-coherent sheaves on $U_\sigma \times S$ is equivalent to the category of σ-families over S.*

Proof. Let $S = \mathrm{Spec}(A)$ be affine and $(\mathcal{F}, \Phi)$ an equivariant coherent sheaf. We have a regular action of T on $F^\sigma = \Gamma(U_\sigma \times S, \mathcal{F})$, inducing a decomposition into weight spaces $F^\sigma = \bigoplus_{m\in M} F^\sigma_m$ (Complete Reducibility Theorem, [Per1, Thm. 2.30]). This time however, the F^σ_m are A-modules instead of k-vector spaces. This gives the desired equivalence of categories. It is easy to see that the same holds for arbitrary S by gluing. $\square$

In the context of the previous proposition, a σ-family $\hat{\mathcal{F}}^\sigma$ over S is called finite if all $\mathcal{F}^\sigma_m$ are coherent sheaves on S, there are integers $A_1, \ldots, A_r$ such that $\mathcal{F}^\sigma(\lambda_1, \ldots, \lambda_r) = 0$

unless $A_1 \leq \lambda_1$, …, $A_r \leq \lambda_r$ and there are only finitely many $m \in M$ such that the morphism

$$\bigoplus_{m' <_\sigma m} \mathcal{F}^\sigma_{m'} \longrightarrow \mathcal{F}^\sigma_m,$$

is not surjective.

Proposition 1.2.5. *Let U_σ be a nonsingular affine toric variety defined by a cone σ and let S be a k-scheme of finite type. The category of equivariant S-flat families is equivalent to the category of finite σ-families $\hat{\mathcal{F}}^\sigma$ over S with all $\mathcal{F}^\sigma_m$ locally free sheaves on S of finite rank.*

Proof. It is not difficult to derive that the equivalence of Proposition 1.2.4 restricts to an equivalence between the category of equivariant coherent sheaves on $X \times S$ and the category of finite σ-families over S. In the proof of Proposition 1.2.2, we saw that S-flatness gives rise to locally free of finite rank. □

In the context of the previous proposition, let $\hat{\mathcal{F}}^\sigma$ be a σ-family over S corresponding to an equivariant S-flat family $\mathcal{F}$. For each connected component $C \subset S$, the characteristic function $\chi_{\mathcal{F}_s} : M \longrightarrow \mathbb{Z}$, gives us the ranks of the $\mathcal{F}^\sigma_m$ on C.

Proposition 1.2.6. *Let U_σ be a nonsingular affine toric variety defined by a cone σ in a lattice N of rank r. Let $\tau \prec \sigma$ and let $(\rho_1, \ldots, \rho_s) \subset (\rho_1, \ldots, \rho_r)$ be the rays of τ respectively σ. Let S be a k-scheme of finite type and $\chi \in \mathcal{X}^\tau$. Let $\mathcal{F}$ be an equivariant S-flat family such that $\chi_{\mathcal{F}_s} = \chi$ for all $s \in S$ and let $\hat{\mathcal{F}}^\sigma$ be the corresponding σ-family over S. Then the fibres $\mathcal{F}_s$ are pure equivariant with support $V(\tau) \times k(s)$ if and only if the following properties are satisfied:*

(i) *There are integers $A_1 \leq B_1, \ldots, A_s \leq B_s, A_{s+1}, \ldots, A_r$ such that $\mathcal{F}^\sigma(\lambda_1, \ldots, \lambda_r) = 0$ unless $A_1 \leq \lambda_1 \leq B_1$, …, $A_s \leq \lambda_s \leq B_s$, $A_{s+1} \leq \lambda_{s+1}$, …, $A_r \leq \lambda_r$ and for $\lambda_i \neq \lambda_1, \ldots, \lambda_s$ there is no such upper bound.*

(ii) *For any $s \in S$, $\mathcal{F}_s$ has a corresponding σ-family $\hat{F}_s^{\sigma}$ as in Proposition 1.1.8 (over ground field $k(s)$) with bounding integers $A_1 \leq B_1, \ldots, A_s \leq B_s, A_{s+1}, \ldots, A_r$.*

Proof. Note first of all that the entire theory of subsection 1.1.1 and 1.1.2 works over any ground field of characteristic 0, so we can replace k by $k(s)$ in all the results ($s \in S$). Note that if F_m^σ is the A-module corresponding to $\mathcal{F}_m^\sigma|_V$ for $V = \mathrm{Spec}(A) \subset S$ an affine open subset, then $(F_s)_m^\sigma \cong F_m^\sigma \otimes_A k(s)$ for all $s \in V$. In particular, $\chi(m) = \mathrm{rk}(\mathcal{F}_m^\sigma) = \dim_{k(s)}((F_s)_m^\sigma)$ for all $m \in M, s \in S$. The result easily follows from Proposition 1.1.8. $\square$

Before we proceed, we need a technical result.

Proposition 1.2.7. *Let S be a k-scheme of finite type and let $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism of locally free sheaves of finite rank on S. Let $\iota_s : k(s) \longrightarrow S$ be the natural morphism for all $s \in S$. Then ϕ is injective and* $\mathrm{coker}(\phi)$ *is S-flat if and only if $\iota_s^* \phi$ is injective for all $s \in S$.*

Proof. By taking an open affine cover over which both locally free sheaves trivialise, it is easy to see we are reduced to proving the following:

Claim. Let $(R, \mathfrak{m})$ be a local ring. Let $k = R/\mathfrak{m}$ be the residue field and let $\phi : R^{\oplus a} \longrightarrow R^{\oplus b}$ be an R-module homomorphism. Then ϕ is injective and $\mathrm{coker}(\phi)$ is free of finite rank if and only if the induced map $\overline{\phi} : k^{\oplus a} \longrightarrow k^{\oplus b}$ is injective.

Proof of Claim: $\Leftarrow$. Let $M = R^{\oplus b}/\mathrm{im}(\phi)$, then we have an exact sequence $R^{\oplus a} \xrightarrow{\phi} R^{\oplus b} \longrightarrow M \longrightarrow 0$. Applying $- \otimes_R k$ and using the assumption, we obtain a short exact sequence $0 \longrightarrow k^{\oplus a} \xrightarrow{\overline{\phi}} k^{\oplus b} \longrightarrow M/\mathfrak{m}M \longrightarrow 0$. Here $M/\mathfrak{m}M$ is a $c = b - a$ dimensional k-vector space. Take c basis elements of $M/\mathfrak{m}M$, then their representatives in M generate M as an R-module (Nakayama's Lemma). Take preimages $x_1, \ldots, x_c$ in $R^{\oplus b}$. Denote the standard generators of $R^{\oplus c}$ by $e_1, \ldots, e_c$ and define $\psi : R^{\oplus c} \longrightarrow R^{\oplus b}$, $e_i \mapsto x_i$. We get a diagram

$$\begin{tikzcd} 0 \arrow[r] & R^{\oplus a} \arrow[r, "\iota"] \arrow[d, "1"'] & R^{\oplus a} \oplus R^{\oplus c} \arrow[r, "\pi"] \arrow[d, "\phi+\psi"'] & R^{\oplus c} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & R^{\oplus a} \arrow[r, "\phi"] & R^{\oplus b} \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

Here the top sequence is split exact and in the lower sequence we still have to verify ϕ is injective. Furthermore, all squares commute and $\phi + \psi$ is an isomorphism, because

$\phi + \psi$ is surjective [Eis, Cor. 4.4]. Therefore all vertical maps are isomorphisms. The statement now follows.

Proof of Claim: $\Rightarrow$. We have a short exact sequence $0 \longrightarrow R^{\oplus a} \xrightarrow{\phi} R^{\oplus b} \longrightarrow M \longrightarrow 0$, where $M = R^{\oplus b}/\mathrm{im}(\phi) \cong R^{\oplus c}$ for some c. The long exact sequence of $\mathrm{Tor}_i(k, -)$ reads

$$\cdots \longrightarrow \mathrm{Tor}_1(k, M) \longrightarrow k^{\oplus a} \longrightarrow k^{\oplus b} \longrightarrow M/\mathfrak{m}M \longrightarrow 0.$$

But clearly $\mathrm{Tor}_1(k, M) = 0$, since $M \cong R^{\oplus c}$ for some c. □

We can now derive a combinatorial description of the type of families we are interested in for the affine case.

Proposition 1.2.8. *Let U_σ be a nonsingular affine toric variety defined by a cone σ in a lattice N of rank r. Let $\tau \prec \sigma$ and let $(\rho_1, \ldots, \rho_s) \subset (\rho_1, \ldots, \rho_r)$ be the rays of τ respectively σ. Let S be a k-scheme of finite type and $\chi \in \mathcal{X}^\tau$. The category of equivariant S-flat families $\mathcal{F}$ with fibres $\mathcal{F}_s$ pure equivariant with support $V(\tau) \times k(s)$ and characteristic function χ is equivalent to the category of σ-families $\hat{\mathcal{F}}^\sigma$ over S having the following properties:*

(i) *There are integers $A_1 \leq B_1, \ldots, A_s \leq B_s, A_{s+1}, \ldots, A_r$ such that $\mathcal{F}^\sigma(\lambda_1, \ldots, \lambda_r) = 0$ unless $A_1 \leq \lambda_1 \leq B_1$, ..., $A_s \leq \lambda_s \leq B_s$, $A_{s+1} \leq \lambda_{s+1}$, ..., $A_r \leq \lambda_r$.*

(ii) *For all integers $A_1 \leq \Lambda_1 \leq B_1$, ..., $A_s \leq \Lambda_s \leq B_s$, there is a locally free sheaf $\mathcal{F}^\sigma(\Lambda_1, \ldots, \Lambda_s, \infty, \ldots, \infty)$ on S of finite rank (not all zero) having the following properties. All $\mathcal{F}^\sigma(\Lambda_1, \ldots, \Lambda_s, \lambda_{s+1}, \ldots, \lambda_r)$ are quasi-coherent subsheaves of $\mathcal{F}^\sigma(\Lambda_1, \ldots, \Lambda_s, \infty, \ldots, \infty)$, the maps $x_{s+1}, \ldots, x_r$ are inclusions with S-flat cokernels and there are integers $\lambda_{s+1}, \ldots, \lambda_r$ such that $\mathcal{F}^\sigma(\Lambda_1, \ldots, \Lambda_s, \lambda_{s+1}, \ldots, \lambda_r) = \mathcal{F}^\sigma(\Lambda_1, \ldots, \Lambda_s, \infty, \ldots, \infty)$.*

(iii) *For any $m \in M$, we have $\chi(m) = \mathrm{rk}(\mathcal{F}^\sigma_m)$.*

Proof. Note that if we have a σ-family $\hat{\mathcal{F}}^{\sigma}$ as in (i), (ii), (iii), then all $\mathcal{F}^{\sigma}_{m}$ are locally free of finite rank [Har1, Prop. III.9.1A(e)]. The statement immediately follows from Propositions 1.2.5, 1.2.6 and 1.2.7. $\square$

The general combinatorial description of the kind of families we are interested in easily follows.

Theorem 1.2.9. *Let X be a nonsingular toric variety with fan Δ in a lattice N of rank r. Let $\tau \in \Delta$, then $V(\tau)$ is covered by those U_{σ} where $\sigma \in \Delta$ has dimension r and $\tau \prec \sigma$. Denote these cones by $\sigma_1, \ldots, \sigma_l$. For each $i = 1, \ldots, l$, let $\left(\rho_1^{(i)}, \ldots, \rho_s^{(i)}\right) \subset \left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ be the rays of τ respectively σ_i. Let S be a k-scheme of finite type and $\vec{\chi} \in \mathcal{X}^{\tau}$. The category of equivariant S-flat families $\mathcal{F}$ with fibres $\mathcal{F}_s$ pure equivariant sheaves with support $V(\tau) \times k(s)$ and characteristic function $\vec{\chi}$ is equivalent to the category $\mathcal{C}^{\tau}_{\vec{\chi}}(S)$, which can be described as follows. An object $\hat{\mathcal{F}}^{\Delta}$ of $\mathcal{C}^{\tau}_{\vec{\chi}}(S)$ consists of the following data:*

(i) *For each $i = 1, \ldots, l$ we have a σ_i-family $\hat{\mathcal{F}}^{\sigma_i}$ over S as described in Proposition 1.2.8.*

(ii) *Let $i, j = 1, \ldots, l$. Let $\left\{\rho_{i_1}^{(i)}, \ldots, \rho_{i_p}^{(i)}\right\} \subset \left\{\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right\}$ resp. $\left\{\rho_{j_1}^{(j)}, \ldots, \rho_{j_p}^{(j)}\right\} \subset \left\{\rho_1^{(j)}, \ldots, \rho_r^{(j)}\right\}$ be the rays of $\sigma_i \cap \sigma_j$ in σ_i respectively σ_j, labeled in such a way that $\rho_{i_k}^{(i)} = \rho_{j_k}^{(j)}$ for all $k = 1, \ldots, p$. Now let $\lambda_1^{(i)}, \ldots, \lambda_r^{(i)} \in \mathbb{Z} \cup \{\infty\}$, $\lambda_1^{(j)}, \ldots, \lambda_r^{(j)} \in \mathbb{Z} \cup \{\infty\}$ be such that $\lambda_{i_k}^{(i)} = \lambda_{j_k}^{(j)} \in \mathbb{Z}$ for all $k = 1, \ldots, p$ and $\lambda_n^{(i)} = \lambda_n^{(j)} = \infty$ otherwise. Then*

$$\mathcal{F}^{\sigma_i}\left(\sum_{k=1}^{r} \lambda_k^{(i)} m\left(\rho_k^{(i)}\right)\right) = \mathcal{F}^{\sigma_j}\left(\sum_{k=1}^{r} \lambda_k^{(j)} m\left(\rho_k^{(j)}\right)\right),$$
$$\chi_n^{\sigma_i}\left(\sum_{k=1}^{r} \lambda_k^{(i)} m\left(\rho_k^{(i)}\right)\right) = \chi_n^{\sigma_j}\left(\sum_{k=1}^{r} \lambda_k^{(j)} m\left(\rho_k^{(j)}\right)\right), \ \forall n = 1, \ldots, r.$$

The morphisms of $\mathcal{C}^{\tau}_{\vec{\chi}}(S)$ are described as follows. If $\hat{\mathcal{F}}^{\Delta}$, $\hat{\mathcal{G}}^{\Delta}$ are two objects, then a morphism $\hat{\phi}^{\Delta} : \hat{\mathcal{F}}^{\Delta} \longrightarrow \hat{\mathcal{G}}^{\Delta}$ is a collection of morphisms of σ-families $\{\hat{\phi}^{\sigma_i} : \hat{\mathcal{F}}^{\sigma_i} \longrightarrow$

$\hat{\mathcal{G}}^{\sigma_i}\}_{i=1,\dots,l}$ *over* S *such that for all* i, j *as in* (ii) *one has*

$$\phi^{\sigma_i}\left(\sum_{k=1}^{r} \lambda_k^{(i)} m\left(\rho_k^{(i)}\right)\right) = \phi^{\sigma_j}\left(\sum_{k=1}^{r} \lambda_k^{(j)} m\left(\rho_k^{(j)}\right)\right).$$

Proof. This theorem follows from combining Proposition 1.2.8 and an obvious analogue of Proposition 1.1.9. □

Note that in the context of the above theorem, we can define the following moduli functor

$$\mathfrak{C}_{\vec{\chi}}^{\tau} : (Sch/k)^o \longrightarrow Sets,$$
$$S \mapsto \mathfrak{C}_{\vec{\chi}}^{\tau}(S) = \mathcal{C}_{\vec{\chi}}^{\tau}(S),$$
$$(f : S' \longrightarrow S) \mapsto \mathfrak{C}_{\vec{\chi}}^{\tau}(f) = f^* : \mathcal{C}_{\vec{\chi}}^{\tau}(S) \longrightarrow \mathcal{C}_{\vec{\chi}}^{\tau}(S').$$

For later purposes, we need to define another moduli functor. If S is a k-scheme of finite type, then we define $\mathcal{C}_{\vec{\chi}}^{\tau,fr}(S)$ to be the full[7] subcategory of $\mathcal{C}_{\vec{\chi}}^{\tau}(S)$ consisting of those objects $\hat{\mathcal{F}}^{\Delta}$ with each limiting sheaf $\mathcal{F}^{\sigma_i}(\Lambda_1, \dots, \Lambda_s, \infty, \dots, \infty)$ equal to a sheaf of the form $\mathcal{O}_S^{\oplus n(\Lambda_1,\dots,\Lambda_s)}$ for some $n(\Lambda_1, \dots, \Lambda_s) \in \mathbb{Z}_{\geq 0}$. We would like to think of the objects of this full subcategory as framed objects. This gives rise to a moduli functor $\mathfrak{C}_{\vec{\chi}}^{\tau,fr} : (Sch/k)^o \longrightarrow Sets$.

1.2.3 GIT Quotients

Our goal is to find k-schemes of finite type corepresenting the moduli functors $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau,ss}$ and $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau,s}$. We will achieve this using GIT.

Let X be a nonsingular toric variety, use notation as in Theorem 1.1.10 and fix $\vec{\chi} \in \mathcal{X}^{\tau}$. The integers $A_1^{(i)} \leq B_1^{(i)}, \dots, A_s^{(i)} \leq B_s^{(i)}, A_{s+1}^{(i)}, \dots, A_r^{(i)}$ for $i = 1, \dots, l$ in Theorem 1.1.10 of any pure equivariant sheaf $\mathcal{E}$ on X with support $V(\tau)$ and characteristic function $\vec{\chi}$ are uniquely determined by $\vec{\chi}$ (if we choose them in a maximal respectively minimal way). Note that we have $A_k^{(1)} = \cdots = A_k^{(l)} =: A_k$ and $B_k^{(1)} = \cdots = B_k^{(l)} =: B_k$ for all

[7]Note that we do insist on the full subcategory, meaning we keep the notion of morphism from the category $\mathcal{C}_{\vec{\chi}}^{\tau}(S)$. This allows us to mod out by isomorphisms and relate to GIT later (see next subsection).

$k = 1, \ldots, s$, because of the gluing conditions in Theorem 1.1.10. For all $A_1 \leq \Lambda_1 \leq B_1$, $\ldots$, $A_s \leq \Lambda_s \leq B_s$ we define

$$n(\Lambda_1, \ldots, \Lambda_s) := \lim_{\lambda_{s+1}, \ldots, \lambda_r \to \infty} \chi^{\sigma_i} \left(\sum_{k=1}^{s} \Lambda_k m \left(\rho_k^{(i)} \right) + \sum_{k=s+1}^{r} \lambda_k m \left(\rho_k^{(i)} \right) \right),$$

which is independent of $i = 1, \ldots, l$, because of the gluing conditions in Theorem 1.1.10. For all other values of $\Lambda_1, \ldots, \Lambda_s \in \mathbb{Z}$, we define $n(\Lambda_1, \ldots, \Lambda_s) = 0$. In general, denote by $\mathrm{Gr}(m, n)$ the Grassmannian of m-dimensional subspaces of k^n and by $\mathrm{Mat}(m, n)$ the affine space of $m \times n$ matrices with coefficients in k. Define the following ambient nonsingular quasi-projective variety

$$\begin{aligned} \mathcal{A} = & \prod_{\substack{A_1 \leq \Lambda_1 \leq B_1 \\ \cdots \\ A_s \leq \Lambda_s \leq B_s}} \prod_{i=1}^{l} \prod_{m \in M} \mathrm{Gr}\left(\chi^{\sigma_i}(m), n(\Lambda_1, \ldots, \Lambda_s)\right) \\ & \times \prod_{\substack{A_1 \leq \Lambda_1 \leq B_1 \\ \cdots \\ A_s \leq \Lambda_s \leq B_s}} \mathrm{Mat}(n(\Lambda_1, \ldots, \Lambda_s), n(\Lambda_1 + 1, \Lambda_2 \ldots, \Lambda_s)) \times \cdots \\ & \times \prod_{\substack{A_1 \leq \Lambda_1 \leq B_1 \\ \cdots \\ A_s \leq \Lambda_s \leq B_s}} \mathrm{Mat}(n(\Lambda_1, \ldots, \Lambda_s), n(\Lambda_1, \ldots, \Lambda_{s-1}, \Lambda_s + 1)). \end{aligned} \tag{1.7}$$

There is a natural closed subscheme $\mathcal{N}_{\vec{\chi}}^{\tau}$ of $\mathcal{A}$ with closed points precisely the framed pure Δ-families with support $V(\tau)$ and characteristic function $\vec{\chi}$. This closed subscheme is cut out by requiring the various subspaces of any $k^{n(\Lambda_1, \ldots, \Lambda_s)}$ to form a multi-filtration and by requiring the matrices between the limiting vector spaces $k^{n(\Lambda_1, \ldots, \Lambda_s)}$ to commute and be compatible with the multi-filtrations (see Theorem 1.1.10). Using the standard atlases of Grassmannians, it is not difficult to see that these conditions cut out a closed subscheme. Define the reductive algebraic group

$$G = \Big\{ M \in \prod_{\substack{A_1 \leq \Lambda_1 \leq B_1 \\ \cdots \\ A_s \leq \Lambda_s \leq B_s}} \mathrm{GL}(n(\Lambda_1, \ldots, \Lambda_s), k) \mid \det(M) = 1 \Big\}. \tag{1.8}$$

There is a natural regular action of G on $\mathcal{A}$ leaving $\mathcal{N}_{\vec{\chi}}^{\tau}$ invariant. Two closed points of $\mathcal{N}_{\vec{\chi}}^{\tau}$ correspond to isomorphic elements if and only if they are in the same G-orbit. For any choice of G-equivariant line bundle $\mathcal{L} \in \mathrm{Pic}^G(\mathcal{N}_{\vec{\chi}})$, we get the notion of GIT (semi)stable elements of $\mathcal{N}_{\vec{\chi}}$ with respect to $\mathcal{L}$ [MFK, Sect. 1.4]. We denote the G-invariant open subset of GIT semistable respectively stable elements by $\mathcal{N}_{\vec{\chi}}^{\tau,ss}$ respectively $\mathcal{N}_{\vec{\chi}}^{\tau,s}$. We call a pure equivariant sheaf $\mathcal{E}$ on X with support $V(\tau)$ and characteristic function $\vec{\chi}$ GIT semistable, respectively GIT stable, if its corresponding framed pure Δ-family $\hat{E}^{\Delta}$ is GIT semistable, respectively GIT stable. Using [MFK, Thm. 1.10], we obtain that there exists a categorical quotient $\pi : \mathcal{N}_{\vec{\chi}}^{\tau,ss} \longrightarrow \mathcal{N}_{\vec{\chi}}^{\tau,ss} /\!\!/ G$, where $\mathcal{N}_{\vec{\chi}}^{\tau,ss} /\!\!/ G$ is a quasi-projective scheme of finite type over k which we denote by $\mathcal{M}_{\vec{\chi}}^{\tau,ss}$. Moreover, there exists an open subset $U \subset \mathcal{M}_{\vec{\chi}}^{\tau,ss}$, such that $\pi^{-1}(U) = \mathcal{N}_{\vec{\chi}}^{\tau,s}$ and $\varpi = \pi|_{\mathcal{N}_{\vec{\chi}}^{\tau,s}} : \mathcal{N}_{\vec{\chi}}^{\tau,s} \longrightarrow U = \mathcal{N}_{\vec{\chi}}^{\tau,s}/G$ is a geometric quotient. We define $\mathcal{M}_{\vec{\chi}}^{\tau,s} = \mathcal{N}_{\vec{\chi}}^{\tau,s}/G$. The fibres of closed points of ϖ are precisely the G-orbits of closed points of $\mathcal{N}_{\vec{\chi}}^{\tau,s}$, or equivalently, the equivariant isomorphism classes of GIT stable pure equivariant sheaves on X with support $V(\tau)$ and characteristic function $\vec{\chi}$. It seems natural to think of $\mathcal{M}_{\vec{\chi}}^{\tau,ss}$ and $\mathcal{M}_{\vec{\chi}}^{\tau,s}$ as moduli spaces. Before making this more precise, we would like to make the problem more geometric. Assume in addition X is projective and fix an ample line bundle $\mathcal{O}_X(1)$ on X. The natural notion of stability of coherent sheaves on X is Gieseker stability, which depends on the choice of $\mathcal{O}_X(1)$ [HL, Sect. 1.2].

Assumption 1.2.10. *Let X be a nonsingular projective toric variety defined by a fan Δ. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and $\tau \in \Delta$. Then for any $\vec{\chi} \in \mathcal{X}^{\tau}$, let $\mathcal{L}_{\vec{\chi}}^{\tau} \in \mathrm{Pic}^G(\mathcal{N}_{\vec{\chi}}^{\tau})$ be an equivariant line bundle such that any pure equivariant sheaf $\mathcal{E}$ on*

X with support $V(\tau)$ and characteristic function $\vec{\chi}$ is GIT semistable respectively GIT stable w.r.t. $\mathcal{L}^{\tau}_{\vec{\chi}}$ if and only if $\mathcal{E}$ is Gieseker semistable respectively Gieseker stable.

We will refer to equivariant line bundles as in this assumption as equivariant line bundles matching Gieseker and GIT stability. So far, the author cannot prove the existence of such equivariant line bundles in full generality. However, in subsection 1.2.5, we will construct such (ample) equivariant line bundles for the case $\tau = 0$, i.e. torsion free equivariant sheaves (Theorem 1.2.22). As a by-product, for reflexive equivariant sheaves, we can always construct particularly simple ample equivariant line bundles matching GIT stability and μ-stability (subsection 1.3.4). For pure equivariant sheaves of lower dimension, the existence of equivariant line bundles matching Gieseker and GIT stability can be proved in specific examples. Note that in the classical construction of moduli spaces of Gieseker (semi)stable sheaves, one also needs to match GIT stability of the underlying GIT problem to Gieseker stability (see [HL, Thm. 4.3.3]).

We can now prove the following results regarding representability and corepresentability.

Proposition 1.2.11. *Let X be a nonsingular projective toric variety defined by a fan Δ. Let $\mathcal{O}_X(1)$ be an ample line bundle on X, $\tau \in \Delta$ and $\vec{\chi} \in \mathcal{X}^{\tau}$. Then $\mathfrak{C}^{\tau,fr}_{\vec{\chi}}$ is represented by $\mathcal{N}^{\tau}_{\vec{\chi}}$. Assume we have an equivariant line bundle $\mathcal{L}^{\tau}_{\vec{\chi}}$ matching Gieseker and GIT stability. Let $\mathfrak{C}^{\tau,ss,fr}_{\vec{\chi}}$ respectively $\mathfrak{C}^{\tau,s,fr}_{\vec{\chi}}$ be the moduli subfunctors of $\mathfrak{C}^{\tau,fr}_{\vec{\chi}}$ with Gieseker semistable respectively geometrically Gieseker stable fibres. Then $\mathfrak{C}^{\tau,ss,fr}_{\vec{\chi}}$ is represented*[8] *by $\mathcal{N}^{\tau,ss}_{\vec{\chi}}$ and $\mathfrak{C}^{\tau,s,fr}_{\vec{\chi}}$ is represented by $\mathcal{N}^{\tau,s}_{\vec{\chi}}$.*

Proof. Recall that for V a k-vector space of dimension n and $0 \leq m \leq n$, one has a moduli functor of Grassmannians $\mathcal{G}r(m, V) : (Sch/k)^o \longrightarrow Sets$ (e.g. [HL, Exm. 2.2.2]), where $\mathcal{G}r(m, V)(S)$ consists of quasi-coherent subsheaves $\mathcal{E} \subset V \otimes \mathcal{O}_S$ with S-flat cokernel of rank $n - m$ and $\mathcal{G}r(m, V)(f) = f^*$ is pull-back. Let $\mathcal{U}$ be the sheaf of sections of the tautological bundle $U \longrightarrow \mathrm{Gr}(m, V)$, then it is not difficult to see

[8] In this setting, it is understood we use a choice of $\mathcal{L}^{\tau}_{\vec{\chi}}$ as in Assumption 1.2.10 to define our notion of GIT stability and hence $\mathcal{N}^{\tau,ss}_{\vec{\chi}}$, $\mathcal{N}^{\tau,s}_{\vec{\chi}}$, $\mathcal{M}^{\tau,ss}_{\vec{\chi}}$, $\mathcal{M}^{\tau,s}_{\vec{\chi}}$.

that $\mathcal{U}$ is a universal family. Consequently, $\mathcal{G}r(m, V)$ is represented by $\mathrm{Gr}(m, V)$. Likewise, for m, n arbitrary nonnegative integers, one has a moduli functor of matrices $\mathcal{M}at(m, n) : (Sch/k)^o \longrightarrow Sets$, where $\mathcal{M}at(m, n)(S)$ consists of all morphisms $\phi : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{O}_S^{\oplus m}$ and $\mathcal{M}at(m, n)(f) = f^*$ is pull-back. Let (x_{ij}) be a matrix of coordinates on $\mathrm{Mat}(m, n)$. Then (x_{ij}) induces a morphism $\xi : \mathcal{O}_{\mathrm{Mat}(m,n)}^{\oplus n} \longrightarrow \mathcal{O}_{\mathrm{Mat}(m,n)}^{\oplus m}$. Again, it is easy to see that ξ is a universal family. Consequently, $\mathcal{M}at(m, n)$ is represented by $\mathrm{Mat}(m, n)$. Now consider $\mathcal{N}_{\vec{\chi}}^{\tau}$ as a closed subscheme of $\mathcal{A}$, where $\mathcal{A}$ is defined in equation (1.7). Consider the universal families on each of the components $\mathrm{Gr}(\chi^{\sigma_i}(m), n(\Lambda_1, \ldots, \Lambda_s))$ and $\mathrm{Mat}(n(\Lambda_1, \ldots, \Lambda_s), n(\Lambda_1, \ldots, \Lambda_i{+}1, \ldots, \Lambda_s))$ of $\mathcal{A}$. Pulling back these universal families along projections $\mathcal{A} \longrightarrow \mathrm{Gr}(\chi^{\sigma_i}(m), n(\Lambda_1, \ldots, \Lambda_s))$ respectively $\mathcal{A} \longrightarrow \mathrm{Mat}(n(\Lambda_1, \ldots, \Lambda_s), n(\Lambda_1, \ldots, \Lambda_i + 1, \ldots, \Lambda_s))$ gives a universal family on $\mathcal{A}$. Pulling this universal family back along the closed immersion $\mathcal{N}_{\vec{\chi}}^{\tau} \hookrightarrow \mathcal{A}$ gives a universal family of $\mathfrak{C}_{\vec{\chi}}^{\tau, fr}$. Consequently, $\mathfrak{C}_{\vec{\chi}}^{\tau, fr}$ is represented by $\mathcal{N}_{\vec{\chi}}^{\tau}$. Since $\mathcal{N}_{\vec{\chi}}^{\tau, s} \subset \mathcal{N}_{\vec{\chi}}^{\tau, ss} \subset \mathcal{N}_{\vec{\chi}}^{\tau}$ are open subschemes defined by properties on the fibres, the rest is easy. $\square$

Theorem 1.2.12. *Let X be a nonsingular projective toric variety defined by a fan Δ. Let $\mathcal{O}_X(1)$ be an ample line bundle on X, $\tau \in \Delta$ and $\vec{\chi} \in \mathcal{X}^{\tau}$. Assume we have an equivariant line bundle matching Gieseker and GIT stability. Then $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau, ss}$ is corepresented by the quasi-projective k-scheme of finite type $\mathcal{M}_{\vec{\chi}}^{\tau, ss}$. Moreover, there is an open subset $\mathcal{M}_{\vec{\chi}}^{\tau, s} \subset \mathcal{M}_{\vec{\chi}}^{\tau, ss}$ such that $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau, s}$ is corepresented by $\mathcal{M}_{\vec{\chi}}^{\tau, s}$ and $\mathcal{M}_{\vec{\chi}}^{\tau, s}$ is a coarse moduli space.*

Proof. Define the moduli functor

$$\begin{aligned}
&\mathcal{G} : (Sch/k)^o \longrightarrow Sets, \\
&\mathcal{G}(S) = \Big\{ \Phi \in \prod_{\substack{A_1 \leq \Lambda_1 \leq B_1 \\ \cdots \\ A_s \leq \Lambda_s \leq B_s}} \mathrm{Aut}\left(\mathcal{O}_S^{\oplus n(\Lambda_1, \ldots, \Lambda_s)}\right) \ | \ \det(\Phi) = 1 \Big\}, \\
&\mathcal{G}(f) = f^*.
\end{aligned}$$

It is easy to see that $\mathcal{G}$ is naturally represented by G defined in equation (1.8). For any $S \in Sch/k$ we have a natural action of $\mathcal{G}(S)$ on $\mathfrak{C}_{\vec{\chi}}^{\tau, ss, fr}(S)$ and a natural action of

$\mathrm{Hom}(S,G)$ on $\mathrm{Hom}(S,\mathcal{N}_{\vec{\chi}}^{\tau,ss})$. Since we have canonical isomorphisms $\mathcal{G} \cong \mathrm{Hom}(-,G)$ and $\mathfrak{C}_{\vec{\chi}}^{\tau,ss,fr} \cong \mathrm{Hom}(-,\mathcal{N}_{\vec{\chi}}^{\tau,ss})$ (Proposition 1.2.11), we get an isomorphism of functors

$$\mathfrak{C}_{\vec{\chi}}^{\tau,ss,fr}/\mathcal{G} \cong \mathrm{Hom}(-,\mathcal{N}_{\vec{\chi}}^{\tau,ss})/\mathrm{Hom}(-,G).$$

Since $\mathcal{M}_{\vec{\chi}}^{\tau,ss} = \mathcal{N}_{\vec{\chi}}^{\tau,ss}/\!\!/G$ is a categorical quotient [HL, Def. 4.2.1], we conclude that $\mathcal{M}_{\vec{\chi}}^{\tau,ss}$ corepresents $\mathrm{Hom}(-,\mathcal{N}_{\vec{\chi}}^{\tau,ss})/\mathrm{Hom}(-,G)$ and therefore $\mathfrak{C}_{\vec{\chi}}^{\tau,ss,fr}/\mathcal{G}$. We also have obvious natural transformations[9]

$$\mathfrak{C}_{\vec{\chi}}^{\tau,ss,fr}/\mathcal{G} = \left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss,fr}/\cong\right) \Longrightarrow \left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss}/\cong\right) \overset{\cong}{\Longrightarrow} \underline{\mathcal{M}}_{\vec{\chi}}^{\prime\tau,ss} \Longrightarrow \underline{\mathcal{M}}_{\vec{\chi}}^{\tau,ss},$$

where the first natural transformation is injective over all $S \in Sch/k$ and we use Theorem 1.2.9 to obtain the isomorphism $\left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss}/\cong\right) \cong \underline{\mathcal{M}}_{\vec{\chi}}^{\prime\tau,ss}$. The moduli functors $\underline{\mathcal{M}}_{\vec{\chi}}^{\prime\tau,ss}$, $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau,ss}$ have been introduced in subsection 1.2.1.

We have to show that $\mathcal{M}_{\vec{\chi}}^{\tau,ss}$ also corepresents $\left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss}/\cong\right) \cong \underline{\mathcal{M}}_{\vec{\chi}}^{\prime\tau,ss}$ and $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau,ss}$. This can be done by using open affine covers on which locally free sheaves respectively equivariant invertible sheaves trivialise. More precisely, we know $\mathcal{M}_{\vec{\chi}}^{\tau,ss}$ corepresents $\left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss,fr}/\cong\right)$

$$\Phi : \left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss,fr}/\cong\right) \longrightarrow \mathrm{Hom}(-,\mathcal{M}_{\vec{\chi}}^{\tau,ss}).$$

For a fixed k-scheme S of finite type, define a morphism

$$\tilde{\Phi}_S : \left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss}/\cong\right)(S) \longrightarrow \mathrm{Hom}(S,\mathcal{M}_{\vec{\chi}}^{\tau,ss}),$$

as follows. Let $[\hat{\mathcal{F}}^{\Delta}] \in \left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss}/\cong\right)(S)$ and let $\{\iota_\alpha : U_\alpha \hookrightarrow S\}_{\alpha\in I}$ be an open affine cover of S on which the limiting locally free sheaves of $\hat{\mathcal{F}}^{\Delta}$ (i.e. the $\mathcal{F}^{\sigma_i}(\Lambda_1,\ldots,\Lambda_s,\infty,\ldots,\infty)$) trivialise. Then $[\iota_\alpha^*\hat{\mathcal{F}}^{\Delta}] \in \left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss,fr}/\cong\right)(U_\alpha)$ and therefore we get a morphism $F_\alpha = \Phi_{U_\alpha}([\iota_\alpha^*\hat{\mathcal{F}}^{\Delta}]) : U_\alpha \longrightarrow \mathcal{M}_{\vec{\chi}}^{\tau,ss}$ for all $\alpha \in I$. From the fact that Φ is a natural transforma-

[9]Here the symbol $\cong$ in the quotients refers to taking isomorphism classes in the corresponding categories.

tion, it is easy to see that $\{F_\alpha\}_{\alpha\in I}$ glues to a morphism $F : S \longrightarrow \mathcal{M}_{\vec{\chi}}^{\tau,ss}$ independent of the choice of open affine cover. This defines $\tilde{\Phi}_S([\hat{\mathcal{F}}^\Delta])$. One readily verifies this defines a natural transformation $\tilde{\Phi}$ fitting in the commutative diagram

$$\begin{tikzcd}
\left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss,fr}/\cong\right) \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\Phi"'] & \left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss}/\cong\right) \arrow[dl, Rightarrow, "\tilde{\Phi}"] \\
\mathrm{Hom}(-,\mathcal{M}_{\vec{\chi}}^{\tau,ss}). &
\end{tikzcd}$$

The fact that $\mathcal{M}_{\vec{\chi}}^{\tau,ss}$ corepresents $\left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss,fr}/\cong\right)$ implies that it corepresents $\left(\mathfrak{C}_{\vec{\chi}}^{\tau,ss}/\cong\right) \cong \underline{\mathcal{M}}_{\vec{\chi}}^{\prime\tau,ss}$ too. Similarly, but easier, one proves $\mathcal{M}_{\vec{\chi}}^{\tau,ss}$ corepresents $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau,ss}$.

The proof up to now also holds in the case "Gieseker stable". By saying $\mathcal{M}_{\vec{\chi}}^{\tau,s}$ is a coarse moduli space, we mean $\mathcal{M}_{\vec{\chi}}^{\tau,s}$ corepresents $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau,s}$ and $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau,s}(k) \longrightarrow \mathrm{Hom}(k,\mathcal{M}_{\vec{\chi}}^{\tau,s})$ is bijective for any algebraically closed field k of characteristic 0. This is clearly the case since the closed points of $\mathcal{M}_{\vec{\chi}}^{\tau,s}$ are precisely the equivariant isomorphism classes of Gieseker stable equivariant sheaves on X with support $V(\tau)$ and characteristic function $\vec{\chi}$. □

We end this subsection by discussing how the theory developed in section 1.2 so far generalises to the case of possibly reducible support. Again, no essentially new ideas will occur, only the notation will become more cumbersome. Let X be a nonsingular toric variety defined by a fan Δ. Let $\tau_1, \ldots, \tau_a \in \Delta$ be some cones of some dimension s. Let $\sigma_1, \ldots, \sigma_l \in \Delta$ be all cones of maximal dimension having a cone τ_α as a face. In subsection 1.1.2, we discussed how to describe pure equivariant sheaves on X with support $V(\tau_1) \cup \cdots \cup V(\tau_a)$. We define characteristic functions of such sheaves as in Definition 1.2.1, we denote the set of all such characteristic functions by $\mathcal{X}^{\tau_1,\ldots,\tau_a}$ and we define the moduli functors $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,ss}$, $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,s}$ as in subsection 1.2.1. The obvious analogue of Theorem 1.2.9 holds. The only new condition discussed in subsection 1.1.2 is condition (1.6) in Proposition 1.1.11. This is an open condition on matrix coefficients so it can be easily incorporated. We can define a k-scheme of finite type $\mathcal{N}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a}$ and a reductive algebraic group G acting regularly on it as earlier in this subsection. Performing the GIT quotients w.r.t. an equivariant line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{N}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a})$ gives rise to a categorical quotient $\mathcal{M}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,ss}$ and a geometric quotient $\mathcal{M}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,s}$ (both are quasi-projective schemes of finite type over k). It is straightforward to prove the following result.

Theorem 1.2.13. *Let X be a nonsingular projective toric variety defined by a fan Δ. Let $\mathcal{O}_X(1)$ be an ample line bundle on X, $\tau_1, \ldots, \tau_a \in \Delta$ some cones of dimension s and $\vec{\chi} \in \mathcal{X}^{\tau_1,\ldots,\tau_a}$. Assume we have an equivariant line bundle matching Gieseker and GIT stability. Then $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,ss}$ is corepresented by the quasi-projective k-scheme of finite type $\mathcal{M}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,ss}$. Moreover, there is an open subset $\mathcal{M}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,s} \subset \mathcal{M}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,ss}$ such that $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,s}$ is corepresented by $\mathcal{M}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,s}$ and $\mathcal{M}_{\vec{\chi}}^{\tau_1,\ldots,\tau_a,s}$ is a coarse moduli space.*

It is important to note that the moduli spaces of Theorems 1.2.12 and 1.2.13 are explicit and combinatorial in nature and their construction is very different from the construction of general moduli spaces of Gieseker (semi)stable sheaves, which makes use of Quot schemes and requires boundedness results [HL, Ch. 1–4].

1.2.4 Chern Characters of Equivariant Sheaves on Toric Varieties

The Hilbert polynomial of a pure equivariant sheaf on a nonsingular projective toric variety with ample line bundle is entirely determined by the characteristic function of that sheaf. We will prove this by a short general argument in the following proposition.

Proposition 1.2.14. *Let X be a nonsingular projective toric variety defined by a fan Δ. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and let $\tau_1, \ldots, \tau_a \in \Delta$ be some cones of dimension s. Then the Hilbert polynomial of any pure equivariant sheaf on X with support $V(\tau_1) \cup \cdots \cup V(\tau_a)$ and characteristic function $\vec{\chi}$ is the same. We refer to this polynomial as the Hilbert polynomial associated to $\vec{\chi}$ and denote the collection of all characteristic functions $\vec{\chi} \in \mathcal{X}^{\tau_1,\ldots,\tau_a}$ having the same associated Hilbert polynomial P by $\mathcal{X}_P^{\tau_1,\ldots,\tau_a}$.*

Proof. Assume the fan Δ lies in a lattice N of rank r. Let $\mathcal{E}$ be a pure equivariant sheaf on X with support $V(\tau_1) \cup \cdots \cup V(\tau_a)$ and characteristic function $\vec{\chi}$. The Hilbert polynomial of $\mathcal{E}$ is the unique polynomial $P_{\mathcal{E}}(t) \in \mathbb{Q}[t]$ satisfying

$$P_{\mathcal{E}}(t) = \chi(\mathcal{E} \otimes \mathcal{O}_X(t)) = \sum_{i=0}^{r} (-1)^i \mathrm{dim}(H^i(X, \mathcal{E} \otimes \mathcal{O}_X(t))),$$

for all $t \in \mathbb{Z}$. Clearly, for a fixed $t \in \mathbb{Z}$, $\chi(\mathcal{E} \otimes \mathcal{O}_X(t))$ only depends on the equivariant isomorphism class $[\mathcal{E}]$ hence only on the isomorphism class $[\hat{E}^{\Delta}]$, where $\hat{E}^{\Delta}$ is the pure Δ-family corresponding to $\mathcal{E}$. Note that $\chi(\mathcal{E} \otimes \mathcal{O}_X(t))$ does not vary if we vary the module structure of $\mathcal{E}$ [Har1, Prop. III.2.6]. The module structure of $\mathcal{E}$ is encoded in the k-linear maps $\chi_1^{\sigma_i}(\vec{\lambda}), \ldots, \chi_r^{\sigma_i}(\vec{\lambda})$, where $i = 1, \ldots, l$ and $\vec{\lambda} \in \mathbb{Z}^r \cong M$. Here $\sigma_1, \ldots, \sigma_l$ are all cones of maximal dimension having a τ_α ($\alpha = 1, \ldots, a$) as a face. Therefore, $\chi(\mathcal{E} \otimes \mathcal{O}_X(t))$ can only depend on the dimensions of the weight spaces of $\hat{E}^{\Delta}$, i.e. only on $\vec{\chi}$. $\square$

The fact that the Hilbert polynomial of a pure equivariant sheaf on a nonsingular projective toric variety with ample line bundle is entirely determined by the characteristic function of that sheaf can be made more specific by using a formula due to Klyachko. Klyachko gives an explicit formula for the Chern character of a torsion free equivariant sheaf on a nonsingular toric variety [Kly4, Sect. 1.2, 1.3]. We will now discuss Klyachko's Formula. The reader has to be aware of the fact that we follow Perling's convention of ascending directions for the maps of σ-families, as opposed to Klyachko's convention of descending directions. This results in some minus signs compared to Klyachko's formulae.

Definition 1.2.15. Let $\{E(\lambda_1, \ldots, \lambda_r)\}_{(\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r}$ be a collection of finite-dimensional k-vector spaces. For each $i = 1, \ldots, r$, we define a $\mathbb{Z}$-linear operator Δ_i on the free abelian group generated by the vector spaces $\{E(\lambda_1, \ldots, \lambda_r)\}_{(\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r}$ determined by

$$\Delta_i E(\lambda_1, \ldots, \lambda_r) = E(\lambda_1, \ldots, \lambda_r) - E(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_r),$$

for any $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}$. This allows us to define $[E](\lambda_1, \ldots, \lambda_r) = \Delta_1 \cdots \Delta_r E(\lambda_1, \ldots, \lambda_r)$ for any $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}$. One can then define dimension dim as a $\mathbb{Z}$-linear operator on the free abelian group generated by the vector spaces $\{E(\lambda_1, \ldots, \lambda_r)\}_{(\lambda_1,\ldots,\lambda_r)\in\mathbb{Z}^r}$ in the obvious way. It now makes sense to consider $\dim([E](\lambda_1, \ldots, \lambda_r))$ for any $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}$. For example

$$
\begin{aligned}
\dim([E](\lambda)) &= \dim(E(\lambda)) - \dim(E(\lambda - 1)), \\
\dim([E](\lambda_1, \lambda_2)) &= \dim(E(\lambda_1, \lambda_2)) - \dim(E(\lambda_1 - 1, \lambda_2)) - \dim(E(\lambda_1, \lambda_2 - 1)) \\
&\quad + \dim(E(\lambda_1 - 1, \lambda_2 - 1)),
\end{aligned}
$$

for any $\lambda, \lambda_1, \lambda_2 \in \mathbb{Z}$. $\oslash$

Proposition 1.2.16 (Klyachko's Formula)**.** *Let X be a nonsingular quasi-projective toric variety with fan Δ in a lattice N of rank r. Let $\sigma_1, \ldots, \sigma_l \in \Delta$ be the cones of dimension r and for each $i = 1, \ldots, l$, let $\left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ be the rays of σ_i. Let $\tau_1, \ldots, \tau_a \in \Delta$ be some cones of dimension s. Then for any pure equivariant sheaf $\mathcal{E}$ on X with support $V(\tau_1) \cup \cdots \cup V(\tau_a)$ and corresponding pure Δ-family $\hat{E}^\Delta$, we have*

$$
\mathrm{ch}(\mathcal{E}) = \sum_{\sigma \in \Delta,\ \vec{\lambda} \in \mathbb{Z}^{\dim(\sigma)}} (-1)^{\mathrm{codim}(\sigma)} \dim([E^\sigma](\vec{\lambda})) \exp\left(- \sum_{\rho \in \sigma(1)} \langle \vec{\lambda}, n(\rho) \rangle V(\rho) \right).
$$

In this proposition, $\sigma(1)$ means the collection of rays of σ. Likewise, we denote the collection of all rays of Δ by $\Delta(1)$. Any cone $\sigma \in \Delta$ is a face of a cone σ_i of dimension r. Assume σ has dimension t. Let $\hat{E}^\sigma$ denotes the σ-family corresponding to the equivariant coherent sheaf $\mathcal{E}|_{U_\sigma}$. Note that if σ has no τ_α as a face, $\mathcal{E}|_{U_\sigma} = 0$ and $\mathcal{E}|_{U_\sigma}$ is pure equivariant otherwise by Propositions 1.1.3, 1.1.9. Let $\left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ be the rays of σ_i, with first integral lattice points $\left(n\left(\rho_1^{(i)}\right), \ldots, n\left(\rho_r^{(i)}\right)\right)$, and let without loss of generality $\left(\rho_1^{(i)}, \ldots, \rho_t^{(i)}\right) \subset \left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ be the rays of σ. Then $E^\sigma(\lambda_1, \ldots, \lambda_t) = E^{\sigma_i}(\lambda_1, \ldots, \lambda_t, \infty, \ldots, \infty)$ for all $\lambda_1, \ldots, \lambda_t \in \mathbb{Z}$ by Proposition 1.1.9. Although Klyachko's Formula as presented in this proposition was originally stated for torsion free equivariant sheaves on nonsingular toric varieties only [Kly4, Sect. 1.2, 1.3], it holds for pure equivariant sheaves on nonsingular quasi-projective toric varieties in general. This can be seen by noting that any equivariant coherent sheaf on a nonsingular quasi-projective toric variety admits a finite locally free equivariant resolution ([CG, Prop. 5.1.28]) and applying Klyachko's Formula to the resolution. Finally, note that Proposition 1.2.14 now follows from Klyachko's Formula (Proposition 1.2.16) and the Hirzebruch–Riemann–Roch Theorem [Har1, Thm. A.4.1].

We end this subsection by proving a combinatorial result we will use in the next subsection. As a nice aside, applying this combinatorial result for $s = 1$ to the above proposition, we recover a simple formula for the first Chern class due to Klyachko [Kly4, Sect. 1.2, 1.3].

Proposition 1.2.17. *Let Δ be a simplicial fan in a lattice N of rank r with support $|\Delta| = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\tau \in \Delta$ be a cone of dimension s. Then*

$$(-1)^{r-s} \sum_{a=0}^{r-s} (-1)^a \#\{\sigma \in \Delta \mid \tau \prec \sigma \text{ and } \dim(\sigma) = a + s\} = 1.$$

Proof. Choose a basis for $N \otimes_{\mathbb{Z}} \mathbb{R}$ such that the first s basis vectors generate τ and let $N \otimes_{\mathbb{Z}} \mathbb{R}$ be endowed with the standard inner product. Let x be in the relative interior of τ and fix $\epsilon > 0$. Define a normal space $N_x\tau \cong \mathbb{R}^{r-s}$ to τ at x and a sphere $S^{r-s-1} \subset N_x\tau$ using the standard inner product

$$N_x\tau = \{x + v \mid v \perp \tau\}, \ S^{r-s-1} = \{x + v \mid v \perp \tau, \ |v| = \epsilon\}.$$

By definition, the union of all cones of Δ is $N \otimes_{\mathbb{Z}} \mathbb{R}$. Choosing $\epsilon > 0$ sufficiently small

$$\{\sigma \cap S^{r-s-1} \mid \sigma \in \Delta, \ \tau \prec \sigma, \ \dim(\sigma) > s\},$$

forms a triangulation of S^{r-s-1}. Therefore

$$\sum_{a=1}^{r-s} (-1)^{a-1} \#\{\sigma \in \Delta \mid \tau \prec \sigma \text{ and } \dim(\sigma) = a + s\} = e(S^{r-s-1}),$$

where $e(S^{r-s-1})$ is the Euler characteristic of S^{r-s-1} [Mun, Sect. 22], which satisfies $e(S^{r-s-1}) = 0$ when $r-s$ is even and $e(S^{r-s-1}) = 2$ when $r-s$ is odd [Mun, Thm. 31.8]. □

Corollary 1.2.18. *Let X be a nonsingular projective toric variety with fan Δ in a lattice N of rank r. Let $\sigma_1, \ldots, \sigma_l \in \Delta$ be the cones of dimension r and for each $i = 1, \ldots, l$, let $\left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ be the rays of σ_i. Let $\tau_1, \ldots, \tau_a \in \Delta$ be some cones of dimension s. Then for any pure equivariant sheaf $\mathcal{E}$ on X with support $V(\tau_1) \cup \cdots \cup V(\tau_a)$ and corresponding pure Δ-family $\hat{E}^\Delta$, we have*

$$c_1(\mathcal{E}) = -\sum_{\rho \in \Delta(1),\ \lambda \in \mathbb{Z}} \lambda \dim([E^\rho](\lambda)) V(\rho).$$

Proof. Using Klyachko's Formula (Proposition 1.2.16), we obtain

$$\begin{aligned} c_1(\mathcal{E}) &= -\sum_{\sigma \in \Delta,\ \vec{\lambda} \in \mathbb{Z}^{\dim(\sigma)}} \sum_{\rho \in \sigma(1)} (-1)^{\operatorname{codim}(\sigma)} \dim([E^\sigma](\vec{\lambda})) \langle \vec{\lambda}, n(\rho) \rangle V(\rho) \\ &= -\sum_{\rho \in \Delta(1)} \sum_{\lambda \in \mathbb{Z}} \sum_{\rho \prec \sigma \in \Delta} (-1)^{\operatorname{codim}(\sigma)} \lambda \dim([E^\rho](\lambda)) V(\rho). \end{aligned}$$

The corollary follows from applying Proposition 1.2.17 with $\tau = \rho \in \Delta(1)$ and $s = 1$. □

1.2.5 Matching Stability

In this subsection, we will prove the existence of ample equivariant line bundles matching Gieseker and GIT stability for torsion free equivariant sheaves on nonsingular projective toric varieties (Theorem 1.2.22). Along the way, we derive a number of important preparatory results as well as some results which are interesting on their own.

As we have seen in Proposition 1.1.5, for a G-equivariant coherent sheaf, it is enough to test purity just for G-equivariant coherent subsheaves. It is natural to ask whether an analogous property holds for Gieseker stability.

Proposition 1.2.19. *Let X be a projective variety with ample line bundle $\mathcal{O}_X(1)$. Let G be an affine algebraic group acting regularly on X. Let $\mathcal{E}$ be a pure G-equivariant sheaf on X. Then $\mathcal{E}$ is Gieseker semistable if and only if $p_{\mathcal{F}} \leq p_{\mathcal{E}}$ for any proper G-equivariant coherent subsheaf $\mathcal{F}$. Now assume $G = T$ is an algebraic torus. Then $\mathcal{E}$ is Gieseker stable if and only if $p_{\mathcal{F}} < p_{\mathcal{E}}$ for any proper equivariant coherent subsheaf $\mathcal{F}$.*

Proof. The statement on Gieseker semistability is clear by noting that the Harder–Narasimhan filtration of $\mathcal{E}$ is G-equivariant. For the definition of the Harder–Narasimhan filtration see [HL, Def. 1.3.2]. Now assume $G = T$ is an algebraic torus and for any proper equivariant coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ one has $p_{\mathcal{F}} < p_{\mathcal{E}}$. We have to prove $\mathcal{E}$ is Gieseker stable. Since $\mathcal{E}$ is clearly Gieseker semistable, it contains a unique nontrivial maximal Gieseker polystable subsheaf $\mathcal{S}$ with the same reduced Hilbert polynomial as $\mathcal{E}$. The sheaf $\mathcal{S}$ is called the socle of $\mathcal{E}$ [HL, Lem. 1.5.5]. Because of uniqueness, $\mathcal{S}$ is an equivariant coherent subsheaf hence $\mathcal{E} = \mathcal{S}$. Therefore, there are $n \in \mathbb{Z}_{>0}$ mutually non-isomorphic Gieseker stable sheaves $\mathcal{E}_1, \ldots, \mathcal{E}_n$, positive integers $m_1, \ldots, m_n$ and an isomorphism of coherent sheaves $\theta : \bigoplus_{i=1}^n \mathcal{E}_i^{\oplus m_i} \xrightarrow{\cong} \mathcal{E}$. Clearly, $p_{\mathcal{E}_1} = \cdots = p_{\mathcal{E}_n} = p_{\mathcal{E}}$. We claim that each $\mathcal{E}_i$ is isomorphic to an equivariant coherent subsheaf of $\mathcal{E}$. This would prove the proposition. We proceed in two steps. First we show $\mathcal{E}_i$ a priori admits an equivariant structure for each $i = 1, \ldots, n$. Subsequently, we use representation theory of the algebraic torus T. Denote by Φ the equivariant structure on $\mathcal{E}$, by $\sigma : T \times X \longrightarrow X$ the regular action of T on X, by $p_2 : T \times X \longrightarrow X$ projection and by T_{cl} the set of closed points of T.

We claim each $\mathcal{E}_i$ admits an equivariant structure. By Proposition 1.3.4 of subsection 1.3.2, it is enough to prove $\mathcal{E}_i$ is invariant, i.e. $\sigma^*\mathcal{E}_i \cong p_2^*\mathcal{E}_i$. By Propositions 1.3.2, 1.3.3 of subsection 1.3.1, this is equivalent to $t^*\mathcal{E}_i \cong \mathcal{E}_i$ for all $t \in T_{cl}$. Note that we use $G = T$ is an algebraic torus. We now prove $t^*\mathcal{E}_i \cong \mathcal{E}_i$ for any $i = 1, \ldots, n$, $t \in T_{cl}$. Since each $\mathcal{E}_i$ is indecomposable, the Krull–Schmidt property of the category of coherent sheaves on X [Ati, Thm. 2] implies for any $i = 1, \ldots, n$ and $t \in T_{cl}$ there is an isomorphism $t^*\mathcal{E}_i \cong \mathcal{E}_j$

for some $j = 1, \ldots, n$. Note that for $i, j = 1, \ldots, n$ we have [HL, Prop. 1.2.7]

$$\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) = \begin{cases} k & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Fix $i = 1, \ldots, n$ and define $\Gamma_j = \{t \in T_{cl} \mid t^*\mathcal{E}_i \cong \mathcal{E}_j\}$ for each $j = 1, \ldots, n$. Each Γ_j can be written as

$$\Gamma_j = \{t \in T_{cl} \mid \dim(\mathrm{Hom}(t^*\mathcal{E}_i, \mathcal{E}_j)) \geq 1\},$$

by using the fact that any morphism between Gieseker stable sheaves with the same reduced Hilbert polynomial is zero or an isomorphism [HL, Prop. 1.2.7]. We deduce each Γ_j is a closed subset by a semicontinuity argument. But each Γ_j is also open, because its complement is the disjoint union $\coprod_{k \neq j} \Gamma_k$. Connectedness of T_{cl} implies $T_{cl} = \Gamma_i$, since $1 \in \Gamma_i$. Therefore, we obtain an equivariant structure $\Psi^{(i)}$ on $\mathcal{E}_i$ for each $i = 1, \ldots, n$.

Fix $i = 1, \ldots, n$. Using θ, we obtain $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}) \cong k^{\oplus m_i}$ and any nonzero element of $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E})$ is injective. The equivariant structures Φ, $\Psi^{(i)}$ give us a regular representation of T_{cl}

$$\begin{aligned} &T_{cl} \times \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}) \longrightarrow \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}), \\ &t \cdot f = \Phi_1^{-1} \circ \Phi_{t^{-1}} \circ (t^{-1})^*(f) \circ \Psi_{t^{-1}}^{(i)-1} \circ \Psi_1^{(i)}, \end{aligned}$$

where we define $\Phi_t = i_t^*\Phi$, $\Psi_t^{(i)} = i_t^*\Psi^{(i)}$ using the natural morphism $i_t : X \longrightarrow T \times X$ for any $t \in T_{cl}$. Now use that we are dealing with an algebraic torus T to deduce there are 1-dimensional k-vector spaces $V_1^{(i)} = k \cdot v_1^{(i)}$, …, $V_{m_i}^{(i)} = k \cdot v_{m_i}^{(i)}$ and characters $\chi_1^{(i)}, \ldots, \chi_{m_i}^{(i)} \in X(T)$ such that (Complete Reducibility Theorem [Per1, Thm. 2.30])

$$\begin{aligned} &\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}) = V_1^{(i)} \oplus \cdots \oplus V_{m_i}^{(i)}, \\ &\qquad t \cdot v_a^{(i)} = \chi_a^{(i)}(t) \cdot v_a^{(i)}, \ \forall t \in T_{cl} \ \forall a = 1, \ldots, m_i. \end{aligned}$$

Redefine $\tilde{\chi}_a^{(i)}(-) = \chi_a^{(i)}((-)^{-1}) \in X(T)$, $\tilde{v}_a^{(i)} = \Phi_1 \circ v_a^{(i)} \circ \Psi_1^{(i)-1} \in \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}) \backslash 0$ and define

$\tilde{\Psi}_a^{(i)}$ to be the equivariant structure on $\tilde{v}_a^{(i)}(\mathcal{E}_i)$ induced by $\Psi^{(i)}$ for all $a = 1, \ldots, m_i$. We deduce

$$\mathcal{E} \cong \bigoplus_{i=1}^{n} \left(\tilde{v}_1^{(i)}(\mathcal{E}_i) \oplus \cdots \oplus \tilde{v}_{m_i}^{(i)}(\mathcal{E}_i) \right),$$

and the equivariant structure Φ induces an equivariant structure on each $\tilde{v}_a^{(i)}(\mathcal{E}_i)$, denoted by $\Phi|_{\tilde{v}_a^{(i)}(\mathcal{E}_i)}$ such that

$$\Phi|_{\tilde{v}_a^{(i)}(\mathcal{E}_i)} = \mathcal{O}(\tilde{\chi}_a^{(i)}) \otimes \tilde{\Psi}_a^{(i)}, \ \forall i = 1, \ldots, n \ \forall a = 1, \ldots, m_i,$$

where $\mathcal{O}(\tilde{\chi}_a^{(i)})$ is the equivariant structure induced by the character $\tilde{\chi}_a^{(i)}$. $\square$

As an exercise, we prove the following proposition matching Gieseker and GIT stability for torsion free equivariant sheaves of rank $r = 2$ on nonsingular projective toric varieties of dimension $d = 2$. Although we will generalise this proposition to any r, d later (Theorem 1.2.22), its proof is nevertheless instructive because of its explicit nature[10].

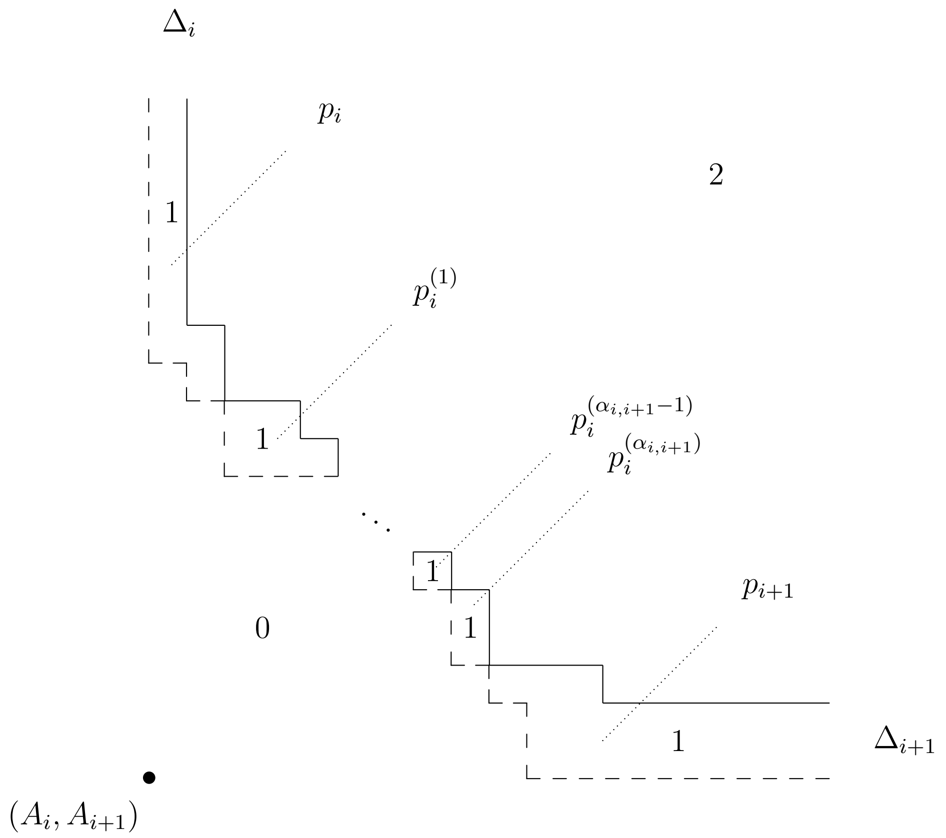
Proposition 1.2.20. *Let X be a nonsingular complete toric surface with ample divisor H. Let $\vec{\chi} \in \mathcal{X}^0$ be the characteristic function of a rank 2 torsion free equivariant sheaf on X. Then there exists an ample equivariant line bundle $\mathcal{L}_{\vec{\chi}}^0 \in \mathrm{Pic}(\mathcal{N}_{\vec{\chi}}^0)$ such that any torsion free equivariant sheaf $\mathcal{E}$ on X with characteristic function $\vec{\chi}$ is Gieseker semistable resp. Gieseker stable if and only if $\mathcal{E}$ is GIT semistable resp. properly[11] GIT stable w.r.t. $\mathcal{L}_{\vec{\chi}}^0$.*

Proof. We consider X a nonsingular complete toric surface with ample divisor H (corresponding to an ample line bundle $\mathcal{O}_X(1)$) and $\vec{\chi} \in \mathcal{X}^0$ the characteristic function of a rank 2 torsion free sheaf on X. Let $\mathcal{E}$ be a rank 2 torsion free equivariant sheaf on X with characteristic function $\vec{\chi}$. We use the notation of subsection 2.2.1. Denoting the corresponding framed torsion free Δ-family by $\hat{E}^{\Delta}$, subsection 2.2.1 introduces integers

[10] In this proposition, we work over ground field $\mathbb{C}$, as we do in chapter 2. In the proof of the proposition, we use the notation introduced in subsection 2.2.1. We identify $A^2(X) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z}$.

[11] Since we want to apply [Dol, Thm. 11.1] and the notion of GIT stable points in [Dol] corresponds to the notion of *properly* GIT stable points in [MFK] (compare [Dol, Sect. 8.1] and [MFK, Def. 1.8]), we match "Gieseker stable" with "*properly* GIT stable". Note that the results of section 1.2 so far continue to hold analogously in this setting.

$A_i \in \mathbb{Z}$, $\Delta_i = \Delta_i(1) \in \mathbb{Z}_{\geq 0}$, 2D partitions $\pi_i(1)$, $\pi_i(2)$ and elements $p_i = p_i(1) \in \mathbb{P}^1$ for all $i = 1, \ldots, N$. We want to compute the Hilbert polynomial $P_{\mathcal{E}}(t)$ of $\mathcal{E}$. For all $i = 1, \ldots, N$, we introduce further elements $p_{i,i+1}^{(1)}, \ldots, p_{i,i+1}^{(\alpha_{i,i+1})} \in \mathbb{P}^1$, which correspond to the 1-dimensional subspaces "in the corners" of the double-filtrations as indicated in the following diagram.



Note that $\alpha_{i,i+1} \in \mathbb{Z}_{\geq 0}$. In Proposition 2.2.2 of subsection 2.2.1, we give a formula for the Chern character $\mathrm{ch}(\mathcal{E})$ of $\mathcal{E}$ in terms of the $A_i \in \mathbb{Z}$, $\Delta_i \in \mathbb{Z}_{\geq 0}$ and number of blocks $\#\pi_i(1)$, $\#\pi_i(2)$ of the 2D partitions $\pi_i(1)$, $\pi_i(2)$ ($i = 1, \ldots, N$). Now write the ample line bundle as $\mathcal{O}_X(1) \cong \mathcal{O}_X(\alpha_3 D_3 + \cdots + \alpha_N D_N)$, where $\alpha_3, \ldots, \alpha_N$ are certain integers and define $\alpha_1 = \alpha_2 = 0$ (see subsection 2.2.1 for the notation). Then for any $t \in \mathbb{Z}$

$$\mathrm{ch}(\mathcal{O}_X(t)) = 1 + \sum_{i=1}^{N} t\alpha_i D_i + \frac{1}{2}\left(\sum_{i=1}^{N} t\alpha_i D_i\right)^2.$$

Using the Euler sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus(N-2)} \longrightarrow \bigoplus_{i=1}^{N} \mathcal{O}_X(D_i) \longrightarrow \mathcal{T}_X \longrightarrow 0,$$

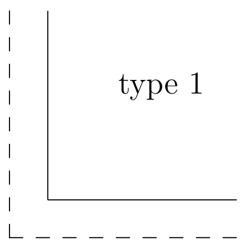
for the tangent bundle $\mathcal{T}_X$ of X and using the expression for the Todd class in terms of the Chern class [Har1, App. A], we obtain

$$\begin{aligned} \operatorname{td}(\mathcal{T}_X) &= 1 + \frac{1}{2}\sum_{i=1}^{N} D_i + \frac{1}{12}\left(3N - \sum_{i=1}^{N} a_i\right) pt \\ &= 1 + \frac{1}{2}\sum_{i=1}^{N} D_i + pt. \end{aligned}$$

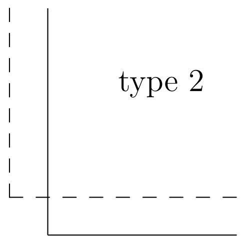
Here we use [Ful, Sect. 2.5] in order to obtain the second equality. Combining Proposition 2.2.2, the Hirzebruch–Riemann–Roch Theorem [Har1, Thm. A.4.1] and the expression for $\operatorname{td}(\mathcal{T}_X)$, one can compute the Hilbert polynomial

$$\begin{aligned} P_{\mathcal{E}}(t) = {} & \left(\sum_{i=1}^{N} \alpha_i D_i\right)^2 t^2 + \left[\left(\sum_{i=1}^{N} \alpha_i D_i\right)\left(\sum_{i=1}^{N} D_i\right) - \left(\sum_{i=1}^{N} \alpha_i D_i\right)\left(\sum_{i=1}^{N} (2A_i + \Delta_i) D_i\right)\right] t \\ & - \frac{1}{2}\left(\sum_{i=1}^{N} D_i\right)\left(\sum_{i=1}^{N} (2A_i + \Delta_i) D_i\right) + \frac{1}{2}\left(\sum_{i=1}^{N} A_i D_i\right)^2 + \frac{1}{2}\left(\sum_{i=1}^{N} (A_i + \Delta_i) D_i\right)^2 \\ & + 2 - \sum_{i=1}^{N} (\#\pi_i(1) + \#\pi_i(2)). \end{aligned}$$

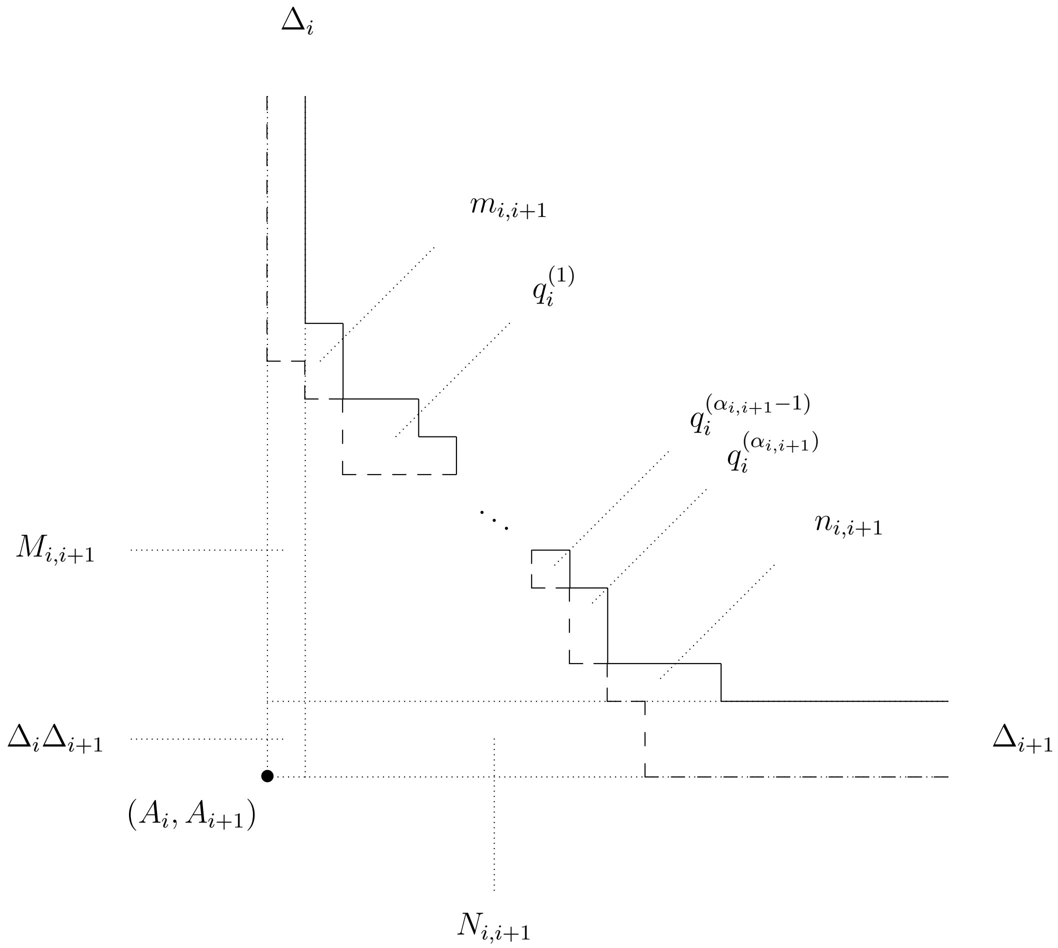
For each component χ^{σ_i} of the characteristic function, there are two possibilities. Either the shape of χ^{σ_i} is such that necessarily $p_i = p_{i+1}$, because they are connected (in particular $\Delta_i > 0$ and $\Delta_{i+1} > 0$), which we symbolically represent by:



Or this is not the case, which we symbolically represent by:



where we allow $\Delta_i = 0$ or $\Delta_{i+1} = 0$. Suppose χ^{σ_i} is of type 2. Then we introduce non-negative integers $M_{i,i+1}$, $N_{i,i+1}$, $m_{i,i+1}$, $n_{i,i+1}$, $\alpha_{i,i+1}$, $q_{i,i+1}^{(1)}$, $\ldots$, $q_{i,i+1}^{(\alpha_{i,i+1})}$ giving the area of the various regions of χ^{σ_i} as indicated in the following diagram.



Note that

$$\#\pi_i(1) = \Delta_i\Delta_{i+1} + M_{i,i+1} + N_{i,i+1} + \#\pi_i(2) - m_{i,i+1} - n_{i,i+1} - \sum_{j=1}^{\alpha_{i,i+1}} q_{i,i+1}^{(j)}.$$

Now let $0 \neq W \subsetneq \mathbb{C}^{\oplus 2}$ be a 1-dimensional linear subspace. Let $\hat{F}_W^{\Delta} = \hat{E}^{\Delta} \cap W$ be the corresponding torsion free Δ-family. Let $\mathcal{F}_W$ be the corresponding equivariant coherent subsheaf of $\mathcal{E}$. Let $\pi_1^W, \ldots, \pi_N^W$ be the 2D partitions corresponding to the torsion free Δ-family $\hat{F}_W^{\Delta}$. Proceeding similarly to before, one can compute

$$\begin{aligned}
P_{\mathcal{F}_W}(t) =& \frac{1}{2}\left(\sum_{i=1}^{N} \alpha_i D_i\right)^2 t^2 + \left[\frac{1}{2}\left(\sum_{i=1}^{N} \alpha_i D_i\right)\left(\sum_{i=1}^{N} D_i\right)\right. \\
& \left. - \left(\sum_{i=1}^{N} \alpha_i D_i\right)\left(\sum_{i=1}^{N} (A_i + (1 - \dim(p_i \cap W))\Delta_i) D_i\right)\right] t \\
& + 1 - \frac{1}{2}\left(\sum_{i=1}^{N} D_i\right)\left(\sum_{i=1}^{N} (A_i + (1 - \dim(p_i \cap W))\Delta_i) D_i\right) \\
& + \frac{1}{2}\left(\sum_{i=1}^{N} (A_i + (1 - \dim(p_i \cap W))\Delta_i) D_i\right)^2 - \sum_{i=1}^{N} \#\pi_i^W.
\end{aligned}$$

Here for any $i = 1, \ldots, N$

$$\#\pi_i^W = \#\pi_i(1)\dim(p_i \cap W) + \#\pi_i(2)(1 - \dim(p_i \cap W)),$$

if χ^{σ_i} is of type 1 and

$$\begin{aligned}
\#\pi_i^W =& \Delta_i \Delta_{i+1} \dim(p_i \cap W)\dim(p_{i+1} \cap W) + M_{i,i+1}\dim(p_i \cap W) + N_{i,i+1}\dim(p_{i+1} \cap W) \\
& + \#\pi_i(2) - m_{i,i+1}\dim(p_i \cap W) - n_{i,i+1}\dim(p_{i+1} \cap W) \\
& - \sum_{j=1}^{\alpha_{i,i+1}} q_{i,i+1}^{(j)} \dim(p_{i,i+1}^{(j)} \cap W),
\end{aligned}$$

if χ^{σ_i} is of type 2. Before we continue, we make two remarks regarding positivity. Firstly, since the leading coefficient of any Hilbert polynomial is positive, we deduce that $\left(\sum_{i=1}^{N} \alpha_i D_i\right)^2 > 0$. Secondly, using the definition of degree of a coherent sheaf [HL, Def. 1.2.11] and the Nakai–Moishezon Criterion [Har1, Thm. A.5.1], we deduce that for any $j = 1, \ldots, N$

$$\deg(D_j) := \deg(\mathcal{O}_X(D_j)) = \left(\sum_{i=1}^{N} \alpha_i D_i\right) D_j > 0.$$

Suppose all χ^{σ_i} are of type 2. The general case can be treated similarly. Since we are dealing with rank 2 sheaves $\dim(p_i \cap W)^2 = \dim(p_i \cap W)$ for all $i = 1, \ldots, N$. Also recall the intersection numbers $D_i^2 = -a_i \; pt$ for all $i = 1, \ldots, N$ discussed in subsection 2.2.1. Using Proposition 1.2.19, we deduce that $\mathcal{E}$ is Gieseker semistable (resp. Gieseker stable) if and only if for any linear subspace $0 \neq W \subsetneq \mathbb{C}^{\oplus 2}$ and $t \gg 0$

$$\begin{aligned}
0 \leq & (\text{resp.} <) \left(\sum_{i=1}^{N} \alpha_i D_i \right)^2 (p_{\mathcal{E}}(t) - p_{\mathcal{F}_W}(t)) \\
= & \sum_{i=1}^{N} \left(\frac{1}{2} - \dim(p_i \cap W) \right) \Big\{ \deg(D_i)\Delta_i t - M_{i,i+1} - N_{i-1,i} + m_{i,i+1} + n_{i-1,i} \\
& + \left(1 - \frac{1}{2} a_i - A_{i+1} + a_i A_i - A_{i-1} \right) \Delta_i + \left(-\Delta_{i+1} + \frac{1}{2} a_i \Delta_i - \Delta_{i-1} \right) \Delta_i \Big\} \\
& + \sum_{i=1}^{N} \sum_{j=1}^{\alpha_{i,i+1}} \left(\frac{1}{2} - \dim(p_{i,i+1}^{(j)} \cap W) \right) q_{i,i+1}^{(j)}.
\end{aligned}$$

Note that $\mathcal{N}_{\vec{\chi}}^{0} = \prod_{i=1}^{N} \mathbb{P}^1 \times \prod_{i=1}^{N} \prod_{j=1}^{\alpha_{i,i+1}} \mathbb{P}^1$, where one omits the ith copy of $\mathbb{P}^1$ in the first product if $\Delta_i = 0$. The framed torsion free Δ-family $\hat{E}^{\Delta}$ corresponds to the closed point $(p_i; p_{i,i+1}^{(j)})_{i=1,\ldots,N, j=1,\ldots,\alpha_{i,i+1}}$. Equivariant line bundles on $\mathcal{N}_{\vec{\chi}}^{0}$ (up to equivariant isomorphism) correspond to arbitrary sequences of integers $(k_i; k_{i,i+1}^{(j)})_{i=1,\ldots,N, j=1,\ldots,\alpha_{i,i+1}}$ and such an equivariant line bundle is ample if and only if all the integers are positive [Dol, Lem. 11.1, Sect. 11.1]. Choose an integer $R \gg 0$ and define the ample equivariant line bundle $\mathcal{L}_{\vec{\chi}}^{0}$ by

$$\begin{aligned}
k_i = & \deg(D_i)\Delta_i R - M_{i,i+1} - N_{i-1,i} + m_{i,i+1} + n_{i-1,i} \\
& + \left(1 - \frac{1}{2} a_i - A_{i+1} + a_i A_i - A_{i-1} \right) \Delta_i + \left(-\Delta_{i+1} + \frac{1}{2} a_i \Delta_i - \Delta_{i-1} \right) \Delta_i, \\
k_{i,i+1}^{(j)} = & q_{i,i+1}^{(j)},
\end{aligned}$$

where $i = 1, \ldots, N$, $j = 1, \ldots, \alpha_{i,i+1}$. The notion of GIT stability determined by such an ample equivariant line bundle is made explicit in [Dol, Thm. 11.1]. Using [Dol, Thm. 11.1], we see that the point $(p_i; p_{i,i+1}^{(j)})_{i=1,\ldots,N, j=1,\ldots,\alpha_{i,i+1}}$ is GIT semistable w.r.t. $\mathcal{L}_{\vec{\chi}}^{0}$

(resp. properly GIT stable w.r.t. $\mathcal{L}^0_{\vec{\chi}}$) if and only if for any linear subspace $0 \neq W \subsetneq \mathbb{C}^{\oplus 2}$

$$\sum_{i=1}^{N} k_i \dim(p_i \cap W) + \sum_{i=1}^{N} \sum_{j=1}^{\alpha_{i,i+1}} k_{i,i+1}^{(j)} \dim(p_{i,i+1}^{(j)} \cap W) \leq (\text{resp.} <) \frac{1}{2} \sum_{i=1}^{N} k_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{\alpha_{i,i+1}} k_{i,i+1}^{(j)}.$$

By choosing $R \gg 0$, we see that $\mathcal{E}$ is Gieseker semistable (resp. Gieseker stable) if and only if $\mathcal{E}$ is GIT semistable w.r.t. $\mathcal{L}^0_{\vec{\chi}}$ (resp. properly GIT stable w.r.t $\mathcal{L}^0_{\vec{\chi}}$). This establishes the proof in the case χ^{σ_i} is of type 2 for all $i = 1, \ldots, N$. The general case works in a similar way. □

The following proposition relates μ-stability and GIT stability of torsion free equivariant sheaves on nonsingular projective toric varieties. Although we do not need this proposition for the proof of Theorem 1.2.22, which matches Gieseker and GIT stability for torsion free equivariant sheaves on nonsingular projective toric varieties in general, the proof is instructive. Moreover, the ample equivariant line bundles $\mathcal{L}^{0,\mu}_{\vec{\chi}}$ constructed in the proof are of a particularly simple form as opposed to the more complicated ample equivariant line bundles $\mathcal{L}^0_{\vec{\chi}}$ matching Gieseker and GIT stability of Proposition 1.2.20 and Theorem 1.2.22. Furthermore, the reasoning in the proof will be used later to construct particularly simple ample equivariant line bundles matching μ-stability and GIT stability for reflexive equivariant sheaves on nonsingular projective toric varieties (subsection 1.3.4). Recall that a torsion free sheaf $\mathcal{E}$ on a nonsingular projective variety X with ample line bundle $\mathcal{O}_X(1)$ is μ-semistable, resp. μ-stable, if $\mu_{\mathcal{F}} \leq \mu_{\mathcal{E}}$, resp. $\mu_{\mathcal{F}} < \mu_{\mathcal{E}}$, for any coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$ [HL, Def. 1.2.12]. Denoting the Hilbert polynomial of $\mathcal{E}$ by $P_{\mathcal{E}}(t) = \sum_{i=0}^{n} \frac{\alpha_i(\mathcal{E})}{i!} t^i$, where $n = \dim(X)$, the rank of $\mathcal{E}$ is defined to be $\mathrm{rk}(\mathcal{E}) = \frac{\alpha_n(\mathcal{E})}{\alpha_n(\mathcal{O}_X)}$, the degree of $\mathcal{E}$ is defined to be $\deg(\mathcal{E}) = \alpha_{n-1}(\mathcal{E}) - \alpha_{n-1}(\mathcal{O}_X) \cdot \mathrm{rk}(\mathcal{E})$ and the slope of $\mathcal{E}$ is defined to be $\mu_{\mathcal{E}} = \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$ [HL, Def. 1.2.2, 1.2.11].

Proposition 1.2.21. *Let X be a nonsingular projective toric variety and let $\mathcal{O}_X(1)$ be an ample line bundle on X. Then for any $\vec{\chi} \in \mathcal{X}^0$, there is an ample equivariant line bundle $\mathcal{L}^{0,\mu}_{\vec{\chi}} \in \mathrm{Pic}^G(\mathcal{N}^0_{\vec{\chi}})$, such that any torsion free equivariant sheaf $\mathcal{E}$ on X with characteristic* *function $\vec{\chi}$ satisfies*

$$\mathcal{E} \text{ is } \mu\text{-stable} \Longrightarrow \mathcal{E} \text{ is properly GIT stable w.r.t. } \mathcal{L}^{0,\mu}_{\vec{\chi}} \Longrightarrow \mathcal{E} \text{ is } \mu\text{-semistable.}$$

Proof. We note that if $\mathcal{E}$ is a torsion free equivariant sheaf on X, then $\mathcal{E}$ is μ-semistable if and only if for any equivariant coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. This can be seen by noting that the Harder–Narasimhan filtration of $\mathcal{E}$ is equivariant. As an aside: note that we do *not* prove $\mathcal{E}$ is μ-stable if and only if for any equivariant coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$ we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$. We will prove this in the case $\mathcal{E}$ is reflexive in Proposition 1.3.13 of subsection 1.3.4. For $\mathcal{E}$ torsion free equivariant, the problem is that if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for any equivariant coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$, then indeed $\mathcal{E}$ is μ-semistable and has a μ-Jordan–Hölder filtration [HL, Sect. 1.6], but the graded object $gr^{JH}(\mathcal{E})$ is only unique in codimension 1 [HL, Sect. 1.6]. Consequently, in the case of μ-stability, we cannot mimic the proof of Proposition 1.2.19, which uses the socle of $\mathcal{E}$ and its uniqueness.

Let X be defined by a fan Δ in a lattice N of rank r. Let $\mathcal{E}$ be a torsion free equivariant sheaf on X with characteristic function $\vec{\chi}$ and corresponding framed torsion free Δ-family $\hat{E}^{\Delta}$. Assume $\mathcal{E}$ has rank M (we can assume $M \geq 2$ otherwise the proposition is trivial). Then $\hat{E}^{\Delta}$ consists of multi-filtrations $\{E^{\sigma_i}(\vec{\lambda})\}_{\vec{\lambda} \in \mathbb{Z}^r}$ of $k^{\oplus M}$, for each $i = 1, \ldots, l$, such that each multi-filtration reaches $k^{\oplus M}$ (see Theorem 1.1.10 and use the notation of this theorem). Moreover, for each $i = 1, \ldots, l$, there are integers $A_1^{(i)}, \ldots, A_r^{(i)}$ such that $E^{\sigma_i}(\vec{\lambda}) = 0$ unless $A_1^{(i)} \leq \lambda_1$, $\ldots$, $A_r^{(i)} \leq \lambda_r$ (let $A_1^{(i)}, \ldots, A_r^{(i)}$ be maximally chosen with this property). These multi-filtrations satisfy certain gluing conditions (see Theorem 1.1.10). Let $(\rho_1, \ldots, \rho_N)$ be all rays and let $A_1, \ldots, A_N$ be the corresponding integers among the $A_j^{(i)}$ (this makes sense because of the gluing conditions). Fix $j = 1, \ldots, N$ and let σ_i be some cone of maximal dimension having ρ_j as a ray. Let $\left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ be the rays of σ_i and let $\rho_k^{(i)} = \rho_j$. Consider the filtration

$$\{\beta_\lambda\}_{\lambda \in \mathbb{Z}} = \left\{ \lim_{\substack{\lambda_k \to \lambda \\ \lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1} \ldots, \lambda_r \to \infty}} E^{\sigma_i} \left(\sum_{\alpha=1}^{r} \lambda_\alpha m(\rho_\alpha) \right) \right\}_{\lambda \in \mathbb{Z}},$$

of $k^{\oplus M}$. Define integers $\Delta_j(1), \Delta_j(2), \ldots, \Delta_j(M-1) \in \mathbb{Z}_{\geq 0}$ and elements $p_j(1) \in \mathrm{Gr}(1, M)$, $p_j(2) \in \mathrm{Gr}(2, M)$, $\ldots$, $p_j(M-1) \in \mathrm{Gr}(M-1, M)$, such that the filtration changes value as follows

$$\beta_\lambda = \begin{cases} 0 & \text{if } \lambda < A_j \\ p_j(1) \in \mathrm{Gr}(1, M) & \text{if } A_j \leq \lambda < A_j + \Delta_j(1) \\ p_j(2) \in \mathrm{Gr}(2, M) & \text{if } A_j + \Delta_j(1) \leq \lambda < A_j + \Delta_j(1) + \Delta_j(2) \\ \cdots & \cdots \\ k^{\oplus M} & \text{if } A_j + \Delta_j(1) + \Delta_j(2) + \cdots + \Delta_j(M-1) \leq \lambda. \end{cases}$$

Note that $\Delta_j(k) = 0$ is allowed. These definitions are independent of the cone σ_i chosen, because of the gluing conditions. Denote the toric divisor $V(\rho_j)$ corresponding to the ray ρ_j by D_j. Using the formula for first Chern class of Corollary 1.2.18, we easily obtain

$$\mathrm{ch}(\mathcal{E}) = M - \sum_{j=1}^{N} \left(M A_j + (M-1)\Delta_j(1) + (M-2)\Delta_j(2) + \cdots + \Delta_j(M-1) \right) D_j + O(2),$$

where $O(2)$ means terms of degree ≥ 2 in the Chow ring $A(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Using the Euler sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus(N-r)} \longrightarrow \bigoplus_{j=1}^{N} \mathcal{O}_X(D_j) \longrightarrow \mathcal{T}_X \longrightarrow 0,$$

for the tangent bundle $\mathcal{T}_X$ of X and using the expression of the Todd class in terms of the Chern class [Har1, App. A], we obtain

$$\mathrm{td}(\mathcal{T}_X) = 1 + \frac{1}{2} \sum_{j=1}^{N} D_j + O(2).$$

Now $c_1(\mathcal{O}_X(1)) = \sum_{j=1}^N \alpha_j D_j$ for some integers α_j, since $D_1, \ldots, D_N$ generate $A^1(X)$ [Ful, Sect. 5.2]. Consequently,

$$\mathrm{ch}(\mathcal{O}_X(t)) = \frac{1}{r!}\left(\sum_{j=1}^N \alpha_j D_j\right)^r t^r + \frac{1}{(r-1)!}\left(\sum_{j=1}^N \alpha_j D_j\right)^{r-1} t^{r-1} + \cdots .$$

From the Hirzebruch–Riemann–Roch Theorem [Har1, Thm. A.4.1], we obtain the two leading terms of the Hilbert polynomial of $\mathcal{E}$

$$\begin{aligned} P_{\mathcal{E}}(t) =& \frac{M}{r!} \deg\left\{\left(\sum_{j=1}^N \alpha_j D_j\right)^r\right\}_r t^r + \frac{1}{(r-1)!} \deg\left\{\frac{M}{2}\left(\sum_{j=1}^N \alpha_j D_j\right)^{r-1}\left(\sum_{j=1}^N D_j\right)\right. \\ &\left. - \left(\sum_{j=1}^N \alpha_j D_j\right)^{r-1}\left(\sum_{j=1}^N \left[MA_j + \sum_{k=1}^{M-1}(M-k)\Delta_j(k)\right] D_j\right)\right\}_r t^{r-1} + \cdots , \end{aligned}$$

where $\cdots$ means terms of degree $< r-1$ in t, $\{-\}_r$ denotes the component of degree r in $A(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\deg : A^r(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}$ is the degree map. Let $0 \neq W \subset k^{\oplus M}$ be an m-dimensional subspace and let $\hat{F}^\Delta = \hat{E}^\Delta \cap W \subset \hat{E}^\Delta$ be the corresponding torsion free Δ-family. Let $\mathcal{F}_W \subset \mathcal{E}$ be the corresponding equivariant coherent subsheaf. Analogous to the previous reasoning, one computes the two leading terms of the Hilbert polynomial of $\mathcal{F}_W$ to be

$$\begin{aligned} P_{\mathcal{F}_W}(t) = \frac{m}{r!} \deg\left\{\left(\sum_{j=1}^N \alpha_j D_j\right)^r\right\}_r t^r + \frac{1}{(r-1)!} \deg\left\{\frac{m}{2}\left(\sum_{j=1}^N \alpha_j D_j\right)^{r-1}\left(\sum_{j=1}^N D_j\right)\right. \\ \left. - \left(\sum_{j=1}^N \alpha_j D_j\right)^{r-1}\left(\sum_{j=1}^N \left[mA_j + \sum_{k=1}^{M-1}\left(m - \dim(p_j(k) \cap W)\right)\Delta_j(k)\right] D_j\right)\right\}_r t^{r-1} + \cdots , \end{aligned}$$

where the term on the second line can be straightforwardly derived by using induction on M. Before we continue, we make two remarks regarding positivity. Firstly, since the leading coefficient of any Hilbert polynomial is positive, we deduce $\deg\left\{\left(\sum_{j=1}^N \alpha_j D_j\right)^r\right\}_r > 0$. Secondly, using the definition of degree of a coherent sheaf [HL, Def. 1.2.11] and the

Nakai–Moishezon Criterion [Har1, Thm. A.5.1], we deduce that for any $j = 1, \ldots, N$

$$\deg(D_j) := \deg(\mathcal{O}_X(D_j)) = \deg\left\{ \left(\sum_{k=1}^{N} \alpha_k D_k \right)^{r-1} D_j \right\}_r > 0.$$

Combining our results so far and using the definition of slope of a coherent sheaf [HL, Def. 1.2.11], we obtain that $\mathcal{E}$ is μ-semistable if and only if for any subspace $0 \neq W \subsetneq k^{\oplus M}$ we have

$$\sum_{j=1}^{N} \sum_{k=1}^{M-1} \Delta_j(k) \deg(D_j) \dim(p_j(k) \cap W) \leq \frac{\dim(W)}{M} \sum_{j=1}^{N} \sum_{k=1}^{M-1} \Delta_j(k) \deg(D_j) \dim(p_j(k)).$$

We are now ready to prove the proposition. Let $\vec{\chi} \in \mathcal{X}^0$ be arbitrary. From $\vec{\chi}$ we read off the integers A_j, the rank M (we assume $M \geq 2$ otherwise the proposition is trivial) and the non-negative integers $\Delta_j(k) \in \mathbb{Z}$. Without loss of generality, we can assume not all $\Delta_j(k) = 0$ (otherwise there are no μ-stable torsion free equivariant sheaves on X with characteristic function $\vec{\chi}$ and the proposition is trivial). In subsection 1.2.3, we defined the closed subscheme[12]

$$\mathcal{N}_{\vec{\chi}}^{0} \subset \mathcal{A}' = \prod_{j=1}^{N} \prod_{k=1}^{M-1} \mathrm{Gr}(k, M) \times \prod_{\alpha=1}^{a} \mathrm{Gr}(n_\alpha, M).$$

Here $a \in \mathbb{Z}_{\geq 0}$ and $0 < n_1, \ldots, n_a < M$ are some integers. A closed point of $\mathcal{N}_{\vec{\chi}}^{0}$ is of the form $(p_j(k); q_\alpha)_{j=1,\ldots,N, k=1,\ldots,M-1, \alpha=1,\ldots,a}$, where there are certain compatibilities among the $p_j(k)$, q_α dictated by the shape of $\vec{\chi}$. An equivariant line bundle $\mathcal{L}_{\vec{\chi}}^{0,\mu\prime}$ on $\mathcal{A}'$ (up to equivariant isomorphism) is of the form $(\kappa_{jk}; \kappa_\alpha)_{j=1,\ldots,N, k=1,\ldots,M-1, \alpha=1,\ldots,a}$, where $\kappa_{jk}, \kappa_\alpha$ can be any integers [Dol, Lem. 11.1]. Such an equivariant line bundle is ample if and only if all $\kappa_{jk}, \kappa_\alpha > 0$ [Dol, Sect. 11.1]. The notion of GIT stability determined by such an ample equivariant line bundle is made explicit in [Dol, Thm. 11.1]. Suppose $a = 0$. Choose $\kappa_{jk} = \Delta_j(k) \deg(D_j)$ for all j, k and $\mathcal{L}_{\vec{\chi}}^{0,\mu} = \mathcal{L}_{\vec{\chi}}^{0,\mu\prime}|_{\mathcal{N}_{\vec{\chi}}^{0}}$. The proposition

[12] To be be precise: for those $\Delta_j(k)$ which are zero, the corresponding term $\mathrm{Gr}(k, M)$ in the product of $\mathcal{A}'$ should be left out.

now follow easily from [Dol, Thm. 11.1] and [MFK, Thm. 1.19]. Now assume $a > 0$. Let R be a positive integer satisfying $0 < \sum_{\alpha=1}^{a} \frac{n_\alpha}{R} < \frac{1}{M^2}$. Choose $\kappa_{jk} = \Delta_j(k)\ \deg(D_j)R$ and $\kappa_\alpha = 1$ for all j, k, α. Note that any μ-stable torsion free equivariant sheaf on X with characteristic function $\vec{\chi}$ and corresponding framed torsion free Δ-family defined by $(p_j(k); q_\alpha)_{j=1,\ldots,N,k=1,\ldots,M-1,\alpha=1,\ldots,a}$ satisfies

$$\frac{\dim(W)}{M}\sum_{j=1}^{N}\sum_{k=1}^{M-1}\Delta_j(k)\deg(D_j)\dim(p_j(k)) - \sum_{j=1}^{N}\sum_{k=1}^{M-1}\Delta_j(k)\deg(D_j)\dim(p_j(k)\cap W) \geq \frac{1}{M}$$

for any subspace $0 \neq W \subsetneq k^{\oplus M}$. Using [Dol, Thm. 11.1] and [MFK, Thm. 1.19] finishes the proof[13]. $\square$

Here is our main result of this subsection, which explicitly matches Gieseker and GIT stability for torsion free sheaves in full generality.

Theorem 1.2.22. *Let X be a nonsingular projective toric variety defined by a fan Δ. Let $\mathcal{O}_X(1)$ be an ample line bundle on X. Then for any $\vec{\chi} \in \mathcal{X}^0$, there is an ample equivariant line bundle $\mathcal{L}_{\vec{\chi}}^0 \in \mathrm{Pic}^G(\mathcal{N}_{\vec{\chi}}^0)$ such that any torsion free equivariant sheaf $\mathcal{E}$ on X with characteristic function $\vec{\chi}$ is GIT semistable resp. properly GIT stable w.r.t. $\mathcal{L}_{\vec{\chi}}^0$ if and only if $\mathcal{E}$ is Gieseker semistable resp. Gieseker stable.*

Proof. Let the fan Δ lie in a lattice $N \cong \mathbb{Z}^r$. Fix $\vec{\chi} \in \mathcal{X}^0$ and let $\mathcal{E}$ be a torsion free equivariant sheaf on X with characteristic function $\vec{\chi}$. Denote the corresponding framed torsion free Δ-family by $\hat{E}^\Delta$. Let $\Delta(1)$ be the collection of rays of Δ, then the corresponding divisors $\{V(\rho)\}_{\rho\in\Delta(1)}$ generate $A^1(X)$ ([Ful, Sect. 5.2]), therefore we can write

$$\mathcal{O}_X(1) \cong \mathcal{O}\left(\sum_{\rho\in\Delta(1)} \alpha_\rho V(\rho)\right).$$

Using the Hirzebruch–Riemann–Roch Theorem [Har1, Thm. A.4.1] and Klyachko's For-

[13]Note that in this proof, we are not allowed to deduce "$\mathcal{E}$ is properly GIT stable w.r.t. $\mathcal{L}_{\vec{\chi}}^{0,\mu} \Longrightarrow \mathcal{E}$ is μ-stable".

mula (Proposition 1.2.16), we obtain the Hilbert polynomial of $\mathcal{E}$

$$
\begin{aligned}
P_{\mathcal{E}}(t) &= \sum_{\sigma \in \Delta,\ \vec{\lambda} \in \mathbb{Z}^{\dim(\sigma)}} \Phi_{\sigma,\vec{\lambda}}(t) \dim([E^{\sigma}](\vec{\lambda})), \\
\Phi_{\sigma,\vec{\lambda}}(t) &= (-1)^{\mathrm{codim}(\sigma)} \deg \left\{ e^{-\sum_{\rho \in \sigma(1)} \langle \vec{\lambda}, n(\rho) \rangle V(\rho) + \sum_{\rho \in \Delta(1)} t \alpha_{\rho} V(\rho)} \mathrm{td}(\mathcal{T}_X) \right\}_r ,
\end{aligned}
$$

where $\mathrm{td}(\mathcal{T}_X)$ is the Todd class of the tangent bundle $\mathcal{T}_X$ of X, $\{-\}_r$ projects to the degree r component in the Chow ring $A(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\deg : A^r(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}$ is the degree map. Let $\sigma_1, \ldots, \sigma_l$ be the cones of dimension r and for each σ_i denote by $(\rho_1^{(i)}, \ldots, \rho_r^{(i)})$ the rays of σ_i. For a fixed $j = 0, \ldots, r$, we denote the $\binom{r}{j}$ faces of σ_i of dimension j by σ_{ijk}. In other words, we choose a bijection between integers $k \in \{1, \ldots, \binom{r}{j}\}$ and j-tuples $i_1 < \cdots < i_j \in \{1, \ldots, r\}$ and the cone σ_{ijk} is by definition generated by the rays $(\rho_{i_1}^{(i)}, \ldots, \rho_{i_j}^{(i)})$. This allows us to rewrite

$$
P_{\mathcal{E}}(t) = \sum_{i=1}^{l} \sum_{j=0}^{r} \sum_{k=1}^{\binom{r}{j}} \sum_{\vec{\lambda} \in \mathbb{Z}^{\dim(\sigma_{ijk})}} \Phi_{\sigma_{ijk},\vec{\lambda}}(t) \dim([E^{\sigma_{ijk}}](\vec{\lambda})).
$$

This sum is imprecise as it stands. For fixed (i, j, k), there can be distinct (i', j', k') such that $\sigma_{ijk} = \sigma_{i'j'k'}$. In this case, we call (i, j, k) and (i', j', k') equivalent. This induces an equivalence relation on the set of such triples. We now choose exactly one representative (i', j', k') of each equivalence class $[(i, j, k)]$, for which the sequence of polynomials $\{\Phi_{\sigma_{i'j'k'},\vec{\lambda}}(t)\}_{\vec{\lambda} \in \mathbb{Z}^{\dim(\sigma_{i'j'k'})}}$ is possibly nonzero. This defines the above sum. Fix a component χ^{σ_i} of the characteristic function $\vec{\chi} = (\chi^{\sigma_1}, \ldots, \chi^{\sigma_l})$. There exists a box

$$
\mathcal{B}^{\sigma_i} = (-\infty, C_1^{(i)}] \times \cdots \times (-\infty, C_r^{(i)}],
$$

with the following properties. Write the box as

$$
\begin{aligned}
\mathcal{B}^{\sigma_i} =& \mathcal{B}_r^{\sigma_i} \sqcup \mathcal{B}_{r-1}^{\sigma_i} \sqcup \cdots \sqcup \mathcal{B}_0^{\sigma_i}, \\
\mathcal{B}_r^{\sigma_i} =& (-\infty, C_1^{(i)}) \times \cdots \times (-\infty, C_r^{(i)}),
\end{aligned}
$$

$$\begin{aligned}
\mathcal{B}^{\sigma_i}_{r-1} =&(\{C^{(i)}_1\} \times (-\infty, C^{(i)}_2) \times \cdots \times (-\infty, C^{(i)}_r)) \sqcup \cdots \\
&\sqcup ((-\infty, C^{(i)}_1) \times \cdots \times (-\infty, C^{(i)}_{r-1}) \times \{C^{(i)}_r\}), \\
\cdots& \\
\mathcal{B}^{\sigma_i}_0 =&\{C^{(i)}_1\} \times \cdots \times \{C^{(i)}_r\},
\end{aligned}$$

where $\mathcal{B}^{\sigma_i}_j$ is the disjoint union of $\binom{r}{j}$ sets $\mathcal{B}^{\sigma_i}_j(k)$. Here the labeling is such that the j components of $\mathcal{B}^{\sigma_i}_j(k)$ in which we have an open interval correspond precisely to the j rays $(\rho^{(i)}_{i_1}, \dots, \rho^{(i)}_{i_j})$ that generate the cone σ_{ijk}, where $i_1 < \cdots < i_j \in \{1, \dots, r\}$ corresponds to k. Then for any $j = 0, \dots, r$, $k = 1, \dots, \binom{r}{j}$ (corresponding to $i_1 < \cdots < i_j \in \{1, \dots, r\}$) and any polynomial $f(\lambda_{i_1}, \dots, \lambda_{i_j}) \in k[\lambda_{i_1}, \dots, \lambda_{i_j}]$

$$\begin{aligned}
&\sum_{(\lambda_{i_1},\dots,\lambda_{i_j}) \in \mathbb{Z}^{\dim(\sigma_{ijk})}} f(\lambda_{i_1}, \dots, \lambda_{i_j}) \dim([E^{\sigma_{ijk}}](\lambda_{i_1}, \dots, \lambda_{i_j})) \\
=& \sum_{(\lambda_{i_1},\dots,\lambda_{i_j}) \in \overline{\mathcal{B}^{\sigma_i}_j(k)}} f(\lambda_{i_1}, \dots, \lambda_{i_j}) \dim([E^{\sigma_{ijk}}](\lambda_{i_1}, \dots, \lambda_{i_j})),
\end{aligned} \tag{1.9}$$

and if we let $a_1 < \cdots < a_{r-j} \in \{1, \dots, r\} \setminus \{i_1, \dots, i_j\}$, then for all $\lambda_{i_1}, \dots, \lambda_{i_j} \in \mathbb{Z}$

$$\lim_{\lambda_{a_1} \to C^{(i)}_{a_1}, \dots, \lambda_{a_{r-j}} \to C^{(i)}_{a_{r-j}}} E^{\sigma_{ijk}} \left(\sum_{i=1}^r \lambda_i m(\rho^{(i)}) \right) = \lim_{\lambda_{a_1} \to \infty, \dots, \lambda_{a_{r-j}} \to \infty} E^{\sigma_{ijk}} \left(\sum_{i=1}^r \lambda_i m(\rho^{(i)}) \right).$$

The bar in equation (1.9) denotes closure in $\mathbb{R}^r$. Let $f(\lambda_{i_1}, \dots, \lambda_{i_j}) \in k[\lambda_{i_1}, \dots, \lambda_{i_j}]$ and assume without loss of generality $(i_1, \dots, i_j) = (1, \dots, j)$. Then equation (1.9) can be rewritten as

$$\begin{aligned}
&\sum_{\lambda_1=-\infty}^{C^{(i)}_1 - 1} \cdots \sum_{\lambda_j=-\infty}^{C^{(i)}_j - 1} [f(\lambda_1, \dots, \lambda_j) - f(\lambda_1 + 1, \lambda_2, \dots, \lambda_j) - \cdots - f(\lambda_1, \dots, \lambda_{j-1}, \lambda_j + 1) \\
&+ \cdots + (-1)^j f(\lambda_1 + 1, \dots, \lambda_j + 1)] \dim(E^{\sigma_i}(\lambda_1, \dots, \lambda_j, \infty, \dots, \infty)) \\
&+ \sum_{\lambda_2=-\infty}^{C^{(i)}_2 - 1} \cdots \sum_{\lambda_j=-\infty}^{C^{(i)}_j - 1} \Big[f(C^{(i)}_1, \lambda_2, \dots, \lambda_j) - f(C^{(i)}_1, \lambda_2 + 1, \lambda_3, \dots, \lambda_j) - \cdots
\end{aligned} \tag{1.10}$$

$$-f(C_1^{(i)}, \lambda_2, \ldots, \lambda_{j-1}, \lambda_j + 1) + \cdots + (-1)^{j-1} f(C_1^{(i)}, \lambda_2 + 1, \ldots, \lambda_j + 1)\Big]$$

$$\cdot \dim(E^{\sigma_i}(\infty, \lambda_2, \ldots, \lambda_j, \infty, \ldots, \infty))$$

$$+ \cdots$$

$$+ \sum_{\lambda_1=-\infty}^{C_1^{(i)}-1} \cdots \sum_{\lambda_{j-1}=-\infty}^{C_{j-1}^{(i)}-1} \Big[f(\lambda_1, \ldots, \lambda_{j-1}, C_j^{(i)}) - f(\lambda_1 + 1, \lambda_2, \ldots, \lambda_{j-1}, C_j^{(i)}) - \cdots$$

$$-f(\lambda_1, \ldots, \lambda_{j-2}, \lambda_{j-1} + 1, C_j^{(i)}) + \cdots + (-1)^{j-1} f(\lambda_1 + 1, \ldots, \lambda_{j-1} + 1, C_j^{(i)})\Big]$$

$$\cdot \dim(E^{\sigma_i}(\lambda_1, \ldots, \lambda_{j-1}, \infty, \ldots, \infty))$$

$$+ \cdots$$

$$+ f(C_1^{(i)}, \ldots, C_j^{(i)}) \dim(E^{\sigma_i}(\infty, \ldots, \infty)).$$

Using these manipulations only, we can write

$$P_{\mathcal{E}}(t) = \sum_{i=1}^{l} \sum_{j=0}^{r} \sum_{k=1}^{\binom{r}{j}} \sum_{\vec{\lambda} \in \mathcal{B}_j^{\sigma_i}(k)} \Psi_{\sigma_{ijk}, \vec{\lambda}}(t) \dim(E^{\sigma_i}(\vec{\lambda})).$$

We rewrite this expression one more time. For each $i = 1, \ldots, l$, $j = 0, \ldots, r$, $k = 1, \ldots, \binom{r}{j}$ we define $\Psi_{\sigma_{ijk}, \vec{\lambda}}(t) = 0$ in the case $\vec{\lambda} \notin \mathcal{B}_j^{\sigma_i}(k)$. Then we can write

$$\begin{aligned} P_{\mathcal{E}}(t) &= \sum_{i=1}^{l} \sum_{j=0}^{r} \sum_{k=1}^{\binom{r}{j}} \sum_{\vec{\lambda} \in \mathcal{B}_j^{\sigma_i}(k)} \Psi_{\sigma_{ijk}, \vec{\lambda}}(t) \dim(E^{\sigma_i}(\vec{\lambda})) \\ &= \sum_{\substack{\text{equivalence classes} \\ [(i', j', k')]}} \sum_{\vec{\lambda} \in \mathbb{Z}^{\dim(\sigma_{i'j'k'})}} \underbrace{\left(\sum_{(i,j,k) \in [(i',j',k')]} \Psi_{\sigma_{ijk}, \vec{\lambda}}(t) \right)}_{\Xi_{\sigma_{i'j'k'}, \vec{\lambda}}(t)} \dim(E^{\sigma_{i'j'k'}}(\vec{\lambda})). \end{aligned}$$

Note that summing over all equivalence classes $[(i', j', k')]$ corresponds to summing over all cones of Δ and if $(i, j, k) \in [(i', j', k')]$, then $j = j'$. Recall that for fixed $i = 1, \ldots, r$, there are integers $A_1^{(i)}, \ldots, A_r^{(i)}$ such that $E^{\sigma_i}(\lambda_1, \ldots, \lambda_r) = 0$ unless $\lambda_1 \geq A_1^{(i)}$, $\ldots$, $\lambda_r \geq A_r^{(i)}$ (Proposition 1.1.8). Therefore, by construction, there are only finitely many $\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t)$ that are possibly non-zero. Consequently, for a fixed equivalence class $[(i',j',k')]$, there is a finite subset $\mathcal{R}([(i',j',k')]) \subset \mathbb{Z}^{\dim(\sigma_{i'j'k'})}$ such that

$$P_{\mathcal{E}}(t) = \sum_{\substack{\text{equivalence classes} \\ [(i',j',k')]}} \sum_{\vec{\lambda} \in \mathcal{R}([(i',j',k')])} \Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t) \dim(E^{\sigma_{i'j'k'}}(\vec{\lambda})). \tag{1.11}$$

Note that the polynomials $\Phi_{\sigma,\vec{\lambda}}(t)$, the boxes $\mathcal{B}_j^{\sigma_i}(k)$ and hence the polynomials $\Psi_{\sigma_{ijk},\vec{\lambda}}(t)$, $\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t)$ and the regions $\mathcal{R}([i',j',k'])$ can be chosen such that they only depend on the characteristic function $\vec{\chi}$. Therefore, equation (1.11) holds for any torsion free equivariant sheaf $\mathcal{E}$ on X with characteristic function $\vec{\chi}$ and corresponding framed torsion free Δ-family $\hat{E}^{\Delta}$. Now let $\mathcal{E}$ be a torsion free equivariant sheaf on X with characteristic function $\vec{\chi}$ and corresponding framed torsion free Δ-family $\hat{E}^{\Delta}$. Assume the rank of $\mathcal{E}$ is M. Let $0 \neq W \subsetneq k^{\oplus M} = E^{\sigma_i}(\infty,\ldots,\infty)$ be a linear subspace. Consider the torsion free Δ-family $\hat{F}^{\Delta}_W = \hat{E}^{\Delta} \cap W$ and denote the corresponding torsion free equivariant sheaf by $\mathcal{F}_W$. It is not difficult to see that

$$P_{\mathcal{F}_W}(t) = \sum_{\substack{\text{equivalence classes} \\ [(i',j',k')]}} \sum_{\vec{\lambda} \in \mathcal{R}([(i',j',k')])} \Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t) \dim(E^{\sigma_{i'j'k'}}(\vec{\lambda}) \cap W).$$

Using Proposition 1.2.19, we see that $\mathcal{E}$ is Gieseker semistable if and only if for any linear subspace $0 \neq W \subsetneq k^{\oplus M}$ and $t \gg 0$

$$\begin{aligned} &\frac{1}{\dim(W)} \sum_{\substack{\text{equivalence classes} \\ [(i',j',k')]}} \sum_{\vec{\lambda} \in \mathcal{R}([(i',j',k')])} \Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t) \dim(E^{\sigma_{i'j'k'}}(\vec{\lambda}) \cap W) \\ &\leq \frac{1}{M} \sum_{\substack{\text{equivalence classes} \\ [(i',j',k')]}} \sum_{\vec{\lambda} \in \mathcal{R}([(i',j',k')])} \Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t) \dim(E^{\sigma_{i'j'k'}}(\vec{\lambda})). \end{aligned}$$

Moreover, $\mathcal{E}$ is Gieseker stable if and only if the same holds with strict inequality.

We now have to study the polynomials $\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t)$ in more detail. Fix $i' = 1, \ldots, l$, $j' = 1, \ldots, r$ and $k' = 1, \ldots, \binom{r}{j'}$. We will now show that for any $\vec{\lambda} \in \mathcal{R}([(i', j', k')])$, we have $\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(R) \in \mathbb{Z}_{>0}$ for integers $R \gg 0$. From the fact that the polynomials $\Phi_{\sigma,\vec{\lambda}}(t)$ are integer-valued for $t \in \mathbb{Z}$ it easily follows that the polynomials $\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t)$ are integer-valued for $t \in \mathbb{Z}$. Let $\mathcal{E}$ be an arbitrary torsion free equivariant sheaf on X with characteristic function $\vec{\chi}$ and corresponding framed torsion free Δ-family $\hat{E}^{\Delta}$. Consider the face $\sigma_{i'j'k'} \prec \sigma_{i'}$ and assume without loss of generality that it is spanned by the rays $(\rho_1^{(i')}, \ldots, \rho_{j'}^{(i')}) \subset (\rho_1^{(i')}, \ldots, \rho_r^{(i')})$. Consider the expression

$$\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t) \dim(E^{\sigma_{i'}}(\lambda_1, \ldots, \lambda_{j'}, \infty, \ldots, \infty)),$$

for fixed $\vec{\lambda} \in \mathcal{R}([(i', j', k')])$. We first claim $\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t)$ is a polynomial in t of degree at most $r - j'$. To see this, consider expression (1.10) for $i = i'$ and $j \geq j'$. Then for any monomial f of degree $< j'$, expression (1.10) does not contribute to $\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t)$. We now want to show $\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t)$ is of degree $r - j'$ in t with positive leading coefficient. Fix i, j as before and consider expression (1.10) for any monomial f of degree j'. We only get a contribution to the leading term of $\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t)$ for $f(\lambda_1, \ldots, \lambda_j) = \lambda_1 \cdots \lambda_{j'}$. From this, it is easy to see that the leading term of $\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(t)$ is

$$\sum_{a=0}^{r-j'} \frac{(-1)^{r-(j'+a)}}{(r-j')!} \#\{\sigma \in \Delta \mid \sigma_{i'j'k'} \prec \sigma, \ \dim(\sigma) = a + j'\}$$
$$\cdot \left(H^{r-j'} \cdot V\left(\rho_1^{(i')}\right) \cdots V\left(\rho_{j'}^{(i')}\right)\right) t^{r-j'} = \frac{1}{(r-j')!} \left(H^{r-j'} \cdot V\left(\rho_1^{(i')}\right) \cdots V\left(\rho_{j'}^{(i')}\right)\right) t^{r-j'},$$

where we use Proposition 1.2.17. Let ν be the cone generated by the rays $\rho_1^{(i')}, \ldots, \rho_{j'}^{(i')}$, then $V\left(\rho_1^{(i')}\right) \cap \cdots \cap V\left(\rho_{j'}^{(i')}\right) = V(\nu)$ is a nonsingular closed subvariety of X of dimension $r - j'$ [Ful, Sect. 3.1]. We deduce that $H^{r-j'} \cdot V\left(\rho_1^{(i')}\right) \cdots V\left(\rho_{j'}^{(i')}\right) = H^{r-j'} \cdot V(\nu) > 0$ by the Nakai–Moishezon Criterion [Har1, Thm. A.5.1].

Let $[(i', j', k')]$ be an equivalence class and let $k' = 1, \ldots, \binom{r}{j'}$ correspond to $i_1 < \cdots < i_{j'} \in \{1, \ldots, r\}$. Assume $a_1 < \cdots < a_{r-j'} \in \{1, \ldots, r\} \setminus \{i_1, \ldots, i_{j'}\}$. Define $\chi^{\sigma_{i',j',k'}}(\vec{\lambda}) =$

$\lim_{\lambda_{a_1}\to\infty,\ldots,\lambda_{a_{r-j'}}\to\infty} \chi^{\sigma_{i'}}(\vec{\lambda})$ for all $\vec{\lambda} = (\lambda_{i_1}, \ldots, \lambda_{i_{j'}}) \in \mathbb{Z}^{\dim(\sigma_{i'j'k'})}$. Consider the product of Grassmannians

$$\prod_{\substack{\text{equivalence classes}\\ [(i',j',k')]}} \prod_{\vec{\lambda}\in\mathcal{R}([i',j',k'])} \mathrm{Gr}(\chi^{\sigma_{i'j'k'}}(\vec{\lambda}), M). \tag{1.12}$$

Referring to [Dol, Sect. 11.1], the equivariant line bundles (up to equivariant isomorphism) on the product of Grassmannians (1.12) correspond to arbitrary sequences of integers $\{k_{[(i',j',k')],\vec{\lambda}}\}$, where $([(i',j',k')],\vec{\lambda}) \in \coprod_{\substack{\text{equivalence classes}\\ [(i',j',k')]}} \mathcal{R}([(i',j',k')])$. Such an equivariant line bundle $\{k_{[(i',j',k')],\vec{\lambda}}\}$ is ample if and only if all $k_{[(i',j',k')],\vec{\lambda}} > 0$ [Dol, Sect. 11.1]. We conclude that by choosing an integer $R \gg 0$, the sequence $\{\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(R)\}$ forms an ample equivariant line bundle on the product of Grassmannians (1.12). The notion of GIT stability determined by such an ample equivariant line bundle is made explicit in [Dol, Thm. 11.1]. By definition, $\mathcal{N}_{\vec{\chi}}^{0}$ is a closed subscheme of the product of Grassmannians (1.12). Using [Dol, Thm. 11.1], we see that a the closed point $\hat{E}^{\Delta}$ of $\mathcal{N}_{\vec{\chi}}^{0}$ is GIT semistable w.r.t. $\{\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(R)\}$ if and only if for any linear subspace $0 \neq W \subsetneq k^{\oplus M}$

$$\begin{aligned} &\frac{1}{\dim(W)} \sum_{\substack{\text{equivalence classes}\\ [(i',j',k')]}} \sum_{\vec{\lambda}\in\mathcal{R}([(i',j',k')])} \Xi_{\sigma_{i'j'k'},\vec{\lambda}}(R)\, \dim(E^{\sigma_{i'j'k'}}(\vec{\lambda}) \cap W) \\ &\leq \frac{1}{M} \sum_{\substack{\text{equivalence classes}\\ [(i',j',k')]}} \sum_{\vec{\lambda}\in\mathcal{R}([(i',j',k')])} \Xi_{\sigma_{i'j'k'},\vec{\lambda}}(R)\, \dim(E^{\sigma_{i'j'k'}}(\vec{\lambda})). \end{aligned}$$

Moreover, $\hat{E}^{\Delta}$ is properly GIT stable w.r.t. $\{\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(R)\}$ if and only if the same holds with strict inequality. By choosing R sufficiently large, we conclude any torsion free equivariant sheaf $\mathcal{E}$ on X with characteristic function $\vec{\chi}$ and framed torsion free Δ-family $\hat{E}^{\Delta}$ is Gieseker semistable resp. Gieseker stable if and only if $\hat{E}^{\Delta}$ is GIT semistable resp. properly GIT stable w.r.t. $\{\Xi_{\sigma_{i'j'k'},\vec{\lambda}}(R)\}$. Pulling back the ample equivariant line bundle $\{\Xi_{\sigma_{i'j'k'},\vec{\chi}}(R)\}$ to $\mathcal{N}^0_{\vec{\chi}}$ defines the desired ample equivariant line bundle $\mathcal{L}^0_{\vec{\chi}}$ and finishes the proof by [MFK, Thm. 1.19]. $\square$

1.3 Fixed Point Loci of Moduli Spaces of Sheaves on Toric Varieties

In this section, we study how the explicit moduli spaces of pure equivariant sheaves of the previous section relate to fixed point loci of moduli spaces of all Gieseker stable sheaves on nonsingular projective toric varieties. We start by studying the torus action on moduli spaces of Gieseker semistable sheaves on projective toric varieties. Subsequently, we study relations between equivariant and invariant simple sheaves. We prove a theorem expressing fixed point loci of moduli spaces of all Gieseker stable sheaves on an arbitrary nonsingular projective toric variety in terms of the explicit moduli spaces of pure equivariant sheaves of the previous section in the case one can match Gieseker and GIT stability. Since this match can always be achieved for torsion free equivariant sheaves, we obtain the theorem discussed in the introduction (Theorem 1.0.1). After discussing some examples appearing more detailed in the next chapter, where we specialise to X a nonsingular complete toric surface over $\mathbb{C}$, we prove how the fixed point locus of any moduli space of μ-stable sheaves on a nonsingular projective toric variety can be expressed combinatorially.

1.3.1 Torus Actions on Moduli Spaces of Sheaves on Toric Varieties

Let us briefly recall some notions concerning moduli spaces of Gieseker (semi)stable sheaves in general as discussed in [HL, Ch. 4]. Let X be a connected projective k-scheme, $\mathcal{O}_X(1)$ an ample line bundle and P a choice of Hilbert polynomial. Let S be a k-scheme of finite type. Any two S-flat families $\mathcal{F}_1, \mathcal{F}_2$ are said to be equivalent if there exists a line bundle $L \in \mathrm{Pic}(S)$ and an isomorphism $\mathcal{F}_1 \cong \mathcal{F}_2 \otimes p_S^* L$, where $p_S : X \times S \longrightarrow S$ is projection. Let $\underline{\mathcal{M}}_P^{ss}(S)$ be the collection of equivalence classes of Gieseker semistable S-flat families with Hilbert polynomial P. Likewise, let $\underline{\mathcal{M}}_P^{s}(S)$ be the collection of equivalence classes of geometrically Gieseker stable S-flat families with Hilbert polynomial P. These give rise to moduli functors $\underline{\mathcal{M}}_P^{ss}$, $\underline{\mathcal{M}}_P^{s}$. One can prove that there is a projective k-scheme of finite type $\mathcal{M}_P^{ss}$ corepresenting $\underline{\mathcal{M}}_P^{ss}$ [HL, Thm. 4.3.4]. Moreover, there is an open subset $\mathcal{M}_P^{s}$ of $\mathcal{M}_P^{ss}$ corepresenting $\underline{\mathcal{M}}_P^{s}$. The closed points of $\mathcal{M}_P^{ss}$ are in bijection with S-equivalence classes of Gieseker semistable sheaves on X with Hilbert polynomial P. The closed points of $\mathcal{M}_P^{s}$ are in bijection with isomorphism classes of Gieseker stable sheaves on X with Hilbert polynomial P (hence $\mathcal{M}_P^{s}$ is a coarse moduli space). If X is a toric variety with torus T, then we can define a natural regular action of T on $\mathcal{M}_P^{ss}$, $\mathcal{M}_P^{s}$ as is expressed by the following proposition.

Proposition 1.3.1. *Let X be a projective toric variety with torus T. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and P a choice of Hilbert polynomial. Choose an equivariant structure on $\mathcal{O}_X(1)$. Then there is a natural induced regular action $\sigma : T \times \mathcal{M}_P^{ss} \longrightarrow \mathcal{M}_P^{ss}$ defined using the equivariant structure, which restricts to $\mathcal{M}_P^{s}$ and on closed points is given by*

$$\sigma : T_{cl} \times \mathcal{M}_{P,cl}^{ss} \longrightarrow \mathcal{M}_{P,cl}^{ss},$$
$$\sigma(t, [\mathcal{E}]) \mapsto [t^*\mathcal{E}].$$

Proof. Denote the action of the torus by $\sigma : T \times X \longrightarrow X$. Let m be an integer such that any Gieseker semistable sheaf on X with Hilbert polynomial P is m-regular [HL, Sect. 4.3]. Let $V = k^{\oplus P(m)}$ and $\mathcal{H} = V \otimes_k \mathcal{O}_X(-m)$. We start by noting that any line bundle on X admits a T-equivariant structure, since $\mathrm{Pic}(T) = 0$ (e.g. [Dol, Thm. 7.2]). Let Φ be a T-equivariant structure on $\mathcal{O}_X(1)$. The T-equivariant structure Φ induces a T-equivariant structure on $\mathcal{O}_X(-m)$ and therefore it induces a T-equivariant structure $\Phi_{\mathcal{H}} : \sigma^*\mathcal{H} \longrightarrow p_2^*\mathcal{H}$ (where we let T act trivially on V). Let $\mathcal{Q} = \underline{\mathrm{Quot}}_{X/k}(\mathcal{H}, P)$ be the Quot functor and let $Q = \mathrm{Quot}_{X/k}(\mathcal{H}, P)$ be the Quot scheme. Here Q is a projective

k-scheme representing $\mathcal{Q}$ [HL, Sect. 2.2]

$$\Xi : \mathcal{Q} \overset{\cong}{\Longrightarrow} \underline{Q},$$

where for any k-scheme S we denote the contravariant functor $\mathrm{Hom}(-, S)$ by $\underline{S}$. Now let $[\mathcal{H}_Q \xrightarrow{u} \mathcal{U}]$ be the universal family. Here $\mathcal{H}_Q$ is the pull-back of $\mathcal{H}$ along projection $X \times Q \longrightarrow X$, $\mathcal{U}$ is a Q-flat coherent sheaf on $X \times Q$ with Hilbert polynomial P and u is a surjective morphism. Let $p_{12} : T \times X \times Q \longrightarrow T \times X$ be projection, then it is easy to see that precomposing $(\sigma \times 1_Q)^* u$ with $p_{12}^* \Phi_{\mathcal{H}}^{-1}$ gives an element of $\mathcal{Q}(T \times Q)$. Applying $\Xi_{T \times Q}$ gives a morphism $\sigma : T \times Q \longrightarrow Q$, our candidate regular action. Note that σ depends on the choice of T-equivariant structure on $\mathcal{H}$. For any closed point $t \in T$, let $i_t : X \longrightarrow T \times X$ be the induced morphism and consider $\Phi_{\mathcal{H},t} = i_t^* \Phi_{\mathcal{H}} : t^*\mathcal{H} \xrightarrow{\cong} \mathcal{H}$. Let $p = [\mathcal{H} \xrightarrow{\rho} \mathcal{F}]$ be a closed point of Q. Using the properties of the universal family and the definition of σ, it is easy to see that the closed point corresponding to $\sigma(t, p) = t \cdot p$ is given by

$$[\mathcal{H} \xrightarrow{\Phi_{\mathcal{H},t}^{-1}} t^*\mathcal{H} \xrightarrow{t^*\rho} t^*\mathcal{F}].$$

Using the properties of the universal family, a somewhat tedious yet straightforward exercise shows that $\sigma : T \times Q \longrightarrow Q$ satisfies the axioms of an action. Let $R \subset Q$ be the open subscheme with closed points those elements $[\mathcal{H} \xrightarrow{\rho} \mathcal{F}] \in Q$, where $\mathcal{F}$ is Gieseker semistable and the induced map $V \longrightarrow H^0(X, \mathcal{F}(m))$ is an isomorphism. Since the T-equivariant structure on $\mathcal{H}$ comes from a T-equivariant structure on $\mathcal{O}_X(1)$ and a trivial action of T on V, σ restricts to a regular action on R. Let $G = \mathrm{PGL}(V)$, then there is a natural (right) action $\mathcal{Q} \times \underline{G} \Longrightarrow \mathcal{Q}$. This induces a regular (right) action of G on Q, which restricts to R. The moduli space $M^{ss} = \mathcal{M}_P^{ss}$ can be formed as a categorical quotient $\pi : R \longrightarrow M^{ss}$ [HL, Sect. 4.3]. Consider the diagram

$$\begin{tikzcd} T \times R \arrow[r, "\sigma"] \arrow[d, "1_T \times \pi"'] & R \arrow[d, "\pi"] \\ T \times M^{ss} & M^{ss}. \end{tikzcd}$$

The morphism $\sigma : T \times R \longrightarrow R$ is G-equivariant (where we let G act trivially on T). Again, this can be seen by using that the T-equivariant structure on $\mathcal{H}$ comes from a T-equivariant structure on $\mathcal{O}_X(1)$ and a trivial action of T on V. Consequently, $\pi \circ \sigma$ is G-invariant. From the definition of a categorical quotient, we get an induced morphism $\sigma : T \times M^{ss} \longrightarrow M^{ss}$. Again, using the definition of a categorical quotient, we obtain that $\sigma : T \times M^{ss} \longrightarrow M^{ss}$ is a regular action of T on M^{ss} acting on closed points as stated in the proposition. Let $R^s \subset R$ be the open subscheme with closed points Gieseker stable sheaves and denote the corresponding geometric quotient by $\varpi : R^s \longrightarrow M^s$. It is clear the regular action $\sigma : T \times M^{ss} \longrightarrow M^{ss}$ will restrict to M^s. □

Proposition 1.3.2. *Let X be a projective toric variety with torus action $\sigma : T \times X \longrightarrow X$. Denote projection to the second factor by $p_2 : T \times X \longrightarrow X$. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and P a choice of Hilbert polynomial. Choose an equivariant structure on $\mathcal{O}_X(1)$. Then the closed points of the fixed point locus*[14] *of the natural induced regular action of T on $\mathcal{M}_P^s$ (defined, using the equivariant structure, in Proposition 1.3.1) are*

$$(\mathcal{M}_P^s)_{cl}^T = \left\{ [\mathcal{E}] \in \mathcal{M}_{P,cl}^s \mid \sigma^* \mathcal{E} \cong p_2^* \mathcal{E} \right\}.$$

Proof. From the definition of $\sigma : T \times \mathcal{M}_P^s \longrightarrow \mathcal{M}_P^s$, it is clear that the fixed point locus can be characterised as [Fog, Rmk. 4]

$$(\mathcal{M}_P^s)_{cl}^T = \left\{ [\mathcal{E}] \in \mathcal{M}_{P,cl}^s \mid t^* \mathcal{E} \cong \mathcal{E} \ \forall t \in T_{cl} \right\}.$$

However, we claim that moreover

$$(\mathcal{M}_P^s)_{cl}^T = \left\{ [\mathcal{E}] \in \mathcal{M}_{P,cl}^s \mid \sigma^* \mathcal{E} \cong p_2^* \mathcal{E} \right\}.$$

The inclusion "$\supset$" is trivial. Conversely, let $\mathcal{E}$ be Gieseker stable sheaf on X with Hilbert

[14] We use the notion of fixed point locus as defined in [Fog].

polynomial P such that $t^*\mathcal{E} \cong \mathcal{E}$ for all closed points $t \in T$. Since $\mathcal{E}$ is simple[15] [HL, Cor. 1.2.8], the result follows from the following proposition[16] applied to $\sigma^*\mathcal{E}$ and $p_2^*\mathcal{E}$. $\square$

Proposition 1.3.3. *Let X be a projective k-scheme of finite type and T an algebraic torus. Let $\mathcal{E}$, $\mathcal{F}$ be T-flat coherent sheaves on $T \times X$ such that and $\mathcal{E}_t \cong \mathcal{F}_t$ is simple for all closed points $t \in T$. Then $\mathcal{E} \cong \mathcal{F}$.*

Proof. Denote projection to the first component by $p_1 : T \times X \longrightarrow X$. For any closed point $t \in T$, we denote the induced morphism by $i_t : k \longrightarrow T$. We consider the coherent sheaf $\mathcal{L} = p_{1*}\mathcal{H}om_{\mathcal{O}_{T\times X}}(\mathcal{E}, \mathcal{F})$ and will prove it is a line bundle on T. There exists a coherent sheaf $\mathcal{N}$ on T and an isomorphism

$$p_{1*}\mathcal{H}om_{\mathcal{O}_{T\times X}}(\mathcal{E}, \mathcal{F} \otimes_{\mathcal{O}_{T\times X}} p_1^*(-)) \cong \mathcal{H}om_{\mathcal{O}_T}(\mathcal{N}, -),$$

of functors $\mathrm{Qco}(T) \longrightarrow \mathrm{Qco}(T)$ [EGA3, Cor. 7.7.8]. We deduce $\mathcal{L} \cong \mathcal{N}^\vee$. However, the construction of $\mathcal{N}$ commutes with base change [EGA3, Rem. 7.7.9], so

$$(i_t^*\mathcal{N})^\vee \cong \mathrm{Hom}_X(\mathcal{E}_t, \mathcal{F}_t) \cong k,$$

for all closed points $t \in T$. Here we use that $\mathcal{E}_t \cong \mathcal{F}_t$ is simple for all closed points $t \in T$. Consequently, $\mathcal{N}$ and $\mathcal{L}$ are line bundles on T [Har1, Exc. II.5.8]. Since $\mathrm{Pic}(T) = 0$, we deduce $\mathcal{L} \cong \mathcal{O}_T$. Now consider

$$H^0(T, \mathcal{L}) = \mathrm{Hom}_{T\times X}(\mathcal{E}, \mathcal{F}).$$

Since $\mathcal{L} \cong \mathcal{O}_T$, there exists a nowhere vanishing section $f \in H^0(T, \mathcal{L})$. This section corresponds to a morphism $f : \mathcal{E} \longrightarrow \mathcal{F}$ having the property that $f_t : \mathcal{E}_t \longrightarrow \mathcal{F}_t$ is nonzero for any closed point $t \in T$. Similarly, $\mathcal{L}' = p_{1*}\mathcal{H}om_{\mathcal{O}_{T\times X}}(\mathcal{F}, \mathcal{E}) \cong \mathcal{O}_T$ and

[15] A simple sheaf $\mathcal{E}$ on a k-scheme X of finite type is by definition a coherent sheaf $\mathcal{E}$ on X such that $\mathrm{End}(\mathcal{E}) \cong k$.

[16] The following proposition and its proof are due to Tom Bridgeland. The author's original proof of Proposition 1.3.2 was much more complicated and involved Luna's Étale Slice Theorem and descent for quasi-coherent sheaves.

we can take a nowhere vanishing section $f' \in H^0(T, \mathcal{L}')$ corresponding to a morphism $f' : \mathcal{F} \longrightarrow \mathcal{E}$. Now consider the composition $g = f' \circ f$. There is a canonical map

$$H^0(T, \mathcal{O}_T) \xrightarrow{\cong} \mathrm{Hom}_{T \times X}(\mathcal{E}, \mathcal{E}),$$

which is an isomorphism by the arguments above. It is easy to see that $g_t \neq 0$ for any closed point $t \in T$, from which we deduce that g corresponds to $c\chi \in H^0(T, \mathcal{O}_T)$ for some $c \in k^*$ and $\chi \in X(T)$ a character. Therefore $(c^{-1}\chi^{-1}f') \circ f = \mathrm{id}_{\mathcal{E}}$. Similarly, we get a right inverse for f, showing f is an isomorphism. $\square$

Note that if we are in the situation of Propositions 1.3.1 and 1.3.2, the regular action of T on $\mathcal{M}_P^{ss}$, $\mathcal{M}_P^s$ a priori depends on choice of equivariant structure on $\mathcal{O}_X(1)$. However, the set $(\mathcal{M}_P^s)_{cl}^T$ is independent of this choice and our future constructions will not depend on this choice either. Hence, whenever we are in the situation of these propositions, we assume we fix an arbitrary equivariant structure on $\mathcal{O}_X(1)$ and induced torus action on $\mathcal{M}_P^{ss}$ without further notice.

1.3.2 Equivariant versus Invariant

Let G be an affine algebraic group acting regularly on a k-scheme X of finite type. Denote the action by $\sigma : G \times X \longrightarrow X$ and projection by $p_2 : G \times X \longrightarrow X$. From Proposition 1.3.2, we see that it is natural to define a G-invariant sheaf on X to be a sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ on X for which there is an isomorphism $\sigma^*\mathcal{E} \cong p_2^*\mathcal{E}$. Clearly, any G-equivariant sheaf on X is G-invariant, but the converse is not true in general (for an example, see [DOPR, App. A]). In the situation of Proposition 1.3.2, the isomorphism classes of Gieseker stable invariant sheaves on X with Hilbert polynomial P are in bijection with the closed points of $(\mathcal{M}_P^s)^T$. We have the following results.

Proposition 1.3.4. *Let G be a connected affine algebraic group acting regularly on a scheme X of finite type over k. Let $\mathcal{E}$ be a simple sheaf on X. Then $\mathcal{E}$ is G-invariant if and only if $\mathcal{E}$ admits a G-equivariant structure.*

Proof. Denote the action by $\sigma : G \times X \longrightarrow X$ and projection to the second component by $p_2 : G \times X \longrightarrow X$. Assume $\mathcal{E}$ is a simple sheaf on X and we have an isomorphism $\Phi : \sigma^*\mathcal{E} \longrightarrow p_2^*\mathcal{E}$. We would like Φ to satisfy the cocycle condition (see Definition 1.1.1). In order to achieve this, we use an argument similar to the proof of [Dol, Lem. 7.1]. For any closed point $g \in G$ given by a morphism $k \longrightarrow G$, let $i_g : X \longrightarrow G \times X$ be the induced map and define $\Phi_g = i_g^*\Phi : g^*\mathcal{E} \longrightarrow \mathcal{E}$. By Proposition 1.1.4, it is enough to prove $\Phi_{hg} = \Phi_g \circ g^*\Phi_h$, for all closed points $g, h \in G$. By redefining Φ, i.e. replacing Φ by $p_2^*(\Phi_1^{-1}) \circ \Phi$, we might just as well assume $\Phi_1 = 1$. Now define the morphism

$$F : G_{cl} \times G_{cl} \longrightarrow \mathrm{Aut}(\mathcal{E}) \cong k^*,$$
$$F(g,h) = \Phi_g \circ g^*(\Phi_h) \circ \Phi_{hg}^{-1},$$

where $(-)_{cl}$ means taking the closed points. We know $F(g,1) = F(1,h) = 1$ and we have to prove $F(g,h) = 1$, for all closed points $g, h \in G$. Since G_{cl} is an irreducible algebraic variety over an algebraically closed field k and $F \in \mathcal{O}(G_{cl} \times G_{cl})^*$, we can use a theorem by Rosenlicht [Dol, Rmk. 7.1], to conclude that $F(g,h) = F_1(g)F_2(h)$, where $F_1, F_2 \in \mathcal{O}(G_{cl})^*$, for all closed points $g, h \in G$. The result now follows immediately. $\square$

Proposition 1.3.5. *Let G be an affine algebraic group acting regularly on a scheme X of finite type over k. Let $\mathcal{E}$ be a simple G-equivariant sheaf on X. Then all G-equivariant structures on $\mathcal{E}$ are given by $\mathcal{E} \otimes \mathcal{O}_X(\chi)$, where $\mathcal{O}_X(\chi)$ is the structure sheaf of X endowed with the G-equivariant structure induced by the character $\chi \in X(G)$.*

Proof. Let $\Phi, \Psi : \sigma^*(\mathcal{E}) \longrightarrow p_2^*(\mathcal{E})$ be two G-equivariant structures on $\mathcal{E}$. Consider the automorphism $\Psi \circ \Phi^{-1} : p_2^*(\mathcal{E}) \longrightarrow p_2^*(\mathcal{E})$. For all closed points $g \in G$

$$\Psi_g \circ \Phi_g^{-1} \in \mathrm{Aut}(\mathcal{E}) \cong k^*.$$

We obtain a morphism of varieties $\chi : G_{cl} \longrightarrow k^*$ defined by $\chi(g) = \Psi_g \circ \Phi_g^{-1}$. In fact, from the fact that Φ, Ψ satisfy the cocycle condition (see Definition 1.1.1), we see that χ is a character. The result follows from applying Proposition 1.1.4. $\square$

This last proposition suggests we should study the effect of tensoring an equivariant sheaf on a toric variety by an equivariant line bundle. We start with a brief recapitulation of equivariant line bundles and reflexive equivariant sheaves on toric varieties. On a general normal variety X, a coherent sheaf $\mathcal{F}$ is said to be reflexive if the natural morphism $\mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$ is an isomorphism, where $(-)^{\vee} = \mathcal{H}om(-, \mathcal{O}_X)$ is the dual. Let X be a nonsingular toric variety defined by a fan Δ in a lattice N of rank r. Take $\tau = 0$ and let $\sigma_1, \ldots, \sigma_l$ be the cones of dimension r. For each $i = 1, \ldots, l$, let $\left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ be the rays of σ_i. The equivariant line bundles on X are precisely the rank 1 reflexive equivariant sheaves on X. In general, reflexive equivariant sheaves on X are certain torsion free equivariant sheaves on X and they admit a particularly nice combinatorial description in terms of filtrations associated to the rays of Δ. Denote the collection of rays by $\Delta(1)$. Let E be a nonzero finite-dimensional k-vector space. For each ray $\rho \in \Delta(1)$ specify k-vector spaces

$$\cdots \subset E^{\rho}(\lambda - 1) \subset E^{\rho}(\lambda) \subset E^{\rho}(\lambda + 1) \subset \cdots ,$$

such that there is an integer A_ρ with $E^{\rho}(\lambda) = 0$ if $\lambda < A_\rho$ and there is an integer B_ρ such that $E^{\rho}(\lambda) = E$ if $\lambda \geq B_\rho$. There is an obvious notion of morphisms between such collections of filtrations $\{E^{\rho}(\lambda)\}_{\rho \in \Delta(1)}$. Suppose we are given such a collection of filtrations $\{E^{\rho}(\lambda)\}_{\rho \in \Delta(1)}$. From it we obtain a torsion free Δ-family by defining

$$E^{\sigma_i}(\lambda_1, \ldots, \lambda_r) = E^{\rho_1^{(i)}}(\lambda_1) \cap \cdots \cap E^{\rho_r^{(i)}}(\lambda_r), \tag{1.13}$$

for all $i = 1, \ldots, l$, $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}$. Denote the full subcategory of torsion free Δ-families obtained in this way by $\mathcal{R}$. The equivalence of categories of Theorem 1.1.10, restricts to an equivalence between the the full subcategory of reflexive equivariant sheaves on X and the full subcategory $\mathcal{R}$ (see [Per1, Thm. 4.21]). This equivalence further restricts to an equivalence between the category of equivariant line bundles on X and the category of filtrations of $E = k$ associated to the rays of Δ as above. We obtain a canonical isomorphism $\mathrm{Pic}^T(X) \cong \mathbb{Z}^{\Delta(1)}$, where $\mathbb{Z}^{\Delta(1)} = \mathbb{Z}^{\#\Delta(1)}$. In particular, if $\Delta(1) = (\rho_1, \ldots, \rho_N)$, then the integers $\vec{k} = (k_1, \ldots, k_N) \in \mathbb{Z}^{\Delta(1)}$ correspond to the filtrations $\{L^{\rho}_{\vec{k}}(\lambda)\}_{\rho \in \Delta(1)}$ defined by[17]

$$L^{\rho_j}_{\vec{k}}(\lambda) = \begin{cases} k & \text{if } \lambda \geq -k_j \\ 0 & \text{if } \lambda < -k_j, \end{cases}$$

for all $j = 1, \ldots, N$. Denote the corresponding equivariant line bundle by $\mathcal{L}_{\vec{k}}$. Note that the first Chern class of $\mathcal{L}_{\vec{k}}$ is given by $c_1(\mathcal{L}_{\vec{k}}) = \sum_j k_j V(\rho_j)$ (Corollary 1.2.18), so as a line bundle $\mathcal{L}_{\vec{k}} \cong \mathcal{O}_X(\sum_j k_j V(\rho_j))$. Finally, when we consider $\vec{k}$, $\{L^{\rho}_{\vec{k}}(\lambda)\}_{\rho \in \Delta(1)}$, $\mathcal{L}_{\vec{k}}$ as above, then the corresponding torsion free Δ-family is given by (equation (1.13))

$$L^{\sigma_i}_{\vec{k}}(\lambda_1, \ldots, \lambda_r) = \begin{cases} k & \text{if } \lambda_1 \geq -k_1^{(i)}, \ldots, \lambda_r \geq -k_r^{(i)} \\ 0 & \text{otherwise,} \end{cases}$$

for all $i = 1, \ldots, l$, where σ_i has rays $\left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ and we denote the corresponding integers where the filtrations $L^{\rho_1^{(i)}}_{\vec{k}}(\lambda), \ldots, L^{\rho_r^{(i)}}_{\vec{k}}(\lambda)$ jump by $-k_1^{(i)}, \ldots, -k_r^{(i)}$.

Proposition 1.3.6. *Let X be a nonsingular toric variety with fan Δ in a lattice N of rank r. Let $\tau_1, \ldots, \tau_a$ be some cones of Δ of dimension s. Let $\sigma_1, \ldots, \sigma_l$ be all cones of Δ of maximal dimension having a cone τ_α as a face. For each $i = 1, \ldots, l$, let $\left(\rho_1^{(i)}, \ldots, \rho_r^{(i)}\right)$ be the rays of σ_i. Let $\mathcal{E}$ be a pure equivariant sheaf on X with support $V(\tau_1) \cup \cdots \cup V(\tau_a)$ and corresponding pure Δ-family $\hat{E}^\Delta$. Consider the equivariant line bundle $\mathcal{L}_{\vec{k}}$ for some $\vec{k} \in \mathbb{Z}^{\Delta(1)}$. Then $\mathcal{F} = \mathcal{E} \otimes \mathcal{L}_{\vec{k}}$ is a pure equivariant sheaf on X with support $V(\tau_1) \cup \cdots \cup V(\tau_a)$ and its pure Δ-family $\hat{F}^\Delta$ is given by*

$$F^{\sigma_i}(\lambda_1, \ldots, \lambda_r) = E^{\sigma_i}(\lambda_1 + k_1^{(i)}, \ldots, \lambda_r + k_r^{(i)}),\ \forall i = 1, \ldots, l$$
$$\chi^{\sigma_i}_{F,n}(\lambda_1, \ldots, \lambda_r) = \chi^{\sigma_i}_{E,n}(\lambda_1 + k_1^{(i)}, \ldots, \lambda_r + k_r^{(i)}),\ \forall i = 1, \ldots, l,\ \forall n = 1, \ldots, r.$$

Proof. One can compute the M-grading of $\Gamma(U_{\sigma_i}, \mathcal{F}) \cong \Gamma(U_{\sigma_i}, \mathcal{E}) \otimes_{k[S_{\sigma_i}]} \Gamma(U_{\sigma_i}, \mathcal{L}_{\vec{k}})$ along the same lines as in the proof of Proposition 1.1.9. The result easily follows. □

[17] Do not be confused by the unfortunate clash of notation: k is the ground field and the k_j are integers.

1.3.3 Combinatorial Description of the Fixed Point Loci $(\mathcal{M}_P^s)^T$

We are now ready to prove the theorem stated in the introduction (Theorem 1.0.1). An analogous result to Theorem 1.0.1 turns out to hold without any assumption on the Hilbert polynomial if we assume the existence of equivariant line bundles matching Gieseker and GIT stability. Therefore, we will first prove a general combinatorial expression for the fixed point locus of any moduli space of Gieseker stable sheaves on a nonsingular projective toric variety making this assumption (Theorem 1.3.9). Theorem 1.0.1 then follows as a trivial corollary of this result by combining with Theorem 1.2.22. Let X be a nonsingular projective toric variety defined by a fan Δ in a lattice N of rank r. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and let P be a choice of Hilbert polynomial. The degree d of P is the dimension d of any coherent sheaf on X with Hilbert polynomial P. Let $s = r - d$ and let $\tau_1, \ldots, \tau_a$ be all cones of Δ of dimension s. For any $i_1 < \cdots < i_\alpha \in \{1, \ldots, a\}$, we have defined $\mathcal{X}_P^{\tau_{i_1}, \ldots, \tau_{i_\alpha}} \subset \mathcal{X}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}}$ to be the subset of all characteristic functions with associated Hilbert polynomial P (see Proposition 1.2.14). Assume that for any $\vec{\chi} \in \mathcal{X}_P^{\tau_{i_1}, \ldots, \tau_{i_\alpha}}$, we can pick an equivariant line bundle matching Gieseker and GIT stability (e.g. for P of degree $\dim(X)$ this can always be done by Theorem 1.2.22). For any $\vec{\chi} \in \mathcal{X}_P^{\tau_{i_1}, \ldots, \tau_{i_\alpha}}$ the obvious forgetful natural transformations $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, ss} \Longrightarrow \underline{\mathcal{M}}_P^{ss}$, $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, s} \Longrightarrow \underline{\mathcal{M}}_P^{s}$ induce morphisms $\mathcal{M}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, ss} \longrightarrow \mathcal{M}_P^{ss}$, $\mathcal{M}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, s} \longrightarrow \mathcal{M}_P^{s}$ (by Theorem 1.2.13). We obtain morphisms

$$\coprod_{\alpha=1}^{a} \coprod_{i_1 < \cdots < i_\alpha \in \{1, \ldots, a\}} \coprod_{\vec{\chi} \in \mathcal{X}_P^{\tau_{i_1}, \ldots, \tau_{i_\alpha}}} \mathcal{M}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, ss} \longrightarrow \mathcal{M}_P^{ss},$$

$$\coprod_{\alpha=1}^{a} \coprod_{i_1 < \cdots < i_\alpha \in \{1, \ldots, a\}} \coprod_{\vec{\chi} \in \mathcal{X}_P^{\tau_{i_1}, \ldots, \tau_{i_\alpha}}} \mathcal{M}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, s} \longrightarrow \mathcal{M}_P^{s},$$

where the second morphism, on closed points, is just the map forgetting the equivariant structure. Consequently, we could expect the second morphism to factor through an isomorphism onto the fixed point locus $(\mathcal{M}_P^s)^T$. Indeed it maps to the fixed point locus on closed points

$$\coprod_{\alpha=1}^{a} \coprod_{i_1 < \cdots < i_\alpha \in \{1,\ldots,a\}} \coprod_{\vec{\chi} \in \mathcal{X}_P^{\tau_{i_1},\ldots,\tau_{i_\alpha}}} \mathcal{M}_{\vec{\chi},cl}^{\tau_{i_1},\ldots,\tau_{i_\alpha},s} \longrightarrow (\mathcal{M}_P^s)_{cl}^T,$$

and this map is surjective (Propositions 1.3.2 and 1.3.4). However, it is not injective (Proposition 1.3.5). Indeed, if $\mathcal{E}$ is an invariant simple sheaf on X, then it admits an equivariant structure (fix one) and all equivariant structures are given by $\mathcal{E} \otimes \mathcal{O}_X(\chi)$, $\chi \in M$. So we have the character group acting as a gauge group. Therefore, we need to do gauge-fixing and take a gauge slice before we can expect the morphism to factor through an isomorphism onto the fixed point locus $(\mathcal{M}_P^s)^T$. In view of Proposition 1.3.6, this might be achieved as follows. Let $\sigma_1, \ldots, \sigma_l$ be all cones of maximal dimension. Let $\alpha = 1, \ldots, a$ and $i_1 < \cdots < i_\alpha \in \{1, \ldots, a\}$. Let σ_n be a cone among $\sigma_1, \ldots, \sigma_l$ having at least one of $\tau_{i_1}, \ldots, \tau_{i_\alpha}$ as a face. For definiteness, we choose σ_n the cone among $\sigma_1, \ldots, \sigma_l$ with this property and smallest index n. Let $\vec{\chi} \in \mathcal{X}_P^{\tau_{i_1},\ldots,\tau_{i_\alpha}}$, then there are integers $A_1^{(n)}, \ldots, A_r^{(n)}$ such that $\chi^{\sigma_n}(\lambda_1, \ldots, \lambda_r) = 0$ unless $\lambda_1 \geq A_1^{(n)}$, …, $\lambda_r \geq A_r^{(n)}$ (see section 1.1). Assume $A_1^{(n)}, \ldots, A_r^{(n)}$ are chosen maximally with this property. We define $\vec{\chi}$ to be gauge-fixed if $A_1^{(n)} = \cdots = A_r^{(n)} = 0$. We denote the subset of gauge-fixed characteristic functions of $\mathcal{X}_P^{\tau_{i_1},\ldots,\tau_{i_\alpha}}$ by $\left(\mathcal{X}_P^{\tau_{i_1},\ldots,\tau_{i_\alpha}}\right)^{gf}$. We get a morphism

$$F : \coprod_{\alpha=1}^{a} \coprod_{i_1 < \cdots < i_\alpha \in \{1,\ldots,a\}} \coprod_{\vec{\chi} \in \left(\mathcal{X}_P^{\tau_{i_1},\ldots,\tau_{i_\alpha}}\right)^{gf}} \mathcal{M}_{\vec{\chi}}^{\tau_{i_1},\ldots,\tau_{i_\alpha},s} \longrightarrow \mathcal{M}_P^s. \tag{1.14}$$

Claim. The map induced by F on closed points maps bijectively onto $(\mathcal{M}_P^s)_{cl}^T$.

Proof of Claim. This can be seen as follows. We need to characterise all equivariant line bundles $\mathcal{O}_X(\chi)$, $\chi \in X(T)$ (introduced in Proposition 1.3.5). By [Dol, Cor. 7.1, Thm. 7.2], they are precisely the elements of the kernel of the forgetful map in the short exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{Pic}^T(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

Suppose $\mathcal{L}_{\vec{k}}$, $\vec{k} \in \mathbb{Z}^{\Delta(1)}$ is an equivariant line bundle (notation as in subsection 1.3.2). Then its underlying line bundle is trivial if and only if $\sum_{j=1}^{N} k_j V(\rho_j) = 0$ in the Chow group $A_{r-1}(X)$. The Chow group $A_{r-1}(X)$ is the free abelian group on $V(\rho_1), \ldots, V(\rho_N)$ modulo the following relations [Ful, Sect. 5.2]

$$\sum_{j=1}^{N} \langle u, n(\rho_j) \rangle V(\rho_j) = 0, \ \forall u \in M.$$

Let σ_n be any cone of maximal dimension and take $m\left(\rho_1^{(n)}\right), \ldots, m\left(\rho_r^{(n)}\right)$ as a basis for M. From this we see that for arbitrary $k_1^{(n)}, \ldots, k_r^{(n)} \in \mathbb{Z}$, there are unique other $k_1^{(i)}, \ldots, k_r^{(i)}$, for all $i = 1, \ldots, l$, $i \neq n$ such that $\sum_{j=1}^{N} k_j V(\rho_j) = 0$. In particular, if $\mathcal{L}_{\vec{k}}$, $\vec{k} \in \mathbb{Z}^{\Delta(1)}$ is an equivariant line bundle with underlying line bundle trivial and $k_1^{(n)} = \cdots = k_r^{(n)} = 0$, then also $k_1^{(i)} = \cdots = k_r^{(i)} = 0$ for all $i = 1, \ldots, l$, $i \neq n$. Now note that for any two distinct sequences $i_1 < \cdots < i_\alpha$, $j_1 < \cdots < j_\beta \in \{1, \ldots, a\}$, we have $\mathcal{X}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}} \cap \mathcal{X}^{\tau_{j_1}, \ldots, \tau_{j_\beta}} = \varnothing$. Using Propositions 1.3.2, 1.3.4, 1.3.5 and 1.3.6, the claim follows. $\square$

We note that the above claim crucially depends on Propositions 1.3.2, 1.3.4, 1.3.5, which are about *simple* sheaves. This is one of the main reasons we have to focus attention on *Gieseker stable* sheaves only in this section. The above claim provides good evidence that the morphism F of equation (1.14) indeed factors through an isomorphism onto the fixed point locus $(\mathcal{M}_P^s)^T$. We will prove this using the following two technical results.

Proposition 1.3.7. *Let X, Y be schemes of finite type and separated over k. Let $f : X \longrightarrow Y$ be a morphism and $\iota : Y' \hookrightarrow Y$ a closed immersion. Assume for any local artinian k-algebra A with residue field k, the map $f \circ -$ factors bijectively*

$$\begin{tikzcd} \mathrm{Hom}(A, X) \arrow[r, "f\circ -"] \arrow[dr, dashed, "\cong"'] & \mathrm{Hom}(A, Y) \\ & \mathrm{Hom}(A, Y'). \arrow[u, hook, "\iota\circ -"'] \end{tikzcd}$$

Then f factors isomorphically and uniquely onto Y'

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "\cong"'] & Y \\ & Y'. \arrow[u, hook, "\iota"'] \end{tikzcd}$$

Proof. First we prove the proposition while assuming $f \circ -$ factors (not necessarily as a bijection) and conclude f factors through Y' (not necessarily as an isomorphism). It is clear that if f factors, then it factors uniquely, because ι is a closed immersion. By taking an appropriate open affine cover we get the following diagram of finitely generated k-algebras

$$\begin{tikzcd} R & S \arrow[l, "f^{\#}"'] \arrow[d, "\iota^{\#}"] \\ & S/I, \end{tikzcd}$$

where $I \subset S$ is some ideal. It is enough to prove $f^{\#}(I) = 0$. Suppose this is not the case. Then there is some $0 \neq s \in I$ such that $f^{\#}(s) = r \neq 0$. There exists a maximal ideal $\mathfrak{m} \subset R$ such that r is not mapped to zero by the localisation map $R \longrightarrow R_{\mathfrak{m}}$ (use [AM, Exc. 4.10]). Moreover, there is an integer $n > 0$ such that the canonical map $R_{\mathfrak{m}} \longrightarrow R_{\mathfrak{m}}/(\mathfrak{m}R_{\mathfrak{m}})^n$ maps $r/1 \in R_{\mathfrak{m}}$ to a nonzero element (this follows from a corollary of Krull's Theorem [AM, Cor. 10.19]). Now $R_{\mathfrak{m}}/(\mathfrak{m}R_{\mathfrak{m}})^n$ is a local artinian k-algebra with residue field k [AM, Prop. 8.6]. We obtain a k-algebra homomorphism $R \longrightarrow R_{\mathfrak{m}}/(\mathfrak{m}R_{\mathfrak{m}})^n$ such that precomposition with $f^{\#}$ maps s to a nonzero element. But by assumption, this composition has to factor through S/I and since $s \in I$ this is a contradiction.

The second part of the proof of the proposition consists of proving the statement of the proposition in the case $Y' = Y$. Together with the first part of the proof, this proves the proposition. This part can be proved similarly by again taking an appropriate open affine cover and using the same corollary of Krull's Theorem in a similar fashion applied to the kernel and cokernel of $f^{\#}$. In dealing with the open affine covers, it is useful to first prove that f is everywhere locally a closed immersion, consequently proper and quasi-finite, hence finite [EGA2, Cor. 4.4.11] and therefore affine. $\square$

Proposition 1.3.8. *Let X be a nonsingular projective toric variety defined by a fan Δ in a lattice N of rank r. Let $\mathcal{O}_X(1)$ be an ample line bundle on X, let P be a choice of Hilbert polynomial of degree d and let $\tau_1, \ldots, \tau_a$ be all cones of Δ of dimension $s = r - d$. Let $i_1 < \cdots < i_\alpha \in \{1, \ldots, a\}$ and assume for any $\vec{\chi} \in \mathcal{X}_P^{\tau_{i_1}, \ldots, \tau_{i_\alpha}}$, we can pick an equivariant line bundle matching Gieseker and GIT stability. Then for any local artinian k-algebra A with residue field k the moduli functors and their moduli spaces induce bijections*

$$\underline{\mathcal{M}}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, s}(A) \xrightarrow{\cong} \mathrm{Hom}(A, \mathcal{M}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, s}),$$
$$\underline{\mathcal{M}}_P^s(A) \xrightarrow{\cong} \mathrm{Hom}(A, \mathcal{M}_P^s).$$

Proof. Let us prove the first bijection first. Denote the moduli functor $\underline{\mathcal{M}}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, s}$ by $\mathcal{M}$. Recall that $M = \mathcal{M}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, s}$ was formed by considering the regular action of the reductive algebraic group G on $N = \mathcal{N}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, s}$ (where GIT stability is defined by the G-equivariant line bundle $L = \mathcal{L}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}}$) and the induced geometric quotient $\varpi : N \longrightarrow M = N/G$ (see subsection 1.2.3). Here G is the closed subgroup of elements of determinant 1 of an algebraic group of the form

$$H = \prod_{i=1}^{n} \mathrm{GL}(n_i, k).$$

We would like to use a corollary of Luna's Étale Slice Theorem to conclude that ϖ is a principal G-bundle [HL, Cor. 4.2.13]. Unfortunately, the stabiliser of a closed point of N is the group μ_p of pth roots of unity, where $p = \sum_{i=1}^{n} n_i$, hence not trivial. Consider the diagonal closed subgroup $\mathbb{G}_m \lhd H$ and define $\tilde{G} = H/\mathbb{G}_m$. There is a natural regular action of the reductive algebraic group $\tilde{G}$ on $\mathcal{N}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}}$ giving rise to the same orbits as G. The natural morphism $G \longrightarrow \tilde{G}$ of algebraic groups gives rise to an isomorphism $G/\mu_p \cong \tilde{G}$ of algebraic groups. If we fix a G-equivariant line bundle L, then it is easy to see that $L^{\otimes p}$ admits a $\tilde{G}$-equivariant structure. For both choices, the sets of GIT semistable respectively stable points will be the same and the categorical and geometric quotients will be the same. In particular $M = N/G = N/\tilde{G}$. The stabiliser in $\tilde{G}$ of any GIT stable closed point of N is trivial. Consequently, $\varpi : N \longrightarrow M$ is a principal $\tilde{G}$-bundle, i.e. there is an étale surjective morphism $\pi : Y \longrightarrow M$ and a $\tilde{G}$-equivariant isomorphism $\psi : \tilde{G} \times Y \longrightarrow N \times_M Y$, such that the following diagram commutes

$$\begin{tikzcd} \tilde{G} \times Y \arrow[r, "\psi"] \arrow[d] & N \times_M Y \arrow[dl] \\ Y. & \end{tikzcd}$$

Let $P = [\mathcal{E}] \in M$ be a closed point and let $Q \in Y$ be a closed point such that $\pi(Q) = P$. Let A be any local artinian k-algebra with residue field k, let $\mathrm{Hom}(A, M)_P$ be the set of morphisms $A \longrightarrow M$ where the point is mapped to P and let $\mathrm{Hom}(A, Y)_Q$ be the set of morphisms $A \longrightarrow Y$ where the point is mapped to Q. Using induction on the length[18] of A and using the definition of formally étale [EGA4, Def. 17.1.1], it is easy to see that composition with π gives a bijection

$$\mathrm{Hom}(A, Y)_Q \xrightarrow{\cong} \mathrm{Hom}(A, M)_P.$$

As an aside, we note that this implies in particular that the Zariski tangent spaces at P and Q are isomorphic $T_Q Y \cong T_P M$, by taking A the ring of dual numbers. We have $\mathcal{M}(A) \cong \mathrm{Hom}(A, N)/\mathrm{Hom}(A, G) = \mathrm{Hom}(A, N)/\mathrm{Hom}(A, \tilde{G})$. The first isomorphism follows from the definition of $\mathcal{M}$ (see proof of Theorem 1.2.12). The second equality can be deduced from the fact that the morphism $G \longrightarrow \tilde{G}$ is étale and surjective on closed points. Using these facts, together with the isomorphism ψ, we obtain a natural injection $\mathrm{Hom}(A, M)_P \hookrightarrow \mathcal{M}(A)$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{M}(A) \arrow[r] & \mathrm{Hom}(A, M) \\ & \mathrm{Hom}(A, M)_P. \arrow[ul, hook, dashed] \arrow[u, hook, "\subset"'] \end{tikzcd}$$

[18]Note that for any local artinian k-algebra $(A', \mathfrak{m}')$ of length $l \geq 2$ with residue field k there is a surjective local k-algebra homomorphism $\sigma : A' \longrightarrow A$, where A is a local artinian k-algebra of length $< l$ with residue field k and kernel J a principal ideal satisfying $\mathfrak{m}' J = 0$. Such surjective morphisms are called small extensions [Sch].

Let $\mathcal{M}(A)_P$ be the image of the injection $\mathrm{Hom}(A, M)_P \hookrightarrow \mathcal{M}(A)$. Consider the natural morphism $\iota : k \longrightarrow A$ (we actually mean $\iota : \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(A)$). It is easy to see that $\mathcal{M}(A)_P$ is the set of equivariant isomorphism classes of A-flat equivariant coherent sheaves $\mathcal{F}$ on $X \times A$ such that there exists an equivariant isomorphism $(1_X \times \iota)^*\mathcal{F} \cong \mathcal{E}$. We obtain a natural bijection

$$\mathcal{M}(A)_P = \{[\mathcal{F}] \in \mathcal{M}(A) \mid (1_X \times \iota)^*\mathcal{F} \cong \mathcal{E}\} \xrightarrow{\cong} \mathrm{Hom}(A, \mathcal{M})_P.$$

Taking a union over all closed points P gives the required bijection. The second bijection of the proposition can be proved entirely analogously. For the definition of the moduli functor and moduli space in this case, we refer to [HL, Ch. 4]. $\square$

We can now formulate and prove the following theorem.

Theorem 1.3.9. *Let X be a nonsingular projective toric variety defined by a fan Δ in a lattice N of rank r. Let $\mathcal{O}_X(1)$ be an ample line bundle on X, let P be a choice of Hilbert polynomial of degree d and let $\tau_1, \ldots, \tau_a$ be all cones of Δ of dimension $s = r - d$. Assume for any $i_1 < \cdots < i_\alpha \in \{1, \ldots, a\}$ and $\vec{\chi} \in \mathcal{X}_P^{\tau_{i_1}, \ldots, \tau_{i_\alpha}}$, we can pick an equivariant line bundle matching Gieseker and GIT stability. Then there is a natural isomorphism of quasi-projective schemes of finite type over k*

$$\coprod_{\alpha=1}^{a} \coprod_{i_1 < \cdots < i_\alpha \in \{1, \ldots, a\}} \coprod_{\vec{\chi} \in \left(\mathcal{X}_P^{\tau_{i_1}, \ldots, \tau_{i_\alpha}}\right)^{gf}} \mathcal{M}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, s} \cong \left(\mathcal{M}_P^s\right)^T. \tag{1.15}$$

Proof. Consider the morphism F of equation (1.14). We start by noting that there are only finitely many characteristic functions $\vec{\chi}$ in the disjoint union of the left hand side of equation (1.15) for which $\mathcal{M}_{\vec{\chi}}^{\tau_{i_1}, \ldots, \tau_{i_\alpha}, s} \neq \varnothing$. This follows from the fact that the morphism F_{cl} on closed points is bijective and the disjoint union is over a countable set. As a consequence, the left hand side of (1.15) is a quasi-projective k-scheme of finite type over k (see Theorem 1.2.13). We now want to apply Proposition 1.3.7 to the morphism F of equation (1.14) and the closed subscheme $\iota : \left(\mathcal{M}_P^s\right)^T \hookrightarrow \mathcal{M}_P^s$. We proceed by induction on the length of local artinian k-algebras with residue field k. For length 1 (i.e. $A \cong k$), the hypothesis of Proposition 1.3.7 is satisfied. This is the content of the claim at the beginning of this subsection. Assume we have proved the hypothesis of Proposition 1.3.7 for all lengths $1, \ldots, l$ and let A' be a local artinian k-algebra of length $l+1$ with residue field k. Then it fits in a small extension $0 \longrightarrow J \longrightarrow A' \xrightarrow{\sigma} A \longrightarrow 0$, where A is a local artinian k-algebra of length $\leq l$ with residue field k. Using [Fog, Thm. 2.3], one can show the image of $\mathrm{Hom}(A', (\mathcal{M}_P^s)^T)$ in $\mathrm{Hom}(A', \mathcal{M}_P^s)$ is $\mathrm{Hom}(A', \mathcal{M}_P^s)^{T_{cl}}$. Define abbreviations

$$\underline{\mathcal{M}} = \coprod_{\alpha=1}^{a} \coprod_{i_1 < \cdots < i_\alpha \in \{1,\ldots,a\}} \coprod_{\vec{\mathcal{X}} \in \left(\mathcal{X}_P^{\tau_{i_1},\ldots,\tau_{i_\alpha}}\right)^{gf}} \underline{\mathcal{M}}_{\vec{\mathcal{X}}}^{\tau_{i_1},\ldots,\tau_{i_\alpha},s},$$

$$\underline{\mathcal{N}} = \underline{\mathcal{M}}_P^s.$$

Using Proposition 1.3.8, it is enough to prove that the map $\underline{\mathcal{M}}(A') \longrightarrow \underline{\mathcal{N}}(A')$ maps bijectively onto the fixed point locus $\underline{\mathcal{N}}(A')^{T_{cl}}$. (Note that T_{cl} act naturally on the set $\underline{\mathcal{N}}(A')$. We will drop the subscript cl referring to closed points from now on.) By the induction hypothesis, we know $\underline{\mathcal{M}}(A) \longrightarrow \underline{\mathcal{N}}(A)$ maps bijectively onto $\underline{\mathcal{N}}(A)^T$. Before we continue, we need to study the deformations and obstructions associated to the moduli functors $\underline{\mathcal{M}}$, $\underline{\mathcal{N}}$.

In general, let $\mathcal{E}_0$ be a simple coherent sheaf on X and $\mathcal{F}_0$ a simple equivariant coherent sheaf on X. Let $Artin/k$ be the category of local artinian k-algebras with residue field k. Consider the deformation functor $\mathcal{D}_{\mathcal{E}_0} : Artin/k \longrightarrow Sets$, where $\mathcal{D}_{\mathcal{E}_0}(A)$ is defined to be the set of isomorphism classes of A-flat coherent sheaves on $X \times A$ such that $\mathcal{F} \otimes_k A \cong \mathcal{E}_0$. Similarly, we define the deformation functor $\mathcal{D}_{\mathcal{F}_0}^{eq} : Artin/k \longrightarrow Sets$, where $\mathcal{D}_{\mathcal{F}_0}^{eq}(A)$ is defined to be the set of equivariant isomorphism classes of A-flat equivariant coherent sheaves on $X \times A$ such that $\mathcal{F} \otimes_k A \cong \mathcal{F}_0$ (equivariant isomorphism). In our setting, we have a small extension $0 \longrightarrow J \longrightarrow A' \xrightarrow{\sigma} A \longrightarrow 0$. We now consider the maps $\mathcal{D}_{\mathcal{E}_0}(\sigma) : \mathcal{D}_{\mathcal{E}_0}(A') \longrightarrow \mathcal{D}_{\mathcal{E}_0}(A)$, $\mathcal{D}_{\mathcal{F}_0}^{eq}(\sigma) : \mathcal{D}_{\mathcal{F}_0}^{eq}(A') \longrightarrow \mathcal{D}_{\mathcal{F}_0}^{eq}(A)$. There is a natural map

$$\mathfrak{o}(\sigma) : \mathcal{D}_{\mathcal{E}_0}(A) \longrightarrow \mathrm{Ext}^2(\mathcal{E}_0, \mathcal{E}_0) \otimes_k J,$$

called the obstruction map, such that $\mathfrak{o}(\sigma)^{-1}(0) = \operatorname{im}(\mathcal{D}_{\mathcal{E}_0}(\sigma))$. The construction of this map can be found in [Art, Sect. 2]. Moreover, for any $[\mathcal{F}] \in \operatorname{im}(\mathcal{D}_{\mathcal{E}_0}(\sigma))$, the fibre $\mathcal{D}_{\mathcal{E}_0}(\sigma)^{-1}([\mathcal{F}])$ is naturally an $\operatorname{Ext}^1(\mathcal{E}_0, \mathcal{E}_0) \otimes_k J$-torsor. This can be seen by noting that Proposition 1.3.8 implies $\mathcal{D}_{\mathcal{E}_0}$ is pro-representable by the completion $\widehat{\mathcal{O}_{\mathcal{M}^s_P,[\mathcal{E}_0]}}$ of the noetherian local k-algebra $\mathcal{O}_{\mathcal{M}^s_P,[\mathcal{E}_0]}$ and using Schlessinger's Criterion [Sch, Thm. 2.11]. Entirely analogously, one can construct an obstruction map[19]

$$\mathfrak{o}^{eq}(\sigma) : \mathcal{D}_{\mathcal{F}_0}(A) \longrightarrow T\text{-}\operatorname{Ext}^2(\mathcal{F}_0, \mathcal{F}_0) \otimes_k J,$$

also called the obstruction map, such that $\mathfrak{o}^{eq}(\sigma)^{-1}(0) = \operatorname{im}(\mathcal{D}^{eq}_{\mathcal{F}_0}(\sigma))$. Moreover, for any $[\mathcal{F}] \in \operatorname{im}(\mathcal{D}^{eq}_{\mathcal{F}_0}(\sigma))$, the fibre $\mathcal{D}^{eq}_{\mathcal{F}_0}(\sigma)^{-1}([\mathcal{F}])$ is naturally a $T\text{-}\operatorname{Ext}^1(\mathcal{F}_0, \mathcal{F}_0) \otimes_k J$-torsor.

Rewriting the moduli functors in terms of deformation functors, we obtain

$$\underline{\mathcal{M}}(A') = \coprod_{[\mathcal{F}] \in \underline{\mathcal{M}}(A)} \mathcal{D}_{\mathcal{F} \otimes_A k}(\sigma)^{-1}([\mathcal{F}]),$$
$$\underline{\mathcal{N}}(A') = \coprod_{[\mathcal{F}] \in \underline{\mathcal{N}}(A)} \mathcal{D}^{eq}_{\mathcal{F} \otimes_A k}(\sigma)^{-1}([\mathcal{F}]).$$

The remarks on obstructions and deformations together with the induction hypothesis, show that it is enough to relate the T-equivariant Ext groups to the invariant part of the ordinary Ext groups. It is enough to prove that for any equivariant coherent sheaves $\mathcal{A}, \mathcal{B}$ on X and for any $i \in \mathbb{Z}$ there is a canonical bijection

$$T\text{-}\operatorname{Ext}^i(\mathcal{A}, \mathcal{B}) \xrightarrow{\cong} \operatorname{Ext}^i(\mathcal{A}, \mathcal{B})^T \subset \operatorname{Ext}^i(\mathcal{A}, \mathcal{B}).$$

This can be seen by using the following spectral sequence [Toh, Thm. 5.6.3]

$$II_2^{p,q}(\mathcal{B}) = H^p(T, \operatorname{Ext}^q(\mathcal{A}, \mathcal{B})) \Longrightarrow T\text{-}\operatorname{Ext}^n(\mathcal{A}, \mathcal{B}),$$

[19] By $\operatorname{Ext}^{\cdot}(-,-)$ we denote the Ext groups formed in the category $\operatorname{Qco}(X)$ of quasi-coherent sheaves on X. By $T\text{-}\operatorname{Ext}^{\cdot}(-,-)$ we denote the Ext groups formed in the category $\operatorname{Qco}^T(X)$ of T-equivariant quasi-coherent sheaves on X.

together with $H^p(T, \mathrm{Ext}^q(\mathcal{A}, \mathcal{B})) = 0$ for any $p > 0$, $q \in \mathbb{Z}$ [Jan, Lem. 4.3]. Note that $H^p(T, -)$ denotes rational cohomology. $\square$

Corollary 1.3.10 (Theorem 1.0.1)**.** *Let X be a nonsingular projective toric variety. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and let P be a choice of Hilbert polynomial of degree* $\dim(X)$*. Then there is a canonical isomorphism*[20]

$$(\mathcal{M}_P^s)^T \cong \coprod_{\vec{\chi} \in (\mathcal{X}_P^0)^{gf}} \mathcal{M}_{\vec{\chi}}^{0,s}.$$

Proof. Immediate from Theorems 1.3.9 and 1.2.22. $\square$

The advantage of this result is that for any nonsingular projective toric variety X with ample line bundle $\mathcal{O}_X(1)$ and Hilbert polynomial P of degree $\dim(X)$, we now have a combinatorial description of $(\mathcal{M}_P^s)^T$ in terms of the explicit moduli spaces of torsion free equivariant sheaves of section 1.2. Explicit knowledge of $(\mathcal{M}_P^s)^T$ is useful for computing invariants associated to $\mathcal{M}_P^s$, e.g. the Euler characteristic of $\mathcal{M}_P^s$, using localisation techniques. This will be exploited in the next chapter, where we take X to be a nonsingular complete toric surface over $\mathbb{C}$. We will derive expressions for generating functions of Euler characteristics of moduli spaces of μ-stable torsion free sheaves on X, keeping track of the dependence on choice of ample line bundle $\mathcal{O}_X(1)$. This will give rise to wall-crossing formulae. We will give the easiest two examples occurring in the next chapter, without further discussion. In these examples, wall-crossing phenomena are absent.

Example 1.3.11. Let X be a nonsingular complete toric surface over $\mathbb{C}$ and let H be an ample divisor on X. Let $e(X)$ be the Euler characteristic of X. Denote by $M_X^H(r, c_1, c_2)$ the moduli space of μ-stable torsion free sheaves on X of rank $r \in H^0(X, \mathbb{Z}) \cong \mathbb{Z}$, first Chern class $c_1 \in H^2(X, \mathbb{Z})$ and second Chern class $c_2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$. Then for rank

[20] Note that in the context of Theorem 1.2.22, "Gieseker stable" is equivalent to "*properly* GIT stable". Therefore, one should take *properly* GIT stable points on the right hand side.

$r = 1$, we have

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(1, c_1, c_2)) q^{c_2} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^{e(X)}}.$$

This result is known for general nonsingular projective surfaces X over $\mathbb{C}$ by work of Göttsche, using very different techniques, i.e. using his expression for the Poincaré polynomial of Hilbert schemes of points computed using the Weil Conjectures. $\oslash$

Example 1.3.12. Using the notation of the previous example, let $X = \mathbb{P}^2$ and rank $r = 2$. Then

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(2, 1, c_2)) q^{c_2} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{mn}}{1-q^{m+n-1}}$$

$$= q + 9q^2 + 48q^3 + 203q^4 + 729q^5 + 2346q^6 + 6918q^7 + 19062q^8 + 49620q^9 + O(q^{10}).$$

Another expression for the same generating function has been obtained by Yoshioka, who obtains an expression for the Poincaré polynomial using the Weil Conjectures. In [Kly4], Klyachko also computes this generating function expressing it in terms of Hurwitz class numbers. In fact, the current chapter lays the foundations for many ideas appearing in [Kly4] and generalises them to pure equivariant sheaves of any dimension on any nonsingular toric variety. The next chapter can be seen as a systematic application to torsion free sheaves on nonsingular complete toric surfaces. $\oslash$

1.3.4 Fixed Point Loci of Moduli Spaces of Reflexive Sheaves on Toric Varieties

We end this section by discussing how our theory so far can be adapted to give combinatorial descriptions of fixed point loci of moduli spaces of μ-stable reflexive sheaves on a nonsingular projective toric variety X with ample line bundle $\mathcal{O}_X(1)$. We will start by describing how sections 1.1 and 1.2 analogously hold in the setting of reflexive equivariant sheaves on nonsingular toric varieties. In fact, we will construct a particularly simple ample equivariant line bundle in the GIT problem, which precisely recovers μ-stability.

Subsequently, we will quickly construct a general theory of moduli spaces of μ-stable reflexive sheaves on any normal projective variety X with ample line bundle $\mathcal{O}_X(1)$ in a form useful for our purposes. Combining the results, gives the desired combinatorial description of the fixed point loci.

Let X be a nonsingular toric variety defined by a fan Δ in a lattice N of rank r. In subsection 1.3.2, we mentioned the combinatorial description of reflexive equivariant sheaves on X originally due to Klyachko (see for instance [Kly4]). As we discussed, the category of reflexive equivariant sheaves on X is equivalent to the category $\mathcal{R}$ of filtrations $\{E^\rho(\lambda)\}_{\rho\in\Delta(1)}$ of finite-dimensional nonzero k-vector spaces. Let $\mathcal{X}^r$ be the collection of characteristic functions of reflexive equivariant sheaves on X, which is a subset of the collection $\mathcal{X}^0$ of characteristic functions of torsion free equivariant sheaves on X. Note that the characteristic function of a reflexive equivariant sheaf can also occur as the characteristic function of a torsion free equivariant sheaf that is not reflexive. Now assume X is projective and $\mathcal{O}_X(1)$ is an ample line bundle on X. Let $\vec{\chi} \in \mathcal{X}^r$, then we can introduce natural moduli functors

$$\underline{\mathcal{N}}_{\vec{\chi}}^{\mu ss} : (Sch/k)^o \longrightarrow Sets,$$
$$\underline{\mathcal{N}}_{\vec{\chi}}^{\mu s} : (Sch/k)^o \longrightarrow Sets,$$

where $\underline{\mathcal{N}}_{\vec{\chi}}^{\mu ss}(S)$ consists of equivariant S-flat families $\mathcal{F}$ on $X \times S$ such that the fibres $\mathcal{F}_s$ are μ-semistable reflexive equivariant sheaves on $X \times k(s)$ with characteristic function $\vec{\chi}$, where we identify two such families $\mathcal{F}_1, \mathcal{F}_2$ if there is a line bundle L on S (with trivial equivariant structure) and an equivariant isomorphism $\mathcal{F}_1 \cong \mathcal{F}_2 \otimes p_S^* L$. The definition of $\underline{\mathcal{N}}_{\vec{\chi}}^{\mu s}$ is analogous using geometric μ-stability[21]. Taking $\tau = 0$, Theorem 1.2.9 tells us how to describe equivariant S-flat families with fibres torsion free equivariant sheaves with fixed characteristic function $\vec{\chi}$. Let $\mathcal{F}$ be such a family with corresponding object

[21] On a projective k-scheme X (for k *any* field, not necessarily algebraically closed of characteristic zero) with ample line bundle $\mathcal{O}_X(1)$, a torsion free sheaf $\mathcal{E}$ is called geometrically μ-stable if $\mathcal{E} \otimes_k K$ is torsion free and μ-stable on $X \times_k K$ for any field extension K/k. If k is algebraically closed, a torsion free sheaf $\mathcal{E}$ on X is μ-stable if and only if geometrically μ-stable [HL, Exm. 1.6.5, Thm. 1.6.6, Cor. 1.5.11].

$\hat{\mathcal{F}}^{\Delta} \in \mathcal{C}_{\vec{\chi}}^{0}(S)$. We see that $\mathcal{F}$ has reflexive fibres if and only if for all $\sigma \in \Delta$ a cone of maximal dimension and $s \in S$ we have

$$\mathcal{F}^{\sigma}(\lambda_1, \ldots, \lambda_r)_s = \mathcal{F}^{\sigma}(\lambda_1, \infty, \ldots, \infty)_s \cap \cdots \cap \mathcal{F}^{\sigma}(\infty, \ldots, \infty, \lambda_r)_s,$$

for all $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}$, or equivalently for all $\sigma \in \Delta$ a cone of maximal dimension and $s \in S$ we have

$$\dim_{k(s)} \left(\mathcal{F}^{\sigma}(\lambda_1, \infty, \ldots, \infty)_s \cap \cdots \cap \mathcal{F}^{\sigma}(\infty, \ldots, \infty, \lambda_r)_s\right) = \chi^{\sigma}(\lambda_1, \ldots, \lambda_r), \tag{1.16}$$

for all $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}$. This gives rise to a subcategory $\mathcal{C}_{\vec{\chi}}^{r}(S) \subset \mathcal{C}_{\vec{\chi}}^{0}(S)$ and the category of equivariant S-flat families with fibres reflexive equivariant sheaves with characteristic function $\vec{\chi}$ is equivalent to $\mathcal{C}_{\vec{\chi}}^{r}(S)$. Using a framing, we get a moduli functor $\mathfrak{C}_{\vec{\chi}}^{r,fr}$ which is a subfunctor of the functor $\mathfrak{C}_{\vec{\chi}}^{0,fr}$ introduced in subsection 1.2.2. Now let N be the number of rays of Δ and $M = \chi^{\sigma}(\infty, \ldots, \infty)$ the rank, where σ can be chosen to be any cone of maximal dimension. Referring to Proposition 1.2.11 and using the notation occurring in the proof of Proposition 1.2.21, we recall $\mathfrak{C}_{\vec{\chi}}^{0,fr}$ is represented by a closed subscheme

$$\mathcal{N}_{\vec{\chi}}^{0} \subset \mathcal{A}' = \prod_{j=1}^{N} \prod_{k=1}^{M-1} \mathrm{Gr}(k, M) \times \prod_{\alpha=1}^{a} \mathrm{Gr}(n_{\alpha}, M).$$

The new conditions on the fibres (1.16) determine an open subset $\mathcal{N}_{\vec{\chi}}^{r} \subset \mathcal{N}_{\vec{\chi}}^{0}$ which represents $\mathfrak{C}_{\vec{\chi}}^{r,fr}$. This can be proved by noting that for any finite product of Grassmannians $\prod_i \mathrm{Gr}(n_i, N)$ the map $\{p_i\} \mapsto \dim_k \left(\bigcap_i p_i\right)$ is upper semicontinuous. In fact, $\mathcal{N}_{\vec{\chi}}^{r}$ is naturally a locally closed subscheme of just $\prod_{j=1}^{N} \prod_{k=1}^{M-1} \mathrm{Gr}(k, M)$. This subscheme is invariant under the natural regular action of $G = \mathrm{SL}(M, k)$ on $\prod_{j=1}^{N} \prod_{k=1}^{M-1} \mathrm{Gr}(k, M)$. We need the following variation on Proposition 1.2.19.

Proposition 1.3.13. *Let X be a projective variety with ample line bundle $\mathcal{O}_X(1)$. Let G be an affine algebraic group acting regularly on X. Let $\mathcal{E}$ be a torsion free G-equivariant sheaf on X. Then $\mathcal{E}$ is μ-semistable if and only if $\mu_{\mathcal{F}} \leq \mu_{\mathcal{E}}$ for any G-equivariant coherent* *subsheaf $\mathcal{F}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$. Now assume in addition $\mathcal{E}$ is reflexive and $G = T$ is an algebraic torus. Then $\mathcal{E}$ is μ-stable if and only if $\mu_{\mathcal{F}} < \mu_{\mathcal{E}}$ for any equivariant coherent subsheaf $\mathcal{F}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$.*

Proof. We can copy the proof of Proposition 1.2.19, but need $\mathcal{E}$ to be reflexive and $G = T$ an algebraic torus for the second part. The reason is that for reflexive sheaves we have the following three claims (see also the discussion at the start of the proof of Proposition 1.2.21). Let X be any projective normal variety with ample line bundle $\mathcal{O}_X(1)$.

Claim 1. Let $\mathcal{E}$ be a reflexive sheaf on X. Then $\mathcal{E}$ is μ-semistable if and only if $\mu_{\mathcal{F}} \leq \mu_{\mathcal{E}}$ for any proper reflexive subsheaf $\mathcal{F} \subset \mathcal{E}$. Moreover, $\mathcal{E}$ is μ-stable if and only if $\mu_{\mathcal{F}} < \mu_{\mathcal{E}}$ for any proper reflexive subsheaf $\mathcal{F} \subset \mathcal{E}$.

Claim 2. A reflexive μ-polystable sheaf on X is a μ-semistable sheaf on X isomorphic to a (finite, nontrivial) direct sum of reflexive μ-stable sheaves. Let $\mathcal{E}$ be a μ-semistable reflexive sheaf on X. Then $\mathcal{E}$ contains a unique maximal reflexive μ-polystable subsheaf of the same slope as $\mathcal{E}$. This subsheaf we refer to as the reflexive μ-socle of $\mathcal{E}$.

Claim 3. Let $\mathcal{E}$, $\mathcal{F}$ be reflexive μ-stable sheaves on X with the same slope. Then

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = \begin{cases} k & \text{if } \mathcal{E} \cong \mathcal{F} \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Claim 1. By [OSS], one only has to test μ-semistability and μ-stability of $\mathcal{E}$ for reflexive subsheaves $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$. The claim follows from the statement that for any reflexive sheaf $\mathcal{E}$ on X and reflexive subsheaf $\mathcal{F} \subset \mathcal{E}$ with $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{E})$ and $\mu_{\mathcal{F}} = \mu_{\mathcal{E}}$ one has $\mathcal{F} = \mathcal{E}$. This can be seen as follows. Suppose $\varnothing \neq Y = \mathrm{Supp}(\mathcal{E}/\mathcal{F})$, then Y is a closed subset with $\mathrm{codim}(Y) \geq 2$. Since $\mathcal{E}|_{X \setminus Y} = \mathcal{F}|_{X \setminus Y}$ and for any open subset $U \subset X$, we have a commutative diagram [Har2, Prop. 1.6],

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\cong"'] & \mathcal{E}(U) \arrow[d, "\cong"] \\ \mathcal{F}(U \setminus Y) \arrow[r, "\cong"'] & \mathcal{E}(U \setminus Y), \end{tikzcd}$$

we reach a contradiction.

Proof of Claim 2. Note that the collection $\{\mathcal{F}_i \mid i \in I\}$ of μ-stable reflexive subsheaves of $\mathcal{E}$ with the same slope as $\mathcal{E}$ is nonempty (first remark in the proof of Claim 1). Consider the subsheaf $\mathcal{S} = \sum_{i\in I} \mathcal{F}_i$, which can be written as $\mathcal{S} = \sum_{i\in J} \mathcal{F}_i$ for some finite subset $\varnothing \neq J \subset I$. Assume $J = \{1, \dots, m\}$ and $\mathcal{F}_{i+1} \nsubseteq \mathcal{F}_1 + \dots + \mathcal{F}_i$ for all $i = 1, \dots, m-1$. It is enough to prove $(\mathcal{F}_1 + \dots + \mathcal{F}_i) \cap \mathcal{F}_{i+1} = 0$ for all $i = 1, \dots, m-1$. Suppose we know the statement for $1, \dots, k-1$, but $(\mathcal{F}_1 + \dots + \mathcal{F}_k) \cap \mathcal{F}_{k+1} \neq 0$. By the short exact sequence

$$0 \longrightarrow (\mathcal{F}_1 + \dots + \mathcal{F}_k) \cap \mathcal{F}_{k+1} \longrightarrow (\mathcal{F}_1 + \dots + \mathcal{F}_k) \oplus \mathcal{F}_{k+1} \longrightarrow \mathcal{F}_1 + \dots + \mathcal{F}_{k+1} \longrightarrow 0,$$

and $\mathcal{F}_1 + \dots + \mathcal{F}_k = \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_k$, we see that if $\mu_{(\mathcal{F}_1+\dots+\mathcal{F}_k)\cap\mathcal{F}_{k+1}} < \mu_{\mathcal{F}_1\oplus\dots\oplus\mathcal{F}_{k+1}}$, then

$$\mu_{\mathcal{F}_1+\dots+\mathcal{F}_{k+1}} > \mu_{\mathcal{F}_1\oplus\dots\oplus\mathcal{F}_{k+1}} = \mu_{\mathcal{F}_1} = \dots = \mu_{\mathcal{F}_{k+1}} = \mu_{\mathcal{E}},$$

which contradicts $\mathcal{E}$ being μ-semistable. Therefore

$$\mu_{\mathcal{E}} = \mu_{\mathcal{F}_1} = \dots = \mu_{\mathcal{F}_{k+1}} = \mu_{\mathcal{F}_1\oplus\dots\oplus\mathcal{F}_{k+1}} \leq \mu_{(\mathcal{F}_1+\dots+\mathcal{F}_k)\cap\mathcal{F}_{k+1}}.$$

On the other hand, $(\mathcal{F}_1 + \dots + \mathcal{F}_k) \cap \mathcal{F}_{k+1} = (\mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_k) \cap \mathcal{F}_{k+1}$ is reflexive by [Har2, Prop. 1.6], so Claim 1 implies

$$\mu_{(\mathcal{F}_1+\dots+\mathcal{F}_k)\cap\mathcal{F}_{k+1}} < \mu_{\mathcal{F}_{k+1}},$$

which yields a contradiction.

Proof of Claim 3. Let $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism. It suffices to prove ϕ is zero or an isomorphism, because $\mathcal{E}$, $\mathcal{F}$ are simple. Let $\mathcal{K}$ be the kernel and $\mathcal{I}$ the image of ϕ. In the case $\mathcal{K} = \mathcal{E}$ we are done. In the case $\mathcal{K} = 0$, the possibility $0 \neq \mathcal{I} \subsetneq \mathcal{F}$ is ruled out by Claim 1 and we are done in the other cases. Suppose $0 \neq \mathcal{K} \subsetneq \mathcal{E}$, then $\mathcal{K}$ can easily seen to be reflexive by [Har2, Prop. 1.6]. Consequently, $\mu_{\mathcal{K}} < \mu_{\mathcal{E}}$ by Claim 1. In the case

$\mathcal{I} = 0$, we are done. In the case $\mathcal{I} = \mathcal{F}$, we reach a contradiction since $\mu_{\mathcal{E}} = \mu_{\mathcal{K}}$. In the case $0 \neq \mathcal{I} \subsetneq \mathcal{F}$ we reach a contradiction since $\mu_{\mathcal{E}} \leq \mu_{\mathcal{K}}$. $\square$

Using Proposition 1.3.13 and the proof of Proposition 1.2.21, it easy to see we can choose an ample equivariant line bundle $\mathcal{L}^{r}_{\vec{\chi}}$ on $\mathcal{N}^{r}_{\vec{\chi}}$ such that the GIT semistable points of $\mathcal{N}^{r}_{\vec{\chi}}$ are precisely the μ-semistable elements and the properly GIT stable points of $\mathcal{N}^{r}_{\vec{\chi}}$ are precisely the μ-stable elements. This ample equivariant line bundle can be deduced from the $a = 0$ case of the proof of Proposition 1.2.21 and is particularly simple. We choose such an ample equivariant line bundle and denote the GIT quotients by $\mathcal{N}^{\mu ss}_{\vec{\chi}}$, $\mathcal{N}^{\mu s}_{\vec{\chi}}$. We obtain the following theorem.

Theorem 1.3.14. *Let X be a nonsingular projective toric variety. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and $\vec{\chi} \in \mathcal{X}^r$ a characteristic function of a reflexive equivariant sheaf on X. Then $\underline{\mathcal{N}}^{\mu ss}_{\vec{\chi}}$ is corepresented by the quasi-projective k-scheme of finite type $\mathcal{N}^{\mu ss}_{\vec{\chi}}$. Moreover, there is an open subset $\mathcal{N}^{\mu s}_{\vec{\chi}} \subset \mathcal{N}^{\mu ss}_{\vec{\chi}}$ such that $\underline{\mathcal{N}}^{\mu s}_{\vec{\chi}}$ is corepresented by $\mathcal{N}^{\mu s}_{\vec{\chi}}$ and $\mathcal{N}^{\mu s}_{\vec{\chi}}$ is a coarse moduli space.*

We now discuss how to define moduli spaces of μ-stable reflexive sheaves on normal projective varieties in general in a way useful for our purposes. Let X be a normal projective variety with ample line bundle $\mathcal{O}_X(1)$ (not necessarily nonsingular or toric). Let P be a choice of Hilbert polynomial of a reflexive sheaf on X. Then there are natural moduli functors $\underline{\mathcal{M}}^{ss}_{P}$, $\underline{\mathcal{M}}^{s}_{P}$ of flat families with fibres Gieseker semistable resp. geometrically Gieseker stable sheaves with Hilbert polynomial P as we discussed in subsection 1.3.1 referring to [HL]. The moduli functor $\underline{\mathcal{M}}^{ss}_{P}$ is corepresented by a projective k-scheme $\mathcal{M}^{ss}_{P}$ of finite type and $\mathcal{M}^{ss}_{P}$ contains an open subset $\mathcal{M}^{s}_{P}$, which corepresents $\underline{\mathcal{M}}^{s}_{P}$ and is in fact a coarse moduli space. Let $\mathcal{P}$ be a property of coherent sheaves on k-schemes of finite type. We say $\mathcal{P}$ is an open property if for any projective morphism $f : Z \longrightarrow S$ of k-schemes of finite type and $\mathcal{F}$ an S-flat coherent sheaf on Z, the collection of points $s \in S$ such that the fibre $\mathcal{F}_s$ satisfies property $\mathcal{P}$ is open (see [HL, Def. 2.1.9]). We claim that if $\mathcal{P}$ is an open property, then the moduli functor

$$\underline{\mathcal{M}}^s_{P,\mathcal{P}} \subset \underline{\mathcal{M}}^s_P,$$

defined as the subfunctor of all families with every fibre satisfying $\mathcal{P}$, is corepresented by an open subset $\mathcal{M}^s_{P,\mathcal{P}} \subset \mathcal{M}^s_P$ and $\mathcal{M}^s_{P,\mathcal{P}}$ is a coarse moduli space. This is immediate in the case we have a universal family for $\underline{\mathcal{M}}^s_P$ and on the level of Quot schemes we can always define obvious subfunctors represented by obvious open subsets. In the general case, one can prove the claim using arguments involving Luna's Étale Slice Theorem and local artinian k-algebras with residue field k as in Propositions 1.3.7 and 1.3.8. We now would like to take $\mathcal{P}$ to be the property "geometrically μ-stable and reflexive". Using an argument analogous to the proof of [HL, Prop. 2.3.1] (which uses a boundedness result by Grothendieck), it is easy to see that geometrically μ-stable is an open property. Using a result by Kollár [Kol, Prop. 28] and a semicontinuity argument, we see that reflexive is also an open property. Therefore, it makes sense to define a moduli functor $\underline{\mathcal{N}}^{\mu s}_P \subset \underline{\mathcal{M}}^s_P$ consisting of those families where all fibres are geometrically μ-stable and reflexive. The moduli functor $\underline{\mathcal{N}}^{\mu s}_P$ is corepresented by an open subset $\mathcal{N}^{\mu s}_P \subset \mathcal{M}^s_P$ and $\mathcal{N}^{\mu s}_P$ is a coarse moduli space coming from a geometric quotient of an open subset of the Quot scheme. In the case X is a nonsingular projective toric variety, we get a regular action $\sigma : T \times \mathcal{N}^{\mu s}_P \longrightarrow \mathcal{N}^{\mu s}_P$ and the fixed point locus is a closed subscheme $(\mathcal{N}^{\mu s}_P)^T \subset \mathcal{N}^{\mu s}_P$. We define $(\mathcal{X}^r_P)^{gf} = (\mathcal{X}^0_P)^{gf} \cap \mathcal{X}^r$ to be the collection of gauge-fixed characteristic functions of reflexive equivariant sheaves on X giving rise to Hilbert polynomial P. Completely analogous to subsections 1.3.1, 1.3.2, 1.3.3, we obtain the following theorem.

Theorem 1.3.15. *Let X be a nonsingular projective toric variety. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and let P be a choice of Hilbert polynomial of a reflexive sheaf on X. Then there is a canonical isomorphism*

$$(\mathcal{N}^{\mu s}_P)^T \cong \coprod_{\vec{\chi} \in (\mathcal{X}^r_P)^{gf}} \mathcal{N}^{\mu s}_{\vec{\chi}}.$$

Chapter 2

Euler Characteristics of Moduli Spaces of Sheaves on Toric Surfaces

The moduli space of Gieseker stable sheaves is a complicated object[1]. For example, it satisfies Murphy's Law, meaning every singularity type of finite type over $\mathbb{Z}$ appears on it [Vak]. Nevertheless, we need a reasonable understanding of components of it, when we want to compute invariants associated to these components. Examples are motivic invariants like virtual Hodge polynomials, virtual Poincaré polynomials and Euler characteristics of components of the moduli space of Gieseker stable sheaves. Other examples are (generalised) Donaldson–Thomas invariants of a Calabi–Yau threefold.

This leads us to consider the following situation. Let X be a nonsingular projective toric variety with torus T, let $\mathcal{O}_X(1)$ be an ample line bundle on X and let P be a choice of Hilbert polynomial[2]. We can lift the action of the torus T on X to a regular action on the moduli space $\mathcal{M}_P^s$ of Gieseker stable sheaves on X with Hilbert polynomial P. In the previous chapter, we gave a combinatorial description of the fixed locus $(\mathcal{M}_P^s)^T$. As a by-product, for any Hilbert polynomial P of a reflexive sheaf on X, we found a combinatorial description of the fixed point locus $(\mathcal{N}_P^s)^T$ of the moduli space $\mathcal{N}_P^s$ of

[1] The notion of Gieseker stability is defined in [HL, Def. 1.2.4].

[2] In this chapter, we will work with varieties, schemes and stacks over ground field $\mathbb{C}$ unless specified otherwise.

Gieseker stable reflexive sheaves on X with Hilbert polynomial P. In the present chapter, we apply the combinatorial descriptions of the fixed point loci to the case of torsion free sheaves on nonsingular complete toric surfaces. Many of the guiding ideas of the previous chapter and the present chapter come from Klyachko's remarkable preprint [Kly4] (see also [Kly1], [Kly2], [Kly3]). The previous chapter lays the foundations for many ideas appearing in [Kly4] and generalises them to pure equivariant sheaves of any dimension on any nonsingular toric variety. The present chapter can be seen as a systematic application to torsion free sheaves on nonsingular complete toric surfaces.

The main goal of the present chapter is to derive an expression for the generating function of Euler characteristics of moduli spaces of μ-stable[3] torsion free sheaves of rank r and first Chern class c_1 on a nonsingular complete toric surface X with ample divisor H. We will obtain an expression for this generating function in terms of Euler characteristics of moduli spaces of stable configurations of linear subspaces in Theorem 2.2.7, keeping X, H, r and c_1 completely arbitrary[4]. The expression in Theorem 2.2.7 can be further simplified in examples. The dependence on H allows us to study wall-crossing phenomena. Note that we compute Euler characteristics of moduli spaces of μ-stable torsion free sheaves *only*, even in the presence of strictly μ-semistable torsion free sheaves.

This chapter is organised as follows. In section 2.1, we gather some rudimentary information about motivic invariants and torus localisation. In section 2.2, we start by giving an explicit expression of the Chern character of a torsion free equivariant sheaf on a nonsingular complete toric surface in terms of certain 2D partitions associated to the sheaf. Here we use a formula due to Klyachko. Subsequently, using a result by Göttsche and Yoshioka, we note that in the surface case it is sufficient to compute generating functions of moduli spaces of μ-stable reflexive sheaves only. Using these results, we derive a formula for any generating function of Euler characteristics of moduli spaces of

[3]The notion of μ-stability is defined in [HL, Def. 1.2.12].

[4]Configurations of linear subspaces and their moduli spaces are a classical topic in GIT. See [Dol, Ch. 11] for a discussion.

μ-stable torsion free sheaves on a nonsingular complete toric surface (Theorem 2.2.7). In section 2.3, we simplify this formula in examples and compare these examples to the literature. We consider the case rank 1 and trivially retrieve a result by Ellingsrud and Strømme [ES] and Göttsche [Got1]. We consider the case rank 2 and $X = \mathbb{P}^2$ and compare to work of Klyachko [Kly4] and Yoshioka [Yos]. We consider rank 2 and $X = \mathbb{P}^1 \times \mathbb{P}^1$ or any Hirzebruch surface $\mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 1}$), where we make the dependence on choice of ample divisor H explicit. This allows us to study wall-crossing phenomena and compare to work of Göttsche [Got2] and Joyce [Joy2]. We perform various consistency checks. Finally, we give a formula for rank 3 and $X = \mathbb{P}^2$, which we are not able to write in a short form[5]. This formula allows for numerical computations. It should be noted that Ellingsrud and Strømme [ES] and Klyachko [Kly4] use the torus action/techniques of toric geometry, whereas Göttsche [Got1], [Got2] and Yoshioka [Yos] use very different techniques namely the Weil Conjectures to compute virtual Poincaré polynomials. Also Joyce [Joy2] uses very different techniques namely his theory of wall-crossing for motivic invariants counting (semi)stable objects.

The entire theory of chapter 1 has been developed for arbitrary moduli spaces of Gieseker stable sheaves on nonsingular projective toric varieties, *except* for matching GIT stability of the moduli spaces of pure equivariant sheaves to Gieseker stability. This has only been achieved in full generality for torsion free sheaves. However, for pure equivariant sheaves of lower dimension, one can still match GIT and Gieseker stability in many examples. We exploit this in section 2.4 to compute generating functions of moduli spaces of μ-stable pure dimension 1 sheaves on $X = \mathbb{P}^2$ with first Chern class $c_1 = 1, 2, 3$. Davesh Maulik suggested looking at this example. Jinwon Choi has also considered this example using [Koo1], i.e. chapter 1 (private communication). He found the same results as the author and in addition considered the case $c_1 = 4$. In section 2.5, we describe how various generating functions obtained in sections 2.3–2.4 can be seen

[5]During the finishing of subsections 2.1–2.3, the author found out about recent independent work of Weist [Wei], where he also computes the case rank 3 and $X = \mathbb{P}^2$ using techniques of toric geometry and quivers.

as generating functions of Donaldson–Thomas invariants of the canonical bundle K_X. In the case of pure dimension 1 sheaves on $X = \mathbb{P}^2$, (generalised) Donaldson–Thomas invariants of K_X are conjectured to correspond to genus zero Gopakumar–Vafa invariants of K_X [Kat], [JS]. The examples we consider are consistent with this conjecture.

2.1 Introduction

In this section, we will briefly discuss motivic invariants and torus localisation. As a warming-up we treat a trivial application, viz. generating functions of Euler characteristics of moduli spaces of μ-stable torsion free sheaves on $\mathbb{P}^1$.

2.1.1 Motivic Invariants

One can define the virtual Poincaré polynomial $P(X, z) \in \mathbb{Q}[z]$ of any quasi-projective variety X (this is summarised in [Joy1, Exm. 4.3, 4.4] and also [Got2, Sect. 1(c)]). The definition is elaborate and involves Deligne's weight filtration. The value $e(X) = P(X, -1)$ is the Euler characteristic of X. In the case X is nonsingular and projective, the virtual Poincaré polynomial reduces to the ordinary Poincaré polynomial $P(X, z) = \sum_{k=0}^{2\dim(X)} b^k(X) z^k$, where $b^k(X)$ are the Betti numbers. The virtual Poincaré polynomial (and therefore the Euler characteristic) satisfies the following properties:

(i) If $Y \subset X$ is a closed subvariety of a quasi-projective variety, then $P(X, z) = P(X \setminus Y, z) + P(Y, z)$.

(ii) If X, Y are quasi-projective varieties, then $P(X \times Y, z) = P(X, z)P(Y, z)$.

(iii) If $f : X \longrightarrow Y$ is a bijective morphism of quasi-projective varieties, then $P(X, z) = P(Y, z)$.

As a consequence, a Zariski locally trivial fibration $\phi : X \longrightarrow Y$ of quasi-projective varieties with fibre a quasi-projective variety F satisfies $P(X, z) = P(F, z)P(Y, z)$ [Joy1, Lem. 4.2]. One can also define the virtual Hodge polynomial $H(X; x, y)$, but we will not go into this. All of these objects are called motivic invariants. By restricting to the reduced subscheme, all of these motivic invariants can be extended to quasi-projective $\mathbb{C}$-schemes of finite type and the aforementioned properties continue to hold. Note that $P(\mathbb{A}^1, z) = z^2$ and $P(pt, z) = 1$. The following result is well-known (e.g. see [CG]).

Proposition 2.1.1 (Torus Localisation)**.** *Let X be a quasi-projective $\mathbb{C}$-scheme of finite type. Let T be an algebraic torus acting regularly on X. Then $e(X) = e(X^T)$.*

2.1.2 The Case $\mathbb{P}^1$

Let X be a nonsingular projective (irreducible) variety of dimension n. Instead of Hilbert polynomial, it is better for computational purposes to fix Chern characters $\mathrm{ch}_k \in H^{2k}(X, \mathbb{Q})$ (for all $k = 0, \ldots, n$) or, equivalently, rank $r \in H^0(X, \mathbb{Z})$ and Chern classes $c_k \in H^{2k}(X, \mathbb{Z})$ (for all $k = 1, \ldots, n$). Therefore, we will proceed to do this instead. Note that $H^0(X, \mathbb{Z}) \cong \mathbb{Z} \cong H^{2n}(X, \mathbb{Z})$. The combinatorial descriptions of fixed point loci of the previous chapter, Theorem 1.3.9, Corollary 1.3.10 and Theorem 1.3.15, hold analogously in this setting[6].

Consider the combinatorial description in Corollary 1.3.10 of fixed point loci of moduli spaces of torsion free sheaves on nonsingular projective toric varieties in the simplest case, i.e. when $\dim(X) = 1$. The only nonsingular projective toric variety of dimension 1 is $X = \mathbb{P}^1$ with fan:

$$\sigma_2 \quad \bullet \quad \sigma_1$$

Let D be a point on X and $H = \alpha D$ an ample divisor on X (i.e. $\alpha \in \mathbb{Z}_{>0}$). A coherent sheaf $\mathcal{E}$ on X is torsion free if and only if reflexive if and only if locally free. Let $\mathcal{E}$ be a rank r equivariant vector bundle on X with corresponding framed torsion free Δ-family $\hat{E}^{\Delta}$. Then $\hat{E}^{\Delta}$ is described by a pair of filtrations $(\{E^{\sigma_1}(\lambda)\}_{\lambda \in \mathbb{Z}}, \{E^{\sigma_2}(\lambda))\}_{\lambda \in \mathbb{Z}})$ where $E^{\sigma_i}(\lambda)$ is 0 for λ sufficiently small and $\mathbb{C}^{\oplus r}$ for λ sufficiently large for each $i = 1, 2$

[6]Note that when X is in addition toric, $H^{2*}(X, \mathbb{Z}) \cong A^*(X)$ and $H^{2*}(X, \mathbb{Q}) \cong A^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, so it does not matter whether we work in cohomology or the Chow ring [Ful, Sect. 5.2].

(Theorem 1.1.10). Let $N_X^H(r, c_1)$ be the moduli space of μ-stable vector bundles on X of rank r and first Chern class c_1.

Case 1: $r = 1$. In this case $\mathcal{E}$ is always a line bundle and $\hat{E}^{\Delta}$ is described by two integers A_1, A_2 indicating where the filtrations $E^{\sigma_1}(\lambda)$, $E^{\sigma_2}(\lambda)$ jump dimension. From Theorem 1.3.15 and Klyachko's Formula (Proposition 1.2.16), we obtain $N_X^H(1, c_1)^T = pt$. Using torus localisation (Proposition 2.1.1), we deduce that

$$\sum_{c_1 \in \mathbb{Z}} e(N_X^H(1, c_1)) q^{c_1} = \sum_{k \in \mathbb{Z}} q^k.$$

Case 2: $r > 1$. In this case, it is easy to see $\mathcal{E}$ always decomposes, since the pair of filtrations $(\{E^{\sigma_1}(\lambda)\}_{\lambda \in \mathbb{Z}}, \{E^{\sigma_2}(\lambda))\}_{\lambda \in \mathbb{Z}})$ decomposes. Hence there cannot be any μ-stable equivariant vector bundles on X of rank r, so Theorem 1.3.15 implies $N_X^H(1, c_1)^T = \varnothing$. Using torus localisation (Proposition 2.1.1), we deduce that

$$\sum_{c_1 \in \mathbb{Z}} e(N_X^H(r, c_1)) q^{c_1} = 0.$$

Note that this result trivially follows from [HL, Thm. 1.3.1].

2.2 Euler Characteristics of Moduli Spaces of Torsion Free Sheaves on Toric Surfaces

In the rest of this chapter, we will consider a nonsingular complete[7] toric surface X with ample divisor H. For fixed rank $r > 0$, first Chern class c_1 and second Chern class c_2, we denote the moduli space of μ-stable torsion free sheaves on X of rank r, first Chern class c_1 and second Chern class c_2 by $M_X^H(r, c_1, c_2)$. Since geometric μ-stability is an open condition, this moduli space is naturally an open subset of the moduli space of Gieseker stable torsion free sheaves on X of rank r, first Chern class c_1 and second

[7] Note that for 2-dimensional toric varieties the notion of complete (i.e. proper) and projective are the same [Ful, Sect. 3.4].

Chern class c_2 (see subsection 1.3.4 for a more detailed discussion). Our goal is to use the combinatorial description of fixed point loci of the previous chapter, i.e. Corollary 1.3.10, Theorem 1.3.15, to compute the generating function

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(r, c_1, c_2)) q^{c_2}.$$

Note that this generating function is an element of $\mathbb{Z}((q))$, i.e. a formal Laurent series in q, by the Bogomolov Inequality [HL, Thm. 3.4.1]. We derive a general formula for this generating function expressing it in terms of Euler characteristics of moduli spaces of stable configurations of linear subspaces (Theorem 2.2.7). Note that we compute Euler characteristics of moduli spaces of μ-stable torsion free sheaves $M_X^H(r, c_1, c_2)$ *only* and ignore strictly μ-semistable torsion free sheaves. The reason is that the combinatorial descriptions of fixed point loci of the previous chapter, Theorem 1.3.9, Corollary 1.3.10 and Theorem 1.3.15, use simpleness in an essential way (see section 1.3). In section 2.3, we simplify the general formula of Theorem 2.2.7 and compare to the literature in the examples X arbitrary and rank $r = 1$, $X = \mathbb{P}^2$ and rank $r = 1, 2, 3$, and $X = \mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$) and rank $r = 1, 2$. Here we write $\mathbb{F}_a$ for the bundle[8] $p : \mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}) \longrightarrow \mathbb{P}^1$. We insist on keeping H and c_1 arbitrary in these examples.

2.2.1 Chern Characters of Torsion Free Equivariant Sheaves on Toric Surfaces

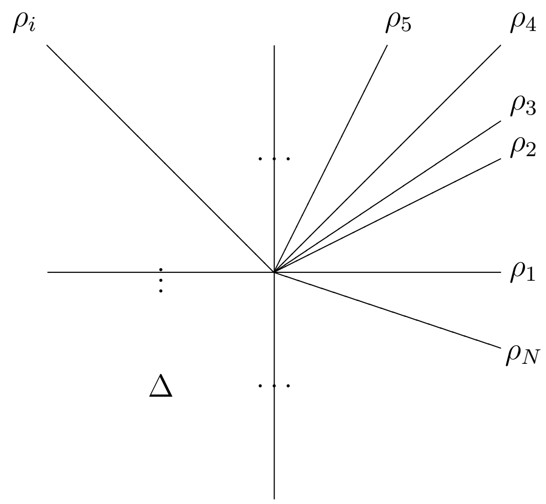
We will start by recalling some well-known facts. A classification of all nonsingular complete toric surfaces is given by the following proposition [Ful, Sect. 2.5].

Proposition 2.2.1. *All nonsingular complete toric surfaces are obtained by successive blow-ups of $\mathbb{P}^2$ and $\mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$) at fixed points.*

Combinatorially, such blow-ups are described by stellar subdivisions, i.e. creating a fan $\tilde{\Delta}$ out of Δ by subdividing a fixed cone through the sum of the two integral lattice

[8] Note that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and the $\mathbb{F}_a$ for $a \in \mathbb{Z}_{\geq 1}$ are the Hirzebruch surfaces.

vectors of its rays. Let Δ be a fan obtained in such a way out of one of the fans of $\mathbb{P}^2$, $\mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$). Let $\sigma_1, \ldots, \sigma_N$ be its 2-dimensional cones and let $\rho_1, \ldots, \rho_N$ be its rays numbered counterclockwise as follows:



where cone σ_i has rays ρ_i, ρ_{i+1} for all $i = 1, \ldots, N$ (the index i is understood modulo N, so cone σ_N has rays ρ_N, ρ_1). Note that we take $N = \mathbb{Z}^2$ as the underlying lattice, $M = \mathbb{Z}^2$ as the dual lattice and $\langle -, - \rangle : M \times N \longrightarrow \mathbb{Z}$ as the canonical pairing[9]. Denote the primitive lattice vectors corresponding to the rays $\rho_1, \ldots, \rho_N$ by $v_1, \ldots, v_N$. Since v_1, v_2 form a basis for N, we can assume without loss of generality that $v_1 = e_1$, $v_2 = e_2$ are the standard basis vectors. Denote the corresponding divisors by $D_1, \ldots, D_N \cong \mathbb{P}^1$ ([Ful, Sect. 2.5]). Consider the Chow ring $A(X) = A^0(X) \oplus A^1(X) \oplus A^2(X)$. Using [Ful, Sect. 5.2], we get $A(X) = \mathbb{Z}[D_1, \ldots, D_N]/I$, where I is the ideal generated by

$$D_1 + \sum_{i=3}^{N} \langle e_1, v_i \rangle D_i = 0, \;\; D_2 + \sum_{i=3}^{N} \langle e_2, v_i \rangle D_i = 0,$$

$$D_i D_j = 0, \;\text{unless}\; i = 1, \ldots, N, \; j = i+1,$$

$$D_i D_j D_k = 0, \;\text{for all}\; i, j, k = 1, \ldots, N.$$

Since X is a complete toric variety, $A^2(X) \cong \mathbb{Z}$ so $D_1 D_2 = D_2 D_3 = \cdots = D_{N-1} D_N = D_N D_1 \neq 0$ in $A(X)$ ([Ful, Sect. 2.5]). Denote this element, which generates $A^2(X)$,

[9]We use $N = \mathbb{Z}^2$ for the lattice and $N \in \mathbb{Z}_{\geq 3}$ for the number of 2-dimensional cones $\sigma_1, \ldots, \sigma_N$ of Δ, but from the context no confusion in notation will arise.

by pt. Finally, the self-intersections are given by $D_i^2 = -a_i\ pt$, where a_i is defined to be the integer satisfying $v_{i-1} + v_{i+1} = a_i v_i$ for all $i = 1, \ldots, N$ ([Ful, Sect. 2.5]). We define $\xi_i = -\langle e_1, v_i \rangle$, $\eta_i = -\langle e_2, v_i \rangle$ for all $i = 3, \ldots, N$. The integers $\{a_i\}_{i=1}^N$, $\{\xi_i\}_{i=3}^N$, $\{\eta_i\}_{i=3}^N$ are entirely determined by the fan Δ. Note that $e(X) = N$ by torus localisation (Proposition 2.1.1).

Let $\mathcal{O}_X(1)$ be an ample line bundle on X. There are isomorphisms $\mathbb{Z}^{N-2} \cong A^1(X) \cong \mathrm{Pic}(X)$, which map integers $(\alpha_3, \ldots, \alpha_N)$ to the divisor $\alpha_3 D_3 + \cdots + \alpha_N D_N$ and to the line bundle $\mathcal{O}_X(\alpha_3 D_3 + \cdots + \alpha_N D_N)$. Let $(\alpha_3, \ldots, \alpha_N)$ be the integers corresponding to $\mathcal{O}_X(1)$ and define $\alpha_1 = \alpha_2 = 0$. Let $\mathcal{E}$ be a torsion free equivariant sheaf on X of rank r with corresponding framed torsion free Δ-family $\hat{E}^\Delta$. Using Theorem 1.1.10, we see such a family is described by N double-filtrations $\{E^{\sigma_i}(\lambda_1, \lambda_2)\}_{(\lambda_1,\lambda_2) \in \mathbb{Z}^2}$ of $\mathbb{C}^{\oplus r}$

$$
\begin{aligned}
&E^{\sigma_i}(\lambda_1, \lambda_2) \subset E^{\sigma_i}(\lambda_1 + 1, \lambda_2), \text{ for all } (\lambda_1, \lambda_2) \in \mathbb{Z}^2, \\
&E^{\sigma_i}(\lambda_1, \lambda_2) \subset E^{\sigma_i}(\lambda_1, \lambda_2 + 1), \text{ for all } (\lambda_1, \lambda_2) \in \mathbb{Z}^2,
\end{aligned}
$$

such that for each $i = 1, \ldots, N$ there are integers A_i, B_i with the property $E^{\sigma_i}(\lambda_1, \lambda_2) = 0$ unless $\lambda_1 \geq A_i$, $\lambda_2 \geq B_i$ and there are integers λ_1, λ_2 such that $E^{\sigma_i}(\lambda_1, \lambda_2) = \mathbb{C}^{\oplus r}$. These double-filtrations satisfy gluing conditions

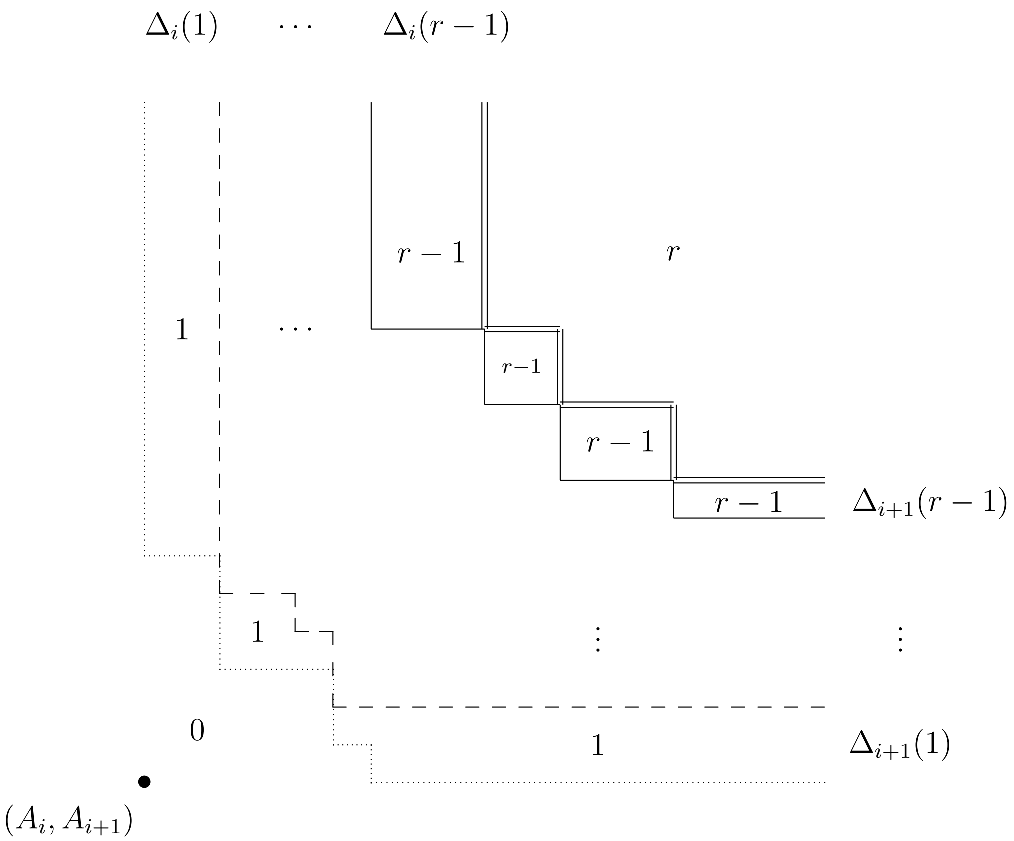
$$
E^{\sigma_i}(\infty, \lambda) = E^{\sigma_{i+1}}(\lambda, \infty), \text{ for all } \lambda \in \mathbb{Z},
$$

for all $i = 1, \ldots, N$. We introduce notation for the limiting filtrations $\{E^{\sigma_i}(\lambda, \infty)\}_{\lambda \in \mathbb{Z}}$ associated to any ray ρ_i

$$
E^{\sigma_i}(\lambda, \infty) = \begin{cases}
0 & \text{if } \lambda < A_i \\
p_i(1) \in \mathrm{Gr}(1, r) & \text{if } A_i \leq \lambda < A_i + \Delta_i(1) \\
p_i(2) \in \mathrm{Gr}(2, r) & \text{if } A_i + \Delta_i(1) \leq \lambda < A_i + \Delta_i(1) + \Delta_i(2) \\
\cdots & \cdots \\
\mathbb{C}^{\oplus r} & \text{if } A_i + \Delta_i(1) + \Delta_i(2) + \ldots + \Delta_i(r-1) \leq \lambda.
\end{cases}
$$

Note that the $A_i \in \mathbb{Z}$ and the $\Delta_i(j) \in \mathbb{Z}_{\geq 0}$.

Let $\vec{\chi} \in \mathcal{X}^0$ be the characteristic function of a torsion free equivariant sheaf on X of rank r. Then for any $i = 1, \ldots, l$, the dimension profile of χ^{σ_i} looks as follows, where we use notation A_i, $\Delta_i(j)$ as just introduced.

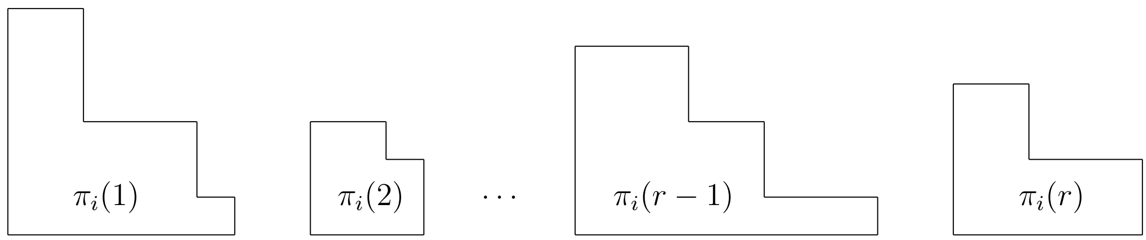


Let $j = 1, \ldots, r$ and define for all $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$

$$\phi_i(j)(\lambda_1, \lambda_2) = \begin{cases} 1 & \text{if } \chi^{\sigma_i}(\lambda_1, \lambda_2) \geq j \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_i(j)(\lambda_1, \lambda_2) = \begin{cases} 1 & \text{if } \phi_i(j)(\lambda_1, \lambda_2') = \phi_i(j)(\lambda_1', \lambda_2) = 1 \text{ for some } \lambda_1' \geq \lambda_1 \text{ and } \lambda_2' \geq \lambda_2 \\ 0 & \text{otherwise.} \end{cases}$$

Subsequently, we define $\pi_i(j)$ to be the union of all blocks $[\lambda_1, \lambda_1 + 1] \times [\lambda_2, \lambda_2 + 1]$ for $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ such that $\psi_i(j)(\lambda_1, \lambda_2) - \phi_i(j)(\lambda_1, \lambda_2) = 1$. In other words, from the diagram of χ^{σ_i}, we obtain 2D partitions $\pi_i(1), \ldots, \pi_i(r) \subset M \otimes_{\mathbb{Z}} \mathbb{R}$:



We denote by $\#\pi_i(j)$ the number of blocks of the 2D partition $\pi_i(j)$

$$\#\pi_i(j) = \sum_{(\lambda_1,\lambda_2)\in\mathbb{Z}^2} (\psi_i(j)(\lambda_1,\lambda_2) - \phi_i(j)(\lambda_1,\lambda_2)) \, .$$

Proposition 2.2.2. *Let $\mathcal{E}$ be a torsion free equivariant sheaf of rank r on a nonsingular complete toric surface X with Euler characteristic N. Suppose the characteristic function $\vec{\chi}_{\mathcal{E}}$ of $\mathcal{E}$ gives rise to integers A_i for all $i = 1, \ldots, N$, nonnegative integers $\Delta_i(j)$ for all $i = 1, \ldots, N$ and $j = 1, \ldots, r-1$ and 2D partitions $\pi_i(j)$ for all $i = 1, \ldots, N$ and $j = 1, \ldots, r$. Then*

$$\begin{aligned} \mathrm{ch}(\mathcal{E}) = r &- \sum_{i=1}^{N}\left(rA_i + \sum_{j=1}^{r-1}(r-j)\Delta_i(j)\right) D_i \\ &+ \frac{1}{2}\left(\sum_{i=1}^{N} A_i D_i\right)^2 + \frac{1}{2}\sum_{j=1}^{r-1}\left(\sum_{i=1}^{N}\left(A_i + \sum_{k=1}^{j}\Delta_i(k)\right) D_i\right)^2 - \sum_{i=1}^{N}\sum_{j=1}^{r} \#\pi_i(j) \; pt. \end{aligned}$$

Proof. Step 1. Assume $r = 1$ and $A_1 = \cdots = A_r = 0$. For each $i = 1, \ldots, N$, the double-filtration $\{E^{\sigma_i}(\lambda_1, \lambda_2)\}_{(\lambda_1,\lambda_2)\in\mathbb{Z}^2}$ gives rise to a 2D partition π_i consisting of $\#\pi_i$ blocks. Referring to Klyachko's Formula (Proposition 1.2.16), for each $i = 1, \ldots, N$ we have to compute

$$\sum_{\lambda\in\mathbb{Z}} f(\lambda) \left[\dim(E^{\sigma_i}(\lambda, \infty)) - \dim(E^{\sigma_i}(\lambda - 1, \infty))\right],$$

$$\sum_{\lambda_1,\lambda_2\in\mathbb{Z}} g(\lambda_1,\lambda_2)\left[\dim(E^{\sigma_i}(\lambda_1,\lambda_2)) - \dim(E^{\sigma_i}(\lambda_1-1,\lambda_2)) - \dim(E^{\sigma_i}(\lambda_1,\lambda_2-1))\right.$$
$$\left. +\dim(E^{\sigma_i}(\lambda_1-1,\lambda_2-1))\right],$$

where $f(\lambda)$ is λ or λ^2 and $g(\lambda_1,\lambda_2)$ is λ_1, λ_2, λ_1^2, $\lambda_1\lambda_2$ or λ_2^2. For each $i = 1,\ldots,N$, define $a^{(i)}$ to be the smallest integer where $E^{\sigma_i}(\lambda,0)$ jumps dimension and define $b^{(i)}$ to be the smallest integer where $E^{\sigma_i}(0,\lambda)$ jumps dimension. Since $A_1 = \cdots = A_N = 0$, the first sum will be zero for both choices of $f(\lambda)$. The second sum can be rewritten as

$$\sum_{\lambda_1=0}^{a^{(i)}-1}\sum_{\lambda_2=0}^{b^{(i)}-1}\left[g(\lambda_1,\lambda_2) - g(\lambda_1+1,\lambda_2) - g(\lambda_1,\lambda_2+1) + g(\lambda_1+1,\lambda_2+1)\right]\dim(E^{\sigma_i}(\lambda_1,\lambda_2))$$
$$+ g(a^{(i)},b^{(i)}) + \sum_{\lambda_2=0}^{b^{(i)}-1} g(a^{(i)},\lambda_2) + \sum_{\lambda_1=0}^{a^{(i)}-1} g(\lambda_1,b^{(i)}) - \sum_{\lambda_1=0}^{a^{(i)}-1} g(\lambda_1+1,b^{(i)})$$
$$- \sum_{\lambda_2=0}^{b^{(i)}-1} g(a^{(i)},\lambda_2+1).$$

It is easy to see this sum only contributes for $g(\lambda_1,\lambda_2) = \lambda_1\lambda_2$, in which case the contribution is

$$-a^{(i)}b^{(i)} + \sum_{\lambda_1=0}^{a^{(i)}-1}\sum_{\lambda_2=0}^{b^{(i)}-1}\dim(E^{\sigma_i}(\lambda_1,\lambda_2)) = -\#\pi_i.$$

We obtain

$$\mathrm{ch}(\mathcal{E}) = 1 - \sum_{i=1}^{N}\#\pi_i\ pt.$$

Step 2. Assume $r = 1$ and $A_1,\ldots,A_r$ arbitrary. This case can be reduced to Step 1 by tensoring with an equivariant line bundle and using Proposition 1.3.6. One immediately obtains the following formula

$$\begin{aligned}\mathrm{ch}(\mathcal{E}) &= \left(1 - \sum_{i=1}^{N}\#\pi_i\ pt\right)e^{-\sum_{i=1}^{N}A_iD_i}\\ &= 1 - \sum_{i=1}^{N}A_iD_i + \frac{1}{2}\left(\sum_{i=1}^{N}A_iD_i\right)^2 - \sum_{i=1}^{N}\#\pi_i\ pt.\end{aligned}$$

Step 3. Now let r be general. Let $\vec{\chi}_{\mathcal{E}}$ be the characteristic function of $\mathcal{E}$, then $\mathrm{ch}(\mathcal{E})$ depends only on $\vec{\chi}_{\mathcal{E}}$ (Proposition 1.2.16). Let $\mathcal{F} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$ be the sum of r rank 1 torsion free equivariant sheaves $\mathcal{L}_a$ defined by torsion free Δ-families $\{L_a^{\sigma_i}(\lambda_1, \lambda_2)\}_{(\lambda_1,\lambda_2)\in\mathbb{Z}^2}$

$$L_a^{\sigma_i}(\lambda_1, \lambda_2) = \begin{cases} \mathbb{C} & \text{if } \dim(E^{\sigma_i}(\lambda_1, \lambda_2)) \geq a \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\vec{\chi}_{\mathcal{E}} = \vec{\chi}_{\mathcal{F}}$, so the result follows from $\mathrm{ch}(\mathcal{E}) = \sum_{i=1}^r \mathrm{ch}(\mathcal{L}_i)$ and Step 2. □

2.2.2 Vector Bundles on Toric Surfaces

In this subsection, we will discuss in more detail reflexive equivariant sheaves on nonsingular complete toric surfaces. Recall that on a nonsingular surface a coherent sheaf is reflexive if and only if locally free [Har2, Cor. 1.4]. We will derive an expression for generating functions of Euler characteristics of moduli spaces of μ-stable vector bundles on nonsingular complete toric surfaces. This will yield an expression for generating functions of Euler characteristics of moduli spaces of μ-stable torsion free sheaves on nonsingular complete toric surfaces by the following proposition of Göttsche and Yoshioka [Got3, Prop. 3.1].

Proposition 2.2.3. *Let X be a nonsingular projective surface, H an ample divisor, $r \in \mathbb{Z}_{>0}$ and $c_1 \in H^2(X, \mathbb{Z})$. Then*

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(r, c_1, c_2)) q^{c_2} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^{re(X)}} \sum_{c_2 \in \mathbb{Z}} e(N_X^H(r, c_1, c_2)) q^{c_2}.$$

In this proposition, $N_X^H(r, c_1, c_2)$ is the moduli space of μ-stable vector bundles on X of rank r, first Chern class c_1 and second Chern class c_2 (see subsection 1.3.4). Note that $N_X^H(r, c_1, c_2)$ is an open subset of $M_X^H(r, c_1, c_2)$, since reflexive is an open condition (see subsection 1.3.4). Combining this proposition with torus localisation (Proposition

2.1.1) and the combinatorial description of fixed point loci of moduli spaces of μ-stable reflexive sheaves on nonsingular projective toric varieties (Theorem 1.3.15), we obtain the following result.

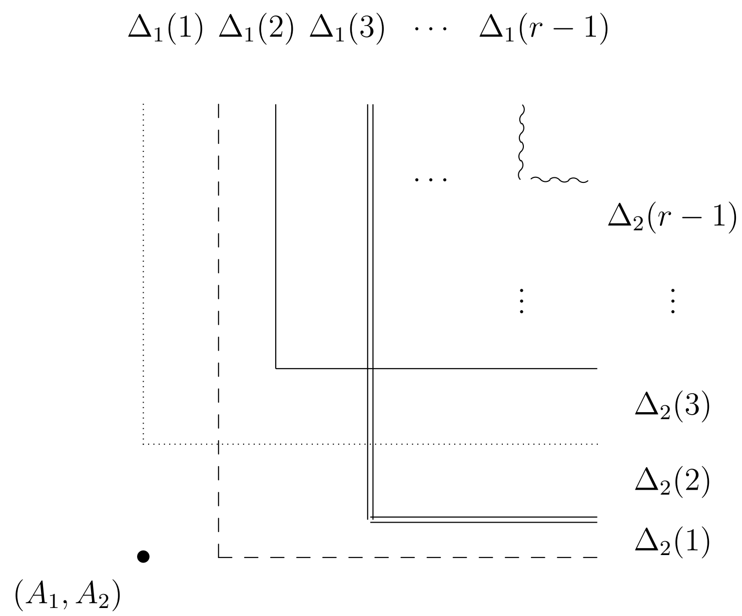
Proposition 2.2.4. *Let X be a nonsingular complete toric surface, H an ample divisor, $r \in \mathbb{Z}_{>0}$ and $c_1 \in H^2(X, \mathbb{Z})$. Then*

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(r, c_1, c_2)) q^{c_2} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^{re(X)}} \sum_{c_2 \in \mathbb{Z}} \sum_{\vec{\chi} \in \left(\mathcal{X}^r_{(r,c_1,c_2)}\right)^{gf}} e(N_{\vec{\chi}}^{\mu s}) q^{c_2}.$$

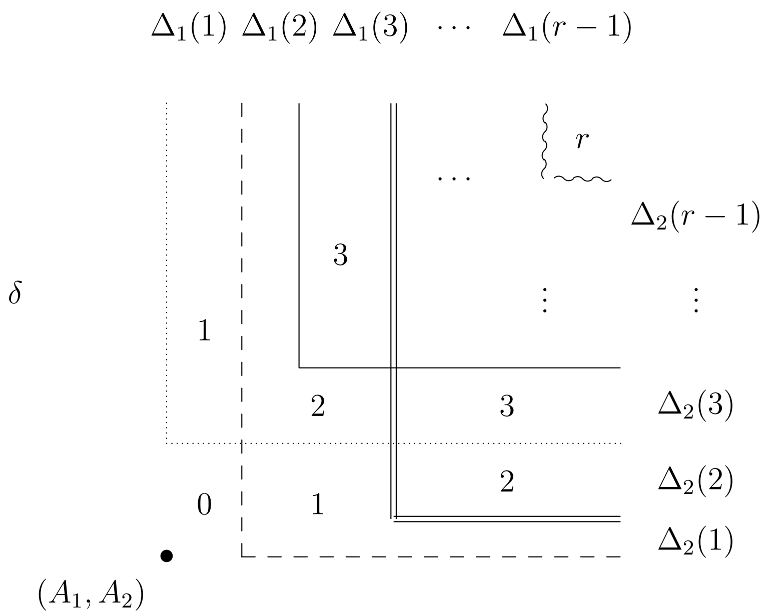
In this proposition, $N_{\vec{\chi}}^{\mu s}$ denotes the moduli space of μ-stable reflexive equivariant sheaves on X with characteristic function $\vec{\chi}$ (see subsection 1.3.4).

The goal of this subsection is to simplify the expression in the previous proposition by studying more closely how characteristic functions and Chern classes of equivariant vector bundles on nonsingular complete toric surfaces are related. Here we will make use of Proposition 2.2.2. The notion of characteristic function of an equivariant vector bundle on a nonsingular complete toric surface can be rephrased by using the notion of display named after Klyachko's similar notion introduced in [Kly4, Def. 1.3.6].

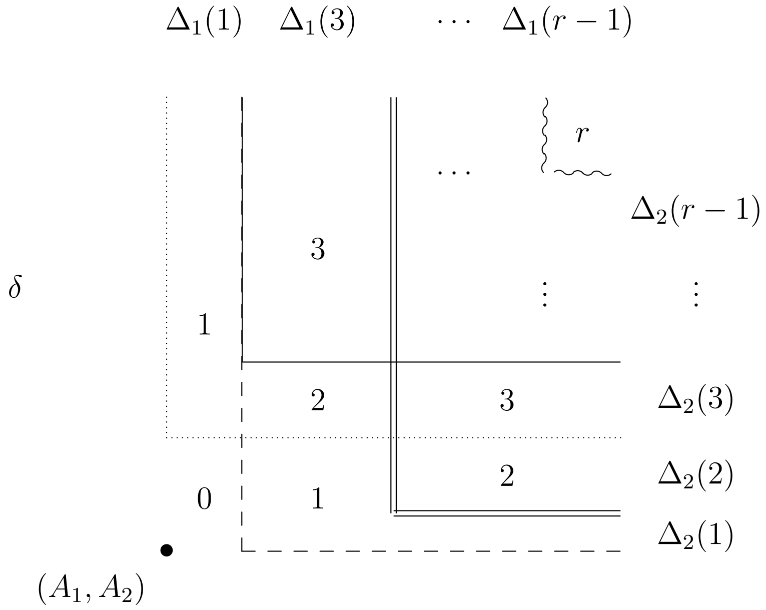
Definition 2.2.5. Let r be a positive integer, let A_1, A_2 be integers and let $\Delta_1(1), \ldots,$ $\Delta_1(r-1)$, $\Delta_2(1), \ldots, \Delta_2(r-1)$ be positive integers. A *display* δ located at (A_1, A_2) of widths $(\Delta_1(1), \ldots, \Delta_1(r-1); \Delta_2(1), \ldots, \Delta_2(r-1))$ and rank r is a diagram δ obtained as follows. Consider the lines $x = A_1$, $x = A_1 + \Delta_1(1)$, …, $x = A_1 + \Delta_1(1) + \cdots + \Delta_1(r-1)$, $y = A_2$, $y = A_2 + \Delta_2(1)$, …, $y = A_2 + \Delta_2(1) + \cdots + \Delta_2(r-1)$ in $\mathbb{R}^2$ an choose a permutation $\sigma \in S_r$. From this we construct a diagram. For example, in the case of a permutation $\sigma \in S_r$ sending $1 \to 3$, $2 \to 1$, $3 \to 4$, $4 \to 2$, …, we draw the following diagram, where we refer to the lines as the *edges*.



It is clear how the edges arise from the permutation σ. One can uniquely put numbers, called *dimensions*, in this diagram as follows. Put the number r in the upper right region $x > A_1 + \Delta_1(1) + \cdots + \Delta_1(r-1)$, $y > A_2 + \Delta_2(1) + \cdots + \Delta_2(r-1)$ and every time one crosses a horizontal or vertical edge, one decreases the dimension of the corresponding region by one as indicated by the following diagram.



The resulting diagram is called a *display* δ located at (A_1, A_2) of widths $(\Delta_1(1), \ldots, \Delta_1(r-1); \Delta_2(1), \ldots, \Delta_2(r-1))$ and rank r. Next, we want to allow degeneracies i.e. allow the $\Delta_i(j)$ in the definition of a display to be zero. For any $A_1, A_2 \in \mathbb{Z}$ and $\Delta_1(1), \ldots, \Delta_1(r-1) \in \mathbb{Z}_{\geq 0}$, $\Delta_2(1), \ldots, \Delta_2(r-1) \in \mathbb{Z}_{\geq 0}$, we define a display δ located at (A_1, A_2) of widths $(\Delta_1(1), \ldots, \Delta_1(r-1); \Delta_2(1), \ldots, \Delta_2(r-1))$ and rank r as follows. Let $\sigma \in S_r$ be a permutation. Consider the display located at (A_1, A_2) of widths $(1, \ldots, 1; 1, \ldots, 1)$ and rank r defined by σ. Then separate (or join) two adjacent horizontal or vertical lines according to the widths $\Delta_1(1), \ldots, \Delta_1(r-1)$, $\Delta_2(1), \ldots, \Delta_2(r-1)$, where several lines are allowed to coincide. The diagram of such a display δ typically looks like:



We denote the collection of displays located at (A_1, A_2) of widths $(\Delta_1(1), \ldots, \Delta_1(r-1); \Delta_2(1), \ldots, \Delta_2(r-1))$ and rank r by $\mathcal{D}(A_1, A_2; \Delta_1(1), \ldots, \Delta_1(r-1); \Delta_2(1), \ldots, \Delta_2(r-1))$. $\oslash$

The main use of displays is as follows. Consider the affine plane $\mathbb{A}^2$ as a toric variety with canonical torus action. Characteristic functions of equivariant vector bundles (or, equivalently, reflexive equivariant sheaves) of rank r on $\mathbb{A}^2$ are in 1-1 correspondence with displays of rank r (see subsections 1.3.2 and 2.2.1). As such, given a display δ of rank r, we can associate 2D partitions $\pi(1), \ldots, \pi(r)$ to it as in subsection 2.2.1. We define the size of δ to be $\#\delta = \sum_{i=1}^{r} \#\pi(i)$.

Let X be a nonsingular complete toric surface. Let $r \in \mathbb{Z}_{>0}$, $A_1, \ldots, A_N \in \mathbb{Z}$ and $\Delta_i(1), \ldots, \Delta_i(r-1) \in \mathbb{Z}_{\geq 0}$ for all $i = 1, \ldots, N$. For each $i = 1, \ldots, N$, define

$\mathrm{Flag}(\Delta_i(1),\ldots,\Delta_i(r-1))$ to be the closed subscheme of $\prod_{j=1}^{r-1}\mathrm{Gr}(j,r)$ defined by closed points $(p_i(1),\ldots,p_i(r-1))$ satisfying $p_i(1)\subset\cdots\subset p_i(r-1)$. Note that we omit factors $\mathrm{Gr}(j,r)$ in the product $\prod_{j=1}^{r-1}\mathrm{Gr}(j,r)$ corresponding to $\Delta_i(j)=0$. Suppose $\vec{\delta}\in\prod_{i=1}^{N}\mathcal{D}(A_i,A_{i+1};\Delta_i(1),\ldots,\Delta_i(r-1);\Delta_{i+1}(1),\ldots,\Delta_{i+1}(r-1))$. We also refer to $\vec{\delta}$ as a display. We define an associated locally closed subscheme

$$\mathcal{D}_{\vec{\delta}}\subset\prod_{i=1}^{N}\mathrm{Flag}(\Delta_i(1),\ldots,\Delta_i(r-1))\subset\prod_{i=1}^{N}\prod_{j=1}^{r-1}\mathrm{Gr}(j,r),$$

where a closed point $\{p_i(j)\}$ of $\prod_{i=1}^{N}\mathrm{Flag}(\Delta_i(1),\ldots,\Delta_i(r-1))$ is defined to belong to $\mathcal{D}_{\vec{\delta}}$ whenever for any $i=1,\ldots,N$, $j_1,j_2=1,\ldots,r-1$ we have $\dim(p_i(j_1)\cap p_{i+1}(j_2))$ is equal to the dimension of the region $A_i+\Delta_i(1)+\cdots+\Delta_i(j_1-1)<x<A_i+\Delta_i(1)+\cdots+\Delta_i(j_1-1)+\Delta_i(j_1)$, $A_{i+1}+\Delta_{i+1}(1)+\cdots+\Delta_{i+1}(j_2-1)<y<A_{i+1}+\Delta_{i+1}(1)+\cdots+\Delta_{i+1}(j_2-1)+\Delta_{i+1}(j_2)$ of the display δ_i. Note that these conditions are locally closed conditions[10] in $\prod_{i=1}^{N}\prod_{j=1}^{r-1}\mathrm{Gr}(j,r)$. We can write[11]

$$\prod_{i=1}^{N}\mathrm{Flag}(\Delta_i(1),\ldots,\Delta_i(r-1))=\coprod_{\vec{\delta}\in\prod_{i=1}^{N}\mathcal{D}(A_i,A_{i+1};\Delta_i(1),\ldots,\Delta_i(r-1);\Delta_{i+1}(1),\ldots,\Delta_{i+1}(r-1))}\mathcal{D}_{\vec{\delta}}.$$

Note that for some $\vec{\delta}$, one can have $\mathcal{D}_{\vec{\delta}}=\varnothing$. We introduce the notation $\#\vec{\delta}=\sum_{i=1}^{N}\#\delta_i$. As we have seen, the category of reflexive equivariant sheaves of rank r on X is equivalent to the category of N filtrations of $\mathbb{C}^{\oplus r}$ (see subsection 1.3.2 for details). The objects of the latter category are precisely the closed points of the following $\mathbb{C}$-scheme[11]

$$\coprod_{A_1,\ldots,A_N\in\mathbb{Z}}\coprod_{i=1}^{N}\coprod_{\Delta_i(1),\ldots,\Delta_i(r-1)\in\mathbb{Z}_{\geq0}}\prod_{i=1}^{N}\mathrm{Flag}(\Delta_i(1),\ldots,\Delta_i(r-1)) \tag{2.1}$$

$$=\coprod_{A_1,\ldots,A_N\in\mathbb{Z}}\coprod_{i=1}^{N}\coprod_{\Delta_i(1),\ldots,\Delta_i(r-1)\in\mathbb{Z}_{\geq0}}\coprod_{\vec{\delta}\in\prod_{i=1}^{N}\mathcal{D}(A_i,A_{i+1};\Delta_i(1),\ldots,\Delta_i(r-1);\Delta_{i+1}(1),\ldots,\Delta_{i+1}(r-1))}\mathcal{D}_{\vec{\delta}}.$$

---

[10]Here it is useful to note that for any finite product of Grassmannians $\prod_i\mathrm{Gr}(n_i,N)$, the map $\{p_i\}\mapsto\dim(\bigcap_i p_i)$ is upper semicontinuous.

[11]Strictly speaking, the equality sign means there is a canonical bijective morphism of $\mathbb{C}$-schemes from LHS to RHS.

Let H be an ample divisor on X and let $r \in \mathbb{Z}_{>0}$, $c_1 \in H^2(X, \mathbb{Z})$, $c_2 \in \mathbb{Z}$. Let $\mathcal{E}$ be any equivariant vector bundle of rank r on X with corresponding framed torsion free Δ-family $\hat{E}^{\Delta}$ considered as a closed point of the $\mathbb{C}$-scheme (2.1). If the point lies in the component indexed by $A_1, \ldots, A_N$, $\Delta_1(1), \ldots, \Delta_1(r-1)$, ..., $\Delta_N(1), \ldots, \Delta_N(r-1)$, $\vec{\delta}$, then its first Chern class is entirely determined by $A_1, \ldots, A_N$, $\Delta_1(1), \ldots, \Delta_1(r-1)$, ..., $\Delta_N(1), \ldots, \Delta_N(r-1)$ and its second Chern class by $A_1, \ldots, A_N$, $\Delta_1(1), \ldots, \Delta_1(r-1)$, ..., $\Delta_N(1), \ldots, \Delta_N(r-1)$ and $\vec{\delta}$ (see Proposition 2.2.2). Therefore, it makes sense to speak about $A_1, \ldots, A_N \in \mathbb{Z}$, $\Delta_1(1), \ldots, \Delta_1(r-1) \in \mathbb{Z}_{\geq 0}$, ..., $\Delta_N(1), \ldots, \Delta_N(r-1) \in \mathbb{Z}_{\geq 0}$ giving rise to c_1 and given such about $\vec{\delta} \in \prod_{i=1}^{N} \mathcal{D}(A_i, A_{i+1}; \Delta_i(1), \ldots, \Delta_i(r-1); \Delta_{i+1}(1), \ldots, \Delta_{i+1}(r-1))$ giving rise to c_2 by the formula in Proposition 2.2.2. We immediately obtain that the objects of the category of N filtrations of $\mathbb{C}^{\oplus r}$ corresponding to equivariant vector bundles on X of rank r, first Chern class c_1 and second Chern class c_2 are in 1-1 correspondence with the closed points of the $\mathbb{C}$-scheme

$$\coprod_{\substack{A_1, \ldots, A_N \in \mathbb{Z} \\ \Delta_1(1), \ldots, \Delta_1(r-1) \in \mathbb{Z}_{\geq 0} \\ \cdots \\ \Delta_N(1), \ldots, \Delta_N(r-1) \in \mathbb{Z}_{\geq 0} \\ \text{giving rise to } c_1}} \quad \coprod_{\substack{\vec{\delta} \in \prod_{i=1}^{N} \mathcal{D}(A_i, A_{i+1}; \Delta_i(1), \ldots, \Delta_i(r-1); \Delta_{i+1}(1), \ldots, \Delta_{i+1}(r-1)) \\ \text{giving rise to } c_2}} \mathcal{D}_{\vec{\delta}}.$$

There is a natural regular action of the reductive algebraic group $\mathrm{SL}(r, \mathbb{C})$ on the ambient variety $\prod_{i=1}^{N} \prod_{j=1}^{r-1} \mathrm{Gr}(j, r)$ leaving each of the locally closed subschemes $\mathcal{D}_{\vec{\delta}}$ invariant. Equivariant isomorphism classes of ample equivariant line bundles on $\prod_{i=1}^{N} \prod_{j=1}^{r-1} \mathrm{Gr}(j, r)$ are in 1-1 correspondence with sequences of positive integers $\{\kappa_{ij}\}_{i=1,\ldots,N, j=1,\ldots,r-1}$ [Dol, Sect. 11.1]. We consider the ample equivariant line bundle $\{\Delta_i(j)(H \cdot D_i)\}_{i=1,\ldots,N, j=1,\ldots,r-1}$, where we recall that $H \cdot D_i > 0$ for each $i = 1, \ldots, r$ by the Nakai–Moishezon Criterion [Har1, Thm. A.5.1]. It is proved in subsection 1.3.4 (referring to Proposition 1.2.21) that the pull-back of this ample equivariant line bundle to each $\mathcal{D}_{\vec{\delta}}$ gives a notion of GIT stability which precisely coincides with μ-stability. More precisely, we use these pull-backs to define our notion of GIT stability on each $\mathcal{D}_{\vec{\delta}}$ and any equivariant vector bundle $\mathcal{E}$ of rank r on X with corresponding collection of N filtrations $\hat{E}^{\Delta}$ of $\mathbb{C}^{\oplus r}$ is μ-semistable if and only if $\hat{E}^{\Delta}$ corresponds to a GIT semistable point in the $\mathbb{C}$-scheme (2.1) and $\mathcal{E}$ is μ-stable if and only if $\hat{E}^{\Delta}$ corresponds to a properly GIT stable point in the $\mathbb{C}$-scheme (2.1). The previous discussion combined with Theorem 1.3.15 yields the following proposition.

Proposition 2.2.6. *Let X be a nonsingular complete toric surface, H an ample divisor on X, $r \in \mathbb{Z}_{>0}$ and $c_1 \in H^2(X, \mathbb{Z})$. Then for any $c_2 \in \mathbb{Z}$, there is a canonical isomorphism*

$$N_X^H(r, c_1, c_2)^T \cong \coprod_{\substack{A_3, \ldots, A_N \in \mathbb{Z} \\ \Delta_1(1), \ldots, \Delta_1(r-1) \in \mathbb{Z}_{\geq 0} \\ \cdots \\ \Delta_N(1), \ldots, \Delta_N(r-1) \in \mathbb{Z}_{\geq 0} \\ \text{giving rise to } c_1 \\ \vec{\delta} \in \prod_{i=1}^N \mathcal{D}(A_i, A_{i+1}; \Delta_i(1), \ldots, \Delta_i(r-1); \Delta_{i+1}(1), \ldots, \Delta_{i+1}(r-1)) \\ \text{giving rise to } c_2}} \mathcal{D}_{\vec{\delta}}^s \,/\, \mathrm{SL}(r, \mathbb{C}),$$

where $\mathcal{D}_{\vec{\delta}}^s$ is the open subset of properly GIT stable elements with respect to the ample equivariant line bundle $\{\Delta_i(j)(H \cdot D_i)\}_{i=1,\ldots,N, j=1,\ldots,r-1}$ and the quotient is a good geometric quotient.

Note that in the above proposition, we take $A_1 = A_2 = 0$ because of the disjoint union over *gauge-fixed* characteristic functions in Theorem 1.3.15. Moreover, for any $A_3, \ldots, A_N \in \mathbb{Z}$, $\Delta_1(1), \ldots, \Delta_1(r-1), \ldots, \Delta_N(1), \ldots, \Delta_N(r-1) \in \mathbb{Z}_{\geq 0}$ giving rise to c_1, the integers $A_3, \ldots, A_N$ are unique with this property (i.e. determined by $\Delta_1(1), \ldots, \Delta_1(r-1), \ldots, \Delta_N(1), \ldots, \Delta_N(r-1)$). The isomorphism of the proposition is nothing but the isomorphism of schemes of Theorem 1.3.15 (with Hilbert polynomial replaced by rank and Chern classes). The displays $\vec{\delta}$ in the disjoint union for which $\mathcal{D}_{\vec{\delta}} \neq \varnothing$ precisely correspond to the characteristic functions $\vec{\chi} \in \left(\mathcal{X}^r_{(r,c_1,c_2)}\right)^{gf}$ and the $\mathcal{D}_{\vec{\delta}}^s \,/\, \mathrm{SL}(r, \mathbb{C})$ precisely correspond to the $N_{\vec{\chi}}^{\mu s}$.

Let $a, b \in \mathbb{Z}$. If $a \neq 0$, then we write $a \mid b$ whenever $b = ak$ for some $k \in \mathbb{Z}$. Also, for later convenience, we write $(a,b) = 1$ whenever a and b are coprime. Recall the notation introduced in subsection 2.2.1. Using Propositions 2.1.1, 2.2.2, 2.2.3, 2.2.6, a now straightforward computation yields an expression for the generating function.

Theorem 2.2.7. *Let X be a nonsingular complete toric surface, H an ample divisor on X, $r \in \mathbb{Z}_{>0}$ and $c_1 = \sum_{i=3}^N f_i D_i \in H^2(X,\mathbb{Z})$. Then*

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(r,c_1,c_2))q^{c_2} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^{re(X)}} \cdot \sum_{\substack{\Delta_1(1),\ldots,\Delta_1(r-1) \in \mathbb{Z}_{\geq 0} \\ \cdots \\ \Delta_N(1),\ldots,\Delta_N(r-1) \in \mathbb{Z}_{\geq 0} \\ \text{such that } \forall i = 3,\ldots,N \\ r \mid -f_i + \sum_{j=1}^{r-1} j\left(\Delta_1(j)\xi_i + \Delta_2(j)\eta_i + \Delta_i(j)\right)}} q^{\frac{1}{2}\left(\sum_{i=3}^N f_i D_i\right)^2}$$

$$\cdot\, q^{\begin{array}{l} -\frac{1}{2r^2}\sum_{j=0}^{r-1}\Big[\sum_{i=3}^N\Big(-f_i - \sum_{k=1}^{r-1}(r-k)\Delta_i(k) + \{-\sum_{k=1}^{r-1}(r-k)\Delta_1(k) + \sum_{k=1}^{j} r\Delta_1(k)\}\xi_i \\ \qquad +\{-\sum_{k=1}^{r-1}(r-k)\Delta_2(k) + \sum_{k=1}^{j} r\Delta_2(k)\}\eta_i + \sum_{k=1}^{j} r\Delta_i(k)\Big) D_i\Big]^2 \end{array}}$$

$$\cdot \sum_{\vec{\delta} \in \prod_{i=1}^N \mathcal{D}(\Delta_i(1),\ldots,\Delta_i(r-1);\Delta_{i+1}(1),\ldots,\Delta_{i+1}(r-1))} e\left(\mathcal{D}_{\vec{\delta}}^s \,/\, \mathrm{SL}(r,\mathbb{C})\right) q^{\#\vec{\delta}},$$

where $\mathcal{D}_{\vec{\delta}}^s$ is the open subset of properly GIT stable elements with respect to the ample equivariant line bundle $\{\Delta_i(j)(H \cdot D_i)\}_{i=1,\ldots,N,j=1,\ldots,r-1}$ and the quotient is a good geometric quotient.

2.3 Examples

Theorem 2.2.7 gives an expression for the generating function of Euler characteristics of moduli spaces of μ-stable torsion free sheaves of rank r and first Chern class c_1 on an arbitrary nonsingular complete toric surface X with ample divisor H. Although the expression in Theorem 2.2.7 is general, further simplifications can be obtained in examples as we will see in this section. We apply Theorem 2.2.7 to the examples X

arbitrary and rank $r = 1$, $X = \mathbb{P}^2$ and rank $r = 1, 2, 3$ and $X = \mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$) and rank $r = 1, 2$. Various authors have considered some of these cases individually including Ellingsrud and Strømme, Göttsche, Klyachko, Yoshioka and Weist. We will compare our results to their work and Joyce's general theory of wall-crossing for motivic invariants counting (semi)stable objects.

2.3.1 Rank 1 on Toric Surfaces

Let us consider the expression in Theorem 2.2.7 for rank $r = 1$.

Corollary 2.3.1. *Let X be a nonsingular complete toric surface and let H be an ample divisor on X. Then for any $c_1 \in H^2(X, \mathbb{Z})$*

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(1, c_1, c_2)) q^{c_2} = \frac{1}{\prod_{k=1}^{\infty} (1 - q^k)^{e(X)}}.$$

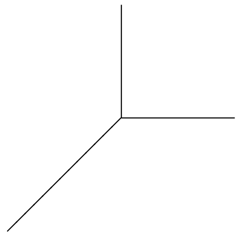
This was first shown by Ellingsrud and Strømme [ES] for the projective plane and the Hirzebruch surfaces using a natural $\mathbb{C}^*$-action. Subsequently, Göttsche proved it for general nonsingular projective surfaces using the Weil Conjectures [Got1]. In fact, he computes an expression for Poincaré polynomials, not just Euler characteristics.

2.3.2 Rank 2 on $\mathbb{P}^2$, $\mathbb{F}_a$

Consider the expression in Theorem 2.2.7 for rank $r = 2$. This time the occurrence of Euler characteristics of moduli spaces of stable configurations of points on $\mathbb{P}^1$ makes the expression for the generating function significantly more complicated. Note that these moduli spaces of stable configurations of points on $\mathbb{P}^1$ depend on the ample divisor H. We will simplify the formula for $X = \mathbb{P}^2$ and $X = \mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$). We will also study wall-crossing for $X = \mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$). Throughout, we will pay special attention to the case $X = \mathbb{P}^1 \times \mathbb{P}^1$.

2.3.2.1 Rank 2 on $\mathbb{P}^2$

Consider the fan of $\mathbb{P}^2$.



Let D be the toric divisor corresponding to any of the rays, then for $\alpha \in \mathbb{Z}$ we have αD is ample if and only if α is a positive integer. Note that for $X = \mathbb{P}^2$ and ample divisor $H = \alpha D$, the generating function in Theorem 2.2.7 is independent of α, so without loss of generality, we can choose $\alpha = 1$. In the rank $r = 2$ case, the spaces $\mathcal{D}_{\vec{\delta}}$ in Theorem 2.2.7 are locally closed subschemes of $(\mathbb{P}^1)^N$, where N is the Euler characteristic of the surface X. For $X = \mathbb{P}^2$, we have $N = 3$. We introduce some graphical notation. Denote by:

$$\overset{1}{\bullet} \!-\! \overset{2}{\bullet} \!-\! \overset{3}{\bullet}$$

the locally closed subset determined by points $(p_1, p_2, p_3) \in (\mathbb{P}^1)^3$ with $p_1 \neq p_2$, $p_2 \neq p_3$ and $p_1 \neq p_3$. Similarly, denote by:

$$\overset{1,2}{\bullet} \!-\! \overset{3}{\bullet}$$

the locally closed subset determined by points $(p_1, p_2, p_3) \in (\mathbb{P}^1)^3$ with $p_1 = p_2 \neq p_3$. We use similar notation for similar locally closed subsets. We refer to such locally closed subschemes as incidence spaces. For completeness, we will write out all terms of the expression in Theorem 2.2.7, though most will trivially be zero. We choose the first Chern class $c_1 = f_3 D_3 = f_3 D$ arbitrary and define $f = f_3 \in \mathbb{Z}$

$$\prod_{k=1}^{\infty}(1-q^k)^6 \sum_{c_2 \in \mathbb{Z}} e(M_X^H(2, c_1, c_2)) q^{c_2} = \sum_{\substack{\Delta_1, \Delta_2, \Delta_3 \in \mathbb{Z}_{>0} \\ 2 \,\mid\, -f + \Delta_1 + \Delta_2 + \Delta_3}} q^{\frac{1}{4}f^2 - \frac{1}{4}(\Delta_1+\Delta_2+\Delta_3)^2}$$

$$\cdot \left\{ e\left(\overset{1}{\bullet} \!-\! \overset{2}{\bullet} \!-\! \overset{3}{\bullet} \Big/_{(\Delta_1,\Delta_2,\Delta_3)} \mathrm{SL}(2,\mathbb{C}) \right) q^{\Delta_1\Delta_2+\Delta_2\Delta_3+\Delta_3\Delta_1} \right.$$

$$+e\left(-\overset{1,2}{\bullet}-\overset{3}{\bullet}-/_{(\Delta_1+\Delta_2,\Delta_3)}\mathrm{SL}(2,\mathbb{C})\right)q^{\Delta_2\Delta_3+\Delta_3\Delta_1}$$

$$+e\left(-\overset{2,3}{\bullet}-\overset{1}{\bullet}-/_{(\Delta_1,\Delta_2+\Delta_3)}\mathrm{SL}(2,\mathbb{C})\right)q^{\Delta_1\Delta_2+\Delta_3\Delta_1}$$

$$+e\left(-\overset{1,3}{\bullet}-\overset{2}{\bullet}-/_{(\Delta_1+\Delta_3,\Delta_2)}\right)q^{\Delta_1\Delta_2+\Delta_2\Delta_3}$$

$$+e\left(-\overset{1,2,3}{\bullet}-/_{\Delta_1+\Delta_2+\Delta_3}\mathrm{SL}(2,\mathbb{C})\right)\Bigg\}+\sum_{\substack{\Delta_2,\Delta_3\in\mathbb{Z}_{>0}\\ 2\,|\,-f+\Delta_2+\Delta_3}} q^{\frac{1}{4}f^2-\frac{1}{4}(\Delta_2+\Delta_3)^2}$$

$$\cdot\Bigg\{e\left(-\overset{2}{\bullet}-\overset{3}{\bullet}-/_{(\Delta_2,\Delta_3)}\mathrm{SL}(2,\mathbb{C})\right)q^{\Delta_2\Delta_3}$$

$$+e\left(-\overset{2,3}{\bullet}-/_{\Delta_2+\Delta_3}\mathrm{SL}(2,\mathbb{C})\right)\Bigg\}+\sum_{\substack{\Delta_1,\Delta_3\in\mathbb{Z}_{>0}\\ 2\,|\,-f+\Delta_1+\Delta_3}} q^{\frac{1}{4}f^2-\frac{1}{4}(\Delta_1+\Delta_3)^2}$$

$$\cdot\Bigg\{e\left(-\overset{1}{\bullet}-\overset{3}{\bullet}-/_{(\Delta_1,\Delta_3)}\mathrm{SL}(2,\mathbb{C})\right)q^{\Delta_3\Delta_1}$$

$$+e\left(-\overset{1,3}{\bullet}-/_{\Delta_1+\Delta_3}\mathrm{SL}(2,\mathbb{C})\right)\Bigg\}+\sum_{\substack{\Delta_1,\Delta_2\in\mathbb{Z}_{>0}\\ 2\,|\,-f+\Delta_1+\Delta_2}} q^{\frac{1}{4}f^2-\frac{1}{4}(\Delta_1+\Delta_2)^2}$$

$$\cdot\Bigg\{e\left(-\overset{1}{\bullet}-\overset{2}{\bullet}-/_{(\Delta_1,\Delta_2)}\mathrm{SL}(2,\mathbb{C})\right)q^{\Delta_1\Delta_2}$$

$$+e\left(-\overset{1,2}{\bullet}-/_{\Delta_1+\Delta_2}\mathrm{SL}(2,\mathbb{C})\right)\Bigg\}+\sum_{\substack{\Delta_1\in\mathbb{Z}_{>0}\\ 2\,|\,-f+\Delta_1}} e\left(-\overset{1}{\bullet}-/_{\Delta_1}\mathrm{SL}(2,\mathbb{C})\right)q^{\frac{1}{4}f^2-\frac{1}{4}\Delta_1^2}$$

$$+\sum_{\substack{\Delta_2\in\mathbb{Z}_{>0}\\ 2\,|\,-f+\Delta_2}} e\left(-\overset{2}{\bullet}-/_{\Delta_2}\mathrm{SL}(2,\mathbb{C})\right)q^{\frac{1}{4}f^2-\frac{1}{4}\Delta_2^2}+\sum_{\substack{\Delta_3\in\mathbb{Z}_{>0}\\ 2\,|\,-f+\Delta_3}} e\left(-\overset{3}{\bullet}-/_{\Delta_3}\mathrm{SL}(2,\mathbb{C})\right)q^{\frac{1}{4}f^2-\frac{1}{4}\Delta_3^2}$$

$$=\sum_{\substack{\Delta_1,\Delta_2,\Delta_3\in\mathbb{Z}_{>0}\\ 2\,|\,-f+\Delta_1+\Delta_2+\Delta_3\\ \Delta_1<\Delta_2+\Delta_3\\ \Delta_2<\Delta_1+\Delta_3\\ \Delta_3<\Delta_1+\Delta_2}} q^{\frac{f^2}{4}+\frac{\Delta_1\Delta_2}{2}+\frac{\Delta_2\Delta_3}{2}+\frac{\Delta_3\Delta_1}{2}-\frac{\Delta_1^2}{4}-\frac{\Delta_2^2}{4}-\frac{\Delta_3^2}{4}}. \tag{2.2}$$

Here the quotients $-\overset{1}{\bullet}-\overset{2}{\bullet}-\overset{3}{\bullet}-/_{(\Delta_1,\Delta_2,\Delta_3)}\mathrm{SL}(2,\mathbb{C})$, ..., are just another way of writing the quotients $\mathcal{D}^s_{\vec{\delta}}$ / $\mathrm{SL}(r,\mathbb{C})$ in Theorem 2.2.7. Therefore, the subscript of "/" refers to the ample equivariant line bundle with respect to which we take the geometric quotient (see subsection 2.2.2). Clearly, all these spaces are empty except for possibly $-\overset{1}{\bullet}-\overset{2}{\bullet}-\overset{3}{\bullet}-/_{(\Delta_1,\Delta_2,\Delta_3)}\mathrm{SL}(2,\mathbb{C})$, which satisfies

$$e\left(-\overset{1}{\bullet}-\overset{2}{\bullet}-\overset{3}{\bullet}-/_{(\Delta_1,\Delta_2,\Delta_3)}\mathrm{SL}(2,\mathbb{C})\right) = \begin{cases} 1 & \text{if } \Delta_i < \Delta_j + \Delta_k \text{ for all } \{i,j,k\} = \{1,2,3\} \\ 0 & \text{otherwise.} \end{cases}$$

Let X be any nonsingular projective surface, H an ample divisor, $r \in \mathbb{Z}_{>0}$, $c_1 \in H^2(X,\mathbb{Z})$ and $c_2 \in \mathbb{Z}$. Let a be a Weil divisor. Applying $- \otimes \mathcal{O}_X(a)$, we obtain an isomorphism

$$M_X^H(r,c_1,c_2) \cong M_X^H(r,c_1+ra,(r-1)c_1a+\frac{1}{2}r(r-1)a^2+c_2).$$

For this, we note that $- \otimes \mathcal{O}_X(a)$ preserves μ-stability. We deduce

$$\sum_{c_2\in\mathbb{Z}} e(M_X^H(r,c_1+ra,c_2))q^{c_2} = q^{(r-1)c_1a+\frac{1}{2}r(r-1)a^2} \sum_{c_2\in\mathbb{Z}} e(M_X^H(r,c_1,c_2))q^{c_2}. \tag{2.3}$$

So for $X = \mathbb{P}^2$ and $r = 2$, the only two interesting values for f are 0 and 1. We can now prove the following corollary.

Corollary 2.3.2. *Let $X = \mathbb{P}^2$ and let H be an ample divisor on X. Then*

$$\sum_{c_2\in\mathbb{Z}} e(M_X^H(2,0,c_2))q^{c_2} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^6} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{q^{mn+m+n}}{1-q^{m+n}}$$
$$= q^3+6q^4+30q^5+116q^6+399q^7+1233q^8+3539q^9+9519q^{10}+O(q^{11}),$$
$$\sum_{c_2\in\mathbb{Z}} e(M_X^H(2,1,c_2))q^{c_2} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^6} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{q^{mn}}{1-q^{m+n-1}}$$
$$= q+9q^2+48q^3+203q^4+729q^5+2346q^6+6918q^7+19062q^8+49620q^9$$
$$+123195q^{10}+O(q^{11}).$$

Proof. Refer to the previous computation, in particular equation (2.2). Summing over

$$\Delta_1, \Delta_2, \Delta_3 \in \mathbb{Z},\ \Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 > 0,\ \Delta_1 < \Delta_2 + \Delta_3,\ \Delta_2 < \Delta_1 + \Delta_3,$$
$$\Delta_3 < \Delta_1 + \Delta_2,\ 2 \mid -f + \Delta_1 + \Delta_2 + \Delta_3,$$

is equivalent to summing over

$$\xi, \eta, \zeta \in \mathbb{Q}_{>0},\ \xi + \eta \in \mathbb{Z},\ \xi + \zeta \in \mathbb{Z},\ \eta + \zeta \in \mathbb{Z},\ 2 \mid -f + 2\xi + 2\eta + 2\zeta,$$

by using the substitutions $\xi = \frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3)$, $\eta = \frac{1}{2}(\Delta_1 - \Delta_2 + \Delta_3)$, $\zeta = \frac{1}{2}(-\Delta_1 + \Delta_2 + \Delta_3)$. This in turn is equivalent to summing over

$$k, m, n \in \mathbb{Z},\ k > \frac{f}{2},\ m > k - \frac{f}{2},\ n > k - \frac{f}{2},$$

by using the substitutions $\xi = \frac{2k-f}{2}$, $\eta = m - \frac{2k-f}{2}$, $\zeta = n - \frac{2k-f}{2}$. We obtain

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(2,0,c_2))q^{c_2} = \frac{1}{\prod_{p=1}^{\infty}(1-q^p)^6} \sum_{k=1}^{\infty} \sum_{m=k+1}^{\infty} \sum_{n=k+1}^{\infty} q^{mn-k^2},$$
$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(2,1,c_2))q^{c_2} = \frac{1}{\prod_{p=1}^{\infty}(1-q^p)^6} \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} \sum_{n=k}^{\infty} q^{mn-k(k-1)},$$

from which the result follows by using the geometric series. $\square$

In [Yos], Yoshioka derives an expression for the generating function of Poincaré polynomials of $M_X^H(2,1,c_2)$ for $X = \mathbb{P}^2$ and H any ample divisor on X using the Weil Conjectures. Specialising to Euler characteristics, his result is

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(2,1,c_2))q^{c_2} =$$
$$\frac{1}{\prod_{k=1}^{\infty}(1-q^k)^6} \left(\frac{1}{2\sum_{m\in\mathbb{Z}} q^{m^2}} \right) \sum_{n=0}^{\infty} \left(\frac{2-4n}{1-q^{2n+1}} + \frac{8q^{2n+1}}{(1-q^{2n+1})^2} \right) q^{(n+1)^2}.$$

Equating to the formula obtained in Corollary 2.3.2, we have proved an interesting equality of expressions. Although it does not seem to be easy to show the equality directly, one can numerically check agreement of the coefficients up to large order by making expansions of both series. In [Kly4], Klyachko computes $\sum_{c_2 \in \mathbb{Z}} e(M_X^H(2,1,c_2))q^{c_2}$ for $X = \mathbb{P}^2$ and H any ample divisor on X, essentially using the same methods as in this chapter. In fact, this thesis is based on the philosophy of Klyachko. As mentioned in the introduction to this chapter, the previous chapter lays the foundations for many ideas appearing in [Kly4] and generalises them to pure equivariant sheaves of any dimension on any nonsingular toric variety. The present chapter can be seen as a systematic application to torsion free sheaves on nonsingular complete toric surfaces. Klyachko expresses his answer as

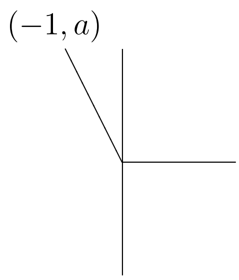
$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(2,1,c_2))q^{c_2} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^6} \sum_{m=1}^{\infty} 3H(4m-1)q^m,$$

where $H(D)$ is the Hurwitz class number

$$H(D) = \left(\begin{array}{c} \text{number of integer binary quadratic forms } Q \text{ of} \\ \text{discriminant } -D \text{ counted with weight } \frac{2}{\mathrm{Aut}(Q)} \end{array} \right).$$

2.3.2.2 Rank 2 on $\mathbb{F}_a$

Let us repeat the computation for rank 2 on $\mathbb{P}^2$ in the more complicated case of rank 2 on $X = \mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$). The fan of $\mathbb{F}_a$ is:



We obtain relations $D_1 = D_3$ and $D_4 = D_2 + aD_3$. Define $E = D_1$, $F = D_2$, then the Chow ring is given by

$$A(X) = \mathbb{Z}[E,F]/(E^2, F^2 + aEF, F^3).$$

Any line bundle (up to isomorphism) is of the form $\mathcal{O}(\alpha E + \beta F)$, where $\alpha, \beta \in \mathbb{Z}$. Such a line bundle is ample if and only if $\beta > 0$, $\alpha' := \alpha - a\beta > 0$ [Ful, Sect. 3.4]. Fix such an ample line bundle and denote the corresponding ample divisor by $H = \alpha E + \beta F$. We note $H \cdot D_1 = \beta$, $H \cdot D_2 = \alpha'$, $H \cdot D_3 = \beta$ and $H \cdot D_4 = \alpha$. Choose an arbitrary first Chern class $c_1 = f_3 D_3 + f_4 D_4 \in H^2(X, \mathbb{Z})$. By formula (2.3), the only interesting cases are $(f_3, f_4) = (0,0), (1,0), (0,1), (1,1)$. Using the same notation for incidence spaces as before, it is easy to see that exactly 11 incidence spaces contribute to the generating function of Theorem 2.2.7, namely for any i, j, k, l such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

$$\begin{array}{ccc} i,j & k & l \\ \bullet & \bullet & \bullet \end{array}$$

$$\begin{array}{ccc} i & j & k \\ \bullet & \bullet & \bullet \end{array}$$

We proceed entirely analogously to the derivation of equation (2.2) in the rank 2 on $\mathbb{P}^2$ case. Note that this time, the geometric quotients $\mathcal{D}^s_{\vec{\delta}} \ / \ \mathrm{SL}(r, \mathbb{C})$ of Theorem 2.2.7 can split up as a disjoint union of various incidence spaces. We obtain

$$\prod_{k=1}^{\infty}(1-q^k)^8 \sum_{c_2 \in \mathbb{Z}} e(M_X^H(2, c_1, c_2)) q^{c_2} = \tag{2.4}$$

$$- \sum_{\substack{\Delta_1, \Delta_2, \Delta_3, \Delta_4 \in \mathbb{Z}_{>0} \\ 2 \,|\, -f_3 + \Delta_1 - a\Delta_2 + \Delta_3 \\ 2 \,|\, -f_4 + \Delta_2 + \Delta_4 \\ \beta\Delta_1 < \alpha'\Delta_2 + \beta\Delta_3 + \alpha\Delta_4 \\ \alpha'\Delta_2 < \beta\Delta_1 + \beta\Delta_3 + \alpha\Delta_4 \\ \beta\Delta_3 < \beta\Delta_1 + \alpha'\Delta_2 + \alpha\Delta_4 \\ \alpha\Delta_4 < \beta\Delta_1 + \alpha'\Delta_2 + \beta\Delta_3}} q^{\frac{1}{2}f_3 f_4 + \frac{a}{4}f_4^2 + \frac{1}{2}(\Delta_2 + \Delta_4)(\Delta_1 + \frac{a}{2}\Delta_2 + \Delta_3 - \frac{a}{2}\Delta_4)}$$

$$+\sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Delta_4\in\mathbb{Z}_{>0}\\ 2\,|\,-f_3+\Delta_1-a\Delta_2+\Delta_3\\ 2\,|\,-f_4+\Delta_2+\Delta_4\\ \beta\Delta_1+\beta\Delta_3<\alpha'\Delta_2+\alpha\Delta_4\\ \alpha'\Delta_2<\beta\Delta_1+\beta\Delta_3+\alpha\Delta_4\\ \alpha\Delta_4<\beta\Delta_1+\alpha'\Delta_2+\beta\Delta_3}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}(\Delta_2+\Delta_4)(\Delta_1+\frac{a}{2}\Delta_2+\Delta_3-\frac{a}{2}\Delta_4)}$$

$$+\sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Delta_4\in\mathbb{Z}_{>0}\\ 2\,|\,-f_3+\Delta_1-a\Delta_2+\Delta_3\\ 2\,|\,-f_4+\Delta_2+\Delta_4\\ \alpha'\Delta_2+\alpha\Delta_4<\beta\Delta_1+\beta\Delta_3\\ \beta\Delta_1<\alpha'\Delta_2+\beta\Delta_3+\alpha\Delta_4\\ \beta\Delta_3<\beta\Delta_1+\alpha'\Delta_2+\alpha\Delta_4}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}(\Delta_2+\Delta_4)(\Delta_1+\frac{a}{2}\Delta_2+\Delta_3-\frac{a}{2}\Delta_4)}$$

$$+\sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Delta_4\in\mathbb{Z}_{>0}\\ 2\,|\,-f_3+\Delta_1-a\Delta_2+\Delta_3\\ 2\,|\,-f_4+\Delta_2+\Delta_4\\ \beta\Delta_1+\alpha'\Delta_2<\beta\Delta_3+\alpha\Delta_4\\ \beta\Delta_3<\beta\Delta_1+\alpha'\Delta_2+\alpha\Delta_4\\ \alpha\Delta_4<\beta\Delta_1+\alpha'\Delta_2+\beta\Delta_3}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2-\frac{1}{2}(\Delta_2+\Delta_4)(\Delta_1-\frac{a}{2}\Delta_2+\Delta_3+\frac{a}{2}\Delta_4)+\Delta_2\Delta_3+\Delta_3\Delta_4+\Delta_4\Delta_1}$$

$$+\sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Delta_4\in\mathbb{Z}_{>0}\\ 2\,|\,-f_3+\Delta_1-a\Delta_2+\Delta_3\\ 2\,|\,-f_4+\Delta_2+\Delta_4\\ \beta\Delta_1+\alpha\Delta_4<\alpha'\Delta_2+\beta\Delta_3\\ \alpha'\Delta_2<\beta\Delta_1+\beta\Delta_3+\alpha\Delta_4\\ \beta\Delta_3<\beta\Delta_1+\alpha'\Delta_2+\alpha\Delta_4}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2-\frac{1}{2}(\Delta_2+\Delta_4)(\Delta_1-\frac{a}{2}\Delta_2+\Delta_3+\frac{a}{2}\Delta_4)+\Delta_1\Delta_2+\Delta_2\Delta_3+\Delta_3\Delta_4}$$

$$+\sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Delta_4\in\mathbb{Z}_{>0}\\ 2\,|\,-f_3+\Delta_1-a\Delta_2+\Delta_3\\ 2\,|\,-f_4+\Delta_2+\Delta_4\\ \alpha'\Delta_2+\beta\Delta_3<\beta\Delta_1+\alpha\Delta_4\\ \beta\Delta_1<\alpha'\Delta_2+\beta\Delta_3+\alpha\Delta_4\\ \alpha\Delta_4<\beta\Delta_1+\alpha'\Delta_2+\beta\Delta_3}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2-\frac{1}{2}(\Delta_2+\Delta_4)(\Delta_1-\frac{a}{2}\Delta_2+\Delta_3+\frac{a}{2}\Delta_4)+\Delta_1\Delta_2+\Delta_3\Delta_4+\Delta_4\Delta_1}$$

$$
+\sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Delta_4\in\mathbb{Z}_{>0}\\ 2\,|\,-f_3+\Delta_1-a\Delta_2+\Delta_3\\ 2\,|\,-f_4+\Delta_2+\Delta_4\\ \beta\Delta_3+\alpha\Delta_4<\beta\Delta_1+\alpha'\Delta_2\\ \beta\Delta_1<\alpha'\Delta_2+\beta\Delta_3+\alpha\Delta_4\\ \alpha'\Delta_2<\beta\Delta_1+\beta\Delta_3+\alpha\Delta_4}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2-\frac{1}{2}(\Delta_2+\Delta_4)(\Delta_1-\frac{a}{2}\Delta_2+\Delta_3+\frac{a}{2}\Delta_4)+\Delta_1\Delta_2+\Delta_2\Delta_3+\Delta_4\Delta_1}
$$

$$
+\sum_{\substack{\Delta_2,\Delta_3,\Delta_4\in\mathbb{Z}_{>0}\\ 2\,|\,-f_3-a\Delta_2+\Delta_3\\ 2\,|\,-f_4+\Delta_2+\Delta_4\\ \alpha'\Delta_2<\beta\Delta_3+\alpha\Delta_4\\ \beta\Delta_3<\alpha'\Delta_2+\alpha\Delta_4\\ \alpha\Delta_4<\alpha'\Delta_2+\beta\Delta_3}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}(\Delta_2+\Delta_4)(\frac{a}{2}\Delta_2+\Delta_3-\frac{a}{2}\Delta_4)}
$$

$$
+\sum_{\substack{\Delta_1,\Delta_3,\Delta_4\in\mathbb{Z}_{>0}\\ 2\,|\,-f_3+\Delta_1+\Delta_3\\ 2\,|\,-f_4+\Delta_4\\ \beta\Delta_1<\beta\Delta_3+\alpha\Delta_4\\ \beta\Delta_3<\beta\Delta_1+\alpha\Delta_4\\ \alpha\Delta_4<\beta\Delta_1+\beta\Delta_3}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}\Delta_4(\Delta_1+\Delta_3-\frac{a}{2}\Delta_4)}
+\sum_{\substack{\Delta_1,\Delta_2,\Delta_4\in\mathbb{Z}_{>0}\\ 2\,|\,-f_3+\Delta_1-a\Delta_2\\ 2\,|\,-f_4+\Delta_2+\Delta_4\\ \beta\Delta_1<\alpha'\Delta_2+\alpha\Delta_4\\ \alpha'\Delta_2<\beta\Delta_1+\alpha\Delta_4\\ \alpha\Delta_4<\beta\Delta_1+\alpha'\Delta_2}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}(\Delta_2+\Delta_4)(\Delta_1+\frac{a}{2}\Delta_2-\frac{a}{2}\Delta_4)}
$$

$$
+\sum_{\substack{\Delta_1,\Delta_2,\Delta_3\in\mathbb{Z}_{>0}\\ 2\,|\,-f_3+\Delta_1-a\Delta_2+\Delta_3\\ 2\,|\,-f_4+\Delta_2\\ \beta\Delta_1<\alpha'\Delta_2+\beta\Delta_3\\ \alpha'\Delta_2<\beta\Delta_1+\beta\Delta_3\\ \beta\Delta_3<\beta\Delta_1+\alpha'\Delta_2}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}\Delta_2(\Delta_1+\frac{a}{2}\Delta_2+\Delta_3)}.
$$

Using equation (2.4), we can now prove the following corollary.

Corollary 2.3.3. *Let $X = \mathbb{F}_a$, where $a \in \mathbb{Z}_{\geq 0}$. Let $H = \alpha D_1 + \beta D_2$ be an ample divisor, i.e. α, β are integers such that $\alpha > a\beta$, $\beta > 0$. Let $c_1 = f_3 D_3 + f_4 D_4 \in H^2(X, \mathbb{Z})$. Define $\lambda = \frac{\alpha}{\beta}$, then*

$$
\begin{aligned}
&\prod_{k=1}^{\infty}(1-q^k)^8 \sum_{c_2 \in \mathbb{Z}} e(M_X^H(2,c_1,c_2))q^{c_2} = \\
&- \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ 2 \mid f_3+i,\ 2 \mid f_4+j \\ 2 \mid i+k,\ 2 \mid j+l \\ \lambda j = i,\ -j<l<j \\ -\lambda j + a(j+l) < k < \lambda j}} q^{\frac{1}{2}f_3f_4 + \frac{a}{4}f_4^2 + \frac{1}{2}j(i-\frac{a}{2}j)} \\
&+ 2\Bigg(\sum_{\substack{i,j,k,l \in \mathbb{Z} \\ 2 \mid f_3+i,\ 2 \mid f_4+j \\ 2 \mid i+k,\ 2 \mid j+l \\ k<\lambda l<i,\ l<j \\ -i-a(j-l)<k,\ -\lambda j<k}} + \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ 2 \mid f_3+i,\ 2 \mid f_4+j \\ 2 \mid i+k,\ 2 \mid j+l \\ k<\lambda l<i,\ l<j \\ -i+a(j+l)<k,\ -\lambda j+a(j+l)<k}} \Bigg) q^{\frac{1}{2}f_3f_4 + \frac{a}{4}f_4^2 + \frac{1}{4}ij - \frac{1}{4}jk + \frac{1}{4}il + \frac{1}{4}kl - \frac{a}{4}l^2} \\
&+ \Bigg(2 \sum_{\substack{i,j,k \in \mathbb{Z} \\ 2 \mid f_3+i,\ 2 \mid f_4+j \\ 2 \mid j+k \\ i<\lambda j,\ \frac{a}{2}(j+k)<i \\ -\frac{i}{\lambda-a}+\frac{aj}{\lambda-a}<k<\lambda^{-1}i}} + \sum_{\substack{i,j,k \in \mathbb{Z} \\ 2 \mid f_3+i,\ 2 \mid f_4+j \\ 2 \mid i+k \\ \lambda j<i,\ -\lambda j<k<\lambda j}} + \sum_{\substack{i,j,k \in \mathbb{Z} \\ 2 \mid f_3+i,\ 2 \mid f_4+j \\ 2 \mid i+k \\ \lambda j<i,\ j>0 \\ -\lambda j+2aj<k<\lambda j}} \Bigg) q^{\frac{1}{2}f_3f_4 + \frac{a}{4}f_4^2 + \frac{1}{2}j(i-\frac{a}{2}j)}.
\end{aligned}
$$

Proof. We start by rewriting the first three terms of equation (2.4). In fact, these three terms will combine to give the first term of the expression in the corollary. By using the substitutions $i = \Delta_1 + \Delta_3 + a\Delta_2$, $j = \Delta_2 + \Delta_4$, $k = \Delta_1 - \Delta_3 + a\Delta_2$ and $l = \Delta_2 - \Delta_4$, the first term of equation (2.4) can be rewritten as

$$-\sum_{\substack{i,j,k,l\in\mathbb{Z}\\ 2\mid f_3+i,\ 2\mid f_4+j\\ 2\mid i+k,\ 2\mid j+l\\ 0<\lambda j\leq i,\ -j<l<j\\ -\lambda j+a(j+l)<k<\lambda j}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}j(i-\frac{a}{2}j)}-\sum_{\substack{i,j,k,l\in\mathbb{Z}\\ 2\mid f_3+i,\ 2\mid f_4+j\\ 2\mid i+k,\ 2\mid j+l\\ 0<i<\lambda j,\ -\frac{i}{\lambda-a}+\frac{aj}{\lambda-a}<l<\lambda^{-1}i\\ -i+a(j+l)<k<i}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}j(i-\frac{a}{2}j)}.$$

Using the same substitutions, the second and third term of equation (2.4) reduce to

$$\sum_{\substack{i,j,k,l\in\mathbb{Z}\\ 2\mid f_3+i,\ 2\mid f_4+j\\ 2\mid i+k,\ 2\mid j+l\\ 0<i<\lambda j,\ -\frac{i}{\lambda-a}+\frac{aj}{\lambda-a}<l<\lambda^{-1}i\\ -i+a(j+l)<k<i}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}j(i-\frac{a}{2}j)}+\sum_{\substack{i,j,k,l\in\mathbb{Z}\\ 2\mid f_3+i,\ 2\mid f_4+j\\ 2\mid i+k,\ 2\mid j+l\\ 0<\lambda j<i,\ -j<l<j\\ -\lambda j+a(j+l)<k<\lambda j}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}j(i-\frac{a}{2}j)}.$$

Therefore the first three terms of equation (2.4) combine to give

$$-\sum_{\substack{i,j,k,l\in\mathbb{Z}\\ 2\mid f_3+i,\ 2\mid f_4+j\\ 2\mid i+k,\ 2\mid j+l\\ \lambda j=i,\ -j<l<j\\ -\lambda j+a(j+l)<k<\lambda j}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}j(i-\frac{a}{2}j)}.$$

We now prove the fourth to seventh terms of equation (2.4) combine to give terms two and three of the expression in the corollary. Using the substitutions $i=\Delta_1+\Delta_3-a\Delta_2$, $j=\Delta_2+\Delta_4$, $k=\Delta_1-\Delta_3-a\Delta_2$ and $l=-\Delta_2+\Delta_4$, the fourth term of equation (2.4) rewrites as

$$\sum_{\substack{i,j,k,l\in\mathbb{Z}\\ 2\mid f_3+i,\ 2\mid f_4+j\\ 2\mid i+k,\ 2\mid j+l\\ k<\lambda l<i,\ l<j\\ -i-a(j-l)<k,\ -\lambda j<k}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{4}ij-\frac{1}{4}jk+\frac{1}{4}il+\frac{1}{4}kl-\frac{a}{4}l^2}.$$

The fifth to seventh terms reduce in a similar way.

Finally, we claim the eighth to eleventh terms of equation (2.4) reduce to the fourth, fifth and sixth term of the expression in the corollary. Using the substitutions $i = \Delta_3 + a\Delta_2$, $j = \Delta_2 + \Delta_4$ and $k = \Delta_2 - \Delta_4$, the eighth term becomes

$$\sum_{\substack{i,j,k\in\mathbb{Z}\\ 2\mid f_3+i,\ 2\mid f_4+j\\ 2\mid j+k\\ i<\lambda j,\ \frac{a}{2}(j+k)<i\\ -\frac{i}{\lambda-a}+\frac{aj}{\lambda-a}<k<\lambda^{-1}i}} q^{\frac{1}{2}f_3f_4+\frac{a}{4}f_4^2+\frac{1}{2}j(i-\frac{a}{2}j)}.$$

The ninth to eleventh terms simplify similarly. $\square$

Specialising to $a = 0$ in Corollary 2.3.3 immediately yields the following result.

Corollary 2.3.4. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Let $H = \alpha D_1 + \beta D_2$ be an ample divisor, i.e. α, β are positive integers. Let $c_1 = f_3 D_3 + f_4 D_4 \in H^2(X, \mathbb{Z})$. Define $\lambda = \frac{\alpha}{\beta}$, then*

$$\prod_{k=1}^{\infty}(1-q^k)^8 \sum_{c_2\in\mathbb{Z}} e(M_X^H(2,c_1,c_2))q^{c_2} = -\sum_{\substack{i,j,k,l\in\mathbb{Z}\\ 2\mid f_3+i,\ 2\mid f_4+j\\ 2\mid i+k,\ 2\mid j+l\\ \lambda j=i,\ -j<l<j\\ -\lambda j<k<\lambda j}} q^{\frac{1}{2}f_3f_4+\frac{1}{2}ij}$$

$$+4 \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ 2 \mid f_3+i,\ 2 \mid f_4+j \\ 2 \mid i+k,\ 2 \mid j+l \\ k < \lambda l < i,\ l < j \\ -i < k,\ -\lambda j < k}} q^{\frac{1}{2}f_3 f_4 + \frac{1}{4}ij - \frac{1}{4}jk + \frac{1}{4}il + \frac{1}{4}kl} + 2 \sum_{\substack{i,j,k \in \mathbb{Z} \\ 2 \mid f_3+i,\ 2 \mid f_4+j \\ 2 \mid j+k,\ i < \lambda j \\ -\lambda^{-1} i < k < \lambda^{-1} i}} q^{\frac{1}{2}f_3 f_4 + \frac{1}{2}ij} + 2 \sum_{\substack{i,j,k \in \mathbb{Z} \\ 2 \mid f_3+i,\ 2 \mid f_4+j \\ 2 \mid i+k,\ \lambda j < i \\ -\lambda j < k < \lambda j}} q^{\frac{1}{2}f_3 f_4 + \frac{1}{2}ij}.$$

In [Got2], Göttsche derives an expression for generating functions of Hodge polynomials of moduli spaces of μ-stable torsion free sheaves of rank 2 on ruled surfaces X with $-K_X$ effective [Got2, Thm. 4.4]. Assume $X = \mathbb{F}_a$, where $a \in \mathbb{Z}_{\geq 0}$. Recall that X is naturally a ruled surface over $\mathbb{P}^1$ and $-K_X$ is effective. In particular, D_1 is a fibre and D_2 is a section. Let $c_1 = \epsilon D_1 + D_2$ ($\epsilon \in \{0,1\}$), H an ample divisor and $c_2 \in \mathbb{Z}$. Denote by $M_X^{H,ss}(2, c_1, c_2)$ the moduli space of Gieseker semistable (w.r.t. H) torsion free sheaves on X of rank 2 with first Chern class c_1 and second Chern class c_2. Note that in our case, $\mathrm{Num}(X) = \mathrm{Pic}(X)$. Göttsche and Qin have proved that the ample cone C_X in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ has a chamber/wall structure such that the moduli space $M_X^{H,ss}(2, c_1, c_2)$ stays constant while varying H in any fixed chamber of type (c_1, c_2) [Got2], [Qin]. In our current example, the non-empty walls of type (c_1, c_2) are precisely the sets

$$W^{\xi} = \{x \in \mathrm{Pic}(X) \text{ ample} \mid x \cdot \xi = 0\},$$

where $\xi = (2n+\epsilon)D_1 + (2m+1)D_2 \in \mathrm{Pic}(X)$ for any integers m, n satisfying $m \geq 0$, $n < 0$, $c_2 - m(m+1)a + (2m+1)n + m\epsilon \geq 0$ [Got2, Sect. 4]. By writing elements of $\mathbb{Q}_{>a}$ as $\frac{\alpha}{\beta}$ for $\alpha, \beta \in \mathbb{Z}_{>0}$ coprime, we can identify them with ample divisors $H = \alpha D_1 + \beta D_2$ on X with α, β coprime and without loss of generality we can restrict attention to these ample divisors. Let Λ be the set of elements in $\mathbb{Q}_{>a}$ which can be written as $\frac{\alpha}{\beta}$, where α, β are coprime positive integers such that $(2, c_1 \cdot H) = 1$. We denote the complement by $W = \mathbb{Q}_{>a} \setminus \Lambda$ and refer to W as the collection of walls[12]. The elements $\lambda \in \Lambda$

[12] The terminology "wall" in this context might be slightly confusing as W lies dense in $\mathbb{Q}_{>a}$ or can even be equal to $\mathbb{Q}_{>a}$.

have corresponding ample divisor H for which there are no strictly μ-semistable torsion free sheaves of rank 2 and first Chern class c_1 on X [HL, Lem. 1.2.13, 1.2.14], in which case $M_X^{H,ss}(2,c_1,c_2) = M_X^H(2,c_1,c_2)$ for any $c_2 \in \mathbb{Z}$. The elements of W are precisely the rational numbers corresponding to ample divisors lying on a wall of type (c_1,c_2) for some $c_2 \in \mathbb{Z}$. Let $H = \alpha D_1 + \beta D_2$ be an ample divisor, i.e. $\alpha, \beta \in \mathbb{Z}_{>0}$ such that $\alpha > a\beta$. Assume $(\alpha,\beta) = 1$ and define $\lambda = \frac{\alpha}{\beta}$. If H does not lie on a wall, in other words $\lambda \in \Lambda$ (i.e. $(2,(\alpha - a\beta) + \epsilon\beta) = 1$), then applying [Got2, Thm. 4.4] gives

$$\begin{aligned}
&\sum_{c_2 \in \mathbb{Z}} e(M_X^H(2,c_1,c_2))q^{c_2} = \\
&\frac{1}{\prod_{k=1}^\infty (1-q^k)^8} \sum_{(m,n) \in L(H)} [a + 2ma - 2(2m+2n+\epsilon+1)]q^{(m+1)ma-(2m+1)n-m\epsilon}, \qquad (2.5)\\
&L(H) := \left\{ (m,n) \in \mathbb{Z}^2 \mid m \geq 0,\ a - \lambda > \frac{2n+\epsilon}{2m+1} \right\}.
\end{aligned}$$

Although Göttsche's formula (2.5) is equal to the result in Corollary 2.3.3, it does not seem easy to obtain equality of both formulae by direct manipulations. However, it is instructive to make expansions of both expressions for various values of a, c_1, H with H not lying on a wall and compare the first few coefficients. One finds a perfect agreement.

We end by simplifying the expression in Corollary 2.3.4 in the case $\lambda = 1$ by splitting up inequalities and using geometric series. Note that by equation (2.3), there are only four interesting cases $(f_3,f_4) = (0,0), (0,1), (1,0), (1,1)$.

Corollary 2.3.5. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, $H = D_1 + D_2$ and $c_1 = f_3 D_3 + f_4 D_4 \in H^2(X,\mathbb{Z})$. Then:*

(i) *If $(f_3,f_4) = (0,0)$, then*

$$\begin{aligned}
\sum_{c_2 \in \mathbb{Z}} e(M_X^H(2,c_1,c_2))q^{c_2} = \frac{1}{\prod_{k=1}^\infty (1-q^k)^8} \Bigg(&-\sum_{m=1}^\infty (2m-1)^2 q^{2m^2} + \sum_{m=1}^\infty \frac{4(2m-1)q^{2m(m+1)}}{1-q^{2m}} \\
&+ \sum_{m=1}^\infty \sum_{n=1}^{2m} \frac{4q^{2m(m-n+2)+1}(q^{(2m+1)n} - q^{n^2})}{(1-q^n)(q^{2m+1} - q^{n-1})} \\
&+ \sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{p=1}^{2m} \frac{4q^{(2m+1)(m-p+2)+n-m}((q^{n+p})^p - (q^{n+p})^{(2m+1)})}{1-q^{n+p}} \Bigg)
\end{aligned}$$

$$= -q^2 - 8q^3 - 40q^4 - 160q^5 - 538q^6 - 1596q^7 - 4237q^8 - 10160q^9 - 21825q^{10} + O(q^{11}).$$

(ii) *If* $(f_3, f_4) = (1,0)$ *or* $(0,1)$, *then*

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(2,c_1,c_2))q^{c_2} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^8} \left(\sum_{m=1}^{\infty} \sum_{n=1}^{2m} \frac{4q^{(2m+3)m-2mn+1}(q^{(2m+1)n} - q^{n^2})}{(1-q^n)(q^{2m+1}-q^n)} \right.$$
$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{2m-1} \frac{4q^{(2m+1)m-2mp+1}((q^{n+p-1})^p - (q^{n+p-1})^{2m})}{q - q^{n+p}} + \sum_{m=1}^{\infty} \frac{2(2m-1)q^{(2m-1)m}}{1-q^{2m-1}}$$
$$\left. + \sum_{m=1}^{\infty} \frac{4mq^{(2m+1)m}}{1-q^{2m}} \right)$$
$$= 2q + 22q^2 + 146q^3 + 742q^4 + 3174q^5 + 11988q^6 + 41150q^7 + 130834q^8 + 390478q^9$$
$$+ 1104724q^{10} + O(q^{11}).$$

(iii) *If* $(f_3, f_4) = (1,1)$, *then*

$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(2,c_1,c_2))q^{c_2} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^8} \left(-\sum_{m=1}^{\infty} 4m^2 q^{2m(m+1)+1} + \sum_{m=1}^{\infty} \frac{8mq^{2(m+1)^2}}{1-q^{2m+1}} \right.$$
$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{2m-1} \frac{4q^{(2m+1)(m-n)+m+2n+1}(q^{2mn} - q^{n^2})}{(1-q^n)(q^{2m+1}-q^n)}$$
$$\left. + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{2m-1} \frac{4q^{(2m+1)(m-p)+m+n+p+1}((q^{n+p})^p - (q^{n+p})^{2m})}{1-q^{n+p}} \right)$$
$$= 4q^4 + 28q^5 + 152q^6 + 656q^7 + 2504q^8 + 8620q^9 + 27520q^{10} + O(q^{11}).$$

2.3.2.3 Wall-Crossing for Rank 2 on $\mathbb{F}_a$

So far we have been applying Theorem 2.2.7 to compute expressions for generating functions in examples. We can also use Theorem 2.2.7 to get expressions for wall-crossing formulae in examples. We start with a few simple definitions. Let $\mathbb{Z}((q))$ be the ring of formal Laurent series. It is clear that for all values $\lambda \in \mathbb{Q}_{>a}$ the six sums on the RHS in Corollary 2.3.3 are all formal Laurent series. Therefore the RHS in Corollary 2.3.3 defines a map $\mathbb{Q}_{>a} \longrightarrow \mathbb{Z}((q))$. We define the following notion of limit.

Definition 2.3.6. Let $a \in \mathbb{Z}_{\geq 0}$ and let $F : \mathbb{Q}_{>a} \longrightarrow \mathbb{Z}((q))$, $\lambda \mapsto F(\lambda)$ be a map. Let $\lambda_0 \in \mathbb{Q}_{>a}$ and let $F_0 \in \mathbb{Z}((q))$. We define

$$\lim_{\epsilon,\epsilon' \searrow 0} (F(\lambda_0 + \epsilon) - F(\lambda_0 - \epsilon')) = F_0,$$

to mean for any $N \in \mathbb{Z}$, there are $\epsilon, \epsilon' \in \mathbb{Q}_{>0}$ such that $a < \lambda_0 - \epsilon'$ and

$$F(\lambda_0 + \epsilon) - F(\lambda_0 - \epsilon') = F_0 + O(q^N).$$

Note that if the limit exists, it is unique. We refer to the expression

$$\lim_{\epsilon,\epsilon' \searrow 0} (F(\lambda_0 + \epsilon) - F(\lambda_0 - \epsilon')) = F_0,$$

as an *infinitesimal wall-crossing formula.* $\oslash$

By using this notion of limit and applying it to the four terms of the expression in Corollary 2.3.4, it is not difficult to derive the following result.

Corollary 2.3.7. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Let $H = \alpha_0 D_1 + \beta_0 D_2$ be an ample divisor, i.e. α_0, β_0 are positive integers and suppose $(\alpha_0, \beta_0) = 1$. Let $c_1 = f_3 D_3 + f_4 D_4 \in H^2(X, \mathbb{Z})$. Define $\lambda_0 = \frac{\alpha_0}{\beta_0}$, then*

$$\prod_{k=1}^{\infty}(1-q^k)^8 \lim_{\epsilon,\epsilon' \searrow 0} \left(\sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0+\epsilon}(2,c_1,c_2))q^{c_2} - \sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0-\epsilon'}(2,c_1,c_2))q^{c_2} \right)$$
$$= 4 \sum_{\substack{i,j,k \in \mathbb{Z},\ \beta_0 \mid k \\ 2 \mid f_3 + i,\ 2 \mid f_4 + j \\ 2 \mid i + \lambda_0 k,\ 2 \mid j + k \\ 0 < \lambda_0 k < i,\ 0 < k < j}} q^{\frac{1}{2}f_3 f_4 + \frac{1}{4}ij - \frac{\lambda_0}{4}jk + \frac{1}{4}ik + \frac{\lambda_0}{4}k^2} - 4 \sum_{\substack{i,j,k \in \mathbb{Z},\ \beta_0 \mid k \\ 2 \mid f_3 + i,\ 2 \mid f_4 + j \\ 2 \mid i + \lambda_0 k,\ 2 \mid j + k \\ -i < \lambda_0 k < 0,\ -j < k < 0}} q^{\frac{1}{2}f_3 f_4 + \frac{1}{4}ij - \frac{\lambda_0}{4}jk + \frac{1}{4}ik + \frac{\lambda_0}{4}k^2}$$

$$- 4 \sum_{\substack{i,j,k \in \mathbb{Z},\ \beta_0 \mid k \\ 2 \mid f_3 + \lambda_0 k,\ 2 \mid f_4 + j \\ 2 \mid i + \lambda_0 k,\ 2 \mid j + k \\ -\lambda_0 k < i < \lambda_0 k,\ k < -j}} q^{\frac{1}{2} f_3 f_4 + \frac{1}{4} ij - \frac{\lambda_0}{4} jk + \frac{1}{4} ik + \frac{\lambda_0}{4} k^2} + 4 \sum_{\substack{i,j,k \in \mathbb{Z},\ \beta_0 \mid k \\ 2 \mid f_3 + i,\ 2 \mid f_4 + k \\ 2 \mid i + \lambda_0 k,\ 2 \mid j + k \\ -k < j < k,\ \lambda_0 k < i}} q^{\frac{1}{2} f_3 f_4 + \frac{1}{4} ij - \frac{\lambda_0}{4} jk + \frac{1}{4} ik + \frac{\lambda_0}{4} k^2}$$

$$+ 2 \sum_{\substack{i,j \in \mathbb{Z},\ \beta_0 \mid i \\ 2 \mid f_3 + \lambda_0 i,\ 2 \mid f_4 + i \\ 2 \mid i + j,\ -i < j < i}} q^{\frac{1}{2} f_3 f_4 + \frac{\lambda_0}{2} i^2} - 4 \sum_{\substack{i,j \in \mathbb{Z},\ \beta_0 \mid j \\ 2 \mid f_3 + \lambda_0 j,\ 2 \mid f_4 + i \\ 2 \mid i + j,\ 0 < j < i}} q^{\frac{1}{2} f_3 f_4 + \frac{\lambda_0}{2} ij}$$

$$- 2 \sum_{\substack{i,j \in \mathbb{Z},\ \alpha_0 \mid i \\ 2 \mid f_4 + \lambda_0^{-1} i,\ 2 \mid f_3 + i \\ 2 \mid i + j,\ -i < j < i}} q^{\frac{1}{2} f_3 f_4 + \frac{\lambda_0^{-1}}{2} i^2} + 4 \sum_{\substack{i,j \in \mathbb{Z},\ \alpha_0 \mid j \\ 2 \mid f_4 + \lambda_0^{-1} j,\ 2 \mid f_3 + i \\ 2 \mid i + j,\ 0 < j < i}} q^{\frac{1}{2} f_3 f_4 + \frac{\lambda_0^{-1}}{2} ij}.$$

Roughly, the formula of the previous corollary is obtained by considering all possible ways of changing in a term of the formula in Corollary 2.3.4 one or more inequalities containing λ into equalities and summing these modified terms with appropriate signs. Note that the expression is only possibly non-zero in the case $2 \mid \alpha_0 f_4 + \beta_0 f_3$ or equivalently $(2, c_1 \cdot H) \neq 1$, i.e. H lies on a wall. It is easy to derive a nice infinitesimal wall-crossing formula from Göttsche's formula (2.5). Let $X = \mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$), $c_1 = \epsilon D_1 + D_2$ ($\epsilon \in \{0,1\}$) and $H = \alpha_0 D_1 + \beta_0 D_2$ an ample divisor, i.e. $\alpha_0, \beta_0 \in \mathbb{Z}_{>0}$ such that $\alpha_0 > a\beta_0$. Assume $(\alpha_0, \beta_0) = 1$ and define $\lambda_0 = \frac{\alpha_0}{\beta_0}$. Using Definition 2.3.6, one immediately obtains

$$\lim_{\epsilon,\epsilon' \searrow 0} \left(\sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0 + \epsilon}(2, c_1, c_2)) q^{c_2} - \sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0 - \epsilon'}(2, c_1, c_2)) q^{c_2} \right)$$
$$= \frac{1}{\prod_{k=1}^{\infty} (1 - q^k)^8} \sum_{\substack{m \in \mathbb{Z}_{\geq 1} \\ \frac{1}{2}(\lambda_0 - a)(2m-1) - \frac{1}{2}\epsilon \in \mathbb{Z}}} 2 \left(1 + \frac{a}{2} - \lambda_0 \right) (2m-1)\, q^{\frac{1}{2}(\lambda_0 - \frac{a}{2})(2m-1)^2 - \frac{1}{4} a + \frac{1}{2}\epsilon}. \tag{2.6}$$

A priori we derive the above formula for $\lambda_0 \in \Lambda$, i.e. H not lying on a wall, in which case there are no strictly μ-semistables and the result is 0. However, equation (2.6) holds for any $\lambda_0 \in \mathbb{Q}_{>a}$, because $\Lambda \subset \mathbb{Q}_{>a}$ lies dense in $\mathbb{Q}_{>a}$.

We can also derive equation (2.6) using Joyce's machinery for wall-crossing of motivic invariants counting (semi)stable objects [Joy2]. Joyce gives a wall-crossing formula for virtual Poincaré polynomials of moduli spaces of Gieseker semistable torsion free sheaves on an arbitrary nonsingular projective surface X with $-K_X$ nef [Joy2, Thm. 6.21][13]. The surfaces $X = \mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$) with anticanonical divisor nef are precisely $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$, $\mathbb{F}_2$ (as can be shown by an easy computation using [Ful, Sect. 4.3] and the Nakai–Moishezon Criterion [Har1, Thm. A.5.1]). However, in our computations, we will keep $a \in \mathbb{Z}_{\geq 0}$ arbitrary. Let $c_1 = f_3 D_3 + f_4 D_4 \in H^2(X, \mathbb{Z})$ and $H = \alpha_0 D_1 + \beta_0 D_2$ be a choice of ample divisor. We take $(\alpha_0, \beta_0) = 1$. Part of Joyce's philosophy is that one should study wall-crossing phenomena for motivic invariants of moduli spaces of (semi)stable objects, where the moduli spaces should be constructed as Artin stacks instead of schemes coming from a GIT construction. Keeping track of the stabilisers of (semi)stable objects will enable one to derive nice wall-crossing formulae. Nevertheless, for the purposes of this thesis, we want to study wall-crossing phenomena of Euler characteristics of moduli spaces of stable objects defined as schemes coming from a GIT construction. Hence, in order to use Joyce's theory for our purposes, we need the following formula. For any nonsingular projective surface X, ample divisor H on X, $r \in \mathbb{Z}_{>0}$, $c_1 \in H^2(X, \mathbb{Z})$ and $c_2 \in \mathbb{Z}$

$$e(M_X^H(r, c_1, c_2)) = \lim_{z \to -1} \left((z^2 - 1) P(\mathrm{Obj}_s^{(r, c_1, \frac{1}{2}(c_1 - 2c_2))}(\mu), z) \right). \tag{2.7}$$

Here P is the virtual Poincaré polynomial (see subsection 2.1.1) and $\mathrm{Obj}_s^{(r, c_1, \frac{1}{2}(c_1 - 2c_2))}(\mu)$ the Artin stack of μ-stable torsion free sheaves on X of rank r, first Chern class c_1 and second Chern class c_2 (notation of [Joy2]). Joyce proves that one can uniquely extend the definition of virtual Poincaré polynomial to Artin stacks of finite type over $\mathbb{C}$ with affine geometric stabilisers requiring that for any special algebraic group G acting regularly on a quasi-projective variety Y one has $P([Y/G], z) = P(Y, z)/P(G, z)$ [Joy1, Thm. 4.10]. Equation (2.7) can be proved as follows. Recall that $M_X^H(r, c_1, c_2)$ is constructed as the

[13]Note that the cited theorem also holds for slope stability instead of Gieseker stability.

geometric quotient $\varpi : R^s \longrightarrow M_X^H(r, c_1, c_2)$ of some open subset of the Quot scheme with a regular action of some $\mathrm{PGL}(n, \mathbb{C})$. In fact, ϖ is a principal $\mathrm{PGL}(n, \mathbb{C})$-bundle [HL, Cor. 4.3.5] and we have isomorphisms of stacks [Gom, Prop. 3.3]

$$M_X^H(r, c_1, c_2) \cong [R^s/\mathrm{PGL}(n, \mathbb{C})], \ \mathrm{Obj}_s^{(r, c_1, \frac{1}{2}(c_1 - 2c_2))}(\mu) \cong [R^s/\mathrm{GL}(n, \mathbb{C})].$$

Now the difficulty is that $\mathrm{PGL}(n, \mathbb{C})$ is in general *not* special. Define $\mathrm{P}((\mathbb{C}^*)^n) = (\mathbb{C}^*)^n/\mathbb{C}^* \cdot \mathrm{id}$ and consider the geometric quotient $R^s/\mathrm{P}((\mathbb{C}^*)^n)$. This gives a morphism $R^s/\mathrm{P}((\mathbb{C}^*)^n) \longrightarrow R^s/\mathrm{PGL}(n, \mathbb{C})$, where all fibres on closed points are isomorphic to $F = \mathrm{PGL}(n, \mathbb{C})/\mathrm{P}((\mathbb{C}^*)^n)$. We deduce

$$\begin{aligned} e\left(M_X^H(r, c_1, c_2)\right) &= \frac{e\left(R^s/\mathrm{P}((\mathbb{C}^*)^n)\right)}{e(F)} = \frac{e\left(R^s/\mathrm{P}((\mathbb{C}^*)^n)\right)}{n!} = \lim_{z \to -1} \frac{P(R^s, z)}{n!(z^2-1)^{n-1}} \\ &= \lim_{z \to -1} \frac{(z^2-1)P(R^s, z)}{P(\mathrm{GL}(n, \mathbb{C}), z)} \cdot \frac{(z^2)^{\frac{n(n-1)}{2}} \prod_{k=1}^n ((z^2)^k - 1)}{n!(z^2-1)^n} = \lim_{z \to -1} (z^2-1) P([R^s/\mathrm{GL}(n, \mathbb{C})], z), \end{aligned}$$

where we apply [Joy2, Thm. 2.4], [Joy1, Lem. 4.6, Thm. 4.10] and we use the limit $\lim_{z \to -1} \frac{(z^2)^{\frac{n(n-1)}{2}} \prod_{k=1}^n ((z^2)^k - 1)}{(z^2-1)^n} = n!$. This proves formula (2.7). Combining equation (2.7), the generating function for the rank 1 case (Corollary 2.3.1) and [Joy2, Thm. 6.21] gives

$$\begin{aligned} &\lim_{\epsilon, \epsilon' \searrow 0} \left(\sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0 + \epsilon}(2, c_1, c_2)) q^{c_2} - \sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0 - \epsilon'}(2, c_1, c_2)) q^{c_2} \right) \qquad (2.8) \\ &= \frac{1}{\prod_{k=1}^\infty (1-q^k)^8} \sum_{\substack{m \in \mathbb{Z}_{> \frac{1}{2} f_4} \\ \frac{1}{2}(\lambda_0 - a)(2m - f_4) - \frac{1}{2}(f_3 + af_4) \in \mathbb{Z}}} 2\left(1 + \frac{a}{2} - \lambda_0\right)(2m - f_4)\, q^{\frac{1}{2}(\lambda_0 - \frac{a}{2})(2m - f_4)^2 - \frac{1}{4} a f_4^2 + \frac{1}{2}(f_3 + af_4) f_4}. \end{aligned}$$

The computation is slightly tedious and uses the Bogomolov Inequality [HL, Thm. 3.4.1] to show that the limit exists and equation (2.3) to split off the rank 1 contributions. Note that [Joy2, Thm. 6.21] is a wall-crossing formula for Artin stacks of *semi*stable objects, whereas we have been dealing with Artin stacks of stable objects only. However, we claim equation (2.8) holds for any $\lambda_0 \in \mathbb{Q}_{>a}$. If not both $f_3 \equiv 0 \bmod 2$ and $f_4 \equiv 0 \bmod 2$, then $\Lambda \subset \mathbb{Q}_{>a}$ is dense. Since we know there are no strictly μ-semistables for ample divisors in Λ, we see equation (2.8) holds in these cases. The case $f_3 \equiv f_4 \equiv 0 \mod 2$ is harder to see, because this time $\Lambda = \varnothing$. Therefore we consider the following more general argument to prove this case. Let $\mathcal{E}$ be a rank 2 torsion free sheaf on $X = \mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$) with arbitrary first Chern class c_1 and second Chern class c_2. Let H, H' be two ample divisors not lying on a wall of type (c_1, c_2). Then $\mathcal{E}$ is strictly μ-semistable w.r.t. H if and only if $\mathcal{E}$ is strictly μ-semistable w.r.t. H' (compare [Got2, Thm. 2.9]). This can be seen as follows. Suppose $\mathcal{E}$ is strictly μ-semistable w.r.t. H, then there is a saturated coherent subsheaf $\mathcal{F}_1 \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}_1) < \mathrm{rk}(\mathcal{E})$ such that $\mu^H_{\mathcal{F}_1} = \mu^H_{\mathcal{E}}$. Denote the quotient by $\mathcal{F}_2$, then $\mu^H_{\mathcal{F}_1} = \mu^H_{\mathcal{E}} = \mu^H_{\mathcal{F}_2}$. Since H is not lying on a wall, we have $c_1(\mathcal{F}_1) = c_1(\mathcal{F}_2)$ so in particular $\mu^{H'}_{\mathcal{F}_1} = \mu^{H'}_{\mathcal{E}} = \mu^{H'}_{\mathcal{F}_2}$. Since $\mathcal{F}_1, \mathcal{F}_2$ have rank 1, they are automatically μ-stable and using [HL, Prop. 1.2.7] it is not difficult to see $\mathcal{E}$ has to be μ-semistable w.r.t. H'. Therefore $\mathrm{Obj}_{ss}^{(2,c_1,\frac{1}{2}(c_1-2c_2))}(\mu) \setminus \mathrm{Obj}_{s}^{(2,c_1,\frac{1}{2}(c_1-2c_2))}(\mu)$ is the same for any ample divisor not on a wall of type (c_1, c_2) as desired. Note that equations (2.6) and (2.8) are consistent. In fact, they are even consistent in the case $a > 2$ suggesting [Joy2, Thm. 6.21] holds more generally.

Although we have now proved equation (2.8) to coincide with the expression in Corollary 2.3.7 in the case $a = 0$, it seems difficult to prove equality directly by manipulation of the formulae. It is instructive to make expansions up to a fixed order for specific values of c_1, λ_0 and verify the coefficients of the expansion are the same. More generally, we know equation (2.6) coincides with infinitesimal wall-crossing of the expression in Corollary 2.3.3. Various numerical experiments by making expansions up to a fixed order again show consistency. In order to give an idea of the kind of expressions one obtains from Corollary 2.3.7, we compute the cases $\lambda_0 = \frac{1}{2}$, $\lambda_0 = 1$ and $\lambda_0 = 2$.

Corollary 2.3.8. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Let $H = \alpha_0 D_1 + \beta_0 D_2$ be an ample divisor, i.e. α_0, β_0 are positive integers. Assume $(\alpha_0, \beta_0) = 1$ and let $c_1 = f_3 D_3 + f_4 D_4 \in H^2(X, \mathbb{Z})$. Define $\lambda_0 = \frac{\alpha_0}{\beta_0}$, then:*

(i) *If $\lambda_0 = \frac{1}{2}$ and $(f_3, f_4) = (0, 0)$, then*

$$\lim_{\epsilon,\epsilon'\searrow 0}\left(\sum_{c_2\in\mathbb{Z}} e(M_X^{\lambda_0+\epsilon}(2,c_1,c_2))q^{c_2} - \sum_{c_2\in\mathbb{Z}} e(M_X^{\lambda_0-\epsilon'}(2,c_1,c_2))q^{c_2}\right)$$
$$= \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^8}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4q^{4m(m+1)+n}}{1-q^{4m+n}} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{-4q^{2m(2m+n)+n}}{1-q^n}\right.$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4q^{2m(2m+n)}(1-q^{2mn})}{1-q^n} + \sum_{m=1}^{\infty}4mq^{4m^2} + \sum_{m=1}^{\infty}\frac{-4q^{2m(2m+1)}}{1-q^{2m}}$$
$$\left.+\sum_{m=1}^{\infty}\frac{4q^{4m(m+1)}}{1-q^{4m}}\right)$$
$$= 4q^4 + 32q^5 + 176q^6 + 768q^7 + 2904q^8 + 9856q^9 + 30816q^{10} + O(q^{11}).$$

(ii) *If* $\lambda_0 = \frac{1}{2}$ *and* $(f_3, f_4) = (1, 0)$, *then*

$$\lim_{\epsilon,\epsilon'\searrow 0}\left(\sum_{c_2\in\mathbb{Z}} e(M_X^{\lambda_0+\epsilon}(2,c_1,c_2))q^{c_2} - \sum_{c_2\in\mathbb{Z}} e(M_X^{\lambda_0-\epsilon'}(2,c_1,c_2))q^{c_2}\right)$$
$$= \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^8}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4q^{4m^2+n+1}}{q^2-q^{4m+n}} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{-4q^{(2m-1)^2+2mn}}{1-q^n}\right.$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{-4q^{(2m+1)^2+n}(1-q^{2mn})}{1-q^n} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4q^{(2m-1)^2+n}(1-q^{(4m-3)n})}{1-q^n}$$
$$\left.+\sum_{m=1}^{\infty}2(2m-1)q^{(2m-1)^2} + \sum_{m=1}^{\infty}\frac{-4q^{4m^2-2m+1}}{q-q^{2m}} + \sum_{m=1}^{\infty}\frac{4q^{4m^2+1}}{q^2-q^{4m}}\right)$$
$$= 2q + 16q^2 + 88q^3 + 384q^4 + 1452q^5 + 4928q^6 + 15408q^7 + 45056q^8 + 124680q^9$$
$$+ 329168q^{10} + O(q^{11}).$$

(iii) *If* $\lambda_0 = \frac{1}{2}$ *and* $(f_3, f_4) = (0, 1)$ *or* $(1, 1)$, *then*

$$\lim_{\epsilon,\epsilon'\searrow 0}\left(\sum_{c_2\in\mathbb{Z}} e(M_X^{\lambda_0+\epsilon}(2,c_1,c_2))q^{c_2} - \sum_{c_2\in\mathbb{Z}} e(M_X^{\lambda_0-\epsilon'}(2,c_1,c_2))q^{c_2}\right) = 0.$$

(iv) *If* $\lambda_0 = 1$ *and* $(f_3, f_4) = (0, 0), (1, 0), (0, 1)$ *or* $(1, 1)$, *then*

$$\lim_{\epsilon,\epsilon'\searrow 0}\left(\sum_{c_2\in\mathbb{Z}} e(M_X^{\lambda_0+\epsilon}(2,c_1,c_2))q^{c_2} - \sum_{c_2\in\mathbb{Z}} e(M_X^{\lambda_0-\epsilon'}(2,c_1,c_2))q^{c_2}\right) = 0.$$

(v) *If* $\lambda_0 = 2$ *and* $(f_3, f_4) = (0,0)$*, then*

$$\lim_{\epsilon,\epsilon' \searrow 0} \left(\sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0+\epsilon}(2,c_1,c_2))q^{c_2} - \sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0-\epsilon'}(2,c_1,c_2))q^{c_2} \right)$$
$$= -\left(\text{formula for } \lambda_0 = \frac{1}{2} \text{ and } (f_3,f_4) = (0,0) \right).$$

(vi) *If* $\lambda_0 = 2$ *and* $(f_3, f_4) = (0,1)$*, then*

$$\lim_{\epsilon,\epsilon' \searrow 0} \left(\sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0+\epsilon}(2,c_1,c_2))q^{c_2} - \sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0-\epsilon'}(2,c_1,c_2))q^{c_2} \right)$$
$$= -\left(\text{formula for } \lambda_0 = \frac{1}{2} \text{ and } (f_3,f_4) = (1,0) \right).$$

(vii) *If* $\lambda_0 = 2$ *and* $(f_3, f_4) = (1,0)$ *or* $(1,1)$*, then*

$$\lim_{\epsilon,\epsilon' \searrow 0} \left(\sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0+\epsilon}(2,c_1,c_2))q^{c_2} - \sum_{c_2 \in \mathbb{Z}} e(M_X^{\lambda_0-\epsilon'}(2,c_1,c_2))q^{c_2} \right) = 0.$$

Referring to the first three cases and last three cases of the previous corollary, we note that changing $\lambda_0 \leftrightarrow \frac{1}{\lambda_0}$ and $(f_3, f_4) \leftrightarrow (f_4, f_3)$ indeed changes the expression of the infinitesimal wall-crossing formula by a sign as a priori expected.

2.3.3 Rank 3 on $\mathbb{P}^2$

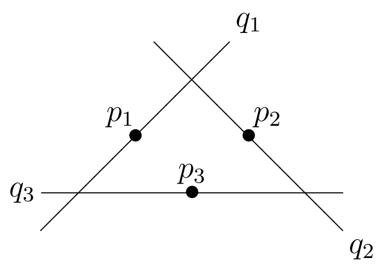
In this subsection, we consider Theorem 2.2.7 for the case[14] rank $r = 3$ and $X = \mathbb{P}^2$. Similar computations can be done in the case $X = \mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$), but the computations become very lengthy.

In the case $X = \mathbb{P}^2$, the expression in Theorem 2.2.7 does not depend on the choice of ample divisor, so we take ample divisor $H = D$ (see 2.3.2.1). Let $c_1 = f_3 D_3 = fD \in H^2(X, \mathbb{Z})$. Define $\Delta_i = \Delta_i(1)$, $\Gamma_i = \Delta_i(2)$ for each $i = 1,2,3$. If $\Delta_1, \Delta_2, \Delta_3, \Gamma_1, \Gamma_2, \Gamma_3 \in$

[14]During the finishing of subsections 2.1–2.3, the author found out about recent independent work of Weist [Wei], where he also computes the case rank 3 and $X = \mathbb{P}^2$ using techniques of toric geometry and quivers. Weist has communicated to the author that his results are consistent with Corollary 2.3.9 of this subsection.

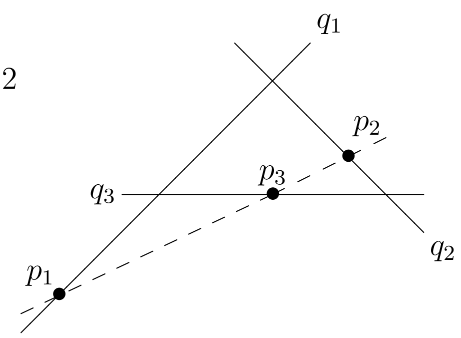
$\mathbb{Z}_{>0}$ and $\vec{\delta} = (\delta_1, \delta_2, \delta_3)$ are displays of widths $(\Delta_1, \Gamma_1; \Delta_2, \Gamma_2)$, $(\Delta_2, \Gamma_2; \Delta_3, \Gamma_3)$ and $(\Delta_3, \Gamma_3; \Delta_1, \Gamma_1)$, then $\mathcal{D}_{\vec{\delta}} \subset \{(p_1, p_2, p_3; q_1, q_2, q_3) \mid p_i \subset q_i \ \forall i = 1, 2, 3\} \subset \mathrm{Gr}(1,3)^3 \times \mathrm{Gr}(2,3)^3 = (\mathbb{P}^2)^3 \times (\mathbb{P}^{2\vee})^3$. We also consider all degenerations, e.g. $\Delta_1, \Delta_2, \Delta_3, \Gamma_1, \Gamma_2 \in \mathbb{Z}_{>0}$, $\Gamma_3 = 0$ and $\vec{\delta} = (\delta_1, \delta_2, \delta_3)$ displays with widths $(\Delta_1, \Gamma_1; \Delta_2, \Gamma_2)$, $(\Delta_2, \Gamma_2; \Delta_3, \Gamma_3)$ and $(\Delta_3, \Gamma_3; \Delta_1, \Gamma_1)$, in which case $\mathcal{D}_{\vec{\delta}} \subset \{(p_1, p_2, p_3; q_1, q_2) \mid p_i \subset q_i \ \forall i = 1, 2\} \subset (\mathbb{P}^2)^3 \times (\mathbb{P}^{2\vee})^2$. In a similar way as for the derivations of equations (2.2), (2.4) in 2.3.2.1, 2.3.2.2, let us describe the incidence spaces which contribute to the expression in Theorem 2.2.7. All other incidence spaces can easily seen to never have properly GIT stable closed points (for ample equivariant line bundles as in subsection 2.2.2). Denote by:

incidence space 1

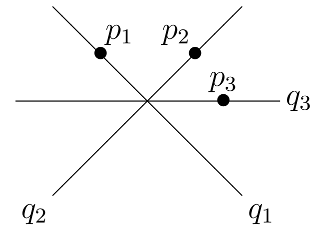


the incidence space of three lines q_1, q_2, q_3 in $\mathbb{P}^2$ and three points $p_1 \in q_1$, $p_2 \in q_2$, $p_3 \in q_3$ on those lines such that q_1, q_2, q_3 are mutually distinct, their intersection points $q_1 \cap q_2$, $q_2 \cap q_3$, $q_3 \cap q_1$ are mutually distinct, p_1, p_2, p_3 are not equal to $q_1 \cap q_2$, $q_2 \cap q_3$, $q_3 \cap q_1$ and p_1, p_2, p_3 are not colinear. This is a locally closed subscheme of $(\mathbb{P}^2)^3 \times (\mathbb{P}^{2\vee})^3$. Likewise, we introduce the incidence spaces:

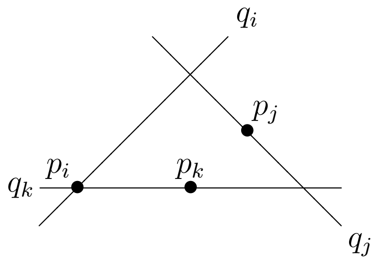
incidence space 2



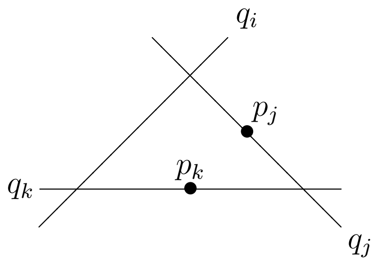
incidence space 3



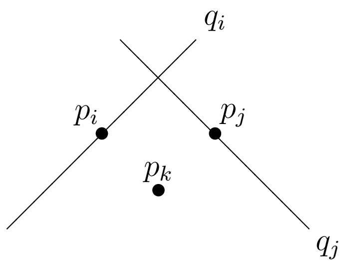
incidence spaces 4–9



incidence spaces $10, 11, 12$



incidence spaces $13, 14, 15$



for all $\{i, j, k\} = \{1, 2, 3\}$, where for the first space the points p_1, p_2, p_3 are colinear, as indicated by the dashed line, and for the second and last space p_1, p_2, p_3 are not colinear. Take one of these incidence spaces. Suppose we have an ample equivariant line bundle as in subsection 2.2.2 such that all closed points of the incidence space are properly GIT stable w.r.t. this ample equivariant line bundle and we form the geometric quotient by $\mathrm{SL}(3, \mathbb{C})$. The resulting Euler characteristics of the geometric quotients are $e = -1$ for the first incidence space and $e = 1$ for the remaining incidence spaces. Here it is useful to note that any four distinct points x_1, x_2, x_3, x_4 in the projective plane no three of which are colinear can be mapped to respectively $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, $[1 : 1 : 1]$ by an element of $\mathrm{SL}(3, \mathbb{C})$ and this element is unique up to multiplication by a 3rd root of unity. The incidence spaces 4–9 all give the same contribution to the expression of

Theorem 2.2.7. This also holds for the incidence spaces 10, 11, 12 as well as the incidence spaces 13, 14, 15. As an aside, note that the first three incidence spaces all give rise to the same display. We deduce[15]

$$
\begin{aligned}
&q^{-\frac{1}{2}f^2}\prod_{k=1}^{\infty}(1-q^k)^9\sum_{c_2\in\mathbb{Z}} e(M_X^H(3,c_1,c_2))q^{c_2} = \\
&- \sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Gamma_1,\Gamma_2,\Gamma_3\in\mathbb{Z}_{>0} \qquad 3\,\mid\, -f+\Delta_1+\Delta_2+\Delta_3+2\Gamma_1+2\Gamma_2+2\Gamma_3 \\
\Delta_1+2\Gamma_1<2\Delta_2+2\Delta_3+\Gamma_2+\Gamma_3 \qquad \Delta_1+\Delta_2<2\Delta_3+\Gamma_1+\Gamma_2+\Gamma_3 \\
\Delta_2+2\Gamma_2<2\Delta_1+2\Delta_3+\Gamma_1+\Gamma_3 \qquad \Delta_2+\Delta_3<2\Delta_1+\Gamma_1+\Gamma_2+\Gamma_3 \\
\Delta_3+2\Gamma_3<2\Delta_1+2\Delta_2+\Gamma_1+\Gamma_2 \qquad \Delta_1+\Delta_3<2\Delta_2+\Gamma_1+\Gamma_2+\Gamma_3 \\
\Gamma_1+2\Delta_1<2\Gamma_2+2\Gamma_3+\Delta_2+\Delta_3 \qquad \Gamma_1+\Gamma_2<2\Gamma_3+\Delta_1+\Delta_2+\Delta_3 \\
\Gamma_2+2\Delta_2<2\Gamma_1+2\Gamma_3+\Delta_1+\Delta_3 \qquad \Gamma_2+\Gamma_3<2\Gamma_1+\Delta_1+\Delta_2+\Delta_3 \\
\Gamma_3+2\Delta_3<2\Gamma_1+2\Gamma_2+\Delta_1+\Delta_2 \qquad \Gamma_1+\Gamma_3<2\Gamma_2+\Delta_1+\Delta_2+\Delta_3}}
q^{\begin{array}{r}
-\frac{1}{18}(-f-2\Delta_1-2\Delta_2-2\Delta_3-\Gamma_1-\Gamma_2-\Gamma_3)^2 \\
-\frac{1}{18}(-f+\Delta_1+\Delta_2+\Delta_3-\Gamma_1-\Gamma_2-\Gamma_3)^2 \\
-\frac{1}{18}(-f+\Delta_1+\Delta_2+\Delta_3+2\Gamma_1+2\Gamma_2+2\Gamma_3)^2 \\
+\Gamma_1\Gamma_2+\Gamma_2\Gamma_3+\Gamma_1\Gamma_3+\Delta_1\Gamma_2+\Delta_2\Gamma_1+\Delta_1\Delta_2 \\
+\Delta_2\Gamma_3+\Delta_3\Gamma_2+\Delta_2\Delta_3+\Delta_1\Gamma_3+\Delta_3\Gamma_1+\Delta_1\Delta_3
\end{array}} \\
&+ \sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Gamma_1,\Gamma_2,\Gamma_3\in\mathbb{Z}_{>0} \qquad 3\,\mid\, -f+\Delta_1+\Delta_2+\Delta_3+2\Gamma_1+2\Gamma_2+2\Gamma_3 \\
\Delta_1+2\Gamma_1<2\Delta_2+2\Delta_3+\Gamma_2+\Gamma_3 \qquad \Gamma_3+2\Delta_3<2\Gamma_1+2\Gamma_2+\Delta_1+\Delta_2 \\
\Delta_2+2\Gamma_2<2\Delta_1+2\Delta_3+\Gamma_1+\Gamma_3 \qquad \Delta_1+\Delta_2+\Delta_3<\Gamma_1+\Gamma_2+\Gamma_3 \\
\Delta_3+2\Gamma_3<2\Delta_1+2\Delta_2+\Gamma_1+\Gamma_2 \qquad \Gamma_1+\Gamma_2<2\Gamma_3+\Delta_1+\Delta_2+\Delta_3 \\
\Gamma_1+2\Delta_1<2\Gamma_2+2\Gamma_3+\Delta_2+\Delta_3 \qquad \Gamma_1+\Gamma_3<2\Gamma_2+\Delta_1+\Delta_2+\Delta_3 \\
\Gamma_2+2\Delta_2<2\Gamma_1+2\Gamma_3+\Delta_1+\Delta_3 \qquad \Gamma_2+\Gamma_3<2\Gamma_1+\Delta_1+\Delta_2+\Delta_3}}
q^{\begin{array}{r}
-\frac{1}{18}(-f-2\Delta_1-2\Delta_2-2\Delta_3-\Gamma_1-\Gamma_2-\Gamma_3)^2 \\
-\frac{1}{18}(-f+\Delta_1+\Delta_2+\Delta_3-\Gamma_1-\Gamma_2-\Gamma_3)^2 \\
-\frac{1}{18}(-f+\Delta_1+\Delta_2+\Delta_3+2\Gamma_1+2\Gamma_2+2\Gamma_3)^2 \\
+\Gamma_1\Gamma_2+\Gamma_2\Gamma_3+\Gamma_1\Gamma_3+\Delta_1\Gamma_2+\Delta_2\Gamma_1+\Delta_1\Delta_2 \\
+\Delta_2\Gamma_3+\Delta_3\Gamma_2+\Delta_2\Delta_3+\Delta_1\Gamma_3+\Delta_3\Gamma_1+\Delta_1\Delta_3
\end{array}}
\end{aligned}
$$

[15] Do not be confused by the number of inequalities over which we sum or the number of terms in the powers of q.

$$
+ \sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Gamma_1,\Gamma_2,\Gamma_3 \in \mathbb{Z}_{>0} \qquad 3 \mid -f+\Delta_1+\Delta_2+\Delta_3+2\Gamma_1+2\Gamma_2+2\Gamma_3 \\ \Delta_1+2\Gamma_1 < 2\Delta_2+2\Delta_3+\Gamma_2+\Gamma_3 \qquad \Gamma_3+2\Delta_3 < 2\Gamma_1+2\Gamma_2+\Delta_1+\Delta_2 \\ \Delta_2+2\Gamma_2 < 2\Delta_1+2\Delta_3+\Gamma_1+\Gamma_3 \qquad \Delta_1+\Delta_2 < 2\Delta_3+\Gamma_1+\Gamma_2+\Gamma_3 \\ \Delta_3+2\Gamma_3 < 2\Delta_1+2\Delta_2+\Gamma_1+\Gamma_2 \qquad \Delta_2+\Delta_3 < 2\Delta_1+\Gamma_1+\Gamma_2+\Gamma_3 \\ \Gamma_1+2\Delta_1 < 2\Gamma_2+2\Gamma_3+\Delta_2+\Delta_3 \qquad \Delta_1+\Delta_3 < 2\Delta_2+\Gamma_1+\Gamma_2+\Gamma_3 \\ \Gamma_2+2\Delta_2 < 2\Gamma_1+2\Gamma_3+\Delta_1+\Delta_3 \qquad \Gamma_1+\Gamma_2+\Gamma_3 < \Delta_1+\Delta_2+\Delta_3}} q^{\begin{array}{c} -\frac{1}{18}(-f-2\Delta_1-2\Delta_2-2\Delta_3-\Gamma_1-\Gamma_2-\Gamma_3)^2 \\ -\frac{1}{18}(-f+\Delta_1+\Delta_2+\Delta_3-\Gamma_1-\Gamma_2-\Gamma_3)^2 \\ -\frac{1}{18}(-f+\Delta_1+\Delta_2+\Delta_3+2\Gamma_1+2\Gamma_2+2\Gamma_3)^2 \\ +\Gamma_1\Gamma_2+\Gamma_2\Gamma_3+\Gamma_1\Gamma_3+\Delta_1\Gamma_2+\Delta_2\Gamma_1+\Delta_1\Delta_2 \\ +\Delta_2\Gamma_3+\Delta_3\Gamma_2+\Delta_2\Delta_3+\Delta_1\Gamma_3+\Delta_3\Gamma_1+\Delta_1\Delta_3 \end{array}}
$$

$$
+ 6 \sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Gamma_1,\Gamma_2,\Gamma_3 \in \mathbb{Z}_{>0} \qquad 3 \mid -f+\Delta_1+\Delta_2+\Delta_3+2\Gamma_1+2\Gamma_2+2\Gamma_3 \\ \Delta_1+2\Gamma_1 < 2\Delta_2+2\Delta_3+\Gamma_2+\Gamma_3 \qquad \Gamma_3+2\Delta_3 < 2\Gamma_1+2\Gamma_2+\Delta_1+\Delta_2 \\ \Delta_2+2\Gamma_2 < 2\Delta_1+2\Delta_3+\Gamma_1+\Gamma_3 \qquad \Delta_1+\Delta_2 < 2\Delta_3+\Gamma_1+\Gamma_2+\Gamma_3 \\ \Delta_1+\Delta_3+2\Gamma_3 < 2\Delta_2+\Gamma_1+\Gamma_2 \qquad \Delta_2+\Delta_3 < 2\Delta_1+\Gamma_1+\Gamma_2+\Gamma_3 \\ \Gamma_1+\Gamma_3+2\Delta_1 < 2\Gamma_2+\Delta_2+\Delta_3 \qquad \Gamma_1+\Gamma_2 < 2\Gamma_3+\Delta_1+\Delta_2+\Delta_3 \\ \Gamma_2+2\Delta_2 < 2\Gamma_1+2\Gamma_3+\Delta_1+\Delta_3 \qquad \Gamma_2+\Gamma_3 < 2\Gamma_1+\Delta_1+\Delta_2+\Delta_3}} q^{\begin{array}{c} -\frac{1}{18}(-f-2\Delta_1-2\Delta_2-2\Delta_3-\Gamma_1-\Gamma_2-\Gamma_3)^2 \\ -\frac{1}{18}(-f+\Delta_1+\Delta_2+\Delta_3-\Gamma_1-\Gamma_2-\Gamma_3)^2 \\ -\frac{1}{18}(-f+\Delta_1+\Delta_2+\Delta_3+2\Gamma_1+2\Gamma_2+2\Gamma_3)^2 \\ +\Gamma_1\Gamma_2+\Gamma_2\Gamma_3+\Gamma_1\Gamma_3+\Delta_1\Gamma_2+\Delta_2\Gamma_1+\Delta_1\Delta_2 \\ +\Delta_2\Gamma_3+\Delta_3\Gamma_2+\Delta_2\Delta_3+\Delta_3\Gamma_1+\Delta_1\Delta_3 \end{array}}
$$

$$
+3 \sum_{\substack{\Delta_2,\Delta_3,\Gamma_1,\Gamma_2,\Gamma_3 \in \mathbb{Z}_{>0} \\ 3 \,\mid\, -f+\Delta_2+\Delta_3+2\Gamma_1+2\Gamma_2+2\Gamma_3 \\ 2\Gamma_1 < 2\Delta_2+2\Delta_3+\Gamma_2+\Gamma_3 \\ \Delta_2+\Delta_3 < \Gamma_1+\Gamma_2+\Gamma_3 \\ \Delta_2+2\Gamma_2 < 2\Delta_3+\Gamma_1+\Gamma_3 \\ \Gamma_1+\Gamma_2 < 2\Gamma_3+\Delta_2+\Delta_3 \\ \Delta_3+2\Gamma_3 < 2\Delta_2+\Gamma_1+\Gamma_2 \\ \Gamma_2+\Gamma_3 < 2\Gamma_1+\Delta_2+\Delta_3 \\ \Gamma_2+2\Delta_2 < 2\Gamma_1+2\Gamma_3+\Delta_3 \\ \Gamma_1+\Gamma_3 < 2\Gamma_2+\Delta_2+\Delta_3 \\ \Gamma_3+2\Delta_3 < 2\Gamma_1+2\Gamma_2+\Delta_2}} q^{\begin{array}{c} -\frac{1}{18}(-f-2\Delta_2-2\Delta_3-\Gamma_1-\Gamma_2-\Gamma_3)^2 \\ -\frac{1}{18}(-f+\Delta_2+\Delta_3-\Gamma_1-\Gamma_2-\Gamma_3)^2 \\ -\frac{1}{18}(-f+\Delta_2+\Delta_3+2\Gamma_1+2\Gamma_2+2\Gamma_3)^2 \\ +\Gamma_1\Gamma_2+\Gamma_2\Gamma_3+\Gamma_1\Gamma_3 \\ +\Delta_2\Gamma_1+\Delta_2\Gamma_3+\Delta_3\Gamma_2+\Delta_2\Delta_3+\Delta_3\Gamma_1 \end{array}}
$$

$$
+3 \sum_{\substack{\Delta_1,\Delta_2,\Delta_3,\Gamma_2,\Gamma_3 \in \mathbb{Z}_{>0} \\ 3 \,\mid\, -f+\Delta_1+\Delta_2+\Delta_3+2\Gamma_2+2\Gamma_3 \\ \Delta_2+2\Gamma_2 < 2\Delta_1+2\Delta_3+\Gamma_3 \\ \Delta_1+\Delta_2 < 2\Delta_3+\Gamma_2+\Gamma_3 \\ \Delta_3+2\Gamma_3 < 2\Delta_1+2\Delta_2+\Gamma_2 \\ \Delta_2+\Delta_3 < 2\Delta_1+\Gamma_2+\Gamma_3 \\ 2\Delta_1 < 2\Gamma_2+2\Gamma_3+\Delta_2+\Delta_3 \\ \Delta_1+\Delta_3 < 2\Delta_2+\Gamma_2+\Gamma_3 \\ \Gamma_2+2\Delta_2 < 2\Gamma_3+\Delta_1+\Delta_3 \\ \Gamma_2+\Gamma_3 < \Delta_1+\Delta_2+\Delta_3 \\ \Gamma_3+2\Delta_3 < 2\Gamma_2+\Delta_1+\Delta_2}} q^{\begin{array}{c} -\frac{1}{18}(-f-2\Delta_1-2\Delta_2-2\Delta_3-\Gamma_2-\Gamma_3)^2 \\ -\frac{1}{18}(-f+\Delta_1+\Delta_2+\Delta_3-\Gamma_2-\Gamma_3)^2 \\ -\frac{1}{18}(-f+\Delta_1+\Delta_2+\Delta_3+2\Gamma_2+2\Gamma_3)^2 \\ +\Delta_1\Delta_2+\Delta_2\Delta_3+\Delta_1\Delta_3 \\ +\Delta_1\Gamma_2+\Delta_3\Gamma_2+\Delta_2\Gamma_3+\Gamma_2\Gamma_3+\Delta_1\Gamma_3 \end{array}} .
$$

Referring to equation (2.3), we see the only relevant values for f are $f = -1, 0, 1$. It is now easy to numerically compute the first ten Euler characteristics for these values.

Corollary 2.3.9. *Let $X = \mathbb{P}^2$ and let H be an ample divisor on X. Then:*

$$
\begin{aligned}
\sum_{c_2 \in \mathbb{Z}} e(M_X^H(3,-1,c_2))q^{c_2} =& 3q^2 + 42q^3 + 333q^4 + 1968q^5 + 9609q^6 + 40881q^7 + 156486q^8 \\
&+ 550392q^9 + 1805283q^{10} + O(q^{11}),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{c_2 \in \mathbb{Z}} e(M_X^H(3,0,c_2))q^{c_2} =& -q^3 - 9q^4 - 60q^5 - 309q^6 - 1362q^7 - 5322q^8 - 18957q^9 \\
& - 62574q^{10} + O(q^{11}), \\
\sum_{c_2 \in \mathbb{Z}} e(M_X^H(3,1,c_2))q^{c_2} =& 3q^2 + 42q^3 + 333q^4 + 1968q^5 + 9609q^6 + 40881q^7 + 156486q^8 \\
& + 550392q^9 + 1805283q^{10} + O(q^{11}).
\end{aligned}
$$

The corollary suggests that the generating functions $\sum_{c_2 \in \mathbb{Z}} e(M_X^H(3, \pm 1, c_2))q^{c_2}$ are the same. Indeed, it is not difficult to see that changing $\Delta_i \leftrightarrow \Gamma_i$ and $f \leftrightarrow -f$ interchanges terms two and three and terms five and six of the expression for the generating function, while leaving terms one and four unchanged. This proves the generating functions $\sum_{c_2 \in \mathbb{Z}} e(M_X^H(3, \pm c_1, c_2))q^{c_2}$ are the same for any $c_1 \in H^2(X, \mathbb{Z})$. This fact can be easily understood as follows. Let X be a nonsingular projective surface with ample divisor H. Let $r \in \mathbb{Z}_{>0}$, $c_1 \in H^2(X, \mathbb{Z})$, $c_2 \in \mathbb{Z}$ and denote by $N_X^H(r, c_1, c_2)$ the moduli space of μ-stable vector bundles on X of rank r, first Chern class c_1 and second Chern class c_2. Then taking the dual gives an isomorphism

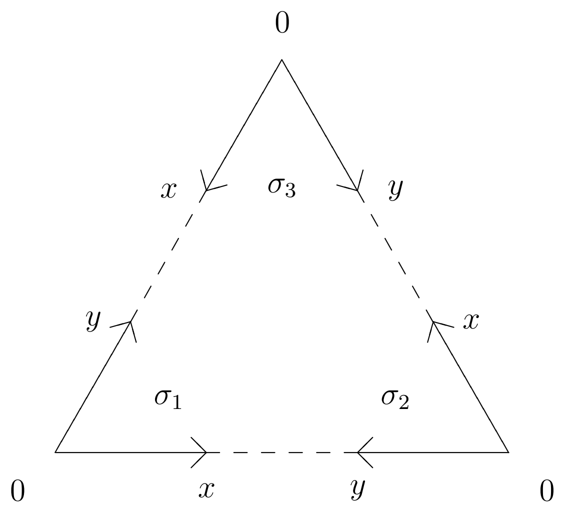
$$
\begin{aligned}
N_X^H(r, c_1, c_2) &\xrightarrow{\cong} N_X^H(r, -c_1, c_2) \\
\mathcal{E} &\mapsto \mathcal{E}^\vee.
\end{aligned}
$$

2.4 Euler Characteristics of Moduli Spaces of Pure Dimension 1 Sheaves on $\mathbb{P}^2$

In Theorem 1.3.9, we gave a combinatorial description of the fixed point locus of an arbitrary moduli space of Gieseker stable sheaves with fixed Hilbert polynomial on an arbitrary nonsingular projective toric variety with ample line bundle. The theorem assumes we can find equivariant line bundles of the GIT problem matching Gieseker and GIT stability. We have shown the existence of such (ample) equivariant line bundles in the case of torsion free sheaves in Theorem 1.2.22. In the case of surfaces and torsion free sheaves, we used Theorem 1.3.15 to give expressions for the generating functions of Euler characteristics of moduli spaces of μ-stable torsion free sheaves on arbitrary nonsingular complete toric surfaces (Theorem 2.2.7). We focused on torsion free sheaves as opposed to pure sheaves of lower dimension, since this allowed us to compare to many results in the literature. As we discussed, the results in the literature were mostly obtained by different means. It is however also interesting to apply Theorem 1.3.9 to surfaces and pure sheaves of dimension 1. Although we will not study this in a systematic way, we will compute some generating functions of μ-stable pure dimension 1 sheaves on $\mathbb{P}^2$ with first Chern class c_1 fixed. We will treat the cases $c_1 = 1, 2, 3$. The computations of this section have also been carried out by Jinwon Choi using [Koo1], i.e. chapter 1, and our results coincide (private communication). Jinwon Choi has also considered the case $c_1 = 4$ (see section 2.5 for a discussion).

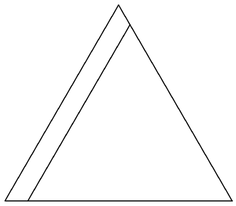
Consider the projective plane $X = \mathbb{P}^2$. In this section, we will use the notation introduced in subsection 2.2.1 and in 2.3.2.1. Let $\mathcal{E}$ be a pure dimension 1 equivariant sheaf on $\mathbb{P}^2$. Then its support is one of $V(\rho_1)$, $V(\rho_2)$, $V(\rho_3)$, $V(\rho_1) \cup V(\rho_2)$, $V(\rho_1) \cup V(\rho_3)$, $V(\rho_2) \cup V(\rho_3)$ or $V(\rho_1) \cup V(\rho_2) \cup V(\rho_3)$. The pure Δ-family $\hat{E}^{\Delta}$ of such a sheaf is described in Theorems 1.1.10, 1.1.12 and subsection 1.1.2. We draw $E^{\sigma_i}(\lambda_1, \lambda_2)$ for $i = 1, 2, 3$ in a copy of $\mathbb{Z}^2$ and draw these copies in such a way as to easily encode the gluing conditions

$$E^{\sigma_i}(\infty, \lambda) = E^{\sigma_{i+1}}(\lambda, \infty),\ \chi_j^{\sigma_i}(\infty, \lambda) = \chi_j^{\sigma_{i+1}}(\lambda, \infty),\ \forall \lambda \in \mathbb{Z},\ i = 1, 2, 3 \bmod 3,\ j = 1, 2.$$

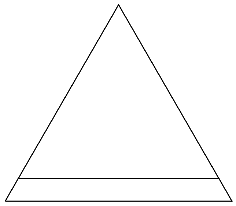


We can graphically represent the different types of support as follows.

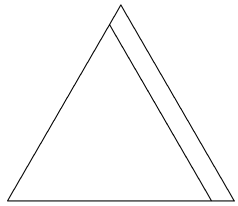
type 1 : $V(\rho_1)$



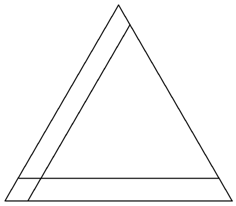
type 2 : $V(\rho_2)$



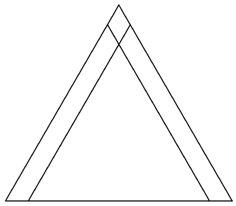
type 3 : $V(\rho_3)$



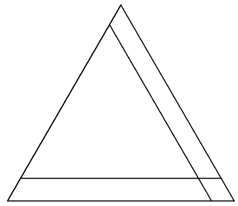
type 4 : $V(\rho_1) \cup V(\rho_2)$



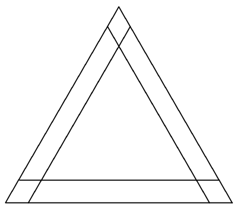
type 5 : $V(\rho_1) \cup V(\rho_3)$



type 6 : $V(\rho_2) \cup V(\rho_3)$



type 7 : $V(\rho_1) \cup V(\rho_2) \cup V(\rho_3)$



Recall there is a canonical isomorphism $A(X) \cong \mathbb{Z}[D]/(D^3)$, where D is the toric divisor corresponding to any of the rays. This induces canonical isomorphisms $H^0(X, \mathbb{Z}) \cong A^0(X) \cong \mathbb{Z}$, $H^2(X, \mathbb{Z}) \cong A^1(X) \cong \mathbb{Z}$ and $H^4(X, \mathbb{Z}) \cong A^2(X) \cong \mathbb{Z}$. For a fixed ample divisor H, first Chern class $c_1 \in \mathbb{Z}_{>0}$ and second Chern class $c_2 \in \mathbb{Z}$, let $M_X^H(0, c_1, c_2)$ be the moduli space of Gieseker stable sheaves on $X = \mathbb{P}^2$ of rank 0, first Chern class c_1 and second Chern class c_2. Note that such sheaves are pure of dimension 1. Also note that μ-stability and Gieseker stability coincide for dimension 1 coherent sheaves on X ([HL, Def. 1.2.4, Def.–Cor. 1.6.9]). Finally, note that $M_X^H(0, c_1, c_2)$ does not depend on the choice of ample divisor H. We want to study the generating function

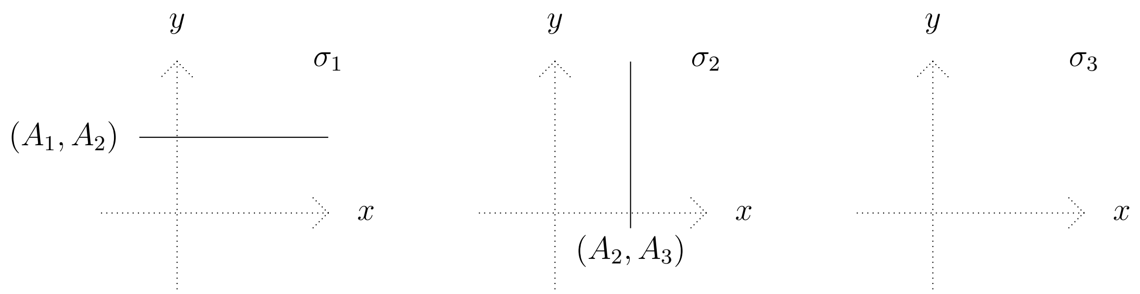
$$\sum_{c_2 \in \mathbb{Z}} e(M_X^H(0, c_1, c_2)) q^{c_2}.$$

We will prove the following theorem.

Theorem 2.4.1. *Let $X = \mathbb{P}^2$ and let H be an ample divisor on X, then*

$$\begin{aligned}
\sum_{c_2 \in \mathbb{Z}} e(M_X^H(0,1,c_2))q^{c_2} &= \sum_{k \in \mathbb{Z}} 3q^k, \\
\sum_{c_2 \in \mathbb{Z}} e(M_X^H(0,2,c_2))q^{c_2} &= \sum_{k \in \mathbb{Z}} 6q^{2k} + \sum_{k \in \mathbb{Z}} 0q^{2k+1}, \\
\sum_{c_2 \in \mathbb{Z}} e(M_X^H(0,3,c_2))q^{c_2} &= \sum_{k \in \mathbb{Z}} 9q^{3k} + \sum_{k \in \mathbb{Z}} 27q^{3k+1} + \sum_{k \in \mathbb{Z}} 27q^{3k+2}.
\end{aligned}$$

Proof. Without loss of generality, we can take $H = D$. Let $\mathcal{L}_1$ be a pure dimension 1 equivariant sheaf on X with corresponding pure Δ-family $\hat{L}_1^{\Delta}$ with characteristic function of the form ($A_1, A_2, A_3 \in \mathbb{Z}$ arbitrary):



Let us explain the notation we use for depicting characteristic functions in this proof. A solid line will mean each lattice point on that line has 1 associated to it and all lattice points outside the solid line will have 0 associated to it. We will also occasionally use a solid dot to depict a lattice point with a 2 associated to it (see later in the proof). We define $\hat{L}_1^{\Delta}$ to have complex vector space $\mathbb{C}$ on the solid line and identity maps between them. All other complex vector spaces and $\mathbb{C}$-linear maps are zero. Using Klyachko's Formula (Proposition 1.2.16), it is straightforward to compute

$$\mathrm{ch}(\mathcal{L}_1) = D + \left(-\frac{1}{2} - A_1 - A_2 - A_3\right) D^2.$$

Now let $\mathcal{L}_2$ be an equivariant coherent sheaf of dimension 0 with corresponding pure Δ-family $\hat{L}_2^{\Delta}$. Then there are only finitely many $L_2^{\sigma_i}(\lambda_1, \lambda_2) \neq 0$ (see Proposition 1.1.8).

Using Klyachko's Formula (Proposition 1.2.16), we see

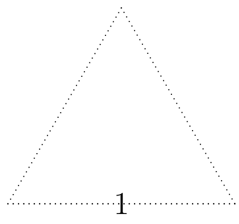
$$\mathrm{ch}(\mathcal{L}_2) = \sum_{i=1}^{3} \sum_{\lambda_1, \lambda_2 \in \mathbb{Z}} \dim(L_2^{\sigma_i}(\lambda_1, \lambda_2)) D^2.$$

Using the same trick as in Step 3 of the proof of Proposition 2.2.2, one can easily deduce the Chern character of any pure dimension 1 equivariant sheaf $\mathcal{E}$ on X with explicitly given characteristic function $\vec{\chi}$. Note in particular that

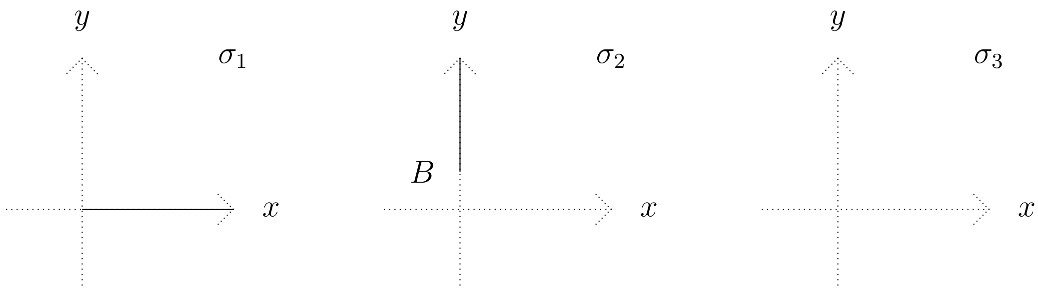
$$c_1(\mathcal{E}) = \sum_{i=1}^{3} \sum_{\lambda \in \mathbb{Z}} \dim(E^{\sigma_i}(\infty, \lambda)). \tag{2.9}$$

We now want to apply Theorem 1.3.9. In each of the cases $c_1 = 1, 2, 3$ and $c_2 \in \mathbb{Z}$ arbitrarily fixed, we will compute all gauge-fixed characteristic functions $\vec{\chi}$ giving rise to c_1, c_2, for which there exists a μ-stable pure dimension 1 equivariant sheaf $\mathcal{E}$ on X with characteristic function $\vec{\chi}$. For the definition of gauge-fixed characteristic function, see subsection 1.3.3. Except in one case, the sheaf $\mathcal{E}$ will turn out to be unique up to equivariant isomorphism. In these cases, the corresponding component of the fixed point locus is an isolated fixed point Spec $\mathbb{C}$. The one exception will be for $c_1 = 3$ and $c_2 \equiv 0 \bmod 3$, in which case we will get a characteristic function giving rise to a component C of the fixed point locus for which there is a morphism $\mathbb{C}^* \longrightarrow C$ that is bijective on closed points. This component will not contribute, since $e(C) = e(\mathbb{C}^*) = 0$ (see subsection 2.1.1). Therefore, counting the isolated fixed points will give the generating functions by torus localisation (Proposition 2.1.1). Note that the computation of the relevant characteristic functions uses Proposition 1.2.19, which tells us we only need to test μ-stability for equivariant coherent subsheaves.

Case 1: $c_1 = 1$. Let $c_2 \in \mathbb{Z}$. Using equation (2.9), we first list the shapes the characteristic functions a priori can have (up to the obvious symmetries of the problem). The number indicates the dimension of the limiting complex vector spaces along the axes.

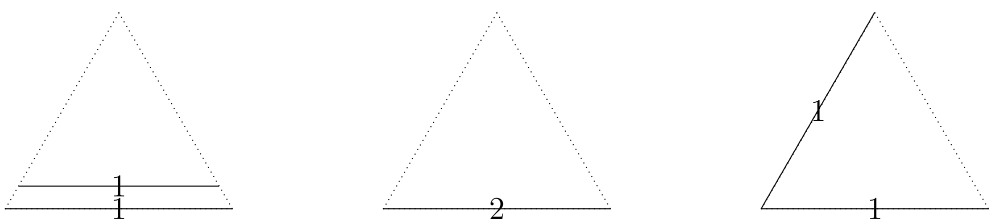


The following is a gauge-fixed characteristic function $\vec{\chi}$ giving rise to c_1, c_2 for which there exists a μ-stable pure dimension 1 equivariant sheaf on X with characteristic function $\vec{\chi}$:

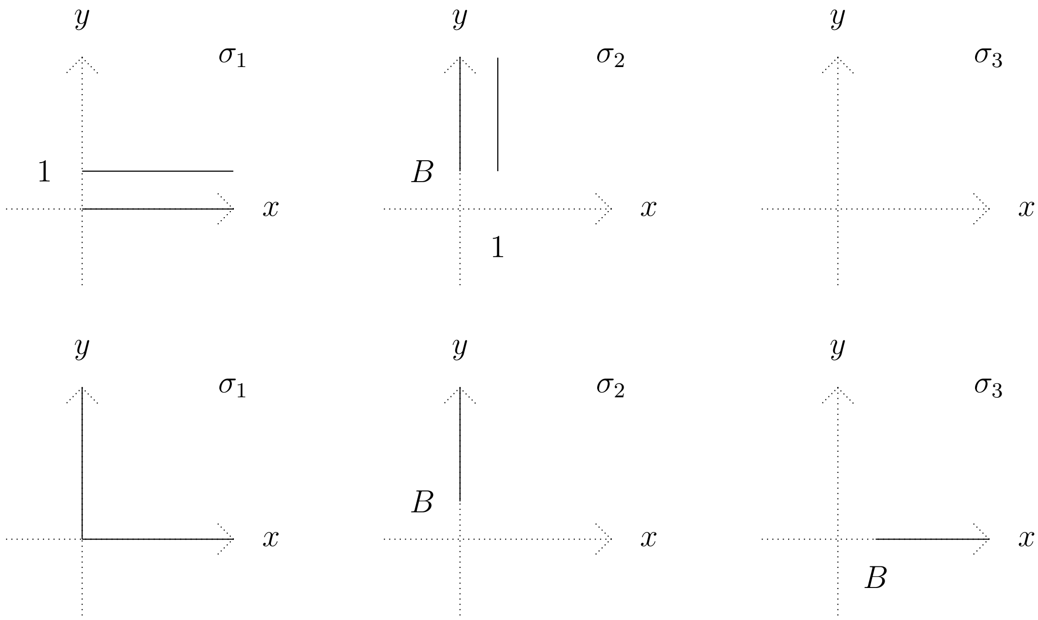


where $c_2 = B + 1$. Here the B in the diagram denotes a position along y-axis, i.e. the solid line terminates at $(0, B)$. We will use similar notation later in the proof. There are two more such gauge-fixed characteristic functions obtained from the above by the obvious symmetries of the problem. For each of these characteristic functions $\vec{\chi}$, there is precisely one μ-stable pure dimension 1 equivariant sheaf on X with characteristic function $\vec{\chi}$ up to equivariant isomorphism. We obtain $e(M_X^H(1, c_2)) = 3 \cdot 1 = 3$.

Case 2: $c_1 = 2$. Let $c_2 \in \mathbb{Z}$. Referring to equation (2.9), we again start by listing the shapes of the characteristic functions that are a priori allowed (up to the obvious symmetries of the problem). As before, the numbers indicate the dimensions of the limiting complex vector spaces along the axes.

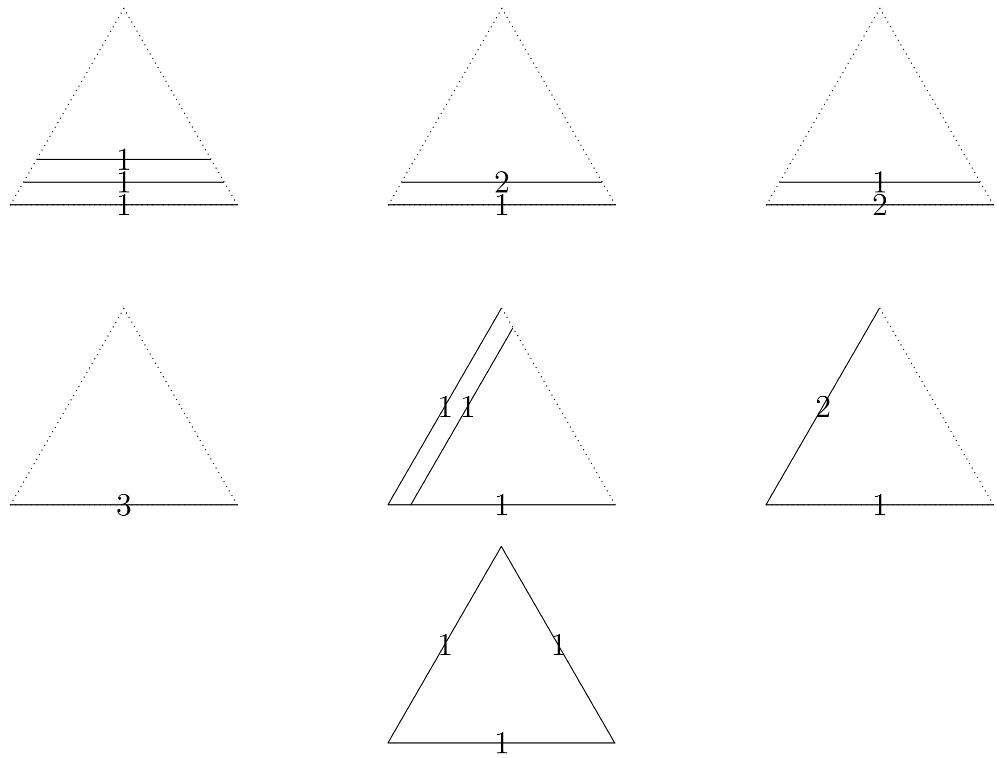


Referring to the last type of characteristic function, we note that in the corner where the two legs that have 1 as their limit meet, numbers ≤ 2 can occur (see subsection 1.1.2, in particular condition (1.6)). Not all of these types of characteristic functions will have a μ-stable pure dimension 1 equivariant sheaf on X with that characteristic function. In the case of the second type of characteristic function, a pure dimension 1 equivariant sheaf on X with that characteristic function is always equivariantly decomposable, hence not μ-stable. The following are all gauge-fixed characteristic functions $\vec{\chi}$ giving rise to c_1, c_2 for which there exists a μ-stable pure dimension 1 equivariant sheaf on X with characteristic function $\vec{\chi}$ (up to symmetry):

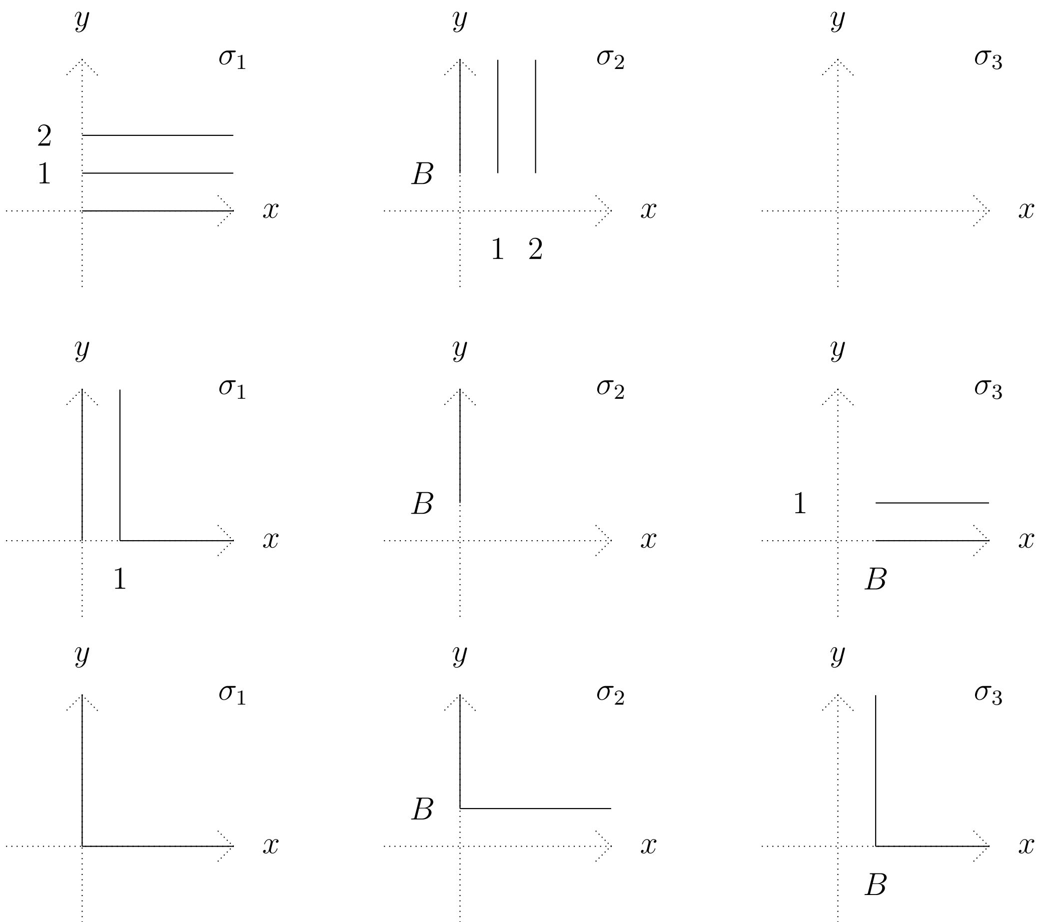


where $B \in \mathbb{Z}$ satisfies $c_2 = 2B + 4$. For each such characteristic function $\vec{\chi}$, there is precisely one μ-stable pure dimension 1 equivariant sheaf on X with characteristic function $\vec{\chi}$ up to equivariant isomorphism. In the case $c_2 \equiv 0 \bmod 2$, we obtain $e(M_X^H(2, c_2)) = 3 \cdot 1 + 3 \cdot 1 = 6$. Note that there are no gauge-fixed characteristic functions $\vec{\chi}$ giving rise to $c_1 = 2$, $c_2 \equiv 1 \bmod 2$, for which there exists a μ-stable pure dimension 1 equivariant sheaf on X with characteristic function $\vec{\chi}$. So in the case $c_2 \equiv 1 \bmod 2$, we obtain $e(M_X^H(2, c_2)) = 0$.

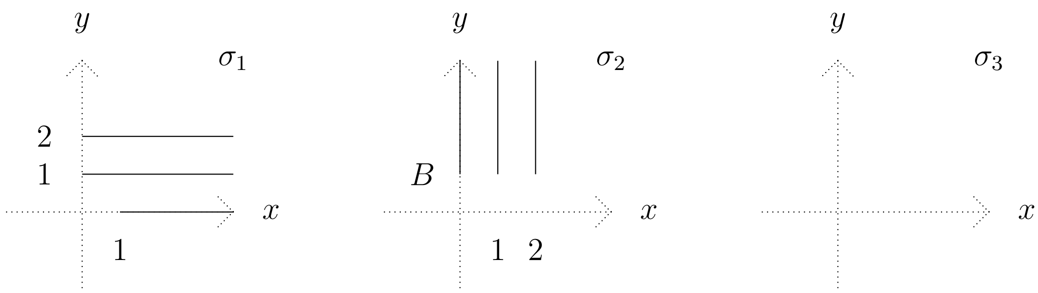
Case 3: $c_1 = 3$. Let $c_2 \in \mathbb{Z}$. Again, using equation (2.9), we start by listing the shapes of the characteristic functions that are a priori allowed (up to the obvious symmetries of the problem). The numbers again indicate the dimensions of the limiting complex vector spaces along the axes.

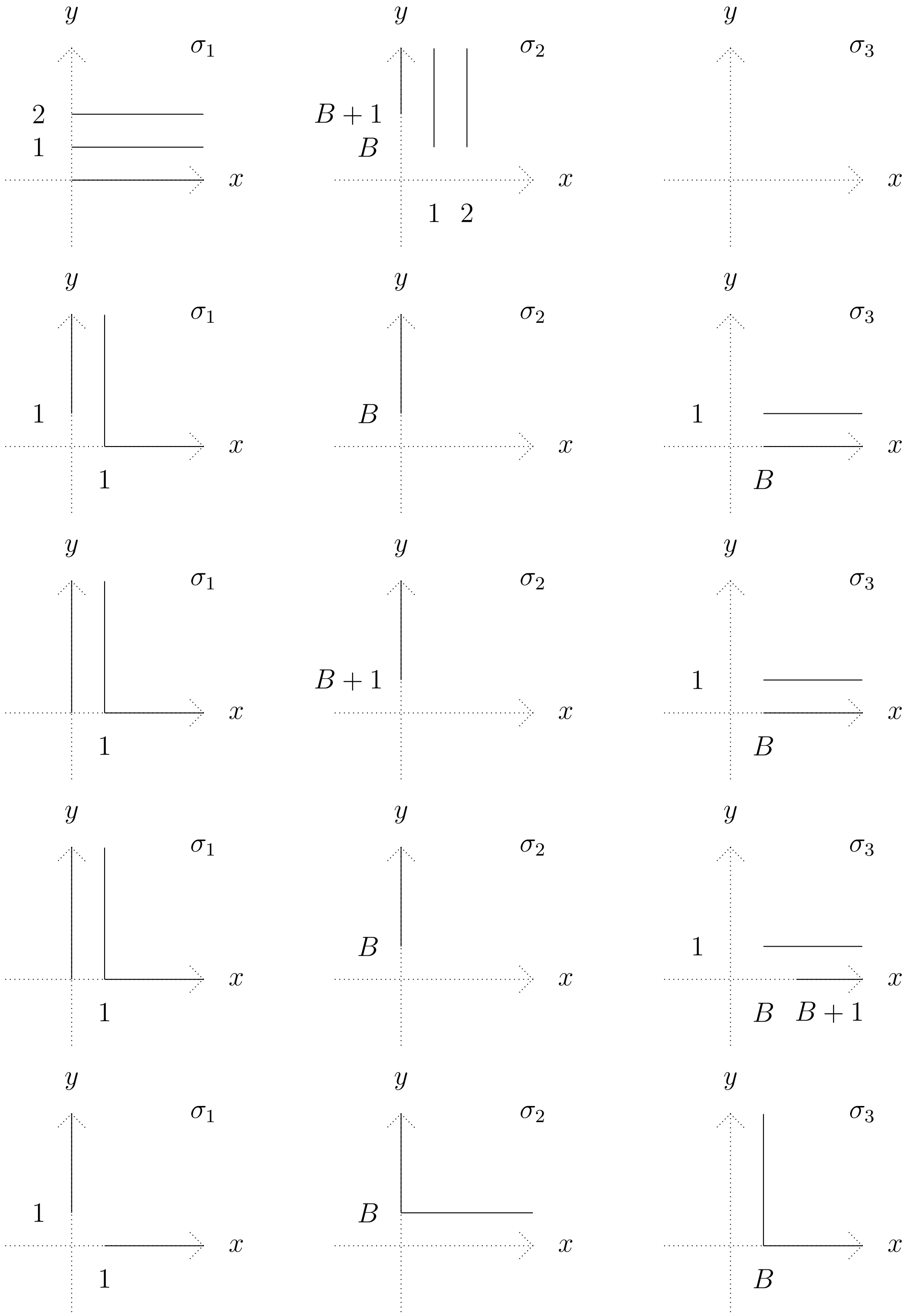


Note that in the corners of the 5th and 7th diagram where the legs that have 1 as their limit meet, numbers ≤ 2 can occur (see subsection 1.1.2, in particular condition (1.6)). In the corner of the 6th diagram where the legs with 1 resp. 2 as their limit meet, numbers ≤ 3 can occur (see subsection 1.1.2, in particular condition (1.6)). Not for all these types of characteristic functions do there exist μ-stable pure dimension 1 equivariant sheaves on X with that characteristic function. A case by case analysis shows that we only have to consider characteristic functions of type 1, 5 and 7. We now list all gauge-fixed characteristic functions $\vec{\chi}$ giving rise to c_1, c_2, for which there exists a μ-stable pure dimension 1 equivariant sheaf on X with characteristic function $\vec{\chi}$. The list is only up to the obvious symmetries of the problem. Recall that a solid line means each lattice point on that line has 1 associated to it, a solid dot depicts a lattice point with a 2 associated to it and all other lattice points have 0 associated to it. Also recall that the numbers along the coordinate axes indicate the positions of the solid lines:

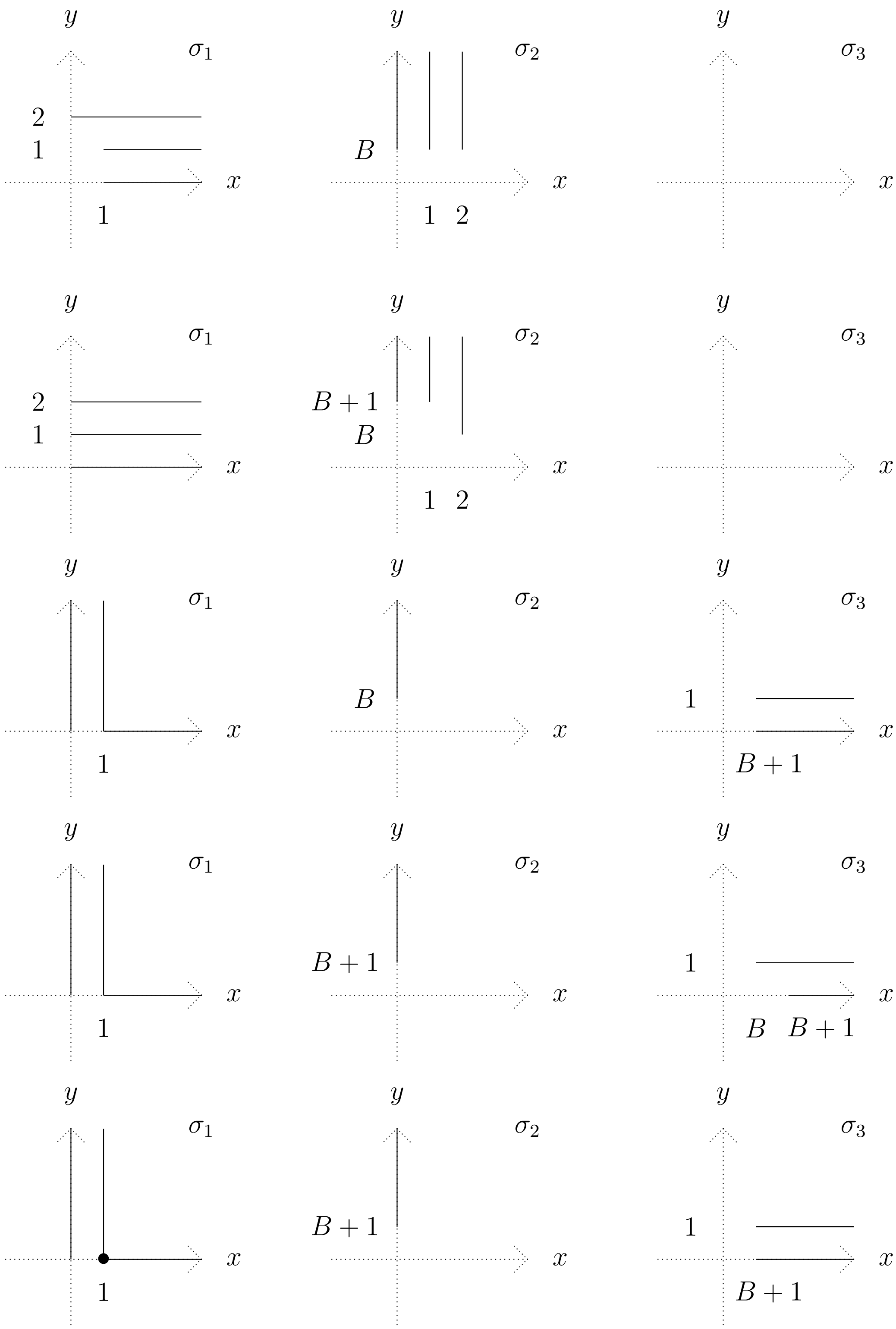


where $B \in \mathbb{Z}$ satisfies $c_2 = 3B + 9$. For each such characteristic function $\vec{\chi}$ other than the last, there is precisely one μ-stable pure dimension 1 equivariant sheaf on X with characteristic function $\vec{\chi}$ up to equivariant isomorphism. For the last characteristic function $\vec{\chi}$ and the corresponding component C of the fixed point locus, it is easy to construct a morphism $\mathbb{C}^* \longrightarrow C$ which is bijective on closed points. Since $e(C) = e(\mathbb{C}^*) = 0$, it does not contribute. We continue the list:





where $B \in \mathbb{Z}$ satisfies $c_2 = 3B + 10$. For each such characteristic function $\vec{\chi}$, there is precisely one μ-stable pure dimension 1 equivariant sheaf on X with characteristic function $\vec{\chi}$ up to equivariant isomorphism. We continue the list: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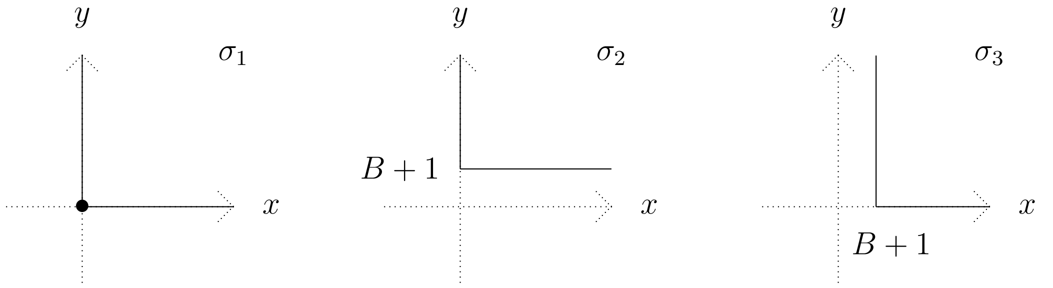

where $B \in \mathbb{Z}$ satisfies $c_2 = 3B + 11$. For each such characteristic function $\vec{\chi}$, there is precisely one μ-stable pure dimension 1 equivariant sheaf on X with characteristic function $\vec{\chi}$ up to equivariant isomorphism. In the case $c_2 \equiv 0 \bmod 3$, we obtain $e(M_X^H(3, c_2)) = 3 \cdot 1 + 6 \cdot 1 + 1 \cdot 0 = 9$. In the case $c_2 \equiv 1 \bmod 3$, we obtain $e(M_X^H(3, c_2)) = 3 \cdot 1 + 3 \cdot 1 + 6 \cdot 1 + 6 \cdot 1 + 6 \cdot 1 + 3 \cdot 1 = 27$. Finally, in the case $c_2 \equiv 2 \bmod 3$, we obtain $e(M_X^H(3, c_2)) = 3 \cdot 1 + 3 \cdot 1 + 6 \cdot 1 + 6 \cdot 1 + 6 \cdot 1 + 3 \cdot 1 = 27$. $\square$

2.5 Application to Donaldson–Thomas Invariants

In this chapter, we have computed various generating functions of Euler characteristics of moduli spaces of μ-stable pure sheaves on nonsingular complete toric surfaces S. In the case there are no strictly μ-semistables and assuming some conditions on S, we will show how these can be seen as generating functions of Donaldson–Thomas invariants of the canonical bundle $X = K_S$, which is a noncompact Calabi–Yau threefold. The computations of the previous section will give examples supporting a conjecture by Katz relating Donaldson–Thomas invariants and genus zero Gopakumar–Vafa invariants.

2.5.1 Generalised Donaldson–Thomas Invariants

Donaldson–Thomas invariants were defined by Thomas [Tho], following a proposal of Donaldson and Thomas [DT]. The (original) Donaldson–Thomas invariants were only defined in the case there are no strictly semistables [Tho]. Recent work of Joyce–Song [JS] and Kontsevich–Soibelman [KoSo] extends the theory to a more general setting including strictly semistables. Even though we only use (original) Donaldson–Thomas invariants in the next subsection, we use the opportunity to give a short overview of the main properties of generalised Donaldson–Thomas invariants as introduced by Joyce–Song [JS]. We discuss their relation to genus zero Gopakumar–Vafa invariants following Joyce–Song [JS] and Katz [Kat]. Since we will be interested in applications to noncompact Calabi–Yau threefolds in the following subsection, we pay special attention to this case.

Let X be a Calabi–Yau threefold over $\mathbb{C}$, i.e. a nonsingular projective threefold X over $\mathbb{C}$ with trivial canonical bundle $\omega_X \cong 0$ and $H^1(X, \mathcal{O}_X) = 0$. We also refer to these as compact Calabi–Yau threefolds. Let $\mathcal{O}_X(1)$ be a very ample line bundle on X and let $(\tau, T, \leq)$ be a Gieseker stability condition induced by $\mathcal{O}_X(1)$. Let $K_0(X) = K_0(\mathrm{coh}(X))$ denote the Grothendieck group over the abelian category $\mathrm{coh}(X)$ of coherent sheaves on X. By Serre Duality and the Calabi–Yau property, we have a well-defined antisymmetric bilinear form

$$\begin{aligned} &\overline{\chi}(-,-) : K_0(X) \times K_0(X) \longrightarrow \mathbb{Z}, \\ &\overline{\chi}([\mathcal{E}], [\mathcal{F}]) = \sum_{i=0}^{3} (-1)^i \dim(\mathrm{Ext}^i(\mathcal{E}, \mathcal{F})). \end{aligned}$$

This map is called the Euler form. We define the numerical Grothendieck group $K^{num}(X)$ to be $K_0(X)$ modulo the kernel of $\overline{\chi}$. In this setting, we also denote the numerical Grothendieck group $K^{num}(X)$ by $K(X)$. We get an induced non-degenerate antisymmetric bilinear form

$$\begin{aligned} &\overline{\chi}(-,-) : K(X) \times K(X) \longrightarrow \mathbb{Z}, \\ &\overline{\chi}([\mathcal{E}], [\mathcal{F}]) = \sum_{i=0}^{3} (-1)^i \dim(\mathrm{Ext}^i(\mathcal{E}, \mathcal{F})). \end{aligned}$$

We define the cone $C(X) = \{[\mathcal{E}] \in K(X) \mid \mathcal{E} \in \mathrm{coh}(X),\ \mathcal{E} \ncong 0\}$. Since $K(X)$ can equivalently be defined as the quotient of $K_0(X)$ by the kernel of the Chern character map $\mathrm{ch} : K_0(X) \longrightarrow H^{even}(X, \mathbb{Q})$, we can think of $K(X)$ as a subgroup of $H^{even}(X, \mathbb{Q})$. Hence $K(X)$ is a lattice of finite rank and we can write $C(X) \subset K(X) \leq H^{even}(X, \mathbb{Q})$. For any $\alpha \in C(X)$, denote by $\mathcal{M}^{\alpha}_{ss}(\tau)$ the moduli space of Gieseker semistable (w.r.t. $\mathcal{O}_X(1)$)

sheaves $\mathcal{E}$ of class $[\mathcal{E}] = \alpha \in C(X)$ on X. This is a projective $\mathbb{C}$-scheme of finite type as discussed in [HL, Ch. 4] and its closed points $\mathcal{M}_{ss}^{\alpha}(\tau)(\mathbb{C})$ are in bijective correspondence with S-equivalence classes of Gieseker semistable sheaves of class α on X. We denote the open subscheme of Gieseker stable sheaves of class α on X by $\mathcal{M}_{st}^{\alpha}(\tau)$. Its closed points $\mathcal{M}_{st}^{\alpha}(\tau)(\mathbb{C})$ are in bijective correspondence with isomorphism classes of Gieseker stable sheaves of class α on X. Joyce–Song [JS] define generalised Donaldson–Thomas invariants $\bar{DT}^{\alpha}(\tau) \in \mathbb{Q}$ of X for all $\alpha \in C(X)$. These generalised Donaldson–Thomas invariants have the following properties:

(i) In the case $\mathcal{M}_{ss}^{\alpha}(\tau) = \mathcal{M}_{st}^{\alpha}(\tau)$, the generalised Donaldson–Thomas invariant coincides with the (original) Donaldson–Thomas invariant $\bar{DT}^{\alpha}(\tau) = DT^{\alpha}(\tau)$ [JS, Prop. 5.15]. The Donaldson–Thomas invariant was introduced by Thomas [Tho], in the case $\mathcal{M}_{ss}^{\alpha}(\tau) = \mathcal{M}_{st}^{\alpha}(\tau)$, as the degree of the virtual class $[\mathcal{M}_{ss}^{\alpha}(\tau)]^{vir} \in A_0(\mathcal{M}_{st}^{\alpha}(\tau))$

$$DT^{\alpha}(\tau) = \int_{[\mathcal{M}_{st}^{\alpha}(\tau)]} 1. \tag{2.10}$$

One can express the Donaldson–Thomas invariant $DT^{\alpha}(\tau)$ in terms of the Euler characteristic of $\mathcal{M}_{st}^{\alpha}(\tau)$ weighted by the Behrend function $\nu_{\mathcal{M}_{st}^{\alpha}(\tau)} : \mathcal{M}_{st}^{\alpha}(\tau)(\mathbb{C}) \longrightarrow \mathbb{Z}$, which is a $\mathbb{Z}$-valued constructible function on $\mathcal{M}_{st}^{\alpha}(\tau)$

$$DT^{\alpha}(\tau) = \int_{\mathcal{M}_{st}^{\alpha}(\tau)(\mathbb{C})} \nu_{\mathcal{M}_{st}^{\alpha}(\tau)} \mathrm{d}e, \tag{2.11}$$

where e denotes the Euler characteristic discussed in subsection 2.1.1. This result was proved by Behrend [Beh]. Note that the Behrend function $\nu_X : X(\mathbb{C}) \longrightarrow \mathbb{Z}$ is a constructible function that is defined for any $\mathbb{C}$-scheme X of finite type. In the case $p \in X$ is a nonsingular closed point, the Behrend function is given by $\nu_X = (-1)^{\dim(T_p X)}$ [JS, Thm. 4.3]. So in the simplest case, where $\mathcal{M}_{ss}^{\alpha}(\tau) = \mathcal{M}_{st}^{\alpha}(\tau)$ is nonsingular and each connected component has the same dimension, we have

$$DT^{\alpha}(\tau) = (-1)^{\dim(\mathcal{M}_{st}^{\alpha}(\tau))} e(\mathcal{M}_{st}^{\alpha}(\tau)).$$

(ii) Suppose $\mathcal{O}_X(1)'$ is another very ample line bundle on X and $(\tau', T', \leq)$ is a Gieseker stability condition w.r.t. $\mathcal{O}_X(1)'$. Then there is an explicit wall-crossing formula expressing $\bar{DT}^\alpha(\tau')$ in terms of various $\bar{DT}^\beta(\tau)$, where $\beta \in C(X)$ [JS, Thm. 5.16]. The precise statement is a bit more subtle, see [JS, Thm. 5.16, Cor. 5.17].

(iii) The generalised Donaldson–Thomas invariants $\bar{DT}^\alpha(\tau)$ are independent of continuous deformations of X [JS, Cor. 5.25].

We now replace "projective" by "quasi-projective and not projective" in the previous definition of Calabi–Yau threefolds and drop the condition $H^1(X, \mathcal{O}_X) = 0$. We refer to these as noncompact Calabi–Yau threefolds over $\mathbb{C}$. Joyce–Song discuss the extension of their theory to noncompact Calabi–Yau threefolds over $\mathbb{C}$ in [JS, Sect. 6.7]. Let X be a noncompact Calabi–Yau threefold with very ample line bundle $\mathcal{O}_X(1)$. In order to have finite-dimensional Ext groups and Serre Duality, we should consider the abelian category $\mathrm{coh}_{cs}(X)$ of coherent sheaves on X with compact (i.e. proper) support. There exists a version of Chern character with compact support $\mathrm{ch}_{cs} : K_0(\mathrm{coh}_{cs}(X)) \longrightarrow H^{even}_{cs}(X, \mathbb{Q})$. We have a well-defined bilinear form called the Euler form

$$\begin{aligned} &\overline{\chi}(-,-) : K_0(\mathrm{coh}(X)) \times K_0(\mathrm{coh}_{cs}(X)) \longrightarrow \mathbb{Z}, \\ &\overline{\chi}([\mathcal{E}],[\mathcal{F}]) = \sum_{i=0}^{3} (-1)^i \mathrm{dim}(\mathrm{Ext}^i(\mathcal{E},\mathcal{F})), \end{aligned} \tag{2.12}$$

which is antisymmetric on $K_0(\mathrm{coh}_{cs}(X)) \times K_0(\mathrm{coh}_{cs}(X))$. The numerical Grothendieck group $K^{num}(\mathrm{coh}_{cs}(X))$ is $K_0(\mathrm{coh}_{cs}(X))$ modulo the kernel of $\overline{\chi}(-,-) : K_0(\mathrm{coh}_{cs}(X)) \times K_0(\mathrm{coh}_{cs}(X)) \longrightarrow \mathbb{Z}$, but this group is often too small and can be trivial in important applications. Therefore, we should consider the group $K(\mathrm{coh}_{cs}(X))$, which we define to be $K_0(\mathrm{coh}_{cs}(X))$ modulo the kernel of the Euler form of equation (2.12) (see [JS, Sect. 6.7]). In this case, $K(\mathrm{coh}_{cs}(X))$ can also be formed as $K_0(\mathrm{coh}_{cs}(X))$ modulo the kernel of ch_{cs}. As before, we define the cone $C(\mathrm{coh}_{cs}(X)) = \{[\mathcal{E}] \in K(\mathrm{coh}_{cs}(X)) \mid \mathcal{E} \in \mathrm{coh}_{cs}(X),\ \mathcal{E} \ncong 0\}$. Hence we have $C(\mathrm{coh}_{cs}(X)) \subset K(\mathrm{coh}_{cs}(X)) \leq H^{even}_{cs}(X, \mathbb{Q})$. For any nonzero coherent sheaf $\mathcal{E}$ with compact support on X, one can now define the Hilbert polynomial $P_{\mathcal{E}}(t) = \overline{\chi}(\mathcal{O}_X(-t), \mathcal{E})$, which only depends on the class $[\mathcal{E}] \in C(\mathrm{coh}_{cs}(X))$. Joyce–Song introduce a further technical condition on X [JS, Def. 6.26].

Definition 2.5.1. Let X be a noncompact Calabi–Yau threefold over $\mathbb{C}$. We call X *compactly embeddable* if for any compact subset $K \subset X$ in the complex analytic topology, there exists an open neighbourhood U of K in X in the complex analytic topology, a compact Calabi–Yau threefold Y over $\mathbb{C}$, an open subset V of Y in the complex analytic topology and an isomorphism of complex manifolds $\phi : U \longrightarrow V$. $\oslash$

If X is compactly embeddable, the entire discussion of the compact case holds similarly except for the following notable differences [JS, Sect. 6.7]. The moduli space $\mathcal{M}_{ss}^{\alpha}(\tau)$ can be defined similarly, but is not necessarily proper. Similarly, certain moduli spaces of pairs used in the theory of generalised Donaldson–Thomas invariants [JS, Sect. 6.7] do not have to be proper anymore. As a consequence, one does not have access to virtual cycle technology. This does not affect the definition of the generalised Donaldson–Thomas invariants. The pair invariants used in the theory of generalised Donaldson–Thomas invariants should now be defined using Behrend functions [JS, Sect. 6.7]. Similarly, equation (2.10) does not make sense, but one can use equation (2.11) as the definition of (original) Donaldson–Thomas invariants. With these changes, all the above statements hold similarly in this setting, except that the (generalised) Donaldson–Thomas invariants and pair invariants are not known to be invariant under continuous deformations of X, essentially because one cannot use virtual cycle technology anymore. Nevertheless, the (generalised) Donaldson–Thomas invariants and pair invariants of X are still very interesting. In short, (i) and (ii) still hold similarly, but we do not expect (iii) to hold.

Let X be a Calabi–Yau threefold (so for the moment compact). For any $\beta \in H_2(X, \mathbb{Z})$ and $g \in \mathbb{Z}_{\geq 0}$, let $GW_g(\beta) \in \mathbb{Q}$ be the genus g Gromov–Witten invariants of X. They are conjectured to satisfy the following identity [Kat]

$$\sum_{\beta,g} GW_g(\beta) q^{\beta} \lambda^{2g-2} = \sum_{\beta,g,m} \frac{GV_g(\beta)}{m} \left(2\sin\left(\frac{m\lambda}{2}\right)\right)^{2g-2} q^{m\beta}, \tag{2.13}$$

where the $GV_g(\beta)$ are certain integers. These integers are called Gopakumar–Vafa invariants and were first introduced as an integer-valued index arising from D-branes and M2-branes wrapping holomorphic curves in string theory and M-theory. One can use relation (2.13) as the definition of the Gopakumar–Vafa invariants in which case they are a priori only known to be rational. The statement that the Gopakumar–Vafa invariants are integers is then known as the Integrality Conjecture for Gopakumar–Vafa Invariants [Kat]. From relation (2.13), one deduces that the genus zero Gopakumar–Vafa invariants satisfy

$$GW_0(\beta) = \sum_{m\in\mathbb{Z}_{\geq 1}, m|\beta} \frac{1}{m^3} GV_0(\beta/m).$$

Using the Möbius inversion formula, we can invert this relation and take this as our definition of the genus zero Gopakumar–Vafa invariants. A priori genus zero Gopakumar–Vafa invariants are rational numbers and the Integrality Conjecture for Genus Zero Gopakumar–Vafa Invariants states they are integers. Similarly, for $(\tau, T, \leq)$ a Gieseker stability condition on $\mathrm{coh}(X)$ w.r.t. a very ample line bundle $\mathcal{O}_X(1)$ and for any $\alpha \in C(X)$, Joyce–Song define BPS invariants $\hat{DT}^{\alpha}(\tau)$ through the relation

$$\bar{DT}^{\alpha}(\tau) = \sum_{m\in\mathbb{Z}_{\geq 1}, m|\alpha} \frac{1}{m^2} \hat{DT}^{\frac{\alpha}{m}}(\tau).$$

A priori BPS invariants are rational numbers and the Integrality Conjecture for BPS Invariants states they are integers in the case τ is generic [JS, Conj. 6.12]. Here generic means for all $\alpha, \beta \in C(X)$ satisfying $\tau(\alpha) = \tau(\beta)$, one has $\overline{\chi}(\alpha, \beta) = 0$. In fact, Katz conjectures genus zero Gopakumar–Vafa invariants and BPS invariants are related [Kat]. This is phrased by Joyce–Song in the following conjecture [JS, Conj. 6.20].

Conjecture 2.5.2 (Katz' Conjecture). *Let X be a Calabi–Yau threefold and $(\tau, T, \leq)$ a Gieseker stability condition on* $\mathrm{coh}(X)$. *Then for any $\gamma \in H_2(X, \mathbb{Z})$ with $\beta \in H^4(X, \mathbb{Z})$ Poincaré dual to γ and $k \in \mathbb{Z}$, we have $\hat{DT}^{(0,0,\beta,k)}(\tau) = GV_0(\gamma)$. In particular, the BPS invariants $\hat{DT}^{(0,0,\beta,k)}(\tau)$ for $\beta \in H^4(X, \mathbb{Z})$, $k \in \mathbb{Z}$ are independent of τ, k.*

Note that this entire discussion of Gopakumar–Vafa invariants, BPS invariants and Katz' Conjecture (Conjecture 2.5.2) also makes sense similarly for X a noncompact compactly embeddable Calabi–Yau threefold, where we have to use coherent sheaves with compact support and (co)homology with compact support everywhere.

In the next subsection, we will be interested in the case X is the canonical bundle K_S of a nonsingular projective surface S. Then X is a noncompact Calabi–Yau threefold. Although, it is known that X is compactly embeddable for certain surfaces like $S = \mathbb{P}^2$ or $S = \mathbb{P}^1 \times \mathbb{P}^1$, this is not known for general S [JS, Sect. 6.7]. However, for the definition of generalised Donaldson–Thomas invariants, BPS invariants, Gromov–Witten invariants and Gopakumar–Vafa invariants only, we do not need to know whether X is compactly embeddable. Therefore, we will not address this question. In this setting, the discussion of (i) still holds similarly except that we have no virtual fundamental class at our disposal. Hence we take the expression of (original) Donaldson–Thomas invariants in terms of Behrend functions (2.11) as the definition of (original) Donaldson–Thomas invariants. In short, (i) still holds similarly, but it is not known whether (ii) holds and (iii) is not expected to hold in this setting. Also note that Katz' Conjecture (Conjecture 2.5.2) can still be phrased in this setting.

2.5.2 Some Generating Functions of Donaldson–Thomas Invariants

Let X be a compact/non-compact Calabi–Yau threefold over $\mathbb{C}$. Let $\mathcal{O}_X(1)$ be a very ample line bundle on X and $(\tau, T, \leq)$ a Gieseker stability condition on $\mathrm{coh}(X)$ w.r.t. $\mathcal{O}_X(1)$. Consider $\alpha \in C(X) \subset K(X) \leq H^{even}(X, \mathbb{Q})$ (or $\alpha \in C(\mathrm{coh}_{cs}(X)) \subset K(\mathrm{coh}_{cs}(X)) \leq H^{even}_{cs}(X, \mathbb{Q})$). We can write $\alpha = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$. Fixing $\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2$ and varying ch_3, one is interested in the generating function for the BPS invariants $\hat{DT}^{(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)}(\tau)$. Since knowledge of Chern character ch on the one hand and rank r and Chern class c on the other hand is equivalent (i.e. $(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3) = (r, c_1, \frac{1}{2}(c_1^2 - 2c_2), \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3))$ [Har1, App. A]), we denote the corresponding BPS invariants by $\hat{DT}^{(r, c_1, c_2, c_3)}(\tau)$. Fixing rank r, first Chern class c_1 and second Chern class c_2 arbitrary, we are interested in the generating function of BPS invariants

$$\sum_{\mathrm{c}_3\in\mathbb{Z}} \hat{DT}^{(\mathrm{c}_0,\mathrm{c}_1,\mathrm{c}_2,\mathrm{c}_3)}(\tau)q^{\mathrm{c}_3}.$$

Note that we use $H^6(X,\mathbb{Z})\cong\mathbb{Z}$ (or $H^6_{cs}(X,\mathbb{Z})\cong\mathbb{Z}$). This generating function is of great interest to both mathematicians and string theorists. The simplest case to consider is $(r,c_1,c_2)=(1,0,0)$. In this case, BPS invariants, generalised Donaldson–Thomas invariants and Donaldson–Thomas invariants coincide. In this context, it is somewhat more natural to consider Donaldson–Thomas invariants $\tilde{N}_{n,0}$ associated to the moduli space $I_n(X,0)$ of ideal sheaves $\mathcal{I}$ on compact X determining a subscheme Y of dimension 0 such that $\chi(\mathcal{O}_Y)=n$. An elegant expression for the generating function was first conjectured by Maulik, Nekrasov, Okounkov and Pandharipande [MNOP] and later proved by Behrend and Fantechi [BF]

$$\sum_{n=0}^{\infty}\tilde{N}_{n,0}q^n=M(-q)^{e(X)},$$

where $M(q)=\prod_{k\geq 1}(1-q^k)^{-k}$ is the MacMahon function counting 3D partitions. Note that in this case, there is no dependence on choice of stability condition $(\tau,T,\leq)$. It is interesting to compute the generating function of BPS invariants for other ranks $r=0$ and $r>1$. We generically expect a dependence on choice of stability condition $(\tau,T,\leq)$.

Let $X=K_S$ be the canonical bundle of a nonsingular projective surface S. Then X is a noncompact Calabi–Yau threefold. Consider the map to the base and the inclusion of the zero section

$$\pi: X\longrightarrow S,$$

$$\iota: S\hookrightarrow X.$$

Using Poincaré Duality PD, we get induced maps on the level of cohomology

$$\mathrm{PD}^{-1} \circ \iota_* \circ \mathrm{PD} : H^0(S, \mathbb{Z}) \longrightarrow H^2_{cs}(X, \mathbb{Z}),$$
$$\mathrm{PD}^{-1} \circ \iota_* \circ \mathrm{PD} : H^2(S, \mathbb{Z}) \longrightarrow H^4_{cs}(X, \mathbb{Z}),$$
$$\mathrm{PD}^{-1} \circ \iota_* \circ \mathrm{PD} : H^4(S, \mathbb{Z}) \longrightarrow H^6_{cs}(X, \mathbb{Z}).$$

Under the canonical isomorphisms $H^4(S, \mathbb{Z}) \cong \mathbb{Z}$ and $H^6_{cs}(X, \mathbb{Z}) \cong \mathbb{Z}$, the last map is the identity map. We will be somewhat sloppy in our notation an write ι_* to mean $\mathrm{PD}^{-1} \circ \iota_* \circ \mathrm{PD}$, unless stated otherwise. Let H be an ample divisor on S, then π^*H is an ample divisor on X since π is affine [EGA1, Prop. 5.1.12]. Consider μ-stability $(\mu_H, M, \leq)$ on $\mathrm{coh}(S)$ w.r.t H and μ-stability $(\mu_{\pi^*H}, M, \leq)$ on $\mathrm{coh}_{cs}(X)$ w.r.t. π^*H. Let $r \in \mathbb{Z}_{\geq 0}$ and $c_1 \in H^2(S, \mathbb{Z})$ be arbitrary. We are interested in the generating function[16]

$$\sum_{\mathrm{c}_3 \in \mathbb{Z}} \hat{DT}^{(0, \iota_* r, \iota_* c_1, \mathrm{c}_3)}(\mu_{\pi^*H}) q^{\mathrm{c}_3}.$$

For a fixed $c_2 \in \mathbb{Z}$, denote by $M_S^H(r, c_1, c_2)$ the moduli space of μ-stable pure sheaves on S of rank r, first Chern class c_1 and second Chern class c_2. In the case $r = 0$, Gieseker and μ-stability coincide. For a fixed $c_3 \in \mathbb{Z}$, denote by $M_X^{\pi^*H}(0, \iota_* r, \iota_* c_1, c_3)$ the moduli space of μ-stable pure sheaves on X of rank 0, first Chern class $\iota_* r$, second Chern class $\iota_* c_1$, third Chern class c_3 and with proper support. Recall that the notion of μ-stability for coherent sheaves in general is defined in [HL, Def.–Cor. 1.6.9] using quotient categories. The notion of pure is as usual (Definition 1.1.2 and [HL, Def. 1.1.2]). Before we proceed, we need the following propositions about these moduli spaces.

Proposition 2.5.3. *Let S be a nonsingular projective surface. Let $r \in \mathbb{Z}_{>0}$, $c_1 \in H^2(S, \mathbb{Z})$, $c_2 \in \mathbb{Z}$. Let H an ample divisor on S such that $-K_S \cdot H > 0$. Then $M_S^H(r, c_1, c_2)$ is a nonsingular quasi-projective variety of dimension $(1-r)c_1^2 + 2rc_2 - r^2\chi(\mathcal{O}_S) + 1$ unless it is empty*[17].

[16] Generalised Donaldson–Thomas invariants and BPS invariants are also defined for μ-stability conditions [JS, Sect. 5, 6].

[17] In this proposition and the next one, varieties are allowed to be reducible. We do not address the question of irreducibility.

Proof. This proposition is proved for $S = \mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$) and $r = 2$ in [Nak, Prop. 1.2]. We proceed similarly. Let $[\mathcal{E}] \in M_S^H(r, c_1, c_2)$ be a closed point. It is sufficient to prove $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$, in which case $M_S^H(r, c_1, c_2)$ is nonsingular at $[\mathcal{E}]$ of dimension $(1-r)c_1^2 + 2rc_2 - r^2\chi(\mathcal{O}_S) + 1$ by [Mar, Prop. 6.9], [HL, Cor. 4.5.2]. By Serre Duality, it is sufficient to show $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_S) = 0$, where ω_S is the canonical bundle of S. As $\mathcal{E}$ is μ-stable torsion free, it is easy to see that $\mathcal{E} \otimes \omega_S$ is a μ-stable torsion free sheaf on S. Using the Hirzebruch–Riemann–Roch Theorem [Har1, Thm. A.4.1], we see the Hilbert polynomials of $\mathcal{E}$, $\mathcal{E} \otimes \omega_S$ are related by

$$P_{\mathcal{E} \otimes \omega_S}(t) = P_{\mathcal{E}}(t) + (K_S \cdot H)rt + \frac{r}{2}K_S^2 + K_S \cdot \left(c_1 + \frac{r}{2}c_1(\mathcal{T}_S)\right),$$

where $\mathcal{T}_S$ is the tangent bundle of S and K_S denotes the canonical divisor of S. Consequently, $\mu_{\mathcal{E}} > \mu_{\mathcal{E} \otimes \omega_S}$ and the result follows from [HL, Prop. 1.2.7]. $\square$

Proposition 2.5.4. *Let $S = \mathbb{P}^2$ and H an ample divisor on S. Let*[18] *$r = 0$, $c_1 \in \mathbb{Z}_{>0}$ and $c_2 \in \mathbb{Z}$. Then $M_S^H(r, c_1, c_2)$ is a nonsingular quasi-projective variety of dimension $c_1^2 + 1$.*

Proof. This proposition is proved in [Pot, Prop. 2.3]. We show the proof except for the computation of the dimension. Let $[\mathcal{E}] \in M_S^H(r, c_1, c_2)$ be a closed point. Proceeding as in the proof of the previous proposition, we see it is sufficient to prove $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$, in which case $M_S^H(r, c_1, c_2)$ is nonsingular at $[\mathcal{E}]$. Using Serre Duality, it is sufficient to show $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_S) = 0$. For an arbitrary coherent sheaf $\mathcal{G}$ of dimension 1 on X, which consequently satisfies $c_1(\mathcal{G}) \cdot H > 0$, an explicit computation as in the proof of the previous proposition shows

$$\mu_{\mathcal{G} \otimes \omega_S} = \mu_{\mathcal{G}} + \frac{c_1(\mathcal{G}) \cdot K_S}{c_1(\mathcal{G}) \cdot H}.$$

Now let H be $\alpha \in \mathbb{Z}_{>0}$ times the class of a hyperplane section. Then $\mu_{\mathcal{G} \otimes \omega_S} = \mu_{\mathcal{G}} - \frac{3c_1}{\alpha}$.

[18] Here we use the usual isomorphisms $H^0(S, \mathbb{Z}) \cong \mathbb{Z}$, $H^2(S, \mathbb{Z}) \cong \mathbb{Z}$, $H^4(S, \mathbb{Z}) \cong \mathbb{Z}$ described in section 2.4.

From this it is easy to deduce that $\mathcal{E} \otimes \omega_S$ is μ-stable with $\mu_{\mathcal{E}} > \mu_{\mathcal{E}\otimes\omega_S}$. The result follows from [HL, Prop. 1.2.7]. $\square$

Proposition 2.5.5. *Let S be a nonsingular projective surface, H an ample divisor on S, $r \in \mathbb{Z}_{\geq 0}$, $c_1 \in H^2(S, \mathbb{Z})$ and $c_2 = k \in \mathbb{Z}$. Assume $\omega_S^{\vee}$ is generated by global sections and for all closed points $[\mathcal{E}] \in M_S^H(r, c_1, k)$ one has $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$. Then the canonical regular action $\sigma : \mathbb{C}^* \times K_S \longrightarrow K_S$ on the fibres, lifts to a regular action of $\mathbb{C}^*$ on $M_X^{\pi^*H}(0, \iota_* r, \iota_* c_1, k)$ and there is an isomorphism of $\mathbb{C}$-schemes*

$$M_X^{\pi^*H}(0, \iota_* r, \iota_* c_1, k)^{\mathbb{C}^*} \cong M_S^H(r, c_1, k).$$

Moreover, $M_X^{\pi^*H}(0, \iota_* r, \iota_* c_1, k)^{\mathbb{C}^*}$ is both open and closed in $M_X^{\pi^*H}(0, \iota_* r, \iota_* c_1, k)$.

Proof. We will use the Grothendieck spectral sequence repeatedly in this proof. For any two coherent sheaves $\mathcal{A}$, $\mathcal{B}$ on S, we apply it as follows ([Wbl, Sect. 5.8])

$$E_2^{pq} = \mathcal{H}^p \mathbf{R}\mathrm{Hom}(L_q \iota^* \iota_* \mathcal{A}, \mathcal{B}) \Rightarrow \mathcal{H}^{p+q} \mathbf{R}\mathrm{Hom}(\mathbf{L}\iota^* \iota_* \mathcal{A}, \mathcal{B}). \tag{2.14}$$

Here $L_q \iota^* \iota_* \mathcal{A}$ is only nonzero for $q = 0, 1$, in which case $L_0 \iota^* \iota_* \mathcal{A} \cong \mathcal{B}$ and $L_1 \iota^* \iota_* \mathcal{A} \cong \mathcal{A} \otimes \mathcal{N}_{S/X}^{\vee}$ [KM, Lem. 1.3.1]. The Calabi–Yau property $\omega_X \cong \mathcal{O}_X$ implies $L_1 \iota^* \iota_* \mathcal{A} \cong \mathcal{A} \otimes \omega_S^{\vee}$ [Har1, Prop. II.8.20]. We obtain an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\mathcal{A}, \mathcal{B}) \xrightarrow{\iota_*} \mathrm{Ext}^1(\iota_* \mathcal{A}, \iota_* \mathcal{B}) \longrightarrow \mathrm{Hom}(\mathcal{A}, \mathcal{B} \otimes \omega_S) \longrightarrow \cdots,$$

where we denote the map $\mathrm{Ext}^1(\iota_* \mathcal{A}, \iota_* \mathcal{B}) \longrightarrow \mathrm{Hom}(\mathcal{A}, \mathcal{B} \otimes \omega_S)$ by α.

Define abbreviations $M = M_S^H(r, c_1, k)$ and $N = M_X^{\pi^*H}(0, \iota_* r, \iota_* c_1, k)$ for the moduli spaces and $\underline{M} = \underline{M}_S^H(r, c_1, k)$ and $\underline{N} = \underline{M}_X^{\pi^*H}(0, \iota_* r, \iota_* c_1, k)$ for the moduli functors. We start by constructing a natural transformation ι_* on the level of moduli functors

$$\iota_* : \underline{M} \Rightarrow \underline{N},$$
$$\iota_{*,B} : \underline{M}(B) \longrightarrow \underline{N}(B), \ [\mathcal{F}] \mapsto [(\iota \times 1_B)_* \mathcal{F}].$$

It is easy to verify this is a well-defined natural transformation. Here we use that ι and π are affine morphisms. Consequently, we have an induced morphism $\iota_* : M \longrightarrow N$, which on the level of closed points is just push-forward along ι. Now consider the $\mathbb{C}^*$-fixed point locus, which is a closed subscheme $j : N^{\mathbb{C}^*} \hookrightarrow N$ (compare to Proposition 1.3.1). We claim ι_* factors as an isomorphism through the fixed point locus. We want to apply Proposition 1.3.7, i.e. prove that for any local artinian $\mathbb{C}$-algebra with residue field $\mathbb{C}$, the map $\iota_* \circ -$ factors bijectively

$$\begin{tikzcd} \mathrm{Hom}(A, M) \arrow[r, "\iota_* \circ -"] \arrow[dr, dashed, "\cong"'] & \mathrm{Hom}(A, N) \\ & \mathrm{Hom}(A, N^{\mathbb{C}}). \arrow[u, hook, "j \circ -"'] \end{tikzcd}$$

We proceed similarly as in the proof of Theorem 1.3.9, i.e. by induction on the length l of local artinian $\mathbb{C}$-algebras with residue field $\mathbb{C}$.

Suppose $l = 1$, then $A \cong \mathbb{C}$. Since $t \circ \iota = \iota$ for all closed points $t \in \mathbb{C}^*$, it is clear that $\iota_* \circ -$ maps closed points into the fixed point locus. Clearly, $\iota_* \circ -$ is injective on closed points since $\pi \circ \iota = \mathrm{id}_S$. We now prove $\iota_* \circ -$ is surjective on closed points. We will prove that any simple sheaf $\mathcal{E} \neq 0$ on X with support contained in the zero section is isomorphic to $\iota_* \mathcal{F}$ for some coherent sheaf $\mathcal{F}$ on S. This suffices since any closed point $[\mathcal{E}] \in N^{\mathbb{C}^*}$ has these properties. Using a trivialisation $U_\alpha = \mathrm{Spec}\ A_\alpha$ of the canonical bundle, the inclusion $\iota : U_\alpha \hookrightarrow U_\alpha \times \mathbb{A}^1$ looks like $\iota^\# = \mathrm{ev}_0 : A_\alpha[x] \longrightarrow A_\alpha$, where x corresponds to the coordinate on the fibre. The coherent sheaf $\mathcal{E}|_{U_\alpha}$ corresponds to a finitely generated $A_\alpha[x]$-module E_α and since the support of $\mathcal{E}$ lies in the zero section, we can form a filtration of E_α

$$0 \subset \ker x \subset \ker x^2 \subset \cdots \subset \ker x^{N_\alpha} = E_\alpha,$$

for some $N_\alpha \in \mathbb{Z}_{\geq 1}$. We can glue these to a filtration of $\mathcal{E}$

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_N = \mathcal{E},$$

for some $N \in \mathbb{Z}_{\geq 1}$. By throwing out terms, we can assume all inclusions are strict. It suffices to prove $N = 1$. Clearly, $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \iota_* \mathcal{F}_i$ for some coherent sheaf $\mathcal{F}_i$ on S for all $i = 1, \ldots, N$. For any $i = 2, \ldots, N$, consider the extension

$$0 \longrightarrow \iota_* \mathcal{F}_{i-1} \longrightarrow \mathcal{E}_i/\mathcal{E}_{i-2} \longrightarrow \iota_* \mathcal{F}_i \longrightarrow 0,$$

which we denote by ϵ_i. Using the Grothendieck spectral sequence (2.14), we get a morphism $\alpha(\epsilon_i) : \mathcal{F}_i \longrightarrow \mathcal{F}_{i-1} \otimes \omega_S$. We claim $\alpha(\epsilon_i)$ is injective. This can be seen as follows. Note that the kernel $\ker \alpha(\epsilon_i)$ is part of a short exact sequence

$$0 \longrightarrow \iota_* \mathcal{F}_{i-1} \longrightarrow \mathcal{G}_i/\mathcal{E}_{i-2} \longrightarrow \iota_* \ker \alpha(\epsilon_i) \longrightarrow 0,$$

for some coherent sheaf $\mathcal{G}_i$ containing $\mathcal{E}_{i-1}$. By construction, this extension maps to zero under α. Hence we deduce from the Grothendieck spectral sequence (2.14) that $\mathcal{G}_i/\mathcal{E}_{i-2} \cong \iota_* \mathcal{H}_i$ for some coherent sheaf $\mathcal{H}_i$ on S. From the definition of the $\mathcal{E}_i$, we deduce $\iota_* \mathcal{F}_{i-1} \cong \mathcal{G}_i/\mathcal{E}_{i-2}$, so $\ker \alpha(\epsilon_i) = 0$. We get morphisms

$$\mathcal{E} \twoheadrightarrow \iota_* \mathcal{F}_N \hookrightarrow \iota_*(\mathcal{F}_{N-1} \otimes \omega_S) \hookrightarrow \cdots \hookrightarrow \iota_*(\mathcal{F}_1 \otimes \omega_S^{N-1}) \xrightarrow{s} \iota_* \mathcal{F}_1 \hookrightarrow \mathcal{E},$$

where we take $s \in \Gamma(S, \omega_S^{-(N-1)})$ some global section. Here the first morphism is surjective and all other morphisms are injective except the one induced by s. Using the fact that $\omega_S^{-(N-1)}$ is generated by global sections and choosing the right section s, this composition $\mathcal{E} \longrightarrow \mathcal{E}$ is non-zero. But since $\mathcal{E}$ is simple, this can only be the case when $N = 1$.

Now assume we have proved the induction hypothesis for all local artinian $\mathbb{C}$-algebras with residue field $\mathbb{C}$ and length $1, \ldots, l$. Let A' be a local artinian $\mathbb{C}$-algebra with residue field $\mathbb{C}$ and length $l+1$. Then A' fits in a small extension $0 \longrightarrow J \longrightarrow A' \xrightarrow{\sigma} A \longrightarrow 0$, where A is a local artinian $\mathbb{C}$-algebra with residue field $\mathbb{C}$ and length $\leq l$. Using [Fog, Thm. 2.3], one can show that the image of $\mathrm{Hom}(A', N^{\mathbb{C}^*})$ in $\mathrm{Hom}(A', N)$ is $\mathrm{Hom}(A', N)^{\mathbb{C}^*_{cl}}$, where cl means closed points. We drop the subscript cl from now on. It is sufficient to prove $\underline{M}(A') \longrightarrow \underline{N}(A')$ maps bijectively onto $\underline{N}(A')^{\mathbb{C}^*}$ (compare to Proposition 1.3.8). Recall the notation for deformation functors in the proof of Theorem 1.3.9. We rewrite the sets $\underline{M}(A')$, $\underline{N}(A')$ in terms of deformation functors

$$\underline{M}(A') = \coprod_{[\mathcal{F}] \in \underline{M}(A)} \mathcal{D}_{\mathcal{F} \otimes_{\mathbb{C}} A}(\sigma)^{-1}([\mathcal{F}]),$$

$$\underline{N}(A') = \coprod_{[\mathcal{F}] \in \underline{N}(A)} \mathcal{D}_{\mathcal{F} \otimes_{\mathbb{C}} A}(\sigma)^{-1}([\mathcal{F}]).$$

By the induction hypothesis, we know $\underline{M}(A) \longrightarrow \underline{N}(A)$ maps bijectively onto $\underline{N}(A)^{\mathbb{C}^*}$. Let $\mathcal{F} \in \underline{M}(A)$, we have to prove that $\mathcal{D}_{\mathcal{F} \otimes_{\mathbb{C}} A}(\sigma)^{-1}([\mathcal{F}])$ is non-empty if and only if $\mathcal{D}_{\iota_*(\mathcal{F} \otimes_{\mathbb{C}} A)}(\sigma)^{-1}([(\iota \times 1_A)_*(\mathcal{F})])$ is non-empty and if this is the case $\mathcal{D}_{\mathcal{F} \otimes_{\mathbb{C}} A}(\sigma)^{-1}([\mathcal{F}])$ maps canonically bijectively onto $\mathcal{D}_{\iota_*(\mathcal{F} \otimes_{\mathbb{C}} A)}(\sigma)^{-1}([(\iota \times 1_A)_*(\mathcal{F})])^{\mathbb{C}^*}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{D}_{\mathcal{F} \otimes_A \mathbb{C}}(A) \arrow[r, "\mathfrak{o}(\sigma)"] \arrow[d, "(\iota \times 1_A)_*"'] & \operatorname{Ext}^2(\mathcal{F} \otimes_A \mathbb{C}, \mathcal{F} \otimes_A \mathbb{C}) \otimes_{\mathbb{C}} J \arrow[d, "\iota_*"] \\
\mathcal{D}_{\iota_*(\mathcal{F} \otimes_A \mathbb{C})}(A) \arrow[r, "\mathfrak{o}(\sigma)"'] & \operatorname{Ext}^2(\iota_*(\mathcal{F} \otimes_A \mathbb{C}), \iota_*(\mathcal{F} \otimes_A \mathbb{C})) \otimes_{\mathbb{C}} J.
\end{tikzcd}$$

Here we recall ι_* is exact, because ι is affine. By assumption, $\operatorname{Ext}^2(\mathcal{F} \otimes_A \mathbb{C}, \mathcal{F} \otimes_A \mathbb{C}) = 0$, so $\mathcal{F}$ and $(\iota \times 1_A)_* \mathcal{F}$ are automatically unobstructed. Therefore, $\mathcal{D}_{\mathcal{F} \otimes_{\mathbb{C}} A}(\sigma)^{-1}([\mathcal{F}])$ and $\mathcal{D}_{\iota_*(\mathcal{F} \otimes_{\mathbb{C}} A)}(\sigma)^{-1}([(\iota \times 1_A)_*(\mathcal{F})])$ are automatically non-empty. The former is an $\operatorname{Ext}^1(\mathcal{F} \otimes_{\mathbb{C}} A, \mathcal{F} \otimes_{\mathbb{C}} A) \otimes_{\mathbb{C}} J$-torsor and the latter is a $\operatorname{Ext}^1(\iota_*(\mathcal{F} \otimes_{\mathbb{C}} A), \iota_*(\mathcal{F} \otimes_{\mathbb{C}} A)) \otimes_{\mathbb{C}} J$-torsor. It suffices to construct a canonical isomorphism

$$\operatorname{Ext}^1(\mathcal{F} \otimes_{\mathbb{C}} A, \mathcal{F} \otimes_{\mathbb{C}} A) \cong \operatorname{Ext}^1(\iota_*(\mathcal{F} \otimes_{\mathbb{C}} A), \iota_*(\mathcal{F} \otimes_{\mathbb{C}} A))^{\mathbb{C}^*}.$$

In fact, for any closed point $[\mathcal{E}] \in M_S^H(r, c_1, k)$, the Grothendieck spectral sequence (2.14) yields an exact sequence

$$0 \longrightarrow \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) \xrightarrow{\iota_*} \operatorname{Ext}^1(\iota_*\mathcal{E}, \iota_*\mathcal{E}) \longrightarrow \operatorname{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_S) \longrightarrow \cdots .$$

Serre Duality and the assumption $\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) = 0$ implies $\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \cong \mathrm{Ext}^1(\iota_*\mathcal{E},\iota_*\mathcal{E})^{\mathbb{C}^*} = \mathrm{Ext}^1(\iota_*\mathcal{E},\iota_*\mathcal{E})$. As a by-product, we get $\mathrm{Ext}^1(\iota_*\mathcal{E},\iota_*\mathcal{E}) = \mathrm{Ext}^1(\iota_*\mathcal{E},\iota_*\mathcal{E})^{\mathbb{C}^*}$, since $\mathrm{Ext}^1(\mathcal{E},\mathcal{E})$ maps into $\mathrm{Ext}^1(\iota_*\mathcal{E},\iota_*\mathcal{E})^{\mathbb{C}^*}$. This implies $N^{\mathbb{C}^*}$ is not only closed, but also open in N. □

Proposition 2.5.6. *Let S be a nonsingular projective surface satisfying one of the following two conditions:*

(i) *Assume $\omega_S^\vee$ is generated by global sections. Let H be an ample divisor on S such that $-K_S \cdot H > 0$. Let $r \in \mathbb{Z}_{>0}$, $c_1 \in H^2(S,\mathbb{Z})$, $c_2 = k \in \mathbb{Z}$. Assume $(r, c_1 \cdot H) = 1$, so there are no strictly μ-semistable torsion free sheaves on S with rank r, first Chern class c_1 and second Chern class c_2.*

(ii) *Let $S = \mathbb{P}^2$ and H an ample divisor on S. Let[19] $r = 0$, $c_1 \in \mathbb{Z}_{>0}$ and $c_2 = k \in \mathbb{Z}$. Then the Hilbert polynomial is $(c_1 \cdot H)t + \chi$, where $\chi = \frac{1}{2}c_1(c_1+3) - c_2$. Assume $(c_1, \chi) = 1$, so there are no strictly μ-semistable pure sheaves on S with rank r, first Chern class c_1 and second Chern class c_2.*

Let $\pi : X = K_S \longrightarrow S$ be the canonical bundle. Then π^*H is an ample divisor on X and there are no strictly μ-semistable sheaves on X w.r.t. π^*H with proper support, rank 0, first Chern class ι_*r, second Chern class ι_*c_1 and third Chern class k. Furthermore,

$$\hat{DT}^{(0,\iota_*r,\iota_*c_1,k)}(\mu_{\pi^*H}) = DT^{(0,\iota_*r,\iota_*c_1,k)}(\mu_{\pi^*H}) = (-1)^{rc_1+c_1+r\chi(\mathcal{O}_S)+1}e(M_S^H(r,c_1,k)).$$

Proof. Using the Hirzebruch–Riemann–Roch Theorem [Har1, Thm. A.4.1], it is easy to see that for numerical reasons there cannot be any strictly μ-semistable pure sheaves on S with the required topological invariants. If $\mathcal{E}$ is a coherent sheaf on X with proper support, then $\pi_*\mathcal{E}$ is a coherent sheaf on S and $P_{\mathcal{E}}(t) = P_{\pi_*\mathcal{E}}(t)$ since π is affine and by the Projection Formula. Using $\pi_* \circ \iota_* = (\pi \circ \iota)_* = 1$ on singular homology, it is easy to see that again for numerical reasons there cannot be any strictly μ-semistable pure sheaves on X with the required topological invariants. Consequently, BPS invariants reduce to

[19] Here we use the usual isomorphisms $H^0(S,\mathbb{Z}) \cong \mathbb{Z}$, $H^2(S,\mathbb{Z}) \cong \mathbb{Z}$, $H^4(S,\mathbb{Z}) \cong \mathbb{Z}$ described in section 2.4.

Donaldson–Thomas invariants (see subsection 2.5.1, property (i) and [JS, Prop. 6.11]). Since $M = M_S^H(r, c_1, c_2)$ is a nonsingular projective variety of dimension $(1-r)c_1^2+2rc_2-r^2\chi(\mathcal{O}_S)+1$ (or empty) by Propositions 2.5.3, 2.5.4, its Behrend function is given by $\nu_M = (-1)^{rc_1+c_1+r\chi(\mathcal{O}_S)+1}$ (see subsection 2.5.1, property (i)). Denote $N = M_X^{\pi^*H}(0, \iota_* r, \iota_* c_1, k)$, then the regular action of $\mathbb{C}^*$ on the fibres of X lifts to a regular action of $\mathbb{C}^*$ on N and $N^{\mathbb{C}^*} \cong M$ by Proposition 2.5.5. Since the Behrend function ν_N is $\mathbb{C}^*$-invariant, we have

$$\hat{DT}^{(0,\iota_* r,\iota_* c_1,k)}(\mu_{\pi^*H}) = DT^{(0,\iota_* r,\iota_* c_1,k)}(\mu_{\pi^*H}) = \int_N \nu_N \mathrm{d}e = \int_{N^{\mathbb{C}^*}} \nu_N|_{N^{\mathbb{C}^*}} \mathrm{d}e.$$

In our case, $N^{\mathbb{C}^*}$ is both open and closed in N by Proposition 2.5.5, so $\nu_N|_{N^{\mathbb{C}^*}} = \nu_{N^{\mathbb{C}^*}}$, which concludes the proof. $\square$

Note that the above proposition gives the following expressions for the generating functions in case (i) resp. case (ii)

$$\begin{aligned}
\sum_{c_3\in\mathbb{Z}} \hat{DT}^{(0,\iota_* r,\iota_* c_1,c_3)}(\mu_{\pi^*H})q^{c_3} &= \sum_{c_3\in\mathbb{Z}} DT^{(0,\iota_* r,\iota_* c_1,c_3)}(\mu_{\pi^*H})q^{c_3} \\
&= (-1)^{rc_1+c_1+r\chi(\mathcal{O}_S)+1} \sum_{c_2\in\mathbb{Z}} e(M_S^H(r, c_1, c_2))q^{c_2}, \\
\sum_{\substack{c_3\in\mathbb{Z} \\ (c_1, \frac{1}{2}c_1(c_1+3)-c_3)=1}} \hat{DT}^{(0,0,\iota_* c_1,c_3)}(\mu_{\pi^*H})q^{c_3} &= \sum_{\substack{c_3\in\mathbb{Z} \\ (c_1, \frac{1}{2}c_1(c_1+3)-c_3)=1}} DT^{(0,0,\iota_* c_1,c_3)}(\mu_{\pi^*H})q^{c_3} \\
&= (-1)^{c_1+1} \sum_{\substack{c_2\in\mathbb{Z} \\ (c_1, \frac{1}{2}c_1(c_1+3)-c_2)=1}} e(M_S^H(0, c_1, c_2))q^{c_2}.
\end{aligned}$$

Let S be the projective plane $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or one of the Hirzebruch surfaces $\mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 1}$). For $\mathbb{P}^2$, take H any ample divisor, $(r, c_1) = (1, c_1)$ for any $c_1 \in \mathbb{Z}$, or $(r, c_1) = (2, 1)$, or $(r, c_1) = (3, \pm 1)$. This data satisfies the conditions[20] of Proposition 2.5.6(i). Consequently, we obtain generating functions for Donaldson–Thomas invariants of $X = K_S$ (Corollaries 2.3.1, 2.3.2, 2.3.9, subsection 2.3.3). These generating functions

[20]Strictly speaking, in Proposition 2.5.6(i) we need $(r, c_1 D \cdot H) = 1$, where D is the class of any projective line in $\mathbb{P}^2$. However for $S = \mathbb{P}^2$, Proposition 2.5.6(i) clearly also holds for $(r, c_1) = 1$.

do not depend on choice of ample divisor H on $\mathbb{P}^2$. Using [Ful, Sect. 3.4], $\mathbb{F}_a$ ($a \in \mathbb{Z}_{\geq 0}$) has anticanonical bundle generated by global sections if and only if $a = 0, 1, 2$ (which are exactly the cases where the anticanonical divisor is nef). For $\mathbb{F}_a$ ($a = 0, 1, 2$), take H an ample divisor, $(r, c_1) = (1, c_1)$ for any $c_1 \in H^2(S, \mathbb{Z})$, or $(r, c_1) = (2, c_1)$ for any $c_1 \in H^2(S, \mathbb{Z})$ such that $(2, c_1 \cdot H) = 1$. Recall that a divisor $H = \alpha D_1 + \beta D_2$ ($\alpha, \beta \in \mathbb{Z}$) is ample if and only if $\alpha > a\beta$ and $\beta > 0$ (see 2.3.2.2). Also, for an arbitrary first Chern class $c_1 = f_3 D_3 + f_4 D_4$ ($f_3, f_4 \in \mathbb{Z}$), the condition $(2, c_1 \cdot H) = 1$ reads $(2, \beta f_3 + \alpha f_4) = 1$ (see 2.3.2.2). This data satisfies the conditions of Proposition 2.5.6(i). Hence, we obtain generating functions for Donaldson–Thomas invariants of $X = K_S$ and see the explicit dependence on ample divisor $\pi^* H$ (Corollaries 2.3.1, 2.3.3, 2.3.4, 2.3.5). Finally, we take $r = 0$, $S = \mathbb{P}^2$ and $c_1 = 1, 2, 3 \in \mathbb{Z}_{>0}$. From Proposition 2.5.6(ii) and Theorem 2.4.1, we obtain

$$\sum_{\mathrm{c}_3 \in \mathbb{Z}} DT^{(0,0,\iota_* 1, \mathrm{c}_3)}(\mu_{\pi^* H}) q^{\mathrm{c}_3} = \sum_{k \in \mathbb{Z}} 3 q^k,$$

$$\sum_{\mathrm{c}_3 \in \mathbb{Z},\ \mathrm{c}_3 \equiv 0 \bmod 2} DT^{(0,0,\iota_* 2, \mathrm{c}_3)}(\mu_{\pi^* H}) q^{\mathrm{c}_3} = \sum_{k \in \mathbb{Z}} -6 q^{2k},$$

$$\sum_{\mathrm{c}_3 \in \mathbb{Z},\ \mathrm{c}_3 \equiv \pm 1 \bmod 3} DT^{(0,0,\iota_* 3, \mathrm{c}_3)}(\mu_{\pi^* H}) q^{\mathrm{c}_3} = \sum_{k \in \mathbb{Z}} 27 q^{3k-1} + \sum_{k \in \mathbb{Z}} 27 q^{3k+1}.$$

Katz' Conjecture (Conjecture 2.5.2) implies these can be seen as generating functions of genus zero Gopakumar–Vafa invariants of $X = K_{\mathbb{P}^2}$

$$\sum_{\substack{c_3 \in \mathbb{Z} \\ (c_1, \frac{1}{2} c_1 (c_1 + 3) - c_3) = 1}} \hat{DT}^{(0,0,\iota_* c_1, c_3)}(\mu_{\pi^* H}) q^{c_3} = \sum_{\substack{c_3 \in \mathbb{Z} \\ (c_1, \frac{1}{2} c_1 (c_1 + 3) - c_3) = 1}} DT^{(0,0,\iota_* c_1, c_3)}(\mu_{\pi^* H}) q^{c_3} = GV_0(\iota_* c_1) \sum_{k \in \mathbb{Z}} q^k.$$

Here $\iota_* c_1$ in the last generating function actually means its Poincaré dual. Note that the fact that 27 appears twice in the $c_1 = 3$ case supports part of Katz' Conjecture as formulated by Joyce–Song in this example (Conjecture 2.5.2). In the $c_1 = 3$ case, it is interesting to see how the various fixed points of the moduli space of μ-stable pure dimension 1 sheaves on $\mathbb{P}^2$ have different types of characteristic functions depending on $c_2 \equiv \pm 1 \bmod 3$. For example, the case $c_2 \equiv -1 \bmod 3$ has fixed points with a characteristic function with a 2 appearing, whereas the case $c_2 \equiv 1 \bmod 3$ has no fixed points with such a characteristic function (see proof of Theorem 2.4.1). Nevertheless, both cases give rise to 27. In this very specific setting, i.e. $X = \mathbb{P}^2$, $r = 0$, $c_1 = 3$, Le Potier has proved that the moduli spaces $M_X^H(0,3,c_2)$ are isomorphic for all $c_2 \equiv \pm 1 \bmod 3$ [Pot, Thm. 5.1]. This has been pointed out by Jinwon Choi (private communication). He also pointed out that the numbers 3, -6, 27 are compatible with a list of Gopakumar–Vafa invariants $GV_g(\iota_* c_1)$ of $K_{\mathbb{P}^2}$ for $g = 0, \ldots, 5$, $c_1 = 1, \ldots, 10$ appearing in a string theory paper by Katz, Klemm and Vafa [KKV] (see table 2.1). Jinwon Choi has communicated to the author that in the case of Hilbert polynomial $P = 4t + 1$ (which satisfies the conditions of Proposition 2.5.6(ii)), he uses the methods of chapter 1 (i.e. [Koo1]) to obtain $e(M_X^H(0,4,c_2)) = 192$, which is the next number appearing in the list of [KKV] (see table 2.1). Unfortunately, these type of computations involve increasingly complicated combinatorics for increasing c_1.

$GV_g(\iota_* c_1)$	$g = 0$	1	2	3	4	5
$c_1 = 1$	3	0	0	0	0	0
2	-6	0	0	0	0	0
3	27	-10	0	0	0	0
4	-192	231	-102	15	0	0
5	1695	-4452	5430	-3672	1386	-270
6	-17064	80948	-194022	290853	-290400	196857
7	188454	-1438086	5784837	-15363990	29056614	-40492272
8	-2228160	25301295	-155322234	649358826	-2003386626	4741754985
9	27748899	-443384578	3894455457	-23769907110	109496290149	-396521732268
10	-360012150	7760515332	-93050366010	786400843911	-5094944994204	26383404443193

Table 2.1: List of Gopakumar–Vafa invariants of $K_{\mathbb{P}^2}$ from [KKV].

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