Calabi–Yau Categories and Quivers with Superpotential

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To my parents,

Vivien Wei Ngan Chu
and
Yuen Hing Lam.

學海無涯
回頭無岸

佚名

The sea of knowledge knows no bounds,
No shore's in sight even if the sailor turns around.

Anonymous.

以有涯隨無涯
殆已

莊子《庖丁解牛》

To pursue the unlimited with the limited,
Is it not perilous?

Master Chong, *Carving up an Ox.*

莫記此中得失
不記恨愛相纏
只記共你當年
曾經相識過

黃霑《强人》

Remember not the gains and losses,
Nor the love and hatred.
Remember only the good old days,
Once we're acquaintances.

Jim Wong, *The Giant.*

回首向來蕭瑟處
歸去
也無風雨也無晴

蘇軾《定風波》

Looking back at where the winds and rains began,
– I'll go home –
Now there's no more wind, rain, nor sunshine therein.

So Sik, *Calming the Winds and Waves.*

Abstract

This thesis studies derived equivalences between total spaces of vector bundles and dg-quivers.

A dg-quiver is a graded quiver whose path algebra is a dg-algebra. A quiver with superpotential is a dg-quiver whose differential is determined by a "function" Φ. It is known that the bounded derived category of representations of quivers with superpotential with finite dimensional cohomology is a Calabi–Yau triangulated category. Hence quivers with superpotential can be viewed as noncommutative Calabi–Yau manifolds.

One might then ask if there are derived equivalences between Calabi–Yau manifolds and quivers with superpotential. In this thesis, we answer this question and, generalizing Bridgeland [15], give a recipe on how to construct such derived equivalences.

Let $\pi : V \to X$ be an anti-semiample vector bundle over a smooth projective variety X, i.e., $S^k V^\vee$ is globally generated for $k \gg 0$. Given a full exceptional sequence $\mathcal{E}$ on $\mathrm{D}^b(\mathrm{Coh}\,(X))$, under some cohomological vanishing conditions, we construct a dg-quiver $Q_\mathcal{E}$ in terms of the dual exceptional sequence of $\mathcal{E}$ such that $\mathrm{D}^b(\mathrm{Coh}\,(V)) \cong \mathrm{D}^b_{fg}(\mathrm{Rep}(Q_\mathcal{E}))$. Moreover, this equivalence restricts to an equivalence between $\mathrm{D}^b_{cs}(\mathrm{Coh}\,(V))$, the full subcategory containing complexes of compact support, and $\mathrm{D}^b_{fd}(\mathrm{Rep}(Q_\mathcal{E}))$, the full subcategory containing complexes with finite dimensional cohomology. If V is non-compact Calabi–Yau, we show that $Q_\mathcal{E}$ is equipped with a superpotential Φ, i.e., the differential on $Q_\mathcal{E}$ is determined by the "function" Φ. In this case, the triangulated categories $\mathrm{D}^b_{cs}(\mathrm{Coh}\,(V))$ and $\mathrm{D}^b_{fd}(\mathrm{Rep}(Q_\mathcal{E}))$ are both Calabi–Yau.

We can also construct derived equivalences equivariantly. Suppose a finite group G acts on X and this action lifts to V, endowing $\pi : V \to X$ the structure of an equivariant vector bundle. Suppose further that each object in the exceptional sequence $\mathcal{E}$ is equipped with a G-linearization. Then we can construct a quotient dg-quiver $Q_\mathcal{E}/G$ from $Q_\mathcal{E}$, generalizing the construction of the McKay quiver, such that $\mathrm{D}^b(\mathrm{Coh}^G(V)) \cong \mathrm{D}^b(\mathrm{Rep}_{fg}(Q_\mathcal{E}/G))$. If V is non-compact Calabi–Yau equivariantly, then $Q_\mathcal{E}/G$ is also equipped with a superpotential.

We also give a product construction for derived equivalences. Suppose we have vector bundles $\pi_V : V \to X$ and $\pi_W : W \to Y$, with full exceptional sequences $\mathcal{E}$ on $\mathrm{D}^b(\mathrm{Coh}\,(V))$ (resp. $\mathcal{F}$ on $\mathrm{D}^b(\mathrm{Coh}\,(W))$), then we can construct a product dg-quiver $Q_\mathcal{E} \times Q_\mathcal{F}$ such that $\mathrm{D}^b(\mathrm{Coh}\,(V \times W)) \cong \mathrm{D}^b(\mathrm{Rep}_{fg}(Q_\mathcal{E} \times Q_\mathcal{F}))$. If both V and W are Calabi–Yau, then $Q_\mathcal{E} \times Q_\mathcal{F}$ is also equipped with a superpotential.

Using these constructions, we can produce a lot of beautiful pictures of quivers with superpotential derived equivalent to the total spaces of vector bundles which are Calabi–Yau. Examples include $T^\vee_{\mathbb{P}^2}$, $K_{\mathbb{P}^n}$, and $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ etc.

Finally, we try to connect quivers with superpotential to the recent work by Pantev, Toën, Vaquié and Vezzosi [58] and Ben-Bassat, Brav, Bussi and Joyce [4] on shifted symplectic structures. We outline a strategy of proof for the existence of shifted symplectic structures in a standard 'Darboux form' on the derived moduli stack of representations of quivers with superpotential.

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May I end this note by quoting a dialogue from the *Ashes of Time*, one of my favourite movies, directed by Wong Kar-Wai:

"*What's beyond this desert?*"
"*Just another desert.*"

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Introduction

In Calabi–Yau geometry, one important object is the category of coherent sheaves. Coherent sheaves on m-Calabi–Yau manifolds enjoy a special form of Serre duality: for any coherent sheaves E and F, $\mathrm{Ext}^i(E, F) \cong \mathrm{Ext}^{m-i}(F, E)^\vee$. This property is writtenly solely in terms of the Ext groups and hence can be axiomatised to a definition of Calabi–Yau abelian categories. As algebraic geometry progresses, we soon know that it is more flexible to work in the derived category of coherent sheaves (which is a triangulated category) rather than the abelian category. Kontsevich [46] later modified the definition of Calabi–Yau abelian categories to give a notion of Calabi–Yau triangulated categories.

The earliest examples of Calabi–Yau categories other than coherent sheaves on Calabi–Yau manifolds were probably given by physicists Berenstein and Douglas [5], Braun [12], Douglas and Moore [23] and later by mathematicians Ginzburg [27] and Derksen, Weyman and Zelevinsky [22] as representations of some quivers with relations coming from the derivatives of a linear sum of closed paths (the superpotential). One problem of this definition of quivers with superpotential, however, is that they do not always produce Calabi–Yau categories. This problem was later solved by Ginzburg [27], Keller [41] and van den Bergh [71] (see also van den Bergh [70]) by adding derived structures, i.e., considering dg-quivers (graded quivers whose path algebra is endowed with the structure of dg-algebras) rather than quivers with relations. This new definition of quivers with superpotential as dg-quivers always produces Calabi–Yau categories and yields the old definition of quivers with superpotential as quivers with relations when taking the zeroth cohomology of the dg-quiver.

As is well known, using the notion of exceptional sequences, one can construct derived equivalences between varieties and quivers with relations. One then ask if there are derived equivalences between Calabi–Yau varieties and quivers with superpotential. This thesis is a study on how to construct such derived equivalences. There are three major difficulties. To start off, compact Calabi–Yau manifolds do not have any exceptional sequences due to Serre duality and therefore, one cannot directly employ exceptional sequences to construct such derived equivalences. Bridgeland [15] found a slick way of getting around this problem by considering noncompact examples, namely, the total space of vector bundles, and pulling back exceptional sequences on the base manifolds to produce derived equivalences between the total spaces of vector bundles with quivers with relations. This is essentially Corollary 5.2.9.

The second difficulty is to resolve quivers with relations by dg-quivers. This is done by using A_∞-Koszul duality. Any dg-quiver can be characterized as the Koszul dual of an augmented A_{fin}-category (A_∞-category with $m_n = 0$ for $n \gg 0$). This construction sends an A_{fin}-category A to the quiver constructed by taking the vertices to be $\mathrm{Obj}\,(A)$ and degree i edges between vertices u and v to be a basis of $\bar{A}^{1-i}(u, v)^\vee$, where $\bar{A}$ denotes the kernel of the augmentation map. The path algebra of the quiver is then given by the Koszul dual dg-algebra $E(A)$, with differential given by $d = \bigoplus m_n^\vee$. In Theorem 5.4.5, we produce an A_{fin}-category in terms of dual exceptional sequence and show that its Koszul dual is the desired dg-quiver. Readers familiar with the theory of A_∞-algebras would perhaps find it awkward to work with A_{fin}-categories as the property of being A_{fin} is not invariant under homotopy perturbation. Indeed, much of the trouble here is to show that we always end up with an A_{fin}-category rather than just an A_∞-category. This is done by considering an additional grading on the A_∞-category known as the Adams grading.

The last difficulty is to show that the dg-quiver built this way is a quiver with superpotential. We use the characterization that quivers with superpotential are precisely the Koszul dual of positively graded augmented A_{fin}-categories with cyclic structure. We then show the existence of a cyclic structure on our A_{fin}-category by using a result of van den Bergh [70] characterizing a class of algebras known as exact Calabi–Yau algebras. This is Proposition 5.5.2.

This yields the following theorem, which is the basis of this thesis and is essentially a generalization of Bridgeland [15, Proposition 4.1].

Theorem A (Theorem 5.4.5 and Proposition 5.5.2)**.** *Let X be a smooth projective variety and $\pi : V \to X$ be an anti-semiample vector bundle, i.e., $S^k V^\vee$ is globally generated for $k \gg 0$. Let $\mathcal{E} = (E_0, \dots, E_n)$ be an exceptional sequence on $\mathrm{D}^b(\mathrm{Coh}\,(X))$. Suppose the vanishing condition*

$$\mathrm{Hom}^\ell(E_i, E_j \otimes S^k V^\vee) = 0$$

is satisfied for all i, j, k and all $\ell \geq 1$. Denote by $\mathcal{F} = (F_n, \dots, F_0)$ the dual exceptional sequence to $\mathcal{E}$. Then there is an A_{fin}-category $A_\mathcal{E}$ where $\mathrm{Obj}\,(A_\mathcal{E}) = \mathcal{F}$ and

$$A^\ell_\mathcal{E}(F_i, F_j) = \bigoplus_{k \geq 0} \mathrm{Hom}^{\ell-k}(F_i, F_j \otimes \wedge^k V)$$

such that $E(A_\mathcal{E})$ is cohomologically Noetherian and $\mathrm{D}^b(\mathrm{Coh}\,(V)) \cong \mathrm{D}^b_{fg}(E(A_\mathcal{E})^{\mathrm{op}})$. Moreover, this equivalence restricts to a derived equivalence $\mathrm{D}^b_{cs}(\mathrm{Coh}\,(V)) \cong \mathrm{D}^b_{fd}(E(A_\mathcal{E})^{\mathrm{op}})$. Furthermore, if V is noncompact Calabi–Yau, $A_\mathcal{E}$ has a cyclic structure.

Here, by cohomologically Noetherian, we mean that the algebra $H^\bullet(E(A)^{\mathrm{op}})$ is Noetherian. The derived category $\mathrm{D}^b_{fg}(E(A_\mathcal{E})^{\mathrm{op}})$ denotes the full triangulated subcategory of $\mathrm{D}^b(E(A_\mathcal{E})^{\mathrm{op}})$ consisting of complexes whose cohomologies are finitely generated over $H^\bullet(E(A_\mathcal{E})^{\mathrm{op}})$; the derived category $\mathrm{D}^b_{fd}(E(A_\mathcal{E})^{\mathrm{op}})$ denotes the full triangulated subcategory of $\mathrm{D}^b(E(A_\mathcal{E})^{\mathrm{op}})$ consisting of complexes whose cohomologies are finite dimensional; and $\mathrm{D}^b_{cs}(\mathrm{Coh}\,(V))$ denotes the full triangulated subcategory of $\mathrm{D}^b(\mathrm{Coh}\,(V))$ consisting of complexes whose cohomologies are compactly supported.

In other words, we can produce a dg-quiver derived equivalent to the total space of the vector bundle, and in the case when the total space of the vector bundle is Calabi–Yau, the dg-quiver is equipped with a superpotential. The underlying graded quiver can be described explicitly in terms of the dual exceptional sequence and the vector bundle, although the differential (or the superpotential in the Calabi–Yau case) is not readily known. However, in the Calabi–Yau examples of dimension no greater than 4, knowing the classical quiver with relations derived equivalent to the vector bundle is enough to determine the superpotential, since there are enough constraints. One can generalize Theorem A and remove the vanishing condition. But then we no longer have a concrete description for the underlying graded quiver.

We can also construct derived equivalences equivariantly. Let G be a finite group and A an A_{fin}-category, with G acting on A by strict A_{fin}-isomorphisms. Then one can construct a quotient A_{fin}-category A/G. We have the following

Theorem B (Theorem 5.6.5)**.** *In the situation in Theorem A, suppose there is a finite group G acting on X, and this action lifts to V. Assume further that each object $E_i \in \mathcal{E}$ can be equipped with a G-linearization. Then there is a G-action on $A_\mathcal{E}$ by strict A_{fin}-isomorphisms and an equivalence $\mathrm{D}^b(\mathrm{Coh}^G(V)) \cong \mathrm{D}^b_{fg}(E(A_\mathcal{E}/G)^{\mathrm{op}})$. Furthermore, if K_V is trivial as an equivariant vector bundle, or equivalently if $\det V \cong K_X$ equivariantly, then $A_\mathcal{E}/G$ also has a cyclic structure.*

In other words, if we know the dg-quiver derived equivalent to V, we can construct the dg-quiver derived equivalent equivariantly to V. The action of the differential (or the superpotential in the Calabi–Yau case) on the equivariant dg-quiver is also determined. In fact, this quotient construction on quiver is a generalization of the McKay quiver.

We also have the following recipe for products. Given two A_∞-algebras A and A', there is an A_∞-tensor product $A \otimes A'$ defined by Amorim [2] which preserves cyclic structures in the sense that if both A and A' have cyclic structures, then so does $A \otimes A'$.

Theorem C (Theorem 5.7.3)**.** *Let $\pi_V : V \to X$ and $\pi_W : W \to Y$ be anti-semiample vector bundles. Let $\mathcal{E}$ be an exceptional sequence on X and $\mathcal{F}$ be an exceptional sequence on Y. Suppose they satisfy the vanishing conditions*

$$\mathrm{Hom}^\ell(E_i, E_j \otimes S^k V^\vee) = 0 \quad \text{and} \quad \mathrm{Hom}^\ell(F_i, F_j \otimes S^k W^\vee) = 0$$

for all i, j, k and $\ell \geq 1$. Then the A_∞-structure on $A_\mathcal{E} \otimes A_\mathcal{F}$ is A_{fin} and there is an equivalence $\mathrm{D}^b(\mathrm{Coh}\,(V \times W)) \cong \mathrm{D}^b_{fg}(E(A_\mathcal{E} \otimes A_\mathcal{F})^{\mathrm{op}})$. If both V and W are Calabi–Yau, then $A_\mathcal{E} \otimes A_\mathcal{F}$ has a cyclic structure.

In other words, if we know the dg-quivers derived equivalent to vector bundles, we also know the dg-quiver derived equivalent to their product. However, the differential on this product quiver is not uniquely determined as tensor product of A_∞-categories are only defined up to A_∞-quasi-isomorphisms. In the case when one of the A_{fin}-category has $m_n = 0$ for $m \geq 3$, then there is a natural choice of A_{fin}-structure on the tensor product, and hence a natural choice of differential on the product quiver. In the Calabi–Yau case, this means that when one of the superpotentials is cubic, then there is a natural choice of superpotential on the product quiver.

Using these three theorems, we can produce a lot of beautiful pictures of quivers with superpotential derived equivalent to the total space of vector bundles which are Calabi–Yau. Examples include $T^\vee_{\mathbb{P}^2}$, $K_{\mathbb{P}^n}$, and $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ etc.

Quivers with superpotential also connect to the recent work by Pantev, Toën, Vaquié and Vezzosi [58] and Ben-Bassat, Brav, Bussi and Joyce [4] on shifted symplectic structures in that the moduli space of representations of quiver with superpotential seems to be equipped with a shifted symplectic structure which is in a standard Darboux form. The precise statement is stated in Conjecture 7.4.1, and an attempt to prove the conjecture is sketched in Section 7.4. The main difficulty here is finding a way to describe symplectic forms on the moduli space, which is a quotient stack of a derived scheme by a linear algebraic group, by invariant symplectic forms on the atlas. We propose a way to do this via Lie algebra cohomology in Section 7.3.

A Guide to the Chapters

Chapter 1 is a review on noncommutative geometry.

In Section 1.1, we introduce quivers and their representations, and discuss how they can be thought of as objects in noncommutative geometry.

Section 1.2 reviews some calculus on noncommutative space.

Chapter 2 is a review on the theory of triangulated categories.

Section 2.1-2.3 define triangulated category, derived categories and t-structures.

Section 2.4 defines Serre functors and Calabi–Yau triangulated categories.

Section 2.5-2.7 give a brief review of different notions of compact generators, admissible subcategories and mutation functors.

Section 2.8-2.9 define exceptional sequences and tilting objects, which are the main ingredients to construct derived equivalences.

Chapter 3 is a survey on A_∞-algebras and operations on them.

In section 3.1, we introduce A_∞-algebras and other related notions such as minimal models, A_∞-modules, and their derived categories.

Section 3.2 defines the notion of minimal A_∞-algebras, and discusses how to construct minimal models by using homotopy perturbation.

Section 3.3 defines the notion of cyclic structure on A_∞-algebras and describes how it gives rise to Calabi–Yau categories.

Section 3.4 defines the Koszul functor, which is essentially a way of producing dg-quivers from A_∞-algebras. There are two versions of this functor: the completed one and the incomplete one. The completed one is defined on A_∞-algebras and yields the completed path algebra of a dg-quiver. The incomplete one is only defined on A_{fin}-algebras, and yields the incomplete path algebra of a dg-quiver. The difference between the two versions is similar to the difference between power series and polynomials. Admittedly, working with A_{fin}-algebras and hence the incomplete Koszul functor is awkward in the world of A_∞-algebras, as being A_{fin} is not a homotopy invariant property. For example, the minimal model of an A_{fin}-algebra is not necessarily A_{fin}. However, as we will see in Chapter 5, the incomplete Koszul functor is central to our problem of constructing derived equivalences between dg-quivers and total spaces of vector bundles.

Section 3.5 defines the quotient of an A_∞-algebra by a finite group, and the smash product of an A_∞-algebra by a finite group. Although the definition of quotient construction is straightforward, it appears to be new. This quotient construction is central to constructing derived equivalences equivariantly as described in Section 5.6. We then prove a relation between the quotient construction and the smash product, and shows how these two constructions commute with the Koszul dual functor, i.e., the Koszul dual of the quotient (resp. smash product) of an A_{fin}-algebra is the quotient (resp. smash product) of the Koszul dual of an A_{fin}-algebra. This section is inspired by the work of Bocklandt, Schedler, and Wemyss [7].

Section 3.6 surveys different constructions of A_∞-tensor product. Since A_∞-tensor products are only unique up to A_∞-quasi-isomorphisms, there is in general no natural formulae for computing the tensor product, although there is one in the case when one of the A_∞-algebras is A_2, i.e., a dg-algebra. Particularly important to us is the tensor product constructed by Amorim and Tu [2], since their construction preserves cyclic structures. We then prove that, under some local finiteness conditions, the Koszul functor commutes with the tensor product, i.e., Koszul dual of tensor product of A_∞-algebras is quasi-isomorphic to tensor product of Koszul duals of A_∞-algebras as dg-algebras.

Chapter 4 is devoted to the study of quivers with superpotential.

In section 4.1, we define quivers with superpotentials. Our definition of quivers with superpotential is taken from van den Bergh [70], where the completed path algebra of a quiver with superpotential is known as a deformed DG-preprojective algebra there.

Section 4.2 gives a correspondence between quivers with superpotential and the Koszul dual of A_{fin}-categories with cyclic structures. Using this correspondence, we define the notion of product of quivers with superpotential and the notion of quotient of quivers with superpotential by finite groups.

In section 4.3, we follow van den Bergh [71] and prove that the path algebras of quivers with superpotential are Calabi–Yau algebras, and hence the categories of representations of quivers with superpotential are also Calabi–Yau.

Finally, Section 4.4 describes quivers with superpotential of dimensions 1 to 4. In particular, we describe in dimension 3 how our definition of quivers with superpotential as dg-quivers is connected to the old definition of quivers with superpotential as quivers with relations given by physicists Berenstein and Douglas [5], Braun [12], Douglas and Moore [23] and later by mathematicians Ginzburg [27] and Derksen, Weyman and Zelevinsky [22].

Chapter 5 is the heart of the thesis where we prove our main results.

Section 5.1 gives a review on equivariant sheaves.

In Section 5.2, we generalize a result by Bridgeland [15, Proposition 4.1] and show that if $\pi : V \to X$ is an anti-semiample vector bundle on a smooth projective manifold with an exceptional poset $\mathcal{E}$, then under some cohomological vanishing conditions, the total space V is derived equivalent to an algebra $\Lambda_{\mathcal{E}}$ which is the path algebra of a quiver with relations. If we remove the cohomological vanishing condition, we end up with an A_{fin}-algebra rather than a quiver with relations.

Section 5.3 tries to resolve $\Lambda_{\mathcal{E}}$, the path algebra of a quiver with relations (or more generally the A_{fin}-algebra) by a dg-quiver $Q_{\mathcal{E}}$.

Section 5.4 gives a concrete description of the underlying graded quiver of the dg-quiver $Q_{\mathcal{E}}$ in terms of the dual exceptional poset of $\mathcal{E}$.

Section 5.5 proves the existence of a superpotential on $Q_{\mathcal{E}}$ when V is noncompact Calabi–Yau.

Section 5.6 considers the G-equivariant situation and constructs from $Q_{\mathcal{E}}$ a quotient quiver $Q_{\mathcal{E}}/G$, generalizing the construction of the McKay quiver, which is derived equivalent to $\mathrm{D}^b(\mathrm{Coh}^G(V))$. In the case when V is equivariantly Calabi–Yau, $Q_{\mathcal{E}}/G$ is also equipped with a superpotential.

Section 5.7 proves the product construction. We start with two dg-quivers $Q_{\mathcal{E}}$ and $Q_{\mathcal{F}}$ derived equivalent to vector bundles V and W respectively, and construct a product quiver $Q_{\mathcal{E}} \times Q_{\mathcal{F}}$ which is derived equivalent to $V \times W$. When both V and W are Calabi–Yau, we show that the product quiver $Q_{\mathcal{E}} \times Q_{\mathcal{F}}$ is also equipped with a superpotential.

Chapter 6 is a list of examples illustrating theorems in Chapter 5.

Section 6.1 contains examples illustrating Theorem 5.2.9 which produces the quivers with relations derived equivalent to total space of vector bundles.

Section 6.2 and 6.3 contain examples illustrating Theorem 5.4.5. Section 6.2 introduces a class of algebras called Koszul algebras whose dg-resolution is particularly easy to describe. We also gives some examples of vector bundles whose classical tilting algebras are Koszul.

Section 6.3 contains some worked out examples of derived equivalences between total spaces of vector bundles and dg-quivers, and if the total spaces of vector bundles are Calabi–Yau, quivers with superpotential. These examples are calculated by first determining the classical tilting algebras, then try to work out the dg-resolutions to determine the dg-quiver. In Calabi–Yau examples of dimension no greater than 4, there are enough constraints and hence the classical tilting algebras determine their dg-resolutions.

Section 6.4 contains a list of examples by applying the product construction in Theorem 5.7.3. Since the general formulae for the cyclic A_∞-tensor product defined by Amorim and Tu [2] is not known, we only work with the case when one of the A_∞-algebras is a classical algebra.

Section 6.5 contains a list of examples illustrating the quotient construction in Theorem 5.6.5.

Chapter 7 contains some unfinished work which aims to make a connection between quivers with superpotential and the recent work on shifted symplectic structures by Pantev, Toën, Vaquié and Vezzosi [58], and Ben-Bassat, Brav, Bussi and Joyce [4].

Section 7.1 reviews the theory on derived algebraic geometry developed by Töen and Vezzosi [64, 65, 66] and Pantev, Toën, Vaquié and Vezzosi [58].

Section 7.2 develops the Lie algebra cohomology for dg-modules by modifying the usual Lie algebra cohomology theory.

Section 7.3 defines the G-invariant de Rham complex of on a derived scheme $\operatorname{Spec} R$ by using the Lie algebra cohomology developed in Section 7.2. We conjecture that the G-invariant de Rham complex should describe forms and closed forms on the quotient stack $[\operatorname{Spec} R/G]$ and outline a strategy of proof.

Section 7.4 describes the moduli space of representations of quiver with superpotential, and outline a strategy of proof on showing the existence of a shifted symplectic structure which is in a standard Darboux form by using the G-invariant de Rham complex introduced in Section 7.3.

Chapter 8 discusses possible future research directions following the thesis.

Appendix A gives some cohomological formulae for computing examples in Chapter 6.

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Chapter 1

Noncommutative Geometry

This chapter is a review on noncommutative geometry. In Section 1.1, we introduce quivers and their representations, and discuss how they can be viewed as objects in noncommutative geometry. Section 1.2 reviews some calculus on noncommutative space.

1.1 Quivers and Representations

We review the definition of quivers and their representations. The lecture notes by Crawley-Boevey [19] are a good reference.

Definition 1.1.1 (Quiver)**.** A finite $\mathbb{Z}$-graded quiver Q consists of the following data:

- A finite set of vertices V_Q;
- For any vertices $v, w \in V_Q$, a finite set of edges

$$E_Q(v,w) = \coprod_{i\in\mathbb{Z}} E_Q^i(v,w),$$

If $e \in E_Q^i(v,w)$, v and w are called the tail and head of e, denoted by $t(e)$ and $h(e)$ respectively, and i is called the degree of e, denoted by $\deg(e)$. Pictorially, we view e as an arrow going from v to w

$$\overset{w}{\bullet} \xleftarrow{e} \overset{v}{\bullet}\ .$$

Definition 1.1.2 (Path)**.** A path p of length n in Q is a sequence of edges $e_n e_{n-1}\cdots e_2 e_1$ with $t(e_{i+1}) = h(e_i)$ for $1 \leq i \leq n-1$. The tail of p is $t(p) = v_0$ and the head is $h(p) = v_n$. Pictorially, we view p as a sequence of arrows going from v_0 to v_n

$$\overset{v_n}{\bullet} \xleftarrow{e_n} \overset{v_{n-1}}{\bullet} \longleftarrow \cdots \longleftarrow \overset{v_1}{\bullet} \xleftarrow{e_1} \overset{v_0}{\bullet}$$

The degree of path is the sum of degrees of its component edges. Each vertex v will also be viewed as a path of both length and degree 0 going from v to v.

Definition 1.1.3 (Path Category)**.** Let $\mathbb{K}$ be a field. The path category of Q over $\mathbb{K}$, denoted by $\mathbb{K}\mathcal{Q}$, is the $\mathbb{K}$-linear category defined by

- $\mathrm{Obj}\,(\mathcal{Q}) = V$, and
- $\mathbb{K}\mathcal{Q}(v,w) = \mathbb{K}\{\text{paths going from } v \text{ to } w\}$,
- the composition map $\circ : \mathbb{K}\mathcal{Q}(u,v) \times \mathbb{K}\mathcal{Q}(v,w) \to \mathbb{K}\mathcal{Q}(u,w)$ is given by concatenation of paths, with
- the identity given by the empty path at each vertex.

Definition 1.1.4 (Path algebra)**.** The path algebra of Q over $\mathbb{K}$, denoted by $\mathbb{K}Q$, is the unital associative $\mathbb{K}$-algebra spanned over $\mathbb{K}$ by all paths of length $k \geq 0$, with multiplication of paths p and q given by the concatenation qp if $h(p) = t(q)$ and zero otherwise. The identity is the sum of empty paths over the set of vertices. The vector subspace spanned by all paths of length $k \geq n$ is a two-sided ideal and is denoted by $\mathbb{K}Q_{(n)}$.

The path category and the path algebra are essentially the same thing, as from the path category we can get the path algebra by taking the direct sum of all morphism spaces and define multiplication to be composition if two morphisms are composable and zero otherwise. If we view the path algebra as an algebra over the discrete $\mathbb{K}$-algebra generated by the vertices, one can recover the path category by defining the morphism space from v to w to be the vector space $w\mathbb{K}Qv$ and composition of morphism by multiplication.

Definition 1.1.5 (Quiver with relations)**.** A quiver with relation (Q, I) is a quiver Q with a two-sided ideal I in $\mathbb{K}Q$ with $I \subseteq \mathbb{K}Q_{(2)}$. The path algebra of (Q, I) is the unital associative algebra $\mathbb{K}Q/I$.

Definition 1.1.6 (Differential-graded Quiver)**.** A dg-quiver is a graded quiver together with a S-linear differential $d : \mathbb{K}Q \to \mathbb{K}Q$ of degree 1.

Path algebras as tensor algebras. Let S be the discrete $\mathbb{K}$-algebra over V_Q, i.e., the path algebra of the quiver with vertex set V_Q and no edges, and $\mathbb{K}E_Q$ be the $\mathbb{K}$-vector space spanned by E_Q. Then $\mathbb{K}E_Q$ is naturally a S-bimodule with scalar multiplication given by path multiplication and $\mathbb{K}Q$ is isomorphic as a unital associative $\mathbb{K}$-algebra to $T_S(\mathbb{K}E_Q)$.

Path categories as tensor categories. Analogously, path categories can be written in the form of tensor categories [41, §3.5].

Definition 1.1.7 (Representation of Quiver)**.** Let Q be a quiver. A finite dimensional representation (W, ρ) of Q consists of finite dimensional $\mathbb{K}$-vector spaces W_v for each vertex $v \in V_Q$ and linear maps $\rho_e : W_{t(e)} \to W_{h(e)}$ for each edge $e \in E_Q$. A finite dimensional representation (W, ρ) of a quiver with relations (Q, I) is a finite dimensional representation of Q such that for all $r = \sum a_{e_n \cdots e_1} e_n \cdots e_1 \in I$ linear combinations of paths having common head and tail vertices, the corresponding linear maps are trivial:

$$\sum a_{e_n \cdots e_1} \rho_{e_n} \circ \cdots \circ \rho_{e_1} = 0.$$

A morphism of representations $\phi : (W, \rho) \to (U, \sigma)$ consists of linear maps $\phi_v : W_v \to U_v$ for each $v \in V_Q$ such that $\phi_{h(e)} \circ \rho_e = \sigma_e \circ \phi_{t(e)}$ for all $e \in E_Q$.

Proposition 1.1.8. *Let (Q, I) be a quiver with relations. The category of representations of (Q, I) is equivalent to the category of finite dimensional left $\mathbb{K}Q/I$-module.*

Proof. Refer to [19]. ■

Definition 1.1.9 (Representation of dg-quiver)**.** A representation of a dg-quiver consists of chain complexes W_v for each vertex v and graded linear maps $\rho_e : W_{t(e)} \to W_{h(e)}$ for each edge e such that for any $m \in W_{t(e)}$, the following identity holds:

$$d_{W_{h(e)}}(\rho_e(m)) = (d_{\mathbb{K}Q}\rho_e)(m) + (-1)^{|e|}\rho_e(d_{W_{h(e)}} m).$$

Proposition 1.1.10. *The category of dg-representations of a dg-quiver (Q, d) is equivalent to the category of dg-modules over $(\mathbb{K}Q, d)$.*

Path algebra as noncommutative analogue of affine variety. One may view the path algebra of a quiver as a noncommutative analogue of a polynomial algebra. For example, let Q be a vertex with 2 loops x and y, then $\mathbb{K}Q = \mathbb{K}\langle x, y\rangle$. Path algebra of a quiver with relations is then analogous to finitegly generated commutative algebras, for instance, $\mathbb{K}\langle x, y\rangle/I$ for some ideal I. Representations of quivers are then the noncommutative analogue of coherent sheaves. Dg-quivers can be think of as the noncommutative analogue of derived schemes.

1.2 Noncommutative Calculus

In this section, we review the tools of noncommutative calculus. Proofs in this section are omitted. For proofs and further details, please refer to the paper by Crawley-Boevey, Etingof and Ginzburg [20], the lecture notes by Ginzburg [26] and the paper by van den Bergh [69].

Notation and Convention. We will work in the relative setting. Fix a ground field $\mathbb{K}$. Let S be a $\mathbb{K}$-algebra, A be a $\mathbb{Z}$-graded unital associative S-algebra and M, N be $\mathbb{Z}$-graded A-bimodules. First, we fix some notation and convention. Anything unadorned will always mean relative to the ground field $\mathbb{K}$, e.g., unadorned tensor product $\otimes$ means $\otimes_{\mathbb{K}}$. The shifted module ΣM is defined by $(\Sigma M)^i = M^{i+1}$. Throughout this thesis, the Koszul sign rule will be enforced: when moving an element a past another element b, the sign $(-1)^{|a||b|}$ will appear. For example, the tensor product $\varphi \otimes \phi$ of two morphisms of graded A-bimodule is defined as:

$$(\varphi \otimes \phi)(a \otimes b) = (-1)^{|\phi||a|}\varphi(a) \otimes \phi(b).$$

We will employ the Sweedler's notation [63] and write any element $b \in A \otimes A$ as $b = b' \otimes b''$ instead of the more accurate $\sum_i b'_i \otimes b''_i$. For instance, if Θ is a $\mathbb{K}$-linear map with target in $A \otimes A$, say $\Theta : A \to A \otimes A$, we will write $\Theta(a) = \Theta'(a) \otimes \Theta''(a)$. The outer and inner A-bimodule structure on the free A-bimodule $A \otimes A$ will be denoted respectively by

$$a(b' \otimes b'')c = (ab') \otimes (b''c)$$
$$a * (b' \otimes b'') * c = (-1)^{|a||b'|+|c||a|+|c||b''|}(b'c) \otimes (ab'').$$

If we denote the interchange operator $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$ by σ, the two A-bimodule structures are then related by

$$\sigma(a(b' \otimes b'')c) = a * \sigma(b' \otimes b'') * c.$$

The multiplication map $a \otimes b \mapsto ab$ will always be denoted by m.

Definition 1.2.1 (Derivations)**.** Let $\mathbb{K}$ be a field, S be a $\mathbb{K}$-algebra, A be a $\mathbb{Z}$-graded S-algebra and M be an $\mathbb{Z}$-graded A-bimodule. A S-linear derivation of degree n from A to M is a S-bimodule morphism $f : A \to M$ of degree n which satisfy Leibnitz's rule, i.e., for all $a, b \in A$,

$$f(ab) = f(a)b + (-1)^{n|a|} af(b).$$

The set of all S-linear derivations from A to M is denoted by $\mathrm{Der}_S(A, M)$. In the special case $M = A \otimes A$ with its outer bimodule structure, such a derivation is said to be a S-linear double derivation of A. The set of all S-linear double derivations of A is denoted by $\mathbb{D}\mathrm{er}_S(A)$.

Remark 1.2.2. In general, $\mathrm{Der}_S(A, M)$ is only an abelian group. However, for double derivation, $\mathbb{D}\mathrm{er}(A)$ is still an A-bimodule due to the inner bimodule structure of $A \otimes A$. In other words, for any double derivation f and $a, b \in A$, we define the double derivation afb by $(afb)(-) = a * f(-) * b$.

Example 1.2.3. To see how double derivations arise naturally, consider $A = \mathbb{K}Q$ for some quiver Q. For any edge e in Q, we can define a double derivation ∂_e acting on any edge f by

$$\partial_e(f) = \begin{cases} h(e) \otimes t(e) & \text{if } f = e \\ 0 & \text{otherwise.} \end{cases}$$

From the double derivation ∂_e, one can define a derivation $\partial_e^\circ : \mathbb{K}Q \to \mathbb{K}Q$ by $\partial_e^\circ = m \circ \sigma \circ \partial_e$. This derivation vanishes on commutators and hence descends to a derivation $\partial_e^\circ : \mathbb{K}Q/[\mathbb{K}Q, \mathbb{K}Q] \to \mathbb{K}Q$.

Definition 1.2.4 (Noncommutative cotangent bundle)**.** The A-algebra $\Theta_S^\bullet(A) = T_A(\mathbb{D}\mathrm{er}_S(A))$ is called the S-relative noncommutative cotangent bundle of A.

Definition 1.2.5 (Differential 1-form)**.** The A-bimodule of noncommutative 1-forms relative to S, denoted by $\Omega_S^1(A)$, is the A-bimodule generated by symbols of the form da for any $a \in A$, subject to the relation

$$d_{dR}(ab) = (d_{dR}a)b + a(d_{dR}b).$$

Proposition 1.2.6. *The functor $M \mapsto \mathrm{Der}_S(A, M)$ is representable by the A-bimodule $\Omega^1_S(A)$. In other words, there are canonical isomorphisms*

$$\mathrm{Der}_S(A, M) \cong \mathrm{Hom}_{A-\mathrm{Bimod}}(\Omega^1_S, M).$$

Proposition 1.2.7. *The following sequence is exact:*

$$0 \to \Omega^1_S(A) \xrightarrow{\varphi} A \otimes_S A \xrightarrow{m} A \to 0,$$

where $\varphi(da) = a \otimes 1 - 1 \otimes a$ and $m(a \otimes b) = ab$.

Definition 1.2.8 (Noncommutative tangent bundle)**.** The A-algebra $\Omega^\bullet_S(A) = T_A\Omega^1_S(A)$ is called the S-relative noncommutative tangent bundle of A. The algebra $\Omega^\bullet_S(A)$ is $\mathbb{Z} \times \mathbb{Z}$-graded, with the first $\mathbb{Z}$-grading $|-|$ coming from the $\mathbb{Z}$-grading on A, and the second one $\|-\|$ coming from the "form" degree, i.e., the number of d_{dR}'s appearing in the element. The Koszul sign rule will be enforced according the following rule: when moving an element $\alpha \in \Omega^\bullet_S(A)$ past another element $\beta \in \Omega^\bullet_S(A)$, the sign $(-1)^{|\alpha||\beta|+\|\alpha\|\|\beta\|}$ will appear. For example, the commutator is defined as

$$[\alpha, \beta] = \alpha\beta - (-1)^{|\alpha||\beta|+\|\alpha\|\|\beta\|}\beta\alpha.$$

Proposition 1.2.9. *The de Rham differential map $d_{dR} : A \to \Omega^1_S A$ extends to an S-linear derivation $d_{dR} : \Omega^\bullet_S(A) \to \Omega^\bullet_S(A)$of degree $(0, 1)$ which satisfies $d^2_{dR} = 0$ and*

$$d_{dR}(a_0 d_{dR}a_1 \cdots d_{dR}a_n) = d_{dR}a_0 d_{dR}a_1 \cdots d_{dR}a_n.$$

Definition 1.2.10 (Noncommutative de Rham complex)**.** The S-relative noncommutative de Rham complex of A is the graded vector space defined by

$$\mathrm{DR}^\bullet_S(A) = \Omega^\bullet_S(A)/[\Omega^\bullet_S(A), \Omega^\bullet_S(A)].$$

Definition 1.2.11 (Contraction)**.** Let $\Theta \in \mathbb{D}\mathrm{er}_S(A)$. The contraction map $i_\Theta : \Omega^1_S(A) \to A \otimes A$ defined to be the A-bimodule morphism given by

$$i_\Theta(d_{dR}a) = \Theta(a).$$

This map extends to a derivation of degree (-1) $i_\Theta : \Omega^\bullet_S(A) \to \Omega^\bullet_S(A) \otimes \Omega^\bullet_S(A)$. The reduced contraction map $\imath_\Theta : \Omega^\bullet_S(A) \to \Omega^\bullet_S(A)$ is the degree (-1) derivation defined by $\imath_\Theta = m \circ \sigma \circ i_\Theta$.

Definition 1.2.12 (Lie derivative)**.** The Lie derivative $L_\Theta : \Omega^\bullet_S(A) \to \Omega^\bullet_S(A) \otimes \Omega^\bullet_S(A)$ is by the Cartan formula

$$L_\Theta = d_{dR}i_\Theta + i_\Theta d_{dR}.$$

The reduced Lie derivative $\mathcal{L}_\Theta : \Omega^\bullet_S(A) \to \Omega^\bullet_S(A)$ is defined by $\mathcal{L}_\Theta = m \circ \sigma \circ L_\Theta$.

Proposition 1.2.13. *Given any double derivation $\Theta \in \mathbb{D}\mathrm{er}(A)$, the reduced contraction $\imath_\Theta : \Omega^\bullet_S(A) \to \Omega^\bullet_S(A)$ descends to $\imath_\Theta : \mathrm{DR}^\bullet_S(A) \to \Omega^\bullet_S(A)$*

Definition 1.2.14 (Symplectic 2-form)**.** A closed noncommutative 2-form $\omega \in \mathrm{DR}^2_S(A)$ is said to be sympletic if the map $\imath_\omega : \mathbb{D}\mathrm{er}_S(A) \to \Omega^1_S(A)$ defined by $\Theta \mapsto \imath_\Theta\omega$ is an isomorphism.

Definition 1.2.15 (Double Poisson bracket)**.** A double Poisson bracket of degree n is a linear map $\{\!\{-, -\}\!\} : A \otimes A \to A \otimes A$ of degree n which satisfies

$$\begin{aligned} \{\!\{a, b\}\!\} &= -(-1)^{(|a|+n)(|b|+n)}\sigma\{\!\{b, a\}\!\}, \\ \{\!\{a, bc\}\!\} &= \{\!\{a, b\}\!\}c + (-1)^{(|a|+n)|b|}b\{\!\{a, c\}\!\}. \end{aligned}$$

and

$$\begin{aligned} 0 = \{\!\{a, \{\!\{b, c\}\!\}'\}\!\} \otimes \{\!\{b, c\}\!\}'' &+ (-1)^{(|a|+n)(|b|+|c|)}\tau\{\!\{b, \{\!\{c, a\}\!\}'\}\!\} \otimes \{\!\{c, a\}\!\}'' \\ &+ (-1)^{(|c|+n)(|a|+|b|)}\tau^2\{\!\{c, \{\!\{a, b\}\!\}'\}\!\} \otimes \{\!\{a, b\}\!\}'', \end{aligned}$$

where $\tau : A \otimes A \otimes A \to A \otimes A \otimes A$ is the map sending $a \otimes b \otimes c \mapsto (-1)^{(|a|+|b|)|c|}c \otimes a \otimes b$. If A is a dg-algebra, then a double Poisson bracket of degree n is a dg-double Poisson bracket of degree n if it further satisfies

$$d\{\!\{a, b\}\!\} = \{\!\{da, b\}\!\} + (-1)^{|a|+n}\{\!\{a, db\}\!\}.$$

Remark 1.2.16. Note that a double bracket also satisfies

$$\{\!\{ab, c\}\!\} = a * \{\!\{b, c\}\!\} + (-1)^{|b|(|c|+n)}\{\!\{a, c\}\!\} * b.$$

Proposition 1.2.17. *Every symplectic 2-form ω give rise to a double Poisson bracket.*

Proof. For any $a \in A$, let H_a be the corresponding Hamiltonian double derivation, i.e., H_a is the unique double derivation satisfying

$$\imath_{H_a}\omega = d_{dR}a.$$

Define $\{\!\{a, b\}\!\} = H_a(b) = i_{H_a}(d_{dR}b) = i_{H_a}\imath_{H_b}\omega$. ∎

Proposition 1.2.18 ([69] Proposition 1.4)**.** *A double Poisson bracket of degree n defines a Kontsevich bracket of degree n*

$$\{-, -\} = m \circ \{\!\{-, -\}\!\} : A \otimes A \to A$$

which satisfy the following properties:

1. $\{-, -\}$ *is a derivation in the second argument, i.e.,*

$$\{a, bc\} = \{a, b\}c + (-1)^{(|a|+n)|b|}b\{a, c\}.$$

2. $\{-, -\}$ *vanishes on commutators in the first argument and hence descends to map*

$$\{-, -\} : A/[A, A] \otimes A \to A.$$

3. $\{-, -\}$ *satisfy the Jacobi identity*

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+n)(|b|+n)}\{b, \{a, c\}\}.$$

4. $\{-, -\}$ *descends to a Lie bracket*

$$\{-, -\} : A/[A, A] \otimes A/[A, A] \to A/[A, A].$$

The double Poisson bracket and the Kontsevich bracket are related by the following

Proposition 1.2.19 ([69] Proposition 2.4.2)**.** *We have the following identity:*

$$\{a, \{\!\{b, c\}\!\}\} = \{\!\{\{a, b\}, c\}\!\} + (-1)^{|b|+n}\{\!\{b, \{a, c\}\}\!\}.$$

Chapter 2

Triangulated Categories

This chapter is a review of the theory on triangulated categories. Sections 2.1-2.3 define triangulated category, derived categories and t-structures. Section 2.4 defines Serre functors and Calabi–Yau triangulated categories. Sections 2.5-2.7 give a brief review of different notions of compact generators, admissible subcategories, and mutation functors. Sections 2.8-2.9 define exceptional sequences and tilting objects, which are the main ingredients to construct derived equivalences.

2.1 Triangulated Categories

We review the definition of triangulated categories. Further details can be found in Gelfand and Manin [25].

Definition 2.1.1 ($\mathbb{K}$-categories)**.** Let $\mathbb{K}$ be a field. A $\mathbb{K}$-*category* is a category $\mathcal{A}$ where for any $X, Y \in \operatorname{Obj} \mathcal{A}$, $\mathcal{A}(X, Y)$ is endowed with the structure of a $\mathbb{K}$-module such that the composition maps

$$\mathcal{A}(X, Y) \times \mathcal{A}(Y, Z) \to \mathcal{A}(X, Z)$$

are $\mathbb{K}$-bilinear. A $\mathbb{K}$-category $\mathcal{A}$ is said to be $\mathbb{K}$-*linear* if $\mathcal{A}$ has a zero object and the product of any two objects in $\mathcal{A}$ exists.

Definition 2.1.2 (Graded $\mathbb{K}$-categories)**.** A *graded* $\mathbb{K}$-*category* $(\mathcal{A}, \Sigma)$ is a k-linear category $\mathcal{A}$ together with an automorphism Σ of $\mathcal{A}$. We define the *graded Hom-sets* by

$$\mathcal{A}^i(X, Y) = \mathcal{A}(A, \Sigma^i B).$$

Definition 2.1.3 (Graded functors)**.** A *graded functor* $(F, \eta) : (\mathcal{A}, \Sigma_{\mathcal{A}}) \to (\mathcal{B}, \Sigma_{\mathcal{B}})$ between graded categories is a k-linear functor $F : \mathcal{A} \to \mathcal{B}$ together with a natural isomorphism $\eta : F\Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{B}} F$.

Definition 2.1.4 (Triangles)**.** A *triangle* in a graded k-category $(\mathcal{A}, \Sigma)$ is a sequence $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$. A *morphism between two triangles* is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & B \arrow[r, "v"] \arrow[d] & C \arrow[r, "w"] \arrow[d] & \Sigma A \arrow[d] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \Sigma A' \end{tikzcd} .$$

Definition 2.1.5 (Triangulated categories)**.** A *triangulated k-category* is a graded k-linear category $(\mathcal{A}, \Sigma)$ equipped with a set of distinguished triangles which is stable under isomorphisms and satisfying the following axioms:

T0. For any $A \in \operatorname{Obj} \mathcal{A}$, the triangle

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow \Sigma A$$

is distinguished.

T1. Any morphism $\phi : A \to B$ can be completed to a distinguished triangle

$$A \xrightarrow{\phi} B \longrightarrow C \longrightarrow \Sigma A$$

T2. A triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

is distinguished if and only if the triangle

$$B \xrightarrow{v} C \xrightarrow{w} \Sigma A \xrightarrow{-\Sigma u} \Sigma B$$

is distinguished.

T3. If there is a commutative diagram of distinguished triangles with vertical morphisms $a : A \to A'$ and $b : B \to B'$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, dotted, "c"'] & \Sigma A \arrow[d, "\Sigma a"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A' \end{tikzcd},$$

there exists a morphism $c : C \to C'$ making the diagram commute.

T4. The triangles satisfy the octahedral axiom: Given distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{k} \Sigma X,$$
$$Y \xrightarrow{v} Z \xrightarrow{\ell} X' \xrightarrow{i} \Sigma Y,$$
$$X \xrightarrow{v \circ u} Z \xrightarrow{m} Y' \xrightarrow{n} \Sigma X,$$

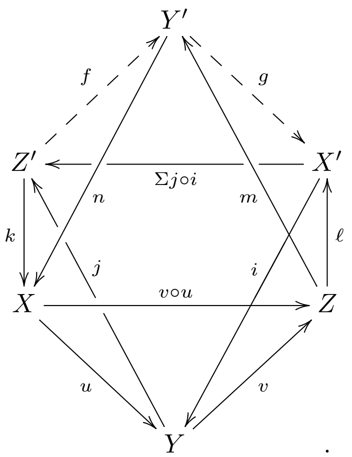
there exists a distinguished triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{h} \Sigma Z'$$

such that

$$\ell = g \circ m, \quad k = n \circ f, \quad h = \Sigma j \circ i, \quad i \circ g = \Sigma u \circ n, \quad f \circ j = m \circ v.$$

The name "octahedral axiom" comes from the fact that the above distinguished triangles can be packed into an octahedron:



.

Remark 2.1.6. Note that we do not assume a priori that two morphisms in a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A$ compose to zero. However, it is a consequence of axioms $T1$ and $T3$ that they do: there exists a morphism h making the diagram commutes

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, dotted, "h"'] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & \Sigma A, \end{tikzcd}$$

i.e., the composition $g \circ f$ factors through the zero object, and hence must be a zero morphism.

Remark 2.1.7. Axiom $T2$ postulates every morphism $A \xrightarrow{f} B$ fits into a distinguished triangle $A \xrightarrow{f} B \to C \to \Sigma A$. It is a consequence of the axioms [25, Corollary IV.1.4] that the object C is unique up to non-unique isomorphism. This object C is called a cone of the morphism f. We can then rephrase Axiom $T4$ in the following way: Let f, g be morphisms in a triangulated category $\mathcal{A}$ and $C(f)$, $C(g)$ and $C(g \circ f)$ be a cone of the morphism f, g and $g \circ f$ respectively. Then there exists a distinguished triangle

$$C(f) \to C(g \circ f) \to C(g) \to \Sigma C(f).$$

Definition 2.1.8 (Triangle Functor)**.** Let $(\mathcal{A}, \Sigma_{\mathcal{A}})$ and $(\mathcal{B}, \Sigma_{\mathcal{B}})$ be triangulated k-categories. A *triangle functor* is a graded k-linear functor $(F, \eta) : (\mathcal{A}, \Sigma_{\mathcal{A}}) \to (\mathcal{B}, \Sigma_{\mathcal{B}})$ preserving distinguished triangles, i.e., for each distinguished triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A,$$

the triangle

$$FA \xrightarrow{Fu} FB \xrightarrow{Fv} FC \xrightarrow{\eta_C \circ (Fw)} \Sigma FC$$

is also distinguished.

Remark 2.1.9. In general, $(\Sigma, \mathrm{id}_{\Sigma^2})$ is not a triangle functor. However, $(\Sigma, -\mathrm{id}_{\Sigma^2})$ is a triangle functor. If $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$ is a distinguished triangle, by $(T2)$, so is $\Sigma A \xrightarrow{-\Sigma u} \Sigma B \xrightarrow{-\Sigma v} \Sigma C \xrightarrow{-\Sigma w} \Sigma^2 A$. The following isomorphism of triangle

$$\begin{tikzcd}
\Sigma A \arrow[r, "-\Sigma u"] \arrow[d, "\mathrm{id}_{\Sigma A}"'] & \Sigma B \arrow[r, "-\Sigma v"] \arrow[d, "-\mathrm{id}_{\Sigma B}"'] & \Sigma C \arrow[r, "-\Sigma w"] \arrow[d, "\mathrm{id}_{\Sigma C}"'] & \Sigma^2 A \arrow[d, "\Sigma \mathrm{id}_{\Sigma A}"] \\
\Sigma A \arrow[r, "\Sigma u"] & \Sigma B \arrow[r, "\Sigma v"] & \Sigma C \arrow[r, "-\Sigma w"] & \Sigma^2 A
\end{tikzcd}$$

shows $\Sigma A \xrightarrow{\Sigma u} \Sigma B \xrightarrow{\Sigma v} \Sigma C \xrightarrow{-\Sigma w} \Sigma^2 A$ is a distinguished triangle. Hence $(\Sigma, -\mathrm{id}_{\Sigma^2})$ is a triangle functor.

Definition 2.1.10 (Morphism of triangle functors)**.** Let $(\mathcal{A}, \Sigma_{\mathcal{A}})$, $(\mathcal{B}, \Sigma_{\mathcal{B}})$ be triangulated category and (F, ϕ), $(G, \psi) : (\mathcal{A}, \Sigma_{\mathcal{A}}) \to (\mathcal{B}, \Sigma_{\mathcal{B}})$ be triangle functors. A *morphism of triangle functors* $\alpha : (F, \phi) \to (G, \psi)$ is a natural transformation $\alpha : F \to G$ such that for all $X \in \mathrm{Obj}\,\mathcal{A}$, the following square commutes

$$\begin{tikzcd}
F\Sigma X \arrow[r, "\phi"] \arrow[d, "\alpha_{\Sigma X}"'] & \Sigma F X \arrow[d, "\Sigma \alpha_X"] \\
G\Sigma X \arrow[r, "\psi"'] & \Sigma G X.
\end{tikzcd}$$

Definition 2.1.11 (Triangulated subcategory)**.** A *triangulated subcategory* of a triangulated category $\mathcal{A}$ is a subcategory $\mathcal{B}$ of $\mathcal{A}$ such that the inclusion functor $i : \mathcal{B} \to \mathcal{A}$ is a triangle functor. A triangulated subcategory $\mathcal{B}$ is said to be *thick* if it is stable under taking direct summands, i.e., $A \oplus B \in \mathrm{Obj}\,\mathcal{B}$ implies $\mathcal{A}, \mathcal{B} \in \mathrm{Obj}\,\mathcal{B}$.

Definition 2.1.12 (Orthogonal Complement)**.** Let $\mathcal{B}$ be a triangulated subcategory of a triangulated category $\mathcal{A}$. The *right orthogonal complement* of $\mathcal{B}$ is the full subcategory $\mathcal{B}^{\perp}$ of $\mathcal{A}$ containing all objects $A \in \mathrm{Obj}\,(\mathcal{A})$ such that $\mathrm{Hom}(B, A) = 0$ for all $B \in \mathrm{Obj}\,(\mathcal{B})$. Similarly, the *left orthogonal complement* of $\mathcal{B}$ is the full subcategory ${}^{\perp}\mathcal{B}$ of $\mathcal{A}$ containing all objects $A \in \mathrm{Obj}\,(\mathcal{A})$ such that $\mathrm{Hom}(A, B) = 0$ for all $B \in \mathrm{Obj}\,(\mathcal{B})$.

2.2 Derived Categories

The main examples of triangulated categories come from deriving abelian categories as we will describe in this section. We will only sketch the constructions. Readers are referred to Gelfand and Manin [25] for details and proofs.

Let $\mathcal{A}$ be an abelian category. Denote by $K(\mathcal{A})$ the *homotopy category* of $\mathcal{A}$, whose objects are chain complexes in $\mathcal{A}$ and morphisms are chain maps modulo chain homotopies. The collection of quasi-isomorphisms in $K(\mathcal{A})$, i.e., chain maps which induce isomorphisms between homology, forms a *multiplication system* which satisfies the following three axioms:

1. All identity morphisms are quasi-isomorphisms and compositions of quasi-isomorphisms are quasi-isomorphisms;

2. If $t : Z \to Y$ is a quasi-isomorphism, then for any morphism $g : X \to Y$ in $K(\mathcal{A})$, there exists $f : W \to Z$ and a quasi-isomorphism $s : W \to X$ making the diagram commutative

$$\begin{tikzcd} W \arrow[r, dashed, "f"] \arrow[d, dashed, "s"] & Z \arrow[d, "t"] \\ X \arrow[r, "g"] & Y. \end{tikzcd}$$

   Similarly, if $s : W \to X$ is a quasi-isomorphism and $f : W \to Z$ any morphism in $K(\mathcal{A})$, there exists $g : X \to Y$ and a quasi-isomorphism $t : Z \to Y$ making the diagram commutative

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "s"] & Z \arrow[d, dashed, "t"] \\ X \arrow[r, dashed, "g"] & Y. \end{tikzcd}$$

3. If $f, g : X \to Y$ are morphisms in $K(\mathcal{A})$, then the following two conditions are equivalent:

   (a) $sf = sg$ for some quasi-isomorphism s,

   (b) $ft = gt$ for some quasi-isomorphism t.

The derived category $\mathrm{D}(\mathcal{A})$ is constructed from $K(\mathcal{A})$ by formally inverting all quasi-isomorphisms: Objects in $\mathrm{D}(\mathcal{A})$ are the same as $K(\mathcal{A})$, i.e., chain complexes of objects in $\mathcal{A}$. Morphisms between an object X and Y are given by equivalence classes of diagrams in the form $X \leftarrow Z \to Y$ where $s : Z \to X$ is a quasi-isomorphism and $f : Z \to Y$ is a chain map. Two diagrams $X \overset{s}{\leftarrow} Z \overset{f}{\to} Y$ and $X \overset{t}{\leftarrow} W \overset{g}{\to} Y$ are equivalent if there is a diagram $W \overset{r}{\leftarrow} U \overset{h}{\to} Z$ which fits into a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & U \arrow[u, "r"] \arrow[d, "h"] & Y. \\ & Z \arrow[ul, "s"] \arrow[ur, "f"'] & \end{tikzcd}$$

Composition of maps are given by

$$(X \overset{s}{\leftarrow} U \overset{f}{\to} Y) \circ (Y \overset{t}{\leftarrow} V \overset{g}{\to} Z) \mapsto (X \overset{sr}{\leftarrow} W \overset{gh}{\to} Z)$$

where

$$\begin{tikzcd} W \arrow[r, dashed, "h"] \arrow[d, dashed, "r"] & V \arrow[d, "t"] \\ U \arrow[r, "f"] & Y \end{tikzcd}$$

is a commutative diagram we get by Axiom 2, with r a quasi-isomorphism. One can check this definition is well-defined. The set $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(X, Y)$ forms a vector space over $\mathbb{K}$:

1. $k \cdot (X \overset{s}{\leftarrow} Z \overset{f}{\to} Y) = (X \overset{s}{\leftarrow} Z \overset{kf}{\to} Y)$ for any $k \in \mathbb{K}$;

2. $(X \xleftarrow{s} Z \xrightarrow{f} Y) + (X \xleftarrow{t} W \xrightarrow{g} Y) = (X \xleftarrow{r} Z \xrightarrow{fu+gv} Y)$ where

$$\begin{tikzcd} U \arrow[r, dashed, "v"] \arrow[d, dashed, "u"'] & W \arrow[d, "t"] \\ Z \arrow[r, "s"] & X \end{tikzcd}$$

is a commutative diagram we get by Axiom 2, with u and $r = su = tv$ quasi-isomorphisms.

This shows $\mathrm{D}(\mathcal{A})$ is a $\mathbb{K}$-linear category. A triangle $X \to Y \to Z \to \Sigma X$ in $\mathrm{D}(\mathcal{A})$ is said to be distinguished if it is isomorphic to a triangle of the form $A \xrightarrow{f} B \to \mathrm{cone}\,(f) \to \Sigma A$. Equipped with this class of distinguished triangles, $\mathrm{D}(\mathcal{A})$ has the structure of a triangulated category. The bounded derived category $\mathrm{D}^b(\mathcal{A})$ is defined as the smallest triangulated subcategory of $\mathrm{D}^b(\mathcal{A})$ containing all bounded complexes.

Example 2.2.1. Let X be a variety. Then $\mathrm{Coh}\,(X)$ is an abelian category and $\mathrm{D}^b(\mathrm{Coh}\,(X))$ is the bounded derived category of coherent sheaves on X. We will denote by $\mathrm{D}^b_{cs}(\mathrm{Coh}\,(X))$ the full triangulated subcategory of $\mathrm{D}^b(\mathrm{Coh}\,(X))$ consisting of complexes whose cohomologies are compactly supported.

Example 2.2.2. Let A be a dg-algebra. We denote by $\mathrm{D}^b(A)$ the bounded derived category of dg-modules over A. We also denote by $\mathrm{D}^b_{fd}(A)$ the full subcategory of $\mathrm{D}^b(A)$ consisting of complexes whose cohomologies are finite dimensional. A dg-algebra A is called cohomologically Noetherian if $H^\bullet(A)$ is Noetherian. In this case, we denote by $\mathrm{D}^b_{fg}(A)$ the full triangulated subcategory of $\mathrm{D}^b(A)$ consisting of complexes whose cohomologies are finitely generated modules over $H^\bullet(A)$. When A is an ordinary Noetherian algebra, the category of finitely generated A-modules A-mod is abelian, and we have an equivalence $\mathrm{D}^b(A\text{-mod}) \cong \mathrm{D}^b_{fg}(A)$.

2.3 t-structures

In this section, we discuss t-structures. We follow Manin [25, §IV.4]. It is known that two different abelian categories might yield the same triangulated category. The formalism of t-structure was invented to see different abelian subcategories inside a triangulated category.

Definition 2.3.1. Let $\mathcal{T}$ be a triangulated category. A *t-structure* on $\mathcal{T}$ is a pair of strictly full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ satisfying

1. $\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 1} \subseteq \mathcal{T}^{\geq 0}$,
2. $\mathrm{Hom}(X, Y) = 0$ for $X \in \mathrm{Obj}\,\mathcal{T}^{\leq 0}$ and $Y \in \mathrm{Obj}\,\mathcal{T}^{\geq 1}$,
3. For any $X \in \mathrm{Obj}\,\mathcal{T}$ there exists a distinguished triangle $A \to X \to B \to A[1]$ with $A \in \mathrm{Obj}\,\mathcal{T}^{\leq 0}$ and $B \in \mathrm{Obj}\,\mathcal{T}^{\geq 1}$, where $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$.

The full subcategory $\mathcal{T}^{\geq 0} \cap \mathcal{T}^{\leq 0}$ is called the *heart* of the t-structure.

Theorem 2.3.2. *The heart of any t-structure on a triangulated category is an abelian category.*

Proof. See [25, §IV.4, Theorem 4]. ■

Remark 2.3.3. In general, given a triangulated category $\mathcal{T}$ with a t-structure whose heart is $\mathcal{A}$, the derived category $\mathrm{D}(\mathcal{A})$ might not be equivalent to $\mathcal{T}$. Moreover, in general, there is no obvious relation of $\mathcal{T}$ with the category of complexes over $\mathcal{A}$. This is caused by the non-functorality of the cone [25, §IV.4 Remark 13].

2.4 Serre Functors

We define Serre functors and Calabi–Yau triangulated categories. In this section, all triangulated categories are assumed to be $\mathbb{K}$-linear and Hom-finite, i.e., all the morphism spaces are finite dimensional over $\mathbb{K}$. References for this section are the paper by Keller [45, §2] and the book by Huybrechts [34, §1].

Definition 2.4.1 (Serre functor)**.** A *Serre functor* on a Hom-finite triangulated category $\mathcal{A}$ is a triangle autoequivalence $(S,\sigma):\mathcal{A}\to\mathcal{A}$ together with a family of isomorphisms

$$\eta_{X,Y}:\mathcal{A}(X,Y)\to\mathcal{A}(Y,SX)^{\vee}$$

natural in both X and Y, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}(X,Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, equal] & \mathcal{A}(Y,SX)^{\vee} \arrow[r, "(\Sigma^{-1})^*"] & \mathcal{A}(\Sigma Y,\Sigma SX)^{\vee} \arrow[d, "(\sigma_X)^*"] \\
\mathcal{A}(X,Y) \arrow[r, "-\Sigma"] & \mathcal{A}(\Sigma X,\Sigma Y) \arrow[r, "\eta_{\Sigma X,\Sigma Y}"] & \mathcal{A}(\Sigma Y,S\Sigma X)^{\vee}.
\end{tikzcd}$$

In other words, for any $X,Y\in\operatorname{Obj}\mathcal{A}$, $f\in\mathcal{A}(X,Y)$ and $g\in\mathcal{A}(\Sigma Y,S\Sigma X)$, we have

$$\langle\eta_{X,Y}(f),\Sigma^{-1}(\sigma_X\circ g)\rangle=-\langle\eta_{\Sigma X,\Sigma Y}(\Sigma f),g\rangle.$$

Proposition 2.4.2. *Let $(\mathcal{A},\Sigma_{\mathcal{A}})$ be a Hom-finite triangulated category and (S,σ) be an autoequivalence. Then S is a Serre functor if and only if there is a family of linear maps $\operatorname{tr}_X:\mathcal{A}(X,SX)\to k$ such that the family of induced pairings $\mathcal{A}(X,Y)\times\mathcal{A}(Y,SX)\to k$ given by $(f,g)\to\operatorname{tr}_X(g\circ f)$ are nondegenerate and they satisfy*

$$\operatorname{tr}_X(g\circ f)=\operatorname{tr}_Y(Sf\circ g) \tag{2.4.1}$$

and for all $h:\Sigma X\to S\Sigma X$

$$\operatorname{tr}_X(\Sigma^{-1}(\sigma_X\circ h))=-\operatorname{tr}_{\Sigma X}(h). \tag{2.4.2}$$

Proof. Suppose (S,σ) is a Serre functor. Define $\operatorname{tr}_X=\eta_{X,X}(\mathrm{id}_X)$. By naturality of η, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{A}(X,X) \arrow[r, "f_*"] \arrow[d, "\eta_{X,X}"'] & \mathcal{A}(X,Y) \arrow[d, "\eta_{X,Y}"] & \mathcal{A}(Y,Y) \arrow[l, "f^*"'] \arrow[d, "\eta_{Y,Y}"] \\
\mathcal{A}(X,SX)^{\vee} \arrow[r, "f_*"'] & \mathcal{A}(Y,SX)^{\vee} & \mathcal{A}(Y,SY)^{\vee}. \arrow[l, "(Sf)^*"]
\end{tikzcd}$$

Then

$$\operatorname{tr}_X(g\circ f)=\langle g\circ f,\eta_{X,X}(\mathrm{id}_X)\rangle=\langle g,\eta_{X,Y}(f)\rangle=\langle Sf\circ g,\eta_{Y,Y}(\mathrm{id}_Y)\rangle=\operatorname{tr}_Y(Sf\circ g).$$

Also, from the above equation, we see that for a fixed f, $\operatorname{tr}_X(g\circ f)=0$ for all g implies $\eta_{X,Y}(f)=0$ which in turn implies $f=0$ since $\eta_{X,Y}$ is an isomorphism. Similarly, if for a fixed g, $\operatorname{tr}_X(g\circ f)=0$ for all f, then $g=0$ since $\eta_{X,Y}$ is an isomorphism and hence $\eta_{X,Y}(f)$ is arbitrary.

For the second equality, by definition, for all $h:\Sigma X\to S\Sigma X$, we have

$$\langle\eta_{X,X}(\mathrm{id}_X),\Sigma^{-1}(\sigma_X\circ h)\rangle=-\langle\eta_{\Sigma X,\Sigma X}(\Sigma\mathrm{id}_X),h\rangle$$

which implies

$$\operatorname{tr}_X(\Sigma^{-1}(\sigma_X\circ h))=-\operatorname{tr}_{\Sigma X}(h).$$

Conversely, we define $\eta_{X,Y}:\mathcal{A}(X,Y)\to\mathcal{A}(Y,SX)^{\vee}$ by $\eta_{X,Y}(f)=\operatorname{tr}_X(-\circ f)$. Since the induced pairing is assumed to be non-degenerate and both vector spaces are finite dimensional, $\eta_{X,Y}$ is an isomorphism. Naturality of $\eta_{X,Y}$ in both X and Y is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{A}(Z,Y) \arrow[r, "h^*"] \arrow[d, "\eta_{Z,Y}"'] & \mathcal{A}(X,Y) \arrow[r, "k_*"] \arrow[d, "\eta_{X,Y}"'] & \mathcal{A}(X,W) \arrow[d, "\eta_{X,W}"'] \\
\mathcal{A}(Y,SZ)^{\vee} \arrow[r, "(Sh)^*"] & \mathcal{A}(Y,SX)^{\vee} \arrow[r, "k_*"] & \mathcal{A}(W,SX)^{\vee}
\end{tikzcd}$$

for arbitrary $W, Z \in \operatorname{Obj}\mathcal{A}$ and $h \in \mathcal{A}(X,Z)$ and $k \in \mathcal{A}(Y,W)$. This is checked readily by unwinding the definition: for all $u \in \mathcal{A}(Z,Y)$

$$((Sh)^* \circ \eta_{Z,Y})(u) = \mathrm{tr}_Z(Sh \circ - \circ u) = \mathrm{tr}_X(- \circ u \circ h) = \eta_{X,Y}(u \circ h) = (\eta_{X,Y} \circ h^*)(u)$$

and for all $v \in \mathcal{A}(X,Y)$,

$$(\eta_{X,W} \circ k_*)(v) = \eta_{X,W}(k \circ v) = \mathrm{tr}_X(- \circ k \circ v) = (k_* \circ \eta_{X,Y})(v).$$

Also, by (2.4.2),

$$\begin{aligned}\langle \eta_{X,Y}(f), \Sigma^{-1}(\sigma_X \circ g)\rangle &= \mathrm{tr}_X(\Sigma^{-1}(\sigma_X \circ g) \circ f)\\ &= \mathrm{tr}_X(\Sigma^{-1}(\sigma_X \circ g \circ \Sigma f))\\ &= -\mathrm{tr}_{\Sigma X}(g \circ \Sigma f)\\ &= -\langle \eta_{\Sigma X, \Sigma Y}(\Sigma f), g\rangle.\end{aligned}$$

■

Remark 2.4.3. The negative sign in Equation (2.4.2) is explained by van den Bergh in [68, Remark A.4.2].

Definition 2.4.4 (Calabi–Yau triangulated category)**.** A triangulated category is said to be *d-Calabi–Yau* if $(\Sigma, -\mathrm{id}_{\Sigma^2})^d$ is a Serre functor.

Proposition 2.4.5. *A triangulated $\mathbb{K}$-category $(\mathcal{A}, \Sigma)$ is d-Calabi–Yau if and only if for each $X \in \operatorname{Obj}\mathcal{A}$ there is a linear map $\mathrm{tr}_X : \mathcal{A}(X, \Sigma^d X) \to \mathbb{K}$ such that for all $X, Y \in \operatorname{Obj}\mathcal{A}$, and integers p, q with $p+q = d$, the induced pairings $\langle \cdot, \cdot \rangle : \mathcal{A}(X, \Sigma^p Y) \times \mathcal{A}(Y, \Sigma^q X) \to \mathbb{K}$ given by $(f, g) \to \mathrm{tr}_X((\Sigma^p g) \circ f)$ are nondegenerate and they satisfy*

$$\mathrm{tr}_X((\Sigma^p g) \circ f) = (-1)^{pq}\mathrm{tr}_Y((\Sigma^q f) \circ g).$$

Proof. Suppose $(\mathcal{A}, \Sigma)$ is d-Calabi–Yau. Without loss of generality, we may assume the Serre functor is given by $(\Sigma, -\mathrm{id})^d = (\Sigma^d, (-1)^d\mathrm{id})$. Thus $(-1)^d\mathrm{tr}_X(\Sigma^{-1}h) = -\mathrm{tr}_{\Sigma X}(h)$ for any $h : \Sigma X \to S\Sigma X$, and

$$\begin{aligned}\mathrm{tr}_X((\Sigma^p f) \circ g) &= \mathrm{tr}_X(\Sigma^{-q}(\Sigma^d f \circ \Sigma^q g))\\ &= (-1)^{dq+q}\mathrm{tr}_{\Sigma^q X}(\Sigma^d f \circ \Sigma^q g) && \text{by equation (2.4.2) in Proposition 2.4.2}\\ &= (-1)^{pq}\mathrm{tr}_{\Sigma^q X}(\Sigma^d f \circ \Sigma^q g)\\ &= (-1)^{pq}\mathrm{tr}_Y((\Sigma^q g) \circ f) && \text{by equation (2.4.1) in Proposition 2.4.2}\end{aligned}$$

We show the converse by showing equation (2.4.1) and (2.4.2) in Proposition 2.4.2. Equation (2.4.1) is a special case for $p = 0$ and $q = d$. For equation (2.4.2), given any $h : \Sigma X \to \Sigma^{d+1}X$, if we put $p = -1$ and $q = d+1$, and view the identity map on X as $\mathrm{id}_X : X \to \Sigma^{-1}(\Sigma X)$, then equation (2.4.2) is verified:

$$\begin{aligned}\mathrm{tr}_X(\Sigma^{-1}((-1)^d h)) &= (-1)^d\mathrm{tr}_X(\Sigma^{-1}h \circ \mathrm{id}_X)\\ &= (-1)^{d+d+1}\mathrm{tr}_{\Sigma X}(\Sigma^{d+1}\mathrm{id}_X \circ h)\\ &= -\mathrm{tr}_{\Sigma X}(h).\end{aligned}$$

■

Example 2.4.6 (Serre duality)**.** Let X be a smooth quasi-projective variety. Denote by $\mathrm{D}^b_{cs}(\mathrm{Coh}\,(X))$ as the smallest full triangulated subcategory of $\mathrm{D}^b(\mathrm{Coh}\,(X))$ which contains all complexes with compact support. Then $(-) \otimes K_X[\dim X] : \mathrm{D}^b_{cs}(\mathrm{Coh}\,(X)) \to \mathrm{D}^b_{cs}(\mathrm{Coh}\,(X))$ is a Serre functor. In particular, if K_X is trivial, $\mathrm{D}^b_{cs}(\mathrm{Coh}\,(X))$ is a Calabi–Yau triangulated category.

Example 2.4.7 ([45], Lemma 4.1)**.** Let A be a dg-algebra which is homologically smooth, i.e., $A \in \mathrm{Per}(A^{\mathrm{op}} \otimes A)$. Denote by $\mathrm{D}^b_{fd}(A)$ as the full triangulated subcategory of $\mathrm{D}^b(A)$ which contains all complexes with finite dimensional cohomologies. Define the dualizing complex $\Omega = \mathrm{RHom}_{A^{\mathrm{op}} \otimes A}(A, A^{\mathrm{op}} \otimes A)$. Then $(-) \otimes \Omega : \mathrm{D}^b_{fd}(A) \to \mathrm{D}^b_{fd}(A)$ is a Serre functor. In particular, if we have a isomorphism $\Omega \cong \Sigma^{-d}A$ as objects in $\mathrm{D}(A^{\mathrm{op}} \otimes A)$, then $\mathrm{D}^b_{fd}(A)$ is d-Calabi–Yau.

2.5 Compact Generators

This section surveys different notions of generators in a triangulated category and results that we will be using in the remaining thesis. It is essentially a summary of Bondal and van den Bergh [9].

Definition 2.5.1 (Compact Objects)**.** Let $\mathcal{D}$ be a triangulated category which admits arbitrary direct sums. In general, for any object $X \in \mathrm{Obj}\,(\mathcal{D})$, the functor $\mathrm{Hom}(X, -)$ only commutes with finite direct sums. An object X is said to be *compact* if $\mathrm{Hom}(X, -)$ commutes with arbitrary direct sums. The full subcategory containing all compact objects is denoted by $\mathcal{D}^c$.

Example 2.5.2 ([9], Thm 3.1.1)**.** Let X be a projective variety and G a finite group acting on X. The compact objects in $\mathrm{D}(\mathrm{QCoh}^G(X))$ are precisely the G-equivariant perfect complexes, i.e., complexes that are locally quasi-isomorphic to a bounded complex of equivariant vector bundles. If X is smooth, all complexes in $\mathrm{D}^b(\mathrm{Coh}^G(X))$ are perfect, hence $\mathrm{D}(\mathrm{QCoh}^G(X))^c = \mathrm{D}^b(\mathrm{Coh}^G(X))$.

Example 2.5.3 ([42], Prop 8.3)**.** Let A be a dg-algebra and $\mathrm{D}(A)$ the derived category of dg-modules over A. Then $\mathrm{D}(A)^c = \mathrm{Per}(A)$, where $\mathrm{Per}(A)$ is the smallest thick subcategory in $\mathrm{D}(A)$ containing the free dg-module A.

Definition 2.5.4 (Generators)**.** A set of objects $\mathcal{E}$ in $\mathcal{D}$ *classically generates* $\mathcal{D}$ if $\mathcal{D}$ is the smallest thick subcategory in $\mathcal{D}$ containing $\mathcal{E}$. We say $\mathcal{E}$ *generates* $\mathcal{D}$ if $\mathcal{E}^{\perp} = 0$. We say $\mathcal{D}$ is *compactly generated* if $\mathcal{D}$ is generated by $\mathcal{D}^c$.

Theorem 2.5.5 ([9], Thm 2.1.2)**.** *Let $\mathcal{D}$ be a compactly generated triangulated category. Then a set of compact objects in $\mathcal{D}$ classically generates $\mathcal{D}^c$ if and only if it generates $\mathcal{D}$.*

Example 2.5.6 ([9], Thm 3.1.1)**.** Let X be a variety and G be a finite group acting on X. Then $\mathrm{D}(\mathrm{QCoh}^G(X))$ is compactly generated.

Corollary 2.5.7. *Let X be a smooth variety and G be a finite group acting on X by automorphisms. Then a set of objects $\mathcal{E}$ in $\mathrm{D}^b(\mathrm{Coh}^G(X))$ classically generates $\mathrm{D}^b(\mathrm{Coh}^G(X))$ if and only if $\mathcal{E}$ generates $\mathrm{D}(\mathrm{QCoh}^G(X))$.*

Proof. This is a consequence of Example 2.5.2, Theorem 2.5.5 and Example 2.5.6. ■

2.6 Admissible Subcategories

Definition 2.6.1 (Admissible Subcategories)**.** A full triangulated subcategory $\mathcal{B} \subseteq \mathcal{D}$ is said to be *left admissible* (resp. *right admissible*) if the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{D}$ has a left (resp. right) adjoint. A full triangulated subcategory is said to be *admissible* if it is both left and right admissible.

Proposition 2.6.2 ([8], Lemma 3.1)**.** *Let $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{D}$. The following are equivalent:*

1. *$\mathcal{B}$ and $\mathcal{B}^{\perp}$ classically generate $\mathcal{D}$;*
2. *For any object $X \in \mathcal{D}$, there exists $B \in \mathcal{B}$ and $C \in \mathcal{B}^{\perp}$ and a distinguished triangle*
$$B \to X \to C \to B[1];$$
3. *$\mathcal{B}$ is right admissible, and the right adjoint $q : \mathcal{D} \to \mathcal{B}$ sends X to B;*
4. *$\mathcal{B}^{\perp}$ is left admissible, and the left adjoint $p : \mathcal{D} \to \mathcal{B}^{\perp}$ sends X to C.*

There is of course the similar

Proposition 2.6.3. *The following are equivalent:*

1. *${}^{\perp}\mathcal{B}$ and $\mathcal{B}$ classically generate $\mathcal{D}$;*

2. For any object $X \in \mathcal{D}$, there exists $B \in \mathcal{B}$ and $C \in {}^{\perp}\mathcal{B}$ and a distinguished triangle

$$C \to X \to B \to C[1];$$

3. $\mathcal{B}$ is left admissible, and the left adjoint $p : \mathcal{D} \to \mathcal{B}$ sends X to B;

4. $\mathcal{B}^{\perp}$ is right admissible, and the right adjoint $q : \mathcal{D} \to {}^{\perp}\mathcal{B}$ sends X to C.

Corollary 2.6.4. *Let $\mathcal{B}$ be a right (resp. left) admissible subcategory of $\mathcal{D}$. Then $\mathcal{B} = \mathcal{D}$ if and only if $\mathcal{B}^{\perp} = 0$ (resp. ${}^{\perp}\mathcal{B} = 0$).*

2.7 Mutation Functors

This section defines mutation functors. We follow the convention of Bridgeland and Stern [16], which differs from the more standard convention of Bondal and Kapranov [8, 10] by a shift functor, but it simplifies some of our formulae.

Let $\mathcal{B}$ be an admissible full triangulated subcategory of $\mathcal{D}$. By Propositions 2.6.2 and 2.6.3, there are left adjoint $p : \mathcal{D} \to \mathcal{B}^{\perp}$ to the inclusion $i : \mathcal{B}^{\perp} \to \mathcal{D}$, and right adjoint $q : \mathcal{D} \to {}^{\perp}\mathcal{B}$ to the inclusion $j : {}^{\perp}\mathcal{B} \to \mathcal{D}$.

Definition 2.7.1. The *left mutation functor* $L_{\mathcal{B}} : {}^{\perp}\mathcal{B} \to \mathcal{B}^{\perp}$ is defined to be $L_{\mathcal{B}} = p \circ j$. Similarly, the *right mutation functor* $R_{\mathcal{B}} : \mathcal{B}^{\perp} \to {}^{\perp}\mathcal{B}$ is defined to be $R_{\mathcal{B}} = q \circ i$.

$$\begin{tikzcd} {}^{\perp}\mathcal{B} \arrow[r, bend left, "j"] & \mathcal{D} \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] & \mathcal{B}^{\perp} \arrow[l, bend left, "i"] \end{tikzcd}$$

If E is an object in $\mathcal{D}$, we define $L_E = L_{\langle E \rangle}$ and $R_E = R_{\langle E \rangle}$, where $\langle E \rangle$ is the smallest full triangulated subcategory in $\mathcal{D}$ containing E.

Proposition 2.7.2. *Let $X \in {}^{\perp}\mathcal{B}$ and $Y \in \mathcal{B}^{\perp}$. Then $Y = L_{\mathcal{B}}(X)$ if and only if there is an object $B \in \mathcal{B}$ and a triangle*

$$B \to X \to Y \to B[1].$$

Similarly, $X = R_{\mathcal{B}}(Y)$ if and only if there is an object $B \in \mathcal{B}$ and a triangle

$$X \to Y \to B \to X[1].$$

The two mutation functors $L_{\mathcal{B}}$ and $R_{\mathcal{B}}$ are inverse to each other.

Proof. The first two claims are immediate from Proposition 2.6.2 and 2.6.3. The last claim follows from the first two claims. ∎

2.8 Exceptional Sequences

This section introduces the notion of exceptional poset in a $\mathbb{K}$-linear triangulated category $\mathcal{D}$. For any objects $A, B \in \mathcal{D}$, we denote $\mathrm{Hom}^k(A, B) = \mathrm{Hom}(A, B[k])$ for $k \in \mathbb{Z}$ and

$$\mathrm{Hom}^{\bullet}(A, B) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k(A, B)[-k] \tag{2.8.1}$$

the chain complex of vector spaces with trivial differential.

Definition 2.8.1 (Exceptional Poset)**.** An object $E \in \mathcal{D}$ is said to be *exceptional* if

$$\mathrm{Hom}^{\bullet}(E, E) = \mathbb{K}.$$

Let $(I, \preccurlyeq)$ be a finite poset. A finite set of exceptional objects $\mathcal{E} = \{E_i\}_{i \in I}$ in $\mathcal{D}$ indexed by $(I, \preccurlyeq)$ is an *exceptional poset* if

$$\mathrm{Hom}^{\bullet}(E_i, E_j) = 0 \text{ unless } i \preccurlyeq j.$$

If $(I, \preccurlyeq)$ is a totally ordered set, i.e., in the form $(\{1, \dots, n\}, \leq)$, we say $\mathcal{E}$ is an *exceptional sequence*. An exceptional poset is said to be *strong* if

$$\mathrm{Hom}^k(E_i, E_j) = 0 \text{ for all } k \geq 1 \text{ and all } i, j \in I.$$

It is said to be *full* if $\mathcal{E}$ classically generates $\mathcal{D}$, i.e., $\mathcal{D}$ is the smallest full triangulated category containing $\mathcal{E}$. We say an exceptional poset $\mathcal{E}$ has *length* n if it has n objects. For any $j \in I$ the subset $\{E_i : i \prec j\} \subseteq \mathcal{E}$ is also an exceptional poset and will be denoted by $\mathcal{E}_{\prec j}$. The exceptional poset $\mathcal{E}_{\preccurlyeq j}$ and $\mathcal{E}_{\not\preccurlyeq j}$ are similarly defined.

Remark 2.8.2. Note that every poset $(I, \preccurlyeq)$ can be refined into a totally ordered set, i.e., there exists a (non-unique) monotone bijection $(I, \preccurlyeq) \to (\{1, 2, \cdots, |I|\}, \leq)$. Note also that if i and j are incomparable elements in I, then one can always find total order refinements ϕ and ψ such that $\phi(i) < \phi(j)$ whilst $\psi(i) > \psi(j)$. Hence, by considering exceptional posets with all total order refinements, there is no loss of generality by considering only exceptional sequences. However, since sometimes we have only a natural partial order instead of a total order on I, we will stick to the notion of exceptional poset.

The following proposition tells us that the length of a full exceptional poset is an invariant of the derived category.

Proposition 2.8.3 ([16], Lemma C.2)**.** *Let $\mathcal{E}$ be a full exceptional poset on $\mathcal{D}$ with length n. Then $[\mathcal{E}] = \{[E_i]\}_{i \in I}$ form a $\mathbb{Z}$-linear basis of $K(\mathcal{D})$. In particular,* $\operatorname{length} \mathcal{E} = \operatorname{rank} K(\mathcal{D})$.

Proposition 2.8.4. *The full triangulated subcategory $\langle \mathcal{E} \rangle$ classically generated by an exceptional poset $\mathcal{E}$ is admissible. Moreover, if E is an exceptional object, then $L_E X$ is the cone of the evaluation map*

$$\mathrm{Hom}^\bullet(E, X) \otimes E \to X \to L_E X.$$

If $\mathcal{E}$ is an exceptional poset and $\phi : (I, \preccurlyeq) \to (\{1, \dots, n\}, \leq)$ is a monotone bijection, then

$$L_{\mathcal{E}} = L_{E_{\phi^{-1}(1)}} \cdots L_{E_{\phi^{-1}(n)}}.$$

Proof. Without loss of generality, we may assume $\mathcal{E}$ is an exceptional sequence. We induct on the length of exceptional poset $\mathcal{E}$. Suppose $n = 1$ and $\mathcal{E} = \{E\}$. For any $X \in \mathcal{D}$, we have a natural evaluation map $\mathrm{Hom}^\bullet(E, X) \otimes E \to X$. Extending it to a triangle

$$\mathrm{Hom}^\bullet(E, X) \otimes E \to X \to Y$$

and applying $\mathrm{Hom}(E, -)$, we get $Y \in \langle E \rangle^\perp$. By Proposition 2.6.2, $\langle E \rangle$ is right admissible and by Proposition 2.7.2, $Y = L_E X$. Left admissibility of $\langle E \rangle$ is similarly proven. Now suppose $\mathcal{E} = \{E_1, \dots, E_n\}$. By induction assumption, $\langle \mathcal{E}_{\geq 2} \rangle$ is admissible, and $L_{\mathcal{E}_{\geq 2}} = L_{E_2} \cdots L_{E_n}$. By Proposition 2.6.2, for any X, there are triangles

$$A \to X \xrightarrow{\alpha} L_{\mathcal{E}_{\geq 2}} X$$

$$\mathrm{Hom}^\bullet(E_1, L_{\mathcal{E}_{\geq 2}} X) \otimes E_1 \to L_{\mathcal{E}_{\geq 2}} X \xrightarrow{\beta} L_{E_1} L_{\mathcal{E}_{\geq 2}} X$$

where $A \in \langle \mathcal{E}_{\geq 2} \rangle$. Applying $\mathrm{Hom}^\bullet(E_i, -)$ to the second sequence, we get $L_{E_1} L_{\mathcal{E}_{\geq 2}} X \in \langle E_i \rangle^\perp$ for all i and hence in $\langle \mathcal{E} \rangle^\perp$. Extend the map $X \xrightarrow{\beta \circ \alpha} L_{E_1} L_{\mathcal{E}_{\geq 2}} X$ to a triangle

$$B \to X \xrightarrow{\beta \circ \alpha} L_{E_1} L_{\mathcal{E}_{\geq 2}} X.$$

By the octahedral axiom, $B \in \langle \mathcal{E} \rangle$ and by Proposition 2.6.2 $\langle \mathcal{E} \rangle$ is right admissible. By Proposition 2.7.2, $L_{\mathcal{E}} X = L_{E_1} L_{\mathcal{E}_{\geq 2}} X = L_{E_1} \cdots L_{E_n} X$. ■

Proposition 2.8.5 (Dual Exceptional Poset)**.** *Let $\mathcal{E} = \{E_i\}_{i \in I}$ be a full exceptional poset. Then there is a unique set of objects $\mathcal{F} = \{F_i\}_{i \in I^{\mathrm{op}}}$ such that*

$$\mathrm{Hom}^\bullet(E_i, F_j) = \begin{cases} \mathbb{K} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the object F_i is given by the formula $F_i = L_{\mathcal{E}_{\prec i}} E_i$, and the set $\mathcal{F}$ is a full exceptional poset which is called the full exceptional poset dual to $\mathcal{E}$.

Proof. The proof is a simple modification of [16, Lemma 2.5]. First, we show existence: By the definition of exceptional poset, $E_j \in \langle \mathcal{E}_{\prec j} \rangle^{\perp}$, hence $F_j = L_{\mathcal{E}_{\prec j}}(E_j)$ is well defined and we have a chain of inclusion

$$F_j \in \langle \mathcal{E}_{\prec j} \rangle^{\perp} \subseteq \langle \mathcal{E}_{\preccurlyeq j} \rangle \subseteq \langle \mathcal{E}_{\not\succcurlyeq j} \rangle^{\perp}.$$

In particular,

$$\mathrm{Hom}^{\bullet}(E_i, F_j) = 0$$

whenever $i \prec j$ or $i \not\preccurlyeq j$, i.e., whenever $i \neq j$. By Proposition 2.7.2, there exists $Y \in \langle \mathcal{E}_{\prec j} \rangle$ and a triangle

$$Y \to E_j \to F_j \to Y[1].$$

Applying the exact functor $\mathrm{Hom}^{\bullet}(E_j, -)$, since $\mathrm{Hom}^{\bullet}(E_j, Y) = 0$, we conclude

$$\mathrm{Hom}^{\bullet}(E_j, F_j) = \mathrm{Hom}^{\bullet}(E_j, E_j) = \mathbb{K}.$$

To show $\mathcal{F} = \{F_i\}_{i \in I^{\mathrm{op}}}$ is an exceptional poset, we have to show $j \not\preccurlyeq i$ implies $\mathrm{Hom}^{\bullet}(F_i, F_j) = 0$. Since $F_i \in \langle \mathcal{E}_{\preccurlyeq i} \rangle$, we are done if $\mathrm{Hom}^{\bullet}(E_k, F_j) = 0$ for all $k \preccurlyeq i$, which is true since $j \not\preccurlyeq i$ and $k \preccurlyeq i$ implies $j \neq k$. For uniqueness, choose a nonzero map $E_j \to F_j$ and extend it to a triangle

$$Y \to E_j \to F_j \to Y[1].$$

Applying the functor $\mathrm{Hom}^{\bullet}(E_i, -)$ shows $\mathrm{Hom}^{\bullet}(E_i, Y) = 0$ for all $i \not\preccurlyeq j$ and $i = j$. Hence $Y \in \langle \mathcal{E}_{\prec j} \rangle$. Proposition 2.7.2 now shows $F_j = L_{\mathcal{E}_{\prec j}}(E_j)$. ∎

Remark 2.8.6. Note that the partial order on $\mathcal{F}$ is reversed. Note also that in general, the dual exceptional poset of a strong exceptional poset is NOT strong.

Example 2.8.7. Take $\mathbb{P}^n$ and the full strong exceptional sequence $\mathcal{E} = (\mathcal{O}, \dots, \mathcal{O}(n))$. By computing cohomologies using the Bott's formula [11], one can check the dual full exceptional poset is given by

$$\mathcal{F} = (\Omega^n(n)[n], \Omega^{n-1}(n-1)[n-1] \dots, \Omega^1(1)[1], \mathcal{O}),$$

where Ω^i denotes the i-th wedge power of the cotangent sheaf.

Example 2.8.8. Take $\mathbb{P}^1 \times \mathbb{P}^1$ and the full strong exceptional poset

$$\mathcal{E} = \{\mathcal{O} < \mathcal{O}(1,0), \mathcal{O}(0,1) < \mathcal{O}(1,1)\}.$$

By computing cohomologies, one can check the dual full exceptional poset is given by

$$\mathcal{F} = \{\mathcal{O}(-1,-1)[2] < \mathcal{O}(0,-1)[1], \mathcal{O}(-1,0)[1] < \mathcal{O}\}.$$

Remark 2.8.9. Exceptional posets do not exist on $\mathrm{D}^b(\mathrm{Coh}\,(X))$ for any Calabi–Yau X. This is because by Serre duality,

$$\mathrm{Hom}^{\dim X}(E, E) = \mathrm{Hom}^0(E, E)$$

which contains at least a copy of $\mathbb{K}$, hence cannot vanish.

2.9 Tilting Objects

In the rest of this thesis, we will be working with algebraic triangulated categories in the sense of Keller [43, §3.6]. The precise definition of an algebraic triangulated category will not bother us much, but let me point out that, in Keller's words, " 'all' triangulated categories occuring in algebra and geometry are algebraic." The main examples we have in our mind will be the derived category of equivariant sheaves on smooth varieties and the derived category of dg-modules over dg-algebras.

Definition 2.9.1 (Tilting Object)**.** An object T in an algebraic triangulated category $\mathcal{D}$ is said to be *tilting* if it is

- compact, i.e., the functor $\mathrm{Hom}(T, -)$ commutes with arbitrary coproduct, and
- generating, i.e., the only object X with $\mathrm{Hom}^\bullet(T, X) = 0$ is the zero object.

A tilting object T is said to be *classical* if $\mathrm{Hom}^k(T, T) = 0$ for all $k \neq 0$.

Here is the main theorem of the section.

Theorem 2.9.2 ([42], Theorem 8.7)**.** *Let T be a tilting object in an algebraic triangulated category $\mathcal{D}$ which admits all set-indexed coproducts. Then there is a dg-algebra $\mathrm{RHom}(T, T)$ with cohomology $\mathrm{Hom}^\bullet(T, T)$ and a triangle equivalence*

$$\Phi : \mathcal{D} \to \mathrm{D}(\mathrm{RHom}(T, T)^{\mathrm{op}})$$

which takes T to the free module of rank one, and whose composition with cohomology is given by

$$H^\bullet \circ \Phi : \mathcal{D} \to \mathrm{Grmod}(\mathrm{Hom}^\bullet(T, T)^{\mathrm{op}}), \quad X \mapsto \mathrm{Hom}^\bullet(T, X).$$

Furthermore, this equivalence restrict to an equivalence between the perfect derived categories

$$\Phi : \mathrm{Per}(T) \to \mathrm{Per}(\mathrm{RHom}(T, T)^{\mathrm{op}}).$$

One way to construct tilting objects is from full exceptional posets.

Proposition 2.9.3. *Let $\mathcal{E} = \{E_i\}_{i \in I}$ be a full exceptional poset of compact objects in $\mathcal{D}$. Then $E = \bigoplus_{i \in I} E_i$ is a tilting object in $\mathcal{D}$. If $\mathcal{E}$ is further assumed to be strong, then E is classical tilting.*

Proof. Since $\mathcal{E}$ is an exceptional poset, $\langle \mathcal{E} \rangle$ is admissible by Proposition 2.8.4, $\langle \mathcal{E} \rangle = \mathcal{D}$ if and only if $\langle \mathcal{E} \rangle^\perp = 0$. Hence E is tilting since $\mathcal{E}$ is full. If $\mathcal{E}$ is strong, then $\mathrm{Hom}^k(E_i, E_j) = 0$ for all $k \geq 1$ and thus E is classical tilting. ■

Chapter 3

A_∞-Algebras

This chapter is a survey on A_∞-algebras and operations on them.

In section 3.1, we introduce A_∞-algebras and other related notions such as minimal models, A_∞-modules and their derived categories.

Section 3.2 defines the notion of minimal A_∞-algebras, and discusses how to construct minimal models by using homotopy perturbation.

Section 3.3 defines the notion of cyclic structure on A_∞-algebras and describes how it gives rise to Calabi–Yau categories.

Section 3.4 defines the Koszul functor, which is essentially a way of producing dg-quivers from A_∞-algebras. There are two versions of this functor: the completed one and the incomplete one. The completed one is defined on A_∞-algebras and yields the completed path algebra of a dg-quiver. The incomplete one is only defined on A_{fin}-algebras, and yields the incompleted path algebra of a dg-quiver. The difference between the two versions is similar to the difference between power series and polynomials. Admittedly, working with A_{fin}-algebras and hence the incomplete Koszul functor is awkward in the world of A_∞-algebras as being A_{fin} is not a homotopy invariant property. For example, the minimal model of an A_{fin}-algebra is not necessarily A_{fin}. However, as we will see in Chapter 5, the incomplete Koszul functor is central to our problem of constructing derived equivalences between dg-quivers and total spaces of vector bundles.

Section 3.5 defines the quotient of an A_∞-algebra by a finite group, and the smash product of an A_∞-algebra by a finite group. Although the definition of quotient construction is straightforward, it seems to be new. This quotient construction is central to constructing derived equivalences equivariantly as described in Section 5.6. We then prove a relation between the quotient construction and the smash product, and shows how these two constructions commute with the Koszul dual functor, i.e., the Koszul dual of the quotient (resp. smash product) of an A_{fin}-algebra is the quotient (resp. smash product) of its Koszul dual. This section is inspired by the work of Bocklandt, Schedler, and Wemyss [7].

Section 3.6 surveys different constructions of A_∞-tensor product. Since A_∞-tensor products are only unique up to A_∞-quasi-isomorphisms, there is in general no natural formulae for computing the tensor product, although there is one in the case when one of the A_∞-algebras is A_2, i.e., a dg-algebra. Particularly important to us is the tensor product constructed by Amorim and Tu [2], since their construction preserves cyclic structures. We then prove that, under some local finiteness conditions, the Koszul functor commutes with the tensor product, i.e., Koszul dual of tensor product of A_∞-algebras is quasi-isomorphic to tensor product of Koszul duals of A_∞-algebras as dg-algebras.

3.1 A_∞-Algebras and A_∞-Modules

This section introduces A_∞-algebras. We follows the sign conventions of Lefèvre-Hasegawa [48]. We refer to Keller [44] for a short introduction and Lefèvre-Hasegawa [48] for a comprehensive reference.

Definition 3.1.1 (A_∞-algebras)**.** Let $\mathbb{K}$ be a field and S be a semisimple algebra over $\mathbb{K}$. An A_∞-algebra over S is a $\mathbb{Z}$-graded S-bimodule,

$$A = \bigoplus_{i\in\mathbb{Z}} A^i$$

together with, for each $n \in \mathbb{N}$, an S-bimodule homomorphisms of degree $2-n$

$$m_n : A^{\otimes_S n} \to A$$

satisfying the the A_∞-relations

$$\sum_{\substack{a+b+c=n\\ b\geq 1}} (-1)^{ab+c} m_{a+1+c} \circ (\mathrm{id}^{\otimes a} \otimes m_b \otimes \mathrm{id}^{\otimes c}) = 0.$$

The first few A_∞-relations read as follows:

- When $n = 1$, we have $m_1 \circ m_1 = 0$, i.e., (A, m_1) is a chain complex.
- When $n = 2$, we have $m_1 \circ m_2 = m_2 \circ (m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1)$, i.e., m_1 is a derivation with respect to m_2.
- When $n = 3$, we have

  $$m_2 \circ (\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) = m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes m_1),$$

  i.e., m_2 is associative up to homotopy given by m_3.

A morphism $f : A \to B$ of A_∞-algebras over S is a family of S-bimodule morphisms of degree $1-n$

$$f_n : A^{\otimes_S n} \to B,$$

satisfying the A_∞-relations

$$\sum_{\substack{a+b+c=n\\ b\geq 1}} (-1)^{ab+c} f_{a+1+c} \circ (\mathrm{id}^{\otimes a} \otimes m_b \otimes \mathrm{id}^{\otimes c}) = \sum_{r=1}^{n} \sum_{i_1+\cdots+i_r=n} (-1)^s m_r \circ (f_{i_1} \otimes \cdots \otimes f_{i_r}),$$

where $s = \displaystyle\sum_{u=2}^{r}\left((1-i_u)\sum_{v=1}^{u} i_v\right)$. Composition of two morphisms $f : B \to C$ and $g : A \to B$ is defined by

$$(f \circ g)_n = \sum_{r=1}^{n} \sum_{i_1+\cdots+i_r=n} (-1)^s f_r \circ (g_{i_1} \otimes \cdots \otimes g_{i_r}).$$

A morphism of A_∞-algebra is called strict if $f_n = 0$ for $n \geq 2$. In this case, the A_∞-relations simplifies to $f_1 m_n = m_n \circ f_1^{\otimes n}$. The identity morphism of an A_∞-algebra is the strict A_∞-morphism with $f_1 = \mathrm{id}$. An A_∞-quasi-isomorphism f is an A_∞-morphism whose f_1 induce isomorphism on the homology on the chain complex (A, m_1).

An A_∞-algebra is strictly unital if there is an element $1_A \in A$ of degree 0 such that $m_2(1_A, a) = a = m_2(a, 1_A)$ for all $a \in A$ and $m_n(a_1, \cdots, a_n) = 0$ whenever $n \neq 2$ and one of the $a_i = 1_A$. A morphism of strictly unital A_∞-algebra $f : A \to B$ is strictly unital if $f_1(1_A) = 1_B$ and $f_n(a_1, \cdots, a_n) = 0$ whenever $n \neq 1$. Note that for any strictly unital A_∞-algebra, there is a canonical strict morphism $\eta : S \to A$ mapping 1_S to 1_A. An A_∞-algebra is augmented if it is strictly unital and there is a strictly unital morphism $\epsilon : A \to S$ such that $\epsilon \circ \eta = \mathrm{id}_S$. A morphism of augmented A_∞-algebras is a strictly unital morphism $f : A \to B$ such that $\epsilon_B \circ f = \epsilon_A$. When an A_∞-algebra is augmented, there is a decomposition $A = S \oplus \bar{A}$, where $\bar{A} = \ker \epsilon$ is called the augmentation ideal.

An A_∞-algebra is said to be A_{fin} if $m_i = 0$ for $i \gg 0$. An A_∞-algebra is said to be A_n if $m_i = 0$ for all $i \geq n$. In particular, an A_1-algebra is a chain complex, and an A_2-algebra is a dg-algebra. Morphisms between chain complexes (resp. dg-algebras) are the same as strict A_∞-morphisms between A_1-algebras (resp. A_2-algebras). A morphism is said to be A_n if $f_r = 0$ for $r > n$ and is said to be A_{fin} if $f_r = 0$ for $r \gg 0$.

Definition 3.1.2 (A_∞-homotopy)**.** Let A and B be two A_∞-algebras and $f, g : A \to B$ be A_∞-morphisms. An A_∞-homotopy between f and g is a family of morphisms of degree $-n$

$$h_n : A^{\otimes n} \to B$$

satisfying the equation

$$\begin{aligned} f_n - g_n = &\sum_{\substack{i_1+\cdots+i_r+k \\ +j_1+\cdots j_t=n}} (-1)^s m_{r+1+t} \circ (f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}) \\ &+ \sum_{j+k+\ell=n} (-1)^{jk+\ell} h_{j+1+\ell} \circ (\mathrm{id}^{\otimes j} \otimes m_k \otimes \mathrm{id}^{\otimes \ell}), \end{aligned}$$

where

$$s = t + \sum_{1\leq\alpha\leq t} (1-j_\alpha)(n - \sum_{u\geq\alpha} j_u) + k \sum_{1\leq u\leq r} i_u + \sum_{2\leq\alpha\leq r} (1-i_\alpha) \sum_{u<\alpha} i_u.$$

Similarly, an A_∞-homotopy is said to be A_n if $h_r = 0$ for $r > n$ and A_{fin} if $h_r = 0$ for $r \gg 0$.

One of the salient features of A_∞-algebras is that all A_∞-quasi-isomorphisms are invertible up to homotopy:

Theorem 3.1.3 ([48] Corollary 1.3.1.3)**.** *An A_∞-quasi-isomorphism is an A_∞-homotopy equivalence.*

Definition 3.1.4 (A_∞-modules)**.** An A_∞-right module over A is a $\mathbb{Z}$-graded S-bimodule M, together with a family of S-bimodule morphisms of degree $2-n$

$$m_n^M : M \otimes_S A^{\otimes_S n-1} \to M$$

satisfying the A_∞-relations

$$\sum_{\substack{a+b+c=n \\ a,b\geq 1}} (-1)^{ab+c} m^M_{a+1+c} \circ (\mathrm{id}^{\otimes a} \otimes m_b \otimes \mathrm{id}^{\otimes c}) + \sum_{\substack{b+c=n \\ b\geq 1}} (-1)^c m^M_{1+c} \circ (m^M_b \otimes \mathrm{id}^{\otimes c}) = 0.$$

Definition 3.1.5 (A_∞-morphism)**.** Let A be an A_∞-algebra and M, N be A_∞-modules. An A_∞-morphism $f : M \to N$ is a family of S-bimodule morphisms of degree $1-n$

$$f_n : M \otimes_S A^{\otimes n-1} \to N$$

which satisfy the equations

$$\sum_{\substack{a+b+c=n \\ b\geq 1}} (-1)^{ab+c} f_{a+1+c} \circ (\mathrm{id}^{\otimes a} \otimes m_b \otimes \mathrm{id}^{\otimes c}) = \sum_{r+s=n} m_{s+1} \circ (f_r \otimes \mathrm{id}^{\otimes s}).$$

An A_∞-morphism is said to be an A_∞*-quasi-isomorphism* if f_1 induces an isomorphism on cohomology. An A_∞-morphism is said to be *strict* if $f_i = 0$ for all $i \geq 2$.

Definition 3.1.6 (A_∞-homotopy)**.** Let A be an A_∞-algebra and M, N be A_∞-modules. Let $f, g : M \to N$ be A_∞-morphisms. An A_∞-homotopy between f and g is a family of morphisms of degree $-n$

$$h_n : M \otimes A^{\otimes n-1} \to N$$

satisfying the equations

$$f_n - g_n = \sum_{r+s=n} (-1)^s m_{1+s}(h_r \otimes \mathrm{id}^{\otimes s}) + \sum_{\substack{a+b+c=n \\ b\geq 1}} (-1)^{ab+c} h_{a+1+c}(\mathrm{id}^{\otimes a} \otimes m_b \otimes \mathrm{id}^{\otimes c}).$$

Theorem 3.1.7 ([48] Proposition 2.4.1.1)**.** *An A_∞-quasi-isomorphism between A_∞-modules is an A_∞-homotopy equivalence.*

Derived Category of A_∞-modules. Recall that for an honest algebra A, the derived category $\mathrm{D}(A)$ is obtained by localizing the homotopy category of A-modules $K(\mathcal{A})$ at the class of quasi-isomorphism. One would like to define the derived category of an A_∞-algebra in a similar way. Since by Theorem 3.1.7, A_∞-quasi-isomorphisms are already homotopy equivalence, we have

Definition 3.1.8 (Derived category of A_∞-modules)**.** The derived category $\mathrm{D}_\infty(A)$ of an A_∞-algebra A is defined to be the homotopy category of A_∞-modules over A, i.e., objects in $\mathrm{D}_\infty(A)$ are A_∞-modules over A and morphisms between two A_∞-modules M and N are A_∞-morphisms modulo A_∞-homotopies.

Definition 3.1.9 (Perfect derived category)**.** Let A be an A_∞-algebra. Then A can be regarded as an A_∞-module over itself. The smallest triangulated category generated by the A-module A is called the perfect derived category of A and is denoted by $\mathrm{Per}_\infty(A)$.

A_∞-categories

Definition 3.1.10. An A_∞-category $\mathcal{A}$ consists of the following data:

1. a set of objects $\mathrm{Obj}\,(\mathcal{A})$,
2. for any $X, Y \in \mathrm{Obj}\,(\mathcal{A})$, a $\mathbb{Z}$-graded vector space
$$\mathcal{A}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}^i(X,Y),$$
3. for each $n = 1, 2, 3, \dots,$ and any $X_0, \dots, X_n \in \mathrm{Obj}\,(\mathcal{A})$, a linear map homogeneous of degree $2-n$
$$m_n : \mathcal{A}(X_{n-1}, X_n) \otimes \mathcal{A}(X_{n-2}, X_{n-1}) \otimes \cdots \otimes \mathcal{A}(X_0, X_1) \to \mathcal{A}(X_0, X_n),$$
satisfying the equations
$$\sum_{a+b+c=n} (-1)^{ab+c} m_{a+1+c} \circ (\mathrm{id}^{\otimes a} \otimes m_b \otimes \mathrm{id}^{\otimes c}) = 0.$$

If $m_1 = 0$, then $\mathcal{A}$ is said to be minimal. An A_∞-category is said to be strictly unital if for each object $X \in \mathrm{Obj}\,\mathcal{A}$, there is $e_X \in \mathcal{A}(X,X)$ such that $m_2(e_X \otimes b) = b$, $m_2(a \otimes e_Y) = a$ for any $a \in \mathcal{A}(Y,X)$ and $b \in \mathcal{A}(X,Y)$, and $m_n(a_n \otimes \cdots \otimes a_1) = 0$ whenever some $a_j = e_X$ for some X. Let $\mathcal{S}$ be the discrete $\mathbb{K}$-category on $\mathrm{Obj}\,\mathcal{A}$, i.e., $\mathrm{Obj}\,\mathcal{S} = \mathrm{Obj}\,\mathcal{A}$ and $\mathcal{S}(X,X) = \mathbb{K}$ and $\mathcal{S}(X,Y) = 0$ whenever $X \neq Y$. Then each strictly unital A_∞-category is endowed with an A_∞-functor $\epsilon : S \to \mathcal{A}$ which sends each $1_X \in \mathcal{S}(X,X)$ to the identity element $e_X \in \mathcal{A}(X,X)$. If there is an A_∞-functor $\eta : \mathcal{A} \to \mathcal{S}$ such that $\eta \circ \epsilon = \mathrm{id}_{\mathcal{S}}$, then $\mathcal{A}$ is said to be augmented. An A_∞-category is said to be finite if $\mathrm{Obj}\,(\mathcal{A})$ is finite and for any objects X, Y, $\mathcal{A}(X,Y)$ is finite dimensional.

Conventions. When the base algebra of an A_∞-algebra is in the form $S = \mathbb{K}^r$, the data of an A_∞-algebra is the same as an A_∞-category $\mathcal{A}$ with r objects: Let e_i be the vector in $S = \mathbb{K}^r$ with 1 in the i-th place and zero elsewhere. Given an A_∞-algebra over S, we can define an A_∞-category $\mathcal{A}$ by taking $\mathrm{Obj}\,(\mathcal{A}) = \{e_1, \dots, e_r\}$ and $\mathcal{A}(e_i, e_j) = e_i A e_j$. Conversely, given an A_∞-category with r objects, choose a bijection between $\mathrm{Obj}\,(\mathcal{A})$ and $\{e_1, \dots, e_r\}$ and take $A = \bigoplus_{i,j} \mathcal{A}(e_i, e_j)$. Each e_i acts on left on A by projecting to the vector subspace $\bigoplus_j \mathcal{A}(e_i, e_j)$. This define a left S-module structure on A. Right action is defined similarly. Since in this thesis we will only deal with A_∞-category with finite objects, we will not distinguish A_∞-algebras and A_∞-categories and will use the two terminologies interchangeably.

3.2 Minimal Model

An A_∞-algebra is said to be minimal if $m_1 = 0$. In this case the multiplication map m_2 is associative. By Kadeishvili's theorem, every A_∞-algebra is quasi-isomorphic to a minimal one:

Theorem 3.2.1 (Kadeishvili [36], see also [48] Corollary 1.4.1.4)**.** *Let A be an A_∞-algebra. Then the cohomology $H^\bullet(A)$ has a unique (up to isomorphism of A_∞-algebras) A_∞-structure such that*

- *$m_1 = 0$ and m_2 is induced by m_2^A,*
- *there is an A_∞-quasi-isomorphism $i : H^\bullet(A) \to A$ lifting the identity map on $H^\bullet(A)$.*

The A_∞-structure on the minimal model can be described more explicitly. The following construction is given by Markl [54, §7] and is known as homotopy perturbation. Denote by B and Z the S-bimodule of coboundaries and cocycles in A. Since S is semisimple, we can choose a splitting (S-subbimodule) H and L such that

$$Z = B \oplus H \text{ and } A = Z \oplus L = B \oplus H \oplus L,$$

where $H(A) \cong H$. Denote by $p : A \to H$ the projection map and by $i : H \to A$ the inclusion map. Define a linear map $h : A \to A$ by $h = 0$ on $L \oplus H$ and $h = (m_1^A|_L)^{-1}$ on B. It follows that hm_1^A (resp. $m_1^A h$) is the projection to L (resp. B), and forms a homotopy from id_A to $i \circ p$ i.e., $\mathrm{id}_A - i \circ p = m_1^A h + hm_1^A$.

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, shift left, "p"] & H \arrow[l, shift left, "i"] \end{tikzcd}$$

Define a sequence of linear maps $\lambda_n : A^{\otimes n} \to A$ of degree $(2-n)$ for all $n \geq 2$ inductively as follow: Take $\lambda_2 = m_2^A$, and for $n \geq 3$, take

$$\lambda_n = \sum_I (-1)^{\theta(r_1,\ldots,r_k)} m_k^A \circ ((h \circ \lambda_{r_1}) \otimes \cdots \otimes (h \circ \lambda_{r_k})), \tag{3.2.1}$$

where the sum is over the set

$$I = \{(k, r_1, \cdots, r_k) : 2 \leq k \leq n, r_1, \ldots r_k \geq 1, r_1 + \cdots + r_k = n\},$$

the sign is given by

$$\theta(r_1, \ldots, r_k) = \sum_{1 \leq \alpha < \beta \leq k} r_\alpha(r_\beta + 1)$$

and $h\lambda_1$ is defined formally to be id_A. Now, define $m_n^{H^\bullet(A)} : H^\bullet(A)^{\otimes n} \to H^\bullet(A)$ of degree $2-n$ by

$$m_n^{H^\bullet(A)} = p \circ \lambda_n \circ i^{\otimes n}.$$

These maps satisfy the A_∞-relations and $H^\bullet(A)$ equipped with linear maps $m_i^{H^\bullet(A)}$ for all $i \geq 2$ is the desired minimal model of A. The A_∞-quasi-isomorphism $i : H^\bullet(A) \to A$ is given by

$$i_n = h \circ \lambda_n \circ i^{\otimes n}.$$

Remark 3.2.2. As pointed out by Markl [54, §4], the recursive formula in Equation 3.2.1 can be reformulated as a sum of trees: Let P_n be the set of all rooted planar directed trees with n leaves and each internal vertex has at least two incoming edges. For each $T \in P_n$, one can assign a linear map $F_T : A^{\otimes n} \to A$ by interpreting each internal vertex with k incoming edges by m_k, and each internal edge by h. Next, we would like to define a number $\theta(T)$. Let v be an internal vertex of T with k incoming edges. Denote by r_i the number of paths going from any leaves of T to the root of T which passes through the i-th edge of v. Define $\theta_T(v) = \theta(r_1, \cdots, r_k)$ and let $\theta(T) = \sum_{\text{internal vertices}} \theta_T(v)$. Then Equation 3.2.1 can be rewritten as

$$\lambda_n = \sum_{T \in P_n} (-1)^{\theta(T)} F_T.$$

3.3 Cyclic Structure

Let $\mathcal{A}$ be a minimal A_∞-category whose morphism spaces are finite dimensional. A cyclic structure of degree d on $\mathcal{A}$ consists of, for each $X, Y \in \operatorname{Obj}\mathcal{A}$, a supersymmetric nondegenerate bilinear form homogeneous of degree $(-d)$

$$\langle -, - \rangle : \mathcal{A}(X,Y) \times \mathcal{A}(Y,X) \to \mathbb{K}$$

satisfying the Koszul sign rule

$$\langle m_n(a_1 \otimes \cdots \otimes a_n), a_{n+1} \rangle = (-1)^{n+|a_1|(|a_2|+\cdots+|a_{n+1}|)} \langle m_n(a_2 \otimes \cdots \otimes a_{n+1}), a_1 \rangle.$$

In general, cyclic structure is not preserved by A_∞-quasi-isomorphisms, i.e., if A and A' are A_∞-quasi-isomorphic, and A has a cyclic structure, A' might not have a cyclic structure. To preserve the cyclic structure, we need a class of A_∞-morphisms which respect the cyclic structure. This gives rise to the notion of cyclic A_∞-morphism, which was first defined by Kajiura [37, Definition 2.13].

Definition 3.3.1 (Cyclic A_∞-morphism)**.** Let A and A' be cyclic algebras. An A_∞-morphism $f : A \to A'$ is said to be cyclic if for all $a_1, \ldots, a_n \in A$,

$$\langle a_1, a_2 \rangle_A = \langle f_1(a_1), f_1(a_2) \rangle_{A'}$$

and for $n \geq 3$

$$\sum_{i+j=n} (-1)^{i+\sum\limits_{\ell=1}^{i=1}(i-\ell+1)|a_\ell| + \sum\limits_{\ell=i+1}^{k-1}(k-\ell)|x_\ell|} \langle f_i(a_1, \cdots, a_i), f_j(a_{i+1}, \ldots, a_n) \rangle = 0$$

Following Keller [45, §5] , we describe a way of producing Calabi–Yau triangulated categories from A_∞-categories with cyclic structure. We denote by $\mathcal{D}(\mathcal{A})$ the derived category of A_∞-module over $\mathcal{A}$. The perfect derived category $\operatorname{per}(\mathcal{A})$ is the thick triangulated subcategory of $\mathcal{D}(\mathcal{A})$ generated by the representable A_∞-modules $\mathcal{A}(-,X)$ for all $X \in \operatorname{Obj}\mathcal{A}$. In other words, $\operatorname{per}(\mathcal{A})$ is the smallest full triangulated subcategory which is stable under taking direct summands which contains all representable A_∞-modules $\mathcal{A}(-,X)$. In case $\mathcal{A}$ is an ordinary $\mathbb{K}$-algebra, then $\operatorname{Per}(\mathcal{A})$ is the full subcategory of $\mathcal{D}(\mathcal{A})$ formed by perfect complexes, i.e., those quasi-isomorphic to a bounded complex of finitely generated projective modules.

Proposition 3.3.2 ([45], §5)**.** *Let $\mathcal{A}$ be a Hom-finite minimal A_∞-category with a cyclic structure of degree d. Then* $\operatorname{Per}(\mathcal{A})$ *is Hom-finite d-Calabi–Yau triangulated category.*

3.4 Koszul Functor

In this section, we will introduce the Koszul functor which produces quasi-free dg-algebras from A_∞-algebras. There are potentially two ways to do this. The first one is to take the bar construction followed by taking dual. The second one is to take dual followed by the cobar construction. If we start with an A_{fin}-algebra, then under some locally finiteness conditions, the two constructions end up giving the same dg-algebra.

For our purpose, the second approach seems to be conceptually simpler and this is the road we will take. In this case when our A_{fin}-algebra is finite dimensional, this is all good and product a dg-algebra which is the path algebra of a dg-quiver. However, it runs into problem as soon as we consider algebras which are not finite dimensional. This is because in general $(V \otimes V)^*$ and $V^* \otimes V^*$ are not isomorphic, and hence taking the dual of an A_∞-algebra does not necessarily produce an A_∞-coalgebra.

Following Lu, Palmieri, Wu and Zhang [51], it is useful to impose some local finiteness condition by equipping A_∞-algebras with an additional grading, called the Adams grading, by an abelian group G. We will write the degree of a bihomogeneous element a in the form $\deg a = (\deg_1 a, \deg_2 a) \in \mathbb{Z} \times G$. The (i,j)-th component of A will be denoted by A^i_j. Henceforth in this section, all A_∞-algebra will be assumed to be locally finite in the sense that each (i,j)-th component A^i_j is finite dimensional. The multiplication maps m_n's will be assumed to preserve the Adams grading, i.e., of degree $(2-n,0)$. A_∞-morphisms, A_∞-homotopies will also be assumed to preserve the Adams grading. The suspension functor Σ will only shift the first grading and ignore the Adams grading, i.e., $(\Sigma A)^i_j = A^{i+1}_j$.

For any Adams graded A_∞-algebra, its minimal model $H^\bullet(A)$ can be chosen to be Adams graded by choosing the splitting $A = B \oplus H \oplus L$ described in Section 3.2 to be Adams graded.

For any locally finite $\mathbb{Z} \times G$-graded vector space $V = \bigoplus V^i_j$, one can take the graded dual $V^\# = \bigoplus \mathrm{Hom}(V^{-i}_j, \mathbb{K}) = \bigoplus (V^{-i}_j)^*$. The graded dual is better behaved than the usual dual since they enjoy the isomorphisms $(V^\#)^\# \cong V$ and $(V^{\otimes n})^\# \cong (V^\#)^{\otimes n}$.

In what follows, we will assume all A_∞-algebras are augmented over the semiample algebra $\mathbb{K}^n$. The kernel of the augmentation map will be denoted by $\bar{A}$.

Adams graded A_{fin}-algebras. We will say an Adams graded A_∞-algebra A_{fin} if for any $j \in G$ in the Adams grading, there is an $r_j \in \mathbb{N}$ such that the j-th component of the map m_n, i.e.,

$$m_n : \bigoplus_{i_1,\dots,i_n} \bigoplus_{j_1+\cdots j_n = j} A^{i_1}_{j_1} \otimes \cdots \otimes A^{i_n}_{j_n} \to \bigoplus_i A^i_j$$

is zero for $n > r_j$. Similarly, for two A_∞-algebras Adams graded by the same abelian group G, an A_∞-morphism $f : A \to B$ is said to be A_{fin} if for any $j \in G$, there is an $r_j \in \mathbb{N}$ such that the j-th component of the map f_n is zero for $n > r_j$. The notion of A_{fin}-homotopy is similarly defined.

Taking graded dual followed by cobar construction. Let A be an Adams graded A_∞-algebra. Taking graded dual of the multiplication maps m_n's, we get linear maps of degree $(2-n,0)$

$$m_n^\# : \bar{A}^\# \longrightarrow (\bar{A}^{\otimes_S n})^\# \cong (\bar{A}^\#)^{\otimes_S n}.$$

Shifting degree, we define linear maps b_n of degree $(1,0)$ via the following commutative diagram

$$\begin{tikzcd}
(\Sigma\bar{A})^\# \arrow[r, "b_n^\#"] \arrow[d, "\Sigma^\#"] & ((\Sigma\bar{A})^\#)^{\otimes_S n} \arrow[d, "(\Sigma^\#)^{\otimes n}"] \\
\bar{A}^\# \arrow[r, "m_n^\#"] & (\bar{A}^\#)^{\otimes_S n}
\end{tikzcd}$$

Putting them together we get a linear map of degree $(1,0)$

$$d = \prod b_n^\# : (\Sigma\bar{A})^\# \longrightarrow \prod ((\Sigma\bar{A})^\#)^{\otimes_S n} = \hat{T}_S(\Sigma\bar{A})^\#.$$

which extends to a (continuous) derivation

$$d : \hat{T}_S(\Sigma\bar{A})^\# \longrightarrow \hat{T}_S(\Sigma\bar{A})^\#.$$

By the A_∞-relations of the m_n's, the map d is a differential, i.e., $d^2 = 0$. We thus get a dg-algebra $\Omega(A^\#) = (\hat{T}_S(\Sigma\bar{A})^\#, d)$. This construction can also be applied to A_∞-morphisms to get dg-algebra morphisms. Thus, we have set up a functor, which we call the completed Koszul functor,

$$\hat{E} : (\text{locally finite } A_\infty\text{-algebras}) \longrightarrow (\text{quasi-free dg-algebras})^{\text{op}}.$$

This functor sends A_∞-morphisms to dg-algebra morphisms and A_∞-homotopies to dg-homotopies.

A variant of the Koszul functor. There is a variant of the Koszul functor for A_{fin}-algebras: instead of taking product, we take direct sum of the b_n's to get a linear map of degree $(1,0)$

$$d = \bigoplus b_n^{\#} : (\Sigma\bar{A})^{\#} \longrightarrow \bigoplus ((\Sigma\bar{A})^{\otimes_S n})^{\#} = T_S(\Sigma\bar{A})^{\#},$$

which extends to a differential

$$d : T_S(\Sigma\bar{A})^{\#} \longrightarrow T_S(\Sigma\bar{A})^{\#}.$$

The component of d which maps $((\Sigma\bar{A})^{\#})^{\otimes p} \to ((\Sigma\bar{A})^{\#})^{\otimes q}$ is given by

$$\sum_{\substack{r+1+t=p\\ r+s+t=q}} \mathrm{id}^{\otimes r} \otimes b_s^{\#} \otimes \mathrm{id}^{\otimes t}.$$

This set up a functor

$$E : (\text{locally finite } A_{\text{fin}}\text{-algebras}) \longrightarrow (\text{quasi-free dg-algebras})^{\mathrm{op}}$$

which sends A_{fin}-morphisms to dg-algebra morphisms and A_{fin}-homotopies to dg-homotopies. For an A_{fin}-algebra, the difference between $\hat{E}(A)$ and $E(A)$ is analogous to the difference between formal power series and polynomials. The dg-algebra $E(A)$ will be called the Koszul dual of A.

Koszul functor as a construction of dg-quiver. Recall that an A_∞-category is said to be finite if A has a finite set of objects and all morphism spaces are finite dimensional. Note that a finite A_{fin}-category is characterised by the property $m_n = 0$ for $n \gg 0$. When the Koszul functor is applied to a finite augmented A_{fin}-category $\mathcal{A}$, it can be viewed as a construction which produces a dg-quiver $Q_{\mathcal{A}}$: $\mathrm{Obj}\,(\mathcal{A})$ correspond to vertex set of $Q_{\mathcal{A}}$; degree i edges between two vertices u and v correspond to a basis of the vector space $\bar{\mathcal{A}}^{1-i}(u,v)^\vee$. Then $\mathbb{K}Q_{\mathcal{A}} = E(\mathcal{A})$ and $\mathbb{K}\hat{Q}_{\mathcal{A}} = \hat{E}(\mathcal{A})$ and the dg-structure on $E(\mathcal{A})$ turns $Q_{\mathcal{A}}$ into a dg-quiver. We may sometimes abuse notation and denote $Q_{\mathcal{A}}$ by $E(\mathcal{A})$, i.e., we are identifying a quiver with its path algebra.

Conversely, every dg-quiver can be constructed this way. Given a dg-quiver Q, we construct an augmented A_{fin}-category by taking

$$\mathrm{Obj}\,(\mathcal{A}) = \{\text{vertices in } Q\},$$

and for any vertices u, v, we take the augmentation ideal $\bar{\mathcal{A}}$ to be

$$\bar{\mathcal{A}}^i(u,v) = \mathbb{K}\{\text{degree } (1-i) \text{ edges in } Q\}^\vee,$$

or in other words,

$$\mathcal{A}^i(u,v) = \begin{cases} \mathbb{K}v \oplus \mathbb{K}\{\text{degree } (1-i) \text{ edges in } Q\}^\vee & \text{if } u = v \text{ and } i = 0,\\ \mathbb{K}\{\text{degree } (1-i) \text{ edges in } Q\}^\vee & \text{if } u \neq v \text{ and } i = 0,\\ \mathbb{K}\{\text{degree } (1-i) \text{ edges in } Q\}^\vee & \text{if } i \neq 0. \end{cases}$$

We then define the shifted higher multiplication maps b_n by

$$b_n(e_1^\vee, \ldots, e_n^\vee) = \sum_{e \text{ edges in } Q} (\text{Coefficient of } e_1 \cdots e_n \text{ in } de) e^\vee,$$

$$b_n(e_i^\vee, \ldots, v, \ldots, e_n^\vee) = 0 \quad \text{for } n \neq 2 \text{ and}$$
$$b_2(v, e^\vee) = (-1)^{|e^\vee|} b_2(e^\vee, v) = e^\vee$$

where e_i are edges in Q and v are vertices in Q. The higher multiplication maps m_n are then given by

$$m_n(e_1^\vee, \ldots, e_n^\vee) = (-1)^{(n-1)|e_1^\vee| + \cdots + 2|e_{n-2}^\vee| + |e_{n-1}^\vee|} \sum_{e \text{ edges in } Q} (\text{Coefficient of } e_1 \cdots e_n \text{ in } de) e^\vee,$$

$$m_n(e_i^\vee, \ldots, v, \ldots, e_n^\vee) = 0 \quad \text{for } n \neq 2 \text{ and}$$
$$m_2(v, e^\vee) = m_2(e^\vee, v) = e^\vee.$$

The A_∞-relations follows from $d^2 = 0$, and by construction we have $E(A) = \mathbb{K}Q$.

Bar construction followed by taking graded dual. As mentioned in the beginning of this section, there is another approach to produce quasi-free dg-algebras from A_∞-algebras. Shifting the degree of the multiplication map m_n, we get a linear map b_n of degree $(1,0)$ by the following commutative diagram

$$\begin{tikzcd} \bar{A}^{\otimes_S n} \arrow[r, "m_n"] \arrow[d, "\Sigma^{\otimes n}"] & \bar{A} \arrow[d, "\Sigma"] \\ (\Sigma\bar{A})^{\otimes_S n} \arrow[r, "b_n"] & \Sigma\bar{A} \end{tikzcd}$$

In other words, $b_n : (\Sigma\bar{A})^{\otimes_S n} \longrightarrow \Sigma\bar{A}$ is the map which sends

$$a_1 \otimes \cdots \otimes a_n \longmapsto (-1)^{(n-1)a_1+\cdots+2a_{n-2}+a_{n-1}} m_n(a_1, \cdots, a_n),$$

where the sign comes from the Koszul sign rule. Putting them together, we get

$$b = \bigoplus b_n : T_S(\Sigma\bar{A}) = \bigoplus (\Sigma\bar{A})^{\otimes_S n} \longrightarrow \Sigma\bar{A}$$

which extends to a coderivation map

$$b : T_S(\Sigma\bar{A}) \longrightarrow T_S(\Sigma\bar{A}).$$

whose component mapping $(\Sigma\bar{A})^{\otimes_S q} \to (\Sigma\bar{A})^{\otimes_S p}$ is given by

$$\sum_{\substack{r+1+t=p \\ r+s+t=q}} \mathrm{id}^{\otimes r} \otimes b_s \otimes \mathrm{id}^{\otimes t}$$

By the A_∞-relations of m_n's, this map is a codifferential, i.e., $b^2 = 0$. We will denote this dg-coalgebra by BA. Taking graded dual, we get a differential

$$b^\# : (BA)^\# \longrightarrow (BA)^\#$$

and a dg-algebra $(BA)^\#$.

Equivalence of the two constructions. Suppose A is A_{fin} and $E(A)$ is locally finite. Then since $E(A)^i_j$ is finite dimensional, the right hand side of

$$\begin{aligned} [E(A)]^i_j &= \bigoplus_{n\geq 1} \bigoplus_{\substack{i_1+\cdots+i_n=i \\ j_1+\cdots+j_n=j}} [(\Sigma\bar{A})^\#]^{i_1}_{j_1} \otimes \cdots \otimes [(\Sigma\bar{A})^\#]^{i_n}_{j_n} \\ &= \bigoplus_{n\geq 1} \bigoplus_{\substack{i_1+\cdots+i_n=i \\ j_1+\cdots+j_n=j}} [(\Sigma\bar{A})^{i_1}_{j_1} \otimes \cdots \otimes (\Sigma\bar{A})^{i_n}_{j_n}]^* \end{aligned}$$

is a finite sum. Hence

$$\begin{aligned} E(A)^\# &= \bigoplus_{i,j} \mathrm{Hom}(E(A)^i_j, \mathbb{K}) \\ &= \bigoplus_{i,j} \bigoplus_{n\geq 1} \bigoplus_{\substack{i_1+\cdots+i_n=i \\ j_1+\cdots+j_n=j}} \mathrm{Hom}([(\Sigma\bar{A})^{i_1}_{j_1} \otimes \cdots \otimes (\Sigma\bar{A})^{i_n}_{j_n}]^*, \mathbb{K}) \\ &= \bigoplus_{i,j} \bigoplus_{n\geq 1} \bigoplus_{\substack{i_1+\cdots+i_n=i \\ j_1+\cdots+j_n=j}} [(\Sigma\bar{A})^{i_1}_{j_1} \otimes \cdots \otimes (\Sigma\bar{A})^{i_n}_{j_n}] \\ &= T_S(\Sigma\bar{A}) \\ &= BA. \end{aligned}$$

Remark 3.4.1. Note that in general $E(A)^\# \ncong BA$ since for infinite sum, we have $\mathrm{Hom}(\bigoplus V_i, W) = \prod \mathrm{Hom}(V_i, W)$. Hence the assumption $E(A)$ is locally finite is crucial.

Comparing the differential, we arrive at

Proposition 3.4.2. *Let A be a locally finite A_{fin}-algebra. Suppose $E(A)$ is also locally finite. Then $E(A) = (BA)^{\#}$.*

Proof. We have already shown $E(A) = (BA)^{\#}$ as a vector space. To see the two maps $b^{\#}$ and d are the same, it suffices to compare their components mapping $((\Sigma\bar{A})^{\#})^{\otimes p} \to ((\Sigma\bar{A})^{\#})^{\otimes q}$ and we are done. ∎

Koszul duality. The following theorem justifies calling $E(A)$ the Koszul dual of A.

Theorem 3.4.3 ([51], Theorem 2.4)**.**

1. *Let A be a locally finite A_{fin}-algebra. Suppose $E(A)$ is locally finite. Then $E(E(A))$ is A_∞-quasi-isomorphic to A.*
2. *Let A be a locally finite dg-algebra. Suppose $E(A)$ is also locally finite. Then there is a quasi-isomorphism of dg-algebras $E(E(A)) \to A$.*

Proof. Denote by Ω the cobar construction on dg-algebras. Then by definition, $E(\Lambda) = \Omega(\Lambda^{\#})$ for any locally finite dg-algebras Λ. By Proposition 3.4.2, $E(E(A)) = E((BA)^{\#}) = \Omega BA$. By [48, Lemma 2.3.4.3], ΩBA and A are A_∞-quasi-isomorphic and we are done.

Now if A is a dg-algebra, we have a quasi-isomorphism of dg-algebras $\Omega BA \to A$ by [48, Lemma 1.3.2.3]. By Proposition 3.4.2, we are done. ∎

3.5 Quotient and Smash Product

In this section, we will fix a finite group G and an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \nmid \mathrm{ord(G)}$, and try to construct a quotient A_∞-category. The reason for the assumption on the characteristic of $\mathbb{K}$ is the following theorem which can be found in almost any standard textbooks on representation theory of finite groups.

Theorem 3.5.1 (Maschke)**.** *Every finite dimensional representation over $\mathbb{K}$ of G is completely reducible, i.e., every G-invariant subspace has a G-invariant complement.*

Proof. Let V be a finite dimensional representation and W an invariant subspace. Choose a projection $\pi : V \to W$. Define a G-equivariant map $\pi_G : V \to W$ by

$$\pi_G(v) = \frac{1}{|G|}\sum_{g\in G} g\pi(g^{-1}v).$$

Take $W^\perp = \ker \pi_G$ and we are done. ∎

The set of all isomorphism classes of irreducible representations of G will be denoted by $\mathrm{Irr}\,(G)$, and irreducible representations by Greek alphabets ρ, σ, τ etc.

Quotient construction. The quotient construction described below is essentially an incarnation of the McKay quiver. Suppose G acts on an A_∞-category $\mathcal{A}$ by fixing all objects of $\mathcal{A}$ and acting on the morphism spaces by strict A_∞-isomorphisms. One can construct a quotient A_∞-category $\mathcal{A}/G$ as follows:

$$\mathrm{Obj}\,(\mathcal{A}/G) = \mathrm{Irr}\,(G) \times \mathrm{Obj}\,(\mathcal{A})$$
$$(\mathcal{A}/G)^i(\rho\times u, \sigma\times v) = \mathrm{Hom}_G(\rho, \mathcal{A}^i(u,v)\otimes\sigma)$$

To define the multiplication maps $m_n^{\mathcal{A}/G}$, observe that $\mathcal{A}^i(\rho\times u, \sigma\times v) = [\mathrm{Hom}(\rho,\sigma)\otimes\mathcal{A}^i(u,v)]^G$. We have natural maps of degree $2-n$

$$\circ\otimes m_n^{\mathcal{A}} : [\mathrm{Hom}(\rho_{n-1},\rho_n)\otimes\mathcal{A}(v_{n-1},v_n)]\otimes\cdots\otimes[\mathrm{Hom}(\rho_0,\rho_1)\otimes\mathcal{A}(v_0,v_1)]\longrightarrow\mathrm{Hom}(\rho_0,\rho_n)\otimes\mathcal{A}(v_0,v_n)$$
$$(\alpha_n\otimes a_n)\otimes\cdots\otimes(\alpha_1\otimes a_1)\mapsto(\alpha_n\circ\cdots\circ\alpha_1)\otimes m_n^{\mathcal{A}}(a_n,\cdots,a_1),$$

which satisfy the A_∞-relations. Since G acts on $\mathcal{A}$ through strict A_∞-isomorphisms,

$$\begin{aligned} g\cdot(\circ\otimes m_n^{\mathcal{A}})((\alpha_n\otimes a_n)\otimes\cdots\otimes(\alpha_1\otimes a_1)) &= g\cdot[(\alpha_n\circ\cdots\circ\alpha_1)\otimes m_n^{\mathcal{A}}(a_n,\cdots,a_1)]\\ &= [g\cdot(\alpha_n\circ\cdots\circ\alpha_1)]\otimes g\cdot m_n^{\mathcal{A}}(a_n,\cdots,a_1)\\ &= [(g\cdot\alpha_n)\circ\cdots\circ(g\cdot\alpha_1)]\otimes m_n^{\mathcal{A}}(g\cdot a_n,\cdots,g\cdot a_1)\\ &= (\circ\otimes m_n^{\mathcal{A}})(g\cdot(\alpha_n\otimes a_n),\cdots,g\cdot(\alpha_1\otimes a_1)). \end{aligned}$$

This shows $\circ\otimes m_n^{\mathcal{A}}$ descends to the G-invariant part to linear maps of degree $2-n$

$$m_n^{\mathcal{A}/G}:(\mathcal{A}/G)(\rho_{n-1}\times v_{n-1},\rho_n\times v_n)\otimes(\mathcal{A}/G)(\rho_0\times v_0,\rho_1\times v_1)\to(\mathcal{A}/G)(\rho_0\times v_0,\rho_n\times v_n)$$

satisfying the A_∞-relations.

Thus we obtained a new A_∞-category $\mathcal{A}/G$. Note that the quotient construction preserves most properties of the original A_∞-category: if $\mathcal{A}$ is finite dimensional/A_n/A_{fin}/unital/augmented/connected, then so is $\mathcal{A}/G$. If $\mathcal{A}$ has a G-invariant cyclic structure of degree m, then so does $\mathcal{A}/G$: the bilinear forms on $\mathcal{A}$

$$\langle -,-\rangle_{\mathcal{A}}:\mathcal{A}(u,v)\otimes\mathcal{A}(v,u)\to\mathbb{K}$$

induce bilinear forms

$$\langle -,-\rangle:[\operatorname{Hom}(\rho,\sigma)\otimes\mathcal{A}(u,v)]\otimes[\operatorname{Hom}(\sigma,\rho)\otimes\mathcal{A}(v,u)]\to\mathbb{K}$$

$$(\alpha\otimes a)\otimes(\beta\otimes b)\mapsto\operatorname{tr}(\alpha\beta)\langle a,b\rangle$$

which restricts to the G-invariant part

$$\langle -,-\rangle_{\mathcal{A}/G}:(\mathcal{A}/G)(\rho\times u,\sigma\times v)\otimes(\mathcal{A}/G)(\sigma\times v,\rho\times u)\to\mathbb{K}.$$

These bilinear forms are also non-degenerate and cyclically invariant since the trace maps tr are.

A variant of the quotient construction. There is a variant of the quotient construction. Given an A_∞-algebra $\mathcal{A}$ over a semisimple algebra $S=\mathbb{K}^r$, on which G acts by strict A_∞-isomorphism, one can define an A_∞-algebra over $S\otimes\mathbb{K}G$ as follows. The underlying $\mathbb{Z}$-graded $S\otimes\mathbb{K}G$-bimodule is given by $\operatorname{Hom}_G(\mathbb{K}G,\mathcal{A}\otimes\mathbb{K}G)$, with $S\otimes\mathbb{K}G$ acting on the left and right by

$$((u\otimes g)\varphi(v\otimes h))(-)=u\varphi(-g)(v\otimes h).$$

The multiplication maps m_n are defined similarly as in $\mathcal{A}/G$. Since $\mathbb{K}G=\bigoplus_{\rho\in\operatorname{Irr}(G)}\rho^{\oplus\dim\rho}$, we see that

$$\operatorname{Hom}_G(\mathbb{K}G,\mathcal{A}\otimes\mathbb{K}G)=\bigoplus_{\rho,\sigma\in\operatorname{Irr}(G)}\operatorname{Hom}_G(\rho,\mathcal{A}\otimes\sigma)^{\oplus(\dim\rho)(\dim\sigma)},$$

i.e., this construction is a variant of $\mathcal{A}/G$ which takes into account the multiplicity of each irreducible representation in the regular representation of G. Note that when G is abelian, the two constructions coincide since every irreducible representation of G are 1-dimensional. In general, the two constructions are related by a Morita functor. Recall that by Maschke's theorem, $\mathbb{K}G$ is a semisimple algebra. In fact, $\mathbb{K}G\cong\bigoplus_{\rho\in\operatorname{Irr}(G)}\operatorname{End}\rho$ canonically as an algebra. Denote by e_ρ the matrix in $\operatorname{End}\rho$ with 1 in the $(1,1)$-entry and 0 in all other entries. Then $\rho=\mathbb{K}Ge_\rho$ and all the e_ρ's are orthogonal idempotents, i.e., $(e_\rho)^2=e_\rho$ and $e_\rho e_\sigma=0$ if $\rho\neq\sigma$. The element $e=\sum_{\rho\in\operatorname{Irr}(G)}e_\rho\in\mathbb{K}G$ is an idempotent element which is full in the sense that $\mathbb{K}Ge\mathbb{K}G=\mathbb{K}G$. Moreover, the algebra $S_G:=e\mathbb{K}Ge$ is commutative and is spanned by all the e_ρ's. Since $\tilde{e}=(1_S\otimes e)$ is also a full idempotent in the semisimple algebra $S\otimes\mathbb{K}G$ with $\tilde{e}(S\otimes\mathbb{K}G)\tilde{e}=S\otimes S_G$, we have the following

Theorem 3.5.2 (Morita Equivalence, [7] Lemma 2.2). *The functor*

$$\mathcal{F} : \mathrm{Bimod}\,(S \otimes \mathbb{K}G) \to \mathrm{Bimod}\,(S \otimes S_G)$$
$$M \mapsto \tilde{e}M\tilde{e}$$

is an equivalence which commutes with tensor product in the sense that

$$\mathcal{F}(M \underset{S\otimes\mathbb{K}G}{\otimes} N) \cong \mathcal{F}(M) \underset{S\otimes S_G}{\otimes} \mathcal{F}(N)$$

are naturally isomorphic through the isomorphism

$$\tilde{e}(m \underset{S\otimes\mathbb{K}G}{\otimes} n)\tilde{e} \mapsto \tilde{e}m\tilde{e} \underset{S\otimes S_G}{\otimes} \tilde{e}n\tilde{e}.$$

Since the Morita functor $\mathcal{F} : \mathrm{Bimod}\ S \otimes \mathbb{K}G \to\ \mathrm{Bimod}\ S \otimes S_G$ commutes with tensor product, the $(S \otimes S_G)$-bimodule $\mathcal{F}(\mathrm{Hom}_G(\mathbb{K}G, \mathcal{A} \otimes \mathbb{K}G))$ has an A_∞-structure given by $\mathcal{F}(m_n)$.

Proposition 3.5.3. *The A_∞-algebras $\mathcal{F}(\mathrm{Hom}_G(\mathbb{K}G, \mathcal{A} \otimes \mathbb{K}G))$ and $\mathcal{A}/G$ are strictly A_∞-isomorphic.*

Proof. This follows by observing that

$$\mathcal{F}(\mathrm{Hom}_G(\mathbb{K}G, \mathcal{A} \otimes \mathbb{K}G)) = \bigoplus_{\rho,\sigma\in \mathrm{Irr}\,(G)} \mathrm{Hom}_G(\mathbb{K}Ge_\rho, \mathcal{A} \otimes \mathbb{K}Ge_\sigma) = \bigoplus_{\rho,\sigma\in \mathrm{Irr}\,(G)} \mathrm{Hom}(\rho, \mathcal{A} \otimes \sigma) = \mathcal{A}/G$$

and that the m_n's of the two A_∞-algebras are defined in the same way. ■

Quotient of minimal model. We show that if G acts on A by strict A_∞-isomorphisms, then it also acts on its minimal model $H^\bullet(A)$ by strict A_∞-isomorphisms and moreover, they give A_∞-isomorphic quotient. Recall that in the construction of minimal model (Proposition 3.2.1), one has to choose a splitting

$$Z = B \oplus H \text{ and } A = Z \oplus L = B \oplus H \oplus L. \tag{3.5.1}$$

Since the group action commutes with $m_1^{\mathcal{A}}$, the space of cocycles and coboundaries Z and B are G-subrepresentations. If we choose the splitting equivariantly, which is possible by the Maschke Theorem 3.5.1, we get the diagram

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, shift left, "p"] & H \arrow[l, shift left, "i"] \end{tikzcd}$$

where all maps are equivariant. Since the A_∞-structure on $H^\bullet(\mathcal{A})$ is given by $m_n^{H^\bullet(\mathcal{A})} = p \circ \lambda_n \circ i^{\otimes n}$, where λ_n is defined inductively by $\lambda_2 = m_2^{\mathcal{A}}$ and equation 3.2.1

$$\lambda_n = \sum_I (-1)^{\theta(r_1,\ldots,r_k)} m_k^{\mathcal{A}} \circ ((h \circ \lambda_{r_1}) \otimes \cdots \otimes (h \circ \lambda_{r_k})),$$

we see that G also acts on $H^\bullet(\mathcal{A})$ by strict A_∞-isomorphisms. Hence we can form the quotient $H^\bullet(\mathcal{A})/G$. We show that one can choose an A_∞-structure on $H^\bullet(\mathcal{A}/G)$ which is strictly A_∞-isomorphic to $H^\bullet(\mathcal{A})/G$. Recall that $m_1^{\mathcal{A}/G}$ is the restriction of

$$\mathrm{id} \otimes m_1^{\mathcal{A}} : \bigoplus_{\rho,\sigma\in \mathrm{Irr}\,(G)} \mathrm{Hom}(\rho, \sigma) \otimes \mathcal{A} \to \bigoplus_{\rho,\sigma\in \mathrm{Irr}\,(G)} \mathrm{Hom}(\rho, \sigma) \otimes \mathcal{A}$$

to its G-invariant part. Hence if we choose the same equivariant splitting (equation 3.5.1), the space of cocycles and coboundaries of $(\mathcal{A}/G, m_1^{\mathcal{A}/G})$ are

$$Z/G = \bigoplus_{\rho,\sigma\in \mathrm{Irr}\,(G)} \mathrm{Hom}_G(\rho, \sigma \otimes Z) \quad \text{and} \quad B/G = \bigoplus_{\rho,\sigma\in \mathrm{Irr}\,(G)} \mathrm{Hom}_G(\rho, \sigma \otimes B)$$

respectively. Similarly, if we denote

$$H/G = \bigoplus_{\rho,\sigma\in \mathrm{Irr}\,(G)} \mathrm{Hom}_G(\rho, \sigma\otimes H) \quad \text{and} \quad L/G = \bigoplus_{\rho,\sigma\in \mathrm{Irr}\,(G)} \mathrm{Hom}_G(\rho,\sigma\otimes L),$$

we have splittings

$$Z/G = B/G \oplus H/G \text{ and } \mathcal{A}/G = Z/G \oplus L/G = B/G \oplus H/G \oplus L/G$$

and a diagram

$$\begin{tikzcd} \mathcal{A}/G \arrow[loop left, "(\mathrm{id}\otimes h)^G"] \arrow[r, shift left, "(\mathrm{id}\otimes p)^G"] & H/G \arrow[l, shift left, "(\mathrm{id}\otimes i)^G"] \end{tikzcd}$$

satisfying

$$\mathrm{id}_{\mathcal{A}/G} - (\mathrm{id}\otimes i)^G \circ (\mathrm{id}\otimes p)^G = m_1^{\mathcal{A}/G} \circ (\mathrm{id}\otimes h)^G + (\mathrm{id}\otimes h)^G \circ m_1^{\mathcal{A}/G}.$$

We then have as in Equation 3.5.1 a family of linear maps $\tilde{\lambda_n} : (\mathcal{A}/G)^{\otimes n} \to \mathcal{A}/G$. One can then show by induction that $\tilde{\lambda_n}$ is the restriction of the linear maps

$$\circ\otimes\lambda_n : \left(\bigoplus_{\rho,\sigma\in \mathrm{Irr}\,(G)} \mathrm{Hom}(\rho,\sigma)\otimes\mathcal{A}\right)^{\otimes n} \to \bigoplus_{\rho,\sigma\in \mathrm{Irr}\,(G)} \mathrm{Hom}(\rho,\sigma)\otimes\mathcal{A}$$

to its G-invariant part. Hence $m_n^{H^\bullet(\mathcal{A}/G)}$ is the restriction of $\circ\otimes m_n^{H^\bullet(\mathcal{A})}$ i.e., $m_n^{H^\bullet(\mathcal{A}/G)} = m_n^{H^\bullet(\mathcal{A})/G}$. We have thus proved the following

Proposition 3.5.4. *Let $\mathcal{A}$ be an A_∞-category on which a finite group G acts by A_∞-isomorphisms. Then one can choose a minimal model $H^\bullet(\mathcal{A})$ of $\mathcal{A}$ on which G acts by A_∞-isomorphisms, and a minimal model $H^\bullet(\mathcal{A}/G)$ of $\mathcal{A}/G$, such that there is an A_∞-isomorphism $H^\bullet(\mathcal{A})/G \cong H^\bullet(\mathcal{A}/G)$.*

Smash product. Suppose G acts on an A_∞-algebra $\mathcal{A}$ over $S = \mathbb{K}^r$ by strict A_∞-isomorphisms. One can construct the smash product A_∞-algebra $\mathcal{A}\#G$ as follows. As a vector space, $\mathcal{A}\#G = \mathcal{A}\otimes\mathbb{K}G$. There is a $(S\otimes\mathbb{K}G)$-bimodule structure on $\mathcal{A}\otimes\mathbb{K}G$ by

$$(u\otimes g)(a\otimes x)(v\otimes h) = u(ga)v\otimes gxh,$$

for any $u, v \in S_{\mathcal{A}}$, $g, h\in G$ and $x\in\mathbb{K}G$. The multiplication maps $m_n^{\mathcal{A}\#G}$ are defined by, for $g_i\in G$,

$$m_n^{\mathcal{A}\#G}(a_1\otimes g_1,\dots,a_n\otimes g_n) = m_n^{\mathcal{A}}(a_1, g_1a_2, g_1g_2a_3,\dots,g_1\cdots g_{n-1}a_n)\otimes g_1\cdots g_n.$$

Using the isomorphism

$$\begin{aligned} (\mathcal{A}\otimes\mathbb{K}G)^{\otimes_{S\otimes\mathbb{K}G} n} &\to \mathcal{A}^{\otimes_S n}\otimes\mathbb{K}G \\ (a_1\otimes g_1)\otimes\ldots\otimes(a_n\otimes g_n) &\mapsto (a_1\otimes g_1a_2\otimes\ldots\otimes g_1\cdots g_{n-1}a_n)\otimes g_1\cdots g_n, \end{aligned}$$

we have the commutative diagram

$$\begin{tikzcd} (\mathcal{A}\otimes\mathbb{K}G)^{\otimes_{S\otimes\mathbb{K}G} n} \arrow[r, "\cong"] \arrow[d, "m_n^{\mathcal{A}\#G}"'] & \mathcal{A}^{\otimes_S n}\otimes\mathbb{K}G \arrow[d, "m_n^{\mathcal{A}}\otimes\mathrm{id}_G"] \\ \mathcal{A}\otimes\mathbb{K}G \arrow[r, equal] & \mathcal{A}\otimes\mathbb{K}G \end{tikzcd} . \tag{3.5.2}$$

Again, the smash product preserves most properties of the original A_∞-algebra: if $\mathcal{A}$ is finite dimensional/A_N/unital/augmented/connected/cyclic of degree m, then so is $\mathcal{A}\#G$.

Proposition 3.5.5. *The two dg-algebras $E(\mathcal{A}\#G)$ and $E(\mathcal{A})\#G$ are isomorphic. The two algebras $H^\bullet(E(\mathcal{A}\#G))$ and $H^\bullet(E(\mathcal{A}))\#G$ are isomorphic.*

Proof. The first claim comes from the commutative diagram (3.5.2). Taking cohomology of the first claim, we have $H^\bullet(E(\mathcal{A}\#G)) = H^\bullet(E(\mathcal{A})\#G)$. Since the functor $(-)\otimes\mathbb{K}G$ is exact, and hence preserves cohomologies, we have $H^\bullet(E(\mathcal{A})\#G) = H^\bullet(E(\mathcal{A}))\#G$ and the second claim follows. ∎

Relation between Quotient and Smash Product. The following proposition relates the smash product and the quotient construction with multiplicities.

Proposition 3.5.6. *The A_∞-algebras $\mathrm{Hom}_G(\mathbb{K}G, \mathcal{A}\otimes\mathbb{K}G)$ and $(A^{\mathrm{op}}\#G)^{\mathrm{op}}$ are strictly A_∞-isomorphic.*

Proof. Observe that $\mathrm{Hom}_G(\mathbb{K}G, \mathcal{A}\otimes\mathbb{K}G) \cong \mathcal{A}\otimes\mathbb{K}G$ as an $S_\mathcal{A}\otimes\mathbb{K}G$-bimodule via the map $\varphi\mapsto\varphi(1)$, with inverse given by sending $a\otimes g\in\mathcal{A}\otimes\mathbb{K}G$ to the map $(h\mapsto ha\otimes hg)$ for all $h\in G$. We wish to write down the induced A_∞-structure on $\mathcal{A}\otimes\mathbb{K}G$ under this isomorphism. To do this, we first write down explicitly the map $h\mapsto ha\otimes hg\in\mathrm{Hom}_G(\mathbb{K}G,\mathcal{A}\otimes\mathbb{K}G)$ as an element in $[\mathcal{A}\otimes\mathrm{Hom}(\mathbb{K}G,\mathbb{K}G)]^G$. Define the $\mathbb{K}$-linear map $\varphi_k^g:\mathbb{K}G\to\mathbb{K}G$ by

$$\varphi_k^g(h) = \begin{cases} kg & \text{if } h=k \\ 0 & \text{otherwise.}\end{cases}$$

Then the map $h\mapsto ha\otimes hg$ correspond to the element $\sum\limits_{k\in G} ka\otimes\varphi_k^g$. We will need the following simple lemma for computation.

Lemma 3.5.7. *1. The group G acts on φ_k^g by $\ell\cdot\varphi_k^g = \varphi_{\ell k}^g$.*

2. Compositions are given by

$$\varphi_k^g\circ\varphi_\ell^h = \begin{cases} \varphi_\ell^{hg} & \textit{if } k=\ell h \\ 0 & \textit{otherwise.}\end{cases}$$

Now we can compute the induced $m_n^{\mathcal{A}\otimes\mathbb{K}G}$ on $\mathcal{A}\otimes\mathbb{K}G$. Since

$$\begin{aligned}\sum_{k_1,\ldots,k_n\in G} m_n(k_na_n\otimes\varphi_{k_n}^{g_n},\ldots,k_1a_1\otimes\varphi_{k_1}^{g_1}) &= \sum_{k_1,\ldots,k_n\in G} m_n(k_na_n,\ldots,k_1a_1)\otimes\varphi_{k_n}^{g_n}\circ\cdots\circ\varphi_{k_1}^{g_1} \\ &= \sum_{k_1\in G} m_n^{\mathcal{A}}(k_1g_1\cdots g_{n-1}a_n,\ldots,k_1a_1)\otimes\varphi_{k_1}^{g_1\cdots g_n} \\ &= \sum_{k_1\in G} k_1m_n^{\mathcal{A}}(g_1\cdots g_{n-1}a_n,\ldots,a_1)\otimes\varphi_{k_1}^{g_1\cdots g_n},\end{aligned}$$

we conclude that

$$m_n^{\mathcal{A}\otimes\mathbb{K}G}(a_n\otimes g_n,\ldots,a_1\otimes g_1) = m_n^{\mathcal{A}}(g_1\cdots g_{n-1}a_n,\ldots,a_1)\otimes(g_1\cdots g_n).$$

Now observe that $(\mathcal{A}^{\mathrm{op}}\#G)^{\mathrm{op}}$ has the same objects and morphism spaces as $\mathcal{A}\otimes\mathbb{K}G$, with A_∞-structure given by

$$\begin{aligned} m_n^{(\mathcal{A}^{\mathrm{op}}\#G)^{\mathrm{op}}}(a_n\otimes g_n,\ldots,a_1\otimes g_1) &= m_n^{\mathcal{A}^{\mathrm{op}}\#G}(a_1\otimes g_1,\ldots,a_n\otimes g_n) \\ &= m_n^{\mathcal{A}^{\mathrm{op}}}(a_1,g_1a_2,\ldots,g_1\cdots g_{n-1}a_n)\otimes(g_1\cdots g_n) \\ &= m_n^{\mathcal{A}}(g_1\cdots g_{n-1}a_n,\ldots,a_1)\otimes(g_1\cdots g_n)\end{aligned}$$

which is the same as that of $\mathcal{A}\otimes\mathbb{K}G$, as desired. ∎

Corollary 3.5.8. *The A_∞-algebras $\mathcal{A}/G$ and $\mathcal{F}((\mathcal{A}^{\mathrm{op}}\#G)^{\mathrm{op}})$ are strictly A_∞-isomorphic.*

Proof. This is immediate since $\mathcal{A}/G\cong\mathcal{F}(\mathrm{Hom}_G(\mathbb{K}G,\mathcal{A}\otimes\mathbb{K}G))$ by Proposition 3.5.3. ∎

Proposition 3.5.9. *The two dg-algebras $E(\mathcal{A}/G)$ and $E(\mathcal{A})/G$ are isomorphic. The two A_∞-algebras $H^\bullet(E(\mathcal{A}/G))$ and $H^\bullet(E(\mathcal{A}))/G$ are A_∞-quasi-isomorphic.*

Proof. By Corollary 3.5.8, $E(\mathcal{A}/G)\cong E(\mathcal{F}((\mathcal{A}^{\mathrm{op}}\#G)^{\mathrm{op}}))$. Since the Morita equivalence functor $\mathcal{F}$ in Theorem 3.5.2 commutes with tensor product, it also commutes with the Koszul functor E, and we have $E(\mathcal{F}((\mathcal{A}^{\mathrm{op}}\#G)^{\mathrm{op}}))\cong\mathcal{F}(E(\mathcal{A}^{\mathrm{op}}\#G)^{\mathrm{op}})$. By Proposition 3.5.5, $E(\mathcal{A}^{\mathrm{op}}\#G)^{\mathrm{op}}\cong(E(A)^{\mathrm{op}}\#G)^{\mathrm{op}}$. By Corollary 3.5.8 again, $\mathcal{F}((E(A)^{\mathrm{op}}\#G)^{\mathrm{op}})\cong E(\mathcal{A})/G$ and we arrive at the first claim. Taking cohomology, we conclude $H^\bullet(E(\mathcal{A}/G))$ and $H^\bullet(E(\mathcal{A})/G)$ are A_∞-quasi-isomorphic. By Proposition 3.5.4, $H^\bullet(E(\mathcal{A})/G)$ and $H^\bullet(E(\mathcal{A}))/G$ are A_∞-quasi-isomorphic and we are done. ∎

3.6 Tensor Product

In this section, we give a review on how to construct tensor product of A_∞-algebras. Except Proposition 3.6.1, this is essentially a summary of Amorim and Tu [2].

Tensor product of a dg-algebra with an A_∞-algebra Before looking at the general case, let us look at the special case where one of them is only a dg-algebra rather than a full-fledged A_∞-algebra. When both of them are dg-algebras, this is well-known and we take

$$m_1^{A\otimes A'} = m_1^A \otimes \mathrm{id}_{A'} + \mathrm{id}_A \otimes m_1^{A'} \quad \text{and} \quad m_2^{A\otimes A'} = m_2^A \otimes m_2^{A'} \tag{3.6.1}$$

If A is a dg-algebra and A' is an A_∞-algebra, there is a natural generalization:

$$m_1^{A\otimes A'} = m_1^A \otimes \mathrm{id}'_A + \mathrm{id}_A \otimes m_1^{A'} \quad \text{and for } n \geq 2 \quad m_n^{A\otimes A'} = \underbrace{m_2^A \circ (m_2^A \circ (m_2^A \circ (\cdots)))}_{n-1 \text{ iterations}} \otimes m_n^{A'}. \tag{3.6.2}$$

For simplicity, we will denote the n-th iteration of m_2^A by $(m_2^A)^{\circ n}$. Direct computation using chain rule on A and the A_∞-relations on A' shows the above definition of $m_n^{A\otimes A'}$ satisfies the A_∞-relations:

$$\begin{aligned}
&\sum_{a+b+c=n} (-1)^{ab+c} m_{a+1+c}^{A\otimes A'} \circ \left(\mathrm{id}_{A\otimes A'}^{\otimes a} \otimes m_b^{A\otimes A'} \otimes \mathrm{id}_{A\otimes A'}^{\otimes c}\right) \\
&= m_1^{A\otimes A'} \circ m_n^{A\otimes A'} + \sum_{a+c=n-1} (-1)^{n-1} m_n^{A\otimes A'} \circ \left(\mathrm{id}_{A\otimes A'}^{\otimes a} \otimes m_1^{A\otimes A'} \otimes \mathrm{id}_{A\otimes A'}^{\otimes c}\right) \\
&\quad + \sum_{\substack{a+b+c=n \\ 2\leq b\leq n-1}} (-1)^{ab+c} m_{a+1+c}^{A\otimes A'} \circ \left(\mathrm{id}_{A\otimes A'}^{\otimes a} \otimes m_b^{A\otimes A'} \otimes \mathrm{id}_{A\otimes A'}^{\otimes c}\right) \\
&= \sum_{a+c=n-1} \left(m_1^A \circ (m_2^A)^{\circ n-1} - (m_2^A)^{\circ n-1} \circ (\mathrm{id}_A^{\otimes a} \otimes m_1^A \otimes \mathrm{id}_A^{\otimes c})\right) \otimes m_n^{A'} \\
&\quad + \sum_{a+c=n-1} (-1)^{n-1} (m_2^A)^{\circ n-1} \otimes \left(m_n^{A'} \circ (\mathrm{id}_{A'}^{\otimes a} \otimes m_1^{A'} \otimes \mathrm{id}_{A'}^{\otimes c})\right) \\
&\quad + \sum_{\substack{a+b+c=n \\ 2\leq b\leq n-1}} (-1)^{ab+c} (m_2^A)^{\circ n-1} \otimes \left(m_n^{A'} \circ (\mathrm{id}_{A'}^{\otimes a} \otimes m_1^{A'} \otimes \mathrm{id}_{A'}^{\otimes c})\right) \\
&= 0.
\end{aligned}$$

Tensor product of A_∞-algebras. For general A_∞-algebras, there is no natural way to define tensor product. There are various ways to construct an A_∞-structure on $A \otimes A'$ which are A_∞-quasi-isomorphic to each other, but in general not strictly A_∞-quasi-isomorphic. In the following, we describe some of the constructions. Saneblidze and Umble [61] were the first to construct an A_∞-structure on $A \otimes A'$ whose $m_n^{A\otimes A'}$ are given by a closed formula in terms of m_j^A and $m_j^{A'}$ where $j \leq n$. Markl and Shnider [55] later reformulated their construction as a diagonal map on the A_∞-operad $\mathcal{A}_\infty$: Given a chain complex (A, d), there is an associated operad End_A. Any A_∞-structure m_n^A on A with $m_1^A = d$ can then be described as an operad homomorphism $\rho : \mathcal{A}_\infty \to \mathrm{End}_A$. Now, given any two chain complexes A and A', there is a natural map of operads $\mathrm{End}_A \otimes \mathrm{End}_{A'} \to \mathrm{End}_{A\otimes A'}$. The problem of constructing an A_∞-structure on $A \otimes A'$ then becomes the problem of constructing a "canonical" diagonal $\Delta : \mathcal{A}_\infty \to \mathcal{A}_\infty \otimes \mathcal{A}_\infty$. For if such a diagonal exists, one can simply take the composition

$$\rho_{A\otimes A'} : \mathcal{A}_\infty \xrightarrow{\Delta} \mathcal{A}_\infty \otimes \mathcal{A}_\infty \xrightarrow{\rho_A \otimes \rho_{A'}} \mathrm{End}_A \otimes \mathrm{End}_{A'} \longrightarrow \mathrm{End}_{A\otimes A'}$$

to give an A_∞-structure on $A \otimes A'$. Amorim [1] also reformulated the Saneblidze-Umble construction in terms of dg-algebras: For every A_∞-algebra A, one can construct a dg-algebra $\mathrm{Hom}(A, A)$ whose space of cycles are the A_∞-endomorphisms of A and whose homology are A_∞-endomorphisms of A up to A_∞-homotopy. This dg-algebra $\mathrm{Hom}(A, A)$ is A_∞-quasi-isomorphic to A. One then forms the tensor product dg-algebra $\mathrm{Hom}(A, A) \otimes \mathrm{Hom}(A', A')$ and uses homology perturbation to transfer the dg-structure on $\mathrm{Hom}(A, A) \otimes \mathrm{Hom}(A', A')$ to an A_∞-structure on $A \otimes A'$. All A_∞-structure on tensor products defined above are A_∞-quasi-isomorphic.

Tensor product for Adams-graded A_∞-algebras. In the case of Adams-graded A_∞-algebras, the tensor product A_∞-structure also preserves the tensor product Adams-grading on $A \otimes A'$ as $m_n^{A\otimes A'}$ are given by a closed formula in terms of m_j^A and $m_j^{A'}$ where $j \leq n$.

Cyclic structure. Given two cyclic A_∞-algebras A and A', there is a natural inner product on $A \otimes A'$ defined by

$$\langle a_1 \otimes \alpha_1, a_2 \otimes \alpha_2 \rangle_{A\otimes A'} = (-1)^{|\alpha_1||a_2|} \langle a_1, a_2 \rangle_A \langle \alpha_1, \alpha_2 \rangle_{A'}. \tag{3.6.3}$$

One might ask whether the tensor product constructions described above are cyclic with respect to this natural inner product. In the special case where one of the A_∞-algebra is a dg-algebra, it can be checked directly that the natural tensor product structure given by Equation 3.6.2 is cyclic. However, for full-fledged A_∞-algebras, all the tensor product constructions described above do not preserve cyclicity in general. As Tradler pointed out in [67], $m_3^{A\otimes A'}$ in the Saneblidze-Umble construction is already not cyclic. The first construction of tensor product of cyclic A_∞-algebras which respect the natural inner product as in Equation 3.6.3 seems to be given by Amorim and Tu in [2]. There, they constructed a cyclic diagonal on the A_∞-operad and showed that any tensor product A_∞-structure defined by a cyclic diagonal are cyclically A_∞-quasi-isomorphic to each other. As the A_∞-structure is constructed by a diagonal, the $m_n^{A\otimes A'}$ can in principle be written as a sum of tensor products and compositions of m_j^A and $m_j^{A'}$ for $j \leq n$. However, Amorim and Tu only gave explicit formulae for $m_n^{A\otimes A'}$ up to $n = 4$ and stated that the general formulae appeared to be a very complicated combinatorial problem.

Convention. As all A_∞-structures on tensor products we described above are A_∞-quasi-isomorphic, in principle it makes no difference to which one uses. Hence we will not distinguish them and only write $A \otimes A'$ to denote A_∞-tensor product. However, the two constructions given by Amorim and Tu [2] and Amorim [1] would be the most important in this thesis as the first one preserves cyclic structures and the second one reduces tensor product of A_∞-algebras to tensor product on dg-algebras.

Koszul functor and tensor product. The following proposition essentially says that under some locally finite conditions, the Koszul functor commutes with the A_∞-tensor product. Recall the bar construction functor B sending A_∞-algebras to dg-coalgebras defined in Section 3.4.

Proposition 3.6.1.

1. *Let A and A' be two A_∞-algebras. Then there is a quasi-isomophism of dg-algebras*
$$B(A \otimes A')^\# \to B(A)^\# \otimes B(A')^\#$$
and an A_∞-quasi-isomorphism of A_∞-algebras
$$H^\bullet(B(A \otimes A')^\#) \cong H^\bullet((BA)^\#) \otimes H^\bullet((BA')^\#).$$

2. *Let A and A' be locally finite Adams graded A_{fin}-algebras. Suppose one of the A_∞-tensor product structure on $A \otimes A'$ is A_{fin} and that $E(A)$, $E(A')$ and $E(A \otimes A')$ are all locally finite. Then there is a quasi-isomorphism of dg-algebras*
$$E(A \otimes A') \cong E(A) \otimes E(A')$$
and an A_∞-quasi-isomorphism of A_∞-algebras
$$H^\bullet(E(A \otimes A')) \cong H^\bullet(E(A)) \otimes H^\bullet(E(A')).$$

Proof. Using Amorim's version of A_∞-tensor product which reduces A_∞-tensor product to dg tensor product in [1], it suffices to prove the proposition when both A and A' are dg-algebras. Both

$$B(A) \otimes A \otimes B(A') \otimes A' \to S_A \otimes S_{A'} \text{ and } B(A \otimes A') \otimes A \otimes A' \to S_A \otimes S_{A'}$$

are semi-free resolution of $S_A \otimes S_{A'}$. Hence there is a chain homotopy equivalence between

$$B(A \otimes A') \otimes A \otimes A' \underset{g}{\overset{f}{\rightleftarrows}} B(A) \otimes A \otimes B(A') \otimes A' .$$

Now,

$$B(A \otimes A') = B(A \otimes A') \otimes (A \otimes A')_{A \otimes A'}(S_A \otimes S'_A)$$

and similarly

$$B(A) \otimes B(A') = (B(A) \otimes A \otimes B(A') \otimes A')_{A \otimes A'}(S_A \otimes S'_A).$$

Hence we have a chain homotopy equivalence

$$B(A \otimes A') \underset{g \otimes \mathrm{id}_{A \otimes A'}}{\overset{f \otimes \mathrm{id}_{A \otimes A'}}{\rightleftarrows}} B(A) \otimes B(A') \; .$$

By MacLane [53, Chapter X, Theorem 12.2], the map $g \otimes \mathrm{id}_{A \otimes A'}$ is given by inclusion maps followed by the shuffle product on $B(A \otimes A')$

$$B(A) \otimes B(A') \to B(A \otimes A') \otimes B(A \otimes A') \overset{sh}{\to} B(A \otimes A')$$

$$a \otimes a' \mapsto (a \otimes 1) \otimes (1 \otimes a') \mapsto (a \otimes 1) * (1 \otimes a')$$

which is a coalgebra map. Taking dual, we obtain a dg-algebra morphism

$$\phi : B(A \otimes A')^{\#} \to B(A)^{\#} \otimes B(A')^{\#}$$

which is a quasi-isomorphism. By homology perturbation, we have the following chain of A_∞-quasi-isomorphisms

$$H^\bullet(B(A \otimes A')^{\#}) \cong B(A \otimes A')^{\#} \cong BA^{\#} \otimes BA'^{\#} \cong H^\bullet(BA^{\#}) \otimes H^\bullet(BA'^{\#}).$$

The second statement follows from the observation that under the assumptions, we have $E(A) = (BA)^{\#}$, $E(A') = (BA)^{\#}$ and $E(A \otimes A') = (B(A \otimes A'))^{\#}$ as dg-algebras by Proposition 3.4.2. ■

Chapter 4

Quivers with Superpotential

This chapter is devoted to the study of quivers with superpotential.

In section 4.1, we define quivers with superpotentials. Our definition of quivers with superpotential is taken from van den Bergh [70], where the completed path algebra of a quiver with superpotential is known as a deformed DG-preprojective algebra there.

Section 4.2 gives a correspondence between quivers with superpotential and the Koszul dual of A_{fin}-categories with cyclic structures. Using this correspondence, we define the notion of product of quivers with superpotential and the notion of quotient of quivers with superpotential by finite groups.

In section 4.3, we follow van den Bergh [71] and prove that the path algebras of quivers with superpotential are Calabi–Yau algebras, and hence the categories of representations of quivers with superpotential are also Calabi–Yau.

Finally, Section 4.4 describes quivers with superpotential of dimensions 1 to 4. In particular, we describe in dimension 3 how our definition of quivers with superpotential as dg-quivers is connected to the old definition of quivers with superpotential as quivers with relations given by physicists Berenstein and Douglas [5], Braun [12], Douglas and Moore [23] and later by mathematicians Ginzburg [27] and Derksen, Weyman and Zelevinsky [22].

4.1 Quivers with Superpotential

In this section, we introduce the notion of quiver with superpotential. The presentation here essentially follows van den Bergh [70, 71].

Notations and Conventions. We will fix a field $\mathbb{K}$ of characteristic zero. By a cycle in a quiver Q, we will mean a closed path forgetting the starting and ending points, or more precisely, a closed path as an element in the vector space $\mathbb{K}Q/[\mathbb{K}Q, \mathbb{K}Q]$. For simplicity, we will write $\mathbb{K}Q_{\text{cyc}} = \mathbb{K}Q/[\mathbb{K}Q, \mathbb{K}Q]$. For any edge e in a graded quiver Q, we will denote by $\partial_e : \mathbb{K}Q \to \mathbb{K}Q \otimes \mathbb{K}Q$ the double derivation of degree $|e|$ acting on any edge f in Q by

$$\partial_e f = \begin{cases} h(e) \otimes t(e) & \text{if } e = f \\ 0 & \text{otherwise.} \end{cases}$$

We will also denote by $\partial_e^\circ : \mathbb{K}Q \to \mathbb{K}Q$ the derivation $m \circ \sigma \circ \partial_e$, where m is the multiplication map in $\mathbb{K}Q$ and $\sigma : \mathbb{K}Q \otimes \mathbb{K}Q$ is the interchange operator sending $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$. This derivation vanishes on commutators and hence descends to a derivation $\partial_e^\circ : \mathbb{K}Q_{\text{cyc}} \to \mathbb{K}Q$. For other notions in noncommutative calculus such as noncommutative symplectic form and double Poisson bracket, please refer to Chapter 1.

Construction of Quivers with Superpotential. Let $\tilde{Q}$ be a graded quiver with degree of all edges lying in the interval $[2-m, 0]$. Suppose $\tilde{Q}$ is equipped with a pairing $(-,-)$ of degree $2-m$ on the set of edges in $\tilde{Q}$ such that

- $(a,b) = -(-1)^{|a||b|}(b,a)$
- $(a,b) = 0$ unless $t(a) = h(b)$ and $t(b) = h(a)$
- the matrix $\{(a,b)\}$ is invertible.

Let $\langle -,- \rangle$ denote the dual quadratic form, i.e., the pairing on the dual space of the set of edges in $\tilde{Q}$ such that the matrix $\{\langle a^\vee, b^\vee \rangle\}$ is given by the inverse of $\{(a,b)\}$. Then there is a noncommutative symplectic 2-form of degree $2-m$

$$\omega = \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle d_{dR}x d_{dR}y.$$

Proposition 4.1.1. *The 2-form ω is a noncommutative symplectic form.*

Proof. We show the map $\iota_\omega : \mathbb{D}\mathrm{er}_S(\mathbb{K}\tilde{Q}_A) \to \Omega^1_S(\mathbb{K}\tilde{Q}_A)$ defined by $\Theta \mapsto \iota_\Theta \omega$ is an isomorphism by exhibiting an inverse. For any edge $a \in \tilde{Q}_A$,

$$\begin{aligned}\iota_\omega(\partial_a) = \iota_{\partial_a}\omega &= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}_A} \langle x^\vee, y^\vee \rangle \left((\partial_a^\circ x) d_{dR} y + (-1)^{1+|a||x|} d_{dR} x (\partial_a^\circ y) \right) \\ &= \sum_{y \text{ edges in } \tilde{Q}_A} \langle a^\vee, y^\vee \rangle d_{dR} y.\end{aligned}$$

By direct calculation, one can see that the inverse of i_ω is given by

$$d_{dR} a \mapsto \sum_{b \text{ edges in } \tilde{Q}_A} (a,b)\partial_b.$$

■

The symplectic form ω induces a Poisson double bracket $\{\!\{-,-\}\!\} : \mathbb{K}\tilde{Q} \otimes \mathbb{K}\tilde{Q} \to \mathbb{K}\tilde{Q} \otimes \mathbb{K}\tilde{Q}$.

Proposition 4.1.2. *For any edge $a \in \tilde{Q}$, we have the formula*

$$\{\!\{a,-\}\!\} = \sum_{b \text{ edges in } \tilde{Q}} (a,b)\partial_b.$$

Proof. By definition of the double bracket, this follows from

$$\begin{aligned}&\sum_{b \text{ edges in } \tilde{Q}} (a,b)\iota_{\partial_b}\omega \\ &= \sum_{b \text{ edges in } \tilde{Q}} \sum_{x,y \text{ edges in } \tilde{Q}_A} \frac{1}{2}(a,b)\langle x^\vee, y^\vee \rangle \left((\partial_b^\circ x) d_{dR} y + (-1)^{1+|b||x|} d_{dR} x (\partial_b^\circ y) \right) \\ &= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}_A} (a,x)\langle x^\vee, y^\vee \rangle d_{dR} y + (a,y)\langle y^\vee, x^\vee \rangle d_{dR} x \\ &= \sum_{x,y \text{ edges in } \tilde{Q}} (a,x)\langle x^\vee, y^\vee \rangle d_{dR} y \\ &= d_{dR} a.\end{aligned}$$

■

The double Poisson bracket induces the Kontsevich bracket (c.f. Section 1) $\{-,-\} : \mathbb{K}\tilde{Q} \otimes \mathbb{K}\tilde{Q}_{\text{cyc}} \to \mathbb{K}\tilde{Q}$, which descends to a Lie bracket $\{-,-\} : \mathbb{K}\tilde{Q}_{\text{cyc}} \otimes \mathbb{K}\tilde{Q}_{\text{cyc}} \to \mathbb{K}\tilde{Q}_{\text{cyc}}$. Let $\Phi \in \mathbb{K}\tilde{Q}_{\text{cyc}}$ be a sum of cycles of degree $3-m$ which satisfies the master equation $\{\Phi, \Phi\} = 0$ in $\mathbb{K}\tilde{Q}_{\text{cyc}}$. Then one can define $d : \mathbb{K}\tilde{Q} \to \mathbb{K}\tilde{Q}$ by $d = \{\Phi, -\}$.

Proposition 4.1.3. *The linear map d is a differential of degree 1. For any edge a in $\tilde{Q}$, we have*

$$da = (-1)^{1+a+m} \sum_{b \text{ edges in } \tilde{Q}} (a,b)\partial_b^\circ \Phi.$$

Proof. Since Φ is of degree $3-m$ and $\{-,-\}$ is of degree $m-2$, d is a map of degree 1. Now, by Jacobi identity,

$$d^2(a) = \{\Phi, \{\Phi, a\}\} = \frac{1}{2}\{\{\Phi, \Phi\}, a\} = 0.$$

Hence d is a differential. For any edge a, we have

$$\{\Phi, a\} = m \circ \{\!\{\Phi, a\}\!\} = -(-1)^{(|a|+2-m)(|\Phi|+2-m)} m \circ \sigma \{\!\{a, \Phi\}\!\} = (-1)^{1+a+m} \sum_{b \text{ edges in } \tilde{Q}_A} (a,b)\partial_b^\circ \Phi$$

where the last equality is by Proposition 4.1.2. ∎

This differential d on $\mathbb{K}\tilde{Q}$ is compatible with the double Poisson bracket.

Proposition 4.1.4. *The double Poisson bracket $\{\!\{-,-\}\!\}$ is a dg-double Poisson bracket of degree $m-2$, i.e.,*

$$d\{\!\{a,b\}\!\} = \{\!\{da, b\}\!\} + (-1)^{|a|+m-2}\{\!\{a, db\}\!\}.$$

Proof. The proposition follows readily from Proposition 1.2.19 and the definition $d = \{\Phi, -\}$. ∎

Let Q be the quiver constructed from $\tilde{Q}$ by adding to each vertex v in Q a loop v^* of degree $1-m$. Define a differential on $\mathbb{K}Q$ by

$$d(v^*) = v\ell v \quad \text{for any vertex } v, \text{ where } \ell = \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle xy,$$
$$d(a) = \{a, \Phi\} \quad \text{for any edge } a \text{ with degree } a \leq 2-m.$$

The following lemma shows that we have indeed defined a differential on $\mathbb{K}Q$.

Lemma 4.1.5. $d^2(v^*) = 0$.

Proof. It suffices to show $d\ell = 0$, for then $d^2(v^*) = d(v\ell v) = v(d\ell)v = 0$. Using Proposition 4.1.3,

$$\begin{aligned}
d\ell &= \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle \Big((dx)y + (-1)^{|x|} x dy \Big) \\
&= \sum_{b,x,y \text{ edges in } \tilde{Q}} (-1)^{1+|x||y|+1+m+|x|} \langle y^\vee, x^\vee \rangle (x,b)(\partial_b^\circ \Phi) y + (-1)^{|x|+1+m+|y|} \langle x^\vee, y^\vee \rangle (y,b) x (\partial_b^\circ \Phi) \\
&= \sum_{y \text{ edges in } \tilde{Q}} (-1)^{(3-m-|y|)|y|} (\partial_y^\circ \Phi) y - \sum_{x \text{ edges in } \tilde{Q}} x(\partial_x^\circ \Phi) \\
&= (\text{sum of all cyclic permutations of } \Phi) - (\text{sum of all cyclic permutations of } \Phi) \\
&= 0.
\end{aligned}$$

∎

The dg-quiver Q built in this way is called a quiver with superpotential Φ of dimension m.

Definition 4.1.6. A *quiver with superpotential of dimension* m is a dg-quiver Q together with an element $\Phi \in \mathbb{K}Q_{\text{cyc}}$ of degree $3-m$ in the following form:

1. The degrees of all the edges of Q lie in the interval $[1-m, 0]$, i.e.,
$$E_Q^i = \varnothing \quad \text{unless} \quad 1-m \leq i \leq 0.$$
2. For each vertex v there is exactly 1 loop v^* of degree $1-m$ at the vertex v, and there are no other edges of degree $1-m$, i.e.,
$$E_Q^{1-m} = \coprod_{v \in V_Q} E_Q^{1-m}(v,v) \quad \text{with} \quad E_Q^{1-m}(v,v) = \{v^*\}.$$
3. For any $i \in [2-m, 0]$, there is a pairing
$$(-,-) : E_Q^i \times E_Q^{2-m-i} \to \mathbb{K}$$
between edges of degree i and edges of degree $2-m-i$ such that
   - $(a,b) = -(-1)^{|a||b|}(b,a)$
   - $(a,b) = 0$ unless $t(a) = h(b)$ and $t(b) = h(a)$
   - the matrix $\{(a,b)\}$ is invertible.
4. For any edge a with $\deg a \in [2-m, 0]$, the differential is given by
$$da = \{\Phi, a\}.$$
5. For any degree $1-m$ loop v^*,
$$dv^* = v \left(\sum_{\substack{x,y \text{ edges in } Q \\ \deg x, \deg y \in [2-m,0]}} \langle x^\vee, y^\vee \rangle xy \right) v,$$
where $\langle -,- \rangle$ is the dual pairing of $(-,-)$.

Remark 4.1.7. Note that the above definition automatically implies that the superpotential Φ satisfies the master equation $\{\Phi, \Phi\} = 0$ in $\mathbb{K}Q_{\text{cyc}}$ as $d^2(a) = \{\Phi, \{\Phi, a\}\} = \frac{1}{2}\{\{\Phi, \Phi\}, a\} = 0$ implies $\partial_a^\circ\{\Phi, \Phi\} = 0$ for all edges a in Q. Hence the data of the subquiver $\tilde{Q}$ together with the antisymmetric pairing $(-,-)$ and the superpotential Φ satisfying the master equation $\{\Phi, \Phi\} = 0$ in $\mathbb{K}Q_{\text{cyc}}$ uniquely determine the quiver with superpotential Q by the above construction.

4.2 Characterisation of Quivers with Superpotential

In this section, we would like to characterize quivers with superpotential in terms of A_{fin}-algebras. Recall from Section 3 that for any dg-quiver Q, its path algebra $\mathbb{K}Q$ is isomorphic to $E(A)$ for a unique (up to strict A_∞-isomorphism) A_{fin}-algebra A. We would like to show that for a quiver with superpotential Q, the corresponding A_{fin}-algebra is equipped with a cyclic structure. Conversely, given any A_{fin}-algebra with cyclic structure, its Koszul dual $E(A)$ is the path algebra of a quiver with superpotential.

All A_{fin}-algebras in this section will be assumed to be augmented, finite (it has finite number of objects and finite dimensional morphism space) and positively graded ($\bar{\mathcal{A}}^i = 0$ for $i \leq 0$, where $\bar{\mathcal{A}}$ is the kernel of the augmentation map).

Let Q be a quiver with superpotential. Recall the A_{fin}-category $\mathcal{A}$ such that $\mathbb{K}Q = E(\mathcal{A})$ is constructed by
$$\text{Obj}\,(\mathcal{A}) = \{\text{vertices in } Q\}.$$

$$\mathcal{A}^i(v,w) = \begin{cases} \mathbb{K}v & \text{if } i=0 \text{ and } v=w \\ \mathbb{K}\{\text{degree } (1-i) \text{ edges in } Q \text{ from } v \text{ to } w\}^\vee & \text{if } i \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

with the A_∞-structure given by

$$m_2 : \mathcal{A}^0(v,v) \otimes \mathcal{A}^i(v,w) \to \mathcal{A}^i(v,w) \quad v \otimes a \mapsto a$$

$$m_2 : \mathcal{A}^0(v,w) \otimes \mathcal{A}^i(w,w) \to \mathcal{A}^i(v,w) \quad a \otimes w \mapsto a$$

$$m_n : \mathcal{A}^{i_n}(v_{n-1},v_n) \otimes \cdots \otimes \mathcal{A}^{i_1}(v_0,v_1) \to \mathcal{A}^{i_1+\cdots+i_n+2-n}(v_0,v_n) \text{ for } i_j \geq 1$$

$$a_n^\vee \otimes \cdots \otimes a_1^\vee \mapsto (-1)^{(n-1)|a_n^\vee|+\cdots+2|a_3^\vee|+|a_2^\vee|} \sum_{a \text{ edges in } Q} (\text{Coefficient of } a_n \cdots a_1 \text{ in } da) a^\vee$$

and zero otherwise. For any edge $a \in Q$, denote by $|a^\vee|$ the degree of $a^\vee$ in the A_∞-category, i.e., $|a^\vee| = 1 - |a|$. Define a pairing $\langle -,- \rangle_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathbb{K}$ of degree m by

$$\langle a^\vee, b^\vee \rangle_{\mathcal{A}} = (-1)^{|a^\vee|} \langle a^\vee, b^\vee \rangle.$$

Then for $a_0, \ldots, a_n$ edges in Q,

$$\begin{aligned}
&\langle m_n(a_n^\vee, \ldots, a_1^\vee), a_0^\vee \rangle_{\mathcal{A}} \\
&= (-1)^{(n-1)|a_n^\vee|+\cdots+2|a_3^\vee|+|a_2^\vee|} \sum_{a \text{ edges in } Q} (\text{Coefficient of } a_n \cdots a_1 \text{ in } da) \langle a^\vee, a_0^\vee \rangle_{\mathcal{A}} \\
&= \sum_{a,b \text{ edges in } Q} (-1)^{(n-1)|a_n^\vee|+\cdots+2|a_3^\vee|+|a_2^\vee|+|a^\vee|+1+|a|+m} (a,b)(\text{Coefficient of } a_n \cdots a_1 \text{ in } \partial_b^\circ \Phi) \langle a^\vee, a_0^\vee \rangle \\
&= (-1)^{(n-1)|a_n^\vee|+\cdots+2|a_3^\vee|+|a_2^\vee|+(m+1)(|a_0|+1)} (\text{Coefficient of } a_n \cdots a_1 \text{ in } \partial_{a_0}^\circ \Phi) \\
&= (-1)^{(n-1)|a_n^\vee|+\cdots+2|a_3^\vee|+|a_2^\vee|+(m+1)(|a_0|+1)} (\text{Coefficient of } a_0 a_n \cdots a_1 \text{ in } \tilde{\Phi}) \\
&= (-1)^{(n-1)|a_n^\vee|+\cdots+2|a_3^\vee|+|a_2^\vee|+1+|a_0|+m} (\text{Coefficient of } a_n \cdots a_1 a_0 \text{ in } \tilde{\Phi}),
\end{aligned}$$

where $\tilde{\Phi}$ = sum of all cyclic permutations of Φ. Similarly,

$$\langle m_n(a_{n-1}^\vee, \ldots, a_0^\vee), a_n^\vee \rangle_{\mathcal{A}} = (-1)^{(n-1)|a_{n-1}|+\cdots+2|a_2|+|a_1|+(m+1)(|a_n|+1)} (\text{Coefficient of } a_n \cdots a_1 a_0 \text{ in } \tilde{\Phi}).$$

Since Φ has degree $3-m$, we have $\langle m_n(a_n^\vee, \ldots, a_1^\vee), a_0^\vee \rangle_{\mathcal{A}} = \langle m_n(a_{n-1}^\vee, \ldots, a_0^\vee), a_n^\vee \rangle_{\mathcal{A}} = 0$ unless $|a_n| + \cdots + |a_0| = 3-m$. If $|a_n| + \cdots + |a_0| = 3-m$, using $|a^\vee| = 1-|a|$, we can verify that

$$\langle m_n(a_n^\vee, \ldots, a_1^\vee), a_0^\vee \rangle_{\mathcal{A}} = (-1)^{n+|a_n^\vee|(|a_{n-1}|+\cdots+|a_0^\vee|)} \langle m_n(a_{n-1}^\vee, \ldots, a_0^\vee), a_n^\vee \rangle_{\mathcal{A}}.$$

Hence $\langle -,- \rangle_{\mathcal{A}}$ is a cyclic structure. Recall that an A_∞-category is said to be *finite* if $\mathrm{Obj}\,(\mathcal{A})$ is finite and $\mathcal{A}(u,v)$ is finite dimensional for all $u,v \in \mathrm{Obj}\,(\mathcal{A})$. Also, an augmented A_∞-category $\mathcal{A}$ is said to be *positively graded* if $\bar{\mathcal{A}}^i = 0$ for all $i \leq 0$. Summarizing, we have

Proposition 4.2.1. *Let Q be a dg-quiver. Then there exists a finite and positively graded A_{fin}-category $\mathcal{A}$ such that $\mathbb{K}Q = E(\mathcal{A})$. If Q is a quiver with superpotential of dimension m, then there is a cyclic structure of degree m on $\mathcal{A}$.*

Next, we would like to prove the converse of Proposition 4.2.1. Let us start with a finite and positively graded A_{fin}-category $\mathcal{A}$ with a cyclic structure of degree m. Recall from Section 3 that one can construct from $\mathcal{A}$ a quiver Q as follows: take the vertex set of Q to be $\mathrm{Obj}\,(\mathcal{A})$ and for any vertices u,v, the set of degree i edges from u to v is given by a fixed basis of $\bar{\mathcal{A}}^{1-i}(u,v)^\vee$. We will denote elements in the chosen basis of $\bar{\mathcal{A}}$ by $a^\vee$, and its dual basis (as elements in Q) by a. The degrees of a and $a^\vee$ are related by $|a| = 1 - |a^\vee|$. By the cyclic structure of degree m on $\mathcal{A}$, we have $\mathcal{A}^i \cong \mathcal{A}^{m-i}$. Since A is positively graded, we conclude that the degree of all edges in Q lies in the interval $[1-m, 0]$, for each vertex v there is exactly one loop v^* of degree $1-m$, and that these are precisely all the degree $1-m$ edges in

Q. The cyclic structure $\langle -, - \rangle_{\mathcal{A}}$ on $\mathcal{A}$ determines a super-antisymmetric pairing $(-,-)$ of degree $2-m$ on the set of edges of Q by defining its dual pairing $\langle -,- \rangle$ to be

$$\langle a^\vee, b^\vee \rangle = (-1)^{|a^\vee|} \langle a^\vee, b^\vee \rangle_{\mathcal{A}}.$$

Let $\tilde{Q}$ be the subquiver of Q which contains all vertices of Q and all edges in Q with degree in the interval $[2-m, 0]$. Proposition 4.1.1 then shows

$$\omega = \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle d_{dR} x d_{dR} y$$

defines a noncommutative symplectic form on $\tilde{Q}$, and the corresponding double Poisson bracket $\{\!\{-,-\}\!\}$ is given by Proposition 4.1.2. It remains to show that the differential on Q is in the form of a quiver with superpotential. Denote by b_n the shift of the A_∞-structure m_n on $\mathcal{A}$. By the cyclic invariance of $\langle -,- \rangle_{\mathcal{A}}$, we conclude that the quadratic form $\langle -,- \rangle$ is invariant under cyclic symmetry:

$$\langle b_n(a_1^\vee, \ldots, a_n^\vee), a_{n+1}^\vee \rangle = \langle b_n(a_2^\vee, \ldots, a_{n+1}^\vee), a_1^\vee \rangle.$$

Define

$$\Phi = \sum_{n \geq 1} \sum_{a_1, \ldots, a_{n+1} \text{ edges in } \tilde{Q}} \frac{\langle b_n(a_1^\vee, \ldots, a_n^\vee), a_{n+1}^\vee \rangle}{n+1} a_1 \cdots a_{n+1},$$

which is a finite sum of closed paths of since $\mathcal{A}$ is A_{fin}, hence $b_n = 0$ for $n \gg 0$. Note that Φ is of degree $3-m$.

Proposition 4.2.2. *We have the following descriptions on the differential d on $\mathbb{K}\tilde{Q}$:*

1. *For any edge a in $\tilde{Q}$, we have*

$$da = (-1)^{1+a+m} \sum_{b \text{ edges in } \tilde{Q}} (a,b) \partial_b^\circ \Phi.$$

2. *The differential can be written in terms of the Kontsevich bracket by*

$$d = \{\Phi, -\}.$$

3. *The differential d is a Hamiltonian vector field with Hamiltonian Φ, i.e.,*

$$d_{dR} \Phi = i_d \omega.$$

Proof. Denote by $(-,-)$ the inverse quadratic form of $\langle -,- \rangle$. Then

$$a = \sum_{b \text{ edges in } \tilde{Q}} (b,a) \langle -, b^\vee \rangle.$$

By definition,

$$\begin{aligned}
da &= \sum_n b_n^\#(a) \\
&= \sum_{n \in \mathbb{N}} \sum_{\substack{b, a_1, \ldots, a_n \\ \text{edges in } \tilde{Q}}} (b,a) \langle b_n(a_1^\vee, \ldots, a_n^\vee), b^\vee \rangle a_1 \cdots a_n \\
&= \sum_{\substack{b, a_1, \ldots, a_n \\ \text{edges in } \tilde{Q}}} (-1)^{|b|(3-m-|b|)} (b,a) \frac{\langle b_n(a_1^\vee, \ldots, a_n^\vee), b^\vee \rangle}{n+1} \partial_b^\circ (a_1 \cdots a_n b) \\
&= \sum_{b \text{ edges in } \tilde{Q}} (-1)^{m|b|+1+|a||b|} (a,b) \partial_b^\circ \Phi \\
&= \sum_{b \text{ edges in } \tilde{Q}} (-1)^{1+|a|+m} (a,b) \partial_b^\circ \Phi,
\end{aligned}$$

where we have used that $(a,b) \neq 0$ if and only if $|a| = |b| + m \mod 2$. The third equality perhaps requires more explanation. If the loop has cyclic symmetry of degree $n+1$, i.e., $a_1 \cdots a_n b = e^{n+1}$ for some edge e, then $\partial_e^\circ(e^{n+1}) = (n+1)e^n$. Up to cyclic permutation, there is only one such loop in the sum in the third line hence the $(n+1)$ factor in the denominator is cancelled. If the loop has no cyclic symmetry, then up to cyclic permutation, there are $n+1$ such loops in the sum and hence the $n+1$ factor in the denominator is also cancelled. In the general case where a loop has a cyclic symmetry of degree k, then we can write $a_1 \cdots a_n b = p^k$, where p has no cyclic symmetry. Up to cyclic permutation there are $(n+1)/k$ such loop in the sum, and $\partial_b^\circ(p^k) = k(\partial_b^{''} p)p^{k-1}(\partial_b^{'} p)$. Hence the $n+1$ factor in the denominator is also cancelled. This proves (a). Now, for any edge a, we have

$$\{\Phi, a\} = m \circ \{\!\{\Phi, a\}\!\} = -(-1)^{(|a|+2-m)(|\Phi|+2-m)} m \circ \sigma \{\!\{a, \Phi\}\!\} = (-1)^{1+a+m} \sum_{b \text{ edges in } \tilde{Q}} (a,b)\partial_b^\circ \Phi = da,$$

where the second last equality is by part (a) of the statement and Proposition 4.1.2. Since both $\{\Phi, -\}$ and d are derivations, we conclude that $d = \{\Phi, -\}$. This proves part (b). For part (c), we have

$$\begin{aligned}
i_d\omega &= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle ((dx)(d_{dR}y) + (-1)^{1+|x|}(d_{dR}x)(dy)) \\
&= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} (-1)^{1+|x||y|+|dx||y|} \langle y^\vee, x^\vee \rangle (d_{dR}y)(dx) + (-1)^{1+|x|} \langle x^\vee, y^\vee \rangle (d_{dR}x)(dy) \\
&= \sum_{x,y \text{ edges in } \tilde{Q}} (-1)^{1+|x|} \langle x^\vee, y^\vee \rangle (d_{dR}x)(dy) \\
&= \sum_{b,x,y \text{ edges in } \tilde{Q}} (-1)^{1+|x|+1+|y|+m} \langle x^\vee, y^\vee \rangle (y,b)(d_{dR}x)\partial_b^\circ \Phi \\
&= \sum_{y \text{ edges in } \tilde{Q}} (d_{dR}x)\partial_x^\circ \Phi \\
&= d_{dR}\Phi.
\end{aligned}$$

■

It remains to describe the action of the differential on $\mathbb{K}Q$ on the degree $1-m$ loops attached to each vertex. For this purpose, we first prove the following

Lemma 4.2.3. *Let $\mathcal{A}$ be an A_∞-category with a cyclic structure of degree m. Then for all $n \geq 3$, $\deg m_n(a_1, \cdots, a_n) = m$ implies $m_n(a_1, \cdots, a_n) = 0$.*

Proof. For any $v \in \mathcal{A}^0$ and $n \geq 3$,

$$\langle m_n(a_1, \ldots, a_n), v \rangle_{\mathcal{A}} = \pm \langle m_n(v, a_1 \ldots, a_{n-1}), a_n \rangle_{\mathcal{A}} = 0$$

since $m_n(v, a_1 \ldots, a_{n-1}) = 0$. Since $\langle -, - \rangle_{\mathcal{A}}$ is nondegenerate, we conclude $m_n(a_1, \ldots, a_n) = 0$. ■

Proposition 4.2.4. *Let v^* be the degree $1-d$ loop at the vertex v in Q. Then*

$$dv^* = v\ell v \quad \textit{where} \quad \ell = \sum_{x,y \textit{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle xy.$$

Proof. Write $t = \sum_{v \text{ vertices in } \tilde{Q}} v^*$. It suffices to show $dt = \ell$, for then $dv^* = d(vtv) = v(dt)v = v\ell v$. We have

$$
\begin{aligned}
dt &= \sum_{v \text{ vertices in } \tilde{Q}} \sum_{n \in \mathbb{N}} b_n^{\#}(v^*) \\
&= \sum_{v \text{ vertices in } \tilde{Q}} \sum_{a_1,\ldots,a_n \text{ edges in } \tilde{Q}} \sum_{n \in \mathbb{N}} \langle v^\vee, b_n(a_1^\vee, \ldots, a_n^\vee)\rangle a_1 \cdots a_n \\
&= \sum_{v \text{ vertices in } \tilde{Q}} \sum_{a_1,a_2 \text{ edges in } \tilde{Q}} \langle v^\vee, b_2(a_1^\vee, a_2^\vee)\rangle a_1 a_2 \\
&= \sum_{v \text{ vertices in } \tilde{Q}} \sum_{a_1,a_2 \text{ edges in } \tilde{Q}} \langle a_1^\vee, b_2(a_2^\vee, v^\vee)\rangle a_1 a_2 \\
&= \sum_{a_1,a_2 \text{ edges in } \tilde{Q}} \langle a_1^\vee, a_2^\vee\rangle a_1 a_2 \\
&= \ell
\end{aligned}
$$

where in the third equality, we have used Lemma 4.2.3. ■

Remark 4.2.5. Note here that ℓ is an element homogeneous of degree $2 - m$.

Summarizing, we have proven the converse of Proposition 4.2.1 and arrived at

Proposition 4.2.6. *A dg-quiver Q is a quiver with superpotential if and only if $\mathbb{K}Q = E(\mathcal{A})$ for a finite positively graded cyclic A_{fin}-category $\mathcal{A}$.*

With the characterisation of quivers with superpotential by A_{fin}-categories with cyclic structure, we can define products and quotients of quivers with superpotential.

Products of Quivers. Products of quiver are not always defined. But as we will see in Chapter 5, they are always defined for quivers arising from exceptional sequences. Recall from Chapter 3 that for every dg-quiver Q, one can construct an A_{fin}-algebra augmented over the discrete $\mathbb{K}$-algebra spanned by the vertices of Q, such that $\mathbb{K}Q = E(A)$. For any two quivers Q and their associated A_{fin}-algebras A and A', one can form the A_∞-tensor product $A \otimes A'$. In general, this tensor product is not A_{fin}. If it is, one can define the product dg-quiver of Q and Q' to be the dg-quiver arising from $A \otimes A'$. In other words, even if the product is well-defined, the product dg-quiver is only defined up to quasi-isomorphism. The underlying product quiver is given by

$$
\begin{aligned}
\{\text{Degree } i \text{ edge in } Q \times Q'\} = {} & \{a \otimes v : a \text{ degree } i \text{ edges in } Q \text{ and } v \text{ vertex in } Q'\} \\
& \cup \{a \otimes b : a \text{ edge in } Q \text{ and } b \text{ edge in } Q' \text{ with } \deg a + \deg b - 1 = i\} \\
& \cup \{u \otimes b : u \text{ vertex in } Q' \text{ and } b \text{ degree } i \text{ edge in } Q'\}
\end{aligned}
$$

The differential, however, is not uniquely determined. If both quivers Q and Q' admit superpotentials, then Amorim's construction of tensor product of A_∞-algebras shows that one can choose a differential on $Q \times Q'$ which comes from a superpotential. In general, to write down the product superpotential, one has to know an explicit formula for the cyclic tensor product A_∞-structure. Unfortunately, the author does not know of such a general formula. When one of the superpotentials is cubic, i.e., one of the A_∞-algebras involved is A_2, i.e., $m_n = 0$ for $n \geq 3$, one can use the formula given by Equation (3.6.2). Examples of product quivers are given in Chapter 6.

Quotients of Quivers. Let G be a finite group and Q be a dg-quiver. Suppose G acts on $\mathbb{K}Q$ by dg-isomorphisms. Write $\mathbb{K}Q = E(A)$ for an A_{fin}-algebra. Then G acts on A by strict A_{fin}-isomorphisms and hence one can form the quotient A/G. The quotient quiver Q/G is by definition the quiver associated to A/G. Now suppose Q is a quiver with superpotential and the antisymmetric pairing on Q is G-invariant, then the cyclic structure on A is also G-invariant. Thus A/G also inherits a cyclic structure, making Q/G a quiver with superpotential. Examples of quotients of quiver with superpotential can be found in Chapter 6.

4.3 Quivers with Superpotential are n-Calabi–Yau

In this section, we would like to show that the path algebra of a quiver with superpotential is Calabi–Yau, and that as a consequence, its derived category is Calabi–Yau.

Calabi–Yau Algebras. Following Kontsevich, Ginzburg defined in [27, §3] the notion of Calabi–Yau algebras which arise naturally in the geometry of Calabi–Yau manifolds to transplant most of traditional Calabi–Yau geometry to the noncommutative setting.

Recall that in Example 2.4.7, for a homologically smooth dg-algebra A, we have the notion of dualizing complex: Denote by $\mathrm{D}^b(A)$ the bounded derived category of dg-modules over A and $\mathrm{D}^b_{fd}(A)$ the full subcategory of $\mathrm{D}^b(A)$ consisting of those dg A-modules whose homology is of finite total dimension. Define the dualizing complex $\Omega = \mathrm{RHom}_{A^{\mathrm{op}}\otimes A}(A, A^{\mathrm{op}} \otimes A)$. Then $(-)\otimes\Omega : \mathrm{D}^b_{fd}(A) \to \mathrm{D}^b_{fd}(A)$ is a Serre functor. In particular, if we have an isomorphism $\Omega \cong \Sigma^{-d}A$ as objects in $\mathrm{D}(A^{\mathrm{op}} \otimes A)$, then $\mathrm{D}^b_{fd}(A)$ is d-Calabi–Yau. This motivates the following definition:

Definition 4.3.1 (Calabi–Yau algebras)**.** A dg-algebra A is said to be m-Calabi–Yau if

- A is homologically smooth, i.e., $A \in \mathrm{Per}(A \otimes A)$, and
- there is a quasi-isomorphism of A-bimodules

$$\eta : \mathrm{RHom}_{A-\mathrm{Bimod}}(A, A \otimes A) \to \Sigma^{-m}A.$$

In the original definition, Ginzburg requires in addition that η is self dual, but it was later shown by van den Bergh [70, Prop. C.1] that this is automatic. We have the following

Proposition 4.3.2. *If A is an m-Calabi–Yau dg-algebra, then $\mathrm{D}^b_{fd}(A)$ is a m-Calabi–Yau triangulated category.*

Proof. See [41, Lemma 3.4] and [45, Lemma 4.1]. ∎

Let Q be a quiver with superpotential of dimension m. Our proof that $\mathbb{K}Q$ is m-Calabi–Yau is a direct adaptation of van den Bergh's proof [71] in the 3-Calabi–Yau case. First, we show that $\mathbb{K}Q$ is homologically smooth.

Proposition 4.3.3. *Let Q be a quiver and S be the discrete $\mathbb{K}$-algebra on the vertices of Q. The exact sequence*

$$0 \to \Omega^1_S(\mathbb{K}Q) \xrightarrow{\varphi} \mathbb{K}Q \otimes_S \mathbb{K}Q \xrightarrow{m} \mathbb{K}Q \to 0,$$

where $\varphi(dp) = p \otimes_S 1 - 1 \otimes_S p$ and $m(p \otimes_S q) = pq$, is a resolution of $\mathbb{K}Q$ by modules in $\mathrm{Per}(\mathbb{K}Q \otimes \mathbb{K}Q)$. In other words, $\mathbb{K}Q$ is homologically smooth.

Proof. The sequence in the proposition is exact by Proposition 1.2.7. Let E be the $\mathbb{K}$-vector space spanned by all edges in Q and E_{jk} the $\mathbb{K}$-vector space spanned by all edges from k to j. Then

$$\begin{aligned}
\mathbb{K}Q \otimes E \otimes \mathbb{K}Q &\cong \bigoplus_{i,j,k,\ell \in V_Q} \mathbb{K}Qi \otimes E_{jk} \otimes \ell\mathbb{K}Q \\
&\cong \bigoplus_{i,\ell \in V_Q} \mathbb{K}Qi \otimes E_{i\ell} \otimes \ell\mathbb{K}Q \oplus \bigoplus_{i\neq j \text{ or } k\neq \ell} \mathbb{K}Qi \otimes E_{jk} \otimes \ell\mathbb{K}Q \\
&\cong \mathbb{K}Q \otimes_S E \otimes_S \mathbb{K}Q \oplus \bigoplus_{i\neq j \text{ or } k\neq \ell} \mathbb{K}Qi \otimes E_{jk} \otimes \ell\mathbb{K}Q \\
&\cong \Omega^1_S(\mathbb{K}Q) \oplus \bigoplus_{i\neq j \text{ or } k\neq \ell} \mathbb{K}Qi \otimes E_{jk} \otimes \ell\mathbb{K}Q
\end{aligned}$$

Hence Ω^1_S is a summand of the free $\mathbb{K}Q$-bimodule $\mathbb{K}Q \otimes E \otimes \mathbb{K}Q$, i.e., $\Omega^1_S(\mathbb{K}Q) \in \mathrm{Per}(\mathbb{K}Q \otimes \mathbb{K}Q)$. Similarly,

$$\mathbb{K}Q \otimes \mathbb{K}Q \cong \bigoplus_{i,j\in V} \mathbb{K}Qi \otimes j\mathbb{K}Q \cong \mathbb{K}Qi \otimes i\mathbb{K}Q \oplus \bigoplus_{i\neq j} \mathbb{K}Qi \otimes j\mathbb{K}Q \cong \mathbb{K}Q \otimes_S \mathbb{K}Q \oplus \bigoplus_{i\neq j} \mathbb{K}Qi \otimes j\mathbb{K}Q$$

shows $\mathbb{K}Q \otimes_S \mathbb{K}Q$ is in $\mathrm{Per}(\mathbb{K}Q \otimes \mathbb{K}Q)$. Since $\mathrm{Per}(\mathbb{K}Q \otimes \mathbb{K}Q)$ is closed under extension, $\mathbb{K}Q$ is also in $\mathrm{Per}(\mathbb{K}Q \otimes \mathbb{K}Q)$, i.e., $\mathbb{K}Q$ is homologically smooth. ■

If we define

$$d_{\mathbb{K}Q\otimes_S\mathbb{K}Q} : \mathbb{K}Q \otimes_S \mathbb{K}Q \to \mathbb{K}Q \otimes_S \mathbb{K}Q, \quad a \otimes_S b \mapsto d(a) \otimes_S b + (-1)^{|a|} a \otimes_S d(b),$$

$$d_{\Omega^1_S(\mathbb{K}Q)} : \Omega^1_S(\mathbb{K}Q) \to \Omega^1_S(\mathbb{K}Q), \quad d_{\Omega^1_S(\mathbb{K}Q)}(d_{dR}a) = d_{dR}(da),$$

and assign a grading on $\Omega^1_S(\mathbb{K}Q)$ by $|d_{dR}a|_{\Omega^1_S(\mathbb{K}Q)} = |a|_{\mathbb{K}Q}$, then $d_{\mathbb{K}Q\otimes_S\mathbb{K}Q}$ and $d_{\Omega^1_S(\mathbb{K}Q)}$ are differentials of degree (-1) on both spaces, and the resolution becomes a dg-bimodule resolution, i.e., both φ and m are morphisms of dg-bimodules.

Cofibrant replacement of $\mathbb{K}Q$. Denote the subquiver of Q consisting of all edges of degree in the interval $[2-m, 0]$ by $\tilde{Q}$. Let v^* be the degree $1-m$ loop at vertex v and t be the sum of all degree $1-m$ loops. The resolution of $\mathbb{K}Q$ in Proposition 4.3.3 shows [25, Prop. III.3.5] $\mathbb{K}Q$ is quasi-isomorphic to

$$\begin{aligned} P &= \mathrm{cone}(\varphi) \\ &= \mathbb{K}Q \otimes_S \mathbb{K}Q \oplus \Sigma\Omega^1_S(\mathbb{K}Q) \\ &= \mathbb{K}Q \otimes_S \mathbb{K}Q \oplus \Sigma(\mathbb{K}Q \otimes_{\mathbb{K}Q} \Omega_S(\mathbb{K}\tilde{Q}) \otimes_{\mathbb{K}\tilde{Q}} \mathbb{K}Q)) \oplus \Sigma(\mathbb{K}Q \otimes_S \Omega^1_S(S[t]) \otimes_S \mathbb{K}Q), \end{aligned} \tag{4.3.1}$$

which is in $\mathrm{Per}(\mathbb{K}Q\otimes\mathbb{K}Q)$. Hence P is a cofibrant replacement of $\mathbb{K}Q$ as a $\mathbb{K}Q$-bimodule. The differential of the cone d_P is given by

$$d_P = \begin{pmatrix} d_{\mathbb{K}Q\otimes\mathbb{K}Q} & 0 \\ \Sigma\varphi & \Sigma d_{\Omega^1_S(\mathbb{K}Q)} \end{pmatrix}$$

Observe that $|a \otimes b|_P = |a|_{\mathbb{K}Q} + |b|_{\mathbb{K}Q}$ and $|d_{dR}a|_P = |a|_{\mathbb{K}Q} - 1$ for any $a \in \mathbb{K}Q$. To avoid too many subscripts, in what follows, the subscript in $d_{\mathbb{K}Q\otimes\mathbb{K}Q}$ and $d_{\Omega^1_S(\mathbb{K}Q)}$ will be suppressed, but, to avoid confusion, the subscript P in d_P is always written.

Since P is a cofibrant replacement of $\mathbb{K}Q$, we have

$$\mathrm{RHom}_{\mathbb{K}Q\text{-Bimod}}(\mathbb{K}Q, \mathbb{K}Q \otimes \mathbb{K}Q) = \mathrm{Hom}_{\mathbb{K}Q\text{-Bimod}}(P, \mathbb{K}Q \otimes \mathbb{K}Q).$$

Hence to construct a quasi-isomorphism

$$\eta : \mathrm{RHom}_{\mathbb{K}Q\text{-Bimod}}(\mathbb{K}Q, \mathbb{K}Q \otimes \mathbb{K}Q) \to \Sigma^{-m}\mathbb{K}Q,$$

it suffices to construct an isomorphism of dg-modules

$$\mathrm{Hom}_{\mathbb{K}Q\text{-Bimod}}(P, \mathbb{K}Q \otimes \mathbb{K}Q) \cong \Sigma^{-m}P.$$

This is done by constructing a nondegenerate pairing of bimodules in the following sense:

Pairing of Bimodules. We follow the conventions of van den Bergh [71]. A pairing of degree n between A-bimodules M and N is a bilinear homogeneous map of degree n

$$\langle\!\langle -, - \rangle\!\rangle : M \times N \to A \otimes A$$

such that $\langle\!\langle p, - \rangle\!\rangle$ is linear for the outer bimodule structure on $A \otimes A$ and $\langle\!\langle -, q \rangle\!\rangle$ is linear for the inner bimodule structure on $A \otimes A$, i.e.,

$$\begin{aligned} \langle\!\langle apb, q \rangle\!\rangle &= (-1)^{|b|(|q|+n)} a * \langle\!\langle p, q \rangle\!\rangle * b \\ \langle\!\langle p, aqb \rangle\!\rangle &= (-1)^{|a|(|p|+n)} a \langle\!\langle p, q \rangle\!\rangle b. \end{aligned}$$

If M and N modules in $\mathrm{Per}(A \otimes A)$, we say the pairing is nondegenerate if the map

$$M \to \Sigma^n \mathrm{Hom}_{A-\mathrm{Bimod}}(N, A \otimes A), \quad p \mapsto \langle\!\langle p, - \rangle\!\rangle$$

is an isomorphism. If A is a dg-algebra and M, N are dg-bimodules, then a dg-pairing is a pairing which satisfy Leibniz's rule

$$\xi\langle\langle p,q\rangle\rangle = \langle\langle \xi p, q\rangle\rangle + (-1)^{|p|+n}\langle\langle p, \xi q\rangle\rangle.$$

A dg-pairing which is nondegenerate induces an isomorphism of dg-modules

$$M \cong \Sigma^n \mathrm{Hom}_{A-\mathrm{Bimod}}(N, A\otimes A).$$

In the special case $M = N$, a pairing is said to be (super)-symmetric if

$$\langle\langle p,q\rangle\rangle = (-1)^{(n+|p|)(n+|q|)}\sigma\langle\langle q,p\rangle\rangle.$$

Example 4.3.4. Let Q be a quiver and S be the discrete $\mathbb{K}$-algebra on the vertices of Q. Then $\mathbb{K}Q \otimes_S \mathbb{K}Q \in \mathrm{Per}(\mathbb{K}Q\otimes\mathbb{K}Q)$ by Proposition 4.3.3. Define a pairing

$$\Sigma^n(\mathbb{K}Q\otimes_S\mathbb{K}Q)\times(\mathbb{K}Q\otimes_S\mathbb{K}Q)\to\mathbb{K}Q\otimes\mathbb{K}Q$$

by

$$\langle\langle 1\otimes_S 1, 1\otimes_S 1\rangle\rangle = \sum_{v\in V_Q} v\otimes v.$$

Then it is a nondegenerate pairing of degree n. To see this, observe that any $\mathbb{K}Q$-bimodule morphism $\varphi : \Sigma^n(\mathbb{K}Q\otimes_S\mathbb{K}Q)\to\mathbb{K}Q\otimes\mathbb{K}Q$ is determined on its value on $1\otimes_S 1$. Let $\varphi(1\otimes_S 1) = a'\otimes a''$. Then

$$\varphi(1\otimes_S 1) = \sum_{v\in V_Q}\varphi(v\otimes_S v) = \sum_{v\in V_Q} v\varphi(1\otimes_S 1)v = \sum_{v\in V_Q} va'\otimes a''v = a'' * \left(\sum_{v\in V_Q} v\otimes v\right) * a'.$$

In other words, $\varphi = \langle\langle a''(1\otimes_S 1)a', -\rangle\rangle$.

Now, we return to our proof of that path algebra of quiver with superpotential are Calabi–Yau. Recall from Equation (4.3.1) that $P = \mathrm{cone}\,\varphi$ is a cofibrant replacement of $\mathbb{K}Q$. We define a symmetric pairing $\langle\langle -,-\rangle\rangle : P\times P\to\mathbb{K}Q\otimes\mathbb{K}Q$ of degree $-m$ by defining on generators

$$\begin{aligned}\langle\langle d_{dR}v^*, 1\otimes_S 1\rangle\rangle &= (-1)^m v\otimes v\\ \langle\langle 1\otimes_S 1, d_{dR}v^*\rangle\rangle &= (-1)^m v\otimes v\\ \langle\langle d_{dR}a, d_{dR}b\rangle\rangle &= (-1)^{|d_{dR}a|_P}\{\!\{a,b\}\!\}\end{aligned} \tag{4.3.2}$$

for all a, b edges in $\mathbb{K}\tilde{Q}$ and assigning the value zero for all other combinations. Note that the first two equations of Equation (4.3.2) is equivalent to

$$\begin{aligned}\langle\langle d_{dR}t, 1\otimes_S 1\rangle\rangle &= (-1)^m\sum_{v\in V_Q} v\otimes v\\ \langle\langle 1\otimes_S 1, d_{dR}t\rangle\rangle &= (-1)^m\sum_{v\in V_Q} v\otimes v\end{aligned}$$

by linearity of the pairing.

Proposition 4.3.5. *The pairing $\langle\langle -,-\rangle\rangle$ is a symmetric nondegenerate dg-pairing of degree $-m$.*

Proof. First, we check symmetry:

$$\langle\langle dt, 1\otimes_S 1\rangle\rangle = (-1)^m\sum_{v\in V_Q} v\otimes v = (-1)^{(|1\otimes_S 1|_P - m)(|dt|_P - m)}\sigma\langle 1\otimes_S 1, dt\rangle$$

since $|d_{dR}t|_P = |t|_{\mathbb{K}Q} - 1 = -m$.

$$\begin{aligned}\langle\langle d_{dR}a, d_{dR}b\rangle\rangle &= (-1)^{|d_{dR}a|_P}\{\!\{a,b\}\!\}\\ &= -(-1)^{|d_{dR}a|_P + (|a|_{\mathbb{K}Q}-m+2)(|b|_{\mathbb{K}Q}-m+2)}\sigma\{\!\{b,a\}\!\}\\ &= (-1)^{(|d_{dR}a|_P - m)(|d_{dR}b|_P - m)}\sigma\langle\langle d_{dR}b, d_{dR}a\rangle\rangle.\end{aligned}$$

since $|d_{dR}a|_P + (|a|_{\mathbb{K}Q} - m + 2)(|b|_{\mathbb{K}Q} - m + 2) + 1 + |d_{dR}b|_P = (|d_{dR}a|_P - m)(|d_{dR}b|_P - m) \mod 2.$

Next we show the pairing is nondegenerate. Since for any edge a,

$$\langle\langle d_{dR}a, -\rangle\rangle = (-1)^{|a|-1}\{\!\{a, -\}\!\} = \sum_b (-1)^{|a|-1}(a, b)\partial_b,$$

we see that the pairing is nondegenerate on $\Omega^1_S(\mathbb{K}\tilde{Q})$. Since $\mathbb{K}Q \otimes_S \Omega^1_S(S[t]) \otimes_S \mathbb{K}Q \cong \mathbb{K}Q \otimes_S \mathbb{K}Q$, together with Example 4.3.4, this shows the pairing is nondegenerate on P.

It remains to check the compatibility of $\langle\langle -, -\rangle\rangle$ with d_P, i.e., we have the Leibnitz's rule

$$d\langle\langle p, q\rangle\rangle = \langle\langle d_P p, q\rangle\rangle + (-1)^{|p|_P - m}\langle\langle p, d_P q\rangle\rangle.$$

This is a direct calculation divided into six cases.

1. We have

$$\begin{aligned} d\langle\langle d_{dR}t, d_{dR}t\rangle\rangle &= 0, \\ \langle\langle d_P d_{dR}t, d_{dR}t\rangle\rangle &= \langle\langle t \otimes_S 1 - 1 \otimes_S t - d_{dR}\ell, d_{dR}t\rangle\rangle \\ &= t * \langle\langle 1 \otimes_S 1, d_{dR}t\rangle\rangle - \langle\langle 1 \otimes_S 1, d_{dR}t\rangle\rangle * t \\ &= (-1)^m \sum_{v \text{ vertices in } Q} (v \otimes t_v - t_v \otimes v) \end{aligned}$$

   and

$$\begin{aligned} \langle\langle d_{dR}t, d_P d_{dR}t\rangle\rangle &= \langle\langle d_{dR}t, t \otimes_S 1 - 1 \otimes_S t - d_{dR}\ell\rangle\rangle \\ &= t\langle\langle d_{dR}t, 1 \otimes_S 1\rangle\rangle - \langle\langle d_{dR}t, 1 \otimes_S 1\rangle\rangle t \\ &= (-1)^m \sum_{v \text{ vertices in } Q} (t_v \otimes v - v \otimes t_v) \end{aligned}$$

   Hence

$$d\langle\langle d_{dR}t, d_{dR}t\rangle\rangle = \langle\langle d_P d_{dR}t, d_{dR}t\rangle\rangle + (-1)^{|d_{dR}t|_P - m}\langle\langle d_{dR}t, d_P d_{dR}t\rangle\rangle.$$

2. We have for any $a \in \mathbb{K}\tilde{Q}$,

$$\begin{aligned} d\langle\langle d_{dR}a, d_{dR}t\rangle\rangle &= 0, \\ \langle\langle d_P d_{dR}a, d_{dR}t\rangle\rangle &= \langle a \otimes_S 1 - 1 \otimes_S a - d_{dR}(da), d_{dR}t\rangle \\ &= a * \langle\langle 1 \otimes_S 1, d_{dR}t\rangle\rangle - \langle\langle 1 \otimes_S 1, d_{dR}t\rangle\rangle * a \\ &= (-1)^m \sum_{v \text{ vertices in } Q} (v \otimes av - va \otimes v) \\ &= (-1)^m (t(a) \otimes a - a \otimes h(a)) \end{aligned}$$

   and

$$\begin{aligned} \langle\langle d_{dR}a, d_P d_{dR}t\rangle\rangle &= \langle\langle d_{dR}a, t \otimes_S 1 - 1 \otimes_S t - d_{dR}\ell\rangle\rangle \\ &= -\langle\langle d_{dR}a, d_{dR}\ell\rangle\rangle \\ &= (-1)^{|d_{dR}a|_P - 1}\{\!\{a, \ell\}\!\}. \end{aligned}$$

   Now

$$\begin{aligned} \{\!\{a, \ell\}\!\} &= \sum_{b \text{ edges in } \tilde{Q}} (a, b)\partial_b \ell \\ &= \sum_{b,x,y \text{ edges in } \tilde{Q}} (a, b)\langle x^\vee, y^\vee\rangle \partial_b(xy) \\ &= \sum_{x,y \text{ edges in } \tilde{Q}} (a, x)\langle x^\vee, y^\vee\rangle h(x) \otimes t(x)y + (-1)^{|x||y|}(a, y)\langle x^\vee, y^\vee\rangle xh(y) \otimes t(y) \\ &= t(a) \otimes a - a \otimes h(a). \end{aligned}$$

   Hence

$$d\langle\langle d_{dR}a, d_{dR}t\rangle\rangle = \langle\langle d_P d_{dR}a, d_{dR}t\rangle\rangle + (-1)^{|d_{dR}a|_P - m}\langle\langle d_{dR}t, d_P d_{dR}a\rangle\rangle.$$

3. We have
$$\xi\langle dt, 1\otimes_S 1\rangle = 0,$$
$$\langle \xi_P dt, 1\otimes_S 1\rangle = \langle t\otimes_S 1 - 1\otimes_S t - d\ell, 1\otimes_S 1\rangle = 0$$
and
$$\langle dt, \xi(1\otimes_S 1)\rangle = 0.$$
Hence Leibnitz's rule is verified.

4. For $a \in \mathbb{K}\bar{Q}$,
$$d\langle\!\langle d_{dR}a, 1\otimes_S 1\rangle\!\rangle = 0 = \langle\!\langle d_{dR}a, d(1\otimes_S 1)\rangle\!\rangle$$
and
$$\langle\!\langle d_P(d_{dR}a), 1\otimes_S 1\rangle\!\rangle = \langle\!\langle a\otimes_S 1 - 1\otimes_S a - d(d_{dR}a), 1\otimes_S 1\rangle\!\rangle = 0.$$
Hence Leibnitz's rule is verified.

5. The case
$$d\langle\!\langle 1\otimes_S 1, 1\otimes_S 1\rangle\!\rangle = \langle\!\langle d_P(1\otimes_S 1), 1\otimes_S 1\rangle\!\rangle + (-1)^{|1\otimes_S 1|+m}\langle\!\langle 1\otimes_S 1, d_P(1\otimes_S 1)\rangle\!\rangle$$
is trivial since all terms are zero.

6. We have for $a, b \in \mathbb{K}\bar{Q}$, since $\{\!\{-,-\}\!\}$ is a double dg-bracket,
$$\begin{aligned} d\langle\!\langle d_{dR}a, d_{dR}b\rangle\!\rangle &= (-1)^{|d_{dR}a|_P} d\{\!\{a,b\}\!\} \\ &= (-1)^{|d_{dR}a|_P}\{\!\{da,b\}\!\} + (-1)^{|d_{dR}a|_P+|a|_{\mathbb{K}\bar{Q}}+m}\{\!\{a,db\}\!\} \\ &= (-1)^{|d_{dR}a|_P}\{\!\{da,b\}\!\} + (-1)^{1+m}\{\!\{a,db\}\!\}, \end{aligned}$$
$$\begin{aligned} \langle\!\langle d_P d_{dR}a, d_{dR}b\rangle\!\rangle &= \langle\!\langle a\otimes_S 1 - 1\otimes_S a - d_{dR}(da), d_{dR}b\rangle\!\rangle \\ &= -(-1)^{|d_{dR}da|_P}\{\!\{\xi a, b\}\!\} \\ &= (-1)^{|d_{dR}a|_P}\{\!\{\xi a, b\}\!\} \end{aligned}$$
and
$$\begin{aligned} \langle\!\langle d_{dR}a, d_P d_{dR}b\rangle\!\rangle &= \langle\!\langle d_{dR}a, b\otimes_S 1 - 1\otimes_S b - d_{dR}(db)\rangle\!\rangle \\ &= -(-1)^{|d_{dR}a|_P}\{\!\{a, db\}\!\}. \end{aligned}$$
Hence
$$d\langle\!\langle d_{dR}a, d_{dR}b\rangle\!\rangle = \langle\!\langle d_P d_{dR}a, d_{dR}b\rangle\!\rangle + (-1)^{|d_{dR}a|_P - m}\langle\!\langle d_{dR}a, d_P d_{dR}b\rangle\!\rangle.$$

■

Theorem 4.3.6. *$\mathbb{K}Q$ is a m-Calabi–Yau dg-algebra.*

Proof.
$$\begin{aligned} \mathrm{RHom}(\mathbb{K}Q, \mathbb{K}Q\otimes\mathbb{K}Q) &\cong \mathrm{Hom}(P, \mathbb{K}Q\otimes\mathbb{K}Q) \\ &\cong \Sigma^{-m}P \\ &\cong \Sigma^{-m}\mathbb{K}Q, \end{aligned}$$
where the first and third isomorphisms hold since P is a cofibrant replacement for $\mathbb{K}Q$ and the second isomorphism is induced by the nondegenerate dg-pairing $\langle\!\langle -,-\rangle\!\rangle$. ■

Theorem 4.3.7. *The bounded derived category $\mathrm{D}^b_{fd}(\mathbb{K}Q)$ with its standard t-structure is a m-Calabi–Yau triangulated category whose heart is equivalent to the category of finite dimensional modules over $H^0(\mathbb{K}Q)$.*

Proof. By Proposition 4.3.2, $\mathrm{D}^b_{fd}(\mathbb{K}Q)$ is an m-Calabi–Yau triangulated category. The heart of the standard t-structure on $\mathrm{D}^b_{fd}(\mathbb{K}Q)$ consists of finite dimensional dg $\mathbb{K}Q$-modules concentrated at degree 0. We show that the action of $\mathbb{K}Q$ on such $\mathbb{K}Q$-modules factors through $H^0(\mathbb{K}Q)$. For any $a \in (\mathbb{K}Q)^1$, $d(am) = (da)m + (-1)^{|a|}adm$ implies $(da)m = 0$ since the other two terms are of degree (-1) and vanish automatically. This shows $d(\mathbb{K}Q)^1$ acts trivially, and hence the action factors through $H^0(\mathbb{K}Q)$. This gives a finite dimensional $H^0(\mathbb{K}Q)$-module and yields the required equivalence. ∎

Calabi–Yau algebras not coming from quivers with superpotential. We have shown that every quiver with superpotential produces a Calabi–Yau algebra. One might be interested in the converse, i.e, if all Calabi–Yau algebras arise in this way. Davison [21] has shown that fundamental group algebras of compact hyperbolic manifolds of dimension greater than one are Calabi–Yau algebras which do not arise in this way. Thus it is not true that every Calabi–Yau algebra comes from a quiver with superpotential. In fact, van den Bergh [70] has defined a class of Calabi–Yau algebras, which he called exact Calabi–Yau algebras, using cyclic and Hochschild homology, and characterized them as Calabi–Yau algebras which are quasi-isomorphic to quivers with superpotential (which he called deformed dg-preprojective algebras).

4.4 Quivers with Superpotential of Low Dimensions

In this section, we describe quivers with superpotential of dimensions 1 to 4.

Quivers with Superpotential of Dimension 1. Quivers with superpotential of dimension 1 are given by a finite number of vertices, and for each vertex v, a loop v^* of degree 0, with trivial differential.

Quivers with Superpotential of Dimension 2. Quiver with superpotential Q of dimension 2 are in the following form: for each vertex v, there is a loop v^* of degree -1, and these are all the degree -1 edges in Q. There is an antisymmetric nondegenerate pairing on degree 0 edges $(-,-)$. The symplectic form on $\mathbb{K}\tilde{Q}$ reads

$$\omega = \frac{1}{2} \sum_{e,f \text{ edges of degree } 0} \langle e^\vee, f^\vee \rangle d_{dR} e d_{dR} f.$$

The differential reads

$$\begin{aligned} d(e) &= 0 \quad \text{for any degree 0 edge } e; \\ d(v^*) &= v\ell v \quad \text{for any degree } -1 \text{ loop } v^*, \text{ where } \ell = \sum \langle e^\vee, f^\vee \rangle ef. \end{aligned}$$

Quivers with Superpotential of Dimension 3. We will put quivers with superpotential of dimension 3 into a standard form and show that they are precisely the Ginzburg algebras introduced by Ginzburg in [27]. There is an anti-symmetric nondegenerate pairing between degree 0 edges and degree -1 edges. By changing basis if necessary, we may assume that for each degree 0 edge e, there is a degree -1 edge e^* in the opposite direction, such that the pairing is given by $(e, f^*) = \delta_{ef}$. The symplectic form on $\mathbb{K}\tilde{Q}$ then reads

$$\omega = \sum_{e \text{ edges of degree 0 in } Q} d_{dR} e d_{dR} e^*,$$

and the superpotential Φ is of degree 0, i.e., only depends on the degree 0 edges e's. The differential $d : \mathbb{K}Q \to \mathbb{K}Q$ reads

$$\begin{aligned} d(e) &= 0 \quad \text{for any degree 0 edge } e; \\ d(e^*) &= -\partial_e \Phi; \\ d(v^*) &= v\ell v \quad \text{for any degree } -2 \text{ loop } v^*, \text{ where } \ell = \sum [e, e^*]. \end{aligned} \tag{4.4.1}$$

In particular, the dg-quiver Q is determined by its degree 0 sub-quiver $Q^\star$ and the superpotential Φ which is an element in $\mathbb{K}Q^\star_{\text{cyc}}$ as follow: Given $Q^\star$ and Φ, one can construct a 3-Calabi–Yau quiver with superpotential Q by first adding to $Q^\star$ a reversed arrow e^* of degree -1 for each arrow e and define a degree 1 map $d : \mathbb{K}Q \to \mathbb{K}Q$ as in equation 4.4.1 using Φ. The map d such defined is a differential as $\{\Phi, \Phi\}$ has degree 1, hence the master equation $\{\Phi, \Phi\} = 0$ is empty. This reverse construction is exactly the one given by Ginzburg in [27]. Thus our definition of quiver with superpotential is a generalization of the Ginzburg construction. Note that $H^0(\mathbb{K}Q, \Phi) = \mathbb{K}Q^\star/\langle\, \partial_e\Phi : e \text{ edges in } Q^\star\rangle$ is the so called Jacobi algebra of $(Q^\star, \Phi)$. Theorem 4.3.7 then readily yields

Theorem 4.4.1. *The bounded derived category* $\mathrm{D}^b_{fd}(\mathbb{K}Q)$ *with its standard t-structure is a 3-Calabi–Yau triangulated category whose heart is equivalent to the category of finite dimensional modules over the Jacobi algebra* $\mathbb{K}Q^\star/\langle\, \partial_e\Phi : e \text{ edges in } Q^\star\rangle$.

Quivers with Superpotential of Dimension 4. We give a standard form for 4-Calabi–Yau quiver with superpotential and a Ginzburg algebra-like construction for dimension 4.

There is an anti-symmetric nondegenerate pairing between degree 0 edges and degree -2 edges. By a change of basis if necessary, we may assume that for each degree 0 edge e, there is a reversed degree -2 edge e^*, such that the pairing is given by $(e, f^*) = \delta_{ef}$. There is a symmetric nondegenerate quadratic form q on the degree -1 edges. The symplectic form on $\mathbb{K}\tilde{Q}_A$ then reads

$$\omega = \sum_{e \text{ edges in degree } 0} d_{dR}e d_{dR}e^* + \frac{1}{2}\sum_{r,\, s \text{ edges of degree } -1} q(r,s) d_{dR}r d_{dR}s.$$

and the superpotential Φ is of degree -1, i.e., only depends on the degree 0 edges e's and the degree -1 edges r's. If we denote by $A_r = dr$ for degree -1 edges r, then the superpotential Φ can be written in the form

$$\Phi = \sum_{s \text{ edges of degree } -1} s(\partial^\circ_s\Phi) = \sum_{u,s,r \text{ edges of degree } -1} (u,r)\langle r^\vee, s^\vee\rangle s\partial^\circ_s\Phi = \sum_{r,s \text{ edges of degree } -1} q(r,s) s A_r.$$

The differential $d : \mathbb{K}Q \to \mathbb{K}Q$ reads

$$\begin{aligned} d(e) &= 0 \quad \text{for any degree 0 edge } e; \\ d(r) &= A_r; \\ d(e^*) &= \partial_e\Phi; \\ d(v^*) &= v\ell v \quad \text{for any degree } -3 \text{ loop } v^*, \text{ where} \\ \ell &= \sum_{e \text{ edges of degree } 0} [e, e^*] + \sum_{r,s \text{ edges of degree } -1} q(r,s) rs. \end{aligned} \tag{4.4.2}$$

The condition $d^2e^* = 0$ gives rise to an equation:

$$\begin{aligned} &\partial^\circ_e\left(\sum_{r,s \text{ edges of degree } -1} q(r,s)A_rA_s\right) \\ &= \sum_{r,s \text{ edges of degree } -1} q(r,s)(\partial''_eA_r)A_s(\partial'_eA_r) + q(r,s)(\partial''_eA_s)A_r(\partial'_eA_s) \\ &= d\left(\sum_{r,s \text{ edges of degree } -1} q(r,s)(\partial''_eA_r)s(\partial'_eA_r) + q(r,s)(\partial''_eA_s)r(\partial'_eA_s)\right) \\ &= 2d\partial^\circ_e\left(\sum_{r,s \text{ edges of degree } -1} q(r,s)A_rs\right) \\ &= 2d\partial^\circ_e\Phi = 2d^2(e^*) = 0 \end{aligned}$$

Since this is true for all degree 0 edges e in Q, we conclude

$$\sum_{r,\, s \text{ edges of degree } -1} q(r,s)A_rA_s = 0 \text{ in } \mathbb{K}Q_{\mathrm{cyc}}, \text{ i.e., modulo cyclic permutation.}$$

Note that the above equation is the equation $\{\Phi, \Phi\} = 0$ written out in coordinates. Now, if we take $Q^\star$ to be the subquiver of Q consisting of all vertices and all degree 0 edges in Q, R to be the set of degree -1 edges, we get the following data:

1. A finite quiver $Q^\star$,
2. A finite set R (of indices),
3. Maps $h : R \to V_{Q^\star}$ and $t : R \to V_{Q^\star}$ (heads and tails),
4. A map $A : R \to \mathbb{K}Q$ such that $A_r = A(r) \in h(r)\mathbb{K}Q^\star t(r)$,
5. A symmetric function $q : R \times R \to \mathbb{K}$ with the following properties:
   (a) $q(r, s) = 0$ unless $h(r) = t(s)$ and $h(s) = t(r)$,
   (b) q is nondegenerate in the sense that the matrix $\{q(r, s)\}_{r,s\in R}$ is invertible,
   (c) $\sum_{r,s\in R} q(r, s)A_rA_s = 0 \mod [\mathbb{K}Q^\star, \mathbb{K}Q^\star]$.

Conversely, if one starts with the above data, one can reverse the construction and produce a quiver with superpotential Q of dimension 4 as follows: Take Q to be the quiver constructed by adding to Q^* a degree -1 edge r from $t(r)$ to $h(r)$ for each element $r \in R$, a degree -2 edge e^* in the reverse direction for each edge e in $Q^\star$, and a degree -3 loop v^* for each vertex v in $Q^\star$. Let $\Phi \in \mathbb{K}Q_{\text{cyc}}$ denotes the element $\sum_{r,s\in R} q(r, s)A_rs$. Define a degree 1 map $d : \mathbb{K}Q \to \mathbb{K}Q$ by Equation (4.4.2). This map d is a differential as we have assume our data satisfy the master equation

$$\{\Phi, \Phi\} = \sum_{r,s\in R} q(r, s)A_rA_s = 0 \mod [\mathbb{K}Q^\star, \mathbb{K}Q^\star].$$

Then Q is a quiver with superpotential of dimension 4 with superpotential Φ and a super-antisymmetric pairing $(-, -)$ on the subquiver $\tilde{Q}$ containing all vertices and all edges with degree $0, -1, -2$ given by

$$\begin{aligned} (e, e^*) &= 1 \\ (r, s) &= q(r, s) \\ (e^*, e) &- 1 \end{aligned}$$

and zero otherwise. Since $H^0(\mathbb{K}Q) = \mathbb{K}Q^\star/\langle A_r : r \in R\rangle$, Theorem 4.3.7 readily yields

Theorem 4.4.2. *The bounded derived category $\mathrm{D}^b_{fd}(\mathbb{K}Q)$ with its standard t-structure is a 4-Calabi–Yau triangulated category whose heart is equivalent to the category of finite dimensional modules over the algebra $\mathbb{K}Q^\star/\langle A_r : r \in R\rangle$.*

Chapter 5

Derived Equivalences between Vector Bundles and DG-Quivers

This chapter is the heart of the thesis where we prove our main results.

Section 5.1 gives a review on equivariant sheaves.

In Section 5.2, we generalize a result by Bridgeland [15, Proposition 4.1] and show that if $\pi : V \to X$ is an anti-semiample vector bundle on a smooth projective manifold with an exceptional poset $\mathcal{E}$, then under some cohomological vanishing conditions, the total space V is derived equivalent to an algebra $\Lambda_{\mathcal{E}}$ which is the path algebra of a quiver with relations. If we remove the cohomological vanishing condition, we end up with an A_{fin}-algebra rather than a quiver with relations.

Section 5.3 tries to resolve $\Lambda_{\mathcal{E}}$, the path algebra of a quiver with relations (or more generally the A_{fin}-algebra) by a dg-quiver $Q_{\mathcal{E}}$.

Section 5.4 gives a concrete description of the underlying graded quiver of the dg-quiver $Q_{\mathcal{E}}$ in terms of the dual exceptional poset of $\mathcal{E}$.

Section 5.5 proves the existence of a superpotential on $Q_{\mathcal{E}}$ when V is noncompact Calabi–Yau.

Section 5.6 considers the G-equivariant situation and constructs from $Q_{\mathcal{E}}$ a quotient quiver $Q_{\mathcal{E}}/G$, generalizing the construction of the McKay quiver, which is derived equivalent to $\mathrm{D}^b(\mathrm{Coh}^G(V))$. In the case when V is equivariantly Calabi–Yau, $Q_{\mathcal{E}}/G$ is also equipped with a superpotential.

Section 5.7 proves the product construction. We start with two dg-quivers $Q_{\mathcal{E}}$ and $Q_{\mathcal{F}}$ derived equivalent to vector bundles V and W respectively, and construct a product quiver $Q_{\mathcal{E}} \times Q_{\mathcal{F}}$ which is derived equivalent to $V \times W$. When both V and W are Calabi–Yau, we show that the product quiver $Q_{\mathcal{E}} \times Q_{\mathcal{F}}$ is also equipped with a superpotential.

5.1 Equivariant Sheaves

In this section, we define G-equivariant sheaves. We will fix a finite group G acting on a smooth variety X of finite type over $\mathbb{K}$ by automorphisms, with $|G| \nmid \mathrm{char}(\mathbb{K})$. We will denote the multiplication map on G by $\mu : G \times G \to G$ and the group action map by $\sigma : G \times X \to X$. We will also need the projection map $\pi_X : G \times X \to X$ and the map $\pi_{23} : G \times G \times X \to G \times X$ projecting onto the last two factors.

Definition 5.1.1 (G-linearization)**.** A *G-linearization* of a quasi-coherent sheaf E on X is an isomorphism $\lambda^E : \pi_X^* E \to \sigma^* E$ satisfying the cocycle condition

$$(\mu \times \mathrm{id}_X)^* \lambda = \pi_{23}^* \lambda \circ (\mathrm{id}_G \times \sigma)^* \lambda.$$

This amounts to the following data: For each $g \in G$, there is an isomorphism $\lambda_g : E \to g^* E$, satisfying the cocycle condition $\lambda_{hg}^E = g^*(\lambda_h^E) \circ \lambda_g^E$.

Definition 5.1.2 (Equivariant sheaves)**.** A *G-equivariant quasi-coherent sheaf* (resp. *coherent sheaf*) on X is a quasi-coherent sheaf (resp. coherent sheaf) E on X together with a G-linearization λ^E. If (E, λ^E), (F, λ^F) are G-equivariant sheaves, then G acts on $\mathrm{Hom}_X(E, F)$ by $g \cdot \phi = (\lambda^F_{g^{-1}})^{-1} \circ (g^{-1})^* \phi \circ \lambda^E_{g^{-1}}$. The Hom space in the categories $\mathrm{QCoh}^G(X)$ and $\mathrm{Coh}^G(X)$ are given by the G-equivariant maps, i.e., $\mathrm{Hom}_G(E, F) = [\mathrm{Hom}_X(E, F)]^G$. In other words, a morphism $\phi : E \to F$ is equivariant if for all $g \in G$ we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\lambda^E_g"] & F \arrow[d, "\lambda^F_g"] \\ g^*E \arrow[r, "g^*\phi"] & g^*F. \end{tikzcd}$$

By Grothendieck [30, Proposition 5.1.2], $\mathrm{QCoh}^G(X)$ has enough injectives. Hence we can resolve any equivariant sheaf by equivariant injective resolution. Moreover, since X is projective and G is finite, one can find a G-invariant ample line bundle on X, and hence every G-equivariant sheaf has a G-equivariant locally free resolution. There is a forgetful functor $U : \mathrm{QCoh}^G(X) \to \mathrm{QCoh}(X)$ which sends an equivariant sheaf (E, λ^E) to its underlying sheaf E and a equivariant morphism f to itself, now regarded as a morphism between sheaves. Clearly, the forgetful functor also restricts to a functor $U : \mathrm{Coh}^G(X) \to \mathrm{Coh}(X)$.

Next, we would like to review how to derive the Hom_G functor.

Proposition 5.1.3. *The functor* $[-]^G : G\text{-Mod} \to \mathbb{K}\text{-Mod}$ *taking G-invariant part is exact.*

Proof. Let $0 \to U \to V \to W$ be an exact sequence of G-representations. By Maschke's theorem, we have a decomposition

$$0 \to U^G \oplus U' \to V^G \oplus V' \to W^G \oplus W' \to 0$$

where each summand is a subrepresentation. Thus taking G-invariant part yields an exact sequence. ∎

Corollary 5.1.4. *For any* $(E, \lambda^E), (F, \lambda^F) \in \mathrm{D}(\mathrm{QCoh}^G(X))$*, we have*

$$\mathrm{RHom}_G((E, \lambda^E), (F, \lambda^F)) = \mathrm{RHom}(E, F)^G.$$

Proof. Write $\mathrm{Hom}_G = [-]^G \circ \mathrm{Hom} \circ U$. The corollary follows from exactness of $[-]^G$. ∎

Should there be no confusion, from now on we will, by abusing notation, denote an equivariant sheaf (E, λ^E) by only its underlying sheaf E.

5.2 Tilting Objects on Equivariant Vector Bundles

In this section, we will be working over the following setting: Let G be a finite group and $\mathbb{K}$ be a field with $\mathrm{ord}(G) \nmid \mathrm{char}(\mathbb{K})$. Let X be a smooth projective variety over $\mathbb{K}$ with G acting by automorphisms. Let $\pi : V \to X$ be a G-equivariant vector bundle. Unless otherwise stated, throughout this section, $\mathcal{E} = \{E_i\}_{i\in I}$ will denote a full exceptional poset on $\mathrm{D}^b(\mathrm{Coh}^G(X))$. We will also write $E = \bigoplus_{i\in I} E_i$.

Lemma 5.2.1.
1. *The map π^* is exact.*
2. *The map π_* is exact.*
3. *The functor $\pi_*\pi^*$ is naturally isomorphic to $- \otimes S^\bullet V^\vee$.*
4. *The functor $\pi_*\pi^*$ preserves injectives.*

Proof. Statement 1 follows from flatness of π. For statement 2, by Hartshorne [31, §Ex. II.5.17] and that π is affine, the map π_* defines an equivalence between the category of quasi-coherent $\mathcal{O}_V$-modules and the category of quasi-coherent $\pi_*\mathcal{O}_V$-modules, which is a subcategory of quasi-coherent $\mathcal{O}_X$-modules. Thus π_* is exact. Statement 3 follows from projection formula: For any $\mathcal{O}_X$-modules M, we have functorial isomorphisms

$$\pi_*\pi^*M = \pi_*(\pi^*M \otimes \mathcal{O}_V) = M \otimes \pi_*\mathcal{O}_V = M \otimes S^\bullet V^\vee.$$

Statement 4 follows from statement 3 since for any injective I and any $\mathcal{O}_X$-modules M,

$$\operatorname{Ext}^i(M, \pi_*\pi^*I) = \operatorname{Ext}^i(M, I \otimes S^\bullet V^\vee) = \operatorname{Ext}^i(M \otimes S^\bullet V, I) = 0$$

whenever $i \neq 0$. ■

Bridgeland pointed out in [15, Prop. 4.1] (see also [16, Thm. 3.6]) that one can construct tilting objects on vector bundles by pulling back tilting objects on the base variety.

Lemma 5.2.2. *If T is a tilting object in $\mathrm{D}(\mathrm{QCoh}^G(X))$, then π^*T is a tilting object in $\mathrm{D}(\mathrm{QCoh}^G(V))$.*

Proof. By the adjunction $\pi^* \dashv \pi_*$, we have $\operatorname{Hom}^\bullet_V(\pi^*T, F) = \operatorname{Hom}^\bullet_X(T, \pi_*F)$ for all objects F. In particular, for an arbitrary coproduct $\bigoplus_{i\in I} F_i$,

$$\operatorname{Hom}^\bullet_V(\pi^*T, \bigoplus_{i\in I} F_i) = \operatorname{Hom}^\bullet_X(T, \pi_* \bigoplus_{i\in I} F_i) = \bigoplus_{i\in I} \operatorname{Hom}^\bullet_X(T, \pi_* F_i) = \bigoplus_{i\in I} \operatorname{Hom}^\bullet_V(\pi^*T, F_i)$$

and hence π^*T is compact. To show π^*T is generating, suppose F has the property $\operatorname{Hom}^\bullet_V(\pi^*T, F) = 0$. Then $\operatorname{Hom}^\bullet_X(T, \pi_*F) = 0$ and hence $\pi_*F = 0$. Since π_* is exact, it preserves cohomologies and this implies F has no cohomologies, i.e., $F = 0$ as desired. ■

Lemma 5.2.3. *Suppose a finite set of objects $\mathcal{E} = \{E_i\}_{i\in I}$ in $\mathrm{D}^b(\mathrm{Coh}^G(X))$ classically generates $\mathrm{D}^b(\mathrm{Coh}^G(X))$, then the finite set of objects $\pi^*\mathcal{E} = \{\pi^*E_i\}_{i\in I}$ classically generates $\mathrm{D}^b(\mathrm{Coh}^G(V))$.*

Proof. By Example 2.5.2, $\mathcal{E}$ are compact objects in $D(\mathrm{QCoh}^G(X))$. By Corollary 2.5.7, $\mathcal{E}$ classically generates $\mathrm{D}^b(\mathrm{Coh}^G(X))$ if and only if $\mathcal{E}$ generates $\mathrm{D}(\mathrm{QCoh}^G(X))$. Thus $E = \bigoplus_{i\in I} E_i$ is a tilting object in $\mathrm{D}(\mathrm{QCoh}^G(X))$. By Lemma 5.2.2, π^*E is a tilting object in $\mathrm{D}(\mathrm{QCoh}^G(V))$. By Corollary 2.5.7 again, the set $\pi^*\mathcal{E}$ classically generates $\mathrm{D}^b(\mathrm{Coh}^G(V))$. ■

The following theorem is a generalization of Bridgeland and Stern [16, Thm. 3.6] and is essentially an application of Theorem 2.9.2.

Theorem 5.2.4. *Let X be a smooth projective variety together with an action by a finite group G. Let $\pi : V \to X$ be a G-equivariant vector bundle and $\mathcal{E} = \{E_i\}_{i\in I}$ be a full exceptional poset on $\mathrm{D}^b(\mathrm{Coh}^G(X))$. Write $E = \bigoplus_{i\in I} E_i$. Then there is a dg-algebra R whose underlying chain complex represents $\operatorname{RHom}_G(E, \pi_*\pi^*E) = \operatorname{RHom}_G(E, E \otimes S^\bullet V^\vee)$ and an equivalence*

$$\Psi : \mathrm{D}^b(\mathrm{Coh}^G(V)) \to \operatorname{Per}(R^{\mathrm{op}})$$

which after composing with the forgetful functor $U : \operatorname{Per}(R^{\mathrm{op}}) \to \mathrm{D}(\mathbb{K})$ yields $\operatorname{RHom}_G(E, \pi_*(-))$, i.e., $U \circ \Psi = \operatorname{RHom}_G(E, \pi_*(-))$.

Proof. For each $i \in I$, choose a G-equivariant injective resolution I_{E_i} for E_i. Write $I_E = \bigoplus_{i\in I} I_{E_i}$. Define a dg-algebra $R = \mathcal{H}om_G(I_E, I_E \otimes S^\bullet V^\vee)$ as follows: As a chain complex, the i-th graded piece is given by

$$R^i = \mathcal{H}om^i_G(I_E, I_E \otimes S^\bullet V^\vee) = \bigoplus_{p\in\mathbb{Z}} \operatorname{Hom}_G(I^p_E, I^{i+p}_E \otimes S^\bullet V^\vee),$$

with differential

$$d_R(f) = (d_{I_E} \otimes \mathrm{id}_{S^\bullet V^\vee}) \circ f - (-1)^i f \circ d_{I_E}.$$

The multiplication map on R is given by composing morphisms followed by symmetric product on the $S^\bullet V^\vee$ factor, i.e.,

$$R^i \otimes R^j \to R^{i+j}$$

$$f \otimes g \mapsto (\mathrm{id}_{I_E} \otimes m)(g \otimes \mathrm{id}_{S^\bullet V^\vee}) \circ f,$$

where the map $m : S^\bullet V^\vee \otimes S^\bullet V^\vee \to S^\bullet V^\vee$ is the symmetric product.

The underlying chain complex of R represents $\mathrm{RHom}(E, E \otimes S^\bullet V^\vee)$: Since $- \otimes S^\bullet V^\vee$ is exact, $\mathrm{Hom}(E, - \otimes S^\bullet V^\vee)$ is a left exact functor. Hence the right derived functor is given by

$$\mathrm{RHom}_G(E, E \otimes S^\bullet V^\vee) = \mathrm{RHom}_G(I_E, E \otimes S^\bullet V^\vee) = \mathcal{H}om_G(I_E, I_E \otimes S^\bullet V^\vee)$$

since I_E is an injective resolution of E.

Next, we define a functor

$$\Psi : \mathrm{D}^b(\mathrm{Coh}^G(V)) \to \mathrm{D}(R^{\mathrm{op}}\text{-Mod}).$$

Recall that by the natural isomorphism $\pi_*\pi^* \cong - \otimes S^\bullet V^\vee$ in Lemma 5.2.1, we have an isomorphism $R = \mathcal{H}om(I_E, \pi_*\pi^* I_E)$ as a dg-algebra. For any $M \in \mathrm{D}^b(\mathrm{Coh}^G(V))$, choose an injective resolution I_M. The chain complex $\mathcal{H}om_G(I_E, \pi_* I_M)$ has a right dg-module structure over R given by

$$\mathcal{H}om_G(I_E, \pi_* I_M) \otimes \mathcal{H}om_G(I_E, \pi_*\pi^* I_E) \to \mathcal{H}om_G(I_E, \pi_* I_M)$$

$$(g, f) \mapsto \left(I_E \xrightarrow{f} \pi_*\pi^* I_E \xrightarrow{\pi_*\pi^* g} \pi_*\pi^*\pi_* I_M \cong \pi_*\mathcal{O}_V \otimes \pi_* I_M \xrightarrow{\pi_*\mu_M} \pi_* I_M \right),$$

where $\mu_M : \mathcal{O}_V \otimes I_M \to I_M$ is the $\mathcal{O}_V$-module structure map on I_M.

We then define

$$\Psi(M) = \mathcal{H}om_G(I_E, \pi_* I_M)$$

$$\Psi(M \xrightarrow{f} N) \mapsto (\mathcal{H}om_G(I_E, \pi_* I_M) \xrightarrow{\pi_* I_f} \mathcal{H}om_G(I_E, \pi_* I_N)).$$

This functor is exact since it is the composition of the following four exact functors:

$$I : \mathrm{D}^b(\mathrm{Coh}^G(V)) \to \mathrm{K}(\mathrm{Coh}^G(V)), \quad M \mapsto I_M$$

$$\pi_* : \mathrm{D}^b(\mathrm{Coh}^G(V)) \to \mathrm{D}^b(\mathrm{Coh}^G(X))$$

$$\mathcal{H}om_G(I_E, -) : \mathrm{K}(\mathrm{Coh}^G(X)) \to \mathrm{K}(R^{\mathrm{op}}\text{-Mod})$$

and the natural projection functor

$$Q : \mathrm{K}(R^{\mathrm{op}}\text{-Mod}) \to \mathrm{D}(R^{\mathrm{op}}\text{-Mod}).$$

By the same reason as R represents $\mathrm{RHom}(E, E \otimes S^\bullet V^\vee)$, we see that $U \circ \Psi = \mathrm{RHom}(E, \pi_*(-))$. Next, we show Ψ is fully faithful. Recall that by Lemma 5.2.1, $\pi_*\pi^*$ preserves injectives. For any $E_i, E_j \in \mathcal{E}$,

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}^G(V))}(\pi^* E_i, \pi^* E_j) \\
&= \mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}^G(X))}(E_i, \pi_*\pi^* E_j) \\
&= \mathrm{Hom}_{\mathrm{K}(\mathrm{Coh}^G(X))}(I_{E_i}, \pi_*\pi^* I_{E_j}) \\
&= H^0(\mathcal{H}om_G(I_{E_i}, \pi_*\pi^* I_{E_j})) \\
&= \mathrm{Hom}_R(\mathcal{H}om_G(I_E, \pi_*\pi^* I_{E_i}), \mathcal{H}om_G(I_E, \pi_*\pi^* I_{E_j}))/\{\text{homotopy equivalence}\} \\
&= \mathrm{Hom}_{\mathrm{K}(R^{\mathrm{op}}\text{-Mod})}(\mathcal{H}om_G(I_E, \pi_*\pi^* I_{E_i}), \mathcal{H}om_G(I_E, \pi_*\pi^* I_{E_j})) \\
&= \mathrm{Hom}_{\mathrm{D}(R^{\mathrm{op}}\text{-Mod})}(\Psi(\pi^* E_i), \Psi(\pi^* E_j))
\end{aligned}$$

By Lemma 5.2.3, the set $\{\pi^* E_i\}_{i \in I}$ classically generates $\mathrm{D}^b(\mathrm{Coh}^G(V))$. Hence Ψ is fully faithful. The essential image is the triangulated category classically generated by $\Psi(\pi^* E_i)$. Since each $\Psi(\pi^* E_i)$ is a direct summand of $\Psi(\pi^* E) = R$, the essential image of Ψ is the triangulated category classically generated by R, i.e., $\mathrm{Per}(R^{\mathrm{op}})$. Hence we have our desired equivalence $\Psi : \mathrm{D}^b(\mathrm{Coh}^G(V)) \to \mathrm{Per}(R^{\mathrm{op}})$. ■

By homology perturbation, we have

Corollary 5.2.5. *Let $\pi : V \to X$ be a G-equivariant vector bundle and $\mathcal{E}$ a full exceptional collection on $\mathrm{D}^b(\mathrm{Coh}^G(V))$, then there is a minimal A_{fin}-algebra*

$$\Lambda_{\mathcal{E}} = \mathrm{Hom}^\bullet_G(E, E \otimes S^\bullet V^\vee)$$

and an equivalence

$$\Phi : \mathrm{D}^b(\mathrm{Coh}^G(V)) \to \mathrm{Per}_\infty(\Lambda^{\mathrm{op}}_{\mathcal{E}}).$$

Proof. Since the underlying chain complex of R represents $\mathrm{RHom}(E, E \otimes S^\bullet V^\vee)$, its cohomology is given by

$$\Lambda_{\mathcal{E}} = \mathrm{Hom}^\bullet_G(E, E \otimes S^\bullet V^\vee).$$

By Theorem 3.2.1, there is a minimal A_∞-structure on $\Lambda_{\mathcal{E}}$ making it quasi-isomorphic to R as an A_∞-algebra. Lemma 5.3.4 below shows that this A_∞-stucture is indeed A_{fin}. Since $\mathrm{D}(R^{\mathrm{op}}) \cong \mathrm{D}_\infty(R^{\mathrm{op}})$, and this equivalence restricts to an equivalence between perfect derived categories, we have an equivalence $\mathrm{D}^b(\mathrm{Coh}^G(V)) \cong \mathrm{Per}(R^{\mathrm{op}}) \cong \mathrm{Per}_\infty(\Lambda_{\mathcal{E}}^{\mathrm{op}})$ as desired. ■

Definition 5.2.6. Let $\pi : V \to X$ be a G-equivariant vector bundle. A full exceptional poset $\mathcal{E} = \{E_i\}_{i\in I}$ on $\mathrm{D}^b(\mathrm{Coh}^G(X))$ is said to be if $\mathrm{Hom}^k_G(E_i, E_j \otimes S^\bullet V^\vee) = 0$ for all $k \neq 0$ and all $i, j \in I$.

Remark 5.2.7. Note that a V-geometric exceptional poset is necessarily strong.

Remark 5.2.8. When G is trivial and $V = K_X$ is the canonical bundle, a K_X-geometric full exceptional collection is the same as a geometric helix defined in Bridgeland and Stern [16]. This is where our terminology comes from.

Corollary 5.2.9. *Let $\pi : V \to X$ be a G-equivariant vector bundle which is anti-semiample i.e., $S^kV^\vee$ is globally generated for $k \gg 0$. Let $\mathcal{E}$ be a full V-geometric exceptional poset on $\mathrm{D}^b(\mathrm{Coh}^G(V))$. Then $\Lambda_{\mathcal{E}}$ is an ordinary algebra which is Noetherian and there is an equivalence*

$$\Phi : \mathrm{D}^b(\mathrm{Coh}^G(V)) \to \mathrm{D}^b_{fg}(\Lambda_{\mathcal{E}}^{\mathrm{op}}),$$

where $\mathrm{D}^b_{fg}(\Lambda_{\mathcal{E}}^{\mathrm{op}})$ is the full triangulated subcategory of $\mathrm{D}^b(\Lambda_{\mathcal{E}}^{\mathrm{op}})$ consisting of complexes whose cohomologies are finitely generated $\Lambda_{\mathcal{E}}^{\mathrm{op}}$-modules.

Proof. Since $\mathcal{E}$ is V-geometric, $\Lambda_{\mathcal{E}}$ is an ordinary algebra. It remains to show $\Lambda_{\mathcal{E}}$ is Noetherian and of finite global dimension, for then $\mathrm{Per}(\Lambda_{\mathcal{E}}^{\mathrm{op}}) = \mathrm{D}^b(\Lambda_{\mathcal{E}}^{\mathrm{op}}\text{-mod}) = \mathrm{D}^b_{fg}(\Lambda_{\mathcal{E}}^{\mathrm{op}})$. Since $S^kV^\vee$ is globally generated for $k \gg 0$, we have an exact sequence

$$\mathcal{O} \otimes H^0(X, S^kV^\vee) \to S^kV^\vee \to 0$$

Taking dual, we see that $S^kV \hookrightarrow \mathcal{O} \otimes H^0(X, S^kV^\vee)^\vee$ embeds S^kV into a trivial bundle as a subbundle. Now choose, $k, \ell \gg 0$ and coprime. Then we have an embedding

$$V \hookrightarrow S^kV \oplus S^\ell V \hookrightarrow \mathcal{O} \otimes \big(H^0(X, S^kV^\vee)^\vee \oplus H^0(X, S^\ell V^\vee)^\vee\big)$$

of V into a trivial bundle. Composing this embedding with the map

$$\mathcal{O} \otimes \big(H^0(X, S^kV^\vee)^\vee \oplus H^0(X, S^\ell V^\vee)^\vee\big) \to H^0(X, S^kV^\vee)^\vee \oplus H^0(X, S^\ell V^\vee)^\vee$$

which projects the trivial bundle to its fiber yields a projective morphism

$$V \to H^0(X, S^kV^\vee)^\vee \oplus H^0(X, S^\ell V^\vee)^\vee.$$

Thus V is projective over an affine scheme of finite type. The algebra $\Lambda_{\mathcal{E}} \cong \mathrm{End}_V(\pi^*E)$ is then a finitely generated module over a finitely generated algebra, thus itself Noetherian. By Hille and van den Bergh [32, Thm 7.6], $\Lambda_{\mathcal{E}}$ has finite global dimension as desired. ■

Corollary 5.2.10. *The above equivalence restricts to an equivalence*

$$\Phi : \mathrm{D}^b_{cs}(\mathrm{Coh}^G(V)) \to \mathrm{D}^b_{fd}(\Lambda_{\mathcal{E}}^{\mathrm{op}}).$$

Proof. If $M \in \mathrm{D}^b_{cs}(\mathrm{Coh}^G(V))$, then $\mathrm{Hom}^\bullet(\pi^*E, M)$ is finite dimensional. As

$$H^\bullet(\Phi(M)) = \mathrm{Hom}^\bullet(\Lambda_{\mathcal{E}}, \Phi(M)) = \mathrm{Hom}^\bullet(\pi^*E, M),$$

we conclude $\Phi(M) \in \mathrm{D}^b_{fd}(\Lambda_{\mathcal{E}}^{\mathrm{op}})$. Now suppose $\Phi(M) \in \mathrm{D}^b_{fd}(\Lambda_{\mathcal{E}}^{\mathrm{op}})$. Then by assumption, $\mathrm{Hom}^\bullet(\pi^*E, M)$ is finite dimensional. Since M lies in the thick category generated by π^*E, we conclude that $\mathrm{Hom}^\bullet(M, M)$ is also finite dimensional. Via the multiplication map

$$S^\bullet H^0(X, S^k V^\vee) \otimes S^\bullet H^0(X, S^\ell V^\vee) \to \bigoplus_{r \geq 0} H^0(X, S^r V^\vee) = H^0(V, \mathcal{O}_V),$$

$\mathrm{Hom}^\bullet(M, M)$ becomes a finite dimensional module over $S^\bullet H^0(X, S^k V^\vee) \otimes S^\bullet H^0(X, S^\ell V^\vee)$, and thus it is supported in finitely many points $\{p_1, \ldots, p_r\}$ on $H^0(X, S^k V^\vee)^\vee \oplus H^0(X, S^\ell V^\vee)^\vee$. Recall that we have a projective morphism $\phi : V \to H^0(X, S^k V^\vee)^\vee \oplus H^0(X, S^\ell V^\vee)^\vee$. We would like to show $\operatorname{supp} M \subseteq \phi^{-1}(\{p_1, \ldots, p_r\})$. If this is true, $\operatorname{supp} M$ would be compact as ϕ is a proper morphism. Let $q \notin \phi^{-1}(\{p_1, \ldots, p_r\})$ be a $\mathbb{K}$-point on the vector bundle V. Then $\phi(q) \notin \operatorname{supp} \mathrm{Hom}^\bullet(M, M)$. Since $\operatorname{supp} \mathrm{Hom}^\bullet(M, M)$ is defined by the annihilator ideal of $\mathrm{Hom}^\bullet(M, M)$, this means there exists $f \in S^\bullet H^0(X, S^k V^\vee) \otimes S^\bullet H^0(X, S^\ell V^\vee)$ with $f \cdot \mathrm{id}_M = 0$ but $f(\phi(q)) \neq 0$. In particular, we see that the identity map on M localizes to the zero map at the point q. Hence M must be supported in $\phi^{-1}(\{p_1, \ldots, p_r\})$ as desired. ■

Tilting algebra as quiver with relations. As pointed out by Bridgeland and Stern [16], in the case when $\Lambda_{\mathcal{E}}$ is an ordinary algebra, one can construct a quiver with relations (Q, I) whose path algebra is isomorphic to $\Lambda_{\mathcal{E}}$. The vertex set of Q corresponds to the full exceptional poset $\mathcal{E}$, while the edges between two vertices v and w corresponding to exceptional object $E_v, E_w \in \mathcal{E}$ is given by a basis of the cokernel of the map

$$\bigoplus_{\substack{E_i \in \mathcal{E}: \\ E_v \preccurlyeq E_i \preccurlyeq E_w}} \mathrm{Hom}(E_v, E_i \otimes S^\bullet V^\vee) \otimes \mathrm{Hom}(E_i, E_w \otimes S^\bullet V^\vee) \to \mathrm{Hom}(E_v, E_w \otimes S^\bullet V^\vee).$$

There is then a natural surjective map $\varphi : \mathbb{K}Q \to \Lambda_{\mathcal{E}}$ and we can take our desired quiver with relations to be $(Q, \ker \varphi)$.

The following examples are applications of Corollary 5.2.9.

Example 5.2.11. Let $X = \operatorname{Spec} \mathbb{K}$ and $V = \mathbb{K}^n$. Then $\mathcal{E} = \mathcal{O}_X$ is a V-geometric exceptional collection on $\mathrm{D}^b(\mathrm{Coh}\,(\operatorname{Spec} \mathbb{K}))$. The classical tilting algebra is given by

$$\Lambda_{\mathcal{E}} = \mathrm{Hom}^\bullet(\mathcal{O}, \mathcal{O} \otimes S^\bullet V^\vee) = S^\bullet V^\vee = \mathbb{K}[x_1, \ldots, x_n]$$

and the classical tilting quiver is given by one vertex with n loops x_i, with relations given by $x_i x_j = x_j x_i$.

Example 5.2.12. If we take V to be the zero vector bundle, we recover the tilting quiver for X. In the case $X = \mathbb{P}^n$, with the full exceptional collection $(\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n))$, we have the Beilinson quiver

$$\underset{v_0}{\bullet} \;\underset{a_{1n}}{\overset{a_{10}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}}}}\; \underset{v_1}{\bullet} \;\underset{a_{2n}}{\overset{a_{20}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}}}}\; \underset{v_2}{\bullet} \cdots\cdots\cdots \underset{v_{n-1}}{\bullet} \;\underset{a_{nn}}{\overset{a_{n0}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}}}}\; \underset{v_n}{\bullet}$$

with relations $a_{i+1\,j} a_{ik} = a_{i+1\,k} a_{i\,j}$. The vertex v_i corresponds to the bundle $\mathcal{O}(i)$, and the arrows a_{ij} correspond to $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \cong H^0(\mathbb{P}^n, \mathcal{O}(1))$, i.e, the homogeneous coordinates on $\mathbb{P}^n$. This also explain the relations $a_{i+1\,j+1} a_{ij} = a_{i+1\,j} a_{i\,j+1}$: they correspond to the relation $Z_j Z_k = Z_k Z_j$ in the homogeneous coordinate ring.

More examples will be given in Section 6.1.

5.3 Quasi-free Resolution of Tilting Algebra

In this section, we will resolve the tilting A_∞-algebra $\Lambda_{\mathcal{E}}$ by an augmented quasi-free dg-algebra.

To simplify notations, we will drop the subscript $\mathcal{E}$ and write $\Lambda = \Lambda_{\mathcal{E}}$. The idea is to use Koszul duality, which is not new and has been explored by Keller [40], Lu, Palmieri, Wu and Zhang [50], Segal

[62], and van den Bergh [70]. By Koszul duality, if $E(\Lambda)$ is locally finite, then $E(E(\Lambda))$ is A_∞-quasi-isomorphic to Λ. However, since $E(\Lambda)$ is a very large dg-algebra, $E(E(\Lambda))$ cannot be a minimal dg model of Λ. To make it smaller, we take its cohomology $H^\bullet(E(\Lambda))$ and transfer the dg-structure on $E(\Lambda)$ to an A_∞-structure on $H^\bullet(E(\Lambda))$. It turns out that $H^\bullet(E(\Lambda)) = \mathrm{Ext}^\bullet_\Lambda(S,S)$ is finite dimensional and $E(\mathrm{Ext}^\bullet_\Lambda(S,S))$ would be the desired quasi-free dg-algebra resolving Λ. Making all of these rigorous involves an Adams grading on Λ. In the following, we will denote by S the semisimple algebra $\mathbb{K}^n$, where n is the length of $\mathcal{E}$. Recall that there is no loss of generality to choose a total order refining the partial order on $\mathcal{E}$, so we will assume further that $\mathcal{E}$ is in fact an exceptional sequence.

Lemma 5.3.1. *Let $\mathcal{E} = \{E_1, \dots, E_n\}$ be a full exceptional sequence on $\mathrm{D}^b(\mathrm{Coh}^G(X))$. The tilting A_∞-algebra*

$$\Lambda = \bigoplus_{k=0}^{\infty} \bigoplus_{\ell \in \mathbb{Z}} \bigoplus_{a=-n}^{n} \bigoplus_{j-i=a} \mathrm{Hom}^\ell_G(E_i, E_j \otimes S^k V^\vee)$$

is Adams graded by $(a,k) \in \mathbb{Z} \times \mathbb{Z}$, and is a locally finite augmented algebra over $\Lambda_{0,0} = S$. The Adams grading of the augmentation ideal $\bar{\Lambda}$ is supported in

$$J = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : -(n-1) \le a \le n-1 \text{ and } b \ge 1\} \cup \{(a,0) \in \mathbb{Z} \times \mathbb{Z} : 1 \le a \le n-1\}.$$

Proof. The tilting algebra Λ is locally finite since $\Lambda^\ell_{a,k} = \bigoplus_{j-i=a} \mathrm{Hom}^\ell_G(E_i, E_j \otimes S^k V^\vee)$ is finite dimensional. The rest follows from the fact that $(E_0, \dots, E_n)$ is an exceptional sequence. ■

A *partition* of $j \in J$ is a way of writing j as a finite sum of elements in J. Denote by $p(j)$ the number of such partitions.

Lemma 5.3.2. *The number $p(j)$ is finite.*

Proof. A general partition of $j = (a,b) \in J$ is in the form

$$(a,b) = (a_1,b_1) + \cdots + (a_k,b_k) + (c_1,0) + \cdots + (c_\ell,0),$$

where $-n \le a_i \le n$, $b_i \ge 1$ and $c_i \ge 1$. Rewriting, we have

$$\begin{aligned} b &= b_1 + \cdots + b_k \\ a - a_1 - \cdots - a_k &= c_1 + \cdots + c_\ell \end{aligned}$$

which are both partitions of natural numbers. Since $0 \le a - a_1 - \cdots - a_k \le a + (n-1)k \le a + (n-1)b$, we conclude

$$p(j) = p(a,b) \le (\text{number of partitions of } b)(\text{number of partitions of } a + (n-1)b).$$

■

Lemma 5.3.3. *Let A and A' be augmented Adams graded A_∞-algebras Adams graded by $\mathbb{Z} \times \mathbb{Z}$. Suppose their augmentation ideals are both supported in J. Then*

1. *A and A' are both A_{fin}.*
2. *Any Adams-graded A_∞-morphism $f : A \to A'$ is A_{fin}.*
3. *Any Adams-graded A_∞-homotopy $h : A \to A'$ between A_∞-morphisms $f, g : A \to A'$ is A_{fin}.*
4. *The Adams-graded minimal model $H^\bullet(A)$ is A_{fin} and is A_{fin}-homotopic to A.*

Proof. For statement 1, since m_r preserves the Adams degree, for any fixed $j \in J$,

$$m_r : A_{j_1}^{\ell_1} \otimes_S \cdots \otimes_S A_{j_r}^{\ell_r} \to A_j^{\ell}$$

can be nonzero only if $j = j_1 + \cdots + j_r$, with $j, j_1, \ldots, j_n \in J$. By Lemma 5.3.2, there can be only a finite number of partitions of j. Hence the j-th component of m_r vanishes for $r \gg 0$, i.e., A is A_{fin}. The other second and third statement are proven similarly.

For the last statement, observe that the Adams grading on A induce an Adams grading on $H^\bullet(A)$, and recall that by choosing a spliting respecting the Adams grading in Section 3.2, one can get an A_∞-structures on $H^\bullet(A)$ preserving the induced Adams grading , and which is A_∞-quasi-isomorphic to A. By Theorem 3.1.3, $H^\bullet(A)$ is A_∞-homotopic to A. By statement 3, this A_∞-homotopy is A_{fin} and we are done. ■

Lemma 5.3.4. *The A_∞-structure on Λ is in an A_{fin}-structure, i.e., $m_r = 0$ for $r \gg 0$.*

Proof. This is a special case of Lemma 5.3.3. ■

The Adams grading on Λ induces an Adams grading on its Koszul dual $E(\Lambda)$, and the combinatorial Lemma 5.3.2 is crucial in showing the following lemma describing the Adams grading on $E(\Lambda)$.

Lemma 5.3.5. *1. The dg-algebra $E(\Lambda)$ is a $\mathbb{Z} \times \mathbb{Z}$-graded locally finite algebra augmented over S. The $\mathbb{Z} \times \mathbb{Z}$-grading of its augmentation ideal is supported in J.*

2. The same also holds for the dg-algebra $E(E(\Lambda))$.

Proof. The j-th component of the augmentation ideal of $E(\Lambda)$ is

$$\begin{aligned}\bigoplus_m [((\Sigma\bar{\Lambda})^{\#})^{\otimes_S^m}]_j^{\ell} &= \bigoplus_m \bigoplus_{\substack{\ell_1+\cdots+\ell_m=\ell \\ j_1+\cdots+j_m=j \\ j_i \in J}} [(\Sigma\bar{\Lambda})^{\#}]_{j_1}^{\ell_1} \otimes_S \cdots \otimes_S [(\Sigma\bar{\Lambda})^{\#}]_{j_m}^{\ell_m} \\ &= \bigoplus_m \bigoplus_{\substack{\ell_1+\cdots+\ell_m=\ell \\ j_1+\cdots+j_m=j \\ j_i \in J}} \left((\Sigma\bar{\Lambda})_{j_1}^{\ell_1} \otimes_S \cdots \otimes_S (\Sigma\bar{\Lambda})_{j_m}^{\ell_m}\right)^{\#}.\end{aligned}$$

If $j \in J$, the direct sum on the right hand side is a finite sum since the number of partitions of $j \in J$ is finite. Since Λ is locally finite, each $\bar{\Lambda}_{j_i}$ is finite dimensional and hence the j-th component of the augmentation ideal of $E(\Lambda)$ is also finite dimensional.

If $j \notin J$, we would like to show the j-th component vanishes, which is equivalent to the vanishing of each vector space $\bar{\Lambda}_{j_1} \otimes_S \cdots \otimes_S \bar{\Lambda}_{j_m}$ whenever $j_1 + \cdots + j_m \notin J$. We show the case when $m = 2$, i.e., $\bar{\Lambda}_j \otimes_S \bar{\Lambda}_k = 0$ whenever $j + k \notin J$. The proof for general m is similar. Write $j = (j', j'')$ and $k = (k', k'')$. Note that $j + k \notin J$ is equivalent to $j' + k' \notin [-(n-1), n-1]$, where n is the length of the exceptional sequence $\mathcal{E}$. Now

$$\begin{aligned}\bar{\Lambda}_j \otimes_S \bar{\Lambda}_k &= \bigoplus_{\substack{0 \le i, i+j' \le n \\ 0 \le \ell, \ell+k' \le n}} \operatorname{Hom}(E_i, E_{i+j'} \otimes S^{j''} V^\vee) \otimes_S \operatorname{Hom}(E_\ell, E_{\ell+k'} \otimes S^{k''} V^\vee) \\ &= \bigoplus_{0 \le i, i+j', i+j'+k' \le n} \operatorname{Hom}(E_i, E_{i+j'} \otimes S^{j''} V^\vee) \otimes_S \operatorname{Hom}(E_{i+j'}, E_{i+j'+k'} \otimes S^{k''} V^\vee).\end{aligned}$$

If $j' + k' \ge n$, then $i + j' + k' \ge n$ for all $i = 1, \ldots, n$. If $j' + k' \le -n$, then $i + j' + k' < 0$ for all $i = 1, \ldots, n$. Hence $\bar{\Lambda}_j \otimes_S \bar{\Lambda}_k = 0$. In particular, the support of the augmentation ideal lies in J and we have proven the first statement.

Now, in proving the first statement, we have only used the fact that Λ is locally finite with Adams grading of its augmented ideal supported in J. Since this is also true for $E(\Lambda)$, the same also holds for $E(E(\Lambda))$. ■

Lemma 5.3.6 ([52] Lemma 11.1)**.** $H^\bullet(E(\Lambda)) = \mathrm{Ext}^\bullet_\Lambda(S,S)$.

Proof. By Lemma 5.3.5 and Koszul duality (Theorem 3.4.3), $E(E(\Lambda)) = \Omega B\Lambda$ is a dg-algebra which is A_∞-quasi-isomorphic to Λ. By Lemma 5.3.3, they are A_{fin}-homotopic. Since the Koszul functor E sends A_{fin}-homotopies to dg-homotopies, $E(\Omega B\Lambda)$ is dg-homotopic to $E(\Lambda)$ and hence $H^\bullet(E(\Omega B\Lambda)) \cong H^\bullet(E(\Lambda))$. By [48, Lemma 2.3.4.4 and 2.4.2.3], we have an equivalence $\mathrm{D}(\Omega B\Lambda) \cong \mathrm{D}_\infty(\Lambda)$ which sends $S \mapsto S[-1]$. Hence $\mathrm{Ext}^\bullet_\Lambda(S,S) = \mathrm{Ext}^\bullet_{\Omega BA}(S,S)$. Thus, passing to $\Omega B\Lambda$ if necessary, we may assume Λ is a dg-algebra. Since $E(\Lambda)$ is locally finite, $E(\Lambda) = (B\Lambda)^\#$. Now, by [24, Prop. 19.2], $B(\Lambda) \otimes_S \Lambda$ is a semi-free resolution of the right Λ-module S. Then

$$\mathrm{RHom}(S,S) = \mathrm{Hom}_\Lambda(B\Lambda \otimes_S \Lambda, S) = \mathrm{Hom}_S(B\Lambda, S) = (B\Lambda)^\# = E(A).$$

Taking cohomologies on both sides, we are done. ∎

Lemma 5.3.7. *Any A_∞-structure on the Yoneda algebra $\mathrm{Ext}^\bullet_\Lambda(S,S)$ preserving the Adams grading is A_{fin}. In particular, the A_{fin}-structure obtained on $\mathrm{Ext}^\bullet_\Lambda(S,S)$ by taking minimal model of $E(\Lambda)$ is turns $\mathrm{Ext}^\bullet_\Lambda(S,S)$ into an augmented A_{fin}-algebra over S which is A_{fin}-homotopic equivalent to $E(\Lambda)$.*

Proof. This is a consequence of Lemma 5.3.3 and 5.3.6. ∎

Henceforth the Yoneda algebra will always be equipped with this natural A_{fin}-structure coming from taking minimal model of $E(\Lambda)$.

Proposition 5.3.8. *1. There is a A_{fin}-quasi-isomorphism $E(\mathrm{Ext}^\bullet_\Lambda(S,S)) \cong \Lambda$.*

2. If Λ is a classical algebra, then there is a quasi-isomorphism

$$E(\mathrm{Ext}^\bullet_\Lambda(S,S)) \to H^0(E(\mathrm{Ext}^\bullet_\Lambda(S,S))) \cong \Lambda,$$

i.e., $E(\mathrm{Ext}^\bullet_\Lambda(S,S))$ is a quasi-free dg-algebra resolving Λ.

Proof. By Lemma 5.3.7, $\mathrm{Ext}^\bullet_\Lambda(S,S)$ is A_{fin}-homotopic equivalent to $E(\Lambda)$. Since the Koszul functor E sends A_{fin}-homotopies to dg-homotopies, $E(\mathrm{Ext}^\bullet_\Lambda(S,S))$ is dg-homotopic to $E(E(\Lambda))$. Now $E(E(\Lambda))$ is A_∞-quasi-isomorphic to Λ by Koszul duality (Theorem 3.4.3), and hence A_{fin}-quasi-isomorphic to Λ by Lemma 5.3.3 and 5.3.5. This proves the first statement.

If Λ is a classical algebra, by the same argument, $E(\mathrm{Ext}^\bullet_\Lambda(S,S))$ is dg-homotopic to $E(E(\Lambda))$. By Koszul duality of dg-algebras (Theorem 3.4.3), there is a quasi-isomorphism of dg-algebras $E(E(\Lambda)) \to \Lambda$. Hence there is a quasi-isomorphism as desired. ∎

5.4 Computing $\mathrm{Ext}^\bullet_{\Lambda_\mathcal{E}}(S,S)$

In this section, we compute $\mathrm{Ext}^\bullet_{\Lambda_\mathcal{E}}(S,S)$ in terms of the dual exceptional sequence to $\mathcal{E}$ and the vector bundle V.

When $\Lambda_\mathcal{E}$ is an algebra rather than an A_∞-algebra, which happens for example when $\mathcal{E}$ is V-geometric, we can compute $\mathrm{Ext}^\bullet_{\Lambda_\mathcal{E}}(S,S)$ in terms of the dual exceptional sequence $\mathcal{F}$ of $\mathcal{E}$. Among all modules over $\Lambda_\mathcal{E}$, the following projective modules and simple modules are of utmost importance for us as they correspond to objects in $\mathcal{E}$ and $\mathcal{F}$. Let $P_i = \mathrm{Hom}_G(E, E_i \otimes S^\bullet V^\vee)$. Then P_i is a right module over $\Lambda_\mathcal{E}$ which is a direct summand of $\Lambda_\mathcal{E}$. Let $S_i = \mathrm{Hom}_G(E_i, E_i \otimes S^0 V^\vee) = \mathrm{Hom}_G(E_i, E_i) = \mathbb{K}$. Then S_i is a simple right module over $\Lambda_\mathcal{E}$. When we write $\Lambda_\mathcal{E}$ in the form of a quiver with relations (Q, I), then P_i corresponds to the vector space of all paths starting at the vertex i, and S_i corresponds to the 1-dimensional vector space sitting at the vertex i. We will denote $S = \bigoplus_{i\in I} S_i$. Then $\Lambda_\mathcal{E}$ is an A_{fin}-algebra augmented over S.

The following proposition describes the preimage of the modules P_i and S_i under the isomorphism $\Phi : \mathrm{D}^b(\mathrm{Coh}^G(V)) \to \mathrm{Per}(\Lambda_\mathcal{E}^{\mathrm{op}})$.

Proposition 5.4.1 ([16], Lemma 3.7). *Let $\pi : V \to X$ be a G-equivariant vector bundle. Let $\mathcal{E} = \{E_i\}_{i\in I}$ be a full V-geometric exceptional poset on $\mathrm{D}^b(\mathrm{Coh}^G(X))$ and $\mathcal{F} = \{F_i\}_{i\in I^{\mathrm{op}}}$ be the dual exceptional poset of $\mathcal{E}$. The equivalence $\Phi : \mathrm{D}^b(\mathrm{Coh}^G(V)) \to \mathrm{Per}(\Lambda_{\mathcal{E}}^{\mathrm{op}})$ sends*

$$\Phi(\pi^* E_i) = P_i \quad \text{and} \quad \Phi(s_* F_i) = S_i,$$

where $s : X \to V$ is the zero section.

Proof. The equivalence $\Psi : \mathrm{D}^b(\mathrm{Coh}^G(V)) \to \mathrm{Per}(R^{\mathrm{op}})$ in Theorem 5.2.4 sends

$$\Psi(E_i) = \mathcal{H}om(I_E, I_{E_i} \otimes S^\bullet V^\vee).$$

Now

$$H^\bullet(\mathcal{H}om(I_E, I_{E_i} \otimes S^\bullet V^\vee)) = \mathrm{Hom}^\bullet(E, E_i \otimes S^\bullet V^\vee).$$

Hence $\Phi(\pi^* E_i) = P_i$.

Next, we choose an injective resolution I_{F_j} for F_j. Then $s_* I_{F_j}$ is an injective resolution for $s_* F_j$ and hence $\Psi(s_* F_j) = \mathcal{H}om(I_E, \pi_* s_* I_{F_j}) = \mathcal{H}om(I_E, I_{F_j})$. Taking cohomology, we conclude $\Phi(s_* F_j) = \mathrm{Hom}^\bullet(E, F_j) = S_j$ as vector spaces. To show they are isomorphic as $\Lambda_{\mathcal{E}}$-modules, it suffices to show that $\mathrm{Hom}(E, E \otimes S^k V^\vee)$ acts trivially on $\Phi(s_* F_j)$ whenever $k \geq 1$. The action of R on $\Psi(s_* F_j)$ factors through the map $S^\bullet V^\vee \otimes I_{F_j} \to I_{F_j}$ which in turns comes from the pushforward under π of the $\mathcal{O}_V$-module structure on $s_* F_j$. This shows that the action of $S^k V^\vee$ on I_{F_j}, and hence the action of $\mathcal{H}om(I_E, I_E \otimes S^k V^\vee)$ on $\mathcal{H}om(I_E, I_{F_j})$ is trivial for $k \geq 1$. Taking cohomology, we see that $\mathrm{Hom}(E, E \otimes S^k V^\vee)$ also acts trivially on $\Phi(s_* F_j)$ whenever $k \geq 1$, as desired. ■

Remark 5.4.2. This proposition is the only place where we need $\Lambda_{\mathcal{E}}$ to be an algebra rather than an A_∞-algebra in order to calculate $\mathrm{Ext}^\bullet_{\Lambda_{\mathcal{E}}}(S, S)$ in terms of the dual exceptional collection of $\mathcal{E}$. In the general A_∞-algebra case, we do not know if the higher multiplication maps acts trivially, and hence we do not know if $\Phi(s_* F_j) = S_j$.

Proposition 5.4.3. *Let $s : X \to V$ be the zero section. Then for all $E \in \mathrm{D}(\mathrm{QCoh}\,(X))$,*

$$\mathrm{L}s^* s_* E = \bigoplus_{k=0}^{\mathrm{rank}\, V} \bigwedge^k V^\vee[k] \otimes E = S^\bullet(V[-1])^\vee \otimes E.$$

Proof. Consider X as a subvariety of V via the zero section. Then X is given by the zero locus of the tautological section σ of the tautological vector bundle $\pi^* V$ on V. The sheaf $s_* \mathcal{O}_X$ can be resolved by the Koszul complex

$$0 \to \bigwedge^{\mathrm{rank}\, V} (\pi^* V)^\vee \to \cdots \to (\pi^* V)^\vee \to \mathcal{O}_V \to s_* \mathcal{O}_X \to 0,$$

where the maps are given by contraction with the section σ. In other words, we have a quasi-isomorphism $\bigwedge^\bullet (\pi^* V)^\vee \cong s_* \mathcal{O}_X$. Since $\bigwedge^\bullet (\pi^* V)^\vee$ is locally free, its derived pullback under s is the same as the underived pullback, i.e., restriction to X. Since the maps in the complex $\bigwedge^\bullet (\pi^* V)^\vee$ is given by contraction with σ, they restrict to zero on X. This yields

$$s^* s_* \mathcal{O}_X \cong s^*(\bigwedge^\bullet (\pi^* V)^\vee) = \bigwedge^\bullet (\pi^* V)^\vee|_X = \bigoplus_{k=0}^{\mathrm{rank}\, V} \bigwedge^k V^\vee[k].$$

The general case follows from

$$\begin{aligned} s^* s_* E &= \pi_* s_* s^* s_* E && (\pi s = \mathrm{id}_X) \\ &= \pi_*(s_* \mathcal{O}_X \otimes s_* E) && \text{(Projection formula)} \\ &= \pi_* s_*(s^* s_* \mathcal{O}_X \otimes E) && \text{(Projection formula)} \\ &= \bigoplus_{k=0}^{\mathrm{rank}\, V} \bigwedge^k V^\vee[k] \otimes E \\ &= S^\bullet(V[-1])^\vee \otimes E. \end{aligned}$$

■

Corollary 5.4.4.

$$\operatorname{Hom}^\bullet_G(s_*E, s_*F) = \bigoplus_{\ell\in\mathbb{Z}} \bigoplus_{k=0}^{\operatorname{rank} V} \operatorname{Hom}^\ell_G(\wedge^k V^\vee[k] \otimes E, F) = \bigoplus_{k=0}^{\operatorname{rank} V} \operatorname{Hom}^\bullet_G(E, F \otimes S^\bullet(V[-1]))$$

Proof. The first equality is by the adjunction $s^* \dashv s_*$, the second by Proposition 5.4.3, the third by locally-freeness of V. ■

We therefore arrived at the following

Theorem 5.4.5. *Let X be a smooth G-variety and $\pi : V \to X$ be a G-equivariant anti-semiample vector bundle. Suppose $\mathcal{E} = \{E_i\}_{i\in I}$ is a full V-geometric exceptional poset on $\mathrm{D}^b(\operatorname{Coh}^G(X))$ with dual exceptional poset $\mathcal{F} = \{F_i\}_{i\in I^{\mathrm{op}}}$. Then there is an A_{fin}-structure on*

$$A_\mathcal{E} = \bigoplus_{i,j\in I} \bigoplus_{\ell\in\mathbb{Z}} \bigoplus_{k=0}^{\operatorname{rank} V} \operatorname{Hom}^{\ell-k}_G(F_i, F_j \otimes \wedge^k V),$$

making it a finite dimensional A_{fin}-algebra augmented over S, such that $E(A_\mathcal{E})$ is a dg-algebra resolving $\Lambda_\mathcal{E}$ and hence $\mathrm{D}^b(\operatorname{Coh}^G(V)) \cong \mathrm{D}^b_{fg}(E(A_\mathcal{E})^{\mathrm{op}})$, where $\mathrm{D}^b_{fg}(E(A_\mathcal{E})^{\mathrm{op}})$ denotes the full triangulated subcategory of $\mathrm{D}^b(E(A_\mathcal{E})^{\mathrm{op}})$ consisting of complexes whose cohomologies are finitely generated over $H^\bullet(E(\mathcal{A}_\mathcal{E})^{\mathrm{op}})$. Moreover, this equivalence restricts to an equivalence $\mathrm{D}^b_{cs}(\operatorname{Coh}^G(V)) \cong \mathrm{D}^b_{fd}(E(A_\mathcal{E})^{\mathrm{op}})$.

Proof. Theorem 5.2.9 gives an A_{fin}-algebra $\Lambda_\mathcal{E}$ and a derived equivalence $\mathrm{D}^b(\operatorname{Coh}^G(V)) \cong \mathrm{D}^b_{fg}(\Lambda_\mathcal{E}^{\mathrm{op}})$. Proposition 5.3.7 and 5.3.8 shows there is an A_{fin}-structure on $H^\bullet(E(\Lambda_\mathcal{E})) = \operatorname{Ext}^\bullet_\Lambda(S, S)$ such that $E(\operatorname{Ext}^\bullet_{\Lambda_\mathcal{E}}(S, S))$ is A_{fin}-isomorphic to $\Lambda_\mathcal{E}$. By Proposition 5.4.1 and 5.4.4,

$$\operatorname{Ext}^\bullet_{\Lambda_\mathcal{E}}(S, S) \cong \bigoplus_{i,j} \operatorname{Hom}^\bullet_G(s_*F_i, s_*F_j) \cong A_\mathcal{E}$$

as graded S-bimodules. Transferring the A_{fin}-structure on $\operatorname{Ext}^\bullet_{\Lambda_\mathcal{E}}(S, S)$ to $A_\mathcal{E}$, we have derived equivalences

$$\mathrm{D}^b(\operatorname{Coh}^G(V)) \cong \mathrm{D}^b_{fg}(\Lambda^{\mathrm{op}}_\mathcal{E}) \cong \mathrm{D}^b_{fg}(E(\operatorname{Ext}^\bullet_{\Lambda_\mathcal{E}}(S, S))^{\mathrm{op}}) \cong \mathrm{D}^b_{fg}(E(A_\mathcal{E})^{\mathrm{op}}).$$

The fact that this equivalence restrict to an equivalence $\mathrm{D}^b_{cs}(\operatorname{Coh}^G(V)) \cong \mathrm{D}^b_{fd}(E(A_\mathcal{E})^{\mathrm{op}})$ follows from Corollary 5.2.10. ■

Remark 5.4.6. One can also transfer the Adams grading on $\operatorname{Ext}^\bullet_{\Lambda_\mathcal{E}}(S, S)$ to $A_\mathcal{E}$. A natural guess for this Adams grading on $A_\mathcal{E}$ would be given by $(j - i, k)$. Since we will not need this explicit description, we will not prove the statement.

Recall $E(A_\mathcal{E})$ is the path algebra of a dg-quiver $Q_\mathcal{E}$. Hence we have shown that V is derived equivalent to a dg-quiver.

Corollary 5.4.7. *Let X be a smooth G-variety and $\pi : V \to X$ be a G-equivariant vector bundle. Suppose $\mathcal{E} = \{E_i\}_{i\in I}$ is a full V-geometric exceptional poset on $\mathrm{D}^b(\operatorname{Coh}^G(X))$, then there exists a dg-quiver $Q_\mathcal{E}$ such that $\mathrm{D}^b(\operatorname{Coh}^G(V)) \cong \mathrm{D}^b_{fg}(Q_\mathcal{E})$ and $\mathrm{D}^b_{cs}(\operatorname{Coh}^G(V)) \cong \mathrm{D}^b_{fd}(Q_\mathcal{E})$.*

Example 5.4.8. Let $X = \mathbb{P}^2$ and $V = 0$. Let $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ be the Beilinson sequence. The classical tilting quiver is given by

$$\underset{v_0}{\bullet} \overset{a_0}{\underset{a_1}{\xrightarrow{\;a_1\;}}} \underset{v_1}{\bullet} \overset{b_0}{\underset{b_2}{\xrightarrow{\;b_1\;}}} \underset{v_2}{\bullet} \tag{5.4.1}$$

with relations $b_i a_j = b_j a_i$. Using the cohomology formula in the Appendix (Section A.2), the A_∞-category $A_\mathcal{E}$ is given by

$$\mathcal{A}^\ell(v_0, v_0) = \operatorname{Hom}^\ell(\Omega^2(2)[2], \Omega^2(2)[2]) = H^\ell(\mathbb{P}^2, \mathcal{O}) = \begin{cases} \mathbb{C} & \text{if } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{A}^\ell(v_0, v_1) = \mathrm{Hom}^\ell(\Omega^2(2)[2], \Omega(1)[1]) = H^{\ell-1}(\mathbb{P}^2, \Omega(2)) = \begin{cases} \mathbb{C}^3 & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{A}^\ell(v_0, v_2) = \mathrm{Hom}^\ell(\Omega^2(2)[2], \mathcal{O}) = H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(1)) = \begin{cases} \mathbb{C}^3 & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{A}^\ell(v_1, v_0) = \mathrm{Hom}^\ell(\Omega(1)[1], \Omega^2(2)[2]) = H^{\ell+1}(\mathbb{P}^2, \mathcal{T}(-2)) = 0$$

$$\mathcal{A}^\ell(v_1, v_1) = \mathrm{Hom}^\ell(\Omega(1)[1], \Omega(1)[1]) = H^{\ell}(\mathbb{P}^2, \mathcal{T} \otimes \Omega) = \begin{cases} \mathbb{C} & \text{if } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{A}^\ell(v_1, v_2) = \mathrm{Hom}^\ell(\Omega(1)[1], \mathcal{O}) = H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-1)) = \begin{cases} \mathbb{C}^3 & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{A}^\ell(v_2, v_0) = \mathrm{Hom}^\ell(\mathcal{O}, \Omega^2(2)[2]) = H^{\ell+2}(\mathbb{P}^2, \mathcal{O}(-1)) = 0.$$

$$\mathcal{A}^\ell(v_2, v_1) = \mathrm{Hom}^\ell(\mathcal{O}, \Omega(1)[1]) = H^{\ell+1}(\mathbb{P}^2, \Omega(1)) = 0.$$

$$\mathcal{A}^\ell(v_2, v_2) = \mathrm{Hom}^\ell(\mathcal{O}, \mathcal{O}) = H^{\ell}(\mathbb{P}^2, \mathcal{O}) = \begin{cases} \mathbb{C} & \text{if } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the dg-quiver is given by

$$\begin{tikzcd} \underset{v_0}{\bullet} \arrow[r, Rrightarrow, "a"'] \arrow[rr, Rrightarrow, bend left, red, "p"] & \underset{v_1}{\bullet} \arrow[r, Rrightarrow, "b"'] & \underset{v_2}{\bullet} \end{tikzcd}$$

where black edges are of degree 0 and red edges are of degree -1. Next, we would like to determine the differential. Since $H^0(\mathbb{K}Q)$ is the classical tilting quiver in Equation (5.4.1), by a change of basis if necessary, we may assume the differential sends

$$dp_i = b_{i+1}a_{i+2} - b_{i+2}a_{i+1}, \quad da_i = 0, \quad db_i = 0.$$

More examples will be given in Section 6.3.

5.5 Superpotential

In this section, we construct a superpotential on the dg-tilting quiver $Q_{\mathcal{E}}$ in the case when the total space of V is Calabi–Yau.

Serre Functor on Equivariant Derived Category We describe how to obtain equivariant Serre duality on a equivariant vector bundle V, which the author learnt from Bridgeland-King-Reid [14]. Since X is projective and G is finite, one can find a G-invariant ample line bundle $\mathcal{O}_X(1)$ on X. Consider the embedding $i : V \hookrightarrow \mathbb{P}(V \oplus \mathcal{O}_X(1))$. Then i_* embeds $\mathrm{D}^b_{cs}(\mathrm{Coh}^G(V))$ into $\mathrm{D}^b_{cs}(\mathrm{Coh}^G\mathbb{P}(V \oplus \mathcal{O}_X(1)))$ as a full subcategory. The Serre functor $(-)\otimes K_{\mathbb{P}(V\oplus\mathcal{O}_X(1))}[\dim V]$ on $\mathrm{D}^b_{cs}(\mathrm{Coh}^G\mathbb{P}(V\oplus\mathcal{O}_X(1)))$ then restricts to a Serre functor $(-) \otimes K_V[\dim V]$ on $\mathrm{D}^b_{cs}(\mathrm{Coh}^G(V))$. In particular, if K_V is trivial as a G-equivariant vector bundle, $\mathrm{D}^b_{cs}(\mathrm{Coh}^G(V))$ becomes a Calabi–Yau category.

Proposition 5.5.1. *Let $\pi : V \to X$ be a G-equivariant vector bundle. Then K_V is trivial as an equivariant vector bundle if and only if $\det V = K_X$ as an equivariant vector bundle. In particular, when G is trivial, V is (non-compact) Calabi–Yau if and only if $\det V = K_X$.*

Proof. We have a short exact sequence

$$0 \to \pi^*V \to TV \to \pi^*TX \to 0.$$

We have therefore

$$\det TV = \det \pi^*V \otimes \det \pi^*TX = \pi^*(\det V \otimes K_X^\vee).$$

If $\det V = K_X$, then $\det TV = \pi^*\mathcal{O}_X$ which is trivial. If V is Calabi-Yau, we have

$$\mathcal{O}_X = \mathcal{O}_V|_X = \pi^*(\det V \otimes K_X^\vee)|_X = \det V \otimes K_X^\vee.$$

Hence $\det V = K_X$. ∎

Pairing on $A_\mathcal{E}$. If V is noncompact Calabi–Yau, there is a pairing on $A_\mathcal{E}$ as follows: consider the composition map

$$\mathrm{Hom}^{\ell-k}(E, F \otimes \wedge^k V) \otimes \mathrm{Hom}^{\dim X+k-\ell}(F, E \otimes \wedge^{\mathrm{rank}\, V-k} V) \to \mathrm{Hom}^{\dim X}(E, E \otimes \det V)$$

$$f \otimes g \mapsto (\mathrm{id} \otimes \wedge) \circ (g \otimes \mathrm{id}_{\wedge^k V}) \circ f$$

Since V is Calabi–Yau, $\det V = K_X$. Composing with the trace map of Serre duality

$$\mathrm{Hom}^{\dim X}(E, E \otimes K_X) \to \mathbb{K},$$

we obtain a pairing on $A_\mathcal{E}$ of degree $\dim V$

$$A_\mathcal{E}^\ell \otimes A_\mathcal{E}^{\dim V-\ell} \to \mathbb{K}.$$

One might ask whether the A_{fin}-structure on $A_\mathcal{E}$ comes from a cyclic structure. The following theorem answers the question:

Proposition 5.5.2. *Let $\pi : V \to X$ be an anti-semiample vector bundle which is m-Calabi–Yau. Let $\mathcal{E}$ be a V-geometric exceptional poset on $\mathrm{D}^b(\mathrm{Coh}\,(X))$. Then $A_\mathcal{E}$ has a cyclic structure of degree m and hence $E(A)$ has a superpotential, i.e., V is derived equivariant to a quiver with superpotential.*

Proof. The tilting object $\bigoplus \pi^* E_i$ is classical since

$$\mathrm{Hom}^\ell(\pi^*E, \pi^*E) = \bigoplus_{i,j\in I} \mathrm{Hom}^\ell(\pi^*E_i, \pi^*E_j) = \mathrm{Hom}^\bullet(E_i, E_j \otimes S^\bullet V^\vee) = 0$$

By [45, Prop. 3.3.1], $\Lambda_\mathcal{E}$ is an Calabi–Yau algebra. Now, since $\Lambda_\mathcal{E}$ is graded, by [70, Corollary 9.3, Theorem 12.1], $A_\mathcal{E}$ is equipped with a cyclic structure. ∎

Examples of quivers with superpotential constructed this way will be given in Section 6.4.

5.6 Quotient Construction

In the last section, we have shown that if $V \to X$ is an anti-semiample vector bundle over a smooth projective variety, and $\mathcal{E}$ is a V-geometric exceptional poset on $\mathrm{D}^b(\mathrm{Coh}\,(X))$, then there exists a dg-quiver $Q_\mathcal{E}$ such that $\mathrm{D}^b(\mathrm{Coh}\,(X)) \cong \mathrm{D}^b_{fg}(Q_\mathcal{E})$. In this section we study the following problem: Suppose there is a finite group G acting on X by automorphisms, and the group action lifts to an action on the vector bundle V and the exceptional poset $\mathcal{E}$. We ask if we can construct a new quiver $Q_\mathcal{E}/G$ such that $\mathrm{D}^b(\mathrm{Coh}^G(V)) \cong \mathrm{D}^b_{fg}(Q_\mathcal{E}/G)$. Theorem 5.6.5 answers this affirmatively: recall the $Q_\mathcal{E}$ is the Koszul dual of an A_{fin}-algebra $A_\mathcal{E}$. The correct construction for $Q_\mathcal{E}/G$ is to apply the quotient construction of A_∞-algebras to $A_\mathcal{E}$ and take its Koszul dual, i.e., $Q_\mathcal{E}/G = E(A_\mathcal{E}/G)$. This quotient construction generalizes the McKay quiver.

Proposition 5.6.1. *Let $\mathcal{E} = \{E_i\}_{i\in I}$ be a finite poset of objects on $\mathrm{D}^b(\mathrm{Coh}^G(X))$ whose underlying objects form a full exceptional poset on $\mathrm{D}^b(\mathrm{Coh}\,(X))$. Then $\mathcal{E} \otimes \mathrm{Irr}\,(G) = \{E_i \otimes \rho\}_{i\in I, \rho\in \mathrm{Irr}\,(G)}$, with the partial order $E_i \otimes \rho \prec E_j \otimes \sigma$ if and only if $i \prec j$, is a full exceptional poset on $\mathrm{D}^b(\mathrm{Coh}^G(X))$. Moreover, if $\mathcal{E}$ is strong, then so is $\mathcal{E} \otimes \mathrm{Irr}\,(G)$.*

Proof. From

$$\mathrm{Hom}^k_G(E_i \otimes \rho, E_j \otimes \sigma) = [\mathrm{Hom}^k_X(E_i \otimes \rho, E_j \otimes \sigma)]^G = [\mathrm{Hom}^k_X(E_i, E_j) \otimes \mathrm{Hom}(\rho, \sigma)]^G,$$

and Schur's lemma, we see that each object $E_i \otimes \rho$ is exceptional, and that $\mathrm{Hom}^\bullet(E_i \otimes \rho, E_j \otimes \sigma)$ vanishes unless $i \preccurlyeq j$, hence $\mathcal{E} \otimes \mathrm{Irr}\,(G)$ is an exceptional poset. Also, if $\mathcal{E}$ is strong, then so is $\mathcal{E} \otimes \mathrm{Irr}\,(G)$. It remains to show $\mathcal{E} \otimes \mathrm{Irr}\,(G)$ classically generates $\mathrm{D}^b(\mathrm{Coh}^G(X))$ if $\mathcal{E}$ classically generates $\mathrm{D}^b(\mathrm{Coh}^G(X))$. Let $\mathcal{F} \in \mathrm{D}(\mathrm{QCoh}\,(X))$ be such that $\mathrm{Hom}^\bullet_G(E_i \otimes \rho, \mathcal{F}) = 0$ for all $i \in I$ and $\rho \in \mathrm{Irr}\,(G)$. Then $\mathrm{Hom}_G(\rho, \mathrm{Hom}^\bullet(E_i, \mathcal{F})) = [\rho^* \otimes \mathrm{Hom}^\bullet(E_i, \mathcal{F})]^G = \mathrm{Hom}^\bullet_G(E_i \otimes \rho, \mathcal{F}) = 0$. Now, decomposing $\mathrm{Hom}^\bullet(E_i, \mathcal{F})$ into a direct sum of irreducible representations of G, we see that $\mathrm{Hom}^\bullet(E_i, \mathcal{F}) = 0$ for all $i \in I$. Since $\mathcal{E}$ is full, this shows $\mathcal{F} = 0$. Hence $\mathcal{E} \otimes \mathrm{Irr}\,(G)$ generates $\mathrm{D}^b(\mathrm{Coh}^G(X))$. ∎

Proposition 5.6.2. *Let $\mathcal{F}$ be the dual sequence to $\mathcal{E}$ in $\mathrm{D}^b(\mathrm{Coh}\,(X))$. Then $\mathcal{F}$ has a natural lift to $\mathrm{D}^b(\mathrm{Coh}^G(X))$ and $\mathcal{F} \otimes \mathrm{Irr}\,(G)$ is the dual sequence to $\mathcal{E} \otimes \mathrm{Irr}\,(G)$ in $\mathrm{D}^b(\mathrm{Coh}^G(X))$.*

Proof. Without loss of generality, we may assume $\mathcal{E}$ is an exceptional sequence. Recall that by Proposition 2.8.5, $F_i = L_{\mathcal{E}_{<i}} E_i = L_{E_1} \cdots L_{E_{i-1}} E_i$. Hence to show F_i has a natural lift to $\mathrm{D}^b(\mathrm{Coh}^G(X))$, it suffices to show that if any $E, X \in \mathrm{D}^b(\mathrm{Coh}\,(X))$ lift to $\mathrm{D}^b(\mathrm{Coh}^G(X))$, then so does $L_E X$. This follows from G-equivariance of the evaluation map $\mathrm{Hom}^\bullet(E, X) \otimes E \to X$. Since $F_i = L_{\mathcal{E}_{<i}} E_i$, we have a triangle

$$A \to E_i \to F_i \to A[1]$$

where both maps are equivariant and $A \in \langle \mathcal{E}_{<i} \rangle$. Applying $\mathrm{Hom}^\bullet(E_i, -)$, we see that $\mathrm{Hom}^\bullet(F_i, F_i) = \mathrm{Hom}^\bullet(E_i, E_i) = \mathbb{K}$ is the trivial representation. Hence

$$\begin{aligned}\mathrm{Hom}^\bullet_G(E_i \otimes \rho, F_j \otimes \sigma) &= [\mathrm{Hom}^\bullet(E_i, F_j) \otimes \mathrm{Hom}(\rho, \sigma)]^G \\ &= \begin{cases} \mathrm{Hom}_G(\rho, \sigma) & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \mathbb{K} & \text{if } i = j \text{ and } \rho = \sigma \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

By uniqueness of dual exceptional sequence in Proposition 2.8.5, $\mathcal{F} \otimes \mathrm{Irr}\,(G)$ is the dual exceptional sequence to $\mathcal{E} \otimes \mathrm{Irr}\,(G)$. ∎

Corollary 5.6.3. *Let $\mathcal{E}$ be a finite poset of objects in $\mathrm{D}^b(\mathrm{Coh}^G(V))$. Suppose the underlying objects of $\mathcal{E}$ in $\mathrm{D}^b(\mathrm{Coh}\,(V))$ form a V-geometric full exceptional poset on $\mathrm{D}^b(\mathrm{Coh}\,(V))$, then $\mathcal{E} \otimes \mathrm{Irr}\,(G)$ is a full V-geometric exceptional poset on $\mathrm{D}^b(\mathrm{Coh}^G(V))$.*

Proof. It suffices to show $\mathrm{Hom}^\ell_G(E_i \otimes \rho, E_j \otimes \sigma \otimes S^k V^*)$ vanishes for all $\ell \geq 1$, $k \geq 0$ and all $\rho, \sigma \in \mathrm{Irr}\,(G)$. This follows immediately since

$$\mathrm{Hom}^\ell_G(E_i \otimes \rho, E_j \otimes \sigma \otimes S^k V^*) = [\mathrm{Hom}(\rho, \sigma) \otimes \mathrm{Hom}^\ell_X(E_i, E_j \otimes S^k V^*)]^G.$$

∎

Proposition 5.6.4. *Let $\pi : V \to X$ be a G-equivariant vector bundle and $\mathcal{E}$ be a finite poset of objects in $\mathrm{D}^b(\mathrm{Coh}^G(X))$ such that the underlying objects in $\mathrm{D}^b(\mathrm{Coh}\,(X))$ form a full exceptional poset. Then there exists an A_{fin}-structure on $\Lambda_{\mathcal{E}}$ such that*

1. *there is an equivalence $\Phi : \mathrm{D}^b(\mathrm{Coh}\,(V)) \cong \mathrm{Per}_\infty(\Lambda_{\mathcal{E}}^{\mathrm{op}})$.*
2. *If $M, N \in \mathrm{D}^b(\mathrm{Coh}\,(V))$ lift to $\mathrm{D}^b(\mathrm{Coh}^G(V))$, then G acts on $\Phi(M)$ and $\Phi(N)$ and the isomorphism $\Phi : \mathrm{Hom}(M, N) \cong \mathrm{Hom}(\Phi(M), \Phi(N))$ is equivariant.*
3. *The finite group G acts on $\Lambda_{\mathcal{E}}$ by strict A_∞-isomorphisms, and there is a derived equivalence $\mathrm{D}^b(\mathrm{Coh}^G(V)) \cong \mathrm{Per}_\infty((\Lambda_{\mathcal{E}}/G)^{\mathrm{op}})$.*

4. If $\mathcal{E}$ is V-geometric and $V^\vee$ is semiample, there is an equivalence $\mathrm{D}^b(\mathrm{Coh}^G(V)) \cong \mathrm{D}^b_{fg}((\Lambda_\mathcal{E}/G)^{\mathrm{op}})$*. Moreover, this equivalence restricts to an equivalence* $\mathrm{D}^b_{cs}(\mathrm{Coh}^G(V)) \cong \mathrm{D}^b_{fd}((\Lambda_\mathcal{E}/G)^{\mathrm{op}})$.

Proof. Statement 1 is the content of Proposition 5.2.4. For statement 2, recall that in Theorem 5.2.4, we constructed a dg-algebra $R = \mathcal{H}om(I_E, I_E \otimes S^\bullet V^\vee)$, I_E is a G-equivariant injective resolution for E, and an equivalence $\Psi : \mathrm{D}^b(\mathrm{Coh}(V)) \cong \mathrm{Per}(R^{\mathrm{op}})$ by setting

$$\Psi(M) = \mathcal{H}om(I_E, \pi_* I_M)$$

$$\Psi(M \xrightarrow{\phi} N) = (\mathcal{H}om(I_E, \pi_* I_M) \xrightarrow{\pi_* I_\phi} \mathcal{H}om(I_E, \pi_* I_N)),$$

where I_M and I_N are G-equivariant injective resolution for M and N respectively. Since I_E and I_M are both equipped with a G-linearization, $\Psi(M) = \mathcal{H}om(I_E, \pi_* I_M)$ is equipped with a G-action. We will choose all maps equivariantly: for example, we will choose the lifting $I_{\lambda_g^M}$ to be $\lambda_g^{I_M}$ etc. For any $\phi \in \mathrm{Hom}(M, N)$,

$$\Psi(g \cdot (M \xrightarrow{\phi} N)) = \Psi(M \xrightarrow{\lambda^M_{g^{-1}}} (g^{-1})^* M \xrightarrow{(g^{-1})^*\phi} (g^{-1})^* N \xrightarrow{(\lambda^N_{g^{-1}})^{-1}} N)$$

sends the map $h \in \mathcal{H}om(I_E, \pi_* I_M)$ to the morphism in $\mathcal{H}om(I_E, \pi_* I_N)$ given by

$$I_E \xrightarrow{h} \pi_* I_M \xrightarrow{\pi_* I_{\lambda^M_{g^{-1}}}} \pi_* I_{(g^{-1})^* M} \xrightarrow{\pi_* I_{(g^{-1})^*\phi}} \pi_* I_{(g^{-1})^* N} \xrightarrow{\pi_* I_{(\lambda^N_{g^{-1}})^{-1}}} \pi_* I_N.$$

On the other hand, $g \cdot \Psi(M \xrightarrow{\phi} N)$ sends $h \in \mathcal{H}om(I_E, \pi_* I_M)$ to the morphism in $\mathcal{H}om(I_E, \pi_* I_N)$ given by

$$\begin{aligned}
& g \cdot (I_E \xrightarrow{g^{-1}\cdot h} \pi_* I_M \xrightarrow{\pi_* I_\phi} \pi_* I_N) = g \cdot (I_E \xrightarrow{I_{\lambda^E_g}} g^* I_E \xrightarrow{g^* h} g^* \pi_* I_M \xrightarrow{I_{\pi_*(\lambda^M_g)^{-1}}} \pi_* I_M \xrightarrow{\pi_* I_\phi} \pi_* I_N) \\
&= (I_E \xrightarrow{I_{\lambda^E_{g^{-1}}}} (g^{-1})^* I_E \xrightarrow{(g^{-1})^* I_{\lambda^E_g}} I_E \xrightarrow{h} \pi_* I_M \xrightarrow{(g^{-1})^* I_{(\lambda^M_g)^{-1}}} (g^{-1})^* \pi_* I_M \\
&\qquad \xrightarrow{(g^{-1})^* \pi_* I_\phi} (g^{-1})^* \pi_* I_N \xrightarrow{I_{\pi_*(\lambda^N_g)^{-1}}} \pi_* I_N) \\
&= (I_E \xrightarrow{h} \pi_* I_M \xrightarrow{(g^{-1})^* I_{\pi_*(\lambda^M_g)^{-1}}} (g^{-1})^* \pi_* I_M \xrightarrow{(g^{-1})^* \pi_* I_\phi} (g^{-1})^* \pi_* I_N \xrightarrow{I_{\pi_*(\lambda^N_g)^{-1}}} \pi_* I_N) \\
&= (I_E \xrightarrow{h} \pi_* I_M \xrightarrow{\pi_* I_{\lambda^M_{g^{-1}}}} \pi_* I_{(g^{-1})^* M} \xrightarrow{\pi_* I_{(g^{-1})^*\phi}} \pi_* I_{(g^{-1})^* N} \xrightarrow{\pi_* I_{(\lambda^N_{g^{-1}})^{-1}}} \pi_* I_N.)
\end{aligned}$$

since everything is equivariant. Hence the isomorphism $\Psi : \mathrm{Hom}(M, N) \cong \mathrm{Hom}(\Psi(M), \Psi(N))$ is equivariant. Taking cohomology, we arrive at statement 2. For statement 3, choose G-invariant injective resolution I_{E_i} for each E_i. Then for any $\rho \in \mathrm{Irr}(G)$, $I_{E_i} \otimes \rho$ is an injective resolution for $E_i \otimes \rho$. Write $I_E = \bigoplus_{i \in I} I_{E_i}$ and let $R = \mathcal{H}om(I_E, I_E \otimes S^\bullet V^\vee)$. Then

$$R/G = \bigoplus_{\rho, \sigma \in \mathrm{Irr}(G)} \mathcal{H}om_G(I_E \otimes \rho, I_E \otimes S^\bullet V^\vee \otimes \sigma)$$

and by Theorem 5.2.4 and Corollary 5.6.3, there is an equivalence $\mathrm{D}^b(\mathrm{Coh}^G(V)) \cong \mathrm{Per}((R/G)^{\mathrm{op}})$. By Proposition 3.5.4, we can choose minimal models of R/G and R such that there are A_∞-quasi-isomorphisms

$$R/G \cong H^\bullet(R/G) \cong H^\bullet(R)/G = \Lambda_\mathcal{E}/G$$

and we have the desired equivalence. For statement 4, by Proposition 5.2.9, $\Lambda_\mathcal{E}$ is an algebra. It suffices to show $\Lambda_\mathcal{E}/G$ has finite global dimension, for then $\mathrm{Per}((\Lambda_\mathcal{E}/G)^{\mathrm{op}}) = \mathrm{D}^b((\Lambda_\mathcal{E}/G)^{\mathrm{op}})$. By Proposition 5.2.9, $\Lambda_\mathcal{E}$ has finite global dimension. By [3, III Theorem 4.4], the global dimension of $\Lambda_\mathcal{E}^{\mathrm{op}} \# G$ is the same as that of $\Lambda_\mathcal{E}^{\mathrm{op}}$, which is the same as that of $\Lambda_\mathcal{E}$, hence also finite. Now by Theorem 3.5.2 and Proposition 3.5.8, $(\Lambda_\mathcal{E}^{\mathrm{op}} \# G)^{\mathrm{op}}$ and $\Lambda_\mathcal{E}/G$ are Morita equivalent, and hence have the same global dimension. Thus $\Lambda_\mathcal{E}/G$ has finite global dimension. ∎

G-action on A_∞-category. We would like to show G acts on

$$A_{\mathcal{E}} = \bigoplus_{i,j \in I} \bigoplus_{\ell \in \mathbb{Z}} \bigoplus_{k=0}^{\operatorname{rank} V} \operatorname{Hom}_G^{\ell-k}(F_i, F_j \otimes \wedge^k V)$$

by strict A_∞-isomorphisms. Recall the A_{fin}-structure on $A_{\mathcal{E}}$ comes from $\operatorname{Ext}_\Lambda^\bullet(S,S)$. Since by Proposition 5.6.4, the isomorphism $A_{\mathcal{E}} \cong \operatorname{Hom}_\Lambda^\bullet(S,S) = \operatorname{Ext}^\bullet(S,S)$ is equivariant, it suffices to show that G acts on $\operatorname{Ext}_\Lambda^\bullet(S,S)$ by strict A_∞-isomorphisms. Now, G acts on $\Lambda_{\mathcal{E}}$ by isomorphisms. By the construction of the Koszul functor, G acts on $E(\Lambda_{\mathcal{E}})$ by dg-algebra isomorphisms. By homology perturbation Proposition 3.5.4, G acts on $H^\bullet(E(\Lambda_{\mathcal{E}}))$ by strict A_∞-isomorphisms. Since $H^\bullet(E(\Lambda_{\mathcal{E}})) \cong \operatorname{Ext}_\Lambda^\bullet(S,S)$ equivariantly, we are done.

Theorem 5.6.5. *Let X be a smooth G-variety and $\pi : V \to X$ be a G-equivariant anti-semiample vector bundle. Suppose $\mathcal{E}$ is a finite set of objects in $\mathrm{D}^b(\operatorname{Coh}^G(X))$ such that the underlying objects form a full exceptional poset on $\mathrm{D}^b(\operatorname{Coh}(X))$. Then G acts on $A_{\mathcal{E}}$ by strict A_{fin}-isomorphisms and $\mathrm{D}^b(\operatorname{Coh}^G(V)) = \mathrm{D}^b_{fg}(E(A_{\mathcal{E}}/G)^{\text{op}})$.*

Proof. By Proposition 5.6.4, $\mathrm{D}^b(\operatorname{Coh}^G(V)) \cong \mathrm{D}^b_{fg}((\Lambda_{\mathcal{E}}/G)^{\text{op}})$. Now by Corollary 3.5.9 and Proposition 3.5.4, $H^\bullet(E(A_{\mathcal{E}}/G)) = H^\bullet(E(A_{\mathcal{E}}))/G = \Lambda_{\mathcal{E}}/G$. Hence $E(A_{\mathcal{E}}/G)$ is quasi-isomorphic to $\Lambda_{\mathcal{E}}/G$ and we have $\mathrm{D}^b(\operatorname{Coh}^G(V)) = \mathrm{D}^b_{fg}(E(A_{\mathcal{E}}/G)^{\text{op}})$. ∎

Examples illustrating this section will be given in Section 6.5.

5.7 Product Construction

In this section we study the following problem: Suppose we have vector bundles $\pi_V : V \to X$ and $\pi_W : W \to Y$, together with exceptional poset $\mathcal{E}$ on X and $\mathcal{F}$ on Y which are respectively V-geometric and W-geometric. Then we can construct dg-quivers $Q_{\mathcal{E}} = E(\mathcal{A}_{\mathcal{E}})$ and $Q_{\mathcal{F}} = E(\mathcal{A}_{\mathcal{F}})$ which are derived equivalent to V and W respectively. We ask if there is a product construction of dg-quivers such that the product vector bundle $\pi_V \times \pi_W : V \times W \to X \times Y$ is derived equivalent to $Q_{\mathcal{E}} \times Q_{\mathcal{F}}$. Theorem 5.7.3 answers this question. The correct product construction should be applied at the A_∞-algebra level: We should set $Q_{\mathcal{E}} \times Q_{\mathcal{F}} = E(\mathcal{A}_{\mathcal{E}} \otimes \mathcal{A}_{\mathcal{F}})$.

Note in the following that the projection map from $X \times Y$ to both X and Y are flat, hence pullback by both maps are exact.

Proposition 5.7.1. *Let X and Y be smooth projective varieties and $\mathcal{E} = \{E_i\}_{i \in I}$ and $\mathcal{F} = \{F_j\}_{j \in J}$ be full exceptional posets on X and Y respectively. Then $\mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{F} = \{E_i \boxtimes^{\mathrm{L}} F_j\}_{I \times J}$ is a full exceptional poset on $X \times Y$, where the partial order on $I \times J$ is given by $(i,j) \preccurlyeq (i',j')$ if and only if $i \preccurlyeq i'$ and $j \preccurlyeq j'$. If both $\mathcal{E}$ and $\mathcal{F}$ are strong, then so is $\mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{F}$.*

Proof. Using the Künneth formula

$$\begin{aligned}
\operatorname{RHom}(E_i \boxtimes^{\mathrm{L}} F_j, E_{i'} \boxtimes^{\mathrm{L}} F_{j'}) &= \mathrm{R}\Gamma\,(X \times Y, \mathrm{R}\mathcal{H}om(E_i \boxtimes^{\mathrm{L}} F_j, E_{i'} \boxtimes^{\mathrm{L}} F_{j'})) \\
&= \mathrm{R}\Gamma\,(X \times Y, (E_i^\vee \otimes^{\mathrm{L}} E_{i'})^\vee \boxtimes^{\mathrm{L}} (F_j^\vee \otimes^{\mathrm{L}} E_{j'})) \\
&= \mathrm{R}\Gamma\,(X, E_i^\vee \otimes^{\mathrm{L}} E_{i'})) \otimes \mathrm{R}\Gamma\,(Y, F_j^\vee \otimes^{\mathrm{L}} F_{j'}) \\
&= \mathrm{R}\Gamma\,(X, \mathrm{R}\mathcal{H}om(E_i^\vee, E_{i'})) \otimes \mathrm{R}\Gamma\,(Y, \mathrm{R}\mathcal{H}om(F_j^\vee, F_{j'})) \\
&= \operatorname{RHom}(E_i, E_{i'}) \otimes \operatorname{RHom}(F_j, F_{j'}),
\end{aligned}$$

we have

$$\operatorname{Hom}^\ell_{X \times Y}(E_i \boxtimes^{\mathrm{L}} F_j, E_{i'} \boxtimes^{\mathrm{L}} F_{j'}) = \bigoplus_{p+q=\ell} \operatorname{Hom}^p_X(E_i, E_{i'}) \otimes \operatorname{Hom}^q_Y(F_j, F_{j'}).$$

Hence $\mathcal{E} \boxtimes^L \mathcal{F}$ is an exceptional poset. Also, if both $\mathcal{E}$ and $\mathcal{F}$ are strong, then so is $\mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{F}$. Next, we show $\mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{F}$ generates $\mathrm{D}(\operatorname{QCoh}(X))$. The following proof is given by Bondal and van den Bergh [9, Lemma

3.4.1]. Write $E = \bigoplus_{i\in I} E_i$ and $F = \bigoplus_{j\in J} F_j$. Let $A \in \mathrm{D}(\mathrm{QCoh}\,(X))$ be such that $\mathrm{Hom}^\bullet(E \boxtimes^{\mathrm{L}} F, A) = 0$. Let π_X and π_Y be projections of $X \times Y$ on the first and second factor. We have a commutative diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & \operatorname{Spec} \mathbb{K} & \end{tikzcd}$$

Since

$$0 = \mathrm{RHom}(E \boxtimes^{\mathrm{L}} F, A) = \mathrm{RHom}(\pi_X^* E, \mathrm{R}\mathcal{H}om(\pi_Y^* F, A))) = \mathrm{RHom}(E, \mathrm{R}\pi_{X*}\mathrm{R}\mathcal{H}om(\pi_Y^* F, A))$$

and E generates $\mathrm{D}(\mathrm{QCoh}\,(X))$, we have $\mathrm{R}\pi_{X*}\mathrm{R}\mathcal{H}om(\pi_Y^* F, A) = 0$. This implies that

$$0 = \mathrm{R}\Gamma\,(X, \mathrm{R}\pi_{X*}\mathrm{R}\mathcal{H}om(\pi_Y^* F, A)) = \mathrm{RHom}(\pi_Y^* F, A) = \mathrm{RHom}(F, \mathrm{R}\pi_{Y*}A)$$

and hence $\mathrm{R}\pi_{Y*}A = 0$ since F generates $\mathrm{D}(\mathrm{QCoh}\,(Y))$. Now, from

$$\mathrm{R}\Gamma\,(X \times Y, A) = \mathrm{R}\Gamma\,(Y, \mathrm{R}\pi_{Y*}A) = 0,$$

we conclude $A = 0$ and hence $\mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{F}$ generates $\mathrm{D}(\mathrm{QCoh}\,(X \times Y))$. ■

Proposition 5.7.2. *Let $\mathcal{E}$ and $\mathcal{F}$ be full exceptional posets and $\mathcal{M}$ and $\mathcal{N}$ be their dual exceptional posets. Then $\mathcal{M} \boxtimes^{\mathrm{L}} \mathcal{N}$ is the dual exceptional poset of $\mathcal{E} \boxtimes^{\mathrm{L}} \mathcal{F}$.*

Proof. This follows from the Kunneth theorem

$$\mathrm{RHom}(E_i \boxtimes^{\mathrm{L}} F_j, M_k \boxtimes^{\mathrm{L}} N_\ell) = \mathrm{RHom}(E_i, M_k) \otimes \mathrm{RHom}(F_j, N_\ell).$$

■

Theorem 5.7.3. *Let $\pi_V : V \to X$ and $\pi_W : W \to Y$ be vector bundles. Let $\mathcal{E}$ be a V-geometric full exceptional poset on $\mathrm{D}^b(\mathrm{Coh}\,(X))$ and $\mathcal{F}$ be a W-geometric full exceptional poset on Y. Then $\mathcal{E} \boxtimes \mathcal{F}$ is $V \times W$-geometric, the tensor product A_∞-structure on $A_\mathcal{E} \otimes A_\mathcal{F}$ is A_{fin}, and there is an equivalence*

$$\mathrm{D}^b(\mathrm{Coh}\,(V \times W)) \cong \mathrm{D}^b_{fg}(E(A_\mathcal{E} \otimes A_\mathcal{F})^{\mathrm{op}}).$$

Proof. Since

$$\begin{aligned} \mathrm{Hom}^\bullet(E_i \boxtimes^{\mathrm{L}} F_j, E_{i'} \boxtimes^{\mathrm{L}} F_{j'} \otimes S^\bullet(V \times W)^\vee) &= \mathrm{Hom}^\bullet(E_i \boxtimes^{\mathrm{L}} F_j, E_{i'} \boxtimes^{\mathrm{L}} F_{j'} \otimes S^\bullet V^\vee \boxtimes S^\bullet W^\vee) \\ &= \mathrm{Hom}^\bullet(E_i, E_{i'} \otimes S^\bullet V^\vee) \otimes \mathrm{Hom}^\bullet(F_j, F_{j'} \otimes S^\bullet W^\vee) \end{aligned}$$

we conclude that $\mathcal{E} \boxtimes \mathcal{F}$ is $V \times W$-geometric and $\Lambda_{\mathcal{E}\boxtimes\mathcal{F}} = \Lambda_\mathcal{E} \otimes \Lambda_\mathcal{F}$ as graded algebras.

By Lemma 5.3.5, $\Lambda_\mathcal{E}$, $\Lambda_\mathcal{F}$ and $\Lambda_{\mathcal{E}\boxtimes\mathcal{F}} = \Lambda_\mathcal{E} \otimes \Lambda_\mathcal{F}$ are all locally finite. Hence by Proposition 3.6.1, we have a chain of A_∞-quasi-isomorphisms

$$A_{\mathcal{E}\boxtimes\mathcal{F}} \cong H^\bullet(E(\Lambda_{\mathcal{E}\boxtimes\mathcal{F}})) \cong H^\bullet(E(\Lambda_\mathcal{E} \otimes \Lambda_\mathcal{F})) \cong H^\bullet(E(\Lambda_\mathcal{E})) \otimes H^\bullet(E(\Lambda_\mathcal{F})) = A_\mathcal{E} \otimes A_\mathcal{F}.$$

To see that the tensor product A_∞-structure on $A_\mathcal{E} \otimes A_\mathcal{F}$ is A_{fin}, observe that since $A_\mathcal{E} \otimes A_\mathcal{F}$ is isomorphic to $A_{\mathcal{E}\boxtimes\mathcal{F}}$ as vector spaces, $A_\mathcal{E} \otimes A_\mathcal{F}$ inherits the same Adams grading on $A_{\mathcal{E}\boxtimes\mathcal{F}}$ (which in turns is inherited from the algebra $\mathrm{Ext}^\bullet_{\Lambda_{\mathcal{E}\boxtimes\mathcal{F}}}(S, S)$). Since the tensor product A_∞-structure is given by a finite sum of tensor products and compositions of the two A_∞-structures, the tensor product A_∞-structures also preserves the Adams grading and by Lemma 5.3.7, the tensor product A_∞-structure is also A_{fin}. ■

Examples illustrating this theorem will be given in Section 6.4.

Chapter 6

Examples

This chapter is a list of examples illustrating theorems in Chapter 5.

Section 6.1 contains examples illustrating Theorem 5.2.9 which produces quivers with relations derived equivalent to total spaces of vector bundles.

Section 6.2 introduces a class of algebras called Koszul algebras whose dg-resolution is particularly easy to describe. We also gives some examples of vector bundles whose classical tilting algebras are Koszul.

Section 6.3 contains some worked out examples illustrating Theorem 5.4.5, constructing derived equivalences between total spaces of vector bundles and dg-quivers, and if the total space of vector bundles are Calabi–Yau, quivers with superpotential. These examples are calculated by first determining the classical tilting algebras, then try to work out the dg-resolutions to determine the dg-quiver. In Calabi–Yau examples of dimension no greater than 4, there are enough constraints and hence the classical tilting algebras determine their dg-resolutions.

Section 6.4 contains a list of examples by applying the product construction in Theorem 5.7.3. Since the general formulae for the cyclic A_∞-tensor product defined by Amorim and Tu [2] are not known, we only work with the case when one of the A_∞-algebras is an honest algebra.

Section 6.5 contains a list of examples illustrating the quotient construction in Theorem 5.6.5.

Unless otherwise specified, all cohomological formulae can be found in the Appendix.

6.1 Classical Tilting Quivers

Example 6.1.1. Take $X = \mathbb{P}^n$, the full exceptional collection $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n))$ and $V = T_{\mathbb{P}^n}^\vee$. Write $W = \mathbb{C}^{n+1}$. The Euler exact sequence

$$0 \to \mathcal{O} \to W(1) \to T_{\mathbb{P}^n} \to 0$$

induces a short exact sequence via Koszul resolution [35, 12.12] for all $k \geq 1$

$$0 \to S^{k-1}W(k-1) \to S^k W(k) \to S^k T_{\mathbb{P}^n} \to 0. \tag{6.1.1}$$

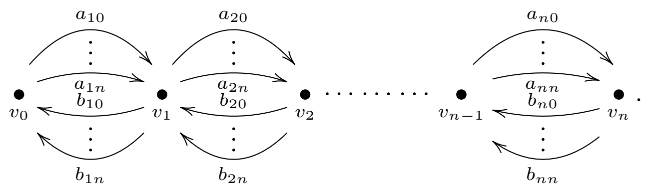
Tensoring the above sequence with $\mathcal{O}(j-i)$ and using the vanishing $H^i(\mathbb{P}^n, \mathcal{O}(p)) = 0$ whenever $i \geq 1$ and $p \geq -n$, we conclude that for all $k \geq 0$, $\ell \geq 1$ and $0 \leq i, j \leq n$

$$\mathrm{Ext}^\ell_{\mathbb{P}^n}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}) = H^\ell(\mathbb{P}^n, S^k T_{\mathbb{P}^n}(j-i)) = 0.$$

Hence $\mathcal{E}$ is $T_{\mathbb{P}^n}^\vee$-geometric. Since $T_{\mathbb{P}^n}$ is ample, by Corollary 5.2.9, we have a derived equivalence $\mathrm{D}^b(\mathrm{Coh}\,(T_{\mathbb{P}^n}^\vee)) \cong \mathrm{D}^b(\Lambda_{\mathcal{E}}^{\mathrm{op}})$, where

$$\Lambda_{\mathcal{E}} = \bigoplus_{i,j=0}^{n} \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^\bullet T_{\mathbb{P}^n}).$$

We claim that $\Lambda_{\mathcal{E}}$ is isomorphic to the path algebra of the quiver



with relations

$$\begin{array}{cc} \sum_{j=0}^{n} a_{ij}b_{ij} = 0 & \sum_{j=0}^{n} b_{ij}a_{ij} = 0 \\ a_{i+1\,k}a_{ij} = a_{i+1\,j}a_{ik} & b_{i\,j+1}b_{i+1\,j} = b_{i\,j}b_{i+1\,j+1} \\ b_{i+1\,j}a_{ik} = a_{i-1\,k}b_{ij}. & \end{array}$$

First, we show that $\Lambda_{\mathcal{E}}$ is generated as a $\bigoplus_{i=0}^{n} \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i))$ algebra by the vector space

$$U = \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathcal{O}(i-1), \mathcal{O}(i)) \oplus \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i-1) \otimes T_{\mathbb{P}^n}).$$

We will need the following lemma.

Lemma 6.1.2. *The multiplication map*

$$H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n}(\ell)) \otimes H^0(\mathbb{P}^n, S^{k'} T_{\mathbb{P}^n}(\ell')) \to H^0(\mathbb{P}^n, S^{k+k'} T_{\mathbb{P}^n}(\ell + \ell'))$$

is surjective if $k + \ell \geq 0$ *and* $k' + \ell' \geq 0$.

Proof. Twisting the short exact sequence (6.1.1) by $\mathcal{O}(\ell)$, we get a short exact sequence

$$0 \to S^{k-1}W(k-1+\ell) \to S^k W(k+\ell) \to S^k T_{\mathbb{P}^n}(\ell) \to 0.$$

By naturality, we have the following commutative diagram

$$\begin{tikzcd} H^0(\mathbb{P}^n, S^k W(k+\ell)) \otimes H^0(\mathbb{P}^n, S^{k'} W(k'+\ell')) \arrow[r] \arrow[d] & H^0(\mathbb{P}^n, S^{k+k'} W(k+k'+\ell+\ell')) \arrow[d] \\ H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n}(\ell)) \otimes H^0(\mathbb{P}^n, S^{k'} T_{\mathbb{P}^n}(\ell')) \arrow[r] & H^0(\mathbb{P}^n, S^{k+k'} T_{\mathbb{P}^n}(\ell+\ell')). \end{tikzcd}$$

The top horizontal map is surjective since both

$$S^k W \otimes S^{k'} W \to S^{k+k'} W \quad \text{and} \quad H^0(\mathbb{P}^n, \mathcal{O}(k+\ell)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k'+\ell')) \to H^0(\mathbb{P}^n, \mathcal{O}(k+k'+\ell+\ell'))$$

are surjective maps. The two vertical maps are also surjective since, as $k + \ell \geq 0$,

$$H^1(\mathbb{P}^n, S^{k-1}W(k-1+\ell)) = S^{k-1}W \otimes H^1(\mathbb{P}^n, \mathcal{O}(k-1+\ell)) = 0.$$

Hence the bottom horizontal map is also surjective. ∎

With the lemma, we can show that $\Lambda_{\mathcal{E}}$ is generated by U. We separate the problem into three cases.

Case 1 $i = j$**.** By the lemma,

$$H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1)) \to H^0(\mathbb{P}^n, T_{\mathbb{P}^n})$$

is surjective, and hence so is the multiplication map

$$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i+1)) \otimes \mathrm{Hom}(\mathcal{O}(i+1), T_{\mathbb{P}^n}(-1)) \to \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i) \otimes T_{\mathbb{P}^n}).$$

Again by the lemma, the map

$$H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n}) \otimes H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) \to H^0(\mathbb{P}^n, S^{k+1} T_{\mathbb{P}^n})$$

is surjective, and hence so is

$$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i) \otimes S^k T_{\mathbb{P}^n}) \otimes \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i) \otimes T_{\mathbb{P}^n}) \to \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i) \otimes S^{k+1} T_{\mathbb{P}^n}).$$

This shows the vector subspace $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i) \otimes S^k T_{\mathbb{P}^n})$ is generated by U.

Case 2 $i < j$**.** Surjectivity of

$$H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n}) \otimes H^0(\mathbb{P}^n, \mathcal{O}(j-i)) \to H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n}(j-i))$$

implies surjectivity of

$$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \otimes \mathrm{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}) \to \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}).$$

Since both $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j))$ and $\mathrm{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n})$ are generated by U, so is $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n})$.

Case 3 $i > j$**.** Since

$$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T^\vee_{\mathbb{P}^n}) = H^0(\mathbb{P}^n, S^k T^\vee_{\mathbb{P}^n}(j-i)) = 0,$$

we may assume $k \geq i - j$. If $k = i - j$, surjectivity of

$$H^0(\mathbb{P}^n, S^{k-1} T_{\mathbb{P}^n}(-k+1)) \otimes H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1)) \to H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n}(-k))$$

implies surjectivity of

$$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j+1) \otimes S^k T_{\mathbb{P}^n}) \otimes \mathrm{Hom}(\mathcal{O}(j+1), \mathcal{O}(j) \otimes T_{\mathbb{P}^n}) \to \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}).$$

Hence $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n})$ is generated by U. For general $k \geq i - j$, surjectivity of

$$H^0(\mathbb{P}^n, S^{i-j} T_{\mathbb{P}^n}(j-i)) \otimes H^0(\mathbb{P}^n, S^{k-i+j} T_{\mathbb{P}^n}) \to H^0(\mathbb{P}^n, S^k T_{\mathbb{P}^n} \otimes \mathcal{O}(j-i))$$

implies surjectivity of

$$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^{i-j} T_{\mathbb{P}^n}) \otimes \mathrm{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes S^{k-i+j} T_{\mathbb{P}^n}) \to \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n}).$$

This implies $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k T_{\mathbb{P}^n})$ is generated by U.

Example 6.1.3. Let us look at the special case $X = \mathbb{P}^2$ and $V = T^\vee_{\mathbb{P}^2}$, which is Calabi–Yau 4. The tilting quiver becomes

$$\begin{tikzcd}
v_0 \arrow[r, bend left=30, "a_0"] \arrow[r, bend left=20, "a_1"] \arrow[r, bend left=10, "a_2"'] & v_1 \arrow[l, bend left=10, "b_0"'] \arrow[l, bend left=20, "b_1"] \arrow[l, bend left=30, "b_2"] \arrow[r, bend left=30, "c_0"] \arrow[r, bend left=20, "c_1"] \arrow[r, bend left=10, "c_2"'] & v_2 \arrow[l, bend left=10, "d_0"'] \arrow[l, bend left=20, "d_1"] \arrow[l, bend left=30, "d_2"]
\end{tikzcd}$$

with relations

$$\begin{array}{cc}
\sum_{j=0}^{2} a_j b_j = 0 & \sum_{j=0}^{2} b_j a_j = 0 \\
\sum_{j=0}^{2} c_j d_j = 0 & \sum_{j=0}^{2} d_j c_j = 0 \\
c_{j+1} a_j = c_j a_{j+1} & b_{j+1} d_j = b_j d_{j+1} \\
a_j b_k = d_k c_j. &
\end{array}$$

The relations listed above are not all independent: the relations $a_j b_k = d_k c_j$ and $\sum_{j=0}^{2} a_j b_j = 0$ together imply $\sum_{j=0}^{2} d_j c_j = 0$.

Example 6.1.4. Take $X = \mathbb{P}^n$, the full strongly exceptional collection $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n))$ and $V = \bigoplus_{m=1}^{p} \mathcal{O}(-a_m)$, where $0 \leq a_1 \leq \cdots \leq a_m$. Since

$$S^k V^\vee = \bigoplus_{\substack{\sum k_m = k \\ k_m \geq 0}} \mathcal{O}\left(\sum_{m=1}^{p} k_m a_m\right) \text{ for all } \ell \geq 1,\ k \geq 0 \text{ and } 1 \leq i, j \leq n,$$

the exceptional collection $\mathcal{E}$ is V-geometric:

$$\operatorname{Ext}^\ell(\mathcal{O}(i), \mathcal{O}(j) \otimes S^k V^\vee) = \bigoplus_{\substack{\sum k_m = k \\ k_m \geq 0}} H^\ell(\mathbb{P}^n, \mathcal{O}(j - i + \sum_{m=1}^{p} k_m a_m)) = 0.$$

Since $a_m \geq 0$, $V^\vee$ is globally generated. Hence by Corollary 5.2.9, we have a derived equivalence $\mathrm{D}^b(\mathrm{Coh}\,(V)) \cong \mathrm{D}^b_{fg}(\Lambda_{\mathcal{E}}^{\mathrm{op}})$, where

$$\begin{aligned}
\Lambda_{\mathcal{E}} &= \bigoplus_{i,j=0}^{n} \operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(j) \otimes S^\bullet V^\vee) \\
&= \bigoplus_{k_1, \ldots, k_m \geq 0} \bigoplus_{i,j=0}^{n} \operatorname{Hom}\left(\mathcal{O}(i), \mathcal{O}(j) \otimes \mathcal{O}\left(\sum_{m=1}^{p} k_m a_m\right)\right).
\end{aligned}$$

Note that $\Lambda_{\mathcal{E}}$ is $\mathbb{Z} \times \mathbb{Z}^p$-graded by $(j - i, k_1, \ldots, k_p)$. Now, let q be such that $a_1 \leq \cdots \leq a_q \leq n < a_{q+1} \leq \cdots \leq a_p$. We claim that $\Lambda_{\mathcal{E}}$ is generated as a $\bigoplus_{i=1}^{n} \operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(i))$ algebra by the vector subspace

$$U = \bigoplus_{i=1}^{n} \operatorname{Hom}(\mathcal{O}(i-1), \mathcal{O}(i)) \oplus \bigoplus_{m=q+1}^{p} \operatorname{Hom}(\mathcal{O}(n), \mathcal{O} \otimes \mathcal{O}(a_m)) \oplus \bigoplus_{m=1}^{q} \bigoplus_{j=a_m}^{n} \operatorname{Hom}(\mathcal{O}(j), \mathcal{O}(j - a_m) \otimes \mathcal{O}(a_m)).$$

First, observe that the multiplication map,

$$\operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \otimes \bigotimes_{m=1}^{p} \operatorname{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes \mathcal{O}(a_m))^{\otimes k_m} \to \operatorname{Hom}\left(\mathcal{O}(i), \mathcal{O}(j) \otimes \mathcal{O}(\sum_{m=1}^{p} k_m a_m)\right)$$

which corresponds to multiplication of homogeneous polynomials, is surjective.

Now, $\operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(j))$ is generated by $\bigoplus_{i=1}^{n} \operatorname{Hom}(\mathcal{O}(i-1), \mathcal{O}(i))$. For $a_m \geq n$, the map

$$\operatorname{Hom}(\mathcal{O}(j), \mathcal{O}(n)) \otimes \operatorname{Hom}(\mathcal{O}(n), \mathcal{O} \otimes \mathcal{O}(a)) \to \operatorname{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes \mathcal{O}(a_m))$$

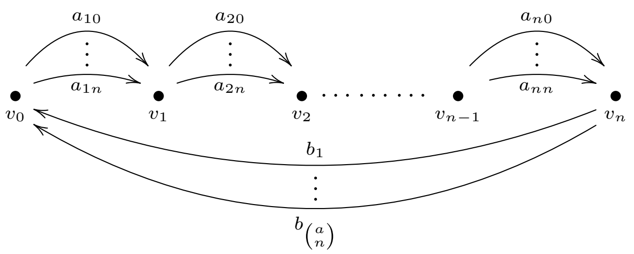
is surjective, and for $a_m \leq n$, the map

$$\operatorname{Hom}(\mathcal{O}(j), \mathcal{O}(j - a) \otimes \mathcal{O}(a)) \otimes \operatorname{Hom}(\mathcal{O}(j - a), \mathcal{O}(a)) \to \operatorname{Hom}(\mathcal{O}(j), \mathcal{O}(j) \otimes \mathcal{O}(a))$$

is surjective. Hence $\Lambda_{\mathcal{E}}$ is generated by U.

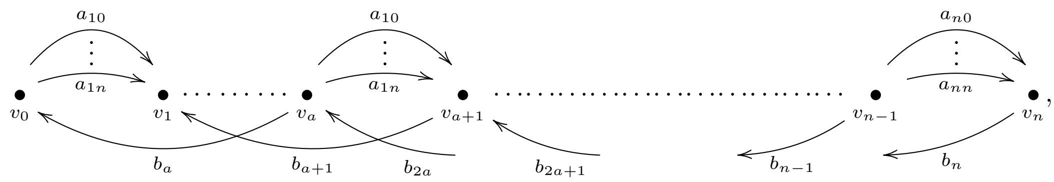
Example 6.1.5. Let us consider the special case $V = \mathcal{O}_{\mathbb{P}^n}(-a)$ over $\mathbb{P}^n$, where $a \geq 0$.

Case 1 $a \geq n$**.** The tilting quiver is



with relations $a_{i+1\,j+1}a_{ij} = a_{i+1\,j}a_{i\,j+1}$ The vertex v_i correspond to $\mathcal{O}(i)$, the arrows a_{ij} corresponds to $\operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(i+1))$, and the arrows b_i correspond to $\operatorname{Hom}(\mathcal{O}(n), \mathcal{O} \otimes \mathcal{O}(a))$.

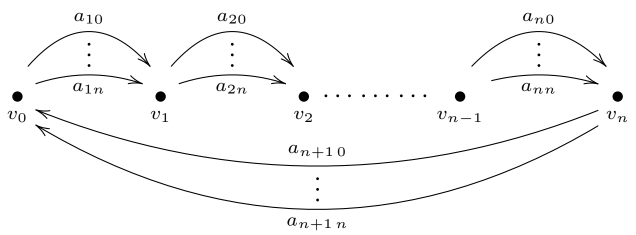
Case 2 $0 \le a \le n$**.** The tilting quiver is



that is, $n+1$ arrows from v_i to v_{i+1} and one arrow from v_i to v_{i-a}. The vertex v_i correspond to $\mathcal{O}(i)$, the arrows a_{ij} corresponds to $\operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(i+1))$, and the arrows b_i correspond to $\operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(i-a) \otimes \mathcal{O}(a))$.

The general case $V = \bigoplus_{m=1}^{p} \mathcal{O}(-a_m)$ is then given by the superposition of the tilting quivers corresponding to $\mathcal{O}(-a_1), \cdots, \mathcal{O}(-a_p)$.

Example 6.1.6. Consider the special case $V = \mathcal{O}_{\mathbb{P}^n}(-n-1) = K_{\mathbb{P}^n}$. The tilting quiver becomes



with relations $a_{i+1\,k} a_{ij} = a_{i+1\,j} a_{ik}$.

Example 6.1.7. Consider the Hirzebruch surface $\mathbb{F}_r = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$. It is a toric surface defined by the complete fan in $\mathbb{R}^2$ given by $u_1 = (-1, r)$, $u_2 = (0, 1)$, $u_3 = (1, 0)$, $u_4 = (0, -1)$. Denote by D_i the divisor defined by u_i. Then D_1 and D_3 are both linearly equivalent to the divisor defined by the fiber of the projection $\mathbb{F}_r \to \mathbb{P}^1$. The divisor D_2 is the exceptional divisor $\mathbb{P}(\mathcal{O}(n))$ and D_2 is linearly equivalent to the divisor defined by $\mathbb{P}(\mathcal{O})$. We have the linear relation $\mathcal{O}(D_1) = \mathcal{O}(D_3)$ and $\mathcal{O}(D_2) = \mathcal{O}(D_4 - rD_3)$. The picard group $\operatorname{Pic}(\mathbb{F}_r)$ is thus the free abelian group generated by $\mathcal{O}(D_3)$ and $\mathcal{O}(D_4)$. A line bundle $\mathcal{O}(a_3, a_4) = \mathcal{O}(a_3 D_3 + a_4 D_4)$ is globally generated (resp. ample) if and only if $a_i \ge 0$ (resp. $a_i > 0$) for $i = 3, 4$. From the generalized Euler sequence

$$0 \to \mathcal{O}^{\oplus 2} \to \bigoplus_{i=1}^{4} \mathcal{O}(D_i) \to T_{\mathbb{F}_r} \to 0,$$

we see that

$$K_{\mathbb{F}_r}^{-1} = \mathcal{O}(D_1 + D_2 + D_3 + D_4) = \mathcal{O}(2 - r, 2).$$

For all $s \ge -1$,

$$(\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(s+1, 1), \mathcal{O}(s+2, 1))$$

is a strongly full exceptional sequence [33]. In particular, when $s = -1$, we have the strongly full exceptional sequence

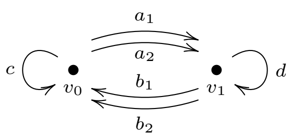
$$\mathcal{E} = (\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)).$$

We claim that $\mathcal{E}$ is geometric with respect to the vector bundle $V = \bigoplus_{m=1}^{p} \mathcal{O}(-a_m, -b_m)$, where $a_m, b_m \ge 0$. In particular, it is geometric with respect to $K_{\mathbb{F}_0}$, $K_{\mathbb{F}_1}$ and $K_{\mathbb{F}_2}$. Indeed, the vanishing of higher Ext groups are reduced to the following vanishing of cohomology of line bundles.

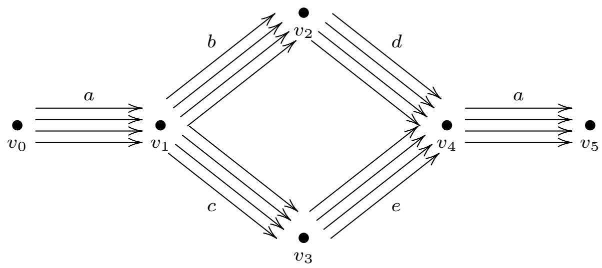
Lemma 6.1.8. *For* $\ell \ge 1$, $H^\ell(\mathbb{F}_r, \mathcal{O}(a, b)) = 0$ *whenever* $a, b \ge -1$.

Proof. The case $a, b \ge 0$ follows from Demazure vanishing theorem [18, Theorem 9.2.3]. The case $-1 \le a, b \le 0$ follows from Batyrev-Borisov vanishing theorem [18, Theorem 9.2.7]. ∎

We will draw explicitly the tilting quiver of $V = \mathcal{O}(-1,-1)$. It is given by



Example 6.1.9. In [38], Kapranov constructed a strongly full exceptional sequence for Grassmannians. Let $G = \mathrm{Gr}(k,n)$ denote the Grassmannian of k-dimensional subspace in a complex vector space W of dimension n. Denote by U the tautological vector bundle of rank k on G and by $Y_{k,n}$ the set of Young diagrams inscribed in a rectangle of size $k \times (n-k)$. Note that $Y_{k,n}$ is partially ordered by inclusion. The set $\mathcal{E} = \{\Sigma^\lambda U^\vee : \lambda \in Y_{k,n}\}$ is a full strongly exceptional collection on G. In the case $G = \mathrm{Gr}(2,4)$, the sequence $(\mathcal{O}, U^\vee, \Sigma^{2,0}U^\vee, \Sigma^{1,1}U^\vee, \Sigma^{2,1}U^\vee, \Sigma^{2,2}U^\vee) = (\mathcal{O}, U^\vee, S^2U^\vee, \mathcal{O}(1), U^\vee(1), \mathcal{O}(2))$ is full and strongly exceptional, with tilting quiver



Example 6.1.10. Take the vector bundle $V = \bigoplus_{i=1}^{m} \mathcal{O}(-a_i)$ over $\mathrm{Gr}(k,n)$, where $a_i \geq 0$. Since $\mathcal{O}(1)$ defines the plucker embedding, it is ample. Hence $V^\vee$ is globally generated. To calculate the Ext group, we need

Lemma 6.1.11. *If $n-k \geq \alpha_1 \geq \cdots \geq 0$ and $\beta_1 \geq \cdots \geq \beta_k \geq 0$, then for all $\ell \geq 1$,*

$$\mathrm{Ext}^\ell(\Sigma^\alpha U^\vee, \Sigma^\beta U^\vee) = 0.$$

Proof. Let $\bar{\alpha} = (\alpha_1 - \alpha_n, \cdots, \alpha_1 - \alpha_2, 0)$. Then $\Sigma^\alpha U \otimes \Sigma^\beta U^\vee = \Sigma^{\bar{\alpha}} U^\vee \otimes \Sigma^\beta U^\vee \otimes \mathcal{O}(-\alpha_1)$. By the Littlewood-Richardson rule, $\Sigma^{\bar{\alpha}}U^\vee \otimes \Sigma^\beta U^\vee$ can be decomposed into a direct sum of vector bundles of the form $\Sigma^\nu U^\vee$, where $\nu_i \geq \bar{\alpha}_i = \alpha_1 - \alpha_{k+1-i}$. Now, for any such $\Sigma^\nu U^\vee$, we have $\Sigma^\nu U^\vee \otimes \mathcal{O}(-\alpha_1) = \Sigma^{(\nu_1-\alpha_1,\cdots,\nu_n-\alpha_1)}U^\vee$. Since $\nu_i - \alpha_1 \geq -(n-k)$, by [38, Lemma 3.2], $H^\ell(\mathrm{Gr}(k,n), \Sigma^{(\nu_1-\alpha_1,\cdots,\nu_n-\alpha_1)}U^\vee) = 0$ for all $\ell \geq 1$ and we get the required vanishing. ∎

Then

$$\mathrm{Ext}^\ell(\Sigma^\alpha U^\vee, \Sigma^\beta U^\vee \otimes S^\bullet V^\vee) = \bigoplus_{k_1,\cdots,k_m \geq 0} \mathrm{Ext}^\ell(\Sigma^\alpha U^\vee, \Sigma^{(\beta_1 + \sum k_i a_i, \cdots, \beta_k + \sum k_i a_i)} U^\vee) = 0.$$

Hence $\mathcal{E} = \{\Sigma^\lambda : \lambda \in Y_{k,n}\}$ is V-geometric. In particular, it is compatible with the canonical bundle $K_{\mathrm{Gr}(k,n)} = \mathcal{O}(-n)$.

6.2 Koszul Algebras

In section 5.4, we produced an A_{fin}-structure on $A = \mathrm{Ext}^\bullet_\Lambda(S,S)$ such that $E(A)$ is a quasi-free model of Λ. This A_{fin}-structure is not readily known in general. In this section, we introduce a class of algebras, called Koszul algebras, whose A_{fin}-structure on A are readily computable. A good introduction is given by Martínez-Villa [56].

Definition 6.2.1. A $\mathbb{Z}$-graded algebra Λ over a semisimple algebra $S = \mathbb{K}^r$ is called *Koszul* if there is a linear graded projective resolution of S, i.e., an exact sequence of Λ-modules

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to S \to 0,$$

where each P_i is projective and concentrated in degree i.

It is known that Koszul algebras are always quadratic, so Koszulness is in fact a rather restrictive property. Below are the main examples of vector bundles producing Koszul algebras we have in mind:

Proposition 6.2.2. *Let X be a smooth projective variety whose K_X^{-1} is semi-ample. Let $\mathcal{E}$ be a K_X-geometric full exceptional collection on $\mathrm{D}^b(\mathrm{Coh}\,(X))$. If $\operatorname{length}\mathcal{E} = \dim X + 1$, then $\Lambda_{\mathcal{E}}$ is Koszul.*

Proof. By Proposition 2.8.3, $\operatorname{length}\mathcal{E} = \operatorname{rank}\mathrm{D}^b(\mathrm{Coh}\,(X))$. The rest follows from [15, Proposition 4.2]. ∎

Remark 6.2.3. Examples of varieties with full exceptional collection $\mathcal{E}$ whose $\operatorname{length}\mathcal{E} = \dim X + 1$ include projective spaces, odd-dimensional quadrics [39] and certain Fano threefolds [57].

Koszul algebras are stable under taking tensor products and quotients.

Proposition 6.2.4. *1. If both Λ and Λ' are Koszul, then so is $\Lambda \otimes \Lambda'$.*

2. Λ^{op} is Koszul.

3. $\Lambda \# G$ is Koszul.

4. Λ/G is Koszul.

Proof. For the first claim, one takes the total complex of the tensor product of the two linear graded resolutions. The proof for the second statement can be found in Woodcock's paper [73]. For the third claim, one smashes the whole linear resolution of S by G. Since tensoring by $\mathbb{K}G$ is an exact functor, this produces a linear resolution of $S \otimes \mathbb{K}G$. The last follows from Corollary 3.5.8 by applying the Morita functor in Theorem 3.5.2 to the linear resolution of that of $(\Lambda^{\mathrm{op}} \# G)^{\mathrm{op}}$ to get a linear resolution for Λ/G. ∎

As a result, if vector bundles V and W produce Koszul algebras, then so does $V \boxtimes W$. If V is equipped with a G-action making it G-equivariant, then its corresponding tilting algebra is also Koszul.

Here is the main proposition which makes the A_{fin}-structure on $\mathrm{Ext}^{\bullet}_{\Lambda}(S,S)$ simple.

Proposition 6.2.5. *If Λ is a Koszul algebra, then any A_∞-structure on $\mathrm{Ext}^{\bullet}_{\Lambda}(S,S)$ which preserve the grading induced by Λ is an ordinary algebra structure, i.e., $m_n = 0$ for $n \neq 2$.*

Proof. Each component of $\mathrm{Ext}^{i}_{\Lambda}(S,S)$ is a $\mathbb{Z}$-graded Λ-module. By the definition of Koszulness, $\mathrm{Ext}^{i}_{\Lambda}(S,S)$ is concentrated in degree i, i.e., $\mathrm{Ext}^{i}_{\Lambda}(S,S) = [\mathrm{Ext}^{i}_{\Lambda}(S,S)]_i$. Since by assumption m_n preserve the grading induced by Λ, it must map

$$m_n : [\mathrm{Ext}^{i_1}_{\Lambda}(S,S)]_{i_1} \otimes \cdots \otimes [\mathrm{Ext}^{i_n}_{\Lambda}(S,S)]_{i_n} \to [\mathrm{Ext}^{i_1+\cdots i_n+2-n}_{\Lambda}(S,S)]_{i_1+\cdots+i_n}.$$

Hence m_n must vanish since the right hand side is zero unless $n = 2$. ∎

Now, we can determine the A_{fin}-structure in Theorem 5.4.5.

Theorem 6.2.6. *Let X be a smooth projective variety and $\pi : V \to X$ be an anti-semiample vector bundle. Suppose $\mathcal{E} = \{E_i\}_{i \in I}$ is a full V-geometric exceptional poset on $\mathrm{D}^b(\mathrm{Coh}\,(X))$ with dual exceptional poset $\mathcal{F} = \{F_i\}_{i \in I^{\mathrm{op}}}$. If the classical tilting algebra*

$$\Lambda_{\mathcal{E}} = \bigoplus_{i,j \in I} \mathrm{Hom}(E_i, E_j \otimes S^{\bullet} V^{\vee})$$

is Koszul, then the A_{fin}-algebra

$$A^{\ell}_{\mathcal{E}} = \bigoplus_{i,j \in I} \bigoplus_{k=0}^{\operatorname{rank} V} \mathrm{Hom}^{\ell-k}(F_i, F_j \otimes \wedge^k V)$$

given in Theorem 5.4.5 is an ordinary algebra, i.e., $m_n = 0$ for $n \neq 2$, with m_2 given by composition of maps.

Proof. This follows from Theorem 5.4.5 and Proposition 6.2.5. ■

We can describe the algebra $A_{\mathcal{E}} \cong \operatorname{Ext}^\bullet_\Lambda(S,S)$ more explicitly. Recall that we can write Λ as a path algebra with relations $\mathbb{K}Q/I$. If Λ is Koszul, then I is homogeneous and is generated by $I_2 = I \cap (\mathbb{K}Q)_2$. Define a pairing $\langle -,- \rangle : (\mathbb{K}Q)_2 \times (\mathbb{K}Q^{\mathrm{op}})_2 \to \mathbb{K}$ by

$$\langle a, b^{\mathrm{op}} \rangle = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

where a and b are paths of length 2 in Q. The orthogonal subspace of I_2 is defined to be

$$I_2^\perp = \{b \in (\mathbb{K}Q^{\mathrm{op}})_2 : \langle a, b \rangle = 0 \text{ for all } a \in I_2\}.$$

We have the following

Theorem 6.2.7 ([29] Theorem 2.2)**.** *Let $\mathbb{K}Q/I$ be a Koszul algebra. Then the Yoneda algebra* $\operatorname{Ext}^\bullet_\Lambda(S,S)$ *of Λ is isomorphic to $\mathbb{K}Q^{\mathrm{op}}/\langle I_2^\perp \rangle$.*

Example 6.2.8. Let Q be the quiver with one vertex and n loops. Then $\mathbb{K}Q = \mathbb{K}\langle x_1, \dots, x_n \rangle$. Let I be the ideal generated by elements in the form $x_i x_j - x_j x_i$ for all $1 \le i,j \le n$. Then $\mathbb{K}Q/I = \mathbb{K}[x_1, \dots, x_n]$. The orthogonal subspace I_2^{op} is spanned by elements in the form $x_i x_j + x_j x_i$ and x_i^2 for all $1 \le i,j \le n$. Hence $\mathbb{K}Q^{\mathrm{op}}/\langle I_2^{\mathrm{op}} \rangle = \bigwedge^\bullet \mathbb{K}^n$. The standard Koszul resolution

$$0 \to \mathbb{K}[x_1, \dots, x_n] \to \mathbb{K}[x_1, \dots, x_n] \otimes \mathbb{K}^n \to \cdots \to \mathbb{K}[x_1, \dots, x_n] \otimes \bigwedge^n \mathbb{K}^n \to \mathbb{K},$$

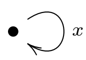
where the first n maps are given by wedging with the vector $(x_1, \dots, x_n)$ and the last map given by evaluation of polynomials at zero, shows that $\mathbb{K}[x_1, \dots, x_n]$ is Koszul. Hence $\operatorname{Ext}^\bullet_{\mathbb{K}Q/I}(\mathbb{K},\mathbb{K}) = \bigwedge^\bullet \mathbb{K}^n$. The A_∞-structure is given by $m_i = 0$ for $i \neq 2$, and m_2 by the Yoneda product, which is the wedge product on $\wedge^\bullet \mathbb{K}^n$.

6.3 DG-Tilting Quivers

In this section, we compute some examples illustrating Theorem 5.4.5 and Proposition 5.5.2. All formulae for cohomologies can be found in the Appendix A.

Example 6.3.1. Let $X = \operatorname{Spec} \mathbb{K}$ and $V = \mathbb{K}^n$. Then $\mathcal{E} = \mathcal{O}_X$ is a V-geometric exceptional collection on $\mathrm{D}^b(\mathrm{Coh}\,(\operatorname{Spec}\mathbb{K}))$. The classical tilting algebra is given by $\Lambda_{\mathcal{E}} = \mathbb{K}[x_1, \dots, x_n]$ and the classical tilting quiver is given by one vertex with n loops x_i, with relations given by $x_i x_j = x_j x_i$. By Example 6.2.8, $A^\ell_{\mathcal{E}} = \bigwedge^\ell \mathbb{K}^n$, with m_2 given by the wedge product and all other $m_n = 0$. Below is the quasi-free dg-quiver derived equivalent to $\mathbb{K}^n$ for $n = 1, 2, 3$.

When $n = 1$, the quasi-free dg-quiver is given by

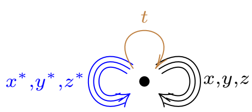


with $\deg x = 0$ and $dx = 0$.

When $n = 2$, we have

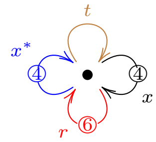
with $\deg x = \deg y = 0$, $\deg t = -1$, $dx = dy = 0$ and $dt = xy - yx$.

When $n = 3$, we have



with $\deg x = \deg y = \deg z = 0$, $\deg x^* = \deg y^* = \deg z^* = -1$, $\deg t = -2$ and $dx = dy = dz = 0$, $dx^* = yz - zy$, $dy^* = zx - xz$, $dz^* = xy - yx$, and $dt = [x, x^*] + [y, y^*] + [z, z^*]$. This dg-quiver has a superpotential given by $\Phi = xyz - xzy$.

When $n = 4$, we have



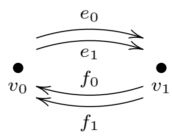
where the black edges are of degree 0, red edges are of degree -1, blue edges are of degree -2 and brown edges are of degree -3. We will denote the four degree 0 (resp. degree -2) edges by x_i (resp. by x_i^*), where $1 \le i \le 4$ and the six degree -1 edges by r_{ij} where $1 \le i < j < 4$. If $i > j$, by r_{ij} we will mean $-r_{ji}$. The pairing of degree -1 edges is then given by

$$\langle r_{ij}, r_{k\ell}\rangle = \epsilon_{ijk\ell} = \begin{cases} 1 & \text{if } (i,j,k,\ell) \text{ is an even permutation of } (1,2,3,4) \\ -1 & \text{if } (i,j,k,\ell) \text{ is an odd permutation of } (1,2,3,4) \\ 0 & \text{otherwise,} \end{cases}$$

with superpotential given by

$$\Phi = \sum \epsilon_{ijk\ell}(x_i x_j - x_j x_i) r_{k\ell}.$$

Example 6.3.2. Consider $X = \mathbb{P}^1$ and $V = K_{\mathbb{P}^1}$ which is Calabi–Yau 2. Take the exceptional sequence $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1))$ which is $K_{\mathbb{P}^1}$-geometric. The classical tilting quiver is given by



with relations $f_j e_i = f_i e_j$ and $e_j f_i = e_i f_j$.

The corresponding A_∞-category is given by

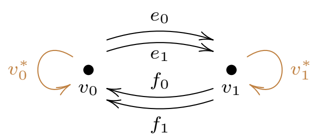
$$\begin{aligned} \mathcal{A}^\ell(v_0, v_0) &= \mathrm{Hom}^\ell(\mathcal{O}, \mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O}, \mathcal{O} \otimes K_{\mathbb{P}^1}) \\ &= H^\ell(\mathbb{P}^1, \mathcal{O}) \oplus H^\ell(\mathbb{P}^1, \mathcal{O}(-2)) \\ &= \begin{cases} \mathbb{C} & \text{if } \ell = 0,1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$\begin{aligned} \mathcal{A}^\ell(v_0, v_1) &= \mathrm{Hom}^\ell(\mathcal{O}, \mathcal{O}(1)) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O}, \mathcal{O}(1) \otimes K_{\mathbb{P}^1}) \\ &= H^\ell(\mathbb{P}^1, \mathcal{O}(1)) \oplus H^\ell(\mathbb{P}^1, \mathcal{O}(-1)) \\ &= \begin{cases} \mathbb{C}^2 & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$\begin{aligned} \mathcal{A}^\ell(v_1, v_0) &= \mathrm{Hom}^\ell(\mathcal{O}(1), \mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O}(1), \mathcal{O} \otimes K_{\mathbb{P}^1}) \\ &= H^\ell(\mathbb{P}^1, \mathcal{O}(-1)) \oplus H^\ell(\mathbb{P}^1, \mathcal{O}(-3)) \\ &= \begin{cases} \mathbb{C}^2 & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$\begin{aligned} \mathcal{A}^\ell(v_1, v_1) &= \mathrm{Hom}^\ell(\mathcal{O}(1), \mathcal{O}(1)) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O}(1), \mathcal{O}(1) \otimes K_{\mathbb{P}^1}) \\ &= H^\ell(\mathbb{P}^1, \mathcal{O}) \oplus H^\ell(\mathbb{P}^1, \mathcal{O}(-2)) \\ &= \begin{cases} \mathbb{C} & \text{if } \ell = 0,1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus the dg-tilting quiver is given by



where black edges are of degree 0 and brown edges are of degree -1. Since H^0 of the dg-tilting quiver gives the classical tilting quiver, by a change of basis if necessary, we may assume the differential sends

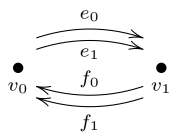
$$dv_0^* = f_1e_0 - f_0e_1, \quad dv_1^* = e_1f_0 - e_0f_1 \quad \text{and} \quad de_i = df_i = 0.$$

The A_∞-structure is given by

$$\begin{aligned} m_2(v_0, f_i) = m_2(f_i, v_1) &= f_i & m_2(v_1, e_i) = m_2(e_i, v_0) &= e_i \\ m_2(f_1, e_0) = -m_2(f_0, e_1) &= v_0^* & m_2(e_1, f_0) = -m_2(e_0, f_1) &= v_1^* \\ m_2(v_0, v_0^*) = m_2(v_0^*, v_0) &= v_0^* & m_2(v_1, v_1^*) = m_2(v_1^*, v_1) &= v_1^* \end{aligned}$$

and zero otherwise.

Example 6.3.3. Consider $X = \mathbb{P}^1$ and $V = \mathcal{O}(-1)^{\oplus 2}$ which is Calabi–Yau 3. Take the exceptional sequence $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1))$ which is V-geometric. The classical tilting quiver is given by



with relations $e_1f_je_0 = e_0f_je_1$ and $f_1e_jf_0 = f_0e_jf_1$.

The corresponding A_∞-category is given by

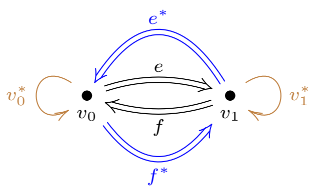
$$\begin{aligned} \mathcal{A}(v_0, v_0) &= \mathrm{Hom}^\ell(\Omega(1)[1], \Omega(1)[1]) \oplus \mathrm{Hom}^{\ell-1}(\Omega(1)[1], \Omega(1)[1] \otimes V) \oplus \mathrm{Hom}^{\ell-2}(\Omega(1)[1], \Omega(1)[1] \otimes \wedge^2 V) \\ &= H^\ell(\mathbb{P}^1, \mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^1, \mathcal{O}(-1)^{\oplus 2}) \oplus H^{\ell-2}(\mathbb{P}^1, \mathcal{O}(-2)) \\ &= \begin{cases} \mathbb{C} & \text{if } \ell = 0, 3 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$\begin{aligned} \mathcal{A}(v_0, v_1) &= \mathrm{Hom}^\ell(\Omega(1)[1], \mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\Omega(1)[1], \mathcal{O} \otimes V) \oplus \mathrm{Hom}^{\ell-2}(\Omega(1)[1], \mathcal{O} \otimes \wedge^2 V) \\ &= H^{\ell-1}(\mathbb{P}^1, \mathcal{O}(1)) \oplus H^{\ell-2}(\mathbb{P}^1, \mathcal{O}^{\oplus 2}) \oplus H^{\ell-3}(\mathbb{P}^1, \mathcal{O}(-1)) \\ &= \begin{cases} \mathbb{C}^2 & \text{if } \ell = 1, 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$\begin{aligned} \mathcal{A}(v_1, v_0) &= \mathrm{Hom}^\ell(\mathcal{O}, \Omega(1)[1]) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O}, \Omega(1)[1] \otimes V) \oplus \mathrm{Hom}^{\ell-2}(\mathcal{O}, \Omega(1)[1] \otimes \wedge^2 V) \\ &= H^{\ell+1}(\mathbb{P}^1, \mathcal{O}(-1)) \oplus H^{\ell}(\mathbb{P}^1, \mathcal{O}(-2)^{\oplus 2}) \oplus H^{\ell-1}(\mathbb{P}^1, \mathcal{O}(-3)) \\ &= \begin{cases} \mathbb{C}^2 & \text{if } \ell = 1, 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$\begin{aligned} \mathcal{A}(v_1, v_1) &= \mathrm{Hom}^\ell(\mathcal{O}, \mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O}, \mathcal{O} \otimes V) \oplus \mathrm{Hom}^{\ell-2}(\mathcal{O}, \mathcal{O} \otimes \wedge^2 V) \\ &= H^\ell(\mathbb{P}^1, \mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^1, \mathcal{O}(-1)^{\oplus 2}) \oplus H^{\ell-2}(\mathbb{P}^1, \mathcal{O}(-2)) \\ &= \begin{cases} \mathbb{C}^2 & \text{if } \ell = 0, 3 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus the dg-quiver is given by



.

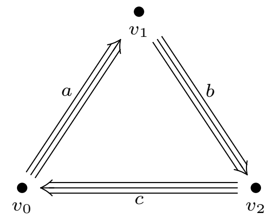
Since H^0 of the dg-quiver gives back the classical tilting quiver, by a change of basis, we may assume

$$de_1 = f_0e_0f_1 - f_1e_0f_0, \quad de_0 = f_1e_1f_0 - f_0e_1f_1,$$

$$df_1 = e_1f_0e_0 - e_0f_0e_1, \quad df_0 = e_0f_1e_1 - e_1f_1e_0.$$

The superpotential is thus given by $\Phi = f_1e_0f_0e_1 - f_1e_1f_0e_0$.

Example 6.3.4. Consider $X = \mathbb{P}^2$, $V = K_{\mathbb{P}^2}$ which is Calabi–Yau 3, with exceptional sequence $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ which is $K_{\mathbb{P}^2}$-geometric. The classical tilting quiver is given by



with relations $b_{i+1}a_i = b_ia_{i+1}$, $c_{i+1}b_i = c_ib_{i+1}$, and $a_{i+1}c_i = a_ic_{i+1}$.

The corresponding A_{fin}-category is given by

$$\begin{aligned}
\mathcal{A}^\ell(v_0, v_0) &= \mathrm{Hom}^\ell(\Omega^2(2)[2], \Omega^2(2)[2]) \oplus \mathrm{Hom}^{\ell-1}(\Omega^2(2)[2], \Omega^2(2)[2] \otimes K_{\mathbb{P}^2}) \\
&= H^\ell(\mathbb{P}^2, \mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^2, \mathcal{O}(-3)) \\
&= \begin{cases} H^0(\mathbb{P}^2, \mathcal{O}) & \text{if } \ell = 0 \\ H^2(\mathbb{P}^2, \mathcal{O}(-3)) & \text{if } \ell = 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathbb{C} & \text{if } \ell = 0, 3 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_0, v_1) &= \mathrm{Hom}^\ell(\Omega^2(2)[2], \Omega(1)[1]) \oplus \mathrm{Hom}^{\ell-1}(\Omega^2(2)[2], \Omega(1)[1] \otimes K_{\mathbb{P}^2}) \\
&= H^{\ell-1}(\mathbb{P}^2, \Omega(2)) \oplus H^{\ell-2}(\mathbb{P}^2, \Omega(-1)) \\
&= H^{\ell-1}(\mathbb{P}^2, \Omega(2)) \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_0, v_2) &= \mathrm{Hom}^\ell(\Omega^2(2)[2], \mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\Omega^2(2)[2], \mathcal{O} \otimes K_{\mathbb{P}^2}) \\
&= H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(1)) \oplus H^{\ell-3}(\mathbb{P}^2, \mathcal{O}(-2)) \\
&= H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(1)) \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_1, v_0) &= \mathrm{Hom}^\ell(\Omega(1)[1], \Omega^2(2)[2]) \oplus \mathrm{Hom}^{\ell-1}(\Omega(1)[1], \Omega^2(2)[2] \otimes K_{\mathbb{P}^2}) \\
&= H^{\ell+1}(\mathbb{P}^2, \mathcal{T}(-2)) \oplus H^\ell(\mathbb{P}^2, \mathcal{T}(-5)) \\
&= H^\ell(\mathbb{P}^2, \mathcal{T}(-5)) \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_1, v_1) &= \mathrm{Hom}^\ell(\Omega(1)[1], \Omega(1)[1]) \oplus \mathrm{Hom}^{\ell-1}(\Omega(1)[1], \Omega(1)[1] \otimes K_{\mathbb{P}^2}) \\
&= H^\ell(\mathbb{P}^2, \mathcal{T} \otimes \Omega) \oplus H^{\ell-1}(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-3)) \\
&= \begin{cases} H^0(\mathbb{P}^2, \mathcal{T} \otimes \Omega) & \text{if } \ell = 0 \\ H^2(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-3)) & \text{if } \ell = 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathbb{C} & \text{if } \ell = 0, 3 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

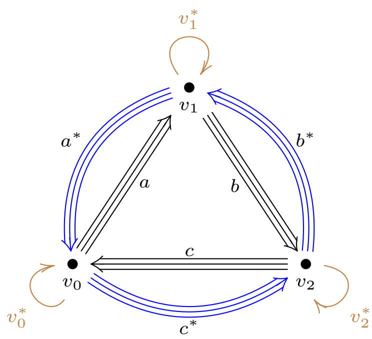
$$
\begin{aligned}
\mathcal{A}^\ell(v_1, v_2) &= \mathrm{Hom}^\ell(\Omega(1)[1], \mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\Omega(1)[1], \mathcal{O} \otimes K_{\mathbb{P}^2}) \\
&= H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-1)) \oplus H^{\ell-2}(\mathbb{P}^2, \mathcal{T}(-4)) \\
&= H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-1)) \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_2, v_0) &= \mathrm{Hom}^\ell(\mathcal{O}, \Omega^2(2)[2]) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O}, \Omega^2(2)[2] \otimes K_{\mathbb{P}^2}) \\
&= H^{\ell+2}(\mathbb{P}^2, \mathcal{O}(-1)) \oplus H^{\ell+1}(\mathbb{P}^2, \mathcal{O}(-4)) \\
&= H^{\ell+1}(\mathbb{P}^2, \mathcal{O}(-4)) \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_2, v_1) &= \mathrm{Hom}^\ell(\mathcal{O}, \Omega(1)[1]) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O}, \Omega(1)[1] \otimes K_{\mathbb{P}^2}) \\
&= H^{\ell+1}(\mathbb{P}^2, \Omega(1)) \oplus H^\ell(\mathbb{P}^2, \Omega(-2)) \\
&= H^\ell(\mathbb{P}^2, \Omega(-2)) \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_2, v_2) &= \mathrm{Hom}^\ell(\mathcal{O}, \mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O}, \mathcal{O} \otimes K_{\mathbb{P}^2}) \\
&= H^\ell(\mathbb{P}^2, \mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^2, \mathcal{O}(-3)) \\
&= \begin{cases} H^0(\mathbb{P}^2, \mathcal{O}) & \text{if } \ell = 0 \\ H^2(\mathbb{P}^2, \mathcal{O}(-3)) & \text{if } \ell = 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathbb{C} & \text{if } \ell = 0, 3 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

The quiver with superpotential is given by



(6.3.1)

where black edges are of degree 0, blue edges are of degree -1, brown loops are of degree -2. Since H^0 of the quiver must be isomorphic to the quiver with relation in (6.3.4), by a change of basis if necessary, we may assume the differential d sends

$$da_i^* = b_{i+1}c_{i+2} - b_{i+2}c_{i+1} \quad db_i^* = c_{i+1}a_{i+2} - c_{i+2}a_{i+1} \quad dc_i^* = a_{i+1}b_{i+2} - a_{i+2}b_{i+1}.$$

Thus the superpotential is given by $\Phi = \sum_{i=0}^{2} a_i(b_{i+1}c_{i+2} - b_{i+2}c_{i+1})$.

Example 6.3.5. Consider $V = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ over $\mathbb{P}^2$ which is Calabi–Yau 4. The classical tilting quiver is given by

$$\begin{tikzcd}
v_0 \arrow[r, shift left=2, "a_0"] \arrow[r, "a_1"] \arrow[r, shift right=2, "a_2"'] & v_1 \arrow[l, bend right, "u"'] \arrow[r, shift left=2, "b_0"] \arrow[r, "b_1"] \arrow[r, shift right=2, "b_2"'] & v_2 \arrow[l, bend right, "v"'] \arrow[ll, bend left, "w"]
\end{tikzcd}$$

with relations

$$\begin{aligned} a_i u &= v b_i & \quad a_{i+1} w b_i &= a_i w b_{i+1} \\ b_{i+1} a_i &= b_i a_{i+1} & \quad w b_i v &= u a_i w. \end{aligned}$$

We calculate its A_∞-category:

$$\begin{aligned}
\mathcal{A}^\ell(v_0, v_0) &= \operatorname{Hom}^\ell(\Omega^2(2)[2], \Omega^2(2)[2]) \oplus \operatorname{Hom}^{\ell-1}(\Omega^2(2)[2], \Omega^2(2)[2] \otimes V) \\
&\quad \oplus \operatorname{Hom}^{\ell-2}(\Omega^2(2)[2], \Omega^2(2)[2] \otimes \wedge^2 V) \\
&= H^\ell(\mathbb{P}^2, \mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^2, \mathcal{O}(-1) \oplus \mathcal{O}(-2)) \oplus H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(-3)) \\
&= \begin{cases} \mathbb{C} & \text{if } \ell = 0, 4 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_0, v_1) &= \operatorname{Hom}^\ell(\Omega^2(2)[2], \Omega(1)[1]) \oplus \operatorname{Hom}^{\ell-1}(\Omega^2(2)[2], \Omega(1)[1] \otimes V) \\
&\quad \oplus \operatorname{Hom}^{\ell-2}(\Omega^2(2)[2], \Omega(1)[1] \otimes \wedge^2 V) \\
&= H^{\ell-1}(\mathbb{P}^2, \Omega(2)) \oplus H^{\ell-2}(\mathbb{P}^2, \Omega \oplus \Omega(1)) \oplus H^{\ell-3}(\mathbb{P}^2, \Omega(-1)) \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell = 1 \\ \mathbb{C} & \text{if } \ell = 3 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_0, v_2) &= \operatorname{Hom}^\ell(\Omega^2(2)[2], \mathcal{O}) \oplus \operatorname{Hom}^{\ell-1}(\Omega^2(2)[2], \mathcal{O} \otimes V) \oplus \operatorname{Hom}^{\ell-2}(\Omega^2(2)[2], \mathcal{O} \otimes \wedge^2 V) \\
&= H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(1)) \oplus H^{\ell-3}(\mathbb{P}^2, \mathcal{O} \oplus \mathcal{O}(-1)) \oplus H^{\ell-4}(\mathbb{P}^2, \mathcal{O}(-2)) \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell = 2 \\ \mathbb{C} & \text{if } \ell = 3 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_1, v_0) &= \operatorname{Hom}^\ell(\Omega(1)[1], \Omega^2(2)[2]) \oplus \operatorname{Hom}^{\ell-1}(\Omega(1)[1], \Omega^2(2)[2] \otimes V) \\
&\quad \oplus \operatorname{Hom}^{\ell-2}(\Omega(1)[1], \Omega^2(2)[2] \otimes \wedge^2 V) \\
&= H^{\ell+1}(\mathbb{P}^2, \mathcal{T}(-2)) \oplus H^{\ell}(\mathbb{P}^2, \mathcal{T}(-3) \oplus \mathcal{T}(-4)) \oplus H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-5)) \\
&= \begin{cases} \mathbb{C} & \text{if } \ell = 1 \\ \mathbb{C}^3 & \text{if } \ell = 3 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_1, v_1) &= \operatorname{Hom}^\ell(\Omega(1)[1], \Omega(1)[1]) \oplus \operatorname{Hom}^{\ell-1}(\Omega(1)[1], \Omega(1)[1] \otimes V) \oplus \operatorname{Hom}^{\ell-2}(\Omega(1)[1], \Omega(1)[1] \otimes \wedge^2 V) \\
&= H^\ell(\mathbb{P}^2, \mathcal{T} \otimes \Omega) \oplus H^{\ell-1}(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-1) \oplus \mathcal{T} \otimes \Omega(-2)) \oplus H^{\ell-2}(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-3)) \\
&= \begin{cases} \mathbb{C} & \text{if } \ell = 0, 4 \\ \mathbb{C}^6 & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

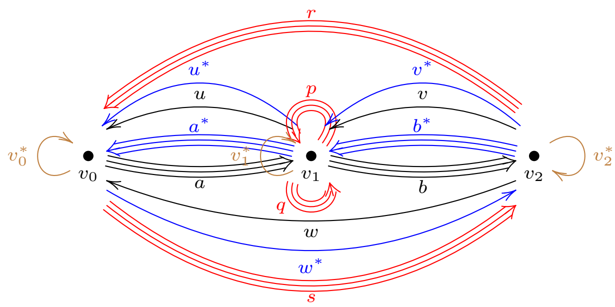
$$
\begin{aligned}
\mathcal{A}^\ell(v_1,v_2) &= \mathrm{Hom}^\ell(\Omega(1)[1],\mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\Omega(1)[1],\mathcal{O}\otimes V) \oplus \mathrm{Hom}^{\ell-2}(\Omega(1)[1],\mathcal{O}\otimes \wedge^2 V)\\
&= H^{\ell-1}(\mathbb{P}^2,\mathcal{T}(-1)) \oplus H^{\ell-2}(\mathbb{P}^2,\mathcal{T}(-2)\oplus\mathcal{T}(-3)) \oplus H^{\ell-3}(\mathbb{P}^2,\mathcal{T}(-4))\\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell=1\\ \mathbb{C} & \text{if } \ell=3\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_2,v_0) &= \mathrm{Hom}^\ell(\mathcal{O},\Omega^2(2)[2]) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O},\Omega^2(2)[2]\otimes V) \oplus \mathrm{Hom}^{\ell-2}(\mathcal{O},\Omega^2(2)[2]\otimes \wedge^2 V)\\
&= H^{\ell+2}(\mathbb{P}^2,\mathcal{O}(-1)) \oplus H^{\ell+1}(\mathbb{P}^2,\mathcal{O}(-2)\oplus\mathcal{O}(-3)) \oplus H^{\ell}(\mathbb{P}^2,\mathcal{O}(-4))\\
&= \begin{cases} \mathbb{C} & \text{if } \ell=1\\ \mathbb{C}^3 & \text{if } \ell=2\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_2,v_1) &= \mathrm{Hom}^\ell(\mathcal{O},\Omega(1)[1]) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O},\Omega(1)[1]\otimes V) \oplus \mathrm{Hom}^{\ell-2}(\mathcal{O},\Omega(1)[1]\otimes \wedge^2 V)\\
&= H^{\ell+1}(\mathbb{P}^2,\Omega(1)) \oplus H^{\ell}(\mathbb{P}^2,\Omega\oplus\Omega(-1)) \oplus H^{\ell-1}(\mathbb{P}^2,\Omega(-2))\\
&= \begin{cases} \mathbb{C} & \text{if } \ell=1\\ \mathbb{C}^3 & \text{if } \ell=3\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_2,v_2) &= \mathrm{Hom}^\ell(\mathcal{O},\mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O},\mathcal{O}\otimes V) \oplus \mathrm{Hom}^{\ell-2}(\mathcal{O},\mathcal{O}\otimes \wedge^2 V)\\
&= H^{\ell}(\mathbb{P}^2,\mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^2,\mathcal{O}(-1)\oplus\mathcal{O}(-2)) \oplus H^{\ell-2}(\mathbb{P}^2,\mathcal{O}(-3))\\
&= \begin{cases} \mathbb{C} & \text{if } \ell=0,4\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

The underlying graded quiver of $E(\mathcal{A}_\mathcal{E})$ is given by



where black edges are of degree 0, red edges are of degree -1, blue edges are of degree -2 and brown edges are of degree -3. Since $H^0(E(\mathcal{A}_\mathcal{E})) = \Lambda_\mathcal{E}$, by a linear change of basis if necessary, we may assume the differential d in $E(\mathcal{A}_\mathcal{E})$ is given by

$$
\begin{aligned}
dp_i &= a_i u - v b_i & dq_i &= a_{i+2} w b_{i+1} - a_{i+1} w b_{i+2}\\
dr_i &= w b_i v - u a_i w & ds_i &= b_{i+2} a_{i+1} - b_{i+1} a_{i+2},
\end{aligned}
$$

where $i \in \mathbb{Z}_3$. Hence the only nonvanishing maps in the form $m_n : \mathcal{A}^1 \otimes \cdots \otimes \mathcal{A}^1 \to \mathcal{A}^2$ are

$$
m_2(a_i,u) = p_i, \quad m_2(v,b_i) = -p_i, \quad m_3(a_{i+2},w,b_{i+1}) = q_i, \quad m_3(a_{i+1},w,b_{i+2}) = -q_i
$$

$$
m_3(w,b_i,v) = r_i, \quad m_3(u,a_i,w) = -r_i, \quad m_2(b_{i+2},a_{i+1}) = s_i, \quad m_2(b_{i+1},a_{i+2}) = -s_i.
$$

Next, we would like to calculate the cyclic structure $\langle -,-\rangle : \mathcal{A}^2 \otimes \mathcal{A}^2 \to \mathbb{C}$. Recall that the pairing is cyclic: $\langle m_n(e_1,\ldots,e_n), e_{n+1}\rangle = (-1)^{(|e_1|+1)n}\langle e_1, m_n(e_2,\ldots,e_{n+1})\rangle$. We have

$$
\begin{aligned}
\langle p_i,q_j\rangle &= \langle m_2(a_i,u), m_3(a_{j+2},w,b_{j+1})\rangle = \langle a_i, m_2(u, m_3(a_{j+2},w,b_{j+1}))\rangle = -\langle a_i, m_2(r_{j+2},b_{j+1})\rangle\\
&= -\langle (m_2(a_i,r_{j+2}), b_{j+1}\rangle = \langle b_{j+1}, m_2(a_i,r_{j+2})\rangle = \langle m_2(b_{j+1},a_i), r_{j+2}\rangle = \delta_{ij}\langle s_{i+2}, r_{i+2}\rangle.
\end{aligned}
$$

In particular, this implies $\langle p_i, q_i\rangle = \langle s_i, r_i\rangle$ for all i, and

$$\langle p_i, p_j\rangle = \langle m_2(a_i, u), m_2(a_j, u)\rangle = \langle a_i, m_2(u, m_2(a_j, u))\rangle = \langle a_i, m_2(m_2(u, a_j), u)\rangle = 0.$$

$$\langle q_i, q_j\rangle = \langle m_3(a_{i+2}, w, b_{i+1}), m_3(a_{j+2}, w, b_{j+1})\rangle = \pm\langle a_{i+2}, m_3(w, b_{i+1}, m_3(a_{j+2}, w, b_{j+1}))\rangle.$$

By the A_∞-relation for $n = 5$, we get

$$m_3(w, b_{i+1}, m_3(a_{j+2}, w, b_{j+1})) = \pm m_4(w, m_2(b_{i+1}, a_{j+2}), w, b_{j+1}))$$

since all other terms vanish. Then

$$\langle q_i, q_j\rangle = \pm\langle a_{i+2}, m_4(w, m_2(b_{i+1}, a_{j+2}), w, b_{j+1}))\rangle = \pm\langle m_2(b_{i+1}, a_{j+2}), m_4(w, b_{j+1}, a_{i+2}, w)\rangle = 0.$$

This shows that the only (possibly) nonvanishing pairings are $\langle p_i, q_i\rangle = \langle s_i, r_i\rangle$. Since the pairing is nondegenerate, we may normalize this number to 1. Thus the superpotential is given by

$$\Phi = (a_{i+2}wb_{i+1} - a_{i+1}wb_{i+2})p_i + (a_iu - vb_i)q_i + (b_{i+2}a_{i+1} - b_{i+1}a_{i+2})r_i + (wb_iv - ua_iw)s_i.$$

Example 6.3.6. Consider $X = \mathbb{P}^2$ and $V = T^\vee_{\mathbb{P}^2}$ which is Calabi–Yau 4. The classical quiver is given by

$$\begin{tikzcd}
\underset{v_0}{\bullet} \arrow[r, shift left=3, "a_0"] \arrow[r, shift left=2, "a_1"] \arrow[r, shift left=1, "a_2"'] & \underset{v_1}{\bullet} \arrow[l, shift left=1, "b_0"] \arrow[l, shift left=2, "b_1"] \arrow[l, shift left=3, "b_2"'] \arrow[r, shift left=3, "c_0"] \arrow[r, shift left=2, "c_1"] \arrow[r, shift left=1, "c_2"'] & \underset{v_2}{\bullet} \arrow[l, shift left=1, "d_0"] \arrow[l, shift left=2, "d_1"] \arrow[l, shift left=3, "d_2"']
\end{tikzcd}$$

with relations

$$\begin{array}{cc}
\sum_{j=0}^{2} a_jb_j = 0 & \sum_{j=0}^{2} b_ja_j = 0 \\
\sum_{j=0}^{2} c_jd_j = 0 & \sum_{j=0}^{2} d_jc_j = 0 \\
c_{j+1}a_j = c_ja_{j+1} & b_{j+1}d_j = b_jd_{j+1} \\
a_jb_k = d_kc_j. &
\end{array}$$

The relations listed above are not all independent: the relation $a_jb_k = d_kc_j$ and $\sum_{j=0}^{2} a_jb_j = 0$ together implies $\sum_{j=0}^{2} d_jc_j = 0$.

Next, we would like to calculate $\mathcal{A}_\mathcal{E}$. Recall the dual sequence to $\mathcal{E}$ is given by

$$\mathcal{F} = (\Omega^2(2)[2], \Omega(1)[1], \mathcal{O}) = (\mathcal{O}(-1)[2], \Omega(1)[1], \mathcal{O}).$$

We have

$$\begin{aligned}
\mathcal{A}^\ell(v_0, v_0) &= \mathrm{Hom}^\ell(\Omega^2(2)[2], \Omega^2(2)[2]) \oplus \mathrm{Hom}^{\ell-1}(\Omega^2(2)[2], \Omega^2(2)[2] \otimes T^\vee_{\mathbb{P}^2}) \\
&\quad \oplus \mathrm{Hom}^{\ell-2}(\Omega^2(2)[2], \Omega^2(2)[2] \otimes \wedge^2 T^\vee_{\mathbb{P}^2}) \\
&= H^\ell(\mathbb{P}^2, \mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^2, \Omega) \oplus H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(-3)) \\
&= \begin{cases} \mathbb{C} & \text{if } \ell = 0, 2, 4 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_0, v_1) &= \mathrm{Hom}^\ell(\Omega^2(2)[2], \Omega(1)[1]) \oplus \mathrm{Hom}^{\ell-1}(\Omega^2(2)[2], \Omega(1)[1] \otimes T^\vee_{\mathbb{P}^2}) \\
&\quad \oplus \mathrm{Hom}^{\ell-2}(\Omega^2(2)[2], \Omega(1)[1] \otimes \wedge^2 T^\vee_{\mathbb{P}^2}) \\
&= H^{\ell-1}(\mathbb{P}^2, \Omega(2)) \oplus H^{\ell-2}(\mathbb{P}^2, \Omega \otimes \Omega(2)) \oplus H^{\ell-3}(\mathbb{P}^2, \Omega(-1)) \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell = 1, 3 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_0,v_2) &= \mathrm{Hom}^\ell(\Omega^2(2)[2],\mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\Omega^2(2)[2],\mathcal{O}\otimes T^\vee_{\mathbb{P}^2}) \oplus \mathrm{Hom}^{\ell-2}(\Omega^2(2)[2],\mathcal{O}\otimes \wedge^2 T^\vee_{\mathbb{P}^2})\\
&= H^{\ell-2}(\mathbb{P}^2,\mathcal{O}(1)) \oplus H^{\ell-3}(\mathbb{P}^2,\Omega(1)) \oplus H^{\ell-4}(\mathbb{P}^2,\mathcal{O}(-2))\\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell=2\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_1,v_0) &= \mathrm{Hom}^\ell(\Omega(1)[1],\Omega^2(2)[2]) \oplus \mathrm{Hom}^{\ell-1}(\Omega(1)[1],\Omega^2(2)[2]\otimes T^\vee_{\mathbb{P}^2})\\
&\quad \oplus \mathrm{Hom}^{\ell-2}(\Omega(1)[1],\Omega^2(2)[2]\otimes \wedge^2 T^\vee_{\mathbb{P}^2})\\
&= H^{\ell+1}(\mathbb{P}^2,\mathcal{T}(-2)) \oplus H^{\ell}(\mathbb{P}^2,\mathcal{T}\otimes\Omega(-2)) \oplus H^{\ell-1}(\mathbb{P}^2,\mathcal{T}(-5))\\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell=1,3\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_1,v_1) &= \mathrm{Hom}^\ell(\Omega(1)[1],\Omega(1)[1]) \oplus \mathrm{Hom}^{\ell-1}(\Omega(1)[1],\Omega(1)[1]\otimes T^\vee_{\mathbb{P}^2})\\
&\quad \oplus \mathrm{Hom}^{\ell-2}(\Omega(1)[1],\Omega(1)[1]\otimes \wedge^2 T^\vee_{\mathbb{P}^2})\\
&= H^{\ell}(\mathbb{P}^2,\mathcal{T}\otimes\Omega) \oplus H^{\ell-1}(\mathbb{P}^2,\mathcal{T}\otimes\Omega\otimes\Omega) \oplus H^{\ell-2}(\mathbb{P}^2,\mathcal{T}\otimes\Omega(-3))\\
&= \begin{cases} \mathbb{C} & \text{if } \ell=0,4\\ \mathbb{C}^{10} & \text{if } \ell=2\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

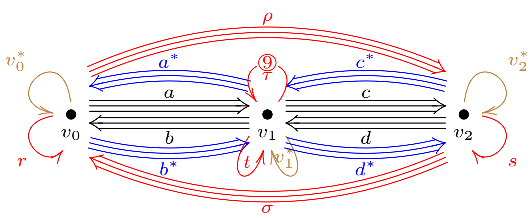
$$\begin{aligned}
\mathcal{A}^\ell(v_1,v_2) &= \mathrm{Hom}^\ell(\Omega(1)[1],\mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\Omega(1)[1],\mathcal{O}\otimes T^\vee_{\mathbb{P}^2}) \oplus \mathrm{Hom}^{\ell-2}(\Omega(1)[1],\mathcal{O}\otimes \wedge^2 T^\vee_{\mathbb{P}^2})\\
&= H^{\ell-1}(\mathbb{P}^2,\mathcal{T}(-1)) \oplus H^{\ell-2}(\mathbb{P}^2,\mathcal{T}\otimes\Omega(-1)) \oplus H^{\ell-3}(\mathbb{P}^2,\mathcal{T}(-4))\\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell=1,3\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_2,v_0) &= \mathrm{Hom}^\ell(\mathcal{O},\Omega^2(2)[2]) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O},\Omega^2(2)[2]\otimes T^\vee\mathbb{P}^2) \oplus \mathrm{Hom}^{\ell-2}(\mathcal{O},\Omega^2(2)[2]\otimes \wedge^2 T^\vee\mathbb{P}^2)\\
&= H^{\ell+2}(\mathbb{P}^2,\mathcal{O}(-1)) \oplus H^{\ell+1}(\mathbb{P}^2,\Omega(-1)) \oplus H^{\ell}(\mathbb{P}^2,\mathcal{O}(-4))\\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell=2\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_2,v_1) &= \mathrm{Hom}^\ell(\mathcal{O},\Omega(1)[1]) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O},\Omega(1)[1]\otimes T^\vee_{\mathbb{P}^2}) \oplus \mathrm{Hom}^{\ell-2}(\mathcal{O},\Omega(1)[1]\otimes \wedge^2 T^\vee_{\mathbb{P}^2})\\
&= H^{\ell+1}(\mathbb{P}^2,\Omega(1)) \oplus H^{\ell}(\mathbb{P}^2,\Omega\otimes\Omega(1)) \oplus H^{\ell-1}(\mathbb{P}^2,\Omega(-2))\\
&= \begin{cases} \mathbb{C}^3 & \text{if } \ell=1,3\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_2,v_2) &= \mathrm{Hom}^\ell(\mathcal{O},\mathcal{O}) \oplus \mathrm{Hom}^{\ell-1}(\mathcal{O},\mathcal{O}\otimes T^\vee\mathbb{P}^2) \oplus \mathrm{Hom}^{\ell-2}(\mathcal{O},\mathcal{O}\otimes \wedge^2 T^\vee\mathbb{P}^2)\\
&= H^{\ell}(\mathbb{P}^2,\mathcal{O}) \oplus H^{\ell-1}(\mathbb{P}^2,\Omega) \oplus H^{\ell-2}(\mathbb{P}^2,\mathcal{O}(-3))\\
&= \begin{cases} \mathbb{C} & \text{if } \ell=0,2,4\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

The underlying graded quiver of the dg-quiver $E(\mathcal{A}_\mathcal{E})$ is thus given by



where black edges are of degree 0, red edges are of degree -1, blue edges are of degree -2 and brown edges are of degree -3. The 10 loops at v_1 will be labelled by τ_{ij} for $1 \le i,j \le 3$ and t. Abusing notation, we will denote an edge e in the quiver and its corresponding element in the A_∞-category $\mathcal{A}_\mathcal{E}$ by the same symbol e, instead of the more correct $e^\vee$. Since $T^\vee_{\mathbb{P}^2}$ is noncompact 4-Calabi–Yau, $\mathcal{A}_\mathcal{E}$ has a cyclic structure of degree 4. By a linear change of basis if necessary, we may assume

$$m_2(a^*, a) = v_0^*, \quad m_2(b^*, b) = m_2(c^*, c) = v_1^*, \quad m_2(d^*, d) = v_2^*.$$

Since $H^0(E(\mathcal{A}_\mathcal{E})) = \Lambda_\mathcal{E}$, by a linear change of basis if necessary, we may assume the differential d in $E(\mathcal{A}_\mathcal{E})$ sends

$$d\rho_i = c_{i+2}a_{i+1} - c_{i+1}a_{i+2}, \quad d\sigma_i = b_{i+2}d_{i+1} - b_{i+1}d_{i+2}$$
$$dr = \sum_{i=0}^{2} b_i a_i, \quad dt = \sum_{i=0}^{2} a_i b_i + d_i c_i, \quad ds = \sum_{i=0}^{2} c_i d_i$$
$$d\tau_{ij} = a_i b_j - d_j c_i.$$

Hence the only nonvanishing maps in the form $m_n : \mathcal{A}^1 \otimes \cdots \otimes \mathcal{A}^1 \to \mathcal{A}^2$ are

$$m_2(c_{i+2}, a_{i+1}) = \rho_i, \quad m_2(c_{i+1}, a_{i+2}) = -\rho_i,$$
$$m_2(b_{i+2}, d_{i+1}) = \sigma_i, \quad m_2(b_{i+1}, d_{i+2}) = -\sigma_i,$$
$$m_2(b_i, a_i) = r, \quad m_2(c_i, d_i) = s,$$
$$m_2(a_i, b_j) = \tau_{ij} + \delta_{ij}t, \quad m_2(d_j, c_i) = -\tau_{ij} + \delta_{ij}t$$

where δ_{ij} is the Kronecker delta, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

We would like to calculate the cyclic structure $\langle -, - \rangle : \mathcal{A}^2 \otimes \mathcal{A}^2 \to \mathbb{C}$. Recall that the pairing is cyclic: $\langle m_n(e_1, \ldots, e_n), e_{n+1} \rangle = (-1)^{(|e_1|+1)n} \langle e_1, m_n(e_2, \ldots, e_{n+1}) \rangle$. If $i \neq j$,

$$\begin{aligned} 2\langle t, \tau_{ij} \rangle &= \langle m_2(a_k, b_k) + m_2(d_k, c_k), \tau_{ij} \rangle \\ &= \langle m_2(a_k, b_k), m_2(a_i, b_j) \rangle - \langle m_2(d_k, c_k), m_2(d_j, c_i) \rangle \\ &= \langle a_k, m_2(m_2(b_k, a_i), b_j) \rangle - \langle d_k, m_2(m_2(c_k, d_j), c_i) \rangle \\ &= 0 \end{aligned}$$

by choosing $k \neq i$ and $k \neq j$. Also,

$$\begin{aligned} 4\langle t, \tau_{ii} \rangle &= \langle m_2(a_k, b_k) + m_2(d_k, c_k), m_2(a_i, b_i) - m_2(d_i, c_i) \rangle \\ &= \langle m_2(a_k, b_k), m_2(a_i, b_i) \rangle + \langle m_2(d_k, c_k), m_2(a_i, b_i) \rangle \\ &\quad - \langle m_2(a_k, b_k), m_2(d_i, c_i) \rangle - \langle m_2(d_k, c_k), m_2(d_i, c_i) \rangle \\ &= \begin{cases} \langle a_i, m_2(r, b_i) \rangle - \langle d_i, m_2(s, c_i) \rangle & \text{if } k = i \\ \langle d_{i+1}, m_2(\rho_{i+2}, b_i) \rangle - \langle a_{i+1}, m_2(\sigma_{i+2}, c_i) \rangle & \text{if } k = i+1 \end{cases} \\ &= \begin{cases} -\langle r, m_2(b_i, a_i) \rangle + \langle s, m_2(c_i, d_i) \rangle & \text{if } k = i \\ -\langle \rho_{i+2}, m_2(b_i, d_{i+1}) \rangle + \langle \sigma_{i+2}, m_2(c_i, a_{i+1}) \rangle & \text{if } k = i+1 \end{cases} \\ &= \begin{cases} -\langle r, r \rangle + \langle s, s \rangle & \text{if } k = i \\ -\langle \rho_{i+2}, \sigma_{i+2} \rangle + \langle \sigma_{i+2}, \rho_{i+2} \rangle & \text{if } k = i+1 \end{cases} \\ &= \begin{cases} -\langle r, r \rangle + \langle s, s \rangle & \text{if } k = i \\ 0 & \text{if } k = i+1 \end{cases} \end{aligned}$$

Since k is arbitrary, $\langle t, \tau_{ij} \rangle = 0$ for all i, j and $\langle r, r \rangle = \langle s, s \rangle$.

For $i \neq j$ and $k \neq \ell$,

$$\langle \tau_{ij}, \tau_{k\ell} \rangle = \langle m_2(a_i, b_j), m_2(a_k, b_\ell) \rangle = \langle a_i, \delta_{jk} m_2(r, b_\ell) \rangle = -\delta_{jk} \langle r, m_2(b_\ell, a_i) \rangle = -\delta_{jk}\delta_{i\ell} \langle r, r \rangle$$

For $i \neq j$,

$$\begin{aligned} 2\langle \tau_{ij}, \tau_{kk} \rangle &= \langle \tau_{ij}, m_2(a_k, b_k) - m_2(d_k, c_k) \rangle = \langle m_2(a_i, b_j), m_2(a_k, b_k) \rangle + \langle m_2(d_j, c_i), m_2(d_k, c_k) \rangle \\ &= \delta_{jk} \langle a_i, m_2(r, b_k) \rangle + \delta_{ik} \langle d_j, m_2(s, c_k) \rangle = -\delta_{jk}\delta_{ik} \langle r, r \rangle - \delta_{jk}\delta_{ik} \langle s, s \rangle = -2\delta_{jk}\delta_{ik} \langle r, r \rangle = 0 \end{aligned}$$

since $i \neq j$. We have

$$\begin{aligned}
4\langle \tau_{ii}, \tau_{jj}\rangle &= \langle m_2(a_i,b_i) - m_2(d_i,c_i), m_2(a_j,b_j) - m_2(d_j,c_j)\rangle \\
&= \langle m_2(a_i,b_i), m_2(a_j,b_j)\rangle - \langle m_2(d_i,c_i), m_2(a_j,b_j)\rangle \\
&\quad - \langle m_2(a_i,b_i), m_2(d_j,c_j)\rangle + \langle m_2(d_i,c_i), m_2(d_j,c_j)\rangle \\
&= \langle a_i, m_2(m_2(b_i,a_j),b_j)\rangle - \langle d_i, m_2(m_2(c_i,a_j),b_j)\rangle \\
&\quad - \langle a_i, m_2(m_2(b_i,d_j),c_j)\rangle + \langle d_i, m_2(m_2(c_i,d_j),c_j)\rangle \\
&= \begin{cases} -\langle m_2(b_i,a_i), m_2(b_i,a_i)\rangle - \langle m_2(c_i,d_i), m_2(c_i,d_i)\rangle & \text{if } j = i \\ \langle m_2(c_i,a_{i+1}), m_2(b_{i+1},d_i)\rangle + \langle m_2(b_i,d_{i+1}), m_2(c_{i+1},a_i)\rangle & \text{if } j = i+1 \\ \langle m_2(c_i,a_{i-1}), m_2(b_{i-1},d_i)\rangle + \langle m_2(b_i,d_{i-1}), m_2(c_{i-1},a_i)\rangle & \text{if } j = i-1 \end{cases} \\
&= \begin{cases} -\langle r,r\rangle - \langle s,s\rangle & \text{if } j = i \\ \langle -\rho_{i+2}, \sigma_{i+2}\rangle + \langle -\sigma_{i+2}, \rho_{i+2}\rangle & \text{if } j = i+1 \\ \langle \rho_{i+1}, -\sigma_{i+1}\rangle + \langle \sigma_{i+1}, -\rho_{i+1}\rangle & \text{if } j = i-1 \end{cases} \\
&= \begin{cases} -2\langle r,r\rangle & \text{if } j = i \\ -2\langle \rho_{i+2}, \sigma_{i+2}\rangle & \text{if } j = i+1 \\ -2\langle \rho_{i+1}, \sigma_{i+1}\rangle & \text{if } j = i-1 \end{cases}
\end{aligned}$$

$$\begin{aligned}
4\langle t,t\rangle &= \langle m_2(a_{j+1},b_{j+1}) + m_2(d_{j+1},c_{j+1}), m_2(a_j,b_j) + m_2(d_j,c_j)\rangle \\
&= \langle a_{j+1}, m_2(m_2(b_{j+1},a_j),b_j)\rangle + \langle a_{j+1}, m_2(m_2(b_{j+1},d_j),c_j)\rangle \\
&\quad \langle d_{j+2}, m_2(m_2(c_{j+1},a_j),b_j)\rangle + \langle d_{j+1}, m_2(m_2(c_{j+1},d_j),c_j)\rangle \\
&= 0 + \langle a_{j+1}, m_2(\sigma_{j+2},c_j)\rangle + \langle d_{j+1}, m_2(\rho_{j+2},b_j)\rangle + 0 \\
&= -2\langle \rho_{j+2}, \sigma_{j+2}\rangle
\end{aligned}$$

$$\begin{aligned}
\langle \rho_i, \sigma_j\rangle &= \langle m_2(c_{i+2},a_{i+1}), m_2(b_{j+2},d_{j+1})\rangle \\
&= \langle c_{i+2}, m_2(a_{i+1}, m_2(b_{j+2},d_{j+1}))\rangle \\
&= \langle c_{i+2}, m_2(m_2(a_{i+1},b_{j+2}),d_{j+1}))\rangle \\
&= \begin{cases} \langle c_{i+2}, m_2(\tau_{i+1,i+2}, d_{i+1})\rangle & \text{if } i = j \\ \langle c_{i+2}, m_2(\tau_{i+1,i}, d_{i+2}))\rangle & \text{if } i = j-1 \\ \langle c_j, m_2(\tau_{j+2,j+2} + t, d_{i+2}))\rangle & \text{if } i = j+1 \end{cases} \\
&= \begin{cases} \langle \tau_{i+1,i+2}, \tau_{i+2,i+1}\rangle & \text{if } i = j \\ \langle \tau_{i+1,i}, -\tau_{i+2,i+2} + t\rangle & \text{if } i = j-1 \\ \langle \tau_{j+2,j+2} + t, -\tau_{j,j+1}\rangle & \text{if } i = j+1 \end{cases} \\
&= \begin{cases} -\langle r,r\rangle & \text{if } i = j \\ 0 & \text{if } i = j-1 \\ 0 & \text{if } i = j+1 \end{cases}
\end{aligned}$$

Summarizing, we conclude

$$\langle r,r\rangle = \langle s,s\rangle = -2\langle t,t\rangle = -\langle \rho_i, \sigma_i\rangle = -\langle \sigma_i, \rho_i\rangle,$$

$$\langle \tau_{ij}, \tau_{ji}\rangle = -\langle r,r\rangle \quad \text{for } i \neq j,$$

$$\langle \tau_{ii}, \tau_{ii}\rangle = \langle t,t\rangle, \quad \langle \tau_{ii}, \tau_{i+1,i+1}\rangle = \langle \tau_{i+1,i+1}, \tau_{ii}\rangle = -\langle t,t\rangle$$

and all other pairings are zero. Since the pairing is nondegenerate, we conclude $\langle t,t\rangle \neq 0$. Normalizing if necessary, we may assume $\langle t,t\rangle = 1$. Then the pairing is given by

$$\langle t,t\rangle = 1, \quad \langle r,r\rangle = \langle s,s\rangle = -2, \quad \langle \rho_i, \sigma_i\rangle = \langle \sigma_i, \rho_i\rangle = 2$$

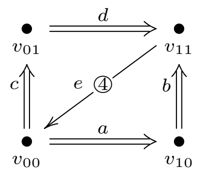
$$\langle \tau_{ij}, \tau_{ji}\rangle = 2, \text{ for } i \neq j, \quad \langle \tau_{ii}, \tau_{ii}\rangle = 1, \quad \langle \tau_{ii}, \tau_{i+1,i+1}\rangle = \langle \tau_{i+1,i+1}, \tau_{ii}\rangle = -1$$

and all other pairings are zero. Thus the superpotential is given by

$$\Phi = \sum_{i=0}^{2} \Big((a_i b_i + d_i c_i)\, t - 2\,(b_i a_i)\, r - 2\,(c_i d_i)\, s + 2\,(c_{i+2}a_{i+1} - c_{i+1}a_{i+2})\,\sigma_i + 2\,(b_{i+2}d_{i+1} - b_{i+1}d_{i+2})\,\rho_i$$
$$+ 2\,(a_i b_{i+1} - d_{i+1}c_i)\,\tau_{i,i+1} + 2\,(a_i b_{i-1} - d_{i-1}c_i)\,\tau_{i,i-1} + (a_i b_i - d_i c_i)(\tau_{ii} - \tau_{i+1,i+1} - \tau_{i-1,i-1}) \Big).$$

The remaining examples are calculated using the same method. We will skip the calculations and simply state the answers.

Example 6.3.7. Take $X = \mathbb{P}^1 \times \mathbb{P}^1$, $V = K_{\mathbb{P}^1\times\mathbb{P}^1} = \mathcal{O}(-2,-2)$, which is Calabi–Yau 3, and $\mathcal{E} = (\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1))$. The classical tilting quiver is given by

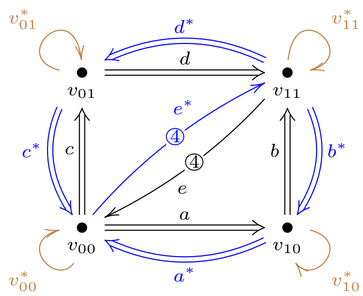


with relations

$$\begin{aligned} e_{i,j+1}b_j &= e_{i,j}b_{j+1}, & e_{i+1,j}d_i &= e_{i,j}d_{i+1},\\ c_i e_{j,i+1} &= c_{i+1}e_{j,i}, & a_{i+1}e_{i,j} &= a_i e_{i+1,j},\\ b_j a_i &= d_i c_j, \end{aligned}$$

where the four edges e are indexed by subscript e_{ij}, edges a, b, c, d indexed by a_i, b_i, c_i, d_i, with $0 \le i, j \le 1$.

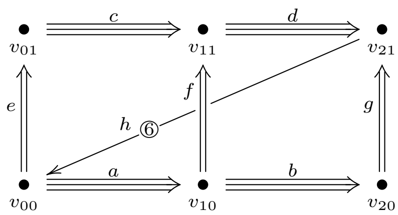
The quiver with superpotential is given by



with superpotential

$$\Phi = e_{00}(b_0a_0 - d_0c_0) + e_{01}(b_1a_0 - d_0c_1) + e_{10}(b_0a_1 - d_1c_0) + e_{11}(b_1a_1 - d_1c_1).$$

Example 6.3.8. Take $X = \mathbb{P}^2 \times \mathbb{P}^1$, $V = K_{\mathbb{P}^2\times\mathbb{P}^1}$ which is Calabi–Yau 4, with exceptional sequence $\mathcal{E} = (\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1), \mathcal{O}(2,0), \mathcal{O}(2,1))$. The classical tilting quiver is given by

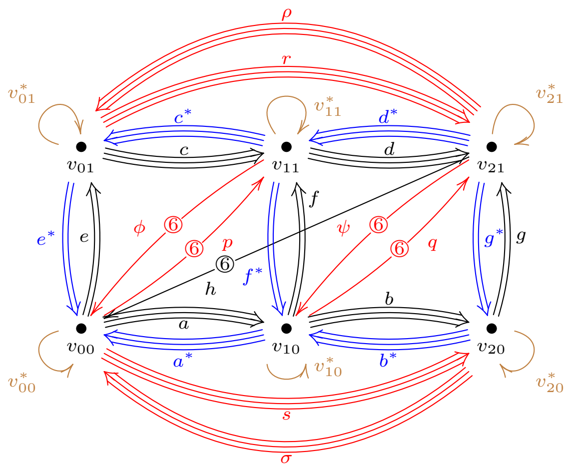


with relations

$$\begin{aligned} b_{i+1}a_{i+2} &= b_{i+2}a_{i+1}, & d_{i+1}c_{i+2} &= d_{i+2}c_{i+1},\\ h_{i0}g_1 &= h_{i1}g_0, & e_0h_{i1} &= e_1h_{i0},\\ c_ie_\alpha &= f_\alpha a_i, & d_if_\alpha &= g_\alpha b_i,\\ h_{i+1,\alpha}d_{i+2} &= h_{i+2,\alpha}d_{i+1}, & a_{i+1}h_{i+2,\alpha} &= a_{i+2}h_{i+1,\alpha}, \end{aligned}$$

where $0 \le i \le 2$ and $0 \le \alpha \le 1$.

The dg-quiver is given by



where black edges are of degree 0, red edges of degree -1, blue of degree -2, and brown of degree -3. Since by H^0 of the dg-tilting quiver is the classical tilting quiver, by a change of basis if necessary, we may assume the differential reads

$$\begin{aligned} dr_i &= d_{i+1}c_{i+2} - d_{i+2}c_{i+1}, & d\rho_i &= e_0h_{i1} - e_1h_{i0}, \\ ds_i &= b_{i+1}a_{i+2} - b_{i+2}a_{i+1}, & d\sigma_i &= h_{i0}g_1 - h_{i1}g_0, \end{aligned}$$

$$\begin{aligned} dp_{i\alpha} &= c_ie_\alpha - f_\alpha a_i, & d\phi_{i\alpha} &= h_{i+1,\alpha+1}d_{i+2} - h_{i+2,\alpha+1}d_{i+1}, \\ dq_{i\alpha} &= d_if_\alpha - g_\alpha b_i, & d\psi_{i\alpha} &= a_{i+1}h_{i+2,\alpha+1} - a_{i+2}h_{i+1,\alpha+1}, \end{aligned}$$

The pairing on degree -1 (red) edges is given by

$$\langle r_i, \rho_i\rangle = \langle s_i, \sigma_i\rangle = 1 \quad \text{and} \quad \langle p_{i\alpha}, \phi_{i\alpha}\rangle = \langle q_{i\alpha}, \psi_{i\alpha}\rangle = (-1)^{\alpha+1}$$

and zero otherwise. Hence the superpotential is given by

$$\begin{aligned} \Phi = \sum_{i=0}^{2} & r_i(e_0h_{i1} - e_1h_{i0}) + \rho_i(d_{i+1}c_{i+2} - d_{i+2}c_{i+1}) + s_i(h_{i0}g_1 - h_{i1}g_0) + \sigma_i(b_{i+1}a_{i+2} - b_{i+2}a_{i+1}) \\ & + \sum_{i=0}^{2}\sum_{\alpha=0}^{1}(-1)^{\alpha+1}p_{i\alpha}(h_{i+1,\alpha+1}d_{i+2} - h_{i+2,\alpha+1}d_{i+1}) + (-1)^{\alpha+1}\phi_{i\alpha}(c_ie_\alpha - f_\alpha a_i) \\ & + \sum_{i=0}^{2}\sum_{\alpha=0}^{1}(-1)^{\alpha+1}q_{i\alpha}(a_{i+1}h_{i+2,\alpha+1} - a_{i+2}h_{i+1,\alpha+1}) + (-1)^{\alpha+1}\psi_{i\alpha}(d_if_\alpha - g_\alpha b_i). \end{aligned}$$

6.4 Product Construction

In this section, we compute some examples illustrating the product construction given in Proposition 5.7.3. The first example is of course the simplest case:

Example 6.4.1. Take $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$, both regarded as vector bundle over the point $\operatorname{Spec}\mathbb{C}$. By Example 6.3.1, the A_∞-category corresponding to V and W are respectively given by $\wedge^\bullet V$ and $\wedge^\bullet W$ with wedge product as the only nonvanishing A_∞-structure. The A_∞-category corresponding to $V \times W \cong V \oplus W = \mathbb{C}^{n+m}$ is given by $\wedge^\bullet(V \oplus W)$. This verifies our product construction since

$$\wedge^\bullet(V \oplus W) \cong \wedge^\bullet V \otimes \wedge^\bullet W$$

as algebras.

Example 6.4.2. Take $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $V = 0$. This example can be viewed as the product of the zero vector bundle over $\mathbb{P}^1$ with itself. Take the exceptional sequence $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1))$ on $\mathrm{D}^b(\mathrm{Coh}\,(\mathbb{P}^1))$. The product sequence is then $\mathcal{E} \boxtimes \mathcal{E} = (\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1))$ which is strong and full. The dg-tilting quiver for $\mathbb{P}^1$ is given by

$$\begin{tikzcd} \underset{u_0}{\bullet} \arrow[r, Rightarrow, "e"] & \underset{u_1}{\bullet} \end{tikzcd}.$$

The corresponding A_∞-category $\mathcal{A}$ for $\mathbb{P}^1$ is given by

$$\mathcal{A}^0(u_0, u_0) = \mathrm{span}_{\mathbb{C}}\{u_0\}, \quad \mathcal{A}^0(u_1, u_1) = \mathrm{span}_{\mathbb{C}}\{u_1\}, \quad \mathcal{A}^1(u_0, u_1) = \mathrm{span}_{\mathbb{C}}\{e_0, e_1\}.$$

and zero otherwise; with the only nonvanishing A_∞-structure given by

$$m_2(u_1, e_i) = m_2(e_i, u_0) = e_i, \quad m_2(u_i, u_i) = u_i.$$

We will use the following naming scheme:

$$v_{ij} = u_i \otimes u_j, \quad a_i = e_i \otimes u_0, \quad b_i = u_1 \otimes e_i, \quad c_i = u_0 \otimes e_i, \quad d_i = e_i \otimes u_1, \quad r_{ij} = e_i \otimes e_j$$

The product dg-tilting quiver, i.e., the dg-tilting quiver for $\mathbb{P}^1 \times \mathbb{P}^1$, is thus given by

$$\begin{tikzcd} \underset{v_{01}}{\bullet} \arrow[r, Rightarrow, "d"] & \underset{v_{11}}{\bullet} \\ \underset{v_{00}}{\bullet} \arrow[u, Rightarrow, "c"] \arrow[r, Rightarrow, "a"'] \arrow[ur, "4\ r"] & \underset{v_{10}}{\bullet} \arrow[u, Rightarrow, "b"'] \end{tikzcd}$$

where black edges are of degree 0, red edges are of degree -1. Since $\mathcal{A}$ has only nonvanishing m_2, so is $\mathcal{A} \otimes \mathcal{A}$. The only nonvanishing $m_2 : (\mathcal{A} \otimes \mathcal{A})^1 \otimes (\mathcal{A} \otimes \mathcal{A})^1 \to (\mathcal{A} \otimes \mathcal{A})^2$ is given by

$$m_2(b_j, a_i) = -m_2(u_1, e_i) \otimes m_2(e_j, u_0) = e_i \otimes e_j = -r_{ij},$$

$$m_2(d_i, c_j) = m_2(e_i, u_0) \otimes m_2(u_1, e_j) = e_i \otimes e_j = r_{ij}.$$

Thus the differential d is given by

$$dr_{ij} = d_i c_j - b_j a_i \quad \text{and} \quad da_i = db_i = dc_i = dd_i = 0.$$

The classical tilting quiver is given by

$$\begin{tikzcd} \underset{v_{01}}{\bullet} \arrow[r, Rightarrow, "d"] & \underset{v_{11}}{\bullet} \\ \underset{v_{00}}{\bullet} \arrow[u, Rightarrow, "c"] \arrow[r, Rightarrow, "a"'] & \underset{v_{10}}{\bullet} \arrow[u, Rightarrow, "b"'] \end{tikzcd}$$

with relations $b_j a_i = d_i c_j$.

Example 6.4.3. Consider $X = \mathbb{P}^1$ and $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ which is Calabi–Yau 4. Then V can be viewed as the product of the trivial bundle $\mathbb{C}^2 \to \mathrm{Spec}\,\mathbb{C}$ over a point and the canonical bundle $K_{\mathbb{P}^1} \to \mathbb{P}^1$ over the projective line. Take the exceptional sequences $\mathcal{O}_{\mathrm{Spec}\,\mathbb{C}}$ on $\mathrm{D}^b(\mathrm{Coh}\,(\mathbb{C}^2))$ and $(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1))$ on $\mathrm{D}^b(\mathrm{Coh}\,(\mathbb{P}^1))$. Recall from Examples 6.3.1 and 6.3.2 that the quivers with superpotential derived equivalent to $\mathbb{C}$ and $K_{\mathbb{P}^1}$ are respectively

$$\begin{tikzcd} \underset{u}{\bullet} \arrow[loop left, "u^*"] \arrow[loop right, Rightarrow, "x"] \end{tikzcd}$$

with $\deg x_0 = \deg x_1 = 0$, $\deg u^* = -1$, $dx_i = 0$ and $du^* = x_1 x_0 - x_0 x_1$; and

$$\begin{tikzcd} \underset{v_0}{\bullet} \arrow[loop left, "v_0^*"] \arrow[r, Rightarrow, shift left, "e"] & \underset{v_1}{\bullet} \arrow[l, Rightarrow, shift left, "f"] \arrow[loop right, "v_1^*"] \end{tikzcd}$$

where black edges are of degree 0 and brown edges are of degree -1, with differential given by

$$dv_0^* = f_1e_0 - f_0e_1, \quad dv_1^* = e_1f_0 - e_0f_1 \quad \text{ and } \quad de_i = df_i = 0.$$

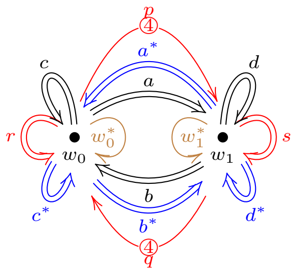
Using the A_∞-categories corresponding to $\mathbb{C}^2$ and $K_{\mathbb{P}^1}$ given in Examples 6.3.1 and 6.3.2, we calculate the tensor product category $\mathcal{A}$. We will use the following naming scheme for simplicity:

$$w_i = u \otimes v_i, \quad c_i = x_i \otimes v_0, \quad d_i = x_i \otimes v_1, \quad a_i = u \otimes e_i, \quad b_i = u \otimes f_i,$$

$$r_0 = u^* \otimes v_0, \quad r_1 = u \otimes v_0^*, \quad s_0 = u^* \otimes v_1, \quad s_1 = u \otimes v_1^*, \quad p_{ij} = x_i \otimes e_j, \quad q_{ij} = x_i \otimes f_j$$

$$a_i^* = u^* \otimes f_{i+1}, \quad b_i^* = u^* \otimes e_{i+1}, \quad c_i^* = x_{i+1} \otimes v_0^*, \quad d_i^* = x_{i+1} \otimes v_1^*, \quad w_i^* = u^* \otimes v_i^*.$$

Thus the dg-tilting quiver is given by



where black edges are of degree 0, red of degree -1, blue of degree -2 and brown of degree -3.

Since $m_n = 0$ for all $n \neq 2$ for both A_∞-category, m_n of their tensor product also vanishes for $n \neq 2$. The action of $m_2 : \mathcal{A}^1 \otimes \mathcal{A}^1 \to \mathcal{A}^2$ is given by

$$\begin{aligned}
m_2(c_1, c_0) &= m_2(x_1, x_0) \otimes m_2(v_0, v_0) = u^* \otimes v_0 = r_0, \\
m_2(c_0, c_1) &= m_2(x_0, x_1) \otimes m_2(v_0, v_0) = -u^* \otimes v_0 = -r_0, \\
m_2(b_0, a_1) &= m_2(u, u) \otimes m_2(f_0, e_1) = -u \otimes v_0^* = -r_1, \\
m_2(b_1, a_0) &= m_2(u, u) \otimes m_2(f_1, e_0) = u \otimes v_0^* = r_1, \\
m_2(d_1, d_0) &= m_2(x_1, x_0) \otimes m_2(v_1, v_1) = u^* \otimes v_1 = s_0, \\
m_2(d_0, d_1) &= m_2(x_0, x_1) \otimes m_2(v_1, v_1) = -u^* \otimes v_1 = -s_0, \\
m_2(a_0, b_1) &= m_2(u, u) \otimes m_2(e_0, f_1) = -u \otimes v_1^* = -s_1, \\
m_2(a_1, b_0) &= m_2(u, u) \otimes m_2(e_1, f_0) = u \otimes v_1^* = s_1, \\
m_2(a_i, c_j) &= -m_2(u, x_j) \otimes m_2(e_i, v_0) = -x_j \otimes e_i = -p_{ji}, \\
m_2(d_j, a_i) &= m_2(x_j, u) \otimes m_2(v_1, e_i) = x_j \otimes e_i = p_{ji}, \\
m_2(c_i, b_j) &= m_2(x_i, u) \otimes m_2(v_0, f_j) = x_i \otimes f_j = q_{ij}, \\
m_2(b_j, d_i) &= -m_2(u, x_i) \otimes m_2(f_j, v_1) = -x_i \otimes f_j = -q_{ij}.
\end{aligned}$$

and zero otherwise. Hence the differential in the dg-quiver sends

$$dr_0 = c_1c_0 - c_0c_1, \quad dr_1 = b_1a_0 - b_0a_1,$$

$$ds_0 = d_1d_0 - d_0d_1, \quad ds_1 = a_1b_0 - a_0b_1,$$

$$dp_{ij} = d_ia_j - a_jc_i, \quad dq_{ij} = c_ib_j - b_jd_i.$$

The cyclic structure on the tensor product $\langle -, - \rangle : \mathcal{A}^2 \otimes \mathcal{A}^2 \to \mathbb{C}$ is given by

$$\begin{aligned}
\langle p_{ij}, q_{k\ell} \rangle = \langle q_{k\ell}, p_{ij} \rangle &= \langle x_k \otimes f_\ell, x_i \otimes e_j \rangle = -\langle x_k, x_i \rangle \langle f_\ell, e_j \rangle \\
&= \begin{cases} 1 & \text{if } (i,j,k,\ell) = (1,0,0,1) \text{ or } (0,1,1,0) \\ -1 & \text{if } (i,j,k,\ell) = (0,0,1,1) \text{ or } (1,1,0,0) \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

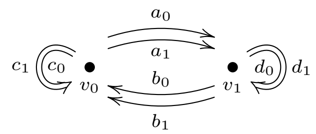
$$\begin{aligned}\langle r_1, r_0\rangle = \langle r_0, r_1\rangle = \langle u^*, u\rangle\langle v_0, v_0^*\rangle = 1, &\quad \langle s_1, s_0\rangle = \langle s_0, s_1\rangle = \langle u^*, u\rangle\langle v_1, v_1^*\rangle = 1\\ \langle r_0, r_0\rangle = \langle u^*, u^*\rangle\langle v_0, v_0\rangle = 0, &\quad \langle s_0, s_0\rangle = \langle u^*, u^*\rangle\langle v_1, v_1\rangle = 0\\ \langle r_1, r_1\rangle = \langle u, u\rangle\langle v_0^*, v_0^*\rangle = 0, &\quad \langle s_1, s_1\rangle = \langle u, u\rangle\langle v_1^*, v_1^*\rangle = 0\end{aligned}$$

and zero otherwise.

Hence the superpotential is given by

$$\begin{aligned}\Phi = {} & r_0(b_1a_0 - b_0a_1) + r_1(c_1c_0 - c_0c_1) + s_0(a_1b_0 - a_0b_1) + s_1(d_1d_0 - d_0d_1)\\ & + p_{01}(c_1b_0 - b_0d_1) + q_{10}(d_0a_1 - a_1c_0) + p_{10}(c_0b_1 - b_1d_0) + q_{01}(d_1a_0 - a_0c_1)\\ & - p_{00}(c_1b_1 - b_1d_1) - q_{11}(d_1a_1 - a_1c_1) - p_{11}(c_0b_0 - b_0d_0) - q_{00}(d_0a_0 - a_0c_0).\end{aligned}$$

Taking H^0 of the dg-quiver, we see that the classical tilting quiver is given by



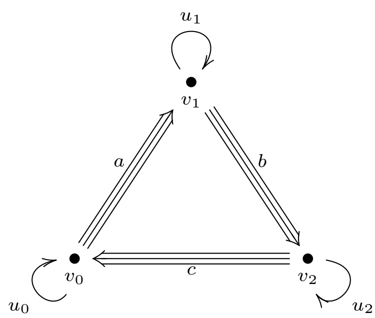
with relations

$$\begin{array}{ll} a_0b_1 = a_1b_0 & d_0d_1 = d_1d_0\\ b_0a_1 = b_1a_0 & c_0c_1 = c_1c_0\\ a_ic_j = d_ja_i & c_ib_j = b_jd_i,\end{array}$$

where $0 \leq i, j \leq 1$.

The remaining examples are calculated using the same method. We will skip the calculations and simply state the answers.

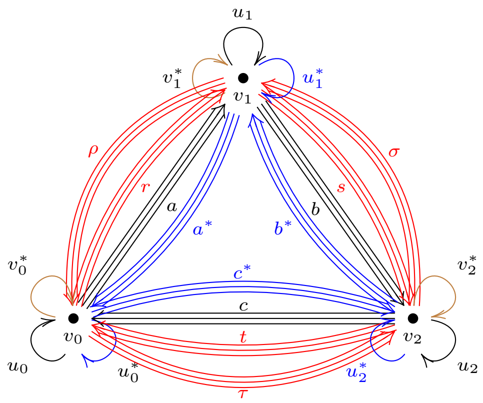
Example 6.4.4. Consider the vector bundle $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$ which is Calabi–Yau 4. The classical tilting quiver is given by



with relations

$$\begin{array}{ll} b_{i+1}a_i = b_ia_{i+1} & c_iu_2 = u_0c_i\\ c_{i+1}b_i = c_ib_{i+1} & a_iu_0 = u_1a_i\\ a_{i+1}c_i = a_ic_{i+1} & b_iu_1 = u_2b_i.\end{array}$$

Now, we construct a quasi-free resolution of the quiver. Observe that $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) = \mathbb{C} \times \mathcal{O}_{\mathbb{P}^2}(-3)$. Hence the dg-quiver for $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$ is the product of dg-quivers $\mathbb{C}$ and $\mathcal{O}_{\mathbb{P}^2}(-3)$ in Examples 6.3.1 and 6.5.1, which is given by



where black edges are of degree 0, red edges are of degree -1, blue edges are of degree -2 and brown edges are of degree -3. The differential sends

$$\begin{aligned} d\tau_i &= b_{i+2}a_{i+1} - b_{i+1}a_{i+2} & dt_i &= c_i u_2 - u_0 c_i \\ d\rho_i &= c_{i+2}b_{i+1} - c_{i+2}b_i & dr_i &= a_i u_0 - u_1 a_i \\ d\sigma_i &= a_{i+2}c_{i+1} - a_{i+1}c_{i+2} & ds_i &= b_i u_1 - u_2 b_i \end{aligned}$$

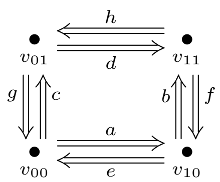
and the symmetric pairing on degree -1 edges is given by

$$\langle \tau_i, t_i \rangle = \langle s_i, \sigma_i \rangle = \langle \rho_i, r_i \rangle = 1.$$

The superpotential is given by

$$\begin{aligned} \Phi = \sum_{i=1}^{3} \Big(& (b_{i+2}a_{i+1} - b_{i+1}a_{i+2})t_i + (c_i u_2 - u_0 c_i)\tau_i \\ & + (c_{i+2}b_{i+1} - c_{i+2}b_i)r_i + (a_i u_0 - u_1 a_i)\rho_i \\ & + (a_{i+2}c_{i+1} - a_{i+1}c_{i+2})s_i + (b_i u_1 - u_2 b_i)\sigma_i \Big). \end{aligned}$$

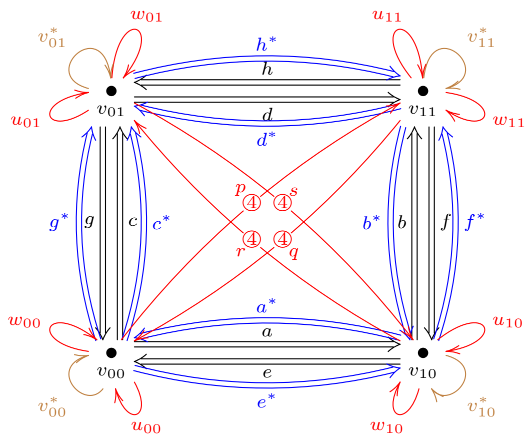
Example 6.4.5. Take $\mathbb{P}^1 \times \mathbb{P}^1$, $V = T^*_{\mathbb{P}^1 \times \mathbb{P}^1}$ which is Calabi–Yau 4. We can view V as the product of canonical bundle on $\mathbb{P}^1$ with itself. Take the exceptional sequence $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1))$ on $\mathrm{D}^b(\mathrm{Coh}\,(\mathbb{P}^1))$. Then $\mathcal{E}\boxtimes = (\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1))$. The tilting quiver is



with relations

$$\begin{aligned} b_j a_i &= d_i c_j, & g_j h_i &= e_i f_j, \\ c_j e_i &= h_i b_j, & f_j d_i &= a_i g_j, \\ a_j e_i &= a_i e_j, & e_j a_i &= e_i a_j, \\ b_j f_i &= b_i f_j, & f_j b_i &= f_i b_j, \\ c_j g_i &= c_i g_j, & g_j c_i &= g_i c_j, \\ d_j h_i &= d_i h_j, & h_j d_i &= h_i d_j, \end{aligned}$$

where $0 \leq i, j \leq 1$. The dg-tilting quiver is given by



with differential given by

$$\begin{array}{rlrl}
dp_{ij} &= b_j a_i - d_i c_j & dq_{ij} &= g_j h_i - e_i f_j \\
dr_{ij} &= c_j e_i - h_i b_j & ds_{ij} &= f_j d_i - a_i g_j \\
du_{00} &= e_1 a_0 - e_0 a_1 & dw_{00} &= g_1 c_0 - g_0 c_1 \\
du_{01} &= c_1 g_0 - c_0 g_1 & dw_{01} &= h_1 d_0 - h_0 d_1 \\
du_{10} &= f_1 b_0 - f_0 b_1 & dw_{10} &= a_1 e_0 - a_0 e_1 \\
du_{11} &= d_1 h_0 - d_0 h_1 & dw_{11} &= b_1 f_0 - b_0 f_1
\end{array}$$

The pairing on degree -1 edges is given by

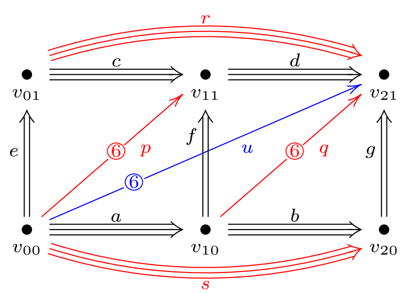
$$\langle p_{i,i+1}, q_{i+1,i}\rangle = \langle r_{i,i+1}, s_{i+1,i}\rangle = \langle u_{ij}, w_{ij}\rangle = 1,$$

$$\langle p_{ii}, q_{i+1,i+1}\rangle = \langle r_{ii}, s_{i+1,i+1}\rangle = -1.$$

and zero otherwise, where we treat $i \in \mathbb{Z}/2\mathbb{Z}$. Thus the superpotential is given by

$$\begin{aligned}
\Phi = {} & u_{00}(g_1c_0 - g_0c_1) + w_{00}(e_1a_0 - e_0a_1) + u_{01}(h_1d_0 - h_0d_1) + w_{01}(c_1g_0 - c_0g_1) \\
& + u_{10}(a_1e_0 - a_0e_1) + w_{10}(f_1b_0 - f_0b_1) + u_{11}(b_1f_0 - b_0f_1) + w_{11}(d_1h_0 - d_0h_1) \\
& + \sum_{i=0}^{1} \Big(p_{i,i+1}(g_ih_{i+1} - e_{i+1}f_i) + q_{i+i,i}(b_{i+1}a_i - d_ic_{i+1}) \\
& \qquad\qquad + r_{i,i+1}(f_id_{i+1} - a_{i+1}g_i) + s_{i+1,i}(c_{i+1}e_i - h_ib_{i+1}) \Big) \\
& - \sum_{i=0}^{1} \Big(p_{ii}(g_{i+1}h_{i+1} - e_{i+1}f_{i+1}) + q_{i+1,i+1}(b_ia_i - d_ic_i) \\
& \qquad\qquad + r_{ii}(f_{i+1}d_{i+1} - a_{i+1}g_{i+1}) + s_{i+1,i+1}(c_ie_i - h_ib_i) \Big).
\end{aligned}$$

Example 6.4.6. Take $X = \mathbb{P}^2 \times \mathbb{P}^1$ and $V = 0$. It can be view as the product of the zero vector bundle on $\mathbb{P}^2$ and the zero vector bundle on $\mathbb{P}^1$. Take $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ an exceptional sequence on $\mathrm{D}^b(\mathrm{Coh}\,(\mathbb{P}^2))$ and $\mathcal{F} = (\mathcal{O}, \mathcal{O}(1))$ an exceptional sequence on $\mathrm{D}^b(\mathrm{Coh}\,(\mathbb{P}^1))$. Then $\mathcal{E} \boxtimes \mathcal{F} = (\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1), \mathcal{O}(2,0), \mathcal{O}(2,1))$. The dg-tilting quiver is given by



where black edges are of degree 0, red edges are of degree -1, blue edges are of degree -2. We will index edges p, q, u by $p_{\alpha i}, q_{\alpha i}, u_{\alpha i}$, where $0 \le i \le 2$ and $0 \le \alpha \le 1$. The differential is given by

$$\begin{array}{cc} de_\alpha = df_\alpha = dg_\alpha = 0, & da_i = db_i = dc_i = dd_i = 0, \\ dp_{\alpha i} = c_i e_\alpha - f_\alpha a_i, & dq_{\alpha i} = d_i f_\alpha - g_\alpha b_i, \\ dr_i = d_{i+1}c_{i+2} - d_{i+2}c_{i+1}, & ds_i = b_{i+1}a_{i+2} - b_{i+2}a_{i+1}, \end{array}$$

$$du_{\alpha i} = d_{i+1}p_{\alpha,i+2} - d_{i+2}p_{\alpha,i+1} + q_{\alpha,i+1}a_{i+2} - q_{\alpha,i+2}a_{i+1} + g_\alpha s_i - r_i e_\alpha.$$

The classical tilting quiver is thus given by

$$\begin{tikzcd} \bullet_{v_{01}} \arrow[r, Rrightarrow, "c"] & \bullet_{v_{11}} \arrow[r, Rrightarrow, "d"] & \bullet_{v_{21}} \\ \bullet_{v_{00}} \arrow[u, Rightarrow, "e"] \arrow[r, Rrightarrow, "a"] & \bullet_{v_{10}} \arrow[u, Rightarrow, "f"] \arrow[r, Rrightarrow, "b"] & \bullet_{v_{20}} \arrow[u, Rightarrow, "g"] \end{tikzcd}$$

with relations

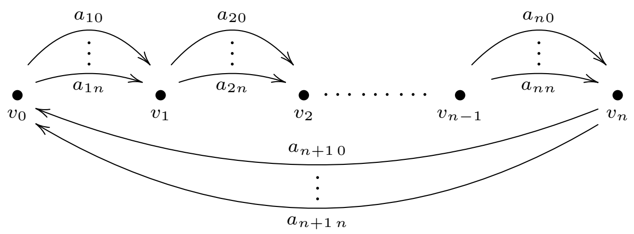
$$\begin{array}{cc} b_j a_i = b_i a_j, & d_j c_i = d_i c_j, \\ c_i e_\alpha = f_\alpha a_i, & d_i f_\alpha = g_\alpha b_i, \end{array}$$

where $0 \le i, j \le 2$ and $0 \le \alpha, \beta \le 1$.

6.5 Quotient Construction

In this section, we compute some examples illustrating the quotient construction given by Proposition 5.6.5.

Example 6.5.1. Consider $V = \mathcal{O}_{\mathbb{P}^n}(-n-1) = K_{\mathbb{P}^n}$ and $X = \mathbb{P}^n$. By Example 6.1.6, the classical tilting quiver is



with relations $a_{i+1\,k}a_{ij} = a_{i+1\,j}a_{ik}$. Note that this quiver with relations has path algebra isomorphic to $\mathbb{C}[x_0, \ldots, x_n]\#\mathbb{Z}_{n+1}$, with $\mathbb{Z}_{n+1}$ acting on each variable by multiplying ω^{-1}, where $\omega = e^{\frac{2\pi i}{n+1}}$ is an n-th root of unity. Hence its corresponding A_∞-algebra is A_∞-isomorphic to $\bigoplus_{\rho,\sigma\in \operatorname{Irr}\mathbb{Z}_{n+1}} \operatorname{Hom}_{\mathbb{Z}_{n+1}}(\rho, \bigwedge^\bullet \mathbb{K}^{n+1} \otimes \sigma)$, which is cyclic since $\bigwedge^\bullet \mathbb{K}^{n+1}$ is. This corresponds to the fact that $K_{\mathbb{P}^n}$ is Calabi-Yau. This example is an incarnation of the McKay correspondence as given by Bridgeland, King and Reid [14]: $K_{\mathbb{P}^n} \to \mathbb{K}^{n+1}/\mathbb{Z}_{n+1}$ is a crepant resolution and we have $\mathrm{D}^b(\operatorname{Coh}(K_{\mathbb{P}^n})) \cong \mathrm{D}^b(\operatorname{Coh}^{\mathbb{Z}_{n+1}}(\mathbb{K}^{n+1}))$.

The group $\mathbb{Z}_{n+1}$ has $n+1$ irreducible representations $\rho_0, \ldots, \rho_n$, where ρ_i is the one dimensional representation on which $\mathbb{Z}_{n+1}$ acts by multiplication by ω^i. We will denote by c_i a basis for ρ_i. We will also choose a basis $u_1, \ldots, u_{n+1}$ for $\mathbb{K}^{n+1}$. For any $0 \le \ell \le n+1$, we have isomorphisms $\rho_j^{\oplus\binom{n+1}{\ell}} \cong \wedge^\ell \mathbb{K}^{n+1} \otimes \rho_{j+\ell}$ via the sum of the following maps:

$$x^j_{i_1\ldots i_\ell} : \rho_j \to \wedge^\ell \mathbb{K}^{n+1} \otimes \rho_{j+\ell}, \quad c_j \mapsto u_{i_1} \wedge \cdots \wedge u_{i_\ell} \otimes c_{j+\ell}$$

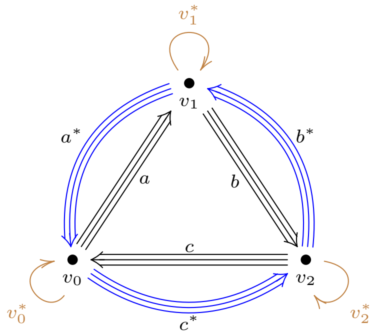
where $1 \le i_1 < i_2 < \cdots < i_{\ell+1} \le n+1$.

Hence the dg-quiver of $K_{\mathbb{P}^n}$ has $n+1$ vertices $v_0, \ldots, v_{n+1}$ and $\binom{n+1}{\ell}$ arrows of degree $1-\ell$ from vertex v_j to $v_{j+\ell}$, where we think of $j \in \mathbb{Z}_{n+1}$. If we extend the definition of $x^j_{i_1\ldots i_\ell}$ to all $1 \le i_k \le n+1$ by setting $x^j_{i_1\ldots i_\ell} = (\operatorname{sgn}\alpha)x^j_{i_{\alpha(1)}\cdots i_{\alpha(\ell)}}$ where α is the permutation such that $i_{\alpha(1)} < \ldots < i_{\alpha(\ell)}$ if all the indices i_k are distinct and setting $x^j_{i_1\ldots i_\ell} = 0$ if two indices i_k are equal, then we can describe the A_∞-structure by

$$m_2(x^{j+\ell}_{k_1\cdots k_{\ell'}}, x^j_{i_1\cdots i_\ell}) = x^j_{i_1\cdots i_\ell k_1\cdots k_{\ell'}}$$

and all other m_n's are zero.

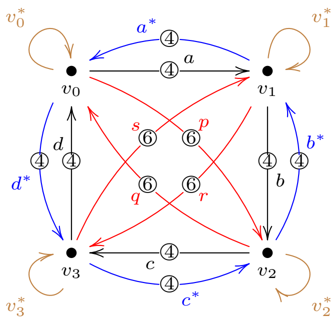
When $n = 2$, we recover the quiver with superpotential in (6.3.1):



(6.5.1)

where black edges are of degree 0, blue edges are of degree -1, brown loops are of degree -2, with superpotential $\Phi = \sum_{i=0}^{2} a_i(b_{i+1}c_{i+2} - b_{i+2}c_{i+1})$.

When $n = 3$, the dg-tilting quiver is given by



Here, the black edges are of degree 0, red edges of degree -1, blue edges of degree -2 and brown edges of degree -3. We will label the red edges by (i, j), where $0 \le i < j \le 3$. The differential in the dg-quiver sends

$$dp_{ij} = b_i a_j - b_j a_i, \quad dq_{ij} = d_i c_j - d_j c_i$$
$$dr_{ij} = c_i b_j - c_j b_i, \quad ds_{ij} = a_i d_j - a_j d_i.$$

The symmetric pairing between the red (degree -1) edges is given by

$$\langle p_{ij}, q_{k\ell}\rangle = \langle r_{ij}, s_{k\ell}\rangle = \epsilon_{ijk\ell} = \begin{cases} 1 & \text{if } (i,j,k,\ell) \text{ is an even permutation of (1,2,3,4)} \\ -1 & \text{if } (i,j,k,\ell) \text{ is an odd permutation of (1,2,3,4)} \\ 0 & \text{otherwise.} \end{cases}$$

The superpotential is given by

$$\Phi = \sum_{i,j,k,\ell} \epsilon_{ijk\ell}\Big(q_{ij}(b_k a_\ell - b_\ell a_k) + p_{ij}(d_k c_\ell - d_\ell c_k) + s_{ij}(c_k b_\ell - c_\ell b_k) + r_{ij}(a_k d_\ell - a_\ell d_k)\Big).$$

Example 6.5.2. Suppose $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ acts on $\mathbb{P}^2$ by multiplication of $e^{4\pi i/3}$ on homogeneous coordinate. This $\mathbb{Z}_3$-action lifts to $\mathcal{O}(1)$ and turn $\mathcal{O}(1)$ into a $\mathbb{Z}_3$-bundle which will be denoted by E. Let ρ_k be the irreducible representation of $\mathbb{Z}_3$ corresponding to multiplication of $e^{2\pi ik/3}$. The sequence $\mathcal{E} = (\mathcal{O}, E, E^{\otimes 2})$ is a sequence of $\mathbb{Z}_3$-sheaves whose underlying sequence form an full strong exceptional sequence. To calculate the corresponding A_∞-category, we need the following

Lemma 6.5.3.

$$H^\ell(\mathbb{P}^2,\mathcal{O}) = \begin{cases}\rho_0 & \text{if } \ell=0\\ 0 & \text{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2,\mathcal{O}(-3)) = \begin{cases}\rho_0 & \text{if } \ell=2\\ 0 & \text{otherwise.}\end{cases}$$

$$H^\ell(\mathbb{P}^2,\mathcal{O}(1)) = \begin{cases}\rho_2^{\oplus 3} & \text{if } \ell=0\\ 0 & \text{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2,\mathcal{O}(-4)) = \begin{cases}\rho_1^{\oplus 3} & \text{if } \ell=2\\ 0 & \text{otherwise.}\end{cases}$$

$$H^\ell(\mathbb{P}^2,\Omega(2)) = \begin{cases}\rho_1^{\oplus 3} & \text{if } \ell=0\\ 0 & \text{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2,\mathcal{T}(-5)) = \begin{cases}\rho_2^{\oplus 3} & \text{if } \ell=2\\ 0 & \text{otherwise.}\end{cases}$$

$$H^\ell(\mathbb{P}^2,\mathcal{T}(-1)) = \begin{cases}\rho_1^{\oplus 3} & \text{if } \ell=0\\ 0 & \text{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2,\Omega(-2)) = \begin{cases}\rho_2^{\oplus 3} & \text{if } \ell=2\\ 0 & \text{otherwise.}\end{cases}$$

$$H^\ell(\mathbb{P}^2,\Omega) = \begin{cases}\rho_0 & \text{if } \ell=1\\ 0 & \text{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2,\mathcal{T}(-3)) = \begin{cases}\rho_0 & \text{if } \ell=1\\ 0 & \text{otherwise.}\end{cases}$$

$$H^\ell(\mathbb{P}^2,\mathcal{T}(-1)) = \begin{cases}\rho_1^{\oplus 3} & \text{if } \ell=0\\ 0 & \text{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2,\Omega(-2)) = \begin{cases}\rho_2^{\oplus 3} & \text{if } \ell=2\\ 0 & \text{otherwise.}\end{cases}$$

$$H^\ell(\mathbb{P}^2,\Omega) = \begin{cases}\rho_0 & \text{if } \ell=1\\ 0 & \text{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2,\mathcal{T}(-3)) = \begin{cases}\rho_0 & \text{if } \ell=1\\ 0 & \text{otherwise.}\end{cases}$$

$$H^\ell(\mathbb{P}^2,\mathcal{T}\otimes\Omega) = \begin{cases}\rho_0 & \text{if } \ell=0\\ 0 & \text{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2,\mathcal{T}\otimes\Omega(-3)) = \begin{cases}\rho_0 & \text{if } \ell=2\\ 0 & \text{otherwise.}\end{cases}$$

$$H^\ell(\mathbb{P}^2,\mathcal{T}\otimes\Omega(-1)) = \begin{cases}\rho_1^{\oplus 3} & \text{if } \ell=1\\ 0 & \text{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2,\mathcal{T}\otimes\Omega(-2)) = \begin{cases}\rho_2^{\oplus 3} & \text{if } \ell=1\\ 0 & \text{otherwise.}\end{cases}$$

Proof. It suffices to show the lemma for the nonvanishing cases. Since $\mathcal{O}$ is trivial $\mathbb{Z}_3$-equivariantly, $H^0(\mathbb{P}^2,\mathcal{O}) = \rho_0$. By Serre duality, $H^2(\mathbb{P}^2,\mathcal{O}(-3)) \cong H^0(\mathbb{P}^2,\mathcal{O})^\vee = \rho_0$. By the definition of the $\mathbb{Z}_3$-action on $\mathbb{P}^2$, $H^0(\mathbb{P}^2,\mathcal{O}(1)) = \rho_2^{\oplus 3}$. By Serre duality, $H^2(\mathbb{P}^2,\mathcal{O}(-4)) = H^0(\mathbb{P}^2,\mathcal{O}(1))^\vee = (\rho_2^{\oplus 3})^\vee = \rho_1^{\oplus 3}$.

Since $\mathbb{P}^2 = \mathbb{P}(H^0(\mathbb{P}^2,\mathcal{O}(1))^\vee)$, we have the short exact sequence

$$0 \to \Omega(2) \to \mathcal{O}(1)\otimes H^0(\mathbb{P}^2,\mathcal{O}(1)) \to \mathcal{O}(2) \to 0,$$

and hence the exact sequence

$$0 \to H^0(\mathbb{P}^2,\Omega(2)) \to H^0(\mathbb{P}^2,\mathcal{O}(1))\otimes H^0(\mathbb{P}^2,\mathcal{O}(1)) \to H^0(\mathbb{P}^2,\mathcal{O}(2)) \to 0.$$

By the isomorphisms

$$H^0(\mathbb{P}^2,\mathcal{O}(2)) \cong S^2H^0(\mathbb{P}^2,\mathcal{O}(1)) \text{ and } H^0(\mathbb{P}^2,\mathcal{O}(1))\otimes H^0(\mathbb{P}^2,\mathcal{O}(1)) \cong S^2H^0(\mathbb{P}^2,\mathcal{O}(1))\oplus\wedge^2H^0(\mathbb{P}^2,\mathcal{O}(1)),$$

we conclude $H^0(\mathbb{P}^2,\Omega(2)) \cong \wedge^2H^0(\mathbb{P}^2,\mathcal{O}(1)) \cong \rho_1^{\oplus 3}$. By Serre duality, we have

$$H^0(\mathbb{P}^2,\mathcal{T}(-5)) = \rho_2^{\oplus 3}.$$

From the short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}\otimes H^0(\mathbb{P}^2,\mathcal{O}(1))^\vee \to \mathcal{T}(-1) \to 0$$

we conclude $H^0(\mathbb{P}^2,\mathcal{T}(-1)) \cong H^0(\mathbb{P}^2,\mathcal{O}(1))^\vee \cong \rho_1^{\oplus 3}$. By Serre duality, we have

$$H^2(\mathbb{P}^2,\Omega(-2)) \cong H^0(\mathbb{P}^2,\mathcal{T}(-1))^\vee \cong \rho_2^{\oplus 3}.$$

From the short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)\otimes H^0(\mathbb{P}^2,\mathcal{O}(1)) \to \mathcal{O} \to 0,$$

we have an isomorphism $H^1(\mathbb{P}^2, \Omega) \cong H^0(\mathbb{P}^2, \mathcal{O}) = \rho_0$. By Serre duality, we have

$$H^1(\mathbb{P}^2, \mathcal{T}(-3)) \cong H^0(\mathbb{P}^2, \mathcal{O})^\vee = \rho_0.$$

Twisting the Euler sequence with $\Omega(k)$, we have a short exact sequence

$$0 \to \Omega(k) \to H^0(\mathbb{P}^2, \mathcal{O}(1))^\vee \otimes \Omega(k+1) \to \mathcal{T} \otimes \Omega(k) \to 0.$$

We have isomorphisms $H^0(\mathbb{P}^2, \mathcal{T} \otimes \Omega) \cong H^1(\mathbb{P}^2, \Omega) = \rho_0$ and

$$H^1(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-1)) \cong H^0(\mathbb{P}^2, \mathcal{O}(1))^\vee \otimes H^1(\mathbb{P}^2, \Omega) = \rho_1^{\oplus 3} \otimes \rho_0 = \rho_1^{\oplus 3}.$$

By Serre duality, we have

$$H^2(\mathbb{P}^2, \mathcal{T} \otimes (-2)) = H^0(\mathbb{P}^2, \mathcal{T} \otimes \Omega)^\vee = \rho_0 \quad \text{and} \quad H^1(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-2)) = H^1(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-1))^\vee = \rho_2^{\oplus 3}.$$

■

We calculate its A_∞-category:

$$\begin{aligned}
\mathcal{A}^\ell(v_{0\alpha}, v_{0\beta}) &= \mathrm{Hom}^\ell_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta) = [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^\ell(\mathbb{P}^2, \mathcal{O})]^G \\
&= \begin{cases} \mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_{0\alpha}, v_{1\beta}) &= \mathrm{Hom}^\ell_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega(1)[1] \otimes \rho_\beta) = [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-1}(\mathbb{P}^2, \Omega(2))]^G \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \alpha - 1 = \beta \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \\
\mathcal{A}^\ell(v_{0\alpha}, v_{2\beta}) &= \mathrm{Hom}^\ell_G(\Omega^2(2)[2] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta) = [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(1))]^G \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \alpha + 1 = \beta \text{ and } \ell = 2 \\ \mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 3 \\ 0 & \text{otherwise.} \end{cases} \\
\mathcal{A}^\ell(v_{1\alpha}, v_{0\beta}) &= \mathrm{Hom}^\ell_G(\Omega(1)[1] \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta) = [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell+1}(\mathbb{P}^2, \mathcal{T}(-2))]^G = 0. \\
\mathcal{A}^\ell(v_{1\alpha}, v_{1\beta}) &= \mathrm{Hom}^\ell_G(\Omega(1)[1] \otimes \rho_\alpha, \Omega(1)[1] \otimes \rho_\beta) = [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^\ell(\mathbb{P}^2, \mathcal{T} \otimes \Omega)]^G \\
&= \begin{cases} \mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 0 \\ 0 & \text{otherwise.} \end{cases} \\
\mathcal{A}^\ell(v_{1\alpha}, v_{2\beta}) &= \mathrm{Hom}^\ell_G(\Omega(1)[1] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta) = [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-1))]^G \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \beta = \alpha - 1 \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \\
\mathcal{A}^\ell(v_{2\alpha}, v_{0\beta}) &= \mathrm{Hom}^\ell_G(\mathcal{O} \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta) = [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell+2}(\mathbb{P}^2, \mathcal{O}(-1))]^G = 0 \\
\mathcal{A}^\ell(v_{2\alpha}, v_{1\beta}) &= \mathrm{Hom}^\ell_G(\mathcal{O} \otimes \rho_\alpha, \Omega(1)[1] \otimes \rho_\beta) = [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell+1}(\mathbb{P}^2, \Omega(1))]^G = 0. \\
\mathcal{A}^\ell(v_{2\alpha}, v_{2\beta}) &= \mathrm{Hom}^\ell_G(\mathcal{O} \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta) = [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^\ell(\mathbb{P}^2, \mathcal{O})]^G \\
&= \begin{cases} \mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{tikzcd}
v_{02} \arrow[r, Rrightarrow, "a_2"'] \arrow[rr, bend left, Rrightarrow, red, "s_2"] & v_{11} \arrow[r, Rrightarrow, "b_1"'] & v_{20} \\
v_{01} \arrow[r, Rrightarrow, "a_1"'] \arrow[rr, bend left, Rrightarrow, red, "s_1"] & v_{10} \arrow[r, Rrightarrow, "b_0"'] & v_{22} \\
v_{00} \arrow[r, Rrightarrow, "a_0"'] \arrow[rr, bend left, Rrightarrow, red, "s_0"] & v_{12} \arrow[r, Rrightarrow, "b_2"'] & v_{21}
\end{tikzcd}$$

with differential $ds_{\alpha,i} = b_{\alpha-1,i+1}a_{\alpha,i+2} - b_{\alpha-1,i+2}a_{\alpha,i+1}$. Here the vertex v_{ij} corresponds to the bundle $E^{\otimes i} \otimes \rho_j$. Note that this quiver is a disjoint union of three Beilinson quivers of $\mathbb{P}^2$ (Example 5.2.12). This corresponds to the fact that $\mathrm{Coh}^{\mathbb{Z}_3}(\mathbb{P}^2) = \mathrm{Coh}(\mathbb{P}^2)^{\oplus 3}$ since $\mathbb{Z}_3$ is acting trivially on $\mathbb{P}^2$.

Example 6.5.4. Following Example 6.5.2, we take the vector bundle $V = E^\vee \oplus E^{\vee\otimes 2}$, which is Calabi–Yau 4. Since the underlying bundle of E is $\mathcal{O}(-1)\oplus\mathcal{O}(-2)$. This example is the $\mathbb{Z}_3$-equivariant version of Example 6.3.5. Since the underlying sequence of $\mathcal{E}$ is compatible with $\mathcal{O}(-1)\oplus\mathcal{O}(-2)$, the $\mathbb{Z}_3$-exceptional sequence $\mathcal{E} \otimes \mathrm{Irr}\,(G)$ is $\mathbb{Z}_3$-compatible with V. Using Lemma 6.5.3, we calculate its A_∞-category:

$$\begin{aligned}
\mathcal{A}^\ell(v_{0\alpha}, v_{0\beta}) &= \mathrm{Hom}^\ell_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta) \oplus \mathrm{Hom}^{\ell-1}_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega^2(2)[2] \otimes \rho_\beta \otimes V) \\
&\quad \oplus \mathrm{Hom}^{\ell-2}_G(\Omega^2(2) \otimes \rho_\alpha[2], \Omega^2(2)[2] \otimes \rho_\beta \otimes \wedge^2 V) \\
&= [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^\ell(\mathbb{P}^2, \mathcal{O})]^G \oplus [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-1}(\mathbb{P}^2, \mathcal{O}(-1) \oplus \mathcal{O}(-2))]^G \\
&\quad \oplus [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(-3))]^G \\
&= \begin{cases} \mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 0, 4 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_{0\alpha}, v_{1\beta}) &= \mathrm{Hom}^\ell_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega(1)[1] \otimes \rho_\beta) \oplus \mathrm{Hom}^{\ell-1}_G(\Omega^2(2)[2]\rho_\alpha, \Omega(1)[1] \otimes \rho_\beta \otimes V) \\
&\quad \oplus \mathrm{Hom}^{\ell-2}_G(\Omega^2(2)[2] \otimes \rho_\alpha, \Omega(1)[1] \otimes \rho_\beta \otimes \wedge^2 V) \\
&= [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-1}(\mathbb{P}^2, \Omega(2))]^G \oplus [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-2}(\mathbb{P}^2, \Omega \oplus \Omega(1))]^G \\
&\quad \oplus [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-3}(\mathbb{P}^2, \Omega(-1))]^G \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \alpha - 1 = \beta \text{ and } \ell = 1 \\ \mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 3 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^\ell(v_{0\alpha}, v_{2\beta}) &= \mathrm{Hom}^\ell_G(\Omega^2(2)[2] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta) \oplus \mathrm{Hom}^{\ell-1}_G(\Omega^2(2)[2] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta \otimes V) \\
&\quad \oplus \mathrm{Hom}^{\ell-2}_G(\Omega^2(2)[2] \otimes \rho_\alpha, \mathcal{O} \otimes \rho_\beta \wedge^2 V) \\
&= [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(1))]^G \oplus [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-3}(\mathbb{P}^2, \mathcal{O} \oplus \mathcal{O}(-1))]^G \\
&\quad \oplus [\mathrm{Hom}(\rho_\alpha, \rho_\beta) \otimes H^{\ell-4}(\mathbb{P}^2, \mathcal{O}(-2))]^G \\
&= \begin{cases} \mathbb{C}^3 & \text{if } \alpha + 1 = \beta \text{ and } \ell = 2 \\ \mathbb{C} & \text{if } \alpha = \beta \text{ and } \ell = 3 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_{1\alpha}, v_{0\beta}) &= \mathrm{Hom}^\ell_G(\Omega(1)[1]\otimes\rho_\alpha, \Omega^2(2)[2]\otimes\rho_\beta)\oplus\mathrm{Hom}^{\ell-1}_G(\Omega(1)[1]\otimes\rho_\alpha, \Omega^2(2)[2]\otimes\rho_\beta\otimes V)\\
&\quad\oplus\mathrm{Hom}^{\ell-2}_G(\Omega(1)[1]\otimes\rho_\alpha, \Omega^2(2)[2]\otimes\rho_\beta\otimes\wedge^2 V)\\
&= [\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell+1}(\mathbb{P}^2,\mathcal{T}(-2))]^G\oplus[\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell}(\mathbb{P}^2,\mathcal{T}(-3)\oplus\mathcal{T}(-4))]^G\\
&\quad\oplus[\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell-1}(\mathbb{P}^2,\mathcal{T}(-5))]^G\\
&= \begin{cases}\mathbb{C} & \text{if } \alpha=\beta \text{ and } \ell=1\\ \mathbb{C}^3 & \text{if } \alpha+1=\beta \text{ and } \ell=3\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_{1\alpha}, v_{1\beta}) &= \mathrm{Hom}^\ell_G(\Omega(1)[1]\otimes\rho_\alpha, \Omega(1)[1]\otimes\rho_\beta)\oplus\mathrm{Hom}^{\ell-1}_G(\Omega(1)[1]\otimes\rho_\alpha, \Omega(1)[1]\otimes\rho_\beta\otimes V)\\
&\quad\oplus\mathrm{Hom}^{\ell-2}_G(\Omega(1)[1]\otimes\rho_\alpha, \Omega(1)[1]\otimes\rho_\beta\otimes\wedge^2 V)\\
&= [\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell}(\mathbb{P}^2,\mathcal{T}\otimes\Omega)]^G\\
&\quad\oplus[\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell-1}(\mathbb{P}^2,\mathcal{T}\otimes\Omega(-1)\oplus\mathcal{T}\otimes\Omega(-2))]^G\\
&\quad\oplus[\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell-2}(\mathbb{P}^2,\mathcal{T}\otimes\Omega(-3))]^G\\
&= \begin{cases}\mathbb{C} & \text{if } \alpha=\beta \text{ and } \ell=0,4\\ \mathbb{C}^3 & \text{if } \ell=2 \text{ and } \beta=\alpha+1 \text{ or } \beta=\alpha-1\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

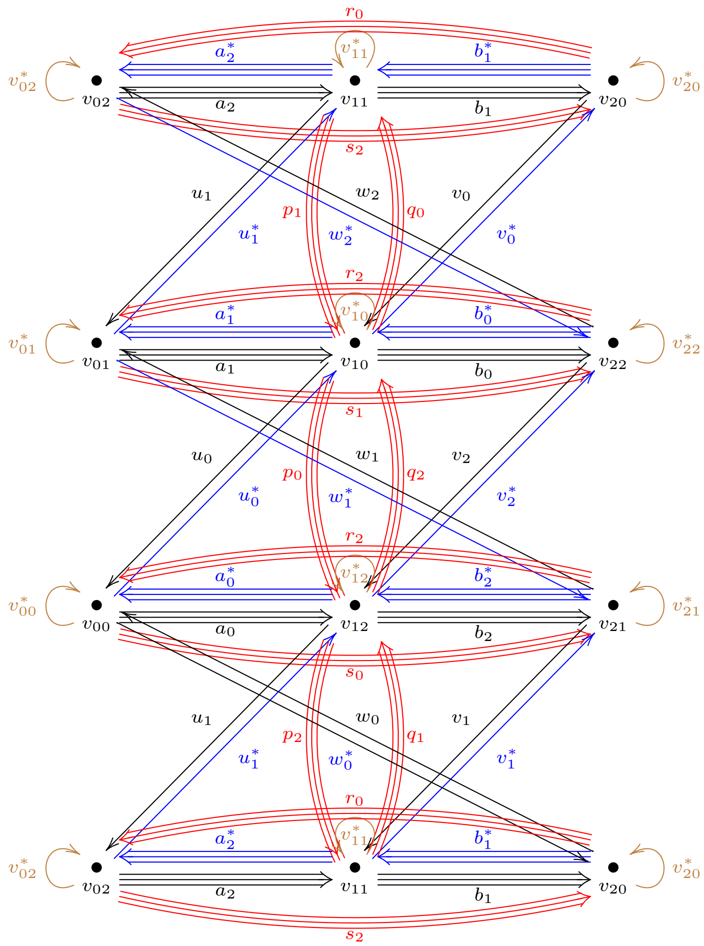
$$
\begin{aligned}
\mathcal{A}^\ell(v_{1\alpha}, v_{2\beta}) &= \mathrm{Hom}^\ell_G(\Omega(1)[1]\otimes\rho_\alpha, \mathcal{O}\otimes\rho_\beta)\oplus\mathrm{Hom}^{\ell-1}_G(\Omega(1)[1]\otimes\rho_\alpha, \mathcal{O}\otimes\rho_\beta\otimes V)\\
&\quad\oplus\mathrm{Hom}^{\ell-2}_G(\Omega(1)[1]\otimes\rho_\alpha, \mathcal{O}\otimes\rho_\beta\otimes\wedge^2 V)\\
&= [\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell-1}(\mathbb{P}^2,\mathcal{T}(-1))]^G\oplus[\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell-2}(\mathbb{P}^2,\mathcal{T}(-2)\oplus\mathcal{T}(-3))]^G\\
&\quad\oplus[\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell-3}(\mathbb{P}^2,\mathcal{T}(-4))]^G\\
&= \begin{cases}\mathbb{C}^3 & \text{if } \beta=\alpha-1 \text{ and } \ell=1\\ \mathbb{C} & \text{if } \alpha=\beta \text{ and } \ell=3\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_{2\alpha}, v_{0\beta}) &= \mathrm{Hom}^\ell_G(\mathcal{O}\otimes\rho_\alpha, \Omega^2(2)[2]\otimes\rho_\beta)\oplus\mathrm{Hom}^{\ell-1}_G(\mathcal{O}\otimes\rho_\alpha, \Omega^2(2)[2]\otimes\rho_\beta\otimes V)\\
&\quad\oplus\mathrm{Hom}^{\ell-2}_G(\mathcal{O}\otimes\rho_\alpha, \Omega^2(2)[2]\otimes\rho_\beta\otimes\wedge^2 V)\\
&= [\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell+2}(\mathbb{P}^2,\mathcal{O}(-1))]^G\oplus[\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell+1}(\mathbb{P}^2,\mathcal{O}(-2)\oplus\mathcal{O}(-3))]^G\\
&\quad\oplus[\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell}(\mathbb{P}^2,\mathcal{O}(-4))]^G\\
&= \begin{cases}\mathbb{C} & \text{if } \alpha=\beta \text{ and } \ell=1\\ \mathbb{C}^3 & \text{if } \beta=\alpha-1 \text{ and } \ell=2\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^\ell(v_{2\alpha}, v_{1\beta}) &= \mathrm{Hom}^\ell_G(\mathcal{O}\otimes\rho_\alpha, \Omega(1)[1]\otimes\rho_\beta)\oplus\mathrm{Hom}^{\ell-1}_G(\mathcal{O}\otimes\rho_\alpha, \Omega(1)[1]\otimes\rho_\beta\otimes V)\\
&\quad\oplus\mathrm{Hom}^{\ell-2}_G(\mathcal{O}\otimes\rho_\alpha, \Omega(1)[1]\otimes\rho_\beta\otimes\wedge^2 V)\\
&= [\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell+1}(\mathbb{P}^2,\Omega(1))]^G\oplus[\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell}(\mathbb{P}^2,\Omega\oplus\Omega(-1))]^G\\
&\quad\oplus[\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell-1}(\mathbb{P}^2,\Omega(-2))]^G\\
&= \begin{cases}\mathbb{C} & \text{if } \alpha=\beta \text{ and } \ell=1\\ \mathbb{C}^3 & \text{if } \beta=\alpha+1 \text{ and } \ell=3\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

$$\begin{aligned}
\mathcal{A}^\ell(v_{2\alpha}, v_{2\beta}) &= \mathrm{Hom}^\ell_G(\mathcal{O}\otimes\rho_\alpha, \mathcal{O}\otimes\rho_\beta) \oplus \mathrm{Hom}^{\ell-1}_G(\mathcal{O}\otimes\rho_\alpha, \mathcal{O}\otimes\rho_\beta\otimes V) \\
&\quad \oplus \mathrm{Hom}^{\ell-2}_G(\mathcal{O}\otimes\rho_\alpha, \mathcal{O}\otimes\rho_\beta\otimes\wedge^2 V) \\
&= [\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^\ell(\mathbb{P}^2,\mathcal{O})]^G \oplus [\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell-1}(\mathbb{P}^2,\mathcal{O}(-1)\oplus\mathcal{O}(-2))]^G \\
&\quad \oplus [\mathrm{Hom}(\rho_\alpha,\rho_\beta)\otimes H^{\ell-2}(\mathbb{P}^2,\mathcal{O}(-3))]^G \\
&= \begin{cases} \mathbb{C} & \text{if } \alpha=\beta \text{ and } \ell = 0,4 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The tilting quiver is



where the top and bottom row are identified. The differential sends

$$ds_{\alpha,i} = b_{\alpha-1,i+1}a_{\alpha,i+2} - b_{\alpha-1,i+2}a_{\alpha,i+1} \quad dr_{\alpha,i} = w_{\alpha-1}b_iv_\alpha - u_{\alpha-1}b_iw_\alpha$$

$$dp_{\alpha,i} = a_{\alpha,i}u_\alpha - v_{\alpha-1}b_{\alpha,i} \quad dq_{\alpha,i} = a_{\alpha-1,i+2}w_{\alpha-1}b_{\alpha,i+1} - a_{\alpha-1,i+1}w_{\alpha-1}b_{\alpha,i+2}.$$

The pairing on degree -1 edges is given by

$$\langle s_{\alpha,i}, r_{\alpha+1,i}\rangle = \langle p_{\alpha,i}, q_{\alpha-1,i}\rangle = 1$$

and zero otherwise. Thus the superpotential is given by

$$\begin{aligned}
\Phi = \sum_{\alpha,i=0}^{2} &(a_{\alpha-2,i+2}w_{\alpha-2}b_{\alpha-1,i+1} - a_{\alpha-2,i+1}w_{\alpha-2}b_{\alpha-1,i+2})p_{\alpha,i} + (a_{\alpha+1,i}u_{\alpha+1} - v_\alpha b_{\alpha+1,i})q_{\alpha,i} \\
&+ (b_{\alpha-2,i+1}a_{\alpha-1,i+2} - b_{\alpha-2,i+2}a_{\alpha-1,i+1})r_{\alpha,i} + (w_\alpha b_i v_{\alpha+1} - u_\alpha b_i w_{\alpha+1})s_{\alpha,i}.
\end{aligned}$$

Example 6.5.5. This example is essentially the dg version of Example 5.5 in Bocklandt, Schedler and Wemyss [7]. Let $G = \langle a,\, b | a^7 = b^3 = 1,\, ab^{-1} = b^{-1}a^4 \rangle$. This group has five irreducible representations:

1. ρ_0 the trivial one dimensional representation.
2. ρ_1 the one dimensional representation where a acts as identity and b by multiplication by $\eta = e^{2\pi i/3}$.
3. ρ_2 the one dimensional representation where a acts as identity and b by multiplication by $\eta^2 = e^{4\pi i/3}$.
4. ρ_3 the three dimensional representation with
$$a = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
where $\omega = e^{2\pi i/7}$.
5. ρ_4 the three dimensional representation with
$$a = \begin{pmatrix} \omega^6 & 0 & 0 \\ 0 & \omega^5 & 0 \\ 0 & 0 & \omega^3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
where $\omega = e^{2\pi i/7}$.

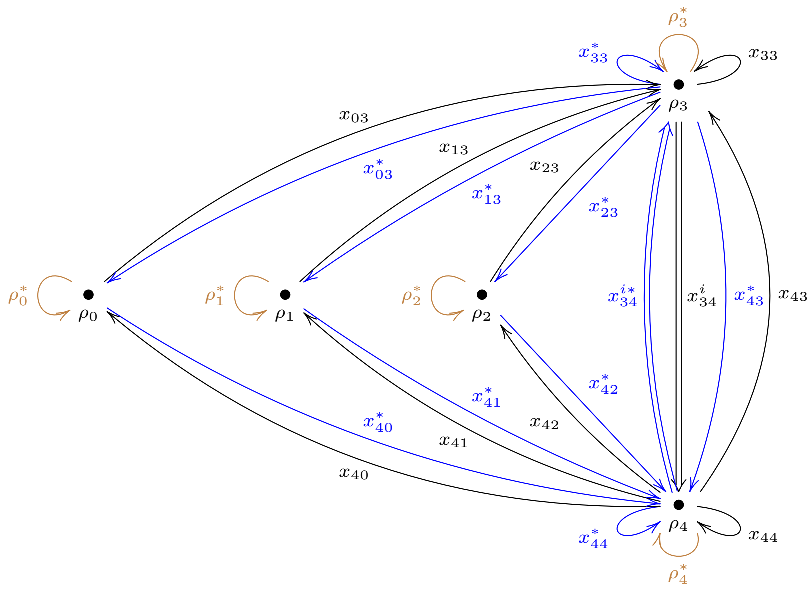
Now we suppose G acts on $\mathbb{C}^3$ through ρ_4. Denote $\mathbb{C}^3$ with this G-action by V. Then $V = \rho_4$, $\wedge^2 V = \rho_3$ and $\wedge^3 V = \rho_0$. We would like to calculate its quiver by the quotient construction. We have decompositions

$$\begin{array}{ll}
\rho_4 \otimes \rho_0 = \rho_4 & \rho_3 \otimes \rho_0 = \rho_3 \\
\rho_4 \otimes \rho_1 = \rho_4 & \rho_3 \otimes \rho_1 = \rho_3 \\
\rho_4 \otimes \rho_2 = \rho_4 & \rho_3 \otimes \rho_2 = \rho_3 \\
\rho_4 \otimes \rho_3 = \rho_0 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_4 & \rho_3 \otimes \rho_3 = \rho_3 \oplus \rho_4 \oplus \rho_4 \\
\rho_4 \otimes \rho_4 = \rho_3 \oplus \rho_3 \oplus \rho_4 & \rho_3 \otimes \rho_4 = \rho_0 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_4
\end{array}$$

We then have

$$(\mathcal{A}/G)^\ell(\rho_i, \rho_j) = \mathrm{Hom}_G(\rho_i, \wedge^\ell V \otimes \rho_j) = \begin{cases} \mathbb{C} & \text{if } i = j \text{ and } \ell = 0, 3 \\ \mathbb{C} & \text{if } j = 3 \text{ and } \ell = 1 \\ \mathbb{C} & \text{if } i = 3 \text{ and } \ell = 2 \\ \mathbb{C} & \text{if } i = 4 \text{ and } \ell = 1 \\ \mathbb{C} & \text{if } j = 4 \text{ and } \ell = 2 \\ \mathbb{C}^2 & \text{if } i = 3,\ j = 4 \text{ and } \ell = 1 \\ \mathbb{C}^2 & \text{if } i = 4,\ j = 3 \text{ and } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The underlying graded quiver is then given by



Next we would like to calculate the superpotential. For this, we need the following explicit isomorphisms of representations. Denote by c_i a basis element for the 1-dimensional representation ρ_i, where $0 \leq i \leq 2$, let u_1, u_2, u_3 be the standard basis for ρ_3, and v_1, v_2, v_3 be the standard basis for ρ_4. Then $\rho_4 \cong \rho_4 \otimes \rho_i$ for $0 \leq i \leq 2$ via the G-equivariant map

$$x_{4i} : \rho_4 \to \rho_4 \otimes \rho_i, \quad v_j \mapsto \eta^{-ij} v_j \otimes c_i.$$

Meanwhile, $\rho_0 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_4 \cong \rho_4 \otimes \rho_3$ via the sum of the G-equivariant maps

$$x_{i3} : \rho_i \to \rho_4 \otimes \rho_3, \quad c_i \mapsto \sum_{j=1}^{3} \eta^{ij} v_j \otimes u_j, \quad \text{for } 0 \leq i \leq 2,$$

$$x_{33} : \rho_3 \to \rho_4 \otimes \rho_3, \quad u_j \mapsto v_j \otimes u_{j+1}, \quad \text{where } j \in \mathbb{Z}/3\mathbb{Z},$$

$$x_{43} : \rho_4 \to \rho_4 \otimes \rho_3, \quad v_j \mapsto v_{j+1} \otimes u_j, \quad \text{where } j \in \mathbb{Z}/3\mathbb{Z}.$$

Also, $\rho_3 \oplus \rho_3 \oplus \rho_4 \cong \rho_4 \otimes \rho_4$ via the sum of the G-equivariant maps

$$x_{34}^1 : \rho_3 \to \rho_4 \otimes \rho_4, \quad u_i \mapsto v_{i+1} \otimes v_{i-1}, \quad \text{where } i \in \mathbb{Z}/3\mathbb{Z},$$

$$x_{34}^2 : \rho_3 \to \rho_4 \otimes \rho_4, \quad u_i \mapsto v_{i-1} \otimes v_{i+1}, \quad \text{where } i \in \mathbb{Z}/3\mathbb{Z},$$

$$x_{44} : \rho_4 \to \rho_4 \otimes \rho_4, \quad v_i \mapsto v_{i-1} \otimes v_{i-1}, \quad \text{where } i \in \mathbb{Z}/3\mathbb{Z}.$$

Since $\wedge^\bullet V$ is A_2, i.e., $m_n = 0$ for $n \geq 3$, $\mathrm{Hom}_G(\rho_i, \rho_j \otimes \wedge^\bullet V)$ is also A_2. Thus the superpotential on the quiver is cubic.

$$m_2(x_{03}, x_{40}) : \rho_4 \to \wedge^2 V \otimes \rho_3, \quad v_j \mapsto v_j \otimes c_0 \mapsto \sum_{k=1}^{3} v_j \otimes v_k \otimes u_k \mapsto \sum_{k=1}^{3} v_j \wedge v_k \otimes u_k$$

$$m_2(x_{34}^1, m_2(x_{03}, x_{40})) = \left(v_j \mapsto \sum_{k=1}^{3} v_j \wedge v_k \otimes u_k \mapsto \sum_{k=1}^{3} v_j \wedge v_k \wedge v_{k+1} \otimes v_{k-1} = v_j \wedge v_{j+1} \wedge v_{j+2} \otimes v_j \right)$$

Thus

$$\langle x_{34}^1, m_2(x_{03}, x_{40}) \rangle = \mathrm{tr}\,(v_j \mapsto \langle v_j, v_{j+1} \wedge v_{j+2} \rangle v_j = v_j) = 3$$

$$m_2(x_{34}^2, m_2(x_{03}, x_{40})) = \left(v_j \mapsto \sum_{k=1}^{3} v_j \wedge v_k \otimes u_k \mapsto \sum_{k=1}^{3} v_j \wedge v_k \wedge v_{k-1} \otimes v_{k+1} = v_j \wedge v_{j-1} \wedge v_{j-2} \otimes v_j \right)$$

Thus

$$\langle x_{34}^2, m_2(x_{03}, x_{40})\rangle = \operatorname{tr}(v_j \mapsto \langle v_j, v_{j-1} \wedge v_{j-2}\rangle v_j = -v_j) = -3$$

$$m_2(x_{13}, x_{41}) : \rho_4 \to \wedge^2 V \otimes \rho_3, \quad \left(v_j \mapsto \eta^{-j} v_j \otimes c_1 \mapsto \sum_{k=1}^{3} \eta^{k-j} v_j \wedge v_k \otimes u_k\right)$$

$$m_2(x_{34}^1, m_2(x_{13}, x_{41})) = \left(v_j \mapsto \sum_{k=1}^{3} \eta^{k-j} v_j \wedge v_k \otimes u_k \mapsto \eta v_j \wedge v_{j+1} \wedge v_{j+2} \otimes v_j\right)$$

$$\langle x_{34}^1, m_2(x_{13}, x_{41})\rangle = \operatorname{tr}(v_j \mapsto \eta v_j) = 3\eta$$

$$m_2(x_{34}^2, m_2(x_{13}, x_{41})) = \left(v_j \mapsto \sum_{k=1}^{3} \eta^{k-j} v_j \wedge v_k \otimes u_k \mapsto \eta^{-1} v_j \wedge v_{j-1} \wedge v_{j-2} \otimes v_j\right)$$

$$\langle x_{34}^2, m_2(x_{13}, x_{41})\rangle = \operatorname{tr}(v_j \mapsto -\eta^{-1} v_j) = -3\eta^{-1} = -3\eta^2$$

$$m_2(x_{23}, x_{42}) : \rho_2 \to \wedge^2 V \otimes \rho_3, \quad v_j \mapsto \eta^{-2j} v_j \otimes c_2 \mapsto \sum_{k=1}^{3} \eta^{2(k-j)} v_j \wedge v_k \otimes u_k$$

$$m_2(x_{34}^1, m_2(x_{23}, x_{42})) = \left(v_j \mapsto \sum_{k=1}^{3} \eta^{2(k-j)} v_j \wedge v_k \otimes u_k \mapsto \eta^2 v_j \wedge v_{j+1} \wedge v_{j+2} \otimes v_j\right)$$

$$\langle x_{34}^1, m_2(x_{23}, x_{42})\rangle = \operatorname{tr}\left(v_j \mapsto \eta^2 v_j\right) = 3\eta^2$$

$$m_2(x_{34}^2, m_2(x_{23}, x_{42})) = \left(v_j \mapsto \sum_{k=1}^{3} \eta^{2(k-j)} v_j \wedge v_k \otimes u_k \mapsto \eta^{-2} v_j \wedge v_{j-1} \wedge v_{j-2} \otimes v_j\right)$$

$$\langle x_{34}^2, m_2(x_{23}, x_{42})\rangle = \operatorname{tr}\left(v_j \mapsto -\eta^{-2} v_j\right) = -3\eta^{-2} = -3\eta$$

$$m_2(x_{33}, x_{43}) : \rho_4 \to \wedge^2 V \otimes \rho_3, \quad (v_j \mapsto v_{j+1} \otimes u_j \mapsto v_{j+1} \wedge v_j \otimes u_{j+1})$$

$$m_2(x_{34}^1, m_2(x_{33}, x_{43})) = (v_j \mapsto v_{j+1} \wedge v_j \otimes u_{j+1} \mapsto v_{j+1} \wedge v_j \wedge v_{j+2} \otimes v_j)$$

$$\langle x_{34}^1, m_2(x_{33}, x_{43})\rangle = \operatorname{tr}(v_j \mapsto -v_j) = -3$$

$$m_2(x_{34}^2, m_2(x_{33}, x_{43})) = (v_j \mapsto v_{j+1} \wedge v_j \otimes u_{j+1} \mapsto v_{j+1} \wedge v_j \wedge v_j \otimes v_{j+2} = 0) = 0$$

$$\langle x_{34}^2, m_2(x_{33}, x_{43})\rangle = 0$$

$$m_2(x_{43}, x_{44}) : \rho_4 \to \wedge^2 V \otimes \rho_3, \quad (v_j \mapsto v_{j-1} \otimes v_{j-1} \mapsto v_{j-1} \wedge v_j \otimes u_{j-1})$$

$$m_2(x_{34}^1, m_2(x_{43}, x_{44})) = (v_j \mapsto v_{j-1} \wedge v_j \otimes u_{j-1} \mapsto v_{j-1} \wedge v_j \wedge v_j \otimes v_{j-2} = 0) = 0$$

$$\langle x_{34}^1, m_2(x_{43}, x_{44})\rangle = 0$$

$$m_2(x_{34}^2, m_2(x_{43}, x_{44})) = (v_j \mapsto v_{j-1} \wedge v_j \otimes u_{j-1} \mapsto v_{j-1} \wedge v_j \wedge v_{j-2} \otimes v_j)$$

$$\langle x_{34}^2, m_2(x_{43}, x_{44})\rangle = \operatorname{tr}(v_j \mapsto v_j) = 3$$

$$m_2(x_{33}, x_{33}) : \rho_3 \to \wedge^2 V \otimes \rho_3, \quad (u_j \mapsto v_j \otimes u_{j+1} \mapsto v_j \wedge v_{j+1} \otimes u_{j+2})$$

$$m_2(x_{33}, m_2(x_{33}, x_{33})) = (u_j \mapsto v_j \wedge v_{j+1} \otimes u_{j+2} \mapsto v_j \wedge v_{j+1} \wedge v_{j+2} \otimes u_j)$$

$$\langle x_{33}, m_2(x_{33}, x_{33})\rangle = \operatorname{tr}(u_j \mapsto u_j) = 3$$

$$m_2(x_{44}, x_{44}) : \rho_4 \to \wedge^2 V \otimes \rho_4, \quad (v_j \mapsto v_{j-1} \otimes v_{j-1} \mapsto v_{j-1} \wedge v_{j-2} \otimes v_{j-2})$$

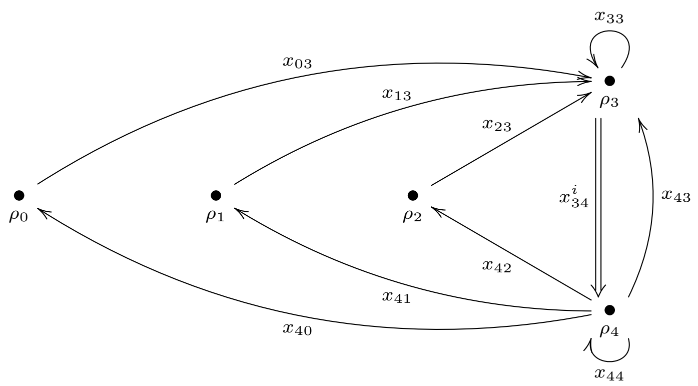
$$m_2(x_{44}, m_2(x_{44}, x_{44})) = (v_j \mapsto v_{j-1} \wedge v_{j-2} \otimes v_{j-2} \mapsto v_{j-1} \wedge v_{j-2} \wedge v_j \otimes v_j)$$

$$\langle x_{44}, m_2(x_{44}, x_{44})\rangle = \operatorname{tr}(v_j \mapsto -v_j) = -3$$

Therefore the superpotential is given by

$$\begin{aligned}\Phi &= 3x_{34}^1x_{03}x_{40} - 3x_{34}^2x_{03}x_{40} + 3\eta x_{34}^1x_{13}x_{41} - 3\eta^2x_{34}^2x_{13}x_{41} + 3\eta^2x_{34}^1x_{23}x_{42} - 3\eta x_{34}^2x_{23}x_{42}\\ &\quad - 3x_{34}^1x_{33}x_{43} + 3x_{34}^2x_{43}x_{44} + x_{33}x_{33}x_{33} - x_{44}x_{44}x_{44}\\ &= 3x_{03}x_{40}(x_{34}^1 - x_{34}^2) + 3\eta x_{13}x_{41}(x_{34}^1 - \eta x_{34}^2) + 3\eta x_{23}x_{42}(\eta x_{34}^1 - x_{34}^2)\\ &\quad - 3x_{34}^1x_{33}x_{43} + 3x_{34}^2x_{43}x_{44} + x_{33}x_{33}x_{33} - x_{44}x_{44}x_{44}.\end{aligned}$$

The classical quiver is thus given by the H^0 of the dg-quiver:



with relation

$$x_{40}x_{34}^1 = x_{40}x_{34}^2 \quad x_{41}x_{34}^1 = \eta x_{41}x_{34}^2 \quad \eta x_{42}x_{34}^1 = x_{42}x_{34}^2$$
$$x_{34}^1x_{03} = x_{34}^2x_{03} \quad x_{34}^1x_{13} = \eta x_{34}^2x_{13} \quad \eta x_{34}^1x_{23} = x_{34}^2x_{23}$$

$$x_{33}x_{33} = x_{43}x_{34}^1 \quad x_{44}x_{44} = x_{34}^2x_{43}$$

$$x_{03}x_{40} + \eta x_{13}x_{41} + \eta^2x_{23}x_{42} = x_{33}x_{43} \quad x_{03}x_{40} + \eta^2x_{13}x_{41} + \eta x_{23}x_{42} = x_{43}x_{44}$$

Example 6.5.6. The action of G on $\mathbb{C}^3$ induce an action of G on $\mathbb{P}^2$ and an equivariant action on $\mathcal{O}(1)$. Take the exceptional sequence $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ on $\mathrm{D}^b(\mathrm{Coh}\,(\mathbb{P}^2))$ and take $V = \mathcal{O}(-3) = K_{\mathbb{P}^2}$, which is the canonical bundle on $\mathbb{P}^2$ equipped with an G-equivariant action.

Lemma 6.5.7.

$$H^\ell(\mathbb{P}^2, \mathcal{O}(1)) = \begin{cases}\rho_3 & \textit{if } \ell = 0\\ 0 & \textit{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2, \mathcal{O}(-4)) = \begin{cases}\rho_4 & \textit{if } \ell = 2\\ 0 & \textit{otherwise.}\end{cases}$$
$$H^\ell(\mathbb{P}^2, \Omega(2)) = \begin{cases}\rho_4 & \textit{if } \ell = 0\\ 0 & \textit{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2, \mathcal{T}(-5)) = \begin{cases}\rho_3 & \textit{if } \ell = 2\\ 0 & \textit{otherwise.}\end{cases}$$
$$H^\ell(\mathbb{P}^2, \mathcal{T}(-1)) = \begin{cases}\rho_4 & \textit{if } \ell = 0\\ 0 & \textit{otherwise.}\end{cases} \qquad H^\ell(\mathbb{P}^2, \Omega(-2)) = \begin{cases}\rho_3 & \textit{if } \ell = 2\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. It suffices to show the lemma for all the nontrivial cases. Since G acts on $\mathbb{P}^2$ through $V = \rho_4$,

$$H^0(\mathbb{P}^2, \mathcal{O}(1)) = \rho_4^\vee = \rho_3.$$

By Serre duality,

$$H^2(\mathbb{P}^2, \mathcal{O}(-4)) \cong H^0(\mathbb{P}^2, \mathcal{O}(1))^\vee = \rho_3^\vee = \rho_4.$$

From the short exact sequence

$$0 \to \Omega(2) \to \mathcal{O}(1) \otimes V^\vee \to \mathcal{O}(2) \to 0,$$

we have an exact sequence

$$0 \to H^0(\mathbb{P}^2, \Omega(2)) \to H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes V^\vee \to H^0(\mathbb{P}^2, \mathcal{O}(2)) \to 0.$$

Since $H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes V^\vee \cong \rho_3 \otimes \rho_3$, $H^0(\mathbb{P}^2, \mathcal{O}(2)) = S^2V^\vee = S^2\rho_3$, and $\rho_3 \otimes \rho_3 \cong \wedge^2\rho_3 \oplus S^2\rho_3$, we conclude $H^0(\mathbb{P}^2, \Omega(2)) \cong \wedge^2\rho_3 \cong \rho_4$. By Serre duality, we have

$$H^2(\mathbb{P}^2, \mathcal{T}(-5)) \cong H^0(\mathbb{P}^2, \Omega(2))^\vee \cong \rho_3.$$

From the short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \otimes V \to \mathcal{T}(-1) \to 0,$$

we conclude

$$H^0(\mathbb{P}^2, \mathcal{T}(-1)) \cong V \cong \rho_4.$$

By Serre duality,

$$H^2(\mathbb{P}^2, \Omega(-2)) \cong H^0(\mathbb{P}^2, \mathcal{T}(-1))^\vee \cong \rho_3$$

■

$$\begin{aligned}
(\mathcal{A}/G)^\ell(v_{0i}, v_{1j}) &= \mathrm{Hom}_G^\ell(\Omega^2(2)[2] \otimes \rho_i, \Omega(1)[1] \otimes \rho_j) \oplus \mathrm{Hom}_G^{\ell-1}(\Omega^2(2)[2] \otimes \rho_i, \Omega(1)[1] \otimes \rho_j \otimes K_{\mathbb{P}^2}) \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell-1}(\mathbb{P}^2, \Omega(2))]^G \oplus [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell-2}(\mathbb{P}^2, \Omega(-1))]^G \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell-1}(\mathbb{P}^2, \Omega(2))]^G \\
&= \begin{cases} \mathrm{Hom}_G(\rho_i, \rho_4 \otimes \rho_j) & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \mathbb{C} & \text{if } j = 3 \text{ and } \ell = 1 \\ \mathbb{C} & \text{if } i = 4 \text{ and } \ell = 1 \\ \mathbb{C}^2 & \text{if } i = 3,\ j = 4 \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

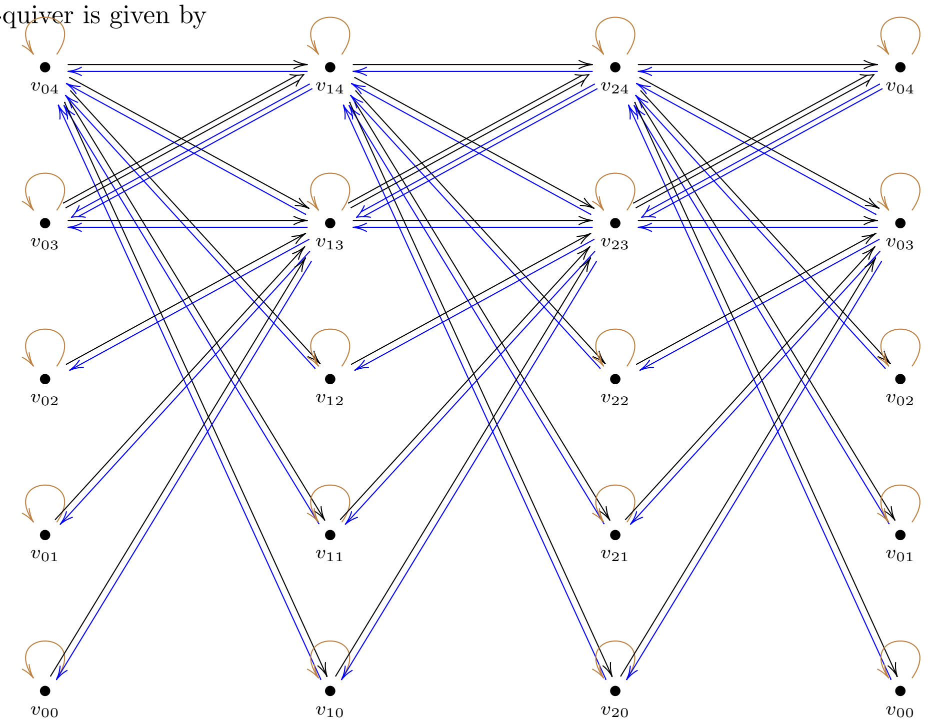
$$\begin{aligned}
(\mathcal{A}/G)^\ell(v_{0i}, v_{2j}) &= \mathrm{Hom}_G^\ell(\Omega^2(2)[2] \otimes \rho_i, \mathcal{O} \otimes \rho_j) \oplus \mathrm{Hom}_G^{\ell-1}(\Omega^2(2)[2] \otimes \rho_i, \mathcal{O} \otimes \rho_j \otimes K_{\mathbb{P}^2}) \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(1))]^G \oplus [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell-3}(\mathbb{P}^2, \mathcal{O}(-2))]^G \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell-2}(\mathbb{P}^2, \mathcal{O}(1))]^G \\
&= \begin{cases} \mathrm{Hom}_G(\rho_i, \rho_3 \otimes \rho_j) & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \mathbb{C} & \text{if } i = 3 \text{ and } \ell = 2 \\ \mathbb{C} & \text{if } j = 4 \text{ and } \ell = 2 \\ \mathbb{C}^2 & \text{if } i = 4,\ j = 3 \text{ and } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{A}/G)^\ell(v_{1i}, v_{0j}) &= \mathrm{Hom}_G^\ell(\Omega(1)[1] \otimes \rho_i, \Omega^2(2)[2] \otimes \rho_j) \oplus \mathrm{Hom}_G^{\ell-1}(\Omega(1)[1] \otimes \rho_i, \Omega^2(2)[2] \otimes \rho_j \otimes K_{\mathbb{P}^2}) \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell+1}(\mathbb{P}^2, \mathcal{T}(-2))]^G \oplus [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell}(\mathbb{P}^2, \mathcal{T}(-5))]^G \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell}(\mathbb{P}^2, \mathcal{T}(-5))]^G \\
&= \begin{cases} \mathrm{Hom}_G(\rho_i, \rho_3 \otimes \rho_j) & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \mathbb{C} & \text{if } i = 3 \text{ and } \ell = 2 \\ \mathbb{C} & \text{if } j = 4 \text{ and } \ell = 2 \\ \mathbb{C}^2 & \text{if } i = 4,\ j = 3 \text{ and } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{A}/G)^\ell(v_{1i}, v_{2j}) &= \mathrm{Hom}_G^\ell(\Omega(1)[1] \otimes \rho_i, \mathcal{O} \otimes \rho_j) \oplus \mathrm{Hom}_G^{\ell-1}(\Omega(1)[1] \otimes \rho_i, \mathcal{O} \otimes \rho_j \otimes K_{\mathbb{P}^2}) \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-1))]^G \oplus [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell-2}(\mathbb{P}^2, \mathcal{T}(-4))]^G \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell-1}(\mathbb{P}^2, \mathcal{T}(-1))]^G \\
&= \begin{cases} \mathrm{Hom}_G(\rho_i, \rho_4 \otimes \rho_j) & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \mathbb{C} & \text{if } j = 3 \text{ and } \ell = 1 \\ \mathbb{C} & \text{if } i = 4 \text{ and } \ell = 1 \\ \mathbb{C}^2 & \text{if } i = 3,\ j = 4 \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \\
(\mathcal{A}/G)^\ell(v_{2i}, v_{0j}) &= \mathrm{Hom}_G^\ell(\mathcal{O} \otimes \rho_i, \Omega^2(2)[2] \otimes \rho_j) \oplus \mathrm{Hom}_G^{\ell-1}(\mathcal{O} \otimes \rho_i, \Omega^2(2)[2] \otimes \rho_j \otimes K_{\mathbb{P}^2}) \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell+2}(\mathbb{P}^2, \mathcal{O}(-1))]^G \oplus [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell+1}(\mathbb{P}^2, \mathcal{O}(-4))]^G \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell+1}(\mathbb{P}^2, \mathcal{O}(-4))]^G \\
&= \begin{cases} \mathrm{Hom}_G(\rho_i, \rho_4 \otimes \rho_j) & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \mathbb{C} & \text{if } j = 3 \text{ and } \ell = 1 \\ \mathbb{C} & \text{if } i = 4 \text{ and } \ell = 1 \\ \mathbb{C}^2 & \text{if } i = 3,\ j = 4 \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{A}/G)^\ell(v_{2i}, v_{1j}) &= \mathrm{Hom}_G^\ell(\mathcal{O} \otimes \rho_i, \Omega(1)[1] \otimes \rho_j) \oplus \mathrm{Hom}_G^{\ell-1}(\mathcal{O} \otimes \rho_i, \Omega(1)[1] \otimes \rho_j \otimes K_{\mathbb{P}^2}) \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell+1}(\mathbb{P}^2, \Omega(1))]^G \oplus [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell}(\mathbb{P}^2, \Omega(-2))]^G \\
&= [\mathrm{Hom}(\rho_i, \rho_j) \otimes H^{\ell}(\mathbb{P}^2, \Omega(-2))]^G \\
&= \begin{cases} \mathrm{Hom}_G(\rho_i, \rho_3 \otimes \rho_j) & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \mathbb{C} & \text{if } i = 3 \text{ and } \ell = 2 \\ \mathbb{C} & \text{if } j = 4 \text{ and } \ell = 2 \\ \mathbb{C}^2 & \text{if } i = 4,\ j = 3 \text{ and } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The dg-quiver is given by

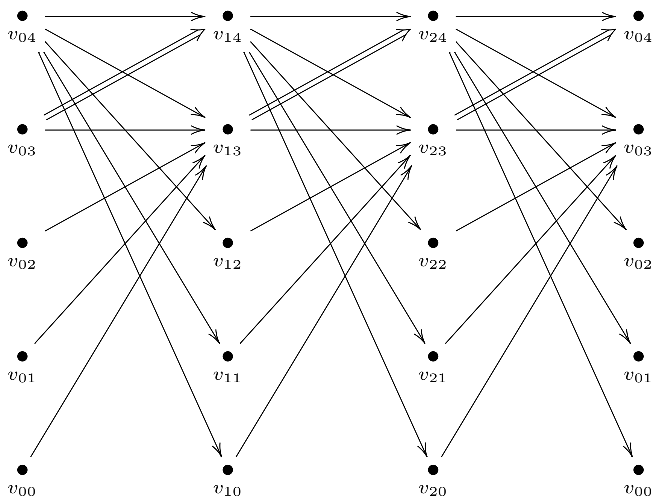


where the vertices on the left and that on the right in the same row are identified. We will use the following scheme to label the edges: the black (degree 0) edges going from vertex v_{ij} to $v_{k\ell}$ will be denoted by $x_{ij,k\ell}$. In case there are two arrows, we will denote by $x^1_{ij,k\ell}$, $x^2_{ij,k\ell}$. The blue (degree -1) arrows going from $v_{k\ell}$ to v_{ij} is denoted by $x^*_{ij,k\ell}$. The brown (degree -2) loops at vertex v_{ij} will be denoted by v^*_{ij}.

The superpotential is given by

$$\begin{aligned}\Phi = \sum_{i\in\mathbb{Z}_3} & 3x_{i0,(i+1)3}x_{(i+2)4,i0}(x^1_{(i+1)3,(i+2)4} - x^2_{(i+1)3,(i+2)4}) \\ & + 3\eta x_{i1,(i+1)3}x_{(i+2)4,01}(x^1_{(i+1)3,(i+2)4} - \eta x^2_{(i+1)3,(i+2)4}) \\ & + 3\eta x_{i2,(i+1)3}x_{(i+2)4,i2}(\eta x^1_{(i+1)3,(i+2)4} - x^2_{(i+1)3,(i+2)4}) \\ & - 3x^1_{i3,(i+1)4}x_{(i+2)3,i3}x_{(i+1)4,(i+2)3} + 3x^2_{i3,(i+1)4}x_{(i+2)4,i3}x_{(i+1)4,(i+2)4} \\ & + x_{i3,(i+1)3}x_{(i+2)3,i3}x_{(i+1)3,(i+2)3} - x_{i4,(i+1)4}x_{(i+2)4,i4}x_{(i+1)4,(i+2)4}.\end{aligned}$$

The classical quiver is given by



where the vertices on the left and that on the right in the same row are identified, with relations

$$\begin{array}{ll}
x_{(i+2)4,(i+1)0}x^1_{(i+1)3,i4} = x_{(i+2)4,(i+1)0}x^2_{(i+1)3,i4} & x_{(i+2)4,(i+1)1}x^1_{(i+1)3,i4} = \eta x_{(i+2)4,(i+1)1}x^2_{(i+1)3,i4} \\
\eta x_{(i+2)4,(i+1)2}x^1_{(i+1)3,i4} = x_{(i+2)4,(i+1)2}x^2_{(i+1)3,i4} & x^1_{(i+2)3,(i+1)4}x_{(i+1)0,i3} = x^2_{(i+2)3,(i+1)4}x_{(i+1)0,i3} \\
x^1_{(i+2)3,(i+1)4}x_{(i+1)1,i3} = \eta x^2_{(i+2)3,(i+1)4}x_{(i+1)1,i3} & \eta x^1_{(i+2)3,(i+1)4}x_{(i+1)2,i3} = x^2_{(i+2)3,(i+1)4}x_{(i+1)2,i3}
\end{array}$$

$$x_{(i+2)3,(i+1)3}x_{(i+1)3,i3} = x_{(i+2)4,(i+1)3}x^1_{(i+1)3,i4} \qquad x_{(i+2)4,(i+1)4}x_{(i+1)4,i4} = x^2_{(i+2)3,(i+1)4}x_{(i+1)4,i3}$$

$$x_{(i+2)0,(i+1)3}x_{(i+1)4,i0} + \eta x_{(i+2)1,(i+1)3}x_{(i+1)4,i1} + \eta^2 x_{(i+2)2,(i+1)3}x_{(i+1)4,i2} = x_{(i+2)3,(i+1)3}x_{(i+1)4,i3}$$

$$x_{(i+2)0,(i+1)3}x_{(i+1)4,i0} + \eta^2 x_{(i+2)1,(i+1)3}x_{(i+1)4,i1} + \eta x_{(i+2)2,(i+1)3}x_{(i+1)4,i2} = x_{(i+2)4,(i+1)3}x_{(i+1)4,i4}.$$

Chapter 7

Shifted Symplectic Structures on Moduli Spaces

This chapter contains some unfinished work which aims to make a connection between quivers with superpotential and the recent work on shifted symplectic structures by Pantev, Toën, Vaquié and Vezzosi [58], and Ben-Bassat, Brav, Bussi and Joyce [4].

Section 7.1 reviews the theory on derived algebraic geometry developed by Töen and Vezzosi [64, 65, 66] and Pantev, Toën, Vaquié and Vezzosi [58].

Section 7.2 develops the Lie algebra cohomology for dg-modules by modifying the usual Lie algebra cohomology theory.

Section 7.3 defines the G-invariant de Rham complex of on a derived scheme $\operatorname{Spec} R$ by using the Lie algebra cohomology developed in Section 7.2. We conjecture that the G-invariant de Rham complex should describe forms and closed forms on the quotient stack $[\operatorname{Spec} R/G]$ and outline a strategy of proof.

Section 7.4 describes the moduli space of representations of quiver with superpotential, and outline a strategy of proof on showing the existence of a shifted symplectic structure which is in a standard Darboux form by using the G-invariant de Rham complex introduced in Section 7.3.

7.1 Derived Schemes

The section is an outline of the theory of derived algebraic geometry needed to state the results in the next section, and is essentially a summary of Brav, Bussi and Joyce [13, §3]. For our purpose, we would not need the general definition of derived stacks. The main point here is that an affine derived scheme is essentially a commutative dg-algebra, in other words, there is a functor

$$\operatorname{Spec} : \{\text{commutative dg-algebra}\}^{\mathrm{op}} \to \{\text{derived stacks}\}.$$

For any derived Artin stack X, Toen and Vezzosi constructed a triangulated category $L_{qcoh}(X)$ with a t-structure whose heart is the category of quasi-coherent sheaves on X and defined a cotangent complex $\mathbb{L}_X$ in $L_{qcoh}(X)$. If $f : X \to Y$ is a morphism of derived Artin stacks, they constructed a morphism $\mathbb{L}_f : f^*\mathbb{L}_Y \to \mathbb{L}_X$ in $L_{qcoh}(X)$ and a relative cotangent complex $\mathbb{L}_{X/Y}$ which fits into a distinguished triangle

$$f^*\mathbb{L}_Y \to \mathbb{L}_X \to \mathbb{L}_{X/Y} \to f^*\mathbb{L}_Y[1].$$

In the case when X is an affine derived scheme, i.e., $X \cong \operatorname{Spec} R$ for some commutative dg-algebra R, we have a derived equivalence $L_{qcoh}(X) \cong D(R\text{-mod})$ which identifies $\mathbb{L}_X \cong \mathbb{L}_R$. If R is further assumed to be quasi-free, then the Kähler differential Ω_R gives a model for $\mathbb{L}_R$. Next, we introduce the notion of p-forms, closed p-forms, and symplectic forms on a affine derived scheme defined by Toën and Vezzossi, reinterpreted in the case for quasi-free affine derived scheme as per Brav, Bussi and Joyce [13, §5].

The grading on the dga R induces a grading on Ω^1_R, and we denote by $(\Omega^1_R)^k$ the k-th piece. The de Rham algebra of R is defined to be a doubled graded algebra

$$\mathrm{DR}(A) = \bigwedge^{\bullet} \Omega^1_R = \bigoplus_{p,k\in\mathbb{Z}} (\bigwedge^{p} \Omega^1_R)^k,$$

where each summand $\bigwedge^p \Omega^1_R$ is of 'form degree' p. There are two differential on the de Rham algebra: the differential induced by the differential on the dg-algebra R

$$d : (\bigwedge^{p} \Omega^1_R)^k \to (\bigwedge^{p} \Omega^1_R)^{k+1}$$

and the de Rham differential d_{dR}

$$d_{dR} : (\bigwedge^{p} \Omega^1_R)^k \to (\bigwedge^{p+1} \Omega^1_R)^k$$

Definition 7.1.1. A p-form of degree k on $\operatorname{Spec} R$ is an element in $H^k(\bigwedge^p \Omega^1_A, d)$. In other words, a p-form of degree k can be represented by an element $\omega \in (\bigwedge^p \Omega^1_R)^k$ with $d\omega = 0$. Two such representatives ω and ω' are equivalent if there exists $\alpha \in (\bigwedge^p \Omega^1_R)^{k-1}$ such that $\omega - \omega' = d\alpha$.

Definition 7.1.2. A closed p-form is an element in $H^k(\prod_{i\geq 0} \bigwedge^{p+i} \Omega^1_R[-i], d + d_{dR})$. In other words, a closed p-form of degree k can be represented by an sequence $\omega = (\omega^0, \omega^1, \omega^2, \ldots)$ with $\omega^i \in (\bigwedge^{p+i} \Omega^1_A)^{k-i}$ for $i = 0, 1, 2, \ldots$ satisfying the equations

$$\begin{aligned} d\omega^0 &= 0 && \text{in } (\bigwedge^{p} \Omega^1_R)^{k+1}, \text{ and} \\ d_{dR}\omega^i + d\omega^{i+1} &= 0 && \text{in } (\bigwedge^{p+i+1} \Omega^1_R)^{k-i} \text{ for all } i \geq 0. \end{aligned}$$

Two such representations ω, ω' are equivalent if there exists $\alpha = (\alpha^0, \alpha^1, \ldots)$ with $\alpha^i \in (\bigwedge^{p+i} \Omega^1_R)^{k-i-1}$ satisfying

$$\begin{aligned} \omega^0 - \omega'^0 &= d\alpha^0 && \text{in } (\bigwedge^{p} \Omega^1_R)^{k}, \text{ and} \\ \omega^{i+1} - \omega'^{i+1} &= d_{dR}\alpha^i + d\alpha^{i+1} && \text{in } (\bigwedge^{p+i+1} \Omega^1_R)^{k-i-1} \text{ for all } i \geq 0. \end{aligned}$$

Definition 7.1.3. A closed 2-form $\omega = (\omega^0, \omega^1, \ldots)$ of degree k is called k-shifted symplectic if ω^0 is a nondegenerate 2-form of degree k.

7.2 Lie Algebra Cohomology

This section develops the Lie algebra cohomology theory on dg-modules and is essentially an adaptation of the usual Lie algebra cohomology discussed, for instance, in Weibel [72, Chapter 7].

Let G be a linear algebraic group of finite type over $\mathbb{K}$ and (R, d_R) be a dg-algebra over $\mathbb{K}$ together with a G-action which is compatible with the dg-algebra structure in the sense that

$$\begin{aligned} \deg(gm) &= \deg m, \\ g(rr') &= (gr)(gr'), \text{ and} \\ d_R(gr) &= g(d_R r) \end{aligned}$$

for any $g \in G$, $r, r' \in R$. Let M be an R-module together with a G-action which is compatible with the dg-module structure in the sense that

$$\begin{aligned} \deg gm &= \deg m, \\ g(rm) &= (gr)(gm), \text{ and} \\ d_M(gm) &= g(d_M m) \end{aligned}$$

for any $g \in G$, $r \in R$, $m \in M$. Then the Lie algebra $\mathfrak{g}$ also acts on M as a derivation (Lie derivative) which satisfies

$$\begin{aligned} d_M(x \cdot m) &= x \cdot d_M(m) \\ x \cdot (rm) &= (x \cdot r)m + r(x \cdot m) \end{aligned}$$

for any $x \in \mathfrak{g}$, $r \in R$ and $m \in M$. We will introduce the Chevalley–Eilenberg double-complex $\mathrm{Hom}_{\mathbb{K}}(\wedge^{\bullet}\mathfrak{g}, M)$ with two anticommuting differentials

$$\delta_{M+} : \mathrm{Hom}_{\mathbb{K}}(\wedge^n \mathfrak{g}, M) \to \mathrm{Hom}_{\mathbb{K}}(\wedge^{n+1} \mathfrak{g}, M)$$

$$\begin{aligned} (\delta_{M+} f)(x_1, \dots, x_{n+1}) = &\sum_{i=1}^{n+1} (-1)^{i+1} x_i f(x_1, \dots, \hat{x_i}, \dots, x_{n+1}) \\ &+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_{n+1}) \end{aligned}$$

and

$$\begin{aligned} \delta_{M-} : \mathrm{Hom}_{\mathbb{K}}(\wedge^n \mathfrak{g}, M^i) &\to \mathrm{Hom}_{\mathbb{K}}(\wedge^n \mathfrak{g}, M^{i+1}) \\ f &\mapsto (-1)^n d_M \circ f. \end{aligned}$$

Proposition 7.2.1. $(\mathrm{Hom}_{\mathbb{K}}(\wedge^{\bullet}\mathfrak{g}, M), \delta_{M+}, \delta_{M-})$ *is double complex.*

Proof. First, we show that $\delta_+^2 = 0$.

$$
\begin{aligned}
&(\delta_{M+}^2 f)(x_1,\dots,x_{n+2})\\
&= \sum_{i=1}^{n+2}(-1)^{i+1}x_i(\delta_+ f)(x_1,\dots,\hat{x_i},\dots,x_{n+2}) + \sum_{i<j}(-1)^{i+j}(\delta_+ f)([x_i,x_j],x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{n+2})\\
&= \sum_{j<i}(-1)^{i+1+j+1}x_ix_jf(x_1,\dots,\hat{x_j},\dots,\hat{x_i},\dots,x_{n+2})\\
&\quad+\sum_{i<j}(-1)^{i+1+j}x_ix_jf(x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{n+2})\\
&\quad+\sum_{j<k<i}(-1)^{i+1+j+k}x_if([x_j,x_k],x_1,\dots\hat{x_j},\dots,\hat{x_k},\dots,\hat{x_i},\dots,x_{n+2})\\
&\quad+\sum_{j<i<k}(-1)^{i+1+j+k-1}x_if([x_j,x_k],x_1,\dots\hat{x_j},\dots,\hat{x_i},\dots,\hat{x_k},\dots,x_{n+2})\\
&\quad+\sum_{i<j<k}(-1)^{i+1+j-1+k-1}x_if([x_j,x_k],x_1,\dots\hat{x_i},\dots,\hat{x_j},\dots,\hat{x_k},\dots,x_{n+2})\\
&\quad+\sum_{i<j}(-1)^{i+j}[x_i,x_j]f(x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{n+2})\\
&\quad+\sum_{k<i<j}(-1)^{i+j+k+2}x_kf([x_i,x_j],x_1,\dots,\hat{x_k}\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{n+2})\\
&\quad+\sum_{i<k<j}(-1)^{i+j+k+1}x_kf([x_i,x_j],x_1,\dots,\hat{x_i}\dots,\hat{x_k},\dots,\hat{x_j},\dots,x_{n+2})\\
&\quad+\sum_{i<j<k}(-1)^{i+j+k}x_kf([x_i,x_j],x_1,\dots,\hat{x_i}\dots,\hat{x_j},\dots,\hat{x_k},\dots,x_{n+2})\\
&\quad+\sum_{k<i<j}(-1)^{i+j+1+k+1}f([[x_i,x_j],x_k],x_1,\dots,\hat{x_k},\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{n+2})\\
&\quad+\sum_{i<k<j}(-1)^{i+j+1+k}f([[x_i,x_j],x_k],x_1,\dots,\hat{x_i},\dots,\hat{x_k},\dots,\hat{x_j},\dots,x_{n+2})\\
&\quad+\sum_{i<j<k}(-1)^{i+j+1+k-1}f([[x_i,x_j],x_k],x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,\hat{x_k},\dots,x_{n+2})\\
&\quad+\sum_{k<\ell<i<j}(-1)^{i+j+k+1+\ell+1}f([x_k,x_\ell],[x_i,x_j],x_1,\dots,\hat{x_k},\dots,\hat{x_\ell},\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{n+2})\\
&\quad+\sum_{k<i<\ell<j}(-1)^{i+j+k+1+\ell}f([x_k,x_\ell],[x_i,x_j],x_1,\dots,\hat{x_k},\dots,\hat{x_i},\dots,\hat{x_\ell},\dots,\hat{x_j},\dots,x_{n+2})\\
&\quad+\sum_{k<i<j<\ell}(-1)^{i+j+k+1+\ell-1}f([x_k,x_\ell],[x_i,x_j],x_1,\dots,\hat{x_k},\dots,\hat{x_i},\dots,\hat{x_j},\dots,\hat{x_\ell},\dots,x_{n+2})\\
&\quad+\sum_{i<k<j<\ell}(-1)^{i+j+k+\ell-1}f([x_k,x_\ell],[x_i,x_j],x_1,\dots,\hat{x_i},\dots,\hat{x_k},\dots,\hat{x_j},\dots,\hat{x_\ell},\dots,x_{n+2})\\
&\quad+\sum_{i<k<\ell<j}(-1)^{i+j+k+\ell}f([x_k,x_\ell],[x_i,x_j],x_1,\dots,\hat{x_i},\dots,\hat{x_k},\dots,\hat{x_\ell},\dots,\hat{x_j},\dots,x_{n+2})\\
&\quad+\sum_{i<j<k<\ell}(-1)^{i+j+k-1+\ell-1}f([x_k,x_\ell],[x_i,x_j],x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,\hat{x_k},\dots,\hat{x_\ell},\dots,x_{n+2})\\
&= 0,
\end{aligned}
$$

where we have used the Jacobi identity, antisymmetry of f, and that $x_i(x_jm) - x_j(x_im) = [x_i,x_j]m$ for any $x_i, x_j \in \mathfrak{g}$ and $m \in M$ in the last step.

Next, we would like to show d and δ anticommute. For any $f \in \mathrm{Hom}(\wedge^n \mathfrak{g}, M)$ and $x_i \in \mathfrak{g}$,

$$\begin{aligned}
&(\delta_{M-}\delta_{M+}f)(x_1, \dots, x_{n+1}) \\
&= \sum_{i=1}^{n+1} (-1)^{n+1+i+1} d_M \left(x_i f(x_1, \dots, \hat{x_i}, \dots, x_{n+1})\right) \\
&\quad + \sum_{i<j} (-1)^{n+1+i+j} d_M \left(f([x_i, x_j], x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_{n+1})\right) \\
&= -\sum_{i=1}^{n+1} (-1)^{i+1+n} x_i d_M \left(f(x_1, \dots, \hat{x_i}, \dots, x_{n+1})\right) \\
&\quad - \sum_{i<j} (-1)^{i+j+n} d_M \left(f([x_i, x_j], x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_{n+1})\right) \\
&= -(\delta_{M+}\delta_{M-}f)(x_1, \dots x_{n+1})
\end{aligned}$$

Hence $\delta_{M-}\delta_{M+} + \delta_{M+}\delta_{M-} = 0$. The last condition $\delta_{M-}^2 = 0$ follows immediately from $d_M^2 = 0$ and hence $(\mathrm{Hom}_{\mathbb{K}}(\wedge^\bullet \mathfrak{g}, M), \delta_{M+}, \delta_{M-})$ is double complex. ■

The total complex of the double complex $(\mathrm{Hom}_{\mathbb{K}}(\wedge^\bullet \mathfrak{g}, M), \delta_{M+}, \delta_{M-})$ computes the Lie algebra cohomology $H^\bullet(\mathfrak{g}, M)$. The total differential of the double complex will be denoted by δ_M, i.e., $\delta_M = \delta_{M+} + \delta_{M-}$.

7.3 G-invariant de Rham Complex

This section aims to define the notion of G-invariant de Rham complex on a derived scheme $\mathrm{Spec}\, R$, where R is a quasi-free dg-algebra equipped with a G-action. Towards the end of this section, we outline a strategy to prove that the G-invariant de Rham complex should describe forms and closed forms on the quotient stack $[\mathrm{Spec}\, R/G]$.

G-invariant sections. A *generalized G-invariant section of degree k* of the module M is an element in the Lie algebra cohomology $[\alpha] \in H^k(\mathfrak{g}, M)$. A generalized G-invariant section $[\alpha]$ is thus represented by an element α of degree k in $(\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, M), \delta_M)$, henceforth called the *complex of G-invariant sections of M*, which satisfies $\delta_M \alpha = (\delta_{M+} + \delta_{M-})\alpha = 0$. If we decompose α into homogeneous terms and write $\alpha = \alpha^0 + \ldots + \alpha^{\dim \mathfrak{g}}$, where $\alpha^i \in \mathrm{Hom}_{\mathbb{K}}(\wedge^i \mathfrak{g}, M^{k-i})$, then the condition $\delta_M \alpha = 0$ becomes the system of equations

$$\begin{aligned}
\delta_{M-}\alpha^0 = 0 &\quad \text{in } \mathrm{Hom}_{\mathbb{K}}(\wedge^0 \mathfrak{g}, M^{k+1}) = M^{k+1}, \text{ and} \\
\delta_{M+}\alpha^i + \delta_{M-}\alpha^{i+1} = 0 &\quad \text{in } \mathrm{Hom}_{\mathbb{K}}(\wedge^{i+1} \mathfrak{g}, M^{k-i}) \text{ for } i = 0, 1, \dots, \dim \mathfrak{g}.
\end{aligned}$$

From these equations, we can see that in the case when M is concentrated in degree 0 and when $k = 0$,

$$H^0(\mathfrak{g}, M) = M^{\mathfrak{g}} = \{m \in M : a \cdot m = 0 \text{ for all } a \in \mathfrak{g}\},$$

i.e., a generalized G-invariant section is an honest G-invariant section.

G-invariant functions. When $M = R$, we can endow an associative product structure on the vector space $\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R)$, turning it into a dg-algebra, and regard it as the *complex of G-invariant functions* on $\mathrm{Spec}\, R$. To construct such a product, recall that there is a coproduct structure Δ on $\wedge^\bullet \mathfrak{g}$, which is the dual of the wedge product on $\wedge^\bullet \mathfrak{g}^\vee$, defined by

$$\Delta : \wedge^\bullet \mathfrak{g} \to \wedge^\bullet \mathfrak{g} \otimes \wedge^\bullet \mathfrak{g},$$

$$\Delta(x_1 \wedge \ldots \wedge x_n) = \sum_{k=0}^{n} \sum_{\sigma \in Sh_{k,n-k}} (\mathrm{sgn}\, \sigma)(x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \wedge \ldots \wedge x_{\sigma(i+k)}),$$

where $Sh_{k,n-k}$ denote the set of all $(k, n-k)$-shuffles. We then define a product on $\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R)$ by

$$\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R) \otimes \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R) \to \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R)$$

$$f \cdot g : \wedge^\bullet \mathfrak{g} \xrightarrow{\Delta} \wedge^\bullet \mathfrak{g} \otimes \wedge^\bullet \mathfrak{g} \xrightarrow{f\otimes g} R \otimes R \longrightarrow R,$$

where $R \otimes R \to R$ is the product on R. The product so defined is associative since the product on R is associative, and the coproduct Δ is coassociative. Explicitly, when $f \in \mathrm{Hom}(\wedge^i \mathfrak{g}, R^j)$ and $g \in \mathrm{Hom}(\wedge^k \mathfrak{g}, R^\ell)$, their product $f \cdot g \in \mathrm{Hom}(\wedge^{i+k} \mathfrak{g}, R^{j+\ell})$ is defined by

$$(f \cdot g)(x_1, \dots, x_{i+k}) = (-1)^{jk} \sum_{\sigma \in Sh_{i,k}} (\operatorname{sgn} \sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(i)}) g(x_{\sigma(i+1)}, \dots, x_{\sigma(i+k)}).$$

Proposition 7.3.1. $(\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R), \delta_R)$ *is a dg-algebra.*

Proof. It remains to check the Leibniz's rule $\delta_R(f \cdot g) = (\delta_R f) \cdot g + (-1)^{i+j} f \cdot (\delta_R g)$. Since $\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R)$ is generated by $\mathrm{Hom}(\mathfrak{g}, R)$ as an algebra, it suffices to show Leibniz's rule holds when $i = 1$, for the general case follows from induction on i.

$$\begin{aligned}
&(\delta_{R^+}(f \cdot g))(x_1, \dots, x_{k+2}) \\
&= \sum_{u=1}^{k+2} (-1)^{u+1} x_u((f \cdot g)(x_1, \dots, \hat{x_u}, \dots, x_{k+2})) \\
&\quad + \sum_{1 \le u < v \le k+2} (-1)^{u+v} (f \cdot g)([x_u, x_v], x_1, \dots, \hat{x_u}, \dots, \hat{x_v}, \dots, x_{k+2}) \\
&= (-1)^{jk} \Bigg(\sum_{u=1}^{k+2} \sum_{v<u} x_u((-1)^{u+v} f(x_v) g(x_1, \dots, \hat{x_v}, \dots, \hat{x_u}, \dots, x_{k+2})) \\
&\quad + \sum_{u=1}^{k+2} \sum_{v>u} x_u((-1)^{u+v+1} f(x_v) g(x_1, \dots, \hat{x_u}, \dots, \hat{x_v}, \dots, x_{k+2})) \\
&\quad + \sum_{1 \le u < v \le k+2} (-1)^{u+v} f([x_u, x_v]) g(x_1, \dots, \hat{x_u}, \dots, \hat{x_v}, \dots, x_{k+2}) \\
&\quad + \sum_{1 \le w < u < v \le k+2} (-1)^{u+v+w} f(x_w) g([x_u, x_v], x_1, \dots, \hat{x_w}, \dots, \hat{x_u}, \dots, \hat{x_v}, \dots, x_{k+2}) \\
&\quad + \sum_{1 \le u < w < v \le k+2} (-1)^{u+v+w-1} f(x_w) g([x_u, x_v], x_1, \dots, \hat{x_u}, \dots, \hat{x_w}, \dots, \hat{x_v}, \dots, x_{k+2}) \\
&\quad + \sum_{1 \le u < v < w \le k+2} (-1)^{u+v+w} f(x_w) g([x_u, x_v], x_1, \dots, \hat{x_u}, \dots, \hat{x_v}, \dots, \hat{x_w}, \dots, x_{k+2}) \Bigg)
\end{aligned}$$

$$\begin{aligned}
&((\delta_{R^+} f) \cdot g)(x_1, \dots, x_{k+2}) \\
&= (-1)^{jk} \left(\sum_{1 \le u < v \le k+2} (-1)^{u+v-1} (\delta_{R^+} f)(x_u, x_v) g(x_1, \dots, \hat{x_u}, \dots, \hat{x_v}, \dots, x_{k+2}) \right) \\
&= (-1)^{jk} \left(\sum_{1 \le u < v \le k+2} (-1)^{u+v-1} (x_u f(x_v) - x_v f(x_u) - f([x_u, x_v])) g(x_1, \dots, \hat{x_u}, \dots, \hat{x_v}, \dots, x_{k+2}) \right)
\end{aligned}$$

$$
\begin{aligned}
&(f \cdot (\delta_{R+} g))(x_1, \ldots, x_{k+2}) \\
&= (-1)^{j(k+1)} \sum_{u=1}^{k+2} (-1)^{u-1} f(x_u)(\delta_{R+} g)(x_1, \ldots, \hat{x_u}, \ldots, x_{k+2}) \\
&= (-1)^{j(k+1)} \left(\sum_{u=1}^{k+2} \sum_{v<u} (-1)^{u+v} f(x_u) x_v g(x_1, \ldots, \hat{x_v}, \ldots, \hat{x_u}, \ldots, x_{k+2}) \right. \\
&\quad + \sum_{u=1}^{k+2} \sum_{u<v} (-1)^{u+v-1} f(x_u) x_v g(x_1, \ldots, \hat{x_u}, \ldots, \hat{x_v}, \ldots, x_{k+2}) \\
&\quad + \sum_{1 \le v < w < u \le k+2} (-1)^{u+v+w-1} f(x_u) g([x_v, x_w], x_1, \ldots, \hat{x_v}, \ldots, \hat{x_w}, \ldots, \hat{x_u}, \ldots, x_{k+2}) \\
&\quad + \sum_{1 \le u < v < w \le k+2} (-1)^{u+v+w} f(x_u) g([x_v, x_w], x_1, \ldots, \hat{x_u}, \ldots, \hat{x_v}, \ldots, \hat{x_w}, \ldots, x_{k+2}) \\
&\quad \left. + \sum_{1 \le v < w < u \le k+2} (-1)^{u+v+w-1} f(x_u) g([x_v, x_w], x_1, \ldots, \hat{x_v}, \ldots, \hat{x_w}, \ldots, \hat{x_u}, \ldots, x_{k+2}) \right)
\end{aligned}
$$

Hence $\delta_{R+}(f \cdot g) = (\delta_{R+} f) \cdot g + (-1)^{1+j} f \cdot (\delta_{R+} g)$. On the other hand,

$$
\begin{aligned}
&(\delta_{R-}(f \cdot g))(x_1, \ldots, x_{k+1}) \\
&= (-1)^{jk+1+k} \sum_{u=1}^{k+1} (-1)^{u-1} d_R \left(f(x_u) g(x_1, \ldots, \hat{x_u}, \ldots, x_{k+1}) \right) \\
&= (-1)^{jk+1+k} \sum_{u=1}^{k+1} (-1)^{u-1} \left((d_R f(x_u)) g(x_1, \ldots, \hat{x_u}, \ldots, x_{k+1}) + (-1)^j f(x_u) d_R(g(x_1, \ldots, \hat{x_u}, \ldots, x_{k+1})) \right) \\
&= -((d_R \circ f) \cdot g)(x_1, \ldots, x_{1+k}) + (-1)^{1+j+k} (f \cdot (d_R \circ g))(x_1, \ldots, x_{1+k}) \\
&= ((\delta_{R-} f) \cdot g)(x_1, \ldots, x_{k+1}) + (-1)^{1+j} (f \cdot (\delta_{R-} g))(x_1, \ldots, x_{k+1}).
\end{aligned}
$$

Hence Leibniz's rule holds. ∎

If R is graded commutative and we endow a grading on $\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R)$ by declaring $f \in \mathrm{Hom}(\wedge^i \mathfrak{g}, R^j)$ has total degree $i + j$, then the product on $\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R)$ is also graded commutative, i.e.,

$$f \cdot g = (-1)^{(i+j)(k+\ell)} g \cdot f,$$

since the coproduct Δ is cocommutative and the product on R is graded commutative.

The complex of G-invariant sections of M also acquire a structure of $\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, M)$-module through the map

$$\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R) \otimes \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, M) \to \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, M),$$

$$f \cdot m : \wedge^\bullet \mathfrak{g} \xrightarrow{\Delta} \wedge^\bullet \mathfrak{g} \otimes \wedge^\bullet \mathfrak{g} \xrightarrow{f \otimes m} R \otimes M \longrightarrow M,$$

where the map $R \otimes M \to M$ is given by the R-module structure on M.

G-invariant Kähler differentials. Next, we would like to describe the Kähler differentials of the dg-algebra $\mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R)$, i.e., the complex of G-invariant functions. If we regard G-invariant functions on R as "functions" on the stack $[\mathrm{Spec}\, R/G]$, then G-invariant Kähler differentials should correspond to Kähler differentials on the stack $[\mathrm{Spec}\, R/G]$. Denote the Kähler differentials on R by Ω^1_R. Recall that since R is a $\mathfrak{g}$-module and $\mathfrak{g}$ acts on R by derivation, the map $R \to \mathrm{Hom}(\mathfrak{g}, R)$ sending $r \mapsto (a \mapsto a \cdot r)$ is a derivation which factorizes to a R-linear map $\alpha : \Omega^1_R \to \mathrm{Hom}(\mathfrak{g}, R)$ by the universal property of Ω^1_R. Both R-modules Ω^1_R and $\mathrm{Hom}(\mathfrak{g}, R)$ have a natural $\mathfrak{g}$ action which turns α into a $\mathfrak{g}$-module morphism: $\mathfrak{g}$ acts on the Ω^1_R component by

$$x \cdot d_{dR} r = d_{dR}(x \cdot r)$$

and on the $\mathrm{Hom}(\mathfrak{g}, R)$ component by enforcing the Leibniz rule:

$$(x \cdot f)(y) = x \cdot (f(y)) - f([x, y]).$$

Both $\mathfrak{g}$-action commutes with the internal differential on the two R-modules, and α is a $\mathfrak{g}$-module morphism since for all $x, y \in \mathfrak{g}$ and $r \in R$, we have

$$\alpha(x \cdot d_{dR}r)(y) - (x \cdot \alpha(d_{dR}r))(y) = y \cdot (x \cdot r) - x \cdot (y \cdot r)) + [x, y] \cdot r = 0.$$

The atlas $\varphi : \operatorname{Spec} R \to [\operatorname{Spec} R/G]$ is a principal G-bundle. Hence the relative cotangent complex is given by $\mathbb{L}_{[\operatorname{Spec} R/G]} \cong \mathfrak{g}^\vee \otimes R \cong \mathrm{Hom}(\mathfrak{g}, R)$. We have a distinguished triangle in $\mathrm{D}(R\text{-mod})$

$$\varphi^*\mathbb{L}_{[\operatorname{Spec} R/G]} \to \mathbb{L}_R \xrightarrow{\alpha} \mathrm{Hom}(\mathfrak{g}, R) \to \varphi^*\mathbb{L}_{[\operatorname{Spec} R/G]}[1].$$

Since R is assumed to be quasi-free, the Kähler differentials Ω^1_R gives a model for $\mathbb{L}_R$. Thus a model for $\varphi^*\mathbb{L}_{[\operatorname{Spec} R/G]}$ is given by $\operatorname{cone} \alpha[-1]$. Recall the cone of the map $\alpha[-1]$ is given by the R-module $\operatorname{cone} \alpha[-1] = \Omega^1_R \oplus \mathrm{Hom}(\mathfrak{g}, R)[-1]$ together with the differential

$$d_{\text{cone}} = \begin{pmatrix} d_{\Omega^1_R} & 0 \\ \alpha & -d_{\mathrm{Hom}(\mathfrak{g},R)} \end{pmatrix},$$

where $d_{\mathrm{Hom}(\mathfrak{g},R)}$ is the differential on $\mathrm{Hom}(\mathfrak{g}, R)$ which maps $f \mapsto d_R \circ f$. The direct sum $\mathfrak{g}$-module structure on $\operatorname{cone} \alpha[-1]$ is compatible with the internal differential d_{cone} since $d_{\Omega^1_R}$, $d_{\mathrm{Hom}(\mathfrak{g},R)}$ and α all commutes with $\mathfrak{g}$-action. We are going to construct a universal derivation $d_{dR} : \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R) \to \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, \operatorname{cone} \alpha[-1])$. Consider the following two maps

$$\begin{gathered} d_{dR+} : \mathrm{Hom}(\wedge^\ell \mathfrak{g}, R) \to \mathrm{Hom}(\wedge^\ell \mathfrak{g}, \Omega^1_R) \\ f \mapsto (-1)^\ell d_{dR} \circ f, \\ d_{dR-} : \mathrm{Hom}(\wedge^\ell \mathfrak{g}, R) \to \mathrm{Hom}(\wedge^{\ell-1} \mathfrak{g}, \mathrm{Hom}(\mathfrak{g}, R)[-1]), \\ (d_{dR-} f)(x_1, \dots, x_{\ell-1}) = (x \mapsto -f(x, x_1, \dots, x_{\ell-1})). \end{gathered}$$

The direct sum of these two maps defines a map

$$d_{dR} = d_{dR+} \oplus d_{dR-} : \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R) \to \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, \operatorname{cone} \alpha[-1])$$

of degree 1 satisfying the Leibnitz's rule which graded-commutes with the internal differentials:

Proposition 7.3.2. *The map $d_{dR} : \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, R) \to \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, \operatorname{cone} \alpha[-1])$ is a derivation which satisfies $d_{dR}\delta_R = -\delta_{\text{cone}}\, d_{dR}$.*

Proof. First, we check Leibnitz's rule:

$$\begin{aligned} &(d_{dR-}(f \cdot g))(x_1, \dots, x_{i+k-1}) \\ &= (x_0 \mapsto -(f \cdot g)(x_0, x_1, \dots, x_{i+k-1})) \\ &= \left(x_0 \mapsto -(-1)^{jk} \sum_{\sigma \in Sh_{i,k}} (\operatorname{sgn} \sigma) f(x_{\sigma(0)}, \dots, x_{\sigma(i-1)}) g(x_{\sigma(i)}, \dots, x_{\sigma(i+k-1)}) \right) \\ &= \left(x_0 \mapsto -(-1)^{jk} \sum_{\sigma \in Sh_{i-1,k}} (\operatorname{sgn} \sigma) f(x_0, x_{\sigma(1)}, \dots, x_{\sigma(i-1)}) g(x_{\sigma(i)}, \dots, x_{\sigma(i+k-1)}) \right) \\ &\quad + \left(x_0 \mapsto -(-1)^{i+jk} \sum_{\sigma \in Sh_{i,k-1}} (\operatorname{sgn} \sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(i)}) g(x_0, x_{\sigma(i+1)}, \dots, x_{\sigma(i+k-1)}) \right) \\ &= ((d_{dR-} f) \cdot g)(x_1, \dots, x_{i+k}) + (-1)^{i+j} (f \cdot (d_{dR-} g))(x_1, \dots, x_{i+k}), \end{aligned}$$

$$\begin{aligned}
&(d_{dR+}(f\cdot g))(x_1,\dots,x_{i+k})\\
&=(-1)^{i+k+jk}d_{dR}\left(\sum_{\sigma\in Sh_{i,k}}(\operatorname{sgn}\sigma)f(x_{\sigma(1)},\dots,x_{\sigma(i)})g(x_{\sigma(i+1)},\dots,x_{\sigma(i+k)})\right)\\
&=(-1)^{i+k+jk}\sum_{\sigma\in Sh_{i,k}}(\operatorname{sgn}\sigma)(d_{dR}f(x_{\sigma(1)},\dots,x_{\sigma(i)}))g(x_{\sigma(i+1)},\dots,x_{\sigma(i+k)})\\
&\quad+(-1)^{i+k+jk+j}\sum_{\sigma\in Sh_{i,k}}(\operatorname{sgn}\sigma)f(x_{\sigma(1)},\dots,x_{\sigma(i)})d_{dR}(g(x_{\sigma(i+1)},\dots,x_{\sigma(i+k)}))\\
&=((d_{dR+}f)\cdot g)(x_1,\dots,x_{i+k})+(-1)^{i+j}(f\cdot(d_{dR+}g))(x_1,\dots,x_{i+k}).
\end{aligned}$$

Hence d_{dR} is a derivation. The equation $d_{dR}\delta_R=-\delta_{\text{cone}}\,d_{dR}$ amounts to three equations

$$\begin{aligned}
d_{dR+}\delta_{R+}&=-\delta_{\text{cone}\,+}d_{dR+},\\
d_{dR+}\delta_{R-}+d_{dR-}\delta_{R+}&=-(\delta_{\text{cone}\,+}d_{dR-}+\delta_{\text{cone}\,-}d_{dR+}),\ \text{and}\\
d_{dR-}\delta_{R-}&=-\delta_{\text{cone}\,-}d_{dR-}.
\end{aligned}$$

Given $f\in\operatorname{Hom}(\wedge^\ell\mathfrak{g},R)$,

$$\begin{aligned}
&(d_{dR+}\delta_{R+}f)(x_1,\dots,x_{\ell+1})\\
&=\sum_{i=1}^{\ell+1}(-1)^{i+1+\ell+1}d_{dR}(x_i(f(x_1,\dots,\hat{x_i},\dots,x_{\ell+1})))\\
&\quad+\sum_{i<j}(-1)^{i+j+\ell+1}d_{dR}(f([x_i,x_j],x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{\ell+1}))\\
&=-\sum_{i=1}^{\ell+1}(-1)^{i+1}x_i((d_{dR+}f)(x_1,\dots,\hat{x_i},\dots,x_{\ell+1}))\\
&\quad-\sum_{i<j}(-1)^{i+j+\ell}(d_{dR+}f)([x_i,x_j],x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{\ell+1})\\
&=-(\delta_{\text{cone}\,+}d_{dR+}f)(x_1,\dots,x_{\ell+1}),
\end{aligned}$$

$$\begin{aligned}
(d_{dR-}\delta_{R-}f)(x_1,\dots,x_{\ell-1})&=-\left(x\mapsto(\delta_{R-}f)(x,x_1,\dots,x_{\ell-1})\right)\\
&=-\left(x\mapsto(\delta_{R-}f)(x,x_1,\dots,x_{\ell-1})\right)\\
&=-\left(x\mapsto(-1)^\ell d_R(f(x,x_1,\dots,x_{\ell-1}))\right)\\
&=(-1)^{\ell-1}d_{\text{cone}}\left(x\mapsto f(x,x_1,\dots,x_{\ell-1})\right)\\
&=-(-1)^{\ell-1}d_{\text{cone}}\left((d_{dR-}f)(x_1,\dots,x_{\ell-1})\right)\\
&=(-\delta_{\text{cone}\,-}d_{dR-}f)(x_1,\dots,x_{\ell-1}),
\end{aligned}$$

$$(d_{dR+}\delta_{R-}f)(x_1,\dots,x_\ell)=(d_{dR}\circ d_R\circ f)(x_1,\dots,x_\ell),$$

$$\begin{aligned}
&(d_{dR-}\delta_{R+}f)(x_1,\dots,x_\ell)\\
&=-\left(x_0\mapsto(\delta_{R+}f)(x_0,x_1,\dots,x_\ell)\right)\\
&=-\left(x_0\mapsto\sum_{i=0}^{\ell}(-1)^ix_if(x_0,x_1,\dots,\hat{x_i},\dots,x_\ell)+\sum_{0\le i<j\le\ell}(-1)^{i+j}f([x_i,x_j],x_0,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_\ell)\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&-(\delta_{\mathrm{cone}+}d_{dR-}f)(x_1,\dots,x_\ell)\\
&=-\sum_{i=1}^{\ell}(-1)^{i+1}x_i((d_{dR-}f)(x_1,\dots,\hat{x_i},\dots,x_\ell))\\
&\quad-\sum_{1\le i<j\le\ell}(-1)^{i+j}(d_{dR-}f)([x_i,x_j],x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_\ell)\\
&=\sum_{i=1}^{\ell}(-1)^{i+1}x_i\,(x_0\mapsto f(x_0,x_1,\dots,\hat{x_i},\dots,x_\ell))\\
&\quad+\left(x_0\mapsto\sum_{1\le i<j\le\ell}(-1)^{i+j}f(x_0,[x_i,x_j],x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_\ell)\right)\\
&=\sum_{i=1}^{\ell}(-1)^{i+1}\,(x_0\mapsto x_if(x_0,\dots,\hat{x_i},\dots,x_\ell)-f([x_i,x_0],x_1,\dots,\hat{x_i},\dots,x_\ell))\\
&\quad-\left(x_0\mapsto\sum_{1\le i<j\le\ell}(-1)^{i+j}f([x_i,x_j],x_0,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_\ell)\right)\\
&=-\left(x_0\mapsto\sum_{i=1}^{\ell}(-1)^{i}x_if(x_0,\dots,\hat{x_i},\dots,x_\ell)+\sum_{0\le i<j\le\ell}(-1)^{i+j}f([x_i,x_j],x_0,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_\ell)\right).
\end{aligned}$$

$$\begin{aligned}
-(\delta_{\mathrm{cone}-}d_{dR+}f)(x_1,\dots,x_\ell)&=-d_{\mathrm{cone}}\,(d_{dR}(f(x_0,\dots,x_\ell))\\
&=-\,(x_0\mapsto x_0\cdot f(x_1,\dots,x_\ell))-d_{\Omega^1_R}(d_{dR}(f(x_1,\dots,x_\ell))\\
&=-\,(x_0\mapsto x_0\cdot f(x_1,\dots,x_\ell))+d_{dR}(d_R(f(x_1,\dots,x_\ell))
\end{aligned}$$

Hence $d_{dR+}\delta_{R-}+d_{dR-}\delta_{R+}=-(\delta_{\mathrm{cone}+}d_{dR-}+\delta_{\mathrm{cone}-}d_{dR+})$. ■

Proposition 7.3.3. *The Kähler differentials* $(\Omega_{\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)},d_\Omega)$ *of the dg-algebra* $\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)$ *is isomorphic to* $(\mathrm{Hom}(\wedge^\bullet\mathfrak{g},\mathrm{cone}\,\alpha[-1]),\delta_{\mathrm{cone}})$.

Proof. Let $d'_{dR}:\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)\to\Omega_{\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)}$ denote the universal derivation. Since

$$\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)=\wedge^\bullet\mathfrak{g}^\vee\otimes R\text{ and }\Omega^1_{\wedge^\bullet\mathfrak{g}^\vee}=\wedge^\bullet\mathfrak{g}^\vee\otimes\mathfrak{g}^\vee[-1],$$

$$\Omega^1_{\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)}=\Omega^1_{\wedge^\bullet\mathfrak{g}^\vee\otimes R}=\wedge^\bullet\mathfrak{g}^\vee\otimes\mathfrak{g}^\vee[-1]\otimes R\oplus\wedge^\bullet\mathfrak{g}^\vee\otimes\Omega^1_R=\mathrm{Hom}(\wedge^\bullet\mathfrak{g},\Omega^1_R\oplus\mathrm{Hom}(\mathfrak{g},R)[-1])$$

as a $\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)$-module. Explicitly, for

$$f\in\mathrm{Hom}(\mathfrak{g},\mathbb{K})\subseteq\mathrm{Hom}(\mathfrak{g},R)\quad\text{and}\quad r\in R\cong\mathrm{Hom}(\mathbb{K},R)\subseteq\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R),$$

the isomorphism

$$\Phi:\Omega^1_{\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)}\cong\mathrm{Hom}(\wedge^\bullet\mathfrak{g},\Omega^1_R\oplus\mathrm{Hom}(\mathfrak{g},R)[-1])$$

sends

$$d'_{dR}f\mapsto(k\mapsto kf)\in\mathrm{Hom}(\mathbb{K},\mathrm{Hom}(\mathfrak{g},R)[-1])\quad\text{and}\quad d'_{dR}r\mapsto(k\mapsto kd_{dR}r)\in\mathrm{Hom}(\mathbb{K},\Omega^1_R).$$

Since $\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)$ is generated as an algebra by elements of the form

$$f\in\mathrm{Hom}(\mathfrak{g},\mathbb{K})\subseteq\mathrm{Hom}(\mathfrak{g},R)\quad\text{and}\quad r\in R\cong\mathrm{Hom}(\mathbb{K},R)\subseteq\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R),$$

the square

$$\begin{tikzcd}
\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R) \arrow[r, "d'_{dR}"] \arrow[d, equal] & \Omega^1_{\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)} \arrow[d, "\Phi"] \\
\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R) \arrow[r, "d_{dR}"] & \mathrm{Hom}(\wedge^\bullet\mathfrak{g},\mathrm{cone}\,\alpha[-1])
\end{tikzcd}$$

commutes. Finally, we would like to show Φ commutes with the internal differentials of $\Omega^1_{\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)}$ and $\mathrm{Hom}(\wedge^\bullet\mathfrak{g},\mathrm{cone}\,\alpha[-1])$: For any $u\in\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)$,

$$\Phi d_{\Omega^1_{\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)}}(d'_{dR}u) = -\Phi d'_{dR}(\delta_R u) = -d_{dR}(\delta_R u) = -\delta_{\mathrm{cone}}\, d_{dR}u = \delta_{\mathrm{cone}}\,\Phi d'_{dR}u.$$

Hence $(\Omega_{\mathrm{Hom}(\wedge^\bullet\mathfrak{g},R)}, d_\Omega)$ is isomorphic to $(\mathrm{Hom}(\wedge^\bullet\mathfrak{g},\mathrm{cone}\,\alpha[-1]),\delta_{\mathrm{cone}})$ as desired. ∎

We will call $(\mathrm{Hom}(\wedge^\bullet\mathfrak{g},\mathrm{cone}\,\alpha[-1]),\delta_{\mathrm{cone}})$ the *complex of G-invariant 1-forms.* Taking the n-th wedge product, we get the *complex of G-invariant n-forms*

$$\mathrm{Hom}(\wedge^\bullet\mathfrak{g},\wedge^n\mathrm{cone}\,\alpha[-1]) = \bigoplus_{j+k=n}\mathrm{Hom}(\wedge^\bullet\mathfrak{g},\mathrm{Hom}(S^j\mathfrak{g},\wedge^k\Omega^1_R)[-j]).$$

G-invariant de Rham complex. We would like to extend the de Rham differential to all G-invariant forms. We define

$$\begin{array}{ccc} d_{dR+}:\mathrm{Hom}(\wedge^\ell\mathfrak{g},\mathrm{Hom}(S^j\mathfrak{g},\wedge^k\Omega^1_R)[-j]) & \longrightarrow & \mathrm{Hom}(\wedge^\ell\mathfrak{g},\mathrm{Hom}(S^j\mathfrak{g},\wedge^{k+1}\Omega^1_R)[-j]) \\ \| & & \| \\ d_{dR+}:\mathrm{Hom}(\wedge^\ell\mathfrak{g}\otimes S^j\mathfrak{g},\wedge^k\Omega^1_R[-j]) & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}(\wedge^\ell\mathfrak{g}\otimes S^j\mathfrak{g},\wedge^{k+1}\Omega^1_R[-j]) \end{array}$$

$$f\mapsto(-1)^\ell d_{dR}\circ f$$

where here $d_{dR}:\wedge^k\Omega^1_R\to\wedge^{k+1}\Omega^1_R$ is the de Rham differential on the complex $\wedge^\bullet\Omega^1_R$,

$$\begin{array}{ccc} d_{dR-}:\mathrm{Hom}(\wedge^\ell\mathfrak{g},\mathrm{Hom}(S^j\mathfrak{g},\wedge^k\Omega^1_R)[-j]) & \longrightarrow & \mathrm{Hom}(\wedge^{\ell-1}\mathfrak{g},\mathrm{Hom}(S^{j+1}\mathfrak{g},\wedge^k\Omega^1_R)[-j-1]) \\ \| & & \| \\ d_{dR-}:\mathrm{Hom}(\wedge^\ell\mathfrak{g}\otimes S^j\mathfrak{g},\wedge^k\Omega^1_R[-j]) & \longrightarrow & \mathrm{Hom}(\wedge^{\ell-1}\mathfrak{g}\otimes S^{j+1}\mathfrak{g},\wedge^k\Omega^1_R[-j-1]) \end{array}$$

$$(d_{dR-}f)(x_1,\dots,x_{\ell-1},y_1,\dots,y_{j+1}) = \sum_{i=1}^{j+1} f(y_i,x_1,\dots,x_{\ell-1},y_1,\dots,\hat{y_i},\dots,y_{j+1}).$$

Proposition 7.3.4. $(\mathrm{Hom}(\wedge^\bullet\mathfrak{g},\mathrm{Hom}(S^*\mathfrak{g},\wedge^\bullet\Omega^1_R)[-*]),d_{dR+},d_{dR-})$ *is a double complex.*

Proof. $d^2_{dR+}=0$ since $d^2_{dR}=0$ on $\wedge^\bullet\Omega^1_R$. Given $f\in\mathrm{Hom}(\wedge^\ell\mathfrak{g}\otimes S^j\mathfrak{g},\wedge^k\Omega^1_R[-j])$,

$$\begin{aligned} &(d^2_{dR-}f)(x_1,\dots,x_{\ell-2},y_1,\dots,y_{j+2}) \\ &=\sum_{u=1}^{j+2}(d_{dR-}f)(y_u,x_1,\dots,x_{\ell-2},y_1,\dots,\hat{y_u},\dots,y_{j+2}) \\ &=\sum_{1\le u<v\le j+2} f(y_v,y_u,x_1,\dots,x_{\ell-2},y_1,\dots,\hat{y_u},\dots,\hat{y_v},\dots,y_{j+2}) \\ &\quad+\sum_{1\le v<u\le j+2} f(y_v,y_u,x_1,\dots,x_{\ell-2},y_1,\dots,\hat{y_v},\dots,\hat{y_u},\dots,y_{j+2}) \\ &=0 \end{aligned}$$

since f is antisymmetric in the first two variables.

$$\begin{aligned} (d_{dR+}d_{dR-}f)(x_1,\dots,x_{\ell-1},y_1,\dots,y_{j+1}) &= (-1)^\ell d_{dR}\left(\sum_{i=1}^{j+1} f(y_i,x_1,\dots,x_{\ell-1},y_1,\dots,\hat{y_i},\dots,y_{j+1})\right) \\ &= -\sum_{i=1}^{j+1}(-1)^{\ell-1}d_{dR}(f(y_i,x_1,\dots,x_{\ell-1},y_1,\dots,\hat{y_i},\dots,y_{j+1})) \\ &= -(d_{dR-}d_{dR+}f)(x_1,\dots,x_{\ell-1},y_1,\dots,y_{j+1}). \end{aligned}$$

Hence the d_{dR+} and d_{dR-} anticommute. ∎

The total complex of

$$(\mathrm{Hom}(\wedge^\bullet\mathfrak{g}, \wedge^\bullet \mathrm{cone}\,\alpha[-1]), d_{dR+}, d_{dR-}) = (\mathrm{Hom}(\wedge^\bullet\mathfrak{g}, \otimes S^*\mathfrak{g}, \wedge^\bullet\Omega^1_R[-*]), d_{dR+}, d_{dR-})$$

is called the *G-invariant de Rham complex* and the total differential d_{dR} is called the de Rham differential.

There is a wedge product on the de Rham complex

$$\wedge : \mathrm{Hom}(\wedge^\bullet\mathfrak{g}, \wedge^\bullet \mathrm{cone}\,\alpha[-1]) \otimes \mathrm{Hom}(\wedge^\bullet\mathfrak{g}, \wedge^\bullet \mathrm{cone}\,\alpha[-1]) \to \mathrm{Hom}(\wedge^\bullet\mathfrak{g}, \wedge^\bullet \mathrm{cone}\,\alpha[-1])$$

defined by

$$f\wedge g : \wedge^\bullet\mathfrak{g} \xrightarrow{\Delta} \wedge^\bullet\mathfrak{g}\otimes\wedge^\bullet\mathfrak{g} \xrightarrow{f\otimes g} \wedge^\bullet \mathrm{cone}\,\alpha[-1] \otimes \wedge^\bullet \mathrm{cone}\,\alpha[-1] \xrightarrow{\wedge} \wedge^\bullet \mathrm{cone}\,\alpha[-1].$$

Under the isomorphism

$$\wedge^n \mathrm{cone}\,\alpha[-1] = \bigoplus_{j+k=n} \mathrm{Hom}(S^j\mathfrak{g}, \wedge^k\Omega^1_R)[-j]),$$

the wedge product on $\wedge^\bullet \mathrm{cone}\,\alpha[-1]$ is given by

$$\mathrm{Hom}(S^j\mathfrak{g}, \wedge^k\Omega^1_R)[-j] \otimes \mathrm{Hom}(S^\ell\mathfrak{g}, \wedge^m\Omega^1_R)[-\ell] \to \mathrm{Hom}(S^{j+\ell}\mathfrak{g}, \wedge^{k+m}\Omega^1_R)[-j-\ell]$$

$$f\wedge g : S^{j+\ell}\mathfrak{g} \xrightarrow{\Delta'} S^j\mathfrak{g}\otimes S^\ell\mathfrak{g} \xrightarrow{f\otimes g} \wedge^k\Omega^1_R\otimes\wedge^m\Omega^1_R \xrightarrow{\wedge} \wedge^{k+m}\Omega^1_R,$$

where Δ' is the coproduct on $S^\bullet\mathfrak{g}$. Hence, when

$$f \in \mathrm{Hom}(\wedge^i\mathfrak{g}\otimes S^j\mathfrak{g}, \wedge^k\Omega^1_R[-j]) \cong \mathrm{Hom}(\wedge^i\mathfrak{g}, \mathrm{Hom}(S^j\mathfrak{g}, \wedge^k\Omega^1_R)[-j]) \subseteq \mathrm{Hom}(\wedge^i\mathfrak{g}, \wedge^{j+k}\Omega^1_R)$$

and

$$g \in \mathrm{Hom}(\wedge^\ell\mathfrak{g}\otimes S^m\mathfrak{g}, \wedge^n\Omega^1_R[-m]) \cong \mathrm{Hom}(\wedge^\ell\mathfrak{g}, \mathrm{Hom}(S^m\mathfrak{g}, \wedge^n\Omega^1_R)[-m]) \subseteq \mathrm{Hom}(\wedge^\ell g, \wedge^{m+n}\Omega^1_R)$$

their wedge product $f\wedge g \in \mathrm{Hom}(\wedge^{i+\ell}\mathfrak{g}\otimes S^{j+m}\mathfrak{g}, (\wedge^\bullet\Omega^1_R)^{k+n}[-j-m])$ is explicitly given by

$$\begin{aligned}
&(f\wedge g)(x_1,\dots,x_{i+\ell},y_1,\dots,y_{j+m})\\
&= (-1)^{k\ell} \sum_{\sigma\in Sh_{i,\ell}} \sum_{\tau\in Sh_{j,m}} (\mathrm{sgn}\,\sigma) f(x_{\sigma(1)},\dots,x_{\sigma(i)},y_{\tau(1)},\dots,y_{\tau(j)})\\
&\qquad\qquad\qquad\qquad \wedge g(x_{\sigma(i+1)},\dots,x_{\sigma(i+k)},y_{\tau(j+1)},\dots,y_{\tau(j+m)}).
\end{aligned}$$

If we endow a grading on $\mathrm{Hom}(\wedge^\bullet\mathfrak{g}, \wedge^\bullet \mathrm{cone}\,\alpha[-1])$ by declaring

$$f \in \mathrm{Hom}(\wedge^i\mathfrak{g}\otimes S^j\mathfrak{g}, (\wedge^\bullet\Omega^1_R)^k[-j]) \subseteq \mathrm{Hom}(\wedge^i\mathfrak{g}, (\wedge^\bullet \mathrm{cone}\,\alpha[-1])^{j+k})$$

and

$$g \in \mathrm{Hom}(\wedge^\ell\mathfrak{g}\otimes S^m\mathfrak{g}, (\wedge^\bullet\Omega^1_R)^n[-m]) \subseteq \mathrm{Hom}(\wedge^\ell\mathfrak{g}, (\wedge^\bullet \mathrm{cone}\,\alpha[-1])^{m+n})$$

has total degree $i+2j+k$ and $\ell+2m+n$ respectively, then the wedge product is graded commutative: $f\wedge g = (-1)^{(i+2j+k)(\ell+2m+n)} g\wedge f$.

Proposition 7.3.5. *The de Rham differential satisfies the Leibniz's rule*

$$d_{dR}(f\wedge g) = (d_{dR}f)\wedge g + (-1)^{i+2j+k} f\wedge(d_{dR}g).$$

Proof.

$$\begin{aligned}
&(d_{dR-}(f\wedge g))(x_1,\dots,x_{i+\ell-1},y_1,\dots,y_{j+m+1})\\
&= \sum_{u=1}^{j+m+1} (f\wedge g)(y_u,x_1,\dots,x_{i+\ell-1},y_1,\dots,\hat{y_u},\dots,y_{j+m+1})\\
&= (-1)^{k\ell} \sum_{\sigma\in Sh_{i-1,\ell}} \sum_{\tau\in A} (\mathrm{sgn}\,\sigma) f(y_{\tau(1)},x_{\sigma(1)},\dots,x_{\sigma(i-1)},y_{\tau(2)},\dots,y_{\tau(j+1)})\\
&\qquad\qquad\qquad\qquad \wedge g(x_{\sigma(i)},\dots,x_{\sigma(i+\ell-1)},y_{\tau(j+2)},\dots,y_{\tau(j+m+1)})\\
&\quad + (-1)^{k\ell} \sum_{\sigma\in Sh_{i,\ell-1}} \sum_{\tau\in B} (-1)^i (\mathrm{sgn}\,\sigma) f(x_{\sigma(1)},\dots,x_{\sigma(i)},y_{\tau(1)},\dots,y_{\tau(j)})\\
&\qquad\qquad\qquad\qquad \wedge g(y_{\tau(j+1)},x_{\sigma(i+1)},\dots,x_{\sigma(i+\ell-1)},y_{\tau(j+2)},\dots,y_{\tau(j+m+1)}),
\end{aligned}$$

where

$$A = \{\tau \in S_{j+m+1} : \tau(2) < \cdots \tau(j+1) \text{ and } \tau(j+2) < \cdots < \tau(j+m+1)\}, \tag{7.3.1}$$
$$B = \{\tau \in S_{j+m+1} : \tau(1) < \cdots \tau(j) \text{ and } \tau(j+2) < \cdots < \tau(j+m+1)\}, \tag{7.3.2}$$

with S_{j+m+1} denoting the symmetric group on $j+m+1$ letters.

$$\begin{aligned}
&((d_{dR-}f)\wedge g)(x_1,\ldots,x_{i+\ell-1},y_1,\ldots,y_{j+m+1})\\
&= (-1)^{k\ell}\sum_{\sigma\in Sh_{i-1,\ell}}\sum_{\tau\in Sh_{j+1,m}}(\operatorname{sgn}\sigma)(d_{dR-}f)(x_{\sigma(1)},\ldots,x_{\sigma(i-1)},y_{\tau(1)},\ldots,y_{\tau(j+1)})\\
&\qquad\qquad \wedge g(x_{\sigma(i)},\ldots,x_{\sigma(i+\ell-1)},y_{\tau(j+2)},\ldots,y_{\tau(j+m+1)})\\
&= (-1)^{k\ell}\sum_{\sigma\in Sh_{i-1,\ell}}\sum_{\tau\in Sh_{j+1,m}}(\operatorname{sgn}\sigma)\sum_{u=1}^{j+1}f(y_{\tau(u)},x_{\sigma(1)},\ldots,x_{\sigma(i-1)},y_{\tau(1)},\ldots,\hat{y}_{\tau(u)},\ldots,y_{\tau(j+1)})\\
&\qquad\qquad \wedge g(x_{\sigma(i)},\ldots,x_{\sigma(i+\ell-1)},y_{\tau(j+2)},\ldots,y_{\tau(j+m+1)})\\
&= (-1)^{k\ell}\sum_{\sigma\in Sh_{i-1,\ell}}\sum_{\tau\in A}(\operatorname{sgn}\sigma)f(y_{\tau(1)},x_{\sigma(1)},\ldots,x_{\sigma(i-1)},y_{\tau(2)},\ldots,\hat{y}_{\tau(u)},\ldots,y_{\tau(j+1)})\\
&\qquad\qquad \wedge g(x_{\sigma(i)},\ldots,x_{\sigma(i+\ell-1)},y_{\tau(j+2)},\ldots,y_{\tau(j+m+1)})
\end{aligned}$$

$$\begin{aligned}
&(f\wedge(d_{dR-}g))(x_1,\ldots,x_{i+\ell-1},y_1,\ldots,y_{j+m+1})\\
&= (-1)^{k(\ell-1)}\sum_{\sigma\in Sh_{i,\ell-1}}\sum_{\tau\in Sh_{j,m+1}}(\operatorname{sgn}\sigma)f(x_{\sigma(1)},\ldots,x_{\sigma(i)},y_{\tau(1)},\ldots,y_{\tau(j)})\\
&\qquad\qquad \wedge (d_{dR-}g)(x_{\sigma(i+1)},\ldots,x_{\sigma(i+\ell-1)},y_{\tau(j+1)},\ldots,y_{\tau(j+m+1)})\\
&= (-1)^{k(\ell-1)}\sum_{\sigma\in Sh_{i,\ell-1}}\sum_{\tau\in Sh_{j,m+1}}\sum_{u=j+1}^{j+m+1}(\operatorname{sgn}\sigma)f(x_{\sigma(1)},\ldots,x_{\sigma(i)},y_{\tau(1)},\ldots,y_{\tau(j)})\\
&\qquad\qquad \wedge g(y_{\tau(u)},x_{\sigma(i+1)},\ldots,x_{\sigma(i+\ell-1)},y_{\tau(j+1)},\ldots,\hat{y}_{\tau(u)},\ldots,y_{\tau(j+m+1)})\\
&= (-1)^{k(\ell-1)}\sum_{\sigma\in Sh_{i,\ell-1}}\sum_{\tau\in B}(\operatorname{sgn}\sigma)f(x_{\sigma(1)},\ldots,x_{\sigma(i)},y_{\tau(1)},\ldots,y_{\tau(j)})\\
&\qquad\qquad \wedge g(y_{\tau(j+1)},x_{\sigma(i+1)},\ldots,x_{\sigma(i+\ell-1)},y_{\tau(j+2)},\ldots,y_{\tau(j+m+1)})
\end{aligned}$$

Hence $d_{dR-}(f\wedge g) = (d_{dR-}f)\wedge g + (-1)^{i+2j+k}f\wedge(d_{dR-}g)$.

Also,

$$\begin{aligned}
&(d_{dR+}(f\wedge g))(x_1,\ldots,x_{i+\ell},y_1,\ldots,y_{j+m})\\
&= (-1)^{i+\ell}d_{dR}\Bigg((-1)^{k\ell}\sum_{\sigma\in Sh_{i,\ell}}\sum_{\tau\in Sh_{j,m}}(\operatorname{sgn}\sigma)f(x_{\sigma(1)},\ldots,x_{\sigma(i)},y_{\tau(1)},\ldots,y_{\tau(j)})\\
&\qquad\qquad \wedge g(x_{\sigma(i+1)},\ldots,x_{\sigma(i+\ell)},y_{\tau(j+1)},\ldots,y_{\tau(j+m)}\Bigg)\\
&= (-1)^{i+\ell+k\ell}d_{dR}\Bigg(\sum_{\sigma\in Sh_{i,\ell}}\sum_{\tau\in Sh_{j,m}}(\operatorname{sgn}\sigma)f(x_{\sigma(1)},\ldots,x_{\sigma(i)},y_{\tau(1)},\ldots,y_{\tau(j)})\Bigg)\\
&\qquad\qquad \wedge g(x_{\sigma(i+1)},\ldots,x_{\sigma(i+\ell)},y_{\tau(j+1)},\ldots,y_{\tau(j+m)}\\
&\quad + (-1)^{i+\ell+k+k\ell}\sum_{\sigma\in Sh_{i,\ell}}\sum_{\tau\in Sh_{j,m}}(\operatorname{sgn}\sigma)f(x_{\sigma(1)},\ldots,x_{\sigma(i)},y_{\tau(1)},\ldots,y_{\tau(j)})\\
&\qquad\qquad \wedge d_{dR}\Bigg(g(x_{\sigma(i+1)},\ldots,x_{\sigma(i+\ell)},y_{\tau(j+1)},\ldots,y_{\tau(j+m)}\Bigg)\\
&= \big(((d_{dR+}f)\wedge g) + (-1)^{i+2j+k}f\wedge d_{dR+}g\big)(x_1,\ldots,x_{i+\ell},y_1,\ldots,y_{j+m}).
\end{aligned}$$

Hence we have the Leibniz's rule.

∎

The internal differential δ_{cone} on $\mathrm{Hom}(\wedge^\bullet\mathfrak{g}, \mathrm{cone}\,\alpha[-1])$ also extends to the G-invariant de Rham complex $\mathrm{Hom}(\wedge^\bullet\mathfrak{g}, \wedge^\bullet\mathrm{cone}\,\alpha[-1])$ by imposing the Leibniz's rule

$$\delta_{\text{cone}}\,(x \wedge y) = (\delta_{\text{cone}}\,x) \wedge y + (-1)^{|x|} x \wedge \delta_{\text{cone}}\,y.$$

Proposition 7.3.6. *The de Rham differential d_{dR} commutes with the internal differential of the de Rham complex, i.e., $\delta_{\text{cone}}\,d_{dR} + d_{dR}\delta_{\text{cone}} = 0$.*

Proof. This follows from Leibniz's rule and Proposition 7.3.3. ∎

We arrive at the following definition, motivated by Pantev, Toën, Vaquié and Vezzosi [58].

Definition 7.3.7 (G-invariant de Rham complex)**.** The double complex

$$(\mathrm{Hom}(\wedge^\bullet\mathfrak{g}, \wedge^\bullet\mathrm{cone}\,\alpha[-1]), \delta_{\text{cone}}\,, d_{dR})$$

is called the *G-invariant de Rham differential complex* on $\mathrm{Spec}\,R$. A *G-invariant differential p-form of degree k* on $\mathrm{Spec}\,R$ is an element in the cohomology group $H^k(\mathfrak{g}, \wedge^p\mathrm{cone}\,\alpha[-1], \delta_{\text{cone}}\,)$. In other words, it is represented by an element

$$\omega^k \in \bigoplus_{i=0}^{\dim\mathfrak{g}} \mathrm{Hom}(\wedge^i\mathfrak{g}, (\wedge^p\mathrm{cone}\,\alpha[-1])^{k-i}) \quad \text{satisfying} \quad \delta_{\text{cone}}\,\omega^k = 0.$$

Two such representations ω^k, ω'^k are equivalent if there exists α^k such that $\omega^k - \omega'^k = \delta_{\text{cone}}\,\alpha^k$.

A *G-invariant closed differential p-form of degree k* is an element in the cohomology

$$H^k(\mathfrak{g}, \prod_{j\geq 0}(\wedge^{p+j}\mathrm{cone}\,\alpha[-1])[-j], \delta_{\text{cone}}\, + d_{dR}).$$

In other words, it is represented by a sequence $\omega = (\omega^0, \omega^1, \ldots)$, where

$$\omega^j \in \bigoplus_{i=0}^{\dim\mathfrak{g}} \mathrm{Hom}(\wedge^i\mathfrak{g}, (\wedge^{p+j}\mathrm{cone}\,\alpha[-1])^{k-i-j})$$

satisfying $(\delta_{\text{cone}}\, + d_{dR})\omega = 0$. Equivalently, the sequence ω satisfies the system of equations

$$\begin{aligned} \delta_{\text{cone}}\,\omega^0 &= 0 && \text{in } \bigoplus_{i=0}^{\dim\mathfrak{g}} \mathrm{Hom}(\wedge^i\mathfrak{g}, (\wedge^p\mathrm{cone}\,\alpha[-1])^{k-i+1}), \text{ and} \\ d_{dR}\omega^k + \delta_{\text{cone}}\,\omega^{k+1} &= 0 && \text{in } \bigoplus_{i=0}^{\dim\mathfrak{g}} \mathrm{Hom}(\wedge^i\mathfrak{g}, (\wedge^{p+j}\mathrm{cone}\,\alpha[-1])^{k-i-j+1}) \text{ for all } k \geq 0. \end{aligned}$$

Two such representations are equivalent if there exists $\alpha = (\alpha^0, \alpha^1, \ldots)$ such that

$$\begin{aligned} \omega^0 - \omega'^0 &= \delta_{\text{cone}}\,\alpha^0 && \text{in } \bigoplus_{i=0}^{\dim\mathfrak{g}} \mathrm{Hom}(\wedge^i\mathfrak{g}, (\wedge^{p+j}\mathrm{cone}\,\alpha[-1])^{k-i-j}), \text{ and} \\ \omega^{k+1} - \omega'^{k+1} &= d_{dR}\alpha^k + \delta_{\text{cone}}\,\alpha^{k+1} && \text{in } \bigoplus_{i=0}^{\dim\mathfrak{g}} \mathrm{Hom}(\wedge^i\mathfrak{g}, (\wedge^{p+j+1}\mathrm{cone}\,\alpha[-1])^{k-i-j+1}) \text{ for all } i \geq 0. \end{aligned}$$

A G-invariant closed 2-form $\omega = (\omega^0, \omega^1, \ldots)$ of degree k is a *G-invariant k-shifted symplectic form* if ω^0 is a nondegenerate G-invariant 2-form of degree k.

Relations between G-invariant forms and forms on quotient stacks. We conjecture the G-invariant forms and G-invariant closed forms in Definition 7.3.7 are equivalent to forms and closed forms on the stack $[\operatorname{Spec} R/G]$ in the sense of Pantev, Toën, Vaquié and Vezzosi [58] (c.f. Section 7.1, Definitions 7.1.1 and 7.1.2). A possible way of showing this equivalence is to use the language of simplicial schemes in the sense of Pridham [59] to describe derived stacks. Presumably the quotient stack $[\operatorname{Spec} R/G]$ can be described as the simplicial scheme

$$\left[\begin{tikzcd} \cdots\cdots \arrow[r, shift left=2] \arrow[r, shift right=2] & G^n \times \operatorname{Spec} R \arrow[l] \arrow[r, shift left=2] \arrow[r, shift right=2] & \arrow[l] \cdots\cdots \arrow[r, shift left=2] \arrow[r, shift right=2] & G \times G \times \operatorname{Spec} R \arrow[l] \arrow[r, shift left=3, "\partial_0"] \arrow[r, "\partial_1" description] \arrow[r, shift right=3, "\partial_2"'] & G \times \operatorname{Spec} R \arrow[l, shift right=1.5, "\sigma_1"'] \arrow[l, shift left=1.5, "\sigma_0"] \arrow[r, shift left=2, "\partial_0"] \arrow[r, shift right=2, "\partial_1"'] & \operatorname{Spec} R \arrow[l, "\sigma_0" description] \end{tikzcd}\right],$$

where

$$\partial_i(g_1, \ldots, g_n, x) = \begin{cases} (g_2, \ldots, g_n, x) & \text{if } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n, x) & \text{if } 1 \leq i \leq n-1, \\ (g_1, \ldots, g_{n-1}, g_n x) & \text{if } i = n, \text{ and} \end{cases}$$
$$\sigma_i(g_1, \ldots, g_n, x) = (g_1, \ldots g_i, 1, g_{i+1}, \ldots, g_n, x).$$

Pridham [59, Definition 7.7] then gives a chain complex which represents the cotangent complex $\mathbb{L}_{\operatorname{Spec} R/G}$ which we expect to be quasi-isomorphic to the G-invariant de Rham complex in Definition 7.3.7.

7.4 Moduli Spaces of Representations of Quivers with Superpotential

In this section, we study the moduli space of representations of a quiver with superpotential, and outline a strategy of proof of the existence of a shifted symplectic form which is in a standard Darboux form on the moduli space.

Let Q be a dg-quiver. A *dg-representation of Q* consists of chain complexes W_v for each vertex v in Q and linear maps $\rho_e : W_{t(e)} \to W_{h(e)}$ of degree i for each degree i edge e in Q. Let S denote the $\mathbb{K}$-algebra spanned by the vertices of Q. Then a dg-representation of Q is the same as a chain complex (W, d_W) over S together with a morphism $\mathbb{K}Q \to \operatorname{End}^\bullet_S(V)$ of dg-algebras over S, where the differential on $\operatorname{End}^\bullet_S(W)$ is given by

$$df = d_W f - (-1)^{\deg f} f d_W.$$

Thus given a fixed chain complex (W, d_W) over S, we can write down a moduli functor for dg-representations of Q:

$$\begin{aligned} \operatorname{Rep}_W(Q) : \{\text{commutative dg-algebras}\} &\longrightarrow \{\text{Sets}\}, \\ C &\longmapsto \operatorname{Hom}_{\mathrm{dga/S}}(\mathbb{K}Q, \operatorname{End}^\bullet_S(W) \otimes C). \end{aligned}$$

Berest, Khachatryan and Ramadoss proved in [6, Theorem 2.1] that this functor is representable. In the special case when Q is quasi-free and the chain complex W is concentrated in degree 0 with dimension vector $\boldsymbol{d}$, the commutative dg-algebra R representing $\operatorname{Rep}_W(Q)$ can be explicitly described: the underlying graded algebra of R is the coordinate algebra of the graded vector space [6, Theorem 2.8]:

$$\bigoplus_{e \text{ edges in Q}} \operatorname{Hom}(W_{t(e)}, W_{h(e)})[\deg e].$$

In other words, for each edge e in Q, there is an associated $d_{t(e)} \times d_{h(e)}$ matrix $M_e = (e_{ij})$, and

$$R = \bigotimes_{e \text{ edges in } Q} \mathbb{K}[M_e] = \bigotimes_{e \text{ edges in } Q} \mathbb{K}[e_{ij}],$$

where each e_{ij} has degree $\deg e$. If $p = e_1 \cdots e_n$ is a path in Q, we write $M_p = M_{e_1} \cdots M_{e_n}$ where the product on the right hand side is the matrix product. For general $p \in \mathbb{K}Q$, we extend bilinearly as usual to define M_p. Then the differential on R can be described as

$$dM_e = M_{de}.$$

Note also that the differential on R defines a cohomological vector field on the graded vector space

$$\bigoplus_{e \text{ edges in Q}} \mathrm{Hom}(W_{t(e)}, W_{h(e)})[\deg e],$$

and turns it into a dg-manifold.

The moduli stack of dg-representations of Q concentrated in degree 0 and with dimension vector $\boldsymbol{d} = (\boldsymbol{d}(v))$ is then given by

$$\begin{aligned}\mathrm{Rep}_{\boldsymbol{d}}(Q) &= \left[\left(\bigoplus_{e \text{ edges in Q}} \mathrm{Hom}(W_{t(e)}, W_{h(e)})[\deg e], d\right) \Big/ \prod_{v \text{ vertices in Q}} \mathrm{GL}\left(\boldsymbol{d}(v)\right)\right] \\ &= \left[\mathrm{Spec}\, R \Big/ \prod_{v \text{ vertices in Q}} \mathrm{GL}\left(\boldsymbol{d}(v)\right)\right],\end{aligned}$$

where $\mathrm{GL}\left(\boldsymbol{d}(v)\right)$ acts on each vector space W_v by conjugation. Notice here that although representations are concentrated in degree 0, the higher degree edges do not act trivially in the moduli functor, hence the higher degree edges are not redundant. For simplicity, we will denote the product group $\prod_{v \text{ vertices in Q}} \mathrm{GL}\left(\boldsymbol{d}(v)\right)$ by G and write elements in G in the form $g = (g_v)$. In local coordinates, the G-action on R is described by

$$g \cdot M_e = g_{h(e)}^{-1} M_e g_{t(e)}.$$

The associated $\mathfrak{g}$-action on R is then given by, for any $\xi = (\xi_v) \in \prod \mathfrak{gl}(\boldsymbol{d}(v)) = \mathfrak{g}$,

$$\xi \cdot M_e = -\xi_{h(e)} M_e + M_e \xi_{t(e)}.$$

Now, suppose our dg-quiver admits Q a superpotential Φ. Then Pantev, Toen, Vaquié and Vezzosi [58, p.9-10] claimed without proof that $\mathrm{Rep}_{\boldsymbol{d}}(Q)$ admits a shifted symplectic structure. We outline a strategy of proof by explicitly writing down a G-invariant shifted symplectic form.

Recall that a quiver with superpotential (Q, Φ) of dimension m has a dg-subquiver $\tilde{Q}$ such that $\mathbb{K}\tilde{Q}$ has a noncommutative symplectic 2-form and that Q can be constructed from $\tilde{Q}$ by adding a degree $1-m$ loop on each vertex on $\tilde{Q}$. Then $\tilde{Q}$ correspond to a dg-subalgebra

$$\tilde{R} = \bigotimes_{e \text{ edges in } \tilde{Q}} \mathbb{K}[M_e] \overset{\iota}{\hookrightarrow} R,$$

and we have a diagram

$$\mathrm{Spec}\, \tilde{R} \xleftarrow{i = \mathrm{Spec}\, \iota} \mathrm{Spec}\, R \xrightarrow{\varphi} \mathrm{Rep}_{\boldsymbol{d}}(Q)\ .$$

Let us choose some good models for the cotangent complexes for these three spaces.

Models for cotangent complexes. The atlas $\mathrm{Spec}\, R \xrightarrow{\varphi} \mathrm{Rep}_d(Q)$ is a principal G-bundle. Hence the relative cotangent complex $\mathbb{L}_{\mathrm{Spec}\, R/\mathrm{Rep}_{\boldsymbol{d}}(Q)} \cong \mathfrak{g}^\vee \otimes R$. We thus have a distinguished triangle in $\mathrm{D}(R\text{-mod})$

$$\varphi^* \mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)} \to \mathbb{L}_R \xrightarrow{\alpha} \mathfrak{g}^\vee \otimes R \to \varphi^* \mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}[1].$$

On the other hand, we also have a distinguished triangle

$$i^* \mathbb{L}_{\tilde{R}} \to \mathbb{L}_R \to \mathbb{L}_{R/\tilde{R}} \to i^* \mathbb{L}_{\tilde{R}}[1].$$

Since Q and $\tilde{Q}$ are both dg-quivers, both R and $\tilde{R}$ are quasi-free as commutative dg-algebras. Hence the Kähler differentials Ω^1_R (resp. $\Omega^1_{\tilde{R}}$) gives a model for $\mathbb{L}_R$ (resp. $\mathbb{L}_{\tilde{R}}$). For each edge e in Q, we will denote the matrix of 1-forms $d_{dR}M_e = (d_{dR}e_{ij})$.

The cone of $\alpha[-1] : \Omega^1_R[-1] \to \mathfrak{g}^\vee \otimes R[-1]$ then gives a model for $\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}$. Recall that

$$\mathrm{cone}(\alpha[-1]) = \Omega^1_R \oplus (\mathfrak{g}^\vee \otimes R[-1]) \,, \text{ with differential } d_{\mathrm{cone}(\alpha[-1])} = \begin{pmatrix} d_{\Omega^1_R} & 0 \\ \alpha & \mathrm{id} \otimes d_{R[-1]} \end{pmatrix}.$$

The map α comes from the G-action on R as follows: The G-action on R induces a linear map $\mathfrak{g} \otimes R \to R$, $\xi \otimes r \mapsto \xi \cdot r$ which satisfies Leibniz rule $\xi \cdot (r_1 r_2) = (\xi \cdot r_1) r_2 + r_1(\xi \cdot r_2)$. The dual map $R \to \mathfrak{g}^\vee \otimes R \cong \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, R)$, $r \mapsto (\xi \mapsto \xi \cdot r)$ is a derivation, hence it factorizes into a linear map $\alpha : \Omega^1_R \to \mathfrak{g}^\vee \otimes R$. In local coordinates, the map α is given by

$$d_{dR}M_e \mapsto -a_{h(e)}M_e + M_e a_{t(e)},$$

where for any vertex v, the (i,j)-th element $(a_v)_{ij}$ of the matrix a_v denotes the linear map in $\mathfrak{gl}(\boldsymbol{d}(v))^\vee$ which projects a $\boldsymbol{d}(v) \times \boldsymbol{d}(v)$ matrix to its (i,j)-th element.

The $\mathfrak{g}$-module structure on $\mathrm{cone}\,\alpha[-1]$ can be described as follows: Let $\xi = (\xi_v) \in \prod \mathfrak{gl}(\boldsymbol{d}(v)) = \mathfrak{g}$. Then $\mathfrak{g}$ acts on $\mathrm{cone}\,\alpha[-1] = \Omega^1_R \oplus \mathfrak{g}^\vee \otimes R[-1]$ by

$$\begin{aligned} \xi \cdot d_{dR}M_e &= -\xi_{h(e)}(d_{dR}M_e) + (d_{dR}M_e)\xi_{t(e)} \\ \xi \cdot a_v &= -\xi_v a_v + a_v \xi_v. \end{aligned}$$

Symplectic form on $\mathrm{Spec}\,\tilde{R}$**.** The noncommutative symplectic form $\omega = \sum_{x,y \text{ edges in } \tilde{Q}} \langle x, y\rangle d_{dR}x d_{dR}y$ on $\mathbb{K}\tilde{Q}$ induces a shifted symplectic form $\omega_{\tilde{R}} = (\omega^0_{\tilde{R}}, 0, \ldots)$ on $\mathrm{Spec}\,\tilde{R}$ by

$$\begin{aligned} \omega^0_{\tilde{R}} &= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle \mathrm{tr}(d_{dR}M_x d_{dR}M_y) \\ &= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle \sum_{ij} (d_{dR}x_{ij} d_{dR}y_{ji}), \end{aligned}$$

where $d_{dR}M_x$ denotes the matrix of 1-form $(d_{dR}x_{ij})$.

The superpotential Φ on Q also induces a function $\Phi_R = \mathrm{tr}(M_\Phi)$ which satisfy the equation

$$i_d \omega^0_{\tilde{R}} = d_{dR}\Phi_R.$$

With these models for the cotangent complexes, we can proceed to write down some symplectic forms.

Symplectic form on $\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}$**.** Recall $G = \prod_{v \text{ vertices in Q}} \mathrm{GL}\,(\boldsymbol{d}(v))$, where $\boldsymbol{d}(v)$ is the dimension of the vector space at the vertex v, and $\mathfrak{gl}(\boldsymbol{d}(v))$ is given by all $\boldsymbol{d}(v) \times \boldsymbol{d}(v)$ matrices. Let $(a_v)_{ij}$ denote the linear map in $\mathrm{Hom}(\mathfrak{gl}(\boldsymbol{d}(v)), R)$ which maps a $\boldsymbol{d}(v) \times \boldsymbol{d}(v)$ matrix to its (i,j)-th element. Let a_v denote a matrix whose (i,j)-th element is given by $(a_v)_{ij}$. Define

$$\begin{aligned} \omega^0_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} &= \omega^0_{\tilde{R}} + \sum_{v \text{ vertices in } Q} \mathrm{tr}(a_v d_{dR}M_{v^*}) \\ &= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle \mathrm{tr}(d_{dR}M_x d_{dR}M_y) + \sum_{v \text{ vertices in } Q} \mathrm{tr}(a_v d_{dR}M_{v^*}). \end{aligned}$$

Now,

$$\begin{aligned}
&\delta_- \omega^0_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} = d_{\text{cone}}\,(\omega^0_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}}) \\
&= d_{\Omega^1_R}\omega^0_{\tilde{R}} + \alpha(\omega_{\tilde{R}}) + \sum_{v \text{ vertices in } Q} \operatorname{tr}\left(a_v d_{\Omega^1_R} d_{dR} M_{v^*}\right) \\
&= \alpha(\omega_{\tilde{R}}) + \sum_{v \text{ vertices in } Q} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle a_v d_{dR}(M_x M_y) \\
&= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle \operatorname{tr}\left(\left(-a_{h(x)}M_x + M_x a_{t(x)}\right) d_{dR} M_y + (-1)^{|x|} d_{dR} M_x \left(-a_{h(y)} M_y + M_y a_{t(y)}\right)\right) \\
&\quad + \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle \operatorname{tr}\left(a_{h(x)} d_{dR}(M_x M_y)\right) \\
&= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} (1 + (-1)^{|x|+1+|x||y|+1+|x|(|y|+1)}) \langle x^\vee, y^\vee\rangle \operatorname{tr}\left(\left(-a_{h(x)}M_x + M_x a_{t(x)}\right) d_{dR} M_y\right) \\
&\quad + \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle \operatorname{tr}\left(a_{h(x)} d_{dR}(M_x M_y)\right) \\
&= \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle \operatorname{tr}\left(\left(-a_{h(x)}M_x + M_x a_{t(x)}\right) d_{dR} M_y + a_{h(x)}(d_{dR}M_x)M_y + a_{h(x)} M_x d_{dR} M_y\right) \\
&= 0
\end{aligned}$$

since $|x| + 1 + |x||y| + 1 + |x|(|y|+1) = 0 \mod 2$. For any $\xi \in \mathfrak{g}$,

$$\begin{aligned}
&\delta_+ \omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}}(\xi) = \xi \cdot \omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} \\
&= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle \operatorname{tr}(d_{dR}M_x d_{dR} M_y) + \sum_{v \text{ vertices in } Q} \operatorname{tr}(a_v d_{dR} M_{v^*}) \\
&= \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle \operatorname{tr}((-\xi_{h(x)} d_{dR} M_x + d_{dR} M_x \xi_{t(x)}) d_{dR} M_y) \\
&\quad + \frac{1}{2} \sum_{x,y \text{ edges in } \tilde{Q}} \langle x^\vee, y^\vee\rangle \operatorname{tr}(d_{dR} M_x(-\xi_{h(y)} d_{dR} M_y + d_{dR} M_y \xi_{t(y)})) \\
&\quad + \sum_{v \text{ vertices in } Q} \operatorname{tr}((-\xi_v a_v + a_v \xi_v) d_{dR} M_{v^*}) + \sum_{v \text{ vertices in } Q} \operatorname{tr}(a_v(-\xi_v d_{dR} M_{v^*} + d_{dR} M_{v^*} \xi_v)) \\
&= 0.
\end{aligned}$$

Hence $\delta\omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} = (\delta_- + \delta_+)\omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} = 0$ and $\omega^0_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}}$ is a G-invariant 2-form. The sequence $\omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} = (\omega^0_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} 0, 0, \dots)$ is then a G-invariant closed 2-form as

$$d_{dR}\omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} = d_{dR-}\omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} + d_{dR+}\omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} = 0.$$

Since $\omega^0_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}}$ is non-degenerate, $\omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}} = (\omega^0_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}}, 0, 0, \dots)$ is a G-invariant symplectic form.

Assuming our definition of G-invariant symplectic forms in Definition 7.3.7 is equivalent to the definition of symplectic forms on $\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}$ in the sense of Pantev, Toën, Vaquié and Vezzosi [58], $\omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}}$ is then a shifted symplectic structure on $\mathrm{Rep}_{\boldsymbol{d}}(Q)$. Observe that the 2-form $\omega^0_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}}$ on $\mathrm{Rep}_{\boldsymbol{d}}(Q)$ and $\omega^0_{\tilde{R}}$ on $\operatorname{Spec}\tilde{R}$ differ only by a term. A natural guess would be that this extra term would drop out and the two 2-forms would be equal when both are pulled back to $\operatorname{Spec} R$. The shifted symplectic structure would then be in the Darboux form described by Ben-Bassat, Brav, Bussi and Joyce [4, Theorem 2.10] as follows:

Conjecture 7.4.1 (Shifted symplectic structure)**.** *The moduli space $\mathrm{Rep}_{\boldsymbol{d}}(Q)$ of representations of a quiver with superpotential Q of dimension m has a $(2-m)$-shifted symplectic structure $\omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}}$ such* *that* $\varphi^*(\omega_{\varphi^*\mathbb{L}_{\mathrm{Rep}_{\boldsymbol{d}}(Q)}}) = i^*\omega_{\tilde{R}}$*, where* $i : \operatorname{Spec}\tilde{R} \to \operatorname{Spec} R$ *is the inclusion map. Moreover,* $\operatorname{Spec} R$ *and* ω *are in a standard Darboux form in the sense of Ben-Bassat, Brav, Bussi and Joyce [4, Theorem 2.10] described as follows:*

1. *The degree* 0 *part* R^0 *is a smooth algebra of dimension* $\sum_{\textit{degree } 0 \textit{ edges in } Q} \boldsymbol{d}(t(e))\boldsymbol{d}(h(e))$ *generated by the entries of the matrices* M_e*, where* $\deg e = 0$*, and that entries in* $d_{dR}M_e$ *form a basis of* $\Omega^1_{R^0}$ *over* R^0.

2. R *is freely generated over* R^0 *by the entries of the matrices* M_e *with* $\deg e = -1, \dots, 1-m$*, and*

$$(i^*\omega_{\tilde{R}})^0 = i^*(\omega^0_{\tilde{R}}) = \frac{1}{2} \sum_{x,y \textit{ edges in } \tilde{Q}} \langle x^\vee, y^\vee \rangle \operatorname{tr}(d_{dR}M_x d_{dR}M_y).$$

3. *The superpotential on* Q *induces a function* $\Phi_R = \operatorname{tr}(M_\Phi)$ *which satisfies the equation known as the classical master equation* $\{\Phi_R, \Phi_R\} = 0$*, with differential* d *on* $\tilde{R}$ *given by*

$$da_{ij} = (-1)^{1+|a|+m}(a,b)\partial_{b_{ji}}\Phi_R,$$

where a_{ij} *denotes the* (i,j)*-th entry of the matrix* M_a *for any edge* a *in* $\tilde{Q}$.

Chapter 8

Future Directions

This chapter discusses several future research directions.

One possible direction is to generalize Theorem 5.4.5 by removing the vanishing condition (or equivalently the assumption that the exceptional poset $\mathcal{E}$ is V-geometric):

Conjecture 8.0.2. *Let X be a smooth variety and $\pi : V \to X$ be a vector bundle. Let $\mathcal{E}$ be an exceptional poset on $\mathrm{D}^b(\mathrm{Coh}\,(X))$ with dual exceptional poset $\mathcal{F}$. Then there is a natural A_{fin}-structure on*

$$A_{\mathcal{E}} = \bigoplus_{i,j\in I} \bigoplus_{\ell\in\mathbb{Z}} \bigoplus_{k=0}^{\operatorname{rank} V} \mathrm{Hom}^{\ell-k}(F_i, F_j \otimes \wedge^k V),$$

making it a finite dimensional A_{fin}-algebra augmented over S, such that $\mathrm{D}^b(\mathrm{Coh}\,(V)) \cong \mathrm{Per}(E(A_{\mathcal{E}}))$.

In fact, in Chapter 5, although not explicitly stated, we have already proven a derived equivalence $\mathrm{D}^b(\mathrm{Coh}\,(X)) \cong \mathrm{Per}(E(\mathrm{Ext}^\bullet_\Lambda(S,S)))$. However, we do not know if $\mathrm{Ext}^\bullet_\Lambda(S,S) = A_{\mathcal{E}}$ (see Remark 5.4.2).

In another direction, we might be able to generalize the quotient construction (Theorem 5.6.5) by relaxing the assumption that each object in the exceptional sequence admits a G-linearization. First, we outline a way to generalize the quotient construction for A_∞-categories in Section 3 by allowing G to permute the objects in $\mathcal{A}$. Let G be a finite group and $\mathcal{A}$ be an A_∞-category with a finite set of objects. Suppose G acts on $\mathrm{Obj}\,(\mathcal{A})$ and for each $g \in G$, the action $g : \mathcal{A}(u,v) \to \mathcal{A}(g\cdot u, g\cdot v)$ is a strict A_∞-isomorphism. Then $\mathrm{Obj}\,(\mathcal{A})$ is partitioned into G-orbits $[u],[v],[w],\ldots$ etc. Denote by $\mathrm{Stab}\,(u)$ the stabilizer group of the orbit $[u]$. A candidate for the quotient A_∞-category $\mathcal{A}/G$ would be given by

$$\mathrm{Obj}\,(\mathcal{A}/G) = \{([u],\rho) : [u] \text{ is a } G\text{-orbit in } \mathrm{Obj}\,(\mathcal{A}) \text{ and } \rho \text{ is a representation of } \mathrm{Stab}\,(u)\},$$

$$\mathcal{A}/G(([u],\rho),([v],\sigma)) = \mathrm{Hom}_G(\rho, \mathcal{A}(u,v)\otimes\sigma),$$

with A_∞-structures induced from that of $\mathcal{A}$ in a way similar to that described in Section 3. With this, suppose H is a normal subgroup of G, then we should also have $(\mathcal{A}/H)/(G/H) \cong \mathcal{A}/G$ strictly A_∞-isomorphic. On the geometry side, we assume that the finite group G permutes the objects in the exceptional poset $\mathcal{E}$ and partitions $\mathcal{E}$ into disjoint G-orbits $\mathcal{E}_{\mathbf{u}} = \{E_{u_1}, \cdots, E_{u_k}\}$ where $\mathbf{u} = (u_1, \ldots, u_k)$. Let $E_{\mathbf{u}} = E_{u_1} \oplus \cdots \oplus E_{u_k}$. Then $E_{\mathbf{u}}$ is equipped with a natural G-linearization. Let

$$\mathcal{E}/G = \{E_{\mathbf{u}} \otimes \rho : \mathcal{E}_{\mathbf{u}} \text{ is a } G\text{-orbit of } \mathcal{E} \text{ and } \rho \text{ is a representation of } \mathrm{Stab}\,(\mathcal{E}_{\mathbf{u}})\}.$$

Then we should be able to define on $\mathcal{E}/G$ a partial order by declaring

$$E_{\mathbf{u}} \prec E_{\mathbf{v}} \quad \text{if and only if} \quad \mathrm{Hom}_G(E_{\mathbf{u}}, E_{\mathbf{v}}) \neq 0 \quad \text{and} \quad \mathbf{u} \neq \mathbf{v},$$

making $\mathcal{E}/G$ an exceptional poset on $\mathrm{D}^b(\mathrm{Coh}^{\,G}(X))$. We should be able to prove:

Conjecture 8.0.3. *Let G be a finite group, X be a smooth variety with G acting by automorphisms, and $\pi : V \to X$ an equivariant vector bundle. Let $\mathcal{E}$ be an exceptional poset on $\mathrm{D}^b(\mathrm{Coh}\,(X))$ and suppose the action of G on X induces a permutation on the objects in $\mathcal{E}$. Then there is an equivalence $\mathrm{D}^b(\mathrm{Coh}^G(V)) \cong \mathrm{Per}(E(A_{\mathcal{E}}/G))$. Furthermore, when V is anti-semiample and $\mathrm{Hom}(E_i, E_j \otimes S^\bullet V^\vee) = 0$, then the above equivalence becomes $\mathrm{D}^b(\mathrm{Coh}^G(V)) \cong \mathrm{D}^b(E(A_{\mathcal{E}}/G))$.*

Thus in this situation, if we know the dg-quiver derived equivalent to $\mathrm{D}^b(\mathrm{Coh}\,(V))$, we would also know the dg-quiver derived equivalent to $\mathrm{D}^b(\mathrm{Coh}^G(V))$ by applying the quotient construction. Moreover, this quotient construction could be factorized: if there is a normal subgroup H in G, we can compute the quotient by first taking quotient by H, followed by taking quotient by G/H.

One might also try to remove the vanishing condition for the product construction (Theorem 5.7.3). To do this, recall that in Theorem 5.2.4, we have constructed from the exceptional poset $\mathcal{E}$ a dg-algebra

$$R_{\mathcal{E}} = \mathcal{H}om(I_E, I_E \otimes S^\bullet V^\vee),$$

where $E = \bigoplus E_i$ and I_E is an injective resolution for E. In the product situation, one starts with two exceptional posets $\mathcal{E}$ and $\mathcal{F}$ and ends up with two dg-algebras $R_{\mathcal{E}}$ and $R_{\mathcal{F}}$. For the product construction to work without the vanishing condition, one needs to prove $R_{\mathcal{E}} \otimes R_{\mathcal{F}} \cong R_{\mathcal{E}\boxtimes\mathcal{F}}$ as a dg-algebra. One possible way to do this is to show that $I_E \boxtimes I_F$ is an injective resolution for $E \boxtimes F$. However, we do not know how to do this.

The above directions are more or less straightforward generalizations on the results of this thesis. A more ambitious direction is perhaps the following. Pantev, Toën, Vaquié and Vezzosi has shown in [58, Theorem 2.9] that if we have an n-shifted symplectic derived stack (X, ω) together with Lagrangians $f_i : L_i \to X$, for $i = 1, 2$, then the derived fiber product $L_1 \times_X L_2$ has an $(n-1)$-shifted symplectic structure. Suppose Conjecture 7.4.1 is true, then quivers with superpotential correspond to shifted symplectic derived stacks via the moduli construction. One might then ask if one can define a similar notion of "Lagrangian" and "fiber product" in the quiver picture, which should be in the following form: Let (Q, Φ) be a quiver with superpotential of dimension n. A Lagrangian should be given by a quiver Q_i with some additional structures encoding homotopy information, together with a morphism of dg-algebras $f_i : \mathbb{K}Q \to \mathbb{K}Q_i$ satisfying some compatibility conditions. The "fiber product" $Q_1 \times_{(Q,\Phi)} Q_2$ should be a quiver with superpotential of dimension $n-1$, whose path algebra is given by the "tensor product" $\mathbb{K}Q_1 \otimes_{\mathbb{K}Q} \mathbb{K}Q_2$. All these constructions, after taking moduli, should yield their corresponding counterparts in the derived stack picture.

Appendix A

Some Cohomological Formulae

This chapter computes some cohomological formulae for computing examples in Chapter 6.

A.1 Cohomologies of Tangent and Cotangent Sheaf of $\mathbb{P}^n$

In this section, we calculate some sheaf cohomologies for the tangent sheaf $\mathcal{T}$ and cotangent sheaf Ω of $\mathbb{P}^n = \mathbb{P}(V)$, where V is an $(n+1)$-dimensional vector space. We will write $\mathcal{T}^p = \wedge^p \mathcal{T}$ and $\Omega^p = \wedge^p \Omega$.

Lemma A.1.1. *Let $0 \to L \to U \to W \to 0$ be a sequence of vector bundles with $\operatorname{rank} L = 1$. Then for $p \geq 1$ we have an exact sequence*

$$0 \to L \otimes \bigwedge^{p-1} W \to \bigwedge^{p} U \to \bigwedge^{p} W \to 0. \tag{A.1.1}$$

Proof. Taking wedge product of the exact sequence, and since L is a line bundle, we have an exact sequence

$$\cdots \to L^{\otimes 2} \otimes \bigwedge^{p-2} U \to L \otimes \bigwedge^{p-1} U \to \bigwedge^{p} U \to \bigwedge^{p} W \to 0.$$

We claim that this exact sequence factorizes into the short exact sequence (A.1.1). We induct on p. The case $p = 1$ holds tautologically. For general p, by induction assumption and twisting with L, we have an exact sequence

$$0 \to L^{\otimes 2} \otimes \bigwedge^{p-2} W \to L \otimes \bigwedge^{p-1} U \to L \otimes \bigwedge^{p-1} W \to 0.$$

The map $L \otimes \bigwedge^{p-1} U \to \bigwedge^{p} U$ vanishes on $L^{\otimes 2} \otimes \bigwedge^{p-2} W$ and thus descends to form an exact sequence

$$L \otimes \bigwedge^{p-1} W \to \bigwedge^{p} U \to \bigwedge^{p} W \to 0.$$

To show the above exact sequence is also exact on the left, it suffices to show

$$\ker\left(\bigwedge^{p} U \to \bigwedge^{p} W\right) = \operatorname{coker}\left(L^{\otimes 2} \otimes \bigwedge^{p-2} U \to L \otimes \bigwedge^{p-1} U\right) = L \otimes \bigwedge^{p-1} W.$$

This follows since by induction assumption

$$\operatorname{coker}\left(L \otimes \bigwedge^{p-2} U \to \bigwedge^{p-1} U\right) = \bigwedge^{p-1} W.$$

■

Proposition A.1.2. *The following sequence of vector bundles on $\mathbb{P}^n$ is exact:*

$$0 \to \mathcal{T}^{p-1}(k) \to \bigwedge^p V \otimes \mathcal{O}(p+k) \to \mathcal{T}^p(k) \to 0.$$

Proof. Apply Lemma A.1.1 to the Euler exact sequence $0 \to \mathcal{O} \to V \otimes \mathcal{O}(1) \to \mathcal{T} \to 0$. ∎

Lemma A.1.3. *For $2 \leq q \leq n-1$,*

$$H^q(\mathbb{P}^n, \mathcal{T}^p(k)) = H^{q-1}(\mathbb{P}^n, \mathcal{T}^{p+1}(k)).$$

Proof. We have an exact sequence

$$\bigwedge^p V \otimes H^{q-1}(\mathbb{P}^n, \mathcal{O}(p+k)) \to H^{q-1}(\mathbb{P}^n, \mathcal{T}^p(k)) \to H^q(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) \to \bigwedge^p V \otimes H^q(\mathbb{P}^n, \mathcal{O}(p+k)).$$

For $2 \leq q \leq n-1$, the flanking terms vanish and we have the isomorphism as desired. ∎

Lemma A.1.4. *For $n \geq 2$ and $p \neq n-1$, $H^1(\mathbb{P}^n, \mathcal{T}^p(k)) = 0$ for all k.*

Proof. We may assume $0 \leq p \leq n$. For $0 \leq p \leq n-2$, applying Lemma A.1.3 repeatedly, we obtain

$$H^1(\mathbb{P}^n, \mathcal{T}^p(k)) = H^{p+1}(\mathbb{P}^n, \mathcal{O}(k)) = 0.$$

For $p = n-1$,

$$H^1(\mathbb{P}^n, \mathcal{T}^n(k)) = H^1(\mathbb{P}^n, \mathcal{O}(n+1+k)) = 0.$$

∎

Lemma A.1.5. *For $n \geq 2$,*

$$h^0(\mathbb{P}^n, \mathcal{T}^p(k)) = \begin{cases} \binom{n+k}{p+k}\binom{n+k+p+1}{p} & \textit{if } 0 \leq p \leq n,\ k \geq -p, \\ 1 & \textit{if } p = n,\ k = -n-1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For $p = 0$,

$$h^0(\mathbb{P}^n, \mathcal{O}(k)) = \begin{cases} \binom{n+k}{k} & \text{if } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $p = n$,

$$h^0(\mathbb{P}^n, \mathcal{T}^n(k)) = h^0(\mathbb{P}^n, \mathcal{O}(n+1+k)) = \begin{cases} \binom{2n+1+k}{n} & \text{if } k \geq -n-1 \\ 0 & \text{otherwise.} \end{cases}$$

For $1 \leq p \leq n-1$, $H^1(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) = 0$ by Lemma A.1.4. Hence we have short exact sequences

$$0 \to H^0(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) \to \bigwedge^p V \otimes H^0(\mathbb{P}^n, \mathcal{O}(p+k)) \to H^0(\mathbb{P}^n, \mathcal{T}^p(k)) \to 0.$$

If $k < -p$, we have $H^0(\mathbb{P}^n, \mathcal{T}^p(k)) = 0$. If $k \geq -p$, we use the equation

$$\begin{aligned} h^0(\mathbb{P}^n, \mathcal{T}^p(k)) &= \left(\dim \bigwedge^p V\right) h^0(\mathbb{P}^n, \mathcal{O}(p+k)) - h^0(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) \\ &= \binom{n+1}{p}\binom{n+p+k}{p+k} - h^0(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) \end{aligned}$$

to induct on p and obtain the desired formula. ∎

Lemma A.1.6. *For $n \geq 2$,*

$$H^1(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) = \begin{cases} \mathbb{C} & \text{if } k = -n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have short exact sequences

$$0 \to H^0(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) \to \bigwedge^n V \otimes H^0(\mathbb{P}^n, \mathcal{O}(n+k)) \to H^0(\mathbb{P}^n, \mathcal{T}^n(k)) \to H^1(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) \to 0.$$

If $k < -n$, we have $H^0(\mathbb{P}^n, \mathcal{O}(n+k)) = 0$ and hence

$$H^1(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) = H^0(\mathbb{P}^n, \mathcal{T}^n(k)) = H^0(\mathbb{P}^n, \mathcal{O}(n+1+k)) = \begin{cases} \mathbb{C} & \text{if } k = -n-1 \\ 0 & \text{if } k < -n-1. \end{cases}$$

If $k \geq -n$, using Lemma A.1.5, we have

$$\begin{aligned} h^1(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) &= h^0(\mathbb{P}^n, \mathcal{T}^n(k)) - \left(\dim \bigwedge^n V\right) h^0(\mathbb{P}^n, \mathcal{O}(n+k)) + h^0(\mathbb{P}^n, \mathcal{T}^{n-1}(k)) \\ &= \binom{2n+1+k}{n} - \binom{n+1}{n}\binom{2n+k}{n+k} + \binom{n+k}{n-1+k}\binom{2n+k}{n-1} \\ &= 0. \end{aligned}$$

■

Lemma A.1.7. *For $1 \leq q \leq n-1$,*

$$H^q(\mathbb{P}^n, \mathcal{T}^p(k)) = \begin{cases} \mathbb{C} & \text{if } p+q = n \text{ and } k = -n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This holds by Lemma A.1.3, A.1.4 and A.1.6. ■

Lemma A.1.8.

$$h^n(\mathbb{P}^n, \mathcal{T}^p(k)) = \begin{cases} \binom{-n-k-2}{p}\binom{-k-p-1}{n-p} & \text{if } 0 \leq p \leq n,\ k \leq -n-p-2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the vanishing $H^{n-2}(\mathbb{P}^n, \mathcal{O}(p+k)) = 0$ and $H^{n+1}(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) = 0$ (since $\dim \mathbb{P}^n = n$), we have exact sequences

$$0 \to H^{n-1}(\mathbb{P}^n, \mathcal{T}^p(k)) \to H^n(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) \to \bigwedge^p V \otimes H^n(\mathbb{P}^n, \mathcal{O}(p+k)) \to H^n(\mathbb{P}^n, \mathcal{T}^p(k)) \to 0.$$

If $k \geq -n-p$, $H^n(\mathbb{P}^n, \mathcal{O}(p+k)) = 0$, hence $H^n(\mathbb{P}^n, \mathcal{T}^p(k)) = 0$.

If $k \leq -n-1-p$, then $H^{n-1}(\mathbb{P}^n, \mathcal{T}^p(k)) = 0$. We then have the equation

$$\begin{aligned} h^n(\mathbb{P}^n, \mathcal{T}^p(k)) &= \left(\dim \bigwedge^p V\right) h^n(\mathbb{P}^n, \mathcal{O}(p+k)) - h^n(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) \\ &= \binom{n+1}{p}\binom{-p-k-1}{-p-k-1-n} - h^n(\mathbb{P}^n, \mathcal{T}^{p-1}(k)) \end{aligned}$$

and induction on p gives us the desired formula. ■

Putting Lemmas A.1.5, A.1.7 and A.1.8 together, we obtain

Proposition A.1.9.

$$h^q(\mathbb{P}^n, \mathcal{T}^p(k)) = \begin{cases} \binom{n+k}{p+k}\binom{n+k+p+1}{p} & \textit{if } q = 0,\ 0 \le p \le n,\ k \ge -p, \\ 1 & \textit{if } p+q = n,\ 0 \le p, q \le n,\ k = -n-1, \\ \binom{-n-k-2}{p}\binom{-k-p-1}{n-p} & \textit{if } q = n,\ 0 \le p \le n,\ k \le -n-p-2, \\ 0 & \textit{otherwise.} \end{cases}$$

By the isomorphism $\mathcal{T}^p \cong \mathcal{T}^n \otimes \Omega^{n-p} \cong \Omega^{n-p}(n+1)$, one can calculate the cohomologies of exterior powers of cotangent sheaf of $\mathbb{P}^n$.

Proposition A.1.10.

$$h^q(\mathbb{P}^n, \Omega^p(k)) = \begin{cases} \binom{n+k-p}{k}\binom{k-1}{p} & \textit{if } q = 0,\ 0 \le p \le n,\ k \ge p+1, \\ 1 & \textit{if } 0 \le p = q \le n,\ k = 0, \\ \binom{-k+p}{-k}\binom{-k-1}{n-p} & \textit{if } q = n,\ 0 \le p \le n,\ k \le p-n-1, \\ 0 & \textit{otherwise.} \end{cases}$$

A.2 Cohomologies of Tangent and Cotangent Sheaf of $\mathbb{P}^2$

In this section, we specialize to $\mathbb{P}^2$. Let V be a three dimensional vector space over $\mathbb{C}$. Denote by Ω and $\mathcal{T}$ the cotangent and tangent sheaf on $\mathbb{P}(V) = \mathbb{P}^2$. From the last section, we have the formulae

$$h^q(\mathbb{P}^2, \mathcal{O}(k)) = \begin{cases} \frac{1}{2}(k+1)(k+2) & \text{if } q = 0 \text{ and } k \ge 0 \\ \frac{1}{2}(k+1)(k+2) & \text{if } q = 2 \text{ and } k \le -3 \\ 0 & \text{otherwise.} \end{cases}$$

$$h^q(\mathbb{P}^2, \mathcal{T}(k)) = \begin{cases} (k+2)(k+4) & \text{if } q = 0 \text{ and } k \ge -1 \\ 1 & \text{if } q = 1 \text{ and } k = -3 \\ (k+2)(k+4) & \text{if } q = 2 \text{ and } k \le -5 \\ 0 & \text{otherwise.} \end{cases}$$

$$h^q(\mathbb{P}^2, \Omega(k)) = \begin{cases} (k+1)(k-1) & \text{if } q = 0 \text{ and } k \ge 2 \\ 1 & \text{if } q = 1 \text{ and } k = 0 \\ (k+1)(k-1) & \text{if } q = 2 \text{ and } k \le -2 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition A.2.1.

$$h^q(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) = \begin{cases} 2k^2 + 6k + 1 & \textit{if } q = 0 \textit{ and } k \ge 0 \\ 3 & \textit{if } q = 1 \textit{ and } k = -1 \textit{ or } k = -2 \\ 2k^2 + 6k + 1 & \textit{if } q = 2 \textit{ and } k \le -3 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We have a long exact sequence

$$\begin{aligned} 0 &\to H^0(\mathbb{P}^2, \Omega(k)) \to V \otimes H^0(\mathbb{P}^2, \Omega(k+1)) \to H^0(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) \\ &\to H^1(\mathbb{P}^2, \Omega(k)) \to V \otimes H^1(\mathbb{P}^2, \Omega(k+1)) \to H^1(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) \\ &\to H^2(\mathbb{P}^2, \Omega(k)) \to V \otimes H^2(\mathbb{P}^2, \Omega(k+1)) \to H^2(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) \to 0. \end{aligned}$$

For $k \ge 1$, from the vanishing $H^i(\mathbb{P}^2, \Omega(k)) = H^i(\mathbb{P}^2, \Omega(k+1)) = 0$, we immediately have the vanishing $H^i(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) = 0$ for $i \ge 1$. Also, we have

$$h^0(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) = 3h^0(\mathbb{P}^2, \Omega(k+1)) - h^0(\mathbb{P}^2, \Omega(k)) = 2k^2 + 6k + 1.$$

For $k = 0$, from $H^i(\mathbb{P}^2, \Omega(1)) = 0$ for all i, we have

$$H^i(\mathbb{P}^2, \mathcal{T} \otimes \Omega) = H^{i+1}(\mathbb{P}^2, \Omega) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For $k = -1$, from $H^i(\mathbb{P}^2, \Omega(-1)) = 0$ for all i, we have

$$H^i(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-1)) = V \otimes H^i(\mathbb{P}^2, \Omega) = \begin{cases} \mathbb{C}^3 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The rest follows from Serre duality

$$h^i(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k)) = h^{2-i}(\mathbb{P}^2, \Omega \otimes \mathcal{T}(-k) \otimes \mathcal{O}(-3)) = h^{2-i}(\mathbb{P}^2, \mathcal{T} \otimes \Omega(-3-k)).$$

■

Using the isomorphism $\mathcal{T} \cong \Omega(3)$, we also have

$$h^q(\mathbb{P}^2, \Omega \otimes \Omega(k)) = h^q(\mathbb{P}^2, \mathcal{T} \otimes \Omega(k-3)) = \begin{cases} 2k^2 + 6k + 1 & \text{if } q = 0 \text{ and } k \geq 3 \\ 3 & \text{if } q = 1 \text{ and } k = 1 \text{ or } k = 2 \\ 2k^2 + 6k + 1 & \text{if } q = 2 \text{ and } k \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Twisting and dualizing the Euler sequence again, we get short exact sequences

$$0 \to \mathcal{T} \otimes \Omega \otimes \Omega(k) \to V^* \otimes \mathcal{T} \otimes \Omega(k-1) \to \mathcal{T} \otimes \Omega(k) \to 0$$

$$0 \to \Omega \otimes \Omega(k) \to V \otimes \Omega \otimes \Omega(k+1) \to \mathcal{T} \otimes \Omega \otimes \Omega(k) \to 0.$$

Proposition A.2.2.

$$h^q(\mathbb{P}^2, \mathcal{T} \otimes \Omega \otimes \Omega) = \begin{cases} \mathbb{C}^{10} & \textit{if } q = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows from the long exact sequence

$$\begin{aligned} 0 &\to H^0(\mathbb{P}^2, \Omega \otimes \Omega) \to V \otimes H^0(\mathbb{P}^2, \Omega \otimes \Omega(1)) \to H^0(\mathbb{P}^2, \mathcal{T} \otimes \Omega \otimes \Omega) \\ &\to H^1(\mathbb{P}^2, \Omega \otimes \Omega) \to V \otimes H^1(\mathbb{P}^2, \Omega \otimes \Omega(1)) \to H^1(\mathbb{P}^2, \mathcal{T} \otimes \Omega \otimes \Omega) \\ &\to H^2(\mathbb{P}^2, \Omega \otimes \Omega) \to V \otimes H^2(\mathbb{P}^2, \Omega \otimes \Omega(1)) \to H^2(\mathbb{P}^2, \mathcal{T} \otimes \Omega \otimes \Omega) \to 0 \end{aligned}$$

■

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