$Spin(7)$ Instantons

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0.1 Abstract

In this thesis, we study the solutions of a first-order differential equation on manifolds with holonomy $Spin(7)$. This differential equation is seen to be analogous to the equations for a connection on a 4-manifold being a self-dual or anti-self-dual instanton. The solutions of the differential equation are shown to be solutions of the Yang-Mills equations, and we will call them $Spin(7)$ instantons.

A variety of elementary results about these $Spin(7)$ instantons are proven, including an energy bound on an instanton and an index-theoretical calculation of the dimension of the moduli space at a generic point. We also investigate the relationship of $Spin(7)$ connections to the more well-known Hermitian-Yang-Mills connections on a complex manifold with complex dimension 4.

We prove a non-existence result concerning finite-energy $Spin(7)$ instantons on $\mathbf{R}^8$, using various analytical results of Uhlenbeck. The thesis then goes on to consider the phenomenon of "bubbling" of a family of $Spin(7)$ instantons, concluding, under mild assumptions, that "bubbling" will occur around a Cayley submanifold of the $Spin(7)$ manifold.

Finally, in the concluding three chapters, we construct a non-trivial example of a $Spin(7)$ instanton on a compact 8-manifold with holonomy $Spin(7)$. This is accomplished using a gluing construction, and successive improvement of an initial "almost $Spin(7)$ instanton".

Contents

0.2 Acknowledgements

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Chapter 1

Introduction

The study of connections which are extrema of the Yang Mills functional

$$\int_M |F_A|^2$$

on a compact 4-manifold M is an area which has revolutionised the subject of 4-dimensional differential geometry. There is little doubt that the study of the resulting self-dual and anti-self-dual instantons has aided geometers greatly in their study of the underlying manifolds. One of the reasons for the existence of solutions to the instanton problem is that the 2-forms on a 4-manifold split up into two $SO(4)$-invariant 3-dimensional components: the self-dual and the anti-self dual 2-forms under the Hodge operator $*$. Thus the problem of finding a connection lying within one of these two components, together with a gauge fixing condition becomes an elliptic problem.

The construction of compact examples of holonomy $Spin(7)$ [J2] opens the question of the behaviour of extrema of the Yang-Mills functional on such manifolds. In 4 dimensions, the 3-dimensional constraint that the curvature of a connection be self-dual (or anti-self-dual) together with the 1-dimensional gauge fixing give rise to the possibility that the operator taking a connection 1-form to the projection of its curvature, together with a gauge fixing map being elliptic, as the operator acts between spaces of the same dimension. Similarly, if we consider a 7-dimensional curvature constraint on a connection on an 8-manifold, then this, together with a gauge-fixing condition has the possibility of having an elliptic linearisation.

These "$Spin(7)$ instantons" (as I will call them throughout the thesis) will clearly have many properties analogous to those of self-dual and anti-self-dual instantons, although they will undoubtedly have many new characteristics also.

Note that they should not be confused with instantons on manifolds of other holonomy group with gauge group $Spin(7)$; the instantons I shall call $Spin(7)$ instantons will have gauge group $SU(2)$, but will lie on a compact manifold with holonomy $Spin(7)$.

The first two chapters of this thesis (i.e. this one and the following one) are meant as an introduction. This chapter gives a brief description of each of the chapters of this thesis, including their aims, and the next contains background mathematics which is used in the later chapters, and which I believe is of use collected together. The topics covered in it are the holonomy group $Spin(7)$, Yang-Mills instantons, Chern-Weil theory, some analysis, Cayley submanifolds and spinors. All of these topics hold central roles in this thesis, and some familiarity with them is necessary to understand much of what follows.

The third chapter contains the analogous results for $Spin(7)$ instantons of some elementary results on 4-manifold instantons. These include an a priori energy bound, an index-theory calculation of the dimension of the moduli space of $Spin(7)$ instantons and some analysis of the moduli space. It also contains an interesting (in my opinion) result on the relationship between $Spin(7)$ instantons on manifolds with holonomy $SU(4)$, and the Hermitian-Yang-Mills connections on the manifold. (We may consider $Spin(7)$ instantons on such a manifold as $SU(4) \subset Spin(7)$.) At first, it appears that the condition for a connection to be a Hermitian-Yang-Mills connection is far stronger than the condition for its being a $Spin(7)$ instanton. However, under the existence of a single Hermitian-Yang-Mills connection on the vector bundle over the manifold, the two conditions become equivalent.

The fourth chapter, however, is of a different flavour, in that rather than dealing with similarities between 4-manifold instantons and $Spin(7)$ instantons, it deals with one of the differences. The finite energy instanton centred at the origin of $\mathbf{R}^4$ is a well-known ingredient in the study of 4-manifold instantons, and so we may at first try to construct a similar instanton on $\mathbf{R}^8$. The main theorem of this chapter, however, is that such a construction is impossible. It does this by considering the asymptotic behaviour of any finite energy instanton, and then using analytic results of Uhlenbeck.

On a 4-manifold, the phenomenon of a family of instantons "bubbling" around a point helps us investigate the limit points of the moduli space of anti-self-dual instantons. Chapter 4 makes it look unlikely that bubbling around a point occurs on the manifolds with holonomy $Spin(7)$, at least if we restrict our attention to the gauge group $SU(4)$. Chapter 5 deals with the limit of families of $Spin(7)$ instantons. Assuming a result of Nakajima [Nak, p.389], we show that the bubbling occurs around a space of Hausdorff dimension 4 in the 8-manifold. We further show that if this is a submanifold, then it must be a Cayley submanifold. Thus we suggest an equivalence between the role of points in the study of 4-manifold instantons, and the role of Cayley submanifolds in the theory of $Spin(7)$ instantons. It also leads me to conjecture that the moduli space of $Spin(7)$ instantons on a manifold of holonomy $Spin(7)$ will have a boundary very closley related to the moduli space of Cayley submanifolds of the 8-manifold.

The proof running through the final three chapters (Chapters 6-8) is that of the existence of a $Spin(7)$ instanton on a particular manifold, constructed by Joyce [J2]. Note that though the proof applies only to this specific manifold, it is easily adjusted to cover many other simliarly constructed manifolds with holonomy $Spin(7)$, and the analytic part needs no adjustment; it may be applied to any such case. This proof is split into three chapters; the first consisting of the geometric component of the argument, the second containing analytic estimate for the linearisation of the problem, and the final chapter deals with the construction, via an iterative sequence, of the solution to the instanton problem. It uses the geometric construction descibed in Chapter 6, and the analytic bounds obtained in Chapter 7. The idea behind the construction is analogous to Taubes' gluing construction [Tau], in that we construct an almost $Spin(7)$ instanton (i.e one in which the component of the curvature lying in the 7-dimensional $Spin(7)$-invariant component of the 2-forms is small) and then make successive amendments to this, getting, in the limit, a true $Spin(7)$ instanton. Of course, Chapters 4 and 5 suggest that though Taubes glues his instanton around a point, we may well have more success in the 8-manifold case if we attempted to glue around a Cayley submanifold. This indeed does prove to be the case. The key idea in Chapter 7 is the choice of Sobolev space with which to work. We use L^2_3; it is the smallest space in which we may apply the iterative step in an elementary manner. The other advantage of using L^2_3 rather than L^p_k is that setting $p = 2$ ensures that we still have some notion of L^2-orthogonality to work with, and may split any function up into the relevant orthogonal components.

Thus we finish with, to the best of my knowledge, the first known example of a $Spin(7)$ instanton on a manifold with holonomy $Spin(7)$.

Finally, before I begin, a point on the notation I intend to use. I will use C to denote a constant independent of any parameters being used. It will not necessarily denote the same constant throughout the thesis, but I do not think this should cause any confusion, and it should prevent needless subscripts, as C_{153}, for example, isn't particularly pleasant or informative. Those constants whose value is of importance will be labelled separately.

Chapter 2

Introductory Material

Before embarking upon the new material in this thesis, I think it would be helpful if I dealt with some introductory material which will be used throughout the thesis. Of course, this material is to be found elsewhere, but I believe the convenience of collecting it together in this background chapter justifies its presence here.

2.1 The Holonomy Group $Spin(7)$

In the list of the possible holonomy groups for a non-symmetric, irreducible riemannian manifold, there are two exceptional cases: G_2 and $Spin(7)$ [Berg]. (A description of their behaviour as Lie Groups may be found in [Sal].) It is the latter of the two which I hope to study in this dissertation and in this chapter I will give a few of its basic properties.

There are several ways of defining the group $Spin(7)$, often involving octonians. However, I believe that one of the more natural, and certainly one of the more useful, definitions, is the one given below.

Choose $\tau_1, \tau_2, ..., \tau_8$ to be an oriented, orthonormal basis of $\mathbf{R}^8$ (with the metric induced from the natural one of $\mathbf{R}^8$).

Define a 4-form Ω_0 on $\mathbf{R}^8$ by

$$\begin{aligned}
\Omega_0 &= \tau_1 \wedge \tau_2 \wedge \tau_5 \wedge \tau_6 + \tau_1 \wedge \tau_3 \wedge \tau_5 \wedge \tau_7 + \tau_1 \wedge \tau_4 \wedge \tau_5 \wedge \tau_8 \\
&+ \tau_2 \wedge \tau_3 \wedge \tau_6 \wedge \tau_7 + \tau_2 \wedge \tau_4 \wedge \tau_6 \wedge \tau_8 + \tau_3 \wedge \tau_4 \wedge \tau_7 \wedge \tau_8 \\
&+ \tau_1 \wedge \tau_2 \wedge \tau_3 \wedge \tau_4 - \tau_1 \wedge \tau_2 \wedge \tau_7 \wedge \tau_8 - \tau_3 \wedge \tau_4 \wedge \tau_5 \wedge \tau_6 + \tau_5 \wedge \tau_6 \wedge \tau_7 \wedge \tau_8 \\
&+ \tau_2 \wedge \tau_4 \wedge \tau_5 \wedge \tau_7 - \tau_2 \wedge \tau_3 \wedge \tau_5 \wedge \tau_8 - \tau_1 \wedge \tau_4 \wedge \tau_6 \wedge \tau_7 + \tau_1 \wedge \tau_3 \wedge \tau_6 \wedge \tau_8.
\end{aligned}$$

(It is perhaps worth noting at this point that Ω_0 is self-dual i.e. $\Omega_0 = *\Omega_0$)

We may define $Spin(7)$ as the subgroup of $GL(8, \mathbf{R}^8)$ preserving Ω_0. This definition is equivalent to the many other definitions of the Lie group $Spin(7)$, including the more usual one as the double cover of $SO(7)$. Note that since the metric may be reconstructed from the 4-form Ω_0,[J2, p. 510] then we have $Spin(7) \subset SO(8)$. Now let M be an 8 dimensional manifold and consider AM, defined as a sub-bundle of $\Lambda^4 T^*M$ by

$$AM_m = \{\lambda \in (\Lambda^4 T^*M)_m : \exists \text{ an isomorphism } \phi : T_m M \to \mathbf{R}^8, \text{ taking } \lambda \text{ to } \Omega_0\}.$$

Now a smooth section of AM gives rise to a $Spin(7)$ structure, and where no confusion will occur, we will call it a $Spin(7)$ structure.

If Ω is a $Spin(7)$ structure for M, then Ω induces a natural metric, g, on M and thus a natural Levi-Civita connection ∇. We say that Ω is a torsion-free $Spin(7)$ structure if $\nabla\Omega = 0$. However, we do have that $\nabla\Omega$ is determined by $d\Omega$, and thus a $Spin(7)$ structure is torsion free if and only if it is closed. [Sal, p. 176]

Of course, as with other holonomy groups, the action of $Spin(7)$ on $\Lambda^k(\mathbf{R}^8)^*$ leads to the splitting of $\Lambda^k(\mathbf{R}^8)^*$ into the orthogonal direct sum of irreducible $Spin(7)$ representations. Similarly, $\Lambda^k T^*M$ splits into an orthogonal direct sum of sub-bundles, with irreducible $Spin(7)$ representations as fibres.

These are [J2, p. 511]

$$\begin{aligned}
\Lambda^1 T^*M &= \Lambda^1_8, \\
\Lambda^1 T^*M &= \Lambda^2_7 \oplus \Lambda^2_{21}, \\
\Lambda^3 T^*M &= \Lambda^3_8 \oplus \Lambda^3_{48}, \\
\Lambda^4 T^4 M &= \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}.
\end{aligned}$$

and since $\Lambda^{8-i} \equiv \Lambda^i$ as $Spin(7)$ representations, the isomorphism being given by the Hodge star, we automatically have the splitting for $\Lambda^5 T^*M, \Lambda^6 T^*M$ and $\Lambda^7 T^*M$.

We shall denote the orthogonal projection from $\Lambda^k T^*M$ to Λ^k_l by π^k_l, or more usually, when no confusion will occur, by π_l.

$Spin(7)$ contains 4 connected subgroups that are possible holonomy groups for 8-dimensional manifolds in Berger's classification, that act non-trivially on nonzero vectors. These are $Spin(7)$ itself, $SU(4)$, $Sp(2)$ and $SU(2) \times SU(2)$. [J2, p. 548]

Note that if M is a compact, simply-connected 8-manifold, with Ω a torsion-free $Spin(7)$ structure with corresponding metric g, then $\mathrm{Hol}(g)$ is one of the four groups listed above, and which of these it is may be determined by the $\hat{A}$ genus of the manifold, $\hat{A}(M)$, defined in [Hirz, p.197].

$$
\begin{aligned}
\hat{A}(M) &= 1 \iff Hol(g) = Spin(7),\\
\hat{A}(M) &= 2 \iff Hol(g) = SU(4),\\
\hat{A}(M) &= 3 \iff Hol(g) = Sp(2),\\
\hat{A}(M) &= 4 \iff Hol(g) = SU(2) \times SU(2).
\end{aligned}
$$

$\hat{A}(M)$ may be defined with either of the following 2 equations

$$45.2^7 \hat{A}(M) = 7p_1(M)^2 - 4p_2(M),$$

or

$$24\hat{A}(M) = -1 + b^1 + b^3 + b^4_+ - 2b^4_-.$$

where $p_1(M), p_2(M)$ are the first and second Pontrjagin numbers of M and b^i is the i th Betti numbers (i.e. the dimension of $H^i(M, \mathbf{R})$) and b^4_+, b^4_- are the dimensions of the self-dual and anti-self-dual subspaces of $H^4(M, \mathbf{R})$ respectively.

2.2 Yang-Mills Instantons

It is in 4 dimensions that the study of Yang-Mills connections has been most developed.

This is due to a convenient splitting of the 2-forms into two three dimensional subspaces.

For consider the Hodge star operator, $* : \Lambda^i \to \Lambda^{4-i}$ and in particular, its action on the 2-forms, $* : \Lambda^2 \to \Lambda^2$.

This satisfies $*^2 = id$, where id is the identity operator. Thus Λ^2 splits up into two *-eigenspaces, corresponding to $+1$ and -1. These are called Λ^2_+ and Λ^2_- respectively.

Thus $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ and we have

$$\Omega^2 = \Omega^+ \oplus \Omega^-,$$

the corresponding splitting of sections of the above bundles.

This splitting extends to bundle valued 2-forms, so that we have

$$\Omega^2(\mathcal{G}_E) = \Omega^+(\mathcal{G}_E) \oplus \Omega^-(\mathcal{G}_E),$$

$\mathcal{G}_E$ is a subbundle of the bundle of endomorphisms of E, as defined in [DK, p.32].

Of course, the curvature of a connection on a bundle E over a 4-manifold X, F_A is an element of $\Omega^2(\mathcal{G}_E)$, so that we have $F_A = F_A^+ + F_A^-$, with $F_A^+ \in \Omega^+(\mathcal{G}_E)$ and $F_A^- \in \Omega^-(\mathcal{G}_E)$.

Now we define a connection A on a bundle E to be self-dual (or, respectively, anti-self-dual) if $F_A^- = 0$ (or, respectively, $F_A^+ = 0$).

Now, since the splitting of $\Omega^2(\mathcal{G}_E)$ is orthogonal, we have, for any connection A

$$||F_A||^2 = \int_X |F_A|^2 d\mu = \int_X (|F_A^+|^2 + |F_A^-|^2) d\mu = ||F_A^+||^2 + ||F_A^-||^2,$$

where $||.||$ denotes an L^2 norm.

If E is a Hermitian bundle, we also have

$$\int Tr(F_A^2) = ||F_A^-||^2 - ||F_A^+||^2$$

This can be shown either by using Schur's lemma, or by observing that $Tr(\xi^2) = -|\xi|^2$ on the Lie algebra $\mathcal{U}(n)$ of skew adjoint matrices. [DK, p.40]

However $\int Tr(F_A^2)$ is an invariant polynomial in the curvature of a connection, and thus, by Chern-Weil theory, may be written as a certain polynomial in the Chern classes of the bundle.

In this case

$$\int Tr(F_A^2) = \langle 8\pi^2 c_2(E) - 4\pi^2 c_1(E)^2, [M]\rangle = 8\pi^2 \kappa(E).$$

Thus $||F_A^-||^2 - ||F_A^+||^2 = 8\pi^2\kappa(E)$ and so self-dual and anti-self-dual connections are minima for the Yang-Mills functional

$$YM(A) = ||F_A||^2.$$

(This is clear from the two equations

$$||F_A^-||^2 - ||F_A^+||^2 = 8\pi^2\kappa(E),$$

$$||F_A^-||^2 + ||F_A^+||^2 = ||F_A||^2.$$

Thus we see that

$$||F_A||^2 = 8\pi^2\kappa(E) + 2||F_A^+||^2,$$

and so that if $F_A^+ = 0$, then the Yang-Mills functional is at a minimum. A similar method shows that if $F_A^- = 0$, then the same is true.)

In the $Spin(7)$ case, we do not have such a natural splitting of the 2-forms involving the Hodge star operator, as of course, $*$ behaves as an involution with respect to the 4-forms in the 8 dimensional case.

There is, however, a natural definition of Yang-Mills type instantons on $Spin(7)$ 8-manifolds.

As we have seen, the 2-forms split into two irreducible $Spin(7)$ representations, Λ_7 and Λ_{21}.

The most natural constraint to apply to a "$Spin(7)$ type Yang-Mills instanton" is that $\pi_7(F_A) = 0$, that is, the part of the curvature in Λ_7 disappears.

I shall investigate some of the properties of the Yang-Mills type instantons on $Spin(7)$ manifolds, which will hopefully be as interesting as the equivalent self-dual and anti-self-dual connections in 4 dimensions.

2.3 Basic Chern-Weil Theory

Suppose $P : M_n(\mathbf{C}) \to \mathbf{C}$ is a polynomial in the entries of the matrix. Then we say P is an invariant polynomial if

$$P(TXT^{-1}) = P(X)$$

for all non singular matrices T.

Now suppose that $E \to M$ is a complex vector bundle over some manifold M. Then near any point $m \in M$ we may choose a basis $e_1, ..., e_n$ for the sections of E near m.

Given a connection A for E over M, we may consider its curvature

$$F_A \in \Lambda^2 \otimes End(E)$$

Thus locally we may write

$$F_A = \Sigma F_{ij} \otimes e_j \otimes e_i^*,$$

where e_i^* is the dual of e_i. Now $F_{ij} \in \Lambda^2 \subseteq \Sigma\Lambda^{2r}$, the commutative algebra of all exterior forms of even degree. Thus we may form powers of F_A, writing

$$F_A^k = \underbrace{F_A \wedge F_A \wedge ... F_A}_{k \text{ times}}$$

with $F_A^k \in \Lambda^{2k} \otimes End(E)$ and thus evaluate any polynomial at F_A.

In particular, we may consider the result of applying the polynomial P at F_A. It will, of course, be an element of $\Sigma\Lambda^{2r}$, the aforementioned commutative algebra.

Note that if P should be homogeneous of degree r, then $P(F_A)$ will be an exterior form of degree $2r$.

Basic Chern-Weil theory shows that the element so obtained is, in fact, closed i.e. $dP(F_A) = 0$, [MS, page 296] and so $P(F_A)$ gives rise to a class in the De Rham cohomology ring

$$[P(F_A)] \in \bigoplus H^i(M, \mathbf{C})$$

The next point worth noting from basic Chern-Weil theory is that the cohomology class is independent of the connection chosen, and thus is only dependent on the bundle E, and the manifold M.

To see this, consider two connections A_1 and A_2 on the bundle E over M. Now consider $M \times \mathbf{R}$, together with the induced bundle E' and the connection $A' = tA_1 + (1-t)A_2$.

Then $[P(F_A)] \in \bigoplus H^i(M \times \mathbf{R}, \mathbf{C})$

Now consider the pair of maps

$$i_n : M \to M \times \mathbf{R}, i_n(m) = (m, n), \text{ where } n = \mathit{0} \text{ or } \mathit{1}$$

Clearly, the connection $(i_n)^*(A')$ can be identified with A_n on E. Thus

$$(i_n)^*[P(F_{A'})] = [P(F_{A_n})]$$

But i_0 and i_1 are homotopic maps, hence

$$[P(F_{A_0})] = [P(F_{A_1})],$$

as required.

Now it is well known that every symmetric polynomial can be written as polynomial function in the elementary polynomial functions $\sigma_1, \sigma_2, \sigma_3, \ldots$ [MS, p.299]

These can be obtained from the equation

$$\det(I_n + t\mathrm{Diag}(\lambda_1, \lambda_2, \ldots)) = 1 + t\sigma_1(\lambda_1, \ldots, \lambda_n) + t^2\sigma_2(\lambda_1, \ldots, \lambda_n) + \ldots + t^n\sigma_n(\lambda_1, \ldots, \lambda_n)$$

The first few are

$$\begin{aligned} \sigma_1 &= \Sigma\lambda_i, \\ \sigma_2 &= \Sigma_{i<j}\lambda_i\lambda_j, \\ \sigma_3 &= \Sigma_{i<j<k}\lambda_i\lambda_j\lambda_k. \end{aligned}$$

Now it can also be shown that any invariant polynomial P on $M_n(\mathbf{C})$ can be expressed as a symmetric polynomial function in the eigenvalues of the matrix. Putting these two results together, we obtain the result that any invariant polynomial may be written as a polynomial function in the eigenvalues of the matrix. Now since

$$[(P_1P_2)(F)] = [P_1(F)][P_2(F)],$$

and

$$[(P_1 + P_2)(F)] = [P_1(F)] + [P_2(F)]$$

(i.e. the map $\Theta : P \to [P(F)]$ is a graded algebra homomorphism), we have that, provided we know $[\sigma_i]$, we may determine $[P(F)]$ for any invariant polynomial P.

The images of σ_i under the map may be determined by considering the two elements of $\bigoplus H^i(M, \mathbf{C})$ given by

$$\begin{aligned} c(E) &= [1 + c_1(E) + c_2(E) + \ldots] \\ c'(E) &= [p(F)], \end{aligned}$$

$$\text{where } p(X) = \det(I + \frac{1}{2\pi i}X)$$

We note that $c(L) = c'(L)$, where L is a line bundle, and, using a standard argument, we may extend this result to any bundle. [MS, 307] Thus $c(E) = c'(E)$ and by reading off the component of each in H^{2k}, we may see that

$$[\sigma_k(F)] = (2\pi i)^k c_k(E). \tag{2.1}$$

In particular, for future use, note that $Tr(F_A \wedge F_A)$ corresponds to the polynomial $P(X) = Tr(X^2) = \Sigma\lambda_i^2$, where λ_i are the eigenvalues of X.

2.4 Some Preliminary Analysis

When considering the space of sections of a vector bundle E over a manifold M, it is often useful to impose an analytic structure upon it, in order to make the space into a Banach space.

This structure will be particularly important when we have two vector bundles, E and F over the same manifold M, and must consider an elliptic operator mapping between them.

Of course, in dealing with these analytic structures, it is essential that we possess some preliminary results about them.

First let us consider four different types of Banach spaces.

Suppose M is a compact Riemannian manifold, with metric g, and E is a vector bundle over M with metrics in the fibres. Let ∇ be a connection preserving the metrics in the fibres of E.

Then, firstly, we define $C^k(E)$, the space of continuous sections of E with at least k continuous derivatives. We define the norm of $v \in C^k(E)$ by

$$||v||_{C^k(E)} = \Sigma_{i=0}^k \sup_M |\nabla^i v|$$

A closely related Banach space is $C^{k,\alpha}(E)$, (with $0 < \alpha < 1$), the Holder spaces.

For a section v of a vector bundle F, we define

$$[v]_\alpha = \sup_{\gamma \in G} \frac{|v(\gamma(1)) - v(\gamma(0))|}{l(\gamma)^\alpha},$$

where G is the set of all smooth geodesic paths in M, and $l(G)$ denotes the length of the geodesic path γ. Note that $F_{\gamma(0)}$ and $F_{\gamma(1)}$ are identified using parallel translation using ∇ along the geodesic.

We say a section v is Holder continuous with exponent α if $[v]_\alpha$ is finite.

Finally, we define $C^{k,\alpha}$ to be those elements of C^k whose k-th order partial derivatives are Holder continuous with exponent α. The norm used is

$$||v||_{C^{k,\alpha}(E)} = ||v||_{C^k(E)} + [\nabla^k v]_\alpha$$

The next family of Banach spaces we must consider are the Lebesgue spaces, $L^p(E)$, with $p \geq 1$.

This is merely the sections of E for which the Lebesgue norm

$$||v||_{L^p(E)} = (\int_M |v|^p d\mu)^{\frac{1}{p}}$$

is finite, where $d\mu$ is the volume form for g on M. Note that if $p = 2$, this is the Hilbert space.

The final family of Banach spaces we should consider are the Sobolev spaces, $L^p_k(E)$, with $1 \leq p < \infty$, and k a positive integer.

In a similar manner to Lebesgue spaces, we define the Sobolev space $L^p_k(E)$ to be those sections of E for which the Sobolev norm

$$||v||_{L^p_k(E)} = \Sigma_{j=0}^k ||\nabla^j v||_{L^p(E)}$$

is finite.

Now I will give some basic analytic results relating some of these Banach spaces, that may come in useful later.

First let us consider the Sobolev embedding theorem, where M is a compact manifold of dimension n, and E a bundle over it.

Theorem 2.1 *Sobolev Embedding Theorem. [Bes, p.458]*

a) If $k - l < \frac{n}{p}$ and q satisfies

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{l}{n}$$

then there is a continuous inclusion $L^p_k(E) \to L^q_l(E)$. Moreover, if $l < k$, and the above inequality is strict, then this inclusion is compact.

b) If $(k - l) - 1 < \frac{n}{p} < k - l$, letting $\alpha = k - \frac{n}{p} - l$, we have that there is a continuous inclusion

$$L^p_k \to C^{l,\alpha}$$

Now let us consider results concerning elliptic regularity. Consider $P : C^\infty(E) \to C^\infty(F)$ a linear elliptic differential operator of order k. Then we have the following relationships between the norms in various Banach spaces, where $c_1, ..., c_6$ are positive constants. [Bes, p. 463]

$$\begin{aligned} ||u||_{C^{k+l,\alpha}(E)} &\leq c_1||P(u)||_{C^{l,\alpha}(F)} + c_2||u||_{C(E)} &&\leq c_3||u||_{C^{k+l,\alpha}(E)}, \\ ||u||_{L^p_{k+l}(E)} &\leq c_4||Pu||_{L^p_l(F)} + c_5||u||_{L^1(E)} &&\leq c_6||u||_{L^p_{k+l}(E)}. \end{aligned}$$

Note that if u is not a member of $C^\infty(E)$ then we have made use of a result that states:

Theorem 2.2 *If $P : C^\infty(E) \to C^\infty(F)$ is a differential operator of order m, then P has a unique continuous extension $P^p_{l+m} : L^p_{l+m}(E) \to L^p_l(F)$, and a similar result involving extension to Holder spaces.*

Now suppose $Pu \equiv \Sigma_{|\beta| \leq k} a_\beta(x)\partial^\beta u$ where k is the order of P.

The theorem of elliptic regularity, a consequence of the above inequalities, states

Theorem 2.3 *: [Bes, p.466] If P is elliptic, and $a_\beta(x)$ are C^l functions, and $f(x)$ is a L^p_l function, then any function $u \in L^p_k$ satisfying*

$$Pu = f$$

almost everywhere, lies in L^p_{k+l}.

Note that these elliptic regularity results hold only for the Sobolev and Holder spaces, and not for the C^k spaces mentioned before, which is why we often use Holder and Sobolev spaces in problems involving elliptic operators.

2.5 Cayley Submanifolds

Cayley 4-manifolds are a special type of submanifold of $Spin(7)$ 8-manifolds. They belong to a type of submanifold known as a calibrated submanifold. Cayley 4-manifolds, let us proceed as follows. Riemannian manifold, and ϕ a closed exterior p-form on M. Then ϕ is said to be a calibration if, for every tangent p-plane ζ to M, we have

$$\phi|_\zeta \leq vol_\zeta,$$

where vol_ζ is, of course, the volume form of the p-plane ζ, induced by the metric g.

Now suppose N is a submanifold of M, such that

$$\phi|_N = vol_N,$$

that is, the p-form restricted to N is N's volume form. (This, of course, implies that N must be a p-dimensional submanifold of M) . Then N is said to be a calibrated submanifold of M, with calibration ϕ.

Now let us suppose that M is a $Spin(7)$ manifold with torsion free $Spin(7)$ structure Ω.

Then note that Ω is a calibration for M. For it is clear that Ω is closed, from one of the two equivalent definitions of torsion free.

To see that $\Omega|_\zeta \leq vol_\zeta$ for any 4-plane ζ, it is perhaps best to resort to the octonion definition of Ω.

Consider $\Phi(x, y, z, w) =< x, y \times z \times w >$, where the triple product is defined by

$$y \times z \times w = \frac{1}{2}(y(\bar{z}w) - w(\bar{z}y)),$$

defined on $x, y, z, w \in \mathbf{O}$.

Considering $\mathbf{O}$ as a real 8 dimensional space, and seeing that Φ is alternating we see that Φ is a 4-form on $\mathbf{R}^8$. Using the orientation given by the basis $\{1, i, j, k, e, ie, je, ke\}$, we have that $\Phi = \Omega_0$, which is most easily seen by checking the value of each on elements of the basis, or by expanding Φ by substituting for x, y, z, w in component form. [HL, p.120]

Now, using Cauchy Schwartz, we have immediately

$$\Phi(x, y, z, w) \leq |x||y \times z \times w| = |x||y||z||w|,$$

and thus

$$\Phi_\zeta \leq vol_\zeta,$$

for any four plane ζ.

Thus, as required, Ω is a calibration.

Now we define N to be a Cayley 4-manifold of M if N is a calibrated submanifold with respect to the calibration Ω.

Note that for any compact 4 submanifold of M, say N', we have

$$< [\Omega], [N'] >= \int_{N'} \Omega|_{N'} \leq \int_{N'} vol_{N'} = vol(N').$$

Equality would imply that N' is a Cayley 4-submanifold, and thus we have that a Cayley 4 manifold is globally volume minimising amongst its homology class. Note that this result works for any calibration.

2.6 Spin Bundles and the Dirac Operator

To consider spin bundles over a $Spin(7)$ manifold M, it is usually best to first consider Clifford algebras.

Let V be a finite dimensional vector space with an inner product defined upon it. Let $e_1, e_2, ..., e_n$ be an orthonormal basis for V.

Then the Clifford algebra, C_n, of V is defined to be the algebra generated by the elements $e_1, ..., e_n$ subject to the relations

$$\begin{aligned} e_i^2 &= -1, \\ e_i e_j + e_j e_i &= 0 \text{ for } i \neq j \end{aligned}$$

Considered as a vector space C_n is of dimension 2^n, spanned by elements of the form

$$e_1^{\delta_1} e_2^{\delta_2} ... e_n^{\delta_n},$$

where $\delta_i = 0$ or 1.

Now consider the case $n = 8$. In this case it can be shown that

$$C_8 \equiv \mathbf{R}(16),$$

the algebra of 16×16 matrices with values in $\mathbf{R}$. [Sal, p.171]

Thus we may consider $\mathbf{R}^{16}$ as a C_8 module.

We may define the group $Spin(8)$ as the subset of C_8 consisting of all even products $x_1 x_2 ... x_{2r-1} x_{2r}$ of elements of V, with each $||x_i|| = 1$. (Similarly we might have defined $Spin(7)$ as the subset of C_7 consisting of all even products $x_1 x_2 ... x_{2r-1} x_{2r}$ of elements of $\mathbf{R}^7$ with each $||x_i|| = 1$.)

Now let us consider the element $v = e_1 e_2 ... e_8$ of C_8. Then v is a involution of C_8, and commutes with every element of $Spin(8)$, and hence $\mathbf{R}^{16}$ splits as a $Spin(8)$ module into the eigenspaces of v.

Thus $\mathbf{R}^{16} = \Delta_+ \oplus \Delta_-$, where Δ_+ is the $+1$ eigenspace of v, and Δ_- is the -1 eigenspace of v. We call them the positive and negative spin representations of $Spin(8)$.

Now suppose that M is a $Spin(7)$ manifold. Then we have that M is a spin manifold i.e. there exists a spin structure of M, a principal $Spin(8)$ bundle $\tilde{E}$ covering the $SO(8)$ bundle of frames for the tangent bundle.

Now since we have a principal $Spin(8)$ bundle, and the two $Spin(8)$ modules (namely Δ_+ and Δ_-), we may form two vector bundles associated to the principal spin bundles by means of the two spin representations.

We call these bundles S_+ and S_-, the positive and negative spinor bundles, and their sections are known as positive and negative spinors. It is perhaps worth noting at this point that the group $Spin(7)$ is the subgroup of $SO(8)$ preserving a spinor, and hence the manifold M will possess a constant spinor. Thus we will have isomorphisms $S_+ \equiv \Lambda^0 \oplus \Lambda^2_7$ and $S_- \equiv \Lambda^1$.

To define the Dirac operator, we should first consider Clifford multiplication.

Let $u \in \mathbf{R}^8$, and $x \in \mathbf{R}^{16}$. Then since $\mathbf{R}^8 \subseteq \mathrm{C}_8$, and $\mathbf{R}^{16}$ is a C_8 module, then we may let u act upon x.

Thus we have a map

$$\mathbf{R}^8 \otimes \mathbf{R}^{16} \to \mathbf{R}^{16},$$

which is known as Clifford multiplication.

Now suppose $x \in \Delta_+$, then $v.x = x$. Now since $v.u = -u.v$, since v anti-commutes with every element of V, then we have

$$v.(u.x) = (v.u).x = -u.(v.x) = -u.x.$$

Thus $u.x \in \Delta_-$.

So we have that Clifford multiplication interchanges the positive and negative spinors.

We may extend the Clifford multiplication from a map

$$\mathbf{R}^8 \otimes \Delta_+ \to \Delta_-$$

to a map

$$C^\infty(TM) \otimes C^\infty(S_+) \to C^\infty(S_-),$$

by defining the multiplication pointwise.

Now to define the positive component of the Dirac operator, we should choose locally an orthonormal basis of sections of TM, $e_1, e_2, ..., e_8$, and then for a section s of S_+ we define

$$Ds = \Sigma_{i=1}^8 e_i.\nabla_{e_i}(s),$$

where ∇ is the Levi-Civita connection, and . denotes Clifford multiplication. [Roe, p.26]

Since the definition is independent of the basis chosen, the local definitions of the Dirac operator agree and hence we have a map

$$D : C^\infty(S_+) \to C^\infty(S_-).$$

Note that the adjoint of D,

$$D^* : C^\infty(S_-) \to C^\infty(S_+)$$

the negative component of the Dirac operator, is also often denoted by D, and if it is necessary to avoid confusion we shall use

$$D_+ : C^\infty(S_+) \to C^\infty(S_-)$$

$$D_- : C^\infty(S_-) \to C^\infty(S_+).$$

The actual Dirac operator is defined by $D : S_+ \oplus S_- \to S_+ \oplus S_-$,

$$D = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}.$$

Chapter 3

Some results about $Spin(7)$ instantons

Before beginning this chapter, let us define precisely what we mean by a $Spin(7)$ instanton.

Definition 3.1 *We say that a connection A on a bundle E over a compact manifold with holonomy contained in $Spin(7)$ is a $Spin(7)$ instanton if $\pi_7(F_A) = 0$.*

3.1 An A Priori Energy bound On $Spin(7)$ Instantons

In this section, I will obtain energy bounds for the quantity $\int_M |F_A|^2$, where F_A is a connection with curvature in the Λ^2_{21} part of the 2-forms, in terms of the bundle $E \to M$ alone. This is closely related to the well-known energy bound on self-dual and anti-self-dual connections on a 4-manifold, corresponding to a multiple of the second Chern class of the bundle [DK, p.40].

Proposition 3.1 *Let $E \to M$ be a complex vector bundle over a compact $Spin(7)$ manifold, and let A be a $Spin(7)$ connection on E. Then*

$$||F_A||^2 = ||\pi_{21}F_A||^2 = \int_M Tr(F_A \wedge F_A) \wedge \Omega = 4\pi^2[2c_2(E) - c_1(E)^2] \cup [\Omega].$$

Proof.

Consider the quadratic form defined on 2-forms with values in $\mathrm{End}(E)$ as follows

$$Q(\lambda) = \int_M Tr(\lambda \wedge \lambda) \wedge \Omega.$$

Now this is a $Spin(7)$ invariant quadratic form, and since we may write the map

$$\lambda \to \mathrm{Tr}(\lambda \wedge \lambda) \wedge \Omega$$

pointwise as

$$\mathrm{Tr}(\lambda \wedge \lambda) \wedge \Omega = *(k_1|\pi_7\lambda|^2 + k_2|\pi_7\lambda|^2),$$

using the $Spin(7)$ decomposition of the 2-forms, we may also write

$$Q(\lambda) = k_1||\pi_7\lambda||^2 + k_2||\pi_{21}\lambda||^2,$$

where π_7, π_{21} are projections onto the 7 and 21 dimensional parts respectively.

Note, first of all, that the trace of the pointwise map must be zero. This is most easily seen by computing the trace in the orthonormal basis

$$\tau_1 \wedge \tau_2, \tau_1 \wedge \tau_3, \tau_1 \wedge \tau_4, \ldots$$

Thus we have $k_1 + 3k_2 = 0$.

Now, unlike the 4 dimensional case, we do not have any obvious bases for our two subspaces of the 2-forms.

However, in a later section, we shall see that

$$\omega = \tau_1 \wedge \tau_5 + \tau_2 \wedge \tau_6 + \tau_3 \wedge \tau_7 + \tau_4 \wedge \tau_8 \in \Lambda^2_7.$$

Thus we may find the required constants.

$$Q(\omega) = \int_M \omega \wedge \omega \wedge \Omega = -12vol(M),$$

and we also have

$$Q(\omega) = k_1||\omega||^2 = 4k_1 vol(M).$$

So $k_1 = -3$, and $k_2 = 1$, i.e.

$$\int_M Tr(\lambda \wedge \lambda) \wedge \Omega = ||\pi_{21}\lambda||^2 - 3||\pi_7\lambda||^2.$$

If A is a connection on E over M, then F_A, the curvature of A is a 2-form with values in $\mathrm{End}(E)$. Thus we may use the above results with F_A in place of λ.

Thus

$$\int_M Tr(F_A \wedge F_A) \wedge \Omega = ||\pi_{21} F_A||^2 - 3||\pi_7 F_A||^2.$$

But the left hand side of this equation is a $Spin(7)$ invariant polynomial in the curvature of a bundle over a manifold, and thus we may apply Chern-Weil theory to it, to get a topological invariant.

We have

$$\int_M Tr(F_A \wedge F_A) \wedge \Omega = [4\pi^2(2c_2(E) - c_1(E)^2)] \cup [\Omega]. \tag{3.1}$$

Thus we have an energy bound on $Spin(7)$ instantons

$$||F_A||^2 = ||\pi_{21} F_A||^2 = \int_M Tr(F_A \wedge F_A) \wedge \Omega = 4\pi^2[2c_2(E) - c_1(E)^2] \cup [\Omega].$$

0

It will be useful later on if we note that in the case of E being an $SU(2)$ bundle, then the energy bound takes a simpler form

$$||F_A||^2 = 8\pi^2[c_2(E)] \cup [\Omega].$$

In any case, we may form a basis for the $Spin(7)$ invariant 7 dimensional subspace for the 2 -forms by looking for 2-forms that satisfy the above equation. This will give us

$$\begin{aligned}
&\frac{1}{2} \quad (\tau_1 \wedge \tau_2 - \tau_3 \wedge \tau_4 - \tau_5 \wedge \tau_6 + \tau_7 \wedge \tau_8),\\
&\frac{1}{2} \quad (\tau_1 \wedge \tau_3 + \tau_2 \wedge \tau_4 - \tau_5 \wedge \tau_7 - \tau_6 \wedge \tau_8),\\
&\frac{1}{2} \quad (\tau_1 \wedge \tau_4 - \tau_2 \wedge \tau_3 - \tau_5 \wedge \tau_8 + \tau_6 \wedge \tau_7),\\
&\frac{1}{2} \quad (\tau_1 \wedge \tau_5 + \tau_2 \wedge \tau_6 + \tau_3 \wedge \tau_7 + \tau_4 \wedge \tau_8),\\
&\frac{1}{2} \quad (\tau_1 \wedge \tau_6 - \tau_2 \wedge \tau_5 + \tau_3 \wedge \tau_8 - \tau_4 \wedge \tau_7),\\
&\frac{1}{2} \quad (\tau_1 \wedge \tau_7 - \tau_2 \wedge \tau_8 - \tau_3 \wedge \tau_5 + \tau_4 \wedge \tau_6),\\
&\frac{1}{2} \quad (\tau_1 \wedge \tau_8 + \tau_2 \wedge \tau_7 - \tau_3 \wedge \tau_6 - \tau_4 \wedge \tau_5).
\end{aligned} \tag{3.2}$$

We may see that these are the coefficients of the standard basis for the imaginary octonions in the expression:

$$d\tau \wedge d\bar{\tau},$$

where

$$\tau = \sum_{n=1}^{8} \tau_n \mathbf{i}_n,$$

each $\mathbf{i}_n$ representing one of the standard basis of octonions. It may be of interest to compare with the basis for the anti-self-dual 2-forms on 4-manifolds given by

$$d\tau \wedge d\bar{\tau},$$

where this time $\tau = \sum_{n=1}^{4} \tau_n \mathbf{i}_n$ can be considered as a quaternion.

3.2 Relationship with Hermitian-Yang-Mills Connections

The inclusion $SU(4) \subset Spin(7)$, using the identification $z_n = x_n + ix_{n+4}$, allows us to consider the relationship between the $Spin(7)$ geometry and the $SU(4)$ geometry of a manifold with holonomy a subgroup of $SU(4)$.

As we have seen earlier, the splitting of the 2-forms of a $Spin(7)$ manifold into $Spin(7)$ irreducible components is

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{21}.$$

Similarly, for an $SU(4)$ manifold, an irreducible splitting is

$$\Lambda^2 \otimes \mathbf{C} = \Lambda_0^{1,1} \oplus <\omega> \oplus \Lambda^{2,0} \oplus \Lambda^{0,2},$$

where ω is the Kahler form of the manifold, and $\Lambda_0^{1,1}$ is the space of trace free (1,1) forms.[Sal, p.33]

A Hermitian-Yang-Mills connection is defined to be a connection whose curvature lies entirely within the $\Lambda_0^{1,1}$ component of the splitting.

Note, however, that the splitting above refers to the complexified space of 2-forms, whilst the corresponding space of real 2-forms decomposes as shown below

$$\Lambda^2 = \Lambda_0^{1,1} \oplus <\omega> \oplus C \oplus B,$$

where C and B are 6 dimensional real subspaces of $Re(\Lambda^{2,0} \oplus \Lambda^{0,2})$.

The dimensions of the subspaces are

$$\dim \Lambda_0^{1,1} = 15$$

$$\dim <\omega> = 1$$

$$\dim C = \dim B = 6.$$

We must have that the $SU(4)$ invariant subspaces are also subspaces of the $Spin(7)$ invariant subspaces mentioned above. Comparing dimensions, we have

$$\Lambda^2_{21} = \Lambda^{1,1}_0 \oplus B$$
$$\Lambda^2_7 = <\omega> \oplus C.$$

(We may assume without loss of generality $C \subset \Lambda^2_7$).

Bases for $\Lambda^{1,1}_0$ and $<\omega>$ are easy to construct, so let us also construct bases for C and B.

We have already seen a basis for C, albeit in passing[Equation 3.2]. The basis for Λ^2_7, with ω removed, of course, is also a basis for C.

A similar basis for B may be constructed

$$\frac{1}{2}(\tau_1 \wedge \tau_2 + \tau_3 \wedge \tau_4 - \tau_5 \wedge \tau_6 - \tau_7 \wedge \tau_8)$$
$$\frac{1}{2}(\tau_1 \wedge \tau_3 - \tau_2 \wedge \tau_4 - \tau_5 \wedge \tau_7 + \tau_6 \wedge \tau_8)$$
$$\frac{1}{2}(\tau_1 \wedge \tau_4 + \tau_2 \wedge \tau_3 - \tau_5 \wedge \tau_8 - \tau_6 \wedge \tau_7)$$
$$\frac{1}{2}(\tau_1 \wedge \tau_6 - \tau_2 \wedge \tau_5 - \tau_3 \wedge \tau_8 + \tau_4 \wedge \tau_7)$$
$$\frac{1}{2}(\tau_1 \wedge \tau_7 + \tau_2 \wedge \tau_8 - \tau_3 \wedge \tau_5 - \tau_4 \wedge \tau_6)$$
$$\frac{1}{2}(\tau_1 \wedge \tau_8 - \tau_2 \wedge \tau_7 + \tau_3 \wedge \tau_6 - \tau_4 \wedge \tau_5).$$

Now it is clear that a Hermitian-Yang-Mills connection is also a $Spin(7)$ instanton, since $\Lambda^{1,1}_0 \subset \Lambda^2_{21}$.

Now I shall prove a partial converse.

Theorem 3.1 *Let E be a bundle over a manifold M, with holonomy contained in* $SU(4)$. *Suppose that E admits a Hermitian-Yang-Mills connection, then any* $Spin(7)$ *instanton on the bundle E is also a Hermitian-Yang-Mills connection.*

Proof.

Consider the 4 form Ω. Now $\Omega = -\frac{1}{2}\omega \wedge \omega + Re(dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4)$, and so Ω consists of a (2,2) part, a (0,4) part and a (4,0) part.

Let us consider Ω', the (0,4)+(4,0) part of Ω. Now this is $SU(4)$ invariant, and it is also closed. (Since $d\Omega = 0$, and $d\Omega'$ is the component of $d\Omega$ in $\Lambda^{5,0} \oplus \Lambda^{4,1} \oplus \Lambda^{1,4} \oplus \Lambda^{0,5}$).

Thus we may consider the $SU(4)$ invariant quadratic Q', defined on the 2 forms by

$$Q'(\lambda) = \int_M Tr(\lambda \wedge \lambda) \wedge \Omega'$$

Now, as before with Q, Q' can be written in the following form :

$$Q'(\lambda) = c_1||\pi_{<\omega>}\lambda||^2 + c_2||\pi_{\Lambda_0^{1,1}}\lambda||^2 + c_3||\pi_C\lambda||^2 + c_4||\pi_B\lambda||^2$$

where *lru* is the orthogonal projection onto the subspace U.

The constants are easily found, since we have a basis for each subspace.

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 2$$

$$c_4 = -2$$

Thus we have

$$Q'(\lambda) = \int_M Tr(\lambda \wedge \lambda) \wedge \Omega' = 2(||\pi_B\lambda||^2 - ||\pi_C\lambda||^2).$$

Now suppose $\lambda = F_A$, the curvature form of a connection. Then

$$\int_M Tr(F_A \wedge F_A) \wedge \Omega' = 2(||\pi_B F_A||^2 - ||\pi_C F_A||^2)$$

Now if A is a $Spin(7)$ instanton, $\pi_C F_A = 0$, and thus

$$\int_M Tr(F_A \wedge F_A) \wedge \Omega' = 2||\pi_B F_A||^2.$$

But we may apply a standard result of Chern Weil theory, [see equation 2.1] in exactly the way we did in the previous section [equation 3.1] to get

$$\int_M Tr(F_A \wedge F_A) \wedge \Omega' = [4\pi^2(2c_2(E) - c_1(E)^2), \Omega'] = 4\pi^2[2c_2(E) - c_1(E)^2] \cup [\Omega']$$

Thus $2||\pi_B F_A||^2 = 4\pi^2[2c_2(E) - c_1(E)^2] \cup [\Omega']$, for A a $Spin(7)$ instanton.

Thus $||\pi_B F_A||^2$ is a constant depending on the topology of the bundle E, and not upon the particular connection.

In particular,$||\pi_B F_A||^2 = ||\pi_B F_{A_0}||^2$, where A_0 is the Hermitian-Yang-Mills connection, that exists by hypothesis.

Of course, $\pi_B F_{A_0} = 0$, by definition.

Thus $||\pi_B F_A||^2 = 0$, for our $Spin(7)$ instanton, and so $\pi_B F_A = 0$. Together with $\pi_C F_A = \pi_{<\omega>} F_A = 0$, which come from the definition of a

$Spin(7)$ instanton, we have that A must be Hermitian-Yang-Mills.

□

Note that this result, together with results in later sections, gives us an interesting corollary.

On an $SU(4)$ manifold, the equations for a connection to be a Hermitian-Yang-Mills connection are over-determined elliptic. Thus we do not have automatically that the moduli space of Hermitian-Yang-Mills connections in a given bundle is a manifold of positive dimension.

However, if the index of the problem (calculated in section 3.3) is positive, this result that a bundle E possessing a Hermitian-Yang-Mills connection has the property that every $Spin(7)$ instanton connection is Hermitian-Yang-Mills, together with the later results on the moduli space of $Spin(7)$ instantons tell us that this is indeed the case, under some mild assumptions.

3.3 The Linearisation of the Instanton Problem

If A is a connection satisfying the $Spin(7)$ instanton condition

$$\pi_7(F_A) = 0,$$

then nearby (i.e. in some open neighbourhood), we have that a connection $A + \tau$ satisfies the $Spin(7)$ instanton condition iff

$$\pi_7(F_{A+\tau}) = 0.$$

But

$$F_{A+\tau} = F_A + d_A\tau + \frac{1}{2}[\tau, \tau]$$

Thus $A + \tau$ is a $Spin(7)$ instanton iff

$$\pi_7(d_A\tau + \frac{1}{2}[\tau, \tau]) = 0.$$

Adding in the gauge-fixing condition $d_A^*\tau = 0$, we see that locally the moduli space of $Spin(7)$ instantons is

$$\Phi^{-1}((0,0))$$

where Φ is given by

$$\Phi(\tau) = (d_A^*\tau, \pi_7(d_A\tau + \frac{1}{2}[\tau, \tau])).$$

Now the linearisation of this operator is T, where

$$T(\tau) = (d_A^*\tau, \pi_7(d_A\tau)).$$

Now T is a first order differential operator,

$$T : C^\infty(T^*M \otimes E) \to C^\infty((\Lambda_7^2 \oplus \lambda^0) \otimes E).$$

By considering local equations for the differential operators d_A^* and d_A, we may see that the symbol for T is

$$\sigma_\xi = \sigma_\xi^0 \otimes Id_E,$$

where Id_E is the identity operator on E, and σ^0 is given by

$$\sigma_\xi^0(\lambda) = - < \xi, \lambda > \oplus \pi_7(\xi \wedge \lambda)$$

for ξ in the sphere bundle over x, and $\lambda \in \Lambda^1(M)_x$. [BB, p.375]

Note in particular that σ^0 is the symbol for the operator T_0 given by

$$T_0 : \Omega^1(M) \to \Omega^0(M) \oplus \Omega_7^2(M)$$

$$T_0(\lambda) = d^*\lambda \oplus \pi_7(d\lambda).$$

Thus provided we may find the index of the operator T_0 (showing it is elliptic on the way), then we may use index theory to calculate the index of T. (Of course, we shall have $\sigma_\xi = \sigma_\xi^0 \otimes Id_E$ is an isomorphism if σ_ξ^0 is.)

Now we have that the Spin bundles of the manifold Δ_+ and Δ_- satisfy isomorphisms

$$\Delta_+ \equiv \Lambda^0 \oplus \Lambda_7^2,$$

$$\Delta_- \equiv \Lambda^1.$$

Thus we have that

$$T_0 : \Delta_- \to \Delta_+$$

and by rescaling the factors of Λ_0 and Λ_7^2 we can assume that $T_0 = D_-$, the Dirac operator, and T is the twisted Dirac operator, that is the Dirac operator with values in the bundle E.

Thus $\text{ind}(T_0) = -\hat{A}(M)$.

By index theory, we have

$$\text{ind}(T) = -\langle \hat{A}(M)\text{ch}(E), [M] \rangle,$$

where $\text{ch}(E)$ is the Chern character of the bundle E. [Roe, p.145]

Now $\hat{A}(M)$ can be expressed as a polynomial in the Pontrjagin classes of M.

Using a subscript i to denote the component of an element of $\Sigma_k H^k(M)$ in $H^k(M)$ we have [Hirz, p.197]

$$\hat{A}_0 = 1$$

$$\hat{A}_4 = -\frac{1}{3.2^3} p_1(M)$$

$$\hat{A}_8 = \frac{1}{45.2^7}(7p_1(M)^2 - 4p_2(M))$$

Also

$$ch(E)_0 = dim(E)$$

$$ch(E)_4 = \frac{1}{2}(c_1(E)^2 - 2c_2(E))$$

$$ch(E)_8 = \frac{1}{24}(c_1(E)^4 - 4c_1(E)^2 c_2(E) + 2c_2(E)^2 + 4c_1(E)c_3(E) - 4c_4(E))$$

Thus $\text{ind}(T) = -\langle \frac{\dim(E)}{45.2^7}(-4p_2(M)+7p_1(M)^2) - \frac{1}{3.2^4} p_1(M)(c_1(E)^2 - 2c_2(E))$

$+\frac{1}{24}(c_1(E)^4 - 4c_1(E)^2 c_2(E) + 2c_2(E)^2 + 4c_1(E)c_3(E) - 4c_4(E)), [M] >$

Now suppose M is a manifold with holonomy $Spin(7)$. Then

$$\langle -4p_2(M) + 7p_1(M)^2, [M] \rangle = 45.2^7$$

Theorem 3.2

$$ind(T) = \underbrace{-dim(E)} - \langle -\frac{1}{3.2^4} p_1(M)(c_1(E)^2 - 2c_2(E))$$

$$+\frac{1}{24}(c_1(E)^4 - c_1(E)^2 c_2(E) + 2c_2(E)^2 + 4c_1(E)c_3(E) - 4c_4(E)), [M] \rangle,$$

and if in addition we have that E is an $SU(2)$ bundle, then $c_1(E) = 0$, and thus

$$ind(T) = -\underline{2} - \langle \frac{1}{3.2^3} p_1(M).c_2(E) + \frac{1}{12} c_2(E)^2, [M] \rangle.$$

Proof.
See above.

□

In the following section, we will see that this is the dimension of the moduli space, provided certain conditions, discussed in the next section, are satisfied.

3.4 An Analytic Result

In the previous section, we considered the linearised equation for instanton connections. In this section, we shall look at how this applies to the full equation.

If we have a connection A that is a $Spin(7)$ instanton i.e. $\pi_7(F_A) = 0$, the 7 dimensional component of the curvature of A vanishes.

Now we will study the moduli space of $Spin(7)$ instanton connections near A, with equivalence classes produced by gauge fixing. The gauge fixing condition ensures that the resulting manifold is finite dimensional.

Thus we shall consider the following submanifold of $A + C^\infty(\Lambda^1 \otimes E)$, the affine space of connection 1-forms

$$M_A = \{A + \tau : \tau \in C^\infty(\Lambda^1 \otimes E), d_A^* \tau = 0, \pi_7(F_{A+\tau}) = 0\}$$

The condition $\pi_7(F_{A+\tau}) = 0$ is, of course, the condition that the resulting connection is also a $Spin(7)$ instanton, whilst the condition $d_A^*\tau = 0$ is the gauge fixing one.

Note that for the gauge fixing condition to be 1-1 (i.e. that each element of M_A correspond to exactly one family of gauge equivalent $Spin(7)$ instanton connections), we must impose a condition on A.

That is, if the bundle E is a G-vector bundle, where G is a Lie group, and GA is the group of gauge transformations of E, then those elements of GA fixing A

$$I(A) = \{g \in GA : g.A = A\}$$

consists of only those elements of GA in the centre of G. If this condition holds, then we say that A is weakly irreducible [BB, p.374]. From this point on, we shall assume this to be the case.

Now

$$F_{A+\tau} = F_A + d_A\tau + \frac{1}{2}[\tau, \tau]$$

and thus

$$\pi_7(F_{A+\tau}) = \pi_7(d_A\tau) + \pi_7(\frac{1}{2}[\tau, \tau]).$$

Thus $M_A = \{\tau : \tau \in C^\infty(\Lambda^1 \otimes E), d_A^*\tau = \pi_7(d_A\tau + \frac{1}{2}[\tau,\tau]) = 0\}$.

As we have seen, the linearisation of the operator Φ given by $\Phi(\tau) = d_A^*\tau \oplus \pi_7(d_A\tau + \frac{1}{2}[\tau,\tau])$ is

$$T : C^\infty(\Lambda^1 \otimes E) \to C^\infty((\Lambda^0 \oplus \Lambda^2_7) \otimes E)$$

$$T(\tau) = d_A^*\tau + \pi_7 d_A \tau$$

Now $M_A = \Phi^{-1}(0, 0)$.

We will look at M_A, but first we must quote some analytic theorems.

Lemma 3.1 *Suppose $Q : E_1 \times E_2 \to E_3$ is a smooth bilinear map of Hermitian vector bundles over M. Then Q induces a bounded bilinear map*

$$Q' : L^p_k(E_1) \times L^p_j(E_2) \to L^p_j(E_3)$$

for $k > \frac{n}{p}, k \geq j$. [BB, p.381]

Lemma 3.2 *Suppose $D : C^\infty(E) \to C^\infty(F)$ is a linear elliptic operator of order m, with $(D^*)_{p,k+m}$ the Sobolev extension of D^*, the adjoint of D.*

Then $L^p_k(E) = Ker D \oplus (D^*)_{p,k+m}(L^p_{k+m}(F))$ is the direct sum of closed subspaces. [BB, p.381]

Theorem 3.3 *Implicit Function Theorem [BB, p.381]: Suppose $F : E_1 \to E_2$ is a C^k map ($1 \leq k \leq \infty$) between Banach spaces and assume*

i) $dF : (TB_1)_x \to (TB_2)_{F(x)}$ is a surjection,

ii) $(TB_1)_x = Ker(dF) \oplus H$, where H is closed.

Then $F^{-1}(F(x))$ is a C^k submanifold of B_1 in a neighbourhood of x, with $Ker(dF)$ as tangent space at x.

This allows us to state and prove the main theorem of the section.

Theorem 3.4 *Suppose A satisfies $\pi_7(F_A) = 0$, and is such that $Ker(T^*) = 0$, where T is as given above. Then for $1 \leq p < \infty$, $0 < k < \infty$, with $p(k+1) > 8$, then M_A is a finite dimensional submanifold of $L^p_{k+1}(E \otimes \Lambda^1(M))$ in a neighbourhood U of 0, with dimension as given in Section 3.2.*

Proof.

Now for $\tau \in L^p_{k+1}(E \otimes \Lambda^1(M))$, then we have $[\tau, \tau] \in L^p_{k+1}(E \otimes \Lambda^2(M))$ by the above Lemma 3.1 , and thus $\tau \to [\tau, \tau]$ defines a bounded bilinear map to $L^p_k(E \otimes \Lambda^2(M))$.

Now recalling that

$$\Phi(\tau) = d_A^* \tau \oplus \pi_7(d_A \tau + \frac{1}{2}[\tau, \tau]),$$

we have a continuous Sobolev extension

$$\Phi_{p,k+1} : L^p_{k+1}(E \otimes \Lambda^1(M)) \to L^p_k(E \otimes (\Lambda^0(M) \oplus \Lambda^2_7(M)))$$

Note that $\Phi_{p,k+1}$ is C^∞ since it is the sum of $(d_A^*)_{p,k+1}$, $\pi_7(d_A)_{p,k+1}$, and $\frac{1}{2}\pi_7[\tau, \tau]$.

The differential of $\Phi_{p,k+1}$ at 0 is the linear part of $\Phi_{p,k+1}$, namely the Sobolev extension of T, an elliptic operator of order 1.

Thus by Lemma 3.2, we have that $L^p_{k+1}(\Lambda^1 \otimes E) = KerT \oplus T^*(L^p_{k+2}(E \otimes (\Lambda^2_7 \oplus \Lambda^0)))$ is the direct sum of closed subspaces. By another use of Lemma 2, with $D = T^*$ we have that $T_{p,k+1}$ is surjective.

Thus we may apply the Implicit Function Theorem to $\Phi_{p,k+1}$ to get that

$$M_A = (\Phi_{p,k+1})^{-1}(0,0)$$

is a smooth submanifold of $L^p_{k+1}(E \otimes \Lambda^1)$ in a neighbourhood of 0.

To see that $\tau \in M_A \Rightarrow \tau$ is C^∞, we must make use of some of the theorems stated in Chapter 2.

Now

$$\begin{aligned} ||\tau||_{L^p_{k+2}} &\leq C(||T_{p,k+1}\tau||_{L^p_{k+1}} + ||\tau||_{L^p_{k+1}}) \\ &= C(||\frac{1}{2}[\tau,\tau]||_{L^p_{k+1}} + ||\tau||_{L^p_{k+1}}) \\ &\leq C(C'||\tau||^2_{L^p_{k+1}} + ||\tau||_{L^p_{k+1}}) \end{aligned}$$

and thus $\tau \in L^p_{k+2}$ and by induction

$$\tau \in L^p_{k+j}\ \forall j > 0$$

Hence $\tau \in C^\infty$, using the Sobolev Embedding Theorem.

Also note that since the map Ψ defined by $\Psi(\tau) = T_A$ is continuous, where T_A is the map T given above, and the property of being surjective is an open one, we have that in a sufficiently small neighbourhood U of A, we have that each $A + \tau \in U$ also has the property that

$$Ker(T_{A+\tau})^* = Ker((d^*_{A+\tau} \oplus \pi_7(d_{A+\tau}))^*) = 0.$$

Thus we have that in a neighbourhood of any $Spin(7)$ instanton connection A, with $\mathrm{Ker}(T_A)^* = 0$, then the moduli space near A is a smooth manifold of the expected dimension.

□

We may note that the assumptions made (i.e. T^* is suurjective and that A is weakly irreducible) certainly seem likely to hold at generic points of the moduli space for which the index is positive. This would tell us that, as is the case for self-dual and anti-self-dual instantons, the moduli space is a smooth manifold of the expected dimension at generic points.

Chapter 4

Finite Energy Instantons on $\mathbf{R}^8$

Amongst the most well known constructions in differential geometry is the ADHM construction of an instanton on $\mathbf{R}^4$. [ADHM] This, together with Taubes' gluing constuction, [Tau] allows us to construct instantons centred around a point of a 4-manifold.

Thus in attempting to construct $Spin(7)$ instantons on compact 8-manifolds with holonomy $Spin(7)$, our first attempt may well be an imitation, in some way, of these two constructions. However, in this chapter we shall see that such a method is doomed to failure at the first hurdle; before even a consideration of the analysis involved in the gluing, we may see that there is, in fact, no finite energy $Spin(7)$ instanton on $\mathbf{R}^8$, in contrast to the 4 dimensional case.

Let us, before embarking upon the proof of this result, consider some of the more relevant properties of the well-known ADHM instanton.[DK, p.116]

1.
$$A = \frac{1}{1+|x|^2}(\theta_1\mathbf{i} + \theta_2\mathbf{j} + \theta_3\mathbf{k})$$

where

$$\begin{aligned}
\theta_1 &= x_1dx_2 - x_2dx_1 - x_3dx_4 + x_4dx_3 \\
\theta_2 &= x_1dx_3 - x_3dx_1 - x_4dx_2 + x_2dx_4 \\
\theta_3 &= x_1dx_4 - x_4dx_1 - x_2dx_3 + x_3dx_2,
\end{aligned}$$

and $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are considered as the basis for $SU(2)$.

2.
$$F = \left(\frac{1}{1+|x|^2}\right)^2(d\theta_1\mathbf{i} + \mathrm{d}\theta_2\mathbf{j} + \mathrm{d}\theta_3\mathbf{k}).$$

3. The instanton is centred at the origin.

4. $|F|$ dies away like x^{-4}, i.e.

$$|F| = O(x^{-4}).$$

Note that this last fact tells us that the ADHM instanton is of finite energy, i.e. $\int |F|^2 < \infty$. It is also worth noting that by using the dilation $x \to \lambda x$ we may "rescale" the instanton, obtaining an instanton of different concentration, but of the same energy.

This instanton may be more succinctly described in quaternionic notation. For if we use the notation:

$$x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k},$$

then we may write

$$A = \frac{1}{1+|x|^2} Im(\bar{x}dx)$$

and

$$F = \frac{1}{(1+|x|^2)^2} \bar{x} \wedge x.$$

Note that the use of $\mathbf{H}$ to define an instanton on $\mathbf{R}^4$ suggests that we should consider the use of the octonions, $\mathbf{O}$, to construct a similar instanton on $\mathbf{R}^8$. However, an attempt of

$$A = f(|x|^2)\pi(\bar{x}dx),$$

with f a function and π a projection gives several major problems. Firstly, for the resulting instanton to be of finite energy, the asymptotic decay of A must be greater than $o(|x|^{-2})$; secondly, unlike the case of $\mathbf{H}$, there is no natural projection of $\mathbf{O}$ onto a 3 dimensional subspace, and hence whilst we may well be able to use an approximation of this method for construction of spherically symmetric instantons with other gauge groups, its seems unlikely that this method will aid us in the $SU(2)$ case; and thirdly, and perhaps most importantly, whilst the imaginary quaternions form a Lie algebra, the same cannot be said of the imaginary octonions. (The first part of this suggestion is motivated by the fact that if we write:

$$x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + \ldots + x_8\mathbf{ke}$$

then the coefficients of the standard basis of the imaginary octonions, in fact, give a basis for Λ^2_7, the seven dimensional $Spin(7)$ invariant subspace of the two-forms.)

Let us now go on to prove that there are no non-trivial finite energy $Spin(7)$ instantons with gauge group $SU(2)$ on $\mathbf{R}^8$. The method of proof I follow is splitting the result into three lemmas: the first obtains an asymptotic estimate of the curvature, the second uses this estimate and converts this to an estimate on the norm of the connection in a certain gauge, and finally, we use these bounds to show that any such connection must be flat.

Before embarking on proving this lemma, I will quote a theorem of Uhlenbeck that will be useful in completing the proof.

Theorem 4.1 *[U2, p.8]*

Let $D^*F = Q$, with $\max_{x\in M}|Q(x)| \leq Q_\infty$. Then there exists $\epsilon_0 > 0$ such that if $4\rho < \sigma_0, Q_\infty\rho < \epsilon \leq \epsilon_0$ and $f_2(\xi, 4\rho) < \epsilon \leq \epsilon_0$, then

$$\max_{x\in B_\rho(\xi)} \rho^2|F(x)| \leq K_n\epsilon.$$

Note that $D^*F = 0$ corresponds to the full Yang-Mills equation, so in our case $Q = 0$. σ_0 is equivalent to an injectivity radius: it corresponds to the radius of ball within which we can accurately estimate the metric using the Euclidean metric. f_2 is the L^2-norm of F made scale invariant.

Lemma 4.1 *Let $E \to \mathbf{R}^8$ be an $SU(2)$ bundle with Spin(7) instanton connection A, such that A has finite energy, i.e. $\int_{\mathbf{R}^8}|F_A|^2d\mu < \infty$.*

Then $|F_A(x)| = o(|x|^{-4})$

Proof. There exists an R such that $\int_{\mathbf{R}^8-\mathrm{B_R}(0)}|F_A|^2 < \epsilon_0$, where ϵ_0 is given as in [U2].

Now for x such that $|x| > \max(2R, 10)$ we have

$$\int_{B_{\frac{|x|}{2}}(x)}|F_A|^2 < \epsilon_0$$

since $B_{\frac{|x|}{2}}(x) \subset \mathbf{R}^8 - \mathrm{B_R}(0)$.

Thus we may apply Theorem 4.1 with $\xi = x$, $\rho = \frac{|x|}{8}$ and

$$\epsilon = 2(\int_{B_{\frac{|x|}{2}}}|F_A|^2)^{\frac{1}{2}}(4\rho)^{-2} = 2(\int_{B_{\frac{|x|}{2}}}|F_A|^2)^{\frac{1}{2}}(\frac{|x|}{2})^{-2}$$

to obtain that there exists a K such that

$$\max_{y\in B_{\frac{|x|}{8}}(x)}\frac{|x|^2}{64}|F(y)| \leq K.2.(\int_{B_{\frac{|x|}{2}}(x)}|F_A|^2)^{\frac{1}{2}}(\frac{|x|}{2})^{-2}$$

i.e. $\max_{y\in B_{\frac{|x|}{8}}(x)}|F(y)| \leq 512K(\int_{B_{\frac{|x|}{2}}(x)}|F_A|^2)^{\frac{1}{2}}|x|^{-4}$.

Now as $x \to \infty$,

$$(\int_{B_{\frac{|x|}{2}}(x)}|F_A|^2)^{\frac{1}{2}} \to 0.$$

since

$B_{\frac{|x|}{2}}(x) \subset \mathbf{R}^8 - \mathrm{B}_{\frac{|x|}{2}}(0)$.

Thus, in particular $|F(x)| = o(|x|^{-4})$.

□

Lemma 4.2 *With the same assumption as in Lemma 4.1, there exists a gauge such that $|A| = o(|x|^{-3})$, for $\mathbf{R}^8$ outside a certain ball.*

Proof. Consider the series of annuli

$$U_n = \{x : 2^n \leq |x| \leq 2^{n+1}\}$$

For an L^∞ bound on the curvature in the rescaled problem in the annulus U_0 , we require a bound on $|F||x|^2$ in the original problem. Since $|F||x|^2 \to 0$, there exists an N such that for all $n \geq N$, $|F||x|^2 \leq \gamma'$ on U_n , where γ' is as given in [U1, p.26].

Thus in one of these annuli, we may rescale the problem, by

$$x \to x/2^n,$$

taking U_n to U_0, with $F|_{U_n}$ going to a connection on U_0, and thus obtain a bound $||A||_\infty \leq K'||F||_\infty$, on the rescaled problem, which translates into the inequality $|A|_{max} \leq K'|x||F|_{max}$. These gauges fit together, as they are unique up to multiplication by constant elements of G.

□

Lemma 4.3 *If the gauge extends to all of $\mathbf{R}^8$, then A is flat.*

Proof. Consider a ball of radius r. Then from Chern Simons theory we have:

$$\int_{B_r(0)} Tr(F_A \wedge F_A) \wedge \Omega = \frac{1}{8\pi^2}\int_{S_r(0)} Tr(dA \wedge A + \frac{2}{3}A \wedge A \wedge A) \wedge \Omega$$

Also, for a Spin(7) instanton, we have $Tr(F_A \wedge F_A) \wedge \Omega = *|F_A|^2$, since $F_A \in \Lambda^2_{21}$.

Thus $\int_{B_r(0)}|F_A|^2 = \frac{1}{8\pi^2}\int_{S_r(0)} Tr(dA \wedge A + \frac{2}{3}A \wedge A \wedge A) \wedge \Omega$

But $A \wedge A \wedge A = o(|x|^{-9})$ (by Lemma 4.2), and $F_A \wedge A = o(|x|^{-7})$ (by Lemma 4.1 and Lemma 4.2), giving $dA \wedge A = o(|x|^{-7})$, and thus $Tr(dA \wedge A + \frac{2}{3}A \wedge A \wedge A) \wedge \Omega = o(|x|^{-7})$, and so $\frac{1}{8\pi^2}\int_{S_r(0)} Tr(dA \wedge A + \frac{2}{3}A \wedge A \wedge A) \wedge \Omega \to 0$ as $r \to \infty$. But $\int_{B_r(0)} |F_A|^2$ is an increasing, non negative function on r. Thus $\int_{B_r(0)} |F_A|^2 = 0$, and A is flat.

□

Theorem 4.2 *There is no non-flat, finite energy connection on* $\mathbf{R}^8$ *with curvature contained in the* Λ^2_7 *part of the* $Spin(7)$ *invariant splitting.*

Proof. Since we may choose a gauge for a connection outside a ball of fixed radius with the properties required above (i.e $|A| = o(|x|^{-3})$, and also one for the interior of the ball, we may consider how these two gauge choices patch together. Note that this will be determined by the transition function from the shared boundary of these two spaces (i.e S^7) to the gauge group, $SU(2) \equiv S^3$.

The fact that $\pi_7(S^3) = Z_2$ [Toda, p.186], gives us that any map from the seven sphere in $\mathbf{R}^8$ into the structure group $SU(2)$ are either homotopically trivial, or torsion of order two. The Lemma 4.3 allows us to see that the trivial case will give no instantons. Suppose that the second case (i.e non-trivial transition function from $S^7 \to S^3$) produced an instanton of finite energy, say E. Then consider the sum of two such instantons. This would have energy $2E$, and thus could not correspond to either of the classes of transition functions in $\pi_7(S^3)$.

Thus there are no finite energy $Spin(7)$ instantons on $\mathbf{R}^8$.

□

Note that despite this fact, we do have that there are spherically symmetric instantons on $\mathbf{R}^8$. They are to be found described in the papers [FN]. They are, however, not instantons with gauge group $SU(2)$, but rather with $Spin(7)$. I do not know whether there exist any non-trivial spherically symmetric solutions to the instanton equation with gauge group $SU(2)$ at present, although as the aforementioned example shows, any proof of the non-existence of such an object would depend on the properties of the Lie group $SU(2)$.

In any case, these results above do suggest that in looking for a method of constucting an instanton on a $Spin(7)$ manifold, that we will not find much hope in looking to construct one based around a point.

Chapter 5

On the Limit of a sequence of Instantons

A natural question to ask regarding a family of instantons of a particular type is concerning the 'bubbling off' phenomenon.

The limiting behaviour of a family of self-dual, or anti self dual, Yang-Mills instantons is well known (at least amongst differential geometers!) [DK, p.117] on four dimensional manifolds. The keys features of a series of instantons on a bundle $E \to M$ are the existence of a subsequence with the following properties:

• Convergence outside a finite set of points, after a suitable sequence of gauge transformations.
• Within this set, convergence to a connection on a different bundle $E' \to M$.
• The behaviour near one of the finite set of points is approximately the same as the behaviour of an ADHM instanton in $\mathbf{R}^4$ that is allowed to become more concentrated.

Such limiting behaviour is known colloquially as 'bubbling' around the finite set of points.

And so we may well consider whether there are analagous properties of families of $Spin(7)$ instantons on 8-manifolds. This is the question I hope to answer, at least partially, in this chapter.

It is perhaps worth noticing at this point that if our compact 8-manifold with holonomy contained in $Spin(7)$, M, can be written as the product of two 4-manifolds, $M = N_1 \times N_2$, with metric and $Spin(7)$ structure splitting compatibly with this, then a self dual instanton on N_1, constant in the N_2 direction, is, in fact, a $Spin(7)$ instanton.

This, via naive reasoning, may well lead us to conjecture that if bubbling occurs on 8 manifolds, that it will do so around a submanifold of dimension 4.

However, the construction of the instanton on $\mathbf{R}^8$ with gauge group $Spin(7)$ in the last chapter suggests that the behaviour may not always be as simple as this.

Naive reasoning, though, does not lead us as far astray as we may believe. Indeed, we may prove that bubbling does occur around a set of finite Hausdorff 4-measure, provided we assume a result given in a paper by Nakajima. [Nak, p.389]; however, I believe the proof in this paper to be slightly flawed, though I do not think this affects the validity of the result.

Theorem 5.1 *Let $n \geq 4$, and let $\{D(j)\}$ be a sequence of Yang-Mills connections which energy bounded by $R < \infty$. Then there exists a subsequence $\{j\} \subset \{i\}$, a compact subset S with finite $(n-4)$-dimensional Hausdorff measure $H_{n-4}(S) < \infty$, a G-principal bundle Q over $M - S$, and a Yang-Mills connection $D(\infty)$ on Q such that for each compact subset $K \subset M - S$ there are bundle equivalences*

$$g_K(j) : P|K \to Q|K$$

so that

$$g_K(j)^*(D(j)) \to D(\infty)$$

in C^∞-topology on K.

From this it is easy to deduce the following result, as all $Spin(7)$ instantons are Yang-Mills connections.

Theorem 5.2 *Let $\{A_i\}$ be a sequence of $Spin(7)$ instantons on a bundle $E \to M$. Then there exists a subsequence $\{A_{j_i}\}$, a closed set of finite Hausdorff 4-measure S, a bundle $E' \to M - S$, and, for each closed subset $K \subset M - S$, a sequence of bundle equivalences $\{g_{j_i}\}$ such that*

$$g_{j_i}(A_{j_i}|_K) \to A'$$

where A' is a limit connection on $E'|_K$.

Thus we see that indeed the property observed of convergence outside a set of Hausdorff dimension $n-4$ carries over from 4 dimensional manifolds to 8-dimensional ones. The first major difference to note between the two cases is that unlike the 4 dimensional case, in which we have that the singular set is a finite number of points, we know very little about the singular set. Another major difference is that we do not a priori have that the bundle over $M - S$ in which the limit connection lies can be extended, together with an extension of the connection, to a bundle over M. As for the final analogy, the behaviour of the instanton near the singular set, I will consider this in later chapters.

However, now let us presume that the set of Hausdorff dimension four around which the family bubbles is indeed a submanifold of dimension 4, N, and let us further assume that the bundle E' over $M - N$ may be extended to a bundle E over the whole of M together with a limit connection. (Note that, in four dimensions, this corresponds to the removal of singularities, an operation described in [U1]). Now we may now ask about any special properties held by this new submanifold.

This manifold in fact holds the very special property of being Cayley. To show this I will use two lemmas, whose purpose I hope will be self-explanatory.

Lemma 5.1 *Suppose E, E' are $SU(2)$ vector bundles on an 8-manifold M, such that there exists a submanifold N of dimension 4, with a bundle equivalence between $E|_{M-N}$ and $E'|_{M-N}$. Then $c_2(E) - c_2(E')$ is Poincare dual to some multiple of $[N]$.*

Proof. For any neighbourhood of N, call it T_ϵ, we may choose connections A and A' on E and E' respectively, such that $A|_{M-T_\epsilon}$ and $A'|_{M-T_\epsilon}$ are gauge equivalent.

Thus $F_A = F_{A'}$ outside T_ϵ. Now

$$\begin{aligned} c_2(E) &= \frac{1}{8\pi^2} Tr(F_A \wedge F_A), \\ c_2(E') &= \frac{1}{8\pi^2} Tr(F_{A'} \wedge F_{A'}) \end{aligned}$$

agreeing on $M - T_\epsilon$, and hence $c_2(E) - c_(E')$ can be supported in T_ϵ.

Noting that $H^4_{cs}(T_\epsilon)$ is generated by the Poincare dual of $[N]$, provided the neighbourhood is small enough, we have that $c_2(E)$ is Poincare dual to $k[N]$, for some $k \in \mathbf{R}$.

□

Suppose, now, that we have a sequence of connections A_i on E with constant energy, and that there is a bundle E' on M, and a submanifold N of dimension 4, such that a subsequence of these connections tends to a connection A on E' on the set $M - N$. (i.e. A is a connection on $E' \to M$, and $A_i \to A$ on the set $M - N$.) Let us now consider a splitting $TM = V_1 \oplus V_2$ near N with $V_1 = TN$ at N, and V_2 the orthogonal complement of V_1. With this, let us choose a closed 4-form η on M which approximates the volume form of V_1, and is equal to it on N.

Lemma 5.2 *$\int_N \eta \leq \int_N \Omega$, where Ω is the $Spin(7)$ structure.*

Proof. Since both η and $\mathrm{vol}(V_1)$ are continuous, and they are equal at N, for all $\epsilon > 0$, we may choose a tubular neighbourhood of N such that

$$|\eta - \mathrm{vol}(V_1)| \leq \frac{\epsilon}{6 \int_M |F_{A_i}|^2}$$

We may also restrict this neighbourhood further, if necessary, to ensure that it has volume less than $\epsilon / \max_M |F_A|^2$. Call this neighbourhood T_ϵ.

Now since outside T_ϵ, $A_j \to A$ as $j \to \infty$, there exists an i such that

$$\begin{aligned} &|Tr(F_{A_i} \wedge F_{A_i}) - Tr(F_A \wedge F_A)| \\ &\leq \frac{\epsilon}{6\mathrm{vol}(M) \max_M(|\Omega|, |\eta|)} \end{aligned}$$

outside T_ϵ.

Now $c_2(E) - c_2(E')$ is dual to $k[N]$ for some constant k, from Lemma 5.1, and furthermore, $k > 0$.

Thus

$$k\mathrm{vol}N = \int_N k\eta = \int_M (c_2(E) - c_2(E')) \wedge \eta = \int_M (Tr(F_{A_i} \wedge F_{A_i}) - Tr(F_A \wedge F_A)) \wedge \eta.$$

We have

$$\begin{aligned} k\mathrm{vol}(N) &= \int_{T_\epsilon} (Tr(F_{A_i} \wedge F_{A_i}) - Tr(F_A \wedge F_A)) \wedge \eta \\ &+ \int_{M - T_\epsilon} (Tr(F_{A_i} \wedge F_{A_i}) - Tr(F_A \wedge F_A)) \wedge \eta \\ &\leq \int_{T_\epsilon} (Tr(F_{A_i} \wedge F_{A_i}) - Tr(F_A \wedge F_A)) \wedge \eta + \frac{\epsilon}{6} \\ &\leq \int_{T_\epsilon} Tr(F_{A_i} \wedge F_{A_i}) \wedge \eta + \frac{\epsilon}{3}. \end{aligned}$$

Now near the 4-manifold N,

$$Tr(F_{A_i} \wedge F_{A_i}) \wedge \mathrm{vol}(V_1) = Tr(F_{A_i}|_{V_2} \wedge F_{A_i}|_{V_2}) \wedge \mathrm{vol}(V_1) = |F_{A_i}|^+_{V_2}|^2 - |F_{A_i}|^-_{V_2}|^2$$

where the $+$ and $-$ indices indicate the self dual and anti-self dual parts respectively.

Now $|\dot{\eta} - \mathrm{vol}(V_1)| \leq \epsilon/6 \int_M |F_{A_i}|^2$ in T_ϵ, and so

$$\int_{T_\epsilon} Tr(F_{A_i} \wedge F_{A_i}) \wedge \eta \leq \int_{T_\epsilon} (|F_{A_i}|_\nu^+|^2 - |F_{A_i}|_\nu^-|^2) + \frac{\epsilon}{6} \leq \int_{T_\epsilon} |F_{A_i}|^2 + \frac{\epsilon}{6}.$$

But $\int_{T_\epsilon} |F_{A_i}|^2 = \int_{T_\epsilon} Tr(F_{A_i} \wedge F_{A_i}) \wedge \Omega$, since F_{A_i} is a Spin(7) instanton. We may argue as before to obtain that

$$\int_{T_\epsilon} Tr(F_{A_i} \wedge F_{A_i}) \wedge \Omega \leq \int_M (Tr(F_{A_i} \wedge F_{A_i}) - Tr(F_A \wedge F_A)) \wedge \Omega + \frac{\epsilon}{3}.$$

But $\int_M (Tr(F_{A_i} \wedge F_{A_i}) - Tr(F_A \wedge F_A)) \wedge \Omega = \int_N k\Omega$
Thus for all $\epsilon > 0$, we have

$$\int_N k\eta \leq \int_N k\Omega + \epsilon.$$

Dividing through by the positive constant k we get

$$\int_N \eta \leq \int_N \Omega.$$

□

Theorem 5.3 *N is a Cayley submanifold of M.*

Proof. From Lemma 5.2 above, we have that

$$\int_N \eta \leq \int_N \Omega.$$

Also, from the definition of a calibration, we have

$$\eta \geq \Omega,$$

on the submanifold N. Hence these two four forms are equal on N, which is hence a Cayley submanifold.

□

Thus we have that, as far as the "bubbling" phenomenon is concerned, that Cayley submanifolds play a very similar role to points in the 4 dimensional case. This information will be put to use in the following chapters.

Chapter 6

Geometrical Description of the Instanton Construction

For the final chapters, I will consider a problem I have referred to earlier in this thesis: that of the construction of a finite energy $Spin(7)$ instanton on a compact manifold with holonomy $Spin(7)$.

Note that what I shall present will not be, unfortunately, a general method for the construction of $Spin(7)$ instantons on any manifold of holonomy $Spin(7)$, but rather a specific example of such an instanton on a particular $Spin(7)$ manifold.

The structure of this construction will follow, to some extent, the method of Taubes' gluing of an anti-self dual instanton around a point of 4-manifolds [Tau]. That is, I shall start with an approximately $Spin(7)$ instanton, and shall then make successively smaller (in some norm) corrections to this, to ensure the sequence converges to a true $Spin(7)$ instanton. I believe that these two steps form a natural division of an otherwise long result, and so have divided the construction along these lines. This chapter, containing mostly geometric ideas, will terminate once the method of construction of the approximate instanton has been spelled-out. The next two chapters, of a more analytic flavour, will then deal with the convergence of the iterative sequence.

Perhaps a natural suggestion to take from the previous chapters is that rather than attempt to construct an instanton centred at a point, it may be far easier to "glue" the instanton in around a Cayley submanifold. Thus we are presented with the first two ingredients in our recipe: a compact $Spin(7)$ manifold, together with a Cayley submanifold within it.

6.1 The Manifold

The manifold, M, upon which the construction will take place will be that described in [J2, p.524]. It is constructed as the resolution of the orbifold obtained from T^8 acted on by the 16 element group $\langle\alpha,\beta,\gamma,\delta\rangle$, where

$$\begin{aligned}
\alpha(x_1,\ldots,x_8) &= (-x_1,-x_2,-x_3,-x_4,x_5,x_6,x_7,x_8),\\
\beta(x_1,\ldots,x_8) &= (x_1,x_2,x_3,x_4,-x_5,-x_6,-x_7,-x_8),\\
\gamma(x_1,\ldots,x_8) &= (\frac{1}{2}-x_1,\frac{1}{2}-x_2,x_3,x_4,\frac{1}{2}-x_5,\frac{1}{2}-x_6,x_7,x_8),\\
\delta(x_1,\ldots,x_8) &= (-x_1,x_2,\frac{1}{2}-x_3,x_4,\frac{1}{2}-x_5,x_6,\frac{1}{2}-x_7,x_8).
\end{aligned}$$

It will aid us somewhat to be able to think of the manifold as occurring from two separate resolutions: firstly, from resolving the singularities of the orbifold $T^8/\langle\alpha,\beta\rangle$ to form the manifold $K3\times K3$, and then resolving $(K3\times K3)/\langle\gamma,\delta\rangle$ to obtain M.

Let us now consider the Cayley submanifold of this $Spin(7)$ manifold. First we should consider the involution defined by:

$$\epsilon(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)=(\frac{1}{2}-x_1,-x_2,-x_3,\frac{1}{2}-x_4,x_5,x_6,x_7,x_8).$$

We may see that ϵ preserves the 4-form Ω defining the $Spin(7)$ structure, and commutes with the group $\langle\alpha,\beta,\gamma,\delta\rangle$. In addition, we have that the fixed points of ϵ, being

$$(a_1,a_2,a_3,a_4)\times T^4,$$

where $a_1,a_4\in\{-\frac{1}{4},+\frac{1}{4}\}$ and $a_2,a_3\in\{0,\frac{1}{2}\}$, are disjoint from the fixed points of $\langle\alpha,\gamma,\delta\rangle$.

Both resolutions (i.e. that with respect to the group $\langle\alpha,\beta\rangle$ and that with respect to the group $\langle\gamma,\delta\rangle$) may be done in an ϵ invariant way, and hence we may ensure that our final manifold M possesses a family of $Spin(7)$ structures that are invariant under ϵ.

(Note that here I am guilty of a mild abuse of notation, by denoting both the isometry of T^8 and the resulting isometry of the final manifold M by ϵ. This should cause no real confusion.)

On M, the fixed points of ϵ will be two copies of $K3$. These will be, in fact, Cayley submanifold of M. [J1, p.368].

Thus we are armed with the manifold, and a Cayley submanifold, around which we plan to 'glue' the instanton. Thus let us now consider the bundle containing the instanton connection.

Considering ϵ acting on the intermediate manifold $K3\times K3$, we may see that it has fixed points $\{p_1,p_2,\ldots,p_8\}\times K3$, where $p_1,\ldots,p_8$ are the points of $K3$ corresponding to those points listed above.

6.2 The Instanton

Restricting our attention, for the moment, to the first $K3$ in the product, we may glue in an $SU(2)$ instanton, A, on a bundle $E \to K3$ such that the curvature of the instanton is concentrated near these 8 points. The method for this follows from Taubes' method; we shall, around each of the points, have two regions: a ball of small, but fixed radius (e.g. $r = 1/100$ will do), together with an annulus around this (say, $1/200 < r < 1/100$). In the inner region, we have that the connection is identical to a standard ADHM instanton, with small characteristic radius, s. In the annulus, we use 'gluing' data, to ensure that the instanton decays to zero on the exterior of the annulus. (See figure 1). The gluing data and the size of these two regions will remain fixed throughout this construction. We can also require that the instanton is $\langle\gamma, \delta\rangle$ invariant.

Now we may pull the bundle, together with the connection, back to the 8-manifold $K3 \times K3$, to obtain an $SU(2)$ bundle, with a $Spin(7)$ connection on it, over the manifold. It will be almost a $Spin(7)$ instanton, because it is almost a 4-instanton in the direction of the first $K3$, and flat in the second $K3$ direction. The $Spin(7)$ structure on this manifold comes from the inclusion $SU(2) \times SU(2) \subset Spin(7)$. [J2, p.548]

Since everything has been done invariantly with respect to the group (J, $\delta\rangle$, we have that there is a corresponding $SU(2)$ bundle $F \to (K3 \times K3)/\langle\gamma, \delta\rangle$, together with a connection, A, with small Λ^2_7 component of curvature.

Resolving the singularities of this orbifold, with a resolving map $\pi : M \to (K3 \times K3)/\langle\gamma, \delta\rangle$, we may pull the $SU(2)$ bundle F back to another $SU(2)$ bundle F' over M, together with a connection $A' = \pi^*(A)$. Note that if we have ensured that E was trivial over the fixed points of $\langle\gamma, \delta\rangle$, then we have that F is a well-defined $SU(2)$-bundle which is also trivial over the fixed points of $\langle\gamma, \delta\rangle$, from which we may go on to deduce that the pull back bundle F' is trivial near the resolutions of the singular points of $(K3 \times K3)/\langle\gamma, \delta\rangle$.

We have now introduced a second sort of error into the connection A'; as well as the original error from the gluing data, we also have that A' deviates from being a $Spin(7)$ metric due to the differences the product $Spin(7)$ structure on $(K3 \times K3)/\langle\gamma, \delta\rangle$, from the family of almost $Spin(7)$ structures introduced to resolve the singularities.

To our advantage, though, is the fact that the largest differences in the metrics occur near the singularities of the orbifold $(K3 \times K3)/\langle\gamma, \delta\rangle$, which we have been careful to ensure occur away from the concentration of the curvature of the almost instanton.

With this geometric recipe for the construction of instantons, we shall proceed to the next chapter to obtain some analytic results.

Chapter 7

Analytic Estimates

Let us now attempt to make precise analytically the construction described in the last chapter. We have, in fact, two independent small parameters to consider: s, the characteristic radius of the instanton glued in, and t, the parameter describing the family of almost *Spin(7)* structures on the manifold M. For the sake of clarity, I should point out that for almost all of the remainder of this thesis, we will choose, in our construction, to make these quantities equal. It will be made clear when we do this. Each of these will contribute to $\pi_7(F_A)$, the error of A from being a true $Spin(7)$ instanton.

Let us consider the error profile near a Cayley submanifold. This is displayed in figure 2 .

Note that in the diagram B and C are the regions independent of s, t, with B representing the smaller of the two regions, that is, a 4-ball of small radius around the Cayley 4-manifold, and C representing the annulus surrounding this. The error in area C will be of the order s^2. This is due to the gluing data, and the error in the instanton on $(K3 \times K3)/\langle\gamma,\delta\rangle$ to be a $Spin(7)$ instanton.

The error in area B is not present in the connection on $(K3 \times K3)/\langle\gamma,\delta\rangle$, and is the result of the difference in the $Spin(7)$ structure on the final manifold and that on $(K3 \times K3)/\langle\gamma,\delta\rangle$. Hence it is of the order of $|F_A|.|\Omega_t - \Omega|$.

Lemma 7.1 $\|\pi_7(F_A)\|_{L^q} \leq O(\max(\mathrm{s}^2, \mathrm{s}^{4/q-2}\mathrm{t}^4))$.

Proof.

First note that for each q,

$$\|\pi_7(F_A)\|_{L^q} \leq (\|\pi_7(F_A|_C)\|_{L^q} + \|\pi_7(F_A|_B)\|_{L^q}).$$

Thus it is necessary only to obtain bounds on these norms. As stated above, $\|\pi_7(F_A|_C)\|_{L^q} = O(s^2)$.

If we let r represent the distance of a point away from the centre of curvature, we may deduce that

$$\|\pi_7(F_A|_B)\|_{L^q} \leq \|\pi_7(F_A|_{r<s})\|_{L^q} + \|\pi_7(F_A|_{s<r<1/200})\|_{L^q}.$$

Now since the curvature of an standard 4-instanton satisfied

$$|F_A| = C\frac{s^2}{(r^2+s^2)^2},$$

we may say that for $r < s$, $|F_A| = O(s^2)$, and that for $r > s$, $|F_A| \leq O(s^2 r^{-4})$.

We have, in addition, that $|\Omega - \Omega_t| \leq O(t^4)$ from [J2, p.535], and from this we may calculate

$$\begin{aligned} \|\pi_7(F_A|_{r<s})\|_{L^q} &\leq K(\int_{r<s}(t^4 s^{-2})^q)^{1/q} \\ &\leq O(t^4 s^{4/q-2}), \end{aligned}$$

since we are integrating over an area of order s^4.

Similarly,

$$\begin{aligned} \|\pi_7(F_A|_{s<r<1/200})\|_{L^q} &\leq K(\int_{s<r<1/200}(t^4 s^2 r^{-4})^q)^{1/q} \\ &\leq K' t^4 s^2([r^{4-4q}]_s^{1/200})^{1/q} \\ &\leq O(t^4 s^{4/q-2}). \end{aligned}$$

Thus $\|\pi_7(F_A)\|_{L^q} \leq O(t^4 s^{4/q-2})$, and finally we may deduce

$$\|\pi_7(F_A)\|_{L^q} \leq O(\max(\mathrm{s}^2, \mathrm{s}^{4/\mathrm{q}-2}\mathrm{t}^4)).$$

□

From this point on, let us set $s = t$, and thus we obtain that

$$||\pi_7(F_A)|| \leq Cs^2.$$

The method to be employed in producing a true $Spin(7)$ instanton will be that of successive iteration of a certain equation. An instanton A is a $Spin(7)$ instanton if it satisfies the equation $\pi_7(F_A) = 0$.

If we write $A = A' + a$, where A' is a fixed connection, and a a 1-form with values in $\mathrm{End}(E)$, then we have $F_A = F_{A'} + d_{A'}a + a \wedge a$, [DK, p.135] and hence we may rewrite the above equation as

$$\pi_7(d_{A'}a) + \pi_7(a \wedge a) = -\pi_7(F_{A'}).$$

We shall write $P = \pi_7 d_{A'} \oplus d^*_{A'}$ for the linear operator, $Q(a) = -\pi_7(a \wedge a)$ for the quadratic function, and ϵ for the error, $\pi_7(F_A)$. The term $d^*_{A'}$ is included for the purpose of gauge-fixing, and turns P into an elliptic operator. The equation becomes

$$Pa = (Q(a) + \epsilon, 0),$$

if we include the gauge-fixing condition $d^*_{A'} a = 0$.

Iteratively, we shall set $Pa_{j+1} = (Q(a_j) + \epsilon, 0)$, with the further condition that a_{j+1} be L^2-orthogonal to the kernel of P. If a_j converges to a limit, a, then we shall have a solution to our equation, and hence a $Spin(7)$ instanton. The rest of this thesis will be concerned with obtaining analytical results concerning this convergence.

We shall not attempt to obtain analytical results on the whole manifold immediately, but shall rather build them up from initial results on simpler spaces. The scheme I will follow will run as stated below:

- Results for compactly supported, N-invariant sections on $B^4 \times N$ with an instanton centred at the origin.

- Results for compactly supported function on manifolds of the type $B^4 \times N$ (where N is a 4-manifold), with an instanton centred at the origin of B^4.

- Results for $Z = M - (X \cup Y)$, where X,Y represent regions near the points of curvature of the instanton and points of resolution of the orbifold respectively.

- Results for the whole manifold M.

The reason for following this procedure becomes clear when we consider that points of curvature of the instanton may be modelled as $B^4 \times N$, and points of high curvature of the manifold may be modelled as $U \times N$, where U is an open subset of the Eguchi-Hanson space. The results obtained for $B^4 \times N$ may easily be transfered to similar results for $U \times N$, in which the situation is easier to handle, as there is no kernel to deal with on U. Hence the last step of this scheme corresponds to the "gluing" of the previous two results together.

7.1 Results on N-invariant functions

The following discussion holds for N any compact 4-manifold with fixed $SU(2)$ structure. However, we shall only need it for N being either $K3$ or T^4.

Although it is obvious intuitively what we mean by an N-invariant function, let us now make the definition precise.

For a smooth 1-form a, on a bundle $E \to B^4 \times N$, we have that

$$\begin{aligned} a &\in C^\infty(\mathrm{End}(E) \otimes T^*(B^4 \times N)) \\ &= C^\infty((\mathrm{End}(E) \otimes T^*B^4) \oplus (\mathrm{End}(E) \otimes T^*N)). \end{aligned}$$

For the bundles we are considering, by their construction we have that $\mathrm{End}(E)$ is the lift to $B^4 \times N$ of a vector bundle over B^4, and hence the idea of an N-invariant section of $C^\infty(\mathrm{End}(E) \otimes T^*B^4)$ is clearly defined, being the lift to $B^4 \times N$ of a section of the bundle over B^4.

We may also define a section, s of $C^\infty(\mathrm{End}(E) \otimes T^*N)$ to be N-invariant if

$$\nabla_N s = 0,$$

that is, the Levi-Civita connection is zero in the N direction on the T^*N component of the 1-forms.

Now let us consider our operator P acting on such 1-forms. By considering only N-invariant 1-forms, we may consider P_1, the component of P involving differentiation in the B^4 directions, together with an appropriate gauge-fixing condition, as an elliptic operator over the 4-manifold B^4, and hence can use results concerning such operators.

Considering P_1 acting on $\mathbf{R}^4 \times \mathrm{N}$ with the appropriate decay, instead of $B^4 \times N$, we see that P_1 has a kernel, corresponding to the first-order deformations of the 4-instanton on $\mathbf{R}^4$, within the family of instantons created using the ADHM construction.

Thus we may obtain some asymptotic bounds on elements in $\mathrm{Ker}P_1$, when P_1 is considered acting on $\mathbf{R}^4$. Let $v \in \mathrm{Ker}P_1$. Then

$$\begin{aligned} v &= O(s^2 r^{-4}), \quad r \geq s, \\ v &= O(s^{-2}), \quad r \leq s, \end{aligned}$$

as we have stated above. Also, we have

$$\begin{aligned} \nabla^j v &= O(s^2 r^{-4-j}), \quad r \geq s, \\ \nabla^j v &= O(s^{-2-j}), \quad r \leq s. \end{aligned}$$

Note that we have normalised to get

$$\|v\|_{L^2} = O(1),$$

i.e. it is independent of s.

Let us now attempt to approximate these sections of $\mathrm{Ker}(P_1)$ acting on $\mathbf{R}^4$ with compactly supported 1-forms on B^4. Denoting the dimension of $\mathrm{Ker}(P_1)$ by k, we may choose L^2-orthogonal sections $w_1, \ldots, w_k$ of the bundle of N-invariant sections satisfying the following conditions:

- w_j is compactly supported on B^4.
- $$\|w_j\|_{L^2} = 1.$$
- w_j approximate elements of $\mathrm{Ker}P_1$ acting on $\mathbf{R}^4$.
- $$\begin{aligned} w_j &= O(s^{-2}), \quad r \leq s, \\ w_j &= O(s^2 r^{-4}), \quad s \leq r \leq 1, \\ w_j &= 0, \quad r \geq 1. \end{aligned}$$
- $$\begin{aligned} P_1(w_j) &= 0, \quad r \leq 1/2, \\ P(w_j) &= O(s^2), \quad 1/2 \leq r \leq 1. \end{aligned}$$

As we calculated the norms of F_A above, we may calculate the norms of w_j in a similar fashion.

We obtain

$$\begin{aligned} \|w_j\|_{L^2} &= 1, \\ ||\nabla^i w_j||_{L^2} &= O(s^{-i}), \\ ||\nabla^i w_j||_{L^q} &= O(s^{-i+2-4/q}). \end{aligned}$$

Since we have a pointwise bound on $P(w_j)$ independent of r, it is even more straightforward to obtain bounds on these:

$$\|P_1(w_j)\|_{L^q_k} = O(s^2).$$

We shall now consider some elliptic regularity results, and use these to obtain some results on B^4.

For f a compactly supported function on $\mathbf{R}^4$, with a metric and connection independent of any small parameters, we have that

$$\|f\|_{L^4} \leq C\|\nabla f\|_{L^2},$$

where C is a constant, also independent of any small parameters.

We now wish to consider the eigenvalues of P_1 when considered acting on the the space of sections L^2-orthogonal to W. We expect that, having removed those elements that would cause us most problems, that the remaining eigenvalues will not tend to zero with s.

By elliptic regularity, we see that if a is a 1-form L^2-orthogonal to $\mathrm{Ker}P_1$, then

$$\begin{aligned}
\|a\|_{L^4} &\leq C\|P_1 a\|_{L^2}, \\
\|\nabla a\|_{L^2} &\leq C\|P_1 a\|_{L^2}, \\
\|\nabla^2 a\|_{L^2} &\leq C\|\nabla P_1 a\|_{L^2}, \\
\|\nabla^3 a\|_{L^2} &\leq C\|\nabla^2 P_1 a\|_{L^2}.
\end{aligned}$$

By making s small, we may approximate $\mathrm{Ker}(P_1)$ by $W = \langle w_1, \ldots, w_k\rangle$ as closely as we choose. In particular, these exists an s_0 such that for all $s \leq s_0$, the following is true:

If a is a 1-form on $\mathbf{R}^4$ orthogonal to the space W, then

$$\begin{aligned}
\|a\|_{L^4} &\leq C\|P_1 a\|_{L^2}, &(7.1)\\
\|\nabla a\|_{L^2} &\leq C\|P_1 a\|_{L^2}, &(7.2)\\
\|\nabla^2 a\|_{L^2} &\leq C\|\nabla P_1 a\|_{L^2}, &(7.3)\\
\|\nabla^3 a\|_{L^2} &\leq C\|\nabla^2 P_1 a\|_{L^2}.
\end{aligned}$$

Let us restrict our attention to such s. Considering now only those compactly supported a that are L^2-orthogonal to W, we have the following result.

Lemma 7.2 *If a is a compactly supported 1-form on B^4, orthogonal to $w_1, \ldots w_k$, then there exists a constant C', independent of $s \leq s_0$, such that*

$$\begin{aligned}
\|a\|_{L^4} &\leq C'\|P_1 a\|_{L^2}, \\
\|\nabla a\|_{L^2} &\leq C'\|P_1 a\|_{L^2}, \\
\|\nabla^2 a\|_{L^2} &\leq C'\|\nabla P_1\|_{L^2}, \\
\|\nabla^3 a\|_{L^2} &\leq C'\|\nabla^2 P_1 a\|_{L^2}.
\end{aligned}$$

Proof.

It is clear that these correspond to the above inequalities 7.1 for $\mathbf{R}^4$. They are obtained from them by considering scaling. Note that both sides of all the above equations scale in a similar manner and hence the constants C' remain invariant.

Thus in considering those 1-forms that are compactly supported within a ball of radius s^{-1} in $\mathbf{R}^4$ with a fixed metric and connection , we may obtain the desired result for the ball of radius 1 (i.e B^4) with the instanton described above by scaling by the appropriate factor.

□

Note also that since on B^4, a region of finite volume, $\|a\|_{L^2} \leq K\|a\|_{L^4}$, we have also $\|a\|_{L^2} \leq C'\|Pa\|_{L^2}$, where, again C' is an constant independent of s.

7.2 Results on $B^4 \times N$.

In attempting to extend our results for compactly supported N-invariant 1-forms to, more generally, any compactly supported 1-form on $B^4 \times N$, it is perhaps a sound strategy to split the 1-form up into several pieces, and use analytic results for each of these pieces.

For let $a \in C^\infty(\mathrm{End}(E) \otimes T^*(B^4 \times N))$. Then we may write

$$a = a' + a'',$$

where a' is the N-invariant part of a, and a'' is L^2-orthogonal to all N-invariant forms. Note that this will imply

$$\int_{\{x\}\times N} a'' = 0,$$

for each $x \in B^4$.

We shall find it useful if we further split a' into $a_0 + a_1$, where a_0 lies within the space $W = \langle w_1, \ldots, w_k \rangle$, and a_1 is L^2-orthogonal to this space. Similarly we may split a'' into $a_2 + a_3$, where at each point $n \in N$, $a_2|_{B^4\times\{n\}}$ lies in the space $W|_{B^4\times\{n\}}$, and a_3 is L^2-orthogonal to forms of this type.

Note that

$$\begin{aligned} a_0 &= \sum_1^k \langle a', w_i \rangle w_i, \\ a_1 &= a' - a_0, \\ a_2(x,y) &= \sum_1^k \left(\int_{z\in B^4} \langle w_i(z), a''(z,y) \rangle \right) w_i(x), \end{aligned}$$

$$a_3 = a'' - a_2.$$

Let us write $f_i(y) = \int_{x \in B^4} \langle w_i(x), a''(x,y) \rangle$ for the functions on N. We may observe that $\int_N f_i = 0$.

We may also consider the splitting of the differential operators ∇ and P. Let us write

$$\nabla = \nabla_1 + \nabla_2,$$

where ∇_1 corresponds to the derivative in the direction of B^4, and ∇_2 corresponds to the derivative in the N directions. It is worth noting at this point that ∇_1 and ∇_2 commute. Similarly, we may write

$$P = P_1 + P_2.$$

Before continuing, let us consider some properties of the above splitting. Clearly, we obtain that $P^* = P_1^* + P_2^*$.

Since P_2 is an fixed elliptic operator on a fixed four-manifold, N, we have that

$$P_2^* P_2 = K \Delta_2,$$

that is, a constant multiple of the Laplacian, using an equivalent of a Weitzenbock formula.

We should also see that since they involve differentiation in different directions, we have that

$$P_1^* P_2 + P_2^* P_1 = 0,$$

and hence

$$\|Pa\|_{L^2}^2 = \|P_1 a\|_{L^2}^2 + \|P_2 a\|_{L^2}^2.$$

We may now use elliptic regularity results for the three components of a orthogonal to W. We have, from the earlier section[Lemma 7.2], that

$$\|a_1\|_{L^2} \leq C \|P a_1\|_{L^2}.$$

Similarly, by first restricting our attention to the cross-section $B^4 \times \{n\}$, and then integrating the inequality we get over the manifold N, we may see that

$$\|a_3\|_{L^2} \leq C \|P_1 a_3\|_{L^2} \leq C \|P a_3\|_{L^2}.$$

Now it merely remains for us to get a similar bounds on $\|a_2\|$. Let us first note that since $\|Pa_2\|_{L^2}^2 = \|P_1 a_2\|_{L^2}^2 + \|P_2 a_2\|_{L^2}^2$, we obviously have $\|P_2 a_2\|_{L^2} \leq \|P a_2\|_{L^2}$. Note that

$$\|P_2 a_2\|_{L^2}^2 = \langle a_2, P_2^* P_2 a_2 \rangle$$

$$\begin{aligned} &= \sum_{i,j=1}^{k} \langle f_i w_i, K\Delta_2(f_j) w_j\rangle \\ &= \sum_{i=1}^{k} \|w_i\|^2_{L^2(B^4)} \langle f_i, K\Delta_2 f_i\rangle_N \\ &= K^{1/2} \sum_{1}^{k} \|df_i\|^2_{L^2}, \end{aligned}$$

the last two equalities following from the fact that the w_i's are orthonormal.

On any compact Riemannian manifold without boundary, it is true that for a function f such that the integral over the manifold is zero, then there is a constant, C, dependent only on the manifold such that $\|f\|_{L^2} \leq C\|df\|_{L^2}$. Applying this fact to the function f_i on N, we have that

$$\sum_{i=4}^{k} \|f_i\|^2_{L^2} \leq C\|P_2 a_2\|^2_{L^2}. \tag{7.4}$$

As $\|a_2\|^2_{L^2} = \sum_1^k \|f_i\|^2_{L^2}$, since the norms of the w_i's are 1, we may conclude that there exists a constant C such that

$$\|a_2\|_{L^2} \leq C\|Pa_2\|_{L^2}.$$

Having these bounds on the components of a, let us continue by attempting to obtain similar bounds on the components of ∇a. Again, for two of the components, this follows in a straightforward manner from the bound on N-invariant functions L^2-orthogonal to W we obtained earlier [Lemma 7.2]. We have $||\nabla a_1\|_{L^2} \leq C\|Pa_1||_{L^2}$ immediately from this, and, as before, by first restricting out attention to a cross-section and then integrating over the whole manifold, we can obtain $\|a_3\|_{L^2} \leq C\|Pa_3\|_{L^2}$. The only difficulty that arises is in dealing with a_2.

We may use a similar approach, noticing that ∇_2 acts only on the f_i part of a_2, leaving the w_i, the N-invariant part alone. Thus we have that

$$\|\nabla_2 a_2\|_{L^2} \leq C\|Pa_2\|_{L^2}.$$

Continuing in the same manner, using elliptic regularity, we may deduce the following lemma.

Lemma 7.3 *There exists a constant C independent of s such that the following all hold.*

$$\|a_1\|_{L^2} \;\leq\; C\|Pa_1\|_{L^2},$$

$$
\begin{aligned}
\|a_2\|_{L^2} &\le C\|Pa_2\|_{L^2}, \\
\|a_3\|_{L^2} &\le C\|Pa_3\|_{L^2}, \\
\|\nabla a_1\|_{L^2} &\le C\|Pa_1\|_{L^2}, \\
\|\nabla_2 a_2\|_{L^2} &\le C\|Pa_2\|_{L^2}, \\
\|\nabla a_3\|_{L^2} &\le C\|Pa_3\|_{L^2}, \\
\|\nabla^2 a_1\|_{L^2} &\le C\|\nabla Pa_1\|_{L^2}, \\
\|\nabla_2^2 a_2\|_{L^2} &\le C\|\nabla_2 Pa_2\|_{L^2}, \\
\|\nabla^2 a_3\|_{L^2} &\le C\|\nabla Pa_3\|_{L^2}, \\
\|\nabla^3 a_1\|_{L^2} &\le C\|\nabla^2 Pa_1\|_{L^2}, \\
\|\nabla_2^3 a_2\|_{L^2} &\le C\|\nabla_2^2 Pa_2\|_{L^2}, \\
\|\nabla^3 a_3\|_{L^2} &\le C\|\nabla^2 Pa_3\|_{L^2}.
\end{aligned}
$$

Proof. We see that we have already proved the assertions about the components of a.

We may also see that since $\nabla^j a_1 = \nabla_1^j a_1$, since a_1 is a independent of the variables in the direction of N. Thus we may use Lemma 7.2 to prove our result, since the ∇ in Lemma 7.2 refers to ∇_1, the derivative in the B^4 direction.

Considering the $\nabla_2^j a_2$ terms, we see that by elliptic regularity on the manifold N, which is independent of any small parameters, we see that

$$
\begin{aligned}
\|\nabla_2^{j+1} a_2\|_{L^2} &\le C\|\nabla_2^j P_2 a_2\|_{L^2} \\
&\le C\|\nabla_2^j Pa_2\|_{L^2}.
\end{aligned}
$$

Finally, let us consider the derivatives of a_3. We have

$$\nabla^j = (\nabla_1 + \nabla_2)^j.$$

Thus we may obtain bounds on $\nabla^{j+1} a_3$ from bounds upon $\nabla_2^k \nabla_1^{j+1-k} a_3$.

Firstly, let us consider the cases when either $k = 0$ or $k = j+1$. In these cases, we may proceed as above to obtain

$$\|\nabla_1^{j+1} a_3\|_{L^2} \le C\|\nabla_1^j P_1 a_3\|_{L^2},$$

from Lemma 7.2 again, and

$$\|\nabla_2^{j+1} a_3\|_{L^2} \le C\|\nabla_2^j P_2 a_3\|_{L^2},$$

from the fact that the manifold N is independent of any small parameters.

In the other cases, we have

$$\|\nabla_1^{j+1-k} b\|_{L^2} \leq C \|\nabla_1^{j-k} P_1 b\|_{L^2},$$

for any 1-form b that is L^2-orthogonal to W, where the integration is over $B^4 \times \{n\}$. But if we let $e_1, \ldots, e_4$ be an orthonormal basis for $T_n^* N$, then we see that

$$\nabla_2^k \nabla_1^{j+1-k} a_3 = \sum v_{i(1),\ldots,i(k)} \otimes e_{i(1)} \otimes e_{i(2)} \otimes \cdots \otimes e_{i(k)},$$

and

$$\nabla_2^k \nabla_1^{j-k} P_1 a_3 = \sum u_{i(1),\ldots,i(k)} \otimes e_{i(1)} \otimes e_{i(2)} \otimes \cdots \otimes e_{i(k)},$$

where

$$\begin{aligned} v_{i(1),\ldots,i(k)} &= \nabla_1^{j+1-k} \left(\nabla_{e_{i(1)}} \nabla_{e_{i(2)}} \cdots \nabla_{e_{i(k)}} a_3 \right), \\ u_{i(1),\ldots,i(k)} &= \nabla_1^{j-k} \left(\nabla_{e_{i(1)}} \nabla_{e_{i(2)}} \cdots \nabla_{e_{i(k)}} P_1 a_3 \right). \end{aligned}$$

Thus we may apply Lemma 7.2 to the v and u terms (note that they are L^2-orthogonal to W since a_3 is), and remembering that $e_1, \ldots, e_4$ is an orthonormal basis, by integrating over N, we obtain

$$\|\nabla_1^{j+1-k} \nabla_2^k a_3\|_{L^2} \leq C \|\nabla_2^k \nabla_1^{j-k} P_1 a_3\|_{L^2}.$$

Putting these results on the components of $\nabla^{j+1} a_3$ together, we do indeed obtain the results stated in the Lemma.

□

Let us also consider ∇_1 acting on the a_2 component of a. Since ∇_1 acts only on the w_i part of $a_2 = \sum f_i w_i$, we may obtain bounds on $\nabla_1^i a_2$ by considering the equivalent bounds on $\nabla_1^i w_i$. We obtain the following:

$$\begin{aligned} \|\nabla_1 a_2\|_{L^2} &\leq Cs^{-1} \|a_2\|_{L^2}, \\ \|\nabla_1 \nabla_2 a_2\|_{L^2} &\leq Cs^{-1} \|\nabla_2 a_2\|_{L^2}, \\ \|\nabla_2 \nabla_1 a_2\|_{L^2} &\leq Cs^{-1} \|\nabla_2 a_2\|_{L^2}, \\ \|\nabla_1^2 a_2\|_{L^2} &\leq Cs^{-2} \|a_2\|_{L^2}, \\ \|\nabla_1^3 a_2\|_{L^2} &\leq Cs^{-3} \|a_2\|_{L^2}. \end{aligned}$$

We may obtain the same results with Pa_2 in place of a_2.

The next lemma will begin to tackle a problem that we would otherwise encounter in attempting to estimate $\|a\|$ by $\|Pa\|$; that, in splitting a, we may find it difficult in moving from bounds of $\|a\|$ by $\|Pa_1\| + \|Pa_2\| + \|Pa_3\|$ to the bounds by the simpler $\|Pa\|$.

It is worth saying at this point that much of this section will be dealing with problems very similar to this one, in trying to deal with the fact that $\nabla^j Pa_2$ and $\nabla^j Pa_3$ are not necessarily L^2-orthogonal. Many of the lemmas in this chapter are concerned with approximating the L^2 inner product between these quantities, and hence in relating $\|\nabla^j Pa\|_{L^2}$ and the pair $\|\nabla^j Pa_2\|_{L^2}$ and $\|\nabla^j Pa_3\|_{L^2}$.

The next lemma can be seen as a first step in relating the two quantities Pa_2 and Pa_3.

Lemma 7.4 $|\langle Pa_2, Pa_3\rangle| \leq Cs^2\|a_2\|_{L^2}\|a_3\|_{L^2}$.

Proof.

$$\begin{aligned}\langle Pa_2, Pa_3\rangle &= \langle P^*Pa_2, a_3\rangle \\ &= \langle (P_1^*P_1 + P_2^*P_2)a_2, a_3\rangle \\ &= \langle P_1^*P_1a_2, a_3\rangle + \langle P_2^*P_2a_2, a_3\rangle.\end{aligned}$$

But by definition, $a_2 = \sum_1^k f_i w_i$. Since f_i is a function of the N-variables only, and hence independent of the B^4 variables, we have that

$$P_1^*P_1a_2 = P_1^*P_1(\sum_1^k f_i w_i) = \sum_1^k f_i(P_1^*P_1w_i).$$

Also, since w_i is independent of the N-variables, we have that

$$P_2^*P_2a_2 = \sum_1^k g_j w_i,$$

where $g_j = \Delta_2 f_j$, and is independent of the B^4 variables. Thus we have $\langle g_j w_i, a_3\rangle = 0$, by definition of a_3. Thus

$$\begin{aligned}\langle Pa_2, Pa_3\rangle &= \sum_1^k \langle f_i(P_1^*P_1w_i), a_3\rangle \\ &\leq \sum_1^k \|a_3\|_{L^2}\|f_i\|_{L^2(N)}\|P_1^*P_1w_i\|_{L^2(B^4)}.\end{aligned}$$

As we stated in the construction of the w_i's any norm of Pw_i will be $O(s^2)$. Hence there exists K_1 such that

$$\|P_1^*P_1w_i\|_{L^2(B^4)} \leq K_1s^2.$$

Now we have that

$$\|a_2\|_{L^2}^2 = \sum_1^k \|f_i\|_{L^2(N)}^2 \|w_i\|_{L^2(B^4)}^2,$$

by definition. Now from construction of w_i, we have that $\|w_i\|_{L^2(B^4)}^2 = 1$, and hence we may deduce that

$$||a_2||_{L^2} \geq C(k) \sum_1^k \|f_i\|_{L^2(N)},$$

for some constant $C(k)$, depending only on k.

Hence

$$|\langle Pa_2, Pa_3 \rangle| \leq Cs^2 \|a_2\|_{L^2} \|a_3\|_{L^2}.$$

□

Lemma 7.5

$$\|Pa_1\|_{L^2}^2 + \|Pa_2\|_{L^2}^2 + \|Pa_3\|_{L^2}^2 \leq \|Pa\|_{L^2}^2 + Cs^2 \|a_2\|_{L^2} ||a_3\|_{L^2}.$$

Proof.

$$\|Pa\|_{L^2}^2 = \|Pa_1\|_{L^2}^2 + \|Pa_2\|_{L^2}^2 + \|Pa_3\|_{L^2}^2 + 2\langle Pa_2, Pa_3 \rangle,$$

since Pa_1 is N-invariant, and both Pa_2 and Pa_3 are L^2-orthogonal to N-invariant forms, these properties being preserved under applying P.

Hence from Lemma 7.4, we deduce our result.

□

Lemma 7.6 *With the same notation as above, for small enough s there exists a constant C such that*

$$\begin{aligned}
||a_1||_{L^2} &\leq C\|Pa\|_{L^2}, \\
||a_2||_{L^2} &\leq C\|Pa\|_{L^2}, \\
||a_3||_{L^2} &\leq C\|Pa\|_{L^2}, \\
\|\nabla a_1\|_{L^2} &\leq C\|Pa||_{L^2}, \\
\|\nabla a_3\|_{L^2} &\leq C\|Pa||_{L^2}, \\
\|\nabla_2 a_2\|_{L^2} &\leq C\|Pa||_{L^2}.
\end{aligned}$$

If a is L^2-orthogonal to W, i.e. $a_0 = 0$, then in addition we have

$$||a||_{L^2} \leq C\|Pa\|_{L^2}.$$

Proof.
This follows immediately from the lemmas 7.3 and 7.5 .

□

Next let us consider $\nabla Pa = \nabla Pa_1 + \nabla Pa_2 + \nabla Pa_3$.

Lemma 7.7

$$|\langle \nabla_2 Pa_2, \nabla_2 Pa_3 \rangle| \le Cs^2 ||Pa||_{L^2} (||Pa||_{L^2} + ||\nabla Pa_3||_{L^2}).$$

Proof.
We have

$$\langle \nabla_2 Pa_2, \nabla_2 Pa_3 \rangle = \sum_{i,j=1}^{2} \langle \nabla_2 P_i a_2, \nabla_2 P_j a_3 \rangle.$$

Firstly, we may note that

$$\langle \nabla_2 P_2 a_2, \nabla_2 P_2 a_3 \rangle = \langle (P_2^* \nabla_2^* \nabla_2 P_2) a_2, a_3 \rangle,$$

and on observing that $(P_2^* \nabla_2^* \nabla_2 P_2 2)$ acts only on the f_i's in $a_2 = \sum f_i w_i$, and hence that the resulting form is a also a combination of w_i's, we see that a_3 is L^2-orthogonal to this, by definition. Hence $\langle \nabla_2 P_2 a_2, \nabla_2 P_2 a_3 \rangle = 0$.

We may also see that

$$\langle \nabla_2 P_1 a_2, \nabla_2 P_2 a_3 \rangle + \langle \nabla_2 P_2 a_2, \nabla_2 P_1 a_3 \rangle$$

$$= \langle P_1 a_2, P_2 (\nabla_2^* \nabla_2) a_3 \rangle + \langle P_2 a_2, P_1 (\nabla_2^* \nabla_2) a_3 \rangle + \langle P_1 a_2, [\nabla_2^* \nabla_2, P_2] a_3 \rangle.$$

We are permitted to perform the above operation because P_1 and $\nabla_2^* \nabla_2$ commute, being differentiation in independent directions. But

$$\langle P_1 a_2, P_2 (\nabla_2^* \nabla_2) a_3 \rangle + \langle P_2 a_2, P_1 (\nabla_2^* \nabla_2) a_3 \rangle = 0,$$

since $P_2^* P_1 + P_1^* P_2 = 0$. As for the $[(\nabla_2^* \nabla_2), P_2] a_3$ term, it must consist of a bilinear combination of the manifold curvature in the ∇_2-direction, together with a_3 and its derivatives. Noting the necessary degree, and the fact that the instanton is flat in the ∇_2-direction, we see that

$$[(\nabla_2^* \nabla_2), P_2] a_3 = Q_1(R_N, \nabla_2 a_3) + Q_2(\nabla_2 R_N, a_3),$$

where Q_i denotes a bilinear map and R_N denotes the manifold curvature in the direction of the submanifold N.

We may also note that in the direction in which ∇_2 acts, both the manifold curvature and its derivative are bounded. Hence

$$\begin{aligned}
|\langle\nabla_2 P_1 a_2, \nabla_2 P_2 a_3\rangle + \langle\nabla_2 P_2 a_2, \nabla_2 P_1 a_3\rangle| &\leq C\|P_1 a_2\|_{L^2}(\|a_3\|_{L^2} + \|\nabla_2 a_3\|_{L^2}) \\
&\leq Cs^2\|a_2\|_{L^2}(\|a_3\|_{L^2} + \|\nabla_2 a_3\|_{L^2}) \\
&\leq Cs^2\|Pa\|_{L^2}(\|Pa\|_{L^2} + \|Pa\|_{L^2}).
\end{aligned}$$

Finally, we shall consider $\langle\nabla_2 P_1 a_2, \nabla_2 P_1 a_3\rangle$. The P_1 acts only on the w_i part of $a_2 = \sum f_i w_i$, and we have that all norms of $P_1 w_i$ are $O(s^2)$. Thus we may deduce that

$$\|\nabla P_1 a_2\| \leq Cs^2(\|\nabla_2 a_2\|_{L^2} + \|a_2\|_{L^2}).$$

Hence

$$\begin{aligned}
|\langle\nabla_2 P_1 a_2, \nabla_2 P_1 a_3\rangle| &\leq Cs^2(\|\nabla a_2\|_{L^2} + \|a_2\|_{L^2})\|\nabla_2 P_1 a_3\|_{L^2} \\
&\leq Cs^2\|Pa_2\|_{L^2}\|\nabla_2 P_1 a_3\|_{L^2} \\
&\leq Cs^2\|Pa\|_{L^2}\|\nabla Pa_3\|_{L^2}.
\end{aligned}$$

□

We have

$$\|\nabla Pa - \nabla_1 Pa_2\|_{L^2}^2 = \|\nabla Pa_1\|_{L^2}^2 + \|\nabla_2 Pa_2\|_{L^2}^2 + \|\nabla Pa_3\|_{L^2}^2 + \langle\nabla_2 Pa_2, \nabla_2 Pa_3\rangle,$$

since the images of ∇_1 and ∇_2 are L^2-orthogonal.

So for small enough s, we see that

$$\begin{aligned}
\|\nabla Pa_1\|_{L^2}^2 + \|\nabla P_2 a_2\|_{L^2}^2 + \|\nabla Pa_3\|_{L^2}^2 &\leq C\left(\|\nabla Pa\|_{L^2}^2 + \|\nabla_1 Pa_2\|_{L^2}^2\right) \\
&\leq C\left(\|\nabla Pa\|_{L^2}^2 + s^{-2}\|Pa\|_{L^2}^2\right).
\end{aligned}$$

Now from the elliptic results

$$\begin{aligned}
\|\nabla^2 a_1\| &\leq C\|\nabla Pa_1\|_{L^2}, \\
\|\nabla_2^2 a_2\|_{L^2} &\leq C\|\nabla Pa_2\|_{L^2}, \\
\|\nabla^2 a_3\| &\leq C\|\nabla Pa_3\|_{L^2},
\end{aligned}$$

together with the above results, we may deduce bounds on the second derivatives of the components of a in terms of $\|\nabla Pa\|$ and $\|Pa\|$ alone.

Proposition 7.1 *For small enough s, there exists a C independent of s such that*

$$
\begin{aligned}
\|\nabla^2 a_1\|_{L^2} &\leq C\|\nabla Pa\|_{L^2}, \\
\|\nabla_2^2 a_2\|_{L^2} &\leq C\|\nabla Pa\|_{L^2}, \\
\|\nabla^2 a_3\|_{L^2} &\leq C\|\nabla Pa\|_{L^2}.
\end{aligned}
$$

The third derivative follows a similar method.

Lemma 7.8

$$
|\langle \nabla_2^2 Pa_2, \nabla^2 Pa_3\rangle| \leq Cs^2 \left(\|\nabla Pa\|_{L^2}^2 + \|Pa\|_{L^2}\left(\|Pa\|_{L^2} + \|\nabla Pa\|_{L^2} + \|\nabla^2 Pa_3\|_{L^2}\right)\right).
$$

Proof.

$$
\langle \nabla_2^2 Pa_2, \nabla^2 Pa_3\rangle = \sum_{i,j=1}^{2} \langle \nabla_2^2 P_i a_2, \nabla_2^2 P_j a_3\rangle.
$$

Similarly to last time, we get

$$
\langle \nabla_2^2 P_2 a_2, \nabla_2^2 P_2 a_3\rangle = \langle (P_2^*(\nabla_2^2)^*\nabla_2^2 P_2)a_2, a_3\rangle = 0.
$$

Also,

$$
\begin{aligned}
& \langle \nabla_2^2 P_1 a_2, \nabla_2^2 P_2 a_3\rangle + \langle \nabla_2^2 P_2 a_2, \nabla_2^2 P_1 a_3\rangle \\
= \ & \langle P_1 a_2, P_2((\nabla_2^2)^*\nabla_2^2)a_3\rangle + \langle P_2 a_2, P_1((\nabla_2^2)^*\nabla_2^2)a_3\rangle + \langle P_1 a_2, [(\nabla_2^2)^*\nabla_2^2, P_2]a_3\rangle \\
= \ & \langle P_1 a_2, [(\nabla_2^2)^*\nabla_2^2, P_2]a_3\rangle.
\end{aligned}
$$

Again we see that the sum of the first two terms disappear, and $[(\nabla_2^2)^*\nabla_2^2, P_2]a_3$ will consist of derivatives of the Riemannian curvature in the ∇_2-direction and derivatives of a_3. That is,

$$
[(\nabla_2^2)^*\nabla_2^2, P_2]a_3 = Q_1(R_N, \nabla_2^3 a_3) + Q_2(\nabla_2 R_N, \nabla_2^2 a_3) + Q_3(\nabla_2^2 R_N, \nabla_2 a_3) + Q_4(\nabla_2^3 R_N, a_3),
$$

for some bilinear functions Q_i. Since these curvature derivatives are bounded, independently of s, we have that this term is bounded by some constant times

$$
\text{lla311L2} + \|\nabla_2 a_3\|_{L^2} + \|\nabla_2^2 a_3\|_{L^2} + \|\nabla_2^3 a_3\|_{L^2}
$$

$$
\leq C\left(\|Pa\|_{L^2} + \|\nabla Pa\|_{L^2} + \|\nabla^2 Pa_3\|_{L^2}\right).
$$

Hence we have

$$
|\langle \nabla_2^2 P_1 a_2, \nabla_2^2 P_2 a_3\rangle + \langle \nabla_2^2 P_2 a_2, \nabla_2^2 P_1 a_3\rangle|
$$

$$\leq C\|P_1 a_2\|_{L^2}\left(\|Pa\|_{L^2} + \|\nabla Pa\|_{L^2} + \|\nabla^2 Pa_3\|_{L^2}\right).$$

The last component is $\langle \nabla_2^2 P_1 a_2, \nabla_2^2 P_2 a_3\rangle$. We can see, as before, that ∇_2^2 acts only on the w_i of a_2. Thus as before, we see

$$\|\nabla_2^2 P_1 a_2\| \leq Cs^2\left(\|a_2\| + \|\nabla_2 a_2\| + \|\nabla_2^2 a_2\|\right).$$

We may then deduce

$$\begin{aligned}
|\langle \nabla_2^2 P_1 a_2, \nabla_2^2 P_1 a_3\rangle| &\leq Cs^2\|\nabla Pa_2\|_{L^2}\|\nabla_2 Pa_3\|_{L^2} \\
&\leq Cs^2\|\nabla Pa\|_{L^2}^2
\end{aligned}$$

These results then allow us to prove the lemma.

□

Thus we may consider

$$\|\nabla^2 Pa - \nabla_1^2 Pa_2 - \nabla_1\nabla_2 Pa_2 - \nabla_2\nabla_1 Pa_2\|_{L^2}^2$$

$$= \|\nabla^2 Pa_1\|_{L^2}^2 + \|\nabla_2^2 Pa_2\|_{L^2}^2 + \|\nabla^2 Pa_3\|_{L^2}^2 + \langle \nabla_2^2 Pa_2, \nabla_2^2 Pa_3\rangle.$$

Lemma 7.8 then gives us that, for small enough s,

$$\|\nabla^2 Pa_1\|_{L^2}^2 + \|\nabla^2 Pa_2\|_{L^2}^2 + \|\nabla^2 Pa_3\|_{L^2}^2$$

$$\leq C\|\nabla^2 Pa\|_{L^2}^2 + s^{-2}\|\nabla Pa\|_{L^2}^2 + s^{-4}\|Pa\|_{L^2}^2.$$

We may state this as a proposition similar to the one above.

Proposition 7.2 *For small enough s, there exists a C independent of s such that*

$$\begin{aligned}
\|\nabla^3 a_1\|_{L^2} &\leq C\|\nabla^2 Pa\|_{L^2}, \\
\|\nabla_2^3 a_2\|_{L^2} &\leq C\|\nabla^2 Pa\|_{L^2}, \\
\|\nabla^3 a_3\|_{L^2} &\leq C\|\nabla^2 Pa\|_{L^2}.
\end{aligned}$$

This will be the last result we need on $B^4 \times N$.

7.3 Results on the remainder of the Manifold

Now let us consider the manifold without the regions of high curvature. If we remove a region of radius t^α around the points of concentration of the instanton curvature, and the manifold curvature, calling the resulting manifold $\dot{Z}_\alpha$, then we may use elliptic regularity results to get similar bounds on a and its derivatives in terms of Pa.

Note first that if we restrict our attention to compactly supported a then P has no kernel. This is because the only elements in the kernel of P when acting on T^8/Γ are constant 1-forms, and, of course, when we require these to vanish in a certain region of the manifold, we ensure that they are uniformly zero.

It is clear that, since the decay of the manifold curvature and the decay of the instanton curvature is of the form $t^m r^{-n}$ for some constants n and m, we may select $\alpha = \alpha_0 > 0$ so that the curvatures and all other relevant quantities are below a constant value, independent of t.

We may cover Z_{α_0} with balls of fixed, constant radius, independent of t, and then apply standard elliptic regularity results to these balls to obtain the following result:

Proposition 7.3 *There exists an α_0 with $0 < \alpha_0 < 1$ such that for a a compactly supported 1-form on Z_{α_0}, we have the following inequalities:*

$$\begin{aligned} \|a\|_{L^2} &\leq C\|Pa\|_{L^2}, \\ \|\nabla a\|_{L^2} &\leq C\|Pa\|_{L^2}, \\ \|\nabla^2 a\|_{L^2} &\leq C\|\nabla Pa\|_{L^2}, \\ \|\nabla^3 a\|_{L^2} &\leq C\|\nabla^2 Pa\|_{L^2}, \end{aligned}$$

where C is independent of t.

Proof.

The existence of such a C follows from elliptic regularity, and the fact that the kernel of P will be 0 for small enough α.

We may obtain the result that C can be chosen independently of t by choosing α_0 as described above, so that the curvature, and other relevant quantities, are smaller than a constant independent of t.

□

Note that we may also use this result if instead of Z_α, we use Z, the manifold obtained by removing regions of fixed, small radius (say, the radius is $1/100$) around the points of centre of curvature.

7.4 Results on the Whole Manifold

Armed with these results, I shall glue them together to obtain a result on the whole manifold.

Let $\beta : \mathbf{R} \to [0,1]$ have the following properties:

- $\beta(x) = 0$ for $x \leq 0$.
- $\beta(x) = 1$ for $x \geq 1$.
- β is smooth.

Let r denote the radius function of B^4. Now define a function $\nu : M \to [0,1]$ by

$$\nu = \beta\left(\frac{\log r}{\alpha_0 \log t}\right), \text{ on } B^4 \times N,$$
$$\nu = 0 \text{ outside } B^4 \times N.$$

We have

$$\begin{aligned} \nu &= 0 \text{ outside } B^4 \times N, \\ 1 - \nu &= 0 \text{ outside } Z'. \end{aligned}$$

Lemma 7.9

$$\left(\int_M |\nabla \nu|^4\right)^{1/4} \leq C|\log t|^{-3/4}.$$

Proof.

$$\nabla \nu = \frac{\partial \beta}{\partial x} \frac{\nabla r}{r \alpha_0 \log t},$$

and we have that $\partial \beta / \partial x$ is independent of all small parameters. Now we have

$$\int_{r=a}^{b} \frac{1}{r^4} = C(\log b - \log a),$$

where it should be amde clear that we are integrating on that region of $B^4 \times N$ with distance from the submanifold $\{0\} \times N$ between a and B. Since we are integrating between $r = t^{\alpha_0}$ and $r = 1$, we see that

$$\|\nabla \nu\|_{L^4} = C \frac{|(\log t)|^{1/4}}{|\log t|}.$$

Thus

$$\|\nabla \nu\|_{L^4} = O(|\log t|^{-3/4}).$$

$\square$

Consider an a L^2-orthogonal to W. Since $a = \nu a + (1-\nu)a$, we may split a up into a piece compactly supported on $B^4 \times N$ and a piece compactly supported on Z_{α_0}. Thus we may apply our previous results to both.

Note that we must have νa being L^2-orthogonal to W to apply the original results directly. However, we may see that $\langle \nu a, w\rangle$ for $w \in W$, is small compared to $\|a\|_{L^2}$, as

$$\begin{aligned}
|\langle \nu a, w\rangle| &= |\langle a, \nu w\rangle| \\
&= |\langle a, (1-\nu)w\rangle| \\
&\leq \|a\|_{L^2}\|(1-\nu)w\|_{L^2} \\
&\leq \|a\|_{L^2}\left(\int_{t^{\alpha_0}}^{1/100}|w|^2\right)^{1/2} \\
&\leq C\|a\|_{L^2}\left(\int_{t^{\alpha_0}}^{1/100}(t^2r^{-4})^2\right)^{1/2} \\
&\leq Ct^2\|a\|_{L^2}\left([r^{-4}]_{t^{\alpha_0}}^{1/100}\right)^{1/2}.
\end{aligned} \tag{7.7}$$

(Note that the second inequality holds as the region intergrated over is the intersection of the supports of the two functions.)

So we see that provided α_0 has been chosen small enough, and we set t small enough, we may also apply the results we obtained for a that are L^2-orthogonal to W to a that are L^2-orthogonal to νW.

Thus we have

$$\|\nu a\|_{L^2} \leq C||P(\nu a)\|_{L^2},$$

from Lemma 7.6.

From Lemma 7.3 , we also have

$$\|(1-\nu)a\|_{L^2} \leq C||P(1-\nu)a\|_{L^2}.$$

Adding these two results, we obtain

$$\begin{aligned}
\|a\|_{L^2} &\leq C(||P(\nu a)\|_{L^2} + ||P(1-\nu)a\|_{L^2}) \\
&\leq C\left(\|Pa\| + \|\nabla\nu.a\|_{L^2}\right).
\end{aligned}$$

From this point on, let us denote the annulus of B^4 given as the projection of $B^4 \times N$'s intersection with Z_{α_0} by D.

Let us consider the final term.

$$\|\nabla\nu.a\|_{L^2}^2 = \int_{y\in N}\int_{x\in D}|\nabla\nu(x)|^2|a(x,y)|^2dxdy.$$

Consider y fixed, we have

$$\int_{x\in D}|\nabla\nu(x)|^2|a(x,y)|^2dx$$

$$\leq \left(\int_{x\in D} |\nabla\nu|^4\right)^{1/2} \left(\int_{x\in D} |a(x,y)|^4\right)^{1/2},$$

by Holder's inequality.

But from lemma 7.9

$$\left(\int_{x\in D} |\nabla\nu|^4\right)^{1/2} \leq C|\log t|^{-3/4}.$$

Hence

$$\|\nabla\nu.a\|_{L^2}^2 \leq C|\log t|^{-3/2} \int_{y\in N} \left(\int_{x\in D} |a|^4 dx\right)^{1/2} dy.$$

On the annulus, D, considered as a 4-manifold, with fixed y, we have $L_1^2 \to L^4$ by Sobolev embedding. So

$$\left(\int_{x\in D} |a|^4 dx\right)^{1/2} \leq C \int_{x\in D} |\nabla_1 a|^2 + |a|^2,$$

and hence

$$\|\nabla\nu.a\|_{L^2}^2 \leq C|\log t|^{-3/2} \int_{y\in N} \int_{x\in D} \left(|\nabla_1 a|^2 + |a|^2\right) dx dy.$$

Hence we deduce

$$\|a\|_{L^2} \leq C \left(\|Pa\|_{L^2} + |\log t|^{-3/4} \|\nabla_1 a|_D\|_{L^2} + |\log t|^{-3/4} \|a|_D\|_{L^2}\right).$$

On the annulus, by using an argument with balls of radius independent of the small parameters involved, and a method similar to those used on the manifold with the highly curved regions removed, we have

$$\|\nabla_1 a|_D\|_{L^2} \leq C \left(\|Pa|_D\|_{L^2} + \|a|_D\|_{L^2}\right).$$

Thus

$$\|a\|_{L^2} \leq C \left(\|Pa\|_{L^2} + |\log t|^{-3/4} \|a|_D\|_{L^2}\right).$$

So for small enough t, that is provided $C|\log t|^{-3/4} \leq 1/2$, we have the following theorem.

Thus we have proved the following theorem:

Theorem 7.1 *For small enough t, if a is L^2-orthogonal to νW, then*

$$||a||_{L^2} \leq C\|Pa\|_{L^2}.$$

We may continue in a similar fashion to obtain such bounds on the derivatives of the components of a. If we were to try to obtain these using the previous cut-off function, we may well run into the problem of dealing with ν's higher derivatives, which are not independent of the small parameters used. Since we have already obtained a bound on a norm of a, we need not use this cut-off function again. Thus let us first define another cutoff function μ as follows:

$$\mu = \beta\left(\frac{\log r}{\log(1/100)}\right).$$

From this point onwards, we shall denote that which we have previously called a_2 as b, and write $c = a - b$. That is, we shall split μa into components $a_1 + a_2 + a_3$ and write $b = a_2$.

For $r > 1$, we have $\mu = 0$, and for $r < 1/2$, we have $\mu = 1$. Note that μa is compactly supported on $B^4 \times N$, and $(1-\mu)a$ is compactly supported on Z, and hence we may apply our known results on the derivatives of a to them. As above (equation 7.7), we may show that results for a that are L^2-orthogonal to W may be applied to a that are L^2-orthogonal to μW,(i.e. μa is L^2-orthogonal to W), by an identical method. We may also see, by the definition of f.J., that its derivatives are independent of t. We may use the results obtained above for a_1 and a_3 to gain similar results for c.

We have bounds for $\|\nabla \mu c\|_{L^2}$ and for $\|\nabla(1-\mu)c\|_{L^2}$ from earlier work [Lemmas 7.6 and 7.3].

$$\begin{aligned}
\|\nabla \mu c\|_{L^2} &\leq C\|P(\mu a)\|_{L^2}, \\
\|\nabla(1-\mu)c\|_{L^2} &\leq C\|P((1-\mu)a)\|_{L^2}.
\end{aligned}$$

Thus since $\|\nabla c\|_{L^2} \leq \|\nabla \mu c\|_{L^2} + \|\nabla(1-\mu)c\|_{L^2}$, we have

$$\|\nabla c\|_{L^2} \leq C(\|P(\mu a)\|_{L^2} + \|P((1-\mu)a)\|_{L^2}).$$

But

$$\|P(\mu a)\|_{L^2} + \|P((1-\mu)a)\|_{L^2} \leq \|Pa\|_{L^2} + C(\|a\|_{L^2}\|\nabla\mu\|_{C^0})$$

.

Thus we obtain, since $\|\nabla\mu\|_{C^0}$ is independent of t, and we already have the result $\|a\|_{L^2} \leq C\|Pa\|_{L^2}$,

$$\|\nabla c\|_{L^2} \leq C\|Pa\|_{L^2}.$$

Similarly, we get from the earlier bounds [Lemma 7.6] on $\|\nabla_2 a_2\|_{L^2}$ a bound on $\|\nabla_2 b\|_{L^2}$, namely

$$\|\nabla_2 b\|_{L^2} \leq C\|Pa\|_{L^2}.$$

The higher derivatives of c follow similarly:

$$\begin{aligned}\|\nabla^2 c\| &\leq \|\nabla^2 \mu c\|_{L^2} + \|\nabla^2 (1-\mu)c\|_{L^2} \\ &\leq C(\|\nabla P(\mu a)\|_{L^2} + t^{-1}\|P(\mu a)\|_{L^2} + \|\nabla P((1-\mu)a)\|_{L^2} + t^{-1}\|P((1-\mu)a)\|_{L^2}).\end{aligned}$$

Now

$$\|\nabla P(\mu a)\|_{L^2} + \|\nabla P((1-\mu)a)\|_{L^2} \leq \|\nabla P a\|_{L^2} + C(\|a\|_{L^2}\|\nabla^2\mu\|_{C^0} + \|\nabla a\|_{L^2}\|\nabla\mu\|_{C^0}).$$

Using this, together with the previous bounds on $\|a\|_{L^2}$ and $\|\nabla a\|_{L^2}$, remembering that the C^0 norms of μ are independent of t, we have

$$\|\nabla^2 c\|_{L^2} \leq C(\|\nabla P a\|_{L^2} + t^{-1}\|Pa\|_{L^2}).$$

A similar argument gives

$$\|\nabla_2^2 b\|_{L^2} \leq C(\|\nabla P a\|_{L^2} + t^{-1}\|Pa\|_{L^2}).$$

For $\nabla^3 a$, we see that

$$\begin{aligned}\|\nabla^3 c\|_{L^2} &\leq \|\nabla^3 \mu c\|_{L^2} + \|\nabla^3(1-\mu)c\|_{L^2} \\ &\leq C(\|\nabla^2 P(\mu a)\|_{L^2} + t^{-1}\|\nabla P(\mu a)\|_{L^2} + t^{-2}\|P(\mu a)\|_{L^2} \\ &+ \|\nabla^2 P((1-\mu)a)\|_{L^2} + t^{-1}\|\nabla P((1-\mu)a)\|_{L^2} + t^{-2}\|P((1-\mu)a)\|_{L^2}).\end{aligned}$$

But

$$\begin{gathered}\|\nabla^2 P(\mu a)\|_{L^2} + \|\nabla^2 P((1-\mu)a)\|_{L^2} \\ \leq \|\nabla^2 P a\|_{L^2} + C(\|\nabla^2 a\|_{L^2}\|\nabla\mu\|_{C^0} + \|\nabla a\|_{L^2}\|\nabla^2\mu\|_{C^0} + \|a\|_{L^2}\|\nabla^2\mu\|_{C^0}).\end{gathered}$$

Thus we may deduce

$$\|\nabla^3 c\|_{L^2} \leq C(\|\nabla^2 P a\|_{L^2} + t^{-1}\|\nabla P a\|_{L^2} + t^{-2}\|Pa\|_{L^2}).$$

Once again, we may obtain the same bound for $\|\nabla_2^3 b\|_{L^2}$. We may summarise these results in the following theorem.

Theorem 7.2 *The following bounds hold on the whole manifold with the splitting $a = b + c$ as above:*

$$\begin{aligned}\|a\|_{L^2} &\leq C\|Pa\|_{L^2}, \\ \|\nabla c\|_{L^2} &\leq C\|Pa\|_{L^2}, \\ \|\nabla_2 b\|_{L^2} &\leq C\|Pa\|_{L^2}, \\ \|\nabla^2 c\|_{L^2} &\leq C(\|\nabla P a\|_{L^2} + t^{-1}\|Pa\|_{L^2}), \\ \|\nabla_2^2 b\|_{L^2} &\leq C(\|\nabla P a\|_{L^2} + t^{-1}\|Pa\|_{L^2}), \\ \|\nabla^3 c\|_{L^2} &\leq C(\|\nabla^2 P a\|_{L^2} + t^{-1}\|\nabla P a\|_{L^2} + t^{-2}\|Pa\|_{L^2}), \\ \|\nabla_2^3 b\|_{L^2} &\leq C(\|\nabla^2 P a\|_{L^2} + t^{-1}\|\nabla P a\|_{L^2} + t^{-2}\|Pa\|_{L^2}).\end{aligned}$$

7.5 Results about Quadratics

Let us proceed by obtaining estimates for the quadratic function Q, along with its derivatives.

We may see that there exists a constant C such that

$$\|Q(a)\|_{L^2} \leq C\|a\|_{L^4}^2.$$

But

$$\|a\|_{L^4}^2 \leq 2(\|c\|_{L^4}^2 + \|b\|_{L^4}^2).$$

Before using any Sobolev embedding theorems, it is worth noting that the constants involved will be independent of t. This is a result we will use greatly in this and the following chapter. The sort of result I would like to be able to make use of is

$$\|a\|_{L^{8/3}} \leq C\|a\|_{L_1^2},$$

with the constant C independent of the small parameter t.

Note that this result is equivalent to the independence of the Sobolev constants for functions, since

$$\|\nabla^j|\nabla^k a|\|_{L^2} \leq \|\nabla^{j+k}a\|_{L^2},$$

and thus we may use the result for $f =$ IY'kal.

The result for functions on the part of the manifold with high instanton curvature follows from the fact that the function norm is independent of the instanton.

Thus we are left to consider the Eguchi-Hanson parts. For any function f compactly supported on $T^4 \times E - H$, where the $E - H$ is scaled by a factor t, we may write

$$f = f_0 + f_1,$$

where f_0 is a function independent of T^4, and f_1 has integral zero over T^4.

From scaling results on the Eguchi-Hanson space, we see that

$$\left(\int_{T^4 \times E - H} |f_0|^4\right)^{1/2} \leq C \int |\nabla_2 f|^2,$$

where ∇_1 denotes differentiation in the T^4 direction, and ∇_2 denotes differentiation in the $E - H$ direction.

For any fixed $x \in E - H$, we have that

$$\left(\int_{T^4 \times \{x\}} |f_1|^4\right)^{1/2} \leq C \int_{T^4 \times \{x\}} |\nabla_1 f_1|^2.$$

Integrating over the Eguchi-Hanson space, we see

$$\int_{E-H}\left(\int_{T^4}|f_1|^4\right)^{1/2} \leq C\int_{T^4\times E-H}|\nabla_1 f_1|^2.$$

Since $\nabla_1 f_0 = 0$, we have $\nabla_1 f_1 = \nabla_1 f$, and hence

$$\int_{E-H}\left(\int_{T^4}|f_1|^4\right)^{1/2} \leq C\int_{T^4\times E-H}|\nabla_1 f|^2.$$

Similarly, by first restricting to $y \in T^4$, and then integrating the inequality we obtain, we see

$$\int_{T^4}\left(\int_{E-H}|f|^4\right)^{1/2} \leq C\int_{T^4\times E-H}|\nabla_2 f|^2.$$

This equation is conformally invariant, and henec the constant C is independent of t.

Now we use a proposition that I will state without proof, as the proof is long and uninstructive.

Proposition 7.4 *Let M and N be manifolds. Then*

$$\left(\int_{M\times N}|f|^{8/3}\right)^{3/4} \leq \left(\int_N\left(\int_M|f|^4\right)^{1/2}\right)^{1/2}\left(\int_M\left(\int_N|f|^4\right)^{1/2}\right)^{1/2}.$$

Thus we may deduce that for a compactly supported function f on $T^4 \times (E - H$ scaled by $t)$, we have

$$\|f\|_{L^{8/3}} \leq C\|\nabla f\|_{L^2},$$

with C having no t-dependence.

Now let us consider a cut-off function β with bounded derivatives in an annulus around the Eguchi-Hanson space. We have, by the remarks above, that

$$\begin{aligned}
\|\beta f\|_{L^{8/3}} &\leq C\|\nabla(\beta f)\|_{L^2} \\
&\leq C\left(\|\nabla f\|_{L^2} + \|d\beta\|_{C^0}.\|f\|_{L^2}\right) \\
&\leq C\|f\|_{L^2_1},
\end{aligned}$$

with C independent of t.

We may obtain a similar bound upon $\|(1-\beta)f\|_{L^{8/3}}$, since the injectivity radius is independent of t on the support of $(1-\beta)$. Thus we obtain the result that the Sobolev constant in the Sobolev embedding $L^2_1 \to L^{8/3}$ is independent of t.

Following the same method, and considering higher derivatives, we may see that other Sobolev constants, in the embeddings $L^2_2 \to L^4$ and $L^2_3 \to L^8$ are also independent of t.

There exists a constant C independent of t such that

$$\|f\|_{L^{8/3}} \leq C\|f\|_{L^2_1},$$

$$\|f\|_{L^4} \leq C\|f\|_{L^2_2},$$

$$\|f\|_{L^8} \leq C\|f\|_{L^2_3}.$$

In 8 dimensions, we have the Sobolev embedding $L^2_2 \to L^4$, and hence we see that

$$\begin{aligned}
\|c\|_{L^4} &\leq C\left(\|c\|_{L^2} + \|\nabla c\|_{L^2} + \|\nabla^2 c\|_{L^2}\right) \\
&\leq C\left(\|Pa\|_{L^2} + \|Pa\|_{L^2} + \|\nabla Pa\|_{L^2} + t^{-1}\|Pa\|_{L^2}\right) \\
&\leq C\left(\|\nabla Pa\|_{L^2} + t^{-1}\|Pa\|_{L^2}\right).
\end{aligned}$$

With b, however, we need not use the Sobolev embedding in 8 dimensions, but rather the more advantageous 4 dimensional Sobolev embedding, $L^2_1 \to L^4$.

This is because

$$\begin{aligned}
\|b\|_{L^4} &= \left(\int_N \int_{B^4} \sum |f_i w_i|^4 dx dy\right)^{1/4} \\
&= \sum \left(\int_N |f_i|^4 dx\right)^{1/4} \left(\int_{B^4} |w_i|^4 dy\right)^{1/4} \\
&= \sum \|f_i\|_{L^4(N)} \|w_i\|_{L^4(B^4)} \\
&\leq \sum C\|f_i\|_{L^2_1(N)} t^{-1} \\
&\leq Ct^{-1}\left(\|\nabla_2 b\|_{L^2} + \|b\|_{L^2}\right).
\end{aligned}$$

We have $\|\nabla_2 b\|_{L^2}$ and $\|b\|_{L^2}$ are both bounded by a constant multiple of $\|Pa\|_{L^2}$.

So we get

$$\|a\|_{L^4} \leq C\left(\|\nabla Pa\|_{L^2} + t^{-1}\|Pa\|_{L^2}\right), \tag{7.8}$$

and thus we get our required bound on $\|Q(a)\|_{L^2}$.

Lemma 7.10

$$\|Q(a)\|_{L^2} \leq C\left(\|\nabla Pa\|_{L^2} + t^{-1}\|Pa\|_{L^2}\right)^2.$$

We may continue by obtaining a similar bound on $\nabla Q(a)$. First, we note that

$$\begin{aligned}\|\nabla Q(a)\|_{L^2} &\leq C\|\nabla a.a\|_{L^2}\\ &\leq C\|\nabla a\|_{L^4}\|a\|_{L^4}.\end{aligned}$$

Of course, we may use the bound obtained earlier [equation 7.8] for $\|a\|_{L^4}$. Also, we have

$$\|\nabla a\|_{L^4} \leq C\left(\|\nabla_1 b\|_{L^4} + \|\nabla_2 b\|_{L^4} + \|\nabla c\|_{L^4}\right).$$

Using the Sobolev embedding $L^2_2 \to L^4$ in 8 dimensions, we see that

$$\begin{aligned}\|\nabla c\|_{L^4} &\leq C\left(\|\nabla^3 c\|_{L^2} + \|\nabla^2 c\|_{L^2} + \|\nabla c\|_{L^2}\right)\\ &\leq C\left(\|\nabla^2 Pa\|_{L^2} + t^{-1}\|\nabla Pa\|_{L^2} + t^{-2}\|Pa\|_{L^2}\right).\end{aligned}$$

Now

$$\begin{aligned}\|\nabla_1 b\|_{L^4} &\leq C\sum\|\nabla_1 f_i w_i\|_{L^4}\\ &\leq C\sum\|f_i\|_{L^4(N)}\|\nabla_1 w_i\|_{L^4}\\ &\leq Ct^{-2}\left(\|b\|_{L^2} + \|\nabla_2 b\|_{L^2}\right)\\ &\leq Ct^{-2}\|Pa\|_{L^2}.\end{aligned}$$

Finally, again using the Sobolev embedding $L^2_1 \to L^4$ in 4 dimensions,

$$\begin{aligned}\|\nabla_2 b\|_{L^4} &\leq C\sum\|\nabla_2 f_i w_i\|_{L^4}\\ &\leq C\sum\|\nabla_2 f_i\|_{L^4}\|w_i\|_{L^4}\\ &\leq C\sum t^{-1}\|\nabla_2 f_i\|_{L^4}\\ &\leq C\sum t^{-1}\left(\|\nabla_2^2 f_i\|_{L^2} + \|\nabla_2 f_i\|_{L^2}\right)\\ &\leq Ct^{-1}\left(\|\nabla_2^2 b\|_{L^2} + \|\nabla_2 b\|_{L^2}\right)\\ &\leq Ct^{-1}\left(\|\nabla Pa\|_{L^2} + t^{-1}\|Pa\|_{L^2}\right).\end{aligned}$$

These results together give

$$\|\nabla a\|_{L^4} \leq C\left(\|\nabla^2 Pa\|_{L^2} + t^{-1}\|\nabla Pa\|_{L^2} + t^{-2}\|Pa\|_{L^2}\right).$$

This gives us the lemma:

Lemma 7.11

$\|\nabla Q(a)\|_{L^2} \leq C\left(\|\nabla Pa\|_{L^2} + t^{-1}\|Pa\|_{L^2}\right)\left(\|\nabla^2 Pa\|_{L^2} + t^{-1}\|\nabla Pa\|_{L^2} + t^{-2}\|Pa\|_{L^2}\right).$

The last of our requirements to begin the iteration will be a bound on $\nabla^2 Q(a)$.

Note that

$$\begin{aligned}\|\nabla^2 Q(a)\|_{L^2} &\leq C\|\nabla a.\nabla a + a.\nabla^2 a\|_{L^2} \\ &\leq C\left(\|\nabla a\|_{L^4}^2 + \|c\|_{L^8}\|\nabla^2 c\|_{L^{8/3}} + \|b.\nabla^2 b\|_{L^2}\right),\end{aligned}$$

by Holder inequalities.

We already have bounds on $\|\nabla a\|_{L^2}$, so let us proceed by calculating bounds on the other two parts.

Now since $L_3^2 \to L^8$, we have that

$$\begin{aligned}\|c\|_{L^8} &\leq C\left(\|\nabla^3 c\|_{L^2} + \|\nabla^2 c\|_{L^2} + \|\nabla c\|_{L^2} + \|c\|_{L^2}\right) \\ &\leq C\left(\|\nabla^2 Pa\|_{L^2} + t^{-1}\|\nabla Pa\|_{L^2} + t{-}2\|Pa\|_{L^2}\right).\end{aligned}$$

In 8 dimensions, we also have $L_1^2 \to L^{8/3}$, and hence can deduce

$$\begin{aligned}\|\nabla^2 c\|_{L^{8/3}} &\leq C\left(\|\nabla^3 c\|_{L^2} + \|\nabla^2 c\|_{L^2}\right) \\ &\leq C\left(\|\nabla^2 Pa\|_{L^2} + t^{-1}\|\nabla Pa\|_{L^2} + t^{-2}\|Pa\|_{L^2}\right).\end{aligned}$$

Considering the b part, we see

$$\begin{aligned}\|b.\nabla^2 b\|_{L^2} &\leq C\left(\|b.\nabla_1^2 b\|_{L^2} + \|b.\nabla_1\nabla_2 b\|_{L^2} + \|b.\nabla_2^2 b\|_{L^2}\right) \\ &\leq C\left(t^{-2}\|b\|_{L^4}^2 + t^{-1}\|b\|_{L^4}\|\nabla_2 b\|_{L^4} + \|b\|_{L^4}\|\nabla_2^2 b\|_{L^4}\right).\end{aligned}$$

From earlier [Equation 7.8] work, we have

$$\begin{aligned}\|b\|_{L^4} &\leq Ct^{-1}\left(\|\nabla_2 b\|_{L^2} + \|b\|_{L^2}\right), \\ \|\nabla_2 b\|_{L^4} &\leq Ct^{-1}\left(\|\nabla_2^2 b\|_{L^2} + \|\nabla_2 b\|_{L^2}\right), \\ \|\nabla_2^2 b\|_{L^4} &\leq Ct^{-1}\left(\|\nabla_2^3 b\|_{L^2} + \|\nabla_2^2 b\|_{L^2}\right).\end{aligned}$$

Hence

$$\begin{aligned}&\|b.\nabla^2 b\|_{L^2} \\ \leq\ & C[t^{-4}\left(\|\nabla_2 b\|_{L^2} + \|b\|_{L^2}\right)^2 + t^{-3}(\|\nabla_2 b\|_{L^2} + \|b\|_{L^2})\left(\|\nabla_2^2 b\|_{L^2} + \|\nabla_2 b\|_{L^2}\right) \\ +\ & t^{-2}\left(\|\nabla_2 b\|_{L^2} + \|b\|_{L^2}\right)\left(\|\nabla_2^3 b\|_{L^2} + \|\nabla_2^2 b\|_{L^2}\right)] \\ \leq\ & C\left(t^{-4}\|Pa\|_{L^2}^2 + t^{-3}\|Pa\|_{L^2}\left(\|\nabla Pa\|_{L^2} + t^{-1}\|Pa\|_{L^2}\right)\right. \\ +\ & t^{-2}\|Pa\|_{L^2}\left(\|\nabla^2 Pa\|_{L^2} + t^{-1}\|\nabla Pa\|_{L^2} + t^{-2}\|Pa\|_{L^2}\right)).\end{aligned}$$

Thus we may deduce the following lemma:

Lemma 7.12

$$\|Q(a)\|_{L^2} \leq C\left(\|\nabla^2 Pa\|_{L^2} + t^{-1}\|\nabla Pa\|_{L^2} + t^{-2}\|Pa\|_{L^2}\right)^2.$$

We may summarise our results in the following theorem.

Theorem 7.3 *For a that is L^2-orthogonal to W we have*

$$\begin{aligned}
\|Q(a)\|_{L^2} &\leq C\left(\|\nabla Pa\|_{L^2} + t^{-1}\|Pa\|_{L^2}\right)^2, \\
\|\nabla Q(a)\|_{L^2} &\leq C\left(\|\nabla Pa\|_{L^2} + t^{-1}\|Pa\|_{L^2}\right)\left(\|\nabla^2 Pa\|_{L^2} + t^{-1}\|\nabla Pa\|_{L^2} + t^{-2}\|Pa\|_{L^2}\right), \\
\|\nabla^2 Q(a)\|_{L^2} &\leq C\left(\|\nabla^2 Pa\|_{L^2} + t^{-1}\|\nabla Pa\|_{L^2} + \|Pa\|_{L^2}\right)^2.
\end{aligned}$$

With these analytic bounds on norms, let us proceed with the iterative solution.

Chapter 8

Iterative Solution to the Instanton Equation

With the results from the last chapter, we may now obtain our pay-off: the construction of a $Spin(7)$ instanton. We do this, as I have mentioned before, through the method of successive iteration of the equation:

$$P(a_{j+1}) = Q(a_j) + \epsilon.$$

If P is sujective, then it is clear that this is always possible.

Note that we may select a_{j+1} so that it is L^2-orthogonal to the space W mentioned in previous chapters, so that all our estimates hold. We may do this by selecting an arbitrary solution to the above equation, and then adding on an element of $\mathrm{Ker}(P)$ to ensure that a is L^2-orthogonal to W. Of course, adding on an element of $\mathrm{Ker}(P)$ will not change the fact that a satisfies the above equation.

We shall consider the sequence ai with $a_0 = 0$.

8.1 Some Estimates on a_1

Lemma 8.1 *There exists a constant C such that*

$$\langle P_2^* \epsilon, b_1 \rangle \leq C t^4 \|b_1\|_{L^2}.$$

Proof.

First, we may note that we have that $P_2^* \epsilon$ is of the same order as ϵ itself. Thus we have

$$
\begin{aligned}
|P_2^* \epsilon| &\leq Ct^2, \qquad r < t \\
&\leq Ct^6 r^{-4}, \qquad t < r < 1/2 \\
&\leq Ct^2, \qquad 1/2 < r.
\end{aligned}
$$

Now $b_1 = \sum f_i w_i$, and we know the size of $|w_i|$:

$$
\begin{aligned}
|w_i| &\leq Ct^{-2}, \qquad r < t \\
&\leq Ct^2 r^{-4}, \qquad t < r.
\end{aligned}
$$

Thus we may calculate

$$
\begin{aligned}
\langle P_2^* \epsilon, w_i \rangle &\leq C \left(\int_{r<t} 1 + \int_{t<r<1/2} t^8 r^{-8} r^3 dr + \int_{1/2<r} t^4 r^{-4} r^3 dr \right) \\
&\leq Ct^4.
\end{aligned}
$$

But

$$
\begin{aligned}
\langle P_2^* \epsilon, b_1 \rangle &\leq C \sum_i \|f_i\|_{L^2} \langle P_2^* \epsilon, w_i \rangle \\
&\leq Ct^4 \|b_1\|_{L^2}.
\end{aligned}
$$

□

Using this, we may calculate estimates for the components of a_1.

Theorem 8.1 *The following bounds hold for $a_1 = b_1 + c_1$, the splitting of a_1 being as mentioned before.*

$$
\begin{aligned}
||c_1||_{L^2} &\leq Ct^2, \\
||\nabla c_1||_{L^2} &\leq Ct^2, \\
||\nabla^2 c_1||_{L^2} &\leq Ct^2, \\
||\nabla^3 c_1||_{L^2} &\leq Ct^2, \\
||b_1||_{L^2} &\leq Ct^4, \\
||\nabla_2 b_1||_{L^2} &\leq Ct^4, \\
||\nabla b_1||_{L^2} &\leq Ct^3, \\
||\nabla^2 b_1||_{L^2} &\leq Ct^2, \\
||\nabla^3 b_1||_{L^2} &\leq Ct.
\end{aligned}
$$

Proof.

Since $Pa_1 = \epsilon$, we may obtain bounds on it and its derivatives, using estimates obtained earlier,[7.1] since all bounds on ϵ are of the order t^2. Thus we have

$$\begin{aligned} ||Pa_1||_{L^2} &\leq Ct^2, \\ ||\nabla Pa_1||_{L^2} &\leq Ct^2 \\ ||\nabla^2 Pa_1||_{L^2} &\leq Ct^2. \end{aligned}$$

Considering the bounds on b_1, we shall now use Lemma 8.1.

Now

$$\epsilon = Pc_1 + P_1 b_1 + P_2 b_1,$$

and hence

$$P_2^* \epsilon = P_2^* P c_1 + P_2^* P_1 b_1 + P_2^* P_2 b_1.$$

Thus

$$\begin{aligned} \langle P_2^* \epsilon, b_1 \rangle &= \langle P_2^* P_2 c_1, b_1 \rangle + \langle P_2^* P_1 c_1, b_1 \rangle + \langle P_2^* P_2 b_1, b_1 \rangle + \langle P_2^* P_1 b_1, b_1 \rangle \\ &= \langle c_1, P_2^* P_2 b_1 \rangle + \langle P_2^* P_1 c_1, b_1 \rangle + \langle P_2^* P_1 b_1, b_1 \rangle + \|P_2 b_1\|_{L^2}^2. \end{aligned}$$

We may note that $P_2^* P_2 b_1$ is orthogonal to c_1, since P_2 acts only on the f_i part of b_1. Thus the inner product is zero. Using the fact that $P_2^* P_1 + P_1^* P_2 = 0$, we see

$$\langle P_2^* \epsilon, b_1 \rangle = \|P_2 b_1\|_{L^2}^2 + \langle P_2 b_1, P_1 b_1 \rangle - \langle P_2^* P_1 b_1, c_1 \rangle.$$

Looking first at the middle term on the right hand side, we see

$$\langle P_2 b_1, P_1 b_1 \rangle = \langle P_1^* P_2 b_1, b_1 \rangle$$

$$= \langle -P_2^* P_1 b_1, b_1 \rangle = -\langle P_2 b_1, P_1 b_1 \rangle.$$

Hence it is zero.

Let us consider now the last of these terms. We have that

$$\begin{aligned} \|P_2^* P_1 b_1\|_{L^2} &\leq Ct^2 \|\nabla_2 b_1\| \\ &\leq Ct^2 \|P_2 b_1\|_{L^2}, \end{aligned}$$

since the P_1 acts only on the w_i parts of b.

Thus we have

$$\begin{aligned} \langle P_2^* P_1 b_1, c_1 \rangle &\leq Ct^2 \|P_2 b_1\|_{L^2} \|c_1\|_{L^2} \\ &\leq Ct^4 \|P_2 b_1\|_{L^2}. \end{aligned}$$

Hence

$$\|P_2 b_1\|_{L^2}^2 \leq Ct^4 \|P_1 b_1\|_{L^2} + Ct^4 + Ct^4 \|b_1\|_{L^2},$$

using the preceeding lemma.

We have that, since P_1 acts only on the w_i part of b_1,

$$\|P_1 b_1\|_{L^2} \leq Ct^2 \|b_1\|_{L^2}.$$

From earlier work (see Lemma 7.4), we see that

$$\|b_1\|_{L^2} \leq C\|P_2 b_1\|_{L^2}.$$

Hence we may deduce that

$$\|P_2 b_1\|_{L^2}^2 \leq C\left(t^2 \|P_2 b_1\|_{L^2}^2 + t^4 \|P_2 b_1\|_{L^2}\right).$$

Thus we must have

$$\|P_2 b_1\|_{L^2} \leq Ct^4,$$

and so we have

$$\|b_1\|_{L^2} \leq Ct^4.$$

From this, we may easily deduce the bounds on the derivatives of b_1 by splitting up ∇ as $\nabla = \nabla_1 + \nabla_2$, and using the bounds on the derivatives of Pa_1.

We see that

$$\|\nabla b_1\|_{L^2} \leq \|\nabla_1 b_1\|_{L^2} + \|\nabla_2 b_1\|_{L^2},$$

and since

$$\|\nabla_1 b_1\|_{L^2} \leq Ct^{-1} \|b_1\|_{L^2},$$

and

$$\|\nabla_2 b_1\|_{L^2} \leq C\|P_2 b_1\|_{L^2},$$

we have our result that

$$\|\nabla b_1\| \leq Ct^3.$$

Higher derivatives follow similarly via the method of proof of Lemma 7.2.

The bounds for c_1 and its derivatives follow immediately from the proof of Theorem 7.2 in the previous chapter, together with the bounds on b_1, since

$$\begin{aligned}
\|c_1\|_{L^2} &\leq C\|Pa_1\|_{L^2}, \\
\|\nabla c_1\|_{L^2} &\leq C\|Pa_1\|_{L^2}, \\
\|\nabla^2 c_1\|_{L^2} &\leq C\left(\|\nabla Pa_1\| + t^{-1}\|Pb_1\|_{L^2}\right) \\
&\leq C\left(\|\nabla Pa_1\|_{L^2} + t^{-1}\left(\|P_1 b_1\|_{L^2} + \|P_2 b_1\|_{L^2}\right)\right), \\
\|\nabla^3 c_1\|_{L^2} &\leq C\left(\|\nabla^2 Pa_1\|_{L^2} + t^{-1}\|\nabla_2 Pb_1\|_{L^2} + t^{-2}\|Pb_1\|_{L^2}\right) \\
&\leq Cbigl[\|\nabla^2 Pa_1\|_{L^2} + t^{-1}\left(\|\nabla_2 P_1 b_1\|_{L^2} + \|\nabla_2 P_2 b_2\|_{L^2}\right) \\
&+ t^{-2}\left(\|P_1 b_1\|_{L^2} + \|P_2 b_1\|_{L^2}\right)].
\end{aligned}$$

□

So we may deduce

$$\begin{aligned} ||Q(a_1)||_{L^2} &\leq Ct^2, \\ ||\nabla Q(a_1)||_{L^2} &\leq Ct, \\ ||\nabla^2 Q(a_1)||_{L^2} &\leq C. \end{aligned}$$

8.2 The Inductive Hypothesis

Using the notation $b_j^- = b_j - b_{j-1}$, $c_j^- = c_j - c_{j-1}$, whilst noting that

$$P(a_{j+1} - a_j) = Q(a_j) - Q(a_{j-1}),$$

and that for any quadratic function Q, there exist C_1, C_2 and C_3 such that

$$||Q(x) - Q(y)||_{L^2} \leq C_1 ||x - y||_{L^4} \left(||x||_{L^4} + ||y||_{L^4}\right),$$

we have that

$$\begin{aligned} \|Q(a_j) - Q(a_{j-1})\|_{L^2} \leq\ & C_1 \Big(||b_j^-||_{L^4} (||b_j||_{L^4} + ||b_{j-1}||_{L^4}) + ||c_j^-||_{L^4} (||b_j||_{L^4} + ||b_{j-1}||_{L^4}) \\ +\ & \Big| ||b_j^-||_{L^4} (||c_j||_{L^4} + ||c_{j-1}||_{L^4}) + ||c_j^-||_{L^4} (||c_j||_{L^4} + ||c_{j-1}||_{L^4}) \Big). \end{aligned}$$

Similarly,

$$||\nabla(Q(a_j) - Q(a_{j-1}))||_{L^2}$$

$$\begin{aligned} \leq\ & C_2(||\nabla b_j^-||_{L^4} (||b_j||_{L^4} + ||b_{j-1}||_{L^4}) + ||\nabla c_j^-||_{L^4} (||b_j||_{L^4} + ||b_{j-1}||_{L^4}) \\ +\ & ||\nabla b_j^-||_{L^4} (||c_j||_{L^4} + ||c_{j-1}||_{L^4}) + ||\nabla c_j^-||_{L^4} (||c_j||_{L^4} + ||c_{j-1}||_{L^4}) \\ +\ & ||b_j^-||_{L^4} (||\nabla b_j||_{L^4} + ||\nabla b_{j-1}||_{L^4}) + ||c_j^-||_{L^4} (||\nabla b_j||_{L^4} + ||\nabla b_{j-1}||_{L^4}) \\ +\ & ||b_j^-||_{L^4} (||\nabla c_j||_{L^4} + ||\nabla c_{j-1}||_{L^4}) + ||c_j^-||_{L^4} (||\nabla c_j||_{L^4} + ||\nabla c_{j-1}||_{L^4})), \end{aligned}$$

and

$$||\nabla^2(Q(a_j) - Q(a_{j-1}))||_{L^2}$$

$$\begin{aligned} \leq\ & C_3 \Big(||(b_j^- + c_j^-)||_{L^8} (||\nabla^2(b_j)||_{L^{8/3}} + ||\nabla^2(b_{j-1})||_{L^{8/3}} + ||\nabla^2(c_j)||_{L^{8/3}} + ||\nabla^2(c_{j-1})||_{L^{8/3}}) \\ +\ & ||\nabla(b_j^- + c_j^-)||_{L^4} (||\nabla(b_j)||_{L^4} + ||\nabla(b_{j-1})||_{L^4} + ||\nabla(c_j)||_{L^4} + ||\nabla(c_{j-1})||_{L^4}) \\ +\ & ||\nabla^2(b_j^- + c_j^-)||_{L^{8/3}} (||b_j||_{L^8} + ||b_{j-1}||_{L^8} + ||c_j||_{L^8} + ||c_{j-1}||_{L^8}). \end{aligned}$$

(8.2)

Let us state our inductive hypothesis on the values of the iterative terms b_j and c_j. It is somewhat long, but I hope it will be self explanatory, and the necessity of each term will become clearer later on.

Hypothesis 8.1 *Assume that for $j < k$ there exists a K_1 such that*

$$\begin{aligned}
||c_j||_{L^2} &\leq K_1 t^2, \quad ||c_j^-||_{L^2} \leq K_1 t^4 2^{-j}, \\
||\nabla c_j||_{L^2} &\leq K_1 t^2, \quad ||\nabla c_j^-||_{L^2} \leq K_1 t^4 2^{-j}, \\
||\nabla^2 c_j||_{L^2} &\leq K_1 t^2, \quad ||\nabla^2 c_j^-||_{L^2} \leq K_1 t^3 2^{-j}, \\
||\nabla^3 c_j||_{L^2} &\leq K_1 t^2, \quad ||\nabla^3 c_j^-||_{L^2} \leq K_1 t^2 2^{-j}, \\
||b_j||_{L^2} &\leq K_1 t^4, \quad ||b_j^-||_{L^2} \leq K_1 t^4 2^{-j}, \\
||\nabla_1 b_j||_{L^2} &\leq K_1 t^3, \quad ||\nabla_1 b_j^-||_{L^2} \leq K_1 t^3 2^{-j}, \\
||\nabla_2 b_j||_{L^2} &\leq K_1 t^4, \quad ||\nabla_2 b_j^-||_{L^2} \leq K_1 t^4 2^{-j}, \\
||\nabla_2^2 b_j||_{L^2} &\leq K_1 t^3, \quad ||\nabla_2^2 b_j^-||_{L^2} \leq K_1 t^3 2^{-j}, \\
||\nabla^2 b_j||_{L^2} &\leq K_1 t^2, \quad ||\nabla^2 b_j^-||_{L^2} \leq K_1 t^2 2^{-j}, \\
||\nabla^3 b_j||_{L^2} &\leq K_1 t, \quad ||\nabla^3 b_j^-||_{L^2} \leq K_1 t 2^{-j}.
\end{aligned}$$

8.3 Some Sobolev Embedding Results

Note that we then have the following:

Lemma 8.2 *With the assumptions of Hypothesis 8.1 , when $j < k$ there exists a constant K_2 such that*

$$\begin{aligned}
||c_j||_{L^{8/3}} &\leq K_2 t^2, \quad ||c_j^-||_{L^{8/3}} \leq K_2 t^4 2^{-j}, \\
||\nabla c_j||_{L^{8/3}} &\leq K_2 t^2, \quad ||\nabla c_j^-||_{L^{8/3}} \leq K_2 t^3 2^{-j}, \\
||\nabla^2 c_j||_{L^{8/3}} &\leq K_2 t^2, \quad ||\nabla^2 c_j^-||_{L^{8/3}} \leq K_2 t^2 2^{-j}, \\
||b_j||_{L^{8/3}} &\leq C t^3, \quad ||b_j^-||_{L^{8/3}} \leq K_2 t^3 2^{-j}, \\
||\nabla b_j||_{L^{8/3}} &\leq K_2 t^2, \quad ||\nabla b_j^-||_{L^{8/3}} \leq K_2 t^2 2^{-j}, \\
||\nabla^2 b_j||_{L^{8/3}} &\leq K_2 t, \quad ||\nabla^2 b_j^-||_{L^{8/3}} \leq K_2 t 2^{-j}.
\end{aligned}$$

Proof.

This is a consequence of the fact that in 8 dimensions the L_1^2 norms bounds the $L^{8/3}$ norm. Thus for any function x, we have

$$||x||_{L^{8/3}} \leq C_1 \left(||x||_{L^2} + ||\nabla x||_{L^2}\right).$$

Putting b and c and their derivatives as x, we obtain our result.

□

Lemma 8.3 *Again with the assumptions of Hypothesis 8.1 , with $j < k$ we have*

$$\begin{aligned} ||c_j||_{L^4} &\leq K_3 t^2, \quad ||c_j^-||_{L^4} \leq K_3 t^3 2^{-j}, \\ ||\nabla c_j||_{L^4} &\leq K_3 t^2, \quad ||\nabla c_j^-||_{L^4} \leq C t^2 2^{-j}, \\ ||b_j||_{L^4} &\leq K_3 t^3, \quad ||b_j^-||_{L^4} \leq K_3 t^3 2^{-j}, \\ ||\nabla b_j||_4 &\leq K_3 t^2, \quad ||\nabla b_j^-||_{L^4} \leq K_3 t^2 2^{-j}. \end{aligned}$$

Proof.

The inequalities for c follow from the Sobolev embedding $L_2^2 \to L^4$ in the same way as the lemma above.

Remembering $b = \sum f_i w_i$, we have

$$||b||_{L^4} \leq C \sum ||f_i||_{L^4} ||w_i||_{L^4}.$$

But f_i is a function on the 4-manifold N, hence we may use the Sobolev embedding $L_1^2 \to L^4$ on it to obtain

$$||f_i||_{L^4} \leq K \left(||\nabla_2 f_i||_{L^2} + ||f_i||_{L^2}\right).$$

Hence we have, since $||w_i||_{L^4} = O(t^{-1})$,

$$||b||_{L^4} \leq C t^{-1} \left(||b||_{L^2} + ||\nabla_2 b||\right),$$

which gives us the required result.

□

Lemma 8.4 *Again with the assumptions of Hypothesis 8.1 and $j < k$, we have*

$$\begin{aligned} ||c_j||_{L^8} &\leq K_4 t^2, \quad ||c_j^-||_{L^8} \leq K_4 t^2 2^{-j}, \\ ||b_j||_{L^8} &\leq K_4 t^{3/2}, \quad ||b_j^-||_{L^8} \leq K_4 t^{3/2} 2^{-j}. \end{aligned}$$

Proof.

We have the inequalities involving c from the Sobolev embedding $L_3^2 \to L^8$ in 8 dimensions.

Also, since

$$||b||_{L^8} \leq C \sum ||f_i||_{L^8} ||w_i||_{L^8},$$

the 4 dimensional Sobolev embedding $L_2^2 \to L^8$ becomes of use to us. (This is not the optimal embedding in 4 dimensions; however, it is sufficient for out purposes.) Noting that

$$||f_i||_{L^8} \leq C \left(||f_i||_{L^2} + ||\nabla_2 f_i||_{L^2} + ||\nabla_2^2 f_i||_{L^2} \right),$$

and that $||w_i||_{L^8} = O(t^{-3/2})$, we may obtain

$$||b||_{L^8} \leq C t^{-3/2} \left(||b||_{L^2} + ||\nabla_2 b||_{L^2} + ||\nabla_2^2 b||_{L^2} \right).$$

□

8.4 Initial estimates for the hypothesis

Now let us work towards showing that Hypothesis 8.1 holds in the case $j = 2$.We shall use the bounds obtained on a_1.

Proposition 8.1 *Hypothesis 8.1 holds for $j = 2$.*

Proof.

Firstly, we note that we need only prove the right-hand inequalities, as these, together with the above inequalities on a_1 will give us the results we require.

Noting $P(a_2 - a_1) = Q(a_1)$, we may obtain estimates for $P(a_2 - a_1)$ and its derivatives using the above estimates (Theorm 8.1) for norms of a_1.

$$\begin{aligned} ||P(a_2 - a_1)||_{L^2} &= \|Qa_1\|_{L^2} \\ &\leq C||a_1||_{L^4}^2 \\ &\leq Ct^2, \end{aligned}$$

by the Sobolev embedding $L_2^2 \to L^4$ in 8 dimensions.

In addition,

$$\begin{aligned} ||\nabla P(a_2 - a_1)||_{L^2} &= \|\nabla Q a_1\|_{L^2} \\ &\leq C||a_1||_{L^4} ||\nabla a_1||_{L^4}. \end{aligned}$$

But

$$||\nabla c_1||_{L^4} \leq O(t^2),$$

by the same Sobolev embedding, and

$$\begin{aligned}
||\nabla b_1||_{L^4} &\leq C\left(||\nabla_1 b_1||_{L^4} + ||\nabla_2 b_1||_{L^4}\right) \\
&\leq C\left(t^{-1}||b_1||_{L^4} + t^{-1}\left(||\nabla_2^2 b_1||_{L^2} + ||\nabla_2 b_1||_{L^2}\right)\right) \\
&\leq \left(t^{-2}\left(\|\nabla_2 b_1\|_{L^2} + \|b_1\|_{L^2}\right) + t^{-1}\left(\|\nabla_2 b_1\|_{L^2} + \|b_1\|_{L^2}\right)\right. \\
&\leq Ct^2.
\end{aligned}$$

Hence

$$||\nabla P(a_2 - a_1)||_{L^2} \leq Ct^4.$$

Also,

$$\begin{aligned}
||\nabla^2 P(a_2 - a_1)||_{L^2} &= \|\nabla^2 Q a_1\| \\
&\leq C\left(||\nabla a_1||_{L^4}^2 + ||a_1||_{L^8}.||\nabla^2 a_1||_{L^{8/3}}\right).
\end{aligned}$$

Now we have bounds on $||\nabla a_1||_{L^4}^2$. Bounds on the c_1 component of a_1 are easy to obtain: we use the Sobolev embeddings $L_3^2 \to L^8$ and $L_1^2 \to L^{8/3}$ to obtain that both are of the order t^2.

Looking at

$$||b_1||_{L^8} \leq Ct^{-3/2}\left(||\nabla_2^2 b_1||_{L^2} + ||\nabla_2 b_1||_{L^2} + \|b_1\|_{L^2}\right),$$

as $L_2^2 \to L^8$ in 4 dimensions. Thus $||b_1||_{L^8}$ is of the order $t^{3/2}$. From the inequality

$$||\nabla^2 b_1||_{L^{8/3}} \leq C\left(||\nabla^3 b_1||_{L^2} + ||\nabla^2 b_1||_{L^2}\right),$$

we obtain that $\nabla^2 P(a_2 - a_1)$ is of the order $t^{5/2}$.

Proceeding using the inequalities relating the components of a with Pa, we obtain

$$\begin{aligned}
||c_2 - c_1||_{L^2} &\leq C||P(a_2 - a_1)||_{L^2} \leq K_1 t^4, \\
||\nabla(c_2 - c_1)||_{L^2} &\leq C||P(a_2 - a_1)||_{L^2} \leq K_1 t^4, \\
||\nabla^2(c_2 - c_1)||_{L^2} &\leq C\left(||\nabla P(a_2 - a_1)||_{L^2} + t^{-1}||P(a_2 - a_1)||_{L^2}\right) \leq K_1 t^3, \\
||\nabla^3(c_2 - c_1)||_{L^2} &\leq C\left(||\nabla^2 P(a_2 - a_1)||_{L^2} + t^{-1}||\nabla P(a_2 - a_1)||_{L^2} + t^{-2}||P(a_2 - a_1)||_{L^2}\right) \\
&\leq K_1 t^2, \\
||b_2 - b_1||_{L^2} &\leq C||P(a_2 - a_1)||_{L^2} \leq K_1 t^4, \\
||\nabla_1(b_2 - b_1)||_{L^2} &\leq Ct^{-1}||b_2 - b_1||_{L^2} \leq K_1 t^3,
\end{aligned}$$

$$\begin{aligned}
||\nabla_2(b_2-b_1)||_{L^2} &\le C||P(a_2-a_1)||_{L^2} \le K_1 t^4,\\
||\nabla_2^2(b_2-b_1)||_{L^2} &\le C\left(||\nabla P(a_2-a_1)|| + t^{-1}||P(a_2-a_1)||_{L^2}\right) \le K_1 t^3,\\
||\nabla^2(b_2-b_1)||_{L^2} &\le C\left(||\nabla_2^2(b_2-b_1)||_{L^2} + ||\nabla_1\nabla_2(b_2-b_1)||_{L^2} + ||\nabla_2^2(b_2-b_1)||_{L^2}\right)\\
&\le C\left(t^{-2}||b_2-b_1||_{L^2} + t^1||\nabla_2(b_2-b_1)||_{L^2} + t^3\right) \le K_1 t^2,\\
||\nabla^3(b_2-b_1)||_{L^2} &\le C(t^{-3}||b_2-b_1||_{L^2} + t^{-2}||\nabla_2(b_2-b_1)||_{L^2}\\
&+ t^{-1}||\nabla_2^2(b_2-b_1)||_{L^2} + ||\nabla_2^3(b_2-b_1)||_{L^2})\\
&\le C\left(||\nabla^2 P(a_2-a_1)||_{L^2} + t^{-1}||\nabla P(a_2-a_1)||_{L^2} + t^{-2}||P(a_2-a_1)||_{L^2}\right)\\
&\le K_1 t,
\end{aligned}$$

for some constant K_1 independent of t. Hence Hypothesis 8.1 holds when $j=2$.

□

8.5 Inductive Step

With the base step of our proof done, let us proceed with the inductive step.

Proposition 8.2 *For small enough t (required size independent of k) if Hypothesis 8.1 is true for $j=k$, then it is true for $j=k+1$, with the same constant K_1.*

Proof.

Clearly, once more it suffices to show this with only the right hand inequalities, as these will imply the left hand inequalities of the Hypothesis 8.1.

Since $P(a_{k+1}-a_k) = Q(a_k) - Q(a_{k-1})$, we may obtain bounds on the various derivatives of the components of $a_{k+1}-a_k$ from estimates on $a_k - a_{k-1}$, a_k and a_{k-1}.

Again, we shall start by obtaining bounds on $P(a_{k+1}-a_k)$ and its derivatives.

$$\begin{aligned}
||P(a_{k+1}-a_k)||_{L^2} &\le C_1||a_k-a_{k-1}||_{L^4}\,(||a_k||_{L^4} + ||a_{k-1}||_{L^4})\\
&\le 2C_1K_3^2t^5 2^{-k},
\end{aligned}$$

from Lemma 8.3.

$$\begin{aligned}
||\nabla P(a_{k+1}-a_k)||_{L^2} &\le C_2(||a_k-a_{k-1}||_{L^4}\,(||\nabla a_k||_{L^4} + ||a_{k-1}||_{L^4})\\
&+ ||\nabla(a_k-a_{k-1})||_{L^4}\,(||a_k||_{L^4} + ||a_{k-1}||_{L^4}))\\
&\le 8C_2K_3^2t^4 2^{-k},
\end{aligned}$$

again from Lemma 8.3.

$$
\begin{aligned}
||\nabla^2 P(a_{k+1}-a_k)||_{L^2} &\leq C_3\,(||\nabla(a_k-a_{k-1})||_{L^4}\,(||\nabla a_k||_{L^4}+||\nabla a_{k-1}||_{L^4})\\
&+ ||\nabla^2(a_k-a_{k-1})||_{L^{8/3}}\,(||a_k||_{L^8}+||a_{k-1}||_{L^8})\\
&+ ||a_k-a_{k-1}||_{L^8}\left(||\nabla^2 a_k||_{L^{8/3}}+||\nabla^2 a_{k-1}||_{L^{8/3}}\right)\\
&\leq C_3(2K_3^2+4K_2K_4)t^{5/2}2^{-k},
\end{aligned}
$$

by Lemmas 8.2, 8.3 and 8.4.

Thus we may estimate the components of $a_{k+1}-a_k$ and their various derivatives.

$$
\begin{aligned}
||c_{k+1}^-||_{L^2} &\leq C||Pa_{k+1}^-||_{L^2}\leq K_5 2^{-k}t^5,\\
||\nabla c_{k+1}^-||_{L^2} &\leq C||Pa_{k+1}^-||_{L^2}\leq K_5 2^{-k}t^5,\\
||\nabla^2 c_{k+1}^-||_{L^2} &\leq C\left(||\nabla Pa_{k+1}^-||_{L^2}+t^{-1}||Pa_{k+1}^-||_{L^2}\right)\leq K_5 2^{-k}t^4,\\
||\nabla^3 c_{k+1}^-||_{L^2} &\leq C(||\nabla^2 Pa_{k+1}||_{L^2}+t^{-1}||\nabla Pa_{k+1}^-||_{L^2}\\
&+ t^{-2}||Pa_{k+1}^-||_{L^2})\\
&\leq K_5 2^{-k}t^{5/2},\\
||b_{k+1}^-||_{L^2} &\leq C||Pa_{k+1}^-||_{L^2}\leq K_5 2^{-k}t^5,\\
||\nabla_1 b_{k+1}^-||_{L^2} &\leq Ct^{-1}||b_{k+1}^-||_{L^2}\leq K_5 2^{-k}t^4,\\
||\nabla_2 b_{k+1}^-||_{L^2} &\leq C||Pa_{k+1}^-||_{L^2}\leq K_5 t^5 2^{-k}\\
||\nabla_2^2 b_{k+1}^-||_{L^2} &\leq C\left(||\nabla Pa_{k+1}^-||_{L^2}+t^{-1}||Pa_{k+1}^-||_{L^2}\right)\\
&\leq K_5 2^{-k}t^4,\\
||\nabla^2 b_{k+1}^-||_{L^2} &\leq C(||\nabla_1^2 b_{k+1}^-||_{L^2}+||\nabla_1\nabla_2 b_{k+1}^-||_{L^2}\\
&+ ||\nabla_2^2 b_{k+1}^-||_{L^2})\\
&\leq K_5 2^{-k}t^3,\\
||\nabla^3 b_{k+1}^-||_{L^2} &\leq C(||\nabla_1^3 b_{k+1}^-||_{L^2}+||\nabla_1^2\nabla_2 b_{k+1}^-||_{L^2}\\
&+ ||\nabla_1\nabla_2^2 b_{k+1}^-||_{L^2}+||\nabla_2^3 b_{k+1}^-||_{L^2})\\
&\leq C(t^{-3}||b_{k+1}^-||_{L^2}+t^{-2}||\nabla_2 b_{k+1}^-||_{L^2}\\
&+ t^{-1}||\nabla_2^2 b_{k+1}^-||_{L^2}+||\nabla^2 Pa_{k+1}^-||+t^{-1}||\nabla Pa_{k+1}^-||+t^{-2}Pa_{k+1}^-||)\\
&\leq K_5 2^{-k}t^{5/2},
\end{aligned}
$$

where K_5 is clearly independent of k, being a function of the numbers K_1, K_2, K_3, K_4, C_1, C_2, C_3 and the size of the bounds on the norms of the derivatives of a in terms of Pa obtained earlier[Lemma 7.2].

From these equations, it is clear that we have a finite number of constraints on t, each of which it is easy to see a bound on t such that the hypothesis will follow inductively. Consider the last but one equation, for example. If we choose t such that $t \leq K_1/(2K_5)$, then it is clear that if the equation $\|^2 b_i^-\|_{L^2} \leq K_1 2^{-i} t^2$ holds for $i = k$ then it will hold for $i = k+1$.

Thus may choose t small enough, independently of k such that Hypothesis 8.1 holding for k implies it holding for $k+1$, with the same constant, K_1.

□

Thus we may conclude that the series a_i does indeed converge to a limit in L_3^2. Our next task is to show that this solution is, in fact, smooth.

First let us consider the space $L_4^{16/9}$.

Lemma 8.5

$$\|Q(a) - Q(b)\|_{L_3^{16/9}} \leq C\|a-b\|_{L_3^2} \left(\|a\|_{L_4^{16/9}} + \|b\|_{L_4^{16/9}} \right).$$

Proof.

Note that

$$\begin{aligned} \|\nabla^3(Q(a) - Q(b))\|_{L^{16/9}} &\leq C(\left(\|\nabla^3 a\|_{L^{16/7}} + \|\nabla^3 b\|_{L^{16/7}}\right) \|a-b\|_{L^8} \\ &+ \left(\|\nabla^2 a\|_{L^{16/5}} + \|\nabla^2 b\|_{L^{16/5}}\right) \|\nabla(a-b)\|_{L^4}) \\ &+ (\|\nabla a\|_{L^{16/3}} + \|\nabla b\|_{L^{16/3}}) \|\nabla^2(a-b)\|_{L^{8/3}} \\ &+ (\|a\|_{L^{16}} + \|b\|_{L^{16}}) \|\nabla^3(a-b)\|_{L^2}. \end{aligned}$$

because of the Sobolev embeddings $L_k^2 \to L^{8/(4-k)}$ and $L_k^{16/9} \to L^{8/(9/2-k)}$ which all hold in 8-dimensions.

The lower derivatives of Q also follow from similar methods.

□

We have

$$\begin{aligned} \|a_j - a_i\|_{L_4^{16/9}} &\leq C\|P(a_j - a_i)\|_{L_3^{16/9}} \\ &\leq C\|Q(a_{j-1}) - Q(a_{i-1})\|_{L_3^{16/9}} \\ &\leq C\|a_{j-1} - a_{i-1}\|_{L_3^2} \left(\|a_{j-1}\|_{L_4^{16/9}} + \|a_{i-1}\|_{L_4^{16/9}} \right), \end{aligned}$$

and also we know that the sequence $\{a_i\}$ converges in L_3^2. Thus there exists an N such that for all $i, j \geq N$, we have

$$\|a_{j-1} - a_{i-1}\|_{L_3^2} \leq 1/(100C).$$

Thus for any $i \geq N$, we have

$$\|a_j - a_N\|_{L_4^{16/9}} \leq 1/100 \left(\|a_{j-1}\|_{L_4^{16/9}} + \|a_{N-1}\|_{L_4^{16/9}} \right).$$

Hence we may deduce the following lemma:

Lemma 8.6 *$\{a_j\}$ converges in $L_4^{16/9}$.*

Proof.
Note that the above equation gives us that

$$\|a_j\|_{L_4^{16/9}} \leq C + 1/100\|a_{j-1}\|_{L_4^{16/9}},$$

for large enough j.

Using induction, we see that we may see that we may obtain an aprior bound for $\|a_j\|_{L_4^{16/9}}$, and hence we see that this sequence must converge.

□

Lemma 8.7

$$a \in L_5^{8/5}.$$

Proof.
We have that

$$a \in L_4^{16/9},$$

from above, and hence we may deduce that a is in any of the Sobolev spaces L^p, with $p \leq 16$, using the Sobolev embedding theorem. [DK, p.166] . Thus we see

$$\begin{aligned}
\|a\|_{L_5^{8/5}} &\leq C\|Pa\|_{L_4^{8/5}} \\
&\leq C\left(\|Qa\|_{L_4^{8/5}} + \|\epsilon\|_{L_4^{8/5}}\right) \\
&\leq C\left(\|\nabla^4 a\|_{L^{16/9}}\|a\|_{L^{16}} + \|\nabla^3 a\|_{L^{16/7}}\|\nabla a\|_{L^{16/3}} + \|\nabla^2 a\|^2_{L^{16/5}} + \|\epsilon\|_{L_4^{8/5}}\right) \\
&\leq C\left(\|a\|^2_{L_4^{16/9}} + \|\epsilon\|_{L_4^{9/5}}\right).
\end{aligned}$$

As the right hand side is finite, we have a bound for the left hand side, and hence $a \in L_5^{8/5}$.

□

We may then use the Sobolev embedding $L_5^{8/5} \to L_4^2$ to obtain that $a \in L_4^2$. An identical argument to the lemma above will then give $a \in L_5^{9/5}$.

Now we may see, using a standard bootstrapping argument, that a will in fact be smooth.

Theorem 8.2 *a is smooth.*

Proof.
Assume, as an inductive hypothesis, that a is in L^2_k. Then we have that

$$\begin{aligned}
\|a\|_{L^2_{k+1}} &\leq C\|Pa\|_{L^2_k} \\
&\leq C\left(\|Qa\|_{L^2_k} + \|\epsilon\|_{L^2_k}\right) \\
&\leq C\left(\|a\|_{L^2_k}\|a\|_{C^0} + \|\epsilon\|_{L^2_k}\right) \\
&\leq C\left(\|a\|_{L^2_k}\|a\|_{L^{9/5}_5} + \|\epsilon\|_{L^2_k}\right),
\end{aligned}$$

because of the Sobolev embedding $L^{9/5}_5 \to C^0$. Hence we have that $a \in L^2_k$ for all k, and hence that it is smooth.

□

Theorem 8.3 *There exists a $Spin(7)$ instanton.*

Proof. See above.

□

This is, to the best of my knowledge, the first construction of a connection with curvature in the Λ^2_{21}-component of the 2-forms on a manifold with holonomy $Spin(7)$.

Again, I will point out that whilst the method only applies to the one specific case of a particular manifold, much of the analysis will generalise quite easily to other cases.

The existence of such an instanton opens up many questions about the moduli space of $Spin(7)$ instantons. We have seen that it is not compact, as we have bubbling around a Cayley submanifold a possible occurance. Whether this is the only case of a family of $Spin(7)$ instantons tending to a limit other that another instanton, I am unsure. If this were the only possiblity, we could 'compactify' the moduli space by attaching a copy of the moduli space of Cayley submanifolds. This, however, would raise the question as to what the behaviour of a family of $Spin(7)$ instantons is like near a degenerate Cayley submanifold. This is a question which has no parallel in the theory of 4-dimensional self- and anti-self-dual instantons, as here the bubbling occurs around a point.

We may also ask what properties of a compact $Spin(7)$ manifold will help determine whether a $Spin(7)$ instanton can exist upon it, and, in addition, if properties of the moduli space of $Spin(7)$ instantons will tell us much about the underlying manifold. This, contrasting with the previous question, is a subject which has been studied at depth on 4-manifolds.

Further, we could ask about gauge groups other that $SU(2)$. It was early on in this thesis that I restricted my attention to this gauge group, but by considering other gauge groups, we have that finite energy $Spin(7)$ instantons exist on $\mathbf{R}^8$. This suggests a great difference in the nature of the moduli space when considering different gauge groups.

There are many unanswered questions about $Spin(7)$ instantons, and whilst it would be extremely optimistic of me to believe that the study of instantons of this type will be as influencial on area of differential geometry as the study of 4-manifold instantons has been, I do believe that there is a rich, undiscovered area of mathematics that will yield great rewards.

Chapter 9

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