D-orbifolds and d-bordism

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This thesis is dedicated to
my beloved wife Ramona

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Abstract

The purpose of this thesis is to study d-manifolds and d-orbifolds and their bordism groups. D-manifolds and d-orbifolds were recently introduced by Joyce [35] as a new class of geometric objects to study moduli problems in algebraic and symplectic geometry. In the spirit of Joyce we will introduce the notion of *(stable) nearly and homotopy complex structures* on these 2-categories and study their unitary bordism groups. Fukaya and Ono [20] proved that the moduli space of n-pointed, genus g, J-holomorphic curves $\overline{\mathcal{M}}_{g,n}(M, J, \beta)$ carries a so called stably almost complex structure, and as Kuranishi spaces are closely related to d-orbifolds, the introduction of complex structures will be essential in studying symplectic Gromov-Witten invariants using d-orbifolds. We furthermore introduce the notion of *representable d-orbifolds* and prove that these representable d-orbifolds can be embedded into an orbifold. We will then explain how a result of Kresch [38] can be used to show that many important moduli spaces in algebraic geometry, are representable and thus embeddable d-orbifolds. Moreover we will sketch how one could prove an analogous result in the symplectic case.

We then prove as one of our main results, that for a compact manifold the unitary d-bordism group is isomorphic to its 'classical' unitary bordism group. This result extends a result by Joyce [35] who proved a similar statement for oriented manifolds and d-manifolds. Furthermore we will introduce the notion of blowups in the 2-category of d-manifolds and prove that these *d-blowups* satisfy a universal property. Finally, we sketch how our results may be used to make a step towards a proof of the Gopakumar–Vafa integrality conjecture and a "resolution of singularities" theorem for d-orbifolds.

Contents

Chapter 1

Introduction

Many important problems in symplectic geometry involve studying the moduli spaces $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$ of J-holomorphic curves in some symplectic manifold (M, ω). For instance symplectic Gromov–Witten theory is about "counting" J-holomorphic curves in a compact symplectic manifold (M, ω). In order to get reasonable results, it is essential that $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$ behaves as much as possible like a compact, oriented manifold with known dimension. The reason for this is, that we want our "counting invariant" to be just dependent on the underlying symplectic manifold and not on a choice of compatible almost complex structure J. This independence of J depends on treating $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$ as if it was a compact oriented manifold and compute its dimension. Unfortunately the moduli space $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$ is in general not a compact, oriented manifold and not even an orbifold, as it can have bad singularities. In order to resolve this problem, there are basically two approaches: The first approach is to make rather strong assumptions on the geometry, as for example taking (M, ω) to be closed monotone and J to be generic. (See McDuff and Salamon [42] for more details.) The second possibility to get around this problem without losing generality, is to find a nice geometric structure on $\overline{\mathcal{M}}_{g,m}(M, J)$ which admits a virtual class $[\overline{\mathcal{M}}]_{\mathrm{vir}} \in H_k(\overline{\mathcal{M}}; \mathbb{Q})$, where k should be the *expected dimension* of $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$. This virtual class allows one to define counting invariants, which are independent of all choices, and so just depend on the underlying symplectic manifold.

In algebraic Gromov–Witten theory the method for defining such a virtual class on the moduli space of m-pointed, stable genus g curves in a projective complex algebraic manifold (M, J) is called a *(perfect) obstruction theory.* This additional structure on the, in the algebraic case, proper, separated $\mathbb{C}$-scheme $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$ enables one to define a virtual class in the Chow homology of $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$, as for example Behrend and Fantechi have shown [7]. The notion of (perfect) obstruction theory is rigorously defined and provides a well-established and well-defined geometric structure on $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$.

On the symplectic side of Gromov–Witten theory, there are basically two approaches to define a nice geometric structure on $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$: *Kuranishi spaces*, introduced by Fukaya and Ono in 1999 [20] (see also [18] for a revised definition and [19] for the most up to date treatment of the subject) and *polyfolds*, introduced by Hofer, Wysocki and Zehnder in 2005 [29]. The theory of polyfolds, a kind of general Fredholm theory, is philosophically opposed to the theory of Kuranishi spaces, as Kuranishi spaces remember only minimal information about the moduli problem, whereas polyfolds remember essentially everything. The reason for this is that polyfolds do not localize, and therefore no information is forgotten by localizing. Although this means that polyfolds do not require the usage of higher categories, it also means that the theory of polyfolds is in some sense unwieldy.

Kuranishi structures on the other hand, were introduced as a geometric structure on the moduli space $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$ and were used to attack some problems in Lagrangian Floer cohomology and Fukaya categories.

Although the definition of a Kuranishi structure was sufficient for the applications of [20] and [18], there are some issues with this theory. One issue with Kuranishi spaces is that they do not give a very satisfactory notion of geometric structure, since for instance notions like "being the same" or morphisms between Kuranishi structures are not well behaved. Another problem is that there is no commonly agreed definition of what a Kuranishi structure actually should be, and there are several not necessarily compatible definitions floating around. (See [20],[18], [33].)

Very recently Joyce invented new classes of geometric objects, which he called "d-manifolds" and "d-orbifolds" [35]. These "derived" smooth manifolds and orbifolds form 2-categories and were originally designed to fix the problems with the notion of Kuranishi structures, but turned out to provide a kind of unified framework for studying enumerative invariants and moduli spaces. D-manifolds are related to David Spivak's "derived manifolds [50]", as they should roughly speaking be a kind of 2-truncation of the ∞-category of derived manifolds. Spivak's 'derived manifolds' form an ∞-category and the definition involves complicated and heavy usage of derived algebraic geometry, in particular the extensive work of Lurie [40]. Borisov and Noel [9] showed that an equivalent ∞-category can be defined using much simpler techniques. Moreover Borisov [8] proved that there exists a strict 2-functor $F^{\mathbf{dMan}}_{\mathbf{DerMan}}$ from a 2-category truncation of the ∞-category of Spivak's 'derived manifolds to the 2-category of d-manifolds.

D-manifolds are particularly nicely behaved as for instance the (1-)category of smooth manifolds **Man** can be embedded as a full subcategory into the 2-category of d-manifolds, and many differential geometric concepts generalize nicely to d-manifolds.

Despite their own beauty, d-manifolds and d-orbifolds are interesting because almost any moduli space used in enumerative invariant theory over $\mathbb{R}$ or $\mathbb{C}$ has the structure of a d-manifold or a d-orbifold. Moreover there are truncation functors from already established geometric structure like $\mathbb{C}$-schemes with obstruction theory in algebraic geometry and Kuranishi spaces or polyfolds in symplectic geometry. In particular, the"correct" notion of Kuranishi space should be "d-orbifold with corners". In [35] Joyce establishes the theory of d-manifolds and d-orbifolds and virtual vector bundles over d-manifolds. A d-manifold is defined as a fibre product of manifolds in the 2-category of so called d-spaces, which can be thought of a C^∞-scheme equipped with additional "derived" data. Joyce proves for example that there is a well behaved and canonical notion of "virtual cotangent bundle", a d-manifold analogue of the ordinary cotangent bundle, and that the oriented d-bordism group, which is a d-manifold analogue of the usual bordism group, for a manifold Y is isomorphic to the usual oriented bordism group. This isomorphism allows one to define virtual classes for d-manifolds, and one can therefore study moduli problems and define for example Gromov–Witten type invariants.

In chapter 2 we start by recalling some basic theory about C^∞-rings, C^∞-schemes, d-spaces and d-manifolds and modules and quasicoherent sheaves over them. C^∞-rings have their origin in in synthetic differential geometry, and standard references are for example Dubuc [17] or Moerdijk and Reyes [45]. A C^∞-ring can be thought of as a generalization of the algebra of smooth functions on a smooth manifold, and will be the underlying structure of d-spaces and d-manifolds. We will follow Joyce [35], [34], who refined in 2010 a version of (Hartshorne) algebraic geometry over C^∞-rings. We will briefly give the basic definitions and state the for our purposes important results, but refer for the details and a much more complete discussion to [35]. Furthermore, we recapitulate the 2-categorical analogue of vector bundles and sheaves, so called *virtual vector bundles* and *virtual quasicoherent sheaves*. In doing so we prove as a new result that each virtual vector bundle over a compact, sufficiently nice C^∞-scheme is equivalent in the 2-category of virtual vector bundles to a virtual vector bundle consisting of actual vector bundles. This result will be crucial for studying unitary bordism for stable nearly complex d-manifolds.

Chapter 3 will discuss similar background for C^∞-stacks, d-stacks and d-orbifolds. Although the actual definitions of C^∞-stacks, d-stacks and d-orbifolds require some background from stack theory, the for us important point is that Deligne–Mumford C^∞-stacks are related to C^∞-schemes in the same way as Deligne–Mumford stacks in algebraic geometry are related to schemes. Many concepts and definitions from the C^∞-space world extend nicely to the C^∞-stack case, but there are some subtleties involved which in some cases will prevent results from being true in the d-orbifold case.

We then follow Joyce [35, §14] in chapter 4 and explain how d-manifolds and d-orbifolds are related to other, established geometric structures. We focus on the for us most important geometric structures and describe in section 4.1 the relation between Kuranishi structures due to Fukaya, Ono [20] and Fukaya, Oh, Ohta and Ono [18] and d-manifolds and d-orbifolds. We will in the spirit of Joyce [35, Remark 14.15] provide a sort of "dictionary" between Fukaya, Oh, Ohta and Ono's Kuranishi spaces. In section 4.2 we discuss the relation between $\mathbb{C}$-schemes and $\mathbb{C}$-stacks with obstruction theory and d-manifolds and d-orbifolds as in [35, §14.5].

Chapter 5 then defines a new class of d-manifolds and d-orbifolds, by introducing *(stable) homotopy and nearly complex structures* on d-manifolds and d-orbifolds. Stable homotopy and nearly complex structures can be thought of an analogue of stable almost complex structures for manifolds and orbifolds. After introducing (stable) homotopy complex structures on d-manifolds and d-orbifolds in §5.1 and (stable) nearly complex structures in §5.2, we will prove that any nearly complex d-manifold (d-orbifold) is locally equivalent to a nearly complex standard model d-manifold (d-orbifold). This result will play a crucial role in studying unitary d-manifold and d-orbifold bordism in chapter 7. In §5.4 we then prove that given a stable homotopy complex d-manifold, we can construct a stable nearly complex d-manifold, and vice versa. Homotopy complex structures are closely related to complex structures on Kuranishi spaces of Fukaya and Ono [20], and we sketch in subsection 5.4.1, assuming a result of Fukaya and Ono [20], why the moduli space of n-pointed, genus g, J-holomorphic curves $\overline{\mathcal{M}}_{g,n}(M,J,\beta)$ is a stable nearly complex d-orbifold.

We then introduce another new class of d-orbifolds in chapter 6, which we call *representable d-orbifolds*. Representable d-orbifolds are d-orbifolds $\boldsymbol{\mathcal{X}}$ which admit a 1-morphism $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}=F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ in $\mathbf{dOrb}$ into an effective orbifold $\mathcal{Y}$, which is representable, that is the underlying C^∞-stack morphism $f_*:\mathrm{Iso}_{\mathcal{X}}([x])\to\mathrm{Iso}_{\mathcal{Y}}([y])$ is injective for all $[x]\in\mathcal{X}_{\mathrm{top}}$ with $f_*([x])=[y]\in\mathcal{Y}_{\mathrm{top}}$. We will prove as a new result that any representable d-orbifold can be embedded into some smooth orbifold, and is thus an *embeddable d-orbifold*. This will allow us to use a result by Kresch [38] to conclude that many important algebraic moduli spaces, like the moduli stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of n-pointed, genus g stable maps to a projective target variety X, are in fact embeddable d-orbifolds. Moreover we will sketch how one could prove that the moduli space of n-pointed, genus g, J-holomorphic curves $\overline{\mathcal{M}}_{g,n}(M,J,\beta)$ is a representable d-orbifold.

In chapter 7, we first recall some basic results about d-(co)bordism due to Joyce [35, §13]. The major theorem ([35, Theorem 13.15]) is here, that oriented d-manifold bordism of a manifold (considered as a d-manifold) is isomorphic to oriented manifold bordism. It is crucial in the whole theory, as one consequence of this theorem is, that oriented compact d-manifolds admit virtual classes, and can therefore be used to study moduli problems in for instance symplectic geometry.

In section 7.3 we then define a unitary version of d-manifold bordism using nearly complex structures introduced in section 5.2. We then prove as a new major result that for a stable almost complex manifold its unitary d-bordism group is isomorphic to the "ordinary" unitary bordism group. This theorem is an extension of [35, Theorem 13.15], and as Fukaya and Ono proved in [20] that the moduli space of n-pointed, genus g, J-holomorphic curves $\overline{\mathcal{M}}_{g,n}(M, J, \beta)$ carries a stable nearly complex structure, it will potentially play a crucial role in studying symplectic Gromov-Witten invariants. We then explain when and how this material on d-manifolds can be extended to the d-orbifold case. The situation in the d-orbifold case is much more subtle, as the straightforward d-orbifold generalization of [35, Theorem 13.15] is false. The problem is, that whereas any d-manifold can be perturbed into a manifold, the analogous result for d-orbifolds is false. To see this note that in a standard model $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ at a point $v \in s^{-1}(0) \subseteq \mathcal{V}$, the orbifold group $\Gamma = \mathrm{Iso}_{\mathcal{V}}(v)$ acts on the tangent space $T_v\mathcal{V}$ and the obstruction space $\mathcal{E}|_v$. So if the nontrivial part of the Γ-representation on $\mathcal{E}|_v$ is not a subrepresentation of $T_v\mathcal{V}$, small deformations $\tilde{s}$ of s cannot be transverse near v, and so $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde{s}}$ cannot be an orbifold.

However, restricting oneself to (semi)effective d-orbifolds, Joyce was able to prove that in this case the oriented d-orbifold bordism groups are isomorphic to the "classical" orbifold bordism groups. (Compare [35, Theorem 13.23].) The major point is that (semi)effective d-orbifolds can be perturbed to (effective (in the effective d-orbifold case)) orbifolds, as (semi)effectiveness prevents the phenomena described above from happening.

Chapter 8 then introduces the notion of *d-blowups*. We will motivate how the classical real (complex) blowup of a manifold along a submanifold can be extended to d-manifolds. The basic idea is to imitate what happens in the classical case: blowing up a manifold along a submanifold at a point (x, λ), where $x \in W$ and $0 \neq \lambda \in \mathcal{N}_{W/V}|_x$, affects the tangent bundle of V by twisting the part of the normal bundle orthogonal to $\langle\lambda\rangle$ by the inverse of the line bundle associated to the exceptional divisor. Imitating this behaviour in the d-manifold case, that is given a closed w-immersion of standard model d-manifolds $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{W,F,t} \to \boldsymbol{S}_{V,E,s}$ we twist the "W-part" of the bundle E by the inverse of the to line bundle associated to the exceptional divisor of the (manifold) blowup $\tilde{V} = \mathrm{Bl}_W V$. Using this idea, we can define the notion of *standard model d-blowups*. We will then prove in subsection 8.2.1, that similarly to the classical case of manifolds or schemes, these standard model d-blowups satisfy an universal property. In contrast to the classical universal property of blowups of manifolds or schemes (as for example in [27, Proposition II.7.14]), the universal property of standard model d-blowups is characterized by pairs of closed w-embedded standard model d- submanifolds with 1-morphisms between them. In section 8.3 we explain how one can extend this local notion of standard-model d-blowups to the general d-manifold case, using the results proven in section 8.2. Moreover, we briefly explain how the results of this chapter can be extended to the d-orbifold case.

Finally, in chapter 9 we briefly sketch how all of the previous results could be used to study integral Gromov-Witten invariants. In particular, we sketch how by using nearly complex structures and blowups, one could make a step towards proving the Gopakumar–Vafa integrality conjecture and how our results lead to a d-orbifold 'resolution of singularities' theorem (in the spirit of Hironaka [28]).

Chapter 2

Background on d-manifolds

We will start by recalling some basic material on d-manifolds in this chapter. D-manifolds were recently introduced by Joyce [35], and can be thought of as a 2-categorical generalization of manifolds. As the precise definition involves material from synthetic differential geometry, we start by introducing C^∞-rings and C^∞-schemes, which will be the foundations for our later discussions on d-spaces and d-manifolds. Most of the covered material can be found in [35] and [34], which we found to be valuable references.

2.1 C^∞-rings and C^∞-schemes

2.1.1 C^∞-rings

We will recall the basic definitions and properties of C^∞-rings and C^∞-schemes. We follow here closely [34] and refer to it as a much more complete and rigorous source. C^∞-rings are a part of *synthetic differential geometry* and were first studied in the 1960s. References for this subject are among others Dubuc [17] on C^∞-schemes, and the book of Moerdijk and Reyes [45]. More recently Joyce [34] re-established the subject and provided new ideas, which lead to Algebraic Geometry over C^∞-rings.

The basic idea behind this theory, is that each smooth manifold X comes naturally equipped with an $\mathbb{R}$-algebra $C^\infty(X)$ of smooth functions $c : X \to \mathbb{R}$. This $\mathbb{R}$-algebra has a much richer structure than just the "ordinary" algebra structure, as for example given any arbitrary smooth map $c : X \to \mathbb{R}$, we can concatenate it with the exponential function, which yields another smooth function $\exp(c) : X \to \mathbb{R}$, and defines therefore an operation $\exp : C^\infty(X) \to C^\infty(X)$, which cannot be expressed by the $\mathbb{R}$-algebra structure (at least without introducing a topology and taking limits of series). These additional structures motivate the definition of a C^∞-ring:

Definition 2.1.1. A C^∞*-ring* is a set $\mathfrak{C}$ together with operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all $n \geq 0$ and smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, where by convention $\mathfrak{C}^0$ is defined to be a single point $\{\emptyset\}$. These operations have to satisfy the following relations:

(1) For $m, n \geq 0$ and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ smooth functions, define a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$ the following holds

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

(2) For all $1 \leq j \leq n$ define $\pi_j : \mathbb{R}^n \to R$ by $\pi_j(x_1, \ldots, x_n) = x_j$. Then

$$\Phi_{\pi_j}(c_1, \ldots, c_n) = c_j \quad \text{for all } (c_1, \ldots, c_n) \in \mathfrak{C}^n.$$

Given two C^∞-rings $(\mathfrak{C}, \Phi_f)$ and $(\mathfrak{D}, \Psi_d)$, a *morphism* between C^∞-rings is given by a map $\phi : \mathfrak{C} \to \mathfrak{D}$ such that $\Psi_f(\phi(c_1), \ldots, \phi(c_n)) = \phi \circ \Phi_f(c_1, \ldots, c_n)$ for all $c_1, \ldots, c_n$ and all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$. The resulting category of C^∞-rings will be denoted by $C^\infty\mathbf{Rings}$.

The following example is somehow the "motivating example" for a C^∞-ring, and is discussed in much more detail in [35, §1.2.1].

Example 2.1.2. Let X be a manifold, possibly with boundary, and write $C^\infty(X)$ for the set of smooth functions $c : X \to \mathbb{R}$. Given a smooth map $f : \mathbb{R}^n \to \mathbb{R}$ for $n \geq 0$, define $\Phi_f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$ by

$$\Phi_f(c_1, \ldots, c_n)(x) = f(c_1(x), \ldots c_n(x))$$

for all $c_1, \ldots, c_n \in C^\infty(X)$ and $x \in X$.

It is immediate that $C^\infty(X)$ together with Φ_f defined above is a C^∞-ring. Every smooth map between manifolds $f : X \to Y$, induces a morphism of C^∞-schemes, given by the pull back $f^* : C^\infty(Y) \to C^\infty(X), f^*(c) = c \circ f$.

If we denote the category of smooth manifolds without boundary by **Man**, and write $C^\infty\mathbf{Rings}^{\mathrm{op}}$ for the category of C^∞-rings with direction of morphisms reversed, we get a full and faithful functor $F_{\mathbf{Man}}^{C^\infty\mathbf{Rings}} : \mathbf{Man} \to C^\infty\mathbf{Rings}^{\mathrm{op}}$ acting on objects by $F_{\mathbf{Man}}^{C^\infty\mathbf{Rings}}(X) = C^\infty(X)$ and on morphisms by $F_{\mathbf{Man}}^{C^\infty\mathbf{Rings}}(f) = f^*$. Under this functor **Man** can be obtained as a fully embedded subcategory of $C^\infty\mathbf{Rings}^{\mathrm{op}}$.

One consequence of the definition of a C^∞-ring structure is that every C^∞ ring $\mathfrak{C}$ has an underlying commutative $\mathbb{R}$-algebra structure. This $\mathbb{R}$-algebra structure allows one to establish notions like ideal of a C^∞-ring, module over a C^∞-ring, and so on.

Definition 2.1.3. Any C^∞-ring $\mathfrak{C}$ carries in a natural way the structure of a *commutative* $\mathbb{R}$*-algebra* as follows:

- Define addition '+' on $\mathfrak{C}$ by $c + c' = \Phi_f(c, c')$ for $c, c' \in \mathfrak{C}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = x + y$.
- Define multiplication '$\cdot$' on $\mathfrak{C}$ by $c \cdot c' = \Phi_g(c, c')$ for $c, c' \in \mathfrak{C}$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is given by $g(x, y) = xy$.
- Define scalar multiplication by $\lambda \in \mathbb{R}$ by $\lambda c = \Phi_{\lambda'}(c)$, where $\lambda' : \mathbb{R} \to \mathbb{R}$ is given by $\lambda'(x) = \lambda x$.
- Define elements 0 and 1 in $\mathfrak{C}$ by $0 = \Phi_{0'}(\emptyset)$ and $1 = \Phi_{1'}(\emptyset)$, where $0' : \mathbb{R}^0 = \{\emptyset\} \to \mathbb{R}$ is given by $0' : \emptyset \mapsto 0$, and $1' : \mathbb{R}^0 \to \mathbb{R}$ is given by $1' : \emptyset \mapsto 1$.

Using the relations on the Φ_f, it is immediate that the definitions above make $\mathfrak{C}$ into a commutative $\mathbb{R}$-algebra. Applying this definition to Example 2.1.2, recovers the usual commutative $\mathbb{R}$-algebra structure in the ring of smooth function $c : X \to \mathbb{R}$. Although being a commutative $\mathbb{R}$-algebra provides a rich set of algebraic structures, it is worth noting, that the C^∞-ring structure has far more structure and operations than a commutative $\mathbb{R}$-algebra.

Since every C^∞-ring has the structure of a commutative $\mathbb{R}$-algebra we can define what an ideal of a C^∞-ring should be.

Definition 2.1.4. An *ideal* I of a C^∞-ring $\mathfrak{C}$ is an ideal $I \subset \mathfrak{C}$ in $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra. The quotient $\mathfrak{C}/I$ can be equipped with a C^∞-ring structure as follows. For any smooth function $f : \mathbb{R}^n \to \mathbb{R}$, define $\Phi_f^I : (\mathfrak{C}/I)^n \to \mathfrak{C}/I$ by

$$(\Phi_f^I(c_1 + I, \ldots, c_n + I))(x) = \Phi_f(c_1(x), \ldots, c_n(x)) + I.$$

For Φ_f^I to be well-defined, we have to check that it is independent of the choice of representatives $c_1, \ldots, c_n$ in $\mathfrak{C}$ for $c_1 + I, \ldots, c_n + I$ in $\mathfrak{C}/I$. To show this, note that Hadamard's Lemma implies the existence of smooth functions $g_i : \mathbb{R}^{2n} \to \mathbb{R}$ for $i + 1, \ldots, n$, satisfying

$$f(y_1, \ldots, y_n) - f(x_1, \ldots x_n) = \sum_{i=1}^{n} (y_i - x_i) g_i(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$. If we now have two choices $c_1, \ldots, c_n$ and $c_1', \ldots, c_n'$, so that $c_i' + I = c + I$ for $i = 1, \ldots, n$ and $c_i' - c_i \in I$, we have

$$\begin{aligned} &\Phi_f(c_1', \ldots, c_n') - \Phi_f(c_1, \ldots c_n) \\ &\qquad = \sum_{i=1}^{n} (c_i' - c_i) \Phi_{g_i}(c_1', \ldots, c_n', c_1, \ldots, c_n). \end{aligned}$$

But the second line lies in I as $c_i' - c_i \in I$ and I is an ideal. This implies that Φ_f^I is indeed well-defined, which makes $(\mathfrak{C}/I, \Phi_f^I)$ into a C^∞-ring.

If $\mathfrak{C}$ is a C^∞-ring, denote by $(f_a : a \in A)$ the ideal generated by a collection of elements $f_a, a \in A$ in $\mathfrak{C}$, that is

$$(f_a : a \in A) = \big\{ \sum_{i=1}^{n} f_{a_i} \cdot c_i : n \geq 0, a_1, \ldots, a_n \in A, c_1, \ldots, c_n \in \mathfrak{C} \big\}.$$

In many situations it will be convenient to describe a C^∞-ring $\mathfrak{C}$ by its generators and relations.

Definition 2.1.5. We call a C^∞-ring $\mathfrak{C}$ *finitely generated*, if there exist $c_1, \ldots, c_n \in \mathfrak{C}$ such that for each $c \in \mathfrak{C}$, there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with $c = \Phi_f(c_1, \ldots, c_n)$. Note that being finitely generated over all C^∞-relations is a much weaker condition than being finitely generated as a commutative $\mathbb{R}$-algebra.

As shown in ([45], Proposition 1.1) the ring $C^\infty(\mathbb{R}^n)$ of smooth function in n variables is the free C^∞-ring on n generators. So for any C^∞-ring with generators $c_1, \dots, c_n$ we have a surjective C^∞-ring morphism $\phi : C^\infty(\mathbb{R}^n) \to \mathfrak{C}$ given by $\phi(f) = \Phi_f(c_1, \dots, c_n)$ for any $f : \mathbb{R}^n \to \mathbb{R}$ smooth. The kernel $I = \ker(\phi)$ is an ideal in $C^\infty(\mathbb{R}^n)$ and $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ is a C^∞-ring. Therefore any finitely generated ring $\mathfrak{C}$ can be written as $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some $n \geq 0$ and ideal I in $C^\infty(\mathbb{R}^n)$, and vice versa.

Definition 2.1.6. An ideal I in $C^\infty(\mathbb{R}^n)$ is called *finitely generated*, if there exists $f_1, \dots, f_k \in C^\infty(\mathbb{R}^n)$ such that $I = (f_1, \dots, f_k)$. Call a C^∞-ring $\mathfrak{C}$ *finitely presented*, if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some $n \geq 0$, with I being a finitely generated ideal in $C^\infty(\mathbb{R}^n)$.

A C^∞-ring $\mathfrak{C}$ is called a *C^∞-local ring* if regarded as an $\mathbb{R}$-algebra, $\mathfrak{C}$ has a unique maximal ideal $\mathfrak{m}_{\mathfrak{C}}$ such that $\mathfrak{C}/\mathfrak{m}_{\mathfrak{C}} \cong \mathbb{R}$.

Denote the full subcategories of finitely generated and finitely presented C^∞-rings in $C^\infty\mathbf{Rings}$ by $C^\infty\mathbf{Rings}^{\mathrm{fg}}$ and $C^\infty\mathbf{Rings}^{\mathrm{fp}}$.

2.1.2 C^∞-schemes

We want now to recall some material on C^∞-schemes and refer again to [35, §1.2.2] for a much more complete and detailed discussion. The basic idea is to adapt "conventional scheme theory" over a ring to the case of C^∞-rings.

Definition 2.1.7. A *C^∞-ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X together with a sheaf $\mathcal{O}_X$ of C^∞-rings on X, the so called *structure sheaf*. A *morphism* $\underline{f} = (f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of C^∞-ringed spaces consists a continuous map $f : X \to Y$ between topological spaces and a morphism of sheaves of C^∞-rings on X, $f^\# : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$, where $f^{-1}\mathcal{O}_Y$ is the inverse image sheaf. These C^∞-ringed spaces form a category, which we will denote by $C^\infty\mathbf{RS}$.

A *local C^∞-ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a C^∞-ringed space for which the stalks $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ are C^∞-local rings for all $x \in X$. Morphisms of C^∞-local rings are automatically local morphisms, and so morphisms of local C^∞-ringed spaces $\underline{X} = (X, \mathcal{O}_X), \underline{Y} = (Y, \mathcal{O}_Y)$ are just morphisms of C^∞-ringed spaces without any additional locality condition. The full subcategory of local C^∞-ringed spaces in $C^\infty\mathbf{RS}$, will be denoted by $\mathbf{L}C^\infty\mathbf{RS}$.

As in the case of "ordinary" rings, one can explicitly define a *spectrum functor* $\mathrm{Spec} : C^\infty\mathbf{Rings}^{\mathrm{op}} \to \mathbf{L}C^\infty\mathbf{RS}$ (compare [34, §6.2]). A local C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme*, if it is isomorphic to $\mathrm{Spec}\,\mathfrak{C}$ in $C^\infty\mathbf{Rings}$ for some C^∞-ring $\mathfrak{C}$, and a local C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$ is called *C^∞-scheme*, if X can be covered by open sets $U \subseteq X$, such that $(U, \mathcal{O}_X|_U)$ is an affine C^∞-scheme. The full subcategory of C^∞-schemes in $\mathbf{L}C^\infty\mathbf{Rings}$ will be denoted by $C^\infty\mathbf{Sch}$.

We call a C^∞-scheme $\underline{X}$ *locally fair*, if X can be covered by open $U \subseteq X$ with $(U, \mathcal{O}_X|_U) \cong \mathrm{Spec}\,\mathfrak{C}$ for some finitely generated C^∞-ring $\mathfrak{C}$, and write $C^\infty\mathbf{Sch}^{\mathrm{lf}}$ for the full subcategory of locally fair C^∞-schemes in $C^\infty\mathbf{Sch}$.

A C^∞-scheme $\underline{X}$ is called *separated, second countable, compact* or *paracompact*, if the underlying topological space X is Hausdorff, second countable, compact or paracompact.

Example 2.1.8. Let X be a manifold. We can define a C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$, where the topological space is just the manifold X itself, and $\mathcal{O}_X(U) = C^\infty(U)$ for each open $U \subseteq X$, where as usual, $C^\infty(U)$ denotes the C^∞-ring of smooth maps $c : U \to \mathbb{R}$. Now, if $V \subseteq U \subseteq X$ are open subsets of X, define the restriction map $\rho_{UV} : C^\infty(U) \to C^\infty(V)$ by $\rho_{UV}(c) = c|_V$. This makes $\underline{X} = (X, \mathcal{O}_X)$ a local C^∞-ringed space, which is canonically isomorphic to $\mathrm{Spec}\,C^\infty(X)$, and so is an affine C^∞-scheme. Moreover it is immediate that $\underline{X}$ is locally fair.

As in the case of C^∞-rings, we can define a full and faithful functor $F_{\mathbf{Man}}^{C^\infty\mathbf{Sch}} : \mathbf{Man} \to C^\infty\mathbf{Sch}^{\mathrm{lf}} \subset C^\infty\mathbf{Sch}$ by $F_{\mathbf{Man}}^{C^\infty\mathbf{Sch}} = \mathrm{Spec} \circ F_{\mathbf{Man}}^{C^\infty\mathbf{Sch}}$, and so the category $\mathbf{Man}$ embeds as a full subcategory of $C^\infty\mathbf{Sch}$.

The following theorem summarizes some important facts of C^∞-schemes, and we refer to [34, §4] for individual proofs of the statements.

Theorem 2.1.9. *(a) All fibre products exist in the categories $C^\infty\mathbf{RS}$, of C^∞-ringed spaces and $C^\infty\mathbf{Sch}$ of C^∞-schemes.*
(b) The subcategory $C^\infty\mathbf{Sch}^{\mathrm{lf}}$ of locally fair C^∞-schemes is closed under fibre products and all finite limits in $C^\infty\mathbf{Sch}$.
(c) The functor $F_{\mathbf{Man}}^{C^\infty\mathbf{Sch}}$ takes transverse fibre products in $\mathbf{Man}$ to fibre products in $C^\infty\mathbf{Sch}$.
(d) Let $(X, \mathcal{O}_X)$ be a fair affine C^∞-scheme, and $U \subseteq X$ be an open subset. Then $(U, \mathcal{O}_X|_U)$ is a fair affine C^∞-scheme.

Note that this does not hold for general C^∞-schemes.

(e) For any separated, paracompact, locally fair C^∞-scheme $\underline{X}$, and open cover $\{U_a : a \in A\}$, there exists a partition of unity $\{\eta_a : a \in A\}$ on $\underline{X}$ subordinate to $\{U_a : a \in A\}$.

2.1.3 Modules over C^∞-rings and cotangent modules

In the following, we will recall some material on modules over C^∞-rings and refer to [35, §1.2.3] or [34, §5] for a more detailed and complete discussion.

Definition 2.1.10. Let $\mathfrak{C}$ be a C^∞-ring. A $\mathfrak{C}$-module M is a module over $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra. We will write $\mathfrak{C}$-mod for the abelian category of $\mathfrak{C}$-modules. Let $\phi : \mathfrak{C} \to \mathfrak{D}$ be a morphism of C^∞-rings. If M is a $\mathfrak{C}$-module, then $\phi_*(M) = M \otimes_{\mathfrak{C}} \mathfrak{D}$ is a $\mathfrak{D}$-module. This induces a functor $\phi_* : \mathfrak{C}\text{-mod} \to \mathfrak{D}\text{-mod}$.

Example 2.1.11. Let X be a manifold, possibly with boundary or corners, and let $E \to X$ be a vector bundle. Denote by $C^\infty(E)$ the vector space of smooth sections $e : X \to E$, and define $\mu_E : C^\infty(X) \times C^\infty(E) \to C^\infty(E)$ by $\mu_E(c, e) = c \cdot e$. Then $(C^\infty(E), \mu_E)$ is a $C^\infty(X)$-module. If E is the trivial rank k vector bundle $E \cong X \times \mathbb{R}^k$, then $(C^\infty(E), \mu_E) \cong (C^\infty(X) \otimes_{\mathbb{R}} \mathbb{R}^k, \mu_{\mathbb{R}^k})$, and so $(C^\infty(E), \mu_E)$ is a free $C^\infty(X)$-module.

Given $E, F \to X$ vector bundles and $\lambda : E \to F$ a bundle morphisms between them, then $\lambda_* : C^\infty(E) \to C^\infty(F)$ defined by $\lambda_* : e \mapsto \lambda \circ e$ is a morphism of $C^\infty(X)$-modules.

Let X, Y be manifolds and $f : X \to Y$ be a smooth map. Then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphisms of C^∞-rings. If $E \to Y$ is a vector bundle, then the pull back bundle $f^*(E)$ is a vector bundle over X. Using the functor $(f^*)_* : C^\infty(Y)\text{-mod} \to C^\infty(X)\text{-mod}$ from Definition 2.1.10, we see that $(f^*)_*(C^\infty(E)) = C^\infty(E) \otimes_{C^\infty(Y)} C^\infty(X)$ is isomorphic as a $C^\infty(X)$-module to $C^\infty(f^*(E))$.

One particularly important example of a module over a C^∞-ring $\mathfrak{C}$, is the *cotangent module* $(\Omega_{\mathfrak{C}}, \mu_{\mathfrak{C}})$ of $\mathfrak{C}$.

Definition 2.1.12. Let $\mathfrak{C}$ be a C^∞-ring, and M a $\mathfrak{C}$-module. A C^∞-*derivation* is a $\mathbb{R}$-linear map $d : \mathfrak{C} \to M$ such that whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and

$c_1, \ldots, c_n \in \mathfrak{C}$, we have

$$d\Phi_f(c_1, \ldots, c_n) = \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot dc_i.$$

The pair (M, d) is called a *cotangent module* for $\mathfrak{C}$, if it has the universal property that for any $\mathfrak{C}$-module M' and C^∞-derivation $d' : \mathfrak{C} \to M'$, there exists a unique morphism of $\mathfrak{C}$-modules $\phi : M \to M'$ with $d' = \phi \circ d$. Let $\Omega_{\mathfrak{C}}$ be the quotient of the free $\mathfrak{C}$-module with basis of symbols dc for $c \in \mathfrak{C}$ by the $\mathfrak{C}$-submodule spanned by all expressions of the form $d\Phi_f(c_1, \ldots, c_n) - \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot dc_i$ for $f : \mathbb{R}^n \to \mathbb{R}$ smooth and $c_1, \ldots, c_n \in \mathfrak{C}$, and define $d_{\mathfrak{C}} : \mathfrak{C} \to \Omega_{\mathfrak{C}}$ by $d_{\mathfrak{C}} : c \mapsto dc$. Then the pair $(\Omega_{\mathfrak{C}}, d_{\mathfrak{C}})$ is a cotangent module for $\mathfrak{C}$. Thus cotangent modules always exist, and are unique up to unique isomorphism.

Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings with cotangent modules $(\Omega_{\mathfrak{C}}, d_{\mathfrak{C}}), (\Omega_{\mathfrak{D}}, d_{\mathfrak{D}})$, and let $\phi : \mathfrak{C} \to \mathfrak{D}$ be a morphism of C^∞-rings. Then ϕ induces an action on $\Omega_{\mathfrak{D}}$, which makes $\Omega_{\mathfrak{D}}$ into a $\mathfrak{C}$-module with C^∞-derivative $d_{\mathfrak{D}} \circ \phi : \mathfrak{C} \to \Omega_{\mathfrak{D}}$. Hence by the universal property of $\Omega_{\mathfrak{C}}$, there exists a unique morphism of $\mathfrak{C}$-modules $\Omega_\phi : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{D}}$ such that $d_{\mathfrak{D}} \circ \phi = \Omega_\phi \circ d_{\mathfrak{C}}$. This morphism induces a morphism of $\mathfrak{D}$-modules $(\Omega_\phi)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{D} \to \Omega_{\mathfrak{D}}$ satisfying $(\Omega_\phi)_* \circ (d_{\mathfrak{C}} \otimes \mathrm{id}_{\mathfrak{D}}) = d_{\mathfrak{D}}$. Moreover, given C^∞-ring morphisms $\phi : \mathfrak{C} \to \mathfrak{D}, \psi : \mathfrak{D} \to \mathfrak{E}$ we have $\Omega_{\psi \circ \phi} = \Omega_\phi \circ \Omega_\phi : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{E}}$.

Example 2.1.13. Let X be a manifold. Then the cotangent bundle T^*X is a vector bundle over X and admits therefore a $C^\infty(X)$-module $C^\infty(T^*X)$. The exterior derivative $d : C^\infty(X) \to C^\infty(T^*X)$ is a C^∞-derivation. The pair $(C^\infty(T^*X), d)$ has the universal property in Definition 2.1.12, and so forms a cotangent module for $C^\infty(X)$.

Now let X, Y be manifolds, and $f : X \to Y$ be a smooth map. Then TX, as well as the pulled back tangent bundle $f^*(TY)$, is a vector bundle over X, and we have a vector bundle morphism $df : TX \to f^*(TY)$ between them. The dual of this morphism is $df^* : f^*(T^*Y) \to T^*X$, and this morphism induces a morphism of $C^\infty(X)$-modules $(df^*)_* : C^\infty(f^*(T^*Y)) \to C^\infty(T^*X)$. This $(df^*)_*$ can be identified with $(\Omega_{f^*})_*$ in Definition 2.1.12 under the natural isomorphism $C^\infty(f^*(T^*Y)) \cong C^\infty(T^*Y) \otimes_{C^\infty(Y)} C^\infty(X)$.

2.1.4 Quasicoherent sheaves on C^∞-schemes

As in [35, §1.2.4] and [34, §6] we can now discuss quasicoherent sheaves on C^∞-schemes.

Definition 2.1.14. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-scheme. An *$\mathcal{O}_X$-module* $\mathcal{E}$ on $\underline{X}$ assigns a module $\mathcal{E}(U)$ over $\mathcal{O}_X(U)$ for each open set $U \subseteq X$, with $\mathcal{O}_X(U)$-action $\mu_U : \mathcal{O}_X(U) \times \mathcal{E}(U) \to \mathcal{E}(U)$, and a linear map $\mathcal{E}_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ for each inclusion of open sets $V \subseteq U \subseteq X$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \mathcal{E}(U) \arrow[r, "\mu_U"] \arrow[d, "\rho_{UV} \times \mathcal{E}_{UV}"] & \mathcal{E}(U) \arrow[d, "\mathcal{E}_{UV}"] \\
\mathcal{O}_X(V) \times \mathcal{E}(V) \arrow[r, "\mu_V"] & \mathcal{E}(V).
\end{tikzcd}$$

All this data $\mathcal{E}(U), \mathcal{E}_{UV}$ must satisfy the usual sheaf axioms (see for example [27, §II.1]).

A morphism of $\mathcal{O}_X$-modules $\phi : \mathcal{E} \to \mathcal{F}$, we can assign a morphism of $\mathcal{O}_X(U)$-modules $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ for each open set $U \subseteq X$, such that $\phi(V) \circ \mathcal{E}_{UV} = \mathcal{F}_{UV} \circ \phi(U)$ for each inclusion of sets $V \subseteq U \subseteq X$. The abelian category of $\mathcal{O}_X$-modules will be denoted by $\mathcal{O}_X$-mod.

Similarly to the spectrum functor Spec $: C^\infty\mathbf{Rings}^{\mathrm{op}} \to C^\infty\mathbf{Sch}$, which assigns to each C^∞-ring an affine C^∞-scheme, there is a *module spectrum functor* MSpec : $\mathfrak{C}$-mod $\to \mathcal{O}_X$-mod, which assigns to each module over a C^∞-ring $\mathfrak{C}$ a sheaf of $\mathcal{O}_X$-modules over Spec $(\mathfrak{C})$.

Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-scheme, and $\mathcal{E}$ an $\mathcal{O}_X$-module. $\mathcal{E}$ is called *quasicoherent*, if there exists an open cover of $\underline{X}$ by $\underline{U}$, where $\underline{U} \cong \operatorname{Spec} \mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$, and under this identification $\mathcal{E}|_U \cong \operatorname{MSpec} M$ for some $\mathfrak{C}$-module M. We call $\mathcal{E}$ *coherent*, if furthermore the $\mathfrak{C}$-modules can be taken to be finitely presented.

$\mathcal{E}$ is called a *vector bundle of rank* $n \geq 0$, if $\underline{X}$ can be covered by open $\underline{U}$ such that $\mathcal{E}|_U \cong \mathcal{O}_U \otimes_{\mathbb{R}} \mathbb{R}^n$.

We will write $\mathrm{qcoh}(\underline{X}), \mathrm{coh}(\underline{X}), \mathrm{vect}(\underline{X})$ for the full subcategories of quasicoherent sheaves, coherent sheaves and vector bundles in $\mathcal{O}_X$-mod. Note that in the case of $\underline{X}$ being a locally fair C^∞-scheme, every $\mathcal{O}_X$-module $\mathcal{E}$ on $\underline{X}$ is quasicoherent, and therefore $\mathrm{qcoh}(\underline{X}) = \mathcal{O}_X$-mod.

Definition 2.1.15. Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of C^∞-schemes, and let $\mathcal{E}$ be an $\mathcal{O}_Y$ -module. The *pullback* $\underline{f}^*(\mathcal{E})$ of $\mathcal{E}$ by $\underline{f}$, is the $\mathcal{O}_X$-module, defined by $\underline{f}^*(\mathcal{E}) = f^{-1}(\mathcal{E}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, where $f^{-1}(\mathcal{E}), f^{-1}(\mathcal{O}_Y)$ are inverse image sheaves.

Now, if $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of $\mathcal{O}_Y$-modules, we have an induced morphism $\underline{f}^*(\phi) = f^{-1}(\phi) \otimes \mathrm{id}_{\mathcal{O}_X} : \underline{f}^*(\mathcal{E}) \to \underline{f}^*(\mathcal{F})$ in $\mathcal{O}_X$-mod. The so defined functor $\underline{f}^* : \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod}$ is a right exact functor between abelian categories and restricts to a right exact functor $\underline{f}^* : \mathrm{qcoh}(\underline{Y}) \to \mathrm{qcoh}(\underline{X})$.

Remark 2.1.16. Pullbacks $\underline{f}^*(\mathcal{E})$ can be characterised by a universal property, as they are closely related to fibre products. It is therefore convenient to regard pullbacks as being unique up to canonical isomorphism rather than unique.

It is possible to construct pullbacks explicitly, using the Axiom of Choice to choose the $\underline{f}^*(\mathcal{E})$ for all $\underline{f}, \mathcal{E}$, but it may not be possible to do this in a strictly functorial way in $\underline{f}$. So in other words, given morphisms $\underline{f} : \underline{X} \to \underline{Y}$, $\underline{g} : \underline{Y} \to \underline{Z}$ and an $\mathcal{O}_Z$-module $\mathcal{E}$, then $(\underline{g} \circ \underline{f})^*(\mathcal{E})$ and $\underline{f}^*(\underline{g}^*(\mathcal{E}))$ are canonically isomorphic as $\mathcal{O}_X$-modules, but may not be equal. These canonical isomorphisms will be denoted by $I_{\underline{f},\underline{g}}(\mathcal{E}) : (\underline{g} \circ \underline{f})^*(\mathcal{E}) \to \underline{f}^*(\underline{g}^*(\mathcal{E}))$ and the 2-morphism $I_{\underline{f},\underline{g}} : (\underline{g} \circ \underline{f})^* \Rightarrow \underline{f}^* \circ \underline{g}^*$ is then a natural isomorphism of functors.

Example 2.1.17. Let X be a manifold and $\underline{X}$ its associated C^∞-scheme, so that $\mathcal{O}_X(U) = C^\infty(U)$ for all open subsets $U \subseteq X$. Let $E \to X$ be a vector bundle. Define an $\mathcal{O}_X$-module $\mathcal{E}$ on $\underline{X}$ by $\mathcal{E}(U) = C^\infty(E|_U)$, where $C^\infty(E|_U)$ denotes the smooth sections of the vector bundle $E_U \to U$, and for $V \subseteq U \subseteq X$ define $\mathcal{E}_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ by $\mathcal{E}_{UV} : e_U \mapsto e_U|_V$. The so defined $\mathcal{O}_X$ module $\mathcal{E}$ turns out to be a vector bundle on $\underline{X}$, that is $\mathcal{E} \in \mathrm{vect}(\underline{X})$, which can be thought of as a lift of E from manifolds to C^∞-schemes.

Let now $f : X \to Y$ be a smooth map of manifolds, $\underline{f} : \underline{X} \to \underline{Y}$ the corresponding morphism of C^∞-schemes and let $F \to Y$ be a vector bundle over Y, so that the pullback $f^*(F) \to X$ is a vector bundle over X. Denote by $\mathcal{F} \in \mathrm{vect}(\underline{Y})$ the vector bundle over $\underline{Y}$ lifting F. Then $\underline{f}^*(\mathcal{F})$ is as a vector bundle over $\underline{X}$ canonically isomorphic to the lifting of the pullback bundle $f^*(F)$.

In the same manner as the notion of modules over C^∞-rings lifts to $\mathcal{O}_X$-modules over C^∞-schemes, we can define the sheaf version of cotangent modules and obtain *cotangent sheaves* of C^∞-schemes.

Definition 2.1.18. Let $\underline{X}$ be a C^∞-scheme. Define a presheaf of $\mathcal{O}_X$-modules $\mathcal{P}T^*\underline{X}$ on $\underline{X}$, by assigning to each open set $U \subseteq X$ the cotangent module $\Omega_{\mathcal{O}_X(U)}$, and to each inclusion of open sets $U \subseteq V \subseteq X$ the morphism of $\mathcal{O}_X(U)$-modules $\Omega_{\rho_{UV}} : \Omega_{\mathcal{O}_X(U)} \to \Omega_{\mathcal{O}_X(V)}$ associated to the morphism of C^∞-rings $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. The *cotangent sheaf* $T^*\underline{X}$ of $\underline{X}$ is then defined as the sheafification of $\mathcal{P}T^*\underline{X}$, as an $\mathcal{O}_X$-module.

If $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes, then by definition of the pullback, $\underline{f}^*(T^*\underline{Y})$ is the sheafification of the presheaf $\underline{f}^*(\mathcal{P}T^*\underline{Y})$. Moreover, similarly to the case of cotangent modules over C^∞-rings, there exists a morphism $\Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$. This morphism $\Omega_{\underline{f}}$ can be thought of as the C^∞-scheme analogue of the morphism $(df)^* : f^*(T^*Y) \to T^*X$ induced by a smooth map of manifolds $f : X \to Y$.

The following theorem explains why it will be much more convenient to work with quasicoherent sheaves instead of coherent sheaves, and can be found in [35, Theorem A.37], or [34, Cor. 6.11 & Prop. 6.12].

Theorem 2.1.19. *(a) Let $\underline{X}$ be a C^∞-scheme. Then* $\mathrm{qcoh}(\underline{X})$ *is closed under kernels, cokernels and extensions in O_X-mod, making it into an abelian category. The category* $\mathrm{coh}(\underline{X})$ *of coherent sheaves is closed under cokernels ad extensions in $\mathcal{O}_X$-mod, but may not be closed under kernels in $\mathcal{O}_X$-mod, so* $\mathrm{coh}(\underline{X})$ *is in general not an abelian category.*
(b) Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of C^∞-schemes. Then the pullback functor $\underline{f}^* : \mathcal{O}_Y$-mod $\to \mathcal{O}_X$-mod preserves the subcategories $\mathrm{qcoh}(\underline{Y}), \mathrm{coh}(\underline{Y}), \mathrm{vect}(\underline{Y})$ *and furthermore* $\underline{f}^* : \mathrm{qcoh}(\underline{Y}) \to \mathrm{qcoh}(\underline{X})$ *is a right exact functor.*
(c) Let $\underline{X}$ be a locally fair C^∞-scheme. Then every $\mathcal{O}_X$-module $\mathcal{E}$ on $\underline{X}$ is quasicoherent. So in other words we have $\mathrm{qcoh}(\underline{X}) = \mathcal{O}_X$*-mod for locally fair C^∞-schemes.*

The next proposition characterizes pullbacks $\underline{f}^*$ of quasicoherent sheaves on C^∞-schemes purely in terms of modules over the corresponding C^∞-rings.

Proposition 2.1.20. *Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings, $\phi : \mathfrak{D} \to \mathfrak{C}$ be a morphism, M, N be $\mathfrak{D}$-modules and $\alpha : M \to N$ be a morphism of $\mathfrak{D}$-modules. Consider* $\underline{X} = \mathrm{Spec}\,(\mathfrak{C}), \underline{Y} = \mathrm{Spec}\,(\mathfrak{D}), \underline{f} = \mathrm{Spec}\,(\phi) : \underline{X} \to \underline{Y}$ *and* $\mathcal{E} = \mathrm{MSpec}\,(M), \mathcal{F} =$

MSpec (N). *Then there exist natural isomorphisms* $\underline{f}^*(\mathcal{E}) \cong \mathrm{MSpec}\,(M \otimes_{\mathfrak{D}} \mathfrak{C})$ *and* $\underline{f}^*(\mathcal{F}) \cong \mathrm{MSpec}\,(N \otimes_{\mathfrak{D}} \mathfrak{C})$ *in* $\mathcal{O}_Y$*-mod. Under these isomorphisms* $\mathrm{MSpec}\,(\alpha \otimes \mathrm{id}_{\mathfrak{C}}) : \mathrm{MSpec}\,(M \otimes_{\mathfrak{D}} \mathfrak{C}) \to \mathrm{MSpec}\,(N \otimes_{\mathfrak{D}} \mathfrak{C})$ *is identified with* $\underline{f}^*(\mathrm{MSpec}\,(\alpha)) : \underline{f}^*(\mathcal{E}) \to \underline{f}^*(\mathcal{F})$.

2.1.5 Virtual quasicoherent sheaves and virtual vector bundles

A general principle in Joyce's theory of derived differential geometry is that 1-categories in the classical picture should be replaced by 2-categories.

In classical differential geometry, the vector bundles over a manifold with their morphisms form a 1-category $\mathrm{vect}(X)$. A particularly important example of a vector bundle over a given manifold X, is the cotangent bundle T^*X. Given a smooth map $f : X \to Y$, pulling back gives a natural functor $f^* : \mathrm{vect}(Y) \to \mathrm{vect}(X)$, and taking the differential provides a natural morphism $(df)^* : f^*(T^*Y) \to T^*X$. We will follow here Joyce [35, §3.1] and describe the 2-categorical "derived" analogues of these notions, and then prove in Proposition 2.1.24 as a new result that a virtual vector bundle is globally equivalent to a morphism of actual vector bundles.

Definition 2.1.21. Let $\underline{X}$ be a C^∞-scheme. Define a 2-category $\mathbf{vqcoh}(\underline{X})$ of *virtual quasicoherent sheaves* on $\underline{X}$. *Objects* in $\mathbf{vqcoh}(\underline{X})$ are given by morphisms $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ in $\mathrm{qcoh}(\underline{X})$, which we will also denote by $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ or $(\mathcal{E}^\bullet, \phi)$. The 1-*morphisms* in $\mathbf{vqcoh}(\underline{X})$ are given by $(f^1, f^2) : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ a pair of morphism $f^1 : \mathcal{E}^1 \to \mathcal{F}^1, f^2 : \mathcal{E}^2 \to \mathcal{F}^2$ in $\mathrm{qcoh}(\underline{X})$ satisfying $\psi \circ f^1 = f^2 \circ \phi$, where $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ and $\psi : \mathcal{F}^1 \to \mathcal{F}^2$ are objects. We will use $f^\bullet$ as an abbreviation for (f^1, f^2).

The *identity* 1-*morphism* of $(\mathcal{E}^\bullet, \phi)$ is defined as $(\mathrm{id}_{\mathcal{E}^1}, \mathrm{id}_{\mathcal{E}^2})$ and *composition of* 1-*morphisms* $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ and $g^\bullet : (\mathcal{F}^\bullet, \psi) \to (\mathcal{G}^\bullet, \zeta)$ as $g^\bullet \circ f^\bullet = (g^1 \circ f^1, g^2 \circ f^2) : (\mathcal{E}^\bullet, \phi) \to (\mathcal{G}^\bullet, \xi)$.

Let $f^\bullet, g^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ be 1-morphisms. A 2-*morphism* $\eta : f^\bullet \Rightarrow g^\bullet$ is a morphism $\eta : \mathcal{E}^2 \to \mathcal{F}^1$ in $\mathrm{qcoh}(\underline{X})$ such that $g^1 = f^1 + \eta \circ \phi$ and $g^2 = f^2 + \psi \circ \eta$.

Given 1-morphisms $f^\bullet, g^\bullet, h^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ and 2-morphisms $\eta : f^\bullet \Rightarrow g^\bullet, \zeta : g^\bullet \Rightarrow h^\bullet$, the *vertical composition* of 2-morphism $\zeta \odot \eta : f^\bullet \Rightarrow h^\bullet$ is defined by $\zeta \odot \eta = \zeta + \eta$.

If $f^\bullet, \tilde{f}^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ and $g^\bullet, \tilde{g}^\bullet : (\mathcal{F}^\bullet, \psi) \to (\mathcal{G}^\bullet, \xi)$ are 1-morphisms and $\eta : f^\bullet \Rightarrow \tilde{f}^\bullet, \zeta : g^\bullet \Rightarrow \tilde{g}^\bullet$ are 2-morphisms, the *horizontal composition* of 2-morphisms $\zeta * \eta : g^\bullet \circ f^\bullet \Rightarrow \tilde{g}^\bullet \circ \tilde{f}^\bullet$ is defined as $\zeta * \eta = g^1 \circ \eta + \zeta \circ f^2 + \zeta \circ \psi \circ \eta$. The resulting strict 2-category will be denoted by $\mathbf{vqcoh}(\underline{X})$.

If $\underline{U} \subseteq \underline{X}$ is an open C^∞-subscheme then restriction from $\underline{X}$ to $\underline{U}$ defines a strict 2-functor $|_{\underline{U}} : \mathbf{vqcoh}(\underline{X}) \to \mathbf{vqcoh}(\underline{U})$.

An object $(\mathcal{E}^\bullet, \phi)$ in $\mathbf{vqcoh}(\underline{X})$ is called a *virtual vector bundle of rank* $d \in \mathbb{Z}$ if $\underline{X}$ may be covered by open $\underline{U} \subseteq \underline{X}$ such that $(\mathcal{E}^\bullet, \phi)|_{\underline{U}}$ is equivalent in $\mathbf{vqcoh}(\underline{U})$ to some $(\mathcal{F}^\bullet, \psi)$ for $\mathcal{F}^1, \mathcal{F}^2$ vector bundles on $\underline{U}$ with $\operatorname{rank}\mathcal{F}^2 - \operatorname{rank}\mathcal{F}^1 = d$. We will write $\operatorname{rank}(\mathcal{E}^\bullet, \phi) = d$. If $\underline{X} \neq \emptyset$ then $\operatorname{rank}(\mathcal{E}^\bullet, \phi)$ depends only on $\mathcal{E}^1, \mathcal{E}^2, \phi$, so it is well-defined. The full 2-subcategory of virtual vector bundles in $\mathbf{vqcoh}(\underline{X})$ will be denoted by $\mathbf{vvect}(\underline{X})$.

Definition 2.1.22. Let $\underline{X}$ be a C^∞-scheme. A virtual vector bundle $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ on $\underline{X}$ is called a *vector bundle* if it is equivalent in $\mathbf{vvect}(\underline{X})$ to $(0, \mathcal{E}, 0)$ for some vector bundle $\mathcal{E}$ on $\underline{X}$.

One can show that a virtual vector bundle $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ is a vector bundle if and only if ϕ has a left inverse in $\operatorname{qcoh}(\underline{X})$.

The following proposition due to Joyce [34, Proposition 3.5], will be a valuable tool in determining when a 1-morphism in $\mathbf{vqcoh}(\underline{X})$ is an equivalence.

Proposition 2.1.23. *Let $\underline{X}$ be a C^∞-scheme, $(\mathcal{E}^\bullet, \phi)$, $(\mathcal{F}^\bullet, \psi)$ be virtual quasicoherent sheaves on $\underline{X}$ and $f^\bullet = (f^1, f^2) : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ be a 1-morphism in $\mathbf{vqcoh}(\underline{X})$. Then $f^\bullet$ is an equivalence if and only if the following complex is a split short exact sequence in* $\operatorname{qcoh}(\underline{X})$*:*

$$0 \longrightarrow \mathcal{E}^1 \xrightarrow{f^1 \oplus -\phi} \mathcal{F}^1 \oplus \mathcal{E}^2 \xrightarrow{\psi \oplus f^2} \mathcal{F}^2 \longrightarrow 0. \tag{2.1}$$

In particular, if $f^\bullet$ is an equivalence then $\mathcal{E}^1 \oplus \mathcal{F}^2 \cong \mathcal{F}^1 \oplus \mathcal{E}^2$ in $\operatorname{qcoh}(\underline{X})$.

As a new result, we will show in the following proposition that every virtual vector bundle over a compact C^∞-scheme $\underline{X}$ is equivalent to a virtual vector bundle, consisting of vector bundles. This result will play an important role, when we later on study stable nearly complex d-manifolds, and their bordism groups.

Proposition 2.1.24. *Let $(\mathcal{E}^\bullet, \phi)$ be a virtual vector bundle over a separated, compact, locally fair C^∞-scheme $\underline{X}$. Then there exists a virtual vector bundle $(\mathcal{G}^\bullet, \psi)$, where $\mathcal{G}^1, \mathcal{G}^2$ are (global) vector bundles over $\underline{X}$, and an equivalence $f^\bullet = (f^1, f^2)$ between $(\mathcal{G}^\bullet, \psi)$ and $(\mathcal{E}^\bullet, \phi)$.*

Proof. We will prove that $(\mathcal{E}^\bullet, \phi)$ is equivalent in $\mathbf{vvect}(\underline{X})$ to some virtual vector bundle $(\mathcal{G}^\bullet, \psi)$, where $\mathcal{G}^1, \mathcal{G}^2$ are vector bundles, by explicitly constructing $\mathcal{G}^1, \mathcal{G}^2$. Consider therefore the vector bundle $\mathcal{G}^2 := \mathbb{R}^N \otimes \mathcal{O}_X$. We will prove in the following, that there exists a morphism $\alpha : \mathbb{R}^N \otimes \mathcal{O}_X \to \mathcal{E}^2$ for some $N \gg 0$ large enough such that $\phi \oplus \alpha : \mathcal{E}^1 \oplus (\mathbb{R}^N \otimes \mathcal{O}_X) \to \mathcal{E}^2$ has a right inverse. Defining $\mathcal{G}^1$ as

$$\mathcal{G}^1 := \mathrm{Ker}(\mathcal{E}^1 \oplus \mathbb{R}^N \otimes \mathcal{O}_X \xrightarrow{\phi\oplus\alpha} \mathcal{E}^2),$$

yields the following commutative diagram in $\mathbf{vqcoh}(\underline{X})$:

$$\begin{tikzcd}
\mathcal{E}^1 \arrow[r, "\phi"] & \mathcal{E}^2 \\
\mathcal{G}^1 \arrow[u, dotted, "\beta"] \arrow[r, dotted, "\psi"] & \mathbb{R}^N \otimes \mathcal{O}_X, \arrow[u, "\alpha"']
\end{tikzcd}$$

where $\beta : \mathcal{G}^1 \to \mathcal{E}^1$ and $\psi : \mathcal{G}^1 \to \mathcal{G}^2 = \mathbb{R}^N \otimes \mathcal{O}_X$ are by the kernel induced morphisms in $\mathrm{qcoh}(\underline{X})$. If we denote the 1-morphism between $(\mathcal{G}^\bullet, \psi)$ and $(\mathcal{E}^\bullet, \phi)$ in $\mathbf{vvect}(\underline{X})$ by $f^\bullet = (\beta, \alpha)$, the following exact sequence in $\mathrm{qcoh}(\underline{X})$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}^1 \arrow[r, shift left, "\beta\oplus-\psi"] & \mathcal{E}^1 \oplus (\mathbb{R}^N \otimes \mathcal{O}_X) \arrow[l, dotted, shift left, "\gamma"] \arrow[r, shift left, "\phi\oplus\alpha"] & \mathcal{E}^2 \arrow[l, dotted, shift left, "\delta"] \arrow[r] & 0,
\end{tikzcd} \tag{2.2}$$

is then a split exact sequence in $\mathrm{qcoh}(\underline{X})$, since $\phi \oplus \alpha$ has a right inverse, and therefore, by Proposition 2.1.23, $f^\bullet$ is an equivalence in $\mathbf{vvect}(\underline{X})$.

Let therefore $x \in \underline{X}$. As $\mathcal{E}^1 \xrightarrow{\phi} \mathcal{E}^2$ is a virtual vector bundle on $\underline{X}$ there exists an open neighbourhood $\underline{U}$ of x in $\underline{X}$, an object $(\mathcal{F}^\bullet, \rho)$ in $\mathbf{vvect}(\underline{U})$ with $\mathcal{F}^1, \mathcal{F}^2$ vector bundles on $\underline{U}$, and an equivalence $i^\bullet = (i^1, i^2) : (\mathcal{F}^\bullet, \rho) \to (\mathcal{E}^\bullet, \phi)|_{\underline{U}}$. Hence we have the following complex in $\mathrm{qcoh}(\underline{U})$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^1 \arrow[r, shift left, "i^1\oplus-\rho"] & \mathcal{E}^1|_{\underline{U}} \oplus \mathcal{F}^2 \arrow[l, dotted, shift left, "\tilde{\gamma}_{\underline{U}}=(\tilde{\gamma}_1,\tilde{\gamma}_2)"] \arrow[r, shift left, "\phi\oplus i^2"] & \mathcal{E}^2|_{\underline{U}} \arrow[l, dotted, shift left, "\tilde{\delta}_{\underline{U}}=(\tilde{\delta}_1,\tilde{\delta}_2)"] \arrow[r] & 0.
\end{tikzcd}$$

Since $i^\bullet$ is an equivalence, this complex is a split exact sequence, and we get in particular a morphism $\tilde{\delta}_{\underline{U}} = (\tilde{\delta}_1, \tilde{\delta}_2) : \mathcal{E}^2|_{\underline{U}} \to \mathcal{E}^1|_{\underline{U}} \oplus \mathcal{F}^2$, satisfying $(\phi \oplus i^2) \circ \tilde{\delta}_{\underline{U}} =$

$\mathrm{id}_{\mathcal{E}^2|_{\underline{U}}}$. Choose a surjective morphism $\tilde{\alpha}_{\underline{U}} : (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}} \to \mathcal{F}^2$ for some $N \gg 0$ large enough (this is possible since $\mathcal{F}^2$ is a vector bundle on $\underline{U}$), and define a morphism $\alpha_{\underline{U}} : (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}} \to \mathcal{E}^2|_{\underline{U}}$ by $\alpha_{\underline{U}} = i^2 \circ \tilde{\alpha}_{\underline{U}}$.

We claim, that there exists a morphism $\sigma_{\underline{U}} : \mathcal{F}^2 \to (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}}$ such that $\alpha_{\underline{U}} \circ \sigma_{\underline{U}} = i^2$, that is we have the following commutative diagram in $\mathbf{vvect}(\underline{X})$:

$$\begin{tikzcd}
\mathcal{F}^2 \arrow[rr, "i^2"] \arrow[dr, "\sigma_{\underline{U}}"'] & & \mathcal{E}^2|_{\underline{U}} \\
& (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}}. \arrow[ur, "\alpha_{\underline{U}}"'] &
\end{tikzcd}$$

This can be seen as follows: the vector bundle $\mathcal{F}^2$ can locally be written as $\mathbb{R}^k \otimes \mathcal{O}_y$. So every section ε of $\mathcal{F}^2$ can be written as $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$, where $\varepsilon_i \in \mathcal{H}^0(\mathcal{E}^2|_{\underline{U}})$ for all $i = 1 \dots k$. Since $\alpha_{\underline{U}}$ is surjective, the sections $H^0(\mathcal{E}^2|_{\underline{U}})$ are generated by $H^0(\mathcal{E}^2|_{\underline{U}}) = \langle (\phi_1|_{\underline{U}}, \dots, \phi_N|_{\underline{U}}) \rangle_{\mathcal{O}_X(\underline{U})}$, that is for each $j = 0 \dots k$ we have $\varepsilon_j = \sum_{i=1}^{k} a_{ij} \phi_i|_{\underline{U}}$, where $a_{ij} \in \mathcal{O}_k(\underline{U})$. Then $(a_{ij})_{i=1,\dots,N}^{j=1,\dots,k}$ is the matrix corresponding to a morphism $\sigma_{\underline{U}} : \mathcal{F}^2 \to (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}}$. Since a vector bundle is locally free, the claim follows.

We want now to use the morphism $\tilde{\delta}_{\underline{U}}$ to construct a morphism $\delta_{\underline{U}}$ in (2.2), such that $\delta_{\underline{U}}$ is a right inverse to $(\phi \oplus \alpha)|_{\underline{U}}$, that is $(\phi \oplus \alpha)|_{\underline{U}} \circ \delta_{\underline{U}} = \mathrm{id}_{\mathcal{E}^2|_{\underline{U}}}$.

We claim that $\delta_{\underline{U}} := (\tilde{\delta}_1, \sigma_{\underline{U}} \circ \tilde{\delta}_2) : \mathcal{E}^2|_{\underline{U}} \to \mathcal{E}^1|_{\underline{U}} \oplus (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}}$ is a right inverse for $(\phi \oplus \alpha)|_{\underline{U}}$:

$$(\phi \quad \alpha_{\underline{U}}) \begin{pmatrix} \tilde{\delta}_1 \\ \sigma_{\underline{U}} \circ \tilde{\delta}_2 \end{pmatrix} = \phi \circ \tilde{\delta}_1 + \alpha_{\underline{U}} \circ \sigma_{\underline{U}} \circ \tilde{\delta}_2 = \phi \circ \tilde{\delta}_1 + i^2 \circ \tilde{\delta}_2 = id_{\mathcal{E}^2|_{\underline{U}}}.$$

Here we used that $i^2 = \alpha_{\underline{U}} \circ \sigma_{\underline{U}}$ and that $\tilde{\delta}_{\underline{U}} = (\tilde{\delta}_1, \tilde{\delta}_2)$ is right inverse to $\phi \oplus i^2$. As $\underline{X}$ can be covered by such open $\underline{U}$, and admits a partition of unity subordinated to such an open cover (this follows from $\underline{X}$ being compact and locally fair, so that we can use Proposition 4.22 in [34]), we can patch all these local data together, and get a morphism $\alpha : \mathbb{R}^N \otimes \mathcal{O}_X \to \mathcal{E}^2$ and a global right inverse morphism of $\phi \oplus \alpha$, which we will denote by $\delta : \mathcal{E}^2 \to \mathcal{E}^1 \oplus (\mathbb{R}^N \otimes \mathcal{O}_X)$. But as the category of quasicoherent sheaves is an abelian category, the existence of δ implies that (2.2) is an split exact sequence (see Lemma A.4.1), and therefore that $f^\bullet = (\beta, \alpha)$ is an equivalence.

The remaining bit, is to show, that $\mathcal{G}^1$ is a vector bundle over $\underline{X}$. Note therefore that on $\underline{U}$ we have the following commutative diagram in $\mathbf{vqcoh}(\underline{U})$:

$$\begin{tikzcd} \mathcal{F}^1 \arrow[rr, "\phi"] & & \mathcal{F}^2 \\ \mathcal{G}^1|_{\underline{U}} \arrow[u, dotted, "\tilde{i}^1"'] \arrow[rr, dotted, "\psi"] & & (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}}, \arrow[u, "\tilde{i}^2"'] \end{tikzcd}$$

where $\tilde{i}^\bullet = (\tilde{i}^1, \tilde{i}^2) : (\mathcal{G}^\bullet, \psi)|_{\underline{U}} \to (\mathcal{F}^\bullet, \rho)$ is the equivalence given by composition of the equivalence $f^\bullet$ with the equivalence $(i^\bullet)^{-1} : (\mathcal{E}^\bullet, \phi)|_{\underline{U}} \to (\mathcal{F}^\bullet, \rho)$. The corresponding complex in $\mathrm{qcoh}(\underline{U})$ is then of the form

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}^1|_{\underline{U}} \arrow[r, shift left, "\tilde{i}^1 \oplus -\psi"] & \mathcal{F}^1 \oplus (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}} \arrow[l, shift left, dotted, "\tilde{\tilde{\gamma}}"] \arrow[r, shift left, "\phi \oplus \tilde{i}^2"] & \mathcal{F}^2 \arrow[l, shift left, dotted, "\tilde{\tilde{\delta}}"] \arrow[r] & 0. \end{tikzcd}$$

But as $\tilde{i}^\bullet$ is an equivalence, this is in fact an exact sequence and the virtual vector bundle

$$\mathcal{F}^1 \oplus (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}} \xrightarrow{\phi \oplus \tilde{i}^2} \mathcal{F}^2$$

has a left inverse $\tilde{\tilde{\delta}}$, which shows that $\mathcal{F}^1 \oplus (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}} \to \mathcal{F}^2$ is a vector bundle. (Compare [34, Proposition 3.9]). But $\mathcal{G}^1|_{\underline{U}} \cong \ker(\mathcal{F}^1 \oplus (\mathbb{R}^N \otimes \mathcal{O}_X)|_{\underline{U}} \to \mathcal{F}^2)$, and hence also a vector bundle. Again, since we can cover $\underline{X}$ by such open $\underline{U}$, this shows that $\mathcal{G}^1$ is a vector bundle, which completes the proof. $\square$

2.1.6 Square zero extensions of C^∞-rings

A *square zero extension of C^∞-rings* is a surjective morphism of C^∞-rings $\phi : \mathfrak{C}' \to \mathfrak{C}$ such that the kernel I of ϕ in $\mathfrak{C}'$ is a square zero ideal. Recall, that a square zero ideal in a commutative $\mathbb{R}$-algebra A is an ideal I satisfying $i \cdot j = 0$ for all $i, j \in I$. Thus, every square zero extension fits into an exact sequence

$$0 \longrightarrow I \xrightarrow{\kappa_\phi} \mathfrak{C}' \xrightarrow{\phi} \mathfrak{C} \longrightarrow 0, \tag{2.3}$$

where $\kappa_\phi : I \to \mathfrak{C}'$ denotes the kernel of ϕ. The ideal I in $\mathfrak{C}'$, has not just the structure of a $\mathfrak{C}'$-module, but since ϕ is surjective also that of a $\mathfrak{C}$-module. This $\mathfrak{C}$-module structure is well-defined, as I is a square zero ideal.

Given two square zero extensions $\phi : \mathfrak{C}' \to \mathfrak{C}$ and $\psi : \mathfrak{D}' \to \mathfrak{D}$, a *morphism of square zero extensions* $(\alpha, \alpha') : \phi \to \psi$ is given by a pair of morphisms of C^∞-rings $\alpha : \mathfrak{C} \to \mathfrak{D}$ and $\alpha' : \mathfrak{C}' \to \mathfrak{D}'$ satisfying $\alpha \circ \psi = \phi \circ \alpha'$. Such a pair (α, α') induces a morphisms α'' between the kernel I of ϕ and the kernel J of ψ, by $\alpha'' := \alpha'|_I : I \to J$, and we get thus a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\kappa_\phi"] \arrow[d, "\alpha''"'] & \mathfrak{C}' \arrow[r, "\phi"] \arrow[d, "\alpha'"'] & \mathfrak{C} \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & J \arrow[r, "\kappa_\psi"] & \mathfrak{D}' \arrow[r, "\psi"] & \mathfrak{D} \arrow[r] & 0\ .
\end{tikzcd}$$

The following definition will associate an exact sequence of $\mathfrak{C}$-modules to a square zero extension. This exact sequence will be particularly useful when investigating 2-morphisms of d-spaces.

Definition 2.1.25. Let $\phi : \mathfrak{C}' \to \mathfrak{C}$ be a square zero extension of C^∞-rings, with kernel $\kappa_\phi : I \to \mathfrak{C}'$. As we already know, I has the structure of a $\mathfrak{C}$-module. As in Definition 2.1.12, we have cotangent modules $\Omega_{\mathfrak{C}}, \Omega_{\mathfrak{C}'}$ and a morphism of $\mathfrak{C}$-modules $(\Omega_\phi)_* : \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C} \to \Omega_{\mathfrak{C}}$. We can therefore define a linear map $\Xi_\phi : I \to \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}$ to be the composition

$$I \xrightarrow{\kappa_\phi} \mathfrak{C}' \xrightarrow{d_{\mathfrak{C}'}} \Omega_{\mathfrak{C}'} = \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}' \xrightarrow{\mathrm{id} \otimes \phi} \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}. \tag{2.4}$$

A not obvious fact about this composition is, that Ξ_ϕ is a $\mathfrak{C}$-module morphism, although none of $\kappa_\phi, d_{\mathfrak{C}'}, \mathrm{id} \otimes \phi$ are $\mathfrak{C}$-module morphisms. This fact and the existence of an exact sequence of $\mathfrak{C}$-modules is proven in the following proposition. (For a proof of this proposition see [35, Proposition 2.4].)

Proposition 2.1.26. *The linear map $\Xi_\phi : I \to \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}$ is a $\mathfrak{C}$-module morphism and fits into an exact sequence of $\mathfrak{C}$-modules:*

$$I \xrightarrow{\Xi_\phi} \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C} \xrightarrow{(\Omega_\phi)_*} \Omega_{\mathfrak{C}} \longrightarrow 0. \tag{2.5}$$

The exact sequence (2.5) extends in a straightforward way to a commutative diagram when we consider morphisms between square zero extensions.

Lemma 2.1.27. *Let $\phi : \mathfrak{C}' \to \mathfrak{C}$ and $\psi : \mathfrak{D}' \to \mathfrak{D}$ be square zero extensions and $(\alpha, \alpha') : \phi \to \psi$ be a morphism of square zero extensions. Then the exact sequence (2.5) extends to a commutative diagram*

$$\begin{tikzcd} I \arrow[r, "\Xi_\phi"] \arrow[d, "\alpha''"'] & \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C} \arrow[r, "(\Omega_\phi)_*"] \arrow[d, "\Omega_{\alpha'} \otimes \alpha"'] & \Omega_{\mathfrak{C}} \arrow[r] \arrow[d, "\Omega_\alpha"'] & 0 \\ J \arrow[r, "\Xi_\psi"] & \Omega_{\mathfrak{D}'} \otimes_{\mathfrak{D}'} \mathfrak{D} \arrow[r, "(\Omega_\psi)_*"] & \Omega_{\mathfrak{D}} \arrow[r] & 0\ . \end{tikzcd}$$

The next proposition is crucial for defining 2-morphisms of d-spaces and will play an important role in the construction of a tangent d-space.

Proposition 2.1.28. *Let $\phi : \mathfrak{C}' \to \mathfrak{C}$ and $\psi : \mathfrak{D}' \to \mathfrak{D}$ be square zero extensions of C^∞-rings with kernels I, J and let $(\alpha, \alpha_1'), (\alpha, \alpha_2')$ be morphisms between square zero extensions inducing morphisms $\alpha_1'', \alpha_2'' : I \to J$. So we have the following diagram*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\kappa_\phi"] \arrow[d, shift right, "\alpha_1''"'] \arrow[d, shift left, "\alpha_2''"] & \mathfrak{C}' \arrow[r, "\phi"] \arrow[d, shift right, "\alpha_1'"'] \arrow[d, shift left, "\alpha_2'"] & \mathfrak{C} \arrow[r] \arrow[d, "\alpha"'] & 0 \\ 0 \arrow[r] & J \arrow[r, "\kappa_\psi"'] & \mathfrak{D}' \arrow[r, "\psi"'] & \mathfrak{D} \arrow[r] & 0\ . \end{tikzcd}$$

Then there exists a unique $\mathfrak{D}$-module morphism $\mu : \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{D} \to J$ such that

$$\alpha_2' = \alpha_1' + \kappa_\psi \circ \mu \circ (\mathrm{id} \otimes (\alpha \circ \phi)) \circ d_{\mathfrak{C}'}. \tag{2.6}$$

Here the morphisms come from the following sequence

$$\mathfrak{C}' \xrightarrow{d_{\mathfrak{C}'}} \Omega_{\mathfrak{C}'} = \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{C}' \xrightarrow{\mathrm{id} \otimes (\alpha \circ \phi)} \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{D} \xrightarrow{\mu} J \xrightarrow{\kappa_\psi} \mathfrak{D}'.$$

Moreover we have

$$\alpha_2'' = \alpha_1'' + \mu \circ (\mathrm{id} \otimes (\alpha \circ \phi)) \circ d_{\mathfrak{C}'} \circ \kappa_\phi$$
$$\text{and} \quad \Omega_{\alpha_2'} = \Omega_{\alpha_1'} + d_{\mathfrak{D}'} \circ \kappa_\psi \circ \mu \circ (\mathrm{id} \otimes (\alpha \circ \phi)).$$

The converse also holds that is, if $(\alpha, \alpha_1') : \phi \to \psi$ is a morphism, and $\mu : \Omega_{\mathfrak{C}'} \otimes_{\mathfrak{C}'} \mathfrak{D} \to J$ a $\mathfrak{D}$-module morphism, then defining α_2' by (2.6) gives a C^∞-ring morphism $\alpha_2' : \mathfrak{C}' \to \mathfrak{D}'$ with $(\alpha, \alpha_2') : \phi \to \psi$ being a morphism of square zero extensions.

The proof of this proposition uses basically the universal property of the cotangent module and can be found in [35, Proposition 2.8] .

Applying the spectrum functor we get an analogous notion and description of *square zero extensions of C^∞-schemes.* The C^∞-rings are basically replaced by sheaves of C^∞-rings, and the modules over C^∞-rings, are replaced by sheaves of $\mathcal{O}_X$-modules. (See [35, Definition 2.9] for more details.)

Definition 2.1.29. A *square zero extension* $(\mathcal{O}'_X, \imath_X)$ of a locally fair C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ is given by a sheaf of C^∞-rings $\mathcal{O}'_X$ on X, such that $\underline{X}' = (X, \mathcal{O}'_X)$ is a C^∞-scheme, and a morphism $i_X : \mathcal{O}'_X \to \mathcal{O}_X$ of sheaves of C^∞-rings on X, which is a sheaf of square zero extensions of C^∞-rings. The tuple $\underline{\imath}_X = (\mathrm{id}_X, \imath_X)$ is a morphism of C^∞-schemes $\underline{\imath}_X : \underline{X} \to \underline{X}'$, and the triple $(\underline{X}, \mathcal{O}'_X, \imath_X)$ is called *square zero extension of C^∞-schemes.*

The C^∞-scheme analogue of equation (2.3) is then given by

$$0 \longrightarrow \mathcal{I}_X \xrightarrow{\kappa_X} \mathcal{O}'_X \xrightarrow{\imath_X} \mathcal{O}_X \longrightarrow 0, \tag{2.7}$$

where we denoted again the kernel of $\imath_X$ by $\kappa_X : \mathcal{I}_X \to \mathcal{O}'_X$.

As the sheaf of C^∞-rings $\mathcal{O}'_X$ has a sheaf of cotangent modules $\Omega_{\mathcal{O}'_X}$ (recall that this is a sheaf of $\mathcal{O}'_X$-modules with exterior derivative $d : \mathcal{O}'_X \to \Omega_{\mathcal{O}'_X}$), we can define $\mathcal{F}_X = \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}_X$ to be the associated sheaf of $\mathcal{O}_X$-modules. $\mathcal{F}_X$ is actually a quasicoherent sheaf on the C^∞-scheme $\underline{X}$, and we get a natural morphism $\psi_X = \Omega_{\imath_X} \otimes \mathrm{id} : \mathcal{F}_X \to T^*\underline{X}$ in $\mathrm{qcoh}(\underline{X})$ to the cotangent sheaf of $\underline{X}$. As in (2.4), we can define a morphism of sheaves of abelian groups $\xi_X : \mathcal{I}_X \to \mathcal{F}_X$ as the composition

$$\mathcal{I}_X \xrightarrow{\kappa_X} \mathcal{O}'_X \xrightarrow{d} \Omega_{\mathcal{O}'_X} = \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}'_X \xrightarrow{\mathrm{id}\otimes\imath_X} \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}_X = \mathcal{F}_X. \tag{2.8}$$

Using Proposition 2.1.26, we get then that ξ_X is a morphism of quasicoherent sheaves on $\underline{X}$ and the sequence

$$\mathcal{I}_X \xrightarrow{\xi_X} \mathcal{F}_X \xrightarrow{\psi_X} T^*\underline{X} \longrightarrow 0, \tag{2.9}$$

is exact in $\mathrm{qcoh}(\underline{X})$.

All the results from the C^∞-ring world translate nicely to the C^∞-scheme world and will not be stated explicitly here, and we will refer instead to [35, §2.1] for a complete discussion.

2.2 D-spaces

We are now in the position to define the 2-category of d-spaces. D-spaces will be the surrounding environment for d-manifolds and many important properties of d-manifolds will already be present in the d-space world. We once again refer to [35, §2] for more details and proofs of the results.

Definition 2.2.1. A *d-space* $\boldsymbol{X}$ is a quintuple $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$, consisting of a separated, second countable, locally fair C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ and an exact sequence of sheaves on X

$$\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{O}'_X \xrightarrow{\imath_X} \mathcal{O}_X \longrightarrow 0, \tag{2.10}$$

satisfying the following conditions:

(a) $\mathcal{O}'_X$ is a sheaf of C^∞ - rings on X, with $\underline{X}' = (X, \mathcal{O}'_X)$ a C^∞-scheme.

(b) $\imath_X : \mathcal{O}'_X \to \mathcal{O}_X$ is a surjective morphism of sheaves of C^∞-rings on X. Its kernel $\mathcal{I}_X$ is a sheaf of ideals in $\mathcal{O}'_X$ which should be a sheaf of square zero ideals. Recall, that a square zero ideal in a commutative $\mathbb{R}$-algebra A is an ideal I satisfying $i \cdot j = 0$ for all $i, j \in I$. $\mathcal{I}_X$ is thus an $\mathcal{O}'_X$-module, but as $\mathcal{I}_X$ consists of square zero ideals and $\imath_X$ is surjective, the action of $\mathcal{O}'_X$ factors through an action of $\mathcal{O}_X$. Hence I_X is an $\mathcal{O}_X$-module, and thus a quasicoherent sheaf on $\underline{X}$, as the C^∞-scheme $\underline{X}$ is locally fair.

(c) $\mathcal{E}_X$ is a quasicoherent sheaf on $\underline{X}$, and $\jmath_X : \mathcal{E}_X \to \mathcal{I}_X$ is a surjective morphism in qcoh($\underline{X}$).

As the C^∞-scheme $\underline{X}$ is locally fair, the underlying topological space X is locally homeomorphic to a closed subset of $\mathbb{R}^n$ and therefore *locally* compact. This together with the Hausdorffness and second countable property implies that $\underline{X}$ is *paracompact.*

Now, the sheaf of C^∞-rings $\mathcal{O}'_X$ has a sheaf of cotangent modules $\Omega_{\mathcal{O}'_X}$, which is an $\mathcal{O}'_X$-module with exterior derivative $d : \mathcal{O}'_X \to \Omega_{\mathcal{O}'_X}$. Define $\mathcal{F}_X = \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}_X$ to be the associated $\mathcal{O}_X$-module. This associated $\mathcal{O}_X$-module is a quasicoherent sheaf on $\underline{X}$, and if we set $\psi_X = \Omega_{\imath_X} : \mathcal{F}_X \to T^*\underline{X}$, we get a morphism in qcoh($\underline{X}$).

Moreover, define $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ to be the composition of morphism of sheaves of abelian groups on X:

$$\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{I}_X \xrightarrow{d|_{\mathcal{I}_X}} \Omega_{\mathcal{O}'_X} = \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}'_X \xrightarrow{id\otimes\imath_X} \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}_X = \mathcal{F}_X.$$

One can show that ϕ_X is a morphism of $\mathcal{O}_X$-modules, and that the following sequence is exact in qcoh($\underline{X}$)

$$\mathcal{E}_X \xrightarrow{\phi_X} \mathcal{F}_X \xrightarrow{\psi_X} T^*\underline{X} \longrightarrow 0. \tag{2.11}$$

It turns out the morphism $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ is in fact a virtual vector bundle in the sense of 2.1.5 , and we refer thus to $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ as the *virtual cotangent sheaf* of $\boldsymbol{X}$.

Given two d-spaces $\boldsymbol{X}, \boldsymbol{Y}$, a 1-*morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ between d-spaces is given by a triple $\boldsymbol{f} = (\underline{f}, f', f'')$, where $\underline{f} = (f, f^{\#}) : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes, $f' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$ a morphism of sheaves of C^∞-rings on X and $f'' : \underline{f}^*(\mathcal{E}_Y) \to \mathcal{E}_X$ a morphism of quasicoherent sheaves on $\underline{X}$, such that the following diagram commutes:

$$\begin{tikzcd}
f^{-1}(\mathcal{E}_Y)\otimes^{\mathrm{id}}_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{O}_Y) \arrow[d, "\mathrm{id}\otimes f^{\#}"'] \arrow[r, phantom, "="] & f^{-1}(\mathcal{E}_Y) \arrow[rr, "f^{-1}(\jmath_Y)"] & & f^{-1}(\mathcal{O}'_Y) \arrow[r, "f^{-1}(\imath_Y)"] \arrow[d, "f'"'] & f^{-1}(\mathcal{O}_Y) \arrow[r] \arrow[d, "f^{\#}"'] & 0 \\
\underline{f}^*(\mathcal{E}_Y) \arrow[rr, "f''"] & & \mathcal{E}_X \arrow[r, "\jmath_X"] & \mathcal{O}'_X \arrow[r, "\imath_X"] & \mathcal{O}_X \arrow[r] & 0\ . \\
= f^{-1}(\mathcal{E}_Y)\otimes^{f^{\#}}_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X
\end{tikzcd} \tag{2.12}$$

One can also define composition of 1-morphisms, 2-morphisms, the identity 1-morphism, the identity 2-morphism and composition of 2-morphisms, and thus define a 2-category of d-spaces, which we will denote by **dSpa**. For all the details of the construction we refer to [35, §2.2].

The following theorem summarises some properties of the category of d-spaces.

Theorem 2.2.2. *(a) D-spaces form a strict* 2*-category* **dSpa***, in which all* 2*-morphism are* 2*-isomorphisms.*
(b) Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *be a* 1*-morphism in* **dSpa***. Then the* 2*-morphisms* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ *form an abelian group under vertical composition, and in fact a real vector space.*
(c) $F^{\mathbf{dSpa}}_{C^\infty\mathbf{Sch}}$ *and* $F^{\mathbf{dSpa}}_{\mathbf{Man}}$ *are full and faithful strict* 2*-functors.*

There exists a 2-functor $F^{\mathbf{dSpa}}_{C^\infty\mathrm{Sch}} : C^\infty\mathbf{Sch}^{\mathrm{lf}}_{\mathrm{ssc}} \to \mathbf{dSpa}$ from the category of separated, second countable, locally fair C^∞-schemes $C^\infty\mathbf{Sch}^{\mathrm{lf}}_{\mathrm{ssc}}$ (regarded as a 2-category) to the 2-category of d-spaces $\mathbf{dSpa}$ and, using this functor, one can define a 2-functor $F^{\mathbf{dSpa}}_{\mathbf{Man}} : \mathbf{Man} \to \mathbf{dSpa}$ given by $F^{\mathbf{dSpa}}_{\mathbf{Man}} = F^{\mathbf{dSpa}}_{C^\infty\mathrm{Sch}} \circ F^{C^\infty\mathrm{Sch}}_{\mathbf{Man}}$.

Using these functors we will write $\hat{C}^\infty\mathbf{Sch}^{\mathrm{lf}}_{\mathrm{ssc}}$ for the 2-subcategory of objects $\boldsymbol{X}$ in $\mathbf{dSpa}$ equivalent to $F^{\mathbf{dSpa}}_{C^\infty\mathrm{Sch}}(\underline{X})$ for some $\underline{X}$ in $C^\infty\mathbf{Sch}^{\mathrm{lf}}_{\mathrm{ssc}}$, and $\hat{\mathbf{M}}\mathbf{an}$ for the full 2-subcategory of objects $\boldsymbol{X}$ in $\mathbf{dSpa}$ equivalent to $F^{\mathbf{dSpa}}_{\mathbf{Man}}(X)$ for X being some manifold.

2.2.1 Gluing d-spaces by equivalences

In many situations it will be convenient to have a gluing procedure for d-spaces on hand. This procedure should satisfy that the "glued" d-spaces are again a d-space and this resulting d-space should be independent of all choices up to equivalence.

Before stating the theorem which provides such a gluing procedure, we will first define what we will mean by an open d-subspace.

Definition 2.2.3. Let $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ be a d-space. We call a d-space $\boldsymbol{U} = (\underline{U}, \mathcal{O}'_X|_{\underline{U}}, \mathcal{E}_X|_{\underline{U}}, \imath_X|_{\underline{U}}, \jmath_X|_{\underline{U}})$ an *open d-subspace* of $\boldsymbol{X}$, if $\underline{U}$ is an open C^∞-subscheme in $\underline{X}$. An *open cover* of a d-space $\boldsymbol{X}$ is a family $\{\boldsymbol{U}_a : a \in A\}$ of open d-subspaces of $\boldsymbol{X}$, where A is some indexing set, such that $\underline{X} = \bigcup\limits_{a\in A} \underline{U}_a$.

The following theorem is proven in [35, Theorem 2.28], and explains how one can glue a collection of d-spaces $\boldsymbol{X}_i, i \in I$ along open d-subspaces $\boldsymbol{U}_{ij} \subseteq \boldsymbol{X}_i, i, j \in I$ and equivalences $e_{ij}\colon \boldsymbol{U}_{ij} \xrightarrow{\sim} \boldsymbol{U}_{ji}$, satisfying some conditions on the overlaps. The so obtained new d-space $\boldsymbol{Z}$ has open subspaces $\hat{\boldsymbol{X}}_i$ equivalent to $\boldsymbol{X}_i$ for $i \in I$, with $\boldsymbol{Z} = \bigcup_{i\in I} \hat{\boldsymbol{X}}_i$ and $\hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j \simeq \boldsymbol{U}_{ij} \simeq \boldsymbol{U}_{ji}$.

Theorem 2.2.4. *Let $\boldsymbol{X}, \boldsymbol{Y}$ be d-spaces, $\boldsymbol{U} \subseteq \boldsymbol{X}, \boldsymbol{V} \subseteq \boldsymbol{Y}$ open d-subspaces and $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{V}$ an equivalence in* $\mathbf{dSpa}$*. On the underlying level of topological spaces we have open subsets $U \subseteq X, V \subseteq Y$ and a homeomorphism $f : U \to V$, and we can form the quotient topological space $Z := X \amalg_f Y = (X \amalg Y)/\sim$, where $\sim$ identifies $u \in U \subseteq X$ with $f(u) \in V \subseteq Y$.*

Suppose that Z is Hausdorff. Then there exists a d-space $\boldsymbol{Z}$, open d-subspaces $\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}$ in $\boldsymbol{Z}$ with $\boldsymbol{Z} = \hat{\boldsymbol{X}} \cup \hat{\boldsymbol{Y}}$, equivalences $\boldsymbol{g} : \boldsymbol{X} \to \hat{\boldsymbol{X}}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \hat{\boldsymbol{Y}}$ in

dSpa *such that* $\boldsymbol{g}|_{\boldsymbol{U}} : \boldsymbol{U} \to \hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$ *and* $\boldsymbol{h}|_{\boldsymbol{U}} : \boldsymbol{U} \to \hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$ *are equivalences, and a 2-morphism* $\eta : \boldsymbol{g}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f} : \boldsymbol{U} \to \hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$. *Furthermore, the d-space* $\boldsymbol{Z}$ *is independent of all choices up to equivalence.*

This theorem can be stated in a more general setting, gluing not just two, but an arbitrary (not necessarily finite) number of d-subspaces together. We will not state this theorem here, but refer to [35, Theorem 2.31] instead.

2.2.2 Fibre products in dSpa

As we will see in section 2.3, a d-manifold will locally be defined as a fibre product of manifolds in the 2-category **dSpa**. (The definition of fibre product in a 2-category can be found in Appendix §A.2.) It is therefore crucial that all fibre products exist in in the 2-category of d-spaces **dSpa**, and that transverse fibre products of manifolds are preserved under the functor $F_{\mathbf{Man}}^{\mathbf{dSpa}}$. This is exactly the statement of the following theorem due to Joyce (compare [35, Theorem 2.36]).

Theorem 2.2.5. *(a) All fibre products in* **dSpa** *exist.*
(b) Let $f : X \to Z$ *and* $h : Y \to Z$ *be smooth maps of manifolds without boundary and let* $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{g}, \boldsymbol{h} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(X,Y,Z,g,h)$. *If* g *and* h *are transverse the fibre product* $X \times_{g,Z,h} Y$ *in* **Man** *exists, and its image under* $F_{\mathbf{Man}}^{\mathbf{dSpa}}$ *is equivalent in* **dSpa** *to the fibre product* $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *in* **dSpa**. *In the case of* g, h *not being transverse, then* $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *exists in* **dSpa**, *but is not a manifold.*

This theorem can be proven by writing down an explicit construction for the fibre product, that is writing down a d-space $\boldsymbol{W}$, 1-morphisms $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}, \boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Z}$ and a 2-morphisms $\eta : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$ and verifying that the so obtained square

$$\begin{tikzcd} \boldsymbol{W} \arrow[r, "\boldsymbol{f}"'] \arrow[d, "\boldsymbol{e}"] \arrow[dr, phantom, "\Uparrow\eta"] & \boldsymbol{Y} \arrow[d, "\boldsymbol{h}"'] \\ \boldsymbol{X} \arrow[r, "\boldsymbol{g}"] & \boldsymbol{Z} \end{tikzcd}$$

in **dSpa** is in fact a 2-Cartesian square, that is, fulfils a universal property.

Note that on the underlying C^∞-scheme level, the d-space $\boldsymbol{W}$ is just the fibre product of the underlying C^∞-schemes, that is we have $\underline{W} = \underline{X} \times_{\underline{Z}} \underline{Y}$.

2.3 D-manifolds

We are now in a situation where we can define the notion of d-manifolds. D-manifolds were recently introduced by Joyce in [35] and can be interpreted as a 2-category truncated version of the 'derived manifolds' of David Spivak [50]. Spivak's 'derived manifolds' form an ∞-category and the definition involves complicated and heavy usage of derived algebraic geometry, in particular the work of Lurie [40]. Borisov and Noel [9] showed that an equivalent ∞-category can be defined using much simpler techniques. Moreover Borisov [8] proved that there exists a strict 2-functor $F_{\mathrm{DerMan}}^{\mathbf{dMan}}$ from a 2-category truncation of the ∞-category of Spivak's 'derived manifolds' to the 2-category of d-manifolds.

The basic idea in defining the 2-category **dMan** of d-manifolds without boundary is to define it as a full 2-subcategory of the category of d-spaces **dSpa**. We follow here closely [35, §3.2] and refer to it for a much more complete and detailed treatment.

Definition 2.3.1. A *principal d-manifold* is a d-space $\boldsymbol{W}$ which is equivalent in the category **dSpa** to a fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$, where $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \hat{\mathbf{Man}}$. The underlying C^∞-scheme $\underline{W}$ of a principal d-manifold $\boldsymbol{W} \simeq (\underline{W}, \mathcal{O}'_W, \mathcal{E}_W, \imath_W, \jmath_W)$ is given by the fibre product $\underline{X} \times_{\underline{Z}} \underline{Y}$, where $\underline{X}, \underline{Y}, \underline{Z} = F_{\mathbf{Man}}^{C^\infty\mathbf{Sch}}(X, Y, Z)$. Since $\underline{X}, \underline{Y}, \underline{Z}$ are finitely presented affine C^∞-schemes, and these are closed under fibre products, $\underline{W}$ is a finitely presented affine C^∞-scheme.

Given a manifold X we can take $Y = Z = *$, a point, and $g = \pi : X \to *, h = \mathrm{id}_* : * \to *$, and get $\boldsymbol{W} \simeq \boldsymbol{X} \times_* * \simeq \boldsymbol{X}$. So the image of every manifold X under $F_{\mathbf{Man}}^{\mathbf{dSpa}}$ is a principal d-manifold, and so is any object in $\hat{\mathbf{Man}}$.

The *virtual dimension* vdim $\boldsymbol{W}$ of $\boldsymbol{W}$ is defined as

$$\mathrm{vdim}\, \boldsymbol{W} = \dim X + \dim Y - \dim Z,$$

where X, Y, Z are manifolds representing $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, that is $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(X, Y, Z)$. Note that vdim $\boldsymbol{W}$ is independent of the choice of X, Y, Z, g, h for $\boldsymbol{W} \neq \emptyset$, and depends only on the d-space $\boldsymbol{W}$. This statement is proven in [34, Proposition 1.4.11] and shows that the integer vdim $\boldsymbol{W}$ is well-defined.

Definition 2.3.2. A *d-manifold of virtual dimension* $n \in \mathbb{Z}$ is a d-space $\boldsymbol{W}$, which can be covered by nonempty open d-subspaces $\boldsymbol{U}$, which are principal d-manifolds of virtual dimension vdim $\boldsymbol{U} = n$.

We will denote by **dMan** the full 2-subcategory of d-manifolds in **dSpa**.

Note, that if $\boldsymbol{X} \in \hat{\mathbf{Man}}$ then $\boldsymbol{X} \simeq \boldsymbol{X} \times_* *$, that is $\boldsymbol{X}$ is a principal d-manifold and thus a d-manifold. Thus $\hat{\mathbf{Man}}$ is a 2-subcategory of **dMan** and we will call a d-manifold $\boldsymbol{X}$ a manifold, if it lies in $\hat{\mathbf{Man}}$. The 2-functor $F_{\mathbf{Man}}^{\mathbf{dSpa}}$ defined above maps actually into **dMan**, and we will therefore write $F_{\mathbf{Man}}^{\mathbf{dMan}} = F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \to \mathbf{dMan}$.

An alternative description of principal d-manifolds, which also motivates the relation to Kuranishi spaces (see [33] or [20]), is the following:

Proposition 2.3.3. *A d-space $\boldsymbol{W}$ is a principal d-manifold if one of the following equivalent statements hold*

(a) $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *for* $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \hat{\mathbf{Man}}$.

(b) $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{i},\boldsymbol{Z},\boldsymbol{j}} \boldsymbol{Y}$*, where* X, Y, Z *are manifolds,* $i : X \to Z$ *and* $j : Y \to Z$ *embeddings, and* $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{i}, \boldsymbol{j} = F_{Man}^{dSpa}(X, Y, Z, i, j)$*. In other words,* $\boldsymbol{W}$ *is an intersection (in the sense of d-spaces) of two submanifolds* $X, Y \subseteq Z$*.*

(c) $\boldsymbol{W} \simeq \boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$*, where* V *is a manifold,* $E \to V$ *a vector bundle,* $s : V \to E$ *a smooth section,* $0 : V \to E$ *the zero section, and* $\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{S}, \boldsymbol{0} = F_{\boldsymbol{Man}}^{dSpa}(V, E, s, 0)$*. In other words,* $\boldsymbol{W}$ *is the zero set* $s^{-1}(0)$ *of a smooth section* s *of a vector bundle* E*, in the sense of d-spaces.*

2.3.1 Local properties of d-manifolds

We want now to investigate the local structure of d-manifolds. We will in particular be interested in the description provided by Proposition 2.3.3 (c) and start with the following definition.(Compare [35, §3.3, §3.4].)

Definition 2.3.4. Let V be a manifold, and $E \to V$ a vector bundle with a smooth section $s \in C^\infty(E)$. To this data, we can assign by an explicit construction, a d-manifold $\boldsymbol{S} = (\underline{S}, \mathcal{O}'_S, \mathcal{E}_S, \imath_S, \jmath_S)$ which is equivalent to $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$, and which will be called the *standard model* $\boldsymbol{S}_{V,E,s}$ of (V, E, s).

In order to construct $\boldsymbol{S}_{V,E,s}$, denote by $C^\infty(V)$ the C^∞-ring of smooth functions $c : V \to \mathbb{R}$, and let $C^\infty(E), C^\infty(E^*)$ be the vector spaces of smooth sections of E, E^* over V. Then $s \in C^\infty(E)$, and $C^\infty(E), C^\infty(E^*)$ are modules over $C^\infty(V)$. Moreover there is a natural bilinear product $\cdot : C^\infty(E^*) \times C^\infty(E) \to C^\infty(V)$. Define $I_s \subseteq C^\infty(V)$ to be the ideal generated by s, that is

$$I_s = \{\alpha \cdot s | \alpha \in C^\infty(E^*)\} \subseteq C^\infty(V). \tag{2.13}$$

Let $I_s^2 = \{\beta \cdot (s \otimes s) | \beta \in C^\infty(E^* \otimes E^*)\} \subseteq C^\infty(V)$ be the square ideal of I_s. Then I_s^2 is an ideal in $C^\infty(V)$ generated by $s \otimes s \in C^\infty(E \otimes E)$, or in other word

$$I_s^2 = \{\beta \cdot (s \otimes s) | \beta \in C^\infty(E^* \otimes E^*)\} \subseteq C^\infty(V).$$

Define C^∞-rings $\mathfrak{C} = C^\infty(V)/I_s, \mathfrak{C}' = C^\infty(V)/I_s^2$, and let $\pi : \mathfrak{C}' \to \mathfrak{C}$ be the natural projection from the inclusion $I_s^2 \subseteq I_s$. Define a topological space $S = s^{-1}(0) \subseteq V$, as the zero set of the section s. Now $s(v) = 0$ if and only if $(s \otimes s)(v) = 0$. Thus S is the underlying topological space of both $\operatorname{Spec} \mathfrak{C}'$ and $\operatorname{Spec} \mathfrak{C}$. This means $\operatorname{Spec} \mathfrak{C} = \underline{S} = (S, \mathcal{O}_s), \operatorname{Spec} \mathfrak{C}' = \underline{S}' = (S, \mathcal{O}'_S)$ and $\operatorname{Spec} \pi = \underline{\imath}_S = (\mathrm{id}_S, \imath_S) : \underline{S}' \to \underline{S}$, where $\underline{S}, \underline{S}'$ are fair affine C^∞-schemes, and $\mathcal{O}_S, \mathcal{O}'_S$ are sheaves of C^∞-rings on S, and $\imath_S : \mathcal{O}'_S \to \mathcal{O}_S$ is a morphism of sheaves of C^∞-rings. Since π is surjective with kernel the square zero ideal I_s/I_s^2 we get that $\imath_S$ is surjective with kernel $\mathcal{I}_S$, a sheaf of square zero ideals in $\mathcal{O}'_S$.

Equation (2.13) yields a surjective $C^\infty(V)$-module morphism $\mu : C^\infty(E^*) \to I_s$ given by $\mu(\alpha) = \alpha \cdot s$. This morphism induces a surjective morphism of $\mathfrak{C}$-modules:

$$\begin{aligned} \sigma \colon C^\infty(E^*)/(I_s \cdot C^\infty(E^*)) &\to I_s/I_s^2 \\ \alpha + (I_s \cdot C^\infty(E^*)) &\mapsto \alpha \cdot s + I_s^2. \end{aligned}$$

Now define $\mathcal{E}_S = \operatorname{MSpec}((C^\infty(E^*))/(I_s \cdot C^\infty(E^*)))$. Note that $\mathcal{I}_S = \operatorname{MSpec}(I_s/I_s^2)$, and so $\jmath_S = \operatorname{MSpec} \sigma$ is a surjective morphism of quasicoherent sheaves on S, $\jmath_S : \mathcal{E}_S \to \mathcal{I}_S$. This implies that $\boldsymbol{S}_{V,E,s}$, defined by $\boldsymbol{S}_{V,E,s} = \boldsymbol{S} = (\underline{S}, \mathcal{O}'_S, \mathcal{E}_S, \imath_S, \jmath_S)$, is a d-space. The remaining bit is to show that $\boldsymbol{S}$ is in fact a d-manifold. This can be seen as follows: First note that $\mathcal{E}_S$ is a vector bundle on $\underline{S}$, which is naturally isomorphic to $\mathcal{E}^*|_{\underline{S}}$, where $\mathcal{E}$ is the vector bundle on $\underline{V} = \mathcal{F}_{\mathbf{Man}}^{C^\infty\mathbf{Sch}}(V)$ corresponding to $E \to V$. Secondly $\mathcal{F}_S \cong T^*\underline{V}|_S$. The morphism $\phi_S : \mathcal{E}_S \to \mathcal{F}_S$ can be interpreted as follows: Choose a connection ∇ on $E \to V$. Then $\nabla s \in C^\infty(E \otimes T^*V)$, so we can regard ∇s as a morphism of vector bundle $E^* \to T^*V$ on V. This lifts to a morphism of vector bundles $\hat{\nabla} s = \mathcal{E}_S^* \to T^*\underline{V}$ on the C^∞-scheme $\underline{V}$, and ϕ_S is identified with $\hat{\nabla} s|_{\underline{S}} : \mathcal{E}^*|_{\underline{S}} \to T^*\underline{V}|_{\underline{S}}$ under the isomorphisms $\mathcal{E}_S \cong \mathcal{E}^*|_{\underline{S}}$ and $\mathcal{F}_S \cong T^*\underline{V}|_{\underline{S}}$.

Note that although ∇s depends on the choice of ∇, its restriction to $\underline{S}$ is independent of the chosen connection ∇.

The following result due to Joyce ([35, Corollary 2.36]) shows, that every d-manifold $\boldsymbol{X}$ is locally equivalent in the 2-category **dMan** to a standard model d-manifold $\boldsymbol{S}_{V,E,s}$ for some manifold V, a vector bundle $E \to V$ and a smooth section $s \in C^\infty(E)$. Moreover, the data (V, E, s) just depends on the underlying C^∞-scheme structure $\underline{X}$ and the virtual dimension of $\boldsymbol{X}$.

Theorem 2.3.5. *For every d-manifold $\boldsymbol{X}$ with $x \in \boldsymbol{X}$, there exists an open neighbourhood $\boldsymbol{U}$ of x in $\boldsymbol{X}$ and an equivalence $\boldsymbol{U} \simeq \boldsymbol{S}_{V,E,s}$ in* **dMan** *for some manifold V, vector bundle $E \to V$ and $s \in C^\infty(E)$ which identifies $x \in \boldsymbol{U}$ with a point $v \in V$ such that $s(v) = ds(v) = 0$. The triple (V, E, s) is determined up to non-canonical isomorphism near v by $\boldsymbol{X}$ near by x, and depends only on the underlying C^∞-scheme $\underline{X}$ and the integer vdim $\boldsymbol{X}$.*

We will end this section by the following proposition ([35, Proposition 3.28]), which gives criteria for when a d-manifold is a manifold.

Proposition 2.3.6. *A d-manifold $\boldsymbol{X}$ is a manifold, that is $\boldsymbol{X} \in \hat{\mathbf{Man}}$, if and only if its virtual cotangent bundle $T^*\boldsymbol{X}$ is a vector bundle. Equivalently, $\boldsymbol{X}$ is a manifold if and only if the morphism $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ has a left inverse.*

2.3.2 1- and 2-morphisms in terms of differential geometric data

The goal of this subsection is to interpret a 1-morphism between standard model principal d-manifolds $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$ and $\boldsymbol{Y} = \boldsymbol{S}_{W,F,t}$ in terms of a pair $(f, \hat{f})$, where $f : V \to W$ is a smooth map and $\hat{f} : E \to f^*(F)$ a vector bundle morphism, and a 2-morphism $\eta : f \Rightarrow g$ as a relation between two such pairs $(f, \hat{f})$ and $(g, \hat{g})$. In order to do this we first fix some notation, which will simplify things tremendously. We follow here closely the work of Joyce [35, §3.4], to which we refer as a much more complete and rigorous source.

Definition 2.3.7. Let V be a manifold, $E \to V$ a vector bundle over V and $s : V \to E$ a smooth section. If $\tilde{E} \to V$ is another vector bundles over V and $\tilde{s}_1, \tilde{s}_2 \in C^\infty(E)$ are smooth sections, we will use the notation $\tilde{s}_1 = \tilde{s}_2 + O(s)$ if there exists an $\alpha \in C^\infty(E^* \otimes F)$ such that $\tilde{s}_1 = \tilde{s}_2 + \alpha \cdot s$ in $C^\infty(F)$. Here $\alpha \cdot s$ is formed using the natural pairing of vector bundles $(E^* \otimes F) \times E \to F$ over V. Similarly, we will use the notation $\tilde{s}_1 = \tilde{s}_2 + O(s^2)$ if there exists an $\alpha \in C^\infty(E^* \otimes E^* \otimes F)$ such that $\tilde{s}_1 = \tilde{s}_2 + \alpha \cdot (s \otimes s)$ in $C^\infty(F)$, where again $\alpha \cdot (s \otimes s)$ is formed using the pairing $(E^* \otimes E^* \otimes F) \times (E \otimes E) \to F$.

If now W is another manifold and $f, g : V \to W$ are smooth maps, we will write $f = g + O(s)$ if whenever $h : W \to \mathbb{R}$ is a smooth map, there exists $\alpha \in C^\infty(E^*)$ such that $h \circ f = h \circ g + \alpha \cdot s$. Moreover, we will write $f = g + O(s^2)$ if whenever $h : W \to \mathbb{R}$ is a smooth map, there exists $\alpha \in C^\infty(E^* \otimes E^*)$ such that $h \circ f = h \circ g + \alpha \cdot (s \otimes s)$.

Now suppose that $f, g : V \to W$ are smooth maps satisfying $f = g + O(s^2)$, and W carries in addition a vector bundle $F \to W$ with two sections $t_1, t_2 \in C^\infty(W)$.

We will write $f^*(t_1) = g^*(t_2) + O(s)$, if $f^*(t_1) = f^*(t_2) + O(s)$ and $f^*(t_1) = g^*(t_2) + O(s^2)$, if $f^*(t_1) = f^*(t_2) + O(s^2)$.

Note that strictly speaking this does not make sense, since $f^*(t_1)$ is a section of $f^*(F)$, and $g^*(t_2)$ as a section of $g^*(F)$, are sections of different vector bundles, but as $f = g + O(s^2)$, we make the convention that $f^*(t_2) = g^*(t_2) + O(s^2)$ for any t_2. This implies, at least informally,

$$f^*(t_1) - g^*(t_2) = (f^*(t_1) - f^*(t_2)) + (f^*(t_2) - g^*(t_2)) = f^*(t_2) - f^*(t_1) + O(s^2).$$

This $O(s)$ and $O(s^2)$ notation has a nice interpretation at the level of C^∞-schemes: let $\underline{V} = F_{\mathbf{Man}}^{C^\infty\mathbf{Sch}}(V)$ be the corresponding C^∞-scheme to the manifold V and $\underline{X}, \underline{X}'$ be C^∞-subschemes in $\underline{V}$ defined by the equations $s = 0$ and $s \otimes s = 0$. Then, using the notation $\underline{f}, \underline{g}$ for the corresponding maps of f, g on the C^∞-scheme level,

(a) $\tilde{s}_1 = \tilde{s}_2 + O(s), f = g + O(s)$ mean that $\tilde{s}_1|_{\underline{X}} = \tilde{s}_2|_{\underline{X}'}, \underline{f}|_{\underline{X}} = \underline{g}|_{\underline{X}'}$.

(b) When $f = g + O(s^2), f^*(t_1) = g^*(t_2) + O(s^2)$ means that $(\underline{f}|_{\underline{X}})^*(t_1) = (\underline{g}|_{\underline{X}})^*(t_2)$.

(c) $f^*(t_1) = g^*(t_2) + O(s^2)$ means $(\underline{f}|_{\underline{X}'})^*(t_1) = (\underline{g}|_{\underline{X}'})^*(t_2)$, which makes sense as $\underline{f}|_{\underline{X}} = \underline{g}|_{\underline{X}}$ and $\underline{f}|_{\underline{X}} = \underline{g}|_{\underline{X}'}$.

Definition 2.3.8. Let V, W be manifolds, $E \to V, F \to W$ be vector bundles and $s : V \to E, t : W \to F$ be smooth sections. Write $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}, \boldsymbol{Y} = \boldsymbol{S}_{W,F,t}$ for the standard model principal d-manifolds. Let $f : V \to W$ be a smooth map and $\hat{f} : E \to f^*(F)$ be a morphism of vector bundles, satisfying

$$\hat{f} \circ s \equiv f^*(t) + O(s^2) \quad \text{in } C^\infty(f^*(F)). \tag{2.14}$$

Using the data $f, \hat{f}$ one can define a 1-morphism $\boldsymbol{g} = (g, g', g'') : \boldsymbol{X} \to \boldsymbol{Y}$ between d-manifolds. This 1-morphism is called *standard model* 1*-morphism* and will also be denoted by $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. Note therefore, that if $x \in X$ then $x \in V$ satisfying $s(x) = 0$ and therefore by (2.14) we get

$$t(f(x)) = (f^*(t))(x) = \hat{f}(s(x)) + O(s(x)^2) = 0$$

which means $f(x) \in Y \subseteq W$. Thus we can define $g := f|_X : X \to Y$.

Now define morphisms of C^∞-rings

$$\begin{aligned} \phi : C^\infty(W)/I_t \to C^\infty(V)/I_s, \quad & \phi' : C^\infty(W)/I_t^2 \to C^\infty(V)/I_s^2, \\ \text{by } \phi : c + I_t \mapsto c \circ f + I_s, \quad & \phi' : c + \mathcal{I}_t^2 \mapsto c \circ f + I_s^2. \end{aligned}$$

Note that ϕ is well-defined, since if $c \in I_t$ then $c = \lambda \cdot t$ for some $\lambda \in C^\infty F^*$, which means

$$\begin{aligned} c \circ f + (\lambda \cdot t) \circ f = f^*(\lambda) \cdot f^*(t) &= f^*(\lambda) \cdot (\hat{f} \circ s + O(s^2)) \\ &= (\hat{f} \circ f^*(\lambda)) \cdot s + O(s^2) \in I_s. \end{aligned}$$

A similar argument holds also for ϕ', so ϕ and ϕ^p are well defined. Thus, we have C^∞-scheme morphisms $\underline{g} = (g, g^\#) = \operatorname{Spec} \phi : \underline{X} \to \underline{Y}$ and $\underline{g}' = (g, g') = \operatorname{Spec}(\phi') : (X, \mathcal{O}'_X) \to (Y, \mathcal{O}'_Y)$, both with underlying continuous map g. Hence

$g^\# : g^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ and $g' : g^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$ are morphisms of sheaves of C^∞-rings on X. In order to define g'', note that $\underline{g}^*(\mathcal{E}_Y) = \mathrm{MSpec}\,(C^\infty(f^*(F^*)/(I_s \cdot C^\infty(f^*(F^*)))))$. Thus, define $g'' : \underline{g}^*(\mathcal{E}_Y) \to \mathcal{E}_X$ by $g'' = \mathrm{MSpec}\,(G'')$, where

$$\begin{aligned} G'' &: C^\infty(f^*(F^*))/(I_s \cdot C^\infty(F^*(F^*))) \to C^\infty(E^*)/(I_s \cdot C^\infty(E^*)) \\ \text{is given by} \qquad G'' &: \lambda + I_s \cdot C^\infty(f^*(F^*)) \mapsto \lambda \circ \hat{f} + I_s \cdot C^\infty(E^*). \end{aligned}$$

This definition of $\boldsymbol{g} = (g, g', g'')$ is indeed a 1-morphism of d-manifolds, which we will also denote by $\boldsymbol{S}_{f.\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$.

Note that if $\tilde{V} \subseteq V$ is an open neighbourhood of $s^{-1}(0)$ in V, with inclusion map $\imath_{\tilde{V}} : \tilde{V} \to V$, we can define $\tilde{E} = E|_V = \imath_{\tilde{V}}^*(E)$ and $\tilde{s} = s|_t V$. We then get a 1-morphism $\boldsymbol{i}_{\tilde{V},V} = \boldsymbol{S}_{\imath_{\tilde{V}},\mathrm{id}_{\tilde{E}}} : \boldsymbol{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \boldsymbol{S}_{V,E,s}$. It is easy to show $\imath_{\tilde{V},V}$ is a 1-morphism with inverse $\boldsymbol{i}_{\tilde{V},V}^{-1}$, which means that making V smaller, without changing $s^{-1}(0)$ does not really change $\boldsymbol{S}_{V,E,s}$, or in other words: the d-manifold $\boldsymbol{S}_{V,E,s}$ depends only on E, s on an arbitrary small open neighbourhood of $s^{-1}(0)$ in V.

The following lemma ([35, Lemma 3.32]) gives a criterion when two standard model 1-morphisms are the same and together with Theorem 2.3.11 below (see [35, Theorem 3.34] for a proof), we get a complete differential geometric classification of standard model 1-morphisms.

Lemma 2.3.9. *Let V, W be manifolds, $E \to V, F \to W$ vector bundles, $s : V \to E, t : W \to F$ smooth sections, $f_1, f_2 : V \to W$ a smooth maps and $\hat{f}_1 : E \to f_1^*(F), \hat{f}_2 : E \to f_2^*(F)$ vector bundle morphisms with $\hat{f}_1 \circ s = f_1^*(t) + O(s^2)$ and $\hat{f}_2 \circ s = f_2^*(t) + O(s^2)$. Definition 2.3.8 yields standard model 1-morphisms $\boldsymbol{S}_{f_1,\hat{f}_1}, \boldsymbol{S}_{f_2,\hat{f}_2} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. Then $\boldsymbol{S}_{f_1,\hat{f}_1} = \boldsymbol{S}_{f_2,\hat{f}_2}$ if and only if $f_1 = f_2 + O(s^2)$ and $\hat{f}_1 = \hat{f}_2 + O(s)$.*

Lemma 2.3.10. *Let V be manifolds, $E \to VW$ vector bundles and $s : V \to E$ a smooth section, and let $\tilde{V} \subseteq V$ be open with restrictions $\tilde{E} = E|_{\tilde{V}}$ and $\tilde{s} = s|_{\tilde{V}}$.*

Then $\boldsymbol{i}_{\tilde{V},V} : \boldsymbol{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \boldsymbol{S}_{V,E,s}$ is a 1-isomorphism with an open d-submanifold of $\boldsymbol{S}_{V,E,s}$, and if additionally $s^{-1}(0) \subseteq V$ then $\boldsymbol{i}_{\tilde{V},V}$ is a 1-isomorphism itself.

Theorem 2.3.11. *Let V, W be manifolds, $E \to V, F \to W$ vector bundles, $s : V \to E, t : W \to F$ smooth sections. Let $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ be the associated standard model 1-morphism.*

Then there exists an open neighbourhood $\tilde{V}$ of $s^{-1}(0)$ in V, a smooth map $f : \tilde{V} \to W$, and a morphism of vector bundles $\hat{f} : E \to f^*(F)$, satisfying $\hat{f} \circ \tilde{s} = f^*(t)$, where $\tilde{E} = E|_{\tilde{V}}$ and $\tilde{s} = s|_{\tilde{V}}$ denote the restrictions of E, s to $\tilde{V}$, satisfying $\boldsymbol{g} = \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{i}_{\tilde{V},V}^{-1}$. Here $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \boldsymbol{S}_{W,F,t}$ and $\boldsymbol{i}_{\tilde{V},V} : \boldsymbol{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \boldsymbol{S}_{V,E,s}$ and the inverse $\boldsymbol{i}_{\tilde{V},V}^{-1}$ exists by Lemma 2.3.10.

Similarly to the 1-morphism case, the next definition will give a differential-geometric characterization of 2-morphisms $\lambda : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ between standard model 1-morphisms $\boldsymbol{S}_{f,\hat{f}}, \boldsymbol{S}_{g,\hat{g}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. We refer to [35, Definition 3.35] for a more detailed treatment.

Definition 2.3.12. Let V, W be manifolds, $E \to V, F \to W$ vector bundles, $s : V \to E, t : W \to F$ smooth sections, and $f, g : V \to W$ smooth maps. Moreover let $\hat{f} : E \to f^*(F), \hat{g} : E \to g^*(F)$ be morphisms of vector bundles on V, satisfying $\hat{f} \circ s = f^*(t) + O(s^2)$ and $\hat{g} \circ s = g^*(t) + O(s^2)$, and let $\boldsymbol{S}_{f,\hat{f}}, \boldsymbol{S}_{g,\hat{g}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ be the standard model 1-morphisms given by Definition 2.3.8.

By choosing a complete Riemannian metric h on W, and a connection ∇^F on $F \to W$ one may write for each $v \in V$ and sufficiently close maps f, g, $g(v) = \exp_{f(v)}(\gamma(v))$ for some $\gamma(v) \in T_{f(v)}W$, where $\exp_{f(v)} : T_{f(v)}W \to W$ is the geodesic exponential map. Furthermore, using parallel transport along the unique short geodesic from $f(v)$ to $g(v)$ for each $v \in V$, we may define an isomorphism $\Theta_{f,g} : f^*(F) \to g^*(F)$.

Given a morphism of vector bundles on V, $\Lambda : E \to f^*(TW)$, we can concatenate it with s, and get a section $\Lambda \circ s \in C^\infty(f^*(TW))$, so that we can require $g = \exp_f(\Lambda \circ s) \circ f$. In addition, $\nabla^F t$ is a section of $T^*W \otimes F \to W$, and so $f^*(\nabla^F t)$ is a section of $f^*(T^*W) \otimes f^*(F) \to V$ and hence a morphism $f^*(TW) \to f^*(F)$. So $f^*(\nabla^F t) \circ \Lambda$ is a morphism $E \to f^*(F)$ and hence we may require that $\hat{g} = \Theta_{f,g} \circ (\hat{f} + f^*(\nabla^F t) \circ \Lambda)$. Taking the dual of Λ and restricting to the C^∞-subscheme $\underline{X} = s^{-1}(0)$ in $\underline{V}$ gives $\lambda = \Lambda^*|_{\underline{X}} : \underline{f}^*(\mathcal{F}_Y) \cong \underline{f}^*(T^*\underline{W})|_{\underline{X}} \to \mathcal{E}^*|_{\underline{X}} = \mathcal{E}_X$. It can be shown that this λ is a 2-morphism $\boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ if and only if

$$g = f + \Lambda \cdot s + O(s^2) \qquad \text{and} \qquad \hat{g} = \hat{f} + \Lambda \cdot f^*(dt) + O(s),$$

which is an informal way of writing $g = \exp_f(\Lambda \circ s) \circ f + O(s^2)$ and $\hat{g} = \Theta_{f,g} \circ (\hat{f} + f^*(\nabla^F t) \circ \Lambda) + O(s)$. The 2-morphism λ will be denoted by $\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ and called a *standard model 2-morphism*.

Moreover, it can be shown that every 2-morphism $\eta : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ in **dSpa** is a 'standard model' 2-morphism and that $\boldsymbol{S}_{\Lambda'} = \boldsymbol{S}_{\Lambda} : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ if and only if $\Lambda' = \Lambda + O(s)$.

The following theorem due to Joyce ([35, Theorem 3.39]) gives a condition when a 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ between two principal d-manifolds $\boldsymbol{S}_{V,E,s}$ and $\boldsymbol{S}_{W,F,t}$ is étale, respectively an equivalence.

Theorem 2.3.13. *Let V, W be manifolds, $E \to V, F \to W$ vector bundles, $s : V \to E, t : W \to F$ smooth sections, $f : V \to W$ a smooth map and $\hat{f} : E \to f^*(F)$ be a morphism of vector bundles on V, satisfying the following condition: $\hat{f} \circ s = f^*(t) + O(s^2)$. Then Definitions 2.3.4 and 2.3.8 give principal d-manifolds $\boldsymbol{S}_{V,E,s}, \boldsymbol{S}_{W,F,t}$ and a 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. This 1-morphism $\boldsymbol{S}_{f,\hat{f}}$ is étale if and only if for each $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_vV \xrightarrow{ds(v) \oplus df(v)} E_v \oplus T_wW \xrightarrow{\hat{f}(v) \oplus -dt(w)} F_w \longrightarrow 0. \tag{2.15}$$

Moreover $\boldsymbol{S}_{f,\hat{f}}$ is an equivalence if and only if in addition $f|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection.

We will end this section with two results, which characterise how 1-morphisms between principal d-manifolds are reflected in the underlying local data V, E, s and refer for a proof to [35, §3.4].

Lemma 2.3.14. *Let V be a manifold, $E \to V$ be a vector bundle, $s \in C^\infty(E)$ be a smooth section and $\tilde{V} \subseteq V$ open. Then the 1-morphism $\boldsymbol{i}_{\tilde{V},V} : \boldsymbol{S}_{\tilde{V},\mathcal{E},\tilde{s}} \to \boldsymbol{S}_{V,E,s}$ is an 1-isomorphism with an open d-submanifold of $\boldsymbol{S}_{V,E,s}$. If $\tilde{V}$ is an open neighbourhood of $s^{-1}(0)$ in V, then $\boldsymbol{i}_{\tilde{V},V} : \boldsymbol{S}_{\tilde{V},\mathcal{E},\tilde{s}} \to \boldsymbol{S}_{V,E,s}$ is a 1-isomorphism.*

Theorem 2.3.15. *Let V, W be manifolds, $E \to V, F \to W$ be vector bundles, and $s \in C^\infty(E), t \in C^\infty(F)$ be smooth sections. Define principal d-manifolds $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$ and $\boldsymbol{Y} = \boldsymbol{S}_{W,F,t}$, with topological space $X = s^{-1}(0)$ and $Y = t^{-1}(0)$. Let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism.*

Then there exist an open neighbourhood $\tilde{V}$ of X in V, a smooth map $f : \tilde{V} \to W$, and a morphism of vector bundles $\hat{f} : \tilde{E} \to f^*(F)$ with $\hat{f} \circ \tilde{s} \equiv f^*(t)$., where $\tilde{E} = E|_{\tilde{V}}$ and $\tilde{s} = s|_{\tilde{V}}$ denotes the restriction of E respectively s to $\tilde{V}$, satisfying $\boldsymbol{g} = \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{i}_{\tilde{V},V}^{-1}$, where $\boldsymbol{i}^{-1} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{\tilde{V},\tilde{E},\tilde{s}}$ exists by Lemma 2.3.14.

2.3.3 Equivalences and gluing by equivalences

As in the case of d-spaces, it will in some situation be important to have a gluing procedure for d-manifolds. We will here just briefly state the basic definitions and theorems and refer once again to [35, §3.5, §3.6] for the details.

Definition 2.3.16. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in **dMan**. We call $\boldsymbol{f}$ *étale*, if it is a *local equivalence*, meaning that for each $x \in \boldsymbol{X}$ there exists an open subset $\boldsymbol{U} \subseteq \boldsymbol{X}$ containing x, and an open subset $\boldsymbol{V} \subseteq \boldsymbol{Y}$ containing $f(x)$ such that $\boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{V}$ and $\boldsymbol{f}|_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ is an equivalence.

The following Theorem due to Joyce ([35, Theorem 3.36]) provides useful criteria when a 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ between d-manifolds is étale .

Theorem 2.3.17. *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism of d-manifolds. Then the following are equivalent:*

(i) $\boldsymbol{f}$ *is étale*

(ii) $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ *is an equivalence in* $\mathbf{vqcoh}(\underline{X})$

(iii) the following is a split short exact sequence in $\mathrm{qcoh}(\underline{X})$*:*

$$0 \longrightarrow \underline{f}^*(\mathcal{E}_Y) \xrightarrow{f'' \oplus -\underline{f}^*(\phi_Y)} \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y) \xrightarrow{\phi_X \oplus f^2} \mathcal{F}_X \longrightarrow 0$$

If in addition $f : X \to Y$ is a bijection, then $\boldsymbol{f}$ is an equivalence in **dMan**.

The next theorem is the analogue of Theorem 2.2.4 for d-manifolds and is proven in [35, Theorem 3.41].

Theorem 2.3.18. *Suppose $\boldsymbol{X}, \boldsymbol{Y}$ are d-manifolds with vdim $\boldsymbol{X}$ = vdim $\boldsymbol{Y} = n \in \mathbb{Z}$, and let $\boldsymbol{U} \subseteq \boldsymbol{X}, \boldsymbol{V} \subseteq \boldsymbol{Y}$ be open d-submanifolds, and $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{V}$ an equivalence in* **dMan**. *Thus, on the underlying topological spaces we have an homeomorphism $f : U \to V$, where $U \subseteq X, V \subseteq Y$ are open, and can therefore form the quotient topological space $Z := X \amalg_f Y = (X \amalg Y)/\sim$. Here the equivalence relation $\sim$ on $X \amalg Y$ identifies $u \in U \subseteq X$ with $f(u) \in V \subseteq Y$.*

Suppose that Z is Hausdorff. Then there exists a d-manifold $\boldsymbol{Z}$ with vdim $\boldsymbol{Z} = n$, open d-submanifolds $\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}$ in $\boldsymbol{Z}$, satisfying $\boldsymbol{Z} = \hat{\boldsymbol{X}} \cup \hat{\boldsymbol{Y}}$, equivalences $\boldsymbol{g} : \boldsymbol{X} \to$

$\hat{\boldsymbol{X}}$ *and* $\boldsymbol{h} : \boldsymbol{Y} \to \hat{\boldsymbol{Y}}$ *such that* $\boldsymbol{g}|_{\boldsymbol{U}}$ *and* $\boldsymbol{h}|_{\boldsymbol{V}}$ *are both equivalences with* $\hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$ *and a 2-morphism* $\eta : \boldsymbol{g}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f} : \boldsymbol{U} \to \hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$. *Furthermore, the d-manifold* $\boldsymbol{Z}$ *is independent of all the choices up to equivalence.*

We will end this section with the following theorem ([35, Theorem 3.42]), which shows that given certain differential-geometric or topological data, there exists an up to equivalence unique d-manifold coming from this data.

Theorem 2.3.19. *Suppose we are given the following data:*

(a) *an integer* n,

(b) *a Hausdorff, second countable topological space* Y,

(c) *an indexing set* I, *and a total order* $<$ *on* I,

(d) *for each* $i \in I$, *a manifold* V_i, *a vector bundle* $E_i \to V_i$, *a smooth section* $s_i : V_i \to E_i$, *and a homeomorphism* $\psi_i : X_i \to \hat{X}_i$, *where* $X_i = \{v_i \in V_i : s_i(v_i) = 0\}$ *and* $\hat{X}_i \subseteq Y$ *is an open set,*

(e) *for all* $i < j$ *in* I, *open submanifolds* $V_{ij} \subseteq V_i, V_{ji} \subseteq V_j$,*a smooth map* $e_{ij} : V_{ij} \to V_{ji}$, *and a morphism of vector bundles* $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$,

satisfying

(i) $Y = \bigcup_{i \in I} \hat{X}_i$,

(ii) *if* $i \in I$ *then* $\dim V_i - \operatorname{rank} E_i = n$,

(iii) *if* $i < j$ *in* I, *then* $\hat{e}_{ij} \circ s_i|_{V_{ij}} \equiv e_{ij}^*(s_j)$ *and* $\psi_i(X_i \cap V_{ij}) = \psi_j(X_j \cap V_j i) = \hat{X}_i \cap \hat{X}_j$, *and* $\psi_i|_{X_i \cap V_{ij}} = \psi_j \circ e_{ij}|_{X_i \cap V_{ij}}$, *and if* $v_i \in V_i$ *with* $s_i(v_i) = 0$ *and* $v_j = e_{ij}(v_i)$ *then the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{ds_i(v_i) \oplus de_{ij}(v_i)} \mathcal{E}_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -ds_j(v_j)} \mathcal{E}_j|_{v_j} \longrightarrow 0.$$

(iv) *if* $i < j < k$ *in* I *then* $e_{ik}|_{V_{ij} \cap V_{ik}} \equiv e_{jk} \circ e_{ij}|_{V_{ij} \cap V_{ik}} + O(s_i^2)$ *and* $\hat{e}_{ik}|_{V_{ij} \cap V_{ik}} \equiv e_{ij}^*|_{V_{ij} \cap V_{ik}}(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{V_{ij} \cap V_{ik}} + O(s_i)$.

Then there exists a d-manifold $\boldsymbol{Y}$ with vdim $\boldsymbol{Y} = n$ and underlying topological space Y, and a 1-morphism $\boldsymbol{f}_i : \boldsymbol{S}_{V_i,E_i,s_i} \to \boldsymbol{Y}$ which is an equivalence with the open submanifold $\hat{\boldsymbol{X}}_i \subseteq \boldsymbol{Y}$ corresponding to $\hat{X}_i \subseteq Y$ for all $i \in I$, such that for all $i < j$ in I there exists a 2-morphism $\eta_{ij} : \boldsymbol{f}_j \circ \boldsymbol{S}_{e_{ij},\hat{e}_{ij}} \Rightarrow \boldsymbol{f}_i \circ \boldsymbol{i}_{V_{ij},V_i}$, where $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}} : \boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V ij}} \to \boldsymbol{S}_{V_j,E_j,s_j}$ and $\boldsymbol{i}_{V_{ij},V_i} : \boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}} \to \boldsymbol{S}_{V_i,E_i,s_i}$. This d-manifold $\boldsymbol{Y}$ is unique up to equivalence in **dMan**.
Furthermore, given a manifold Z and $g_i : V_i \to Z$ smooth maps for all $i \in I$, and $g_j \circ e_{ij} = g_i|_{V_{ij}} + O(s_i^2)$ for all $i < j$ in I, there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ unique up to 2-morphism, where $\boldsymbol{Z} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Z) = \boldsymbol{S}_{Z,0,0}$, and 2-morphisms $\zeta_i : \boldsymbol{h} \circ \boldsymbol{f}_i \Rightarrow \boldsymbol{S}_{g_i,0}$ for all $i \in I$. Here $\boldsymbol{S}_{Z,0,0}$ is from Definition 2.3.8 with vector bundle E and the section s both zero, and $\boldsymbol{S}_{g_i,0} : \boldsymbol{S}_{V_i,E_i,s_i} \to \boldsymbol{S}_{Z,0,0} = \boldsymbol{Z}$ is from Definition 2.3.4.

2.3.4 Submersion, immersions and embeddings

In this section we will follow [35, §4.1] and state some basic definitions and theorems about immersions, submersions and embeddings of d-manifolds.
Submersions and immersions of smooth manifolds can be described by injectivity and surjectivity of the differential. In the same spirit one can define what immersions, submersion and embeddings for d-manifolds should be.

Definition 2.3.20. Let $\underline{X}$ be a C^∞-scheme, $\mathcal{E}^1, \mathcal{E}^2, \phi$ and $\mathcal{F}^1, \mathcal{F}^2, \psi$ be virtual vector bundles on $\underline{X}$ and $(f^1, f^2) : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ be a 1-morphism in $\mathbf{vvect}(\underline{X})$. We have the following complex in qcoh$(\underline{X})$:

$$0 \longrightarrow \mathcal{E}^1 \underset{\gamma}{\overset{f^1 \oplus -\phi}{\rightleftarrows}} \mathcal{F}^1 \oplus \mathcal{E}^2 \underset{\delta}{\overset{\psi \oplus f^2}{\rightleftarrows}} \mathcal{F}^2 \longrightarrow 0\,. \tag{2.16}$$

Proposition 2.1.23 shows, that $f^\bullet$ is an equivalence in $\mathbf{vvect}(\underline{X})$ if and only if (2.16) is a *split short exact sequence* in qcoh$(\underline{X})$, which means that there exist morphism γ, δ as in (2.16) satisfying

$$\gamma \circ \delta = 0, \qquad \gamma \circ (f^1 \oplus -\phi) = \mathrm{id}_{\mathcal{E}^1},$$
$$(f^1 \oplus -\phi) \circ \gamma + \delta \circ (\psi \oplus f^2) = \mathrm{id}_{\mathcal{F}^1 \oplus \mathcal{E}^2}, \qquad (\psi \oplus f^2) \circ \delta = \mathrm{id}_{\mathcal{F}^2}.$$

Weakening some of these conditions leads to the following definitions:

(a) $f^\bullet$ is called *weakly injective* if there exist $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ in qcoh($\underline{X}$) with $\gamma \circ (f^1 \oplus -\phi) = \mathrm{id}_{\mathcal{E}^1}$.

(b) $f^\bullet$ is called *injective* if there exist $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ and $\delta : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ with $\gamma \circ \delta = 0$, $\gamma \circ (f^1 \oplus -\phi) = \mathrm{id}_{\mathcal{E}^1}$ and $(f^1 \oplus -\phi) \circ \gamma + \delta \circ (\psi \oplus f^2) = \mathrm{id}_{\mathcal{F}^1 \oplus \mathcal{E}^2}$.

(c) $f^\bullet$ is called *weakly surjective* if there exist $\delta : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ in qcoh($\underline{X}$) with $(\psi \oplus f^2) \circ \delta = \mathrm{id}_{\mathcal{F}^2}$.

(d) $f^\bullet$ is called *surjective* if there exist $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ and $\delta : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ with $\gamma \circ \delta = 0$, $\gamma \circ (f^1 \oplus -\phi) = \mathrm{id}_{\mathcal{E}^1}$ and $(\psi \oplus f^2) \circ \delta = \mathrm{id}_{\mathcal{F}^2}$.

Using these notions of injectivity and surjectivity one can define the following.

Definition 2.3.21. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism of d-manifolds and denote by $\Omega_{\boldsymbol{f}} : \underline{f}^\star(T^\star \boldsymbol{Y}) \to T^\star \boldsymbol{X}$ the corresponding 1-morphism in $\mathbf{vvect}(\underline{X})$. Then

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\Omega_{\boldsymbol{f}}$ is weakly injective.

(b) We call $\boldsymbol{f}$ a *submersion* if $\Omega_{\boldsymbol{f}}$ is injective.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $\Omega_{\boldsymbol{f}}$ is weakly surjective.

(d) We call $\boldsymbol{f}$ an *immersion* if $\Omega_{\boldsymbol{f}}$ is surjective.

(e) We call $\boldsymbol{f}$ a *w-embedding* if it is a w-immersion and $f : X \to f(X)$ is a homeomorphism, which in particular implies f is injective.

(f) We call $\boldsymbol{f}$ an *embedding* if it is an immersion and $f : X \to f(X)$ is homeomorphism.

Note that all of the conditions above concern the existence of suitable morphisms γ, δ in the following complex in qcoh($\underline{X}$):

$$\begin{tikzcd} 0 \arrow[r] & \underline{f}^*(\mathcal{E}_Y) \arrow[r, "f'' \oplus -\underline{f}^*(\phi_Y)"] & \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y) \arrow[l, dotted, bend left, "\gamma"] \arrow[r, "\phi_X \oplus f^2"] & \mathcal{F}_X \arrow[l, dotted, bend left, "\delta"] \arrow[r] & 0\,. \end{tikzcd} \tag{2.17}$$

Using $(c) - (f)$ from above, one can define the notion of *d-submanifolds* of a d-manifold. A 1-morphism $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{Y}$ between two d-manifolds $\boldsymbol{X}$ and $\boldsymbol{Y}$ is called a *w-immersed,* or *immersed,* or *w-embedded,* or *embedded d-submanifold* of $\boldsymbol{Y}$, if $\boldsymbol{i}$ is a w-immersion, immersion, w-embedding, or embedding respectively.

It will be important to have a good understanding of the cohomology of the complex (2.16). The following proposition characterised this cohomology and is proven in [35, Proposition 4.3].

Proposition 2.3.22. *Let $\underline{X}$ be a separated, paracompact, locally fair C^∞-scheme and $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ be a 1-morphism in $\mathbf{vvect}(\underline{X})$, so that (2.16) is a complex in $\mathrm{qcoh}(\underline{X})$. Define the cohomology of (2.16) at the second, third and fourth terms by $\mathcal{G}, \mathcal{H}, \mathcal{I} \in \mathrm{qcoh}(\underline{X})$, as follows:*

$$\mathcal{G} = Ker(f^1 \oplus -\phi : \mathcal{E}^1 \to \mathcal{F}^1 \oplus \mathcal{E}^2), \tag{2.18}$$

$$\mathcal{H} = \frac{Ker(\psi \oplus f^2 : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{F}^2)}{Im(f^1 \oplus -\phi : \mathcal{E}^1 \to \mathcal{F}^1 \oplus \mathcal{E}^2)}, \tag{2.19}$$

$$\mathcal{I} = Coker(\psi \oplus f^2 : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{F}^2). \tag{2.20}$$

Then

(i) Let $f^\bullet, \tilde{f}^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ be 1-morphisms, $\eta : f^\bullet \Rightarrow \tilde{f}^\bullet$ a 2-morphism and $\mathcal{G}, \mathcal{H}, \mathcal{I}$ and $\tilde{\mathcal{G}}, \tilde{\mathcal{H}}, \tilde{\mathcal{I}}$ be as above for $f^\bullet, \tilde{f}^\bullet$. Then there are canonical isomorphisms $\tilde{\mathcal{G}} \cong \mathcal{G}, \tilde{\mathcal{H}} \cong \mathcal{H}, \tilde{\mathcal{I}} \cong \mathcal{I}$ in $\mathrm{qcoh}(\underline{X})$.

(ii) Let $i^\bullet : (\tilde{\mathcal{E}}^\bullet, \tilde{\phi}) \to (\mathcal{E}^\bullet, \phi), j^\bullet : (\mathcal{F}^\bullet, \psi) \to (\tilde{\mathcal{F}}^\bullet, \tilde{\psi})$ be equivalences in $\mathbf{vvect}(\underline{X})$, and denote by $\tilde{f}^\bullet = j^\bullet \circ f^\bullet \circ i^\bullet : (\tilde{\mathcal{E}}^\bullet, \tilde{\phi}) \to (\tilde{\mathcal{F}}^\bullet, \tilde{\psi})$ the concatenation of $j^\bullet, f^\bullet, i^\bullet$. Let $\mathcal{G}, \mathcal{H}, \mathcal{I}$ and $\tilde{\mathcal{G}}, \tilde{\mathcal{H}}, \tilde{\mathcal{I}}$ be as in (2.18) for $f^\bullet, \tilde{f}^\bullet$. Then there are canonical isomorphisms $\tilde{\mathcal{G}} \cong \mathcal{G}, \tilde{\mathcal{H}} \cong \mathcal{H}, \tilde{\mathcal{I}} \cong \mathcal{I}$ in $\mathrm{qcoh}(\underline{X})$.

(iii) If $f^\bullet$ is weakly injective, then $\mathcal{G} = 0$.

(iv) If $f^\bullet$ is injective, then $\mathrm{rank}\,(\mathcal{E}^\bullet, \phi) \leq \mathrm{rank}\,(\mathcal{F}^\bullet, \psi)$ and $\mathcal{G} = \mathcal{H} = 0$, and $\mathcal{I}$ is a vector bundle on $\underline{X}$ of rank $\mathrm{rank}\,(\mathcal{F}^\bullet, \psi) - \mathrm{rank}\,(\mathcal{E}^\bullet, \phi)$. Moreover, if $\mathrm{rank}\,(\mathcal{E}^\bullet, \phi) = \mathrm{rank}\,(\mathcal{F}^\bullet, \psi)$ then $\mathcal{I} = 0$ and $f^\bullet$ is an equivalence.

(v) If $f^\bullet$ is weakly surjective, then $\mathcal{I} = 0$.

(vi) If $f^\bullet$ is surjective, then $\mathrm{rank}\,(\mathcal{E}^\bullet, \phi) \geq \mathrm{rank}\,(\mathcal{F}^\bullet, \psi)$ and $\mathcal{G} = \mathcal{I} = 0$, and $\mathcal{H}$ is a vector bundle on $\underline{X}$ of rank $\mathrm{rank}\,(\mathcal{E}^\bullet, \phi) - \mathrm{rank}\,(\mathcal{F}^\bullet, \psi)$. Moreover, if $\mathrm{rank}\,(\mathcal{E}^\bullet, \phi) = \mathrm{rank}\,(\mathcal{F}^\bullet, \psi)$ then $\mathcal{H} = 0$ and $f^\bullet$ is an equivalence.

Proposition 2.3.22 yields to the following proposition (compare [35, Proposition 4.5]).

Proposition 2.3.23. *(a) Any equivalence of d-manifolds is a w-submersion, submersion, w-immersion, immersion, w-embedding and embedding.*

(b) For 2-isomorphic 1-morphisms $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{f}$ is a w-submersion, submersion, w-immersion, immersion, w-embedding or embedding if and only if $\boldsymbol{g}$ is.

(c) Compositions of w-submersions, submersions, w-immersions, immersions, w-embeddings or embeddings are 1-morphisms of the same type.

(d) The condition on a 1-morphisms of d-manifolds $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ to be a w-submersion, submersion, w-immersion or immersion are local in $\boldsymbol{X}$ and $\boldsymbol{Y}$. That is, for each $x \in \boldsymbol{X}$ with $y = f(x) \in \boldsymbol{Y}$, it suffices to check the conditions for $\boldsymbol{f}|_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ where $\boldsymbol{U}$ is an open neighbourhood of x in $\boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$ and $\boldsymbol{V}$ an open neighbourhood of y in $\boldsymbol{Y}$.

Theorem 2.3.17 needed a rather strong condition on $\boldsymbol{f}$ being an étale 1-morphism and Theorem 2.3.13 provided a differential-geometric criterion for when a standard 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ is étale. As Definitions 2.3.20 and 2.3.21 introduce weaker notions of $\boldsymbol{f}$ being a w-submersion, submersion, w-immersion or immersion, the following theorem due to Joyce [35, Theorem 4.8] provides criteria for when $\boldsymbol{S}_{f,\hat{f}}$ is a a w-submersion, submersion, w-immersion or immersion.

Theorem 2.3.24. *Let V, W be manifold, $E \to V, F \to W$ be vector bundles of V and W and $s \in C^\infty(V, E), t \in C^\infty(W, F)$ be smooth sections. Let $f : V \to W$ be a a smooth map and $\hat{f} : E \to f^*(F)$ be a morphism of vector bundles on V satisfying $\hat{f} \circ s = f^*(t) + O(s^2)$. Then Definitions 2.3.4 and 2.3.8 define principal d-manifolds $\boldsymbol{S}_{V,E,s}, \boldsymbol{S}_{W,F,t}$ and a 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. As in (2.15) we have the following complex of vector spaces*

$$0 \longrightarrow T_vV \xrightarrow{ds(v) \oplus df(v)} E_v \oplus T_wW \xrightarrow{\hat{f}(v) \oplus -dt(w)} F_w \longrightarrow 0, \tag{2.21}$$

for each $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$.

(a) $\boldsymbol{S}_{f,\hat{f}}$ *is then a w-submersion if and only if for all* $v \in V$ *with* $s(v) = 0$ *and* $w = f(v) \in W$ *equation* (2.21) *is exact at the fourth position, that is* $\hat{f}(v) \oplus -dt(w)$ *is surjective.*

(b) $\boldsymbol{S}_{f,\hat{f}}$ *is then a submersion if and only if for all* $v \in V$ *with* $s(v) = 0$ *and* $w = f(v) \in W$ *equation* (2.21) *is exact at the third and fourth position.*

(c) $\boldsymbol{S}_{f,\hat{f}}$ *is then a w-immersion if and only if for all* $v \in V$ *with* $s(v) = 0$ *and* $w = f(v) \in W$ *equation* (2.21) *is exact at the second position, that is* $ds(v) \oplus df(v)$ *is injective.*

(d) $\boldsymbol{S}_{f,\hat{f}}$ *is then an immersion if and only if for all* $v \in V$ *with* $s(v) = 0$ *and* $w = f(v) \in W$ *equation* (2.21) *is exact at the second and fourth position.*

Note that all of the above conditions are open condition on v *in* $\{v \in V : s(v) = 0\}$.

The following theorem ([35, Theorem 4.9]) gives a local characterisation in terms of standard models and standard model 1-morphism of the above defined 1-morphism types of d-manifolds.

Theorem 2.3.25. *Let* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ *be a 1-morphism and* $x \in \boldsymbol{X}$ *with* $\boldsymbol{g}(x) = y \in \boldsymbol{Y}$. *Then there exist open d-submanifolds* $\boldsymbol{T} \subseteq \boldsymbol{X}$ *and* $\boldsymbol{U} \subseteq \boldsymbol{Y}$ *with* $x \in \boldsymbol{T}, y \in \boldsymbol{U}$ *and* $g(\boldsymbol{T}) \subseteq \boldsymbol{U}$, *manifolds* V, W, *vector bundles* $E \to V$, $F \to W$, *smooth sections* $s \in C^\infty(E), t \in C^\infty(F)$, *a smooth map* $f : V \to W$, *a morphism of vector bundles* $\hat{f} : E \to f^*(F)$ *with* $\hat{f} \circ s \equiv f^*(t)$, *equivalences* $\boldsymbol{i} : \boldsymbol{T} \to \boldsymbol{S}_{V,E,s}, \boldsymbol{j} : \boldsymbol{U} \to \boldsymbol{S}_{W,F,t}$, *and a 2-morphism* $\eta : \boldsymbol{j} \circ \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{i} \Rightarrow \boldsymbol{g}|_T$, *where* $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$.

(a) *If* $\boldsymbol{g}$ *is a* ***w-submersion****, we can chose the data* $\boldsymbol{T}, \ldots, \boldsymbol{j}$ *such that* $f : V \to W$ *is a submersion in* **Man**, *and* $\hat{f} : E \to f^*(F)$ *is a surjective vector bundle morphism.*

(b) *If* $\boldsymbol{g}$ *is a* ***submersion****, we can chose the data* $\boldsymbol{T}, \ldots, \boldsymbol{j}$ *such that* $f : V \to W$ *is a submersion and* $\hat{f} : E \to f^*(F)$ *is an isomorphism.*

(c) *In the case of* $\boldsymbol{g}$ *being a* ***w-immersion****, we can chose the data* $\boldsymbol{T}, \ldots, \boldsymbol{j}$ *such that* V *is a submanifold of* W, $f : V \hookrightarrow W$ *is the inclusion, and* $F|_V = E \oplus G$ *for some vector bundle* $G \to V$, *and* $\hat{f} = \mathrm{id}_E \oplus 0 : E \to E \oplus G = f^*(F), t|_V = s \oplus 0$.

(d) *In the case of* $\boldsymbol{g}$ *being an* ***immersion****, we can choose* $F|_V = E$, $\hat{f} = \mathrm{id}_E, t|_V = s$.

2.3.5 Embedding theorems for d-manifolds

The following section discusses embedding theorems for d-manifolds. We will just state the important results and refer for proofs to [35, §4.4]. The following lemma (see [35, Lemma 1.4.27]) follows easily from the description of principal d-manifolds in terms of "standard models", and shows that any principal d-manifold can be embedded into some manifold.

Lemma 2.3.26. *Let* $\boldsymbol{U}$ *be a principal d-manifold. Then there exists an embedding* $\boldsymbol{i} : \boldsymbol{U} \to \boldsymbol{V}$ *of* $\boldsymbol{U}$ *into a manifold* $\boldsymbol{V}$.

As this lemma shows, in the case of principal d-manifolds there are no restrictions on the d-manifold to obtain an embedding. In the case of general d-manifolds however, it is no longer true that any d-manifold $\boldsymbol{X}$ can be embedded into some manifold Y, and as Theorem 2.3.29 will show, this is the case if and only if $\boldsymbol{X}$ is a principal d-manifold. So Theorem 2.3.29 is the converse of Lemma 2.3.26.

Theorem 2.3.28 below will generalize the following well-known classical result by Whitney [53]:

Theorem 2.3.27 (Whitney [53])**.** *(a) A generic smooth map* $f : X \to \mathbb{R}^n$ *from a* m*-dimensional manifold into* $\mathbb{R}^n$ *for some* $n \geq 2m + 1$ *is an immersion.*

(b) For any m*-dimensional manifold* X*, there exists an embedding* $f : X \to \mathbb{R}^n$ *for some* $n \geq 2m + 1$ *and* f *can be chosen such that* $f(X)$ *is closed in* $\mathbb{R}^n$*. Moreover, generic smooth maps* $f : X \to \mathbb{R}^n$ *are embeddings.*

The d-manifold version of this theorem will play a central role in defining and studying bordism theory of d-manifolds. We will state the proof of this theorem (and therefore an implicit proof of Lemma 2.3.26) as we will later on imitate this proof when studying representable d-orbifolds. The proof follows closely [35, Theorem 4.29], to which we refer for a more complete and detailed discussion.

Theorem 2.3.28. *Let* $\boldsymbol{X}$ *be a compact d-manifold. Then there exists an embedding* $\boldsymbol{f} : \boldsymbol{X} \hookrightarrow \mathbb{R}^{\boldsymbol{n}}$ *for some* $n \gg 0$.

Proof. Let $x \in X$ and let $\boldsymbol{U}_x$ be a principal open neighbourhood of x in $\boldsymbol{X}$ with equivalence $\boldsymbol{i} : \boldsymbol{U}_x \to \boldsymbol{S}_{V_x,E_x,s_x}$ for some triple V_x, E_x, s_x. So in particular, $\boldsymbol{i}(x) = v_x \in V_x$ and $s_x(v_x) = 0$. As X is paracompact and Hausdorff, we can choose an open neighbourhood U'_x of x in U_x, such that the closure $\overline{U'_x}$ of U'_x in X is a subset of U_x. Denote by $\boldsymbol{U}'_x \subseteq \boldsymbol{U}_x$ be the corresponding open d-submanifold, and choose an open $V'_x \subseteq V_x$ such that $\boldsymbol{i}(\boldsymbol{U}'_x) = \boldsymbol{S}_{V'_x,E'_x,s'_x} \subseteq \boldsymbol{S}_{V_x,E_x,s_x}$, where $E'_x = E|_{V'_x}$ and $s^p_x = s|_{V'_x}$ denote the restrictions of E_x and s_x to V'_x.

For some $n_x > \dim V_x$ we can choose an open neighbourhood V''_x of v_x in V'_x and a smooth cut off function $g_x : V_x \to \mathbb{R}^{n_x}$, such that $g_x|_{V_x \setminus V'_x} = 0, g_x|_{V''_x} \to \mathbb{R}^{n_x}$ is an embedding, $g_x(V''_x)$ doesn't contain 0 and $g_x(V''_x) \cap g_x(V_x \setminus V''_x) = \emptyset$. Set $E''_x = E_x|_{V''_X}, s''_x = s_x|_{V''_X}$ and $\boldsymbol{U}''_x = \boldsymbol{i}^{-1}(\boldsymbol{S}_{V''_x,E''_x,s''_x})$, so that $\boldsymbol{U}''_x$ is an open neighbourhood of x in $\boldsymbol{U}'_x \subseteq \boldsymbol{X}$ and $\boldsymbol{i}|_{\boldsymbol{U}''_x} : \boldsymbol{U}''_x \to \boldsymbol{S}_{V''_x,E''_x,s''_x}$ is an equivalence.

Using the cut off function $g_x$we get a 1-morphism $\boldsymbol{S}_{g_x,0} \circ \boldsymbol{i} : \boldsymbol{U}_x \to \boldsymbol{S}_{\mathbb{R}^{n_x},0,0} = F^{\mathbf{dMan}}_{\mathbf{Man}}(\mathbb{R}^{n_x}) = \mathbb{R}^{n_x}$. On $\boldsymbol{U}_x \setminus \overline{\boldsymbol{U}'_x}$ this 1-morphism is identically 0, as $g_x|_{V_x \setminus V'_x} = 0$, and hence we can write $\boldsymbol{S}_{g_x,0} \circ \boldsymbol{i}|_{\boldsymbol{U}_x \setminus \overline{\boldsymbol{U}'_x}} = \boldsymbol{0} \circ \boldsymbol{\pi}$, where $\boldsymbol{\pi} : \boldsymbol{U}_x \setminus \overline{\boldsymbol{U}'_x} \to \boldsymbol{*}$ and $\boldsymbol{0} : \boldsymbol{*} \to \mathbb{R}^{n_x} = F^{\mathbf{dSpa}}_{\mathbf{Man}}(0 : * \to \mathbb{R}^{n_x})$. As $\overline{U'_x} \subseteq U_x$, we can extend $\boldsymbol{S}_{g_x,0} \circ \boldsymbol{i}$ uniquely by zero to all of $\boldsymbol{X}$, and we get therefore a unique 1-morphism $\boldsymbol{f}_x : \boldsymbol{X} \to \mathbb{R}^{n_x}$, satisfying $\boldsymbol{f}_x|_{\boldsymbol{U}_x} = \boldsymbol{S}_{g_x,0} \circ \boldsymbol{i}$ and $\boldsymbol{f}_x|_{\boldsymbol{X} \setminus \overline{\boldsymbol{U}'_x}} = \boldsymbol{0} \circ \boldsymbol{\pi}$.

As $0 \notin g_x(V''_x)$ and $g_x(V''_x) \cap g_x(V_x \setminus V''_x)$, we can conclude that $\boldsymbol{f}_x(\boldsymbol{U}''_x) \cap f_x(\boldsymbol{X} \setminus \boldsymbol{U}''_x) = \emptyset$. To see that $\boldsymbol{f}_x|_{\boldsymbol{U}''_x} : \boldsymbol{U}''_x \to \mathbb{R}^{n_x}$ is an embedding, note that $\boldsymbol{f}_x|_{\boldsymbol{U}''_x} = \boldsymbol{S}_{g_x|_{V''_x},0} \circ \boldsymbol{i}|_{\boldsymbol{U}''_x}$, where $\boldsymbol{i}|_{\boldsymbol{U}''_x}$ is the equivalence given as above and $\boldsymbol{S}_{g_x|_{V''_x},0} : bS_{V''_x,E''_x,s''_x} \to bS_{\mathbb{R}^{n_x},0,0} = \mathbb{R}^{n_x}$ with $g_x|_{V''_x} : V''_x \to \mathbb{R}^{n_x}$ an embedding.

Equation (2.21) yields for $\boldsymbol{S}_{g_x|_{V''_x},0}$ the following sequence

$$0 \longrightarrow T_x V''_x \xrightarrow{ds_x(v) \oplus dg_x(v)} E_v \oplus \mathbb{R}^{n_x} \xrightarrow{0 \oplus 0} 0 \longrightarrow 0,$$

which is exact at the second and fourth terms as $dg_x(v)$ is injective. Hence, by Theorem 2.3.24(d), $\boldsymbol{S}_{g_x|_{V''_x},0}$ is an immersion and thus and embedding as $g_x|_{V''_x}$ is an embedding and therefore a homeomorphism with its image. Hence $\boldsymbol{f}_x|_{\boldsymbol{U}''_x}$ is an embedding by Proposition 2.3.23(a),(c).

Choosing $n_x, \boldsymbol{U}''_x, \boldsymbol{f}_x$ for all $x \in \boldsymbol{X}$, we get an open cover $\{\boldsymbol{U}''_x; x \in \boldsymbol{X}\}$ of $\boldsymbol{X}$, and as $\boldsymbol{X}$ is compact, there exists a finite subcover $\{\boldsymbol{U}''_{x_i} : i = 1, \ldots, k\}$. Defining $n = n_{x_1} + \cdots + n_{x_k}$ we may define a 1-morphism $\boldsymbol{f}$ by $\boldsymbol{f} = \boldsymbol{f}_{x_1} \times \cdots \times \boldsymbol{f}_{x_k} : \boldsymbol{X} \to$

$\mathbb{R}^{n_{x_1}} \times \cdots \times \mathbb{R}^{n_{x_k}} = \mathbb{R}^n$ which we claim is an embedding. To see this, note that as $\boldsymbol{f}_{x_i}|_{\boldsymbol{U}''_{x_i}}$ is an embedding, $\boldsymbol{f}|_{\boldsymbol{U}''_{x_i}}$ is an immersion for $i = 1, \dots, k$ and therefor $\boldsymbol{f}$ is an immersion as $\boldsymbol{X} = \boldsymbol{U}''_{x_1} \cup \cdots \cup \boldsymbol{U}''_{x_k}$. Suppose now that $x \neq y \in X$. Then $x \in U''_{x_i}$ for some $i = 1, \dots, k$. If $y \in U''_{x_i}$ then $f_{x_i}(x) \neq f_{x_i}(y)$ as $f_{x_i}|_{U''_{x_i}}$ is injective. If $y \neq U''_{x_i}$ then $f_{x_i}(y)$ as $f_{x_i}|_{U''_{x_i}}$ as $\boldsymbol{f}_{x_i}(\boldsymbol{U}''_{x_i}) \cap \boldsymbol{f}_{x_i}(\boldsymbol{X} \setminus \boldsymbol{U}''_{x_i}) = \emptyset$. Hence $f(x) \neq f(y)$ and $f : X \to \mathbb{R}^n$ is injective. As f is locally an embedding and X is compact, we get that f is a homeomorphism with its image and therefore that $\boldsymbol{f}$ is an embedding. $\square$

We will end this section with the following theorem ([35, Theorem 4.34]), which proves that if a d-manifold $\boldsymbol{X}$ can be embedded into a manifold Y, then $\boldsymbol{X}$ can be written as the zero set of a smooth section of a vector bundle over the manifold Y near its image.

Theorem 2.3.29. *Let $\boldsymbol{X}$ be a d-manifold, Y a manifold and $f : \boldsymbol{X} \to \boldsymbol{Y}$ an embedding, in the sense of d-manifolds. Then there exist an open subset $V \subseteq Y$, with $\boldsymbol{f}(\boldsymbol{X}) \subseteq \boldsymbol{V}$, a vector bundle $E \to V$ and a smooth section $s : V \to E$ fitting into a 2-Cartesian diagram in the category of d-spaces* **dSpa***:*

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r, "\boldsymbol{f}"'] \arrow[d, "\boldsymbol{f}"] \arrow[dr, phantom, "\Uparrow\eta"] & \boldsymbol{V} \arrow[d, "\mathbf{0}"'] \\
\boldsymbol{V} \arrow[r, "\boldsymbol{s}"] & \boldsymbol{E},
\end{tikzcd}$$

for some 2-morphism $\eta : \boldsymbol{s} \circ \boldsymbol{f} \Rightarrow \mathbf{0} \circ \boldsymbol{f}$. Here $0 : V \to E$ is the zero section, and $\boldsymbol{Y}, \boldsymbol{V}, \boldsymbol{E}, \boldsymbol{s}, \mathbf{0} = F^{\mathbf{dMan}}_{\mathbf{Man}}(Y, Y, E, s, 0)$. Hence $\boldsymbol{X}$ is equivalent to the standard model d-manifold $\boldsymbol{S}_{V,E,s}$, and is therefore a principal d-manifold.

As a consequence of Theorems 2.3.28 and 2.3.29 and Lemma 2.3.26 we get the following corollary which shows that any compact d-manifold is principal.

Corollary 2.3.30. *A d-manifold $\boldsymbol{X}$ is principal if and only if $\dim T^*_x\underline{X}$ is bounded above for all $x \in \underline{X}$. In particular, if $\boldsymbol{X}$ is compact then $\boldsymbol{X}$ is principal.*

2.3.6 D-transversality and fibre products

We have seen in Theorem 2.2.5, that in the 2-category of d-spaces **dSpa** all fibre products exist. Since the 2-category **dMan** is a full 2-subcategory of **dSpa**, we know that given 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ the fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in **dSpa**. We will now follow [35, §4.3] and investigate under which circumstances this fibre product will exist in **dMan**. As it will turn out, a sufficient condition for $\boldsymbol{W}$ being a d-manifold will be *d-transversality* of $\boldsymbol{g}$ and $\boldsymbol{h}$. As the name suggests, the notion of d-transversality is motivated by the notion of transversality between smooth maps of manifolds.

Recall, that in the 'classical' manifold case, the fibre product $W = X \times_{g,Z,h} Y$ of smooth manifolds X, Y with smooth maps $g : X \to Z$ and $h : Y \to Z$ exists in **Man**, if g and h are *transverse maps*, that is the tangent bundle $T_z Z$ can be split into $T_z Z = dg|_x(T_x X) + dh|_y(T_y Y)$ for all $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$. This can be reformulated into g and h are transverse if the following morphism of vector bundles on the topological space W is injective:

$$e^*(dg^*) \oplus f^*(dh^*) : (g \circ e)^*(T^*Z) \to e^*(T^*X) \oplus f^*(T^*Y),$$

where $e : W \to X$ and $f : W \to Y$ denote the natural projections satisfying $g \circ e = h \circ f$.

Definition 2.3.31 below (compare [35, Definition 4.16]) imitates this condition for the d-manifold case, but on the obstruction bundle rather than the cotangent bundle.

Definition 2.3.31. Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be d-manifolds and $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphism. Let $\underline{W} = \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y}$ be the fibre product of the underlying C^∞-schemes and define $\underline{e} : \underline{W} \to \underline{X}$, $\underline{f} : \underline{W} \to \underline{Y}$ to be the projection morphisms.

We call $\boldsymbol{g}, \boldsymbol{h}$ *d-transverse*, if the following morphism in $\mathrm{qcoh}(\underline{W})$ has a left inverse:

$$\alpha = \begin{pmatrix} \underline{e}^*(g'') \circ \mathcal{I}_{\underline{e},\underline{g}}(\mathcal{E}_Z) \\ -\underline{f}^*(h'') \circ \mathcal{I}_{\underline{f},\underline{h}}(\mathcal{E}_Z) \\ (\underline{g} \circ \underline{e})^*(\phi_Z) \end{pmatrix} : (\underline{g} \circ \underline{e})^*(\mathcal{E}_Z) \to \underline{e}^*(\mathcal{E}_X) \oplus \underline{f}^*(\mathcal{E}_Y) \oplus (\underline{g} \circ \underline{e})^*(\mathcal{F}_Z).$$

The following theorem is the d-manifold analogue of the classical result that fibre products between smooth manifolds exist in **Man** if the involved smooth maps are transverse and is proven in [35, Theorem 4.21].

Theorem 2.3.32. *Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be d-manifolds and $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be d-transverse* 1*-morphisms.*

Then the d-space fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in **dMan***, that is $\boldsymbol{W}$ is a d-manifold with*

$$\mathit{vdim}\, \boldsymbol{W} = \mathit{vdim}\, \boldsymbol{X} + \mathit{vdim}\, \boldsymbol{Y} - \mathit{vdim}\, \boldsymbol{Z}.$$

Moreover, Joyce [35, Theorem 4.22] gives sufficient conditions for two 1-morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}, \boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ between d-manifolds $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ to be d-transverse.

Theorem 2.3.33. *Let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be* 1*-morphisms of d-manifolds.*

Then the following are sufficient conditions for $\boldsymbol{g}, \boldsymbol{h}$ to be d-transverse:

(a) $\boldsymbol{Z} \in \hat{\mathbf{M}}\mathbf{an}$*, that is $\boldsymbol{Z}$ is a manifold,*

(b) $\boldsymbol{g}$ or $\boldsymbol{h}$ is a w-submersion.

In the case of smooth manifolds, it is sufficient for either $g : X \to Z$ or $h : Y \to Z$ to be a submersion, as this implies that g and h are transverse. An analogous result in the d-manifold world is the following theorem again proven in [35, Theorem 4.23].

Theorem 2.3.34. *Let $\boldsymbol{Y}$ be a manifold, $\boldsymbol{X}, \boldsymbol{Z}$ be d-manifolds, and $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be* 1*-morphism with $\boldsymbol{g}$ being a submersion.*

Then $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ is a manifold of dimension $\dim \boldsymbol{W} = \mathit{vdim}\, \boldsymbol{X} + \dim \boldsymbol{Y} - \mathit{vdim}\, \boldsymbol{Z}$.

Fibre products with $\mathbb{R}^n$ in **dMan** can be used to locally characterise embeddings and immersion in **dMan** (see [35, Proposition 4.26]) and vice versa ([35, Proposition 4.27]).

Theorem 2.3.35. *(a) Let* $\boldsymbol{X}$ *be a d-manifold and* $\boldsymbol{g} : \boldsymbol{X} \to \mathbb{R}^n$ *a* 1*-morphism of d-manifolds. Then the fibre product* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\mathbb{R}^n,\boldsymbol{0}} *$ *exists in* **dMan** *by Theorem 2.3.33(a) and the projection map* $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$ *is an embedding of d-manifolds.*

(b) Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *be an immersion of d-manifolds, and* $x \in \boldsymbol{X}$ *with* $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$*. Then there exist open d-submanifolds* $x \in \boldsymbol{U} \subseteq \boldsymbol{X}$ *and* $y \in \boldsymbol{V} \subseteq \boldsymbol{Y}$ *with* $\boldsymbol{f}(\boldsymbol{U}) \subseteq \boldsymbol{V}$*, and a* 1*-morphism* $\boldsymbol{g} : \boldsymbol{V} \to \mathbb{R}^n$ *satisfying* $\boldsymbol{g}(y) = 0$*, where* $n = \operatorname{vdim} \boldsymbol{Y} - \operatorname{vdim} \boldsymbol{X}$*. These data fits into the following* 2*-Cartesian square in* **dMan***:*

$$\begin{tikzcd} \boldsymbol{U} \arrow[r] \arrow[d, "\boldsymbol{f}|_{\boldsymbol{U}}"'] \arrow[dr, phantom, "\Uparrow"] & * \arrow[d, "\boldsymbol{0}"] \\ \boldsymbol{V} \arrow[r, "\boldsymbol{g}"] & \mathbb{R}^n. \end{tikzcd}$$

In the case of $\boldsymbol{f}$ *being an embedding,* $\boldsymbol{U}$ *can be taken as* $\boldsymbol{U} = \boldsymbol{f}^{-1}(\boldsymbol{V})$*.*

2.3.7 Orientations on d-manifolds

This section will quickly review some material on orientations on d-manifolds. The notion of orientation on a d-manifold is the d-manifold analogue of the notion of orientation in the 'classical' manifold case, and we advise the reader to consult [35, §4.5, §4.6] for an in depth treatment of the subject.

Definition 2.3.36. Let $\boldsymbol{X}$ be a d-manifold. Then the virtual cotangent bundle $T^*\boldsymbol{X} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X)$ is a virtual vector bundle on $\underline{X}$. As shown in [35, §4.5], one can construct a line bundle $\mathcal{L}_{T^*\boldsymbol{X}}$ on $\underline{X}$, which we will call the *orientation line bundle* of $\boldsymbol{X}$.

Note that this construction holds more generally on the C^∞-scheme level, that is for a given C^∞-scheme $\underline{X}$ and a virtual vector bundle $(\mathcal{E}^\bullet, \phi)$ on $\underline{X}$, one can construct a real line bundle $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ on $\underline{X}$, the so called *orientation line bundle* of $(\mathcal{E}^\bullet, \phi)$.

An *orientation* ω on $\boldsymbol{X}$ is then an orientation on $\mathcal{L}_{T^*\boldsymbol{X}}$, that is, ω is an equivalence class $[\tau]$ of isomorphism $\tau : \mathcal{O}_X \to \mathcal{L}_{T^*\boldsymbol{X}}$, where τ is equivalent to τ' if and only if they are proportional by a positive function on the underlying scheme $\underline{X}$.

We will call $\boldsymbol{X}$ *orientable* if it admits an orientation, that is, $\boldsymbol{X}$ is orientable if and only if $\mathcal{L}_{T^*\boldsymbol{X}}$ is trivilizable.

An *oriented d-manifold* is a pair $(\boldsymbol{X}, \omega)$, where $\boldsymbol{X}$ is a d-manifold and ω an orientation on $\boldsymbol{X}$.

The *opposite orientation* to a given orientation $\omega = [\tau]$ on a d-manifold $\boldsymbol{X}$, is given by $-\omega = [-\tau]$, which changes the sign of the isomorphism $\tau : \mathcal{O}_X \to \mathcal{L}_{T^*\boldsymbol{X}}$. Using the shorthand notation $\boldsymbol{X}$ for an oriented d-manifold $(\boldsymbol{X}, \omega)$, we will write $-\boldsymbol{X}$ for $\boldsymbol{X}$ with the opposite orientation, that is $-\boldsymbol{X}$ is short for $(\boldsymbol{X}, -\omega)$.

The following theorem summarizes some important properties of orientation line bundles. (Compare [35, §4.5, §4.6] for proofs and more details).

Theorem 2.3.37. *Let $\underline{X}$ be a C^∞-scheme, $(\mathcal{E}^\bullet, \phi)$ a virtual vector bundle on $\underline{X}$ and $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$ be the orientation line bundle. Then*

(a) Let $\mathcal{E}^1, \mathcal{E}^2$ be vector bundles on $\underline{X}$ with ranks k_1, k_2 and $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ be a morphism of vector bundles. Then $(\mathcal{E}^\bullet, \phi)$ is a virtual vector bundle of rank $k_2 - k_1$, and the orientation line bundle $\mathcal{L}_{\mathcal{E}^\bullet,\phi}$ is canonically isomorphic to the tensor product of the determinant line bundles of $(\mathcal{E}^1)^*$ and $\mathcal{E}^2$, that is $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)} \cong \Lambda^{k_1}(\mathcal{E}^1)^* \otimes \Lambda^{k_2}\mathcal{E}^2$.

(b) If $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ is an equivalence in $\mathbf{vqcoh}(\underline{X})$, then there exists a canonical isomorphism $\mathcal{L}_{f^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet,\phi)} \to \mathcal{L}_{(\mathcal{F}^\bullet,\psi)}$ in $\mathrm{qcoh}(\underline{X})$.

(c) If $(\mathcal{E}^\bullet, \phi)$ is a virtual vector bundle on $\underline{X}$, that is $(\mathcal{E}^\bullet, \phi) \in \mathbf{vvect}(\underline{X})$, then $\mathcal{L}_{\mathrm{id}_\phi} = \mathrm{id}_{\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}} : \mathcal{L}_{(\mathcal{E}^\bullet,\phi)} \to \mathcal{L}_{(\mathcal{E}^\bullet,\phi)}$.

(d) Suppose $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ and $g : (\mathcal{F}^\bullet, \psi) \to (\mathcal{G}^\bullet, \xi)$ are equivalences in $\mathbf{vqcoh}(\underline{X})$, then $\mathcal{L}_{g^\bullet \circ f^\bullet} = \mathcal{L}_{g^\bullet} \circ \mathcal{L}_{f^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet,\phi)} \to \mathcal{L}_{(\mathcal{G}^\bullet,\xi)}$.

(e) If $f^\bullet, g^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ are 2-isomorphic equivalences in $\mathbf{vqcoh}(\underline{X})$, then $\mathcal{L}_{f^\bullet} = \mathcal{L}_{g^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet,\phi)} \to \mathcal{L}_{(\mathcal{F}^\bullet,\psi)}$.

(f) Suppose $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes, and $(\mathcal{E}^\bullet, \phi) \in \mathbf{vqcoh}(\underline{Y})$. Then there is a canonical isomorphism $I_{\underline{f},(\mathcal{E}^\bullet,\phi)} : \underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet,\phi)}) \to \mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet,\phi)}$ between the pulled backed line bundle and the line bundle associated to the pulled back virtual quasi coherent scheme.

The next theorem (see [35, Theorem 4.50] for a proof) shows that the fibre product of d-transverse oriented d-manifolds itself carries an orientation.

Theorem 2.3.38. *Given d-manifolds $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ and d-transverse 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$, Theorem 2.3.32 shows that the fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists and is a d-manifold. Denote by $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$ and $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ the projection morphisms. Then we have orientation line bundles $\mathcal{L}_{T^*\boldsymbol{W}}, \dots, \mathcal{L}_{T^*\boldsymbol{Z}}$ on $\underline{W}, \dots, \underline{Z}$ and so $\mathcal{L}_{T^*\boldsymbol{W}}, \underline{e}^*(\mathcal{L}_{T^*\boldsymbol{X}}), \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}), (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\boldsymbol{Z}})$ are line bundles on $\underline{W}$. A suitable choice of an orientation convention, yields a canonical isomorphism*

$$\boldsymbol{\Phi} : \mathcal{L}_{T^*\boldsymbol{W}} \to \underline{e}^*(\mathcal{L}_{T^*\boldsymbol{X}}) \otimes_{\mathcal{O}_W} \underline{f}^*(\mathcal{L}_{T^*\boldsymbol{Y}}) \otimes_{\mathcal{O}_W} (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*\boldsymbol{Z}})^*. \tag{2.22}$$

Thus, given oriented d-manifolds $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, the fibre product $\boldsymbol{W}$ also has a natural orientation, since trivializations of $\mathcal{L}_{T^*\boldsymbol{X}}, \mathcal{L}_{T^*\boldsymbol{Y}}, \mathcal{L}_{T^*\boldsymbol{Z}}$ induce a trivialization of $\mathcal{L}_{T^*\boldsymbol{W}}$ by (2.22).

2.3.8 D-manifolds with boundary

As we want to study d-bordism later on, we will in the following give a short summary of d-manifolds with boundary. We follow here the exposition of the material in [37, §6, §7] and refer to [35, §7] for a much more general, rigorous and complete approach.

In a similar spirit to the definition of **dSpa** and **dMan**, Joyce defines in [35],§7-§8 the 2-categories $\mathbf{dSpa}^{\boldsymbol{b}}, \mathbf{dSpa}^{\boldsymbol{c}}$ of d-spaces with boundary and with corners, and full 2-subcategories $\mathbf{dMan}^{\boldsymbol{b}}, \mathbf{dMan}^{\boldsymbol{c}}$ of d-manifolds with boundary and corners. The objects in $\mathbf{dSpa}^{\boldsymbol{b}}, \mathbf{dSpa}^{\boldsymbol{c}}, \mathbf{dMan}^{\boldsymbol{b}}, \mathbf{dMan}^{\boldsymbol{c}}$ are quadruples $\boldsymbol{X} = (\boldsymbol{X}, \partial\boldsymbol{X}, \boldsymbol{i}_{\boldsymbol{X}}, \omega_{\boldsymbol{X}})$, where $\boldsymbol{X}, \partial\boldsymbol{X}$ are d-spaces, and $\boldsymbol{i}_{\boldsymbol{X}} : \partial\boldsymbol{X} \to \boldsymbol{X}$ is a 1-morphism, such that $\partial\boldsymbol{X}$ is locally equivalent to a fibre product $\boldsymbol{X} \times_{\boldsymbol{[0,\infty)}} \boldsymbol{*}$ in **dSpa**.

The following theorem summarizes some of the properties of d-manifolds with boundary and corners. For proofs of these statements and a much more detailed and complete approach to d-manifolds with boundary and corners we refer to [35, §7].

Theorem 2.3.39. *The 2-categories $\mathbf{dMan}^{\boldsymbol{b}}$ and $\mathbf{dMan}^{\boldsymbol{c}}$ have the following properties:*

(a) There exist full and faithful functors $F_{\mathbf{Man}^b}^{\mathbf{dMan}^b} : \mathbf{Man}^{\boldsymbol{b}} \to \mathbf{dMan}^{\boldsymbol{b}}$ *and* $F_{\mathbf{Man}^c}^{\mathbf{dMan}^c} : \mathbf{Man}^{\boldsymbol{c}} \to \mathbf{dMan}^{\boldsymbol{c}}$*. The full* 2*-subcategories of objects in* $\mathbf{dMan}^{\boldsymbol{b}}$ *and* $\mathbf{dMan}^{\boldsymbol{c}}$*, which are equivalent to objects in the image of* $F_{\mathbf{Man}^b}^{\mathbf{dMan}^b}$ *and* $F_{\mathbf{Man}^c}^{\mathbf{dMan}^c}$ *will be denoted by* $\overline{\mathbf{Man}}^{\boldsymbol{b}}$ *and* $\overline{\mathbf{Man}}^{\boldsymbol{c}}$*.*

(b) Each object $\boldsymbol{X} = (\boldsymbol{X}, \partial\boldsymbol{X}, \boldsymbol{i_X}, \omega_{\boldsymbol{X}})$ *in* $\mathbf{dMan}^{\boldsymbol{b}}$ *or* $\mathbf{dMan}^{\boldsymbol{c}}$ *has a* virtual dimension *vdim* $\boldsymbol{X} \in \mathbb{Z}$*. Moreover, the virtual cotangent sheaf* $T^*\boldsymbol{X}$ *of the underlying d-space* $\boldsymbol{X}$ *is a virtual vector bundle on* $\underline{X}$ *with rank vdim* $\boldsymbol{X}$*.*

(c) If $\boldsymbol{X} \in \mathbf{dMan}^{\boldsymbol{b}}$*, then* $\partial\boldsymbol{X} \in \mathbf{d\overline{Man}}$*, and if* $\boldsymbol{X} \in \mathbf{d\overline{Man}}$*, then* $\partial\boldsymbol{X} = \emptyset$*. Here* $\mathbf{d\overline{Man}}$ *denotes the image of* $\mathbf{dMan}$ *under the full and faithful* 2*-functor* $F_{\mathbf{Man}^c}^{\mathbf{dMan}^c} : \mathbf{Man}^{\boldsymbol{c}} \to \mathbf{dMan}^{\boldsymbol{c}}$*.*

(d) Boundaries in $\mathbf{dMan}^{\boldsymbol{b}}, \mathbf{dMan}^{\boldsymbol{c}}$ *have strong functorial properties. For example, if* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is a simple* 1*-morphism in* $\mathbf{dMan}^{\boldsymbol{b}}$*, which roughly speaking means* $\boldsymbol{f}$ *maps* $\partial^k\boldsymbol{X} \to \partial^k\boldsymbol{Y}$ *for all* k*, then there exists a unique simple* 1*-morphism* $\boldsymbol{f}_- : \partial\boldsymbol{X} \to \partial\boldsymbol{Y}$ *with* $f \circ \imath_{\boldsymbol{X}} = \imath_{\boldsymbol{Y}} \circ \boldsymbol{f}_-$*, and the following diagram is* 2*-Cartesian in* $\mathbf{dMan}^{\boldsymbol{c}}$

$$\begin{tikzcd} \partial\boldsymbol{X} \arrow[rr, "\boldsymbol{f}_-"] \arrow[d, "\imath_{\boldsymbol{X}}"'] & \Uparrow \mathrm{id}_{f\circ\imath_{\boldsymbol{X}}} & \partial\boldsymbol{Y} \arrow[d, "\imath_{\boldsymbol{Y}}"] \\ \boldsymbol{X} \arrow[rr, "\boldsymbol{f}"'] & & \boldsymbol{Y}, \end{tikzcd}$$

so that $\partial\boldsymbol{X} \simeq \boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Y},\imath_{\boldsymbol{Y}}} \partial\boldsymbol{Y}$ *in* $\mathbf{dMan}^{\boldsymbol{c}}$*.*

(e) An orientation *on a d-manifold with corners* $\boldsymbol{X} \in \mathbf{dMan}^{\boldsymbol{c}}$*, is an orientation on the line bundle* $\mathcal{L}_{T^*\boldsymbol{X}}$ *on* $\underline{X}$*. Moreover, an orientation on* $\boldsymbol{X}$ *induces a natural orientation on* $\partial\boldsymbol{X}$*.*

Chapter 3

Background on d-orbifolds

We want now, in a similar way to chapter 2, review some basic material on d-orbifolds. If one thinks of d-manifolds as aderived generalization of manifolds, one can think of d-orbifolds as a derived generalization of orbifolds. The basic idea in defining d-orbifolds is very similar to the d-manifold case, but we have to replace C^∞-schemes and d-spaces, by Deligne–Mumford C^∞-stacks and d-stacks. We will start by recalling some theory on 'classical' orbifolds, and would like to refer to [35, §11] for an in depth treatment of the material covered in this chapter.

3.1 Some orbifold background

Orbifolds were introduced by Satake [49] in 1956, who called them "V-manifolds". Thurston [52] studied them later in his work on 3-manifolds, and gave them the name "orbifold". He proved that orbifolds admit well-behaved notions of fundamental group and universal cover.

We will start by briefly recalling some basic definitions and properties and refer to the book of Adem, Leida and Ruan [1, §1] for a much more complete introduction.

Definition 3.1.1. A n-dimensional *orbifold chart* on a topological space X is given by a connected open subset $\tilde{U} \subseteq \mathbb{R}^n$, a finite group G of smooth automorphisms of $\tilde{U}$ and a G-invariant morphism $\phi : \tilde{U} \to X$, which induces a homeomorphism of $\tilde{U}/G$ to an open subset $U \subseteq X$.

As in the case of manifolds an *orbifold atlas* on X is then given by a family $\mathcal{U} = \{(\tilde{U}, G, \phi)\}$ of locally compatible orbifold charts covering X. *Locally compatible* means here that for any two charts $(\tilde{U}, G, \phi)$ and $(\tilde{V}, H, \psi)$ and a given point $x \in \phi(\tilde{U}) \cap \psi(\tilde{V}) =: U \cap V$ there exists an open neighbourhood $W \subseteq Y \cap V$ of x and an orbifold chart $(\tilde{W}, K, \rho)$ such that there exist embeddings $(\tilde{W}, K, \rho) \hookrightarrow (\tilde{U}, G, \phi)$ and $(\tilde{W}, K, \rho) \hookrightarrow (\tilde{V}, H, \psi)$. (Here an embedding $e : (\tilde{W}, K, \rho) \hookrightarrow (\tilde{U}, G, \phi)$ is a smooth embedding $e : \tilde{W} \hookrightarrow \tilde{U}$ with $\phi \circ e = \rho$.)

An atlas $\mathcal{U}$ is called a *refinement* of another atlas $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding in some chart of $\mathcal{V}$. Two atlases $\mathcal{U}$ and $\mathcal{V}$ are called *equivalent* if there exists a common refinement $\mathcal{W}$.

Definition 3.1.2. A second countable Hausdorff space X, equipped with an equivalence class $[\mathcal{U}]$ of n-dimensional orbifold atlases is called an *effective orbifold*, written $\mathcal{X}$.

As in classical differential geometry, we can define what smooth maps between orbifolds should be:

Definition 3.1.3. let $\mathcal{X} = (X, \mathcal{U})$ and $\mathcal{Y} = (Y, \mathcal{V})$ be orbifolds. A morphism $f : X \to Y$ is called *smooth* if for all $x \in X$ there exists a chart $\tilde{U}, G, \phi$ around x and $(\tilde{V}, H, \psi)$ around $f(x)$, such that $f(U) \subseteq V$, where $U = \phi(\tilde{U})$ and $V = \psi(\tilde{V})$, and f can be lifted to a smooth map $\tilde{f} : \tilde{U} \to \tilde{V}$ with $\psi \circ \tilde{f} = f \circ \phi$.

This immediately yields the notion of a *diffeomorphism* between orbifolds.

Definition 3.1.4. Two orbifolds $\mathcal{X}$ and $\mathcal{Y}$ are called *diffeomorphic* if there exist smooth maps of orbifold $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_X$.

Definition 3.1.5. For each $x \in X$, with $\mathcal{X} = (X, \mathcal{U})$ being an orbifold, define the *local group* at x as

$$G_x = \{g \in G | gy = y\},$$

whenever $(\tilde{U}, G, \psi)$ is a local chart around $x = \psi(y)$. Note that G_x is determined up to conjugacy in G, which follows as in [1, §1].

Definition 3.1.6. For an orbifold $\mathcal{X} = (X,\mathcal{U})$, the *singular set* of $\mathcal{X}$ is defined as

$$\Sigma(\mathcal{X}) = \{x \in X | G_x \neq 1\}.$$

The most natural source of orbifolds are compact transformation groups: consider a compact Lie group G acting smoothly, effectively and almost freely (that is, with finite stabilizers) on a smooth manifold M. Then, as smooth actions on manifolds are locally smooth, we can conclude that for each $x \in M$ with isotropy group G_x there exists a G_x-invariant chart $U \cong \mathbb{R}^n$ containing x. The orbifold charts are then simply given by (U, G_x, π), where $\pi : U \to U/G_x$ is the projection map. The quotient space $X = M/G$ is automatically second countable and Hausdorff.

Definition 3.1.7. We call an orbifold $\mathcal{X} = (X,\mathcal{U})$ an *effective quotient orbifold*, if it is given as the quotient of a smooth, effective, almost free action of a compact Lie group G on a smooth manifold M, that is $X = M/G$ with $\mathcal{U}$ being constructed from a manifold atlas using the local smooth structure from above.

In the case of G being finite, this yields:

Definition 3.1.8. If G is finite in the situation of Definition 3.1.7, $\mathcal{X} = (M/G,\mathcal{U})$ will be called *effective global quotient* orbifold.

3.2 C^∞-stacks

In this section we briefly review the basic theory of *Deligne–Mumford C^∞-stacks* due to Joyce ([34] and [35]). We will just recall the for us most important definitions ans concepts and refer a much more detailed and complete discussion about C^∞-stacks and orbifolds to Joyce [34, §8].

Definition 3.2.1. A C^∞-stack is a geometric stack on the site $(\mathbf{C^\infty Sch}, \mathcal{J})$, where $\mathcal{J}$ is a *Grothendieck topology* on the category $C^\infty\mathbf{Sch}$. (For more details see [35, Definition C.1]).

The 2-category of C^∞-stacks will be denoted $C^\infty\mathbf{Sta}$. For any C^∞-scheme $\underline{X}$, $\overline{\underline{X}}$ is a C^∞-stack.

The following definition defines open and closed C^∞-substacks and can be found in [34, Definition 8.13].

Definition 3.2.2. A C^∞*-substack* $\mathcal{Y}$ in a C^∞-stack $\mathcal{X}$ is a substack of $\mathcal{X}$ (as for instance in [34, Definition 7.4]), which is also a C^∞-stack. There exists a natural inclusion 1-morphism $i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{X}$.

$\mathcal{Y}$ is called *open* C^∞*-substack* of $\mathcal{X}$ if $i_{\mathcal{Y}}$ is a representable open embedding, *closed* C^∞*-substack* of $\mathcal{X}$ if $i_{\mathcal{Y}}$ is a representable closed embedding, and *locally closed* C^∞*-substack* of $\mathcal{X}$ if $i_{\mathcal{Y}}$ is a representable embedding.

A collection $\{\mathcal{Y}_a : a \in A\}$ of open C^∞-substacks $\mathcal{Y}_a$ in $\mathcal{X}$ with $\coprod_{a\in A} i_{\mathcal{Y}_a} : \coprod_{a\in A} \mathcal{Y}_a \to \mathcal{X}$ being surjective, is called *open cover* of $\mathcal{X}$.

We will now recall some material on quotient C^∞-stacks as in [35, §C.4].

Definition 3.2.3. Consider a separated C^∞-scheme $\underline{X}$, a finite group G, and an action of G on $\underline{X}$ by isomorphisms $\underline{r} : G \to \mathrm{Aut}(\underline{X})$. Then the *quotient* C^∞*-stack* $\mathcal{X} = [\underline{X}/G]$ can be defined as follows:

Define a category $\mathcal{X}$ to have objects septuples $(A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})$. Here A is a finite group, $\mu : A \to G$ is a group morphism, $\underline{T}, \underline{U}$ are C^∞-schemes, $\underline{t} : A \to \mathrm{Aut}(\underline{T})$ is a free action of A on $\underline{T}$ by isomorphisms, $\underline{u} : \underline{T} \to \underline{X}$ is a morphism satisfying $\underline{u} \circ \underline{t}(\alpha) = \underline{r}(\mu(\alpha)) \circ \underline{u} : \underline{T} \to \underline{X}$ for all $\alpha \in A$, and $\underline{v} : \underline{T} \to \underline{U}$ is a morphism which makes $\underline{T}$ into a principal A-bundle over $\underline{U}$, i.e. $\underline{v}$ is proper, étale and surjective, and its fibres are A-orbits in $\underline{T}$.

Morphisms between objects $(\underline{a}, \tilde{\underline{a}}) : (A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \to (A', \mu', \underline{T}', \underline{U}', \underline{t}', \underline{u}', \underline{v}')$ can be defined to be pairs of morphisms, where $\underline{a} : \underline{U} \to \underline{U}'$ is a morphism of C^∞-schemes, and $\tilde{\underline{a}} : \underline{T} \times_A G \to \underline{T}' \times_{A'} G'$ satisfying some compatibility conditions as in [35, Definition C.26]. Joyce shows in [35, Definition C.26], that this makes $\mathcal{X}$ into a category. The functor $\rho_{\mathcal{X}} : \mathcal{X} \to C^\infty\mathbf{Sch}$ defined by $\rho_{\mathcal{X}} : (A, \mu, \underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \mapsto \underline{U}$ on objects and $\rho_{\mathcal{X}} : (\underline{a}, \tilde{\underline{a}}) \mapsto \underline{a}$ on morphism makes $\mathcal{X}$ into a Deligne–Mumford C^∞-stack, which will also be denoted by $[\underline{X}/G]$.

In Algebraic Geometry, the notion of *Deligne–Mumford stack* plays an important role in studying moduli problems. Deligne–Mumford stacks are locally modelled on quotient stacks $[X/G]$, where X is an affine scheme and G a finite group acing on X. In the same spirit, Joyce [34, Definition 8.16] defines *Deligne–Mumford* C^∞*-stacks.*

Definition 3.2.4. A *Deligne–Mumford C^∞-stack* is a C^∞-stack $\mathcal{X}$ admitting an open cover $\{\mathcal{Y}_a : a \in A\}$, in the sense of Definition 3.2.2, where each $\mathcal{Y}_a$ is equivalent to a quotient stack $[\underline{U}_a/G]$ for $\underline{U}_a$ an affine C^∞-scheme and G_a a finite group. $\mathcal{X}$ is called *locally fair,locally finitely presented* if it admits an open cover with each $\underline{U}_a$ a fair, or finitely presented affine C^∞-scheme, respectively.

We write $\mathbf{DMC^\infty Sta}, \mathbf{DMC^\infty Sta}^{\mathrm{lf}}, \mathbf{DMC^\infty Sta}^{\mathrm{lg}}, \mathbf{DMC^\infty Sta}^{\mathrm{lfp}}$ for the full 2-subcategories of locally fair, locally good, locally finitely presented, and all Deligne–Mumford C^∞-stacks in $\mathbf{C^\infty Sta}$, respectively.

Proposition 3.2.5. *Let $\mathcal{X}$ be a C^∞-stack with $\mathcal{X} = [\underline{X}/G]$, where $\underline{X}$ a separated C^∞-scheme and G is finite. Then $\mathcal{X}$ is a separated Deligne–Mumford C^∞-stack.*

As the following example will show, the condition that $\underline{X}$ is separated cannot be weakened:

Example 3.2.6. Consider the non-separated C^∞-scheme $\underline{X} = (\underline{\mathbb{R}} \amalg \underline{\mathbb{R}})/\sim$, where $\sim$ is the equivalence relation identifying the two copies of $\underline{\mathbb{R}}$ along $(0, \infty)$. Consider furthermore the group $G = \mathbb{Z}_2$ acting on $\underline{X}$ by exchanging the two copies of $\underline{\mathbb{R}}$. The quotient C^∞-stack $\mathcal{X} = [\underline{X}/G]$ can be thought of as a copy of $\mathbb{R}$, with stabilizer group $\{1\}$ for $x \in (-\infty, 0]$ and $\mathbb{Z}_2$ for $x \in (0, \infty)$. Then as in [34, Example 8.19], $\mathcal{X}$ is not a Deligne–Mumford C^∞-stack.

3.2.1 The underlying topological space of a C^∞-stack

Given a C^∞-stack $\mathcal{X}$, we follow Joyce [34, §8.6] and explain briefly how one can associate a topological space $\mathcal{X}_{\mathrm{top}}$ to $\mathcal{X}$. It can then be shown, that for a given Deligne–Mumford C^∞-stack $\mathcal{X}$, its underlying topological space $\mathcal{X}_{\mathrm{top}}$ can be given the structure of a C^∞-scheme.

Definition 3.2.7. Let $\mathcal{X}$ be a C^∞-stack. Write $\underline{*}$ for the point $\operatorname{Spec}\mathbb{R}$ in $\mathbf{C^\infty Sch}$ and $\bar{\underline{*}}$ for the associated point in $\mathbf{C^\infty Sta}$. The *underlying topological space* ($\mathcal{X}_{\mathrm{top}}$, $\mathcal{T}_{\mathcal{X}_{\mathrm{top}}}$) of $\mathcal{X}$ is then defined as the set of 2-isomorphism classes $[x]$ of 1-morphisms $x : \bar{\underline{*}} \to \mathcal{X}$, denoted by $\mathcal{X}_{\mathrm{top}}$ and the topology

$$\mathcal{T}_{\mathcal{X}_{\mathrm{top}}} = \{\mathcal{U}_{\mathcal{X},\mathrm{top}} : i_{\mathcal{U}} : \mathcal{U} \to \mathcal{X} \text{ is an open } C^\infty\text{-substack in } \mathcal{X}_{\mathrm{top}}\},$$

where

$$\mathcal{U}_{\mathcal{X},\mathrm{top}} = \{[u \circ i_{\mathcal{U}}] \in \mathcal{X}_{\mathrm{top}} : u : \underline{*} \to \mathcal{U} \text{ is a 1-morphism}\} \subseteq \mathcal{X}_{\mathrm{top}}.$$

To see that $\mathcal{T}_{\mathcal{X}_{\mathrm{top}}}$ is indeed a topology, note first that taking $\mathcal{U} = \mathcal{X}$ or $\mathcal{U} = \emptyset$ gives $\mathcal{X}_{\mathrm{top}}, \emptyset \in \mathcal{T}_{\mathcal{X}_{\mathrm{top}}}$. Let now $i_{\mathcal{U}} : \mathcal{U} \to \mathcal{X}, i_{\mathcal{V}} : \mathcal{V} \to \mathcal{X}$ be open C^∞-substacks of $\mathcal{X}$. Then $\mathcal{W} := \mathcal{U} \times_{i_{\mathcal{U}},\mathcal{X},i_{\mathcal{V}}} \mathcal{V}$ is an open C^∞-substack of $\mathcal{X}$ satisfying $\mathcal{W}_{\mathcal{X},\mathrm{top}} = \mathcal{U}_{\mathcal{X},\mathrm{top}} \cap \mathcal{V}_{\mathcal{X},\mathrm{top}}$, which shows that $\mathcal{T}_{\mathcal{X}_{\mathrm{top}}}$ is closed under finite intersections. To see that $\mathcal{T}_{\mathcal{X}_{\mathrm{top}}}$ is closed under arbitrary unions, note that given a family of open C^∞-substacks in $\mathcal{X}$, $\{\mathcal{U}_a : a \in A\}$ for some index set A, each $\mathcal{U}_a$ is a subcategory of $\mathcal{X}$ and so the union $\mathcal{V} = \bigcup_{a \in A} \mathcal{U}_a$ is a subcategory of $\mathcal{X}$. This subcategory $\mathcal{V}$ can be shown to be a prestack and the associated stack $\hat{\mathcal{V}}$ turns out to be an open C^∞-substack of $\mathcal{X}$ satisfying $\hat{\mathcal{V}}_{\mathcal{X},\mathrm{top}} = \bigcup_{a \in A} \mathcal{U}_{a\mathcal{X},\mathrm{top}}$.

The underlying topological space $(\mathcal{X}_{\mathrm{top}}, \mathcal{T}_{\mathcal{X}_{\mathrm{top}}})$, or $\mathcal{X}_{\mathrm{top}}$ for short, has the following properties:

- Given a 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ of C^∞-stacks, there exists a natural continuous map $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ defined by $f_{\mathrm{top}}([x]) = [f \circ x]$.
- Given 1-morphism $f, g : \mathcal{X} \to \mathcal{Y}$ and a 2-isomorphism $\eta : f \Rightarrow g$ we have $f_{\mathrm{top}} = g_{\mathrm{top}}$.

Viewing the category of topological spaces as a 2-category with only identity 2-morphisms, we can define a 2-functor $F^{\mathbf{Top}}_{C^\infty\mathbf{Sta}} : C^\infty\mathbf{Sta} \to \mathbf{Top}$, by mapping $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{top}}, f \mapsto f_{\mathrm{top}}$ and any 2-morphism to the identity.

3.2.2 Quasicoherent sheaves on C^∞-stacks

In the following section we will recall the definition of quasicoherent sheaves on C^∞-stacks (as in [35, §C.2] and [34, §9]). The here presented material will later on in section 3.4.1 be extended to the notions of virtual quasicoherent sheaf and virtual vector bundle on C^∞-stacks.

We start by defining $\mathcal{O}_{\mathcal{X}}$-modules, (quasi)coherent sheaves and vector bundles on a Deligne–Mumford C^∞-stack and refer to [35, Definition C.12] or [34, Definition 9.1] for an in depth treatment of the subject.

Definition 3.2.8. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. We can define a category $\mathcal{C}_\mathcal{X}$ with objects being pairs $(\underline{U}, u)$, where $\underline{U}$ is a C^∞-scheme and $u : \overline{\underline{U}} \to \mathcal{X}$ is an étale 1-morphism, and morphisms being pairs $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$, where $\underline{f} : \underline{U} \to \underline{V}$ is an étale morphism of C^∞-schemes, and $\eta : u \Rightarrow v \circ \underline{f}$ is a 2-isomorphism. Define the composition $(\underline{g}, \zeta) \circ (\underline{f}, \eta)$ of two morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ and $(\underline{g}, \zeta) : (\underline{V}, v) \to (\underline{W}, w)$ in $\mathcal{C}_\mathcal{X}$ to be

$$(\underline{g} \circ \underline{f}, \theta) : (\underline{U}, u) \to (\underline{W}, w),$$

where θ is the composition of 2-morphisms across the following diagram

$$\begin{tikzcd}
\overline{\underline{U}} \arrow[dr, "\overline{\underline{f}}"] \arrow[drr, bend left, "u"] \arrow[dd, "\overline{\underline{g}\circ\underline{f}}"'] & \arrow[d, phantom, "\Swarrow\eta"] & \\
\arrow[r, phantom, "\Leftarrow\mathrm{id}"] & \overline{\underline{V}} \arrow[r, "v"] \arrow[dl, "\overline{\underline{g}}"] \arrow[d, phantom, "\Swarrow\zeta"] & \mathcal{X}. \\
\overline{\underline{W}} \arrow[urr, bend right, "w"'] & &
\end{tikzcd}$$

We can define a *sheaf of $\mathcal{O}_\mathcal{X}$-modules* $\mathcal{E}$, or just an $\mathcal{O}_\mathcal{X}$-module $\mathcal{E}$, to assign for all objects $(\underline{U}, u)$ in $\mathcal{C}_\mathcal{X}$ a sheaf of $\mathcal{O}_U$-modules $\mathcal{E}(\underline{U}, u)$ on $\underline{U} = (U, \mathcal{O}_U)$, and for all morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_\mathcal{X}$ an isomorphism of $\mathcal{O}_U$-modules $\mathcal{E}_{(\underline{f},\eta)} : \underline{f}^*(\mathcal{E}(\underline{V}, v)) \to \mathcal{E}(\underline{U}, u)$ such that for all $(\underline{f}, \eta), (\underline{g}, \zeta), (\underline{g} \circ \underline{f}, \theta)$ as above, the following diagram of isomorphisms of sheaves of $\mathcal{O}_U$-modules commutes:

$$\begin{tikzcd}
(\underline{g} \circ \underline{f})^*(\mathcal{E}(\underline{W}, w)) \arrow[rr, "\mathcal{E}_{(\underline{g}\circ\underline{f},\theta)}"] \arrow[dr, "I_{\underline{f},\underline{g}}(\mathcal{E}(\underline{W},w))"'] & & \mathcal{E}(\underline{U}, u). \\
& \underline{f}^*(\underline{g}^*(\mathcal{E}(\underline{W}, w))) \arrow[r, "\underline{f}^*(\mathcal{E}_{(\underline{g},\zeta)})"] & \underline{f}^*(\mathcal{E}(\underline{V}, v)) \arrow[u, "\mathcal{E}(\underline{f},\eta)"']
\end{tikzcd} \tag{3.1}$$

Here $I_{\underline{f},\underline{g}}(\mathcal{E}(\underline{W}, w))$ is a natural isomorphism of functors as in Remark 2.1.16.

We call $\phi : \mathcal{E} \to \mathcal{F}$ a *morphism of sheaves of $\mathcal{O}_\mathcal{X}$-modules* if it assigns a morphism of $\mathcal{O}_U$-modules $\phi(\underline{U}, u) : \mathcal{E}(\underline{U}, u) \to \mathcal{F}(\underline{U}, u)$ for each object $(\underline{U}, u)$ in $\mathcal{C}_\mathcal{X}$ such that for all morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_\mathcal{X}$ the following diagram commutes

$$\begin{tikzcd}
\underline{f}^*(\mathcal{E}(\underline{V}, v)) \arrow[r, "\mathcal{E}_{(\underline{f},\eta)}"] \arrow[d, "\underline{f}^*(\phi(\underline{V},v))"'] & \mathcal{E}(\underline{U}, u) \arrow[d, "\phi(\underline{U},u)"] \\
\underline{f}^*(\mathcal{F}(\underline{V}, v)) \arrow[r, "\mathcal{F}_{(\underline{f},\eta)}"] & \mathcal{F}(\underline{U}, u).
\end{tikzcd} \tag{3.2}$$

The sheaf of $\mathcal{O}_{\mathcal{X}}$-modules is called *quasicoherent*, or *coherent*, or a *vector bundle of rank* n, if $\mathcal{E}(\underline{U}, u)$ is quasicoherent, or coherent or a vector bundle of rank n for all objects $(\underline{U}, u)$ in $\mathcal{C}_{\mathcal{X}}$. The category of $\mathcal{O}_{\mathcal{X}}$-modules will be denoted by $\mathcal{O}_{\mathcal{X}}$-mod, the full subcategories of quasicoherent and coherent sheaves will be denoted by $\mathrm{qcoh}(\mathcal{X})$ and $\mathrm{coh}(\mathcal{X})$ respectively.

The following proposition (see [34, Proposition 9.3]) shows some nice properties of the categories defined above.

Proposition 3.2.9. *Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Then the category $\mathcal{O}_{\mathcal{X}}$-mod of $\mathcal{O}_{\mathcal{X}}$-modules is an abelian category and the full subcategory* $\mathrm{qcoh}(\mathcal{X})$ *of quasicoherent sheaves is closed under kernels, cokernels and extensions in $\mathcal{O}_{\mathcal{X}}$-mod, and so is itself an abelian category. Moreover the category of coherent sheaves* $\mathrm{coh}(\mathcal{X})$ *is closed under cokernels and extensions in $\mathcal{O}_{\mathcal{X}}$-mod, but may not be closed under kernels in $\mathcal{O}_{\mathcal{X}}$-mod, and so it may not be abelian. In the case of $\mathcal{X}$ being a locally fair Deligne–Mumford C^∞-stack, the categories $\mathcal{O}_{\mathcal{X}}$-mod and* $\mathrm{qcoh}(\mathcal{X})$ *coincide, that is we have* $\mathrm{qcoh}(\mathcal{X}) = \mathcal{O}_{\mathcal{X}}$*-mod.*

3.2.3 Sheaves of abelian groups and C^∞-rings on C^∞-stacks

In this section we briefly review material on *sheaves of abelian groups and sheaves of C^∞-rings* on Deligne–Mumford C^∞-stacks, as in [35, §C.3]. This section can be seen as an extension of the previous section 3.2.2, and we start with the following definition, where we use the same notation as in §3.2.2.

Definition 3.2.10. Given the data of Definition 3.2.8, define a *sheaf of abelian groups* $\mathcal{E}$ on $\mathcal{X}$ which assigns a sheaf of abelian groups $\mathcal{E}(\underline{U}, u)$ on U for all objects $(\underline{U}, u)$ in $\mathcal{C}_{\mathcal{X}}$, and an isomorphism of sheaves of abelian group $\mathcal{E}_{(\underline{f},\eta)} : f^{-1}(\mathcal{E}(\underline{V}, v)) \to \mathcal{E}(\underline{U}, u)$ for all morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_{\mathcal{X}}$ with $\underline{f} = (f, f^\sharp)$ such that for all morphism $(\underline{f}, \eta), (\underline{g}, \zeta), (\underline{g} \circ \underline{f}, \theta)$ the analogue of equation (3.1) commutes:

$$\begin{tikzcd}
(g\circ f)^{-1}(\mathcal{E}(\underline{W},w)) \arrow[rr, "\mathcal{E}_{(\underline{g}\circ\underline{f},\theta)}"] \arrow[dr, "I_{f,g}(\mathcal{E}(\underline{W},w))"'] & & \mathcal{E}(\underline{U},u). \\
& f^{-1}(g^{-1}(\mathcal{E}(\underline{W},w))) \arrow[r, "f^{-1}(\mathcal{E}_{(\underline{g},\zeta)})"] & f^{-1}(\mathcal{E}(\underline{V},v)) \arrow[u, "\mathcal{E}_{(\underline{f},\eta)}"']
\end{tikzcd} \tag{3.3}$$

Here $I_{f,g}(\mathcal{E}(\underline{W},w))$ is the natural isomorphism as $I_{\underline{f},\underline{g}}(\mathcal{E})$ in Definition 3.2.8 and f^{-1} denotes the pullbacks for sheaves of abelian group.

Given two sheaves of abelian groups $\mathcal{E}$ and $\mathcal{F}$, a *morphism of sheaves of abelian groups* $\phi : \mathcal{E} \to\ F$, assigns a morphism of sheaves of abelian groups $\phi(\underline{U},u)$: $\mathcal{E}(\underline{U},u) \to \mathcal{F}(\underline{U},u)$ on U for each object $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$, such that for all morphism $(\underline{f},\eta) : (\underline{U},u) \to (\underline{V},v)$ in $\mathcal{C}_{\mathcal{X}}$ the analogue of equation (3.2) commutes:

$$\begin{tikzcd} f^{-1}(\mathcal{E}(\underline{V},v)) \arrow[r, "\mathcal{E}_{(\underline{f},\eta)}"] \arrow[d, "f^{-1}(\phi(\underline{V},v))"'] & \mathcal{E}(\underline{U},u) \arrow[d, "\phi(\underline{U},u)"] \\ f^{-1}(\mathcal{F}(\underline{V},v)) \arrow[r, "\mathcal{F}_{(\underline{f},\eta)}"] & \mathcal{F}(\underline{U},u). \end{tikzcd} \tag{3.4}$$

Sheaves of C^∞-rings on $\mathcal{X}$ and their morphisms are defined in the exact same way, where sheaves of abelian groups are replaced by sheaves of C^∞-rings.

Remark 3.2.11. Any quasicoherent sheaf $\mathcal{E} \in \mathrm{qcoh}(\underline{X})$ on a C^∞-scheme $\underline{X}$ has an underlying sheaf of abelian groups, by regarding $\mathcal{E}(U)$ as a abelian group for open subsets $U \subseteq X$ and forgetting about its $\mathcal{O}_{\mathcal{X}}(U)$-module structure. In the same way, any quasicoherent sheaf $\mathcal{E}$ on a Deligne–Mumford C^∞-stack $\mathcal{X}$ has an underlying sheaf of abelian groups, which in the following will also be denoted by $\mathcal{E}$. The only subtlety in the Deligne–Mumford C^∞-stack case, is that $\mathcal{E}$ being a quasicoherent sheaf requires $\mathcal{E}_{(\underline{f},\eta)} : \underline{f}^*(\mathcal{E}(\underline{V},v)) \to \mathcal{E}(\underline{U},u)$, but in the case where $\mathcal{E}$ is a sheaf of abelian groups, we need $\mathcal{E}_{(\underline{f},\eta)} : f^{-1}(\mathcal{E}(\underline{V},v)) \to \mathcal{E}(\underline{U},u)$. But $\underline{f}^*(\mathcal{E}(\underline{V},v))$ and $f^{-1}(\mathcal{E}(\underline{V},v))$ can be related by the following morphism :

$$\begin{aligned}(\mathrm{id} \otimes f^\sharp) : f^{-1}(\mathcal{E}(\underline{V},v)) &= f^{-1}(\mathcal{E}(\underline{V},v)) \otimes_{f^{-1}(\mathcal{O}_V)} f^{-1}(\mathcal{O}_V) \\ &\to f^{-1}(\mathcal{E}(\underline{V},v)) \otimes_{f^{-1}(\mathcal{O}_V)} \mathcal{O}_U = \underline{f}^*(\mathcal{E}(\underline{V},v)),\end{aligned}$$

where the tensor products use the fact that we have an $\mathcal{O}_V$-module structure on $\mathcal{E}(\underline{V},v) \in \mathrm{qcoh}(\underline{V})$.

The following example will define the *structure sheaf* $\mathcal{O}_{\mathcal{X}}$ on a Deligne–Mumford C^∞-stack, and can be found in [35, Example C.23].

Example 3.2.12. Given a Deligne–Mumford C^∞-stack $\mathcal{X}$, the *structure sheaf* $\mathcal{O}_\mathcal{X}$ is a sheaf of C^∞-rings on $\mathcal{X}$, defined by $\mathcal{O}_\mathcal{X}(\underline{U},u) = \mathcal{O}_U$ for all objects $(\underline{U},u)$ in $\mathcal{C}_\mathcal{X}$ with $\underline{U} = (U,\mathcal{O}_U)$, and $(\mathcal{O}_\mathcal{X})_{\underline{f},\eta} = f^\sharp : f^{-1}(\mathcal{O}_V) \to \mathcal{O}_U$ for all morphisms $(\underline{f},\eta) : (\underline{U},u) \to (\underline{V},v)$ in $\mathcal{C}_\mathcal{X}$ with $\underline{f} = (f,f^\sharp)$.

3.2.4 Effective Deligne–Mumford C^∞-stacks

In this section we recall basic definitions and properties of *effective Deligne–Mumford C^∞-stacks*. We refer to [35, §C.5] for a more detailed discussion of the subject.

Definition 3.2.13. A Deligne–Mumford C^∞-stack $\mathcal{X}$ is called *effective* if whenever we have $[x] \in \mathcal{X}_{\rm top}$ and $\mathcal{X}$ is near $[x]$ locally modelled on a quotient C^∞-stack $[\underline{U}/G]$, where $G = \mathrm{Iso}_\mathcal{X}([x])$, then G acts effectively on $\underline{U}$ near u, where $u \in \underline{U}$ is fixed by G. So for each $1 \neq \gamma \in G$ and the G-action $\underline{r} : G \to \mathrm{Aut}(\underline{U})$ we have $\underline{r}(\gamma) \not\equiv \underline{\mathrm{id}}_U$ near u in $\underline{U}$.

The C^∞-scheme $\underline{U}$ is determined up to G-equivariant isomorphism by $\mathcal{X}, [x]$ locally near u, which implies that in order to test $\mathcal{X}$ being effective, it is enough to consider one choice $[\underline{U}/G]$ for each $[x] \in \mathcal{X}_{\rm top}$.

A quotient C^∞-stack $[\underline{X}/G]$ is *effective* if and only if the action $\underline{r} : G \to \mathrm{Aut}(\underline{X})$ of G on $\underline{X}$ is *locally effective*, which means that for each $1 \neq \gamma \in G$ we have $r(\gamma)|_{\underline{U}} \not\equiv \underline{\mathrm{id}}_U$ for every open C^∞-subscheme $\emptyset \neq \underline{U} \subseteq \underline{X}$.

Note that any Deligne–Mumford C^∞-stack $\mathcal{X}$ that is a C^∞-scheme is automatically effective. Examples of non-effective Deligne–Mumford C^∞-stacks are for instance quotients of the form $\underline{*}/\mathcal{G}$ for any nontrivial group $G \neq \{1\}$.

The following proposition summarizes important uniqueness properties of 2-morphisms of effective Deligne–Mumford C^∞-stacks. (See [35, Proposition C.32] for a proof.)

Proposition 3.2.14. *Let $\mathcal{X}, \mathcal{Y}$ be Deligne–Mumford C^∞-stacks and $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms between $\mathcal{X}$ and $\mathcal{Y}$. If any one of the following conditions hold:*

(a) $\mathcal{X}$ is effective and f is an embedding of C^∞-stacks; (note that this implies that $f_* : \mathrm{Iso}_\mathcal{X}([x]) \to \mathrm{Iso}_\mathcal{Y}(f_{top}([x]))$ is an isomorphism for each $[x] \in \mathcal{X}_{top}$.)

(b) *$\mathcal{Y}$ is effective and $f^\sharp : f^{-1}(\mathcal{O}_\mathcal{Y}) \to \mathcal{O}_\mathcal{X}$ injective; (for example f is an étale morphism, an equivalence, or a submersion of orbifolds)*

(c) *$\mathcal{Y}$ is a C^∞-scheme;*

then there exists at most one 2-morphism $\eta : f \Rightarrow g$.

3.2.5 Orbifold strata of C^∞-stacks

We will in this section describe orbifold strata of C^∞-stacks and refer once again to [35, §C.8] for a detailed discussion of the subject.

Given a Deligne–Mumford C^∞-stack $\mathcal{X}$, with topological space $\mathcal{X}_{\mathrm{top}}$, each point $[x] \in \mathcal{X}_{\mathrm{top}}$ has an orbifold group $\mathrm{Iso}_\mathcal{X}([x])$, that is a finite group defined up to isomorphism. For each finite group Γ, we will write $\tilde{\mathcal{X}}^\Gamma_{\mathrm{o,top}} = \{[x] \in \mathcal{X}_{\mathrm{top}} : \mathrm{Iso}_\mathcal{X}([x]) \cong \Gamma\}$. Note that $\tilde{\mathcal{X}}^\Gamma_{\mathrm{o,top}}$ is a locally closed subset of $\mathcal{X}_{\mathrm{top}}$ coming from a locally closed C^∞-substack $\tilde{\mathcal{X}}^\Gamma_{\mathrm{o}}$ of $\mathcal{X}$ with inclusion $\tilde{\mathcal{O}}^\Gamma_{\mathrm{o}}(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma_{\mathrm{o}} \to \mathcal{X}$. Furthermore we get the following decomposition of $\mathcal{X}_{\mathrm{top}}$:

$$\mathcal{X}_{\mathrm{top}} = \coprod_{\text{isomorphism classes of finite groups } \Gamma} \tilde{\mathcal{X}}^\Gamma_{\mathrm{o,top}}. \tag{3.5}$$

For each Γ, the closure $\overline{\tilde{\mathcal{X}}}^\Gamma_{\mathrm{o,top}}$ of $\tilde{\mathcal{X}}^\Gamma_{\mathrm{o,top}}$ in $\mathcal{X}_{\mathrm{top}}$ can be shown to satisfy

$$\overline{\tilde{\mathcal{X}}}^\Gamma_{\mathrm{o,top}} \subseteq \coprod_{\substack{\text{isomorphism classes of finite groups } \Delta: \\ \Gamma \text{ is isomorphic to a subgroup of } \Delta}} \tilde{\mathcal{X}}^\Delta_{\mathrm{o,top}}.$$

Therefore (3.5) is a stratification of $\mathcal{X}_{\mathrm{top}}$, and we will call $\tilde{\mathcal{X}}^\Gamma_{\mathrm{o}}$ *orbifold strata* of $\mathcal{X}$.

There exist six variations of this idea, the Deligne–Mumford C^∞-stacks $\mathcal{X}^\Gamma$, $\tilde{\mathcal{X}}^\Gamma$, $\hat{\mathcal{X}}^\Gamma$ and their open C^∞-substacks $\mathcal{X}^\Gamma_{\mathrm{o}} \subseteq \mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma_{\mathrm{o}} \subseteq \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma_{\mathrm{o}} \subseteq \hat{\mathcal{X}}^\Gamma$. The geometric points and orbifold groups of these strata are given by:

(i) Points of $\mathcal{X}^\Gamma$ are isomorphism classes $[x, \rho]$, with $[x] \in \mathcal{X}_{\mathrm{top}}$ and $\rho : \Gamma \to \mathrm{Iso}_\mathcal{X}([x])$ is an injective morphism. The orbifold group $\mathrm{Iso}_{\mathcal{X}^\Gamma}([x, \rho])$ is given by the centralizer of $\rho(\Gamma)$ in $\mathrm{Iso}_\mathcal{X}([x])$. The points of $\mathcal{X}^\Gamma_{\mathrm{o}}$ are pairs $[x, \rho]$ as above, where ρ is an isomorphism, and $\mathrm{Iso}_{\mathcal{X}^\Gamma_{\mathrm{o}}}([x, \rho]) \cong C(\Gamma)$, where $C(\Gamma)$ is the centre of Γ.

(ii) Points of $\tilde{\mathcal{X}}^\Gamma$ are pairs $[x,\Delta]$, with $[x]\in\mathcal{X}_{\mathrm{top}}$ and $\Delta\subseteq\mathrm{Iso}_{\mathcal{X}}([x])$ is a subgroup of $\mathrm{Iso}_{\mathcal{X}}([x])$, isomorphic to Γ. The orbifold group $\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])$ is given by the normalizer of Δ in $\mathrm{Iso}_{\mathcal{X}}([x])$. The points of $\tilde{\mathcal{X}}^\Gamma_{\mathrm{o}}$ are pairs $[x,\Delta]$ as above, where $\Delta=\mathrm{Iso}_{\mathcal{X}}([x])$, and $\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma_{\mathrm{o}}}([x,\Delta])\cong\{1\}$.

(iii) Points $[x,\Delta]$ of $\hat{\mathcal{X}}^\Gamma$ and $\hat{\mathcal{X}}^\Gamma_{\mathrm{o}}$ are the same as for $\tilde{\mathcal{X}}^\Gamma$ and $\tilde{\mathcal{X}}^\Gamma_{\mathrm{o}}$, but with orbifold groups $\mathrm{Iso}_{\hat{\mathcal{X}}^\Gamma}([x,\Delta])\cong\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x,\Delta])/\Delta$ and $\mathrm{Iso}_{\hat{\mathcal{X}}^\Gamma_{\mathrm{o}}}([x,\Delta])\cong\{1\}$.

The 1-morphisms $\mathcal{O}^\Gamma(\mathcal{X}),\ldots,\hat{\Pi}^\Gamma_{\mathrm{o}}(\mathcal{X})$ between the different strata, form a strictly commutative diagram as follows:

$$\begin{tikzcd}
\mathcal{X}_o^\Gamma \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[rr, "\tilde{\Pi}_o^\Gamma(\mathcal{X})"] \arrow[dd, "\subset"'] \arrow[dr, "O_o^\Gamma(\mathcal{X})"] & & \tilde{\mathcal{X}}_o^\Gamma \arrow[rr, "\hat{\Pi}_o^\Gamma(\mathcal{X})"] \arrow[dd, "\subset"] \arrow[dl, "\tilde{O}_o^\Gamma(\mathcal{X})"'] & & \hat{\mathcal{X}}_o^\Gamma \cong \overline{\underline{X}}_o^{\Gamma} \arrow[dd, "\subset"] \\
& \mathcal{X} & & & \\
\mathcal{X}^\Gamma \arrow[loop left, "\mathrm{Aut}(\Gamma)"] \arrow[ur, "O^\Gamma(\mathcal{X})"] \arrow[rr, "\tilde{\Pi}^\Gamma(\mathcal{X})"] & & \tilde{\mathcal{X}}^\Gamma \arrow[ul, "\tilde{O}^\Gamma(\mathcal{X})"'] \arrow[rr, "\hat{\Pi}^\Gamma(\mathcal{X})"] & & \hat{\mathcal{X}}^\Gamma.
\end{tikzcd} \tag{3.6}$$

Here the columns are inclusions of open C^∞-substacks and the automorphism group $\mathrm{Aut}(\Gamma)$ acts on $\mathcal{X}^\Gamma,\mathcal{X}^\Gamma_o$ such that $\tilde{\mathcal{X}}^\Gamma\simeq[\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$ and $\tilde{\mathcal{X}}_o^{\Gamma}\simeq[\mathcal{X}^\Gamma_o/\mathrm{Aut}(\Gamma)]$.

Definition 3.2.15. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, and Γ a finite group. Then the Deligne–Mumford C^∞-stack $\mathcal{X}^\Gamma$ can be defined as follows:

First recall, that $\mathcal{X}$ being a stack on the site $(C^\infty\mathbf{Sch},\mathcal{J})$ means that $\mathcal{X}$ is a category with functor $p_{\mathcal{X}}:\mathcal{X}\to C^\infty\mathbf{Sch}$ satisfying various conditions. In order to define $\mathcal{X}^\Gamma$ we have therefore to define a category $\mathcal{X}^\Gamma$ and a functor $p_{\mathcal{X}^\Gamma}:\mathcal{X}^\Gamma\to C^\infty\mathbf{Sch}$.

The objects of $\mathcal{X}^\Gamma$ are pairs (A,ρ) satisfying the following three conditions:

(1) A is an object in $\mathcal{X}$, with $p_{\mathcal{X}}(A)=\underline{U}$ for some C^∞-scheme $\underline{U}\in C^\infty\mathbf{Sch}$.

(2) $\rho:\Gamma\to\mathrm{Aut}(A)$ is a group morphism, with $\mathrm{Aut}(A)$ denoting the isomorphism group of A. That is the elements in $\mathrm{Aut}(A)$ are given by isomorphism $a: A\to A$ in $\mathcal{X}$, satisfying $p_X\circ\rho(\gamma)=\underline{\mathrm{id}}_{\underline{U}}$ for all $\gamma\in\Gamma$.

(3) Consider $u \in \underline{U}$ with corresponding morphism $\underline{u} : \underline{*} \to \underline{U}$ in $C^\infty\mathbf{Sch}$. As in [34, Definition C. 45], there exists a morphism $a_u : A_u \to A$ in $\mathcal{X}$ with $p_{\mathcal{X}}(A_u) = \underline{*}$ and $p_{\mathcal{X}}(a_u) = \underline{u}$ where A_u is unique up to isomorphism. Given such data A_u, a_u, [34, Definition 7.2] implies furthermore that for each $\gamma \in \Gamma$ there exists a group morphism $\rho_u : \Gamma \to \mathrm{Aut}(A_u)$, which we require to be injective for all $u \in \underline{U}$. This condition is independent of the choice of A_u, a_u.

The *morphisms* $c : (A, \rho) \to (B, \sigma)$ in $\mathcal{X}^\Gamma$ are defined to be morphisms $c : A \to B$ in $\mathcal{X}$, satisfying $\sigma(\gamma) \circ c = c \circ \rho(\gamma)$ for all $\gamma \in \Gamma$. Given two morphisms $c : (A, \rho) \to (B, \sigma), d : (B, \sigma) \to (C, \tau)$ define the composition $d \circ c : (A, \rho) \to (C, \tau)$ to be the composition $c \circ d : A \to C$ in $\mathcal{X}$. For each $(A, \rho) \in \mathcal{X}^\Gamma$, the *identity morphism* $\mathrm{id}_{(A,\rho)} : (A, \rho) \to (A, \rho)$ in $\mathcal{X}^\Gamma$ is defined to be $\mathrm{id}_A : A \to A$ in $\mathcal{X}$. The functor $p_{\mathcal{X}^\Gamma} : \mathcal{X}^\Gamma \to C^\infty\mathbf{Sch}$ is defined by $p_{\mathcal{X}^\Gamma} : (A, \rho) \mapsto \underline{U} = p_{\mathcal{X}}(A)$ and $p_{\mathcal{X}^\Gamma} : c \mapsto p_{\mathcal{X}}(c)$.

The full subcategory of objects (A, ρ) in $\mathcal{X}^\Gamma$ such that $\rho_u : \Gamma \to \mathrm{Aut}(A_u)$ is an isomorphism for all $u \in \underline{U}$ is denoted by $\mathcal{X}^\Gamma_{\circ}$ with functor $p_{\mathcal{X}^\Gamma_\circ} = p_{\mathcal{X}}|_{\mathcal{X}^\Gamma_\circ} : \mathcal{X}^\Gamma_\circ \to C^\infty\mathbf{Sch}$. As [35, Theorem C.45] shows, $\mathcal{X}^\Gamma$ is in fact a Deligne–Mumford C^∞-stack and $\mathcal{X}^\Gamma_\circ$ an open C^∞-substack in $\mathcal{X}^\Gamma$.

3.2.6 Orbifolds as C^∞-stacks

We follow here closely Joyce [35, §8.2] in associating the classical theory of orbifolds to the theory of C^∞-stacks.

Definition 3.2.16. We call a C^∞-stack $\mathcal{X}$ an *orbifold (without boundary)*, if it is equivalent to a groupoid stack $[\underline{V} \rightrightarrows \underline{U}]$ for some groupoid $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{i}, \underline{m})$ in $C^\infty\mathbf{Sch}$ which is the image of a groupoid (U, V, s, t, u, i, m) in $\mathbf{Man}$ under $F^{C^\infty\mathbf{Sch}}_{\mathbf{Man}}$. Here $s : V \to U$ should be an étale smooth map and $s \times t : V \to U \times U$ is proper and smooth. So in other words, $\mathcal{X}$ is the C^∞-stack associated to a proper étale Lie groupoid in $\mathbf{Man}$, and in particular every orbifold $\mathcal{X}$ is a separable, second countable, locally compact, paracompact, locally finitely presented Deligne–Mumford C^∞-stack.

Another definition, which is more in the spirit of Satake and Thurston is the following:

Definition 3.2.17. A separable, second countable Deligne–Mumford C^∞-stack $\mathcal{X}$ is called an *orbifold* of dimension n, if for every $[x] \in \mathcal{X}_{\mathrm{top}}$ there exists a linear action $G = \mathrm{Iso}_{\mathcal{X}}([x])$ on $\mathbb{R}^n$, a G-invariant open neighbourhood $0 \in U \subseteq \mathbb{R}^n$ and a 1-morphism $i : [\underline{U}/G] \to \mathcal{X}$, which is an equivalence with an open neighbourhood $\mathcal{U} \subseteq \mathcal{X}$ of $[x]$ in $\mathcal{X}$, with $i_{\mathrm{top}}([0]) = [x]$. Here $\underline{U} = F_{\mathbf{Man}}^{C^\infty\mathbf{Sch}}(U)$.

Definition 3.2.17 states that an orbifold is a Deligne–Mumford C^∞-stack $\mathcal{X}$ which is locally modelled on $\mathbb{R}^n/G$ for some finite group G. In contrast to the "classical" theory of orbifolds, it follows immediately that orbifolds form a 2-category **Orb**, which is a full 2-subcategory in $\mathbf{DMC^\infty Sta}$. The 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ in **Orb** will also be called *smooth maps of orbifolds* and by concatenating $F_{\mathbf{Man}}^{C^\infty\mathbf{Sch}}$ with $F_{C^\infty\mathbf{Sch}}^{C^\infty\mathbf{Sta}}$ we get a functor $F_{\mathbf{Man}}^{\mathbf{Orb}} : \mathbf{Man} \to \mathbf{Orb}$ in the obvious way.

As in Joyce [35, Theorem 8.2] it can be shown that the above defined 2-category **Orb** is (weakly) equivalent to various (weak) 2-categories of orbifolds studied by other recent authors like [44] or [39].

Many differential geometric constructions and ideas, like submanifolds, transverse fibre products and orientations, extend nicely to the orbifold world and will be used in the further without specifically mentioning them. Other notions like immersion, submersion or embeddings need to be slightly adjusted, as the following definition shows. (Compare [35, Definition 8.3].)

Definition 3.2.18. A smooth map $f : \mathcal{X} \to \mathcal{Y}$ between orbifolds $\mathcal{X}$ and $\mathcal{Y}$ is called

(a) *representable*, if $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is an injective morphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, i.e. f acts injectively on orbifold groups;

(b) *immersion*, if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is a surjective morphism of vector bundles, that is has a right inverse in $\mathrm{qcoh}(\mathcal{X})$;

(c) *embedding*, if it is an immersion and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is not just an injective morphism, but an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism on its image;

(d) *submersion*, if $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is an injective morphism of vector bundles, that is has a left inverse;

(e) *étale*, if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is an isomorphism.

Note that in contrast to the other notions, submersions between orbifold need not be representable. Note furthermore, that there are equivalent definitions of representable and étale morphism as follows:

(a') f is called *representable*, if it is a representable 1-morphism of C^∞-stacks, that is whenever $\underline{V}$ is a C^∞-scheme and $\Pi : \underline{V} \to \mathcal{Y}$ is a 1-morphism, then the fibre product $\mathcal{X} \times_{f,\mathcal{Y},\Pi} \overline{\underline{V}}$ in $C^\infty\mathbf{Sta}$ is a C^∞-scheme.

(e') f is called *étale* if it is étale as a 1-morphism of C^∞-stack.

3.2.7 Orbifold strata and effective orbifolds

As shown in Joyce [35, §C.8], there are six different variations of the idea of *orbifold strata* of Deligne–Mumford C^∞-stack. For each finite group Γ, he defined C^∞-stacks $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$, and open C^∞-substacks $\mathcal{X}_o^\Gamma \subseteq \mathcal{X}^\Gamma, \tilde{\mathcal{X}}_o^\Gamma \subseteq \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_o^\Gamma \subseteq \hat{\mathcal{X}}_o^\Gamma$. Note that as $\hat{\mathcal{X}}_o^\Gamma$ is a C^∞-scheme, we get $\hat{\mathcal{X}}_o^\Gamma \simeq \underline{\hat{\mathcal{X}}_o^\Gamma}$.

We want now to define these strata for orbifolds. The geometric points and orbifold groups of $\mathcal{X}^\Gamma, \dots, \hat{\mathcal{X}}_o^\Gamma$ are given by:

(i) Geometric points of $\mathcal{X}^\Gamma$ are isomorphism classes $[x, \rho]$, where $[x] \in \mathcal{X}_{\mathrm{top}}$ and $\rho : \Gamma \to \mathrm{Iso}_{\mathcal{X}}([x])$ is an injective morphism. Moreover we require $\mathrm{Iso}_{\mathcal{X}^\Gamma}([x, \rho])$ to be the centralizer of $\rho(\Gamma)$ in $\mathrm{Iso}_{\mathcal{X}}([x])$. Points of $\mathcal{X}_o^\Gamma \subseteq \mathcal{X}^\Gamma$ are given by isomorphism classes $[x, \rho]$, where ρ is an isomorphism and $\mathrm{Iso}_{\mathcal{X}_o^\Gamma}([x, \rho]) \cong C(\Gamma)$, with $C(\Gamma)$ being the centre of Γ.

(ii) Geometric points of $\tilde{\mathcal{X}}^\Gamma$ are pairs $[x, \Delta]$, where $[x] \in \mathcal{X}_{\mathrm{top}}$ and $\Delta \subseteq \mathrm{Iso}_{\mathcal{X}}([x])$ is a subgroup of $\mathrm{Iso}_{\mathcal{X}}([x])$ isomorphic to Γ. Moreover we require $\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x, \Delta])$ to be the normalizer of Δ in $\mathrm{Iso}_{\mathcal{X}}([x])$. Points of $\tilde{\mathcal{X}}_o^\Gamma \subseteq \tilde{\mathcal{X}}^\Gamma$ are given by pairs $[x, \Delta]$, where $\Delta = \mathrm{Iso}_{\mathcal{X}}([x])$ and $\mathrm{Iso}_{\tilde{\mathcal{X}}_o^\Gamma}([x, \Delta]) \cong \Gamma$.

(iii) Geometric points $[x, \Delta]$ of $\hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_o^\Gamma$ are the same as for $\tilde{\mathcal{X}}^\Gamma, \tilde{\mathcal{X}}_o^\Gamma$, except the orbifold groups are given by $\mathrm{Iso}_{\hat{\mathcal{X}}}([x, \Delta]) \cong \mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x, \Delta])/\Delta$ and $\mathrm{Iso}_{\hat{\mathcal{X}}_o^\Gamma}([x, \Delta]) \cong \{1\}$.

As in 3.2.5, there exist 1-morphisms $O^\Gamma(\mathcal{X}), \ldots, \hat{\Pi}^\Gamma_o(\mathcal{X})$ forming the strictly commutative diagram (3.6). (Compare [35, §8.4].)

Definition 3.2.19. Let Γ be a finite group. A representation (V, ρ) of Γ, where V is a finite-dimensional real vector space and $\rho : \Gamma \to \mathrm{Aut}(V)$ a group morphism, is called *nontrivial* if $V^{\rho(\Gamma)} = \{0\}$. The abelian category of nontrivial Γ-representations (V, ρ) will be denoted by $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$, and its Grothendieck group by $\Lambda^\Gamma := K_0(\mathrm{Rep}_{\mathrm{nt}}(\Gamma))$. Define the *positive cone* Λ^Γ_+ of Λ^Γ by $\Lambda^\Gamma_+ = \{[V, \rho] : (V, \rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)\} \subseteq \Gamma^\Gamma$.

By an elementary result in representation theory, Γ has, up to isomorphism, finitely many irreducible representations. Denoting some choices of representations in these isomorphism classes by $R_0, R_1, \ldots, R_k$, with R_0 being the trivial irreducible representation and $R_1, \ldots, R_k$ being nontrivial irreducible representations, Λ^Γ and Λ^Γ_+ can be described as follows:

Λ^Γ is the freely generated group over $\mathbb{Z}$ by $[R_1], \ldots, [R_k]$, and Λ^Γ_+ the subgroup generated over $\mathbb{Z}_{\geq 0}$. So in other words

$$\Lambda^\Gamma_+ = \{a_1[R_1] + \cdots + a_k[R_k]; a_1, \ldots, a_k \in \mathbb{Z}\}, \quad \text{and} \tag{3.7}$$

$$\Lambda^\Gamma_+ = \{a_1[R_1] + \cdots + a_k[R_k]; a_1, \ldots, a_k \in \mathbb{Z}_{\geq 0}\} \subseteq \Lambda^\Gamma, \tag{3.8}$$

and therefore $\Lambda^\Gamma \cong \mathbb{Z}^k$ and $\Lambda^\Gamma_+ \cong \mathbb{Z}^k_{\geq 0}$.

The dimension of Λ^Γ can be defined by the group morphism $\dim : \Lambda^\Gamma \to \mathbb{Z}, a_1[R_1] + \cdots + a_k[R_k] \mapsto a_1 \dim R_1 + \cdots + a_k \dim R_k$. Then $\dim : [(V, \rho)] \mapsto \dim V$ and $\dim(\Lambda^\Gamma_+) \subseteq \mathbb{Z}_{\geq 0}$.

3.2.8 Effective orbifolds

In section 3.2.4 we followed Joyce ([35, §C.5]) in defining *effective* Deligne–Mumford C^∞-stacks. Since orbifolds can be seen as examples of Deligne–Mumford C^∞-stacks (section 3.2.6), we get the notion of *effective orbifold*. The following proposition due to Joyce ([35, Proposition 8.13]) characterizes effective orbifolds in different ways.

Proposition 3.2.20. *An orbifold $\mathcal{X}$ is* effective*, if any of the following equivalent conditions hold:*

(i) $\mathcal{X}$ *is locally modelled near each* $[x] \in \mathcal{X}_{top}$ *on* $\mathbb{R}^n/G$*, where* G *acts effectively on* R^n*. Here an action is called effective, if every* $1 \neq \gamma \in G$ *acts nontrivially on* $\mathbb{R}^n$*.*

(ii) *Generic points* $[x] \in \mathcal{X}_{top}$ *have trivial stabilizer group* $\mathrm{Iso}_{\mathcal{X}}([x]) = \{1\}$*.*

(iii) *Whenever* $\Gamma \neq \{1\}$ *is a nontrivial finite group and* $\lambda \in \Lambda_+^\Gamma$*, with* $\lambda \neq [R]$ *for* R *an effective representation of* Γ*, then the orbifold stratum* $\mathcal{X}^{\Gamma,\lambda} = \emptyset$ *is empty.*

(iv) *Whenever* $\Gamma \neq \{1\}$ *is a nontrivial finite group, then the orbifold stratum* $\mathcal{X}^{\Gamma,0} = \emptyset$ *is empty.*

Given effective orbifolds $\mathcal{X}, \mathcal{Y}$ and 1-morphism $f.g : \mathcal{X} \to \mathcal{Y}$, the following proposition (compare [35, Proposition 8.14]) gives criterions when there exists at most one 2-morphism $\eta : f \Rightarrow g$.

Proposition 3.2.21. *Let* $f, g : \mathcal{X} \to \mathcal{Y}$ *be* 1*-morphisms between effective orbifolds* $\mathcal{X}, \mathcal{Y}$*. Then there exists at most one* 2*-morphisms* $\eta : f \Rightarrow g$*, if one of the following conditions is satisfied:*

(i) f *is an embedding.*

(ii) f *is a submersion.*

(iii) $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{top}([x]))$ *is surjective for all* $[x] \in \mathcal{X}_{top}$*.*

(iv) $\mathrm{Iso}_{\mathcal{Y}}(f_{top}([x])) \cong \{1\}$ *for generic* $[x] \in \mathcal{X}_{top}$*.*

(v) $\mathcal{Y}$ *is a manifold.*

Note that effective orbifolds play an important role in questions of integrality in homology and cohomology. Consider an oriented, n-dimensional orbifold $\mathcal{X}$. The *fundamental class* $[\mathcal{X}]$ of an arbitrary orbifold $\mathcal{X}$ is naturally defined as an element in $H_n(\mathcal{X}_{\mathrm{top}}, \mathbb{Q})$, as we get for each point $[x] \in \mathcal{X}_{\mathrm{top}}$ a rational "weight" contribution $1/|\,\mathrm{Iso}_{\mathcal{X}}(x)|$. If however, $\mathcal{X}$ is an effective oriented n-orbifold, the fundamental class $[\mathcal{X}]$ is actually defined in $H_n(\mathcal{X}_{\mathrm{top}}, \mathbb{Z})$.

3.2.9 Vector bundles on orbifolds

As defined in section 3.2.2, a vector bundle $\mathcal{E}$ on an orbifold $\mathcal{X}$ is a special kind of quasicoherent sheaf on $\mathcal{X}$.

A *smooth section* s of a vector bundle $\mathcal{E}$ on an orbifold $\mathcal{X}$ is a morphism $s : \mathcal{O}_{\mathcal{X}} \to \mathcal{E}$ in the category $\mathrm{qcoh}(\mathcal{X})$. The vector space of smooth sections will be denoted by $C^\infty(\mathcal{E})$. As in the manifold case, any morphism $\phi : \mathcal{E} \to \mathcal{F}$ of vector bundles on $\mathcal{X}$, induces a linear map $\phi_* : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{F})$ by sending $s \mapsto \phi \circ s$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a smooth map (1-morphism) of orbifolds and $\mathcal{F}$ a vector bundle over $\mathcal{Y}$, then as in the manifold case, the pullback bundle $f^*(\mathcal{F})$ is a vector bundle over $\mathcal{X}$. Moreover, f induces a linear map $f^* : C^\infty(\mathcal{F}) \to C^\infty(f^*(\mathcal{F}))$ by sending $s \mapsto f^*(s) \circ \imath$, where $\imath : \mathcal{O}_{\mathcal{X}} \to f^*(\mathcal{O}_{\mathcal{Y}}) = f^{-1}(\mathcal{O}_{\mathcal{Y}}) \otimes_{f^{-1}(\mathcal{O}_{\mathcal{Y}})} \mathcal{O}_{\mathcal{X}}$ is the natural isomorphism.

The cotangent sheaf $T^*\mathcal{X}$ of an n-orbifold $\mathcal{X}$ is a vector bundle on $\mathcal{X}$ of rank n, and is called the *cotangent bundle* of $\mathcal{X}$, and as in the manifold case the tangent bundle $T\mathcal{X}$ of $\mathcal{X}$ is defined as $T\mathcal{X} = (T^*\mathcal{X})^*$.

Contrary to the manifold case, vector bundles on orbifolds can have fibres which are equipped with a non-trivial representation of $\mathrm{Iso}_{\mathcal{X}}([x])$. This can be seen as follows: Let $\mathcal{X}$ be an n-orbifold and $\mathcal{E} \to \mathcal{X}$ be a rank k vector bundle on $\mathcal{X}$. Let $[x] \in \mathcal{X}_{\mathrm{top}}$ be a geometric point of $\mathcal{X}$, and $G = \mathrm{Iso}_{\mathcal{X}}([x])$ its orbifold group. Then $\mathcal{X}$ is locally modelled near $[x]$ on $\mathbb{R}^n/G$ near 0, where G acts linearly on $\mathbb{R}^n$ and $\mathcal{E}$ is locally modelled near $[x]$ on the orbifold vector bundle $(\mathbb{R}^k \times \mathbb{R}^n)/G \to \mathbb{R}^n/G$, where G acts linearly on $\mathbb{R}^k$. However, this action of G needs not to be trivial, and so at each geometric point $[x] \in \mathcal{X}_{\mathrm{top}}$ the fibre $\mathcal{E}_x$ is a vector space isomorphic to $\mathbb{R}^k$, which is equipped with a not necessarily trivial representation of $\mathrm{Iso}_{\mathcal{X}}([x])$.

Smooth sections $s : \mathcal{X} \to \mathcal{E}$ are locally modelled near $[x]$ on G-equivariant smooth maps $s : \mathbb{R}^n \to \mathbb{R}^k$ near 0. As $s(0)$ must take values in the G-invariant subspace $(\mathbb{R}^k)^G$ of $\mathbb{R}^k$, a smooth section s of a vector bundle $\mathcal{E}$ over an orbifold $\mathcal{X}$ must take values in the $\mathrm{Iso}_{\mathcal{X}}([x])$ invariant subspace of $\mathcal{E}_x$ at each geometric point $[x] \in \mathcal{X}_{\mathrm{top}}$.

3.2.10 Sheaves on orbifold strata

This section briefly describes some basic properties and definitions of sheaves on orbifold strata. In particular we will sketch that for any quasicoherent sheaf $\mathcal{E} \in \mathrm{qcoh}(\mathcal{X})$, there exists a natural representation of Γ on $\mathcal{E}^\Gamma := O^\Gamma(\mathcal{X})^*(\mathcal{E}) \in \mathrm{qcoh}(\mathcal{X}^\Gamma)$ and that the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ lifts in an equivariant way to $\mathcal{E}^\Gamma$. The material used here can be found in more detail in [35, §C.9].

Definition 3.2.22. Consider a Deligne–Mumford C^∞-stack $\mathcal{X}$, a finite group Γ acting on $\mathcal{X}$. Section 3.2.5 defines the orbifold stratum $\mathcal{X}^\Gamma$, a 1-morphism $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$ a natural action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ by 1-isomorphisms $L^\Gamma(\Lambda, \mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}^\Gamma$ for $\Lambda \in \mathrm{Aut}(\Gamma)$, and a natural action on $O^\Gamma(\mathcal{X})$ by 2-isomorphisms $E^\Gamma(\gamma, \mathcal{X}) \Rightarrow O^\Gamma(\mathcal{X})$, where $\gamma \in \Gamma$.

For a quasicoherent sheaf $\mathcal{E} \in \mathrm{qcoh}(\mathcal{X})$ we can define $\mathcal{E}^\Gamma := O^\Gamma(\mathcal{X})^*(\mathcal{E}) \in \mathrm{qcoh}(\mathcal{X}^\Gamma)$ and an action of Γ on $\mathcal{E}^\Gamma$ as follows: for each $\gamma \in \Gamma$ define a morphism $R^\Gamma(\gamma, \mathcal{E}) : \mathcal{E}^\Gamma \to \mathcal{E}^\Gamma$ by

$$R^\Gamma(\gamma, \mathcal{E}) = E^\Gamma(\gamma, \mathcal{X})^*(\mathcal{E}) : \mathcal{O}^\Gamma(\mathcal{X})^*(\mathcal{E}) \to \mathcal{O}^\Gamma(\mathcal{X})^*(\mathcal{E}).$$

As $E^\Gamma(1, \mathcal{X}) = \mathrm{id}_{\mathcal{O}^\Gamma(\mathcal{X})}$ and $E^\Gamma(\gamma, \mathcal{X}) \odot E^\Gamma(\delta, \mathcal{X}) = E^\Gamma(\gamma \circ \delta, \mathcal{X})$ for $\gamma, \delta \in \Gamma$, we can conclude that

$$\begin{aligned} &R^\Gamma(1, \mathcal{E}) = \mathrm{id}_{\mathcal{E}^\Gamma} \quad \text{and} \\ &R^\Gamma(\gamma, \mathcal{E}) \circ R^\Gamma(\delta, \mathcal{E}) = R^\Gamma(\gamma \circ \delta, \mathcal{E}) \qquad \text{for all } \gamma, \delta \in \Gamma. \end{aligned}$$

Therefore $R^\Gamma(-, \mathcal{E})$ defines *an action* of Γ on $\mathcal{E}^\Gamma$ *by isomorphisms.*

In a similar fashion one can show that $S^\Gamma(\Lambda, \mathcal{E}) : L^\Gamma(\Lambda, \mathcal{X})^*(\mathcal{E}^\Gamma) \to \mathcal{E}^\Gamma$, $S^\Gamma(\Lambda, \mathcal{E}) = I_{L^\Gamma(\Lambda, \mathcal{X}), \mathcal{O}^\Gamma(\mathcal{X})}(\mathcal{E})^{-1} : L^\Gamma(\Lambda, \mathcal{X})^* \circ \mathcal{O}^\Gamma(\mathcal{X})^*(\mathcal{E}) \to \mathcal{O}^\Gamma(\mathcal{X})^*(\mathcal{E})$ for $\Lambda \in \mathrm{Aut}(\Gamma)$ defines a lift of the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ to $\mathcal{E}^\Gamma$.

Moreover it can be shown that the actions of $\mathrm{Aut}(\Gamma)$ and Γ on $\mathcal{X}^\Gamma$ are in fact compatible so that the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ lifts to an action of $\mathrm{Aut}(\Gamma) \ltimes \Gamma$ on $\mathcal{E}^\Gamma$.

Furthermore, $E^\Gamma(\gamma, \mathcal{X})^* : \mathcal{O}^\Gamma(\mathcal{X})^* \Rightarrow \mathcal{O}^\Gamma(\mathcal{X})^*$ being a natural isomorphism of functors implies that $\mathbb{R}^\Gamma(\gamma, -)$ and $S(\Lambda, -)$ are natural isomorphisms of functors, that is for a morphism $\alpha : \mathcal{E}_1 \to \mathcal{E}_2$ in $\mathrm{qcoh}(\mathcal{X}^\Gamma)$ we have

$$\begin{aligned} \alpha^\Gamma \circ \mathbb{R}^\Gamma(\gamma, \mathcal{E}_1) &= R^\Gamma(\gamma, \mathcal{E}_2) \circ \alpha^\Gamma \quad \text{for} \gamma \in \Gamma \\ \alpha \circ S^\Gamma(\Lambda, \mathcal{E}_1) &= S^\Gamma(\Lambda, \mathcal{E}_2) \circ L^\Gamma(\Lambda, \mathcal{X})^*(\alpha^\Gamma) \quad \text{for} \Lambda \in \mathrm{Aut}(\Gamma). \end{aligned}$$

Here α^Γ denotes the by α induced morphism $\alpha^\Gamma : \mathcal{O}^\Gamma(\mathcal{X})^*(\alpha) : \mathcal{E}_1^\Gamma \to \mathcal{E}_2^\Gamma$.

Let $\mathcal{X}, \Gamma, \mathcal{X}^\Gamma, \mathcal{E}, \mathcal{E}^\Gamma$ be as above and denote by $R_0 = \mathbb{R}, \ldots, R_k$ representatives of the irreducible representations of Γ over $\mathbb{R}$. As $R^\Gamma(-, \mathcal{E})$ is an action of Γ on $\mathcal{E}^\Gamma$ by isomorphism, an elementary result in representation theory yields a splitting

$$\mathcal{E}^\Gamma \cong \bigoplus_{i=0}^{k} \mathcal{E}_i^\Gamma \otimes R_i \quad \text{for } \mathcal{E}_0^\Gamma, \ldots, \mathcal{E}_k^\Gamma \in \mathrm{qcoh}(\mathcal{X}^\Gamma). \tag{3.9}$$

We can split $\mathcal{E}^\Gamma$ into *trivial* and *nontrivial* representations of Γ

$$\mathcal{E}^\Gamma \cong \mathcal{E}_{\mathrm{tr}}^\Gamma \oplus \mathcal{E}_{\mathrm{nt}}^\Gamma, \tag{3.10}$$

where the subsheaf $\mathcal{E}_{\mathrm{tr}}^\Gamma$ of trivial representations of $\mathcal{E}^\Gamma$ corresponds to the factor $\mathcal{E}_0^\Gamma \otimes R_0$ in 3.9 and the subsheaf $\mathcal{E}_{\mathrm{nt}}^\Gamma$ of nontrivial representations of $\mathcal{E}^\Gamma$ to $\bigoplus_{i=0}^{k} \mathcal{E}_i^\Gamma \otimes R_i$.

If we denote the action of Γ on R_i by $\rho_i : \Gamma \to \mathrm{Aut}(\Gamma)$ then we get for every automorphism $\Lambda \in \mathrm{Aut}(\Gamma)$ that $\rho_i \circ \Lambda^{-1} : \Gamma \to \mathrm{Aut}(R_i)$ is an irreducible representation of Γ, and is therefore isomorphic to $R_{\Lambda(i)}$ for some unique $\Lambda(i) = 0, \ldots, k$. Hence we get an action of $\mathrm{Aut}(\Gamma)$ on $0, \ldots, k$ given by permutations. As in Joyce [35, Definition C.50] one can show that the the lift $S^\Gamma(\Lambda, \mathcal{E})$ preserves the splitting 3.10.

The following definition will describe what morphisms of square zero extension should be. We refer once again to [35, Definition 9.2] for a much more detailed discussion.

Definition 3.2.23. Given square zero extensions of C^∞-stacks $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$, we call a pair (f, f'), where $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks, and $f' : f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \to \mathcal{O}'_{\mathcal{X}}$ a morphism of sheaves of C^∞-rings on $\mathcal{X}$, a *morphism of square zero extensions* from $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}})$ to $(\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$, if the following condition is satisfied:

$$f^\sharp \circ f^{-1}(\imath_{\mathcal{Y}}) = \imath_{\mathcal{X}} \circ f' : f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \to \mathcal{O}_{\mathcal{X}}.$$

3.3 D-stacks

In this section we follow Joyce [35, §9] in defining the 2-category of d-stacks **dSta**. A d-stack can be thought of as a derived version of Deligne–Mumford C^∞-stack, and play a similar role in the d-orbifold world, as d-spaces in the d-manifold world. We will start by providing the d-orbifold analogue of §2.1.6 and define what square zero extensions of C^∞-stacks should be.

3.3.1 Square zero extensions of C^∞-stacks

This section will be the d-orbifold analogue of section 2.1.6, and extend square zero extensions of C^∞-schemes to Deligne–Mumford C^∞-stacks. We refer to Joyce [35, §9.1] for a more detailed discussion.

Definition 3.3.1. Given a locally fair Deligne–Mumford C^∞-stack, it can be shown that all $\mathcal{O}_\mathcal{X}$-modules are quasicoherent (compare [35, Proposition C.13]).

A tuple $(\mathcal{O}'_\mathcal{X}, \imath_\mathcal{X})$, consisting of a sheaf of C^∞-rings $\mathcal{O}'_\mathcal{X}$ on $\mathcal{X}$ and a morphism of sheaves of C^∞-rings $\imath_\mathcal{X} : \mathcal{O}'_\mathcal{X} \to \mathcal{O}_\mathcal{X}$ on $\mathcal{X}$, where $\mathcal{O}_\mathcal{X}$ is the structure sheaf of $\mathcal{X}$ (see [35, Example C.23]), is called a *square zero extension of* $\mathcal{X}$, if for all objects $(\underline{U}, u)$ in the category $\mathcal{C}_\mathcal{X}$ (sheaves on $\mathcal{X}$ are defined in terms of this category ; compare [35, Definition C.12])

$$\imath_\mathcal{X}(\underline{U}, u) : \mathcal{O}'_\mathcal{X}(\underline{U}, u) \to \mathcal{O}_\mathcal{X}(\underline{U}, u) = \mathcal{O}_U$$

is a square zero extension of C^∞-schemes on $\underline{U}$ in the sense of § 2.1.6. The triple $(\mathcal{X}, \mathcal{O}'_\mathcal{X}, \imath_\mathcal{X})$ is called a *square zero extension of C^∞-stacks.*

Given an object $(\underline{U}, u)$ in $\mathcal{C}_\mathcal{X}$, we can define quasicoherent sheaves $I_\mathcal{X}(\underline{U}, u)$ and $\mathcal{F}_\mathcal{X}(\underline{U}, u)$ on $\underline{U}$, a morphism $\kappa_\mathcal{X}(\underline{U}, u) : \mathcal{I}_\mathcal{X}(\underline{U}, u) \to \mathcal{O}'_\mathcal{X}(\underline{U}, u)$ of sheaves of abelian groups on U, and morphisms $\xi_\mathcal{X}(\underline{U}, u) : \mathcal{I}_\mathcal{X}(\underline{U}, u) \to \mathcal{F}_\mathcal{X}(\underline{U}, u), \psi_\mathcal{X}(\underline{U}, u) : \mathcal{F}_\mathcal{X}(\underline{U}, u) \to T^*\underline{U} = (T^*\mathcal{X})(\underline{U}, u)$ of quasicoherent sheaves on $\underline{U}$, which full fill the role of $\mathcal{I}_X, \mathcal{F}_X, \kappa_X, \xi_x, \psi_X$ in the definition of square zero extension of C^∞-schemes. (Compare [35, Definition 2.9].)

Given a morphism $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_\mathcal{X}$ we can define a morphism of sheaves of C^∞-rings on U $f' := (\mathcal{O}'_\mathcal{X})_{(\underline{f},\eta)} : f^{-1}(\mathcal{O}'_\mathcal{X}(\underline{V}, v)) \to \mathcal{O}'_\mathcal{X}(\underline{U}, u)$, such that

$(\underline{f}, f')$ is a morphism of square zero extensions of C^∞-schemes $(f, f') : (\underline{U}, \mathcal{O}'_{\mathcal{X}}(\underline{U}, u), \imath_{\mathcal{X}}(\underline{U}, u)) \to (\underline{V}, \mathcal{O}'_{\mathcal{X}}(\underline{V}, v), \imath_{\mathcal{X}}(\underline{V}, v))$.

As in §2.1.6, this morphism of square zero extensions of C^∞-schemes defines morphisms f^1, f^2, f^3 in $\mathrm{qcoh}(\underline{U})$, which are isomorphism as $\underline{f}$ is étale and f' an isomorphism. Using these isomorphisms, and setting

$$\mathcal{I}_{\mathcal{X}}(\underline{f}, \eta) = f^1 : \underline{f}^*(\mathcal{I}_{\mathcal{X}}(\underline{V}, v)) \to \mathcal{I}_{\mathcal{X}}(\underline{U}, u),$$
$$\mathcal{F}_{\mathcal{X}}(\underline{f}, \eta) = f^2 : \underline{f}^*(\mathcal{F}_{\mathcal{X}}(\underline{V}, v)) \to \mathcal{F}_{\mathcal{X}}(\underline{U}, u),$$

one can check that the data $\mathcal{I}_{\mathcal{X}}(\underline{U}, u), \mathcal{F}_{\mathcal{X}}(\underline{U}, u), (\mathcal{I}_{\mathcal{X}})_{(\underline{f},\eta)}, (\mathcal{F}_{\mathcal{X}}(\underline{f}, \eta))$ defines quasicoherent sheaves $\mathcal{I}_{\mathcal{X}}, \mathcal{F}_{\mathcal{X}}$ on $\mathcal{X}$ and $\xi_{\mathcal{X}}(\underline{U}, u), \psi_{\mathcal{X}}(\underline{U}, u)$ defines morphisms of quasicoherent sheaves $\xi_{\mathcal{X}} : \mathcal{I}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}, \psi_{\mathcal{X}} : \mathcal{F}_{\mathcal{X}} \to T^*\mathcal{X}$. Moreover, if one regards $\mathcal{I}_{\mathcal{X}}$ as a sheaf of abelian groups on $\mathcal{X}$, then $\kappa_{\mathcal{X}}(\underline{U}, u)$ defines a morphism $\kappa_{\mathcal{X}} : \mathcal{I}_{\mathcal{X}} \to \mathcal{O}'_{\mathcal{X}}$ of sheaves of abelian groups on $\mathcal{X}$.

Equation (2.7) yields for each $(\underline{U}, u)$ an exact sequence of sheaves of abelian groups on $\mathcal{X}$

$$0 \longrightarrow \mathcal{I}_X \xrightarrow{\kappa_{\mathcal{X}}} \mathcal{O}'_{\mathcal{X}} \xrightarrow{\imath_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}} \longrightarrow 0, \tag{3.11}$$

and equation (2.9) implies for each $(\underline{U}, u)$ the existence of an exact sequence of sheaves of quasicoherent sheaves on $\mathcal{X}$:

$$\mathcal{I}_{\mathcal{X}} \xrightarrow{\xi_{\mathcal{X}}} \mathcal{F}_{\mathcal{X}} \xrightarrow{\psi_{\mathcal{X}}} T^*\mathcal{X} \longrightarrow 0. \tag{3.12}$$

3.3.2 The 2-category of d-stacks

We will now define the 2-category **dSta** of *d-stacks*, which can be thought as the Deligne–Mumford C^∞-stack analogue of d-spaces. (The C^∞-schemes $\underline{X}, \underline{X}'$ are replaced by Deligne–Mumford C^∞-stacks $\mathcal{X}, \mathcal{X}'$.) We will not prove any results in the following, but refer to Joyce [35, §9.2] for a more detailed discussion instead.

Definition 3.3.2. A *d-stack* $\boldsymbol{\mathcal{X}}$ is given by a quintuple $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$ consisting of a separated, second countable, locally fair Deligne–Mumford C^∞-stack $\mathcal{X}$, a square zero extension $(\mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}})$ of $\mathcal{X}$ with kernel $\kappa_{\mathcal{X}} : \mathcal{I}_{\mathcal{X}} \to \mathcal{O}'_{\mathcal{X}}$, such that $I_{\mathcal{X}} \in \mathrm{qcoh}(\mathcal{X})$, and a quasicoherent sheaf $\mathcal{E}_{\mathcal{X}} \in \mathrm{qcoh}(\mathcal{X})$, and a surjective morphism $\jmath_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{I}_{\mathcal{X}}$ in $\mathrm{qcoh}(\mathcal{X})$.

Using (3.11), equation (2.10) translates into the following exact sequence of sheaves of abelian groups on $\mathcal{X}$:

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\kappa_{\mathcal{X}}\circ\jmath_{\mathcal{X}}} \mathcal{O}'_{\mathcal{X}} \xrightarrow{\imath_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}} \longrightarrow 0, \tag{3.13}$$

and by setting $\phi_{\mathcal{X}} = \xi_{\mathcal{X}} \circ \jmath_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ and using (3.12), the d-stack analogue of equation (2.11) is given by

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\phi_{\mathcal{X}}} \mathcal{F}_{\mathcal{X}} \xrightarrow{\psi_{\mathcal{X}}} T^*\mathcal{X} \longrightarrow 0. \tag{3.14}$$

The morphism $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ is called *virtual cotangent sheaf* of $\mathcal{X}$.

Denote the kernel of $\jmath_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{I}_{\mathcal{X}}$ in qcoh($\mathcal{X}$) by $\lambda_{\mathcal{X}}$ and let $\mu_{\mathcal{X}} : \mathcal{D}_{\mathcal{X}} \to \mathcal{E}_{\mathcal{X}}$ be the kernel of $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ in qcoh($\mathcal{X}$). Then there exists a unique morphism $\nu_{\mathcal{X}} : \mathcal{C}_{\mathcal{X}} \to \mathcal{D}_{\mathcal{X}}$ such that $\lambda_{\mathcal{X}} = \mu_{\mathcal{X}} \circ \nu_{\mathcal{X}}$, making the following diagram commutative with exact diagonals

$$\begin{tikzcd}
 & & & & & 0 \\
0 \arrow[dr] & & & & T^*\mathcal{X} \arrow[ur] & \\
 & \mathcal{C}_{\mathcal{X}} \arrow[dr, "\lambda_{\mathcal{X}}"] \arrow[dd, "\nu_{\mathcal{X}}"'] & & \mathcal{F}_{\mathcal{X}} \arrow[ur, "\psi_{\mathcal{X}}"] & & \\
 & & \mathcal{E}_{\mathcal{X}} \arrow[ur, "\phi_{\mathcal{X}}"] \arrow[dr, "\jmath_{\mathcal{X}}"'] & & & \\
 & \mathcal{D}_{\mathcal{X}} \arrow[ur, "\mu_{\mathcal{X}}"'] & & \mathcal{I}_{\mathcal{X}} \arrow[uu, "\xi_{\mathcal{X}}"'] \arrow[dr] & & \\
0 \arrow[ur] & & & & 0. &
\end{tikzcd}$$

Given two d-stacks $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$, a 1-*morphism* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a triple $\boldsymbol{f} = (f, f', f'')$, where $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks, $f' : f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \to \mathcal{O}'_{\mathcal{X}}$ a morphism of sheaves of C^∞-rings on $\mathcal{X}$ such that (f, f') is a morphism of square zero extensions $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \imath_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}}, \imath_{\mathcal{Y}})$ as in section 3.3.1, and $f'' : f^*(\mathcal{E}_{\mathcal{Y}}) \to \mathcal{E}_{\mathcal{X}}$ is a morphism in qcoh($\mathcal{X}$) satisfying

$$\jmath_{\mathcal{X}} \circ f'' = f^1 \circ f^*(\jmath_{\mathcal{Y}}) : f^*(\mathcal{E}_{\mathcal{Y}}) \to \mathcal{I}_{\mathcal{X}},$$

with f^1, f^2, f^3 as in section 3.3.1.

One can also define composition of 1-morphisms, 2-morphisms, the identity 1-morphism, the identity 2-morphism and composition of 2-morphisms, and thus define a 2-category of d-stacks, which we will denote by **dSta**. For all the details of the construction we refer to [35, Definition 9.6].

3.3.3 Gluing d-stacks by equivalences

In this section we generalize the material of section 2.2.1 to the d-stack case. We will not discuss the material in detail, and refer for an in depth discussion of the subtleties and details to Joyce [35, §9.4] instead.

The following theorem is the d-stack analogue of Theorem 2.2.4.

Theorem 3.3.3. *Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ be d-stacks, $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{Y}}$ be open d-substacks and $\boldsymbol{f} : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{V}}$ an equivalence in* **dSta***. For the underlying topological spaces this means that we have open subspaces $\mathcal{U}_{top} \subseteq \mathcal{X}_{top}, \mathcal{V}_{top} \subseteq \mathcal{Y}_{top}$, with a homeomorphism $f_{top} : \mathcal{U}_{top} \to \mathcal{V}_{top}$. Hence we can form the quotient topological space $\mathcal{Z}_{top} := \mathcal{X}_{top} \amalg_{f_{top}} \mathcal{Y}_{top} = (\mathcal{X}_{top} \amalg \mathcal{Y}_{top})/\sim$, where the equivalence relation $\sim$ on $\mathcal{X}_{top} \amalg \mathcal{Y}_{top}$ identifies an element $[u] \in \mathcal{U}_{top} \subseteq \mathcal{X}_{top}$ with $f_{top}([u]) \in \mathcal{V}_{top} \subseteq \mathcal{Y}_{top}$.*

Assume that the quotient space $\mathcal{Z}_{top}$ is Hausdorff, or that on the C^∞-stack level, the quotient C^∞-stack $\mathcal{Z} = \mathcal{X} \amalg_f \mathcal{Y}$ is separated. Then there exist a d-stack $\boldsymbol{\mathcal{Z}}$, open d-substacks $\hat{\boldsymbol{\mathcal{X}}}, \hat{\boldsymbol{\mathcal{Y}}}$ in $\boldsymbol{\mathcal{Z}}$ with $\boldsymbol{\mathcal{Z}} = \hat{\boldsymbol{\mathcal{X}}} \cup \hat{\boldsymbol{\mathcal{Y}}}$, equivalences $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \hat{\boldsymbol{\mathcal{X}}}$ and $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \hat{\boldsymbol{\mathcal{Y}}}$ such that $\boldsymbol{g}|_{\boldsymbol{\mathcal{U}}}$ and $\boldsymbol{h}|_{\boldsymbol{\mathcal{V}}}$ are equivalences with $\hat{\boldsymbol{\mathcal{X}}} \cup \hat{\boldsymbol{\mathcal{Y}}}$, and a 2-morphism $\boldsymbol{\eta} : \boldsymbol{g}|_{\boldsymbol{\mathcal{U}}} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f} : \boldsymbol{\mathcal{U}} \to \hat{\boldsymbol{\mathcal{X}}} \cup \hat{\boldsymbol{\mathcal{Y}}}$. Moreover, the d-stack $\boldsymbol{\mathcal{Z}}$ is independent of choices up to equivalence.

3.4 D-orbifolds

We want now to recapitulate some basic definitions and properties of d-orbifolds. As orbifolds are an extension of manifolds, d-orbifolds will play the same role as an extension of d-manifolds. We follow throughout this section Joyce [35, §10], and refer to him for a much more complete treatment of d-orbifolds.

3.4.1 Virtual quasicoherent sheaves on C^∞-stacks

In this section we will briefly extend the material of section 3.2.2 to virtual quasicoherent sheaves and virtual vector bundles on C^∞-stacks. As it turns out, most of the concepts of section 2.1.5 on virtual quasicoherent sheaves and virtual vector bundles on C^∞-schemes extend nicely to the C^∞-stack case. However there are some differences and subtleties in the C^∞-stack case, which we will explain in Definition 3.4.1 below. (Compare [35, §10.1.1].)

Definition 3.4.1. The C^∞-stack analogue of Definition 2.1.21 (the definition of the 2-category $\mathbf{vqcoh}(\mathcal{X})$) is exactly the same as in the C^∞-scheme case. A virtual quasicoherent sheaf $(\mathcal{E}^\bullet, \phi)$ in $\mathbf{vqcoh}(\mathcal{X})$ is called a *virtual vector bundle of rank* $d \in \mathbb{Z}$ if $\mathcal{X}$ may be covered by Zariski open C^∞-substacks $\mathcal{U}$ such that $(\mathcal{E}^\bullet, \phi)|_{\mathcal{U}}$ is equivalent in $\mathbf{vqcoh}(\mathcal{U})$ to some virtual quasicoherent sheaf $(\mathcal{F}^\bullet, \psi)$ for $\mathcal{F}^1, \mathcal{F}^2$ vector bundles on $\mathcal{U}$ with $\operatorname{rank}\mathcal{F}^2 - \operatorname{rank}\mathcal{F}^1 = d$. The difference between this definition and Definition 2.1.21 is that the vector bundles $\mathcal{F}^1, \mathcal{F}^2$ on $\mathcal{U}$ need only be locally trivial in the *étale topology*, so that the orbifold groups $\mathrm{Iso}_{\mathcal{U}}([u])$ of $\mathcal{U}$ can act nontrivially on the fibres of $\mathcal{F}^1, \mathcal{F}^2$. The full 2-subcategory of virtual vector bundles in $\mathbf{vqcoh}(\mathcal{X})$ will be denoted as in the C^∞-scheme case by $\mathbf{vvect}(\mathcal{X})$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of Deligne–Mumford C^∞-stacks then the pullback f^* defines a strict 2-functors $f^*\colon \mathbf{vqcoh}(\mathcal{Y}) \to \mathbf{vqcoh}(\mathcal{X})$ and $f^*\colon \mathbf{vvect}(\mathcal{Y}) \to \mathbf{vvect}(\mathcal{X})$, as for C^∞-schemes. In contrast to the C^∞-scheme case, the "pullback" of a 2-morphism exists, that is if $f, g\colon \mathcal{X} \to \mathcal{Y}$ are 1-morphisms of Deligne–Mumford C^∞-stacks and $\eta\colon f \Rightarrow g$ is a 2-morphism, then $\eta^*\colon f^* \Rightarrow g^*$ is a strict 2-natural transformation.

As in the d-space case (compare section 2.2), one can define the virtual cotangent sheaf $T^*\boldsymbol{\mathcal{X}}$ of a d-stack $\boldsymbol{\mathcal{X}}$ to be the morphism $\phi_{\mathcal{X}}\colon \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ in $\mathrm{qcoh}(\mathcal{X})$ as in Definition 3.3.2. If $f\colon \mathcal{X} \to \mathcal{Y}$ is a 1-morphism in $\mathbf{dSta}$ then $\Omega_f := (f'', f^2)$ is a 1-morphism, mapping $f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ in $\mathbf{vqcoh}(\mathcal{X})$. The first subtleties in the d-stack case arise in the 2-morphism picture in $\mathbf{dSta}$. Suppose $f, g\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ are 1-morphisms and $\eta = (\eta, \eta')\colon f \Rightarrow g$ is a 2-morphism in $\mathbf{dSta}$. Then we have beside the induced 1-morphisms $\Omega_f\colon f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$, $\Omega_g\colon g^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$, a 1-isomorphism

$\eta^*(T^*\boldsymbol{\mathcal{Y}})\colon f^*(T^*\boldsymbol{\mathcal{Y}}) \to g^*(T^*\boldsymbol{\mathcal{Y}})$ in $\mathrm{qcoh}(\mathcal{X})$ and a 2-morphism $\eta'\colon \Omega_f \Rightarrow \Omega_g \circ \eta^*(T^*\boldsymbol{\mathcal{Y}})$ in $\mathbf{vqcoh}(\mathcal{X})$.

Most of the other results in the C^∞-scheme and d-space case, like Propositions 2.1.23 and 2.1.24 carry over nicely to the C^∞-stack and d-stack world and can be proven in the exact same manner.

3.4.2 The definition of d-orbifolds

As in the d-manifold case, we start by defining *principal d-orbifolds.* (Compare [35, Definition 10.1],)

Definition 3.4.2. We call a d-stack $\boldsymbol{\mathcal{W}}$ a *principal d-orbifold (without boundary)*, if it is equivalent in **dSta** to a fibre product $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ with $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}} \in \hat{\mathbf{Orb}}$. On the underlying C^∞-stack level, the fibre product $\mathcal{W} \simeq \mathcal{X} \times_{g,\mathcal{Z},h} \mathcal{Y}$ is locally finitely presented, as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are, and similar to the d-manifold case, any object in $\hat{\mathbf{Orb}}$ is a principal d-orbifold.

Let $\boldsymbol{\mathcal{W}}$ now be a nonempty principal d-orbifold. Then, as we have seen before, the virtual cotangent sheaf $T^*\boldsymbol{\mathcal{W}}$ is a virtual vector bundle on the C^∞-stack $\mathcal{W}$. We can therefore define as in the d-manifold case the *virtual dimension* of $\boldsymbol{\mathcal{W}}$ as vdim $\boldsymbol{\mathcal{W}} = \operatorname{rank} T^*\boldsymbol{\mathcal{W}} \in \mathbb{Z}$. Similar to the d-manifold case, the virtual dimension is well-defined, and if $\boldsymbol{\mathcal{W}} \simeq \boldsymbol{\mathcal{X}} \times_{\boldsymbol{\mathcal{Z}}} \boldsymbol{\mathcal{Y}}$, with $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ being orbifolds, then vdim $\boldsymbol{\mathcal{W}} = \dim \mathcal{X} + \dim \mathcal{Y} - \dim \mathcal{Z}$.

A d-stack $\boldsymbol{\mathcal{W}}$ is called a *d-orbifold (without boundary) of virtual dimension* $n \in Z$ if it can be covered by open d-substacks $\boldsymbol{\mathcal{W}}$, which are principal d-orbifolds with vdim $\boldsymbol{\mathcal{W}} = n$. The underlying C^∞-stack $\mathcal{X}$ is separated, second countable, locally compact, paracompact, and locally finitely presented. As in the d-manifold case, the virtual cotangent sheaf $T^*\boldsymbol{\mathcal{X}} = (\mathcal{E}_{\mathcal{X},\mathcal{F}_{\mathcal{X}},\phi_{\mathcal{X}}})$ of $\boldsymbol{\mathcal{X}}$ is a virtual vector bundle of rank vdim $\boldsymbol{\mathcal{X}} = n$, and is therefore called the *virtual cotangent bundle* of $\boldsymbol{\mathcal{X}}$. The empty d-stack $\boldsymbol{\emptyset}$ is defined to be a d-orbifold of any virtual dimension $n \in \mathbb{Z}$, and hence vdim $\boldsymbol{\emptyset}$ is undefined.

Write **dOrb** for the full 2-subcategory of d-orbifolds in **dSta**. Then, as in the d-manifold case, the image of the 2-functor $F_{\mathbf{Orb}}^{\mathbf{dSta}} \colon \mathbf{Orb} \to \mathbf{dSta}$ is **dOrb**, and we write $F_{\mathbf{Orb}}^{\mathbf{dOrb}} \colon \mathbf{Orb} \to \mathbf{dOrb}$ instead. Moreover, $\hat{\mathbf{Orb}}$ is a 2-subcategory of **dOrb**. If we restrict the 2-functor $F_{\mathbf{dSpa}}^{\mathbf{dSta}}$ to **dMan**, we obtain a 2-functor $F_{\mathbf{dMan}}^{\mathbf{dOrb}} = F_{\mathbf{dSpa}}^{\mathbf{dSta}}|_{\mathbf{dMan}} \colon \mathbf{dMan} \to \mathbf{dOrb}$. Then $F_{\mathbf{dMan}}^{\mathbf{dOrb}} \circ F_{\mathbf{Man}}^{\mathbf{dMan}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}} \circ F_{\mathbf{Man}}^{\mathbf{Orb}} \colon \mathbf{Man} \to \mathbf{dOrb}$.

We can hence write $\hat{\mathbf{dMan}}$ for the full 2-subcategory of objects $\boldsymbol{\mathcal{X}}$ in **dOrb** being equivalent to $F_{\mathbf{dMan}}^{\mathbf{dOrb}}(\boldsymbol{X})$, for some d-manifold $\boldsymbol{X}$. We will refer to a d-orbifold $\boldsymbol{\mathcal{X}}$ as a *d-manifold*, if $\boldsymbol{\mathcal{X}} \in \hat{\mathbf{dMan}}$.

As in the d-manifold case, there are several characterizations of when a d-stack is a principal d-orbifold. This again, builds a bridge to Kuranishi neighbourhoods, as we will see later on. The following proposition is the d-orbifold analogue of Proposition 2.3.3.

Proposition 3.4.3. *A d-stack $\boldsymbol{\mathcal{W}}$ is a principal d-orbifold, if one of the following equivalent characterizations hold*

(a) $\boldsymbol{\mathcal{W}} \simeq \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g},\boldsymbol{\mathcal{Z}},\boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ *for* $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}} \in \hat{\mathbf{Orb}}$.

(b) $\boldsymbol{\mathcal{W}} \simeq \boldsymbol{\mathcal{X}} \times_{\boldsymbol{i},\boldsymbol{\mathcal{Z}},\boldsymbol{j}} \boldsymbol{\mathcal{Y}}$, *where* $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ *are orbifolds,* $i : \mathcal{X} \to \mathcal{Z}, j : \mathcal{Y} \to \mathcal{Z}$ *are orbifold embeddings and* $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}, \boldsymbol{i}, \boldsymbol{j} = F^{\mathbf{dSta}}_{\mathbf{Orb}}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, i, j)$. *So in other words* $\boldsymbol{\mathcal{W}}$ *is an intersection of two suborbifolds* $\mathcal{X}, \mathcal{Y}$ *in* $\mathcal{Z}$ *in the sense of d-stacks.*

(c) $\boldsymbol{\mathcal{W}} \simeq \boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}} \boldsymbol{\mathcal{V}}$, *where* $\mathcal{V}$ *is an orbifolds,* $\mathcal{E} \in \mathbf{vvect}(\mathcal{V})$ *a vector bundle on* $\mathcal{V}$ *(as in §3.2.9),* $s \in C^\infty(\mathcal{E})$ *a smooth orbifold section of* $\mathcal{E}$ *and* $0 \in C^\infty(\mathcal{E})$ *is the zero section. Here* $\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{E}}, \boldsymbol{s}, \boldsymbol{0} = F^{\mathbf{dSta}}_{\mathbf{Orb}}(\mathcal{V}, \mathrm{Tot}(\mathcal{E}), \mathrm{Tot}(s), \mathrm{Tot}(0))$, *where* Tot *is given by the 'total space functor' as in [35, Definition 8.4]. So* $\boldsymbol{\mathcal{W}}$ *is the zero set* $s^{-1}(0)$ *of an orbifold section* s *of an orbifold bundle* $\mathcal{E}$, *in the sense of d-stacks.*

The only difference in the proof is that in the manifold case one could take Φ to be a diffeomorphism with an open neighbourhood U' of the diagonal in $Z \times Z$. In the orbifold case Φ is no longer a diffeomorphism, but rather a $|\operatorname{Iso}_{\mathcal{Z}}(z)|$-fold branched cover near $(z, 0) \in U$ and $(z, z) \in \mathcal{Z} \times \mathcal{Z}$.

Many concepts and properties of d-manifolds extend nicely to the d-orbifold case, like the following lemma, which shows that open substacks of d-orbifolds are d-orbifolds themself. (Compare [35, Lemma 10.3].)

Lemma 3.4.4. *An open substack* $\boldsymbol{\mathcal{U}}$ *of a d-orbifold* $\boldsymbol{\mathcal{W}}$ *is also a d-orbifold with vdim* $\boldsymbol{\mathcal{U}} =$ *vdim* $\boldsymbol{\mathcal{W}}$.

The following lemma gives a nice characterisation when a d-orbifold is a d-manifold.

Lemma 3.4.5. *A d-orbifold* $\boldsymbol{\mathcal{X}}$ *is a d-manifold, that is equivalent in* **dOrb** *to* $F_{\mathbf{dMan}}^{\mathbf{dOrb}}(\boldsymbol{X})$ *for some d-manifold* $\boldsymbol{X}$*, if and only if* $\mathrm{Iso}_{\mathcal{X}} \cong \{1\}$ *for all* $[x]$ *in* $\mathcal{X}_{top}$*.*

Proposition 3.4.6. *Let* $\boldsymbol{\mathcal{X}}$ *be a d-orbifold,* $[x] \in \mathcal{X}_{top}$ *and* $G = \mathrm{Iso}_{\mathcal{X}}([x])$*. As in [35, Theorem C.25 (a)], there exists a quotient* C^∞*-stack* $[\underline{U}/G]$*, where* $\underline{U}$ *is an affine* C^∞*-scheme, and a* 1*-morphism* $i : [\underline{U}/G] \to \mathcal{X}$*, which is an equivalence with an open* C^∞*-substack* $\mathcal{U} \subseteq \mathcal{X}$*, such that on the underlying topological spaces* $i_{top} : [u] \mapsto [x] \in \mathcal{U}_{top} \subseteq \mathcal{X}_{top}$ *for some* $u \in \underline{U}$*, fixed by* G*.*

Then $a := \dim T_U^* \underline{U} - \dim \mathcal{O}_u \underline{U} - vdim\, \mathcal{X} \geq 0$ *and* $\boldsymbol{\mathcal{X}}$ *is determined up to non-canonical equivalence near* $[x]$ *by* $\mathcal{X}$*, vdim* $\boldsymbol{\mathcal{X}}$ *and a choice of representation of* G *on* $\mathbb{R}^a$*, up to automorphisms of* $\mathbb{R}^a$*.*

3.4.3 Local description of d-orbifolds

Similarly to the d-manifold case (see §2.3.1), there exist local descriptions of d-orbifolds in terms of orbifolds and vector bundles. We refer to Joyce [35, §10.1.3] for a more detailed discussion of the subject.

Definition 3.4.7. Let $\mathcal{V}$ be an orbifold, $\mathcal{E} \in \mathrm{vect}\mathcal{V}$ a vector bundle over $\mathcal{V}$, as in § 3.2.9, and $s : \mathcal{V} \to \mathcal{E}$ a smooth section, that is $s : \mathcal{O}_{\mathcal{V}} \to \mathcal{E}$ is a morphism in $\mathrm{vect}\mathcal{V}$. Then the *'standard model'* d-orbifold, is defined to be the principal d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$.

Consider the C^∞-substack $\mathcal{S}$ in $\mathcal{V}$, defined by the equation $s = 0$, so that roughly speaking $\mathcal{S}$ can be thought of being $\mathcal{S} = s^{-1}(0) \subset \mathcal{V}$. It turns out (see for instance [35, Definition 10.5]) that $\mathcal{S}$ is then a Deligne–Mumford C^∞-stack.

The inclusion of categories $i_{\mathcal{S}} : \mathcal{S} \to \mathcal{V}$ is a closed embedding in $C^\infty\mathbf{Sta}$ and it can be shown that

$$\mathcal{S} \simeq \mathcal{V} \times_{\mathrm{Tot}(s),\mathrm{Tot}(\mathcal{E}),\mathrm{Tot}(0)} \mathcal{V},$$

that is $\mathcal{S}$ is equivalent to the C^∞-stack fibre product consisting of the orbifolds $\mathcal{V}, \mathrm{Tot}(\mathcal{E})$ and the orbifold 1-morphisms $\mathrm{Tot}(s), \mathrm{Tot}(0) : \mathcal{V} \to \mathrm{Tot}(\mathcal{E})$

As $i_{\mathcal{S}} : \mathcal{S} \to \mathcal{V}$ is the inclusion of C^∞-stacks, the morphism of sheaves of C^∞-rings on $\mathcal{S}$, $i_{\mathcal{V}}^\sharp : i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}}) \to \mathcal{O}_{\mathcal{S}}$ is surjective. If we denote the kernel of $i_{\mathcal{V}}^\sharp$ by $\mathcal{I}_s$, the corresponding sheaf of square ideals by $\mathcal{I}_s^2$, the quotient sheaf of C^∞-rings by

$\mathcal{O}'_{\mathcal{S}} = i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2$, and the natural projection $i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2 \twoheadrightarrow i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s \cong \mathcal{O}_{\mathcal{S}}$ by $i_{\mathcal{S}} : \mathcal{O}'_{\mathcal{S}} \to \mathcal{O}_{\mathcal{S}}$, $(\mathcal{O}'_{\mathcal{S}}, i_{\mathcal{S}})$ is a square zero extension of $\mathcal{S}$.

The vector bundle $\mathcal{E}_{\mathcal{S}}$ is then given by $\mathcal{E}_{\mathcal{S}} = i_{\mathcal{V}}^*(\mathcal{E}^*)$, where $\mathcal{E}^* \in \mathrm{vect}(\mathcal{V})$ is the dual vector bundle of $\mathcal{E}$.

As in the d-manifold case, it can then be shown that $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$ indeed a d-stack which is equivalent in **dSta** to $\boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}} \boldsymbol{\mathcal{V}}$. Hence $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is a principal d-orbifold, and every principal d-orbifold is equivalent in **dSta** to some $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$.

We want now to outline briefly how one could use an alternative description of standard model d-orbifolds in terms of quotient of standard model d-manifolds. We want to refer to [35, §9.3 and §10.1.3] for a much more complete treatment.

The following theorem (compare [35, Theorem 9.16(a)]) shows that a d-stack $\boldsymbol{\mathcal{X}}$ is equivalent to a quotient d-stack $\boldsymbol{\mathcal{U}}/\mathcal{G}$ near each $[x] \in \mathcal{X}_{\mathrm{top}}$.

Theorem 3.4.8. *Let $\boldsymbol{\mathcal{X}}$ be a d-stack and $[x] \in \mathcal{X}_{top}$. Then there exists a quotient d-stack $[\boldsymbol{\mathcal{U}}/G]$, where $G = \mathrm{Iso}_{\mathcal{X}}([x])$, and a 1-morphism $\boldsymbol{i} : [\boldsymbol{\mathcal{U}}/G] \to \boldsymbol{\mathcal{X}}$, which is an equivalence with an open d-substack $\boldsymbol{\mathcal{U}}$ in $\boldsymbol{\mathcal{X}}$. Moreover we have $i_{top} : [u] \mapsto [x] \in \mathcal{U}_{top} \subseteq \mathcal{X}_{top}$ for some fixed point u of G in U.*

Using this theorem we can study d-orbifolds $\boldsymbol{\mathcal{X}}$ locally near a point $[x] \in \mathcal{X}_{\mathrm{top}}$, as quotients $\boldsymbol{\mathcal{X}} \simeq [\boldsymbol{U}/G]$, where $\boldsymbol{U}$ is a d-manifold and $G = \mathrm{Iso}_{\mathcal{X}}([x])$. This equivalence identifies $[x]$ with a fixed point u of G in $\boldsymbol{U}$, and as $\boldsymbol{U} \simeq \boldsymbol{S}_{V,E,s}$ in **dMan** near u for some standard model d-manifold $\boldsymbol{S}_{V,E,s}$, we can take G to act on V, E, s and the equivalence to be G-equivariant. Hence $\boldsymbol{\mathcal{X}} \simeq [\boldsymbol{S}_{V,E,s}/G]$ near $[x]$ and every result from §2.3.1 extends nicely, provided we can show where necessary that the proofs work equivariantly with respect to G.

The next proposition is the d-orbifold analogue of Theorem 2.3.5 and Proposition 2.3.6. A proof can be found in [35, Proposition 10.7].

Proposition 3.4.9. *Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold and $[x] \in \mathcal{X}_{top}$. Then there exists an open neighbourhood $\boldsymbol{\mathcal{U}}$ of $[x]$ in $\boldsymbol{\mathcal{X}}$ and an equivalence $\boldsymbol{\mathcal{U}} \simeq \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ in* **dOrb** *such that $[x]$ is identified with $[v] \in \mathcal{V}_{top}$, where $s(v) = ds(v) = 0$. Moreover $\boldsymbol{\mathcal{X}}$ near $[x]$ determines $\mathcal{V}$, $\mathcal{E}$, s up to non-canonical equivalence near $[v]$.*

Proposition 3.4.10. *A d-orbifold $\boldsymbol{\mathcal{X}}$ is an orbifold, that is $\boldsymbol{\mathcal{X}} \in \hat{\mathbf{O}}\mathbf{rb}$ if and only if $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ has a left inverse, or equivalently if and only if $T^*\boldsymbol{\mathcal{X}}$ is a vector bundle.*

3.4.4 1- and 2-morphisms in terms of differential geometric data

The following section is the d-orbifold analogue of section 2.3.2 in studying 1 and 2 morphisms of d-orbifolds in terms of 'standard models'. A much more complete discussion of this subject can be found in [35, §10.1.4].

Definition 3.4.11. Let $\mathcal{V}, \mathcal{W}$ be orbifolds, $\mathcal{E}, \mathcal{F}$ be vector bundles on $\mathcal{V}, \mathcal{W}$ and $s \in C^\infty(\mathcal{E}), t \in C^\infty(\mathcal{F})$ be smooth sections. Definition 3.4.7 defines 'standard model' principal d-orbifolds $\boldsymbol{S}_{\mathcal{V},\mathcal{E},s}$ and $\boldsymbol{S}_{\mathcal{W},\mathcal{F},t}$, which we will abbreviate in the following by $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$ and $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t} = (\mathcal{T}, \mathcal{O}'_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}}, \imath_{\mathcal{T}}, \jmath_{\mathcal{T}})$. Furthermore, assume that $f : \mathcal{V} \to \mathcal{W}$ is a 1-morphism between orbifolds and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ a morphism in $\mathrm{vect}(\mathcal{V})$, which satisfies

$$\hat{f} \circ s = f^*(t) \circ \imath : \mathcal{O}_{\mathcal{X}} \to f^*(\mathcal{F}),$$

where $\imath : \mathcal{O}_{\mathcal{S}} \to f^*(\mathcal{O}_{\mathcal{T}})$ is the natural isomorphism. The *standard model* 1-morphism $\boldsymbol{\mathcal{S}}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ can then be defined as a 1-morphism $\boldsymbol{g} = (g, g', g'') : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{T}}$in **dSta** as in [35, Definition 10.9].

In the case of $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ being a open suborbifold with inclusion 1-morphism $\imath_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \to \mathcal{V}$ and vector bundle $\tilde{\mathcal{E}} = \mathcal{E}|_{\tilde{\mathcal{V}}} = \imath^*(\mathcal{V})(\mathcal{E})$ and section $\tilde{s} = s|_{\tilde{\mathcal{V}}}$, the standard model 1-morphism $\boldsymbol{\imath}_{\tilde{\mathcal{V}},\mathcal{V}} = \boldsymbol{\mathcal{S}}_{\imath_{\tilde{\mathcal{V}}},\mathrm{id}_{\tilde{\mathcal{E}}}} : \boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}} \to \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is an 1-isomorphism, if $s^-1(0) \subseteq \tilde{\mathcal{V}}$.

It is sometimes useful to contemplate another form of 'standard models' and not to build the 'standard models' $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}, \boldsymbol{\mathcal{S}}_{f,\hat{f}}$ using orbifolds $\mathcal{V}$. One can instead use the 'standard model' descriptions $\boldsymbol{S}_{V,E,s}, \boldsymbol{S}_{f,\hat{f}}, \boldsymbol{S}_\Lambda$ for d-manifolds (as in §2.3.1) and the quotient d-stack notation as in [35, §9.3].

The following example due to Joyce [35, Example 10.11] explains this alternative form of 'standard model' for d-orbifolds.

Example 3.4.12. Consider a manifold V, a vector bundle E on V, a finite group Γ acting smoothly on V, E preserving the vector bundle structure, and $s \in C^\infty(E)$ a smooth, Γ-equivariant section of E. For $\gamma \in \Gamma$, let $r(\gamma) : V \to V$ and $\hat{r}(\gamma) : E \to r(\gamma)^*(E)$ be the Γ-action on V, E respectively. Definitions 2.3.4 and 2.3.8 define principal d-manifolds $\boldsymbol{S}_{V,E,s}$ and 1-morphism $\boldsymbol{S}_{r(\gamma),\hat{r}(\gamma)} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{V,E,s}$ for $\gamma \in \Gamma$ which can be understood as an action of Γ on $\boldsymbol{S}_{V,E,s}$. Hence we get a quotient d-stack $[\boldsymbol{S}_{V,E,s}/\Gamma]$ as in [35, Definition 9.15].

The quotient $\tilde{\mathcal{V}} = [\underline{V}/\Gamma]$ is an orbifold and it can be shown that E, s induce a vector bundle $\tilde{\mathcal{E}}$ on $\tilde{\mathcal{V}}$ and a section $\tilde{s} \in C^\infty(\tilde{\mathcal{E}})$ so that Definition 3.4.7 defines a 'standard model' principal d-orbifold $\boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}}$. It can be shown that the quotient d-stack is indeed equivalent to the so defined principal d-orbifold, that is we have $[\boldsymbol{S}_{V,E,s}/\Gamma] \simeq \boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}}$ and $[\boldsymbol{S}_{V,E,s}/\Gamma]$ is a principal d-orbifold.

Note however, that not all principal d-orbifolds $\boldsymbol{\mathcal{W}}$ can be represented by a quotient d-stack $\boldsymbol{S}_{V,E,s}/\Gamma$, as not all orbifolds $\mathcal{V}$ can be represented as a quotient of a manifold by a finite group, so $\mathcal{V} \simeq [\underline{V}/\Gamma]$ for some manifold V and finite group Γ does not hold in general.

Example 3.4.13 below explains what 1-morphism between two quotient standard models $[\boldsymbol{S}_{V,E,s}/\Gamma], [\boldsymbol{S}_{W,F,t}/\Delta]$ look like. (See [35, Example 10.12].)

Example 3.4.13. Consider two standard model quotient d-orbifolds $[\boldsymbol{S}_{V,E,s}/\Gamma]$, $[\boldsymbol{S}_{W,F,t}/\Delta]$ as in Example 3.4.12, where the action of Γ on V, E is given by $q(\gamma) : V \to V$ and $\hat{q} : E \to q(\gamma)^*(E)$ for $\gamma \in \Gamma$, and the action of Δ on W, F by $r(\delta) : W \to W$ and $\hat{r}$"$F \to r(\delta)^*(F)$ for $\delta \in \Delta$. Let $f : V \to W$ be a smooth map between manifolds and $\hat{f} : E \to f^*(F)$ a morphism of vector bundles of V, satisfying $\hat{f} \circ s = f^*(t) + O(s^2)$. Moreover, let $\rho : \Gamma \to \Delta$ be a group morphism satisfying $f \circ q(\gamma) = r(\rho(\gamma)) \circ f : V \to W$ and $q(\gamma)^*(\hat{f}) \circ \hat{q}(\gamma) = f^*(\hat{r}(\rho(\gamma))) \circ \hat{f} : E \to (f\ circq(\gamma))^*(F)$ for all $\gamma \in \Gamma$, so that $f, \hat{f}$ are equivariant under Γ, Δ, ρ. Definition 2.3.8 defines a 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ in **dSpa**. Since $f, \hat{f}$ are equivariant under Γ, Δ, ρ, we get that

$$\boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{S}_{q(\gamma),\hat{q}(\gamma)} = \boldsymbol{S}_{r(\rho(\gamma)),\hat{r}(\rho(\gamma))} \circ \boldsymbol{S}_{f,\hat{f}}$$

for all $\gamma \in \Gamma$. This data defines then a quotient 1-morphism $[\boldsymbol{S}_{f,\hat{f},\rho}] : [\boldsymbol{S}_{V,E,s}/\Gamma] \to [\boldsymbol{S}_{W,F,t}/\Delta]$ as in [35, Definition 9.15].

3.4.5 Submersions, immersion and embeddings

In this section we want to imitate section 2.3.4 and introduce the notions of submersion, immersion and embeddings of d-orbifolds. We follow here closely Joyce and refer to [35, §10.3] for a much more detailed approach including the proofs of the cited theorem and propositions.

Before stating the first definition, note that given a Deligne–Mumford C^∞-stack $\mathcal{X}$ and a 1-morphism $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ in $\mathbf{vvect}\mathcal{X}$ one can define $f^\bullet$ to be *weakly injective, injective, weakly surjective or surjective* in the exact same way as in Definition 2.3.20. Moreover Proposition 2.3.22 holds for $\mathcal{X}$ being a Deligne–Mumford C^∞-stack.

The following definition ([35, Definition 10.22]) is the d-orbifold analogue of Definition 2.3.21.

Definition 3.4.14. Let $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a 1-morphism of d-orbifolds and denote by $\Omega_{\boldsymbol{f}} : \underline{f}^\star(T^\star\boldsymbol{\mathcal{Y}}) \to T^\star\boldsymbol{\mathcal{X}}$ the corresponding 1-morphism in $\mathbf{vvect}(\mathcal{X})$. ($T^*\boldsymbol{\mathcal{X}} = (\mathcal{E}_\mathcal{X}, \mathcal{F}_\mathcal{X}, \phi_\mathcal{X})$ and $f^*(T^*\boldsymbol{\mathcal{Y}}) = (f^*(\mathcal{E}_\mathcal{Y}), f^*(\mathcal{F}_\mathcal{Y}), f^*(\phi_\mathcal{Y}))$ are virtual vector bundles on $\mathcal{X}$ of ranks vdim $\boldsymbol{\mathcal{X}}$ and vdim $\boldsymbol{\mathcal{Y}}$.) Then:

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\Omega_{\boldsymbol{f}}$ is weakly injective.

(b) We call $\boldsymbol{f}$ a *submersion* if $\Omega_{\boldsymbol{f}}$ is injective.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $\Omega_{\boldsymbol{f}}$ is weakly surjective and the 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ is representable, that is $f_* : \mathrm{Iso}_\mathcal{X}([x]) \to \mathrm{Iso}_\mathcal{Y}(f_{\mathrm{top}}([x]))$ is injective for all $[x] \in \mathcal{X}_{\mathrm{top}}$.

(d) We call $\boldsymbol{f}$ an *immersion* if $\Omega_{\boldsymbol{f}}$ is surjective and $f : \mathcal{X} \to \mathcal{Y}$ is representable.

(e) We call $\boldsymbol{f}$ a *w-embedding* if it is a w-immersion and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to f_{\mathrm{top}}(\mathcal{X}_{\mathrm{top}}) \subseteq \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism, which in particular implies f_{top} is injective, and $f_* : \mathrm{Iso}_\mathcal{X}([x]) \to \mathrm{Iso}_\mathcal{Y}(f_{\mathrm{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$.

(f) We call $\boldsymbol{f}$ an *embedding* if it is an immersion and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to f_{\mathrm{top}}(\mathcal{X}_{\mathrm{top}}) \subseteq \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism, and $f_* : \mathrm{Iso}_\mathcal{X}([x]) \to \mathrm{Iso}_\mathcal{Y}(f_{\mathrm{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$.

Using (c)-(f) from above, one can define the notion of *d-suborbifolds* of a d-orbifold. A 1-morphism $\boldsymbol{i} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ between two d-orbifolds $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ is called a *w-immersed,* or *immersed,* or *w-embedded,* or *embedded d-suborbifold* of $\boldsymbol{\mathcal{Y}}$, if $\boldsymbol{i}$ is a w-immersion, immersion, w-embedding, or embedding respectively.

The next theorem is the d-orbifold analogue of Theorem 2.3.24 and shows under what circumstances d-orbifold standard model 1-morphisms are w-submersions, submersion, w-immersions or immersion. We refer once again for more details to [35, Theorem 10.23].

Theorem 3.4.15. *Let $\boldsymbol{\mathcal{S}}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ be a 'standard model' 1-morphism in* **dOrb** *as in Definition 3.4.11, and consider for each $[v] \in \mathcal{V}_{top}$ with $s(v) = 0$ and $[w] = f_{top}([v]) \in \mathcal{W}_{top}$ the following complex*

$$0 \longrightarrow T_v\mathcal{V} \xrightarrow{ds(v)\oplus df(v)} \mathcal{E}_v \oplus T_v\mathcal{W} \xrightarrow{\hat{f}(v)\oplus -dt(w)} \mathcal{F}_w \longrightarrow 0. \tag{3.15}$$

Then

(a) $\boldsymbol{\mathcal{S}}_{f,\hat{f}}$ is a w-submersion, if and only if for all $[v] \in \mathcal{V}_{top}$ with $s(v) = 0$ and $[w] = f_{top}([v]) \in \mathcal{W}_{top}$ 3.15 is exact at the fourth term, that is $\hat{f}(v) \oplus -dt(w)$ is surjective.

(b) $\boldsymbol{\mathcal{S}}_{f,\hat{f}}$ is a submersion, if and only if for all $[v] \in \mathcal{V}_{top}$ with $s(v) = 0$ and $[w] = f_{top}([v]) \in \mathcal{W}_{top}$ 3.15 is exact at the third and fourth term.

(c) $\boldsymbol{\mathcal{S}}_{f,\hat{f}}$ is a w-immersion, if and only if for all $[v] \in \mathcal{V}_{top}$ with $s(v) = 0$ and $[w] = f_{top}([v]) \in \mathcal{W}_{top}$ 3.15 is exact at the second term and $f_* : \mathrm{Iso}_{\mathcal{V}}([v]) \to \mathrm{Iso}_{\mathcal{W}}([w])$ is injective.

(d) $\boldsymbol{\mathcal{S}}_{f,\hat{f}}$ is a immersion, if and only if for all $[v] \in \mathcal{V}_{top}$ with $s(v) = 0$ and $[w] = f_{top}([v]) \in \mathcal{W}_{top}$ 3.15 is exact at the second and forth term and $f_* : \mathrm{Iso}_{\mathcal{V}}([v]) \to \mathrm{Iso}_{\mathcal{W}}([w])$ is injective.

As in Theorem 2.3.24, conditions (a)-(d) are open conditions on $[v]$ in $\{[v] \in \mathcal{V}_{top} : s(v) = 0\}$.

We will end this section by stating the d-orbifold analogue of Theorem 2.3.25. (Compare [35, Theorem 10.24].)

Theorem 3.4.16. *Let $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a 1-morphism of d-orbifolds, and $[x] \in \mathcal{X}_{top}$ with $g_{top}([x]) = [y] \in \mathcal{Y}_{top}$. Then there exists open d-suborbifolds $\boldsymbol{\mathcal{T}} \subseteq \boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{Y}}$ with $[x] \in \mathcal{T}_{top}, [y] \in \mathcal{U}_{top}$ and $\boldsymbol{g}(\boldsymbol{\mathcal{T}}) \subseteq \boldsymbol{\mathcal{U}}$, 'standard model' 1-morphisms $\boldsymbol{\mathcal{S}}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},E,s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$ in* **dOrb**, *equivalences $\boldsymbol{i} : \boldsymbol{\mathcal{T}} \to \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}, \boldsymbol{j} : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$, and a 2-morphism $\boldsymbol{\eta} : \boldsymbol{j} \circ \boldsymbol{\mathcal{S}}_{f,\hat{f}} \circ \boldsymbol{i} \Rightarrow \boldsymbol{g}|_{\boldsymbol{\mathcal{T}}}$. Moreover*

*(a) If $\boldsymbol{g}$ is a **w-submersion**, $\boldsymbol{T}, \ldots, \boldsymbol{j}$ can be chosen such that $f : \mathcal{V} \to \mathcal{W}$ is a submersion in* **Orb**, *and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ is a surjective vector bundle morphism.*

*(b) If $\boldsymbol{g}$ is a **submersion**, $\boldsymbol{T}, \ldots, \boldsymbol{j}$ can be chosen such that $f : \mathcal{V} \to \mathcal{W}$ is a submersion and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ is an isomorphism.*

*(c) In the case of $\boldsymbol{g}$ being a **w-immersion**, $\boldsymbol{T}, \ldots, \boldsymbol{j}$ can be chosen such that $\mathcal{V}$ is an immersed suborbifold of $\mathcal{W}$, that is $f : \mathcal{V} \to \mathcal{W}$ is an immersion in* **Orb**, *and $\hat{f} : \mathcal{E} \oplus f^*(\mathcal{F})$ is an injective morphism of vector bundles.*

*(d) In the case of $\boldsymbol{g}$ being an **immersion**, $\boldsymbol{T}, \ldots, \boldsymbol{j}$ can be chosen such that $\mathcal{V}$ such that $f : \mathcal{V} \to \mathcal{W}$ is an immersion in* **Orb** *and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ is an isomorphism.*

The following corollary (see [35, Corollary 10.25]) is a consequence of Theorem 3.4.16(b) from above.

Corollary 3.4.17. *Given a submersion of d-orbifolds $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, where $\boldsymbol{\mathcal{Y}}$ is an orbifold, the source d-orbifold $\boldsymbol{\mathcal{X}}$ is actually an orbifold.*

3.4.6 Embedding d-orbifolds into orbifolds

In the following we will discuss when d-orbifolds can be embedded into orbifolds. In contrast to the d-manifold case, where Theorems 2.3.28 and 2.3.29 prove that any compact d-manifold can be embedded into some $\mathbb{R}^n$ and is therefore a principal d-manifold, we will in general not get a similar result.

The reason why a d-orbifold $\boldsymbol{\mathcal{X}}$ does in general not admit an embedding $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \mathbb{R}^n$ is that if $\mathcal{X}$ is an orbifold and $[x] \in \mathcal{X}_{\text{top}}$ a point, such that $\text{Iso}_{\mathcal{X}}([x])$ acts nontrivially on $T_x\mathcal{X}$, then for any 1-morphism $f : \mathcal{X} \to \mathbb{R}^n$ the map $df|_x : T_x\mathcal{X} \to$

$\mathbb{R}^n$ is not injective, as the kernel contains nontrivial representation parts of the action of $\mathrm{Iso}_{\mathcal{X}}([x])$ on $T_x\mathcal{X}$.

Note that it is also not possible to alter the situation a bit and try to embed a d-orbifold $\boldsymbol{\mathcal{X}}$ into $[\mathbb{R}^n/G]$ for some finite group G, acting linearly on $\mathbb{R}^n$. The reason why this approach does not succeed in general, is that there one can show that there exist representable 1-morphisms $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to [\mathbb{R}^n/G]$ if and only if $\boldsymbol{\mathcal{X}} \simeq [\boldsymbol{X}/G]$ for some d-manifold $\boldsymbol{X}$. However most orbifolds and d-orbifolds do not admit a representation as a global quotient, as the example of the weighted projective space $\mathbb{C}P^1_{1,k}$ for some $k \geq 2$ shows: the 2-dimensional orbifold $\mathbb{C}P^1_{1,k}$ is homeomorphic to S^2 with one orbifold point at $[0,1]$, whose orbifold group is $\mathbb{Z}_k$. But $\mathbb{C}P^1_{1,k} \setminus \{[0,1]\}$ is simply-connected, and one can show that $\mathbb{C}P^1_{1,k} \not\simeq [V/G]$ for any manifold V and finite group G.

We follow here closely Joyce [35, §10.5] and refer to him for a more detailed discussion and proofs of the following results.

Theorem 3.4.18. *Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold, $\mathcal{Y}$ be an orbifold and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be an embedding in* **dOrb**. *Then there exist an open suborbifold $\mathcal{V} \subseteq \mathcal{Y}$ with $\boldsymbol{f}(\boldsymbol{\mathcal{X}}) \subseteq \boldsymbol{\mathcal{V}}$, a vector bundle $\mathcal{E}$ on $\mathcal{V}$ and a smooth section $s \in C^\infty(\mathcal{E})$, such that the following* 2-*Cartesian diagram in* **dOrb** *commutes:*

$$\begin{tikzcd} \boldsymbol{\mathcal{X}} \arrow[r, "\boldsymbol{f}"'] \arrow[d, "\boldsymbol{f}"] \arrow[dr, phantom, "\Uparrow"] & \boldsymbol{\mathcal{V}} \arrow[d, "\boldsymbol{0}"'] \\ \boldsymbol{\mathcal{V}} \arrow[r, "\boldsymbol{s}"] & \boldsymbol{\mathcal{E}}, \end{tikzcd}$$

where $\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{E}}, \boldsymbol{s}, \boldsymbol{0} = F^{\mathbf{dOrb}}_{\mathbf{Orb}}(\mathcal{Y}, \mathcal{V}, \mathrm{Tot}(\mathcal{E}), \mathrm{Tot}(s), \mathrm{Tot}(0))$. Therefore, $\boldsymbol{\mathcal{X}}$ is equivalent to the "standard model" d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ of Definition 3.4.7 and is a principal d-orbifold.

Building upon Joyce's idea (see [35, Proposition 10.34]), we will later on in section 6.1 provide a useful criterion for the existence of an embedding from a d-orbifold into an orbifold.

3.4.7 Semieffective and effective d-orbifolds

In this section we follow Joyce [35, §10.9] in defining semieffective and effective d-orbifolds. Semieffective and effective d-orbifolds will play a prominent role in the bordism theory of d-orbifolds, as it will turn out, that in contrast to general d-orbifolds, semieffective and effective d-orbifolds can be generically perturbed to orbifolds and effective orbifolds. This property will be crucial in showing that (semi)effective d-orbifold bordism is isomorphic to classical (effective) orbifold bordism.

Definition 3.4.19. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. Let $[x] \in \mathcal{X}_{\mathrm{top}}$ be so that $x : \underline{\bar{*}} \to \mathcal{X}$ is a C^∞-stack 1-morphism. Applying the right exact operator x^* to the exact sequence

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\phi_{\mathcal{X}}} \mathcal{F}_{\mathcal{X}} \xrightarrow{\psi_{\mathcal{X}}} T^*\mathcal{X} \longrightarrow 0,$$

yields an exact sequence in $\mathrm{qcoh}(\underline{\bar{*}})$

$$0 \longrightarrow K_{[x]} \longrightarrow x^*(\mathcal{E}_{\mathcal{X}}) \xrightarrow{x^*(\phi_{\mathcal{X}})} x^*(\mathcal{F}_{\mathcal{X}}) \xrightarrow{x^*(\psi_{\mathcal{X}})} x^*(T^*\mathcal{X}) \cong T^*_x\mathcal{X} \longrightarrow 0. \qquad (3.16)$$

Here $K_{[x]} = \mathrm{Ker}(x^*(\phi_{\mathcal{X}}))$, and we may think of (3.16) as an exact sequence of real vector spaces, where $K_{[x]}$ and $T^*_x\mathcal{X}$ are finite-dimensional with $\mathrm{vdim}\, \boldsymbol{\mathcal{X}} = \dim T^*\mathcal{X} - \dim K_{[x]}$.

The orbifold group $\mathrm{Iso}_{\mathcal{X}}([x])$ is the group of 2-morphisms $\eta : x \Rightarrow x$, and we get an induced isomorphism $\eta^*(\mathcal{E}_{\mathcal{X}}) : x^*(\mathcal{E}_{\mathcal{X}}) \to x^*(\mathcal{E}_{\mathcal{X}})$ in $\mathrm{qcoh}(\underline{\bar{*}})$, which makes $x^*(\mathcal{E}_{\mathcal{X}})$ a representation of $\mathrm{Iso}_{\mathcal{X}}([x])$. Similarly $x^*(\mathcal{E}_{\mathcal{X}})$ and $x^*(T^*\mathcal{X})$ are representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ and moreover $x^*(\phi_{\mathcal{X}})$ and $x^*(\psi_{\mathcal{X}})$ turn out to be equivariant. Therefore $K_{[x]}$ and $T^*_x\mathcal{X}$ are $\mathrm{Iso}_{\mathcal{X}}([x])$-representations.

A d-orbifold $\boldsymbol{\mathcal{X}}$ is called *semieffective*, if $K_{[x]}$ is a trivial $\mathrm{Iso}_{\mathcal{X}}([x])$-representation for all $[x] \in \mathcal{X}_{\mathrm{top}}$ and we call $\boldsymbol{\mathcal{X}}$ *effective*, if in addition $T^*_x\mathcal{X}$ is an effective $\mathrm{Iso}_{\mathcal{X}}([x])$-representation for all $[x] \in \mathcal{X}_{\mathrm{top}}$.

The most important property of (semi-)effective d-orbifolds is that a generic perturbation yields an (effective) orbifold. The following proposition shows, that if the standard model $\mathcal{S}_{\mathcal{V},\mathcal{E},s}$ is a semieffective d-orbifold, then any generic, sufficiently small perturbation of $\int$ is an orbifold. We refer to [35, Proposition 10.58] for a proof.

Definition 3.4.20. Let $\mathcal{V}$ be an orbifold, $\mathcal{E}$ a vector bundle over $\mathcal{V}$ and $s, \tilde{s} \in C^\infty(\mathcal{V})$ be smooth section. We say that $\tilde{s} - s$ is *sufficiently small in C^1 locally*, if $|\tilde{s} - s|([v]) + |\nabla(\tilde{s} - s)|([v]) \leq C([v])$ for all $[v] \in \mathcal{V}_{\mathrm{top}}$, for some choice of connection ∇ on $\mathcal{E}$ and metrics $|\cdot|$ on $\mathcal{E}, \mathcal{E}\otimes T^*\mathcal{V}$ and some continuous function $C : \mathcal{V}_{\mathrm{top}} \to (0,\infty)$.

Proposition 3.4.21. *Let $\mathcal{V}$ be an orbifold, $\mathcal{E}$ a vector bundle over $\mathcal{V}$ and $s : \mathcal{V} \to \mathcal{E}$ a smooth section. Suppose the standard model $\mathcal{S}_{\mathcal{V},\mathcal{E},s}$ is a semieffective d-orbifold. Then for any generic, smooth perturbation $\tilde{s}$ of s, with $\tilde{s}-s$ sufficiently small in C^1 locally, the d-orbifold $\mathcal{S}_{\mathcal{V},\mathcal{E},\tilde{s}}$ is an orbifold, that is it lies in $\hat{\mathbf{O}}\mathbf{rb} \subset \mathbf{dOrb}$.*

If $\mathcal{S}_{\mathcal{V},\mathcal{E},s}$ is effective, then the perturbed $\mathcal{S}_{\mathcal{V},\mathcal{E},\tilde{s}}$ is an effective orbifold.

Using good coordinate systems, one can extend Proposition 3.4.21 to general (semi)effective d-orbifolds $\boldsymbol{\mathcal{X}}$ as in [35, §10.9].

Remark 3.4.22. As we have seen, being a semieffective (effective) d-orbifold is a sufficient condition for a small, generic perturbation to be an (effective) orbifold. However, it is not a necessary condition. To get a more precise description of sufficient and necessary conditions, let $[x] \in \mathcal{X}_{\mathrm{top}}$, the representations $K_{[x]}, T_x^*\mathcal{X}$ of $\mathrm{Iso}_{\mathcal{X}}([x])$, and the splittings $K_{[x]} = K_{[x],\mathrm{tr}} \oplus K_{[x],\mathrm{nt}}$ and $T_x^*\mathcal{X} = (T_x^*)_{\mathrm{tr}} \oplus (T_x^*\mathcal{X})_{\mathrm{nt}}$ be as in Definition 3.4.19.

Let $\mathrm{Hom}(K_{[x],\mathrm{nt}}, (T_x^*\mathcal{X})_{\mathrm{nt}})$ be the finite-dimensional vector space of morphisms of $\mathrm{Iso}_{\mathcal{X}}([x])$-representations $\lambda : K_{[x],\mathrm{nt}} \to (T_x^*\mathcal{X})_{\mathrm{nt}}$, and $\mathrm{Hom}_0(K_{[x],\mathrm{nt}}, (T_x^*\mathcal{X})_{\mathrm{nt}})$ for the closed subset of $\mathrm{Iso}_{\mathcal{X}}([x])$-representations λ, which are not injective. Note that $\mathrm{Hom}_0(K_{[x],\mathrm{nt}}, (T_x^*\mathcal{X})_{\mathrm{nt}})$ will in general be singular.

Then, a small generic perturbation of $\boldsymbol{\mathcal{X}}$ is an orbifold if and only if for all $[x] \in \mathcal{X}_{\mathrm{top}}$, either $K_{[x],\mathrm{nt}} = 0$ or the codimension of $\mathrm{Hom}_0(K_{[x],\mathrm{nt}}, (T_x^*\mathcal{X})_{\mathrm{nt}})$ in $\mathrm{Hom}(K_{[x],\mathrm{nt}}, (T_x^*\mathcal{X})_{\mathrm{nt}})$ is strictly greater than $\dim(T_x^*\mathcal{X})_{\mathrm{tr}} - \dim K_{[x],\mathrm{tr}}$. A small generic perturbation of $\boldsymbol{\mathcal{X}}$ is an effective orbifold if and only if in addition to the condition above for each $[x] \in \mathcal{X}_{\mathrm{top}}$, either $[(T_x^*\mathcal{X})_{\mathrm{nt}}] - [K_{[x],\mathrm{nt}}] = [R]$ in Λ^Γ for some effective representation R of $\mathrm{Iso}_{\mathcal{X}}([x])$, or $\dim(T_x^*\mathcal{X})_{\mathrm{tr}} < \dim K_{[x],\mathrm{tr}}$.

The next lemmas summarise some properties of (semi)effective d-orbifolds.

Lemma 3.4.23. *Let $\mathcal{X}$ be an orbifold and $\boldsymbol{\mathcal{X}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{X})$ its associated d-orbifold. Then $\boldsymbol{\mathcal{X}}$ is a semieffective d-orbifold, and if $\mathcal{X}$ is effective $\boldsymbol{\mathcal{X}}$ is effective.*

This lemma is an immediate consequence of the fact that (semi-)effectiveness is preserves by equivalences $\boldsymbol{i} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in **dOrb**, as these induce isomorphisms $K_{[x]} \cong K_{[y]}, T_x^*\mathcal{X} \cong T_y^*\mathcal{Y}$ for $i_{\mathrm{top}}([x]) = [y]$. The Lemma follows, as if $\mathcal{X}$ is an orbifold and $\boldsymbol{\mathcal{X}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{X})$ then $\mathcal{E}_{\mathcal{X}} = K_{[x]} = 0$ in Definition 3.4.19. Moreover, the last part of Definition 3.4.19 yields the following

Lemma 3.4.24. *Let $\boldsymbol{\mathcal{X}}$ be an effective d-orbifold. Then the underlying C^∞-stack $\mathcal{X}$ is effective.*

Note that the converse of Lemma 3.4.24 is not true, that is the condition that the underlying C^∞-stack $\mathcal{X}$ of a d-orbifold $\boldsymbol{\mathcal{X}}$ is effective does not imply that $\boldsymbol{\mathcal{X}}$ is an effective d-orbifold.

Lemma 3.4.25. *Let $\boldsymbol{\mathcal{X}}$ be a semieffective d-orbifold, Γ a finite group and $\lambda \in \Lambda^\Gamma$. Then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda} = \emptyset$ unless $\lambda \in \Lambda_+^\Gamma \subset \Lambda^\Gamma$. In the case of $\boldsymbol{\mathcal{X}}$ being effective then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda} = \emptyset$ unless $\lambda = [R]$ for R an effective Γ-representation.*

Again, the converse is false: $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda} = \emptyset$ for $\lambda \in \Lambda^\Gamma \setminus \Lambda_+^\Gamma$ does not imply $\boldsymbol{\mathcal{X}}$ to be semieffective and similarly for effective d-orbifolds.

Lemma 3.4.26. *Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ be (semi)effective d-orbifolds, then the product $\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}$ is also (semi)effective. More generally, a fibre product $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{\mathcal{Z}}} \boldsymbol{\mathcal{Y}}$ in* **dOrb** *with $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ (semi)effective and $\boldsymbol{\mathcal{Z}}$ a manifold is (semi)effective.*

Proposition 3.4.27. *Let $\boldsymbol{\mathcal{X}}$ be an oriented, semieffective d-orbifold, and Γ a finite group. Then there exist orientations on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \boldsymbol{\mathcal{X}}_0^{\Gamma,\lambda}$ for all $\lambda \in \Lambda_{ev,+}^\Gamma$. These depend on orientations $R_1, \ldots, R_k$ for representatives $(R_1, \rho_1), \ldots, (R_k, \rho_k)$ of the nontrivial irreducible, even-dimensional Γ-representations.*

Proposition 3.4.28. *Let Γ be a finite group and $\lambda \in \Lambda_{ev,+}^\Gamma$ with $\Phi^\Gamma(\delta, \lambda) = 1$ for all $\delta \in Aut(\Gamma)$ with $\lambda \cdot \delta = \lambda$. Then, for any oriented, semieffective d-orbifold $\boldsymbol{\mathcal{X}}$, the orbifold strata $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \tilde{\boldsymbol{\mathcal{X}}}_0^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}_0^{\Gamma,\mu}$ are oriented, where $\mu = \lambda \cdot Aut(\Gamma)$ in $\Lambda_{ev,+}^\Gamma / Aut(\Gamma)$.*

3.4.8 D-orbifold strata

This section is the d-orbifold analogue of section 3.2.7 and discusses orbifold strata of d-orbifolds. We want once again refer to [35, §10.7] for a much more rigorous and complete approach.

Definition 3.4.29. Let Γ be a finite group and $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$, $\Lambda^\Gamma = K_0(\mathrm{Rep}_{\mathrm{nt}}(\Gamma))$, $\Lambda^\Gamma_+ \subseteq \Lambda^\Gamma$ and $\dim : \Lambda^\Gamma \to \mathbb{Z}$ be as in Definition 3.2.19. Denote by $R_0, \dots, R_k$ the representatives for the isomorphism classes of irreducible Γ-representations, where $R_0 = \mathbb{R}$ is given by the trivial irreducible representation, so that $R_1, \dots, R_k$ are nontrivial. Using this setup, Λ^Γ is freely generated over $\mathbb{Z}$ by $[R_1], \dots, [R_k]$ and hence equation 3.7 in Definition 3.2.19 yields isomorphisms $\Lambda^\Gamma \cong \mathbb{Z}^k, \Lambda^\Gamma_+ \cong \mathbb{N}^k$.

Let now $\boldsymbol{\mathcal{X}}$ be a d-orbifold. As $\boldsymbol{\mathcal{X}}$ is a d-stack, it inherits, as in [35, Definitions 9.24 and 9.25], a d-stack $\boldsymbol{\mathcal{X}}^\Gamma$ and a 1-morphisms $\boldsymbol{\mathcal{O}}^\Gamma(\boldsymbol{\mathcal{X}}) : \boldsymbol{\mathcal{X}}^\Gamma \to \boldsymbol{\mathcal{X}}$. The virtual cotangent bundle $T^*\boldsymbol{\mathcal{X}} = (\mathcal{E}_\mathcal{X}, \mathcal{F}_\mathcal{X}, \phi_\mathcal{X})$ of $\boldsymbol{\mathcal{X}}$ is a virtual vector bundle of rank $\mathrm{vdim}\, \boldsymbol{\mathcal{X}}$ on $\mathcal{X}$. Hence the pullback $\mathcal{O}^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})$ of $T^*\boldsymbol{\mathcal{X}}$ is a virtual vector bundle on $\mathcal{X}^\Gamma$. As in Definition 3.2.22 the virtual quasicoherent sheaves $\mathcal{O}^\Gamma(\mathcal{X})^*(\mathcal{E}_\mathcal{X}), \mathcal{O}^\Gamma(\mathcal{X})^*(\mathcal{F}_\mathcal{X})$ admit decompositions of the form 3.10 and 3.9 and as $\mathcal{O}^\Gamma(\mathcal{X})^*(\phi_\mathcal{X})$ is Γ-equivariant and preserves therefore these splitting,we get the following decomposition in $\mathbf{vqcoh}(\mathcal{X}^\Gamma)$:

$$
\begin{aligned}
&\mathcal{O}^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}}) \cong \bigoplus_{i=0}^{k} (T^*\boldsymbol{\mathcal{X}})_i^\gamma \otimes R_i \quad \text{for} (T^*\boldsymbol{\mathcal{X}})_i^\Gamma \in \mathbf{vqcoh}(\mathcal{X}^\Gamma) \\
&\mathcal{O}^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})_0^\Gamma = (T^*\boldsymbol{\mathcal{X}})_{\mathrm{tr}}^\Gamma \oplus (T^*\boldsymbol{\mathcal{X}})_{\mathrm{nt}}^\Gamma.
\end{aligned}
\tag{3.17}
$$

Here $(T^*\boldsymbol{\mathcal{X}})_{\mathrm{tr}}^\Gamma \cong (T^*\boldsymbol{\mathcal{X}})_0^\Gamma \otimes R_0$ and $(T^*\boldsymbol{\mathcal{X}})_{\mathrm{nt}}^\Gamma \cong \bigoplus_{i=0}^{k} (T^*\boldsymbol{\mathcal{X}})_i^\Gamma \otimes R_i$.

Moreover it can be shown that $T^*(\boldsymbol{\mathcal{X}}^\Gamma) \cong (T^*\boldsymbol{\mathcal{X}})_{\mathrm{tr}}^\Gamma$.

The splitting 3.17 implies that since $\mathcal{O}^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})$ is a virtual vector bundle, the $(T^*\boldsymbol{\mathcal{X}})_i^\Gamma$ are *virtual vector bundles of mixed rank*, where the ranks may vary on different connected components of $\boldsymbol{\mathcal{X}}^\Gamma$. (Compare [35, Definition 10.38] for more details.)

Example 3.4.30. Given an orbifold $\mathcal{V}$, a vector bundle $\mathcal{E} \to \mathcal{V}$ and a smooth section $s : \mathcal{V} \to \mathcal{E}$, we can form the 'standard model' d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ as in Definition 3.4.7. Given a finite group Γ, we will define the orbifold strata $(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s})^{\Gamma,\lambda}$

of $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ as follows: consider the decomposition $\mathcal{V}^\Gamma = \coprod_{\lambda_1\in\Lambda_+^\Gamma} \mathcal{V}^{\Gamma,\lambda_1}$. As in [35, Appendix C], the vector bundle $\mathcal{E}^\Gamma = \mathcal{O}^\Gamma(\mathcal{V})^*(\mathcal{E})$ on $\mathcal{V}^\Gamma$ has a Γ-representation and can be split as follows:

$$\mathcal{E}^\Gamma \cong \bigoplus_{i=0}^{k} \mathcal{E}_i^\Gamma \otimes R_i,$$
$$\mathcal{E}^\Gamma = \mathcal{E}_{\mathrm{tr}}^\Gamma \oplus \mathcal{E}_{\mathrm{nt}}^\Gamma.$$

Here R_i are irreducible Γ-representation over $\mathbb{R}$ with $R_0 = \mathbb{R}$ being the trivial representation, and $\mathcal{E}_{tr}^\Gamma, \mathcal{E}_{\mathrm{nt}}^\Gamma$ denote the subsheaves of $\mathcal{E}^\Gamma$ corresponding to $\mathcal{E}_0^\Gamma \otimes R_0$ and $\bigoplus_{i=1}^k \mathcal{E}_i^\Gamma \otimes R_i$.

3.4.9 Good coordinate systems

In this section we want to briefly review good coordinate systems of d-orbifolds and Kuranishi neighbourhoods on d-orbifolds. All of the following can be found in much greater detail and rigour in Joyce [35, §10.8].

We start by defining what *type A Kuranishi neighbourhoods* and *type A coordinate changes* are. (Compare [35, Definitions 10.45 & 10.46].)

Definition 3.4.31. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. A *type A Kuranishi neighbourhood* on $\boldsymbol{\mathcal{X}}$ is a quintuple $(V, E, \Gamma, s, \boldsymbol{\psi})$ consisting of a manifold V, a vector bundle $E \to V$, a finite group Γ acting smoothly and locally effectively (in the sense of Definition 3.2.13) on V, E, and a smooth, Γ-equivariant section $s : V \to E$. If we denote the actions of Γ on V and E by $r(\gamma) : V \to V$ and $\hat{r}(\gamma) : E \to r(\gamma)^*(E)$ for $\gamma \in \Gamma$, we can define a principal d-orbifold $[\boldsymbol{\mathcal{S}}_{V,E,s}/\Gamma]$ as in Example 3.4.12.

We require that $\boldsymbol{\psi} : [\mathcal{S}_{V,E,s}/\Gamma] \to \boldsymbol{\mathcal{X}}$ is a 1-morphism of d-orbifolds which is an equivalence with a nonempty open d-suborbifold $\psi([\mathcal{S}_{V,E,s}/\Gamma]) \subseteq \boldsymbol{\mathcal{X}}$. We call such a quintuple $(V, E, \Gamma, s, \boldsymbol{\psi})$ a *type A Kuranishi neighbourhood* of $[x] \in \mathcal{X}_{\mathrm{top}}$, if $[x] \in \boldsymbol{\psi}([\boldsymbol{\mathcal{S}}_{V,E,s}/\Gamma])_{\mathrm{top}}$.

Definition 3.4.32. Let $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ be type A Kuranishi neighbourhoods on a d-orbifold $\boldsymbol{\mathcal{X}}$ with

$$\boldsymbol{\emptyset} \neq \boldsymbol{\psi}_i([\boldsymbol{\mathcal{S}}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\boldsymbol{\mathcal{S}}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \boldsymbol{\mathcal{X}}.$$

A *type A coordinate change from* $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ to $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ is given by a quintuple $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$, where

(a) $\emptyset \neq V_{ij} \subseteq V_i$ is a Γ_i-invariant open submanifold, which satisfies

$$\boldsymbol{\psi}_i([\boldsymbol{\mathcal{S}}_{V_{ij}}, E_i|_{V_{ij}}, s_i|_{V_{ij}}/\Gamma_i]) = \boldsymbol{\psi}_i([\boldsymbol{\mathcal{S}}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\boldsymbol{\mathcal{S}}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \boldsymbol{\mathcal{X}}.$$

(b) $\rho_{ij} : \Gamma_i \to \Gamma_j$ is an injective morphism of groups.

(c) $e_{ij} : V_{ij} \to V_j$ is an embedding of manifolds compatible with the action, that is $e_{ij} \circ r_i(\gamma) \equiv r_j(\rho_{ij}(\gamma) \circ e_{ij}) : V_{ij} \to V_j$ for all $\gamma \in \Gamma_i$. Moreover, if $v_i, v_i' \in V_{ij}$ and $\delta \in \Gamma_j$ with $r_j(\delta) \circ e_{ij}(v_i') = e_{ij}(v_i)$, then there exists a $\gamma \in \Gamma_i$ with $\rho_{ij}(\gamma) = \delta$ and $r_i(\gamma)(v_i') = v_i$.

(d) $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$ is an embedding of vector bundles (in other words, $\hat{e}_{ij}$ has a left inverse) such that $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j)$ and $r_i(\gamma)^*(\hat{e}_{ij}) \circ \hat{r}_i(\gamma) \equiv e_{ij}^*(\hat{r}_j(\rho_{ij}(\gamma))) \circ \hat{e}_{ij} : E_i|_{V_{ij}} \to (e_{ij} \circ r_i(\gamma))^*(E_j)$ for all $\gamma \in \Gamma_i$. Hence, Example 3.4.13 yields an 1-morphism

$$[\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}] : [\boldsymbol{\mathcal{S}}_{V_{ij}}, E_i|_{V_{ij}}, s_i|_{V_{ij}}/\Gamma_i] \to [\boldsymbol{\mathcal{S}}_{V_j,E_j,s_j}/\Gamma_j],$$

where $[\boldsymbol{\mathcal{S}}_{V_{ij}}, E_i|_{V_{ij}}, s_i|_{V_{ij}}/\Gamma_i]$ is an open d-suborbifold in $[\boldsymbol{\mathcal{S}}_{V_i,E_i,s_i}/\Gamma_i]$.

(e) For all $v_i \in V_{ij}$ with $s_i(v_i) = 0$ and $v_j = (e_{ij}(v_i)) \in V_j$ the following linear map is an isomorphism

$$(ds_j(v_j))_* : (T_{v_j}V_j)/(de_{ij}(v_i)[T_{v_i}V_i]) \to (E_j|_{v_j})/(\hat{e}_{ij}(v_i)[E_i|_{v_i}]).$$

This implies then that $[\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}]$ is an equivalence with an open d-suborbifold of $[\boldsymbol{\mathcal{S}}_{V_i,E_i,s_i}/\Gamma_i]$.

(f) $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ [\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}] \Rightarrow \boldsymbol{\psi}_i|_{[\boldsymbol{\mathcal{S}}_{V_{ij}}, E_i|_{V_{ij}}, s_i|_{V_{ij}}/\Gamma_i]}$ is a 2-morphism in **dOrb**.

(g) The quotient topological space $V_i \amalg_{V_{ij}} V_j = (V_i \amalg V_j)/\sim$ is Hausdorff, where $\sim$ is the equivalence relation identifying $v \in V_{ij} \subseteq V_i$ with $e_{ij}(v) \in V_j$.

Definition 3.4.33. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. A *type A good coordinate system* on $\boldsymbol{\mathcal{X}}$ consists of the following data satisfying $(a) - (e)$

(a) An index set I, together with a total order $<$ on I, making $(I,<)$ into a well-ordered set.

(b) For each $i \in I$, there exists a type A Kuranishi neighbourhood $(V_i,E_i,\Gamma_i,s_i,\boldsymbol{\psi}_i)$ on $\boldsymbol{\mathcal{X}}$, such that the following holds: if $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i])$, so that $\boldsymbol{\mathcal{X}}_i \subseteq \boldsymbol{\mathcal{X}}$ is an open d-suborbifold, and $\boldsymbol{\psi}_i : [\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i] \to \boldsymbol{\mathcal{X}}_i$ is an equivalence, then we require $\bigcup_{i\in I} \boldsymbol{\mathcal{X}}_i = \boldsymbol{\mathcal{X}}$, making $\{\boldsymbol{\mathcal{X}}_i; i \in I\}$ an open cover of $\boldsymbol{\mathcal{X}}$.

(c) For all $i < j$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j = \boldsymbol{\emptyset}$, there exists a type A coordinate change $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$ from $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ to $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$.

(d) For all $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k = \boldsymbol{\emptyset}$, there exists $\gamma_{ijk} \in \Gamma_k$ satisfying $\rho_{ij} = \gamma_{ijk}\rho_{jk}(\rho_{ij}(\gamma))\gamma_{ijk}^{-1}$ for all $\gamma \in \Gamma_i$, and

$$\begin{aligned} e_{ij}|_{V_{ij}\cap e_{ij}^{-1}(V_{jk})} &= r_k(\gamma_{ijk}) \circ e_{jk} \circ e_{ij}|_{V_{ik}\cap e_{ij}^{-1}(V_{ij})}, \\ \hat{e}_{ij}|_{V_{ij}\cap e_{ij}^{-1}(V_{jk})} &= (e_{ij}^*(e_{jk}^*(\hat{r}_k(\gamma_{ijk}))) \circ e_{jk}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{V_{ik}\cap e_{ij}^{-1}(V_{ij})}). \end{aligned}$$

Moreover this γ_{ijk} is uniquely determined.

(e) For all $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j = \boldsymbol{\emptyset}$ and $\boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k = \boldsymbol{\emptyset}$ the following holds: let $v_i \in V_{ik}, v_j \in V_{jk}$ and $\delta \in \Gamma_k$ with $e_{jk}(v_j) = r_k(\delta) \circ e_{ik}(v_i) \in V_k$. Then $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k = \boldsymbol{\emptyset}$, and $v_i \in V_{ij}$, and there exists a $\gamma \in \Gamma_j$ with $\rho_{jk}(\gamma) = \delta\gamma_{ijk}$ and $v_j = r_j(\gamma) \circ e_{ij}(v_i)$.

Let Y now be a manifold and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Man}}^{\mathbf{dOrb}}(Y)$ a 1-morphism in **dOrb**. A *type A good coordinate system for* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ consists of a type A good coordinate system $(I, <, \ldots, \gamma_{ijk})$ for $\boldsymbol{\mathcal{X}}$ satisfying $(a) - (e)$, together with the following data satisfying $(f) - (g)$:

(f) Let $i \in I$. Then there exists a smooth map $g_i : V_i \to Y$, with $g_i \circ r_i(\gamma) = g_i$ for all $\gamma \in \Gamma_i$, so that

$$[\boldsymbol{S}_{g_i,0,\pi}] \colon [\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i] \to [\boldsymbol{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}},$$

where $\pi : \Gamma_i \to \{1\}$ denotes the projection. Moreover, there exists a 2-morphism $\eta_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow [\boldsymbol{S}_{g_i,0,\pi}]$ in **dOrb**, which will sometimes required to be a submersion.

(g) For all $i < j$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \neq \emptyset$, the following equality holds $g_i \circ e_{ij} = g_i|_{V_{ij}}$. Note that this equation implies that

$$[\boldsymbol{S}_{g_i,0,\pi}] \circ [\boldsymbol{S}_{e_{ij},\hat{e}_{ij},\rho_{ij}}] = [\boldsymbol{S}_{g_i,0,\pi}]|_{[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]} :$$
$$[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i] \to [\boldsymbol{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}}.$$

The following theorem due to Joyce [35, Theorem 10.48] shows that there exists for any d-orbifold $\boldsymbol{\mathcal{X}}$ a type A good coordinate system. The proof of this theorem is rather complex and lengthy, and we refer to [35, Appendix D].

Theorem 3.4.34. *Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. Then there exists a type A good coordinate system $(I, <, (V_i, E_i, s_i, \boldsymbol{\psi}_i), (V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij}), \gamma_{ijk})$ for $\boldsymbol{\mathcal{X}}$. If $\boldsymbol{\mathcal{X}}$ is compact, I can be taken to be finite.*

Let $\{\boldsymbol{\mathcal{U}}_j : j \in J\}$ be an open cover of $\boldsymbol{\mathcal{X}}$. Then the $\boldsymbol{\mathcal{X}}_i$ can be taken as $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i/\Gamma_i}]) \subseteq \boldsymbol{\mathcal{U}}_i$ for each $i \in I$ and some $j_i \in J$.

If Y is a manifold and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Man}}^{\mathbf{dOrb}}$ is a 1-morphism in $\mathbf{dOrb}$. Then all of the above extends to type A good coordinate systems for $\boldsymbol{H} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, and moreover the g_i in Definition 3.4.33 can be taken to submersions.

Chapter 4

Relation between d-manifolds and d-orbifolds and other geometric structures

This chapter briefly summarizes [35, §14] in relating d-manifolds and d-orbifolds to other classes of spaces. We will in particular focus on the relationship of d-manifolds and d-orbifolds to Kuranishi spaces (in the sense of Fukaya, Oh, Ohta and Ono [18]) and to $\mathbb{C}$-schemes and $\mathbb{C}$-stacks with obstruction theories, as these spaces play an important role in several moduli problems.

4.1 Fukaya–Oh–Ohta–Ono's Kuranishi spaces

We begin by recalling the basic definitions and start with the following definition, which is analogues to that of a 'Kuranishi space with tangent bundle' in [18, §A1.1]. We refer to the original work of Fukaya and Ono [20], the extensive treatment of the subject in Fukaya, Oh, Ohta and Ono [18] and their most recent work [19] for a much more detailed and complete resource.

Definition 4.1.1. Let X be a topological space and $p \in X$. A *Kuranishi neighbourhood* of p in X is a quintuple $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ consisting of a manifold V_p, a vector bundle $E_p \to V_p$, a finite group Γ_p acting structure preserving, smoothly and locally effectively on V_p and E_p, a Γ_p-equivariant section s_p, and a homeomorphism $\psi_p : s_p^{-1}(0)/\Gamma_p \to U \subseteq X$, where U is an open neighbourhood of p in X.

Definition 4.1.2. Let X be a topological space, and consider Kuranishi neighbourhoods $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ and $(V_q, E_q, \Gamma_q, s_q, \psi_q)$ of $p, q \in X$. We call a quadruple $(V_{pq}, e_{pq}, \hat{e}_{pq}, \rho_{pq})$ a *coordinate change* from $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ to $(V_q, E_q, \Gamma_q, s_q, \psi_q)$, if the following conditions are satisfied:

(a) $V_{pq} \subseteq V_p$ is a non-empty, Γ_p-invariant open submanifold, satisfying

$$p \in \psi_p(s_p|_{V_{pq}}^{-1}(0)/\Gamma_p) \subseteq \psi_q(s_q^{-1}(0)/\Gamma_q) \subseteq X.$$

(b) $\rho_{pq} : \Gamma_p \to \Gamma_q$ is injective.

(c) $e_{pq} : V_{pq} \to V_q$ is an embedding of manifolds, such that $e_{pq} \circ r_p(\gamma) = r_q(\rho_{pq}(\gamma)) \circ e_{pq}$ for all $\gamma \in \Gamma_p$. Here $r_p(\gamma) : V_p \to V_p$ denotes the action of Γ_p on V_p, where $\gamma \in \Gamma_p$. Moreover, if $v_p, v_p' \in V_{pq}$ and $\gamma \in \Gamma_q$ with $r_q(\delta) \circ e_{pq}(v_p') = e_{pq}(v_p)$, there exists $\gamma \in \Gamma_p$ with $\rho_{pq}(\gamma) = \delta$ and $r_p(\gamma)(v_p') = v_p$.

(d) $\hat{e}_{pq} : E_p|_{V_{pq}} \to e_{pq}^*(E_q)$ is an embedding of vector bundles, such that $\hat{e}_{pq} \circ s_p|_{V_{pq}} = e_{pq}^*(s_q)$ and $r_p(\gamma)^*(\hat{e}_{pq}) \circ \hat{r}_p(\gamma) = e_{pq}^*(\hat{r}_q(\rho_{pq}(\gamma))) \circ \hat{e}_{pq} : E_p|_{V_{pq}} \to (e_{pq} \circ r_p(\gamma))^*(E_q)$ for all $\gamma \in \Gamma_p$. Here $\hat{r}_p(\gamma) : E_p \to r_p(\gamma)^*(E_p)$ denotes the action of Γ_p on E_p, where $\gamma \in \Gamma_p$.

(e) Let $v_p \in V_{pq}$ with $s_p(v_p) = 0$ and $v_q = e_{pq}(v_p) \in V_q$. Then we require

$$(ds_q(v_q))_* : (T_{v_q}V_q)/(de_{pq}(v_p)[T_{v_p}V_p]) \to (E_q|_{v_q})/(\hat{e}_{pq}(v_p)[E_p|_{v_p}]),$$

to be an isomorphism.

(f) $\psi_q \circ (e_{pq})_*|_{s_p|_{V_{pq}}^{-1}(0)/\Gamma_p} = \psi_p|_{s_p|_{V_{pq}}^{-1}(0)/\Gamma_p} : s_p|_{V_{pq}}^{-1}(0)/\Gamma_p \to X$, where $(e_{pq})_* : V_{pq}/\Gamma_p \to V_q/\Gamma_q$ is the induced map by $e_{pq} : V_{pq} \to V_q$.

With these ingredients, we can finally define what a *Kuranishi structure* κ on a topological space should be. We omit here some technical details, as this will not be important for the applications we have in mind.

Definition 4.1.3. Let X be a second countable, topological Hausdorff space. A *Kuranishi structure* κ on X of dimension $n \in \mathbb{Z}$ assigns for each $p \in X$ with $\dim V_p - \operatorname{rank} E_p = n$ a Kuranishi neighbourhood $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ and a coordinate change $(V_{pq}, e_{pq}, \hat{e}_{pq}, \rho_{pq})$ for all $p, q \in X$ with $p \in \psi_q(s_q^{-1}(0)/\Gamma_q)$ satisfying some 'associativity' condition.

A *Kuranishi space* (X, κ) *of virtual dimension* n is a second countable, topological Hausdorff space X, admitting a Kuranishi structure κ of dimension n.

As there is currently no definition of morphisms between Kuranishi spaces, they do not form a category. However Fukaya, Oh, Ohta and Ono define morphisms from Kuranishi spaces to manifolds (compare [18, Definition A1.13]).

Definition 4.1.4. Given a Kuranishi space (X, κ) and a manifold Y, a *strongly smooth map* $(f, \lambda) : (X, \kappa) \to Y$ is a continuous map $f : X \to Y$ of topological spaces together with additional data λ, which assigns to each Kuranishi neighbourhood $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ in κ for $p \in X$ a smooth map $f_p : V_p \to Y$ such that $f \circ \psi_p = (f_p)_* : s_p^{-1}(0)/\Gamma_p \to Y$ and f is compatible with coordinate changes. (See [35, Definition 14.14].)

If $f_p : V_p \to Y$ is a submersion for all $p \in X$, (f, λ) is called *weakly submersive.*

The following remark will summarize the similarities and differences of Kuranishi spaces and d-orbifolds. We will follow here closely Joyce and refer to [35, Remark 14.15] for a more complete and detailed resource on the topic.

Remark 4.1.5. (a) Given a Kuranishi space (X, κ), Fukaya, Oh, Ohta, Ono [18, Definition A1.17] define an orientation on (X, κ) as an orientation of the line bundle $\Lambda^{\mathrm{top}} E_p \otimes \Lambda^{\mathrm{top}} T^* V_p$ on V_p for $p \in X$ which is compatible under coordinate changes. This definition corresponds to the definition of orientation on a standard model d-manifold $\boldsymbol{S}_{V,E,s}$. (Compare [35, Definition 4.48].)

(b) The *good coordinate systems* of Fukaya et al. [18, Lemma A1.11] and their claimed existence are very similar to type A good coordinate systems defined in section 3.4.9 and Theorem 3.4.34.

(c) Given a compact, oriented Kuranishi space (X, κ) of virtual dimension n, a manifold Y and a strongly smooth map $(f, \lambda) : (X, \kappa) \to Y$, Fukaya et al. define after choosing some extra data a *virtual chain* $[(X, \kappa)]_{\mathrm{virt}} \in C_n^{\mathrm{si}}(Y; \mathbb{Q})$ for (X, κ), and for Kuranishi spaces without boundary a *virtual class* $[(X, \kappa)]_{\mathrm{virt}} \in H_n^{\mathrm{si}}(Y; \mathbb{Q})$. The proof of the existence (see [18, Theorem A1.23]) uses good coordinate systems and is similar to the proof of Theorem 7.4.5 in section 7 below.

This corresponds to the explanation that compact oriented d-manifolds and d-orbifolds admit virtual classes. (Compare chapter 7 for more details.)

(d) Fukaya and Ono define in [20, Definition 5.13] *bundle systems* on Kuranishi spaces (although the sections s_p in Kuranishi neighbourhoods in [20] are just assumed to be continuous and not smooth like in [18]). These bundle systems correspond to virtual vector bundles defined in section 3.2.2.

(e) For a compact, symplectic manifold (X, ω) with compatible almost complex structure J, Fukaya and Ono [20, §12-§16] construct an oriented Kuranishi structure (with different definition of Kuranishi space) on the moduli space $\overline{\mathcal{M}}_{g,n}(X, J, \beta)$ of n-pointed, genus g stable J-holomorphic curves in X.

The following theorem due to Joyce [35, Theorem 14.17] provides the connection between Kuranishi spaces due to Fukaya, Ono, Ohta, Oh [18] and d-orbifolds.

Theorem 4.1.6. *(a) Given a Kuranishi space (X, κ), one can construct a d-orbifold with corners $\mathfrak{X}$ with the same underlying topological space and virtual dimension, which is unique up to equivalence in $\mathbf{dOrb^c}$. Similarly strong smooth maps $(f, \lambda) : (X, \kappa) \to Y$ from (X, κ) into a manifold Y induce 1-morphisms $\boldsymbol{f} : \mathfrak{X} \to \mathfrak{Y} = F_{\mathbf{Man}}^{\mathbf{dOrb^c}}(Y)$ in $\mathbf{dOrb^c}$ with the same continuous map $f : X \to Y$, which are unique up to 2-isomorphism in $\mathbf{dOrb^c}$.*

(b) Vice versa: given a d-orbifold with corners $\mathfrak{X}$, one can define a Kuranishi space (X, κ) from $\mathfrak{X}$ with the same topological space $X = \mathcal{X}_{top}$ and virtual dimension. The Kuranishi structure κ depends on many arbitrary choices. Moreover, given a 1-morphism $\boldsymbol{f} : \mathfrak{X} \to \mathfrak{Y} = F_{\mathbf{Man}}^{\mathbf{dOrb^c}}(Y)$ in $\mathbf{dOrb^c}$, one can construct a strongly smooth map $(f, \lambda) : (X, \kappa) \to Y$ with the same

underlying continuous maps $f = f_{top} : \mathcal{X}_{top} \to Y$, where λ depends on many choices.

(c) *Construction (a) is 'left inverse' up to equivalence to construction (b). So in other words, given a d-orbifold with corners $\boldsymbol{\mathfrak{X}}$, applying (b) yields a Kuranishi structure (X, κ). Applying (a) to this Kuranishi structure yields then a d-orbifold with corners $\boldsymbol{\mathfrak{X}}'$ which is equivalent to the d-orbifold $\boldsymbol{\mathfrak{X}}'$ in* $\mathbf{dOrb^c}$*. A similar result holds for the morphisms $\boldsymbol{f}, (f, \lambda)$.*

4.2 $\mathbb{C}$-schemes and $\mathbb{C}$-stacks with obstruction theories

4.2.1 Cotangent complexes

We follow here Joyce [35, §14.5], and begin by briefly reviewing the theory of cotangent complexes.

Let $f : X \to Y$ be a morphism of $\mathbb{C}$-schemes. As in [27, §II.8], one can define the *cotangent sheaf* (*sheaf of relative differentials*) $\Omega_{X/Y} \in \mathrm{coh}(X)$. In the case $Y = \operatorname{Spec}\mathbb{C}$ and $f : X \to \operatorname{Spec}\mathbb{C}$ being the unique projection, we write Ω_X for the cotangent sheaf. If X is a smooth $\mathbb{C}$-scheme, than Ω_X is a vector bundle (locally fee sheaf) of rank $\dim X$ on X, the cotangent bundle T^*X.

Morphisms of $\mathbb{C}$-schemes, $X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z$ induce an exact sequence

$$f^*(\Omega_{Y/Z})X \xrightarrow{\Omega_f} \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

in $\mathrm{coh}(X)$. This sequence may not be a short exact sequence, as the morphism $f^*(\Omega_{Y/Z})X \xrightarrow{\Omega_f} \Omega_{X/Z}$ need not to be injective.

The *cotangent complex* $\mathbb{L}_{X/Y}$ of a morphism $f : X \to Y$ is an object in the derived category $D(\mathrm{qcoh}(X))$ of quasicoherent sheaves on X, which can be constructed as in Illusie [32]. As in the cotangent sheaf situation we write $\mathbb{L}_X$ in the case where $Y = \operatorname{Spec}\mathbb{C}$ and $\phi : X \to \operatorname{Spec}\mathbb{C}$ is the projection. We will not discuss the multi-faceted theory of cotangent complexes here in detail, but will highlight the following points instead.

- $h^i(\mathbb{L}_{X/Y}) = 0$ for $i > 0$, and $h^0(\mathbb{L}_{X/Y}) \cong \Omega_{X/Y}$. Moreover, if $f : X \to Y$ is smooth, then $\mathbb{L}_{X/Y} \cong \Omega_{X/Y}$.

- There exist truncation functors $\tau_{<k}, \tau_{\geq k} : D(\mathrm{qcoh}(X)) \to D(\mathrm{qcoh}(X))$ for each $k \in \mathbb{Z}$, satisfying

$$h^i(\tau_{<k}(E^\bullet)) \simeq \begin{cases} h^i(E^\bullet), & i < k, \\ 0, & i \geq k, \end{cases} \qquad h^i(\tau_{\geq k}(E^\bullet)) \simeq \begin{cases} 0, & i < k, \\ h^i(E^\bullet), & i \geq k, \end{cases}$$

for any $E^\bullet \in D(\mathrm{qcoh}(X))$ and $i, k \in \mathbb{Z}$. Moreover there is a distinguished triangle

$$\tau^{<k} E^\bullet \xrightarrow{\tau^{<k}} E^\bullet \xrightarrow{\tau^{\geq k}} \tau^{\geq k} E^\bullet \longrightarrow (\tau^{<k} E^\bullet)[1].$$

4.2.2 Perfect obstruction theories

We will now briefly review some material on *perfect obstruction theories*. Perfect obstruction theories play a major role in algebraic geometry and are used to construct virtual cycles on moduli spaces and define enumerative invariants such as Gromov–Witten invariants. Behrend and Fantechi [7] introduced obstruction theories as morphism $\phi : E^\bullet \to \mathbb{L}_X$, whereas Huybrechts and Thomas [31] introduced a weaker definition of a morphism $\phi : E^\bullet \to \tau_{\geq -1}(\mathbb{L}_X)$. We follow here Joyce [35, §14.5] in discussing the by Huybrechts and Thomas introduced obstruction theories.

Definition 4.2.1. Let X be a $\mathbb{C}$-scheme (Deligne–Mumford $\mathbb{C}$-stack).

(i) A complex $E^\bullet \in D(\mathrm{qcoh}(X))$ is called *perfect* (or *amplitude contained in* $[a, b]$*)*, if locally on X, $E^\bullet$ is quasi-isomorphic to a complex $F^\bullet$ of vector bundles (locally free sheaves) of finite rank in degrees $a, a+1, \dots, b$. Here locally means Zariski locally, if X is a $\mathbb{C}$-scheme, and étale locally if X is a Deligne–Mumford $\mathbb{C}$-stack.

The *virtual rank* of $E^\bullet$ is a locally constant function $\mathrm{rank}\, E^\bullet : X \to \mathbb{Z}$ defined locally (Zariski or étale) by $\mathrm{rank}\, E^\bullet = \sum_{k=a}^{b} (-1)^k \mathrm{rank}\, F^k$, where $F^\bullet$ is the complex from above. If $\mathrm{rank}\, E^\bullet = n$, we say $E^\bullet$ has *constant rank* $n \in \mathbb{Z}$.

(ii) An *obstruction theory* for X is a morphism $\phi : E^\bullet \to \tau_{\geq -1}(\mathbb{L}_X)$ in $D(\mathrm{qcoh}(X))$, where $\mathbb{L}_X$ is the cotangent complex of X, and $\tau_{\geq -1}(\mathbb{L}_X)$ its truncation. Moreover E has to satisfy the following conditions:

(q) $h^i(E^\bullet) = 0$ for all $i > 0$.

(b) ${}^i(E^\bullet)$ is coherent for $i = 0, -1$.

(c) $h^0(\phi) : h^0(E^\bullet) \to h^0(\tau_{\geq -1}(\mathbb{L}_X)) \simeq h^0(\mathbb{L}_X)$ is an isomorphism.

(d) $h^{-1}(\phi) : h^{-1}(E^\bullet) \to h^{-1}(\tau_{\geq -1}(\mathbb{L}_X)) \simeq h^{-1}(\mathbb{L}_X)$ is surjective.

(iii) We call $\phi : E^\bullet \to \tau_{\geq -1}(\mathbb{L}_X)$ is called a *perfect obstruction theory*, if $E^\bullet$ is perfect of amplitude contained in $[-1, 0]$.

In the same way, one can define a *relative (perfect) obstruction theory* $\phi : E^\bullet \to \tau_{\geq -1}(\mathbb{L}_{X/Y})$ for a morphism of $\mathbb{C}$-schemes $f : X \to Y$.

The following theorem due to Behrend and Fantechi [7, §5] equips $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-stacks with virtual fundamental classes.

Theorem 4.2.2 (Behrend and Fantechi [7])**.** *Let X be a proper $\mathbb{C}$-scheme or Deligne–Mumford $\mathbb{C}$-stacks with perfect obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ of constant rank $n \in \mathbb{Z}$. Then one can construct a* virtual fundamental class $[X]_{virt}$ *in the Chow homology $A_n(X)$. In the case where X is smooth of dimension n and ϕ is the identity on the cotangent bundle $\mathrm{id}_{T^*X} : T^*X \to \mathbb{L}_X \simeq T^*X$, this virtual fundamental class $[X]_{virt}$ is just the usual fundamental class of X.*

One particular important example of a moduli space that admits an obstruction theory, is the Deligne–Mumford moduli $\mathbb{C}$-stack $\overline{\mathcal{M}}_{g,m}(X, \beta)$ of m-pointed, genus g stable maps to a projective target variety X, and we will explain later on in section 6 how we can think of $\overline{\mathcal{M}}_{g,m}(X, \beta)$ as a special kind of d-orbifold.

4.2.3 $\mathbb{C}$-schemes with perfect obstruction theories as a category

In contrast to Kuranishi spaces ([20],[18]) $\mathbb{C}$-schemes with perfect obstruction theory can be made into a category as the following definition (compare [35, Definition 14.25]) will show. Another reference where schemes and stacks with obstruction theories are treated as a category is the work of Manolache [41].

Definition 4.2.3. The category of $\mathbb{C}$-schemes with perfect obstruction theory $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$ is defined as follows:

- Objects are given by $(X, E^\bullet, \phi)$, where X is a separated, second countable $\mathbb{C}$-scheme and $\phi : E^\bullet \to \tau_{\geq -1}(\mathbb{L}_X)$ is a perfect obstruction theory on X with constant rank.

- Morphisms between two objects $(X_1, E_1^\bullet, \phi_1)$ and $(X_2, E_2^\bullet, \phi_2)$ are given by a pair $(f, \hat{f}) : (X_1, E_1^\bullet, \phi_1) \to (X_2, E_2^\bullet, \phi_2)$ consisting of a morphism $f : X_1 \to X_2$ of $\mathbb{C}$-schemes, and a morphism $\hat{f} : f^*(E_2^\bullet) \to E^\bullet$ in $D(\text{qcoh}(X))$ making the following diagram commute

$$\begin{tikzcd} f^*(E_2^\bullet) \arrow[rr, "\hat{f}"] \arrow[d, "f^*(\phi_2)"'] & & E^\bullet \arrow[d, "\phi_1"] \\ f^*(\tau_{\geq -1}(\mathbb{L}_{X_2})) \arrow[rr, "\tau_{\geq -1}(\mathbb{L}_f)"] & & \tau_{\geq -1}(\mathbb{L}_{X_1}). \end{tikzcd}$$

- Composition of morphisms $(f, \hat{f}) : (X_1, E_1^\bullet, \phi_1) \to (X_2, E_2^\bullet, \phi_2)$ and $(g, \hat{g}) : (X_2, E_2^\bullet, \phi_2) \to (X_3, E_3^\bullet, \phi_3)$ is given by

$$(g, \hat{g}) \circ (f, \hat{f}) = (g \circ f, \hat{f} \circ f^*(\hat{g}) \circ I_{f,g}(E_3^\bullet)),$$

where $I_{f,g}(G^\bullet) : (g \circ f)^*(E_3^\bullet) \to f^*(g^*(E_3^\bullet))$ is the canonical isomorphism. The identity morphism for $(X, E^\bullet, \phi)$ is given by $(\text{id}_X, \delta_{E^\bullet})$, where $\delta_{E^\bullet} : \text{id}_X^*(E^\bullet) \to E^\bullet$ is the natural isomorphism.

In the same spirit one can define a 2-category $\mathbf{Sta}_{\mathbb{C}}\mathbf{Obs}$ of Deligne–Mumford $\mathbb{C}$-stacks. The objects are given by $(X, E^\bullet, \phi)$, where X is now a second countable Deligne–Mumford $\mathbb{C}$-stack, and $\phi : E^\bullet \to \tau_{\geq -1}(\mathbb{L}_X)$ is a perfect obstruction theory on X with constant rank. 1-morphism, composition and identities are defined as for $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$.

Let $(f, \hat{f}), (g, \hat{g}) : (X_1, E_1^\bullet, \phi_1) \to (X_2, E_2^\bullet, \phi_2)$ be 1-morphisms in $\mathbf{Sta}_{\mathbb{C}}\mathbf{Obs}$. A 2-morphism $\eta : (f, \hat{f}) :\Rightarrow (g, \hat{g})$ in $\mathbf{Sta}_{\mathbb{C}}\mathbf{Obs}$ is a 2-morphism $\eta : f \Rightarrow g$ in $\mathbf{DMSta}_{\mathbb{C}}$, such that $\hat{g} \circ \eta^*(E_2^\bullet) = \hat{f}$, where $\eta^*(E_2^\bullet) : f^*(E_2^*) \to g^*(E_2^\bullet)$ is the natural isomorphism in $D(\text{qcoh}(X))$. All composition and identities involving 2-morphism (horizontal and vertical composition, and identity 2-morphism) are induced from the compositions and identities in $\mathbf{DMSta}_{\mathbb{C}}$.

4.2.4 Truncation functors from $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-stacks with perfect obstruction theories to d-manifolds and d-orbifolds

The following theorem due to Joyce ([34, Theorem 14.27]) relates $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-Stacks with perfect obstruction theories to d-manifolds and d-orbifolds.

Theorem 4.2.4. *(a) Let X be a separated, second countable $\mathbb{C}$-scheme and $\phi : E^\bullet \to \tau_{\geq -1}(\mathbb{L}_X)$ a perfect obstruction theory of virtual rank $n \in \mathbb{Z}$ on X. Then there exists a, up to oriented equivalence in* **dMan***, natural oriented d-manifold $\boldsymbol{X}$ with vdim $\boldsymbol{X} = 2n$, whose underlying topological space is given by the set $X(\mathbb{C})$ of $\mathbb{C}$-points of X, with the complex analytic topology. This d-manifold $\boldsymbol{X}$ can be explicitly constructed.*

(b) Let $(f, \hat{f}) : (X_1, E_1^\bullet, \phi_1) \to (X_2, E_2^\bullet, \phi_2)$ be a morphism in $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$ *and let $\boldsymbol{X}_1, \boldsymbol{X}_2$ be (choices of) the d-manifolds constructed from $X_1, E_1^\bullet, \phi_1$ and $X_2, E_2^\bullet, \phi_2$. Then one can construct a 1-morphism $\boldsymbol{f} : \boldsymbol{X}_1 \to \boldsymbol{X}_2$ in* **dMan***, which is natural up to 2-isomorphism and whose underlying continuous map is given by $f(\mathbb{C}) : X_1(\mathbb{C}) \to X_2(\mathbb{C})$ induced by f on the sets of $\mathbb{C}$-points in X_1, X_2.*

(c) Combining (a) and (b) one can define a functor $\Pi^{\mathbf{dMan}}_{\mathbf{SchObs}} : \mathbf{Sch}_{\mathbb{C}}\mathbf{Obs} \to Ho(\mathbf{dMan})$, where $Ho(\mathbf{dMan})$ is the homotopy category of the 2-category **dMan***.*

(d) (a)–(b) also hold for separated, second countable Deligne–Mumford $\mathbb{C}$-stacks $\mathcal{X}$ with perfect obstruction theories $\phi : E^\bullet \to \tau_{\geq -1}(\mathbb{L}_X)$ of virtual rank $n \in Z$ and oriented d-orbifolds $\boldsymbol{\mathcal{X}}$ with vdim $\boldsymbol{\mathcal{X}} = 2n$. Part (c) yields then a functor $\Pi^{\mathbf{dOrb}}_{\mathbf{StaObs}} : Ho(\mathbf{Sta}_{\mathbb{C}}\mathbf{Obs}) \to Ho(\mathbf{dOrb})$.

Corollary 4.2.5. *The moduli stack $\overline{\mathcal{M}}_{g,m}(X, \beta)$ of m-pointed, genus g stable maps to a projective target variety X, with fixed topological data β, admits the structure of an oriented d-orbifold.*

Corollary 4.2.6. *There are natural truncation functors*

$$\begin{aligned}\Pi^{\mathbf{dMan}}_{\mathbf{QsDSch}} = \Pi^{\mathbf{dMan}}_{\mathbf{SchObs}} \circ \Pi^{\mathbf{SchObs}}_{\mathbf{QsDSch}} : Ho(\mathbf{QsDSch}_{\mathbb{C}}) \to Ho(\mathbf{dMan})\\ \Pi^{\mathbf{dOrb}}_{\mathbf{QsDSta}} = \Pi^{\mathbf{dOrb}}_{\mathbf{StaObs}} \circ \Pi^{\mathbf{StaObs}}_{\mathbf{QsDSta}} : Ho(\mathbf{QsDSta}_{\mathbb{C}}) \to Ho(\mathbf{dOrb}),\end{aligned}$$

from the ∞-categories of separated, second countable, quasi-smooth derived $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-stacks of constant dimension to d-manifolds and d-orbifolds.

Chapter 5

Nearly and homotopy complex structures

This section will introduce the notions of *nearly complex structures* and *homotopy complex structures*. As we have seen in section 2.3, d-manifolds are a 2-categorical generalization of manifolds. In the same manner as d-manifolds generalize manifolds, and virtual vector bundles generalize vector bundles, nearly and homotopy complex structures and nearly and homotopy complex d-manifolds are 2-categorical generalizations of almost complex structures and complex manifolds.

5.1 Homotopy complex structures

In this section we want to establish the notion of a stable homotopy complex structure on a d-manifold.

We first want to recall some basic properties and definitions of stable (almost) complex structures on vector bundles and manifolds. In contrast to the usual definition in the literature, it will be convenient for us to define (stable) almost complex structures on the cotangent bundle T^*M of a manifold M, instead of on the tangent bundle TM. We will therefore start with the following definition: (For more details in the 'classical' case, see for example [26, Appendix D].)

Definition 5.1.1. A *stable complex structure* on a real vector bundle $E \to M$ over a manifold M, is a fiberwise complex structure on the Whitney sum $E \oplus \mathbb{R}^k$ for some $k \in \mathbb{Z}_{\geq 0}$. Here $\mathbb{R}^k$ denotes the trivial bundle $M \times \mathbb{R}^k$ with fibre $\mathbb{R}^k$. We will call a triple (E, J, k) *stable complex vector bundle*, if E is a real vector bundle, and J a complex structure on $E \oplus \mathbb{R}^k$.

A stable almost complex structure on a manifold M is a stable complex structure on its cotangent bundle T^*M, and we will call the triple (M, J, k) a *stable almost complex manifold* if J is a complex structure on $T^*M \oplus \mathbb{R}^k$. We will sometimes refer to $T^*M \oplus \mathbb{R}^k$ as the *stable cotangent bundle* of M or the *stabilization* of T^*M.

An *almost complex structure* on a manifold M is a fiberwise complex structure on the cotangent bundle T^*M, that is, an automorphism of real vector bundles $J : T^*M \to T^*M$ such that $J^2 = -\mathrm{id}$. Note that an almost complex structure is a stable almost complex structure with $k = 0$.

It is usually convenient to work not with a special choice of stable complex structure, but with an *equivalence class* of stable complex structures, where we will use the following notion of *equivalence*:

Definition 5.1.2. Let $E = E_0 = E_1$ be vector bundles over a manifold M, carrying stable complex structures J_0, J_1. We say that the stable complex structures (E, J_0, k) and (E, J_1, l) are *homotopic* if there exist $a, b \in \mathbb{Z}_{\geq 0}$ such that $k + 2a = l + 2b$ and such that the resulting almost complex structures on the vector bundle $E \oplus \mathbb{R}^m$, where $m = k + 2a = l + 2b$, obtained from the identifications $(E \oplus \mathbb{R}^k) \oplus \mathbb{C}^a \cong E \oplus \mathbb{R}^m \cong (E \oplus \mathbb{R}^l) \oplus \mathbb{C}^b$ are homotopic through a family of almost complex structures J_t, for t in $[0, 1]$.

The following proposition (compare [26, Proposition D.14]) plays an important role in defining unitary bordism groups:

Proposition 5.1.3. *Let M be a manifold with boundary ∂M. A stable almost complex structure on M induces one on ∂M, which is canonical up to homotopy. Homotopic structures on M induce homotopic structures on ∂M.*

Proof. We will prove the proposition for the 'classical' case in which the tangent bundle TM carries the (stable) almost complex structure, but as the tangent bundle TM is isomorphic to the cotangent bundle T^*M, the statement of the proposition is true for both definitions. For any manifold M with boundary, we have the following exact sequence of vector bundles over the boundary ∂M

$$0 \longrightarrow T(\partial M) \longrightarrow TM|_{\partial M} \longrightarrow \mathcal{N}(\partial M) \longrightarrow 0, \tag{5.1}$$

where $T(\partial M)$ is the tangent bundle to the boundary and $\mathcal{N}(\partial M)$ its normal bundle.

As a one-dimensional real vector space, the normal bundle $\mathcal{N}_{\partial M}$ can be given an orientation by choosing the "outward" pointing direction to be positive. This choice of orientation determines an isomorphism of vector bundles $\mathcal{N}(\partial M) \cong \partial M \times \mathbb{R}$, and this isomorphism is unique up to homotopy.

Now, as a sequence of vector bundles, (5.1) splits and this splitting is unique up to homotopy, since the space of all splittings can be identified with the connected space of sections of the vector bundle $\mathrm{Hom}(\mathcal{N}(\partial M), T(\partial M))$. We therefore obtain an isomorphism

$$TM|_{\partial M} = \mathbb{R} \oplus T(\partial M),$$

which is canonical up to homotopy, and the proposition follows. $\square$

We want now to generalize this definition to the d-manifold world, and start by defining what a complex structure on a virtual vector bundle is.

Definition 5.1.4. A *complex virtual quasicoherent sheaf* over a C^∞-scheme $\underline{X}$ is given by the following data: a virtual quasicoherent sheaf $(\mathcal{F}^\bullet, \psi) \in \mathbf{vqcoh}(\underline{X})$ on $\underline{X}$, a 1-morphism of virtual quasicoherent sheaves $J^\bullet = (J^1, J^2) : (\mathcal{F}^\bullet, \psi) \to (\mathcal{F}^\bullet, \psi)$, and a 2-isomorphism $\eta : (J^\bullet)^2 \Rightarrow -\mathrm{id}_{\mathcal{F}^\bullet}$ in $\mathbf{vqcoh}(\underline{X})$, such that the following compatibility condition between $J^\bullet$ and η is fulfilled:

$$\eta \circ J^2 = J^1 \circ \eta. \tag{5.2}$$

Note that since $J^\bullet$ is a 1-morphism, ψ is complex linear, that is we have $\psi \circ J^1 = J^2 \circ \psi$. We will call $(J^\bullet, \eta)$ a *complex structure* on the virtual quasicoherent sheaf $(\mathcal{F}^\bullet, \psi)$, and we can make these complex virtual quasicoherent sheaves into a 2-category $\mathbf{vqcoh^{cx}}(\underline{X})$ as follows: *Objects* are given by complex virtual quasicoherent sheaves $((\mathcal{E}^\bullet, \phi), J^\bullet, \eta_\mathcal{E})$. The 1-*morphisms* between two objects $((\mathcal{E}^\bullet, \phi), J^\bullet_\mathcal{E}, \eta_\mathcal{E})$

and $((\mathcal{F}^\bullet,\psi),J^\bullet_{\mathcal{F}},\eta_{\mathcal{F}})$ can be characterized by the following commutative diagram in $\mathbf{vqcoh}(\underline{X})$:

$$\begin{tikzcd}
\mathcal{E}^1 \arrow[rr, "\phi"] \arrow[dd, "f^1"'] \arrow[dr, "J^1_{\mathcal{E}}"] & & \mathcal{E}^2 \arrow[dd, "f^2" near end] \arrow[dr, "J^2_{\mathcal{E}}"] \arrow[ddl, dotted, "\theta"'] & \\
& \mathcal{E}^1 \arrow[rr, crossing over, "\phi" near start] & & \mathcal{E}^2 \arrow[dd, "f^2"] \\
\mathcal{F}^1 \arrow[rr, "\psi" near start] \arrow[dr, "J^1_{\mathcal{F}}"'] & & \mathcal{F}^2 \arrow[dr, "J^2_{\mathcal{F}}"] & \\
& \mathcal{F}^1 \arrow[rr, "\psi"] \arrow[from=uu, crossing over, "f^1" near start] & & \mathcal{F}^2.
\end{tikzcd}$$

That is 1-morphisms in $\mathbf{vqcoh^{cx}}(\underline{X})$ are pairs $(f^\bullet,\theta)$, where $f^\bullet$ is a 1-morphism in $\mathbf{vqcoh}(\underline{X})$ and $\theta : J^\bullet_{\mathcal{F}}\circ f^\bullet \Rightarrow f^\bullet\circ J^\bullet_{\mathcal{E}}$ is a 2-morphism in $\mathbf{vqcoh}(\underline{X})$ satisfying the following compatibility condition:

$$\eta_{\mathcal{F}} * \mathrm{id}_{f^\bullet} = (\mathrm{id}_{f^\bullet} * \eta_{\mathcal{E}}) \odot (\theta_f * \mathrm{id}_{J^\bullet_{\mathcal{E}}}) \odot (\mathrm{id}_{J^\bullet_{\mathcal{F}}} * \theta_f). \tag{5.3}$$

Note that condition (5.3) comes from the fact that we want the following diagram of 2-morphisms in $\mathbf{vqcoh}(\underline{X})$ to commute:

$$\begin{tikzcd}
f^\bullet\circ(J^\bullet_{\mathcal{E}})^2 \arrow[d, Rightarrow, "\mathrm{id}_{f^\bullet}*\eta_{\mathcal{E}}"'] & & (J^\bullet_{\mathcal{F}})^2\circ f^\bullet \arrow[ll, Rightarrow, "(\theta_f*\mathrm{id}_{J^\bullet_{\mathcal{E}}})\odot(\mathrm{id}_{J^\bullet_{\mathcal{F}}}*\theta_f)"'] \arrow[d, Rightarrow, "\eta_{\mathcal{F}}*\mathrm{id}_{f^\bullet}"] \\
f^\bullet\circ -\mathrm{id}_{(\mathcal{E}^\bullet,\psi)} \arrow[rr, equal] & & -\mathrm{id}_{(\mathcal{F}^\bullet,\psi)}\circ f^\bullet.
\end{tikzcd}$$

The 2-*morphisms* $\lambda : f^\bullet \Rightarrow g^\bullet$ between two 1-morphisms $(f^\bullet,\theta_f) : (\mathcal{E}^\bullet,\phi)\to(\mathcal{F}^\bullet,\psi)$ and $(g^\bullet,\theta_g) : (\mathcal{E}^\bullet,\phi)\to(\mathcal{F}^\bullet,\psi)$ are given by 2-morphisms $\lambda : f^\bullet\Rightarrow g^\bullet$ in $\mathbf{vqcoh}(\underline{X})$ such that

$$\theta_f \odot (\lambda * \mathrm{id}_{J^\bullet_{\mathcal{E}}}) = (\mathrm{id}_{J^\bullet_{\mathcal{F}}} * \lambda) \odot \theta_g.$$

We call a complex virtual quasicoherent sheaf $((\mathcal{F}^\bullet,\psi),J^\bullet,\eta)$ a *complex virtual vector bundle*, if $(\mathcal{F}^\bullet,\psi)$ is a virtual vector bundle, $J^\bullet$ a 1-morphism in $\mathbf{vvect}(\underline{X})$ and η a 2-morphism in $\mathbf{vvect}(\underline{X})$, and refer to the corresponding 2-category as $\mathbf{vvect^{cx}}(\underline{X})$.

Remark 5.1.5. Note that the compatibility condition (5.2) can be expressed in 2-categorical terms as

$$\eta * \mathrm{id}_{J^\bullet} = \mathrm{id}_{J^\bullet} * \eta,$$

where $\mathrm{id}_{J^\bullet}$ denotes the identity 2-morphism of $J^\bullet$ and $*$ the horizontal composition of 2-morphisms. Note further, that this means nothing else than that the resulting 2-isomorphism from $J^{\bullet 3}$ to $-J^\bullet$ is canonical , as $\eta * \mathrm{id}_{J^\bullet} : J^{\bullet 3} \Rightarrow -J^\bullet$ and $\mathrm{id}_{J^\bullet} * \eta : J^{\bullet 3} \Rightarrow -J^\bullet$.

The following definition will use complex virtual vector bundles to define a notion of homotopy complex structure on a d-manifold. In the classical manifold case, an almost complex manifold carries an almost complex structure on its tangent bundle, in the d-manifold case however we need a slightly weaker notion for the application we have in mind.

Definition 5.1.6. A *homotopy complex structure* $((\mathcal{E}^\bullet, \phi), J^\bullet, i^\bullet, j^\bullet)$ on a d-manifold $\boldsymbol{X}$ consists of a virtual vector bundle $(\mathcal{E}^\bullet, \phi)$ on $\underline{X} \times [0,1]$ and equivalences $i^\bullet : (\mathcal{E}^\bullet, \phi)|_{\underline{X} \times \{0\}} \to \tilde{T}^*\boldsymbol{X}$ and $j^\bullet : (\mathcal{E}^\bullet, \phi)|_{\underline{X} \times \{1\}} \to (\mathcal{F}^\bullet, \psi)$ in $\mathbf{vvect}(\underline{X})$, where $((F^\bullet, \psi), J^\bullet, \eta)$ is a complex virtual vector bundle on $\underline{X}$. We will sometime leave the equivalences and the virtual vector bundle implicit and refer to $J^\bullet$ as a homotopy complex structure.

Note that the basic idea of this definition is the following: although the virtual cotangent bundle $\tilde{T}^*\boldsymbol{X}$ may not admit a complex structure itself, it can be deformed to a complex virtual vector bundle.

The next definition introduces the stabilization of the cotangent bundle of a d-manifold, which will allow us to introduce the more general notion of stable homotopy complex structures.

Definition 5.1.7. For each positive integer $a \in \mathbb{Z}_{\geq 0}$, define a *stabilization* of the cotangent bundle $T^*\boldsymbol{X}$ to a d-manifold $\boldsymbol{X}$, to be the virtual vector bundle $\tilde{T}^*\boldsymbol{X}$, given by

$$\mathcal{E}_X \xrightarrow{\phi_X \oplus *} \mathcal{F}_X \oplus (\mathbb{R}^a \otimes \mathcal{O}_X),$$

where the map $* : \mathcal{E}_X \to \mathbb{R}^a \otimes \mathcal{O}_X$ is arbitrary. Note that we will in the following, for brevity, sometimes suppress the $\otimes\mathcal{O}_X$-part if it is clear from the context, and just write $\mathcal{E}_X \xrightarrow{\phi_X \oplus *} \mathcal{F}_X \oplus \mathbb{R}^a$ instead.

Definition 5.1.8. Fix a positive integer $a \in \mathbb{Z}_{\geq 0}$, and let $\boldsymbol{X}$ be a d-manifold with underlying C^∞-scheme $\underline{X}$. Let $\tilde{T}^*\boldsymbol{X}$ be a stablilization of the cotangent bundle of $\boldsymbol{X}$. A *stable homotopy complex structure* $((\mathcal{E}^\bullet, \phi), J^\bullet, a)$ on a d-manifold $\boldsymbol{X}$ consists of a virtual vector bundle $(\mathcal{E}^\bullet, \phi)$ on $\underline{X} \times [0,1]$ and equivalences $i^\bullet : (\mathcal{E}^\bullet, \phi)|_{\underline{X}\times\{0\}} \to \tilde{T}^*\boldsymbol{X}$ and $j^\bullet : (\mathcal{E}^\bullet, \phi)|_{\underline{X}\times\{1\}} \to (\mathcal{F}^\bullet, \psi)$ in $\mathbf{vvect}(\underline{X})$, where $((F^\bullet, \psi), J^\bullet, \eta)$ is a complex virtual vector bundle on $\underline{X}$.

We call a quintuple $(\boldsymbol{X}, (\mathcal{E}^\bullet, \phi), J^\bullet, a)$, consisting of a d-manifold $\boldsymbol{X}$ and a stable homotopy complex structure $((\mathcal{E}^\bullet, \phi), J^\bullet, a)$ on $\boldsymbol{X}$ a *stable homotopy complex d-manifold.*

The following proposition is the homotopy complex analogue of Proposition 2.1.24 and can be proven in a similar way.

Proposition 5.1.9. *Let $(\mathcal{E}^\bullet, \phi, J^\bullet, \eta)$ be a complex virtual vector bundle over a separated, compact, locally fair C^∞-scheme $\underline{X}$. Then there exists a complex virtual vector bundle $(\mathcal{G}^\bullet, \psi, \tilde{J}^\bullet, \tilde{\eta} = 0)$, where $\mathcal{G}^1, \mathcal{G}^2$ are global complex vector bundles over $\underline{X}$, and an equivalence $(f^\bullet, \theta_f)$ in $\mathbf{vvect^{cx}}(\underline{X})$ between $(\mathcal{G}^\bullet, \psi, \tilde{J}^\bullet, \tilde{\eta})$ and $(\mathcal{E}^\bullet, \phi, J^\bullet, \eta)$.*

Proof. Similar to the situation in the proof of Proposition 2.1.24, we consider the complex vector bundle $\mathcal{G}^2 = (\mathbb{C}^N \otimes \mathcal{O}_X, J_{\mathbb{C}^N})$ for some $N \gg 0$ large enough, where $J_{\mathbb{C}^N}$ denotes the standard complex structure on $\mathbb{C}^N$. We will first construct the "f^2-part" of the equivalence $f^\bullet$ and the associated 2-morphism $\theta_f : \mathbb{C}^N \otimes \mathcal{O}_X \to \mathcal{E}^1$ and show that the equation

$$f^2 \circ J_{\mathbb{C}^N} = J^2 \circ f^2 + \phi \circ \theta_f, \tag{5.4}$$

is satisfied.

Note therefore that since $(\mathcal{E}^\bullet, \phi)$ is a complex virtual vector bundle, we have the following commutative diagrams in qcoh$(\underline{X})$

$$\begin{tikzcd}
\mathcal{E}^1 \arrow[r, "\phi"] \arrow[d, "J^1"] & \mathcal{E}^2 \arrow[d, "J^2"] \\
\mathcal{E}^1 \arrow[r, "\phi"] & \mathcal{E}^2,
\end{tikzcd} \tag{5.5}$$

$$\begin{tikzcd}
\mathcal{E}^1 \arrow[r, "\phi"] \arrow[d, "(J^1)^2"] & \mathcal{E}^2 \arrow[d, "(J^2)^2"] \arrow[dl, dotted, "\eta"'] \\
\mathcal{E}^1 \arrow[r, "\phi"] & \mathcal{E}^2.
\end{tikzcd} \tag{5.6}$$

Recall that in the diagrams (5.5) and (5.6) the following 1- and 2-morphism equations are encoded:

$$J^2 \circ \phi = \phi \circ J^1, \tag{5.7}$$

$$\eta \circ J^2 = J^1 \circ \eta, \tag{5.8}$$

$$(J^1)^2 = -\mathrm{id}_{\mathcal{E}^1} + \eta \circ \phi, \tag{5.9}$$

$$(J^2)^2 = -\mathrm{id}_{\mathcal{E}^2} + \phi \circ \eta. \tag{5.10}$$

In order to construct the f^2-part of $f^\bullet$ and the associated 2-morphism θ_f, consider the splitting $\mathbb{C}^N = \mathbb{R}^N \oplus i\mathbb{R}^N$ and write $f^2 = f_1^2 + if_2^2$ and $\theta_f = \theta_{f_1} + i\theta_{f_2}$ according to this splitting. Now let $f_1^2 : \mathbb{R}^N \otimes \mathcal{O}_X \to \mathcal{E}^2$ be as in Proposition 2.1.24, and define $f_2^2 : \mathbb{R}^N \otimes \mathcal{O}_X \to \mathcal{E}^2$ by $-f_2^2 = J^2 \circ f_1^2$. Furthermore, set $\theta_{f_1} = 0$ and $\theta_{f_2} = \eta \circ f_1^2$. We claim, that these choices of f^2, θ_f satisfy equation (5.4) from above. Note that this is nothing else than to prove that the following equations hold:

$$f_1^2 = J^2 \circ f_2^2 + \phi \circ \theta_{f_2}, \tag{5.11}$$

$$-f_2^2 = J^2 \circ f_1^2. \tag{5.12}$$

But equation (5.12) is true by definition and equation (5.11) simplifies, by using the definitions of θ_{f_2} and f_2^2, to $f_1^2 = -(J^2)^2 f_1^2 + \phi \circ \eta \circ f_1^2$, which in turn is nothing else than equation (5.10).

As in the proof of Proposition 2.1.24, define $\mathcal{G}^1$ to be the kernel of $\phi \oplus f^2$, and denote the induced map by $f^1 : \mathcal{G}^1 \to \mathcal{E}^1$. As we have seen, $\mathcal{G}^1$ is a vector bundle over $\underline{X}$. We claim that there is a natural complex structure $\tilde{J}^1$ on $\mathcal{G}^1 = \ker(\phi \oplus f^2)$, making it into a complex vector bundle.

Consider therefore the following commutative diagram in $\mathrm{qcoh}(\underline{X})$:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}^1 \arrow[r, "f^1\oplus\psi"] \arrow[d, dotted, "\tilde{J}^1"'] & \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X) \arrow[r, "\phi\oplus f^2"] \arrow[d, "\begin{pmatrix} J^1 & -\theta_f \\ 0 & J_{\mathbb{C}^N} \end{pmatrix}"] & \mathcal{E}^2 \arrow[r] \arrow[d, "J^2"] & 0 \\ 0 \arrow[r] & \mathcal{G}^1 \arrow[r, "f^1\oplus\psi"] & \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X) \arrow[r, "\phi\oplus f^2"] & \mathcal{E}^2 \arrow[r] & 0. \end{tikzcd} \tag{5.13}$$

Here $\tilde{J}^1 : \mathcal{G}^1 \to \mathcal{G}^1$ is the unique morphism induced by the commutativity and exactness of the diagram.

Analogous to (5.4), one can show that f^1 and θ_f satisfy the equation

$$f^1 \circ \tilde{J}^1 = J^1 \circ f^1 + \theta_f \circ \psi, \tag{5.14}$$

and that the compatibility condition between θ_f and η, as in Definition 5.1.4, is satisfied. Hence $(f^\bullet, \theta_f)$ is indeed a complex 1-morphism from $(\mathcal{G}^\bullet, \psi)$ to $(\mathcal{E}^\bullet, \phi)$.

We want now to show that $(\tilde{J}^1)^2 = -\mathrm{id}_{\mathcal{G}}$, that is $\tilde{J}^1$ is a complex structure on the vector bundle $\mathcal{G}^1$. Using diagrams (5.13), (5.9) and the definition of θ_f, we get the following commutative diagram in $\mathrm{qcoh}(\underline{X})$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}^1 \arrow[r, "f^1\oplus\psi"] \arrow[d, dotted, "(\tilde{J}^1)^2"'] & \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X) \arrow[r, "\phi\oplus f^2"] \arrow[d, "\begin{pmatrix} -\mathrm{id}_{\mathcal{E}^1}+\eta\circ\phi & -J^1\circ\theta_f-\theta_f\circ J_{\mathbb{C}^N} \\ 0 & -\mathrm{id}_{\mathbb{C}^N} \end{pmatrix}"] & \mathcal{E}^2 \arrow[r] \arrow[d, "J_2^2"] & 0 \\ 0 \arrow[r] & \mathcal{G}^1 \arrow[r, "f^1\oplus\psi"] & \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X) \arrow[r, "\phi\oplus f^2"'] & \mathcal{E}^2 \arrow[r] & 0. \end{tikzcd} \tag{5.15}$$

Defining $\gamma = \gamma_1 + \gamma_2 : \mathcal{E}^2 \to \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X)$ by $\gamma_1 = \eta, \gamma_2 = 0$, yields the following equation:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} (\phi \quad f^2) = \begin{pmatrix} \eta\circ\phi & \eta\circ f^2 \\ 0 & 0 \end{pmatrix}.$$

But as $\eta \circ f_1^2 = \theta_{f_2}$ by definition of θ_f, and $\eta \circ f_2^2 = -\eta \circ J^2 \circ f_1^2 = -J^1 \circ \eta \circ f_1^2 = -J^1 \circ \theta_{f_2}$ by equations (5.12) and (5.8), we get for $\eta \circ f^2$:

$$\eta \circ f^2 = -J^1 \circ \theta_f - \theta_f \circ J_{\mathbb{C}^N},$$

and thus

$$\begin{pmatrix} -\mathrm{id}_{\mathcal{E}^1} + \eta \circ \phi & -J^1 \circ \theta_f - \theta_f \circ J_{\mathbb{C}^N} \\ 0 & -\mathrm{id}_{\mathbb{C}^N} \end{pmatrix} = \begin{pmatrix} -\mathrm{id}_{\mathcal{E}^1} & 0 \\ 0 & -\mathrm{id}_{\mathbb{C}^N} \end{pmatrix} + \begin{pmatrix} \eta \circ \phi & \eta \circ f^2 \\ 0 & 0 \end{pmatrix}$$
$$= -\mathrm{id}_{\mathcal{E}^1 \oplus \mathbb{C}^N} + \gamma \circ (\phi \oplus f^2).$$

Hence, the morphism

$$\begin{pmatrix} -\mathrm{id}_{\mathcal{E}^1} + \eta \circ \phi & -J^1 \circ \theta_f - \theta_f \circ J_{\mathbb{C}^N} \\ 0 & -\mathrm{id}_{\mathbb{C}^N} \end{pmatrix} : \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X) \to \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X)$$

from diagram (5.15) factors through $\gamma : \mathcal{E}^2 \to \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X)$:

$$\begin{tikzcd} \dots \arrow[r] & \mathcal{E}^1 \oplus \mathbb{C}^N \otimes \mathcal{O}_X \arrow[r, "\phi \oplus f^2"] \arrow[d, "\left(\begin{smallmatrix} -\mathrm{id}_{\mathcal{E}^1}+\eta\circ\phi & -J^1\theta_f-\theta_f J_{\mathbb{C}^N} \\ 0 & -\mathrm{id}_{\mathbb{C}^N} \end{smallmatrix}\right)"'] & \mathcal{E}^2 \arrow[r] \arrow[d, "J_2^2"] \arrow[dl, dotted, "\gamma"'] & \dots \\ \dots \arrow[r] & \mathcal{E}^1 \oplus \mathbb{C}^N \otimes \mathcal{O}_X \arrow[r, "\phi \oplus f^2"'] & \mathcal{E}^2 \arrow[r] & \dots \end{tikzcd} \tag{5.16}$$

But uniqueness of (5.15) yields then that $(\tilde{J}^1)^2 = -\mathrm{id}_{\mathcal{G}^1}$, which proves that $\mathcal{G}^1$ is a complex vector bundle.

Moreover it follows immediately that $\psi : \mathcal{G}^1 \to \mathbb{C}^N \otimes \mathcal{O}_X$ is complex linear, that is $J_{\mathbb{C}^N} \circ \psi = \psi \circ J^1$, which completes the proof of the proposition. □

Corollary 5.1.10. *Every compact, stable homotopy complex d-manifold* $(\boldsymbol{X},((\mathcal{E}^\bullet,\phi), J^\bullet),a)$ *has an orientation.*

Proof. Let $(\mathcal{E}^\bullet, \phi)$ be the virtual vector bundle associated to the stable homotopy complex structure $J^\bullet$ and let $i_2 : (\mathcal{E}^\bullet, \phi)|_{\underline{X}\times\{1\}} \to (\mathcal{F}^\bullet, \psi, J^\bullet, \eta)$ the corresponding equivalence in **vvect**$(\underline{X})$. Proposition 5.1.9 shows that $(\mathcal{F}^\bullet, \psi, J^\bullet, \eta)$ is equivalent to a complex virtual vector bundle $(\mathcal{G}^\bullet, \rho, \tilde{J}^\bullet, \tilde{\eta})$, where $\mathcal{G}^1, \mathcal{G}^2$ are complex vector bundles. Thus, Theorem 2.3.37(a) shows that $\mathcal{L}_{(\mathcal{G}^\bullet,\rho)} \cong \Lambda_{\mathbb{R}}^{\mathrm{rank}_{\mathbb{R}}\mathcal{G}^1}(\mathcal{G}^1)^* \otimes \Lambda_{\mathbb{R}}^{\mathrm{rank}_{\mathbb{R}}\mathcal{G}^2}\mathcal{G}^2$. But since $\mathcal{G}^1, \mathcal{G}^2$ are complex vector bundles they are oriented, and this isomorphism induces an orientation on $\mathcal{L}_{(\mathcal{G}^\bullet,\rho)}$. Part (b) of Theorem 2.3.37 shows, that this orientation induces one on $(\mathcal{F}^\bullet, \psi, J^\bullet, \eta)$ and we get therefore an orientation on $(\mathcal{E}^\bullet, \phi)|_{\underline{X}\times\{1\}}$ which determines an orientation on $(\mathcal{E}^\bullet, \phi)$ and therefore one on $(\mathcal{E}^\bullet, \phi)|_{\underline{X}\times\{0\}}$, which in turn gives an orientation on $\boldsymbol{X}$. □

Remark 5.1.11. As the definition of a (stable) homotopy complex d-manifold just involves the virtual cotangent bundle and a homotopy complex structure, we can define in exactly the same way what a *(stable) homotopy complex d-orbifold* should be. Moreover, all the results in this section, like Proposition 5.1.9, extend nicely to d-orbifolds, as they are results about complex virtual vector bundles and do not involve d-orbifold specific properties.

5.2 Nearly complex structures

In this section we will define the notion of *nearly complex structures* on d-manifolds and d-orbifolds. The basic idea is, that given a virtual quasicoherent sheaf (virtual vector bundle) $(\mathcal{E}^\bullet, \phi)$ over a C^∞-scheme $\underline{X}$, a nearly complex structure on $(\mathcal{E}^\bullet, \phi)$ is given by complex structures on $\mathcal{E}^1, \mathcal{E}^2$ which do *not* necessarily make ϕ into a complex linear morphism. The advantage in working with nearly complex structures over homotopy complex structures (defined in the previous section), is that the cotangent bundle will be equipped directly with a nearly complex structure and we do not have to use perturbation arguments.

Definition 5.2.1. A *nearly complex virtual quasicoherent sheaf* $((\mathcal{E}^\bullet, \phi), J^\bullet)$ over a C^∞-scheme $\underline{X}$ is given by the following data: a virtual quasi coherent sheaf $(\mathcal{E}^\bullet, \phi) \in \mathbf{vqcoh}(\underline{X})$ on $\underline{X}$ and a pair of morphisms $J^\bullet = (J^1, J^2)$ with $J^1 : \mathcal{E}^1 \to \mathcal{E}^1$, $J^2 : \mathcal{E}^2 \to \mathcal{E}^2$ in $\mathrm{qcoh}(\underline{X})$, satisfying the condition $(J^i)^2 = -\mathrm{id}_{\mathcal{E}^i}$ for $i = 1, 2$. We will call $J^\bullet$ a *nearly complex structure* on the virtual quasicoherent sheaf $(\mathcal{E}^\bullet, \phi)$.

We want to emphasise that we do not require any compatibility of ϕ with $J^\bullet$ whatsoever.

We can make these nearly complex virtual quasicoherent sheaves into a 2-category $\mathbf{vqcoh^{nc}}(\underline{X})$ as follows: *Objects* are given by nearly complex virtual quasicoherent sheaves $((\mathcal{E}^\bullet, \phi), J^\bullet)$. The 1-*morphisms* between two objects $((\mathcal{E}^\bullet, \phi), J^\bullet_{\mathcal{E}})$ and $((\mathcal{F}^\bullet, \psi), J^\bullet_{\mathcal{F}})$ are given by the following commutative diagram in $\mathbf{vqcoh}(\underline{X})$:

$$\begin{tikzcd}
(\mathcal{E}^1, J^1_{\mathcal{E}}) \arrow[r, "\phi"] \arrow[d, "f^1"'] & (\mathcal{E}^2, J^2_{\mathcal{E}}) \arrow[d, "f^2"] \\
(\mathcal{F}^1, J^1_{\mathcal{F}}) \arrow[r, "\psi"] & (\mathcal{F}^2, J^2_{\mathcal{F}}),
\end{tikzcd}$$

that is 1-morphisms in $\mathbf{vqcoh^{nc}}(\underline{X})$ are 1-morphisms of virtual quasicoherent sheaves $f^\bullet = (f^1, f^2) : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ such that $f^i \circ J^i_{\mathcal{E}} = J^i_{\mathcal{F}} \circ f^i$ for $i = 1, 2$. The 2-*morphisms* $\eta : f^\bullet \Rightarrow g^\bullet$ between two 1-morphisms $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ and $g^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ are given by 2-morphisms $\eta : f^\bullet \Rightarrow g^\bullet$ in $\mathbf{vqcoh}(\underline{X})$ such that $\eta \circ J^2_{\mathcal{E}} = J^1_{\mathcal{F}} \circ \eta$.

$$
\begin{tikzcd}
(\mathcal{E}^1, J^1_{\mathcal{E}}) \arrow[r, "\phi"] \arrow[d, shift right, "f^1"'] \arrow[d, shift left, "g^1"] & (\mathcal{E}^2, J^2_{\mathcal{E}}) \arrow[dl, dotted, "\eta"'] \arrow[d, shift right, "f^2"'] \arrow[d, shift left, "g^2"] \\
(\mathcal{F}^1, J^1_{\mathcal{F}}) \arrow[r, "\psi"] & (\mathcal{F}^2, J^2_{\mathcal{F}}),
\end{tikzcd}
\tag{5.17}
$$

Note that η being a 2-morphism in $\mathbf{vqcoh}(\underline{X})$ implies $g^1 = f^1 + \eta \circ \phi$ and $g^2 = f^2 + \psi \circ \eta$, which yields that although ϕ (and ψ) need not be $J^1_{\mathcal{E}}$-$J^2_{\mathcal{E}}$ complex linear ($J^1_{\mathcal{F}}$-$J^2_{\mathcal{F}}$ complex linear), $\eta \circ \phi$ (and $\psi \circ \eta$) are complex linear, as f^1 and g^1 are.

We will call a diagram of the form (5.17) an *equivalence diagram* in the future.

As in the non-complex case, we call $((\mathcal{E}^\bullet, \phi), J^\bullet)$ a *nearly complex virtual vector bundle*, if it is locally equivalent in $\mathbf{vqcoh^{nc}}(\underline{X})$ to some $((\mathcal{F}^\bullet, \psi), K^\bullet)$, for $\mathcal{F}^1, \mathcal{F}^2$ being complex vector bundles with almost complex structures K^1, K^2, and we denote the corresponding 2-category by $\mathbf{vvect^{nc}}(\underline{X})$.

A nearly complex virtual vector bundle over a d-manifold $\boldsymbol{X}$ is given by a nearly complex virtual vector bundle $((\mathcal{E}^\bullet, \phi), J^\bullet) \in \mathbf{vvect^{nc}}(\underline{X})$ on its underlying C^∞-scheme $\underline{X}$.

Remark 5.2.2. Given a nearly complex virtual quasicoherent sheaf $((\mathcal{E}^\bullet, \phi), J^\bullet)$ on a C^∞-scheme $\underline{X}$, we can define a complex virtual quasicoherent sheaf $((\mathcal{E}^\bullet, \tilde{\phi}), J^\bullet, \eta)$ on $\underline{X}$ by setting

$$\tilde{\phi} = \frac{1}{2}(\phi - J^2 \circ \phi \circ J^1) \quad \text{and} \quad \eta = 0.$$

Moreover, given a 1-morphism $f^\bullet$ in $\mathbf{vqcoh^{nc}}(\underline{X})$, we can define a 1-morphism in $\mathbf{vqcoh^{cx}}(\underline{X})$ by $(f^\bullet, \theta_f = 0)$, and given a 2-morphism $\lambda : f^\bullet \Rightarrow g^\bullet$ in $\mathbf{vqcoh^{nc}}(\underline{X})$ we get a 2-morphism $\lambda : (f^\bullet, \theta_f = 0) \Rightarrow (g^\bullet, \theta_g = 0)$ in $\mathbf{vqcoh^{cx}}(\underline{X})$.

It is easy to check that using the above, we get a strict 2-functor $F^{\mathbf{cx}}_{\mathbf{nc}} : \mathbf{vqcoh^{nc}}(\underline{X}) \to \mathbf{vqcoh^{cx}}(\underline{X})$ from the 2-category of nearly complex virtual quasicoherent sheaves to the 2-category of complex virtual quasicoherent sheaves.

Using the notion of nearly complex virtual vector bundles we can now define what a nearly complex d-manifold should be.

Definition 5.2.3. A *nearly complex structure* on a d-manifold $\boldsymbol{X}$ is given by a quadruple $((\mathcal{E}^\bullet, \phi), J^\bullet, i^\bullet)$ consisting of a nearly complex virtual vector bundle $((\mathcal{E}^\bullet, \phi), J^\bullet)$ on $\underline{X}$ and an equivalence $i^\bullet : (\mathcal{E}^\bullet, \phi) \to T^*\boldsymbol{X}$ in $\mathbf{vvect}(\underline{X})$. For the sake of brevity, we will sometimes leave the equivalence $i^\bullet$ implicit, and call the triple $((\mathcal{E}^\bullet, \phi), J^\bullet)$ a nearly complex structure.

A quadruple $(\boldsymbol{X}, ((\mathcal{E}^\bullet, \phi), J^\bullet))$, consisting of a d-manifold $\boldsymbol{X}$ and a nearly complex structure $((\mathcal{E}^\bullet, \phi), J^\bullet)$ will be called a *nearly complex d-manifold.*

The next definition introduces the more general notion of stable nearly complex structures.

Definition 5.2.4. As in the homotopy complex case, fix a non-negative integer $a \in \mathbb{Z}_{\geq 0}$, and let $\boldsymbol{X}$ be a d-manifold with underlying C^∞-scheme $\underline{X}$. Let $\tilde{T}^*\boldsymbol{X}$ be a choice of stabilization of the cotangent bundle of $\boldsymbol{X}$. A *stable nearly complex structure* $((\mathcal{E}^\bullet, \phi), J^\bullet, a, i^\bullet)$ on a d-manifold $\boldsymbol{X}$ consists then of a nearly complex virtual vector bundle $((\mathcal{E}^\bullet, \phi), J^\bullet)$ on $\underline{X}$ and an equivalence $i^\bullet : ((\mathcal{E}^\bullet, \phi), J^\bullet) \to \tilde{T}^*\boldsymbol{X}$ in $\mathbf{vvect}(\underline{X})$. We will sometimes leave the equivalence $i^\bullet$ implicit and refer to $((\mathcal{E}^\bullet, \phi), J^\bullet, a)$ as a stable nearly complex structure. We want to emphasise that the choice of stabilization of $\tilde{T}^*\boldsymbol{X}$ is a part of the data of a stable nearly complex structure.

A quintuple $(\boldsymbol{X}, ((\mathcal{E}^\bullet, \phi), J^\bullet, a))$, consisting of a d-manifold $\boldsymbol{X}$ and a stable nearly complex structure $((\mathcal{E}^\bullet, \phi), J^\bullet, a)$ will be called *stable nearly complex d-manifold.*

As a consequence of the definition of stable nearly complex d-manifold, we can prove that stable nearly complex d-manifolds are oriented.

Proposition 5.2.5. *Every stable nearly complex d-manifold $(\boldsymbol{X}, ((\mathcal{E}^\bullet, \phi), J^\bullet), a)$ has a natural orientation.*

Proof. Proposition 2.3.37(a) (see [35, Proposition 4.40] for a proof) shows that for a virtual vector bundle $(\mathcal{E}^\bullet, \phi)$, the associated orientation line bundle $\mathcal{L}_{\mathcal{E}^\bullet,\phi}$ is canonically isomorphic to the tensor product of the determinant line bundles of

$(\mathcal{E}^1)^*$ and $\mathcal{E}^2$, that is $\mathcal{L}_{(\mathcal{E}^\bullet,\phi)} \cong \Lambda^{k_1}(\mathcal{E}^1)^* \otimes \Lambda^{k_2}\mathcal{E}^2$, where $k_1 = \operatorname{rank}\mathcal{E}_1, k_2 = \operatorname{rank}\mathcal{E}_2$. Given a nearly complex virtual vector bundle $((\mathcal{E}^\bullet,\phi),J^\bullet)$, we can use the functor $F^{\mathbf{cx}}_{\mathbf{nc}}$ from Remark 5.2.2 and get a complex virtual vector bundle $((\mathcal{E}^\bullet,\tilde{\phi}),J^\bullet,\eta = 0)$. It is clear that the proof of Proposition 2.3.37 extends nicely to complex determinant line bundles and hence we get the following isomorphism:

$$\mathcal{L}^{\mathbb{C}}_{((\mathcal{E}^\bullet,\tilde{\phi}),J^\bullet)} \cong \Lambda^{k_1}_{\mathbb{C}}(\mathcal{E}^1)^* \otimes \Lambda^{k_2}_{\mathbb{C}}\mathcal{E}^2.$$

Since $\mathcal{E}^1,\mathcal{E}^2$ carry almost complex structures, they are oriented and hence we get an orientation on $\mathcal{L}^{\mathbb{C}}_{((\mathcal{E}^\bullet,\tilde{\phi}),J^\bullet)}$. But as

$$\mathcal{L}_{(\mathcal{E}^\bullet,\phi)} \cong \mathcal{L}_{(\mathcal{E}^\bullet,\tilde{\phi})} \cong \Lambda^2_{\mathbb{R}}(\mathcal{L}^{\mathbb{C}}_{((\mathcal{E}^\bullet,\phi),J^\bullet)})$$

the orientation line bundle $\mathcal{L}_{\mathcal{E}^\bullet,\phi}$ is oriented. Part (b) of Theorem 2.3.37 shows finally, that this orientation induces one on $\boldsymbol{X}$, as $T^*\boldsymbol{X} \cong ((\mathcal{E}^\bullet,\phi),J^\bullet)$. □

The following proposition is the nearly complex analogue of Proposition 2.1.24 and will play a central role in the following applications. To fix some notation, let $(\mathcal{E},J),(\mathcal{F},K)$ be quasicoherent sheaves with complex structures. We will call a morphism $g:\mathcal{E}\to\mathcal{F}$ *"complex linear"* if it is J-K-linear, that is $g\circ J = K\circ g$.

Proposition 5.2.6. *Let $((\mathcal{E}^\bullet,\phi),J^\bullet)$ be a nearly complex virtual vector bundle over a separated, compact, locally fair C^∞-scheme $\underline{X}$. Then there exists a nearly complex virtual vector bundle $((\mathcal{G}^\bullet,\psi),K^\bullet)$, where $\mathcal{G}^1,\mathcal{G}^2$ are complex vector bundles over $\underline{X}$, and an equivalence $f^\bullet = (f^1,f^2)$ in $\mathbf{vvect}^{nc}(\underline{X})$ between $((\mathcal{E}^\bullet,\phi),J^\bullet)$ and $((\mathcal{G}^\bullet,\psi),K^\bullet)$.*

Proof. Take an open, finite cover $(\underline{Y_i} : i\in I)$ of $\underline{X}$ and a partition of unity $(\alpha_i : i\in I)$ subordinated to this cover, such that on each $\underline{Y_i}$ the nearly complex virtual vector bundle $((\mathcal{E}^\bullet,\phi),J^\bullet)$ is equivalent to a nearly complex virtual vector bundle consisting of trivial complex vector bundles. This means, we get for each $i\in I$ the following diagram in $\mathrm{qcoh}(\underline{Y_i})$:

$$\begin{tikzcd} \mathcal{E}^1|_{\underline{Y_i}} \arrow[r, "\Phi|_{\underline{Y_i}}"] \arrow[d, shift right, "a_i^1"'] & \mathcal{E}^2|_{\underline{Y_i}} \arrow[l, bend right, "\zeta_i"'] \arrow[d, shift right, "a_i^2"'] \\ \mathbb{C}^{n_i^1}_{\underline{Y_i}} \arrow[r, "\beta_{\underline{Y_i}}"] \arrow[u, shift right, "b_i^1"'] & \mathbb{C}^{n_i^2}_{\underline{Y_i}}, \arrow[l, bend left, "\eta_i"] \arrow[u, shift right, "b_i^2"'] \end{tikzcd} \tag{5.18}$$

where $\zeta_i : \mathcal{E}^2|_{\underline{Y}_i} \to \mathcal{E}^1|_{\underline{Y}_i}$ and $\eta_i : \mathbb{C}^{n_i^2}_{\underline{Y}_i} \to \mathbb{C}^{n_i^1}_{\underline{Y}_i}$ are the 2-morphisms induced by the equivalences $a_i^\bullet = (a_i^1, a_i^2)$ and $b^\bullet = (b_i^1, b_i^2)$ for each $i \in I$. Note further that except for $\phi|_{\underline{Y}_i}$ and $\beta_{\underline{Y}_i}$, all the morphisms are complex linear with respect to the appropriate complex structures.

Now, define $N := \sum_{i\in I} n_i^2$, such that $\mathbb{C}^N = \bigoplus_{i\in I} \mathbb{C}^{n_i^2}$. By defining $G^2 := \mathbb{C}^N \otimes \mathcal{O}_X$ we get a morphism $f^2 : G^2 \to \mathcal{E}^2, f^2 = \sum_{i\in I} \alpha_i b_i^2$.

Define $\mathcal{G}^1 := \ker(\phi \oplus f^2)$ and let $f^1 : \mathcal{G}^1 \to \mathcal{E}^1$ and $\psi : \mathcal{G}^1 \to \mathcal{G}^2$ be the morphisms making the following sequence in $\mathbf{vqcoh}(\underline{X})$ exact :

$$0 \longrightarrow \mathcal{G}^1 \xrightarrow{f^1 \oplus -\psi} \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X) \xrightarrow{\phi \oplus f^2} \mathcal{E}^2 \longrightarrow 0. \tag{5.19}$$

We claim that $\mathcal{G}^1$ is a complex vector bundle over $\underline{X}$ and that $f^\bullet = (f^1, f^2)$ is an equivalence in $\mathbf{vvect}(\underline{X})$.

In order to prove this, we have to show that there exists a morphism $\chi \oplus e^2 : \mathcal{E}^2 \to \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X)$ which is complex linear with respect to the complex structures J^2 on $\mathcal{E}^2$ and $J^1 \oplus J_{\mathbb{C}^N}$ on $\mathcal{E}^1 \oplus \mathbb{C}^N \otimes \mathcal{O}_X$, so that (5.19) would become a split exact sequence in qcoh$(\underline{X})$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}^1 \arrow[r, shift left, "f^1 \oplus -\psi"] & \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X) \arrow[l, dotted, shift left, "e^1 \oplus \xi"] \arrow[r, shift left, "\phi \oplus f^2"] & \mathcal{E}^2 \arrow[l, dotted, shift left, "\chi \oplus e^2"] \arrow[r] & 0. \end{tikzcd} \tag{5.20}$$

Proposition 2.1.23 then implies that $f^\bullet$ is an equivalence in $\mathbf{vvect}(\underline{X})$ and we get the existence of a unique complex structure K^1 on $\mathcal{G}^1$ and a complex linear morphism $e^1 \oplus \xi : \mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X) \to \mathcal{G}^1$.

Define $\chi \oplus e^2 := \sum_{i\in I} \alpha_i \eta_i \oplus \sum_{i\in I} \alpha_i b_i^2$, where $\eta_i : \mathbb{C}^{n_i^2}_{\underline{Y}_i} \to \mathbb{C}^{n_i^1}_{\underline{Y}_i}$ and $b_i^2 : \mathbb{C}^{n_i^2}_{\underline{Y}_i} \to \mathcal{E}^2|_{\underline{Y}_i}$ are the morphisms from (5.18). As η_i and b_i^2 are complex linear for all $i \in I$, $\chi \oplus e^2$ is complex linear. Indeed

$$\begin{aligned}(\phi \oplus f^2) \circ (\chi \oplus e^2) &= \sum_{i\in I} \alpha_i (\phi \circ \eta_i + b_i^2 \circ a_i^2) \\ &= \sum_{i\in I} \alpha \mathrm{id}_{\mathcal{E}^2|_{\underline{Y}_i}} = \mathrm{id}_{\mathcal{E}^2},\end{aligned}$$

where we used the fact that $a^\bullet, b^\bullet$ is an equivalence. Using Proposition 2.1.23, we get a morphism $e^1 \oplus \xi$ fitting into (5.20), and as $\mathcal{G}^1 \cong \mathrm{Coker}(\chi \oplus e^2 : \mathcal{E}^2 \to$

$\mathcal{E}^1 \oplus (\mathbb{C}^N \otimes \mathcal{O}_X))$, we get a unique complex structure on $\mathcal{G}^1$ making $e^1 \oplus \xi$ complex linear.

To complete the proof, we need to show that $\mathcal{G}^1$ is a vector bundle over $\underline{X}$. But this was already proven in the proof of Proposition 2.1.24. $\square$

Remark 5.2.7. As the definition of a (stable) nearly complex d-manifold, again just involves the virtual cotangent bundle and a nearly complex structure, we get as in the (stable) homotopy complex case, a nice extension of the definitions and results to d-orbifolds.

5.3 Local nearly complex standard model equivalence

In the following we will prove as a new result that locally the cotangent bundle of a nearly-complex d-manifold behaves well in the sense that there exists a nearly complex analogue to the local description $T^*\boldsymbol{X}|_{\boldsymbol{U}} \simeq (E^*|_{s^{-1}(0)} \xrightarrow{ds|_{s^{-1}(0)}} T^*_{s^{-1}(0)}V)$ as in section 2.3.1.

We start with the definition of a *stable nearly complex standard model d-manifold.*

Definition 5.3.1. Let $a \in \mathbb{Z}_{\geq 0}$ be an integer, V be a manifold, $E \to V$ a vector bundle on V with $\mathrm{rank}\,(E) = 2k$ for some $k \in \mathbb{Z}_{\geq 0}$, $s \in C^\infty(E)$ a smooth section, $J \in \mathrm{End}(TV \oplus \mathbb{R}^a)$ a stable almost complex structure on V and $K \in C^\infty(E \otimes E^*)$ an almost complex structure on the fibres of E.

Then the standard model d-manifold $\boldsymbol{S}_{V,E,s}$, defined in Definition 2.3.4, admits a nearly complex structure

$$(E^*, K) \xrightarrow{ds \oplus *} (T^*V \oplus \mathbb{R}^a, J).$$

We call the quadruple $(\boldsymbol{S}_{V,E,s}, J, K, a)$ a *stable nearly complex standard model d-manifold.*

The following proposition will play a crucial role in studying unitary d-manifold bordism, as it will allow us to switch to a local picture.

Proposition 5.3.2. *Let $(\boldsymbol{X}, (\mathcal{E}^\bullet, \phi), J^\bullet, a)$ be a stable nearly complex d-manifold. Then near each $x \in \underline{X}$, $\boldsymbol{X}$ is equivalent as a stable nearly complex d-manifold to a stable nearly complex standard model d-manifold $(\boldsymbol{S}_{V,E,s}, J, K, a)$.*

If $\boldsymbol{X}$ is compact, then $(\boldsymbol{X}, ((\mathcal{E}^\bullet, \phi), J^\bullet, a))$ is globally equivalent to a stable nearly complex standard model d-manifold $(\boldsymbol{S}_{V,E,s}, J, K, a)$.

Proof. As shown in section 2.3.1, for each $x \in \underline{X}$, $\boldsymbol{X}$ there exists an open neighbourhood $x \in \boldsymbol{U}$ and an equivalence of $\boldsymbol{U}$ to a standard model d-manifold $\boldsymbol{U} \simeq \boldsymbol{S}_{V,E,s}$, for some manifold V, a vector bundle $E \to V$ and a smooth section $s \in C^\infty(E)$.

Proposition 5.2.6 allows us to replace the nearly complex virtual vector bundle $(\mathcal{E}^\bullet, \phi)$, by an equivalent nearly complex virtual vector bundle $(\mathcal{F}^\bullet, \psi)$, where $\mathcal{F}^1, \mathcal{F}^2$ are complex vector bundles, and so we may assume w.l.o.g. that $\mathcal{E}^1, \mathcal{E}^2$ are complex vector bundles.

On the open neighbourhood $\boldsymbol{U}$ of x, we have the following equivalence in $\mathbf{vvect}(\underline{X})$

$$\tilde{T}^*\boldsymbol{X}|_{\boldsymbol{U}} \simeq (E^*|_{s^{-1}(0)} \xrightarrow{ds|_{s^{-1}(0)} \oplus *} T^*V|_{s^{-1}(0)} \oplus \mathbb{R}^a),$$

and therefore the following equivalence diagram in $\mathbf{vvect}(\underline{X})$

$$\begin{tikzcd}
E^*|_{s^{-1}(0)} \arrow[rr, "ds|_{s^{-1}(0)} \oplus *"] \arrow[d, shift right, "a^1"'] & & T^*V|_{s^{-1}(0)} \oplus \mathbb{R}^a \arrow[ll, bend right, "\zeta"'] \arrow[d, shift right, "a^2"'] \\
\mathcal{E}^1|_{\underline{U}} \arrow[rr, "\phi"] \arrow[u, shift right, "b^1"'] & & \mathcal{E}^2|_{\underline{U}}. \arrow[ll, bend left, "\eta"] \arrow[u, shift right, "b^2"']
\end{tikzcd}$$

Here $\zeta : T^*V|_{s^{-1}(0)} \to E^*|_{s^{-1}(0)}$ and $\eta : \mathcal{E}^2|_{\underline{U}} \to \mathcal{E}^1|_{\underline{U}}$ are the 2-morphisms corresponding to the equivalences $a^\bullet, b^\bullet$.

The idea of the proof is first to show, that after choosing suitable, equivalent replacements for $(\mathcal{E}^\bullet, \phi)$ and $a^\bullet$, we get the following equivalences: $E^*|_{s^{-1}(0)} \oplus \mathcal{G} \cong \mathcal{E}^1|_{\underline{U}}$ and $T^*V|_{s^{-1}(0)} \oplus \mathcal{G} \cong \mathcal{E}^2|_{\underline{U}}$, where $\mathcal{G}$ denotes the cokernel of $E^*|_{s^{-1}(0)}$ in $\mathcal{E}^1|_{\underline{U}}$. Then, after extending $\mathcal{G}$ to V, we can replace V by the total space of $\mathcal{G}^*$ and pullback E and s under the projection map $\pi : \mathrm{Tot}(\mathcal{G}^*) \to V$.

In order to achieve this, we first have to show that we can replace $a^\bullet$ by an equivalent equivalence $\tilde{a}^\bullet$, where $\tilde{a}^1, \tilde{a}^2$ are injective. This implies then the existence of left-inverse morphisms and we get therefore that the following short exact sequence is split exact:

$$0 \longrightarrow E^*|_{s^{-1}(0)} \xrightarrow{\tilde{a}^1} \mathcal{E}^1|_{\underline{U}} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Let therefore $\alpha : T^*V \oplus \mathbb{R}^a \to \mathcal{E}^1$ be a generic morphism and define

$$\begin{aligned} \tilde{a}^1 &:= a^1 + \alpha \circ ds|_{s^{-1}(0)}, \\ \tilde{a}^2 &:= a^2 + \phi \circ \alpha. \end{aligned}$$

We claim that $\tilde{a}^1$ and $\tilde{a}^2$ are injective for $\operatorname{rank} \mathcal{E}^1, \operatorname{rank} \mathcal{E}^2$ sufficiently large. To see this, let $x \in s^{-1}(0)$ and consider the following diagram in $\operatorname{qcoh}(\underline{X})$

$$\begin{tikzcd}
0 \arrow[r] & K_x \arrow[r] \arrow[d, "\cong"'] & E^*|_x \arrow[r, "ds|_x"] \arrow[d, shift right, "a^1"'] \arrow[d, shift left, "\tilde{a}^1"] & T^*V|_x \oplus \mathbb{R}^a \arrow[r] \arrow[d, shift right, "a^2"'] \arrow[d, shift left, "\tilde{a}^2"] \arrow[dl, "\alpha"'] & C_x \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & K'_x \arrow[r] & \mathcal{E}^1|_x \arrow[r, "\phi_x"] & \mathcal{E}^2_x \arrow[r] & C'_x \arrow[r] & 0,
\end{tikzcd}$$

where K_x, K'_x denote the kernels of $ds|_x, \phi_x$ and C_x, C'_x the respective cokernels.

We have to ensure that $\alpha \circ ds|_x : \operatorname{im}(ds|_x) \to \mathcal{E}^1|_x$ is injective. Let therefore $\dim K_x = c, \operatorname{rank} E^* = k, \dim T^*V|_x \oplus \mathbb{R}^a = n, \dim C_x = d$, and $\operatorname{rank} \mathcal{E}^1 = N + k, \operatorname{rank} \mathcal{E}^2 = N + n$. We know that $\alpha \circ ds|_x \in \operatorname{Hom}(\operatorname{im}(ds|_x), \mathcal{E}^1|_x)$ and that $\dim \operatorname{Hom}(\operatorname{im}(ds|_x), \mathcal{E}^1|_x) = (N+k)(k-c)$, as $\dim \operatorname{im}(ds|_x) = k - c$.

The non-injective morphisms have at least 1-dimensional kernel and so the dimension of non-injective maps $\operatorname{Hom}_{\text{non-inj}}(\operatorname{im}(ds|_x), \mathcal{E}^1|_x) \subseteq \operatorname{Hom}(\operatorname{im}(ds|_x), \mathcal{E}^1|_x)$ is at most

$$\dim_{\max} \operatorname*{Hom}_{\text{non-inj}}(\operatorname{im}(ds|_x), \mathcal{E}^1|_x) = (k - c - 1) + (k - c - 1)(N + k).$$

Therefore the codimension of the non-injective morphisms within all morphisms is given by

$$\begin{aligned} \operatorname{codim} \operatorname*{Hom}_{\text{non-inj}}(\operatorname{im}(ds|_x), \mathcal{E}^1|_x) &= (N + k) - (k - c - 1) \\ &= N + c + 1 \geq N + 1. \end{aligned}$$

This shows, that in order to ensure that $\alpha \circ ds|_x$ is injective, we require $n < N+1$, or equivalently $N \geq n$.

By adding sufficiently large trivial bundles to $\mathcal{E}^1$ and $\mathcal{E}^2$ and thus replacing $(\mathcal{E}^1, \mathcal{E}^2, \phi, J^1, J^2)$ with the equivalent nearly complex virtual vector bundle $(\mathcal{E}^1 \oplus \underline{\mathbb{C}}^m, \mathcal{E}^2 \oplus \underline{\mathbb{C}}^m, \phi \oplus \mathrm{id}_{\underline{\mathbb{C}}^m}, J^1 \oplus J_{\mathbb{C}^m}, J^2 \oplus J_{\mathbb{C}^m})$, we can assume w.l.o.g that $\operatorname{rank} \mathcal{E}^1 \geq n$, which in turn ensures that there exists a generic morphism $\alpha : TV^* \oplus \mathbb{R}^a \to \mathcal{E}^1$ making $\tilde{a}^1, \tilde{a}^2$ injective.

As $\tilde{a}^1, \tilde{a}^2$ are injective, there exist left inverse morphisms $\tilde{b}^1 : \mathcal{E}^1|_{\underline{U}} \to E^*|_{s^{-1}(0)}$ and $\tilde{b}^2 : \mathcal{E}^2|_{\underline{U}} \to TV^*|_{s^{-1}(0)} \oplus \mathbb{R}^a$.

Replacing $\tilde{a}^1, \tilde{a}^2, \tilde{b}^1, \tilde{b}^2$ by a^1, a^2, b^1, b^2 we get the following diagram in $\mathrm{vect}(\underline{X})$

$$\begin{tikzcd}
0 \arrow[d, shift right] & & 0 \arrow[d, shift right] \\
E^*|_{s^{-1}(0)} \arrow[u, shift right] \arrow[rr, "ds|_{s^{-1}(0)}"] \arrow[d, shift right, "a^1"'] & & T^*V|_{s^{-1}(0)} \oplus \mathbb{R}^a \arrow[u, shift right] \arrow[lld, "\alpha"'] \arrow[d, shift right, "a^2"'] \\
\mathcal{E}^1|_{\underline{U}} \arrow[u, shift right, "b^1"'] \arrow[rr, "\phi"] \arrow[d, shift right, "c^1"'] & & \mathcal{E}^2|_{\underline{U}} \arrow[u, shift right, "b^2"'] \arrow[d, shift right, "c^2"'] \\
\mathcal{G} \arrow[u, shift right, "d^1"'] \arrow[rr, "\mathrm{id}_{\mathcal{G}}", "\cong"'] \arrow[d, shift right] & & \mathcal{G} \arrow[u, shift right, "d^2"'] \arrow[d, shift right] \\
0 \arrow[u, shift right] & & 0, \arrow[u, shift right]
\end{tikzcd} \tag{5.21}$$

where $\mathcal{G} = \operatorname{coker} a^1 \cong \operatorname{coker} a^2$. a^1 and a^2 being injective imply that the vertical exact sequences are split exact, and we get therefore the following isomorphisms

$$\begin{tikzcd}
E^*|_{s^{-1}(0)} \oplus \mathcal{G} \arrow[rr, "\begin{pmatrix} ds|_{s^{-1}(0)} & 0 \\ * & 0 \\ 0 & \mathrm{id}_{\mathcal{G}} \end{pmatrix}"] \arrow[d, "a^1 \oplus d^1"', "\cong"] & & T^*V|_{s^{-1}(0)} \oplus \mathbb{R}^a \oplus \mathcal{G} \arrow[d, "a^2 \oplus d^2"', "\cong"] \\
\mathcal{E}^1|_{\underline{U}} \arrow[rr, "\phi"] & & \mathcal{E}^2|_{\underline{U}}.
\end{tikzcd}$$

The complex structures $J^1|_{\underline{U}}, J^2|_{\underline{U}}$ on $\mathcal{E}^1|_{\underline{U}}, \mathcal{E}^2|_{\underline{U}}$ induce complex structures K^1 and K^2 on $E^*|_{s^{-1}(0)} \oplus \mathcal{G}$ and $T^*V|_{s^{-1}(0)} \oplus \mathbb{R}^a \oplus \mathcal{G}$.

Extending $\mathcal{G}$ from $s^{-1}(0)$ to V, we can define $\tilde{V} := \mathrm{Tot}(\mathcal{G}^*)$, where $\mathrm{Tot}(\mathcal{G}^*)$ denotes the total space of $\mathcal{G}^*$, and let $\pi : \tilde{V} \to V$ be the projection map. Moreover, let $\tilde{E} := \pi^*(E) \oplus \pi^*(\mathcal{G}^*)$ and $\tilde{s} := \pi^*(s) \oplus \mathrm{id}_{\mathcal{G}^*}$. We then get the following commutative diagram in $\mathrm{qcoh}(\underline{X})$

$$\begin{tikzcd}
\tilde{E}^*|_{\tilde{s}^{-1}(0)} \arrow[rr, "d\tilde{s}|_{\tilde{s}^{-1}(0)} \oplus *"] \arrow[d, "\cong"'] & & T^*\tilde{V}|_{\tilde{s}^{-1}(0)} \oplus \mathbb{R}^a \arrow[d, "\cong"] \\
E^*|_{s^{-1}(0)} \oplus \mathcal{G} \arrow[rr, "\begin{pmatrix} ds|_{s^{-1}(0)} & 0 \\ * & 0 \\ 0 & \mathrm{id}_{\mathcal{G}} \end{pmatrix}"] \arrow[d, "\cong"'] & & T^*V|_{s^{-1}(0)} \oplus \mathbb{R}^a \oplus \mathcal{G} \arrow[d, "\cong"] \\
\mathcal{E}^1|_{\underline{U}} \arrow[rr, "\phi"] & & \mathcal{E}^2|_{\underline{U}},
\end{tikzcd}$$

and therefore we get almost complex structures $\tilde{J}^1, \tilde{J}^2$ on $\tilde{E}^*|_{\tilde{s}^{-1}(0)}$ and $T^*\tilde{V}|_{\tilde{s}^{-1}(0)} \oplus \mathbb{R}^a$. Hence, by changing V to $\tilde{V}$, E to $\tilde{E}$ and s to $\tilde{s}$ we get that $(\boldsymbol{U}, (\mathcal{E}^\bullet, \phi)|_{\underline{U}}, J^\bullet|_{\underline{U}}, a)$ is equivalent to $(\boldsymbol{S}_{\tilde{V},\tilde{E},\tilde{s}}, K^1, K^2, a)$ as claimed.

In the case where $\boldsymbol{X}$ is compact, it is principal by Corollary 2.3.30 and therefore (globally) equivalent to a standard model d-manifold $\boldsymbol{S}_{V,E,s}$. Using this global standard model, and the fact that Proposition 5.2.6 allows us to replace the nearly complex virtual vector bundle $(\mathcal{E}^\bullet, \phi)$ globally by an equivalent nearly complex virtual vector bundle, consisting of complex vector bundles, the proof is the same as before. $\square$

In the d-orbifold case, the same proposition holds and can be proven using the same proof, except that we have to ensure that not just the rank of $\mathcal{E}^1, \mathcal{E}^2$ are large enough, but also that $\mathcal{E}^1, \mathcal{E}^2$ contain "large enough representations" of orbifold groups at each point. The reason for this is, that $a^1 : E^*|_{s^{-1}(0)} \to \mathcal{E}^1|_{\underline{U}}$ needs to be $\mathrm{Iso}(x)$-equivariant and hence we have to ensure that there are enough copies of each $\mathrm{Iso}(x)$ representation in $E^*|_{s^{-1}(0)}$ in $\mathcal{E}^1|_{\underline{U}}$. We call a d-orbifold $\boldsymbol{\mathcal{X}}$ *embeddable*, if there exists an embedding $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ in $\mathbf{dOrb}$, with $\mathcal{Y}$ being an orbifold.

Proposition 5.3.3. *Let $(\boldsymbol{\mathcal{X}}, ((\mathcal{E}^\bullet, \phi), J^\bullet, a))$ be a stable nearly complex d-orbifold. Then near each $[x] \in \mathcal{X}_{top}$, $\boldsymbol{\mathcal{X}}$ is equivalent as a stable nearly complex d-orbifold to a stable nearly complex standard model d-orbifold $(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}, J, K, a)$.*

If $\boldsymbol{\mathcal{X}}$ is compact and embeddable, then $(\boldsymbol{\mathcal{X}}, ((\mathcal{E}^\bullet, \phi), J^\bullet, a))$ is globally equivalent to a stable nearly complex standard model d-orbifold $(\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}, J, K, a)$.

The only adjustment of the proof in the d-orbifold case, is that instead of choosing $\mathcal{E}^1|_{\underline{U}}, \mathcal{E}^2|_{\underline{U}}$ with sufficiently large rank, we have to choose $\mathcal{E}^1_{[x]}, \mathcal{E}^2_{[x]}$ as follows:

$$\tilde{\mathcal{E}}^1_{[x]} = \mathcal{E}^1_{[x]} \oplus (\mathcal{E}^*_{[x]} \otimes_{\mathbb{R}} \mathbb{C}^{N_1}) \oplus ((T^*_{[x]}\mathcal{V} \oplus \mathbb{R}^a) \otimes_{\mathbb{R}} \mathbb{C}^{N_2}) \oplus \mathbb{C}^{N_3},$$
$$\tilde{\mathcal{E}}^2_{[x]} = \mathcal{E}^2_{[x]} \oplus (\mathcal{E}^*_{[x]} \otimes_{\mathbb{R}} \mathbb{C}^{N_1}) \oplus ((T^*_{[x]}\mathcal{V} \oplus \mathbb{R}^a) \otimes_{\mathbb{R}} \mathbb{C}^{N_2}) \oplus \mathbb{C}^{N_3}.$$

Here N_1, N_2, N_3 are chosen sufficiently large as in the proof of Proposition 5.3.2. With this choice of $\tilde{\mathcal{E}}^1_{[x]}, \tilde{\mathcal{E}}^2_{[x]}$ the proof of Proposition 5.3.3 is essentially the same as the proof of Proposition 5.3.2 and will therefore be omitted.

5.4 The relation between nearly and homotopy complex structures

The major difference between the notion of nearly complex d-manifolds (Definition 5.2.4) and homotopy complex d-manifolds (Definition 5.1.6), is roughly speaking that for a stable homotopy complex d-manifold, the stabilization of the virtual cotangent bundle $\tilde{T}^*\boldsymbol{X}$ does not admit a complex structure itself (but can be deformed to a complex virtual vector bundle), whereas for a nearly complex d-manifold the virtual cotangent bundle is equivalent to a nearly complex virtual vector bundle without any deformation. On the other hand, though, the notion of homotopy complex structure requires the morphism ϕ to be complex linear (which is a very strong condition on ϕ), whereas in the nearly complex case ϕ will in general be not complex linear.

In the following, we will prove that there exists a (partly non-canonical) 1-1-correspondence between these two notions. What we mean by partly non-canonical is that although the direction from stable nearly complex d-manifold to stable homotopy complex d-manifold is canonical, the reverse direction from stable homotopy complex d-manifold to stable nearly complex d-manifold is not.

Lemma 5.4.1. *Given a stable nearly complex d-manifold $(\boldsymbol{X}, (\mathcal{F}^\bullet, \psi), K^\bullet, a)$, there exists a canonical homotopy complex structure $((\mathcal{E}^\bullet, \phi), J^\bullet, a)$, making $(\boldsymbol{X}, (\mathcal{E}^\bullet, \phi), J^\bullet, a)$ into a stable homotopy complex d-manifold.*

On the other hand, given a stable homotopy complex d-manifold $(\boldsymbol{X}, (\mathcal{E}^\bullet, \phi), J^\bullet, a)$*, there exists a non-canonical stable nearly complex structure* $((\mathcal{F}^\bullet, \psi), K^\bullet, a)$*, making* $(\boldsymbol{X}, (\mathcal{F}^\bullet, \psi), K^\bullet, a)$ *into a stable nearly complex d-manifold.*

Proof. Let therefore $\boldsymbol{X}$ be a stable nearly complex d-manifold, with stable nearly complex structure $((\mathcal{F}^\bullet, \psi), K^\bullet, a)$. We can then define an homotopy complex structure $((\mathcal{E}^\bullet, \phi), J^\bullet, \eta)$ on $\boldsymbol{X}$ by setting

$$\begin{aligned} \mathcal{E}^i &= \pi_{\underline{X}}^*(\mathcal{F}^i) \quad \text{for } i = 1,2 \\ \phi &= t\pi_{\underline{X}}^*(\psi) - (1-t)J^2 \circ \pi_{\underline{X}}^*(\psi) \circ J^1 \qquad \text{for } t \in [0,1] \\ J^i &= \pi_{\underline{X}}^*(K^i) \quad \text{for } i = 1,2 \\ \eta &= 0. \end{aligned}$$

This defines then a homotopy complex structure $((\mathcal{E}^\bullet, \phi), J^\bullet, \eta)$ on $\boldsymbol{X}$ and makes it into a stable homotopy complex d-manifold.

For the other direction, that is given a stable homotopy complex d-manifold we can construct a stable nearly complex d-manifold, consider a stable homotopy complex d-manifold $\boldsymbol{X}$, with stable homotopy complex structure $((\mathcal{E}^\bullet, \phi), J^\bullet, a)$. Using Proposition 2.1.24, we can conclude that the virtual vector bundle $(\mathcal{E}^\bullet, \phi)$ on $\underline{X} \times [0,1]$ is equivalent to a virtual vector bundle $(\mathcal{G}^\bullet, \chi)$, where $\mathcal{G}^1, \mathcal{G}^2$ are vector bundles on $\underline{X} \times [0,1]$.

On $\underline{X} \times \{1\}$ the virtual vector bundle $(\mathcal{E}^\bullet, \phi)|_{\underline{X}\times\{1\}}$ is equivalent to a complex virtual vector bundle $((\mathcal{H}^\bullet, \xi), J^\bullet_{\mathcal{H}}, \eta)$, and by using Proposition 5.1.9 we can assume with out loss of generality that $\mathcal{H}^1$ and $\mathcal{H}^2$ are complex vector bundles on $\underline{X} \times \{1\}$ with almost complex structures $J_{\mathcal{H}^i}$ for $i = 1, 2$.

Choosing connections in the $[0,1]$ directions on the vector bundles $\mathcal{G}^1 \to \underline{X} \times [0,1]$ and $\mathcal{G}^2 \to \underline{X} \times [0,1]$ allows us through parallel transport, to identify $\mathcal{G}^i|_{\underline{X}\times\{t\}}$ with $\mathcal{G}^i|_{\underline{X}\times\{1\}} \simeq \mathcal{H}^i$ for all $t \in [0,1]$ and $i = 1, 2$. Hence we get for each $t \in [0,1]$ an equivalence of virtual vector bundles on $\underline{X} \times [0,1]$:

$$(\mathcal{G}^1 \xrightarrow{\chi} \mathcal{G}^2)|_{\underline{X}\times\{t\}} \simeq (\pi_{\underline{X}}^*(\mathcal{H}^1) \xrightarrow{\xi(t)} \pi_{\underline{X}}^*(\mathcal{H}^2)).$$

Note that although $\pi_{\underline{X}}^*(\mathcal{H}^1)$ and $\pi_{\underline{X}}^*(\mathcal{H}^2)$ do not depend on $t \in [0,1]$, the morphism $\xi(t)$ does. Denote the so obtained virtual vector bundles on $\underline{X} \times \{0\}$ by

$(\mathcal{F}^1 \xrightarrow{\psi} \mathcal{F}^2)$ and let K^i be the complex structures induced by $J_{\mathcal{H}^i}$, for $i = 1, 2$. Although the induced morphism ψ depends on $t \in [0,1]$ and is therefore not complex linear, the resulting virtual vector bundle $((\mathcal{F}^\bullet, \psi), K^\bullet)$ is a nearly complex virtual vector bundle over $\underline{X} \times \{0\}$.

Furthermore, on $\underline{X} \times \{0\}$ we have $(\mathcal{E}^\bullet, \phi)|_{\underline{X}\times\{0\}} \simeq \tilde{T}^*\boldsymbol{X}$, and as all the above identifications preserve equivalences, we get that

$$((\mathcal{F}^\bullet, \psi), K^\bullet, a) \simeq \tilde{T}^*\boldsymbol{X},$$

where $a \in \mathbb{Z}_{\geq 0}$ is the same as before. Hence we get a nearly complex d-manifold $(\boldsymbol{X}, (\mathcal{F}^\bullet, \psi), K^\bullet, a)$, completing the argument that we can construct a stable nearly complex structure from an homotopy complex structure. □

Note that the lemma above is not independent of choices we made. In order to get a canonical 1-1-correspondence in both directions one has to make sure that the resulting nearly complex virtual vector bundle does not depend on the various choices, like the choice of connection, involved. One way how one could tackle this problem is by introducing a kind of K-theory for nearly complex virtual vector bundles and show that the constructed nearly complex virtual vector bundles lie all in one fixed K-theory class. For our purposes however, the statement of the lemma will be enough.

5.4.1 The relation between nearly and homotopy complex structures and Kuranishi structures

In this subsection we want in a similar way to section 4.1 explain some results from Fukaya and Ono [20] in terms of nearly and homotopy complex structures on d-manifolds and d-orbifolds. The following remark can be thought of as an extension of Remark 4.1.5.

Remark 5.4.2. **(a)** Fukaya and Ono [20, Definition 5.15] define *K-groups* $KO(X)$, $KSO(X)$ and $K(X)$ of a Kuranishi structure X. (Where again the sections s_p in the definition of Kuranishi neighbourhood in [20] are just assumed to be continuous and not smooth like in [18].) These K-groups are defined as the free abelian group generated by the set of all isomorphism classes of bundle systems, oriented bundle systems and complex bundle systems divided by some relation. There is then an obvious map $K(X) \to KSO(X) \to KO(X)$, which corresponds in the d-orbifold world to the fact that "homotopy complex structure on $T^*\boldsymbol{\mathcal{X}} \to$ orientation on $T^*\boldsymbol{\mathcal{X}} \to T^*\boldsymbol{\mathcal{X}}$". Note that here (in an extension of Remark 4.1.5(d)), Fukaya and Ono's bundle systems correspond to virtual vector bundles, oriented bundle systems correspond to oriented virtual vector bundles, and complex bundle systems to complex virtual vector bundles.

(b) In [20, §16] Fukaya and Ono prove that for a compact, symplectic manifold (X, ω) with compatible almost complex structure J, the moduli space $\bar{\mathcal{M}}_{g,n}(X, J, \beta)$ of n-pointed, genus g stable J-holomorphic curves in X carries a *stably almost complex structure*. Here a stably almost complex structure on a Kuranishi space X (see [20, Definition 5.17]) is a complex structure on the tangent bundle TX. (To be more precise, X carries a stably almost complex structure if $[TX] \in KO(X)$ lies in the image of $K(X)$). In the language of d-manifolds and d-orbifolds, Fukaya and Ono construct in the proof of [20, Proposition 16.5] a virtual vector bundle $(\mathcal{E}^\bullet, \phi)$ (a bundle system) on $\bar{\mathcal{M}}_{g,n}(X, J, \beta) \times [0, 1]$ which restricted to $\bar{\mathcal{M}}_{g,n}(X, J, \beta) \times \{0\}$ is equivalent to the tangent bundle $T\bar{\mathcal{M}}_{g,n}(X, J, \beta)$ and restricted to $\bar{\mathcal{M}}_{g,n}(X, J, \beta) \times \{1\}$ admits a complex structure. So in other words, Fukaya and Ono prove that $\bar{\mathcal{M}}_{g,n}(X, J, \beta)$ is an homotopy complex d-orbifold.

Chapter 6

Representable d-orbifolds

In the following we will study d-orbifolds, which admit a representable map into an effective orbifold. We will then prove that these *representable d-orbifolds* as we call them, will have the property that they can be embedded into an orbifold. As discussed earlier in section 3.4.6 it is not known, whether d-orbifolds do in general admit an embedding into an orbifold. The theorem we will prove in the following, makes a step towards answering this question and provides a useful criterion for the existence of such embeddings.

We will then briefly discuss a result of Kresch [38] about the embeddability of Deligne–Mumford stacks, and show how this result can be used to prove as a new result that a large class of 'interesting' moduli spaces in algebraic geometry can be given the structure of representable d-orbifolds. This result can be seen as a justification of the relevance of representable d-orbifolds, and will potentially be useful for future applications.

At the end of this chapter we will then sketch how one could prove the same result for the moduli spaces of n-pointed, genus g, J-holomorphic curves, using symplectic geometry. The idea we present is not fully worked out, but should rather provide a rough sketch how ideas of Cieliebak and Mohnke [11] and Donaldson [14] could be used to prove a theorem along these lines.

6.1 The definition of representable d-orbifolds

We will start by defining the new subclass of *representable d-orbifolds* in the class of d-orbifolds, which will have the property that they can be embedded into some orbifold. As it will turn out, many important examples in algebraic geometry, like the moduli stack $\overline{\mathcal{M}}_{g,m}(X,\beta)$, do not just possess a d-orbifold counterpart, but a representable d-orbifold counterpart.

Definition 6.1.1. A d-orbifold $\boldsymbol{\mathcal{X}}$ is called *representable*, if there exists an effective orbifold $\mathcal{Y}$ and a 1-morphism between d-orbifolds $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ which is representable, i.e. the underlying C^∞-stack morphism $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$ is injective for all $[x] \in \mathcal{X}_{\mathrm{top}}$ with $f_*([x]) = [y] \in \mathcal{Y}_{\mathrm{top}}$. $\boldsymbol{f}$ will be called the *representation map.*

We call a d-orbifold $\boldsymbol{\mathcal{X}}$ *embeddable*, if $\boldsymbol{\mathcal{X}}$ can be embedded into an orbifold $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ as d-orbifolds.

We will now prove as a new result that compact representable d-orbifolds are embeddable. Our proof is based on ideas of Joyce (compare [35, Proposition 10.34]) and will imitate the proof of Theorem 2.3.29. The following theorem can therefore be considered as an analogue of Theorem 2.3.29.

Theorem 6.1.2. *For a compact d-orbifold $\boldsymbol{\mathcal{X}}$, the following are equivalent:*

(i) $\boldsymbol{\mathcal{X}}$ admits a representable 1-morphism $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$, where $\mathcal{Y}$ is an effective, smooth orbifold.

(ii) $\boldsymbol{\mathcal{X}}$ admits an embedding $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ in **dOrb**, *where $\mathcal{Y}$ is an effective orbifold.*

Note also, that that this means that $\boldsymbol{\mathcal{X}}$ is a principal d-orbifold by Theorem 3.4.18.

Proof. (ii)⇒(i): Is immediate by the definition of embedding (Definition 3.4.14 (f)).
(i)⇒(ii): Note first that that it is sufficient to prove the theorem for the C^∞-stack case. The reason for this is that given an embedding $f : \mathcal{X} \to \mathcal{Y}$ from a C^∞-stack $\mathcal{X}$ into an effective orbifold $\mathcal{Y}$, we get a 2-isomorphism class of 1-morphism

embeddings $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ in **dOrb**. This can be seen by using local models (and joining the local choices by a partition of unity) and noting that two different standard model 1-morphisms on a coordinate chart differ by an $O(s)$-term.

Suppose now first that $f : \mathcal{X} \to \mathcal{Y}$ is a representable 1-morphism between smooth orbifolds, where $\mathcal{Y}$ is effective. Let $\mathcal{E}$ be a vector bundle over $\mathcal{Y}$, and let $s : \mathcal{X} \to f^*(\mathcal{E})$ be a generic section of $f^*(\mathcal{E})$. Assume further that for each $x \in X$, each irreducible representation R_i ($i \in I, I = \{0, \dots k\}$ for some $k \in \mathbb{Z}_{\geq 0}$) of $\mathrm{Iso}_{\mathcal{X}}([x])$ is contained in the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on $f^*(\mathcal{E})$ and, moreover, that for each $i \in I$

$$\mathrm{mult}_{T_x\mathcal{X}}(R_i) \leq \mathrm{mult}_{f^*(\mathcal{E})|_x}(R_i),$$

where $\mathrm{mult}_{T_x\mathcal{X}}(R_i)$ denotes the multiplicity of the representation R_i in $T_x\mathcal{X}$ and $\mathrm{mult}_{f^*(\mathcal{E})|_x}(R_i)$ the multiplicity of R_i in $f^*(\mathcal{E})|_x$. Then we claim (that after modifying $\mathcal{E}$):

Claim. There exists a lift of $f : \mathcal{X} \to \mathcal{Y}$ to $f' : \mathcal{X} \to \mathrm{Tot}(\mathcal{E})$ which is an embedding.

For the proof of this claim, fix a finite group Γ and consider the orbifold strata $\mathcal{X}_o^\Gamma$, as in section 3.2.7. For a point $x \in \mathcal{X}_o^\Gamma$ we can split $T_x\mathcal{X}$ and $\mathcal{E}|_x$ into the irreducible representations of Γ, that is $T_x\mathcal{X} = \bigoplus_{i=0}^{k} R_i^{n_i}$ and $\mathcal{E}|_x = \bigoplus_{i=0}^{k} R_i^{m_i}$, where $n_i = \mathrm{mult}_{T_x\mathcal{X}}(R_i), m_i = \mathrm{mult}_{f^*(\mathcal{E})|_x}(R_i)$.

The tangent bundle of $\mathrm{Tot}(\mathcal{E})$ is given by $T(\mathrm{Tot}(\mathcal{E})) = T\mathcal{Y} \oplus \mathcal{E}$. Hence we can split $ds : T\mathcal{X} \to T(\mathrm{Tot}(\mathcal{E}))$ for a fixed $x \in X_o^\Gamma$ into

$$df' = \nabla f \oplus ds,$$

and as ∇f is fixed (for a fixed $x \in \mathcal{X}_o^\Gamma$), df' is injective if $ds_i : n_iR_i \to m_iR_i$ is injective for all $i \in I$. But as ds is Γ-equivariant, and $m_i \geq n_i$ for all $i \in I$, genericity of s shows that $f' : \mathcal{X} \to \mathrm{Tot}(\mathcal{E})$ is an immersion at a fixed point $x \in \mathcal{X}_0^\Gamma$.

In general, we require a generic family of maps $n_iR_i \to m_iR_i$, for all $i \in I$, of $\dim \mathcal{X}_o^\Gamma$ to be injective, that is the following condition has to be satisfied for all $i \in I$:

$$(\dim \mathcal{X}_o^\Gamma) < \mathrm{codim}(\underset{\text{non-inj}}{\mathrm{Hom}}(n_iR_i, m_iR_i)),$$

where $\mathrm{Hom}_{\text{non-inj}}(n_iR_i, m_iR_i) \subseteq \mathrm{Hom}(n_iR_i, m_iR_i)$ denotes the non-injective map from n_iR_i to m_iR_i. The non-injective maps have at least 1-dimensional kernel and the dimension of $\mathrm{Hom}_{\text{non-inj}}(n_iR_i, m_iR_i)$ is at most

$$\dim_{\max} \underset{\text{non-inj}}{\mathrm{Hom}}(n_iR_i, m_iR_i) = (n_i - 1) + (n_i - 1)(m_i)$$

and hence the codimension of the non-injective maps within all maps is given by

$$\begin{aligned} \operatorname{codim} \underset{\text{non-inj}}{\mathrm{Hom}}(n_iR_i, m_iR_i) &= (n_im_i) - (n_i - 1) - (n_i - 1)(m_i) \\ &= m_i - n_i + 1. \end{aligned}$$

So in order to ensure that f' is an immersion, the condition $\dim \mathcal{X}_o^\Gamma < (m_i - n_i + 1)$ has to be satisfied for all $i = 0, \dots k$. So by replacing $\mathcal{E}$ with $\mathcal{E}^{\oplus 2 \dim \mathcal{X}}$ we can guarantee that $m_i \geq n_i + \dim \mathcal{X}$, which in turn implies that $f' : \mathcal{X} \to \mathrm{Tot}(\mathcal{E})$ is an immersion.

Moreover, by making $\mathrm{rank}\,(\mathcal{E})$ large enough we can ensure that $f' : \mathcal{X} \to \mathcal{E}$ is injective and thus an embedding.

In the case where $\mathcal{X}$ is not a smooth orbifold, but a singular C^∞-stack, there exists locally a smooth orbifold $\tilde{\mathcal{X}}$ in which $\mathcal{X}$ can be embedded. Denote this embedding by $\imath_{\mathcal{X}} : \mathcal{X} \to \tilde{\mathcal{X}}$ and note that $\tilde{\mathcal{X}}$ exists provided that $\mathcal{X}$ is locally fair.

Making $\tilde{\mathcal{X}}$ smaller if necessary, the morphism $f : \mathcal{X} \to \mathcal{Y}$ factors through $\tilde{\mathcal{X}}$ as $\mathcal{Y}$ is smooth, and we get a 1-morphism $\tilde{f} : \tilde{\mathcal{X}} \to \mathcal{Y}$. We are then in the situation as before and get an embedding $\tilde{f}' : \tilde{\mathcal{X}} \to \mathcal{E}$ and therefore an embedding $f' = \tilde{f}' \circ \imath_{\mathcal{X}} : \mathcal{X} \to \mathcal{Y}$.

To prove now the theorem, choose for each $x \in \boldsymbol{\mathcal{X}}$ an open neighbourhood $x \in \boldsymbol{U}_x \subseteq \boldsymbol{\mathcal{X}}$ and an equivalence $\boldsymbol{U}_x \simeq \boldsymbol{S}_{\mathcal{V}_x, \mathcal{E}_x, s_x}$. Consider the vector bundle $\mathcal{E} = \bigoplus_{j=0}^{N} T\mathcal{Y}^{\otimes j}$ over $\mathcal{Y}$. Let $[x] \in \mathcal{X}_{\text{top}}$ with $f_{\text{top}}([x]) = [y]$. As f is representable $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$ is injective and the representation of $\mathrm{Iso}_{\mathcal{Y}}([y])$ on $\mathcal{E}$ is effective, as $\mathcal{Y}$ is effective. Note that the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on $f_x^*(\mathcal{E})$ contains all irreducible representations of $\mathrm{Iso}_{\mathcal{X}}([x])$ for $N \gg 0$. This is true as f is representable and the representation of $\mathrm{Iso}_{\mathcal{Y}}([y])$ on $T^*\mathcal{Y}$ is effective, as $\mathcal{Y}$ is effective. (Compare [2, §7, Theorem 1] for a proof that every irreducible representation is a subrepresentation of an n-fold tensor product of an effective representation.)

So in particular, the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on $f_x^*(\mathcal{E})$ contains the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on $T_x\mathcal{X}$ for $n_0, \ldots, n_N \gg 0$.

Compactness of $\boldsymbol{\mathcal{X}}$ guarantees that first of all $\dim T_x\mathcal{X}$ is bounded and that the size of the orbifold groups $|\,\mathrm{Iso}\,\mathcal{X}([x])|$ is bounded. Hence we are in the situation of before and we get an embedding $g_x : \mathcal{X} \to \mathcal{E}$. By choosing a partition of unity subordinated to a covering of $\boldsymbol{\mathcal{X}}$ by open neighbourhoods $\boldsymbol{U}_x$, we can join these local embeddings and end up with an embedding $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{E}}$ as claimed. $\square$

As we have seen in the proof of Theorem 6.1.2, we do not actually require compactness of $\boldsymbol{\mathcal{X}}$, as we just used the facts that $\dim T_x\mathcal{X}$ and $|\,\mathrm{Iso}_{\mathcal{X}}([x])|$ are bounded. This gives us the following stronger result:

Theorem 6.1.3. *Let $\boldsymbol{\mathcal{X}}$ be a representable d-orbifold. Assume further that* $\dim T_x\mathcal{X}$ *and* $|\,\mathrm{Iso}_{\mathcal{X}}([x])|$ *are bounded for x in X. Then $\boldsymbol{\mathcal{X}}$ is embeddable.*

6.2 Kresch's Theorem

We want now to discuss Kresch's theorem about the embeddability of Deligne–Mumford stacks and use his result to conclude that many important moduli spaces in algebraic geometry can be thought of as representable d-orbifolds. Let us begin by recalling some facts about Deligne–Mumford stacks, and stating the embeddability theorem due to Kresch [38]. For more background on Deligne–Mumford stacks we refer to [38].

An *algebraic orbifold* is a smooth Deligne–Mumford $\mathbb{C}$-stack $\mathcal{X}$ of finite type, that has a dense open subset isomorphic to an algebraic variety. This is equivalent to $\mathcal{X}$ having trivial generic stabilizer, which implies by a well-known theorem that $\mathcal{X}$ is a *quotient stack*, that is $\mathcal{X} \cong [P/G]$ for some algebraic space P and a linear algebraic group G. Over $\mathbb{C}$ one can take P to be the frame bundle associated with the tangent bundle $T\mathcal{X}$ of $\mathcal{X}$ and G to be GL_d, where $d = \dim \mathcal{X}$ (see for example Satake [49, §1.5] as a reference).

In [38, §5] Kresch proves the following theorem about the embeddability of Deligne–Mumford stacks. We will show how this result can be used to conclude that every moduli space of n-pointed genus g curves is an embeddable d-orbifold.

Theorem 6.2.1 (Kresch)**.** *For a proper Deligne–Mumford stack $\mathcal{X}$ over $\mathbb{C}$, the following are equivalent:*

(i) $\mathcal{X}$ is a quotient stack and has projective coarse moduli space.

(ii) $\mathcal{X}$ possesses a generating sheaf and has projective coarse moduli space.

(iii) $\mathcal{X}$ can be embedded into a smooth, proper Deligne–Mumford stack with projective coarse moduli space.

A Deligne–Mumford stack $\mathcal{X}$ is called *quasi-projective* (*projective)* if there exists a locally closed embedding (closed embedding) to a smooth Deligne–Mumford stack which is proper over $\mathbb{C}$ and has projective coarse moduli space.

Examples of projective moduli stacks include the moduli stack $\overline{\mathcal{M}}_{g,m}(X,\beta)$ of m-pointed, genus g stable maps to a projective target variety X or the moduli stacks $\mathcal{K}_{g,n}(\mathcal{X},d)$ of twisted stable maps, where $\mathcal{X}$ is a proper Deligne–Mumford stack having projective coarse moduli space.

The following theorem is an addition to Theorem 4.2.4(d) and will show that projective Deligne–Mumford $\mathbb{C}$-stacks can be thought of as not just d-orbifolds, but as representable d-orbifolds.

Theorem 6.2.2. *Let $\mathcal{X}$ be a separated, second countable Deligne–Mumford $\mathbb{C}$-stack with perfect obstruction theory $\phi : E^\bullet \to \tau_{\geq -1}(\mathbb{L}_{\mathcal{X}})$ of virtual rank $n \in \mathbb{Z}$. Then, as in Theorem 4.2.4(d), one can construct an up to equivalence unique, oriented d-orbifold $\boldsymbol{\mathcal{X}}$ with vdim $\boldsymbol{\mathcal{X}} = 2n$.*

If $\mathcal{X}$ is also a projective Deligne–Mumford stack, which implies that it admits an embedding into a smooth Deligne–Mumford stack, then the above constructed d-orbifold $\boldsymbol{\mathcal{X}}$ is representable, and therefore principal.

The reason why Theorem 6.2.2 is true, is that there exists a "functor" $F_{\mathbb{C}-\mathrm{alg}}^{C^\infty}$ from algebraic geometry over $\mathbb{C}$ to algebraic geometry over C^∞-rings. What we mean by this, is that there exists a functor $F_{\mathbf{Sch}}^{C^\infty\mathbf{Sch}}$ from the category of $\mathbb{C}$-schemes to the category of C^∞-schemes, a functor $F_{\mathbf{Sta}}^{C^\infty\mathbf{Sta}}$ from the category of Deligne–Mumford $\mathbb{C}$-stacks to the 2-category of Deligne–Mumford C^∞-stacks, a functor from smooth Deligne–Mumford $\mathbb{C}$-stacks to smooth Deligne–Mumford C^∞-stacks, that is orbifolds, … . So for every algebraic object over $\mathbb{C}$ we get a C^∞-analogue, and this in a functorial way.

6.3 Symplectic case

In this section we want to outline a proof, that the moduli space of n-pointed, genus g, J-holomorphic curves $\overline{\mathcal{M}}_{g,n}(M,J,\beta)$ is representable, that is, it admits a representable 1-morphism to an effective orbifold $\mathcal{Y}$. The obvious candidate for a target space $\mathcal{Y}$ is the Deligne–Mumford moduli space of stable, n-pointed, genus g curves $\overline{\mathcal{M}}_{g,n}$, which can be shown to be an effective Hausdorff, compact complex orbifold of complex dimension $3g+n-3$, provided $(g,n)\neq(1,1)$.

Restricting ourselves to triples $[\Sigma,\vec{z},u]$, where $(\Sigma,\vec{z})$ is a stable n-pointed, genus g curve gives an indication why $\overline{\mathcal{M}}_{g,n}$ should indeed be the "correct" choice of target space. In the case of $(\Sigma,\vec{z})$ being stable, the projection map $\pi:\overline{\mathcal{M}}_{g,n}(M,J,\beta)\to\overline{\mathcal{M}}_{g,n}$, $[\Sigma,\vec{z},u]\mapsto[\Sigma,\vec{z}]$ is indeed a representable morphism, which implies that $\overline{\mathcal{M}}^{\text{stable}}_{g,n}(M,J,\beta)$ is a representable, and therefore embeddable d-orbifold.

In general however, this is not true, as for any element $[\Sigma,\vec{z},u]\in\overline{\mathcal{M}}_{g,n}(M,J,\beta)$ where $(\Sigma,\vec{z})$ is not stable, the map π maps $[\Sigma,\vec{z},u]$ to its stabilization. The unstable component of $(\Sigma,\vec{z},u)$ can for example be given by a k-fold cover of $\mathbb{C}P^1$ with only one node, so that $\mathrm{Iso}(\Sigma,\vec{z},u)$ is given by $\mathbb{Z}_k$. But $\pi(\Sigma,\vec{z},u)$ is then just a $\mathbb{C}P^1$ with trivial orbifold group, and therefore the corresponding representation map cannot be injective.

The idea how to resolve this issue, is to contemplate J-holomorphic hypersurfaces intersecting the unstable parts in exactly "size of the orbifold group"-points. The idea presented in the following is similar to the idea of *Donaldson pairs* of Cieliebak and Mohnke [11]. They use a result of Donaldson [14], which provides, for a sufficiently large $D\in\mathbb{Z}_{\geq 0}$, the existence of a degree D, symplectic hypersurface for a given ω-compatible almost complex structure J. We refer therefore to Cieliebak and Mohnke [11] and Donaldson [14] for more detailed discussions of this idea.

To be more specific, given an almost complex manifold (M,J) and a J- holomorphic hypersurface H, consider n-pointed, genus g, J-holomorphic curves $u:\Sigma\to M$ representing the homology class $[u(\Sigma)]=\beta\in H_2(M;\mathbb{Z})$ and satisfying $u_*([\Sigma])\cdot H=k>0$, where $k\in\mathbb{Z}_{\geq 0}$ denotes the maximum order of the stabilizer groups of the unstable components.

Denote the moduli space of such curves by $\overline{\mathcal{M}}_{g,m}(M,J,\beta,H)$, where $m = n+k$. If everything is 'nice', we would expect that every curve in $\overline{\mathcal{M}}_{g,n}(M,J,\beta)$ intersects H in exactly k points, counted with multiplicity. We could then define $\overline{\mathcal{M}}_{g,n,k}$ to be the moduli space of n-pointed prestable, genus g curves $(\Sigma, \vec{z}, \vec{z}')$ with additional marked points $\vec{z}'$, where these additional marked points are allowed to repeat, that is $z_i' = z_j'$ for $i \neq j$, and are allowed to be nodes or coincide with the already given marked points z_i.

The map $\overline{\mathcal{M}}_{g,m}(M,J,\beta,H) \to \overline{\mathcal{M}}_{g,n,k}$, which treats the intersection points $u(\Sigma) \cap H$ as additional marked points $\vec{z}'$, is then a representable morphism. Moreover we would get the following diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n}(M,J,\beta) \arrow[rr, dotted, shift left, "\alpha(\cong)"] \arrow[dr, "a"'] & & \overline{\mathcal{M}}_{g,m}(M,J,\beta,H) \arrow[ll, shift left, "\pi_H"] \arrow[dl, "b"] \\
& \overline{\mathcal{M}}_{g,n,k} &
\end{tikzcd},$$

where $\alpha : \overline{\mathcal{M}}_{g,n}(M,J,\beta) \to \overline{\mathcal{M}}_{g,m}(M,J,\beta,H)$ is given by $\alpha([\Sigma,u,\vec{z}]) = [\Sigma, u, \vec{z}, u^{-1}(H)]$ and $\pi_H : \overline{\mathcal{M}}_{g,m}(M,J,\beta,H) \to \overline{\mathcal{M}}_{g,n}(M,J,\beta)$ by $\pi_H([\Sigma,u,\vec{z},\vec{z}']) = [\Sigma,u,\vec{z}]$.

In order to make this approach rigorous, one have to deal among others with the following problems that can arise:

(A) For a given hypersurface H, we may get $u(\Sigma) \subseteq H$ for some curves $\Sigma \subseteq H$.

(B) The condition that all $\vec{z}'$ occur "in the right multiplicity" might not be open in the set of all $(\Sigma, u, \vec{z}, \vec{z}')$.

(C) Does $\overline{\mathcal{M}}_{g,m}(M,J,\beta,H)$ carry the structure of a d-orbifold?

One possible way how problem (A) could be resolved on the common domain of symplectic and algebraic geometry (for J integrable and (M,J) being projective algebraic), is to choose H as a generic smooth hypersurface in $\mathcal{O}(N)$ for some $N \gg 0$. For a generic choice of such a hyperplane H any fixed curve $u(\Sigma)$ should satisfy

$$u(\Sigma) \not\subset H. \tag{6.1}$$

The reason why (6.1) should be true is the following: Denote by $\pi : H^0(\mathcal{O}(N)) \to H^0(u^*(\mathcal{O}(N)))$ the projection map. Then, assuming that $u(\Sigma) \subset H$, we find for N large enough that $H^0(\mathcal{O}(n))$ admits a smooth section $s_H \in H^0(\mathcal{O}(N))$ near $\pi \circ s_H = 0$ in $H^0(u^*(\mathcal{O}(N)))$, which for $N \gg 0$ should not happen.

Chapter 7

D-(co)bordism

In this chapter we want to introduce the notion of unitary d-(co)bordism theory. In the same way classical unitary bordism theory extends oriented bordism theory, unitary d-(co)bordism theory can be thought of an extension of the oriented d-(co)bordism theory due to Joyce [35, §13]. One of the major theorems in Joyce's book, is that the d-(co)bordism group of a manifold is isomorphic to the "usual" oriented (co)bordism group, or in other words, that d-(co)bordism for a manifold is "the same as classical" (co)bordism. This result is crucial, as it shows that d-manifolds admit virtual cycles and can therefore be used as a geometric structure in enumerative invariant problems, like symplectic Gromov–Witten theory. We will prove in section 7.3 as a new result that the same result holds for unitary d-bordism, that is, given a compact manifold without boundary, its unitary d-bordism group is isomorphic to its classical unitary bordism group. We then discuss in secion 7.4 how these results can be extended to d-orbifolds.

7.1 Classical cobordism and bordism theory for manifolds

In the following we want to briefly review some classical (co)bordism theory. Classical bordism groups were introduced by Atiyah [5] and a good introduction can be found in Conner [12]. We will closely follow Joyce in describing his "non-standard approach" and refer for more details to [35, §13.1].

Definition 7.1.1. Let Y be a compact manifold without boundary and $k \in \mathbb{Z}$. Consider pairs (X, f), where X is a compact, oriented k-manifold without boundary, and $f : X \to Y$ is a smooth map. By convention, $\emptyset$ is an oriented manifold of any dimension $k \in \mathbb{Z}$, and $\emptyset : \emptyset \to Y$ is smooth. In particular the pair $(\emptyset, \emptyset)$ is the only such pair for $k < 0$.

Define a binary relation $\sim$ between (X, f) and (X', f') by $(X, f) \sim (X', f')$ if there exists a compact, oriented $(k+1)$-manifold W with boundary ∂W, a smooth map $e : W \to Y$, and a diffeomorphism of oriented manifolds $j : -X \amalg X' \to \partial W$, such that $f \amalg f' = e \circ i_W \circ j$, where $i_W : \partial W \to W$ denotes the inclusion map. Here $-X$ is given by X with reversed orientation and the orientation of ∂W is induced from that on W.

As for example shown in Conner [12, Th. 1.2.1] the binary relation $\sim$ is an equivalence relation, and is called *bordism relation.*

Denote by $[X, f]$ the $\sim$-equivalence class of such a pair (X, f) and define for each $k \in \mathbb{Z}$, the *k-th bordism group* $B_k(Y)$ of Y to be the set of all such bordism classes $[X, f]$ with $\dim X = k$. $B_k(Y)$ can be given the structure of an abelian group, with zero element $0_Y = [\emptyset, \emptyset]$, addition $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$ and additive inverse $-[X, f] = [-X, f]$. Note that if $k < 0$ then $B_k(Y) = 0$, as the only element is 0_Y.

The following definition can be found in [35, §2.8 and §6.7] and will be important in defining the (co)bordism group.

Definition 7.1.2. Let $f : X \to Y$ be a smooth map between manifolds with dimensions $\dim Y = n$ and $\dim X = n - k$. A *coorientation* for f is an orientation on the line bundle $\Lambda^{n-k} T^* X \otimes f^*(\Lambda^n T^* Y)^*$ over X.

In the spirit of Definition 7.1.1 one can define the *cobordism group* of a manifold Y as follows:

Definition 7.1.3. Let Y be a compact n-dimensional manifold without boundary, and $k \in \mathbb{Z}$ fixed. Consider pairs (X, f), where X is a compact, oriented $n - k$-manifold without boundary, and $f : X \to Y$ is a cooriented smooth map.

Define a binary relation $\sim$ between (X, f) and (X', f') by $(X, f) \sim (X', f')$ if there exists a compact $(n - k + 1)$-manifold with boundary ∂W, a cooriented smooth map $e : W \to Y$, and a diffeomorphism $j : X \amalg X' \to \partial W$, such that $f \amalg f' = e \circ i_W \circ j$, and j identifies the given coorientation on $e \circ i_W : \partial W \to Y$ with the disjoint union of the opposite coorientation on $f : X \to Y$, and the coorientation $f' : X' \to Y$.

The inclusion map $i_W : \partial W \to W$ has a natural coorientation coming from the identification $\Lambda^{n-k} T^* \partial W \otimes i_W^* (\Lambda^{n-k+1} T^* W)^* \cong \mathcal{N}_{\partial W} W$, where $\mathcal{N}_{\partial W} W$ is the normal bundle of ∂W in W, and the orientation on $\mathcal{N}_{\partial W} W$ is given by outward-pointing normal vectors.

Once again, one can show (see for example Conner [12, Theorem 1.2.1]) that the binary relation $\sim$ is an equivalence relation, which is called *cobordism*.

Denote by $[X, f]$ the $\sim$-equivalence class of such a pair (X, f), and define for each $k \in \mathbb{Z}$, the *k-th cobordism group* $B^k(Y)$ of Y to be the set of all such cobordism classes $[X, f]$ with $\dim X = n - k$. $B^k(Y)$ can be given the structure of an abelian group, with zero element $0_Y = [\emptyset, \emptyset]$, addition $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$ and additive inverse $-[X, f] = [-X, f]$. Note that for $k > n, B^k(Y) = 0$ as the only element is 0_Y, but for $k < 0$ it can happen that $B^k(Y) \neq 0$.

The bordism and cobordism groups carry much more structure than just that of an abelian group. We will follow Joyce [35] and define products on (co)bordism, identities and fundamental classes.

Definition 7.1.4. Let Y be a compact manifold of dimension n. Then we can define a biadditive *cup product* $\cup : B^k(Y) \times B^l(Y) \to B^{k+l}(Y)$ on cobordism, a biadditive *cap product* $\cap : B^k(Y) \times B_l(Y) \to B_{l-k}(Y)$ mixing bordism and cobordism, and in the case of Y being oriented the *intersection product* $\bullet : B_k(Y) \times B_l(Y) \to B_{k+l}(Y)$ on bordism.

All of these operations can be defined by the same formula: given suitable classes $[X, f], [X', f']$, we can deform f, f' within their (co)bordism classes to make $f : X \to Y$ and $f' : X' \to Y$ transverse smooth maps. We can then define

$$\begin{aligned} [X, f] \cup [X', f'] &= [X \times_{f,Y,f'} X', f \circ \pi_X], \\ [X, f] \cap [X', f'] &= [X \times_{f,Y,f'} X', f \circ \pi_X], \\ [X, f] \bullet [X', f'] &= [X \times_{f,Y,f'} X', f \circ \pi_X]. \end{aligned}$$

Note that since f and f' are transverse, the fibre product $X \times_{f,Y,f'} X'$ exists in **Man**. Moreover, the orientations on X, X', Y (or coorientations on f, f') combine to an orientation on $X \times_{f,Y,f'} X'$ (or a coorientation on $f \circ \pi_X : X \times_{f,Y,f'} X' \to Y$).

The fibre product and the orientations fulfil certain associativity and commutativity properties and these imply that $\cup, \bullet$ are associative and supercommutative and that $(B_*(Y), \cap)$ is a graded module over $(B^*(Y), \cup)$.

The identity $\mathrm{id}_Y : Y \to Y$ on Y inherits a natural coorientation from $\Lambda^n T^*Y \otimes \mathrm{id}_Y^*(\Lambda^n T^*Y)^* \cong \mathcal{O}_Y$ and we can therefore define the *identity element* $1_Y = [Y, \mathrm{id}_Y] \in B^0(Y)$. If $f : X \to Y$ is smooth, we have $X \times_{f,Y,\mathrm{id}_Y} Y \cong Y \times_{\mathrm{id}_Y,Y,f} X \cong X$, and hence $[X, f] \cup 1_Y = 1_Y \cup [X, f] = [X, f]$, which means that 1_Y is the identity for $\cup$. This makes $B^*(Y)$ into a supercommutative graded ring.

If Y is oriented we can define the *fundamental class* $[Y] \in B_n(Y)$ by $[Y] = [Y, \mathrm{id}_Y]$. The fundamental class is the identity for the intersection product $\bullet$ on $B_*(Y)$.

If $g : Y \to Z$ is a smooth map of compact manifolds without boundary, we can define pushforwards g_*, pullbacks g^* and projections to (co)homology.

Definition 7.1.5. Let $g : Y \to Z$ be a smooth map of compact manifolds without boundary. Define the *pushforward* $g_* : B_k(Z) \to B_k(Y)$ of a class $[X, f]$ by $g_*([X, f]) = [X, g \circ f]$. Define the *pullback* $g^* : B^k(Z) \to B^k(Y)$ of a class $[X, f] \in B^k(Z)$ as follows: perturb the map $f : X \to Z$ within its cobordism class so that f, g are transverse. Then the fibre product $X \times_{f,Z,g} Y$ exists in **Man**, and is compact as X, Y, Z are. The coorientation on f induces a coorientation for $\pi_Y : X \times_{f,Z,g} Y \to Y$ and we define $g^*([X, f]) = [X \times_{f,Z,g} Y, \pi_Y]$. The so defined morphism g^* preserves the cup product $\cup$, and satisfies $g^*(1_Z) = 1_Y$, which makes g^* a morphism of graded rings.

Define *projection morphisms* $\Pi_{\mathrm{bo}}^{\mathrm{hom}} : B_k(Y) \to H_k(Y; \mathbb{Z})$ by $\Pi_{\mathrm{bo}}^{\mathrm{hom}} : [X, f] \mapsto f_*([X])$, where $[X] \in H_k(X; \mathbb{Z})$ is the fundamental class of X.

Let Y be a compact manifold of dimension n. Then there exists a unique morphism $\Pi_{\mathrm{cob}}^{\mathrm{coh}} : B^k(Y) \to H^k(Y;\mathbb{Z})$ such that the following diagram commutes

$$\begin{tikzcd} B^k(Y) \arrow[r, "\Pi_{\mathrm{cob}}^{\mathrm{coh}}"] \arrow[d, "\cong"', "\cap[Y]"] & H^k(Y;\mathbb{Z}) \arrow[d, "\cong"', "\cap[Y]"] \\ B_{n-k}(Y) \arrow[r, "\Pi_{\mathrm{bo}}^{\mathrm{hom}}"'] & H_{n-k}(Y;\mathbb{Z}). \end{tikzcd}$$

Here the columns are just the Poincaré duality isomorphisms, and are therefore invertible.

If Y is not oriented, one can still define a projection morphism $\Pi_{\mathrm{cob}}^{\mathrm{coh}} : B^*(Y) \to H^*(Y;\mathbb{Z})$ using the non-oriented version of Poincaré duality, which relates cohomology of Y to the homology of Y twisted by the orientation line bundle $\Lambda^n T^*Y$ of Y. (Compare [35, Remark 13.4].)

The projection morphisms $\Pi_{\mathrm{bo}}^{\mathrm{hom}}, \Pi_{\mathrm{cob}}^{\mathrm{coh}}$ are structure-preserving, that is they take $\cup, \cap, \bullet$, identities and fundamental classes in $B_*(Y), B^*(Y)$ to $\cup, \cap, \bullet$, identities and fundamental classes in $H_*(Y;\mathbb{Z}), H^*(Y;\mathbb{Z})$. Hence, $\Pi_{\mathrm{cob}}^{\mathrm{coh}} : B^*(Y) \to H^*(Y;\mathbb{Z})$ is a morphism of graded rings.

For a point $*$, the bordism ring is understood completely, as the following theorem due to Thom [51] shows:

Theorem 7.1.6 (Thom [51])**.** *The bordism ring $B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ of a point is the free commutative $\mathbb{Q}$-algebra generated by $\zeta_{4k} = [\mathbb{C}P^{2k}, \pi] \in B_{4k}(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ for $k \geq 1$. Therefore $B_n(*) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$ if and only if $n = 4k$ for $k = 0, 1, 2, \ldots$.*

As discussed for example in [12, §1.5 & §1.13] bordism and cobordism satisfy all of the Eilenberg–Steenrod axioms except for the dimension axiom, which makes them into *generalized homology and cohomology theories.*

7.2 D-manifold (co)bordism

We want now to generalize this notion of bordism and cobordism to the case of d-manifolds. Most of the basic definitions in the classical approach generalize nicely to the d-manifold world and as we will see, it can be shown that the d-(co)bordism groups $\mathrm{dB}_*(Y)$ are in fact isomorphic to the classical (co)bordism groups when Y is a manifold. We follow here again Joyce, and refer to [34, §13] for further details and discussion.

Definition 7.2.1. Let Y be a compact manifold without boundary, and $k \in \mathbb{Z}$. Consider pairs $(\boldsymbol{X}, \boldsymbol{f})$, where $\boldsymbol{X} \in \mathbf{dMan}$ is a compact, oriented d-manifold without boundary of virtual dimension $\operatorname{vdim} \boldsymbol{X} = k$, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism between d-manifolds. Here $\boldsymbol{Y} = \boldsymbol{F}_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$.

Define a binary relation $\sim$ between pairs $(\boldsymbol{X}, \boldsymbol{f}) \sim (\boldsymbol{X}', \boldsymbol{f}')$, if there exists a compact, oriented d-manifold with boundary $\boldsymbol{W}$ of virtual dimension $\operatorname{vdim} \boldsymbol{W} = k+1$, a 1-morphism $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{Y}$ in $\mathbf{dMan}^{\boldsymbol{b}}$, an equivalence of oriented d-manifolds $\boldsymbol{j} : -\boldsymbol{X} \amalg \boldsymbol{X}' \to \partial \boldsymbol{W}$, and a 2-morphism $\eta : \boldsymbol{f} \amalg \boldsymbol{f}' \Rightarrow \boldsymbol{e} \circ \boldsymbol{i}_{\boldsymbol{W}} \circ \boldsymbol{j}$. This binary relation is in fact an equivalence relation, as for example proven in [35].

Denote by $[\boldsymbol{X}, \boldsymbol{f}]$ the equivalence class (the *d-bordism class*) of a pair $(\boldsymbol{X}, \boldsymbol{f})$ and define for every $k \in \mathbb{Z}$ the *k-th d-manifold bordism group*, or for short *d-bordism group* $\mathrm{dB}_k(Y)$ of Y as the set of all such d-bordism classes $[\boldsymbol{X}, \boldsymbol{f}]$ with $\operatorname{vdim} \boldsymbol{X} = k$. Defining $0_Y = [\emptyset, \emptyset]$, $[\boldsymbol{X}, \boldsymbol{f}] + [\boldsymbol{X}', \boldsymbol{f}'] = [\boldsymbol{X} \amalg \boldsymbol{X}', \boldsymbol{f} \amalg \boldsymbol{f}']$ and $-[\boldsymbol{X}, \boldsymbol{f}] = [-\boldsymbol{X}, \boldsymbol{f}]$ gives $\mathrm{dB}_k(Y)$ the structure of an abelian group with zero element.

Definition 7.2.2. Let Y be a compact n-dimensional manifold without boundary, and $k \in \mathbb{Z}$. Consider pairs $(\boldsymbol{X}, \boldsymbol{f})$, where $\boldsymbol{X} \in \mathbf{dMan}$ is a compact d-manifold without boundary of virtual dimension $\operatorname{vdim} \boldsymbol{X} = n - k$, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a cooriented 1-morphism between d-manifolds, where again $\boldsymbol{Y} = \boldsymbol{F}_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$.

Define a binary relation $\sim$ between pairs $(\boldsymbol{X}, \boldsymbol{f}), (\boldsymbol{X}', \boldsymbol{f}')$, if there exists a compact d-manifold with boundary $\boldsymbol{W}$ of virtual dimension $\operatorname{vdim} \boldsymbol{W} = n - k + 1$, a cooriented 1-morphism $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{Y}$ in $\mathbf{dMan}^{\boldsymbol{b}}$, an equivalence of d-manifolds $\boldsymbol{j} : \boldsymbol{X} \amalg \boldsymbol{X}' \to \partial \boldsymbol{W}$, and a 2-morphism $\eta : \boldsymbol{f} \amalg \boldsymbol{f}' \Rightarrow \boldsymbol{e} \circ \boldsymbol{i}_{\boldsymbol{W}} \circ \boldsymbol{j}$, such that $\boldsymbol{j}, \eta$ identify the coorientation on $\boldsymbol{e} \circ \boldsymbol{i}_{\boldsymbol{W}} : \partial \boldsymbol{W} \to \boldsymbol{Y}$ with the union of the reversed coorientation on $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and the coorientation on $\boldsymbol{f}' : \boldsymbol{X}' \to \boldsymbol{Y}$. Once again, this binary relation turns out to be an equivalence relation, which we call *d-cobordism*.

The *d-cobordism class* $[\boldsymbol{X}, \boldsymbol{f}]$ is the equivalence class of a pair $(\boldsymbol{X}, \boldsymbol{f})$. For each $k \in \mathbb{Z}$ define the *k-th d-manifold cobordism group*, or just *d-cobordism group* $\mathrm{dB}^k(Y)$ of Y to be the set of all such d-bordism classes $[\boldsymbol{X}, \boldsymbol{f}]$ with $\operatorname{vdim} \boldsymbol{X} = n-k$. As in the d-bordism case, $\mathrm{dB}^k(Y)$ can also be given the structure of an abelian group with zero element $0_Y = [\emptyset, \emptyset]$, by defining addition as $[\boldsymbol{X}, \boldsymbol{f}] + [\boldsymbol{X}', \boldsymbol{f}'] = [\boldsymbol{X} \amalg \boldsymbol{X}', \boldsymbol{f} \amalg \boldsymbol{f}']$ and additive inverses $-[\boldsymbol{X}, \boldsymbol{f}] = [-\boldsymbol{X}, \boldsymbol{f}]$.

As in the classical case, we can define a cup product, a cap product, an intersection product, identities and fundamental classes on $\mathrm{dB}^*(Y)$ and $\mathrm{dB}_*(Y)$.

Definition 7.2.3. Let Y be a compact n-manifold without boundary. Define the *cup product* $\cup : \mathrm{dB}^k(Y) \times \mathrm{dB}^l(Y) \to \mathrm{dB}^{k+l}(Y)$ on d-cobordism, the *cap product* $\cap : \mathrm{dB}^k(Y) \times \mathrm{dB}_l(Y) \to \mathrm{dB}_{k-l}(Y)$ mixing d-cobordism and d-bordism and, for oriented Y, the *intersection product* $\bullet : \mathrm{dB}_k(Y) \times \mathrm{dB}_l(Y) \to \mathrm{dB}_{k+l-n}(Y)$ on d-bordism as follows:

if $[\boldsymbol{X}, \boldsymbol{f}], [\boldsymbol{X}', \boldsymbol{f}']$ are classes, define

$$\begin{aligned}
[\boldsymbol{X}, \boldsymbol{f}] \cup [\boldsymbol{X}', \boldsymbol{f}'] &= [\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Y},\boldsymbol{f}'} \boldsymbol{X}', \boldsymbol{f} \circ \boldsymbol{\pi}_{\boldsymbol{X}}],\\
[\boldsymbol{X}, \boldsymbol{f}] \cap [\boldsymbol{X}', \boldsymbol{f}'] &= [\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Y},\boldsymbol{f}'} \boldsymbol{X}', \boldsymbol{f} \circ \boldsymbol{\pi}_{\boldsymbol{X}}],\\
[\boldsymbol{X}, \boldsymbol{f}] \bullet [\boldsymbol{X}', \boldsymbol{f}'] &= [\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Y},\boldsymbol{f}'} \boldsymbol{X}', \boldsymbol{f} \circ \boldsymbol{\pi}_{\boldsymbol{X}}].
\end{aligned}$$

Note that the fibre product $\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Y},\boldsymbol{f}'} \boldsymbol{X}'$ exists by Theorem 2.3.33(a) as a d-manifold, and is oriented when $\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{Y}$ are oriented, by Theorem 2.3.38.

Moreover, using [35, §6.6 – §6.7] one can use the orientations and coorientations on $\boldsymbol{X}, \boldsymbol{f}, \boldsymbol{X}', \boldsymbol{f}'$ to define an orientation on $\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{X},\boldsymbol{f}'} \boldsymbol{X}'$ or a coorientation on $\boldsymbol{f} \circ \boldsymbol{\pi}_{\boldsymbol{X}}$. Using these results, one gets similarly to the classical case, that $\cup, \bullet$ are supercommutative and associative, and that $\cap$ makes $\mathrm{dB}_*(Y)$ into a module over $(dB^*(Y), \cup)$.

Again, $\mathbf{id}_{\boldsymbol{Y}} : \boldsymbol{Y} \to \boldsymbol{Y}$ carries a natural coorientation, and we can define the *identity* $1_Y = [\boldsymbol{Y}, \mathrm{id}_{\boldsymbol{Y}}] \in \mathrm{dB}^0(Y)$. This identity satisfies $[\boldsymbol{X}, \boldsymbol{f}] \cap 1_Y = 1_Y \cap [\boldsymbol{X}, \boldsymbol{f}] = [\boldsymbol{X}, \boldsymbol{f}]$ for any class $[\boldsymbol{X}, \boldsymbol{f}]$ in $\mathrm{dB}^*(Y)$. If Y is in addition oriented, define the *fundamental class* $[Y] \in \mathrm{dB}_n(Y)$ by $[Y] = [\boldsymbol{Y}, \mathbf{id}_{\boldsymbol{Y}}]$. It turns out that the fundamental class is the identity for $\bullet$ on $\mathrm{dB}_*(Y)$.

Definition 7.2.4. Let $g : Y \to Z$ be a smooth map of compact manifolds without boundary. Define a morphism $g_* : \mathrm{dB}_k(Y) \to \mathrm{dB}_k(Z)$ by $g_*([\boldsymbol{X}, \boldsymbol{f}]) = [\boldsymbol{X}, \boldsymbol{g} \circ \boldsymbol{f}]$, where $\boldsymbol{g} = F_{\mathbf{Man}}^{\mathbf{dMan}}(g)$.

Define $g^* : \mathrm{dB}^k(Z) \to \mathrm{dB}^k(Y)$ by $g^* : [\boldsymbol{X}, \boldsymbol{f}] \mapsto [\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Z},\boldsymbol{g}} \boldsymbol{Y}, \boldsymbol{\pi}_{\boldsymbol{Y}}]$. Here the fibre product $\boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Z},\boldsymbol{g}} \boldsymbol{Y}$ exists in **dMan** by Theorem 2.3.33(a), and is compact as $\boldsymbol{X}, Y, Z$ are. Moreover, the coorientation for $\boldsymbol{f}$ induces one for $\boldsymbol{\pi} : \boldsymbol{X} \times_{\boldsymbol{f},\boldsymbol{Z},\boldsymbol{g}} \boldsymbol{Y} \to \boldsymbol{Y}$.

There are natural projections $\Pi_{\mathrm{bo}}^{\mathrm{dbo}} : B_k(Y) \to \mathrm{dB}_k(Y)$ and $\Pi_{\mathrm{cob}}^{\mathrm{dco}} : B^k(Y) \to \mathrm{dB}^k(Y)$ given by $[X,f] \mapsto [\boldsymbol{X}, \boldsymbol{f}]$, where $\boldsymbol{X}, \boldsymbol{f} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X,f)$. Note that the orientations on X correspond to orientations on $\boldsymbol{X}$, and coorientations for $f : X \to Y$ correspond to coorientations for $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$. These projections are well-defined and preserve all the structures, that is they take $\cup, \cap, \bullet$, identities and fundamental classes in $B_*(Y), B^*(Y)$ to $\cup, \cap, \bullet$, identities and fundamental classes $\mathrm{dB}_*(Y), \mathrm{dB}^*(Y)$.

Remark 7.2.5. In the case of classical (co)bordism one had to perturb the classes $[X,f], [X',f']$ to make $f : X \to Y$ and $f' : X' \to Y$ transversal to define $\cup, \cap, \bullet$ and pullbacks. In the case of d-(co)bordism one can define these products and the pullback without making a perturbation, since the appropriate fibre products exist in **dMan** for all $(\boldsymbol{X}, \boldsymbol{f}), (\boldsymbol{X}', \boldsymbol{f}')$.

The next theorem is crucial for the whole theory of d-(co)bordism. It says that for a compact manifold without boundary, the d-bordism group is isomorphic to the ordinary bordism group, and that all the structures like the intersection product are preserved under that isomorphism. One consequence of this theorem, is that oriented compact d-manifolds admit virtual classes, and can therefore be used to study moduli problems in symplectic geometry.

Theorem 7.2.6. *Let Y be a compact manifold without boundary. Then the morphisms $\Pi_{\mathrm{bo}}^{\mathrm{dbo}} : B_k(Y) \to dB_k(Y)$ and $\Pi_{\mathrm{cob}}^{\mathrm{dco}} : B^k(Y) \to dB^k(Y)$ are structure preserving $(\cap, \cup, \bullet, 1_Y, [Y], g_*, g^*)$ isomorphisms for all $k \in \mathbb{Z}$.*

We will not give the full proof of this theorem here, since the proof of Theorem 7.3.3, which can be seen as a complex analogue, 'contains' the proof of Theorem 7.2.6. Hence, we will just sketch the basic idea and refer for the full proof to Theorem 7.3.3 or to our standard reference [35, Theorem 13.15].

Sketch proof. Let $[\boldsymbol{X}, \boldsymbol{f}]$ be an element of $\mathrm{dB}_k(Y)$. Then by Theorem 2.3.28 there exists an embedding $\boldsymbol{g} : \boldsymbol{X} \to \mathbb{R}^n$, for n big enough. The direct product $(\boldsymbol{f}, \boldsymbol{g}) : \boldsymbol{X} \to \boldsymbol{Y} \times \mathbb{R}^n$ is also an embedding. Hence, Theorem 2.3.29 yields the existence of an open subset $V \subset Y \times \mathbb{R}^n$, a vector bundle $E \to V$ over V and a smooth section $s \in C^\infty(E)$ such that $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$. A generic perturbation $\tilde{s} \in C^\infty(E)$

of s intersects the zero section in E transversely and $\tilde{X} := \tilde{s}^{-1}(0)$ is therefore a k-manifold for $k \geq 0$, and $\tilde{X} = \emptyset$ for $k < 0$. Choosing $\tilde{s} - s$ small, one can ensure that $\tilde{X}$ is actually compact and oriented. Defining $\tilde{f} := \pi_Y|_{\tilde{X}} : \tilde{X} \to Y$, one gets $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}([\tilde{X}, \tilde{f}]) = [\boldsymbol{X}, \boldsymbol{f}]$, that is, $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}$ is surjective.

A similar argument for $\boldsymbol{W}, \boldsymbol{e}$ yields injectivity of $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}$ and proves the theorem. □

Corollary 7.2.7. *By Theorem 7.2.6 there exist projection maps from d-bordism and d-cobordism to ordinary homology with integer coefficients:*

$$\Pi_{\mathrm{dbo}}^{\mathrm{hom}} : dB_k(Y) \to H_k(Y; \mathbb{Z})$$
$$\Pi_{\mathrm{dbo}}^{\mathrm{hom}} = \Pi_{\mathrm{bo}}^{\mathrm{hom}} \circ \left(\Pi_{\mathrm{bo}}^{\mathrm{dbo}}\right)^{-1}$$

and

$$\Pi_{\mathrm{dco}}^{\mathrm{coh}} : dB^k(Y) \to H^k(Y; \mathbb{Z})$$
$$\Pi_{\mathrm{dco}}^{\mathrm{coh}} = \Pi_{\mathrm{cob}}^{\mathrm{coh}} \circ \left(\Pi_{\mathrm{cob}}^{\mathrm{dco}}\right)^{-1}.$$

The main conclusion we want to draw from this, is that oriented compact d-manifolds admit virtual classes, as one can think of $\Pi_{\mathrm{dbo}}^{\mathrm{hom}}$ and $\Pi_{\mathrm{dco}}^{\mathrm{coh}}$ as *virtual class maps*. Due to this fact d-manifolds can be used as geometric structures on moduli spaces in invariant problems like symplectic Gromov–Witten theory.

7.3 Unitary d-manifold bordism

In this section we extend the oriented d-manifold bordism theory of Joyce [35, §13] to the case of stable nearly complex d-manifolds, and obtain a *unitary d-bordism* theory. The main result in this section is that, similarly to the oriented case, the unitary d-bordism group of a stable almost complex manifold is isomorphic to its "ordinary" unitary bordism group. As we will see, unitary d-bordism can be thought of as oriented d-bordism with some extra structure in form of a stable nearly complex structure. Since this stable nearly complex structure is encoded in virtual vector bundles and morphisms between them, unitary d-bordism is in some sense oriented d-bordism which keeps track of certain nearly complex virtual vector bundles.

We begin first by recalling the basic definition of "ordinary" unitary bordism. (Compare for instance [26, Definition D22].)

Definition 7.3.1. Let Y be a manifold, (X_0, J_0, a_0) and (X_1, J_1, a_1) be oriented stable almost complex manifolds of dimension k, and $f_0 : X_0 \to Y$ and $f_1 : X_1 \to Y$ be smooth maps. Define an equivalence relation $\sim$ between two such quadruples (X_0, J_0, a_0, f_0) and (X_1, J_1, a_1, f_1) by $(X_0, J_0, a_0, f_0) \sim (X_1, J_1, a_1, f_1)$ if there exist integers $b_0, b_1 \in \mathbb{Z}_{\geq 0}$, satisfying $a_0 + 2b_0 = a_1 + 2b_1 =: m + 1$, a compact oriented stable almost complex manifold (W, J_W, m) of dimension $(k + 1)$ and a smooth map $g : W \to Y$ such that $\partial W \cong -X_0 \amalg X_1$ as oriented manifolds, $g|_{\partial W} \cong f_0 \amalg f_1$, and that $J_W|_{X_0} \cong J_0 \oplus J_{\mathbb{C}^{b_0}}$ and $J_W|_{X_1} \cong J_1 \oplus J_{\mathbb{C}^{b_1}}$.

Define for all $k \in \mathbb{Z}$ the *k-th unitary bordism group* $\mathrm{BU}_k(Y)$ of Y to be the set of all such equivalence classes $[X, J, a, f]$, with $\dim X = k$. We will sometimes leave a implicit and refer to $[X, J, a, f]$ as $[X, J, f]$.

We want now to adapt this definition to the d-manifold level.

Definition 7.3.2. Let Y be a manifold. Consider compact stable nearly complex d-manifolds $(\boldsymbol{X}_0, ((\mathcal{E}_0^\bullet, \phi_0), J_0^\bullet), a_0)$ and $(\boldsymbol{X}_1, ((\mathcal{E}_1^\bullet, \phi_1), J_1^\bullet), a_1)$, of virtual dimension k, and let $\boldsymbol{f}_0 : \boldsymbol{X}_0 \to \boldsymbol{Y}$ and $\boldsymbol{f}_1 : \boldsymbol{X}_1 \to \boldsymbol{Y}$ be 1-morphisms between d-manifold, where $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$. Define an equivalence relation $\sim$ between two such sextuples as follows: $(\boldsymbol{X}_0, ((\mathcal{E}_0^\bullet, \phi_0), J_0^\bullet), a_0, \boldsymbol{f}_0) \sim (\boldsymbol{X}_1, ((\mathcal{E}_1^\bullet, \phi_1), J_1^\bullet), a_1, \boldsymbol{f}_1)$ if $[\boldsymbol{X}_1, \boldsymbol{f}_1] \sim [\boldsymbol{X}_0, \boldsymbol{f}_0]$ as compact oriented d-manifolds in the sense of Definition 7.2.1 and the following conditions are satisfied: let $\boldsymbol{W}$ be an oriented d-manifold with boundary of virtual dimension $k+1$ having $-\boldsymbol{X}_0 \amalg \boldsymbol{X}_1$ as boundary, fulfilling the criteria of Definition 7.2.1. Then there should exist integers $b_0, b_1 \in \mathbb{Z}_{\geq 0}$, satisfying $a_0 + 2b_0 = a_1 + 2b_1 =: m + 1$, a stable nearly complex structure $((\tilde{\mathcal{E}}^\bullet, \tilde{\phi}), \tilde{J}^\bullet, m)$ on $\boldsymbol{W}$ and equivalences of nearly complex virtual vector bundles

$$\begin{aligned} &e_0^\bullet : ((\tilde{\mathcal{E}}^\bullet, \tilde{\phi}), \tilde{J}^\bullet)|_{\underline{X}_0} \to (\mathcal{E}_0^1 \xrightarrow{\phi_0 \oplus *} \mathcal{E}_0^2 \oplus \mathbb{C}^{b_0}, (J_0^1, J_0^2 \oplus J_{\mathbb{C}^{b_0}})), \\ \text{and} \quad &e_1^\bullet : ((\tilde{\mathcal{E}}^\bullet, \tilde{\phi}), \tilde{J}^\bullet)|_{\underline{X}_1} \to (\mathcal{E}_1^1 \xrightarrow{\phi_1 \oplus *} \mathcal{E}_1^2 \oplus \mathbb{C}^{b_1}, (J_1^1, J_1^2 \oplus J_{\mathbb{C}^{b_1}})). \end{aligned}$$

As $(\mathcal{E}_i^1 \xrightarrow{\phi_i \oplus *} \mathcal{E}_i^2 \oplus \mathbb{C}^{b_i}) \simeq (\mathcal{E}_{X_i} \xrightarrow{\phi_{X_i} \oplus *} \mathcal{F}_{X_i} \oplus (\mathbb{R}^{a_i} \oplus \mathbb{C}^{b_i}))$ for $i = 0, 1$ we get equivalences from the stable nearly complex structure $((\tilde{\mathcal{E}}^\bullet, \tilde{\phi}), \tilde{J}^\bullet, m)$ on $\boldsymbol{W}$ to the stabilized cotangent bundles of $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$.

Define for all $k \in \mathbb{Z}$ the *k-th unitary d-bordism group* $\mathrm{dBU}_k(Y)$ of Y to be the set of all such equivalence classes $[\boldsymbol{X}, ((\mathcal{E}^\bullet, \phi)J^\bullet), a, \boldsymbol{f}]$, with $\operatorname{vdim} \boldsymbol{X} = k$. Using the obvious virtual vector bundles, one can show as in Definition 7.2.1, that $\mathrm{dBU}_k(Y)$ is in fact an abelian group with zero element.

Given a stable almost complex manifold (X, J, a) we can define a stable nearly complex d-manifold as follows: Let $\boldsymbol{X} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X)$ be the image of X under the functor $F_{\mathbf{Man}}^{\mathbf{dMan}}$. Define a virtual vector bundle $(\mathcal{E}^\bullet, \phi, J^\bullet)$ on $\boldsymbol{X}$ by $0 \xrightarrow{0} T^*X \oplus \mathbb{R}^a$. The equivalence is just given by the natural isomorphism, and the complex structure $J^\bullet = (0, J)$ on $0 \xrightarrow{0} T^*X \oplus \mathbb{R}^a$ is given by the given stable almost complex structure $J : T^*X \oplus \mathbb{R}^a \to T^*X \oplus \mathbb{R}^a$ on X.

We can therefore define a projection map $\Pi_{\mathrm{bu}}^{\mathrm{dbu}} : \mathrm{BU}_k(Y) \to \mathrm{dBU}_k(Y)$ for all $k \in \mathbb{Z}_{\geq 0}$ by $\Pi_{\mathrm{bu}}^{\mathrm{dbu}}([X, J, a, f]) = [\boldsymbol{X}, ((\mathcal{E}^\bullet, \phi), J^\bullet), a, \boldsymbol{f}]$, where $\boldsymbol{X}, \boldsymbol{f} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X, f)$ and $((\mathcal{E}^\bullet, \phi), J^\bullet)$ are defined as above.

Since the stable nearly complex structure of a d-manifold is encoded in terms of virtual vector bundles, we can think of unitary d-bordism as oriented d-bordism with a kind of "virtual vector bundle bordism". On the underlying d-manifold, unitary d-bordism equals oriented d-bordism, but we have to keep track of the virtual vector bundle defining the stable nearly complex structure.

The following theorem is an analogue of Theorem 7.2.6, and shows that for a compact manifold Y, the unitary d-bordism groups are isomorphic to the usual unitary bordism groups. The proof follows basically the proof of Theorem 7.2.6 (which can be found as Theorem 13.15 in [35]), except that we have to show, that the virtual vector bundle associated to the nearly complex structure behaves "nicely".

Theorem 7.3.3. *Let Y be a compact manifold without boundary. Then the morphism $\Pi_{\mathrm{bu}}^{\mathrm{dbu}} : BU_k(Y) \to dBU_k(Y)$ is an isomorphism for all $k \in \mathbb{Z}$.*

Proof. Let $[\boldsymbol{X}, ((\mathcal{E}^\bullet, \phi), J^\bullet), a, \boldsymbol{f}] \in \mathrm{dBU}_k(Y)$. Since $\boldsymbol{X}$ is compact, Theorem 2.3.28 yields an embedding $\boldsymbol{g} : \boldsymbol{X} \to \mathbb{R}^n$ for some $n \gg 0$. Moreover, the direct product $(\boldsymbol{f}, \boldsymbol{g}) : \boldsymbol{X} \to \boldsymbol{Y} \times \mathbb{R}^n$ is also an embedding, and thus Theorem 2.3.28 gives an open set $V \subseteq Y \times \mathbb{R}^n$, a vector bundle $E \to V$, a smooth section $s \in C^\infty(E)$, an equivalence $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{S}_{V,E,s}$ and a 2-morphism $\zeta : \boldsymbol{S}_{\pi_Y,0} \circ \boldsymbol{i} \Rightarrow \boldsymbol{f}$. Since a stable nearly complex d-manifold is oriented, we can require $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{S}_{V,E,s}$ to be orientation preserving, and get a unique orientation on $\boldsymbol{S}_{V,E,s}$. Furthermore, compactness of $\boldsymbol{X}$ yields, that in the standard model description $s^{-1}(0) \subseteq V$ is compact, although V will be non-compact.

Corollary 2.3.30 and the proof of Proposition 5.3.2 show that we can assume without loss of generality that the standard model d-manifold $\boldsymbol{S}_{V,E,s}$ is a stable nearly complex standard model d-manifold $(\boldsymbol{S}_{V,E,s}, J, K, a)$, that is we have almost complex structures K on E^* and J on $T^*V \oplus \mathbb{R}^a$ and an equivalence

$$\tilde{T}^*\boldsymbol{X} \simeq (E^* \xrightarrow{ds} \tilde{T}^*V \oplus \mathbb{R}^a).$$

To simplify the notation, we will in the following identify equivalent (nearly complex) virtual vector bundles.

Next, choose an open neighbourhood U of $s^{-1}(0)$ in V whose closure $\overline{U} \subset V$ is compact, and a generic perturbation $\tilde{s} : V \to E$ of s, such that $|\tilde{s} - s| \leq \frac{1}{2}|s|$ on $V \setminus U$, where $| \cdot |$ is computed using some choice of metric on E. For this generic choice $\tilde{s}^{-1}(0)$ is closed in V and contained in the compact subset $\tilde{U}$, and so $\tilde{s}^{-1}(0)$ is compact. This implies, that the standard model $\boldsymbol{S}_{V,E,\tilde{s}}$ is a compact d-manifold, which inherits an orientation from the orientation on the fibres of $\Lambda^{\mathrm{rank}\, E} E \otimes \Lambda^{\dim V} T^*V$. Genericity of $\tilde{s}$ guarantees $\tilde{s}$ to be transverse, and so $\tilde{X} = \tilde{s}^{-1}(0)$ is a compact submanifold of dimension $k = \dim V - \mathrm{rank}\, E$ and $\tilde{f} = \pi_Y|_{\tilde{X}} : \tilde{X} \to Y$ is a smooth map.

We now need to show that there exists a stabilization of the cotangent bundle of $\tilde{X}$, $\tilde{T}^*\tilde{X} = T^*\tilde{X} \oplus \mathbb{R}^{\tilde{a}}$, that carries an almost complex structure. To prove this note that as $\tilde{X}$ is a manifold, the stabilization of the cotangent bundle of $\tilde{\boldsymbol{X}} = \Pi^{\mathbf{dMan}}_{\mathbf{Man}}(\tilde{X})$ is a vector bundle (in the sense of Definition 2.1.22), that is $\tilde{T}^*\tilde{\boldsymbol{X}} \simeq (0 \longrightarrow \tilde{T}^*\tilde{X})$. Restricting $(E^* \xrightarrow{ds} T^*V \oplus \mathbb{R}^a)$ to $\tilde{X}$ yields then the following equivalence of virtual vector bundles

$$(E^*|_{\tilde{X}} \xrightarrow{ds|_{\tilde{X}}} T^*V \oplus \mathbb{R}^a|_{\tilde{X}}) \simeq (0 \xrightarrow{0} T^*\tilde{X} \oplus \mathbb{R}^c).$$

As in the proof of Proposition 5.2.6, the nearly complex virtual vector bundle $(E^* \xrightarrow{ds} T^*V \oplus \mathbb{R}^a)$ is equivalent as a nearly complex virtual vector bundle to

$$(E^*|_{\tilde{X}} \xrightarrow{ds|_{\tilde{X}}} T^*V \oplus \mathbb{R}^a|_{\tilde{X}}) \simeq (\mathcal{G}^1|_{\tilde{X}} \xrightarrow{\psi} \mathbb{C}^N \otimes \mathcal{O}_{\tilde{X}}),$$

for some $N \in \mathbb{Z}_{\geq 0}$. We get therefore an equivalence between (real) virtual vector bundles

$$(\mathcal{G}^1|_{\tilde{X}} \xrightarrow{\psi} \mathbb{C}^N \otimes \mathcal{O}|_{\tilde{X}}) \simeq (0 \xrightarrow{0} T^*\tilde{X} \oplus \mathbb{R}^a),$$

which we will denote by $\alpha = (\alpha_1, \alpha_2)$.

Proposition 2.1.23 yields then the following split exact sequence

$$0 \longrightarrow \mathcal{G}^1|_{\tilde{X}} \xrightarrow{\psi} \mathbb{C}^N \otimes \mathcal{O}|_{\tilde{X}} \xrightarrow{\alpha_2} \tilde{T}^*\tilde{X} \longrightarrow 0, \tag{7.1}$$

where

$$\mathbb{C}^N \otimes \mathcal{O}|_{\tilde{X}} \cong \tilde{T}^*\tilde{X} \oplus \mathcal{G}^1|_{\tilde{X}} \tag{7.2}$$

as a real vector bundle. Adding $\tilde{T}^*\tilde{X}$ on both sides yields $\tilde{T}^*\tilde{X} \oplus \mathbb{C}^m \otimes \mathcal{O}_{\tilde{X}} \cong \tilde{T}^*\tilde{X} \oplus \tilde{T}^*\tilde{X} \oplus \tilde{\mathcal{G}}^1|_{\tilde{X}}$, which allows us to define a complex structure $\tilde{J}_{\mathrm{ct}}$ on $\tilde{T}^*\tilde{X} \oplus \mathbb{C}^m \otimes \mathcal{O}_{\tilde{X}}$ by

$$\tilde{J}_{\mathrm{ct}} = \begin{pmatrix} 0 & \mathrm{id}_{T^*\tilde{X}} & 0 \\ -\mathrm{id}_{T^*\tilde{X}} & 0 & 0 \\ 0 & 0 & J_{\tilde{\mathcal{G}}^1} \end{pmatrix},$$

where, $J_{\tilde{\mathcal{G}}^1}$ denotes the almost complex structure on $\tilde{\mathcal{G}}^1$.

Rewriting $\tilde{T}^*\tilde{X} \oplus \mathbb{C}^m = T^*\tilde{X} \oplus \mathbb{R}^{a+2m}$, we can define $\tilde{a} = a + 2m \in \mathbb{Z}_{\geq 0}$ and we obtain $[\tilde{X}, \tilde{J}_{\mathrm{ct}}, \tilde{a}, \tilde{f}] \in \mathrm{BU}_k(Y)$.

Note that we will in the following, for brevity, suppress the $\otimes\mathcal{O}_{\tilde{X}}$-part if it is clear from the context, and just write $\mathbb{C}^m$ instead.

Denote the by the projection map $\Pi^{\mathrm{dbu}}_{\mathrm{bu}}$ induced nearly complex d-manifold by $[\tilde{\boldsymbol{X}}, \tilde{J}^{\bullet}, \tilde{a}, \tilde{\boldsymbol{f}}] = \Pi^{\mathrm{dbu}}_{\mathrm{bu}}([\tilde{X}, \tilde{J}_{\mathrm{ct}}, \tilde{a}, \tilde{f}])$. Recall, that $\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{f}} = F^{\mathbf{dMan}}_{\mathbf{Man}}(\tilde{X}, \tilde{f})$ and that the stable nearly complex structure $\tilde{J}^{\bullet}$ is encoded in the virtual vector bundle

$$0 \xrightarrow{0} \tilde{T}^*\tilde{X} \oplus \mathbb{C}^m \quad \text{on } \underline{\tilde{X}},$$

with almost complex structure $\tilde{J}_{\mathrm{ct}}$ on $\tilde{T}^*\tilde{X} \oplus \mathbb{C}^m$.

We have now to show $[\tilde{\boldsymbol{X}}, \tilde{J}^{\bullet}, \tilde{a}, \tilde{\boldsymbol{f}}] \sim [\boldsymbol{X}, J^{\bullet}, a, \boldsymbol{f}]$. Consider therefore the manifold with boundary $W = V \times [0, 1]$, the vector bundle $F = \pi_V^*(E)$ over W, and define a smooth section $t : W \to F$ by $t = (1 - z)\pi_V^*(s) + z\pi_V^*(\tilde{s})$, where z is the coordinate on $[0,1]$. Here π_V denotes the projection $\pi_V : W \to V$ on V. The boundary ∂W of W is isomorphic to two disjoint copies of V, $\partial W \cong V \times \{0\} \amalg V \times \{1\}$. The vector bundle F restricted to the boundary is isomorphic to E, $F|_{\partial W} \cong E$, on each of the copies of V, and we have $t|_{\partial W} \cong s \in C^\infty(E)$ on $V \times \{0\}$ and $t|_{\partial W} \cong \tilde{s} \in C^\infty(E)$ on $V \times \{1\}$. The standard model d-manifolds $\boldsymbol{S}_{W,F,t}$ and $\boldsymbol{S}_{V,E,\tilde{s}}$ inherit an orientation by the orientations on the fibres of $\Lambda^{\operatorname{rank} E} \otimes \Lambda^{\dim V} T^*V$ and on $[0,1]$, and we get an equivalence $\partial \boldsymbol{S}_{W,F,t} \simeq -\boldsymbol{S}_{V,E,s} \amalg \boldsymbol{S}_{V,E,\tilde{s}}$ of oriented d-manifolds. The perturbation $\tilde{s}$ of s was chosen such that $|\tilde{s} - s| \leq \frac{1}{2}|s|$ on $V \setminus U$, which implies that $t^{-1}(0) \subseteq U \times [0,1]$, and so $t^{-1}(0)$ is closed in $V \times [0,1]$ and contained in the compact subset $\overline{U} \times [0,1]$. This implies that $t^{-1}(0)$ is compact and thus that $\boldsymbol{S}_{W,F,t}$ is compact.

We have now a compact, oriented d-manifold with boundary $\boldsymbol{S}_{W,F,t}$, a 1-morphism $\boldsymbol{S}_{\pi_Y,0} : \boldsymbol{S}_{W,F,t} \to \boldsymbol{Y}$ and equivalences of oriented d-manifolds $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$, $\tilde{\boldsymbol{X}} \simeq \boldsymbol{S}_{V,E,\tilde{s}}$ and $\partial \boldsymbol{S}_{W,F,t} \simeq -\boldsymbol{S}_{V,E,s} \amalg \boldsymbol{S}_{V,E,\tilde{s}}$. Putting these equivalences together yields an equivalence $\boldsymbol{j} : -\boldsymbol{X} \amalg \tilde{\boldsymbol{X}} \to \partial \boldsymbol{S}_{W,F,t}$. Moreover, the 2-morphism $\eta : \boldsymbol{S}_{\pi_Y,0} \circ \boldsymbol{i} \Rightarrow \boldsymbol{f}$ and the definition of $\tilde{f} = \pi_Y|_{\tilde{X}}$ imply that there exists a 2-morphism $\tilde{\eta} : \boldsymbol{f} \amalg \tilde{\boldsymbol{f}} \Rightarrow \boldsymbol{S}_{\pi_Y,0} \circ \boldsymbol{i}_{\boldsymbol{S}_{W,F,t}} \circ \boldsymbol{j}$. Hence $(\boldsymbol{X}, \boldsymbol{f}) \sim (\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{f}})$ by Definition 7.2.1 and therefore $[\boldsymbol{X}, \boldsymbol{f}] = [\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{f}}]$ in oriented d-bordism.

The remaining bit is to show that the virtual vector bundle $(\mathcal{G}^\bullet, \psi)$ on $\underline{X}$ and $(0, \tilde{T}^*\tilde{X} \oplus \mathbb{C}^m, 0)$ on $\underline{\tilde{X}}$ satisfy

$$(\mathcal{G}^1 \xrightarrow{\psi} \mathbb{C}^m) \sim (0 \xrightarrow{0} \tilde{T}^*\tilde{X} \oplus \mathbb{C}^m).$$

Consider therefore the virtual vector bundle $(\mathcal{H}^\bullet, \xi)$ on $\boldsymbol{S}_{W,F,t}$ given by

$$\mathcal{H}^1 \xrightarrow{\xi} \mathbb{C}^m \oplus \mathbb{C}^m,$$

where $\mathcal{H}^1 = \pi_V^*(\mathcal{G}^1_{\text{ext}})$, with $\mathcal{G}^1_{\text{ext}}$ being an extension of $\mathcal{G}^1$ to V, and ξ being an extension of ψ, such that $\mathcal{G}^1_{\text{ext}}|_{\underline{\tilde{X}}} = \tilde{\mathcal{G}}^1$ and $\xi|_{\underline{\tilde{X}}} = \tilde{\psi}$.

Using $\mathcal{H}^1|_{\underline{\tilde{X}}} = \tilde{\mathcal{G}}^1$ and equation (7.2), we obtain that on $\underline{\tilde{X}}$ the virtual vector bundle $(\mathcal{H}^\bullet, \xi)|_{\underline{\tilde{X}}}$ is equivalent in $\mathbf{vvect}(\underline{\tilde{X}})$ to $\tilde{\mathcal{G}}^1 \xrightarrow{\tilde{\psi}} \tilde{\mathcal{G}}^1 \oplus T^*\tilde{X} \oplus \mathbb{C}^m$, which in turn is equivalent to $0 \xrightarrow{0} \tilde{T}^*\tilde{X} \oplus \mathbb{C}^m$.

On the other hand, on $\underline{X}$ we have $\mathcal{H}^1|_{\underline{X}} = \mathcal{G}^1$ and hence $(\mathcal{H}^\bullet, \xi)|_{\underline{X}}$ is equivalent in $\mathbf{vvect}(\underline{X})$ to $\mathcal{G}^1 \xrightarrow{\psi} \mathbb{C}^m \oplus \mathbb{C}^m$.

So choosing $b_0 = 0$ and $b_1 = m$, we obtain

$$\begin{aligned} &(\mathcal{H}^\bullet, \xi)|_{\underline{\tilde{X}}} \simeq \mathcal{G}^1 \xrightarrow{\psi} \mathbb{C}^m + \mathbb{C}^{b_1}, \\ \text{and} \quad &(\mathcal{H}^\bullet, \xi)|_{\underline{X}} \simeq 0 \xrightarrow{0} \tilde{T}^*\tilde{X} \oplus \mathbb{C}^m. \end{aligned}$$

and therefore

$$(\mathcal{G}^1 \xrightarrow{\psi} \mathbb{C}^m) \sim (0 \xrightarrow{0} \tilde{T}^*\tilde{X} \oplus \mathbb{C}^m),$$

as claimed.

This shows that $\Pi_{\mathrm{bu}}^{\mathrm{dbu}}([\tilde{X}, \tilde{J}_{\mathrm{ct}}, \tilde{a}, \tilde{f}]) = [\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{J}}^\bullet, a, \tilde{\boldsymbol{f}}] = [\boldsymbol{X}, ((\mathcal{E}^\bullet, \phi), J^\bullet), a, \boldsymbol{f}]$ and therefore that $\Pi_{\mathrm{bu}}^{\mathrm{dbu}} : \mathrm{BU}_k(Y) \to \mathrm{dBU}_k(Y)$ is surjective.

To prove injectivity of $\Pi_{\mathrm{bu}}^{\mathrm{dbu}}$, suppose that $[X, J, a, f], [X', J', a', f'] \in \mathrm{BU}_k(Y)$ with $\Pi_{\mathrm{bu}}^{\mathrm{dbu}}([X, J, a, f]) = \Pi_{\mathrm{bu}}^{\mathrm{dbu}}([X', J', a', f'])$. Then Joyce shows in [35, Theorem 14.15], that $[X, f] = [X', f']$ in oriented bordism. We will not give the proof of this statement here, since the proof requires more theory on d-manifolds with boundary and corners, in particular the notion of sf-embeddings, but refer the reader to the proof of [35, Theorem 14.15] instead.

Using the notation of Definition 7.3.1, we have to show that there exists $b, b' \in \mathbb{Z}_{\geq 0}$ with $a + 2b = a' + 2b' =: k$, and an almost complex structure J_W on $T^*W \oplus \mathbb{R}^k$ such that $J_W|_X = J \oplus J_{\mathbb{C}^m}$ and $J_W|_{X'} = J' \oplus J_{\mathbb{C}^m}$. But since $\Pi_{\mathrm{bu}}^{\mathrm{dbu}}([X, J, a, f]) = \Pi_{\mathrm{bu}}^{\mathrm{dbu}}([X', J', a', f'])$, we have $(\boldsymbol{X}', (J^\bullet)', a', \boldsymbol{f}') \sim (\boldsymbol{X}, J^\bullet, a, \boldsymbol{f})$, and the associated virtual vector bundles $0 \xrightarrow{0} T^*X \oplus \mathbb{R}^a$ and $0 \xrightarrow{0} T^*X' \oplus \mathbb{R}^{a'}$ satisfy the condition above by definition.

Hence $\Pi_{\mathrm{bu}}^{\mathrm{dbu}} : \mathrm{BU}_k(Y) \to \mathrm{dBU}_k(Y)$ is injective for all $k \in \mathbb{Z}$, and therefore an isomorphism, which completes the proof of the theorem. □

Remark 7.3.4. Using similar definitions and methods, one could have contemplated stable homotopy complex d-manifolds in section 7.3 instead of stable nearly complex d-manifolds and by a slight alteration of the proof, one could have proved Theorem 7.3.3 considering unitary stable homotopy complex d-bordism groups instead.

7.4 Bordism and d-bordism for orbifolds and d-orbifolds

We want now to elaborate a bit on what concepts and results can be extended to the orbifold and d-orbifold case. We will follow here in major parts Joyce [35, §13.3 -13.4] and will later on explain how our new results concerning unitary bordism can be extended to the d-orbifold case.

7.4.1 Orbifold bordism

Many definitions and concepts from manifold and d-manifold bordism can be extended to the orbifold and d-orbifold case. Using the 2-categories of orbifolds **Orb**, and of orbifolds with boundary $\mathbf{Orb}^b$ (as in [35, §8.2, §8.5]) the definition of orbifold bordism is essentially the same as the definition of d-manifold bordism. (Compare [35, Definition 13.12]).

Definition 7.4.1. Given an orbifold $\mathcal{Y}$ and an integer $k \in \mathbb{Z}$, consider pairs $(\mathcal{X}, f)$, where $\mathcal{X}$ is a compact, oriented orbifold of dimension $\dim \mathcal{X} = k$ and $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism in **Orb**. Define an equivalence relation $\sim$ between such pairs by $(\mathcal{X}, f) \sim (\mathcal{X}', f')$ if there exists a compact, oriented $(k+1)$-dimensional orbifold with boundary $\mathcal{W}$, a 1-morphism $g : \mathcal{W} \to \mathcal{Y}$, an orientation-preserving equivalence $j : -\mathcal{X} \amalg \mathcal{X}' \to \partial\mathcal{W}$ and a 2-morphism $\eta : f \amalg f' \Rightarrow g \circ i_{\mathcal{W}} \circ j$.

As in the manifold case, it can be shown that the *k-th bordism group* $B_k^{Orb}(\mathcal{Y})$ is an abelian group satisfying $B_k^{Orb}(\mathcal{Y}) = 0$ for $k < 0$.

Requiring the orbifolds $\mathcal{X}$ and $\mathcal{W}$ to be effective, one can define in exactly the same way, the *effective orbifold bordism group* $B_k^{\mathrm{eff}}(\mathcal{Y})$.

Many concepts from the manifold case generalize to the orbifold case (by introducing some additional conditions), like cup product, cap product, pullbacks and fundamental classes. Moreover, given an orbifold $\mathcal{Y}$, there exist similarly to the manifold case several projection morphisms:

$$\begin{aligned} \Pi_{\mathrm{eff}}^{\mathrm{orb}} : B_k^{\mathrm{eff}}(\mathcal{Y}) &\to B_k^{\mathrm{orb}}(\mathcal{Y}), & \Pi_{\mathrm{orb}}^{\mathrm{hom}} : B_k^{\mathrm{orb}}(\mathcal{Y}) &\to H_k(\mathcal{Y}_{\mathrm{top}}; \mathbb{Q}), \\ [\mathcal{X}, f] &\mapsto [\mathcal{X}, f], & [\mathcal{X}, f] &\mapsto (f_{\mathrm{top}})_*([\mathcal{X}]), \end{aligned} \tag{7.3}$$

$$\text{and} \qquad \Pi^{\mathrm{hom}}_{\mathrm{eff}} : B^{\mathrm{eff}}_k(\mathcal{Y}) \to H_k(\mathcal{Y}_{\mathrm{top}}; \mathbb{Z}),$$
$$[\mathcal{X}, f] \mapsto (f_{\mathrm{top}})_*([\mathcal{X}]),$$

where $[\mathcal{X}]$ is the fundamental class of the compact, oriented k-dimensional orbifold $\mathcal{X}$. Note that this fundamental class lies in $H_k(\mathcal{X}_{\mathrm{top}}; \mathbb{Q})$ for general orbifolds $\mathcal{X}$, and in $H_k(\mathcal{X}_{\mathrm{top}}; \mathbb{Z})$ when $\mathcal{X}$ is effective.

If Y is a manifold and $\mathcal{Y} = F^{\mathbf{Orb}}_{\mathbf{Man}}(Y)$, we can define a morphism

$$\Pi^{\mathrm{eff}}_{\mathrm{bo}} : B_k(Y) \to B^{\mathrm{eff}}_k(\mathcal{Y}) \text{ by } \Pi^{\mathrm{eff}}_{\mathrm{bo}} : [\mathcal{X}, f] \mapsto [F^{\mathbf{Orb}}_{\mathbf{Man}}(X), F^{\mathbf{Orb}}_{\mathbf{Man}}(f)]. \tag{7.4}$$

The morphisms defined in (7.3) and (7.4) commute with pushforwards g_*, and preserve fundamental classes $[\mathcal{Y}]$ when defined.

In contrast to the manifold case, Poincaré duality will in general not hold for (effective) orbifolds, as the definition of the cobordism groups requires $(\mathcal{X}, f)$ to satisfy an additional condition, which may not be satisfied in the bordism groups. (For more details we refer to [35, Remark 13.14]).

As the maximal effective open suborbifold $\mathcal{W}'$ of a compact orbifold with effective boundary $\mathcal{W}$ satisfies $\partial\mathcal{W}' \simeq \mathcal{W}$, one can conclude that

Lemma 7.4.2. $\Pi^{orb}_{eff} : B^{eff}_*(\mathcal{Y}) \to B^{orb}_*(\mathcal{Y})$ *in injective for any orbifold* $\mathcal{Y}$

In the manifold case results by Thom, Milnor, Wall and others determined the bordism ring $B_*(*)$ completely. The following theorem will characterise the effective orbifold bordism ring $B_*(*)$ of the point $*$, and summarizes results by Druschel [15] and Angel [4].

Theorem 7.4.3. ***(a) (Druschel [15]).*** *The morphism* $\Pi^{eff}_{bo} : B_*(*) \to B^{eff}_*(*)$ *from* (7.4) *induces a morphism between* $\mathbb{Q}$*-algebras* $B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q} \to B^{eff}_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$, *where* $B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ *can be described using Theorem 7.1.6. Regarding* $B^{eff}_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ *as a* $B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$*-module, we get the following:*

$$B^{eff}_*(*) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (B_*(*) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \bigoplus_{\Gamma \subset SO(n)} H_*(B(N_{O(n)}(\Gamma)/\Gamma); \hat{\mathbb{Q}})^*. \tag{7.5}$$

Here the direct sum is over all conjugacy classes of finite subgroups $\Gamma \subset SO(n)$ *for* $n \geq 0$ *with trivial fixed point set* $(\mathbb{R})^\Gamma = \{0\}$, *and* $N_{O(n)}(\Gamma)$ *is the normalizer* *of* Γ *in* $SO(n)$. *Moreover,* $B(N_{O(n)}(\Gamma)/\Gamma)$ *denotes the classifying space of the quotient subgroup* $N_{O(n)}(\Gamma)/\Gamma$*, and* $\hat{Q}$ *is a local system on* $B(N_{O(n)}(\Gamma)/\Gamma)$ *with fibre* $\mathbb{Q}$ *induced by the orientations on the fibres of the universal* $\mathbb{R}^n/\Gamma$*-bundles over* $B(N_{O(n)}(\Gamma)/\Gamma)$.

(b) (Druschel [15]). $B^{\mathit{eff}}_{2k+1}(*) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ *for all* $k \geq 0$. *Note that contrary to the manifold case,* $B^{\mathit{eff}}_{4k+2}(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ *might be non zero, as the example* $B^{\mathit{eff}}_{70}(*) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$. *shows.*

(c) (Druschel [15]). $B^{\mathit{eff}}_{k}(*) = \{0\}$ *for* $k = 1, 2, 3$.

(d) (Angel [4]). *The torsion (that is the elements of finite order) in* $B_*(*)$ *is given by the kernel of* $\Pi^{\mathit{eff}}_{\mathrm{bo}} : B_*(*) \to B^{\mathit{eff}}_*(*)$.

7.4.2 D-orbifold bordism

As in the manifold case it is possible to extend some results from the classical orbifold bordism theory to the d-orbifold case.

The definition of the d-orbifold bordism group $dB^{\mathrm{orb}}_k(\mathcal{Y})$ is very similar to the definition of d-manifold bordism. We will skip some minor technical details, like the identification of the 2-category of d-orbifolds with that of d-orbifolds without boundary, and refer for more details to [35, Definition 13.21].

Definition 7.4.4. Let $\mathcal{Y}$ be an orbifold, and $k \in \mathbb{Z}$. Consider pairs $(\boldsymbol{\mathcal{X}}, \boldsymbol{f})$, where $\boldsymbol{\mathcal{X}} \in \mathbf{dOrb}$ is a compact, oriented d-orbifold of virtual dimension $\mathrm{vdim}\, \boldsymbol{\mathcal{X}} = k$ without boundary, and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism in $\mathbf{dOrb}$, where $\boldsymbol{\mathcal{Y}} = F^{\mathbf{dOrb}}_{\mathbf{Orb}}(\mathcal{Y})$.

As in the d-manifold case, we can define a binary relation $\sim$ between such pairs by $(\boldsymbol{\mathcal{X}}, \boldsymbol{f}) \sim (\boldsymbol{\mathcal{X}}', \boldsymbol{f}')$ if there exists a compact, oriented d-orbifold with boundary $\boldsymbol{\mathcal{W}}$ of virtual dimension $\mathrm{vdim}\, \boldsymbol{\mathcal{W}} = k + 1$, a 1-morphism $\boldsymbol{e} : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb}^b$, an equivalence of oriented d-orbifolds $\boldsymbol{j} : -\boldsymbol{\mathcal{X}} \amalg \boldsymbol{\mathcal{X}}' \to \partial \boldsymbol{\mathcal{W}}$, and a 2-morphism $\eta : \boldsymbol{f} \amalg \boldsymbol{f}' \Rightarrow \boldsymbol{e} \circ \boldsymbol{i_{\mathcal{W}}} \circ \boldsymbol{j}$, where $\boldsymbol{i_{\mathcal{W}}}$ denotes the inclusion of $\partial \boldsymbol{\mathcal{W}}$ into $\boldsymbol{\mathcal{W}}$. This binary relation $\sim$ can be shown to be an equivalence relation as in the d-manifold case.

For each $k \in \mathbb{Z}$ the k^{th} *d-orbifold bordism group* $dB^{\mathrm{orb}}_k(\mathcal{Y})$ is then defined as the set of all $\sim$-equivalence classes $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$ with $\mathrm{vdim}\, \boldsymbol{\mathcal{X}} = k$. Similarly to the d-manifold case, dB^{orb}_k can again be given the structure of an abelian group.

Taking $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{W}}$ to be (semi)effective d-orbifolds, defines the *semieffective* and *effective d-orbifold bordism groups* $\mathrm{dB}_k^{\mathrm{sef}}(\mathcal{Y})$ and $\mathrm{dB}_k^{\mathrm{eff}}(\mathcal{Y})$.

If $\mathcal{Y}$ is an orbifold, we get the following projection maps:

$$\Pi_{\mathrm{orb}}^{\mathrm{sef}} : B_k^{\mathrm{orb}}(\mathcal{Y}) \to dB_k^{\mathrm{sef}}(\mathcal{Y}), \Pi_{\mathrm{orb}}^{\mathrm{sef}} : [\mathcal{X}, f] \mapsto [\boldsymbol{\mathcal{X}}, \boldsymbol{f}],$$
$$\text{and } \Pi_{\mathrm{orb}}^{\mathrm{deff}} : B_k^{\mathrm{eff}}(\mathcal{Y}) \to dB_k^{\mathrm{eff}}(\mathcal{Y}), \Pi_{\mathrm{orb}}^{\mathrm{deff}} : [\mathcal{X}, f] \mapsto [\boldsymbol{\mathcal{X}}, \boldsymbol{f}],$$

where $\boldsymbol{\mathcal{X}}, \boldsymbol{f} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{X}, f)$.

In the d-manifold case, Theorem 7.2.6 provides an isomorphism between d-bordism and classical bordism in the case where Y is a manifold. In the d-orbifold setting this statement is no longer true, as not every d-orbifold can be deformed to an orbifold, and $dB_k^{\mathrm{orb}}(\mathcal{Y})$ will in general be much bigger than $B_k^{\mathrm{orb}}(\mathcal{Y})$. The reason for this is that given a standard model d-orbifold $\boldsymbol{S}_{\mathcal{V},\mathcal{E},s}$, at a point $v \in \mathcal{V}$ with $s(v) = 0$, we have an action of the stabilizer group $\mathrm{Iso}_{\mathcal{V}}(v)$ on $T_v\mathcal{V}$ and on $\mathcal{E}|_v$. If the nontrivial part of the $\mathrm{Iso}_{\mathcal{V}}(v)$-representation on $\mathcal{E}|_v$ is not a subrepresentation of $T_v\mathcal{V}$, then small deformations $\tilde{s}$ of the section s will not be transverse near v, and so the deformed standard model $\boldsymbol{S}_{\mathcal{V},\mathcal{E},\tilde{s}}$ will not be an orbifold.

Restricting oneself to (semi)effective d-orbifolds $\boldsymbol{\mathcal{X}}$, these can be perturbed to (effective) orbifolds and Joyce [35, Theorem 13.23] was able to prove the following analogous result:

Theorem 7.4.5. *Let $\mathcal{Y}$ be an orbifold. Then the maps $\Pi_{orb}^{sef} : B_k^{orb}(\mathcal{Y}) \to dB_k^{sef}(\mathcal{Y})$ and $\Pi_{orb}^{deff} : B_k^{eff}(\mathcal{Y}) \to dB_k^{eff}(\mathcal{Y})$ are isomorphisms for all $k \in \mathbb{Z}$.*

We want now briefly discuss which aspects of section 7.3 on unitary d-manifold bordism can be extend to the d-orbifold case. As Definition 7.3.2 shows, we can think of unitary d-manifold bordism as oriented d-manifold bordism with the additional structure of a kind of “virtual vector bundle bordism”. But since we have a nice notion of oriented d-orbifold bordism (Definition 7.4.4) and of stable nearly complex structures on d-orbifold (section 5), it becomes obvious that one can define unitary d-orbifold bordism in the exact same ways as unitary d-manifold bordism in Definition 7.3.2. Moreover, all the different kinds of d-orbifold bordism, like effective or semieffective d-orbifold bordism carry over to effective or semieffective unitary d-orbifold bordism.

Considering the structure of the proof of Theorem 7.3.3, we notice that the proof consists basically of two parts:

(1) The proof that oriented d-manifold bordism is isomorphic to 'usual' oriented manifold bordism.

(2) The proof that the virtual vector bundle structure, which encodes the complex data, is preserved.

Part (1) can be dealt with by using Theorem 7.4.5 and for part (2) we note that almost all the techniques and definitions used have a nice d-orbifold counterparts. The key ingredient in the proof of part (2) is the use of Proposition 5.3.2, but as we have seen earlier, using an additional assumption, namely that the d-orbifold $\boldsymbol{\mathcal{X}}$ is embeddable, we get a d-orbifold version of this proposition in Proposition 5.3.3.

As we have seen in section 6.1, the embeddability of a compact d-orbifold is equivalent to the existence of a representable morphism $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ in **dOrb** for some effective orbifold $\mathcal{Y}$. But the key point in the proof of Theorem 7.4.5 is that one can perturb an effective d-orbifold into an effective orbifold. Hence, after the perturbation step we get a representable, compact d-orbifold $\boldsymbol{\mathcal{X}}$ and the rest of the proof extends nicely using Proposition 5.3.3 instead of Proposition 5.3.2.

If we denote the 'classical' unitary orbifold bordism (which is defined in exactly the same way as unitary manifold bordism) by $\mathrm{BU}_k^{\mathrm{orb}}$ and the effective version by $\mathrm{BU}_k^{\mathrm{eff}}$, we then get the following d-orbifold analogue of Theorem 7.3.3:

Theorem 7.4.6. *Let $\mathcal{Y}$ be a compact, orbifold. Then the morphism $\Pi_{\mathrm{bu}}^{\mathrm{dbu}deff}$: $BU_k^{eff}(\mathcal{Y}) \to dBU_k^{deff}(\mathcal{Y})$ is an isomorphism for all $k \in \mathbb{Z}$.*

Chapter 8

D-blowups

In this chapter we define what a blowup in the 2-category of d-manifolds **dMan** is, and show that the definition we give is well-defined. We will define the d-blowup locally and show that this local construction does not depend on the choice of local data. The basic idea is to use the local description of (w-)embeddings of d-manifolds as in Theorem 2.3.25 and twist a part of the vector bundle E by L_D^{-1}.

8.1 Classical blowups

We will briefly review some classical theory on (differential geometric) blowups and motivate how this can be generalized to the d-manifold case.

Recall the following theorem (compare [30, §2.5] or [25, §6] for a more detailed discussion), which summarizes some important properties of blow-ups.

Theorem 8.1.1. *Let W be a complex submanifold of a complex manifold V. Then there exists a complex manifold $\tilde{V} = Bl_W V$, the so called* blow-up of V along W, *together with a holomorphic map $\pi : \tilde{V} \to V$, such that $\pi : \tilde{V} \setminus \pi^{-1}(W) \xrightarrow{\cong} V \setminus W$ and $\pi : \pi^{-1}(W) \to W$ is isomorphic to the projectivization of the normal bundle of W in V, $\mathbb{P}(\mathcal{N}_{W/V}) \to W$.*

In the case of real differential geometry, where we want to blow up a real manifold V along a real submanifold W, the map $\pi : \tilde{V} \to V$ is a diffeomorphism from $\tilde{V} \setminus \pi^{-1}(W)$ to $V \setminus W$ instead and we have as in the complex case that $\pi : \pi^{-1}(W) \to W$ is isomorphic to the projectivization of the normal bundle of W

in V, $\mathbb{P}(\mathcal{N}_{W/V}) \to W$. The hypersurface $D = \pi^{-1}(W) = \mathbb{P}(\mathcal{N}_{W/V}) \subset Bl_W V$ will be called the exceptional divisor *of the blow-up $\pi : Bl_W V \to V$.*

Considering that the blowup $\tilde{V}$ of a manifold V along a submanifold W is itself a manifold and isomorphic to V outside the exceptional divisor, one can ask how the tangent space of $\tilde{V}$ at the exceptional divisor changes.

Let therefore $x \in W$ and $0 \neq \lambda \in \mathcal{N}_{W/V}|_x$. We can then split (in a non canonical way) the tangent space of V as follows:

$$\begin{aligned} T_x V &= T_x W \oplus \mathcal{N}_{W/V}|_x \\ &= T_x W \oplus \langle\lambda\rangle \oplus \langle\lambda\rangle^{\perp}, \end{aligned}$$

where $\langle\lambda\rangle$ denotes the span of λ in $\mathcal{N}_{W/V}|_x$ and $\langle\lambda\rangle^{\perp} = \mathcal{N}_{W/V}|_x/\langle\lambda\rangle$.

The exceptional divisor can locally be written as $D \cong W \times \mathbb{P}(\mathcal{N}_{W/V})$ and as the normal bundle to D in $\tilde{V}$ has fibre $\langle\lambda\rangle$ at $(x, \langle\lambda\rangle)$, and the tangent space of $\mathbb{P}(\mathcal{N}_{W/V})$ at $\langle\lambda\rangle$ is given by $T_{\langle\lambda\rangle}\mathbb{P}(N_{W/V}) = Hom(\langle\lambda\rangle, \mathcal{N}_{W/V}/\langle\lambda\rangle) = \langle\lambda\rangle^{-1} \otimes \mathcal{N}_{W/V}/\langle\lambda\rangle$, we get

$$\begin{aligned} T_{x,\langle\lambda\rangle}\tilde{V} &= T_x W \oplus \langle\lambda\rangle \oplus T_{\langle\lambda\rangle}\mathbb{P}(\mathcal{N}_{W/V}) \\ &= T_x W \oplus \langle\lambda\rangle \oplus (\langle\lambda\rangle^{\perp} \otimes \langle\lambda\rangle^{-1}). \end{aligned}$$

Hence, blowing up a manifold along a submanifold has the effect on tangent spaces of twisting the normal bundle part orthogonal to $\langle\lambda\rangle$ by $\langle\lambda\rangle^{-1}$.

Apart from that, we have a line bundle L_D associated to the exceptional divisor D of $\tilde{V}$, and as D is a divisor there exists a nontrivial section $s_D : \tilde{V} \to L_D$ satisfying $s_D \neq 0$ on $\tilde{V} \setminus D$ and s_D vanishes to first order along D. (Compare for instance [30, §2.3]). Moreover it is not to difficult to see, that on the exceptional divisor D this line bundle is in fact isomorphic to the tautological line, that is we have $L_D|_{\langle\lambda\rangle} \cong \langle\lambda\rangle$. (See for instance [30, Proposition 2.3.18] for a proof in the complex case).

Hence blowing up a manifold along a submanifold at a point (x, λ) affects the tangent bundle of V by twisting the part of the normal bundle orthogonal to $\langle\lambda\rangle$ by the inverse of the line bundle associated to the exceptional divisor.

Now in the d-manifold case, we do not just have a manifold V and a submanifold W in V, but also a vector bundle $E \to V$ which can near W be split into a W-part and a complement as $E = F \oplus G$. The idea for the blowup in the d-manifold case is now to blowup V along W, pullback $E = F \oplus G$ to the blowup and twist the non W-part G by L_D^{-1} accordingly.

8.2 D-blowup of standard models

As we have seen in section 8.1, blowing up a manifold along a submanifold has the effect of twisting a part of the normal bundle. We want now to translate this fact to the d-manifold case and define what the blowup of a d-manifold $\boldsymbol{Y}$ along a d-submanifold $\boldsymbol{X}$ should be. To do this, we will first define what the blow up of a standard model d-manifold along a standard model d-submanifold is, and then define the blowup of a d-manifold along a d-submanifold by gluing blown up standard model d-manifolds together.

Definition 8.2.1. Let $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{W,F,t} \to \boldsymbol{S}_{V,E,s}$ be a closed w-embedding of standard model d-manifolds. Assume further, that W is a closed embedded submanifold of V, $f : W \hookrightarrow V$ is the inclusion of submanifolds, $E|_W = F \oplus H$ for some vector bundle $H \to W$, $\hat{f} = \mathrm{id} \oplus 0 : F \to F \oplus H = f^*(E)$ and $s|_W = t \oplus 0$. Given a sufficiently small open neighbourhood V' of W in V we can extend F to V', and furthermore split the bundle $E|_{V'} = F \oplus G$ and the section $s|_{V'} = t' \oplus u$ with $u|_W = 0$. We will in the following sometimes omit the restriction to V' if it is clear from the context, and write $V' = V, E|_{V'} = E$, and $s|_{V'} = s$.

Define *the blowup of* $\boldsymbol{S}_{V,E,s}$ *along* $\boldsymbol{S}_{W,F,t}$ as the standard model d-manifold $\boldsymbol{S}_{\tilde{V},\tilde{E},\tilde{s}}$, where $\tilde{V} := Bl_W V$ is the blowup of V along W in **Man** with projection map $\pi : \tilde{V} \to V$, and $\tilde{E}$, $\tilde{s}$ are defined by $\tilde{E} = \pi^*(F) \oplus (\pi^*(G) \otimes L_D^{-1})$ and $\tilde{s} = \pi^*(t) \oplus (\pi^*(u) \otimes s_D^{-1})$ on V', and $\tilde{E} = E$ and $\tilde{s} = s$ on $V \setminus V'$. Here D denotes the exceptional divisor associated to the the blowup $\tilde{V}$, L_D the corresponding line bundle, and $s_D : \tilde{V} \to L_D$ the associated non-trivial section satisfying $s_D|_D = 0$. The *blowdown map* $\boldsymbol{S}_{\pi,\hat{\pi}} : \boldsymbol{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \boldsymbol{S}_{V,E,s}$ is then given by the blowdown map $\pi : \tilde{V} \to V$ and by $\hat{\pi} = \begin{pmatrix} \mathrm{id}_{\pi^*(F)} & 0 \\ 0 & s_D \cdot \mathrm{id}_G \end{pmatrix} : \pi^*(F) \oplus (\pi^*(G) \otimes L_D^{-1}) \to \pi^*(F) \oplus \pi^*(G)$.

The following lemma will show that the definition of the standard model d-blowup is independent of the choice of splitting of the vector bundle E.

Lemma 8.2.2. *Let $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{W,F,t} \to \boldsymbol{S}_{V,E,s}$ be a closed w-embedding of standard model d-manifolds, and let $E = F_1 \oplus G_1, s = t_1 \oplus u_1$ and $E = F_2 \oplus G_2, s = t_2 \oplus u_2$ be two different choices of splitting of E near W. Then there exists a unique isomorphism $I : \tilde{E}_1 \xrightarrow{\cong} \tilde{E}_2$ with $I(\tilde{s}_1) = \tilde{s}_2$, making the resulting standard model d-blowups $\boldsymbol{S}_{\tilde{V},\tilde{E}_1,\tilde{s}_1}$ and $\boldsymbol{S}_{\tilde{V},\tilde{E}_2,\tilde{s}_2}$ 1-isomorphic through the canonical 1-isomorphism $\boldsymbol{S}_{\mathrm{id}_{\tilde{V}},I} : \boldsymbol{S}_{\tilde{V},\tilde{E}_1,\tilde{s}_1} \to \boldsymbol{S}_{\tilde{V},\tilde{E}_2,\tilde{s}_2}$.*

Proof. Consider two different splittings of E near W:

$$
\begin{aligned}
E &= F_1 \oplus G_1, && \text{with } F_1|_W = F \\
s &= t_1 \oplus u_1, && \text{with } u_1|_W = 0 \quad \text{and} \\
E &= F_2 \oplus G_2, && \text{with } F_2|_W = F \\
s &= t_2 \oplus u_2, && \text{with } u_2|_W = 0,
\end{aligned}
$$

and the resulting standard model d-blowups

$$
\begin{aligned}
\tilde{E}_1 &= \pi^*(F_1) \oplus \pi^*(G_1) \otimes L_D^{-1}, \\
\tilde{s}_1 &= \pi^*(t_1) \oplus \pi^*(u_1) \otimes s_D^{-1}, \\
\tilde{E}_2 &= \pi^*(F_2) \oplus \pi^*(G_2) \otimes L_D^{-1}, \\
\tilde{s}_2 &= \pi^*(t_2) \oplus \pi^*(u_2) \otimes s_D^{-1}.
\end{aligned}
$$

Away from the exceptional divisor D of the blowup $\tilde{V} = \mathrm{Bl}_W V$ we have that $\tilde{V} \setminus D \cong V \setminus W$ and as the line bundle L_D associated to the exceptional divisor D and the corresponding section s_D, is trivial on $\tilde{V} \setminus D$, we get that $\tilde{E}_1 = E = \tilde{E}_2$ and $\tilde{s}_1 = s = \tilde{s}_2$. Therefore on $\tilde{V} \setminus D$ the isomorphism I is given by the identity.

To prove the statement on D, note first that near W in V, we can view F_2 as a graph over F_1 and G_2 as a graph over G_1. Hence there exist unique vector bundle morphisms $\alpha : F_1 \to G_1$ and $\beta : G_1 \to F_1$, such that

$$
\begin{aligned}
F_2 &= \Gamma_\alpha = \{(f_1, \alpha(f_1)) : f_1 \in F_1\} \quad \text{and} \\
G_2 &= \Gamma_\beta = \{(\beta(g_1), g_1) : g_1 \in G_1\}.
\end{aligned}
$$

Since $F_2|_W = F = F_1|_W$, we get that $\alpha|_W = 0$. Moreover, using the different splittings of E, we can split $\mathrm{id}_E : F_2 \oplus G_2 = \Gamma_\alpha \oplus \Gamma_\beta \to F_1 \oplus G_1$ near W into

$$
\mathrm{id}_E = \begin{pmatrix} \mathrm{id}_{F_1} & \beta \\ \alpha & \mathrm{id}_{G_1} \end{pmatrix}.
$$

Lifting this to $\tilde{E}$ yields the following map $I : \tilde{E}_2 \to \tilde{E}_1$

$$I = \begin{pmatrix} \pi^*(\mathrm{id}_{F_1}) & \pi^*(\beta) \cdot s_D \\ \pi^*(\alpha) \cdot s_D^{-1} & \pi^*(\mathrm{id}_{G_1}) \cdot \mathrm{id}_{L_D^{-1}} \end{pmatrix}.$$

The crucial point is, to note that the bit $\pi^*(\alpha) \cdot s_D^{-1}$ extends smoothly over D, as $\pi^*(\alpha)$ lies in the ideal of D. Hence, I is a well-defined, smooth isomorphism and it is clear from the construction of $\boldsymbol{S}_{\tilde{V},\tilde{E}_1,\tilde{s}_1}$ and $\boldsymbol{S}_{\tilde{V},\tilde{E}_2,\tilde{s}_2}$ that we get an isomorphic standard model d-blowups. $\square$

8.2.1 Universal property

We will now show that the standard model d-blowup satisfies a universal property in the sense that there exists an up to 2-isomorphism unique standard model 1-morphism between a pair of standard model d-blowups. In contrast to the classical universal property of blowups of manifolds or schemes (as for example in [27, Proposition II.7.14]), this universal property of standard model d-blowups is characterized by pairs of closed w-embedded standard model d-submanifolds with 1-morphisms between them. We start by defining a property of w-embeddings of standard model d-manifolds which we will need for the universal property.

Definition 8.2.3. Let $\boldsymbol{S}_{h_1,\hat{h}_1} : \boldsymbol{S}_{W_1,F_1,t_1} \to \boldsymbol{S}_{V_1,E_1,s_1}$ and $\boldsymbol{S}_{h_2,\hat{h}_2} : \boldsymbol{S}_{W_2,F_2,t_2} \to \boldsymbol{S}_{V_2,E_2,s_2}$ be closed w-embeddings of standard model d-manifolds and suppose that the following diagram is 2-commutative

$$\begin{tikzcd} \boldsymbol{S}_{W_1,F_1,t_1} \arrow[r, "\boldsymbol{S}_{g,\hat{g}}"] \arrow[d, "\boldsymbol{S}_{h_1,\hat{h}_1}"'] & \boldsymbol{S}_{W_2,F_2,t_2} \arrow[d, "\boldsymbol{S}_{h_2,\hat{h}_2}"] \\ \boldsymbol{S}_{V_1,E_1,s_1} \arrow[r, "\boldsymbol{S}_{f,\hat{f}}"'] \arrow[ur, Rightarrow, "\eta"] & \boldsymbol{S}_{V_2,E_2,s_2}. \end{tikzcd} \tag{8.1}$$

We say that (8.1) *satisfies condition* (†) if the corresponding complex

$$(\hat{f} \circ \hat{h}_1)^*(E_2^*) \xrightarrow{\beta_1} (\hat{h}_1)^*(E_1^*) \oplus (\hat{g})^*(F_2^*) \oplus (f \circ h_1)^*(T^*V_2) \xrightarrow{\beta_2} F_1^* \oplus h_1^*(T^*V_1) \oplus g^*(T^*W_2) \xrightarrow{\beta_3} T^*W_1 \longrightarrow 0, \tag{8.2}$$

is exact at the third position. (Compare [35, Proposition 2.41] for more details and the definition of $\beta_i, i = 1, 2, 3$). Using the splittings $E_i = F_i \oplus G_i$ and $T^*V_i = T^*W_i \oplus \mathcal{N}^*_{W_i/V_i}$ for $i = 1, 2$, equation (8.2) has the same cohomology as

$$(\hat{f} \circ \hat{h}_1)^*(G_2^*) \longrightarrow (\hat{h}_1)^*(G_1^*) \oplus (f \circ h_1)^*(\mathcal{N}^*_{W_2/V_2}) \longrightarrow h_1^*(\mathcal{N}^*_{W_1/V_1}) \longrightarrow 0 \longrightarrow 0, \tag{8.3}$$

and therefore being exact at the third position is (by taking the dual) equivalent of

$$h_1^*(\mathcal{N}_{W_1/V_1}) \longrightarrow (f \circ h_1)^*(\mathcal{N}_{W_2/V_2}) \oplus (\hat{h}_1)^*(G_1)$$

being injective.

The following theorem will construct a 1-morphism between a pair of standard model d-blowups. We will then show that this morphism is unique up to 2-isomorphism and that the standard model d-blowup thus fulfils a universal property.

Theorem 8.2.4. *Consider two standard model d-blowups $\boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1}$ and $\boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2}$ of closed w-embedded standard modeld d-submanifolds $\boldsymbol{S}_{h_1,\hat{h}_1} : \boldsymbol{S}_{W_1,F_1,t_1} \to \boldsymbol{S}_{V_1,E_1,s_1}$ and $\boldsymbol{S}_{h_2,\hat{h}_2} : \boldsymbol{S}_{W_2,F_2,t_2} \to \boldsymbol{S}_{V_2,E_2,s_2}$, with standard model 1-morphisms $\boldsymbol{S}_{g,\hat{g}} : \boldsymbol{S}_{W_1,F_1,t_1} \to \boldsymbol{S}_{W_2,F_2,t_2}$ and $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V_1,E_1,s_1} \to \boldsymbol{S}_{V_2,E_2,s_2}$, such that the resulting diagram*

$$\begin{tikzcd} \boldsymbol{S}_{W_1,F_1,t_1} \arrow[r, "\boldsymbol{S}_{g,\hat{g}}"] \arrow[d, "\boldsymbol{S}_{h_1,\hat{h}_1}"'] & \boldsymbol{S}_{W_2,F_2,t_2} \arrow[d, "\boldsymbol{S}_{h_2,\hat{h}_2}"] \\ \boldsymbol{S}_{V_1,E_1,s_1} \arrow[r, "\boldsymbol{S}_{f,\hat{f}}"] \arrow[ur, Rightarrow, "\eta"] & \boldsymbol{S}_{V_2,E_2,s_2}, \end{tikzcd} \tag{8.4}$$

is 2-commutative and satisfies (†).

Then there exists a natural standard model 1-morphism $\boldsymbol{S}_{\tilde{f},\tilde{\hat{f}}} : \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1} \to \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2}$ making the following diagram 2-commutative

$$\begin{tikzcd} \boldsymbol{S}_{W_1,F_1,t_1} \arrow[r, "\boldsymbol{S}_{g,\hat{g}}"] \arrow[d, "\boldsymbol{S}_{h_1,\hat{h}_1}"'] & \boldsymbol{S}_{W_2,F_2,t_2} \arrow[d, "\boldsymbol{S}_{h_2,\hat{h}_2}"] \\ \boldsymbol{S}_{V_1,E_1,s_1} \arrow[r, "\boldsymbol{S}_{f,\hat{f}}"] \arrow[ur, Rightarrow, "\eta"] & \boldsymbol{S}_{V_2,E_2,s_2} \\ \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1} \arrow[r, dotted, "\boldsymbol{S}_{\tilde{f},\tilde{\hat{f}}}"'] \arrow[u, "\boldsymbol{S}_{\pi_1,\hat{\pi}_1}"] \arrow[ur, Rightarrow, "\zeta=\mathrm{id}"] & \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2} \arrow[u, "\boldsymbol{S}_{\pi_2,\hat{\pi}_2}"'] \end{tikzcd}. \tag{8.5}$$

Note that $\tilde{f}, \tilde{\hat{f}}$ are actually only defined on an open neighbourhood $\tilde{V}_1'$ of $\tilde{s}_1^{-1}(0)$ in $\tilde{V}_1$, but we can concatenate $\boldsymbol{S}_{\tilde{f},\tilde{\hat{f}}} : \boldsymbol{S}_{\tilde{V}_1',\tilde{E}_1|_{\tilde{V}_1'},\tilde{s}_1|_{\tilde{V}_1'}} \to \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2}$ with the inverse of the inclusion 1-morphism $\boldsymbol{S}_{i,\hat{i}} : \boldsymbol{S}_{\tilde{V}_1',\tilde{E}_1|_{\tilde{V}_1'},\tilde{s}_1|_{\tilde{V}_1'}} \to \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1}$ to get the desired standard model 1-morphism.

Proof. Denote by $\mathcal{N}_1, \mathcal{N}_2$ the normal bundles of W_1, W_2 in V_1, V_2 respectively, so that we get the following diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & T_{x_1}W_1 \arrow[r] \arrow[d, "dg"] & T_{x_1}V_1 \arrow[r] \arrow[d, "df"] & \mathcal{N}_1|_{x_1} \arrow[r] \arrow[d, dotted, "\exists!\alpha"] & 0 \\
0 \arrow[r] & T_{x_2}W_2 \arrow[r] & T_{x_2}V_2 \arrow[r] & \mathcal{N}_2|_{x_2} \arrow[r] & 0.
\end{tikzcd}$$

Here $\alpha : \mathcal{N}_1|_{x_1} \to \mathcal{N}_2|_{x_2}$ is the unique lift making the diagram commutative. As (8.4) satisfies condition (†), we have that

$$\alpha \oplus du_1|_{x_1} : \mathcal{N}_1|_{x_1} \to \mathcal{N}_2|_{x_2} \oplus G_1|_{x_1}$$

is injective, where we used the fact that we can split $E_1 = F_1 \oplus G_1$ and $s_1 = t_1 \oplus u_1$. As $\mathbb{P}(\ker(\alpha)) \subseteq D \subseteq \tilde{V}_1$, we cannot define $\tilde{f}, \tilde{\hat{f}}$ on all of $\tilde{V}_1$, but we claim that $\tilde{s}_1 \neq 0$ on $\mathbb{P}(\ker\alpha)$, which means that we can define $\tilde{f}, \tilde{\hat{f}}$ on an open neighbourhood $\tilde{V}_1'$ of $\tilde{s}_1^{-1}(0)$ in $\tilde{V}_1$ not containing $\mathbb{P}(\ker(\alpha))$. To prove this claim, note that we have

$$\tilde{s}_1|_{x_1,\langle\lambda\rangle} = (du_1|_{x_1})|_{\langle\lambda\rangle}.$$

Therefore, $\alpha \oplus du_1|_{x_1}$ being injective gives us either $\langle\lambda\rangle \nsubseteq \ker(\alpha)$, or $du_1|_{\langle\lambda\rangle} \neq 0$, which implies that $\tilde{s}_1|_{x_1,\langle\lambda\rangle} \neq 0$. In both cases we get an induced map between $\tilde{f} : \tilde{V}_1' \to \tilde{V}_2$. Using the definitions of $\tilde{E}_1$ and $\tilde{E}_2$ we get in a similar fashion a morphism $\tilde{\hat{f}} : \tilde{E}_1 \to \tilde{E}_2$:

Split the morphism $\hat{f} : E_1 \to E_2$ according to the splittings $E_1 = F_1 \oplus G_1, E_2 = F_2 \oplus G_2$ into $\hat{f} = \begin{pmatrix} \hat{g}_{\mathrm{ext}} & k \\ l & m \end{pmatrix}$, where $\hat{g}_{\mathrm{ext}}$ denotes an extension of $\hat{g}$ to the extended vector bundle F_1 on V and $l : F_1 \to G_2$ satisfies $l|_{W_1} = 0$. Then we can define a morphism $\tilde{\hat{f}} : \pi^*(F_1) \oplus \pi^*(G_1) \otimes L_{D_1}^{-1} \to \pi^*(F_2) \oplus \pi^*(G_2) \otimes L_{D_2}^{-1}$ as follows:

$$\tilde{\hat{f}} = \begin{pmatrix} \pi_1^*(\hat{g}_{\mathrm{ext}}) & \pi_1^*(k) \cdot s_{D_1} \\ \pi_1^*(l) \cdot \delta^{-1} \cdot s_{D_1}^{-1} & \pi_1^*(m) \cdot \delta^{-1} \end{pmatrix}.$$

Here $\delta : L_{D_1} \to L_{D_2}$ denotes the by α induced isomorphism between L_{D_1} and L_{D_2}, and $s_{D_i}, i = 1, 2$ the associated sections to the line bundles L_{D_1}, L_{D_2}. Moreover, $\hat{\tilde{f}}$ is well defined as $l|_{W_1} = 0$, and as δ is an isomorphism it is clear from the definition that we have the identity $\hat{\tilde{f}} \circ \tilde{s}_1 = \tilde{s}_2 \circ \tilde{f}$. Hence, we get a standard model 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{\tilde{V}_1', \tilde{E}_1|_{\tilde{V}_1'}, \tilde{s}_1|_{\tilde{V}_1'}} \to \boldsymbol{S}_{\tilde{V}_2, \tilde{E}_2, \tilde{s}_2}$, and after concatenating with the inverse of the inclusion 1-morphism $\boldsymbol{S}_{i,\hat{i}}^{-1} : \boldsymbol{S}_{\tilde{V}_1', \tilde{E}_1|_{\tilde{V}_1'}, \tilde{s}_1|_{\tilde{V}_1'}} \to \boldsymbol{S}_{\tilde{V}_1, \tilde{E}_1, \tilde{s}_1}$ the claimed standard model 1-morphism.

To see that diagram (8.5) is 2-commutative, note that the following diagrams commute strictly:

$$\begin{tikzcd} \tilde{V}_1 \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"'] & \tilde{V}_2 \arrow[d, "\pi_2"'] \\ V_1 \arrow[r, "f"] & V_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \tilde{E}_1 \arrow[r, "\hat{\tilde{f}}"] \arrow[d, "\hat{\pi}_1"'] & \tilde{E}_2 \arrow[d, "\hat{\pi}_2"'] \\ E_1 \arrow[r, "\hat{f}"] & E_2 \end{tikzcd} .$$

Hence diagram (8.5) is 2-commutative with $\zeta = \mathrm{id}$. □

We will now show that the constructed standard model 1-morphism $\boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}} : \boldsymbol{S}_{\tilde{V}_1, \tilde{E}_1, \tilde{s}_1} \to \boldsymbol{S}_{\tilde{V}_2, \tilde{E}_2, \tilde{s}_2}$ is unique up to 2-isomorphism and thus that a pair of d-blow ups satisfies a universal property.

Theorem 8.2.5. *Given the assumptions of Theorem 8.2.4, let* $\boldsymbol{S}_{h,\hat{h}} : \boldsymbol{S}_{\tilde{V}_1, \tilde{E}_1, \tilde{s}_1} \to \boldsymbol{S}_{\tilde{V}_2, \tilde{E}_2, \tilde{s}_2}$ *and* $\boldsymbol{S}_{k,\hat{k}} : \boldsymbol{S}_{\tilde{V}_1|_{D_1}, \tilde{E}_1|_{D_1}, \tilde{s}_1|_{D_1}} \to \boldsymbol{S}_{\tilde{V}_2|_{D_2}, \tilde{E}_2|_{D_2}, \tilde{s}_2|_{D_2}}$ *be* 1*-morphisms in* **dMan** *and* $\boldsymbol{S}_{\Lambda_1} : \boldsymbol{S}_{g,\hat{g}} \circ \boldsymbol{S}_{\pi_1|_{D_1}, \hat{\pi}_1|_{D_1}} \Rightarrow \boldsymbol{S}_{\pi_2|_{D_2}, \hat{\pi}_2|_{D_2}} \circ \boldsymbol{S}_{k,\hat{k}}$, $\boldsymbol{S}_{\Lambda_2} : \boldsymbol{S}_{\mathrm{inc}_2, \widehat{\mathrm{inc}}_2} \circ \boldsymbol{S}_{k,\hat{k}} \Rightarrow \boldsymbol{S}_{h,\hat{h}} \circ \boldsymbol{S}_{\mathrm{inc}_1, \widehat{\mathrm{inc}}_1}$ *and* $\boldsymbol{S}_{\Lambda_3} : \boldsymbol{S}_{\pi_2,\hat{\pi}_2} \circ \boldsymbol{S}_{h,\hat{h}} \Rightarrow \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{S}_{\pi_1,\hat{\pi}_1}$ *be* 2*-morphisms making the following diagram* 2*-commutative*

$$\begin{tikzcd}[column sep=small]
\boldsymbol{S}_{D_1, \tilde{E}_1|_{D_1}, \tilde{s}_1|_{D_1}} \arrow[rr, "\boldsymbol{S}_{\mathrm{inc}_1, \widehat{\mathrm{inc}}_1}"'] \arrow[dr, "\boldsymbol{S}_{\tilde{f}|_{D_1}, \hat{\tilde{f}}|_{D_1}}"'] \arrow[dd] \arrow[dr, bend left=60, "\boldsymbol{S}_{k,\hat{k}}"] \arrow[r, phantom, "\Downarrow \exists\, \Xi'"] & & \boldsymbol{S}_{\tilde{V}_1, \tilde{E}_1, \tilde{s}_1} \arrow[dr, "\boldsymbol{S}_{\tilde{f}, \hat{\tilde{f}}}"] \arrow[dd, "\boldsymbol{S}_{\pi_1,\hat{\pi}_1}"'] \arrow[dr, bend left=60, "\boldsymbol{S}_{h,\hat{h}}"] \arrow[r, phantom, "\Downarrow \exists\, \Xi"] & \\
& \boldsymbol{S}_{D_2, \tilde{E}_2|_{D_2}, \tilde{s}_2|_{D_2}} \arrow[rr, "\boldsymbol{S}_{\mathrm{inc}_2, \widehat{\mathrm{inc}}_2}"'] \arrow[dd] \arrow[ur, Rightarrow, "\boldsymbol{S}_{\Lambda_2}"] & & \boldsymbol{S}_{\tilde{V}_2, \tilde{E}_2, \tilde{s}_2} \arrow[dd, "\boldsymbol{S}_{\pi_2,\hat{\pi}_2}"] \arrow[dl, Rightarrow, "\boldsymbol{S}_{\Lambda_3}"] \\
\boldsymbol{S}_{W_1, F_1, t_1} \arrow[rr, "\boldsymbol{S}_{h_1,\hat{h}_1}"] \arrow[dr, "\boldsymbol{S}_{g,\hat{g}}"] \arrow[ur, Rightarrow, "\boldsymbol{S}_{\Lambda_1}"] & & \boldsymbol{S}_{V_1, E_1, s_1} \arrow[dr, "\boldsymbol{S}_{f,\hat{f}}"] & \\
& \boldsymbol{S}_{W_2, F_2, t_2} \arrow[rr, "\boldsymbol{S}_{h_2,\hat{h}_2}"] & & \boldsymbol{S}_{V_2, E_2, s_2}
\end{tikzcd} , \qquad (8.6)$$

such that it satisfies the composition round the cube condition (compare section A.3)

$$
\begin{aligned}
&(\mathrm{id}_{front} * \mathrm{id}_{\boldsymbol{S}_{k,\hat{k}}}) \odot (\mathrm{id}_{\boldsymbol{S}_{h_2,\hat{h}_2}} * \boldsymbol{S}_{\Lambda_2}) \odot (\mathrm{id}_{bottom} * \mathrm{id}_{\boldsymbol{S}_{\pi_1|_{D_1},\hat{\pi}_1|_{D_1}}}) \odot \\
&(\mathrm{id}_{\boldsymbol{S}_{f,\hat{f}}} * \mathrm{id}_{back}) \odot (\boldsymbol{S}_{\Lambda_3} * \mathrm{id}_{\boldsymbol{S}_{\mathrm{inc}_1,\hat{\mathrm{inc}}_1}}) \odot (\mathrm{id}_{\boldsymbol{S}_{\pi_2,\hat{\pi}_2}} * \boldsymbol{S}_{\Lambda_2}) \\
&= \mathrm{id}_{\boldsymbol{S}_{\pi_2,\hat{\pi}_2} \circ \boldsymbol{S}_{\mathrm{inc}_2,\hat{\mathrm{inc}}_2} \circ \boldsymbol{S}_{k,\hat{k}}}.
\end{aligned}
\tag{8.7}
$$

Note that here $\boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}} : \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1} \to \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2}$ *and* $\boldsymbol{S}_{\tilde{f}|_{D_1},\hat{\tilde{f}}|_{D_1}} : \boldsymbol{S}_{\tilde{D}_1,\tilde{E}_1|_{D_1},\tilde{s}_1|_{D_1}} \to \boldsymbol{S}_{\tilde{D}_2,\tilde{E}_2|_{D_2},\tilde{s}_2|_{D_2}}$ *denote the* 1*-morphisms constructed in Theorem 8.2.4 and that we, for the sake of readability, did not include the identity* 2*-morphisms for the strictly commutative diagrams at the bottom, on the front and on the back. (Compare the* 2*-morphisms* η_{FG}, η_{GD} *and* η_{BE} *in diagram* (A.1) *in Appendix A.3).*

Given these assumptions, there exist unique 2*-isomorphism* $\boldsymbol{S}_{\Xi} : \boldsymbol{S}_{h,\hat{h}} \Rightarrow \boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}}$, $\boldsymbol{S}_{\Xi'} : \boldsymbol{S}_{k,\hat{k}} \Rightarrow \boldsymbol{S}_{\tilde{f}|_{D_1},\hat{\tilde{f}}|_{D_1}}$ *completing the diagram. In particular, the morphism* $\boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}} : \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1} \to \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2}$ *constructed in Theorem 8.2.4 is unique up to* 2*-isomorphism.*

Proof. We will prove the existence of a 2-isomorphism $\boldsymbol{S}_{\Xi} : \boldsymbol{S}_{h,\hat{h}} \Rightarrow \boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}}$ by showing that the 2-morphism $\boldsymbol{S}_{\Lambda_3} : \boldsymbol{S}_{\pi_2,\hat{\pi}_2} \circ \boldsymbol{S}_{h,\hat{h}} \Rightarrow \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{S}_{\pi_1,\hat{\pi}_1}$ satisfies

$$\boldsymbol{S}_{\Lambda_3} = \mathrm{id}_{\boldsymbol{S}_{\pi_2,\hat{\pi}_2}} * \boldsymbol{S}_{\Xi}.$$

Using the description of 2-morphisms in terms of maps as in Definition 2.3.12, this translates to showing that the to $\boldsymbol{S}_{\Lambda_3}$ associated map $\Lambda_3 : \tilde{E}_1 \to (\pi_2 \circ h)^*(T\tilde{V}_2)$ factors through $\Xi : \tilde{E}_1 \to h^*(T\tilde{V}_2)$, that is we have the following commutative diagram

$$
\begin{tikzcd}
 & h^*(T\tilde{V}_2) \arrow[d, "h^*(d\pi_2)"] \\
\tilde{E}_1 \arrow[ur, dotted, "\Xi"] \arrow[r, "\Lambda_3"] & h^*(TV_2).
\end{tikzcd}
$$

To do this, we first show that we can change $\boldsymbol{S}_{h,\hat{h}}$ by a 2-isomorphism, such that $\boldsymbol{S}_{k,\hat{k}} = \boldsymbol{S}_{h|_{D_1},\hat{h}|_{D_1}}$ and the diagram on top is strictly commutative.

Choose therefore a standard model 1-morphism $\boldsymbol{S}_{h',\hat{h}'} : \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1} \to \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2}$ and a 2-isomorphism $\boldsymbol{S}_{\Delta} : \boldsymbol{S}_{h,\hat{h}} \Rightarrow \boldsymbol{S}_{h',\hat{h}'}$ such that $(\boldsymbol{S}_{\Delta} * \mathrm{id}_{\boldsymbol{S}_{\mathrm{inc}_1,\hat{\mathrm{inc}}_1}}) = -\boldsymbol{S}_{\Lambda_2}$. This can for example be done, by setting $\boldsymbol{S}_\Delta = -\boldsymbol{S}_{\Lambda_2}$ on D_1 and arbitrary away from D_1, and then choose

$$\begin{aligned} h' &= h + \Delta \cdot \tilde{s}_1 + O(\tilde{s}_1^2) \\ \hat{h}' &= \hat{h} + \Delta \cdot h^*(d\tilde{s}_2) + O(\tilde{s}_1), \end{aligned}$$

accordingly to make $\boldsymbol{S}_\Delta : \boldsymbol{S}_{h,\hat{h}} \Rightarrow \boldsymbol{S}_{h',\hat{h}'}$ into a 2-isomorphism. We can think of $\Delta : \tilde{E}_1 \to h^*(T\tilde{V}_2)$ as an extension of Λ_2 to $\tilde{E}_1$ in some sense. The by $\boldsymbol{S}_\Delta$ induced 2-morphism $\boldsymbol{S}_{\tilde{\Lambda}_2} : \boldsymbol{S}_{\mathrm{inc}_2,\hat{\mathrm{inc}}_2} \circ \boldsymbol{S}_{k,\hat{k}} \Rightarrow \boldsymbol{S}_{h',\hat{h}'} \circ \boldsymbol{S}_{\mathrm{inc}_1,\hat{\mathrm{inc}}_1}$ satisfies

$$\begin{aligned} \boldsymbol{S}_{\tilde{\Lambda}_2} &= (\boldsymbol{S}_\Delta * \mathrm{id}_{\boldsymbol{S}_{\mathrm{inc}_1,\hat{\mathrm{inc}}_1}}) \odot \boldsymbol{S}_{\Lambda_2} \\ &= \mathrm{id}, \end{aligned}$$

which makes the diagram on top

$$\begin{tikzcd} \boldsymbol{S}_{D_1,\tilde{E}_1|_{D_1},\tilde{s}_1|_{D_1}} \arrow[r, "\boldsymbol{S}_{\mathrm{inc}_1,\hat{\mathrm{inc}}_1}"] \arrow[d, "\boldsymbol{S}_{k,\hat{k}}"'] & \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1} \arrow[d, "\boldsymbol{S}_{h',\hat{h}'}"] \\ \boldsymbol{S}_{D_2,\tilde{E}_2|_{D_2},\tilde{s}_2|_{D_2}} \arrow[r, "\boldsymbol{S}_{\mathrm{inc}_2,\hat{\mathrm{inc}}_2}"'] \arrow[ur, Rightarrow, "\mathrm{id}_{\mathrm{top}}"'] & \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2} \end{tikzcd} \tag{8.8}$$

strictly commutative.

In terms of the standard model description of 2-morphisms (Definition 2.3.12), we choose a map $\Delta : \tilde{E}_1 \to h^*(T\tilde{V}_2)$ such that $\Delta|_{D_1} = \Lambda_2$ and set

$$\begin{aligned} h' &= h + \Delta \cdot \tilde{s}_1 + O(\tilde{s}_1^2), \\ \hat{h}' &= \hat{h} + \Delta \cdot h^*(d\tilde{s}_2) + O(\tilde{s}_1). \end{aligned}$$

Note that if we replace $\boldsymbol{S}_{k,\hat{k}}$ in diagram (8.8) with $\boldsymbol{S}_{h|_{D_1},\hat{h}|_{D_1}}$, we also get a strictly commutative diagram, and as $\mathrm{inc}_i : D_i \to \tilde{V}_i$, $i = 1, 2$ is injective, we can conclude that

$$\begin{aligned} k &= h'|_{D_1} + O(\tilde{s}_1|_{D_1}^2) \\ \hat{k} &= \hat{h}'|_{D_1} + O(\tilde{s}_1|_{D_1}). \end{aligned}$$

But $\boldsymbol{S}_{h',\hat{h}'}$ only depends on h' up to order $O(\tilde{s}_1^2)$, and so we can arrange by adding suitable $O(\tilde{s}_1^2)$-terms to h', that we have a strict equality above without changing

$\boldsymbol{S}_{h',\hat{h}'}$, that is we get

$$\begin{aligned} k &= h'|_{D_1} \\ \hat{k} &= \hat{h}'|_{D_1}. \end{aligned}$$

This then simplifies diagram (8.8) to

$$\begin{tikzcd} \boldsymbol{S}_{D_1,\tilde{E}_1|_{D_1},\tilde{s}_1|_{D_1}} \arrow[r, "\boldsymbol{S}_{\mathrm{inc}_1,\hat{\mathrm{inc}}_1}"] \arrow[d, "\boldsymbol{S}_{h'|_{D_1},\hat{h}'|_{D_1}}"'] & \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1} \arrow[d, "\boldsymbol{S}_{h',\hat{h}'}"] \\ \boldsymbol{S}_{D_2,\tilde{E}_2|_{D_2},\tilde{s}_2|_{D_2}} \arrow[r, "\boldsymbol{S}_{\mathrm{inc}_2,\hat{\mathrm{inc}}_2}"'] \arrow[ur, Rightarrow, "\mathrm{id}"'] & \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2}. \end{tikzcd} \tag{8.9}$$

If we denote by $\boldsymbol{S}_{\tilde{\Lambda}_3} = \boldsymbol{S}_{\Lambda_3} \odot (\mathrm{id}_{\boldsymbol{S}_{\pi_2,\hat{\pi}_2}} * \boldsymbol{S}_{\Delta}^{-1})$ the by $\boldsymbol{S}_{\Delta}$ induced 2-morphism, the round the cube condition (8.7), written in terms of maps, becomes

$$\Lambda_1 + \tilde{\Lambda}_3|_{D_1} = 0 + O(\tilde{s}_1^2). \tag{8.10}$$

On the exceptional divisor D_1, the projecting of equation (8.10) to the normal bundle $(f \circ \pi_1)^*(TV_2/TW_2)$ is zero

$$\pi_{TV_2/TW_2}(\tilde{\Lambda}_3) = 0 + O(\tilde{s}_1^2),$$

as $\Lambda_1 : \tilde{E}_1|_{D_1} \to TW_2$. Hence, the factorization

$$\begin{tikzcd} \tilde{E}_1|_{D_1} \arrow[r, "\tilde{\Lambda}_3|_{D_1}"] \arrow[dr, "\pi_{TV_2/TW_2}(\tilde{\Lambda}_3)"'] & (f \circ \pi_1|_{D_1})^*(T\tilde{V}_2) \arrow[d, "(f\circ\pi_1|_{D_1})^*(\pi_{TV_2/TW_2})"] \\ & (f \circ \pi_1|_{D_1})^*(TV_2/TW_2) \end{tikzcd}$$

shows that the image of $\tilde{\Lambda}_3|_{D_1}$ on D_1 is $(f \circ \pi_1|_{D_1})^*(TW_2)$ up to order $O(\tilde{s}_1^2)$.

On the other hand, using the local splitting of $TV_2 = TW_2 \oplus \langle\lambda\rangle \oplus \langle\lambda\rangle^{\perp}$ and $T\tilde{V}_2 = TW_2 \oplus \langle\lambda\rangle \oplus (\langle\lambda\rangle^{\perp} \otimes \langle\lambda\rangle^{-1})$ as in §8.1, we can split the morphism $\Lambda_3 = (\Lambda_3)_1 \oplus (\Lambda_3)_2 \oplus (\Lambda_3)_3$ accordingly and write $d\pi_2 : T\tilde{V}_2 \to TV_2$ as

$$d\pi_2 = \begin{pmatrix} \mathrm{id}_{TW_2} & * & * \\ 0 & \mathrm{id}_{\langle\lambda\rangle} & * \\ 0 & 0 & \otimes s_D \end{pmatrix}.$$

The morphism $\tilde{\Lambda}_3|_{D_1}$ factors then through a morphism $\Xi|_{D_1} : \tilde{E}_1|_{D_1} \to \tilde{f}|^*_{D_1}(T\tilde{V}_2)$, which near $(x, \langle\lambda\rangle)$, is given by $\Xi = (\Lambda_3)_1 \oplus (\Lambda_3)_2 \oplus ((\Lambda_3)_3 \otimes s_D^{-1})$. But as the projection to the normal bundle satisfies $\pi_{TV_2/TW_2}(\tilde{\Lambda}_3) = 0 + O(\tilde{s}_1^2)$, we can conclude that

$$
\begin{aligned}
(\Lambda_3)_2 &= B_2 \otimes s_D + O(\tilde{s}_1^2) \\
(\Lambda_3)_3 &= B_3 \otimes s_D + O(\tilde{s}_1^2),
\end{aligned}
$$

for some smooth maps B_2, B_3. Hence $(\Lambda_3)_3 \otimes s_D^{-1}$ is smooth up to $O(\tilde{s}_1^2)$ and we get the following commutative diagram near $(x, \langle\lambda\rangle)$:

$$
\begin{tikzcd}
 & \tilde{f}|^*_{D_1}(TW_2 \oplus \langle\lambda\rangle \oplus (\langle\lambda\rangle^{\perp} \otimes \langle\lambda\rangle^{-1})) \arrow[d, "(f\circ\pi_1|_{D_1})^*(\begin{pmatrix} \mathrm{id}_{TW_2} & * & * \\ 0 & \mathrm{id}_{\langle\lambda\rangle} & * \\ 0 & 0 & \otimes s_D \end{pmatrix})"] \\
\tilde{E}_1|_{D_1} \arrow[ur, dotted, "(\Lambda_3)_1\oplus(\Lambda_3)_2\oplus B_3+O(\tilde{s}_1^2)"] \arrow[r, "(\Lambda_3)_1\oplus(\Lambda_3)_2\oplus(\Lambda_3)_3"] \arrow[dr, "0"'] & (f\circ\pi_1|_{D_1})^*(TW_2 \oplus \langle\lambda\rangle \oplus \langle\lambda\rangle^{\perp}) \arrow[d, "(f\circ\pi_1|_{D_1})^*(\pi_{TV_2/TW_2})"] \\
 & (f\circ\pi_1|_{D_1})^*(TV_2/TW_2).
\end{tikzcd}
$$

Hence the morphism $\tilde{\Lambda}_3|_{D_1}$ factors (up to $O(\tilde{s}_1^2)$) through a morphism $\Xi|_{D_1} : \tilde{E}_1|_{D_1} \to \tilde{f}|^*_{D_1}(T\tilde{V}_2)$ making the following diagram commutative:

$$
\begin{tikzcd}
 & \tilde{f}|^*_{D_1}(T\tilde{V}_2) \arrow[d, "(f\circ\pi_1|_{D_1})^*(d\pi_2|_{D_1})"] \\
\tilde{E}_1|_{D_1} \arrow[ur, dotted, "\Xi|_{D_1}"] \arrow[r, "\tilde{\Lambda}_3|_{D_1}"] \arrow[dr, "\pi_{TV_2/TW_2}(\tilde{\Lambda}_3)"'] & (f\circ\pi_1|_{D_1})^*(TV_2) \arrow[d, "(f\circ\pi_1|_{D_1})^*(\pi_{TV_2/TW_2})"] \\
 & (f\circ\pi_1|_{D_1})^*(TV_2/TW_2).
\end{tikzcd}
$$

The morphism $\Xi|_{D_1} : \tilde{E}_1|_{D_1} \to \tilde{f}|^*_{D_1}(T\tilde{V}_2)$ extends then uniquely to a 2-isomorphism $\boldsymbol{S}_{\Xi} : \boldsymbol{S}_{h',\hat{h}'} \Rightarrow \boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}}$ on $\tilde{V}_1$, satisfying $\boldsymbol{S}_{\tilde{\Lambda}_3} = \mathrm{id}_{\boldsymbol{S}_{\pi_2,\hat{\pi}_2}} * \boldsymbol{S}_{\Xi}$, which then completes the proof. $\square$

Remark 8.2.6. Note that the obvious approach in defining the universal property as the existence of an unique 2-isomorphism in the diagram (8.5) between $\boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}}$ and another standard model 1-morphism $\boldsymbol{S}_{h,\hat{h}} : \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1} \to \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2}$, which makes (8.5) 2-commutative, fails. Setting all the sections to be 0 produces the following counterexample: let $t_i = s_i = \tilde{s}_i = 0$ for $i = 1, 2$, but let E_1 be non-trivial. Then $\Lambda : \tilde{E}_1 \to (\pi_2 \circ \tilde{f})^*(TV_2)$ satisfies no conditions, but a general such Λ does not factor through a $\Lambda' : \tilde{E}_1 \to \tilde{f}^*(T\tilde{V}_2)$

$$
\begin{tikzcd}
\tilde{E}_1 \arrow[rr, "\Lambda"] \arrow[dr, "\Lambda'"'] & & (\pi_2 \circ \tilde{f})^*(TV_2) \\
& \tilde{f}^*(T\tilde{V}_2). \arrow[ur, "\tilde{f}^*(d\pi_2)"'] &
\end{tikzcd}
$$

Hence there does not exist a $\Lambda' : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{h,\hat{h}}$ with $\Lambda = \mathrm{id}_{\boldsymbol{S}_{\pi_2,\hat{\pi}_2}} * \Lambda'$, and therefore $\boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}}$ is not unique up to 2-isomorphism.

The following lemma will prove, that if the standard model 1-morphisms $\boldsymbol{S}_{g,\hat{g}}$ and $\boldsymbol{S}_{f,\hat{f}}$ in Theorem 8.2.4 are equivalences so is $\boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}}$.

Lemma 8.2.7. *Assume that in diagram* (8.5) *of Theorem 8.2.4 the standard model* 1*-morphisms $\boldsymbol{S}_{g,\hat{g}}$ and $\boldsymbol{S}_{f,\hat{f}}$ are equivalences. Then $\boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}}$ is an equivalence.*

Proof. Let $x_1 \in t_1^{-1}(0)$, and $\lambda \in \mathcal{N}_1|_{x_1}$ be such that $\pi_1^*(u_1) \otimes s_D^{-1}|_{[\lambda]} = 0$. Here $[\lambda] \in \mathbb{P}(\mathcal{N}_1|_{x_1})$ denotes the equivalence class of λ in the projectivization of $\mathcal{N}_1$. Furthermore, let $x_2 = f(x_1) \in t_2^{-1}(0)$ and let $\mu = \alpha(\lambda) \in \mathcal{N}_2|_{x_2}$, with associated equivalence class $[\mu] \in \mathbb{P}(\mathcal{N}_2|_{x_2})$. Note further that $\alpha : \mathcal{N}_1|_{x_1} \to \mathcal{N}_2|_{x_2}$ being injective implies that $\langle\lambda\rangle \cong \langle\mu\rangle$, and we will denote this isomorphism by $\delta : \langle\lambda\rangle \xrightarrow{\cong} \langle\mu\rangle$.

Now, to prove that $\boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}} : \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1} \to \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2}$ is an equivalence, consider the following diagram:

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \langle\lambda\rangle^{\perp} \arrow[r] \arrow[u] & G_1 \oplus \langle\mu\rangle^{\perp} \arrow[r] \arrow[u] & G_2 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & T_{x_1}W_1 \oplus \langle\lambda\rangle \oplus \langle\lambda\rangle^{\perp} \arrow[r, "ds_1 \oplus df"] \arrow[u, "\begin{pmatrix} 0 & 0 & \mathrm{id} \end{pmatrix}"] & \begin{gathered} (F_1 \oplus G_1) \\ \oplus \\ (T_{x_2}W_2 \oplus \langle\mu\rangle \oplus \langle\mu\rangle^{\perp}) \end{gathered} \arrow[r, "\hat{f} \oplus -ds_2"] \arrow[u] & F_2 \oplus G_2 \arrow[r] \arrow[u, "\begin{pmatrix} 0 & \mathrm{id} \end{pmatrix}"] & 0 \\
0 \arrow[r] & \begin{gathered} T_{x_1}W_1 \oplus \langle\lambda\rangle \\ \oplus \\ \langle\lambda\rangle^{\perp} \otimes \langle\lambda\rangle^{-1} \end{gathered} \arrow[r, "d\tilde{s}_1 \oplus d\tilde{f}"] \arrow[u, "\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 \end{pmatrix}"] & \begin{gathered} (F_1 \oplus G_1 \otimes \langle\lambda\rangle^{-1}) \\ \oplus \\ (T_{x_2}W_2 \oplus \langle\mu\rangle \\ \oplus \\ \langle\mu\rangle^{\perp} \otimes \langle\mu\rangle^{-1}) \end{gathered} \arrow[r, "\tilde{\hat{f}} \oplus -d\tilde{s}_2"] \arrow[u] & \begin{gathered} F_2 \\ \oplus \\ G_2 \otimes \langle\mu\rangle^{-1} \end{gathered} \arrow[r] \arrow[u, "\begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}"] & 0 \\
0 \arrow[r] & \langle\lambda\rangle^{\perp} \otimes \langle\lambda\rangle^{-1} \arrow[r] \arrow[u, "0 \oplus 0 \oplus \mathrm{id}"] & \begin{gathered} G_1 \otimes \langle\lambda\rangle^{-1} \\ \oplus \\ \langle\mu\rangle^{\perp} \otimes \langle\mu\rangle^{-1} \end{gathered} \arrow[r] \arrow[u] & G_2 \otimes \langle\lambda\rangle^{-1} \arrow[r] \arrow[u, "0 \oplus \delta^{-1}"] & 0. \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{8.11}
$$

We claim that all the horizontal and vertical sequences are exact.

Note first, that the first vertical sequence is clearly exact. In a similar fashion the obvious morphisms make the second and the third vertical sequences exact, and therefore all the vertical sequences are exact.

For the horizontal sequences note that the second row is exact by Theorem 2.3.13, as $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V_1,E_1,s_1} \xrightarrow{\simeq} \boldsymbol{S}_{V_2,E_2,s_2}$ is an equivalence. As the fourth horizontal sequence is just the first row tensored by $\langle\lambda\rangle^{-1} \cong \langle\mu\rangle^{-1}$, exactness of the first row implies exactness of the fourth row.

To prove that the first row is exact, consider the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & TW_1 \arrow[r] \arrow[d] & F_1 \oplus TW_2 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TW_1 \oplus \mathcal{N}_1 \arrow[r] \arrow[d] & F_1 \oplus G_1 \oplus TW_2 \oplus \mathcal{N}_2 \arrow[r] \arrow[d] & F_2 \oplus G_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{N}_1 \arrow[r] \arrow[d] & G_1 \oplus \mathcal{N}_2 \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{8.12}$$

The first and second rows are exact as the second row is just the exact sequence associated to the equivalence $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V_1,E_1,s_1} \xrightarrow{\simeq} \boldsymbol{S}_{V_2,E_2,s_2}$, and the first row the restriction of this equivalence to $\boldsymbol{S}_{W_1,F_1,t_1}$. But as all the vertical sequences are exact for obvious reasons, we get that the third row is exact making (8.12) into an exact diagram.

Using this exact sequence we can deduce that

$$0 \longrightarrow \langle\lambda\rangle^\perp \longrightarrow G_1 \oplus \langle\mu\rangle^\perp \longrightarrow G_2 \longrightarrow 0$$

is exact as follows: We claim that the following diagram is exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \langle\lambda\rangle \arrow[r] \arrow[d] & \langle\mu\rangle \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \langle\lambda\rangle \oplus \langle\lambda\rangle^\perp \arrow[r] \arrow[d] & G_1 \oplus \langle\mu\rangle \oplus \langle\mu\rangle^\perp \arrow[r] \arrow[d] & F_2 \oplus G_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \langle\lambda\rangle \arrow[r] \arrow[d] & G_1 \oplus \langle\mu\rangle^\perp \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{8.13}$$

Indeed, the first row is exact as $\langle\lambda\rangle \cong \langle\mu\rangle$, the second row is exact, because it is just a reformulation of the third row in diagram (8.12), and as all the vertical sequences are exact for obvious reasons, we may conclude that

$$0 \longrightarrow \langle\lambda\rangle^\perp \longrightarrow G_1 \oplus \langle\mu\rangle^\perp \longrightarrow G_2 \longrightarrow 0$$

is an exact sequence.

This means, that in diagram (8.11) all vertical sequences, as well as the first, the second and the fourth row are exact. We can therefore conclude that the third row

$$0 \longrightarrow \begin{matrix} T_{x_1}W_1 \oplus \langle\lambda\rangle \\ \oplus \\ \langle\lambda\rangle^\perp \otimes \langle\lambda\rangle^{-1} \end{matrix} \longrightarrow \begin{matrix} (F_1 \oplus G_1 \otimes \langle\lambda\rangle^{-1}) \\ \oplus \\ (T_{x_2}W_2 \oplus \langle\mu\rangle \\ \oplus \\ \langle\mu\rangle^\perp \otimes \langle\mu\rangle^{-1}) \end{matrix} \longrightarrow \begin{matrix} F_2 \\ \oplus \\ G_2 \otimes \langle\mu\rangle^{-1} \end{matrix} \longrightarrow 0$$

is an exact sequence. Hence, by Theorem 2.3.13 this means that the induced standard model morphism $\boldsymbol{S}_{\tilde{f},\hat{\tilde{f}}} : \boldsymbol{S}_{\tilde{V}_1,\tilde{E}_1,\tilde{s}_1} \xrightarrow{\simeq} \boldsymbol{S}_{\tilde{V}_2,\tilde{E}_2,\tilde{s}_2}$ between the two d-blowups is an equivalence. $\square$

8.3 D-blowup of general d-manifolds

We want now to use the standard model d-blowups from Definition 8.2.1 to define the blowup of a d-manifold $\boldsymbol{Y}$ along a closed w-embedded d-submanifold $\boldsymbol{X}$. In order to do this, we want to use Theorem 2.3.19 and glue the standard model d-blowups along equivalences.

Definition 8.3.1. Let $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{X}$ be a closed w-embedding of d-manifolds. As shown in section 2.3.1, for each $y \in \boldsymbol{Y}$ there exist open neighbourhoods $y \in \boldsymbol{U}$ and $x = \underline{f}(y) \in \boldsymbol{V}$ and equivalences from $\boldsymbol{U}$,$\boldsymbol{V}$ to some standard d-manifolds $\boldsymbol{S}_{W,F,t}$ and $\boldsymbol{S}_{V,E,s}$, with standard model morphism $\boldsymbol{S}_{h,\hat{h}} : \boldsymbol{S}_{W,F,t} \to \boldsymbol{S}_{V,E,s}$. As $\boldsymbol{h}$ is a w-embedding, Theorem 2.3.25(c) shows that W can be taken as a submanifold of V, $f : W \hookrightarrow V$ can be taken to be the inclusion of submanifolds, $E|_W = F \oplus H$ for some vector bundle $H \to W$, $\hat{f} = \mathrm{id} \oplus 0 : F \to F \oplus H = f^*(E)$ and $s|_W = t \oplus 0$.

To define the blowup of $\boldsymbol{X}$ along $\boldsymbol{Y}$, chose an open covering $\boldsymbol{U}_i$ of $\boldsymbol{Y}$ for some countable indexing set I, as above. That is, for each $i \in I$ we have equivalences from $\boldsymbol{U}_i$, $\boldsymbol{V}_i$ to standard model d-manifolds $\boldsymbol{S}_{W_i,F_i,t_i}$ and $\boldsymbol{S}_{V_i,E_i,s_i}$, with standard model morphism $\boldsymbol{S}_{h_i,\hat{h}_i} : \boldsymbol{S}_{W_i,F_i,t_i} \to \boldsymbol{S}_{V_i,E_i,s_i}$. By making $\boldsymbol{U}_i$, $\boldsymbol{V}_i$ smaller if necessary, we can assume that on the overlaps $\boldsymbol{U}_i \cap \boldsymbol{U}_j$ and $\boldsymbol{V}_i \cap \boldsymbol{V}_j$ the induced maps $\boldsymbol{S}_{g_{ij},\hat{g}_{ij}} : \boldsymbol{S}_{W_i,F_i,t_i} \to \boldsymbol{S}_{W_j,F_j,t_j}$, and $\boldsymbol{S}_{f_{ij},\hat{f}_{ij}} : \boldsymbol{S}_{V_i,E_i,s_i} \to \boldsymbol{S}_{V_j,E_j,s_j}$ are equivalences. For each $i \in I$, we can then define standard model d-blowups $\boldsymbol{Z}_i := \boldsymbol{S}_{\tilde{V}_i,\tilde{E}_i,\tilde{s}_i}$ as in Definition 8.2.1. As we have equivalences on the overlaps $\boldsymbol{S}_{g_{ij},\hat{g}_{ij}} : \boldsymbol{S}_{W_i,F_i,t_i} \to \boldsymbol{S}_{W_j,F_j,t_j}$, and $\boldsymbol{S}_{f_{ij},\hat{f}_{ij}} : \boldsymbol{S}_{V_i,E_i,s_i} \to \boldsymbol{S}_{V_j,E_j,s_j}$, Lemma 8.2.7 shows that the resulting standard model d-blowup morphisms $\boldsymbol{S}_{\tilde{f}_{ij},\hat{\tilde{f}}_{ij}} : \boldsymbol{S}_{\tilde{V}_i,\tilde{E}_i,\tilde{s}_i} \to \boldsymbol{S}_{\tilde{V}_j,\tilde{E}_j,\tilde{s}_j}$ are equivalences.

We are now in the situation that we fulfil all the prerequisites of [35, Theorem 2.31]. This theorem is just the infinite countable generalization of Theorem 2.2.4, and although said theorem is just stated for d-spaces, the proof extends straightforward to the d-manifold case. We get therefore an up to equivalence unique d-manifold $\boldsymbol{\pi} : \tilde{\boldsymbol{X}} \to \boldsymbol{X}$, which we will call *the blowup of $\boldsymbol{X}$ along $\boldsymbol{Y}$*.

All the properties and results discussed previously extend nicely to the general d-manifold case. We will not repeat the statements and the results here as one simply can exchange the standard model d-manifolds and standard model morphisms by d-manifolds and morphisms between d-manifolds throughout. We want to highlight however, that in particular the material discussed in §8.2.1 can be extended to the general d-manifold case, which gives us a universal property for blowups of d-manifolds.

Remark 8.3.2. Similarly to the d-manifold case, we can define what the blowup of standard model d-orbifolds should be, by using the local description of d-orbifolds $\boldsymbol{\mathcal{X}}$ in terms of quotients of d-manifolds $\boldsymbol{\mathcal{X}} \simeq [\boldsymbol{S}_{V,E,s}/G]$ by the stabilizer group $G = \mathrm{Iso}_{\mathcal{X}}([x])$ as in §3.4.3. Instead of just considering w-embedded d-submanifolds, we consider w-embeddings of d-orbifolds which are isomorphisms on the stabilizer groups and make all the constructions G-equivariant. This then yields to a notion of standard model d-orbifold blowup, and by using the d-orbifold analogue of Theorem 2.3.19 (compare Theorem 3.3.3 for the d-stack case and [35, Theorem 10.19] for an extensive discussion), we can glue the local blowups to get a notion of d-blowup for general d-orbifolds.

Chapter 9

Towards a resolution of singularities and integral Gromov–Witten invariants

We want now to outline, how the material above could be used to tackle some problems in symplectic Gromov–Witten theory. (Symplectic) Gromov–Witten invariants are invariants 'counting' J-holomorphic, genus g curves Σ with marked points in a complex symplectic manifold. Despite being 'curve counting' invariants, (symplectic) Gromov–Witten invariants lie in rational homology instead of integral homology, as points $[\Sigma, \vec{z}, u] \in \overline{\mathcal{M}}_{g,m}(M, J)$ have to be counted with rational weight $|\mathrm{Aut}(\Sigma, \vec{z}, u)|^{-1}$. Hence, the reason that Gromov–Witten invariants are defined over $\mathbb{Q}$ rather than $\mathbb{Z}$ comes down to the fact that there exists points $[\Sigma, \vec{z}, u] \in \overline{\mathcal{M}}_{g,m}(M, J, \beta)$ with non-trivial finite automorphism groups $\mathrm{Aut}(\Sigma, \vec{z}, u)$. Equivalently, using the notion of d-orbifolds, it is because of non-trivial d-orbifold strata $\overline{\mathcal{M}}_{g,m}(M, J, \beta)^{\Gamma,\rho} \subseteq \overline{\mathcal{M}}_{g,m}(M, J, \beta)$ in the sense of §3.4.8.

It is therefore natural to ask, whether (symplectic) Gromov–Witten invariants can be expressed in terms of other (Gromov–Witten type) integral invariants. In the case of semi-positive symplectic manifolds and genus zero invariants for example, it is indeed true that one can define genus zero Gromov–Witten invariants $GW_{0,m}(M, \omega, u)$ in integral homology. This is proved in detail in the book of McDuff and Salamon [43, §7], and the reason why this is true from the viewpoint of d-orbifolds, is that the codimension of all non-trivial, non-empty orbifold strata is at least 2, which is enough to define virtual cycles over $\mathbb{Z}$ in a similar way as in the semi-effective and effective d-orbifold case. (Compare 7.4.2 or [35, §13.4].) Another prominent example, where integrality questions arose, is the case of symplectic Calabi–Yau 3-folds, that is compact symplectic 6-manifolds (M, ω) with $c_1(M) = 0$. Let therefore the number of marked points m to be zero. This then implies that vdim $\overline{\mathcal{M}}_{g,0}(M, J, \beta) = 0$ for all g, β and therefore that the Gromov-Witten invariants $GW_{g,0}(M, \omega, u)$ are in this case rational numbers.

The String Theorists Gopakumar and Vafa [22],[23] used physical reasoning about counting so called BPS states, and conjectured the existence of invariants $GV_g(M, \omega, u) \in \mathbb{Z}$ for Calabi–Yau 3-folds, which roughly speaking 'count' embedded J-holomorphic curves of genus g representing a homology class β in M. These integral counting invariants are known as *Gopakumar–Vafa invariants* and by expressing J-holomorphic curves in M as branched covers of embedded curves, Gopakumar and Vafa conjecturally expressed Gromov–Witten invariants in terms of Gopakumar–Vafa invariants, and vice versa, by the following equation in formal power series

$$\sum_{g,\beta} GW_{g,0}(M, \omega, \beta) t^{2g-2} q^{\beta} = \sum_{k>0,g,\beta} GV_g(M, \omega, \beta) \frac{1}{k} (2 \sin(kt/2))^{2g-2} q^{k\beta}. \quad (9.1)$$

The *Gopakumar-Vafa Integrality Conjecture* says, that Gromov-Witten invariants of Calabi-Yau 3-folds satisfy (9.1) for some integers $GV_g(M, \omega, \beta)$. Moreover Gopakumar and Vafa conjectured that $GV_g(M, \omega, \beta) = 0$ for all fixed classes β and $g \gg 0$.

There are two obvious approaches to tackle this conjecture. The first approach is to define an integral curve-counting invariant $GV_g(M, \omega, u)$ and prove that these invariants satisfy equation (9.1). In the context of algebraic geometry, important steps in this direction where undertaken by Pandharipande and Thomas [47],[48], by defining integer-valued invariants counting 'stable pairs' (F, s) of a coherent sheaf F supported on a curve in M and a section $s \in H^0(F)$. It is still not yet totally understood how to prove that these *Pandharipande–Thomas invariants* are equivalent to Gromov–Witten invariants.

The second approach is to regard equation (9.1) as a definition of numbers $GV_g(M, \omega, \beta) \in \mathbb{Q}$ and prove that these $GV_g(M, \omega, \beta)$ are actually integers. We will sketch in the following how one could try to use the previous material on d-orbifolds to tackle this problem and make the second approach work. The techniques and ideas are similar to the ideas of Fukaya and Ono [21] in the Kuranishi space framework.

Another useful resource on this subject is Pandharipande [46], who extends Gopakumar–Vafa invariants and their integrality conjecture to all smooth projective complex algebraic 3-folds.

The following procedure will outline, how one can prove the existence of a procedure which modifies a general (nearly complex) d-orbifold to a (semi-)effective (nearly complex) d-orbifold (or even a d-manifold). The induced functor from unitary d-orbifold bordism (as in 7.3 and 7.4.2) to unitary (semi-)effective d-orbifold bordism can then be applied to the (d-orbifold) Gromov–Witten invariants which yields integral Gromov–Witten type invariants. We will explain the procedure for general d-orbifolds (and not just $\overline{\mathcal{M}}_{g,m}(M,J,\beta)$) as it might be interesting to have a "resolution of singularities"-type result for other future applications.

Let therefore $\boldsymbol{\mathcal{X}}$ be a nearly complex d-orbifold with orbifold strata $\boldsymbol{\mathcal{X}}_\Gamma$, where Γ is an abelian group.

Step (1): Make the d-orbifold strata $\boldsymbol{\mathcal{X}}_\Gamma$ abelian. This can be done by using the 'wonderful blowup' argument of Borisov and Gunnels [10]. Instead of successively blowing up the strata of smallest dimension classically like in the original paper, we perform d-blowups as defined in chapter 8.

Step (2): Choose a type A good coordinate system $(I,<,(V_i,E_i,s_i,\psi_i),(V_{ij},e_{ij},\hat{e}_{ij},\rho_{ij},\eta_{ij}),\gamma_{ijk})$ on $\boldsymbol{\mathcal{X}}$ as in Definition 3.4.33. Note that this type A good coordinate systems exists on $\boldsymbol{\mathcal{X}}$ because of Theorem 3.4.34.

Step (3): Choose nearly complex structures on the good coordinate system and make the 'real' good coordinate system of step (2) into a 'nearly complex' good coordinate system. This can be done by using the techniques of Proposition 5.3.2.

Step(4): For each relevant subgroup $\Delta_i \subseteq \Gamma_i$, choose a tubular neighbourhood $T_i^{\Delta_i}$ of V^{Δ_i} in V_i plus an identification of this tubular neighbourhood with an open neighbourhood of the 0-section in the total space of the normal bundle $T_i^{\Delta_i} \cong U_i^{\Delta_i} \subseteq \mathcal{N}_{V^{\Delta_i}/V_i}$. Moreover, choose a nearly complex structure on $\mathcal{N}_{V_i^{\Delta_i}/V_i}$ and a splitting $E_i|_{T_i^{\Delta_i}} \cong E_i^{\Delta_i,\mathrm{tr}} \oplus E_i^{\Delta_i,\mathrm{nt}}$ of E_i on $T_i^{\Delta_i}$ into trivial and nontrivial Δ_i representations on $V_i^{\Delta_i}$.

This data has to satisfy various compatibility conditions, like the compatibility with coordinate changes of the good coordinate system and the compatibility with change of subgroup.

Step (5): Perturb the section s_i near the 0-section in $T_i^{\Delta_i}$, such that the following conditions are satisfied:

(a) The component of s_i in $E_i^{\Delta_i,\mathrm{nt}}$ is ($\mathbb{C}$-)linear in $\mathcal{N}_{V_i^{\Delta_i}/V_i}$, that is there exist morphisms $\alpha_i \in \mathrm{Hom}_(\mathbb{C})(\mathcal{N}_{V_i^{\Delta_i}/V_i}, E_i^{\Delta_i,\mathrm{nt}})$ such that we have near the 0-section in $T_i^{\Delta_i}$

$$s_i^{\mathrm{nt}}(v,n) = \alpha(n),$$

where $v \in V_i^{\Delta_i}$ and $n \in \mathcal{N}_{V_i^{\Delta_i}/V_i}|_v$.

(b) The perturbation satisfying (a) is compatible with coordinate changes.

(c) The perturbation makes the s_i generic under condition (a), that is the α_i are generic.

Step (6): Define and use a simultaneous toric resolution process for all V_i/Γ_i. The rough idea is to view the standard model d-orbifolds $\boldsymbol{S}_{V,E,s}/\Gamma$ as toric objects, and use a slight modification of the following 'classical' toric resolution theorem (see [13, Theorem 11.1.9] for a proof):

Theorem 9.0.3. *Let X_Σ be a toric variety coming from a fan Σ. Then there exists a refinement Σ' of Σ satisfying the following:*

(i) *Σ' is smooth.*

(ii) *$\Sigma^0 \subseteq (\Sigma')^0$, where $\Sigma^0, (\Sigma')^0$ are the smooth loci of Σ and Σ'.*

(iii) *Σ' is obtained from Σ by a sequence of star subdivisions.*

(iv) *The toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ is a projective resolution of singularities.*

Using the notation of [13], the alteration one has to make to this theorem, is that there is some ambiguity in choosing an element $v \in P_{\sigma_0} \cap \Lambda \setminus \{0\}$ to star-subdivide through.

We want to address this ambiguity and make Theorem 9.0.3 algorithmic to get a step-by-step procedure for resolving toric singularities. Consider therefore a cone σ_0 of maximal multiplicity.[13, Proposition 11.1.8 (ii)] shows that the multiplicity of a cone can be expressed as the number of points in $\mathrm{mult}(\sigma_0) = \#(P_{\sigma_0} \cap \Lambda)$, where $P_\sigma = \{\sum_{i=1}^{d} \lambda_i v_i : 0 \leq \lambda_i < 1\}$. Any element $v \in P_{\sigma_0} \cap \Lambda \setminus \{0\}$ is of the form $v = \sum_{i=1}^{d_0} \lambda_i v_i$, where d_0 denotes the dimension of σ_0 and $0 < \lambda_i < 1$ for all $i = 1, \ldots, d_0$. We can define an order $\leq$ on $P_{\sigma_0} \cap \Lambda$ as follows: for $v = \sum_{i=1}^{d_0} \lambda_i v_i, w = \sum_{i=1}^{d_0} \mu_i v_i \in P_{\sigma_0} \cap \Lambda$ we say $v \leq w$ if and only if $\sum_{i=1}^{d_0} \lambda_i \leq \sum_{i=1}^{d_0} \mu_i$, with equality $v = w$ if and only if $\sum_{i=1}^{d_0} \lambda_i = \sum_{i=1}^{d_0} \mu_i$. It is easy to check that this definition makes $P_{\sigma_0} \cap \Lambda$ into a totally ordered set.

Using $\leq$, we can order the elements in $P_{\sigma_0} \cap \Lambda \setminus \{0\}$. If there exists a unique minimal element $v \in P_{\sigma_0} \cap \Lambda \setminus \{0\}$ (i.e. $v < w$ for all $v \neq w \in P_{\sigma_0} \cap \Lambda \setminus \{0\}$) star subdivide through this v.

Otherwise let $\alpha_1 = \sum_{i=1}^{d_0} \lambda_i^1 v_i, \ldots, \alpha_k = \sum_{i=1}^{d_0} \lambda_i^k v_i \in P_{\sigma_0} \cap \Lambda \setminus \{0\}$ be the elements realising the minimum.

Consider $\beta \in \mathbb{Q}$ minimal, such that $\alpha := \beta \sum_{i=1}^{k} \alpha_i \in \Lambda$. Star subdividing through α, divides the initial cone σ into k different subcones. But not all of the $\alpha_i, i = 1, \ldots, k$ can lie in one new subcone, so we improve the situation in one of the following two ways:

(1) In each newly introduced subcone of σ_0, the number of minimal elements is reduced. The star subdivision through α "separated" therefore the minimal points in σ_0.

(2) Minimal elements of σ_0 lay in a lower dimensional cone $\mathrm{Cone}(\tau, \alpha)$, which also reduces the number of minimal elements.

In both cases we reduced the number of minimal elements, and thus we get rid of the ambiguity in choosing an element to star-subdivide through.

As the toric resolution process can be expresses as a sequence of blowups, the idea is to use the d-blowups defined in §8 to imitate a toric resolution for d-orbifolds.

Step (6): Patch the resulting Kuranishi neighbourhoods together and get a d-manifold (effective d-orbifold).

In the case of $\boldsymbol{\mathcal{X}}$ being an embeddable d-orbifold, we know that $\boldsymbol{\mathcal{X}}$ admits a global Kuranishi neighbourhood, and so the situation could potentially be simplified, as we just have to worry about one "Kuranishi-patch" and do not have to consider step (6) from above for example.

Hence one should be able to prove a "resolution of singularities"-type theorem as follows:

"Theorem" 9.0.4 (Resolution theorem for d-orbifolds)**.** *Let $\boldsymbol{\mathcal{X}}$ be an (embeddable) nearly complex d-orbifold with orbifold strata $\boldsymbol{\mathcal{X}}_\Gamma$. Then there exists a "resolution" $\tilde{\boldsymbol{\mathcal{X}}}$ of $\boldsymbol{\mathcal{X}}$, such that $\tilde{\boldsymbol{\mathcal{X}}}$ is a nearly complex d-manifold (or a nearly complex effective d-orbifold). Moreover $\tilde{\boldsymbol{\mathcal{X}}}$ has the following properties:*

(a) The d-manifold (effective d-orbifold) $\tilde{\boldsymbol{\mathcal{X}}}$ can be defined by an algorithm involving small perturbations and blow-ups (or other resolution-type modifications) of $\boldsymbol{\mathcal{X}}$.

(b) If $\boldsymbol{\mathcal{X}}$ is compact, so is $\tilde{\boldsymbol{\mathcal{X}}}$.

(c) If $\boldsymbol{\mathcal{X}}$ admits a morphism to a manifold $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$, so does $\tilde{\boldsymbol{\mathcal{X}}}$.

(d) If $\boldsymbol{\mathcal{X}}$ is a d-manifold, then we have $\tilde{\boldsymbol{\mathcal{X}}} = \boldsymbol{\mathcal{X}}$.

(e) All of the above is compatible with unitary d-orbifold bordism over $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$. That is, if $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{X}}'$ are compact, (stable) nearly complex d-orbifolds with (stable) nearly complex structures $((\mathcal{E}^\bullet, \phi), J^\bullet, a), ((\tilde{\mathcal{E}}^\bullet, \tilde{\phi}), \tilde{J}^\bullet, \tilde{a})$ and 1-morphisms $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{Y}, \boldsymbol{f}' : \boldsymbol{\mathcal{X}}' \to \boldsymbol{Y}$ which lie in the same unitary bordism class

$$[\boldsymbol{\mathcal{X}}, ((\mathcal{E}^\bullet, \phi), J^\bullet), a, \boldsymbol{f}] = [\boldsymbol{\mathcal{X}}', ((\mathcal{E}'^\bullet, \phi'), J'^\bullet), a', \boldsymbol{f}'],$$

then the resolutions $\tilde{\boldsymbol{\mathcal{X}}}, \tilde{\boldsymbol{\mathcal{X}}}'$ *of* $\boldsymbol{\mathcal{X}}$ *and* $\boldsymbol{\mathcal{X}}'$ *satisfy*

$$[\tilde{\boldsymbol{\mathcal{X}}}, ((\tilde{\mathcal{E}}^\bullet, \tilde{\phi}), \tilde{J}^\bullet), \tilde{a}, \tilde{\boldsymbol{f}}] = [\tilde{\boldsymbol{\mathcal{X}}}', ((\tilde{\mathcal{E}}'^\bullet, \tilde{\phi}'), \tilde{J}'^\bullet), \tilde{a}', \tilde{\boldsymbol{f}}'].$$

In other words, if two compact, (stable) nearly complex d-orbifolds are equal in unitary d-orbifold bordism, then their resolutions are equal in unitary d-manifold bordism (or (semi-)effective unitary d-orbifold bordism).

Moreover the construction of $\tilde{\boldsymbol{\mathcal{X}}}$ *yields a group morphism* $dBU_k^{orb}(Y) \to dBU_k(Y)$ *(or* $dBU_k^{orb}(Y) \to dBU_k^{eff}(Y)$*) which can be composed with the integral virtual cycle map* $dBU_k(Y) \to H_k(Y, \mathbb{Z})$ *(or* $dBU_k^{eff}(Y) \to H_k(Y, \mathbb{Z})$*) to get an integral virtual cycle map* $dBU_k^{orb}(Y) \to H_k(Y, \mathbb{Z})$*.*

As we explained before, we do not claim to have proven the above "theorem", but we feel confident that by following the outlined steps, a proof of "Theorem" 9.0.4 is within reach and that the techniques and results developed in this thesis may be useful to get a step closer in proving the Gopakumar–Vafa integrality conjecture.

Appendix A

Basics of 2-categories

A.1 2-categories

Definition A.1.1. A 2-*category* $\mathfrak{C}$ consists of a proper class of objects $\mathrm{Obj}(\mathfrak{C})$, for all $X, Y \in \mathrm{Obj}(\mathfrak{C})$ a category of morphisms $\mathrm{Hom}(X, Y)$, for all $X \in \mathrm{Obj}(\mathfrak{C})$ an object $\mathrm{id}_X \in \mathrm{Hom}(X, Y)$, the so called *identity* 1-*morphism*, and for all $X, Y, Z \in \mathrm{Obj}(\mathfrak{C})$ a functor $\mu_{X,Y,Z} : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$.

These data has to satisfy the following properties:

(a) *identity property:* $\mu_{X,X,Y}(\mathrm{id}_X, -) = \mu_{X,Y,Y}(-, \mathrm{id}_Y) = \mathrm{id}_{\mathrm{Hom}(X,Y)}$ as functors $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y)$.

(b) *associativity property:* $\mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}_{\mathrm{Hom}(Y,Z)}) = \mu_{W,X,Z} \circ (\mathrm{id}_{\mathrm{Hom}(W,X)} \times \mu_{X,Y,Z})$ as functors $\mathrm{Hom}(W, X) \times \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(W, X)$ for all objects W, X, Y, Z.

Objects f of $\mathrm{Hom}(X, Y)$ are called *1-morphisms* and will be written as $f : X \to Y$. Given two 1-morphisms $f, gX \to Y$, we call morphisms $\eta \in \mathrm{Hom}_{\mathrm{Hom}(X,Y)}(f, g)$ 2-*morphisms* and write $\eta : f \Rightarrow g$.

In a nutshell, a 2-category $\mathfrak{C}$ consists of objects $\mathrm{Obj}(\mathfrak{C})$, 1-morphisms $f : X \to Y$ between objects X, Y, and 2-morphisms $\eta : f \Rightarrow g$ between 1-morphisms.

There are three different compositions in a 2-category:

(1) Let $f : X \to Y$ and $g : Y \to Z$ be 1-morphisms, then $\mu_{X,Y,Z}(f, g)$ is the *horizontal composition of* 1-*morphisms*, written as $g \circ f : X \to Z$.

(2) If $f, g, h : X \to Y$ are 1-morphisms and $\eta : f \Rightarrow g, \zeta : g \Rightarrow h$ are 2-morphisms, then composition of η, ζ in $\mathrm{Hom}(X,Y)$ gives the *vertical composition of 2-morphisms* of η, ζ, written $\zeta \odot \eta : f \Rightarrow h$, or as a diagram

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, "g" description] \arrow[r, bend right, "h"'] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow \zeta \odot \eta"] & Y. \end{tikzcd}$$

(3) If $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are 1-morphisms and $\eta : f \Rightarrow \tilde{f}, \zeta : g \Rightarrow \tilde{g}$ are 2-morphisms then $\mu_{X,Y,Z}(\eta, \zeta)$ is called *horizontal composition of 2-morphisms*, written as $\zeta * \eta : g \circ f \Rightarrow \tilde{g} \circ \tilde{f}$, or as a diagram

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "\tilde{f}"'] \arrow[r, phantom, "\Downarrow \eta"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "\tilde{g}"'] \arrow[r, phantom, "\Downarrow \zeta"] & Z \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, bend left, "g \circ f"] \arrow[r, bend right, "\tilde{g} \circ \tilde{f}"'] \arrow[r, phantom, "\Downarrow \zeta * \eta"] & Z. \end{tikzcd}$$

Moreover there are 2 different kinds of identity in a 2-category, *identity 1-morphisms* $\mathrm{id}_X : X \to X$, and *identity 2-morphisms* $\mathrm{id}_f : f \Rightarrow f$.

In contrast to the 1-category case, there are several notions of when objects X, Y in $\mathfrak{C}$ are "the same":

(i) *equality* $X = Y$,

(ii) *isomorphism*, where two objects $X, Y \in \mathrm{Obj}(\mathfrak{C})$ are called *isomorphic*, if there exist 1-morphisms $f : X \to Y, g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$,

(iii) *equivalence*, where two objects $X, Y \in \mathrm{Obj}(\mathfrak{C})$ are called *equivalent*, if there exist 1-morphism $f : X \to Y, g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$.

From these different notions of "being the same", equivalence is usually the correct notion.

Example A.1.2. The basic example of a 2-category is the category of categories $\mathfrak{Cat}$. Here the objects are categories $\mathcal{C}$, 1-morphisms are functors $F : \mathcal{C} \to \mathcal{D}$ and 2-morphisms are natural transformations $\eta : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$.

A.2 Fibre products in 2-categories

Commutative diagrams in a 2-category $\mathfrak{C}$ should in general only commute up to 2-*isomorphisms* rather than strictly. For example the following commutative diagram

$$\begin{tikzcd} & Y \arrow[d, phantom, "\Downarrow \eta"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & C, \arrow[ul, "g"'] \end{tikzcd}$$

means that X, Y, Z are objects in $\mathfrak{C}$, $f : X \to Y, g : Y \to Z$ and $h : X \to Z$ are 1-morphisms in $\mathfrak{C}$ and $\eta : g \circ f \Rightarrow h$ is a 2-isomorphism.

Moreover, there is also the notion of *fibre product* in a 2-category.

Definition A.2.1. Let $\mathfrak{C}$ be a 2-category and $g : X \to Z, h : Y \to Z$ be 1-morphism in $\mathfrak{C}$. A *fibre product* $X \times_Z Y$ in $\mathfrak{C}$ consists of the following data: an object W, 1-morphisms $\pi_X : W \to X$ and $\pi_Y : W \to Y$ and 2-isomorphisms $\eta : g \circ \pi_X \Rightarrow h \circ \pi_Y$ in $\mathfrak{C}$. These data satisfies the following universal property: Let $\pi'_X : W' \to X$ and $\pi'_Y : W' \to Y$ be 1-morphisms and $\eta' : g \circ \pi'_X \Rightarrow h \circ \pi'_Y$ be a 2-isomorphism. Then there exist a 1-morphism $b : W' \to W$, and 2-isomorphisms $\zeta_X : \pi_X \circ b \Rightarrow \pi'_X, \zeta_Y : \pi_Y \circ b \Rightarrow \pi'_Y$ such that the following diagram of 2-isomorphisms commutes

$$\begin{tikzcd} g \circ \pi_X \circ b \arrow[r, Rightarrow, "\eta * \mathrm{id}_b"] \arrow[d, Rightarrow, "\mathrm{id}_g * \zeta_X"'] & h \circ \pi_Y \circ b \arrow[d, Rightarrow, "\mathrm{id}_h * \zeta_Y"] \\ g \circ \pi'_X \arrow[r, Rightarrow, "\eta'"'] & h \circ \pi'_Y. \end{tikzcd}$$

Moreover, if $\tilde{b}, \tilde{\zeta}_X, \tilde{\zeta}_Y$ are alternative choices of b, ζ_X, ζ_Y, there exists a unique 2-isomorphism $\theta : \tilde{b} \Rightarrow b$ with

$$\tilde{\zeta}_X = \zeta_X \odot (\mathrm{id}_{\pi_X} * \theta) \quad \text{and} \quad \tilde{\zeta}_Y = \zeta_Y \odot (\mathrm{id}_{\pi_Y} * \theta).$$

A.3 2-Commutative Cubes

Consider a 2-commutative cube as follows

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& C \arrow[rr] \arrow[dd] \arrow[ur, phantom, "\eta_{CB}"] \arrow[r, phantom, "\eta_{BE}"] & & D \arrow[dd] \arrow[dl, phantom, "\eta_{DF}"] \\
E \arrow[rr] \arrow[dr] \arrow[ur, phantom, "\eta_{EC}"] & & F \arrow[dr] \arrow[dl, phantom, "\eta_{FG}"] \arrow[r, phantom, "\eta_{GD}"] & \\
& G \arrow[rr] & & H.
\end{tikzcd}
\tag{A.1}
$$

The *composition round the cube condition* is then given by the following identity:

$$
\begin{aligned}
&(\eta_{GD} * \mathrm{id}_{A\to C}) \odot (\mathrm{id}_{G\to H} * \eta_{EC}) \odot (\eta_{FG} * \mathrm{id}_{A\to E}) \odot \\
&(\mathrm{id}_{F\to H} * \eta_{BE}) \odot (\eta_{DF} * \mathrm{id}_{A\to B}) \odot (\mathrm{id}_{D\to H} * \eta_{CB}) \\
&= \mathrm{id}_{(D\to H)\circ(C\to D)\circ(A\to C)},
\end{aligned}
\tag{A.2}
$$

where $\mathrm{id}_{\bullet\to\bullet}$ denotes the respective identity 2-morphisms.

A.4 Splitting Lemma

The following well known and easy to prove lemma is a categorical generalization of the rank-nullity theorem in Linear Algebra.

Lemma A.4.1 (Splitting Lemma). *Let $\mathfrak{C}$ be an abelian category, and let $0 \longrightarrow A \xrightarrow{\;i\;} B \xrightarrow{\;j\;} C \longrightarrow 0$ be a short exact sequence of objects $A, B, C \in \mathrm{Obj}(\mathfrak{C})$. Then the following are equivalent:*

(i) There exists a morphism $p : B \to A$, such that $p \circ i = \mathrm{id}_A$.

(ii) There exists a morphism $q : C \to B$, such that $j \circ q = \mathrm{id}_C$.

(iii) There exists an isomorphisms $B \cong A \oplus C$.

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