A 'twistor transform' for complex manifolds with connection

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In this note we will briefly describe the geometry of a class of complex manifolds, to be called *complex-flat manifolds*, that have a connection ∇ satisfying a curvature condition given in §1, which is the curvature condition satisfied by the Levi-Civita connection of a Kähler manifold. The structure has a sort of twistor transform: in §2, ∇ will be used to define an almost complex structure J on the tangent bundle of X, and it will be shown that J is integrable exactly when the curvature condition holds.

It therefore gives a miniature picture of the Penrose transform for conformal 4-manifolds, where the Cartan conformal connection is used to define a complex structure on a bundle, and the integrability condition is a condition on the conformal curvature. In §3 we give some examples of complex-flat manifolds.

1. Connections, curvature and complex structures

We begin by recalling how to decompose tensors relative to a complex structure I. Let X be a complex manifold, with complex structure I, which will be written with indices as I_j^k with respect to some real coordinate system $(x^1, \ldots, x^{2n})$. Let $K = K^{a\cdots}_{\cdots}$ be a tensor on X, taking values in $\mathbb{C}$. Here a is a contravariant index of K, and any other indices of K are represented by dots. The Greek characters $\alpha, \beta, \gamma, \delta, \epsilon$, and the starred characters $\alpha^*, \beta^*, \gamma^*, \delta^*, \epsilon^*$, will be used in place of the Roman indices a, b, c, d, e respectively. They are tensor indices with respect to $(x^1, \ldots, x^{2n})$ in the normal sense, and their use is actually a shorthand indicating a modification to the tensor itself.

Define $K^{\alpha\cdots}_{\cdots} = (K^{a\cdots}_{\cdots} + iI^a_j K^{j\cdots}_{\cdots})/2$ and $K^{\alpha^*\cdots}_{\cdots} = (K^{a\cdots}_{\cdots} - iI^a_j K^{j\cdots}_{\cdots})/2$. In the same way, if b is a covariant index on a complex-valued tensor $L^{\cdots}_{b\cdots}$, define $L^{\cdots}_{\beta\cdots} = (L^{\cdots}_{b\cdots} - iI^j_b L^{\cdots}_{j\cdots})/2$ and $L^{\cdots}_{\beta^*\cdots} = (L^{\cdots}_{b\cdots} + iI^j_b L^{\cdots}_{j\cdots})/2$. Then $K^{\alpha\cdots}_{\cdots}$ and $L^{\cdots}_{\beta\cdots}$ are the components of K and L that are *complex linear* w.r.t. I, and the starred versions are the components that are *complex antilinear* w.r.t. I. These operations are projections, and satisfy $K^{a\cdots}_{\cdots} = K^{\alpha\cdots}_{\cdots} + K^{\alpha^*\cdots}_{\cdots}$ and $L^{\cdots}_{b\cdots} = L^{\cdots}_{\beta\cdots} + L^{\cdots}_{\beta^*\cdots}$. The complex decomposition of a real-valued tensor is *self-adjoint*. This means that changing round starred and unstarred indices has the same effect as complex conjugation. All the tensors we deal with will be self-adjoint.

Let ∇ be a torsion-free connection on X satisfying $\nabla I = 0$. The connection will be written in the usual way as Γ^a_{bc}, relative to the coordinate system $(x^1, \ldots, x^{2n})$. In this fixed coordinate system, Γ may be decomposed into components relative to I as above, but as Γ is not a tensor this decomposition does depend on the coordinate system. Therefore, we shall consider only coordinate systems $(x^1, \ldots, x^{2n})$ with the property that I is constant in coordinates, i.e. $\partial I^a_b/\partial x^c = 0$ for all a, b, c.

As $\nabla I = 0$ we have $\Gamma^a_{bc} = \Gamma^\alpha_{\beta c} + \Gamma^{\alpha^*}_{\beta^* c}$, and as ∇ is torsion-free $\Gamma^a_{bc} = \Gamma^a_{cb}$. Together these imply that $\Gamma^a_{bc} = \Gamma^\alpha_{\beta\gamma} + \Gamma^{\alpha^*}_{\beta^*\gamma^*}$. Now the curvature $R^a{}_{bcd}$ of ∇ is given by $R^a{}_{bcd} = \partial\Gamma^a_{bd}/\partial x^c - \partial\Gamma^a_{bc}/\partial x^d + \Gamma^a_{jc}\Gamma^j_{bd} - \Gamma^a_{jd}\Gamma^j_{bc}$. Substituting in for Γ gives $R^a{}_{bcd} = R^\alpha{}_{\beta cd} + R^{\alpha^*}{}_{\beta^* cd}$.

Because ∇ is torsion-free, R satisfies the Bianchi identity $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$, and thus $R^\alpha{}_{\beta\gamma^*\delta^*} + R^\alpha{}_{\gamma^*\delta^*\beta} + R^\alpha{}_{\delta^*\beta\gamma^*} = 0$. But from above the last two terms are zero, and so $R^\alpha{}_{\beta\gamma^*\delta^*} = 0$, and similarly $R^{\alpha^*}{}_{\beta^*\gamma\delta} = 0$. Therefore

$$R^a{}_{bcd} = R^\alpha{}_{\beta\gamma\delta} + R^\alpha{}_{\beta\gamma^*\delta} + R^\alpha{}_{\beta\gamma\delta^*} + R^{\alpha^*}{}_{\beta^*\gamma^*\delta^*} + R^{\alpha^*}{}_{\beta^*\gamma^*\delta} + R^{\alpha^*}{}_{\beta^*\gamma\delta^*}. \tag{1}$$

Now by [Bo], Lemma 5, the curvature tensor of a Kähler manifold satisfies

$$R^a{}_{bcd} = R^{\alpha^*}{}_{\beta^*\gamma\delta^*} + R^{\alpha^*}{}_{\beta^*\gamma^*\delta} + R^\alpha{}_{\beta\gamma\delta^*} + R^\alpha{}_{\beta\gamma^*\delta}. \tag{2}$$

So the curvature of the Levi-Civita connection of a Kähler metric satisfies (2), whereas the curvature of a torsion-free $GL(n,\mathbb{C})$- connection ∇ only need satisfy (1), which is weaker. A torsion-free connection ∇ will be called *complex-flat* if $\nabla I = 0$ and its curvature satisfies (2). In fact, as the curvature of ∇ already satisfies (1) it is necessary and sufficient that it should satisfy the additional condition $R^\alpha{}_{\beta\gamma\delta} = 0$.

2. The twistor transform

Let X be a complex manifold, with complex structure I, equipped with a torsion-free connection ∇ satisfying $\nabla I = 0$. The tangent bundle TX of X is naturally a complex manifold, with complex structure also denoted I. Using ∇, a second almost complex structure J will be defined upon the total space of TX, which will turn out to be integrable exactly when $R^\alpha{}_{\beta\gamma\delta} = 0$. So J is a complex structure if and only if ∇ is a complex-flat connection.

Let $x \in X$ and $y \in T_x X$. Then (x,y) is a point in TX. The tangent space $T_{(x,y)}(TX)$ splits naturally into a direct sum $H \oplus V$, where H is the horizontal subspace of the connection ∇ at (x,y), and V is the tangent space of the fibre of TX over x. Now V is closed under I as TX is a holomorphic bundle, and H is closed under I as $\nabla I = 0$. Let v be a vector in $T_{(x,y)}(TX)$. Under the splitting $T_{(x,y)}(TX) = H \oplus V$, we may write $v = (v_1, v_2)$. Define $Jv = (Iv_1, -Iv_2)$ for all vectors v, and for all $x \in X, y \in T_x X$. This defines an almost complex structure J on the total space of TX, commuting with I and projecting down to I on X.

We will write J out explicitly in terms of the connection components Γ, and calculate the Nijenhuis tensor N_J of J, which will give the condition for J to be integrable. Let $(x^1, \ldots, x^{2n})$ be a coordinate system as in §1, for some open set $U \subset X$. Let $(y^1, \ldots, y^{2n})$ be coordinates w.r.t. the basis $(\partial/\partial x^1, \ldots, \partial/\partial x^{2n})$ for the fibres of TU. Then $(x^1, \ldots, x^{2n}, y^1, \ldots, y^{2n})$ are coordinates for TU. In these coordinates, J is

$$J\left(p^a \frac{\partial}{\partial x^a} + q^a \frac{\partial}{\partial y^a}\right) = I^a_b p^b \frac{\partial}{\partial x^a} - I^a_b q^b \frac{\partial}{\partial y^b} - 2 I^d_a \Gamma^a_{bc} y^b p^c \frac{\partial}{\partial y^d}.$$

Decomposing this expression w.r.t. I leads to some simplifications, as we may use the facts that $\Gamma^a_{bc} = \Gamma^\alpha_{\beta\gamma} + \Gamma^{\alpha^*}_{\beta^*\gamma^*}$ and $I^a_b = i\delta^\alpha_\beta - i\delta^{\alpha^*}_{\beta^*}$. So we have

$$\begin{aligned} J\left(p^a \frac{\partial}{\partial x^a} + q^a \frac{\partial}{\partial y^a}\right) &= ip^\alpha \frac{\partial}{\partial x^\alpha} - ip^{\alpha^*} \frac{\partial}{\partial x^{\alpha^*}} - iq^\alpha \frac{\partial}{\partial y^\alpha} + iq^{\alpha^*} \frac{\partial}{\partial y^{\alpha^*}} \\ &\quad - 2i\Gamma^\alpha_{\beta\gamma} y^\beta p^\gamma \frac{\partial}{\partial y^\alpha} + 2i\Gamma^{\alpha^*}_{\beta^*\gamma^*} y^{\beta^*} p^{\gamma^*} \frac{\partial}{\partial y^{\alpha^*}}. \end{aligned}$$

Theorem. *The almost complex structure J is integrable if and only if $R^\alpha{}_{\beta\gamma\delta} = 0$.*

Proof. By the Newlander-Nirenberg theorem, a necessary and sufficient condition for the integrability of J is the vanishing of the Nijenhuis tensor N_J of J, which is given by $N_J(v,w) = [v,w] + J\big([Jv,w] + [v,Jw]\big) - [Jv,Jw]$. We shall evaluate N_J with $v = p^a\partial/\partial x^a + q^a\partial/\partial y^a$ and $w = r^a\partial/\partial x^a + s^a\partial/\partial y^a$, where p^a, q^a, r^a and s^a are constants independent of x^a, y^a. It is easy to see that $[v,w] = 0$. Using the fact that J acts as $-I$ on V, one calculates that

$$J([Jv,w]) = 2r^d\frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^d}y^\beta p^\gamma\frac{\partial}{\partial y^\alpha} + 2r^d\frac{\partial\Gamma^{\alpha^*}_{\beta^*\gamma^*}}{\partial x^d}y^{\beta^*}p^{\gamma^*}\frac{\partial}{\partial y^{\alpha^*}} + 2\Gamma^a_{bc}s^bp^c\frac{\partial}{\partial y^a},$$

$$J([v,Jw]) = -2p^d\frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^d}y^\beta r^\gamma\frac{\partial}{\partial y^\alpha} - 2p^d\frac{\partial\Gamma^{\alpha^*}_{\beta^*\gamma^*}}{\partial x^d}y^{\beta^*}r^{\gamma^*}\frac{\partial}{\partial y^{\alpha^*}} - 2\Gamma^a_{bc}q^br^c\frac{\partial}{\partial y^a},$$

and $[Jv,Jw] =$

$$\begin{aligned}
&\left(ip^\delta\frac{\partial}{\partial x^\delta} - ip^{\delta^*}\frac{\partial}{\partial x^{\delta^*}}\right)\left(-2i\Gamma^\alpha_{\beta\gamma}y^\beta r^\gamma\frac{\partial}{\partial y^\alpha} + 2i\Gamma^{\alpha^*}_{\beta^*\gamma^*}y^{\beta^*}r^{\gamma^*}\frac{\partial}{\partial y^{\alpha^*}}\right)\\
&-\left(ir^\delta\frac{\partial}{\partial x^\delta} - ir^{\delta^*}\frac{\partial}{\partial x^{\delta^*}}\right)\left(-2i\Gamma^\alpha_{\beta\gamma}y^\beta p^\gamma\frac{\partial}{\partial y^\alpha} + 2i\Gamma^{\alpha^*}_{\beta^*\gamma^*}y^{\beta^*}p^{\gamma^*}\frac{\partial}{\partial y^{\alpha^*}}\right)\\
&-4\Gamma^\delta_{\beta\gamma}y^\beta p^\gamma\Gamma^\alpha_{\delta\epsilon}r^\epsilon\frac{\partial}{\partial y^\alpha} - 4\Gamma^{\delta^*}_{\beta^*\gamma^*}y^{\beta^*}p^{\gamma^*}\Gamma^{\alpha^*}_{\delta^*\epsilon^*}r^{\epsilon^*}\frac{\partial}{\partial y^{\alpha^*}}\\
&+4\Gamma^\delta_{\beta\gamma}y^\beta r^\gamma\Gamma^\alpha_{\delta\epsilon}p^\epsilon\frac{\partial}{\partial y^\alpha} + 4\Gamma^{\delta^*}_{\beta^*\gamma^*}y^{\beta^*}r^{\gamma^*}\Gamma^{\alpha^*}_{\delta^*\epsilon^*}p^{\epsilon^*}\frac{\partial}{\partial y^{\alpha^*}}\\
&-2\Gamma^\alpha_{\beta\gamma}q^\beta r^\gamma\frac{\partial}{\partial y^\alpha} - 2\Gamma^{\alpha^*}_{\beta^*\gamma^*}q^{\beta^*}r^{\gamma^*}\frac{\partial}{\partial y^{\alpha^*}} + 2\Gamma^\alpha_{\beta\gamma}s^\beta p^\gamma\frac{\partial}{\partial y^\alpha} + 2\Gamma^{\alpha^*}_{\beta^*\gamma^*}s^{\beta^*}p^{\gamma^*}\frac{\partial}{\partial y^{\alpha^*}}.
\end{aligned}$$

Combining the above gives

$$N_J(v,w) = 4R^\alpha{}_{\beta\gamma\delta}y^\beta r^\gamma p^\delta\frac{\partial}{\partial y^\alpha} + 4R^{\alpha^*}{}_{\beta^*\gamma^*\delta^*}y^{\beta^*}r^{\gamma^*}p^{\delta^*}\frac{\partial}{\partial y^{\alpha^*}},$$

using the expression for R in §1. As this holds for all v, w and y^a for each fixed x, $N_J = 0$ identically if and only if $R^\alpha{}_{\beta\gamma\delta} = 0$. □

3. Examples

The simplest examples of complex-flat manifolds are Kähler manifolds, taking ∇ to be the Levi-Civita connection of the Kähler metric. However, there are many other examples of complex-flat manifolds with no compatible Kähler metric. We shall comment briefly on three such families. Firstly, using the work of [J] for hypercomplex manifolds it is possible to define a quotient construction for complex-flat manifolds analogous to the Kähler quotient. Starting with a flat complex-flat structure one may produce non-Kähler complex-flat structures by choosing a moment map not compatible with any Kähler metric.

Another way of constructing examples is to consider complex submanifolds of complex-flat manifolds. To induce a connection on the tangent bundle of a submanifold M of X we need a splitting $TX|_M = TM \oplus V$ for some vector bundle V; for the induced connection to be complex-flat, it turns out that V must be a holomorphic subbundle w.r.t. J. In the case, say, of projective varieties in $X = \mathbb{CP}^n$, there may be many different choices of V satisfying this condition, and each will give a distinct complex-flat connection on M.

Our final family of examples are hypercomplex manifolds. A hypercomplex manifold is a manifold M^{4n} with complex structures I_1, I_2 and I_3 satisfying $I_1 I_2 = I_3$. By [S], §6, there is a unique connection ∇ on M called the Obata connection, that is torsion-free and satisfies $\nabla I_j = 0$. We shall show that ∇ is a complex-flat connection for each of the complex structures I_1, I_2, I_3. Thus hypercomplex manifolds are examples of complex-flat structures that in general do not come from Kähler structures.

Proposition. *Let M and ∇ be as above. Then the curvature $R^a{}_{bcd}$ of ∇ satisfies $R^\alpha{}_{\beta\gamma\delta} = 0$ in the complex decomposition with respect to each complex structure I_j. Thus (M, I_j, ∇) is a complex-flat manifold.*

Proof. We shall prove the result for I_1, for by symmetry it then holds for I_2, I_3. As ∇ is torsion-free and $\nabla I_2 = 0$, from §1 the curvature R satisfies $R^a{}_{bcd} = R^\alpha{}_{\beta cd} + R^{\alpha^*}{}_{\beta^* cd}$ in the complex decomposition w.r.t. I_2, and so $R^a{}_{bcd} = -(I_2)^a_j (I_2)^k_b R^j{}_{kcd}$. Also, from §1 the component $R^{\alpha^*}{}_{\beta^*\gamma\delta}$ is zero in the complex decomposition w.r.t. I_1. Therefore

$$\begin{aligned} 0 &= (1 - iI_1)^a_p (1 + iI_1)^q_b (1 - iI_1)^r_c (1 - iI_1)^s_d R^p{}_{qrs} \\ &= (1 - iI_1)^a_p (1 + iI_1)^q_b (1 - iI_1)^r_c (1 - iI_1)^s_d (I_2)^p_j (I_2)^k_q R^j{}_{krs} \\ &= (I_2)^a_j (I_2)^k_b (1 + iI_1)^j_p (1 - iI_1)^q_k (1 - iI_1)^r_c (1 - iI_1)^s_d R^p{}_{qrs}, \end{aligned}$$

where $I_1 I_2 = -I_2 I_1$ is used in the last line. So

$$\frac{1}{16}(1 + iI_1)^a_p (1 - iI_1)^q_b (1 - iI_1)^r_c (1 - iI_1)^s_d R^p{}_{qrs} = R^\alpha{}_{\beta\gamma\delta} = 0 \tag{3}$$

in the complex decomposition w.r.t. I_1, which is the condition for (I_1, ∇) to be a complex-flat structure on M. $\square$

Thus the results of §2 apply to hypercomplex manifolds, and lead to some new ideas about the Obata connection and complex structures on the tangent and cotangent bundles of a hypercomplex manifold.

References

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