D-manifolds and d-orbifolds: a theory of derived differential geometry. I.

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UK-Japan Mathematical Forum, July 2012.

Based on survey paper:

arXiv:1206.4207, 44 pages

and preliminary version of book which may be downloaded from

`people.maths.ox.ac.uk/`

`~joyce/dmanifolds.html.`

These slides available at

`people.maths.ox.ac.uk/~joyce/talks.html.`

1. Introduction

Many important areas in both differential and algebraic geometry involve forming ‘moduli spaces’ $\mathcal{M}$ of some geometric objects, and then ‘counting’ the points in $\mathcal{M}$ to get an ‘invariant’ $I(\mathcal{M})$ with interesting properties, for example Donaldson, Seiberg–Witten, Gromov–Witten and Donaldson–Thomas invariants. Taking the ‘invariant’ to be a vector space, category, . . . , rather than a number, Floer homology theories, contact homology, Symplectic Field Theory, and Fukaya categories also fit in this framework.

All these ‘invariants’ theories have some common features:

- You start with some geometrical space X you want to study.
- You define a moduli space $\mathcal{M}$ of auxiliary geometric objects E on X.
- This $\mathcal{M}$ is a topological space, hopefully compact and Hausdorff, but generally not a manifold – it may have bad singularities.
- Nevertheless, $\mathcal{M}$ behaves *as if* it is a compact, oriented manifold of known dimension k. One defines a *virtual class* $[\mathcal{M}]_{\text{vir}}$ in $H_k(\mathcal{M};\mathbb{Q})$, which ‘counts’ the points in $\mathcal{M}$.
- This $[\mathcal{M}]_{\text{vir}}$ is then independent of choices in the construction, deformations of X etc.

Methods for defining $[\mathcal{M}]_{\text{vir}}$ vary. In good cases, with generic initial data $\mathcal{M}$ is smooth. Otherwise, we prove $\mathcal{M}$ has some extra geometric structure $\mathcal{G}$, and use $\mathcal{G}$ to define $[\mathcal{M}]_{\text{vir}}$.

• In algebraic geometry problems $\mathcal{M}$ is a scheme or Deligne–Mumford stack with obstruction theory.

• In areas of symplectic geometry based on moduli of J-holomorphic curves – Gromov–Witten theory, Lagrangian Floer cohomology, Symplectic Field Theory, Fukaya categories – there are two main geometric structures: *Kuranishi spaces* (Fukaya–Oh–Ohta–Ono) and *polyfolds* (Hofer–Wysocki–Zehnder).

2. D-manifolds and d-orbifolds

I will describe a new class of geometric objects I call *d-manifolds* — 'derived' smooth manifolds. Some properties of d-manifolds:

• They form a *strict 2-category* $\mathbf{dMan}$. That is, we have objects $\boldsymbol{X}$, the d-manifolds, 1-morphisms $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$, the smooth maps, and also 2-morphisms $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$.

• Smooth manifolds embed into d-manifolds as a full (2)-subcategory.

• There are also 2-categories $\mathbf{dMan^b}$ of d-manifolds *with boundary* and $\mathbf{dMan^c}$ of *d-manifolds with corners*, and orbifold versions $\mathbf{dOrb}, \mathbf{dOrb^b}$, $\mathbf{dOrb^c}$ of these, *d-orbifolds*.

• Many concepts of differential geometry extend nicely to d-manifolds: submersions, immersions, orientations, submanifolds, transverse fibre products, cotangent bundles, ...

• Almost any moduli space used in any enumerative invariant problem over $\mathbb{R}$ or $\mathbb{C}$ has a d-manifold or d-orbifold structure, natural up to equivalence. There are truncation functors to d-manifolds and d-orbifolds from structures currently used – $\mathbb{C}$-schemes with obstruction theories, Kuranishi spaces, polyfolds.

• Virtual classes/cycles/chains can be constructed for compact oriented d-manifolds and d-orbifolds.

So, d-manifolds and d-orbifolds provide a unified framework for studying enumerative invariants and moduli spaces. They also have other applications, and are interesting and beautiful in their own right.

D-manifolds and d-orbifolds are related to other classes of spaces already studied, in particular to the *Kuranishi spaces* of Fukaya–Oh–Ohta–Ono in symplectic geometry, and to David Spivak's *derived manifolds*, from Jacob Lurie's 'derived algebraic geometry' programme.

2.1. Kuranishi spaces

Kuranishi spaces were defined by Fukaya–Ono 1999 and Fukaya–Oh–Ohta–Ono 2009 as the geometric structure on moduli spaces $\mathcal{M}$ of J-holomorphic curves in symplectic geometry. A Kuranishi space is locally modelled on the zeroes $s^{-1}(0)$ of a smooth section s of a vector bundle $E \to V$ over an orbifold V.

The theory has problems, and is basically incomplete.

My starting point for this project was to find the 'right' definition of Kuranishi space. I claim that this is: *a Kuranishi space is (should really be) a d-orbifold with corners*.

2.2. Derived manifolds

Derived manifolds were defined by David Spivak (Duke Math. J. 153, 2010), a student of Jacob Lurie. A lot of my ideas are stolen from Spivak. D-manifolds are much simpler than derived manifolds. D-manifolds are a 2-category, using Hartshorne-level algebraic geometry. Derived manifolds are an ∞-category, and use very advanced and scary technology – homotopy sheaves, Bousfeld localization, ...

D-manifolds are (roughly) a 2-category truncation of derived manifolds. I claim that this truncation remembers all the geometric information of importance to symplectic geometers, and other real people.

2.3. Why should $\mathbf{dMan}$ be a 2-category?

Here are two reasons why any class of 'derived manifolds' should be (at least) a 2-category. Firstly, one property we want of $\mathbf{dMan}$ is that it contains manifolds $\mathbf{Man}$ as a subcategory, and if X, Y, Z are manifolds and $g : X \to Z$, $h : Y \to Z$ are smooth then a fibre product $\boldsymbol{W} = X \times_{g,Z,h} Y$ should exist in $\mathbf{dMan}$, characterized by a universal property in $\mathbf{dMan}$, and should be a d-manifold of 'virtual dimension'

$$\operatorname{vdim} \boldsymbol{W} = \dim X + \dim Y - \dim Z.$$

Note that g, h need not be transverse, and $\operatorname{vdim} \boldsymbol{W}$ may be negative.

Consider the case $X = Y = *$, the point, $Z = \mathbb{R}$, and $g, h : * \mapsto 0$. If $\mathbf{dMan}$ were an ordinary category then as $*$ is a terminal object, the unique fibre product $* \times_{0,\mathbb{R},0} *$ would be $*$. But this has virtual dimension 0, not -1. So $\mathbf{dMan}$ must be some kind of higher category.

Secondly, two approximations for $\mathbf{dMan}$ are $\mathbb{C}$-schemes X with obstruction theory, and quasi-smooth dg-schemes. Both of these include a 'cotangent complex' in $D^b\,\mathrm{coh}(X)$ concentrated in two degrees $-1, 0$. It seems reasonable to capture the behaviour of such complexes in a 2-category.

2.4. Why is a 2-category enough?

Usually in derived algebraic geometry, one considers an ∞-category of objects (derived stacks, etc.). But we work in a 2-category, effectively a truncation of Spivak's ∞-category of derived manifolds.

Here are two reasons why this truncation does not lose important information. Firstly, d-manifolds correspond to *quasi-smooth* derived schemes X, whose cotangent complex $\mathbb{L}_X$ lies in degrees $[-1,0]$. So $\mathbb{L}_X$ lies in a 2-category of complexes, not an ∞-category. Note that $f : X \to Y$ is étale in $\mathbf{dMan}$ iff $\Omega_f : f^*(\mathbb{L}_Y) \to \mathbb{L}_X$ is an equivalence.

Secondly, the existence of *partitions of unity* in differential geometry means that our structure sheaves $\mathcal{O}_{\boldsymbol{X}}$ are 'fine' or 'soft', which simplifies behaviour. Partitions of unity are also essential in gluing by equivalences in $\mathbf{dMan}$ (see Theorem 7, lecture 2). Our '2-category style derived geometry' probably would not work very well in a conventional algebro-geometric context, rather than a differential-geometric one.

3. The definition of d-manifolds

Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes.

In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces.

There is a little-known theory of schemes in differential geometry, C^∞-*schemes*, going back to Lawvere, Dubuc, Moerdijk and Reyes, . . . in synthetic differential geometry in the 1960s-1980s. This will be the foundation of our d-manifolds.

3.1. C^∞-rings

Let X be a manifold, and $C^\infty(X)$ the set of smooth functions $c : X \to \mathbb{R}$. Then $C^\infty(X)$ is an $\mathbb{R}$-*algebra*, by adding and multiplying smooth functions. But there are many more operations on $C^\infty(X)$, e.g. if $c : X \to \mathbb{R}$ is smooth then $\exp(c) : X \to \mathbb{R}$ is smooth, giving $\exp : C^\infty(X) \to C^\infty(X)$, algebraically independent of addition and multiplication.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by

$$\Phi_f(c_1, \ldots, c_n)(x) = f\bigl(c_1(x), \ldots, c_n(x)\bigr)$$

for all $x \in X$. Addition comes from $f : \mathbb{R}^2 \to \mathbb{R}$, $f : (c_1, c_2) \mapsto c_1 + c_2$, multiplication from $(c_1, c_2) \mapsto c_1 c_2$.

Definition. A C^∞-*ring* is a set $\mathfrak{C}$ together with n-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying the following conditions:
Let $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathfrak{C}$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have $\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j$.
A *morphism* of C^∞-rings is $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Then $C^\infty(X)$ is a C^∞-ring for any manifold X, and from $C^\infty(X)$ we can recover X up to isomorphism. If $f : X \to Y$ is smooth then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings. This gives a *full and faithful functor* $F : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\mathbf{op}}$ by $F : X \mapsto C^\infty(X)$, $F : f \mapsto f^*$. Thus, we think of manifolds as examples of C^∞-rings, and C^∞-rings as generalizations of manifolds. But there are many more C^∞-rings than manifolds, e.g. $C^0(X)$ is a C^∞-ring for any topological space X.

3.2. C^∞-schemes

We can now develop the whole machinery of scheme theory in algebraic geometry, replacing rings or algebras by C^∞-rings throughout — see my arXiv:1001.0023.

We obtain a category $\mathbf{C^\infty Sch}$ of C^∞-*schemes* $\underline{X} = (X, \mathcal{O}_X)$, which are topological spaces X equipped with a sheaf of C^∞-rings $\mathcal{O}_X$ locally modelled on the spectrum of a C^∞-ring.

If X is a manifold, define a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ by $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. This defines a full and faithful embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$.

We also define *vector bundles*, *coherent sheaves* coh($\underline{X}$) and *quasi-coherent sheaves* qcoh($\underline{X}$), and the *cotangent sheaf* $T^*\underline{X}$ on $\underline{X}$. Then qcoh($\underline{X}$) is an abelian category.

Some differences with conventional algebraic geometry:

• affine schemes are Hausdorff. No need to introduce étale topology.

• partitions of unity exist subordinate to any open cover of a (nice) C^∞-scheme $\underline{X}$.

• C^∞-rings such as $C^\infty(\mathbb{R}^n)$ are not noetherian as $\mathbb{R}$-algebras. Causes problems with coherent sheaves: coh($\underline{X}$) is not closed under kernels, so not an abelian category.

3.3. The 2-category of d-spaces

We define d-manifolds as a 2-subcategory of a larger 2-category of *d-spaces*. These are 'derived' versions of C^∞-schemes.

Definition. A *d-space* is a is a quintuple $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ where $\underline{X} = (X, \mathcal{O}_X)$ is a separated, second countable, locally fair C^∞-scheme, $\mathcal{O}'_X$ is a second sheaf of C^∞-rings on X, and $\mathcal{E}_X$ is a quasicoherent sheaf on $\underline{X}$, and $\imath_X : \mathcal{O}'_X \to \mathcal{O}_X$ is a surjective morphism of sheaves of C^∞-rings whose kernel $\mathcal{I}_X$ is a sheaf of *square zero ideals* in $\mathcal{O}'_X$, and $\jmath_X : \mathcal{E}_X \to \mathcal{I}_X$ is a surjective morphism in qcoh($\underline{X}$), so we have an exact sequence of sheaves on X:

$$\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{O}'_X \xrightarrow{\imath_X} \mathcal{O}_X \longrightarrow 0.$$

A 1-*morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a triple $\boldsymbol{f} = (\underline{f}, f', f'')$, where $\underline{f} = (f, f^\sharp) : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes and $f' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$, $f'' : \underline{f}^*(\mathcal{E}_Y) \to \mathcal{E}_X$ are sheaf morphisms such that the following commutes:

$$\begin{array}{ccccccc} f^{-1}(\mathcal{E}_Y) & \xrightarrow{f^{-1}(\jmath_Y)} & f^{-1}(\mathcal{O}'_Y) & \xrightarrow{f^{-1}(\imath_Y)} & f^{-1}(\mathcal{O}_Y) & \longrightarrow & 0 \\ \downarrow f'' & & \downarrow f' & & \downarrow f^\sharp & & \\ \mathcal{E}_X & \xrightarrow{\jmath_X} & \mathcal{O}'_X & \xrightarrow{\imath_X} & \mathcal{O}_X & \longrightarrow & 0. \end{array}$$

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms with $\boldsymbol{f} = (\underline{f}, f', f'')$, $\boldsymbol{f} = (\underline{g}, g', g'')$. Suppose $\underline{f} = \underline{g}$. A 2-*morphism* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a morphism

$$\eta : f^{-1}(\Omega_{\mathcal{O}'_Y}) \otimes_{f^{-1}(\mathcal{O}'_Y)} \mathcal{O}_X \longrightarrow \mathcal{E}_X$$

in qcoh($\underline{X}$), where $\Omega_{\mathcal{O}'_Y}$ is the sheaf of cotangent modules of $\mathcal{O}'_Y$, such that $g' = f' + \jmath_X \circ \eta \circ \Pi_{XY}$ and $g'' = f'' + \eta \circ \underline{f}^*(\phi_Y)$, for natural morphisms Π_{XY}, ϕ_Y.

Theorem 1. *This defines a strict* 2*-category* $\mathbf{dSpa}$*. All fibre products exist in* $\mathbf{dSpa}$.

We can map $\mathbf{C^\infty Sch}$ into $\mathbf{dSpa}$ by taking a C^∞-scheme $\underline{X}=(X,\mathcal{O}_X)$ to the d-space $\boldsymbol{X}=(\underline{X},\mathcal{O}_X,0,\mathrm{id}_{\mathcal{O}_X},0)$, with exact sequence

$$0 \xrightarrow{\;0\;} \mathcal{O}_X \xrightarrow{\mathrm{id}_{\mathcal{O}_X}} \mathcal{O}_X \longrightarrow 0.$$

This embeds $\mathbf{C^\infty Sch}$, and hence manifolds $\mathbf{Man}$, as discrete 2-subcategories of $\mathbf{dSpa}$. For *transverse* fibre products of manifolds, the fibre products in $\mathbf{Man}$ and $\mathbf{dSpa}$ agree. But if $g:X\rightarrow Z$, $h:Y\rightarrow Z$ are non-transverse smooth maps of manifolds, then the fibre product $\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ in $\mathbf{dSpa}$ is not (equivalent to) a manifold.

3.4. The 2-subcategory of d-manifolds

Definition. A d-space $\boldsymbol{W}$ is a *principal d-manifold* if it is a fibre product $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$ in $\mathbf{dSpa}$, where $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are manifolds without boundary. The *virtual dimension* of $\boldsymbol{W}$ is $\mathrm{vdim}\,\boldsymbol{W}=\dim\boldsymbol{X}+\dim\boldsymbol{Y}-\dim\boldsymbol{Z}$. (We can have $\mathrm{vdim}\,\boldsymbol{W}<0$.)

A d-space $\boldsymbol{X}$ is a *d-manifold of dimension* $n\in\mathbb{Z}$ if $\boldsymbol{X}$ may be covered by open d-subspaces $\boldsymbol{W}\subset\boldsymbol{X}$ which are principal d-manifolds with $\mathrm{vdim}\,\boldsymbol{W}=n$.

Write $\mathbf{dMan}$ for the full 2-subcategory of d-manifolds in $\mathbf{dSpa}$.

Think of a d-manifold $\boldsymbol{X}=(\underline{X},\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X)$ as a 'classical' C^∞-scheme $\underline{X}$, with extra 'derived' data $\mathcal{O}'_X,\mathcal{E}_X,\imath_X,\jmath_X$. The extra information in the 'derived' data is like a vector bundle on $\underline{X}$.

The 2-subcategory $\mathbf{dMan}$ is not closed under fibre products in $\mathbf{dSpa}$, but we can say:

Theorem 2. *All fibre products of the form* $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$ *with* $\boldsymbol{X},\boldsymbol{Y}$ *d-manifolds and* $\boldsymbol{Z}$ *a manifold exist in the* 2*-category* $\mathbf{dMan}$*.*

This is a very useful property of d-manifolds and d-orbifolds. For example, moduli spaces $\bar{\boldsymbol{\mathcal{M}}}_k(\gamma)$ of J-holomorphic curves Σ in a symplectic manifold (M,ω) with boundary in a Lagrangian L and k boundary marked points with $[\Sigma]=\gamma\in H_2(M,L;\mathbb{Z})$ are d-orbifolds with boundary satisfying

$$\partial\bar{\boldsymbol{\mathcal{M}}}_k(\gamma)=\coprod_{i+j=k,\,\alpha+\beta=\gamma}\bar{\boldsymbol{\mathcal{M}}}_{i+1}(\alpha)\times_{\boldsymbol{L}}\bar{\boldsymbol{\mathcal{M}}}_{j+1}(\beta),$$

and this formula makes sense as d-orbifold fibre products over the manifold $\boldsymbol{L}$ exist.