D-manifolds and d-orbifolds: a theory of derived differential geometry. II.

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and preliminary version of book which may be downloaded from

people.maths.ox.ac.uk/

∼joyce/dmanifolds.html.

These slides available at

people.maths.ox.ac.uk/∼joyce/talks.html.

4. Properties of d-manifolds

4.1. Virtual vector bundles

Vector bundle and cotangent bundles have good 2-category generalizations. Let $\underline{X}$ be a C^∞-scheme. Define a 2-category vqcoh($\underline{X}$) of *virtual quasicoherent sheaves* to have objects morphisms $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ in qcoh($\underline{X}$). If $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ and $\psi : \mathcal{F}^1 \to \mathcal{F}^2$ are objects, a 1-morphism $(f^1, f^2) : \phi \to \psi$ is morphisms $f^j : \mathcal{E}^j \to \mathcal{F}^j$ in qcoh($\underline{X}$) for $j = 1, 2$ with $\psi \circ f^1 = f^2 \circ \phi$. If $(f^1, f^2), (g^1, g^2)$ are 1-morphisms $\phi \to \psi$, a 2-morphism $\eta : (f^1, f^2) \Rightarrow (g^1, g^2)$ is a morphism $\eta : \mathcal{E}^2 \to \mathcal{F}^1$ in qcoh($\underline{X}$) with $g^1 = f^1 + \eta \circ \phi$ and $g^2 = f^2 + \psi \circ \eta$.

Call $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ a *virtual vector bundle* on $\underline{X}$ of rank $k \in \mathbb{Z}$ if X may be covered by open $\underline{U} \subseteq \underline{X}$ such that $\phi|_{\underline{U}} : \mathcal{E}^1|_{\underline{U}} \to \mathcal{E}^2|_{\underline{U}}$ is equivalent in the 2-category vqcoh($\underline{U}$) to $\psi : \mathcal{F}^1 \to \mathcal{F}^2$, where $\mathcal{F}^1, \mathcal{F}^2$ are vector bundles on $\underline{U}$ with $\operatorname{rank} \mathcal{F}^2 - \operatorname{rank} \mathcal{F}^1 = k$. Write vvect($\underline{X}$) for the full 2-subcategory of virtual vector bundles on vqcoh($\underline{X}$).

If $\boldsymbol{X}$ is a d-manifold, it has a natural *virtual cotangent bundle* $T^*\boldsymbol{X}$ in vvect($\underline{X}$), of rank $\operatorname{vdim} \boldsymbol{X}$.

If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in $\mathbf{dMan}$, there is a natural 1-morphism $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ in vvect($\underline{X}$).

Then f is *étale* (a local equivalence) if and only if $\Omega_{\boldsymbol{f}}$ is an equivalence in vvect($\underline{X}$). Similarly, f is an *immersion* or *submersion* if $\Omega_{\boldsymbol{f}}$ is surjective or injective in a suitable sense. If $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ lies in vvect($\underline{X}$) we can define a line bundle $\mathcal{L}_\phi$ on $\underline{X}$ analogous to the 'top exterior power' of $\phi : \mathcal{E}^1 \to \mathcal{E}^2$. So for a d-manifold $\boldsymbol{X}$, $\mathcal{L}_{T^*\boldsymbol{X}}$ is a line bundle on $\underline{X}$ which we think of as $\Lambda^{\text{top}} T^*\boldsymbol{X}$. An *orientation* on $\boldsymbol{X}$ is an orientation on the line bundle $\mathcal{L}_{T^*\boldsymbol{X}}$. Orientations have the properties one would expect from the manifold case.

4.2 'Standard model' d-manifolds

Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section. Then we can define an explicit principal d-manifold $\boldsymbol{S}_{V,E,s}$ in a 2-Cartesian diagram in $\mathbf{dMan}$:

$$\begin{array}{ccc} \boldsymbol{S}_{V,E,s} & \xrightarrow[\boldsymbol{\pi}]{} & \boldsymbol{V} \\ \Big\downarrow \boldsymbol{\pi} & \eta \Uparrow & \boldsymbol{0} \Big\downarrow \\ \boldsymbol{V} & \xrightarrow{\boldsymbol{s}} & \boldsymbol{E}. \end{array}$$

We call $\boldsymbol{S}_{V,E,s}$ a 'standard model' d-manifold. It is similar to *Kuranishi neighbourhoods* in Fukaya–Oh–Ohta-Ono's Kuranishi spaces. It has dimension $\operatorname{vdim} \boldsymbol{S}_{V,E,s} = \dim V - \operatorname{rank} E$. Every principal d-manifold is equivalent to some $\boldsymbol{S}_{V,E,s}$.

4.3 'Standard model' 1-morphisms

Let V,W be manifolds, $E \to V$, $F \to W$ vector bundles, and $s : V \to E$, $t : W \to F$ smooth sections, so we have d-manifolds $\boldsymbol{S}_{V,E,s}$, $\boldsymbol{S}_{W,F,t}$.

Suppose $f : V \to W$ is a smooth, and $\hat{f} : E \to f^*(F)$ is a morphism of vector bundles on V satisfying $\hat{f} \circ s = f^*(t) + O(s^2)$ in $C^\infty(f^*(F))$. Then we define a 'standard model' 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$.

Two 1-morphisms $\boldsymbol{S}_{f,\hat{f}}, \boldsymbol{S}_{g,\hat{g}}$ are equal iff $g = f + O(s^2)$ and $\hat{g} = \hat{f} + O(s)$.

Theorem 3. *Every* 1-*morphism* $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ *is of the form* $\boldsymbol{S}_{f,\hat{f}}$, *possibly after making* V *smaller.*

Theorem 4. *A 'standard model' 1-morphism* $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \longrightarrow \boldsymbol{S}_{W,F,t}$ *is étale (a local equivalence) in* $\mathbf{dMan}$ *iff for each* $v \in V$ *with* $s(v) = 0$ *and* $w = f(v) \in W$, *the following sequence is exact:*

$$0 \longrightarrow T_v V \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} E_v \oplus T_w W \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)} F_w \longrightarrow 0.$$

$\boldsymbol{S}_{f,\hat{f}}$ *is an equivalence iff also* $f|_{s^{-1}(0)} : s^{-1}(0) \rightarrow t^{-1}(0)$ *is a bijection.*

Example: in Kuranishi spaces, a 'coordinate change' $(f, \hat{f}) : (V, E, s) \rightarrow (W, F, t)$ is embeddings $f : V \hookrightarrow W$ and $\hat{f} : E \hookrightarrow f^*(F)$ with $\hat{f} \circ s = f^*(t)$, $f^*(TW)/TV \cong f^*(F)/E$. Theorem 4 shows $\boldsymbol{S}_{f,\hat{f}}$ is étale, or an equivalence.

4.4 'Standard model' 2-morphisms

Let $\boldsymbol{S}_{V,E,s}, \boldsymbol{S}_{W,F,t}$ be 'standard model' d-manifolds, and $\boldsymbol{S}_{f,\hat{f}}, \boldsymbol{S}_{g,\hat{g}} : \boldsymbol{S}_{V,E,s} \rightarrow \boldsymbol{S}_{W,F,t}$ 'standard model' 1-morphisms. Suppose $\Lambda : E \rightarrow f^*(TW)$ is a morphism of vector bundles on V, with $g = f + \Lambda \cdot s + O(s^2)$ and $\hat{g} = \hat{f} + \Lambda \cdot f^*(\mathrm{d}t) + O(s)$. Then we can define a 'standard model' 2-morphism $\boldsymbol{S}_{\Lambda} : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$. Every 2-morphism $\eta : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ is $\boldsymbol{S}_{\Lambda}$ for some Λ. Also $\boldsymbol{S}_{\Lambda} = \boldsymbol{S}_{\Lambda'}$ iff $\Lambda' = \Lambda + O(s)$.

These 'standard models' give a very explicit picture of objects, 1- and 2-morphisms in $\mathbf{dMan}$. The $O(s), O(s^2)$ notation tells you how much information about V, E, s the d-manifolds and morphisms remember.

4.5 When is a d-manifold a principal d-manifold?

Theorem 4. *A d-manifold $\boldsymbol{X}$ is principal (that is, $\boldsymbol{X}$ is equivalent in* $\mathbf{dMan}$ *to some $\boldsymbol{S}_{V,E,s}$) iff* $\dim T_x^*\underline{X}$ *is bounded above for all $x \in \underline{X}$.*

This holds if $\boldsymbol{X}$ is compact, giving:

Corollary. *All compact d-manifolds are principal.*

Essentially this means that all interesting d-manifolds are principal d-manifolds. The analogue is *not* true for d-orbifolds.

Example $\boldsymbol{X} = \coprod_{n\geqslant 0} \mathbb{R}^n \times_{\mathbf{0},\mathbb{R}^n,\mathbf{0}} *$ is a d-manifold of dimension 0, but is not principal, as $\underline{X} = \coprod_{n\geqslant 0} \mathbb{R}^n$.

Theorem 4 follows from the following two theorems:

Theorem 5. *Suppose $\boldsymbol{X}$ is a d-manifold and $n \geqslant 2 \dim T_x^* \underline{X} + 1$ for all $x \in \underline{X}$. Then generic 1-morphisms $\boldsymbol{f} : \boldsymbol{X} \to \mathbb{R}^{\boldsymbol{n}}$ are embeddings.*

Theorem 6. *Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is an embedding, for $\boldsymbol{X}$ a d-manifold and $\boldsymbol{Y}$ a manifold. Then there exist open $V \subseteq Y$ with $\boldsymbol{f}(\boldsymbol{X}) \subseteq \boldsymbol{V}$, a vector bundle $E \to V$, and a smooth section $s : V \to E$ of E fitting into a 2-Cartesian diagram in* $\mathbf{dMan}$:

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\quad \boldsymbol{f} \quad} & \boldsymbol{V} \\ \downarrow \boldsymbol{f} & \Uparrow & \boldsymbol{0} \downarrow \\ \boldsymbol{V} & \xrightarrow{\quad \boldsymbol{s} \quad} & \boldsymbol{E}. \end{array}$$

Hence $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ in $\mathbf{dMan}$.

4.6. Gluing by equivalences

A 1-morphism $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ in $\mathbf{dMan}$ is an *equivalence* if there exist a 1-morphism $\boldsymbol{g}:\boldsymbol{Y}\rightarrow\boldsymbol{X}$ and 2-morphisms $\eta:\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$ and $\zeta:\boldsymbol{f}\circ\boldsymbol{g}\Rightarrow\mathbf{id}_{\boldsymbol{Y}}$.

Theorem 7. *Let $\boldsymbol{X},\boldsymbol{Y}$ be d-manifolds, $\emptyset\neq\boldsymbol{U}\subseteq\boldsymbol{X}$, $\emptyset\neq\boldsymbol{V}\subseteq\boldsymbol{Y}$ open d-submanifolds, and $\boldsymbol{f}:\boldsymbol{U}\rightarrow\boldsymbol{V}$ an equivalence. Suppose the topological space $Z=X\cup_{U=V}Y$ made by gluing X,Y using f is Hausdorff. Then there exists a d-manifold $\boldsymbol{Z}$, unique up to equivalence, open $\hat{\boldsymbol{X}},\hat{\boldsymbol{Y}}\subseteq\boldsymbol{Z}$ with $\boldsymbol{Z}=\hat{\boldsymbol{X}}\cup\hat{\boldsymbol{Y}}$, equivalences $\boldsymbol{g}:\boldsymbol{X}\rightarrow\hat{\boldsymbol{X}}$ and $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\hat{\boldsymbol{Y}}$, and a 2-morphism $\eta:\boldsymbol{g}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$.*

Equivalence is the natural notion of when two objects in $\mathbf{dMan}$ are 'the same'. In Theorem 7, $\boldsymbol{Z}$ is a *pushout* $\boldsymbol{X}\amalg_{\mathbf{id}_{\boldsymbol{U}},\boldsymbol{U},\boldsymbol{f}}\boldsymbol{Y}$ in $\mathbf{dMan}$. Theorem 7 generalizes to gluing families of d-manifolds $\boldsymbol{X}_i : i \in I$ by equivalences on double overlaps $\boldsymbol{X}_i \cap \boldsymbol{X}_j$, with (weak) conditions on triple overlaps $\boldsymbol{X}_i \cap \boldsymbol{X}_j \cap \boldsymbol{X}_k$.

We can take the $\boldsymbol{X}_i$ to be 'standard model' d-manifolds $\boldsymbol{S}_{V_i,E_i,s_i}$, and the equivalences on overlaps $\boldsymbol{X}_i \cap \boldsymbol{X}_j$ to be 1-morphisms $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}$.

This is very useful for proving existence of d-manifold structures on moduli spaces.

4.7. D-manifold bordism

Let Y be a manifold. Define the *bordism group* $B_k(Y)$ to have elements $\sim$-equivalence classes $[X, f]$ of pairs (X, f), where X is a compact oriented k-manifold and $f : X \to Y$ is smooth, and $(X, f) \sim (X', f')$ if there exists a compact oriented $(k+1)$-manifold with boundary W and a smooth map $e : W \to Y$ with $\partial W \cong X \amalg -X'$ and $e|_{\partial W} \cong f \amalg f'$. It is an abelian group, with $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$.

Similarly, define the *derived bordism group* $dB_k(Y)$ to have elements $\approx$-equivalence classes $[\boldsymbol{X},\boldsymbol{f}]$ of pairs (X,f), where $\boldsymbol{X}$ is a compact oriented d-manifold with $\operatorname{vdim}\boldsymbol{X}=k$ and $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}=F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$ is a 1-morphism in $\mathbf{dMan}$, and $(\boldsymbol{X},\boldsymbol{f})\approx(\boldsymbol{X}',\boldsymbol{f}')$ if there exists a compact oriented d-manifold with boundary $\boldsymbol{W}$ with $\operatorname{vdim}\boldsymbol{W}=k+1$ and a 1-morphism $\boldsymbol{e}:\boldsymbol{W}\rightarrow\boldsymbol{Y}$ in $\mathbf{dMan^b}$ with $\partial\boldsymbol{W}\simeq\boldsymbol{X}\amalg-\boldsymbol{X}'$ and $\boldsymbol{e}|_{\partial\boldsymbol{W}}\cong\boldsymbol{f}\amalg\boldsymbol{f}'$. It is an abelian group, with $[\boldsymbol{X},\boldsymbol{f}]+[\boldsymbol{X}',\boldsymbol{f}']=[\boldsymbol{X}\amalg\boldsymbol{X}',\boldsymbol{f}\amalg\boldsymbol{f}']$.

There is a natural morphism $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}: B_k(Y) \to dB_k(Y)$ mapping $[X,f] \mapsto [F_{\mathbf{Man}}^{\mathbf{dMan}}(X), F_{\mathbf{Man}}^{\mathbf{dMan}}(f)]$.

Theorem 8. $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}: B_k(Y) \to dB_k(Y)$ *is an isomorphism for all* k, *with* $dB_k(Y) = 0$ *for* $k < 0$.

This holds because every d-manifold $\boldsymbol{X}$ can be perturbed to a manifold. To see this, write $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ by Theorem 4, and perturb s to a generic, transverse $\tilde{s} \in C^\infty(E)$.

Composing $(\Pi_{\mathrm{bo}}^{\mathrm{dbo}})^{-1}$ with the projection $B_k(Y) \to H_k(Y,\mathbb{Z})$ gives a morphism $\Pi_{\mathrm{dbo}}^{\mathrm{hom}}: dB_k(Y) \to H_k(Y,\mathbb{Z})$. We can interpret this as a *virtual class map* for compact oriented d-manifolds.

5. D-orbifolds

Orbifolds are generalizations of manifolds locally modelled on $\mathbb{R}^n/G$, for G a finite group. In algebraic geometry, *Deligne–Mumford stacks* (which form a 2-category) are locally modelled on $[X/G]$ for X a scheme and G a finite group.

We define orbifold versions of the whole story so far. C^∞-schemes, d-spaces and d-manifolds generalize to *Deligne–Mumford C^∞-stacks*, *d-stacks*, and *d-orbifolds*, locally modelled on $[\underline{X}/G], [\boldsymbol{X}/G]$ for G a finite group. Orbifolds are special examples of Deligne–Mumford C^∞-stacks (i.e. we have an inclusion of 2-categories $\mathbf{Orb} \subset \mathbf{DMC^\infty Sta}$).

5.1. The 2-category of d-stacks

Definition. A *d-stack* $\boldsymbol{\mathcal{X}}$ is a quintuple $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$, where $\mathcal{X}$ is a where $\mathcal{X}$ is a separated, second countable, locally fair Deligne–Mumford C^∞-stack, $\mathcal{O}'_{\mathcal{X}}$ a sheaf of C^∞-rings on $\mathcal{X}$, and $\mathcal{E}_{\mathcal{X}}$ a quasicoherent sheaf on $\mathcal{X}$, in an exact sequence of sheaves on $\mathcal{X}$:

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\jmath_{\mathcal{X}}} \mathcal{O}'_{\mathcal{X}} \xrightarrow{\imath_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}} \longrightarrow 0.$$

D-stacks form a strict 2-category $\mathbf{dSta}$. 1- and 2-morphisms are defined as for d-spaces in 3.3, except 2-morphisms of d-stacks $\boldsymbol{\eta} = (\eta, \eta')$ also include a 2-morphism η of the Deligne–Mumford C^∞-stacks $\mathcal{X}, \mathcal{Y}$. All fibre products exist in $\mathbf{dSta}$.

5.2. The 2-subcategory of d-orbifolds

There is a full and faithful 2-functor $F_{\mathbf{Orb}}^{\mathbf{dSta}}:\mathbf{Orb}\to\mathbf{dSta}$, so we regard orbifolds as examples of d-stacks. It preserves transverse fibre products in $\mathbf{Orb}$.

Definition. A d-stack $\mathcal{W}$ is a *principal d-orbifold* if it is a fibre product $\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$ in $\mathbf{dSta}$, with $\mathcal{X},\mathcal{Y},\mathcal{Z}$ orbifolds. The *virtual dimension* is $\mathrm{vdim}\,\mathcal{W}=\dim\mathcal{X}+\dim\mathcal{Y}-\dim\mathcal{Z}$.

A d-stack $\mathcal{X}$ is a *d-orbifold of dimension* $n\in\mathbb{Z}$ if $\mathcal{X}$ may be covered by open d-substacks $\mathcal{W}\subset\mathcal{X}$ which are principal d-orbifolds with $\mathrm{vdim}\,\mathcal{W}=n$. Write $\mathbf{dOrb}$ for the full 2-subcategory of d-orbifolds in $\mathbf{dSta}$.

Many properties of d-manifolds extend to d-orbifolds. For example:

- All fibre products $\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$ with $\mathcal{X},\mathcal{Y}$ d-orbifolds and $\mathcal{Z}$ an orbifold exist in $\mathbf{dOrb}$, as in Theorem 2.
- If $\mathcal{V}$ is an orbifold, $\mathcal{E}$ a vector bundle on $\mathcal{E}$, and $s\in C^\infty(\mathcal{E})$ a smooth section, we define explicit 'standard model' d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ fitting into a 2-Cartesian diagram in $\mathbf{dOrb}$:

$$\begin{array}{ccc}\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} & \xrightarrow{\boldsymbol{\pi}} & \boldsymbol{\mathcal{V}} \\ \downarrow{\scriptstyle\boldsymbol{\pi}} & \boldsymbol{\eta}\Uparrow & \downarrow{\scriptstyle\boldsymbol{0}} \\ \boldsymbol{\mathcal{V}} & \xrightarrow{\boldsymbol{s}} & \boldsymbol{\mathcal{E}}.\end{array}$$

Every principal d-orbifold is equivalent to some $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$. We also define 'standard model' 1- and 2-morphisms.

• We can glue d-orbifolds by equivalences, as in Theorem 7. For gluing a family of d-orbifolds $\boldsymbol{\mathcal{X}}_i$, $i \in I$ by equivalences on overlaps $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j$, the conditions are stronger in the d-manifold case (we need 2-morphisms $\boldsymbol{\eta}_{ijk} = (\eta_{ijk}, \eta'_{ijk})$ on triple overlaps $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k$ whose C^∞-stack 2-morphisms η_{ijk} satisfy an identity on quadruple overlaps $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \cap \boldsymbol{\mathcal{X}}_l$).

• Virtual classes exist for compact oriented d-orbifolds, but usually in homology over $\mathbb{Q}$, not $\mathbb{Z}$.

5.3. Differences between d-manifolds and d-orbifolds

Other properties of d-manifolds do not extend well to d-orbifolds. For example: in contrast to Theorem 4, we do not have good criteria for when a d-orbifold $\boldsymbol{\mathcal{X}}$ is principal. This is because the analogue of Theorem 5 is false for d-orbifolds: many d-orbifolds $\boldsymbol{\mathcal{X}}$ do not admit embeddings $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\mathbb{R}^n$ or $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to[\mathbb{R}^n/G]$. The analogue of Theorem 6 holds: if an embedding $\boldsymbol{f}:\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}$ exists for $\boldsymbol{\mathcal{Y}}$ an orbifold, then $\boldsymbol{\mathcal{X}}\simeq\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$. But we do not know when d-orbifolds can be embedded in an orbifold. Similar questions arise for Kuranishi spaces.

As for manifolds and d-manifolds, for any orbifold $\mathcal{Y}$ we can define *orbifold bordism groups* $B_k^{\mathrm{orb}}(\mathcal{Y})$ with elements $[\mathcal{X}, f]$ for $\mathcal{X}$ a compact oriented k-orbifold and $f : \mathcal{X} \to \mathcal{Y}$ a 1-morphism, and *d-orbifold bordism groups* $dB_k^{\mathrm{orb}}(\mathcal{Y})$ with elements $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$ for $\boldsymbol{\mathcal{X}}$ a compact oriented d-orbifold with $\mathrm{vdim}\, \boldsymbol{\mathcal{X}} = k$ and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$ a 1-morphism. There is a natural functor $F_{\mathrm{obo}}^{\mathrm{dobo}} : B_k^{\mathrm{orb}}(\mathcal{Y}) \to dB_k^{\mathrm{orb}}(\mathcal{Y})$. But it is not an isomorphism. Generally $dB_k^{\mathrm{orb}}(\mathcal{Y})$ is much bigger than $B_k^{\mathrm{orb}}(\mathcal{Y})$, and $dB_k^{\mathrm{orb}}(\mathcal{Y})$ may be nonzero for all $k \in \mathbb{Z}$.

Now $F_{\rm obo}^{\rm dobo}: B_k^{\rm orb}(\mathcal{Y}) \rightarrow dB_k^{\rm orb}(\mathcal{Y})$ is not an isomorphism as some d-orbifolds cannot be deformed to orbifolds. In a 'standard model' $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$, at a point $v \in \mathcal{V}$ with $s(v) = 0$, the orbifold group $G = \mathrm{Iso}_{\mathcal{V}}(v)$ acts on the tangent space $T_v\mathcal{V}$ and obstruction space $\mathcal{E}|_v$. If the nontrivial part of the G-representation on $\mathcal{E}|_v$ is not a subrepresentation of $T_v\mathcal{V}$, small deformations $\tilde{s}$ of s are not transverse near v, so $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde{s}}$ is not an orbifold. We can express this in terms of *orbifold strata* of orbifolds and d-orbifolds.

6. Things with corners

I also define categories $\mathbf{Man^b}, \mathbf{Man^c}$ of *manifolds with boundary* and *with corners*, and 2-categories $\mathbf{dMan^b}$, $\mathbf{dMan^c}, \mathbf{Orb^b}, \mathbf{Orb^c}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$ of *d-manifolds, orbifolds, and d-orbifolds with boundary* and *with corners*.

Doing 'things with corners' properly, to get (2-)categories with good properties such as functoriality of boundaries, and existence of fibre products under good conditions, turns out to be almost unexplored, and surprisingly complex. Even for manifolds with corners, the 'right' notion of smooth map $f : X \to Y$ is new, with extra discrete and continuous conditions over $\partial^k X, \partial^l Y$.

A few highlights:

- A d-manifold (or d-orbifold) with corners $\boldsymbol{X}$ has a *boundary* $\partial\boldsymbol{X}$, of dimension vdim $\boldsymbol{X}-1$, with inclusion 1-morphism $\boldsymbol{i_X}:\partial\boldsymbol{X}\rightarrow\boldsymbol{X}$.
- 'simple' 1-morphisms $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ lift uniquely to boundaries, giving $\boldsymbol{f}_-:\partial\boldsymbol{X}\rightarrow\partial\boldsymbol{Y}$ with $\boldsymbol{f}\circ\boldsymbol{i_X}=\boldsymbol{i_Y}\circ\boldsymbol{f}_-$.
- The symmetric group S_k acts freely on $\partial^k\boldsymbol{X}$. Define the k-*corners* $C_k(\boldsymbol{X})$ $=\partial^k\boldsymbol{X}/S_k$, and the *corners* $C(\boldsymbol{X})=\coprod_{k\geqslant 0}C(\boldsymbol{X})$. The map $\boldsymbol{X}\mapsto C(\boldsymbol{X})$ extends to a strict 2-functor $C:\mathbf{dMan^c}\rightarrow\mathbf{d\check{M}an^c}$ which commutes with boundaries, products, many fibre products, etc.
- Need d-orbifolds with corners for symplectic geometry applications.