Kuranishi (co)homology: a new tool in symplectic geometry. II. Kuranishi (co)homology

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These slides available at

`www.maths.ox.ac.uk/∼joyce/talks.html`

II.1. Introduction

Let Y be an orbifold and R a $\mathbb{Q}$-algebra. We will define *Kuranishi homology* $KH_*(Y;R)$, a homology theory of Y with coefficients in R. It is the homology of a chain complex $\bigl(KC_*(Y;R),\partial\bigr)$, spanned by isomorphism classes $[X,\boldsymbol{f},\boldsymbol{G}]$ of triples $(X,\boldsymbol{f},\boldsymbol{G})$, for X a compact, oriented Kuranishi space with boundary and corners, $\boldsymbol{f}:X\rightarrow Y$ a strongly smooth map, and $\boldsymbol{G}$ some *gauge-fixing data* for $(X,\boldsymbol{f})$. The boundary operator is

$$\partial:[X,\boldsymbol{f},\boldsymbol{G}]\mapsto[\partial X,\boldsymbol{f}|_{\partial X},\boldsymbol{G}|_{\partial X}].$$

For Kuranishi cohomology, we add *co-gauge-fixing data* $\boldsymbol{C}$.

II.2. Gauge-fixing data

Chains in $KC_k(Y;R)$ will be R-linear combinations of isomorphism classes $[X,\boldsymbol{f},\boldsymbol{G}]$, where X is a compact, oriented Kuranishi space with corners, $\mathrm{vdim}\, X = k$, $\boldsymbol{f} : X \to Y$ is strongly smooth, and $\boldsymbol{G}$ is a (very nonunique) choice of *gauge fixing data* for $(X,\boldsymbol{f})$. Here are the most important properties of $\boldsymbol{G}$:

(a) Every $(X,\boldsymbol{f})$ has a (nonunique) choice of gauge-fixing data $\boldsymbol{G}$.

(b) The automorphism group $\mathrm{Aut}(X,\boldsymbol{f},\boldsymbol{G})$ is finite for all $(X,\boldsymbol{f},\boldsymbol{G})$.

(c) If $\boldsymbol{G}$ is gauge-fixing data for $(X,\boldsymbol{f})$, it has a restriction $\boldsymbol{G}|_{\partial X}$, which is gauge-fixing data for $(\partial X,\boldsymbol{f}|_{\partial X})$.

(d) There is a natural, orientation-reversing involution $\boldsymbol{\sigma} : \partial^2 X \to \partial^2 X$. Suppose $\boldsymbol{H}$ is gauge-fixing data for $(\partial X, \boldsymbol{f}|_{\partial X})$. Then there exists gauge-fixing data $\boldsymbol{G}$ for $(X, \boldsymbol{f})$ with $\boldsymbol{G}|_{\partial X} = \boldsymbol{H}$ iff $\boldsymbol{H}|_{\partial^2 X}$ is invariant under $\boldsymbol{\sigma}$.

(e) There are good, functorial notions of products, pushforwards, and pullbacks of (co-)gauge-fixing data, as one needs to make cup and cap products, pushforwards, and pullbacks in (co)homology work at the (co)chain level.

It does not matter exactly what (co-)gauge-fixing data is, as long as it has these properties (a)–(e). My definition of gauge-fixing data $\boldsymbol{G}$ for $(X,\boldsymbol{f})$ involves a finite cover of X by Kuranishi neighbourhoods (V^i,E^i,s^i,ψ^i) for $i\in I$, and finite maps $G^i:E^i\rightarrow\mathbb{R}^\infty$ for $i\in I$, satisfying complex conditions. The fact that the G^i are *finite* maps (that is, $(G^i)^{-1}(p)$ is finite for all $p\in\mathbb{R}^\infty$) ensures that $\mathrm{Aut}(X,\boldsymbol{f},\boldsymbol{G})$ is finite.

II.3. Kuranishi homology

Let Y be an orbifold. Consider triples $(X, \boldsymbol{f}, \boldsymbol{G})$, for X a compact oriented Kuranishi space, $\boldsymbol{f} : X \to Y$ strongly smooth, and $\boldsymbol{G}$ gauge-fixing data for $(X, \boldsymbol{f})$. Write $[X, \boldsymbol{f}, \boldsymbol{G}]$ for the isomorphism class of $(X, \boldsymbol{f}, \boldsymbol{G})$. Let R be a $\mathbb{Q}$-algebra.

For each $k \in \mathbb{Z}$, define $KC_k(Y; R)$ to be the R-module of finite R-linear combinations of $[X, \boldsymbol{f}, \boldsymbol{G}]$ with $\operatorname{vdim} X = k$, with the relations:

(i) Write $-X$ for X with the opposite orientation. Then

$$[X, \boldsymbol{f}, \boldsymbol{G}] + [-X, \boldsymbol{f}, \boldsymbol{G}] = 0$$

in $KC_k(Y; R)$, for all $[X, \boldsymbol{f}, \boldsymbol{G}]$.

(ii) Let $[X, \boldsymbol{f}, \boldsymbol{G}]$ be an isomorphism class. Suppose that X may be written as a disjoint union $X = X_+ \amalg X_-$ of compact oriented Kuranishi spaces in a way compatible with $\boldsymbol{G}$. Then in $KC_k(Y; R)$ we have

$$[X, \boldsymbol{f}, \boldsymbol{G}] = [X_+, \boldsymbol{f}|_{X_+}, \boldsymbol{G}|_{X_+}] + [X_-, \boldsymbol{f}|_{X_-}, \boldsymbol{G}|_{X_-}].$$

(iii) Suppose Γ is a finite subgroup of $\mathrm{Aut}(X, \boldsymbol{f}, \boldsymbol{G})$. Then $\tilde{X} = X/\Gamma$ is a compact oriented Kuranishi space, with $\boldsymbol{\pi} : X \to \tilde{X}$, and $\boldsymbol{f}, \boldsymbol{G}$ push down to $\boldsymbol{\pi}_*(\boldsymbol{f}), \boldsymbol{\pi}_*(\boldsymbol{G})$. We require that

$$[X/\Gamma, \boldsymbol{\pi}_*(\boldsymbol{f}), \boldsymbol{\pi}_*(\boldsymbol{G})] = \frac{1}{|\Gamma|}[X, \boldsymbol{f}, \boldsymbol{G}]$$

in $KC_k(Y; R)$.

Elements of $KC_k(Y; R)$ will be called *Kuranishi chains*.

Define the *boundary operator* ∂ : $KC_k(Y;R) \to KC_{k-1}(Y;R)$ by

$$\partial : \sum_{a\in A} \rho_a [X_a, \boldsymbol{f}_a, \boldsymbol{G}_a] \longmapsto \sum_{a\in A} \rho_a [\partial X_a, \boldsymbol{f}_a|_{\partial X_a}, \boldsymbol{G}_a|_{\partial X_a}],$$

where A is a finite indexing set and $\rho_a \in R$ for $a \in A$. Using a natural orientation-reversing involution $\boldsymbol{\sigma}$: $\partial^2 X_a \to \partial^2 X_a$ and relation (i), we find that $\partial^2 = 0$.

— Explain $\boldsymbol{\sigma}$ on the board —

Define the k^{th} *Kuranishi homology group* $KH_k(Y;R)$ to be

$$\frac{\mathrm{Ker}\bigl(\partial : KC_k(Y;R) \to KC_{k-1}(Y;R)\bigr)}{\mathrm{Im}\bigl(\partial : KC_{k+1}(Y;R) \to KC_k(Y;R)\bigr)}.$$

Let Y, Z be orbifolds, and $h : Y \to Z$ a smooth map. Define the *pushforward* $h_* : KC_k(Y;R) \to KC_k(Z;R)$ on Kuranishi chains by

$$h_* : [X, \boldsymbol{f}, \boldsymbol{G}] \mapsto [X, h \circ \boldsymbol{f}, h_*(\boldsymbol{G})].$$

Then $h_* \circ \partial = \partial \circ h_*$, so h_* induces $h_* : KH_k(Y;R) \to KH_k(Z;R)$ on Kuranishi homology.

II.4. Isomorphism with $H_*^{\mathrm{si}}(Y;R)$

For $k \geqslant 0$, the *k-simplex* Δ_k is

$$\Delta_k = \{(x_0,\ldots,x_k) \in \mathbb{R}^{k+1} : x_i \geqslant 0, \quad x_0+\ldots+x_k = 1\}.$$

Let Y be an orbifold, and R a $\mathbb{Q}$-algebra. Write $C_k^{\mathrm{si}}(Y;R)$ for the R-module spanned by smooth maps $\sigma : \Delta_k \to Y$. By identifying $\partial\Delta_k$ with the disjoint union of $k+1$ copies of Δ_{k-1} we define $\partial : C_k^{\mathrm{si}}(Y;R) \to C_{k-1}^{\mathrm{si}}(Y;R)$. *Singular homology* $H_*^{\mathrm{si}}(Y;R)$ is the homology of $(C_*^{\mathrm{si}}(Y;R), \partial)$.

We define a morphism
$\Pi_{\mathrm{si}}^{\mathrm{Kh}} : C_k^{\mathrm{si}}(Y;R) \to KC_k(Y;R)$ by

$$\Pi_{\mathrm{si}}^{\mathrm{Kh}} : \sum_{a\in A} \rho_a\sigma_a \mapsto \sum_{a\in A} \rho_a\big[\Delta_k,\sigma_a,\boldsymbol{G}_{\Delta_k}\big].$$

Here $\boldsymbol{G}_{\Delta_k}$ is an explicit choice of gauge-fixing data for (Δ_k,σ_a). Then $\partial\circ\Pi_{\mathrm{si}}^{\mathrm{Kh}} = \Pi_{\mathrm{si}}^{\mathrm{Kh}}\circ\partial$, and so $\Pi_{\mathrm{si}}^{\mathrm{Kh}}$ induces R-module morphisms

$$\Pi_{\mathrm{si}}^{\mathrm{Kh}} : H_k^{\mathrm{si}}(Y;R) \to KH_k(Y;R).$$

Our main result is:

Theorem 1. *For Y an orbifold and R a $\mathbb{Q}$-algebra, $\Pi_{\mathrm{si}}^{\mathrm{Kh}} : H_k^{\mathrm{si}}(Y;R) \to KH_k(Y;R)$ is an isomorphism.*

This shows Kuranishi homology can be used instead of singular homology in applications, e.g. G–W theory, Lagrangian Floer homology.

To prove Theorem 1 we must construct an inverse $(\Pi_{\mathrm{si}}^{\mathrm{Kh}})^{-1} : KH_k^{\mathrm{Kh}}(Y;R) \to H_k^{\mathrm{si}}(Y;R)$. Basically this is a *virtual cycle construction*: $(\Pi_{\mathrm{si}}^{\mathrm{Kh}})^{-1}$ should take $[X, \boldsymbol{f}, \boldsymbol{G}]$ to a virtual chain for X. We use some ideas of Fukaya–Ono. However, ensuring the virtual chains are functorial, and compatible at boundary and corners of X, makes the proof very complex and difficult. A lot of the technical issues in [FOOO] are transferred to the proof of Theorem 1 in my set-up.

II.5. Kuranishi cohomology

In Kuranishi homology we used triples $(X, \boldsymbol{f}, \boldsymbol{G})$, in which $\boldsymbol{f}: X \to Y$ was strongly smooth, X was oriented, and $\boldsymbol{G}$ was gauge-fixing data. For Kuranishi cohomology we use triples $(X, \boldsymbol{f}, \boldsymbol{C})$, in which $\boldsymbol{f}: X \to Y$ is a *strong submersion*, $(X, \boldsymbol{f})$ is *cooriented*, and $\boldsymbol{C}$ is *co-gauge-fixing data*. Here a *coorientation* of a (strong) submersion is a relative orientation.

If $f: X \to Y$ is a submersion of manifolds, then a coorientation is an orientation on the fibres of the vector bundle $\mathrm{Ker}\big(\mathrm{d}f : TX \to f^*(TY)\big)$ over X.

Let Y be an orbifold. Consider triples $(X, \boldsymbol{f}, \boldsymbol{C})$, for X a compact Kuranishi space, $\boldsymbol{f} : X \to Y$ a cooriented strong submersion, and $\boldsymbol{C}$ co-gauge-fixing data for $(X, \boldsymbol{f})$. Write $[X, \boldsymbol{f}, \boldsymbol{C}]$ for the isomorphism class of $(X, \boldsymbol{f}, \boldsymbol{C})$. Let R be a $\mathbb{Q}$-algebra.
For each $k \in \mathbb{Z}$, define $KC^k(Y; R)$ to be the R-module of finite R-linear combinations of $[X, \boldsymbol{f}, \boldsymbol{C}]$ with $\operatorname{vdim} X = \dim Y - k$, with the analogues of relations (i)–(iii) in §II.2.

Define d : $KC^k(Y;R) \to KC^{k+1}(Y;R)$ by

$$\mathrm{d}: \sum_{a\in A} \rho_a [X_a, \boldsymbol{f}_a, \boldsymbol{C}_a] \longmapsto \sum_{a\in A} \rho_a [\partial X_a, \boldsymbol{f}_a|_{\partial X_a}, \boldsymbol{C}_a|_{\partial X_a}].$$

Then $\mathrm{d}^2 = 0$.
Define the k^{th} *Kuranishi cohomology group* $KH^k(Y;R)$ to be

$$\frac{\mathrm{Ker}\big(\mathrm{d}: KC^k(Y;R) \to KC^{k+1}(Y;R)\big)}{\mathrm{Im}\big(\mathrm{d}: KC^{k-1}(Y;R) \to KC^k(Y;R)\big)}.$$

Remarks. In general, homology or bordism involves structures on X, such as orientations, and cohomology or cobordism involves the corresponding *relative structures* for $f : X \to Y$, such as coorientations.

Also note: for manifolds X, Y, if $f : X \to Y$ is a submersion, then $\dim X \geqslant \dim Y$. So if we defined $KC^k(Y; R)$ using $[X, f, C]$ for X a manifold with $\dim X = \dim Y - k$, f a submersion then $KC^k(Y; R) = 0$ for $k > 0$, which would be no use. However, if X is a Kuranishi space and $f : X \to Y$ a strong submersion, can have $\operatorname{vdim} X < \dim Y$. Strong submersions are easy to produce.

Let Y, Z be orbifolds, and $h : Y \to Z$ a smooth map. Define the *pull-back* $h^* : KC^k(Z;R) \to KC^k(Y;R)$ on Kuranishi chains by

$$h^* : [X, \boldsymbol{f}, \boldsymbol{C}] \mapsto [Y \times_{h,Z,\boldsymbol{f}} X, \boldsymbol{\pi}_Y, h^*(\boldsymbol{C})].$$

Here $Y \times_{h,Z,\boldsymbol{f}} X$ is the *fibre product of Kuranishi spaces*, defined as $\boldsymbol{f}$ is a strong submersion. The coorientation for $(X, \boldsymbol{f})$ pulls back to a coorientation for $(Y \times_{h,Z,\boldsymbol{f}} X, \boldsymbol{\pi}_Y)$. Then $h^* \circ \mathrm{d} = \mathrm{d} \circ h^*$, so h^* induces $h^* : KH^k(Z;R) \to KH^k(Y;R)$ on Kuranishi cohomology.

Define a *cup product* $KC^k(Y;R)\times KC^l(Y;R)\to KC^{k+l}(Y;R)$ by

$$[X,\boldsymbol{f},\boldsymbol{C}]\cup[\tilde X,\tilde{\boldsymbol{f}},\tilde{\boldsymbol{C}}]= \bigl[X\times_{\boldsymbol{f},Y,\tilde{\boldsymbol{f}}}\tilde X,\boldsymbol{\pi}_Y,\boldsymbol{C}\times_Y\tilde{\boldsymbol{C}}\bigr],$$

using fibre products of Kuranishi spaces. This is *associative* and *supercommutative*, at the cochain level, and compatible with d, so it induces $\cup: KH^k(Y;R)\times KH^l(Y;R)\to KH^{k+l}(Y;R)$.
Define *cap products* the same way.
Note: products of co-gauge-fixing data $\boldsymbol{C}\times_Y\tilde{\boldsymbol{C}}$ are very nontrivial – it was difficult to make them associative and commutative.

For Y an oriented n-orbifold and R a $\mathbb{Q}$-algebra we have $H^{\mathrm{si}}_{n-k}(Y;R) \cong H^k_{\mathrm{cs}}(Y;R)$, compactly-supported cohomology. We can also prove Poincaré duality isomorphisms between Kuranishi (co)homology. So using Theorem 1 we show:

Theorem 2. *For Y an orbifold and R a $\mathbb{Q}$-algebra, we have $H^k_{\mathrm{cs}}(Y;R) \cong KH^k(Y;R)$, with $KH^k(Y;R) = 0$ for $k < 0$. The isomorphisms $H^{\mathrm{si}}_*(Y;R) \cong KH_*(Y;R)$, $H^*_{\mathrm{cs}}(Y;R) \cong KH^*(Y;R)$ identify the cup and cap products on $H^*_{\mathrm{cs}}(Y;R), H^{\mathrm{si}}_*(Y;R)$ with those on $KH^*(Y;R), KH_*(Y;R)$.*

II.8. Conclusions.

Kuranishi (co)homology is well adapted for use in symplectic geometry, because we can turn moduli spaces of J-holomorphic curves $\bar{\mathcal{M}}$ with evaluation maps $\mathbf{ev} : \bar{\mathcal{M}} \rightarrow Y$ directly into (co)chains $[\bar{\mathcal{M}}, \mathbf{ev}, \boldsymbol{G}]$ or $[\bar{\mathcal{M}}, \mathbf{ev}, \boldsymbol{C}]$ just by choosing (co-) gauge-fixing data $\boldsymbol{G}, \boldsymbol{C}$. There is *no need to perturb moduli spaces*. Choosing (co-)gauge-fixing data is a much milder process, there is no problem in making infinitely many compatible choices.

Also, Kuranishi (co)homology is *very well behaved at the cochain level*, with cup products that are everywhere defined, associative, and supercommutative on Kuranishi cochains. This is useful in Lagrangian Floer cohomology, in which the moduli spaces $\bar{\mathcal{M}}$ are Kuranishi spaces with boundary and corners, and the boundary $\partial\bar{\mathcal{M}}$ is written as a disjoint union $\coprod_{\bar{\mathcal{M}}',\bar{\mathcal{M}}''}\pm\bar{\mathcal{M}}'\times_L\bar{\mathcal{M}}''$ of fibre products $\bar{\mathcal{M}}'\times_L\bar{\mathcal{M}}''$ of other curve moduli spaces $\bar{\mathcal{M}}',\bar{\mathcal{M}}''$.

We translate this into an *exact algebraic identity* on Kuranishi cochains, roughly of the form

$$\mathrm{d}[\bar{\mathcal{M}},\mathrm{ev},\boldsymbol{C}] = \sum_{\bar{\mathcal{M}}',\bar{\mathcal{M}}''} \pm[\bar{\mathcal{M}}',\mathrm{ev}',\boldsymbol{C}'] \cup_L [\bar{\mathcal{M}}'',\mathrm{ev}'',\boldsymbol{C}''].$$

We can then define a geometric A_∞ algebra directly on the completed Kuranishi cochains $\widehat{KC}^*(Y;\Lambda_{\mathrm{nov}})$, bypassing many steps in [FOOO].