Volume comparison theorems without JACOBI fields

Dominique Bakry

Laboratoire de Statistique et Probabilités

Université Paul Sabatier

118 route de Narbonne

31062 Toulouse, FRANCE

Zhongmin Qian

Mathematical Institute, University of Oxford

24-29 St Giles', Oxford OX1 3LB, ENGLAND

Abstract

Using a generalized curvature-dimension inequality and a new approach, we present a differential inequality for an elliptic second order differential operator acting on distance functions, from which we deduce volume comparison theorems and diameter bounds without the use of the theory of JACOBI fields.

1 Introduction.

The classical volume comparison theorems in Riemannian geometry are among the basic ingredients of the analysis on manifolds. They may be stated as follows. Assume that M is an n-dimensional Riemannian manifold with RICCI curvature being bounded below by some constant ρ. Choose an arbitrary point o in M and let $V(r)$ denote the Riemannian volume of the ball centered at o and radius r. On the other hand, let $V_{\rho,n}(r)$ denote the volume of the ball of the Riemannian model with constant RICCI curvature ρ, that is a sphere if $\rho > 0$, an Euclidean space if $\rho = 0$, and an hyperbolic space if $\rho < 0$. Then, BISHOP-GROMOV comparison theorems assert that $\frac{V'(r)}{V'_{\rho,n}(r)}$ is a decreasing function of r, and, as a consequence, that $\frac{V(r)}{V_{\rho,n}(r)}$ is also a decreasing function of r.

The classical proof of this result relies on the theory of JACOBI fields, that is the precise study of deformations of a geodesic ball along a geodesic.

The aim of this paper is to provide an elementary approach to these volume comparison theorems using only basic differential inequalities on distance functions. In [6] the second author proposed a method to devise a comparison theorem for distance functions using the maximum principle for parabolic equations, but this approach worked only in the non-negative Ricci case. The method presented in this article avoids the maximum principle and the use of the theory of JACOBI fields. It extends the classical results to measures which are different from the Riemannian measures. Also, with this approach, the lower bound on the RICCI curvature may depend on the distance to the starting point o.

The basic tool is a curvature-dimension inequality associated to a generic second order differential operator, which leads to differential inequalities on distance functions, and therefore to information on diameter and on volumes of balls for the invariant measure of this operator. This curvature-dimension inequality may as well be stated for operators in dimension 1, and those 1-dimensional operators appear to be the natural objects to compare with.

The case of Laplacians leads to the fundamental comparison theorem for Riemannian measures [3], [4] and follows easily from our differential inequality.

On the other hand, it is well known that the comparison theorem for Laplacians implies MYER's estimate on the diameter of the manifold, and also yields the comparison theorem for volumes. These results were proved in the literature via the theory of JACOBI fields, or the variation formula for volumes. All comparison theorems have their origins in the STURM-LIOUVILLE theory, and JACOBI fields are used in the reduction from geometric quantities to ordinary differential equations.

The approach in this paper presents a direct STURM-LIOUVILLE type argument dealing with the distance function under a lower bound of the RICCI curvature. There are obvious advantages with this approach. First it applies to general elliptic second order operators, not only Laplacians, not even symmetric operators, in which case one has very little information about the invariant measure. Secondly, since we only use several basic facts about complete manifolds, it may be adopted in the study of metric geometry. Also, the one dimensional models provide a nice setting for comparison theorems with bounds on the RICCI tensor depending on the distance.

2 Curvature-dimension inequalities.

Let M be a smooth manifold with dimension N and L be a second order elliptic differential operator on M, with no 0-order term. In a local system of coordinates, L may be written as

$$Lf(x) = \sum_{ij} g^{ij}(x) \frac{\partial^2_{ij} f}{\partial x^i \partial x^j} + \sum_i b^i(x) \frac{\partial_i f}{\partial x^i}.$$

If we introduce the Riemannian metric $g = (g_{ij})(x)$ which is the inverse of the matrix $(g^{ij})(x)$, then this allows us to rewrite L as

$$L = \Delta + X,$$

where Δ is the LAPLACE-BELTRAMI operator associated with the metric g and X is a vector field. Our basic assumption is that the manifold M is complete for this metric. This is done for simplicity, but if it is not, one should assume at least that the boundary is convex for the Riemannian metric. We shall not deal with this situation here.

We shall denote by μ an invariant measure for the operator L, that is a solution of $L^*(\mu) = 0$. By ellipticity, such an invariant measure has a smooth density with respect to the RIEMANN measure dm.

When X is a gradient field, say for example $X = \nabla f$, then we may choose $d\mu = \exp(h)dm$, but no such simple formula is valid in the general case.

Introduce $\Gamma(u, v) = \nabla u \cdot \nabla v$. It is worthwhile to observe that we may define $\Gamma(u, v)$ as

$$\Gamma(u, v) = \frac{1}{2}(L(uv) - uLv - vLu),$$

and this naturally leads to the following definition of the iterated squared field operator, according to [1]:

$$\Gamma_2(u, v) = \frac{1}{2} \left\{ L\Gamma(u, v) - \Gamma(Lu, v) - \Gamma(u, Lv) \right\} .$$

We shall denote $\Gamma(u)$ and $\Gamma_2(u)$ instead of $\Gamma(u, u)$ and $\Gamma_2(u, u)$ respectively.

In the case of the Laplacian $L = \Delta$, the BÖCHNER identity may be written as

$$\Gamma_2(u) = |\nabla\nabla u|^2 + \mathrm{Ric}\,(\nabla u, \nabla u)\,.$$

We say that the operator L satisfies the curvature-dimension inequality $CD(K, n)$ for some $n \geq N$ and some function K, if, for any $u \in C^2(M)$, one has

$$\Gamma_2(u)(x) \geq \frac{1}{n} (Lu)(x)^2 + K(x)\Gamma(u)(x).$$

In the case of Laplacians, $CD(K, N)$ means exactly that the RICCI curvature at point x is bounded below by $K(x)$, and the dimension $n = N$ of the manifold is the least possible value for which a $CD(K, n)$ inequality may occur.

For a general elliptic operator $L = \Delta + X$, with $X \neq 0$, then the $CD(K,n)$ inequality is valid if and only if $n > N$ and if the tensor

$$\mathrm{Ric} - K(x)g - \nabla_S X - \frac{1}{n-N} X \otimes X$$

is non negative, where g is the metric and $\nabla_S X$ denotes the symmetric version of the covariant derivative of the vector field X.

In dimension 1, if $L(f) = f'' + a(x)f'$, then this boils down to

$$-a'(x) \geq K(x) + \frac{a^2}{n-1},$$

while the invariant measure μ is $\exp(a(x))dx$, which says that, if $V(x)$ denotes the volume (for the invariant measure) of the ball centered at 0 and radius x, then $a(x) = V''(x)/V'(x)$.

In what follows, we shall denote by $a_{K,n}$ the solution of the Ricatti equation

$$-a' = K(x) + \frac{a^2}{n-1}$$

on $(0, \infty)$ such that $\lim_{x \to 0} xa(x) = n-1$. It is easier to change K into $(n-1)K$ and to write $a_{K,n} = (n-1)\theta_K$, such that

$$\theta'_K = -(K + \theta_K^2).$$

We shall mainly consider this equation on $\mathbb{R}_+$, with initial condition at 0 described above. If this equation explodes in finite time (this happens as soon as K is bounded below by some positive constant), then we shall denote by δ_K the explosion point.

If K is constant, then this operator is exactly the radial part of the Laplace-Beltrami operator of a sphere, a Euclidean or a hyperbolic space, according to the sign of K. This operator will serve as a 1-dimensional model in our comparison theorems.

An important fact is that the $CD(K,n)$ inequality has a natural self-improvement, which is optimal for one-dimensional models. More precisely

Lemma 2.1. *Let L satisfy the $CD(K,n)$ inequality. Then, for any smooth u on M, one has*

$$\Gamma_2(u) \geq \frac{1}{n}(Lu)^2 + K\Gamma(u) + \frac{n}{n-1}\left(\frac{Lu}{n} - \frac{\Gamma(\Gamma(u),u)}{2\Gamma(u)}\right)^2 \tag{2.1}$$

on $\Gamma(u) \neq 0$.

For the one dimensional models described above, this inequality is an equality for any smooth function u.

Proof. — We shall not give the details (see [2] for a complete proof). In fact, if we write $\Gamma_2(f)-K\Gamma(f)-\frac{1}{n}(Lf)^2$ with $f=\Phi(u)$, and if we use the chain rule formula for derivation, we end up with a quadratic form in the two variables $\Phi'(u)$ and $\Phi''(u)$. Therefore, the $CD(K,n)$ inequality gives that this expression is non negative. Since in any point, we may choose independently $\Phi'(u)$ and $\Phi''(u)$, the $CD(K,n)$ inequality tells us that this quadratic form is non negative, and the inequality in lemma 2.1 is nothing else than the fact that it's discriminant is non negative.

The last assertion of the lemma is a simple verification. ∎

3 Fundamental inequality.

In what follows, we shall use the 1-dimensional models described above, and give a differential inequality on the function $L(\rho)$, for any function satisfying $\Gamma(\rho)=1$ under a curvature-dimension condition for the operator. This inequality is in fact an identity for 1-dimensional models.

For this, K being a C^1 function defined on the real line (or on interval), we shall define θ_K to be any solution to the equation

$$\theta_K' = -(K+\theta_K^2) \tag{3.2}$$

on an interval (a,b) on which such a solution exists. We may as well set $\theta_K=\frac{\sigma'}{\sigma}$ in which case $\sigma''+K\sigma=0$. Then, let ϕ_K be a solution of

$$\phi_K''+(n-1)\phi_K'\theta_K=1 \tag{3.3}$$

on (a,b).

We are now in a position to state our main result.

Theorem 3.1. *Let $L=\Delta+X$ and ρ be a function which satisfies $\Gamma(\rho)=1$ in an open set $\Omega\subset M$,and that ρ^2 is C^2 in Ω. Suppose that L satisfies $CD((n-1)K(\rho),n)$. Let $F=L\phi_K(\rho)-1$, where ϕ_K satisfies (3.3). Then, on $\Omega\cap\rho^{-1}((a,b))$, one has*

$$F\left[F+(2-(n+1)\phi_K''(\rho))\right]\leq-(n-1)\,\phi_K'(\rho)\Gamma(F,\rho)\ . \tag{3.4}$$

Of course, we shall apply this in particular to the distance function ρ to some point o with $\Omega=M\setminus\mathrm{cut}(o)$. Then, we shall specify the boundary behaviour of the functions θ_K and ϕ_K. But this may apply as well to other function with gradient 1, such as distances to submanifolds, for example. We made an assumption of smoothness on ρ^2 instead of ρ because distance functions satisfy this property outside the cut-locus (see lemma 4.1 below).

Proof. — The proof relies on Lemma 2.1. We shall write ϕ for ϕ_K and θ for θ_K, since there is no possible confusion. Notice that $\phi(\rho)\in C^2(\Omega)$ as soon as ρ^2 is C^2 in Ω. In fact $\phi(\rho) \simeq \rho^2/n$ as $\rho \to 0$, thus $\phi(\rho)$ is smooth at o. Since $\Gamma(\rho) = 1$, $\Gamma(\phi(\rho)) = \phi'^2$, and applying (2.1) to $\phi(\rho)$ we obtain

$$\begin{aligned} \Gamma_2(\phi(\rho)) &\geq \frac{1}{n-1}\left(L(\phi(\rho))\right)^2 - \frac{2}{n-1}\phi'' L(\phi(\rho)) \\ &\quad + (n-1)K\phi'^2 + \frac{n}{n-1}\phi''^2 \ . \end{aligned}$$

On the other hand, by definition

$$\Gamma_2(\phi(\rho)) = \phi'' L\left(\phi(\rho)\right) + \phi'\phi''' - \Gamma(L\phi(\rho), \phi(\rho)) \ .$$

Since ϕ satisfies the differential equation (3.3),

$$\phi''' = -(n-1)\theta\phi'' + (n-1)\theta^2\phi' + (n-1)K\phi'$$

Therefore

$$\begin{aligned} \Gamma_2(\phi(\rho)) &= \phi'' L\left(\phi(\rho)\right) + \left(\phi'' - 1\right)\phi'' + \frac{1}{n-1}\left(\phi'' - 1\right)^2 \\ &\quad + (n-1)K\phi'^2 - \Gamma(L\phi(\rho), \phi(\rho)) \ . \end{aligned}$$

The curvature-dimension inequality thus may be written as

$$\begin{aligned} -\left(n-1\right)\Gamma(L\phi(\rho), \phi(\rho)) &\geq \left(L(\phi(\rho))\right)^2 - \left(n+1\right)\phi'' L(\phi(\rho)) \\ &\quad + \left(n+1\right)\phi'' - 1 \ . \end{aligned}$$

Since $F = L\phi(\rho) - 1$, the previous is thus equivalent to (3.4). ∎

4 Diameter bounds and volume comparison theorems.

It remains to apply the fundamental inequality to the distance function ρ to some point $o \in M$. We shall use very little of the information given by the basic result (3.1). The first elementary ingredient is the following.

Lemma 4.1. *Let o be a point in M, let ρ denotes the distance function from o and let $cut(o)$ denotes the cut-locus with respect to the point o. Then*

1) $M \backslash cut(o)$ is a star domain, and there is an increasing sequence of pre-compact domains D_n of D with smooth boundary ∂D_n such that $\bar{D}_n \subset D_{n+1}$ and $\cup_n D_n = D$. Each D_n is again a star domain, i.e. for each point p on ∂D_n, the segment of the unique minimal geodesic connecting o and p lies within D_n. Let ρ be the distance function from o. Then $\partial\rho/\partial\nu > 0$ on ∂D_n for each n, where ν is the normal vector field pointing outward on ∂D_n.

2) ρ is smooth on $M \backslash \left(cut(o) \cup \{o\}\right)$, ρ^2 is smooth at o.

These basic facts in Lemma 4.1 can be verified by using the exponential maps. Part 1) is called CALABI's Lemma.

Theorem 4.2. *Let ρ be the distance function from o, and suppose that the operator $L = \Delta + X$ satisfies $CD((n-1)K(\rho), n)$. We define the function θ_K to be the solution of equation (3.3) on $(0, \delta_K)$ with $\lim_{x\to 0} x\theta_K(x) = 1$, and δ_K to be the explosion time of this equation, if such exists.*

Then

1. *The diameter of (M, g) is bounded above by $2\delta_K$.*
2. *We have $L(\rho) \leq (n-1)\theta_K(\rho)$ on $M\backslash cut(o)$.*

Proof. — Notice that, thanks to the definition of ϕ_K (equation (3.3)), the equation $L(\rho) \leq (n-1)\theta_K(\rho)$ is equivalent to

$$F = L(\phi_K(\rho)) - 1 \leq 0. \tag{4.5}$$

Once again, we write θ and ϕ instead of θ_K and ϕ_K. We choose the function ϕ on $(0, \delta_K)$ satisfying equation (3.3) with boundary conditions $\phi(0) = \phi'(0) = 0$.

Let $\lambda = (n+1)\,\phi''$. Then the differential inequality for distance functions may be written as

$$F^2 + (2-\lambda)\,F \leq -\,(n-1)\,\phi'\Gamma(F, \rho)\ .$$

The behaviour of $L(\rho)$ near 0 is easy to obtain and comparable to the Euclidean case; therefore $\rho L(\rho)$ goes to $N-1$ as ρ goes to 0. As $\phi''(0) = 1/n$, the fact that $F < 0$ near 0 boils down to the fact that $n > N$. In the case $n = N$, we may replace n by $n+\epsilon$ (for any $\epsilon > 0$) and rescale K accordingly. Therefore, we may choose $r > 0$ so small such that $F < 0$ on $(0, r]$. (The boundary conditions on θ and ϕ at 0 are chosen just to satisfy those properties, and insure that $F < 0$ near 0.)

We claim that $F \leq 0$ still holds for any $x \in B_o(\delta_K) \cap (M \setminus \mathrm{cut}(o))$. To see that, consider the function F restricted on the geodesic line connecting o and x (denoted as I). We may assume that $\rho(x) > r$, since there is nothing to prove otherwise. If we choose ρ as the parameter along the geodesic, then the inequality for F on the line I may be written as

$$F^2 + (2-\lambda)\,F \leq -\,(n-1)\,\phi' F'\ .$$

Any function satisfying this differential inequality on $[r, \delta_K)$, with $F(r) < 0$, may not reach 0 before δ_K. In fact, if $\rho_0 \in (r, \delta_K)$ is the first zero of F, then $F < 0$ on (r, ρ_0), and if we set $G = 1/F$ (so that G explodes to $-\infty$ at ρ_0). The previous inequality becomes

$$1 + (2-\lambda)\,G \leq (n-1)\,\phi' G'$$

which cannot explode to $-\infty$ on (r, ρ_0) as $\phi' > 0$ and ϕ'' are bounded on compact subintervals of $(0, \delta_K)$, a contradiction to the assumption that $\rho_0 \in (r, \delta_K)$. Thus we have proved $L\phi(\rho) \leq 1$ on $B_o(\delta_K) \cap (M \setminus \mathrm{cut}(o))$. That is

$$L\rho \leq (n-1)\theta_K(\rho) \quad \text{on} \quad B_o(\delta_K) \cap (M \setminus \mathrm{cut}(o)) \ .$$

Then we show that if $x \in M$ such that $\rho(x) = \delta_K$, then $x \in \mathrm{cut}(o)$. If this is not the case, then choose y in $B_o(\delta_K)$ close to x. Then at y we have

$$(L\rho)\,(y) \leq (n-1)\theta_K(\rho).$$

Since $\delta_K > 0$ is the explosion point of θ_K, the right-hand side goes to $-\infty$ as $y \to x$. Hence

$$\lim_{y \to x} (L\rho)\,(y) = -\infty$$

which is impossible as ρ is smooth at x. Thus the diameter of M is bounded by $2\delta_K$ and also proves part 2).

Finally, using integration by parts and Calabi's lemma, it is easy to show that (4.5) holds in distribution, that is, for all non-negative function $\varphi \in C^2(M)$ with compact support

$$\int_M (L^*\varphi)\,\rho d\mu \leq \int_M \phi(n-1)\theta_K(\rho)d\mu,$$

with L^* the adjoint of L in $L^2(\mu)$. Now, the invariant measure is entirely characterized by the fact that, for any smooth compactly supported function g,

$$\int_M L(g)d\mu = 0.$$

applying this with a function $\Phi(\rho)$ shows that, if ν denotes the image measure of μ by the function ρ, one has formally

$$\int \phi(\rho)L(\rho)d\mu = -\int \phi'(\rho)d\nu(\rho).$$

This shows that, if $V(\rho)$ denotes the volume of the ball of radius ρ, in the distribution sense, we do have $V''/V'(\rho) = \mathbb{E}(L(\rho)/\rho)$, the expected value of $L(\rho)$ given ρ under the measure μ, or, in other words, we may identify it with the surface measure

$$V''/V'(r) = \int_{\rho(y)=r} L(\rho)(y)d\mu.$$

This leads to the volume comparison theorem $V''(\rho)/V'(\rho) \leq (n-1)\theta_K(\rho)$ in the distribution sense. This is exactly the content of volume comparison theorems.

To justify the computations, let D_n be one of the domains given by Lemma (4.1). First, we have, for any smooth function Φ on the real line, and for the invariant measure μ

$$\int_{D_n} (\Phi'(\rho)L(\rho) + \Phi''(\rho'))d\mu = \int_{\partial D_n} \Phi'(\rho)\frac{\partial \rho}{\partial \nu} d\mu_S,$$

where ν is the exterior normal derivative on the boundary ∂D_n and $d\mu_S$ is the surface measure on the boundary.

Setting $\Phi' = \phi$, and using the upper bound for $L(\rho)$ on D_n given by theorem (4.2),one gets, for any non negative function ϕ,

$$\int_{D_n} (\phi(\rho)(n-1)\theta_K(\rho) + \phi'(\rho))d\mu \geq 0.$$

Then we choose a compactly supported function ϕ and we let n go to infinity: the conclusion is the same, that is, for any smooth non negative compactly supported function ϕ and for the image measure ν of μ through the function ρ, one has

$$\int (\phi(x)(n-1)\theta_K(x) + \phi'(x))d\nu(x) \geq 0,$$

which exactly the content of the volume comparison theorem. ∎

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