Ricci flow on a 3-manifold with positive scalar curvature I

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Abstract

In this paper we consider Hamilton's Ricci flow on a 3-manifold having a metric of positive scalar curvature. We establish several *a priori* estimates for the Ricci flow which we believe are important in understanding possible singularities of the Ricci flow. For Ricci flow with initial metric of positive scalar curvature, we obtain a sharp estimate on the norm of the Ricci curvature in terms of the scalar curvature (which is not trivial even if the initial metric has non-negative Ricci curvature, a fact which is essential in Hamilton's estimates [8]), some L^2-estimates for the gradients of the Ricci curvature, and finally the Harnack type estimates for the Ricci curvature. These results are established through careful (and rather complicated and lengthy) computations, integration by parts and the maximum principles for parabolic equations. (**Revised version**).

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1 Introduction

In the seminal paper [8], R. S. Hamilton has introduced an evolution equation for metrics on a manifold, the Ricci flow equation, in order to obtain a "better metric" by deforming a Riemannian metric in a way of improving positivity of the Ricci curvature. Hamilton devised his heat flow type equation (originally motivated by Eells-Sampson's work [6] on the heat flow method for harmonic mappings) by considering the gradient vector field of the total scalar curvature functional on the space $\mathcal{M}(M)$ of metrics on a manifold M

$$E(g) = \int_M R_g \mathrm{d}\mu_g \ ; \qquad g \in \mathcal{M}(M)$$

where R_g and $d\mu_g$ are the scalar curvature and the volume measure with respect to the metric g, respectively. While Hamilton's Ricci flow is *not* the gradient flow of the energy functional E which is ill-posed, rather Hamilton has normalized the gradient flow equation of E to the following heat flow type equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{n}\sigma_g g_{ij} \tag{1.1}$$

where σ_g denotes the average of the scalar curvature $V_g^{-1}\int_M R\mathrm{d}\mu_g$. The variation characteristic of the Ricc flow (1.1) has been revealed by G. Perelman in recent works [15], [16]. In these papers Perelman presented powerful and substantial new ideas in order to understand the singularities of solutions to (1.1), for further information, see the recent book [5].

It is clear that the Ricci flow (1.1) preserves the total mass. After change of the space variable scale and re-parametrization of t (1.1) is equivalent to

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1.2}$$

which however is not volume-preserving. We will refer this equation as the *Ricci flow.* The parameter t (which has no geometric significance) is suppressed from notations if no confusion may arise.

If we express the Ricci tensor R_{ij} in terms of the metric tensor g_{ij}, we may see that the Ricci flow equation (1.2) is a highly non-linear system of second-order partial differential equations, which is not strictly parabolic, for which there is no general theory in hand to solve this kind of PDE. Even worse, there are known topological obstructions to the existence of a solution to (1.2).

The system (1.2) has a great interest by its own from a point-view of PDE theory, it however has significance in the resolution of the Poincaré conjecture. The Poincaré conjecture claims that a simply-connected 3-manifold is a sphere. If we run the Ricci flow (1.2) on a 3-manifold, and if we could show a limit metric exists with controlled bounds of the curvature tensor, then we could hope the limit metric must have constant Ricci curvature (see (1.1)), therefore the manifold must be a sphere. Thus Ricci flow is a very attractive approach to a possible positive answer to the Poincaré conjecture. This approach was worked out in the classical paper [8] for 3-manifolds with positive Ricci curvature by proving a series of striking *a priori* estimates for solutions of the Ricci flow.

Theorem 1.1 *(Hamilton [8], Main Theorem 1.1, page 255) On any 3-manifold with a metric of positive Ricci curvature, there is an Einstein metric with positive scalar curvature.*

1.1 Description of main results

Hamilton proved his theorem 1.1 through three *a priori* estimates for the Ricci flow with initial metric of positive curvature. The essential feature in this particular case is that the scalar curvature dominates the whole curvature tensor. Indeed if the Ricci tensor R_{ij} is positive then we have the following elementary fact

$$\frac{1}{3}R^2 \leq |R_{ij}|^2 \leq R^2 \ . \tag{1.3}$$

While in general case we still have the lower bound for $|R_{ij}|$, but there is no way to control $|R_{ij}|$ by its trace R. The first estimate Hamilton proved is the expected one: the Ricci flow improves the positivity of the Ricci tensor, thus if the initial metric has positive Ricci curvature then it remains so as long as the Ricci flow alive. This conclusion is proved by using a maximum principle for solutions to the tensor type parabolic equations (which indeed follows from the use of the classical maximum principle in parabolic theory).

One of our results shows that under the Ricci flow on a 3-manifold with positive scalar curvature, the squared norm of the Ricci tensor can be controlled in terms of its scalar curvature and the initial date. Indeed we prove a comparison theorem for the quantity $|R_{ij}|^2/R^2$. We then deduce precise bounds on the scalar curvature for the Ricci flow with positive scalar curvature.

By removing the positivity assumption of the Ricci curvature, three eigenvalues of the Ricci curvature under the Ricci flow may develop into a state of dispersion: one of eigenvalues may go to $-\infty$ while another to $+\infty$ but still keep the scalar curvature R bounded. Our above result just excludes this case if the initial metric has positive scalar curvature.

The second key estimate in [8], also the most striking result in [8], is an estimate which shows (after re-parametrization) the eigenvalues of the Ricci flow at each point approach each other. Hamilton achieved this claim by showing the variance of the three eigenvalues of the Ricci tensor decay like R^κ where $\kappa \in (1,2)$ depending on the positive lower bound of the Ricci curvature of the initial metric. It is not easy to see the curvature explodes in time for normalized Ricci flow (the volumes of the manifold are scaled to tend to zero), the variance of three eigenvalues of the Ricci curvature are easily seen as $|R_{ij}|^2 - R^2/3$, by (1.3) one might guess the variance $|R_{ij}|^2 - R^2/3$ possesses the same order of R^2. The striking fact is that indeed

$$\frac{|R_{ij}|^2 - R^2/3}{R^\kappa}$$

is bounded for some $\kappa < 2$! That is to say, $|R_{ij}|^2 - R^2/3$ explodes much slower then R^2, so that, after re-scaling back to the un-normalized Ricci flow, it shows the variance of the eigenvalues of the Ricci tensor goes to zero, hence proves the claim. The positive Ricci curvature assumption is washed down to an elementary fact recorded in Lemma 5.2. This *core estimate* (see [8], Theorem 10.1, page 283) is definitely false if the positivity assumption on the Ricci tensor is removed, and it seems no replacement could be easily recognized without an assumption on the Ricci curvature.

Finally in order to show a smooth metric does exist, and has constant Ricci curvature, Hamilton [8] established an important gradient estimate for $|\nabla R|$ in terms of R and $|R_{ij}|$ ([8], Theorem 11.1, page 287), which in turn implies that, for the un-normalized Ricci flow, $|\nabla R|$ goes to zero. This estimate was proved by using the previous crucial estimate on the variance $|R_{ij}|^2 - R^2/3$ and a clever use of the Bianchi identity: $|\nabla \mathrm{Ric}|^2 \geq 7|\nabla R|^2/20$, instead of the trivial one $|\nabla \mathrm{Ric}|^2 \geq |\nabla R|^2/3$. The key question we would ask is what kind of gradient estimates for the scalar curvature and for the Ricci curvature can we expect without the essential estimates on the variance of three eigenvalues of the Ricci tensor?

This paper gives some partial answers to this question: we establish a weighted integral estimate for $|\nabla R|^2$ and a Harnack type estimate for the Ricci curvature. We hope these estimates would help us to understand the singularities in the Ricci flow on a 3-manifold with positive scalar curvature, and complete the Hamilton's program [11] for these 3-manifolds. For recent exciting development, see the papers by G. Perelman and Chow and Knopf's excellent recent book [5], and also [18] and Ecker's book for the mean curvature flow.

The main simplification for 3-manifolds comes from the fact that the full curvature tensor R_{ijkl} may be read out from the Ricci tensor (R_{ij}):

$$\begin{aligned} R_{ijkl} &= R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il} \\ &\quad -\frac{1}{2}R\left(g_{jl}g_{ik} - g_{jk}g_{il}\right) \ . \end{aligned} \tag{1.4}$$

In all computations the Bianchi identity will play an essential role. The following are the several forms we will need. The first one is

$$\nabla_m R_{ijkl} + \nabla_k R_{ijlm} + \nabla_l R_{ijmk} = 0 \ . \tag{1.5}$$

Taking trace over indices m and j we obtain

$$g^{ab}\nabla_a R_{ibkl} = \nabla_l R_{ik} - \nabla_k R_{il} \tag{1.6}$$

and taking trace again over i and k we thus have

$$\nabla_k R = 2g^{ij}\nabla_j R_{ik} \ . \tag{1.7}$$

That is $\nabla^a R_{ka} = \frac{1}{2}\nabla_k R$. The equation (1.7) may be write as $\nabla^a G(\mathrm{Ric})_{ak} = 0$ for all k, where

$$G(V)_{ij} = V_{ij} - \frac{1}{2}\mathrm{tr}_g(V)g_{ij}$$

for a symmetric tensor (V_{ij}).

1.2 Example

In order to appreciate the difficulty induced by the topology for the Ricci flow, let us examine a typical example not covered by the theorem of Hamilton [8].

Let $M = S^2 \times S^1$ endowed with the standard product metric. Choose a coordinate system, and write (h_{ij}) to be the standard metric on the sphere S^2 with sectional curvature 1. The standard product metric on M may be written as

$$(g(0)_{ij}) = \begin{pmatrix} (h_{ij}) & 0 \\ 0 & 1 \end{pmatrix}$$

and the Ricci curvature tensor

$$(R(0)_{ij}) = \begin{pmatrix} (h_{ij}) & 0 \\ 0 & 0 \end{pmatrix}$$

so that the scalar curvature $R(0) = 2$, a constant. Consider the Ricci flow with the initial metric $(g(0)_{ij})$:

$$\frac{\partial}{\partial t}(g(t)_{ij}) = -2R(t)_{ij} + \frac{2}{3}\sigma(t)g(t)_{ij} \ .$$

Let us assume that the solution flow has the following form

$$(g(t)_{ij}) = \begin{pmatrix} a(t)(h_{ij}) & 0 \\ 0 & f(t) \end{pmatrix} . \tag{1.8}$$

Since

$$(R(t)_{ij}) = \begin{pmatrix} (h_{ij}) & 0 \\ 0 & 0 \end{pmatrix}$$

so that

$$R(t) = g(t)^{ij} R(t)_{ij} = \frac{2}{a(t)} .$$

Therefore $\sigma(t) = \frac{2}{a(t)}$. Thus the Ricci flow equation is equivalent to the following system of ordinary differential equations

$$\begin{aligned} \frac{\partial}{\partial t} a(t) &= -\frac{2}{3} ; \qquad a(0) = 1 ; \\ \frac{\partial}{\partial t} f(t) &= \frac{4}{3} \frac{f(t)}{a(t)} ; \qquad f(0) = 1 \end{aligned}$$

which have solutions

$$a(t) = 1 - \frac{2}{3} t ; \quad f(t) = \frac{1}{\left(1 - \frac{2}{3} t\right)^2} .$$

The condition that $a > 0$ then yields that $t < \frac{3}{2}$. Notice that as $t \uparrow \frac{3}{2}$, then $\sigma(t)$ explodes to $+\infty$, though the volumes still keep as a constant. On the other hand the Ricci tensor for all t has constant eigenvalues, and therefore $\nabla_k R_{ij} = 0$. Let us write down the solution flow as the following explicit form

$$(g(t)_{ij}) = \begin{pmatrix} (1 - \frac{2t}{3})(h_{ij}) & 0 \\ 0 & \left(1 - \frac{2}{3} t\right)^{-2} \end{pmatrix}$$

or in terms of the mean value of the scalar curvature $\sigma(t) = 2/\left(1 - \frac{2}{3} t\right)$ we have the solution metric

$$(g(t)_{ij}) = \begin{pmatrix} \frac{2}{\sigma(t)} (h_{ij}) & 0 \\ 0 & \frac{\sigma(t)^2}{4} \end{pmatrix} .$$

The Ricci flow $(g(t)_{ij})$ on $S^2 \times S^1$ explains the difficulty about Ricci flow on a general 3-manifold: a limit metric (even after a scaling) to the Ricci flow may not exist.

However we may also consider the following form of Ricci flow

$$\frac{\partial}{\partial t} (g(t)_{ij}) = -2R(t)_{ij} + \frac{2}{3} \left(\alpha \sigma(t) + \beta\right) g(t)_{ij} \tag{1.9}$$

where α and β are two constants to be chosen according to the type of 3-manifolds. We still search for a solution flow with the form (1.8), which leads to again $\sigma(t) = \frac{2}{a(t)}$ and

$$\begin{aligned}\frac{\partial}{\partial t}a(t) &= -2 + \frac{2}{3}(2\alpha + \beta a(t)) \ ; \qquad a(0) = 1 \\ \frac{\partial}{\partial t}f(t) &= \frac{2}{3}\left(\alpha\frac{2}{a(t)} + \beta\right)f(t) \ ; \qquad f(0) = 1 \ .\end{aligned}$$

Obviously $\alpha = \frac{3}{2}$ is a good choice for the manifold $S^2 \times S^1$. With this choice $\alpha = \frac{3}{2}$ and any $\beta \geq 0$ we have

$$a(t) = \exp\left(\frac{2}{3}\beta t\right)$$

and

$$f(t) = \exp\left[\frac{3}{\beta} - \frac{3}{\beta}\exp\left(-\frac{2}{3}\beta t\right) + \frac{2}{3}\beta t\right] \ .$$

Thus the solution flow to (1.9) exists for all time t, the eigenvalues of the Ricci tensor are constant (in space variables), and

$$\sigma(t) = 2\exp\left(-\frac{2}{3}\beta t\right)$$

goes to zero as $t \to \infty$ if $\beta > 0$. In particular if $\alpha = \frac{3}{2}$ and $\beta = 0$, then

$$a(t) = 1 \ ; \qquad f(t) = \exp(2t) \ .$$

The solution flow exists, but no limit exist as $t \to \infty$.

2 Deformation of metrics

Let $(g(t)_{ij})$ be a family of metrics on M satisfying the following equation

$$\frac{\partial}{\partial t}g_{ij} = -2h_{ij}$$

where h_{ij} is a family of symmetric tensors depending on t maybe on $(g(t)_{ij})$ as well. If no confusion may arise, the parameter t will be suppressed. The inverse of (g_{ij}) then evolves according to the equation

$$\frac{\partial}{\partial t}g^{ij} = 2g^{ib}g^{aj}h_{ab} \tag{2.1}$$

which may be written as $\frac{\partial}{\partial t}g^{ij} = 2h^{ij}$ (the indices in $h(t)^{ij}$ are lifted with respect to the metric $g(t)_{ij}$, this remark applies to other similar notations).

Although for each t the Christoffel symbol Γ^j_{ik} is not a tensor, is however $\frac{\partial \Gamma^j_{ik}}{\partial t}$ its derivative in t. Recall that

$$\Gamma^j_{ik} = \frac{1}{2} g^{jp} \left(\frac{\partial g_{ip}}{\partial x_k} + \frac{\partial g_{kp}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_p} \right)$$

so that

$$\frac{\partial \Gamma^j_{ik}}{\partial t} = -g^{jp} \left\{ (\nabla_k h_{ip}) + (\nabla_i h_{kp}) - (\nabla_p h_{ik}) \right\} \ . \tag{2.2}$$

From which it follows immediately the variation for the Ricci curvature tensor, $\frac{\partial R_{ik}}{\partial t}$. Indeed under a normal coordinate system

$$\begin{aligned} \frac{\partial R_{ik}}{\partial t} &= \partial_j \frac{\partial \Gamma^j_{ik}}{\partial t} - \partial_i \frac{\partial \Gamma^j_{jk}}{\partial t} \\ &= \nabla_j \frac{\partial \Gamma^j_{ik}}{\partial t} - \nabla_i \frac{\partial \Gamma^j_{jk}}{\partial t} \end{aligned}$$

which together (2.2) implies the following

$$\frac{\partial}{\partial t} R_{ij} = \Delta h_{ij} + \nabla_i \nabla_j \mathrm{tr}_g(h_{ab}) - (\nabla^a \nabla_j h_{ia} + \nabla^a \nabla_i h_{ja}) \ . \tag{2.3}$$

After taking trace

$$\frac{\partial}{\partial t} R = 2\Delta \mathrm{tr}_g(h_{ab}) + 2h_{ab}R^{ab} - 2\nabla^a \nabla^b h_{ab} \ .$$

We may exchange the order of taking co-variant derivatives in the last term of (2.3) via the following

Lemma 2.1 *Let (h_{ij}) be a symmetric tensor on a 3-manifold. Then*

$$\begin{aligned} & (\nabla^a \nabla_j h_{ia} + \nabla^a \nabla_i h_{ja}) \\ = \ & (\nabla_i \nabla^a h_{ja} + \nabla_j \nabla^a h_{ia}) - 2R^{ab} h_{ab} g_{ij} - 2tr_g(h_{ab}) R_{ij} \\ & -Rh_{ij} + Rtr_g(h_{ab}) g_{ij} + 3g^{ab} \left(h_{ia} R_{jb} + h_{ja} R_{ib} \right) \ . \end{aligned} \tag{2.4}$$

Proof. Indeed, by the Ricci identity for symmetric tensors,

$$\nabla_j \nabla_k h_{il} = \nabla_k \nabla_j h_{il} + h_{al} g^{ab} R_{kjbi} + h_{ia} g^{ab} R_{kjbl}$$

and

$$\nabla_l \nabla_i h_{kj} = \nabla_i \nabla_l h_{kj} + h_{aj} g^{ab} R_{ilbk} + h_{ka} g^{ab} R_{ilbj}$$

we thus have

$$\begin{aligned}
& g^{jl}\left(\nabla_j\nabla_k h_{il}+\nabla_l\nabla_i h_{kj}\right) \\
= & g^{jl}\left(\nabla_i\nabla_l h_{kj}+\nabla_k\nabla_j h_{il}\right) \\
& +g^{jl}h_{aj}g^{ab}\left(R_{ilbk}+R_{iblk}\right)+h_{ka}g^{jl}g^{ab}R_{ijbl}+h_{ia}g^{jl}g^{ab}R_{kjbl} \\
= & \left(g^{jl}\nabla_i\nabla_l h_{kj}+\nabla_k\nabla_j h_{il}\right) \\
& -2h_{aj}g^{jl}g^{ab}R_{ibkl}+h_{ka}g^{jl}g^{ab}R_{ijbl}+h_{ia}g^{jl}g^{ab}R_{kjbl}\ . \qquad (2.5)
\end{aligned}$$

However in 3-manifolds, the full Ricci curvature R_{ijkl} may be expressed in terms of the Ricci tensor, equation (1.4), we may easily verify the followings

$$\begin{aligned}
h_{aj}g^{jl}g^{ab}R_{ibkl} = & \left(g^{jl}g^{ab}h_{aj}R_{bl}-\frac{1}{2}R\mathrm{tr}_g(h_{ab})\right)g_{ik}+\frac{1}{2}Rh_{ik} \\
& -g^{ab}h_{ai}R_{bk}-g^{ab}h_{kb}R_{ia}+\mathrm{tr}_g(h_{ab})R_{ik},
\end{aligned}$$

$$h_{ka}g^{jl}g^{ab}R_{ijbl}=g^{ab}h_{ka}R_{ib}$$

and

$$h_{ia}g^{jl}g^{ab}R_{kjbl}=h_{ia}g^{ab}R_{kb}\ ,$$

substituting these equations into (2.5) to get the conclusion. ■

Therefore

Lemma 2.2 *On 3-manifolds, if* $\frac{\partial}{\partial t}g_{ij}=-2h_{ij}$, *then*

$$\begin{aligned}
\frac{\partial}{\partial t}R_{ij} = & \Delta h_{ij}+\nabla_i\nabla_j tr_g(h_{ab})-\left\{\nabla_i\nabla^a h_{ja}+\nabla_j\nabla^a h_{ia}\right\} \\
& +\left\{2h_{ab}R^{ab}-Rtr_g(h_{ab})\right\}g_{ij}+Rh_{ij} \\
& +2tr_g(h_{ab})R_{ij}-3g^{ab}\left(h_{ia}R_{jb}+h_{ja}R_{ib}\right)\ . \qquad (2.6)
\end{aligned}$$

Corollary 2.3 *On 3-manifolds, if* $\frac{\partial}{\partial t}g_{ij}=-2h_{ij}$, *then*

$$\frac{\partial}{\partial t}R=2\Delta tr_g(h_{ab})+2h_{ab}R^{ab}-2\left(\nabla^b\nabla^a h_{ab}\right)\ . \qquad (2.7)$$

3 Hamilton's Ricci flow

Although most of our results will be stated only for the normalized Ricci flow

$$\frac{\partial}{\partial t}g_{ij}=-2R_{ij}$$

it may be useful to consider a general evolution equation for metrics. Let $\mathcal{M}(M)$ denote the space of all metrics on the manifold M of dimension 3, and let

$$h_{ij} = R_{ij} - \frac{1}{3}\left(\alpha\sigma + \beta\right) g_{ij} \tag{3.1}$$

where α and β are two constant, R_{ij} is the Ricci tensor of g_{ij}, and σ is the mean value of the scalar curvature

$$\sigma = \frac{1}{V_g}\int_M R_g \mathrm{d}\mu_g$$

which is a functional on $\mathcal{M}(M)$. Therefore (h_{ij}) is a symmetric tensor which may be seen as a functional on $\mathcal{M}(M)$. It is obvious by definition

$$\mathrm{tr}_g(h_{ij}) = R - (\alpha\sigma + \beta)$$

The Ricci flow (with parameters α and β) is the following equation on the metrics (g_{ij}):

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2h_{ij} \\ &= -2\left(R_{ij} - \frac{1}{3}\left(\alpha\sigma + \beta\right) g_{ij}\right) . \end{aligned} \tag{3.2}$$

The Ricci flow (3.2) may be written in different forms:

$$\frac{\partial}{\partial t} g^{ij} = 2\left(R^{ij} - \frac{1}{3}\left(\alpha\sigma + \beta\right) g^{ij}\right) \tag{3.3}$$

and

$$\left(\Delta - \frac{\partial}{\partial t}\right) g_{ij} = -2\left(R_{ij} - \frac{1}{3}\left(\alpha\sigma + \beta\right) g_{ij}\right)$$

where Δ denotes the trace Laplacian $g^{ij}\nabla_i\nabla_j$ associated with the metric $(g(t)_{ij})$.

In what follows we assume that $(g(t)_{ij})$ (but t will be suppressed from notations, unless specified) is the maximum solution to the Ricci flow (3.2). For simplicity we use $A(t)$ to denote $\alpha\sigma(t) + \beta$, and V_g denote the volume of (M, g).

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ denote the three eigenvalues of the Ricci tensor (R_{ij}). Then

$$R = \lambda_1 + \lambda_2 + \lambda_3 \; ; \qquad S = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

where $S = |R_{ij}|^2$. We also define

$$T = \lambda_1^3 + \lambda_2^3 + \lambda_3^3, \quad U = \lambda_1^4 + \lambda_2^4 + \lambda_3^4 .$$

Then

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = \frac{1}{2}(R^2 - S) ,$$

$$\lambda_1\lambda_2\lambda_3 = \frac{1}{6}R^3 - \frac{1}{2}RS + \frac{1}{3}T$$

and

$$U = \frac{4}{3}RT - R^2S + \frac{1}{2}S^2 + \frac{1}{6}R^4 \ .$$

The variance of λ_1, λ_2 and λ_3 is $S - \frac{1}{3}R^2$ which will be denoted by Y.

Let $S_{ij} = g^{ab}R_{ib}R_{aj}$ and $T_{ij} = g^{ab}R_{ib}S_{aj}$. Then $S = \mathrm{tr}_g(S_{ij})$ and $T = \mathrm{tr}_g(T_{ij})$.

By equation (2.6) we easily see that

$$\left(\Delta - \frac{\partial}{\partial t}\right) R_{ij} = 6S_{ij} - 3RR_{ij} + \left(R^2 - 2S\right) g_{ij} \tag{3.4}$$

which has a form independent of α or β. From here we may compute the evolution for the scalar curvature

$$\left(\Delta - \frac{\partial}{\partial t}\right) R = -2\left(S - \frac{1}{3}AR\right) \ . \tag{3.5}$$

We next compute the evolution equations for the tensors (S_{ij}) and (T_{ij}).

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right) S_{ij} &= g^{ab}R_{aj}\left(\Delta - \frac{\partial}{\partial t}\right) R_{ib} - R_{ib}R_{aj}\frac{\partial}{\partial t}g^{ab} \\
&\quad + g^{ab}R_{ib}\left(\Delta - \frac{\partial}{\partial t}\right) R_{aj} + 2g^{ab}\langle \nabla R_{ib}, \nabla R_{aj}\rangle \\
&= 6g^{ab}R_{aj}S_{ib} + 6g^{ab}R_{ib}S_{aj} - 2R_{ib}R_{aj}R^{ab} \\
&\quad - 6RS_{ij} + 2\left(R^2 - 2S\right) R_{ij} + \frac{2}{3}AS_{ij} \\
&\quad + 2g^{ab}\langle \nabla R_{ib}, \nabla R_{aj}\rangle
\end{aligned}$$

that is

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right) S_{ij} &= 10T_{ij} - 6RS_{ij} + 2\left(R^2 - 2S\right) R_{ij} \\
&\quad + \frac{2}{3}AS_{ij} + 2g^{ab}(\nabla_k R_{ib})(\nabla^k R_{aj}) \ .
\end{aligned} \tag{3.6}$$

Taking trace we obtain

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right) S &= 2\left(R^3 - 5RS + 4T\right) \\
&\quad + \frac{4}{3}AS + 2|\nabla_k R_{ij}|^2 \ .
\end{aligned} \tag{3.7}$$

These are the evolution equations computed in Hamilton [8] (in the case $\alpha = \beta = 0$) via his evolution equation for the full curvature tensor.

Similarly we have

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) T_{ij} &= 14 g^{ab} R_{ib} T_{aj} - 9 R T_{ij} + 3\left(R^2 - 2S\right) S_{ij} \\ &\quad + \frac{4}{3} A T_{ij} + 2 R^{ab} (\nabla^k R_{ib})(\nabla_k R_{aj}) \\ &\quad + 2 (\nabla^k R^{ab}) \left\{ R_{ib} (\nabla_k R_{aj}) + R_{aj} (\nabla_k R_{ib}) \right\} \ . \qquad (3.8)\end{aligned}$$

It follows that

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) T &= 12 U - 9 R T + 3\left(R^2 - 2S\right) S + 2 A T \\ &\quad + 6 g^{ab} R_{ia} (\nabla^k R^{ij})(\nabla_k R_{bj})\end{aligned}$$

together with the fact that

$$U = \frac{4}{3} R T - R^2 S + \frac{1}{2} S^2 + \frac{1}{6} R^4$$

we thus establish the following evolution equation

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) T &= 7 R T - 9 R^2 S + 2 R^4 + 2 A T \\ &\quad + 6 g^{ab} R_{ia} (\nabla^k R^{ij})(\nabla_k R_{bj}) \ . \qquad (3.9)\end{aligned}$$

In what follows, we only consider 3-manifolds with a metric of positive scalar curvature. Let $(g(t)_{ij})$ be the maximum solution to the Ricci flow

$$\left(\Delta - \frac{\partial}{\partial t}\right) g_{ij} = -2 \left(R_{ij} - \frac{1}{3} A g_{ij} \right)$$

on a 3-manifold M with initial metric $g(0)_{ij}$ of positive *constant* scalar curvature $R(0)$, unless otherwise specified.

Let μ_t denote the volume measure associated with the solution metric $(g(t)_{ij})$ (at time t), and let

$$M_t = \frac{\mathrm{d}\mu_t}{\mathrm{d}\mu_0} = \sqrt{\frac{\det(g(t)_{ij})}{\det(g(0)_{ij})}} \ .$$

Then an elementary computation shows that

$$\begin{aligned}\frac{\partial}{\partial t} \log M_t &= -\mathrm{tr}_g(h_{ij}) \\ &= -R + A \ .\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_{g(t)} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M M_t \mathrm{d}\mu_0 \\ &= \int_M \left(\frac{\mathrm{d}}{\mathrm{d}t}\log M_t\right)\mathrm{d}\mu_t \\ &= \int_M (-R+A)\,\mathrm{d}\mu_t \\ &= (\beta + (\alpha-1)\,\sigma)\,V_{g(t)}\end{aligned}$$

so that the following

Lemma 3.1 *The volume function $t \to V_{g(t)}$ is constant if $\beta = 0$ and $\alpha = 1$. For general α and β we have*

$$\frac{d}{dt}\log V_{g(t)} = \beta + (\alpha-1)\,\sigma(t)$$

where

$$\sigma(t) = \frac{1}{V_{g(t)}}\int_M R_{g(t)}\,d\mu_t\ .$$

Since

$$\begin{aligned}\frac{\partial}{\partial t}\log \mathrm{M}(t) &= -\mathrm{tr}_g\left(R_{ij} - \frac{1}{3}Ag_{ij}\right) \\ &= -R(t,\cdot) + A(t)\end{aligned}$$

and therefore

$$\log \mathrm{M}(t) = -\int_0^t (R(s,\cdot) - A(s))\,\mathrm{d}s\ . \tag{3.10}$$

Taking Laplacian of logM(t) in the last equation we obtain

$$\begin{aligned}\Delta \log \mathrm{M}(t) &= -\int_0^t (\Delta R)(s,\cdot)\mathrm{d}s \\ &= R(0) - R + 2\int_0^t \left(S - \frac{1}{3}AR\right)\end{aligned}$$

so that

$$\left(\Delta - \frac{\partial}{\partial t}\right)\log \mathrm{M}(t) = R(0) - A(t) + 2\int_0^t \left(S - \frac{1}{3}AR\right)\ . \tag{3.11}$$

The scalar curvature R remains positive if the initial metric possesses positive scalar curvature. To see this let us consider the function

$$K = \mathrm{e}^{\frac{2}{3}\int_0^t A(s)\mathrm{d}s}R\ , \tag{3.12}$$

and we can easily see the evolution equation for K is given as

$$\left(\Delta - \frac{\partial}{\partial t}\right) K = -2\mathrm{e}^{\frac{2}{3}\int_0^t A(s)\mathrm{d}s} S \tag{3.13}$$

so the claim follows easily from the maximum principle.

4 Control the curvature tensor

Consider the maximum solution $(g(t)_{ij})$ to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2\left(R_{ij} - \frac{1}{3} A g_{ij}\right) \tag{4.1}$$

where $A(t) = \alpha\sigma(t) + \beta$ as before.

Recall that the scalar curvature R satisfies the following parabolic equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) R = -2S + \frac{2}{3} A R$$

so that

$$\left(\Delta - \frac{\partial}{\partial t}\right)\left(\mathrm{e}^{-\frac{2}{3}\int_0^t A(s)\mathrm{d}s} R\right) = -2\mathrm{e}^{-\frac{2}{3}\int_0^t A(s)\mathrm{d}s} S \ ,$$

therefore by the maximum principle, it follows that R remains positive if for the initial metric $R(0)$ is positive.

Our first result shows that if the initial metric possesses positive scalar curvature, then the full curvature tensor of $(g(t)_{ij})$ may be controlled by its scalar curvature.

Let $V_{ij} = R_{ij} - \varepsilon R g_{ij}$ where ε is a constant. Then by the evolution equations for (R_{ij}), R and the Ricci flow (4.1) we deduce that

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right) V_{ij} &= 6 g^{pq} V_{pj} V_{iq} + (10\varepsilon - 3) R V_{ij} \\ &\quad + \left(2(\varepsilon - 1) S + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2\right) g_{ij} \ . \end{aligned} \tag{4.2}$$

The very nice feature of this identity is that it involves α, β and σ only through the symmetric tensor V_{ij}, which takes the same form for all α, β.

Theorem 4.1 *Let M be a closed 3-manifold. If $\varepsilon \leq 1/3$ is a constant such that $R_{ij} - \varepsilon R g_{ij} \geq 0$ at $t = 0$, so does it remain. If $\varepsilon \geq 1$ and at $t = 0$, $R_{ij} - \varepsilon R g_{ij} \leq 0$, then the inequality remains to hold for all $t > 0$.*

Proof. We prove these conclusions by using the maximum principle to the tensor type parabolic equation (4.2) and (V_{ij}) (see [8], Theorem 9.1, page 279). Let us prove the first conclusion. Suppose $\lambda \geq \mu \geq \nu$ are eigenvalues of the Ricci tensor R_{ij}. Then $S = \lambda^2 + \mu^2 + \nu^2$ and $R = \lambda + \mu + \nu$. If $\xi \neq 0$ such that $V_{ij}\xi^j = 0$ for all i. Then one of the eigenvalues of V_{ij} is zero. Since the eigenvalues of V_{ij} are $\lambda - \varepsilon R$, $\mu - \varepsilon R$, $\nu - \varepsilon R$, so we may assume that $\nu - \varepsilon R = 0$. Hence

$$\lambda + \mu = (1 - \varepsilon)R \ , \quad S = \lambda^2 + \mu^2 + \varepsilon^2 R^2 \ .$$

Since

$$\begin{aligned}
& 2\,(\varepsilon - 1)\,S + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \\
= \ & 2\,(\varepsilon - 1)\left(\lambda^2 + \mu^2 + \varepsilon^2 R^2\right) + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \\
= \ & 2\,(\varepsilon - 1)\,(\lambda^2 + \mu^2) + 2\,(\varepsilon - 1)\,\varepsilon^2 R^2 + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \\
\leq \ & (\varepsilon - 1)\,(\lambda + \mu)^2 + 2\,(\varepsilon - 1)\,\varepsilon^2 R^2 + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \\
= \ & (\varepsilon - 1)\,(1 - \varepsilon)^2 R^2 + 2\,(\varepsilon - 1)\,\varepsilon^2 R^2 + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \\
= \ & (3\varepsilon - 1)\,\varepsilon^2 R^2 \leq 0 \ ,
\end{aligned}$$

where the first inequality follows from the Cauchy inequality and the last one follows our assumption $\varepsilon \leq 1/3$. Now the conclusion follows from equation (4.2) and the maximum principle.

To show the second conclusion, we apply the maximum principle to $-V_{ij}$ and use the fact that $\varepsilon \geq 1$. The only thing then one should notice is the following inequality

$$2\,(\varepsilon - 1)\,S + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \geq 7R^2/16 \geq 0 \ .$$

■

Corollary 4.2 *On a closed 3-manifold M and suppose $R(0) > 0$ (so $R(t) > 0$ for all t).*

1. If $R(0)_{ij} \geq 0$, then $R^2/3 \leq S \leq R^2$. If $\varepsilon > 0$ such that

$$R(0)_{ij} \geq -\varepsilon R(0) g(0)_{ij}$$

then

$$R^2/3 \leq S \leq (1 + 4\varepsilon + 6\varepsilon^2)R^2 \ . \tag{4.3}$$

2. If $R(0)_{ij} \geq bR(0)g(0)_{ij}$ for some constant $0 \leq b \leq 1/3$, then

$$R^2/3 \leq S \leq (1 - 4b + 6b^2)R^2 \ .$$

Proof. By Theorem 4.1, $R_{ij} \geq -\varepsilon R g_{ij}$ as long as the solution exists. Let $\lambda \geq \mu \geq \nu$ be the eigenvalues of the Ricci tensor R_{ij}. Then $\lambda, \mu, \nu \geq -\varepsilon R$. Suppose $\mu,\ \nu < 0$, then

$$\begin{aligned}\lambda^2+\mu^2+\nu^2 &\leq (R+|\mu|+|v|)^2+|\mu|^2+|v|^2 \\ &= (R+2\varepsilon R)^2+2\varepsilon^2R^2 \\ &= \left(1+4\varepsilon+6\varepsilon^2\right)R^2 \ ,\end{aligned}$$

and similarly when $\lambda \geq \mu \geq 0$ and $\nu < 0$ we have

$$\begin{aligned}\lambda^2+\mu^2+\nu^2 &\leq (\lambda+\mu)^2+v^2 \\ &= (R+|\nu|)^2+|v|^2 \\ &\leq (R+\varepsilon R)^2+\varepsilon^2R^2\end{aligned}$$

and while $\nu \geq 0$ we then clearly have $\lambda^2+\mu^2+\nu^2 \leq R^2$. Therefore the conclusion follows. ■

The estimate (4.3) shows that the scalar curvature dominates the full curvature if the initial metric has positive scalar curvature, which however is a rough estimate. We will give a sharp estimate in next section (Theorem 5.6 below).

5 Control the norm of the Ricci tensor

In this section we work with the Ricci flow on a 3-manifold

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

with initial metric such that $R(0) > 0$. Thus $R \geq R(0) > 0$ and the variance of three eigenvalues of the Ricci tensor (R_{ij}), $Y \equiv S - \frac{1}{3}R^2$ is non-negative as well. The third variable in our mind is not T whose sign is difficulty to determined. Motivated by the fundamental work [8], we consider the polynomial of the eigenvalues of (R_{ij}):

$$P = S^2 + \frac{1}{2}R^4 - \frac{5}{2}R^2S + 2RT \tag{5.1}$$

as the third independent variable, which is non-negative, as showed in [8].

Let us begin with a geometric explanation about the polynomial P. In terms of the three eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of the Ricci tensor (R_{ij}) ([8], Lemma 10.6, page 285)

$$P = (\lambda_1-\lambda_2)^2\left[\lambda_1^2+(\lambda_1+\lambda_2)(\lambda_2-\lambda_3)\right]+\lambda_3^2(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\ . \tag{5.2}$$

Hence we establish the following

Lemma 5.1 *We have*

1) $P \geq 0$ *;*

2) $P = 0$ *if and only if 2a)* $\lambda_1 = \lambda_2 = \lambda_3$*; or 2b) one of eigenvalues* λ_i *is zero, the other two are equal number.*

Together with this lemma, the following lemma explains why the assumption of positive Ricci curvature is special.

Lemma 5.2 *Suppose that* $\lambda_1 + \lambda_2 + \lambda_3 \geq 0$ *and*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \varepsilon \left(\lambda_1 + \lambda_2 + \lambda_3\right) \tag{5.3}$$

for some $\varepsilon \in [0, 1/3]$*, then*

$$P \geq \varepsilon^2 S \left(S - R^2/3\right) \ . \tag{5.4}$$

Therefore all evolution equations must be written in terms of these variables.

$$\left(\Delta - \frac{\partial}{\partial t}\right) R = -2S \ \ , \tag{5.5}$$

$$\left(\Delta - \frac{\partial}{\partial t}\right) S = 2R^3 - 10RS + 8T + 2|\nabla_k R_{ij}|^2 \ . \tag{5.6}$$

To take care the homogeneity, we may use R^2 instead of R

$$\left(\Delta - \frac{\partial}{\partial t}\right) R^2 = -4RS + 2|\nabla R|^2 \ ,$$

and for S we prefer to use $Y = S - \frac{1}{3}R^2$ then S:

$$\left(\Delta - \frac{\partial}{\partial t}\right) Y = 2R^3 - \frac{26}{3}RS + 8T + 2|\nabla_k R_{ij}|^2 - \frac{2}{3}|\nabla R|^2 \ .$$

Therefore, for any constant κ,

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right) \frac{Y}{R^\kappa} \ &= \ \frac{1}{R^\kappa}\left(\Delta - \frac{\partial}{\partial t}\right) Y + Y \left(\Delta - \frac{\partial}{\partial t}\right) R^{-\kappa} \\
&\quad +2\langle \nabla Y, \nabla R^{-\kappa}\rangle \\
&= \ \left[R^3 - \frac{13}{3}RS + 4T + \kappa \frac{S}{R} Y\right] \frac{2}{R^\kappa} \\
&\quad + \left[|\nabla_k R_{ij}|^2 - \frac{1}{3}|\nabla R|^2 + \frac{\kappa(\kappa+1)}{2} Y \frac{|\nabla R|^2}{R^2}\right] \frac{2}{R^\kappa} \\
&\quad +2\langle \nabla Y, \nabla R^{-\kappa}\rangle \ .
\end{aligned}$$

Lemma 5.3 *We have the following elementary facts*

$$\langle \nabla Y, \nabla R^{-\kappa}\rangle = \frac{2}{3}\kappa|\nabla R|^2\frac{1}{R^\kappa} - \kappa\frac{1}{R}\langle \nabla S, \nabla R\rangle\frac{1}{R^\kappa} \tag{5.7}$$

and

$$\langle \nabla Y, \nabla R^{-\kappa}\rangle = -\kappa^2 Y\frac{|\nabla R|^2}{R^2}\frac{1}{R^\kappa} - \kappa\langle \nabla \log R, \nabla\frac{Y}{R^\kappa}\rangle\ . \tag{5.8}$$

Therefore we may write the evolution equation for Y/R^κ into different forms. For the one we need, we decompose $2\langle \nabla Y, \nabla R^{-\kappa}\rangle$ into

$$\begin{aligned} & 2\theta\langle \nabla Y, \nabla R^{-\kappa}\rangle + 2(1-\theta)\langle \nabla Y, \nabla R^{-\kappa}\rangle \\ = \; & -\theta\kappa^2 Y\frac{|\nabla R|^2}{R^2}\frac{2}{R^\kappa} - 2\theta\kappa\langle \nabla \log R, \nabla\frac{Y}{R^\kappa}\rangle \\ & + \left(\frac{2}{3}(1-\theta)\kappa|\nabla R|^2 - \kappa(1-\theta)\frac{1}{R}\langle \nabla S, \nabla R\rangle\right)\frac{2}{R^\kappa} \end{aligned}$$

so that

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right)\frac{Y}{R^\kappa} = \; & \left(R^3 - \frac{13}{3}RS + 4T + \kappa\frac{S}{R}Y\right)\frac{2}{R^\kappa} \\ & + \left\{|\nabla_k R_{ij}|^2 - \frac{\kappa(1-\theta)}{R}\langle \nabla S, \nabla R\rangle + \frac{2(1-\theta)\kappa - 1}{3}|\nabla R|^2\right\}\frac{2}{R^\kappa} \\ & + \kappa\left\{(1-2\theta)\kappa + 1\right\}\frac{|\nabla R|^2}{R^2}\frac{Y}{R^\kappa} - 2\theta\kappa\langle \nabla \log R, \nabla\frac{Y}{R^\kappa}\rangle\ . \end{aligned} \tag{5.9}$$

To eliminate the term $\langle \nabla S, \nabla R\rangle$ we use the following elementary fact

Lemma 5.4 *For any constant a we have*

$$|\nabla_k R_{ij} - aR_{ij}\nabla_k R|^2 = |\nabla_k R_{ij}|^2 + a^2 S|\nabla R|^2 - a\langle \nabla R, \nabla S\rangle\ .$$

In order to apply it to (5.9) we choose $a = \frac{\kappa(1-\theta)}{R}$ so that

$$|\nabla_k R_{ij}|^2 - \frac{\kappa(1-\theta)}{R}\langle \nabla R, \nabla S\rangle = |\nabla_k R_{ij} - aR_{ij}\nabla_k R|^2 - \kappa^2(1-\theta)^2\frac{S}{R^2}|\nabla R|^2$$

hence equation (5.9) may be written as

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right)\frac{Y}{R^\kappa} = \; & \left(R^3 - \frac{13}{3}RS + 4T + \kappa\frac{S}{R}Y\right)\frac{2}{R^\kappa} \\ & + \left[\kappa\left((1-2\theta)\kappa + 1\right) - 2\kappa^2(1-\theta)^2\right]\frac{Y}{R^2}\frac{|\nabla R|^2}{R^\kappa} \\ & - \left(\kappa(1-\theta) - 1\right)^2\frac{2}{3}\frac{|\nabla R|^2}{R^\kappa} \\ & + |\nabla_k R_{ij} - aR_{ij}\nabla_k R|^2\frac{2}{R^\kappa} - 2\theta\kappa\langle \nabla \log R, \nabla\frac{Y}{R^\kappa}\rangle\ . \end{aligned} \tag{5.10}$$

Choose θ so that

$$\kappa\,(1-\theta)=1$$

which will eliminate the non-positive term

$$(\kappa\,(1-\theta)-1)^2\,\frac{2}{3}\frac{|\nabla R|^2}{R^\kappa}\ ,$$

that is

$$\theta = 1-\frac{1}{\kappa}\ .$$

With this value of θ,

$$\begin{aligned}\left(\Delta-\frac{\partial}{\partial t}\right)\frac{Y}{R^\kappa} &= \left(R^3-\frac{13}{3}RS+4T+\kappa\frac{S}{R}Y\right)\frac{2}{R^\kappa} \\ &\quad +|\nabla_k R_{ij}-aR_{ij}\nabla_k R|^2\frac{2}{R^\kappa}+(2-\kappa)\,(\kappa-1)\frac{Y}{R^2}\frac{|\nabla R|^2}{R^\kappa} \\ &\quad -2\,(\kappa-1)\,\langle\nabla\log R,\nabla\frac{Y}{R^\kappa}\rangle\ . \end{aligned} \tag{5.11}$$

Finally we replace T by via P, R and S

$$4T=2\frac{P}{R}-2\frac{S^2}{R}-R^3+5RS \tag{5.12}$$

and thus establish

$$\begin{aligned}\left(\Delta-\frac{\partial}{\partial t}\right)\frac{Y}{R^\kappa} &= \left(P\ -\frac{2-\kappa}{2}SY\right)\frac{4}{R}\frac{1}{R^\kappa}+(2-\kappa)\,(\kappa-1)\frac{Y}{R^2}\frac{|\nabla R|^2}{R^\kappa} \\ &\quad -2(\kappa-1)\langle\nabla\log R,\nabla\frac{Y}{R^\kappa}\rangle+|\nabla_k R_{ij}-aR_{ij}\nabla_k R|^2\frac{2}{R^\kappa}\ . \end{aligned} \tag{5.13}$$

Lemma 5.5 *For any constant κ, set $L_\kappa=\Delta+2(\kappa-1)\nabla\log R.\nabla$. Then*

$$\begin{aligned}R^\kappa\left(L_\kappa-\frac{\partial}{\partial t}\right)\frac{Y}{R^\kappa} &= \left[P\ -\frac{2-\kappa}{2}SY\right]\frac{4}{R}+(2-\kappa)\,(\kappa-1)\frac{Y|\nabla R|^2}{R} \\ &\quad +2\left|\nabla_k R_{ij}-\frac{R_{ij}}{R}\nabla_k R\right|^2\ . \end{aligned} \tag{5.14}$$

In particular for any $\kappa\in[1,2]$, the following differential inequality holds

$$R^\kappa\left(L_\kappa-\frac{\partial}{\partial t}\right)\frac{Y}{R^\kappa}\geq\left(P\ -\frac{2-\kappa}{2}SY\right)\frac{4}{R}\ . \tag{5.15}$$

Indeed this lemma was proved in Hamilton [8] in order to prove his most striking estimate ([8], Theorem 10.1, page 283). We give a detailed computation to exhibit the reason why the differential inequality (5.15) takes this form, which we need to prove our estimate on S.

In particular if $\kappa = 2$ then

$$R^2 \left(L_2 - \frac{\partial}{\partial t} \right) \frac{Y}{R^2} = \frac{4}{R} P + 2 \left| \nabla_k R_{ij} - \frac{R_{ij}}{R} \nabla_k R \right|^2$$

which is always non-negative, which allows us to derive a sharp estimate on the scalar function $|R_{ij}|^2$. Indeed the above equation yields the following differential inequality

$$R^2 \left(L_2 - \frac{\partial}{\partial t} \right) \frac{Y}{R^2} \geq 0$$

so that the maximum principle for parabolic equations yields the following

Theorem 5.6 *Under the Ricci flow, and suppose $R(0) > 0$. Then as long as the Ricci flow exists we always have*

$$\frac{S}{R^2} \leq \max_M \frac{S(0)}{R(0)^2} \ . \tag{5.16}$$

The estimate in this theorem is sharp.

Another observation is the following.

Corollary 5.7 *We have for any constants κ and η*

$$\begin{aligned} R^{\kappa+\eta} \left(L_{\kappa+\eta} - \frac{\partial}{\partial t} \right) \frac{Y}{R^{\kappa+\eta}} &= R^{\kappa} \left(L_{\kappa} - \frac{\partial}{\partial t} \right) \frac{Y}{R^{\kappa}} + 2\eta \frac{S}{R} Y \\ &\quad + (3 - 2\kappa - \eta)\, \eta \frac{Y |\nabla R|^2}{R} \ . \end{aligned} \tag{5.17}$$

In particular

$$\frac{d}{d\kappa} R^{\kappa} \left(L_{\kappa} - \frac{\partial}{\partial t} \right) \frac{Y}{R^{\kappa}} = \left\{ 2 \frac{S}{R} - (2\kappa - 3) \frac{|\nabla R|^2}{R} \right\} Y \ . \tag{5.18}$$

5.1 Gradient estimate for scalar curvature

Next we want to treat the gradient of the Ricci tensor, begin with the scalar curvature R. In general if a scalar function F satisfies the following parabolic equation (under the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$)

$$\left(\Delta - \frac{\partial}{\partial t} \right) F = B_F \ , \tag{5.19}$$

then by the Bochner identity and the Ricci flow equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) |\nabla F|^2 = 2|\nabla\nabla F|^2 + 2\langle \nabla B_F, \nabla F\rangle \ . \tag{5.20}$$

It is better to look at the evolution equation for $|\nabla F|^2/R$ which can be found by using the chain rule. It is given indeed as the following

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) \frac{|\nabla F|^2}{R} &= \frac{2}{R}\langle \nabla B_F, \nabla F\rangle + \left\{\frac{2S}{R^2} + \frac{2}{R^3}|\nabla R|^2\right\} |\nabla F|^2 \\ &\quad + \frac{2}{R}|\nabla\nabla F|^2 - \frac{2}{R^2}\langle \nabla R, \nabla |\nabla F|^2\rangle \ .\end{aligned} \tag{5.21}$$

In order to use the hessian term which is non-negative, we observe

$$\nabla_k |\nabla F|^2 = 2g^{ab} \left(\nabla_a F\right) \left(\nabla_k \nabla_b F\right)$$

so that

$$\langle \nabla R, \nabla |\nabla F|^2\rangle = 2(\nabla^k R) \left(\nabla^l F\right) \left(\nabla_k \nabla_l F\right) \ .$$

It follows that

$$\begin{aligned}|\nabla\nabla F|^2 &= |\nabla_k \nabla_l F - a(\nabla_k R) \left(\nabla_l F\right)|^2 \\ &\quad - a^2 |\nabla R|^2 |\nabla F|^2 + a\langle \nabla R, \nabla |\nabla F|^2\rangle \ .\end{aligned}$$

Therefore

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) \frac{|\nabla F|^2}{R} &= \frac{2}{R}\langle \nabla B_F, \nabla F\rangle + \left\{\frac{2S}{R^2} + \frac{2}{R^3}|\nabla R|^2\right\} |\nabla F|^2 \\ &\quad + \frac{2}{R}|\nabla_k \nabla_l F - a(\nabla_k R) \left(\nabla_l F\right)|^2 \\ &\quad - \frac{2a^2}{R}|\nabla R|^2 |\nabla F|^2 + a\frac{2}{R}\langle \nabla R, \nabla |\nabla F|^2\rangle \\ &\quad - \frac{2}{R^2}\langle \nabla R, \nabla |\nabla F|^2\rangle \ .\end{aligned} \tag{5.22}$$

In particular, to eliminate the term $\langle \nabla R, \nabla |\nabla F|^2\rangle$ we choose $a = 1/R$ which then gives us the following

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) \frac{|\nabla F|^2}{R} &= \frac{2}{R}\langle \nabla B_F, \nabla F\rangle + \frac{2S}{R^2}|\nabla F|^2 \\ &\quad + \frac{2}{R}\left|\nabla_k \nabla_l F - R^{-1}(\nabla_k R) \left(\nabla_l F\right)\right|^2 \ .\end{aligned} \tag{5.23}$$

In particular, if we apply the formula to the scalar curvature R, we establish

$$\frac{1}{2}\left(\Delta-\frac{\partial}{\partial t}\right)\frac{|\nabla R|^2}{R} = -\frac{2}{R}\langle\nabla S,\nabla R\rangle+\frac{S}{R^2}|\nabla R|^2 +\frac{1}{R}\left|\nabla_k\nabla_l R-R^{-1}(\nabla_k R)\,(\nabla_l R)\right|^2 . \tag{5.24}$$

Together with the equations

$$\frac{1}{2}\left(\Delta-\frac{\partial}{\partial t}\right)R^2=-2RY-\frac{2}{3}R^3+|\nabla R|^2$$

and

$$\frac{1}{2}\left(\Delta-\frac{\partial}{\partial t}\right)Y=2\frac{P}{R}-2\frac{S}{R}Y+|\nabla_k R_{ij}|^2-\frac{1}{3}|\nabla R|^2 ,$$

for function

$$H=\frac{|\nabla R|^2}{R}+\xi Y+\eta R^2$$

where ξ and η are two constants, we have

$$\begin{aligned}\frac{1}{2}\left(\Delta-\frac{\partial}{\partial t}\right)H &= -\frac{2}{R}\langle\nabla S,\nabla R\rangle+\frac{1}{R}\left|\nabla_k\nabla_l R-R^{-1}(\nabla_k R)\,(\nabla_l R)\right|^2\\ &\quad+\left[2\xi\frac{P}{R}-\frac{2Y}{R}\left(\xi S+\eta R^2\right)-\frac{2}{3}\eta R^3\right]\\ &\quad+\left\{\xi|\nabla_k R_{ij}|^2-\frac{1}{3}\xi|\nabla R|^2+\eta|\nabla R|^2+\frac{S}{R^2}|\nabla R|^2\right\} .\end{aligned}$$

In this formula, the only term we have to deal is $\langle\nabla S,\nabla R\rangle$, which we handle as the following.

$$|b\nabla_k R_{ij}-aR_{ij}\nabla_k R|^2=b^2|\nabla_k R_{ij}|^2+a^2S|\nabla R|^2-ab\langle\nabla S,\nabla R\rangle$$

in which we set $ab=\frac{2}{R}$, i.e. $a=\frac{2}{bR}$, and thus

$$\begin{aligned}-\frac{2}{R}\langle\nabla S,\nabla R\rangle &= \left|b\nabla_k R_{ij}-\frac{2}{bR}R_{ij}\nabla_k R\right|^2\\ &\quad-b^2|\nabla_k R_{ij}|^2-\frac{4}{b^2}\frac{S}{R^2}|\nabla R|^2 .\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{2}\left(\Delta-\frac{\partial}{\partial t}\right)H &= \frac{1}{R}\left|\nabla_k\nabla_l R - R^{-1}(\nabla_k R)\,(\nabla_l R)\right|^2 \\
&\quad + \left|b\nabla_k R_{ij} - \frac{2}{bR}R_{ij}\nabla_k R\right|^2 \\
&\quad + \left[2\xi\frac{P}{R} - \frac{2Y}{R}\left(\xi S+\eta R^2\right) - \frac{2}{3}\eta R^3\right] \\
&\quad + \left(\xi - b^2\right)|\nabla_k R_{ij}|^2 - \frac{1}{3}\xi|\nabla R|^2 + \eta|\nabla R|^2 \\
&\quad + \left(1-\frac{4}{b^2}\right)\frac{S}{R^2}|\nabla R|^2 \ .
\end{aligned} \tag{5.25}$$

We next need to decide the signs of three constants ξ, η and b. It is suggested that $\xi \geq b^2$ (otherwise we can not control $|\nabla_k R_{ij}|^2$). Under this choice, and we are expected with the good choices of these three constants, we will lose nothing from the first two terms on the right-hand side of equation (5.25), we thus simply drop these two, and use the inequality ([8], Lemma 11.6, page 288)

$$|\nabla_k R_{ij}|^2 \geq \frac{7}{20}|\nabla R|^2$$

so that

$$\begin{aligned}
\frac{1}{2}\left(\Delta-\frac{\partial}{\partial t}\right)H &\geq 2\xi\frac{P}{R} - \frac{2Y}{R}\left(\xi S+\eta R^2\right) - \frac{2}{3}\eta R^3 \\
&\quad + \left(\frac{1}{60}\xi - \frac{7}{20}b^2 + \eta\right)|\nabla R|^2 \\
&\quad + \left(1-\frac{4}{b^2}\right)\frac{S}{R^2}|\nabla R|^2 \ .
\end{aligned} \tag{5.26}$$

Obviously a simple choice for b is $b = 2$ so that the last term in (5.26) is dropped. Thus

$$\begin{aligned}
\frac{1}{2}\left(\Delta-\frac{\partial}{\partial t}\right)H &\geq 2\xi\frac{P}{R} - \frac{2}{3}\eta R^3 - 2R\left(\frac{S}{R^2}+\frac{\eta}{\xi}\right)\xi Y \\
&\quad + \left(\frac{1}{60}\xi - \frac{7}{5} + \eta\right)|\nabla R|^2 \ .
\end{aligned} \tag{5.27}$$

This inequality is enough to prove Hamilton's estimate ([8], Theorem 11.1, page 287).

Replace Y by

$$\xi Y = H - \frac{|\nabla R|^2}{R} - \eta R^2$$

to obtain

$$\begin{aligned}\frac{1}{2}\left(\Delta-\frac{\partial}{\partial t}\right)H &\geq 2\xi\frac{P}{R}+2\left(\delta+\frac{\eta}{\xi}-\frac{1}{3}\right)\eta R^3-2R\left(\delta+\frac{\eta}{\xi}\right)H \\ &\quad+\left(\frac{1}{60}\xi+2\delta+2\frac{\eta}{\xi}-\frac{7}{5}+\eta\right)|\nabla R|^2 \qquad (5.28)\end{aligned}$$

in which we choose $\eta = 0$ and $\xi = 84$ we have

$$\frac{1}{2}\left(\Delta-\frac{\partial}{\partial t}\right)H \geq -2\delta RH\ . \qquad (5.29)$$

We may use the Kac formula to deduce an estimate for the scalar curvature.

Let $(X_t, \mathbb{P}^{s,x})$ be the diffusion process associated with the time dependent elliptic operator $\frac{1}{2}\Delta$. Then we have

Theorem 5.8 *Under the Ricci flow* $\frac{\partial}{\partial t}g_{ij} = -2R_{ij}$ *with initial metric of positive scalar curvature* $R(0) > 0$. *Then*

$$H(t,x) \leq \mathbb{P}^{0,x}\left(H(0,X_s)e^{2\delta\int_0^t R(s,X_s)ds}\right)\ , \qquad (5.30)$$

where $H = \frac{|\nabla R|^2}{R} + 84S$. *In particular if* $R(t,\cdot) \leq \theta(t)$ *on* M, *then*

$$H(t,x) \leq e^{2\delta\int_0^t \theta(s)ds}\max_M H(0,\cdot)\ .$$

6 Estimates on the scalar curvature

Thanks to the resolution of the Yamabe problem (e.g. , for a 3-manifold with a metric of positive scalar curvature, we may run the general Ricci flow with an initial metric $(g(0)_{ij})$ such that $R(0)$ is a positive constant. Under such an initial metric, and if we run the Ricci flow

$$\frac{\partial}{\partial t}g_{ij} = -2\left(R_{ij}-\frac{1}{3}\sigma g_{ij}\right)$$

(i.e. for the case $\alpha = 1$ and $\beta = 0$), then all volume measures μ_t associated with the Ricci flow $(g(t)_{ij})$ have the same volume.

We may choose a non-negative constant ε such that $R(0)_{ij} \geq -\varepsilon R(0)g(0)_{ij}$, which is possible since $R(0) > 0$. We will of course choose the least one $\varepsilon \geq 0$ for a given initial data. Then $R > 0$ as long as the Ricci flow exists, and by Theorem 4.1

$$R(t)_{ij} \geq -\varepsilon R(t)g(t)_{ij}$$

for all t. Furthermore there is a constant $\delta \geq \frac{1}{3}$ such that

$$R(t)^2/3 \leq S(t) \leq \delta R(t)^2 \ . \tag{6.1}$$

For example, any $\delta \geq 1 + 4\varepsilon + 6\varepsilon^2$ will do. However in the case of the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, the optimal choice for δ is the maximum of $S(0)/R(0)^2$.

With the last estimate we may deduce several elementary estimates on the scalar curvature R. Define

$$\rho(t) = \exp\left(-\frac{3}{2}\int_0^t A(s)\mathrm{d}s\right) \ , \quad \rho(0) = 1 \ .$$

Lemma 6.1 *Let* $\psi(r) = e^{-\frac{3}{2}\frac{1}{r}}$ *and set*

$$\begin{aligned} F &= \psi(K)\exp\left\{-\int_0^t \rho(s)\,ds\right\} \\ &= \exp\left\{-\left[\frac{3}{2}\frac{1}{R(t)}\rho(t) + \int_0^t \rho(s)\,ds\right]\right\} \end{aligned}$$

where K *is given in (3.12). Then*

$$\left(\Delta - \frac{\psi''(K)}{\psi'(K)}\nabla K - \frac{\partial}{\partial t}\right) F \leq 0 \ . \tag{6.2}$$

Proof. The differential inequality relies on the evolution (3.13) for K. By the chain

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right)\psi(K) &= \psi'(K)\left(\Delta - \frac{\partial}{\partial t}\right)K + \frac{\psi''(K)}{\psi'(K)}\nabla K.\nabla\psi(K) \\ &= -2\psi'(K)\mathrm{e}^{\frac{2}{3}\int_0^t A(s)\mathrm{d}s}S + \frac{\psi''(K)}{\psi'(K)}\nabla K.\nabla\psi(K) \\ &\leq -\frac{2}{3}\psi'(K)K^2\mathrm{e}^{-\frac{2}{3}\int_0^t A(s)\mathrm{d}s} + \frac{\psi''(K)}{\psi'(K)}\nabla K.\nabla\psi(K) \\ &= -\mathrm{e}^{-\frac{2}{3}\int_0^t A(s)\mathrm{d}s}\psi(K) + \frac{\psi''(K)}{\psi'(K)}\nabla K.\nabla\psi(K) \end{aligned}$$

which implies (6.2). ■

By the maximum principle applying to (6.2) and the fact that $R(0)$ is constant, we have

$$\frac{3}{2}\frac{\rho(t)}{R(t)} + \int_0^t \rho(s)\mathrm{d}s \leq \frac{3}{2}\frac{1}{R(0)} \ .$$

Since we always have $\frac{\rho(t)}{R(t)} > 0$ so that

$$\int_0^t \rho(s)\mathrm{d}s \leq \frac{3}{2}\frac{1}{R(0)}$$

for all t and

$$R(t) \geq \frac{3R(0)\rho(t)}{3 - 2R(0)\int_0^t \rho(s)\mathrm{d}s} \ .$$

for all t provides the Ricci flow exists.

Similarly, if we use the fact that $S \leq \delta R^2$ we then have

Lemma 6.2 *Let* $\psi(r) = e^{-\frac{1}{2\delta}\frac{1}{r}}$ *and*

$$\begin{aligned} F &= \psi(K)\exp\left\{-\int_0^t \rho(s)\,ds\right\} \\ &= \exp\left\{-\frac{1}{2\delta R}\rho(t) - \int_0^t \rho(s)\,du\right\} \ . \end{aligned}$$

Then

$$\left(\Delta - \frac{\psi''(K)}{\psi'(K)}\nabla K - \frac{\partial}{\partial t}\right) F \geq 0 \ . \qquad (6.3)$$

Again the maximum principle then implies that

$$\frac{\rho(t)}{2\delta R} + \int_0^t \rho(s)\mathrm{d}s \geq \frac{1}{2\delta R(0)}$$

and therefore

$$R(t) \leq \frac{R(0)\rho(t)}{1 - 2\delta\int_0^t \rho(s)\mathrm{d}s} \ .$$

Theorem 6.3 *Under the general Ricci flow with initial metric having positive constant scalar curvature* $R(0)$ *and*

$$\rho(t) = \exp\left(-\frac{3}{2}\int_0^t A(s)\,ds\right) \ .$$

Then

$$\int_0^t \rho(s)\,ds < \frac{3}{2}\frac{1}{R(0)}$$

and

$$\frac{3R(0)\rho(t)}{3 - 2R(0)\int_0^t \rho(s)\,ds} \leq R(t) \leq \frac{R(0)\rho(t)}{1 - 2\delta\int_0^t \rho(s)\,ds}$$

as long as the Ricci flow $(g(t)_{ij})$ *exists.*

7 L^2-estimates for scalar curvature

If (g_{ij}) is a solution flow to an evolution equation: $\frac{\partial}{\partial t} g_{ij} = -2h_{ij}$, and if $f \in C^{2,1}(\mathbf{R}_+ \times M)$, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_M f(t,\cdot)\mathrm{d}\mu_t &= \int_M \frac{\partial}{\partial t} f(t,\cdot)\mathrm{d}\mu_t + \int_M f(t,\cdot)\frac{\partial}{\partial t}(\log M_t)\,\mathrm{d}\mu_t \\ &= \int_M \frac{\partial}{\partial t} f(t,\cdot)\mathrm{d}\mu_t - \int_M \mathrm{tr}_g(h_{ij}) f(t,\cdot)\mathrm{d}\mu_t \ . \end{aligned} \tag{7.1}$$

Since M is compact, so that $\int_M \Delta f(t,\cdot)\mathrm{d}\mu_t = 0$ (where Δ is the Laplace-Beltrami operator associated with $g(t)_{ij}$, if no confusion may arise we will suppress the parameter t), and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M f(t,\cdot)\mathrm{d}\mu_t = -\int_M \left(\Delta - \frac{\partial}{\partial t}\right) f(t,\cdot)\mathrm{d}\mu_t - \int_M \mathrm{tr}_g(h_{ij}) f(t,\cdot)\mathrm{d}\mu_t \ . \tag{7.2}$$

In particular, under the Ricci flow (3.2), $\mathrm{tr}_g(h_{ij}) = R - A$ so that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_M f(t,\cdot)\mathrm{d}\mu_t &= -\int_M \left(\Delta - \frac{\partial}{\partial t}\right) f(t,\cdot)\mathrm{d}\mu_t \\ &\quad - \int_M R f(t,\cdot)\mathrm{d}\mu_t + A \int_M f(t,\cdot)\mathrm{d}\mu_t \ . \end{aligned} \tag{7.3}$$

For the gradient of f we may use the Bochner identity

$$\begin{aligned} \Delta|\nabla f|^2 &= 2\Gamma_2(f) + 2\langle \nabla\Delta f, \nabla f\rangle \\ &= 2|\nabla\nabla f|^2 + 2\mathrm{Ric}(\nabla f, \nabla f) + 2\langle \nabla\Delta f, \nabla f\rangle \ , \end{aligned}$$

while by using the fact that $\frac{\partial}{\partial t} g_{ij} = -2h_{ij}$ we have

$$\begin{aligned} \frac{\partial}{\partial t}|\nabla f|^2 &= \left(\frac{\partial}{\partial t} g^{ij}\right)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} + 2\langle \nabla\left(\frac{\partial}{\partial t} f\right), \nabla f\rangle \\ &= 2h^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} + 2\langle \nabla\left(\frac{\partial}{\partial t} f\right), \nabla f\rangle \ , \end{aligned}$$

so that

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right)|\nabla f|^2 &= 2|\nabla\nabla f|^2 + 2\left(R^{ij} - h^{ij}\right)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \\ &\quad + 2\langle \nabla\left(\Delta - \frac{\partial}{\partial t}\right) f, \nabla f\rangle \ . \end{aligned} \tag{7.4}$$

Next we consider the solution flow to the Ricci flow (3.2). In this case

$$R^{ij} - h^{ij} = \frac{1}{3} A g^{ij}$$

where $A(t) = \alpha\sigma(t) + \beta$, and therefore

$$\left(\Delta - \frac{\partial}{\partial t}\right) |\nabla f|^2 = 2|\nabla\nabla f|^2 + \frac{2}{3} A |\nabla f|^2 + 2\langle \nabla \left(\Delta - \frac{\partial}{\partial t}\right) f, \nabla f\rangle \ . \tag{7.5}$$

Applying (7.2) to $|\nabla f|^2$ we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2 \mathrm{d}\mu_t &= -\int_M \left(\Delta - \frac{\partial}{\partial t}\right) |\nabla f|^2 \mathrm{d}\mu_t \\ &\quad -\int_M R|\nabla f|^2 \mathrm{d}\mu_t + A\int_M |\nabla f|^2 \mathrm{d}\mu_t \\ &= -2\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t - \int_M R|\nabla f|^2 \mathrm{d}\mu_t + \frac{1}{3}A\int_M |\nabla f|^2 \mathrm{d}\mu_t \\ &\quad -2\int_M \langle \nabla \left(\Delta - \frac{\partial}{\partial t}\right) f, \nabla f\rangle \mathrm{d}\mu_t \ , \end{aligned}$$

and integration by parts we establish the following interesting equality

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2 \mathrm{d}\mu_t &= -2\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t - \int_M R|\nabla f|^2 \mathrm{d}\mu_t \\ &\quad +\frac{1}{3}A\int_M |\nabla f|^2 \mathrm{d}\mu_t + 2\int_M (\Delta f)\left(\Delta - \frac{\partial}{\partial t}\right) f \mathrm{d}\mu_t \ . \end{aligned} \tag{7.6}$$

Theorem 7.1 *Under the Ricci flow (3.2)*

$$\begin{aligned} \frac{d}{dt}\int_M |\nabla f|^2 d\mu_t &\leq -\int_M R|\nabla f|^2 d\mu_t + \frac{1}{3}A\int_M |\nabla f|^2 d\mu_t \\ &\quad +\frac{3}{2}\int_M \left[\left(\Delta - \frac{\partial}{\partial t}\right) f\right]^2 d\mu_t \ . \end{aligned} \tag{7.7}$$

Proof. On 3-manifolds $|\nabla\nabla f|^2 \geq \frac{1}{3}(\Delta f)^2$ so that by (7.6)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2 \mathrm{d}\mu_t &\leq -\int_M R|\nabla f|^2 \mathrm{d}\mu_t + \frac{1}{3}A\int_M |\nabla f|^2 \mathrm{d}\mu_t \\ &\quad -\frac{2}{3}\int_M (\Delta f)^2 \mathrm{d}\mu_t + 2\int_M (\Delta f)\left(\Delta - \frac{\partial}{\partial t}\right) f \mathrm{d}\mu_t \\ &\leq -\int_M R|\nabla f|^2 \mathrm{d}\mu_t + \frac{1}{3}A\int_M |\nabla f|^2 \mathrm{d}\mu_t \\ &\quad +\frac{3}{2}\int_M \left[\left(\Delta - \frac{\partial}{\partial t}\right) f\right]^2 \mathrm{d}\mu_t \ . \end{aligned}$$

■

Since under the Ricci flow (3.2)

$$\left(\Delta - \frac{\partial}{\partial t}\right) R = -2S + \frac{2}{3}AR$$

we therefore have the following

Corollary 7.2 *Under the Ricci flow (3.2) for the scalar curvature R we have the following energy estimate*

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla R|^2 d\mu_t \;\leq\; & -\int_M R|\nabla R|^2 d\mu_t + \frac{1}{3}A\int_M |\nabla R|^2 d\mu_t \\ & +6\int_M S^2 d\mu_t + \frac{2}{3}A^2\int_M R^2 d\mu_t \\ & -4A\int_M RS\, d\mu_t \ . \qquad (7.8)\end{aligned}$$

In particular if $\frac{\partial}{\partial t}g_{ij} = -2R_{ij}$ with initial metric $g(0)_{ij}$ such that $R(0)$ is a positive constant, then

$$\int_M |\nabla R|^2 d\mu_t \leq 6\int_0^t \left\{ e^{-R(0)(t-s)} \int_M S^2 d\mu_s \right\} \ . \qquad (7.9)$$

7.1 Some applications

Another application of equation (7.6) is to obtain information on spectral gaps. Let us prove the following

Theorem 7.3 *Let $(g(t)_{ij})$ be the solution to the Ricci flow $\frac{\partial}{\partial t}g_{ij} = -2R_{ij}$ on the 3-manifold M.*

1) If $\lambda(t)$ denotes the first non-negative eigenvalue of $(M, g(t)_{ij})$, then

$$\frac{d}{dt}\lambda = 2\lambda^2 + \lambda\int_M Rf^2 d\mu_t - 2\int_M |\nabla\nabla f|^2 d\mu_t - \int_M R|\nabla f|^2 d\mu_t \qquad (7.10)$$

where f is an eigenvector: $\Delta f = -\lambda f$ such that $\int_M f^2 d\mu_t = 1$.

2) If $\lambda(t)$ denotes the first non-negative eigenvalue of $\Delta + \frac{1}{4}R$, then

$$\frac{d}{dt}\lambda \leq \frac{19}{8}\lambda^2. \qquad (7.11)$$

Proof. Let us make computations under the Ricci flow (3.2). Indeed, consider an eigenvector f with eigenvalue λ (both depending smoothly on t):

$$(\Delta + V)\, f = -\lambda f \quad ; \qquad \int_M f^2 \mathrm{d}\mu_t = 1$$

where V is some potential (depending on t as well) to be chosen later. Then, by equation (7.6)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2 \mathrm{d}\mu_t &= \int_M (V+\lambda)\frac{\partial f^2}{\partial t}\mathrm{d}\mu_t - 2\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t - \int_M R|\nabla f|^2 \mathrm{d}\mu_t \\ &\quad +2\int_M (V+\lambda)^2 f^2 \mathrm{d}\mu_t + \frac{1}{3}A\int_M |\nabla f|^2 \mathrm{d}\mu_t \ . \end{aligned} \tag{7.12}$$

On the other hand

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2 \mathrm{d}\mu_t &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M (-\Delta f) f \mathrm{d}\mu_t \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M (V+\lambda)\, f^2 \mathrm{d}\mu_t \\ &= \int_M f^2 \frac{\mathrm{d}}{\mathrm{d}t}(V+\lambda)\,\mathrm{d}\mu_t + \int_M (V+\lambda)\frac{\partial f^2}{\partial t}\mathrm{d}\mu_t \\ &\quad + \int_M (V+\lambda)\, f^2 \left(\frac{\partial}{\partial t}\log M_t\right)\mathrm{d}\mu_t \\ &= \int_M f^2 \frac{\mathrm{d}}{\mathrm{d}t}(V+\lambda)\,\mathrm{d}\mu_t + \int_M (V+\lambda)\frac{\partial f^2}{\partial t}\mathrm{d}\mu_t \\ &\quad - \int_M (R-A)(V+\lambda)\, f^2 \mathrm{d}\mu_t \ ,\end{aligned}$$

combining with equation (7.12), the facts that $\int_M f^2 \mathrm{d}\mu_t = 1$ and

$$\begin{aligned}\int_M |\nabla f|^2 \mathrm{d}\mu_t &= -\int_M (\Delta f)\, f \mathrm{d}\mu_t \\ &= \int_M (\lambda + V) f^2 \mathrm{d}\mu_t \ ,\end{aligned}$$

we deduce the following

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda &= \int_M (R-A)(V+\lambda)\, f^2 \mathrm{d}\mu_t - 2\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t - \int_M R|\nabla f|^2 \mathrm{d}\mu_t \\ &\quad +2\int_M (V+\lambda)^2 f^2 \mathrm{d}\mu_t + \frac{1}{3}A\int_M |\nabla f|^2 \mathrm{d}\mu_t - \int_M f^2 \frac{\partial V}{\partial t}\mathrm{d}\mu_t \\ &= -2\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t - \int_M R|\nabla f|^2 \mathrm{d}\mu_t + 2\int_M (V+\lambda)^2 f^2 \mathrm{d}\mu_t \\ &\quad - \int_M f^2 \frac{\partial V}{\partial t}\mathrm{d}\mu_t + \int_M \left(R - \frac{2}{3}A\right)(V+\lambda)\, f^2 \mathrm{d}\mu_t \ .\end{aligned}$$

We then choose $V = -\varphi(t)R(t,\cdot)$ where φ depends on t only. Then

$$\frac{\partial V}{\partial t} = -\varphi' R - \varphi \frac{\partial}{\partial t} R$$

so that

$$\int_M f^2 \frac{\partial V}{\partial t} \mathrm{d}\mu_t = -\int_M \varphi' R f^2 \mathrm{d}\mu_t - \varphi \int_M f^2 \frac{\partial}{\partial t} R \mathrm{d}\mu_t \ .$$

To treat the term $\int_M f^2 \frac{\partial}{\partial t} R \mathrm{d}\mu_t$ we use integration by parts again, thus

$$\begin{aligned}
\int_M R|\nabla f|^2 \mathrm{d}\mu_t &= \int_M \langle \nabla(Rf), \nabla f\rangle \mathrm{d}\mu_t - \frac{1}{2}\int_M \langle \nabla R, \nabla f^2\rangle \mathrm{d}\mu_t \\
&= \int_M R(V+\lambda) f^2 \mathrm{d}\mu_t + \frac{1}{2}\int_M f^2 (\Delta R) \mathrm{d}\mu_t \\
&= \int_M R(V+\lambda) f^2 \mathrm{d}\mu_t + \frac{1}{2}\int_M f^2 \left(\Delta - \frac{\partial}{\partial t}\right) R \mathrm{d}\mu_t + \frac{1}{2}\int_M \frac{\partial R}{\partial t} f^2 \mathrm{d}\mu_t \\
&= \int_M R(V+\lambda) f^2 \mathrm{d}\mu_t + \frac{1}{2}\int_M f^2 \frac{\partial R}{\partial t} \mathrm{d}\mu_t \\
&\quad + \int_M f^2 \left(-S + \frac{1}{3} A R\right) \mathrm{d}\mu_t \ .
\end{aligned}$$

In other words

$$\begin{aligned}
\int_M f^2 \frac{\partial R}{\partial t} \mathrm{d}\mu_t &= 2\int_M f^2 \left(S - \frac{1}{3} A R\right) \mathrm{d}\mu_t + 2\int_M R|\nabla f|^2 \mathrm{d}\mu_t \\
&\quad -2\int_M R(V+\lambda) f^2 \mathrm{d}\mu_t \ ,
\end{aligned}$$

therefore

$$\begin{aligned}
-\int_M f^2 \frac{\partial V}{\partial t} \mathrm{d}\mu_t &= \int_M \varphi' R f^2 \mathrm{d}\mu_t + 2\varphi \int_M f^2 \left(S - \frac{1}{3} A R\right) \mathrm{d}\mu_t \\
&\quad -2\varphi \int_M R|\nabla f|^2 \mathrm{d}\mu_t + 2\varphi \int_M R(\varphi R + \lambda) f^2 \mathrm{d}\mu_t \ .
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\lambda &= -2\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t - (1+2\varphi)\int_M R|\nabla f|^2 \mathrm{d}\mu_t \\
&\quad + \int_M \left(2\varphi S - \frac{2}{3}\varphi A R + \varphi' R\right) f^2 \mathrm{d}\mu_t \\
&\quad + \varphi \int_M R\left(R + 4\varphi R - \frac{2}{3} A + 2\lambda\right) f^2 \mathrm{d}\mu_t \\
&\quad + \lambda \int_M \left(R + 4\varphi R - \frac{2}{3} A + 2\lambda\right) f^2 \mathrm{d}\mu_t \ .
\end{aligned}$$

In particular, under the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, then $A = 0$, and if choose φ to be a constant as well, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\lambda &= -2\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t - (1+2\varphi)\int_M R|\nabla f|^2 \mathrm{d}\mu_t \\
&\quad +2\varphi\int_M Sf^2 \mathrm{d}\mu_t + \varphi\int_M R\left(R+4\varphi R+2\lambda\right) f^2 \mathrm{d}\mu_t \\
&\quad +\lambda\int_M \left(R+4\varphi R+2\lambda\right) f^2 \mathrm{d}\mu_t \ . \qquad (7.13)
\end{aligned}$$

For example, if we choose $\varphi = 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda = 2\lambda^2 + \lambda\int_M Rf^2 \mathrm{d}\mu_t - 2\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t - \int_M R|\nabla f|^2 \mathrm{d}\mu_t \ . \qquad (7.14)$$

However if we choose $\varphi = -1/4$ then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\lambda &= -2\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t - \frac{1}{2}\int_M R|\nabla f|^2 \mathrm{d}\mu_t \\
&\quad +2\lambda^2 - \frac{1}{2}\int_M Sf^2 \mathrm{d}\mu_t - \frac{1}{2}\lambda\int_M Rf^2 \mathrm{d}\mu_t.
\end{aligned}$$

While $S \geq \frac{1}{3}R^2$ so that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\lambda &= -2\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t - \frac{1}{2}\int_M R|\nabla f|^2 \mathrm{d}\mu_t \\
&\quad +2\lambda^2 - \frac{1}{6}\int_M \left(R^2 - 3\lambda R\right) f^2 \mathrm{d}\mu_t \\
&\leq \frac{19}{8}\lambda^2.
\end{aligned}$$

Let us come back to equation (7.13). By the Bochner identity

$$\Gamma_2(f) = |\nabla\nabla f|^2 + \mathrm{Ric}(\nabla f, \nabla f)$$

so that

$$\int_M \Gamma_2(f)\mathrm{d}\mu_t = \int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t + \int_M \mathrm{Ric}(\nabla f, \nabla f)\mathrm{d}\mu_t,$$

while by integration by parts

$$\begin{aligned}
\int_M \Gamma_2(f)\mathrm{d}\mu_t &= -\int_M \langle\nabla\Delta f, \nabla f\rangle \mathrm{d}\mu_t \\
&= \int_M (\Delta f)^2 \mathrm{d}\mu_t \\
&= \int_M (\varphi R + \lambda)^2 f^2
\end{aligned}$$

hence

$$-\int_M |\nabla\nabla f|^2 \mathrm{d}\mu_t = -\int_M (\varphi R + \lambda)^2 f^2 + \int_M \mathrm{Ric}(\nabla f, \nabla f)\mathrm{d}\mu_t \ .$$

Inserting this fact into (7.13) we obtain, after simplification,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda &= (1+2\varphi)\,\lambda \int_M Rf^2 \mathrm{d}\mu_t - (1+2\varphi)\int_M R|\nabla f|^2 \mathrm{d}\mu_t \\ &\quad + \left(2\varphi^2+\varphi\right)\int_M R^2 f^2 \mathrm{d}\mu_t + 2\varphi\int_M S f^2 \mathrm{d}\mu_t \\ &\quad +2\int_M \mathrm{Ric}(\nabla f, \nabla f)\mathrm{d}\mu_t \ . \end{aligned} \tag{7.15}$$

While under the Ricci flow

$$\mathrm{Ric}(\nabla f, \nabla f) \geq -\varepsilon R |\nabla f|^2$$

where $\varepsilon \in [0, 1/3]$ so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda &\geq (1+2\varphi)\,\lambda \int_M Rf^2 \mathrm{d}\mu_t - (1+2\varphi+2\varepsilon)\int_M R|\nabla f|^2 \mathrm{d}\mu_t \\ &\quad + \left(2\varphi^2+\varphi\right)\int_M R^2 f^2 \mathrm{d}\mu_t + 2\varphi\int_M S f^2 \mathrm{d}\mu_t \ . \end{aligned}$$

In particular if $\varphi = 0$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda = \lambda\int_M Rf^2 \mathrm{d}\mu_t - \int_M R|\nabla f|^2 \mathrm{d}\mu_t + 2\int_M \mathrm{Ric}(\nabla f, \nabla f)\mathrm{d}\mu_t \ .$$

■

7.2 L^2-estimates for the Ricci tensor

Our next goal is to improve these L^2-estimates to weighted forms. By (7.5) and the evolution equations (3.5, 3.7, 3.9) we may easily establish the following

Lemma 7.4 *Under the Ricci flow (3.2)*

1) For the scalar curvature

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla R|^2 = 2|\nabla\nabla R|^2 - 4\langle \nabla S, \nabla R\rangle + 2A|\nabla R|^2 \ . \tag{7.16}$$

2)

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla S|^2 &= 2|\nabla\nabla S|^2 + \frac{10}{3}A|\nabla S|^2 - 20R|\nabla S|^2 \\ &\quad +4\left\{3R^2 - 5S\right\}\langle \nabla R, \nabla S\rangle \\ &\quad +16\langle \nabla T, \nabla S\rangle + 4\langle \nabla|\nabla_k R_{ij}|^2, \nabla S\rangle \ . \end{aligned} \tag{7.17}$$

3)

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla T|^2 = 2|\nabla\nabla T|^2 + 14\left(\frac{1}{3}A + R\right)|\nabla T|^2 - 18R^2\langle\nabla S, \nabla T\rangle$$
$$+\left\{14T - 36RS + 16R^3\right\}\langle\nabla R, \nabla T\rangle$$
$$+12\langle\nabla\left\{g^{ab}R_{ia}(\nabla^k R^{ij})(\nabla_k R_{bj})\right\}, \nabla T\rangle \ . \tag{7.18}$$

By chain rule, (7.16) and the evolution equation for the scalar curvature we deduce easily the following

$$\left(\Delta - \frac{\partial}{\partial t}\right)\left(R^\gamma|\nabla R|^2\right) = \left(-2\gamma\frac{S}{R} + \left(\frac{2\gamma}{3} + 2\right)A\right)|\nabla R|^2 R^\gamma$$
$$+\gamma\left(\gamma - 1\right)R^{\gamma-2}|\nabla R|^4 + 2|\nabla\nabla R|^2 R^\gamma$$
$$-4\langle\nabla S, \nabla R\rangle R^\gamma + 2\langle\nabla R^\gamma, \nabla|\nabla R|^2\rangle \ .$$

Then we apply (7.3) to function $R^\gamma|\nabla R|^2$, after simplification we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla R|^2 R^\gamma \mathrm{d}\mu_t = \int_M \left[2\gamma\frac{S}{R} - \left(\frac{2\gamma}{3} + 1\right)A - R\right]|\nabla R|^2 R^\gamma \mathrm{d}\mu_t$$
$$-\gamma\left(\gamma - 1\right)\int_M R^{\gamma-2}|\nabla R|^4 \mathrm{d}\mu_t - 2\int_M |\nabla\nabla R|^2 R^\gamma \mathrm{d}\mu_t$$
$$+4\int_M \langle\nabla S, \nabla R\rangle R^\gamma \mathrm{d}\mu_t - 2\int_M \langle\nabla R^\gamma, \nabla|\nabla R|^2\rangle \mathrm{d}\mu_t \ . \tag{7.19}$$

Using integration by parts in the last two integrals we then deduce

$$4\int_M \langle\nabla S, \nabla R\rangle R^\gamma \mathrm{d}\mu_t = -4\int_M \left(S\Delta R\right)\mathrm{d}\nu_t - 4\gamma\int_M \frac{S}{R}|\nabla R|^2 \mathrm{d}\nu_t$$

and

$$-2\int_M \langle\nabla R^\gamma, \nabla|\nabla R|^2\rangle \mathrm{d}\mu_t = 2\gamma\int_M R^{-1}\left(\Delta R\right)|\nabla R|^2 \mathrm{d}\nu_t$$
$$+2\gamma(\gamma - 1)\int_M R^{-2}|\nabla R|^4 \mathrm{d}\nu_t$$

so that, with $\mathrm{d}\nu_t = R^\gamma \mathrm{d}\mu_t$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla R|^2 \mathrm{d}\nu_t = -\int_M \left[2\gamma\frac{S}{R} + \left(\frac{2\gamma}{3} + 1\right)A + R\right]|\nabla R|^2 \mathrm{d}\nu_t$$
$$+\gamma(\gamma - 1)\int_M R^{-2}|\nabla R|^4 \mathrm{d}\nu_t - 2\int_M |\nabla\nabla R|^2 \mathrm{d}\nu_t$$
$$+2\gamma\int_M R^{-1}|\nabla R|^2\left(\Delta R\right)\mathrm{d}\nu_t - 4\int_M S\left(\Delta R\right)\mathrm{d}\nu_t \ .$$

It follows that, since $|\nabla\nabla R|^2 \geq \frac{1}{3}(\Delta R)^2$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla R|^2 \mathrm{d}\nu_t \ &\leq \ -\int_M \left[2\gamma\frac{S}{R} + \left(\frac{2\gamma}{3}+1\right)A + R\right]|\nabla R|^2 \mathrm{d}\nu_t \\
&\quad +\gamma(\gamma-1)\int_M R^{-2}|\nabla R|^4 \mathrm{d}\nu_t \\
&\quad -\frac{2}{3}\int_M \left[(\Delta R)^2 - 2\left(\frac{3\gamma}{2R}|\nabla R|^2 - 3S\right)(\Delta R)\right]\mathrm{d}\nu_t \\
&\leq \ -\int_M \left[8\gamma\frac{S}{R} + \left(\frac{2\gamma}{3}+1\right)A + R\right]|\nabla R|^2 \mathrm{d}\nu_t \\
&\quad -\gamma\left(1-\frac{5}{2}\gamma\right)\int_M \frac{|\nabla R|^4}{R^2}\mathrm{d}\nu_t + 6\int_M S^2 \mathrm{d}\nu_t
\end{aligned}$$

which allows us to establish an estimate for $\int_M |\nabla R|^2 \mathrm{d}\nu_t$, and here again $\gamma = 2/5$ seems to be the best choice.

Theorem 7.5 *Under the Ricci flow (3.2) with initial metric of positive scalar curvature, and let* $\gamma \in [0, 2/5]$*, then*

$$\begin{aligned}
\frac{d}{dt}\int_M |\nabla R|^2 R^\gamma \, d\mu_t \ &\leq \ -\int_M \left[8\gamma\frac{S}{R} + \left(\frac{2}{3}\gamma+1\right)A + R\right]|\nabla R|^2 R^\gamma \, d\mu_t \\
&\quad +6\int_M S^2 R^\gamma \, d\nu_t \ .
\end{aligned}$$

In particular if $A = 0$ *and* $R(0)$ *is a positive constant, then*

$$\int_M |\nabla R|^2 R^\gamma \, d\mu_t \leq 6\int_0^t \left\{ e^{-\left(\frac{8\gamma}{3}+1\right)R(0)(t-s)} \int_M S^2 R^\gamma \, d\mu_s \right\} \ . \tag{7.20}$$

8 Harnack estimate for Ricci flow

We in this section prove Harnack estimates for the Ricci curvature under the Ricci flow.

8.1 Some formulae about Ricci tensors

On a 3-manifold with a family of metrics $(g(t)_{ij})$ evolving with the equation

$$\frac{\partial g_{ij}}{\partial t} = -2h_{ij}$$

and suppose that (V_{ij}) is a symmetric tensor depending smoothly in t, then it is clear that

$$\frac{\partial}{\partial t}\mathrm{tr}_g(V_{ij}) = 2h^{ij}V_{ij} + \mathrm{tr}_g(\frac{\partial V_{ab}}{\partial t}) \ . \qquad (8.1)$$

Since in a local coordinate system

$$\nabla_b V_{ij} = \partial_b V_{ij} - V_{pj}\Gamma^p_{bi} - V_{ip}\Gamma^p_{bj} \ ,$$

$$\frac{\partial \Gamma^p_{bi}}{\partial t} = -g^{pc}\left\{(\nabla_i h_{bc}) + (\nabla_b h_{ic}) - (\nabla_c h_{bi})\right\}$$

and

$$\frac{\partial}{\partial t}\left(\nabla_b V_{ij}\right) = \nabla_b \frac{\partial V_{ij}}{\partial t} - V_{pj}\frac{\partial \Gamma^p_{bi}}{\partial t} - V_{ip}\frac{\partial \Gamma^p_{bj}}{\partial t}$$

we have the following

Lemma 8.1 *If* $\frac{\partial}{\partial t}g_{ij} = -2h_{ij}$ *then*

$$\begin{aligned}\frac{\partial}{\partial t}\left(\nabla_b V_{ij}\right) &= \nabla_b \frac{\partial V_{ij}}{\partial t} + g^{pq}V_{qj}\left(\nabla_i h_{pb} + \nabla_b h_{pi} - \nabla_p h_{ib}\right) \\ &\quad + V_{iq}g^{pq}\left(\nabla_j h_{pb} + \nabla_b h_{pj} - \nabla_p h_{jb}\right) \ . \qquad (8.2)\end{aligned}$$

Lemma 8.2 *If* $\frac{\partial}{\partial t}g_{ij} = -2h_{ij}$ *then*

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_a\nabla_b V_{ij} &= \nabla_a\nabla_b\frac{\partial V_{ij}}{\partial t} + V_{pj}g^{pq}\left(\nabla_a\nabla_i h_{qb} + \nabla_a\nabla_b h_{qi} - \nabla_a\nabla_q h_{ib}\right) \\ &\quad + V_{ip}g^{pq}\left(\nabla_a\nabla_b h_{qj} + \nabla_a\nabla_j h_{qb} - \nabla_a\nabla_q h_{bj}\right) \\ &\quad + g^{pq}\left(\nabla_a V_{pj}\right)\left(\nabla_i h_{qb} + \nabla_b h_{qi} - \nabla_q h_{ib}\right) \\ &\quad + g^{pq}\left(\nabla_a V_{ip}\right)\left(\nabla_b h_{qj} + \nabla_j h_{qb} - \nabla_q h_{bj}\right) \\ &\quad + g^{pq}\left(\nabla_p V_{ij}\right)\left(\nabla_b h_{qa} + \nabla_a h_{qb} - \nabla_q h_{ba}\right) \\ &\quad + g^{pq}\left(\nabla_b V_{pj}\right)\left(\nabla_i h_{qa} + \nabla_a h_{qi} - \nabla_q h_{ia}\right) \\ &\quad + g^{pq}\left(\nabla_b V_{ip}\right)\left(\nabla_a h_{qj} + \nabla_j h_{qa} - \nabla_q h_{aj}\right)\end{aligned}$$

so that

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\Delta V_{ij}\right) = \Delta\frac{\partial V_{ij}}{\partial t} + 2h_{ab}\nabla^a\nabla^b V_{ij} - V_{bj}\nabla^a\nabla^b h_{ia} - V_{ib}\nabla^a\nabla^b h_{aj} \\ &+ V_{ip}g^{pq}\left(\nabla^a\nabla_a h_{qj} + \nabla^a\nabla_j h_{qa}\right) \\ &+ V_{pj}g^{pq}\left(\nabla^a\nabla_i h_{qa} + \nabla^a\nabla_a h_{qi}\right) \\ &+ \left(\nabla^b V_{ij}\right)\left(2\nabla^a h_{ab} - \nabla_b tr_g(h_{kl})\right) \\ &+ 2g^{pq}\left(\nabla^a V_{pj}\right)\left(\nabla_i h_{qa} + \nabla_a h_{qi}\right) - 2\left(\nabla^a V_{bj}\right)\left(\nabla^b h_{ia}\right) \\ &+ 2g^{pq}\left(\nabla^a V_{ip}\right)\left(\nabla_a h_{qj} + \nabla_j h_{qa}\right) - 2\left(\nabla^a V_{ib}\right)\left(\nabla^b h_{aj}\right) \ .\end{aligned}$$

8.2 Harnack inequality for Ricci curvature

In what follows we are working with the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{8.3}$$

with initial metric $(g(0)_{ij})$ such that the scalar curvature $R(0)$ is a positive constant. Then $R \geq R(0)$ and

$$\frac{1}{3} \leq \frac{S}{R^2} \leq \max_M \frac{S(0)}{R(0)^2}$$

as long as the solution flow to (8.3) exists.

Next we compute the evolution equation for $|\nabla_k R_{ij}|^2$.

We consider the Ricci flow $(g(t)_{ij})$: the maximum solution to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

In this section we deduce the evolution equation for $|\nabla_k R_{ij}|^2$.

Let us make computation in a normal coordinate system at the point we evaluate the geometric quantities.

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) |\nabla_k R_{ij}|^2 \;&=\; 2\left(\nabla^k R^{ij}\right)\left(\Delta - \frac{\partial}{\partial t}\right)(\nabla_k R_{ij}) \\ &\quad -2R^{ka}(\nabla_k R_{ij})\left(\nabla_a R^{ij}\right) - 2g^{jq}R^{ip}(\nabla_k R_{ij})\left(\nabla^k R_{pq}\right) \\ &\quad -2g^{ip}R^{jq}(\nabla_k R_{ij})\left(\nabla^k R_{pq}\right) + 2\langle \nabla(\nabla_k R_{ij}), \nabla\left(\nabla^k R^{ij}\right)\rangle \;.\end{aligned}$$

By the Ricci identity for symmetric tensor we have

Lemma 8.3 *If (T_{ij}) is a symmetric tensor, then on 3-manifolds we have*

$$\begin{aligned}&\nabla_c(\Delta T_{ij}) - \Delta(\nabla_c T_{ij}) \\ =\;& T_{aj}(\nabla^a R_{ci}) + T_{ia}(\nabla^a R_{cj}) - g^{ab}T_{ia}(\nabla_j R_{cb}) - g^{ab}T_{aj}(\nabla_i R_{cb}) \\ &+ (\nabla^a T_{ai})(2R_{cj} - Rg_{cj}) + (\nabla^a T_{aj})(2R_{ci} - Rg_{ci}) - (\nabla^a T_{ij})R_{ca} \\ &-2g^{ab}R_{cb}\{(\nabla_i T_{aj}) + (\nabla_j T_{ai})\} + (\nabla_j T_{ci} + \nabla_i T_{cj})R \\ &+2g^{pq}(\nabla^a T_{pi})R_{aq}g_{cj} - 2(\nabla^a T_{ci})R_{aj} \\ &+2g^{pq}(\nabla^a T_{pj})R_{aq}g_{ci} - 2(\nabla^a T_{cj})R_{ai} \;.\end{aligned} \tag{8.4}$$

Proof. In fact a direct computation (though a little bit complicated) shows that

$$\begin{aligned}&\nabla_c\nabla_a\nabla_b T_{ij} - \nabla_a\nabla_b\nabla_c T_{ij} \\ =\;& (\nabla_p T_{ij})R^p_{acb} + (\nabla_a T_{ip})R^p_{bcj} + (\nabla_a T_{pj})R^p_{bci} \\ &+ (\nabla_b T_{ip})R^p_{acj} + (\nabla_b T_{pj})R^p_{aci} + T_{ip}\left(\nabla_a R^p_{bcj}\right) + T_{pj}(\nabla_a R^p_{bci})\end{aligned} \tag{8.5}$$

and

$$
\begin{aligned}
& \nabla_b\nabla_a\nabla_c T_{ij} - \nabla_c\nabla_a\nabla_b T_{ij} \\
= \; & T_{pj}\nabla_a R^p_{cbi} + T_{ip}\nabla_a R^p_{cbj} + \left(\nabla_p T_{ij}\right) R^p_{cba} \\
& + \left(\nabla_a T_{pj}\right) R^p_{cbi} + \left(\nabla_a T_{ip}\right) R^p_{cbj} + \left(\nabla_b T_{pj}\right) R^p_{cai} \\
& + \left(\nabla_b T_{ip}\right) R^p_{caj} + \left(\nabla_c T_{pj}\right) R^p_{abi} + \left(\nabla_c T_{ip}\right) R^p_{abj} \ .
\end{aligned}
$$

Taking trace over indices a and b we thus obtain

$$
\begin{aligned}
& \nabla_c\left(\Delta T_{ij}\right) - \Delta\left(\nabla_c T_{ij}\right) \\
= \; & -\left(\nabla^a T_{ij}\right) R_{ca} + 2\left(\nabla^a T_{ip}\right) R^p_{acj} + 2\left(\nabla^a T_{pj}\right) R^p_{aci} \\
& + T_{ia}\left(\nabla^a R_{cj}\right) + T_{aj}(\nabla^a R_{ci}) - g^{ab} T_{ia}\left(\nabla_j R_{cb}\right) - g^{ab} T_{aj}\left(\nabla_i R_{cb}\right). \qquad (8.6)
\end{aligned}
$$

Now on a 3-manifold we may replace the full curvature tensor by the Ricci curvature via equation (1.4), which then yields (8.4). ■

Applying (8.4) to the Ricci symmetric tensor R_{ij} together with the Bianchi identity $\nabla^a R_{aj} = \frac{1}{2}\nabla_j R$ we obtain

$$
\begin{aligned}
\Delta\left(\nabla_k R_{ij}\right) \;=\; & \nabla_k\left(\Delta R_{ij}\right) - R_{rj}\left(\nabla^r R_{ki}\right) - R_{ir}\left(\nabla^r R_{kj}\right) \\
& + g^{rb} R_{ir}\left(\nabla_j R_{kb}\right) + g^{rb} R_{rj}\left(\nabla_i R_{kb}\right) + \left(\nabla^r R_{ij}\right) R_{kr} \\
& - \left(\nabla_i R\right)\left\{R_{kj} - \frac{1}{2} R g_{kj}\right\} - \left(\nabla_j R\right)\left\{R_{ki} - \frac{1}{2} R g_{ki}\right\} \\
& + 2 g^{rb} R_{kb}\left\{\left(\nabla_i R_{rj}\right) + \left(\nabla_j R_{ri}\right)\right\} - \left(\nabla_j R_{ki} + \nabla_i R_{kj}\right) R \\
& - 2 g^{pq}\left(\nabla^r R_{pi}\right) R_{rq} g_{kj} + 2\left(\nabla^r R_{ki}\right) R_{rj} \\
& - 2 g^{pq}\left(\nabla^r R_{pj}\right) R_{rq} g_{ki} + 2\left(\nabla^r R_{kj}\right) R_{ri} \ .
\end{aligned}
$$

On the other hand, applying (8.2) to R_{ij} we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\nabla_k R_{ij}\right) \;=\; & \nabla_k \frac{\partial R_{ij}}{\partial t} + g^{rb} R_{rj}\left\{\left(\nabla_i R_{bk}\right) + \left(\nabla_k R_{bi}\right)\right\} - g^{rb} R_{rj}(\nabla_b R_{ik}) \\
& + g^{rb} R_{ir}\left\{\left(\nabla_j R_{bk}\right) + \left(\nabla_k R_{bj}\right)\right\} - g^{rb} R_{ir}(\nabla_b R_{jk}) \ .
\end{aligned}
$$

Putting the previous two equations, after simplification, gives us the following

$$
\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right)\left(\nabla_k R_{ij}\right) \;=\; & \nabla_k\left[\left(\Delta - \frac{\partial}{\partial t}\right) R_{ij}\right] - R_{rj}\left(\nabla^r R_{ki}\right) - R_{ir}\left(\nabla^r R_{kj}\right) \\
& - \left(\nabla_i R\right)\left\{R_{kj} - \frac{1}{2} R g_{kj}\right\} - \left(\nabla_j R\right)\left\{R_{ki} - \frac{1}{2} R g_{ki}\right\} \\
& + 2 g^{rb} R_{kb}\left\{\left(\nabla_i R_{rj}\right) + \left(\nabla_j R_{ri}\right)\right\} - \left(\nabla_j R_{ki} + \nabla_i R_{kj}\right) R \\
& - 2 g_{kj} R^{rs}\left(\nabla_r R_{si}\right) - 2 g_{ki} R^{rs}\left(\nabla_r R_{sj}\right) + R_{kr}\left(\nabla^r R_{ij}\right) \\
& + g^{rb} R_{rj}\left\{3(\nabla_b R_{ik}) - (\nabla_k R_{bi})\right\} \\
& + g^{rb} R_{ir}\left\{3(\nabla_b R_{jk}) - (\nabla_k R_{bj})\right\} \ .
\end{aligned}
$$

We are now in a position to show the following

Proposition 8.4 *Under the Ricci flow* $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ *it holds that*

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right)(\nabla_k R_{ij}) \;=\;& g^{ab} R_{aj} \{5(\nabla_k R_{bi}) + 2(\nabla_b R_{ki})\} \\
& + g^{ab} R_{ia} \{5(\nabla_k R_{bj}) + 2(\nabla_b R_{kj})\} \\
& + g^{ab} R_{ka} \{2(\nabla_i R_{bj}) + 2(\nabla_j R_{bi}) + (\nabla_b R_{ij})\} \\
& - 3R_{ij}(\nabla_k R) + 2Rg_{ij}(\nabla_k R) - 2g_{ij}(\nabla_k S) \\
& - R\{(\nabla_j R_{ki}) + (\nabla_i R_{kj}) + 3(\nabla_k R_{ij})\} \\
& - (\nabla_i R)\left\{R_{kj} - \frac{1}{2} R g_{kj}\right\} - (\nabla_j R)\left\{R_{ki} - \frac{1}{2} R g_{ki}\right\} \\
& - 2g_{kj} R^{ab}(\nabla_a R_{bi}) - 2g_{ki} R^{ab}(\nabla_a R_{bj}) \,. \qquad (8.7)
\end{aligned}$$

Proof. Differentiating the evolution equation (3.4) for the Ricci curvature gives that

$$\begin{aligned}
\nabla_k \left(\Delta - \frac{\partial}{\partial t}\right) R_{ij} \;=\;& 6g^{ab} R_{aj}(\nabla_k R_{ib}) + 6g^{ab} R_{ib}(\nabla_k R_{aj}) \\
& - 3R_{ij}(\nabla_k R) - 3R(\nabla_k R_{ij}) \\
& + 2Rg_{ij}(\nabla_k R) - 2g_{ij}(\nabla_k S) \,,
\end{aligned}$$

together with the previous equation for $\left(\Delta - \frac{\partial}{\partial t}\right)(\nabla_k R_{ij})$ yield the result. ■

Taking trace in equation (8.7) we establish the following

Corollary 8.5 *Under the Ricci flow (8.3)*

$$\left(\Delta - \frac{\partial}{\partial t}\right)(\nabla_k R) = -2(\nabla_k S) + g^{ij} R_{kj}(\nabla_i R) \;. \qquad (8.8)$$

The identity (8.8) yields the following

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla R|^2 = 2|\nabla\nabla R|^2 - 4\langle \nabla R, \nabla S\rangle$$

obtained via the Bochner identity.

We now proceed to deduce the evolution for $|\nabla_k R_{ij}|^2$. By (8.7)

$$\begin{aligned}
& 2(\nabla^k R^{ij})\left(\Delta - \frac{\partial}{\partial t}\right)(\nabla_k R_{ij}) \\
=\;& -6R|\nabla_k R_{ij}|^2 - 4R(\nabla^k R^{ij})(\nabla_i R_{kj}) \\
& - 7(\nabla^k R)(\nabla_k S) + 5R|\nabla R|^2 - 8R_{kj}(\nabla^k R^{ij})(\nabla_i R) \\
& + g^{ab} R_{aj}(\nabla^k R^{ij})\{20(\nabla_k R_{bi}) + 8(\nabla_b R_{ki})\} \\
& + g^{ab} R_{ka}(\nabla^k R^{ij})\{8(\nabla_i R_{bj}) + 2(\nabla_b R_{ij})\} \;. \qquad (8.9)
\end{aligned}$$

Therefore

Lemma 8.6 *Under the Ricci flow* $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ *we have*

$$\begin{aligned}
& \left(\Delta - \frac{\partial}{\partial t}\right) |\nabla_k R_{ij}|^2 \\
= \quad & -6R|\nabla_k R_{ij}|^2 - 4R(\nabla^k R^{ij})(\nabla_i R_{kj}) \\
& -7(\nabla^k R)(\nabla_k S) + 5R|\nabla R|^2 - 8R_{kj}(\nabla^k R^{ij})(\nabla_i R) \\
& +16 g^{ab} R_{aj}(\nabla^k R^{ij})\{(\nabla_k R_{bi}) + (\nabla_b R_{ki})\} \\
& +2\langle \nabla(\nabla_k R_{ij}), \nabla(\nabla^k R^{ij})\rangle \ . \qquad (8.10)
\end{aligned}$$

We next deduce the equation for $|\nabla_k R_{ij}|^2/R$. By chain rule

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right) \frac{|\nabla_k R_{ij}|^2}{R} \quad = \quad & \left(\frac{2S}{R^2} - 6\right)|\nabla_k R_{ij}|^2 + 5|\nabla R|^2 - \frac{7}{R}\langle \nabla S, \nabla R\rangle \\
& -4(\nabla^k R^{ij})(\nabla_i R_{kj}) - \frac{8}{R} R_{kj}(\nabla^k R^{ij})(\nabla_i R) \\
& +\frac{16}{R} g^{ab} R_{ai}(\nabla^k R^{ij})\{(\nabla_b R_{jk}) + (\nabla_k R_{jb})\} \\
& +\frac{2}{R}\langle \nabla(\nabla_k R_{ij}), \nabla(\nabla^k R^{ij})\rangle - 2\langle \nabla \log R, \nabla \frac{|\nabla_k R_{ij}|^2}{R}\rangle \qquad (8.11)
\end{aligned}$$

This identity allows us to deduce a gradient estimate for the Ricci curvature. To simplify the notations set $F = |\nabla_k R_{ij}|^2/R$, and use a normal coordinate which diagonalizes the Ricci curvature (R_{ij}), and set $X_{kij} = \nabla_k R_{ij}$, then equation (8.11) may be written as the following

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right) F \quad = \quad & \left(\frac{2S}{R^2} - 6\right) RF + 5|\nabla R|^2 - \frac{7}{R}\langle \nabla S, \nabla R\rangle \\
& +\frac{16}{R}\lambda_i X_{kij}^2 + \frac{16}{R}\lambda_i X_{kij} X_{ikj} - \frac{8}{R}\lambda_i X_{iji} X_{jaa} - 4X_{kij} X_{ikj} \\
& +\frac{2}{R}\langle \nabla(\nabla_k R_{ij}), \nabla(\nabla^k R^{ij})\rangle - 2\langle \nabla \log R, \nabla F\rangle \ . \qquad (8.12)
\end{aligned}$$

We then use the following identity, for any constant $\xi \neq 0$

$$-\langle \nabla S, \nabla R\rangle = \frac{1}{\xi}|\nabla_k R_{ij} - \xi R_{ij} \nabla_k R|^2 - \frac{1}{\xi}|\nabla_k R_{ij}|^2 - \xi S|\nabla R|^2 \qquad (8.13)$$

so that, for $\xi > 0$ we have

$$\begin{aligned}
-\frac{7}{R}\langle\nabla S,\nabla R\rangle &= \frac{1}{\xi}\frac{7}{R}|\nabla_k R_{ij}-\xi R_{ij}\nabla_k R|^2-\frac{1}{\xi}\frac{7}{R}|\nabla_k R_{ij}|^2 \\
&\quad -7\xi\frac{S}{R}|\nabla R|^2 \\
&\geq -\frac{1}{\xi}\frac{7}{R}|\nabla_k R_{ij}|^2-7\xi\frac{S}{R}|\nabla R|^2 \\
&\geq -\frac{1}{\xi}\frac{7}{R}|\nabla_k R_{ij}|^2-7\xi\delta R|\nabla R|^2\ .
\end{aligned}$$

By choosing $7\xi\delta R = 5$, i.e. $\xi = \frac{5}{7\delta R}$ we deduce that

$$\begin{aligned}
\left(\Delta-\frac{\partial}{\partial t}\right)F &\geq \left(\frac{2S}{R^2}-6-\frac{49}{5}\delta\right)RF+\frac{16}{R}\lambda_i X_{kij}^2+\frac{16}{R}\lambda_i X_{kij}X_{ikj} \\
&\quad -\frac{8}{R}\lambda_i X_{iji}X_{jaa}-4X_{kij}X_{ikj}-2\langle\nabla\log R,\nabla F\rangle\ .
\end{aligned}$$

Since

$$\begin{aligned}
|\lambda_i X_{kij}X_{ikj}| &\leq \sqrt{S}\sqrt{\sum_i\left(\sum_{j,k}X_{kij}X_{ikj}\right)^2} \\
&\leq \sqrt{S}\sqrt{\sum_i\sqrt{\sum_{j,k}X_{kij}^2}\sqrt{\sum_{j,k}X_{ikj}^2}} \\
&\leq \sqrt{3S}|\nabla_k R_{ij}|^2
\end{aligned}$$

and

$$\begin{aligned}
|\lambda_i X_{iji}X_{jaa}| &= |\sum_i\lambda_i\left(\sum_{j,a}X_{iji}X_{jaa}\right)| \\
&\leq \sqrt{S}\sqrt{\sum_i\left(\sum_{j,a}X_{iji}X_{jaa}\right)^2} \\
&\leq \sqrt{S}\sqrt{\sum_{j,a}X_{jaa}^2\left(\sum_i\sum_{i,j,a}X_{iji}^2\right)} \\
&\leq \sqrt{S}|\nabla_k R_{ij}|^2\ .
\end{aligned}$$

Therefore, as $\lambda_i \geq -\varepsilon R$,

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) F \;\; &\geq \;\; \left(\frac{2S}{R^2} - 6 - \frac{49}{5}\delta\right) RF - 16\varepsilon RF - \frac{16}{R}\sqrt{3S}|\nabla_k R_{ij}|^2 \\ &\quad -\frac{8}{R}\sqrt{S}|\nabla_k R_{ij}|^2 - 4|\nabla_k R_{ij}|^2 - 2\langle \nabla \log R, \nabla F\rangle \\ &\geq \;\; \left(\frac{2}{3} - 10 - 16\sqrt{3\delta} - 8\sqrt{\delta} - \frac{49}{5}\delta - 16\varepsilon\right) RF \\ &\quad -2\langle \nabla \log R, \nabla F\rangle \ .\end{aligned}$$

Theorem 8.7 *Under the Ricci flow (8.3) on the 3-manifold with initial metric with positive constant scalar curvature* $R(0)$*, such that* $R(0)_{ij} \geq -\varepsilon R(0) g(0)_{ij}$ *for some* $\varepsilon \in [0, 1/3]$*. Then*

$$\left(L - \frac{\partial}{\partial t}\right) F \geq -C(\varepsilon) RF$$

where $L = \Delta + 2\nabla \log R$*,* $F = |\nabla_k R_{ij}|^2/R$ *and*

$$C(\varepsilon) = \frac{28}{3} + 16\sqrt{3\delta} + 8\sqrt{\delta} + \frac{49}{5}\delta + 16\varepsilon \ .$$

In particular, suppose $R(t,\cdot) \leq \theta(t)$ *then*

$$\left(L - \frac{\partial}{\partial t}\right) F \geq -C(\varepsilon)\theta(t) F$$

so that

$$\frac{|\nabla_k R_{ij}|^2}{R} \leq e^{C(\varepsilon)\int_0^t \theta(s) ds} \frac{|\nabla_k R(0)_{ij}|^2}{R(0)} \ . \tag{8.14}$$

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