Quantum Field Theory and Quantum Groups

Oxford, December 2012

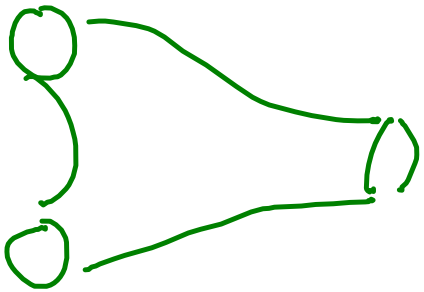
Reshitikhin + Turaev build a 3d TFT from the quantum group.

3 manifold $M^3 \rightsquigarrow$ a number $Z(M)$

2 manifold $\Sigma^2 \rightsquigarrow$ a vector space

1 manifold $S^1 \rightsquigarrow$ the category $\operatorname{Rep} U_q \mathfrak{g}$

Cobordisms



$\rightsquigarrow$ braided monoidal structure on $\operatorname{Rep} U_q \mathfrak{g}$

Witten considers a 3d TFT built from the Chern-Simons action functional:

$$A \in \Omega^1(M^3, \underset{\sim}{g})$$

$$S_{CS}(A) = \int_M \frac{1}{2}\langle A, dA\rangle_g + \frac{1}{6}\langle A, [A, A]\rangle$$

Heuristic defn of a 3d TFT:

$$Z(M) = \int_{A \in \Omega^1(M, g)/\text{Gauge}} e^{\kappa S_{CS}(A)}$$

It's believed that

Witten's TFT = Reshitikhin
-Turaev's TFT

Question: How does the Chern-Simons action functional relate to the quantum group?

Aim of this talk:

1) Derive the quantum group from S_{CS} from first principles (joint with J. Francis).

2) Explain a 4d generalization: a twist of $N=1$ SUSY gauge theory is controlled by the Yangian.

Based on a rigorous approach to QFT similar to deformation quantization:

Classical field theory (Lagrangian) $\rightsquigarrow$ Algebraic Structure

(Joint with O. Gwilliam)

Supersymmetric gauge theory

$\Longrightarrow$ interesting mathematics.

$N=2$: $\xrightarrow[\text{twisting}]{\text{topological}}$ Donaldson theory

Conversely: can perform exact calculations in $N=2$ gauge theory using Donaldson theory.

$N=4$ $\xrightarrow[\text{twisting}]{\text{top.}}$ Geometric Langlands

Program

Consider holomorphic twists of SUSY field theories

Top' Twist $\subseteq$ Hol. twist $\subseteq$ Full, untwisted theory

Give rigorous constructions of holomorphic twisted field theories, using methods of K.C., O. Gwilliam.

Try to prove physics conjectures in this context. (E.g. dualities).

Today: we'll analyze a partially twisted theory.

$N=1$ theory : NO topological twist —
but every SUSY theory in even dimensions has a holomorphic partial twist.

Main result

SUSY gauge theory (twisted, $N=1$ case deformed) $\rightsquigarrow$ the Yangian

as

CS theory $\rightsquigarrow$ the quantum group.

$\Longrightarrow$ new results about SUSY gauge theory!

Axioms for QFT

Definition

Let X be a manifold. A prefactorization algebra $\mathcal{F}$ on X is a cochain complex

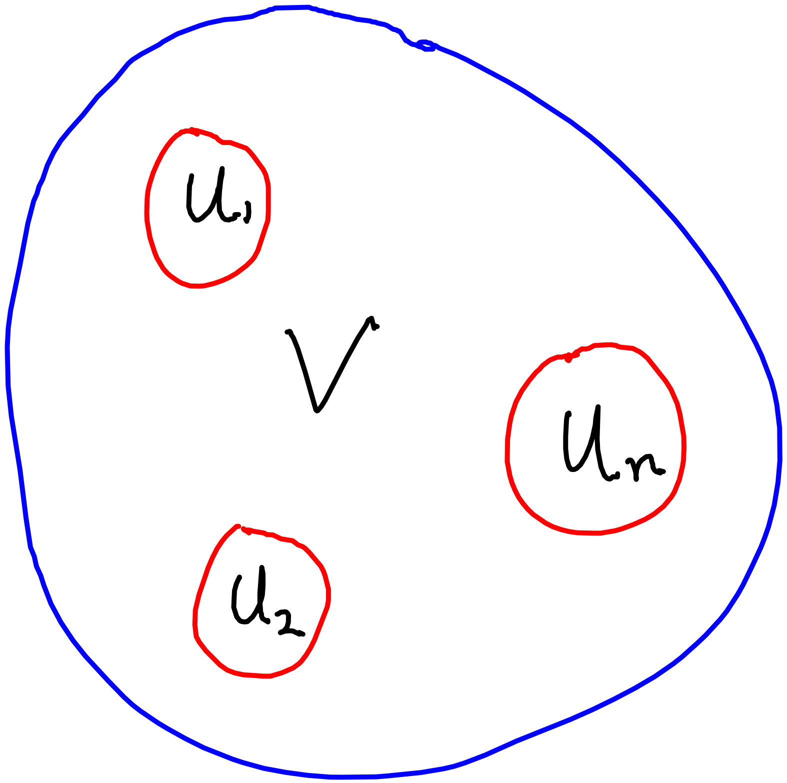
$\mathcal{F}(U)$

assigned to every open subset

$U \subseteq X$

together with some structure maps:

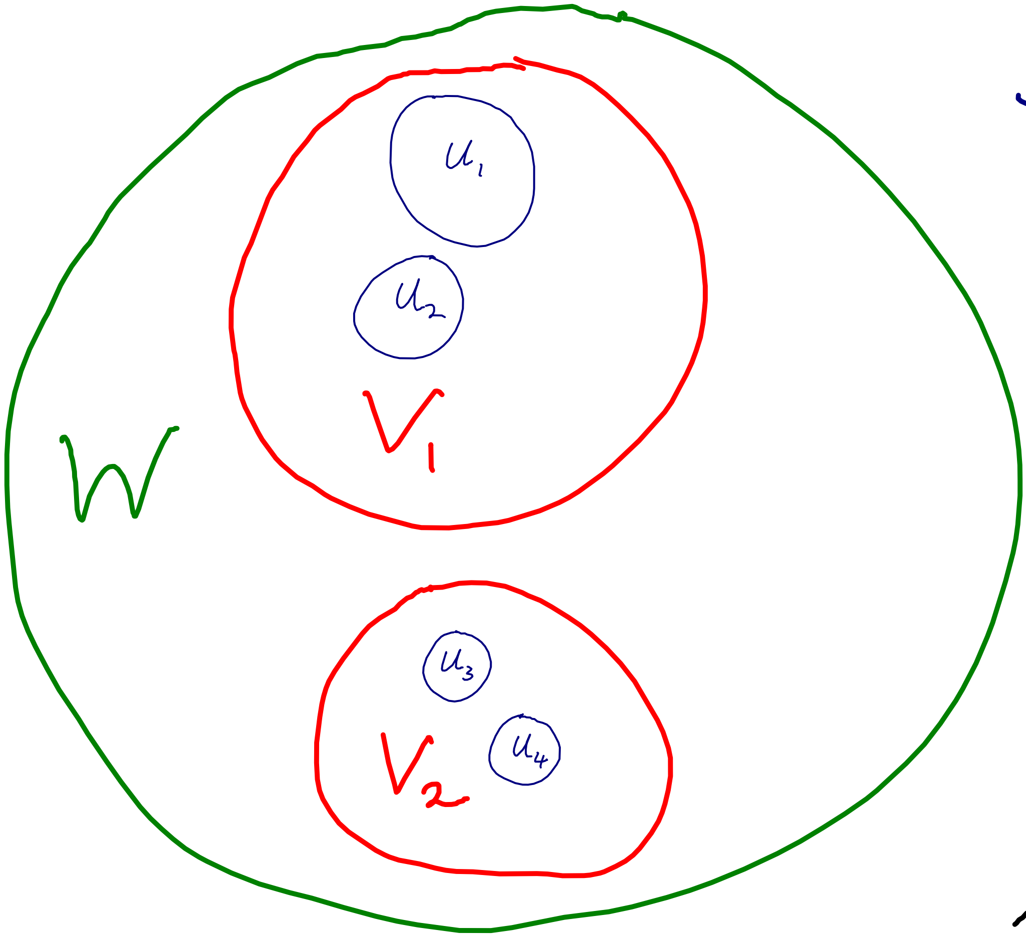
If $U_1, \dots, U_n \subseteq V \subseteq X$ are disjoint opens have a map



$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V)$$

Satisfying a natural associativity constraint:

$$\coprod U_i \subseteq \coprod V_j \subseteq W$$



$$\begin{tikzcd}
 & \bigotimes \mathcal{F}(V_j) \arrow[d] \\
\bigotimes \mathcal{F}(U_i) \arrow[ur] \arrow[r] & \mathcal{F}(W)
\end{tikzcd}$$

commutes.

Physics Interpretation:

$\mathcal{F}(\mathcal{U})$ = "quantum observables on $\mathcal{U}$"

= observations that only depend on behaviour of fields on $\mathcal{U} \subseteq X$

Structure maps = "operator product"

Often use notation

$\mathrm{Obs}^q(\mathcal{U})$ instead of $\mathcal{F}(\mathcal{U})$.

History: Beilinson, Drinfeld

Segal, "Locality of holomorphic bundles ...."

Defn is a specialization of Segal axioms for QFT, to $n-1$ manifolds

$N = \partial M$

which are boundaries.

Factorization algebras can be constructed by deformation quantization.

Classical field theory: suppose we have a space of fields (a sheaf on X) and an action functional.

Classically fields satisfy Euler-Lagrange equations.

Classical observables:

$$\mathrm{Obs}^{cl}(U) = \text{functions on } EL(U)$$

Need to use derived and formal moduli of EL solutions.

Derived = BV formalism

Formal = Perturbative

Quantum theory:

$U \mapsto \mathrm{Obs}^q(U)$ a deformation of $\mathrm{Obs}^{cl}(U)$

BV bracket gives leading order deformation.

Theorem (C., O. Gwilliam)

Can construct factorization algebras quantizing a classical field theory by obstruction theory.

(Proof uses renormalization and Feynman diagrams)

The obstruction-deformation complex built from possible action functionals.

Special classes of fact. algebras:

Locally constant:

$$\mathcal{F}(D) \longrightarrow \mathcal{F}(D')$$

a quasi-isomorphism, $D \subseteq D'$ discs.

Locally constant f.alg. on $\mathbb{R}^n$

$\Longleftrightarrow$ E_n algebras

On manifold M

$\Longleftrightarrow$

theory of configuration spaces with labels

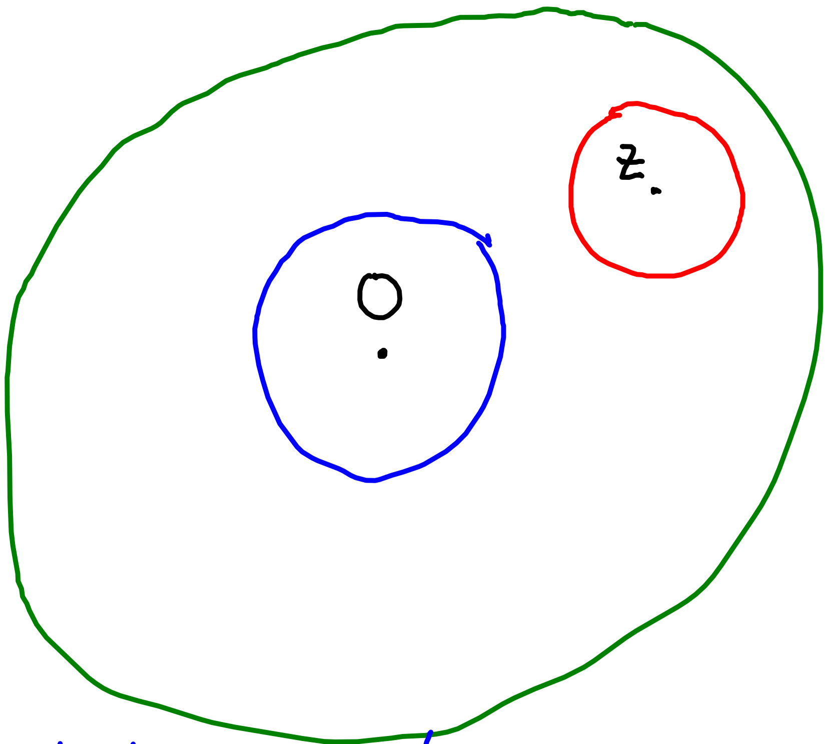
(Segal, McDuff, later Salvatore, Lurie, Francis)

Holomorphic factorization algebras:

on $\mathbb{C}$, require that $\mathcal{F}$ is translation invariant:

$$\mathcal{F}(D(z,r)) \simeq \mathcal{F}(D(0,r))$$

And: product map varies holomorphically:



$$\mathcal{F}(D(z,r_0)) \otimes \mathcal{F}(D(0,r_1)) \longrightarrow \mathcal{F}(D(0,s))$$

varies holomorphically with z in the annulus

$$r_1 + r_0 < |z| < s - r_0$$

Holomorphic version of a vertex algebra.

Example Chern-Simons.

$D \subseteq M$
disk
3-manifold

$$EL(D) = \left\{ \begin{array}{l} \text{flat } G\text{-bundles} \\ \text{on } D \end{array} \right\} = BG \text{ (classifying stack)}$$

Need to use formal moduli space: get

$$EL(D) = B\hat{G} = B\underset{\sim}{g}$$

So,

$$\mathcal{O}(EL(U)) \simeq C^\bullet(\underset{\sim}{g})$$

Chevalley cochain complex.

<u>Theorem</u> The moduli of quantizations of CS theory is $\simeq H^3(\mathfrak{g})[[\hbar]] \simeq$ space of $\hbar$-dependent levels

Each quantization $\Rightarrow$ locally-constant fact. alg. on $\mathbb{R}^3 \Rightarrow$ an E_3 algebra structure on

$$C^*(\mathfrak{g})[[\hbar]]$$

<u>Theorem</u>

This E_3 algebra encodes the quantum group $U_\hbar(\mathfrak{g})$.

(I'll explain how later).

Main Example

$\mathcal{N}=1$ SUSY gauge theory $\Longrightarrow$ the Yangian $Y(\mathfrak{g})$

Fields: $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$

$$B \in \Omega^2_+(\mathbb{R}^4, \mathfrak{g})$$

$$\psi_\pm \in \Omega^0(\mathbb{R}^4, S_\pm \otimes \mathfrak{g})$$

Action:

$$S_{N=1} = \int \langle F(A), B \rangle + c \int \langle B, B \rangle + \int \langle \psi_+, \partial\!\!\!/_A \psi_- \rangle$$

$S_\pm$ are spin representations of
$$Spin(4) = SU(2)_+ \times SU(2)_-$$
Have an action of the $N=1$ SUSY Lie algebra on the space of fields.
$$T^{N=1} = \Pi(S_+ \oplus S_-) \oplus (\mathbb{R}^4 \otimes \mathbb{C})$$
odd $\qquad$ even $\qquad \mathbb{R}$

In Euclidean signature, defined over $\mathbb{C}$

Twisting: Choose $Q \in S_+$. This has
$$[Q, Q] = 0.$$
Twisted theory: add Q to the BRST differential $\Longleftrightarrow$ localized Q-invariants

Have a spectral sequence

Factorization algebra for untwisted theory $\Longrightarrow$ Factorization algebra for twisted theory

Twisted theory knows about OPE of Q-closed observables (modulo Q-exact terms).

Examples: Donaldson theory is a twisted $N=2$ gauge theory.

$\Rightarrow$ can use Donaldson theory to compute things in $N=2$ gauge theory.

The Yangian will encode a twist of a deformed $\mathcal{N}=1$ SUSY gauge theory.

Deformation: Fields as before.

$$S_{\text{deformed}} = S_{\mathcal{N}=1} + \lambda \int dz\, CS(A)$$

$CS(A)$ = Chern-Simons 3-form

dz : choose complex structure on $\mathbb{R}^4$

so $\mathbb{R}^4 = \mathbb{C}^2$, coordinates z, w.

We still have enough SUSY to twist.

Lemma: The twisted, deformed $N=1$ SUSY gauge theory has classical soln^s

$$\left\{ \begin{array}{l} \text{Holomorphic } G\text{-bundles on } \mathbb{C}_z \times \mathbb{C}_w \\ \text{with a flat holomorphic connection} \\ \text{in the } w\text{-direction} \end{array} \right\}$$

Classical Observables = functions on moduli of classical solutions

$$\mathrm{Obs}^{cl}(D_z \times D_w) \simeq C^*(\mathfrak{g} \otimes \mathrm{Hol}(D))$$

Theorem The twisted, deformed $N=1$ gauge theory admits unique quantization (satisfying some natural properties)

Obs^q = factorization algebra of quantum observables.

This is locally constant in the w-direction holomorphic in the z-direction.

$N=2,4$ theories: there's a similar construction, only a twist, no deformation.

$$\mathrm{Obs}_q(\hat{D}_z \times D_w) \simeq C^*(\mathfrak{g}[[z]])[[\hbar]]$$

Operator product in w-direction gives this the structure of an E_2 algebra.

OP in the z-direction gives us the structure of a vertex algebra.

(These two structures are compatible!)

Recall, in Chern-Simons we found

$$\mathrm{Obs}^q_{CS}(D \subseteq \mathbb{R}^3) \simeq C^*(\mathfrak{g})[[\hbar]]$$ has E_3 structure.

$N=1$ observables $\Rightarrow$ the Yangian

Chern-Simons observables $\Rightarrow$ the quantum group.

In both cases the relationship is Koszul duality:

A, an E_2 algebra, has a Koszul dual Hopf algebra $A^!$ (due to Tamarkin)

If A is E_3, then $A^!$ is quasi-triangular.

(Drinfeld: quantum group $U_\hbar(\mathfrak{g})$ is a quasi-triangular Hopf algebra)

A a commutative algebra

$\Longrightarrow$ A-mod is symmetric monoidal

E_2 algebras are partly commutative: there's enough commutativity so A-mod is monoidal (NOT symmetric).

If A is E_3, then A-mod is braided monoidal (results of Lurie).

If B is a Hopf algebra, then B-mod is monoidal.

If B is quasi-triangular, then B-mod is braided monoidal.

Koszul duality:

A an E_2-algebra. $A^!$ is a Hopf algebra.

$$A\text{-mod} \simeq A^!\text{-mod}$$ (equivalence of monoidal categories).

If A is E_3, $A^!$ quasi-triangular have an equivalence of braided monoidal categories.

Quantum group:

$U_\hbar(\mathfrak{g})$ is a quasi-triangular Hopf algebra deforming $U(\mathfrak{g})$ ($\hbar$ is formal)

Theorem

Obs^q_{CS} is Koszul dual to $U_\hbar(\mathfrak{g})$

E_3 alg. of quantum observables of CS

(Possibly after a change of coordinates $\hbar \to f(\hbar)$)

Joint work with J. Francis

Also: Obs^q_{CS} encodes directly CS knot invariants (e.g. the Jones polynomial)

Yangian: $Y(\mathfrak{g})$ is a Hopf algebra deforming $\mathcal{U}(\mathfrak{g}[[z]])$.

Theorem $\mathrm{Obs}^{q}_{N=1}(\hat{D}_z \times D_w)$ is Koszul dual to $Y(\mathfrak{g})$.

$N=2,4$ gauge theories: similar results, NO deformation needed - only a twist.

$N=2$: find $Y(\mathfrak{g} \oplus \mathfrak{g}^*)$

$N=4$: find $Y(\mathfrak{g}[\varepsilon, \delta])$ $|\varepsilon|=1$ $|\delta|=-1$

Can hope for Langlands duality (Kapustin Witten)

So far: Operator product in the ω-direction on $\mathrm{Obs}^q_{N=1}$

$\Longleftrightarrow$ $Y(\mathfrak{g})$.

Operator product in the z-direction:

Theorem $Y(\mathfrak{g})$-mod has the structure of "vertex algebra valued in monoidal categories"

i.e. there's an OPE monoidal functor

$$Y(\mathfrak{g})\text{-mod} \times Y(\mathfrak{g})\text{-mod} \longrightarrow Y(\mathfrak{g})((\lambda))\text{-mod}$$

satisfying an associativity condition

OPE is defined using fact. alg. structure on $\mathrm{Obs}^{cl}_{N=1}$ (i.e. from OPE of $N=1$ gauge theory).

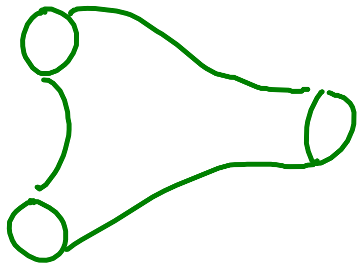
Theorem The OPE on $Y(\mathfrak{g})$-mod is encoded by Drinfeld's R-matrix

$$R \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})((\lambda))$$

Categorical interpretation

Chern-Simons: $S^1 \longrightarrow \mathcal{U}_\hbar(\mathfrak{g})\text{-mod}$

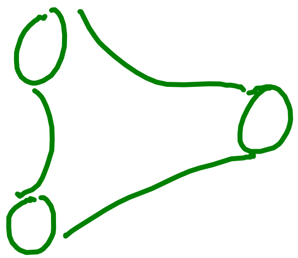
Cobordism



$\longrightarrow$ braided monoidal structure.

$N=1$: $S^1 \subseteq \mathbb{C}_z \longrightarrow$ Morita 2-category of $Y(\mathfrak{g})$-algebras

Cobordism



$\longrightarrow$ "Vertex 2-category structure" (encoded by the R-matrix)

Concrete corollaries

Consider our theory on $\mathbb{C}_z \times \mathbb{C}^\times_w$

If $V \in Y(\mathfrak{g})$-mod is finite dimensional

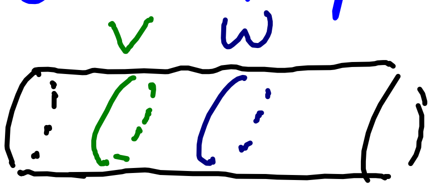
$\Longrightarrow$ we have a Wilson operator

$$\chi_V \in \mathrm{Obs}^q(\hat{D}_z \times \{|w| = 1\})$$

$V, W \in Y(\mathfrak{g})$-mod then

$$\chi_V \cdot \chi_W = \chi_{V \otimes W}$$

operator product of Wilson operators



Consider our theory on $\mathbb{C}_z \times E_\omega$

$\hookrightarrow$ elliptic curve

$A \in \underline{Y(g)}$-mod an associative algebra. If A is semisimple

$\Rightarrow$ we have a surface operator

$$\chi_A(z) \in \mathrm{Obs}^q(z \times E_\omega)$$

Can consider OPE of surface operators:

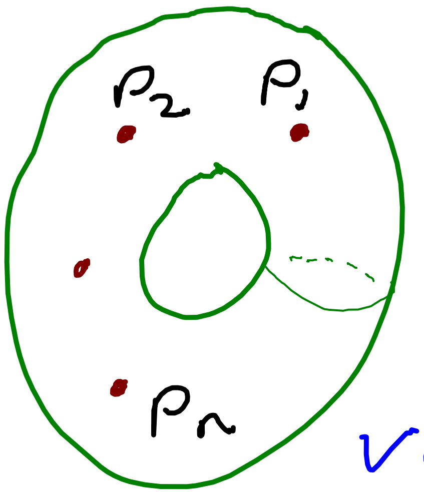
$$\chi_A(z) \cdot \chi_B(z+\lambda) \underset{\lambda \to 0}{\sim} \chi_{A \cdot B}(z)$$

$A \cdot B =$ explicit $Y(g)((\lambda))$-$\underline{\text{algebra}}$
(defined using the R-matrix)

Compactify in the other direction:

$E_z \times \mathbb{C}_w$, E_z an elliptic curve

$\rightsquigarrow$ monoidal deformation (conjecturally) of $QC(Bun_G(E_z))$. There are "correlation functors"



$$C_{p_1,\dots,p_n} : (Y(\mathfrak{g})\text{-mod})^{\times n} \longrightarrow QC_\hbar(Bun_G(E_z))$$

Monoidal functor, varying holomorphically with p_i

As p_1, p_2 collide, the functor

$c_{p_1,\dots,p_n}$ is described in terms of

$c_{p_2,\dots,p_n}$ and the OPE functor

$$Y(\mathfrak{g})\text{-mod} \times Y(\mathfrak{g})\text{-mod} \to Y(\mathfrak{g})\text{-mod}$$

encoded by the R-matrix.

Integrability Compactify in w-direction:

$H^0(\mathrm{Obs}^q(\hat{D}_z \times E))$ is a vertex algebra.

It's an analog of the "Verlinde ring" in Chern-Simons.

Theorem This vertex algebra is completely integrable: it contains a maximal commutative subalgebra. (Type A only).

(all OPEs are non-singular)

$$H^0(\mathrm{Obs}^q(\hat{D} \times S^1)) \subseteq H^0(\mathrm{Obs}^q(\hat{D} \times E))$$

is the subalgebra ($S^1 \subseteq E$ an a-cycle)

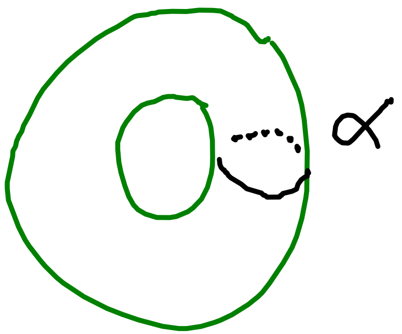
Algebra of integrals of motion

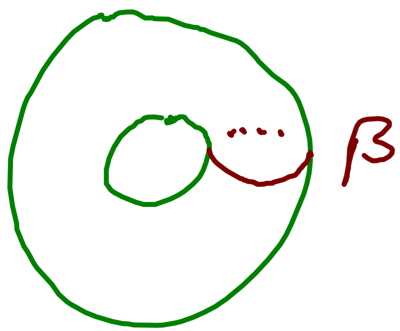
= "Bethe subalgebra of the dual Yangian"

= integrals of motion for spin-chain integrable system.

Question: Are there other relationships with the spin chain system?

Commutativity:

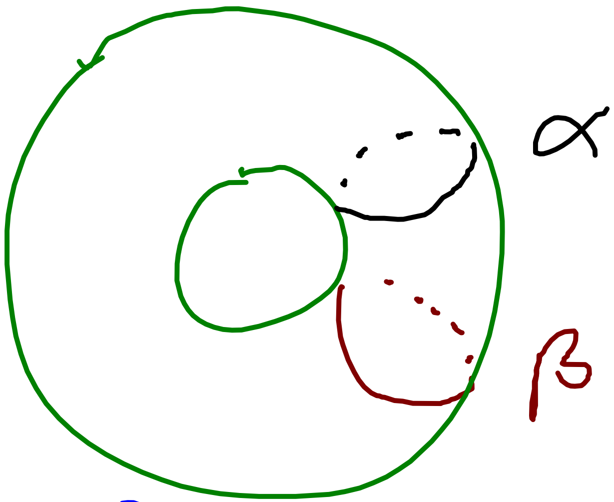




$\dot{z}$ $\qquad$ $\dot{z} + \lambda$

Can assume the 2 circles are disjoint.

$\lambda \to 0$ we get



$\dot{z}$