A LINE BUNDLE IN HYPERKÄHLER GEOMETRY

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Graeme Segal's 70th birthday

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HYPERKÄHLER MANIFOLDS

- $I, J, K \in \mathrm{End}\, T : I^2 = J^2 = K^2 = IJK = -1$

- integrable complex structures

- Kähler forms $\omega_1, \omega_2, \omega_3$

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G.B.Segal & A.Selby, *The cohomology of the space of magnetic monopoles,*Comm. Math. Phys. **177** (1996) 775-787.

- symplectic manifold (M, ω)
- $d\omega = 0$
- $[\omega/2\pi] \in H^2(M, \mathbf{Z}) \Rightarrow$ curvature of a connection ...
- ... on the "prequantum" line bundle

- Hamiltonian circle action: vector field X

- $i_X \omega = d\mu$

- moment map μ

- lifting of action: equivariant integrality of $[\omega + u\mu]$

- Kähler manifold (M, ω, I)

- ω Hodge type $(1,1) \sim$ holomorphic line bundle

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- curvature $\omega = dd^c \log \|s\|^2$
- rescale Hermitian metric $\|s\|^2 \mapsto e^f \|s\|^2$

  curvature $F = \omega + dd^c f$

HAYDYS'S RESULT

- hyperkähler with circle action
- action preserves ω_1 and $(\omega_2 + i\omega_3) \mapsto e^{i\theta}(\omega_2 + i\omega_3)$
- moment map μ...
- ... then $\omega_1 + dd^c\mu$ is of type $(1,1)$ with respect to I, J, K

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A.Haydys, *Hyperkähler and quaternionic Kähler manifolds with S^1-symmetries,* J. Geom. Phys. **58** (2008) 293–306.

MULTI-INSTANTONS

EXAMPLE: GIBBONS-HAWKING (1977)

- V harmonic function on $\mathbf{R}^3$
- $dV = *F$, $dF = 0$
- $F = d\alpha$ ($U(1)$-monopole)
- $g = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}(d\theta + \alpha)^2$
- $\omega_1 = V dx_2 \wedge dx_3 + dx_1 \wedge (d\theta + \alpha)$

FLAT SPACE

- $\mathbf{R}^4 = \mathbf{C}^2$, circle action $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2)$

- quotient space $= \mathbf{R}^3$

- $$x_1 = \frac{1}{2}(|z_1|^2 - |z_2|^2), \qquad x_2 + ix_3 = z_1 z_2$$

- $$V = \frac{1}{2r}$$ Dirac monopole

- $$V = \sum_{1}^{k+1} \frac{1}{|\mathbf{x} - \mathbf{a}_i|}$$

- each segment $t\mathbf{a}_i + (1-t)\mathbf{a}_j$ is a minimal S^2

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- circle action induced by rotation about x_1-axis $\Rightarrow \mathbf{a}_i = (a_i, 0, 0)$

- complex structure I:

  resolution of Kleinian singularity $xy = z^{k+1}$

- $\omega_1 = V dx_2 \wedge dx_3 + dx_1 \wedge (d\theta + \alpha)$

- integrality of $\omega_1/2\pi$: integrate ω_1 over the spheres

- $\Rightarrow a_{i+1} - a_i \in \mathbf{Z}$

- dimension 4, hyperholomorphic = anti-self-dual

- S^1-invariant instantons on M^4 = (singular) monopoles on $\mathbf{R}^3$

$$*F_A = \nabla_A \phi \qquad \text{(Kronheimer)}$$

- $\hat{A} = A - \phi V^{-1}(d\theta + \alpha)$

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$$\phi = -\sum_{1}^{k+1} \frac{a_i}{|\mathbf{x} - \mathbf{a}_i|} + c$$

HYPERKÄHLER QUOTIENTS

- symplectic manifold (M, ω)
- group action G
- equivariant moment map $\mu : M \to \mathfrak{g}^*$
- $\mu^{-1}(0)/G$ symplectic

- hyperkähler manifold $(M,\omega_1,\omega_2,\omega_3)$
- triholomorphic group action G: $g^*\omega_i = \omega_i, i = 1,2,3$
- equivariant moment maps $\mu_i : M \to \mathfrak{g}^*$
- $\bigcap \mu_i^{-1}(0)/G$ hyperkähler

- $\mu_c = \mu_2 + i\mu_3$ is I-holomorphic
- $\mu_c^{-1}(0)$ complex submanifold, ω_1 Kähler form
- hyperkähler quotient = symplectic quotient of $\mu_c^{-1}(0)$

- circle action commuting with G

- $(\omega_2 + i\omega_3) \mapsto e^{i\theta}(\omega_2 + i\omega_3) \Rightarrow \mu_c^{-1}(0)$ preserved

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- line bundle = prequantum line bundle

  + rescaling by f (moment map for $U(1)$)

- quantum line bundle descends in symplectic quotient $\mu_c^{-1}(0)/\!\!/G$

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- f G-invariant $\Rightarrow$ hyperholomorphic bundle descends
  to hyperkähler quotient

- $G \subset U(n) \subset Sp(n)$ acting on $\mathbf{H}^n$ commutes with S^1

- $\dim Z(G) > 0 \Rightarrow$ choose moment maps to get smooth quotient

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- e.g. $(e^{i\theta_1}, \ldots, e^{i\theta_k}, e^{i\theta_1 + \ldots + i\theta_k}) \subset Sp(k+1)$

- $\Rightarrow$ multi-instanton metrics

THE LINE BUNDLE ON $\mathbf{H}^n$

- $\mathbf{H}^n = \mathbf{C}^n \oplus j\mathbf{C}^n = V \oplus V^*$

- circle action $(z, w) \mapsto (z, e^{i\theta} w)$

- line bundle I-holomorphically trivial,

  Hermitian metric $\exp((|z|^2 - |w|^2)/2)$

TWISTOR SPACES

- $Z = M \times S^2$ twistor space

- complex structure $(\mathbf{I_u}, I_{\mathbf{P}^1})$, $I_{\mathbf{u}} = u_1 I + u_2 J + u_3 K$

- $Z \to \mathbf{P}^1$ holomorphic family of complex symplectic manifolds

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- $\zeta \in \mathbf{C} \subset \mathbf{P}^1$, I: $\zeta = 0, -I : \zeta = \infty$

- $(\omega_2 + i\omega_3) + 2i\omega_1\zeta + (\omega_2 - i\omega_3)\zeta^2$ fibrewise holomorphic symplectic form

FLAT SPACE

- $z_i + \zeta\bar{w}_i, w_i - \zeta\bar{z}_i$ holomorphic functions for $\zeta \neq \infty$
- $\zeta^{-1}z_i + \bar{w}_i, \zeta^{-1}w_i - \bar{z}_i$ holomorphic for $\zeta \neq 0$
- $Z = \mathbf{C}^{2n}(1) \to \mathbf{P}^1$
- C^∞ product $(z, w, \zeta) \mapsto (z + \zeta\bar{w}, w - \zeta\bar{z}, \zeta)$

HYPERHOLOMORPHIC BUNDLES

- F curvature $(1,1)$ wrt all complex structures

- $p : Z = M \times S^2 \to M$

- p^*F type $(1,1)$ on Z

- hyperholomorphic bundle on $M \Leftrightarrow$ holomorphic bundle on Z

FLAT SPACE

- $Z = V(1) \oplus V^*(1)$
- hyperholomorphic line bundle
- $(v, \xi, \zeta) \sim (v/\zeta, w/\zeta, 1/\zeta) = (\tilde{v}, \tilde{w}, \tilde{\zeta})$
- transition function

$$\exp(-\langle v, \xi \rangle / 2\zeta)$$

HYPERKÄHLER QUOTIENTS

- G acts on M preserving $\omega_1, \omega_2, \omega_3$ and I, J, K

- holomorphic action on Z preserving fibres of $Z \to \mathbf{P}^1$

- holomorphic moment map $\nu = (\mu_2 + i\mu_3) + 2i\mu_1\zeta + (\mu_2 - i\mu_3)\zeta^2$

- twistor space of quotient $\bar{Z} = \nu^{-1}(0)/G^c$

- $\nu^{-1}(0)$ is a principal G^c-bundle over $\bar{Z}$

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- for each homomorphism $\chi : G \to U(1)$, we obtain

- .... a hyperholomorphic line bundle on $\bar{M}$.

T.Gocho & H.Nakajima, *Einstein-Hermitian connections on hyperkähler quotients,* J. Math. Soc. Japan **44** (1992) 43–51.

EXAMPLE: CALABI METRIC

- hyperkähler quotient of $\mathbf{H}^{n+1}$ by $U(1) \cong T^*\mathbf{CP}^n$
- twistor space $V(1) \oplus V^*(1)$, action $(v, \xi) \mapsto (e^{i\theta}v, e^{-i\theta}\xi)$
- moment map $\nu = \langle v, \xi \rangle + 2(2\pi n)i\zeta$

EXAMPLE: CALABI METRIC

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- moment map $\nu = \langle v, \xi \rangle + 2(2\pi n)i\zeta$
- on $\nu^{-1}(0)$ transition function

$$\exp(-\langle v, \xi \rangle / 2\zeta) = \exp 2\pi i n = 1$$

- line bundle holomorphically trivial on $\nu^{-1}(0)$

- non-trivial action

- $\Rightarrow$ holomorphic line bundle on quotient

PREQUANTUM LINE BUNDLE

- twistor space $Z \to \mathbf{P}^1$

- fibrewise symplectic form $(\omega_2 + i\omega_3) + 2i\omega_1\zeta + (\omega_2 - i\omega_3)\zeta^2$

- $(\omega_2 + i\omega_3) \mapsto e^{i\theta}(\omega_2 + i\omega_3)$

- $d(i_X(\omega_2 + i\omega_3)) = \mathcal{L}_X(\omega_2 + i\omega_3) = i(\omega_2 + i\omega_3)$

- twistor space $Z \to \mathbf{P}^1$
- fibrewise symplectic form $(\omega_2 + i\omega_3) + 2i\omega_1\zeta + (\omega_2 - i\omega_3)\zeta^2$
- $(\omega_2 + i\omega_3) \mapsto e^{i\theta}(\omega_2 + i\omega_3)$
- $d(i_X(\omega_2 + i\omega_3)) = \mathcal{L}_X(\omega_2 + i\omega_3) = i(\omega_2 + i\omega_3)$

$$\frac{1}{2\zeta}(\omega_2 + i\omega_3) + i\omega_1 + \frac{1}{2}(\omega_2 - i\omega_3)\zeta$$

$2\pi\times$ integral cohomology class for all ζ

PROP: The line bundle over Z defining the hyperholomorphic bundle on M admits a meromorphic connection such that

- there are simple poles at $\zeta = 0$ and $\zeta = \infty$
- the curvature restricts to
$$\frac{1}{2\zeta}(\omega_2 + i\omega_3) + i\omega_1 + \frac{1}{2}(\omega_2 - i\omega_3)\zeta$$
on each fibre over $\mathbf{C}^* \subset \mathbf{P}^1$
- the curvature form is annihilated by the holomorphic vector field on Z generated by the circle action.

- holomorphic “prequantum line bundle”: descends in a quotient

- $\mathcal{F}$ = curvature, $i_X \mathcal{F} = 0$

- $\Rightarrow \mathbf{C}^*$-action gives a holomorphic symplectic identification of the fibres over $\mathbf{C}^*$

NJH *On the hyperkähler/quaternion Kähler correspondence*, arXiv 1210.0424

EXAMPLE: FLAT SPACE

- connection: 1-forms A_U, A_V : $A_V = A_U + g_{UV}^{-1} dg_{UV}$

- twistor space $V(1) \oplus V^*(1)$

- $$A_U = \frac{1}{2\zeta} \sum_i v_i dw_i, \qquad A_V = -\frac{1}{2\tilde{\zeta}} \sum_i \tilde{w}_i d\tilde{v}_i$$

- $$A_V - A_U = -\frac{\zeta}{2} \sum_i \frac{w_i}{\zeta} d\frac{v_i}{\zeta} - \frac{1}{2\zeta} \sum_i v_i dw_i = -d\left(\frac{1}{2\zeta} \sum_i v_i w_i\right)$$

INFINITE DIMENSIONAL QUOTIENTS

- compact Riemann surface Σ

- principal G-bundle P

- $\mathcal{A}$ affine space of connections on P

- infinite-dimensional flat Kähler manifold

- $\mathcal{G}$ group of gauge transformations

- $(A, \Phi) \in T^*\mathcal{A} = \mathcal{A} \times \Omega^{1,0}(\Sigma, \mathfrak{g})$ flat hyperkähler manifold

- moment map $(F_A + [\Phi, \Phi*], \bar{\partial}_A \Phi)$

- $\mathcal{G}$ group of gauge transformations
- $(A, \Phi) \in T^*\mathcal{A} = \mathcal{A} \times \Omega^{1,0}(\Sigma, \mathfrak{g})$ flat hyperkähler manifold
- moment map $(F_A + [\Phi, \Phi*], \bar{\partial}_A \Phi)$
- quotient moduli space of Higgs bundles
- $(A, \Phi) \in T^*\mathcal{A}$, circle action $\Phi \mapsto e^{i\theta}\Phi$

S.K.Donaldson, *Boundary value problems for Yang-Mills fields,*
J. Geom. Phys. **8** (1992) 89–122.

- $\Sigma =$ unit disc D

- given a map (metric) $f : \partial D \to G^c/G$ and holomorphic $\Phi : D \to \mathfrak{g}^c$

- **Thm** (Donaldson)... *there is a unique reduction to G on D such that* $F_A + [\Phi, \Phi^*] = 0$

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- $\Phi = 0 \Rightarrow F_A = 0$

- constant sections on D restrict to a G-framing on ∂D

- moduli space with $\Phi = 0 \Rightarrow LG/G$

- general Φ: moduli space $= T^*(LG/G)$

- hyperkähler manifold (complex structure I)

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- circle action $\Phi \mapsto e^{i\theta}\Phi$

- moment map $\mu \sim \|\Phi\|^2$

- $F_A + [\Phi, \Phi*] = 0, \bar{\partial}_A \Phi = 0$

- $\Rightarrow \nabla_A + \Phi + \Phi^*$ flat G^c connection

- LG^c/G^c (complex structure J)

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- **Thm** (Hamilton) *Given* $f : \partial D \to G^c/G$ *there is a unique harmonic extension to* D

- $\mu = \|\Phi\|^2 =$ energy $=$ minimum energy over all extensions.

- infinite-dimensional hyperkähler manifold

- $T^*(LG/G)$ or LG^c/G^c

- circle action

- **Problem:** *Describe the hyperholomorphic line bundle.*

COADJOINT ORBITS

- $\gamma : S^1 \to G$, $e^{i\theta} \cdot g(z) = \gamma(e^{i\theta} z)\gamma(e^{i\theta})^{-1}$

- fixed points in LG/G: coadjoint orbits

COADJOINT ORBITS

- $\gamma : S^1 \to G$, $e^{i\theta} \cdot g(z) = \gamma(e^{i\theta} z)\gamma(e^{i\theta})^{-1}$

- fixed points in LG/G: coadjoint orbits of G

- on $T^*(LG/G)$ triholomorphic circle action

- $\Rightarrow$ hyperkähler metric on cotangent bundle
  (complex structure I)

- complex coadjoint orbit (complex structure J)

- equations $F + [\Phi, \Phi^*] = 0, \bar{\partial}_A \Phi = 0$

- invariant $\Rightarrow$ ODE Nahm's equations

- $T_1' = [T_2, T_3]$ etc. on $(-\infty, 0]$

- P.B. Kronheimer, *A hyperkähler structure on coadjoint orbits of a semi-simple complex group*, J. London Math. Soc. **42** (1990) 193–208.

COADJOINT ORBITS: TWISTOR SPACE

D.Burns, *Some examples of the twistor construction,* in "Contributions to several complex variables", 5167, Aspects Math., Vieweg, (1986)

- $z \in \mathfrak{g}$ centralizer H

- parabolic subgoups $P_+, P_- \subset G^c, P_+ \cap P_- = H^c$

- real coadjoint orbit $G/H \cong G^c/P_+ \cong G^c/P_-$

- complex coadjoint orbit G^c/H^c

- $\mathfrak{p}_+ = \mathfrak{h} + \mathfrak{n}_+, \quad z \in \mathfrak{h}$

- $Z_0 = G^c \times_{P_+} \{\mathbf{C} \cdot z + \mathfrak{n}_+\} \quad Z_\infty = G^c \times_{P_-} \{\mathbf{C} \cdot z + \mathfrak{n}_-\}$

- $T^*(G^c/P_+) \cong G^c \times_{P_+} \mathfrak{n}_+$

- $\zeta \neq 0, G^c \times_{P_+} \{\zeta z + \mathfrak{n}_+\}$ affine bundle over G^c/P_+

- $(g, \zeta z + x_+) \mapsto (\mathrm{Ad}\, g(\zeta z + x_+), \zeta)$

- G^c-orbit of ζz

- $z \mapsto \zeta z$ isomorphism of orbits

- symplectic for $\omega_{\mathrm{can}}/\zeta$

- twistor space: identify Z_0, Z_∞ over $\zeta \in \mathbf{C}^*$ by
- $(x, \zeta) \mapsto (\zeta^{-2}x, \zeta^{-1})$
- $Z \to \mathbf{P}^1$

- $p_0 : Z_0 \to G^c/P_+, \quad p_\infty : Z_\infty \to G^c/P_-$

- prequantum line bundles L_+, L_- on $G/H = G^c/P_\pm$

  defined by $\chi_\pm : P_\pm \to \mathbf{C}^*$

- $p_0 : Z_0 \to G^c/P_+, \quad p_\infty : Z_\infty \to G^c/P_-$

- prequantum line bundles L_+, L_- on $G/H = G^c/P_\pm$

  defined by $\chi_\pm : P_\pm \to \mathbf{C}^*$

- $\chi_\pm$ agree on $H^c = P_+ \cap P_-$

- isomorphism $p_+^* L_+ \cong p_-^* L_-$ on $Z_0 \cap Z_\infty \cong G^c/H^c \times \mathbf{C}^*$

HERMITIAN SYMMETRIC SPACES

O. Biquard, P. Gauduchon, *Hyperkähler metrics on cotangent bundles of Hermitian symmetric spaces,* in Lecture Notes in Pure and Appl. Math **184**, 287–298, Dekker (1996)

- $p : T^*(G/H) \to G/H$

- $\omega_1 = p^*\omega + dd^c h$

- $h = (f(IR(IX, X))X, X)$, R curvature tensor, $X \in T^*$

- $$f(u) = \frac{1}{u}\left(\sqrt{1+u} - 1 - \log \frac{1+\sqrt{1+u}}{2}\right)$$

- hyperholomorphic line bundle $F = \omega_1 + dd^c \mu$

- $F = p^* \omega + dd^c \tilde{h}$

- $\tilde{h} = (\tilde{f}(IR(IX, X))X, X)$

- $\tilde{f}(u) = \frac{1}{u}\left(-\log \frac{1 + \sqrt{1+u}}{2} \right)$

A.Tumpach, *Hyperkähler structures and infinite-dimensional Grassmannians,* J. Func. Anal. **243**, 158–206 (2007)

- restricted Grassmannian

- infinite-dimensional Hermitian symmetric space

- same formula


MAISON FONDÉE
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APPELLATION D'ORIGINE
CHAMPAGNE CONTRÔLÉE
HEIDSIECK & Co
MONOPOLE
BLUE TOP
CHAMPAGNE
BRUT
ÉLABORÉ PAR HEIDSIECK & CO MONOPOLE ÉPERNAY FRANCE


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