Operads In the Infrared

Oxford, December 18, 2012

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Collaboration with Davide Gaiotto & Edward Witten

… *work in progress* ….

Thanks to Nick Sheridan for useful discussions.

A talk for Graeme

Instead of talking with more confidence about finished work I chose this work in progress because the topic is right.

Over the years I've learned from Graeme about many many things, and several of them show up prominently in this talk, including the relation of Morse theory and quantum field theory, the theory of determinant lines and eta invariants, and, most relevant to this talk, the theory of operads.

Newton Institute: August 1992.

I've always felt ashamed that after he taught me all that I never used them in my work.

But it always seemed wise to wait for the operads to come to me, rather than the other way round.

Somewhat surprisingly just this has happened in the course of an investigation involving massive QFT in 1+1 dimensions.

Motivations

1. 1+1 dimensional Landau-Ginzburg models with (2,2) supersymmetry: Boundary conditions and D-branes.

2. Knot homology:

Witten reformulated knot homology in terms of Morse complexes. This formulation can be further refined to a problem in categorification of Witten indices in certain LG models.

3. Higgs bundles & Hitchin systems on Riemann surfaces:

GMN studied wall-crossing of BPS degeneracies. An important special case is related to Hitchin systems. It is clear there should be a "categorification" of our nonabelianization map, and of the KSWCF, and understanding LG models is an important first step.

Outline

- Introduction & Motivations
- Webology
- Landau-Ginzburg Models & Morse Theory
- Fans of solitons & Webs
- A_∞ categories of branes
- Supersymmetric Interfaces
- Summary & Outlook

Definition of a Plane Web

We begin with a purely mathematical construction.

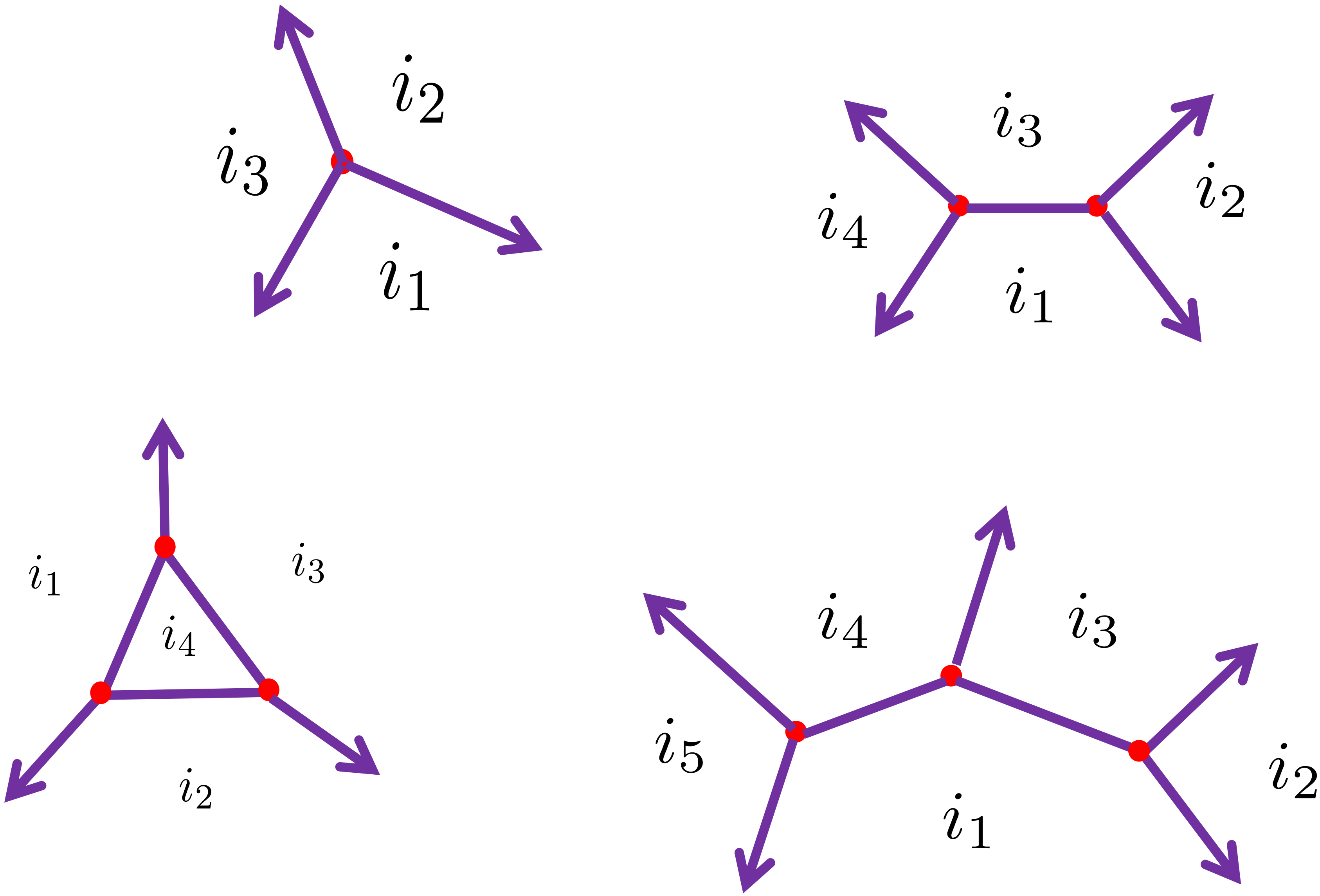
We show later how it emerges from LG field theory.

Basic data:

1. A finite set of ``vacua'': $i, j, k, \cdots \in \mathbb{V}$

2. A set of weights $z : \mathbb{V} \to \mathbb{C}$

Definition: A *plane web* is a graph in $\mathbb{R}^2$, together with a labeling of faces by vacua so that across edges labels differ and if an edge is oriented so that *i* is on the left and *j* on the right then the edge is parallel to $z_{ij} = z_i - z_j$.



i_2
i_3
i_1
i_3
i_4
i_2
i_1
i_1
i_3
i_4
i_2
i_4
i_3
i_5
i_2
i_1

Remarks & Definitions

Useful intuition: We are joining together straight strings under a tension z_{ij}. At each vertex there is a no-force condition:

$$z_{i_1,i_2} + z_{i_2,i_3} + \cdots z_{i_n,i_1} = 0$$

Definition: A cyclic fan of vacua is an ordered set

$$I = \{i_1, \ldots, i_n\}$$

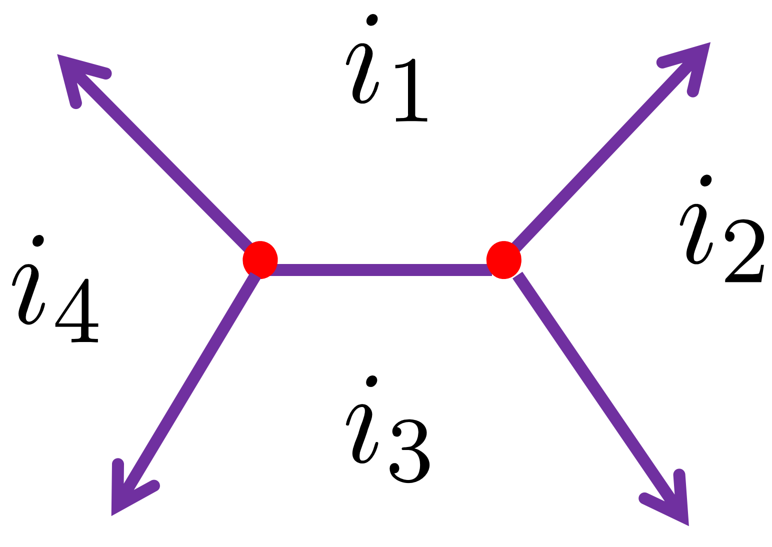
so that the rays $z_{i_k,i_{k+1}}\mathbb{R}_+$ are ordered counterclockwise

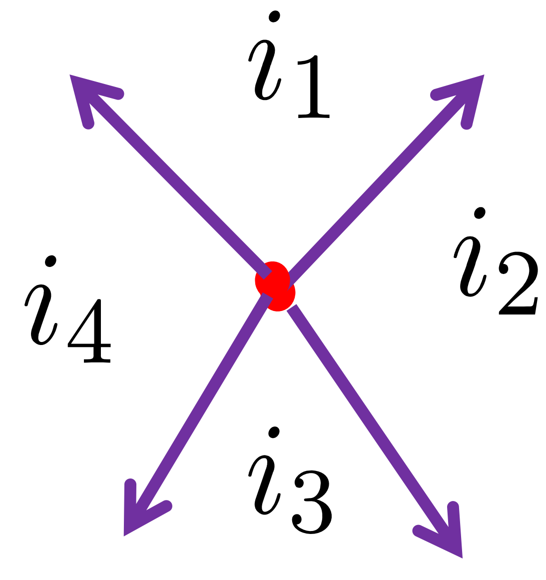
The set of vertices of a web $\mathfrak{w}$ is denoted $\mathcal{V}(\mathfrak{w})$

Local fan of vacua at a vertex *v*: $I_v(\mathfrak{w})$ and at ∞ $I_\infty(\mathfrak{w})$

Deformation Type

Equivalence under translation and stretching (but not rotating) of strings subject to no-force constraint defines ***deformation type***.





Moduli of webs with fixed deformation type

$$\mathcal{D}(\mathfrak{w}) \cong \mathbb{R}^{2V(\mathfrak{w})-E(\mathfrak{w})}$$

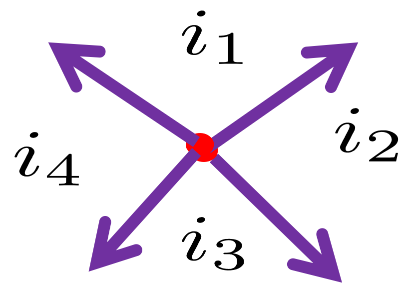
$$\mathcal{D}^{\mathrm{red}}(\mathfrak{w}) = \mathcal{D}(\mathfrak{w})/\mathbb{R}^2_{\mathrm{transl}}$$

$$\mathcal{D}^{\mathrm{red}}(\mathfrak{w}) \cong \mathbb{R}^{d(\mathfrak{w})}_{>0}$$

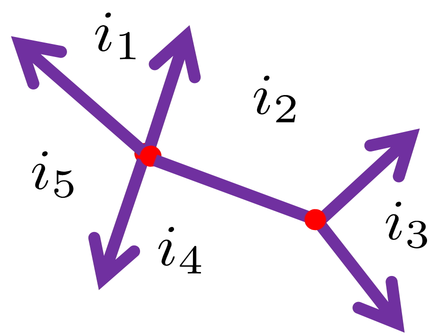
$$d(\mathfrak{w}) := 2V(\mathfrak{w}) - E(\mathfrak{w}) - 2$$

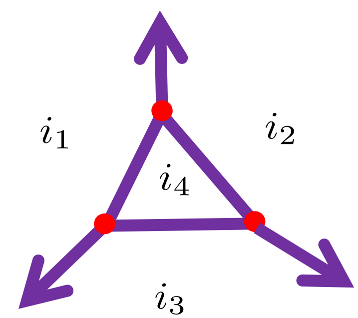
Rigid, Taut, and Sliding

A *rigid web* has d($\boldsymbol{\mathfrak{w}}$) = 0.
It has one vertex:

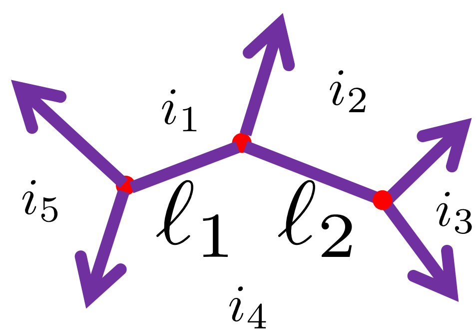


A *taut web* has d($\boldsymbol{\mathfrak{w}}$) = 1:





A *sliding web* has d($\boldsymbol{\mathfrak{w}}$) = 2

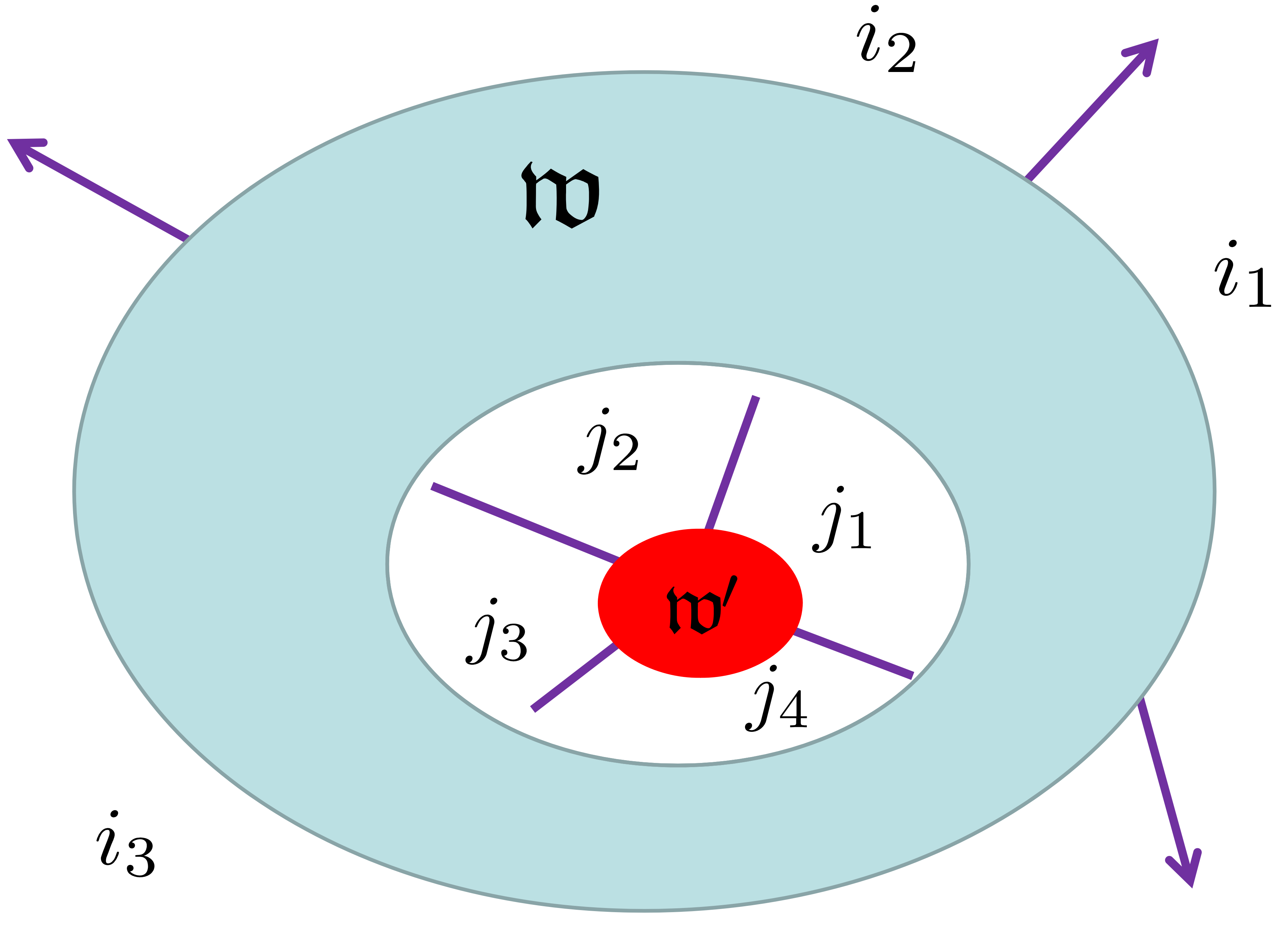


Convolution of Webs

$\mathcal{M}^{\rm red}(I_\infty)$ Reduced moduli space of all webs with specified fan of vacua I_∞ at infinity. It is a manifold with corners, made of cells $\mathcal{D}^{\rm red}(\mathfrak{w})$

Definition: Suppose $\mathfrak{w}$ and $\mathfrak{w}$' are two plane webs and $v \in \mathcal{V}(\mathfrak{w})$ such that $I_v(\mathfrak{w}) = I_\infty(\mathfrak{w}')$

The *convolution of $\mathfrak{w}$ and $\mathfrak{w}$'*, denoted $\mathfrak{w} *_v \mathfrak{w}$' is the deformation type where we glue in a copy of $\mathfrak{w}$' into a small disk cut out around *v*.



i_2
$\mathfrak{w}$
i_1
j_2
j_1
$\mathfrak{w}'$
j_3
j_4
i_3

Boundaries & Convolution

Reduced dimensions add under convolution:

$$d(\mathfrak{w} *_v \mathfrak{w}') = d(\mathfrak{w}) + d(\mathfrak{w}')$$

Near the boundaries of the closure of $\mathcal{D}^{\text{red}}(\mathfrak{w})$

$\mathfrak{w}$ can be written as a convolution

$$\mathfrak{w}_1 *_v \mathfrak{w}_2$$

The Web Group

$\mathcal{W}$ Free abelian group generated by oriented deformation types.

``oriented'': Choose an orientation o($\mathfrak{w}$) of $\mathcal{D}^{\text{red}}(\mathfrak{w})$

$$* : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$$

$$I_v(\mathfrak{w}_1) \neq I_\infty(\mathfrak{w}_2) \Rightarrow \mathfrak{w}_1 *_v \mathfrak{w}_2 = 0$$

$$\mathfrak{w}_1 * \mathfrak{w}_2 = \sum_{v \in \mathcal{V}(\mathfrak{w}_1)} \mathfrak{w}_1 *_v \mathfrak{w}_2 = 0$$

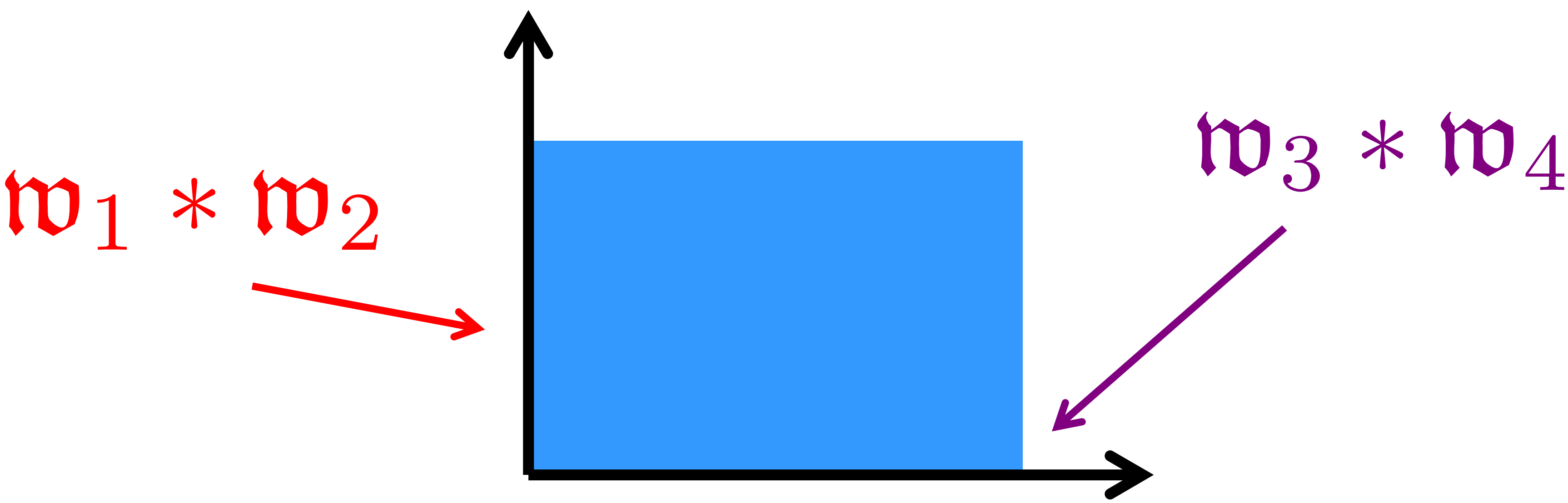
$$o(\mathfrak{w} *_v \mathfrak{w}') = o(\mathfrak{w}) \wedge o(\mathfrak{w}')$$

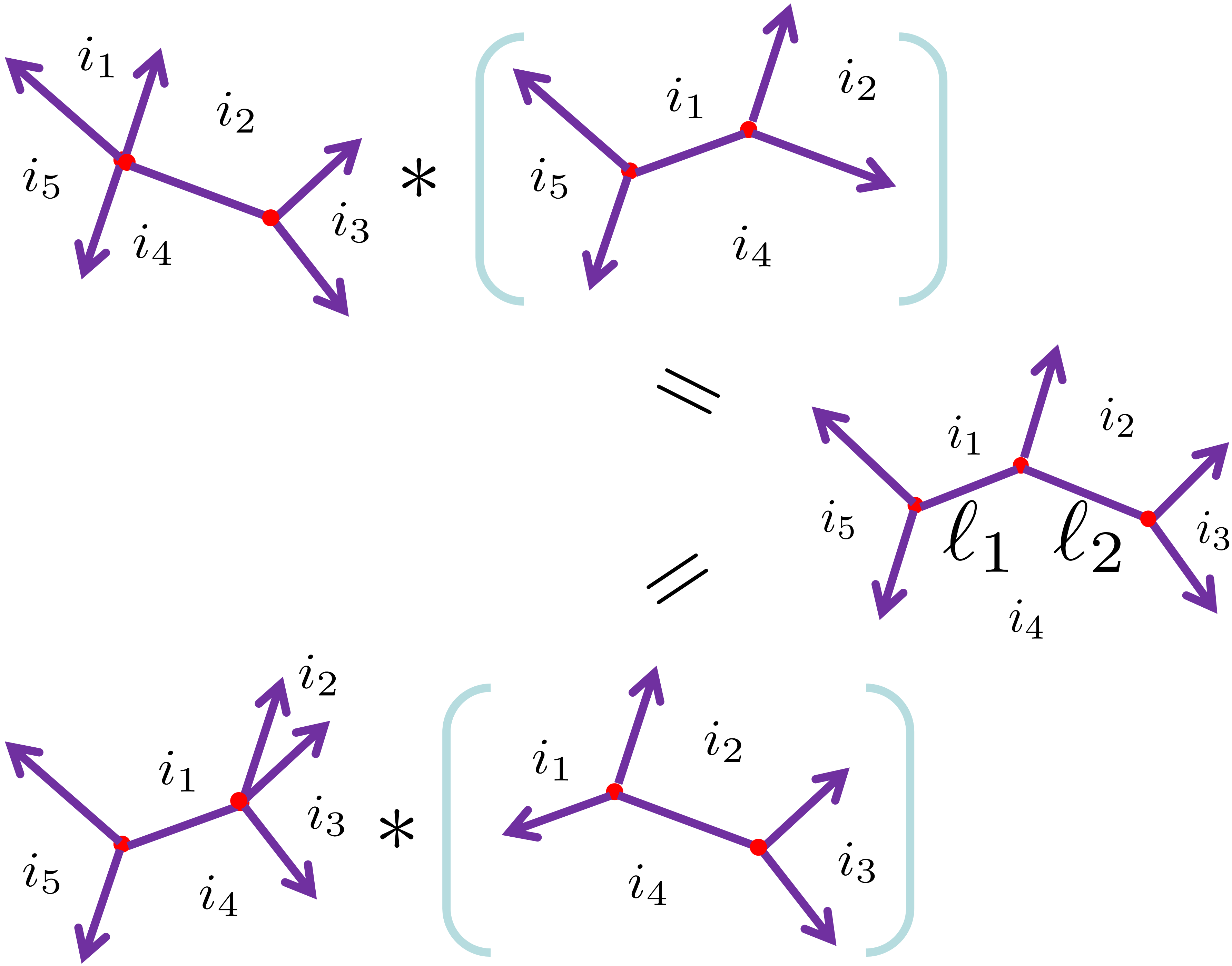
The taut element

Definition: The taut element $\mathfrak{t}$ is the sum of all taut webs with standard orientation

$$\mathfrak{t} := \sum_{d(\mathfrak{w})=1} \mathfrak{w}$$

Theorem: $\mathfrak{t} * \mathfrak{t} = 0$





i_1
i_2
i_3
i_4
i_5
$*$
$=$
ℓ_1
ℓ_2

An Associative Multiplication

Convolution is not associative.

Define an associative operation by taking an unordered set $\{v_1, \ldots, v_m\}$ and an ordered set $\{\mathfrak{w}_1, \ldots, \mathfrak{w}_m\}$ and saying

$$\mathfrak{w} *_{\{v_1,\ldots,v_m\}} \{\mathfrak{w}_1, \ldots, \mathfrak{w}_m\}$$

vanishes unless there is some ordering of the v_i so that the fans match up.

When the fans match up we take the appropriate convolution.

$$T\mathcal{W} := \mathcal{W} \oplus \mathcal{W}^{\otimes 2} \oplus \mathcal{W}^{\otimes 3} \oplus \cdots$$

$$T(\mathfrak{w})[\mathfrak{w}_1 \otimes \cdots \otimes \mathfrak{w}_n] := \mathfrak{w} *_{\mathcal{V}(\mathfrak{w})} \{\mathfrak{w}_1, \ldots, \mathfrak{w}_n\}$$

L-∞ Relations

Now $*$ and T are compatible in the sense that

$$T(\mathfrak{w} * \mathfrak{w}')[\mathfrak{w}_1, \dots, \mathfrak{w}_n] = \sum_{S_1 \amalg S_2} \epsilon \, T\mathfrak{w}[T\mathfrak{w}'[S_1], S_2] \qquad (7.22)$$

where we sum over 2-shuffles of the ordered set $\{\mathfrak{w}_1, \dots, \mathfrak{w}_n\}$. That is we sum over ordered disjoint decompositions

$$\{\mathfrak{w}_1, \dots, \mathfrak{w}_n\} = S_1 \amalg S_2 \qquad (7.23)$$

$$\sum_{S_1 \amalg S_2} \epsilon \, T(\mathfrak{t})[T(\mathfrak{t})[S_1], S_2] = 0 \qquad (7.24)$$

where we sum over 2-shuffles of any ordered set of plane webs. These are the L_∞ relations.

This makes $\mathcal{W}$ into an L∞ algebra

Half-Plane Webs & Fans -1

Same as plane webs, but they sit in a left- or right half-plane.

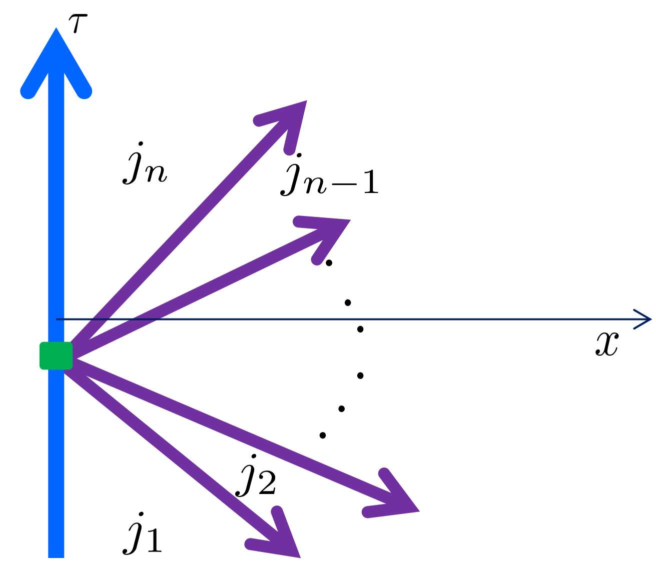
Some vertices (but no edges) are allowed on the boundary.

A half-plane fan is an ordered set of vacua, rays through

$$J = \{j_1, \dots, j_n\}$$

$$z_{j_s, j_{s+1}}$$

ordered counterclockwise.



Half-Plane Webs & Fans - 2

$\mathcal{V}_i(\mathfrak{u})$ Interior vertices

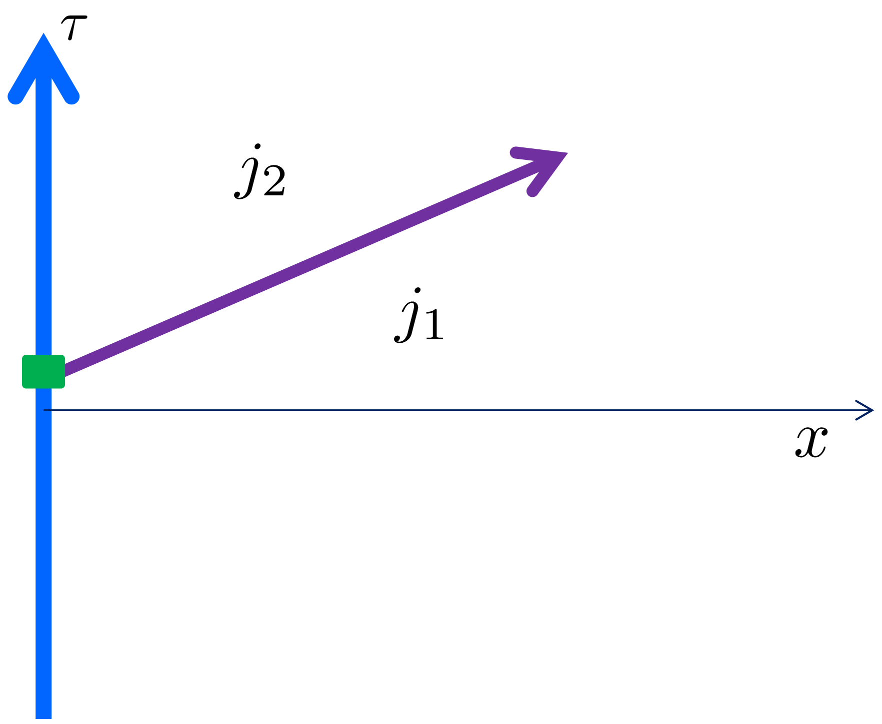
$\mathcal{V}_\partial(\mathfrak{u}) = \{v_1, \dots, v_n\}$ ***time-ordered*** boundary vertices.

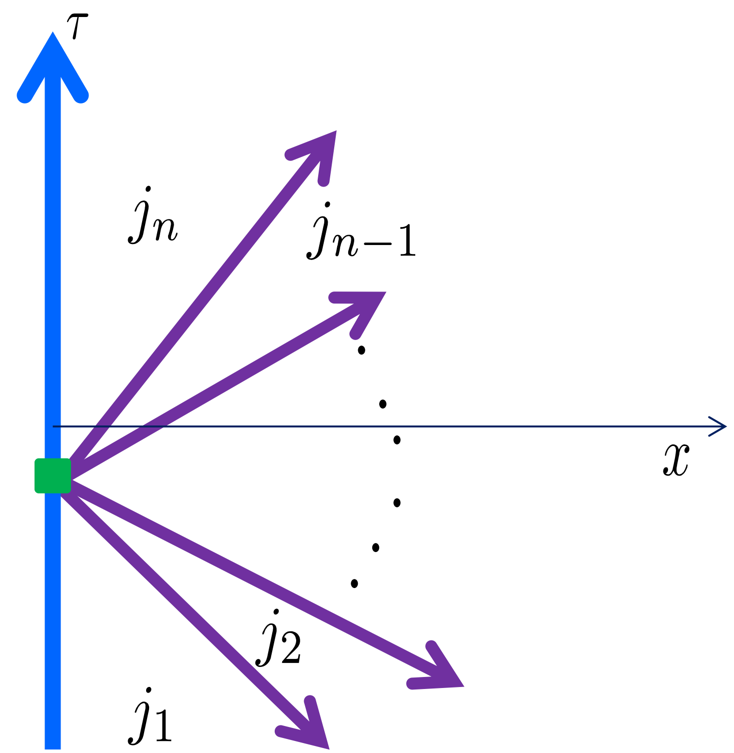
Local half-plane fan at a boundary vertex *v*: $J_v(\mathfrak{u})$

Half-plane fan at infinity: $J_\infty(\mathfrak{u})$

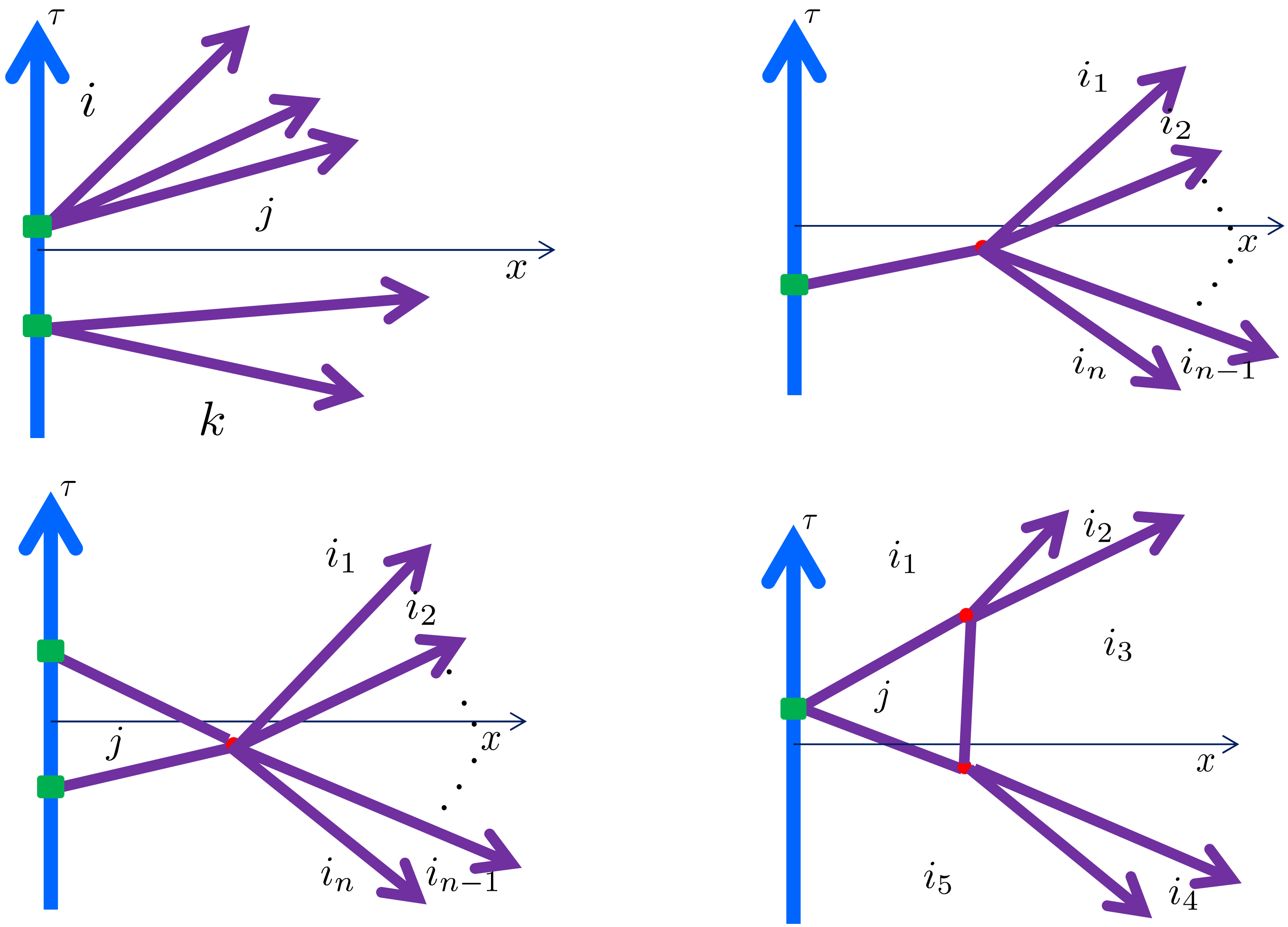
$$d(\mathfrak{u}) := 2V_i(\mathfrak{u}) + V_\partial(\mathfrak{u}) - E(\mathfrak{u}) - 1$$

Rigid Half-Plane Webs

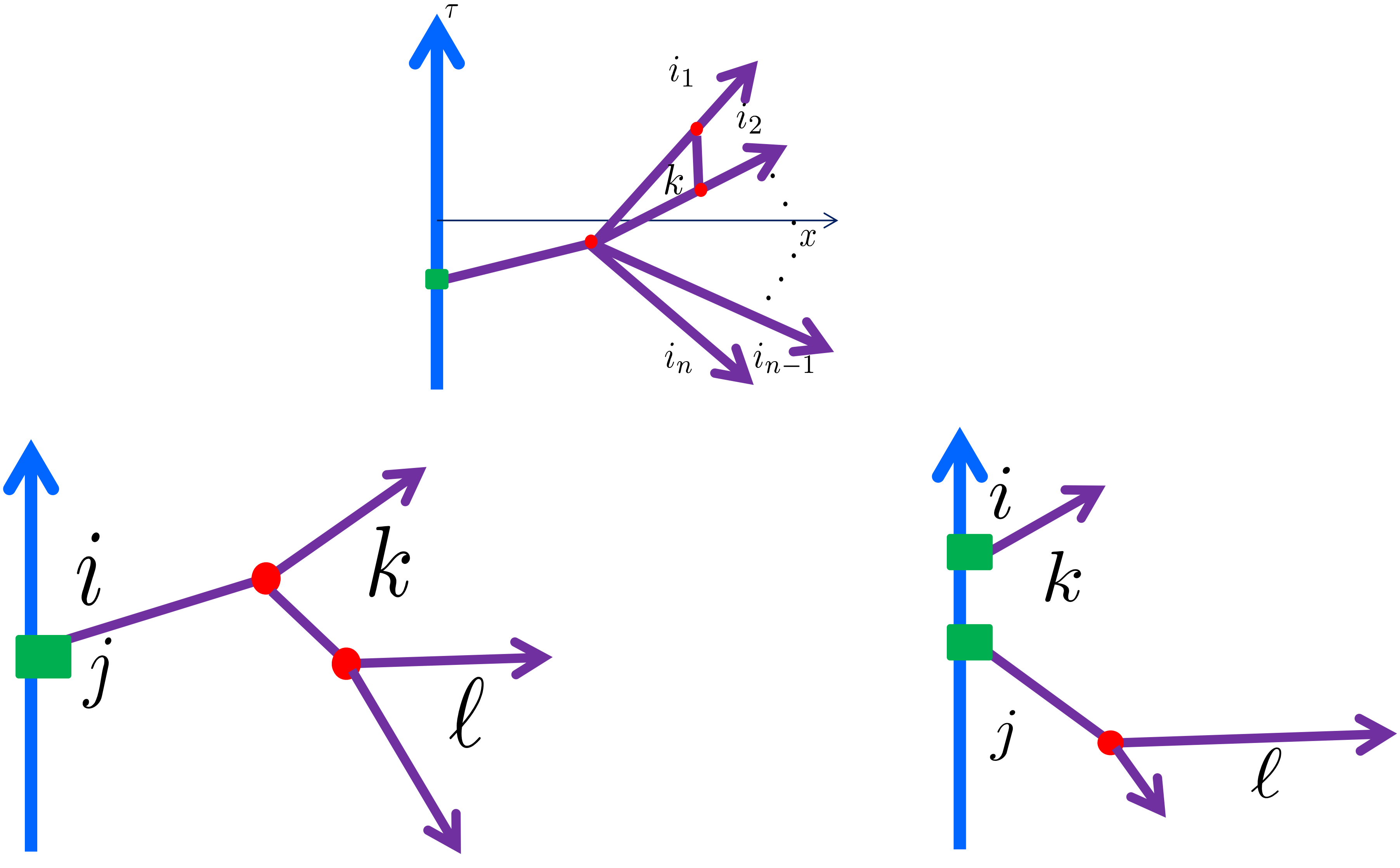




Taut Half-Plane Webs



Sliding Half-Plane webs



Convolution Theorem for Half-Plane Webs

$\mathcal{BW}$ Free abelian group generated by half-plane webs

There are now two convolutions:

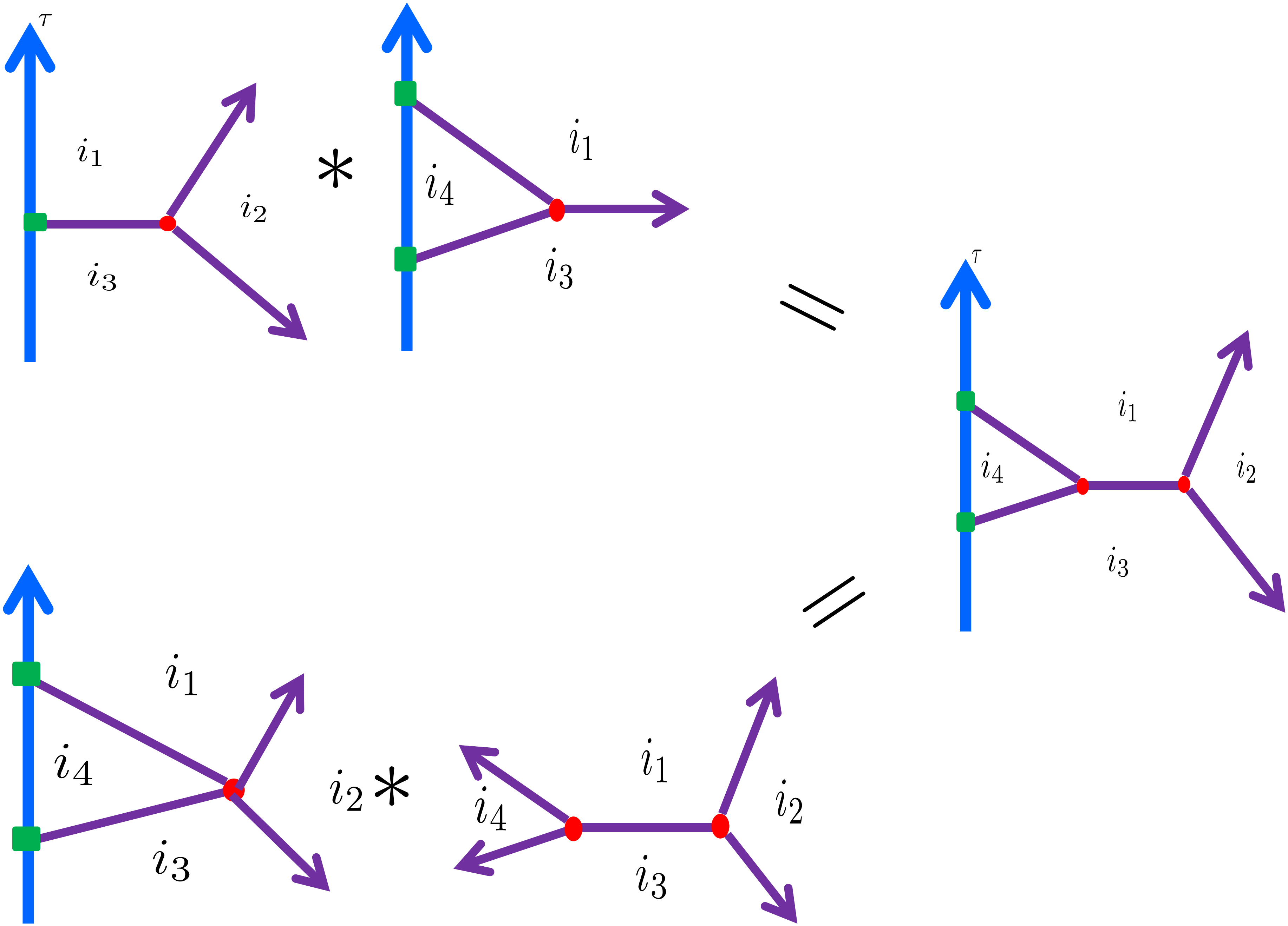
$$\mathcal{BW} \times \mathcal{BW} \to \mathcal{BW}$$

$$\mathcal{BW} \times \mathcal{W} \to \mathcal{BW}$$

Define the half-plane taut element:

$$\mathfrak{t}^{\partial} = \sum_{d(\mathfrak{u})=1} \mathfrak{u}$$

Theorem: $$\mathfrak{t}^{\partial} * \mathfrak{t}^{\partial} + \mathfrak{t}^{\partial} * \mathfrak{t} = 0$$



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i_2
i_3
=
i_1
i_4
i_2
i_3
*
i_1
i_4
i_2
i_3

Extension to the tensor algebra

$$T(\mathfrak{u}) : T\mathcal{B}\mathcal{W} \otimes T\mathcal{W} \to \mathcal{B}\mathcal{W}$$

· declaring $T(\mathfrak{u})[\mathfrak{u}_1 \otimes \cdots \otimes \mathfrak{u}_n; \mathfrak{w}_1 \otimes \cdots \otimes \mathfrak{w}_m]$ to be zero unless $n = V_\partial(\mathfrak{u})$ and $m = V_i(\mathfrak{u})$

$$T(\mathfrak{u})[\mathfrak{u}_1, \ldots, \mathfrak{u}_n; \mathfrak{w}_1, \ldots, \mathfrak{w}_m] := (\mathfrak{u} *_{\mathcal{V}_i(\mathfrak{u})} \{\mathfrak{w}_1, \ldots, \mathfrak{w}_m\}) *_{\mathcal{V}_\partial(\mathfrak{u})} \{\mathfrak{u}_1, \ldots, \mathfrak{u}_n\}$$

$$\begin{aligned} &\sum_{S_1 \amalg S_2} \sum_{r=0}^{n-1} \epsilon\, T(\mathfrak{t}^\partial)[\mathfrak{u}_1, \ldots, \mathfrak{u}_r, T(\mathfrak{t}^\partial)[\mathfrak{u}_{r+1}, \ldots, \mathfrak{u}_{r+n'}; S_1], \mathfrak{u}_{r+n'+1}, \ldots, \mathfrak{u}_{n''}; S_2] \\ &+ \sum_{S_1 \amalg S_2} \epsilon\, T(\mathfrak{t}^\partial)[\mathfrak{u}_1, \ldots, \mathfrak{u}_n; T(\mathfrak{t})[S_1], S_2] = 0 \end{aligned} \tag{7.42}$$

$$\{\mathfrak{w}_1, \ldots, \mathfrak{w}_m\} = S_1 \amalg S_2$$

Web Representations

Definition: A representation of webs is

a.) A choice of $\mathbb{Z}$-graded $\mathbb{Z}$-module R_{ij} for every ordered pair ij of vacua.

b.) A degree = -1 pairing $$K : R_{ij} \otimes R_{ji} \to \mathbb{Z}$$

For every cyclic fan of vacua introduce a fan representation:

$$I = \{i_1, \dots, i_n\}$$

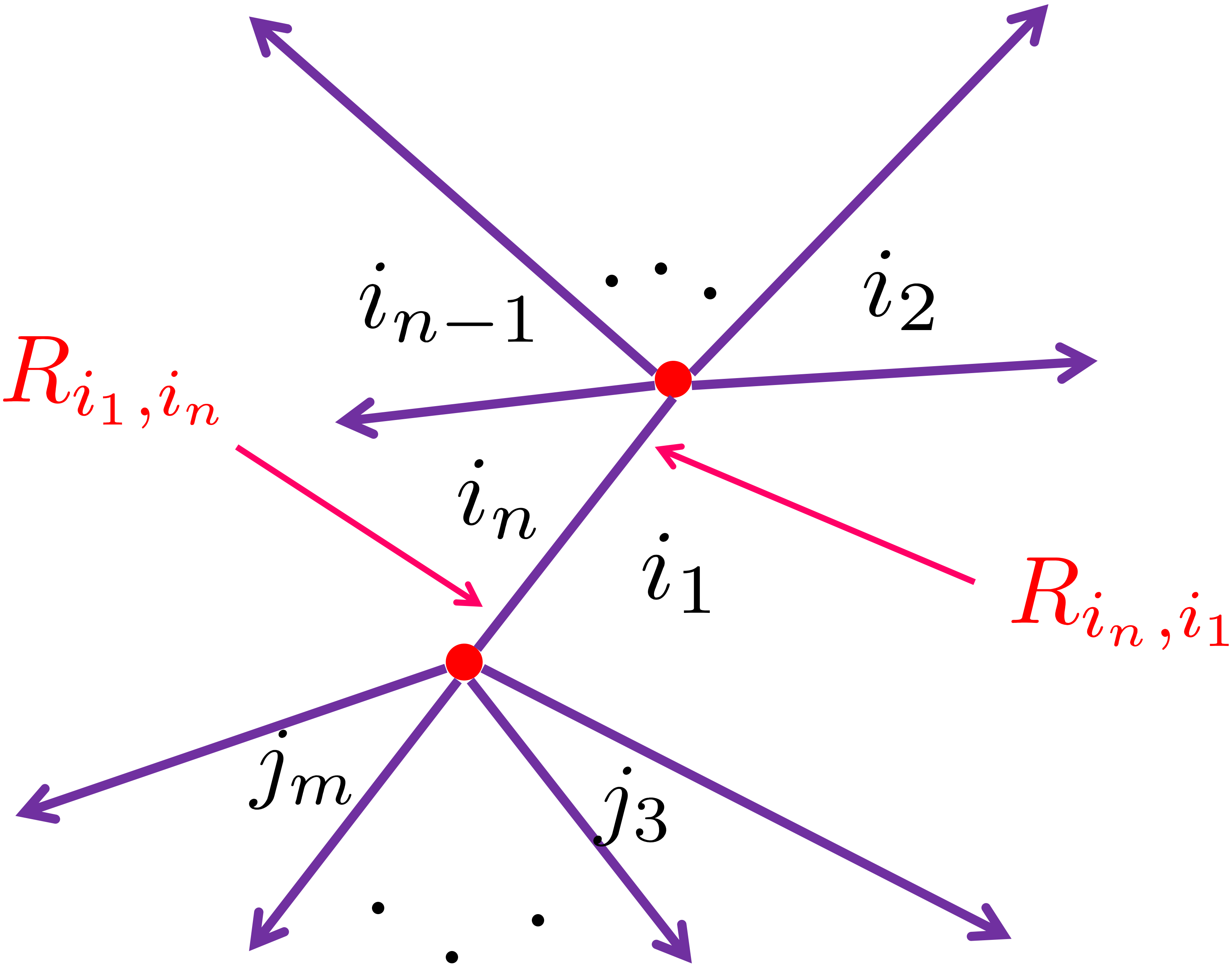
$$R_I := R_{i_1,i_2} \otimes \cdots \otimes R_{i_n,i_1}$$

Contraction

Given a rep and a deformation type $\mathfrak{w}$ we define the contraction operation:

$$\rho(\mathfrak{w}) : \otimes_{v \in \mathcal{V}(\mathfrak{w})} R_{I_v(\mathfrak{w})} \to R_{I_\infty(\mathfrak{w})}$$

by applying the contraction K to the pairs R_{ij} and R_{ji} on each edge:



i_{n-1}
i_2
R_{i_1,i_n}
i_n
i_1
R_{i_n,i_1}
j_m
j_3

Half-Plane Contractions

Similarly for half-plane fans: $J = \{j_1, \ldots, j_n\}$

$$R_J := R_{j_1,j_2} \otimes \cdot \otimes R_{j_{n-1},j_n}$$

$\rho(\mathfrak{u})$ now contracts

$$\bigotimes_{v \in \mathcal{V}_\partial(\mathfrak{u})} R_{J_v(\mathfrak{u})} \bigotimes_{v \in \mathcal{V}_i(\mathfrak{u})} R_{I_v(\mathfrak{u})}$$

$$\to R_{J_\infty(\mathfrak{u})}$$

time ordered!

Definition of an Interior Amplitude

$$R^{\text{int}} := \oplus_I R_I$$

where the sum is over all cyclic fans of vacua. We define

$$\rho(\mathfrak{w}) : TR^{\text{int}} \to R^{\text{int}}$$

Definition An *interior amplitude* is an element $\beta \in R^{\text{int}}$ so that if we define $e^\beta \in TR^{\text{int}} \otimes \mathbb{Q}$ by

$$e^\beta := \beta + \frac{1}{2!}\beta \otimes \beta + \frac{1}{3!}\beta \otimes \beta \otimes \beta + \cdots \tag{7.50}$$

then

$$\rho(\mathfrak{t})(e^\beta) = 0 \tag{7.51}$$

The A-∞ Category

An interior amplitude β defines an A∞ category $\mathfrak{Vac}^{\beta}$

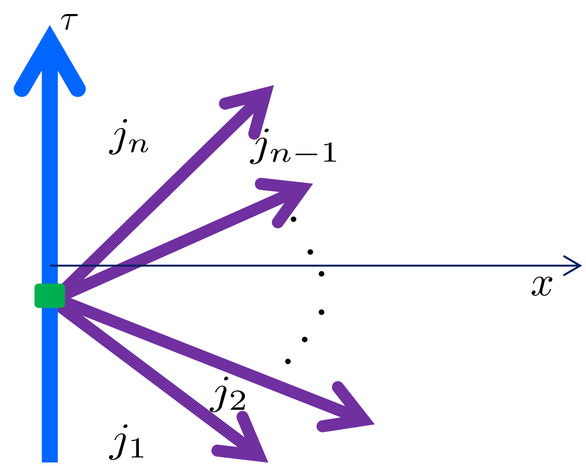
Objects: $i \in \mathbb{V}$.

Morphisms:

$$\mathrm{Hom}(i,j) = \begin{cases} \widehat{R}_{ij} & Re(z_{ij}) > 0 \\ \mathbb{C} & i = j \\ 0 & Re(z_{ij}) < 0 \end{cases}$$

$$\widehat{R}_{j_1, j_n} = \oplus_J R_J$$

$$J = \{j_1, \dots, j_n\}$$



$$m_\ell(\alpha_{J_1}, \dots, \alpha_{J_\ell}) = \rho(\mathfrak{t}^{\partial})(\alpha_{J_1} \otimes \cdots \otimes \alpha_{J_\ell} \otimes e^{\beta})$$

Proof of A-∞ Relations

$$\begin{aligned}&\sum_{S_1 \amalg S_2}\sum_{r=0}^{n-1} \epsilon\, T(\mathrm{t}^{\partial})[\mathrm{u}_1,\ldots,\mathrm{u}_r, T(\mathrm{t}^{\partial})[\mathrm{u}_{r+1},\ldots,\mathrm{u}_{r+n'};S_1],\mathrm{u}_{r+n'+1},\ldots,\mathrm{u}_{n''};S_2] \\ &+ \sum_{S_1 \amalg S_2} \epsilon\, T(\mathrm{t}^{\partial})[\mathrm{u}_1,\ldots,\mathrm{u}_n; T(\mathrm{t})[S_1],S_2] = 0\end{aligned} \tag{7.42}$$

Apply ρ and evaluate on exp[β], then $\rho(S) \to \frac{1}{p!}\beta^{\otimes p}$

and the second line vanishes.

Hence we obtain the A∞ relations:

$$\sum_{r+s+t=n,q=r+t+1} \epsilon \;\; m_q^{\beta}(\alpha_1,\ldots,\alpha_r, m_s^{\beta}(\alpha_{r+1},\ldots,\alpha_{r+s}),\alpha_{r+s+1},\ldots,\alpha_{r+s+t}) = 0$$

Remark 1

The morphism spaces can be defined by a Kontsevich-Soibelman-like product as follows:

Suppose $\mathbb{V}$ = { 1, …, K}. Introduce the elementary KxK matrices e_{ij}

$$1 + \oplus_{\mathrm{Re}(z_{ij})>0} \widehat{R}_{ij} e_{ij} = \underbrace{\otimes_{\mathrm{Re}(z_{ij})>0} (1 + R_{ij} e_{ij})}_{\text{phase ordered!}}$$

Remark 2: Chan-Paton Factors

Given any A_∞ category we can always "add Chan-Paton factors". What this means is that to every object we assign a vector space $i \to \mathcal{E}_i$ and we modify the hom-spaces by

$$\mathrm{Hom}(i,j) \to \mathrm{Hom}(i,j) \otimes \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) \cong \mathcal{E}_i^* \otimes \mathrm{Hom}(i,j) \otimes \mathcal{E}_j \tag{7.69}$$

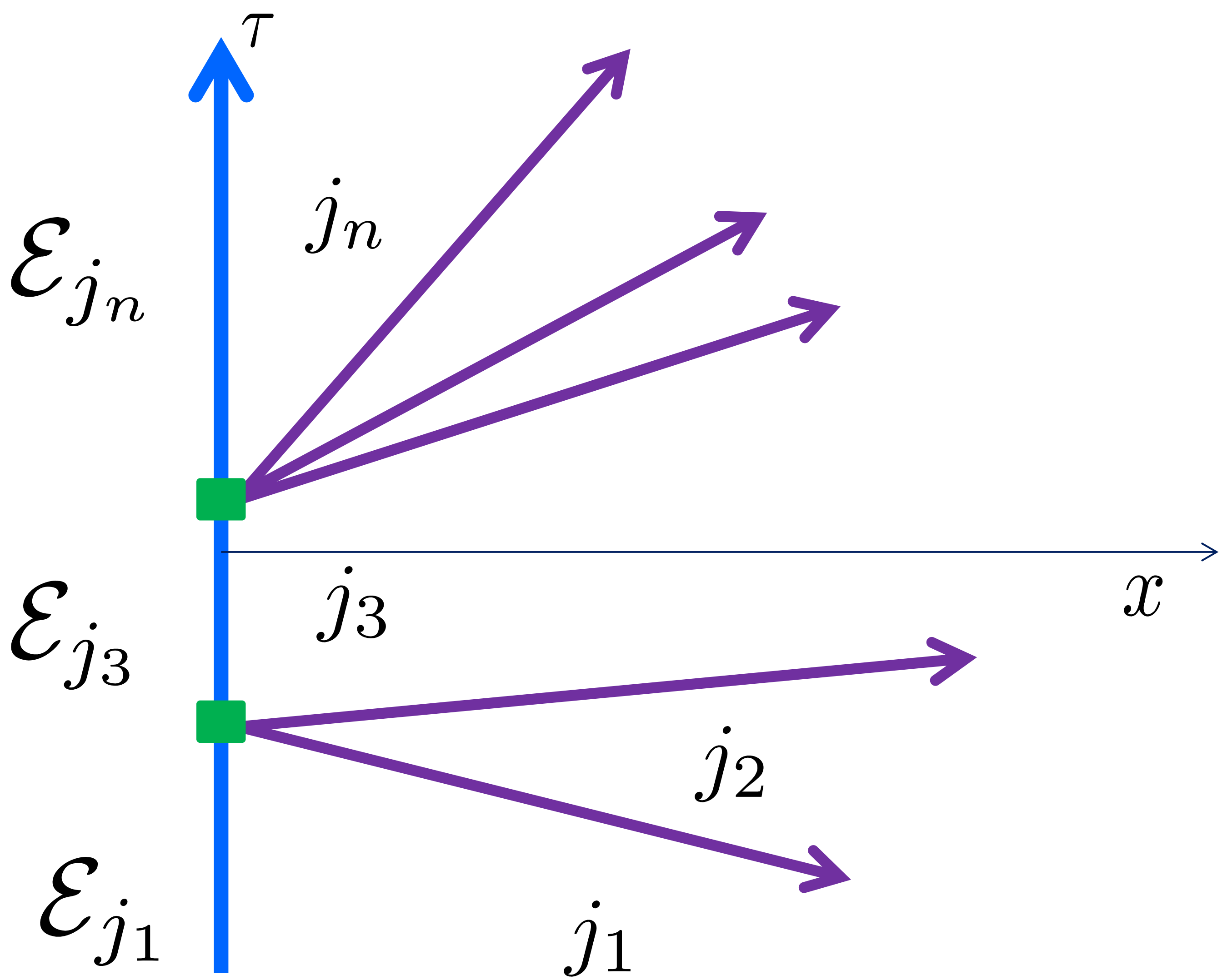
The new composition maps are given by

$$m_n^{\mathfrak{Vac}^\beta} \otimes m_2^{\mathrm{Chan-Paton}} \tag{7.70}$$

and the A_∞ relations continue to hold. We will denote this A_∞ category enhanced by Chan-Paton spaces as

$$\mathfrak{Vac}^\beta(\mathcal{E}_*) \tag{7.71}$$

Picturing Chan-Paton Factors



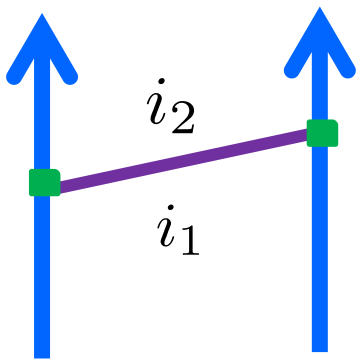
Strip-Webs

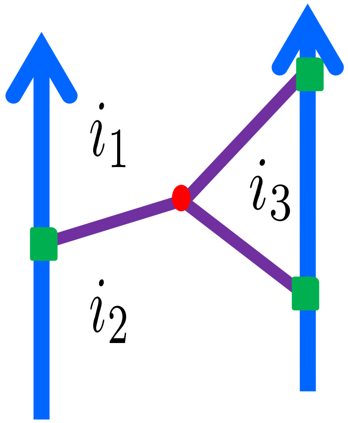
Now consider webs in the strip $\mathbb{R} \times [x_\ell, x_r]$

$$d(\mathfrak{s}) := 2V_i(\mathfrak{s}) + V_\partial(\mathfrak{s}) - E(\mathfrak{s}) - 1$$

Now ***taut*** and ***rigid strip-webs*** are the same, and have d($\mathfrak{s}$)=0.

sliding strip-webs have d($\mathfrak{s}$)=1.





Convolution Identity for Strip $\mathfrak{t}$'s

$$\mathfrak{t}^s = \sum_{d(\mathfrak{s})=0} \mathfrak{s}$$

Convolution theorem:

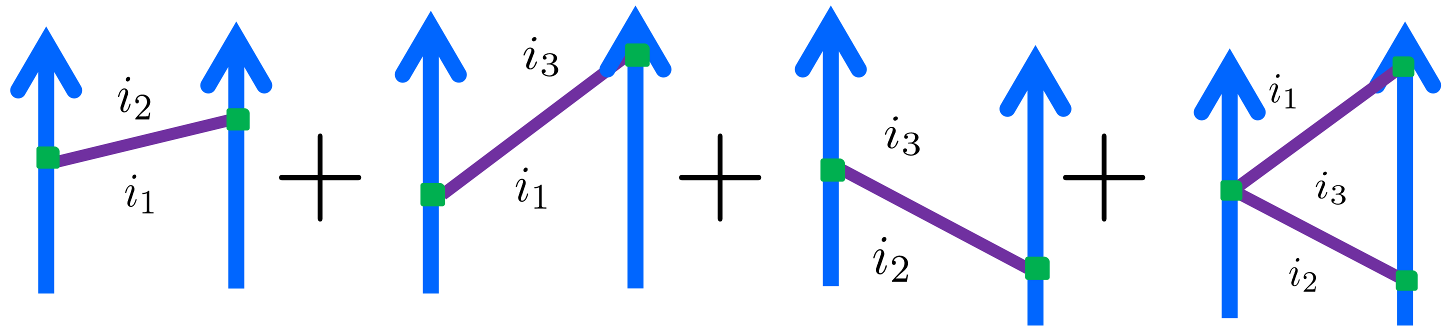
$$\mathfrak{t}^s * \mathfrak{t}^{\partial}_L + \mathfrak{t}^s * \mathfrak{t}^{\partial}_R + \mathfrak{t}^s * \mathfrak{t} + \mathfrak{t}^s \circ \mathfrak{t}^s = 0$$

where for strip webs we denote time-concatenation by

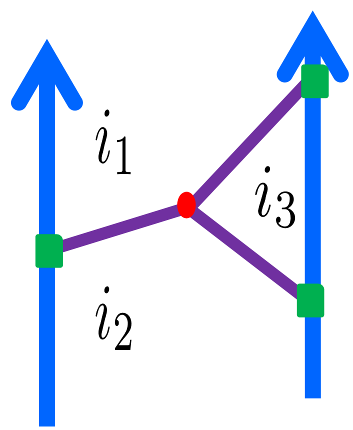
$$\mathfrak{s}_1 \circ \mathfrak{s}_2$$

$$d(\mathfrak{s}_1 \circ \mathfrak{s}_2) = d(\mathfrak{s}_1) + d(\mathfrak{s}_2) + 1$$

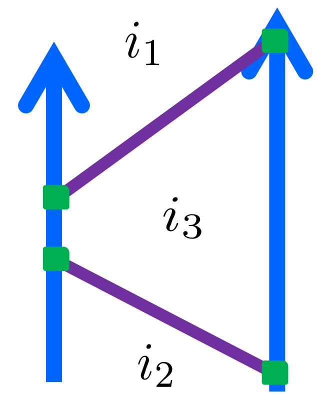
$\mathfrak{t}^s =$



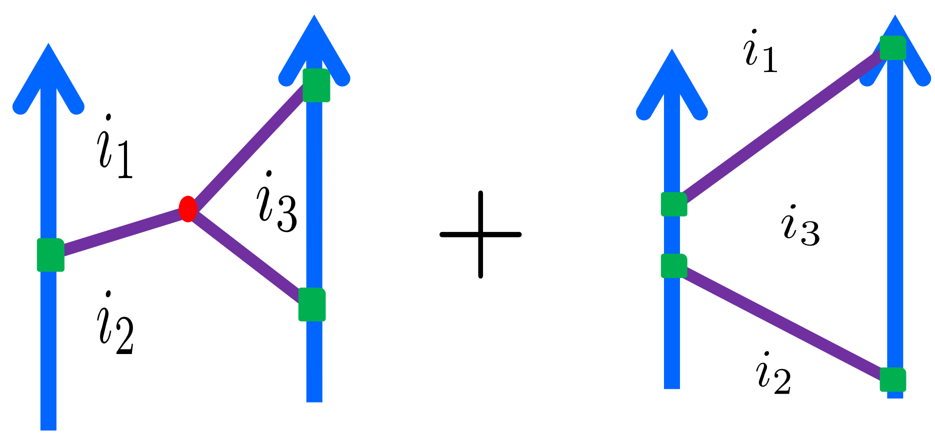
$\mathfrak{t}^s * \mathfrak{t}^\partial_R =$



$\mathfrak{t}^s \circ \mathfrak{t}^s =$



$\mathfrak{t}^s * \mathfrak{t}^\partial_L =$



Convolution Identity on the Tensor Algebra

$$\sum_{S_1 \amalg S_2} \sum_{r=0}^{n_L-1} \sum_{n'} \epsilon\, T(\mathrm{t}^s)[\mathrm{u}_1, \ldots, \mathrm{u}_r, T(\mathrm{t}_L^\partial)[\mathrm{u}_{r+1}, \ldots, \mathrm{u}_{r+n'}; S_1], \mathrm{u}_{r+n'+1}, \ldots, \mathrm{u}_{n_L}; S_2; \tilde{\mathrm{u}}_{n_R}, \ldots, \tilde{\mathrm{u}}_1]$$
$$+ \sum_{S_1 \amalg S_2} \sum_{r=0}^{n_R-1} \sum_{n'} \epsilon\, T(\mathrm{t}^s)[\mathrm{u}_1, \ldots, \mathrm{u}_{n_L}; S_1; \tilde{\mathrm{u}}_{n_R}, \ldots, \tilde{\mathrm{u}}_{r+n'+1}, T(\mathrm{t}_R^\partial)[\tilde{\mathrm{u}}_{r+n'}, \ldots, \tilde{\mathrm{u}}_{r+1}; S_2], \tilde{\mathrm{u}}_r, \ldots, \tilde{\mathrm{u}}_1]$$
$$+ \sum_{S_1 \amalg S_2} \epsilon\, T(\mathrm{t}^s)[\mathrm{u}_1, \ldots, \mathrm{u}_{n_L}; T(\mathrm{t})[S_1], S_2; \tilde{\mathrm{u}}_{n_R}, \ldots, \tilde{\mathrm{u}}_1]$$
$$+ \sum_{r_L=1}^{n_L} \sum_{r_R=0}^{n_R} \sum_{S_1 \amalg S_2} \epsilon\, T(\mathrm{t}^s)[\mathrm{u}_1, \ldots, \mathrm{u}_{r_L}; S_1; \tilde{\mathrm{u}}_{r_R}, \ldots, \tilde{\mathrm{u}}_1] \circ T(\mathrm{t}^s)[\mathrm{u}_{r_L+1}, \ldots, \mathrm{u}_{n_L}; S_2; \tilde{\mathrm{u}}_{n_R}, \ldots, \tilde{\mathrm{u}}_{r_R+1}]$$
$$= 0$$

So, what does it mean?

A∞ Bimodules

Applying a representation of webs and inserting an interior amplitude exp[β] one term drops out and we can interpret the above identity as defining an A∞ bimodule.

If we add Chan-Paton spaces on the left and right the bimodule is

$$\oplus_{i\in\mathbb{V}}\mathcal{E}_i^L \otimes \mathcal{E}_i^R$$

Proof of Bimodule Identity

$$\sum_{S_1 \amalg S_2} \sum_{r=0}^{n_L-1} \sum_{n'} \epsilon\, T(\mathfrak{t}^s)[\mathfrak{u}_1, \ldots, \mathfrak{u}_r, T(\mathfrak{t}_L^{\partial})[\mathfrak{u}_{r+1}, \ldots, \mathfrak{u}_{r+n'}; S_1], \mathfrak{u}_{r+n'+1}, \ldots, \mathfrak{u}_{n_L}; S_2; \tilde{\mathfrak{u}}_{n_R}, \ldots, \tilde{\mathfrak{u}}_1]$$

$$+ \sum_{S_1 \amalg S_2} \sum_{r=0}^{n_R-1} \sum_{n'} \epsilon\, T(\mathfrak{t}^s)[\mathfrak{u}_1, \ldots, \mathfrak{u}_{n_L}; S_1; \tilde{\mathfrak{u}}_{n_R}, \ldots, \tilde{\mathfrak{u}}_{r+n'+1}, T(\mathfrak{t}_R^{\partial})[\tilde{\mathfrak{u}}_{r+n'}, \ldots, \tilde{\mathfrak{u}}_{r+1}; S_2], \tilde{\mathfrak{u}}_r, \ldots, \tilde{\mathfrak{u}}_1]$$

$$+ \sum_{S_1 \amalg S_2} \epsilon\, T(\mathfrak{t}^s)[\mathfrak{u}_1, \ldots, \mathfrak{u}_{n_L}; T(\mathfrak{t})[S_1], S_2; \tilde{\mathfrak{u}}_{n_R}, \ldots, \tilde{\mathfrak{u}}_1]$$

$$+ \sum_{r_L=1}^{n_L} \sum_{r_R=0}^{n_R} \sum_{S_1 \amalg S_2} \epsilon\, T(\mathfrak{t}^s)[\mathfrak{u}_1, \ldots, \mathfrak{u}_{r_L}; S_1; \tilde{\mathfrak{u}}_{r_R}, \ldots, \tilde{\mathfrak{u}}_1] \circ T(\mathfrak{t}^s)[\mathfrak{u}_{r_L+1}, \ldots, \mathfrak{u}_{n_L}; S_2; \tilde{\mathfrak{u}}_{n_R}, \ldots, \tilde{\mathfrak{u}}_{r_R+1}]$$

$$= 0$$

Apply ρ

$$\rho(S) \to \frac{1}{p!} \beta^{\otimes p}$$

Maurer-Cartan & Differential

If, moreover, we use for left and right morphisms a solution of the Maurer-Cartan equation

$$\sum_{n=1}^{\infty} m_n^{\beta}(\alpha^{\otimes n}) = 0$$

$$d = \rho(\mathfrak{t}^s)(\sum \alpha^{\otimes n}; e^{\beta}; \sum \tilde{\alpha}^{\otimes n})$$

becomes a differential on the complex

$$\bigoplus_{i \in \mathbb{V}} \mathcal{E}_i^L \otimes \mathcal{E}_i^R$$

How Convolution Identity Gives a Differential:

$$\sum_{S_1 \amalg S_2} \sum_{r=0}^{n_L-1} \sum_{n'} \epsilon\, T(\mathfrak{t}^s)[\mathfrak{u}_1,\dots,\mathfrak{u}_r, T(\mathfrak{t}_L^\partial)[\mathfrak{u}_{r+1},\dots,\mathfrak{u}_{r+n'}, S_1], \mathfrak{u}_{r+n'+1},\dots,\mathfrak{u}_{n_L}, S_2, \tilde{\mathfrak{u}}_{n_R},\dots,\tilde{\mathfrak{u}}_1]$$

$$+ \sum_{S_1 \amalg S_2} \sum_{r=0}^{n_R-1} \sum_{n'} \epsilon\, T(\mathfrak{t}^s)[\mathfrak{u}_1,\dots,\mathfrak{u}_{n_L}; S_1; \tilde{\mathfrak{u}}_{n_R},\dots,\tilde{\mathfrak{u}}_{r+n'+1}, T(\mathfrak{t}_R^\partial)[\tilde{\mathfrak{u}}_{r+n'},\dots,\tilde{\mathfrak{u}}_{r+1}; S_2], \tilde{\mathfrak{u}}_r,\dots,\tilde{\mathfrak{u}}_1]$$

$$+ \sum_{S_1 \amalg S_2} \epsilon\, T(\mathfrak{t}^s)[\mathfrak{u}_1,\dots,\mathfrak{u}_{n_L}; T(\mathfrak{t})[S_1], S_2; \tilde{\mathfrak{u}}_{n_R},\dots,\tilde{\mathfrak{u}}_1]$$

$$+ \sum_{r_L=1}^{n_L} \sum_{r_R=0}^{n_R} \sum_{S_1 \amalg S_2} \epsilon\, T(\mathfrak{t}^s)[\mathfrak{u}_1,\dots,\mathfrak{u}_{r_L}; S_1; \tilde{\mathfrak{u}}_{r_R},\dots,\tilde{\mathfrak{u}}_1] \circ T(\mathfrak{t}^s)[\mathfrak{u}_{r_L+1},\dots,\mathfrak{u}_{n_L}; S_2; \tilde{\mathfrak{u}}_{n_R},\dots,\tilde{\mathfrak{u}}_{r_R+1}]$$

$$= 0$$

Apply ρ $\quad \rho(S) \to \frac{1}{p!}\beta^{\otimes p} \qquad \rho(\mathfrak{u}) \to \alpha_{J_\infty(\mathfrak{u})}$

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SQM & Morse Theory (Witten: 1982)

M: Riemannian; h: $M \longrightarrow \mathbb{R}$, Morse function

SQM: $q : \mathbb{R}_t \to M \quad \chi \in \Gamma(q^*(TM \otimes \mathbb{C}))$

$$L = \frac{1}{2} g_{IJ} \partial_t q^I \partial_t q^J + i g_{IJ} \bar{\chi}^I D_t \chi^J - \frac{1}{2} R_{IJKL} \bar{\chi}^I \chi^J \bar{\chi}^K \chi^L - \frac{1}{2} g^{IJ} \partial_I h \partial_J h - D_I \partial_J h \bar{\chi}^I \chi^J \quad (2.2)$$

$$\mathbb{M}^{\bullet} = \oplus_{p:\nabla h=0} \mathbb{Z} \cdot \Psi(p)$$

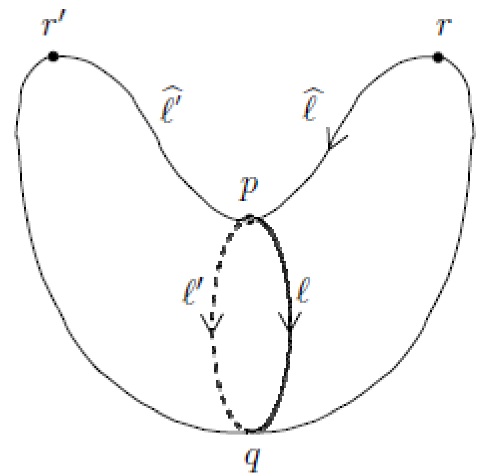
$$F(\Psi(p)) = \tfrac{1}{2}(d_{\uparrow}(p) - d_{\downarrow}(p))$$

SQM Differential

$$d(\Psi(p)) = \sum_{p':F(p')-F(p)=1} n(p,p')\Psi(p')$$

n(p,p') counts instantons

Why $d^2 = 0$



1+1 LG Model as SQM

X: Kähler manifold

W: X $\longrightarrow \mathbb{C}$ Holomorphic Morse function

Target space for SQM:

$$M = \mathrm{Map}(D, X) = \{\phi : D \to X\}$$

$$D = \mathbb{R}, [x_\ell, \infty), (-\infty, x_r], [x_\ell, x_r], S^1$$

$$h = \int_D \left(pdq + \mathrm{Re}(\zeta^{-1} W) dx \right)$$

$$dpdq = \phi^*(\omega)$$

1+1 Dimensional Action & BC's

Take X = $\mathbb{C}$ with its Euclidean metric, for simplicity.

$$
\begin{aligned}
L_{SQM} = & \frac{1}{2}\int dx \left[\partial_\mu \bar{\phi}\partial^\mu \phi - \frac{1}{4}|\frac{\partial W}{\partial \phi}|^2\right] \\
& + \frac{1}{2}\int dx \left[\mathrm{i}\bar{\psi}_-(\partial_t + \partial_x)\psi_- + \mathrm{i}\bar{\psi}_+(\partial_t - \partial_x)\psi_+ - \frac{1}{2}W''\psi_+\psi_- - \frac{1}{2}\bar{W}''\bar{\psi}_-\bar{\psi}_+\right] \\
& + \int dx \frac{d}{dx}\left[\frac{1}{2}\mathrm{Im}(\zeta^{-1}W) + \epsilon_{ab}\bar{\chi}^a\chi^b\right]
\end{aligned}
\tag{B.28}
$$

Need to constrain fieldspace:

$$\phi \longrightarrow \phi_i \qquad x \to -\infty$$

$$\phi \to \phi_j \qquad x \to +\infty$$

At finite boundaries ϕ sits in a Lagrangian subvariety

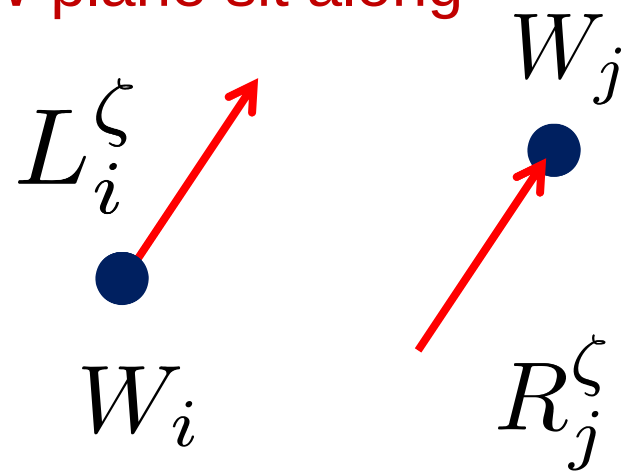
$$\mathcal{L} \subset X$$

Lefshetz Thimbles

Stationary points of h are solutions to the differential equation

$$\frac{\partial}{\partial x}\phi = \frac{\mathrm{i}\zeta}{2}\frac{\partial \bar{W}}{\partial \bar{\phi}}$$

The projection of solutions to the complex W plane sit along straight lines of slope iζ



If D contains x $\longrightarrow$ -∞ $\quad \phi \to \phi_i$

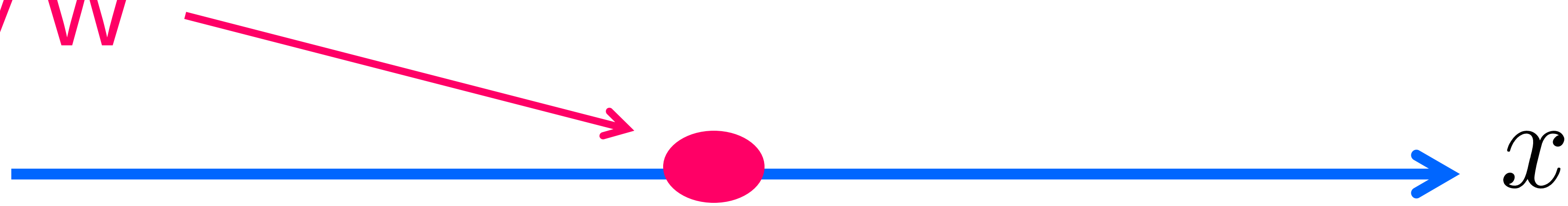
If D contains x $\longrightarrow$ +∞ $\quad \phi \to \phi_j$

Inverse image in X defines left and right Lefshetz thimbles

They are maximal Lagrangian subvarieties of X

Solitons For D=$\mathbb{R}$

Scale set by W

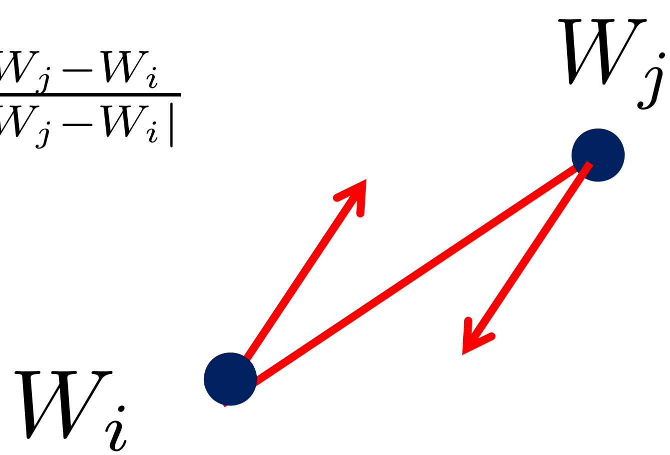


$\phi \cong \phi_i$ $\phi \cong \phi_j$

For general ζ there is no solution

But for a suitable phase there is a solution

$$\mathrm{i}\zeta = \xi_{ji} = \frac{W_j - W_i}{|W_j - W_i|}$$



This is the classical soliton. There is one for each intersection

$$p \in L_i^{\zeta} \cap R_j^{\zeta}$$

(in the fiber of a regular value)

Fermionic Vacua

These critical points are almost but not quite nondegenerate. translation symmetry leads to a zeromode of the linearization:

$$\frac{\partial}{\partial x}\delta\phi = \frac{\mathrm{i}\zeta}{2}\frac{\partial^2 \bar{W}}{\partial \bar{\phi}^2}\delta\bar{\phi}$$

This is just the equation of motion of the fermions Dψ= 0

$$D = \sigma^3 \mathrm{i}\frac{d}{dx} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\frac{\zeta^{-1}}{2}W'' + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\frac{\zeta}{2}\bar{W}''$$

Quantization of the fermion zeromodes gives a twofold-degenerate groundstate with fermionic vacua

$$\Psi_{ij}^{f}(p) \qquad \Psi_{ij}^{f+1}(p)$$

Morse Complex

$$\mathbb{M}_{ij} = \oplus_{p \in L_i^\zeta \cap R_j^\zeta} \left(\mathbb{Z}\Psi_{ij}^{f}(p) \oplus \mathbb{Z}\Psi_{ij}^{f+1}(p) \right)$$

$$D = \sigma^3 \mathrm{i} \frac{d}{dx} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\zeta^{-1}}{2} W'' + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\zeta}{2} \bar{W}''$$

$$f = \frac{1}{2} \eta (D - \epsilon)$$

Witten index: $$\mu_{ij} = e^{\mathrm{i}\pi f_0} \# L_i^\zeta \cap R_j^\zeta = e^{\mathrm{i}\pi f_0} \sum_{p \in L_i^\zeta \cap R_j^\zeta} (-1)^{\iota(p)}$$

Instantons

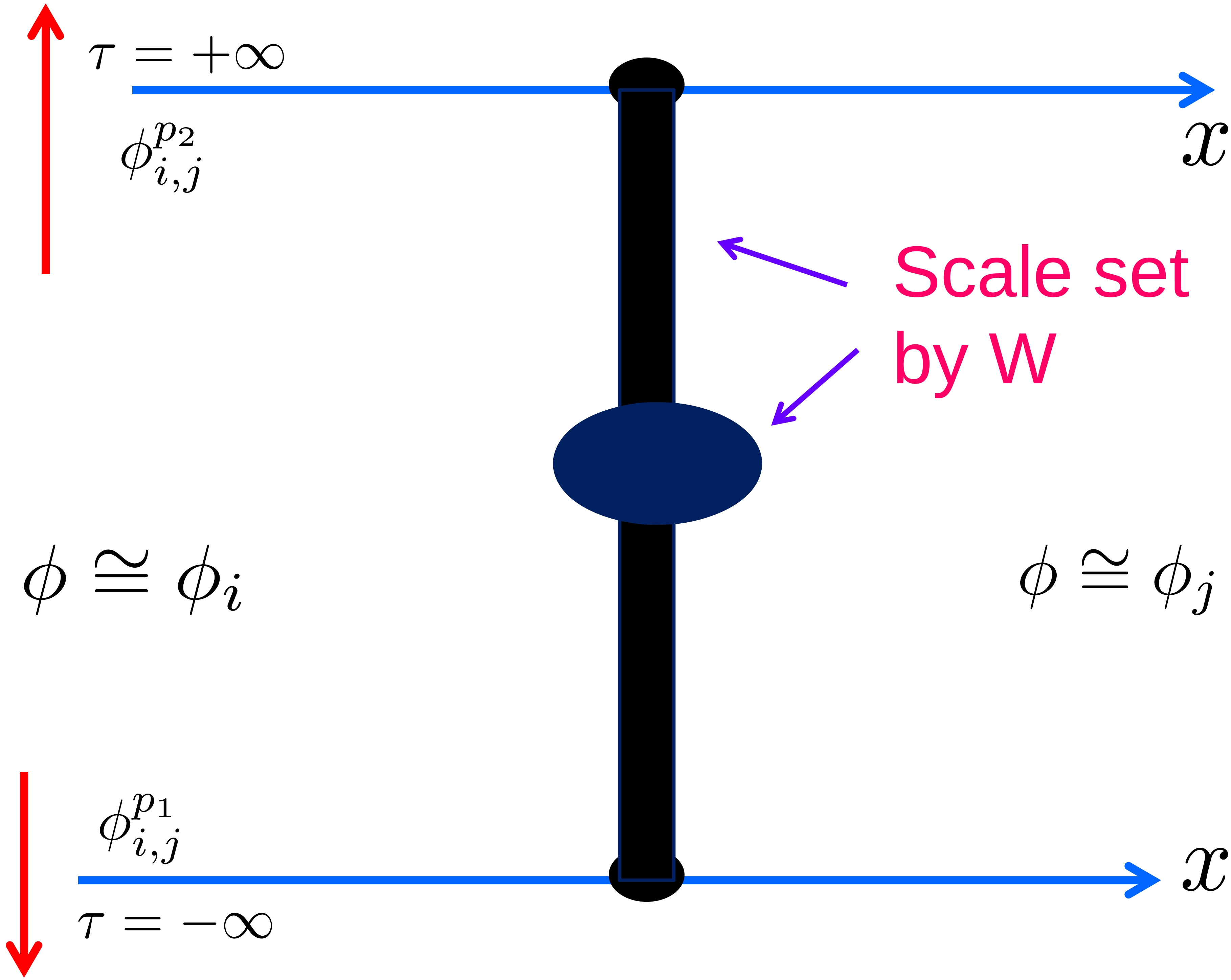
Instanton equation $$\frac{d\phi}{d\tau} = -\frac{\delta h}{\delta \bar{\phi}}$$

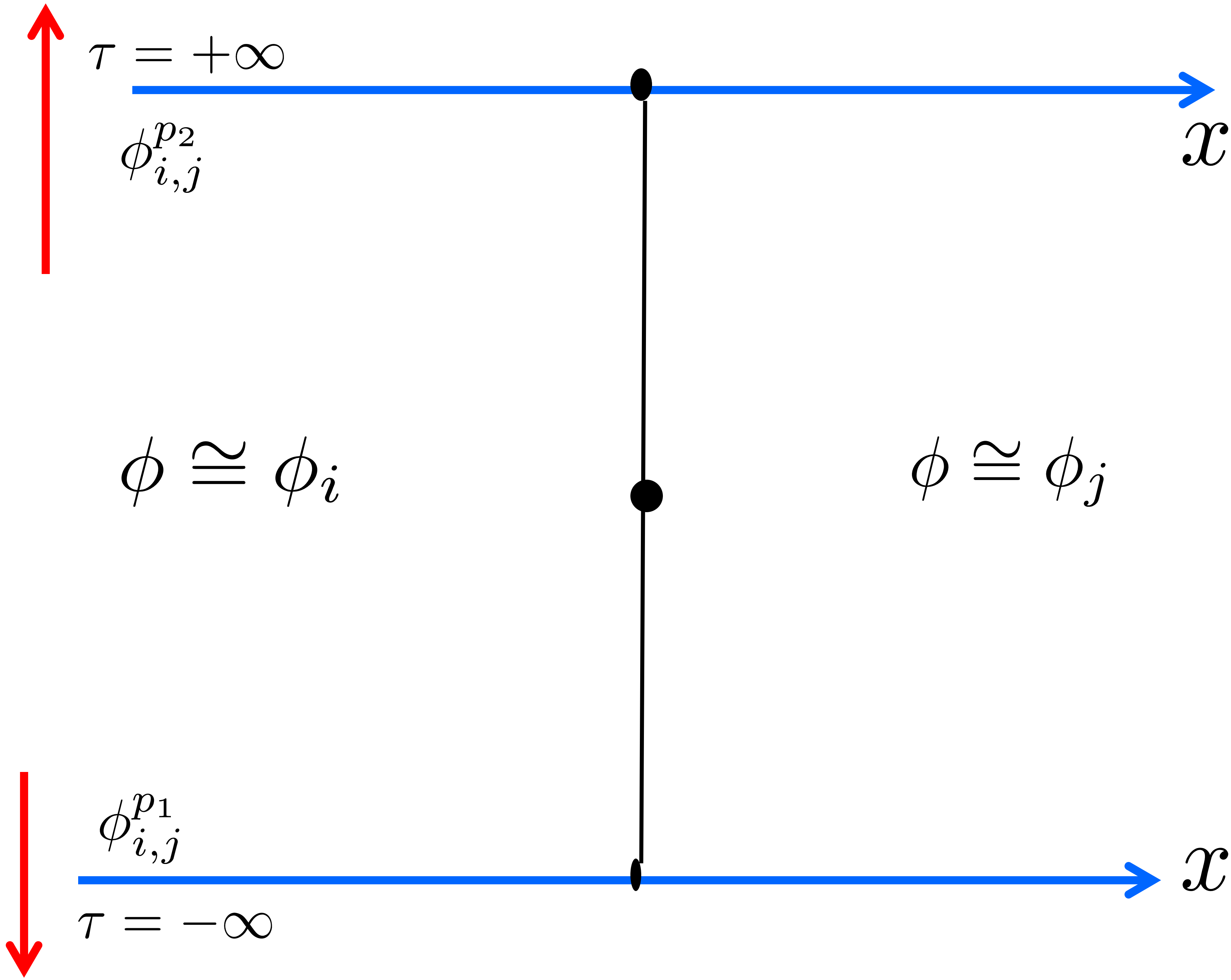
$$\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\Big)\,\phi = \frac{\mathrm{i}\zeta}{2}\,\frac{\partial \bar{W}}{\partial \bar{\phi}}$$

$$\bar{\partial}\phi = \frac{\mathrm{i}\zeta}{4}\,\frac{\partial \bar{W}}{\partial \bar{\phi}}$$

At short distance scales W is irrelevant and we have the usual holomorphic map equation.

At long distances the theory is almost trivial since it has a mass scale, and it is dominated by the vacua of W.


$\tau = +\infty$
$\phi_{i,j}^{p_2}$
x
Scale set
by W
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi_{i,j}^{p_1}$
x
$\tau = -\infty$


$\tau = +\infty$
$\phi_{i,j}^{p_2}$
x
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi_{i,j}^{p_1}$
x
$\tau = -\infty$

Half-Line Solitons

Classical solitons on the right half-line are labeled by: $p \in \mathcal{L} \cap R_j^{\zeta}$

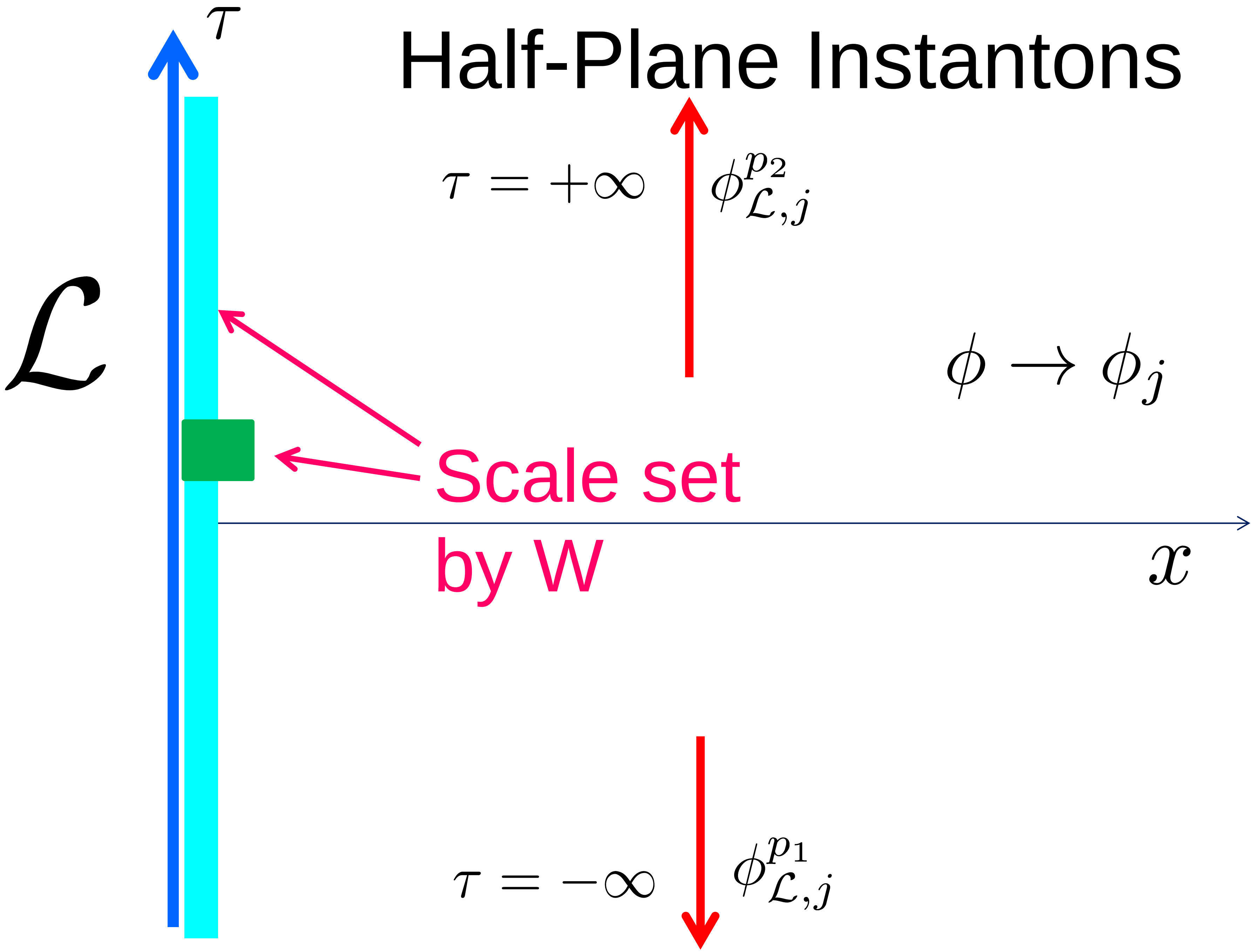
Morse complex: $\mathbb{M}_{\mathcal{L},j} = \oplus_p \mathbb{Z} \cdot \Psi_{\mathcal{L},j}(p)$

Grading the complex: Assume X is CY and that we can find a logarithm:

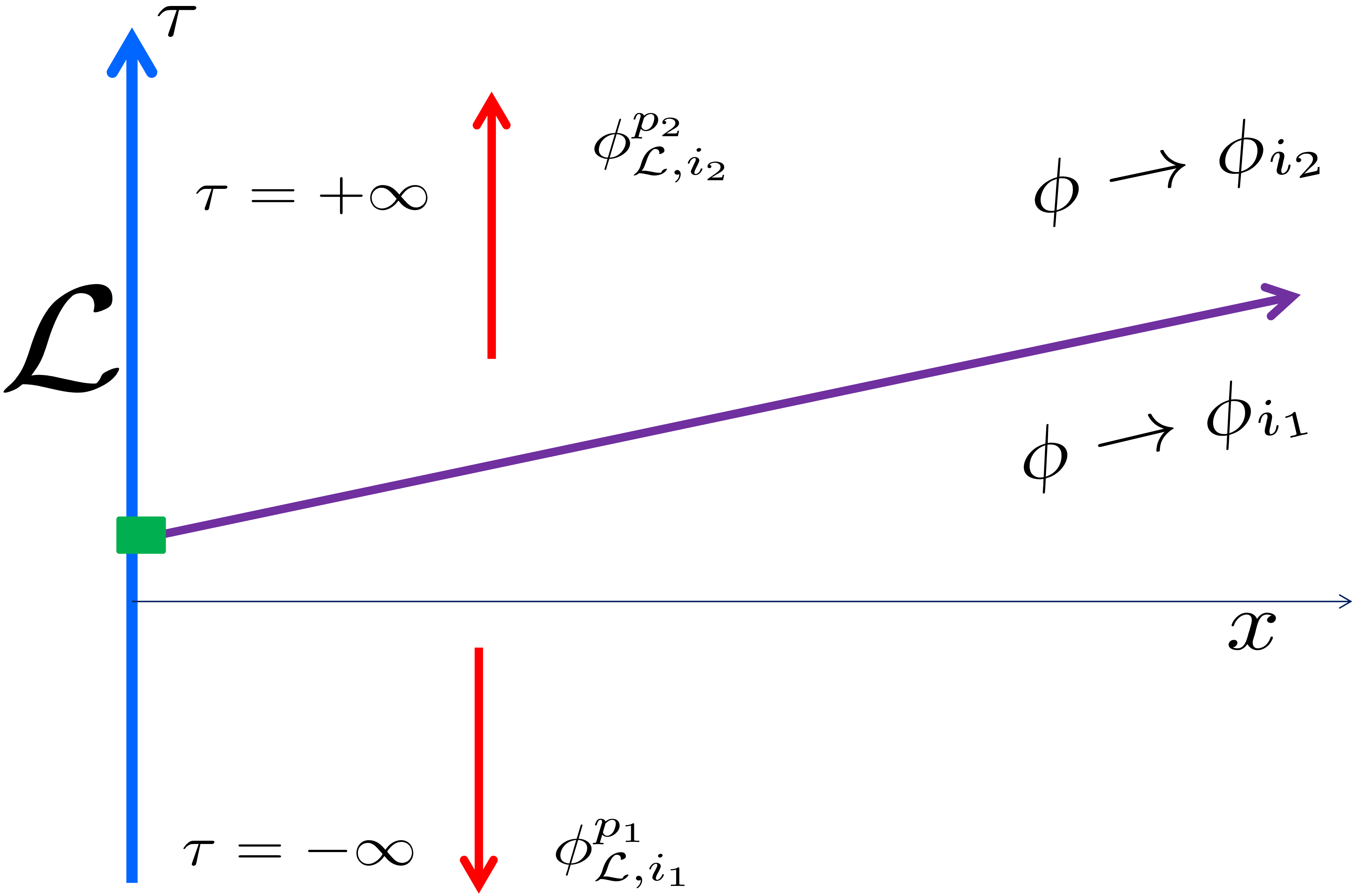
$$w = \mathrm{Im}\log \frac{\iota^*(\Omega^{d,0})}{\mathrm{vol}(\mathcal{L})}$$

Then the grading is by $f = \eta(D) - w$

Half-Plane Instantons



A Natural Generalization



The Boosted Soliton - 1

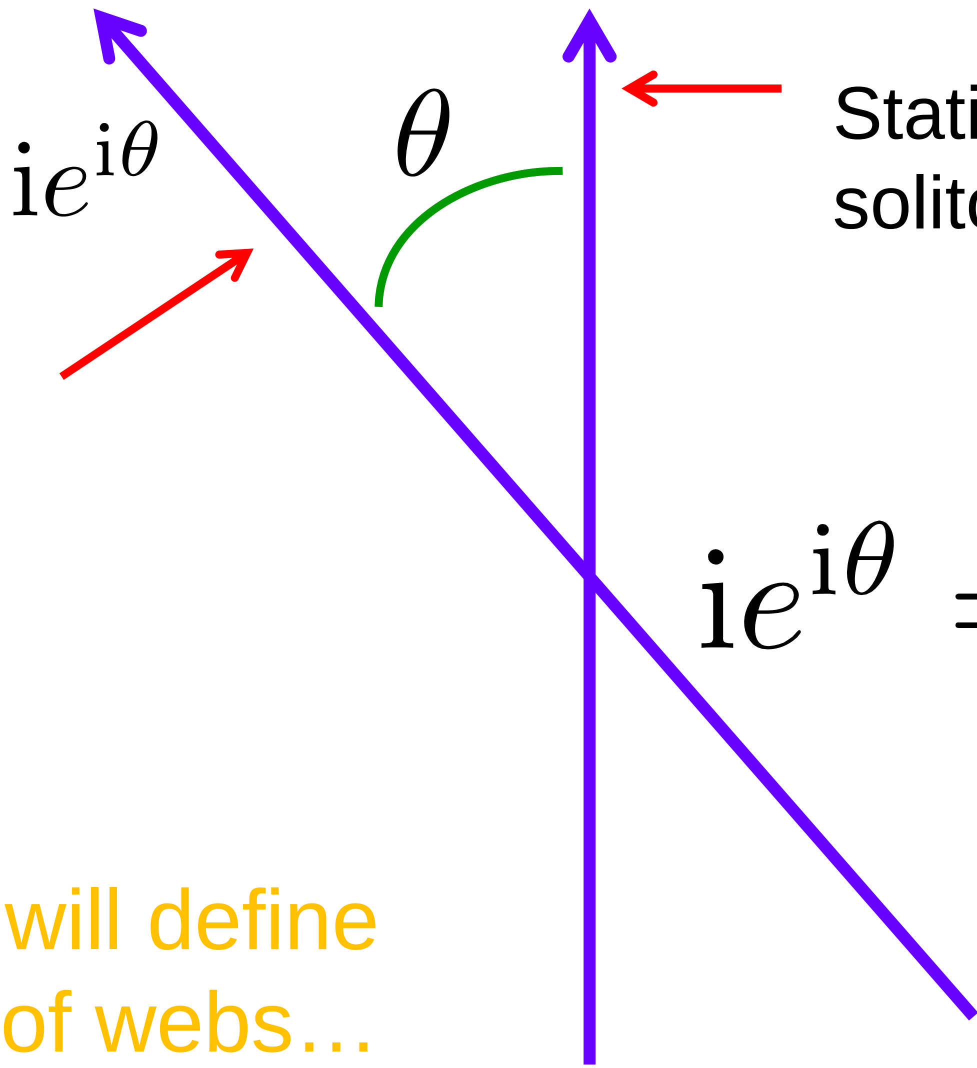
$$\phi_{ij}^{\rm inst}(x,\tau) := \phi_{ij}^{\rm sol}(\cos\theta x + \sin\theta\tau)$$

$$\left(\frac{\partial}{\partial x} + {\rm i}\frac{\partial}{\partial \tau}\right)\phi_{ij}^{\rm inst}(x,\tau) = \frac{e^{{\rm i}\theta}\xi_{ji}}{2}\frac{\partial \bar W}{\partial \bar\phi}$$

Therefore we produce a solution of the instanton equation with phase ζ if

$$e^{{\rm i}\theta}\xi_{ji} = {\rm i}\zeta \qquad \xi_{ji} = \frac{W_j - W_i}{|W_j - W_i|}$$

The Boosted Soliton -2



Stationary soliton

$$\phi \cong \phi_j$$

Boosted soliton

$$\phi \cong \phi_i$$

$$\mathrm{i}e^{\mathrm{i}\theta} = \frac{z_i - z_j}{|z_i - z_j|}$$

$$z_i = \zeta \overline{W}_i$$

These will define edges of webs…

Solitons On The Interval

Now consider the finite interval $[x_l, x_r]$ with boundary conditions $\mathcal{L}_l$, $\mathcal{L}_r$

When the interval is much longer than the scale set by W the Morse complex is

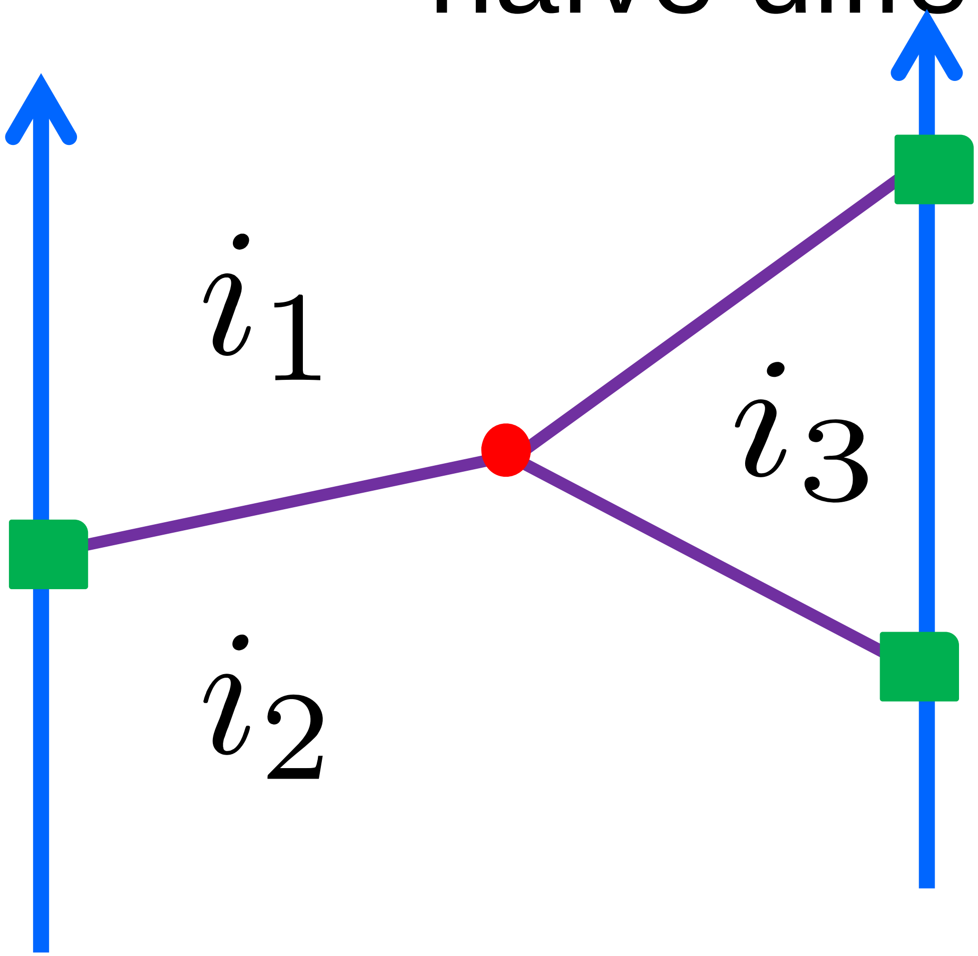
$$\mathbb{M}_{\mathcal{L}_\ell,\mathcal{L}_r} = \oplus_{i\in\mathbb{V}}\mathbb{M}_{\mathcal{L}_\ell,i} \otimes \mathbb{M}_{i,\mathcal{L}_r}$$

The Witten index factorizes nicely: $\mu_{\mathcal{L}_\ell,\mathcal{L}_r} = \sum_i \mu_{\mathcal{L}_\ell,i}\mu_{i,\mathcal{L}_r}$

But the differential $d_{\mathcal{L}_\ell,i} \otimes 1 + 1 \otimes d_{i,\mathcal{L}_r}$

is too naïve !

Instanton corrections to the naïve differential



There will be instanton corrections which, at long distances, are made by gluing together boosted solitons.

Now we will make this more precise….

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- Introduction & Motivations
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The Morse Complex on $\mathbb{R}$ Gives a Web Representation

$$R_{ij} := \mathrm{Hom}(\mathbb{M}_{ij}, \mathbb{Z})$$

If I = {i_1, …, i_n} is a cyclic fan define

$$R_I = \mathrm{Hom}(\mathbb{M}_I, \mathbb{Z})$$

$$\mathbb{M}_I := \mathbb{M}_{i_1,i_2} \otimes \cdots \otimes \mathbb{M}_{i_n,i_1}$$

A typical basis

$$\Psi^{f_1}_{i_1,i_2} \otimes \cdots \otimes \Psi^{f_n}_{i_n,i_1}$$

Distinguishes a set of solitons

$$\phi^{p_1}_{i_1,i_2}, \dots, \phi^{p_n}_{i_n,i_1}$$

Fans of Solitons

So we define a ***cyclic fan of solitons*** to be an ordered set

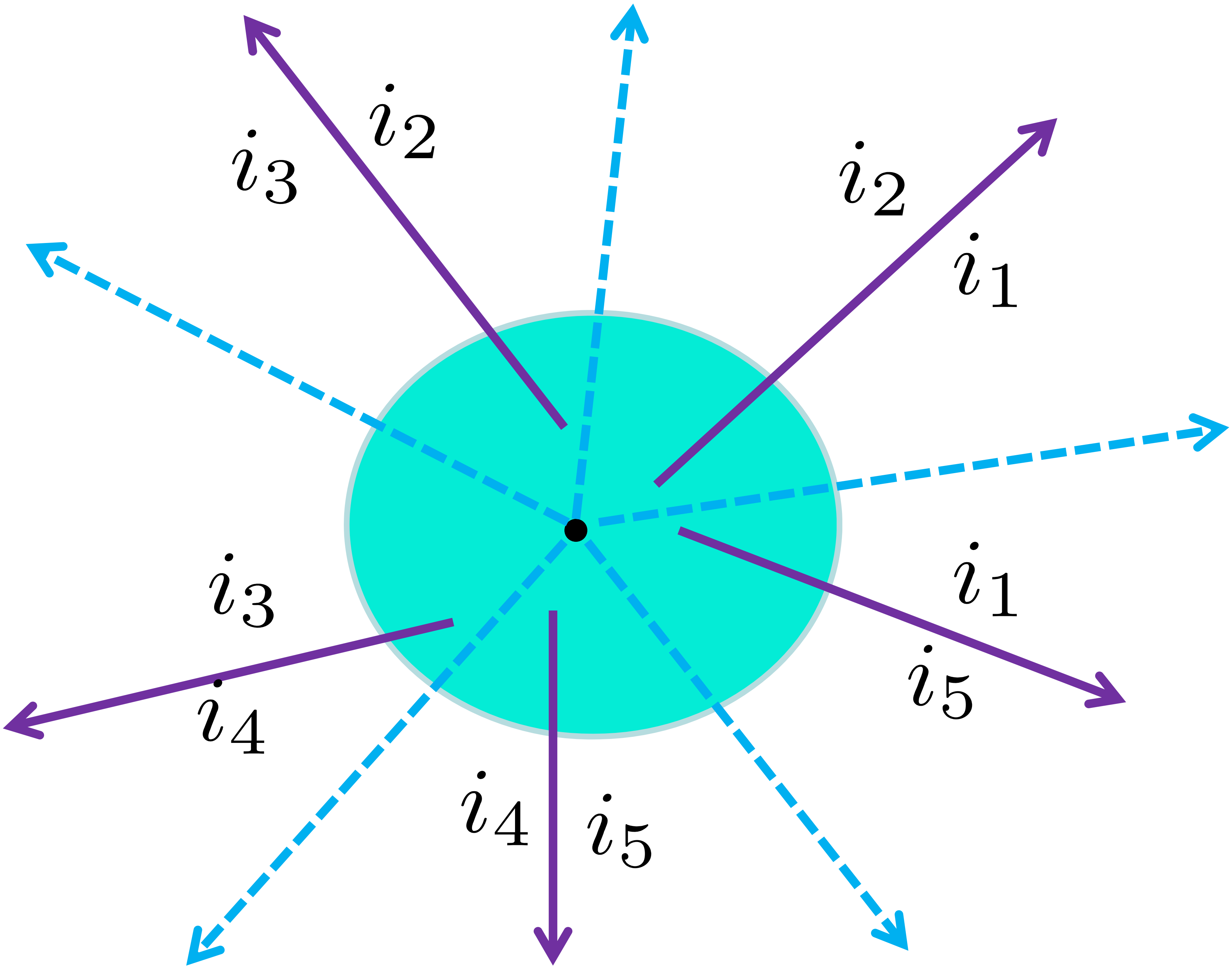
$$\mathcal{F} = \phi^{p_1}_{i_1,i_2}, \cdots, \phi^{p_n}_{i_n,i_1}$$

Use these to define boundary conditions on the instanton equation:

For (x,τ) large, near a ray parallel to z_{i_{k+1},i_k}

the instanton is approximately given by the boosted soliton for that ray:

$$\phi_{\mathcal{F}}(x,\tau) \sim \phi^{p_k}_{i_k,i_{k+1}}(\cos\theta_k x + \sin\theta_k \tau)$$


i_2
i_3
i_2
i_1
i_1
i_5
i_3
i_4
i_4
i_5

Counting Instantons

$\mathcal{M}^{\mathrm{red}}(\mathcal{F})$ Moduli of solutions of $\bar{\partial}\phi = \frac{\mathrm{i}\zeta}{4}\frac{\partial \bar{W}}{\partial \bar{\phi}}$

With fan boundary condition $\mathcal{F}$ at ∞

$$\mathcal{N}(\mathcal{F}) := \begin{cases} \#\mathcal{M}^{\mathrm{red}}(\mathcal{F}) & \dim \mathcal{M}^{red} = 0 \\ 0 & \mathrm{else} \end{cases}$$

$$\mathcal{N}_I \in \mathrm{Hom}(\mathbb{M}_I, \mathbb{Z}) = R_I$$

Instanton Counting Defines an Interior Amplitude

Theorem 1: If we define

$$\beta := \sum_I \mathcal{N}_I$$

then β is an *interior amplitude*, that is

$$\rho(\mathfrak{t})(e^{\beta}) = 0$$

Idea of proof: We look at the contributions to $d^2=0$ for one-dimensional reduced moduli spaces of instantons. The boundaries look like taut webs.

The Vacuum Category

Thanks to webology we get an A_∞ category

$$\mathfrak{Vac}^{\beta}$$

Intrinsically associated to the holomorphic Morse function W

Define $\mathfrak{Vac}[W]$ to be this A_∞ category.

The Morse Complex on $\mathbb{R}_+$ Gives Chan-Paton Factors

Now introduce Lagrangian boundary conditions $\mathcal{L}$:

$$\mathcal{E}_j := \mathbb{M}_{\mathcal{L},j} \qquad \tilde{\mathcal{E}}_j := \mathbb{M}_{j,\mathcal{L}}$$

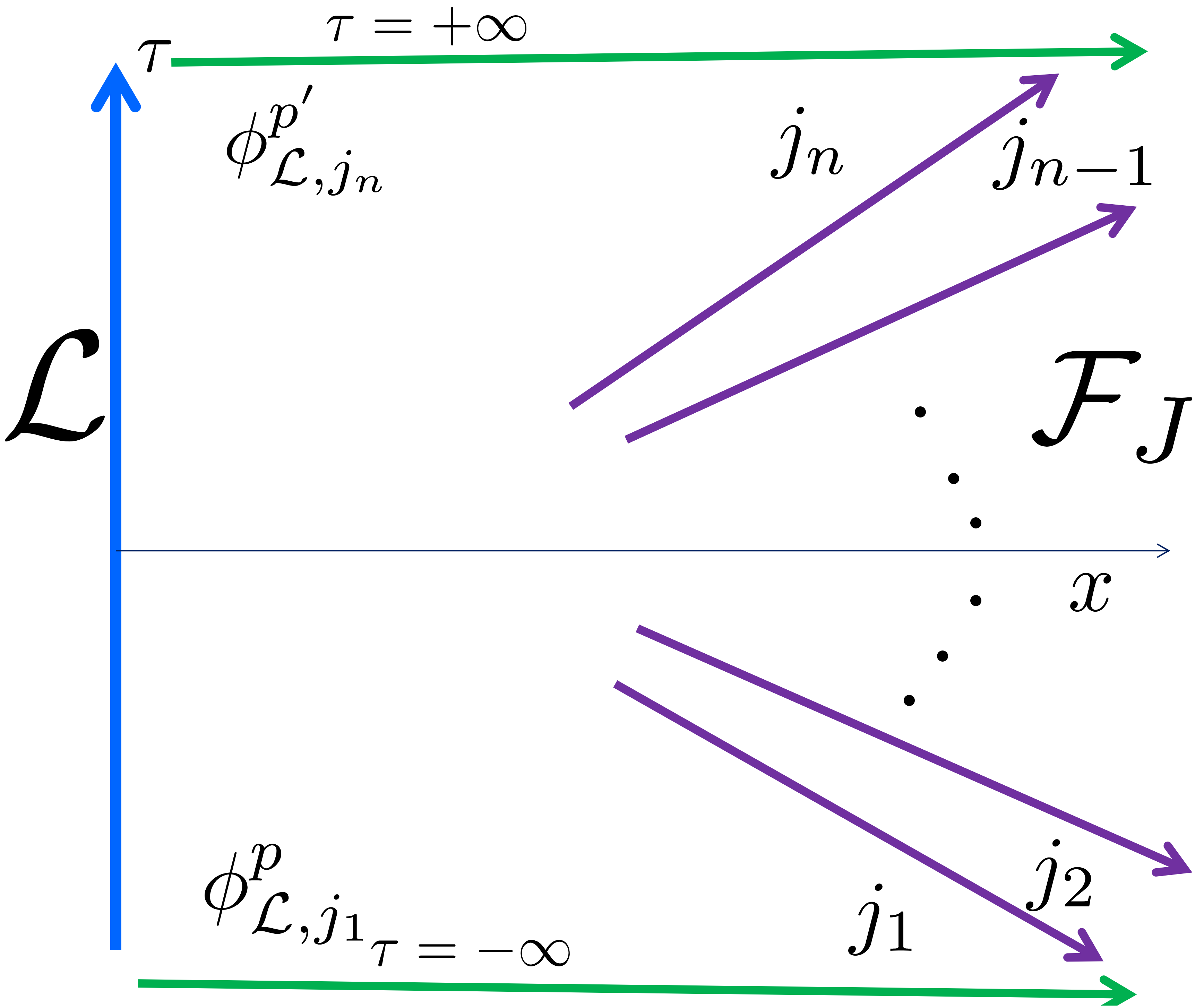
For a half-plane fan J = {j_1, … , j_n} define

$$\mathbb{M}_J := \mathbb{M}_{j_1,j_2} \otimes \cdots \otimes \mathbb{M}_{j_{n-1},j_n}$$

then we define

$$\mathcal{N}_J \in \mathrm{Hom}(\mathcal{E}_{j_1}, \mathcal{E}_{j_n}) \otimes \mathrm{Hom}(\mathbb{M}_J, \mathbb{Z}) = \mathrm{Hom}(\mathbb{M}_J, \mathrm{Hom}(\mathcal{E}_{j_1}, \mathcal{E}_{j_n}))$$

by instanton counting:


$\tau = +\infty$
τ
$\phi^{p'}_{\mathcal{L},j_n}$
j_n
j_{n-1}
$\mathcal{L}$
$\mathcal{F}_J$
x
j_2
$\phi^{p}_{\mathcal{L},j_1}$
j_1
$\tau = -\infty$

Half-Space Instanton Counting

$$\mathcal{N}(\mathcal{L}, \mathcal{F}_J, \Psi_{\mathcal{L},j_1}(p), \Psi_{\mathcal{L},j_n}(p')) = \begin{cases} \#\mathcal{M}^{\mathrm{red}}(\mathcal{L}, \dots) & \dim = 0 \\ 0 & \mathrm{else} \end{cases}$$

These are the matrix elements of

$$\mathcal{N}_J \in \mathrm{Hom}(\mathcal{E}_{j_1}, \mathcal{E}_{j_n}) \otimes \mathrm{Hom}(\mathbb{M}_J, \mathbb{Z}) = \mathrm{Hom}(\mathbb{M}_J, \mathrm{Hom}(\mathcal{E}_{j_1}, \mathcal{E}_{j_n}))$$

Instanton Amplitudes Solve MC

Theorem 2: The instanton amplitudes $\mathcal{N}_J$ define a solution to the Maurer-Cartan equation for $\mathfrak{Vac}^{\beta}$ enhanced by the Chan-Paton spaces $\mathcal{E}_{\mathcal{L},j}$.

Proof: Again consider $d^2=0$ for the half-plane instantons with reduced dimension =1.

Now we can apply webology again: Using the interior amplitude and the solutions of MC provided by instanton counting we get a differential on the strip.

Conjecture: The cohomology of this differential is the space of BPS states on the strip.

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The Brane Category

Suppose A is an A∞ category

Define a new A∞ category Br[A] whose objects are solutions of the MC equation of A[$\mathcal{E}$] for *some* set of Chan-Paton factors $\mathcal{E}$

$$M_n : \mathrm{Hom}(a_0, a_1) \otimes \cdots \otimes \mathrm{Hom}(a_{n-1}, a_n) \to \mathrm{Hom}(a_0, a_n)$$

$$M_n(\delta_1, \dots, \delta_n) := \sum_{\vec{r} \in \mathbb{Z}_+^{n+1}} (-1)^{x_n(\vec{r})} m_{n+|\vec{r}|}(a_0^{r_0}, \delta_1, a_1^{r_1}, \delta_2, \dots, a_{n-1}^{r_{n-1}}, \delta_n, a_n^{r_n})$$
$$= m_n(\delta_1, \dots, \delta_n) + m_{n+1}(a_0, \delta_1, \dots, \delta_n) + m_{n+1}(\delta_1, a_1, \delta_2, \dots, \delta_n) + \cdots$$

Same as “twisted complexes construction” – an analog of the derived category for A∞ categories

A Natural Conjecture

Following constructions used in the Fukaya category, Paul Seidel constructed an A_∞ category FS[W] associated to a holomorphic Morse function W: X to $\mathbb{C}$.

Br[FS[W]] is meant to be the category of A-branes of the LG model.

But, we also think that Br[$\mathfrak{Vac}$[W]] is the category of A-branes of the LG model!

So it is natural to conjecture an equivalence of A_∞ categories:

$$\mathrm{Br}[\mathrm{FS}[W]] \cong \mathrm{Br}[\mathfrak{Vac}[W]]$$

"ultraviolet" "infrared"

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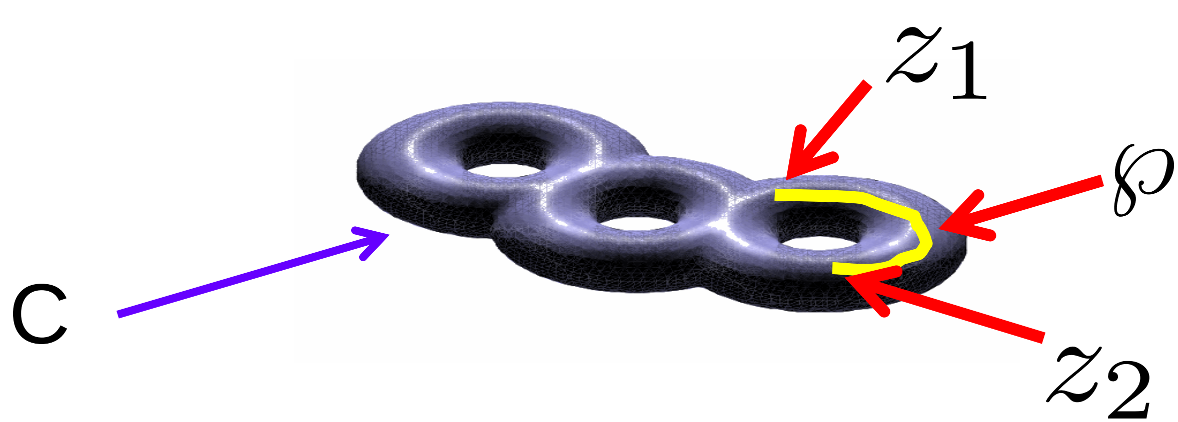
Families of Theories

Now consider a *family* of Morse functions

$$W(\phi; z) \qquad z \in C$$

Let $\wp$ be a path in C connecting z_1 to z_2.

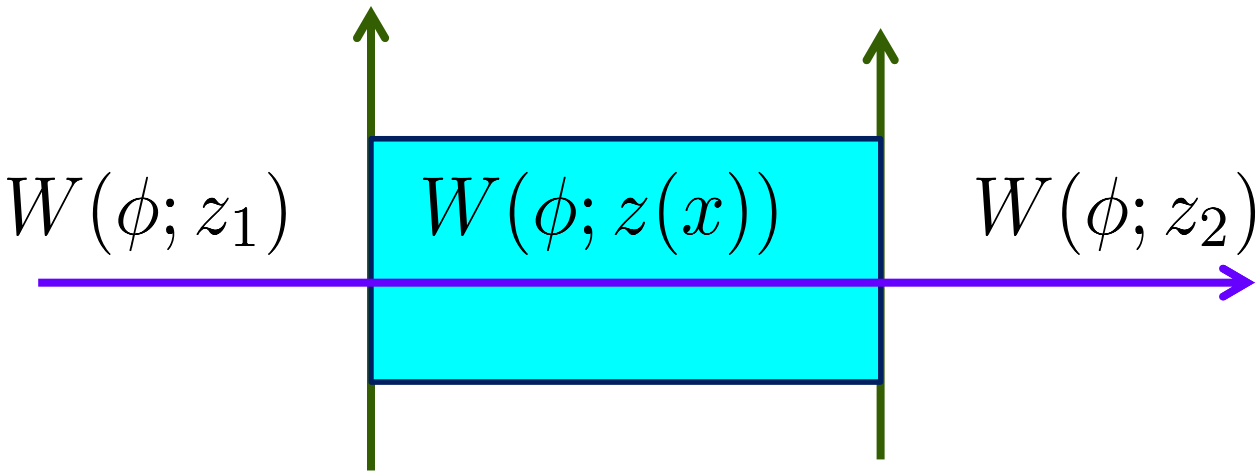
View it as a map z: $[x_l, x_r] \longrightarrow C$ with $z(x_l) = z_1$ and $z(x_r) = z_2$



Domain Wall/Interface

Using z(x) we can still formulate our SQM!

$$h = \int_D \left(pdq + \mathrm{Re}(\zeta^{-1} W(\phi; z(x)) dx \right)$$



From this construction it manifestly preserves two supersymmetries.

Parallel Transport of Categories

To $\wp$ we associate an A_∞ functor

$$\mathbb{F}(\wp) : Br[Vac[W_1]] \to Br[Vac[W_2]]$$

(Relation to GMN: "Categorification of S-wall crossing")

To a composition of paths we associate a composition of A_∞ functors:

$$\mathbb{F}(\wp_1 \circ \wp_2) = \mathbb{F}(\wp_1) \circ \mathbb{F}(\wp_2)$$

To a homotopy of $\wp_1$ to $\wp_2$ we associate an equivalence of A_∞ functors.

Outline

- Introduction & Motivations
- Webology
- Landau-Ginzburg Models & Morse Theory
- Fans of solitons & Webs
- A-∞ categories of branes
- Supersymmetric Interfaces
- Summary & Outlook

Summary

1. We gave a viewpoint on instanton corrections in 1+1 dimensional LG models based on IR considerations.

2. This naturally leads to L_∞ and A_∞ structures.

3. As an application, one can construct the (nontrivial) differential which computes BPS states on the interval.

4. When there are families of LG superpotentials there is a notion of parallel transport of the A_∞ categories.

Outlook

1. Finish proofs of parallel transport statements.

2. Interpretation of the convolution identities in terms of an L_∞ morphism from $\mathcal{W}$ to the Hochschild cohomology of $\mathfrak{Vac}^{\beta}$

3. Are these examples of universal identities for massive 1+1 QFT?

4. Generalization to 2d4d systems: Categorification of the KSWCF

5. Computability of Witten's approach to knot homology? Relation to other approaches to knot homology?