Langlands Duality in 2D gauge theory

1. Extended 2D TQFT and the space of states
2. Locally trivial actions. First approx. of answer.
3. Concrete problem: the space of states and twisted sectors
4. Correct formulation and "honest" answer
5. "Derivation" from Langlands duality (after Ed Witten et al)
6. Gauged TQFT and its space of states. Quantum Kirwan map

1. TQFT (after Segal, Atiyah & Witten) is a symmetric monoidal functor $Z: \mathrm{Bord}_n \longrightarrow (\mathrm{Vect}, \otimes)$
Extended TQFT: symm. monoidal functor from $\mathrm{Bord}_{0,\dots,n}$ to some symm. monoidal n-category delooping $(\mathrm{Vect}, \otimes)$.
Eg.: $n=2$, target can be (Alg, Bimod, Intertw) or Linear cats, linear functors, Nat Transformations, or DG versions
Work of Costello, Kontsevich, Hopkins-Lurie shows that $Z|_{\mathrm{Bord}^{fr}}$ is completely determined by $Z(+)$ — subject to finiteness conditions while $Z|_{\mathrm{Bord}^{or}}$ requires an additional "Frobenius structure" on it.
$Z(S^1) = HH_* Z(+)$.
Examples:
1. B-model for a CY manifold Y with potential $W: Y \to \mathbb{C}$ with proper critical locus. $B(+) = DMF(Y, W)$.
2. A-model for a compact symplectic manifold X.
$A(+) = \mathcal{F}(X)$. The associated 1,2 theory is $GW(X)$.

When X is not compact, eg T^*M, this is more safely defined but not quite a TQFT (string topology)

Mirror symmetry is an interchange $1 \leftrightarrow 2$.
Example: (Givental - Hori - Vafa)
X toric variety (smooth, proj). Can write $X = \mathbb{C}^N /\!\!/ K_{\mathbb{C}}$,

$$1 \to K_{\mathbb{C}} \to (\mathbb{C}^*)^N \to T_{\mathbb{C}} \to 1$$

Dually, $T_{\mathbb{C}}^\vee \to (\mathbb{C}^*)^N \to K_{\mathbb{C}}^\vee$, and the B-model mirror of X is $T_{\mathbb{C}}^\vee$, $W = z_1 + \ldots + z_N$.

Remarks.

- There is a family of mirrors parametrized by $K_{\mathbb{C}}^\vee = H^2(X; \mathbb{C}^*)$.
- Actually the mirror is a T-annulus in $T_{\mathbb{C}}^\vee$, cut by $\{|z_i| < 1\}$ (unitary SYZ mirror).

And the parameter space is a K-annulus defined by the image of the K-moment map on $\mathbb{C}^N$.

2\. Interested in: equivariant version of above, for compact G
Analogy: 1-dim TQFTs based on $H^*(Y; \mathcal{O})$ and $H^*(X; \mathbb{C})$
Gauged versions of these: $H^*(Y; \mathcal{O})^G$ and $H_G^*(X; \mathbb{C})$.
These are the (derived) invariant subspaces in the categories of vector spaces with G-action, resp. locally trivial G-action.

Preliminary Definition: Gauging a 2D TQFT Z means
- defining a G-action on $Z(+)$ (B-model)
- defining a locally trivial G-action on $Z(+)$ (A-model)

This defines a TQFT on surfaces with G-bundles (with flat connection, in the B-case), or a classically gauged QFT. Quantizing the theory = integrating over bundles, to $\mathrm{Bord}^{or}_{0..2}$

<u>Metatheorem:</u> With enough assumptions, the gauged TQFT is generated by $Z(+)^G \cong Z(+)_G$.

<u>Remark</u> This is easy to make precise in the B-model, and gives the "correct" gauge theory. A-model is difficult. In fact, the 'obvious' implementation of these ideas gives the wrong answer. [Problem: $\mathbb{Z}/2$ vs $\mathbb{Z}$-gradings]

<u>Theorem</u> (G connected).
A locally trivial G action on a (A_∞, DG, top.) category $\mathcal{C}$ is (up to coherent homotopy) an E_2 ring homomorphism

$$C_* \Omega G \longrightarrow HH^* \mathcal{C}.$$

The invariant and co-invariant categories are

$$\mathcal{C}^G = \mathrm{RHom}_{C_*\Omega G\text{-mod}}(\mathrm{Vect}_0, \mathcal{C})\ ;\ \mathcal{C}_G = \mathrm{Vect} \otimes_{C_*\Omega G\text{-mod}} \mathcal{C}.$$

Example that works:

* $\mathcal{C} = C^*G$ - mods; localization of $C_*\Omega G$ - mod at Vect.
$\mathcal{C}^G = \mathcal{C}_G$ = Vect, reasonable interpretation of G/G = point.
Note: $C_*\Omega G$ - mod $= \mathcal{F}(T^*G)$, and $T^*G /\!/ G$ = point also
Example that works badly:
* $\mathcal{C}$ = Vect, get $\mathcal{C}^G = C_*G$ - Pontrjagin modules.
This is a localization of the correct answer, $C^*(BG)$ - mods.

3. Why Categories?

The problem of gauging a TQFT cannot be solved from its 1-2 part alone. This is caused by the appearance of the famous "twisted sectors"

If the finite group F acts on X, the space of states of the gauged theory is not $H^*(X)^F$, but

$$\left[\bigoplus_{f \in F} H^*(X^f)\right]^F \quad (\text{Chen-Ruan } H^*).$$

This is the cohomology of LBF with local coefficients not determined from $H^*(X)$ alone.

For connected G, have an analogous result: the space of states is $H^*_G(G; \mathcal{HH}^*\mathcal{C})$, where $\mathcal{HH}^*\mathcal{C}$ is a local system over G, determined from the ΩG - action.
E_2 - ness of the action ensures the existence of a Pontrjagin prod.
Thus, for symplectic X, one must understand this E_2 action of ΩG on $H^*(X)$ to find the space of states.

This action is a quantum feature (no classical limit).

Morally, for compact symplectic X, $H^*(X) = H^{\infty/2}(LX)$, and we are hacking the LG-action on LX. But the only way to make this Floer philosophy precise passes through Fukaya cats.

The only convenient way to compute Fukaya-like things proceeds via finding the complex mirror and computing there.

4. Mirror of an A-model group action

"A clear argument always wins over a correct argument"

Main Theorem, (simplified incorrect form)

* The mirror of a T-action on X is a holomorphic map $Y \longrightarrow T_{\mathbb{C}}^{\vee}$ from the mirror Y to the Langlands dual torus
* The fiber over $1 \in T_{\mathbb{C}}^{\vee}$ is the mirror of the sympl. quot. $X/\!/T$.

(This is correct to 0^{th} order; fixable with standard methods

* For connected G, replace $T_{\mathbb{C}}^{\vee}$ by $T_{\mathbb{C}}^{\vee}/W = G_{\mathbb{C}}^{\vee}/\mathrm{conj}$.

(This is correct after some blowing up.)

The correct answers require a reformulation of the problem.

Main slogan:

Gauged 2D field theories are boundary conditions for Pure 3D gauge theory.

Note the two distinguished ∂ conditions:

* trivial boundary condition "Vect"
* gauge fixing boundary condition C^*G-mods.

Reducing 3D gauge theory along an interval with ∂ conditions your favourite $\mathcal{C}$ + one of the above produces the 2D theory $\mathcal{C}^G$ and $\mathcal{C}$, respectively.

<u>Theorem</u> (Seiberg-Witten; Argyros-Manoussakis; Martinez-Wasma) (- for ∂ conditions; Witten)

The relevant pure 3D gauge theory is equivalent to the Rozansky-Witten theory of a certain hyperkähler manifold:

* $(T^*G^\vee_{\mathbb{C}})^{reg}/\!/G^\vee_{\mathbb{C}}$
* $\mathrm{Spec}\, H^G_*(\Omega G)$
* The moduli space of solutions to Nahm's equations on the interval with principal sl_2 residues at the ends
* For $G = SU(n)$, the reduced moduli space of $SU(2)$ monopoles of charge n

<u>Remark</u>. This space is a desingularization of $(T^*T^\vee_{\mathbb{C}})/W$

Kapustin-Rozansky have partially defined the 2-category

of boundary conditions for RW theory, in terms of $\mathcal{O}$-linear sheaves of categories over holomorphic Lagrangians.
For example, for $M = T^*N$, the localization of the KR 2-category near the 0-section is the 2-cat. of $Coh(\mathcal{O}_N)$-modules, and the Lagrangian (germ) is the singular support.

This explains the simplified picture for the torus; but the G picture needs correction at the singular conj. classes.

5. Witten's derivation from 4D Langlands duality

Kapustin & Witten formulated a version of Langlands duality as an equivalence of 4D $N=2$ gauge theories for dual groups. groups.

The 3D gauge $\Longleftrightarrow$ RW equivalence can be physically deduced from that by compactification along an interval.
We need boundary conditions, and the relevant ones were studied by Gaiotto-Witten and shown to be Langlands dual:
* the Neumann condition (on the gauge field)
Using this leads (in low energy) to 3D gauge theory.
* The "Nahm pole" condition.
Using this leads to the $N=2$ Sigma-model in the space of solutions mentioned above. QED.

6 Further directions

This story correctly explains the GHV toric Ansatz and is confirmed by some simple nonabelian examples. But some difficulties remain:

* $\mathcal{F}(X)$ should spread out naturally over $\mathrm{Spec}\, H_*^G(\Omega G)$. In the abelian case, this is a form of spectral decomposition, but the general case is less clear. Can be done by reduction to T, W + homological algebra correction at the blowup loci; but finiteness, say, is not clear.

* The equivalence of the "gauged Fukaya category" with the $\mathcal{F}$.cat of the GIT (orbifold) quotient is conjectural, but strongly supported by examples and thematic similarity to the unitary SYZ constructions.

* Givental's equivariant GW theory also fits into this context. It is the localization of the gauged theory at $1 \in G$.

* Woodward's construction of the "quantum Kirwan map", an A_∞ morphism $QH_G^*(X) \to QH^*(X/\!\!/G)$, fits into this picture as the Thom map $H_G^*(X) \to H_G^*(G; H^*(X))$ for $1 \in G$, followed by the iso $H_G^*(G; H^*(X)) \cong HH^*(\mathcal{F}(X)^G) \cong HH^*\mathcal{F}(X/\!\!/G)$