

Physics over a finite field and Wick rotation

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Abstract

The paper develops some mathematics supporting an earlier hypothesis that the physical universe is a finite system co-ordinatised by a huge finite field F_p which looks like the field of complex numbers to an observer.

Earlier we constructed a place ('limit' homomorphism) lm from a pseudo-finite field F_p onto the compactified field of complex numbers. In the current paper we construct lm in a more concrete form. In particular, lm sends certain multiplicative subgroups ' \mathbb{R}'_+ ' and ' \mathbb{S}' ' of F_p onto the non-negative reals \mathbb{R}_+ and the unit circle \mathbb{S} in \mathbb{C} . Thus F_p , ' \mathbb{R}'_+ ' and ' \mathbb{S}' ' provide co-ordinates for physical universe.

We introduce two systems of natural units corresponding to ' \mathbb{R}'_+ ' and ' \mathbb{S}' ', respectively, on the logarithmic scale. The passage from the scale of units of ' \mathbb{R}'_+ ' to the scale of units of ' \mathbb{S}' ' corresponds to a multiplication (on the logarithmic scale) by a 'huge' (non-standard) integer i equal approximately to \sqrt{p} . This provides an explanation to the phenomenon of Wick rotation.

In the same model we explain the phenomenon of phase transition in a large finite system

1 Introduction

1.1 The hypothesis that the universe is infinite is an open question. This concerns both the size of the universe and the number of atoms or elements that comprise it. Since it is now accepted that there is a minimal length, the Planck length, assumption of the spatial finiteness of the universe implies the assumption of the finiteness of the number of its elements.

In [1] we discussed the concept of approximation in physics and the suggestion that physics universe is co-ordinatised by a *huge*¹ finite field F_p . It was proved (Proposition 5.2 of [1]) that the only metric field (locally compact field) that can be approximated by finite fields is the field \mathbb{C} of complex numbers. Thus “seen from afar” the huge finite field looks like a field of complex numbers \mathbb{C} .

The current work has been inspired by Hao Hu, the expert in Philosophy of Physics, who approached the author with the suggestion to apply the idea of [1] to statistical mechanics and to attack the well-discussed problem of phase transition (which happen in large finite systems but require the assumption of infinity for its mathematical theory). We suggest here an answer to the problem.

Perhaps a more important outcome of the mathematical theory developed below is the interpretation of the phenomenon of *Wick rotation* as the result of the change of scales in physics.

I would also like to note that a compatible attempt to develop the mathematical background for a theory of finite physics was presented in [2], [3] and [4].

1.2 Let us recall the notion of structural approximation suggested in [1], in the specific context of approximation by huge finite fields F_p . A finite structure is discrete but to see its grainy structure we should be able to detect difference, *inequality*, between its neighbouring elements, which might be impossible with the instruments we use to observe the structure. However, if there is a shape within the n -space F_p^n which is given by an algebraic *equation* then the observer could see it as a shape in a continuous field, say equal to \mathbb{R} , \mathbb{C} or maybe a p -adic field, given by the same equation. This brings us to the idea that such an approximation by an observer is a map (called “limit”) onto a continuous field K ,

$$\text{lm} : F_p \rightarrow K$$

which takes tuples $(x_1, \dots, x_n) \in F_p^n$ satisfying a polynomial equation $f(x_1, \dots, x_n) = 0$ (with integer coefficients) to the tuple $(y_1, \dots, y_n) \in K^n$ satisfying the same equation. In other words, lm is a ring-homomorphism.

¹We use *huge* for numbers which some physics authors call “ridiculously large”, see the discussion in 2.7 below.

In fact, for a finite $F_{\mathfrak{p}}$ this scheme is not going to work verbatim but it works when we assume that $F_{\mathfrak{p}}$ is infinite *pseudo-finite*, which for all intents and purposes replaces a huge finite structure, see 1.5 below for definition.

However, as explained in [1], the requirement that lm is defined on the whole of the discrete structure necessitates that the target structure K must be compact, which for a metric field can be achieved by adding a point ∞ (so for $K = \mathbb{C}$ the compactification gives us the *extended complex numbers* $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, equivalently, the Riemann sphere). In particular, there are non-zero elements $x \in F_{\mathfrak{p}}$ indistinguishable from 0 (that is $\text{lm}(x) = 0$) and so for the inverses $x^{-1} \in F_{\mathfrak{p}}$, $\text{lm}(x^{-1}) = \infty$. Such a map lm between fields is called a *place* in algebra.

The above mentioned key Proposition 5.2 of [1] states:

There exists a place

$$\text{lm} : F_{\mathfrak{p}} \rightarrow \bar{\mathbb{C}} \tag{1}$$

from any pseudo-finite field $F_{\mathfrak{p}}$ of characteristic 0 onto the compactification of the field of complex numbers \mathbb{C} , and $\bar{\mathbb{C}}$ is the only metric (locally compact) field for which such a map exists.

1.3 The relationship with the p-adic approach in physics. The p-adic approach in physics, in particular string theory, has proved quite productive, see e.g. the survey [5]. The recourse to a prime p is motivated by the needs of discretisation and the field \mathbb{Q}_p of p -adic numbers has the advantage of bearing a nice metric and locally compact topology. There is no preferred prime but a very large prime like the above \mathfrak{p} seems to be a reasonable choice. We note that by definition there is a canonical place

$$\mathbb{Q}_p \rightarrow F_{\mathfrak{p}} \tag{2}$$

and thus, combining with (1) we get a place

$$\mathbb{Q}_p \rightarrow \bar{\mathbb{C}}.$$

In other words, p-adic calculations pass via (2) to $F_{\mathfrak{p}}$ which, according to (1), can be reinterpreted as calculations in the complex numbers.

1.4 The main mathematical problem in making a practical use of the idea of a “physics over a finite field $F_{\mathfrak{p}}$ ” is to find a way of representing the common-sense real quantities of physics inside the huge finite field $F_{\mathfrak{p}}$ and to find such

a representation that allows the standard physics calculations. That is to explain how both \mathbb{R} and $i\mathbb{R}$ emerge from a large finite field.

In this regard it is useful to invoke a notion of *feasible numbers* that was discussed by philosophers of mathematics, mathematical logicians and computer scientists, see e.g. [9] for a mathematical treatment of the notion. Roughly speaking, $1, 2, \dots, 1000$, as well as their ratios such as $1/5$ and 0.203 are feasible numbers, but the Avogadro number $\sim 10^{23}$ is not feasible. We think of the latter as *very large* but potentially observable numbers. This contrasts with *huge* numbers, such as \mathfrak{p} , the number of points in $F_{\mathfrak{p}}$, and $\mathfrak{p} \gg 10^{23}$.

Thus, while we work with feasible numbers $1, 2, \dots$ inside $F_{\mathfrak{p}}$ we can think of these as the usual integers but when our integers become very large but still much less than \mathfrak{p} then Im takes such ones to ∞ . Further on this scale, according to the approximation theorem of [1], the integers start to behave like complex numbers, e.g. if an integer i satisfies $i^2 + 1 = 0 \pmod{\mathfrak{p}}$ (huge number) it takes the role of $\sqrt{-1}$.

All this is made precise in the formulation of the Main Theorem in 1.7 below.

1.5 In the current work a “huge finite field” is a *pseudo-finite field* $F_{\mathfrak{p}}$ which can be obtained by considering a non-principal ultrafilter \mathcal{D} on the set of prime numbers $P \subset \mathbb{N}$ (positive integers) and the ultraproduct

$$F_{\mathfrak{p}} = \prod_{p \in P} F_p / \mathcal{D}.$$

Such a construction sees $F_{\mathfrak{p}}$ as a logical limit of finite fields F_p along the ultrafilter: a first order sentence Φ is valid in $F_{\mathfrak{p}}$ if and only if it is valid in almost all, in the sense of \mathcal{D} , finite fields F_p .

Model theory tells us that $F_{\mathfrak{p}}$ can equivalently be obtained as the quotient-ring of the ring ${}^*\mathbb{Z}$ of non-standard integers by the prime ideal $\mathfrak{p} {}^*\mathbb{Z}$, where \mathfrak{p} is the respective non-standard prime number,

$$F_{\mathfrak{p}} \cong {}^*\mathbb{Z}/\mathfrak{p} \text{ where } {}^*\mathbb{Z} = \mathbb{Z}^P / \mathcal{D}.$$

The interpretation of $F_{\mathfrak{p}}$ in non-standard integers allows us to apply, among others, the means of non-standard analysis, in particular the *standard part map*

$$\text{st} : {}^*\mathbb{Q} \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$$

from non-standard rationals $\frac{l}{m}$, $l, m \in {}^*\mathbb{Z}$, to the compactification of the reals.

1.6 Along with \mathfrak{p} and the field $(F_{\mathfrak{p}}; +, \cdot, 0, 1)$ we specify:

- a non-standard *highly divisible* number \mathfrak{l} (each standard integer m divides \mathfrak{l}) satisfying some other assumptions below;
- a two-sorted pseudo-finite structure $(\mathbb{U}_{\mathfrak{p},\mathfrak{l}}, F_{\mathfrak{p}})$ with

$$\mathbb{U}_{\mathfrak{p},\mathfrak{l}} = (*\mathbb{Z}/_{(\mathfrak{p}-1)\mathfrak{l}}; +, \hat{0}, \hat{1})$$

a pseudo-finite additive cyclic² group of order $(\mathfrak{p} - 1)\mathfrak{l}$ with generator $\hat{1}$;

- a surjective group homomorphism

$$\exp_{\mathfrak{p}} : \mathbb{U}_{\mathfrak{p},\mathfrak{l}} \rightarrow F_{\mathfrak{p}}^{\times}; \quad n \cdot \hat{1} \mapsto \epsilon^n$$

where ϵ is a generator of the (pseudo)-cyclic group $F_{\mathfrak{p}}^{\times}$, $n \in *\mathbb{Z}$. It follows that

$$\ker \exp_{\mathfrak{p}} = (\mathfrak{p} - 1) \cdot \mathbb{U}_{\mathfrak{p},\mathfrak{l}},$$

the subgroup generated by $(\mathfrak{p} - 1) \cdot \hat{1}$, (and suggests that $\mathfrak{p} - 1$ of $\mathbb{U}_{\mathfrak{p},\mathfrak{l}}$ should be interpreted as $2\pi i$);

- a pair of surjective “limit” homomorphisms (places) lm which make the diagram commute

$$\begin{array}{ccc} \text{lm}_{\mathbb{U}} : & \mathbb{U}_{\mathfrak{p},\mathfrak{l}} & \twoheadrightarrow & \bar{\mathbb{C}} \\ & \exp_{\mathfrak{p}} \downarrow & & \exp \downarrow \\ \text{lm}_{F} : & F_{\mathfrak{p}} & \rightarrow & \bar{\mathbb{C}} \end{array} \quad (3)$$

There is a natural cyclic order on $\mathbb{U}_{\mathfrak{p},\mathfrak{l}}$ in which $u + \hat{1} > u$ and there is a related cyclic order on $F_{\mathfrak{p}}^{\times}$ in which $\epsilon^{n+1} > \epsilon^n$ for all $n \in *\mathbb{Z}$.

We treat $\hat{1}$ as an infinitesimal and choose two “units of length” \mathbf{u} and \mathbf{v} , elements of $\mathbb{U}_{\mathfrak{p},\mathfrak{l}}$,

$$\hat{1} \ll \mathbf{u} \ll \mathbf{v}.$$

More precisely,

$$\mathbf{u} = \frac{\mathfrak{p} - 1}{\mathfrak{i}} \text{ and } \mathbf{v} = \mathfrak{p} - 1 \quad (4)$$

for some $\mathfrak{i} \in *\mathbb{N}$ such that

$$\mathfrak{i}\mathfrak{l} \mid (\mathfrak{p} - 1), \quad \mathfrak{i} \gg \mathfrak{l} \quad (5)$$

²here and below “cyclic” means in the pseudo-finite sense, i.e. the ultraproduct of cyclic groups

(l divides $\mathfrak{p} - 1$).

Thus, \mathbf{u} and \mathbf{v} are units of two very different scales (see 1.9 below for further comment).

We assume that

$$\mathfrak{i} = \iota^2 \text{ and } \mathfrak{l} = \mu^2 \quad (6)$$

for some $\mu, \iota \in {}^*\mathbb{N}$.

We also need to assume

$$\mathfrak{i}^2 + 1 = \mathfrak{p} \text{ or } \mathfrak{i}, \mathfrak{l} \text{ algebraically independent in } F_{\mathfrak{p}} \quad (7)$$

(the first is the preferable and more elegant assumption but it is not known whether it is consistent with \mathfrak{p} being infinite (the Landau problem)).

It is easy to check that our set of assumptions (5)-(7) along with the assumption of high divisibility of l are consistent. We are going to slightly extend these assumption later, in particular (24) assumes that $l^n < \mathfrak{p}$ for all $n \in \mathbb{N}$.

We also define additive subgroups of $\mathbb{U}_{\mathfrak{p},l}$ called suggestively ' \mathbb{R}' ' and ' $i\mathbb{R}'$ '.

1.7 Main Theorem. *There exists a surjective ring homomorphism (place)*

$$\text{Im}_F : F_{\mathfrak{p}} \twoheadrightarrow \mathbb{C} \cup \{\infty\}$$

and a surjective additive semigroup homomorphism

$$\text{Im}_U : \mathbb{U}_{\mathfrak{p},l} \twoheadrightarrow \mathbb{C} \cup \{\infty\}$$

such that:

the diagram (3) is commutative,

$$\text{Im}_U : \mathbb{R}' \twoheadrightarrow \mathbb{R}; \quad (8)$$

$$\text{Im}_U : i\mathbb{R}' \twoheadrightarrow i\mathbb{R}; \quad (9)$$

where $i = \sqrt{-1}$.

For any $l \in {}^*\mathbb{Z}$ such that $0 < l \leq \mathfrak{l}$,

$$\text{Im}_F : l \cdot \iota^{-1} \mapsto \text{st}\left(\frac{l}{\iota}\right) \in \mathbb{R} \quad (10)$$

$$\text{Im}_F : \exp_{\mathfrak{p}}(l\mathbf{u}) \mapsto e^{-\pi z}, \quad \text{for } z := \text{st}\left(\frac{l}{\iota}\right) \in \mathbb{R} \quad (11)$$

$$\text{lm}_F : \exp_p(l\mathbf{v}) \mapsto e^{i\pi z}, \quad \text{for } z := \text{st}\left(\frac{l}{\mathfrak{l}}\right) \in \mathbb{R} \quad (12)$$

where st is the standard part map.

For $a \in \mathbb{Q}_+$, a complete square and $\mu \in {}^*\mathbb{Z}$ such that $\mu^2 = \mathfrak{l}$:

$$\text{lm}_F : \frac{1}{\mu} \sum_{-i\mathfrak{l}/a \leq n < i\mathfrak{l}/a} \exp_p\left(a \frac{n^2}{2\mathfrak{l}} \mathbf{u}\right) \mapsto \int_{\mathbb{R}} e^{-a\pi x^2} dx \quad (13)$$

and $\int_{\mathbb{R}} e^{-ax^2} dx = \frac{1}{\sqrt{a}}$, according to the standard definition of the Gaussian integral;

$$\text{lm}_F : \frac{1}{\mu} \sum_{-i\mathfrak{l}/a \leq n < i\mathfrak{l}/a} \exp_p\left(a \frac{n^2}{2\mathfrak{l}} \mathbf{v}\right) \mapsto \int_{\mathbb{R}} e^{ia\pi x^2} dx \quad (14)$$

where $\int_{\mathbb{R}} e^{ia\pi x^2} dx = \frac{e^{\frac{\pi i}{4}}}{\sqrt{a}}$ according to the Quantum Mechanics calculus.

Note that (8) and (9) give us subgroups of F_p^\times

$$'R'_+ := \exp_p('R') \text{ and } 'S' := \exp_p('iR')$$

furnish a good analogue of polar coordinate system in F_p .

1.8 Remark. Since lm is a place, (10) determines the values of $\text{lm}(Q(l))$ for any rational function $Q(x)$ over \mathbb{Z} .

1.9 Discussion. The theorem clarifies the relationship between two different scales in $(\mathbb{U}_{p,\mathfrak{l}}, F_p)$ presented by units \mathbf{u} and \mathbf{v} . These should be thought of as units for physics of 'low energy' and 'high energy', respectively, the latter being physics at quantum level and the former the physics at the level of Brownian motion.

The statements (11) and (12) demonstrate that the action of the huge pseudofinite integer \mathfrak{i} which changes the scale of units in $\mathbb{U}_{p,\mathfrak{l}}$ (recall that $\mathbf{v} = \mathfrak{i} \cdot \mathbf{u}$, and $\mathfrak{i} \approx \sqrt{p}$) is seen in \mathbb{C} as the **Wick rotation** $e^{\pi z} \mapsto e^{i\pi z}$.

The Gaussian integrals in (13) and (14) are mathematical manifestation of the same phenomena expressed by the summation formulae over F_p on the left hand side of \mapsto . The difference in the integral expressions on the right comes from the difference in the scale of units that measure the respective processes.

Thus the puzzling regularity of the transition from the Brownian motion integral (13) to the quantum mechanics integral (14) known and exploited by physicists as Wick rotation has an explanation as a mathematical consequence of the change of scales.

1.10 Discussion. The “limit” lm_F maps the discrete field F_p into (the compactification of) the field of complex numbers and thus endows the image with a metric. The Theorem determines lm_F on the subring generated by specific points (see (10), (11) and (12)) but leaves the rest free. This means that the observer, which sees F_p with all the algebraic geometry over it through lm_F , has some freedom in choosing the metric on algebraic varieties. In particular, if $Z \subseteq F_p^m$ is an algebraic subvariety, say a torus, then $\text{lm}_F(Z)$ is a subset of the compactification of \mathbb{C}^m , a compact complex variety. In general, such a compactification is far of being unique.

Thus the freedom in the choice of lm_F implies a respective degree of freedom in the choice of complex/metric version of physics.

2 Statistical physics and phase transition

2.1 Physical units and dimensions

In the formalism of two sorts $\mathbb{U}_{p,l}$ and F_p introduced in 1.6 the sort $\mathbb{U}_{p,l}$ is assumed to be the sort that holds all the physical units (dimensions). It is convenient for each principal unit of measurement to define a special sort \mathbb{D}_i , $i = 1, \dots, k$ which is going to be naturally interpreted in terms of $\mathbb{U}_{p,l}$.

Each sort \mathbb{D}_i is a subgroup of $\mathbb{U}_{p,l}$ and so has a cyclic additive group structure with the unit (generator) $\mathbf{d}_i \in \mathbb{U}_{p,l}$ (\mathbb{D}_i -unit). \mathbb{D}_i is isomorphic to $\mathbb{U}_{p,l}/\ker_i$, where

$$\ker_i = \frac{(p-1)l}{\mathbf{d}_i} \cdot \mathbb{U}_{p,l}, \quad \mathbf{d}_i | (p-1)l.$$

Thus the size of \mathbb{D}_i ,

$$|\mathbb{D}_i| = \frac{(p-1)l}{\mathbf{d}_i}.$$

Between some of the unit sorts there are bilinear maps

$$\mathbb{D}_1 \times \mathbb{D}_2 \rightarrow \mathbb{D}_3; \quad (x_1 \mathbf{d}_1, x_2 \mathbf{d}_2) \mapsto x_1 x_2 \mathbf{d}_3$$

where we assume $\ker_3 = \ker_1 \cap \ker_2$ and

$$x_1 = u_1 + \ker_1, \quad x_2 = u_2 + \ker_2, \quad x_1 \cdot x_2 := u_1 \cdot u_2 + \ker_3$$

in the ring ${}^*\mathbb{Z}/(p-1)l$.

Thus counting in units of \mathbb{D}_3 gives

$$x_3 = x_1 x_2$$

and this can be equivalently written as

$$x_1 = x_2^{-1}x_3.$$

The units \mathbf{u} and \mathbf{v} introduced in 1.6 are examples of \mathbb{D} -units.

2.2 The field sort F_p is assumed to be dimensionless and the exponentiation map \exp_p restricted to a sort \mathbb{D}_i is a homomorphism

$$\exp_p : \mathbb{D}_i \rightarrow F_p^\times; \quad n \cdot \mathbf{d}_i \mapsto \exp_p(n\mathbf{d}_i).$$

2.3 In Statistical Mechanics the dimensions in $\mathbb{U}_{p,l}$ are usually Energy (E), Temperature (T) and (in ferromagnets) magnetic moment (H).

According to this theory probability that the system in temperature T is in a state σ is equal to $\exp(-\frac{E_\sigma}{kT})$, and the probability p_n that the system consists of exactly n atoms, out of possible N , is

$$p_n = \frac{P_n}{Z_N}, \quad P_n = \sum_{\sigma \in \Sigma(n)} \exp(-\frac{E_\sigma}{kT}),$$

where $\Sigma(n)$ are all the states with exactly n atoms and

$$Z_N = \sum_{\sigma} \exp(-\frac{E_\sigma}{kT}),$$

where σ runs in all possible states with at most N atoms.

Setting (with some simplifications) $y := \exp(-\frac{H}{kT})$, the equilibrium state of the system of volume N (that is having up to N particles) is analysed via the polynomial

$$\mathcal{P}_N(y) := \sum_{n=0}^N p_n y^n.$$

which is called the *grand partition function* of the system.

Assuming frequentist probabilities we may regard the P_n and Z_N integers.

Since there are very few restrictions on states in the models the number Z_N is close to the number of all possible subsets, that is

$$\sum_{n=0}^N P_n = Z_N \approx 2^N. \tag{15}$$

(The estimate (15) appears also in [8] on page 4, for N the Avogadro number, and the number Z_N is being characterised as *ridiculously large number*.)

Also note that when n is near $N/2$, P_n is near its maximum

$$P_n \approx \binom{N}{\frac{N}{2}} \approx \frac{2^N}{\sqrt{N}} \quad (16)$$

2.4 We note that since N is supposed to be very large number, by (16) P_n can reach huge values and so modulo \mathfrak{p} will be outside the real part ' \mathbb{R} ' of $\mathbb{F}_{\mathfrak{p}}$. Same is true for Z_N . Hence in general the p_n should be treated as **probability amplitudes** rather than classical probabilities, and \mathcal{P}_N should be treated as a polynomial with complex coefficients rather than polynomial over \mathbb{R} .

2.5 The seminal work of C.N.Yang and T.D.Lee, [6] - [7] (1952), laid the ground for the modern theory of critical points in the evolution of large finite systems such as the ideal gas.

The main theorem of [6] states that the phase transition in the system happens at the point y_{crit} (a critical point) such that

$$\mathcal{P}_N(y_{\text{crit}}) = 0$$

which obviously **can not be a real point**. The paper analyses complex roots of the polynomial which proves very important for the behaviour of the system near the critical point.

The modern theory resolves this paradox by assuming $N \rightarrow \infty$, in which case $\frac{1}{N} \ln \mathcal{P}_N(y)$ converges, away of the critical point, to an analytic function (the passage to the thermodynamical limit) and y_{crit} converges to a real point. This solution of the paradox is not considered to be fully satisfactory as the actual systems are always finite, although very large. Under the passage to thermodynamical limit some information is being lost.

2.6 The hypothesis of the *universe over finite field* suggests a solution to the paradox. Under the hypothesis $y_{\text{crit}} \in \mathbb{F}_{\mathfrak{p}}$, that is y_{crit} is an integer such that

$$\mathcal{P}_N(y_{\text{crit}}) = 0 \pmod{\mathfrak{p}}. \quad (17)$$

One can explore the assumption further using the reduction (24) from [7] together with Theorem 3 therein which states that after the reduction all the

zeroes of \mathcal{P}_N are on the unit circle and the limit of the zeroes is $= 1$. This leads us to the conclusion that

$$\mathcal{P}_N(1) = 0 \pmod{\mathfrak{p}}$$

Equivalently,

$$\sum_{n=1}^N P_n = 0 \pmod{\mathfrak{p}}.$$

Combining this with (15) and assuming that the number of atoms N is $\approx 10^{23}$, the Avogadro number, we can make an estimate on \mathfrak{p} , the upper bound:

$$\mathfrak{p} < 2^N \approx 2^{10^{23}} \tag{18}$$

On the other hand the same argument proves that *there is a low bound on the volume N of gas which allows a phase transition*, that is has a critical point satisfying (17):

$$N > \log \mathfrak{p}.$$

2.7 Discussion. The assumption of physics over a finite field explains the necessity of extending the definition of grand partition function as the function of complex variable as well as explains the phase transition in a large finite system.

In a forthcoming paper we develop an *analytic theory* on $\mathbb{U}_{\mathfrak{p},t}$ which, via $\text{lm}_{\mathbb{U}}$ corresponds to the analytic theory on \mathbb{C} . In particular the expression like

$$\frac{1}{N} \ln \mathcal{P}_N(y) \text{ and } \text{lm}_{\mathbb{U}} \left\{ \frac{1}{N} \ln \mathcal{P}_N(y) \right\}$$

become the lawfull objects of the theory and one can carry on the thermodynamic theory as usual, along with the discrete theory on $\mathbb{F}_{\mathfrak{p}}$.

The rest of the paper is purely mathematical. It provides the construction of the limit map lm and proofs.

3 The pseudo-finite exponentiation

Fix notation for a non-standard model of \mathbb{C}

$$*\mathbb{C} = \mathbb{C}^P / \mathcal{D}$$

where P and the ultrafilter \mathcal{D} on P are defined in 1.5.

Note that by construction ${}^*\mathbb{Z} \subset {}^*\mathbb{C}$ and this allows us to identify elements $l \in \mathbb{F}_p$ which are represented by $l \in {}^*\mathbb{Z}$, $0 \leq l < p$, with $l \in {}^*\mathbb{C}$ in the theorem below.

3.1 Theorem. *Then there is a place $\mathcal{I} : \mathbb{F}_p \rightarrow {}^*\bar{\mathbb{C}}$ such that \mathcal{I} maps:*
for all $l \in {}^*\mathbb{Z}$ satisfying $-l \leq l \leq l$

$$l \mapsto l \tag{19}$$

$$i \mapsto e^{-\frac{\pi i}{4}} \text{ and } i \mapsto e^{-\frac{\pi i}{2}} \tag{20}$$

For all $a \in \mathbb{Q}$, for all $l \in {}^*\mathbb{Z}$, $-l < l \leq l$,

$$\epsilon^{\frac{al(p-1)}{2i}} \mapsto e^{-\frac{al\pi i}{i}} \tag{21}$$

$$\epsilon^{\frac{al(p-1)}{2i}} \mapsto e^{-\frac{al\pi}{i}}. \tag{22}$$

The proof is by Lemmata below.

We consider linear equations of the form

$$\sum_{i=1}^k c_i \cdot X_i = 1$$

where the variables X_i are assumed to be in a specific subset G of the field. A solution x_1, \dots, x_k is said to be **non-degenerate** if for any proper subset $K \subset \{1, \dots, k\}$

$$\sum_{i \in K} c_i \cdot x_i \neq 1.$$

3.2 Lemma. *There is a function $f : \mathbb{N} \rightarrow \mathbb{N}$, an $\eta \in {}^*\mathbb{N}$ and a highly divisible $\nu \in {}^*\mathbb{N}$ such that*

$$2\nu\eta | (p-1),$$

$$\text{for all } n \in \mathbb{N} \nu^n | \eta,$$

and

$$\text{for all } k \in \mathbb{N}, \text{ for all rational functions } c(X, Y) = \langle c_1(X, Y), \dots, c_k(X, Y) \rangle,$$

for any $0 \leq l \leq \nu$

$$\left. \begin{array}{l} \text{any non-degenerate solution } x_1, \dots, x_k \in \mathbb{F}_{\mathfrak{p}} \\ \text{of } \sum_{i=1}^k c_i(l, \eta) \cdot x_i = 1 \text{ \& } \bigwedge_i x_i' = 1 \\ \text{satisfies } \bigwedge_i x_i^{f(k)} = 1. \end{array} \right\} \quad (23)$$

Proof. Recall that $\mathbb{F}_{\mathfrak{p}}$ is a field of characteristic 0. Thus the well-known Theorem of Mann about linear equations in roots of unity with rational coefficients is applicable. A consequence of Mann's Theorem is that there is a function f satisfying (23) for any $\nu, \eta \in \mathbb{N}$ (see [14] for this and other consequences).

We treat the expression x^ν as an arithmetic function of x, ν defined in $({}^*\mathbb{Z}; +, \cdot, \mathfrak{p})$ along with the interpretation of the field $\mathbb{F}_{\mathfrak{p}}$.

Let $\mathcal{M}_c \subset {}^*\mathbb{N}^2$ be the set of $(\nu, \eta) \in {}^*\mathbb{N}^2$ such that (23) holds for given k and $c(X, Y)$. Clearly \mathcal{M}_c is definable in $({}^*\mathbb{Z}; +, \cdot, \mathfrak{p})$. By the above consequence of the Mann Theorem $\mathbb{N}^2 \subseteq \mathcal{M}_c$ and so

$$\mathbb{N}^2 \subseteq \bigcap_c \mathcal{M}_c$$

where c runs in all k -tuples of rational functions $c(X, Y)$.

Since each $n \in \mathbb{N}$ divides $\mathfrak{p} - 1$ it follows that the countable type

$$\bigwedge_c (\nu, \eta) \in \mathcal{M}_c \text{ \& } 2\nu\eta | (\mathfrak{p} - 1) \text{ \& } \bigwedge_{n \in \mathbb{N}} n | \nu \text{ \& } \nu^n | \eta$$

is consistent, thus has a realisation in the \aleph_0 -saturated structure ${}^*\mathbb{Z}$. \square

Below we use notation

$${}^*\mathbb{Z}[\mathfrak{l}] := \{l \in {}^*\mathbb{Z} : -\mathfrak{l} \leq l \leq \mathfrak{l}\}.$$

Assuming $\mathfrak{l} \ll \mathfrak{p}$ we may equally treat ${}^*\mathbb{Z}[\mathfrak{l}]$ as a subset of $\mathbb{F}_{\mathfrak{p}}$.

3.3 Corollary. *We may assume that for all k and $c(X, Y)$ (23) is satisfied when $\nu := \mathfrak{l}$ and, if $\mathfrak{i}^2 + 1 \neq \mathfrak{p}$, $\eta := \mathfrak{i}$. In particular,*

$$\begin{array}{l} \mathfrak{l}^n | \mathfrak{i}, \text{ for all } n \in \mathbb{N} \text{ and} \\ \mathfrak{i}^2 + 1 = \mathfrak{p} \text{ or } \mathfrak{i} \text{ is transcendental in } \mathbb{F}_{\mathfrak{p}} \text{ over } {}^*\mathbb{Z}[\mathfrak{l}] \end{array} \quad (24)$$

3.4 Set

$$'i\mathbb{R}' := \left\{ \frac{\kappa}{\mathfrak{l}} \mathbf{v} : -m\mathfrak{l}/2 \leq \kappa \leq m\mathfrak{l}/2, m \in \mathbb{N} \right\} \subset \mathbb{U}_{\mathfrak{p}, \mathfrak{l}}$$

and let

$$'S' := \exp_{\mathfrak{p}}('i\mathbb{R}') \subset \mathbb{F}_{\mathfrak{p}}.$$

Since $\exp_{\mathfrak{p}}(\mathbf{v}) = 1$, $'S'$ is the group of all the elements γ of $\mathbb{F}_{\mathfrak{p}}^{\times}$ satisfying $\gamma^{\mathfrak{l}} = 1$.

3.5 Lemma. *Let $\gamma_1, \dots, \gamma_n \in 'S'$ be multiplicatively independent. Then $\gamma_1, \dots, \gamma_n$ are algebraically independent over $\mathbb{Q}(*\mathbb{Z}[\mathfrak{l}], \mathfrak{i})$.*

Proof. Suppose not. Then for some k , for some $c_1, \dots, c_k \in \mathbb{Q}(*\mathbb{Z}[\mathfrak{l}], \mathfrak{i})$, for some monomials $x_i = x_i(\gamma_1, \dots, \gamma_n)$, $i = 1, \dots, k$, the equality $\sum_{i=1}^k c_i \cdot x_i = 1$ holds. We assume that k is minimal with this property. Since $\gamma_1, \dots, \gamma_n$ are roots of 1 of order dividing \mathfrak{l} so are their products x_1, \dots, x_{k-1} . Since k is minimal, the solution $x_i(\gamma_1, \dots, \gamma_n)$, $i = 1, \dots, k$, of the equation is non-degenerate. Thus the $x_i(\gamma_1, \dots, \gamma_n)$ satisfy (23) and so are roots of unity of order $\leq f(k)$. This would contradict our assumption on multiplicative independence. \square

3.6 Corollary. *There is a place*

$$\mathcal{I} : \mathbb{Q}(*\mathbb{Z}[\mathfrak{l}], \mathfrak{i}, 'S') \rightarrow *\bar{\mathbb{C}}$$

which satisfies (19), (20) and (21).

Proof. Denote $\mathbb{Q}(*\mathbb{Z}[\mathfrak{l}], \mathfrak{i}, \sqrt[\infty]{1})$ the extension of $\mathbb{Q}(*\mathbb{Z}[\mathfrak{l}], \mathfrak{i})$ by roots of 1.

First define the place $\mathcal{I}_0 : \mathbb{Q}(*\mathbb{Z}[\mathfrak{l}], \sqrt[\infty]{1}) \rightarrow *\bar{\mathbb{C}}$ to be defined as the obvious embedding of the subfield of $F_{\mathfrak{p}}$ generated by $*\mathbb{Z}[\mathfrak{l}]$ and $\sqrt[\infty]{1}$ into $*\bar{\mathbb{C}}$, where $*\mathbb{Z}[\mathfrak{l}]$ is treated as both subset of $F_{\mathfrak{p}}$ and of $*\mathbb{C}$.

In case \mathfrak{i} satisfies the option $\mathfrak{i}^2 + 1 = \mathfrak{p}$ of (24), \mathfrak{i} is a square root of -1 in $F_{\mathfrak{p}}$ and so can be identified with an appropriate element of $\sqrt[\infty]{1}$ and thus included into the domain of \mathcal{I}_0 .

In the alternative option \mathfrak{i} is independent over the domain of \mathcal{I}_0 and so one can extend \mathcal{I}_0 to an \mathcal{I}_1 sending \mathfrak{i} to $\sqrt{-1}$ of $*\mathbb{C}$.

By Lemma 3.5 \mathcal{I}_1 can be extended to \mathcal{I} as required. \square

3.7 Let

$$'R' := \left\{ \frac{\kappa}{\mathfrak{l}} \mathbf{u} : -m\mathfrak{l}/2 \leq \kappa \leq m\mathfrak{l}/2, m \in \mathbb{N} \right\} \subset \mathbb{U}_{\mathfrak{p}, \mathfrak{l}},$$

a subgroup of $\mathbb{U}_{\mathfrak{p}, \mathfrak{l}}$,

and

$$'R'_+ := \exp_{\mathfrak{p}}('R') \subset F_{\mathfrak{p}},$$

a subgroup of $F_{\mathfrak{p}}^{\times}$.

Remark. Note that the definition of $\exp_{\mathfrak{p}}$ in 1.6 depends on the choice of the generator ϵ . So the above $'R'_+$ depends on ϵ as well (but $'S'$ does not).

3.8 Lemma. Let $k, l \in \mathbb{N}$ and $\alpha = \alpha(\epsilon) = \epsilon^{\frac{p-1}{li}}$. Let $\bar{M} = \langle M_1, \dots, M_k \rangle$ in ${}^*\mathbb{Z}^k$, $-ll < M_i < ll$.

Let $\bar{g}(\bar{Z}) = \langle g_1(Z_1, \dots, Z_k), \dots, g_k(Z_1, \dots, Z_k) \rangle$ be a k -tuple of rational functions over $\mathbb{Q}({}^*\mathbb{Z}[\mathfrak{l}], \mathfrak{i})$ and $\bar{c} = \langle c_1, \dots, c_k \rangle$, $c_i \in \mathbb{Q}(\mathbb{Z}[\mathfrak{l}], \mathfrak{i}, \mathfrak{S}')$ of the form

$$c_i = g_i(\bar{s}), \quad \bar{s} := \langle s_1, \dots, s_k \rangle, \quad \text{for } s_1, \dots, s_k \in \mathfrak{S}'.$$

Consider a non-standard Laurent polynomial in \mathbb{F}_p of the form

$$P_{\bar{c}, \bar{M}}(X) := \sum_{i=1}^k c_i X^{M_i} - 1. \quad (25)$$

Then there is a generator ϵ of \mathbb{F}_p such that for all \bar{M} and \bar{s} as above

$$P_{\bar{c}, \bar{M}}(\alpha) \neq 0.$$

Proof. For a given \bar{M} and \bar{c} , $P_{\bar{c}, \bar{M}}(X)$ has at most ll zeroes. There are at most $(ll)^k$ possible tuples $\bar{c} = \bar{g}(\bar{s})$ and $(ll)^k$ of tuples \bar{M} , so at most $(ll)^{2k}$ polynomials (25) altogether, so $(ll)^{2k+1}$ zeroes of the polynomials.

On the other hand $\alpha(\epsilon)$ takes any value of primitive root of order li as ϵ runs through generators of \mathbb{F}_p^\times . Thus there are $\varphi(li)$ (the Euler function) such α and the well-known lower estimate gives us $\varphi(li) > \sqrt{li}$, which is bigger than $(ll)^{2k+1}$ by (24). Thus there is an ϵ such as $\alpha(\epsilon)$ is as required. \square

3.9 Corollary. There is ϵ such that $\alpha(\epsilon)$ is not a zero of any $P_{\bar{c}, \bar{M}}(X)$ for all l, k, \bar{M} and $\bar{c} \in \mathbb{Q}({}^*\mathbb{Z}[\mathfrak{l}], \mathfrak{i}, \mathfrak{S}')^k$ as in 3.8.

Proof. Note that the conclusion of 3.8 can be restated as a formal statement $\exists \epsilon \Phi_{l, \bar{g}}(\alpha(\epsilon), \mathfrak{l}, \mathfrak{i}, \mathfrak{p})$ in the language of arithmetic. The statement of (3.8) readily generalises to the statement that α avoids zeroes of a finite number of polynomials of the form (25), by taking the product of the polynomials. Thus the type

$$\{\Phi_{l, \bar{g}}(\alpha(\epsilon), \mathfrak{l}, \mathfrak{i}, \mathfrak{p}) : l, \bar{g} \text{ as in (3.8)}\}$$

in variable ϵ is consistent. Since ${}^*\mathbb{Z}$ is \aleph_0 -saturated there is an ϵ such that $\alpha(\epsilon) \in \mathbb{F}_p$ is not a zero of any polynomial (25) \square

3.10 Lemma. There is a generator ϵ such that any multiplicatively independent $\gamma_1, \dots, \gamma_m \in \mathfrak{R}'_+$ are algebraically independent over $\mathbb{Q}({}^*\mathbb{Z}[\mathfrak{l}], \mathfrak{i}, \mathfrak{S}')$.

Proof. Let ϵ be as stated in 3.9. Suppose $\gamma_1, \dots, \gamma_m$ are multiplicatively independent and satisfy a polynomial equation $R(\gamma_1, \dots, \gamma_m) = 0$. The equation can be rewritten as

$$\sum_j c_j \gamma^{\bar{m}_j} = 1$$

for some monomials $\gamma^{\bar{m}_j} = \prod_i \gamma_i^{m_{ji}}$ and $c_j \in \mathbb{Q}(\mathbf{l}, \mathbf{i}, \mathbf{S}')$.

Clearly, $\gamma^{\bar{m}_j}$ belong to the group $'\mathbb{R}'_+$ and so $\gamma^{\bar{m}_j} = \alpha(\epsilon)^{M_j}$, for some M_j , $|M_j| < \mathcal{U}$, contradicting our choice of ϵ . \square

3.11 Corollary. *The place \mathcal{I} of 3.6 can be extended to*

$$\mathcal{I} : \mathbb{Q}(*\mathbb{Z}[\mathbf{l}], \mathbf{i}, \mathbf{S}', \mathbf{R}'_+) \rightarrow *\bar{\mathbb{C}}$$

which satisfies (22).

Proof. \mathcal{I} of (22) is an isomorphism of groups. Since the only relations in the language of rings between elements in $'\mathbb{R}'_+$ are multiplicative relations, \mathcal{I} is a place in the language of rings. \square

3.12 Since $*\bar{\mathbb{C}}$ is algebraically closed the \mathcal{I} of 3.11 can be extended to a place

$$\mathcal{I} : \mathbb{F}_p \rightarrow *\bar{\mathbb{C}}.$$

This finishes the proof of Theorem 3.1.

\square

3.13 Proof of the Main Theorem 1.7: (8) - (12).

Define

$$\text{Im}_F := \text{st} \circ \mathcal{I}.$$

In particular, taking into account that, for $\text{st} : *\bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$,

$$\text{st}(e^x) = e^{\text{st}(x)}$$

we get

$$\text{Im}_F : \epsilon^{\frac{1}{2i}\mathbf{u}} \mapsto e^{-\pi \text{st}(\frac{1}{i})}; \quad \epsilon^{\frac{1}{2i}\mathbf{v}} \mapsto e^{-i\pi \text{st}(\frac{1}{i})}.$$

It is easy to check that

$$'R' \cap 'iR' = \{0\}.$$

For $\frac{l}{\mathfrak{l}} \in {}^*\mathbb{Q}$ (non-standard rationals), such that $\frac{l}{\mathfrak{l}}\mathbf{u} \in {}'\mathbb{R}'$, define

$$\mathrm{lm}_{\mathbb{U}}\left(\frac{l}{\mathfrak{l}}\mathbf{u}\right) := -2\pi \mathrm{st}\left(\frac{l}{\mathfrak{l}}\right). \quad (26)$$

This is an additive homomorphism

$${}'\mathbb{R}' \rightarrow \mathbb{R}.$$

Then also $\frac{l}{\mathfrak{l}}\mathbf{v} \in {}'i\mathbb{R}'$, and define

$$\mathrm{lm}_{\mathbb{U}}\left(\frac{l}{\mathfrak{l}}\mathbf{v}\right) := -i2\pi \mathrm{st}\left(\frac{l}{\mathfrak{l}}\right), \quad (27)$$

an additive homomorphism

$${}'i\mathbb{R}' \rightarrow i\mathbb{R}.$$

This defines $\mathrm{lm}_{\mathbb{U}}$ on ${}'\mathbb{R}' + {}'i\mathbb{R}'$ respecting the commutation with \exp . Moreover,

$$\mathrm{lm}_{\mathbb{U}} : {}'\mathbb{R}' + {}'i\mathbb{R}' \rightarrow \mathbb{C}.$$

Note that the definition of the homomorphism $\mathrm{lm}_{\mathbb{U}}$ on ${}'\mathbb{R}'$ and ${}'i\mathbb{R}'$ above extends uniquely on their divisible hulls

$$\mathrm{lm}_{\mathbb{U}} : \frac{1}{n}{}'\mathbb{R}' \rightarrow \mathbb{R} \text{ and } \mathrm{lm}_{\mathbb{U}} : \frac{1}{n}{}'i\mathbb{R}' \rightarrow i\mathbb{R}, \quad \text{for } n \in \mathbb{N}$$

The sum of divisible hulls $H({}'\mathbb{R}') + H({}'i\mathbb{R}')$ is a divisible subgroups of $\mathbb{U}_{p,\mathfrak{l}}$ and so can be complemented by a subgroup $\mathbb{U}_{p,\mathfrak{l}}(\infty)$,

$$\mathbb{U}_{p,\mathfrak{l}} = H({}'\mathbb{R}') + H({}'i\mathbb{R}') + \mathbb{U}_{p,\mathfrak{l}}(\infty).$$

Define $\mathrm{lm}_{\mathbb{U}}(u) = \infty$ for all $u \in \mathbb{U}_{p,\mathfrak{l}}(\infty)$. Using \exp_p and \exp as in the commuting diagram (3) this can be extended to $\mathrm{lm}_{\mathbb{F}}$ so that (8), (9), (10), (11) and (12) are satisfied. \square

The rest of the Main Theorem will be proved in the next section.

4 Integration

Recall that $\mathfrak{l} = \mu^2$ and $\mathfrak{i} = \iota^2$ elements of ${}^*\mathbb{N}$.

4.1 Proposition. Let $a = \frac{d^2}{l^2}$ for some $d, l \in \mathbb{N}$.

$$\sum_{-il/2a \leq n \leq il/2a} \exp_{\mathfrak{p}}\left(-a \frac{n^2}{2l} \mathbf{u}\right) = \mu \frac{l\omega}{\sqrt{a}} \quad (28)$$

$$\sum_{-l/2a \leq n \leq l/2a} \exp_{\mathfrak{p}}\left(-a \frac{n^2}{2l} \mathbf{v}\right) = \mu \frac{\omega}{\sqrt{a}} \quad (29)$$

where $\omega \in \mathbb{F}_{\mathfrak{p}}$ is a primitive root of 1 of order 8. and μ, ι and d, l should be seen as elements of $\mathbb{F}_{\mathfrak{p}}$ corresponding to the respective integers.

Proof. Let $\nu \in {}^*\mathbb{N}$ be even and $2\nu^2 | (\mathfrak{p} - 1)$. Let $\xi \in \mathbb{F}_{\mathfrak{p}}$ be a primitive root of order $2\nu^2$, equivalently, for some $\epsilon \in \mathbb{F}_{\mathfrak{p}}$, a primitive root of order $\mathfrak{p} - 1$,

$$\xi = \epsilon^{\frac{\mathfrak{p}-1}{2\nu^2}}.$$

Writing $n = m\nu + k$, $0 \leq m, k < \nu$, we get

$$\sum_{0 \leq n < \nu^2} \xi^{n^2} = \sum_{0 \leq k < \nu} \xi^{k^2} \cdot \sum_{0 \leq m < \nu} \xi^{m^2\nu^2 + 2mk\nu}. \quad (30)$$

Now we use the fact that $\frac{m^2 - m}{2} \in {}^*\mathbb{Z}$ and get in $\mathbb{F}_{\mathfrak{p}}$

$$\begin{aligned} \sum_{0 \leq m < \nu} \xi^{m^2\nu^2 + 2mk\nu} &= \sum_{0 \leq m < \nu} \xi^{m\nu^2 + 2mk\nu} = \sum_{0 \leq m < \nu} \xi^{2m\nu(\frac{\nu}{2} + k)} = \\ &= \begin{cases} \nu, & \text{if } \frac{\nu}{2} + k \equiv 0 \pmod{\nu}, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

using that $\xi^{2m\nu(\frac{\nu}{2} + k)} = \xi^{2\nu^2} = 1$ in the first line and that

$$\sum_{0 \leq m < \nu} \zeta^m = 0,$$

for $\zeta := \xi^{2\nu(\frac{\nu}{2} + k)}$, $\zeta^{\nu} = 1$ but $\zeta \neq 1$, in the second line.

Hence in the sum (30) only $k = \frac{\nu}{2}$ contributes, and we get

$$\sum_{0 \leq n < \nu^2} \xi^{n^2} = \nu \xi^{\frac{\nu^2}{4}} = \nu \epsilon^{\frac{\mathfrak{p}-1}{8}} = \nu \cdot \omega \quad (31)$$

for $\omega := \epsilon^{\frac{\mathfrak{p}-1}{8}}$, a primitive root of 1 of order 8.

Note that

$$\sum_{0 \leq n < \nu^2} \xi^{n^2} = \sum_{-\nu^2/2 \leq n < \nu^2/2} \xi^{n^2}$$

because of periodicity

$$\xi^{(n+\nu^2)^2} = \xi^{n^2}.$$

Finally, set

$$\nu := \frac{\mu l}{\sqrt{a}} \text{ for (28) and } \nu := \frac{\mu}{\sqrt{a}} \text{ for (29)}$$

and (28) and (29) follow. \square

4.2 Corollary (proof of the Main Theorem, (13) and (14))

$$\text{lm}_F : \frac{1}{\mu} \sum_{-il/a \leq n < il/a} \exp_p\left(a \frac{n^2}{2l} \mathbf{u}\right) \mapsto \frac{1}{\sqrt{a}}$$

$$\text{lm}_F : \frac{1}{\mu} \sum_{-l/a \leq n < l/a} \exp_p\left(a \frac{n^2}{2l} \mathbf{v}\right) \mapsto \frac{e^{\frac{\pi i}{4}}}{\sqrt{a}}$$

This follows from the facts that $\text{lm}_F(l) = e^{-\frac{\pi i}{4}}$ and that $\text{lm}_F(\epsilon^{\frac{p-1}{8}}) = e^{\frac{\pi i}{4}}$, see (26) and (27) together with Theorem 3.1.

4.3 Note that $\frac{1}{\sqrt{a}} = \int_{\mathbb{R}} e^{-ax^2} dx$, the classical Gaussian integral. Analogously, in quantum mechanics $\int_{\mathbb{R}} e^{iax^2} dx := \frac{e^{\frac{\pi i}{4}}}{\sqrt{a}}$, although the integral is not classically defined since e^{iax^2} is oscillating on the whole of \mathbb{R} . One of the ways of justifying the assignment of the value to the integral expression is by referring to the fact that the respective Fresnel integral $\int_{-A}^A e^{iax^2} dx$ is well-defined for any $A > 0$ and

$$\lim_{A \rightarrow \infty} \int_{-A}^A e^{iax^2} dx = \frac{e^{\frac{\pi i}{4}}}{\sqrt{a}}.$$

4.4 The domains of integrations and domains of summation.

Let, for $l \in \mathbb{N}$

$$I_l = \{n \in {}^*\mathbb{Z} : -l\mu \leq n \leq l\mu\} \text{ and } I = \bigcup_{l \in \mathbb{N}} I_l.$$

Let $a = \frac{m}{l}$ and

$$I_{a,\mathbf{u}} = \{n \in {}^*\mathbb{Z} : -il/2a \leq n \leq il/2a\} \text{ and } I_{a,\mathbf{v}} = \{n \in {}^*\mathbb{Z} : -l/2a \leq n \leq l/2a\}$$

the domains of summations of (28) and (29). Clearly, using the assumptions on \mathfrak{l} and μ ,

$$\mathfrak{I} \subset \mathfrak{I}_{a,\mathbf{u}} \text{ and } \mathfrak{I} \subset \mathfrak{I}_{a,\mathbf{v}}$$

and

$$\frac{\mathbf{u}}{\mu} \cdot \mathfrak{I} \subset {}'\mathbb{R}' \text{ and } \frac{\mathbf{v}}{\mu} \cdot \mathfrak{I} \subset {}'i\mathbb{R}'.$$

The application of $\text{lm}_{\mathbb{U}}$ defined in 3.13 gives us

$$\text{lm}_{\mathbb{U}} : \frac{\mathbf{u}}{\mu} \cdot \mathfrak{I}_l \rightarrow \mathbb{R} \cap [-l\pi, l\pi] \text{ and } \frac{\mathbf{u}}{\mu} \cdot \mathfrak{I} \rightarrow \mathbb{R},$$

$$\text{lm}_{\mathbb{U}} : \frac{\mathbf{v}}{\mu} \cdot \mathfrak{I}_l \rightarrow i(\mathbb{R} \cap [-l\pi, l\pi]) \text{ and } \frac{\mathbf{v}}{\mu} \cdot \mathfrak{I} \rightarrow i\mathbb{R},$$

that is $\frac{\mathbf{u}}{\mu} \cdot \mathfrak{I}$ can be seen as a Riemann integration partition of sets $'\mathbb{R}'$ with the infinitesimal mesh $\frac{\mathbf{u}}{\mu}$, and respectively $\frac{\mathbf{v}}{\mu} \cdot \mathfrak{I}$ in $i\mathbb{R}$ with mesh $\frac{\mathbf{v}}{\mu}$.

Note that in (28) and (29)

$$\frac{n^2}{2\mathfrak{l}} = \frac{1}{2} \left(\frac{n}{\mu}\right)^2; \quad \text{lm}_{\mathbb{U}}\left(\frac{n^2}{2\mathfrak{l}} \mathbf{u}\right) = -\pi \frac{x^2}{2}, \quad \text{lm}_{\mathbb{U}}\left(\frac{n^2}{2\mathfrak{l}} \mathbf{v}\right) = -i\pi \frac{x^2}{2}$$

for $x = \text{st}\left(\frac{n}{\mu}\right)$.

4.5 Lemma. *For every $l \in \mathbb{N}$*

$$\text{lm}_{\mathbb{F}} : \frac{1}{\mu} \sum_{n \in \mathfrak{I}_l} \exp_{\mathbb{p}}\left(-a \frac{n^2}{2\mathfrak{l}} \mathbf{u}\right) \mapsto \int_{-l}^l e^{-ax^2} dx$$

$$\text{lm}_{\mathbb{F}} : \frac{1}{\mu} \sum_{n \in \mathfrak{I}_l} \exp_{\mathbb{p}}\left(-a \frac{n^2}{2\mathfrak{l}} \mathbf{v}\right) \mapsto \int_{-l}^l e^{-iax^2} dx$$

and the integrals are well-defined.

Proof. Note that by (21) and (22) the map \mathcal{I} translates the respective elements of the sums from $\mathbb{F}_{\mathbb{p}}$ to the elements of the non-standard model ${}^*\mathbb{C}$ of complex numbers:

$$\mathcal{I} : \exp_{\mathbb{p}}\left(-a \frac{n^2}{2\mathfrak{l}} \mathbf{u}\right) \mapsto e^{-a\pi\left(\frac{n}{\mu}\right)^2}$$

$$\mathcal{I} : \exp_{\mathbb{p}}\left(-a \frac{n^2}{2\mathfrak{l}} \mathbf{v}\right) \mapsto e^{-ia\pi\left(\frac{n}{\mu}\right)^2}$$

and thus

$$\mathcal{I}\left\{\frac{1}{\mu} \sum_{n \in I_l} \exp_{\mathfrak{p}}\left(-a\pi \frac{n^2}{2l} \mathbf{u}\right)\right\} \text{ and } \mathcal{I}\left\{\frac{1}{\mu} \sum_{n \in I_l} \exp_{\mathfrak{p}}\left(-a\pi \frac{n^2}{2l} \mathbf{v}\right)\right\}$$

become non-standard Riemann sums with infinitesimal mesh $\frac{1}{\mu}$

$$\sum_{-l < \frac{n}{\mu} < l} \frac{1}{\mu} e^{-a\pi \left(\frac{n}{\mu}\right)^2} \text{ and } \sum_{-l < \frac{n}{\mu} < l} \frac{1}{\mu} e^{-ia\pi \left(\frac{n}{\mu}\right)^2}$$

Since the summation is over a compact interval $[-l, l] \subset {}^*\mathbb{R}$ the application of the standard part map gives us (see e.g. the integration via non-standard analysis in [12])

$$\text{st}\left(\sum_{-l < \frac{n}{\mu} < l} \frac{1}{\mu} e^{-a\pi \left(\frac{n}{\mu}\right)^2}\right) = \int_{-l}^l e^{-a\pi x^2} dx \text{ and } \text{st}\left(\sum_{-l < \frac{n}{\mu} < l} \frac{1}{\mu} e^{-ia\pi \left(\frac{n}{\mu}\right)^2}\right) = \int_{-l}^l e^{-ia\pi x^2} dx$$

4.6 Corollary.

$$\begin{aligned} \text{lm}_{\mathbb{F}} : \frac{1}{\mu} \sum_{n \in \mathbb{I}} \exp_{\mathfrak{p}}\left(-a \frac{n^2}{2l} \mathbf{u}\right) &\mapsto \int_{\mathbb{R}} e^{-ax^2} dx = \frac{1}{\sqrt{a}} \\ \text{lm}_{\mathbb{F}} : \frac{1}{\mu} \sum_{n \in \mathbb{I}} \exp_{\mathfrak{p}}\left(-a \frac{n^2}{2l} \mathbf{v}\right) &\mapsto \lim_{l \rightarrow \infty} \int_{-l}^l e^{-iax^2} dx = \frac{e^{\frac{\pi}{4}}}{\sqrt{a}} \end{aligned}$$

Proof. In both cases the right-hand side is the limit of integrals in 4.5 since $\mathbb{I} = \bigcup_l I_l$. In the first case the classical Gaussian integral over the whole of \mathbb{R} converges and is equal to the limit. In the second case the right-hand side as the limit of the Fresnel integral is well-defined but the Riemann integral is not.

4.7 Discussion. The left-hand sides of 4.2 and of 4.6 differ in the domains of summations but the right-hand sides are the same. This implies that our definition of $\text{lm}_{\mathbb{F}}$ is such that

$$\text{lm}_{\mathbb{F}} \left(\frac{1}{\mu} \sum_{n \in I_{a, \mathbf{u}} \setminus \mathbb{I}} \exp_{\mathfrak{p}}\left(-a \frac{n^2}{2l} \mathbf{u}\right) \right) = 0 \text{ and } \text{lm}_{\mathbb{F}} \left(\frac{1}{\mu} \sum_{n \in I_{a, \mathbf{v}} \setminus \mathbb{I}} \exp_{\mathfrak{p}}\left(-a \frac{n^2}{2l} \mathbf{v}\right) \right) = 0$$

In these “tail domains” $I_{a, \mathbf{u}} \setminus \mathbb{I}$ and $I_{a, \mathbf{v}} \setminus \mathbb{I}$ the respective values under exponentiation are very large, non-feasible numbers, and according to the interpretation of $F_{\mathfrak{p}}$ in \mathbb{C} the application of \exp to such values oscillates uncontrollably. R.Feynman intuition was that for this reason the sum should be considered negligible, see e.g. [13], 2-3.

References

- [1] B.Zilber, *Perfect infinities and finite approximation*, In: **Infinity and Truth**. IMS Lecture Notes Series, V.25, 2014
- [2] J.A.Cruz Morales and B.Zilber, *The geometric semantics of algebraic quantum mechanics*, Phil. Trans. R. Soc. A 2015 373 20140245; DOI: 10.1098/rsta.2014.0245. Published 29 June 2015
- [3] B.Zilber, *The semantics of the canonical commutation relations* arxiv1604.07745 (see also a 2016 version on author's webpage)
- [4] B.Zilber, *Structural approximation and quantum mechanics*, 2017, author's web-page
- [5] B. Dragovich¹, A. Khrennikov³, S. Kozyrev, I. Volovich and E. Zelenov *p-Adic Mathematical Physics: The First 30 Years*, P-Adic Num. Ultramet. Anal. Appl. 9, 87 - 121, 2017.
- [6] C.Yang and T.Lee, *Statistical Theory of Equations of State and Phase Transition I. Theory of condensation*, Phys.Rev. v87, 3, p.404, 1952
- [7] C.Yang and T.Lee, *Statistical Theory of Equations of State and Phase Transition II. Lattice Gas and Ising Model*, Phys.Rev. v87, 3, p.410, 1952
- [8] D.Tong, **Statistical Physics**, Preprint, Cambridge, <https://www.damtp.cam.ac.uk/user/tong/statphys.html>
- [9] V.Sazonov, *On feasible numbers*, **International Workshop on Logic and Computational Complexity**, LCC 1994: Logic and Computational Complexity, pp 30 - 51
- [10] P. Kustaanheimo, **On the fundamental prime of the finite world**, Annales Academiae scientiarum Fennicae. Series A. 1, Mathematica-Physica, 129, 1952
- [11] E. Welti, **Die Philosophie des strikten Finitismus**. Entwicklungstheoretische und mathematische Untersuchungen ber Unendlichkeitsbegriffe in Ideengeschichte und heutiger Mathematik, Bern: Peter Lang, 1987.

- [12] A. Robinson, **Non-standard analysis**. North-Holland Publishing Co., Amsterdam 1966.
- [13] R.Feynman and A.Hibbs, **Quantum Machanics and Path Integrals**, Dover Publ., 2005
- [14] J.-H.Evertse *The number of solutions of linear equations in roots of unity*, Acta Arithmetica (1999) v89, 1, pp. 45 - 51