Sections and towers

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Abstract

We discuss the towers of finite étale covers which were essentially introduced by A.Tamagawa [5] and used e.g. in [4]. The statement about correspondence between sections and cofinal towers is a folklore but perhaps not in a very explicit form. The last section explains how the "injectivity statement" of Grothendieck section conjecture fails for abelian varieties, which is also known in some form from [2].

The paper is based on [1] which was aimed to reinterpret anabelian setting in model theory terms.

1 A short overview of structure $\tilde{\mathbf{X}}^{et}$

We start with an overview of the key structure introduced and studied in [1]. It is essentially the projective object - the Grothendieck universal étale cover of a smooth k-variety $\mathbb{X}$.

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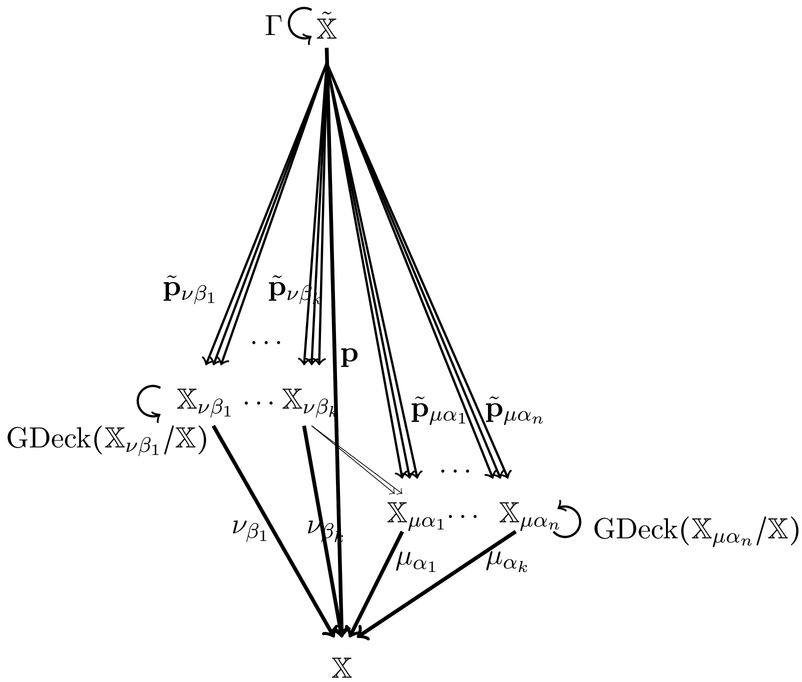

The diagram for $\tilde{\mathbf{X}}^{et}$.

Explaining the picture (see [1], 7.1-7.3 and Corollary 7.11)

1.1 All arrow diagrams commute.

1.2 Each $\mathbb{X}_{\nu,\beta_i}$ is an absolutely irreducible variety over the field $\mathrm{k}[\beta_1] = ... = \mathrm{k}[\beta_k]$, a Galois extension of k. $\mathbb{X}_{\nu,\beta_i}(\mathrm{k}^{alg})$ is the set of its k^{alg}-points, a subset of a projective space.

1.3 Each ν_{β_i} is an étale covering map $\nu_{\beta_i} : \mathbb{X}_{\nu,\beta_i}(\mathrm{k}^{alg}) \to \mathbb{X}(\mathrm{k}^{alg})$.

1.4 $\tilde{\mathbb{X}}(\mathrm{k}^{alg})$ is a set with the regular action of a group Γ.

1.5 Each $\tilde{\mathbf{p}}_{\nu\beta_i}$ is a finite collection of surjective **maps**

$$p : \tilde{\mathbb{X}}(\mathrm{k}^{alg}) \to \mathbb{X}_{\nu\beta_i}(\mathrm{k}^{alg}).$$

In particular, if $\mathbb{X}_{\nu\beta_i}(\mathrm{k}^{alg}) = \mathbb{X}_{\mu\alpha_j}(\mathrm{k}^{alg})$ and $\nu_{\beta_i} = \mu_{\alpha_j}$ then $\tilde{\mathbf{p}}_{\nu\beta_i} = \tilde{\mathbf{p}}_{\mu\alpha_j}$. In case $\mathbb{X}_{\nu,\beta_i}(\mathrm{k}^{alg}) = \mathbb{X}(\mathrm{k}^{alg})$ the collection $\tilde{\mathbf{p}}_{\nu\beta_i}$ consists of one map $\mathbf{p}$.

1.6 Suppose there is a morphism $(\mu_\alpha^{-1}\nu_\beta) : \mathbb{X}_{\nu,\beta}(\mathrm{k}^{alg}) \to \mathbb{X}_{\mu,\alpha}(\mathrm{k}^{alg})$ of étale covers (see notation in [1], section 4) . Then for every $p \in \tilde{\mathbf{p}}_{\mu,\alpha}$ there is $q \in \tilde{\mathbf{p}}_{\nu,\beta}$ such that

$$(\mu_\alpha^{-1}\nu_\beta) = p \circ q^{-1}, \tag{1}$$

and for every $q \in \tilde{\mathbf{p}}_{\nu,\beta}$ there is $p \in \tilde{\mathbf{p}}_{\mu,\alpha}$ such that (1) holds.

1.7 Given $p \in \tilde{\mathbf{p}}_{\nu\beta_i}$

$$\tilde{\mathbf{p}}_{\nu\beta_i} = \{g \circ p : g \in \mathrm{GDeck}(\mathbb{X}_{\nu\beta_i}/\mathbb{X})\}$$

where $\mathrm{GDeck}(\mathbb{X}_{\nu\beta_i}/\mathbb{X})\}$ is the geometric deck-transformation group.

1.8 The fibres of $\mathbf{p}$ are Γ-orbits. The fibres of $p \in \tilde{\mathbf{p}}_{\nu,\beta_i}$ are orbits by a finite index normal subgroup Δ_{ν,β_i} of Γ.

$$\mathrm{GDeck}(\mathbb{X}_{\nu\beta_i}/\mathbb{X}) \cong \Gamma/\Delta_{\nu,\beta_i}.$$

1.9 For each finite collection $\mathbb{X}_{\lambda_1\gamma_1}, \dots \mathbb{X}_{\lambda_m\gamma_m}$, there is a ν_β such that

$$\Delta_{\nu,\beta} \le \bigcap_{0<j\le m} \Delta_{\lambda_j,\gamma_j}.$$

1.10

$$\bigcap_{\text{all } \nu,\beta} \Delta_{\nu,\beta} = \{1\}.$$

1.11

$$\mathrm{Aut}\,\tilde{\mathbf{X}}^{et}(\mathrm{k}^{alg}) \cong \pi_1^{et}(\mathbb{X}, x)$$

2 Sections and towers

2.1 Let

$$T(\mathbb{X}) : \quad \mathbb{X} \leftarrow \mathbb{X}_1 \leftarrow \mathbb{X}_2 \leftarrow \dots \mathbb{X}_i \leftarrow \mathbb{X}_{i+1} \leftarrow \dots$$

be a tower of smooth complex algebraic varieties and unramified covers, all defined over k. Let

$$\Gamma_{T(\mathbb{X})} = \lim_{\leftarrow} \mathrm{GDeck}(\mathbb{X}_i/\mathbb{X}).$$

We call the tower **cofinal** if

$$\Gamma_{T(\mathbb{X})} \cong \hat{\pi}_1^{top}(\mathbb{X})$$

as profinite groups.

2.2 Proposition. *Given* $\mathbb{X}$ *there is a cofinal chain*

$$\Gamma > \Delta_1 > \dots \Delta_i > \Delta_{i+1} > \dots$$

of $\mathrm{Aut}\tilde{\mathbf{X}}^{et}(\mathbb{F})$*-invariant normal finite index subgroups of* $\hat{\pi}_1^{top}(\mathbb{X}) = \Gamma$.

Given a section s *and a cofinal chain* $\{\Delta_i\}$ *of* $\mathrm{Aut}\tilde{\mathbf{X}}^{et}(\mathbb{F})$*-invariant normal finite index subgroups of* Γ *there exists a tower* $T_s(\mathbb{X})$ *over* k *such that*

$$\mathrm{GDeck}(\mathbb{X}_i/\mathbb{X}) \cong \Gamma/\Delta_i.$$

Proof. Let $\Gamma := \hat{\pi}_1^{top}(\mathbb{X})$. $s\mathrm{Gal}_{\mathrm{k}}$ acts on Γ since group Γ is definable in $\tilde{\mathbf{X}}^{et}$, In particular, $s\mathrm{Gal}_{\mathrm{k}}$ acts on the set of all finite index subgroups.

Claim 1. There exists a decreasing sequence $\{\Delta_n : n \in \mathbb{N}\}$ (depending on $\mathbb{X}$ only) of $\mathrm{Aut}\tilde{\mathbf{X}}^{et}(\mathbb{F})$-invariant normal subgroups of Γ of finite index with $\bigcap_n \Delta_n = \{1\}$.

Proof. For each $\mu \in \mathcal{M}_{\mathbb{X}}$ consider the subgroup $\Delta_\mu < \Gamma$

$$\Delta_\mu = \{\gamma \in \Gamma : \forall p \in \tilde{\mathbf{p}}_\mu \forall u \in \mathbb{U}\ p^\gamma(u) = p(u)\}$$

where p^γ is the map $u \mapsto p(\gamma \cdot u)$.

By 4.15 of [1]

$$\Delta_\mu = \bigcup_{\alpha \in \mathrm{Zeros}\mathbf{f}_\mu} \Delta_{\mu,\alpha},$$

the intersection of subgroups of periods of the maps $p : \mathbb{U} \to \mathbb{X}_{\mu,\alpha}$ which are finite index. Hence Δ_μ is of finite index in Γ. It also follows that the intersection of the Δ_μ is trivial. It remains to choose a linearly ordered cofinal subset $\Delta_n : n \in \mathbb{N}$ in $\Delta_\mu : \mu \in \mathcal{M}_{\mathbb{X}}$. Claim proved.

Let

$$\mathbb{U}_n := \Delta_n \backslash \mathbb{U},\ \ \bar{\mathbf{p}}_n : \mathbb{U} \to \mathbb{X},\ \ \bar{\mathbf{p}}_{n,m} : \mathbb{U}_n \to \mathbb{U}_m \tag{2}$$

where $\bar{\mathbf{p}}_n$ is the covering map induced by $\mathbf{p} : \mathbb{U} \to \mathbb{X}$ on $\mathbb{U}_n$ (recall that fibres of $\mathbf{p}$ are Γ-orbits) and $\bar{\mathbf{p}}_{n,m}$ is the map induced by the embedding $\Delta_n \le \Delta_m$.

Note that the $\mathbb{U}_n$, $\bar{\mathbf{p}}_n$ and $\bar{\mathbf{p}}_{n,m}$ are $\mathrm{Aut}\tilde{\mathbf{X}}^{et}(\mathbb{F})$-invariant and so the action of $s\mathrm{Gal}_{\mathrm{k}}$ on $\mathbb{U}$ induces the action on the tower

$$\mathbb{U}_1 \leftarrow \mathbb{U}_2 \leftarrow \dots$$

Claim 2. The $\mathbb{U}_n$ can be given structure of smooth projective algebraic varieties defined over k.

Proof. By the argument in the proof of Claim 1, $\Delta_n = \Delta_{\mu,\alpha} = \operatorname{Per} p$, for some μ_α, $p : \mathbb{U} \to \mathbb{X}_{\mu,\alpha}$. Set $p_n : \mathbb{U}_n \to \mathbb{X}_{\mu,\alpha}$ be the bijective map induced by p on $\mathbb{U}_n$. We may assume that the set $\mathbb{X}_{\mu,\alpha}(\mathbb{F})$ and the map p are $s\mathrm{Gal}_{\mathrm{k}[\alpha]}$-invariant, by possibly extending $\mathrm{k}[\alpha]$ without changing the set and the map. Call $i_{n,\alpha}$ the map $p_n^{-1} : \mathbb{X}_{\mu,\alpha}(\mathbb{F}) \to \mathbb{U}(\mathbb{F})$. Note that by applying Galois conjugation we obtain a finite family

$$\{i_{n,\alpha} : \mathbb{X}_{\mu,\alpha}(\mathbb{F}) \to \mathbb{U}(\mathbb{F});\ \alpha \in \mathrm{Zeros}\mathbf{f}_\mu\}$$

of bijections.

Let

$$\mathbb{Y}_n = \{\langle x, \alpha\rangle : x \in \mathbb{X}_{\mu,\alpha}\ \&\ \alpha \in \mathrm{Zeros}\mathbf{f}_\mu\}$$

the disjoint union of $\mathrm{k}[\alpha]$-varieties isomorphic to $\mathbb{X}_{\mu,\alpha}$. Let $i_n : \mathbb{Y} \to \mathbb{U}_n$ be the surjective map defined as

$$i_n(y) = u \leftrightarrow \exists\alpha\exists x \in \mathbb{X}_{\mu,\alpha}\ y = \langle x, \alpha\rangle\ \&\ i_{n,\alpha}(x) = u.$$

By construction $\mathbb{Y}_n$ and i_n are $\mathrm{Gal}_{\mathrm{k}}$-invariant.

Let G be the group $\mathrm{Gal}(\mathrm{k}[\alpha] : \mathrm{k})$ (recall that by our assumptions $\mathrm{k}[\alpha] : \mathrm{k}$ is Galois. For each $u \in \mathbb{U}_n$, define the action of G on $i_n^{-1}(u)$. Note that by construction

$$i_n^{-1}(u) = \{\langle x_\alpha, \alpha\rangle :\ \alpha \in \mathrm{Zeros}\mathbf{f}_\mu\}$$

for some $x_\alpha \in \mathbb{X}_{\mu,\alpha}$. For $\sigma \in G$ set

$$\sigma : \langle x_\alpha, \alpha\rangle \mapsto \langle x_{\sigma(\alpha)}, \sigma(\alpha)\rangle.$$

By construction $G\backslash\mathbb{Y}_n$ is in bijective k-definable correspondence with $i_n(\mathbb{Y}_n)$ that is with $\mathbb{U}_n$, that is

$$\mathbb{U}_n \cong G\backslash\mathbb{Y}_n.$$

The object on the right is the quotient of smooth projective variety (reducible, in general) by a regular action of a finite group. Hence $G\backslash\mathbb{Y}_n$ is isomorphic[1] to a smooth projective variety $\mathbb{X}_n$ over k via a surjective map $t_n : \mathbb{Y}_n \to \mathbb{X}_n$ with fibres which are G-orbits. Thus there is a $s\mathrm{Gal}_{\mathrm{k}}$-invariant bijective map onto the k-variety

$$\mathbf{i}_n : \mathbb{U}_n \to \mathbb{X}_n.$$

[1] Reference?

Claim proved.

Note that $t_{n,\alpha}$, the restriction of t_n to $\mathbb{X}_{\mu,\alpha} \times \{\alpha\}$, a component of $\mathbb{Y}_n$, is a biregular isomorphism $t_{n,\alpha} : \langle x, \alpha \rangle \mapsto G \cdot \langle x, \alpha \rangle$ on $\mathbb{X}_n$ defined over $\mathrm{k}[\alpha]$. Consider the map

$$t'_{n,\alpha} : x \mapsto G \cdot \langle x, \alpha \rangle, \quad \mathbb{X}_{\mu,\alpha} \to \mathbb{X}_n$$

which for simplicity of notation we call $t_{n,\alpha}$ as well. By construction

$$t_{n,\alpha} \circ p_n = \mathbf{i}_n.$$

Define $\mathbf{j}_{n,m} : \mathbb{X}_n \to \mathbb{X}_m$ to be $\mathbf{j}_{n,m} = \mathbf{i}_m \circ \bar{\mathbf{p}}_{nm} \circ \mathbf{i}_n^{-1}$. This is definable over k since $\mathbf{i}_m$, $\bar{\mathbf{p}}_{nm}$ and $\mathbf{i}_n$ are $s\mathrm{Gal}_{\mathrm{k}}$-invariant. This is also a Zariski regular map since by above

$$\mathbf{j}_{n,m} = t_{m,\beta} \circ p_m \circ \bar{\mathbf{p}}_{n,m} \circ p_n^{-1} \circ t_{n,\alpha}^{-1} = t_{m,\beta} \circ (\nu_\beta^{-1} \mu_\alpha) \circ t_{n,\alpha}^{-1}$$

where $(\nu_\beta^{-1} \mu_\alpha) : \mathbb{X}_{\mu,\alpha} \to \mathbb{X}_{\nu,\beta}$ is an intermediate regular map which can be presented as $p_m \circ \bar{\mathbf{p}}_{n,m} \circ p_n^{-1}$.

This gives us the cofinal tower

$$T_s(\mathbb{X}) : \ \mathbb{X} \leftarrow \mathbb{X}_1 \leftarrow \mathbb{X}_2 \leftarrow \ldots \mathbb{X}_i \leftarrow \mathbb{X}_{i+1} \leftarrow \ldots$$

where the arrows $\mathbb{X}_{i+1} \to \mathbb{X}_i$ stand for the regular maps $\mathbf{j}_{i+1,i}$. □

2.3 Corollary (of the proof). *Given s and the tower $\{\Delta_i : i \in \mathbb{N}\}$ of (2) the tower $T_s(\mathbb{X})$ is determined uniquely up to isomorphism over* k.

The system of bijections $\mathbf{i}_i$

$$\mathbb{X} \leftarrow \mathbb{U}_1 \leftarrow \mathbb{U}_2 \leftarrow \ldots \mathbb{U}_i \leftarrow \mathbb{U}_{i+1} \leftarrow \ldots$$

$$\downarrow \mathbf{i} \ \ \downarrow \mathbf{i}_1 \ \ \ \downarrow \mathbf{i}_2 \ldots \ \ \downarrow \mathbf{i}_i \ldots \downarrow \mathbf{i}_{i+1} \ldots$$

$$\mathbb{X} \leftarrow_{j_1} \mathbb{X}_1 \leftarrow_{j_2} \mathbb{X}_2 \leftarrow \ldots \mathbb{X}_i \leftarrow_{j_{i+1}} \mathbb{X}_{i+1} \leftarrow_{j_{i+2}} \ldots$$

furnishes isomorphism between the structure on the tower of the $\mathbb{U}_i$ induced by the action of $s\mathrm{Gal}_{\mathrm{k}}$ and the tower $T_s(\mathbb{X})$.

Given any other such $s\mathrm{Gal}_{\mathrm{k}}$-invariant tower

$$T'_s(\mathbb{X}) : \ \mathbb{X} \leftarrow \mathbb{X}'_1 \leftarrow \mathbb{X}'_2 \leftarrow \ldots \mathbb{X}'_i \leftarrow \mathbb{X}'_{i+1} \leftarrow \ldots$$

with covering maps $\mathbf{j}'_{i+1} : \mathbb{X}_{i+1} \to \mathbb{X}_i$ there are isomorphism $q_i : \mathbb{X}_i \to \mathbb{X}'_i$ over k *such that*

$$q_i \circ \mathbf{j}_{i+1} = \mathbf{j}'_{i+1} \circ q_{i+1}.$$

2.4 Proposition. *Let*

$$\mathcal{T}(\mathbb{X}) = \{T(\mathbb{X}) : \{\Delta_i : i \in \mathbb{N}\} - \textit{towers}\}$$

the set of all $\{\Delta_i : i \in \mathbb{N}\}$*- towers over* k.[2] *Let*

$$\mathcal{S}(\mathbb{X}) = \{s : \mathrm{Gal}_{\mathrm{k}} \to \mathrm{Aut}\,\tilde{\mathbf{X}}^{et}(\mathrm{k}^{alg})\}$$

the set of all sections of $\mathrm{pr} : \mathrm{Aut}\,\tilde{\mathbf{X}}^{et}(\mathrm{k}^{alg}) \to \mathrm{Gal}_{\mathrm{k}}$.

Then the map

$$s \mapsto T_s(\mathbb{X})$$

induces a bijection

$$\mathcal{S}(\mathbb{X})_{/conj} \to \mathcal{T}(\mathbb{X})_{/iso}$$

between the set of section modulo conjugation and the set of towers modulo isomorphisms over k.

Proof. The map $s \mapsto T_s(\mathbb{X})_{/\mathrm{iso}}$ is constructed above, see 2.3. We construct the inverse map

$$T(\mathbb{X})_{/\mathrm{iso}} \mapsto s_{/\mathrm{conj}};\quad \mathcal{T}(\mathbb{X})_{/\mathrm{iso}} \to \mathcal{S}(\mathbb{X})_{/\mathrm{conj}}.$$

Let $T(\mathbb{X})$ be a $\mathrm{Gal}_{\mathrm{k}}$-invariant $\{\Delta_i : i \in \mathbb{N}\}$- tower. By the construction of $\tilde{\mathbf{X}}^{et}$ the tower can be embedded into $\tilde{\mathbf{X}}^{et}$, that is $\mathbb{X}_i = \mathbb{X}_{\mu_i,\alpha_i}$ for some $\mu_i \in \mathcal{M}_{\mathbb{X}}$, $\alpha_i \in \mathbf{f}_{\mu_i}$ and the $\mathbf{j}_{i+1}$ are appropriate intermediate morphisms. Since the tower is over k we can drop α_i. We also write i for μ_i.

Now we consider the respective sets of covering maps $p : \mathbb{U} \to \mathbb{X}_i$, $p \in \tilde{\mathbf{p}}_i$ for each $i \in \mathbb{N}$.

Claim 1. There is a sequence $\mathbf{p}_i \in \tilde{\mathbf{p}}_i, i \in \mathbb{N}$ of covering maps $\mathbf{p}_i : \mathbb{U} \to \mathbb{X}_i$ such that

$$\mathbf{j}_{i+1} \circ \mathbf{p}_{i+1} = \mathbf{p}_i, \text{ all } i \in \mathbb{N}. \tag{3}$$

Proof. By induction. For $i = 0$, set $\mathbb{X}_0 := \mathbb{X}$ and $\mathbf{p}_0 := \mathbf{p}$. Suppose $\mathbf{p}_n$, $n \le i$ have been constructed satisfying the requirement. We can choose $\mathbf{p}_{i+1}$ by property 1.6. Claim proved

Claim 2. Suppose $\{\mathbf{p}'_i : i \in \mathbb{N}\}$ is another sequence satisfying (3). Then there is $\gamma \in \Gamma$ such that

$$\mathbf{p}'_i = \mathbf{p}_i^{\gamma}, \text{ all } i \in \mathbb{N},$$

[2] That is $\mathrm{GDeck}(\mathbb{X}_i/\mathbb{X}) \cong \Gamma/\Delta_i$. Have to assume here that the tower $\mathrm{GDeck}(\mathbb{X}_l/\mathbb{X})$ has unique, up to isomorphism of Γ, presentation in the form Γ/Δ_i.

where $\mathbf{p}_i^\gamma(u) \coloneqq \mathbf{p}_i(\gamma \cdot u)$ for all $u \in \mathbb{U}$.

Proof. Choose $u \in \mathbb{U}$ and set $u_i \coloneqq \mathbf{p}_i(u)$. First we prove that for each $n \in \mathbb{N}$ there exists $u' \in \mathbb{U}$ such that $\mathbf{p}'_i(u') = u_i$ for all $i \leq n$. And for that it is enough to find u' such that $\mathbf{p}'_n(u') = u_n$, since then

$$\mathbf{p}'_{n-1}(u') = \mathbf{j}_n(\mathbf{p}'_n(u')) = u_{n-1}, \dots \mathbf{p}'_{n-2}(u') = \dots$$

Note that $u' = u$ when $\mathbf{p}'_0 = \mathbf{p} = \mathbf{p}_0$.

By induction we assume that $\mathbf{p}'_n(u') = u_n$ and need to find u'' such that $\mathbf{p}'_{n+1}(u'') = u_{n+1}$. Note that by (3) $\mathbf{j}_{n+1}(\mathbf{p}'_{n+1}(u')) = u_n$ and so

$$\mathbf{p}'_{n+1}(u') = g \cdot u_{n+1} \text{ for some } g \in \mathrm{GDeck}(\mathbb{X}_{n+1}/\mathbb{X}_n).$$

We can find $\gamma \in \Gamma$ such that

$$\mathbf{p}'_{n+1}(\gamma^{-1}u') = g^{-1}\mathbf{p}'_{n+1}(u') = u_{n+1}.$$

Hence $u'' = \gamma^{-1}u'$ satisfies the required.

Since the structure $\tilde{\mathbf{X}}^{et}$ is compact in the profinite topology there is an u' which satisfies $\mathbf{p}'_i(u') = u_i$ for all $i \in \mathbb{N}$. Clealrly, u and u' are in the same fibre of $\mathbf{p}$ and thus $u' = \gamma \cdot u$ for some $\gamma \in \Gamma$. Hence

$$\mathbf{p}'_i(u) = \mathbf{p}_i^\gamma(u) = g_i \cdot \mathbf{p}_i(u).$$

It follows[3] that the equality holds for all u. Claim proved.

Claim 3. Any two sequences $\{\mathbf{p}_i\}$ and $\{\mathbf{p}'_i\}$ satisfying (3) satisfy the same type over the sort $\mathbb{F}$.

Proof. By Claim 2 the sequence are conjugated by an element of $\gamma \in \Gamma$. By construction the map $u \mapsto \gamma \cdot u$ is an automorphism of $\mathrm{Aut}\,\tilde{\mathbf{X}}^{et}(\mathbb{F})$ fixing all elements of sort $\mathbb{F}$.

Claim 4. Let $\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F})$ be the structure $\tilde{\mathbf{X}}^{et}(\mathbb{F})$ with $\{\mathbf{p}_i\}$ named. The definable relation on the sort $\mathbb{F}$ in the structure are exactly those which are definable in $\mathbb{F}_{|\mathrm{k}}$, the field with constants for elements of k.

Proof. By [1], Theorem 7.5, it is enough to prove that the definable relations on $\mathbb{F}$ in $\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F})$ are the same as in $\tilde{\mathbf{X}}^{et}(\mathbb{F})$.

Let $\varphi(\bar{x}, \mathbf{p}_1, \dots, \mathbf{p}_n)$ be a formula in the language $\mathcal{L}_{\mathbb{X}}(\{\mathbf{p}_i\})$ (the language of structure $\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F})$), $\bar{x}$ a tuple of variables of sort $\mathbb{F}$. By Claim 3 there is a formula $\psi_n(p_1, \dots, p_n)$ in language $\mathcal{L}_{\mathbb{X}}$ which is equivalent to a complete type of $\langle \mathbf{p}_1, \dots, \mathbf{p}_n \rangle$ over $\mathbb{F}$. We may assume that

$$\varphi(\bar{x}, p_1, \dots, p_n) \to \psi_n(p_1, \dots, p_n).$$

[3]Use the fact that groups of periods of both $\mathbf{p}_i$ and $\mathbf{p}'_i$ are Δ_i.

Now it is easy to see that in $\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F})$

$$\varphi(\bar{x}, \mathbf{p}_1, \ldots, \mathbf{p}_n) \equiv \exists\, p_1, \ldots, p_n\ \psi_n(p_1, \ldots, p_n)\ \&\ \varphi(\bar{x}, p_1, \ldots, p_n)$$

The formula on the right of $\equiv$ is in the language $\mathcal{L}_{\mathbb{X}}$ and defines the relation $\varphi(\bar{x}, \mathbf{p}_1, \ldots, \mathbf{p}_n)$ in terms of $\tilde{\mathbf{X}}^{et}(\mathbb{F})$. Claim proved.

Claim 5. Let $T(\mathbb{X}) \in \mathcal{T}(\mathbb{X})$ and $\{\mathbf{p}_i\}$ an associated sequence satisfying (3). Any automorphism σ of $\mathbb{F}_{|\mathrm{k}}$ induces a unique automorphism $s(\sigma)$ of $\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F})$.

Proof. First note that σ, being an automorphism of the field $\mathbb{F}$, defines a transformation $\hat{\sigma}$ on algebraic sorts of $\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F})$,

$$\hat{\sigma} : \mathbb{X}_{\mu,\alpha}(\mathbb{F}) \to \mathbb{X}_{\mu,\sigma(\alpha)}.$$

This transformation is an elementary monomorphism of $\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F})$, i.e. it preserves the relation induced on the algebraic sorts in the structure $\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F})$. Indeed, by Claim 4 these relations are just the relations definable in terms of $\mathbb{F}_{|\mathrm{k}}$.

In particular $\hat{\sigma}$ acts on $\mathbb{X}_i(\mathbb{F})$ of $T(\mathbb{X}(\mathbb{F}))$ as an automorphism of $T(\mathbb{X}(\mathbb{F}))$. Now we want to extend the action $\hat{\sigma}$ to the whole of $\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F})$. Note that by (3) the sequence of maps $\mathbf{j_i}$ is definable in $\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F})$, It follows that $\mathbb{U}_i(\mathbb{F}) \subseteq \mathrm{dcl}(\mathbb{X}_i(\mathbb{F}))$ and thus the elementary monomorphism $\hat{\sigma}$ extends uniquely to all the sorts $\mathbb{U}_i(\mathbb{F})$. Now the extension of $\hat{\sigma}$ to $\mathbb{U}(\mathbb{F})$ follows from the fact that $\mathbb{U}(\mathbb{F})$ is the projective limit of the $\mathbb{U}_i(\mathbb{F})$ along $\bar{\mathbf{p}}_i$; each $u \in \mathbb{U}(\mathbb{F})$ is the limit of the sequence $\bar{\mathbf{p}}_i(u) \in \mathbb{U}_i(\mathbb{F})$.

Set $s(\sigma) \coloneqq \hat{\sigma}$. Claim proved.

It follows that s is a homomorphism of $\mathrm{Aut}(\mathbb{F}_{|\mathrm{k}})$ into $\mathrm{Aut}\,\tilde{\mathbf{X}}^{et}_{\{\mathbf{p}_i\}}(\mathbb{F}) \subset \mathrm{Aut}\,\tilde{\mathbf{X}}^{et}(\mathbb{F})$. Thus we have

$$s : \mathrm{Aut}(\mathbb{F}_{|\mathrm{k}}) \to \mathrm{Aut}\,\tilde{\mathbf{X}}^{et}(\mathbb{F})$$

a section associated with $T(\mathbb{X})$. [4]

□

3 Abelian varieties

Let $\mathbb{X}$ be an abelian variety of dimension g over k, (in particular, $\mathbb{X}(\mathrm{k}) \neq \varnothing$) and $\mathrm{J}(\mathbb{X})$ the Jacobi variety of $\mathbb{X}$.

[4] Need also that the tower $\mathrm{GDeck}(\mathbb{X}_l/\mathbb{X})$ has unique presentation in the form Γ/Δ_i.

Our aim here is to construct a class of non-isomorphic cofinal towers $T(\mathbb{X})$ over k.

3.1 For $n \in \mathbb{N}$ and $e \in \mathbb{X}(\mathrm{k})$ define the map

$$[n]_e : \mathbb{X} \to \mathbb{X};\ e + x \mapsto e + n \cdot x.$$

Also fix an element $o \in \mathbb{X}(\mathrm{k})$ and let

$$\mathcal{E}_{\mathbb{X}} = \mathbb{X}(\mathrm{k})^{\mathbb{N}} = \{\{e_i \in \mathbb{X}(\mathrm{k}) : i \in \mathbb{N}\}, e_0 = o\}$$

the set of all sequences of elements of $\mathbb{X}(\mathrm{k})$ beginning with o.

For each $\mathbf{e} = \{e_i\} \in \mathcal{E}_{\mathbb{X}}$ set

$$\mathbb{X}_0 = \mathbb{X},\ \mathbb{X}_i = \mathbb{X} \text{ and } \mathbf{j}_i := [i]_{e_i}; \mathbb{X}_i \to \mathbb{X}_{i-1}, i \in \mathbb{N}.$$

Clearly,

$$\mathrm{GDeck}(\mathbb{X}_i/\mathbb{X}_{i-1}),\ \mathrm{GDeck}(\mathbb{X}_i/\mathbb{X}) \subset \mathrm{J}(\mathbb{X}),$$

$$\mathrm{GDeck}(\mathbb{X}_i/\mathbb{X}_{i-1}) \cong (\mathbb{Z}/i\mathbb{Z})^{2g},\quad \mathrm{GDeck}(\mathbb{X}_i/\mathbb{X}) \cong (\mathbb{Z}/i!\mathbb{Z})^{2g},$$

the $2g$-cartesian powers of cyclic groups of orders i and $i!$ respectively.

It follows,

$$T_{\mathbf{e}}(\mathbb{X}) :\quad \mathbb{X} \leftarrow_{j_{e_1}} \mathbb{X}_1 \leftarrow_{j_{e_2}} \mathbb{X}_2 \leftarrow \dots \mathbb{X}_i \leftarrow_{j_{e_{i+1}}} \mathbb{X}_{i+1} \leftarrow_{j_{e_{i+2}}} \dots$$

is a cofinal Δ_i- tower over k for

$$\Delta_i = i! \cdot \Gamma,\ \text{where } \Gamma = \hat{\pi}_1^{top}(\mathbb{X}(\mathbb{C})).$$

3.2 Lemma. *Suppose* $T_{\mathbf{e}}(\mathbb{X}) \cong T_{\mathbf{e}'}(\mathbb{X})$. *Then* $(e_i - e_i') \in \mathrm{Tors}\,\mathrm{J}(\mathbb{X})$ *for all* $i \in \mathbb{N}$.

Proof. Let $f_i : \mathbb{X}_i \to \mathbb{X}_i'$, $i \in \mathbb{N}$, be the system of isomorphisms which realise the isomorphism $T_{\mathbf{e}}(\mathbb{X}) \cong T_{\mathbf{e}'}(\mathbb{X})$. By definitions, $\mathbb{X}_i = \mathbb{X}_i' = \mathbb{X}$

$$f_{i-1} \circ [i]_{e_i} = [i]_{e_i'} \circ f_i \tag{4}$$

Note that f_i can be seen also as an isomorphism of étale covers $\mathbb{X}_i \to \mathbb{X}_0$ and $\mathbb{X}_i' \to \mathbb{X}_0$ given by compositions $j_1 \circ \dots \circ j_i$ and $j_1' \circ \dots \circ j_i'$, respectively. It follows that f_i has the form $f_i(x) = x + t_i$ for some $t \in \mathrm{Tors}(\mathrm{J}(\mathbb{X}))$.

Applying both sides of (4) to x we get , for $i = 1, 2, \dots$

$$f_{i-1}(i(x - e_i) + e_i) = i \cdot (f_i(x) - e_i') + e_i'.$$

in particular,

$$f_{i-1}(e_i) = i \cdot (f_i(e_i) - e_i') + e_i'$$

and so

$$e_i + t_{i-1} = i \cdot (e_i + t_i - e_i') + e_i'$$

and finally

$$(i-1)(e_i - e_i') = t_{i-1} - i \cdot t_i \in \mathrm{Tors}(\mathrm{J}(\mathbb{X})).$$

It follows $e_i - e_i' \in \mathrm{Tors}(\mathrm{J}(\mathbb{X}))$. □

3.3 Corollary. *Assume that the group of* k*-rational points of* $\mathbb{X}$ *contains non-torsion points. Then there are continuum-many non-isomorphic towers* $T_e(\mathbb{X})$ *and respectively continuum-many non-conjugated sections of the projection* $\pi_1(\mathbb{X}) \to \mathrm{Gal}_\mathrm{k}$.

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