

# A topological $L_{\omega_1, \omega}$ -invariant.

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April 19, 2024

## Abstract

We suggest to look at formal sentences describing complex algebraic varieties together with their universal covers as topological invariants. We prove that for abelian varieties and Shimura varieties this is indeed a complete invariant, i.e. it determines the variety up to complex conjugation.

## 1 Introduction

**1.1** The universal (and more general) cover of a complex algebraic variety as an abstract structure has been a subject of study in model theory in recent years, see e.g. [2], [3], [5], [6]. More specifically, given a smooth complex variety  $\mathbb{X}$  and its cover  $\mathbf{p} : \mathbb{U} \rightarrow \mathbb{X}$  presented in a natural universal language (see 1.2 below), the aim has been to find an  $L_{\omega_1, \omega}$ -sentence  $\Sigma_{\mathbb{X}}$  which axiomatises  $\mathbf{p} : \mathbb{U} \rightarrow \mathbb{X}$  categorically, that is determines the isomorphism type of an uncountable model of  $\Sigma_{\mathbb{X}}$  uniquely in a given cardinality, up to an abstract isomorphism (ignoring topology).

We suggest here that, in cases that the cover is sufficiently non-degenerate, such a  $\Sigma_{\mathbb{X}}$  can serve as a complete topological invariant of respective  $\mathbb{X}$ . More precisely, assuming that in a given class of varieties  $\mathbb{X}$  such a categorical  $L_{\omega_1, \omega}$ -sentence  $\Sigma_{\mathbb{X}}$  exists, then:

*if for a complex variety  $\mathbb{X}'$  its cover structure  $\mathbf{p}' : \mathbb{U}' \rightarrow \mathbb{X}'$  is a model of the same sentence  $\Sigma_{\mathbb{X}}$  then there is a biholomorphic isomorphism either between  $\mathbb{X}$  and  $\mathbb{X}'$  or between  $\mathbb{X}^c$ , the complex conjugate of  $\mathbb{X}$ , and  $\mathbb{X}'$ . (Compare this with interesting speculations in a similar vein presented by M.Gavrilovich in [8].)*

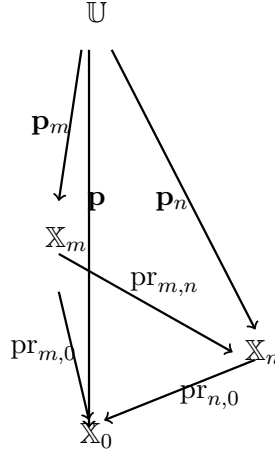
We elaborate the cases of abelian varieties and Shimura varieties.

**1.2** For the general case of cover structure, let  $k \subseteq k^\sharp \subseteq \mathbb{C}$  be a countable subfield of  $\mathbb{C}$  and its algebraic extension invariant under  $\text{Aut}(\mathbb{C}/k)$ . Let  $\{\mathbb{X}_i : i \in I\}$  be a collection of non-singular irreducible complex algebraic varieties (of  $\dim > 0$ ) defined over  $k^\sharp$  and  $I := (I, \geq)$  a lattice with the minimal element 0 determined by unramified  $k^\sharp$ -rational epimorphisms  $\text{pr}_{i',i} : \mathbb{X}_{i'} \rightarrow \mathbb{X}_i$ , for  $i' \geq i$ .  $\mathbb{X}_0 = \mathbb{X}$  defined over  $k$ . It is assumed that the lattice is invariant under  $\text{Aut}(\mathbb{C}/k)$ .

Let  $\mathbb{U}(\mathbb{C})$  be a connected complex manifold and  $\{\mathbf{p}_i : i \in I\}$  a collection of holomorphic covering maps (local bi-holomorphismss)

$$\mathbf{p}_i : \mathbb{U}(\mathbb{C}) \rightarrow \mathbb{X}_i(\mathbb{C}), \quad \text{pr}_{i',i} \circ \mathbf{p}_{i'} = \mathbf{p}_i.$$

as illustrated by the picture:



A model-theoretic treatment of the general case is discussed in [1].

However, in the cases of interest to us we may assume that  $\mathbb{X}_0(k) \neq \emptyset$  (the variety has a  $k$ -rational point) and then the structure can be simplified (bi-interpretible, in an appropriate language) to one with  $(I, \geq)$  a linear order, that is the lattice is reduced to a tower, see [13] for detail.

This multisorted structure we call the **cover structure** of  $\mathbb{X}$  (here  $\mathbb{X} = \mathbb{X}_0$ ) and often write as  $(\mathbf{p} : \mathbb{U} \rightarrow \mathbb{X})$ . The constants and basic relations of  $(\mathbf{p} : \mathbb{U} \rightarrow \mathbb{X})$  are:

- the set of constant symbols  $C$  naming elements of  $k$ , the field of definition of  $\mathbb{X}$ ;
- the Zariski closed relations on sorts  $\mathbb{X}_i$  defined over  $k$ ;
- the rational maps  $\text{pr}_{m,n}$  defined over  $k$ ;

- the maps  $\mathbf{p}_i$ .

**Examples.**

1.  $\mathbb{X} = \mathbb{A}$  an abelian variety. In this case all the  $\mathbb{X}_i$  are equal to  $\mathbb{A}$ ,  $I$  is a subset of  $\mathbb{N}$  linearly ordered by the relation  $n|m$  such that, for each prime power  $p^k$  there is  $m \in I$ ,  $p^k|m$ , and  $\text{pr}_{m,n} : x \mapsto \frac{n}{m}x$ .

2.  $\mathbb{X} = \mathbb{Y}(1) = A^1 \cong \Gamma(1)\backslash\mathbb{H}$  the modular curve,  $\Gamma(1) = \text{SL}_2(\mathbb{Z})$ ,  $k = \mathbb{Q}$ .  $\mathbb{X}_n = \mathbb{Y}(n) \cong \Gamma(n)\backslash\mathbb{H}$  and  $\text{pr}_{m,n}$  are induced by the embeddings  $\Gamma(m) \subseteq \Gamma(n)$ .

In such a cover structure, as noted by M.Gavrilovich [8] and used since multiple times, many more interesting relations and maps are definable. In particular, if  $\mathbb{X}$  is an abelian variety then  $+$  on  $\mathbb{X}$  can be lifted to  $\mathbb{U}$  as a type-definable (and so  $L_{\omega_1,\omega}$ -definable) operation on  $\mathbb{U}$ . In case of  $\mathbb{X}$  a Shimura variety the subgroup  $G(\mathbb{Q})$  of  $G(\mathbb{R})$  is definable along with its action on  $\mathbb{U}$ , for  $G(\mathbb{R})$  the reductive group of the Shimura data for  $\mathbb{X}$ . In general, the topological fundamental group of  $\mathbb{X}$  acting on  $\mathbb{U}$  is definable.

**1.3** For each of the cases of  $\mathbb{X} = \mathbb{X}(\mathbb{C})$  we are interested in an  $L_{\omega_1,\omega}$ -sentence  $\Sigma_{\mathbb{X}}$  which describes the cover structure  $\mathbf{p} : \mathbb{U} \rightarrow \mathbb{X}$ .

We may assume that  $\Sigma_{\mathbb{X}}$  is stronger than the first order theory  $\text{Th}_{\mathbb{X}}$  in the same language. Recall the following folklore fact mentioned e.g. in [12] and proved in detail in [2], Fact A.21.

**1.4 Fact.** Let  $F$  be an algebraically closed field and  $\mathbb{X}(F)$  be  $F$ -points an algebraic variety over  $k \subset F$  considered as the structure with relations corresponding to  $k$ -definable Zariski closed subsets of  $\mathbb{X}^m(F)$ ,  $m \in \mathbb{N}$ .

Then  $\mathbb{X}(F)$  is bi-interpretable with  $(F; +, \cdot, c_a)_{a \in k}$ .

Consequently,

*if  $\mathbb{X}'$  is a variety over  $k' \subset F$  and  $\mathbb{X}'(F)$  is a model of  $\Sigma_{\mathbb{X}}^{f.o.}$ , the first order part of axioms  $\Sigma_{\mathbb{X}}$ , then  $\mathbb{X}'$  is conjugated to  $\mathbb{X}$  by an automorphism  $\sigma$  of field  $F$ , that is  $k' = k^\sigma$  and  $\mathbb{X}^\sigma(F) = \mathbb{X}'(F)$ .*

**1.5** Let  $\mathbb{X}_1$  and  $\mathbb{X}_2$  be two complex algebraic varieties over  $k_1$  and  $k_2$  respectively. We will say that  $\mathbb{X}_1$  and  $\mathbb{X}_2$  have the **same**  $L_{\omega_1,\omega}$ -**type** if there is an uncountably categorical  $L_{\omega_1,\omega}$ -sentence  $\Sigma$  over parameters  $C$  interpreted as elements of  $k_1$  in  $\mathbb{X}_1$  and of  $k_2$  in  $\mathbb{X}_2$  such that the cover structures of both  $\mathbb{X}_1(\mathbb{C})$  and  $\mathbb{X}_2(\mathbb{C})$  are models of  $\Sigma$ .

Our aim is to prove that for certain classes of complex algebraic varieties:

if  $\mathbb{X}_1$  and  $\mathbb{X}_2$  have the same  $L_{\omega_1, \omega}$ -type then

$$\mathbb{X}_1 \cong \mathbb{X}_2 \text{ or } \mathbb{X}_1 \cong \mathbb{X}_2^c$$

where  $\cong$  stands for biholomorphic isomorphism and  $\mathbb{X}^c$  is the variety complex-conjugated to  $\mathbb{X}$ .

This is equivalent to the statement that there is a homeomorphism  $\mathbb{X}_1 \rightarrow \mathbb{X}_2$  which respects the Zariski closed relations.

**1.6 A complementary problem** is also of interest:

Let  $\mathbb{X} = \mathbb{X}(\mathbb{C})$  be a complex algebraic variety and  $\mathbb{X}^\tau$  the variety obtained by applying an (abstract) automorphism  $\tau$  of the field  $\mathbb{C}$  to  $\mathbb{X}(\mathbb{C})$  point-wise. Suppose there exists  $\Sigma_{\mathbb{X}}$ , a categorical  $L_{\omega_1, \omega}$ -sentence for the cover structure of  $\mathbb{X}$ .

Does a categorical  $L_{\omega_1, \omega}$ -sentence for the cover structure of  $\mathbb{X}^\tau$  exist? How one obtains  $\Sigma_{\mathbb{X}^\tau}$  from  $\Sigma_{\mathbb{X}}$ ?

The latter question in application to Shimura varieties is essentially the content of Langlands' conjecture on conjugation of Shimura varieties [10] (without a categoricity statement), proved by M.Borovoi and independently by J.Milne and K.Shih.

The following citation from Langlands's [10] helps to appreciate difficulties in formulating  $\Sigma_{\mathbb{X}}$ , not to mention proving its categoricity:

*The problem of conjugation is formulated in the sixth section as a conjecture, which was arrived at only after a long sequence of revisions. My earlier attempts were all submitted to Rapoport for approval, and found lacking. They were too imprecise, and were not even in principle amenable to proof by Shimuras methods of descent.*

## 2 Abelian varieties

Our references to the standard facts about abelian varieties is [9].

We assume the abelian varieties below to be polarised, that is embedded into a complex projective space.

**2.1** Recall that by [2] for each elliptic non-CM curve  $E$  over a field  $k \subset \mathbb{C}$  there is a categorical  $L_{\omega_1, \omega}$ -sentence  $\Sigma_E$  in the language with parameters  $C$  interpreted as elements of  $k$ , which models the cover structure  $(\mathbf{p} : \mathbb{C} \rightarrow E(\mathbb{C}))$  :

$$(\mathbf{p} : \mathbb{C} \rightarrow E(\mathbb{C})) \models \Sigma_E.$$

The same is true for the general case of abelian varieties  $\mathbb{A}$  by [3]. The sentence  $\Sigma_{\mathbb{A}}$  therein is in the language naming generators  $\bar{\lambda}$  of  $\Lambda$ , so is  $\Sigma_{\mathbb{A}}(\bar{\lambda})$  in fact. But these extra constant symbols can be eliminated by taking instead

$$\Sigma_{\mathbb{A}}^{\sharp} := \exists \bar{\lambda} \Sigma_{\mathbb{A}}(\bar{\lambda}) \ \& \ \text{qftp}(\mathbf{p}(\mathbb{Q} \otimes \Lambda)).$$

**Remark.** Unlike the general case  $\Sigma_{\mathbb{E}}$  for an elliptic curve in [2] is described in quite precise terms, which is important when listing concrete topological invariants as they expressed by the formula.

**2.2 Theorem.** *Let  $\mathbb{A}_a$  and  $\mathbb{A}_b$  be two complex abelian varieties.*

*Suppose  $\mathbb{A}_a$  and  $\mathbb{A}_b$  have the same  $L_{\omega_1, \omega}$ -type. Then there is a biholomorphic isomorphism*

$$\varphi : \mathbb{A}_a \rightarrow \mathbb{A}_b, \text{ or } \varphi : \mathbb{A}_a \rightarrow \mathbb{A}_b^c$$

where  $\mathbb{A}_b^c$  is the abelian variety obtained by complex conjugation from  $\mathbb{A}_b$ .

**Proof.** We assume that  $\mathbb{A}_a$  is over  $k_a$  and  $\mathbb{A}_b$  over  $k_b$ , subfields of  $\mathbb{C}$ . By categoricity and the Fact above there is an abstract isomorphism  $\phi$  between the cover structures, which include the isomorphism

$$\phi : k_a \rightarrow k_b; \quad \mathbb{F} \rightarrow \mathbb{F},$$

and, because the isomorphism type of the respective lattice  $\Lambda$  is definable, the isomorphism

$$\phi : \Lambda_a \otimes \mathbb{Q} \rightarrow \Lambda_b \otimes \mathbb{Q}$$

We continue via Lemmata below.

For simplicity of notation we write  $\mathbb{A}$ ,  $\Lambda$ ,  $\mathbb{U}$  and  $\mathbf{p}$  instead of  $\mathbb{A}_a$ ,  $\Lambda_a$ ,  $\mathbb{U}_a$  and  $\mathbf{p}_a$  and  $\mathbb{A}_b$ ,  $\Lambda_b$ ,  $\mathbb{U}_b$  and  $\mathbf{p}_b$  when it does not lead to confusion.

**2.3** Note that there is a metric on  $\Lambda \otimes \mathbb{Q}$  induced from the standard metric on the field  $\mathbb{Q}$ . Any such metric is equivalent to the metric induced from the complex manifold  $\mathbb{U}(\mathbb{C})$ .

**2.4 Lemma.** *The restriction  $\phi_{\Lambda} : \Lambda_a \rightarrow \Lambda_b$  can be uniquely extended to the topological homeomorphism*

$$\psi : \Lambda_a \otimes \mathbb{R} \rightarrow \Lambda_b \otimes \mathbb{R} \tag{1}$$

which coincides with  $\phi$  on  $\Lambda \otimes \mathbb{Q}$ .

Proof. Immediate from the remark 2.3.  $\square$

Set

$$A^Q := \mathbf{p}(\Lambda \otimes \mathbb{Q}) \subset \mathbb{A}.$$

**2.5 Lemma.**

(A) *The additive structure on  $\Lambda \otimes \mathbb{Q}$  together with a definable point  $0 \in \mathbb{A}$  defines an abelian group structure on  $A^Q \subset \mathbb{A}(\mathbb{C}) \subset \mathbf{P}^N(\mathbb{C})$  which is dense in  $\mathbb{A}(\mathbb{C})$ .*

(B) *The map  $\psi$  can be uniquely extended to a continuous isomorphism of groups in the metric of the ambient space  $\mathbf{P}^N(\mathbb{C})$ ,*

$$\psi : \mathbb{A}_a(\mathbb{C}) \rightarrow \mathbb{A}_b(\mathbb{C}) \tag{2}$$

**Proof.** Immediate from definitions of torus  $\mathbb{A}(\mathbb{C}) = \mathbb{U}/\Lambda$  and  $\Lambda$  is discrete closed in the metric topology of  $\mathbb{C}$ .  $\square$

**2.6 Lemma.** *Let  $A^0$  be a Zariski open  $k$ -definable affine subset of  $\mathbb{A}$  and  $f = (f_1, \dots, f_N) : A^0 \rightarrow \mathbb{C}^N$ , a Zariski embedding by non-constant coordinate functions. The homeomorphism  $\psi$  can be extended to a homeomorphism  $\psi_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$  which coincides with  $\phi$  on the dense subset  $\mathbb{C}^Q := \bigcup_{i=1}^N f_i(A^Q)$ , and satisfies*

$$\psi_{\mathbb{C}} \circ f_i^a = f_i^b \circ \psi, \quad i = 1, \dots, N. \tag{3}$$

**Proof** Let

$$\text{graph}(f) \subset A^0 \times \mathbb{C}^N$$

be the graph of the map.

Note that  $\text{graph}(f) \cap (A^Q \times \mathbb{C}^N)$  is dense in  $\text{graph } f$  because  $f$  is continuous.

Let  $f^a$  and  $f^b$  be the map  $f$  in respective structures.

Now we claim that the map  $\psi : \text{graph}(f^a) \rightarrow \text{graph}(f^b)$  is well-defined by continuous extension from the isomorphism-map

$$\phi : (q, f^a(q)) \mapsto (\phi(q), f^b(\phi(q))), \quad \text{for } q \in A^Q.$$

to

$$\psi : (x, f^a(x)) \mapsto (\psi(x), f^b(\psi(x))), \quad \text{for } x \in A.$$

Indeed, the value on the right is determined by  $\psi(x)$  which is well-defined once  $x$  is given, by 2.4.

Define

$$\psi_{\mathbb{C}}(f^a(x)) := f^b(\psi(x)).$$

This is well-defined since  $f^a$  is bijective.

Thus we have

$$\psi_{\mathbb{C}} \circ f^a = f^b \circ \psi \quad (4)$$

The above implies (3).

Note that for each  $i$ , the co-ordinate map is non-constant continuous,  $f_i(A^0)$  is dense in  $\mathbb{C}$  and hence  $f_i(A^Q \cap A^0)$  is dense in  $\mathbb{C}$ .

□

**2.7 Lemma.** *The continuous map  $\psi_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$  as in 2.6 is an automorphism the field  $(\mathbb{C}; +, \cdot)$ .*

**Proof.** Let  $S \subset \mathbb{C}^3$  and  $P \subset \mathbb{C}^3$  be graphs of  $+$  and  $\times$  on  $\mathbb{C}$  respectively. Let

$$S^Q := \{(x, y, x+y) : x, y \in f_1(A^Q \cap A^0)\}; \quad P^Q := \{(x, y, x \cdot y) : x, y \in f_1(A^Q \cap A^0)\}.$$

$S^Q$  and  $P^Q$  are dense in  $S$  and  $P$  respectively since  $f_1(A^Q)$  is dense in  $\mathbb{C}$  and both  $+$  and  $\times$  are continuous surjections  $\mathbb{C}^2 \rightarrow \mathbb{C}$ .

We have  $\phi : S^Q \rightarrow S^Q$  and  $\phi : P^Q \rightarrow P^Q$ , bijections, since  $\phi$  is an isomorphism of fields. Hence  $\psi_{\mathbb{C}}(S) = S$  and  $\psi_{\mathbb{C}}(P) = P$  which means that  $\psi_{\mathbb{C}}$  preserves the field structure. □

### 2.8 Corollary.

1.  $\psi_{\mathbb{C}}$  is either the identity or a complex conjugation  $z \mapsto z^c$  on  $\mathbb{C}$ .
2.  $\psi$  induces an isomorphism of coordinate rings  $\mathbb{C}[A_a^0] \rightarrow \mathbb{C}[A_b^0]$  for each Zariski open  $A^0 \subset \mathbb{A}$  and thus bijection (2)  $\psi : \mathbb{A}_a \rightarrow \mathbb{A}_b$  in local coordinates is of the form

$$\langle z_1, \dots, z_N \rangle \mapsto \langle z'_1, \dots, z'_N \rangle$$

where  $z'_i = \psi_{\mathbb{C}}(z_i)$  is either identity or complex conjugation.

This completes the proof of Theorem 2.2. □

## 3 Shimura varieties

Let  $\mathbb{X}_a, \mathbb{X}_b$  be connected Shimura varieties (see [11], Def.4.4) over their reflex fields  $k_a, k_b$  given by the Shimura datum  $(G_a, \mathcal{H}_a), (G_b, \mathcal{H}_b)$  and congruence subgroups  $\Gamma_a \subset G_a(\mathbb{Q}), \Gamma_b \subset G_b(\mathbb{Q})$ , respectively.

$$\mathbb{X}_a = \Gamma_a \backslash \mathcal{H}_a \text{ and } \mathbb{X}_b = \Gamma_b \backslash \mathcal{H}_b$$

with respective covering maps

$$j_a : \mathcal{H}_a \rightarrow \mathbb{X}_a \text{ and } j_b : \mathcal{H}_b \rightarrow \mathbb{X}_b.$$

A Shimura datum  $(G, \mathcal{H})$ , also associates with each point  $h \in \mathcal{H}_a$  a homomorphism  $h : U(1) \rightarrow G_{\mathbb{R}}^{\text{ad}}$  (where  $G^{\text{ad}}(\mathbb{R}) = G(\mathbb{R})/Z$ ,  $Z$  subgroup acting trivially) from the 1-dimensional real torus  $U(1)$ , and all the homomorphisms are linked by conjugation by elements of  $G^{\text{ad}}(\mathbb{R})^+$ . The image  $h(U(1)) = T_h$  is a torus subgroup of  $G_{\mathbb{R}}^{\text{ad}}$  which can be identified as a subgroup fixing the point  $h$ . Since there is only one non-identity continuous automorphism of  $U(1)$ ,  $u \mapsto u^{-1}$ , there are exactly two  $G_{\mathbb{R}}^{\text{ad}}$ -conjugacy classes of homomorphisms  $h : U(1) \rightarrow G_{\mathbb{R}}^{\text{ad}}$  which can be called  $\mathcal{H}^+$  and  $\mathcal{H}^-$ .

It follows from Shimura theory, see [4], that  $\mathbb{X}$ , a complex Shimura variety and  $\mathbb{X}^c$  its complex conjugate,

$$\mathbb{X} = \Gamma \backslash \mathcal{H}^+ \Leftrightarrow \mathbb{X}^c = \Gamma \backslash \mathcal{H}^- \quad (5)$$

and the action of complex conjugation corresponds to the automorphism  $u \rightarrow u^{-1}$  on  $U(1)$ .

Below we use the notation  $\mathcal{H}_a$  and  $\mathcal{H}_b$  for sets and  $\mathcal{H}_a^{\pm}$  and  $\mathcal{H}_b^{\pm}$  for their presentations as classes of homomorphisms from  $U(1)$ .

**Fact** (Th 5.4,[11]). Let  $G$  be a connected algebraic group over  $\mathbb{Q}$ . Then  $G(\mathbb{Q})$  is dense in  $G(\mathbb{R})$ .

**Theorem.** *Suppose there is a categorical  $L_{\omega_a, \omega}$ -sentence  $\Sigma$  over parameters  $C$  such that both structures  $(G_a, \mathcal{H}_a, \Gamma_a, \mathbb{X}_a, j_a)$  and  $(G_b, \mathcal{H}_b, \Gamma_b, \mathbb{X}_b, j_b)$  are models of  $\Sigma$  when one interprets  $C$  as  $k_a$  and  $k_b$  respectively. Then there is a biholomorphic isomorphism*

$$\varphi : \mathbb{X}_a(\mathbb{C}) \rightarrow \mathbb{X}_b(\mathbb{C}), \text{ or } \varphi : \mathbb{X}_a(\mathbb{C}) \rightarrow \mathbb{X}_b^c(\mathbb{C}). \quad (6)$$

**Proof.** First we note that it follows from the categoricity assumption and the Fact of section 2 that there exists an abstract isomorphism  $\phi$  between the two structures which is realised as isomorphisms

$$\phi : k_a \rightarrow k_b \quad (7)$$

$$\phi : \mathbb{X}_a \rightarrow \mathbb{X}_b \quad (8)$$

$$\phi : \mathcal{H}_a \rightarrow \mathcal{H}_b \quad (9)$$

$$\phi : G_a(\mathbb{Q}) \rightarrow G_b(\mathbb{Q}); \quad (10)$$



$$\phi : \Gamma_a \rightarrow \Gamma_b \quad (11)$$

$$\forall g \in G_a(\mathbb{Q}), \forall x \in \mathcal{H}_a \quad \phi(g \cdot x) = \phi(g) \cdot \phi(x) \quad (12)$$

and

$$j_b \circ \phi = \phi \circ j_a. \quad (13)$$

Since  $G_a(\mathbb{Q})$  is dense in  $G_a(\mathbb{R})$  and  $G_b(\mathbb{Q})$  in  $G_b(\mathbb{R})$  there is a unique extension of  $\phi$  to the continuous isomorphism

$$\phi_{\mathbb{R}} : G_a(\mathbb{R}) \rightarrow G_b(\mathbb{R}), \quad G_a^{\text{ad}}(\mathbb{R}) \rightarrow G_b^{\text{ad}}(\mathbb{R})$$

together with the isomorphism of the action (12) since it is continuous both in  $g$  and  $x$  :

$$\forall g \in G_a(\mathbb{R}), \forall x \in \mathcal{H}_a \quad \phi_{\mathbb{R}}(g \cdot x) = \phi_{\mathbb{R}}(g) \cdot \phi_{\mathbb{R}}(x)$$

Recall that  $h(U(1))$  is a 1-dimensional real subgroup in  $G(\mathbb{R})$  which fixes a unique element  $x_h \in \mathcal{H}$ .

Thus,  $\phi$  induces

$$\phi_{\pm} : \{h : U(1) \rightarrow G_a^{\text{ad}}(\mathbb{R})\} \rightarrow \{\phi_{\pm}(h) : U(1) \rightarrow G_b^{\text{ad}}(\mathbb{R})\}$$

Since  $h^{-1} \circ \phi_{\mathbb{R}}^{-1} \circ \phi_{\pm}(h) : U(1) \rightarrow U(1)$  is either identity or  $u \rightarrow u^{-1}$ , there are two possibilities:

$$\phi_{\pm} : \mathcal{H}_a^+ \rightarrow \mathcal{H}_b^+$$

and

$$\phi_{\pm} : \mathcal{H}_a^- \rightarrow \mathcal{H}_b^+$$

Respectively, by fact (5), we have two biholomorphic isomorphisms (6).  $\square$

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